Contributions to stochastic control with nonlinear expectations and backward stochastic differential equations

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DAUPHINE UNIVERSITÉ PARIS | CEREMADE UMR CNRS 7534

ÉCOLE DOCTORALE DE DAUPHINE

THÈSE DE DOCTORAT

pour obtenir le grade de

Docteur en sciences de l'université Paris-Dauphine

présentée par

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Contributions au contrôle stochastique avec des espérances non linéaires et aux équations stochastiques rétrogrades

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Soutenue le 28 Septembre 2015 devant le jury composé de MM. :

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Remerciements

J'ai eu la grande chance (tout au long de ma thèse) d'être soutenue par des chercheurs de renommée internationale. J'ai donc toutes les raisons pour me considérer privilegiée. Maintenant j'ai une belle occasion d'adresser un grand "Merci " aux personnes qui ont contribué à ma formation professionnelle et humaine dans l'étape doctorale.

Je tiens à remercier Bruno Bouchard, parce qu'il a dirigé mes activités de recherche dans le cadre du groupe de travail dans la seconde année de master et (ensuite) ma thèse. Il m'a profondément influencée par sa personnalité scientifique et par sa manière exigeante et amicale en même temps de travailler avec les jeunes chercheurs. Ses nombreux conseils en mathématiques, toujours très judicieux, ses encouragements, la confiance qu'il m'a manifestée et sa constante gentillesse m'ont donné l'envie de me perfectionner, de progresser chaque jour. Je tiens également à le remercier de m'avoir offert la très honorante opportunité de donner les travaux dirigés pour son cours de M2 (finance mathématique du Master Masef), ainsi que de m'avoir impliquée dans l'organisation du Séminaire Bachelier.

Je remercie également Romuald Elie pour son investissement dans mon travail, pour le temps qu'il m'a accordé avec grande générosité. Il a contribué pleinement, par ses idées toujours stimulantes, au developpement de mon goût et de mes connaissances en finance mathématique.

Une très profonde gratitude se dirige vers Agnès Sulem que j'ai eu la grande chance de rencontrer comme Professeur pendant le M2. Elle a accepté de diriger mon stage de recherche (du master 2) et, ulterieurement, elle m'a adressé une proposition inesperée : celle de poursuivre des travaux de recherche dans l'équipe Mathrisk, ce qui fut pour moi une réelle opportunité. En tant que directrice de Mathrisk, elle a suivi très attentivement mon évolution professionnelle, elle m'a proposé des sujets variés (sur le plan théorique et numérique) et m'a offert la très honorante possibilité de travailler avec elle et d'intéragir avec d'autres chercheurs. Je la remercie pour son inconditionnelle disponibilité à mon égard et pour toute sa confiance: délicate, très chaleureuse et ferme en même temps, Agnès Sulem a toujours su me donner les meilleurs conseils afin que je puisse m'intégrer dans le monde (si competitif) de la recheche francaise en mathématiques financières.

A son tour, Marie-Claire Quenez a toujours été près de moi, en me soutenant avec un enthousiame qui m'a profondément touchée. Le travail avec Marie-Claire Quenez m'a beaucoup apporté du point de vue scientifique: ses connaissances, son esprit pédagogique et sa rigueur m'ont influencé d'une manière substantielle dans ma formation de jeune mathématicien. C'est un réel plaisir et une grande chance de travailler avec elle. Je lui serai toujours reconnaissante pour sa gentillesse, sa confiance et pour tout le temps qu'elle m'a accordé sans jamais le compter.

Pendant ma thèse, j'ai aussi eu la chance de travailler avec des chercheurs plus jeunes: je tiens les remercier pour leur soutien. Je pense en particulier à Céline Labart: la collaboration avec elle a été réellement très enrichissante. Je remercie aussi Dylan Possamai d'avoir accepté de travailler avec moi.

J'adresse mes plus sincères remerciements à Nizar Touzi qui m'a fait l'honneur d'être examinateur dans mon jury de thèse, ainsi qu'à Dirk Becherer, Philippe Briand et Rainer Buckdahn, pour tout le temps investi à rapporter ma thèse et pour leurs précieux conseils.

Je ne peux pas oublier d'autres personnes qui ont représenté des repères importants dans mon activité, leur permanente disponiblité et leur constant appui étant très importants pour moi:

Massimiliano Gubinelli (directeur du programme doctoral en mathématiques), Jean-Yves Chemin et Jean Dolbeault (directeurs de la Fondation de Sciences Mathématiques), Vlad Bally (professeur à Marne-la-Vallée et membre de l'équipe Mathrisk) et Gabriel Turinici (professeur à Dauphine).

Je suis aussi très reconnaissante envers Dan Crisan et Mete Soner de m'avoir accuellie à l'Imperial College et ETH Zurich, ce qui m'a permis de gagner une riche experience.

Je tiens à remercier très chaleureusement l'équipe de la Fondation de Sciences Mathématiques, avec laquelle j'ai eu une très belle collaboration dès ma premiére année de master et également à Juliette Bavard et Camille Male pour leur soutien et implication dans la création de l'ALUMNI de la FSMP.

Mes meilleures pensées se dirigent aussi vers Adélaide Olivier, avec laquelle j'ai organisé le Séminaire des Jeunes Chercheurs du Ceremade et j'ai partagé de nombreux beaux moments. Evidemment, je ne peux pas oublier les autres doctorants qui one été proches de moi pendant ces trois années de thèses, pour leur chaleureuse amitié (je pense en particulier à Kaniav, Yiyi, Nicolas, Viviana, Géraldine, Ahmed, Fang, Jean-Luc ...).

Je n'aurais pas pu arriver à ce point sans la contribution essentielle de mes professeurs de la Faculté de Mathématiques de l'Université “Al. I. Cuza”, Iasi (Roumanie) et je les remercie pour toute leur confiance. Ma plus sincère reconaissance se dirige vers l'Académicien Viorel Barbu, qui a dirigé mon travail de recherche pendant la licence et qui m'a toujours encouragée de continuer avec une carrière de haut niveau en mathématiques.

Je ne peux pas finir ces remerciements sans penser à ma famille, qui a toujours cru en moi et m'a toujours offert un amour inconditionnel.

Résumé

Cette thèse se compose de deux parties indépendantes qui portent sur le contrôle stochastique avec des espérances non linéaires et les équations stochastiques rétrogrades (EDSR), ainsi que sur les méthodes numériques de résolution de ces équations.
Dans la première partie on étudie une nouvelle classe d'équations stochastiques rétrogrades, dont la particularité est que la condition terminale n'est pas fixée mais vérifie une contrainte non linéaire exprimée en termes de "f-espérances". Ce nouvel objet mathématique est étroitement lié aux problèmes de couverture approchée des options européennes où le risque de perte est quantifié en termes de mesures de risque dynamiques, induites par la solution d'une EDSR non linéaire. Dans le chapitre suivant on s'intéresse aux problèmes d'arrêt optimal pour les mesures de risque dynamiques avec sauts. Plus précisément, on caractérise dans un cadre markovien la mesure de risque minimale associée à une position financière comme l'unique solution de viscosité d'un problème d'obstacle pour une équation intégro-différentielle. Dans le troisième chapitre, on établit un principe de programmation dynamique faible pour un problème mixte de contrôle stochastique et d'arrêt optimal avec des espérances non linéaires, qui est utilisé pour obtenir les EDP associées. La spécificité de ce travail réside dans le fait que la fonction de gain terminal ne satisfait aucune condition de régularité (elle est seulement considérée mesurable), ce qui n'a pas été le cas dans la littérature précédente. Dans le chapitre suivant, on introduit un nouveau problème de jeux stochastiques, qui peut être vu comme un jeu de Dynkin généralisé (avec des espérances non linéaires). On montre que ce jeu admet une fonction valeur et on obtient des conditions suffisantes pour l'existence d'un point selle. On prouve que la fonction valeur correspond à l'unique solution d'une équation stochastique rétrograde doublement réfléchie avec un générateur non linéaire général. Cette caractérisation permet d'obtenir de nouveaux résultats sur les EDSR doublement réfléchies avec sauts. Le problème de jeu de Dynkin généralisé est ensuite étudié dans un cadre markovien.
Dans la deuxième partie, on s'intéresse aux méthodes numériques pour les équations stochastiques rétrogrades doublement réfléchies avec sauts et barrières irrégulières, admettant des sauts prévisibles et totalement inaccessibles. Dans un premier chapitre, on propose un schéma numérique qui repose sur la métode de pénalisation et l'approximation de la solution d'une EDSR par une suite d'EDSR discrètes dirigées par deux arbres binomiaux indépendants (un qui approxime le mouvement brownien et l'autre le processus de Poisson composé). Dans le deuxième chapitre, on construit un schéma en discrétisant directement l'équation stochastique rétrograde doublement réfléchie, schéma qui présente l'avantage de ne plus dépendre du paramètre de pénalisation. On prouve la convergence des deux schémas numériques et on illustre avec des exemples numériques les résultats théoriques.

Abstract

This thesis consists of two independent parts which deal with stochastic control with nonlinear expectations and backward stochastic differential equations (BSDE), as well as with the numerical methods for solving these equations.
We begin the first part by introducing and studying a new class of backward stochastic differential equations, whose characteristic is that the terminal condition is not fixed, but only satisfies a

nonlinear constraint expressed in terms of "f - expectations". This new mathematical object is closely related to the approximative hedging of an European option, when the shortfall risk is quantified in terms of dynamic risk measures, induced by the solution of a nonlinear BSDE. In the next chapter we study an optimal stopping problem for dynamic risk measures with jumps. More precisely, we characterize in a Markovian framework the minimal risk measure associated to a financial position as the unique viscosity solution of an obstacle problem for partial integro-differential equations. In the third chapter, we establish a weak dynamic programming principle for a mixed stochastic control problem / optimal stopping with nonlinear expectations, which is used to derive the associated PDE. The specificity of this work consists in the fact that the terminal reward does not satisfy any regularity condition (it is considered only measurable), which was not the case in the previous literature. In the next chapter, we introduce a new game problem, which can be seen as a generalized Dynkin game (with nonlinear expectations). We show that this game admits a value function and establish sufficient conditions ensuring the existence of a saddle point . We prove that the value function corresponds to the unique solution of a doubly reflected backward stochastic equation (DRBSDE) with a nonlinear general driver. This characterisation allows us to obtain new results on DRBSDEs with jumps. The generalized Dynkin game is finally addressed in a Markovian framework.

In the second part, we are interested in numerical methods for doubly reflected BSDEs with jumps and irregular barriers, admitting both predictable and totally inaccesibles jumps. In the first chapter we provide a numerical scheme based on the penalisation method and the approximation of the solution of a BSDE by a sequence of discrete BSDEs driven by two independent random walks (one approximates the Brownian motion and the other one the compensated Poisson process). In the second chapter, we construct an alternative scheme based on the direct discretisation of the DRBSDE, scheme which presents the advantage of not depending anymore on the penalisation parameter. We prove the convergence of the two schemes and illustrate the theoretical results with some numerical examples.

Contents

Chapter 1

General Introduction

1.1 BSDEs with nonlinear weak terminal condition

This chapter is based on a paper written under the coordination of B. Bouchard [1] and submitted for publication: "BSDEs with nonlinear weak terminal condition" [57].

1.1.1 Preliminaries and overview of the literature

We start by recalling that a Backward Stochastic Differential Equation (in short BSDE) is an equation which takes the following form

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \ \ 0 \leq t \leq T, \tag{1.1.1}$$

where $\{W_t\}_{0\leq t\leq T}$ is a Brownian motion defined on a probability space endowed with the natural complete filtration denoted by $\{\mathcal{F}_t\}_{0\leq t\leq T}$. The data of a such equation are given by the terminal condition ξ, which is a random variable $\mathcal{F}_T$-measurable, valued in $\mathbf{R}$ and a driver g, a random map defined on $[0,T] \times \Omega \times \mathbf{R} \times \mathbf{R}$ and valued in $\mathbf{R}$, which is measurable with respect to the σ-algebras $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$ and $\mathcal{B}(\mathbf{R})$, where $\mathcal{P}$ represents the predictable σ-algebra. To solve this equation means to find a couple of processes $\{(Y_t, Z_t)\}_{0\leq t\leq T}$ satisfying equation (1.1.1) and adapted with respect to the filtration generated by the Brownian motion. We give below a more precise definition.

Definition 1.1.1. *The solution of a BSDE is a couple of processes (Y, Z) valued in $\mathbf{R}\times\mathbf{R}$ such that Y is continuous and adapted, Z is predictable and $\mathbb{P}$-a.s, $t \mapsto Z_t$ belongs to $\mathbf{L}_2(0,T)$, $t \mapsto g(t, Y_t, Z_t)$ belongs to $\mathbf{L}_1(0,T)$ and*

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \ \ 0 \leq t \leq T. \tag{1.1.2}$$

The BSDEs have been first introduced in the case of a linear driver by J.-M.Bismut [24]. The starting point of the theory of nonlinear backward equations is the paper of E. Pardoux and S. Peng [124], since the authors consider BSDEs with nonlinear generator in (y, z). Let us recall this result:

[1] I also express my gratitude to Romuald Elie for all the challenging and very useful discussions he has initiated.

Theorem 1.1.2 (E.Pardoux - S.Peng)**.** *Suppose that the driver g is lipschitz in (y, z), uniformly with respect to (t, ω) and*

$$\mathbb{E}[|\xi|^2 + \int_0^T |g(s, 0, 0)|^2 ds] < +\infty.$$

Then the BSDE (1.1.1) *admits an unique solution (Y, Z) belonging to $\mathbf{S}_2 \times \mathbf{H}_2$.*

Expressed in a "forward form", the resolution of such equations boils down in finding an initial condition Y_0 and a process Z such that the controlled process $(Y_t^{Y_0,Z})_{0 \leq t \leq T}$ satisfies the SDE:

$$Y_t^{Y_0,Z} = Y_0 - \int_0^t g(s, Y_s^{Y_0,Z}, Z_s)ds + \int_0^t Z_s dW_s \tag{1.1.3}$$

and the condition $Y_T^{Y_0,Z} = \xi$ at the terminal time T. From a financial application point of view, the study of these equations is related to the pricing of European options in complete markets, since Y gives the price and Z provides the associated hedging strategy. However, since in incomplete markets it is not always possible to construct a replicating portfolio such that its terminal value coincides with the price of the claim ξ, a weaker formulation is to find an initial condition Y_0 and a control Z such that

$$Y_T^{Y_0,Z} \geq \xi. \tag{1.1.4}$$

In this case, one is interested in finding the minimal initial condition Y_0, which corresponds to the cost of the cheapest super-replication strategy for the contingent claim ξ and the associated control Z (see e.g. [72]).
Since in most cases, the super-hedging price leads to an unbearble cost for the buyer, which is not reasonable in practice, it was suggested to relax the strong constraint (1.1.4) into a weaker one of the form

$$\mathbb{E}[l(Y_T^{Y_0,Z} - \xi)] \geq m, \tag{1.1.5}$$

where m is a given threshold and l is a non-decreasing map.

For $l(x) = \mathbf{1}_{\{x \geq 0\}}$, this corresponds to matching the criteria $Y_T^{Y_0,Z} \geq \xi$ at least with probability m and corresponds to the quantile hedging problem introduced by Föllmer and Leukert [84]. Then, this problem has been studied by Bouchard, Elie and Touzi [32] in a Markovian framework, using the stochastic target techniques developed by Soner and Touzi (see [142]). This approach, based on the primal formulation of the value function and the geometric dynamic programming, allows one for a treatement of this problem in a more general framework, e.g. when the strategy of the agent may influence the value of the risky assets (large investor model). The original treatment of the problem by Föllmer and Leukert relies on the fact that this strategy is linear in the control.

More generally, l may represent a loss function, a classical example being $l(x) := -(x^-)^q$ with $q \geq 1$, see [85] for general non-Markovian but linear dynamics. Another example in financial mathematics could be represented by the case when l plays the role of an utility function.
Very recently, Bouchard, Elie and Reveillac [31] have addressed this problem in a nonlinear non-Markovian setting and to this purpose they introduce a new class of BSDEs whose terminal condition is not fixed as a random variable, but only satisfies the following weak constraint

$$\mathbb{E}[\Psi(Y_T^{Y_0,Z})] \geq m. \tag{1.1.6}$$

The problem can be thus formulated as follows:

$$\text{Find the minimal } Y_0 \text{ such that (1.1.3) and (1.1.6) hold for some } Z. \tag{1.1.7}$$

The key idea is to "transpose" problem (1.1.7) written in terms of BSDEs with weak terminal condition into an equivalent one, expressed as an optimization problem on the solutions of a family of BSDEs with strong terminal conditions, indexed by an additional control α, as we shall explain in the sequel. In order to do it, the authors appeal to the martingale representation theorem. More precisely, if Y_0 and Z are such that (1.1.6) holds, then the martingale Theorem implies that we can find an element α in the set $\mathbf{A}_0$ of predictable square integrable processes, such that

$$\Psi(Y_T^{Y_0,Z}) \geq M_T^{m,\alpha} := m + \int_0^T \alpha_s dW_s. \tag{1.1.8}$$

Since Ψ is non-decreasing, one can define its left-continuous inverse Φ and we get that the solution (Y^α, Z^α) of the following BSDE

$$Y_t^\alpha = \Phi(M_T^{m,\alpha}) + \int_t^T g(s, Y_s^\alpha, Z_s^\alpha)ds - \int_t^T Z_s^\alpha dW_s,\ 0 \leq t \leq T, \tag{1.1.9}$$

solves (1.1.3) and (1.1.6). It is finally proved that the solution of (1.1.7) is given by

$$\mathcal{Y}_0(m) := \inf\{Y_0^\alpha, \alpha \in \mathbf{A}_0\}. \tag{1.1.10}$$

We would like to point out that in a Markovian setting, it is used the same idea of introducing an additional process M and control α and the difficulty relies on the fact that α can take unbounded values, since it comes from a martingale representation Theorem.
Now, in order to study (1.1.10), the authors make the problem dynamic and define

$$\mathcal{Y}^\alpha(\tau) := \operatorname{essinf}\{Y_\tau^{\alpha'},\ \alpha' \in \mathbf{A}_0 \text{ s.t. } \alpha' = \alpha \text{ on } [\![0,\tau]\!]\},\ \ 0 \leq \tau \leq T. \tag{1.1.11}$$

It is shown that the family $\{\mathcal{Y}^\alpha,\ \alpha \in \mathbf{A}_0\}$ satisfies a dynamic programming principle, which can be seen as a counterpart of the geometric dynamic programming principle. It is then provided a representation of the family $\{\mathcal{Y}^\alpha,\ \alpha \in \mathbf{A}_0\}$ in terms of minimal supersolutions to a family of BSDEs with driver g and (strong) terminal conditions $\{\Phi(M_T^{m,\alpha}), \alpha \in \mathbf{A}_0\}$, as well as the existence of an optimal control in the case when g and Φ are convex. Some main properties of the value function given by (1.1.10), as continuity and convexity with respect to the threshold m are obtained. Finally, by using only probabilistic arguments, it is shown that Problem (1.1.10) admits a dual representation which takes the form of a stochastic control problem in Meyer form, extending the results obtained in the case when the driver g is linear (see [32], [84] and [85]).

1.1.2 Contributions

In this Chapter, we introduce a more general class of BSDEs than the one considered by Bouchard, Elie and Reveillac [31], whose terminal condition satisfies the following *nonlinear weak constraint*:

$$\mathcal{E}_{0,T}^f[\Psi(Y_T^{Y_0,Z})] \geq m, \tag{1.1.12}$$

where $\mathcal{E}^f[\xi]$ is the nonlinear operator which gives the solution of the BSDE associated to the terminal condition ξ and the *nonlinear driver* f. We can easily remark that (1.1.6) represents a particular case of (1.1.12) for $f = 0$. The problem under study in this paper is the following:

$$\inf\{Y_0 \text{ such that } \exists Z : \text{ (1.1.3) and (1.1.12) hold}\}. \tag{1.1.13}$$

Following the key idea of [31], we rewrite our problem (1.1.13) into an equivalent one expressed in terms of a family of BSDEs with strong terminal condition. The main difference with respect to [31] is given by the fact that in our case we have to introduce a new controlled diffusion process, which is an $f-$*martingale*, contrary to [31] where it is a classical martingale. Indeed, for a given Y_0 and Z such that (1.1.3) and (1.1.12) are satisfied, using the *BSDE representation* of $\Psi(Y_T^{Y_0,Z})$, we can find $\alpha \in \mathbf{A}_0$ such that:

$$\Psi(Y_T^{Y_0,Z}) \geq \mathcal{M}_T^{m,\alpha} = m - \int_0^T f(s, \mathcal{M}_s^{m,\alpha}, \alpha_s)ds + \int_0^T \alpha_s dW_s. \tag{1.1.14}$$

Thanks to this observation, we show that Problem (1.1.13) is equivalent to (1.1.10), where, in our more general framework, Y_t^α corresponds to the solution at time t of the BSDE with (strong) terminal condition $\Phi(\mathcal{M}_T^{m,\alpha})$. We study the dynamical counterpart of (1.1.10):

$$\mathcal{Y}^\alpha(\tau) := \operatorname{essinf}\{Y_\tau^{\alpha'}, \alpha' \in \mathbf{A}_0 \text{ s.t. } \alpha' = \alpha \text{ on } [\![0, \tau]\!]\}. \tag{1.1.15}$$

We carry out a similar analysis as in [31] of the family $\{\mathcal{Y}^\alpha,\ \alpha \in \mathbf{A}_0\}$. We start by studying for each $\alpha \in \mathbf{A}_0$ the regularity of the family $\{\mathcal{Y}^\alpha(\tau), \tau \in \mathcal{T}\}$. More precisely, we show that it can be aggregated into a right-continuous process, result which requires in our case some subtle arguments of stochastic analysis due to the nonlinearity of the driver f. We then provide that $\mathcal{Y}^\alpha$ corresponds to the unique minimal solution of a BSDE. We show that our value function is continuous and convex (in a probabilistic sense) with respect to the threshold m. In the case of a concave driver f, we obtain the existence of an optimal control, as well as a dual a representation. Indeed, we prove that $\mathcal{Y}_0(m)$ (defined as in (1.1.10)) corresponds to the Fenchel transform of the value function of the following stochastic control problem, that is $\mathcal{Y}_0(m) = \sup_{l>0}(lm - \mathcal{X}_0(l))$, where

$$\mathcal{X}_0(l) := \inf_{(\lambda,\gamma)\in\mathcal{U}\times\mathcal{V}} X_0^{l,\lambda,\gamma}, \tag{1.1.16}$$

with

$$X_0^{l,\lambda,\gamma} := \mathbb{E}\left[\int_0^T \mathcal{L}_s^\lambda \tilde{g}(s,\lambda_s)ds - \int_0^T \mathcal{A}_s^{l,\gamma}\tilde{f}(s,\gamma_s)ds + \mathcal{L}_T^\lambda \tilde{\Phi}(\frac{\mathcal{A}_T^{l,\gamma}}{\mathcal{L}_T^\lambda})\right],$$

with $\tilde{f}$ (respectively $\tilde{g}$, $\tilde{\Phi}$) the concave conjugate of f (respectively the convex conjugates of g and Φ).

The additional nonlinearity f raises important and subtle technical difficulties, since most of the results in [31] are provided using specific techniques to the case of linear constraints, which cannot be adapted to our nonlinear setting. Besides the mathematical interest of our study, this work is also motivated by some financial applications. Indeed, our problem is closely related to the *approximative hedging under dynamic risk measures contraints* of an European option, which can be expressed in the following form:

$$\inf\{Y_0 \text{ such that } \exists\ Z : \ \rho_{0,T}[-(Y_T - \xi)^-] \leq m\}, \tag{1.1.17}$$

where $\rho_{t,T}[\xi]$ represents the risk measure at time t of ξ which is defined as $-\mathcal{E}^f_{t,T}[\xi]$. Note that in the case of a nonlinear concave driver, the associated dynamic risk measure is convex. More details concerning the design of risk measures in a dynamic setting by means of backward stochastic differential equations are presented in the next chapter.

1.2 Optimal Stopping for Dynamic Risk measures with jumps and obstacle problems

This chapter is based on the paper "Optimal Stopping for Dynamic Risk measures with jumps and obstacle problems" [61], joint work with M.C. Quenez and A. Sulem, *J. Optim. Theory Applic.*(2014) DOI 10.1007/s10957-014-0636-2.

1.2.1 Preliminaries and overview of the literature

In the first chapter, we have introduced the Backward Stochastic Differential Equations in the case of a Brownian filtration, which can be seen as a generalization of the conditional expectation of a random variable ξ, since when the driver g is the null function, we have $Y_t = \mathbb{E}[\xi|\mathcal{F}_t]$, and in that case, Z is the process appearing in $(\mathcal{F}_t)_{t\geq 0}$-martingale representation property of $(\mathbb{E}[\xi|\mathcal{F}_t])_{t\geq 0}$.

In the case of a filtered probability space generated by both a Brownian Motion W and a Poisson random measure N with compensator ν, the martingale representation of $(\mathbb{E}[\xi|\mathcal{F}_t])_{t\geq 0}$ becomes:

$$\mathbb{E}[\xi|\mathcal{F}_t] = \xi + \int_t^T Z_s dW_s + \int_t^T \int_{\mathbf{R}^*} U_s(e)(N-\nu)(de,ds), \ \mathbb{P}-\text{a.s.},$$

where U is a predictable function. This leads to the following natural generalization of equation (1.1.1) to the case of jumps. We will say that (Y, Z, U) is a solution of the BSDE with jumps (BSDEJ in short) with generator f and terminal condition ξ if for all $t \in [0,T]$ we have $\mathbb{P}$-a.s.

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbf{R}^*} U_s(e)(N-\nu)(de,ds), \ \ 0 \leq t \leq T. \quad (1.2.1)$$

In 1994, Tang and Li [147] were the first to prove existence and uniqueness of a solution for (1.2.1) in the case when g is Lipschitz in (y, z, u).
The case of a discontinuous framework is more involved, especially concerning the comparison theorem which requires an additional assumption. In 1995, Barles, Buckdahn, Pardoux [9] provided a comparison theorem as well as some links between BSDEs and non-linear parabolic integral-partial differental equations, generalizing some result of [125] to the case of jumps. In 2006, Royer [140] proved a comparison theorem under weaker assumptions, and introduced the nonlinear expectations in this framework.

Furthermore, in 2004-2005, various authors have introduced dynamic risk measures in a Brownian framework, defined as the solutions of BSDEs. More precisely, given a Lipschitz driver $g(t, x, \pi)$ and a terminal time T, the risk measure ρ at time t of a position ξ is given by $-X_t$, where X is the solution of the BSDE driven by a Brownian motion, associated with g and terminal condition ξ. By the comparison theorem, ρ satisfies the *monotonicity property*, which is usually required for a risk measure. Many studies have been recently done on such dynamic risk measures, especially concerning robust optimization problems and optimal stopping problems, in the case of a Brownian filtration and a concave driver (see, among others, Bayraktar and coauthors in [13]). In the case with jumps, the links between BSDEs and dynamic risk measures have been recently studied by Quenez-Sulem in [137].

Reflected backward stochastic differential equations (RBSDEs in short) have been introduced in 1997 by the five authors El Karoui, Kapoudjian, Pardoux, Peng and Quenez [71] in the case of a filtration generated by the Brownian motion. These equations are generalisations of the deterministic Skorokhod problem. Indeed, given an adapted process $\xi := (\xi_t)_{t\leq T}$ which plays the role of the barrier, the solution of a RBSDE associated to data (η, g, ξ) is a triplet of square integrable processes $\{(Y_t, Z_t, A_t);\ \ 0 \leq t \leq T\}$ which satisfy:

$$\begin{cases} Y_t = \eta + \displaystyle\int_t^T g(s, \omega, Y_s, Z_s)ds + A_T - A_t - \int_t^T Z_s dW_s, \ \ 0 \leq t \leq T, \\ Y_t \geq \xi_t, \\ \displaystyle\int_0^T (Y_t - \xi_t)dA_t = 0. \end{cases} \tag{1.2.2}$$

where A is a continuous, increasing process whose role is to push the solution Y such it remains above the barrier ξ. The condition $\int_0^T (Y_t - \xi_t)dA_t = 0$ ensures that the process A acts in a minimal way. More precisely, A increases only on the set $\{Y = \xi\}$.

The development of reflected BSDEs has been motivated in particular by two applications: the pricing and hedging of American options, especially in markets with constraints, and the probabilistic representation of solutions of obstacle problems for nonlinear PDEs .

Concerning the first application in financial mathematics, El Karoui, Pardoux and Quenez were the first to show that in a complete market, the price of an American option with underlying asset $(\xi_t)_{t\leq T}$ and exercise price γ is given by Y_0 where $(Y_t, \pi_t, A_t)_{t\leq T}$ is the solution of the following reflected BSDE:

$$\begin{cases} -dY_t = b(t, Y_t, \pi_t)dt + dA_t - \pi_t dW_t, \ \ Y_T = (\xi_T - \gamma)^+, \\ Y_t \geq (\xi_t - \gamma)^+ \text{ and } \displaystyle\int_0^T (Y_t - (\xi_t - \gamma)^+)dA_t = 0, \end{cases} \tag{1.2.3}$$

for a particular choice of b. The process π gives us the replication strategy and A is the buyer's consumption process. In a standard financial market, the function b is given by $b(t, \omega, y, z) = r_t y + z\theta_t$, where θ_t is the risk premium and r_t represents the interest rate of investement or borrowing.

The generalization to the case of reflected BSDEs with jumps, which is a standard reflected BSDE driven by a Brownian motion and an independent Poisson random measure, has been established by Hamadène and Ouknine in [90]. A solution for such equation, associated with a coefficient f, terminal value η and a barrier ξ, is a quadruple of process (Y, Z, U, A) of adapted solutions which satisfy the following equation:

$$\begin{cases} Y_t = \eta + \displaystyle\int_t^T g(s, \omega, Y_s, Z_s, U_s)ds + A_T - A_t - \int_t^T Z_s dW_s \\ \qquad - \displaystyle\int_t^T \int_{\mathbf{R}^*} U_s(e)\tilde{N}(ds, de), \;\; 0 \le t \le T, \\ Y_t \ge \xi_t, \\ \displaystyle\int_0^T (Y_t - \xi_t)dA_t = 0. \end{cases} \tag{1.2.4}$$

Using two methods - the first one based on the penalization argument and the second one on the snell envelope theory -, the authors have shown the existence and uniqueness of solutions if η is square integrable, g is uniformly lipschitz with respect to (y, z, u) and the barrier ξ is right continuous left- limited (RCLL for short) whose jumping times are inaccessible stopping times. Note that this later condition played a crucial role in their proofs. In this case, the jumping times of the process Y come only from those of its Poisson process and then they are inaccessible.

The general case of RBSDEs with jumps and irregular obstacles has been considered e.g. in [78] and more recently by Quenez-Sulem [138]. The barrier ξ is just rcll and thus the jumping times of process Y come not only from those of its Poisson process (inaccessible jumps) but also from those of the process ξ (predictable jumps), which means that the process Y has two types of jumps: inaccessible and predictable ones. The difficulty here lies in the fact that since the barrier ξ is allowed to have predictable jumps then the reflecting process A is no longer continuous but just RCLL. In this case, the difference with respect to (1.2.4) only appears in the Skorokhod condition which becomes: $\int_0^T (Y_{t^-} - \xi_{t^-})dA_t = 0$.

An important application of reflected BSDEs is its connection to optimal stopping problems and its associated variational inequalities in the Markovian case. More precisely, given an RCLL process $(\xi_t, \; 0 \le t \le T)$ and a Lipschitz driver g satisfying the additional assumption such that the comparison theorem holds, the solution Y of the associated RBSDE satisfies: for each stopping time $S \in \mathcal{T}_0$,

$$Y_S = \operatorname*{esssup}_{\tau \in \mathcal{T}_S} \mathcal{X}_S(\xi_\tau, \tau), \quad \text{a.s.} \tag{1.2.5}$$

where for $\tau \in \mathcal{T}_S$, $\mathcal{X}_\cdot(\xi_\tau, \tau)$ is the solution of the BSDE associated with terminal time τ, terminal condition ξ_τ, and driver g (see [138]). Note that $\mathcal{T}_S$ represents the set of stopping times with values in $[0, T]$, a.s. greater than S.

1.2.2 Contributions

In this chapter, we study the optimal stopping problem for dynamic risk measures with jumps in a Markovian framework.

Let us first formulate our problem. Let $T > 0$ be the terminal time and f be a Lipschitz driver. For each $T' \in [0, T]$ and $\eta \in \mathbf{L}_2(\mathcal{F}_{T'})$, set:

$$\rho_t^f(\eta, T') = \rho_t(\eta, T') := -\mathcal{X}_t(\eta, T'), \;\; 0 \le t \le T', \tag{1.2.6}$$

where $\mathcal{X}_t(\eta, T')$ denotes the solution of the BSDE with driver g and terminal conditions (T', η). If T' represents a given maturity and η a financial position at time T', then $\rho(\eta, T')$ is interpreted as the risk of η at time t. The functional $\rho : (\eta, T') \mapsto \rho.(\eta, T')$ thus represents a *dynamic risk measure* induced by the BSDE with driver f.

Let $(\xi_t,\ 0 \leq t \leq T)$ be an RCLL adapted process in $\mathbf{S}_2$ (which denotes the set of processes ϕ such that $\mathbb{E}[\sup_{t \leq T} \phi_t^2] \leq +\infty$), representing a dynamic financial position. Let $S \in \mathcal{T}_0$. The problem is to minimize the risk measure at time S. Let $v(S)$ be the associated value function, equal to the $\mathcal{F}_S$-measurable random variable (unique for the equality in the almost sure sense) defined by

$$v(S) := \operatorname*{essinf}_{\tau \in \mathcal{T}_S} \rho_S(\xi_\tau, \tau). \tag{1.2.7}$$

This random variable $v(S)$ corresponds to the minimal risk measure at time S.
Since by definition $\rho_S(\xi_\tau, \tau) = -\mathcal{X}_S(\xi_\tau, \tau)$, we have, for each stopping time $S \in \mathcal{T}_0$,

$$v(S) = \operatorname*{essinf}_{\tau \in \mathcal{T}_S} -\mathcal{X}_S(\xi_\tau, \tau) = -\operatorname*{esssup}_{\tau \in \mathcal{T}_S} \mathcal{X}_S(\xi_\tau, \tau). \tag{1.2.8}$$

Now, using the link between reflected BSDEs and optimal stopping (3.2.4), one can relate the value function of the problem defined by (2.2.8) to the solution of the reflected BSDE. More precisely, we have:

$$v(S) = -Y_S. \tag{1.2.9}$$

Since our aim is to characterize this value function in a Markovian framework, we consider the terminal condition, obstacle and driver of the following form:

$$\begin{cases} \xi_s^{t,x} := h(s, X_s^{t,x}),\ \ s < T, \\ \xi_T^{t,x} := g(X_T^{t,x}), \\ g(s, \omega, y, z, k) := g(s, X_s^{t,x}(\omega), y, z, k),\ \ s \leq T, \end{cases} \tag{1.2.10}$$

where (t, x) is a fixed initial condition and $X^{t,x}$ is a state process which has the following dynamic:

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r + \int_t^s \int_{\mathbf{R}^*} \beta(X_{r^-}^{t,x}, e)\tilde{N}(dr, de). \tag{1.2.11}$$

The maps $f, h, g, b, \sigma, \beta$ are deterministic functions satisfying usual Lipschitz assumptions (the reader is refered to the corresponding chapter). In the Markovian setting, for each (t, x), the minimal risk measure $v(t, x)$ is defined as:

$$v(t, x) = -Y_t^{t,x}, \tag{1.2.12}$$

where $Y^{t,x}$ is the reflected BSDE with data given by (1.2.10).

Our main constribution consists in establishing the link between the value function of our optimal stopping problem and parabolic partial integro-differential variational inequalities (PIDVIs). We prove that the minimal risk measure is a viscosity solution of a PIDVI. This provides an existence result for the obstacle problem under relatively weak assumptions. In the Brownian case, this result was obtained by using a penalization method via non-reflected BSDEs. This method could also be adapted to our case with jumps, but would involve heavy computations in order to prove the convergence of the solutions of the penalized BSDEs to the solution of the reflected BSDE. It would also require some convergence results of the viscosity solutions theory in the integro-differential case. We provide a direct and much shorter proof.

Under some additional asumptions, we provide a comparison theorem, relying on a non-local version of Jensen-Ishii Lemma, from which the uniqueness of the viscosity solution follows. We extend the results of [10] to the case of nonlinear BSDEs, which leads to a more complex integro-differential operator in the associated PDE. In the case of integro-differential equations, the difficulty arises from the treatement of nonlocal operators. The main idea is to split them in one operator corresponding to the *small jumps* and one corresponding to the *big jumps* and to use a less classical definition of viscosity solution introduced in [10], adapted to integro-differential equations and equivalent to the two classical ones, which combines the approach with test-functions and sub-superjets (the solution is replaced by the test function only around the singularity of the measure in the nonlocal operator).

1.3 Generalized Dynkin Games and Doubly Reflected BSDEs with jumps

This chapter is based on the paper "Generalized Dynkin games and Doubly Reflected BSDEs with jumps" [62], joint with M.C. Quenez and A. Sulem and submitted for publication.

1.3.1 Preliminaries and overview of the literature

The Dynkin game is a zero-sum, optimal stopping game between two players. Each player can either stop the game or continue. The game is stopped as soon as either player stops, and the payoff depends on who stops first. This stochastic stopping game, nowadays known as the Dynkin game, was first introduced by Dynkin [66] as a generalization of optimal stopping problems. Since then, there has been a considerable amount of research on Dynkin games and related problems. Some examples include Dynkin and Yushkevich (1968) [67], Bensoussan and Friedman (1974) [21], Neveu (1975) [121], Bismut (1977) [23], Stettner (1982) [146], Alario, Lepeltier et Marechal (1982) [1], Morimoto (1984) [119], Lepeltier and Maingueneau (1984) [111], Cvitanic and Karatzas (1996) [52], Karatzas and Wang (2001) [101], Ekstrom and Peskir [77], Laraki and Solan [114], Peskir [135], Rosenberg and al. [139], Touzi and Vieille (2002) [149] etc. Most of the literature focuses on establishing the existence of optimal stopping times as well as value under various models and payoff assumptions. In discrete-time, it is easy to show the existence of optimal stopping times and value using backward induction arguments. In continuous-time, perhaps the most important result is due to Lepeltier and Maingueneau [111], who proved the existence of ε-optimal stopping times as well as the value.

Let us recall the mathematical formulation of a classical Dynkin Game.

The setting of the problem is very simple. There are two players, labeled Player 1 and Player 2, who observe two payoff processes ξ and ζ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Player 1 (resp., 2) chooses a stopping time τ (resp. σ) as a control for this optimal stopping problem.
At (stopping) time $\sigma \wedge \tau$ the game is over, and Player 2 pays the amount $\zeta_\sigma \mathbf{1}_{\tau>\sigma} + \xi_\tau \mathbf{1}_{\tau\leq\sigma}$ to Player 1. Therefore the objective of Player 1 is to maximize this payment, while Player 2 wishes to minimize it. It is then natural to introduce the lower and upper values of the game

$$\underline{V} := \sup_{\sigma} \inf_{\tau} E[\xi_\sigma \mathbf{1}_{\tau>\sigma} + \zeta_\tau \mathbf{1}_{\tau\leq\sigma}]; \; \overline{V} := \inf_{\tau} \sup_{\sigma} E[\xi_\sigma \mathbf{1}_{\tau>\sigma} + \zeta_\tau \mathbf{1}_{\tau\leq\sigma}]. \tag{1.3.1}$$

If the two value functions defined above coincide, then the game is said to admit a value function.

An interesting financial application of the Dynkin game is in the study of game options, also known as Israeli options, as defined by Kifer [103]. A game option is a contract between an issuer and a holder, in which the holder may exercise the option at any time for a payoff and the issuer may cancel the option at any time for a fee. It is one of the few financial contracts in which the issuer also makes meaningful decisions affecting the payoff. If we ignore the dependence on the underlying assets and focus on the relationship between decisions and payoffs, the game option is comparable to a Dynkin game. Moreover, the cancellation fee is typically assumed to be greater than or equal to the exercise payoff, echoing the standard payoff inequalities found in Dynkin games. In both discrete-time and continuous-time models, it was shown by Kifer [103] that the game option has a unique arbitrage price. Further research on game options, as well as more sophisticated game-type financial contracts, includes papers by Bielecki and al. [22], and Dolinsky Kifer [56], Dolinsky and al. [55], Hamadene and Zhang [93], Kallsen and Kuhn [97, 98], and Kifer [103].

We now focus on the relationship between *Classical Dynkin Games* and *Doubly Reflected BSDEs* (in short DRBSDEs), which have been introduced by Cvitanic and Karatzas [52] in the case of a Brownian filtration. The solution is forced to remain between two upper and lower barriers ξ and ζ and it is represented by a quadruple of square integrable processes $\{(Y_t, Z_t, A_t, A_t^{'});\ 0 \leq t \leq T\}$ satisfying:

$$\begin{cases} Y_t = \xi + \displaystyle\int_t^T g(s, \omega, Y_s, Z_s) ds + A_T - A_t - (A'_T - A'_t) - \int_t^T Z_s dW_s, \;\; 0 \leq t \leq T, \\ \xi_t \leq Y_t \leq \zeta_t, \\ \displaystyle\int_0^T (Y_t - \xi_t) dA_t = 0 \text{ et } \int_0^T (\zeta_t - Y_t) dA'_t = 0. \end{cases} \tag{1.3.2}$$

with A and A' two continuous processes, increasing, whose role is to keep the process Y between the two barriers. They proved existence and uniqueness of the solution in the case when the barriers are regular and satisfy the so-called Mokobodski condition which turns into the existence of the difference of two non-negative supermartingales between ξ and ζ.
In the case of a process driver g which only depends on (t, ω), Cvitanic-Karatzas have shown that the existence of a solution (Y, Z, A) to the above BSDE implies that Y corresponds to the value function of a Classical Dynkin Game. We give below their result:

Theorem 1.3.1 (Cvitanic-Karatzas)**.** *Let (Y, Z, A, A') be a solution of the BSDE with $g(t,\omega,y,z) = g(t,\omega)$. For any $0 \leq t \leq T$ and any two stopping times $(\tau,\sigma) \in \mathcal{T} \times \mathcal{T}$, consider the payoff:*

$$I_t(\tau,\sigma) := \int_t^{\tau\wedge\sigma} g(u)du + \xi_\tau \boldsymbol{1}_{\tau\leq\sigma} + \zeta_\sigma \boldsymbol{1}_{\sigma<\tau},$$

as well as the upper and lower values, respectively,

$$\overline{V}(t) := \operatorname*{essinf}_{\sigma\in\mathcal{T}_t} \operatorname*{esssup}_{\tau\in\mathcal{T}_t} \mathbb{E}[I_t(\tau,\sigma)|\mathcal{F}_t], \;\; \underline{V}(t) := \operatorname*{esssup}_{\tau\in\mathcal{T}_t} \operatorname*{essinf}_{\sigma\in\mathcal{T}_t} \mathbb{E}[I_t(\tau,\sigma)|\mathcal{F}_t].$$

of a corresponding Dynkin game. This game has value $V(t)$, given by the state process Y of the solution to the BSDE, that is,

$$V(t) = \overline{V}(t) = \underline{V}(t) = Y_t \quad a.s. \; \forall 0 \leq t \leq T,$$

as well as a saddlepoint $(\hat{\sigma}_t, \hat{\tau}_t) \in \mathcal{T}_t \times \mathcal{T}_t$ given by

$$\hat{\sigma}_t := \inf\{s \in [t,T), \; Y_s = \zeta_s\} \wedge T; \;\; \hat{\tau}_t := \inf\{s \in [t,T), \; Y_s = \xi_s\} \wedge T.$$

namely

$$\mathbb{E}[I_t(\tau_t,\hat{\sigma}_t)] \leq \mathbb{E}[I_t(\hat{\tau}_t,\hat{\sigma}_t)] = Y_t \leq \mathbb{E}[I_t(\hat{\tau}_t,\sigma_t)],$$

for every $(\sigma,\tau) \in \mathcal{T}_t \times \mathcal{T}_t$.

Since the seminal paper of Cvitanic-Karatzas, many authors have explored the existence and the uniqueness of the solution as well as the links with classical Dynkin Games under different assumptions on the coefficient g and regularity of the barriers (see for e.g. Lepeltier-San Martin and [112]).These results have also been extended to the case of DRBSDEs driven by both a Brownian motion and a random Poisson measure (see for e.g. [86], [87], [50]).

The above link between *classical Dynkin games and DRBSDEs* can be extended to the case of general nonlinear DRBSDEs, since given the solution Y of the DRBSDE, it is shown to coincide with the value function of the classical Dynkin game with payoff:

$$I_S(\tau,\sigma) = \int_S^{\sigma\wedge\tau} g(u, Y_u, Z_u, k_u)du + \xi_\tau \mathbf{1}_{\tau\leq\sigma} + \zeta_\sigma \mathbf{1}_{\sigma<\tau}, \tag{1.3.3}$$

where Z, k are the associated processes with Y. However, this characterization is not really tractable because the instantaneous payoff $g(u, Y_u, Z_u, k_u)$ depends on the value function Y of the associated Dynkin game. We shall see in our contribution that we can define a well-posed game problem (in the sense that the criterium does not involve the value function itself), which is shown to admit a value coinciding with the solution of a DRBSDE with general nonlinear driver.

1.3.2 Contributions

In Chapter 3, we introduce a new game problem, which can be seen as a *generalization of the classical Dynkin game*. More precisely, the linear expectation in the performance is replaced by a nonlinear g-conditional expectation, induced by a backward stochastic differential equation (BSDE) with jumps. We describe below very briefly this new game problem.

Let ξ and ζ be two adapted processes only supposed to be RCLL with $\zeta_T = \xi_T$ a.s., $\xi \in \mathbf{S}_2$, $\zeta \in \mathbf{S}_2$, $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s.

For each $\tau, \sigma \in \mathcal{T}_0$, the *payoff* at the stopping time $\tau \wedge \sigma$ is given by:

$$I(\tau, \sigma) := \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau}. \tag{1.3.4}$$

Let $S \in \mathcal{T}_0$. For each $\tau \in \mathcal{T}_S$ and $\sigma \in \mathcal{T}_S$, the associated *criterium* is given by $\mathcal{E}_{S,\tau\wedge\sigma}(I(\tau,\sigma))$, the g-conditional expectation of $I(\tau,\sigma)$. At the stopping time S, the first (resp. the second) player chooses a stopping time τ (resp. σ) after S, in order to maximize (resp. minimize) the criterium.

For each stopping time $S \in \mathcal{T}_0$, the *upper* and *lower value functions* at time S are defined as follows:

$$\overline{V}(S) := \operatorname*{essinf}_{\sigma \in \mathcal{T}_S} \operatorname*{esssup}_{\tau \in \mathcal{T}_S} \mathcal{E}_{S,\tau\wedge\sigma}(I(\tau,\sigma)); \tag{1.3.5}$$

$$\underline{V}(S) := ess \sup_{\tau \in \mathcal{T}_S} \operatorname*{essinf}_{\sigma \in \mathcal{T}_S} \mathcal{E}_{S,\tau\wedge\sigma}(I(\tau,\sigma)). \tag{1.3.6}$$

The game admits a value function if $\underline{V}(S) = \overline{V}(S)$.

Under Mokobodski's condition, we show the existence of a value function for this game, which can be characterized by the unique solution of a nonlinear doubly reflected BSDE (DRBSDE). Up to now, no interpretation of general nonlinear doubly reflected BSDEs in terms of control or game problems (with nonlinear expectation) had been given in the literature.

Using this characterization, we obtain some properties of these DRBSDEs, such as a general comparison theorem and a strict comparison theorem. We also establish new a priori estimates *with universal constant* for DRBSDEs, and the proof is based on the characterization of the solution as the value function of this new game problem. When both obstacles are left upper semicontinuous along stopping times, we show the existence of a saddle point of the generalized Dynkin game. We point out that we do not assume the strict separability of the barriers, assumption which is crucial in the previous literature. We can get rid of it by imposing an additional constraint on the increasing processes A, A' which appear in Definition 1.3.2 (note that in our setting the increasing processes A and A' are no longer continuous). More precisely, we assume that the measures dA and dA' are mutually singular in the probabilistic sense, i.e. there exists $D \in \mathcal{P}$ such that

$$E[\int_0^T \mathbf{1}_D dA_t] = E[\int_0^T \mathbf{1}_{D^c} dA'_t] = 0.$$

This constraint is also important in order to obtain the uniqueness of the increasing processes A and A'. Moreover, it allows us to identify the positive and negative jumps of the solution of the DRBSDE, this identification being used in the proof of the existence of saddle point without assuming the strict separability of the barriers.

We continue by studying a generalized mixed zero-sum game under the g-conditional expectation, in which two players compete by taking two actions: continuous control and stopping. We provide some sufficient conditions (such as the controlled drivers $g^{u,v}$ have a saddle point $g^{\overline{u},\overline{v}}$), which ensure the existence of a value function of the generalized mixed game and characterize the common value function as the solution of a DRBSDE with driver $g^{\overline{u},\overline{v}}$. When both obstacles are left upper semicontinuous along stopping times, the associated generalized mixed game admits a saddle point.

We then address the generalized Dynkin game in the Markovian framework and study its links with parabolic partial integro-differential variational inequalities (PIDVI) with two obstacles. More precisely, we show that the value function of the generalized Dynkin game in the Markovian case is the unique viscosity solution of the corresponding PIDVI. From a PDE point of view, this result provides a new probabilistic interpretation of semi linear PDEs with two barriers in terms of game problems.

1.4 A Weak Dynamic Programming Principle for Combined Stochastic Control/Optimal Stopping with $\mathcal{E}^f$-Expectations

This chapter is based on the paper "A weak dynamic programming principle for Combined Stochastic Control / Optimal Stopping with $\mathcal{E}^f$-expectations" [63], joint with M.C. Quenez and A. Sulem and submitted for publication.

1.4.1 Preliminaries and overview of the literature

The Dynamic Programming Principle (in short DPP) is the main tool in the theory of stochastic control. The basic idea of the method is to consider a family of stochastic control problems with different initial states and to establish relationships between the associated value functions. It was initiated in the fifties by Bellman ([28], [19]), who says that "an optimal policy has the property that, whatever the initial state and control are, the remaining decisions must constitue an optimal policy with regard to the state resulting from the first decision". Typically, a stochastic control problem in a finite horizon time T can be written as follows:

$$V(0,x) := \sup_{\alpha\in\mathcal{A}} \mathbb{E}[\int_0^T f(s, X_s^\alpha, \alpha_s)ds + g(X_T^\alpha)], \tag{1.4.1}$$

where f is the instantaneous reward and g the terminal payoff.

A formal statement of the DPP is

$$V(0,x) = v(0,x) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}[\int_0^\tau f(s, X_s^\alpha, \alpha_s)ds + V(\tau, X_\tau^\alpha)], \tag{1.4.2}$$

where τ is an arbitrary stopping time such that $\tau \in [0,T)$ a.s.

In the case of controlled Markov jump-diffusions, the DPP is used in order to derive the corresponding dynamic programming equation in the sense of viscosity solutions. In the literature, this principle is classically established under assumptions which ensure that the value function satisfies some regularity/measurability properties, see e.g. Fleming-Rischel, Krylov, El Karoui, Bensoussan-Lions, Lions P.-L., Fleming-Soner, Touzi for the case of controlled diffusions and Oksendal and Sulem for the case of Markov jump-diffusions. The statement (1.4.2) of the DPP is very intuitive and can be easily proved in the deterministic framework, or in discrete-time with finite probability space. However, its proof is in general not trivial and requires on the first stage that V is measurable.

The case of a discontinuous value function has been studied in a deterministic framework in the eighties: a *weak* dynamic programming principle has been established for deterministic control by Barles (1993) (see [8]) (see also Barles and Perthame (1986) [11]). More precisely, he proves that the upper semicontinuous envelope V^* and the lower semicontinuous envelope V_* of the value function V satisfy, respectively, the *sub- and super-optimality principle of dynamic programming* of Lions and Souganidis (1985) [114]. He then derived that the (discontinuous) value function is a *weak viscosity* solution of the associated Bellman equation in the sense that V^* is a viscosity subsolution and V_* is a supersolution of the Bellman equation.

More recently, Bouchard and Touzi (2011) (see [35]) have proved a *weak* dynamic programming principle in a stochastic framework, when the value function is not necessarily continuous, not even measurable. They prove that the upper semicontinuous envelope V^* satisfies the sub-optimality principle of dynamic programming, and under an additional regularity (lower semi continuity) assumption of the reward g, they obtain that the lower semicontinuous envelope V_* satisfies the super-optimality principle.

A *weak* dynamic principle has been further established, under some specific regularity assumptions, for problems with state constraints by Bouchard and Nutz (2012) in [34], and for zero-sum stochastic games by Bayraktar and Yao (2013) in [14].

In the sequel, we present the classical statement of the problem for both stochastic control and optimal stopping problems (in a finite horizon time T), in the case when the value function is not a priori continuous, not even measurable. We recall the *weak dynamic programming principle* obtained by Bouchard and Touzi ([35]), as well as the associated HJB equations.

(i) *Stochastic control and weak dynamic programming in the case of classical expectations*

We denote by $\mathcal{A}$ the set of all progressively measurable processes $\alpha = \{\alpha_t, t < T\}$ valued in A, a subset of $\mathbf{R}$, belonging to $\mathbf{H}_2$ (the set of processes ϕ such taht $\mathbb{E}[\int_0^T \phi_s^2 ds] < +\infty$). The elements of $\mathcal{A}$ are called control processes.

For each control process $\alpha \in \mathcal{A}$, we consider the following controlled stochastic differential equation:

$$dX_s^{t,x,\alpha} = b(X_s^{t,x,\alpha}, \alpha_s)ds + \sigma(X_s^{t,x,\alpha}, \alpha_s)dW_s, \tag{1.4.3}$$

where the coefficients b and σ satisfy the usual Lipschitz and linear growth conditions so that the above SDE has a unique strong solution.
For a given initial data (t,x) and control $\alpha \in \mathcal{A}$, the process $X^{t,x,\alpha}$ is called the controlled process, as its dynamic is driven by the action of the control process α.
We define the cost functional J on $[0,T] \times \mathbf{R} \times \mathcal{A}$ by:

$$J(t,x,\alpha) = \mathbb{E}\left[\int_t^T f(s, X_s^{\alpha,t,x}, \alpha_s)ds + g(X_T^{\alpha,t,x})\right],$$

where f is Lipschitz continuous and g Borelian, with quadratic growth.

The purpose is to study the following stochastic control problem:

$$v(t,x) = \sup_{\alpha \in \mathcal{A}_t} J(t,x,\alpha), \tag{1.4.4}$$

where $\mathcal{A}_t$ represents the set of t-admissible controls, which are independent of $\mathcal{F}_t$.

In order to describe the local behavior of the value function V by means of the so-called *dynamic programming equation* or *Hamilton-Jacobi-Bellman*, the key point is the *Dynamic Programming Principle*. Since the DPP involves the value function itself, which may not be measurable under these assumptions, Bouchard and Touzi [35] propose a *Weak version* of the *Dynamic Programming Principle*, which is shown to be sufficient for the derivation of the dynamic programming equation. This weak DPP involves the upper semicontinuous envelope of the value function V, respectively the lower semicontinuous one, which are defined as follows: for each $t \in [0,T]$, for each $x \in \mathbf{R}$,

$$V_*(t,x) := \liminf_{(t',x') \to (t,x)} V(t',x') \quad \text{and} \quad V^*(t,x) := \limsup_{(t',x') \to (t,x)} V(t',x'). \tag{1.4.5}$$

Let us now recall the Weak dynamic Programming Principle.

Theorem 1.4.1 (Weak Dynamic Programming Principle)**.** *1. Let $\{\theta^\alpha,\ \alpha \in \mathcal{U}_t\}$ be a family of finite stopping times independent of $\mathcal{F}_t$, with values in $[t,T]$. Then:*

$$V(t,x) \leq \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}[\int_t^{\theta^\alpha} f(s, X_s^{\alpha,t,x}, \alpha_s)ds + V^*(\theta^\alpha, X_{\theta^\alpha}^{\alpha,t,x})], \tag{1.4.6}$$

2. Assume further that g is lower-semicontinuous and $X_{t,x}^\alpha \boldsymbol{1}_{t,\theta^\alpha}$ is $\mathbf{L}^\infty$-bounded for all $\nu \in \mathcal{A}_t$. Then

$$V(t,x) \geq \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}[\int_t^{\theta^\alpha} f(s, X_s^{\alpha,t,x}, \alpha_s)ds + V_*(\theta^\alpha, X_{\theta^\alpha}^{\alpha,t,x})]. \tag{1.4.7}$$

The above weak DPP is shown without using the abstract theorems of measurable selection. The authors use instead to Vitali's covering lemma. The inequality which is the most difficult to provide is the second one and it requires a lower semicontinuity assumption on the criterium J (which is satisfied in the case when the terminal reward g is lower semicontinuous).

We also point out that, in the case when V is continuous, then $V = V_* = V^*$, and the above weak dynamic programming principle reduces to the classical dynamic programming principle:

$$V(t,x) = \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}\left[\int_t^{\theta^\alpha} f(s, X_s^{\alpha,t,x}, \alpha_s)ds + V(\theta^\alpha, X_{\theta^\alpha}^{\alpha,t,x})\right].$$

As mentioned previously, the dynamic programming principle represents the main step for the derivation of the dynamic programming equation, corresponding to the infinitesimal counterpart of the DPP. It is widely called the *Hamilton-Jacobi-Bellman* equation.The associated HJB equation is provided in the following theorem:

Theorem 1.4.2. *Assume that the value function* $V \in C^{1,2}([0,T), \mathbf{R})$*, and let* $f(\cdot,\cdot,a)$ *be continuous in* (t,x) *for all fixed* $a \in A$*. Then, for all* $(t,x) \in [0,T) \times \mathbf{R}$*:*

$$-\partial_t V(t,x) - \sup_{a \in A}\{b(t,x,a)\partial_x V(t,x) + \frac{1}{2}Tr[\sigma\sigma(t,x,a)D^2_{xx}V(t,x)] + f(t,x,a)\} = 0. \tag{1.4.8}$$

Note that in the case when the value function V is not continuous, then it satisfies the above PDE in the viscosity sense.

We now present the main results concerning optimal stopping, which represent a particular case of stochastic control problems when the control takes the form of a stopping time.

(*ii*) *Optimal stopping and weak dynamic programming in the case of classical expectations*

For $0 \leq t \leq T < +\infty$, we denote by $\mathcal{T}_{[t,T]}$ the collection of all $\mathbb{F}$-stopping times with values in $[t,T]$. The underlying state process $X^{t,x}$, with initial condition (t,x), is defined by the stochastic differential equation:

$$dX^{t,x}_s = b(s, X^{t,x}_s)ds + \sigma(s, X^{t,x}_s)dW_s,$$

where b and σ satisfy the usual Lipschitz and linear growth conditions so that the above SDE has a unique strong solution.
Let g be a measurable function, with polynomial growth, and assume that:

$$\mathbb{E}[\sup_{0 \leq t \leq T} |g(X_t)|] < \infty.$$

For an admissible stopping time, the criterium is defined as follows:

$$J(t,x,\tau) = \mathbb{E}\left[g(X^{t,x}_\tau)\right]. \tag{1.4.9}$$

We now consider the subset of stopping rules:

$$\mathcal{T}^t_t := \{\tau \in \mathcal{T}_{[t,T]} : \tau \text{ independant of } \mathcal{F}_t\}. \tag{1.4.10}$$

The optimal stopping problem is defined by:

$$V(t,x) = \sup_{\tau \in \mathcal{T}^t_t} J(t,x,\tau). \tag{1.4.11}$$

Using the same arguments as for the stochastic control problem presented above, Bouchard and Touzi show the following *Weak Dynamic Programming Principle*:

Theorem 1.4.3. *For* $(t,x) \in [0,T] \times \mathbf{R}$*, let* $\theta \in \mathcal{T}^t_t$ *be a stopping time such that* $X^{t,x}_\theta$ *is bounded. Then:*

$$V(t,x) \leq \sup_{\tau \in \mathcal{T}^t_t} \mathbb{E}\left[\mathbf{1}_{\{\tau<\sigma\}}g(X^{t,x}_\tau) + \mathbf{1}_{\{\tau \geq \theta\}}V^*(\theta, X^{t,x}_\theta)\right], \tag{1.4.12}$$

$$V(t,x) \geq \sup_{\tau \in \mathcal{T}^t_t} \mathbb{E}\left[\mathbf{1}_{\{\tau<\sigma\}}g(X^{t,x}_\tau) + \mathbf{1}_{\{\tau \geq \theta\}}V_*(\theta, X^{t,x}_\theta)\right]. \tag{1.4.13}$$

In the case when the value function V is a priori known to be smooth, the infinitesimal counterpart of the dynamic programming principle is the following:

Theorem 1.4.4. *Assume that $V \in C^{1,2}([0,T),\mathbf{R})$ and let $g : \mathbf{R} \mapsto \mathbf{R}$ be continuous. Then V solves the obstacle problem:*

$$\min\{-(\partial_t + \mathcal{L})V, V - g\} = 0, \tag{1.4.14}$$

where $\mathcal{L}V$ represents the infinitesimal generator of the Markov diffusion process X.

The classical stochastic control problem (1.4.1) has been generalized by Peng to the case when the cost functional is defined through a nonlinear controlled backward stochastic differential equation (see [127] and [128]), under assumptions which ensure that the value function is continuous. He establishes a dynamic programming by using the backward semigroup method and derives the associated HJB equations. These results allow him to obtain a stochastic interpretation for a larger class of nonlinear HJB equations, since the coefficient f also depends on (y,z).

At the end of this section, we would like to mention some developments in the case when the uncertainty impacts only the volatility of the model. Soner, Touzi and Zhang ([145]) recently introduced the notion of second order BSDEs (2BSDEs), whose basic idea is to require that the solution verifies the equations $\mathbb{P}^\alpha$ a.s. for every probability measure in a non dominated class of mutually singular measures. This theory is closely related to the notion of G-expectation of Peng ([129]) and provides a different probabilitic representation of the solutions of fully nonlinear HJB equations.

1.4.2 Contributions

In this chapter, we are interested in generalizing the results obtained by Bouchard and Touzi ([35]) to the case when the linear expectation $\mathbb{E}$ is replaced by a nonlinear expectation induced by a Backward Stochastic Differential Equation with jumps. In a Markovian setting, the value function of our problem is the following:

$$V(t,x) := \sup_{\alpha\in\mathcal{A}} \mathcal{E}^\alpha_{0,T}[g(X_T^{\alpha,t,x})], \tag{1.4.15}$$

where $\mathcal{E}^\alpha$ is the nonlinear conditional expectation associated with a BSDE with jumps with controlled driver $f(\alpha_t, X_t^\alpha, y, z, k)$. We address this study in the case when the reward function g is only Borelian. Moreover, in this chapter, we consider the combined problem when there is an additional control in the form of a stopping time. We thus consider mixed generalized stochastic control/ optimal stopping problems of the form

$$V(t,x) := \sup_{\alpha} \sup_{\tau} \mathcal{E}^\alpha_{0,\tau}[\overline{h}(X_\tau^{\alpha,t,x})], \tag{1.4.16}$$

where $\overline{h}(X_\tau^{\alpha,t,x})$ is an irregular payoff.

In order to characterize the value function as the solution of a HJB variational inequality, we first establish a *Dynamic Programming Principle*, which is obtained using sofisticated techniques of stochastic analysis. We point out that, due to the weak assumptions on the coefficients, the value function of our problem is not necessarily continuous, not even measurable.

As mentioned in the introductory section, since for fixed t, the value function $x \to V(t,x)$ is not necessarily measurable, we cannot a priori establish a classical dynamic programming. We thus provide a *weak dynamic programming* involving the map V_* and the map V^* defined by

$$V^*(t,x) := \limsup_{(t',x')\to(t,x)} V(t',x'),\ \ \forall (t,x) \in [0,T) \times \mathbf{R} \text{ and } V^*(T,x) = g(x), \forall x \in \mathbf{R};$$
$$V_*(t,x) := \liminf_{(t',x')\to(t,x)} V(t',x'),\ \ \forall (t,x) \in [0,T) \times \mathbf{R} \text{ and } V_*(T,x) = g(x), \forall x \in \mathbf{R}.$$

Remark that in our case, the map V^* (resp. V_*) is not necessarily upper (resp. lower) semicontinuous on $[0,T] \times \mathbf{R}$, because the terminal reward g is only Borelian (it is not supposed to satisfy any regularity assumption). This is not the case in the previous literature even in the linear case, where g is supposed to be lower-semicontinuous (see [35]). We give below *the sub-* (*resp. super-*) *optimality principle of dynamic programming* satisfied by V^* (resp. V_*), one of the main results of this chapter.

Theorem 1.4.5 (A *weak* dynamic programming principle)**.** *The function V^* satisfies the* sub–optimality principle of dynamic programming*, that is for each $t \in [0,T]$ and for each stopping time $\theta \in \mathcal{T}_t^t$, that is*

$$V(t,x) \leq \sup_{\alpha\in\mathcal{A}_t^t} \sup_{\tau\in\mathcal{T}_t^t} \mathcal{E}_{t,\theta\wedge\tau}^{\alpha,t,x} \left[h(\tau, X_\tau^{\alpha,t,x}) \boldsymbol{1}_{\tau<\theta} + V^*(\theta, X_\theta^{\alpha,t,x}) \boldsymbol{1}_{\tau\geq\theta}\right], \tag{1.4.17}$$

The function V_* satisfies the super–optimality principle of dynamic programming*, that is for each $t \in [0,T]$ and for each stopping time $\theta \in \mathcal{T}_t^t$, that is*

$$V(t,x) \geq \sup_{\alpha\in\mathcal{A}_t^t} \sup_{\tau\in\mathcal{T}_t^t} \mathcal{E}_{t,\theta\wedge\tau}^{\alpha,t,x} \left[h(\tau, X_\tau^{\alpha,t,x}) \boldsymbol{1}_{\tau<\theta} + V_*(\theta, X_\theta^{\alpha,t,x}) \boldsymbol{1}_{\tau\geq\theta}\right]. \tag{1.4.18}$$

In the above theorem, $\mathcal{A}_t^t$ represents the set of controls independent on $\mathcal{F}_t$ and restricted to $[t,T]$. Similarly, $\mathcal{T}_t^t$ denotes the set of stopping times independent on $\mathcal{F}_t$, with values in $[t,T]$.

The sub-optimality principle is the easiest to prove. It is based on the flow property for both backward and forward SDEs and a splitting property, which basically states that given an intermediary time $t \leq T$ and a fixed path up to time t (corresponding to the realization of the Brownian motion and Poisson random measure), the BSDE can be solved with respect to the t-translated Brownian motion and Poisson random measure. This result is needed in order to be able to use the definition of the value function, which is a deterministic map.

The second inequality is considerably more difficult and relies on the existence of *weak* ε-optimal controls for our mixed control/optimal stopping problem (result requiring some subtle arguments, as an abstract measurable selection theorem), as well as on some new properties of BSDEs (for e.g. a Fatou lemma for reflected BSDEs where the limit involves both terminal time and terminal condition).

Using this weak dynamic programming principle and a new comparison theorem between BSDEs and reflected BSDEs, we derive that the value function is a weak viscosity solution of a nonlinear generalized HJB variational inequality. More precisely, the result is the following:

Theorem 1.4.6. *The function V, defined by* (1.4.16)*, is a* weak viscosity solution *of the HJBVI*

$$\begin{cases} \min(V(t,x) - h(t,x), \inf_{a\in\mathbf{A}}(-\dfrac{\partial V}{\partial t}(t,x) - L^a V(t,x) \\ \qquad -f(a,t,x,V(t,x),(\sigma\dfrac{\partial V}{\partial x})(t,x), B^a V(t,x))) = 0, (t,x) \in [0,T) \times \mathbf{R} \\ V(T,x) = g(x), x \in \mathbf{R} \end{cases} \tag{1.4.19}$$

with $L^a := A^a + K^a$, and for $\phi \in C^2(\mathbf{R})$,

- $A^a\phi(x) := \dfrac{1}{2}\sigma^2(x,\alpha)\dfrac{\partial^2\phi}{\partial x^2}(x) + b(x,\alpha)\dfrac{\partial\phi}{\partial x}(x)$
- $K^a\phi(x) := \displaystyle\int_{\mathbf{E}} \left(\phi(x+\beta(x,\alpha,e)) - \phi(x) - \dfrac{\partial\phi}{\partial x}(x)\beta(x,\alpha,e)\right)\nu(de)$
- $B^a\phi(x) := \phi(x+\beta(x,\alpha,\cdot)) - \phi(x)$,

in the sense that V^* is a viscosity subsolution of (1.4.19) *and V_* is a viscosity supersolution of* (1.4.19)*.*

We conclude this chapter with some financial applications of the theoretical part.

1.5 Numerical methods for Doubly Reflected BSDEs with Jumps and irregular obstacles

This part of the thesis is dedicated to the study of numerical methods for DRBSDEs with jumps and irregular obstacles and is based on two papers written in collaboration with C. Labart: *Numerical approximation for DRBSDEs with jumps and RCLL obstacles* [59] (accepted for publication in *Journal of Mathematical Analysis and Applications*) and *Reflected scheme for DRBSDEs with jumps and RCLL obstacles* [60] (accepted for publication in *Journal of Computational and Applied Mathematics*).

We start this section with a short presentation of the existing numerical methods for backward SDEs.

1.5.1 Preliminaries and overview of the literature

Since Backward Stochastic differential equations provide probabilistic representations of solutions of semilinear PDEs, there are many works on numerical schemes in the Markovian setting, in the case of a filtration generated by a Brownian motion. Among them, we recall the four step algorithm developed by J. Ma, P. Protter and J. Yong ([115], see also [68]), Bouchard-Touzi (see [26]), Zhang ([152]) etc. In the case of reflected and doubly reflected BSDEs, see e.g. [25] and [45].
A relevant problem in the theory of BSDEs is to propose implementable numerical methods to approximate the solutions of such equations and the complexity is due to the computation of conditional expectations. Several efforts have been made in this direction. In [26], Bouchard and Touzi

use the Malliavin calculus to rewrite the conditional expectations as the ratio of two unconditional expectations which can be estimated by standard Monte Carlo methods. In the reflected case, where the driver does not depend on Z, Bally and Pages (see [5], [6]) use a quantization approach. This method is based on the approximation of the continuous time processes on a finite grid, and requires a further estimation of the transition probabilities on the grid. Gobet et al. ([110]) have suggested an adaptation of the so-called Longstaff-Schwartz algorithm based on non-parametric methods and very recently Ph. Briand and C. Labart ([41]) have proposed the Wiener chaos expansion, which, in the spirit, is not so far from the regression techniques. We also recall the cubature methods, used by T. Lyons, D. Crisan and K. Manolarakis (see e.g. [50]).

In the non-markovian setting, in the case of standard BSDEs ([134]), as well as in the case of reflected BSDEs [151], the authors propose another technique which is based on the approximation of the Brownian motion by a random walk. This method allows them to simplify the computation of the conditional expectations involved at each time step and to obtain fully implementable schemes. The BSDE is thus replaced by an appropriate discrete backward stochastic differential equation, which is shown to convege by a result of Briand, Delyon and Memin [37] (see also [38]).

While many authors studied discrete schemes for the approximation of solutions of BSDEs in a purely Brownian setting, in a setting with jumps there is considerably less literature available, and only in the case of nonreflected BSDEs. In the Markovian setting, Bouchard and Elie ([30]) considered numerical schemes for BSDEs in a pure finite activity jump setting based on the dynamic programming equation. Recently, Lejay *et al.* (2014) [109] have extended the results of Briand, Delyon and Memin to the case of jumps. Their method thus relies on the construction of a discrete BSDE with jumps driven by a complete system of three orthogonal discrete time-space martingales.

1.5.2 Contributions

In Chapter 5, we study in a non-markovian setting a discrete time approximation for the solution of Doubly Reflected BSDEs with Jumps, driven by a Brownian motion (denoted by W) and an independant compensated Poisson process of intensity λ (denoted by $\tilde{N}$). Moreover, we assume that the barriers are right continuous left limited processes and admit both totally inaccesible and predictable jumps. The DRBSDE we solve numerically has the dynamics:

$$Y_t = \xi_T + \int_t^T g(s, Y_s, Z_s, U_s)ds + (A_T - A_t) - (K_T - K_t) - \int_t^T Z_s dW_s - \int_t^T U_s d\tilde{N}_s, \tag{1.5.1}$$

and satisfies the following constraints:

$$\begin{cases} (i)\ \forall t \in [0,T],\ \xi_t \leq Y_t \leq \zeta_t\ a.s. \\ (ii)\ \displaystyle\int_0^T (Y_{t^-} - \xi_{t^-})dA_t^c = 0 \ \text{ and } \ \int_0^T (\zeta_{t^-} - Y_{t^-})dK_t^c = 0 \text{ a.s.} \\ (iii)\ \forall \tau \text{ predictable stopping time, } \ \Delta A_\tau^d = \Delta A_\tau^d \mathbf{1}_{Y_{\tau^-} = \xi_{\tau^-}} \\ \qquad\qquad\qquad \text{and } \Delta K_\tau^d = \Delta K_\tau^d \mathbf{1}_{Y_{\tau^-} = \zeta_{\tau^-}}. \end{cases} \tag{1.5.2}$$

As we have mentioned in the previous chapters, since we consider the general setting when the jumps of the obstacles can be either predictable or totally inaccesible, the increasing processes A

and K, whose role is to keep the solution Y between the barriers, are no longer continous. We can thus rewrite the Skorokhod condition separately, for the continuous part A^c (resp. K^c) of A (resp. K) and the discontinuous one, denoted by A^d (resp. K^d).

Our aim is to propose a fully implementable scheme to the above DRBSDE, based on two random binomial trees and the penalization method, which is then shown to converge to the solution of the DRBSDE. We present below the main ideas:

(i) We first introduce a sequence of penalized BSDEs in order to approximate the doubly reflected BSDE (1.5.1; 1.5.2), satisfying:

$$\begin{aligned} Y_t^p = \xi + \int_t^T g(s, Y_s^p, Z_s^p, U_s^p)ds + A_T^p - A_t^p - (K_T^p - K_t^p) \\ - \int_t^T Z_s^p dW_s - \int_t^T U_s^p d\tilde{N}_s, \end{aligned} \tag{1.5.3}$$

with $A_t^p := p\int_0^t (Y_s^p - \xi_s)^- ds$ and $K_t^p := p\int_0^t (\zeta_s - Y_s^p)^- ds$.
We provide the convergence of the penalized equations in the case of a general Poisson random measure and, since we have to deal with a driver which depends on the solution, the penalization method used in previous literature (which treates only the case of a driver process, the general case being obtained by a fixed point argument) cannot be adapted to our general setting. We propose instead a proof which is based on a combination of penalization, Snell envelope theory, comparison theorem for BSDEs with jumps, a generalized monotonic theorem under Mokobodski's condition and stochastic games.

(ii) We approximate the Brownian motion and the Poisson process by two independent random walks, denoted by W^n respectively $\tilde{N}^n$ and defined as follows:

$$W_0^n = 0; \;\; W_t^n = \sqrt{\delta} \sum_{i=1}^{[t/\delta]} e_i^n, \;\; \tilde{N}_0^n = 0, \;\; \tilde{N}_t^n = \sum_{i=1}^{[t/\delta]} \eta_i^n,$$

with $e_i^n, i = 1, n$ independent identically distributed random variables taking the values $\{-1; 1\}$, both with probability $\frac{1}{2}$ and $\eta_i^n, i = 1, n$ defined similarly to (e_i^n), but taking the values $\{\kappa_n - 1; \kappa_n\}$ with probability $1 - \kappa_n$, resp. κ_n, where $\kappa_n = e^{-\frac{\lambda}{n}}$. In the above definition, $\delta_n := \frac{T}{n}$ represents the time step. The couple $(W^n, \tilde{N}^n)$ converges to $(W, \tilde{N})$ in probability for the J_1-Skorokhod topology. Using these approximations, we get the following discrete approximation of the penalized equation defined by (1.5.3):

$$\begin{cases} y_j^{p,n} = y_{j+1}^{p,n} + g(t_j, y_j^{p,n}, z_j^{p,n}, u_j^{p,n})\delta_n + a_j^{p,n} - k_j^{p,n} \\ \qquad\qquad -(z_j^{p,n}\sqrt{\delta_n} e_{j+1}^n + u_j^{p,n}\eta_{j+1}^n + v_j^{p,n}\mu_{j+1}^n) \\ a_j^{p,n} = p\delta_n (y_j^{p,n} - \xi_j^n)^-; \;\; k_j^{p,n} = p\delta_n(\zeta_j^n - y_j^{p,n})^-, \\ y_n^{p,n} := \xi_n^n, \end{cases} \tag{1.5.4}$$

where the third martingale increments sequence $\{\mu_j^n = e_j^n \eta_j^n, \; j = 0, ..., n\}$ is needed in order to obtain the martingale representation (see [109]).

Then, using the above discrete implicit scheme, we can derive the expressions of the coefficients $(z_j^{p,n}, u_j^{p,n}, v_j^{p,n})_{j=1,n}$, involving conditional expectations, which are easy to compute in our framework, thanks to the above approximations of W and $\tilde{N}$. However, the value of $(y_j^{p,n})_{j=1,n}$ is not so easy to deduce, since we have to introduce an operator whose numerical inversion is quite difficult and time consuming. In order to overcome this issue, we introduce an explicit discrete backward equation, which is obtained by replacing in (1.5.4) $y_j^{p,n}$ by $\mathbb{E}[y_{j+1}^{p,n}|\mathcal{F}_j^n]$ in the generator g:

$$\begin{cases} y_j^{p,n} = y_{j+1}^{p,n} + g(t_j, \mathbb{E}[y_{j+1}^{p,n}|\mathcal{F}_j^n], z_j^{p,n}, u_j^{p,n})\delta_n + a_j^{p,n} - k_j^{p,n} \\ \qquad\qquad -(z_j^{p,n}\sqrt{\delta_n}e_{j+1}^n + u_j^{p,n}\eta_{j+1}^n + v_j^{p,n}\mu_{j+1}^n) \\ a_j^{p,n} = p\delta_n(y_j^{p,n} - \xi_j^n)^-; \;\; k_j^{p,n} = p\delta_n(\zeta_j^n - y_j^{p,n})^-, \\ y_n^{p,n} := \xi_n^n, \end{cases} \tag{1.5.5}$$

where $\mathcal{F}^n$ represents the discrete filtration generated by $(e_j^n, \eta_j^n)_{j=1,n}$.

We then introduce the continuous time version $(Y_t^{p,n}, Z_t^{p,n}, U_t^{p,n}, A_t^{p,n}, K_t^{p,n})_{0\leq t\leq T}$ of the solution of this explicite scheme and show its convergence in n to the solution of (1.5.3). Coupling this result with the convergence in p of the penalized equation (see (i)), we obtain the convergence of our scheme in (p,n) to the solution of the DRBSDE.

We finally study numerically our theoretical results, in the case of barriers admiting both predictable and totally inaccesible jumps. The difficulty in the choice of the examples is due to the Mokobodski's condition, that we have to assume and which is difficult to check in practice. We point out that the practical use of our scheme is restricted to low dimensional cases. Indeed, since we use a random walk to approximate the Brownian motion and the Poisson process, the complexity of the algorithm grows very fast in the number of time steps n (more precisely, in n^d, d being the dimension) and, as we will see in the numerical part, the penalization method requires many time steps to be stable.

In Chapter 6, we propose an alternative scheme to (1.5.4) and respectively to (1.5.5) in order to solve the DRBSDE given by (1.5.1; 1.5.2). Compared to the discrete backward equations (1.5.4) and (1.5.5) , the schemes we present in chapter 6, called *implicit reflected scheme* and *explicit reflected scheme* are based on a direct discretization of (1.5.1; 1.5.2). More precisely, there is no penalization step. Then, this method only depends on one parameter of approximation (the number of time steps n), contrary to the schemes proposed in Chapter 5 (see (1.5.4) and (1.5.5)), which also depends on the penalization parameter. We provide the convergence of both schemes. The explicit reflected scheme is the following:

$$\begin{cases} y_j^n = \mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n] + g(t_j, \mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n], z_j^n, u_j^n)\delta_n + a_j^n - k_j^n \\ a_j^n \geq 0, \; k_j^n \leq 0, \; a_j^n k_j^n = 0, \; \xi_j^n \leq y_j^n \leq \zeta_j^n, \;\; (y_j^n - \xi_j^n)a_j^n = (y_j^n - \zeta_j^n)k_j^n = 0. \end{cases} \tag{1.5.6}$$

We illustrate numerically the theoretical results and show they coincide with the ones obtained by using the penalized scheme (1.5.5), for large values of the penalization parameter p.

1.6 Conclusions and perspectives

In this Phd thesis we have investigated some new problems of interest in stochastic analysis, stochastic control, game theory and financial mathematics, from both theoretical and numerical point of views. The main results are the following:

- In Chapter 3 (based on paper [57]), we introduce and study a new class of *BSDEs with nonlinear weak terminal condition*, related to the approximative hedging under dynamic risk measures constraints.

- In Chapter 4 (based on paper [61]), we study an optimal stopping problem for dynamic risk measures induced by BSDEs with jumps and show that the value function corresponds to the unique viscosity solution of an obstacle problem for partial integro-differential equations.

- In Chapter 5 (based on paper [62]), we introduce a new game problem, which can be seen as a generalization of the classical Dynkin game to the case of nonlinear expectations , allowing us to obtain a representation of the solution of general nonlinear doubly reflected BSDEs in terms of stochastic games.

- In Chapter 6 (based on paper [63]), we study in a Markovian framework a mixed stochastic control/optimal stopping problem in the case when the classical expectation in the criterium is replaced by a nonlinear one induced by a solution of a BSDE with jumps and the terminal reward is only measurable. We establish a weak dynamic programming principle and derive the associated nonlinear HJB equations.

- In Chapter 7 (based on paper [59]), we introduce a numerical approximation for the solution of doubly reflected BSDEs with jumps and irregular obstacles, which admit both totally inaccesible and predictable jumps. We propose a fully implementable scheme, based on penalization method and the approximation of the Brownian motion and the Poisson process by two independent random walks, which is shown to converge to the solution of the DRBSDE. We illustrate the theoretical results with some numerical examples.

- In Chapter 8 (based on paper [60]), we introduce an alternative fully implementable scheme to the one presented in Chapter 6, in order to approximate the solution of doubly reflected BSDEs with jumps and irregular obstacles. This scheme is obtained by a direct discretization of the DRBSDE and it thus depends only on the time step n (no more on the penalization parameter p). We provide the convergence of the scheme, as well as some numerical examples.

Regarding the perspectives, there are many directions of research to follow, from both theoretical and numerical point of views. In collaboration with R. Elie and D. Possamai we are finishing a work on BSDEs with weak reflection ([58]), which are related to the approximative hedging for American options. Together with M.C. Quenez and A. Sulem, we address a new mixed stochastic control/ optimal stopping game problem in the Markovian framework [64] and study the links between Generalized Dynkin Games and nonlinear pricing, in complete and incomplete markets ([65]). From a numerical point of view, it would be useful to propose some numerical schemes for the solution of DRBSDEs with RCLL barriers in the case of a general Poisson measure, as well as for BSDEs with weak terminal condition.

Chapter 2

Introduction générale

2.1 EDSR avec condition terminale faible non linéaire

Ce chapitre repose sur un article écrit sous la coordination de Prof B. Bouchard[1] et soumis pour publication: ≪BSDEs with nonlinear weak terminal condition≫ [57].

2.1.1 Préliminaires et vue d'ensemble de la littérature

Nous commençons par rappeler qu'une équation différentielle stochastique rétrograde (EDSR) est une équation de la forme

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \;\; 0 \leq t \leq T, \tag{2.1.1}$$

où $\{W_t\}_{0\leq t\leq T}$ est un mouvement brownien défini sur un espace de probabilité équipé de la filtration naturelle complète notée $\{\mathcal{F}_t\}_{0\leq t\leq T}$. Les données d'une telle équation sont la condition terminale ξ, qui est une variable aléatoire $\mathcal{F}_T$-mesurable à valeurs dans $\mathbf{R}$ et un générateur g qui est une fonction aléatoire définie sur $[0,T]\times\Omega\times\mathbf{R}\times\mathbf{R}$, à valeurs dans $\mathbf{R}$, mesurable par rapport à la tribu $\mathcal{P}\otimes\mathcal{B}(\mathbf{R})\otimes\mathcal{B}(\mathbf{R})$ et $\mathcal{B}(\mathbf{R})$ où $\mathcal{P}$ représente la tribu prévisible. Résoudre cette équation signifie trouver un couple de processus $\{(Y_t, Z_t)\}_{0\leq t\leq T}$ qui satisfont l'équation (2.1.1) et qui sont adaptés à la filtration générée par le mouvement brownien. Une définition plus précise est donnée ci-dessous.

Definition 2.1.1. *La solution d'une EDSR est un couple de processus* (Y, Z) *à valeurs dans* $\mathbf{R}\times\mathbf{R}$ *tel que* Y *est continu et adapté,* Z *est prévisible et* $\mathbb{P}$*-p.s,* $t \mapsto Z_t$ *appartient à* $\mathbf{L}_2(0,T)$*,* $t \mapsto g(t, Y_t, Z_t)$ *appartient à* $\mathbf{L}_1(0,T)$ *et*

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \;\; 0 \leq t \leq T. \tag{2.1.2}$$

Les EDSR ont été introduites pour la première fois avec un générateur linéaire par J.-M.Bismut [24]. Le point de départ de la théorie des EDSR non linaires est l'article de E. Pardoux et S. Peng [124] où les auteurs prouvent l'existence et l'unicité pour des EDSR avec un générateur non linéaire en (y, z). Rappelons ce résultat:

[1] J'exprime aussi ma gratitude envers Prof. Romuald Elie pour toutes les discussions très fructueuses qu'il a initiées.

Theorem 2.1.2 (E.Pardoux - S.Peng). *Supposons le générateur g Lipschitz en (y,z), uniformément par rapport (t,ω) et*

$$\mathbb{E}[|\xi|^2 + \int_0^T |g(s,0,0)|^2 ds] < +\infty.$$

Alors l'EDSR (2.1.1) *admet une unique solution (Y,Z) appartenant $\mathbf{S}_2 \times \mathbf{H}_2$.*

Exprimée dans le ≪sens forward≫, la résolution de ces équations revient à trouver une condition initiale Y_0 et un processus Z tels que le processus contrôlé $(Y_t^{Y_0,Z})_{0\le t\le T}$ satisfait l'EDS:

$$Y_t^{Y_0,Z} = Y_0 - \int_0^t g(s, Y_s^{Y_0,Z}, Z_s)ds + \int_0^t Z_s dW_s \tag{2.1.3}$$

et la condition $Y_T^{Y_0,Z} = \xi$ à la date terminale T. D'un point de vue des application en finance, l'étude de ces équations se rapporte à l'évaluation d'options européennes en marché complet car Y donne le prix et Z fournit la stratégie de couverture associée. Cependant, puisqu'en marché incomplet il n'est pas toujours possible de construire un portefeuille de réplication tel que la valeur terminale coïncide avec le prix de la créance ξ, une formulation plus faible est de chercher une condition initiale Y_0 et un contrôle Z tels que

$$Y_T^{Y_0,Z} \ge \xi. \tag{2.1.4}$$

Dans ce cas, nous sommes intéressés à trouver la condition initiale Y_0 minimale qui correspond au coût de la stratégie de sur-réplication la moins onéreuse pour la créance éventuelle ξ, et le contrôle associé Z (voir e.g. [72]).

Comme dans la plupart des cas, le prix de sur-réplication conduit à coût trop élevé pour l'acheteur, il a été suggéré d'assouplir la forte contrainte (2.1.4) en une version plus faible de la forme

$$\mathbb{E}[l(Y_T^{Y_0,Z} - \xi)] \ge m, \tag{2.1.5}$$

où m est un seuil fixé et l est une fonction croissante.

En particulier, $l(x) = \mathbf{1}_{\{x\ge 0\}}$ correspond au critère $Y_T^{Y_0,Z} \ge \xi$ avec probabilité m au moins ce qui correspond au problème de couverture de quantile introduit par Föllmer et Leukert [84]. Ce problème a ensuite été étudié par Bouchard, Elie et Touzi [32] dans un cadre markovien en utilisant les techniques de cible stochastique développées par Soner et Touzi (voir [142]). Cette approche, reposant sur la formulation primale de la fonction valeur et sur la programmation dynamique géométrique, permet un traitement de ce problème dans un cadre plus général, par exemple quand la stratégie de l'agent peut influencer la valeur des actifs risqués (modèle avec un grand investisseur). La résolution initiale du problème par Föllmer et Leukert s'appuie sur le fait que la stratégie est linéaire dans le contrôle.

Plus généralement, l peut représenter une fonction de perte, un exemple classique est $l(x) := -(x^-)^q$ avec $q \ge 1$, voir [85] pour des dynamiques générales non markoviennes mais linéaires. Un autre exemple en mathématiques financières peut être représenté avec l jouant le rôle d'une fonction d'utilité.

Très récemment, Bouchard, Elie et Reveillac [31] ont traité le problème dans un cadre non linéaire et non markovien. Dans ce but, ils ont introduit une nouvelle classe d'EDSR pour lesquelles la condition terminale n'est pas une variable aléatoire fixée mais satisfait seulement la faible contrainte suivante

$$\mathbb{E}[\Psi(Y_T^{Y_0,Z})] \geq m. \tag{2.1.6}$$

Le problème peut alors être formulé comme suit:

$$\text{Trouver le plus petit } Y_0 \text{ tel que (2.1.3) et (2.1.6) sont vérifiées pour un } Z. \tag{2.1.7}$$

L'idée centrale est de ≪transposer≫ le problème (2.1.7) écrit en termes d'EDSR avec conditions terminales faibles en un problème équivalent, exprimé comme un problème d'optimisation sur les solutions d'une famille d'EDSR avec condition terminale forte, indexées par un contrôle additionnel α, comme cela sera expliqué dans la suite. Afin de mener à bien cet objectif, les auteurs font appel au théorème de représentation des martingales. Plus précisément, si Y_0 et Z sont tels que (2.1.6) est vérifié, le théorème de représentation des martingales implique qu'on peut trouver un élément α dans l'ensemble $\mathbf{A}_0$ des processus prévisibles de carré intégrable tel que

$$\Psi(Y_T^{Y_0,Z}) \geq M_T^{m,\alpha} := m + \int_0^T \alpha_s dW_s. \tag{2.1.8}$$

Puisque Ψ est croissante, on peut définir son inverse continue à gauche Φ et on obtient que la solution (Y^α, Z^α) de l'EDSR suivante

$$Y_t^\alpha = \Phi(M_T^{m,\alpha}) + \int_t^T g(s, Y_s^\alpha, Z_s^\alpha)ds - \int_t^T Z_s^\alpha dW_s,\ 0 \leq t \leq T, \tag{2.1.9}$$

résout (2.1.3) et (2.1.6). Il est finalement prouvé que la solution de (2.1.7) est donnée par

$$\mathcal{Y}_0(m) := \inf\{Y_0^\alpha, \alpha \in \mathbf{A}_0\}. \tag{2.1.10}$$

Nous souhaitons faire remarquer que, dans un cadre markovien, la même idée est utilisée: introduire un processus additionnel M et un contrôle α, la difficulté provenant du fait que α n'est pas borné puisqu'il est donné par un théorème de représentation des martingales.
Maintenant, afin d'étudier (2.1.10), les auteurs rendent le problème dynamique et définissent

$$\mathcal{Y}^\alpha(\tau) := \text{essinf}\{Y_\tau^{\alpha'},\ \alpha' \in \mathbf{A}_0 \text{ t.q. } \alpha' = \alpha \text{ sur } [\![0,\tau]\!]\},\ \ 0 \leq \tau \leq T. \tag{2.1.11}$$

Il est montré que la famille $\{\mathcal{Y}^\alpha,\ \alpha \in \mathbf{A}_0\}$ satisfait un principe de programmation dynamique qui peut être vu comme une contrepartie du principe de programmation dynamique géométrique. Une représentation de la famille $\{\mathcal{Y}^\alpha,\ \alpha \in \mathbf{A}_0\}$ est ensuite obtenue en termes de super-solution minimale d'une famille d'EDSR avec générateur g et des conditions terminales (fortes) $\{\Phi(M_T^{m,\alpha}), \alpha \in \mathbf{A}_0\}$, ainsi que l'existence d'un contrôle optimal lorsque g et Φ sont convexes. Quelques propriétés de la fonction valeur décrite par (2.1.10), comme la continuité et la convexité par rapport au seuil m, sont obtenues. Finalement, par des arguments probabilistiques, il est montré que le problème (2.1.10) admet une représentation duale qui prend la forme d'un problème de contrôle stochastique sous la forme de Meyer, étendant les résultats obtenus lorsque le générateur g est linéaire (voir [32], [84] et [85]).

2.1.2 Contributions

Dans ce chapitre, nous introduisons une classe d'EDSR plus générale que celle considérée par Bouchard, Elie et Reveillac [31] dont la condition terminale satisfait la *contrainte faible et non linéaire* suivante:

$$\mathcal{E}^f_{0,T}[\Psi(Y_T^{Y_0,Z})] \geq m, \tag{2.1.12}$$

où $\mathcal{E}^f[\xi]$ est l'opérateur non linéaire qui donne la solution de l'EDSR associée à la condition terminale ξ et le *générateur non linéaire* f. Nous pouvons facilement remarquer que (2.1.6) représente un cas particulier de (2.1.12) pour $f = 0$. Le problème étudié dans cet article est le suivant:

$$\inf\{Y_0 \text{ tel que } \exists Z : \text{(1.1.3) et (1.1.12) sont vérifiées}\}. \tag{2.1.13}$$

En suivant l'idée centrale de [31], nous réécrivons notre problème (2.1.13) en un autre équivalent exprimé en termes d'une famille d'EDSR avec condition terminale forte. La différence principale par rapport à [31] est que, dans notre cas, nous devons introduire un nouveau processus de diffusion contrôlé qui est une *$f-$martingale*, contrairement à [31] où c'est une martingale classique. En effet, pour Y_0 et Z tels que (2.1.3) et (2.1.12) sont vérifiées, utilisant la *représentation par EDSR* de $\Psi(Y_T^{Y_0,Z})$, nous pouvons trouver $\alpha \in \mathbf{A}_0$ tel que:

$$\Psi(Y_T^{Y_0,Z}) \geq \mathcal{M}_T^{m,\alpha} = m - \int_0^T f(s, \mathcal{M}_s^{m,\alpha}, \alpha_s)ds + \int_0^T \alpha_s dW_s. \tag{2.1.14}$$

Grâce à cette observation, nous montrons que le problème (2.1.13) est équivalent à (2.1.10) où, dans notre cadre général, Y_t^α correspond à la solution à la date t de l'EDSR avec condition terminale (forte) $\Phi(\mathcal{M}_T^{m,\alpha})$. Nous étudions la contrepartie dynamique de (1.1.10):

$$\mathcal{Y}^\alpha(\tau) := \text{essinf}\{Y_\tau^{\alpha'}, \alpha' \in \mathbf{A}_0 \text{ t.q. } \alpha' = \alpha \text{ sur } [\![0,\tau]\!]\}. \tag{2.1.15}$$

Nous réalisons une analyse similaire, comme dans [31], de la famille $\{\mathcal{Y}^\alpha,\ \alpha \in \mathbf{A}_0\}$. Nous commençons par étudier pour chaque $\alpha \in \mathbf{A}_0$ la régularité de la famille $\{\mathcal{Y}^\alpha(\tau), \tau \in \mathcal{T}\}$. Plus précisément, nous montrons qu'elle peut être agrégée en un processus continu à droite, résultat qui nécessiste dans notre cas des arguments fins d'analyse stochastique dus à la non-linéarité du générateur f. Nous obtenons ensuite que $\mathcal{Y}^\alpha$ correspond à l'unique solution minimale d'une EDSR. Nous montrons que notre fonction valeur est continue et convexe (dans un sens probabiliste) par rapport au seuil m. Si le générateur f est concave, nous obtenons l'existence d'un contrôle optimal ainsi qu'une représentation duale. En effet, nous prouvons que $\mathcal{Y}_0(m)$ (défini par (2.1.10)) correspond à la transformée de Fenchel de la fonction valeur du problème de contrôle stochastique suivant, c.à.d. $\mathcal{Y}_0(m) = \sup_{l>0}(lm - \mathcal{X}_0(l))$, où

$$\mathcal{X}_0(l) := \inf_{(\lambda,\gamma)\in\mathcal{U}\times\mathcal{V}} X_0^{l,\lambda,\gamma}, \tag{2.1.16}$$

avec

$$X_0^{l,\lambda,\gamma} := \mathbb{E}\left[\int_0^T \mathcal{L}_s^\lambda \tilde{g}(s,\lambda_s)ds - \int_0^T \mathcal{A}_s^{l,\gamma}\tilde{f}(s,\gamma_s)ds + \mathcal{L}_T^\lambda \tilde{\Phi}(\frac{\mathcal{A}_T^{l,\gamma}}{\mathcal{L}_T^\lambda})\right],$$

et $\tilde{f}$ (respectivement $\tilde{g}$, $\tilde{\Phi}$) la conjuguée concave de f (respectivement la conjuguée convexe de g et Φ).
La non-linéarité additionnelle de f soulève à des difficultés techniques, puisque la plupart des résultats de [31] sont obtenus en utilisant des techniques exploitant la linéarité de la contrainte et ne sont donc pas adaptées à notre cadre non linéaire. Outre l'intérêt mathématique de notre étude, ce travail est aussi motivé par des applications financières. En effet, notre problème est étroitement lié à la *couverture approximative sous des contraintes données par des mesures de risque dynamiques* d'une créance, qui peut être exprimée par la forme suivante:

$$\inf\{Y_0 \text{ such that } \exists\, Z: \ \rho_{0,T}[-(Y_T-\xi)^-] \leq m\}, \tag{2.1.17}$$

où $\rho_{t,T}[\xi]$ représente la mesure de risque à la date t de ξ qui est définie par $-\mathcal{E}^f_{t,T}[\xi]$. Notons que lorsque le générateur est non linéaire et concave, la mesure de risque associée est convexe. Davantage de détails concernant la conception des mesures de risque dans un cadre dynamique au moyen des EDSR sont présentés dans le chapitre suivant.

2.2 Arrêt optimal pour des mesures de risque dynamiques avec sauts et problèmes d'obstacle

Ce chapitre repose sur l'article ≪Optimal Stopping for Dynamic Risk measures with jumps and obstacle problems≫ [61], joint avec M.C. Quenez et A. Sulem, *J. Optim. Theory Applic.*(2014) DOI 10.1007/s10957-014-0636-2.

2.2.1 Préliminaires et vue d'ensemble de la littérature

Dans le premier chapitre, nous avons introduit les équations différentielles stochastiques rétrogrades dans le cadre d'une filtration engendrée par le mouvement brownien, qui peuvent être vues comme une généralisation de l'espérance conditionnelle d'une variable aléatoire ξ, puisque lorsque f est la fonction nulle, nous avons $Y_t = \mathbb{E}[\xi|\mathcal{F}_t]$, et dans ce cas, Z est le processus qui apparait dans la propriété de représentation $(\mathcal{F}_t)_{t\geq 0}$-martingale de $(\mathbb{E}[\xi|\mathcal{F}_t])_{t\geq 0}$.

Dans le cas d'un espace de probabilité filtré généré par un mouvement brownien W et une mesure aléatoire de Poisson N avec compensateur ν, la représentation martingale de $(\mathbb{E}[\xi|\mathcal{F}_t])_{t\geq 0}$ devient:

$$\mathbb{E}[\xi|\mathcal{F}_t] = \xi + \int_t^T Z_s dW_s + \int_t^T \int_{\mathbf{R}^*} U_s(e)(N-\nu)(de,ds), \ \mathbb{P}-\text{p.s.},$$

où U est une fonction prévisible. Ceci conduit à la généralisation naturelle suivante de l'équation (2.1.1) dans le cas à sauts. Nous dirons que (Y,Z,U) est une solution de l'EDSR avec sauts (EDSRS), de générateur g et de condition terminale ξ si, pour tout $t \in [0,T]$, nous avons $\mathbb{P}$-p.s.

$$Y_t = \xi + \int_t^T g(s,Y_s,Z_s,U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbf{R}^*} U_s(e)(N-\nu)(de,ds), \ \ 0 \leq t \leq T. \tag{2.2.1}$$

En 1994, Tang et Li [147] prouvent l'existence et l'unicité d'une solution pour (1.2.1) dans le cas où g est Lipschitz en (y, z, u).
Le cadre discontinu est plus compliqué, particulièrement en ce qui concerne le théorème de comparaison qui nécesssite une hypothèse supplémentaire. En 1995, Barles, Buckdahn, Pardoux [9] obtiennent un théorème de comparaison ainsi que des liens entre les EDSR et les équations intégro-différentielles partielles paraboliques et non linéaires, généralisant des résultats de [125] au cas à sauts. En 2006, Royer [140] prouve un théorème de comparaison sous des hypothèses plus faibles et introduit l'espérance non linéaire dans ce cadre.

En outre, en 2004-2005, différents auteurs ont introduit des mesures de risque dynamiques dans un cadre brownien, définies comme des solutions d'EDSR. Plus précisément, étant donné un générateur Lipschitz $g(t, x, \pi)$ et une condition terminale T, la mesure de risque ρ à la date t d'une position ξ est donnée par $-X_t$ où X est la solution de l'EDSR dirige par un mouvement brownien, associée au driver g et à la condition terminale ξ. Par le théorème de comparaison, ρ satisfait la *propriété de monotonicité* qui est habituellement exigée pour une mesure de risque. Beaucoup d'études ont été faites récemment sur ce type de mesure de risque dynamique, particulièrement concernant les problèmes d'optimisation robuste et les problèmes d'arrêt optimal, dans le cas d'une filtration brownienne et d'un générateur concave (voir, par exemple, Bayraktar et co-auteurs de [13]). Dans le cas avec sauts, les liens entre les EDSR et les mesures de risque dynamiques ont été étudiés recemment par Quenez-Sulem dans [137].

Les Équations Différentielles Stochastiques Rétrogrades Réfléchies (EDSRR) ont été introduites en 1997 par El Karoui, Kapoudjian, Pardoux, Peng et Quenez [71] dans le cas d'une filtration générée par un mouvement brownien. Ces équations sont des généralisations du problème de Skorokhod déterministe. En effet, étant donné un processus adapté $\xi := (\xi_t)_{t \leq T}$ qui joue le rôle d'une barrière, la solution d'une EDSRR associée aux données (η, g, ξ) est un triplet de processus de carré intégrable $\{(Y_t, Z_t, A_t);\ \ 0 \leq t \leq T\}$ qui satisfont:

$$\begin{cases} Y_t = \eta + \displaystyle\int_t^T g(s, \omega, Y_s, Z_s) ds + A_T - A_t - \int_t^T Z_s dW_s, \ \ 0 \leq t \leq T, \\ Y_t \geq \xi_t, \\ \displaystyle\int_0^T (Y_t - \xi_t) dA_t = 0. \end{cases} \tag{2.2.2}$$

où A est un processus continu et croissant dont le rôle est de repousser la solution Y afin qu'elle reste au-dessus de la barrière ξ. La condition $\int_0^T (Y_t - \xi_t) dA_t = 0$ assure que le processus A agit de manière minimale. Plus précisément, A croît seulement sur l'ensemble $\{Y = \xi\}$.

Le développement des EDSR réfléchies a été motivé par deux importantes applications: l'évaluation et la couverture d'options américaines, particulièrement dans les marchés avec contraintes, et la représentation probabiliste des solutions des problèmes d'obstacles pour les EDP non linéaires.

Concernant l'application en mathématiques financières, El Karoui, Pardoux et Quenez [72] montrent qu'en marché complet, le prix d'une option américaine avec actif sous-jacent $(\xi_t)_{t \leq T}$ et prix d'exercice γ est donné par Y_0 où $(Y_t, \pi_t, A_t)_{t\leq T}$ est la solution de l'EDSR réfléchie suivante:

$$\begin{cases} -dY_t = b(t, Y_t, \pi_t)dt + dA_t - \pi_t dW_t, \;\; Y_T = (\xi_T - \gamma)^+, \\ Y_t \geq (\xi_t - \gamma)^+ \text{ et } \displaystyle\int_0^T (Y_t - (\xi_t - \gamma)^+) dA_t = 0, \end{cases} \tag{2.2.3}$$

pour un choix particulier de b. Le processus π donne la stratégie de réplication et A est le processus de consommation de l'acheteur. Dans un marché financier standard, la fonction b est donnée par $b(t, \omega, y, z) = r_t y + z\theta_t$ où θ_t est la prime de risque et r_t représente le taux d'intérêt d'investissement ou d'emprunt.

La généralisation au cas des EDSR réfléchies avec sauts, qui est une EDSR réfléchie standard dirigée par un mouvement brownien et mesure aléatoire de Poisson indépendante, a été établie par Hamadene et Ouknine dans [90]. Une solution pour ce type d'équation, associée à un générateur f, une valeur terminale η et une barrière ξ, est un quadruplet de processus (Y, Z, U, A) de solutions adaptées qui satisfont l'équation suivante:

$$\begin{cases} Y_t = \eta + \displaystyle\int_t^T g(s, \omega, Y_s, Z_s, U_s) ds + A_T - A_t \\ \qquad - \displaystyle\int_t^T Z_s dW_s - \int_t^T \int_{\mathbf{R}^*} U_s(e) \tilde{N}(ds, de), \;\; 0 \leq t \leq T, \\ Y_t \geq \xi_t, \\ \displaystyle\int_0^T (Y_t - \xi_t) dA_t = 0. \end{cases} \tag{2.2.4}$$

Utilisant deux méthodes - la première repose sur la méthode de pénalisaton et la seconde sur la théorie de l'enveloppe de Snell - les auteurs ont montré l'existence et l'unicité des solutions si η est de carré intégrable, g est uniformément Lipschitz par rapport à (y, z, u) et la barrière ξ est continue à droite et limitée à gauche (càdlàg) dont les dates de saut sont des temps d'arrêt inaccessibles. Notons que la deuxième condition joue un rôle crucial dans leurs preuves. Dans ce cas, les dates des sauts du processus Y proviennent uniquement de ceux du processus de Poisson associé et sont donc inaccessibles.

Le cas général des EDSRR avec sauts et obstacles irréguliers a été considéré e.g. dans [78] et plus récemment par Quenez-Sulem [138]. La barrière ξ est seulement càdlàg et donc les dates des sauts du processus Y ne proviennent pas uniquement de ceux du processus de Poisson associé (sauts inaccessibles) mais aussi de ceux du processus ξ (sauts prévisibles) qui implique que le processus Y a deux types de sauts: inaccesibles et prévisibles. La difficulté vient du fait que, puisque la barrière ξ peut avoir des sauts prévisibles, le processus réfléchissant A n'est plus continu mais seulement càdlàg. Dans ce cas, la différence par rapport à (2.2.4) apparaît seulement dans la condition de Skorokhod qui devient: $\int_0^T (Y_{t^-} - \xi_{t^-}) dA_t = 0$.

Une application importante des EDSR réfléchies est leur connexion aux problèmes d'arrêt optimal et aux inégalités variationnelles associées dans le cas markovien. Plus précisément, étant donné un processus càdlàg $(\xi_t, \; 0 \leq t \leq T)$ et un générateur Lipschitz g satisfaisant une hypothèse supplémentaire afin que le théorème de comparaison soit vérifié, la solution Y de l'EDSRR associée satisfait: pour tout temps d'arrêt $S \in \mathcal{T}_0$,

$$Y_S = \operatorname*{esssup}_{\tau \in \mathcal{T}_S} \mathcal{X}_S(\xi_\tau, \tau), \quad \text{p.s.} \tag{2.2.5}$$

où pour $\tau \in \mathcal{T}_S$, $\mathcal{X}_\cdot(\xi_\tau, \tau)$ est la solution de l'EDSR associée à la date terminale τ, condition terminale ξ_τ et générateur g (voir [138]). On précise que $\mathcal{T}_S$ représente l'ensemble des temps d'arrêt à valeurs en $[0, T]$, p.s. plus grands que S.

2.2.2 Contributions

Dans ce chapitre, nous étudions le problème d'arrêt optimal pour des mesures de risque dynamiques avec sauts dans un cadre markovien.

Formulons notre problème. Soit $T > 0$ la date terminale et f un générateur Lipschitz. Pour chaque $T' \in [0, T]$ et $\eta \in \mathbf{L}_2(\mathcal{F}_{T'})$, définissons:

$$\rho_t^f(\eta, T') = \rho_t(\eta, T') := -\mathcal{X}_t(\eta, T'), \;\; 0 \leq t \leq T', \tag{2.2.6}$$

où $\mathcal{X}_t(\eta, T')$ désigne la solution de l'EDSR avec générateur f et condition terminale (T', η). Si T' représente une maturité et η une position financière à la date T', alors $\rho(\eta, T')$ est interprété comme le risque de η à la date t. La fonctionnelle $\rho : (\eta, T') \mapsto \rho_\cdot(\eta, T')$ représente alors une *mesure de risque dynamique* induite par l'EDSR avec générateur g.

Soit $(\xi_t, \; 0 \leq t \leq T)$ un processus càdlàg et adapté dans $\mathbf{S}_2$ qui représente une position financière dynamique. Soit $S \in \mathcal{T}_0$. Le problème est minimiser la mesure de risque à la date S. Soit $v(S)$ la fonction valeur associée, égale à la variable aléatoire $\mathcal{F}_S$-mesurable (unique pour l'égalité au sens presque sûr) définie par

$$v(S) := \operatorname*{essinf}_{\tau \in \mathcal{T}_S} \rho_S(\xi_\tau, \tau), \tag{2.2.7}$$

Cette variable aléatoire $v(S)$ correspond à la mesure de risque minimale à la date S.
Puisque par définition $\rho_S(\xi_\tau, \tau) = -\mathcal{X}_S(\xi_\tau, \tau)$, nous avons, pour chaque temps d'arrêt $S \in \mathcal{T}_0$,

$$v(S) = \operatorname*{essinf}_{\tau \in \mathcal{T}_S} -\mathcal{X}_S(\xi_\tau, \tau) = -\operatorname*{esssup}_{\tau \in \mathcal{T}_S} \mathcal{X}_S(\xi_\tau, \tau). \tag{2.2.8}$$

Maintenant, en utilisant le lien entre les EDSR réfléchies et l'arrêt optimal (2.2.5), on peut relier la fonction valeur du problème défini par (2.2.8) à la solution de l'EDSR réfléchie. Plus précisément, nous avons:

$$v(S) = -Y_S. \tag{2.2.9}$$

Puisque notre objectif est de caractériser cette fonction valeur dans un cadre markovien, nous considérons la condition terminale, l'obstacle et le générateur de la forme suivante:

$$\begin{cases} \xi_s^{t,x} := h(s, X_s^{t,x}), \;\; s < T, \\ \xi_T^{t,x} := g(X_T^{t,x}), \\ g(s, \omega, y, z, k) := g(s, X_s^{t,x}(\omega), y, z, k), \;\; s \leq T, \end{cases} \tag{2.2.10}$$

où (t, x) est une condition initiale fixée et $X^{t,x}$ est un processus d'état qui a la dynamique suivante:

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r + \int_t^s \int_{\mathbf{R}^*} \beta(X_{r^-}^{t,x}, e)\tilde{N}(dr, de). \tag{2.2.11}$$

Les fonctions $f, h, g, b, \sigma, \beta$ sont déterministes et satisfont les conditions de Lipschitz habituelles (le lecteur pourra se référer au chapitre correspondant). Dans le cadre markovien, pour chaque (t, x), la mesure de risque minimale $v(t, x)$ est définie par:

$$v(t, x) = -Y_t^{t,x}, \tag{2.2.12}$$

où $Y^{t,x}$ est l'EDSR réfléchie dont les paramètres sont donnés par (1.2.10).

Notre contribution principale est l'établissement d'un lien entre la fonction valeur de notre problème d'arrêt optimal et les inégalités variationnelles des EDP intégro-différentielles et paraboliques. Nous prouvons que la mesure de risque minimale est une solution de viscosité d'une EDP intégro-différentielle. Cela fournit un résultat d'existence pour le problème d'obstacle sous des hypothèses relativement faibles. Dans le cas brownien, ce résultat est obtenu en utilisant une méthode de pénalisation par les EDSR non réfléchies. Cette méhode pourrait aussi être adaptée dans notre cas avec sauts, mais impliquerait de lourds calculs afin de prouver la convergence des solutions des EDSR pénalisées vers la solution de l'EDSR réfléchie. Cela demanderait également des résultats de convergence des solutions de viscosité dans le cas intégro-différentiel. Nous obtenons une preuve directe et plus courte.

Sous quelques hypothèses supplémentaires, nous obtenons un théorème de comparaison, s'appuyant sur une version non locale du lemme de Jensen-Ishii, duquel l'unicité de la solution de viscosité est une conséquence. Nous étendons les résultats de [10] dans le cas des EDSR non linéaires qui conduisent à un opérateur intégro-différentiel plus complexe dans l'EDP associée. Dans le cas d'équations intégro-différentielles, une difficulté significative réside dans le traitement d'opérateurs non locaux. L'idée principale est de les décomposer en un opérateur qui correspond aux *petits sauts* et un autre qui correspond aux *grands sauts* et d'utiliser une définition moins classique des solutions de viscosité introduite dans [10], adaptée aux équations intégro-différentielles et équivalentes aux deux classiques qui combinent l'approche avec les fonctions tests et les *sub-superjets* (la solution est remplacée par la fonction test seulement autour de la singularité de la mesure dans l'opérateur non local).

2.3 Jeux de Dynkin généralisés et EDSR doublement réfléchies avec sauts

Ce chapitre repose sur l'article ≪Generalized Dynkin games and Doubly Reflected BSDEs with jumps≫ [62], travail en collaboration avec M.C. Quenez et A. Sulem et soumis pour publication.

2.3.1 Préliminaires et vue d'ensemble de la littérature

Le jeu de Dynkin est un jeu à somme nulle d'arrêt optimal entre deux joueurs. Chaque joueur peut soit arrêter le jeu soit le continuer. Le jeu est arrêté dès lors que l'un des deux joueurs arrête et le gain dépend de qui l'a arrêté. Ce jeu d'arrêt stochastique, de nos jours appelé jeu de Dynkin, a été introduit pour la première fois par Dynkin [66] comme une généralisation des problèmes d'arrêt optimal. Depuis, une quantité considérable de travaux de recherche a été réalisée sur les jeux de Dynkin et les problèmes associés. Quelques exemples: Dynkin et Yushkevich (1968) [67], Bensoussan et Friedman (1974) [21], Neveu (1975) [121], Bismut (1977) [23], Stettner (1982) [146], Alario, Lepeltier et Marechal (1982) [1], Morimoto (1984) [119], Lepeltier et Maingueneau (1984) [111], Cvitanic et Karatzas (1996) [52], Karatzas et Wang (2001) [101], Ekstrom et Peskir [77], Laraki et Solan [114], Peskir [135], Rosenberg et al. [139], Touzi et Vieille (2002) [149] etc. La plupart de la littérature s'intéresse à établir l'existence d'arrêts optimaux ainsi que la valeur sous différents modèles et hypothèses de gain. En temps discret, il est facile de montrer l'existence d'arrêt optimaux ainsi que la valeur en utilisant des arguments d'induction rétrograde. En temps continu, un résultat important est dû à Lepeltier et Maingueneau [111] qui prouvent l'existence de temps d'arrêts ε-optimaux ainsi que la valeur.

Rappelons la formulation mathématique d'un jeu de Dynkin classique.

Le cadre associé au problème est très simple. Il y a deux joueurs, nommés Joueur 1 et Joueur 2, qui observent deux processus de gain ξ et ζ définis sur un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$. Joueur 1 (respectivement Joueur 2) choisit un temps d'arrêt τ (respectivement σ) comme contrôle pour ce problème d'arrêt optimal.
Au temps (d'arrêt) $\sigma \wedge \tau$ le jeu est terminé et Joueur 2 paie le montant $\zeta_\sigma \mathbf{1}_{\tau > \sigma} + \xi_\tau \mathbf{1}_{\tau \leq \sigma}$ à Joueur 1. Par conséquent, l'objectif de Joueur 1 est de maximiser son paiement pendant que Joueur 2 essaie de le minimiser. Il est alors naturel d'introduire les valeurs inférieures et supérieures du jeu.

$$\underline{V} := \sup_\sigma \inf_\tau E[\xi_\sigma \mathbf{1}_{\tau>\sigma} + \zeta_\tau \mathbf{1}_{\tau\leq\sigma}]; \ \overline{V} := \inf_\tau \sup_\sigma E[\xi_\sigma \mathbf{1}_{\tau>\sigma} + \zeta_\tau \mathbf{1}_{\tau\leq\sigma}]. \tag{2.3.1}$$

Si les deux fonctions valeurs définies ci-dessus coïncident, on dit que le jeu admet une fonction valeur.

Une application financière intéressante du jeu de Dynkin est l'étude des options jeu, aussi connus sous le nom d'options israéliennes, tels que définies par Kifer [103]. Une option jeu est un contrat entre un émetteur et un détenteur dans lequel le détenteur peut exercer l'option à tout moment pour un gain tandis que l'émetteur peut la résilier à tout moment contre un paiement.

Ceci est l'un des quelques contrats financiers dans lesquels l'émetteur prend également des décisions qui affectent le gain. Si nous ignorons la dépendance par rapport aux actifs sous-jacents et nous concentrons sur les liens entre décisions et gains, l'option jeu est comparable au jeu de Dynkin. De plus, le paiement associé à la résiliation est typiquement supposé supérieur ou égal au gain de l'exercice, faisant écho aux inégalités standard sur le gain existant dans les jeux de Dynkin. À la fois dans les modèles à temps discret et continu, Kifer [103] a montré que l'option jeu a un unique prix d'arbitrage. D'autres recherches associées aux options jeu ainsi qu'à des contrats financiers de type jeu ont été initiées par Bielecki et al. [22], et Dolinsky Kifer [56], Dolinsky et al. [55], Hamadene et Zhang [93], Kallsen et Kuhn [97, 98], et Kifer [103], etc.

Nous nous concentrons maintenant sur le lien entre les *Jeux de Dynkin Classiques* et les *EDSR Doublement Réfléchies* (EDSRDR) introduites par Cvitanic et Karatzas [52] dans le cas d'une filtration brownienne. La solution est contrainte de rester entre une barrière supérieure ζ et une barrière inférieure ξ et est représentée par un quadruplet de processus de carré intégrable $\{(Y_t, Z_t, A_t, A'_t);\ 0 \le t \le T\}$ satisfaisant:

$$\begin{cases} Y_t = \xi + \int_t^T g(s, \omega, Y_s, Z_s)ds + A_T - A_t - (A'_T - A'_t) \\ \qquad - \int_t^T Z_s dW_s, \ \ 0 \le t \le T, \\ \xi_t \le Y_t \le \zeta_t, \\ \int_0^T (Y_t - \xi_t) dA_t = 0 \text{ et } \int_0^T (\zeta_t - Y_t) dA'_t = 0. \end{cases} \tag{2.3.2}$$

avec A et A' deux processus continus, croissants, dont le rôle est de garder le processus Y entre les deux barrières. Ils prouvèrent l'existence et l'unicité de la solution lorsque les barrières sont régulières et satisfont la condition de Mokobodski qui se transforme en l'existence de la différence de deux sur-martingales positives entre ξ et ζ.
Lorsque le générateur g ne dépend que de (t, ω), Cvitanic-Karatzas ont montré que l'existence d'une solution (Y, Z, A) de l'EDSR ci-dessus implique que Y correspond à la fonction valeur d'un Jeu de Dynkin Classique. Enoncons leur résultat:

Theorem 2.3.1 (Cvitanic-Karatzas)**.** *Soit (Y, Z, A, A') une solution de l'EDSR avec $g(t, \omega, y, z) = g(t, \omega)$. Pour tout $0 \le t \le T$ et tout couple de temps d'arrêt $(\tau, \sigma) \in \mathcal{T} \times \mathcal{T}$, considérons le gain:*

$$I_t(\tau, \sigma) := \int_t^{\tau \wedge \sigma} g(u)du + \xi_\tau \boldsymbol{1}_{\tau \le \sigma} + \zeta_\sigma \boldsymbol{1}_{\sigma < \tau},$$

ainsi que les fonctions valeurs supérieures et inférieures, respectivement,

$$\overline{V}(t) := \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}[I_t(\tau, \sigma) | \mathcal{F}_t], \ \ \underline{V}(t) := \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \mathbb{E}[I_t(\tau, \sigma) | \mathcal{F}_t].$$

d'un jeu de Dynkin correspondant. Le jeu a la valeur $V(t)$, donnée par le processus d'état Y de la solition de l'EDSR, qui est:

$$V(t) = \overline{V}(t) = \underline{V}(t) = Y_t \quad a.s. \ \forall 0 \le t \le T,$$

ainsi qu'un point-selle $(\hat{\sigma}_t, \hat{\tau}_t) \in \mathcal{T}_t \times \mathcal{T}_t$ donné par

$$\hat{\sigma}_t := \inf\{s \in [t, T),\ Y_s = \zeta_s\} \wedge T; \ \ \hat{\tau}_t := \inf\{s \in [t, T),\ Y_s = \xi_s\} \wedge T.$$

à savoir

$$\mathbb{E}[I_t(\tau_t, \hat{\sigma}_t)] \le \mathbb{E}[I_t(\hat{\tau}_t, \hat{\sigma}_t)] = Y_t \le \mathbb{E}[I_t(\hat{\tau}_t, \sigma_t)],$$

pour tout $(\sigma, \tau) \in \mathcal{T}_t \times \mathcal{T}_t$.

Depuis l'article séminal de Cvitanic-Karatzas, beaucoup d'auteurs ont exploré l'existence et l'unicité de la solution ainsi que les liens avec les Jeux de Dynkin Classiques avec différentes hypothèses sur le coefficient g et sur la régularité des barrières (voir par exemple Lepeltier-San Martin et [112]). Ces résultats ont également été étendus au cas des EDSRDR dirigées par un mouvement brownien et une mesure aléatoire de Poisson (voir par exemple [86], [87], [50]).

Le lien ci-dessus entre les *Jeux de Dynkin Classiques* et les *EDSRDR* peut être étendu dans le cas plus général des EDSRDR non linéaires, puisque étant donné la solution Y de la EDSRDR, il a été montré qu'elle coïncide avec la fonction valeur d'un jeu de Dynkin classique de gain:

$$I_S(\tau,\sigma) = \int_S^{\sigma\wedge\tau} g(u, Y_u, Z_u, k_u)du + \xi_\tau \mathbf{1}_{\tau\leq\sigma} + \zeta_\sigma \mathbf{1}_{\sigma<\tau}, \tag{2.3.3}$$

où Z, k sont les processus associés à Y. Cependant, cette caractérisation n'est pas vraiment utilisable parce le gain instantané $g(u, Y_u, Z_u, k_u)$ dépend de la fonction valeur Y et du jeu de Dynkin associé. Nous verrons dans la partie originale que nous pouvons définir un problème de jeu bien posé (dans le sens où le critère n'est pas défini à partir de la fonction valeur elle-même) dont nous montrons qu'il admet une valeur qui coïncide avec la solution d'une EDSRDR avec un générateur général non linéaire.

2.3.2 Contributions

Dans le chapitre 3, nous introduisons un nouveau problème de jeu qui peut être vu comme une *généralisation du jeu classique de Dynkin*. Plus précisément, l'espérance linéaire dans le critère est remplacée par une g-espérance conditionnelle non linéaire, induite pas une Équation Différentielle Stochastique Rétrograde (EDSR) avec sauts. Nous décrivons ci-dessous très brièvement ce nouveau problème de jeu.

Soient ξ et ζ deux processus adaptés et qui sont seulement supposés être càdlàg avec $\zeta_T = \xi_T$ p.s., $\xi \in \mathbf{S}_2$ (l'ensemble des processus ϕ tels que $\mathbb{E}[\sup_{t\leq T} \phi_t^2] < +\infty$), $\zeta \in \mathbf{S}_2$, $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ p.s.

Pour chaque $\tau, \sigma \in \mathcal{T}_0$, le *gain* au temps d'arrêt $\tau \wedge \sigma$ est donné par:

$$I(\tau,\sigma) := \xi_\tau \mathbf{1}_{\tau\leq\sigma} + \zeta_\sigma \mathbf{1}_{\sigma<\tau}. \tag{2.3.4}$$

Soit $S \in \mathcal{T}_0$. Pour chaque $\tau \in \mathcal{T}_S$ et $\sigma \in \mathcal{T}_S$, le *critère* associé est donné par $\mathcal{E}_{S,\tau\wedge\sigma}(I(\tau,\sigma))$, la g-espérance conditionnelle de $I(\tau,\sigma)$.

Au temps d'arrêt S, le premier (respectivement le second) joueur choisit un temps d'arret τ (respectivement σ) après S, afin de maximiser (respectivement minimiser) le critère.

Pour chaque temps d'arrêt $S \in \mathcal{T}_0$, les *fonctions valeurs supérieures* et *inférieures* au temps S sont définies comme suit:

$$\overline{V}(S) := \operatorname*{essinf}_{\sigma\in\mathcal{T}_S} \operatorname*{esssup}_{\tau\in\mathcal{T}_S} \mathcal{E}_{S,\tau\wedge\sigma}(I(\tau,\sigma)); \tag{2.3.5}$$

$$\underline{V}(S) := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} \operatorname*{ess\,inf}_{\sigma \in \mathcal{T}_S} \mathcal{E}_{S,\tau\wedge\sigma}(I(\tau,\sigma)). \tag{2.3.6}$$

Le jeu admet une fonction valeur si $\underline{V}(S) = \overline{V}(S)$.

Sous la condition de Mokobodski, nous montrons l'existence d'une fonction valeur pour ce jeu qui peut être caractérisée comme étant l'unique solution d'une EDSR non linéaire doublement réfléchie (EDSRDR). Jusqu'à maintenant, aucune interprétation des EDSR non linéaires doublement réfléchies en termes de contrôle ou de problèmes de jeu (avec espérance non linéaire) n'a été donnée dans la littérature.

Grâce à cette caractérisation, nous obtenons des propriétés de ces EDSRDR comme un théorème de comparaison général et un théorème de comparaison strict. Nous établissons également de nouvelles estimations à priori *avec constante universelle* pour EDSRDR et la preuve s'appuie sur la caractérisation de la solution comme la fonction valeur de ce nouveau problème de jeu. Lorsque les deux obstacles sont semi-continus supérieurement à gauche le long des temps d'arrêts, nous montrons l'existence d'un point-selle du jeu de Dynkin généralisé. Nous soulignons que nous ne supposons pas la stricte séparabilité des barrières, hypothèse qui est cruciale dans la littérature antérieure. Nous pouvons nous en passer en imposant une contrainte supplémentaire sur les processus croissants A, A' qui apparaissent dans la définition (2.3.2) (notons que dans notre cadre les processus croissants A et A' ne sont plus continus). Plus précisément, nous supposons que les mesures dA et dA' sont mutuellement singulières dans le sens probabiliste, i.e. qu'il existe $D \in \mathcal{P}$ tel que

$$E[\int_0^T \mathbf{1}_D dA_t] = E[\int_0^T \mathbf{1}_{D^c} dA'_t] = 0.$$

Cette contrainte est également importante afin d'obtenir l'unicité des processus croissants A et A'. De plus, elle permet d'identifier les sauts positifs et négatifs de la solution de l'EDSRDR.

Nous continuons par l'étude d'un jeu à somme nulle mixte généralisé sous la g-espérance conditionnelle, dans lequel deux joueurs s'affrontent en prenant deux actions: contrôle continu et arrêt. Nous obtenons des conditions suffisantes (par exemple les générateurs contrôlés $g^{u,v}$ ont un point-selle $g^{\overline{u},\overline{v}}$) qui assurent l'existence d'une fonction valeur du jeu mixte généralisé et caractérisent la fonction valeur commune comme la solution d'une EDSRDR avec générateur $g^{\overline{u},\overline{v}}$. Quand les deux obstacles sont semi-continus supérieurement à gauche le long des temps d'arrêt, le jeu mixte généralisé correspondant admet un point-selle.

Nous étudions ensuite le jeu de Dynkin généralisé dans le cadre markovien et ses liens avec les inégalités variationnelles des équations intégro-différentielles partielles paraboliques avec deux obstacles. Plus précisément, nous montrons que la fonction valeur d'un jeu de Dynkin généralisé dans le cas markovien est l'unique solution de viscosité de l'équation intégro-différentielle correspondant. Du point de vue des EDP, ce résultat donne une nouvelle interprétation probabiliste de l'EDP semi-linéaire avec deux barrières en terme de problèmes de jeux.

2.4 Un principe de programmation dynamique faible pour des problèmes combinés de Contrôle Stochastique et d'Arrêt Optimal avec des $\mathcal{E}^f$-Espérances

Ce chapitre repose sur l'article ≪A weak dynamic programming principle for Combined Stochastic Control / Optimal Stopping with $\mathcal{E}^f$-expectations≫ [63] joint avec M.C. Quenez et A. Sulem et soumis pour publication.

2.4.1 Préliminaires et vue d'ensemble de la littérature

Le Principe de Programmation Dynamique (PPD) est l'outil principal dans la théorie du contrôle stochastique. L'idée de base de la méthode est de considérer une famille de problèmes de contrôle stochastique avec différents états initiaux et d'établir des liens entre les fonctions valeurs associées. Cela a été initité dans les années 50 par Bellman ([28], [19]) qui disait que ≪an optimal policy has the property that, whatever the initial state and control are, the remaining decisions must constitue an optimal policy with regard to the state resulting from the first decision≫ qui pourrait se traduire en français par ≪une règle optimale a la propriété que, quel que soit l'état initial et le contrôle, les décisions ultérieures doivent constituer une règle optimale par rapport à l'état résultant de la première décision≫. Typiquement, un problème de contrôle stochastique avec un horizon fini T peut être écrit comme suit:

$$V(0,x) := \sup_{\alpha\in\mathcal{A}} \mathbb{E}[\int_0^T f(s, X_s^\alpha, \alpha_s)ds + g(X_T^\alpha)], \tag{2.4.1}$$

où f est le gain instantané et g est le gain terminal.

Une écriture formelle du PPD est

$$V(0,x) = v(0,x) := \sup_{\alpha\in\mathcal{A}} \mathbb{E}[\int_0^\tau f(s, X_s^\alpha, \alpha_s)ds + V(\tau, X_\tau^\alpha)], \tag{2.4.2}$$

où τ est un temps d'arrêt arbitraire tel que $\tau \in [0,T)$ p.s.

Dans le cas des diffusions à sauts markoviennes contrôlées, le PPD est utilisé afin de calculer l'équation de programmation dynamique correspondante dans le sens des solutions de viscosité. Dans la littérature, le principe est traditionnellement établi sous des hypothèses qui assurent que la fonction valeur satisfait des propriétés de mesurabilité et de régularité, voir e.g. Fleming-Rischel, Krylov, El Karoui, Bensoussan-Lions, Lions P.-L., Fleming-Soner, Touzi pour le cas de diffusions contrôlées et Oksendal et Sulem pour le cas de diffusions markoviennes avec sauts. La relation (2.4.2) du PPD est très intuitive et peut être facilement prouvée dans le cadre déterministe ou en temps discret avec un espace de probabilité fini. Cependant, sa preuve dans le cas général n'est pas triviale et exige dans un premier temps que V soit mesurable.

Le cas d'une fonction valeur discontinue a été étudié dans un cadre déterministe dans les années 80: un principe de programmation dynamique *faible* a été établi pour le contrôle déterministe par Barles (1993) (voir [8], voir aussi Barles et Perthame (1986) [11]).

Plus précisément, il prouve que l'enveloppe semi-continue supérieure V^* et l'enveloppe semi-continue inférieure V_* de la fonction valeur V satisfait, respectivement, le principe de *sous- et sur-optimalité de la programmation dynamique* de Lions et Souganidis (1985) [114]. Il montre ensuite que la fonction valeur (discontinue) est une solution de *viscosité faible* de l'équation de Bellman associée dans le sens où V^* est une sous-solution de viscosité et V_* est une sur-solution de viscosité de l'équation de Bellman.

Plus récemment, Bouchard et Touzi (2011) (voir [35]) ont prouvé un principe de programmation dynamique *faible* dans un cadre stochastique lorsque la fonction valeur n'est pas nécessairement continue, ni même mesurable. Ils prouvent que l'enveloppe semi-continue supérieure V^* satisfait le principe de sous-optimalité de la programmation dynamique et avec une hypothèse supplémentaire de régularité (semi-continuité inférieure) du gain g, ils obtiennent que l'enveloppe semi-continue inférieure V_* satisfait le principe de sur-optimalité.

Un principe de programmation dynamique *faible* a ensuite été établi, sous des hypothèses de régularité spécifiques, pour des problèmes avec contraintes d'état par Bouchard et Nutz (2012) dans [34] et pour les jeux stochastiques à somme nulle par Bayraktar et Yao (2013) dans [14].

Dans la suite, nous présentons les résultats classiques du problème pour le contrôle stochastique et l'arrêt optimal (pour des problèmes à horizon fini) dans le cas où la fonction valeur n'est pas a priori continue, ni même mesurable. Nous rappelons le *principe de programmation dynamique faible* obtenu par Bouchard et Touzi ([35]) ainsi que les équations HJB associées.

(i) *Contrôle stochastique et programmation dynamique faible pour les espérances classiques*

Nous notons $\mathcal{A}$ l'ensemble de tous les processus progessivement mesurables $\alpha = \{\alpha_t, t < T\}$ à valeurs dans A, un sous ensemble de $\mathbf{R}$, appartenant à $\mathbf{H}_2$ (l'ensemble des processus ϕ tels que $\mathbb{E}[\int_0^T \phi_s^2 ds] < +\infty$). Les éléments de $\mathcal{A}$ sont appelés les processus de contrôle.

Pour chaque processus de contrôle $\alpha \in \mathcal{A}$, nous considérons l'équation différentielle stochastique suivante:

$$dX_s^{t,x,\alpha} = b(X_s^{t,x,\alpha}, \alpha_s)ds + \sigma(X_s^{t,x,\alpha}, \alpha_s)dW_s, \tag{2.4.3}$$

où les coefficients b et σ satisfont les conditions habituelles de Lipschitz et de croissance linéaire afin que l'EDS ci-dessus admette une unique solution forte.

Étant donné une condition initiale (t, x), le processus $X^{t,x,\alpha}$ est appelé le processus controlé car sa dynamique est dirigée par l'action du processus de contrôle α.
Nous définissos la fonctionnelle de coût J sur $[0, T] \times \mathbf{R} \times \mathcal{A}$ par:

$$J(t, x, \alpha) = \mathbb{E}\left[\int_t^T f(s, X_s^{\alpha,t,x}, \alpha_s)ds + g(X_T^{\alpha,t,x})\right],$$

où f est continue et Lipschitz et g est borélienne à croissante quadratique.

L'objectif est d'étudier le problème de contrôle stochastique suivant:

$$v(t, x) = \sup_{\alpha \in \mathcal{A}_t} J(t, x, \alpha), \tag{2.4.4}$$

où $\mathcal{A}_t$ représente l'ensemble des contrôles t-admissibles qui sont indépendants de $\mathcal{F}_t$.

Afin de décrire le comportement local de la fonction valeur V au moyen de *l'équation de programmation dynamique* ou *Hamilton-Jacobi-Bellman*, le point clé est le *Principe de Programmation Dynamique.* Puisque le PPD implique la fonction valeur elle-même qui n'est peut-être pas mesurable sous ces hypothèses, Bouchard et Touzi [35] proposent une *version Faible* du *Principe de Programmation Dynamique* laquelle est montrée être suffisante pour obtenir l'équation de programmation dynamique. Ce PPD faible fait apparaître l'enveloppe semi-continue supérieure de la fonction valeur V, respectivement celle semi-continue inférieure, qui sont définies comme suit: pour tout $t \in [0,T]$, pour tout $x \in \mathbf{R}$,

$$V_*(t,x) := \liminf_{(t',x')\to(t,x)} V(t',x') \quad \text{et} \quad V^*(t,x) := \limsup_{(t',x')\to(t,x)} V(t',x'). \tag{2.4.5}$$

Rappelons le Principe de Programmation Dynamique faible.

Theorem 2.4.1 (Principe de Programmation Dynamique Faible)**.** *1. Soit $\{\theta^\alpha,\ \alpha \in \mathcal{U}_t\}$ une famille de temps d'arrêt finis indépendants de $\mathcal{F}_t$ à valeurs dans $[t,T]$. Alors:*

$$V(t,x) \leq \sup_{\alpha\in\mathcal{A}_t} \mathbb{E}[\int_t^{\theta^\alpha} f(s, X_s^{\alpha,t,x}, \alpha_s)ds + V^*(\theta^\alpha, X_{\theta^\alpha}^{\alpha,t,x})], \tag{2.4.6}$$

2. Supposons de plus que g est semi-continue inférieurement et $X_{t,x}^\alpha \mathbf{1}_{t,\theta^\alpha}$ est $\mathbf{L}^\infty$-borné pout tout $\nu \in \mathcal{A}_t$. Alors:

$$V(t,x) \geq \sup_{\alpha\in\mathcal{A}_t} \mathbb{E}[\int_t^{\theta^\alpha} f(s, X_s^{\alpha,t,x}, \alpha_s)ds + V_*(\theta^\alpha, X_{\theta^\alpha}^{\alpha,t,x})]. \tag{2.4.7}$$

Le PPD faible ci-dessus est montré sans utiliser les théorèmes abstraits de sélection mesurable. Les auteurs appellent à la place le *covering lemma* de Vitali. L'inégalité qui est la plus difficile à obtenir est la seconde et elle nécessiste une hypothèse de semi-continuité inférieure sur le critère (qui est satisfaite dans le cas où le gain g est semi-continue inférieurement).

Nous faisons remarquer que lorsque V est continue alors $V = V_* = V^*$ et le principe de programmation dynamique faible ci-dessus se réduit au principe de programmation dynamique classique:

$$V(t,x) = \sup_{\alpha\in\mathcal{A}_t} \mathbb{E}\left[\int_t^{\theta^\alpha} f(s, X_s^{\alpha,t,x}, \alpha_s)ds + V(\theta^\alpha, X_{\theta^\alpha}^{\alpha,t,x})\right].$$

Comme mentionné précédemment, le principe de programmation dynamique faible représente l'étape principale pour obtenir l'équation de programmation dynamique qui correspond à la contrepartie infinitésimale du PPD. Elle est généralement appelée l'équation *Hamilton-Jacobi-Bellman.* L'équation HJB associée est obtenue grâce au théorème suivant:

Theorem 2.4.2. *Supposons que la fonction valeur $V \in C^{1,2}([0,T),\mathbf{R})$ et soit $f(\cdot,\cdot,a)$ continue en (t,x) pour tout $a \in A$ fixé. Alors, pour tout $(t,x) \in [0,T) \times \mathbf{R}$:*

$$-\partial_t V(t,x) - \sup_{a\in A}\{b(t,x,a)\partial_x V(t,x) + \frac{1}{2}Tr[\sigma\sigma(t,x,a)D_{xx}^2 V(t,x)] + f(t,x,a)\} = 0. \tag{2.4.8}$$

Notons que lorsque la fonction valeur V n'est pas continue, alors elle vérifie au sens des solutions de viscoité l'EDP ci-dessus.

Nous présentons maintenant les résultats principaux concernant l'arrêt optimal qui représentent un cas particulier des problèmes de contrôle stochastique lorsque le contrôle prend la forme d'un temps d'arrêt.

(ii) Arrêt optimal et principe de programmation dynamique faible pour les espérances classiques

Pour $0 \leq t \leq T < +\infty$, nous notons $\mathcal{T}_{[t,T]}$ la collection de tous les $\mathbb{F}$-temps d'arrêt à valeurs dans $[t, T]$. Le processus d'état sous-jacent $X^{t,x}$ de condition initiale (t, x) est défini par l'équation différentielle stochastique:

$$dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dW_s,$$

où b et σ satisfont les conditions habituelles de Lipschitz et de croissance linéaire afin que l'EDS ci-dessus ait une unique solution forte.
Soit g une fonction mesurable à croissance polynômiale et supposons que:

$$\mathbb{E}[\sup_{0 \leq t \leq T} |g(X_t)|] < \infty.$$

Pour un temps d'arrêt admissible, le critère est défini comme suit:

$$J(t, x, \tau) = \mathbb{E}\left[g(X_\tau^{t,x})\right] \tag{2.4.9}$$

Nous considérons maintenant le sous-ensemble des temps d'arrêt:

$$\mathcal{T}_t^t := \{\tau \in \mathcal{T}_{[t,T]} : \tau \text{ independents de } \mathcal{F}_t\}. \tag{2.4.10}$$

Le problème d'arrêt optimal est défini par:

$$V(t, x) = \sup_{\tau \in \mathcal{T}_t^t} J(t, x, \tau). \tag{2.4.11}$$

En utilisant les mêmes arguments que pour le problème de contrôle stochastique présenté ci-dessus, Bouchard et Touzi ont montré le *Principe de Programmation Dynamique Faible* suivant:

Theorem 2.4.3. *Pour $(t, x) \in [0, T] \times \mathbf{R}$, soit $\theta \in \mathcal{T}_t^t$ un temps d'arrêt tel que $X_\theta^{t,x}$ soit borné. Alors:*

$$V(t, x) \leq \sup_{\tau \in \mathcal{T}_t^t} \mathbb{E}\left[\mathbf{1}_{\{\tau<\sigma\}} g(X_\tau^{t,x}) + \mathbf{1}_{\{\tau \geq \theta\}} V^*(\theta, X_\theta^{t,x})\right], \tag{2.4.12}$$

$$V(t, x) \geq \sup_{\tau \in \mathcal{T}_t^t} \mathbb{E}\left[\mathbf{1}_{\{\tau<\sigma\}} g(X_\tau^{t,x}) + \mathbf{1}_{\{\tau \geq \theta\}} V_*(\theta, X_\theta^{t,x})\right]. \tag{2.4.13}$$

Lorsqu'on sait a priori que la fonction valeur V est régulière, la contrepartie infinitésimale du principe de programmation dynamique est la suivante:

Theorem 2.4.4. *Supposons que $V \in C^{1,2}([0, T), \mathbf{R})$ et soit $g : \mathbf{R} \mapsto \mathbf{R}$ continue. Alors V est solution du problème d'obstacle:*

$$\min\{-(\partial_t + \mathcal{L})V, V - g\} = 0, \tag{2.4.14}$$

où $\mathcal{L}V$ est le générateur infinitésimal du processus de diffusion markovien X.

Le problème de contrôle stochastique classique (2.4.1) a été généralisé par Peng lorsque la fonctionnelle de coût est définie au moyen d'une équation différentielle stochastique rétrograde non linéaire (voir [127] et [128]) sous des hypothèses qui assurent que la fonction valeur est continue. Il établit une programmation dynamique en utilisant la méthode du semi-groupe rétrograde et obtient les équations HJB associées. Ces résultats permettent d'obtenir une interprétation stochastique pour une plus large classe d'équations HJB non linéaires puisque le coefficient f dépend également de (y, z).

À la fin de cette section, nous aimerions mentionner quelques développements dans le cas où l'incertitude affecte uniquement la volatilité du modèle. Soner, Touzi et Zhang ([145]) ont récemment introduit la notion d'EDSR du second ordre (EDSR2) dont l'idée de base est d'exiger que la solution vérifie les équations $\mathbb{P}^\alpha$ p.s. pour chaque mesure de probabilité dans une classe non dominée de mesures mutuellement singulières. Leur théorie est étroitement liée à la notion de G-espérance de Peng ([129]) et permet d'obtenir une représentation probabiliste différente des solutions d'équations HJB complètement non linéaires.

2.4.2 Contributions

Dans ce chapitre, nous nous intéressons à la généralisation des résultats obtenus par Bouchard et Touzi ([35]) lorsque l'espérance linéaire $\mathbb{E}$ est remplacée par une espérance non linéaire définie par une Équation Différentielle Stochastique avec sauts. Dans le cadre markovien, la fonction valeur de notre problème est la suivante:

$$V(t,x) := \sup_{\alpha \in \mathcal{A}} \mathcal{E}^{\alpha}_{0,T}[g(X_T^{\alpha,t,x})], \tag{2.4.15}$$

où $\mathcal{E}^\alpha$ est l'espérance conditionnelle non linéaire associée à l'EDSR avec sauts et le générateur contrôlé $f(\alpha_t, X_t^\alpha, y, z, k)$. Nous regardons cette étude dans le cas où la fonction de gain g est uniquement borélienne. De plus, dans ce chapitre, nous considérons le problème combiné lorsqu'il y a un contrôle supplémentaire prenant la forme d'un temps d'arrêt. Nous considérons ensuite des problèmes de contrôle stochastique et d'arrêt optimal généralisés de la forme

$$V(t,x) := \sup_{\alpha} \sup_{\tau} \mathcal{E}^{\alpha}_{0,\tau}[\overline{h}(X_\tau^{\alpha,t,x})], \tag{2.4.16}$$

où $\overline{h}(X_\tau^{\alpha,t,x})$ est un gain irrégulié.

Afin de caractériser la fonction valeur comme la solution d'une inégalité variationnelle HJB, nous établissons tout d'abord un *Principe de Programmation Dynamique* qui est obtenu en utilisant des techniques sophistiquées d'analyse stochastique. Nous mettons en avant que, en conséquence des faibles hypothèses sur les coefficients, la fonction valeur de notre problème n'est pas nécessairement continue, ni même mesurable.

Comme mentionné dans la section introductive, puisque pour t fixé, la fonction valeur $x \to V(t,x)$ n'est pas nécessairement mesurable, nous ne pouvons pas établir une programmation dynamique classique. Nous obtenons à la place un *Principe Programmation Dynamique faible* qui implique la fonction V_* et la fonction V^* définies par

$$V^*(t,x) := \limsup_{(t',x')\to(t,x)} V(t',x'),\ \ \forall (t,x) \in [0,T) \times \mathbf{R} \text{ et } V^*(T,x) = g(x), \forall x \in \mathbf{R};$$
$$V_*(t,x) := \liminf_{(t',x')\to(t,x)} V(t',x'),\ \ \forall (t,x) \in [0,T) \times \mathbf{R} \text{ et } V_*(T,x) = g(x), \forall x \in \mathbf{R}.$$

Remarquons que dans notre cas, la fonction V^* (respectivement V_*) n'est pas nécessairement semi-continue supérieurement (respectivement inférieurement) sur $[0,T]\times\mathbf{R}$ car le gain terminal g est seulement borélien (il n'est pas supposé satisfaire une quelconque hypothèse de régularité). Ce n'est pas le cas dans la littérature précédente même dans le cas linéaire où g est supposée être semi-continue inférieurement (voir [35]). Nous donnons ci-dessous *le principe de sous- (respectivement sur-) optimalité de la programmation dynamique* satisfait par V^* (respectivement V_*), l'un de nos principaux résultats de ce chapitre.

Theorem 2.4.5. *(Un principe de programmation dynamique* faible*) La fonction V^* satisfait le* principe de sous-optimalité de la programmation dynamique*, c'est à dire pour tout $t \in [0,T]$ et pour tout temps d'arrêt $\theta \in \mathcal{T}_t^t$,*

$$V(t,x) \leq \sup_{\alpha\in\mathcal{A}_t^t} \sup_{\tau\in\mathcal{T}_t^t} \mathcal{E}^{\alpha,t,x}_{t,\theta\wedge\tau}\left[h(\tau, X^{\alpha,t,x}_\tau)\mathbf{1}_{\tau<\theta} + V^*(\theta, X^{\alpha,t,x}_\theta)\mathbf{1}_{\tau\geq\theta}\right], \tag{2.4.17}$$

La fonction V^* satisfait le principe de sur-optimalité de la programmation dynamique*, c'est à dire pour tout $t \in [0,T]$ et pour tout temps d'arrêt $\theta \in \mathcal{T}_t^t$,*

$$V(t,x) \geq \sup_{\alpha\in\mathcal{A}_t^t} \sup_{\tau\in\mathcal{T}_t^t} \mathcal{E}^{\alpha,t,x}_{t,\theta\wedge\tau}\left[h(\tau, X^{\alpha,t,x}_\tau)\mathbf{1}_{\tau<\theta} + V_*(\theta, X^{\alpha,t,x}_\theta)\mathbf{1}_{\tau\geq\theta}\right]. \tag{2.4.18}$$

Dans le théorème ci-dessus, $\mathcal{A}_t^t$ représente l'ensemble des contrôles indépendents de $\mathcal{F}_t$ et restrictionnés à $[t,T]$. De manière similaire, $\mathcal{T}_t^t$ denote l'ensemble des temps d'arrêt indépendents de $\mathcal{F}_t$, à valeurs dans $[t,T]$.

Le principe de sous-optimalité est le plus facile à obtenir. Il repose sur la propriété de flot pour les EDS rétrogrades et classiques et une propriété de séparation qui dit essentiellement que, étant donné une date intermédiaire $t \leq T$ et une trajectoire fixée jusqu'à la date t (correspondant à la réalisation du mouvement brownien et de la mesure aléatoire de Poisson), l'EDSR peut être résolue par rapport au mouvement brownien et à la mesure de Poisson aléatoire les deux translatés de t. Ce résultat est nécessaire afin de pouvoir utiliser la définition de la fonction valeur qui est une fonction déterministe.

La seconde inégalité est considérablement plus difficile à établir et repose sur l'existence de contrôles ε-optimaux *faibles* pour notre problème mixte de contrôle et temps d'arrêt optimal (résultat qui necessite des arguments fins comme un théorème de sélection mesurable abstrait) ainsi que sur de nouvelles propriétés des EDSR (e.g. un lemme de Fatou pour les EDSR réfléchies où la limite concerne à la fois la date terminale et la condition terminale).

En utilisant ce principe de programmation dynamique faible et un nouveau théorème de comparaison entre les EDSR et les EDSR réfléchies, nous déduisons que la fonction valeur est une solution faible de viscosité d'une inégalité variationnelle HJB non linéaire généralisée. Plus précisément, le résultat est le suivant:

Theorem 2.4.6. *La fonction V, définie dans* (1.4.16)*, est une* solution de viscosité faible *de HJBVI*

$$\begin{cases} \min(V(t,x)-h(t,x), \\ \quad \inf_{\alpha\in\mathbf{A}}(-\frac{\partial V}{\partial t}(t,x)-L^\alpha V(t,x)-f(\alpha,t,x,V(t,x),(\sigma\frac{\partial V}{\partial x})(t,x),B^\alpha V(t,x)))=0, \\ \qquad\quad (t,x)\in[0,T)\times\mathbf{R} \\ V(T,x)=g(x), x\in\mathbf{R} \end{cases} \tag{2.4.19}$$

avec $L^\alpha := A^\alpha + K^\alpha$, et pour $\phi\in C^2(\mathbf{R})$,

- $A^\alpha\phi(x) := \frac{1}{2}\sigma^2(x,\alpha)\frac{\partial^2\phi}{\partial x^2}(x)+b(x,\alpha)\frac{\partial\phi}{\partial x}(x)$
- $K^\alpha\phi(x) := \int_{\mathbf{E}}\left(\phi(x+\beta(x,\alpha,e))-\phi(x)-\frac{\partial\phi}{\partial x}(x)\beta(x,\alpha,e)\right)\nu(de)$
- $B^\alpha\phi(x) := \phi(x+\beta(x,\alpha,\cdot))-\phi(x),$

dans le sens où V^* est une sous-solution de viscosité de (2.4.19) *et V_* est une sur-solution de viscosité de* (2.4.19).

Nous concluons ce chapitre avec des applications financières de la partie théorique.

2.5 Méthodes numériques pour les EDSR Doublement Réfléchies avec Sauts et obstacles irréguliers

Cette partie de la thèse est dédiée aux méthodes numériques pour les EDSRDR avec sauts et obstacles irréguliers et repose sur deux articles écrits en collaboration avec C. Labart: *Numerical approximation for DRBSDEs with jumps and RCLL obstacles* [59] (accepté pour publication dans *Journal of Mathematical Analysis and Applications*) et *Reflected scheme for DRBSDEs with jumps and RCLL obstacles* [60] (accepté pour publication dans *Journal of Computational and Applied Mathematics*).

Nous commençons cette section avec une courte présentation des méthodes numériques existantes pour les EDS rétrogrades.

2.5.1 Préliminaires et vue d'ensemble de la littérature

Les Équations Différentielles Stochastiques Rétrogrades offrant une représentation probabiliste de la solution des EDP semi-linéaires, de nombreux travaux concernent les schémas numériques dans le cadre markovien lorsque la filtration est générée par un mouvement brownien. Parmi ceux-ci, rappelons l'algorithme à quatre étapes developpé par J. Ma, P. Protter et J. Yong ([115], voir aussi [68]), Bouchard-Touzi ([26]), Zhang ([152]) etc. Pour le cas des EDSR réfléchies et doublement réfléchies, voir [25] et [45].

Une approche pertinente dans la théorie des EDSR est de proposer des méthodes numériques implémentables afin d'approximer les solutions de ces équations, la complexité étant due au calcul des espérances conditionnelles. Beaucoup d'efforts ont été faits dans cette direction. Dans [26], Bouchard et Touzi utilisent le calcul de Malliavin pour réécrire les espérances conditionnelles comme le ratio de deux espérances non conditionnelles qui peuvent être estimées par des méthodes standard de Monte Carlo. Dans le cas réfléchi où le générateur ne dépend pas de Z, Bally et Pagès (voir [5], [6]) utilisent une approche de quantification. Cette méthode repose sur l'approximation des processus temporels continus sur un maillage fini et nécessite une autre estimation des probabilités de transition sur le maillage. Gobet et al. ([110]) ont suggéré une adaptation de l'algorithme de Longstaff-Schwartz qui repose sur des méthodes non paramétriques et très récemment Ph. Briand et C. Labart ([41]) ont proposé un développement en chaos de Wiener qui, dans l'esprit, n'est pas très éloigné des techniques de régression. Nous rappelons également les méthodes de cubature, utilisées par T. Lyons, D. Crisan et K. Manolarakis. (voir [50]).

Dans le cadre non markovien, dans le cas des EDSR standard ([134]) et des EDSR réfléchies [151], les auteurs proposent une autre technique qui repose sur l'approximation du mouvement brownien par une marche aléatoire. Cette méthode permet de simplifier le calcul des espérances conditionnelles qui apparaissent à chaque étape et d'obtenir des schémas complètement implémentables. Les EDSRs ont alors été remplacées par une une équation stochastique rétrograde discrète appropriée dont la convergence est obtenue grâce à un résultat de Briand, Delyon et Memin [37] (voir aussi [38]).

Alors que les schémas discrets pour l'approximation des solutions des EDSR dans un cadre purement brownien sont étudiés dans plusieurs travaux, le cadre avec saut a une littéraire moins abondante et réduite au cas des EDSR non réfléchies. Dans le cadre markovien, Bouchard et Elie ([30]) ont considéré des schémas numériques pour les EDSR avec une activité de saut purement finie reposant sur l'équation de la programmation dynamique. Récemment, Lejay *et al.* (2014) [109] ont étendu les résultats de Briand, Delyon et Memin au cas avec sauts. Leur méthode s'appuie sur la construction d'une EDSR avec sauts discrète et dirigée par un système complet de trois martingales orthogonales discrètes en temps et en espace, la première étant une marche aléatoire qui converge vers un mouvement brownien, la deuxième une autre marche aléatoire indépendante de la première, et la troisième converge vers un processus de Poisson.

2.5.2 Contributions

Dans le chapitre 5, nous étudions dans un cadre non markovien un schéma d'approximation en temps discret de la solution des EDSR Doublement Réfléchies avec Sauts dirigées par un mouvement brownien (noté W) et un processus de Poisson compensé indépendant d'intensité λ (noté N). De plus, nous supposons que les barrières sont continues à droite, limitées à gauche et admettent à la fois des sauts inaccessibles et prévisibles. L'EDSRDR que nous résolvons numériquement a la dynamique:

$$Y_t = \xi_T + \int_t^T g(s, Y_s, Z_s, U_s)ds + (A_T - A_t) - (K_T - K_t) - \int_t^T Z_s dW_s - \int_t^T U_s d\tilde{N}_s, \qquad (2.5.1)$$

et satisfait les contraintes suivantes:

$$\begin{cases} (i) \ \forall t \in [0,T], \ \xi_t \leq Y_t \leq \zeta_t \ p.s. \\ (ii) \ \displaystyle\int_0^T (Y_{t^-} - \xi_{t^-}) dA_t^c = 0 \ \text{ et } \ \int_0^T (\zeta_{t^-} - Y_{t^-}) dK_t^c = 0 \text{ p.s.} \\ (iii) \ \forall \tau \text{ temps d'arrêt prévisible, } \ \Delta A_\tau^d = \Delta A_\tau^d \mathbf{1}_{Y_{\tau^-} = \xi_{\tau^-}} \ \text{ et } \Delta K_\tau^d = \Delta K_\tau^d \mathbf{1}_{Y_{\tau^-} = \zeta_{\tau^-}}. \end{cases} \tag{2.5.2}$$

Comme nous avons mentionné dans le chapitre précédent, puisque nous considérons le cadre général dans lequel les sauts des obstacles peuvent être soit prévisibles soit totalement inaccessibles, les processus croissants A et K, dont le rôle est de garder la solution Y entre les barrières, ne sont plus continus. Nous pouvons alors réécrire la condition de séparabilité de Skorokhod, pour la partie continue A^c (respectivement K^c) de A (respectivement K) et la discontinue, notée A^d (respectivement K^d).

Notre objectif est de proposer une schéma complètement implémentable de l'EDSRDR ci-dessus, reposant sur deux arbres binomiaux aléatoires et la métode de pénalisation, qui est alors convergente vers la solution de l'EDSRDR. Nous présentons ci-dessous l'idée principale:

(i) Nous introduisons dans un premier temps une suite d'EDSR prénalisées afin d'approximer l'EDSR doublement réfléchie (1.5.1; 1.5.2) satisfaisant:

$$\begin{aligned} Y_t^p = \xi &+ \int_t^T g(s, Y_s^p, Z_s^p, U_s^p) ds + A_T^p - A_t^p - (K_T^p - K_t^p) \\ &- \int_t^T Z_s^p dW_s - \int_t^T U_s^p d\tilde{N}_s, \end{aligned} \tag{2.5.3}$$

avec $A_t^p := p\int_0^t (Y_s^p - \xi_s)^- ds$ et $K_t^p := p\int_0^t (\zeta_s - Y_s^p)^- ds$.
Nous obtenons la convergence des équations pénalisées dans le cas d'une mesure aléatoire de Poisson générale et, puisque nous devons traiter avec un générateur qui dépend de la solution, la méthode de pénalisation utilisée dans la littérature précédente (qui traite seulement le cas d'un générateur sous la forme d'un processus, le cas général étant obtenu par un argument de point fixe) ne peut être utilisée dans notre cadre général. Nous proposons à la place une preuve qui repose sur une combinaison de pénalisations, la théorie de l'enveloppe de Snell, le théorème de comparaison pour les EDSR avec sauts, un théorème de monotonicité généralisé sous la condition de Mokobodski et des jeux stochastiques.

(ii) Nous approximons le mouvement brownien et le processus de Poisson par deux marches aléatoires indépendantes, notées W^n, respectivement $\tilde{N}^n$, et définies comme suit:

$$W_0^n = 0; \ \ W_t^n = \sqrt{\delta} \sum_{i=1}^{[t/\delta]} e_i^n, \ \ \tilde{N}_0^n = 0, \ \ \tilde{N}_t^n = \sum_{i=1}^{[t/\delta]} \eta_i^n,$$

avec $e_i^n, i = 1, n$ des varables aléatoires indépendantes et identiquement distribuées qui prennent les valeurs $\{-1; 1\}$, chacune avec probabilité $\frac{1}{2}$ et $\eta_i^n, i = 1, n$ définies similairement à (e_i^n) mais prenant les valeurs $\{\kappa_n - 1; \kappa_n\}$ avec probabilités $1 - \kappa_n$, respectivement κ_n, où $\kappa_n = e^{-\frac{\lambda}{n}}$. Dans la définition ci-dessus, $\delta_n := \frac{T}{n}$ représente le pas de temps. Le couple $(W^n, \tilde{N}^n)$ converge vers $(W, \tilde{N})$ en probabilité pour la topologie J_1 de Skorokhod. En utilisant ces approximations, nous obtenons le schéma d'approximation discret suivant de l'équation pénalisée définie par (2.5.3):

$$\begin{cases} y_j^{p,n} = y_{j+1}^{p,n} + g(t_j, y_j^{p,n}, z_j^{p,n}, u_j^{p,n})\delta_n + a_j^{p,n} - k_j^{p,n} \\ \qquad -(z_j^{p,n}\sqrt{\delta_n}e_{j+1}^n + u_j^{p,n}\eta_{j+1}^n + v_j^{p,n}\mu_{j+1}^n) \\ a_j^{p,n} = p\delta_n(y_j^{p,n} - \xi_j^n)^-; \;\; k_j^{p,n} = p\delta_n(\zeta_j^n - y_j^{p,n})^-, \\ y_n^{p,n} := \xi_n^n, \end{cases} \tag{2.5.4}$$

où la troisième suite d'incréments de martingale $\{\mu_j^n = e_j^n\eta_j^n,\ j = 0, ..., n\}$ est nécessaire afin d'obtenir la représentation des martingales (voir [109]).

Ensuite, en utilisant le schéma discret implicite ci-dessus, nous pouvons déduire les expressions des coefficients $(z_j^{p,n}, u_j^{p,n}, v_j^{p,n})_{j=1,n}$ faisant apparaître les espérances conditionnelles qui sont faciles à calculer dans notre cadre grâce aux approximations ci-dessus de W et de $\tilde{N}$. Cependant, la valeur de $(y_j^{p,n})_{j=1,n}$ n'est pas facile à déduire puisque nous devons introduire un opérateur dont l'inversion numérique est difficile et consommatrice de temps. Afin de surmonter ce problème, nous introduisons une équation rétrograde discrète explicite qui est obtenue en remplaçant dans (2.5.4) $y_j^{p,n}$ par $\mathbb{E}[y_{j+1}^{p,n}|\mathcal{F}_j^n]$ dans le générateur g:

$$\begin{cases} y_j^{p,n} = y_{j+1}^{p,n} + g(t_j, \mathbb{E}[y_{j+1}^{p,n}|\mathcal{F}_j^n], z_j^{p,n}, u_j^{p,n})\delta_n + a_j^{p,n} - k_j^{p,n} \\ \qquad -(z_j^{p,n}\sqrt{\delta_n}e_{j+1}^n + u_j^{p,n}\eta_{j+1}^n + v_j^{p,n}\mu_{j+1}^n) \\ a_j^{p,n} = p\delta_n(y_j^{p,n} - \xi_j^n)^-; \;\; k_j^{p,n} = p\delta_n(\zeta_j^n - y_j^{p,n})^-, \\ y_n^{p,n} := \xi_n^n, \end{cases} \tag{2.5.5}$$

où $\mathcal{F}^n$ représente la filtration discrète générée par $(e_j^n, \eta_j^n)_{j=1,n}$.

Nous introduisons ensuite les versions à temps continu $(Y_t^{p,n}, Z_t^{p,n}, U_t^{p,n}, A_t^{p,n}, K_t^{p,n})_{0\leq t\leq T}$ de la solution de ce schéma explicite et nous montrons sa convergence en n vers la solution de (2.5.3). Reliant ce résultat avec la convergence en p de l'équation pénalisée (voir (i)), nous obtenons la convergence de notre schéma en (p, n) de la solution de l'EDSRDR.

Nous terminons par étudier numériquement nos résultats théoriques dans le cas où les barrières admettent à la fois des sauts prévisibles et totalement inaccessibles. La difficulté dans le choix des exemples provient de la condition de Mokobodski que nous devons supposer et qui est difficile à vérifier en pratique. Nous faisons remarquer que l'utilité pratique de nos schémas est restreinte aux petites dimensions. En effet, puisque nous utilisons une marche aléatoire pour approcher le mouvement brownien et le processus de Poisson, la complexité de l'algorithme croît très vite avec le nombre de pas n (plus précisément, en n^d, d étant la dimension) et, comme nous le verrons dans la partie numérique, la méthode de pénalisation nécessite un pas de temps faible afin d'être stable.

Dans le chapitre 6, nous proposons un schéma alternatif à (2.5.4) et respectivement à (2.5.5) afin de résoudre l'EDSRDR donnée par (2.5.1; 2.5.2). Comparés aux équations rétrogrades discrètes

(1.5.4) et (1.5.5), les schémas que nous présentons dans le chapitre 6, appelés *schéma réfléchi implicite* et *schéma réfléchi explicite* reposent sur une discrétisation directe de (2.5.1; 2.5.2). Il n'y a pas d'étape de pénalisation. Cette méthode ne dépend que d'un paramètre d'approximation (le nombre de pas de temps n) contrairement aux schémas proposés dans le chapitre 5 (voir (2.5.4) et (2.5.5)) qui dépendent également du paramètre de pénalisation. Nous obtenons la convergence des deux schémas. Le schéma réfléchi explicite est le suivant:

$$\begin{cases} y_j^n = \mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n] + g(t_j, \mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n], z_j^n, u_j^n)\delta_n + a_j^n - k_j^n \\ a_j^n \geq 0,\ k_j^n \leq 0,\ a_j^n k_j^n = 0,\ \xi_j^n \leq y_j^n \leq \zeta_j^n,\ \ (y_j^n - \xi_j^n)a_j^n = (y_j^n - \zeta_j^n)k_j^n = 0. \end{cases} \tag{2.5.6}$$

Nous illustrons numériquement les résultats théoriques et nous montrons qu'ils coïncident avec ceux obtenus en utilisant le schéma pénalisé (2.5.5) pour de grandes valeurs du paramètre de pénalisation p.

2.6 Conclusions et perspectives

Dans cette thèse, nous avons exploré de nouveaux problèmes en analyse stochastique, contrôle stochastique, théorie de jeux et mathématiques financières, à la fois d'un point de vue théorique et numérique. Les résultats principaux sont les suivants:

- Dans le chapitre 3 (article [57]), nous introduisons une nouvelle classe d'*EDSR avec condition terminale faible non linéaire*, associées à la couverture approximative sous contraintes de mesures de risque dynamiques.

- Dans le chapitre 4 (article [61]), nous étudions un problème d'arrêt optimal pour des mesures de risque dynamiques induites par des EDSR acec sauts et nous montrons que la fonction valeur correspond à l'unique solution de viscosité d'un problème d'obstacle pour les équation partielles intégro-différentielles.

- Dans le chapitre 5 (article [62]), nous introduisons un nouveau problème de jeu, qui généralise le jeu classique de Dynkin au cas d'espérance non linéaire, permettant d'obtenir une représentation de la solution des EDSR doublement réfléchies non linéaires en termes de jeux stochastiques.

- Dans le chapitre 6 (article [63]), nous étudions dans un cadre markovien un problème de contrôle stochastique et d'arrêt optimal mixte dans le cas où l'espérance classique dans le critère est remplacée par une espérance non linéaire induite par la solution d'une EDSR avec sauts et la fonction de profit terminal est seulement mesurable. Nous établissons un principe de programmation dynamique faible et en déduisons les équations HJB non linéaires associées.

- Dans le chapitre 7 (article [63]), nous introduisons une approximation numérique pour la solution d'une EDSRDR avec sauts et obstacles irréguliers qui admet à la fois des sauts totalement inaccessibles et prévisibles. Nous proposons un schéma complètement implémentable, reposant sur une méthode de pénalisation et l'approximation du mouvement brownien et du processus de Poisson par deux marches aléatoires indépendantes dont on montre la convergence vers la solution de l'EDSRDR. Nous illustrons les résultats théoriques avec des exemples numériques.

- Dans le chapitre 8 (article [60]), nous introduisons un schéma complètement implémentable alternatif à celui présenté dans le chapitre 6 afin d'approximer la solution de l'EDSR doublement réfléchie avec sauts et obstacles irréguliers. Ce schéma est obtenu par une discrétisation directe des EDSRDR et ne dépend alors que du nombre de pas de temps n (et plus du paramètre de pénalisation p). Nous obtenons la convergence du schéma et donnons des exemples numériques.

Concernant les perspectives, ils y a plusieurs orientations de recherche à venir, à la fois d'un point de vue théorique et numérique. En collaboration avec R. Elie et D. Possamai nous sommes en train de finir un travail sur les EDSR avec réflexion faible ([58]) qui sont reliées à la couverture approximative des options américaines. Avec M.C. Quenez et A. Sulem, nous travaillons sur un nouveau problème de jeux mixte avec controle stochastique et temps d'arrêt dans le cadre markovien [64] et on étudie les liens entre les jeux de Dynkin Généralisés et le pricing non linéaire, dans des marchés complets et incomplets [65].

D'un pont de vue numérique, il serait utile de proposer des schémas numériques pour la solution des EDSRDR avec barrières càdlàg dans le cas d'une mesure de Poisson générale ainsi que des EDSR avec condition terminale faible.

Part I

Stochastic control and Optimal Stopping with non linear expectations

Chapter 3

BSDEs with nonlinear weak terminal condition

Abstract. In a recent paper, Bouchard, Elie and Reveillac [31] have introduced a new class of Backward Stochastic Differential Equations with weak terminal condition, in which the T-terminal value Y_T of the solution (Y, Z) is not fixed as a random variable, but only satisfies a constraint of the form $E[\Psi(Y_T)] \geq m$. The aim of this paper is to study a more general class of BSDEs, with *nonlinear expectation constraints* on the terminal condition, induced by the solution of a Backward Stochastic Differential Equation. More precisely, the constraint takes the form $\mathcal{E}^f_{0,T}[\Psi(Y_T)] \geq m$, where $\mathcal{E}^f$ represents the f-conditional expectation associated to a *nonlinear driver* f. These BSDEs are called *BSDEs with nonlinear weak terminal solution.* We carry out a similar analysis as in [31] of the value function corresponding to the minimal solutions Y of the BSDE with nonlinear weak terminal condition: we study the regularity, establish the main properties, in particular continuity and convexity with respect to the parameter m, and finally provide a dual representation in the case of concave constraints. From a financial point of view, our study is closely related to the approximative hedging of an European option under dynamic risk measures constraints. The nonlinearity f raises subtle difficulties, highlighted through out the paper, which cannot be handled by the arguments used in the case of classical expectations constraints studied in [31].
Key words : Backward stochastic differential equations, g-expectation, dynamic risk measures, optimal control, stochastic targets.

3.1 Introduction

Linear backward stochastic differential equations (BSDEs) were introduced by Bismut as the adjoint equations associated with Pontryagin maximum principles in stochastic control theory. The general case of non-linear BSDEs was then studied by Pardoux and Peng [131]. They provided Feynman-Kac representations of solutions of non-linear parabolic partial differential equations. The solution of a BSDE consists in a pair of predictable processes (Y, Z) satisfying

$$-dY_t = g(t, Y_t, Z_t)dt - Z_t dW_t; \quad Y_T = \xi. \tag{3.1.1}$$

These equations appear as an useful mathematical tool in various problems in finance, for example in the theory of derivatives pricing. In a complete market - when it is possible to construct a portfolio which attains as final wealth the payoff- the value of the replicating portfolio is given by Y and the hedging strategy by Z. Since in incomplete markets is not always possible to construct a portfolio which attains exactly as final wealth the amount ξ, it was suggested to replace the terminal condition into a weaker one of the form $Y_T \geq \xi$. In this case, the minimal initial value Y_0 defines the smallest initial investment which allows one to superhedge the contingent claim ξ.
Recently, Bouchard, Elie and Reveillac [31] introduced a new class of BSDEs, the so called BSDEs with weak terminal condition, in which the T-terminal value Y_T only satisfies a weak constraint. More precisely, a couple of predictable processes (Y, Z) is said to be a solution of such a BSDE if it satisfies:

$$-dY_t = g(t, Y_t, Z_t)dt - Z_t dW_t; \tag{3.1.2}$$

$$E[\Psi(Y_T)] \geq m, \tag{3.1.3}$$

where m is a given threshold and Ψ a non-decreasing map. The main question in [31] is the following:

$$\text{Find the minimal } Y_0 \text{ such that (3.1.2) and (3.1.3) hold for some } Z. \tag{3.1.4}$$

From a financial point of view, this study is related to the hedging in quantile or more generally to the hedging with *expected loss constraints*. This problem was addressed in the literature for the first time by Follmer and Leukert [85] and then further studied in a Markovian framework in [32] and [118], using stochastic target techniques .
In [31], the key point of the analysis is the reformulation of the problem written in terms of BSDE with weak terminal condition into an optimization problem on a family of BSDEs with strong terminal condition, by using *the martingale representation theorem*. The main observation is that if Y_0 and Z are such that (3.1.3) holds, then the martingale representation Theorem implies that it exists an element $\alpha \in \mathbf{A}_0$, the set of predictable square integrable processes, such that:

$$\Psi(Y_T) \geq M_T^{m,\alpha} = m + \int_0^T \alpha_s dW_s, \tag{3.1.5}$$

It is then shown that the initial problem (3.1.4) is equivalent to:

$$\inf\{Y_0^\alpha,\ \alpha \in \mathbf{A}_0\}, \tag{3.1.6}$$

where Y_t^α corresponds to the solution at time t of the BSDE with (strong) terminal condition $\Phi(M_T^\alpha)$, Φ representing the left-continuous inverse of Ψ.

The aim of this paper is to introduce a new class of *BSDEs with weak nonlinear terminal condition* . We extend the results of [31] to a more general class of constraints which take the form:

$$\mathcal{E}^f_{0,T}[\Psi(Y_T)] \geq m, \tag{3.1.7}$$

where f is a nonlinear driver and $\mathcal{E}^f_{\cdot,T}[\xi]$ the solution of the BSDE with generator f and terminal condition ξ.

We can easily remark that the constraint (3.1.3) is a particular case of (3.1.7) for $f = 0$. The problem under study in this paper is the following:

$$\inf\{Y_0 \text{ such that } \exists Z : \text{(3.1.2) and (3.1.7) hold}\}. \tag{3.1.8}$$

Following the key idea of [31], we rewrite our problem (3.1.8) into an equivalent one expressed in terms of BSDEs with strong terminal condition. The main difference with respect to [31] is given by the fact that in our case we have to introduce a new controlled diffusion process, which is an $f-$*martingale*, contrary to [31] where it is a classical martingale. Indeed, for a given Y_0 and Z such that (3.1.2) and (3.1.7) are satisfied, appealing to the *BSDE representation* of $\Psi(Y_T)$, we can find $\alpha \in \mathbf{A}_0$ such that:

$$\Psi(Y_T) \geq \mathcal{M}_T^{m,\alpha} = m - \int_0^T f(s, \mathcal{M}_s^{m,\alpha}, \alpha_s)ds + \int_0^T \alpha_s dW_s. \tag{3.1.9}$$

Thanks to this observation, we show that Problem (3.1.8) is equivalent to (3.1.6), where, in our more general framework, Y_t^α corresponds to the solution at time t of the BSDE with (strong) terminal condition $\Phi(\mathcal{M}_T^\alpha)$. We study the dynamical counterpart of (3.1.6):

$$\mathcal{Y}^\alpha(\tau) := \text{essinf}\{Y_\tau^{\alpha'}, \alpha' \in \mathbf{A}_0 \text{ s.t.} \alpha' = \alpha \text{ on } [\![0, \tau]\!]\}. \tag{3.1.10}$$

We carry out a similar analysis as in [31] of the family $\{\mathcal{Y}^\alpha,\ \alpha \in \mathbf{A}_0\}$. We start by studying the regularity of the family $\mathcal{Y}^\alpha$ and show that it can be aggregated into a RCLL process, proof which becomes considerably more technical in our context with respect to [31], because we have to deal with the nonlinearity f. We then provide a BSDE representation of $\mathcal{Y}^\alpha$ and show that, under a concavity assumption on the driver f, there exists an optimal control. We also study the main properties of the value function, as continuity and convexity with respect to the threshold m, and propose proofs specific to the nonlinear case. We finally get, in the case of concave constraints, a dual representation of the value function, related to a stochastic control problem in Meyer's form.

Besides the mathematical interest of our study, this work is also motivated by some financial applications, as it provides the *approximative hedging under dynamic risk measures contraints* of an European option, when the shortfall risk is quantified in terms of dynamic risk measures induced by BSDEs.

The paper is organized as follows. In Section 2 we introduce notation, assumptions and the BSDEs with nonlinear weak terminal condition. In Section 3, we study the regularity and the BSDE representation of the value function $\mathcal{Y}^\alpha$. In Section 4, we establish the main properties of the value function and finally we provide a dual representation in Section 5.

3.2 Problem formulation

3.2.1 Notation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a d-dimensional Brownian motion W and $\mathbb{F} := (\mathcal{F}_t)_{t \leq T}$ the completed associated filtration. Fix $T > 0$.

In the sequel, we adopt the following notation:

- $\mathcal{P}$ denotes the predictable σ-algebra on $[0, T] \times \Omega$;
- $\mathbf{L}_2(\mathcal{F}_T)$ is the set of random variables ξ which are $\mathcal{F}_T$-measurable and square-integrable;
- $\mathbf{H}_2$ denotes the set of $\mathbf{R}^d$-valued predictable processes ϕ such that $\|\phi\|^2_{\mathbf{H}_2} := E[(\int_0^T \phi_t^2 dt)] < \infty$;
- $\mathbf{S}_2$ is the set of real-valued RCLL adapted processes ϕ such that $\|\phi\|^2_{\mathbf{S}^2} := E[\sup_{0 \le t \le T} |\phi_t|^2] < \infty$;
- $\mathbf{I}_2$ is the set of non-decreasing adapted processes ϕ such that $\|\phi\|^2_{\mathbf{S}^2} < \infty$;
- For any σ-algebra $\mathcal{G} \subset \mathcal{F}_T$, $\mathbf{L}_0(\mathcal{G})$ denotes the set of random variables measurable with respect to $\mathcal{G}$;
- $\mathcal{T}$ denotes the set of stopping times τ such that $\tau \in [0, T]$ a.s.

3.2.2 BSDEs with nonlinear weak terminal condition

Definition and Assumptions

In this section, we introduce the main object of this paper, the *BSDEs with nonlinear weak terminal condition.*
It is well known that, in the classical case of nonlinear BSDEs introduced by Pardoux-Peng, the data of the BSDE is represented by a *driver* g and a *terminal condition* ξ.

In the recent paper [31], the authors define a new class of BSDEs called *BSDEs with weak terminal condition*. The particularity consists in the fact that the terminal condition is not fixed as a $\mathcal{F}_T$-measurable random variable, but only satisfies a weak constraint expressed in terms of classical expectations. The data of this new class of BSDEs is given by four elements: a *driver* g and a triplet (Ψ, μ, τ) describing the constraint on the terminal condition.

The aim of this work is to introduce a more general class of BSDEs, named *BSDEs with nonlinear weak terminal condition*, whose terminal value verifies a weak constraint defined via a BSDE with a nonlinear driver f, satisfying the following hypothesis:

Assumption 3.2.1. *Let* $f : (\omega, t, y, z) \in \Omega \times [0, T] \times \mathbf{R} \times \mathbf{R}^d \mapsto f_t(\omega, y, z) \in \mathbf{R}$ *be a driver such that* $(f_t(\cdot, y, z))_{t \le T}$ *is* $\mathcal{P}$*-measurable for every* $(y, z) \in \mathbf{R} \times \mathbf{R}^d$ *and*

$$|f_t(\omega, y, z) - f_t(\omega, y', z')| \le C_f \left(|y - y'| + \|z - z'\|_{\mathbf{R}^d}\right),$$

$\forall (y, z), (y', z') \in \mathbf{R} \times \mathbf{R}^d$*, for* $dt \otimes dP$*-a.e.* $(t, \omega) \in [0, T] \times \Omega$*, for some constant number* $C_f > 0$.
We also assume that $(f_t(0, 0, 0))$ *satisfies the following condition*

$$\mathbb{E}\left[\int_0^T |f_t(0, 0)|^2 dt\right] < \infty.$$

Note that the data of this new BSDE are (f, Ψ, μ, τ, g) and the particular case when $f = 0$ corresponds to the class of BSDEs studied in [31]. In the sequel, we shall denote this BSDE with nonlinear weak terminal condition by $BSDE(f, \Psi, \mu, \tau, g)$.

Before defining this new mathematical object, we introduce the nonlinear conditional expectation $\mathcal{E}^f$ associated with f, defined for each stopping time $\tau \in \mathcal{T}$ and for each $\eta \in \mathbf{L}_2(\mathcal{F}_\tau)$ as:

$$\mathcal{E}^f_{t,\tau}[\eta] := Y_r, \qquad 0 \leq t \leq \tau, \tag{3.2.1}$$

where $(Y_t)_{t\leq\tau}$ is the unique solution in $\mathbf{S}_2$ of the BSDE associated with driver f, terminal time τ and terminal condition η, that is satisfying:

$$\begin{cases} -dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t; \\ Y_T = \eta, \end{cases} \tag{3.2.2}$$

with Z the associated process belonging to $\mathbf{H}_2$. Moreover, set $\mathcal{E}^f_{\sigma,\tau}[\eta] := -\infty$, for any $\eta \in \mathbf{L}_0(\mathcal{F}_\tau)$ such that $\mathbb{E}[\eta^-] = +\infty$, for any $\sigma \in \mathcal{T}$ with $\sigma \leq \tau$ a.s.

We are now in position to define the so-called *BSDEs with nonlinear weak terminal condition.*

Definition 3.2.2 (BSDEs with nonlinear weak terminal condition)**.** Given a measurable map $\Psi : \mathbf{R} \times \Omega \to U$, with $U \subset \mathbf{A} \cup \{-\infty\}$, $\mathbf{A}$ a bounded subset of $\mathbf{R}$, $\tau \in \mathcal{T}$, $\mu \in \mathbf{L}_0(\mathbf{R}, \mathcal{F}_\tau)$, a driver f satisfying Assumption 3.2.1 and a measurable function g, we say that $(Y, Z) \in \mathbf{S}_2 \times \mathbf{H}_2$ is a solution of the BSDE (f, Ψ, μ, τ, g) if

$$Y_t = Y_T + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s,\ 0 \leq t \leq T; \tag{3.2.3}$$

$$\mathcal{E}^f_{\tau,T}[\Psi(Y_T)] \geq \mu. \tag{3.2.4}$$

Throughout the paper, we shall assume that the driver g satisfies Assumption 3.2.1, with C_g instead of C_f. To the coefficient g, we associate the nonlinear operator $\mathcal{E}^g$ defined as $\mathcal{E}^f$, with g instead of f.
Let us now precise the hypothesis on the map Ψ and the threshold μ. We then discuss the wellposedness of the $BSDE(f, \Psi, \tau, \mu, g)$ under these assumptions.

Assumption 3.2.3. *For a.e. $\omega \in \Omega$, the map $y \in \mathbf{R} \to \Psi(\omega, y)$ is non-decreasing and valued in $[0; 1] \cup \{-\infty\}$ and its right-inverse $\Phi(\omega, \cdot)$ is such that $\Phi : \Omega \times [0, 1] \to [0, 1]$ and it is measurable.*

This means that $\Psi(\omega, \cdot) \in [0, 1]$ on $[0, \infty)$ and $\Psi(\omega, \cdot) = -\infty$ on $(-\infty, 0)$. In view of the definition of the operator $\mathcal{E}^f$, this implies that $Y_T \geq 0$ a.s. Note that for notational simplicity we have considered the compact $[0, 1]$, as in [31], which can be obviously replaced by an arbitrary compact set belonging to $\mathbf{R}$. Moreover, our analysis is the same if for a.e. ω the map $\Psi(\omega, \cdot)$ is valued in $[G_1(\omega), G_2(\omega)]$, with $G_1, G_2 \in \mathbf{L}_\infty(\mathcal{F}_T)$.

The threshold μ is assumed to belong to $\mathbf{D}_\tau$, where $\mathbf{D}_\tau$ corresponds to the set of random variables $\{\eta \in \mathbf{L}_2(\mathcal{F}_\tau)$ such that $\eta \in [\mathcal{E}^f_{\tau,T}[0], \mathcal{E}^f_{\tau,T}[1]]$ a.s.$\}$. throughout the paper, we shall denote by (Y^i, Z^i), for $i = 1, 2$, the solution of the BSDE associated to driver f and terminal condition ξ^i, where $\xi^1 = 0$ and $\xi^2 = 1$.
Concerning the existence of a solution, remark that any random variable $\Phi(\xi)$, with $\xi \in [0, 1]$ a.s. and $\mathcal{E}^f_{\tau,T}[\xi] \geq \mu$ could serve as terminal condition. However, the constraint is too weak to expect

uniqueness.

We now introduce the value function $\mathcal{V} : \mathbf{D} \to \mathbf{L}_2$; $(\tau, \mu) \to \mathcal{V}(\tau, \mu)$, where $\mathbf{D} := \{(\tau, \mu);\ \tau \in \mathcal{T}$ and $\mu \in \mathbf{D}_\tau\}$ as follows:

$$\mathcal{V}(\tau, \mu) := \text{essinf}\{Y_\tau : (Y, Z) \in \mathbf{S}_2 \times \mathbf{H}_2 \text{ is a solution of BSDE}(f, \Psi, \mu, \tau, g)\}. \tag{3.2.5}$$

The rest of the paper is dedicated to the study of the above map. In order to do it, we shall first establish the link with a control problem for BSDEs with strong terminal condition.

Link with a control problem for BSDEs with strong terminal condition

In the spirit of [31] or [32], we introduce an additional process $\mathcal{M}$ which allows to transform the weak constraint $\mathcal{E}^f_{0,T}[\Psi(Y_T)] \geq \mu$ into a strong one of the form $Y_T \geq \Phi(\mathcal{M}^\mu_T)$. Since our constraint is expressed in terms of nonlinear BSDEs, the process $\mathcal{M}$ is an f-martingale, contrary to [31] and [32] where $\mathcal{M}$ is a classical martingale.
For each $\alpha \in \mathbf{H}_2$, stopping time $\tau \in \mathcal{T}$ and $\mu \in \mathbf{D}_\tau$, let $\mathcal{M}^{\tau,\mu,\alpha}$ be the $\mathbf{R}$-valued solution of the SDE:

$$\mathcal{M}^{\tau,\mu,\alpha}_{t\vee\tau} = \mu - \int_\tau^{t\vee\tau} f(s, \mathcal{M}^{\tau,\mu,\alpha}_s, \alpha_s)ds + \int_\tau^{t\vee\tau} \alpha_s^\top dW_s, \ \ 0 \leq t \leq T.$$

We introduce the set of admissible controls $\mathbf{A}_{\tau,\mu}$, which is defined as follows:

$$\mathbf{A}_{\tau,\mu} := \{\alpha \in \mathbf{H}_2 \text{ such that } \mathcal{M}^{\tau,\mu,\alpha}_t \in \left[\mathcal{E}^f_{t,T}[0], \mathcal{E}^f_{t,T}[1]\right] \ dP \otimes dt \text{ a.s. on } [\![\tau, T]\!]\}.$$

Notice that for all $\alpha \in \mathbf{A}_{\tau,\mu}$, $\Phi(\mathcal{M}^{\tau,\mu,\alpha}_T)$ could serve as terminal condition, since satisfies (3.2.4). We thus introduce for all $\alpha \in \mathbf{A}_{\tau,\mu}$ the BSDE with strong condtion $\Phi(\mathcal{M}^{\tau,\mu,\alpha}_T)$ and driver g and define the value function $\mathcal{Y}(\tau, \mu)$ as follows:

$$\mathcal{Y}(\tau, \mu) := \operatorname*{essinf}_{\alpha \in \mathbf{A}_{\tau,\mu}} \mathcal{E}^g_{\tau,T}[\Phi(\mathcal{M}^{\tau,\mu,\alpha}_T)]. \tag{3.2.6}$$

Our aim now is to link $\mathcal{Y}(\tau, \mu)$ to $\mathcal{V}(\tau, \mu)$, i.e. to prove that for all $\tau \in \mathcal{T}$ and $\mu \in \mathbf{D}_\tau$:

$$\mathcal{V}(\tau, \mu) = \mathcal{Y}(\tau, \mu) \ \text{ a.s.} \tag{3.2.7}$$

In order to explain the above equality between $\mathcal{V}$ and $\mathcal{Y}$, we state the following proposition:

Proposition 3.2.4. *Fix $\tau \in \mathcal{T}$, $\mu \in \boldsymbol{D}_\tau$. Then $(Y, Z) \in \mathbf{S}_2 \times \mathbf{H}_2$ is a solution of $BSDE(f, \Psi, \mu, \tau, g)$ if and only if (Y, Z) satisfies* (3.2.3) *and there exists $\alpha \in \mathbf{A}_{\tau,\mu}$ such that $Y_t \geq \mathcal{E}^g_{t,T}[\Phi(\mathcal{M}^{\tau,\mu,\alpha}_T)]$ for $t \in [0, T]$, $\mathbb{P}$-a.s.*

A sketch of proof is given in Appendix.
We come back to the explanation of equality (3.2.7).

(*i*) Let $(Y, Z) \in \mathbf{S}_2 \times \mathbf{H}_2$ be a solution of the BSDE(f, Ψ, μ, τ, g). Then the above Proposition implies that it exists $\alpha \in \mathbf{A}_{\tau,\mu}$ such that $Y_\tau \geq \mathcal{E}_{\tau,T}[\Phi(\mathcal{M}^{\tau,\mu,\alpha}_T)] \geq \mathcal{Y}(\tau, \mu)$, where the last inequality follows from definition (3.2.6). By arbitrariness of (Y, Z), we get $\mathcal{V}(\tau, \mu) \geq \mathcal{Y}(\tau, \mu)$ a.s.

(*ii*) Fix $\alpha \in \mathbf{A}_{\tau,\mu}$. Let Z^α be the associated process to the BSDE representation of $\Phi(\mathcal{M}_T^{\tau,\mu,\alpha})$. Since $\Phi(\mathcal{M}_T^{\tau,\mu,\alpha})$ is admissible as a terminal condition, we obtain, by the Proposition 3.2.4 that $(\mathcal{E}_{\cdot,T}[\Phi(\mathcal{M}^{\tau,\mu,\alpha})], Z^\alpha)$ is a solution, and thus $\mathcal{E}_{\tau,T}[\Phi(\mathcal{M}^{\tau,\mu,\alpha})] \geq \mathcal{V}(\tau,\mu)$. By arbitrariness of α, we deduce $\mathcal{V}(\tau,\mu) \leq \mathcal{Y}(\tau,\mu)$ a.s.

From now on, we fix an initial condition $\mu_0 \in \mathbf{D}_0$ at time 0. For each $\alpha \in \mathbf{A}_{0,\mu_0}$ (denoted for simplicity $\mathbf{A}_0$), we introduce the process $(\mathcal{M}_t^\alpha)_{t\leq T}$, representing a dynamic threshold controlled by the action of α, which is defined as follows:

$$\mathcal{M}_t^\alpha := \mathcal{M}_t^{0,\mu_0,\alpha}.$$

We introduce for each $\tau \in \mathcal{T}$ the set of admissible controls coinciding with α up to the stopping time τ:

$$\mathbf{A}_\tau^\alpha := \{\alpha' \in \mathbf{A}_{\tau,\mathcal{M}_\tau^\alpha} : \alpha' = \alpha \, dt \otimes dP \text{ on } [\![0,\tau]\!]\}.$$

The associated value is defined by:

$$\mathcal{Y}^\alpha(\tau) := \operatorname*{essinf}_{\alpha' \in \mathbf{A}_\tau^\alpha} \mathcal{E}^g_{\tau,T}[\Phi(\mathcal{M}_T^{\tau,\mathcal{M}_\tau^\alpha,\alpha'})].$$

In the following section, we shall investigate the time regularity of the above function and provide a BSDE representation. Before doing this, note that

$$|\mathcal{Y}^\alpha(\tau)| \leq \eta_\tau \text{ a.s. for all } \tau \in \mathcal{T}, \text{where } \eta \text{ belongs to } \mathbf{S}_2. \tag{3.2.8}$$

Note that η is given by $\eta_t := |\mathcal{E}^g_{t,T}[\Phi(1)]| + |\mathcal{E}^g_{t,T}[\Phi(0)]|$, $t \leq T$.

3.3 Time regularity of the value function $\mathcal{Y}^\alpha$ and BSDE representation

In this section, we study the regularity of the family $\{\mathcal{Y}^\alpha(\tau), \tau \in \mathcal{T}\}$. More precisely, we show that it can be aggregated into a right continuous left limited process. The proof of this result becomes considerably more technical in our nonlinear case. Some comments regarding the main difficulties with respect to the case of linear constraints are provided in Remark 3.3.4.

We first state the following dynamic programming principle.

Lemma 3.3.1. *For any $\alpha \in \boldsymbol{A}_0$, $\mathcal{Y}^\alpha$ satisfies the following dynamic programming principle: for all $\tau_1 \in \mathcal{T}$, $\tau_2 \in \mathcal{T}$ with $\tau_1 \leq \tau_2$ a.s. it holds:*

$$\mathcal{Y}^\alpha(\tau_1) = \operatorname*{essinf}_{\overline{\alpha} \in \boldsymbol{A}_{\tau_1}^\alpha} \mathcal{E}^g_{\tau_1,\tau_2}[\mathcal{Y}^{\overline{\alpha}}(\tau_2)].$$

Since the proof of the dynamic programming principle is based on classical arguments, we refer the reader to [31].

We now make the following hypothesis on the map Φ, under which we provide the time-regularity of our value function $\mathcal{Y}^\alpha$.

Assumption 3.3.2. *The map $m \in [0,1] \to \Phi(\omega, m)$ is continuous for a.e. $\omega \in \Omega$.*

Theorem 3.3.3. *Under the Assumption* 3.3.2*, for each* $\alpha \in \boldsymbol{A}_0$*, there exists a right-continuous left limited process* $(\overline{\mathcal{Y}}_t^\alpha)_{t\leq T}$ *which aggregates the family* $\{\mathcal{Y}^\alpha(\tau),\ \tau \in \mathcal{T}\}$.

Proof. By Lemma 3.3.1, we easily obtain that the family $\{-\mathcal{Y}^\alpha(\tau),\ \tau \in \mathcal{T}\}$ is a $-g(-)$ supermartingale system. Since moreover 3.2.8 holds, we can apply Lemma A.2 in [33] and obtain the existence of an optional ladlag process, denoted by $(\mathcal{Y}_t^\alpha)_{t\leq T}$ which aggregates the family, that is $\mathcal{Y}^\alpha(\tau) = \mathcal{Y}_\tau^\alpha$, for all $\tau \in \mathcal{T}$. Hence, the following limits:

$$\lim_{s\in(t,T]\downarrow t} \mathcal{Y}_s^\alpha \text{ and } \lim_{s\in(t,T]\uparrow t} \mathcal{Y}_s^\alpha.$$

are well-defined and finite.
Now, we define:

$$\overline{\mathcal{Y}}_t^\alpha := \lim_{s\in(t,T]\downarrow t} \mathcal{Y}_s^\alpha,\ t \in [0,T[,\ \ \overline{\mathcal{Y}}_T^\alpha := \mathcal{Y}_T^\alpha. \tag{3.3.1}$$

which is by definition a real-valued RCLL process.

In order to prove the desired regularity property, we have to show that for every stopping time $\tau \in \mathcal{T}$, it holds that:

$$\overline{\mathcal{Y}}_\tau^\alpha = \mathcal{Y}_\tau^\alpha \ \ a.s.$$

The above relation implies that the processes $\overline{\mathcal{Y}}^\alpha$ and $\mathcal{Y}^\alpha$ are indistinguishable. The proof is divided in two steps.

Step 1. Fix $\tau \in \mathcal{T}$. We first prove that $\overline{\mathcal{Y}}_\tau^\alpha \leq \mathcal{Y}_\tau^\alpha$ a.s.
a. Let $\alpha' \in \mathbf{A}_\tau^\alpha$. Fix $k \in \mathbb{N}^*$.
Define $\tilde{\mathcal{M}}_T^{k,\alpha'} := \frac{1}{k} + \mathcal{M}_T^{\alpha'}(1-\frac{1}{k})$. Note that $\tilde{\mathcal{M}}_T^{k,\alpha'} \geq \mathcal{M}_T^{\alpha'}$ and $\tilde{\mathcal{M}}_T^{k,\alpha'} \to \mathcal{M}_T^{\alpha'}$ when $k \to \infty$. In the sequel, we denote by $(\mathcal{E}_{\cdot,T}^f[\tilde{\mathcal{M}}_T^{k,\alpha'}], \tilde{Z}^k)$ the solution of the BSDE associated to $(\tilde{\mathcal{M}}_T^{k,\alpha'}, f)$.
Recall that $\mathcal{M}_T^{\alpha'}$ belongs for a.e. ω to $[0,1]$. Hence, by construction, we have:

$$0 \leq \mathcal{M}_T^{\alpha'} \leq \tilde{\mathcal{M}}_T^{k,\alpha'} \leq 1 \text{ a.s.}$$

By applying the comparison theorem for BSDEs and since $\alpha' \in \mathbf{A}_\tau^\alpha$, we obtain:

$$\mathcal{E}_{\tau,T}^f[0] \leq \mathcal{M}_\tau^\alpha \leq \mathcal{E}_{\tau,T}^f[\tilde{\mathcal{M}}_T^{k,\alpha'}] \text{ a.s.} \tag{3.3.2}$$

We claim that it exists a sequence of stopping times $(\tau_{n,k})_n$ and an admissible control $\tilde{\alpha}_k \in \mathbf{A}_{\tau_{n,k}}^\alpha$ such that: $\tau_{n,k} \to \tau$ when n tends to $+\infty$, $\tau_{n,k} > \tau$ a.s. on $\{\tau < T\}$ for all $n \in \mathbb{N}$ and $\mathcal{M}_T^{\tilde{\alpha}_k} \leq \tilde{\mathcal{M}}_T^{k,\alpha'}$. The proof is postponed to Step 1.b.
Thanks to the above assertion, we can appeal to (3.3.1) and obtain:

$$\overline{\mathcal{Y}}_\tau^\alpha = \lim_{n\to\infty} \mathcal{Y}_{\tau_{n,k}}^\alpha. \tag{3.3.3}$$

Using the definition of $\mathcal{Y}^\alpha$, we get:

$$\mathcal{Y}_{\tau_{n,k}}^\alpha \leq \mathcal{E}_{\tau_{n,k},T}^g[\Phi(\mathcal{M}_T^{\tilde{\alpha}^k})]. \tag{3.3.4}$$

As $\mathcal{M}_T^{\tilde{\alpha}^k} \leq \tilde{\mathcal{M}}_T^{k,\alpha'}$ a.s. and Φ is nondecreasing, by applying the comparison theorem for BSDEs, we get for all n:

$$\mathcal{E}^g_{\tau_{n,k},T}[\Phi(\mathcal{M}_T^{\tilde{\alpha}^k})] \leq \mathcal{E}^g_{\tau_{n,k},T}[\Phi(\tilde{\mathcal{M}}_T^{k,\alpha'})] \quad \text{a.s.}$$

The above inequality together with (3.3.3), (3.3.4) and the continuity of the process $\mathcal{E}^f_{\cdot,T}[\Phi(\tilde{\mathcal{M}}_T^{k,\alpha'})]$ lead to:

$$\overline{\mathcal{Y}}^\alpha_\tau \leq \mathcal{E}^g_{\tau,T}[\Phi(\tilde{\mathcal{M}}_T^{k,\alpha'})].$$

Since $\tilde{\mathcal{M}}_T^{k,\alpha'} \to \mathcal{M}_T^{\alpha'}$ a.s. and Φ is a.s. continuous, by letting k tend to ∞, we obtain:

$$\overline{\mathcal{Y}}^\alpha_\tau \leq \mathcal{E}^g_{\tau,T}[\Phi(\mathcal{M}_T^{\alpha'})] \text{ a.s.}$$

By arbitrariness of $\alpha' \in \mathbf{A}^\alpha_\tau$, we conclude:

$$\overline{\mathcal{Y}}^\alpha_\tau \leq \mathcal{Y}^\alpha_\tau \text{ a.s.}$$

b. *i)* We first construct, for each $k \in \mathbb{N}^*$, the sequence of stopping times $(\tau_{n,k})_n$ such that $\tau_{n,k} \to \tau$ when $n \to \infty$ and $\tau_{n,k} > \tau$ a.s. on $\{\tau > T\}$ for all $n \in \mathbb{N}$.
To do this, we start by defining the following stopping time:

$$\sigma_k := \inf\{\tau \leq t \leq T; \mathcal{M}^\alpha_t = \mathcal{E}^f_{t,T}[\tilde{\mathcal{M}}_T^{k,\alpha'}]\}. \tag{3.3.5}$$

We use the convention $\inf \emptyset = +\infty$.

We introduce $(\tau_n)_n$ a sequence of stopping times which take values in $[0,T]$ a.s. such that $\tau_n > \tau$ on $\{\tau < T\}$ for all n and $\tau_n \to \tau$ a.s. when n tends to $+\infty$.
For each n, we define $\tau_{n,k}$ as follows:

$$\tau_{n,k} := \tau_n \mathbf{1}_{A_k} + (\tau_n \wedge \sigma_k)\, \mathbf{1}_{A_k^c}, \tag{3.3.6}$$

with

$$A_k := \{\mathcal{E}^f_{\tau,T}[\tilde{\mathcal{M}}_T^{k,\alpha'}] - \mathcal{M}^\alpha_\tau = 0\} \in \mathcal{F}_\tau; \;\; A_k^c := \{\mathcal{E}^f_{\tau,T}[\tilde{\mathcal{M}}_T^{k,\alpha'}] - \mathcal{M}^\alpha_\tau > 0\} \in \mathcal{F}_\tau.$$

Remark that by (3.3.2), $P(A_k \cup A_k^c) = 1$ and thus $\tau_{n,k} \downarrow \tau$ a.s. when $n \to \infty$. We precise that we have to introduce the sets A_k and A_k^c because $\sigma_k = \tau$ on A_k. It thus remains to prove that $\tau < \sigma_k$ on A_k^c.

The definition of σ_k together with the continuity of the processes $\mathcal{M}^\alpha$ and $\mathcal{E}^f_{\cdot,T}[\tilde{\mathcal{M}}^{k,\alpha'}]$, imply that almost surely, $\sigma_k = +\infty$ or $\mathcal{E}^f_{\sigma_k,T}[\tilde{\mathcal{M}}_T^{k,\alpha'}] \leq \mathcal{M}^\alpha_{\tilde{\tau}_k}$. Moreover, since on A_k^c we have $\mathcal{E}^f_{\tau,T}[\tilde{\mathcal{M}}_T^{k,\alpha'}] > \mathcal{M}^\alpha_\tau$ and $\tau \leq \sigma_k$ a.s., one can thus conclude that

$$\tau < \sigma_k \text{ a.s. on } A_k^c.$$

ii) We provide the existence of an admissible control $\tilde{\alpha}_k \in \mathbf{A}^\alpha_{\tau_{n,k}}$ such that $\mathcal{M}_T^{\tilde{\alpha}_k} \leq \tilde{\mathcal{M}}_T^{k,\alpha'}$.
The control $\tilde{\alpha}^k$ is defined as follows:

$$\tilde{\alpha}^k := \alpha_s \mathbf{1}_{\{s \leq \tilde{\sigma}_k\}} + \tilde{Z}^k_s \mathbf{1}_{\{s > \tilde{\sigma}_k\}},$$

where $\tilde{\sigma}_k = \sigma_k \wedge T$. Recall that $\tilde{Z}^k$ is the process associated to the BSDE representation of $\tilde{\mathcal{M}}_T^{k,\alpha'}$. Note that the above construction ensures that $0 \leq \mathcal{M}_T^{\tilde{\alpha}^k} \leq \tilde{\mathcal{M}}_T^{k,\alpha'}$ a.s. It remains to show that $\tilde{\alpha}^k \in \mathbf{A}^{\alpha}_{\tau_{n,k}}$. It is clear that we have:

$$\mathcal{M}^{\alpha}_{\tau_n \wedge \sigma_k} = \mathcal{M}^{\tilde{\alpha}^k}_{\tau_n \wedge \sigma_k} \quad \text{a.s.} \tag{3.3.7}$$

and hence

$$\mathcal{M}^{\alpha}_{\tau_n \wedge \sigma_k} = \mathcal{M}^{\tilde{\alpha}^k}_{\tau_n \wedge \sigma_k} \quad \text{a.s. on } A_k^c. \tag{3.3.8}$$

Since $\sigma_k = \tau$ on A_k, it remains to show that $\mathcal{M}^{\alpha}_{\tau_n} = \mathcal{M}^{\tilde{\alpha}^k}_{\tau_n}$ *a.s.* on A_k. Recall that $\alpha' \in \mathbf{A}^{\alpha}_{\tau}$. Hence, by definition of the set A_k, we obtain $\mathcal{M}^{\alpha'}_{\tau} = \mathcal{E}^f_{\tau,T}[\tilde{\mathcal{M}}_T^{k,\alpha'}]$ a.s. on A_k. A strict comparison theorem for BSDEs and the definition of $\tilde{\mathcal{M}}_T^{k,\alpha'}$ lead to

$$\tilde{\mathcal{M}}_T^{k,\alpha'} = \mathcal{M}_T^{\alpha'} = 1, \quad \text{a.s. on } A_k. \tag{3.3.9}$$

The comparison theorem for BSDEs implies:

$$\mathcal{E}^f_{\tau_n,T}[\tilde{\mathcal{M}}_T^{k,\alpha'}] = \mathcal{M}^{\alpha'}_{\tau_n} = \mathcal{E}^f_{\tau_n,T}[1], \quad \text{a.s. on } A_k. \tag{3.3.10}$$

Moreover, by (3.3.9) and the comparison theorem for BSDEs, we have $\mathcal{M}^{\alpha'}_{\tau} = \mathcal{E}^f_{\tau,T}[1]$ a.s. on A_k and since $\alpha' \in \mathbf{A}^{\alpha}_{\tau}$, we get $\mathcal{M}^{\alpha}_{\tau} = \mathcal{E}^f_{\tau,T}[1]$ a.s. on A_k. The strict comparison theorem for BSDEs allows us to conclude that:

$$\mathcal{M}^{\alpha}_{\tau_n} = \mathcal{E}^f_{\tau_n,T}[1] \text{ a.s. on } A_k. \tag{3.3.11}$$

Since $\sigma_k = \tau$ on A_k and since (3.3.10), (3.3.11) hold, we finally obtain:

$$\mathcal{M}^{\alpha}_{\tau_n} = \mathcal{E}^f_{\tau_n,T}[\tilde{\mathcal{M}}_T^{k,\alpha'}] = \mathcal{M}^{\tilde{\alpha}^k}_{\tau_n} \text{ a.s. on } A_k. \tag{3.3.12}$$

By (3.3.6), (3.3.8) and (3.3.12), we deduce that $\mathcal{M}^{\alpha}_{\tau_{n,k}} = \mathcal{M}^{\tilde{\alpha}^k}_{\tau_{n,k}}$ a.s. and hence that $\tilde{\alpha}^k \in \mathbf{A}^{\alpha}_{\tau_{n,k}}$.

Step 2. Let us prove now the converse inequality $\overline{\mathcal{Y}}^{\alpha}_{\tau} \geq \mathcal{Y}^{\alpha}_{\tau}$ a.s.

We apply on $[\tau, \tau_n]$ the stability result for BSDEs with parameters $(\overline{\mathcal{Y}}^{\alpha}_{\tau}, 0)$ and $(\mathcal{Y}^{\alpha}_{\tau_n}, g\boldsymbol{1}_{[0,\tau_n)})$, we obtain:

$$\|\overline{\mathcal{Y}}^{\alpha}_{\tau} - \mathcal{E}^g_{\tau,\tau_n}[\mathcal{Y}^{\alpha}_{\tau_n}]\|_{\mathbf{L}_2} \leq C \left(\|\overline{\mathcal{Y}}^{\alpha}_{\tau} - \mathcal{Y}^{\alpha}_{\tau_n}\|_{\mathbf{L}_2} + \mathbb{E}[\int_{\tau}^{\tau^n} |g(s, \overline{\mathcal{Y}}^{\alpha}_{\tau}, 0)|^2 ds] \right). \tag{3.3.13}$$

Definition 3.3.1, together with the assumptions on the driver g, the convergence of τ_n to τ, the observation on the integrability of $\mathcal{Y}^{\alpha}$ (see 3.2.8), and Lebesgue's Theorem imply that $\mathbb{E}[\int_{\tau}^{\tau^n} |g(s, \overline{\mathcal{Y}}^{\alpha}_{\tau}, 0)|^2 ds] \to 0$. By the same arguments and (3.3.1), we get $\|\overline{\mathcal{Y}}^{\alpha}_{\tau} - \mathcal{Y}^{\alpha}_{\tau_n}\|_{\mathbf{L}_2} \to 0$. Now, we let n tend to ∞ in (3.3.13), and obtain $\mathcal{E}^g_{\tau,\tau_n}[\mathcal{Y}^{\alpha}_{\tau_n}] \to \overline{\mathcal{Y}}^{\alpha}_{\tau}$ a.s., up to a subsequence.

Moreover, Lemma 3.3.1 implies that $\mathcal{E}^g_{\tau,\tau_n}[\mathcal{Y}^{\alpha}_{\tau_n}] \geq \mathcal{Y}^{\alpha}_{\tau}$. This inequality and the above convergence lead to the desired result. □

Remark 3.3.4. In [31], it is provided the existence of a control $\alpha_n \in \boldsymbol{A}^{\alpha}_{\tau_n}$, with $\tau_n \to \tau$ and $\tau_n > \tau$ for all n, such that $M_T^{\alpha_n}$ remains "sufficiently close" to $M_T^{\alpha'}$. The control α_n is obtained by scaling α in an appropriate way. This approach cannot be applied in the case of nonlinear constraints, as being clearly specific to the linear setting.

Using similar arguments as in Theorem 2.1 in [31] (points (iii), (iv)) one can show the following BSDE representation for $\mathcal{Y}^\alpha$:

Theorem 3.3.5. *Assume that Assumption* 3.3.2 *holds. Then there exists a family* $(\mathcal{Z}^\alpha, \mathcal{K}^\alpha)_{\alpha \in \boldsymbol{A}_0}$ *satisfying*

$$\sup_{\alpha \in \boldsymbol{A}_0} \|\mathcal{Y}^\alpha, \mathcal{Z}^\alpha, \mathcal{K}^\alpha\|_{\mathbf{S}_2 \times \boldsymbol{H}_2 \times \boldsymbol{I}_2} < +\infty. \tag{3.3.14}$$

and such that for all $\alpha \in \boldsymbol{A}_0$, *we have*

$$\mathcal{Y}^\alpha_t = \Phi(\mathcal{M}^\alpha_T) + \int_t^T g(s, \mathcal{Y}^\alpha_s, \mathcal{Z}^\alpha_s) ds - \int_t^T \mathcal{Z}^\alpha_s dW_s + \mathcal{K}^\alpha_t - \mathcal{K}^\alpha_T. \tag{3.3.15}$$

$$\mathcal{K}^\alpha_{\tau_1} = \operatorname*{essinf}_{\bar{\alpha} \in \boldsymbol{A}^\alpha_{\tau_1}} E[\mathcal{K}^{\bar{\alpha}}_{\tau_2} | \mathcal{F}_{\tau_1}], \ \forall \tau_1 \in \mathcal{T}, \ \tau_2 \in \mathcal{T}_{\tau_1}, \tag{3.3.16}$$

and

$$(\mathcal{Y}^\alpha, \mathcal{Z}^\alpha, \mathcal{K}^\alpha) \boldsymbol{1}_{[0,\tau]} = (\mathcal{Y}^\alpha, \mathcal{Z}^\alpha, \mathcal{K}^\alpha) \boldsymbol{1}_{[0,\tau]}, \ \forall \tau \in \mathcal{T}, \ \bar{\alpha} \in \boldsymbol{A}^\alpha_\tau. \tag{3.3.17}$$

Moreover, $(\mathcal{Y}^\alpha, \mathcal{Z}^\alpha, \mathcal{K}^\alpha)_{\alpha \in \boldsymbol{A}_0}$ *is the unique family satisfying* (3.3.14), (3.3.15), (3.3.16) *and* (3.3.17).

3.4 Existence of optimal controls in the case of concave constraints

We show that in the case of concave constraints and under convexity assumptions on Φ and g, we get the existence of an optimal control $\hat{\alpha}$, that is $\mathcal{Y}^{\hat{\alpha}}_t = \mathcal{E}^g_{t,T}[\Phi(\mathcal{M}^{\hat{\alpha}}_T)]$.

For all $(\lambda, m_1, m_2, t, y_1, y_2, z_1, z_2) \in [0,1] \times [0,1]^2 \times [0,T] \times \mathbf{R}^2 \times [\mathbf{R}^d]^2$, we assume a.s. the following:

$(\mathbf{H_{conc}})$

$$\lambda f(t, y_1, z_1) + (1-\lambda) f(t, y_2, z_2) \le f(t, \lambda y_1 + (1-\lambda) y_2, \lambda z_1 + (1-\lambda) z_2).$$

$(\mathbf{H_{conv}})$

$$\Phi(\lambda m_1 + (1-\lambda) m_2) \le \lambda \Phi(m_1) + (1-\lambda) \Phi(m_2)$$

$$g(t, \lambda y_1 + (1-\lambda) y_2, \lambda z_1 + (1-\lambda) z_2) \le \lambda g(t, y_1, z_1) + (1-\lambda) g(t, y_2, z_2).$$

Proposition 3.4.1. *Under Hypothesis* $(\mathbf{H_{conv}})$ *and* $(\mathbf{H_{conc}})$, *for any* $(\tau, \alpha) \in \mathcal{T} \times \boldsymbol{H}_2$, *there exists* $\hat{\alpha}^{\tau,\alpha} \in \boldsymbol{A}^\alpha_\tau$ *such that*

$$\mathcal{Y}^\alpha_\tau = \mathcal{E}^g_\tau \left[\Phi(\mathcal{M}^{\hat{\alpha}^{\tau,\alpha}}_T)\right] = \mathcal{E}^g_{\tau,\tau'} \left[\mathcal{Y}^{\hat{\alpha}^{\tau,\alpha}}_{\tau'}\right], \ \forall \tau' \in \mathcal{T}_\tau.$$

Proof. By Lemma 3.7.1 in the Appendix, there exists $(\alpha^n)_n \in \mathbf{A}^\alpha_\tau$ such that:

$$\mathcal{Y}^\alpha_\tau = \lim_{n \to \infty} \mathcal{E}^g_{t,T}[\Phi(\mathcal{M}^{\alpha^n}_T)]. \tag{3.4.1}$$

Recall that $(\mathcal{M}^{\alpha^n}_T)_n$ is valued in [0,1]. By Komlos Theorem, $\tilde{\mathcal{M}}^n_T := \frac{1}{n} \sum_{i \le n} \mathcal{M}^{\alpha^i}_T$ converges a.s. to a random variable $\tilde{\mathcal{M}}_T$ which belongs a.s. to $[0,1]$.
From the concavity assumption on the driver f and the comparison theorem for BSDEs we get:

$$\mathcal{E}^f_{\tau,T}[\tilde{\mathcal{M}}^n_T] \ge \frac{1}{n} \sum_{i \le n} \mathcal{E}^f_{\tau,T}[\mathcal{M}^{\alpha^i}_T] = \mathcal{M}^\alpha_\tau, \tag{3.4.2}$$

since $\alpha^n \in \mathbf{A}_\tau^\alpha$ for all n.
The a priori estimates for BSDEs lead to:

$$|\mathcal{E}^f_{\tau,T}[\tilde{\mathcal{M}}^n_T] - \mathcal{E}^f_{\tau,T}[\tilde{\mathcal{M}}_T]|^2 \leq \mathbb{E}_t[|\tilde{\mathcal{M}}^n_T - \tilde{\mathcal{M}}_T|^2]. \tag{3.4.3}$$

The a.s. convergence $\tilde{\mathcal{M}}^n_T \to \tilde{\mathcal{M}}_T$ and the boundness of the sequence $(\tilde{\mathcal{M}}^n_T)_n$ allow us to apply the Lebesgue's theorem and to derive that the right hand side of the above inequality tends to 0 when n goes to $+\infty$. We thus derive that:

$$\mathcal{E}^f_{\tau,T}[\tilde{\mathcal{M}}^n_T] \to \mathcal{E}^f_{\tau,T}[\tilde{\mathcal{M}}_T] \;\; a.s. \tag{3.4.4}$$

Hence, inequality (3.4.2) combined with (3.4.4) lead to $\mathcal{E}^f_{\tau,T}[\tilde{\mathcal{M}}_T] \geq \mathcal{M}^\alpha_\tau$.

Let us denote by $\tilde{\alpha}$ the control associated to the BSDE with terminal condition $\tilde{\mathcal{M}}_T$ and driver f. We define the following stopping time:

$$\theta^{\tilde{\alpha}} := \inf\{\tau \leq s \leq T : \mathcal{M}^{\tau,\mathcal{M}^\alpha_\tau,\tilde{\alpha}}_s = \mathcal{E}^f_{s,T}[0]\} \wedge T,$$

with the convention $\inf \emptyset = +\infty$. We recall that (Y^0, Z^0) represents the solution of the BSDE associated to driver f and terminal condition 0 and we define the control $\hat{\alpha}$ as follows:

$$\hat{\alpha}_s := \alpha_s \mathbf{1}_{s\leq\tau} + \tilde{\alpha}_s \mathbf{1}_{\{\tau<s\leq\theta^{\tilde{\alpha}}\}} + Z^0_s \mathbf{1}_{\{s>\theta^{\tilde{\alpha}}\}}. \tag{3.4.5}$$

Note that $\hat{\alpha}$ belongs to $\mathbf{A}^\alpha_\tau$. Moreover, by construction, we have:

$$\mathcal{M}^{\hat{\alpha}}_T \leq \tilde{\mathcal{M}}_T \;\; \text{a.s.} \tag{3.4.6}$$

Now, by using hypothesis ($\mathbf{H_{conv}}$) and the comparison theorem, we obtain:

$$\tilde{\mathcal{Y}}^n_\tau := \frac{1}{n}\sum_{i\leq n} \mathcal{E}^g_{\tau,T}\left[\Phi(\mathcal{M}^{\alpha^i}_T)\right] \geq \mathcal{E}^g_{\tau,T}\left[\Phi(\tilde{\mathcal{M}}^n_T)\right]. \tag{3.4.7}$$

By (3.4.1) and Cesaro's Lemma we have $\lim_{n\to\infty} \tilde{\mathcal{Y}}^n_\tau = \mathcal{Y}^\alpha_\tau$ a.s.

Similar arguments as in the explanation of the convergence (3.4.4) allow us to deduce that $\lim_{n\to\infty} \mathcal{E}^g_{\tau,T}\left[\Phi(\tilde{\mathcal{M}}^n_T)\right] = \mathcal{E}^g_{\tau,T}\left[\Phi(\tilde{\mathcal{M}}_T)\right]$a.s. By letting n tend to ∞ in (3.4.7) we conclude:

$$\mathcal{Y}^\alpha_\tau \geq \mathcal{E}^g_{\tau,T}\left[\Phi(\tilde{\mathcal{M}}_T)\right]. \tag{3.4.8}$$

From (3.4.6), (3.4.8), the non-decreasing monotonicity of the map Φ and the comparison theorem for BSDEs, we finally get:

$$\mathcal{Y}^\alpha_\tau \geq \mathcal{E}^g_{\tau,T}\left[\Phi(\mathcal{M}^{\hat{\alpha}}_T)\right]. \tag{3.4.9}$$

The equality follows by definition of $\mathcal{Y}^\alpha_\tau$ and $\hat{\alpha}$ is hence the optimal control.
In order to show the second equality $\mathcal{Y}^\alpha_\tau = \mathcal{E}^g_{\tau,\tau'}\left[\mathcal{Y}^{\hat{\alpha}^{\tau,\alpha}}_{\tau'}\right]$, $\forall \tau' \in \mathcal{T}_\tau$, we first observe that $\mathcal{Y}^\alpha_\tau = \mathcal{E}^g_{\tau,\tau'}\left[\mathcal{E}^g_{\tau',T}[\Phi(\mathcal{M}^{\hat{\alpha}}_T)]\right] \geq \mathcal{E}^g_{\tau,\tau'}\left[\mathcal{Y}^{\hat{\alpha}}_{\tau'}\right]$, by definition of the value function $\mathcal{Y}^{\hat{\alpha}}_{\tau'}$ and the comparison theorem. As above, there exists $(\hat{\alpha}^n) \in \mathbf{A}^{\hat{\alpha}}_{\tau'}$ such that $\mathcal{E}^g_{\tau',T}[\Phi(\mathcal{M}^{\hat{\alpha}^n}_T)] \to \mathcal{Y}^{\hat{\alpha}}_{\tau'}$ a.s. By (3.2.8), the convergence also holds in $\mathbf{L}_2$. The a priori estimates on BSDEs give: $\mathcal{Y}^\alpha_\tau \leq \mathcal{E}^g_{\tau,\tau'}\left[\mathcal{E}^g_{\tau',T}[\Phi(\mathcal{M}^{\hat{\alpha}^n}_T)]\right] \to \mathcal{E}^g_{\tau,\tau'}[\mathcal{Y}^{\hat{\alpha}}_{\tau'}]$. □

Remark 3.4.2. Note that in [31], the optimal control is obtained directly by using the martingale representation of $\tilde{\mathcal{M}}_T$, due to the linearity of the expectation. In our nonlinear case, that is no longer possible and we need a more complicated construction.

3.5 Properties of the value function

In this section, we study the continuity and the convexity (defined in a probabilistic sense) of the map $\mathcal{Y}_t(\mu) := \mathcal{Y}(t, \mu)$ with respect to the threshold μ, for any $t < T$.

3.5.1 Continuity

Fix $t \in [0, T]$. We give below an estimate on the map $\mu \to \mathcal{Y}_t(\mu)$, ensuring its continuity under some weak assumptions on the map Φ (e.g. Φ is Lipschitz continuous with respect to x, uniformly in ω or deterministic continuous). We obtain a nicer and more natural bound for $|\mathcal{Y}_t(\mu_1) - \mathcal{Y}_t(\mu_2)|$ than the one provided in the case of classical expectations constraints ([31]), which is expressed through the spread $|\mu_1 - \mu_2|^{\frac{1}{2}}$ (in [31] it depends on $(1 - \frac{\mu_1}{\mu_2})\mathbf{1}_{\mu_1 < \mu_2} + \frac{\mu_1 - \mu_2}{1 - \mu_2}\mathbf{1}_{\mu_1 > \mu_2}$; $(1 - \frac{\mu_2}{\mu_1})\mathbf{1}_{\mu_2 < \mu_1} + \frac{\mu_2 - \mu_1}{1 - \mu_1}\mathbf{1}_{\mu_1 < \mu_2}$ and on other two terms related to the case when the thresholds take the boundary values 0 and 1). Moreover, our proof is based on BSDEs techniques, allowing to treat the nonlinear case, contrary to [31], where the arguments hold only in the case of linear constraints.

Theorem 3.5.1. *Let $t < T$, and $\mu_1, \mu_2 \in \boldsymbol{D}_t$.*
Then $|\mathcal{Y}_t(\mu_1) - \mathcal{Y}_t(\mu_2)| \leq Err_t(\Delta(\mu_1, \mu_2))$, where $\Delta(\mu_1, \mu_2) = C|\mu_1 - \mu_2|^{\frac{1}{2}}$, with C a constant depending only on (C_f, T) and

$$Err_t(\xi) := ess\sup\{\mathcal{R}_t(M, M') : M, M' \in \boldsymbol{L}_0([0,1]), E_t[|M - M'|^2] \leq \xi\}, \tag{3.5.1}$$

where $\xi \in \boldsymbol{L}_2(\mathbf{R}, \mathcal{F}_t)$ and $\mathcal{R}_t(M, M') := |\mathcal{E}^g_{t,T}[\Phi(M)] - \mathcal{E}^g_{t,T}[\Phi(M')]|$.

Proof. We define $\tilde{\mu_1} := \mu_1 \vee \mu_2$ and $\tilde{\mu_2} := \mu_1 \wedge \mu_2$. By the monotonocity property of the map $\mu \to \mathcal{Y}_t(\mu)$ (3.7.2), we have $\mathcal{Y}_t(\tilde{\mu_1}) \geq \mathcal{Y}_t(\tilde{\mu_2})$ a.s.

By Lemma 3.7.1, it exists $\alpha^n \in \mathbf{A}_{t,\tilde{\mu}_2}$ such that $\lim_{n \to \infty} \mathcal{E}^g_{t,T}[\Phi(\mathcal{M}^{\tilde{\mu}_2, \alpha^n})] = \mathcal{Y}_t(\tilde{\mu}_2)$ a.s. Fix $n \in \mathbb{N}$. We now construct an admissible control $\tilde{\alpha}^n \in \mathbf{A}_{t,\tilde{\mu}_1}$ such that $\mathcal{M}^{\tilde{\mu}_1, \tilde{\alpha}^n}_s \in \left[\mathcal{M}^{\tilde{\mu}_2, \alpha^n}_s, \mathcal{E}^f_{s,T}[1]\right]$, $t \leq s \leq T$, a.s. It is defined as follows:

$$\tilde{\alpha}^n_s := \alpha^n_s \mathbf{1}_{\{s \leq \tau\}} + Z^1_s \mathbf{1}_{\{s > \tau\}},$$

where $\tau := \inf\{s \in [t, T] : \mathcal{M}^{\tilde{\mu}_1, \alpha^n}_s = \mathcal{E}^f_s[1]\} \wedge T$, with the convention $\inf \emptyset = +\infty$. Recall that Z^1 corresponds to the control associated to the BSDE of terminal condition 1 and driver f.

By definition of the value function $\mathcal{Y}_t$, we get:

$$\mathcal{Y}_t(\tilde{\mu_1}) \leq \mathcal{E}^g_{t,T}[\Phi(\mathcal{M}^{\tilde{\mu}_1, \tilde{\alpha}^n}_T)] = \mathcal{E}^g_{t,T}[\Phi(\mathcal{M}^{\tilde{\mu}_1, \tilde{\alpha}^n}_T)] - \mathcal{E}^g_{t,T}[\Phi(\mathcal{M}^{\tilde{\mu}_2, \alpha^n}_T)] + \mathcal{E}^g_{t,T}[\Phi(\mathcal{M}^{\tilde{\mu}_2, \alpha^n}_T)]. \tag{3.5.2}$$

Let us now estimate $E_t[|\mathcal{M}^{\tilde{\mu}_1, \tilde{\alpha}^n}_T - \mathcal{M}^{\tilde{\mu}_2, \alpha^n}_T|^2]$.
Since $\mathcal{M}^{\tilde{\mu}_1, \tilde{\alpha}^n}_T$ and $\mathcal{M}^{\tilde{\mu}_2, \alpha^n}_T$ belong to $[0, 1]$ and by construction $\mathcal{M}^{\tilde{\mu}_1, \tilde{\alpha}^n}_T \geq \mathcal{M}^{\tilde{\mu}_2, \alpha^n}_T$ a.s., we obtain:

$$E_t[|\mathcal{M}^{\tilde{\mu}_1, \tilde{\alpha}^n}_T - \mathcal{M}^{\tilde{\mu}_2, \alpha^n}_T|^2] \leq E_t[\mathcal{M}^{\tilde{\mu}_1, \tilde{\alpha}^n}_T - \mathcal{M}^{\tilde{\mu}_2, \alpha^n}_T]. \tag{3.5.3}$$

A similar linearization technique as in the proof of the Comparison Theorem for BSDEs (see for e.g. [132]) yields:

$$\tilde{\mu}_1 - \tilde{\mu}_2 \geq E_t\left[H^n_{t,T}(\mathcal{M}^{\tilde{\mu}_1, \tilde{\alpha}^n}_T - \mathcal{M}^{\tilde{\mu}_2, \alpha^n}_T)|\mathcal{F}_t\right] \text{ a.s.,} \tag{3.5.4}$$

where $(H^n_{t,s})_{s\in[t,T]}$ is the square integrable process satisfying

$$dH^n_{t,s} = H^n_{t,s}\left[\delta^n_s ds + \beta^n_s dW_s\right]; \;\; H^n_{t,t} = 1,$$

with

$$\begin{cases} \delta^n_t := \dfrac{f(t, \mathcal{M}^{\tilde{\mu}_1,\tilde{\alpha}^n}_t, \tilde{\alpha}^n_t) - f(t, \mathcal{M}^{\tilde{\mu}_2,\alpha^n}_t, \tilde{\alpha}^n_t)}{\mathcal{M}^{\tilde{\mu}_1,\tilde{\alpha}^n}_t - \mathcal{M}^{\tilde{\mu}_2,\alpha^n}_t} \mathbf{1}_{\{\mathcal{M}^{\tilde{\mu}_1,\tilde{\alpha}^n}_t \neq \mathcal{M}^{\tilde{\mu}_2,\alpha^n}_t\}}; \\ \beta^n_t := \dfrac{f(t, \mathcal{M}^{\tilde{\mu}_2,\alpha^n}_t, \tilde{\alpha}^n_t) - f(t, \mathcal{M}^{\tilde{\mu}_2,\alpha^n}_t, \alpha^n_t)}{|\tilde{\alpha}^n_t - \alpha^n_t|^2} (\tilde{\alpha}^n_t - \alpha^n_t)\mathbf{1}_{\tilde{\alpha}^n_t \neq \alpha^n_t}. \end{cases}$$

Now, from (1.4.7) and the Holder inequality, we obtain:

$$\begin{aligned} E_t[\mathcal{M}^{\tilde{\mu}_1,\tilde{\alpha}^n}_T - \mathcal{M}^{\tilde{\mu}_2,\alpha^n}_T] &= E_t[(H^n_{t,T})^{-\frac{1}{2}}(H^n_{t,T})^{\frac{1}{2}}(\mathcal{M}^{\tilde{\mu}_1,\tilde{\alpha}^n}_T - \mathcal{M}^{\tilde{\mu}_2,\alpha^n}_T)] \\ &\leq E_t\left[(H^n_{t,T})^{-1}\right]^{\frac{1}{2}} E_t\left[H^n_{t,T}(\mathcal{M}^{\tilde{\mu}_1,\tilde{\alpha}^n}_T - \mathcal{M}^{\tilde{\mu}_2,\alpha^n}_T)^2\right]^{\frac{1}{2}}. \end{aligned} \tag{3.5.5}$$

Note that $(\delta^n)_n, (\beta^n)_n$ are predictable process bounded by C_f, the Lipschitz constant of f. We thus have for all $n \in \mathbb{N}$, $E_t\left[(H^n_{t,T})^{-1}\right] \leq C$, for some $C > 0$ depending on C_f and T (by the properties of exponential martingales).
The above relation together with (3.5.4) and the fact that $\mathcal{M}^{\tilde{\mu}_1,\tilde{\alpha}^n}_T - \mathcal{M}^{\tilde{\mu}_2,\alpha^n}_T$ takes values in $[0,1]$ a.s., imply:

$$E_t[|\mathcal{M}^{\tilde{\mu}_1,\tilde{\alpha}^n}_T - \mathcal{M}^{\tilde{\mu}_2,\alpha^n}_T|^2] \leq C(\tilde{\mu}_1 - \tilde{\mu}_2)^{\frac{1}{2}}, \tag{3.5.6}$$

where C is a constant depending on the Lipschitz constant of the driver f.
By letting n tend to infinity in inequality (3.5.2) and using (3.5.6), we get:

$$|\mathcal{Y}_t(\tilde{\mu}_1) - \mathcal{Y}_t(\tilde{\mu}_2)| \leq Err_t\left(\Delta(\tilde{\mu}_1, \tilde{\mu}_2)\right). \tag{3.5.7}$$

Same arguments as in Step 2 of the previous theorem lead to:

$$|\mathcal{Y}_t(\mu_1) - \mathcal{Y}_t(\mu_2)| \leq Err_t\left(\Delta(\mu_1, \mu_2)\right). \tag{3.5.8}$$

□

3.5.2 Convexity

In this section, we provide a convexity result adapted to the non-markovian setting which is established for the map $\mu \to \mathcal{Y}_t(\mu)$, for any $t < T$. We extend the results of [31] to the case of nonlinear constraints, which lead to nontrivial additional technicalities. More precisely, in [31] it is used the fact that a threshold μ "admissible" at time t (as it belongs to $[0,1]$ a.s.), it is "admissible" at any time between 0 and T. In our case, due to nonlinearity, the admissibility set is not the same for all t, as it is given by the two processes $\mathcal{E}^f[0]$ and $\mathcal{E}^f[1]$.

We first recall the notion of $\mathcal{F}_t$ - convexity introduced in [31].

Definition 3.5.2 ($\mathcal{F}_t$-convexity). *(i) We say that a subset $D \subset \boldsymbol{L}_2(\mathbf{R}, \mathcal{F}_t)$ is $\mathcal{F}_t$-convex if for all $\mu_1, \mu_2 \in D$ and $\lambda \in \boldsymbol{L}_0([0,1], \mathcal{F}_t)$, $\lambda\mu_1 + (1-\lambda)\mu_2 \in D$.*

(ii) *Let D be an $\mathcal{F}_t$-convex subset of $\boldsymbol{L}_2(\mathbf{R},\mathcal{F}_t)$. A map $\mathcal{J} : D \mapsto \boldsymbol{L}_2(\mathbf{R},\mathcal{F}_t)$ is said to be $\mathcal{F}_t$-convex if*

$$Epi(\mathcal{J}) := \{(\mu, Y) \in D \times \boldsymbol{L}_2(\mathbf{R},\mathcal{F}_t) : Y \geq \mathcal{J}(\mu)\}$$

is $\mathcal{F}_t$-convex.

(iii) *Let $Epi^c(\mathcal{J})$ be the set of elements of the form $\sum_{n\leq N} \lambda_n(\mu_n, Y_n)$ with $(\mu_n, Y_n, \lambda_n)_{n\leq N} \subset Epi(\mathcal{J}) \times \boldsymbol{L}_0([0,1],\mathcal{F}_t)$ such that $\sum_{n\leq N} \lambda_n = 1$, for some $N \geq 1$. We then denote by $\overline{Epi}^c(\mathcal{J})$ its closure in $\boldsymbol{L}_2$. The $\mathcal{F}_t$-convex envelope of $\mathcal{J}_t$ is defined as*

$$\mathcal{J}_t^c(\mu) := ess\inf\{Y \in \boldsymbol{L}_2(\mathbf{R},\mathcal{F}_t) : (\mu, Y) \in \overline{Epi}^c(\mathcal{J}_t)\}. \tag{3.5.9}$$

Assumption 3.5.3. *We assume that the map Φ is Lipschitz continuous in x, uniformly with respect to ω.*

Proposition 3.5.4. *Under Assumption 3.5.3, the map $\mu \in \boldsymbol{D}_t \mapsto \mathcal{Y}_t(\mu)$ is $\mathcal{F}_t$-convex, for all $t < T$.*

The proof is divided in several steps. We follow the arguments used in the proof of Proposition 3.2 in [31] up to non trivial modifications due to the nonlinearity of the driver f. The technical arguments specific to the nonlinear case are mostly needed in Step 5 of the proof. For convienence of the reader, we also present the main ideas of Steps 1-4.

Proof. 1. $(\mu, \mathcal{Y}_t^c(\mu)) \in \overline{Epi}^c(\mathcal{Y}_t)$, for all $\mu \in \mathbf{D}_t$.
For every fixed element $\mu \in \mathbf{D}_t$, the family $F := \{Y \in \mathbf{L}_2(\mathbf{R},\mathcal{F}_t) : (\mu, Y) \in \overline{Epi}^c(\mathcal{Y}_t)\}$ is direct downward since $Y^1\mathbf{1}_{\{Y^1\leq Y^2\}} + Y^2\mathbf{1}_{\{Y^1>Y^2\}} \in F$ for all Y^1, Y^2, by $\mathcal{F}_t$-convexity of $\overline{Epi}^c(\mathcal{Y}_t)$. It then follows that we can find a sequence $(Y^n)_{n\geq 1} \subset F$ such that $Y^n \downarrow \mathcal{Y}_t^c(\mu)$ a.s. Moreover, Y^1 and $\mathcal{Y}_t^c(\mu)$ belong to $\mathbf{L}_2$, and thus the monotone convergence Theorem leads to $Y^n \to \mathcal{Y}_t^c(\mu)$ in $\mathbf{L}_2$, as n goes to infinity. The set $\overline{Epi}^c(\mathcal{Y}_t)$ is closed in $\mathbf{L}_2$ and hence the result follows.

2. Let $\eta \in \mathbf{S}_2$ be as in inequality 3.2.8. Then, $|\mathcal{Y}_t^c(\mu)| \leq \eta_t$, for all $t \leq T$ and $\mu \in \mathbf{D}_t$.
We first show that $\mathcal{Y}_t^c(\mu) \geq -\eta_t$. By Point 1, it follows that $(\mu, \mathcal{Y}_t^c(\mu)) \in \overline{Epi}^c(\mathcal{Y}_t)$ is obtaind as $\mathbf{L}_2$-limit of elements of the form $\sum_{n\leq N} \lambda_n(\mu_n, Y_n)$ with $(\mu_n, Y_n, \lambda_n) \subset Epi(\mathcal{Y}_t) \times \mathbf{L}_0([0,1],\mathcal{F}_t)$, such that $\sum_{n\leq N} \lambda_n = 1$. By 3.2.8, each Y^n of the above family is bounded below by $-\eta_t$ and hence this also holds for $\mathcal{Y}_t^c(\mu)$. The converse inequality $\mathcal{Y}_t^c \leq \eta_t$ is clear since Remark 3.2.8 holds and, by construction, $\mathcal{Y} \geq \mathcal{Y}_t^c$.

3. The map $\mu \in \mathbf{D}_t \mapsto \mathcal{Y}_t^c(\mu)$ is $\mathcal{F}_t$-convex.
We have the show that $Epi(\mathcal{Y}_t^c)$ is $\mathcal{F}_t-$convex. Let us fix $\mu^1, \mu^2 \in \mathbf{D}_t$ and $\lambda \in \mathbf{L}_0([0,1],\mathcal{F}_t)$. Since $\overline{Epi}^c(\mathcal{Y}_t)$ is $\mathcal{F}_t$-convex and $(\mu^i, \mathcal{Y}_t^c(\mu^i)) \in \overline{Epi}^c(\mathcal{Y}_t)$, for $i = 1, 2$, it follows that $(\lambda\mu^1 + (1 - \lambda)\mu^2, \lambda\mathcal{Y}_t^c(\mu^1) + (1-\lambda)\mathcal{Y}_t^c(\mu^2)) \in \overline{Epi}^c(\mathcal{Y}_t)$, and thus $\lambda\mathcal{Y}_t^c(\mu^1) + (1-\lambda)\mathcal{Y}_t^c(\mu^2) \geq \mathcal{Y}_t^c(\lambda\mu^1 + (1-\lambda)\mu^2$, by definition of $\mathcal{Y}_t^c(\mu)$. We obtain that $\lambda Y^1 + (1-\lambda)Y^2 \geq \mathcal{Y}_t^c(\lambda\mu^1 + (1-\lambda)\mu^2)$, for any Y^1, Y^2 such that $(\mu^i, Y^i) \in Epi(\mathcal{Y}_t^c)$, $i = 1, 2$. The result follows.

4. $\mathcal{Y}_t(\mu) \geq \mathcal{Y}_t^c(\mu)$, for all $\mu \in \mathbf{D}_t$.
Let $(\mu_n)_n \in \mathbf{D}_t$ be such that $\mu_n \to \mu$ a.s. when $n \to \infty$. Recall that under Assumption 3.5.3, the

map $\mu \to \mathcal{Y}_t(\mu)$ is a.s. continuous and hence $\mathcal{Y}_t(\mu_n) \to \mathcal{Y}_t(\mu)$ a.s. when $n \to \infty$. Moreover, by 3.2.8 we have $\mathcal{Y}_t(\mu_n) \to \mathcal{Y}_t(\mu)$ in $\mathbf{L}_2$. Note that $Epi(\mathcal{Y}_t) \subset \overline{Epi}^c(\mathcal{Y}_t)$ and thus $(\mu, \mathcal{Y}_t(\mu)) \in \overline{Epi}^c(\mathcal{Y}_t)$. The result follows by using the definition of $\mathcal{Y}_t^c$.

5. $\mathcal{Y}_t^c(\mu) \geq \mathcal{Y}_t(\mu)$, for all $\mu \in \mathbf{D}_t$.

(i) It follows from Point 1, that there exists a sequence

$$(\mu_n, Y_n, \lambda_n^N)_{n\geq 1, N\geq 1} \subset Epi(\mathcal{Y}_t) \times \mathbf{L}_0([0,1], \mathcal{F}_t)$$

such that $\sum_{n\leq N} \lambda_n^N = 1$, for all N, and

$$(\hat{\mu}_N, \hat{Y}_N) := \sum_{n\leq N} \lambda_n^N (\mu_n, Y_n) \mapsto (\mu, \mathcal{Y}_t^c) \in \mathbf{L}_2. \tag{3.5.10}$$

Fix $N \geq 1$ and $M \geq 1$. We claim that $\mathcal{Y}_t(\hat{\mu}_N) \leq \hat{Y}_N$. The proof is postponed to Step 5, point *(ii)*. We deduce:

$$\lim \inf_{N\to\infty} \mathcal{Y}_t(\hat{\mu}_N) \leq \mathcal{Y}_t^c(\mu). \tag{3.5.11}$$

We now define:

$$Z_M(\mu) := ess \inf\{\mathcal{Y}_t(\mu') : \ |\mu' - \mu| \leq \frac{1}{M}\}. \tag{3.5.12}$$

and set $D_\mu^M := \{\mu' \in \mathbf{D}_t : \ |\mu' - \mu| \leq \frac{1}{M}\}$. By Lemma 3.7.1, it exists a sequence $(\mu_n^M)_n$ with $\mu_n^M \in \mathbf{D}_\mu^M$ for all n such that

$$\mathcal{Y}_t(\mu_n^M) \to Z_M(\mu) \text{ a.s. when } n \to \infty. \tag{3.5.13}$$

One can easily remark that under Assumption 3.5.3, the estimate given in Theorem 3.5.1 becomes:

$$|\mathcal{Y}_t(\mu_n^M) - \mathcal{Y}_t(\mu)| \leq Err_t(\Delta|\mu_n^M - \mu|) \leq K|\mu_n^M - \mu|^{\frac{1}{4}} \leq K\frac{1}{M^{\frac{1}{4}}}, \tag{3.5.14}$$

where K is a constnat depending on C_f, T and the Lipschitz constant of Φ.
Coupling the inequality (3.5.14) with (3.5.13) and letting first n and then M to ∞, we get

$$Z_M(\mu) \to \mathcal{Y}_t(\mu) \text{ a.s. when } M \to +\infty. \tag{3.5.15}$$

Now, the convergence $\hat{\mu}_N \to \mu$ a.s. and Lemma 3.7.3 imply that:

$$Z_M(\mu) \leq \lim \inf_{N\to\infty} \mathcal{Y}_t(\bar{\mu}_N) = \lim \inf_{N\to\infty} \left(\mathcal{Y}_t(\hat{\mu}_N)\mathbf{1}_{|\hat{\mu}_N - \mu| \leq \frac{1}{M}} + \mathcal{Y}_t(\mu)\mathbf{1}_{|\hat{\mu}_N - \mu| > \frac{1}{M}}\right) \leq \mathcal{Y}_t^c(\mu), \tag{3.5.16}$$

where:

$$\bar{\mu}_N := \hat{\mu}_N \mathbf{1}_{|\hat{\mu}_N - \mu| \leq \frac{1}{M}} + \mu \mathbf{1}_{|\hat{\mu}_N - \mu| > \frac{1}{M}} \in D_\mu^M.$$

Also, since by (3.5.15), $Z_M(\mu) \uparrow \mathcal{Y}_t(\mu)$ as M goes to $+\infty$, the result follows.

(ii) It remains to prove:

$$\mathcal{Y}_t(\hat{\mu}_N) \leq \hat{Y}_N. \tag{3.5.17}$$

Fix $\varepsilon > 0$. Let us consider a random variable, $\mathcal{F}_{t+\varepsilon}$ measurable ζ_N^ε such that $P\left[\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n} | \mathcal{F}_t\right] = \lambda_n^N$, where $\alpha_n \in \mathbf{A}_{t,\mu_n}$. Clearly, by construction, ζ_N^ε belongs to $\left[\mathcal{E}_{t+\varepsilon,T}^f[0], \mathcal{E}_{t+\varepsilon,T}^f[1]\right]$ a.s. We set:

$$\mu_N^\varepsilon := \mathcal{E}_{t,t+\varepsilon}^f\left[\zeta_N^\varepsilon\right]. \tag{3.5.18}$$

We rewrite $\mathcal{Y}_t(\hat{\mu}_N)$ as follows:

$$\mathcal{Y}_t(\hat{\mu}_N) = \mathcal{Y}_t(\hat{\mu}_N) - \mathcal{Y}_t(\mu_N^\varepsilon) + \mathcal{Y}_t(\mu_N^\varepsilon) \tag{3.5.19}$$

and by appealing to Theorem 3.5.1, we obtain:

$$\mathcal{Y}_t(\hat{\mu}_N) \leq Err_t(\Delta(\hat{\mu}_N - \mu_N^\varepsilon)) + \mathcal{Y}_t(\mu_N^\varepsilon). \tag{3.5.20}$$

We now show that $\limsup_{\varepsilon \to 0}\left[Err_t(\Delta(\hat{\mu}_N - \mu_N^\varepsilon)) + \mathcal{Y}_t(\mu_N^\varepsilon)\right] \leq \hat{Y}_N$.

To this purpose, we split the proof in several steps:

Step a. We prove that $\lim_{\varepsilon \to 0} Err_t(\Delta(\hat{\mu}_N - \mu_N^\varepsilon)) = 0$ a.s.

We start by showing that $\lim_{\varepsilon \to 0} \mu_N^\varepsilon = \hat{\mu}_N$ a.s.

Since $(\mu_n)_{n \leq N}$ are $\mathcal{F}_t$-measurable and $P\left[\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n} | \mathcal{F}_t\right] = \lambda_n^N$, we have

$$\hat{\mu}_N = E_t\Big[\sum_{n \leq N} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}} \mu_n\Big]$$

a.s. We split the difference between μ_N^ε and μ_N in two terms as follows:

$$\begin{aligned}
|\mu_N^\varepsilon - \hat{\mu}_N| &= |\mathcal{E}_{t,t+\varepsilon}^f\Big[\sum_{n \leq N} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}} \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}\Big] - E_t\Big[\sum_{n \leq N} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}} \mu_n\Big]|^2 \\
&\leq 2|\mathcal{E}_{t,t+\varepsilon}^f\Big[\sum_{n \leq N} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}} \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}\Big] - \mathcal{E}_{t,t+\varepsilon}^f\Big[\sum_{n \leq N} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}} \mu_n\Big]|^2 \\
&\quad + 2|\mathcal{E}_{t,t+\varepsilon}^f\Big[\sum_{n \leq N} \mathbf{1}_{A_n^\varepsilon} \mu_n\Big] - E_t\Big[\sum_{n \leq N} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}} \mu_n\Big]|^2.
\end{aligned} \tag{3.5.21}$$

From the a priori estimations on BSDEs, we obtain:

$$\begin{aligned}
&|\mathcal{E}_{t,t+\varepsilon}^f\Big[\sum_{n \leq N} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}} \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}\Big] - \mathcal{E}_{t,t+\varepsilon}^f\Big[\sum_{n \leq N} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}} \mu_n\Big]|^2 \\
&\quad \leq E_t\Big[\sum_{n \leq N} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}} (\mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n} - \mu_n)^2\Big] \leq \sum_{n \leq N} E_t\big[(\mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n} - \mu_n)^2\big].
\end{aligned} \tag{3.5.22}$$

Since for all $n \leq N$ the processes $\mathcal{M}_\cdot^{\mu_n,\alpha_n}$ are continuous and belong to $\mathbf{S}_2$, we can apply Lebesgue's theorem and obtain that the right member of (3.5.22) tends to 0 when $\varepsilon \to 0$. Moreover, by applying Proposition 3.7.4 with $\xi^\varepsilon = \sum_{n\geq 1} \mathbf{1}_{A_n^\varepsilon}\mu_n$, we derive that it exists η_ε, with $\eta_\varepsilon \to 0$ a.s. when $\varepsilon \to 0$ such that:

$$|\mathcal{E}^f_{t,t+\varepsilon}[\sum_{n\leq N} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}}\mu_n] - E_t[\sum_{n\leq N} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}}\mu_n]|^2 \leq \eta_\varepsilon. \tag{3.5.23}$$

From (3.5.21), (3.5.22) and (3.5.23), by letting ε tend to 0, we get that $\lim_{\varepsilon\to 0}\mu_N^\varepsilon = \hat{\mu}_N$ a.s. This implies that $\Delta(\hat{\mu}_N - \mu_N^\varepsilon) = C|\hat{\mu}_N - \mu_N^\varepsilon|^{\frac{1}{2}} \to 0$. Since Φ satisfies Assumption 3.5.3, we get by Theorem 3.5.1, the desired result.

Step b. We prove that for each $n \leq N$, $\lim_{\varepsilon\to 0} E_t[|\mathcal{Y}_{t+\varepsilon}(\mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}) - \mathcal{Y}_t(\mu_n)|] = 0$ a.s.
As Assumption 3.5.3 holds, we can apply Theorem 3.3.3 and get $\mathcal{Y}_{t+\varepsilon}(\mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}) \to \mathcal{Y}_t(\mu_n)$ a.s. This convergence together with inequality 3.2.8 allow us to apply Lebesgue's Theorem and to obtain the desired result.

Step c.
Recall that by (3.5.18) we have $\mu_N^\varepsilon = \mathcal{E}^f_{t,t+\varepsilon}[\zeta_N^\varepsilon]$. Lemma 3.3.1 leads to:

$$\mathcal{Y}_t(\mu_N^\varepsilon) \leq \mathcal{E}^g_{t,t+\varepsilon}[\mathcal{Y}_{t+\varepsilon}(\zeta_N^\varepsilon)] = \mathcal{E}^g_{t,t+\varepsilon}\left(\mathcal{Y}_{t+\varepsilon}(\sum_{n\leq N} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}}\mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n})\right).$$

By Lemma 3.7.3, we obtain:

$$\mathcal{Y}_t(\mu_N^\varepsilon) \leq \mathcal{E}^g_{t,t+\varepsilon}\left(\sum_{n\leq N} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}}\mathcal{Y}_{t+\varepsilon}(\mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n})\right). \tag{3.5.24}$$

We now apply Proposition 3.7.4 with $\xi^\varepsilon := \sum_{n\geq 1} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}}\mathcal{Y}_{t+\varepsilon}(\mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n})$ and derive that it exists η'_ε, with $\eta'_\varepsilon \to 0$ a.s. when $\varepsilon \to 0$ such that:

$$\mathcal{Y}_t(\mu_N^\varepsilon) \leq \mathcal{E}^g_{t,t+\varepsilon}\left(\sum_{n\leq N} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}}\mathcal{Y}_{t+\varepsilon}(\mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n})\right) \leq E_t\left[\sum_{n\leq N} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}}\mathcal{Y}_{t+\varepsilon}(\mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n})\right] + \eta'_\varepsilon.$$

We finally get:

$$\mathcal{Y}_t(\mu_N^\varepsilon) \leq E_t\left[\sum_{n\leq N} \mathbf{1}_{\zeta_N^\varepsilon = \mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}}\mathcal{Y}_t(\mu_n)\right] + \sum_{n\leq N} E_t\left[|\mathcal{Y}_{t+\varepsilon}(\mathcal{M}_{t+\varepsilon}^{\mu_n,\alpha_n}) - \mathcal{Y}_t(\mu_n)|\right] + \eta'_\varepsilon.$$

Letting ε tend to 0 in the above inequality, we obtain, by Step b:

$$\limsup_{\varepsilon\to 0}\mathcal{Y}_t(\mu_N^\varepsilon) \leq \sum_{n\leq N}\lambda_n^N\mathcal{Y}_t(\mu_n) \leq \sum_{n\leq N}\lambda_n^N Y_n = \hat{Y}_n, \tag{3.5.25}$$

where the last inequality follows by definition of the sequence $(\mu_n, Y_n)_n$ and (1.4.4).

The desired result (4.4.18) is obtained by combining (3.5.20) with Step a and (3.5.25). □

3.6 Dual representation in the case of concave constraints

We now provide a dual representation of the value function defined by (3.2.6), which takes the form of a stochastic control problem in Meyer form. The results of this section extend the ones given in [31], but involve technical additional proofs, due to the nonlinearity of the coefficient f.

For each (ω, t), let $\tilde{f}(\omega, t, \cdot, \cdot, \cdot)$ be be the concave conjugate of f with respect to (x, π), defined for each (p, q) in $\mathbf{R} \times \mathbf{R}^d$ as follows:

$$\tilde{f} : (\omega, t, p, q) \in \Omega \times [0, T] \times \mathbf{R} \times \mathbf{R}^d \to \inf_{(x,\pi) \in \mathbf{R} \times \mathbf{R}^d} \left(xp + \pi^\top q - f(\omega, t, x, \pi) \right).$$

For each (ω, t), we denote by $\tilde{g}(\omega, t, \cdot, \cdot, \cdot)$ the convexe conjugate of g with respect to (y, z), defined for each (u, v) in $\mathbf{R} \times \mathbf{R}^d$ as follows:

$$\tilde{g} : (\omega, t, u, v) \in \Omega \times [0, T] \times \mathbf{R} \times \mathbf{R}^d \to \sup_{(y,z) \in \mathbf{R} \times \mathbf{R}^d} \left(yu + z^\top v - g(\omega, t, y, z) \right).$$

We also introduce for each ω, the polar function of Φ with respect to m:

$$\tilde{\Phi} : (\omega, l) \in \Omega \times \mathbf{R} \to \sup_{m \in [0,1]} \left(ml - \Phi(\omega, m) \right).$$

In the sequel, we denote by $\mathcal{U}$ the set of predictable processes valued in D^1, respectively by $\mathcal{V}$ the set of predictable processes valued in D^2_t, where for each $(t, \omega) \in [0, T] \times \Omega$, $D^1_t(\omega)$ and $D^2_t(\omega)$ are defined:

$$D^1_t(\omega) := \{(p, q) : \; \tilde{f}(t, \omega, p, q) > -\infty\}; \;\; D^2_t(\omega) := \{(u, v) : \; \tilde{g}(t, \omega, u, v) < +\infty\}. \tag{3.6.1}$$

Remark 3.6.1. For each (t, ω), $D^1_t(\omega) \subset U$, where U is the closed subset of $\mathbf{R} \times \mathbf{R}^d$ of elements $\alpha = (\alpha_1, \alpha_2)$ such that $|\alpha_1| \leq C_g$ and $|\alpha^i_2| \leq C_g$, $\forall i = \overline{1, d}$. The same remark holds for the elements belonging to $D^2_t(\omega)$, with C_f instead of C_g.

To each $l > 0$, $\gamma = (\kappa, \vartheta) \in \mathcal{V}$ (resp. $\lambda = (\mu, \nu) \in \mathcal{U}$), we associate the processes $\mathcal{A}^{l,\gamma}$ (resp. $\mathcal{L}^\lambda$) defined by

$$\mathcal{A}^{l,\gamma}_t = l + \int_0^t \mathcal{A}^{l,\gamma}_s \kappa_s ds + \int_0^t \mathcal{A}^{l,\gamma}_s \vartheta_s dW_s, \; t \in [0, T];$$
$$\mathcal{L}^\lambda_t = 1 + \int_0^t \mathcal{L}^\lambda_s \mu_s ds + \int_0^t \mathcal{L}^\lambda_s \nu_s dW_s, \; t \in [0, T].$$

The dual formulation of $\mathcal{Y}_0$ is expressed in terms of

$$\mathcal{X}_0(l) := \inf_{(\lambda,\gamma) \in \mathcal{U} \times \mathcal{V}} X^{l,\lambda,\gamma}_0$$

where

$$X^{l,\lambda,\gamma}_0 := E\left[\int_0^T \mathcal{L}^\lambda_s \tilde{g}(s, \lambda_s) ds - \int_0^T \mathcal{A}^{l,\gamma}_s \tilde{f}(s, \gamma_s) ds + \mathcal{L}^\lambda_T \tilde{\Phi}\left(\frac{\mathcal{A}^{l,\gamma}_T}{\mathcal{L}^\lambda_T}\right) \right].$$

Proposition 3.6.2. $\mathcal{Y}_0(m) \geq \sup_{l > 0}(lm - \mathcal{X}_0(l))$, *for all* $m \in \left[\mathcal{E}^f_{0,T}[0], \mathcal{E}^f_{0,T}[1] \right]$.

Proof. Fix $\alpha \in \mathbf{A}_{0,m}$, $\lambda = (\nu, \mu) \in \mathcal{U}$, $l > 0$ and $\gamma = (\kappa, \vartheta) \in \mathcal{V}$. The definition of $\tilde{\Phi}$, together with Ito formula imply:

$$E[Y_T^{m,\alpha}\mathcal{L}_T^\lambda] \le Y_0^{m,\alpha} + E[\int_0^T \mathcal{L}_s^\lambda \tilde{g}(s, \lambda_s)ds] \tag{3.6.2}$$

and

$$\begin{aligned} E[Y_T^{m,\alpha}\mathcal{L}_T^\lambda] = E[\Phi(\mathcal{M}_T^{m,\alpha})\mathcal{L}_T^\lambda] &\ge E[\mathcal{A}_T^{l,\gamma}\mathcal{M}_T^{m,\alpha} - \mathcal{L}_T^\lambda\tilde{\Phi}(\frac{\mathcal{A}_T^{l,\gamma}}{\mathcal{L}_T^\lambda})] \\ &\ge E[lm + \int_0^T \mathcal{A}_s^{l,\gamma}\tilde{f}(s, \gamma_s)ds - \mathcal{L}_T^\lambda\tilde{\Phi}(\frac{\mathcal{A}_T^{l,\gamma}}{\mathcal{L}_T^\lambda})]. \end{aligned} \tag{3.6.3}$$

Note that since $Y^{m,\alpha}, \mathcal{L}^\lambda, \mathcal{M}^{m,\alpha}, \mathcal{A}^{l,\gamma} \in \mathbf{S}_2$, $Z^{m,\alpha}, \alpha \in \mathbf{H}_2$ and Remark 3.6.1 holds, by applying Burkholder-Davis-Gundy inequality, we obtain that the local martingales $\int_0^\cdot Y_s^{m,\alpha}\mathcal{L}_s^\lambda\nu_s^\top dW_s$, $\int_0^\cdot \mathcal{L}_s^\lambda Z_s^{m,\alpha,\top}dW_s$, $\int_0^\cdot \mathcal{M}_s^{m,\alpha}\mathcal{A}_s^{l,\gamma}\vartheta_s^\top dW_s$, $\int_0^\cdot \mathcal{A}_s^{l,\gamma}\alpha_s^\top dW_s$ are in fact martingales. Hence we can cancel their expectations. From the two above inequalities, we derive that:

$$Y_0^{m,\alpha} \ge lm - E\left[\int_0^T \mathcal{L}_s^\lambda\tilde{g}(s, \lambda_s)ds - \int_0^T \mathcal{A}_s^{l,\gamma}\tilde{f}(s, \gamma_s)ds + \mathcal{L}_T^\lambda\tilde{\Phi}(\frac{\mathcal{A}_T^{l,\gamma}}{\mathcal{L}_T^\lambda})\right].$$

By arbitrariness of $(\lambda, \gamma) \in \mathcal{U} \times \mathcal{V}$, we get:

$$Y_0^{m,\alpha} \ge lm - \mathcal{X}_0(l).$$

We then take the essential infimum on $\alpha \in \mathbf{A}_0^m$ and the supremum on $l > 0$. The result follows. $\square$

We now show that equality holds under some additional assumptions.

Assumption 3.6.3. *We make the following assumptions:*

(a) *For each $(t, \omega) \in \Omega \times [0, T]$, the maps $\tilde{\Phi}(\omega, \cdot)$, $\tilde{f}(t, \omega, \cdot)$ and $\tilde{g}(t, \omega, \cdot)$ are of class C_b^1. Also $D_t^1(\omega)$and $D_t^2(\omega)$ are closed.*

(b) $|\nabla\tilde{\Phi}(\omega, \cdot)| + \|\nabla\tilde{f}(\omega, t, \cdot)\|_{\mathbf{R}\times\mathbf{R}^d} + \|\nabla\tilde{g}(\omega, t, \cdot)\|_{\mathbf{R}\times\mathbf{R}^d} \le C_{\tilde{\Phi},\tilde{f},\tilde{g}}$, *for some $C_{\tilde{\Phi},\tilde{f},\tilde{g}} \in \boldsymbol{L}_2(\mathbf{R})$;*

(c) $\Phi(\omega, m) = \sup_{l>0}\Big(lm - \tilde{\Phi}(\omega, l)\Big)$, *for all $m \in [0, 1]$;*

(d) $f(\omega, t, x, \pi) = \min_{(u,v)\in D_t^2(\omega)}\Big(px + \pi^\top q - \tilde{f}(\omega, t, p, q)\Big)$, *for all $(x, \pi) \in \mathbf{R} \times \mathbf{R}^d$;*

(e) $g(\omega, t, y, z) = \max_{(u,v)\in D_t^1(\omega)}\big(yu + z^\top v - \tilde{g}(\omega, t, u, v)\big)$, *for all $(y, z) \in \mathbf{R} \times \mathbf{R}^d$.*

Proposition 3.6.4. *Assume that there exists $\hat{l} > 0$, $\hat{\lambda} \in \mathcal{U}$ and $\hat{\gamma} \in \mathcal{V}$ such that*

$$\sup(lm - \mathcal{X}_0(l)) = \hat{l}m - \mathcal{X}_0(\hat{l}) = \hat{l}m - X_0^{\hat{l},\hat{\lambda},\hat{\gamma}}. \tag{3.6.4}$$

Then there exists $\hat{\alpha} \in \boldsymbol{H}_2$ such that

$$\mathcal{Y}_0(m) = Y_0^{m,\hat{\alpha}} = \hat{l}m - \mathcal{X}_0(\hat{l}). \tag{3.6.5}$$

Also it satisfies

$$\begin{cases} f(\cdot, \mathcal{M}^{m,\hat{\alpha}}, \hat{\alpha}) = \hat{\kappa}\mathcal{M}^{m,\hat{\alpha}} + \hat{\vartheta}^\top \hat{\alpha} - \tilde{f}(\cdot, \hat{\gamma}); \quad \mathcal{M}_T^{m,\hat{\alpha}} = \nabla\tilde{\Phi}\left(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}\right); \\ g(\cdot, Y^{m,\hat{\alpha}}, Z^{m,\hat{\alpha}}) = \hat{\mu}Y^{m,\hat{\alpha}} + \hat{\nu}^\top Z^{m,\hat{\alpha}} - \tilde{g}(\cdot, \hat{\lambda}); \quad \Phi(\mathcal{M}_T^{m,\hat{\alpha}}) = \frac{\mathcal{M}_T^{m,\hat{\alpha}}\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}} - \tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}). \end{cases} \tag{3.6.6}$$

Proof. The proof is divided in two steps.

Step 1. We denote by $\left(\mathcal{E}^f_{\cdot,T}\left[\nabla\tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}})\right], \hat{\alpha}\right)$ the solution of the BSDE associated to the terminal condition $\nabla\tilde{\Phi}\left(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}\right)$ and driver f. We first need to show that $\mathcal{E}^f_{0,T}\left[\nabla\tilde{\Phi}\left(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}\right)\right] = m$.
By the optimality of $\hat{l}$, we get:

$$\begin{aligned} \hat{l}m - E\left[\mathcal{L}_T^{\hat{\lambda}}\tilde{\Phi}\left(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}\right) - \int_0^T \mathcal{A}_s^{\hat{l},\hat{\gamma}}\tilde{f}(s,\hat{\gamma}_s)ds\right] \\ \geq m(\hat{l}+\varepsilon) - E\left[\mathcal{L}_T^{\hat{\lambda}}\tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l}+\varepsilon,\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}) - \int_0^T \mathcal{A}_s^{\hat{l}+\varepsilon,\hat{\gamma}}\tilde{f}(s,\hat{\gamma}_s)ds\right], \end{aligned}$$

for all $\varepsilon > -\hat{l}$. Note that $\mathcal{A}^{l,\gamma} = l\mathcal{A}^{1,\gamma}$ for all $l \in \mathbf{R}$. Since by construction $\tilde{\Phi}$ is a.s. convex, we deduce that:

$$m\varepsilon \leq E\left[-\int_0^T \tilde{f}(s,\hat{\gamma}_s)\mathcal{A}_s^{1,\hat{\gamma}} + \mathcal{A}_T^{1,\hat{\gamma}}\nabla\tilde{\Phi}\left(\frac{\mathcal{A}_T^{\hat{l}+\varepsilon,\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}\right)\right]\varepsilon.$$

We take in the above inequality $\varepsilon = \frac{1}{n}$ and $\varepsilon = -\frac{1}{n}$. By letting n tend to ∞ and using (3.6.3) (a) and Lebesgue's Theorem, we finally get:

$$m = E\left[-\int_0^T \tilde{f}(s,\hat{\gamma}_s)\mathcal{A}_s^{1,\hat{\gamma}} + \mathcal{A}_T^{1,\hat{\gamma}}\nabla\tilde{\Phi}\left(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}\right)\right]. \tag{3.6.7}$$

We now introduce the processes $(\hat{M}, \hat{N}) \in \mathbf{S}_2 \times \mathbf{H}_2$, solution of the BSDE associated to the terminal condition $\nabla\tilde{\Phi}\left(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}\right)$ and driver

$$h(s,\omega,y,z) := -\tilde{f}(s,\hat{\kappa}_s(\omega),\hat{\vartheta}_s(\omega)) + y\hat{\kappa}_s(\omega) + z^\top\hat{\vartheta}_s(\omega). \tag{3.6.8}$$

Note that h is Lipschitz continuous with respect to (y,z), uniformly in (s,ω) (see Remark 3.6.1). Existence and uniqueness of the solution of the above BSDE is thus guaranteed.

We apply Itô formula to $\mathcal{A}^{1,\hat{\gamma}}\hat{M}$ and obtain:

$$\mathcal{A}_t^{1,\hat{\gamma}}\hat{M}_t = \mathcal{A}_T^{1,\hat{\gamma}}\nabla\tilde{\Phi}\left(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}\right) - \int_t^T \tilde{f}(s,\hat{\gamma}_s)\mathcal{A}_s^{1,\hat{\gamma}}ds - \int_t^T \mathcal{A}_s^{1,\hat{\gamma}}\tilde{N}_s dW_s, \tag{3.6.9}$$

where $\tilde{N}$ is defined by $\tilde{N} := \hat{N} + \hat{M}\hat{\vartheta}$. Clearly, $\tilde{N}$ belongs to $\mathbf{H}_2$ since $\hat{N} \in \mathbf{H}_2$, $\hat{M} \in \mathbf{S}_2$ and $\|\hat{\vartheta}\|_{\mathbf{R}^d} \leq C$, by Remark 3.6.1 .

Let us now fix $\gamma = (\kappa, \vartheta) \in \mathcal{V}$. Since $\mathcal{V}$ is convex, we get that for all $\varepsilon \in [0,1]$, $\gamma^\varepsilon := (1-\varepsilon)(\hat{\kappa}, \hat{\vartheta}) + \varepsilon(\kappa, \vartheta) \in \mathcal{V}$.
Using now the optimality condition $X_0^{\hat{l},\gamma^\varepsilon,\hat{\lambda}} \geq X_0^{\hat{l},\hat{\gamma},\hat{\lambda}}$, the fact that $\hat{l} > 0$, the Lagrange's and Lebesgue's Theorems, one can easily show that $\nabla \tilde{f}(\cdot, \hat{\gamma})$ satisfies:

$$
\begin{aligned}
0 \geq E\Bigg[\int_0^T &-\mathcal{A}_s^{1,\hat{\gamma}}\left(\hat{K}_s \tilde{f}(s,\hat{\gamma}_s) + \nabla_p \tilde{f}(s,\hat{\gamma}_s)\delta\kappa_s + \nabla_q \tilde{f}(s,\hat{\gamma}_s)^\top \delta\vartheta_s\right) ds \\
&+ \hat{K}_T \mathcal{A}_T^{1,\hat{\gamma}} \nabla \tilde{\Phi}\left(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}\right)\Bigg],
\end{aligned} \tag{3.6.10}
$$

where $(\delta\kappa, \delta\vartheta) := (\kappa - \hat{\kappa}, \vartheta - \hat{\vartheta})$ and $\hat{K} := \int_0^\cdot \left(\delta\kappa_s - \delta\vartheta_s \hat{\vartheta}_s\right) ds + \int_0^\cdot \delta\vartheta_s dW_s$.
By (3.6.9) we have $\mathcal{A}_T^{1,\hat{\gamma}} \nabla \tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}) = \mathcal{A}_T^{1,\hat{\gamma}} \hat{M}_T$. Hence inequality (3.6.10) can be re-written as follows:

$$
\begin{aligned}
0 \geq E\Bigg[\int_0^T &-\mathcal{A}_s^{1,\hat{\gamma}}\Big(\hat{K}_s \tilde{f}(s,\hat{\gamma}_s) + \nabla_p \tilde{f}(s,\hat{\gamma}_s)\delta\kappa_s \\
&+ \nabla_q \tilde{f}(s,\hat{\gamma}_s)^\top \delta\vartheta_s\Big) ds + \hat{K}_T \mathcal{A}_T^{1,\hat{\gamma}} \hat{M}_T\Bigg].
\end{aligned} \tag{3.6.11}
$$

The definition of $\hat{K}$ together with (3.6.11) and Itô formula implies:

$$
0 \leq E\left[\int_0^T \mathcal{A}_s^{1,\hat{\gamma}}\left((\nabla_p \tilde{f}(s,\hat{\gamma}_s) - \hat{M}_s)\delta\kappa_s + (\nabla_q \tilde{f}(s,\hat{\gamma}_s) - \hat{N}_s)^\top \delta\vartheta_s\right) ds\right]. \tag{3.6.12}
$$

We introduce the map $\Theta : [0,T] \times \Omega \times \mathbf{R} \times \mathbf{R}^d \mapsto \mathbf{R}$ defined as follows:

$$
\Theta : (\omega, t, u, v) \mapsto (\nabla_p \tilde{f}(\omega, t, \hat{\gamma}_t(\omega)) - \hat{M}_t(\omega))(u - \hat{\kappa}_t(\omega)) + (\nabla_q \tilde{f}(\omega, t, \hat{\gamma}_t(\omega)) - \hat{N}_t(\omega))^\top (v - \hat{\vartheta}_t(\omega)).
$$

By Remark 3.6.1, Assumption 3.6.3 (a) and Theorem 18.19, p.605 in [2], there exists a predictable $\bar{\gamma}$ belonging to $\mathcal{V}$ such that $\bar{\gamma}$ =argmin$\{\Theta(\cdot, u, v),\ (u,v) \in D^2\}$. For each $(t,\omega) \in [0,T] \times \Omega$, define the map F as follows:

$$
(p,q) \in D_t^2(\omega) \mapsto F(\omega, t, p, q) := \tilde{f}(\omega, t, p, q) - p\hat{M}_t(\omega) - q^\top \hat{N}_t(\omega). \tag{3.6.13}
$$

Note that we have:

$$
\Theta(t, \omega, u, v) = \nabla_p F(t, \omega, \hat{\gamma}_t(\omega))(u - \hat{\kappa}_t(\omega)) + \nabla_q F(t, \omega, \hat{\gamma}_t(\omega))^\top (v - \hat{\vartheta}_t(\omega)).
$$

Since (3.6.12) holds for all $\gamma \in \mathcal{V}$, we can take $\bar{\gamma}\mathbf{1}_{\Theta(\cdot,\bar{\gamma})>0} + \hat{\gamma}\mathbf{1}_{\Theta(\cdot,\bar{\gamma})\leq 0}$. Hence we derive that, for $dt \otimes dP$- a.e. $(\omega, t) \in \Omega \times [0,T]$, we have:

$$
\Theta(t, \omega, u, v) \leq 0,\ \forall\, (u,v) \in D_t^2(\omega).
$$

By a result of convex analysis, this implies that $\hat{\gamma}_t(\omega)$ maximizes $F(\omega, t, \cdot)$ for $dt \otimes dP$- a.e. $(\omega, t) \in \Omega \times [0,T]$ and thus by Assumption 3.6.3 (d) we get:

$$
\tilde{f}(\cdot, \hat{\gamma}) = \hat{\kappa}\hat{M} + \hat{\vartheta}^\top \hat{N} - f(\cdot, \hat{M}, \hat{N}). \tag{3.6.14}
$$

The above relation together with the definition of h (see (3.6.8)) leads to:

$$h(\cdot,\hat{M},\hat{N}) = f(\cdot,\hat{M},\hat{N}).$$

Recall that $(\hat{M},\hat{N})$ represents the solution of the BSDE of terminal condition $\nabla\tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}})$ and driver h. Hence by applying the comparison theorem for BSDEs, we get

$$(\hat{M},\hat{N}) = (\mathcal{E}^f_{\cdot,T}[\nabla\tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}})],\hat{\alpha}). \tag{3.6.15}$$

Now, we take the conditional expectation in (3.6.9) and we get:

$$\hat{M}_t := (\mathcal{A}_t^{1,\hat{\gamma}})^{-1}E[-\int_t^T \tilde{f}(s,\hat{\gamma}_s)\mathcal{A}_s^{1,\hat{\gamma}} + \mathcal{A}_T^{1,\hat{\gamma}}\nabla\tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\gamma}}})|\mathcal{F}_t]. \tag{3.6.16}$$

We have cancelled the expectation of $\int_\cdot^T \mathcal{A}_s^{1,\hat{\gamma}}\tilde{N}_s dW_s$, since by martingale inequalities, it is a martingale.

From (3.6.7), (3.6.15), (3.6.14) and (3.6.16), we derive that $\mathcal{E}^f_{0,T}\left[\nabla\tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}})\right] = m$ and moreover, that the first statement of (3.6.6) holds .

Since $\tilde{\Phi}$ is a.s.incresing, we derive that $\nabla\tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}) \geq 0$ a.s. Also, by construction, $\tilde{\Phi}$ is a.s. $\mathbf{1}$-Lipschitz, which implies that $\nabla\tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}) \in [-1,1]$ a.s. We thus conclude that $\nabla\tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}) \in [0,1]$ a.s. and $\mathcal{E}^f_{0,T}[\nabla\tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}})] = m$.

Step 2. First, recall that $(Y^{m,\hat{\alpha}}, Z^{m,\hat{\alpha}})$ represents the solution of the BSDE with terminal condition $\Phi(\mathcal{M}_T^{m,\hat{\alpha}})$ and driver g, where by *Step 1*, $\mathcal{M}_T^{m,\hat{\alpha}} = \nabla\tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}})$.

Now, Assumption 3.6.3 (c) yields

$$\Phi(\mathcal{M}_T^{m,\hat{\alpha}}) = \frac{\mathcal{M}_T^{m,\hat{\alpha}}\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}} - \tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}). \tag{3.6.17}$$

Using the optimality of $\hat{\lambda}$, i.e. for all $\varepsilon > 0$, $X_0^{\hat{l},\hat{\gamma},\lambda^\varepsilon} \geq X_0^{\hat{l},\hat{\gamma},\hat{\lambda}}$ and similar arguments as in *Step 1*, we get:

$$(Y^{m,\hat{\alpha}}, Z^{m,\hat{\alpha}}) = (\hat{Y},\hat{Z}), \tag{3.6.18}$$

where $(\hat{Y},\hat{Z})$ corresponds to the solution of the BSDE associated to the terminal condition $\Phi(\mathcal{M}_T^{m,\hat{\alpha}})$ and driver $-\tilde{g}(s,\hat{\mu}_s(\omega),\hat{\nu}_s(\omega)) + y\mu_s(\omega) + z^\top\hat{\nu}_s(\omega)$. Also by the same arguments given at Step 1, $\hat{Y}$ satisfies:

$$\hat{Y} = (\mathcal{L}^{\hat{\lambda}})^{-1}E_\cdot[-\int_\cdot^T \tilde{g}(s,\hat{\lambda}_s)\mathcal{L}_s^{\hat{\lambda}} + \mathcal{L}_T^{\hat{\lambda}}\Phi(\mathcal{M}_T^{m,\hat{\alpha}})]. \tag{3.6.19}$$

Since by (3.6.18) and (3.6.19) we have $\hat{Y}_0 = Y_0^{m,\hat{\alpha}}$ and $\mathcal{L}_0^{\hat{\lambda}} = 1$, we obtain:

$$\begin{aligned} Y_0^{m,\hat{\alpha}} &= E\left[\mathcal{L}_T^{\hat{\lambda}}\Phi(\mathcal{M}_T^{m,\hat{\alpha}}) - \int_0^T \mathcal{L}_s^{\hat{\lambda}}\tilde{g}(s,\hat{\lambda})ds\right] = E\left[\mathcal{M}_T^{m,\hat{\alpha}}\mathcal{A}_T^{\hat{l},\hat{\gamma}}\right] \\ &- E\left[\mathcal{L}_T^{\hat{\lambda}}\tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}) + \int_0^T \mathcal{L}_s^{\hat{\lambda}}\tilde{g}(s,\hat{\lambda})ds\right]. \end{aligned} \tag{3.6.20}$$

Now, we appeal to (3.6.7) and since by *Step 1*, $\mathcal{M}_T^{m,\hat{\alpha}} = \nabla\tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}})$, we get $E\left[\mathcal{M}_T^{m,\hat{\alpha}}\mathcal{A}_T^{\hat{l},\hat{\gamma}}\right] = \hat{l}\left(m + E\left[\int_0^T \hat{\mathcal{A}}_s^{1,\hat{\gamma}}\tilde{f}(s,\hat{\gamma})ds\right]\right) = m\hat{l} + E\left[\int_0^T \hat{\mathcal{A}}_s^{\hat{l},\hat{\gamma}}\tilde{f}(s,\hat{\gamma})ds\right]$. From the two above equalities, we finally obtain

$$Y_0^{m,\hat{\alpha}} = \hat{l}m - E\left[\mathcal{L}_T^{\hat{\lambda}}\tilde{\Phi}(\mathcal{M}_T^{m,\hat{\alpha},\hat{\beta}}) - \int_0^T \hat{\mathcal{A}}_s^{\hat{l},\hat{\gamma}}\tilde{f}(s,\hat{\gamma})ds + \int_0^T \mathcal{L}_s^{\hat{\lambda}}\tilde{g}(s,\hat{\lambda})ds\right].$$

The above equality together with Proposition 3.6.2 give the desired result. □

We now show that the existence of an optimal control in the primal problem implies the existence of an optimal control in the dual problem, under the following assumptions:

Assumption 3.6.5.

(a) *For each* (t,ω), *the maps* $\Phi(\omega)$, $f(\omega,t,\cdot)$ *and* $g(\omega,t,\cdot)$ *are* C_b^1 *on* $[0,1]$ *and* $\mathbf{R}\times\mathbf{R}^d$ *respectively;*

(b) $|\nabla\Phi(\omega,\cdot)| \le C_\Phi(\omega)$, *for some* $C_\Phi \in \boldsymbol{L}_2(\mathbf{R})$.

Proposition 3.6.6. *Let* $l > 0$ *be fixed and assume that there exists* $\hat{m} \in [\mathcal{E}_{0,T}^f[0], \mathcal{E}_{0,T}^f[1]]$ *and* $\hat{\alpha} \in \boldsymbol{A}_{0,\hat{m}}$ *such that*

$$\sup_{m\in[\mathcal{E}_{0,T}^f[0],\mathcal{E}_{0,T}^f[1]]} (ml - \mathcal{Y}_0(m)) = \hat{m}l - Y_0^{\hat{m},\hat{\alpha}}. \tag{3.6.21}$$

Then, there exists $(\hat{\lambda},\hat{\gamma}) \in \mathcal{U}\times\mathcal{V}$ *such that*

$$\mathcal{Y}_0(\hat{m}) = \hat{m}l - \mathcal{X}_0(l) = \hat{m}l - X_0^{l,\hat{\gamma},\hat{\lambda}}. \tag{3.6.22}$$

Proof. The proof is divided in three steps.

Step 1. Let $\mathcal{M}_\cdot$ be an arbitrary f-martingale valued in $[\mathcal{E}_\cdot^f[0], \mathcal{E}_\cdot^f[1]]$ and $\varepsilon \in [0,1]$. We denote by $\mathcal{M}^\varepsilon$ the process defined as $\mathcal{M}_\cdot^\varepsilon := \mathcal{E}_{\cdot,T}^f\left[\hat{\mathcal{M}}_T + \varepsilon(\mathcal{M}_T - \hat{\mathcal{M}}_T)\right]$, where $\hat{\mathcal{M}} := \mathcal{M}^{\hat{m},\hat{\alpha}}$. We set $m_\varepsilon := \mathcal{M}_0^\varepsilon$ and $(\delta\mathcal{M},\delta\alpha) := (\mathcal{M} - \hat{\mathcal{M}}, \alpha - \hat{\alpha})$.
We now consider the BSDE associated to $\delta\mathcal{M}_T$ and generator:

$$h_1(t,\omega,u,v) := \nabla_x f(t,\omega,\hat{\mathcal{M}}_t(\omega),\hat{\alpha}_t(\omega))u + \nabla_\pi f(t,\omega,\hat{\mathcal{M}}_t(\omega),\hat{\alpha}_t(\omega))^\top v.$$

Since $\delta\mathcal{M}_T$ belongs to $\mathbf{L}_2(\mathcal{F}_T)$ and since by Assumption 3.6.5 on the coefficient f, h is uniformly Lipschitz in (u,v) with respect to (t,ω) , we conclude that the above BSDE admits an unique solution. This unique solution will be denoted by $(\nabla M, \nabla\alpha)$.

Our aim is to show that $\varepsilon^{-1}(\delta\mathcal{M}^\varepsilon, \delta\alpha^\varepsilon)$ converges in $\mathbf{S}_2\times\mathbf{H}_2$ as $\varepsilon\to 0$ to $(\nabla M, \nabla\alpha)$.

First, observe that $\varepsilon^{-1}(\delta\mathcal{M}_s^\varepsilon, \delta\alpha_s^\varepsilon)$ solves the following equation:

$$\frac{\delta\mathcal{M}_t^\varepsilon}{\varepsilon} = \delta\mathcal{M}_T + \int_t^T \left(B_s^{\mathcal{M},\varepsilon} \frac{\delta\mathcal{M}_s^\varepsilon}{\varepsilon} + B_s^{\alpha,\varepsilon,\top} \frac{\delta\alpha_s^\varepsilon}{\varepsilon} \right) ds - \int_t^T \frac{\delta\alpha_s^\varepsilon}{\varepsilon}^\top dW_s, \tag{3.6.23}$$

where

$$B_s^{\mathcal{M},\varepsilon} := \int_0^1 \nabla_x f\left(s, \hat{\mathcal{M}}_s + r\delta\mathcal{M}_s^\varepsilon, \hat{\alpha}_s\right) dr; \; B_s^{\alpha,\varepsilon} := \int_0^1 \nabla_\pi f\left(s, \hat{\mathcal{M}}_s, \hat{\alpha}_s + r\delta\alpha_s^\varepsilon\right) dr.$$

We now introduce the processes $\Xi^\varepsilon := \varepsilon^{-1}\delta\mathcal{M}^\varepsilon - \nabla\mathcal{M}$ and $\Pi^\varepsilon := \varepsilon^{-1}\delta\alpha^\varepsilon - \nabla\alpha$. We can remark that $(\Xi^\varepsilon, \Pi^\varepsilon)$ solves the BSDE associated to terminal condition 0 and driver:

$$h_2(t,\omega,u,v) := B_t^{\mathcal{M},\varepsilon}(\omega)u + B_t^{\alpha,\varepsilon}(\omega)^\top v + D_t^\varepsilon(\omega),$$

where $D_t^\varepsilon := \nabla\mathcal{M}_t\left(B_t^{\mathcal{M},\varepsilon} - \nabla_x f(t, \hat{\mathcal{M}}_t, \hat{\alpha}_t)\right) + \nabla\alpha_t^\top\left(B_t^{\alpha,\varepsilon} - \nabla_\pi f(t, \hat{\mathcal{M}}_t, \hat{\alpha}_t)\right)$.
We apply the stability result with BSDE(ξ, h_2) and BSDE$(\xi, 0)$, where $\xi = 0$. We thus get:

$$\|\Xi^\varepsilon\|_{\mathbf{S}_2} + \|\Pi^\varepsilon\|_{\mathbf{H}_2} \leq C\|D^\varepsilon\|_{\mathbf{H}_2}. \tag{3.6.24}$$

In order to show the convergence of $\|D^\varepsilon\|_{\mathbf{H}_2}$ to 0 when $\varepsilon \to 0$, we prove that $(\mathcal{M}^\varepsilon, \alpha^\varepsilon)$ converges to $(\mathcal{M}, \alpha)$ in $\mathbf{S}_2 \times \mathbf{H}_2$. In order to do this, we apply again the stability result for BSDEs and obtain:

$$\|\mathcal{M}^\varepsilon - \mathcal{M}\|_{\mathbf{S}_2}^2 + \|\alpha^\varepsilon - \alpha\|_{\mathbf{H}_2}^2 \leq C(\|\mathcal{M}_T^\varepsilon - \mathcal{M}_T\|_{\mathbf{L}_2}^2) \to_{\varepsilon\to 0} 0. \tag{3.6.25}$$

By (3.6.25), Assumption 3.6.5 and the Lebesgue's Theorem, we get that $\|D^\varepsilon\|_{\mathbf{H}_2} \to 0$ when $\varepsilon \to 0$. Finally, by (3.6.24), we derive that $\varepsilon^{-1}(\delta\mathcal{M}^\varepsilon, \delta\alpha^\varepsilon)$ converges in $\mathbf{S}_2 \times \mathbf{H}_2$ to $(\nabla M, \nabla\alpha)$ as $\varepsilon \to 0$.

Step 2. We denote by $(Y^\varepsilon, Z^\varepsilon)$ the solution of the BSDE$(g, \Phi(\mathcal{M}_T^\varepsilon))$ and we set $(\hat{Y}, \hat{Z}) := (Y^{m,\hat{\alpha}}, Z^{m,\hat{\alpha}})$. Using the same arguments as in *Step 2*, one can show that $(\frac{\delta Y^\varepsilon}{\varepsilon}, \frac{\delta Z^\varepsilon}{\varepsilon}) := (\frac{Y^\varepsilon - \hat{Y}}{\varepsilon}, \frac{Z^\varepsilon - \hat{Z}}{\varepsilon})$ converges in $\mathbf{S}_2 \times \mathbf{H}_2$ to the unique solution $(\nabla Y, \nabla Z)$ of the following BSDE:

$$\begin{aligned}\nabla Y_t = \nabla\Phi(\hat{\mathcal{M}}_T)\delta\mathcal{M}_T &+ \int_t^T \nabla_y g(s, \hat{Y}_s, \hat{Z}_s)\nabla Y_s ds \\ &+ \int_t^T \nabla_z g(s, \hat{Y}_s, \hat{Z}_s)^\top \nabla Z_s ds - \int_t^T \nabla Z_s^\top dW_s.\end{aligned} \tag{3.6.26}$$

Step 3. Since $(\hat{m}, \hat{\alpha})$ is optimal, we have $Y_0^\varepsilon - m_\varepsilon - \hat{Y}_0 + \hat{m}l \geq 0$, for any $\varepsilon > 0$. Dividing now by $\varepsilon > 0$ and sending $\varepsilon \to 0$, we get

$$\begin{aligned}0 \leq \nabla\Phi(\hat{\mathcal{M}}_T)\delta\mathcal{M}_T &+ \int_0^T \nabla g(s, \hat{Y}_s, \hat{Z}_s)^\top(\nabla Y_s, \nabla Z_s)ds - \int_0^T \nabla Z_s^\top dW_s \\ - l&\left(\delta\mathcal{M}_T + \int_0^T \nabla f(s, \hat{\mathcal{M}}_s, \hat{\alpha}_s)^\top(\nabla\mathcal{M}_s, \nabla\alpha_s)ds - \int_0^T \nabla\alpha_s^\top dW_s\right) = \nabla Y_0 - l\nabla\mathcal{M}_0.\end{aligned} \tag{3.6.27}$$

We set $\hat{\gamma}_t := \nabla f(s, \hat{\mathcal{M}}_t, \hat{\alpha}_t)$ and $\hat{\lambda}_t := \nabla g(s, \hat{Y}_t, \hat{Z}_t)$, which belong to $\mathcal{V}$ and, respectively, $\mathcal{U}$. Since $\hat{\gamma}_t$ (resp. $\hat{\lambda}_t$) belongs to the subdifferential of f at $(\mathcal{M}_t, \hat{\alpha}_t)$ (resp. the subdifferential of g at $(\hat{Y}_t, \hat{Z}_t)$) we have (see [16]):

$$f(\cdot, \hat{\mathcal{M}}, \hat{\alpha}) = \hat{\kappa}\hat{\mathcal{M}} + \hat{\vartheta}^\top\hat{\alpha} - \tilde{f}(\cdot, \hat{\gamma}). \tag{3.6.28}$$

and

$$g(\cdot, \hat{Y}, \hat{Z}) = \hat{\mu}\hat{Y} + \hat{\nu}^\top \hat{Z} - \tilde{g}(\cdot, \hat{\lambda}). \tag{3.6.29}$$

Now, by applying Ito's formula, we obtain that $\mathcal{A}^{l,\hat{\gamma}}\nabla\mathcal{M}$ and $\mathcal{L}^{\hat{\lambda}}\nabla Y$ are martingales. As $\mathcal{L}_0^{\hat{\lambda}} = 1$ and (3.6.27) holds, we thus obtain:

$$\begin{aligned}\hat{\mathcal{L}}_0\nabla Y_0 - l\nabla\mathcal{M}_0 &= E\left[\mathcal{L}_T^{\hat{\lambda}}\nabla Y_T - \mathcal{A}_T^{\hat{l},\hat{\gamma}}\nabla\mathcal{M}_T\right] \\ &= E\left[\mathcal{L}_T^{\hat{\lambda}}\delta\mathcal{M}_T\left(\nabla\Phi(\hat{\mathcal{M}}_T) - \frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}\right)\right] \geq 0. \end{aligned} \tag{3.6.30}$$

Since $\mathcal{M}_T$ can be arbitrary choses with values in $[0,1]$, we obtain that $\hat{\mathcal{M}}_T(\omega)$ minimizes the map $m \in [0,1] \mapsto \Phi(\omega, m) - m\frac{\mathcal{A}_T^{l,\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}}(\omega)$. Thus, we obtain: $\hat{\mathcal{M}}_T\mathcal{A}_T^{\hat{l},\hat{\gamma}} - \mathcal{L}_T^{\hat{\lambda}}\Phi(\hat{\mathcal{M}}_T) = \mathcal{L}_T^{\hat{\lambda}}\tilde{\Phi}(\frac{\mathcal{A}_T^{\hat{l},\hat{\gamma}}}{\mathcal{L}_T^{\hat{\lambda}}})$. This inequality together with (6.5.2), (3.6.29) and Ito's formula allow to conclude that $l\hat{m} - \hat{Y}_0 = X_0^{l,\hat{\lambda},\hat{\gamma}}$. The conclusion follows by Proposition 3.6.2. □

3.7 Appendix

Proof of Proposition 3.2.4. The proof is standard. We provide it for completeness. Let (Y, Z) be a supersolution of $BSDE(g, f, \Psi, \mu, \tau)$. Now, the BSDE representation of $\Psi(Y_T)$ implies that it exists $\bar{\alpha} \in \mathbf{A}_{\tau,\rho}$ such that $\Psi(Y_T) = \mathcal{M}_T^{\tau,\rho,\bar{\alpha}}$, where $\rho := \mathcal{E}^f_{\tau,T}[\Psi(Y_T)]$. Since condition (3.2.4) is satisfied, we have $\rho \geq \mu$ a.s. We define the following stopping time

$$\sigma^{\bar{\alpha}} := \inf\{\tau \leq s \leq T : \mathcal{M}_s^{\tau,\mu,\bar{\alpha}} = \mathcal{E}^f_{s,T}[0]\} \wedge T,$$

with the convention $\inf \emptyset = -\infty$. Recall that (Y^0, Z^0) represents the solution of the BSDE associated to driver f and terminal condition 0. We define the control $\tilde{\alpha}$ as follows:

$$\tilde{\alpha}_s := \bar{\alpha}_s \mathbf{1}_{\{s \leq \sigma^{\bar{\alpha}}\}} + Z_s^0 \mathbf{1}_{\{s > \sigma^{\bar{\alpha}}\}}. \tag{3.7.1}$$

Note that $\tilde{\alpha}$ belongs to $\mathbf{A}_{\tau,\mu}$. The control is constructed in such a way that $\mathcal{M}_\cdot^{\tau,\mu,\tilde{\alpha}}$ belongs to $[\mathcal{E}^f_{\cdot,T}[0], \mathcal{E}^f_{\cdot,T}[1]]$. We have not considered the hitting time of the process $\mathcal{E}^f_{\cdot,T}[1]$, since clearly $\mathcal{M}_\cdot^{\tau,\mu,\bar{\alpha}} \leq \mathcal{M}_\cdot^{\tau,\rho,\bar{\alpha}}$. We can easily remark that $\mathcal{M}_T^{\tau,\rho,\tilde{\alpha}} \geq \mathcal{M}_T^{\tau,\mu,\alpha}$ a.s. The monotonocity of Φ and the identity $\Psi(Y_T) = \mathcal{M}_T^{\tau,\rho,\tilde{\alpha}}$ imply that

$$Y_T \geq (\Phi \circ \Psi)(Y_T) \geq \Phi(\mathcal{M}_T^{\tau,\mu,\alpha}). \tag{3.7.2}$$

Hence, by the comparison theorem for BSDEs, we obtain that $Y_t \geq \mathcal{E}^g_{t,T}[\Phi(\mathcal{M}_T^{\tau,\mu,\alpha})]$ for $t \in [0,T]$. Conversely, let $\alpha \in \mathbf{A}_{\tau,\mu}$ be such that $Y_t \geq \mathcal{E}^g_{t,T}[\Phi(\mathcal{M}_T^{\tau,\mu,\alpha})]$ for $t \in [0,T]$ and suppose that (Y,Z) satifies (3.2.3). We thus get

$$\Psi(Y_T) \geq (\Psi \circ \Phi)(\mathcal{M}_T^{\tau,\mu,\alpha}) \geq \mathcal{M}_T^{\tau,\mu,\alpha}.$$

Taking the f-conditional expectation on both sides, the result follows.

Lemma 3.7.1. *Fix $\theta, \nu \in \mathcal{T}$, with $\theta \geq \tau, \mu \in \boldsymbol{D}_\tau$ and $\alpha \in \boldsymbol{A}_{\tau,\mu}$. Then there exists a sequence $(\alpha'_n) \subset \boldsymbol{A}^{\theta,\alpha}_{\tau,\mu} := \{\alpha' \in \boldsymbol{A}_{\tau,\mu}, \alpha' \boldsymbol{1}_{[0,\theta)} = \alpha \boldsymbol{1}_{[0,\theta)}\}$ such that* $\lim_{n\to\infty} \downarrow \mathcal{E}^g_{\theta,T}[\Phi(\mathcal{M}^{\tau,\mu,\alpha'_n}_T)] = \mathcal{Y}^\alpha_\theta(\mathcal{M}^{\tau,\mu,\alpha}_\theta)$ *a.s.*

Proof. In order to obtain the desired result, we only have to prove that

$$\{J(\alpha') := \mathcal{E}^g_{\theta,T}[\Phi(\mathcal{M}^{\tau,\mu,\alpha'}_T)], \;\; \alpha' \in \mathbf{A}^{\theta,\alpha}_{\tau,\mu}\}$$

is directed downward. Set $A := \{J(\alpha'_1) \leq J(\alpha'_2)\} \in \mathcal{F}_\theta$ and fix $\alpha'_1, \alpha'_2 \in \mathbf{A}^{\theta,\alpha}_{\tau,\mu}$. We denote $\tilde{\alpha}' := \alpha \mathbf{1}_{[0,\theta)} + \mathbf{1}_{[\theta,T]}(\alpha'_1 \mathbf{1}_A + \alpha'_2 \mathbf{1}_{A^c})$. Note that $\tilde{\alpha}' \in \mathbf{A}^{\theta,\alpha}_{\tau,\mu}$. We get: $J(\tilde{\alpha}') = \mathcal{E}_{\theta,T}[\Phi(\mathcal{M}^{\tau,\mu,\alpha'_1}_\theta)\mathbf{1}_A + \Phi(\mathcal{M}^{\tau,\mu,\alpha'_2}_\theta)\mathbf{1}_{A^c}] = \min\{J(\alpha'_1), J(\alpha'_2)\}$. □

Theorem 3.7.2. *Fix $t \in [0,T]$. The map $\mathcal{Y}_t : \mu \to \mathcal{Y}_t(\mu)$; $\boldsymbol{D}_t \mapsto \boldsymbol{L}_2$; is non-decreasing, i.e. for all $\mu_1, \mu_2 \in \boldsymbol{D}_t$, we have $\mathcal{Y}_t(\mu_1) \leq \mathcal{Y}_t(\mu_2)$ on $\{\mu_1 \leq \mu_2\}$ and $\mathcal{Y}_t(\mu_1) \geq \mathcal{Y}_t(\mu_2)$ on $\{\mu_1 \geq \mu_2\}$.*

Proof. The proof is divided in two steps.

Step 1. We set $\tilde{\mu}_1 := \mu_1 \wedge \mu_2$ and $\tilde{\mu}_2 := \mu_1 \vee \mu_2$. Remark that $\tilde{\mu}_1$ and $\tilde{\mu}_2$ belong to $\mathbf{D}_t$.

By Lemma 3.7.1, we know that it exists $\alpha^n \in \mathbf{A}_{t,\tilde{\mu}_2}$ s.t. $\mathcal{E}^g_{t,T}[\Phi(\mathcal{M}^{\tilde{\mu}_2,\alpha^n}_T)] \to \mathcal{Y}_t(\tilde{\mu}_2)$ a.s.

Fix $n \in \mathbb{N}$. We define $\tilde{\alpha}^n \in \mathcal{A}_{t,\tilde{\mu}_1}$ as follows:

$$\tilde{\alpha}^n_s := \alpha^n_s \mathbf{1}_{s \leq \tau} + Z^0_s \mathbf{1}_{s > \tau},$$

where $\tau := \inf\{t \leq s \leq T : \mathcal{M}^{\tilde{\mu}_1,\alpha^n}_s = \mathcal{E}^f_{s,T}[0]\} \wedge T$, with the convention $\inf \emptyset = +\infty$. Recall that Z^0 is the associated control to the the BSDE with terminal condition 0 and driver f.

By construction of $\tilde{\alpha}^n$, we have $\mathcal{M}^{\tilde{\mu}_1,\alpha^n}_T \in [0,1]$ a.s. Now, by using the fact that Φ in nondecreasing and the comparison theorem for BSDEs, we obtain:

$$\mathcal{E}^g_{t,T}[\Phi(\mathcal{M}^{\tilde{\mu}_1,\tilde{\alpha}^n}_T)] \leq \mathcal{E}^g_{t,T}[\Phi(\mathcal{M}^{\tilde{\mu}_2,\alpha^n}_T)] \text{ a.s.}$$

which implies

$$\mathcal{Y}_t(\tilde{\mu}_1) \leq \mathcal{E}^g_{t,T}[\Phi(\mathcal{M}^{\tilde{\mu}_2,\alpha^n}_T)] \text{ a.s.} \tag{3.7.3}$$

By letting $n \to \infty$ in the above relation, we obtain $\mathcal{Y}_t(\tilde{\mu}_1) \leq \mathcal{Y}_t(\tilde{\mu}_2)$ a.s.

Step 2. We define $A := \{\mu_1 \leq \mu_2\} \in \mathcal{F}_t$. Let us show that $\mathcal{Y}_t(\tilde{\mu}_1) = \mathcal{Y}_t(\mu_1)\mathbf{1}_A + \mathcal{Y}_t(\mu_2)\mathbf{1}_{A^c}$. For all $\alpha_i \in \mathbf{A}_{t,\mu_i}$, $i = 1, 2$, we set $\tilde{\alpha} := \mathbf{1}_{[t,T]}\,(\alpha_1 \mathbf{1}_A + \alpha_2 \mathbf{1}_{A^c}) \in \mathbf{A}_{t,\tilde{\mu}_1}$. Bt the zero-one law for f- conditional expectations, we get $\mathcal{E}^g_{t,T}[\Phi(\mathcal{M}^{\tilde{\mu}_1,\tilde{\alpha}}_T)] = \mathcal{E}^g_{t,T}[\Phi(\mathcal{M}^{\mu_1,\alpha_1}_T)]\mathbf{1}_A + \mathcal{E}^g_{t,T}[\Phi(\mathcal{M}^{\mu_2,\alpha_2}_T)]\mathbf{1}_{A^c}$ and by arbitrariness of α_i, $i = 1, 2$, we derive that $\mathcal{Y}_t(\tilde{\mu}_1) \leq \mathcal{Y}_t(\mu_1)\mathbf{1}_A + \mathcal{Y}_t(\mu_2)\mathbf{1}_{A^c}$. In order to show that $\mathcal{Y}_t(\tilde{\mu}_1) \geq \mathcal{Y}_t(\mu_1)\mathbf{1}_A + \mathcal{Y}_t(\mu_2)\mathbf{1}_{A^c}$, we use the previous equality with $\alpha_1 := \tilde{\alpha}\mathbf{1}_A + \tilde{\alpha}_1 \mathbf{1}_{A^c}$ and $\alpha_2 := \tilde{\alpha}_2 \mathbf{1}_A + \tilde{\alpha}\mathbf{1}_{A^c}$, for all $\tilde{\alpha} \in \mathbf{A}_{t,\tilde{\mu}_1}$, $\tilde{\alpha}_1 \in \mathbf{A}_{t,\mu_1}$ and $\tilde{\alpha}_2 \in \mathbf{A}_{t,\mu_2}$. Similarly, one can prove that $\mathcal{Y}_t(\tilde{\mu}_2) = \mathcal{Y}_t(\mu_2)\mathbf{1}_A + \mathcal{Y}_t(\mu_1)\mathbf{1}_{A^c}$.

From Step 1 and Step 2, the result follows. □

Using the same arguments as in Step 2 of the above proof, one can easily show:

Lemma 3.7.3. *Fix $t \in [0,T]$. We have $\mathcal{Y}_t(\mu_1 \boldsymbol{1}_A + \mu_2 \boldsymbol{1}_{A^c}) = \mathcal{Y}_t(\mu_1)\boldsymbol{1}_A + \mathcal{Y}_t(\mu_2)\boldsymbol{1}_{A^c}$, for all $A \in \mathcal{F}_t$, $\mu_1, \mu_2 \in \boldsymbol{D}_t$.*

We now recall the following result, which can be found in [31].

Proposition 3.7.4. *Let the Assumption 3.2.1 (with g instead f) holds. Then:*

(i) *There exist $\chi_g \in \mathbf{L}_2$ and $C > 0$ which only depends on C_g and T such that:*

$$ess \sup_{\xi \in \boldsymbol{L}_0([0,1])} |\mathcal{E}^g_{t,T}[\xi] \leq C(1 + E_t[|\chi_g|^2])|^{\frac{1}{2}}), \ \ 0 \leq t \leq T.$$

(ii) *For some $\xi \in \boldsymbol{L}_2$ and $t \in [0,T]$, consider a family $(\xi^\varepsilon)_{\varepsilon \geq 0} \subset \boldsymbol{L}_0(\mathbf{R}^d)$ satisfying $|\xi^\varepsilon| \leq \xi$ and $\xi^\varepsilon \in \boldsymbol{L}_0(\mathcal{F}_{(t+\varepsilon)\wedge T})$, for any $\varepsilon > 0$. Then, there exists a family $(\eta_\varepsilon)_{\varepsilon > 0} \subset \boldsymbol{L}_0(\mathbf{R})$ which converges to 0 $\mathbb{P}$ - a.s. as $\varepsilon \to 0$ such that:*

$$\left|\mathcal{E}^g_{t,t+\varepsilon}[\xi^\varepsilon] - E_t[\xi^\varepsilon]\right| \leq \eta_\varepsilon, \ \ \forall \varepsilon \in [0, T-t].$$

(iii) *Let $(\xi^\varepsilon)_{\varepsilon > 0}$ and $t \in [0,T]$ be as in (ii). Then, there exists a family $(\eta_\varepsilon)_{\varepsilon > 0} \subset \boldsymbol{L}_0(\mathbf{R})$ which converges to 0 a.s. as $\varepsilon \to 0$ such that*

$$\left|\mathcal{E}^g_{t-\varepsilon,t}[\xi^\varepsilon] - E_t[\xi^\varepsilon]\right| \leq \eta_\varepsilon, \ \ \forall \varepsilon \in [0, t].$$

Chapter 4

Optimal stopping for dynamic risk measures with jumps and obstacle problems

Abstract. We study the optimal stopping problem for a monotonous dynamic risk measure induced by a BSDE with jumps in the Markovian case. We show that the value function is a viscosity solution of an obstacle problem for a partial integro-differential variational inequality, and we provide an uniqueness result for this obstacle problem.

4.1 Introduction

In the last years, there has been several studies on dynamic risk measures and their links with nonlinear backward stochastic differential equations (BSDEs). We recall that nonlinear BSDEs have been introduced in [125] in a Brownian framework, in order to provide a probabilistic representation of semilinear parabolic partial-differential equations. BSDEs with jumps and their links with partial integro-differential equations are studied in [9]. A comparison theorem is established in [140] and generalized in [137], where properties of dynamic risk measures induced by BSDEs with jumps are also provided. An optimal stopping problem for such risk measures is addressed in [138], and the value function is characterized as the solution of a reflected BSDE with jumps and RCLL obstacle process.

In the present chapter, we focus on the optimal stopping problem for dynamic risk measures induced by BSDEs with jumps in a Markovian framework. In this case the driver of the BSDE depends on a given state process X, which can represent, for example, an index or a stock price. This process will be assumed to be driven by a Brownian motion and a Poisson random measure.

Our main contribution consists in establishing the link between the value function of our optimal stopping problem and parabolic partial integro-differential variational inequalities (PIDVIs). We prove that the minimal risk measure, which corresponds to the solution of a reflected BSDE with jumps, is a viscosity solution of a PIDVI. This provides an existence result for the obstacle problem under relatively weak assumptions. In the Brownian case, this result was obtained in [71] by using a penalization method via non-reflected BSDEs. Note that this method could also be adapted to our case with jumps, but would involve heavy computations in order to prove the convergence of the solutions of the penalized BSDEs to the solution of the reflected BSDE. It would also require some convergence results of the viscosity solutions theory in the integro-differential case. We provide here instead a direct and shorter proof.

Furthermore, under some additional assumptions, we prove a comparison theorem in the class of bounded continuous functions, relying on a non-local version of Jensen-Ishii Lemma (see [10]), from which the uniqueness of the viscosity solution follows. We point out that our problem is not covered by the study in [10], since we are dealing with nonlinear BSDEs, and this leads to a more complex integro-differential operator in the associated PDE.

The chapter is organized as follows: In Section 4.2 we give the formulation of our optimal stopping problem. In Section 4.3, we prove that the value function is a solution of an obstacle problem for a PIDVI in the viscosity sense. In Section 4.4, we establish an uniqueness result. In the Appendix, we prove some estimates, from which we derive that the value function is continuous and has polynomial growth and provide some complementary results.

4.2 Optimal stopping problem for dynamic risk measures with jumps in the Markovian case

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let W be a one-dimensional Brownian motion and $N(dt, du)$ be a Poisson random measure with compensator $\nu(du)dt$ such that ν is a σ-finite measure on $I\!R^*$ equipped with its Borel field $\mathcal{B}(\mathbf{R}^*)$, and satisfies $\int_{\mathbb{R}^*}(1 \wedge e^2)\nu(de) < \infty$. Let $\tilde{N}(dt, du)$ be its compensated process. Let $I\!F = \{\mathcal{F}_t, t \geq 0\}$ be the natural filtration associated with W and N.

We consider a state process X which may be interpreted as an index, an interest rate process, an economic factor, an indicator of the market or the value of a portfolio, which has an influence on the risk measure and the position. For each initial time $t \in [0, T]$ and each condition $x \in \mathbb{R}$, let $X^{t,x}$ be the solution of the following stochastic differential equation (SDE):

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r + \int_t^s \int_{\mathbb{R}^*} \beta(X_{r^-}^{t,x}, e)\tilde{N}(dr, de), \tag{4.2.1}$$

where $b, \sigma : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous, and $\beta : \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}$ is a measurable function such that for some non negative real C, and for all $e \in \mathbb{R}$

$$\begin{aligned} &|\beta(x, e)| \leq C(1 \wedge |e|), \quad x \in \mathbb{R} \\ &|\beta(x, e) - \beta(x', e)| \leq C|x - x'|(1 \wedge |e|), \quad x, x' \in \mathbb{R}. \end{aligned}$$

We introduce a dynamic risk measure ρ induced by a BSDE with jumps. For this, we consider two functions γ and f satisfying the following assumption:

Assumption $\mathbf{H}_1$

- $\gamma : \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^*)$-measurable,
  $|\gamma(x, e) - \gamma(x', e)| < C|x - x'|(1 \wedge |e|), x, x' \in \mathbb{R}, e \in \mathbb{R}^*$
  $-1 \leq \gamma(x, e) \leq C(1 \wedge |e|)$, $e \in \mathbb{R}^*$

- $f : [0,T] \times \mathbb{R}^3 \times L^2_\nu \to \mathbb{R}$ is continuous in t uniformly with respect to x, y, z, k, and continuous in x uniformly with respect to t, y, z, k.

  (i) $|f(t,x,0,0,0)| \leq C(1+x^p),\ \forall x \in \mathbb{R}$

  (ii) $|f(t,x,y,z,k) - f(t,x',y',z',k')| \leq C(|y-y'| + |z-z'| + \|k-k'\|_{L^2_\nu}),\ \forall\ t \in [0,T]$, $y, y', z, z' \in \mathbb{R},\ k, k' \in L^2_\nu$

  (iii) $f(t,x,y,z,k_1) - f(t,x,y,z,k_2) \geq < \gamma(x,\cdot), k_1 - k_2 >_\nu, \forall t, x, y, z, k_1, k_2$.

Here, L^2_ν denotes the set of Borelian functions $\ell : I\!R^* \to I\!R$ such that $\|\ell\|^2_\nu := \int_{I\!R^*} |\ell(u)|^2 \nu(du) < +\infty$. It is a Hilbert space equipped with the scalar product $\langle \delta, \ell \rangle_\nu := \int_{\mathbb{R}^*} \delta(e)\ell(e)\nu(de)$ for all $\delta, \ell \in L^2_\nu \times L^2_\nu$.

We also introduce the set $\mathbb{H}^2$ (resp. $\mathbb{H}^2_\nu$) of predictable processes (π_t) (resp. $(l_t(\cdot))$) such that $\mathbb{E} \int_0^T \pi_s^2 ds < \infty$ (resp. $\mathbb{E} \int_0^T \|l_s\|^2_{L^2_\nu} ds < \infty$); the set $\mathcal{S}^2$ of real-valued RCLL adapted processes (φ_s) with $\mathbb{E}[\sup_s \varphi_s^2] < \infty$, and the set $L^2(\mathcal{F}_T)$ of $\mathcal{F}_T$-measurable and square-integrable random variables.

Let (t,x) be a fixed intial condition. For each maturity S in $[t,T]$ and each position ζ in $L^2(\mathcal{F}_S)$, the associated risk measure at time $s \in [t,S]$ is defined by

$$\rho_s^{t,x}(\zeta, S) := -\mathcal{E}^{t,x}_{s,S}(\zeta),\ t \leq s \leq S, \tag{4.2.2}$$

where $\mathcal{E}^{t,x}_{\cdot,S}(\zeta)$ denotes the f-conditional expectation, starting at (t,x), defined as the solution in $\mathcal{S}^2$ of the BSDE with Lipschitz driver $f(s, X_s^{t,x}, y, z, k)$, terminal condition ζ and terminal time S, that is the solution $(\mathcal{E}^{t,x}_s)$ of

$$-d\mathcal{E}_s = f(s, X_s^{t,x}, \mathcal{E}_s, \pi_s, l_s(\cdot))ds - \pi_s dW_s - \int_{\mathbb{R}^*} l_s(u)\tilde{N}(dt,du)\,;\ \mathcal{E}_S = \zeta, \tag{4.2.3}$$

where (π_s), (l_s) are the associated processes, which belong to $\mathbb{H}^2$ and $\mathbb{H}^2_\nu$ respectively.
The functional $\rho : (\zeta, S) \to \rho_\cdot(\zeta, S)$ defines then a dynamic risk measure induced by the BSDE with driver f (see [137]). Assumption $\mathbf{H}_1$ implies that the driver $f(s, X_s^{t,x}, y, z, k)$ satisfies Assumption 3.1 in [138], which ensures the monotonocity property of ρ with respect to ζ. More precisely, for each maturity S and for each positions $\zeta_1,\ \zeta_2 \in L^2(\mathcal{F}_S)$, with $\zeta_1 \leq \zeta_2$ a.s., we have $\rho_s^{t,x}(\zeta_1, S) \geq \rho_s^{t,x}(\zeta_2, S)$ a.s.

We now formulate our optimal stopping problem for dynamic risk measures. For each $(t,x) \in [0,T] \times \mathbb{R}$, we consider a dynamic financial position given by the process $(\xi_s^{t,x}, t \leq s \leq T)$, defined via the state process $(X_s^{t,x})$ and two functions g and h such that

- $g \in \mathcal{C}(\mathbb{R})$ with at most polynomial growth at infinity,

- $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous in t, continuous in x uniformly with respect to t, and there exist $p \in \mathbb{N}$ and a real constant C, such that

$$|h(t,x)| \leq C(1+|x|^p), \forall t \in [0,T], x \in \mathbb{R}, \tag{4.2.4}$$

- $h(T,x) \leq g(x),\ \ \forall x \in \mathbb{R}$.

For each initial condition $(t,x) \in [0,T] \times \mathbb{R}$, the dynamic position is then defined by:

$$\begin{cases} \xi_s^{t,x} := h(s, X_s^{t,x}), \quad s < T \\ \xi_T^{t,x} := g(X_T^{t,x}). \end{cases}$$

Let $t \in [0,T]$ be the initial time and let $x \in \mathbb{R}$ be the initial condition. The minimal risk measure at time t is given by:

$$\operatorname*{ess\,inf}_{\tau \in \mathcal{T}_t} \rho_t^{t,x}(\xi_\tau^{t,x}, \tau) = -\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathcal{E}_{t,\tau}^{t,x}(\xi_\tau^{t,x}). \tag{4.2.5}$$

Here $\mathcal{T}_t$ denotes the set of stopping times with values in $[t,T]$.

By Th. 3.2 in [138], the minimal risk measure is characterized via the solution $Y^{t,x}$ in $\mathcal{S}^2$ of the following reflected BSDE (RBSDE) associated with driver f and obstacle ξ:

$$\begin{cases} Y_s^{t,x} = g(X_T^{t,x}) + \displaystyle\int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, K_r^{t,x}(\cdot))dr + A_T^{t,x} - A_s^{t,x} \\ \qquad - \displaystyle\int_s^T Z_r^{t,x} dW_r - \int_s^T \int_{\mathbb{R}^*} K^{t,x}(r,e)\tilde{N}(dr,de) \\ Y_s^{t,x} \geq \xi_s^{t,x}, 0 \leq s \leq T \quad \text{a.s.} \\ A^{t,x} \text{ is a nondecreasing, continuous predictable process in } \mathcal{S}^2 \text{ with} \\ \qquad A_t^{t,x} = 0 \text{ and such that} \\ \displaystyle\int_t^T (Y_s^{t,x} - \xi_s^{t,x}) dA_s^{t,x} = 0 \text{ a.s.} \quad , \end{cases} \tag{4.2.6}$$

with $Z^{t,x}, K^{t,x} \in \mathbb{H}^2$ (resp. $\mathbb{H}^2_\nu$). Note that by the assumptions made on h and g, the obstacle $(\xi_s^{t,x})_{s \geq t}$ is continuous except at the inaccessible jump times of the Poisson measure, and at time T with $\Delta \xi_T^{t,x} \leq 0$ a.s., and this implies the continuity of $A^{t,x}$ by Th. 2.6 in [138]. Moreover, Th. 3.2 in [138] ensures that

$$Y_t^{t,x} = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathcal{E}_{t,\tau}^{t,x}(\xi_\tau^{t,x}) \quad \text{a.s.} \tag{4.2.7}$$

The SDE (4.2.1) and the RBSDE (4.2.6) can be solved with respect to the translated Brownian motion $(W_s - W_t)_{s \geq t}$. Hence $Y_t^{t,x}$ is constant for each t, x. We can thus define a deterministic function u called *value function* of our optimal stopping problem by setting for each t, x

$$u(t,x) := Y_t^{t,x}. \tag{4.2.8}$$

By Lemma 4.6.4 and Lemma 4.6.5 given in Appendix, the function u is continuous and has at most polynomial growth.
The continuity of u implies that $Y_s^{t,x} = u(s, X_s^{t,x})$, $t \leq s \leq T$ a.s.

Moreover, the stopping time $\tau^{*,t,x}$ (also denoted by τ^*), defined by

$$\tau^* := \inf\{s \geq t, \; Y_s^{t,x} = \xi_s^{t,x}\} = \inf\{s \geq t, \; u(s, X_s^{t,x}) = \bar{h}(s, X_s^{t,x})\}$$

is an optimal stopping time for (4.2.5) (see Th. 3.6 in [138]). Here, the function $\bar{h}$ is defined by $\bar{h}(t,x) := h(t,x)\mathbf{1}_{t<T} + g(x)\mathbf{1}_{t=T}$, so that $\xi_s^{t,x} = \bar{h}(s, X_s^{t,x})$, $0 \leq t \leq T$ a.s.

In the next section, we prove that the value function is a viscosity solution of an obstacle problem.

4.3 The value function, viscosity solution of an obstacle problem

We consider the following related obstacle problem for a parabolic PIDE:

$$\begin{cases} \min(u(t,x) - h(t,x), -\dfrac{\partial u}{\partial t}(t,x) - Lu(t,x) - f(t,x,u(t,x),(\sigma\dfrac{\partial u}{\partial x})(t,x), Bu(t,x)) = 0, \\ \quad (t,x) \in [0,T[\times\mathbb{R} \\ u(T,x) = g(x),\ x \in \mathbb{R} \end{cases} \tag{4.3.1}$$

where

$$\begin{aligned} &L := A + K, \\ &A\phi(t,x) := \frac{1}{2}\sigma^2(x)\frac{\partial^2\phi}{\partial x^2}(t,x) + b(x)\frac{\partial\phi}{\partial x}(t,x), \\ &K\phi(t,x) := \int_{\mathbb{R}^*}\left(\phi(t,x+\beta(x,e)) - \phi(t,x) - \frac{\partial\phi}{\partial x}(t,x)\beta(x,e)\right)\nu(de), \\ &B\phi(t,x)(\cdot) := \phi(t,x+\beta(x,\cdot)) - \phi(t,x) \in L^2_\nu. \end{aligned} \tag{4.3.2}$$

The operator B and K are well defined for $\phi \in C^{1,2}([0,T]\times\mathbb{R})$. Indeed, since β is bounded, we have $|\phi(t,x+\beta(x,e)) - \phi(t,x)| \leq C|\beta(x,e)|$ and

$$|\phi(t,x+\beta(x,e)) - \phi(t,x) - \frac{\partial\phi}{\partial x}(t,x)\beta(x,e)| \leq C\beta(x,e)^2.$$

We prove below that the value function u defined by (4.2.8) is a viscosity solution of the above obstacle problem.

Definition 4.3.1. • A continuous function u is said to be a *viscosity subsolution* of (4.3.1) iff $u(T,x) \leq g(x), x \in \mathbb{R}$, and iff for any point $(t_0,x_0) \in (0,T)\times\mathbb{R}$ and for any $\phi \in C^{1,2}([0,T]\times\mathbb{R})$ such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi - u$ attains its minimum at (t_0,x_0), we have

$$\begin{aligned} &\min(u(t_0,x_0) - h(t_0,x_0), \\ &-\frac{\partial\phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial\phi}{\partial x})(t_0,x_0), B\phi(t_0,x_0)) \leq 0. \end{aligned}$$

In other words, if $u(t_0,x_0) > h(t_0,x_0)$, then

$$-\frac{\partial\phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial\phi}{\partial x})(t_0,x_0), B\phi(t_0,x_0)) \leq 0.$$

• A continuous function u is said to be a *viscosity supersolution* of (4.3.1) iff $u(T,x) \geq g(x), x \in \mathbb{R}$, and iff for any point $(t_0,x_0) \in (0,T)\times\mathbb{R}$ and for any $\phi \in C^{1,2}([0,T]\times\mathbb{R})$ such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi - u$ attains its maximum at (t_0,x_0), we have

$$\begin{aligned} &\min(u(t_0,x_0) - h(t_0,x_0), \\ &-\frac{\partial\phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial\phi}{\partial x})(t_0,x_0), B\phi(t_0,x_0)) \geq 0. \end{aligned}$$

In other words, we have both $u(t_0,x_0) \geq h(t_0,x_0)$, and

$$-\frac{\partial \phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial \phi}{\partial x})(t_0,x_0),B\phi(t_0,x_0)) \geq 0.$$

Theorem 4.3.2. *The function u, defined by* (4.2.8), *is a viscosity solution (i.e. both a viscosity sub- and supersolution) of the obstacle problem* (4.3.1).

Proof. • We first prove that u is a subsolution of (4.3.1).

Let $(t_0,x_0) \in (0,T) \times \mathbb{R}$ and $\phi \in C^{1,2}([0,T] \times \mathbb{R})$ be such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi(t,x) \geq u(t,x)$, $\forall (t,x) \in [0,T] \times \mathbb{R}$. Suppose by contradiction that $u(t_0,x_0) > h(t_0,x_0)$ and that

$$-\frac{\partial \phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,\phi(t_0,x_0),(\sigma\frac{\partial \phi}{\partial x})(t_0,x_0),B\phi(t_0,x_0)) > 0.$$

By continuity of $K\phi$ (which can be shown using Lebesgue's theorem) and that of $B\phi : [0,T] \times \mathbb{R} \to L^2_\nu$, we can suppose that there exists $\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that: $\forall (t,x)$ such that $t_0 \leq t \leq t_0 + \eta_\varepsilon < T$ and $|x - x_0| \leq \eta_\varepsilon$, we have: $u(t,x) \geq h(t,x) + \varepsilon$ and

$$-\frac{\partial \phi}{\partial t}(t,x) - L\phi(t,x) - f(t,x,\phi(t,x),(\sigma\frac{\partial \phi}{\partial x})(t,x),B\phi(t,x)) \geq \varepsilon. \tag{4.3.3}$$

Note that $Y_s^{t_0,x_0} = Y_s^{s,X_s^{t_0,x_0}} = u(s,X_s^{t_0,x_0})$ a.s. because X^{t_0,x_0} is a Markov process and u is continuous. We define the stopping time θ as:

$$\theta := (t_0 + \eta_\varepsilon) \wedge \inf\{s \geq t_0, |X_s^{t_0,x_0} - x_0| > \eta_\varepsilon\}. \tag{4.3.4}$$

By definition of the stopping time θ,

$$u(s,X_s^{t_0,x_0}) \geq h(s,X_s^{t_0,x_0}) + \varepsilon > h(s,X_s^{t_0,x_0}), t_0 \leq s < \theta \text{ a.s.}$$

This means that for a.e. ω the process $(Y_s^{t_0,x_0}(\omega), s \in [t_0,\theta(\omega)[)$ stays strictly above the barrier. It follows that for a.e. ω, the function $s \to A_s^c(\omega)$ is constant on $[t_0,\theta(\omega)]$. In other words, $Y_s^{t_0,x_0} = \mathcal{E}_{s,\theta}^{t_0,x_0}(Y_\theta)$, $t_0 \leq s \leq \theta$ *a.s*, that is $(Y_s^{t_0,x_0}, s \in [t_0,\theta])$ is the solution of the classical BSDE associated with driver f, terminal time θ and terminal value $Y_\theta^{t_0,x_0}$. Applying Itô's lemma to $\phi(t,X_t^{t_0,x_0})$, we get:

$$\begin{aligned}\phi(t,X_t^{t_0,x_0}) = \phi(\theta,X_\theta^{t_0,x_0}) - \int_t^\theta \psi(s,X_s^{t_0,x_0})ds - \int_t^\theta (\sigma\frac{\partial \phi}{\partial x})(s,X_s^{t_0,x_0})dW_s \\ - \int_t^\theta \int_{\mathbb{R}^*} B\phi(s,X_{s^-}^{t_0,x_0})\tilde{N}(ds,de)\end{aligned} \tag{4.3.5}$$

where $\psi(s,x) := \frac{\partial \phi}{\partial s}(s,x) + L\phi(s,x)$.

Note that $(\phi(s,X_s^{t_0,x_0}),(\sigma\frac{\partial \phi}{\partial x})(s,X_s^{t_0,x_0}),B\phi(s,X_{s^-}^{t_0,x_0}); s \in [t_0,\theta])$ is the solution of the BSDE associated to terminal time θ, terminal value $\phi(\theta,X_\theta^{t_0,x_0})$ and driver process $-\psi(s,X_s^{t_0,x_0})$.

By (4.3.3) and the definition of the stopping time θ, we have a.s. that for each $s \in [t_0, \theta]$:

$$\begin{aligned} - \frac{\partial \phi}{\partial t}(s, X_s^{t_0,x_0}) - L\phi(s, X_s^{t_0,x_0}) \\ - f\left(s, X_s^{t_0,x_0}, \phi(s, X_s^{t_0,x_0}), (\sigma \frac{\partial \phi}{\partial x})(s, X_s^{t_0,x_0}), B\phi(s, X_s^{t_0,x_0})\right) \geq \varepsilon. \end{aligned} \tag{4.3.6}$$

Using the definition of the function ψ, (4.3.6) can be rewritten: for all $s \in [t_0, \theta]$,

$$\begin{aligned} - \psi(s, X_s^{t_0,x_0}) \\ - f\left(s, X_s^{t_0,x_0}, \phi(s, X_s^{t_0,x_0}), (\sigma \frac{\partial \phi}{\partial x})(s, X_s^{t_0,x_0}), B\phi(s, X_s^{t_0,x_0})\right) \geq \varepsilon. \end{aligned}$$

This gives a relation between the drivers $-\psi(s, X_s^{t_0,x_0})$ and $f(s, X_s^{t_0,x_0}, \cdot)$ of the two BSDEs. Also, $\phi(\theta, X_\theta^{t_0,x_0}) \geq u(\theta, X_\theta^{t_0,x_0}) = Y_\theta^{t_0,x_0}$ a.s.
Consequently, the extended comparison result for BSDEs with jumps given in the Appendix (see Proposition 4.6.2) implies that:

$$\phi(t_0, x_0) = \phi(t_0, X_{t_0}^{t_0,x_0}) > Y_{t_0}^{t_0,x_0} = u(t_0, x_0),$$

which leads to a contradiction.

• We now prove that u is a viscosity supersolution of (4.3.1).

Let $(t_0, x_0) \in (0, T) \times \mathbb{R}$ and $\phi \in C^{1,2}([0, T] \times \mathbb{R})$ be such that $\phi(t_0, x_0) = u(t_0, x_0)$ and $\phi(t, x) \leq u(t, x)$, $\forall (t, x) \in [0, T] \times \mathbb{R}$. Since the solution $(Y_s^{t_0,x_0})$ stays above the obstacle, we have:

$$u(t_0, x_0) \geq h(t_0, x_0).$$

We must prove that:

$$-\frac{\partial \phi}{\partial t}(t_0, x_0) - L\phi(t_0, x_0) - f\left(t_0, x_0, \phi(t_0, x_0), (\sigma \frac{\partial \phi}{\partial x})(t_0, x_0), B\phi(t_0, x_0)\right) \geq 0.$$

Suppose by contradiction that:

$$-\frac{\partial \phi}{\partial t}(t_0, x_0) - L\phi(t_0, x_0) - f\left(t_0, x_0, \phi(t_0, x_0), (\sigma \frac{\partial \phi}{\partial x})(t_0, x_0), B\phi(t_0, x_0)\right) < 0.$$

By continuity, we can suppose that there exists $\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that for each (t, x) such that $t_0 \leq t \leq t_0 + \eta_\varepsilon < T$ and $|x - x_0| \leq \eta_\varepsilon$, we have:

$$-\frac{\partial \phi}{\partial t}(t, x) - L\phi(t, x) - f\left(t, x, \phi(t, x), (\sigma \frac{\partial \phi}{\partial x})(t, x), B\phi(t, x)\right) \leq -\varepsilon. \tag{4.3.7}$$

We define the stopping time θ as:

$$\theta := (t_0 + \eta_\varepsilon) \wedge \inf\{s \geq t_0 / |X_s^{t_0,x_0} - x_0| > \eta_\varepsilon\}.$$

Applying as above Itô's lemma to $\phi(s, X_s^{t_0,x_0})$, we get that $(\phi(s, X_s^{t_0,x_0}), (\sigma \frac{\partial \phi}{\partial x})(s, X_s^{t_0,x_0}), B\phi(s, X_{s^-}^{t_0,x_0}); s \in [t_0, \theta])$ is the solution of the BSDE associated with terminal value $\phi(\theta, X_\theta^{t_0,x_0})$ and driver $-\psi(s, X_s^{t_0,x_0})$.

The process $(Y^{t_0,x_0}, s \in [t_0,\theta])$ is the solution of the classical BSDE associated with terminal condition $Y_\theta^{t_0,x_0} = u(\theta, X_\theta^{t_0,x_0})$ and generalized driver

$$f(s, X_s^{t_0,x_0}, y, z, q)ds + dA_s^{t_0,x_0}.$$

By (5.6.3) and the definition of the stopping time θ, we have :

$$\begin{aligned}&(-\frac{\partial \phi}{\partial t}(s, X_s^{t_0,x_0}) - L\phi(s, X_s^{t_0,x_0}) - f(s, X_s^{t_0,x_0}, \phi(s, X_s^{t_0,x_0}),\\ &\quad(\sigma\frac{\partial \phi}{\partial x})(s, X_s^{t_0,x_0}), B\phi(s, X_s^{t_0,x_0})))ds - dA_s^{t_0,x_0} \le -\varepsilon\ ds, \quad t_0 \le s \le \theta \text{ a.s.}\end{aligned}$$

or, equivalently,

$$\begin{aligned}&-\psi(s, X_s^{t_0,x_0})ds\\ &\le (f(s, X_s^{t_0,x_0}, \phi(s, X_s^{t_0,x_0}), (\sigma\frac{\partial \phi}{\partial x})(s, X_s^{t_0,x_0}), B\phi(s, X_s^{t_0,x_0})))ds\\ &\quad + dA_s^{t_0,x_0} - \varepsilon\ ds, \quad t_0 \le s \le \theta \text{ a.s.}\end{aligned}$$

This gives a relation between the drivers of the two BSDEs.
Also, $\phi(\theta, X_\theta^{t_0,x_0}) \le u(\theta, X_\theta^{t_0,x_0}) = Y_\theta^{t_0,x_0}$ a.s. Consequently, Proposition 6.4.1 in the Appendix implies that:

$$\phi(t_0, x_0) = \phi(t_0, X_{t_0}^{t_0,x_0}) < Y_{t_0}^{t_0,x_0} = u(t_0, x_0),$$

which leads to a contradiction. $\square$

4.4 Uniqueness result for the obstacle problem

We provide a uniqueness result for (4.3.1) in the particular case when for each $\phi \in C^{1,2}([0,T]\times\mathbb{R})$, $B\phi$ is a map valued in $\mathbb{R}$ instead of L^2_ν. More precisely,

$$B\phi(t,x) := \int_{\mathbb{R}^*} (\phi(t, x+\beta(x,e)) - \phi(t,x))\gamma(x,e)\nu(de), \tag{4.4.1}$$

which is well defined since $|\phi(t, x+\beta(x,e)) - \phi(t,x)| \le C|\beta(x,e)|$.
We suppose that Assumption $\mathbf{H}_1$ holds and we make the additional assumptions:

Assumption $\mathbf{H}_2$:

1. $$f(s, X_s^{t,x}(\omega), y, z, k) := \overline{f}\left(s, X_s^{t,x}(\omega), y, z, \int_{\mathbb{R}^*} k(e)\gamma(X_s^{t,x}(\omega), e)\nu(de)\right)\mathbf{1}_{s\ge t},$$

where $\overline{f} : [0,T]\times\mathbb{R}^4 \to \mathbb{R}$ is continuous in t uniformly with respect to x, y, z, k, continuous in x uniformly with respect to y, z, k, and satisfies:

(i) $|\overline{f}(t,x,0,0,0)| \le C$, for all $t\in[0,T], x\in\mathbb{R}$.

(ii) $|\overline{f}(t,x,y,z,k) - \overline{f}(t,x',y',z',k')| \le C(|y-y'| + |z-z'| + |k-k'|)$, for all $t \in [0,T]$, $y, y', z, z', k, k' \in \mathbb{R}$.

(iii) $k \mapsto \overline{f}(t,x,y,z,k)$ is non-decreasing, for all $t \in [0,T]$, $x,y,z \in \mathbb{R}$.

2. For each $R > 0$, there exists a continuous function $m_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that $m_R(0) = 0$ and $|\overline{f}(t,x,v,p,q) - \overline{f}(t,y,v,p,q)| \le m_R(|x-y|(1+|p|))$,
for all $t \in [0,T]$, $|x|, |y| \le R, |v| \le R,\ p,q \in \mathbb{R}$.
3. $|\gamma(x,e) - \gamma(y,e)| \le C|x-y|(1 \wedge e^2)$ and $0 \le \gamma(x,e) \le C(1 \wedge |e|)$, for all $x,y \in \mathbb{R}, e \in \mathbb{R}^*$.
4. There exists $r > 0$ such that for all $t \in [0,T]$, $x,u,v,p,l \in \mathbb{R}$:

$$\overline{f}(t,x,v,p,l) - \overline{f}(t,x,u,p,l) \ge r(u-v) \text{ when } u \ge v.$$

5. $|h(t,x)| + |g(x)| \le C$, for all $t \in [0,T]$, $x \in \mathbb{R}$.

To simplify notation, $\overline{f}$ is denoted by f in the sequel.

We state below a comparison theorem, which uses results of three lemmas. The proofs of these lemmas are given in Subsection 4.4.1.

Theorem 4.4.1 (Comparison principle)**.** *Under the above hypotheses, if U is a viscosity subsolution and V is a viscosity supersolution of the obstacle problem* (4.3.1) *in the class of continuous bounded functions, then $U(t,x) \le V(t,x)$, for each $(t,x) \in [0,T] \times \mathbb{R}$.*

Proof. Set

$$M := \sup_{[0,T]\times\mathbb{R}} (U - V).$$

It is sufficient to prove that $M \le 0$. For each $\varepsilon, \eta > 0$, we introduce the function:

$$\psi^{\varepsilon,\eta}(t,s,x,y) := U(t,x) - V(s,y) - \frac{(x-y)^2}{\varepsilon^2} - \frac{(t-s)^2}{\varepsilon^2} - \eta^2(x^2+y^2),$$

for t,s,x,y in $[0,T]^2 \times \mathbb{R}^2$. Let

$$M^{\varepsilon,\eta} := \max_{[0,T]^2\times\mathbb{R}^2} \psi^{\varepsilon,\eta}.$$

This supremum is reached at some point $(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta})$.

Using that $\psi^{\varepsilon,\eta}(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \ge \psi^{\varepsilon,\eta}(0,0,0,0)$, we obtain:

$$\begin{aligned} U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) - \frac{(t^{\varepsilon,\eta} - s^{\varepsilon,\eta})^2}{\varepsilon^2} - \frac{(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})^2}{\varepsilon^2} - \eta^2((x^{\varepsilon,\eta})^2 + (y^{\varepsilon,\eta})^2) \\ \ge U(0,0) - V(0,0), \end{aligned} \tag{4.4.2}$$

or, equivalently,

$$\frac{(t^{\varepsilon,\eta} - s^{\varepsilon,\eta})^2}{\varepsilon^2} + \frac{(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})^2}{\varepsilon^2} + \eta^2((x^{\varepsilon,\eta})^2 + (y^{\varepsilon,\eta})^2) \le \|U\|_\infty + \|V\|_\infty - U(0,0) - V(0,0). \tag{4.4.3}$$

Consequently, we can find a constant C such that:

$$|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}| + |t^{\varepsilon,\eta} - s^{\varepsilon,\eta}| \le C\varepsilon \tag{4.4.4}$$

$$|x^{\varepsilon,\eta}| \le \frac{C}{\eta}, |y^{\varepsilon,\eta}| \le \frac{C}{\eta}. \tag{4.4.5}$$

Extracting a subsequence if necessary, we may suppose that for each η the sequences $(t^{\varepsilon,\eta})_\varepsilon$ and $(s^{\varepsilon,\eta})_\varepsilon$ converge to a common limit t^η when ε tends to 0, and from (4.4.4) and (4.4.5) we may also suppose, extracting again, that for each η, the sequences $(x^{\varepsilon,\eta})_\varepsilon$ and $(y^{\varepsilon,\eta})_\varepsilon$ converge to a common limit x^η.

Lemma 4.4.2. *We have:*

$$\lim_{\varepsilon\to 0}\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2}=0\,;\quad \lim_{\varepsilon\to 0}\frac{(t^{\varepsilon,\eta}-s^{\varepsilon,\eta})^2}{\varepsilon^2}=0$$
$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0}M^{\varepsilon,\eta}=M.$$

We now introduce the functions:

$$\Psi_1(t,x):=V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})+\frac{(x-y^{\varepsilon,\eta})^2}{\varepsilon^2}+\frac{(t-s^{\varepsilon,\eta})^2}{\varepsilon^2}+\eta^2(x^2+(y^{\varepsilon,\eta})^2)$$

$$\Psi_2(s,y):=U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-\frac{(x^{\varepsilon,\eta}-y)^2}{\varepsilon^2}-\frac{(t^{\varepsilon,\eta}-s)^2}{\varepsilon^2}-\eta^2((x^{\varepsilon,\eta})^2+y^2).$$

As $(t,x)\to(U-\Psi_1)(t,x)$ reaches its maximum at $(t^{\varepsilon,\eta},x^{\varepsilon,\eta})$ and U is a subsolution we have two cases:

- $t^{\varepsilon,\eta}=T$ and then $U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})\le g(x^{\varepsilon,\eta})$,

- $t^{\varepsilon,\eta}\neq T$ and then

$$\min\Bigg(U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-h(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),\;\frac{\partial\Psi_1}{\partial t}(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-L\Psi_1(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-$$
$$-f\left(t^{\varepsilon,\eta},x^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),(\sigma\frac{\partial\Psi_1}{\partial x})(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),B\Psi_1(t^{\varepsilon,\eta},x^{\varepsilon,\eta})\right)\Bigg)\le 0. \tag{4.4.6}$$

As $(s,y)\to(\Psi_2-V)(s,y)$ reaches its maximum at $(s^{\varepsilon,\eta},y^{\varepsilon,\eta})$ and V is a supersolution we have the two following cases:

- $s^{\varepsilon,\eta}=T$ and then $V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})\ge g(y^{\varepsilon,\eta})$,

- $s^{\varepsilon,\eta}\neq T$ and then

$$\begin{aligned}&\min(V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})-h(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),\\&\frac{\partial\Psi_2}{\partial t}(s^{\varepsilon,\eta},y^{\varepsilon,\eta})-L\Psi_2(s^{\varepsilon,\eta},y^{\varepsilon,\eta})-f(s^{\varepsilon,\eta},y^{\varepsilon,\eta},V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),\\&\quad(\sigma\frac{\partial\Psi_2}{\partial x})(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),B\Psi_2(s^{\varepsilon,\eta},y^{\varepsilon,\eta}))\ge 0.\end{aligned} \tag{4.4.7}$$

We now prove that $M\le 0$. Three cases are possible.
1st case: There exists a subsequence of (t^η) such that $t^\eta=T$ for all η (of this subsequence). As U is continuous, for all η and for ε small enough

$$U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})\le U(t^\eta,x^\eta)+\eta\le g(x^\eta)+\eta,$$

and as V is continuous, for all η and for ε small enough

$$V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})\ge V(t^\eta,x^\eta)-\eta\ge g(x^\eta)-\eta.$$

Hence

$$U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})\le 2\eta$$

and

$$M^{\varepsilon,\eta} = U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) - \frac{(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})^2}{\varepsilon^2} - \frac{(t^{\varepsilon,\eta} - s^{\varepsilon,\eta})^2}{\varepsilon^2}$$
$$- \eta^2((x^{\varepsilon,\eta})^2 + (y^{\varepsilon,\eta})^2) \leq U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \leq 2\eta.$$

Letting $\varepsilon \to 0$ and then $\eta \to 0$ one gets, using Lemma 4.4.2, that $M \leq 0$.

2nd case: There exists a subsequence such that $t^\eta \neq T$, and for all η belonging to this subsequence, there exists a subsequence of $(x^{\varepsilon,\eta})_\eta$ such that

$$U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - h(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) \leq 0.$$

As from (4.4.7) one has

$$V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) - h(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \geq 0,$$

it comes that

$$M^{\varepsilon,\eta} \leq U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \leq h(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - h(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}).$$

Letting $\varepsilon \to 0$ and then $\eta \to 0$, using the equality $\lim_{\eta\to 0} \lim_{\varepsilon\to 0} M^{\varepsilon,\eta} = M$ (see Lemma 4.4.2), we derive that $M \leq 0$.

Last case: We are left with the case when, for a subsequence of η, we have $t^\eta \neq T$ and for all η belonging to this subsequence there exists a subsequence of $(x^{\varepsilon,\eta})_\varepsilon$ such that:

$$U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - h(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) > 0.$$

Set

$$\varphi(t, s, x, y) := \frac{(x-y)^2}{\varepsilon^2} + \frac{(t-s)^2}{\varepsilon^2} + \eta^2(x^2 + y^2). \tag{4.4.8}$$

The maximum of the function $\psi^{\varepsilon,\eta}(t, s, x, y) := U(t, x) - V(s, y) - \varphi(t, s, x, y)$ is reached at the point $(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta})$.

Let us fix $\delta > 0$ and consider the ball $\mathcal{B}_\delta = \mathcal{B}(0, \delta)$. We introduce the operators K^δ, $\tilde{K}^\delta, B^\delta, \tilde{B}^\delta$ corresponding to the operators K and B defined in (4.3.2) and (4.4.1), but integrating on $\mathcal{B}_\delta$ or $\mathbb{R}\backslash\mathcal{B}_\delta$ (also denoted by $\mathcal{B}_\delta^c$) only.
They are defined respectively for all $\phi \in C^{1,2}$, $\Phi \in \mathcal{C}$ by

$$K^\delta[t, x, \phi] := \int_{\mathcal{B}_\delta} \left(\phi(t, x + \beta(x, e)) - \phi(t, x) - \frac{\partial \phi}{\partial x}(t, x)\beta(x, e) \right) \nu(de) \tag{4.4.9}$$

$$\tilde{K}^\delta[t, x, \pi, \Phi] := \int_{\mathcal{B}_\delta^c} \left(\Phi(t, x + \beta(x, e)) - \Phi(t, x) - \pi\beta(x, e) \right) \nu(de). \tag{4.4.10}$$

$$B^\delta[t, x, \phi] := \int_{\mathcal{B}_\delta} \left(\phi(t, x + \beta(x, e)) - \phi(t, x) \right) \gamma(x, e)\nu(de) \tag{4.4.11}$$

$$\tilde{B}^\delta[t, x, \Phi] := \int_{\mathcal{B}_\delta^c} \left(\Phi(t, x + \beta(x, e)) - \Phi(t, x) \right) \gamma(x, e)\nu(de) \tag{4.4.12}$$

Here $\mathcal{C}$ denotes the set of bounded continuous functions.

We apply the non-local version of Jensen Ishii's lemma [10] (see also Corollary 1 in [10]) and we obtain that there exists $\bar{\alpha}$, such that for any α such that $0 < \alpha \leq \bar{\alpha}$ there exist:

$$(a, \overline{p}, X) \in \mathcal{P}^{2,+}U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}),\ (b, \overline{q}, Y) \in \mathcal{P}^{2,-}V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta})$$

such that

$$\begin{cases} F(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}), a, \overline{p}, X, K^{\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_{\alpha}(\cdot, s^{\varepsilon,\eta}, y^{\varepsilon,\eta})] \\ \qquad +\tilde{K}^{\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \overline{p}, U], B^{\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_{\alpha}(\cdot, s^{\varepsilon,\eta}, y^{\varepsilon,\eta})] + \tilde{B}^{\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U]) \leq 0 \\ F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, K^{\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_{\alpha}(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \cdot)] \\ \qquad +\tilde{K}^{\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, \overline{q}, V], B^{\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_{\alpha}(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \cdot)] + \tilde{B}^{\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, V]) \geq 0 \end{cases} \tag{4.4.13}$$

where

$$F(t, x, u, a, p, X, l_1, l_2) := -a - \frac{1}{2}\sigma^2(x)X - b(x)p - l_1 - f(t, x, u, p\sigma(x), l_2). \tag{4.4.14}$$

and such that

$$\begin{cases} \overline{p} = p + 2\eta^2 x^{\varepsilon,\eta};\ \ \overline{q} = p - 2\eta^2 y^{\varepsilon,\eta};\ \ p = \frac{2(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})}{\varepsilon^2} \\ a = b = \frac{2(t^{\varepsilon,\eta} - s^{\varepsilon,\eta})}{\varepsilon^2} \\ \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{2}{\varepsilon^2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + (2\eta^2 + O(\alpha)) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{cases}$$

Here, $\mathcal{P}^{2,+}$ (resp. $\mathcal{P}^{2,-}$) is the set of superjets (resp. subjets) defined in [10] (see Definition 3). Note that the operators K^{δ}, $\tilde{K}^{\delta}$, B^{δ} and $\tilde{B}^{\delta}$ satisfy the hypotheses (NLT) of [10] (see Section 2.2 in [10]). Hence we can use the alternative definition for sub-superviscosity solutions expressed in terms of sub-supersolutions and super-subjets given by Definition 4 in [10]. By Lemma 1 in [10], we have $\varphi_{\alpha} := \mathcal{R}^{\alpha}[\varphi]$, with
$\mathcal{R}^{\alpha}[\varphi][(\tilde{x}, \tilde{p})] := \sup_{Z \in B(\tilde{x},\kappa)} \left[\varphi(Z) - r(Z - \tilde{x}) - \dfrac{|Z - \tilde{x}|^2}{2\alpha}\right]$, $(\tilde{x}, \tilde{p}) := ((t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta}), (a, b, \overline{p}, \overline{q}))$
and κ is assumed to be sufficiently small. Proposition 3 in [10] together with the Lipschitz continuity of F with respect to l_1, l_2 lead to:

$$\begin{cases} F(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}), a, \overline{p}, X, K^{\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] \\ \qquad +\tilde{K}^{\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \overline{p}, U], B^{\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] + \tilde{B}^{\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U]) \leq O(\alpha) \\ F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, K^{\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] \\ \qquad +\tilde{K}^{\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, \overline{q}, V], B^{\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] + \tilde{B}^{\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, V]) \geq O(\alpha), \end{cases} \tag{4.4.15}$$

where we denote by φ_x the function $(t, x) \mapsto \varphi(t, x, s^{\varepsilon,\eta}, y^{\varepsilon,\eta})$ and by φ_y the function $(s, y) \mapsto \varphi(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, s, y)$.

Since $(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta})$ is a global maximum of $\psi^{\varepsilon,\eta}$,we have:

$$
\begin{aligned}
\psi^{\varepsilon,\eta}(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e), y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)) &\leq \psi^{\varepsilon,\eta}(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \\
&\Leftrightarrow U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e)) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)) \\
&- \frac{(x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e) - y^{\varepsilon,\eta} - \beta(y^{\varepsilon,\eta}, e))^2}{\varepsilon^2} \\
&- \frac{(t^{\varepsilon,\eta} - s^{\varepsilon,\eta})^2}{\varepsilon^2} - \eta^2((x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e))^2 + (y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e))^2) \\
&\leq U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \\
&- \frac{(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})^2}{\varepsilon^2} - \frac{(t^{\varepsilon,\eta} - s^{\varepsilon,\eta})^2}{\varepsilon^2} - \eta^2((x^{\varepsilon,\eta})^2 + (y^{\varepsilon,\eta})^2).
\end{aligned}
$$

Consequently, we get:

$$
\begin{aligned}
&U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e)) - U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) \leq V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)) \\
&- V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) + \frac{(\beta(x^{\varepsilon,\eta}, e) - \beta(y^{\varepsilon,\eta}, e))^2}{\varepsilon^2} + p(\beta(x^{\varepsilon,\eta}, e) - \beta(y^{\varepsilon,\eta}, e)) \\
&+ \eta^2(\beta^2(x^{\varepsilon,\eta}, e) + 2x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta}, e) + 2y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta}, e) + \beta^2(y^{\varepsilon,\eta}, e)).
\end{aligned}
\tag{4.4.16}
$$

The two following lemmas hold.

Lemma 4.4.3. *Let*

$$
\begin{aligned}
l_K &:= K^\delta[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] + \tilde{K}^\delta[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \overline{p}, U] \\
l'_K &:= K^\delta[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] + \tilde{K}^\delta[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, \overline{q}, V].
\end{aligned}
\tag{4.4.17}
$$

We have

$$
l_K \leq l'_K + O(\frac{(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})^2}{\varepsilon^2}) + O(\eta^2) + (\frac{1}{\varepsilon^2} + \eta^2)O(\delta). \tag{4.4.18}
$$

Lemma 4.4.4. *Let*

$$
\begin{aligned}
l_B &:= B^\delta[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] + \tilde{B}^\delta[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U] \\
l'_B &:= B^\delta[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] + \tilde{B}^\delta[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, V].
\end{aligned}
\tag{4.4.19}
$$

We have

$$
l_B \leq l'_B + (\eta^2 + \frac{1}{\varepsilon^2})O(\delta) + O(\frac{(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})^2}{\varepsilon^2}) + O(|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|) + O(\eta^2). \tag{4.4.20}
$$

We argue now by contradiction by assuming that

$$
M > 0. \tag{4.4.21}
$$

Using Assumption $(\mathbf{H}_2)$.4, we get

$$
\begin{aligned}
0 < \frac{r}{2}M \le rM_{\varepsilon,\eta} &\le r(U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}) - V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}))\\
&\le F(s^{\varepsilon,\eta},y^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),a,\overline{q},Y,l'_K,l'_B)\\
&\quad - F(s^{\varepsilon,\eta},y^{\varepsilon,\eta},V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),a,\overline{q},Y,l'_K,l'_B)\\
&= F(s^{\varepsilon,\eta},y^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),a,\overline{q},Y,l'_K,l'_B)\\
&\quad - F(s^{\varepsilon,\eta},y^{\varepsilon,\eta},U(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),a,\overline{q},Y,l'_K,l'_B)\\
&\quad + F(s^{\varepsilon,\eta},y^{\varepsilon,\eta},U(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),a,\overline{q},Y,l'_K,l'_B)\\
&\quad - F(s^{\varepsilon,\eta},y^{\varepsilon,\eta},U(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),a,\overline{q},Y,l_K,l_B)\\
&\quad + F(s^{\varepsilon,\eta},y^{\varepsilon,\eta},U(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),a,\overline{q},Y,l_K,l_B)\\
&\quad - F(t^{\varepsilon,\eta},x^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),a,\overline{p},X,l_K,l_B)\\
&\quad + F(t^{\varepsilon,\eta},x^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),a,\overline{p},X,l_K,l_B)\\
&\quad - F(s^{\varepsilon,\eta},y^{\varepsilon,\eta},V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),a,\overline{q},Y,l'_K,l'_B)\\
&\le K|U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}) - U(s^{\varepsilon,\eta},y^{\varepsilon,\eta})| + F(s^{\varepsilon,\eta},y^{\varepsilon,\eta},U(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),a,\overline{q},Y,l_K,l_B)\\
&\quad - F(t^{\varepsilon,\eta},x^{\varepsilon,\eta},U(t^{\varepsilon,\eta},X^{\varepsilon,\eta}),a,\overline{p},X,l_K,l_B)\\
&\quad + (\eta^2 + \frac{1}{\varepsilon^2})O(\delta) + O(\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2}) + O(|x^{\varepsilon,\eta}-y^{\varepsilon,\eta}|) + O(\eta^2) + O(\alpha).
\end{aligned}
\tag{4.4.22}
$$

We have used here the (nonlocal) ellipticity of F, the Lipschitz property of F, (4.4.15) and the estimates proven in Lemma 4.4.3 and Lemma 4.4.4. From the hypothesis on b and σ, we have:

$$
\begin{aligned}
\sigma^2(x^{\varepsilon,\eta})X - \sigma^2(y^{\varepsilon,\eta})Y &\le \frac{C(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2} + O(\eta^2) + O(\alpha),\\
b(x^{\varepsilon,\eta})\overline{p} - b(y^{\varepsilon,\eta})\overline{q} &\le \frac{C|x^{\varepsilon,\eta}-y^{\varepsilon,\eta}|}{\varepsilon^2} + O(\eta^2).
\end{aligned}
$$

We thus obtain the inequality:

$$
\begin{aligned}
&F(s^{\varepsilon,\eta},y^{\varepsilon,\eta},U(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),a,\overline{q},Y,l_K,l_B) - F(t^{\varepsilon,\eta},x^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),a,\overline{p},X,l_K,l_B)\\
&\quad \le \frac{C(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2} + O(\eta^2)\\
&\quad + f(t^{\varepsilon,\eta},x^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),(p+2\eta^2)\sigma(x^{\varepsilon,\eta}),l_B)\\
&\quad - f(s^{\varepsilon,\eta},y^{\varepsilon,\eta},U(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),(p-2\eta^2)\sigma(y^{\varepsilon,\eta}),l_B)\\
&\quad \le f(t^{\varepsilon,\eta},x^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),(p+2\eta^2)\sigma(x^{\varepsilon,\eta}),l_B)\\
&\quad - f(s^{\varepsilon,\eta},x^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),(p+2\eta^2)\sigma(x^{\varepsilon,\eta}),l_B)\\
&\quad + m_R(|x^{\varepsilon,\eta}-y^{\varepsilon,\eta}|(1+(p+2\eta^2)\sigma(x^{\varepsilon,\eta})))\\
&\quad + K|U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}) - U(s^{\varepsilon,\eta},y^{\varepsilon,\eta})| + O(\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2}) + O(\eta^2) + O(\alpha).
\end{aligned}
\tag{4.4.23}
$$

The last equality is obtained by some computations similar to those in (4.4.22). From (4.4.22), (4.4.23) we get

$$0 < \frac{r}{2}M \leq rM^{\varepsilon,\eta} \leq f(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}), (p+2\eta^2)\sigma(x^{\varepsilon,\eta}), l_B)$$
$$- f(s^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}), (p+2\eta^2)\sigma(x^{\varepsilon,\eta}), l_B)$$
$$+ m_R(|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|(1 + (p+2\eta^2)\sigma(x^{\varepsilon,\eta}))$$
$$+ K|U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta})| +$$
$$+ O(\frac{(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})^2}{\varepsilon^2}) + O(|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|) + (\eta^2 + \frac{1}{\varepsilon^2})O(\delta) + O(\eta^2) + O(\alpha). \tag{4.4.24}$$

By Lemma 4.4.2, letting successively $\alpha, \delta, \varepsilon$ and η tend to 0 in (4.4.24) we obtain that $0 < \frac{r}{2}M \leq 0$. Hence, the assumption $M > 0$ made above (see (4.4.21)) is wrong. This ends the proof of Theorem 4.4.1. □

Corollary 4.4.5 (Uniqueness)**.** *Under the additional Assumption* ($\boldsymbol{H}_2$)*, the value function is the unique solution of the obstacle problem* (4.3.1) *in the class of bounded continuous functions.*

4.4.1 Proofs of the lemmas

Proof of Lemma 4.4.2. For $\eta > 0$, we introduce the functions:
$\tilde{U}^\eta(t,x) = U(t,x) - \eta^2 x^2$ and $\tilde{V}^\eta(t,x) = V(t,x) + \eta^2 x^2$. Set

$$M^\eta := \sup_{[0,T]\times\mathbb{R}} (\tilde{U}^\eta - \tilde{V}^\eta).$$

The maximum M^η is reached at some point $(\hat{t}^\eta, \hat{x}^\eta)$. From the form of $\psi^{\varepsilon,\eta}$, we have that for fixed η, there exists a subsequence $(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon\eta}, y^{\varepsilon,\eta})_\varepsilon$ which converges to some point $(t^\eta, s^\eta, x^\eta, y^\eta)$ when ε tends to 0.

Since $M^{\varepsilon,\eta}$ is reached at $(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta})$, we have:

$$(\tilde{U}^\eta - \tilde{V}^\eta)(\hat{t}^\eta, \hat{x}^\eta) = (U - V)(\hat{t}^\eta, \hat{x}^\eta) - \eta^2((\hat{x}^\eta)^2 + (\hat{y}^\eta)^2) \leq M^{\varepsilon,\eta}$$
$$= U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) - \frac{(t^{\varepsilon,\eta} - s^{\varepsilon,\eta})^2}{\varepsilon^2}$$
$$- \frac{(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})^2}{\varepsilon^2} - \eta^2((x^{\varepsilon,\eta})^2 + (y^{\varepsilon,\eta})^2).$$

Setting

$$\bar{l}_\eta := \limsup_{\varepsilon\to 0} \frac{(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})^2}{\varepsilon^2}, \quad \underline{l}_\eta := \liminf_{\varepsilon\to 0} \frac{(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})^2}{\varepsilon^2}$$

we get

$$0 \leq \underline{l}_\eta \leq \bar{l}_\eta \leq (\tilde{U}^\eta - \tilde{V}^\eta)(t^\eta, x^\eta) - (\tilde{U}^\eta - \tilde{V}^\eta)(\hat{t}^\eta, \hat{x}^\eta) \leq 0. \tag{4.4.25}$$

We derive that, up to a subsequence, $\lim_{\varepsilon\to 0} \frac{(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})^2}{\varepsilon^2} = 0$ and
$\lim_{\varepsilon\to 0} M^{\varepsilon,\eta} = M^\eta$. Similarly, we get $\lim_{\varepsilon\to 0} \frac{(t^{\varepsilon,\eta} - s^{\varepsilon,\eta})^2}{\varepsilon^2} = 0$.

Let us prove that $\lim_{\eta\to 0} M^\eta = M$. First, note that $M^\eta \le M$, for all η. By definition of M, for all $\delta > 0$ there exists $(t_\delta, x_\delta) \in [0,T] \times \mathbb{R}$ such that $M - \delta \le (U - V)(t_\delta, x_\delta)$. Consequently, we get

$$M - 2\eta^2 x_\delta^2 - \delta \le (U-V)(t_\delta, x_\delta) - 2\eta^2 x_\delta^2 = (\tilde{U}^\eta - \tilde{V}^\eta)(t_\delta, x_\delta) \le M^\eta \le M.$$

By letting η and then δ tend to 0, the result follows. □

Proof of Lemma 4.4.3. We have:

$$K^\delta[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] = \int_{\mathcal{B}_\delta} (\frac{1}{\varepsilon^2} + \eta^2)\beta^2(x^{\varepsilon,\eta}, e)\nu(de) \tag{4.4.26}$$

$$K^\delta[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] = \int_{\mathcal{B}_\delta} (-\frac{1}{\varepsilon^2} - \eta^2)\beta^2(y^{\varepsilon,\eta}, e)\nu(de). \tag{4.4.27}$$

Equations (4.4.26) and (4.4.27) imply:

$$\begin{aligned} K^\delta[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] &\le K^\delta[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] + (\frac{1}{\varepsilon^2} + \eta^2)\int_{\mathcal{B}_\delta} \beta^2(y^{\varepsilon,\eta}, e)\nu(de) \\ + (\frac{1}{\varepsilon^2} + \eta^2)\int_{\mathcal{B}_\delta} \beta^2(x^{\varepsilon,\eta}, e)\nu(de) &\le K^\delta[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] + (\frac{1}{\varepsilon^2} + \eta^2)O(\delta). \end{aligned} \tag{4.4.28}$$

Using inequality (4.4.16) and integrating on $\mathcal{B}_\delta^c$, we obtain:

$$\begin{aligned} &\tilde{K}^\delta[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \overline{p}, U] = \int_{\mathcal{B}_\delta^c} \Big(U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta} + \beta(x^{\varepsilon,\eta}, e)) - U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) \\ &- (p + 2\eta^2 x^{\varepsilon,\eta})\beta(x^{\varepsilon,\eta}, e)\Big)\nu(de) \le \int_{\mathcal{B}_\delta^c} \Big(V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta} + \beta(y^{\varepsilon,\eta}, e)) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \\ &- (p - 2\eta^2 y^{\varepsilon,\eta})\beta(y^{\varepsilon,\eta}, e)\Big)\nu(de) + \int_{\mathcal{B}_\delta^c} \frac{(\beta(x^{\varepsilon,\eta}, e) - \beta(y^{\varepsilon,\eta}, e))^2}{\varepsilon^2}\nu(de) \\ &+ \eta^2 \int_{\mathcal{B}_\delta^c} (\beta^2(x^{\varepsilon,\eta}, e) + \beta^2(y^{\varepsilon,\eta}, e))\nu(de) \\ &\le \tilde{K}^\delta[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, \overline{q}, V] + O(\frac{(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})^2}{\varepsilon^2}) + O(\eta^2). \end{aligned}$$

Using (4.4.17) and (4.4.28), we derive (4.4.18), which ends the proof of Lemma 4.4.3. □

Proof of Lemma 4.4.4. From (4.4.11), we derive that:

$$\begin{aligned} B^\delta[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] = \int_{\mathcal{B}_\delta} \Big(&(\eta^2 + \frac{1}{\varepsilon^2})\beta^2(x^{\varepsilon,\eta}, e) + \frac{2\beta(x^{\varepsilon,\eta}, e)}{\varepsilon^2}(x^{\varepsilon,\eta} - y^{\varepsilon,\eta}) \\ &+ 2\eta^2 x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta}, e)\Big)\gamma(x^{\varepsilon,\eta}, e)\nu(de) \end{aligned} \tag{4.4.29}$$

$$\begin{aligned} B^\delta[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] = \int_{\mathcal{B}_\delta} \Big(&(-\eta^2 - \frac{1}{\varepsilon^2})\beta^2(y^{\varepsilon,\eta}, e) + \frac{2\beta(y^{\varepsilon,\eta}, e)}{\varepsilon^2}(x^{\varepsilon,\eta} - y^{\varepsilon,\eta}) \\ &- 2\eta^2 y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta}, e)\Big)\gamma(y^{\varepsilon,\eta}, e)\nu(de). \end{aligned} \tag{4.4.30}$$

After some computations, we obtain:

$$
\begin{aligned}
&\Big((\eta^2+\frac{1}{\varepsilon^2})\beta^2(x^{\varepsilon,\eta},e)+\frac{2\beta(x^{\varepsilon,\eta},e)}{\varepsilon^2}(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})+2\eta^2x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta},e)\Big)\gamma(x^{\varepsilon,\eta},e)\\
&=(-\eta^2-\frac{1}{\varepsilon^2})\beta^2(y^{\varepsilon,\eta},e)\gamma(y^{\varepsilon,\eta},e)+\frac{2\beta(y^{\varepsilon,\eta},e)}{\varepsilon^2}(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})\gamma(y^{\varepsilon,\eta},e)\\
&-2\eta^2y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta},e)\gamma(y^{\varepsilon,\eta},e)\\
&+(\eta^2+\frac{1}{\varepsilon^2})\Big(\beta^2(y^{\varepsilon,\eta},e)\gamma(y^{\varepsilon,\eta},e)+\beta^2(x^{\varepsilon,\eta},e)\gamma(x^{\varepsilon,\eta},e)\Big)\\
&+\frac{2}{\varepsilon^2}(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})\Big(\beta(x^{\varepsilon,\eta},e)\gamma(x^{\varepsilon,\eta},e)-\beta(y^{\varepsilon,\eta},e)\gamma(y^{\varepsilon,\eta},e)\Big)\\
&+2\eta^2\Big(x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta},e)\gamma(x^{\varepsilon,\eta},e)+y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta},e)\gamma(y^{\varepsilon,\eta},e)\Big).
\end{aligned}\tag{4.4.31}
$$

From (4.4.29), (4.4.30), (4.4.31) and using the hypothesis on β and γ, we get:

$$
B^\delta[t^{\varepsilon,\eta},x^{\varepsilon,\eta},\varphi_x]\le B^\delta[s^{\varepsilon,\eta},y^{\varepsilon,\eta},-\varphi_y]+(\eta^2+\frac{1}{\varepsilon^2})O(\delta)+O(\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2})+O(\eta^2). \tag{4.4.32}
$$

We now estimate the operator $\tilde{B}^\delta$. Inequality (4.4.16) implies:

$$
\begin{aligned}
&\Big(U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}+\beta(x^{\varepsilon,\eta},e))-U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})\Big)\gamma(x^{\varepsilon,\eta},e)\\
&\quad\le\Big(V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}+\beta(y^{\varepsilon,\eta},e))-V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})\\
&\quad+\frac{|\beta(x^{\varepsilon,\eta},e)-\beta(y^{\varepsilon,\eta},e)|^2}{\varepsilon^2}+p(\beta(x^{\varepsilon,\eta},e)-\beta(y^{\varepsilon,\eta},e))\\
&\quad+\eta^2(\beta^2(x^{\varepsilon,\eta},e)+2x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta},e)+2y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta},e)+\beta^2(y^{\varepsilon,\eta},e)\Big)\gamma(x^{\varepsilon,\eta},e)\\
&\quad=\Big(V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}+\beta(y^{\varepsilon,\eta},e))-V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})\Big)\gamma(y^{\varepsilon,\eta},e)\\
&\quad+\Big(V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}+\beta(y^{\varepsilon,\eta},e))-V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})\Big)\Big(\gamma(x^{\varepsilon,\eta},e)-\gamma(y^{\varepsilon,\eta},e)\Big)\\
&\quad+\frac{|\beta(x^{\varepsilon,\eta},e)-\beta(y^{\varepsilon,\eta},e)|^2}{\varepsilon^2}\gamma(x^{\varepsilon,\eta},e)+p\Big(\beta(x^{\varepsilon,\eta},e)-\beta(y^{\varepsilon,\eta},e)\Big)\gamma(x^{\varepsilon,\eta},e)\\
&\quad+\eta^2\Big(\beta^2(x^{\varepsilon,\eta},e)+2x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta},e)+2y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta},e)+\beta^2(y^{\varepsilon,\eta},e)\Big)\gamma(x^{\varepsilon,\eta},e).
\end{aligned}
$$

Now, by (4.4.5), we have $|x^{\varepsilon,\eta}|\le\dfrac{C}{\eta}$ and $|y^{\varepsilon,\eta}|\le\dfrac{C}{\eta}$. Hence, using the hypothesis on β,γ and integrating on $\mathcal{B}_\delta^c$, we get

$$
\tilde{B}^\delta[t^{\varepsilon,\eta},x^{\varepsilon,\eta},U]\le\tilde{B}^\delta[s^{\varepsilon,\eta},y^{\varepsilon,\eta},V]+O(|x^{\varepsilon,\eta}-y^{\varepsilon,\eta}|)+O(\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2})+O(\eta^2). \tag{4.4.33}
$$

Finally, from (4.4.32), (4.4.19) and (4.4.33), we derive inequality (4.4.20). □

4.5 Conclusions

In this chapter, we have studied the optimal stopping problem for a monotonous dynamic risk measure defined by a Markovian BSDE with jumps. We have proven that, under relatively weak hypotheses, the value function is a viscosity solution of an obstacle problem for a partial integro-differential variational inequality. To obtain the uniqueness of the solution under appropriate conditions, we have proven a comparison theorem, based on the nonlocal version of the Jensen Ishii Lemma, which extends some results established in [10] (Section 5.1, Th.3) to the case of a nonlinear BSDE.

The links given in this paper between optimal stopping problems for BSDEs and obstacle problems for PDEs can be extended to a larger class of problems. Among them, we can mention generalized Dynkin games with nonlinear expectation (see [62]), and mixed optimal stopping/stochastic control problems (see [63]). However, the latter case requires to establish a weak dynamic programming principle, which does not follow from the flow property of reflected BSDEs only, and needs rather sophisticated techniques.

4.6 Appendix

4.6.1 Some useful estimates

Let $T > 0$ be a fixed terminal time.

A map $f : [0,T] \times \Omega \times \mathbb{R}^2 \times L^2_\nu \to \mathbb{R}; (t, \omega, y, z, k) \mapsto f(t, \omega, y, z, k)$ is said to be a *Lipschitz driver* if it is predictable, uniformly Lipchitz with respect to y, z, k and such that $f(t, 0, 0, 0) \in \mathbb{H}^2$.

Let $\xi^1_t, \xi^2_t \in \mathcal{S}^2$. Let f^1, f^2 be two admissible Lipschitz drivers with Lipchitz constant C. For $i = 1, 2$, let $\mathcal{E}^i$ be the f^i-conditional expectation associated with driver f^i, and let (Y^i_t) be the adapted process defined for each $t \in [0,T]$,

$$Y^i_t := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathcal{E}^i_{t,\tau}(\xi^i_\tau). \tag{4.6.1}$$

Proposition 4.6.1. *For $s \in [0,T]$, denote $\overline{Y}_s = Y^1_s - Y^2_s$, $\overline{\xi}_s = \xi^1_s - \xi^2_s$ and $\overline{f}_s = \sup_{y,z,k} |f^1(s, y, z, k) - f^2(s, y, z, k)|$. Let $\eta, \beta > 0$ be such that $\beta \geq \dfrac{3}{\eta} + 2C$ and $\eta \leq \dfrac{1}{C^2}$. Then for each t, we have:*

$$e^{\beta t}\overline{Y}^2_t \leq e^{\beta T}(\mathbb{E}[\sup_{s \geq t} \overline{\xi_s}^2 | \mathcal{F}_t] + \eta \mathbb{E}[\int_t^T \overline{f}^2_s ds | \mathcal{F}_t]) \ a.s. \tag{4.6.2}$$

Proof. For $i = 1, 2$ and for each $\tau \in \mathcal{T}_0$, let $(X^{i,\tau}, \pi^{i,\tau}_s, l^{i,\tau}_s)$ be the solution of the BSDE associated with driver f^i, terminal time τ and terminal condition ξ^i_τ. Set $\overline{X}^\tau_s = X^{1,\tau}_s - X^{2,\tau}_s$.

By a priori estimate on BSDEs (see Proposition $A.4$ in [138]), we have:

$$\begin{aligned} e^{\beta t}(\overline{X}^\tau_t)^2 \leq e^{\beta T}\mathbb{E}[\overline{\xi}^2_\tau | \mathcal{F}_t] + \eta \mathbb{E}[\int_t^T e^{\beta s}(f^1(s, X^{2,\tau}_s, \pi^{2,\tau}_s, l^{2,\tau}_s) \\ - f^2(s, X^{2,\tau}_s, \pi^{2,\tau}_s, l^{2,\tau}_s))^2 ds | \mathcal{F}_t] \quad \text{a.s.} \end{aligned} \tag{4.6.3}$$

from which we derive that

$$e^{\beta t}(\overline{X}_t^{\tau})^2 \leq e^{\beta T}(\mathbb{E}[\sup_{s\geq t} \overline{\xi}_s^2 | \mathcal{F}_t] + \eta \mathbb{E}[\int_t^T \overline{f}_s^2 ds | \mathcal{F}_t]). \tag{4.6.4}$$

Now, by definition of Y^i, we have $Y_t^i = \operatorname{ess\,sup}_{\tau\geq t} X_t^{i,\tau}$ a.s. for $i = 1, 2$. We thus get $|\overline{Y}_t| \leq \operatorname{ess\,sup}_{\tau\geq t} |\overline{X}_t^{\tau}|$ a.s. The result follows. □

Let $\xi_t \in \mathcal{S}^2$. Let f be a Lipschitz driver with Lipschitz constant $C > 0$. Set

$$Y_t := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathcal{E}_{t,\tau}(\xi_\tau) \tag{4.6.5}$$

where $\mathcal{E}$ is the f-conditional expectation associated with driver f.

Proposition 4.6.2. *Let $\eta, \beta > 0$ be such that $\beta \geq \frac{3}{\eta} + 2C$ and $\eta \leq \frac{1}{C^2}$. Then for each t, we have:*

$$e^{\beta t} Y_t^2 \leq e^{\beta T}(\mathbb{E}[\sup_{s\geq t} {\xi_s}^2 | \mathcal{F}_t] + \eta \mathbb{E}[\int_t^T f(s,0,0,0)^2 ds | \mathcal{F}_t]) \ a.s. \tag{4.6.6}$$

Proof. Let X_t^{τ} be the solution of the BSDE associated with driver f, terminal time τ and terminal condition ξ_τ. By applying inequality (4.6.3) with $f^1 = f$, $\xi_1 = \xi$, $f^2 = 0$ and $\xi^2 = 0$, we get:

$$e^{\beta t}(X_t^{\tau})^2 \leq e^{\beta T} \mathbb{E}[\xi_\tau^2 | \mathcal{F}_t] + \eta \mathbb{E}[\int_t^T e^{\beta s}(f(s,0,0,0))^2 | \mathcal{F}_t]. \tag{4.6.7}$$

The result follows. □

Remark 4.6.3. If the drivers satisfy Assumption 3.1 in [138], then Y (resp. Y^i) is the solution of the RBSDE associated with driver f (resp.f^i) and obstacle ξ (resp. ξ^i). Hence the above estimates provide some new estimates on RBSDEs. Note that η and β are universal constants, i.e. they do not depend on T, $\xi, \xi^1, \xi^2, f, f^1, f^2$. This was not the case for the estimates given in the previous literature (see e.g. [71]).

4.6.2 Some properties of the value function u

We prove below the continuity and polynomial growth of the function u defined by (4.2.8).

Lemma 4.6.4. *The function u is continuous in (t, x).*

Proof. It is sufficient to show that, when $(t_n, x_n) \to (t, x)$, $|u(t_n, x_n) - u(t, x)| \to 0$.
Let $\bar{h}$ be the map defined by $\bar{h}(t, x) = h(t, x)$ for $t < T$ and $\bar{h}(T, x) = g(x)$, so that, for each (t, x), we have $\xi_s^{t,x} = \bar{h}(s, X_s^{t,x})$, $0 \leq s \leq T$ a.s. By applying Proposition 4.6.1 with $X_s^1 = X_s^{t_n,x_n}$, $X_s^2 = X_s^{t,x}$, $f^1(s,\omega,y,z,q) := \mathbf{1}_{[t,T]}(s) f(s, X_s^{t,x}(\omega), y, z, q)$ and
$f^2(s,\omega,y,z,q) := \mathbf{1}_{[t_n,T]}(s) f(s, X_s^{t_n,x_n}(\omega), y, z, q)$, we obtain:

$$|u(t_n, x_n) - u(t, x)|^2 \leq K_{C,T} \mathbb{E}[\sup_{0\leq s\leq T} |\overline{h}(s, X_s^{t_n,x_n}) - \overline{h}(s, X_s^{t,x})|^2 + \int_0^T (\overline{f}_s^n)^2],$$

where

$$\begin{cases} K_{C,T} := e^{(3C^2+2C)T} \max(1, \dfrac{1}{C^2}) \\ \overline{f}^n_s(\omega) := \sup_{y,z,q} |\mathbf{1}_{[t,T]} f(s, X^{t,x}_s(\omega), y, z, q) - \mathbf{1}_{[t_n,T]} f(s, X^{t_n,x_n}_s(\omega), y, z, q)|. \end{cases}$$

The continuity of u is then a consequence of the following convergences as $n \to \infty$:

$$\mathbb{E}(\sup_{0 \le s \le T} |\overline{h}(s, X^{t,x}_s) - \overline{h}(s, X^{t_n}_s(x_n))|^2) \to 0$$
$$\mathbb{E}[\int_0^T (\overline{f}^n_s)^2 ds] \to 0,$$

which follow from the Lebesgue's theorem, using the continuity assumptions and polynomial growth of f and h. □

Lemma 4.6.5. *The function u has at most polynomial growth at infinity.*

Proof. By applying Prop. 4.6.2 , we obtain the following estimate:

$$u(t,x)^2 \le K_{C,T} (\mathbb{E}(\int_0^T f(s, X^{t,x}_s, 0, 0, 0)^2 ds + \sup_{0 \le s \le T} \overline{h}(s, X^{t,x}_s)^2). \tag{4.6.8}$$

Using now the hypothesis of polynomial growth on f, h, g and the standard estimate

$$\mathbb{E}[\sup_{0 \le s \le T} |X^{t,x}_s|^2] \le C'(1 + x^2),$$

we derive that there exist $\bar{C} \in \mathbb{R}$ and $p \in \mathbb{N}$ such that $|u(t,x)| \le \bar{C}(1+x^p)$, $\forall t \in [0,T]$, $\forall x \in \mathbb{R}$. □

Remark 4.6.6. By (4.6.8), if $(t,x) \mapsto f(t,x,0,0)$, h and g are bounded, then u is bounded.

4.6.3 An extension of the comparison result for BSDEs with jumps

We provide here an extension of the comparison theorem for BSDEs given in [137] which formally states that if two drivers f_1, f_2 satisfy $f_1 \ge f_2 + \varepsilon$, then the associated solutions X^1 and X^2 satisfy $X^1_0 > X^2_0$.

Proposition 4.6.7. *Let $t_0 \in [0,T]$ and let θ be a stopping time such that $\theta > t_0$ a.s. Let ξ_1 and $\xi_2 \in L^2(\mathcal{F}_\theta)$. Let f_1 be a driver. Let f_2 be a Lipschitz driver. For $i = 1,2$, let (X^i_t, π^i_t, l^i_t) be a solution in $S^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ of the BSDE*

$$-dX^i_t = f_i(t, X^i_t, \pi^i_t, l^i_t) dt - \pi^i_t dW_t - \int_{\mathbb{R}^*} l^i_t(u) \tilde{N}(dt, du); \quad X^i_\theta = \xi_i. \tag{4.6.9}$$

Assume that there exists a bounded predictable process (γ_t) such that $dt \otimes dP \otimes \nu(de)$-a.s. $\gamma_t(e) \ge -1$ and $|\gamma_t(e)| \le C(1 \wedge |e|)$, and such that

$$f_2(t, X^2_t, \pi^2_t, l^1_t) - f_2(t, X^2_t, \pi^2_t, l^2_t) \ge \langle \gamma_t \,,\, l^1_t - l^2_t \rangle_\nu, \;\; t_0 \le t \le \theta, \;\; dt \otimes dP \;\; a.s. \tag{4.6.10}$$

Suppose also that

$$\xi_1 \geq \xi_2 \;\; a.s.$$
$$f_1(t, X_t^1, \pi_t^1, l_t^1) \geq f_2(t, X_t^1, \pi_t^1, l_t^1) + \varepsilon, \;\; t_0 \leq t \leq \theta, \;\; dt \otimes dP \;\; a.s.$$

where ε is a real constant. Then,

$$X_{t_0}^1 - X_{t_0}^2 \geq \varepsilon\alpha \quad \text{a.s.}$$

where α is a non negative $\mathcal{F}_{t_0}$-measurable r.v. which does not depend on ε, with $P(\alpha > 0) > 0$.

Proof. From inequality (4.22) in the proof of the Comparison Theorem in [137], we derive that

$$X_{t_0}^1 - X_{t_0}^2 \geq e^{-CT}\mathbb{E}\left[\int_{t_0}^{\theta} H_{t_0,s}\,\varepsilon\, ds|\mathcal{F}_{t_0}\right] \quad \text{a.s.}\,,$$

where C is the Lipschitz constant of f_2, and $(H_{t_0,s})_{s\in[t_0,T]}$ is the square integrable non negative martingale satisfying

$$dH_{t_0,s} = H_{t_0,s^-}\left[\beta_s dW_s + \int_{\mathbb{R}^*} \gamma_s(u)\tilde{N}(ds,du)\right]; \;\; H_{t_0,t_0} = 1,$$

(β_s) being a predictable process bounded by C. We get

$$X_{t_0}^1 - X_{t_0}^2 \geq e^{-CT}\varepsilon\,\mathbb{E}\left[H_{t_0,\theta}\,(\theta - t_0)|\mathcal{F}_{t_0}\right] \quad \text{a.s.}$$

Since $\theta > t_0$ a.s., we have $H_{t_0,\theta}\,(\theta - t_0) \geq 0$ a.s. and $P(H_{t_0,\theta}\,(\theta - t_0) > 0) > 0$. Setting $\alpha := e^{-CT}\,\mathbb{E}\left[H_{t_0,\theta}\,(\theta - t_0)|\mathcal{F}_{t_0}\right]$, the result follows. $\square$

Chapter 5

Generalized Dynkin Games and DRBSDEs with Jumps

Abstract. We introduce a new game problem which can be seen as a generalization of the classical Dynkin game problem to the case of a nonlinear $\mathcal{E}^g$-conditional expectation , induced by a Backward Stochastic Differential Equation (BSDE) with jumps. Let ξ, ζ be two RCLL adapted processes with $\xi \le \zeta$. The criterium is given by

$$\mathcal{J}_{\tau,\sigma} = \mathcal{E}^g_{0,\tau\wedge\sigma}\left(\xi_\tau \mathbf{1}_{\{\tau\le\sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma<\tau\}}\right)$$

where τ and σ are stopping times valued in $[0,T]$. Under Mokobodski's condition, we establish the existence of a value function for this game, i.e. $\inf_\sigma \sup_\tau \mathcal{J}_{\tau,\sigma} = \sup_\tau \inf_\sigma \mathcal{J}_{\tau,\sigma}$. This value can be characterized via a doubly reflected BSDE. Using this characterization, we provide some new results on these equations, such as comparison theorems and a priori estimates. When ξ and ζ are left upper semicontinuous along stopping times, we prove the existence of a saddle point. We also study a generalized mixed game problem when the players have two actions: continuous control and stopping. We then study the generalized Dynkin game in a Markovian framework and its links with parabolic partial integro-differential variational inequalities with two obstacles.

5.1 Introduction

The classical Dynkin game has been widely studied: see e.g. Bismut [23], Alario-Nazaret et al. [1], Kobylanski et al. [105]. Let ξ, ζ be two Right Continuous Left-Limited (RCLL) adapted processes with $\xi \le \zeta$ and $\xi_T = \zeta_T$ a.s. The criterium is given, for each pair (τ, σ) of stopping times valued in $[0,T]$, by

$$J_{\tau,\sigma} = E\left(\xi_\tau \mathbf{1}_{\{\tau\le\sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma<\tau\}}\right).$$

Under Mokobodski's condition, which states that there exist two supermartingales such that their difference is between ξ and ζ, there exists a value function for the Dynkin game, i.e. $\inf_\sigma \sup_\tau J_{\tau,\sigma} = \sup_\tau \inf_\sigma J_{\tau,\sigma}$. When the barriers ξ, ζ are left upper semicontinuous, and $\xi_t < \zeta_t$, $t < T$, there exists a saddle point.

Using a change of variable, these results can be generalized to the case of a criterium with an instantaneous reward process (g_t), of the form

$$E\left(\int_0^{\tau\wedge\sigma} g_s ds + \xi_\tau \mathbf{1}_{\{\tau\leq\sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma<\tau\}}\right). \tag{5.1.1}$$

In the Brownian case and when (ξ_t) and (ζ_t) are continuous processes, Cvitanić and Karatzas have established links between these Dynkin games and doubly reflected Backward stochastic differential equations with driver process (g_t) and barriers (ξ_t) and (ζ_t) (see [52]).

In this chapter, we introduce a new game problem, which generalizes the classical Dynkin game to the case of $\mathcal{E}^g$-conditional expectations. Nonlinear expectations induced by BSDEs have been introduced by S. Peng [130] in the Brownian framework . Given a Lipschitz driver $g(t, y, z)$, a stopping time $\tau \leq T$ and a square integrable $\mathcal{F}_\tau$-measurable random variable η, the associated conditional $\mathcal{E}^g$-expectation process denoted by $(\mathcal{E}^g_{t,\tau}(\eta), 0 \leq t \leq \tau)$ is defined as the solution of the BSDE with driver g, terminal time τ and terminal condition η. The extension to the case with jumps is studied in e.g. [137]. We consider here a *generalized Dynkin game*, where the criterium is given, for each pair (τ, σ) of stopping times valued in $[0, T]$, by

$$\mathcal{J}_{\tau,\sigma} = \mathcal{E}^g_{0,\tau\wedge\sigma}\left(\xi_\tau \mathbf{1}_{\{\tau\leq\sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma<\tau\}}\right)$$

with ξ, ζ two RCLL adapted processes satisfying $\xi \leq \zeta$.

When the driver g does not depend on the solution, that is, when it is given by a process (g_t), the criterium $\mathcal{J}_{\tau,\sigma}$ coincides with (5.1.1). It is well-known that in this case, under Mokobodski's condition, the value function for the Dynkin game problem can be characterized as the solution of the Doubly Reflected BSDE (DRBSDE) associated with driver process (g_t) and barriers (ξ_t) and (ζ_t) (see e.g. [52, 89, 112]). We generalize this result to the case of a nonlinear driver $g(t, y, z, k)$ depending on the solution. More precisely, under Mokobodski's condition, we prove that

$$\inf_\sigma \sup_\tau \mathcal{J}_{\tau,\sigma} = \sup_\tau \inf_\sigma \mathcal{J}_{\tau,\sigma}$$

and we characterize this common value function as the solution of the DRBSDE associated with driver g and barriers ξ and ζ. Moreover, when ξ and ζ are left-upper semicontinuous along stopping times, we show that there exist saddle points. Note that, contrary to the previous existence results given in the case of classical Dynkin games, we do not assume the strict separability of the barriers. We point out that the approach used in the classical case cannot be adapted to our case because of the nonlinearity of the driver.

Using the characterization of the solution of a DRBSDE as the value function of a generalized Dynkin game, we prove some results on DRBSDEs, such as a comparison and a strict comparison theorem, and a priori estimates, which complete those given in the previous literature.

Moreover, we introduce a new mixed game problem expressed in terms of $\mathcal{E}^g$-conditional expectations, when the players have two possible actions: continuous control and stopping. The first (resp. second) player chooses a pair (u, τ) (resp. (v, σ)) of control and stopping time, and aims at maximizing (resp. minimizing) the criterium. This problem has been studied by [41] and [89] in the classical case, that is when the criterium is given, for each quadruple (u, τ, v, σ) of controls and stopping times, by

$$E_{Q^{u,v}}\left[\int_0^{\tau\wedge\sigma} c(t,u_t,v_t)dt + \xi_\tau \mathbf{1}_{\{\tau\leq\sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma<\tau\}}\right], \tag{5.1.2}$$

where $Q^{u,v}$ are a priori probability measures and $c(t,u_t,v_t)$ represents the instantenous reward associated with controls u,v. In [89], Hamadène and Lepeltier have established some links between this mixed game problem and DRBSDEs, when ξ and ζ are regular. Here, we consider a *generalized mixed game* problem, where, for a given family of Lipschitz drivers $g^{u,v}$, the criterium is defined by

$$\mathcal{E}^{u,v}_{0,\tau\wedge\sigma}\left(\xi_\tau \mathbf{1}_{\{\tau\leq\sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma<\tau\}}\right), \tag{5.1.3}$$

where $\mathcal{E}^{u,v}$ denotes the $g^{u,v}$-conditional expectation. Note that the criterium (5.1.3) corresponds to a criterium of the form (5.1.2) when the drivers $g^{u,v}$ are linear. We generalize the results of [89] to the case of nonlinear expectations and irregular payoffs ξ and ζ. We provide some sufficient conditions which ensure the existence of a value function of our *generalized mixed game* problem, and show that the common value function can be characterized as the solution of a DRBSDE. Under additional regularity assumptions on ξ and ζ, we prove the existence of saddle points.

The chapter is organized as follows. In Section 5.2 we introduce notation and definitions and provide some preliminary results. In Section 5.3, we consider a classical Dynkin game problem and study its links with a DRBSDE associated with a driver which does not depend on the solution. We provide an existence result for this game problem under relatively weak assumptions on ξ and ζ. Note that Section 5.3, although it contains new results, mainly situates our work and introduces the tools used in the sequel. In Section 5.4, we introduce the *generalized Dynkin game* with $\mathcal{E}^g$-conditional expectation. We prove the existence of a value function for this game problem. We show that the common value function can be characterized as the solution of a nonlinear DRBSDE with jumps and RCLL barriers ξ and ζ. We then study a *generalized mixed game* problem when the players have two actions: continuous control and stopping. In Section 5.5, using the characterization of the solution of a DRBSDE as the value function of a *generalized Dynkin game*, we prove comparison theorems and a priori estimates for DRBSDEs. Finally, we address the *generalized Dynkin game* in the Markovian case and its links with parabolic partial integro-differential variational inequalities (PIDVI) with two obstacles in Section 5.6. The value function of the generalized Dynkin game is a viscosity solution of a PIDVI. A uniqueness result is obtained under additional assumptions.

5.2 Notation and definitions

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let W be a one-dimensional Brownian motion. Let $\mathbf{E} := I\!R^*$ and $\mathcal{B}(\mathbf{E})$ be its Borelian filtration. Suppose that it is equipped with a σ-finite positive measure ν and let $N(dt,de)$ be a Poisson random measure with compensator $\nu(de)dt$. Let $\tilde{N}(dt,de)$ be its compensated process. Let $I\!F = \{\mathcal{F}_t, t\geq 0\}$ be the completed natural filtration associated with W and N.

Notation. Let $\mathcal{P}$ be the predictable σ-algebra on $[0,T] \times \Omega$.

For each $T > 0$, we use the following notation: $L^2(\mathcal{F}_T)$ is the set of random variables ξ which are $\mathcal{F}_T$-measurable and square integrable; $I\!H^2$ is the set of real-valued predictable processes ϕ such that $\|\phi\|^2_{I\!H^2} := E\left[\int_0^T \phi_t^2 dt\right] < \infty$; $\mathcal{S}^2$ denotes the set of real-valued RCLL adapted processes ϕ such that $\|\phi\|^2_{\mathcal{S}^2} := E(\sup_{0 \le t \le T} |\phi_t|^2) < \infty$; $\mathcal{A}^2$ (resp. $\mathcal{A}^1$) is the set of real-valued non decreasing RCLL predictable processes A with $A_0 = 0$ and $E(A_T^2) < \infty$ (resp. $E(A_T) < \infty$). We also introduce the following spaces:

- L^2_ν is the set of Borelian functions $\ell : \mathbf{E} \to I\!R$ such that $\int_{\mathbf{E}} |\ell(e)|^2 \nu(de) < +\infty$.
  The set L^2_ν is a Hilbert space equipped with the scalar product
  $\langle \ell', \ell \rangle_\nu := \int_{\mathbf{E}} \ell(e)\ell'(e)\nu(de)$ for all $\ell, \ell' \in L^2_\nu \times L^2_\nu$, and the norm $\|\ell\|^2_\nu := \int_{\mathbf{E}} |\ell(e)|^2 \nu(de)$.
- $I\!H^2_\nu$ is the set of all mappings $l : [0,T] \times \Omega \times \mathbf{E} \to I\!R$ that are $\mathcal{P} \otimes \mathcal{B}(\mathbf{E})/\mathcal{B}(I\!R)$ measurable and satisfy $\|l\|^2_{I\!H^2_\nu} := E\left[\int_0^T \|l_t\|^2_\nu \, dt\right] < \infty$, where $l_t(\omega, e) = l(t, \omega, e)$ for all $(t, \omega, e) \in [0,T] \times \Omega \times \mathbf{E}$.

Moreover, $\mathcal{T}_0$ is the set of stopping times τ such that $\tau \in [0,T]$ a.s. and for each S in $\mathcal{T}_0$, we denote by $\mathcal{T}_S$ the set of stopping times τ such that $S \le \tau \le T$ a.s.

Definition 5.2.1 (Driver, Lipschitz driver). *A function g is said to be a* driver *if*

- $g : [0,T] \times \Omega \times I\!R^2 \times L^2_\nu \to I\!R$
  $(\omega, t, y, z, \kappa(\cdot)) \mapsto g(\omega, t, y, z, k(\cdot))$ *is* $\mathcal{P} \otimes \mathcal{B}(I\!R^2) \otimes \mathcal{B}(L^2_\nu)-$ *measurable,*
- $g(., 0, 0, 0) \in I\!H^2$.

A driver g is called a Lipschitz driver *if moreover there exists a constant $C \ge 0$ such that $dP \otimes dt$-a.s., for each (y_1, z_1, k_1), (y_2, z_2, k_2),*

$$|g(\omega, t, y_1, z_1, k_1) - g(\omega, t, y_2, z_2, k_2)| \le C(|y_1 - y_2| + |z_1 - z_2| + \|k_1 - k_2\|_\nu).$$

Recall that for each Lipschitz driver g, and each terminal condition $\xi \in L^2(\mathcal{F}_T)$, there exists a unique solution $(X, \pi, l) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu$ satisfying

$$-dX_t = g(t, X_{t^-}, \pi_t, l_t(\cdot))dt - \pi_t dW_t - \int_{\mathbf{E}} l_t(e)\tilde{N}(dt, de); \qquad X_T = \xi. \tag{5.2.1}$$

The solution is denoted by $(X(\xi, T), \pi(\xi, T), l(\xi, T))$.

This result can be extended when the terminal time T is replaced by a stopping time $\tau \in \mathcal{T}_0$ and when ξ is replaced by a random variable $\eta \in L^2(\mathcal{F}_\tau)$. The solution $X_.(\eta, \tau)$ corresponds to the so-called $\mathcal{E}^g$-conditional expectation of η, denoted by $\mathcal{E}^g_{.,\tau}(\eta)$.

Definition 5.2.2. *Let $A = (A_t)_{0 \le t \le T}$ and $A' = (A'_t)_{0 \le t \le T}$ belonging to $\mathcal{A}^1$. We say that the random measures dA_t and dA'_t are* mutually singular*, and we write $dA_t \perp dA'_t$, if there exists $D \in \mathcal{P}$ such that:*

$$E[\int_0^T \mathbf{1}_{D^c} dA_t] = E[\int_0^T \mathbf{1}_D dA'_t] = 0,$$

which can also be written as $\int_0^T \mathbf{1}_{D_t^c} dA_t = \int_0^T \mathbf{1}_{D_t} dA'_t = 0$ a.s., where for each $t \in [0,T]$, D_t is the section at time t *of D, that is, $D_t := \{\omega \in \Omega, (t, \omega) \in D\}$.*

We define now DRBSDEs with jumps, for which the solution is constrained to stay between two given RCLL processes called barriers $\xi \leq \zeta$. Two nondecreasing processes A and A' are introduced in order to push the solution Y above ξ and below ζ in a minimal way. This *minimality property* of A and A' is ensured by the *Skorohod* conditions (condition (iii) below) together with the additional constraint $dA_t \perp dA'_t$ (condition (ii)).

Definition 5.2.3 (Doubly Reflected BSDEs with Jumps). *Let $T > 0$ be a fixed terminal time and g be a Lipschitz driver. Let ξ and ζ be two adapted RCLL processes with $\zeta_T = \xi_T$ a.s., $\xi \in \mathcal{S}^2$, $\zeta \in \mathcal{S}^2$, $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s.*

A process $(Y, Z, k(.), A, A')$ in $\mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times \mathcal{A}^2 \times \mathcal{A}^2$ is said to be a solution of the doubly reflected BSDE (DRBSDE) associated with driver g and barriers ξ, ζ if

$$-dY_t = g(t, Y_t, Z_t, k_t(\cdot))dt + dA_t - dA_t^{'} - Z_t dW_t - \int_{\mathbf{E}} k_t(e)\tilde{N}(dt, de);\ Y_T = \xi_T, \tag{5.2.2}$$

with

$$(i)\ \ \xi_t \leq Y_t \leq \zeta_t,\ 0 \leq t \leq T \ a.s.,$$

$$(ii)\ \ dA_t \perp dA'_t$$

$$(iii) \int_0^T (Y_t - \xi_t)dA_t^c = 0 \ a.s. \ and \ \int_0^T (\zeta_t - Y_t)dA_t^{'c} = 0 \ a.s.$$

$$\Delta A_\tau^d = \Delta A_\tau^d \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}} \ and \ \Delta A_\tau^{'d} = \Delta A_\tau^{'d} \mathbf{1}_{\{Y_{\tau^-} = \zeta_{\tau^-}\}} \ a.s.\ \forall \tau \in \mathcal{T}_0 \ predictable.$$

Here A^c (resp $A^{'c}$) denotes the continuous part of A (resp $A^{'}$) and A^d (resp $A^{'d}$) its discontinuous part.

Remark 5.2.4. Note that when A and A' are not required to be mutually singular, they can simultaneously increase on $\{\xi_{t^-} = \zeta_{t^-}\}$. The constraint $dA_t \perp dA'_t$ will allow us to obtain the uniqueness of the non decreasing RCLL processes A and A', without the usual strict separability condition $\xi < \zeta$ (see Theorem 5.3.5).

We introduce the following definition.

Definition 5.2.5. *A progressively measurable process (ϕ_t) (resp. integrable) is said to be* left-upper semicontinuous (l.u.s.c.) along stopping times *(resp. along stopping times in expectation) if for all $\tau \in \mathcal{T}_0$ and for each non decreasing sequence of stopping times (τ_n) such that $\tau^n \uparrow \tau$ a.s.,*

$$\phi_\tau \geq \limsup_{n\to\infty} \phi_{\tau_n} \quad a.s. \quad (resp.\ E[\phi_\tau] \geq \limsup_{n\to\infty} E[\phi_{\tau_n}]). \tag{5.2.3}$$

Remark 5.2.6. Note that when (ϕ_t) is left-limited, then (ϕ_t) is *left-upper semicontinuous (l.u.s.c.) along stopping times* if and only if for all predictable stopping time $\tau \in \mathcal{T}_0$, $\phi_\tau \geq \phi_{\tau_-}$ a.s.

5.3 Classical Dynkin games and links with doubly reflected BSDEs with a driver process

In this section, we are given a predictable process $g = (g_t)$ in $\mathbb{H}^2$.

Let ξ and ζ be two adapted processes only supposed to be RCLL with $\zeta_T = \xi_T$ a.s., $\xi \in \mathcal{S}^2$, $\zeta \in \mathcal{S}^2$,

$\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s.
We prove below that the doubly reflected BSDE associated with the driver process (g_t) and the barriers ξ and ζ admits a unique solution $(Y, Z, k(\cdot), A, A')$, which is related to a classical Dynkin game problem. Our results complete previous works on classical Dynkin games and DRBSDEs (see e.g. [52], [86]). In particular, we provide an existence result of saddle points under weaker assumptions than those made in the previous literature.
For any $S \in \mathcal{T}_0$ and any stopping times $\tau, \sigma \in \mathcal{T}_S$, consider the gain (or payoff):

$$I_S(\tau, \sigma) = \int_S^{\sigma \wedge \tau} g(u)du + \xi_\tau \mathbf{1}_{\{\tau \leq \sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma < \tau\}}. \tag{5.3.1}$$

For any $S \in \mathcal{T}_0$, the upper and lower value functions at time S are defined respectively by

$$\overline{V}(S) := \operatorname*{essinf}_{\sigma \in \mathcal{T}_S} \operatorname*{\mathit{ess}\,sup}_{\tau \in \mathcal{T}_S} E[I_S(\tau, \sigma) | \mathcal{F}_S] \tag{5.3.2}$$

$$\underline{V}(S) := \operatorname*{\mathit{ess}\,sup}_{\tau \in \mathcal{T}_S} \operatorname*{essinf}_{\sigma \in \mathcal{T}_S} E[I_S(\tau, \sigma) | \mathcal{F}_S]. \tag{5.3.3}$$

We clearly have the inequality $\underline{V}(S) \leq \overline{V}(S)$ a.s. By definition, we say that there *exists a value function* at time S for the Dynkin game if $\overline{V}(S) = \underline{V}(S)$ a.s.

Definition 5.3.1 (S-saddle point). *Let $S \in \mathcal{T}_0$. A pair $(\tau^*, \sigma^*) \in \mathcal{T}_S^2$ is called an* S-saddle point *if for each $(\tau, \sigma) \in \mathcal{T}_S^2$, we have*

$$E[I_S(\tau, \sigma^*) | \mathcal{F}_S] \leq E[I_S(\tau^*, \sigma^*) | \mathcal{F}_S] \leq E[I_S(\tau^*, \sigma) | \mathcal{F}_S] \;\; a.s.$$

We introduce the following RCLL adapted processes which depend on the process g:

$$\tilde{\xi}_t^g := \xi_t - E[\xi_T + \int_t^T g(s)ds | \mathcal{F}_t], \quad \tilde{\zeta}_t^g := \zeta_t - E[\zeta_T + \int_t^T g(s)ds | \mathcal{F}_t], \quad 0 \leq t \leq T. \tag{5.3.4}$$

They satisfy the property $\tilde{\xi}_T^g = \tilde{\zeta}_T^g = 0$ a.s. Moreover, this change of variables allows us to get rid of the term $\int g(t)dt$, and thus to simplify the notation.

Definition 5.3.2. *A nonnegative process $\phi_. = (\phi_t)$ valued in $[0, +\infty]$ is said to be a* strong supermartingale *if for any $\theta, \theta' \in \mathcal{T}_0$ such that $\theta \geq \theta'$ a.s., $E[\phi_\theta \,|\, \mathcal{F}_{\theta'}] \leq \phi_{\theta'}$ a.s.*

Lemma 5.3.3. *There exist two* strong supermartingales *(J_t^g) and $(J_t^{'g})$ valued in $[0, +\infty]$ such that for all $\theta \in \mathcal{T}_0$,*

$$J_\theta^g = \operatorname*{\mathit{ess}\,sup}_{\tau \in \mathcal{T}_\theta} E\left[J_\tau^{\prime g} + \tilde{\xi}_\tau^g | \mathcal{F}_\theta\right] \;\; a.s. \quad \text{and} \quad J_\theta^{'g} = \operatorname*{\mathit{ess}\,sup}_{\sigma \in \mathcal{T}_\theta} E\left[J_\sigma^g - \tilde{\zeta}_\sigma^g | \mathcal{F}_\theta\right] \;\; a.s., \tag{5.3.5}$$

and satisfying the following minimality property: $J_.^g$ and $J_.^{'g}$ are the smallest strong supermartingales *valued in $[0, +\infty]$ such that*

$$J_.^g \geq J_.^{'g} + \tilde{\xi}_.^g \;\; \text{and} \; J_.^{'g} \geq J_.^g - \tilde{\zeta}_.^g. \tag{5.3.6}$$

If $J_0^g < +\infty$ and $J_0^{'g} < +\infty$, $J_.^g$ and $J_.^{\prime g}$ are indistinguishable from RCLL supermartingales.

The proof is given in the Appendix. Using this lemma, we derive the following result.

Theorem 5.3.4. *Let ξ and ζ be two adapted RCLL processes in $\mathcal{S}^2$ with $\zeta_T = \xi_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. Suppose that $J^g, J'^g \in \mathcal{S}^2$. Let $\overline{Y}$ be the RCLL adapted process defined by*

$$\overline{Y}_t := J_t^g - J_t^{'g} + E[\xi_T + \int_t^T g(s)ds|\mathcal{F}_t];\ 0 \leq t \leq T. \tag{5.3.7}$$

There exist $(Z, k, A, A') \in I\!H^2 \times I\!H_\nu^2 \times \mathcal{A}^2 \times \mathcal{A}^2$ such that $(\overline{Y}, Z, k, A, A')$ is a solution of DRBSDE (5.2.2) associated with the driver process $g(t)$.

Proof. By assumption, J^g and $J^{'g}$ are square integrable supermartingales. The process $\overline{Y}$ is thus well defined. By Lemma 5.3.3, we have $J_T^g = J_T'^g$ a.s. Hence, $\overline{Y}_T = \xi_T$ a.s. By the Doob-Meyer decomposition, there exist two square integrable martingales M and M' and two processes B and $B^{'} \in \mathcal{A}^2$ such that:

$$dJ_t^g = dM_t - dB_t \quad ; \quad dJ_t^{'g} = dM_t^{'} - dB_t^{'}. \tag{5.3.8}$$

Set

$$\overline{M}_t := M_t - M_t^{'} + E[\xi_T + \int_0^T g(s)ds|\mathcal{F}_t].$$

By (5.3.8), (5.3.7), we derive $d\overline{Y}_t = d\overline{M}_t - d\alpha_t - g(t)dt$, with $\alpha := B - B^{'}$.
Now, by the martingale representation theorem, there exist $Z \in \mathbb{H}^2$ and $k \in \mathbb{H}_\nu^2$ such that $d\overline{M}_t = Z_t dW_t + \int_{\mathbf{E}} k_t(e)\tilde{N}(de, dt)$. Hence,

$$-d\overline{Y}_t = g(t)dt + d\alpha_t - Z_t dW_t - \int_{\mathbf{E}} k_t(e)\tilde{N}(dt, de).$$

By the optimal stopping theory (see e.g. Proposition B.7 or B.11 in [104]), the process B^c increases only when the value function J^g is equal to the corresponding reward $J^{'g} + \tilde{\xi}^g$. Now, $\{J_t^g = J_t^{'g} + \tilde{\xi}^g\} = \{\overline{Y_t} = \xi_t\}$. Hence, $\int_0^T (\overline{Y_t} - \xi_t)dB_t^c = 0$ a.s. Similarly the process $B^{'c}$ satisfies $\int_0^T (\overline{Y_t} - \zeta_t)dB_t^{'c} = 0$ a.s. and for each predictable stopping time $\tau \in \mathcal{T}_0$ we have
$\Delta B_\tau^d = \mathbf{1}_{J_{\tau-}^g = J_{\tau-}^{'g} + \tilde{\xi}_{\tau-}^g} \Delta B_\tau^d = \mathbf{1}_{\overline{Y}_{\tau-} = \xi_{\tau-}} \Delta B_\tau^d$ a.s. and $\Delta B_\tau^{'d} = \mathbf{1}_{\overline{Y}_{\tau-} = \zeta_{\tau-}} \Delta B_\tau^{'d}$ a.s.
By the canonical decomposition of an RCLL process with integrable variation (see Proposition 5.7.9), there exist $A, A' \in \mathcal{A}^2$ such that $\alpha = A - A'$ with $dA_t \perp dA'_t$. Also, $dA_t \ll dB_t$. Hence, since $\int_0^T \mathbf{1}_{Y_{t-} > \xi_{t-}} dB_t = 0$ a.s., we get $\int_0^T \mathbf{1}_{Y_{t-} > \xi_{t-}} dA_t = 0$ a.s. Similarly, we obtain $\int_0^T \mathbf{1}_{Y_{t-} < \zeta_{t-}} dA'_t = 0$ a.s. The processes A and A' thus satisfy conditions (5.2.2)(iii). □

From this theorem, we derive the following uniqueness and existence result for the DRBSDE associated with the driver process (g_t), as well as the characterization of the solution as the value function of the above Dynkin game problem.

Theorem 5.3.5. *Let ξ and ζ be two adapted RCLL processes in $\mathcal{S}^2$ with $\zeta_T = \xi_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. Suppose that $J_t^g, J_t'^g \in \mathcal{S}^2$. The doubly reflected BSDE (5.2.2) associated with driver process $g(t)$ admits a unique solution (Y, Z, k, A, A') in $\mathcal{S}^2 \times I\!H^2 \times I\!H_\nu^2 \times (\mathcal{A}^2)^2$.*
For each $S \in \mathcal{T}_0$, Y_S is the common value function of the Dynkin game, that is

$$Y_S = \overline{V}(S) = \underline{V}(S) \quad a.s. \tag{5.3.9}$$

Moreover, if the processes A, A' are continuous, then, for each $S \in \mathcal{T}_0$, the pair of stopping times (τ_s^*, σ_s^*) defined by

$$\sigma_S^* := \inf\{t \geq S,\ Y_t = \zeta_t\}; \quad \tau_S^* := \inf\{t \geq S,\ Y_t = \xi_t\} \tag{5.3.10}$$

is an S-saddle point for the Dynkin game problem associated with the gain I_S.

A short proof is given in the Appendix.

Remark 5.3.6. The condition $dA_t \perp dA'_t$ ensures that for each predictable stopping time $\tau \in \mathcal{T}_0$, we have $\Delta A^d_\tau = (\Delta Y_\tau)^-$ and $\Delta A'^d_\tau = (\Delta Y_\tau)^+$ a.s.

We now provide a sufficient condition on ξ and ζ for the existence of saddle points. By the last assertion of Theorem 5.3.5, it is sufficient to give a condition which ensures the continuity of A and A'.

Theorem 5.3.7 (Existence of S-saddle points)**.** *Suppose that the assumptions of Theorem 5.3.5 are satisfied and that ξ and $-\zeta$ are l.u.s.c. along stopping times.*
Let $(Y, Z, k(.), A, A')$ be the solution of DRBSDE (5.2.2)*.*
The processes A and A' are then continuous. Also, for each $S \in \mathcal{T}_0$, the pair of stopping times (τ_S^*, σ_S^*) defined by (5.3.10) *is an S-saddle point.*

Remark 5.3.8. The assumptions made on ξ and ζ are weaker than the ones made in the literature where it is supposed $\xi_t < \zeta_t, t < T$ a.s. (see e.g. [1], [52], [105]).

Proof. By the second assertion of Theorem 5.3.5, it is sufficient to prove that A and A' are continuous. Let $\tau \in \mathcal{T}_0$ be a predictable stopping time. Let us show $\Delta A_\tau = 0$ a.s.
By Remark 5.3.6, we have $\Delta A_\tau = (\Delta Y_\tau)^-$ a.s. Since $dA_t \perp dA'_t$, there exists $D \in \mathcal{P}$ such that: $\int_0^T \mathbf{1}_{D_t^c} dA_t = \int_0^T \mathbf{1}_{D_t} dA'_t = 0$ a.s. We introduce the set $D_\tau := \{\omega,\ (\tau(\omega), \omega) \in D\}$. Since A satisfies the Skorohod condition, we thus have

$$\Delta A_\tau = \mathbf{1}_{D_\tau \cap \{Y_{\tau^-} = \xi_{\tau^-}\}} (Y_{\tau^-} - Y_\tau)^+ = \mathbf{1}_{D_\tau \cap \{Y_{\tau^-} = \xi_{\tau^-}\}} (\xi_{\tau^-} - Y_\tau)^+ \leq \mathbf{1}_{D_\tau \cap \{Y_{\tau^-} = \xi_{\tau^-}\}} (\xi_\tau - Y_\tau)^+$$

a.s. , where the last inequality follows from the inequality $\xi_{\tau^-} \leq \xi_\tau$ a.s. (see Remark 5.2.6). Since $\xi \leq Y$, we derive that $\Delta A_\tau \leq 0$ a.s. Hence, $\Delta A_\tau = 0$ a.s. , and this holds for each predictable stopping time τ. Consequently, A is continuous. Similarly, A' is continuous. The saddle point property of (τ_S^*, σ_S^*) follows from the second assertion of Theorem 5.3.5. □

Definition 5.3.9 (Mokobodski's condition)**.** *Let ξ and ζ be adapted RCLL processes in $\mathcal{S}^2$ with $\zeta_T = \xi_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. Mokobodski's condition is said to be satisfied when there exist two nonnegative RCLL supermartingales H and H' in $\mathcal{S}^2$ such that:*

$$\xi_t \leq H_t - H'_t \leq \zeta_t \quad 0 \leq t \leq T \quad \text{a.s.} \tag{5.3.11}$$

Proposition 5.3.10. *Let $g \in \mathbb{H}^2$. Let ξ and ζ be two adapted RCLL processes in $\mathcal{S}^2$ with $\zeta_T = \xi_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. The following assertions are equivalent:*

(i) $J^g \in \mathcal{S}^2$

(ii) $J^0 \in \mathcal{S}^2$

(iii) *Mokobodski's condition holds.*

(iv) *DRBSDE* (5.2.2) *with driver process* (g_t) *has a solution.*

A short proof is given in the Appendix.

5.4 Generalized Dynkin games and links with nonlinear doubly reflected BSDEs

In this section, we are given a Lipschitz driver g.

Theorem 5.4.1 (Existence and uniqueness for DRBSDEs)**.** *Suppose ξ and ζ are RCLL adapted process in $\mathcal{S}^2$ such that $\xi_T = \zeta_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. Suppose that $J^0 \in \mathcal{S}^2$ (or equivalently suppose that Mokobodski's condition is satisfied).*
Then, DRBSDE (5.2.2) admits a unique solution $(Y, Z, k(.), A, A') \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times (\mathcal{A}^2)^2$.
If ξ and ζ are l.u.s.c. along stopping times, then the processes A and A' are continuous.

The proof is based on classical arguments and is given in the Appendix.

Remark 5.4.2. Note that the solution Y of the DRBSDE (5.2.2) coincides with the value function of the classical Dynkin game (5.3.2) and (5.3.3) with the gain:

$$I_S(\tau, \sigma) = \int_S^{\sigma \wedge \tau} g(u, Y_u, Z_u, k_u) du + \xi_\tau \mathbf{1}_{\{\tau \leq \sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma < \tau\}}. \tag{5.4.1}$$

where Z, k are the associated processes with Y. However, this characterization is not really usable and exploitable because the instantaneous reward $g(u, Y_u, Z_u, k_u)$ depends on the value function Y of the associated Dynkin game.

We now introduce a new game problem, which can be seen as a *generalized Dynkin game* expressed in terms of $\mathcal{E}^g$-conditional expectations.
In order to ensure that the $\mathcal{E}^g$-conditional expectation is non decreasing, we make the following assumption.

Assumption 5.4.3. *Assume that $dP \otimes dt$-a.s for each $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$,*

$$g(t, y, z, k_1) - g(t, y, z, k_2) \geq \langle \gamma_t^{y,z,k_1,k_2} \, , \, k_1 - k_2 \rangle_\nu,$$

$$\text{with} \quad \gamma : [0, T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2 \to L^2_\nu \, ; \, (\omega, t, y, z, k_1, k_2) \mapsto \gamma_t^{y,z,k_1,k_2}(\omega, .)$$

$\mathcal{P} \otimes \mathcal{B}(I\!R^2) \otimes \mathcal{B}((L^2_\nu)^2)$-measurable, bounded, and satisfying $dP \otimes dt \otimes d\nu(e)$-a.s. , for each $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$,

$$\gamma_t^{y,z,k_1,k_2}(e) \geq -1 \quad \text{ and } \quad |\gamma_t^{y,z,k_1,k_2}(e)| \leq \psi(e), \tag{5.4.2}$$

where $\psi \in L^2_\nu$.

For example, this assumption is satisfied if g is $\mathcal{C}^1$ with respect to k with $\nabla_k g \geq -1$ and $|\nabla_k g| \leq \psi$, where $\psi \in L^2_\nu$. Also if g is of the form $g(\omega, t, y, z, k) := \overline{g}(\omega, t, y, z, \int_{\mathbf{E}} k(e)\psi(e)\nu(de))$ where ψ is a nonnegative function in L^2_ν and $\overline{g} : \Omega \times [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is Borelian and non-decreasing with respect to k, then g satisfies Assumption 6.3.9. (see Proposition 5.7.2 in the Appendix for details).

Assumption 6.3.9 ensures the non decreasing property of $\mathcal{E}^g$ by the comparison theorem for BSDEs with jumps (see Theorem 4.2 in [137]). When in (6.3.12), $\gamma_t > -1$, the strict comparison theorem (see Theorem 4.4 in [137]) implies that $\mathcal{E}^g$ is strictly monotonous.

For each $\tau, \sigma \in \mathcal{T}_0$, the *reward* at time $\tau \wedge \sigma$ is given by the random variable

$$I(\tau, \sigma) := \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau}. \tag{5.4.3}$$

Note that $I(\tau, \sigma)$ is $\mathcal{F}_{\tau\wedge\sigma}$-measurable.

Let $S \in \mathcal{T}_0$. For each $\tau \in \mathcal{T}_S$ and $\sigma \in \mathcal{T}_S$, the associated *criterium* is given by $\mathcal{E}^g_{S,\tau\wedge\sigma}(I(\tau,\sigma))$, the $\mathcal{E}^g$-conditional expectation of the reward $I(\tau, \sigma)$.
Recall that $\mathcal{E}^g_{\cdot,\tau\wedge\sigma}(I(\tau,\sigma)) = X^{\tau,\sigma}_\cdot$, where $(X^{\tau,\sigma}_\cdot, \pi^{\tau,\sigma}_\cdot, l^{\tau,\sigma}_\cdot)$ is the solution of the BSDE associated with driver g, terminal time $\tau \wedge \sigma$ and terminal condition $I(\tau, \sigma)$, that is

$$-dX^{\tau,\sigma}_s = g(s, X^{\tau,\sigma}_s, \pi^{\tau,\sigma}_s, l^{\tau,\sigma}_s)ds - \pi^{\tau,\sigma}_s dW_s - \int_{\mathbf{E}} l^{\tau,\sigma}_s(e)\tilde{N}(ds, de); \qquad X^{\tau,\sigma}_{\tau\wedge\sigma} = I(\tau, \sigma).$$

There are two players with antagonistic objectives. At time S, the first player chooses a stopping time τ greater than S, and aims at maximizing the criterium. The other player chooses a stopping time σ greater than S, and aims at the opposite, that is, minimizing the criterium. For each stopping time $S \in \mathcal{T}_0$, the *upper* and *lower value functions* at time S are defined respectively by

$$\overline{V}(S) := \operatorname*{essinf}_{\sigma\in\mathcal{T}_S} \operatorname*{\mathit{ess}\,sup}_{\tau\in\mathcal{T}_S} \mathcal{E}^g_{S,\tau\wedge\sigma}(I(\tau,\sigma)); \tag{5.4.4}$$

$$\underline{V}(S) := \operatorname*{\mathit{ess}\,sup}_{\tau\in\mathcal{T}_S} \operatorname*{essinf}_{\sigma\in\mathcal{T}_S} \mathcal{E}^g_{S,\tau\wedge\sigma}(I(\tau,\sigma)). \tag{5.4.5}$$

We clearly have the inequality $\underline{V}(S) \leq \overline{V}(S)$ a.s.
By definition, we say that there *exists a value function* at time S for the generalized Dynkin game if $\overline{V}(S) = \underline{V}(S)$ a.s.

We now introduce the definition of an S-saddle point for this game problem.

Definition 5.4.4. *Let $S \in \mathcal{T}_0$. A pair $(\tau^*, \sigma^*) \in \mathcal{T}_S^2$ is called an* S-saddle point *for the generalized Dynkin game if for each $(\tau, \sigma) \in \mathcal{T}_S^2$ we have*

$$\mathcal{E}^g_{S,\tau\wedge\sigma^*}(I(\tau,\sigma^*)) \leq \mathcal{E}^g_{S,\tau^*\wedge\sigma^*}(I(\tau^*,\sigma^*)) \leq \mathcal{E}^g_{S,\tau^*\wedge\sigma}(I(\tau^*,\sigma)) \qquad a.s.$$

We first provide a sufficient condition for the existence of an S-saddle point and for the characterization of the common value function as the solution of the DRBSDE.

Lemma 5.4.5. *Suppose that the driver g satisfies Assumption* (6.3.9). *Let ξ and ζ be RCLL adapted processes in $\mathcal{S}^2$ such that $\xi_T = \zeta_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. Suppose that Mokobodski's condition is satisfied.*

Let $(Y, Z, k(\cdot), A, A')$ be the solution of the DRBSDE (5.2.2). Let $S \in \mathcal{T}_0$. Let $(\hat{\tau}, \hat{\sigma}) \in \mathcal{T}_S$. Suppose that $(Y_t,\, S \leq t \leq \hat{\tau})$ is a strong $\mathcal{E}^g$-submartingale and that $(Y_t,\, S \leq t \leq \hat{\sigma})$ is a strong $\mathcal{E}^g$-supermartingale with $Y_{\hat{\tau}} = \xi_{\hat{\tau}}$ and $Y_{\hat{\sigma}} = \zeta_{\hat{\sigma}}$ a.s.
The pair $(\hat{\tau}, \hat{\sigma})$ is then an S-saddle point for the generalized Dynkin game (5.4.4)-(5.4.5) *and*

$$Y_S = \overline{V}(S) = \underline{V}(S) \text{ a.s.}$$

Proof. Since the process $(Y_t,\, S \leq t \leq \hat{\tau} \wedge \hat{\sigma})$ is a strong $\mathcal{E}^g$-martingale (see Definition 5.7.7) and since $Y_{\hat{\tau}} = \xi_{\hat{\tau}}$ and $Y_{\hat{\sigma}} = \zeta_{\hat{\sigma}}$ a.s. , we have

$$Y_S = \mathcal{E}^g_{S,\hat{\tau}\wedge\hat{\sigma}}(Y_{\hat{\tau}\wedge\hat{\sigma}}) = \mathcal{E}^g_{S,\hat{\tau}\wedge\hat{\sigma}}(\xi_{\hat{\tau}}\mathbf{1}_{\hat{\tau}\leq\hat{\sigma}} + \zeta_{\hat{\sigma}}\mathbf{1}_{\hat{\sigma}<\hat{\tau}}) = \mathcal{E}^g_{S,\hat{\tau}\wedge\hat{\sigma}}(I(\hat{\tau}, \hat{\sigma})) \quad \text{a.s.}$$

Let $\tau \in \mathcal{T}_S$. We want to show that for each $\tau \in \mathcal{T}_S$

$$Y_S \geq \mathcal{E}^g_{S,\tau\wedge\hat{\sigma}}(I(\tau, \hat{\sigma})) \quad \text{a.s.} \tag{5.4.6}$$

Since the process $(Y_t,\, S \leq t \leq \tau \wedge \hat{\sigma})$ is a strong $\mathcal{E}^g$-supermartingale, we get

$$Y_S \geq \mathcal{E}^g_{S,\tau\wedge\hat{\sigma}}(Y_{\tau\wedge\hat{\sigma}}) \quad \text{a.s.} \tag{5.4.7}$$

Since $Y \geq \xi$ and $Y_{\hat{\sigma}} = \zeta_{\hat{\sigma}}$ a.s. , we also have

$$Y_{\tau\wedge\hat{\sigma}} = Y_\tau \mathbf{1}_{\tau\leq\hat{\sigma}} + Y_{\hat{\sigma}}\mathbf{1}_{\hat{\sigma}<\tau} \geq \xi_\tau \mathbf{1}_{\tau\leq\hat{\sigma}} + \zeta_{\hat{\sigma}}\mathbf{1}_{\hat{\sigma}<\tau} = I(\tau, \hat{\sigma}) \quad \text{a.s.}$$

By inequality (5.4.7) and the monotonicity property of $\mathcal{E}^g$, we derive inequality (5.4.6).
Similarly, one can show that for each $\sigma \in \mathcal{T}_S$, we have:

$$Y_S \leq \mathcal{E}^g_{S,\hat{\tau}\wedge\sigma}(I(\hat{\tau}, \sigma)) \quad \text{a.s.}$$

The pair $(\hat{\tau}, \hat{\sigma})$ is thus an S-saddle point and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s. □

We now provide an existence result under an additional assumption.

Theorem 5.4.6 (Existence of S-saddle points)**.** *Suppose that g satisfies Assumption* 5.4.3. *Let ξ and ζ be RCLL adapted processes in $\mathcal{S}^2$ such that $\xi_T = \zeta_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. Suppose that Mokobodski's condition is satisfied.*
Let (Y, Z, k, A, A') be the solution of the DRBSDE (5.2.2). Suppose that A, A' are continuous (which is the case if ξ and $-\zeta$ are l.u.s.c. along stopping times). For each $S \in \mathcal{T}_0$, let

$$\tau^*_S := \inf\{t \geq S,\ Y_t = \xi_t\}; \quad \sigma^*_S := \inf\{t \geq S,\ Y_t = \zeta_t\}.$$

$$\overline{\tau}_S := \inf\{t \geq S,\ A_t > A_S\}; \quad \overline{\sigma}_S := \inf\{t \geq S,\ A'_t > A'_S\}.$$

Then, for each $S \in \mathcal{T}_0$, the pairs of stopping times (τ^*_S, σ^*_S) and $(\overline{\tau}_S, \overline{\sigma}_S)$ are S-saddle points for the generalized Dynkin game and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s.
Moreover, $Y_{\sigma^*_S} = \zeta_{\sigma^*_S}$, $Y_{\tau^*_S} = \xi_{\tau^*_S}$, $A_{\tau^*_S} = A_S$ and $A'_{\sigma^*_S} = A'_S$ a.s. The same properties hold for $\overline{\tau}_S, \overline{\sigma}_S$.

Remark 5.4.7. Note that $\sigma^*_S \leq \overline{\sigma}_S$ and $\tau^*_S \leq \overline{\tau}_S$ a.s. Moreover, by Proposition 5.7.8 in the Appendix, $(Y_t,\, S \leq t \leq \overline{\tau}_S)$ is a strong $\mathcal{E}^g$-submartingale and $(Y_t,\, S \leq t \leq \overline{\sigma}_S)$ is a strong $\mathcal{E}^g$-supermartingale.

Proof. Let $S \in \mathcal{T}_0$. Since Y and ξ are right-continuous processes, we have $Y_{\sigma_S^*} = \zeta_{\sigma_S^*}$ and $Y_{\tau_S^*} = \xi_{\tau_S^*}$ a.s. By definition of τ_S^*, for almost every ω, we have $Y_t(\omega) > \xi_t(\omega)$ for each $t \in [S(\omega), \tau_S^*(\omega)[$. Hence, since Y is solution of the DRBSDE, the continuous process A is constant on $[S, \tau_S^*]$ a.s. because A is continuous. Hence, $A_{\tau_S^*} = A_S$ a.s. Similarly, $A'_{\sigma_S^*} = A'_S$ a.s. By Lemma 5.4.5, (τ_S^*, σ_S^*) is an S-saddle point and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s.
It remains to show that $(\overline{\tau}_S, \overline{\sigma}_S)$ is an S-saddle point. By definition of $\overline{\tau}_S$, $\overline{\sigma}_S$, we have $A_{\overline{\tau}_S} = A_S$ a.s. and $A'_{\overline{\sigma}_S} = A'_S$ a.s. because A and A' are continuous and $\overline{\tau}_S$, $\overline{\sigma}_S$ are predictable stopping times. Moreover, since the continuous process A increases only on $\{Y_t = \xi_t\}$, we have $Y_{\overline{\tau}_S} = \xi_{\overline{\tau}_S}$ a.s. Similarly, $Y_{\overline{\sigma}_S} = \zeta_{\overline{\sigma}_S}$ a.s. The result then follows from Lemma 5.4.5. □

In the case of irregular payoffs ξ and ζ, there does not generally exist a saddle point. However, we will now see that it is not necessary to have the existence of an S-saddle point to ensure the existence of a common value function and its characterization as the solution of a DRBSDE.

Theorem 5.4.8 (Existence of the value function)**.** *Suppose that g satisfies Assumption* (6.3.9). *Let ξ and ζ be RCLL adapted processes in $\mathcal{S}^2$ such that $\xi_T = \zeta_T$ a.s. and $\xi_t \le \zeta_t$, $0 \le t \le T$ a.s. Suppose that Mokobodski's condition is satisfied. Let (Y, Z, k, A, A') be the solution of the DRBSDE (5.2.2). Then, there exists a value function for the generalized Dynkin game, and for each stopping time $S \in \mathcal{T}_0$, we have*

$$Y_S = \overline{V}(S) = \underline{V}(S) \quad a.s. \tag{5.4.8}$$

Proof. For each $S \in \mathcal{T}_0$ and for each $\varepsilon > 0$, let τ_S^ε and σ_S^ε be the stopping times defined by

$$\tau_S^\varepsilon := \inf\{t \ge S,\ Y_t \le \xi_t + \varepsilon\}. \tag{5.4.9}$$

$$\sigma_S^\varepsilon := \inf\{t \ge S,\ Y_t \ge \zeta_t - \varepsilon\}. \tag{5.4.10}$$

We first prove two lemmas.

Lemma 5.4.9.

- *We have*

$$Y_{\tau_S^\varepsilon} \le \xi_{\tau_S^\varepsilon} + \varepsilon \quad a.s. \tag{5.4.11}$$

$$Y_{\sigma_S^\varepsilon} \ge \zeta_{\sigma_S^\varepsilon} - \varepsilon \quad a.s. \tag{5.4.12}$$

- *We have $A_{\tau_S^\varepsilon} = A_S$ a.s. and $A'_{\sigma_S^\varepsilon} = A'_S$ a.s.*

Remark 5.4.10. By the second point and Proposition 5.7.8 in the Appendix, the process $(Y_t,\ S \le t \le \tau_S^\varepsilon)$ is a strong $\mathcal{E}^g$-submartingale and the process $(Y_t,\ S \le t \le \sigma_S^\varepsilon)$ is a strong $\mathcal{E}^g$-supermartingale.

□

The first point follows from the definitions of τ_S^ε and σ_S^ε and the right-continuity of ξ, ζ and Y. Let us show the second point. Note that $\tau_S^\varepsilon \in \mathcal{T}_S$ and $\sigma_S^\varepsilon \in \mathcal{T}_S$. Fix $\varepsilon > 0$. For a.e. ω, if $t \in [S(\omega), \tau_S^\varepsilon(\omega)[$, then $Y_t(\omega) > \xi_t(\omega) + \varepsilon$ and hence $Y_t(\omega) > \xi_t(\omega)$. It follows that almost surely, A^c is constant on $[S, \tau_S^\varepsilon]$ and A^d is constant on $[S, \tau_S^\varepsilon[$. Also, $Y_{(\tau_S^\varepsilon)^-} \ge \xi_{(\tau_S^\varepsilon)^-} + \varepsilon$ a.s. Since $\varepsilon > 0$, it follows that $Y_{(\tau_S^\varepsilon)^-} > \xi_{(\tau_S^\varepsilon)^-}$ a.s., which implies that $\Delta A^d_{\tau_S^\varepsilon} = 0$ a.s. Hence, almost surely, A is constant on $[S, \tau_S^\varepsilon]$. Similarly, A' is a.s. constant on $[S, \sigma_S^\varepsilon]$. □

Lemma 5.4.11. *Let $\varepsilon > 0$. For all $S \in \mathcal{T}_0$ and $(\tau, \sigma) \in \mathcal{T}_S^2$, we have*

$$\mathcal{E}_{S,\tau\wedge\sigma_S^\varepsilon}(I(\tau, \sigma_S^\varepsilon)) - K\varepsilon \ \leq \ Y_S \ \leq \ \mathcal{E}^g_{S,\tau_S^\varepsilon\wedge\sigma}(I(\tau_S^\varepsilon, \sigma)) + K\varepsilon \quad a.s.\,, \tag{5.4.13}$$

where K is a positive constant which only depends on T and the Lipschitz constant C of f.

Proof. Let $\tau \in \mathcal{T}_S$. By Remark 5.4.7, the process $(Y_t,\, S \leq t \leq \sigma_S^\varepsilon)$ is a strong $\mathcal{E}^g$-supermartingale. Hence,

$$Y_S \geq \mathcal{E}^g_{S,\tau\wedge\sigma_S^\varepsilon}(Y_{\tau\wedge\sigma_S^\varepsilon}) \quad \text{a.s.} \tag{5.4.14}$$

Since $Y \geq \xi$ and $Y_{\sigma_S^\varepsilon} \geq \zeta_{\sigma_S^\varepsilon} - \varepsilon$ a.s. (see Lemma 5.4.9), we have:

$$Y_{\tau\wedge\sigma_S^\varepsilon} \geq \xi_\tau \mathbf{1}_{\tau\leq\sigma_S^\varepsilon} + (\zeta_{\sigma_S^\varepsilon} - \varepsilon)\mathbf{1}_{\sigma_S^\varepsilon<\tau} \geq I(\tau, \sigma_S^\varepsilon) - \varepsilon \quad \text{a.s.}$$

where the last inequality follows from the definition of $I(\tau, \sigma)$. Hence, using (5.4.14) and the monotonicity property of $\mathcal{E}^g$, we get

$$Y_S \geq \mathcal{E}^g_{S,\tau\wedge\sigma_S^\varepsilon}(I(\tau, \sigma_S^\varepsilon) - \varepsilon) \quad \text{a.s.} \tag{5.4.15}$$

Now, by a priori estimates on BSDEs (see Proposition A.4, [137]), we have

$$|\mathcal{E}^g_{S,\tau\wedge\sigma_S^\varepsilon}(I(\tau, \sigma_S^\varepsilon) - \varepsilon) - \mathcal{E}^g_{S,\tau\wedge\sigma_S^\varepsilon}(I(\tau, \sigma_S^\varepsilon))| \leq K\varepsilon \quad \text{a.s.}$$

It follows that

$$Y_S \ \geq \ \mathcal{E}^g_{S,\tau\wedge\sigma_S^\varepsilon}(I(\tau, \sigma_S^\varepsilon)) - K\varepsilon \quad \text{a.s.}$$

Similarly, one can show that

$$Y_S \ \leq \ \mathcal{E}^g_{S,\tau_S^\varepsilon\wedge\sigma}(I(\tau_S^\varepsilon, \sigma)) + K\varepsilon \quad \text{a.s.}\,,$$

which ends the proof of Lemma 5.4.11.

□

End of proof of Theorem 5.4.8. Using Lemma 5.4.11, we derive that for each $\varepsilon > 0$,

$$ess\sup_{\tau\in\mathcal{T}_s} \mathcal{E}^g_{S,\tau\wedge\sigma_S^\varepsilon}(I(\tau, \sigma_S^\varepsilon)) - K\varepsilon \ \leq \ Y_S \leq ess\inf_{\sigma\in\mathcal{T}_S} \mathcal{E}^g_{S,\tau_S^\varepsilon\wedge\sigma}(I(\tau, \sigma_S^\varepsilon)) + K\varepsilon \ \ \text{a.s.}\,,$$

which implies

$$\overline{V}(S) - K\varepsilon \ \leq \ Y_S \ \leq \ \underline{V}(S) + K\varepsilon \quad \text{a.s.}$$

Since $\underline{V}(S) \leq \overline{V}(S)$ a.s., we get $\underline{V}(S) = Y_S = \overline{V}(S)$ a.s. The proof of Theorem 5.4.8 is thus complete. □

Remark 5.4.12. Inequality (5.4.13) shows that $(\tau_S^\varepsilon, \sigma_S^\varepsilon)$ defined by (5.4.9) and (5.4.10) is an ε'-saddle point at time S with $\varepsilon' = K\varepsilon$.

Remark 5.4.13. Note that contrary to the classical Dynkin game with payoff (5.4.1) (see Remark 5.4.2), the *generalized Dynkin game* is well-posed in the sense that the criterium does not depend on the value function. The characterization of the solution Y of the DRBSDE (5.2.2) in terms of the value function of the *generalized Dynkin game* is thus more interesting and exploitable than the one given in Remark 5.4.2.

5.4.1 Generalized mixed game problems

We now introduce a new game problem, which can be seen as a generalization of a mixed game problem studied in [41] and [89] to the case of nonlinear $\mathcal{E}^g$-conditional expectations. The players have two actions: continuous control and stopping.
Let $(g^{u,v}; (u,v) \in \mathcal{U} \times \mathcal{V})$ be a family of Lipschitz drivers satisfying Assumption 5.4.3 .
Let $S \in \mathcal{T}_0$. For each quadruple $(u,\tau,v,\sigma) \in \mathcal{U} \times \mathcal{T}_S \times \mathcal{V} \times \mathcal{T}_S$, the *criterium* at time S is given by $\mathcal{E}^{u,v}_{S,\tau\wedge\sigma}(I(\tau,\sigma))$, where $\mathcal{E}^{u,v}$ corresponds to the $g^{u,v}$-conditional expectation. The first (resp. second) player chooses a pair (u,τ) (resp. (v,σ)) of control and stopping time, and aims at maximizing (resp. minimizing) the criterium.
For each stopping time $S \in \mathcal{T}_0$, the *upper* and *lower value functions* at time S are defined respectively by

$$\overline{V}(S) := \operatorname*{essinf}_{v\in\mathcal{V},\sigma\in\mathcal{T}_S} ess \sup_{u\in\mathcal{U},\tau\in\mathcal{T}_S} \mathcal{E}^{u,v}_{S,\tau\wedge\sigma}(I(\tau,\sigma)); \tag{5.4.16}$$

$$\underline{V}(S) := ess \sup_{u\in\mathcal{U},\tau\in\mathcal{T}_S} \operatorname*{essinf}_{v\in\mathcal{V},\sigma\in\mathcal{T}_S} \mathcal{E}^{u,v}_{S,\tau\wedge\sigma}(I(\tau,\sigma)). \tag{5.4.17}$$

We say that there *exists a value function* at time S for the game problem if $\overline{V}(S) = \underline{V}(S)$ a.s. We now introduce the definition of an S-saddle point for this game problem.

Definition 5.4.14. *Let $S \in \mathcal{T}_0$. A quadruple $(\overline{u},\overline{\tau},\overline{v},\overline{\sigma}) \in \mathcal{U} \times \mathcal{T}_S \times \mathcal{V} \times \mathcal{T}_S$ is called an* S-saddle point *for the generalized mixed game problem if for each $(u,\tau,v,\sigma) \in \mathcal{U} \times \mathcal{T}_S \times \mathcal{V} \times \mathcal{T}_S$ we have*

$$\mathcal{E}^{u,\overline{v}}_{S,\tau\wedge\overline{\sigma}}(I(\tau,\overline{\sigma})) \le \mathcal{E}^{\overline{u},\overline{v}}_{S,\overline{\tau}\wedge\overline{\sigma}}(I(\overline{\tau}\wedge\overline{\sigma})) \le \mathcal{E}^{\overline{u},v}_{S,\overline{\tau}\wedge\sigma}(I(\overline{\tau},\sigma)) \quad a.s.$$

We prove below that when the obstacles are l.u.s.c. along stopping times, there exist saddle points for the above *generalized mixed game problem.*

Theorem 5.4.15. *Let $(g^{u,v}; (u,v) \in \mathcal{U}\times\mathcal{V})$ be a family of Lipschitz drivers satisfying Assumptions* 5.4.3*. Let ξ and ζ be RCLL adapted processes in $\mathcal{S}^2$ and l.u.s.c. along stopping times, such that $\xi_T = \zeta_T$ a.s. and $\xi_t \le \zeta_t$, $0 \le t \le T$ a.s. Suppose that Mokobodski's condition is satisfied and that there exist controls $\overline{u} \in \mathcal{U}$ and $\overline{v} \in \mathcal{V}$ such that for each $(u,v) \in \mathcal{U} \times \mathcal{V}$,*

$$g^{u,\overline{v}}(t,Y_t,Z_t,k_t) \le g^{\overline{u},\overline{v}}(t,Y_t,Z_t,k_t) \le g^{\overline{u},v}(t,Y_t,Z_t,k_t) \quad dt\otimes dP \ a.s. \ , \tag{5.4.18}$$

where (Y,Z,k,A,A') is the solution of the DRBSDE (5.2.2) associated with driver $g^{\overline{u},\overline{v}}$. Consider the stopping times

$$\tau^*_S := \inf\{t \ge S : Y_t = \xi_t\} \quad ; \quad \sigma^*_S := \inf\{t \ge S : Y_t = \zeta_t\}.$$

The quadruple $(\overline{u},\tau^*_S,\overline{v},\sigma^*_S)$ is then an S-saddle point for the generalized mixed game *problem* (5.4.16)-(5.4.17)*, and we have $Y_S = \underline{V}(S) = \overline{V}(S)$ a.s.*

Proof. By the last assertion of Theoreom 5.4.6, the process $(Y_t,\ S \le t \le \tau^*_S \wedge \sigma^*_S)$ is a strong $\mathcal{E}^{\overline{u},\overline{v}}$-martingale and $Y_{\tau^*_S} = \xi_{\tau^*_S}$, $Y_{\sigma^*_S} = \zeta_{\sigma^*_S}$ a.s. , which implies

$$Y_S = \mathcal{E}^{\overline{u},\overline{v}}_{S,\tau^*_S\wedge\sigma^*_S}(Y_{\tau^*_S\wedge\sigma^*_S}) = \mathcal{E}^{\overline{u},\overline{v}}_{S,\tau^*_S\wedge\sigma^*_S}(\xi_{\tau^*_S}\mathbf{1}_{\tau^*_S\le\sigma^*_S} + \zeta_{\sigma^*_S}\mathbf{1}_{\sigma^*_S<\tau^*_S}) = \mathcal{E}^{\overline{u},\overline{v}}_{S,\tau^*_S\wedge\sigma^*_S}(I(\tau^*_S,\sigma^*_S)) \quad \text{a.s.}$$

Let $\tau \in \mathcal{T}_S$. Since $Y \geq \xi$ and $Y_{\sigma_S^*} = \zeta_{\sigma_S^*}$ a.s. , we have

$$Y_{\tau\wedge\sigma_S^*} = Y_\tau \mathbf{1}_{\tau\leq\sigma_S^*} + Y_{\sigma_S^*}\mathbf{1}_{\sigma_S^*<\tau} \geq \xi_\tau \mathbf{1}_{\tau\leq\sigma_S^*} + \zeta_{\sigma_S^*}\mathbf{1}_{\sigma_S^*<\tau} = I(\tau,\sigma_S^*) \quad \text{a.s.}$$

Moreover, by Theorem 5.4.6, $A'_{\sigma_S^*} = A'_s$ a.s., which implies that:

$$-dY_t = g^{\overline{u},\overline{v}}(t, Y_t, Z_t, k_t)dt + dA_t - Z_t dW_t - \int_{\mathbf{E}} k_t(e)\tilde{N}(dt, de); \quad S \leq t \leq \sigma_S^*, \quad dt \otimes dP \text{ a.s.}$$

Hence, $(Y_t)_{S\leq t\leq\tau\wedge\sigma_S^*}$ is the solution of the BSDE associated with generalized driver $g^{\overline{u},\overline{v}}(\cdot)dt + dA_t$ and terminal condition $Y_{\tau\wedge\sigma_S^*}$. By using Assumption (5.4.18), the inequality $Y_{\tau\wedge\sigma_S^*} \geq I(\tau,\sigma_S^*)$ and the comparison theorem for BSDEs with jumps, we obtain that for each $u \in \mathcal{U}$:

$$Y_S \geq \mathcal{E}^{u,\overline{v}}_{S,\tau\wedge\sigma_S^*}(I(\tau,\sigma_S^*)) \quad \text{a.s.}$$

Similarly, one can prove that for each $v \in \mathcal{V}, \sigma \in \mathcal{T}_S$, we have:

$$Y_S \leq \mathcal{E}^{\overline{u},v}_{S,\tau_S^*\wedge\sigma}(I(\tau_S^*,\sigma)) \quad \text{a.s.}$$

The quadruple $(\overline{u}, \tau_S^*, \overline{v}, \sigma_S^*)$ is thus an S-saddle point and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s. $\square$

Under less restricted assumptions on the obstacles, we prove below that there exists a value function for the above game problem which can be characterized as the solution of a DRBSDE.

Theorem 5.4.16 (Existence of the value function)**.** *Let $(g^{u,v}; (u, v) \in \mathcal{U}\times\mathcal{V})$ be a family of drivers satisfying Assumptions* 5.4.3 *and uniformly Lipschitz with common Lipchitz constant C. Let ξ and ζ be RCLL adapted processes in $\mathcal{S}^2$ such that $\xi_T = \zeta_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. Suppose that Mokobodski's condition is satisfied and that there exist controls $\overline{u} \in \mathcal{U}$ and $\overline{v} \in \mathcal{V}$ such that for each $u \in \mathcal{U}, v \in \mathcal{V}$:*

$$g^{u,\overline{v}}(t, Y_t, Z_t, k_t) \leq g^{\overline{u},\overline{v}}(t, Y_t, Z_t, k_t) \leq g^{\overline{u},v}(t, Y_t, Z_t, k_t), \quad dt \otimes dP \ \text{a.s.} \tag{5.4.19}$$

where (Y, Z, k, A, A') is the solution of the DRBSDE (5.2.2) associated with driver $g^{\overline{u},\overline{v}}$.
Then, there exists a value function for the generalized mixed game *problem* (5.4.16)-(5.4.17)*, and for each stopping time $S \in \mathcal{T}_0$, we have*

$$Y_S = \overline{V}(S) = \underline{V}(S) \quad \text{a.s.}$$

Proof. For each $S \in \mathcal{T}_0$ and for each $\varepsilon > 0$, let τ_S^ε and σ_S^ε be the stopping times defined by

$$\tau_S^\varepsilon := \inf\{t \geq S,\ Y_t \leq \xi_t + \varepsilon\}; \quad \sigma_S^\varepsilon := \inf\{t \geq S,\ Y_t \geq \zeta_t - \varepsilon\}.$$

Let $\tau \in \mathcal{T}_S$. Since $Y \geq \xi$ and $Y_{\sigma_S^\varepsilon} \geq \zeta_{\sigma_S^\varepsilon} - \varepsilon$ a.s. (see Lemma 5.4.9), we have:

$$Y_{\tau\wedge\sigma_S^\varepsilon} \geq \xi_\tau \mathbf{1}_{\tau\leq\sigma_S^\varepsilon} + (\zeta_{\sigma_S^\varepsilon} - \varepsilon)\mathbf{1}_{\sigma_S^\varepsilon<\tau} \geq I(\tau,\sigma_S^\varepsilon) - \varepsilon \quad \text{a.s.}$$

By Lemma 5.4.9, $A'_{\sigma_S^\varepsilon} = A'_S$ a.s. which implies that:

$$-dY_t = g^{\overline{u},\overline{v}}(t, Y_t, Z_t, k_t)dt + dA_t - Z_t dW_t - \int_{\mathbf{E}} k_t(e)\tilde{N}(dt, de), \quad S \leq t \leq \sigma_S^\varepsilon, \quad dt \otimes dP \text{ a.s.}$$

Hence, $(Y_t)_{S\leq t\leq \tau\wedge\sigma^\varepsilon}$ is the solution of the BSDE associated with generalized driver $f(\cdot)dt + dA_t$ and terminal condition $Y_{\tau\wedge\sigma^\varepsilon}$. By using Assumption (5.4.19), the inequality $Y_{\tau\wedge\sigma^\varepsilon} \geq I(\tau,\sigma^\varepsilon) - \varepsilon$ and the comparison theorem for BSDEs with jumps, we obtain

$$Y_S \geq \mathcal{E}_S^{u,\overline{v}}(I(\tau,\sigma^\varepsilon) - \varepsilon) \geq \mathcal{E}_S^{u,\overline{v}}(I(\tau,\sigma^\varepsilon)) - K\varepsilon \quad \text{a.s. },$$

where the second inequality follows from a priori estimates for BSDEs with jumps. Here, the constant K only depends on T and C, the common Lipschitz constant. Consequently, we get

$$Y_S \geq \operatorname*{essinf}_{v\in\mathcal{V},\sigma\in\mathcal{T}_S} ess \sup_{u\in\mathcal{U},\tau\in\mathcal{T}_S} \mathcal{E}^{u,v}_{S,\tau\wedge\sigma}(I(\tau,\sigma)) - K\varepsilon \quad \text{a.s.}$$

Similarly, one can prove that for each $\varepsilon > 0$,

$$Y_S \leq ess \sup_{u\in\mathcal{U},\tau\in\mathcal{T}_S} \operatorname*{essinf}_{v\in\mathcal{V},\sigma\in\mathcal{T}_S} \mathcal{E}^{u,v}_{S,\tau\wedge\sigma}(I(\tau,\sigma)) + K\varepsilon \quad \text{a.s.}$$

Hence, $\overline{V}(S) \leq \underline{V}(S)$ a.s. Since $\underline{V}(S) \leq \overline{V}(S)$ a.s., the equality follows. $\square$

Remark 5.4.17. Note that Theorem 5.4.16 still holds if $g^{\overline{u},\overline{v}}$ is replaced by any Lipschitz driver g which satisfies (5.4.19).

Application: Let U, V be compact Polish spaces.
We are given a map $F : [0,T]\times\Omega\times U\times V\times I\!R^2\times L^2_\nu \to I\!R$, $(t,\omega,u,v,y,z,k) \mapsto F(t,\omega,u,v,y,z,k)$, supposed to be measurable with respect to $\mathcal{P}\otimes\mathcal{B}(U)\otimes\mathcal{B}(V)\otimes\mathcal{B}(I\!R^2)\otimes\mathcal{B}(L^2_\nu)$, continuous, concave (resp. convex) with respect to u (resp. v), and uniformly Lipchitz with respect to (y,z,k). Suppose that F is $\mathcal{C}^1$ with respect to k with $\nabla_k F \geq -1$, and that $F(t,\omega,u,v,0,0,0)$ is uniformly bounded. Let $\mathcal{U}$ (resp. $\mathcal{V}$) be the set of predictable processes valued in U (resp. V). For each $(u,v)\in\mathcal{U}\times\mathcal{V}$, let $g^{u,v}$ be the driver defined by

$$g^{u,v}(t,\omega,y,z,k) := F(t,\omega,u_t(\omega),v_t(\omega),y,z,k). \tag{5.4.20}$$

Let ξ and ζ be RCLL adapted processes in $\mathcal{S}^2$ such that $\xi_T = \zeta_T$ a.s. and $\xi_t \leq \zeta_t$, $0\leq t\leq T$ a.s. Suppose that Mokobodski's condition is satisfied.
Let us consider the associated *generalized mixed game* problem. Define for each (t,ω,y,z,k) the map

$$g(t,\omega,y,z,k) = \sup_{u\in U}\inf_{v\in V} F(t,\omega,u,v,y,z,k). \tag{5.4.21}$$

Since U and V are Polish spaces, there exist some dense countable subsets $\overline{U}$ (resp. $\overline{V}$) of U (resp. V). Since F is continuous with respect to u, v, the *sup* and the *inf* can be taken over $\overline{U}$ (resp. $\overline{V}$). Hence, g is a Lipschitz driver.
Let $(Y,Z,k,A,A')\in\mathcal{S}^2\times\mathbb{H}^2\times\mathbb{H}^2_\nu\times(\mathcal{A}^2)^2$ be the solution of the DRBSDE associated with driver g and obstacles ξ and ζ. By classical convex analysis, for each (t,ω) there exist $(u^*,v^*)\in(U,V)$ such that

$$\begin{aligned} F(t,\omega,u,v^*,Y_{t^-}(\omega),Z_t(\omega),k_t(\omega)) &\leq F(t,\omega,u^*,v^*,Y_{t^-}(\omega),Z_t(\omega),k_t(\omega)) \\ &\leq F(t,\omega,u^*,v,Y_{t^-}(\omega),Z_t(\omega),k_t(\omega)),\ \forall (u,v)\in\overline{U}\times\overline{V}; \\ g(t,\omega,Y_{t^-}(\omega),Z_t(\omega),k_t(\omega))) &= F(t,\omega,u^*,v^*,Y_{t^-}(\omega),Z_t(\omega),k_t(\omega)) \end{aligned} \tag{5.4.22}$$

Let $(u,v) \in \overline{U} \times \overline{V}$. Since the processes Y_{t^-}, Z_t and k_t are predictable, the map $(t,\omega,u^*,v^*) \mapsto (t,\omega,u,v^*,Y_{t^-}(\omega),Z_t(\omega),k_t(\omega))$ is measurable with respect to the σ-algebras $\mathcal{P} \otimes \mathcal{B}(U) \otimes \mathcal{B}(V)$ and $\mathcal{P} \otimes \mathcal{B}(U) \otimes \mathcal{B}(V) \otimes \mathcal{B}(I\!R^2) \otimes \mathcal{B}(L^2_\nu)$. By using the measurability property of F, it follows by composition that the map $(t,\omega,u^*,v^*) \mapsto F(t,\omega,u,v^*,Y_{t^-}(\omega),Z_t(\omega),k_t(\omega))$ is $\mathcal{P} \otimes \mathcal{B}(U) \otimes \mathcal{B}(V)$-measurable. Similarly, the other maps which appear in (5.4.22) are $\mathcal{P} \otimes \mathcal{B}(U) \otimes \mathcal{B}(V)$-measurable, which implies that the set of all $(t,\omega,u^*,v^*) \in [0,T] \times \Omega \times U \times V$ satisfying conditions (5.4.22) belongs to $\mathcal{P} \otimes \mathcal{B}(U) \otimes \mathcal{B}(V)$. By applying a section theorem (see Section 81 in the Appendix of Ch. III in [53]), we get that there exists a pair of predictable process $(u^*,v^*) \in \mathcal{U} \times \mathcal{V}$ such that $dt \otimes dP$ a.s., for all $(u,v) \in \mathcal{U} \times \mathcal{V}$ we have $dt \otimes dP$ a.s.:

$$F(t,u_t,v^*_t,Y_t,Z_t,k_t) \leq F(t,u^*_t,v^*_t,Y_t,Z_t,k_t) \leq F(t,u^*_t,v_t,Y_t,Z_t,k_t)$$

and $g(t,Y_t,Z_t,k_t) = F(t,u^*_t,v^*_t,Y_t,Z_t,k_t)$. Hence, Assumption (5.4.18) is satisfied. By applying Theorems 5.4.16 and 5.4.15, we derive the following result:

Proposition 5.4.18. *There exists a value function for the generalized mixed game problem associated with the controlled drivers $g^{u,v}$ given by* (5.4.20)*. Let Y be the solution of the DRBSDE associated with obstacles ξ, ζ and the driver g defined by* (5.4.21)*. For each stopping time $S \in \mathcal{T}_0$, we have $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s. Suppose that ξ and ζ are l.u.s.c. along stopping times, and consider the stopping times*

$$\tau^*_S := \inf\{t \geq S : Y_t = \xi_t\} \quad ; \quad \sigma^*_S := \inf\{t \geq S : Y_t = \zeta_t\}.$$

The quadruple $(u^*,\tau^*_S,v^*,\sigma^*_S)$ is then an S-saddle point for this generalized mixed game *problem.*

We give now an example of application of the above proposition.
Example: Consider the particular case when F takes the following form:
$F(t,\omega,u,v,y,z,k) = \beta(t,\omega,u,v)z + < \gamma(t,\omega,u,v,\cdot),k >_\nu + c(t,\omega,u,v)$, with β,γ,c bounded. By classical results on linear BSDEs (see [137]), the criterium can be written

$$\mathcal{E}^{u,v}_{S,\tau\wedge\sigma}(I(\tau,\sigma)) = E_{Q^{u,v}}\left[\int_S^{\tau\wedge\sigma} c(t,u_t,v_t)dt + I(\tau,\sigma)|\mathcal{F}_S\right],$$

with $Q^{u,v}$ the probability measure which admits $Z^{u,v}_T$ as density with respect to P, where $(Z^{u,v}_t)$ is the solution of the following SDE:

$$dZ^{u,v}_t = Z^{u,v}_t[\beta(t,u_t,v_t)dW_t + \int_{\mathbf{E}} \gamma(t,u_t,v_t,e)\tilde{N}(dt,de)]; \quad Z^{u,v}_0 = 1.$$

The process $c(t,u_t,v_t)$ can be interpreted as an instantaneous reward associated with controls u,v. This linear model takes into account some ambiguity on the model via the probability measures $Q^{u,v}$ as well as some ambiguity on the instantaneous reward. This case corresponds to the classical mixed game problems studied in [41] and [137], for which the above analysis provides some alternative short proofs.

5.5 Comparison theorems for DRBSDEs with jumps and a priori estimates

5.5.1 Comparison theorems

Theorem 5.5.1 (Comparison theorem for DRBSDEs.). *Let ξ^1, ξ^2, ζ^1, ζ^2 be processes in $\mathcal{S}^2$ such that $\xi^i_T = \zeta^i_T$ a.s. and $\xi^i_t \leq \zeta^i_t$, $0 \leq t \leq T$ a.s. for $i = 1, 2$. Suppose that for $i = 1, 2$, ξ^i, ζ^i satisfies Mokobodski's condition. Let g^1 and g^2 be Lipschitz drivers satisfying Assumption 5.4.3.*
Suppose that

- $\xi^2_t \leq \xi^1_t$ *and* $\zeta^2_t \leq \zeta^1_t$, $0 \leq t \leq T$ *a.s.*
- $g^2(t, y, z, k) \leq g^1(t, y, z, k)$, *for all* $(y, z, k) \in \mathbb{R}^2 \times \mathcal{L}^2_\nu$; $dP \otimes dt - a.s.$

Let $(Y^i, Z^i, k^i, A^i, A'^i)$ be the solution of the DRBSDE associated with (ξ^i, ζ^i, g^i) , $i = 1, 2$. Then,

$$Y^2_t \leq Y^1_t, \ 0 \leq t \leq T \quad a.s.$$

Remark 5.5.2. Note that a comparison theorem has been provided in [50] in the case of jumps under stronger assumptions.Their proof is different and based on Itô's calculus.

Proof. We give a short proof based on the characterization of solutions of DRBSDEs (Theorem 5.4.8) via *generalized Dynkin games*. Let $t \in [0, T]$. For each $\tau, \sigma \in \mathcal{T}_t$, let us denote by $\mathcal{E}^i_{\cdot,\tau\wedge\sigma}(I^i(\tau, \sigma))$ the unique solution of the BSDE associated with driver g^i, terminal time $\tau \wedge \sigma$ and terminal condition $I^i(\tau, \sigma) := \xi^i_\tau \mathbf{1}_{\tau\leq\sigma} + \zeta^i_\sigma \mathbf{1}_{\sigma<\tau}$ for $i = 1, 2$. Since $g^2 \leq g^1$, and $I^2(\tau, \sigma) \leq I^1(\tau, \sigma)$, by the comparison theorem for BSDEs, the following inequality

$$\mathcal{E}^2_{t,\tau\wedge\sigma}(I^2(\tau, \sigma)) \leq \mathcal{E}^1_{t,\tau\wedge\sigma}(I^1(\tau, \sigma)) \ a.s.$$

holds for each τ, σ in $\mathcal{T}_t$. Hence, by taking the essential supremum over τ in $\mathcal{T}_t$ and the essential infimum over σ in $\mathcal{T}_t$, and by using Theorem 5.4.8, we get

$$Y^2_t = \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_t} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathcal{E}^2_{t,\tau\wedge\sigma}(I^2(\tau, \sigma)) \leq \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_t} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathcal{E}^1_{t,\tau\wedge\sigma}(I^1(\tau, \sigma)) = Y^1_t \ a.s.$$

□

We now provide a strict comparison theorem. Note that no strict comparison theorem exists in the literature even in the Brownian case. The first assertion addresses the particular case when the non decreasing processes are continuous and the second one deals with the general case.

Theorem 5.5.3 (Strict comparison.). *Suppose that the assumptions of Theorem 5.5.1 hold and that the driver g^1 satisfies Assumption 5.4.3 with $\gamma_t > -1$ in* (5.4.2). *Let S in $\mathcal{T}_0$ and suppose that $Y^1_S = Y^2_S$ a.s.*

1. *Suppose that A^i, A'^i, $i = 1, 2$ are continuous. For $i = 1, 2$, let $\overline{\tau}_i = \overline{\tau}_{i,S} := \inf\{s \geq S;\ A^i_s > A^i_S\}$ and $\overline{\sigma}_i = \overline{\sigma}_{i,S} := \inf\{s \geq S;\ A'^i_s > A'^i_S\}$. Then*

   $$Y^1_t = Y^2_t, \ \ S \leq t \leq \overline{\tau}_1 \wedge \overline{\tau}_2 \wedge \overline{\sigma}_1 \wedge \overline{\sigma}_2 \ \ a.s.$$

   and

   $$g^2(t, Y^2_t, Z^2_t, k^2_t) \ = \ g^1(t, Y^2_t, Z^2_t, k^2_t) \quad S \leq t \leq \overline{\tau}_1 \wedge \overline{\tau}_2 \wedge \overline{\sigma}_1 \wedge \overline{\sigma}_2, \ dP \otimes dt - a.s. \quad (5.5.1)$$

2. *Consider the case when A^i, A'^i, $i = 1,2$ are not necessarily continuous. For $i = 1,2$, define for each $\varepsilon > 0$,*

$$\tau_i^\varepsilon := \inf\{t \geq S,\ Y_t^i \leq \xi_t^i + \varepsilon\}\ ;\ \ \sigma_i^\varepsilon := \inf\{t \geq S,\ Y_t^i \geq \zeta_t^i - \varepsilon\}.$$

Setting $\tilde{\tau}_i := \lim_{\varepsilon \downarrow 0} \uparrow \tau_i^\varepsilon$ and $\tilde{\sigma}_i := \lim_{\varepsilon \downarrow 0} \uparrow \sigma_i^\varepsilon$, we have

$$Y_t^1 = Y_t^2,\ \ S \leq t < \tilde{\tau}_1 \wedge \tilde{\tau}_2 \wedge \tilde{\sigma}_1 \wedge \tilde{\sigma}_2. \quad a.s. \tag{5.5.2}$$

Moreover, equality (5.5.1) *holds on* $[S, \tilde{\tau}_1 \wedge \tilde{\tau}_2 \wedge \tilde{\sigma}_1 \wedge \tilde{\sigma}_2]$.

Proof. We adopt the same notation as in the proof of the comparison theorem.
1. Suppose first that A^i, A'^i, $i = 1,2$ are continuous. By Theorem 5.4.6, for $i = 1,2$, $(\overline{\tau}_i, \overline{\sigma}_i)$ is a saddle point for the game problem associated with $g = g^i$, $\xi = \xi^i$ and $\zeta = \zeta^i$. By Remark 5.4.7, $(Y_t^i, S \leq t \leq \overline{\tau}_i \wedge \overline{\sigma}_i)$ is an $\mathcal{E}^i$ martingale. Hence we have

$$Y_t^i = \mathcal{E}^i_{t, \overline{\tau}_i \wedge \overline{\sigma}_i}(I(\overline{\tau}_i, \overline{\sigma}_i)),\ \ S \leq t \leq \overline{\tau}_i \wedge \overline{\sigma}_i \ \text{ a.s.}$$

Setting $\overline{\theta} = \overline{\tau}_1 \wedge \overline{\tau}_2 \wedge \overline{\sigma}_1 \wedge \overline{\sigma}_2$, we thus have

$$Y_t^i = \mathcal{E}^i_{t, \overline{\theta}}(Y^i_{\overline{\theta}}),\ \ S \leq t \leq \overline{\theta} \ \text{ a.s. for } i = 1,2.$$

By hypothesis, $Y_S^1 = Y_S^2$ a.s. Now, we apply the strict comparison theorem for non reflected BSDEs with jumps (see [137], Th 4.4) for terminal time $\overline{\theta}$. Hence, we get $Y_t^1 = Y_t^2$, $S \leq t \leq \overline{\theta}$ a.s., as well as equality (5.5.1), which provides the desired result.

2. Consider now the general case.
Let $\varepsilon > 0$. By Remark 5.4.10, $(Y_t^i, S \leq t \leq \tau_i^\varepsilon \wedge \sigma_i^\varepsilon)$ is an $\mathcal{E}^i$ martingale. Hence we have

$$Y_t^i = \mathcal{E}^i_{t, \tau_i^\varepsilon \wedge \sigma_i^\varepsilon}(I(\tau_i^\varepsilon, \sigma_i^\varepsilon)),\ \ S \leq t \leq \tau_i^\varepsilon \wedge \sigma_i^\varepsilon \ \text{ a.s.}$$

By the same arguments as above with τ_1^*,τ_2^* and σ_1^*,σ_2^* replaced by τ_1^ε,τ_2^ε and σ_1^ε,σ_2^ε respectively, we derive $Y_t^1 = Y_t^2$, $S \leq t \leq \tau_1^\varepsilon \wedge \tau_2^\varepsilon \wedge \sigma_1^\varepsilon \wedge \sigma_2^\varepsilon$ a.s., and equality (5.5.1) holds on $[S, \tau_1^\varepsilon \wedge \tau_2^\varepsilon \wedge \sigma_1^\varepsilon \wedge \sigma_2^\varepsilon]$, $dt \otimes dP$-a.s. By letting ε tend to 0, we obtain the desired result. □

We now give an application of the above comparison theorem to a control game problem for DRBSDEs.

Proposition 5.5.4 (Control game problem for DRBSDEs)**.** *Suppose that the assumptions of Th.* 5.4.16 *hold. For each $(u,v) \in \mathcal{U} \times \mathcal{V}$, let $Y^{u,v}$ be the solution of the DRBSDE* (5.2.2) *associated with driver $g^{u,v}$. Then, for each $S \in \mathcal{T}_0$, $Y_S^{u,\overline{v}} \leq Y_S^{\overline{u},\overline{v}} \leq Y_S^{\overline{u},v}$ a.s.*

Proof. By using Assumption (5.4.18) and by applying the comparison theorem for DRBSDEs (Th. 5.5.1), we get that for each $u \in \mathcal{U}$, $Y_S^{u,\overline{v}} \leq Y_S^{\overline{u},\overline{v}}$ a.s. Similarly, for all $v \in \mathcal{V}$, we have $Y_S^{\overline{u},\overline{v}} \leq Y_S^{\overline{u},v}$ a.s. □

Remark 5.5.5. We point out that the above control game problem for DRBSDEs is different from the *generalized mixed game* problem studied in Section **??**. However, from the above proposition, it follows that, under Assumption (5.4.18), the value functions of these two game problems coincide.

5.5.2 A priori estimates with universal constants

Using the characterization of the solution of the nonlinear DRBSDE as the value function of a generalized Dynkin games and DRBSDEs (see Theorem 5.4.8), we prove the following estimates on the spread of the solutions of two DRBSDEs.

Proposition 5.5.6. *Let $\xi^1, \xi^2, \zeta^1, \zeta^2 \in \mathcal{S}^2$ such that $\xi^i_T = \zeta^i_T$ a.s. and $\xi^i_t \leq \zeta^i_t$, $0 \leq t \leq T$ a.s. Suppose that for $i = 1, 2$, ξ^i and ζ^i satisfy Mokobodski's condition. Let g^1, g^2 be Lipschitz drivers satisfying Assumption 5.4.3 with common Lipschitz constant $C > 0$. For $i = 1, 2$, let Y^i be the solution of the DRBSDE associated with driver g^i, terminal time T and barriers ξ^i, ζ^i.*
For $s \in [0, T]$, let $\overline{Y} := Y^1 - Y^2$, $\overline{\xi} := \xi^1 - \xi^2$, $\overline{\zeta} = \zeta^1 - \zeta^2$ and
$\overline{g}_s := \sup_{y,z,k} |g^1(s, y, z, k) - g^2(s, y, z, k)|$. Let $\eta, \beta > 0$ be such that $\beta \geq \dfrac{3}{\eta} + 2C$ and $\eta \leq \dfrac{1}{C^2}$. Then for each t, we have:

$$\overline{Y}_t^2 \leq e^{\beta(T-t)} E[\sup_{s \geq t} \overline{\xi_s}^2 + \sup_{s \geq t} \overline{\zeta_s}^2 | \mathcal{F}_t] + \eta E[\int_t^T e^{\beta(s-t)} \overline{g}_s^2 ds | \mathcal{F}_t] \ a.s. \tag{5.5.3}$$

Remark 5.5.7. Note that here the constants η and β are universal, i.e. they only depend on the terminal time T and the common Lipschitz constant C. This is not the case for the a priori estimates on DRBSDEs given in the literature (for details see Proposition **??** and Remark 5.7.6 in the Appendix).

Proof. For $i = 1, 2$ and for each $\tau, \sigma \in \tau_0$, let $(X^{i,\tau,\sigma}, \pi^{i,\tau,\sigma}, l^{i,\tau,\sigma})$ be the solution of the BSDE associated with driver g^i, terminal time $\tau \wedge \sigma$ and terminal condition $I^i(\tau, \sigma)$, where $I^i(\tau, \sigma) = \xi^i_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta^i_\sigma \mathbf{1}_{\sigma < \tau}$. Set $\overline{X}^{\tau,\sigma} := X^{1,\tau,\sigma} - X^{2,\tau,\sigma}$ and $\overline{I}^{\tau,\sigma} := I^1(\tau, \sigma) - I^2(\tau, \sigma) = \overline{\xi}_\tau \mathbf{1}_{\tau \leq \sigma} + \overline{\zeta}_\sigma \mathbf{1}_{\sigma < \tau}$.
By a priori estimate on BSDEs (see Proposition $A.4$ in [138]), we have a.s.:

$$(\overline{X}_t^{\tau,\sigma})^2 \leq e^{\beta(T-t)} E[\overline{I}(\tau, \sigma)^2 \mid \mathcal{F}_t] + \eta E[\int_t^T e^{\beta(s-t)} [(g^1 - g^2)(s, X_s^{2,\tau,\sigma}, \pi_s^{2,\tau,\sigma}, l_s^{2,\tau,\sigma})]^2 ds \mid \mathcal{F}_t] \tag{5.5.4}$$

from which we derive that

$$(\overline{X}_t^{\tau,\sigma})^2 \leq e^{\beta(T-t)} E[\sup_{s \geq t} \overline{\xi}_s^2 + \sup_{s \geq t} \overline{\zeta}_s^2 | \mathcal{F}_t] + \eta E[\int_t^T e^{\beta(s-t)} \overline{g}_s^2 ds | \mathcal{F}_t] \quad a.s. \tag{5.5.5}$$

Now, by using inequality (5.4.13), we obtain that for each $\varepsilon > 0$ and for all stopping times τ, σ,

$$Y_t^1 - Y_t^2 \leq X_t^{1,\tau_1^\epsilon,\sigma} - X_t^{2,\tau,\sigma_2^\epsilon} + 2K\epsilon.$$

Applying this inequality to $\tau = \tau_1^\epsilon, \sigma = \sigma_2^\epsilon$ we get

$$Y_t^1 - Y_t^2 \leq X_t^{1,\tau_1^\epsilon,\sigma_2^\epsilon} - X_t^{2,\tau_1^\epsilon,\sigma_2^\epsilon} + 2K\epsilon \leq |X_t^{1,\tau_1^\epsilon,\sigma_2^\epsilon} - X_t^{2,\tau_1^\epsilon,\sigma_2^\epsilon}| + 2K\epsilon. \tag{5.5.6}$$

By (5.5.5) and (5.5.6), we have:

$$Y_t^1 - Y_t^2 \leq \sqrt{e^{\beta(T-t)} E[\sup_{s \geq t} \overline{\xi_s}^2 + \sup_{s \geq t} \overline{\zeta_s}^2 | \mathcal{F}_t] + \eta E[\int_t^T e^{\beta(s-t)} \overline{g}_s^2 ds | \mathcal{F}_t]} + 2K\epsilon.$$

By symmetry, the last inequality is also verified by $Y_t^2 - Y_t^1$. The result follows. $\square$

Remark 5.5.8. Note that the arguments of the above proof are different from those used in the literature. Based on Theorem 5.4.8, they allow us to obtain universal constants.

We also state the following estimate on the common value function Y of our generalized Dynkin game problem (5.4.4)-(5.4.5) (or equivalently the solution of the DRBSDE associated with driver g).

Proposition 5.5.9. *For each t, we have:*

$$Y_t^2 \leq e^{\beta(T-t)} E[\sup_{s\geq t} \xi_s{}^2 + \sup_{s\geq t} \zeta_s{}^2 | \mathcal{F}_t] + \eta E[\int_t^T e^{\beta(s-t)} g(s,0,0,0)^2 ds | \mathcal{F}_t] \ a.s. \tag{5.5.7}$$

Proof. Let $X_t^{\tau,\sigma}$ be the solution of the BSDE associated with driver g, terminal time $\tau \wedge \sigma$ and terminal condition $I(\tau,\sigma)$. By applying inequality (5.5.4) with $g^1 = g$, $\xi_1 = \xi$, $\zeta_1 = \zeta$, $g^2 = 0$, $\xi^2 = 0$ and $\zeta^2 = 0$, we get:

$$(X_t^{\tau,\sigma})^2 \leq e^{\beta(T-t)} E[I(\tau,\sigma)^2 | \mathcal{F}_t] + \eta E[\int_t^T e^{\beta(s-t)} (g(s,0,0,0))^2 | \mathcal{F}_t]. \tag{5.5.8}$$

By using the same procedure as in the proof of Proposition 5.5.6, the result follows. □

5.6 Relation with partial integro-differential variational inequalities (PIDVI)

We consider now the Markovian case, and we study the links between Markovian generalized Dynkin games (or equivalently DRBSDEs) and obstacle problems.

Let $b : \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R} \to \mathbb{R}$ be continuous mappings, globally Lipschitz and $\beta : \mathbb{R} \times \mathbf{E} \to \mathbb{R}$ a measurable function such that for some nonnegative real C, and for all $e \in \mathbf{E}$

$$|\beta(x,e)| \leq C\varphi(e), \qquad |\beta(x,e) - \beta(x',e)| \leq C|x-x'|\varphi(e), \ \ x,x' \in \mathbb{R},$$

where $\varphi \in L^2_\nu$. For each $(t,x) \in [0,T] \times \mathbb{R}$, let $(X_s^{t,x}, t \leq s \leq T)$ be the unique $\mathbb{R}$-valued solution of the SDE with jumps:

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x}) dr + \int_t^s \sigma(X_r^{t,x}) dW_r + \int_t^s \int_{\mathbf{E}} \beta(X_{r^-}^{t,x}, e) \tilde{N}(dr, de),$$

and set $X_s^{t,x} = x$ for $s \leq t$. We consider the DRBSDE associated with obstacles $\xi^{t,x}$, $\zeta^{t,x}$ of the following form: $\xi_s^{t,x} := h_1(s, X_s^{t,x})$, $\zeta_s^{t,x} := h_2(s, X_s^{t,x})$, $s < T$, $\xi_T^{t,x} = \zeta_T^{t,x} := g(X_T^{t,x})$. We suppose that $g \in \mathcal{C}(\mathbb{R})$, $h_1, h_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous with respect to t and Lipschitz continuous with respect to x, uniformly in t and that g, h_1, h_2 have at most polynomial growth with respect to x.
Moreover, the obstacles $\xi_s^{t,x}$ and $\zeta_s^{t,x}$ are supposed to satisfy Mokobodski's condition, which holds if for example h_1 and h_2 are $\mathcal{C}^{1,2}$.
We consider two functions γ and f satisfying Assumption 2.1 in [9]. More precisely, we are given a map $\gamma : \mathbb{R} \times \mathbf{E} \to \mathbb{R}$ which is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{K}$-measurable, such that

$|\gamma(x,e) - \gamma(x',e)| < C|x - x'|\varphi(e)$ and $-1 \leq \gamma(x,e) \leq C\varphi(e)$ for each $x, x' \in I\!R, e \in \mathbf{E}$.
Let $f : [0,T] \times I\!R^3 \times L^2_\nu \to I\!R$ be a map supposed to be continuous in t uniformly with respect to x, y, z, k, uniformly Lipschitz with respect to x, y, z, k uniformly in t, such that $f(t, x, 0, 0, 0)$ at most polynomial growth with respect to x, and such that for each t, x, y, z, k_1, k_2, $f(t, x, y, z, k_1) - f(t, x, y, z, k_2) \geq < \gamma(x, \cdot), k_1 - k_2 >_\nu$.
The driver is defined by $f(s, X^{t,x}_s(\omega), y, z, k)$. By Th. 5.4.1, for each $(t,x) \in [0,T] \times I\!R$, there exists an unique solution $(Y^{t,x}, Z^{t,x}, K^{t,x}, A^{t,x}, A'^{\,t,x})$ of the associated DRBSDE. We define:

$$u(t,x) := Y^{t,x}_t, \quad t \in [0,T],\ x \in I\!R. \tag{5.6.1}$$

which is a deterministic quantity. In the following, the map u is called the *value function* of the generalized Dynkin game. By the a priori estimates (see Propositions 5.5.6 and 5.5.9) and the same arguments as those used in the proofs of Lemma 3.1 and Lemma 3.2 in [9], we derive that the value function u is continuous in (t,x) and has at most polynomial growth at infinity. It follows that the process $Y^{t,x}_s = u(s, X^{t,x}_s)$ admits only totally inaccessible jumps. Hence, the processes $A^{t,x}$, $A'^{\,t,x}$ are continuous.

A solution of the obstacle problem is a function $u : [0,T] \times I\!R \to I\!R$ which satisfies the equality $u(T,x) = g(x)$ and

$$\begin{cases} h_1(t,x) \leq u(t,x) \leq h_2(t,x) \\ \text{if } u(t,x) < h_2(t,x) \text{ then } \mathcal{H}u \geq 0 \\ \text{if } h_1(t,x) < u(t,x) \text{ then } \mathcal{H}u \leq 0 \end{cases} \tag{5.6.2}$$

where $L := A + K$ and

- $A\phi(x) := \dfrac{1}{2}\sigma^2(x)\dfrac{\partial^2\phi}{\partial x^2}(x) + b(x)\dfrac{\partial\phi}{\partial x}(x),\ B\phi(t,x)(\cdot) := \phi(t, x + \beta(x,\cdot)) - \phi(t,x)$,
- $K\phi(x) := \int_{\mathbf{E}} \left(\phi(x + \beta(x,e)) - \phi(x) - \dfrac{\partial\phi}{\partial x}(x)\beta(x,e) \right) \nu(de)$,
- $\mathcal{H}\phi(t,x) := -\dfrac{\partial\phi}{\partial t}(t,x) - L\phi(t,x) - f(t, x, \phi(t,x), (\sigma\dfrac{\partial\phi}{\partial x})(t,x), B\phi(t,x))$.

Definition 5.6.1. • A continuous function u is said to be a *viscosity subsolution* of (5.6.2) if $u(T,x) \leq g(x), x \in I\!R$, and if for any point $(t_0,x_0) \in [0,T) \times I\!R$, we have $h_1(t_0,x_0) \leq u(t_0,x_0) \leq h_2(t_0,x_0)$ and, for any $\phi \in C^{1,2}([0,T] \times I\!R)$ such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi - u$ attains its minimum at (t_0,x_0), if $u(t_0,x_0) > h_1(t_0,x_0)$, then $(\mathcal{H}\phi)(t_0,x_0) \leq 0$.

• A continuous function u is said to be a *viscosity supersolution* of (5.6.2) if $u(T,x) \geq g(x), x \in I\!R$, and if for any point $(t_0,x_0) \in [0,T) \times I\!R$, we have $h_1(t_0,x_0) \leq u(t_0,x_0) \leq h_2(t_0,x_0)$ and, for any $\phi \in C^{1,2}([0,T] \times I\!R)$ such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi - u$ attains its maximum at (t_0,x_0), if $u(t_0,x_0) < h_2(t_0,x_0)$ then $(\mathcal{H}\phi)(t_0,x_0) \geq 0$.

Theorem 5.6.2. *The value function u defined by* (5.6.1) *is a viscosity solution (i.e. both a viscosity sub- and supersolution) of the obstacle problem* (5.6.2).

Proof. The proof is given for the convenience of the reader. We prove that u is a viscosity supersolution of (5.6.2), the proof in the case of subsolution being similar.

Let $(t_0, x_0) \in (0,T) \times \mathbb{R}$ and $\phi \in C^{1,2}([0,T] \times \mathbb{R})$ be such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi(t,x) \le u(t,x)$, $\forall (t,x) \in [0,T] \times \mathbb{R}$. Suppose that $u(t_0,x_0) < h_2(t_0,x_0)$ and that

$$-\frac{\partial}{\partial t}\phi(t,x) - L\phi(t,x) - g\left(t,x,\phi(t,x),(\sigma\frac{\partial \phi}{\partial x})(t,x), B\phi(t,x)\right) < 0.$$

By continuity, we can suppose that there exists $\epsilon > 0$ and $\eta_\epsilon > 0$ such that: $\forall (t,x)$ such that $t_0 \le t \le t + \eta_\epsilon < T$ and $|x - x_0| \le \eta_\epsilon$, we have: $u(t,x) \le h_2(t,x) - \epsilon$ and

$$-\frac{\partial}{\partial t}\phi(t,x) - L\phi(t,x) - g\left(t,x,\phi(t,x),(\sigma\frac{\partial \phi}{\partial x})(t,x), B\phi(t,x)\right) \le -\epsilon. \tag{5.6.3}$$

Let θ be the stopping time defined as follows:

$$\theta := (t_0 + \eta_\epsilon) \wedge \inf\{s \ge t_0 / |X_s^{t_0,x_0} - x_0| > \eta_\epsilon\}.$$

By this definition, we have

$$u(s, X_s^{t_0,x_0}) \le h_2(s, X_s^{t_0,x_0}) - \epsilon < h_2(s, X_s^{t_0,x_0}), t_0 \le s < \theta \text{ a.s.}$$

Hence, the process $(Y_s^{t_0,x_0} = u(s, X_s^{t_0,x_0}), s \in [t_0, \theta[)$ stays strictly below the upper barrier. It follows that the continuous process $A_s^{',t_0,x_0}$ is constant on $[t, \theta]$. The process $(Y^{t_0,x_0}, s \in [t_0,\theta])$ is thus the solution of the classical BSDE associated with terminal condition $Y_\theta^{t_0,x_0} = u(\theta, X_\theta^{t_0,x_0})$ and the generalized driver

$$g(s, X_s^{t_0,x_0}, y, z, q)ds + dA_s^{t_0,x_0}.$$

Our aim now is to use the comparison theorem. We apply as above Itô's lemma to $\phi(s, X_s^{t_0,x_0})$ and we get that $\left(\phi(s,X_s^{t_0,x_0}), (\sigma\frac{\partial\phi}{\partial x})(s,X_s^{t_0,x_0}), \Phi(s, X_{s^-}^{t_0,x_0}, \cdot); s \in [t_0,\theta]\right)$ is the solution of the BSDE associated to the terminal value $\phi(\theta, X_\theta^{t_0,x_0})$ and driver $-\psi(s, X_s^{t_0,x_0})$, where $\psi(s,x) := \frac{\partial \phi}{\partial s}(s,x) + L\phi(s,x)$. By assumption (5.6.3) and the definition of the stopping time, we have :

$$\begin{aligned} &-\psi(s,X_s^{t_0,x_0})ds \le (g(s, X_s^{t_0,x_0}, \phi(s,X_s^{t_0,x_0}),\\ &\quad (\sigma\frac{\partial\phi}{\partial x})(s,X_s^{t_0,x_0}), B\phi(s,X_s^{t_0,x_0})))ds + dA_s^{t_0,x_0} - \epsilon\, ds, \quad \forall s \in [t_0,\theta]. \end{aligned}$$

The above inequality gives a relation between the drivers of the two BSDEs. Moreover, $\phi(\theta, X_\theta^{t_0,x_0}) \le u(\theta, X_\theta^{t_0,x_0}) = Y_\theta^{t_0,x_0}$. By applying the extended comparison theorem for BSDEs with jumps given in [9] (Proposition A.3) we get:

$$\phi(t_0,x_0) = \phi(t_0, X_{t_0}^{t_0,x_0}) < Y_{t_0}^{t_0,x_0} = u(t_0,x_0),$$

which provides a contradiction. □

In the sequel, we suppose that the function φ is defined by $\varphi(e) := 1 \wedge |e|$ and belongs to L^2_ν. We also suppose that g, h_1 and h_2 are bounded, and that Assumption 4.1 in [9] holds. More precisely, we assume:

(i) $\quad f(s, X_s^{t,x}(\omega), y, z, k) := \overline{f}\left(s, X_s^{t,x}(\omega), y, z, \int_{I\!R^*} k(e)\gamma(X_s^{t,x}(\omega), e)\nu(de)\right) \mathbf{1}_{s\ge t}$,

where $\overline{f} : [0,T] \times I\!R^4 \to I\!R$ is a map which is continuous with respect to t uniformly in x, y, z, k, and continuous with respect x uniformly in y, z, k. It is also uniformly Lipschitz with respect to y, z, k and the map $\overline{f}(t, x, 0, 0, 0)$ is uniformly bounded.
The map $k \mapsto \overline{f}(t, x, y, z, k)$ is also non-decreasing, for all $t \in [0,T]$, $x, y, z \in I\!R$.
(ii) For each $R > 0$, there exists a continuous function $m_R : I\!R_+ \to I\!R_+$ with $m_R(0) = 0$ and $|\overline{f}(t, x, y, z, k) - \overline{f}(t, x', y, z, k)| \leq m_R(|x - x'|(1 + |z|))$, for all $t \in [0,T]$, $|x|, |x'| \leq R$, $|y| \leq R$, $z, k \in I\!R$.
(iii) $|\gamma(x, e) - \gamma(y, e)| \leq C|x - y|(1 \wedge e^2)$; $0 \leq \gamma(x, e) \leq C(1 \wedge |e|)$, $x, y \in I\!R, e \in I\!R^*$.
(iv) $\overline{f}(t, x, y, z, l) - \overline{f}(t, x, y, z, l) \geq r(u - v)$, $u \geq v$, $t \in [0,T]$, $x, u, v, p, l \in I\!R$, where $r > 0$.
To simplify notation, in the sequel, $\overline{f}$ is denoted by f.
The operator B has now the following form: $B\phi(x) := \int_{I\!R^*}(\phi(x + \beta(x, e)) - \phi(x))\gamma(x, e)\nu(de)$.

Theorem 5.6.3 (Comparison principle)**.** *Suppose that Assumptions (i) to (iv). If U is a bounded viscosity subsolution and V is a bounded viscosity supersolution of the obstacle problem (5.6.2), then $U(t, x) \leq V(t, x)$, for each $(t, x) \in [0,T] \times I\!R$.*

Proof. The proof is similar to the proof given in [9] (in the case of one barrier). For the convenience of the reader, we give a sketch of proof, where we draw attention to some points which differ from the proof in [9]. Set

$$\psi^{\epsilon,\eta}(t, s, x, y) := U(t, x) - V(s, y) - \frac{|x - y|^2}{\epsilon^2} - \frac{|t - s|^2}{\epsilon^2} - \eta^2(|x|^2 + |y|^2),$$

where ϵ, η are small parameters devoted to tend to 0. Let $M^{\epsilon,\eta}$ be a maximum of $\psi^{\epsilon,\eta}(t, s, x, y)$. This maximum is reached at some point $(t^{\epsilon,\eta}, s^{\epsilon,\eta}, x^{\epsilon,\eta}, y^{\epsilon,\eta})$. We define:

$$\Psi_1(t, x) := V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) + \frac{|x - y^{\epsilon,\eta}|^2}{\epsilon^2} + \frac{|t - s^{\epsilon,\eta}|^2}{\epsilon^2} + \eta^2(|x|^2 + |y^{\epsilon,\eta}|^2);$$

$$\Psi_2(s, y) := U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - \frac{|x^{\epsilon,\eta} - y|^2}{\epsilon^2} - \frac{|t^{\epsilon,\eta} - s|^2}{\epsilon^2} - \eta^2(|x^{\epsilon,\eta}|^2 + |y|^2).$$

As $(t, x) \to (U - \Psi_1)(t, x)$ reaches its maximum at $(t^{\epsilon,\eta}, x^{\epsilon,\eta})$ and U is a subsolution, we have the two following cases:

- $t^{\epsilon,\eta} = T$ and then $U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) \leq g(x^{\epsilon,\eta})$,
- $t^{\epsilon,\eta} \neq T$, $h_1(t^{\epsilon,\eta}, x^{\epsilon,\eta}) \leq U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) \leq h_2(t^{\epsilon,\eta}, x^{\epsilon,\eta})$ and, if $U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) > h_1(t^{\epsilon,\eta}, x^{\epsilon,\eta})$, we then have:

$$\begin{aligned} &-\frac{\partial \Psi_1}{\partial t}(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - L\Psi_1(t^{\epsilon,\eta}, x^{\epsilon,\eta}) \\ &\qquad - f\left(t^{\epsilon,\eta}, x^{\epsilon,\eta}, U(t^{\epsilon,\eta}, x^{\epsilon,\eta}), (\sigma\frac{\partial \Psi_1}{\partial x})(t^{\epsilon,\eta}, x^{\epsilon,\eta}), B\Psi_1(t^{\epsilon,\eta}, x^{\epsilon,\eta})\right) \leq 0. \end{aligned} \tag{5.6.4}$$

As $(s, y) \to (\Psi_2 - V)(s, y)$ reaches its maximum at $(s^{\epsilon,\eta}, y^{\epsilon,\eta})$ and V is a supersolution, we have the two following cases:

- $s^{\epsilon,\eta} = T$ and $V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) \geq g(y^{\epsilon,\eta})$,

- $s^{\epsilon,\eta} \neq T$, $h_1(s^{\epsilon,\eta}, y^{\epsilon,\eta}) \leq V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) \leq h_2(s^{\epsilon,\eta}, y^{\epsilon,\eta})$ and, if $V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) < h_2(s^{\epsilon,\eta}, y^{\epsilon,\eta})$ then

$$-\frac{\partial \Psi_2}{\partial t}(s^{\epsilon,\eta}, y^{\epsilon,\eta}) - L\Psi_2(s^{\epsilon,\eta}, y^{\epsilon,\eta})$$
$$- f(s^{\epsilon,\eta}, y^{\epsilon,\eta}, V(s^{\epsilon,\eta}, y^{\epsilon,\eta}), (\sigma \frac{\partial \Psi_2}{\partial x})(s^{\epsilon,\eta}, y^{\epsilon,\eta})), B\Psi_2(s^{\epsilon,\eta}, y^{\epsilon,\eta}) \geq 0.$$

As in [9], we have: $|x^{\epsilon,\eta} - y^{\epsilon,\eta}| + |t^{\epsilon,\eta} - s^{\epsilon,\eta}| \leq C\epsilon$, $|x^{\epsilon,\eta}| \leq \frac{C}{\eta}$ and $|y^{\epsilon,\eta}| \leq \frac{C}{\eta}$.

Extracting a subsequence if necessary, we may suppose that for each η the sequences $(t^{\epsilon,\eta})_\epsilon$ and $(s^{\epsilon,\eta})_\epsilon$ converge to a common limit t^η, and the sequences $(x^{\epsilon,\eta})_\epsilon$ and $(y^{\epsilon,\eta})_\epsilon$ converge to a common limit x^η. Here, we have to consider four cases.

1st case: there exists a subsequence of (t^η) such that $t^\eta = T$ for all η (of this subsequence)

2nd case: there exists a subsequence of (t^η) such that $t^\eta \neq T$ and for all η belonging to this subsequence, there exist a subsequence of $(x^{\epsilon,\eta})_\epsilon$ and a subsequence of $(t^{\epsilon,\eta})_\epsilon$, such that $U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - h_1(t^{\epsilon,\eta}, x^{\epsilon,\eta}) = 0$.

3rd case: there exists a subsequence such that $t^\eta \neq T$, and for all η belonging to this subsequence, there exist a subsequence of $(y^{\epsilon,\eta})_\epsilon$ and a subsequence of $(s^{\epsilon,\eta})_\epsilon$, such that $V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) - h_2(s^{\epsilon,\eta}, y^{\epsilon,\eta}) = 0$.

Last case: we are left with the case when, for a subsequence of η we have $t^\eta \neq T$, and for all η belonging to this subsequence, there exists a subsequence of $(x^{\epsilon,\eta})_\epsilon$, $(y^{\epsilon,\eta})_\epsilon$, $(t^{\epsilon,\eta})_\epsilon$ and $(s^{\epsilon,\eta})_\epsilon$ such that

$$U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - h_1(t^{\epsilon,\eta}, x^{\epsilon,\eta}) > 0; \quad h_2(s^{\epsilon,\eta}, y^{\epsilon,\eta}) - V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) > 0.$$

We are thus in the case when the solution if strictly between the barriers, that is when there is no reflection. We can then use the same arguments as in the case of one barrier when there is no reflection. For convenience of the reader, we recall below the main arguments. We argue by contradiction by assuming that $M > 0$. We set

$$\varphi(t, s, x, y) := \frac{|x - y|^2}{\epsilon^2} + \frac{|t - s|^2}{\epsilon^2} + \eta^2(|x|^2 + |y|^2). \tag{5.6.5}$$

We know that he maximum of the function $\psi_{\epsilon,\eta} := U(t, x) - V(s, y) - \varphi(t, s, x, y)$ is reached at the point $(t^{\epsilon,\eta}, s^{\epsilon,\eta}, x^{\epsilon,\eta}, y^{\epsilon,\eta})$. We can thus apply the non-local version of Jensen Ishii's lemma in [10], which leads to the desired result, by using exactly the same arguments as in [9] (see Theorem 4.1, last case).

Note that the first, second and fourth case are identical to the three cases considered for reflected BSDEs (see [9]). The third one, which didn't appear in the case of reflected BSDEs, can be treated similarly to the second one. □

We derive that under Assumptions (i) to (iv), there exists an unique solution of the obstacle problem (5.6.2) in the class of bounded continuous functions.

5.7 Appendix

Remark 5.7.1. Note that L^2_ν is a separable Hilbert space. Indeed, by a result of Measure Theory (see e.g. Proposition 3.4.5 of Cohn's book on Measure Theory [47]), given a measurable space

$(Y, \mathcal{B}, \mu)$, if μ is σ-finite and $\mathcal{B}$ is countably generated, then $L^2(Y, \mathcal{B}, \mu)$ is separable. Applying this property to $Y = \mathbf{E}$ (where $\mathbf{E} = \mathbb{R}^*$), $\mathcal{B} = \mathcal{B}(\mathbf{E})$ and $\mu = \nu$, since $\mathcal{B}(\mathbf{E})$ is countably generated, it follows that $L^2_\nu= L^2(\mathbf{E}, \mathcal{B}(\mathbf{E}), \nu)$ is separable.

Proposition 5.7.2. *Let $(X, \mathcal{A})$ be a measurable space.*
Let $f : (X \times L^2_\nu, \mathcal{A} \otimes \mathcal{B}(L^2_\nu) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}));\ (\alpha, k) \to f(\alpha, k)$. Suppose that f satisfies one of the three following conditions:
1. f is of class $\mathcal{C}^1$ with respect to k such that for all $(\alpha, k) \in X \times L^2_\nu$,

$$|\nabla_k f(\alpha, k)(e)| \leq \psi(e) \quad \text{and} \quad \nabla_k f(\alpha, k)(e) \geq -1 \quad d\nu(e) - \text{a.s.} \tag{5.7.1}$$

where $\psi \in L^2_\nu$.
2. f is convex (resp. concave) with respect to k and Gâteaux-differentiable with respect to k such that the Gâteaux-gradiant $\nabla^g_k f(\alpha, k)$, which is also the sub- (resp. super-) differential with respect to k, satisfies (5.7.1).
3. f of the form $f(\alpha, k) := \overline{f}(\alpha, \int_{\mathbf{E}} k(e)\psi(e)\nu(de))$, where ψ is a nonnegative function in L^2_ν and $\overline{f} : X \times \mathbb{R} \to \mathbb{R}$ is a measurable map, supposed to be non-decreasing with respect to its second variable and Lipschitz continuous with Lipschitz constant denoted by C.

Then, there exists a measurable map $\gamma : (X \times (L^2_\nu)^2, \mathcal{A} \otimes \mathcal{B}((L^2_\nu)^2)) \to (L^2_\nu, \mathcal{B}(L^2_\nu))$; $(\alpha, k_1, k_2) \mapsto \gamma(\alpha, k_1, k_2)$ such that $|\gamma(.)(e)| \leq \psi(e)$, where $\psi \in L^2_\nu$; $\gamma(.)(e) \geq -1 \quad \nu(de) - \text{a.s.}$ and

$$f(\alpha, k_2) - f(\alpha, k_1) \geq < \gamma(\alpha, k_1, k_2)\, , \, k_2 - k_1 >_\nu, \quad \forall (\alpha, k_1, k_2) \in X \times (L^2_\nu)^2.$$

Proof. 1. Since L^2_ν is a separable Hilbert space, it admits a countable orthonormal basis $\{e^i, i \in \mathbb{N}\}$. Let $(\alpha, k) \in X \times L^2_\nu$. Since f is differentiable at k, for each h in V we have: $f(\alpha, k+h) = f(\alpha, k) + < \nabla_k f(\alpha, k), h >_\nu + ||h||_\nu \varepsilon(||h||_\nu)$, where $\lim_{x \to 0} \varepsilon(x) = 0$. By taking $h = te_i$, $t \in \mathbb{R}, i \in \mathbb{N}$ we obtain that

$$< \nabla_k f(\alpha, k), e_i >_\nu = \lim_{t \to 0} \frac{f(\alpha, k + te_i) - f(\alpha, k)}{t}.$$

Hence, the map δ_i defined for each $(\alpha, k) \in X \times V$ by $\delta_i(\alpha, k) :=< \nabla_k f(\alpha, k), e_i >$ is $\mathcal{A} \otimes \mathcal{B}(L^2_\nu)$ -measurable. We thus obtain that $\nabla_k f(., .) : (X \times L^2_\nu, \mathcal{A} \otimes \mathcal{B}(L^2_\nu) \to (L^2_\nu, \mathcal{B}(L^2_\nu))$; $(\alpha, k) \mapsto \nabla_k f(\alpha, k) = \sum_{i \in \mathbb{N}} \delta_i(\alpha, k) e_i$ is measurable.
Now, for each $(\alpha, k_1, k_2) \in X \times (L^2_\nu)^2$, the map $t \mapsto f(\alpha, k_1 + t(k_2 - k_1))$ is $\mathcal{C}^1$. Hence, by the mean theorem, we have that

$$\begin{aligned} f(\alpha, k_2) - f(\alpha, k_1) &= \int_0^1 < \nabla_k f(\alpha, k_1 + t(k_2 - k_1))\, , \, k_2 - k_1 >_\nu dt \\ &= \int_0^1 \sum_{i \in \mathbb{N}} \nabla^i_k f(\alpha, k_1 + t(k_2 - k_1))\ (k^i_2 - k^i_1) dt. \end{aligned}$$

where for each $l \in L^2_\nu$, we have denoted its coordinates in the basis $(e^i)_{i \in \mathbb{N}}$ by $(l^i)_{i \in \mathbb{N}}$.
Now, by (5.7.1), $||\nabla_k f(.)||_\nu$ is uniformly bounded. Using this property and Fubini's theorem, one can show that

$$f(\alpha, k_2) - f(\alpha, k_1) = < \gamma(\alpha, k_1, k_2)\, , \, k_2 - k_1 >_\nu,$$

where $\gamma(\alpha, k_1, k_2) := \int_0^1 \nabla_k f(\alpha, k_1 + t(k_2 - k_1))dt$. Here, for each continuous map $F : [0,1] \to L^2_\nu; \, t \mapsto F(t)$, the integral $\int_0^1 F(t)dt$ is defined as $\int_0^1 F(t)dt := \sum_{i\in\mathbb{N}}(\int_0^1 F^i(t)dt)e_i$. The desired result follows.

2. Suppose f is convex. By Proposition 5.4 in [76], since f is convex and Gâteaux-differentiable, f is sub-differentiable. By Proposition 5.3 in [76], the Gâteaux-gradiant $\nabla^g_k f(\alpha, k)$ coincides with the sub-differential at k. Hence, for each k, h in L^2_ν, we have:
$f(\alpha, k+h) \geq \; f(\alpha, k) + \; < \nabla^g_k f(\alpha, k), h >_\nu$. By definition of the Gâteaux-gradiant (see Definition 5.2. in [76]), we have that for each $i \in \mathbb{N}$, $< \nabla^g_k f(\alpha, k), e_i >_\nu = \lim_{t\to 0} \dfrac{f(\alpha, k + te_i) - f(\alpha, k)}{t}$.
Setting $\gamma(\alpha, k_1, k_2) := \nabla^g_k f(\alpha, k_1)$, the result follows.
Suppose f is concave. By applying the previous property to the convex map $-f$ and with (k_2, k_1) instead of (k_1, k_2), we get $-f(\alpha, k_1) + f(\alpha, k_2) \; \geq \; < -\nabla^g_k f(\alpha, k_2) \, , \, k_1 - k_2 >_\nu$, for each $(\alpha, k_1, k_2) \in X \times (L^2_\nu)^2$. Setting $\gamma(\alpha, k_1, k_2) := \nabla^g_k f(\alpha, k_2)$, the result follows.

3. Setting $\gamma(\alpha, k_1, k_2) := C\psi(e)\mathbf{1}_{\{\int_{\mathbf{E}}(k_2(e)-k_1(e))\psi(e)\nu(de) \leq 0\}}$, the result follows. □

Proof of Lemma 5.3.3: The results of this lemma can be derived from the results of [105] obtained in the general framework of admissible families of random variables indexed by stopping times. We give here a sketch of the proof. Set $J^{(0)}_\cdot = 0$ and $J'^{(0)}_\cdot = 0$, and define recursively for each $n \in \mathbb{N}$, the RCLL supermartingale processes:

$$J^{(n+1)}_\cdot := \bar{\mathcal{R}}(J'^{(n)}_\cdot + \tilde{\xi}^g_\cdot); \qquad J'^{(n+1)}_\cdot := \bar{\mathcal{R}}(J^{(n)}_\cdot - \tilde{\zeta}^g_\cdot) \tag{5.7.2}$$

which belong to $\mathcal{S}^2$, where $\bar{\mathcal{R}}$ is the classical Snell envelop operator. For sake of simplicity, in the above definition we have omitted the exponent g in the definition of $J^{(n)}$. Since $\tilde{\xi}^g_T = \tilde{\zeta}^g_T = 0$ a.s., it follows that, for each n, $J^{(n)}_T = J'^{(n)}_T = 0$ a.s.

We have $J^{(0)}_\cdot = 0$ and $J^{'(0)}_\cdot = 0$. Let us prove recursively that for each n, $J^{'(n)}_\cdot, J^{(n)}_\cdot$ are well defined and nonnegative. Suppose that $J^{'(n)}_\cdot, J^{(n)}_\cdot$ are well defined and nonnegative. Then $J^{(n+1)}_\cdot$, $J^{'(n+1)}_\cdot$ are well defined since $(J^{'(n)}_\cdot + \tilde{\xi}^g_\cdot)^-$ and $(J^{(n)}_\cdot - \tilde{\zeta}^g_\cdot)^-$ belong to $\mathcal{S}^2$. Also, $J^{(n+1)}_t \geq E[J^{'(n)}_T + \tilde{\xi}^g_T|\mathcal{F}_t] \geq 0$ a.s. since $\tilde{\xi}^g_T = 0$ a.s. Similarly, because $\tilde{\zeta}^g_T = 0$ a.s., $J^{'(n+1)}_t \geq 0$ a.s. By classical results, $J^{(n)}_\cdot$ and $J^{'(n)}_\cdot$ are RCLL supermartingales.
Let us prove that $J^{(n)}_\cdot$ and $J^{'(n)}_\cdot$ are non decreasing sequences of processes. We have $J^{(1)}_\cdot \geq 0 = J^{(0)}_\cdot$ and $J^{'(1)}_\cdot \geq 0 = J^{'(0)}_\cdot$. Suppose that $J^{(n)}_\cdot \geq J^{(n-1)}_\cdot$ and $J^{'(n)}_\cdot \geq J^{'(n-1)}_\cdot$. We then have:

$$\bar{\mathcal{R}}(J'^{(n)}_\cdot + \tilde{\xi}^g_\cdot) \geq \bar{\mathcal{R}}(J^{'(n-1)}_\cdot + \tilde{\xi}^g_\cdot); \qquad \bar{\mathcal{R}}(J^{(n)}_\cdot - \tilde{\zeta}^g_\cdot) \geq \bar{\mathcal{R}}(J^{(n-1)}_\cdot - \tilde{\zeta}^g_\cdot), \tag{5.7.3}$$

which leads to $J^{(n+1)}_\cdot \geq J^{(n)}_\cdot$ and $J^{'(n+1)}_\cdot \geq J^{'(n)}_\cdot$. Let us introduce the following optional processes valued in $[0, +\infty]$ defined by $J^g_\cdot := \lim \uparrow J^{(n)}_\cdot$ and $J'^g_\cdot := \lim \uparrow J'^{(n)}_\cdot$.
Since for each n, $J^{(n)}_T = J'^{(n)}_T = 0$ a.s. we have $J^g_T = J^{'g}_T = 0$ a.s.
By the monotone convergence theorem, one can show that $J^g_\cdot$ and $J'^g_\cdot$ are strong supermartingales.

We now show equalities (5.3.5). In the following, we use the Snell envelope operator $\mathcal{R}$ which acts on admissible families of random variables (r.v.). The reader is referred to Section 1.1 in [105] for the definition of an admissible family of r.v. indexed by stopping times, as well as the definition of a supermartingale family. Recall that for each admissible family $\phi = (\phi(\theta))_{\theta\in\mathcal{T}_0}$ valued in $\mathbf{R} \cup \{+\infty\}$ with $E\,[ess\sup_{\theta\in\mathcal{T}} \phi(\theta)^-] < +\infty$, $\mathcal{R}(\phi)$ is defined as the smallest supermartingale family greater than ϕ. Note that by some results of optimal stopping (see Section 1.1 in [105]), we have

$$\mathcal{R}(\phi)(\theta) = ess \sup_{\tau \in \mathcal{T}_\theta} E[\phi(\tau) \,|\, \mathcal{F}_\theta] \quad \text{a.s.} \tag{5.7.4}$$

for each stopping time θ. In the following, for each optional process $\phi_. = (\phi_t)_{0 \le t \le T}$ valued in $\mathbf{R} \cup \{+\infty\}$, we denote by ϕ its associated family of r.v. defined by $\phi := (\phi_\theta)_{\theta \in \mathcal{T}_0}$. If $\phi_. \in \mathcal{S}^2$, we then have

$$\mathcal{R}(\phi)(\theta) = ess \sup_{\tau \in \mathcal{T}_\theta} E[\phi_\tau \,|\, \mathcal{F}_\theta] = \bar{\mathcal{R}}(\phi_.)_\theta \quad \text{a.s.} \tag{5.7.5}$$

for each stopping time θ. This property and equalities (5.7.3) lead to the following equalities written in terms of families and the operator $\mathcal{R}$:

$$J^{(n+1)} := \mathcal{R}(J'^{(n)} + \tilde{\xi}^g); \qquad J'^{(n+1)} := \mathcal{R}(J^{(n)} - \tilde{\zeta}^g)$$

As the operator $\mathcal{R}$ is nondecreasing, for each $n \in I\!N$, we have $J^{(n+1)} = \mathcal{R}(J'^{(n)} + \tilde{\xi}^g) \le \mathcal{R}(J'^g + \tilde{\xi}^g)$. By letting n tend to $+\infty$, we get that

$$J^g \le \mathcal{R}(J'^g + \tilde{\xi}^g). \tag{5.7.6}$$

Now, for each $n \in \mathbb{N}$, $J^{(n+1)} \ge J'^{(n)} + \tilde{\xi}^g$. By letting n tend to $+\infty$, we derive that $J^g \ge J'^g + \tilde{\xi}^g$. By the supermartingale property of the family of r.v. $J^g = (J^g_\theta)_{\theta \in \mathcal{T}_0}$ and the characterization of $\mathcal{R}(J'^g + \tilde{\xi}^g)$ as the smallest supermartingale family greater than $J'^g + \tilde{\xi}^g$, it follows that $J^g \ge \mathcal{R}(J'^g + \tilde{\xi}^g)$. This with (5.7.6) yields that $J^g = \mathcal{R}\left(J'^g + \xi\right)$. Similarly, $J'^g = \mathcal{R}(J^g - \tilde{\zeta}^g)$, which, by the property (5.7.4), leads to the desired equalities (5.3.5). Note that the supermartingale property of the families J^g and J'^g corresponds to the strong supermartingale property of the optional processes $J^g_.$ and $J'^g_.$.

We have $J^g_. \ge J'^g_. + \tilde{\xi}^g_.$ and $J'^g_. \ge J^g_. - \tilde{\zeta}^g_.$. The proof of the minimality of $J^g_.$ and $J'^g_.$ follows from Proposition 5.1 in [105].

Moreover, if $J^g_0 < +\infty$ and $J'^g_0 < +\infty$, by Th.18 ch. VI in [54], $J^g_.$ and $J'^g_.$ are indistinguishable from nonnegative RCLL supermartingales, as the non decreasing limits of nonnegative RCLL supermartingales. □

Remark 5.7.3. The property $\tilde{\xi}^g_T = \tilde{\zeta}^g_T = 0$ a.s. ensures that for each n, $J^{(n)}_T = J'^{(n)}_T = 0$ a.s. Note that if we had not made the change of variable (5.3.4), then $\tilde{\xi}^g, \tilde{\zeta}^g$ would be replaced by ξ, ζ in the definitions of $J^{(n)}$ and $J'^{(n)}$. In that case, $\xi_T = \zeta_T$ a.s. but would not necessarily be equal to 0, and we would have $J^{(n)}_T = -J'^{(n)}_T = 0$ a.s. if n is even, and ξ_T otherwise.

Remark 5.7.4. The proof of Th. 5.3.4 together with Lemma 5.3.3 ensures that $B = A$. Indeed, set $H_t := E[A_T - A_t|\mathcal{F}_t]$ (resp. $H'_t := E[A'_T - A'_t|\mathcal{F}_t]$). Since $dA_t << dB_t$ (resp. $dA'_t << dB'_t$), we have $H_t \le J_t = E[B_T - B_t|\mathcal{F}_t]$ (resp. $H'_t \le J'_t = E[B'_T - B'_t|\mathcal{F}_t]$). Moreover, $H - H' = J - J'$. Hence, we have $H \ge H' + \tilde{\xi}^g$ and $H' \ge H - \tilde{\zeta}^g$. By the minimality property of J, J' (5.3.6), we derive that $J = H$ (resp. $J' = H'$).

Proof of Theorem 5.3.5: Theorem 5.3.4 gives the existence. Let (Y, Z, k, A, A') be a solution of the DRBSDE associated with driver process $g(t)$ and obstacles (ξ, ζ). Let us prove that it is unique. We first show the uniqueness of Y. For each $S \in \mathcal{T}_0$ and for each $\varepsilon > 0$, let

$$\tau_S^\varepsilon := \inf\{t \geq S,\ Y_t \leq \xi_t + \varepsilon\} \quad \sigma_S^\varepsilon := \inf\{t \geq S,\ Y_t \geq \zeta_t - \varepsilon\}. \tag{5.7.7}$$

Note that σ_S^ε and $\tau_S^\varepsilon \in \mathcal{T}_S$. Fix $\varepsilon > 0$. By the same arguments as in the proof of Lemma 4.8, the function $t \mapsto A_t$ is constant a.s. on $[S, \tau_S^\varepsilon]$ and $Y_{\tau_S^\varepsilon} \leq \xi_{\tau_S^\varepsilon} + \varepsilon$ a.s. Similarly, A' is constant on $[S, \sigma_S^\varepsilon]$ and $Y_{\sigma_S^\varepsilon} \geq \zeta_{\sigma_S^\varepsilon} - \varepsilon$ a.s.

Let $\tau \in \mathcal{T}_S$. Since A' is constant on $[S, \sigma_S^\varepsilon]$, the process $(Y_t + \int_0^t g(s)ds,\ S \leq t \leq \tau \wedge \sigma_S^\varepsilon)$ is a supermartingale. Hence

$$Y_S \geq E[Y_{\tau\wedge\sigma_S^\varepsilon} + \int_S^{\tau\wedge\sigma_S^\varepsilon} g(s)ds \mid \mathcal{F}_S] \quad \text{a.s.}$$

We also have that $Y_{\tau\wedge\sigma_S^\varepsilon} = Y_\tau \mathbf{1}_{\tau\leq\sigma_S^\varepsilon} + Y_{\sigma_S^\varepsilon}\mathbf{1}_{\sigma_S^\varepsilon<\tau} \geq \xi_\tau \mathbf{1}_{\tau\leq\sigma_S^\varepsilon} + (\zeta_{\sigma_S^\varepsilon} - \varepsilon)\mathbf{1}_{\sigma_S^\varepsilon<\tau}$ a.s. We get
$Y_S \geq E[I_S(\tau, \sigma_S^\varepsilon) \mid \mathcal{F}_S] - \varepsilon$ a.s. Similarly, one can show that for each $\sigma \in \mathcal{T}_S$,
$Y_S \leq E[I_S(\tau_S^\varepsilon, \sigma) \mid \mathcal{F}_S] + \varepsilon$ a.s. It follows that for each $\varepsilon > 0$,

$$ess \sup_{\tau\in\mathcal{T}_s} E[I_S(\tau, \sigma_S^\varepsilon) \mid \mathcal{F}_S] - \varepsilon \ \leq\ Y_S \ \leq\ ess \inf_{\sigma\in\mathcal{T}_S} E[I_S(\tau_S^\varepsilon, \sigma) \mid \mathcal{F}_S] + \varepsilon \ \text{ a.s.},$$

that is $\overline{V}(S) - \varepsilon \leq Y_S \leq \underline{V}(S) + \varepsilon$ a.s. Since $\underline{V}(S) \leq \overline{V}(S)$ a.s. we get $\underline{V}(S) = Y_S = \overline{V}(S)$ a.s. This equality holds of each stopping time $S \in \mathcal{T}_0$, which implies the uniqueness of Y. It remains to show the uniqueness of (Z, k, A, A'). By the uniqueness of the decomposition of the semimartingale $Y_t + \int_0^t g(s)ds$, there exists an unique square integrable martingale M and an unique square integrable finite variation RCLL adapted process α with $\alpha_0 = 0$ such that $dY_t + g(t)dt = dM_t - d\alpha_t$. The martingale representation theorem applied to M ensures the uniqueness of the pair $(Z, k) \in I\!H^2 \times I\!H^2_\nu$.
The uniqueness of the processes A, A' follows from the uniqueness of the canonical decomposition of an RCLL process with integrable variation (see Proposition 5.7.9).

Suppose that A and A' are continuous. Since Y and ξ are right-continuous, we have $Y_{\sigma_S^*} = \xi_{\sigma_S^*}$ and $Y_{\tau_S^*} = \zeta_{\tau_S^*}$ a.s. By definition of τ_S^*, on $[S, \tau_S^*[$, we have $Y_t > \xi_t$ a.s. Since $(Y, Z, k(.), A, A')$ is the solution of the DRBSDE, A is constant on $[S, \tau_S^*[$ a.s. and even on $[S, \tau_S^*]$ because A is continuous. Similarly, A' is constant on $[S, \sigma_S^*]$ a.s. The process $(Y_t + \int_0^t g(s)ds, S \leq t \leq \tau_S^* \wedge \sigma_S^*)$ is thus a martingale. Hence, we have $Y_S = E[I_S(\tau_S^*, \sigma_S^*) \mid \mathcal{F}_S]$ a.s. By similar arguments as above, one can show that for each $\tau, \sigma \in \mathcal{T}_S$, $E[I_S(\tau, \sigma_S^*) \mid \mathcal{F}_S] \leq Y_S$ and $Y_S \leq E[I_S(\tau_S^*, \sigma) \mid \mathcal{F}_S]$ a.s., which yields that (τ_S^*, σ_S^*) is an S-saddle point. □

Proof of Proposition 5.3.10. Since $J^g \geq J'^g + \tilde{\xi}^g$ and $J'^g \geq J^g - \tilde{\zeta}^g$, $J^g \in \mathcal{S}^2 \Leftrightarrow J'^g \in \mathcal{S}^2$. Using the minimality property of J and J' given in Lemma 5.3.3, one can show that $J^g \in \mathcal{S}^2$ if and only if there exist two non-negative supermartingales $H^g, H'^g \in \mathcal{S}^2$ such that

$$\tilde{\xi}_t^g \leq H_t^g - H_t'^g \leq \tilde{\zeta}_t^g \quad 0 \leq t \leq T \quad \text{a.s.} \tag{5.7.8}$$

Since this equivalence holds for all $g \in I\!H^2$, in particular when $g = 0$, we get (ii) $\Leftrightarrow$ (iii).
To prove (i) $\Leftrightarrow$ (ii), it is sufficient to show that (5.3.11) is equivalent to (5.7.8). Suppose that (5.3.11) is satisfied. By setting

$$\begin{cases} H_t^g := H_t - E[\xi_T^+ ds|\mathcal{F}_t] - E[\int_t^T g^+(s)ds|\mathcal{F}_t], 0 \leq t \leq T \\ H_t'^g := H_t' - E[\xi_T^- ds|\mathcal{F}_t] - E[\int_t^T g^-(s)ds|\mathcal{F}_t], 0 \leq t \leq T, \end{cases}$$

(5.7.8) holds. Similarly, (5.7.8) implies (5.3.11). We have that (i) implies (iv). It remains to prove that (iv) implies (i). Let (Y, Z, k, A, A') be the solution of the DRBSDE (5.2.2) associated with driver process (g_t). Let $H_t^g := E[A_T - A_t|\mathcal{F}_t]$ and $H_t^{'g} := E[A_T^{'} - A_t^{'}|\mathcal{F}_t]$. We have $H_t^g - H_t^{'g} = Y_t - E[\int_t^T g(s)ds|\mathcal{F}_t]$. Since $\xi \leq Y \leq \zeta$, condition (5.7.8) holds. $\square$

Proof of Theorem 5.4.1: For $\beta > 0$, $\phi \in I\!H^2$, and $l \in I\!H_\nu^2$, we introduce the norms $\|\phi\|_\beta^2 := E[\int_0^T e^{\beta s}\phi_s^2 ds]$, and $\|l\|_{\nu,\beta}^2 := E[\int_0^T e^{\beta s}\|l_s\|_\nu^2\, ds]$.

Let $I\!H_{\beta,\nu}^2$ (below simply denoted by $I\!H_\beta^2$) the space $I\!H^2 \times I\!H^2 \times I\!H_\nu^2$ equipped with the norm $\|Y, Z, k(\cdot)\|_\beta^2 := \|Y\|_\beta^2 + \|Z\|_\beta^2 + \|k\|_{\nu,\beta}^2$.

We define a mapping Φ from $I\!H_\beta^2$ into itself as follows. Given $(U, V, l) \in I\!H_\beta^2$, by Theorem 5.3.5 there exists a unique process $(Y, Z, k) = \Phi(U, V, l)$ solution of the DRBSDE associated with driver process $g(s) = g(s, U_s, V_s, l_s)$. Note that $(Y, Z, k) \in I\!H_\beta^2$. Let A, A' be the associated non decreasing processes. Let us show that Φ is a contraction and hence admits a unique fixed point (Y, Z, k) in $I\!H_\beta^2$, which corresponds to the unique solution of DRBSDE (5.2.2). The associated finite variation process is then uniquely determined in terms of (Y, Z, k) and the pair (A, A') corresponds to the unique canonical decomposition of this finite variation process. Let (U^2, V^2, l^2) be another element of $I\!H_\beta^2$ and define $(Y^2, Z^2, k^2) = \Phi(U^2, V^2, l^2)$. Let A^2, A'^2 be the associated non decreasing processes. Set $\overline{U} = U - U^2$, $\overline{V} = V - V^2$, $\bar{l} = l - l^2$ and, $\overline{Y} = Y - Y^2$, $\overline{Z} = Z - Z^2$, $\overline{k} = k - k^2$. By Itô's formula, for any $\beta > 0$, we have

$$\begin{aligned}
&\overline{Y}_0^2 + E\int_0^T e^{\beta s}[\beta\overline{Y}_s^2 + \overline{Z}_s^2 + \|\overline{k}_s^2\|]ds + E\sum_{0<s\leq T} e^{\beta s}(\Delta A_s - \Delta A_s^2 - \Delta A_s^{'} + \Delta A_s^{'2})^2 \\
&= 2E\int_0^T e^{\beta s}\overline{Y}_s[g(s, U_s, V_s, l_s) - g(s, U_s^2, V_s^2, l_s^2)]\, ds + 2E[\int_0^T e^{\beta s}\overline{Y}_{s^-}\, dA_s - \int_0^T e^{\beta s}\overline{Y}_{s^-}\, dA_s^2] \\
&- 2E[\int_0^T e^{\beta s}\overline{Y}_{s^-}\, dA_s^{\prime} - \int_0^T e^{\beta s}\overline{Y}_{s^-}\, dA_s^{\prime 2}].
\end{aligned} \tag{5.7.9}$$

Now, we have a.s.

$$\overline{Y}_s dA_s^c = (Y_s - \xi_s)dA_s^c - (Y_s^2 - \xi_s)dA_s^c = -(Y_s^2 - \xi_s)dA_s^c \leq 0$$

and by symmetry, $\overline{Y}_s dA_s^{2c} \geq 0$ a.s. Also, we have a.s.

$$\overline{Y}_{s^-}\Delta A_s^d = (Y_{s^-} - \xi_{s^-})\Delta A_s^d - (Y_{s^-}^2 - \xi_{s^-})\Delta A_s^d = -(Y_{s^-}^2 - \xi_{s^-})\Delta A_s^d \leq 0$$

and $\overline{Y}_{s^-}\Delta A^{2}{}_s^d \geq 0$ a.s. Similarly, we have a.s.

$$\overline{Y}_s dA_s^{\prime c} = (Y_s - \zeta_s)dA_s^{\prime c} - (Y_s^2 - \zeta_s)dA_s^{\prime c} = -(Y_s^2 - \zeta_s)dA_s^{\prime c} \geq 0$$

and by symmetry, $\overline{Y}_s dA_s^{\prime 2c} \leq 0$ a.s. Also, we have a.s.

$$\overline{Y}_{s^-}\Delta A_s^{\prime d} = (Y_{s^-} - \zeta_{s^-})\Delta A_s^{\prime d} - (Y_{s^-}^2 - \zeta_{s^-})\Delta A_s^{\prime d} = -(Y_{s^-}^2 - \zeta_{s^-})\Delta A_s^{\prime d} \geq 0$$

and $\overline{Y}_{s^-}\Delta A'^{2}{}_s^d \leq 0$ a.s.

Consequently, the second and the third term of (5.7.9) are non positive. By using the Lipschitz property of g and the inequality $2Cyu \leq 2C^2y^2 + \frac{1}{2}u^2$, we get

$$\beta\|\overline{Y}\|_\beta^2 + \|\overline{Z}\|_\beta^2 + \|\overline{k}\|_{\nu,\beta}^2 \leq 6C^2\|\overline{Y}\|_\beta^2 + \frac{1}{2}(\|\overline{U}\|_\beta^2 + \|\overline{V}\|_\beta^2 + \|\bar{l}\|_{\nu,\beta}^2).$$

Choosing $\beta = 6C^2 + 1$, we deduce $\|(\overline{Y}, \overline{Z}, \overline{k})\|_\beta^2 \leq \frac{1}{2}\|(\overline{U}, \overline{V}, \overline{l})\|_\beta^2$.
The last assertion of the theorem follows from Theorem 5.3.7. □

By similar arguments as above, one can show the following estimate, which is expressed in terms of the associated increasing processes.

Proposition 5.7.5 (A classical estimate). *Let $\xi^1, \xi^2, \zeta^1, \zeta^2 \in \mathcal{S}^2$ such that for $i = 1, 2$, $\xi_T^i = \zeta_T^i$ a.s. and $\xi_t^i \leq \zeta_t^i$, $0 \leq t \leq T$ a.s. Suppose that for $i = 1, 2$, ξ^i and ζ^i satisfy Mokobodski's condition. Let g^1, g^2 be Lipschitz drivers satisfying Assumption 6.3.9 with Lipschitz constant $C > 0$. For $i = 1, 2$, let $(Y^i, Z^i, k^i, A^i, A'^i)$ be the solution of the DRBSDE associated with driver g^i, terminal time T and barriers ξ^i, ζ^i.*
For $s \in [0, T]$, let $\overline{Y}_s := Y_s^1 - Y_s^2$, $\overline{\xi}_s := \xi_s^1 - \xi_s^2$, $\overline{\zeta}_s = \zeta_s^1 - \zeta_s^2$, $\overline{g}(s) := g^1(s, Y_s^2, Z_s^2, k_s^2) - g(s, Y_s^2, Z_s^2, k_s^2)$. We have:

$$\|\overline{Y}\|_{\mathcal{S}^2}^2 \leq K \Bigg(E[\overline{\xi}_T^2] + E[\int_0^T \overline{g}^2(s)ds] + \|A_T^1 + A_T^2\|_{L^2} \| \sup_{0 \leq s < T} |\overline{\xi}_s| \|_{L^2} + \|A_T'^1 + A_T'^2\|_{L^2} \| \sup_{0 \leq s < T} |\overline{\zeta}_s| \|_{L^2} \Bigg), \quad (5.7.10)$$

where the real constant $K > 0$ is universal, that is, depends only on C and T.

For the proof, see proof of Th. 3.2 in Appendix of [50].

Remark 5.7.6. In [50], in the particular case when for each $i = 1, 2$, the lower barrier ξ^i is of the form $\xi^i = M^i + B^i$, where M^i is a square integrable martingale and B^i is a square integrable RCLL predictable non decreasing process with $B_0^i = 0$, the authors derive from (5.7.10) the following estimate:

$$\|\overline{Y}\|_{\mathcal{S}^2}^2 \leq K \left(E[\overline{\xi}_T^2] + E[\int_0^T \overline{g}^2(s)ds] + \phi(\| \sup_{0 \leq s < T} |\overline{\xi}_s| \|_{L^2} + \| \sup_{0 \leq s < T} |\overline{\zeta}_s| \|_{L^2}) \right), \quad (5.7.11)$$

where the constant $\phi > 0$ is not necessarily universal, depending in particular on $\|\xi^i\|_{\mathcal{S}^2}$, $\|\zeta^i\|_{\mathcal{S}^2}$, $\|g^i(s, 0, 0, 0)\|_{I\!H^2}$ and B^i, for $i = 1, 2$. For details, the reader is refered to estimate (14) of Th. 3.2 in [50] proven in the Appendix.

We now easily show an $\mathcal{E}^g$-Doob-Meyer decomposition of $\mathcal{E}^g$-supermartingales, which generalizes the results given in [130] and [?] under stronger assumptions. Moreover, our proof gives an alternative short proof of these previous results.

Definition 5.7.7. *Let $Y \in \mathcal{S}^2$. The process (Y_t) is said to be a strong $\mathcal{E}^g$-supermartingale (resp $\mathcal{E}^g$-submartingale), if $\mathcal{E}_{\sigma,\tau}^g(Y_\tau) \leq Y_\sigma$ (resp. $\mathcal{E}_{\sigma,\tau}^g(Y_\tau) \geq Y_\sigma$) a.s. on $\sigma \leq \tau$, for all $\sigma, \tau \in \mathcal{T}_0$.*

Proposition 5.7.8. *Suppose that g satisfies Assumption* (5.4.3).

- *Let A be a non decreasing (resp non increasing) RCLL predictable process in $\mathcal{S}^2$ with $A_0 = 0$. Let $(Y, Z, k) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\nu^2$ following the dynamics:*

$$-dY_s = g(s, Y_s, Z_s, k_s)ds + dA_s - Z_s dW_s - \int_{\mathbf{E}} k_s(e)\tilde{N}(ds, de). \quad (5.7.12)$$

Then the process (Y_t) a strong $\mathcal{E}^g$-supermartingale (resp $\mathcal{E}^g$-submartingale).

- *($\mathcal{E}^g$-Doob-Meyer decomposition) Let (Y_t) be a strong $\mathcal{E}^g$-supermartingale (resp. $\mathcal{E}^g$-submartingale). Then, there exists a non decreasing (resp non increasing) RCLL predictable process A in $\mathcal{S}^2$ with $A_0 = 0$ and $(Z, k) \in I\!H^2 \times I\!H^2_\nu$ such that (5.7.12) holds.*

Proof. Suppose A is non decreasing. Let $(X^\tau, \pi^\tau, l^\tau)$ be the solution of the BSDE associated with driver g, terminal time τ, and terminal condition Y_τ, Since g satisfies Assumption 5.4.3 and since $g(s, y, z, k)ds + dA_s \geq g(s, y, z, k)ds$, the comparison theorem for BSDEs (see Theorem 4.2 in [137]) gives that $Y_\sigma \geq X^\tau_\sigma = \mathcal{E}^g_{\sigma,\tau}(Y_\tau)$ a.s. on $\{\sigma \leq \tau\}$. The case when A is non-increasing can be shown similarly.
Let us show the second assertion. Fix $S \in \mathcal{T}_0$. Since (Y_t) is a strong $\mathcal{E}^g$-supermartingale, we derive that for each $\tau \in \mathcal{T}_S$, we have $Y_S \geq \mathcal{E}^g_{S,\tau}(Y_\tau)$ a.s. We thus get $Y_S \geq \operatorname{ess\,sup}_{\tau\in\mathcal{T}_S} \mathcal{E}^g_{S,\tau}(Y_\tau)$ a.s. Now, by definition of the essential supremum, $Y_S \leq \operatorname{ess\,sup}_{\tau\in\mathcal{T}_S} \mathcal{E}^g_{S,\tau}(Y_\tau)$ a.s. because $S \in \mathcal{T}_S$. Hence,

$$Y_S = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_S} \mathcal{E}^g_{S,\tau}(Y_\tau) \quad a.s.$$

By Theorem 3.3 in [138], the process (Y_t) coincides with the solution of the reflected BSDE associated with the RCLL obstacle (Y_t). The result follows. □

We now show the following result on RCLL adapted processes with integrable total variation, which can be seen as a probabilistic version of a well-known analysis result.

Proposition 5.7.9. *Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a completed right-continuous filtration $(\mathcal{F}_t)_{0\leq t\leq T}$. Let $\alpha = (\alpha_t)_{0\leq t\leq T}$ be a RCLL adapted process with integrable total variation, that is, $E(\int_0^T |d\alpha_t|) < \infty$.*
There exists an unique pair $(A, A') \in (\mathcal{A}^1)^2$ such that $\alpha = A - A'$ with $dA_t \perp dA'_t$.
This decomposition is called the canonical decomposition *of the process α.*
Moreover, if $(B, B') \in (\mathcal{A}^1)^2$ satisfies $\alpha = B - B'$, then $dA_t << dB_t$ in the (probabilistic) sense, that is, for each $K \in \mathcal{P}$ with $\int_0^T \mathbf{1}_K dB_t = 0$ a.s., then $\int_0^T \mathbf{1}_K dA_t = 0$ a.s.

Proof. By classical results, the process α can be written as $\alpha = B - B'$ with $B, B' \in \mathcal{A}^1$. Let $C_t := B_t + B'_t$. This process belongs to $\mathcal{A}^1$. For almost every ω, the measures $dB_.(\omega)$ and $dB'_.(\omega)$ on $[0, T]$ are absolutely continuous with respect to $dC_.(\omega)$. By using the Radon-Nikodym Theorem for predictable RCLL non decreasing processes (see Th. 67, Chap. VI in [54]), there exist nonnegative predictable processes H and H' such that for each $t \in [0, T]$,

$$B_t = \int_0^t H_s dC_s \quad \text{and} \quad B'_t = \int_0^t H'_s dC_s \quad \text{a.s}$$

Let A and A' be the processes defined by

$$A_t := \int_0^t (H_s - H'_s)^+ dC_s \quad \text{and} \quad A'_t := \int_0^t (H_s - H'_s)^- dC_s.$$

They belong to $\mathcal{A}^1$. Now, the set D defined by

$$D := \{(t, \omega)\,,\ H_t(\omega) - H'_t(\omega) \geq 0\}$$

belongs to $\mathcal{P}$. We have $\int_0^T \mathbf{1}_{D_t^c} dA_t = \int_0^T \mathbf{1}_{\{H_t - H'_t < 0\}} (H_t - H'_t)^+ dC_t = 0$ a.s. Similarly $\int_0^T \mathbf{1}_{D_t} dA'_t = 0$ a.s., which implies that $dA_t \perp dA'_t$.
It remains to show the uniqueness of this decomposition. Since $dA_t \perp dA'_t$, it follows that, for almost every ω, the deterministic measures $dA_t(\omega)$ and $dA'_t(\omega)$ are mutually singular in the classical analysis sense. Hence, for almost every ω, the non decreasing maps $A_.(\omega)$ and $A'_.(\omega)$ correspond to the unique canonical decomposition of the RCLL bounded variational map $\alpha_.(\omega)$ by a well-known analysis result. This implies the uniqueness of A, A'.
Moreover, since $(H_t - H'_t)^+ \leq H_t$, the last assertion holds. $\square$

Remark 5.7.10. Note that it is obvious that, if the random measures dA_t and dA'_t are mutually singular in the probabilistic sense (see Definition 5.2.2), then for almost every ω, the deterministic measures on $[0, T]$ $dA_t(\omega)$ and $dA'_t(\omega)$ are mutually singular in the classical analysis sense. The converse is not so immediate. However, it holds by the above property.

Chapter 6

A Weak Dynammic Programming Principle for Stochastic Control/Optimal Stopping with f-Expectations.

Abstract. We study combined optimal control/stopping problems with $\mathcal{E}^f$-expectations in the Markovian framework on a finite horizon of time T. We establish a *weak* dynamic programming principle, which extends to the nonlinear case the one obtained in [35] in the case of linear expectations . To this purpose, we prove some measurability properties and a "splitting" result stating that, given an intermediary time $t \leq T$, the problem can be decomposed into two independent parts, one corresponding to the past (before t) and one to the future (after t). Using this *weak* dynamic programming principle and properties of reflected backward stochastic differential equations, we prove that the value function of our combined control problem, which is not necessarily continuous, not even measurable, is a *weak* viscosity solution of a nonlinear Hamilton-Jacobi-Bellman variational inequality.

Some illustrating examples in mathematical finance are provided.

6.1 Introduction

Markovian stochastic control problems on a given horizon of time T can typically be written as

$$u(0,x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}[\int_0^T f(\alpha_s, X_s^\alpha)ds + g(X_T^\alpha)], \tag{6.1.1}$$

where $\mathcal{A}$ is a set of admissible control processes α_s, and (X_s^α) is a controlled process of the form

$$X_s^\alpha = x + \int_0^s b(X_u^\alpha, \alpha_u)du + \int_0^s \sigma(X_u^\alpha, \alpha_u)dW_u + \int_0^s \int_{\mathbb{R}^n} \beta(X_u^\alpha, \alpha_u, e)\tilde{N}(du, de).$$

The random variable $g(X_T^\alpha)$ may represent a terminal reward and $f(\alpha_s, X_s^\alpha)$ an instantaneous reward process. Formally, for all initial time t in $[0,T]$ and initial state y, the associated value function is defined by

$$u(t,y) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}[\int_t^T f(\alpha_s, X_s^\alpha)ds + g(X_T^\alpha) \mid X_t^\alpha = y]. \tag{6.1.2}$$

The dynamic programming principle can formally be stated as

$$u(0,x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}[\int_0^t f(\alpha_s, X_s^\alpha)ds + u(t, X_t^\alpha)], \quad \text{for } t \text{ in } [0,T]. \tag{6.1.3}$$

In the literature, this principle is classically established under assumptions which ensure that the value function u satisfies some regularity/ measurability properties. We refer e.g. to the books by Fleming-Rischel (1975) [82], Krylov (1980) [106], El Karoui (1981) [70], Bensoussan-Lions J. (1988), Lions P.-L. (1983) [113], Fleming-Soner (2006) [83], Oksendal-Sulem (2007) [122], Pham (2009) [131].

The case of a discontinuous value function has been studied in a deterministic framework in the eighties: a *weak* dynamic programming principle has been established for deterministic control by Barles (1993) (see [8]) (see also Barles and Perthame (1986) [11]). More precisely, he proves that the upper semicontinuous envelope u^* and the lower semicontinuous envelope u_* of the value function u satisfy, respectively, the *sub- and super-optimality principle of dynamic programming* of Lions and Souganidis (1985) [114]. He then derived that the (discontinuous) value function is a *weak viscosity* solution of the associated Bellman equation in the sense that u^* is a viscosity subsolution and u_* is a supersolution of the Bellman equation.

More recently, Bouchard and Touzi (2011) (see [35]) have proved a *weak* dynamic programming principle in a stochastic framework, when the value function is not necessarily continuous, not even measurable. They prove that the upper semicontinuous envelope u^* satisfies the suboptimality principle of dynamic programming, and under an additional regularity (lower semi continuity) assumption of the reward g, they obtain that the lower semicontinuous envelope u_* satisfies the super-optimality principle.

A *weak* dynamic principle has been further established, under some specific regularity assumptions, for problems with state constraints by Bouchard and Nutz (2012) in [34], and for zero-sum stochastic games by Bayraktar and Yao (2013) in [14].

In this chapter we are interested in generalizing these results to the case when the linear expectation $\mathbb{E}$ is replaced by a nonlinear expectation induced by a Backward Stochastic Differential Equation (BSDE). Typically, such problems in the Markovian case can be formulated as

$$\sup_{\alpha \in \mathcal{A}} \mathcal{E}^{\alpha}_{0,T}[g(X_T^\alpha)], \tag{6.1.4}$$

where $\mathcal{E}^\alpha$ is the nonlinear conditional expectation associated with a BSDE with jumps with controlled driver $f(\alpha_t, X_t^\alpha, y, z, k)$.

Note that Problem (6.1.1) is a particular case of (6.1.4) when the driver f does not depend on the solution of the BSDE, that is when $f(\alpha_t, X_t^\alpha, y, z, k) \equiv f(\alpha_t, X_t^\alpha)$.

We study here the case when the reward function g is only Borelian. We provide a *weak* dynamic programming principle. To this purpose, we prove some measurability properties and a "splitting" result stating that, given an intermediary time $t \leq T$, the problem can be decomposed into two independent parts, one corresponding to the past (before t) and one to the future (after t). We point out that no regularity conditions on the reward g are required to obtain the sub and super-optimality principles, which is not the case in the previous literature even in the linear case (see [35], [34] and [14]). Using this *weak* dynamic programming principle, we derive that the value function u, which is not necessarily continuous, not even measurable, is a *weak* viscosity solution of an associated nonlinear Hamilton-Jacobi-Bellman (HJB) equation.

Moreover, in this chapter, we consider the combined problem when there is an additional control in the form of a stopping time. We thus consider mixed generalized optimal control/stopping problems of the form

$$\sup_{\alpha\in\mathcal{A}}\sup_{\tau\in\mathcal{T}}\mathcal{E}^{\alpha}_{0,\tau}[\bar{h}(\tau, X^{\alpha}_{\tau})], \tag{6.1.5}$$

where $\mathcal{T}$ denotes the set of stopping times with values in $[0,T]$, and $\bar{h}$ is an irregular reward function.

The chapter is organized as follows: in Section 6.2, we introduce our generalized mixed control-optimal stopping problem. Using results on reflected BSDEs, we express this problem as an optimal control problem for reflected BSDEs. In Section 6.3, we prove a *weak* dynamic programming principle for our mixed problem with f-expectation. This requires some specific techniques of stochastic analysis and BSDEs to handle measurability and other issues due to the nonlinearity of the expectation and the lack of regularity of the terminal reward.

Using the dynamic programming principle and properties of reflected BSDEs, we prove in Section 6.4 that the value function of our mixed problem is a *weak* viscosity solution of a nonlinear Hamilton-Jacobi-Bellman (HJB) variational inequality. In Section 6.5, we give illustrating examples in mathematical finance such as optimization problems involving recursive utilities and dynamic risk measures.

6.2 Formulation of the problem

We consider the product space $\Omega := \Omega_W \otimes \Omega_N$, where $\Omega_W := \mathcal{C}([0,T])$ is the Wiener space, that is the set of continuous functions ω^1 from $[0,T]$ into $\mathbb{R}^p$ such that $\omega^1(0)=0$, and $\Omega_N := \mathbb{D}([0,T])$ is the Skorohod space of right-continuous with left limits (RCLL) functions ω^2 from $[0,T]$ into $\mathbb{R}^d$, such that $\omega^2(0)=0$. Recall that Ω is a Polish space for the topology of Skorohod. Here $p,d\geq 1$, but for notational simplicity, however, we shall consider only $\mathbb{R}$-valued functions, that is the case $p=d=1$.

Let $B=(B^1,B^2)$ be the canonical process defined for each $t\in[0,T]$ and each $\omega=(\omega^1,\omega^2)$ by $B^i_t(\omega)=B^i_t(\omega^i):=\omega^i_t$, for $i=1,2$. Let us denote the first coordinate process B^1 by W. Let P^W be the probability measure on $(\Omega_W,\mathcal{B}(\Omega_W))$ such that W is a Brownian motion. Here $\mathcal{B}(\Omega_W)$ denotes the Borelian σ-algebra on Ω_W.

Set $\mathbf{E} := \mathbb{R}^n\backslash\{0\}$ equipped with its Borelian σ-algebra $\mathcal{B}(\mathbf{E})$, where $n\geq 1$. We define the jump random measure N as follows: for each $t>0$ and each $\mathbf{B}\in\mathcal{B}(\mathbf{E})$,

$$N(.,[0,t]\times\mathbf{B}) := \sum_{0<s\leq t}\mathbf{1}_{\{\Delta B^2_s\in\mathbf{B}\}}. \tag{6.2.1}$$

The measurable set $(\mathbf{E},\mathcal{B}(\mathbf{E}))$ is equipped with a σ-finite positive measure ν such that $\int_{\mathbf{E}}(1\wedge|e|)\nu(de)<\infty$. Let P^N be the probability measure on $(\Omega_N,\mathcal{B}(\Omega_N))$ such that N is a Poisson random measure with compensator $\nu(de)dt$ and such that $B^2_t=\sum_{0<s\leq t}\Delta B^2_s$ a.s. (note that

the sum of jumps is well defined up to a P^N-null set). In the following, we set $\tilde{N}(dr, de) = N(dr, de) - \nu(de)dt$.

Let $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ be the completed filtration associated with the canonical process B. We also define the product probability measure $P := P^W \otimes P^N$.

Let $T > 0$ be fixed. Let $\mathbb{H}^2_T$ (denoted also by $\mathbb{H}^2$) be the set of real-valued predictable processes (Z_t) such that $\mathbb{E}\int_0^T Z_s^2 ds < \infty$ and let $\mathcal{S}^2$ be the set of real-valued RCLL adapted processes (φ_s) with $\mathbb{E}[\sup_{0\leq s\leq T}\varphi_s^2] < \infty$,

Let L^2_ν be the set of measurable functions $l : (\mathbf{E}, \mathcal{B}(\mathbf{E})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\|l\|^2_\nu := \int_{\mathbf{E}} l^2(e)\nu(de) < \infty$. The set L^2_ν is a Hilbert space equipped with the scalar product $\langle l,\, l'\rangle_\nu := \int_{\mathbf{E}} l(e)l'(e)\nu(de)$ for all $l,\, l' \in L^2_\nu \times L^2_\nu$.

Let $\mathbb{H}^2_\nu$ denote the set of predictable real-valued processes $(k_t(\cdot))$ with $\mathbb{E}\int_0^T \|k_s\|^2_{L^2_\nu} ds < \infty$.

Let $\mathcal{A}$ be the set of controls, defined as the set of predictable processes α valued in a compact subset $\mathbf{A}$ of $\mathbb{R}^p$, where $p \in \mathbb{N}^*$. For each $\alpha \in \mathcal{A}$, initial time $t \in [0, T]$ and initial condition x in $\mathbb{R}$, let $(X_s^{\alpha,t,x})_{t\leq s\leq T}$ be the unique $\mathbb{R}$-valued solution in $\mathcal{S}^2$ of the stochastic differential equation (SDE):

$$X_s^{\alpha,t,x} = x + \int_t^s b(X_r^{\alpha,t,x}, \alpha_r)dr + \int_t^s \sigma(X_r^{\alpha,t,x}, \alpha_r)dW_r + \int_t^s \int_{\mathbf{E}} \beta(X_{r^-}^{\alpha,t,x}, \alpha_r, e)\tilde{N}(dr, de),$$

where $b,\ \sigma : \mathbb{R}\times\mathbf{A} \to \mathbb{R}$, are Lipschitz continuous with respect to x and α, and $\beta : \mathbb{R}\times\mathbf{A}\times\mathbf{E} \to \mathbb{R}$ is a measurable bounded function such that for some constant $C \geq 0$, and for all $e \in \mathbf{E}$

$$\begin{aligned}&|\beta(x, \alpha, e)| \leq C\,\Psi(e), \quad x \in \mathbb{R}, \alpha \in \mathbf{A} \quad \text{where} \quad \Psi \in L^2_\nu.\\ &|\beta(x, \alpha, e) - \beta(x', \alpha', e)| \leq C(|x - x'| + |\alpha - \alpha'|)\Psi(e), \quad x, x' \in \mathbb{R}, \alpha, \alpha' \in \mathbf{A}.\end{aligned}$$

The criterion of our mixed control problem, depending on α, is defined via a BSDE with driver function f satisfying the following hypothesis:

Assumption 6.2.1. *$f : \mathbf{A} \times [0, T] \times \mathbb{R}^3 \times L^2_\nu \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is $\mathcal{B}(\mathbf{A}) \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^3) \otimes \mathcal{B}(L^2_\nu)$-measurable and satisfies*

(i) $|f(\alpha, t, x, 0, 0, 0)| \leq C(1 + |x|^p), \forall\alpha \in \mathbf{A}, t \in [0, T], x \in \mathbb{R}$, *where* $p \in \mathbb{N}^*$.

(ii) $|f(\alpha, t, x, y, z, k) - f(\alpha', t, x', y', z', k')| \leq C(|\alpha - \alpha'| + |x - x'| + |y - y'| + |z - z'| + \|k - k'\|_{L^2_\nu})$, $\forall t \in [0, T],\ x, x', y, y', z, z' \in \mathbb{R},\ k, k' \in L^2_\nu, \alpha, \alpha' \in \mathbf{A}$.

(iii) $f(\alpha, t, x, y, z, k_2) - f(\alpha, t, x, y, z, k_1) \geq < \gamma(\alpha, t, x, y, z, k_1, k_2), k_2 - k_1 >_\nu, \forall t, x, y, z, k_1, k_2, \alpha$,

where $\gamma : \mathbf{A}\times[0, T]\times\mathbb{R}^3\times(L^2_\nu)^2 \to (L^2_\nu, \mathcal{B}(L^2_\nu))$ *is* $\mathcal{B}(\mathbf{A})\otimes\mathcal{B}([0, T])\otimes\mathcal{B}(\mathbb{R}^3)\otimes\mathcal{B}((L^2_\nu)^2)$*-measurable,*

$$|\gamma(.)(e)| \leq \Psi(e) \quad \text{and} \quad \gamma(.)(e) \geq -1 \quad d\nu(e) - \text{a.s.} \quad \text{where} \quad \Psi \in L^2_\nu.$$

Remark 6.2.2. Note that if f is of the form $f(\alpha, x, y, z, k) := \overline{f}(\alpha, x, y, z, \int_{\mathbf{E}} k(e)\Psi(e)\nu(de))$, where Ψ is a non negative function belonging to L^2_ν and $\overline{f} : \mathbf{A} \times [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ is Borelian, non-decreasing with respect to its last variable, and Lipschitz continuous with constant C (as in [9]), then f satisfies condition (iii) with $\gamma(k_1, k_2)(e) := C\Psi(e)\mathbf{1}_{\{\int_{\mathbf{E}}(k_2(e)-k_1(e))\Psi(e)\nu(de)\leq 0\}}$.

For all $(t,x) \in [0,T] \times \mathbb{R}$ and all control $\alpha \in \mathcal{A}$, let $f^{\alpha,t,x}$ be the driver defined by

$$f^{\alpha,t,x}(r,\omega,y,z,k) := f(\alpha_r(\omega), r, X_r^{\alpha,t,x}(\omega), y, z, k).$$

Since f satisfies condition (iii), the driver $f^{\alpha,t,x}$ satisfies Assumption 6.3.9, which ensures that the Comparison Theorem for BSDEs with jumps holds (see Section 6.3.3 or [137]).

We introduce the nonlinear conditional expectation $\mathcal{E}^{f^{\alpha,t,x}}$ (denoted more simply by $\mathcal{E}^{\alpha,t,x}$) associated with $f^{\alpha,t,x}$, defined for each stopping time S and for each $\eta \in L^2(\mathcal{F}_S)$ as:

$$\mathcal{E}_{r,S}^{\alpha,t,x}[\eta] := \mathcal{X}_r^{\alpha,t,x}, \ \ t \leq r \leq S,$$

where $(\mathcal{X}_r^{\alpha,t,x})_{t\leq r\leq S}$ is the solution in $\mathcal{S}^2$ of the BSDE associated with driver $f^{\alpha,t,x}$, terminal time S and terminal condition η, that is satisfying:

$$\begin{cases} -d\mathcal{X}_r^{\alpha,t,x} = f(\alpha_r, r, X_r^{\alpha,t,x}, \mathcal{X}_r^{\alpha,t,x}, Z_r^{\alpha,t,x}, K_r^{\alpha,t,x}(\cdot))dr - Z_r^{\alpha,t,x} dW_r - \int_{\mathbf{E}} K_r^{\alpha,t,x}(e)\tilde{N}(dr,de) \\ \mathcal{X}_S^{\alpha,t,x} = \eta, \end{cases}$$

where $(Z_s^{\alpha,t,x})$, $(K_s^{\alpha,t,x})$ are the associated processes, which belong respectively to $\mathbb{H}^2$ and $\mathbb{H}^2_\nu$.

For all $(t,x) \in [0,T] \times \mathbb{R}$ and all control $\alpha \in \mathcal{A}$, we define the reward by $h(s, X_s^{\alpha,t,x})$ for $t \leq s < T$ and $g(X_T^{\alpha,t,x})$ for $t = T$, where

- $g : \mathbb{R} \to \mathbb{R}$ is Borelian.
- $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a function which is continuous with respect to x uniformly in t, and right-continuous and left-limited with respect to t on $[0,T]$.
- $|h(t,x)| + |g(x)| \leq C(1 + |x|^p), \forall t \in [0,T], x \in \mathbb{R}, \text{ with } p \in \mathbb{N}^*$.

Let $\mathcal{T}$ be the set of stopping times with values in $[0,T]$. Suppose the initial time is equal to 0. For each initial condition $x \in \mathbb{R}$, we consider the mixed optimal control/stopping problem:

$$u(0,x) := \sup_{\alpha\in\mathcal{A}} \sup_{\tau\in\mathcal{T}} \mathcal{E}_{0,\tau}^{\alpha,0,x}[\bar{h}(\tau, X_\tau^{\alpha,0,x})], \tag{6.2.2}$$

where

$$\bar{h}(t,x) := h(t,x)\mathbf{1}_{t<T} + g(x)\mathbf{1}_{t=T}.$$

Note that $\bar{h}$ is Borelian but not necessarily regular in (t,x).

We now make the problem dynamic. We define, for $t \in [0,T]$ and each $\omega \in \Omega$ the t-translated path $\omega^t = (\omega_s^t)_{s\geq t} := (\omega_s - \omega_t)_{s\geq t}$. Note that $(\omega_s^{1,t})_{s\geq t} := (\omega_s^1 - \omega_t^1)_{s\geq t}$ corresponds to the realizations of the translated Brownian motion $W^t := (W_s - W_t)_{s\geq t}$ and that the translated Poisson random measure $N^t := N(]t,s],.)_{s\geq t}$ can be expressed in terms of $(\omega_s^{2,t})_{s\geq t} := (\omega_s^2 - \omega_t^2)_{s\geq t}$ similarly to (6.2.1). Let $\mathbb{F}^t = (\mathcal{F}_s^t)_{t\leq s\leq T}$ be the completed filtration associated with W^t and N^t. Let us denote by $\mathcal{T}_t^t$ the set of stopping times with respect to $\mathbb{F}^t$ with values in $[t,T]$. Let $\mathcal{P}^t$ be the predictable σ-algebra on $\Omega \times [t,T]$ equipped with the filtration $\mathbb{F}^t$.

We now introduce the following spaces of processes. Let $t \in [0,T]$. Let $\mathbb{H}_t^2$ be the $\mathcal{P}^t$-measurable processes Z on $\Omega \times [t,T]$ such that $\|Z\|_{\mathbb{H}_t^2} := \mathbb{E}[\int_t^T Z_u^2 du] < \infty$. We define $\mathbb{H}^2_{t,\nu}$ as the set of $\mathcal{P}^t$-measurable processes K on $\Omega \times [t,T]$ such that $\|K\|_{\mathbb{H}^2_{t,\nu}} := \mathbb{E}[\int_t^T ||K_u||_\nu^2 du] < \infty$. Also, we denote by $\mathcal{S}_t^2$ the set of real-valued RCLL processes φ on $\Omega \times [t,T]$, adapted to the filtration $\mathbb{F}^t$, with $\mathbb{E}[\sup_{t\leq s\leq T} \varphi_s^2] < \infty$.

Let $\mathcal{A}_t^t$ be the set of controls $\alpha : \Omega \times [t,T] \mapsto \mathbf{A}$, which are $\mathcal{P}^t$-measurable. For each initial time t and each initial condition x, the associated value function is defined by:

$$u(t,x) := \sup_{\alpha\in\mathcal{A}_t^t} \sup_{\tau\in\mathcal{T}_t^t} \mathcal{E}_{t,\tau}^{\alpha,t,x}[\bar{h}(\tau, X_\tau^{\alpha,t,x})]. \tag{6.2.3}$$

Note that since α and τ depend only on ω^t, the SDE satisfied by $X^{\alpha,t,x}$ and the BSDE satisfied by $\mathcal{E}_{t,\tau}^{\alpha,t,x}[\bar{h}(\tau, X_\tau^{\alpha,0,x})]$ can be solved in $\mathcal{S}_t^2 \times \mathbb{H}_t^2 \times \mathbb{H}_{t,\nu}^2$, with respect to the translated Brownian motion W^t and the translated Poisson random measure N^t and the filtration $\mathbb{F}^t$. Note that the solution depends on ω only through ω^t. Hence the function u is well defined as a deterministic function of t and x.

For each $\alpha \in \mathcal{A}_t^t$, we introduce the function u^α defined as

$$u^\alpha(t,x) := \sup_{\tau\in\mathcal{T}_t^t} \mathcal{E}_{t,\tau}^{\alpha,t,x}[\bar{h}(\tau, X_\tau^{\alpha,t,x})].$$

We thus get

$$u(t,x) = \sup_{\alpha\in\mathcal{A}_t^t} u^\alpha(t,x). \tag{6.2.4}$$

Note that for each α, $u^\alpha(t,x) \geq \bar{h}(t,x)$, and hence $u(t,x) \geq \bar{h}(t,x)$.
Moreover, $u^\alpha(T,x) = u(T,x) = g(x)$.
By Theorem 3.2 in [138], for each α, the value function u^α corresponds to the solution of the reflected BSDE associated with driver $f^{\alpha,t,x} := f(\alpha_\cdot, \cdot, X_\cdot^{\alpha,t,x}, y, z, k)$, (RCLL) obstacle process $\xi_s^{\alpha,t,x} := \bar{h}(s, X_s^{\alpha,t,x})_{t\leq s\leq T}$, and terminal condition $g(X_T^{\alpha,t,x})$, that is

$$u^\alpha(t,x) = Y_t^{\alpha,t,x}, \tag{6.2.5}$$

where $(Y^{\alpha,t,x}, Z^{\alpha,t,x}, K^{\alpha,t,x}) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\nu^2$ is the solution of the reflected BSDE:

$$\begin{cases} Y_s^{\alpha,t,x} = g(X_T^{\alpha,t,x}) + \displaystyle\int_s^T f(\alpha_r, r, X_r^{\alpha,t,x}, Y_r^{\alpha,t,x}, Z_r^{\alpha,t,x}, K_r^{\alpha,t,x}(\cdot))dr + A_T^{\alpha,t,x} - A_s^{\alpha,t,x} \\ \qquad - \displaystyle\int_s^T Z_r^{\alpha,t,x} dW_r - \int_s^T \int_{\mathbf{E}} K^{\alpha,t,x}(r,e)\tilde{N}(dr,de) \\ Y_s^{\alpha,t,x} \geq \xi_s^{\alpha,t,x} = h(s, X_s^{\alpha,t,x}), 0 \leq s < T \ \text{ a.s. }, \\ A^{\alpha,t,x} \text{ is a RCLL nondecreasing predictable process with } A_t^{\alpha,t,x} = 0 \text{ and such that} \\ \displaystyle\int_0^T (Y_s^{\alpha,t,x} - \xi_s^{\alpha,t,x}) dA_s^{\alpha,t,x,c} = 0 \text{ a.s. and } \ \Delta A_s^{\alpha,t,x,d} = -\Delta A_s^{\alpha,t,x} \mathbf{1}_{\{Y_{s^-}^{\alpha,t,x} = \xi_{s^-}^{\alpha,t,x}\}} \quad \text{a.s.} \end{cases} \tag{6.2.6}$$

Here $A^{\alpha,t,x,c}$ denotes the continuous part of A and $A^{\alpha,t,x,d}$ its discontinuous part. In the particular case when $h(T^-,x) \leq g(x)$, then the obstacle $\xi^{\alpha,t,x}$ satisfies for all predictable stopping time τ, $\xi_{\tau^-} \leq \xi_\tau$ a.s. which implies the continuity of the process $A^{\alpha,t,x}$ (see [138]).

In the following, for each $\alpha \in \mathcal{A}_t^t$, $Y_s^{\alpha,t,x}$ will be also denoted by $Y_{s,T}^{\alpha,t,x}[g(X_T^{\alpha,t,x})]$. Using this notation, equality (6.2.5) can be written:

$$u^\alpha(t,x) = Y_t^{\alpha,t,x} = Y_{t,T}^{\alpha,t,x}[g(X_T^{\alpha,t,x})]. \tag{6.2.7}$$

The reflected BSDE (6.2.6) can be solved in $\mathcal{S}_t^2 \times \mathbb{H}_t^2 \times \mathbb{H}_{t,\nu}^2$with respect to the t-translated Brownian motion and the t-translated Poisson random measure. Note that the solution depends on ω only through ω^t.

Our mixed optimal stopping/control problem (6.2.3) can thus be reduced to an optimal control problem for reflected BSDEs:

$$u(t,x) = \sup_{\alpha \in \mathcal{A}_t^t} Y_t^{\alpha,t,x} = \sup_{\alpha \in \mathcal{A}_t^t} Y_{t,T}^{\alpha,t,x}[g(X_T^{\alpha,t,x})].$$

This key property will be used to solve our mixed problem. We point out that in the classical case of linear expectations, this approach allows us to provide alternative proofs of the dynamic programming principle to those given in the previous literature.

Remark 6.2.3. Some mixed optimal control/stopping problems with nonlinear expectations have been studied in [15, 138]. In these papers, the reward process does not depend on the control, which yields the characterization of the value function as the solution of a reflected BSDE. This is not the case here.

6.3 Weak dynamic programming

6.3.1 Splitting properties

Let $s \in [0,T]$. For each ω, let ${}^s\omega := (\omega_{r\wedge s})_{0\leq r\leq T}$ and $\omega^s := (\omega_r - \omega_s)_{s\leq r\leq T}$.
We shall identify the path ω with $({}^s\omega, \omega^s)$, which means that a path can be splitted into two parts: the path before time s and the s-translated path after time s.
Let α be a given control in $\mathcal{A}$. We show below the following: at time s, for fixed past path $\tilde{\omega} :={}^s\omega$, the process $\alpha(\tilde{\omega}, .)$ which only depends on the future path ω^s is an s-admissible control, that is $\alpha(\tilde{\omega}, .) \in \mathcal{A}_s^s$; furthermore, the criterium $Y^{\alpha,0,x}(\tilde{\omega}, .)$ from time s coincides with the solution of the reflected BSDE driven by W^s and $\tilde{N}^s$, controlled by $\alpha(\tilde{\omega}, .)$ and associated with initial time s and initial state condition $X_s^{\alpha,0,x}(\tilde{\omega})$.

We introduce the following random variables defined on Ω by

$$S^s : \omega \mapsto {}^s\omega \ ; \quad T^s : \omega \mapsto \omega^s.$$

Note that they are independent. For each $\omega \in \Omega$, we have

$$\omega = S^s(\omega) + T^s(\omega)\mathbf{1}_{]s,T]},$$

or equivalently $\omega_r = \omega_{r\wedge s} + \omega_r^s \mathbf{1}_{]s,T]}(r)$, fort all $r \in [0,T]$.

We introduce the following notation : for all paths $\omega, \omega' \in \Omega$, $({}^s\omega, T^s(\omega'))$ denotes the path such that the past trajectory before s is that of ω, and the s-translated trajectory after s is that of ω'. This can also be written as:

$$({}^s\omega, T^s(\omega')) := {}^s\omega + T^s(\omega')\mathbf{1}_{]s,T]} = {}^s\omega + {\omega'}^s\mathbf{1}_{]s,T]} = (\omega_r \mathbf{1}_{r\leq s} + (\omega_s + \omega'_r - \omega'_s)\mathbf{1}_{r>s})_{0\leq r\leq T}.$$

Note that for each $\omega \in \Omega$, we have $({}^s\omega, T^s(\omega)) = \omega$.

Lemma 6.3.1. *Let $s \in [0,T]$. Let $Z \in \mathbb{H}^2$. There exists a P-null set $\mathcal{N}$ such that for each ω in the complement $\mathcal{N}^c$ of $\mathcal{N}$, setting $\tilde{\omega} := {}^s\omega = \omega_{.\wedge s}$, the process $Z(\tilde{\omega}, T^s)$ (denoted also be $Z(\tilde{\omega}, .)$) defined by*

$$Z(\tilde{\omega}, T^s) : \Omega \times [s,T] \to \mathbb{R}\,;\, (\omega', r) \mapsto Z_r(\tilde{\omega}, T^s(\omega'))$$

belongs to $\mathbb{H}^2_s$. Moreover, if $Z \in \mathcal{A}$, then $Z(\tilde{\omega}, T^s) \in \mathcal{A}^s_s$.

Proof. By a classical property of conditional expectations, we have
$\mathbb{E}[\int_s^T Z_r^2 dr]= \mathbb{E}[\mathbb{E}[\int_s^T Z_r^2 dr|\mathcal{F}_s]] < +\infty$. Using the independence of T^s with respect to $\mathcal{F}_s$ and the measurability of S^s with respect to $\mathcal{F}_s$, we then derive that

$$\mathbb{E}[\int_s^T Z_r^2 dr|\,\mathcal{F}_s] = \mathbb{E}[\int_s^T Z_r(S^s, T^s)^2 dr|\,\mathcal{F}_s] = F(\,S^s) < +\infty \quad P-\text{a.s.}\,,$$

where $F(\tilde{\omega}) := \mathbb{E}[\int_s^T Z_r(\tilde{\omega}, T^s(\cdot))^2 dr]$.
It remains to prove that the process $Z(\tilde{\omega}, T^s) \,:\, (\omega', r) \mapsto Z_r(\tilde{\omega}, T^s(\omega'))$ is $\mathcal{P}^s$-measurable.
Suppose we have shown that the map

$$\psi : (\Omega \times [s,T], \mathcal{P}^s) \to (\Omega \times [s,T], \mathcal{P})\,;\, (\omega', r) \mapsto ((\tilde{\omega}, T^s(\omega')), r)$$

is measurable. Note that we have $Z(\tilde{\omega}, T^s)(\omega', r) = Z \circ \psi(\omega', r)$ for each $(\omega', r) \in \Omega \times [s,T]$. Since Z is $\mathcal{P}$-measurable, the $\mathcal{P}^s$-measurability of $Z(\tilde{\omega}, T^s)$ thus follows by composition.

It remains to show that the map ψ is $\mathcal{P}^s$-measurable. The proof is based on classical arguments of Integration Theory. Recall that the σ-algebra $\mathcal{P}$ is generated by the sets $H \times]v,T]$, where H is a cylindrical set belonging to $\mathcal{F}_v$, that is of the following form: $H = \{B_{t_i} \in A_i,\ 1 \le i \le n\}$, where $A_i \in \mathcal{B}(\mathbb{R}^2)$ and $t_1 < t_2 < ... \le v$. It is thus sufficient to show that $\psi^{-1}(H \times]v,T]) \in \mathcal{P}^s$. Note that $\psi^{-1}(H \times]v,T]) = H' \times]v,T]$, where $H' = \{\omega' \in \Omega\,,\, (\tilde{\omega}, T^s(\omega')) \in H\}$. We have:

$$H' = \begin{cases} \emptyset \quad \text{if } \exists\ i \text{ such that } t_i \le s \text{ and } \tilde{\omega}_{t_i} \notin A_i \\ \{\omega'_{t_i} - \omega'_s \in A_i,\ \ \forall\ i \text{ such that } t_i > s\} \quad \text{otherwise.} \end{cases}$$

Hence $H' \in \mathcal{F}^s_v$. This implies that $\psi^{-1}(H \times]v,T]) \in \mathcal{P}^s$. The map ψ is thus $\mathcal{P}^s$-measurable. $\square$

Remark 6.3.2. The same proof shows that this property still holds for each initial time $t \in [0,T]$. More precisely, let $s \in [t,T]$. Let $Z \in \mathbb{H}^2_t$ (resp. $\mathcal{A}^t_t$). For almost every $\omega \in \Omega$, the process $Z({}^s\omega, .) = (Z_r({}^s\omega, T^s))_{r \ge s}$ belongs to $\mathbb{H}^2_s$ (resp. $\mathcal{A}^s_s$).

Let $Z \in \mathbb{H}^2$. Let us give an intermediary time $s \in [0,T]$ and a fixed past path ${}^s\omega$. Note that the Lebesgue integral $(\int_s^u Z_r dr)({}^s\omega, .)$ is equal a.s. to the integral $\int_s^u Z_r({}^s\omega, .) dr$. This property is not so clear for a stochastic integral. We now show that the stochastic integral $(\int_s^u Z_r dW_r)({}^s\omega, .)$ coincides with the stochastic integral of the process $Z({}^s\omega, .)$ with respect to the translated Brownian motion W^s, that is $\int_s^u Z_r({}^s\omega, .) dW^s_r$. A similar property holds for the integral with respect to the Poisson random measure.

Lemma 6.3.3 (Splitting properties for stochastic integrals)**.** *Let $t \in [0,T]$. Let $s \in [t,T]$. Let $Z \in \mathbb{H}^2_t$ and $K \in \mathbb{H}^2_{t,\nu}$. There exists a P-null set $\mathcal{N}$ such that for each $\omega \in \mathcal{N}^c$, and $\tilde{\omega} := {}^s\omega$, we* *have* $(Z_r(\tilde{\omega}, T^s))_{r\geq s} \in \mathbb{H}^2_s$ *and* $(K_r(\tilde{\omega}, T^s))_{r\geq s} \in \mathbb{H}^2_{s,\nu}$, *and thus for each* $u \in [s,T]$, $\int_s^u Z_r(\tilde{\omega}, T^s)dW^s_r$ *and* $\int_s^u \int_{\mathbf{E}} K_r(\tilde{\omega}, T^s, e)\tilde{N}^s(dr, de)$ *are well defined, and*

$$\begin{aligned}(\int_s^u Z_r dW_r)(\tilde{\omega}, T^s) &= \int_s^u Z_r(\tilde{\omega}, T^s)dW^s_r \quad P-\text{a.s.} \qquad (6.3.1)\\ (\int_s^u \int_{\mathbf{E}} K_r(e)\tilde{N}(dr, de))(\tilde{\omega}, T^s) &= \int_s^u \int_{\mathbf{E}} K_r(T^s, \cdot, e)\tilde{N}^s(dr, de) \quad P-\text{a.s.}.\end{aligned}$$

Remark 6.3.4. Equality (6.3.1) is equivalent to $(\int_s^u Z_r dW_r)(\tilde{\omega}, T^s(\omega')) = (\int_s^u Z_r(\tilde{\omega}, T^s)dW^s_r)(\omega')$ for P-almost every $\omega' \in \Omega$. A similar property holds for the second equality.

Proof. We shall only prove the first equality with the Brownian motion. The second one with the Poisson random measure can be shown by similar arguments.

Let us first show that equality (6.3.1) holds for a simple process. Let $a < T$ and let $H \in \mathbb{L}^2(\mathcal{F}_a)$. For each $\omega \equiv ({}^s\omega, \omega^s) = (S^s(\omega), T^s(\omega)) \in \Omega$, we have

$$(\int_s^u H\mathbf{1}_{]a,T]}dW_r)({}^s\omega, \omega^s) = H({}^s\omega, \omega^s)(\omega^s_u - \omega^s_{a\wedge u}) = (\int_s^u H({}^s\omega, T^s)\mathbf{1}_{]a,T]}dW^s_r)(\omega).$$

Let now $Z \in \mathbb{H}^2$. Let us show that Z satisfies equality (6.3.1). The idea is to approximate Z by an appropriate sequence of simple processes $(Z^n)_{n\in\mathbb{N}}$ so that the sequence $(Z^n)_{n\in\mathbb{N}}$ converges in $\mathbb{H}^2$ to Z, and that, for almost every past path ${}^s\omega$, the sequence $(Z^n({}^s\omega, T^s))_{n\in\mathbb{N}}$ converges to $Z({}^s\omega, T^s)$ in $\mathbb{H}^2_s$.

For each $n \in \mathbb{N}^*$, define $Z^n_r := n\sum_{i=1}^{n-1}(\int_{\frac{(i-1)T}{n}}^{\frac{iT}{n}} Z_u du)\mathbf{1}_{]\frac{iT}{n}, \frac{(i+1)T}{n}]}(r)$.

By inequality (6.6.1) in the Appendix, we have $\int_s^u (Z^n_r(\omega))^2 dr \leq \int_s^u Z_r(\omega)^2 dr$, and for each $\omega \in \Omega$ and $s \leq u$, $\int_s^u (Z^n_r(\omega) - Z_r(\omega))^2 dr \to 0$. As $\int_s^u Z_r^2 dr \in L^1(\Omega)$, it follows, by the Lebesgue theorem for the conditional expectation, that

$$\mathbb{E}[\int_s^u (Z^n_r - Z_r)^2 dr|\mathcal{F}_s] \to 0 \qquad (6.3.2)$$

excepted on a P-null set N. Since S^s is $\mathcal{F}_s$-measurable and T^s is independant of $\mathcal{F}_s$, there exists a P-null set including the previous one, such that for each $\omega \in N^c$, setting $\tilde{\omega} = {}^s\omega$, we have

$$\begin{aligned}\mathbb{E}[\int_s^u (Z^n_r - Z_r)^2 dr|\mathcal{F}_s](\tilde{\omega}) &= \mathbb{E}[\int_s^u (Z^n_r(\tilde{\omega}, T^s) - Z_r(\tilde{\omega}, T^s))^2 dr]\\ &= \mathbb{E}[(\int_s^u Z^n_r(\tilde{\omega}, T^s)dW^s_r - \int_s^u Z_r(\tilde{\omega}, T^s)dW^s_r)^2]. \qquad (6.3.3)\end{aligned}$$

The second equality follows by the classical isometry property. Now, for each square integrable martingale M, $M^2 - \langle M\rangle$ is a martingale. Hence, for each $\omega \in \mathcal{N}^c$, where $\mathcal{N}$ is a P-null set included the previous one, setting $\tilde{\omega} = {}^s\omega$, we have

$$\begin{aligned}\mathbb{E}[\int_s^u (Z^n_r - Z_r)^2 dr|\mathcal{F}_s](\tilde{\omega}) &= \mathbb{E}[(\int_s^u Z^n_r dW_r - \int_s^u Z_r dW_r)^2|\mathcal{F}_s](\tilde{\omega})\\ &= \mathbb{E}[\Big((\int_s^u Z^n_r dW_r)(\tilde{\omega}, T^s) - (\int_s^u Z_r dW_r)(\tilde{\omega}, T^s)\Big)^2]. \qquad (6.3.4)\end{aligned}$$

For each n, since Z^n is a simple process, it satisfies equality (6.3.1) everywhere, that is

$$(\int_s^u Z_r^n dW_r)(\tilde{\omega}, T^s) = \int_s^u Z_r^n(\tilde{\omega}, T^s) dW_r^s.$$

By the convergence property (6.3.2), equalities (6.3.3) and (6.3.4), and the uniqueness property of the limit in L^2, we derive equality (6.3.1), which ends the proof. $\square$

Using the above lemmas, we now show that for each $s \geq t$, for almost every $\omega \in \Omega$, setting $\tilde{\omega} = {}^s\omega$, the process $Y^{\alpha,t,x}(\tilde{\omega}, T^s)$ coincides with the solution of the reflected BSDE on $\Omega \times [s,T]$, associated with driver $f^{\alpha(\tilde{\omega},T^s),s,\eta(\tilde{\omega})}$, with obstacle $\bar{h}(r, X_r^{\alpha(\tilde{\omega},T^s),s,X_s^{\alpha(\tilde{\omega}),t,x}(\tilde{\omega})})$ and filtration $\mathbb{F}^s$, and driven by W^s and $\tilde{N}^s$.

To simplify notation, T^s will be replaced by $\cdot$ in the following. In particular $Y^{\alpha,t,x}(\tilde{\omega}, T^s)$ will be simply denoted by $Y^{\alpha,t,x}(\tilde{\omega}, .)$.

Theorem 6.3.5 (Splitting properties for the forward-backward "system")**.** *Let $t \in [0,T]$, $\alpha \in \mathcal{A}_t^t$ and $s \in [t,T]$. There exists a P-null set $\mathcal{N}$ such that for each $\omega \in \mathcal{N}^c$, setting $\tilde{\omega} = {}^s\omega$, the following properties hold:*

- *There exists an unique solution $(X_r^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})})_{s \leq r \leq T}$ in $\mathcal{S}_s^2$ of the following SDE:*

$$\begin{aligned} X_u^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})} &= \eta(\tilde{\omega}) + \int_s^u b(X_r^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})}, \alpha_r(\tilde{\omega},\cdot))dr + \int_s^u \sigma(X_r^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})}, \alpha_r(\tilde{\omega},\cdot))dW_r^s \\ &+ \int_s^u \int_{\mathbf{E}} \beta(X_{r^-}^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})}, \alpha_r(\tilde{\omega},\cdot), e)\tilde{N}^s(dr, de), \end{aligned} \tag{6.3.5}$$

  where $\eta(\tilde{\omega}) := X_s^{\alpha(\tilde{\omega}),t,x}(\tilde{\omega})$. We also have $X_r^{\alpha,t,x}(\tilde{\omega}, .) = X_r^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})}$, $s \leq r \leq T$ P-a.s.

- *There exists an unique solution $(Y_r^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})}, Z_r^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})}, K_r^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})})_{s \leq r \leq T}$ in $\mathcal{S}_s^2 \times \mathbb{H}_s^2 \times \mathbb{H}_{s,\nu}^2$ of the reflected BSDE on $\Omega \times [s,T]$ driven by W^s and $\tilde{N}^s$ and associated with filtration $\mathbb{F}^s$, driver $f^{\alpha(\tilde{\omega},.),s,\eta(\tilde{\omega})}$, and with obstacle $\bar{h}(r, X_r^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})})$. We have:*

$$\begin{aligned} Y_r^{\alpha,t,x}(\tilde{\omega}, .) &= Y_r^{\alpha(\tilde{\omega},.),s,\eta(\tilde{\omega})}, \quad s \leq r \leq T, \quad P-\text{a.s.} \\ Z_r^{\alpha,t,x}(\tilde{\omega}, .) &= Z_r^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})}, \quad s \leq r \leq T, \quad dP \otimes dr - \text{a.s.} \\ K_r^{\alpha,t,x}(\tilde{\omega}, ., e) &= K_r^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})}(., e), \quad s \leq r \leq T, \quad dP \otimes dr \otimes d\nu(e) - \text{a.s.} \end{aligned} \tag{6.3.6}$$

$$Y_s^{\alpha,t,x}(\tilde{\omega}) = Y_s^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})} = u^{\alpha(\tilde{\omega},\cdot)}(s, \eta(\tilde{\omega})). \tag{6.3.7}$$

Proof. Recall that by Lemma 6.3.1 the process $\alpha(\tilde{\omega},\cdot) := (\alpha_r(\tilde{\omega},\cdot))_{r \geq s}$ belongs to $\mathcal{A}_s^s$.
Let us show the first assertion. To simplify the exposition, we suppose that there is no Poisson random measure. There exists a P-null set $\mathcal{N}$ such that for each $\omega \in \mathcal{N}^c$, setting $\tilde{\omega} = {}^s\omega$,

$$X_u^{\alpha,t,x}(\tilde{\omega}, .) = \eta(\tilde{\omega}) + \int_s^u b(X_r^{\alpha,t,x}(\tilde{\omega}, .), \alpha_r(\tilde{\omega}, .))du + (\int_s^u \sigma(X_r^{\alpha,t,x}, \alpha_r)dW_u^s)(\tilde{\omega}, .),$$

on $[s,T]$ P-a.s. Now, by the first equality in Lemma 6.3.3, there exists a P-null set $\mathcal{N}$ such that for each $\omega \in \mathcal{N}^c$, setting $\tilde{\omega} = {}^s\omega$, we have

$$(\int_s^u \sigma(X_r^{\alpha,t,x}, \alpha_r)dW_u^s)(\tilde{\omega}, .) = \int_s^u \sigma(X_r^{\alpha,t,x}(\tilde{\omega}, .), \alpha_r(\tilde{\omega},\cdot))dW_u^s \quad P-\text{a.s.},$$

which implies that the process $(X_r^{\alpha,t,x}(\tilde{\omega},\cdot))_{r\in[s,T]}$ is a solution of SDE (6.3.5), and then, by uniqueness of the solution of this SDE, we have $X_r^{\alpha,t,x}(\tilde{\omega},.) = X_r^{\alpha(\tilde{\omega},.),s,\eta(\tilde{\omega})}$, $s \le r \le T$, P-a.s.
Let us show the second assertion. First, note that since the filtration $\mathbb{F}^s$ is generated by W^s and $\tilde{N}^s$, we have a martingale representation theorem for $\mathbb{F}^s$-martingales with respect to W^s and $\tilde{N}^s$. Hence, there exists an unique solution $(Y_r^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})}, Z_r^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})}, K_r^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})})_{s\le r\le T}$ in $\mathcal{S}_s^2\times\mathbb{H}_s^2\times\mathbb{H}_{s,\nu}^2$ of the reflected BSDE on $\Omega\times[s,T]$ driven by W^s and $\tilde{N}^s$ and associated with filtration $\mathbb{F}^s$ and with obstacle $\bar{h}(r, X_r^{\alpha(\tilde{\omega},\cdot),s,\eta(\tilde{\omega})})$. Equalities (6.3.6) then follow from similar arguments as above together with the uniqueness of the solution of a Lipschitz BSDE. Equality (6.3.7) is obtained by taking $r = s$ in equality (6.3.6) and by using the definition of $u^{\alpha(\tilde{\omega},.)}$. □

6.3.2 Existence of *weak* ε-optimal controls

We first show a measurability property of the function $u^\alpha(t,x)$ with respect to control α and initial condition x.

Lemma 6.3.6. *Let $s\in[0,T]$.*
The map $(\alpha,x)\mapsto u^\alpha(s,x)$; $(\mathcal{A}_s^s\times\mathbb{R}\,,\,\mathcal{B}(\mathcal{A}_s^s)\otimes\mathcal{B}(\mathbb{R}))\to(\mathbb{R},\mathcal{B}(\mathbb{R}))$ is measurable.

Proof. Recall that $u^\alpha(s,x) = Y_{s,T}^{\alpha,s,x}[g(X_T^{\alpha,s,x})]$ also denoted by $Y_{s,T}^{\alpha,s,x}[\bar{h}(.,X_.^{\alpha,s,x})]$.
Let $x^1,x^2\in\mathbb{R}$, and $\alpha^1,\alpha^2\in\mathcal{A}_s^s$. By classical estimates on diffusion processes and the assumptions made on the coefficients, we get

$$\mathbb{E}[\sup_{r\ge s}|X_r^{\alpha^1,s,x^1} - X_r^{\alpha^2,s,x^2}|^2] \le C(\|\alpha^1-\alpha^2\|_{\mathbb{H}_s^2}^2 + |x^1-x^2|^2). \tag{6.3.8}$$

We introduce the map $\Phi:\mathcal{A}_s^s\times\mathbb{R}\times\mathcal{S}_s^2\to\mathcal{S}_s^2$; $(\alpha,x,\xi)\mapsto Y_{s,T}^{\alpha,s,x}[\xi_.]$, where
$Y_{s,T}^{\alpha,s,x}[\xi_.]$ denotes here the solution at time s of the reflected BSDE associated with driver $f^{\alpha,s,x} := (f(\alpha_r,r,X_r^{\alpha,s,x},.)\mathbf{1}_{r\ge s})$ and obstacle $\xi_.$.
By the estimates on reflected BSDEs (see the Appendix in [61]), using the Lipschitz property of f with respect to x,α and estimates (6.3.8), for all $x^1,x^2\in\mathbb{R}$, $\alpha^1,\alpha^2\in\mathcal{A}_s^s$ and $\xi_.^1,\xi_.^2\in S_s^2$, we have

$$|Y_s^{\alpha^1,s,x^1}[\xi_.^1] - Y_s^{\alpha^2,s,x^2}[\xi_.^2]|^2 \le C(\|\alpha^1-\alpha^2\|_{\mathbb{H}_s^2}^2 + |x^1-x^2|^2 + \|\xi_.^1-\xi_.^2\|_{S_s^2}^2).$$

The map Φ is thus Lipschitz-continuous with respect to the norm $\|\,.\,\|_{\mathbb{H}_s^2}^2 + |\,.\,|^2 + \|\,.\,\|_{\mathcal{S}_s^2}^2$.
Moreover, the reward map $\bar{h}$ is Borelian, which implies that the map $(\alpha,x)\mapsto(\alpha,x,\bar{h}(.,X_.^{\alpha,s,x}))$ defined on $(\mathcal{A}_s^s\times\mathbb{R},\mathcal{B}(\mathcal{A}_s^s)\otimes\mathcal{B}(\mathbb{R}))$ and valued in $(\mathcal{A}_s^s\times\mathbb{R}\times\mathcal{S}_s^2\,,\,\mathcal{B}(\mathcal{A}_s^s)\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathcal{S}_s^2)$ is measurable.
By composition, it follows that the map $(\alpha,x)\mapsto Y_{s,T}^{\alpha,s,x}[\bar{h}(.,X_.^{\alpha,s,x})] = u^\alpha(s,x)$ is measurable. □

For each (t,s) with $s\ge t$, we introduce the set $\mathcal{A}_s^t$ of restrictions to $[s,T]$ of the controls in $\mathcal{A}_t^t$. They can also be identified to the controls α in $\mathcal{A}_t^t$ which are equal to 0 on $[t,s]$.

Let $\eta\in L^2(\mathcal{F}_s^t)$. Since η is $\mathcal{F}_s$-measurable, it can be written as a measurable map, still denoted by η, of the past trajectory ${}^s\omega$.

For each $\omega\in\Omega^t$, by using the definition of the function u we have:

$$u(s,\eta({}^s\omega)) = \sup_{\alpha\in\mathcal{A}_s^s} u^\alpha(s,\eta({}^s\omega)). \tag{6.3.9}$$

Note that for fixed s, the map $x \mapsto u(s,x)$ is not necessarily Borelian.

We now introduce the map u_* defined by

$$u_*(t,x) := \liminf_{(t',x') \to (t,x)} u(t',x') \ \forall (t,x) \in [0,T[\times \mathbb{R} \quad \text{and} \quad u_*(T,x) = g(x) \ \forall x \in \mathbb{R}. \tag{6.3.10}$$

The map u_* thus coincides with the classical lower-semicontinuous envelope of u on $[0,T[\times\mathbb{R}$. It follows that u_* is Borelian. Note also that $u_* \le u$.

Using the measurability of the maps $(\alpha,x) \mapsto u^\alpha(s,x)$ and $x \mapsto u_*(s,x)$, we derive the existence of nearly optimal controls for (6.3.9) satisfying some specific measurability properties.

Theorem 6.3.7. *(Existence of* weak *ε-optimal controls) Let $t \in [0,T]$, $s \in [t,T]$ and $\eta \in L^2(\mathcal{F}_s^t)$. Let $\varepsilon > 0$. There exists $\alpha^\varepsilon \in \mathcal{A}_s^t$ such that, for almost every $\omega \in \Omega$, $\alpha^\varepsilon({}^s\omega, T^s)$ is* weak *ε-optimal for Problem* (6.3.9)*, in the sense that*

$$u_*(s,\eta({}^s\omega)) \ \le \ u^{\alpha^\varepsilon({}^s\omega,T^s)}(s,\eta({}^s\omega)) + \varepsilon.$$

Proof. Without loss of generality, we may assume that $t = 0$. We introduce the space ${}^s\Omega := \{(\omega_r)_{0\le r\le s}; \omega \in \Omega\}$, equipped with the σ-algebra $\mathcal{F}_s$, that is the σ-algebra associated with the coordinate process, and the probability measure ${}^sP$, which corresponds to the image of P by ${}^sS$ i.e. $P \circ (S^s)^{-1}$.

Let $x \in \mathbb{R}$. From the definition of $u(s,x)$ as a supremum (see (6.2.4)), we derive that for each $\tilde\omega \in {}^s\Omega$, there exists $\bar\alpha^\varepsilon \in \mathcal{A}_s^s$ such that $u(s,\eta(\tilde\omega)) \le u^{\bar\alpha^\varepsilon}(s,\eta(\tilde\omega)) + \varepsilon$ and hence such that

$$u_*(s,\eta(\tilde\omega)) \le u^{\bar\alpha^\varepsilon}(s,\eta(\tilde\omega)) + \varepsilon.$$

Recall that the Hilbert space $\mathbb{H}_s^2$ of square-integrable predictable processes on $\Omega^s \times [s,T]$, equipped with the norm $\|\cdot\|_{\mathbb{H}_s^2}$ is separable. It follows that $\mathcal{A}_s^s$ is a Polish space because it is a closed subset of $\mathbb{H}_s^2$. Also, recall that the space ${}^s\Omega$ of paths (RCLL) before s is Polish for the Skorohod metric. Now, η is $\mathcal{F}_s$-measurable and the map u_* is Borelian. Moreover, by Lemma 6.3.6, u^α is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathcal{A}_s^s)$-measurable with respect to (x,α). We thus have that

$$\mathbb{A} := \{(\tilde\omega,\alpha) \in {}^s\Omega \times \mathcal{A}_s^s, \quad u_*(s,\eta(\tilde\omega)) \le u^\alpha(s,\eta(\tilde\omega)) + \varepsilon\,\} \, \in \, \mathcal{F}_s \otimes \mathcal{B}(\mathcal{A}_s^s). \tag{6.3.11}$$

Now, a measurable selection theorem of [53] in Section 81, Appendix of Ch. III (see also [16]) ensures that if A is a Borel subset of $\Omega \times E$, where Ω is a metrizable space and E a Polish space, and if for each $\omega \in \Omega$, there exists $\alpha \in E$ such that $(\omega,\alpha) \in A$, then there exists an universally measurable map $h : \Omega \to E$ such that for all $\omega \in \Omega$, $(\omega, h(\omega)) \in A$. Also, by a result of Measurable Theory (see Remark 6.6.4), if E is a (separable) Hilbert space, for each probability P on $\mathcal{B}(\Omega)$ there exists a Borelian map $\hat h : \Omega \to E$ such that $h(\omega) = \hat h(\omega)$ for P- almost every ω.

Let us apply these properties to our case with Ω replaced by ${}^s\Omega$, P replaced by ${}^sP$, $E = \mathcal{A}_s^s$, $A = \mathbb{A}$ defined by (6.3.11). We thus obtain that there exists a Borelian map $\hat\alpha^\varepsilon : ({}^s\Omega, \mathcal{F}_s) \mapsto (\mathcal{A}_s^s, \mathcal{B}(\mathcal{A}_s^s))\,;\ \tilde\omega \mapsto \hat\alpha^\varepsilon(\tilde\omega,\cdot)$ such that

$$u_*(s,\eta(\tilde\omega)) \le u^{\hat\alpha^\varepsilon(\tilde\omega,\cdot)}(s,\eta(\tilde\omega)) + \varepsilon \quad \text{for} \quad {}^sP - \text{almost every } \tilde\omega \in {}^s\Omega.$$

Since $\mathbb{H}_s^2$ is a separable Hilbert space, for each $\tilde\omega$, we have $\hat\alpha_u^\varepsilon(\tilde\omega,\omega) = \sum_i \beta^{i,\varepsilon}(\tilde\omega) e_u^i(\omega) \ dP(\omega) \otimes du$-a.s. , where $\beta^{i,\varepsilon}(\tilde\omega) =< \hat\alpha^\varepsilon(\tilde\omega,\cdot), e^i(\cdot) >_{\mathbb{H}_s^2}$ and $\{e^i, i \in \mathbb{N}\}$ is a countable orthonormal basis of $\mathbb{H}_s^2$.

Note that $\beta^{i,\varepsilon}$ is $\mathcal{F}_s$-measurable.
Let $\bar{\alpha}^\varepsilon : ({}^s\Omega, \mathcal{F}_s) \mapsto (\mathcal{A}_s^s, \mathcal{B}(\mathcal{A}_s^s))\,;\ \tilde{\omega} \mapsto \bar{\alpha}^\varepsilon(\tilde{\omega}, \cdot) = \sum_i \beta^{i,\varepsilon}(\tilde{\omega}) e^i(\cdot)$. It is a measurable map.

We now define a process α^ε on $[0,T] \times \Omega$ by $\alpha_r^\varepsilon(\omega) := \sum_i \beta^{i,\varepsilon}(S^s(\omega)) e^i(\omega)$. It remains to prove that it is predictable, that is $\mathcal{P}$-measurable. Note that $\beta^{i,\varepsilon} \circ S^s$ is $\mathcal{F}_s$-measurable by composition. Since the process $(e_u^i)_{s \le u \le T}$ is $\mathcal{P}^s$-measurable, the process $(\beta^{i,\varepsilon} \circ S^s)\, e_u^i$ is $\mathcal{P}$-measurable. Indeed, if we take e^i of the form $e_u^i = H \mathbf{1}_{]r,T]}(u)$ with $r \ge s$ and H a random variable $\mathcal{F}_r^s$-measurable, then the random variable $(\beta^{i,\varepsilon} \circ S^s)\, H$ is $\mathcal{F}_r$-measurable and hence the process $(\beta^{i,\varepsilon} \circ\, S^s)\, H \mathbf{1}_{]r,T]}$ is $\mathcal{P}$-measurable. The process α^ε is thus $\mathcal{P}$-measurable.
Note also that $\alpha^\varepsilon(\tilde{\omega}, T^s(\omega)) = \sum_i \beta^{i,\varepsilon}(\tilde{\omega}) e^i(\tilde{\omega}, \omega)$. Now, we have $e^i(\tilde{\omega}, T^s(\omega)) = e^i(\omega)$ because $e^i(\omega)$ depends on ω only through $T^s(\omega)$. Hence, $\alpha^\varepsilon(\tilde{\omega}, T^s(\omega)) = \bar{\alpha}^\varepsilon(\tilde{\omega}, \omega)$, which completes the proof. □

Remark 6.3.8. Recall that in control theory, selection theorems are closely related to the existence of optimal or nearly optimal controls and their regularity or measurability properties (see, among others, [81] in a deterministic framework, and [17] in a discrete time Markovian stochastic framework).

The above result will be used to prove that the map u_* satisfies a *super–optimality principle of dynamic programming* (see Theorem 6.3.13). In the next section, we provide a Fatou lemma for reflected BSDEs which will be also used to prove this *super–optimality principle.*

6.3.3 A Fatou lemma for reflected BSDEs

In this section, we establish a Fatou lemma for reflected BSDEs, where the limit involves both terminal condition and terminal time. We first introduce some notation.

A function f is said to be a *Lipschitz driver* if
$f : [0,T] \times \Omega \times I\!R^2 \times L_\nu^2 \to I\!R\ (\omega, t, y, z, k(\cdot)) \mapsto f(\omega, t, y, z, k(\cdot))$ is $\mathcal{P} \otimes \mathcal{B}(I\!R^2) \otimes \mathcal{B}(L_\nu^2)-$ measurable, uniformly Lipschitz with respect to $y, z, k(\cdot)$ and such that $f(., 0, 0, 0) \in I\!H^2$.

A Lipschitz driver f is said to satisfy Assumption 6.3.9 if the following holds:

Assumption 6.3.9. *Assume that $dP \otimes dt$-a.s for each $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L_\nu^2)^2$,*

$$f(t, y, z, k_1) - f(t, y, z, k_2) \ge \langle \gamma_t^{y,z,k_1,k_2}\,,\, k_1 - k_2 \rangle_\nu,$$

with $\gamma : [0,T] \times \Omega \times \mathbb{R}^2 \times (L_\nu^2)^2 \to L_\nu^2\,;\, (\omega, t, y, z, k_1, k_2) \mapsto \gamma_t^{y,z,k_1,k_2}(\omega, .)$, supposed to be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}((L_\nu^2)^2)$-measurable, bounded, and satisfying $dP \otimes dt \otimes d\nu(e)$-a.s. , for each (y, z, k_1, k_2) $\in \mathbb{R}^2 \times (L_\nu^2)^2$,

$$\gamma_t^{y,z,k_1,k_2}(e) \ge -1 \quad \text{and} \quad |\gamma_t^{y,z,k_1,k_2}(e)| \le \psi(e), \quad \text{where} \quad \psi \in L_\nu^2. \tag{6.3.12}$$

Recall that this assumption ensures the comparison theorem for BSDEs with jumps (see [137] Th 4.2).

Let (η_t) be a given obstacle RCLL process in $\mathcal{S}^2$ and let f be a given Lipschitz driver. In the following, we will consider the case when the terminal time is a stopping time $\theta \in \mathcal{T}$ and the terminal condition is a random variable ξ in $L^2(\mathcal{F}_\theta)$. In this case, the solution, denoted $(Y_{.,\theta}(\xi), Z_{.,\theta}(\xi), k_{.,\theta}(\xi))$, of the *reflected BSDEs associated with terminal stopping time* θ, driver f, obstacle $(\eta_s)_{s<\theta}$, and terminal condition ξ is defined as the unique solution in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ of the reflected BSDE with terminal time T, driver $f(t,y,z,k)\mathbf{1}_{\{t\leq\theta\}}$, terminal condition ξ and obstacle $\eta_t\mathbf{1}_{t<\theta} + \xi\mathbf{1}_{t\geq\theta}$. Note that $Y_{t,\theta}(\xi) = \xi, Z_{t,\theta}(\xi) = 0, k_{t,\theta}(\xi) = 0$ for $t \geq \theta$.

We first prove a continuity property for reflected BSDEs where the limit involves both terminal condition and terminal time.

Lemma 6.3.10 (A continuity property for reflected BSDEs)**.** *Let $T > 0$. Let (η_t) be an RCLL process in $\mathcal{S}^2$. Let f be a given Lipschitz driver. Let $(\theta^n)_{n\in\mathbb{N}}$ be a non increasing sequence of stopping times in $\mathcal{T}$, converging a.s. to $\theta \in \mathcal{T}$ as n tends to ∞. Let $(\xi^n)_{n\in\mathbb{N}}$ be a sequence of random variables such that $\mathbb{E}[\sup_n(\xi^n)^2] < +\infty$, and for each n, ξ^n is $\mathcal{F}_{\theta^n}$-measurable. Suppose that ξ^n converges a.s. to an $\mathcal{F}_\theta$-measurable random variable ξ as n tends to ∞. Suppose that*

$$\eta_\theta \leq \xi \quad \text{a.s.} \tag{6.3.13}$$

Let $Y_{.,\theta^n}(\xi^n)$; $Y_{.,\theta}(\xi)$ be the solutions of the reflected BSDEs associated with driver f, obstacle $(\eta_s)_{s<\theta^n}$ (resp. $(\eta_s)_{s<\theta}$) , terminal time θ^n (resp. θ), terminal condition ξ^n (resp. ξ). We have

$$Y_{0,\theta}(\xi) = \lim_{n\to+\infty} Y_{0,\theta^n}(\xi^n) \quad a.s.$$

When for each n, $\theta_n = \theta$ a.s., the result still holds without Assumption (6.3.13).

Remark 6.3.11. Compared with the case of non reflected BSDEs (see Proposition A.6 in [137]), there is an additional difficulty due both to the obstacle and to the variation of the terminal time. An additional assumption (Assumption (6.3.13)) is here required to obtain the result.

Proof. Let $n \in \mathbb{N}$. We apply a classical estimate on reflected BSDEs (see Proposition 5.7.6) with $f^1 = f\mathbf{1}_{t\leq\theta^n}$, $f^2 = f\mathbf{1}_{t\leq\theta}$, $\xi^1 = \xi^n$, $\xi^2 = \xi$, $\eta^1_t = \eta_t\mathbf{1}_{t<\theta^n} + \xi^n\mathbf{1}_{\theta^n\leq t<T}$ and $\eta^2_t = \eta_t\mathbf{1}_{t<\theta} + \eta_\theta\mathbf{1}_{\theta\leq t<\theta^n} + \xi\mathbf{1}_{\theta^n\leq t<T}$. Using the notation of Proposition 5.7.6, (Y^i, Z^i, K^i) denotes the solution of the reflected BSDE associated with terminal time T, driver f^i, obstacle (η^i_t) and terminal condition ξ^i. We have $Y^1_. = Y_{.,\theta^n}(\xi^n)$ a.s. Moreover, since by assumption $\eta_\theta \leq \xi$ a.s., we have $Y^2_. = Y_{.,\theta}(\xi)$ a.s. Note that $(Y^2_t, Z^2_t, k^2_t) = (\xi, 0, 0)$ a.s. on $\{t \geq \theta\}$. We thus obtain

$$|Y_{0,\theta^n}(\xi^n) - Y_{0,\theta}(\xi)|^2 \leq K\left(\mathbb{E}[(\xi^n-\xi)^2] + \mathbb{E}[\int_\theta^{\theta^n} f^2(s,\xi,0,0)ds]\right) + \phi \,\|\sup_{\theta\leq s<\theta^n}|\eta_s - \eta_\theta|\|_{L^2}, \tag{6.3.14}$$

where the constant K depends only on the Lipschitz constant C of f and the terminal time T, and where the constant ϕ depends only on C, T, $\|\eta\|_{\mathcal{S}^2}$, $\sup_n \|\xi^n\|_{L^2}$ and $\|f(s,0,0,0)\|_{I\!H^2}$. Since the obstacle (η_t) is right-continuous and $\theta^n \downarrow \theta$ a.s., we have $\lim_{n\to+\infty}\|\sup_{\theta\leq s\leq\theta^n}|\eta_s - \eta_\theta|\|_{L^2} = 0$. The right member of (6.3.14) thus tends to 0 as n tends to $+\infty$. The result follows. □

Using this lemma, we derive a Fatou lemma in the reflected case, where the limit involves both terminal condition and terminal time.

Theorem 6.3.12 (A Fatou lemma for reflected BSDEs)**.** *Let $T > 0$. Let (η_t) be an RCLL process in $\mathcal{S}^2$. Let f be a Lipschitz driver satisfying Assumption* 6.3.9. *Let $(\theta^n)_{n\in\mathbb{N}}$ be a non increasing sequence of stopping times in $\mathcal{T}$, converging a.s. to $\theta \in \mathcal{T}$ as n tends to ∞. Let $(\xi^n)_{n\in\mathbb{N}}$ be a sequence of random variables such that $\mathbb{E}[\sup_n(\xi^n)^2] < +\infty$, and for each n, ξ^n is $\mathcal{F}_{\theta^n}$-measurable.*

Let $Y_{.,\theta^n}(\xi^n)$; $Y_{.,\theta}(\liminf_{n\to+\infty}\xi^n)$ *and* $Y_{.,\theta}(\limsup_{n\to+\infty}\xi^n)$ *be the solution(s) of the reflected BSDE(s) associated with driver* f*, obstacle* $(\eta_s)_{s<\theta^n}$ *(resp.* $(\eta_s)_{s<\theta}$*) , terminal time* θ^n *(resp.* θ*), terminal condition* ξ^n *(resp.* $\liminf_{n\to+\infty}\xi^n$ *and* $\limsup_{n\to+\infty}\xi^n$*).*
Suppose that

$$\liminf_{n\to+\infty}\xi^n \geq \eta_\theta \quad \text{(resp.} \quad \limsup_{n\to+\infty}\xi^n \geq \eta_\theta) \quad \text{a.s.} \tag{6.3.15}$$

$$\text{then} \qquad Y_{0,\theta}(\liminf_{n\to+\infty}\xi^n) \leq \liminf_{n\to+\infty} Y_{0,\theta^n}(\xi^n) \qquad \left(\text{resp.} \quad Y_{0,\theta}(\limsup_{n\to+\infty}\xi^n) \geq \limsup_{n\to+\infty} Y_{0,\theta^n}(\xi^n)\right).$$

When for each n*,* $\theta_n=\theta$ *a.s., the result still holds without Assumption* (6.3.15).

Proof. We present only the proof of the first inequality, since the second one is obtained by similar arguments. For all n, we have by the monotonicity of reflected BSDEs with respect to terminal condition, $Y_{0,\theta^n}(\inf_{p\geq n}\xi^p) \leq Y_{0,\theta^n}(\xi^n)$. We derive that

$$\liminf_{n\to+\infty} Y_{0,\theta^n}(\inf_{p\geq n}\xi^p) \leq \liminf_{n\to+\infty} Y_{0,\theta^n}(\xi^n).$$

By Assumption (6.3.15), since $\lim_{n\to+\infty}\inf_{p\geq n}\xi^p = \liminf_{n\to+\infty}\xi^n$ a.s. , Lemma 6.3.10 yields that

$$\lim_{n\to+\infty} Y_{0,\theta^n}(\inf_{p\geq n}\xi^p) = Y_{0,\theta}(\liminf_{n\to+\infty}\xi^n).$$

The desired result follows. □

6.3.4 A *weak* dynamic programming principle

Since for fixed s, the value function $x\mapsto u(s,x)$ is not necessarily Borelian, we cannot a priori establish a classical dynamic programming principle. We will provide a weak dynamic programming principle involving the map u_* (defined above by (6.3.4)) and the map u^* defined by

$$u^*(t,x) := \limsup_{(t',x')\to(t,x)} u(t',x') \;\forall (t,x)\in[0,T[\times\mathbb{R} \quad \text{and} \quad u^*(T,x)=g(x)\;\forall x\in\mathbb{R}.$$

The map u^* coincides with the classical upper semicontinuous envelope of u on $[0,T[\times\mathbb{R}$. It follows that u^* is Borelian (as u_*). Note that u^* (resp. u_*) is not necessarily upper (resp. lower) semicontinuous on $[0,T]\times\mathbb{R}$, because the terminal reward map g is Borelian but is not supposed to satisfy any regularity assumption. Note also that $u_*\leq u\leq u^*$ and $u_*(T,.)=u(T,.)=u^*(T,.)=g(.)$.

We now prove that the value function satisfies a *weak* dynamic programming principle, in the sense that u^* (resp.u_*) satisfies a *weak sub- (resp. super-) optimality principle of dynamic programming.* In order to do this, we will use the splitting properties (Th. 6.3.5), the existence of *weak* ε-optimal controls (Th. 6.3.7) and the above Fatou lemma for reflected BSDEs, where the limit involves both terminal condition and terminal time (Th. 6.3.12).

Theorem 6.3.13 (A *weak* dynamic programming principle)**.** *The function* u^* *satisfies the* weak sub–optimality principle of dynamic programming*, that is for each* $t\in[0,T]$ *and for each stopping time* $\theta\in\mathcal{T}_t^t$*, that is*

$$u(t,x) \leq \sup_{\alpha\in\mathcal{A}_t^t}\sup_{\tau\in\mathcal{T}_t^t}\mathcal{E}^{\alpha,t,x}_{t,\theta\wedge\tau}\left[h(\tau,X^{\alpha,t,x}_\tau)\mathbf{1}_{\tau<\theta} + u^*(\theta,X^{\alpha,t,x}_\theta)\mathbf{1}_{\tau\geq\theta}\right], \tag{6.3.16}$$

The function u_* satisfies the weak super–optimality principle of dynamic programming*, that is for each $t \in [0,T]$ and for each stopping time $\theta \in \mathcal{T}_t^t$, that is*

$$u(t,x) \geq \sup_{\alpha \in \mathcal{A}_t^t} \sup_{\tau \in \mathcal{T}_t^t} \mathcal{E}_{t,\theta\wedge\tau}^{\alpha,t,x}\left[h(\tau, X_\tau^{\alpha,t,x})\mathbf{1}_{\tau<\theta} + u_*(\theta, X_\theta^{\alpha,t,x})\mathbf{1}_{\tau\geq\theta}\right]. \tag{6.3.17}$$

Remark 6.3.14. We stress that no regularity condition is required on the terminal reward map g to ensure these dynamic programming principles, even the second one, which is the most difficult one to establish. This is not the case in the previous literature even in the case of a classical expectation, where g is supposed to be lower-semicontinuous (see [35], [34] and [14]).

Before giving the proof, we introduce the following notation. For each $\theta \in \mathcal{T}$ and each ξ in $L^2(\mathcal{F}_\theta)$, we denote by $(Y_{\cdot,\theta}^{\alpha,t,x}(\xi), Z_{\cdot,\theta}^{\alpha,t,x}(\xi), k_{\cdot,\theta}^{\alpha,t,x}(\xi))$ the unique solution in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ of the reflected BSDE with driver $f^{\alpha,t,x}\mathbf{1}_{\{s\leq\theta\}}$, terminal time T, terminal condition ξ and obstacle $h(r, X_r^{\alpha,t,x})\mathbf{1}_{r<\theta} + \xi\mathbf{1}_{r\geq\theta}$.

Proof. By estimates for reflected BSDE (see Prop. 5.1 in [61]), the function u has at most polynomial growth at infinity. Hence, the random variables $u^*(\theta, X_\theta^{\alpha,t,x})$ and $u_*(\theta, X_\theta^{\alpha,t,x})$ are square integrable. Without loss of generality, to simplify notation, we suppose that $t = 0$.
We first show the second assertion (which is the most difficult), or equivalently:

$$\sup_{\alpha\in\mathcal{A}} Y_{0,\theta}^{\alpha,0,x}\left[u_*(\theta, X_\theta^{\alpha,0,x})\right] \leq u(0,x), \quad \forall \theta \in \mathcal{T}. \tag{6.3.18}$$

Let $\theta \in \mathcal{T}$. For each $n \in \mathbb{N}$, we define

$$\theta^n := \sum_{k=0}^{2^n-1} t_k \mathbf{1}_{A_k} + T\mathbf{1}_{\theta=T}, \tag{6.3.19}$$

where $t_k := \frac{(k+1)T}{2^n}$ and $A_k := \{\frac{kT}{2^n} \leq \theta < \frac{(k+1)T}{2^n}\}$. Note that $\theta^n \in \mathcal{T}$ and $\theta^n \downarrow \theta$ a.s.
On $\{\theta = T\}$ we have $\theta^n = T$ a.s. for each n. We thus get $u_*(\theta^n, X_{\theta^n}^{\alpha,0,x}) = u_*(\theta, X_\theta^{\alpha,0,x})$ a.s. for each n on $\{\theta = T\}$. Moreover, on $\{\theta < T\}$, the lower semicontinuity of u_* on $[0,T[\times I\!R$ together with the right continuity of the process $X^{\alpha,0,x}$ implies that

$$u_*(\theta, X_\theta^{\alpha,0,x}) \leq \liminf_{n\to+\infty} u_*(\theta^n, X_{\theta^n}^{\alpha,0,x}) \quad \text{a.s.},$$

Hence, by the comparison theorem, we get:

$$Y_{0,\theta}^{\alpha,0,x}\left[u_*(\theta, X_\theta^{\alpha,0,x})\right] \leq Y_{0,\theta}^{\alpha,0,x}\left[\liminf_{n\to+\infty} u_*(\theta^n, X_{\theta^n}^{\alpha,0,x})\right] \quad \text{a.s.}$$

On $\{\theta < T\}$, we have

$$\liminf_{n\to\infty} u_*(\theta^n, X_{\theta^n}^{\alpha,0,x}) \geq \liminf_{n\to\infty} \overline{h}(\theta^n, X_{\theta^n}^{\alpha,0,x}) = \lim_{n\to\infty} h(\theta^n, X_{\theta^n}^{\alpha,0,x}) = h(\theta, X_\theta^{\alpha,0,x}) \quad a.s.$$

by the regularity properties of h on $[0,T[\times\mathbb{R}$.
On $\{\theta = T\}$ we have $\theta^n = T$ a.s. for each n. We thus get

$$u_*(\theta^n, X_{\theta^n}^{\alpha,0,x}) = u_*(T, X_T^{\alpha,0,x}) = g(X_T^{\alpha,0,x}) = \bar{h}(T, X_T^{\alpha,0,x}) \quad a.s.$$

Hence, we have $\liminf_{n\to+\infty} u_*(\theta^n, X_{\theta^n}^{\alpha,0,x}) \geq \bar{h}(\theta, X_\theta^{\alpha,0,x})$ a.s. Condition (6.3.15) is thus satisfied with $\xi^n = u_*(\theta^n, X_{\theta^n}^{\alpha,0,x})$ and $\xi_t = \bar{h}(t, X_t^{\alpha,0,x})$. We can thus apply the above Fatou lemma for reflected BSDEs (Th. 6.3.12). We thus get:

$$Y_{0,\theta}^{\alpha,0,x}\left[u_*(\theta, X_\theta^{\alpha,0,x})\right] \leq Y_{0,\theta}^{\alpha,0,x}\left[\liminf_{n\to+\infty} u_*(\theta^n, X_{\theta^n}^{\alpha,0,x})\right] \leq \liminf_{n\to\infty} Y_{0,\theta^n}^{\alpha,0,x}\left[u_*(\theta^n, X_{\theta^n}^{\alpha,0,x})\right] \quad \text{a.s.}$$

Fix $n \in \mathbb{N}$. Let $\mathcal{A}_{t_k}$ be the set of the restrictions to $[t_k, T]$ of the controls α in $\mathcal{A}$. By Theorem 6.3.7, for each $\varepsilon > 0$, for each k, there exists a *weak* ε-optimal control control $\alpha^{n,\varepsilon,k}$ in $\mathcal{A}_{t_k}^0$ for the control problem at time t_k with initial condition $\eta = X_{t_k}^{\alpha,0,x}$, that is satisfying the inequality

$$u_*(t_k, X_{t_k}^{\alpha,0,x}({}^{t_k}\omega)) \;\leq\; u^{\alpha^{n,\varepsilon,k}({}^{t_k}\omega,\cdot)}(t_k, X_{t_k}^{\alpha,0,x}({}^{t_k}\omega)) + \varepsilon \tag{6.3.20}$$

for almost every $\omega \in \Omega$. By definition of $u^{\alpha^{n,\varepsilon,k}({}^{t_k}\omega,\cdot)}$, we have

$$u^{\alpha^{n,\varepsilon,k}({}^{t_k}\omega,\cdot)}(t_k, X_{t_k}^{\alpha,0,x}({}^{t_k}\omega)) = Y_{t_k,T}^{\alpha^{n,\varepsilon,k}({}^{t_k}\omega,\cdot),t_k,X_{t_k}^{\alpha,0,x}({}^{t_k}\omega)} = Y_{t_k,T}^{\alpha^{n,\varepsilon,k},t_k,X_{t_k}^{\alpha,0,x}}({}^{t_k}\omega)$$

Here, $Y_{\cdot,T}^{\alpha^{n,\varepsilon,k},t_k,X_{t_k}^{\alpha,0,x}} = Y_{\cdot,T}^{f^{\alpha^{n,\varepsilon,k},t_k,X_{t_k}^{\alpha,0,x}}}[\bar{h}(r, X_r^{\alpha^{n,\varepsilon,k},t_k,X_{t_k}^{\alpha,0,x}})]$ denotes the solution of the reflected BSDE associated with terminal time T, obstacle $(\bar{h}(r, X_r^{\alpha^{n,\varepsilon,k},t_k,X_{t_k}^{\alpha,0,x}}))_{t_k\leq r\leq T}$ and driver $f^{\alpha^{n,\varepsilon,k},t_k,X_{t_k}^{\alpha,0,x}}(r,y,z,k) := f(\alpha_r^{n,\varepsilon,k}, r, X_r^{\alpha,t_k,X_{t_k}^{\alpha,0,x}}, y, z, k)$.

Set $\alpha_s^{n,\varepsilon} := \sum_{k=0}^{2^n-1} \alpha_s^{n,\varepsilon,k}\mathbf{1}_{A_k}$. Since for each k, $A_k \in \mathcal{F}_{t_k}$, we have

$$\begin{aligned}
Y_{t_k,T}^{\alpha^{n,\varepsilon,k},t_k,X_{t_k}^{\alpha,0,x}}\mathbf{1}_{A_k} &= Y_{t_k,T}^{f^{\alpha^{n,\varepsilon,k},t_k,X_{t_k}^{\alpha,0,x}}\mathbf{1}_{A_k}}[\bar{h}(r, X_r^{\alpha^{n,\varepsilon,k},t_k,X_{t_k}^{\alpha,0,x}})\mathbf{1}_{A_k}] \\
&= Y_{t_k,T}^{f^{\alpha^{n,\varepsilon},\theta^n,X_{\theta^n}^{\alpha,0,x}}\mathbf{1}_{A_k}}[\bar{h}(r, X_r^{\alpha^{n,\varepsilon},\theta^n,X_{\theta^n}^{\alpha,0,x}})\mathbf{1}_{A_k}] \\
&= Y_{\theta^n,T}^{\alpha^{n,\varepsilon},\theta^n,X_{\theta^n}^{\alpha,0,x}}\mathbf{1}_{A_k} \quad a.s.,
\end{aligned}$$

where, for a given driver f, $Y^{f\mathbf{1}_{A_k}}$ denotes the solution of the reflected BSDE associated with $f\mathbf{1}_{A_k}$. Using inequality (6.3.20), we get

$$u_*(\theta^n, X_{\theta^n}^{\alpha,0,x}) = \sum_{k=0}^{2^n-1} u_*(t_k, X_{t_k}^{\alpha,0,x})\mathbf{1}_{A_k} \leq Y_{\theta^n,T}^{\alpha^{n,\varepsilon},\theta^n,X_{\theta^n}^{\alpha,0,x}} + \varepsilon \quad a.s.$$

We set:

$$\tilde{\alpha}_s^{n,\varepsilon} := \alpha_s \mathbf{1}_{s<\theta^n} + \alpha_s^{n,\varepsilon}\mathbf{1}_{\theta^n\leq s\leq T}.$$

Note that $\tilde{\alpha}^{n,\varepsilon} \in \mathcal{A}$. Using the comparison theorem together with the estimates on reflected BSDEs (see [61]), we obtain

$$Y_{0,\theta^n}^{\alpha,0,x}[u_*(\theta^n, X_{\theta^n}^{\alpha,0,x})] \leq Y_{0,\theta^n}^{\alpha,0,x}[Y_{\theta^n,T}^{\alpha^{n,\varepsilon},\theta^n,X_{\theta^n}^{\alpha,0,x}}] + K\varepsilon = Y_{0,T}^{\tilde{\alpha}^{n,\varepsilon},0,x} + K\varepsilon,$$

where the last equality follows from the flow property. Since $Y_{0,T}^{\tilde{\alpha}^{n,\varepsilon},0,x} \leq u(0,x)$, we have

$$Y_{0,\theta^n}^{\alpha,0,x}[u_*(\theta^n, X_{\theta^n}^{\alpha,0,x})] \leq u(0,x) + K\varepsilon \quad a.s.$$

which holds for all n. Hence, by inequality (6.3.4), we get

$$Y_{0,\theta}^{\alpha,0,x}\left[u_*(\theta, X_\theta^{\alpha,0,x})\right] \leq u(0,x) + K\varepsilon.$$

Taking the supremum on $\alpha \in \mathcal{A}$ and letting ε tend to 0, we obtained inequality (6.3.18).

It remains to show the first assertion, that is, u^* satisfies the sub–optimality principle of dynamic programming (which is the easiest one). It is sufficient to show that for each $\theta \in \mathcal{T}$,

$$u(0,x) \leq \sup_{\alpha \in \mathcal{A}} Y_{0,\theta}^{\alpha,0,x}\left[u^*(\theta, X_\theta^{\alpha,0,x})\right] \ . \tag{6.3.21}$$

Let $\theta \in \mathcal{T}$. As in the proof of the super–optimality principle, we approximate θ by the sequence of stopping times $(\theta^n)_{n\in\mathbb{N}}$. Let $n \in \mathbb{N}$. By applying the flow property for reflected BSDEs, we get $Y_{0,T}^{\alpha,0,x} = Y_{0,\theta^n}^{\alpha,0,x}[Y_{\theta^n,T}^{\alpha,\theta^n,X_{\theta^n}^{\alpha,0,x}}]$. By similar arguments as in the proof of the super–optimality principle (but without using the existence of *weak*-optimal controls), we derive that:

$$Y_{\theta^n,T}^{\alpha,\theta^n,X_{\theta^n}^{\alpha,0,x}} \leq u^*(\theta^n, X_{\theta^n}^{\alpha,0,x}) \text{ a.s.}$$

By the comparison theorem for reflected BSDEs, it follows that

$$Y_{0,T}^{\alpha,0,x} = Y_{0,\theta^n}^{\alpha,0,x}[Y_{\theta^n,T}^{\alpha,\theta^n,X_{\theta^n}^{\alpha,0,x}}] \leq Y_{0,\theta^n}^{\alpha,0,x}[u^*(\theta^n, X_{\theta^n}^{\alpha,0,x})] \text{ a.s.}$$

By taking the limit in n in the above relation and using the Fatou lemma for reflected BSDEs with respect to both terminal time and terminal condition (Th. 6.3.12), we get:

$$Y_{0,T}^{\alpha,0,x} \leq \limsup_{n\to\infty} Y_{0,\theta^n}^{\alpha,0,x}[u^*(\theta^n, X_{\theta^n}^{\alpha,0,x})] \leq Y_{0,\theta}^{\alpha,0,x}[\limsup_{n\to\infty} u^*(\theta^n, X_{\theta^n}^{\alpha,0,x})] \text{ a.s.}$$

By the upper semicontinuity property of u^* on $[0,T[\times\mathbb{R}$ and the fact that $u^*(T,x) = g(x)$, we finally obtain

$$Y_{0,T}^{\alpha,0,x} \leq Y_{0,\theta}^{\alpha,0,x}[\limsup_{n\to\infty} u^*(\theta^n, X_{\theta^n}^{\alpha,0,x})] \leq Y_{0,\theta}^{\alpha,0,x}[u^*(\theta, X_\theta^{\alpha,0,x})] \quad \text{ a.s.}$$

Since $\alpha \in \mathcal{A}$ is arbitrary, we get inequality (6.3.21), which completes the proof. □

Remark 6.3.15. The above proof also shows that the weak dynamic programming principle of Theorem 6.3.13 still holds with θ replaced by θ^α in inequalities (6.3.16) and (6.3.17), given a family of stopping times indexed by controls $\{\theta^\alpha, \alpha \in \mathcal{A}_t^t\}$.

6.4 Nonlinear HJB variational inequalities

6.4.1 Some extensions of comparison theorems for BSDEs and reflected BSDEs

We provide two results which will be used to prove that the value function u, defined by (6.2.3), is a *weak* viscosity solution of some nonlinear Hamilton Jacobi Bellman variational inequalities (see Theorem 6.4.5). We first show a slight extension of the comparison theorem for BSDEs given in [137] which formally states that if two terminal conditions ξ_1, ξ_2 satisfy $\xi_1 \geq \xi_2 + \varepsilon$, then the associated solutions X^1 and X^2 satisfy $X^1 \geq X^2 + \varepsilon K$.

Lemma 6.4.1. *Let $t_0 \in [0,T]$ and let $\theta \in \mathcal{T}_{t_0}$. Let ξ_1 and $\xi_2 \in L^2(\mathcal{F}_\theta)$. Let f_1 be a driver. Let f_2 be a Lipschitz driver with Lipschitz constant $C > 0$, satisfying Assumption 6.3.9. For $i = 1, 2$, let (X_t^i, π_t^i, l_t^i) be a solution in $\mathcal{S}^2 \times I\!H^2 \times I\!H_\nu^2$ of the BSDE associated with driver f_i, terminal time θ and terminal condition ξ_i. Suppose that*

$$\xi_1 \geq \xi_2 + \varepsilon \ a.s. \quad \text{and} \quad f_1(t, X_t^1, \pi_t^1, l_t^1) \geq f_2(t, X_t^1, \pi_t^1, l_t^1) \ \ t_0 \leq t \leq \theta, \ \ dt \otimes dP \ a.s.$$

where ε is a real constant. Then, for each $t \in [t_0, \theta]$, we have $X_t^1 - X_t^2 \geq \varepsilon\, e^{-CT}$ a.s.

Proof. From inequality (4.22) in the proof of the Comparison Theorem in [137], we derive that $X_{t_0}^1 - X_{t_0}^2 \geq e^{-CT}\mathbb{E}\,[H_{t_0,\theta}\,\varepsilon\,|\mathcal{F}_{t_0}]$ a.s., where C is the Lipschitz constant of f_2, and $(H_{t_0,s})_{s\in[t_0,T]}$ is the non negative martingale satisfying $dH_{t_0,s} = H_{t_0,s^-}[\beta_s dW_s + \int_{\mathbf{E}} \gamma_s(u)\tilde{N}(ds,du)]$ with $H_{t_0,t_0} = 1$, (β_s) being a predictable process bounded by C. The result follows. $\square$

From this property, we derive the following comparison result.

Proposition 6.4.2 (A comparison theorem between a BSDE and a reflected BSDE)**.** *Let $t_0 \in [0,T]$ and let $\theta \in \mathcal{T}_{t_0}$. Let $\xi_1 \in L^2(\mathcal{F}_\theta)$. Let f_1 be a driver. Let f_2 be a Lipschitz driver with Lipschitz constant $C > 0$ which satisfies Assumption 6.3.9. Let $(\xi_t^2) \in \mathcal{S}^2$.*
Let (X_t^1, π_t^1, l_t^1) be a solution of the BSDE associated with f_1, terminal time θ and terminal condition ξ^1. Let (Y_t^2) be the solution of the reflected BSDE associated with f_2, terminal time θ and obstacle (ξ_t^2). Suppose that

$$\begin{cases} f_1(t, X_t^1, \pi_t^1, l_t^1) \geq f_2(t, X_t^1, \pi_t^1, l_t^1), \ \ t_0 \leq t \leq \theta, \ \ dt \otimes dP \ a.s. \\ X_t^1 \geq \xi_t^2 + \varepsilon, \ t_0 \leq t \leq \theta \ a.s. \end{cases} \tag{6.4.1}$$

Then for each $t \in [t_0, \theta]$, we have $X_t^1 \geq Y_t^2 + \varepsilon e^{-CT}$ a.s.

Proof. Let $t \in [t_0, \theta]$. By the characterization of the solution of the reflected BSDE as the value function of an optimal stopping problem (see Theorem 3.2 in [137]), $Y_t^2 = ess\sup_{\tau\in\mathcal{T}_{[t,\theta]}} \mathcal{E}_{t,\tau}^2(\xi_\tau^2)$. Now, by Lemma 6.4.1, for each $\tau \in \mathcal{T}_{[t,\theta]}$, $X_t^1 \geq \mathcal{E}_{t,\tau}^2(\xi_\tau^2) + e^{-CT}\varepsilon$ a.s. By taking the supremum over $\tau \in \mathcal{T}_{[t,\theta]}$, the result follows. $\square$

Remark 6.4.3. We stress that unlike to the comparison theorem for two reflected BSDEs where condition $f^1(t,y,z,k) \geq f^2(t,y,z,k)$ is required for all y, z, k, in the above Proposition, this condition is required to be satisfied only along the solution of the BSDE. This point will be used in the proof of Theorem 6.4.5.

6.4.2 The value function, *weak* solution of a nonlinear HJBVI

We introduce the following Hamilton Jacobi Bellman variational inequality (HJBVI):

$$\begin{cases} \min(u(t,x) - h(t,x), \\ \quad \inf_{\alpha\in\mathbf{A}}(-\dfrac{\partial u}{\partial t}(t,x) - L^\alpha u(t,x) - f(\alpha, t, x, u(t,x), (\sigma\dfrac{\partial u}{\partial x})(t,x), B^\alpha u(t,x))) = 0, \\ \qquad (t,x) \in [0,T) \times \mathbb{R} \\ u(T,x) = g(x), x \in \mathbb{R} \end{cases} \tag{6.4.2}$$

where $L^\alpha := A^\alpha + K^\alpha$, and for $\phi \in C^2(\mathbb{R})$,

- $A^\alpha\phi(x) := \dfrac{1}{2}\sigma^2(x,\alpha)\dfrac{\partial^2\phi}{\partial x^2}(x) + b(x,\alpha)\dfrac{\partial\phi}{\partial x}(x)$ and $B^\alpha\phi(x) := \phi(x+\beta(x,\alpha,\cdot)) - \phi(x)$.
- $K^\alpha\phi(x) := \int_{\mathbf{E}}\left(\phi(x+\beta(x,\alpha,e)) - \phi(x) - \dfrac{\partial\phi}{\partial x}(x)\beta(x,\alpha,e)\right)\nu(de)$.

Definition 6.4.4. • A function u is said to be a *viscosity subsolution* of (6.4.2) if it is upper semicontinuous on $[0,T[\times\mathbb{R}$, if $u(T,x) \le g(x), x \in \mathbb{R}$, and if for any point $(t_0,x_0) \in [0,T[\times\mathbb{R}$ and for any $\phi \in C^{1,2}([0,T]\times\mathbb{R})$ such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi - u$ attains its minimum at (t_0,x_0), we have

$$\begin{aligned}&\min(u(t_0,x_0) - h(t_0,x_0),\\ &\inf_{\alpha\in A}(-\frac{\partial\phi}{\partial t}(t_0,x_0) - L^\alpha\phi(t_0,x_0) - f(\alpha,t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial\phi}{\partial x})(t_0,x_0),B^\alpha\phi(t_0,x_0))) \le 0.\end{aligned}$$

In other words, if $u(t_0,x_0) > h(t_0,x_0)$,

$$\inf_{\alpha\in A}(-\frac{\partial\phi}{\partial t}(t_0,x_0) - L^\alpha\phi(t_0,x_0) - f(\alpha,t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial\phi}{\partial x})(t_0,x_0),B^\alpha\phi(t_0,x_0))) \le 0.$$

• A function u is said to be a *viscosity supersolution* of (6.4.2) if it is lower semicontinuous on $[0,T[\times\mathbb{R}$, $u(T,x) \ge g(x), x \in \mathbb{R}$, and if for any point $(t_0,x_0) \in [0,T[\times\mathbb{R}$ and any $\phi \in C^{1,2}([0,T]\times\mathbb{R})$ such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi - u$ attains its maximum at (t_0,x_0), we have

$$\begin{aligned}&\min(u(t_0,x_0) - h(t_0,x_0),\\ &\inf_{\alpha\in A}(-\frac{\partial}{\partial t}\phi(t_0,x_0) - L^\alpha\phi(t_0,x_0) - f(\alpha,t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial\phi}{\partial x})(t_0,x_0),B^\alpha\phi(t_0,x_0))) \ge 0.\end{aligned}$$

In other words, we have both $u(t_0,x_0) \ge h(t_0,x_0)$ and

$$\inf_{\alpha\in A}(-\frac{\partial\phi}{\partial t}(t_0,x_0) - L^\alpha\phi(t_0,x_0) - f(\alpha,t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial\phi}{\partial x})(t_0,x_0),B^\alpha\phi(t_0,x_0))) \ge 0.$$

Note that if a map is both a viscosity subsolution and a viscosity supersolution, then it is continuous and a viscosity solution in the classical sense. Here, since the value function u is not regular, it is not in general a viscosity solution in the classical sense.

Using the *weak* dynamic programming principle (Theorem 6.3.13) and the comparison theorem between a BSDE and a reflected BSDE (Proposition 6.4.2), we now prove that the value function of our problem is a *weak* viscosity solution of the above HJBVI. More precisely, without additional assumptions, the following theorem holds.

Theorem 6.4.5. *Under the same assumptions as those of Theorem 6.3.13, the function u, defined by* (6.2.3)*, is a* weak viscosity solution *of the HJBVI* (6.4.2)*, in the sense that u^* is a viscosity subsolution of* (6.4.2) *and u_* is a viscosity supersolution of* (6.4.2).

Proof. • *We first prove that u^* is a subsolution of* (6.4.2). Let $(t_0,x_0) \in [0,T[\times\mathbb{R}$ and $\phi \in C^{1,2}([0,T]\times\mathbb{R})$ be such that $\phi(t_0,x_0) = u^*(t_0,x_0)$ and $\phi(t,x) \ge u^*(t,x)$, $\forall (t,x) \in [0,T]\times\mathbb{R}$. Without loss of generality, we can suppose that the minimum of $\phi - u^*$ attained at (t_0,x_0) is strict. We suppose that $u^*(t_0,x_0) > h(t_0,x_0)$ and that

$$\inf_{\alpha\in A}(-\frac{\partial}{\partial t}\phi(t_0,x_0) - L^\alpha\phi(t_0,x_0) - f(\alpha,t_0,x_0,\phi(t_0,x_0),(\sigma\frac{\partial\phi}{\partial x})(t_0,x_0),B^\alpha\phi(t_0,x_0))) > 0.$$

By uniform continuity of $K^\alpha\phi$ and $B^\alpha\phi : [0,T]\times\mathbb{R}\to L^2_\nu$ with respect to α, we can suppose that there exists $\epsilon>0$, $\eta_\epsilon>0$ such that: $\forall(t,x)$ such that $t_0\le t\le t_0+\eta_\epsilon<T$ and $|x-x_0|\le\eta_\epsilon$, we have: $\phi(t,x)\ge h(t,x)+\epsilon$ and

$$-\frac{\partial}{\partial t}\phi(t,x)-L^\alpha\phi(t,x)-f(\alpha,t,x,\phi(t,x),(\sigma\frac{\partial\phi}{\partial x})(t,x),B^\alpha\phi(t,x))\ge\epsilon,\ \forall\alpha\in A. \tag{6.4.3}$$

We denote by $B_{\eta_\varepsilon}(t_0,x_0)$ the ball of radius η_ε and center (t_0,x_0). By definition of u^*, there exists a sequence $(t_n,x_n)_n$ in $B_{\eta_\varepsilon}(t_0,x_0)$, such that $(t_n,x_n,u(t_n,x_n))\to(t_0,x_0,u^*(t_0,x_0))$.
Fix $n\in\mathbb{N}$. Let α be an arbitrary control of $\mathcal{A}^{t_n}_{t_n}$ and X^{α,t_n,x_n} the associated state process.
We define the stopping time $\theta^{\alpha,n}$ as:

$$\theta^{\alpha,n}:=(t_0+\eta_\epsilon)\wedge\inf\{s\ge t_n\,,\,|X^{\alpha,t_n,x_n}_s-x_0|\ge\eta_\epsilon\}.$$

Applying Itô's lemma to $\phi(t,X^{\alpha,t_n,x_n}_t)$, we obtain:

$$\begin{aligned}\phi(t_n,X^{\alpha,t_n,x_n}_{t_n})&=\phi(\theta^{\alpha,n},X^{\alpha,t_n,x_n}_{\theta^{\alpha,n}})-\int_{t_n}^{\theta^{\alpha,n}}\psi^{\alpha_s}(s,X^{\alpha,t_n,x_n}_s)ds\\&\quad-\int_{t_n}^{\theta^{\alpha,n}}(\sigma\frac{\partial\phi}{\partial x})(s,X^{\alpha,t_n,x_n}_s)dW_s-\int_{t_n}^{\theta^{\alpha,n}}\int_{\mathbf{E}}B^{\alpha_s}\phi(s,X^{\alpha,t_n,x_n}_{s^-})\tilde N(ds,de)\end{aligned}$$

where $\psi^\alpha(s,x):=\dfrac{\partial}{\partial s}\phi(s,x)+L^\alpha\phi(s,x)$.
Note that $(\phi(s,X^{\alpha,t_n,x_n}_s),(\sigma\dfrac{\partial\phi}{\partial x})(s,X^{\alpha,t_n,x_n}_s),B^{\alpha_s}\phi(s,X^{\alpha,t_n,x_n}_{s^-});s\in[t_n,\theta^{\alpha,n}])$ is the solution of the BSDE associated with the driver process $-\psi^{\alpha_s}(s,X^{\alpha,t_n,x_n}_s)$, terminal time $\theta^{\alpha,n}$ and terminal value $\phi(\theta^{\alpha,n},X^{\alpha,t_n,x_n}_{\theta^{\alpha,n}})$. By (6.4.3) and by definition of $\theta^{\alpha,n}$, we get that for each $s\in[t_n,\theta^{\alpha,n}]$:

$$-\psi^{\alpha_s}(s,X^{\alpha,t_n,x_n}_s)\ge f(\alpha_s,s,X^{\alpha,t_n,x_n}_s,\phi(s,X^{\alpha,t_n,x_n}_s),(\sigma\frac{\partial\phi}{\partial x})(s,X^{\alpha,t_n,x_n}_s),B\phi(s,X^{\alpha,t_n,x_n}_s))+\epsilon \tag{6.4.4}$$

for each $s\in[t_n,\theta^{\alpha,n}]$. This inequality gives a relation between the drivers $-\psi^{\alpha_s}(s,X^{\alpha,t_n,x_n}_s)$ and $f(\alpha_s,\cdot)$ of two BSDEs. Now, since the minimum (t_0,x_0) is strict, there exists γ_ϵ such that:

$$u^*(t,x)-\phi(t,x)\le-\gamma_\epsilon\text{ on }[0,T]\times\mathbb{R}\setminus B_{\eta_\epsilon}(t_0,x_0). \tag{6.4.5}$$

We have

$$\phi(\theta^{\alpha,n}\wedge t,X^{\alpha,t_n,x_n}_{\theta^{\alpha,n}\wedge t})=\phi(t,X^{\alpha,t_n,x_n}_t)\mathbf{1}_{t<\theta^{\alpha,n}}+\phi(\theta^{\alpha,n},X^{\alpha,t_n,x_n}_{\theta^{\alpha,n}})\mathbf{1}_{t\ge\theta^{\alpha,n}},\ \ t_n\le t\le T.$$

To simplify notation, set $\delta_\varepsilon:=\min(\epsilon,\gamma_\epsilon)$. Using (6.4.5) together with the definition of $\theta^{\alpha,n}$, we derive that for each $t\in[t_n,\theta^{\alpha,n}]$:

$$\phi(t,X^{\alpha,t_n,x_n}_t)\ge(h(t,X^{\alpha,t_n,x_n}_t)+\delta_\varepsilon)\mathbf{1}_{t<\theta^{\alpha,n}}+(u^*(\theta^{\alpha,n},X^{\alpha,t_n,x_n}_{\theta^{\alpha,n}})+\delta_\varepsilon)\mathbf{1}_{t=\theta^{\alpha,n}}\quad\text{a.s.}$$

This, together with inequality (6.4.4) on the drivers and the above comparison theorem between a BSDE and a reflected BSDE (see Proposition 6.4.2) lead to:

$$\phi(t_n,x_n)\ge Y^{\alpha,t_n,x_n}_{t_n,\theta^{\alpha,n}}[h(t,X^{\alpha,t_n,x_n}_t)\mathbf{1}_{t<\theta^{\alpha,n}}+u^*(\theta^{\alpha,n},X^{\alpha,t_n,x_n}_{\theta^{\alpha,n}})\mathbf{1}_{t=\theta^\alpha}]+\delta_\varepsilon K,$$

where K is a positive constant which only depends on T and the Lipschitz constant of f.
Now, recall $(t_n, x_n, u(t_n, x_n)) \to (t_0, x_0, u^*(t_0, x_0))$ and ϕ is continuous with $\phi(t_0, x_0) = u^*(t_0, x_0)$. We can thus assume that n is sufficiently large so that $|\phi(t_n, x_n) - u(t_n, x_n)| \leq \frac{\delta_\varepsilon K}{2}$.Hence,

$$u(t_n, x_n) \geq Y^{\alpha, t_n, x_n}_{t_n, \theta^{\alpha,n}}[h(t, X^{\alpha, t_n, x_n}_t)\mathbf{1}_{t<\theta^{\alpha,n}} + u^*(\theta^{\alpha,n}, X^{\alpha, t_n, x_n}_{\theta^{\alpha,n}})\mathbf{1}_{t=\theta^\alpha}] + \frac{\delta_\varepsilon K}{2}.$$

As this inequality holds for all $\alpha \in \mathcal{A}^{t_n}_{t_n}$, we get a contradiction of the sub-optimality principle of dynamic programming principle (6.3.16) satisfied by u^* (see also Remark 6.3.15).

- *We now prove that u_* is a viscosity supersolution of* (6.4.2).

Let $(t_0, x_0) \in [0, T[\times\mathbb{R}$ and $\phi \in C^{1,2}([0, T] \times \mathbb{R})$ be such that $\phi(t_0, x_0) = u_*(t_0, x_0)$ and $\phi(t, x) \leq u_*(t, x)$, $\forall (t, x) \in [0, T] \times \mathbb{R}$. Without loss of generality, we can suppose that the maximum is strict in (t_0, x_0). Since the solution (Y^{α, t_0, x_0}_s) stays above the obstacle, for each $\alpha \in \mathcal{A}$, we have $u_*(t_0, x_0) \geq h(t_0, x_0)$. Our aim is to show that:

$$\inf_{\alpha \in A}(-\frac{\partial}{\partial t}\phi(t_0, x_0) - L^\alpha\phi(t_0, x_0) - f(\alpha, t_0, x_0, \phi(t_0, x_0), (\sigma\frac{\partial \phi}{\partial x})(t_0, x_0), B^\alpha\phi(t_0, x_0))) \geq 0.$$

Suppose for contradiction that this inequality does not hold.
By continuity, we can suppose that there exists $\alpha \in A$, $\epsilon > 0$ and $\eta_\epsilon > 0$ such that:
$\forall (t, x)$ with $t_0 \leq t \leq t_0 + \eta_\epsilon < T$ and $|x - x_0| \leq \eta_\epsilon$, we have:

$$-\frac{\partial}{\partial t}\phi(t, x) - L^\alpha\phi(t, x) - f(\alpha, t, x, \phi(t, x), (\sigma\frac{\partial \phi}{\partial x})(t, x), B^\alpha\phi(t, x)) \leq -\epsilon. \tag{6.4.6}$$

We denote by $B_{\eta_\varepsilon}(t_0, x_0)$ the ball of radius η_ε and center (t_0, x_0). Let $(t_n, x_n)_n$ be a sequence in $B_{\eta_\varepsilon}(t_0, x_0)$ such that $(t_n, x_n, u(t_n, x_n)) \to (t_0, x_0, u_*(t_0, x_0))$. We introduce the state process X^{α, t_n, x_n} associated with the above constant control α and define the stopping time θ^n as:

$$\theta^n := (t_0 + \eta_\epsilon) \wedge \inf\{s \geq t_n\,,\, |X^{\alpha, t_n, x_n}_s - x_0| \geq \eta_\epsilon\}.$$

By Itô's lemma applied to $\phi(s, X^{\alpha, t_n, x_n}_s)$, we have that

$$(\phi(s, X^{\alpha, t_n, x_n}_s), (\sigma\frac{\partial \phi}{\partial x})(s, X^{\alpha, t_n, x_n}_s), B^\alpha\phi(s, X^{\alpha, t_n, x_n}_{s^-}); s \in [t_n, \theta^n])$$

is the solution of the BSDE associated with terminal time θ^n, terminal value $\phi(\theta^n, X^{\alpha, t_n, x_n}_{\theta^n})$ and driver $-\psi^\alpha(s, X^{\alpha, t_n, x_n}_s)$. The definition of the stopping time θ^n and inequality (6.4.6) lead to:

$$-\psi^\alpha(s, X^{\alpha, t_n, x_n}_s) \leq f(\alpha, s, X^{\alpha, t_n, x_n}_s, \phi(s, X^{\alpha, t_n, x_n}_s), (\sigma\frac{\partial \phi}{\partial x})(s, X^{\alpha, t_n, x_n}_s), B^\alpha\phi(s, X^{\alpha, t_n, x_n}_s)), \tag{6.4.7}$$

for $t_n \leq s \leq \theta^n$ $ds \otimes dP$-a.s. Now, since the maximum (t_0, x_0) is strict, there exists γ_ϵ (which depends on η_ϵ) such that $u_*(t, x) \geq \phi(t, x) + \gamma_\epsilon$ on $[0, T] \times \mathbb{R} \backslash B_{\eta_\epsilon}(t_0, x_0)$ which implies $\phi(\theta^n, X^{\alpha, t_n, x_n}_{\theta^n}) \leq u_*(\theta, X^{\alpha, t_n, x_n}_{\theta^n}) - \gamma_\epsilon$. Hence, using inequality (6.4.7) on the drivers, together with the comparison theorem for BSDEs, we derive that:

$$\phi(t_n, x_n) = \mathcal{E}^{-\psi^\alpha}_{t_n, \theta^n}[\phi(\theta^n, X^{\alpha, t_n, x_n}_{\theta^n})] \leq \mathcal{E}^{\alpha, t_n, x_n}_{t_n, \theta^n}[u_*(\theta^n, X^{\alpha, t_n, x_n}_\theta) - \gamma_\epsilon] \leq \mathcal{E}^{\alpha, t_n, x_n}_{t_n, \theta^n}[u_*(\theta^n, X^{\alpha, t_n, x_n}_{\theta^n})] - \gamma_\epsilon K.$$

where the second inequality is obtained by using the above extension of the comparison theorem (Lemma 6.4.1). Now, we can assume that n is sufficient large so that $|\phi(t_n,x_n)-u(t_n,x_n)| \leq \frac{\delta_\varepsilon K}{2}$. Hence, we get:

$$u(t_n,x_n) \leq \mathcal{E}^{\alpha,t_n,x_n}_{t_n,\theta^n}[u_*(\theta^n, X^{\alpha,t_n,x_n}_{\theta^n})] - \frac{\gamma_\epsilon K}{2}. \tag{6.4.8}$$

Since u_* satisfies the super-optimality principle of dynamic programming (Th. 6.3.13), we have $u(t_n,x_n) \geq \mathcal{E}^{\alpha,t_n,x_n}_{t_n,\theta^n}[u_*(\theta^n, X^{\alpha,t_n,x_n}_{\theta^n})]$. This inequality with (6.4.8) lead to a contradiction. □

Remark 6.4.6. We mention the paper [127] which studies stochastic control with nonlinear expectation in the regular case (and when there is no stopping time optimization). The approach is different and relies on the continuity assumption of the reward function. We also mention [145] where relations between some nonlinear HJB equations and second order BSDEs in the Brownian case are studied.

6.5 Examples in mathematical finance

Maximization of recursive utility of terminal wealth. We consider a portfolio optimization problem for an agent with recursive utility. His wealth process $X^{\alpha,t,x}$ is controlled by α, which represents a portfolio-strategy. The recursive utility process is defined via a BSDE associated with a driver $f : [0,T] \times \mathbb{R}^2 \times L^2_\nu \to \mathbb{R}$; $(t,y,z,k) \mapsto f(t,y,z,k)$ satisfying Assumption 6.2.1 and concave with respect to y,z,k. The terminal reward is given by $h(X^{\alpha,t,x}_T)$, where h is a concave non decreasing map. Recall that the recursive utility generalizes the standard additive utilities but in this case, the utility depends on the future utility through the dependance of the driver f on y (see [69]). Also, the recursive utility may depend on the future utility "variability" or "volatility" through the dependance of f with respect to z and k.

If x is the initial wealth, for each strategy α, the associated recursive utility function at initial time t is equal to $\mathcal{E}_{t,T}[h(X^{\alpha,t,x}_T)]$, where $\mathcal{E}$ is the f-conditional expectation associated with the driver f. The aim of the investor is to maximize his recursive utility of wealth over all portfolio-strategies $\alpha \in \mathcal{A}^t_t$. By Theorem 6.4.5, the value function $u(t,x)$ defined by

$$u(t,x) := \sup_{\alpha \in \mathcal{A}^t_t} \mathcal{E}^{t,x}_{t,T}[h(X^{\alpha,t,x}_T)]$$

is a weak viscosity solution of the nonlinear HJB equation:

$$\sup_{\alpha \in A}\Big(\frac{\partial u}{\partial t}(t,x) + L^\alpha u(t,x) + f(t,x,u(t,x),(\sigma\frac{\partial u}{\partial x})(t,x), B^\alpha u(t,x))\Big) = 0, \tag{6.5.1}$$

with $u(T,x) = h(x)$, where the operators are defined by (6.4.2).

An example given in [69] in a Brownian framework is $f(t,x,y,z,k) := -C|z|$. We can generalize this example to the case of jumps by setting $f := f_1$ with

$$f_1(t,z,k) := -C_1|z| - C_2 \int_{\mathbf{E}} |k(e)|\Psi(e)\nu(de). \tag{6.5.2}$$

The constants C_1, C_2 are here positive constants which can be interpreted as risk-aversion coefficients. Note that this driver allows us to model asymmetry in risk-aversion, depending on whether the risk comes from the Brownian random source or from the jumps random source (Poisson random measure). If $C_2 \leq 1$, then, f_1 satisfies Assumption 6.2.1, in particular condition (iii), which ensures the monotonicity of the recursive utility with respect to terminal reward, and thus to terminal wealth because h is non decreasing.

We can also consider an extension of this example to the case of a seller of a European option with payoff $G(S_T)$, where G is an irregular function, for example

$$G(x) = \mathbf{1}_B(x),$$

where B is a Borelian, and S is a Markovian jump-diffusion process representing the price of the underlying asset. He wants to maximize his recursive utility of terminal wealth over all portfolio-strategies. In this case, the value function at time t is then given by

$$u(t,x) := \sup_{\alpha \in \mathcal{A}_t^t} \mathcal{E}_{t,T}^{t,x}[h(X_T^{\alpha,t,x} - G(S_T))].$$

Note that u is not necessarily continuous, not even measurable. By Theorem 6.4.5, one can derive that u is a weak viscosity solution of an associated nonlinear HJB equation with terminal condition $u(T,x) = h(x_1 - G(x_2))$ for $x = (x_1, x_2)$.

Minimization of the risk of terminal wealth. We consider the same model as in the above example and a dynamic risk-measure ρ defined for each position $\xi \in L^2(\mathcal{F}_T)$ by

$$\rho_{t,T}(\xi) := -\mathcal{E}_{t,T}[g(\xi)], \qquad 0 \leq t \leq T,$$

where g is a Borelian non decreasing function with polynomial growth. At time t, for a given initial wealth x, the aim of the investor is to minimize his risk-measure of terminal wealth over all portfolio-strategies $\alpha \in \mathcal{A}_t^t$. The value function $v(t,x)$ is given by

$$v(t,x) := \inf_{\alpha \in \mathcal{A}_t^t} \rho_{t,T}[g(X_T^{\alpha,t,x})] = -u(t,x),$$

where $u(t,x) := \sup_{\alpha \in \mathcal{A}_t^t} \mathcal{E}_{t,T}[g(X_T^{\alpha,t,x})]$. As in the previous example, u is a weak viscosity solution of HJB equation (6.5.1) with $u(T,x) = g(x)$. An example is given by f_1 defined by (6.5.2), where the coefficients C_1 and C_2 can also be interpreted as risk-aversion coefficients when the risk comes from the Brownian random source, respectively from the jumps random source.

We can also consider the minimization problem of shortfall risk for the seller of a European option with payoff $G(S_T)$, where G is an irregular function such as an indicator function. More precisely, at time t, for an initial wealth x, the aim of the seller is to minimize the risk measure associated with the negative part (shortfall) of his terminal position given by $-(X_T^{\alpha,t,x} - G(S_T))^-$. The value function $v(t,x)$ is then given by

$$v(t,x) := \inf_{\alpha \in \mathcal{A}_t^t} \rho_{t,T}[-(X_T^{\alpha,t,x} - G(S_T))^-].$$

The extension to the case when the agent or the seller also acts on stopping times leads to a mixed optimal control/stopping problem.

6.6 Appendix

Lemma 6.6.1. *The function u has at most polynomial growth at infinity.*

Proof. Applying some estimates on the solution of a reflected BSDE (see Prop. 5.1 in [?]), we obtain:

$$|Y_t^{\alpha,t,x}|^2 = |Y_0^{\alpha,t,x}|^2 \leq K(\mathbb{E}(\int_0^T f(\alpha_s, s, X_s^{\alpha,t,x}, 0, 0, 0)^2 ds + \sup_{0\leq s\leq T} h(s, X_s^{\alpha,t,x})^2), \forall \alpha \in \mathcal{A}_t^t,$$

where K is a real constant which depends only on C and T. Using now the hypothesis of polynomial growth on f, h, g and the standard estimate $\mathbb{E}[\sup_{0\leq s\leq T} |X_s^{\alpha,t,x}|^2] \leq C'(1+x^2)$, we derive that there exist $\bar{C} \in \mathbb{R}$ and $p \in \mathbb{N}$ such that $|u^\alpha(t,x)| \leq \bar{C}(1+|x|^p)$, for all t in $[0,T]$ and all x in $\mathbb{R}$. We finally get that $|u(t,x)| \leq \sup_{\alpha\in\mathcal{A}_t^t} |u^\alpha(t,x)| \leq \bar{C}(1+|x|^p)$. □

We recall the following property.

Lemma 6.6.2. *The Hilbert space $\mathbb{H}_t^2$ is separable.*

Proof. A short proof is given for the convenience of the reader. Since the paths are right-continuous, for every $r > t$, $\mathcal{F}_{r^-}^t = \sigma(\{\omega_u^t,\ u \in \mathbb{Q}$ and $t \leq u < r\})$ and is thus *countably generated*, that is generated by a countable subfamily of $\mathcal{F}_{r^-}^t$. Now, the predictable σ-algebra $\mathcal{P}^t$ on $\Omega \times [t,T]$ is generated by the sets of the form $[r,T[\times H$ (or $]r,T] \times H$), where r is rational with $r \geq t$, and H belongs to $\mathcal{F}_{r^-}^t$. It follows that $\mathcal{P}^t$ is countably generated. By an argument of Measure Theory (see e.g. Proposition 3.4.5 in [47]), since $\mathcal{P}^t$ is countably generated, the space $\mathbb{H}_t^2 = L^2([t,T] \times \Omega, \mathcal{P}^t, ds \otimes dP)$ is separable. □

A result of classical analysis. We state a result of classical analysis concerning the approximation of a real-valued function in $L^2([0,T]), dt)$ equipped with the norm $||f||_{L_T^2}^2 = \int_0^T f(r)^2 dr$ by a specific sequence of step functions as well as useful inequalities used in the chapter. For each $n \in \mathbb{N}$, we consider the linear operator $P_n : L^2([0,T], dr) \to L^2([0,T], dr)$ defined for each $f \in L^2([0,T], dr)$ by

$$P^n(f)(t) := n \sum_{i=1}^{n-1} (\int_{\frac{(i-1)T}{n}}^{\frac{iT}{n}} f(r) dr) \mathbf{1}_{]\frac{iT}{n}, \frac{(i+1)T}{n}]}(t).$$

By Cauchy-Schwartz's inequality, we have that for each $t \in]\frac{iT}{n}, \frac{(i+1)T}{n}]$, $1 \leq i \leq n-1$, $P^n(f)^2(t) \leq n \int_{\frac{(i-1)T}{n}}^{\frac{iT}{n}} f^2(r) dr$. Hence,

$$||P_n(f)||_{L_T^2} \leq ||f||_{L_T^2}; \quad ||P_n(f) - f||_{L_T^2} \to 0, \text{ when } n \to \infty. \tag{6.6.1}$$

Indeed, the above convergence clearly holds when f is continuous, and the general case follows by using the uniform continuity of f and the density of $\mathcal{C}([0,T])$ in L_T^2.

A result of Measure Theory (see Lemma 1.2 in [49]) ensures the following property.

Lemma 6.6.3 (A result of Measure Theory)**.** *Let $(X, \mathcal{F}, Q)$ be a probability space. Let $\mathcal{F}_Q$ be the* completion *σ-algebra of $\mathcal{F}$ with respect to Q, that is the class of sets of the form $B \cup M$, with $B \in \mathcal{F}$ and M being a subset of a set N belonging to $\mathcal{F}$ with Q-measure 0. Let E be a separable Hilbert space, equipped with its scalar product $< . , . >$, and its Borel σ-algebra $\mathcal{B}(E)$.*
*Then, for each $\mathcal{F}$-measurable map $f : X \to E$, there exists an $\mathcal{F}$-measurable map f_Q such that $f_Q(x) = f(x)$ for Q-*almost every *x, in the sense that the set $\{x \in X \, , \, f_Q(x) \neq f(x)\}$ is included in a set belonging to $\mathcal{F}$ with Q-measure 0.*

Proof. For completeness, we give the proof of this lemma. Since E is a separable Hilbert space, its admits a countable orthonormal basis $\{e^i, i \in \mathbb{N}\}$. Hence, for each $x \in X$, we have $f(x) = \sum_i f_i(x) e^i$ with $f_i(x) =< f(x), e^i >$. Note that $f_i : (X, \mathcal{F}_Q) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable. It is thus sufficient to show that the result holds for f valued in $E = \mathbb{R}$.

For $r \in \mathbb{Q}$, let $\bar{B}_r := \{x \in X, \, f(x) \leq r\}$. For each $x \in X$, we have $f(x) = \inf\{r \in \mathbb{Q}, \, x \in \bar{B}_r\}$. Since $\bar{B}_r \in \mathcal{F}_Q$, there exists $B_r \in \mathcal{F}$ such that $B_r \subset \bar{B}_r$ and $\bar{B}_r \backslash B_r \subset N_r$, where $N_r \in \mathcal{F}$ with Q-measure 0. For each $x \in N_r^c$, $x \in B_r$ if and only if $x \in \bar{B}_r$, which ensures that $f(x) = \inf\{r \in \mathbb{Q}, \, x \in B_r\}$. Define f_Q for each each $x \in X$ by $f_Q(x) := \inf\{r \in \mathbb{Q}, \, x \in B_r\}$. The map f_Q is $\mathcal{F}$-measurable because for all $x \in X$, we have $f_Q(x) = \inf_{r \in \mathbb{Q}} \varphi_r(x)$, where φ_r is the $\mathcal{F}$-measurable map defined by $\varphi_r(x) := r$ if $x \in B_r$ and $\varphi_r(x) := +\infty$ otherwise. Also, $\{x \in X \, , \, f_Q(x) \neq f(x)\} \subset \cup_{r \in \mathbb{Q}} N_r$. Hence, $f_Q(x) = f(x)$ for Q-*almost every* x. $\square$

Remark 6.6.4. One can immediately derive the following result, used in the proof of Theorem 6.3.7. Let X be a topological space. Let $\mathcal{P}(X)$ be the set of all probability measures on $\mathcal{B}(X)$, the Borelian σ-algebra of X. For each $Q \in \mathcal{P}(X)$, $\mathcal{B}_Q(X)$ denotes *the completion* of $\mathcal{B}(X)$ with respect to Q. The *universal* σ-algebra on X is then defined by $\mathcal{U}(X) := \cap_{Q \in \mathcal{P}(X)} \mathcal{B}_Q(X)$.
Let now E be a separable Hilbert space. Let $f : X \to E$ be an *universally measurable* map, that is $\mathcal{U}(X)$-measurable. By Lemma 6.6.3, for each probability Q on $\mathcal{B}(X)$, since $\mathcal{U}(X) \subset \mathcal{B}_Q(X)$, there exists a Borelian map $f_Q : X \to E$ such that $f(x) = f_Q(x)$ for Q-*almost every* x.

Part II

Numerical methods for Doubly Reflected BSDEs with Jumps and RCLL obstacles

Chapter 1

Numerical approximation of doubly reflected BSDEs with jumps and RCLL obstacles

Abstract. We study a discrete time approximation scheme for the solution of a doubly reflected Backward Stochastic Differential Equation (DRBSDE in short) with jumps, driven by a Brownian motion and an independent compensated Poisson process. Moreover, we suppose that the obstacles are right continuous and left limited (RCLL) processes with predictable and totally inaccessible jumps and satisfy Mokobodski's condition. Our main contribution consists in the construction of an implementable numerical sheme, based on two random binomial trees and the penalization method, which is shown to converge to the solution of the DRBSDE. Finally, we illustrate the theoretical results with some numerical examples in the case of general jumps.

1.1 Introduction

In this chapter, we study in the non-markovian case a discrete time approximation scheme for the solution of a doubly reflected Backward Stochastic Differential Equation (DRBSDE in short) when the noise is given by a Brownian motion and a Poisson random process mutually independent. Moreover, the barriers are supposed to be right-continuous and left-limited (RCLL in short) processes, whose jumps are arbitrary, they can be either predictable or inaccessible. The DRBSDE we solve numerically has the following form:

$$
\begin{cases}
(i)\ Y_t = \xi_T + \int_t^T g(s, Y_s, Z_s, U_s) ds + (A_T - A_t) - (K_T - K_t) \\
\qquad - \int_t^T Z_s dW_s - \int_t^T U_s d\tilde{N}_s, \\
(ii)\ \forall t \in [0, T], \xi_t \leq Y_t \leq \zeta_t a.s., \\
(iii)\ \int_0^T (Y_{t^-} - \xi_{t^-}) dA_t^c = 0 a.s. \text{ and } \int_0^T (\zeta_{t^-} - Y_{t^-}) dK_t^c = 0 a.s. \\
(iv)\ \forall \tau \text{ predictable stopping time },\ \Delta A_\tau^d = \Delta A_\tau^d \mathbf{1}_{Y_{\tau^-} = \xi_{\tau^-}} \text{ and } \Delta K_\tau^d \\
\qquad = \Delta K_\tau^d \mathbf{1}_{Y_{\tau^-} = \zeta_{\tau^-}}.
\end{cases}
\tag{1.1.1}
$$

Here, A^c (resp. K^c) denotes the continuous part of A (resp. K) and A^d (resp. K^d) its discontinuous part, $\{W_t : 0 \leq t \leq T\}$ is a one dimensional standard Brownian motion and $\{\tilde{N}_t := N_t - \lambda t, 0 \leq t \leq T\}$ is a compensated Poisson process. Both processes are independent and they are defined on the probability space $(\Omega, \mathcal{F}_T, \mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$. The processes A and K have the role to keep the solution between the two obstacles ξ and ζ. Since we consider the general setting when the jumps of the obstacles can be either predictable or totally inaccessible, A and K are also discontinuous.

In the case of a Brownian filtration, non-linear backward stochastic differential equations (BSDEs in short) were introduced by Pardoux and Peng [124]. One barrier reflected BSDEs have been firstly studied by El Karoui et al in [71]. In their setting, one of the components of the solution is forced to stay above a given barrier which is a continuous adapted stochastic process. The main motivation is the pricing of American options especially in constrained markets. The generalization to the case of two reflecting barriers has been carried out by Cvitanic and Karatzas in [52]. It is also well known that doubly reflected BSDEs are related to Dynkin games and in finance to the pricing of Israeli options (or Game options, see [102]). The case of standard BSDEs with jump processes driven by a compensated Poisson random measure was first considered by Tang and Li in [147]. The extension to the case of reflected BSDEs and one reflecting barrier with only inaccessible jumps has been established by Hamadène and Ouknine [90]. Later on, Essaky in [78] and Hamadène and Ouknine in [91] have extended these results to a RCLL obstacle with predictable and inaccessible jumps. Results concerning existence and uniqueness of the solution for doubly reflected BSDEs with jumps can be found in [50],[62], [86], [92] and [79].

Numerical shemes for DRBSDEs driven by the Brownian motion and based on a random tree method have been proposed by Xu in [151] (see also [117] and [134]) and, in the Markovian framework, by Chassagneux in [45]. In the case of a filtration driven also by a Poisson process, some results have been provided only in the non-reflected case. In [30], the authors propose a scheme for Forward-Backward SDEs based on the dynamic programming equation and in [109] the authors propose a fully implementable scheme based on a random binomial tree. This work extends the paper [37], where the authors prove a Donsker type theorem for BSDEs in the Brownian case.

Our aim is to propose an implementable numerical method to approximate the solution of DRBSDEs with jumps and RCLL obstacles (2.1.1). As for standard BSDEs, the computation of conditional expectations is an important issue. Since we consider reflected BSDEs, we also have to model the constraints. To do this, we consider the following approximations

- we approximate the Brownian motion and the Poisson process by two independent random walks,
- we introduce a sequence of penalized BSDEs to approximate the reflected BSDE.

These approximations enable us to provide a fully implementable scheme, called *explicit penalized discrete scheme* in the following. We prove in Theorem 2.4.3 that the scheme weakly converges to the solution of (2.1.1). Moreover, in order to prove the convergence of our sheme, we prove, in the case of jump processes driven by a general Poisson random measure, that the solutions of the penalized equations converge to the solution of the doubly reflected BSDE in the case of a driver depending on the solution, which was not the case in the previous literature (see [79], [86], [92]). This gives another proof for the existence of a solution of DRBSDEs with jumps and RCLL barriers. Our method is based on a combination of penalization, Snell envelope theory, stochastic games, comparison theorem for BSDEs with jumps (see [137], [138]) and a generalized monotonic theorem under the Mokobodski's condition. It extends [112] to the case when the solution of the DRBSDE also admits totally inaccessible jumps. Finally, we illustrate our theoretical results with some numerical simulations in the case of general jumps. We point out that the practical use of our scheme is restricted to low dimensional cases. Indeed, since we use a random walk to approximate the Brownian motion and the Poisson process, the complexity of the algorithm grows very fast in the number of time steps n (more precisely, in n^d, d being the dimension) and, as we will see in the numerical part, the penalization method requires many time steps to be stable.

The chapter is organized as follows: in Section 2 we introduce notation and assumptions. In Section 3, we precise the discrete framework and give the numerical scheme. In Section 4 we provide the convergence by splitting the error : the error due to the approximation by penalization and the error due to the time discretization. Finally, Section 5 presents some numerical examples, where the barriers contain predictable and totally inaccessible jumps. In Appendix, we extend the generalized monotonic theorem and prove some technical results for discrete BSDEs to the case of jumps. For the self-containment of the chapter, we also recall some recent results on BSDEs with jumps and reflected BSDEs.

1.2 Notations and assumptions

Although we propose a numerical scheme for reflected BSDEs driven by a Brownian motion and a Poisson process, one part of the proof of the convergence of our scheme is done in the general setting of jumps driven by a Poisson random measure. Then, we first introduce the general framework, in which we prove the convergence of a sequence of penalized BSDEs to the solution of (2.1.1).

1.2.1 General framework

Notation

As said in Introduction, let $(\Omega, \mathbb{F}, P)$ be a probability space, and $\mathcal{P}$ be the predictable σ-algebra on $[0,T] \times \Omega$. W is a one-dimensional Brownian motion and $N(dt, de)$ is a Poisson random measure, independent of W, with compensator $\nu(de)dt$ such that ν is a σ-finite measure on $\mathbb{R}^*$, equipped with its Borel field $\mathcal{B}(\mathbb{R}^*)$. Let $\tilde{N}(dt, du)$ be its compensated process. Let $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ be the natural filtration associated with W and N.

For each $T > 0$, we use the following notations:

- $L^2(\mathcal{F}_T)$ is the set of random variables ξ which are $\mathcal{F}_T$-measurable and square integrable.
- $\mathbb{H}^2$ is the set of real-valued predictable processes ϕ such that $\|\phi\|^2_{\mathbb{H}^2} := \mathbb{E}\left[\int_0^T \phi_t^2 dt\right] < \infty$.

- L^2_ν is the set of Borelian functions $\ell : I\!\!R^* \to I\!\!R$ such that $\int_{I\!\!R^*} |\ell(u)|^2\nu(du) < +\infty$.

  The set L^2_ν is a Hilbert space equipped with the scalar product $\langle \delta,\, \ell\rangle_\nu := \int_{I\!\!R^*} \delta(u)\ell(u)\nu(du)$ for all $\delta,\, \ell \in L^2_\nu \times L^2_\nu$, and the norm $\|\ell\|^2_\nu := \int_{I\!\!R^*} |\ell(u)|^2\nu(du)$.

- $\mathcal{B}(I\!\!R^2)$ (resp $\mathcal{B}(L^2_\nu)$) is the Borelian σ-algebra on $I\!\!R^2$ (resp. on L^2_ν).

- $I\!\!H^2_\nu$ is the set of processes l which are *predictable*, that is, measurable

$$l : ([0,T] \times \Omega \times I\!\!R^*,\ \mathcal{P} \otimes \mathcal{B}(I\!\!R^*)) \to (I\!\!R\ , \mathcal{B}(I\!\!R)); \quad (\omega, t, u) \mapsto l_t(\omega, u)$$

  such that $\|l\|^2_{I\!\!H^2_\nu} := \mathbb{E}\left[\int_0^T \|l_t\|^2_\nu\, dt\right] < \infty$.

- $\mathcal{S}^2$ is the set of real-valued RCLL adapted processes ϕ such that $\|\phi\|^2_{\mathcal{S}^2} := \mathbb{E}(\sup_{0\le t\le T} |\phi_t|^2) < \infty$.

- $\mathcal{A}^2$ is the set of real-valued non decreasing RCLL predictable processes A with $A_0 = 0$ and $\mathbb{E}(A^2_T) < \infty$.

- $\mathcal{T}_0$ is the set of stopping times τ such that $\tau \in [0,T]$ a.s

- For S in $\mathcal{T}_0$, $\mathcal{T}_S$ is the set of stopping times τ such that $S \le \tau \le T$ a.s.

Definitions and assumptions

We start this section by recalling the definition of a driver and a Lipschitz driver. We also introduce DRBSDEs and our working assumptions.

Definition 1.2.1 (Driver, Lipschitz driver)**.** *A function g is said to be a* driver *if*

- $g : \Omega \times [0,T] \times I\!\!R^2 \times L^2_\nu \to I\!\!R$
  $(\omega, t, y, z, \kappa(\cdot)) \mapsto g(\omega, t, y, z, k(\cdot))$ *is* $\mathcal{P} \otimes \mathcal{B}(I\!\!R^2) \otimes \mathcal{B}(L^2_\nu)-$ *measurable,*

- $\|g(.,0,0,0)\|_\infty < \infty$.

A driver g is called a Lipschitz driver *if moreover there exists a constant $C_g \ge 0$ and a bounded, non-decreasing continuous function Λ with $\Lambda(0) = 0$ such that $d\mathbb{P} \otimes dt$-a.s. , for each (s_1, y_1, z_1, k_1), (s_2, y_2, z_2, k_2),*

$$|g(\omega, s_1, y_1, z_1, k_1) - g(\omega, s_2, y_2, z_2, k_2)| \le \Lambda(|s_2 - s_1|) + C_g(|y_1 - y_2| + |z_1 - z_2| + \|k_1 - k_2\|_\nu).$$

In the case of BSDEs with jumps, the coefficient g must satisfy an additional assumption, which allows to apply the comparison theorem for BSDEs with jumps (see Theorem 1.9.1), which extends the result of [140]. More precisely, the driver g satisfies the following assumption:

Assumption 1.2.2. *A Lipschitz driver g is said to satisfy Assumption* 1.2.2 *if the following holds: $dP \otimes dt$ a.s. for each $(y, z, k_1, k_2) \in I\!\!R^2 \times (L^2_\nu)^2$, we have*

$$g(t, y, z, k_1) - g(t, y, z, k_2) \ge \langle \theta^{y,z,k_1,k_2}_t, k_1 - k_2\rangle_\nu,$$

with

$$\begin{aligned}\theta :&\Omega \times [0,T] \times \mathbb{R}^2 \times (L^2_\nu)^2 \longmapsto L^2_\nu;\\ &(\omega, t, y, z, k_1, k_2) \longmapsto \theta^{y,z,k_1,k_2}_t(\omega, \cdot)\end{aligned}$$

$\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^2)\otimes\mathcal{B}((L^2_\nu)^2)$-measurable, bounded, and satisfying $dP\otimes dt\otimes\nu(du)$-a.s., for each $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$,

$$\theta^{y,z,k_1,k_2}_t(u) \geq -1 \text{ and } |\theta^{y,z,k_1,k_2}_t(u)| \leq \psi(u),$$

where $\psi \in L^2_\nu$.

We now recall the "Mokobodski's condition" which is essential in the case of doubly reflected BSDEs, since it ensures the existence of a solution. This condition essentially postulates the existence of a quasimartingale between the barriers.

Definition 1.2.3 (Mokobodski's condition)**.** *Let ξ, ζ be in $\mathcal{S}^2$. There exist two nonnegative RCLL supermartingales H and H' in $\mathcal{S}^2$ such that*

$$\forall t \in [0,T], \;\; \xi_t\mathbf{1}_{t\leq T} \leq H_t - H'_t \leq \zeta_t\mathbf{1}_{t\leq T} \text{ a.s.}$$

Assumption 1.2.4. *ξ and ζ are two adapted RCLL processes with $\xi_T = \zeta_T$ a.s., $\xi \in \mathcal{S}^2$, $\zeta \in \mathcal{S}^2$, $\xi_t \leq \zeta_t$ for all $t \in [0,T]$, the Mokobodski's condition holds and g is a Lipschitz driver satisfying Assumption* 2.2.4.

We introduce the following general reflected BSDE with jumps and two RCLL obstacles

Definition 1.2.5. *Let $T > 0$ be a fixed terminal time and g be a Lipschitz driver. Let ξ and ζ be two adapted RCLL processes with $\xi_T = \zeta_T$ a.s., $\xi \in \mathcal{S}^2$, $\zeta \in \mathcal{S}^2$, $\xi_t \leq \zeta_t$ for all $t \in [0,T]$ a.s. A process (Y, Z, U, α) is said to be a solution of the double barrier reflected BSDE (DRBSDE) associated with driver g and barriers ξ, ζ if*

$$\begin{cases} (i)\; Y \in \mathcal{S}^2, Z \in \mathbb{H}^2, U \in \mathbb{H}^2_\nu \text{ and } \alpha \in \mathcal{S}^2, \text{ where } \alpha = A - K \text{ with } A, K \text{ in } \mathcal{A}^2\\ (ii)\; Y_t = \xi_T + \int_t^T g(s, Y_s, Z_s, U_s)ds + (A_T - A_t) - (K_T - K_t)\\ \qquad - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbb{R}^*} U_s(e)\tilde{N}(ds, de),\\ (iii)\; \forall t \in [0,T], \xi_t \leq Y_t \leq \zeta_t a.s.,\\ (iv)\; \int_0^T (Y_{t^-} - \xi_{t^-})dA_t = 0 a.s. \text{ and } \int_0^T (\zeta_{t^-} - Y_{t^-})dK_t = 0 a.s. \end{cases} \tag{1.2.1}$$

Remark 1.2.6. Condition (iv) is equivalent to the following condition : if $K = K^c + K^d$ and $A = A^c + A^d$, where K^c (resp. K^d) represents the continuous (resp. the discontinous) part of K (the same notation holds for A), then

$$\int_0^T (Y_t - \xi_t)dA^c_t = 0 \text{ a.s.}, \;\; \int_0^T (\zeta_t - Y_t)dK^c_t = 0 \text{ a.s.}$$

and

$$\forall \tau \in \mathcal{T}_0 \text{ predictable}, \;\; \Delta A^d_\tau = \Delta A^d_\tau \mathbf{1}_{Y_{\tau^-}=\xi_{\tau^-}} \text{ and } \Delta K^d_\tau = \Delta K^d_\tau \mathbf{1}_{Y_{\tau^-}=\zeta_{\tau^-}}.$$

Theorem 1.2.7 ([62], Theorem 4.1)**.** *Suppose ξ and ζ are RCLL adapted processes in $\mathcal{S}^2$ such that for all $t \in [0,T]$, $\xi_t \leq \zeta_t$ and Mokobodski's condition holds (see Definition 1.2.3). Then, DRBSDE* (1.2.1) *admits a unique solution (Y, Z, U, α) in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{A}^2$.*

Remark 1.2.8. As said in [62, Remark 4.3], if for all $t \in]0,T]$ $\xi_{t^-} < \zeta_{t^-}$ a.s., [62, Proposition 4.2] gives the uniqueness of $A, K \in (\mathcal{A}^2)^2$.

Definition 1.2.9 (convergence in J1-Skorokhod topology)**.** *ξ^n is said to converge in probability (resp. in L^2) to ξ for the J1-Skorokhod topology, if there exists a family $(\psi^n)_{n\in\mathbb{N}}$ of one-to-one random time changes from $[0,T]$ to $[0,T]$ such that $\sup_{t\in[0,T]} |\psi^n(t) - t| \xrightarrow[n\to\infty]{} 0$ almost surely and $\sup_{t\in[0,T]} |\xi^n_{\psi^n(t)} - \xi_t| \xrightarrow[n\to\infty]{} 0$ in probability (resp. in L^2). Throughout the chapter, we denote this convergence $||\xi^n - \xi||_{J1-\mathbb{P}} \to 0$ (resp. $||\xi^n - \xi||_{J1-L^2} \to 0$).*

1.2.2 Framework for our numerical scheme

In order to propose an implementable numerical scheme we consider that the Poisson random measure is simply generated by the jumps of a Poisson process. We consider a Poisson process $\{N_t : 0 \leq t \leq T\}$ with intensity λ and jumps times $\{\tau_k : k = 0, 1, ...\}$. The random measure is then

$$\tilde{N}(dt, de) = \sum_{k=1}^{N_t} \delta_{\tau_k,1}(dt, de) - \lambda dt \delta_1(de)$$

where δ_a denotes the Dirac measure at the point a. In the following, $\tilde{N}_t := N_t - \lambda t$. Then, the unknown fonction $U_s(e)$ does not depend on the magnitude e anymore, and we write $U_s := U_s(1)$.

In this particular case, (1.2.1) becomes:

$$\begin{cases} (i)\ Y \in \mathcal{S}^2, Z \in \mathbb{H}^2, U \in \mathbb{H}^2 \text{ and } \alpha \in \mathcal{S}^2, \text{ where } \alpha = A - K \text{ with } A, K \text{ in } \mathcal{A}^2 \\ (ii)\ Y_t = \xi + \displaystyle\int_t^T g(s, Y_s, Z_s, U_s) ds + (A_T - A_t) - (K_T - K_t) \\ \qquad - \displaystyle\int_t^T Z_s dW_s - \int_t^T U_s d\tilde{N}_s, \\ (iii)\ \forall t \in [0,T], \xi_t \leq Y_t \leq \zeta_t \text{ a.s.}, \\ (iv)\ \displaystyle\int_0^T (Y_{t^-} - \xi_{t^-}) dA_t = 0 a.s. \text{ and } \int_0^T (\zeta_{t^-} - Y_{t^-}) dK_t = 0 \text{ a.s.} \end{cases} \tag{1.2.2}$$

In view of the proof of the convergence of the numerical scheme, we also introduce the penalized version of (1.2.2):

$$Y^p_t = \xi + \int_t^T g(s, Y^p_s, Z^p_s, U^p_s) ds + A^p_T - A^p_t - (K^p_T - K^p_t) - \int_t^T Z^p_s dW_s - \int_t^T U^p_s d\tilde{N}_s, \tag{1.2.3}$$

with $A^p_t := p \int_0^t (Y^p_s - \xi_s)^- ds$ and $K^p_t := p \int_0^t (\zeta_s - Y^p_s)^- ds$, and $\alpha^p_t := A^p_t - K^p_t$ for all $t \in [0,T]$.

1.3 Numerical scheme

The basic idea is to approximate the Brownian motion and the Poisson process by random walks based on the binomial tree model. As explained in Section 1.3.1, these approximations enable to get a martingale representation whose coefficients, involving conditional expectations, can be easily computed. Then, we approximate $(W, \tilde{N})$ in the penalized version of our DRBSDE (i.e. in (1.2.3)) by using these random walks. Taking conditional expectation and using the martingale representation leads to the *explicit penalized discrete scheme* (1.3.9). In view of the proof of the convergence of this explicit scheme, we introduce an implicit intermediate scheme (1.3.5).

1.3.1 Discrete time approximation

We adopt the framework of [109], presented below.

Random walk approximation of $(W, \tilde{N})$

For $n \in \mathbb{N}$, we introduce $\delta_n := \frac{T}{n}$ and the regular grid $(t_j)_{j=0,\ldots,n}$ with step size δ_n (i.e. $t_j := j\delta_n$) to discretize $[0, T]$. In order to approximate W, we introduce the following random walk

$$\begin{cases} W_0^n = 0 \\ W_t^n = \sqrt{\delta_n} \displaystyle\sum_{i=1}^{[t/\delta_n]} e_i^n \end{cases} \tag{1.3.1}$$

where $e_1^n, e_2^n, \ldots, e_n^n$ are independent identically distributed random variables with the following symmetric Bernoulli law:

$$P(e_1^n = 1) = P(e_1^n = -1) = \frac{1}{2}.$$

To approximate $\tilde{N}$, we introduce a second random walk

$$\begin{cases} \tilde{N}_0^n = 0 \\ \tilde{N}_t^n = \displaystyle\sum_{i=1}^{[t/\delta_n]} \eta_i^n \end{cases} \tag{1.3.2}$$

where $\eta_1^n, \eta_2^n, \ldots, \eta_n^n$ are independent and identically distributed random variables with law

$$P(\eta_1^n = \kappa_n - 1) = 1 - P(\eta_1^n = k_n) = \kappa_n$$

where $\kappa_n = e^{-\frac{\lambda}{n}}$. We assume that both sequences $e_1^n, \ldots, e_n^n$ and $\eta_1^n, \eta_2^n, \ldots, \eta_n^n$ are defined on the original probability space $(\Omega, \mathbb{F}, P)$. The (discrete) filtration in the probability space is $\mathbb{F}^n = \{\mathcal{F}_j^n : j = 0, \ldots, n\}$ with $\mathcal{F}_0^n = \{\Omega, \emptyset\}$ and $\mathcal{F}_j^n = \sigma\{e_1^n, \ldots, e_j^n, \eta_1^n, \ldots, \eta_j^n\}$ for $j = 1, \ldots, n$.

The following result states the convergence of $(W^n, \tilde{N}^n)$ to $(W, \tilde{N})$ for the J_1-Skorokhod topology, and the convergence of W^n to W in any L^p, $p \geq 1$, for the topology of uniform convergence on $[0, T]$. We refer to [109, Section 3] for more results on the convergence in probability of $\mathcal{F}^n$-martingales.

Lemma 1.3.1 ([109], Lemma3, (III), and [37],Proof of Corollary 2.2)**.** *The couple* $(W^n, \tilde{N}^n)$ *converges in probability to* $(W, \tilde{N})$ *for the* J_1*-Skorokhod topology, and*

$$\sup_{0\leq t\leq T} |W_t^n - W_t| \to 0 \text{ as } n \to \infty$$

in probability and in L^p*, for any* $1 \leq p < \infty$*.*

Martingale representation

Let y_{j+1} denote a $\mathcal{F}_{j+1}^n$-measurable random variable. As said in [109], we need a set of three strongly orthogonal martingales to represent the martingale difference $m_{j+1} := y_{j+1} - \mathbb{E}(y_{j+1}|\mathcal{F}_j^n)$. We introduce a third martingale increments sequence $\{\mu_j^n = e_j^n \eta_j^n, j = 0, \cdots, n\}$. In this context there exists a unique triplet (z_j, u_j, v_j) of $\mathcal{F}_j^n$-random variables such that

$$m_{j+1} := y_{j+1} - \mathbb{E}(y_{j+1}|\mathcal{F}_j^n) = \sqrt{\delta_n} z_j e_{j+1}^n + u_j \eta_{j+1}^n + v_j \mu_{j+1}^n,$$

and

$$\begin{cases} z_j = \dfrac{1}{\sqrt{\delta_n}} \mathbb{E}(y_{j+1} e_{j+1}^n | \mathcal{F}_j^n), \\ u_j = \dfrac{\mathbb{E}(y_{j+1}\eta_{j+1}^n|\mathcal{F}_j^n)}{\mathbb{E}((\eta_{j+1}^n)^2|\mathcal{F}_j^n)} = \dfrac{1}{\kappa_n(1-\kappa_n)} \mathbb{E}(y_{j+1}\eta_{j+1}^n|\mathcal{F}_j^n), \\ v_j = \dfrac{\mathbb{E}(y_{j+1}\mu_{j+1}^n|\mathcal{F}_j^n)}{\mathbb{E}((\mu_{j+1}^n)^2|\mathcal{F}_j^n)} = \dfrac{1}{\kappa_n(1-\kappa_n)} \mathbb{E}(y_{j+1}\mu_{j+1}^n|\mathcal{F}_j^n) \end{cases} \tag{1.3.3}$$

Remark 1.3.2 (Computing the conditional expectations)**.** Let Φ denote a function from $\mathbb{R}^{2j+2}$ to $\mathbb{R}$. We use the following formula to compute the conditional expectations

$$\begin{aligned} \mathbb{E}(\Phi(e_1^n, \cdots, e_{j+1}^n, \eta_1^n, \cdots, \eta_{j+1}^n)|\mathcal{F}_j^n) =& \frac{\kappa_n}{2} \Phi(e_1^n, \cdots, e_j^n, 1, \eta_1^n, \cdots, \eta_j^n, \kappa_n - 1) \\ &+ \frac{\kappa_n}{2} \Phi(e_1^n, \cdots, e_j^n, -1, \eta_1^n, \cdots, \eta_j^n, \kappa_n - 1) \\ &+ \frac{1-\kappa_n}{2} \Phi(e_1^n, \cdots, e_j^n, 1, \eta_1^n, \cdots, \eta_j^n, \kappa_n) \\ &+ \frac{1-\kappa_n}{2} \Phi(e_1^n, \cdots, e_j^n, -1, \eta_1^n, \cdots, \eta_j^n, \kappa_n). \end{aligned}$$

1.3.2 Fully implementable numerical scheme

In this Section we present two numerical schemes to approximate the solution of the penalized equation (1.2.3): the first one, (1.3.5), is an implicit intermediate scheme, useful for the proof of convergence. We also introduce the main scheme (1.3.9), which is explicit. The implicit scheme (1.3.5) is not easy to solve numerically, since it involves to inverse a function, as we will see below. However, it plays an important role in the proof of the convergence of the explicit scheme, that's why we introduce it.

In both schemes, we approximate the barrier $(\xi_t)_t$ (resp. $(\zeta_t)_t$) by $(\xi^n_j)_{j=0,\cdots,n}$ (resp. $(\zeta^n_j)_{j=0,\cdots,n}$). We also introduce their continuous time versions:

$$\overline{\xi}^n_t := \xi^n_{[t/\delta_n]}, \ \ \overline{\zeta}^n_t := \zeta^n_{[t/\delta_n]}.$$

These approximations satisfy

Assumption 1.3.3.

(i) *For some* $r > 2$, $\sup_{n\in I\!N} \max_{j\leq n} \mathbb{E}(|\xi^n_j|^r) + \sup_{n\in I\!N} \max_{j\leq n} \mathbb{E}(|\zeta^n_j|^r) + \sup_{t\leq T} \mathbb{E}|\xi_t|^r$ $+ \sup_{t\leq T} \mathbb{E}|\zeta_t|^r < \infty$

(ii) $\overline{\xi}^n$ *(resp* $\overline{\zeta}^n$*) converges in probability to* ξ *(resp.* ζ*) for the J1-Skorokhod topology.*

Remark 1.3.4. Assumption 1.3.3 implies that for all t in $[0,T]$ $\overline{\xi}^n_{\psi^n(t)}$ (resp. $\overline{\zeta}^n_{\psi^n(t)}$) converges to ξ_t (resp. ζ_t) in L^2.

Remark 1.3.5. Let us give different examples of barriers in $\mathcal{S}^2$ satisfying Assumption 1.3.3. In this Remark, X represents either ξ or ζ.

1. X satisfies the following SDE

$$X_t = X_0 + \int_0^t b_X(X_{s^-})ds + \int_0^t \sigma_X(X_{s^-})dW_s + \int_0^t c_X(X_{s^-})d\tilde{N}_s$$

where b_X, σ_X and c_X are Lipschitz functions. We approximate it by

$$\overline{X}^n_t = \overline{X}^n_0 + \sum_{j=0}^{[t/\delta_n]-1} b_X(\overline{X}^n_{j\delta_n})\delta_n + \int_0^t \sigma_X(\overline{X}^n_{s^-})dW^n_s + \int_0^t c_X(\overline{X}^n_{s^-})d\tilde{N}^n_s$$

Since $(W^n, \tilde{N}^n)$ converges in probability to $(W, \tilde{N})$ for the J1-topology, [141, Corollary 1] gives that $\overline{X}^n$ converges to X in probability for the J1-topology (for more details on the convergence of sequences of stochastic integrals on the space of RCLL functions endowed with the J1-Skorokhod topology, we refer to [96]). Then, $\overline{X}^n$ satisfies Assumption 1.3.3 (ii). We deduce from Doob and Burkhölder-Davis-Gundy inequalities that X and $\overline{X}^n$ satisfy Assumption 1.3.3 (i) and that X belongs to $\mathcal{S}^2$.

2. X is defined by $X_t := \Phi(t, W_t, \tilde{N}_t)$, where Φ satisfies the following assumptions

   (a) $\Phi(t,x,y)$ is uniformly continuous in (t,y) uniformly in x, i.e. there exist two continuous non decreasing functions $g_0(\cdot)$ and $g_1(\cdot)$ from $I\!R_+$ to $I\!R_+$ with linear growth and satisfying $g_0(0) = g_1(0) = 0$ such that

   $$\forall\ (t,t',x,y,y'),\ \ |\Phi(t,x,y) - \Phi(t',x,y')| \leq g_0(|t-t'|) + g_1(|y-y'|).$$

   We denote a_0 (resp. a_1) the constant of linear growth for g_0 (resp. g_1) i.e. $\forall\ (t,y) \in (I\!R_+)^2$, $0 \leq g_0(t) + g_1(y) \leq a_0(1+t) + a_1(1+y)$,

   (b) $\Phi(t,x,y)$ is "strongly" locally Lispchitz in x uniformly in (t,y), i.e. there exists a constant K_0 and an integer p_0 such that

   $$\forall\ (t,x,x',y),\ \ |\Phi(t,x,y) - \Phi(t,x',y)| \leq K_0(1 + |x|^{p_0} + |x'|^{p_0})|x - x'|.$$

Then, $\forall(t,x,y)$ we have $|\Phi(t,x,y)| \leq a_0|t|+a_1|y|+K_0(1+|x|^{p_0})|x|+|\Phi(0,0,0)|+a_0+a_1$. From this inequality, we prove that X satisfies Assumption 1.3.3 (i) by standard computations. Since $(\tilde{N}^n)$ converges in probability to $(\tilde{N})$ for the J1-topology and $\lim_{n\to\infty}\sup_t |W_t^n - W_t| = 0$ in L^p for any p (see Lemma 1.3.1), we get that $(X_t^n)_t := (\Phi(\delta_n[t/\delta_n], W_t^n, \tilde{N}_t^n))_t$ converges in probability to X for the J1-topology.

Intermediate penalized implicit discrete scheme

After the discretization of the penalized equation (1.2.3) on time intervals $[t_j, t_{j+1}]_{0\leq j\leq n-1}$, we get the following discrete backward equation. For all j in $\{0,\cdots,n-1\}$

$$\begin{cases} y_j^{p,n} = y_{j+1}^{p,n} + g(t_j, y_j^{p,n}, z_j^{p,n}, u_j^{p,n})\delta_n + a_j^{p,n} - k_j^{p,n} - (z_j^{p,n}\sqrt{\delta_n}e_{j+1}^n + u_j^{p,n}\eta_{j+1}^n + v_j^{p,n}\mu_{j+1}^n) \\ a_j^{p,n} = p\delta_n(y_j^{p,n} - \xi_j^n)^-;\ k_j^{p,n} = p\delta_n(\zeta_j^n - y_j^{p,n})^-, \\ y_n^{p,n} := \xi_n^n. \end{cases} \tag{1.3.4}$$

Following (1.3.3), the triplet $(z_j^{p,n}, u_j^{p,n}, v_j^{p,n})$ can be computed as follows

$$\begin{cases} z_j^{p,n} = \dfrac{1}{\sqrt{\delta_n}}\mathbb{E}(y_{j+1}^{p,n}e_{j+1}^n|\mathcal{F}_j^n), \\ u_j^{p,n} = \dfrac{1}{\kappa_n(1-\kappa_n)}\mathbb{E}(y_{j+1}^{p,n}\eta_{j+1}^n|\mathcal{F}_j^n), \\ v_j^{p,n} = \dfrac{1}{\kappa_n(1-\kappa_n)}\mathbb{E}(y_{j+1}^{p,n}\mu_{j+1}^n|\mathcal{F}_j^n), \end{cases}$$

where we refer to Remark 1.3.2 for the computation of conditional expectations. By taking the conditional expectation w.r.t. $\mathcal{F}_j^n$ in (1.3.4), we get the following scheme, called *implicit penalized discrete scheme*: $y_n^{p,n} := \xi_n^n$ and for $j = n-1,\cdots,0$

$$\begin{cases} y_j^{p,n} = (\Theta^{p,n})^{-1}(\mathbb{E}(y_{j+1}^{p,n}|\mathcal{F}_j^n)), \\ a_j^{p,n} = p\delta_n(y_j^{p,n} - \xi_j^n)^-;\ k_j^{p,n} = p\delta_n(\zeta_j^n - y_j^{p,n})^-, \\ z_j^{p,n} = \dfrac{1}{\sqrt{\delta_n}}\mathbb{E}(y_{j+1}^{p,n}e_{j+1}^n|\mathcal{F}_j^n), \\ u_j^{p,n} = \dfrac{1}{\kappa_n(1-\kappa_n)}\mathbb{E}(y_{j+1}^{p,n}\eta_{j+1}^n|\mathcal{F}_j^n), \end{cases} \tag{1.3.5}$$

where $\Theta^{p,n}(y) = y - g(j\delta_n, y, z_j^{p,n}, u_j^{p,n})\delta_n - p\delta_n(y-\xi_j^n)^- + p\delta_n(\zeta_j^n - y)^-$.

We also introduce the continuous time version $(Y_t^{p,n}, Z_t^{p,n}, U_t^{p,n}, A_t^{p,n}, K_t^{p,n})_{0\leq t\leq T}$ of the solution to (1.3.5):

$$Y_t^{p,n} := y_{[t/\delta_n]}^{p,n}, Z_t^{p,n} := z_{[t/\delta_n]}^{p,n}, U_t^{p,n} := u_{[t/\delta_n]}^{p,n}, A_t^{p,n} := \sum_{i=0}^{[t/\delta_n]} a_i^{p,n}, K_t^{p,n} := \sum_{i=0}^{[t/\delta_n]} k_i^{p,n}. \tag{1.3.6}$$

We also introduce $\alpha_t^{p,n} := A_t^{p,n} - K_t^{p,n}$, for all $t \in [0,T]$.

Main scheme

As said before, the numerical inversion of the operator $\Theta^{p,n}$ is not easy and is time consuming. If we replace $y_j^{p,n}$ by $\mathbb{E}[y_{j+1}^{p,n}|\mathcal{F}_j^n]$ in g, (1.3.4) becomes

$$\begin{cases} \overline{y}_j^{p,n} = \overline{y}_{j+1}^{p,n} + g(t_j, \mathbb{E}(\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n), \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\delta_n + \overline{a}_j^{p,n} \\ \qquad -\overline{k}_j^{p,n} - (\overline{z}_j^{p,n}\sqrt{\delta_n}e_{j+1}^n + \overline{u}_j^{p,n}\eta_{j+1}^n + \overline{v}_j^{p,n}\mu_{j+1}^n) \\ \overline{a}_j^{p,n} = p\delta_n(\overline{y}_j^{p,n} - \xi_j^n)^-;\ \overline{k}_j^{p,n} = p\delta_n(\zeta_j^n - \overline{y}_j^{p,n})^-, \\ \overline{y}_n^{p,n} := \xi_n^n. \end{cases} \tag{1.3.7}$$

Now, by taking the conditional expectation in the above equation, we obtain:

$$\overline{y}_j^{p,n} = \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] + g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\delta_n + \overline{a}_j^{p,n} - \overline{k}_j^{p,n}. \tag{1.3.8}$$

Solving this equation, we get the following scheme, called *explicit penalized scheme*: $\overline{y}_n^{p,n} := \xi_n^n$ and for $j = n-1, \cdots, 0$

$$\begin{cases} \overline{y}_j^{p,n} = \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] + g(t_j, \mathbb{E}(\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n), \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\delta_n + \overline{a}_j^{p,n} - \overline{k}_j^{p,n}, \\ \overline{a}_j^{p,n} = \dfrac{p\delta_n}{1+p\delta_n}\left(\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] + \delta_n g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n}) - \xi_j^n\right)^-, \\ \overline{k}_j^{p,n} = \dfrac{p\delta_n}{1+p\delta_n}\left(\zeta_j^n - \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] - \delta_n g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\right)^- \\ \overline{z}_j^{p,n} = \dfrac{1}{\sqrt{\delta_n}}\mathbb{E}(\overline{y}_{j+1}^{p,n}e_{j+1}^n|\mathcal{F}_j^n), \\ \overline{u}_j^{p,n} = \dfrac{1}{\kappa_n(1-\kappa_n)}\mathbb{E}(\overline{y}_{j+1}^{p,n}\eta_{j+1}^n|\mathcal{F}_j^n). \end{cases} \tag{1.3.9}$$

Remark 1.3.6 (Explanations on the derivation of the main scheme)**.** We give below some explanations concerning the derivation of the values of $\overline{a}_j^{p,n}$ and $\overline{k}_j^{p,n}$. We consider the following cases:

- If $\xi_j^n < \overline{y}_j^{p,n} < \zeta_j^n$, then by (1.3.7) we get $\overline{a}_j^{p,n} = \overline{k}_j^{p,n} = 0$, which corresponds to $\dfrac{p\delta_n}{1+p\delta_n}\left(\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] + \delta_n g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n}) - \xi_j^n\right)^- = \dfrac{p\delta_n}{1+p\delta_n}\left(\overline{y}_j^{p,n} - \xi_j^n\right)^- = 0$ and $\dfrac{p\delta_n}{1+p\delta_n}\left(\zeta_j^n - \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] - \delta_n g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\right)^- = \dfrac{p\delta_n}{1+p\delta_n}(\zeta_j^n - \overline{y}_j^{p,n})^- = 0.$
- If $\xi_j^n \geq \overline{y}_j^{p,n}$, then by (1.3.7) we have $\overline{a}_j^{p,n} = p\delta_n(\xi_j^n - \overline{y}_j^{p,n})$ and $\overline{k}_j^{p,n} = 0$; we then replace $\overline{a}_j^{p,n}$ and $\overline{k}_j^{p,n}$ in (1.3.8) and we get $\overline{a}_j^{p,n} = \dfrac{p\delta_n}{1+p\delta_n}\left(\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] + g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\delta_n - \xi_j^n\right)^-$. We also have $\dfrac{p\delta_n}{1+p\delta_n}\left(\zeta_j^n - \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] - \delta_n g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\right)^- = 0$ and hence $\overline{k}_j^{p,n} = \dfrac{p\delta_n}{1+p\delta_n}\left(\zeta_j^n - \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] - \delta_n g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\right)^-$.
- The case $\zeta_j^n \leq \overline{y}_j^{p,n}$ is symmetric to the one studied above: $\xi_j^n \geq \overline{y}_j^{p,n}$.

As for the implicit scheme, we define the continuous time version $(\overline{Y}^{p,n}_t, \overline{Z}^{p,n}_t, \overline{U}^{p,n}_t, \overline{A}^{p,n}_t, \overline{K}^{p,n}_t)_{0\leq t\leq T}$ of the solution to (1.3.9):

$$\overline{Y}^{p,n}_t = \overline{y}^{p,n}_{[t/\delta_n]}, \quad \overline{Z}^{p,n}_t = \overline{z}^{p,n}_{[t/\delta_n]}, \quad \overline{U}^{p,n}_t = \overline{u}^{p,n}_{[t/\delta_n]}, \quad \overline{A}^{p,n}_t = \sum_{j=0}^{[t/\delta_n]} \overline{a}^{p,n}_j \quad \overline{K}^{p,n}_t = \sum_{j=0}^{[t/\delta_n]} \overline{k}^{p,n}_j. \tag{1.3.10}$$

We also introduce $\overline{\alpha}^{p,n}_t := \overline{A}^{p,n}_t - \overline{K}^{p,n}_t$, for all $t \in [0,T]$.

1.4 Convergence result

The following result states the convergence of $\overline{\Theta}^{p,n} := (\overline{Y}^{p,n}, \overline{Z}^{p,n}, \overline{U}^{p,n}, \overline{\alpha}^{p,n})$ to $\Theta := (Y, Z, U, \alpha)$, the solution of the DRBSDE (1.2.2).

Theorem 1.4.1. *Assume that Assumptions 1.2.4 and 1.3.3 hold. The sequence* $(\overline{Y}^{p,n}, \overline{Z}^{p,n}, \overline{U}^{p,n})$ *defined by* (1.3.10) *converges to* (Y, Z, U)*, the solution of the DRBSDE* (1.2.2)*, in the following sense:* $\forall r \in [1, 2[$

$$\lim_{p\to\infty} \lim_{n\to\infty} \left(\mathbb{E}\left[\int_0^T |\overline{Y}^{p,n}_s - Y_s|^2 ds\right] + \mathbb{E}\left[\int_0^T |\overline{Z}^{p,n}_s - Z_s|^r ds\right] + \mathbb{E}\left[\int_0^T |\overline{U}^{p,n}_s - U_s|^r ds\right]\right) = 0. \tag{1.4.1}$$

Moreover, $\overline{Z}^{p,n}$ *(resp.* $\overline{U}^{p,n}$*) weakly converges in* $I\!H^2$ *to* Z *(resp. to* U*) and for* $0 \leq t \leq T$*,* $\overline{\alpha}^{p,n}_{\psi^n(t)}$ *converges weakly to* α_t *in* $L^2(\mathcal{F}_T)$ *as* $n \to \infty$ *and* $p \to \infty$*.*

In order to prove this result, we split the error in three terms, by introducing $\Theta^{p,n}_t := (Y^{p,n}_t, Z^{p,n}_t, U^{p,n}_t, \alpha^{p,n}_t)$, the solution of the implicit penalized discrete scheme (2.4.3) and $\Theta^p_t := (Y^p_t, Z^p_t, U^p_t, \alpha^p_t)$, the penalized version of (1.2.2), defined by (1.2.3). For the error on Y, we get

$$\mathbb{E}[\int_0^T |\overline{Y}^{p,n}_s - Y_s|^2 ds] \leq 3\left(\mathbb{E}[\int_0^T |\overline{Y}^{p,n}_s - Y^{p,n}_s|^2 ds] + \mathbb{E}[\int_0^T |Y^{p,n}_s - Y^p_s|^2 ds] + \mathbb{E}[\int_0^T |Y^p_s - Y_s|^2 ds]\right),$$

and the same splitting holds for $|\overline{Z}^{p,n} - Z|^r$ and $|\overline{U}^{p,n} - U|^r$. For the increasing processes, we have:

$$\mathbb{E}[|\overline{\alpha}^{p,n}_{\psi^n(t)} - \alpha_t|^2] \leq 3\left(\mathbb{E}[|\overline{\alpha}^{p,n}_{\psi^n(t)} - \alpha^{p,n}_{\psi^n(t)}|^2] + \mathbb{E}[|\alpha^{p,n}_{\psi^n(t)} - \alpha^p_t|^2] + \mathbb{E}[|\alpha^p_t - \alpha_t|^2]\right). \tag{1.4.2}$$

The proof of Theorem 1.4.1 ensues from Proposition 1.4.2, Corollary 1.4.4 and Proposition 1.4.5. Proposition 1.4.2 states the convergence of the error between $\overline{\Theta}^{p,n}$, the explicit penalization scheme defined in (1.3.10), and $\Theta^{p,n}$, the implicit penalization scheme. It generalizes the results of [134]. We refer to Section 1.4.1. Corollary 1.4.4 states the convergence (in n) of $\Theta^{p,n}$ to Θ^p. This is based on the convergence of a standard BSDE with jumps in discrete time setting to the associated BSDE with jumps in continuous time setting, which is proved in [109]. We refer to Section 1.4.2. Finally, Proposition 1.4.5 proves the convergence (in p) of the penalized BSDE with jumps Θ^p to Θ, the solution of the DRBSDE (1.2.2). In fact, we prove a more general result in Section 1.4.3, since we show the convergence of penalized BSDEs to (1.2.1) in the case of jumps driven by a Poisson random measure.

The rest of the Section is devoted to the proof of these results.

1.4.1 Error between explicit and implicit penalization schemes

We prove the convergence of the error between the explicit penalization scheme and the implicit one. The scheme of the proof is inspired from [134, Proposition 5].

Proposition 1.4.2. *Assume Assumption 1.3.3 (i) and g is a Lipschitz driver. We have*

$$\lim_{n\to\infty}\sup_{0\le t\le T}\left(\mathbb{E}[|\overline{Y}_t^{p,n}-Y_t^{p,n}|^2]+\mathbb{E}[\int_0^T|\overline{Z}_s^{p,n}-Z_s^{p,n}|^2ds]+\mathbb{E}[\int_0^T|\overline{U}_s^{p,n}-U_s^{p,n}|^2ds]\right)=0.$$

Moreover, $\lim_{n\to\infty}(\overline{\alpha}_t^{p,n}-\alpha_t^{p,n})=0$ *in* $L^2(\mathcal{F}_t)$*, for* $t\in[0,T]$*.*

Proof. By using the definitions of the implicit and explicit schemes (1.3.4) and (1.3.7), we obtain that:

$$\begin{aligned}y_{j+1}^{p,n}-\overline{y}_{j+1}^{p,n}=&(y_j^{p,n}-\overline{y}_j^{p,n})+(g_p(t_j,\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n],\overline{y}_j^{p,n},\overline{z}_j^{p,n},\overline{u}_j^{p,n})-g(t_j,y_j^{p,n},y_j^{p,n},z_j^{p,n},u_j^{p,n}))\delta_n\\&+(z_j^{p,n}-\overline{z}_j^{p,n})e_{j+1}^n\sqrt{\delta_n}+(u_j^{p,n}-\overline{u}_j^{p,n})\eta_{j+1}^n+(v_j^{p,n}-\overline{v}_j^{p,n})\mu_{j+1}^n\end{aligned}$$

where $g_p(t,y_1,y_2,z,u)=g(t,y_1,z,u)+p(y_2-\overline{\xi}_t^n)^- - p(\overline{\zeta}_t^n-y_2)^-$. It implies that:

$$\begin{aligned}\mathbb{E}[(y_j^{p,n}-\overline{y}_j^{p,n})^2]=&\mathbb{E}[(y_{j+1}^{p,n}-\overline{y}_{j+1}^{p,n})^2]-\mathbb{E}[(g_p(t_j,\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n],\overline{y}_j^{p,n},\overline{z}_j^{p,n},\overline{u}_j^{p,n})\\&-g_p(t_j,y_j^{p,n},y_j^{p,n},z_j^{p,n},u_j^{p,n}))^2]\delta_n^2-\mathbb{E}[(z_j^{p,n}-\overline{z}_j^{p,n})^2]\delta_n\\&-\mathbb{E}[(u_j^{p,n}-\overline{u}_j^{p,n})^2](1-\kappa_n)\kappa_n-\mathbb{E}[(v_j^{p,n}-\overline{v}_j^{p,n})^2](1-\kappa_n)\kappa_n\\&+2\mathbb{E}[(g_p(t_j,y_j^{p,n},y_j^{p,n},z_j^{p,n},u_j^{p,n})\\&-g_p(t_j,\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n],\overline{y}_j^{p,n},\overline{z}_j^{p,n},\overline{u}_j^{p,n}))(y_j^{p,n}-\overline{y}_j^{p,n})]\delta_n.\end{aligned}$$

In the above relation, we take the sum over j from i to $n-1$. We have:

$$\begin{aligned}\mathbb{E}[(y_i^{p,n}-\overline{y}_i^{p,n})^2]+\delta_n\sum_{j=i}^{n-1}\mathbb{E}[(z_j^{p,n}-\overline{z}_j^{p,n})^2]+(1-\kappa_n)\kappa_n\sum_{j=i}^{n-1}\mathbb{E}[(u_j^{p,n}-\overline{u}_j^{p,n})^2]&\\ \le 2\delta_n\sum_{j=i}^{n-1}\mathbb{E}[(g_p(t_j,y_j^{p,n},y_j^{p,n},z_j^{p,n},u_j^{p,n})&\\ -g_p(t_j,\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n],\overline{y}_j^{p,n},\overline{z}_j^{p,n},\overline{u}_j^{p,n}))(y_j^{p,n}-\overline{y}_j^{p,n})].&\end{aligned}$$

Let us introduce $f:y\longmapsto(y-\overline{\xi}_t^n)^- -(\overline{\zeta}_t^n-y)^-$. We have $g_p(t,y_1,y_2,z,u)=g(t,y_1,z,u)+pf(y_2)$. The last expectation of the previous inequality can be written

$$\mathbb{E}[(g(t_j,y_j^{p,n},z_j^{p,n},u_j^{p,n})-g(t_j,\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n],\overline{z}_j^{p,n},\overline{u}_j^{p,n}))(y_j^{p,n}-\overline{y}_j^{p,n})+p(f(y_j^{p,n})-f(\overline{y}_j^{p,n}))(y_j^{p,n}-\overline{y}_j^{p,n})]$$

Since f is decreasing and g is Lipschitz, we obtain:

$$\begin{aligned}&\mathbb{E}[(y_i^{p,n}-\overline{y}_i^{p,n})^2]+\delta_n\sum_{j=i}^{n-1}\mathbb{E}[(z_j^{p,n}-\overline{z}_j^{p,n})^2]+(1-\kappa_n)\kappa_n\sum_{j=i}^{n-1}\mathbb{E}[(u_j^{p,n}-\overline{u}_j^{p,n})^2]\\&\le 2\delta_n\sum_{j=i}^{n-1}\mathbb{E}\left[(C_g|y_j^{p,n}-\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n]|+C_g|z_j^{p,n}-\overline{z}_j^{p,n}|+C_g|u_j^{p,n}-\overline{u}_j^{p,n}|)|y_j^{p,n}-\overline{y}_j^{p,n}|\right].\end{aligned}$$

Consequently, by applying the inequality $2ab \le a^2+b^2$ for $a = C_g|y_j^{p,n} - \overline{y}_j^{p,n}|\sqrt{2\delta_n}$; $b = \sqrt{\frac{\delta_n}{2}}|z_j^{p,n} - \overline{z}_j^{p,n}|$ and $a = C_g|y_j^{p,n} - \overline{y}_j^{p,n}|\sqrt{2}\frac{\delta_n}{\sqrt{\kappa_n(1-\kappa_n)}}$; $b = \sqrt{\frac{\kappa_n(1-\kappa_n)}{2}}|u_j^{p,n} - \overline{u}_j^{p,n}|$ we get that:

$$\begin{aligned}
&\mathbb{E}[(y_i^{p,n} - \overline{y}_i^{p,n})^2] + \delta_n \sum_{j=i}^{n-1} \mathbb{E}[(z_j^{p,n} - \overline{z}_j^{p,n})^2] + (1-\kappa_n)\kappa_n \sum_{j=i}^{n-1} \mathbb{E}[(u_j^{p,n} - \overline{u}_j^{p,n})^2] \\
&\quad \le 2\delta_n C_g^2 \sum_{j=i}^{n-1} \mathbb{E}[(y_j^{p,n} - \overline{y}_j^{p,n})^2] + \frac{\delta_n}{2} \sum_{j=i}^{n-1} \mathbb{E}[(z_j^{p,n} - \overline{z}_j^{p,n})^2] + \frac{2C_g^2\delta_n^2}{\kappa_n(1-\kappa_n)} \sum_{j=i}^{n-1} \mathbb{E}[(y_j^{p,n} - \overline{y}_j^{p,n})^2] \\
&\quad + \frac{(1-\kappa_n)\kappa_n}{2} \sum_{j=i}^{n-1} \mathbb{E}[(u_j^{p,n} - \overline{u}_j^{p,n})^2] + 2C_g\delta_n \mathbb{E}[\sum_{j=i}^{n-1} |y_j^{p,n} - \overline{y}_j^{p,n}||y_j^{p,n} - \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n]|].
\end{aligned}$$

Now, since $\overline{y}_j^{p,n} - \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] = g_p(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\delta_n$, the last term is dominated by

$$\delta_n \sum_{j=i}^{n-1} (2C_g+1)\mathbb{E}[(y_j^{p,n} - \overline{y}_j^{p,n})^2] + C_g^2\delta_n^3 \sum_{j=i}^{n-1} \mathbb{E}[g_p(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{y}_j^{p,n}, \overline{z}_j^{p,n}, \overline{u}_j^{p,n})^2].$$

Using the definition of g_p yields

$$\begin{aligned}
g_p(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{y}_j^{p,n}, \overline{z}_j^{p,n}, \overline{u}_j^{p,n}) &\le |g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})| + p(|\overline{y}_j^{p,n}| + |\xi_j^n| + |\zeta_j^n|), \\
&\le |g(t_j,0,0,0)| + C_g(|\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n]| + |\overline{z}_j^{p,n}| + |\overline{u}_j^{p,n}|) + p(|\overline{y}_j^{p,n}| + |\xi_j^n| + |\zeta_j^n|).
\end{aligned}$$

We get

$$\begin{aligned}
\delta_n^3 \sum_{j=i}^{n-1} \mathbb{E}[g_p(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{y}_j^{p,n}, \overline{z}_j^{p,n}, \overline{u}_j^{p,n})^2] &\le C_0\delta_n^2(\delta_n \sum_{j=i}^{n-1} |g(t_j,0,0,0)|^2 \\
&+ \delta_n \sum_{j=i}^{n-1} |\overline{z}_j^{p,n}|^2 + \delta_n \sum_{j=i}^{n-1} |\overline{u}_j^{p,n}|^2) + C_0(p\delta_n)^2(\max_j \mathbb{E}(|\xi_j^n|^2) \\
&+ \max_j \mathbb{E}(|\zeta_j^n|^2)) + C_0\delta_n^2(1+p^2)\max_j \mathbb{E}(|\overline{y}_j^{p,n}|^2)
\end{aligned}$$

where C_0 denotes a generic constant depending on C_g. Since $\frac{\delta_n}{(1-\kappa_n)\kappa_n} = \frac{1}{\lambda}\frac{\lambda\delta_n}{(1-e^{-\lambda\delta_n})e^{-\lambda\delta_n}}$ and $e^x \le \frac{xe^{2x}}{e^x-1} \le e^{2x}$, we get $\frac{\delta_n}{(1-\kappa_n)\kappa_n} \le \frac{1}{\lambda}e^{2\lambda T}$. Hence, for δ_n small enough such that $(3+2p+2C_g+2C_g^2(1+\frac{1}{\lambda}e^{2\lambda T}))\delta_n < 1$, Lemma 1.8.1 enables to write:

$$\begin{aligned}
\mathbb{E}[(y_i^{p,n} - \overline{y}_i^{p,n})^2] &+ \frac{\delta_n}{2}\mathbb{E}[\sum_{j=i}^{n-1}(z_j^{p,n} - \overline{z}_j^{p,n})^2] + \frac{1}{2}(1-\kappa_n)\kappa_n \mathbb{E}[\sum_{j=i}^{n-1}(u_j^{p,n} - \overline{u}_j^{p,n})^2] \\
&\le \left(1 + 2C_g + 2C_g^2 + \frac{2C_g^2\delta_n}{(1-\kappa_n)\kappa_n}\right)\delta_n \mathbb{E}[\sum_{j=i}^{n-1}(y_j^{p,n} - \overline{y}_j^{p,n})^2] + C_1(p)\delta_n^2, \qquad (1.4.3)
\end{aligned}$$

where $C_1(p) = C_0(\|g(\cdot,0,0,0)\|_\infty^2 + p^2(\sup_n \max_j \mathbb{E}|\xi_j^n|^2 + \sup_n \max_j \mathbb{E}|\zeta_j^n|^2) + (1+p^2)K_{\text{Lem.1.8.1}})$, $K_{\text{Lem.1.8.1}}$ denotes the constant appearing in Lemma 1.8.1. Discrete Gronwall's Lemma (see [134, Lemma 3]) gives

$$\sup_{i\leq n} \mathbb{E}[(y_i^{p,n} - \overline{y}_i^{p,n})^2] \leq C_1(p)\delta_n^2 e^{(1+2C_g+2C_g^2(1+\frac{1}{\lambda}e^{2\lambda T}))T}.$$

Since $\delta_n \leq T$, $(1-\kappa_n)\kappa_n \geq \lambda\delta_n e^{-2\lambda T}$, and Equation (1.4.3) gives

$$\mathbb{E}[\int_0^T |\overline{Z}_s^{p,n} - Z_s^{p,n}|^2 ds] + \mathbb{E}[\int_0^T |\overline{U}_s^{p,n} - U_s^{p,n}|^2 ds] \leq C_1'(p)\delta_n^2,$$

where $C_1'(p)$ is another constant depending on C_g, λ, T and $C_1(p)$. It remains to prove the convergence for the increasing processes. We have

$$\overline{A}_t^{p,n} - \overline{K}_t^{p,n} = \overline{Y}_0^{p,n} - \overline{Y}_t^{p,n} - \int_0^t g(s, \overline{Y}_s^{p,n}, \overline{Z}_s^{p,n}, \overline{U}_s^{p,n})ds + \int_0^t \overline{Z}_s^{p,n} dW_s^n + \int_0^t \overline{U}_s^{p,n} d\tilde{N}_s^n,$$
$$A_t^{p,n} - K_t^{p,n} = Y_0^{p,n} - Y_t^{p,n} - \int_0^t g(s, Y_s^{p,n}, Z_s^{p,n}, U_s^{p,n})ds + \int_0^t Z_s^{p,n} dW_s^n + \int_0^t U_s^{p,n} d\tilde{N}_s^n.$$

Using the Lispchitz property of g and the convergence of $(\overline{Y}_s^{p,n} - Y_s^{p,n}, \overline{Z}_s^{p,n} - Z_s^{p,n}, \overline{U}_s^{p,n} - U_s^{p,n})$, we get the result. □

1.4.2 Convergence of the discrete time setting to the continuous time setting

The following Proposition ensues from [109].

Proposition 1.4.3. *Let g be a Lipschitz driver and assume that Assumption 1.3.3 (ii) holds. For any $p \in \mathbb{N}^*$, the sequence $(Y_t^{p,n}, Z_t^{p,n}, U_t^{p,n})$ converges to (Y_t^p, Z_t^p, U_t^p) in the following sense:*

$$\lim_{n\to\infty} \left(||Y^{p,n} - Y^p||_{J1-L^2}^2 + \mathbb{E}[\int_0^T |Z_s^{p,n} - Z_s^p|^2 ds + \int_0^T |U_s^{p,n} - U_s^p|^2 ds] \right) = 0. \tag{1.4.4}$$

Proof. For a fixed p, we have the following:

$$Y^{p,n} - Y^p = (Y^{p,n} - Y^{p,n,q}) + (Y^{p,n,q} - Y^{p,\infty,q}) + (Y^{p,\infty,q} - Y^p). \tag{1.4.5}$$

where $(Y^{p,\infty,q}, Z^{p,\infty,q}, U^{p,\infty,q})$ is the Picard approximation of (Y^p, Z^p, U^p) and $(Y^{p,n,q}, Z^{p,n,q}, U^{p,n,q})$ represents the continuous time version of the discrete Picard approximation of $(y_k^{p,n}, z_k^{p,n}, u_k^{p,n})$, denoted by $(y_k^{p,n,q}, z_k^{p,n,q}, u_k^{p,n,q})$. Note that $(y_k^{p,n,q+1}, z_k^{p,n,q+1}, u_k^{p,n,q+1})$ is defined inductively as the solution of the backward recursion given by [109, Eq. (3.16)], for the penalized driver $g_n(\omega,t,y,z,u) := g(\omega,t,y,z,u) + p(y - \overline{\xi}_t^n(\omega))^- - p(\overline{\zeta}_t^n(\omega) - y)^-$. Since $\overline{\xi}^n$ and $\overline{\zeta}^n$ satisfy Assumption 1.3.3 (*ii*), $(g_n(\omega,\cdot,\cdot,\cdot,\cdot))_n$ converges uniformly to $g(\omega,\cdot,\cdot,\cdot,\cdot) + p(y - \xi_t(\omega))^- - p(\zeta_t(\omega) - y)^-$ almost surely up to a subsequence (i.e. g_n satisfies [109, Assumption (A')]).
Now, by using (1.4.5), [109, Proposition 1], [109, Proposition 3] and [109, Eq. (3.17)], one can easily show that (1.4.4) holds. □

The following Corollary ensues from Proposition 1.4.3.

Corollary 1.4.4. *Let g be a Lipschitz driver, ξ and ζ belong to $\mathcal{S}^2$, ψ^n is the random mapping introduced in Proposition 1.4.3 and assume that Assumption 1.3.3 holds. For any $p \in \mathbb{N}^*$, the sequence $(Y_t^{p,n}, Z_t^{p,n}, U_t^{p,n})$ converges to (Y_t^p, Z_t^p, U_t^p) in the following sense:*

$$\lim_{n\to\infty} \mathbb{E}[\int_0^T |Y_s^{p,n} - Y_s^p|^2 ds + \int_0^T |Z_s^{p,n} - Z_s^p|^2 ds + \int_0^T |U_s^{p,n} - U_s^p|^2 ds] = 0,$$

Moreover, $A^{p,n}$ (resp. $K^{p,n}$) converges to A^p (resp. K^p) when n tends to infinity in L^2 for the $J1$-Skorohod topology.

Proof. Note that:

$$\int_0^T |Y_s^{p,n} - Y_s^p|^2 ds \leq 2\int_0^T |Y_s^{p,n} - Y_{\eta^n(s)}^p|^2 dt + 2\int_0^T |Y_{\eta^n(s)}^p - Y_s^p|^2 ds,$$

where $\eta^n(s)$ represents the inverse of $\psi^n(s)$.

Proposition 1.4.3 gives that the first term in the right-hand side converges to 0. Concerning the second term, $s \mapsto Y_s^p$ is continuous except at the times at which the Poisson process jumps. Consequently, $Y_{\eta^n(s)}^p$ converges to Y_s^p for almost every s and as Y^p belongs to $\mathcal{S}^2$, we get that $\mathbb{E}[\int_0^T |Y_{\eta^n(s)}^p - Y_s^p|^2 ds] \to 0$ when $n \to \infty$.

Now, remark that we can rewrite $A_t^{p,n}$ and A_t^p as follows:

$$A_t^{p,n} = p\int_0^t (Y_s^{p,n} - \overline{\xi}_s^n)^- ds \qquad A_t^p = p\int_0^t (Y_s^p - \xi_s)^- ds. \tag{1.4.6}$$

Then

$$\begin{aligned}\sup_{t\in[0,T]} |A_{\psi^n(t)}^{p,n} - A_t^p| &= \sup_{t\in[0,T]} |A_t^{p,n} - A_{\eta^n(t)}^p| \\ &= \sup_{k\in\{0,\cdots,n\}} |A_{t_k}^{p,n} - A_{t_k}^p| + \sup_{k\in\{0,\cdots,n\}} \sup_{t\in[t_k,t_{k+1}]} |A_{t_k}^p - A_{\eta^n(t)}^p|.\end{aligned}$$

since ξ and Y^p belong to $\mathcal{S}^2$, we get that the second term in the right hand side tends to 0 in L^2 when $n \to \infty$.

$$\sup_{k\in\{0,\cdots,n\}} |A_{t_k}^{p,n} - A_{t_k}^p| \leq p\int_0^T |Y_s^{p,n} - Y_s^p| + |\overline{\xi}_s^n - \xi_s| ds.$$

Since $\lim_{n\to\infty} \mathbb{E}[\int_0^T |Y_s^{p,n} - Y_s^p|^2 ds] = 0$, $\lim_{n\to\infty} \mathbb{E}|\overline{\xi}_s^n - \xi_{\eta(s)}|^2 = 0$ (see Remark 1.3.4) and $\lim_{n\to\infty} \mathbb{E}[\int_0^T |\xi_{\eta^n(s)} - \xi_s|^2 ds] = 0$ (ξ is RCLL, its jumps are countable), we get that $\sup_{k\in\{0,\cdots,n\}} |A_{t_k}^{p,n} - A_{t_k}^p|$ converges to 0 in L^2 in n, which ends the proof.

□

1.4.3 Convergence of the penalized BSDE to the reflected BSDE

As said in the Introduction, this part of the proof deals with the general case of jumps driven by a random Poisson measure. We state in Proposition 1.4.5 that a sequence of penalized BSDEs converges to the solution to (1.2.1). To do so, we give in Section 1.4.3 an other proof of existence of solutions to reflected BSDEs with jumps and RCLL barriers based on the penalization method. We extend the proof of [112, Section 4] to the case of totally inacessible jumps. We are able to generalize their proof thanks to Mokobodski's condition (which in particular enables to get Lemma 1.4.7, generalizing [112, Lemma 4.1]), to the comparison Theorem for BSDEs with jumps (see Theorem 1.9.1 and Theorem 1.9.2) and to the caracterization of the solution of the DRBSDE as the value function of a stochastic game (proved in Proposition 1.9.5).

We introduce the penalization scheme, generalizing (1.2.3) to the case of random Poisson measure :

$$\begin{aligned} Y_t^p =& \xi_T + \int_t^T g(s, Y_s^p, Z_s^p, U_s^p)ds + p\int_t^T (Y_s^p - \xi_s)^- ds - p\int_t^T (\zeta_s - Y_s^p)^- ds - \int_t^T Z_s^p dW_s \\ & - \int_t^T \int_{\mathbb{R}^*} U_s^p(e)\tilde{N}(ds, de) \end{aligned} \tag{1.4.7}$$

with $A_t^p = p\int_0^t (Y_s^p - \xi_s)^- ds$ and $K_t^p = p\int_0^t (\zeta_s - Y_s^p)^- ds$.

Proposition 1.4.5. *Under Hypothesis 1.2.4, Y^p converges to Y in $\mathbb{H}^2$, Z^p weakly converges in $\mathbb{H}^2$ to Z, U^p weakly converges in $\mathbb{H}^2_\nu$ to U, and $\alpha_t^p := A_t^p - K_t^p$ weakly converges to α_t in $L^2(\mathcal{F}_t)$. Moreover, for all $r \in [1, 2[$, the following strong convergence holds*

$$\lim_{p\to\infty} \mathbb{E}\left[\int_0^T |Y_s^p - Y_s|^2 ds\right] + \mathbb{E}\left[\int_0^T |Z_s^p - Z_s|^r ds + \int_0^T \left(\int_{\mathbb{R}^*} |U_s^p - U_s|^2 \nu(de)\right)^{\frac{r}{2}} ds\right] = 0. \tag{1.4.8}$$

The proof of Proposition 1.4.5 is postponed to Section 1.4.3.

Intermediate result

For each p, q in $\mathbb{N}$, since the driver $g(s, y, z, u) + q(y - \xi_s)^- - p(\zeta_s - y)^-$ is Lipschitz in (y, z, u), the following classical BSDE with jumps admits a unique solution $(Y^{p,q}, Z^{p,q}, U^{p,q})$ (see [?])

$$\begin{aligned} Y_t^{p,q} =& \xi_T + \int_t^T g(s, Y_s^{p,q}, Z_s^{p,q}, U_s^{p,q})ds + q\int_t^T (Y_s^{p,q} - \xi_s)^- ds - p\int_t^T (\zeta_s - Y_s^{p,q})^- ds - \int_t^T Z_s^{p,q} dW_s \\ & - \int_t^T \int_{\mathbb{R}^*} U_s^{p,q}(e)\tilde{N}(ds, de). \end{aligned} \tag{1.4.9}$$

We set $A_t^{p,q} = q\int_0^t (Y_s^{p,q} - \xi_s)^- ds$ and $K_t^{p,q} = p\int_0^t (\zeta_s - Y_s^{p,q})^- ds$.

Theorem 1.4.6. *Let us assume that Assumption 1.2.4 holds. The quadruple $(Y^{p,q}, Z^{p,q}, U^{p,q}, \alpha^{p,q})$, where $\alpha^{p,q} = A^{p,q} - K^{p,q}$, converges to (Y, Z, U, α), the solution of (1.2.1), as $p \to \infty$ then $q \to \infty$ (or equivalently as $q \to \infty$ then $p \to \infty$) in the following sense : $Y^{p,q}$ converges to Y in $\mathbb{H}^2$, $Z^{p,q}$*

weakly converges to Z in $I\!H^2$, $U^{p,q}$ weakly converges to U in $I\!H^2_\nu$, $\alpha_t^{p,q}$ weakly converges to α_t in $L^2(\mathcal{F}_t)$. Moreover, for each $r \in [1,2[$, the following strong convergence holds

$$\begin{aligned}\lim_{p\to\infty}\lim_{q\to\infty} \mathbb{E}\left(\int_0^T |Y_s^{p,q} - Y_s|^2 ds\right)\\ + \mathbb{E}\left(\int_0^T |Z_s^{p,q} - Z_s|^r ds + \int_0^T \left(\int_{I\!R^*} |U_s^{p,q} - U_s|^2 \nu(de)\right)^{\frac{r}{2}} ds\right) = 0. \end{aligned} \tag{1.4.10}$$

The proof of Theorem 1.4.6 is divided in several steps. We prove

1. the quadruple $(Y^{p,q}, Z^{p,q}, U^{p,q}, \alpha^{p,q})$ converges as $q \to \infty$ then $p \to \infty$
2. the quadruple $(Y^{p,q}, Z^{p,q}, U^{p,q}, \alpha^{p,q})$ converges as $p \to \infty$ then $q \to \infty$
3. the two limits are equal (see Lemma 1.4.11)
4. the limit of the penalized BSDE is the solution of the reflected BSDE (1.2.1) (see Theorem 1.4.3)
5. Equation (1.4.10) ensues from (1.4.27) and (1.4.29).

Proof of point 1.

Let us first state the following preliminary result.

Lemma 1.4.7. *Suppose that $H, H' \in \mathcal{S}^2$ are two supermartingales such that Assumption* 1.2.4 *holds. Let Y^* be the RCLL adapted process defined by $Y_t^* := (H_t - H_t')\mathbf{1}_{t<T} + \xi_T \mathbf{1}_{t=T}$. There exists $(Z^*, U^*, A^*, K^*) \in \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{A}^2 \times \mathcal{A}^2$ such that $(Y^*, Z^*, U^*, A^*, K^*)$ solves $(i), (ii), (iii)$ of* (1.2.1).

Proof. By assumption, H and H' are square integrable supermartingales. The process Y^* is thus well defined. By the Doob-Meyer decomposition of supermartingales, there exist two square integrable martingales M and M', two square integrable nondecreasing predictable RCLL processes V and V' with $V_0 = V_0^{'} = 0$ such that:

$$dH_t = dM_t - dV_t; \quad dH_t^{'} = dM_t^{'} - dV_t^{'}. \tag{1.4.11}$$

Define

$$\overline{M}_t := M_t - M_t^{'}.$$

By the above relation and (1.4.11), we derive $dY_t^* = d\overline{M}_t - dV_t + dV_t^{'}$. Now, by the martingale representation theorem, there exist $Z^* \in \mathbb{H}^2, U^* \in \mathbb{H}^2_\nu$ such that:

$$d\overline{M}_t = Z_t^* dW_t + \int_{\mathbb{R}^*} U_t^*(e)\tilde{N}(de, dt). \tag{1.4.12}$$

Consequently, (1.4.11) and (1.4.12) imply that:

$$\begin{aligned}Y_t^* =&\xi_T + \int_t^T g(s, Y_s^*, Z_s^*, U_s^*)ds - \left(\int_t^T g(s, Y_s^*, Z_s^*, U_s^*)ds + (V_T - V_t) - (V_T' - V_t')\right)\\ &- \int_t^T Z_s^* dW_s - \int_t^T \int_{\mathbb{R}^*} U_s^*(e)\tilde{N}(ds, de).\end{aligned}$$

Now let g^+ (resp. g^-) denote the positive (resp. negative) part of the function g. By setting $A_t^* := V_t + \int_0^t g^+(s, Y_s^*, Z_s^*, U_s^*)ds$ and $K_t^* := V_t' + \int_0^t g^-(s, Y_s^*, Z_s^*, U_s^*)ds$, the result follows. □

Proposition 1.4.8. *Suppose Assumption 1.2.4 holds. Then, there exists a constant C, independent of p and q such that we have :*

$$\mathbb{E}\left[\sup_{0\le t\le T}(Y_t^{p,q})^2\right] + \mathbb{E}\left[\int_0^T |Z_t^{p,q}|^2 dt\right] + \mathbb{E}\left[\int_0^T \int_{I\!R^*} |U_t^{p,q}(e)|^2 \nu(de)dt\right] \\ + \mathbb{E}[(A_T^{p,q})^2] + \mathbb{E}[(K_T^{p,q})^2] \le C. \tag{1.4.13}$$

Proof. This proof generalizes the proof of [112, Proposition 4.1] to the case of jumps. Since p and q play symmetric roles, the calculations over p and q are uniform throughout this proof. From Lemma 1.4.7, we know that there exists $(Y^*, Z^*, U^*, A^*, K^*)$ in $\mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times \mathcal{A}^2 \times \mathcal{A}^2$ such that

$$Y_t^* = \xi_T + \int_t^T g(s, \theta_s^*)ds + (A_T^* - A_t^*) - (K_T^* - K_t^*) - \int_t^T Z_s^* dW_s - \int_t^T \int_{I\!R^*} U_s^*(e)\tilde{N}(ds, de)$$

and $\xi_t \le Y_t^* \le \zeta_t$ $dP \otimes dt$ a.s. (θ_s^* denotes (Y_s^*, Z_s^*, U_s^*)). Then, for $p, q \in I\!N$, we also have

$$Y_t^* = \xi_T + \int_t^T g(s, \theta_s^*)ds + (A_T^* - A_t^*) - (K_T^* - K_t^*) + q\int_t^T (\xi_s - Y_s^*)^+ ds - p\int_t^T (Y_s^* - \zeta_s)^+ ds \\ - \int_t^T Z_s^* dW_s - \int_t^T \int_{I\!R^*} U_s^*(e)\tilde{N}(ds, de).$$

Let $\overline{\theta}^{p,q} := (\overline{Y}^{p,q}, \overline{Z}^{p,q}, \overline{U}^{p,q})$ and $\tilde{\theta}^{p,q} = (\tilde{Y}^{p,q}, \tilde{Z}^{p,q}, \tilde{U}^{p,q})$ be the solutions of the following equations

$$\overline{Y}_t^{p,q} = \xi_T + \int_t^T g(s, \overline{\theta}_s^{p,q})ds + (A_T^* - A_t^*) + q\int_t^T (\xi_s - \overline{Y}_s^{p,q})^+ ds - p\int_t^T (\overline{Y}_s^{p,q} - \zeta_s)^+ ds \tag{1.4.14}$$

$$- \int_t^T \overline{Z}_s^{p,q} dW_s - \int_t^T \int_{I\!R^*} \overline{U}_s^{p,q}(e)\tilde{N}(ds, de). \tag{1.4.15}$$

$$\tilde{Y}_t^{p,q} = \xi_T + \int_t^T g(s, \tilde{\theta}_s^{p,q})ds - (K_T^* - K_t^*) + q\int_t^T (\xi_s - \tilde{Y}_s^{p,q})^+ ds - p\int_t^T (\tilde{Y}_s^{p,q} - \zeta_s)^+ ds \tag{1.4.16}$$

$$- \int_t^T \tilde{Z}_s^{p,q} dW_s - \int_t^T \int_{I\!R^*} \tilde{U}_s^{p,q}(e)\tilde{N}(ds, de). \tag{1.4.17}$$

By the comparison theorem for BSDEs with jumps (see Theorem 1.9.1), we get that for all p, q in $I\!N$, $\tilde{Y}_t^{p,q} \le Y_t^{p,q} \le \overline{Y}_t^{p,q}$, $\xi_t \le Y_t^* \le \overline{Y}_t^{p,q}$ and $\tilde{Y}_t^{p,q} \le Y_t^* \le \zeta_t$. Applying this result to (1.4.14) gives that $(\overline{Y}^{p,q}, \overline{Z}^{p,q}, \overline{U}^{p,q})$ is also solution to

$$\overline{Y}_t^{p,q} = \xi_T + \int_t^T g(s, \overline{\theta}_s^{p,q})ds + (A_T^* - A_t^*) - p\int_t^T (\overline{Y}_s^{p,q} - \zeta_s)^+ ds - \int_t^T \overline{Z}_s^{p,q} dW_s \\ - \int_t^T \int_{I\!R^*} \overline{U}_s^{p,q}(e)\tilde{N}(ds, de). \tag{1.4.18}$$

Doing the same with (1.4.16) gives that $(\tilde{Y}^{p,q}, \tilde{Z}^{p,q}, \tilde{U}^{p,q})$ is also solution to

$$\tilde{Y}_t^{p,q} = \xi_T + \int_t^T g(s, \tilde{\theta}_s^{p,q})ds - (K_T^* - K_t^*) + q\int_t^T (\xi_s - \tilde{Y}_s^{p,q})^+ ds$$
$$- \int_t^T \tilde{Z}_s^{p,q} dW_s - \int_t^T \int_{I\!R^*} \tilde{U}_s^{p,q}(e)\tilde{N}(ds, de). \tag{1.4.19}$$

Let us consider the following BSDEs

$$Y_t^+ = \xi_T + \int_t^T g(s, \theta_s^+)ds + (A_T^* - A_t^*) - \int_t^T Z_s^+ dW_s - \int_t^T \int_{I\!R^*} \tilde{U}_s^+(e)\tilde{N}(ds, de), \tag{1.4.20}$$
$$Y_t^- = \xi_T + \int_t^T g(s, \theta_s^-)ds - (K_T^* - K_t^*) - \int_t^T Z_s^- dW_s - \int_t^T \int_{I\!R^*} \tilde{U}_s^-(e)\tilde{N}(ds, de), \tag{1.4.21}$$

where $\theta_s^+ := (Y_s^+, Z_s^+, U_s^+)$ and $\theta_s^- := (Y_s^-, Z_s^-, U_s^-)$. Since $\overline{K}_t^{p,q} := p\int_0^t (\overline{Y}_s^{p,q} - \zeta_s)^+ ds$ and $\tilde{A}_t^{p,q} := q\int_0^t (\xi_s - \tilde{Y}_s^{p,q})^+ ds$ are increasing processes, Theorem 1.9.1 applied to (1.4.18) and (1.4.20) (resp. to (1.4.19) and (1.4.21)) gives $\overline{Y}_t^{p,q} \le Y_t^+$ (resp. $Y_t^- \le \tilde{Y}_t^{p,q}$). Combining theses results with the inequality $\tilde{Y}_t^{p,q} \le Y_t^{p,q} \le \overline{Y}_t^{p,q}$ leads to

$$\forall (p,q) \in I\!N^2,\ \forall t \in [0,T],\ \ Y_t^- \le \tilde{Y}_t^{p,q} \le Y_t^{p,q} \le \overline{Y}_t^{p,q} \le Y_t^+. \tag{1.4.22}$$

Then we have

$$\mathbb{E}[\sup_{0\le t\le T} (Y_t^{p,q})^2] \le \max\{\mathbb{E}[\sup_{0\le t\le T} (Y_t^+)^2], \mathbb{E}[\sup_{0\le t\le T} (Y_t^-)^2]\}. \tag{1.4.23}$$

Since A^* and K^* belong to $\mathcal{A}^2$, Itô's formula, BDG inequality and Gronwall's Lemma give $\mathbb{E}[\sup_{0\le t\le T}(Y_t^+)^2] \le C$ and $\mathbb{E}[\sup_{0\le t\le T}(Y_t^-)^2] \le C$. Then we get

$$\mathbb{E}[\sup_{0\le t\le T} (Y_t^{p,q})^2] \le C. \tag{1.4.24}$$

Let us now prove that $\mathbb{E}[(A_T^{p,q})^2] + \mathbb{E}[(K_T^{p,q})^2] \le C$. Since for all p, q in $I\!N$, $\tilde{Y}_t^{p,q} \le Y_t^{p,q} \le \overline{Y}_t^{p,q}$, then $\tilde{A}_t^{p,q} \ge A_t^{p,q} \ge 0$ and $\overline{K}_t^{p,q} \ge K_t^{p,q} \ge 0$. It boils down to prove $\mathbb{E}[(\tilde{A}_T^{p,q})^2] + \mathbb{E}[(\overline{K}_T^{p,q})^2] \le C$. Let us first prove that $\mathbb{E}[(\tilde{A}_T^{p,q})^2] \le C$. To do so, we apply [78, Equation (17)] to (1.4.19) (as a sequence in q). In the same way, we apply [78, Equation (17)] to (1.4.18) (as a sequence in p). We get $\mathbb{E}[(\overline{K}_T^{p,q})^2] \le C$.

It remains to prove $\mathbb{E}\left[\int_0^T |Z_t^{p,q}|^2 dt\right] + \mathbb{E}\left[\int_0^T \int_{I\!R^*} |U_t^{p,q}(e)|^2 \nu(de)dt\right] \le C$. By applying Itô's formula to $|Y_t^{p,q}|^2$, we get

$$\mathbb{E}\left[|Y_t^{p,q}|^2\right] + \mathbb{E}\left[\int_t^T |Z_s^{p,q}|^2 ds\right] + \mathbb{E}\left[\int_t^T \int_{I\!R^*} |U_s^{p,q}(e)|^2 \nu(de)ds\right]$$
$$= \mathbb{E}(\xi_T^2) + 2\mathbb{E}\left[\int_t^T Y_s^{p,q} g(s, Y_s^{p,q}, Z_s^{p,q}, U_s^{p,q})ds\right]$$
$$+ 2\mathbb{E}\left[\int_t^T Y_s^{p,q} q(Y_s^{p,q} - \xi_s)^- ds\right] - 2\mathbb{E}\left[\int_t^T Y_s^{p,q} p(\zeta_s - Y_s^{p,q})^- ds\right].$$

The third term of the right hand side is null if $Y_s^{p,q} \geq \xi_s$.

Then we can bound it by $2\mathbb{E}\left[\sup_{0\leq t\leq T}|\xi_t|(A_T^{p,q}-A_t^{p,q})\right]$. The last term of the right hand side is bounded in the same way. We bound it by $2\mathbb{E}\left[\sup_{0\leq t\leq T}|\zeta_t|(K_T^{p,q}-K_t^{p,q})\right]$. By using that g is Lipschitz, we bound the second term of the right hand side

$$2\mathbb{E}\left[\int_t^T Y_s^{p,q} g(s,Y_s^{p,q},Z_s^{p,q},U_s^{p,q})ds\right] \leq 2\mathbb{E}\left[\int_t^T |Y_s^{p,q}|(\|g(\cdot,0,0,0)\|_\infty + C_g(|Y_s^{p,q}|+|Z_s^{p,q}|+|U_s^{p,q}|))ds\right].$$

By applying Young's inequality, we get

$$\begin{aligned}
&\mathbb{E}\left[|Y_t^{p,q}|^2\right] + \mathbb{E}\left[\int_t^T |Z_s^{p,q}|^2 ds\right] + \mathbb{E}\left[\int_t^T \int_{\mathbb{R}^*} |U_s^{p,q}(e)|^2 \nu(de)ds\right] \\
&\leq \|g(\cdot,0,0,0)\|_\infty^2 + (1+2C_g+4C_g^2)\mathbb{E}\left[\int_t^T |Y_s^{p,q}|^2 ds\right] \\
&+ \frac{1}{2}\mathbb{E}\left[\int_t^T |Z_s^{p,q}|^2 ds\right] + \frac{1}{2}\mathbb{E}\left[\int_t^T \int_{\mathbb{R}^*} |U_s^{p,q}(e)|^2 \nu(de)ds\right] \\
&+ \mathbb{E}[\sup_{0\leq t\leq T}\xi_t^2] + \mathbb{E}[\sup_{0\leq t\leq T}\zeta_t^2] + \mathbb{E}[(A_T^{p,q})^2] + \mathbb{E}[(K_T^{p,q})^2].
\end{aligned} \tag{1.4.25}$$

By combining the assumptions on ξ, ζ, (1.4.24) and the previous result bounding $\mathbb{E}[(A_T^{p,q})^2] + E[(K_T^{p,q})^2]$, we get $\mathbb{E}[\int_t^T |Z_s^{p,q}|^2 ds] + \mathbb{E}[\int_t^T \int_{\mathbb{R}^*} |U_s^{p,q}(e)|^2 \nu(de)ds] \leq C$. $\square$

In (1.4.9), for fixed p we set $g_p(s,y,z,u) = g(s,y,z,u) - p(\zeta_s - y)^-$. g_p is Lipschitz and

$$\mathbb{E}\left(\int_0^T (g_p(s,0,0,0))^2 ds\right) \leq 2\mathbb{E}\left(\int_0^T (g(s,0,0,0))^2 ds\right) + 2p^2 T\mathbb{E}(\sup_{0\leq t\leq T}(\zeta_t)^2) < \infty.$$

By Theorem 1.9.1, we know that $(Y^{p,q})$ is increasing in q for all p. Thanks to Theorem 1.9.4, we know that $(Y^{p,q}, Z^{p,q}, U^{p,q})_{q\in\mathbb{N}}$ has a limit $(Y^{p,\infty}, Z^{p,\infty}, U^{p,\infty}) := \theta^{p,\infty}$ such that $(Y^{p,q})_q$ converges increasingly to $Y^{p,\infty} \in \mathcal{S}^2$, and thanks to Theorem 1.9.3, we know that there exists $Z^{p,\infty} \in \mathbb{H}^2$, $U^{p,\infty} \in \mathbb{H}_\nu^2$ and $A^{p,\infty} \in \mathcal{A}^2$ such that $(Y^{p,\infty}, Z^{p,\infty}, U^{p,\infty}, A^{p,\infty})$ satisfies the following equation

$$\begin{aligned}
Y_t^{p,\infty} =& \xi_T + \int_t^T g(s,\theta_s^{p,\infty})ds + (A_T^{p,\infty} - A_t^{p,\infty}) - p\int_t^T (\zeta_s - Y_s^{p,\infty})^- ds - \int_t^T Z_s^{p,\infty} dW_s \\
&- \int_t^T \int_{\mathbb{R}^*} U_s^{p,\infty}(e)\tilde{N}(ds,de)
\end{aligned} \tag{1.4.26}$$

$Z^{p,\infty}$ is the weak limit of $(Z^{p,q})_q$ in $\mathbb{H}^2$, $U^{p,\infty}$ is the weak limit of $(U^{p,q})_q$ in $\mathbb{H}_\nu^2$ and $A_t^{p,\infty}$ is the weak limit of $(A_t^{p,q})_q$ in $L^2(\mathcal{F}_t)$. Moreover, for each $r \in [1,2[$, the following strong convergence holds

$$\begin{aligned}
&\lim_{q\to\infty} \mathbb{E}\left(\int_0^T |Y_s^{p,q} - Y_s^{p,\infty}|^2 ds\right) \\
&\quad + \mathbb{E}\left(\int_0^T |Z_s^{p,q} - Z_s^{p,\infty}|^r ds + \int_0^T \left(\int_{\mathbb{R}^*} |U_s^{p,q} - U_s^{p,\infty}|^2 \nu(de)\right)^{\frac{r}{2}} ds\right) = 0.
\end{aligned} \tag{1.4.27}$$

From [78, Theorem 5.1], we also get that $\forall t \in [0,T]$, $Y_t^{p,\infty} \geq \xi_t$ and $\int_0^T (Y_{t^-}^{p,\infty} - \xi_{t^-})dA_t^{p,\infty} = 0$ a.s. Set $K_t^{p,\infty} = p\int_0^t (\zeta_s - Y_s^{p,\infty})^- ds$. Since $Y^{p,q} \nearrow Y^{p,\infty}$ when $q\to\infty$, $K^{p,q} \nearrow K^{p,\infty}$ when $q\to\infty$. By the monotone convergence theorem and (1.4.13), we get that $\mathbb{E}((K_T^{p,\infty})^2) \leq C$. Then we get the following Lemma

Lemma 1.4.9. *There exists a constant C independent of p such that*

$$\mathbb{E}\left[\sup_{0\le t\le T}(Y_t^{p,\infty})^2\right]+\mathbb{E}\left[\int_0^T|Z_t^{p,\infty}|^2dt\right]+\mathbb{E}\left[\int_0^T\int_{I\!R^*}|U_t^{p,\infty}(e)|^2\nu(de)dt\right]$$
$$+\mathbb{E}[(A_T^{p,\infty})^2]+\mathbb{E}[(K_T^{p,\infty})^2]\le C.$$

From Theorem 1.9.2, we have $Y_t^{p,\infty}\ge Y_t^{p+1,\infty}$, then there exists a process Y such that $Y^{p,\infty}\searrow Y$. By using Fatou's lemma, we get

$$\mathbb{E}\left(\sup_{0\le t\le T}(Y_t)^2\right)\le C,$$

and the dominated convergence theorem gives us that $\lim_{p\to\infty}Y^{p,\infty}=Y$ in $I\!H^2$. Since $(Y^{p,q})_p$ is a decreasing sequence, $(A^{p,q})_p$ is an increasing sequence, and by passing to the limit ($(A_t^{p,q})_q$ weakly converges to $A_t^{p,\infty}$), we get $A_t^{p,\infty}\le A_t^{p+1,\infty}$. Then, we deduce from Lemma 1.4.9 that there exists a process A such that $A^{p,\infty}\nearrow A$ and $\mathbb{E}(A_T^2)<\infty$. Since $A_t^{p,q}-A_s^{p,q}=\int_s^t q(\xi_r-Y_r^{p,q})^+dr\le\int_s^t q(\xi_r-Y_r^{p+1,q})^+dr=A_t^{p+1,q}-A_s^{p+1,q}$, we get that

$$A_t^{p,\infty}-A_s^{p,\infty}\le A_t^{p+1,\infty}-A_s^{p+1,\infty}\ \ \forall\,0\le s\le t\le T.$$

Thanks to Lemma 1.4.9, we can apply the "generalized monotonic Theorem" 1.6.1: there exist $Z\in I\!H^2$, $U\in I\!H_\nu^2$ and $K\in\mathcal{A}^2$ such that

$$Y_t=\xi_T+\int_t^T g(s,Y_s,Z_s,U_s)ds+A_T-A_t-(K_T-K_t)-\int_t^T Z_sdW_s$$
$$-\int_t^T\int_{I\!R^*}U_s(e)\tilde N(ds,de),\tag{1.4.28}$$

K_t is the weak limit of $K_t^{p,\infty}$ in $L^2(\mathcal{F}_t)$, Z is the weak limit of $Z^{p,\infty}$ in $I\!H^2$ and U is the weak limit of $U^{p,\infty}$ in $I\!H_\nu^2$. Moreover, $A_t^{p,\infty}$ strongly converges to A_t in $L^2(\mathcal{F}_t)$ and $A\in\mathcal{A}^2$, and we have for each $r\in[1,2[$,

$$\lim_{p\to\infty}\mathbb{E}\left(\int_0^T|Y_s^{p,\infty}-Y_s|^2ds\right)$$
$$+\mathbb{E}\left(\int_0^T|Z_s^{p,\infty}-Z_s|^rds+\int_0^T\left(\int_{I\!R^*}|U_s^{p,\infty}-U_s|^2\nu(de)\right)^{\frac r2}ds\right)=0.\tag{1.4.29}$$

Proof of point 2.

Similarly, $(Y^{p,q})_p$ is decreasing for any fixed q. The same arguments as before give that $(Y^{p,q},Z^{p,q},U^{p,q})_{p\in I\!N}$ has a limit $(Y^{\infty,q},Z^{\infty,q},U^{\infty,q}):=\theta^{\infty,q}$ such that $(Y^{p,q})_p$ converges decreasingly to $Y^{\infty,q}\in\mathcal{S}^2$, and thanks to Theorem 1.9.3, we know that there exists $Z^{\infty,q}\in I\!H^2$, $U^{\infty,q}\in I\!H_\nu^2$ and $K^{\infty,q}\in\mathcal{A}^2$ such that $(Y^{\infty,q},Z^{\infty,q},U^{\infty,q},K^{\infty,q})$ satisfies the following equation

$$Y_t^{\infty,q}=\xi_T+\int_t^T g(s,\theta_s^{\infty,q})ds+q\int_t^T(Y_s^{\infty,q}-\xi_s)^-ds-(K_T^{\infty,q}-K_t^{\infty,q})-\int_t^T Z_s^{\infty,q}dW_s$$
$$-\int_t^T\int_{I\!R^*}U_s^{\infty,q}(e)\tilde N(ds,de)\tag{1.4.30}$$

$Z^{\infty,q}$ is the weak limit of $(Z^{p,q})_p$ in $I\!H^2$, $U^{\infty,q}$ is the weak limit of $(U^{p,q})_p$ in $I\!H^2_\nu$ and $K_t^{\infty,q}$ is the weak limit of $(K_t^{p,q})_p$ in $L^2(\mathcal{F}_t)$. From [78, Theorem 5.1], we also get that $\forall t \in [0,T]$, $Y_t^{\infty,q} \leq \zeta_t$ and $\int_0^T (Y_{t^-}^{\infty,q} - \zeta_{t^-}) dK_t^{\infty,q} = 0$ a.s. Set $A_t^{\infty,q} = q\int_0^t (Y_s^{\infty,q} - \xi_s)^- ds$. Since $Y^{p,q} \searrow Y^{\infty,q}$ when $p \to \infty$, $A^{p,q} \nearrow A^{\infty,q}$ when $p \to \infty$. By the monotone convergence theorem and (1.4.13), we get that $\mathbb{E}((A_T^{\infty,q})^2) \leq C$. We get the following result, equivalent to Lemma 1.4.9

Lemma 1.4.10. *There exists a constant C independent of q such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}(Y_t^{\infty,q})^2\right] + \mathbb{E}\left[\int_0^T |Z_t^{\infty,q}|^2 dt\right] + \mathbb{E}\left[\int_0^T \int_{I\!R^*} |U_t^{\infty,q}(e)|^2 \nu(de) dt\right] + \mathbb{E}[(A_T^{\infty,q})^2] + \mathbb{E}[(K_T^{\infty,q})^2] \leq C.$$

From Theorem 1.9.2, we have $Y_t^{\infty,q} \leq Y_t^{\infty,q+1}$, then there exists a process Y' such that $Y^{\infty,q} \nearrow Y'$. By using Fatou's lemma, we get that Y' belongs to $\mathcal{S}^2$, and the convergence also holds in $I\!H^2$. By using the same proof as before, we can apply Theorem 1.6.1: there exist $Z' \in I\!H^2$, $U' \in I\!H^2_\nu$ and $A' \in \mathcal{A}^2$ such that

$$Y'_t = \xi_T + \int_t^T g(s, Y'_s, Z'_s, U'_s) ds + A'_T - A'_t - (K'_T - K'_t) - \int_t^T Z'_s dW_s - \int_t^T \int_{I\!R^*} U'_s(e) \tilde{N}(ds, de),$$

A'_t is the weak limit of $A_t^{\infty,q}$ in $L^2(\mathcal{F}_t)$, Z' is the weak limit of $Z^{\infty,q}$ in $I\!H^2$ and U' is the weak limit of $U^{\infty,q}$ in $I\!H^2_\nu$. Moreover, $K_t^{\infty,q}$ strongly converges to K'_t in $L^2(\mathcal{F}_t)$ and $K' \in \mathcal{A}^2$. We will now prove that the two limits are equal.

Proof of point 3.

Lemma 1.4.11. *The two limits Y and Y' are equal. Moreover $Z = Z'$, $U = U'$ and $A - K = A' - K'$.*

Proof. Since $Y^{p,q} \nearrow Y^{p,\infty}$ and $Y^{p,q} \searrow Y^{\infty,q}$, we get that for all $p, q \in I\!N$, $Y^{\infty,q} \leq Y^{p,q} \leq Y^{p,\infty}$. Then, since $Y^{p,\infty} \searrow Y$ and $Y^{\infty,q} \nearrow Y'$, we get $Y' \leq Y$. On the other hand, since $Y^{\infty,q} \leq Y^{p,q}$, we get that for all $0 \leq s \leq t \leq T$

$$A_t^{p,q} - A_s^{p,q} \leq A_t^{\infty,q} - A_s^{\infty,q}.$$

Since $(A_t^{p,q})_q$ weakly converges to $A_t^{p,\infty}$ in $L^2(\mathcal{F}_t)$, $(A_t^{\infty,q})_q$ weakly converges to A'_t in $L^2(\mathcal{F}_t)$, and $(A_t^{p,\infty})_p$ strongly converges to A_t in $L^2(\mathcal{F}_t)$, taking limit in q and then limit in p gives

$$A_t - A_s \leq A'_t - A'_s. \tag{1.4.31}$$

Since $Y^{p,q} \leq Y^{p,\infty}$, we get that for all $0 \leq s \leq t \leq T$

$$K_t^{p,q} - K_s^{p,q} \leq K_t^{p,\infty} - K_s^{p,\infty}.$$

Letting $p \to \infty$ and $q \to \infty$ leads to

$$K'_t - K'_s \leq K_t - K_s. \tag{1.4.32}$$

Combining (1.4.31) and (1.4.32) gives that for all $0 \leq s \leq t \leq T$, $A_t - A_s - (K_t - K_s) \leq A'_t - A'_s - (K'_t - K'_s)$. Thanks to Theorem 1.9.1, we get that $Y' \geq Y$. Then $Y' = Y$, and we get $Z' = Z$, $U' = U$, and $A' - K' = A - K$. $\square$

Proof of point 4.

It remains to prove that the limit $(Y, Z, U, A-K)$ of the penalized BSDE is the solution of the reflected BSDE with two RCLL barriers ξ and ζ. To do so, we use stochastic game theory (see Proposition 1.9.5) and Snell envelope theory (see Appendix 1.7).

Theorem 1.4.12. *Let $\alpha := A - K$. The quartuple (Y, Z, U, α) solving (1.4.28) is the unique solution to (1.2.1).*

Proof. We know from Theorem 1.2.7 that (1.2.1) has a unique solution. We already know that (Y, Z, U, A, K) belongs to $\mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times \mathcal{A}^2 \times \mathcal{A}^2$ and satisfies (ii). It remains to check (iii) and (iv). We first check (iii). From (1.4.26), we know that $(Y^{p,\infty}, Z^{p,\infty}, U^{p,\infty}, A^{p,\infty})$ is the solution of a reflected BSDE (RBSDE in the following) with one lower barrier ξ. Let $\alpha^{p,\infty} := A^{p,\infty} - K^{p,\infty}$. Then, $(Y^{p,\infty}, Z^{p,\infty}, U^{p,\infty}, \alpha^{p,\infty})$ can be considered as the solution of a RBSDE with two barriers ξ and $\zeta + (\zeta - Y^{p,\infty})^-$, since we have

$$\xi \leq Y^{p,\infty} \leq \zeta + (\zeta - Y^{p,\infty})^-, \quad \int_0^T (Y_t^{p,\infty} - \xi_t) dA_t^{p,\infty} = 0$$

and

$$\int_0^T (Y_t^{p,\infty} - \zeta_t - (\zeta - Y^{p,\infty})_t^-) dK_t^{p,\infty} = -p \int_0^T (Y_t^{p,\infty} - \zeta_t)^- (\zeta_t - Y_t^{p,\infty})^- dt = 0.$$

From Proposition 1.9.5 we know that

$$\begin{aligned}
Y_t^{p,\infty} =& \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left(\int_t^{\sigma\wedge\tau} g(s, \theta_s^{p,\infty}) ds + \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau} + (\zeta_\sigma - Y_\sigma^{p,\infty})^- \mathbf{1}_{\sigma<\tau} \big| \mathcal{F}_t \right) \\
\geq& \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left(\int_t^{\sigma\wedge\tau} g(s, \theta_s^{p,\infty}) ds + \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau} \big| \mathcal{F}_t \right) \\
\geq& \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left(\int_t^{\sigma\wedge\tau} g(s, \theta_s) ds + \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau} \big| \mathcal{F}_t \right) \\
&- C_g \mathbb{E}\left(\int_0^T |Y_s^{p,\infty} - Y_s| + |Z_s^{p,\infty} - Z_s| + \|U_s^{p,\infty} - U_s\|_\nu ds |\mathcal{F}_t \right).
\end{aligned}$$

Since $Y^{p,\infty} \to Y$ in $I\!H^2$, $Z^{p,\infty} \to Z$ in $I\!H^r$ for $r < 2$, and $U^{p,\infty} \to U$ in $I\!H^r_\nu$ for $r < 2$, there exists a subsequence p_j such that the last conditional expectation converges to 0 a.s. Taking the limit in p in the last inequality gives

$$Y_t \geq \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left(\int_t^{\sigma\wedge\tau} g(s, \theta_s) ds + \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau} \big| \mathcal{F}_t \right). \tag{1.4.33}$$

In the same way, we know that $(Y^{\infty,q}, Z^{\infty,q}, U^{\infty,q}, K^{\infty,q})$ is the solution of a RBSDE with one upper barrier ζ. Let $\alpha^{\infty,q} := A^{\infty,q} - K^{\infty,q}$. Then $(Y^{\infty,q}, Z^{\infty,q}, U^{\infty,q}, \alpha^{\infty,q})$ is the solution of a RBSDE with two barriers $\xi - (Y^{\infty,q} - \xi)^-$ and ζ. By Proposition 1.9.5 we know that

$$\begin{aligned}
Y_t^{\infty,q} \leq& \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \mathbb{E}\left(\int_t^{\sigma\wedge\tau} g(s, \theta_s) ds + \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau} \big| \mathcal{F}_t \right) \\
&+ C_g \mathbb{E}\left(\int_0^T |Y_s^{\infty,q} - Y_s| + |Z_s^{\infty,q} - Z_s| + \|U_s^{\infty,q} - U_s\|_\nu ds |\mathcal{F}_t \right).
\end{aligned}$$

Since $Y^{\infty,q} \to Y$ in $I\!H^2$, $Z^{\infty,q} \to Z$ in $I\!H^r$ for $r < 2$, and $U^{\infty,q} \to U$ in $I\!H^r_\nu$ for $r < 2$, there exists a subsequence q_j such that the last conditional expectation converges to 0 a.s. Taking the limit in q in the last inequality gives

$$Y_t \leq \operatorname*{esssup}_{\tau\in\mathcal{T}_t} \operatorname*{essinf}_{\sigma\in\mathcal{T}_t} \mathbb{E}\left(\int_t^{\sigma\wedge\tau} g(s,\theta_s)ds + \xi_\tau \mathbf{1}_{\tau\leq\sigma} + \zeta_\sigma \mathbf{1}_{\sigma<\tau} \Big| \mathcal{F}_t\right). \tag{1.4.34}$$

Comparing (1.4.33) and (1.4.34) and since esssup essinf $\leq$ essinf esssup, we deduce

$$\begin{aligned}
Y_t &= \operatorname*{esssup}_{\tau\in\mathcal{T}_t} \operatorname*{essinf}_{\sigma\in\mathcal{T}_t} \mathbb{E}\left(\int_t^{\sigma\wedge\tau} g(s,\theta_s)ds + \xi_\tau \mathbf{1}_{\tau\leq\sigma} + \zeta_\sigma \mathbf{1}_{\sigma<\tau} \Big| \mathcal{F}_t\right) \\
&= \operatorname*{essinf}_{\sigma\in\mathcal{T}_t} \operatorname*{esssup}_{\tau\in\mathcal{T}_t} \mathbb{E}\left(\int_t^{\sigma\wedge\tau} g(s,\theta_s)ds + \xi_\tau \mathbf{1}_{\tau\leq\sigma} + \zeta_\sigma \mathbf{1}_{\sigma<\tau} \Big| \mathcal{F}_t\right).
\end{aligned}$$

Let $M_t := \mathbb{E}(\xi_T + \int_0^T g(s,\theta_s)ds|\mathcal{F}_t) - \int_0^t g(s,\theta_s)ds$, $\tilde{\xi}_t = \xi_t - M_t$ and $\tilde{\zeta}_t = \zeta_t - M_t$. We can rewrite Y in the following form

$$\begin{aligned}
Y_t &= \operatorname*{esssup}_{\tau\in\mathcal{T}_t} \operatorname*{essinf}_{\sigma\in\mathcal{T}_t} \mathbb{E}\left(\tilde{\xi}_\tau \mathbf{1}_{\tau\leq\sigma} + \tilde{\zeta}_\sigma \mathbf{1}_{\sigma<\tau} \Big| \mathcal{F}_t\right) + M_t \\
&= \operatorname*{essinf}_{\sigma\in\mathcal{T}_t} \operatorname*{esssup}_{\tau\in\mathcal{T}_t} \mathbb{E}\left(\tilde{\xi}_\tau \mathbf{1}_{\tau\leq\sigma} + \tilde{\zeta}_\sigma \mathbf{1}_{\sigma<\tau} \Big| \mathcal{F}_t\right) + M_t
\end{aligned}$$

Then $Y_t - M_t$ is the value of a stochastic game problem with payoff $I_t(\tau,\sigma) = \tilde{\xi}_\tau \mathbf{1}_{\tau\leq\sigma} + \tilde{\zeta}_\sigma \mathbf{1}_{\sigma<\tau}$. Let us check that $\tilde{\xi}$ and $\tilde{\zeta}$ are in $\mathcal{S}^2$. Since ξ and ζ are in $\mathcal{S}^2$, we only have to check that $M \in \mathcal{S}^2$. Using Doob's inequality

$$\begin{aligned}
\mathbb{E}(\sup_{0\leq t\leq T}(M_t)^2) &\leq 2\mathbb{E}\left(\sup_{0\leq t\leq T}\left(E(\xi + \int_0^T g(s,\theta_s)ds|\mathcal{F}_t)\right)^2 + \left(\int_0^T |g(s,\theta_s)|ds\right)^2\right), \\
&\leq C(1 + \mathbb{E}\int_0^T |Y_s|^2 + |Z_s|^2 + \|U_s\|_\nu^2 ds) < \infty.
\end{aligned}$$

Since $\tilde{\xi}_T = \tilde{\zeta}_T = 0$ and ξ and ζ satisfy Mokobodski'condition, we can apply [112, Theorem 5.1]: there exists a pair of non-negative RCLL supermatingales (X^+, X^-) in $\mathcal{S}^2$ such that

$$X_t^+ = \mathcal{R}_t(X^- + \tilde{\xi}), \quad X_t^- = \mathcal{R}_t(X^+ - \tilde{\zeta}),$$

where $\mathcal{R}_t(\phi)$ denotes the Snell enveloppe of ϕ (see Appendix 1.7). Thanks to [112, Theorem 5.2], we know that $Y_t - M_t = X_t^+ - X_t^-$. Moreover, by the Doob-Meyer decomposition theorem, we get $X_t^+ = \mathbb{E}(A_T^1|\mathcal{F}_t) - A_t^1$, $X_t^- = \mathbb{E}(K_T^1|\mathcal{F}_t) - K_t^1$, where A^1, K^1 are predictable increasing processes belonging to $\mathcal{A}^2$. With the representation theorem for the martingale part we know that there exists $Z^1 \in \mathbb{H}^2$ and $U^1 \in \mathbb{H}^2_\nu$ such that

$$\begin{aligned}
Y_t &= M_t + X_t^+ - X_t^- \\
&= \mathbb{E}(\xi + \int_0^T g(s,\theta_s)ds + A_T^1 - K_T^1|\mathcal{F}_t) - \int_0^t g(s,\theta_s)ds - A_t^1 + K_t^1, \\
&= Y_0 + \int_0^t Z_s^1 dW_s + \int_0^t \int_{I\!R^*} U_s^1(e)\tilde{N}(ds,de) - \int_0^t g(s,\theta_s)ds - A_t^1 + K_t^1.
\end{aligned}$$

Then, we compare the forward form of (1.4.28) and the previous equality, we get

$$(A_t - K_t) - (A_t^1 - K_t^1) = \int_0^t (Z_s - Z_s^1)dW_s + \int_0^t \int_{I\!R^*} (U_s(e) - U_s^1(e))\tilde{N}(ds, de)$$

and then $Z_t = Z_t^1$, $U_t = U_t^1$ and $K_t - A_t = K_t^1 - A_t^1$. By using the properties of the Snell envelope in (1.4.35), we get the $X^+ \geq X^- + \tilde{\xi}$ and $X^- \geq X^+ - \tilde{\zeta}$, which leads to $\xi = M + \tilde{\xi} \leq Y = M + X^+ - X^- \leq M + \tilde{\zeta} = \zeta$ and (iii) follows.
It remains to check (iv). From the theory of the Snell envelope (see Section 1.7), we get that

$$0 = \int_0^T (X_{t^-}^+ - (\tilde{\xi}_{t^-} + X_{t^-}^-))dA_t^1 = \int_0^T (X_{t^-}^+ - X_{t^-}^- - \xi_{t^-} + M_{t^-})dA_t^1 = \int_0^T (Y_{t^-} - \xi_{t^-})dA_t^1,$$
$$0 = \int_0^T (X_{t^-}^- - (X_{t^-}^+ - \tilde{\zeta}_{t^-}))dK_t^1 = \int_0^T (X_{t^-}^- - X_{t^-}^+ + \zeta_{t^-} - M_{t^-})dK_t^1 = \int_0^T (\zeta_{t^-} - Y_{t^-})dK_t^1,$$

which ends the proof. □

Proof of Proposition 1.4.5

In order to prove the convergence of $(Y^p, Z^p, U^p, \alpha^p)$, we rewrite (1.4.26), the solution of the reflected BSDE with one lower obstacle ξ

$$\begin{aligned} Y_t^{p,\infty} =& \xi + \int_t^T g(s, \theta_s^{p,\infty})ds + (A_T^{p,\infty} - A_t^{p,\infty}) - p\int_t^T (\zeta_s - Y_s^{p,\infty})^- ds - \int_t^T Z_s^{p,\infty} dW_s \\ & - \int_t^T \int_{I\!R^*} U_s^{p,\infty}(e)\tilde{N}(ds, de), \end{aligned}$$

and (1.4.30), the solution of the reflected BSDE with one upper obstacle ζ

$$\begin{aligned} Y_t^{\infty,p} =& \xi + \int_t^T g(s, \theta_s^{\infty,p})ds + p\int_t^T (Y_s^{\infty,p} - \xi_s)^- ds - (K_T^{\infty,p} - K_t^{\infty,p}) - \int_t^T Z_s^{\infty,p} dW_s \\ & - \int_t^T \int_{I\!R^*} U_s^{\infty,p}(e)\tilde{N}(ds, de). \end{aligned}$$

Since $Y_t^{p,\infty} \geq \xi_t$ and $Y^{\infty,p} \leq \zeta_t$, we can substract $p\int_t^T (Y_s^{p,\infty} - \xi_s)^- ds$ to the first BSDE and we can add $p\int_t^T (\zeta_s - Y_s^{\infty,p})^- ds$ to the second BSDE. By the comparison theorem we get $Y_t^{\infty,p} \leq Y_t^p \leq Y_t^{p,\infty}$. Since $Y^{p,\infty} \searrow Y$ and $Y^{\infty,p} \nearrow Y$ when $p \to \infty$, we get that $Y_t^p \to Y_t$ almost surely, for all $t \in [0, T]$. From (1.4.29) and the corresponding result for $Y^{\infty,p}$, we get that $\lim_{p\to\infty} \mathbb{E}(\int_0^T |Y_s^p - Y_s|^2 ds) = 0$.
Applying Itô's formula to $\mathbb{E}(|Y_t^p - Y_t|^2)$ between $[\sigma, \tau]$, a pair of stopping times such that $t \leq \sigma \leq \tau \leq T$, we get

$$\begin{aligned} &\mathbb{E}\left(|Y_\sigma^p - Y_\sigma|^2 + \int_\sigma^\tau |Z_s^p - Z_s|^2 ds + \int_\sigma^\tau \int_{I\!R^*} |U_s^p(e) - U_s(e)|^2 \nu(de)ds\right) = \mathbb{E}(|Y_\tau^p - Y_\tau|^2) \\ &+ 2\mathbb{E}(\int_\sigma^\tau (Y_s^p - Y_s)(g(s, \theta_s^p) - g(s, \theta_s))ds) + \sum_{\sigma \leq s \leq \tau} (\Delta_s A)^2 + \sum_{\sigma \leq s \leq \tau} (\Delta_s K)^2 + 2\sum_{\sigma \leq s \leq \tau} \Delta_s A \Delta_s K \\ &+ 2\int_\sigma^\tau (Y_s^p - Y_s)d(A^p - A)_s - 2\int_\sigma^\tau (Y_s^p - Y_s)d(K^p - K)_s. \end{aligned}$$

By using the Cauchy-Schwarz inequality, the convergence of Y^p to Y in $I\!H^2$, and the fact that $g(s,\theta_s^p)$ and $g(s,\theta_s)$ are bounded in $L^2(\Omega\times[0,T])$, we get that the second term of the r.h.s. tends to zero when p tends to ∞. From the dominated convergence theorem the last two terms of the r.h.s. also tend to zero. Since $2\sum_{\sigma\leq s\leq\tau}\Delta_s A\Delta_s K\leq\sum_{\sigma\leq s\leq\tau}(\Delta_s A)^2+\sum_{\sigma\leq s\leq\tau}(\Delta_s K)^2$, we are back to Theorem 1.9.3, which ends the proof of (1.4.8).

It remains to prove that Z^p weakly converges to Z in $I\!H^2$, U^p weakly converges to U in $I\!H^2_\nu$ and α_t^p weakly converges to α in $L^2(\mathcal{F}_t)$. Since $Y_t^{\infty,p}\leq Y_t^p\leq Y_t^{p,\infty}$, we get $A_t^p\leq A_t^{\infty,p}$ and $K_t^p\leq K_t^{p,\infty}$. Then, by using Lemmas 1.4.9 and 1.4.10, we obtain $\mathbb{E}((A_T^p)^2)+\mathbb{E}((K_T^p)^2)\leq C$, where C does not depend on p. By applying Itô's formula to $|Y_t^p|^2$ and by using Young's inequality as in (1.4.25) we get $\mathbb{E}(\int_0^T|Z_t^p|^2dt+\int_0^T(\int_{I\!R^*}|U_s^p(e)|^2\nu(de)ds))\leq C$, where C does not depend on p. The sequences $(Z^p)_{p\geq0}$, $(U^p)_{p\geq0}$, $(A_t^p)_{p\geq0}$ and $(K_t^p)_{p\geq0}$ are bounded in the respective spaces $I\!H^2$, $I\!H^2_\nu$, $L^2(\mathcal{F}_t)$ and $L^2(\mathcal{F}_t)$. Then, we can extract subsequences which weakly converge in the related spaces. Let us denote Z',U',A' and K' the respective limits. Since (Z^p,U^p) strongly converge to (Z,U) for any $q<2$ (see (1.4.8)), we get that $Z=Z'$ and $U=U'$.

Let us prove that $A'-K'=A-K$. We have

$$\begin{aligned}A_t^p-K_t^p&=Y_0^p-Y_t^p-\int_0^t g(s,\theta_s^p)ds+\int_0^t Z_s^p dW_s+\int_0^t\int_{I\!R^*}U_s^p(e)\tilde{N}(ds,de),\\A_t-K_t&=Y_0-Y_t-\int_0^t g(s,\theta_s)ds+\int_0^t Z_s dW_s+\int_0^t\int_{I\!R^*}U_s(e)\tilde{N}(ds,de).\end{aligned}$$

Taking the limit in p in the first equation, we get $A_t'-K_t'=A_t-K_t$.

1.5 Numerical simulations

In this section, we illustrate the convergence of our scheme with two examples. The difficulty in the choice of examples is given by the hypothesis we assume, in particular the Mokobodsi's condition which is difficult to check in practice.

Example 1 : inaccessible jumps

We consider the simulation of the solution of a DRBSDE with obstacles having only totally inaccessible jumps. More precisely, we take the barriers and driver of the following form: $\xi_t:=(W_t)^2+\tilde{N}_t+(T-t),\zeta_t:=(W_t)^2+\tilde{N}_t+3(T-t),g(t,\omega,y,z,u):=-5|y+z|+6u-1$.

Our example satisfies the assumptions assumed in the theoretical part, in particular Hypotheses 2.2.5 and 1.3.3 (see Remark 1.3.5, point 2.). Assumption (1.2.4), which represents the Mokobodski's condition, is fulfilled, since $H_t:=(W_t)^2+\tilde{N}_t+2(T-t)$ satisfies $\xi_t\leq H_t\leq\zeta_t$ and $H_t=M_t+A_t$, where $M_t:=(W_t)^2+\tilde{N}_t+T-t$ is a martingale and $A_t:=T-t$ is a decreasing finite variation process.

Table 2.1 gives the values of Y_0 with respect to parameters n and p of our explicit sheme. We notice that the algorithm converges quite fast in p and n. However, when n is too small ($n=20$

and $n = 50$), the result for $p = 20000$ is quite far from the "reference" result ($n = 600$ and $p = 20000$). Concerning the computational time, we notice that it is low, even for big values of p and n.

Table 1.1: The solution $\overline{y}^{p,n}$ at time $t = 0$

$Y_0^{p,n}$	n=20	n=50	n=100	n=200	n=400	n=500	n=600
p=20	1.1736	1.2051	1.2181	1.2245	1.2277	1.2283	1.2288
p=50	1.2077	1.2482	1.2648	1.2728	1.2767	1.2775	1.2780
p=100	1.2214	1.2634	1.2808	1.2894	1.2936	1.2945	1.2950
p=500	1.2350	1.2753	1.2939	1.3033	1.3079	1.3088	1.3094
p=1000	1.2365	1.2767	1.2957	1.3051	1.3098	1.3107	1.3113
p=5000	1.2376	1.2778	1.2971	1.3066	1.3113	1.3122	1.3129
p=20000	1.2377	1.2780	1.2974	1.3069	1.3116	1.3125	1.3132
CPU time for p=20000	0.00071	0.0084	0.0644	0.6622	6.3560	12.5970	20.0062

Figure 2.1 represents one path of $(\overline{y}_t^{p,n}, \overline{\xi}_t^n, \zeta_t^n)_{t\geq 0}$. We notice that for all t, $\overline{y}_t^{p,n}$ stays between the two obstacles.

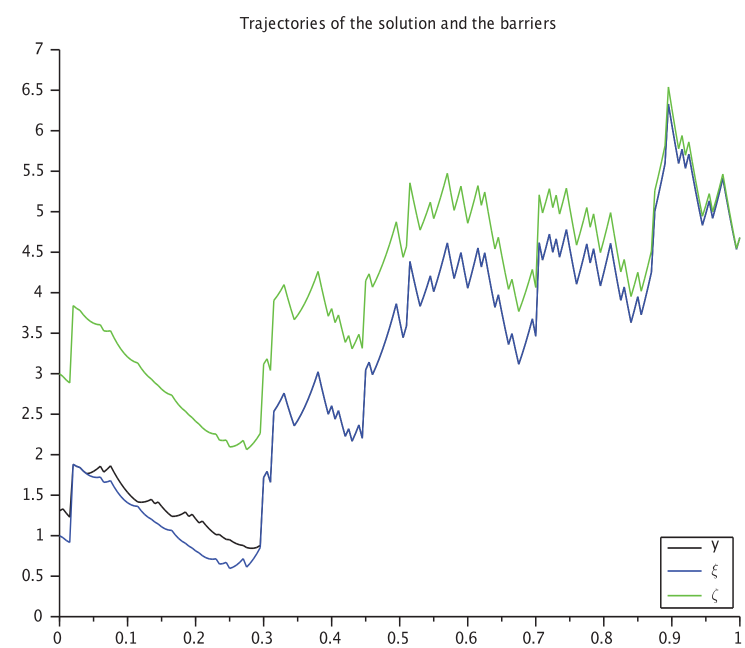

Figure 1.1: Trajectories of the solution $\overline{y}^{p,n}$ and the barriers $\overline{\xi}^n$ and $\overline{\zeta}^n$ for $\lambda = 5$, $N = 200$, $p = 20000$.

Example 2 : predictable and totally inaccessible jumps

We consider now the simulation of the DRBSDE with obstacles having general jumps (totally inaccessible and predictable). More precisely, we take the barriers and driver of the fol-

lowing form: $\xi_t := (W_t)^2 + \tilde{N}_t + (T-t)(1-\mathbf{1}_{W_t \geq a}), \zeta_t := (W_t)^2 + \tilde{N}_t + (T-t)(2+\mathbf{1}_{W_t \geq a})$, $g(t,\omega,y,z,u) := -5|y+z| + 6u - 1$.
We first give the numerical results for two different values of a, in order to show the influence of the predictable jumps given by $\mathbf{1}_{W_t \geq a}$ on the solution Y and also the convergence in n and p of the numerical explicit scheme (see Tables 1.2 and 1.3).

Then, Figures 1.2, 1.3 and 1.4 allow to distinguish the predictable jumps of totally inaccesible ones and their influence on the barriers (for e.g. the first jump of the barriers is totally inaccessible, the second and third ones are predictable). Moreover, we remark, as in the previous example, that the solution Y stays between the two obstacles ξ and ζ.

Table 1.2: The solution Y at time $t=0$ for a=-1

$Y_0^{p,n}$	n=100	n=200	n=400	n=500	n=600
p=20	1.0745	1.0698	1.0782	1.0748	1.0759
p=50	1.1138	1.1103	1.1191	1.1159	1.1170
p=100	1.1266	1.1238	1.1328	1.1297	1.1308
p=500	1.1373	1.1353	1.1448	1.1419	1.1431
p=1000	1.1387	1.1369	1.1465	1.1437	1.1449
p=5000	1.1399	1.1382	1.1481	1.1453	1.1466
p=20000	1.1401	1.1385	1.1484	1.1456	1.1469

Table 1.3: The solution Y at time $t=0$ for a=1

$Y_0^{p,n}$	n=100	n=200	n=400	n=500	n=600
p=20	1.2125	1.2177	1.2203	1.2208	1.2212
p=50	1.2582	1.2647	1.2680	1.2686	1.2690
p=100	1.2738	1.2808	1.2843	1.2850	1.2855
p=500	1.2866	1.2944	1.2982	1.2990	1.2995
p=1000	1.2884	1.2962	1.3001	1.3008	1.3013
p=5000	1.2898	1.2976	1.3016	1.3023	1.3029
p=20000	1.2900	1.2979	1.3018	1.3026	1.3032

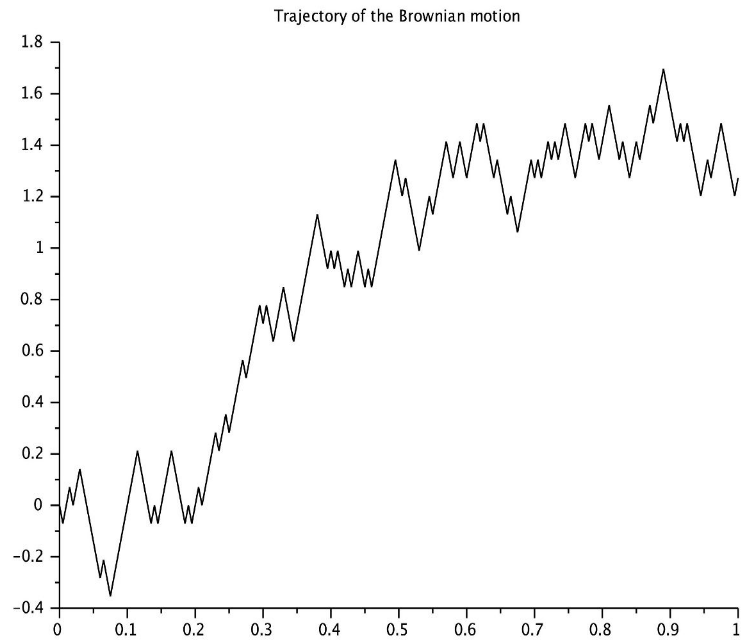

Figure 1.2: Trajectories of the Brownian motion for $a = -0.2$, $N = 200$.

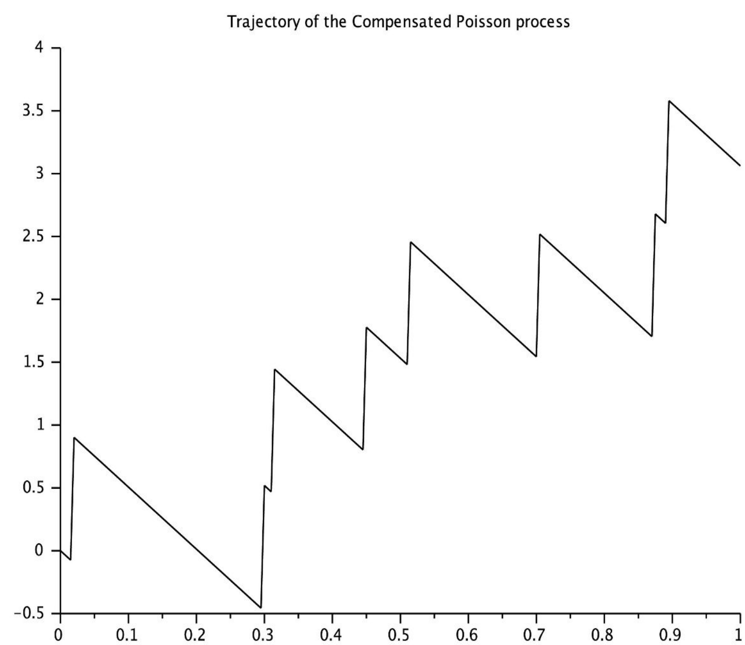

Figure 1.3: Trajectories of the Compensated Poisson process for $\lambda = 5$, $N = 200$.

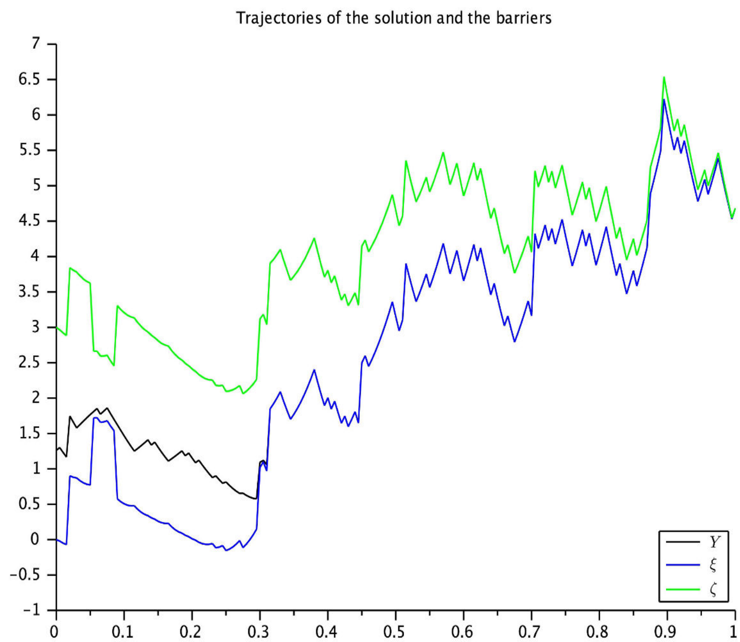

Figure 1.4: Trajectories of the solution Y and the barriers ξ and ζ for a=-0.2, $\lambda = 5$, $N = 200$.

1.6 Generalized monotonic limit theorem

The following Theorem generalizes [133, Theorem 3.1] and Theorem 1.9.3 to the case of doubly reflected BSDEs with jumps.

Theorem 1.6.1 (Monotonic limit theorem)**.** *Assume that g satisfies Assumption 1.2.2, and ξ belongs to $L^2(\mathcal{F}_T)$. We consider the following sequence (in n) of BSDEs :*

$$\begin{aligned} Y_t^n = \xi + \int_t^T g(s, Y_s^n, Z_s^n, U_s^n)ds + (A_T^n - A_t^n) - (K_T^n - K_t^n) - \int_t^T Z_s^n dW_s \\ - \int_t^T \int_{I\!R^*} U_s^n(e)\tilde{N}(ds, de) \end{aligned}$$

such that $Y^n \in \mathcal{S}^2$, A^n and K^n are in $\mathcal{A}^2$, and $\sup_n \mathbb{E}(\int_0^T |Z_s^n|^2 ds) + \sup_n \mathbb{E}(\int_0^T \int_{I\!R^*} |U_s^n(e)|^2 \nu(de)ds) < \infty$. We also assume that for each $n \in I\!N$

1. *$(A^n)_n$ is continuous and increasing and such that $A_0^n = 0$ and $\sup_n \mathbb{E}((A_T^n)^2) < \infty$*

2. *$K_t^j - K_s^j \geq K_t^i - K_s^i$, for all $0 \leq s \leq t \leq T$ and for all $i \leq j$*

3. *for all $t \in [0, T]$, $(K_t^n)_n \nearrow K_t$ and $E(K_T^2) < \infty$*

4. *$(Y_t^n)_n$ increasingly converges to Y_t with $\mathbb{E}(\sup_{0 \leq t \leq T} |Y_t|^2) < \infty$.*

Then $K \in \mathcal{A}^2$ and there exist $Z \in I\!H^2$, $A \in \mathcal{A}^2$ and $U \in I\!H^2_\nu$ such that

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s, U_s)ds + A_T - A_t - (K_T - K_t) - \int_t^T Z_s dW_s - \int_t^T \int_{I\!R^*} U_s(e)\tilde{N}(ds, de).$$

Z is the weak limit of $(Z^n)_n$ in $I\!H^2$, K_t is the strong limit of $(K^n_t)_n$ in $L^2(\mathcal{F}_t)$, A_t is the weak limit of $(A^n_t)_n$ in $L^2(\mathcal{F}_t)$ and U is the weak limit of $(U^n)_n$ in $I\!H^2_\nu$. Moreover, for all $r \in [1, 2[$, the following strong convergence holds

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^T |Y^n_s - Y_s|^2 ds + \int_0^T |Z^n_s - Z_s|^r ds + \int_0^T \left(\int_{I\!R^*} |U^n_s(e) - U_s(e)|^2 \nu(de)\right)^{\frac{r}{2}} ds\right) = 0.$$

Proof of Theorem 1.6.1. This proof follows the proofs of Theorem 1.9.3 and [133, Theorem 3.1]. From the hypotheses, the sequences $(Z^n)_n$, $(U^n)_n$ and $(g(\cdot, Y^n, Z^n, U^n))_n$ are bounded in $I\!H^2$, $I\!H^2_\nu$ and $L^2([0,T] \times \Omega)$, then we can extract subsequences which weakly converge in the related spaces. Let Z, U and g_0 denote the respective weak limits. Thus, for each stopping time $\tau \leq T$, the following weak convergence holds in $L^2(\mathcal{F}_\tau)$

$$\int_0^\tau g(s, Y^n_s, Z^n_s, U^n_s)ds \underset{n\to\infty}{\rightharpoonup} \int_0^\tau g_0(s)ds, \quad \int_0^\tau Z^n_s dW_s \underset{n\to\infty}{\rightharpoonup} \int_0^\tau Z_s dW_s$$

and

$$\int_0^\tau \int_{I\!R^*} U^n_s(e)\tilde{N}(ds, de) \underset{n\to\infty}{\rightharpoonup} \int_0^\tau \int_{I\!R^*} U_s(e)\tilde{N}(ds, de), \quad K^n_\tau \underset{n\to\infty}{\rightharpoonup} K_\tau$$

since $(K^n_t)_n \nearrow K_t$ in $L^2(\mathcal{F}_t)$.

$$A^n_\tau = Y^n_0 - Y^n_\tau - \int_0^\tau g(s, Y^n_s, Z^n_s, U^n_s)ds + K^n_\tau + \int_0^\tau Z^n_s dW_s + \int_0^\tau \int_{I\!R^*} U^n_s(e)\tilde{N}(ds, de)$$

we also have the following weak convergence in $L^2(\mathcal{F}_\tau)$

$$A^n_\tau \rightharpoonup A_\tau := Y_0 - Y_\tau - \int_0^\tau g_0(s)ds + K_\tau + \int_0^\tau Z_s dW_s + \int_0^\tau \int_{I\!R^*} U_s(e)\tilde{N}(ds, de).$$

Then $\mathbb{E}(A^2_T) < \infty$. Since the process $(A^n_t)_t$ is increasing, predictable and such that $A^n_0 = 0$, the limit process A remains an increasing predictable process with $A_0 = 0$. We deduce from [133, Lemma 3.2] that K is a RCLL process, and from [133, Lemma 3.1] that A and Y are RCLL processes. Then Y has the form

$$Y_t = \xi + \int_t^T g_0(s)ds + A_T - A_t - (K_T - K_t) - \int_t^T Z_s dW_s - \int_t^T \int_{I\!R^*} U_s(e)\tilde{N}(ds, de).$$

It remains to prove that for all $r \in [1, 2[$

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^T |Z^n_s - Z_s|^r ds + \int_0^T \left(\int_{I\!R^*} |U^n_s(e) - U_s(e)|^2 \nu(de)\right)^{\frac{r}{2}} ds\right) = 0$$

and for all $t \in [0,T]$

$$\int_0^t g_0(s)ds = \int_0^t g(s, Y_s, Z_s, U_s)ds.$$

Let $N_t = \int_0^t \int_{I\!R^*} U_s(e)\tilde{N}(ds,de)$ and $N_t^n = \int_0^t \int_{I\!R^*} U_s^n(e)\tilde{N}(ds,de)$. We have $\Delta_s(Y^n - Y) = \Delta_s(N^n - N + K^n - K + A)$. We appply Itô's formula to $(Y_t^n - Y_t)^2$ on each subinterval $]\sigma, \tau]$, where σ and τ are two predictable stopping times such that $0 \leq \sigma \leq \tau \leq T$. Let θ_s^n denotes (Y_s^n, Z_s^n, U_s^n)

$$\begin{aligned}
&(Y_\sigma^n - Y_\sigma)^2 + \int_\sigma^\tau |Z_s^n - Z_s|^2 ds + \sum_{\sigma \leq s \leq \tau} \Delta_s(Y^n - Y)^2 \\
&= (Y_\tau^n - Y_\tau)^2 + 2\int_\sigma^\tau (Y_s^n - Y_s)(g(s,\theta_s^n) - g_0(s))ds + 2\int_\sigma^\tau (Y_s^n - Y_s)dA_s^n - 2\int_\sigma^\tau (Y_{s^-}^n - Y_{s^-})dA_s \\
&- 2\int_\sigma^\tau (Y_{s^-}^n - Y_{s^-})d(K_s^n - K_s) - 2\int_\sigma^\tau (Y_{s^-}^n - Y_{s^-})(Z_s^n - Z_s)dW_s \\
&- 2\int_\sigma^\tau (Y_{s^-}^n - Y_{s^-})(U_s^n(e) - U_s(e))\tilde{N}(ds,de).
\end{aligned}$$

Since $\int_\sigma^\tau (Y_s^n - Y_s)dA_s^n \leq 0$, $-2\int_\sigma^\tau (Y_{s^-}^n - Y_{s^-})d(K_s^n - K_s) \leq 0$ and

$$\begin{aligned}
\sum_{\sigma \leq s \leq \tau} \Delta_s(Y^n - Y)^2 &= \sum_{\sigma \leq s \leq \tau} \Delta_s(N^n - N)^2 + \sum_{\sigma \leq s \leq \tau} \Delta_s(K^n - K)^2 + \sum_{\sigma \leq s \leq \tau} (\Delta_s A)^2 \\
&+ 2\sum_{\sigma \leq s \leq \tau} \Delta_s A \Delta_s(K^n - K).
\end{aligned}$$

By taking expectation and using $Y_{s^-}^n - Y_{s^-} = (Y_s^n - Y_s) - \Delta_s(Y^n - Y)$, we get

$$\begin{aligned}
&\mathbb{E}(Y_\sigma^n - Y_\sigma)^2 + \mathbb{E}\int_\sigma^\tau |Z_s^n - Z_s|^2 ds + \mathbb{E}\int_\sigma^\tau \int_{I\!R^*} |U_s^n(e) - U_s(e)|^2 \nu(de)ds + \mathbb{E}\sum_{\sigma \leq s \leq \tau} \Delta_s(K^n - K)^2 \\
&\leq \mathbb{E}(Y_\tau^n - Y_\tau)^2 + 2\mathbb{E}\int_\sigma^\tau (Y_s^n - Y_s)(g(s,\theta_s^n) - g_0(s))ds - 2\mathbb{E}\int_\sigma^\tau (Y_s^n - Y_s)dA_s + \mathbb{E}\sum_{\sigma \leq s \leq \tau} (\Delta_s A)^2.
\end{aligned}$$

It comes down to [78, Equation (10)], we refer to this paper for the end of the proof. □

1.7 Snell envelope theory

Definition 1.7.1. *Any $\mathcal{F}_t$-adapted RCLL process $\eta = (\eta_t)_{0 \leq t \leq T}$ is of class $\mathcal{D}[0,T]$ if the family $\{\eta(\tau)\}_{\tau \in \mathcal{T}_0}$ is uniformly integrable.*

Definition 1.7.2. *Let $\eta = (\eta_t)_{t \leq T}$ be a $\mathcal{F}_t$-adapted RCLL process of class $\mathcal{D}[0,T]$. Its Snell envelope $\mathcal{R}_t(\eta)$ is defined as*

$$\mathcal{R}_t(\eta) = \operatorname*{esssup}_{\nu \in \mathcal{T}_t} \mathbb{E}(\eta_\nu | \mathcal{F}_t).$$

Proposition 1.7.3. *$\mathcal{R}_t(\eta)$ is the lowest RCLL $\mathcal{F}_t$-supermartingale of class $\mathcal{D}[0,T]$ which dominates η, i.e. $\mathbb{P}$-a.s., for all $t \in [0,T]$, $\mathcal{R}(\eta)_t \geq \eta_t$.*

Proposition 1.7.4 (Doob-Meyer decomposition of Snell envelopes). *Let $\eta := (\eta_t)_{t\le T}$ be of class $\mathcal{D}([0,T])$. There exists a unique decomposition of the Snell envelope*

$$\mathcal{R}_t(\eta) = M_t - K_t^c - K_t^d,$$

where M_t is a RCLL $\mathcal{F}_t$-martingale, K^c is a continuous integrable increasing process with $K_0^c = 0$, and K^d is a pure jump integrable increasing predictable RCLL process with $K_0^d = 0$. Moreover, we have

$$\int_0^T (\mathcal{R}_{t^-}(\eta) - \eta_{t^-}) dK_t = 0,$$

where $K := K^c + K^d$.

Proof. The first part of the proposition corresponds to the Doob-Meyer decomposition of supermartingales of class $\mathcal{D}[0,T]$. To prove the second part of the proof, we write

$$\int_0^T (\mathcal{R}_{t^-}(\eta) - \eta_{t^-}) dK_t = \int_0^T (\mathcal{R}_{t^-}(\eta) - \eta_{t^-}) dK_t^d + \int_0^T (\mathcal{R}_{t^-}(\eta) - \eta_{t^-}) dK_t^c.$$

The first term of the right hand side is null, since $\{\Delta K^d > 0\} \subset \{\mathcal{R}(\eta)_- = \eta_-\}$ (see [91, Property A.2, (ii)]). Let us prove that the second term of the r.h.s. is also null. We know that $(\mathcal{R}_t(\eta)+K_t^d)_t = (M_t - K_t^c)_t$ is a supermartingale satisfying $\mathcal{R}_t(\eta) + K_t^d \ge \eta_t + K_t^d$, then $\mathcal{R}_t(\eta) + K_t^d \ge \mathcal{R}(\eta_t + K_t^d)$. On the other hand, for every supermartingale N_t such that $N_t \ge \eta_t + K_t^d$, we have $N_t - K_t^d \ge \eta_t$, and then $N_t - K_t^d \ge \mathcal{R}(\eta)_t$ (since $(N_t - K_t^d)_t$ is a supermartingale), then $N_t \ge \mathcal{R}(\eta)_t + K_t^d$. By choosing $N_t := \mathcal{R}(\eta + K^d)_t$, we get $\mathcal{R}_t(\eta) + K_t^d = \mathcal{R}(\eta_t + K_t^d)$. Since K^c is continuous, $(\mathcal{R}_t(\eta)+K_t^d)_t$ is regular (see [136, Exercise 27]). Then, from [91, Property A3], we get that $\tau_t := \inf\{s \ge t : K_s^c - K_t^c > 0\}$ is optimal after t. This yields $\int_t^{\tau_t} (\mathcal{R}(\eta)_s + K_s^d - (\eta_s + K_s^d)) dK_s^c = 0$ for all $t \le T$. Then, we get $\int_0^T (\mathcal{R}_{t^-}(\eta) - \eta_{t^-}) dK_t^c = 0$. □

1.8 Technical result for standard BSDEs with jumps

Lemma 1.8.1. *We assume that δ_n is small enough such that $(3+2p+2C_g+2C_g^2(1+\frac{1}{\lambda}e^{2\lambda T}))\delta_n < 1$. Then we have:*

$$\sup_{j\le n} \mathbb{E}[|\overline{y}_j^{p,n}|^2] + \delta_n \sum_{j=0}^{n-1} \mathbb{E}[|\overline{z}_j^{p,n}|^2] + (1-\kappa_n)\kappa_n \sum_{j=0}^{n-1} \mathbb{E}[|\overline{u}_j^{p,n}|^2] \le K_{Lem.1.8.1}.$$

where

$$K_{Lem.1.8.1} = (\|g(\cdot,0,0,0)\|_\infty^2 + (p^2 + C_g T)(\sup_n \max_j \mathbb{E}[|\xi_j^n|^2] + \sup_n \max_j \mathbb{E}[|\zeta_j^n|^2])) e^{(3+2p+2C_g+2C_g^2(2+\frac{1}{\lambda}e^{2\lambda T}))}.$$

Proof. From the explicit scheme, we derive that:

$$\begin{aligned}\mathbb{E}[|\overline{y}_j^{p,n}|^2] - \mathbb{E}[|\overline{y}_{j+1}^{p,n}|^2] &= -\delta_n \mathbb{E}[|\overline{z}_j^{p,n}|^2] - (1-\kappa_n)\kappa_n \mathbb{E}[|\overline{u}_j^{p,n}|^2] - (1-\kappa_n)\kappa_n \mathbb{E}[|\overline{v}_j^{p,n}|^2] \\ &- \delta_n^2 \mathbb{E}[g_p^2(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})] + 2\delta_n \mathbb{E}[\overline{y}_j^{p,n} g_p(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})].\end{aligned}$$

Taking the sum for $j = i, ..., n-1$ yields

$$\begin{aligned}
\mathbb{E}[|\overline{y}_i^{p,n}|^2] &\leq \mathbb{E}[|\xi^n|^2] - \delta_n \sum_{j=i}^{n-1} \mathbb{E}[|\overline{z}_j^{p,n}|^2] - (1-\kappa_n)\kappa_n \sum_{j=i}^{n-1} \mathbb{E}[|\overline{u}_j^{p,n}|^2] \\
&\quad + 2\delta_n \sum_{j=i}^{n-1} \mathbb{E}[\overline{y}_j^{p,n} g_p(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})] \\
&\leq \mathbb{E}[|\xi^n|^2] - \delta_n \sum_{j=i}^{n-1} \mathbb{E}[|\overline{z}_j^{p,n}|^2] - (1-\kappa_n)\kappa_n \sum_{j=i}^{n-1} \mathbb{E}[|\overline{u}_j^{p,n}|^2] \\
&\quad + 2\delta_n \sum_{j=i}^{n-1} \mathbb{E}[|\overline{y}_j^{p,n}|(|g(t_j, 0, 0, 0)| \\
&\quad + C_g|\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n]| + C_g|\overline{z}_j^{p,n}| + C_g|\overline{u}_j^{p,n}| + p(|\overline{y}_j^{p,n}| + |\xi_j^n| + |\zeta_j^n|))]
\end{aligned}$$

Hence, we get that:

$$\begin{aligned}
\mathbb{E}[|\overline{y}_i^{p,n}|^2] + \frac{\delta_n}{2} \sum_{j=i}^{n-1} \mathbb{E}[|\overline{z}_j^{p,n}|^2] + \frac{(1-\kappa_n)\kappa_n}{2} \sum_{j=i}^{n-1} \mathbb{E}[|\overline{u}_j^{p,n}|^2] &\leq \delta_n \sum_{j=i}^{n-1} \mathbb{E}[|g(t_j, 0, 0, 0)|^2] \\
&\quad + (p^2 + C_g\delta_n)(\max_j \mathbb{E}[|\xi_j^n|^2] + \max_j \mathbb{E}[|\zeta_j^n|^2]) \\
&\quad + \delta_n \left(3 + 2p + 2C_g + 2C_g^2 + \frac{2C_g^2\delta_n}{(1-\kappa_n)\kappa_n}\right) \sum_{j=i}^{n-1} \mathbb{E}[|\overline{y}_j^{p,n}|^2].
\end{aligned}$$

Since $\dfrac{\delta_n}{\kappa_n(1-\kappa_n)} \leq \dfrac{1}{\lambda} e^{2\lambda T}$, the assumption on δ_n enables to apply Gronwall's Lemma, and the result follows. □

1.9 Some recent results on BSDEs and reflected BSDEs with jumps

For the self-containment of the chapter, we recall in this Section some recent results used several times in the chapter.

1.9.1 Comparison theorem for BSDEs and reflected BSDEs with jumps

Theorem 1.9.1 (Comparison Theorem for BSDEs with jumps ([137], Theorem 4.2)). *Let ξ_1 and ξ_2 be in $L^2(\mathcal{F}_T)$. Let f_1 be a Lipschitz driver and f_2 be a driver. For $i = 1, 2$ let (X_t^i, π_t^i, l_t^i) be a solution in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\nu^2$ of the BSDE*

$$-dX_t^i = f_i(t, X_t^i, \pi_t^i, l_t^i)dt - \pi_t^i dW_t - \int_{\mathbb{R}^*} l_t^i(u)\tilde{N}(dt, du);\ X_T^i = \xi_i. \tag{1.9.1}$$

Assume that there exists a bounded predictable process (γ_t) such that $dt \otimes dP \otimes \nu(du)$-a.s.

$$\gamma_t(u) \geq -1 \quad \text{and } |\gamma_t(u)| \leq \psi(u),$$

where $\psi \in L^2_\nu$ *and such that*

$$f_1(t, X_t^2, \pi_t^2, l_t^1) - f_1(t, X_t^2, \pi_t^2, l_t^2) \geq \langle \gamma_t, l_t^1 - l_t^2 \rangle_\nu, \ t \in [0,T], dt \otimes dP \ a.s. \tag{1.9.2}$$

Assume that

$$\xi_1 \geq \xi_2 \ a.s. \ and \ f_1(t, X_t^2, \pi_t^2, l_t^2) \geq f_2(t, X_t^2, \pi_t^2, l_t^2) \ t \in [0,T], dt \otimes dP \ a.s. \tag{1.9.3}$$

Then we have

$$X_t^1 \geq X_t^2 \ a.s. \ for \ all \ t \in [0,T]. \tag{1.9.4}$$

Moreover, if inequality (1.9.3) *is satisfied for* (X_t^1, π_t^1, l_t^1) *instead of* (X_t^2, π_t^2, l_t^2) *and if* f_2 *(instead of* f_1*) is Lipschitz and satisfies* (1.9.2)*, then* (1.9.4) *still holds.*

Theorem 1.9.2 (Comparison Theorem for reflected BSDEs with jumps ([138], Theorem 5.1))**.** *Let* ξ^1, ξ^2 *be two RCLL obstacle processes in* $\mathcal{S}^2$*. Let* f_1 *and* f_2 *be Lipschitz drivers satisfying Assumption 2.2.4. Suppose that*

$$\begin{aligned}
&\xi_t^2 \leq \xi_t^1, \ 0 \leq t \leq T \ a.s.\\
&f_2(t, y, z, k) \leq f_1(t, y, z, k), \ for \ all \ (y, z, k) \in \mathbb{R}^2 \times L^2_\nu, \ dP \otimes dt \ a.s.
\end{aligned}$$

Let (Y^i, Z^i, k^i, A^i) *be a solution in* $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{S}^2$ *of the reflected BSDE*

$$-dY_t^i = f_i(t, Y_t^i, Z_t^i, k_t^i(\cdot))dt + dA_t^i - Z_t^i dW_t - \int_{\mathbb{R}^*} k_t^i(u) \tilde{N}(dt, du); \ Y_T^i = \xi_T^i, \tag{1.9.5}$$

$$Y_t^i \geq \xi_t^i, \ 0 \leq t \leq T \ a.s. \tag{1.9.6}$$

and A^i *is a non decreasing RCLL predictable process with* $A_0^i = 0$ *and such that*

$$\int_0^T (Y_t^i - \xi_t^i) dA_t^{i,c} = 0 \ a.s. \ and \ \Delta A_t^{i,d} = -\Delta Y_t^i \mathbf{1}_{Y_{t-}^i = \xi_{t-}^i} \ a.s.$$

Then $Y_t^2 \leq Y_t^1$ *for all* t *in* $[0,T]$ *a.s.*

1.9.2 Convergence results on reflected BSDEs with jumps

Theorem 1.9.3 (Monotonic limit theorem for reflected BSDEs with jumps ([78], Theorem 3.1))**.** *Assume that* f *satisfies [78, Assumption A.2],* $\xi \in L^2$ *and* K^n *is a continuous and increasing process such that* $\sup_{n \in \mathbb{N}} \mathbb{E}(K_T^n)^2 < \infty$ *and* $K_0^n = 0$ *for any* $n \in \mathbb{N}$*. Let* (Y^n, Z^n, V^n) *be the solution of the following BSDE*

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n, V_s^n) ds + K_T^n - K_t^n - \int_t^T Z_s^n dW_s - \int_t^T \int_U V_s^n(u) \tilde{N}(ds, du), \ t \leq T,$$

where $\sup_{n \in \mathbb{N}} \mathbb{E} \int_0^T |Z_s^n|^2 ds < \infty$ *and* $\sup_{n \in \mathbb{N}} \mathbb{E} \int_0^T \int_U |V_s^n(u)|^2 \nu(du) ds < \infty$*. If* Y^n *converges increasingly to* Y *with* $\mathbb{E}(\sup_{0 \leq t \leq T} Y_t^2) < \infty$*, then there exists* $Z \in \mathbb{H}^2$*,* $K \in \mathcal{A}^2$ *and* $V \in \mathbb{H}^2_\nu$ *such that the triple* (Z, K, V) *satisfies the following equation*

$$Y_t = \xi + \int_0^T f(s, Y_s, Z_s, V_s) ds + K_T - K_t - \int_t^T Z_s dW_s - \int_t^T \int_U V_s(u) \tilde{N}(ds, du), \ t \leq T.$$

Here Z is the weak limit of $(Z^n)_n$ in $\mathbb{H}^2$, K_t is the weak limit of $(K_t^n)_n$ in $L^2(\mathcal{F}_t)$ and V is the weak limit of $(V^n)_n$ in $\mathbb{H}^2_\nu$. Moreover, for every $p \in [1, 2[$, the following strong convergence holds

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T |Y_s^n - Y_s|^2 ds\right] + \mathbb{E}\left[\int_0^T |Z_s^n - Z_s|^p ds + \int_0^T \left(\int_U |V_s^n(u) - V_s(u)|^2 \nu(du)\right)^{\frac{p}{2}} ds\right] = 0.$$

Now we introduce the following penalized equation

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n, V_s^n) ds + K_T^n - K_t^n - \int_t^T Z_s^n dW_s - \int_t^T \int_U V_s^n(u) \tilde{N}(ds, du),\ t \leq T,$$

where $K_t^n = n \int_0^t (Y_s^n - S_s)^- ds$. We have

Theorem 1.9.4 ([78], Theorem 4.2)**.** *The sequence $(Y^n, Z^n, V^n)_n$ has a limit (Y, Z, V) such that Y^n converges to Y in $\mathcal{S}^2$ and Z is the weak limit in $\mathbb{H}^2$, K_t is the weak limit of $(K_t^n)_n$ in $L^2(\mathcal{F}_t)$ and V is the weak limit in $\mathbb{H}^2_\nu$.*

1.9.3 Stochastic game for DRBSDE

Let us now give the characterization of the solution of the DRBSDE as the value function of a stochastic game we introduce. For more details on stochastic games applied to DRBSDE, we refer to [138].

Proposition 1.9.5. *Let $(Y, Z, U, \alpha) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{A}^2$ be a solution of the DRBSDE* (1.2.1)*. For any $S \in \mathcal{T}_0$ and any stopping times $\tau, \sigma \in \mathcal{T}_S$, consider the payoff:*

$$I_S(\tau, \sigma) = \int_S^{\tau\wedge\sigma} g(s, Y_s, Z_s, U_s(\cdot)) ds + \xi_\tau \mathbf{1}_{\{\tau \leq \sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma < \tau\}}. \tag{1.9.7}$$

The upper and lower value functions at time S associated to the stochastic game are defined respectively by

$$\overline{V}(S) := \operatorname*{essinf}_{\sigma\in\mathcal{T}_S} \operatorname*{esssup}_{\tau\in\mathcal{T}_S} \mathbb{E}[I_S(\tau, \sigma)|\mathcal{F}_S]. \tag{1.9.8}$$

$$\underline{V}(S) := \operatorname*{esssup}_{\tau\in\mathcal{T}_S} \operatorname*{essinf}_{\sigma\in\mathcal{T}_S} \mathbb{E}[I_S(\tau, \sigma)|\mathcal{F}_S] \tag{1.9.9}$$

This game has a value V, given by the state-process Y solution of DRBSDE, i.e.

$$Y_S = \overline{V}(S) = \underline{V}(S). \tag{1.9.10}$$

Proof. For each $S \in \mathcal{T}_0$ and for each $\varepsilon > 0$, let

$$\tau_S^\varepsilon := \inf\{t \geq S,\ Y_t \leq \xi_t + \varepsilon\} \quad \sigma_S^\varepsilon := \inf\{t \geq S,\ Y_t \geq \zeta_t - \varepsilon\}. \tag{1.9.11}$$

Remark that σ_S^ε and $\tau_S^\varepsilon \in \mathcal{T}_S$. Fix $\varepsilon > 0$. We have that almost surely, if $t \in [S, \tau_S^\varepsilon[$, then $Y_t > \xi_t + \varepsilon$ and hence $Y_t > \xi_t$. It follows that the function $t \mapsto A_t^c$ is constant a.s. on $[S, \tau_S^\varepsilon]$ and $t \mapsto A_t^d$ is constant a.s. on $[S, \tau_S^\varepsilon[$. Also, $Y_{(\tau_S^\varepsilon)^-} \geq \xi_{(\tau_S^\varepsilon)^-} + \varepsilon$ a.s. Since $\varepsilon > 0$, it follows that $Y_{(\tau_S^\varepsilon)^-} > \xi_{(\tau_S^\varepsilon)^-}$ a.s. , which implies that $\Delta A^d_{\tau_S^\varepsilon} = 0$ a.s. (see Remark 1.2.6). Hence, the process A is constant on

$[S,\tau_S^\varepsilon]$. Furthermore, by the right-continuity of (ξ_t) and (Y_t), we clearly have $Y_{\tau_S^\varepsilon} \leq \xi_{\tau_S^\varepsilon} + \varepsilon$ a.s. Similarly, one can show that the process K is constant on $[S,\sigma_S^\varepsilon]$ and that $Y_{\sigma_S^\varepsilon} \geq \zeta_{\sigma_S^\varepsilon} - \varepsilon$ a.s. Let us now consider two cases. First, on the set $\{\sigma_S^\varepsilon < \tau\}$, by using the definition of the stopping times and the fact that K is constant on $[S,\sigma_S^\varepsilon]$, we have:

$$\begin{aligned} I_S(\tau,\sigma_S^\varepsilon) &\leq \int_S^{\sigma_S^\varepsilon} g(s,Y_s,Z_s,U_s(\cdot))ds + Y_{\sigma_S^\varepsilon} + \varepsilon - (K_{\sigma_S^\varepsilon} - K_S) + (A_{\sigma_S^\varepsilon} - A_S) \\ &\leq Y_S + \int_S^{\sigma_S^\varepsilon} Z_s dW_s + \int_S^{\sigma_S^\varepsilon} \int_{\mathbb{R}^*} U_s(e)\tilde{N}(ds,de) + \varepsilon. \end{aligned} \tag{1.9.12}$$

On the set $\{\tau \leq \sigma_S^\varepsilon\}$, we obtain:

$$\begin{aligned} I_S(\tau,\sigma_S^\varepsilon) &\leq \int_S^{\tau} g(s,Y_s,Z_s,U_s(\cdot))ds + Y_\tau - (K_\tau - K_S) + (A_\tau - A_S) \\ &\leq Y_S + \int_S^{\tau} Z_s dW_s + \int_S^{\tau} \int_{\mathbb{R}^*} U_s(e)\tilde{N}(ds,de). \end{aligned}$$

The two above inequalities imply:

$$\mathbb{E}[I_S(\tau,\sigma_S^\varepsilon)|\mathcal{F}_S] \leq Y_S + \varepsilon.$$

Similarly, one can show that:

$$\mathbb{E}[I_S(\tau_S^\varepsilon,\sigma)|\mathcal{F}_S] \geq Y_S - \varepsilon.$$

Consequently, we get that for each $\varepsilon > 0$

$$\operatorname*{esssup}_{\tau\in\mathcal{T}_s} E[I_S(\tau,\sigma_S^\varepsilon)|\mathcal{F}_S] - \varepsilon \ \leq \ Y_S \ \leq \ \operatorname*{essinf}_{\sigma\in\mathcal{T}_S} E[I_S(\tau_S^\varepsilon,\sigma)|\mathcal{F}_S] + \varepsilon \quad \text{a.s.},$$

that is $\overline{V}(S) - \varepsilon \ \leq \ Y_S \ \leq \ \underline{V}(S) + \varepsilon$ a.s. Since $\underline{V}(S) \leq \overline{V}(S)$ a.s., the result follows. □

Chapter 2

Reflected scheme for doubly reflected BSDEs with jumps and RCLL obstacles

Abstract. We introduce a discrete time reflected scheme to solve doubly reflected Backward Stochastic Differential Equations with jumps (in short DRBSDEs), driven by a Brownian motion and an independent compensated Poisson process. As in [59], we approximate the Brownian motion and the Poisson process by two random walks, but contrary to this chapter, we discretize directly the DRBSDE, without using a penalization step. This gives us a fully implementable scheme, which only depends on one parameter of approximation: the number of time steps n (contrary to the scheme proposed in [59], which also depends on the penalization parameter). We prove the convergence of the scheme, and give some numerical examples.

2.1 Introduction

Non-linear backward stochastic differential equations (BSDEs in short) have been introduced by Pardoux and Peng in the Brownian framework in their seminal chapter [124] and then extended to the case of jumps by Tang and Li [147]. BSDEs appear as a useful mathematical tool in finance (hedging problems) and in stochastic control. Moreover, these stochastic equations provide a probabilistic representation for the solution of semilinear partial differential equations. BSDEs have been extended to the reflected case by El Karoui et al in [71]. In their setting, one of the components of the solution is forced to stay above a given barrier which is a continuous adapted stochastic process. The main motivation is the pricing of American options especially in constrained markets. The generalization to the case of two reflecting barriers has been carried out by Cvitanic and Karatzas in [52]. It is well known that doubly reflected BSDEs (DRBSDEs in the following) are related to Dynkin games and to the pricing of Israeli options (or Game options). The extension to the case of reflected BSDEs with jumps and one reflecting barrier with only inaccessible jumps has been established by Hamadène and Ouknine [90]. Later on, Essaky in [78] and Hamadène and Ouknine in [91] have extended these results to a right-continuous left limited (RCLL) obstacle with predictable and inaccessible jumps. Results concerning existence and uniqueness of the solution for doubly reflected BSDEs with jumps can be found in [50],[62], [86], [92] and [79].

Numerical shemes for DRBSDEs driven by the Brownian motion have been proposed by Xu in [151] (see also [117] and [134]) and, in the Markovian framework, by Chassagneux in [45]. In this chapter, we are interested in numerically solving DRBSDEs driven by a Brownian motion and an independent Poisson process in the case of RCLL obstacles with only totally inacessible jumps. More precisely, we consider equations of the following form:

$$\begin{cases} (i)\ Y_t = \xi_T + \int_t^T g(s, Y_s, Z_s, U_s)ds + (A_T - A_t) - (K_T - K_t) \\ \qquad - \int_t^T Z_s dW_s - \int_t^T U_s d\tilde{N}_s, \\ (ii)\ \forall t \in [0,T], \xi_t \leq Y_t \leq \zeta_t \text{ a.s.}, \\ (iii)\ \int_0^T (Y_t - \xi_t)dA_t = 0 \text{ a.s. and } \int_0^T (\zeta_t - Y_t)dK_t = 0 a.s. \end{cases} \tag{2.1.1}$$

$\{W_t : 0 \leq t \leq T\}$ is a one dimensional standard Brownian motion and $\{\tilde{N}_t := N_t - \lambda t, 0 \leq t \leq T\}$ is a compensated Poisson process. Both processes are independent and they are defined on the probability space $(\Omega, \mathcal{F}_T, \mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$. The processes A and K have the role to keep the solution between the two obstacles ξ and ζ. Since we consider that the jumps of the obstacles are totally inaccessible, A and K are continuous processes.

In the non-reflected case, some numerical methods have been provided: in [30], the authors propose a scheme for Forward-Backward SDEs based on the dynamic programming equation and in [109], the authors propose a fully implementable scheme based on a random binomial tree. In the reflected case, a fully implementable numerical scheme has been recently provided by Dumitrescu and Labart in [59]. Their method is based on the approximation of the Brownian motion and the Poisson process by two random walks and on the approximation of the reflected BSDE by a sequence of penalized BSDEs.

The aim of this chapter is to propose an alternative scheme to [59] to solve (2.1.1). The scheme proposed here takes the following form:

$$\begin{cases} \overline{y}_j^n = \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n] + g(t_j, \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n], \overline{z}_j^n, \overline{u}_j^n)\delta + \overline{a}_j^n - \overline{k}_j^n \\ \overline{a}_j^n \geq 0,\ \overline{k}_j^n \leq 0,\ \overline{a}_j^n \overline{k}_j^n = 0, \\ \xi_j^n \leq \overline{y}_j^n \leq \zeta_j^n,\ (\overline{y}_j^n - \xi_j^n)\overline{a}_j^n = (\overline{y}_j^n - \zeta_j^n)\overline{k}_j^n = 0. \end{cases} \tag{2.1.2}$$

It generalizes the scheme proposed by [151] to the case of jumps. Compared to the scheme proposed in [59], the scheme proposed here —called *reflected scheme* in the following —is based on the direct discretization of (2.1.1). In particular, there is no penalization step. Then, this method only depends on one parameter of approximation (the number of time steps n), contrary to the scheme proposed in [59] (which also depends on the penalization parameter). We provide here an *explicit reflected scheme* and an *implicit reflected scheme* and we show the convergence of both schemes. We illustrate numerically the theoretical results and show they coincide with the ones obtained by using the penalized scheme presented in [59], for large values of the penalization parameter.

The chapter is organized as follows: in Section 2 we introduce notations and assumptions. In Section 3, we precise the discrete time framework and present the numerical schemes. In Section 4 we provide the convergence of the schemes. Numerical examples are given in Section 5 .

2.2 Notations and assumptions

In this Section we introduce notations and assumptions. We recall the result on existence and uniqueness of solution to (2.1.1). We also introduce some assumptions on the obstacles ξ and ζ specific to this chapter (Assumption 2.2.5).

Let $(\Omega, I\!F, \mathbb{P})$ be a probability space, and $\mathcal{P}$ be the predictable σ-algebra on $[0,T]\times\Omega$. Let W be a one-dimensional Brownian motion and N be a Poisson process with intensity $\lambda > 0$. Let $I\!F = \{\mathcal{F}_t, 0 \leq t \leq T\}$ be the natural filtration associated with W and N.

For each $T > 0$, we use the following notations:

- $L^2(\mathcal{F}_T)$ is the set of $\mathcal{F}_T$-measurable and square integrable random variables.
- $I\!H^2$ is the set of real-valued predictable processes ϕ such that $\|\phi\|^2_{I\!H^2} := \mathbb{E}\left[\int_0^T \phi_t^2 dt\right] < \infty$.
- $\mathcal{B}(I\!R^2)$ is the Borelian σ-algebra on $I\!R^2$.
- $\mathcal{S}^2$ is the set of real-valued RCLL adapted processes ϕ such that $\|\phi\|^2_{\mathcal{S}^2} := \mathbb{E}(\sup_{0\leq t\leq T}|\phi_t|^2) < \infty$.
- $\mathcal{A}^2$ is the set of real-valued non decreasing RCLL predictable processes A with $A_0 = 0$ and $\mathbb{E}(A_T^2) < \infty$.

Definition 2.2.1 (Driver, Lipschitz driver)**.** *A function g is said to be a* driver *if*

- $g : \Omega \times [0,T] \times I\!R^3 \to I\!R$
  $(\omega, t, y, z, u) \mapsto g(\omega, t, y, z, u)$ *is* $\mathcal{P} \otimes \mathcal{B}(I\!R^3)$*-measurable,*
- $\|g(., 0, 0, 0)\|_\infty < \infty$.

A driver g is called a Lipschitz driver *if moreover there exists a constant $C_g \geq 0$ and a bounded, non-decreasing continuous function Λ with $\Lambda(0) = 0$ such that $d\mathbb{P}\otimes dt$-a.s. , for each (s_1, y_1, z_1, u_1), (s_2, y_2, z_2, u_2),*

$$|g(\omega, s_1, y_1, z_1, u_1) - g(\omega, s_2, y_2, z_2, u_2)| \leq \Lambda(|s_2 - s_1|) + C_g(|y_1 - y_2| + |z_1 - z_2| + |u_1 - u_2|).$$

Definition 2.2.2 (Mokobodski's condition)**.** *Let ξ, ζ be in $\mathcal{S}^2$. There exist two nonnegative RCLL supermartingales H and H' in $\mathcal{S}^2$ such that*

$$\forall t \in [0,T], \quad \xi_t \mathbf{1}_{t\leq T} \leq H_t - H'_t \leq \zeta_t \mathbf{1}_{t\leq T} \ \ a.s.$$

The following Theorem states existence and uniqueness of solutions to (2.1.1).

Theorem 2.2.3 ([62],Theorem 4.1)**.** *Suppose ξ and ζ are RCLL adapted processes in $\mathcal{S}^2$ such that for all $t \in [0,T]$, $\xi_t \leq \zeta_t$, Mokobodski's condition holds and g is a Lipschitz driver. Then, DRBSDE* (2.1.1) *admits a unique solution (Y, Z, U, α) in $\mathcal{S}^2 \times I\!H^2 \times I\!H^2 \times \mathcal{S}^2$, where $\alpha := A - K$, A and K in $\mathcal{A}^2$.*

Let us now introduce an additional assumption on g, which ensures the comparison theorem for BSDEs with jumps (see [137, Theorem 4.2]). The comparison theorem plays a key role in the proof of the convergence of the penalized scheme (see [59]), which is useful to prove the convergence of the reflected scheme (see Section 2.4).

Assumption 2.2.4. *A lipschitz driver g is said to satisfy Assumption* 2.2.4 *if the following holds : $d\mathbb{P} \otimes dt$ a.s. for each $(y, z, u_1, u_2) \in I\!R^4$, we have*

$$g(t, y, z, u_1) - g(t, y, z, u_2) \geq \theta(u_1 - u_2), \;\; \textit{with} \;\; -1 \leq \theta \leq \theta_0.$$

We also assume the following hypothesis on the barriers.

Assumption 2.2.5. *ξ and ζ are It processes of the following form*

$$\xi_t = \xi_0 + \int_0^t b_s^{\xi} ds + \int_0^t \sigma_{s^-}^{\xi} dW_s + \int_0^t \beta_{s^-}^{\xi} d\tilde{N}_s \tag{2.2.1}$$

$$\zeta_t = \zeta_0 + \int_0^t b_s^{\zeta} ds + \int_0^t \sigma_{s^-}^{\zeta} dW_s + \int_0^t \beta_{s^-}^{\zeta} d\tilde{N}_s \tag{2.2.2}$$

where b^{ξ}, b^{ζ}, σ^{ξ}, σ^{ζ}, β^{ξ} and β^{ζ} are adapted RCLL processes such that there exists $r > 2$ and a constant $C_{\xi,\zeta}$ such that $\mathbb{E}(\sup_{s\leq T} |b_s^{\xi}|^r) + \mathbb{E}(\sup_{s\leq T} |b_s^{\zeta}|^r) + \mathbb{E}(\sup_{s\leq T} |\sigma_s^{\xi}|^r) + \mathbb{E}(\sup_{s\leq T} |\sigma_s^{\zeta}|^r) + \mathbb{E}(\sup_{s\leq T} |\beta_s^{\xi}|^r) + \mathbb{E}(\sup_{s\leq T} |\beta_s^{\zeta}|^r) \leq C_{\xi,\zeta}$. We also assume $\xi_T = \zeta_T$ a.s., $\xi_t \leq \zeta_t$ for all $t \in [0,T]$ and the Mokobodski's condition holds.

2.3 Discrete time framework and numerical scheme

2.3.1 Discrete time framework

For the numerical part of the chapter, we adopt the framework of [109] and [59], presented below.

Random walk approximation of $(W, \tilde{N})$

For $n \in \mathbb{N}$, we introduce $\delta := \frac{T}{n}$ and the regular grid $(t_j)_{j=0,\dots,n}$ with step size δ (i.e. $t_j := j\delta$) to discretize $[0,T]$. In order to approximate W, we introduce the following random walk

$$\begin{cases} W_0^n = 0 \\ W_t^n = \sqrt{\delta} \sum_{i=1}^{[t/\delta]} e_i^n \end{cases} \tag{2.3.1}$$

where $e_1^n, e_2^n, \dots, e_n^n$ are independent identically distributed random variables with the following symmetric Bernoulli law:

$$\mathbb{P}(e_1^n = 1) = \mathbb{P}(e_1^n = -1) = \frac{1}{2}.$$

To approximate $\tilde{N}$, we introduce a second random walk

$$\begin{cases} \tilde{N}_0^n = 0 \\ \tilde{N}_t^n = \sum_{i=1}^{[t/\delta]} \eta_i^n \end{cases} \tag{2.3.2}$$

where $\eta_1^n, \eta_2^n, ..., \eta_n^n$ are independent and identically distributed random variables with law

$$\mathbb{P}(\eta_1^n = \kappa_n - 1) = 1 - \mathbb{P}(\eta_1^n = \kappa_n) = \kappa_n$$

where $\kappa_n = e^{-\lambda\delta}$. We assume that both sequences $e_1^n, ..., e_n^n$ and $\eta_1^n, \eta_2^n, ..., \eta_n^n$ are defined on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The (discrete) filtration in the probability space is $\mathbb{F}^n := \{\mathcal{F}_j^n : j = 0, ..., n\}$ with $\mathcal{F}_0^n = \{\Omega, \emptyset\}$ and $\mathcal{F}_j^n = \sigma\{e_1^n, ..., e_j^n, \eta_1^n, ..., \eta_j^n\}$ for $j = 1, ..., n$.

The following result states the convergence of $(W^n, \tilde{N}^n)$ in the J_1-Skorokhod topology. We refer to [109, Section 3] for more results on the convergence in probability of $\mathcal{F}^n$-martingales .

Lemma 2.3.1 ([109],Lemma3, (III))**.** *The couple $(W^n, \tilde{N}^n)$ converges in probability to $(W, \tilde{N})$ for the J_1-Skorokhod topology.*

We recall that the process ξ^n converges in probability to ξ in the J1-Skorokhod topology if there exists a family $(\psi^n)_{n\in\mathbb{N}}$ of one-to-one random time changes from $[0, T]$ to $[0, T]$ such that $\sup_{t\in[0,T]} |\psi^n(t) - t| \xrightarrow[n\to\infty]{} 0$ almost surely and $\sup_{t\in[0,T]} |\xi^n_{\psi^n(t)} - \xi_t| \xrightarrow[n\to\infty]{} 0$ in probability.

Martingale representation

Let y_{j+1} denote a $\mathcal{F}_{j+1}^n$-measurable random variable. As said in [109], we need a set of three strongly orthogonal martingales to represent the martingale difference $m_{j+1} := y_{j+1} - \mathbb{E}(y_{j+1}|\mathcal{F}_j^n)$. We introduce a third martingale increment sequence $\{\mu_j^n = e_j^n \eta_j^n, j = 0, \cdots, n\}$. In this context there exists a unique triplet (z_j, u_j, v_j) of $\mathcal{F}_j^n$-random variables such that

$$m_{j+1} := y_{j+1} - \mathbb{E}(y_{j+1}|\mathcal{F}_j^n) = \sqrt{\delta} z_j e_{j+1}^n + u_j \eta_{j+1}^n + v_j \mu_{j+1}^n,$$

and

$$\begin{cases} z_j = \dfrac{1}{\sqrt{\delta}} \mathbb{E}(y_{j+1} e_{j+1}^n | \mathcal{F}_j^n), \\ u_j = \dfrac{\mathbb{E}(y_{j+1}\eta_{j+1}^n|\mathcal{F}_j^n)}{\mathbb{E}((\eta_{j+1}^n)^2|\mathcal{F}_j^n)} = \dfrac{1}{\kappa_n(1-\kappa_n)} \mathbb{E}(y_{j+1}\eta_{j+1}^n|\mathcal{F}_j^n), \\ v_j = \dfrac{\mathbb{E}(y_{j+1}\mu_{j+1}^n|\mathcal{F}_j^n)}{\mathbb{E}((\mu_{j+1}^n)^2|\mathcal{F}_j^n)} = \dfrac{1}{\kappa_n(1-\kappa_n)} \mathbb{E}(y_{j+1}\mu_{j+1}^n|\mathcal{F}_j^n) \end{cases} \tag{2.3.3}$$

The computation of conditional expectations is done in the following way:

Remark 2.3.2 (Computing the conditional expectations)**.** Let Φ denote a function from $\mathbb{R}^{2j+2}$ to $\mathbb{R}$. We use the following formula

$$\begin{aligned} \mathbb{E}(\Phi(e_1^n, \cdots, e_{j+1}^n, \eta_1^n, \cdots, \eta_{j+1}^n)|\mathcal{F}_j^n) = & \frac{\kappa_n}{2} \Phi(e_1^n, \cdots, e_j^n, 1, \eta_1^n, \cdots, \eta_j^n, \kappa_n - 1) \\ & + \frac{\kappa_n}{2} \Phi(e_1^n, \cdots, e_j^n, -1, \eta_1^n, \cdots, \eta_j^n, \kappa_n - 1) \\ & + \frac{1-\kappa_n}{2} \Phi(e_1^n, \cdots, e_j^n, 1, \eta_1^n, \cdots, \eta_j^n, \kappa_n) \\ & + \frac{1-\kappa_n}{2} \Phi(e_1^n, \cdots, e_j^n, -1, \eta_1^n, \cdots, \eta_j^n, \kappa_n). \end{aligned}$$

2.3.2 Reflected schemes

The barriers ξ and ζ given in Assumption 2.2.5 are approximated in the following way: for all $k \in \{1, \cdots, n\}$

$$\xi_k^n = \xi_0 + \sum_{i=0}^{k-1} b_{t_i}^{\xi} \delta + \sum_{i=0}^{k-1} \sigma_{t_i}^{\xi} \sqrt{\delta} e_{i+1}^n + \sum_{i=0}^{k-1} \beta_{t_i}^{\xi} \eta_{i+1}^n \tag{2.3.4}$$

$$\zeta_k^n = \zeta_0 + \sum_{i=0}^{k-1} b_{t_i}^{\zeta} \delta + \sum_{i=0}^{k-1} \sigma_{t_i}^{\zeta} \sqrt{\delta} e_{i+1}^n + \sum_{i=0}^{k-1} \beta_{t_i}^{\zeta} \eta_{i+1}^n \tag{2.3.5}$$

Lemma 2.3.3. *Under Assumption 2.2.5, there exists a constant $C_{\xi,\zeta,T,\lambda}$ depending on $C_{\xi,\zeta}$, T and λ such that*

(i) $\sup_n \max_j \mathbb{E}(|\xi_j^n|^r) + \sup_n \max_j \mathbb{E}(|\zeta_j^n|^r) + \sup_{t \le T} \mathbb{E}(|\xi_t|^r) + \sup_{t \le T} \mathbb{E}(|\zeta_t|^r) \le C_{\xi,\zeta,T,\lambda}$

(ii) ξ^n (resp. ζ^n) converges in probability to ξ (resp. ζ) in J1-Skorokhod topology.

Proof. (i) ensues from Burkholder-Davis-Gundy and Rosenthal inequalities, and (ii) ensues from [96, Theorem 6.22 and Corollary 6.29]. $\square$

Implicit reflected scheme

After the discretization of the time interval, our discrete reflected BSDEs with two RCLL barriers on small interval $[t_j, t_{j+1}[$, for $0 \le j \le n-1$ is

$$\begin{cases} y_j^n = y_{j+1}^n + g(t_j, y_j^n, z_j^n, u_j^n)\delta + a_j^n - k_j^n - z_j^n \sqrt{\delta} \varepsilon_{j+1}^n - u_j^n \eta_{j+1}^n - v_j^n \mu_{j+1}^n \\ a_j^n \ge 0,\ k_j^n \ge 0,\ a_j^n k_j^n = 0,\ \xi_j^n \le y_j^n \le \zeta_j^n,\ (y_j^n - \xi_j^n) a_j^n = (y_j^n - \zeta_j^n) k_j^n = 0. \end{cases} \tag{2.3.6}$$

with terminal condition $y_n^n = \xi_n^n$. By taking the conditional expectation in (2.3.6) w.r.t. $\mathcal{F}_j^n$, we get

$$(\mathcal{S}_1) \begin{cases} y_n^n = \xi_n^n, \\ y_j^n = \mathbb{E}[y_{j+1}^n | \mathcal{F}_j^n] + g(t_j, y_j^n, z_j^n, u_j^n)\delta + a_j^n - k_j^n, \\ a_j^n \ge 0,\ k_j^n \ge 0,\ a_j^n k_j^n = 0, \\ \xi_j^n \le y_j^n \le \zeta_j^n,\ (y_j^n - \xi_j^n) a_j^n = (y_j^n - \zeta_j^n) k_j^n = 0. \end{cases}$$

Lemma 2.3.4. *For δ small enough, $(\mathcal{S}_1)$ is equivalent to*

$$(\mathcal{S}_2) \begin{cases} y_n^n = \xi_n^n, \\ y_j^n = \Psi^{-1}(\mathbb{E}[y_{j+1}^n | \mathcal{F}_j^n] + a_j^n - k_j^n), \\ a_j^n = (\mathbb{E}[y_{j+1}^n | \mathcal{F}_j^n] + g(t_j, \xi_j^n, z_j^n, u_j^n)\delta - \xi_j^n)^-. \\ k_j^n = (\mathbb{E}[y_{j+1}^n | \mathcal{F}_j^n] + g(t_j, \zeta_j^n, z_j^n, u_j^n)\delta - \zeta_j^n)^+. \end{cases}$$

where $\Psi(y) := y - g(t_j, y, z_j^n, u_j^n)\delta$.

Proof. For δ small enough, Ψ is invertible because the Lipschitz property of g leads to $(\Psi(y) - \Psi(y'))(y-y') \geq (1-\delta C_g)(y-y')^2 \geq 0$.
We first prove that $(\mathcal{S}_1)$ implies $(\mathcal{S}_2)$. Let us firstly assume that $\forall j \leq n-1$ $\xi_j^n < \zeta_j^n$. On the set $\{y_j^n = \xi_j^n\}$ we have $k_j^n = 0$, then $a_j^n = \Psi(\xi_j^n) - \mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n] = (\mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n] - \Psi(\xi_j^n))^-$ (since $\mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n] - \Psi(\xi_j^n) = \Psi(y_j^n) - \Psi(\xi_j^n) - a_j^n \leq 0$) and on $\{y_j^n > \xi_j^n\}$ we have $a_j^n = 0$, $(\mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n] - \Psi(\xi_j^n) = \Psi(y_j^n) - \Psi(\xi_j^n) + k_j^n > 0$ (thanks to the monotonicity of Ψ)). Then, $a_j^n = (\mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n] - \Psi(\xi_j^n))^-$. The same type of proof leads to the fourth line of $(\mathcal{S}_2)$. If there exists $j \leq n-1$ such that $\xi_j^n = \zeta_j^n$, we get $\xi_j^n = \zeta_j^n = y_j^n$. Then, we have $a_j^n = 0$ or $k_j^n = 0$. If both are null, we get $\Psi(y_j^n) = \mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n] = \Psi(\xi_j^n) = \Psi(\zeta_j^n)$. This coincides with the definitions of a_j^n and k_j^n given in $(\mathcal{S}_2)$. If $a_j^n > 0$, $k_j^n = 0$ and we get $a_j^n = \Psi(y_j^n) - \mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n] = \Psi(\xi_j^n) - \mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n]$, then $a_j^n = (\mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n] - \Psi(\xi_j^n))^-$. Conversely, assume $(\mathcal{S}_2)$, let us prove $a_j^n k_j^n = 0$, $(y_j^n - \xi_j^n)a_j^n = (y_j^n - \zeta_j^n)k_j^n = 0$ and $\xi_j^n \leq y_j^n \leq \zeta_j^n$. If $a_j^n > 0$, we get $\Psi(\zeta_j^n) \geq \Psi(\xi_j^n) > \mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n]$, then $k_j^n = 0$. Let us prove that $(y_j^n - \xi_j^n)a_j^n = 0$. If $a_j^n > 0$, $\Psi(y_j^n) = \mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n] + a_j^n = \Psi(\xi_j^n)$. Since Ψ is a one to one map, we get $y_j^n = \xi_j^n$. The same argument holds to prove $(y_j^n - \zeta_j^n)k_j^n = 0$. Let us prove that $\xi_j^n \leq y_j^n$. To do so, assume that $y_j^n < \xi_j^n$. In this case $a_j^n = k_j^n = 0$, which gives $\Psi(\xi_j^n) \leq \mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n]$, by definition of a_j^n. Then $\Psi(y_j^n) = \mathbb{E}[y_{j+1}^n|\mathcal{F}_j^n] \geq \Psi(\xi_j^n)$. Ψ being a non decreasing function, this leads to absurdity. □

We also introduce the continuous time version $(Y_t^n, Z_t^n, U_t^n, A_t^n, K_t^n)_{0\leq t\leq T}$ of $(y_j^n, z_j^n, u_j^n, a_j^n, k_j^n)_{j\leq n}$:

$$Y_t^n := y_{[t/\delta]}^n, Z_t^n := z_{[t/\delta]}^n, U_t^n := u_{[t/\delta]}^n, A_t^n := \sum_{i=0}^{[t/\delta]} a_i^n, K_t^n := \sum_{i=0}^{[t/\delta]} k_i^n. \tag{2.3.7}$$

In the following $\Theta^n := (Y^n, Z^n, U^n, A^n - K^n)$.

Explicit reflected scheme

The explicit reflected scheme is introduced by replacing y_j^n by $\mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n]$ in g. We obtain

$$\begin{cases} \overline{y}_j^n = \overline{y}_{j+1}^n + g(t_j, \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n], \overline{z}_j^n, \overline{u}_j^n)\delta + \overline{a}_j^n - \overline{k}_j^n - \overline{z}_j^n\sqrt{\delta}\varepsilon_{j+1}^n - \overline{u}_j^n\eta_{j+1}^n - \overline{v}_j^n\mu_{j+1}^n \\ \overline{a}_j^n \geq 0,\ \overline{k}_j^n \geq 0,\ \overline{a}_j^n\overline{k}_j^n = 0, \\ \xi_j^n \leq \overline{y}_j^n \leq \zeta_j^n,\ (\overline{y}_j^n - \xi_j^n)\overline{a}_j^n = (\overline{y}_j^n - \zeta_j^n)\overline{k}_j^n = 0. \end{cases} \tag{2.3.8}$$

with terminal condition $\overline{y}_n^n = \xi_n^n$. By taking the conditional expectation in (2.3.8) with respect to $\mathcal{F}_j^n$, we derive that:

$$(\overline{\mathcal{S}}_1) \begin{cases} \overline{y}_n^n = \xi_n^n, \\ \overline{y}_j^n = \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n] + g(t_j, \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n], \overline{z}_j^n, \overline{u}_j^n)\delta + \overline{a}_j^n - \overline{k}_j^n \\ \overline{a}_j^n \geq 0,\ \overline{k}_j^n \geq 0,\ \overline{a}_j^n\overline{k}_j^n = 0, \\ \xi_j^n \leq \overline{y}_j^n \leq \zeta_j^n,\ (\overline{y}_j^n - \xi_j^n)\overline{a}_j^n = (\overline{y}_j^n - \zeta_j^n)\overline{k}_j^n = 0. \end{cases}$$

As for the implicit reflected scheme, we get that $(\overline{\mathcal{S}}_1)$ is equivalent to $(\overline{\mathcal{S}}_2)$

$$(\overline{\mathcal{S}}_2) \begin{cases} \overline{y}_n^n = \xi_n^n, \\ \overline{y}_j^n = \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n] + g(t_j, \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n], \overline{z}_j^n, \overline{u}_j^n)\delta + \overline{a}_j^n - \overline{k}_j^n, \\ \overline{a}_j^n = (\mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n] + g(t_j, \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n], \overline{z}_j^n, \overline{u}_j^n)\delta - \xi_j^n)^-. \\ \overline{k}_j^n = (\mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n] + g(t_j, \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n], \overline{z}_j^n, \overline{u}_j^n)\delta - \zeta_j^n)^+. \end{cases}$$

We also introduce the continuous time version $(\overline{Y}_t^n, \overline{Z}_t^n, \overline{U}_t^n, \overline{A}_t^n, \overline{K}_t^n)_{0\leq t\leq T}$ of $(\overline{y}_j^n, \overline{z}_j^n, \overline{u}_j^n, \overline{a}_j^n, \overline{k}_j^n)_{j\leq n}$:

$$\overline{Y}_t^n := \overline{y}_{[t/\delta]}^n, \overline{Z}_t^n := \overline{z}_{[t/\delta]}^n, \overline{U}_t^n := \overline{u}_{[t/\delta]}^n, \overline{A}_t^n := \sum_{i=0}^{[t/\delta]} \overline{a}_i^n, \overline{K}_t^n := \sum_{i=0}^{[t/\delta]} \overline{k}_i^n. \tag{2.3.9}$$

In the following $\overline{\Theta}^n := (\overline{Y}^n, \overline{Z}^n, \overline{U}^n, \overline{A}^n - \overline{K}^n)$.

2.4 Convergence result

We prove in this Section that $\overline{\Theta}^n$ to converges to $\Theta := (Y_t, Z_t, U_t, A_t - K_t)_{0\leq t\leq T}$, the solution to the DRBSDE (2.1.1). The main result is stated in the following Theorem.

Theorem 2.4.1. *Suppose that Assumption 2.2.5 holds and g is a Lipschitz driver satisfying Assumption 2.2.4. Then we have*

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T |\overline{Y}_t^n - Y_t|^2 dt + \int_0^T |\overline{Z}_t^n - Z_t|^2 dt + \int_0^T |\overline{U}_t^n - U_t|^2 dt\right] = 0.$$

Moreover, $\overline{\alpha}_{\psi^n(t)}^n$ converges weakly to α_t in $L^2(\mathcal{F}_T)$.

Proof. To prove this result, we split the error in three terms. The first one is the error $\overline{\Theta}^n - \Theta^n$, the second one is $\Theta^n - \Theta^{p,n}$, where $\Theta^{p,n} := (Y^{p,n}, Z^{p,n}, U^{p,n}, A^{p,n} - K^{p,n})$ represents the solution given by the implicit penalization scheme (see (2.4.3)), and the third error term is $\Theta^{p,n} - \Theta$, whose convergence has already been proved in [59]. The result on the convergence of $\Theta^{p,n}$ to Θ is recalled in Theorem 2.4.3.

We have the following inequality for the error on Y (the same inequality holds for the errors on Z and U)

$$\mathbb{E}[\int_0^T |\overline{Y}_t^n - Y_t|^2 dt] \leq 3\mathbb{E}[\int_0^T |\overline{Y}_t^n - Y_t^n|^2 dt] + 3\mathbb{E}[\int_0^T |Y_t^n - Y_t^{p,n}|^2 dt] + 3[\int_0^T |Y_t^{p,n} - Y_t|^2 dt]$$

For the increasing processes, we have:

$$\mathbb{E}[|\overline{\alpha}_{\psi^n(t)}^n - \alpha_t|^2] \leq 3\left(E[|\overline{\alpha}_{\psi^n(t)}^n - \alpha_{\psi^n(t)}^n|^2] + \mathbb{E}[|\alpha_{\psi^n(t)}^n - \alpha_t^{p,n}|^2] + \mathbb{E}[|\alpha_t^{p,n} - \alpha_t|^2]\right) \tag{2.4.1}$$

Then, combining Propositions 2.4.6, 2.4.7 and Theorem 2.4.3 yields the result. $\square$

Definition 2.4.2 (Definition of c and N_0)**.** *In this Section and in the Appendix, c denotes a generic constant depending on C_g, $\|g(\cdot,0,0,0)\|_\infty$ and $C_{\xi,\zeta,\lambda,T}$. N_0 is defined by $N_0 := 4T(1+C_g+C_g^2+C_g^2\frac{e^{2\lambda T}}{\lambda})$.*

The rest of the Section is organized as follows: Section 2.4.1 recalls the implicit penalization scheme introduced in [59] and the convergence of $\Theta^{p,n}-\Theta$, we give some intermediate results in Section 2.4.2 and we prove the convergence of $\overline{\Theta}^n-\Theta^n$ (see Proposition 2.4.6) and the convergence of $\Theta^n-\Theta^{p,n}$ (see Proposition 2.4.7) in Section 2.4.3.

2.4.1 Implicit penalization scheme

In this Section we recall the *implicit penalization scheme* introduced in [59]. For all j in $\{0,\cdots,n-1\}$ we have

$$\begin{cases} y_j^{p,n} = y_{j+1}^{p,n} + g(t_j, y_j^{p,n}, z_j^{p,n}, u_j^{p,n})\delta + a_j^{p,n} - k_j^{p,n} - (z_j^{p,n}\sqrt{\delta}e_{j+1}^n + u_j^{p,n}\eta_{j+1}^n + v_j^{p,n}\mu_{j+1}^n) \\ a_j^{p,n} = p\delta(y_j^{p,n}-\xi_j^n)^-;\ k_j^{p,n} = p\delta(\zeta_j^n - y_j^{p,n})^-, \\ y_n^{p,n} := \xi_n^n. \end{cases} \tag{2.4.2}$$

Following (2.3.3), the triplet $(z_j^{p,n}, u_j^{p,n}, v_j^{p,n})$ can be computed as follows

$$\begin{cases} z_j^{p,n} = \dfrac{1}{\sqrt{\delta}}\mathbb{E}(y_{j+1}^{p,n}e_{j+1}^n|\mathcal{F}_j^n), \\ u_j^{p,n} = \dfrac{1}{\kappa_n(1-\kappa_n)}\mathbb{E}(y_{j+1}^{p,n}\eta_{j+1}^n|\mathcal{F}_j^n), \\ v_j^{p,n} = \dfrac{1}{\kappa_n(1-\kappa_n)}\mathbb{E}(y_{j+1}^{p,n}\mu_{j+1}^n|\mathcal{F}_j^n). \end{cases}$$

Taking the conditional expectation w.r.t. $\mathcal{F}_j^n$ in (2.4.2), we get

$$\begin{cases} y_j^{p,n} = (\Psi^{p,n})^{-1}(\mathbb{E}(y_{j+1}^{p,n}|\mathcal{F}_j^n)), \\ a_j^{p,n} = p\delta(y_j^{p,n}-\xi_j^n)^-;\ k_j^{p,n} = p\delta(\zeta_j^n - y_j^{p,n})^-, \\ z_j^{p,n} = \dfrac{1}{\sqrt{\delta}}\mathbb{E}(y_{j+1}^{p,n}e_{j+1}^n|\mathcal{F}_j^n), \\ u_j^{p,n} = \dfrac{1}{\kappa_n(1-\kappa_n)}\mathbb{E}(y_{j+1}^{p,n}\eta_{j+1}^n|\mathcal{F}_j^n), \end{cases}$$

where $\Psi^{p,n}(y) = y - g(j\delta, y, z_j^{p,n}, u_j^{p,n})\delta - p\delta(y-\xi_j^n)^- + p\delta(\zeta_j^n - y)^-$.

We also introduce the continuous time version $(Y_t^{p,n}, Z_t^{p,n}, U_t^{p,n}, A_t^{p,n}, K_t^{p,n})_{0\le t\le T}$ of the solution of the discrete equation (2.4.2):

$$Y_t^{p,n} := y_{[t/\delta]}^{p,n}, Z_t^{p,n} := z_{[t/\delta]}^{p,n}, U_t^{p,n} := u_{[t/\delta]}^{p,n}, A_t^{p,n} := \sum_{i=0}^{[t/\delta]} a_i^{p,n}, K_t^{p,n} := \sum_{i=0}^{[t/\delta]} k_i^{p,n}, \tag{2.4.3}$$

and $\alpha^{p,n} := A^{p,n} - K^{p,n}$. The following result ensues from [59, Theorem 4.1 and Proposition 4.2].

Theorem 2.4.3. *Assume that Assumption 2.2.5 holds and g is a Lipschitz driver satisfying Assumption 2.2.4. The sequence $(Y^{p,n}, Z^{p,n}, U^{p,n})$ defined by (2.4.3) converges to (Y, Z, U), the solution of the DRBSDE (2.1.1), in the following sense: $\forall r \in [1, 2[$*

$$\lim_{p\to\infty}\lim_{n\to\infty}\left(\mathbb{E}\left[\int_0^T |Y_s^{p,n} - Y_s|^2 ds\right] + \mathbb{E}\left[\int_0^T |Z_s^{p,n} - Z_s|^r ds\right]\right.$$
$$\left. + \mathbb{E}\left[\int_0^T |U_s^{p,n} - U_s|^r ds\right]\right) = 0. \tag{2.4.4}$$

Moreover, $Z^{p,n}$ (resp. $U^{p,n}$) weakly converges in $\mathbb{H}^2$ to Z (resp. to U) and for $0 \le t \le T$, $\alpha^{p,n}_{\psi^n(t)}$ converges weakly to α_t in $L^2(\mathcal{F}_T)$ as $n \to \infty$ and $p \to \infty$, where $(\psi^n)_{n\in\mathbb{N}}$ is a one-to-one random map from $[0, T]$ to $[0, T]$ such that $\sup_{t\in[0,T]} |\psi^n(t) - t| \xrightarrow[n\to\infty]{} 0$ a.s..

2.4.2 Intermediate results

In this Section we state two intermediate results useful for Section 2.4.3.

Lemma 2.4.4. *Under Assumption 2.2.5 we have*

$$\sup_j \mathbb{E}[|y_j^n|^2] + \mathbb{E}\left[\delta \sum_{j=0}^{n-1} |z_j^n|^2 + \kappa_n(1-\kappa_n)\sum_{j=0}^{n-1} |u_j^n|^2 + \frac{1}{\delta}\sum_{j=0}^{n-1} |a_j^n|^2 + \frac{1}{\delta}\sum_{j=0}^{n-1} |k_j^n|^2\right] \le c.$$

Proof. Since $\xi_j^n \le y_j^n \le \zeta_j^n$, Assumption 2.2.5 gives $\sup_j \mathbb{E}(|y_j^n|^2) \le c$. Let us deal with z_j^n and u_j^n. We apply the discrete Itô's formula and we get:

$$\begin{aligned}
&\mathbb{E}[|y_j^n|^2] + \delta\sum_{i=j}^{n-1}\mathbb{E}[|z_i^n|^2] + \kappa_n(1-\kappa_n)\sum_{i=j}^{n-1}\mathbb{E}[|u_i^n|^2] \\
\le&\mathbb{E}[|\xi_n^n|^2] + 2\delta\sum_{i=j}^{n-1}\mathbb{E}[y_i^n g(t_i, y_i^n, z_i^n, u_i^n)] + 2\sum_{i=j}^{n-1}\mathbb{E}[y_i^n a_i^n] - 2\sum_{i=j}^{n-1}\mathbb{E}[y_i^n k_i^n] \\
\le&\mathbb{E}[|\xi_n^n|^2] + \delta\sum_{i=j}^{n-1} g(t_i,0,0,0)^2 + \delta\left(1 + 2C_g + 2C_g^2 + \frac{2C_g^2\delta}{\kappa_n(1-\kappa_n)}\right)\sum_{i=j}^{n-1}\mathbb{E}[|y_i^n|^2] \\
&+ \frac{\delta}{2}\sum_{i=j}^{n-1}\mathbb{E}[|z_i^n|^2] + \frac{\kappa_n(1-\kappa_n)}{2}\sum_{i=j}^{n-1}\mathbb{E}[|u_i^n|^2] + \frac{2\delta}{\alpha}\sum_{i=j}^{n-1}\mathbb{E}(|y_i^n|^2) + \frac{\alpha}{\delta}\sum_{i=j}^{n-1}\mathbb{E}(|a_i^n|^2) + \frac{\alpha}{\delta}\sum_{i=j}^{n-1}\mathbb{E}(|k_i^n|^2).
\end{aligned}$$

Since $\xi_i^n \le y_i^n \le \zeta_i^n$, we get

$$\begin{aligned}
a_i^n &\le \left(\mathbb{E}(\xi_{i+1}^n|\mathcal{G}_i^n) + \delta g(t_i, \xi_i^n, z_i^n, u_i^n) - \xi_i^n\right)^- = \delta(b_{t_i}^\xi + g(t_i, \xi_i^n, z_i^n, u_i^n))^-, \\
k_i^n &\le \left(\mathbb{E}(\zeta_{i+1}^n|\mathcal{G}_i^n) + \delta g(t_i, \zeta_i^n, z_i^n, u_i^n) - \zeta_i^n\right)^+ = \delta(b_{t_i}^\zeta + g(t_i, \zeta_i^n, z_i^n, u_i^n))^+.
\end{aligned} \tag{2.4.5}$$

Then, using the Lipschitz property of g gives

$$\frac{\alpha}{\delta}\sum_{i=j}^{n-1}\mathbb{E}(|a_i^n|^2) \le 5\alpha\delta\sum_{i=j}^{n-1}\mathbb{E}[|b_i^\xi|^2 + |g(t_i,0,0,0)|^2 + C_g^2(|\xi_i^n|^2 + |z_i^n|^2 + |u_i^n|^2)] \tag{2.4.6}$$

and the same result holds for $\frac{\alpha}{\delta}\sum_{i=j}^{n-1}\mathbb{E}(|k_i^n|^2)$. By Using Assumption 2.2.5 and the inequality $\sup_i \mathbb{E}(|y_i^n|^2) \leq c$, we get

$$\delta\sum_{i=j}^{n-1}\mathbb{E}[|z_i^n|^2] + \kappa_n(1-\kappa_n)\sum_{i=j}^{n-1}\mathbb{E}[|u_i^n|^2] \leq c + \delta\left(\frac{1}{2} + 10\alpha C_g^2\right)\sum_{i=j}^{n-1}\mathbb{E}(|z_i^n|^2)$$
$$+ \kappa_n(1-\kappa_n)\left(\frac{1}{2} + 10\alpha C_g^2\frac{\delta}{\kappa_n(1-\kappa_n)}\right)\sum_{i=j}^{n-1}\mathbb{E}(|u_i^n|^2)$$

Since $\frac{\delta}{(1-\kappa_n)\kappa_n} = \frac{1}{\lambda}\frac{\lambda\delta}{(1-e^{-\lambda\delta})e^{-\lambda\delta}}$ and $e^x \leq \frac{xe^{2x}}{e^x-1} \leq e^{2x}$, we get $\frac{\delta}{(1-\kappa_n)\kappa_n} \leq \frac{1}{\lambda}e^{2\lambda T}$. Then, by taking $\alpha = \frac{1}{40C_g^2}(\lambda e^{-2\lambda T} \wedge 1)$, we get $\delta\sum_{i=j}^{n-1}\mathbb{E}[|z_i^n|^2] + \kappa_n(1-\kappa_n)\sum_{i=j}^{n-1}\mathbb{E}[|u_i^n|^2] \leq c$. Plugging this result in (2.4.6) ends the proof. $\square$

The same type of proof gives the following Lemma

Lemma 2.4.5. *Under Assumption 2.2.5, we have*

$$\sup_j \mathbb{E}[|\overline{y}_j^n|^2] + \mathbb{E}\left[\delta\sum_{j=0}^{n-1}|\overline{z}_j^n|^2 + \kappa_n(1-\kappa_n)\sum_{j=0}^{n-1}|\overline{u}_j^n|^2 + \frac{1}{\delta}\sum_{j=0}^{n-1}|\overline{a}_j^n|^2 + \frac{1}{\delta}\sum_{j=0}^{n-1}|\overline{k}_j^n|^2\right] \leq c.$$

2.4.3 Proof of the convergence of $\overline{\Theta}^n - \Theta^n$ and $\Theta^n - \Theta^{p,n}$

Proposition 2.4.6. *Assume that Assumption 2.2.5 holds and g is a Lipschitz driver. We have*

$$\lim_{n\to\infty}\sup_{0\leq t\leq T}\mathbb{E}[|\overline{Y}_t^n - Y_t^n|^2] + \mathbb{E}[\int_0^T|\overline{Z}_s^n - Z_s^n|^2ds] + \mathbb{E}[\int_0^T|\overline{U}_s^n - U_s^n|^2ds] = 0. \tag{2.4.7}$$

Moreover, $\lim_{n\to\infty}(\overline{\alpha}_t^n - \alpha_t^n) = 0$ *in* $L^2(\mathcal{F}_t)$*, for* $t \in [0,T]$*.*

Proof. Let us consider y_j^n, the solution of the discrete implicit reflected sheme (2.3.6) and $\overline{y}_j^n$, the solution of the explicit reflected scheme (2.3.8). We compute $|y_j^n - \overline{y}_j^n|^2$, we take the expectation and we get:

$$\begin{aligned}\mathbb{E}[|y_j^n - \overline{y}_j^n|^2] \leq& \mathbb{E}[|y_{j+1}^n - \overline{y}_{j+1}^n|^2] - \delta\mathbb{E}[|z_j^n - \overline{z}_j^n|^2] - \kappa_n(1-\kappa_n)\mathbb{E}[|u_j^n - \overline{u}_j^n|^2]\\
&+ 2\delta\mathbb{E}[(y_j^n - \overline{y}_j^n)(g(t_j, y_j^n, z_j^n, u_j^n) - g(t_j, \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n], \overline{z}_j^n, \overline{u}_j^n))]\\
&- \mathbb{E}\left[\delta(g(t_j, y_j^n, z_j^n, u_j^n) - g(t_j, \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n], \overline{z}_j^n, \overline{u}_j^n)) + (a_j^n - \overline{a}_j^n) - (k_j^n - \overline{k}_j^n)\right]^2\\
&+ 2\mathbb{E}[(y_j^n - \overline{y}_j^n)(a_j^n - \overline{a}_j^n)] - 2\mathbb{E}[(y_j^n - \overline{y}_j^n)(k_j^n - \overline{k}_j^n)]\\
&\leq \mathbb{E}[|y_{j+1}^n - \overline{y}_{j+1}^n|^2] - \delta\mathbb{E}[|z_j^n - \overline{z}_j^n|^2] - \kappa_n(1-\kappa_n)\mathbb{E}[|u_j^n - \overline{u}_j^n|^2]\\
&+ 2\delta\mathbb{E}[(y_j^n - \overline{y}_j^n)(g(t_j, y_j^n, z_j^n, u_j^n) - g(t_j, \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n], \overline{z}_j^n, \overline{u}_j^n))].\end{aligned}$$

The last inequality comes from $(y_j^n - \overline{y}_j^n)(a_j^n - \overline{a}_j^n) \leq 0$ and $(y_j^n - \overline{y}_j^n)(k_j^n - \overline{k}_j^n) \geq 0$ (this ensues from the third and fourth lines of $(\mathcal{S}_1)$ and $(\overline{\mathcal{S}}_1)$). Taking the sum from $j = i$ to $n-1$ we get

$$\begin{aligned}
&\mathbb{E}[|y_i^n - \overline{y}_i^n|^2] + \delta \sum_{j=i}^{n-1} \mathbb{E}[|z_j^n - \overline{z}_j^n|^2] + \kappa_n(1-\kappa_n) \sum_{j=i}^{n-1} \mathbb{E}[|u_j^n - \overline{u}_j^n|^2] \\
&\leq 2\delta \sum_{j=i}^{n-1} \mathbb{E}[(y_j^n - \overline{y}_j^n)(g(t_j, y_j^n, z_j^n, u_j^n) - g(t_j, \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n], \overline{z}_j^n, \overline{u}_j^n))] \\
&\leq 2\delta C_g \sum_{j=i}^{n-1} \mathbb{E}\left[|y_j^n - \overline{y}_j^n||y_j^n - \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n]|\right] + 2\delta C_g^2 \left(1 + \frac{\delta}{\kappa_n(1-\kappa_n)}\right) \sum_{j=i}^{n-1} \mathbb{E}[|y_j^n - \overline{y}_j^n|^2] \\
&+ \frac{\delta}{2} \sum_{j=i}^{n-1} \mathbb{E}[|z_j^n - \overline{z}_j^n|^2] + \frac{\kappa_n(1-\kappa_n)}{2} \sum_{j=i}^{n-1} \mathbb{E}[|u_j^n - \overline{u}_j^n|^2]. \qquad (2.4.8)
\end{aligned}$$

Since $y_j^n - \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n] = y_j^n - \overline{y}_j^n + \overline{y}_j^n - \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n] = y_j^n - \overline{y}_j^n + \delta g(t_j, \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n], \overline{z}_j^n, \overline{u}_j^n) + \overline{a}_j^n - \overline{k}_j^n$, we get

$$\begin{aligned}
2\delta C_g \mathbb{E}\left[|y_j^n - \overline{y}_j^n||y_j^n - \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n]|\right] &\leq (2C_g+1)\delta \mathbb{E}[|y_j^n - \overline{y}_j^n|^2] \\
&+ C_g^2 \delta \mathbb{E}\left[\left(|\delta g(t_j, \mathbb{E}[\overline{y}_{j+1}^n|\mathcal{F}_j^n], \overline{z}_j^n, \overline{u}_j^n)| + |\overline{a}_j^n| + |\overline{k}_j^n|\right)^2\right].
\end{aligned}$$

Plugging the previous inequality in (2.4.8) and using Lemma 2.4.5 gives

$$\begin{aligned}
&\mathbb{E}[|y_i^n - \overline{y}_i^n|^2] + \frac{\delta}{2} \sum_{j=i}^{n-1} \mathbb{E}[|z_j^n - \overline{z}_j^n|^2] + \frac{\kappa_n(1-\kappa_n)}{2} \sum_{j=i}^{n-1} \mathbb{E}[|u_j^n - \overline{u}_j^n|^2] \\
&\qquad \leq \left(1 + 2C_g + 2C_g^2 + \frac{2C_g^2\delta}{\kappa_n(1-\kappa_n)}\right) \delta \sum_{j=i}^{n-1} \mathbb{E}[|y_j^n - \overline{y}_j^n|^2] + c\delta^2.
\end{aligned}$$

Let n be bigger than N_0, then $\delta(1 + 2C_g + 2C_g^2 + \frac{2\delta C_g^2 \delta}{\kappa_n(1-\kappa_n)}) < 1$ (for all $n \geq 1$ we have $\frac{\delta}{\kappa_n(1-\kappa_n)} \leq \frac{1}{\lambda} e^{2\lambda T}$).

The assumption on δ enables to apply Gronwall's Lemma to get $\sup_{0 \leq i \leq n} \mathbb{E}[|y_i^n - \overline{y}_i^n|^2] \leq c\delta^2$. Plugging this result in the previous inequality leads to (2.4.7). The convergence of $(A^n - K^n) - (\overline{A}^n - \overline{K}^n)$ ensues from

$$\begin{aligned}
A_t^n - K_t^n &= Y_0^n - Y_t^n - \int_0^t g(s, Y_s^n, Z_s^n, U_s^n) ds + \int_0^t Z_s^n dW_s^n + \int_0^t U_s^n d\tilde{N}_s^n, \\
\overline{A}_t^n - \overline{K}_t^n &= \overline{Y}_0^n - \overline{Y}_t^n - \int_0^t g(s, \overline{Y}_s^n, \overline{Z}_s^n, \overline{U}_s^n) ds + \int_0^t \overline{Z}_s^n dW_s^n + \int_0^t \overline{U}_s^n d\tilde{N}_s^n,
\end{aligned}$$

from the Lipschitz property of g and from (2.4.7). □

Proposition 2.4.7. *Assume that Assumption 2.2.5 holds and g is a Lipschitz driver. For $n \geq N_0$, we get*

$$\sup_{0 \leq t \leq T} \mathbb{E}[|Y_t^n - Y_t^{p,n}|^2] + \mathbb{E}[\int_0^T |Z_s^n - Z_s^{p,n}|^2 ds] + \mathbb{E}[\int_0^T |U_s^n - U_s^{p,n}|^2 ds] \leq \frac{c}{\sqrt{p}}. \qquad (2.4.9)$$

Moreover, $\forall\, t \in [0,T]$, $\mathbb{E}[|\alpha_t^n - \alpha_t^{p,n}|^2] \leq \frac{c}{\sqrt{p}}$.

Proof. Let us first prove (2.4.9). From (2.3.6), (2.4.2) and the discrete Itô's formula applied to $(y_j^n - y_j^{p,n})^2$, we get

$$\begin{aligned}
\mathbb{E}|y_j^n - y_j^{p,n}|^2 + \delta \sum_{i=j}^{n-1} \mathbb{E}|z_i^n - z_i^{p,n}|^2 + (1-\kappa_n)\kappa_n \sum_{i=j}^{n-1} \mathbb{E}[|u_i^n - u_i^{p,n}|^2] + (1-\kappa_n)\kappa_n \sum_{i=j}^{n-1} \mathbb{E}[|v_i^n - v_i^{p,n}|^2] \\
= 2\sum_{i=j}^{n-1} \mathbb{E}[(y_i^n - y_i^{p,n})(g(t_i, y_i^n, z_i^n, u_i^n) - g(t_i, y_i^{p,n}, z_i^{p,n}, u_i^{p,n}))\delta] \\
+ 2\sum_{i=j}^{n-1} \mathbb{E}[(y_i^n - y_i^{p,n})(a_i^n - a_i^{p,n})] - 2\sum_{i=j}^{n-1} \mathbb{E}[(y_i^n - y_i^{p,n})(k_i^n - k_i^{p,n})].
\end{aligned}$$

Let us deal with the last two terms $(y_i^n - y_i^{p,n})(a_i^n - a_i^{p,n}) = (y_i^n - \xi_i^n)a_i^n - (y_i^{p,n} - \xi_i^n)a_i^n - (y_i^n - \xi_i^n)a_i^{p,n} + (y_i^{p,n} - \xi_i^n)a_i^{p,n} \leq (y_i^{p,n} - \xi_i^n)^- a_i^n$. By using same computations, we derive $(y_i^n - y_i^{p,n})(k_i^n - k_i^{p,n}) \geq -(y_i^{p,n} - \zeta_i^n)^+ k_i^n$.

By using the Lipschitz property of g, we get

$$\begin{aligned}
&\mathbb{E}[|y_j^n - y_j^{p,n}|^2] + \frac{1}{2}\delta \mathbb{E}[|z_j^n - z_j^{p,n}|^2] + \frac{\kappa_n(1-\kappa_n)}{2}\mathbb{E}[|u_j^n - u_j^{p,n}|^2] \\
&\leq \left(2C_g + 2C_g^2 + \frac{2C_g^2\delta}{\kappa_n(1-\kappa_n)}\right)\delta \sum_{i=j}^{n-1} \mathbb{E}[(y_i^n - y_i^{p,n})^2] + 2\sum_{i=j}^{n-1} \mathbb{E}[(y_i^{p,n} - \xi_i^n)^- a_i^n + (y_i^{p,n} - \zeta_i^n)^+ k_i^n].
\end{aligned}$$

Using Cauchy-Schwarz inequality gives

$$\begin{aligned}
&\mathbb{E}[|y_j^n - y_j^{p,n}|^2] + \frac{1}{2}\delta \mathbb{E}[|z_j^n - z_j^{p,n}|^2] + \frac{\kappa_n(1-\kappa_n)}{2}\mathbb{E}[|u_j^n - u_j^{p,n}|^2] \\
&\leq \left(2C_g + 2C_g^2 + \frac{2C_g^2\delta}{\kappa_n(1-\kappa_n)}\right)\delta \sum_{i=j}^{n-1} \mathbb{E}[(y_i^n - y_i^{p,n})^2] \\
&+ 2\left(\delta \sum_{i=j}^{n-1} \mathbb{E}\left[\left((y_i^{p,n} - \xi_i^n)^-\right)^2\right]\right)^{\frac{1}{2}} \left(\frac{1}{\delta}\sum_{i=j}^{n-1} \mathbb{E}[(a_i^n)^2]\right)^{\frac{1}{2}} \\
&+ 2\left(\delta \sum_{i=j}^{n-1} \mathbb{E}\left[\left((y_i^{p,n} - \zeta_i^n)^+\right)^2\right]\right)^{\frac{1}{2}} \left(\frac{1}{\delta}\sum_{i=j}^{n-1} \mathbb{E}[(k_i^n)^2]\right)^{\frac{1}{2}} \\
&\leq \left(2C_g + 2C_g^2 + \frac{2C_g^2\delta}{\kappa_n(1-\kappa_n)}\right)\delta \sum_{i=j}^{n-1} E[(y_i^n - y_i^{p,n})^2] \\
&+ \frac{2}{\sqrt{p}}\left(\frac{1}{p\delta}\sum_{i=j}^{n-1} \mathbb{E}[(a_i^{p,n})^2]\right)^{\frac{1}{2}} \left(\frac{1}{\delta}\sum_{i=j}^{n-1} \mathbb{E}[(a_i^n)^2]\right)^{\frac{1}{2}} + \frac{2}{\sqrt{p}}\left(\frac{1}{p\delta}\sum_{i=j}^{n-1} \mathbb{E}[(k_i^{p,n})^2]\right)^{\frac{1}{2}} \left(\frac{1}{\delta}\sum_{i=j}^{n-1} \mathbb{E}[(k_i^n)^2]\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $n \geq N_0$, Lemma 2.4.4, Lemma 2.6.1 and Gronwall inequality give (2.4.9). Concerning $\alpha_t^n - \alpha_t^{p,n}$ we have

$$\begin{aligned}
\alpha_t^n - \alpha_t^{p,n} =& (Y_t^n - Y_t^{p,n}) - (Y_0^n - Y_0^{p,n}) - \int_0^t g(s, Y_s^n, Z_s^n, U_s^n) - g(s, Y_s^{p,n}, Z_s^{p,n}, U_s^{p,n})ds \\
&+ \int_0^t (Z_s^n - Z_s^{p,n})dW_s^n + \int_0^t (U_s^n - U_s^{p,n})d\tilde{N}_s^n.
\end{aligned}$$

It remains to take the square of both sides, then the expectation, and to use the Lipschitz property of g combining with (2.4.9) to get the result. $\square$

2.5 Numerical simulations

We consider the simulation of the solution of a DRBSDE with obstacles and driver of the following form: $\xi_t := (W_t)^2 + 2(1-\frac{t}{T})\tilde{N}_t + \frac{1}{2}(T-t)$, $\zeta_t := (W_t)^2 + (1-\frac{t}{T})((\tilde{N}_t)^2+1) + \frac{1}{2}(T-t)$, $g(t,\omega,y,z,u) := -5|y+z| + 6u$.

Table 2.1 gives the values of Y_0 with respect to n. We notice that the algorithm converges quite fast in n. Moreover, the computational time is low.

Table 2.1: The solution y^n at time $t=0$

n	10	20	50	100	200	300	400
y_0^n	1.2191	1.3238	1.3953	1.4167	1.4293	1.4332	1.4352
CPU time	2.14×10^{-4}	1.5×10^{-3}	0.0211	0.1622	1.4230	5.2770	12.5635

When we use the explicit penalized scheme introduced in [59], we get $y_0^{p,n} = 1.4353$ for $n = 400$ and $p = 20000$. The CPU time is 12.85s.

Figures 2.1, 2.2 and 2.3 represent one path the Brownian motion, one path of the compensated Poisson process (with $\lambda = 5$) and the corresponding path of $(y_i^n, \xi_i^n, \zeta_i^n)_{1\le i\le n}$. We notice that for all i, y_i^n stays between the two obstacles. The values of y_0^n and $y_0^{p,n}$ are almost the same when $n = 400$ and $p = 20000$. The CPU times are also of the same order. The main advantage of the reflected scheme is that there is only one parameter to tune (n).

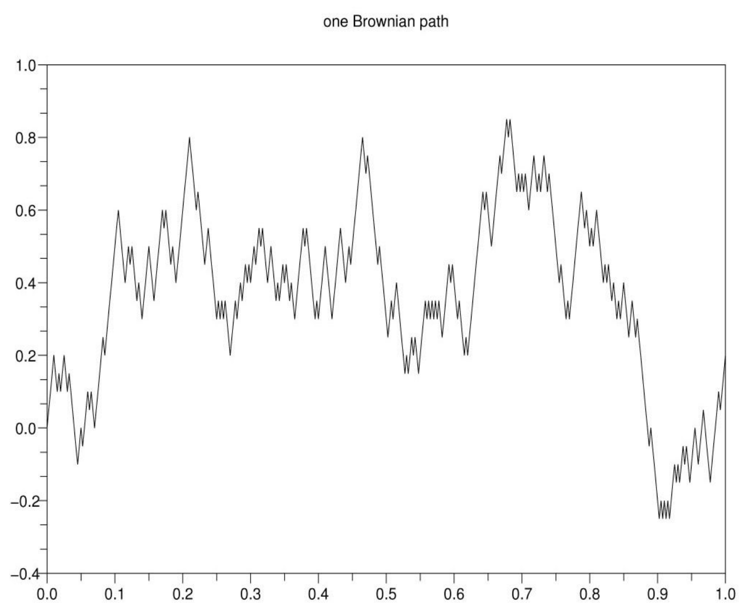

Figure 2.1: One path of the Brownian motion for $n = 400$.

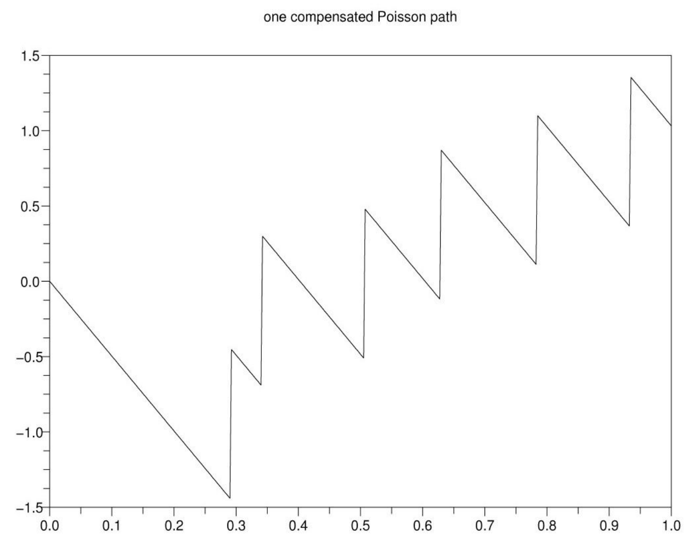

Figure 2.2: One path of the compensated Poisson process for $\lambda = 5$ and $n = 400$.

2.6 Technical result for the implicit penalized scheme

In this Section, we use N_0 and c introduced in Definition 2.4.2.

Lemma 2.6.1. *Suppose Assumption 2.2.5 holds and g is a Lipschitz driver. For each $p \in \mathbb{N}$ and $n \geq N_0$ we have*

$$\sup_j \mathbb{E}[|y_j^{p,n}|^2] + \delta \sum_{j=0}^{n-1} \mathbb{E}[|z_j^{p,n}|^2] + \kappa_n(1-\kappa_n) \sum_{j=0}^{n-1} \mathbb{E}[|u_j^{p,n}|^2] + \frac{1}{p\delta} \sum_{j=0}^{n-1} \mathbb{E}[|a_j^{p,n}|^2] + \frac{1}{p\delta} \sum_{j=0}^{n-1} \mathbb{E}[|k_j^{p,n}|^2] \leq c$$

Proof. By applying the discrete Itô's formula, we get

$$\begin{aligned} \mathbb{E}[|y_j^{p,n}|^2] &+ \delta \sum_{i=j}^{n-1} \mathbb{E}[|z_i^{p,n}|^2] + \kappa_n(1-\kappa_n) \sum_{i=j}^{n-1} \mathbb{E}[|u_i^{p,n}|^2] + \kappa_n(1-\kappa_n) \sum_{i=j}^{n-1} \mathbb{E}[|v_i^{p,n}|^2] \\ &\leq \mathbb{E}[|\xi_n^n|^2] + 2 \sum_{i=j}^{n-1} \mathbb{E}[|y_i^{p,n}||g(t_i, y_i^{p,n}, z_i^{p,n}, u_i^{p,n})\delta|] + 2\mathbb{E}[\sum_{i=j}^{n-1} (y_i^{p,n} a_i^{p,n} - y_i^{p,n} k_i^{p,n})]. \end{aligned}$$

Note that $y_i^{p,n} a_i^{p,n} = -\frac{1}{p\delta}(a_i^{p,n})^2 + \xi_i^n a_i^{p,n}$ and $y_i^{p,n} k_i^{p,n} = \frac{1}{p\delta}(k_i^{p,n})^2 + \zeta_i^n k_i^{p,n}$. We have that:

$$\begin{aligned} \mathbb{E}[|y_j^{p,n}|^2] &+ \frac{\delta}{2} \sum_{i=j}^{n-1} \mathbb{E}[|z_i^{p,n}|^2] + \frac{\kappa_n(1-\kappa_n)}{2} \sum_{i=j}^{n-1} \mathbb{E}[|u_i^{p,n}|^2] + \frac{1}{p\delta} \sum_{i=j}^{n-1} \mathbb{E}[|a_i^{p,n}|^2] + \frac{1}{p\delta} \sum_{i=j}^{n-1} \mathbb{E}[|k_i^{p,n}|^2] \\ &\leq \mathbb{E}[|\xi_n^n|^2] + \delta \mathbb{E}[\sum_{i=j}^{n-1} |g(t_i, 0, 0, 0)|^2] + 2\delta \left(1 + 2C_g + 2C_g^2 + \frac{2C_g^2 \delta}{\kappa_n(1-\kappa_n)}\right) \sum_{i=j}^{n-1} \mathbb{E}[|y_i^{p,n}|^2] \\ &+ 2 \sum_{i=j}^{n-1} \mathbb{E}[(\xi_i^n) a_i^{p,n}] - 2 \sum_{i=j}^{n-1} \mathbb{E}[(\zeta_i^n) k_i^{p,n}]. \end{aligned}$$

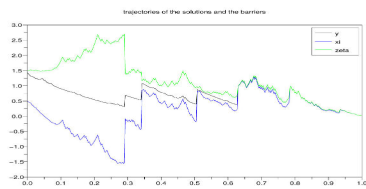

Figure 2.3: Trajectories of the solution y^n and the barriers ξ^n and ζ^n for $\lambda = 5$ and $n = 400$.

We get

$$2\sum_{i=j}^{n-1}\mathbb{E}[(\xi_i^n)a_i^{p,n}] \leq \alpha\mathbb{E}(\sup_i|\xi_i^n|^2) + \frac{1}{\alpha}\mathbb{E}\left(\sum_{i=j}^{n-1}a_i^{p,n}\right)^2$$

and $2\sum_{i=j}^{n-1}\mathbb{E}[(\zeta_i^n)k_i^{p,n}] \leq \beta\mathbb{E}(\sup_i|\zeta_i^n|^2) + \frac{1}{\beta}\mathbb{E}\left(\sum_{i=j}^{n-1}k_i^{p,n}\right)^2$. Following the same type of proof as [109, Lemma 2], we get

$$\mathbb{E}\left(\sum_{i=j}^{n-1}a_i^{p,n}\right)^2 + \mathbb{E}\left(\sum_{i=j}^{n-1}k_i^{p,n}\right)^2 \leq C(c + \mathbb{E}[\sum_{i=j}^{n-1}\delta(|y_i^{p,n}|^2 + |z_i^{p,n}|^2) + \kappa_n(1-\kappa_n)(|u_i^{p,n}|^2 + |v_i^{p,n}|^2)].$$

Finally, by taking $\alpha = \beta = 4C$ and by applying the Gronwall inequality (we recall $n \geq N_0$), we get that:

$$\sup_j \mathbb{E}[|y_j^{p,n}|^2 + \frac{\delta}{4}\sum_{j=0}^{n-1}|z_j^{p,n}|^2 + \frac{\kappa_n(1-\kappa_n)}{4}\sum_{j=0}^{n-1}|u_j^{p,n}|^2 + \frac{1}{p\delta}\sum_{j=0}^{n-1}|a_j^{p,n}|^2 + \frac{1}{p\delta}\sum_{j=0}^{n-1}|k_j^{p,n}|^2] \leq c.$$

$\square$

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