Homomorphisms of (j,k)-mixed graphs

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THÈSE EN COTUTELLE PRÉSENTÉE

POUR OBTENIR LE GRADE DE

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SPÉCIALITÉ: INFORMATIQUE

Par Christopher DUFFY

Homomorphisms of *(j,k)*-mixed Graphs

Sous la direction de Éric SOPENA
et de Gary MACGILLIVRAY

Soutenue le 19 août au Université de Victoria.

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Titre : Homomorphisms of *(j,k)*-mixed Graphs

Résumé : Un graphe mixte est un graphe simple tel que un sous-ensemble des arêtes a une orientation. Pour entiers non négatifs j et k, un graphe mixte-*(j,k)* est un graphe mixte avec j types des arcs and k types des arêtes. La famille de graphes mixte-*(j,k)* contient graphes simple, (graphes mixte$-$*(0,1)*), graphes orienté (graphes mixte$-$*(1,0)*) and graphe coloré arête $-k$ (graphes mixte$-$*(0,k)*).

Un homomorphisme est un application sommet entre graphes mixte$-$*(j,k)* que tel les types des arêtes sont conservés et les types des arcs et leurs directions sont conservés. Le nombre chromatique$-$*(j,k)* d'un graphe mixte$-$*(j,k)* est le moins entier m tel qu'il existe un homomorphisme à une cible avec m sommets. Quand on observe le cas de *(j,k)* = *(0,1)*, on peut déterminer ces définitions correspondent à les définitions usuel pour les graphes.

Dans ce mémoire on etude le nombre chromatique$-$*(j,k)* et des paramètres similaires pour diverses familles des graphes. Aussi on etude les coloration incidence pour graphes and digraphs. On utilise systèmes de représentants distincts et donne une nouvelle caractérisation du nombre chromatique incidence. On define le nombre chromatique incidence orienté et trouves un connexion entre le nombre chromatique incidence orienté et le nombre chromatic du graphe sous-jacent.

Mots clés:

graphe

graphe orienté

graphe orientée coloration

homomorphism

Title : Homomorphisms of *(j,k)*-mixed Graphs

Abstract : A mixed graph is a simple graph in which a subset of the edges have been assigned directions to form arcs. For non-negative integers j and k, a *(j,k)*–mixed graph is a mixed graph with j types of arcs and k types of edges. The collection of *(j,k)*–mixed graphs contains simple graphs (*(0,1)*–mixed graphs), oriented graphs (*(1,0)*–mixed graphs) and k–edge-coloured graphs (*(0,k)*–mixed graphs).
A homomorphism is a vertex mapping from one *(j,k)*–mixed graph to another in which edge type is preserved, and arc type and direction are preserved. The *(j,k)*–chromatic number of a *(j,k)*–mixed graph is the least m such that an m–colouring exists. When *(j,k)=(0,1)*, we see that these definitions are consistent with the usual definitions of graph homomorphism and graph colouring.

In this thesis we study the *(j,k)*–chromatic number and related parameters for different families of graphs, focussing particularly on the *(1,0)*–chromatic number, more commonly called the oriented chromatic number, and the *(0,k)*–chromatic number.

In addition to considering vertex colourings, we also consider incidence colourings of both graphs and digraphs. Using systems of distinct representatives, we provide a new characterisation of the incidence chromatic number. We define the oriented incidence chromatic number and find, by way of digraph homomorphism, a connection between the oriented incidence chromatic number and the chromatic number of the underlying graph. This connection motivates our study of the oriented incidence chromatic number of symmetric complete digraphs.

Keywords :

graph

oriented graph

oriented colouring

homomorphism

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Chapter 0
Résumé

L'histoire des sommets colorés des graphes mixtes commence, indépendamment, avec Gallai, Roy, Hasse et Vitaver ([19], [45], [26], [56]).

Theorem 0.1. *Le nombre chromatique de G est le moins m qu'une orientation acyclique de G existe où la durée la plus longue a m sommets.*

Même si ce résultat célèbre ne construit pas des colorations de graphes orientées qui considère l'orientation des arcs, il utilise néanmoins des graphes orientées dans les plies des graphes colorés. Pour trouver une définition de coloration propres de sommets des graphes orientés qui examine l'orientation des arcs, nous devons revenir au graphe homomorphisme. En traduisant la connexité entre la coloration du graphe et le graphe homomorphisme dans la langue des graphes orientés, nous arrivons à une définition raisonnable des colorants de sommets pour ces graphes. En utilisant cette même idée, nous arrivons à une définition de colorants de sommets qui tient des sortes d'agencements différentes dans le même graphe, et qui inclus des types d'arcs et d'arêtes différents.

Un *graphe mixte*, $G = (V, E, A)$, est un triplé tel que V est un ensemble des sommets, E est un ensemble des arêtes et A est un ensemble de arcs, et tel que pour tout $uv \in E(G)$, $uv, vu \notin A(G)$ et pour tout $uv \in A(G)$, $uv \notin E(G)$. Un graphe mixte est un graphe simple oú un sous-ensemble de les arêtes a une orientation.

Un *graphe coloré arête$-k$* est un graphe simple avec un application, $\Sigma : V(G) \to \{1, 2, 3, \ldots k\}$. On se réfère au graphe colorés arêtes$-k$ en utilisant la notation suivante (G, Σ). Quand le contexte est clair, on peut référer au (G, Σ) avec la notation G.

Pour G est un graphe simple, on peut obtenir une *orientation* de G en assignant à chaque de ces arêtes une direction pour obtenir un digraphe. Si un digraphe D est obtenu dans cette manière on peut dire que D est un *graphe orienté*.

Un *graphe coloré arcs$-j$* est un graphe orienté avec un application, $\alpha : V(G) \to \{1, 2, 3, \ldots k\}$. On se réfère au graphe colorés arcs$-j$ en utilisant la notation suivante (G, α). Quand le contexte est clair, on peut référer au (G, α) avec la notation G.

Si $(j, k) \neq (0, 0)$, une *graphe mixte$-(j, k)$* est

- un graphe coloré arête$-k$, (G, Σ), quand $j = 0$ et $k \neq 0$;
- un graphe coloré arcs$-j$, (G, α), quand $j \neq 0$ et $k = 0$; et
- un triplé (G, α, Σ), tel que $G = (V, E, A)$ est un graphe mixte, $((V(G), A(G)), \alpha)$ est un graphe coloré arcs$-j$, et $((V(G), E(G)), \Sigma)$ est un graphe coloré arête$-k$, sinon.

Quand le contexte est clair, on peut référer au (G, α, Σ) avec la notation G et on démontre le graphe simple sous G avec la notation $U(G)$.

En utilisant graphes mixtes$-(j, k)$ on peut définir une notion d'homomorphisme qui est commun au graphes simples, graphes orientés, graphe coloré arc$-j$, et graphes coloré arêtes$-k$.

Regardons (G, α^G, Σ^G) et (H, α^H, Σ^H) comme graphes mixtes$-(j,k)$. On peut poser comme principe que G *accepte un homomorphisme à* H, dénoté par $G \to H$, si $\phi : V(G) \to V(H)$ existe tel que

- si $k > 0$, pour tout $uv \in \Sigma_i^G$, $\phi(u)\phi(v) \in \Sigma_i^H$ $(1 \leq i \leq k)$, et
- si $j > 0$, pour tout $uv \in \alpha_i^G$, $\phi(u)\phi(v) \in \alpha_i^H$ $(1 \leq i \leq j)$.

Si ϕ est un application comme tel, on peut dire que ϕ est un *homomorphisme*, ou que ϕ a un *colorant*$-H$ de G et on écrit $\phi : G \to H$. Si $|V(H)| = m$, on peut dire que ϕ est un *colorant*$-m$ de G. Pour une classe, $\mathcal{F}$, de graphes mixtes$-(j,k)$, H est une *cible universel* pour $\mathcal{F}$ si pour tout $F \in \mathcal{F}$, on a $F \to H$.

Le *nombre chromatique*$-(j,k)$ d'un graphe mixte$-(j,k)$ G, dénoté $\chi_{j,k}(G)$, est le moins m tel qu'il existe un graphe mixte$-(j,k)$, H, avec m sommets tel que $G \to H$.

Quand on observe le cas de $(j,k) = (0,1)$, on peut déterminer les définitions données ci-dessus pour l'homomorphisme et la coloration correspondent à les définitions usuel pour les graphes. En effet, la définition pour la coloration des graphes mixtes$-(j,k)$ est motivé par la relation entre la coloration du graphe et homomorphisme du graphe. Une étude compréhensive des caractères divers de graphes homomorphisme est donnée par [27].

Malgré le fait que la coloration$-(j,k)$ généralise la coloration convenable des graphes, il n'existe pas ordinairement une relation entre le nombre chromatique$-(j,k)$ d'un graphe et le nombre chromatique du graphe sous-jacent. C'est donc facile de construire des graphes mixtes$-(j,k)$ où la différence entre ces deux paramètres est arbitrairement grande [50].

Rappelons-nous qu'une *coloration acyclique* d'un graphe est une coloration de sommet propre où le graphe induit par n'importe paire de classes de couleurs est acyclique. Étonnamment, il y a une connexité entre le nombre chromatique acyclique et le nombre chromatique mixte$-(j,k)$.

Theorem 0.2. *[42] Si G est un graphe mixte$-(j,k)$ pour qui le nombre chromatique acyclique de $U(G)$ est au plus m, alors*

$$\chi_{j,k}(G) \leq m(2j + k)^{m-1}.$$

Ce résultat unifie des résultats précédents pour les graphes orientés ([44]) et les graphes colorés arête$-k$ ([2]). Ici, les auteurs construisent un cible universel pour la famille de graphes mixtes$-(j,k)$ pour qui le graphe sous-jacent a un nombre chromatique acyclique maximum de m. Ce n'est toutefois pas le cas que la famille $\mathcal{F}$ de graphes mixtes$-(j,k)$ a une cible universel avec $\chi_{j,k}(\mathcal{F})$ sommets. Considérons, par exemple, la classe de tournois avec n sommets. Chacun de ces graphes orientés a un nombre chromatique$-(1,0)$ n, mais une cible universelle pour cette famille a au moins $2^{n/2}$ sommets [38]. Des classes de graphes mixtes$-(j,k)$ pour qui une cible universelle existe sur sommets peut être retrouvé parmi les classes complète de graphe mixte$-(j,k)$ ([50]).

Proposition 0.3. *[50] Si $\mathcal{F}$ est une classe complète de graphes mixte$-(j,k)$ tel que le nombre chromatique-(j,k) de $\mathcal{F}$ est borne, il existe une cible universelle pour $\mathcal{F}$, H, tel que $|V(H)| = \chi_{j,k}(\mathcal{F})$.*

Pour le cas ou $(j,k) = (1,0)$, nos définitions pour l'homomorphisme et la coloration correspondent exactement à les définitions de homomorphisme orienté et coloration orientée. Donc, au lieu d'utiliser $\chi_{1,0}$ et attribuer au nombre chromatique$-(1,0)$, on utilises la notation plus conventionnelle, χ_o, et la phrase coloration orientée.

La définition de coloration avec une homomorphisme a une définition comparable pour les marquages de sommets. Si G est un graphe orienté, une *coloration orientée* de G qui utilise m couleurs est une application, $c : V(G) \to \{1, 2, 3, \ldots, m\}$ tel que

1. si $uv \in E(G)$, $c(u) \neq c(v)$, et
2. si $uv, xy \in E(G)$ et $c(u) = c(y)$, $c(v) \neq c(x)$.

Des colorations orientées (auparavant appelés good colorings) ont été utiliser par Courcelle comme un exemple dans la deuxième-ordre de logique des graphes monadiques [13]. Il a étudié les colorations orientées injectif de les graphe planaires et $k-$arbres. Il démontre que chaque graphe planaire G avec $d^-(x) \leq 3$ pour tout $x \in V(G)$ a une coloration bonne qui utilise au plus $4^3 \cdot 3^{63}$, qui est injectif sur son voisinage. Cette borne a été amélioré par Raspaud et Sopena qui utilisent la connexité entre la coloration acyclique et coloration orientée.

Theorem 0.4. *[44] Si un graphe connexité G a un nombre chromatique acyclique maximum de m, alors le nombre chromatique orienté de n'importe le quelle orientation de G est au plus $m \cdot 2^{m-1}$.*

Observons que ceci est exactement Théorème 0.2 pour $j = 1$ et $k = 0$. Donc, au lieu d'utiliser $\chi_{1,0}$, nous employons la notation χ_o. Kostochka, Sopena et Zhu démontrent, plus tard, que l'inverse, où chaque classe de graphe avec des nombres bornes chromatique orientés ont un nombre borne acyclique chromatique [32].

Les colorations orientées ont été étudié pour des classes de graphes assorti ([53], [32], [14]).

De plus de la coloration orientée, plusieurs affaiblissements des conditions des colorations orientées ont établi d'autres paramètres de colorations pour des graphes orientés: colorations $2-$dipaths [34] , colorations simple [49], et colorations push [30]. Sopena a donné une étude sur les colorations orientées en 2001 [51] et a réactualisé cette étude en 2015 [52].

Quoi que le borne donné pour Théorème 0.4 est précis, lorsqu'on l'applique aux classes de graphes définies par des propriétés autre que leur nombre acyclique chromatique, nous découvrons une borne faible. Particulièrement, nous pouvons s'attendre a une amélioration d'une borne de classes de graphes orientés avec un degré de borne et la classe d'orientations de graphes planaires.

Nous considérons les colorations de graphes orientés qui tiennent des graphes sous-jacents avec un degré maximum de 3 et 4 dans le deuxième chapitre de cette thèse. Nous considérons une proximité utile des propriétés pour les cibles d'homomorphisme de ces graphes orientés. En utilisant ces cibles, nous trouvons de nouveaux bornes pour le nombre chromatique orienté de la classe de graphes orientés qui ont des graphes sous-jacents avec un degré maximum de 3 et la classe de graphes orienté avec des graphes simples sous-jacent qui tiennent un degré maximum de 4.

Theorem 0.5. *Si $\mathcal{F}$ est la famille des graphes orientés qu'ont $\Delta(U(G)) \leq 3$ pour tout $G \in \mathcal{F}$, alors*

$$7 \leq \chi_o(\mathcal{F}) \leq 9.$$

Theorem 0.6. *Si $\mathcal{F}$ est la famille des graphes orientés qu'ont $\Delta(U(G)) \leq 4$ pour tout $G \in \mathcal{F}$, alors*

$$\chi_o(\mathcal{F}) \leq 69.$$

Ces résultats améliorent les précédents meilleures bornes connues.

Les colorations simples de graphes orientés apparaissent lorsque l'on considère les homomorphismes des graphes orientés pour viser ver les graphes où les boucles se retrouvent à chaque sommet. D'autres travaux précédents dans cette domaine ont montré que pour quelques classes de graphes, y compris les graphes planaires, le nombre simple chromatique est égale au nombre chromatique orienté [49]. Dans le quatrième chapitre, nous examinons les implications de ce fait en relation au graphes planaires. De plus, nous considérons un assouplissement de certaines des exigences pour une coloration simple orientée afin d'arriver à une définition raisonnable pour une coloration 2−dipathe simple de graphes orientés. Nous proposons des résultats préliminaires pour ce paramètre de coloration nouveau et nous considérons la difficulté avec laquelle on peut déterminer si un graphe donné peut avoir une coloration 2−dipathe simple avec 2 couleurs.

Theorem 0.7. *SIMPLE* $2-$*DIPATH* $2-$*COLOURING est NP-complete .*

Dans le cinquième chapitre, nous introduisons des colorations simples de graphe coloré arête−k.

Dans la deuxième condition d'une coloration orientée nous découvrons une situation intéressante lorsque $v = x$. Dans ce cas, cette condition indique que les sommets qui sont au bouts d'un 2−dipathe reçoivent des couleurs différentes. Plusieurs auteurs ont été motive par ce lien et ont étudié les colorations de graphes orientes ou les sommets a chaque bout d'un 2−dipath, aussi que les sommets adjacents, doivent recevoir des couleurs différentes ([12], [34]). En utilisant la notation introduite par Griggs et Yeh pour les graphes [22], et adapte au digraphes par Chang et Liaw [11] et les graphes orientés par Gonç alves et al. [21], nous pouvons appeler ces graphes orientes $L(1, 1)$. Dans le troisième chapitre, nous examinons une généralisation d'une coloration de graphes orientes 2−dipathe. En utilisant des idées semblables à [34], nous construisons un modèle homomorphisme de colorations qui exige des sommets au bout d'un chemin au moins k long, pour un k fixé, qui reçoit des couleurs différentes. De plus, nous considérons la complexité de déterminer si un graphe orienté donne a une coloration k−dipathe qui utilise pas plus que m couleurs, pour des valeurs fixes de m et de k.

Theorem 0.8. *Regardons m et $k \geq 3$ comme entiers positifs fixes. La problème* $k-$DIPATH $m-$COLOURING *est NP-complete pour $m > k$. La problème est Polynomial pour tout $m \leq k$.*

Pour le cas $(j, k) = (0, k)$ nos définitions pour l'homomorphisme s'accordent exactement a ceux pour l'homomorphisme et la coloration de graphe coloré arête−k. Donc, au lieu d'utiliser $\chi_{0,k}$, nous employons la notation χ_k. De plus, pour le cas des graphes orientes, la définition pour la coloration homomorphe peut signaler également une définition de sommet-marques.

Si G est une graphe coloré arête−k et $c : V(G) \to \{1, 2, 3, \ldots, k\}$, alors c es une coloration−m de G pourvu que les conditions ci-dessous sont satisfait:

1. si $uv \in E(G)$, $c(u) \neq c(v)$, et
2. pour tout $1 \leq i \leq k$, tel que $uv \in \Sigma_i$ et $xy \in E(G)$, si $c(u) = c(x)$ et $c(v) = c(y)$, $xy \in \Sigma_i$.

Comme pour les graphes orientés, une connexité existe entre le nombre chromatique acyclique du $U(G)$ et le nombre chromatique une graphe coloré arête−k.

Theorem 0.9 (Alon et Marshall [2])**.** *. Si G est un graphe coloré arête−k pour qui le nombre chromatique acyclique du $U(G)$ est au plus m, alors*

$$\chi_k(G) \leq m \cdot k^{m-1}.$$

Observons que ceci est exactement Théorème 0.2 quand $j = 0$. Donc, au lieu d'utiliser $\chi_{0,k}$, nous employons la notation χ_k. Chronologiquement, ce résultat apparat entre celui de Raspaud et Sopena (Théorème 0.4) et celui de Nesetril et Raspaud (Théorème 0.2). Dans [2] les auteurs notent la similarité entre la saveur de leur résultat et la méthode de Raspaud et Sopena. Mais, nous devons aussi prendre note qu'ils n'observe pas une solution pour dériver un ensemble de résultats.

Harary ([25] et [26]) a mentionné, en 1953, les graphes graphe coloré arête-2 (aussi appelé des graphes *signed*, ou graphes *signified*). Il a étudié la structure des cycles de graphes colorés arêtes-2 germé d'un problème dans les sciences humaines. Une notion de coloration pour ces graphes qui est diffèrent de ceux présenté dans ce texte, est donné par Zaslavsky [59]. Récemment, les graphes colorés arêtes-2 se sont présenté dans les thèses de Brewster [8] et Sen [48] et de plusieurs autres auteurs ([33], [37], [40]).

Dans le cinquième chapitre, nous examinons les colorations des graphes coloré arête-2. Nous trouvons une borne inférieure pour le nombre chromatique de la classe de graphe coloré arête-2 avec un degré maximum 3 en considérant un paramètre nouveau pour la coloration de ces graphes, qui exige que les sommets adjacents et les sommets au bout du chemin de la durée 2 où chacun de ces arêtes on des couleurs différentes reçoivent des couleurs différentes. Nous trouvons une borne supérieure pour le nombre de classe de graphe coloré arête-2 avec un degré maximum 3 en construisant une paire de cibles pour les graphes de cette classe.

Theorem 0.10. *Si G est un graphe coloré arête-2 avec un degré maximum 3, alors G accepte un homomorphisme à une cible qui a 11 sommets.*

Les incidents colorés apparu en 1993 quand Brualdi et Massey ont initialement défini l'incident d'un nombre chromatique d'un graphe simple [9]. Dans ce mémoire, ils ont donné les bornes supérieures et inférieures pour le nombre chromatique incident à partir d'un degré maximum. Ces auteurs ont utilisé leurs résultats en tant que méthode pour améliorer une borne pour l'index chromatique fort des graphes bipartis. Depuis, les bornes pour le nombre chromatique incident ont été étudié pour déterminer un assortisse ment de classes de graphes, y compris des graphes planaires, les arbres$-k$, les graphes réguliers$-k$, des grilles torodales et les graphes dégénérés$-k$ ([15], [54], [47], [57]).

Dans le sixième chapitre, nous trouvons une nouvelle caractérisation d'un incident chromatique d'un nombre en utilisant des structures de représentations précis et nous introduisons une version orienté de ce paramètre. En utilisant un digraphe homomorphisme nous trouvons que l'incident orienté du nombre chromatique d'un graphe orienté est liée au nombre chromatique du graphe simple sous-jacent. Ceci motive notre étude des incidences orientées du nombre chromatique de graphes complets symétriques.

Chapter 1
Introduction and Preliminaries

The story of vertex colourings of mixed graphs begins, independently, with Gallai, Roy, Hasse, and Vitaver.

Theorem 1.1 ([19] Gallai, [45] Roy, [26] Hasse, [56] Vitaver)**.** *The chromatic number of G is the least m such that there exists an acyclic orientation of G in which the longest path has m vertices.*

Though this celebrated result does not construct colourings of oriented graphs that take into account the orientation of the arcs, it does welcome oriented graphs into the fold of graph colourings. To find a definition of proper vertex colouring of oriented graphs that takes into account the orientation of the arcs, we must turn to graph homomorphism. By translating the link between graph colouring and graph homomorphism into the language of oriented graphs, we arrive at a reasonable definition of vertex colouring for these graphs. Using this same idea we arrive at a definition of vertex colouring for graphs that have different sorts of adjacency within the same graph, including different arc types and edge types.

In this thesis, we study colourings of such graphs, called $(j,k)-$mixed graphs. We examine the $(j,k)-$chromatic number and related colouring parameters, focussing mainly on $(1,0)-$mixed graphs (oriented graphs) and $(0,k)-$mixed graphs ($k-$edge-coloured graphs).

In Chapter 2, we consider colourings of oriented graphs whose underlying graphs have maximum degree 3 and 4. We consider a useful adjacency property for targets of homomorphisms from these oriented graphs. Using these targets, we find new upper bounds for the oriented chromatic number of the family of oriented graphs whose underlying graphs have maximum degree 3 and the family of oriented graphs whose underlying simple graphs have maximum degree 4.

Simple colourings of oriented graphs arise from considering homomorphisms from oriented graphs to target graphs in which loops are present at each vertex. Previous work in this area has shown for some families of oriented graphs that the simple chromatic number is equal to the oriented chromatic number. In Chapter 4 we examine the implications of this fact for planar graphs. Additionally, we consider an easing of some of the requirements for a simple oriented colouring to arrive at a reasonable definition of simple $2-$dipath colouring for oriented graphs. We give some preliminary results for this new colouring parameter, as well as consider the complexity of determining if a given graph has a simple $2-$dipath colouring using two colours.

In the second condition of an oriented colouring (see Definition 1.17) an interesting situation arises when $v = x$. In this case, this condition implies vertices at the ends of a directed path of length two receive different colours. Motivated by this connection, many authors have studied colourings of oriented graphs in which vertices at the ends of a $2-$dipath, as well as adjacent vertices, must receive different colours ([12], [34]). Using the notation first introduced by Griggs and Yeh for graphs [22], and then adapted to digraphs by Chang and Liaw [11], and to oriented graphs by Gonçalves et al. [21], we may consider these to be $L(1,1)$ labellings of oriented graphs. In Chapter 3 we examine a generalisation of 2−dipath colourings of oriented graphs. Using ideas similar to [34], we construct a homomorphism model for colourings that require vertices at the end of a directed path of length at most k, for fixed k, receive different colours. Additionally, we consider the complexity of determining if a given oriented graph has a $k-$dipath colouring using no more than m colours, for fixed values of m and k.

In Chapter 5 we examine colourings of $k-$edge-coloured graphs. We find a lower bound for the chromatic number of the family 2−edge-coloured graphs with maximum degree 3 by considering a new colouring parameter for these graphs, which requires that adjacent vertices and vertices at the end of a path of length 2 where each of the edges have different colours receive different colours. We find an upper bound for the chromatic number of the family 2−edge-coloured graphs with maximum degree 3 by constructing a pair of targets for graphs in this family.

In the final chapter, we consider colourings of graphs and digraphs that assign colours to incidences, rather than vertices. In Chapter 6, we find a new characterisation of the incidence chromatic number using systems of distinct representatives, as well as introduce a directed version of this parameter. Using digraph homomorphism, we find the oriented incidence chromatic number of a directed graph is closely related to the chromatic number of its underlying simple graph. This motivates our study of the oriented incidence chromatic number of symmetric complete graphs.

We now present definitions and notation regarding various types of graphs, as well as relevant results and commentary that give context to the work presented in later chapters. Special definitions and notation defined and used exclusively in the context of a single chapter are defined in that chapter. A glossary of the colouring parameters used in this thesis appears as an appendix. For all other commonly-used terms and notation we refer to [7].

Definition 1.1. *A* $k-$edge-coloured graph *is a simple graph, G, together with a function $\Sigma : E(G) \to \{1, 2, 3, \dots, k\}$. For $1 \leq i \leq k$, we let*

$$\Sigma_i = \{e \in E(G) | \Sigma(e) = i\}.$$

We refer to a $k-$edge-coloured graph using the notation (G, Σ). When the context is clear, we may refer to (G, Σ) simply as G.

Definition 1.2. *If G is a simple graph, then we obtain an* orientation *of G by assigning to each of the edges a direction to obtain a digraph. If a digraph D is obtained in this manner we say that D is an* oriented graph.

For simplicity, when referring to arcs and the arc set of a oriented graph, G, we use uv to refer to an arc from u to v and $E(G)$ to refer to the set of arcs of G.

Definition 1.3. *A* $j-$arc-coloured graph *is a oriented graph, G, together with a function $\alpha : E(G) \to \{1, 2, 3, \dots, j\}$. For $1 \leq i \leq j$, we let*

$$\alpha_i = \{uv \in E(G) | \alpha(uv) = i\}.$$

We refer to a $j-$arc-coloured graph using the notation (G, α). When the context is clear, we may refer to (G, α) simply as G.

Definition 1.4. *If G is an oriented graph, the* converse *of G is the oriented graph formed by reversing the direction of each arc.*

Definition 1.5. *An oriented graph, G, is* self-converse *if G admits an isomorphism to the converse of G.*

Let $G = (V, E)$ be a directed graph.

Definition 1.6. *If $u, v \in V(G)$ and $uv \in E(G)$, then we call v an* out-neighbour *of u and u an* in-neighbour *of v. The* out-neighbourhood *of $v \in V(G)$, denoted $N^+(v)$, is the set of all out-neighbours of v. The* in-neighbourhood *of $v \in V(G)$, denoted $N^-(v)$, is the set of all in-neighbours of v. The cardinality of $N^+(v)$, denoted $d^+(v)$, is called the* out-degree *of v. The cardinality of $N^-(v)$, denoted $d^-(v)$, is called the* in-degree *of v. A vertex, s, is called a* source *if $d^-(s) = 0$ and $d^+(s) \neq 0$. A vertex, t, is called a* sink *if $d^+(t) = 0$ and $d^-(t) \neq 0$. A source or sink is called* universal *if it adjacent to every vertex in G, other than itself.*

Definition 1.7. *For $u, v \in V(G)$ let $\overrightarrow{d_G}(u, v)$ be the number of arcs in a shortest directed path from u to v, or ∞ if no such path exists. When context allows, we write $d(u, v)$. The* distance between u and v *is the least k such that there exists a directed path of length k from u to v, or from v to u. If no such directed path exists we write $\overrightarrow{d_G}(u, v) = \infty$.*

For brevity we refer to a directed path of length k as a $k-$*dipath*.

Definition 1.8. *If for all $u, v \in V(G)$ at least one of $\overrightarrow{d_G}(u, v)$ and $\overrightarrow{d_G}(v, u) \neq \infty$, then the* weak diameter *of G is the least integer k such that for all pairs, $u, v \in V(G)$, the distance between u and v is no more than k. Otherwise, the* weak diameter *of G is defined to be ∞.*

Definition 1.9. *If G has no directed cycle, we say that G is* acyclic.

Definition 1.10. *The* directed girth *of G is the length of the shortest directed cycle in G. If G is acyclic then the directed girth of G is defined to be ∞.*

Definition 1.11. *A* mixed graph, *$G = (V, E, A)$, is a triple, where V is a set of vertices, E a set of edges and A a set of arcs, so that for all $uv \in E(G)$, $uv, vu \notin A(G)$ and for all $uv \in A(G)$, $uv \notin E(G)$. We may view a mixed graph as a simple graph in which a subset of the edges have been oriented.*

Mixed graphs capture both graphs and oriented graphs. We extend this definition to capture $k-$edge-coloured graphs, and $j-$arc-coloured graphs.

Definition 1.12. *For a pair of non-negative integers $(j, k) \neq (0, 0)$, a $(j, k)-$*mixed graph, *is*

- *a $k-$edge-coloured graph, (G, Σ), when $j = 0$ and $k \neq 0$;*
- *a $j-$arc-coloured graph (G, α), when $j \neq 0$ and $k = 0$; and*
- *a triple (G, α, Σ), where $G = (V, E, A)$ is a mixed graph, $((V(G), A(G)), \alpha)$ is a $j-$arc-coloured graph, and $((V(G), E(G)), \Sigma)$ is a $k-$edge coloured graph, otherwise.*

When the context is clear, we refer to (G, α, Σ) as G, and the simple graph underlying (G, α, Σ) as $U(G)$.

Definition 1.13. *A family of mixed-graphs, $\mathcal{F}$, is* complete *if for all $F_1, F_2 \in \mathcal{F}$ there exists $G \in \mathcal{F}$ containing both F_1 and F_2 as subgraphs.*

Using $(j, k)-$mixed graphs we define a notion of homomorphism that is common to simple graphs, mixed graphs, oriented graphs and $k-$edge-coloured graphs.

Definition 1.14. *Let (G, α^G, Σ^G) and (H, α^H, Σ^H) be $(j,k)-$mixed graphs. We say that (G, α^G, Σ^G)* admits a homomorphism to *(H, α^H, Σ^H), denoted $(G, \alpha^G, \Sigma^G) \to (H, \alpha^H, \Sigma^H)$ or, when the context is clear, $G \to H$, if there exists $\phi : V(G) \to V(H)$ such that*

- *if $k > 0$, then for all $uv \in \Sigma_i^G$, $\phi(u)\phi(v) \in \Sigma_i^H$ $(1 \leq i \leq k)$, and*
- *if $j > 0$, then for all $uv \in \alpha_i^G$, $\phi(u)\phi(v) \in \alpha_i^H$ $(1 \leq i \leq j)$.*

If ϕ is such a mapping, we say that ϕ is a homomorphism*, or that ϕ is an $H-$*colouring of G*, and we write $\phi : G \to H$. If H has order m, we say that ϕ is an $m-$*colouring *of G. For a family, $\mathcal{F}$, of $(j,k)-$mixed graphs we say that a $(j,k)-$mixed graph, H, is a* universal target for *$\mathcal{F}$ if for all $F \in \mathcal{F}$, we have $F \to H$.*

Definition 1.15. *The $(j,k)-$*chromatic number *of a $(j,k)-$mixed graph, denoted $\chi_{j,k}(G)$, is the least m such that there exists a $(j,k)-$mixed graph, H, with m vertices so that $G \to H$. If $\mathcal{F}$ is a family of $(j,k)-$mixed graphs with bounded $(j,k)-$chromatic number then we define $\chi_{j,k}(\mathcal{F})$ to be the least m such that for all $F \in \mathcal{F}$, $\chi_{j,k}(F) \leq m$.*

1.1 Graph Colouring

When considering the case $(j,k) = (0,1)$, we see that the definitions given above for homomorphism and colouring match the usual definitions for graphs. In fact, the definition for colouring of $(j,k)-$mixed graphs is motivated by the relationship between graph colouring and graph homomorphism. A comprehensive study of various aspects of graph homomorphisms is given by [28].

1.1.1 $(j,k)-$colouring

Though $(j,k)-$colouring generalises proper colouring of graphs, in general there is no relationship between the $(j,k)-$chromatic number of a graph and the chromatic number of the underlying graph. It is easy to construct $(j,k)-$mixed graphs for which the difference between these two parameters is arbitrarily large [50].

Recall that an *acyclic colouring* of a graph is a proper vertex colouring where the subgraph induced by any pair of colour classes is acyclic. Surprisingly, there is a connection between the acyclic chromatic number and the $(j,k)-$mixed chromatic number.

Theorem 1.2 (Nešetřil and Raspaud [42])**.** *If G is a $(j,k)-$mixed graph for which the acyclic chromatic number of the underlying undirected graph is at most m, then*

$$\chi_{j,k}(G) \leq m(2j+k)^{m-1}.$$

This result unifies previous results for oriented graphs [44] and $k-$edge-coloured graphs [2]. Here the authors construct a universal target for the family of $(j,k)-$mixed graphs for which the underlying graph has acyclic chromatic number at most m. In general, however, it is not the case that a family, $\mathcal{F}$, of $(j,k)-$mixed graphs has a universal target with $\chi_{j,k}(\mathcal{F})$ vertices. For example, consider the family of tournaments with n vertices. Each of these oriented graphs has $(1,0)-$chromatic number n, however a universal target for this family has at least $2^{\frac{n}{2}}$ vertices [38]. Families of $(j,k)-$mixed graphs for which a universal target exists on $\chi_{j,k}(\mathcal{F})$ vertices may be found amongst complete families of $(j,k)-$mixed graphs.

Proposition 1.3 (Sopena [50])**.** *If $\mathcal{F}$ is a complete family of $(j,k)-$mixed graphs with bounded $(j,k)-$chromatic number, then there exists a universal target for $\mathcal{F}$, H, such that $|V(H)| = \chi_{j,k}(\mathcal{F})$.*

Those $(j, k)-$mixed graphs, G, for which $\chi_{j,k}(G) = |V(G)|$ are of particular interest. For $(j, k) = (0, 1)$, these are just the complete graphs. Motivated by this, we consider the concept of a $(j, k)-$*clique*.

Definition 1.16. *A $(j, k)-$mixed graph, G, is a $(j, k)-$*clique *if $\chi_{j,k}(G) = |V(G)|$.*

Such cliques have been studied for both $(1, 0)-$mixed graphs (called *oriented cliques*, or *ocliques*) ([48], [30], [18] [31]) and $(0, 2)-$mixed graphs (called *signified cliques*, or *scliques*) ([33], [29]).

1.1.2 $(1, 0)-$mixed graphs

For the case $(j, k) = (1, 0)$ our definitions for homomorphism and colouring match exactly those for homomorphism of oriented graphs and oriented colouring. And so rather than using $\chi_{1,0}$ and referring to the $(1, 0)-$chromatic number, we use the more conventional notion of χ_o as well as the phrase *oriented chromatic number*.

When considering oriented graphs, the homomorphism definition of colouring has an equivalent vertex-labelling definition.

Definition 1.17. *Let G be an oriented graph. An* oriented colouring *of G using m* colours *is a mapping $c : V(G) \to \{1, 2, \ldots, m\}$ such that:*

- *$c(u) \neq c(v)$ for all $uv \in E(G)$,*
- *for all $uv, xy \in E(G)$ if $c(u) = c(y)$, then $c(v) \neq c(x)$.*

That this definition of oriented colouring is equivalent to the homomorphism of oriented colouring follows by observing that if the head and tail of an arc are coloured with a and b, then there is an arc ab in the target. Since the target is an oriented graph, if ab is an arc of the target, then ba is not an arc of the target. This implies that no arc will have its tail coloured with b and its head coloured with a. To see the other half of the equivalence, observe that from an oriented colouring that satisfies the vertex labelling definition the target for a homomorphism can be constructed by taking the vertex set to be the set of colours, and for an arc ij to exist in the target there must be an arc in the coloured oriented graph with its tail coloured i and its head coloured j.

Oriented colourings (then called good colourings) were used by Courcelle as an example in the monadic second-order logic of graphs [13]. He studied locally-injective oriented colourings of planar graphs and $k-$trees. He showed that every oriented planar graph G with $d^{-}(x) \leq 3$ for every $x \in V(G)$ has a good colouring that uses at most $4^3 \cdot 3^{63}$ colours, which is injective on in-neighbourhoods. This bound was improved by Raspaud and Sopena using the connection between acyclic colouring and oriented colouring later utilised by Nešetřil and Raspaud.

Theorem 1.4 (Raspaud and Sopena [44])**.** *If a connected graph G has acyclic chromatic number at most m, then the oriented chromatic number of any orientation of G is at most $m \cdot 2^{m-1}$.*

Observe that this is exactly Theorem 1.2 for $j = 1$ and $k = 0$. The converse, that every family of graphs with bounded oriented chromatic number has bounded acyclic chromatic number, was shown later by Kostochka, Sopena and Zhu [32].

Oriented colourings have been studied for a wide variety of families of graphs ([53], [32], [14]). In addition to oriented colouring, various weakenings of the requirements of oriented colourings have led to other colouring parameters for oriented graphs, including $2-$dipath colouring [34], simple colouring [49], and push colouring [31]. A survey on the study of oriented colourings was given by Sopena in 2001 [51] and updated in 2015 [52].

Though the bound given in Theorem 1.4 is known to be tight, when applied to families of graphs defined by properties other than their acyclic chromatic number this bound is weak. In particular, this bound may be improved for families of oriented graphs with bounded degree [32] and it is expected that it may be improved for the family of orientations of planar graphs.

1.1.3 $(0, k)-$mixed graphs

For the case $(j, k) = (0, k)$ our definitions for homomorphism and colouring match exactly those for homomorphism and colouring of $k-$edge-coloured graphs. And so rather than using $\chi_{0,k}$, we use the notation χ_k. Similar to the case for oriented graphs, the homomorphism colouring definition can be equivalently stated as a vertex-labelling definition.

Definition 1.18. *If (G, Σ) is a $k-$edge-coloured graph and $c : V(G) \to \{1, 2, 3, \ldots, m\}$, then c is an $m-$colouring of G provided that the following conditions are met:*

- *for all $uv \in E(G)$, $c(u) \neq c(v)$, and*
- *for all $1 \leq i \leq k$ where $uv \in \Sigma_i$, and $xy \in E(G)$, if $c(u) = c(x)$ and $c(v) = c(y)$, then $xy \in \Sigma_i$.*

As with oriented graphs, a connection exists between the acyclic chromatic number of the underlying graph and the chromatic number of the $k-$edge-coloured graph.

Theorem 1.5 (Alon and Marshall [2])**.** *If G is a $k-$edge-coloured graph for which the acyclic chromatic number of the underlying graph is at most m, then*

$$\chi_k(G) \leq m \cdot k^{m-1}.$$

Observe that this is exactly Theorem 1.2 when $j = 0$. Chronologically this result comes between that of Raspaud and Sopena (Theorem 1.4) and that of Nešetřil and Raspaud (Theorem 1.2). In [2] the authors note the similarity in the flavour of their result and method to that of Raspaud and Sopena. But also note that they see no way to derive one set of results from the other.

An early mention of 2−edge-coloured graphs (also called signed graphs, or signified graphs) was in 1953 by Harary ([25] and [10]). Here he studied the structure of cycles of 2−edge-coloured graphs arising from a problem in the social sciences. A notion of colouring of these graphs, different to the one presented herein, is given by Zaslavsky [59]. More recently, 2−edge-coloured graphs appear in the theses of Brewster [8] and Sen [48], as well as in work by many others ([37], [40], [41]).

1.2 Incidence Colourings

Incidence colouring arose in 1993 when Brualdi and Massey first defined the incidence chromatic number of a simple graph (then called the incidence colouring number) [9]. In this paper they gave upper and lower bounds for the incidence chromatic number based on maximum degree. These authors used their results as a method to improve a bound for the strong chromatic index of bipartite graphs. Since then, bounds for the incidence chromatic number have been investigated for a variety of families of graphs, including planar graphs, $k-$trees, $k-$regular graphs, toroidal grids and $k-$degenerate graphs ([15], [55], [54], [57]). This topic is discussed in further detail in Chapter 6.

Chapter 2
Oriented Colourings of Bounded Degree Graphs

In this chapter we consider oriented colourings of oriented graphs whose underlying graphs have maximum degree 3 or 4. For the case $\Delta \leq 3$, we improve the upper bound given by Sopena and Vignal [53] by constructing $9-$vertex targets for such oriented graphs. For the case $\Delta \leq 4$ we improve the upper bound implied by Theorem 1.4. In this latter case we note that room for improvement certainly exists.

2.1 Background and Preliminaries

When restricted to $(j, k) = (1, 0)$, the definition for homomorphism and colouring given in Chapter 1 give the following.

Definition 2.1. *Let G and H be oriented graphs. We say that G admits a homomorphism to H, denoted $G \to H$, if there exists $\phi : V(G) \to V(H)$ such that if $uv \in E(G)$, then $\phi(u)\phi(v) \in E(H)$. We call ϕ a homomorphism and we write $\phi : G \to H$.*

Definition 2.2. *Let G be an oriented graph. The oriented chromatic number of G, denoted $\chi_o(G)$, is the least integer m such that there exists an oriented graph H with $|V(H)| = m$ and a homomorphism $\phi : G \to H$. We call ϕ an oriented $m-$colouring of G, or an oriented colouring of G using m colours. If $\mathcal{F}$ is a family of oriented graphs with bounded oriented chromatic number, then we define $\chi_k(\mathcal{F})$ to be the least m such that $\chi_k(G) \leq m$ for all $F \in \mathcal{F}$.*

Recall the vertex labelling definition for colouring of oriented coloured graphs.

Definition 2.3. *Let G be an oriented graph. An* oriented colouring of G using m colours *is a mapping $c : V(G) \to \{1, 2, \ldots, m\}$ such that:*

- *$c(u) \neq c(v)$ for all $uv \in E(G)$,*
- *for all $uv, xy \in E(G)$ if $c(u) = c(y)$, then $c(v) \neq c(x)$.*

For proper colourings of graphs a simple argument based on graph degeneracy gives an upper bound of $\Delta + 1$ for the chromatic number of a graph with maximum degree Δ. Brooks' Theorem refines this idea and tightens the upper bound to exactly Δ for all graphs other than odd cycles and complete graphs. In the proofs of these results, vertices are being added one at a time to the graph so that at each step there is an available colour for the newly-added vertex. In trying to replicate this procedure with oriented graphs, a difference arises between the oriented and unoriented case.

Consider the partially coloured oriented graph in Figure 2.1. The uncoloured vertex cannot be coloured with colours 0 or 1. Trying to colour this vertex with another colour,

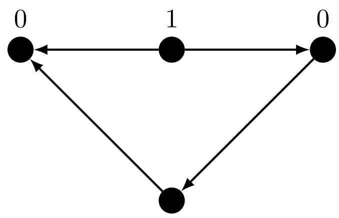


Figure 2.1: A colouring that cannot be extended.

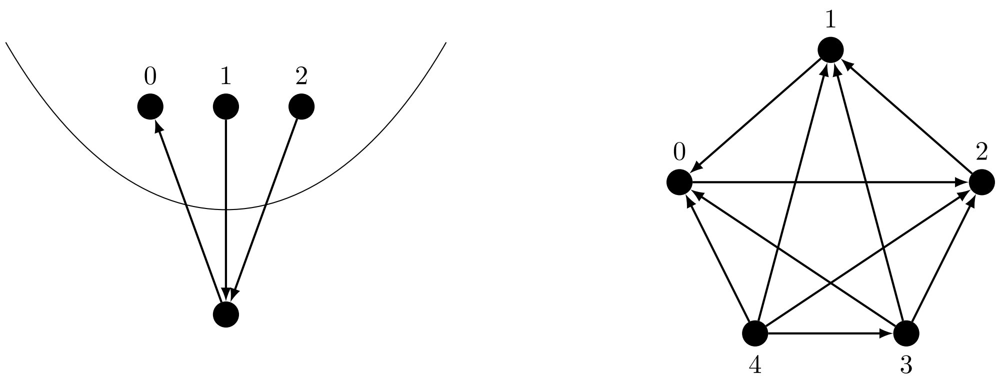


Figure 2.2: Another colouring that cannot be extended.

say 2, will also fail, as there would be an arc with its tail labelled 0 and its head labelled 2, as well as an arc with its tail labelled 2 and its head labelled 0. Consider trying to extend the homomorphism given in Figure 2.2, where the oriented graph on the right is the target and the oriented graph on the left is partially coloured. We wish to extend the homomorphism to include the uncoloured vertex. In the target we are looking for a vertex that is an in-neighbour of 0 and an out-neighbour of both 1 and 2. By inspection we see that no such vertex in the target fits this description. The colouring given by this homomorphism cannot be extended without adding a new vertex to the target graph. Though the uncoloured vertex has degree strictly smaller than the order of the target, this homomorphism cannot be extended. These small examples imply, regardless of the size of the palette of available colours, it is not guaranteed a colouring of a partially coloured oriented graph can be extended.

This second situation leads us to desire the following property in the target of a homomorphism from an oriented graph with bounded degree.

Property $P_{i,j}$. *A tournament, G, has property $P_{i,j}$ if for every subset $X \subset V(G)$ of size i and for every sequence $(z_1, z_2, \ldots, z_i)$, where $z_k \in \{0, 1\}$ $(1 \leq k \leq i)$, there exist j distinct*

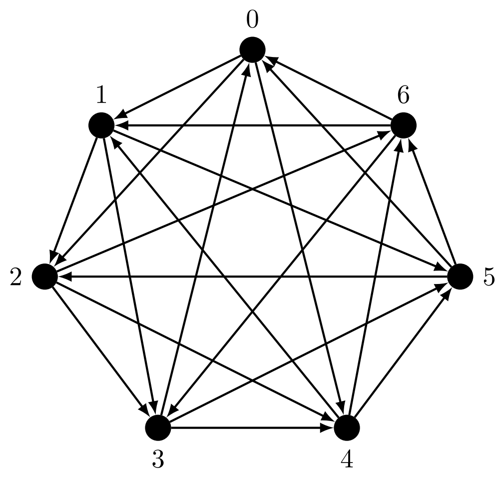


Figure 2.3: The non-zero quadratic residue tournament on 7 vertices.

vertices in $V(G) \setminus X$*,* $\{y_1, y_2, \ldots, y_j\}$*, such that for all* $1 \leq \ell \leq j$*,* $x_i y_\ell \in E(G)$ *if and only if* $z_i = 1$*.*

Property $P_{i,j}$ relates closely to the subject of $n-$*existentially closed* tournaments (see [4], [5] and [6]). We discuss a version of this property for 2−edge coloured graphs in Chapter 5.

A well-studied family of oriented graphs with property $P_{i,j}$ is the *non-zero quadratic residue tournaments*, or *Paley tournaments* (see [4]). Let q be a prime power such that $q \equiv 3 \mod 4$, and let $\mathbf{F}_q^\times$ be the field of order q. The *non-zero quadratic residue tournament on* q *vertices*, QR_q, is the oriented graph with:

- $V(QR_q) = \{0, 1, \ldots, q-1\}$, and
- $E(QR_q) = \{uv | v - u$ is a non-zero quadratic residue in $\mathbf{F}_q^\times$ $\}$.

The oriented graph in Figure 2.3 is QR_7.

We call an oriented graph, G, *subcubic* if $\Delta(G) \leq 3$ and there exists $v \in V(G)$ such that $d(v) < 3$. To see how this property $P_{i,j}$ is useful, consider trying to extend a colouring of a subcubic graph to a target, P, with property $P_{2,2}$. Let H be an orientation of subcubic graph with at least two non-adjacent vertices of degree 2 and let $\phi : H \to P$ be a homomorphism. Let u and v be non-adjacent vertices of degree 2 in H and let H^* be the oriented graph formed from H by adding a new vertex z together with the arcs uz and zv. Let α be the restriction of ϕ to $H \setminus \{u, v\}$. Since P has property $P_{2,2}$, α can be extended such that $\beta(u) \neq \beta(v)$. Since P has property $P_{2,2}$, β can be extended to include z. Strictly speaking, we may not have extended ϕ to be a homomorphism from $H^\star$ to P, as it may be that $\phi(u) \neq \delta(u)$. However, starting from ϕ we have successfully constructed a homomorphism $\delta : H^\star \to P$.

The first upper bound on the oriented chromatic number of oriented graphs with bounded degree was given by Sopena.

Theorem 2.1 (Sopena [50])**.** *An orientation of a graph with maximum degree* Δ *has oriented chromatic number at most* $(2\Delta - 1)2^{2\Delta - 2}$*.*

Using the probabilistic method, this result was later improved by Kostochka, Sopena and Zhu.

Theorem 2.2 (Kostochka, Sopena and Zhu [32])**.** *An orientation of a graph with maximum degree* Δ *has oriented chromatic number at most* $2\Delta^2 2^\Delta$.

2.2 Oriented Cliques with Bounded Degree

Definition 2.4. *An oriented graph, G, is an* oriented clique *or* oclique *if* $\chi_o = |V(G)|$.

As discussed in Chapter 1, oriented cliques have been studied by a variety of authors. Here we find oriented cliques with bounded maximum degree.

Theorem 2.3. *The order of a largest oriented clique in the family of orientations of graphs with maximum degree* 3 *is* 7*.*

Proof. Suppose G is an oriented clique whose underlying graph has maximum degree 3. If $U(G)$ has a vertex of degree 2, then G has at most 7 vertices. As such, we may assume that $U(G)$ is 3−regular. Every vertex of G is the centre vertex of at most two 2−dipaths. Since G is an oriented clique, each vertex has a 2−dipath to each of its non-neighbours in one direction or the other. Therefore the number of 2−dipaths in G is at least $\frac{n(n-4)}{2}$. This implies

$$2n \geq \frac{n(n-4)}{2}.$$

In turn, this implies $n \leq 8$.

The two cubic graphs on eight vertices are given in Figure 2.5. Consider orienting each of them to be an oclique. Without loss of generality we may assume that we have the arcs 23 and 34, as there must be a 2−dipath from 2 to 4. We note that generality is not lost here, as if an oriented graph is an oclique, then its converse is also an oclique. This implies we have the arc 34, as there must be a 2−dipath from 3 to 5. Continuing with this line of reasoning we see that the outer cycle must be a directed cycle. However, if this is the case we cannot successfully orient the edge 26 so that there is a 2−dipath between 2 and 5 and one between 2 and 7. □

Figure 2.4 gives an oriented clique on 7 vertices. A similar technique for orientations of graphs with maximum degree 4 yields the following result, which we state without proof.

Theorem 2.4. *The order of a largest oriented clique in the family of orientations of graphs with maximum degree* 4 *is no more than* 13*.*

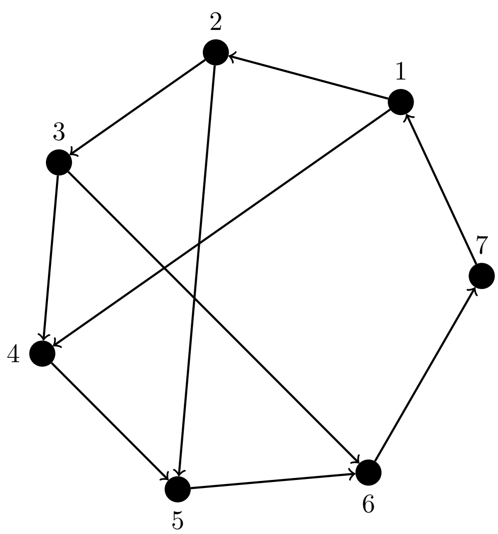


Figure 2.4: An oriented clique on 7 vertices

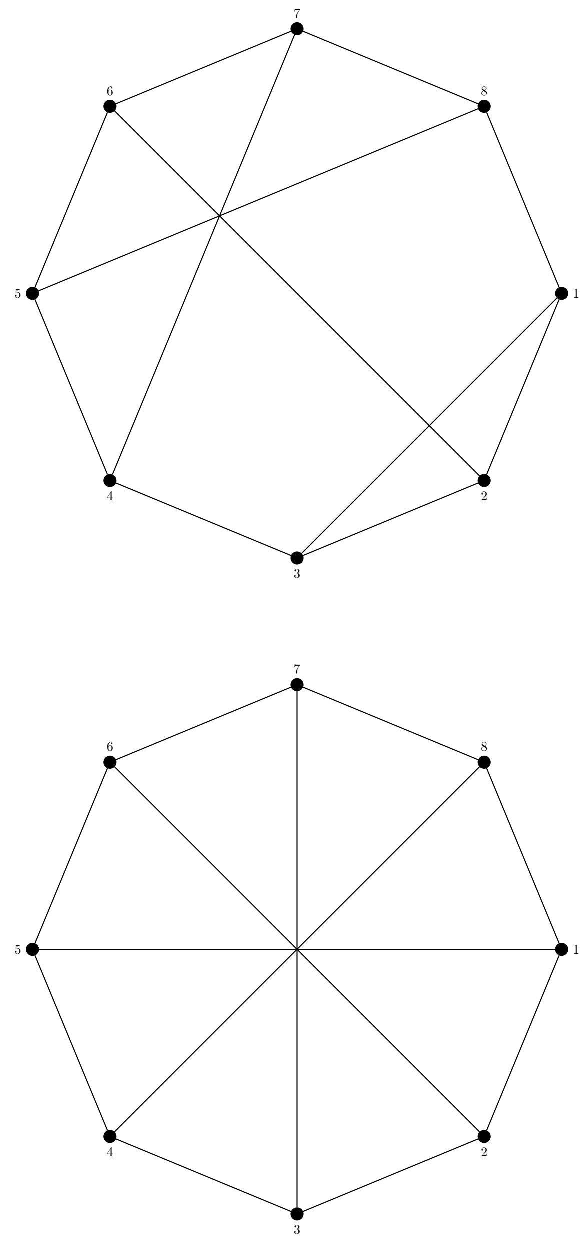


Figure 2.5: Cubic graphs with diameter 2.

2.3 Oriented Colourings of Graphs with Maximum Degree Three

For the family, $\mathcal{F}_3$, of orientations of connected graphs with maximum degree 3, Theorem 2.2 gives $\chi_o(\mathcal{F}_3) \leq 144$. However, for $\mathcal{F}_3$ we can get a better bound by considering the acyclic chromatic number of the underlying graphs. Cubic graphs have acyclic chromatic number at most 4 [23], and so, by Theorem 1.4 in Chapter 1,

$$\chi_o(\mathcal{F}_3) \leq 4 \cdot 2^{4-1} = 32.$$

A series of incremental improvements ([50], [53]) has led to the following upper bound for $\chi_o(\mathcal{F}_3)$.

Theorem 2.5 (Sopena and Vignal [53]). *An orientation of a graph with $\Delta \leq 3$ has oriented chromatic number at most 11.*

Since the oriented graph given in Figure 2.4 is a member of $\mathcal{F}_3$, we have directly that $\chi_o(\mathcal{F}_3) \geq 7$.

In their proof of Theorem 2.5 the authors show that QR_{11} is a universal target for $\mathcal{F}_3$. To improve this bound we show that every oriented subcubic graph that does not contain a subgraph with a particular structure admits a homomorphism to QR_7. We begin by observing some useful properties of QR_7.

Property 2.6. *QR_7 is arc-transitive and vertex-transitive.*

Paley tournaments are a type of Cayley tournament. Since Cayley tournaments are known to be vertex-transitive, it follows that QR_7 is vertex transitive. To see that QR_7 is arc transitive, observe that for any pair of arcs $uv, wx \in E(QR_7)$, the mapping ϕ, defined by

$$\phi(z) = \frac{x-w}{v-u} z + w - u\frac{x-w}{v-u} \pmod 7,$$

is an automorphism that maps uv to wx.

Property 2.7. *QR_7 is self-converse.*

To prove Property 2.7 observe that the arc set of the converse of QR_7, QR_7^c is given by

$$E(QR_7^c) = \{uv | v - u \notin \{0, 1, 2, 4\}\}.$$

The mapping that sends $x \in V(QR_7)$ to $y \in V(QR_7^c)$ such that

$$x + y \equiv 0 \pmod 7$$

is an isomorphism, as if $i - j \in \{1, 2, 4\}$, then $j - i \in \{3, 5, 6\}$.

Property 2.8 ([5]). *QR_7 has property $P_{2,1}$.*

Property 2.9. *For every $x \in V(QR_7)$ and every sequence $(z_u, z_v) \in \{0, 1\}^2$ there exists a pair of arcs $u_1v_1, u_2v_2 \in E(QR_7)$ such that the edge between x and y_i, $y \in \{u, v\}$, $i \in \{1, 2\}$, is oriented as xy_i if and only if $z_y = 1$.*

Property 2.10. *For a given arc ij, there exist vertices $k_1 \neq k_2$ such that ijk_1 and ijk_2 are directed 3−cycles.*

Property 2.11. *For a given arc ij, there are exactly three pairwise distinct vertices, $k_1, k_2, k_3 \in V(G)$ such that*

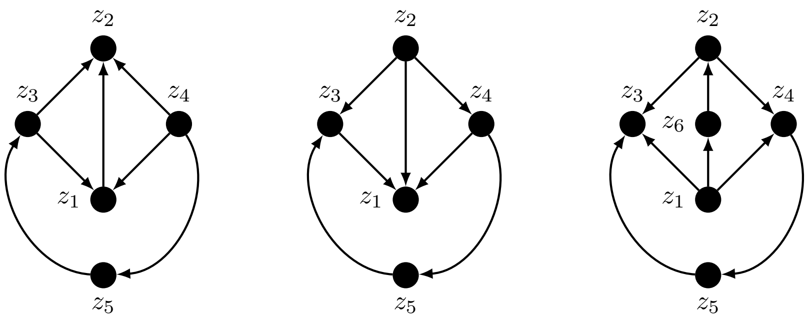


Figure 2.6: Oriented graphs that do not admit a homomorphism to QR_7.

- $ik_1, jk_1 \in E(G)$,
- $k_2i, k_2j \in E(G)$, *and*
- $ik_3, k_3j \in E(G)$.

By Property 2.6, these last three properties can be verified by considering the neighbourhood of 0 and the arc 01.

Property 2.12. *Let G be an oriented graph with a cut arc uv. The oriented graph G admits a homomorphism to QR_7 if and only if each component of $G \setminus \{uv\}$ admits a homomorphism to QR_7.*

This follows directly from the vertex transitivity of QR_7.

In [50] the author conjecture that 7 colours suffice for an oriented colouring of any member of $\mathcal{F}_3$. However it is not the case that QR_7 is a universal target for this family of oriented graphs. Let $\mathcal{Z}$ be the set of oriented graphs given in Figure 2.6 together with the oriented graphs formed by reversing all of the arcs in any pictured graph.

Proposition 2.13. *No oriented graph in $\mathcal{Z}$ admits a homomorphism to QR_7.*

Proof. Let G be an oriented graph in $\mathcal{Z}$ such that there exists $\phi : G \to QR_7$. For each $Z \in \mathcal{Z}$ it must be that $\phi(z_1) \neq \phi(z_2)$. By Property 2.11, $\phi(z_3) = \phi(z_4)$. But then z_3 and z_4 are the ends of a 2−dipath, a contradiction. □

Corollary 2.14. *Any oriented subcubic graph that contains a subgraph from $\mathcal{Z}$ does not admit a homomorphism to QR_7.*

Consider the family, $\mathcal{R}$, of oriented graphs formed from graphs $\mathcal{Z}$ by

- adding a pair of vertices r_1 and r_2,
- adding in the arcs r_1z_3 and z_4r_2, and
- deleting z_5.

For any $R \in \mathcal{R}$, observe that identifying r_1 and r_2 into a single vertex gives the oriented graph from $\mathcal{Z}$ that was used to generate R.

Since no subcubic oriented graph in $\mathcal{Z}$ admits a homomorphism to QR_7, in any subcubic oriented graph that contains a copy of an oriented graph from $\mathcal{R}$ that admits a homomorphism to QR_7 it must be that r_1 and r_2 are assigned different colours. Consider the following reduction to those subcubic graphs in $\mathcal{F}_3$ that contain a copy of an oriented graph from $\mathcal{R}$.

Reduction. *Let G be a subcubic oriented graph such that G contains a subgraph $R \in \mathcal{R}$. The subcubic graph G^R is obtained from G by*

- *deleting the vertices corresponding to z_1, z_2, z_3, z_4 and, if it exists, z_6;*
- *adding a vertex r together with the arcs rr_1 and r_2r.*

We call an oriented subcubic graph *reducible* if it may be reduced and an oriented subcubic graph *reduced* if it cannot be reduced. Since each oriented graph in $\mathcal{R}$ contains either a source or a sink of degree 3, if an oriented subcubic graph has no source and no sink of degree 3, then it is reduced.

Lemma 2.15 (The Reduction Lemma)**.** *Let G be a reducible oriented subcubic graph. Then G admits a homomorphism to QR_7 if and only if G^R admits a homomorphism to QR_7.*

Proof. Let G be a reducible oriented subcubic graph that admits a homomorphism, ϕ, to QR_7. Let x_i be the vertex corresponding to z_i in the copy of $Z \in \mathcal{Z}$ formed by identifying r_1 and r_2 in G. Since ϕ is a homomorphism to QR_7 it must be that $\phi(x_1) \neq \phi(x_2)$. By Property 2.11 of QR_7, we have directly that that $\phi(x_3) = \phi(x_4)$, which in turn implies that $\phi(r_1) \neq \phi(r_2)$. Restricting this homomorphism to the vertices that are common to G^R and G, and colouring r using property $P_{2,1}$ yields a homomorphism from G^R to QR_7.

Assume now that G^R admits a homomorphism, β, to QR_7. By Property 2.6 of QR_7, we may assume that $\beta(r) = 0$. If the vertex x_6 does not exist in G, we see that β can be used to colour G by colouring each of x_3 and x_4 with 0 and then colouring the remaining vertices using Property 2.8. Consider now the case that x_6 does exist. Since G is subcubic and x_6 is adjacent with both x_1 and x_2 we must consider the colour of a potential third neighbour, s, of x_6 in G. Since $s \in G^R$, let $\beta(s) = k$. We wish to extend β to all vertices of G in such a way that the arc between $\beta(s)$ and $\beta(x_6)$ in QR_7 is oriented the same way as the arc between s and x_6 in G. As in the case where x_6 did not exist, we can extend β to colour the vertices x_3 and x_4 each with colour 0. By Property 2.7 we may, without loss of generality, assume that the arcs between x_1 and x_3, and x_1 and x_4 are oriented such that x_3x_1 is an arc. Observe that the colours in the set $\{1, 2, 4\}$ may be assigned to the vertices x_1 and x_2. Therefore colours in the set $\{2, 3, 5, 6\}$ may appear on the vertex x_6. Every vertex in QR_7 appears as an out-neighbour (respectively in-neighbour) of a vertex in the set $\{2, 3, 5, 6\}$. And so the colouring may be extended to be consistent with the colour of s. □

Consider those oriented graphs, G, with the property that a single reduction produces an oriented graph from $\mathcal{Z}$. In G^R the vertex r corresponds to, without loss of generality, z_5. Therefore there exist vertices of G that are configured as in the subgraph shown in Figure 2.7, or the graphs formed by replacing one or both of the 2−dipaths $x_4x_5x_3$ and $y_4y_5y_3$ with a single arc from the start of the 2−dipath to the end of the 2−dipath. In this figure, the direction of the undirected edges can take orientations as the oriented graphs in $\mathcal{Z}$.

Our goal in the remainder of this section is to prove that every connected oriented cubic graph has an oriented colouring that uses no more than 9 colours. First we show that any reduced oriented subcubic graph that does not have a subgraph from $\mathcal{Z}$ admits a homomorphism to QR_7.

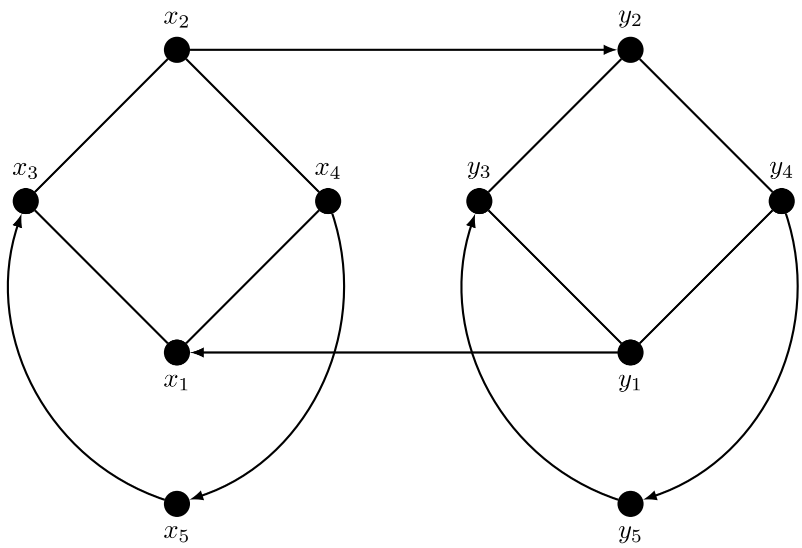


Figure 2.7: A graph that reduces to a graph containing a member of $\mathcal{Z}$ with a single reduction.

Lemma 2.16. *Every reduced connected oriented subcubic graph that does not contain a subgraph isomorphic to an oriented graph in $\mathcal{Z}$ admits a homomorphism to QR_7.*

Proof. Let G be a minimum counter-example with respect to number of vertices and subject to that with respect to the number of arcs. Since G is minimum there exists a vertex of degree 2, z, with neighbours u and v such that uzv is a $2-$dipath. Further in every homomorphism from $G \setminus \{z\}$ to QR_7, u and v receive the same colour, as otherwise z may be coloured using Property 2.8 of QR_7. Notice that if either u or v have degree 1 in $G \setminus \{z\}$, then u and v need not receive the same colour as $G \setminus \{z\}$ has a cut arc and QR_7 is vertex transitive. Let u_1, u_2 (respectively v_1 and v_2) be the neighbours of u (respectively v) in $G \setminus \{z\}$. We proceed by proving various properties about G that eventually allow us to conclude that G does not exist.

Claim 1. *G does not contain a cut arc.*

This follows directly from Property 2.12 of QR_7 and the minimality of G.

Claim 2. *If $\{e_1, e_2\}$ is an edge cut in $G \setminus \{z\}$, then e_1 and e_2 have a common endpoint of degree* 2.

Assume the contrary.

Case I: e_1 and e_2 have a common endpoint of degree 3. Let a be a common endpoint of e_1 and e_2 such that a has degree 3 in G. Let b be the neighbour of a that is not incident with e_1 or e_2. It follows directly that ab is a cut arc of $G \setminus \{z\}$. This violates Claim 1.

Case II: e_1 and e_2 do not have a common endpoint. Since neither e_1 nor e_2 is a cut edge, $G \setminus \{z, e_1, e_2\}$ has exactly two components. Let a_1 and b_1 be the endpoints of e_1 and a_2, and let b_2 be the endpoints of e_2 such that a_1 and a_2 are in the same component, A, of $G \setminus \{z, e_1, e_2\}$. Let $B = G \setminus A$.

Case II.i: u and v are in different components of $G \setminus \{z, e_1, e_2\}$. Let u be in the same component as a_1 and a_2 in $G \setminus \{z, e_1, e_2\}$. We proceed by examining homomorphisms $\phi_A : A \to QR_7$ and ϕ_B: $B \to QR_7$ and the direction of the arcs between A and B. By the minimality of G, such homomorphisms must exist.

If there exist homomorphisms $\phi_A : A \to QR_7$ and ϕ_B: $B \to QR_7$ such that $\phi_A(a_1) \neq \phi_A(a_2)$ and $\phi_B(b_1) \neq \phi_A(b_2)$, then we construct a homomorphism $G \to QR_7$ as follows. Since G is arc transitive, we may assume that $\phi_A(a_1) = 0$ and $\phi_A(a_2) = 1$. Further, since $\phi_B(b_1) \neq \phi_B(b_2)$ we may assume the existence of an arc, in some direction, between b_1 and b_2. If such an arc does not exist we may add it such that it is oriented the same as the arc between $\phi_B(b_1)$ and $\phi_B(b_2)$ in QR_7. The graphs in Table 2.1 give the possibilities for the arcs between A and B. Since QR_7 is arc transitive, we may construct a pair homomorphisms $\phi, \phi' : G \setminus \{z\} \to QR_7$ as follows.

- For all $a \in V(A)$, $\phi(a) = \phi'(a) = \phi_A(a)$.
- For all $b \in V(B)$, $\phi(b) = \alpha_B(b)$ and $\phi'(b) = \alpha'_B(b)$, where each of α_B and α'_B are homomorphisms from B to QR_7 (See Table 2.1). Since QR_7 is arc transitive, there is an automorphism of QR_7 that induces a map from α_b to α'_b.

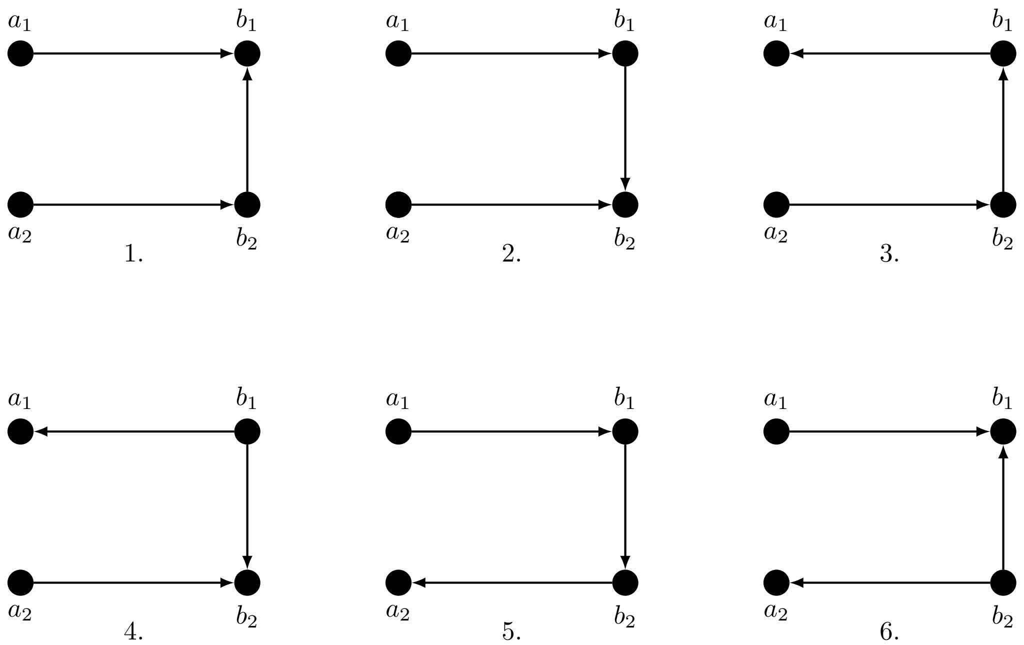
Observe that in each of these cases, the automorphism of QR_7 that maps the arc $\alpha_B(b_1)\alpha_B(b_2)$ to the arc $\alpha'_B(b_1)\alpha'_B(b_2)$ (or $\alpha_B(b_2)\alpha_B(b_1)$ to the arc $\alpha'_B(b_2)\alpha'_B(b_1)$) does not fix any vertex of QR_7. Therefore, if $\phi(u) = \phi(v)$, then $\phi'(u) \neq \phi'(v)$ And so, one of ϕ and ϕ' may be extended to include z.

Assume for all homomorphisms $\phi_A : A \to QR_7$ and ϕ_B: $B \to QR_7$ that $\phi_A(a_1) = \phi_A(a_2)$ and $\phi_B(b_1) = \phi_A(b_2)$. If the arcs between A and B are both oriented to have their head in A (respectively B), then, since QR_7 is vertex transitive, a homomorphism may be constructed from $G \setminus \{z\}$ to QR_7 such that u and v are assigned different colours, as the oriented graph produced by identifying a_1 and a_2, and b_1 and b_2 admits a homomorphism to QR_7 and contains a cut arc.

Therefore we may assume, without loss of generality, that a_1 is the head of e_1 and b_2 is the head of e_2. Consider constructing $A^\star$ from A by adding a vertex, a, together with the arcs a_2a and aa_1. It cannot be that $A^\star$ admits a homomorphism to QR_7, as otherwise such a homomorphism would be one in which a_1 and a_2 are assigned different colours. By minimality, $A^\star$ contains a copy of a graph from $\mathcal{Z}$ or a copy of a graph from $\mathcal{R}$. It must be that a is in this copy. Since a has degree 2, we can assume that it corresponds to either z_5 or z_6 in either case. By symmetry we may assume that it corresponds to z_6. If A contains a copy of a graph from $\mathcal{Z}$, observe that the vertex corresponding to z_5 is a cut vertex. This contradicts that $G \setminus \{z\}$ has no cut arc. Therefore $A^\star$ contains a copy of a graph from $\mathcal{R}$ and, when reduced, contains a copy of a graph from $\mathcal{Z}$. We note that by the minimality of G, a single reduction in $A^\star$ yields a copy of a graph from $\mathcal{Z}$. Therefore $A^\star$ contains a copy of the graph given in Figure 2.7, where a corresponds to a vertex of degree 2. However if this is the case we notice that $G \setminus \{z\}$ is reducible. This is a contradiction.

Finally, assume for all homomorphisms $\phi_A : A \to QR_7$ that $\phi_A(a_1) = \phi_A(a_2)$ and for all homomorphisms ϕ_B: $B \to QR_7$ that $\phi_B(b_1) \neq \phi_A(b_2)$. Since $\phi_B(b_1) \neq \phi_B(b_2)$, we may assume the existence of an arc, in some direction, between b_1 and b_2. If such an arc does not exist we may add it such that it is oriented the same as the arc between $\phi_B(b_1)$ and $\phi_B(b_2)$ in QR_7. By identifying a_1 and a_2 into a single vertex and by applying Property 2.9 of QR_7 we obtain a homomorphism from $G \setminus \{z\}$ to QR_7 in which u and v are assigned different colours.

Case II.ii: u and v are in the same component of $G \setminus \{z, e_1, e_2\}$.



1.	α_B	α'_B
b_1	4	1
b_2	2	6

2.	α_B	α'_B
b_1	1	2
b_2	2	3

3.	α_B	α'_B
b_1	3	6
b_2	2	5

4.	α_B	α'_B
b_1	5	6
b_2	2	3

5.	α_B	α'_B
b_1	2	4
b_2	4	6

6.	α_B	α'_B
b_1	2	1
b_2	0	6

Table 2.1: Colourings for Claim 2

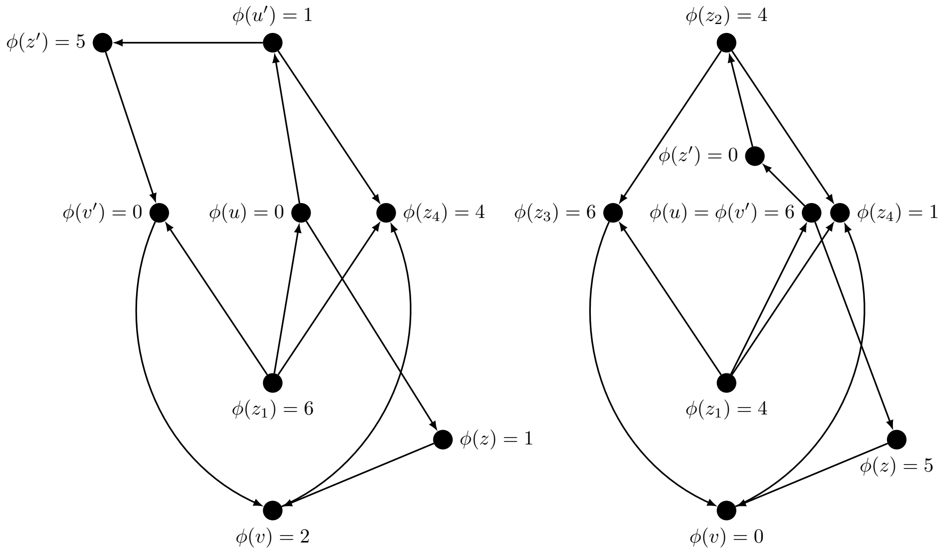


Figure 2.8: A configuration of vertices for Claim 3.

Let u and v be in A. By the minimality of G, observe that B admits a homomorphism to QR_7. Construct A_z by adding the vertex z together with the arcs uz and zv to A. By the minimality of G, A_z admits a homomorphism to QR_7. Regardless of the orientations of the arcs between A and B, these homomorphisms may be combined to be one from G to QR_7 as above, as long as it is not the case that for all $\phi_{A_z} : A_z \to QR_7$ and ϕ_B: $B \to QR_7$ it is the case that $\phi_A(a_1) = \phi_A(a_2)$ and $\phi_B(b_1) = \phi_A(b_2)$, and that a_1 is the head of e_1 and a_2 is the tail of e_2. However in this case we proceed as in Case II.i by constructing $A^\star_z$ and following the argument above.

Therefore if $\{e_1, e_2\}$ is an edge cut in $G \backslash \{z\}$, then e_1 and e_2 have a common neighbour of degree 2.

Claim 3. *G contains a single vertex of degree* 2*.*

Assume there exists $z' \neq z$ with neighbours u' and v' such that $u'z'v'$ form a $2-$dipath. Consider the oriented graph $G^{z'}$ formed by removing z' and adding in the arc $u'v'$. If this graph admits a homomorphism to QR_7, then this homomorphism may be extended to include z'. This is a contradiction. Therefore $G^{z'}$contains either an oriented graph from $\mathcal{Z}$ or $\mathcal{R}$. It must be that both u' and v' appear in this copy. If $G^{z'}$ contains a copy of $R \in \mathcal{R}$, then by the minimality of G it must contain a graph as in Figure 2.7. Since G does not contain this graph, it must be that the newly added arc corresponds to the arc between x_2 and y_2 or the arc between y_1 and x_1. However here we see that G is reducible.

If $G^{z'}$ contains a copy of $Z \in \mathcal{Z}$ and this copy does not contain z_6, then the arc incident with z_5 is a cut arc in G. Therefore Z must contain z_6. In this case we see that the arcs not in Z that are incident with z_6 and z_5 form an edge cut. By Claim 2 these two arcs must have a common endpoint of degree 2. If so, this common endpoint must be z, as all other vertices have degree 3. The oriented graphs given in Figure 2.8 give two of the four possibilities for the configuration of the vertices in G. The other two can be obtained by reversing the orientations of all the arcs. We see that both of these oriented graphs admit a homomorphism to QR_7. This is a contradiction.

Claim 4. *u_1 and u_2 do not have three common neighbours in G setminus$\{z\}$.*

If u_1 and u_2 have three common neighbours in $G\setminus\{z\}$, then $G\setminus\{z\}$ contains a 2−edge cut. This is a violation of Claim 2.

Claim 5. *u_1 and u_2 are not adjacent in G setminus$\{z\}$.*

If u_1 and u_2 are adjacent in $G \setminus \{z\}$, then $G \setminus \{z\}$ contains a 2−edge cut. This is a violation of Claim 2.

Claim 6. *Each of u and v is either a source or sink vertex in $G \setminus \{z\}$*

Assume that u_1uu_2 forms a 2−dipath in $G\setminus\{z\}$. Consider the graph, H_u, formed from $G \setminus \{z\}$ by removing u and adding the arc u_2u_1. If H_u admits a homomorphism to QR_7, then by Property 2.10 there exists a homomorphism from $G \setminus \{z\}$ to QR_7 in which u and v receive different colours. Therefore we may assume that H_u contains either a subgraph from $\mathcal{Z}$ or a subgraph from $\mathcal{R}$.

Assume that H_u contains a subgraph from $\mathcal{Z}$. It must be that the arc u_2u_1 is in this subgraph. And so since $G\setminus\{z\}$ has no cut arc or 2−edge cut, it must be that z_6 exists. Up to symmetry, the arc u_2u_1 corresponds to the one between z_1 and z_3 or the one between z_5 and z_3. However observe that in either case $G \setminus \{z\}$ has either a cut arc or a 2−edge cut.

Assume that H_u contains a subgraph from $\mathcal{R}$. By the minimality of G it must contain a graph as in Figure 2.7. Since $G \setminus \{z\}$ is not reducible it must that u_2u_1 corresponds to, without loss of generality, the arc between x_2 and y_2. However in this case observe the arcs incident with x_5 and y_5 that do not have their other ends at either x_3, x_4, y_3 or y_4 form a 2−edge cut. If these arcs have a common endpoint, it must be an endpoint of degree 2, as otherwise $G \setminus \{z\}$ has a cut arc. Since G has only one vertex of degree 2, this common endpoint must be v. We conclude that the vertices are configured as in Figure 2.9. Since this graph must reduce to one that contains a copy of a graph from $\mathcal{Z}$, we may assume that neither u_1 nor u_2 are the centre of a 2−dipath in $G \setminus \{z, u\}$. This leads to the four possible partial orientations given in Figure 2.10. However in each of these cases, regardless of the orientation of the arcs incident with v, a homomorphism to QR_7 exists in which u and v receive different colours, as shown in Figure 2.11.

Therefore H_u does not reduce to have an oriented graph from $\mathcal{Z}$. Therefore H_u admits a homomorphism to QR_7.

Claim 7. $|\{u_1, u_2, v_1, v_2\}| \neq 2$.

If this is true, then either G is reducible or violates Claim 2.

Claim 8. $|\{u_1, u_2, v_1, v_2\}| \neq 3$.

Suppose $|\{u_1, u_2, v_1, v_2\}| = 3$. Assume, without loss of generality, that $u_1 = v_1$. Since u and v receive the same colour in any QR_7−colouring of $G \setminus \{z\}$, it must be that the arc between u_1 and u, and the arc between u_1 and v are oriented in the same direction with respect to u_1. Consider the subcubic graph, A_u, formed by removing z and v and adding an arc between u and v_2 that is oriented oppositely to the arc between v and v_2, with respect to v_2. If A_u admits a homomorphism to QR_7, then observe that it may be extended to include v by applying Property 2.8 of QR_7. We see that in this case u and v are assigned different colours as uv_2v form a 2−dipath in the graph formed by adding v. The existence of such a homomorphism is a violation of the assumption that G does not admit a homomorphism to QR_7. Therefore it must be that A_u either contains a copy of a graph from $\mathcal{Z}$ or is reducible. Observe that in A_u, $d(u_1) = 2$.

Assume that A_u contains a copy of a graph from $\mathcal{Z}$. Since adding the arc between v_2 and u created this copy, it must be that this arc appears in the copy of the graph from $\mathcal{Z}$. Up to symmetry there are two possibilities for this arc: the arc between z_2 and z_3 or

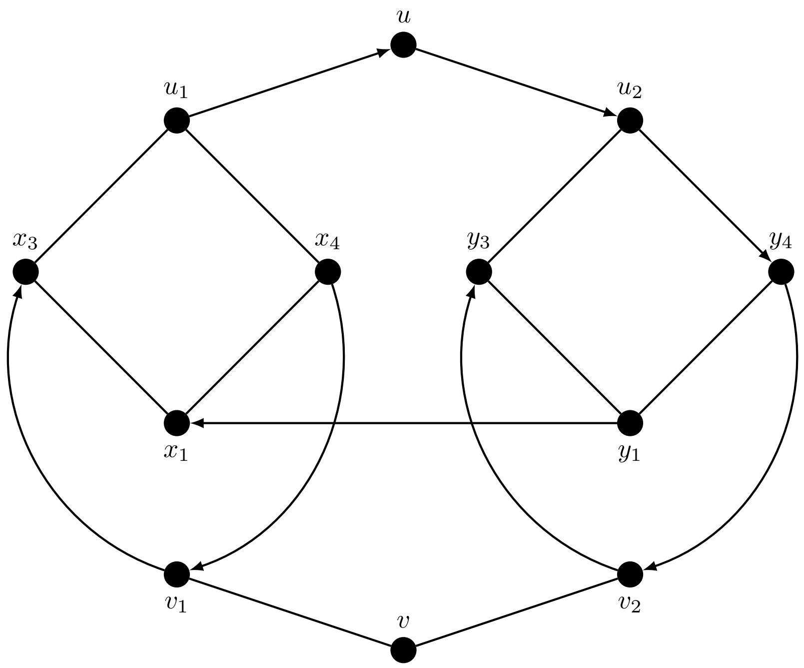


Figure 2.9: A configuration of vertices for Claim 6.

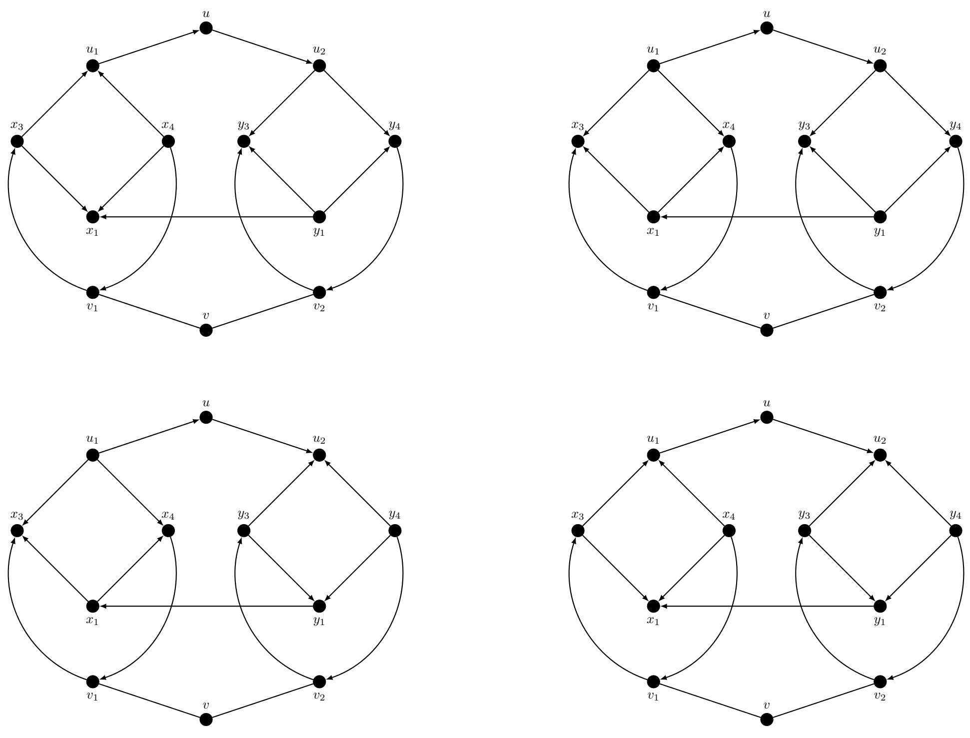


Figure 2.10: A configuration of vertices for Claim 6.

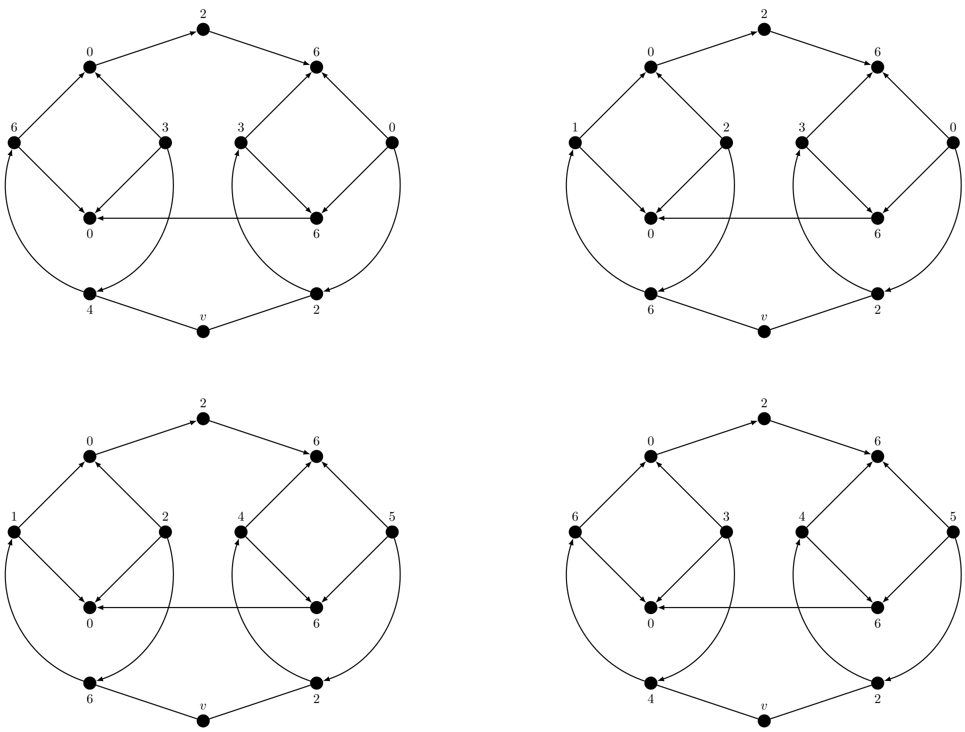


Figure 2.11: Colourings of four orientations for Claim 6.

the arc between z_3 and z_5. We note that although z_5 has degree 2 in this copy, it may have degree 2 or degree 3 in A_u.

Case I: u corresponds to z_3. Since u_1 has degree 2 in A_u, it must be that u_1 corresponds to z_5. This implies that u_2 corresponds to z_1. If z_6 does not exist, then $G \setminus \{z\}$ is the oriented graph given in Figure 2.12, or the one formed by reversing each arc. This oriented graph admits a homomorphism to QR_7 in which u and v receive different colours, a contradiction. If z_6 exists, then observe that the vertex corresponding to z_6 is a cut vertex in $G \setminus \{z\}$, or has degree two. This is a contradiction of Claim 1 or Claim 3.

Case II: u corresponds to z_2 and v_2 corresponds to z_3. Since u_1 has degree 2 in A_u, it must be that u_1 corresponds to z_6. This implies that the vertex corresponding to z_5 is a cut vertex in $G \setminus \{z\}$ or is a vertex of degree 2. This is a contradiction of Claim 1 or Claim 3.

Case III: u corresponds to z_5 and v_2 corresponds to z_3. Since u_1 has degree 2 in A_u, it must be that u_2 corresponds to z_4. If z_6 exists, then vertices are configured as in Figure 2.13. However, here we see that G has a $2-$edge cut. If the arcs in the cut have a common endpoint, then either this endpoint is v, or $G \setminus \{z\}$ has a cut vertex. However, we observe that this endpoint is not v. Therefore $G \setminus \{z\}$ has a cut vertex, this is a contradiction. If z_6 does not exist we see that the arc incident with u_1 that does not have its endpoint at either u or v is a cut arc, a contradiction.

Therefore A_u does not contain a copy of a graph from $\mathcal{Z}$, and so must contain a copy of an oriented graph in $\mathcal{R}$.

Assume that A_u contains a copy of a graph $R \in \mathcal{R}$. Since G is reduced, A_u must reduce to a graph containing a graph from $\mathcal{Z}$ with a single reduction. Therefore A_u contains a copy of the graph in Figure 2.7. Since G is reduced it must be that u corresponds to either x_1 or x_2, as otherwise G would be reducible. However, if this is the case we see that u_1 and u_2 have three common neighbours. This contradicts Claim 4. Therefore $|\{u_1, u_2, v_1, v_2\}| \neq 3$.

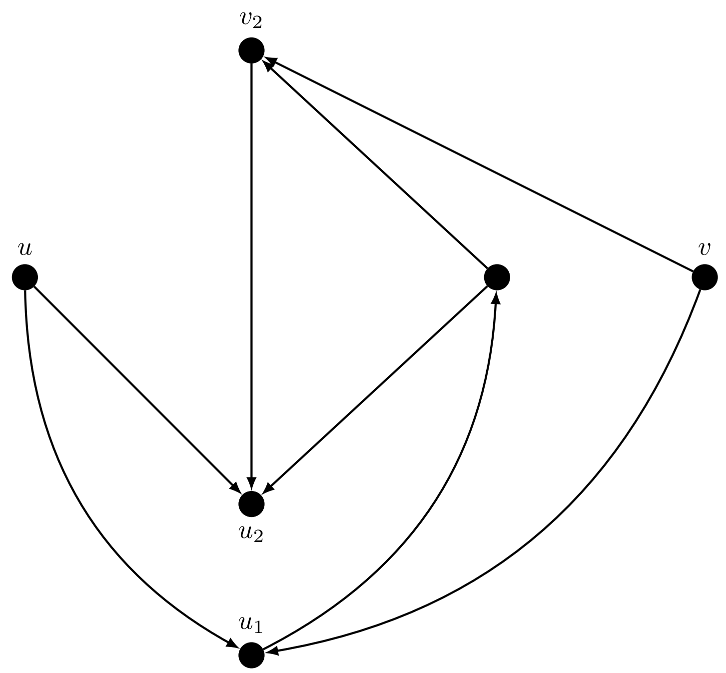


Figure 2.12: A configuration of vertices for Claim 8.

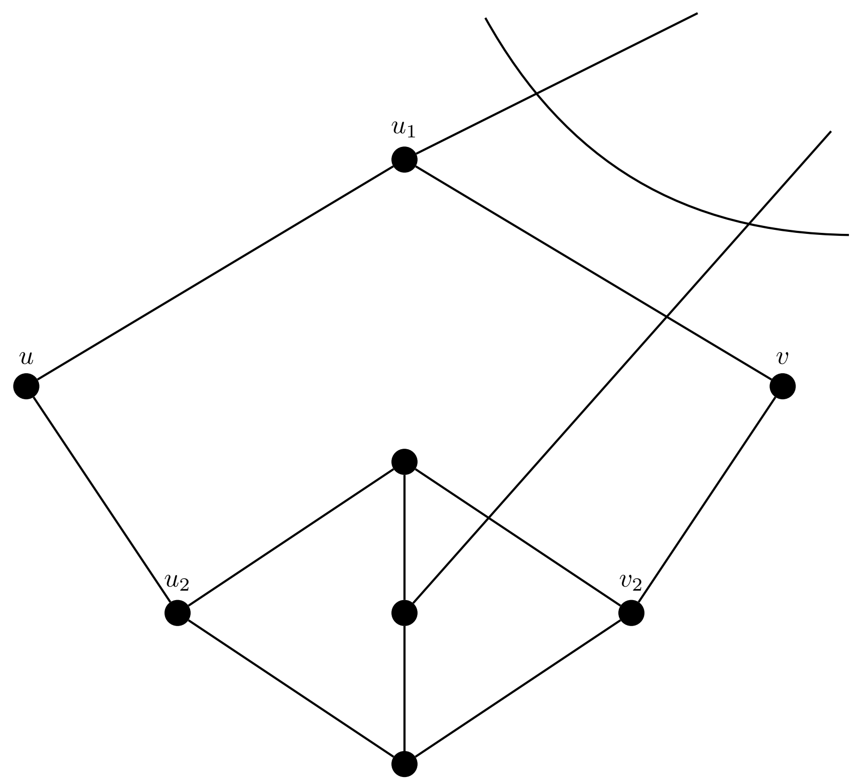


Figure 2.13: A configuration of vertices for Claim 8.

Claim 9. *$|\{u_1, u_2, v_1, v_2\}| \neq 4$.*

Suppose $|\{u_1, u_2, v_1, v_2\}| = 4$. Let A_{v_1} be the oriented graph formed from G by removing z, v and adding the edge between v_1 and u, orienting it oppositely to the arc between v and v_1, with respect to v_1.

If this oriented graph admits a homomorphism to QR_7, then, by Property 2.8, and Claim 6 we can extend this homomorphism to include v. However in this case it cannot be that u and v receive the same colour; there is a $2-$dipath between them. Therefore A_{v_1} does not admit a homomorphism to QR_7. As such it either contains a copy of an oriented graph in $\mathcal{Z}$ or is reducible. Similarly we construct A_{v_2} and assert that it contains a copy of an oriented graph in $\mathcal{Z}$ or is reducible.

We claim that neither of A_{v_2} and A_{v_1} contain a graph from $\mathcal{R}$. Assume that A_{v_1} contains a graph $R \in \mathcal{R}$. By the minimality of G, A_{v_1} must reduce to an oriented graph that contains a graph from $\mathcal{Z}$. Further it must be that a single reduction leads to a copy of a graph from $\mathcal{Z}$, as otherwise G would be reducible. Therefore A_{v_1} contains a copy of the graph in Figure 2.7. Since G is not reducible it must be that, without loss of generality, u corresponds to x_2 and v_1 corresponds to y_2. However, if this is true, u_1 and u_2 have three common neighbours in $G \setminus \{z\}$. This contradicts Claim 4.

Therefore each of A_{v_2} and A_{v_1} contain a graph from $\mathcal{Z}$. Assume that A_{v_1} contains a copy of $Z \in \mathcal{Z}$. We first show that each of u_1 and u_2 has at least two common neighbours with one of v_1 and v_2 in G. We do this by considering the degree of the vertex to which u corresponds in Z. From this fact we then derive a contradiction.

If u corresponds to a vertex of degree 2 in Z it must be that v_1 corresponds to a vertex of degree 3. So we may assume that if u corresponds to z_5, then one of u_1 and u_2 corresponds to z_4. Without loss of generality, we may assume that u_1 corresponds to z_4. We see that u_1 has two common neighbours with v_1 in G.

If u corresponds to a vertex of degree 3 in Z, then it cannot be that v_1 corresponds to a vertex of degree 2, as otherwise u_1 and u_2 would have three common neighbours. Therefore, if u corresponds to z_1, then we may assume that u_1 corresponds to z_3, u_2 corresponds to z_6 and v_1 corresponds to z_4. Notice that u_1 and u_2 each have two common neighbours with v_1 in G.

By considering A_{v_2} and observing that u_1 cannot have two common neighbours with both v_1 and v_2 we conclude that u_2 has two common neighbours with v_2. Thus the vertices are configured as in Figure 2.14. Here we see a $2-$edge cut. This is a contradiction of Claim 2. Therefore one of A_{v_2} or A_{v_1} admits a homomorphism to QR_7.

Claim 10. *G does not exist.*

By the previous claims $|\{u_1, u_2, v_1, v_2\}| > 4$. □

Theorem 2.17. *An orientation of a connected graph with $\Delta \leq 3$ has oriented chromatic number at most 9.*

Proof. Let G be a connected oriented cubic graph. We proceed based on the existence of source and sink vertices of degree 3.

Case I: G has a source or a sink vertex of degree 3. Let $G^\star$ be the oriented graph formed by removing all the source and sink vertices of degree 3. Since $G^\star$ contains no source or sink vertices of degree 3, $G^\star$ is reduced and contains no subgraph from $\mathcal{Z}$. By Lemma 2.16, there exists $\phi^\star : G^\star \to QR_7$. Let QR_7' be the oriented graph formed from QR_7 by adding a universal source vertex, s, and a universal sink vertex, t. We construct a homomorphism $\phi : G \to QR_7'$ given by

- $\phi(u) = \phi'(u)$, for all $u \in V(G)$ such that u has positive in- and out-degree.
- $\phi(u) = s$, for all $u \in V(G)$ such that $d^+(u) = 3$.

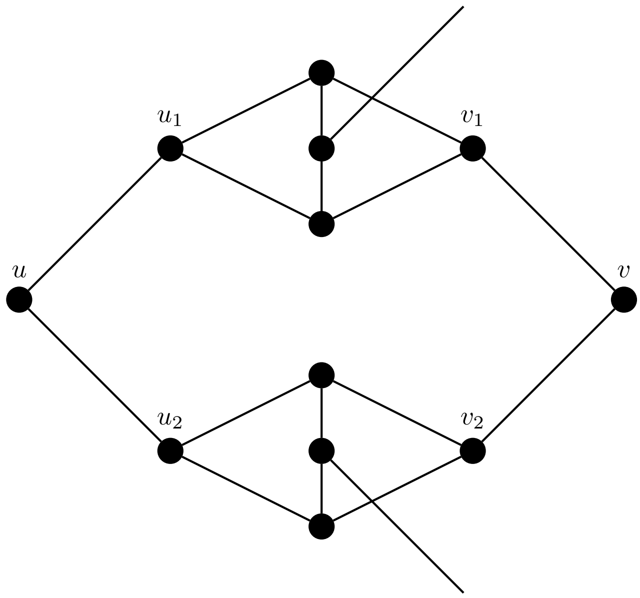


Figure 2.14: A configuration of vertices for Claim 9.

- $\phi(u) = t$, for all $u \in V(G)$ such that $d^-(u) = 3$.

Case II: G has neither a source or a sink vertex of degree 3. Let $uv \in E(G)$. Since G has no source or a sink vertex of degree 3, $G \setminus \{uv\}$ is reduced and contains no subgraph from $\mathcal{Z}$. By Lemma 2.16, there exists $\phi : G \setminus \{uv\} \to QR_7$. We extend ϕ to be an oriented 9−colouring of G by letting $\phi(u) = 7$ and $\phi(v) = 8$.

□

Note that in this theorem the assumption of connectedness is important. We achieve an oriented 9−colouring by either removing an arc, or removing sources and sinks. This technique will fail to produce an oriented 9−colouring in the case where G is not connected, each reduced component is cubic, and not all of these components contain a copy of a graph from $\mathcal{Z}$.

Corollary 2.18. *For the family, $\mathcal{F}_3$, of orientations of connected graphs with maximum degree at most three,* $7 \leq \chi_o(\mathcal{F}_3) \leq 9$.

2.4 Oriented Colourings of Graphs with Maximum Degree Four

For the family, $\mathcal{F}_4$, of orientations of connected graphs with maximum degree 4, Theorem 2.2 gives an upper bound of 512. However, for $\mathcal{F}_4$ we can get a better bound by considering the acyclic chromatic number of graphs with $\Delta \leq 4$. Graphs with maximum degree 4 have acyclic chromatic number at most 5 [23], and so by Theorem 1.4 in Chapter 1 ,

$$\chi_o(\mathcal{F}_4) \leq 5 \cdot 2^{5-1} = 80.$$

As with our improved bound for orientations of cubic graphs, we use a non-zero quadratic residue tournament as a means to construct a target.

Proposition 2.19 (Bonato [5]). *The Paley tournament on* 67 *vertices, QR_{67}, has property $P_{4,1}$ and property $P_{3,2}$.*

Proposition 2.20. *The Paley tournament on* 67 *vertices, QR_{67} is vertex transitive and arc transitive.*

This follows similarly to Property 2.6.

Lemma 2.21. *Every orientation of a* $3-$*degenerate graph with maximum degree at most* 4 *admits a homomorphism to* QR_{67}

Proof. Let G be a minimum counter-example with respect to number of vertices and subject to that with respect to the number of arcs. We consider cases based on the minimum degree of a vertex in G. Let z be a vertex of minimum degree in G. Since G is $3-$degenerate, it must be that z has degree 1, 2 or 3.

Case I: z has degree 1: Since QR_{67} has property $P_{1,1}$ any homomorphism $\phi : G \setminus \{z\} \to QR_{67}$ can be extended.

Case II: z has degree 2: Let u and v be the neighbours of z in G. By the minimality of G, $G \setminus \{z\}$ admits a homomorphism to QR_{67}. If both u and v have z as an out-neighbour (respectively in-neighbour), then any homomorphism from $G \setminus \{z\}$ to QR_{67} can be extended, since QR_{67} has property $P_{2,1}$. Thus, without loss of generality, we may assume that $uz, zv \in E(G)$. We may further assume that in every homomorphism from $G \setminus \{z\}$ to QR_{67}, u and v receive the same colour, as otherwise any homomorphism from $G \setminus \{z\}$ to QR_{67} can be extended, since QR_{67} has property $P_{2,1}$.

Consider a homomorphism $\phi : G \setminus \{z\} \to QR_{67}$. Let ϕ' be the restriction of this homomorphism to $G \setminus \{u, v, z\}$. Since QR_{67} has property $P_{4,1}$ it also has property $P_{3,2}$. This implies that ϕ' may be extended in such a way that u and v receive different colours. This is a contradiction.

Case III: z has degree 3: Let u, v, w be the neighbours of z in G. Following *Case II* we may assume, without loss of generality, that $uz, zv, zw \in E(G)$ and that in every homomorphism from $G \setminus \{z\}$ to QR_{67} that u receives the same colour of at least one of v and w. This implies we may assume that u is adjacent to at most one of v and w. We proceed based on the existence of arcs between u, v and w.

Case III.i: u, v, w form an independent set: Consider a homomorphism $\phi : G \setminus \{z\} \to QR_{67}$. Let ϕ' be the restriction of this homomorphism to $G \setminus \{u, v, w, z\}$. Consider extending ϕ'. Since QR_{67} has property $P_{3,2}$, there are two choices for each of u, v, w, and each of these choices may be made independently. By hypothesis, no matter how these choices are made, it must be that u receives the same colour as at least one of v or w. Consider a graph with vertex set $\{u', v', w'\}$ and edge set $\{u'v', u'w'\}$. If we assign to u' (respectively v' and w') the same list of colours that are available for u (respectively v and w) when extending ϕ', then a list colouring of this constructed graph corresponds exactly to an extension of ϕ' to include u, v and w where u does not receive the same colour as v or w. Since QR_{67} has property $P_{3,2}$ each of these lists has cardinality at least two. Since $K_{2,1}$ is $2-$chooseable, such an extension must exist. This is a contradiction.

Case III.ii: There is an arc, in some direction, between u and v. Consider a homomorphism $\alpha : G \setminus \{z\} \to QR_{67}$. Let α' be the restriction of this homomorphism to $G \setminus \{w, z\}$. Since QR_{67} has property $P_{3,2}$, α' can be extended in such a way that w does not receive the same colour as u. Therefore there is a homomorphism from $G \setminus \{z\}$ to QR_{67} in which u does not receive the same colour as v or w. This is a contradiction. □

Theorem 2.22. *An orientation of a connected graph with $\Delta \leq 4$ has oriented chromatic number at most* 69*.*

Proof. Let G be an oriented graph such that $\Delta(U(G)) \leq 4$ and let $uv \in E(G)$. By Lemma 2.22, $G \setminus \{uv\}$ admits a homomorphism to QR_{67}. An oriented colouring of G using 69 colours can be constructed from this homomorphism by adding the arc uv and recolouring u and v respectively with two new colours. □

2.5 Future Directions and Conclusions

The proof in Case II of Theorem 2.17 presents an interesting case. If it is indeed the case that 9 colours are required, then oriented colourings constructed in this manner have a pair of vertices that are the only elements of their respective colour class; these oriented graphs are critical with respect to oriented chromatic number. Further, for every arc $uv \in V(G)$ it is possible to construct an oriented $9-$colouring so that u and v are the only vertices of their respective colour. In fact, by considering all choices of uv, if an oriented cubic graph does need 9 colours, then the oriented graph formed by reversing any arc of G leads to an oriented cubic graph that requires only 7 colours. It is also worth pointing out that if the QR_7-colouring maps the ends of uv to a correctly oriented arc in QR_7, then u and v need not be recoloured. In such a case an oriented $7-$colouring of G exists.

In [50] the author conjectures that 7 colours suffice for an oriented colouring of an orientation of a connected graph with maximum degree 3. Our result, that 9 colour suffice, improves the previous upper-bound of 11. Though not specifically invoked in the proof, the assumption of connectedness is important. Consider an oriented cubic graph that is formed from the disjoint union of a pair of oriented $3-$regular graphs, where one is oriented to have sources and sinks, and one without. By our main result each of these oriented graphs admits a homomorphism to a target with at most 9 vertices. However, by our construction, these two oriented graphs do not admit a homomorphism to the same target. Similar remarks can be made for the results concerning the family of orientations of graphs with maximum degree 4. When the ends of the arc are coloured with the two new colours, this implicitly defines a 69-vertex target. However, since the target constructed depends on the colours of the neighbours of the ends of the arc that had its ends recoloured, this target is not a universal target for the family of orientations of connected graphs with maximum degree 4.

Chapter 3
$k-$dipath Colourings of Oriented Graphs

In Chapter 3 we examine a generalisation of $2-$dipath colourings of oriented graphs. Using ideas similar to [34] we construct a homomorphism model for colourings that require vertices at the end of a directed path of length at most k receive different colours. Additionally, we consider the complexity of determining if a given oriented graph has a $k-$dipath colouring using no more than m colours, for fixed values of m and k.

3.1 Background and Preliminaries

Recall the vertex labelling definition for oriented colouring:

Let G be an oriented graph. A mapping $c : V(G) \to \{1, 2, 3, \ldots, m\}$ is an oriented colouring of G provided that:

- for all $uv \in E(G)$, $c(u) \neq c(v)$, and
- for all $uv, xy \in E(G)$ such that $c(u) = c(y)$, $c(v) \neq c(x)$.

In the second condition of the labelling definition of oriented colouring an interesting case arises when $v = x$. In this case we observe that it must be that u and y are assigned different colours. In general we see that in any oriented colouring that vertices at the ends of a $2-$dipath are assigned different colours. And so by considering proper vertex colourings of an oriented graph that also assign vertices at distance 2 different colours, we find a lower bound for the oriented chromatic number.

Definition 3.1. *Let G be an oriented graph. A* $2-$dipath colouring *of G* using m colours *is a mapping $c : V(G) \to \{1, 2, 3, \ldots, m\}$ such that:*

- *for all $uv \in E(G)$, $c(u) \neq c(v)$, and*
- *for all $uv, vw \in E(G)$, $c(u) \neq c(w)$.*

The $2-$dipath chromatic number *of G is the least integer m such that there is a $2-$dipath colouring of G using m colours. We use $\chi_{2d}(G)$ to denote this parameter.*

Since every oriented colouring is necessarily a $2-$dipath colouring we observe that if G is an oriented graph, then $\chi_{2d}(G) \leq \chi_o(G)$.

Definition 3.2. *If G a digraph, then define G^k to be the simple graph formed from G as follows:*

- *$V(G^k) = V(G)$, and*
- *$E(G^k) = \{uv |$ there is a directed path of length at most k from u to v in $G\}$*

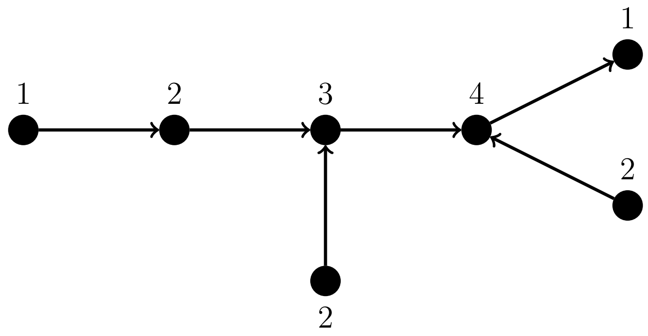


Figure 3.1: A 3−dipath colouring using 4 colours.

When considering 2−dipath colourings of an oriented graph G we are led naturally to the equivalence between a 2−dipath colouring and a proper vertex colouring of the simple graph, G^2 [34]. For 2−dipath colourings we consider oriented graphs, rather than general digraphs, as the inclusion of 2−cycles allows for a vertex to be on both ends of a directed path of length 2. And so when generalising to k−dipath colourings we consider only those graphs with no directed cycles of length at most k.

Definition 3.3. *Let G be an oriented graph with directed girth at least $k+1$. A k−*dipath colouring *of G using m* colours *is a mapping $c : V(G) \to \{1, 2, 3, \ldots, m\}$ such that for all pairs of distinct vertices $u, v \in V(G)$ if there is a directed path of length at most k from u to v, then $c(u) \neq c(v)$. The k−*dipath chromatic number *of G is the least m such that there is a k−dipath colouring of G using m colours. We use $\chi_{kd}(G)$ to denote this parameter.*

Figure 3.1 gives an example of a 3−dipath colouring using 4 colours. Here this colouring is optimal as the directed path on 4 vertices requires 4 colours.

As an analogy to the notion of clique in a simple graph, we consider the notion of a *k− dipath clique*.

Definition 3.4. *Let G be an oriented graph with directed girth at least $k+1$. We say G is a k−*dipath *clique if $\chi_{kd}(G) = |V(G)|$.*

Proposition 3.1. *An oriented graph G is a k−dipath clique if and only if it has weak diameter at most k.*

Proof. Let G be an oriented graph with directed girth at least $k+1$. We observe that G^k is a complete graph if and only if for each pair of non-adjacent vertices, say u and v, there is a directed path of length at most k, in some direction, between u and v. This condition is equivalent to G having weak diameter at most k. □

The first appearance of k−dipath colouring is in a paper giving an upper bound for the 2−dipath chromatic number of Halin graphs [12]. Chen and Wang use directed distance when defining k−dipath colouring, and avoid the directed girth condition mentioned above. And so we note that our definition of k−dipath colouring differs slightly to theirs. Though the k−dipath chromatic number is largely unstudied, the 2−dipath chromatic number has received attention as it gives a lower bound for the oriented chromatic number. In her master's thesis [58] (more recently published as [34]), Sherk (née Young) gives a homomorphism model for 2−dipath colouring, which implies an upper bound for the oriented chromatic number as a function of the 2−dipath chromatic number. This model is discussed further in Section 3.3.

Our goal in this chapter is to construct a homomorphism model similar to that given for 2−dipath colourings in [34]. Additionally, we discuss the complexity of the problem of k−dipath colouring with m colours. To achieve these two goals we first require some preliminary results for k−dipath colourings.

3.2 A Theory of k−dipath Colouring

As discussed above, there is a direct connection between k−dipath colourings of a particular oriented graph, and proper colourings of the k^{th} power of the oriented graph, G^k. And so k−dipath colouring is equivalent to proper colouring of G^k for oriented graphs G with directed girth at least $k+1$.

Proposition 3.2. *If G is an oriented graph with directed girth at least $k+1$, then there is a one-to-one correspondence between k−dipath colourings of G and proper colourings of G^k.*

In [34] the authors observe this correspondence for 2−dipath colourings and also obtain the following result.

Theorem 3.3 (MacGillivray and Sherk [34]). *If G is an oriented graph, then $\chi_o(G) \leq 2^{\chi_{2d}(G)} - 1$.*

This upper bound follows from the construction of a universal target for the family of oriented graphs with 2−dipath chromatic number at most m. This universal target has an oriented colouring using $2^m - 1$ colours. We further explore the topic of universal targets for k−dipath colourings in Section 3.3. Theorem 3.3 implies the following result for the k−dipath chromatic number.

Corollary 3.4. *If G is an oriented graph with directed girth at least $k+1$, then $\chi_o(G) \leq 2^{\chi_{kd}(G)} - 1$.*

Proof. Let G be an oriented graph with directed girth at least $k+1$ such that $\chi_{kd}(G) \leq m$. Let c be a k−dipath colouring of G using at most m colours. Observe that c is a 2−dipath colouring of G. By Theorem 3.3, $\chi_o(G) \leq 2^m - 1$. □

Notice that the lower bound for $\chi_o(G)$ given in Theorem 3.3 does not hold for the k−dipath chromatic number. The oriented chromatic number of P_3, the directed path on four vertices, is 3, however; four colours are required for a 4−dipath colouring.

The proof given in Corollary 3.4 uses the fact that any k−dipath colouring is also a 2−dipath colouring. It is easy to see that any k−dipath colouring is also a k'−dipath colouring for all $k' < k$. The example given in Figure 3.1 is a k'−dipath colouring for $k' = 2, 3$.

Consider the case where G is acyclic and the longest directed path has at most k arcs. Since acyclic oriented graphs have infinite directed girth, the k'-dipath chromatic number is defined for all k'. Here the k'-dipath chromatic number is exactly the k−dipath chromatic number for any $k' > k$. To consider k−dipath colourings of acyclic oriented graphs we require the following result about path length in an acyclic oriented graph.

Theorem 3.5 (Maurer, Sudborough, and Welzl [36]). *If G is a acyclic oriented graph, and T_k is the transitive tournament on k vertices, then $G \to T_k$ if and only if G has no directed path with $k+1$ vertices.*

Theorem 3.6. *If G is an acyclic oriented graph and the longest directed path in G has m vertices, then $\chi_{kd}(G) = m$ for all $k \geq m$.*

Proof. Let G be an oriented acyclic graph graph where the longest directed path in G has m vertices. In any $k-$dipath colouring of G ($m \leq k$), each of the vertices in this longest path must receive a distinct colour, and so $\chi_{kd}(G) \geq m$. By Theorem 3.5, G admits a homomorphism $\phi : G \to T_m$, where T_m is the transitive tournament on m vertices. We claim ϕ is a $k-$dipath colouring of G. If not, then there exist a pair of vertices $u, v \in V(G)$ such that $\phi(u) = \phi(v)$ and a directed path $u = x_1, x_2, x_3, \ldots, x_\ell = v$ ($2 \leq \ell \leq m \leq k$). Since ϕ is a homomorphism it must be $\phi(x_1), \phi(x_2) \ldots, \phi(x_\ell)$ is a closed directed walk in T_m. This is a contradiction, as T_m is acyclic. □

Corollary 3.7. *An acyclic oriented graph, G, has $\chi_{kd}(G) \leq m$ if and only if G has no directed path on at least $m+1$ vertices.*

Corollary 3.8. *An acyclic oriented graph, G, has $\chi_{kd}(G) \leq m$ if and only if G admits a homomorphism to T_m.*

Though not a direct analogue, Theorem 3.6 has a similar flavour to the early results on graph colourings of Gallai, Roy, Hasse, Vitaver ([19], [45], [26], [56]) (see Theorem 1.1). In our version the length of the longest directed path in an acyclic oriented graph gives the value of the $k-$dipath chromatic number for all k such that k is larger than the length of the path. The proof of this theorem relies on T_k being acyclic, so there is no closed directed walk in T_k. This technique may be generalized for when the target has sufficiently large directed girth.

Theorem 3.9. *Let G be an oriented graph and let H be an oriented graph with directed girth at least $k+1$. If $G \to H$, then $\chi_{kd}(G) \leq \chi_{kd}(H)$.*

Proof. Let G and H be oriented graphs such that $\phi : G \to H$. If H has directed girth at least $k+1$, then so must G. Let c be a $k-$dipath colouring of H that uses m colours. Construct the mapping $c' : V(G) \to \{1, 2, 3, \ldots, m\}$ such that for all $v \in V(G)$, $c'(v) = c(\phi(v))$. Consider $u, v \in V(G)$ such that there is an $\ell-$dipath from u to v ($1 \leq \ell \leq k$). If $c(u) = c(v)$, then there is a closed directed walk $\phi(v) = y_1, y_2, \ldots, y_i = \phi(u)$ ($1 \leq i \leq k$) in H. However since $\ell \leq k$ the existence of such a walk violates that H has directed girth at least $k+1$. Therefore c' is a $k-$dipath colouring of G. □

3.3 A Homomorphism Model for $k-$dipath Colouring

In [34] the authors give a homomorphism model for $2-$dipath colouring. That is, for each $m \geq 1$ they describe an oriented graph $G_{2,m}$ with the property that an oriented graph G has $2-$dipath chromatic number m if and only if G admits a homomorphism to $G_{2,m}$. We begin by reviewing this model and offering an improvement to the upper bound in Theorem 3.3 for the cases $m = 3$ and $m = 4$. We then move on to consider a homomorphism model for $k-$dipath colourings.

3.3.1 The $2-$dipath Chromatic Number

The homomorphism model given in [34] is based upon the idea that in any $2-$dipath colouring with m colours and for any particular colour, i, three possibilities arise when considering the colours of vertices in the closed neighbourhood of v:

1. v is coloured with i,

2. there are out-neighbours of v coloured with i, or

3. either there are in-neighbours of v coloured with i or there is no vertex coloured with i in the closed neighbourhood of v.

For a particular vertex and a particular colour, exactly one of these possibilities arises. As such, for any particular $2-$dipath colouring, c, with each vertex, $x \in V(G)$ we can associate a vector of length m, with entries $x_1, x_2, \ldots, x_m$ given by

$$x_i = \begin{cases} \cdot & , \quad \text{if } c(x) = i; \\ 1 & , \quad \text{if } \exists y \in N(x)^+ \text{ such that } c(y) = i; \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Let $\mathcal{A}_{2,m}$ be the set of all such vectors that arise from a $2-$dipath colouring using m colours over all oriented graphs that have a $2-$dipath colouring using m colours. Each element of $\mathcal{A}_{2,m}$ is a vector of length m where exactly one entry is $\cdot$ and all other entries are either 1 or 0.

Using $\mathcal{A}_{2,m}$, construct the oriented graph $G_{2,m}$.

- $V(G_{2,m}) = \mathcal{A}_{2,m}$, and
- for $X = (x_1, x_2, \ldots, x_m)$ and $Y = (y_1, y_2, \ldots, y_m)$ in $V(G_{2,m})$, where $x_i = \cdot$ and $y_j = \cdot$, $XY \in E(G_{2,m})$ when $x_j = 1$ and $y_i = 0$.

Proposition 3.10 (MacGillivray and Sherk [34]). *The oriented graph $G_{2,m}$ is a universal target for the family of oriented graphs with $2-$dipath chromatic number at most m.*

Proposition 3.11 (MacGillivray and Sherk [34]). $\chi_o(G_{2,m}) \leq 2^m - 1$.

Corollary 3.12 (MacGillivray and Sherk [34]). *If G is an oriented graph such that $\chi_{2d}(G) \leq m$, then $\chi_o(G) \leq 2^m - 1$.*

For the cases $m = 3, 4$ we offer the following improvements to Corollary 3.12.

Proposition 3.13. *If G is an oriented graph with $\chi_{2d} \leq m$, then*

- *if $m = 3$, then $\chi_o(G) \leq 5$, and*
- *if $m = 4$, then $\chi_o(G) \leq 12$.*

Proof. Figure 3.2 gives $G_{2,3}$, leaving out arcs between the source vertices on the left and the sink vertices on the right. By inspection this oriented graph admits a homomorphism to the tournament formed from a copy of a directed $3-$cycle together with a universal source vertex and universal sink vertex. Therefore if $\chi_{2d}(G) \leq 3$, then $\chi_o(G) \leq 5$.

Figure 3.3 gives a mapping of all vertices of $G_{2,4}$, excluding sources and sinks, to the ten-vertex target given. This target, together with a universal source and universal sink vertex is a homomorphic image of $G_{2,4}$. Therefore if $\chi_{2d}(G) \leq 4$, then $\chi_o(G) \leq 12$. □

For the cases $m = 3$ and $m = 4$ the number of vertices of $G_{2,m}$ is small enough to be examined by hand. Though here improved bounds are established by separately combining sources and sinks, no general pattern seems to emerge in which such vertices are mapped to the same vertex in the target. Because of this, it does not seem feasible to utilise this method to establish new bounds for larger values of m.

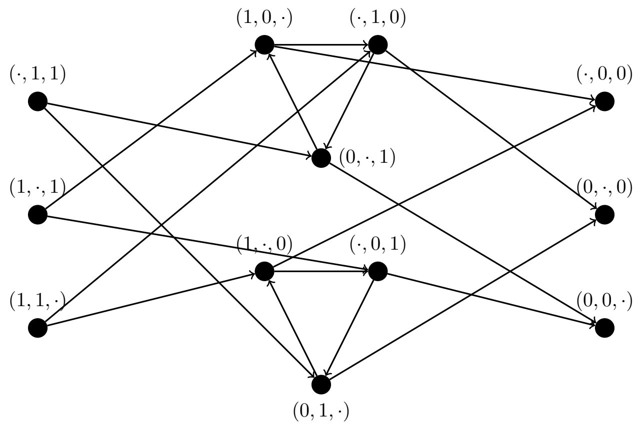


Figure 3.2: The universal target for the family of oriented graphs with $\chi_{2d} \leq 3$.

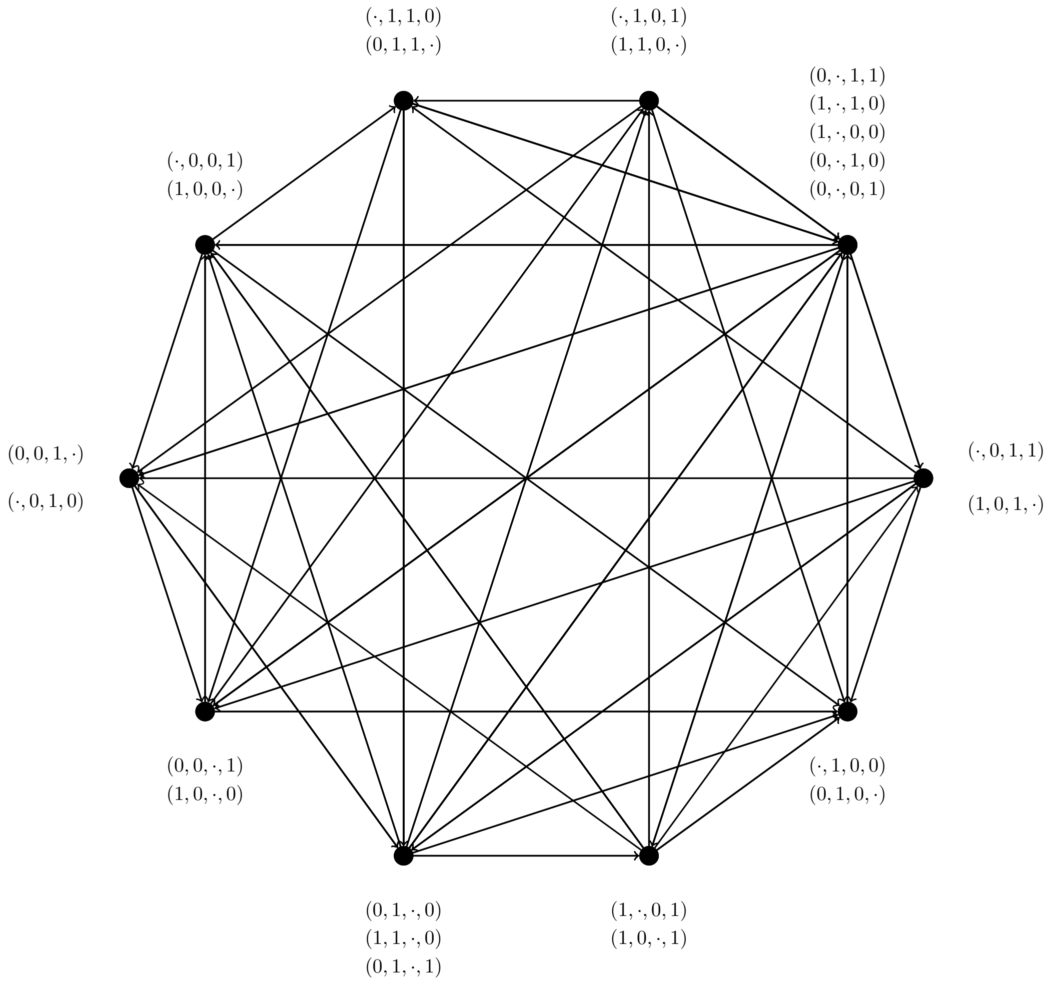


Figure 3.3: A homomorphic image of the universal target for the family of oriented graphs with $\chi_{2d} \leq 4$.

3.3.2 The $k-$dipath Chromatic Number

To construct a universal target for the family of oriented graphs with $k-$dipath chromatic number at most m, we begin by constructing an object similar to the vector constructed for the construction of the universal target for the family of oriented graphs with 2$-$dipath chromatic number at most m. However rather than constructing a vector, we construct a matrix for each vertex.

Let G be an oriented graph with directed girth at least $k+1$. For a vertex $v \in V(G)$ and a $k-$dipath colouring, c, of G using m colours, we define the *$k-$dipath colouring matrix of v, A_v,* to be the $(2k-1) \times m$ $\{0,1\}-$matrix, with rows indexed by the set

$$\{(k-1)^-, (k-2)^- \ldots, 1^-, 0, 1^+, \ldots, (k-2)^+, (k-1)^+\},$$

and columns indexed by $\{1, 2, 3, \ldots, m\}$, that has a 1 in the i^{th} row and j^{th} column if there is a vertex $u \in V(G)$ such that $c(u) = j$, and

- if $i \in \{(k-1)^-, (k-2)^-, \ldots, 1^-\}$, then there is a directed path from u to v of length i;
- if $i \in \{1^+, \ldots, (k-2)^+, (k-1)^+\}$, then there is a directed path from v to u of length i; and
- if $i = 0$, then $v = u$.

Consider the colouring, c, given in Figure 3.1, and the unique vertex v such that $c(v) = 3$. The 3$-$dipath colouring matrix of v is given by:

	1	2	3	4
2^-	1	0	0	0
1^-	0	1	0	0
0	0	0	1	0
1^+	0	0	0	1
2^+	1	0	0	0

For example, the value, 1, in entry $(2^-, 1)$ comes by observing that there is a vertex w such that $c(w) = 1$ and there is a path of length 2 from w to v.

Let $\mathcal{A}_{k,m}$ denote the set of all possible $k-$dipath colouring matrices across all $k-$dipath colourings using m colours over all oriented graphs with directed girth at least $k+1$ and $k-$dipath chromatic number at most m. Each element of $\mathcal{A}_{k,m}$ is a matrix that satisfies the following conditions.

Observation 3.14. *For all $A \in \mathcal{A}_{k,m}$, and all columns a_j of A $(1 \leq j \leq m)$,*

- *if there is a* 1 *in the i^- row of this column, then the entries in the ℓ^+ row are all* 0 $(0 \leq \ell \leq k-i)$*;*
- *if there is a* 1 *in the i^+ row of this column, then the entries in the ℓ^- row are all* 0 $(0 \leq \ell \leq k-i)$*; and*
- *if there is a* 1 *in the* 0 *row of this column, then all other entries of this column are* 0.

Using $\mathcal{A}_{k,m}$ we construct an oriented graph $G_{k,m}$, which is a universal target for the family of oriented graphs with $k-$dipath chromatic number at most m.

- $V(G_{k,m}) = \mathcal{A}_{k,m}$.

- If $A \in \mathcal{A}_{k,m}$ has a 1 in entry $(0, m_1)$, and $B \in \mathcal{A}_{k,m}$ has a 1 in entry $(0, m_2)$, then $AB \in E(G_{k,m})$ provided each of the following conditions hold.
  1. A has a 1 in entry $(1^+, m_2)$;
  2. B has a 1 in entry $(1^-, m_1)$;
  3. if A has a 1 in entry (i^-, m_3), then B has 0 in entry (ℓ^+, m_3) $(0 \leq \ell \leq k - i$, $1 \leq i \leq k - 1)$ and a 1 in entry $((i-1)^-, m_3)$; and
  4. if B has a 1 in entry (i^+, m_3), then A has 0 in entry (ℓ^-, m_3) $(0 \leq \ell \leq k - i$, $1 \leq i \leq k - 1)$ and a 1 in entry $((i+1)^+, m_3)$.

This completes the construction of $G_{k,m}$.

Lemma 3.15. *The oriented graph $G_{k,m}$ has directed girth at least $k + 1$.*

Proof. Let $A_1, A_2, \ldots, A_\ell, A_1$ $(1 \leq \ell \leq k)$ be a directed cycle in $G_{k,m}$ such that A_1 has a 1 in entry $(0, m_1)$. This implies A_2 has a 1 in entry $(1^-, m_1)$ and a 1 in entry $((\ell - 1)^+, m_1)$. This is contrary to Observation 3.14. □

Lemma 3.16. *The $k-$dipath chromatic number of $G_{k,m}$ is at most m.*

Proof. Consider the colouring, c, given by $c(A) = m_1$, where m_1 the is lone column of the matrix of A for which the entry $(0, m_1)$ of A is a 1. We claim that c is a $k-$dipath colouring of $G_{k,m}$. Consider a path of length k in $G_{k,m}$: $A_1A_2, \ldots, A_k$. Assume there exists a pair of indices, $1 \leq i < j \leq k$ such that $c(A_i) = c(A_j)$. This implies A_{i+1} has a 1 in entry $(1^-, c(A_i))$ and a 1 in entry $((j - (i+1))^+, c(A_i))$. This violates Observation 3.14. □

Using these two lemmata we prove our main result in this section.

Theorem 3.17. *Suppose G is an oriented graph with directed girth at least $k + 1$, then $\chi_{kd}(G) \leq m$ if and only if $G \to G_{k,m}$.*

Proof. Let G be an oriented graph with directed girth at least $k + 1$. If $G \to G_{k,m}$, then by Lemmas 3.15 and 3.16, and Theorem 3.9, $\chi_{kd}(G) \leq m$.

Assume $\chi_{kd}(G) \leq m$. Let c be a $k-$dipath colouring of G using m colours. Consider the mapping $\phi : V(G) \to V(G_{k,m})$, where for all $v \in V(G)$, $\phi(v) = A_v$, the $k-$dipath colouring matrix of v (with respect to c). Let uv be an arc of G. We claim A_uA_v is an arc of $G_{k,m}$. Assume A_u has a 1 in entry $(0, m_1)$, and A_v has a 1 in entry $(0, m_2)$. We must show A_u and A_v satisfy the four conditions for an arc to exist in $G_{k,m}$.

1. Since $c(v) = m_2$, A_u has a 1 in entry $(1^+, m_2)$.
2. Since $c(u) = m_1$, A_v has a 1 in entry $(1^-, m_1)$.
3. Assume there exists i such that A_u has a 1 in entry (i^-, m_3). Since c is a $k-$dipath colouring, A_u has a 0 in entry (ℓ^+, m_3) $(i + \ell \leq k,\ \ell > 0)$, as otherwise there would be a pair of vertices coloured m_3 at the ends of a dipath of length at most k in G.
4. Assume there exists i such that A_v has a 1 in entry (i^+, m_3). Since c is a $k-$dipath colouring, A_v has a 0 in entry (ℓ^-, m_3) $(i + \ell \leq k,\ \ell > 0)$, as otherwise there would be a pair of vertices coloured m_3 at the ends of a dipath of length at most k in G.

Therefore $\phi : G \to G_{k,m}$ is a homomorphism. □

3.4 Complexity of $k-$dipath Colourings

In [34] the authors use their homomorphism model for $2-$dipath colouring to discuss the complexity of the following colouring problem.

2−DIPATH $m-$COLOURING
Input: an oriented graph, G.
Question: does G have a $2-$dipath colouring using m colours?

Theorem 3.18 (MacGillivray and Sherk [34]). *Let $m \geq 1$ be a fixed integer. If $m \leq 2$, then 2-DIPATH m-COLOURING is Polynomial. If $m \geq 3$, then 2-DIPATH m-COLOURING is NP-complete.*

Here the authors use the indicator construction and make an argument based on the complexity of graph homomorphism. Our goal in Section 3.4 is to find similar results for the following decision problems.

3−DIPATH $m-$COLOURING
Input: an oriented graph, G.
Question: does G have a $3-$dipath colouring using m colours?

$k-$DIPATH $m-$COLOURING
Input: an oriented graph, G.
Question: does G have a $k-$dipath colouring using m colours?

We construct a gadget that allows us to transform proper $m-$colouring of simple undirected graphs to $3-$dipath colouring (respectively $k-$dipath colouring) using m colours. Formally, the decision problem of $m-$colouring of simple undirected graphs is stated as follows.

$m-$COLOURING
Input: a graph, G.
Question: does G have proper colouring using m colours?

Theorem 3.19 (Garey, Johnson and Stockmeyer [20]). *For any fixed integer $m > 2$, $m-$COLOURING is NP-complete.*

Let G be a simple graph, and let $\tilde{G}$ be an arbitrary orientation of G. We construct H_m ($m > 3$) from $\tilde{G}$.

- For all $v \in V(G)$ add
  - vertices v_i, v_o and x_v;
  - a transitive tournament on $m-2$ vertices with source vertex s_v and sink vertex t_v;
  - a $2-$dipath $(t_v x_v v_i)$, and
  - an arc $v_o s_v$.
- For all $uv \in E(\tilde{G})$ add
  - vertices u_{v_1}, u_{v_2};

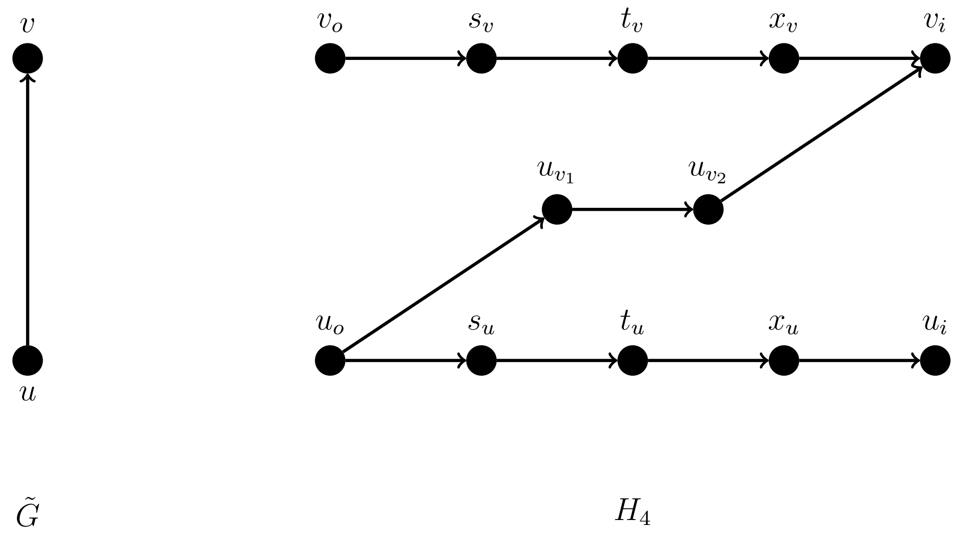


Figure 3.4: Constructing H_4 for the proof of Theorem 3.23 for the case $m = 4$.

- the 3−dipath $(u_o u_{v_1} u_{v_2} v_i)$; and
- an arc $v_o s_v$.

This completes the construction of H_m. See Figure 3.4 for an example of this construction with $m = 4$. Observe H_m is an acyclic oriented graph.

Observation 3.20. $\chi_{3d}(H_m) \geq m$.

For any vertex $v \in V(G)$ observe that the $m-2$ vertices of the transitive tournament constructed for v, together with x_v and v_o form a 3−dipath clique in H_m with m vertices.

Observation 3.21. *If $\chi_{3d}(H_m) = m$, then for every $v \in V(G)$ and every 3−dipath colouring, c, of H_m using m colours, $c(v_o) = c(v_i)$.*

For any vertex $v \in V(G)$ observe that the $m-2$ vertices of the transitive tournament constructed for v, together with x_v and v_o form a 3−dipath clique on m vertices in H_m. Also observe that replacing v_o with v_i also yields a 3−dipath clique on m vertices in H_m. As such, in any 3−dipath colouring of H_m using m colours, v_o and v_i must receive the same colour, as the other $m-1$ colours are used for the transitive tournament and x_v.

Observation 3.21 provides a natural way to construct a proper m−colouring of G given a 3−dipath colouring of H_m using m colours. Given a 3−dipath colouring of H_m we can unambiguously lift back the colour of the vertices $v_i, v_o \in H$ to the vertex $v \in G$. This next proposition shows this lifting can be done in either direction.

Proposition 3.22. *If G is a simple graph and H_m is constructed from G as above, then for all $m \geq 4$, $\chi(G) \leq m$ if and only if $\chi_{3d}(H_m) \leq m$.*

Proof. Let c be a 3−dipath colouring of H_m using m colours. By Observation 3.21, for every vertex $v \in V(G)$ we have $c(v_o) = c(v_i)$. Consider the function $\phi : V(G) \to \{1, 2, 3, \ldots, m\}$ given by $\phi(v) = c(v_i)$. If ϕ is not a proper colouring of G, then there exists $uv \in E(\tilde{G})$ such that $\phi(u) = \phi(v)$. By construction of ϕ, this implies $c(u_o) = c(v_i)$. However this contradicts our hypothesis that c is a 3−dipath colouring of H_m using m colours. Therefore no such arc of $\tilde{G}$ can exist. Therefore ϕ is a proper colouring of G using no more than m colours.

Let ϕ be a proper colouring of G using m colours. Construct a partial colouring of H_m, $c : V(H_m) \to \{1, 2, 3, \ldots, m\}$, given by $c(v_o) = c(v_i) = \phi(v)$, for all $v \in V(G)$. To see that c can be completed to a 3−dipath colouring of H_m using m colours observe that

- for every $v \in V(G)$, if $\phi(v_o) = i$ $(1 \leq i \leq m)$, then vertices of the transitive tournament constructed for v together with x_v can be coloured using the set $\{1, 2, 3 \ldots, m\} \setminus \{i\}$; and
- for every $uv \in E(\tilde{G})$, if $\phi(u_o) = i$ and $\phi(v_i) = j$ $(1 \leq i, j \leq m)$, then vertices $u_{v_1}, u_{v_2} \in V(H_m)$ can be coloured using the set $\{1, 2, 3 \ldots, m\} \setminus \{i, j\}$.

□

Theorem 3.23. *For all fixed integers $m > 3$, 3−DIPATH m−COLOURING is NP-complete. The problem is Polynomial for all $m \leq 3$.*

Proof. For fixed $m > 3$ our transformation is from m−COLOURING. Consider an instance of m−COLOURING, with input graph G. Construct the acyclic oriented graph H_m, as described above. We note this construction can be obtained in polynomial time. By Proposition 3.22, $\chi(G) \leq m$ if and only if $\chi_{3d}(H) \leq m$. Since m−COLOURING is NP-complete it follows that 3−*DIPATH* m−*COLOURING* is NP-complete.

Consider now an instance of 3−*DIPATH* m−*COLOURING* for fixed $m \leq 3$ with input graph G. If G has a directed path with at least four vertices, then at least 4 colours are required. Therefore we may assume the longest directed path in G has no more than m vertices. Since G has directed girth at least 4, we have directly that G is acyclic. By Theorem 3.8, G has a 3−dipath colouring using m colours if and only if G admits a homomorphism to T_m, the transitive tournament on m vertices. Homomorphism to T_m can be checked in polynomial time [3] and so 3−*DIPATH* m−*COLOURING* is *Polynomial* for all fixed $m \leq 3$.

□

Comparing our results to the results for 2−DIPATH m−COLOURING given in [34], the dividing line above is expected. In fact, with only slight modifications to the construction of H_m this same method may be used for 2−DIPATH m−COLOURING. Furthermore, with this same construction we can attack the problem of k−DIPATH m−COLOURING for any fixed m and k.

As before, given a simple graph G we construct an oriented graph, $H_{m,k}$ $(m > k \geq 3)$, such that $\chi_{kd}(H) \leq m$ if and only if $\chi(G) \leq m$. Let G be a simple graph, and let $\tilde{G}$ be an arbitrary acyclic orientation of G.

- For all $v \in V(G)$, add
  - vertices v_i and v_o,
  - a transitive tournament on $m - k + 1$ vertices with source vertex s_v and sink vertex t_v,
  - the vertices and arcs required for a $(k-1)$−dipath: $t_v, x_{v_1}, x_{v_2}, x_{v_3}, \ldots, x_{v_{k-2}}, v_i$, and
  - an arc $v_o s_v$.
- For all $uv \in E(\tilde{G})$, add
  - the vertices and arcs required for a k−dipath $u_o, u_{v_1}, u_{v_2}, u_{v_3}, \ldots, u_{v_{k-1}}, v_i$; and
  - an arc $v_o s_v$.

This completes the construction of $H_{m,k}$.

Observation 3.24. $\chi_{kd}(H_{m,k}) \geq m$.

For any vertex $v \in V(G)$, observe that the $m-k+1$ vertices of the transitive tournament constructed for v, together with the vertices $x_{v_1}, x_{v_2}, x_{v_3}, \ldots, x_{v_{k-2}}, v_i$ form a $k-$dipath clique on m vertices.

Observation 3.25. *If $\chi_{kd}(H_{m,k}) = m$, then for every $v \in V(G)$ and every $k-$dipath colouring, c, of $H_{m,k}$ using m colours, $c(v_o) = c(v_i)$.*

For any $v \in V(G)$ replacing v_i with v_o in the clique formed from the $m-k+1$ vertices of the transitive tournament constructed for v, together with the vertices $x_{v_1}, x_{v_2}, x_{v_3}, \ldots, x_{v_{k-2}}, v_i$ also forms a $k-$dipath clique on m vertices.

Proposition 3.26. *If G is a simple graph and $H_{m,k}$ is constructed from G as above, then for all $m > k \geq 3$, $\chi(G) \leq m$ if and only if $\chi_{kd}(H_{m,k}) \leq m$.*

Proof. Let c be a $k-$dipath colouring of $H_{m,k}$ using m colours. By Observation 3.25 for every vertex $v \in V(G)$ we have $c(v_o) = c(v_i)$. Consider the function $\phi : V(G) \to \{1, 2, 3, \ldots, m\}$ given by $\phi(v) = c(v_i)$. If ϕ is not a proper colouring of G, then there exists $uv \in E(\tilde{G})$ such that $\phi(u) = \phi(v)$. By construction of ϕ this implies $c(u_o) = c(v_i)$. However this contradicts our hypothesis that c is a $k-$dipath colouring of $H_{m,k}$ using m colours. Therefore no such arc of $\tilde{G}$ can exist. Therefore ϕ is a proper colouring of G using no more than m colours.

Let ϕ be a proper colouring of G using m colours. Construct a partial colouring of $H_{m,k}$, $c : V(H_{m,k}) \to \{1, 2, 3, \ldots, m\}$, given by $c(v_o) = c(v_i) = \phi(v)$, for all $v \in V(G)$. To see c can be completed to a $k-$dipath colouring of $H_{m,k}$ using m colours observe that

- for every $v \in V(G)$, if $\phi(v_o) = i$ $(1 \leq i \leq m)$, then vertices of the transitive tournament constructed for v together with the vertices $x_{v_1}, x_{v_2}, x_{v_3}, \ldots, x_{v_{k-2}}$ can be coloured using the set $\{1, 2, 3 \ldots, m\} \setminus \{i\}$; and
- for every $uv \in E(\tilde{G})$, if $\phi(u_o) = i$ and $\phi(v_i) = j$ $(1 \leq i, j \leq m)$, then vertices $u_{v_1}, u_{v_2}, u_{v_3}, \ldots, u_{v_{k-1}} \in V(H_{m,k})$ can be coloured using the set $\{1, 2, 3 \ldots, m\} \setminus \{i, j\}$.

□

Theorem 3.27. *Let m and $k \geq 3$ be fixed positive integers. The problem $k-$*DIPATH *$m-$*COLOURING *is NP-complete for $m > k$. The problem is Polynomial for all $m \leq k$.*

Proof. For fixed $m > k \geq 3$ our transformation is from $m-$COLOURING. Consider an instance of $m-$COLOURING, with input graph G. Construct the acyclic oriented graph $H_{m,k}$, as described above. We note this construction can be obtained in polynomial time. By Proposition 3.26, $\chi(G) \leq m$ if and only if $\chi_{kd}(H) \leq m$. Since $m-$COLOURING is NP-complete for all $m \geq 3$ it follows that $3-$*DIPATH* $m-$*COLOURING* is NP-complete.

Consider now an instance of $k-$*DIPATH* $m-$*COLOURING* for fixed $m \leq k$ with input graph G. If G has a directed path with at least $k+1$ vertices, then at least $k+1$ colours are required. Therefore we may assume the longest directed path in G has no more than m vertices. Since G has directed girth at least $k+1$, we have directly that G is acyclic. By Theorem 3.8, G has a $k-$dipath colouring using m colours if and only if G admits a homomorphism to T_m, the transitive tournament on m vertices. Homomorphism to T_m can be checked in polynomial time [3] and so $k-$*DIPATH* $m-$*COLOURING* is Polynomial for all fixed m and k such that $m \leq k$.

□

3.5 Future Directions and Conclusions

For the cases $m = 3$ and $m = 4$ we have given an improvement for the upper bound for the oriented chromatic number of oriented graphs with $2-$dipath chromatic number m. The number of vertices in $G_{m,k}$ in these cases is small enough to be able to analyse these graphs by hand. For larger values of m, however, such an approach is not likely to yield results. It remains to be seen if the bound of $2^m - 1$ given by [34] is best possible for large values of m.

The homomorphism model presented in Section 3.3, based on the vector model presented in [34], seems as if it may be adaptable to more general situations. Here we are enforcing that no two vertices of the same colour can appear on a path of length at most k. However we can imagine adapting this model to be used in a situation where the distance constraint is different for each colour. Further it may be possible to encode more complicated constraints into this model. For example it may be possible to use this model to construct colourings where vertices of the same colour are permitted to be at some distances from each other, but prohibited at some other distances from each other.

Recall the definition of a graph multi-colouring: If G is a graph, $c : V(G) \to 2^m$ is a multi-colouring using m colours if for all $uv \in E(G)$, we have $c(u) \setminus c(v) \neq 0$ and $c(v) \setminus c(u) \neq 0$. Using multi-colouring rather than enforcing a directed girth condition is a method of avoiding the problem of short cycles in $k-$dipath colourings. Rather than ensuring that a vertex does not lie on a prohibitively short cycle, we can allow for a vertex to be coloured differently from itself by assigning to it a set of colours. This could led to a definition of $k-$dipath colouring that is equivalent not to a proper colouring of the k^{th} power of the graph, but to a multi-colouring of the k^{th} power of the graph. In this formulation, a possible universal target may have all subsets of a $k-$set as the set of vertices. This would be consistent with an exponential upper bound for the oriented chromatic number.

Chapter 4
Simple Colourings of Oriented Graphs

The best upper bound for the oriented chromatic number of the family planar graphs has rested at 80 since 1994, when Raspaud and Sopena provided a connection between the acyclic chromatic number and the oriented chromatic number. Given that the best lower bound is 18 [35], it is likely that there is room for improvement. In this chapter we consider a pair of methods that may be employed in an attempt to improve this bound: simple colouring and simple $2-$dipath colouring.

4.1 Background and Preliminaries

Consider relaxing the requirement of oriented colouring that adjacent vertices must receive different colours. With this constraint relaxed colouring becomes trivial, each vertex can receive the same colour. To avoid trivial colouring let us require that at least two colours be used. Formally, we define *simple $m-$colouring* as follows.

Definition 4.1. *Let G be an oriented graph. A* simple $m-$colouring *of G is a mapping $c : V(G) \to \{1, 2, 3, \dots, m\}$ such that*

- *there exists $u, v \in V(G)$ so that $c(u) \neq c(v)$, and*
- *if there exist uv and xy so that $c(u) = c(y)$, then either $c(v) \neq c(x)$ or $c(u) = c(v) = c(x) = c(y)$.*

For an oriented graph, G, we define the simple chromatic number of G*, denoted $\chi_s(G)$, to be the least m such that G has a simple $m-$colouring. For a family of oriented graphs, $\mathcal{F}$, we define the* simple chromatic number of $\mathcal{F}$*, denoted $\chi_s(\mathcal{F})$, to be the least m such that for all $F \in \mathcal{F}$, $\chi_s(F) \leq m$.*

As with other colouring parameters, we may use graph homomorphism to define the simple chromatic number.

Definition 4.2. *Let G be an oriented graph and let H be a reflexive anti-symmetric digraph. We say G* admits a simple homomorphism *to H if there exists $\phi : G \to H$, and there exists $x, y \in V(G)$ such that $\phi(x) \neq \phi(y)$. The* simple chromatic number *of G, denoted $\chi_s(G)$, is the least m such that there exists H so that G admits a simple homomorphism to H and $|V(H)| = m$.*

As a first example, consider an oriented graph G that contains, as a proper subgraph, a directed $3-$cycle uvw. If u and v receive the same colour in a simple colouring of G, then we observe that w must also receive this same colour, as otherwise the second condition of a simple colouring is violated. If u and v receive different colours in a simple colouring of G, then we observe that w must be assigned a colour that is distinct from both the

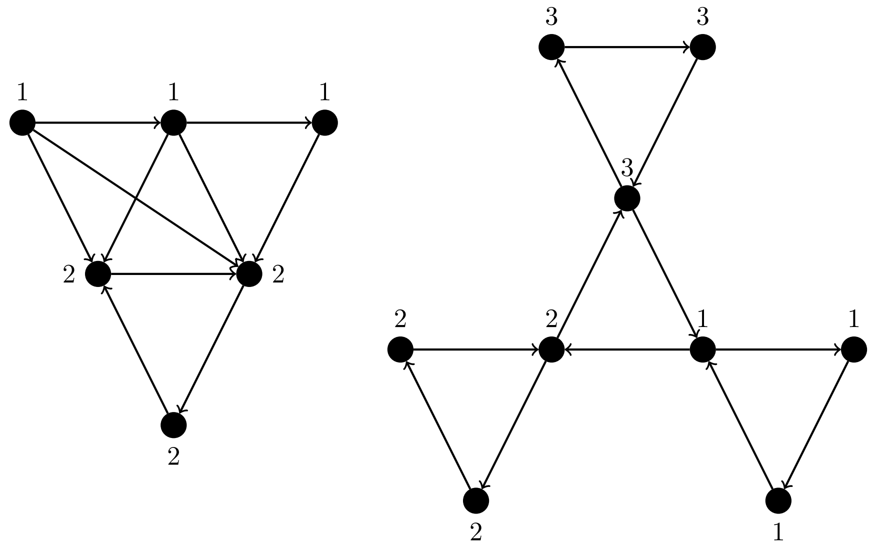


Figure 4.1: Examples of oriented graphs with $\chi_s(G) = 2$ and $\chi_s(G) = 3$.

colour of u and the colour of v, as otherwise the second condition of a simple colouring would be violated.

The oriented graphs given in Figure 4.1 have simple chromatic number 2 and 3, respectively. To see that 3 is optimal for the second example, observe that in any simple $2-$colouring of this oriented graph, there would necessarily be a directed $3-$cycle containing only two colours.

Proposition 4.1 (Smolíková [49])**.** *An oriented graph, G, has simple chromatic number at most two if and only if there exists a set of vertices $X \subset V(G)$ such that every edge of $U(G)$ with an end in X and an end in $\overline{X}$ is oriented in G so that its tail is in X and its head is in $\overline{X}$.*

This characterisation follows by observing the only two-vertex target for such a simple homomorphism is a single arc with a loop at each end.

Corollary 4.2. *Every graph has an orientation with simple chromatic number* 2.

This follows by observing that any oriented graphs with either a source or a sink vertex has simple chromatic number 2 and any graph may be oriented to have either a source or sink vertex.

Simple colourings of oriented graphs were introduced by Smolíková [49]. In her Ph.D. thesis, amongst other things, she considered families of oriented graphs such that $\chi_o(\mathcal{F}) = \chi_s(\mathcal{F})$. For a family of graphs, $\mathcal{F}$, we say $\mathcal{F}$ is *optimally simply colourable* if $\chi_o(\mathcal{F}) = \chi_s(\mathcal{F})$.

Theorem 4.3 (Smolíková [49])**.** *The families of oriented planar graphs and oriented $p-$trees ($p \geq 3$) are optimally simply colourable.*

In general, the difference between the oriented chromatic number and the simple chromatic number may be arbitrarily large; the transitive tournament on m vertices has oriented chromatic number m but simple chromatic number 2. In addition to studying families of optimally simply colourable graphs, Smolíková considered oriented graphs for which $\chi_s(G) = |V(G)|$.

Definition 4.3. *An oriented graph, G, is a* simple clique *if $\chi_s(G) = |V(G)|$. We call a tournament* simple *if it is a simple clique.*

Theorem 4.4 (Smolíková [49])**.** *Let $s(n)$ be the number of simple cliques on n vertices. There is a constant $c > 1$ and $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$*

$$s(n) \geq (1 - c^{-n}) \cdot 3^{\binom{n}{2}}.$$

Simple tournaments arise in an algebraic context by viewing a tournament as a binary algebra. In this context, simple homomorphisms correspond to non-trivial homomorphisms between quasi-trivial algebras [39]. Such an algebra is called *simple* if it has no proper non-trivial sub algebra. The family of simple quasi-trivial algebras corresponds exactly to the family of simple tournaments. And so by applying a result on quasi-trivial algebras, we get the following result.

Theorem 4.5 (Erdős et al. [17])**.** *Almost all tournaments are simple.*

To examine simple cliques more closely, we require the following terminology.

Definition 4.4. *Let G be an oriented graph and let $u, v, w \in V(G)$ such that $uv, vw \in E(G)$. We say v is* between *u and w if uvw is $2-$dipath. We say $C \subseteq V(G)$ is* convex *if for any pair $u, w \in C$ there is no $v \in V(G) \setminus C$ such that v is between u and w.*

Definition 4.5. *The* convex hull *of $C \subseteq V(G)$ is the smallest convex set of vertices of G that has C as a subset. We denote this set as $conv(C)$.*

From this definition it follows directly that if C is a convex set, then $conv(C) = C$.

Definition 4.6. *We call G* complete-convex *if for every connected subgraph on at least two vertices, H, $conv(V(H)) = V(G)$.*

Observation 4.6. *If G is an oriented graph, c a simple colouring of G and u and v a pair of vertices such $c(u) = c(v)$, then for all w between u and v, $c(u) = c(w)$*

Proposition 4.7 (Smolíková [49])**.** *Let G be an oriented graph and c a simple colouring of G. If there exists an arc $uv \in E(G)$ such that $c(u) = c(v)$, then for every $x \in conv(\{u, v\})$, $c(x) = c(u)$.*

Proof. Consider a vertex $x \in conv(\{u, v\})$ and let N be the subset of $conv(\{u, v\})$ such that $x \notin N$ and for all $y, z \in N$ such that if there is a vertex $w \neq x$ between y and z, then $w \in N$. We proceed by induction on the cardinality of N. If $|N| = 2$, then $N = \{u, v\}$. Since N is largest it must be x is between u and v. If c is a simple colouring of G such that $c(u) = c(v)$, then any vertex between u and v must also be assigned this same colour, as otherwise the second condition of a simple oriented colouring would be violated. Therefore $c(x) = c(u)$.

Assume now $|N| = k > 2$. Since N is largest there exists a pair of vertices $y, z \in N$ such that x is between y and z. If c is a simple colouring of G such that $c(u) = c(v)$, then by induction $c(u) = c(y) = c(z)$. Since x is between y and z, it must also be $c(x) = c(u)$. □

Corollary 4.8 (Smolíková [49])**.** *If G is a complete-convex graph, then every simple colouring of G is also an oriented colouring of G.*

Following Sen's characterisation of oriented cliques [48], we arrive at the following characterisation of simple cliques.

Proposition 4.9. *An oriented graph on at least three vertices is a simple clique if and only if it has weak diameter at most two and is complete-convex.*

Let T_m be the transitive tournament on m vertices with source vertex s and sink vertex t. The tournament formed from T_k by reversing the direction of the arc between s and t is a simple clique; this graph has weak diameter 1 and is complete-convex. As mentioned above $\chi_s(T_m) = 2$ and so we see that changing the direction of even a single arc can have a drastic effect on the simple chromatic number.

4.2 Simple Colourings of Planar Graphs

Given the on-going interest in the oriented chromatic number of the family of planar graphs, we examine the implications of Smolíková's result that the family of oriented planar graphs is optimally simply colourable.

Let $\mathcal{P}$ be the family of oriented planar graphs and let $m = \chi_o(\mathcal{P})$. Since $\mathcal{P}$ is optimally simply colourable, $m = \chi_s(\mathcal{P})$. Let $P \in \mathcal{P}$ be the smallest oriented graph in $\mathcal{P}$ so that $\chi_s(P) = m$. By finding the simple chromatic number of P we can improve the upper bound for the oriented chromatic number of planar graphs. Here we give some properties of P that could aid in the search for an oriented graph on fewer than 80 vertices that is a homomorphic image of P.

Property 4.10. *No vertex of P is a source or a sink.*

Any oriented graph with a source or a sink vertex has simple chromatic number 2. There exist oriented planar graphs with oriented chromatic number at least 3, and so it must be that $m > 2$. Therefore P has no source or sink vertex.

Property 4.11. *The planar graph P is complete-convex.*

Proof. Let H be a connected subgraph of P with at least two vertices. Assume $conv(V(H)) \subset V(P)$. Consider the result of identifying the vertices of $conv(V(H))$ into a single vertex, h. Remove all loops and copies of identical arcs, and call this new digraph P_H. Since H is connected this identification can be considered as a sequence of edge contractions. Therefore since P is planar, P_H is also planar. However it may be that P_H contains $2-$cycles. If P_H is an oriented graph, then, by choice of P, $\chi_o(P_H) < \chi_o(P)$. However, an oriented colouring, c, of P_H using $m' < m$ colours can be extended to be a simple $m'-$colouring of P by colouring each vertex in H with $c(h)$. Therefore P_H is not an oriented graph and so must contain $2-$cycles. This implies there exists a vertex not in $conv(V(H))$ that is between a pair of vertices in $conv(V(H))$. This contradicts that $conv(V(H))$ is a convex set. Therefore $conv(V(H)) = V(P)$. □

When taking H to be a single arc, we arrive at the following property of P.

Property 4.12. *For every $xy \in E(P)$ there is an ordering of the vertices of P, $x, y, u_1, u_2, \ldots, u_{n-2}$, so that for every vertex $u_i, 1 \leq i \leq n-2$, there exists a pair of vertices s and t so that both of s and t appear earlier in the ordering than u_i and su_it is $2-$dipath.*

Knowing P is complete-convex allows us to restrict our considerations when seeking to improve the upper bound on the oriented chromatic number of planar graphs. The existing upper bound of 80 comes by way of constructing a universal target for the family of planar graphs. Such a graph necessarily exists because the family of planar graphs is complete (see Proposition 1.3). The same is true for the family of complete-convex planar graphs.

Theorem 4.13. *The family of complete-convex planar graphs is complete.*

Proof. Let G_1 and G_2 be complete-convex planar graphs with at least two vertices. Let uv be an arc on the outer face of a planar embedding G_1 and let xy be an arc on the outer face of a planar embedding G_2. Consider the oriented planar graph G formed from G_1 and G_2 in the following manner:

- $V(G) = V(G_1) \cup V(G_2)$, and
- $E(G) = E(G_1) \cup E(G_2) \cup \{vy, yu, ux\}$.

To show G is complete-convex it suffices to show $x, y \in conv(\{u, v\})$, $u, v \in conv(\{x, y\})$, and that the convex hull of each of the newly added arcs is the entirety of the vertex set of G.

- $x, y \in conv(\{u, v\})$. Observe y is between v and u and x is between u and y.
- $u, v \in conv(\{x, y\})$. Observe u is between y and x and v is between u and y.
- $conv(\{v, y\}) = V(G)$. Observe u is between y and v. Since u and v are elements of $conv\{v, y\}$, and G_1 is complete-convex, then each vertex of G_1 is contained in $conv(\{v, y\})$. Further, x is between u and y and so since x and y are both elements of $conv(\{v, y\})$ and since G_2 is complete-convex, each vertex of G_2 is contained in $conv(\{v, y\})$.
- $conv(\{u, x\}) = V(G)$. This claim follows similarly to the previous claim.
- $conv(\{u, y\}) = V(G)$. Since both v and x are between u and y, each of u, v, x, and y are in $conv(\{u, y\})$. Our claim now follows similarly to the previous claim.

□

Corollary 4.14. *The family of complete-convex planar graphs has a universal target with $\chi_o(\mathcal{P})$ vertices.*

The oriented graph constructed by Raspaud and Sopena [44] is a universal target for the family of complete-convex planar graphs by virtue of it being a universal target for the family of oriented planar graphs. Complete-convex graphs are highly structured and so it is possible this structure can be used to construct a universal target for the family of complete-convex planar graphs on fewer than 80 vertices.

Consider now the smallest non-trivial connected subgraphs of P: single arcs. If uv is an arc of P, then since P is complete-convex it must be $conv(\{u, v\}) = V(P)$. Since there must be a vertex between u and v, we get the following property of arcs of P.

Property 4.15. *The ends of every arc in P are also the ends of some $2-$dipath of P.*

Property 4.16. *Every edge of $U(P)$ is contained in a $3-$cycle.*

Property 4.15 is a necessary condition of all complete-convex oriented graphs.

Definition 4.7. *We call G, an oriented graph,* $2-$convex *if for every arc uv, $|conv(\{u, v\})| > 2$.*

From this definition it follows directly that every complete-convex oriented graph is also $2-$convex. Though every complete-convex graph is also $2-$convex, the opposite is not true.

Proposition 4.17. *There exist planar graphs that are $2-$convex but not complete-convex.*

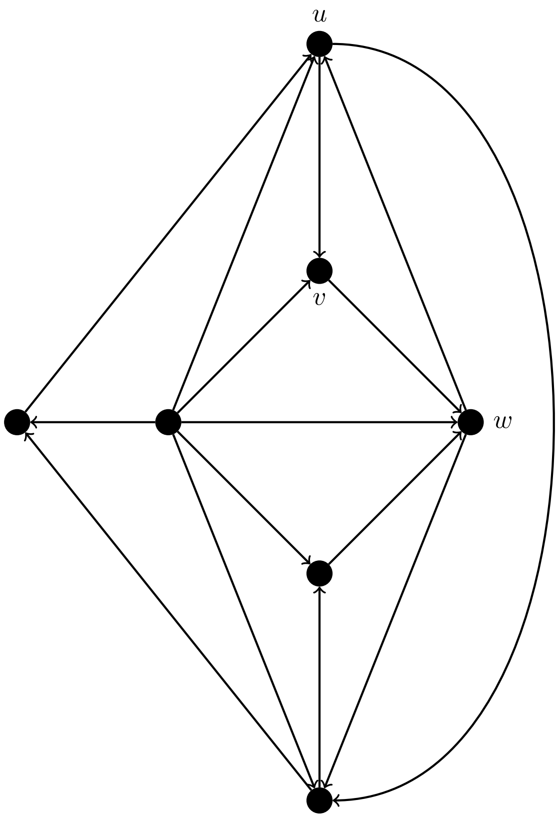


Figure 4.2: A 2−convex graph that is not complete-convex.

Proof. Let G be the oriented graph given in Figure 4.2. The convex hull of $\{u, v\}$ contains w and no other vertex, but the ends of every arc of G are also the ends of a $2-$dipath. Therefore this oriented graph is $2-$convex but not complete-convex. □

Amongst the family of orientations of planar graphs, it is easy to find examples of $2-$convex graphs. Consider a plane triangulated graph G. We show that we can orient the edges so that each arc is a member of a facial directed $3-$cycle. We do this by $3-$colouring the planar dual of G and then orienting the arcs in a face according to the colour assigned to the corresponding vertex in the planar dual. Oriented faces can be classified into one of three categories: transitive triples, clockwise directed $3-$cycles, and anti-clockwise directed $3-$cycles. We begin by observing in which ways the directed $3-$cycles can share an edge.

Observation 4.18. *If f_1 and f_2 are faces of a planar embedding of G with a common arc such that f_1 and f_2 are both directed $3-$cycles, then at most one of f_1 and f_2 are clockwise.*

To show that every triangulated planar graph has a $2-$convex orientation we require the following lemma.

Lemma 4.19. *Let G be an embedding of a maximally-planar graph, and let F be the set of faces of G. If $C = \{f_1, f_2, \ldots, f_k\} \subseteq F$ and $A = \{h_1, h_2, \ldots, h_\ell\} \subseteq F$ such that*

- $C \cap A = \emptyset$,
- *for all $1 \leq i < j \leq k$, f_i and f_j do not share an edge, and*
- *for all $1 \leq p < m \leq \ell$, h_p and h_m do not share an edge,*

then there exists an orientation of G such that each $f_i \in C$ is a clockwise $3-$cycle and each $h_p \in A$ is an anti-clockwise $3-$cycle.

Proof. Since for all $1 \leq i < j \leq k$ f_i and f_j do not share an edge, it is possible to orient the edges that are part of some f_i so that each f_i is a clockwise directed $3-$cycle. Let $1 \leq p \leq \ell$ be the least integer such that each of the edges of the preceding $p-1$ faces in A may be successfully oriented to be a directed $3-$cycle, but the edges of h_p may not be. If h_p shares no edge with some $f_i \in C$, then its edges may be oriented to be an anti-clockwise directed $3-$cycle. If h_p shares a single edge with some f_i, then by our observation above, this shared edge is oriented correctly to be able to orient h_p as an anti-clockwise directed $3-$cycle. The same holds if h_p shares two edges or three edges with faces in C. (See Figure 4.3) □

Theorem 4.20. *Every simple triangulated plane graph other than K_4 has a $2-$convex orientation.*

Proof. Let $G \neq K_4$ be a triangulated planar graph. Since G is not K_4, by Brooks' Theorem the planar dual of G, G^P, admits a $3-$colouring. Let c be a $3-$colouring of G^P. and A be the set of vertices of G^P that are assigned colours 1 and 2, respectively. It follows that every edge of G^P has an end in C or an end in A. By viewing C and A as sets of faces of G and applying Lemma 4.19 we obtain our result. □

Our exploratory work on convexity in oriented planar graphs brings us no closer to improving the upper bound for the oriented chromatic number of planar graphs. However it does provide us with a possible roadmap for improving this bound; rather than considering the family of all oriented planar graphs, we may instead consider the family of complete-convex planar graphs.

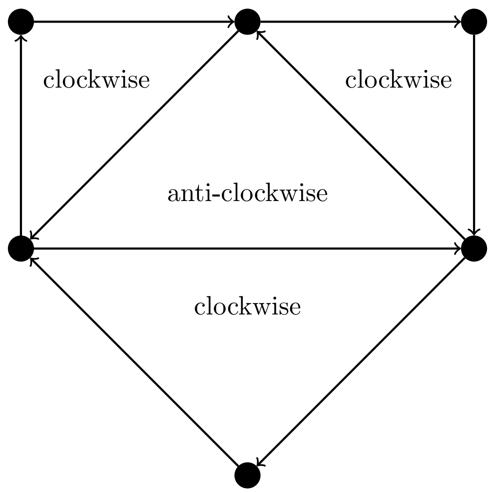


Figure 4.3: An anti-clockwise triangle bordered by clockwise triangles.

4.3 Simple 2−dipath Colourings of Oriented Graphs

The 2−dipath chromatic number, χ_2, stands as a lower bound for the oriented chromatic number, as every oriented colouring is also a 2−dipath colouring. This topic is covered more in depth in Chapter 3. Sherk and MacGillivray show a function of the 2−dipath chromatic number also stands as an upper bound for the oriented chromatic number [34]. Here we consider a similar idea for simple colourings of oriented graphs and introduce the notion of simple 2−dipath colouring of an oriented graph.

Definition 4.8. *Let G be an oriented graph. A* simple 2−dipath *colouring of G with $m > 1$ colours is a surjective function $c : V(G) \to \{1, 2, 3, \ldots, m\}$ such that if $uv, vw \in E(G)$ and $c(u) = c(w)$, then $c(u) = c(v) = c(w)$. The* simple 2−dipath chromatic number *of G, denoted $\chi_{2s}(G)$, is the least m such that G has a simple 2−dipath colouring with m colours.*

From this definition we have directly that for any oriented graph, G, $\chi_{2s}(G) \leq \chi_s(G)$ and $\chi_{2s}(G) \leq \chi_{2d}(G)$.

Figure 4.4 gives examples of oriented graphs that require two and three colours, respectively, in a simple 2−dipath colouring. As with simple colouring, observe that in any oriented graph that contains, as a proper subgraph, a directed 3−cycle, the vertices of this 3−cycle either all receive the same colour, or all receive pairwise distinct colours.

We begin our investigation of this parameter by characterising completely those oriented graphs, G, with $\chi_{2s}(G) = 2$.

Theorem 4.21. *An oriented graph, G, has $\chi_{2s}(G) = 2$ if and only if $U(G)$ contains an edge cut, C, so that the oriented graph induced by C contains no 2−dipath.*

Proof. Let G be an oriented graph so that $\chi_{2s}(G) = 2$, and let c be a simple 2−dipath colouring of G using two colours. Let $X = \{x \in V(G) | c(x) = 1\}$. Let $C \subset E(U(G))$ be the set of edges with exactly one end in X. By definition C is an edge cut of $U(G)$. Consider now a pair of edges $uv, vw \in C$ so that $u, w \in X$. If these arcs are oriented in G as uv and vw, respectively, then since $c(u) = c(w) = 1$, it must also be $c(v) = 1$. This is

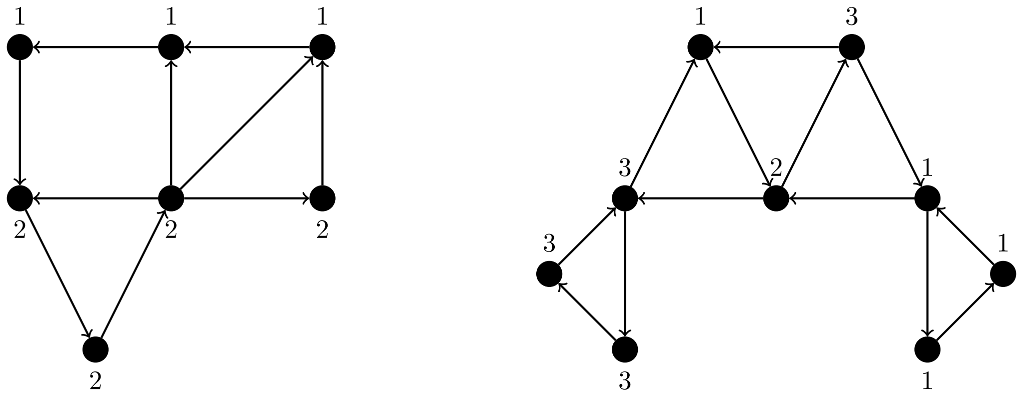


Figure 4.4: Examples of oriented graphs with $\chi_{2s}(G) = 2$ and $\chi_{2s}(G) = 3$.

a contradiction, as $v \notin X$. Therefore the oriented graph induced by C does not contain a 2−dipath.

Let G be an oriented graph, and let C be a minimal edge cut of $U(G)$ so that the oriented graph induced by C contains no 2−dipath. Since C is a minimal edge cut, there exists a non-empty set of vertices $X \subset V(G)$ such that the subgraph induced by X is a component of $V(G) \setminus C$. A colouring that assigns colour 1 to vertices in X and colour 2 to all other vertices is a simple 2−dipath colouring. □

The upper bound for the oriented chromatic number in terms of the 2−dipath chromatic number given in [34] comes by exhibiting a universal target for the family of oriented graphs with 2−dipath chromatic number at most m. However, we show no such upper bound for the simple chromatic number in terms of the simple 2−dipath chromatic number can exist, as there exists a family of oriented graphs with simple 2−dipath chromatic number 2 and arbitrarily large simple chromatic number.

Theorem 4.22. *For each $m > 2$ there exists an oriented graph, H, with $\chi_{2s}(H) = 2$ and $\chi_s(H) \geq m$.*

Proof. Let G_1 and G_2 be simple cliques (see Definition 4.3) on $m > 2$ vertices, where

- $V(G_1) = \{u_1, u_2, \ldots, u_m\}$, and
- $V(G_2) = \{v_1, v_2, \ldots, v_m\}$.

Construct an oriented graph, H, as follows.

- $V(H) = V(G_1) \cup V(G_2)$, and
- $E(H) = E(G_1) \cup E(G_2) \cup \{v_1u_1\} \cup \{u_iv_j | 2 \leq i, j \leq m\}$.

By Theorem 4.21 $\chi_{2s}(H) = 2$, as the set of an arc $\{v_1u_1\} \cup \{u_iv_j | 2 \leq i, j \leq m\}$ is an edge cut that does not induce any 2−dipath.

To show $\chi_s(H) \geq m$, we assume the contrary. Assume $\chi_s(H) < m$ and let d be a simple colouring of H using no more than $m - 1$ colours. Since G_1 (respectively G_2) has m vertices, we may assume a pair of vertices of G_1 (respectively G_2) is assigned the same colour by d. Since G_1 (respectively G_2) is a simple clique it must be that all of the vertices of G_1 (respectively G_2) are assigned this same colour. This implies $\chi_s(H) = 2$. This is a contradiction whenever $m > 2$. □

Corollary 4.23. *For all $k > 2$ there exists no oriented graph G_k such that G admits a simple homomorphism to G_k if and only if $\chi_{2s}(G) \leq k$.*

Though there is no result to be obtained bounding the simple chromatic number from above with a function of the simple 2−dipath chromatic number, we can relate the simple 2−dipath chromatic number to the 2−dipath chromatic number for the family of complete-convex graphs. To do so we require the following result relating convexity and simple 2−dipath colourings. The proof of which follows similarly to the proof of Proposition 4.7.

Proposition 4.24. *Let G be an oriented graph and c a simple 2−dipath colouring of G. If there exists an arc $uv \in E(G)$ such that $c(u) = c(v)$, then for every $x \in conv(\{u, v\})$, $c(x) = c(u)$.*

Corollary 4.25. *If G is a complete-convex graph, then every simple 2−dipath colouring of G is also a 2−dipath colouring of G.*

Corollary 4.26. *If G is a complete-convex graph, then $\chi_{2d}(G) = \chi_{2s}(G)$.*

Corollary 4.27. *If $\mathcal{F}$ is a family of complete-convex graphs, then $\chi_{2s}(\mathcal{F}) = \chi_{2d}(\mathcal{F})$.*

Corollary 4.28. *If $\mathcal{P}_c$ is the family of complete-convex planar graphs, then $\chi_{2s}(\mathcal{P}_c) = \chi_{2d}(\mathcal{P}_c)$.*

Little is known about the 2−dipath chromatic number of the family of planar graphs. It is possible that the ideas utilised by Smolíková, and also utilised here in Chapter 5, may be re-purposed to define a similar notion to optimally simply colourable for 2−dipath colouring and simple 2−dipath colouring. If similar results hold true, then the family of complete-convex planar graphs may be used to find bounds for the 2−dipath chromatic number of the family of planar graphs. In turn, this may provide new insight into the oriented chromatic number for the family of planar graphs.

4.3.1 Complexity of Simple 2−dipath Colouring with Two Colours

In [34] MacGillivray and Sherk show that deciding if an input oriented graph has 2−dipath chromatic number at most k (for fixed k) is NP-complete whenever $k > 2$. The problem is Polynomial when $k = 2$. In Chapter 3 we extend this complexity result for the problem of k−dipath colouring. Here we examine the decision problem for simple 2−dipath colouring and show that deciding if an oriented graph has a simple 2−dipath colouring using two colours is NP-complete.

SIMPLE 2−DIPATH 2−COLOURING
Input: an oriented graph G.
Question: does G have a simple 2−dipath colouring using two colours?

MONOTONE NOT-ALL-EQUAL 3−SATISFIABILITY (MONOTONE-NAE3SAT)
Input: a $3CNF$ formula, F, with variables $x_1, x_2, \ldots, x_k$ and clauses $e_1, e_2, \ldots, e_\ell$ without negated variables.
Question: is F not-all-equal satisfiable?

Theorem 4.29 (Schaefer [46]). *MONOTONE-NAE3SAT is NP-complete.*

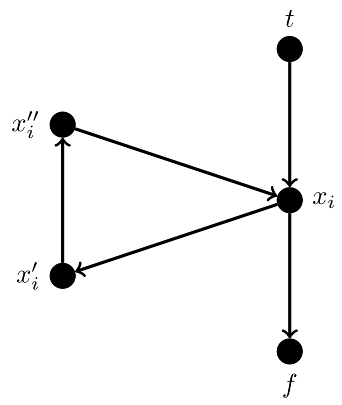


Figure 4.5: The construction for each variable of F in the proof of Theorem 4.37.

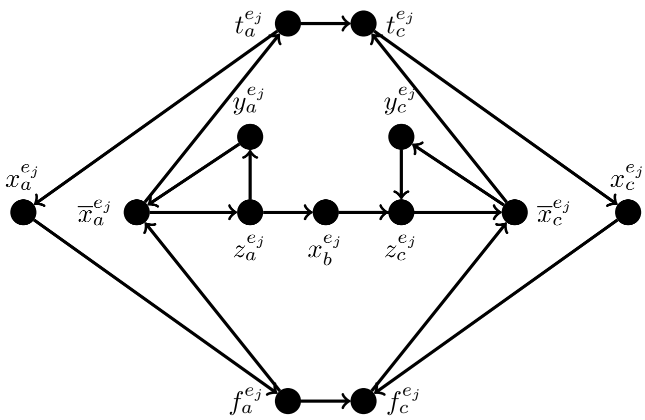


Figure 4.6: The construction for each clause, $e_j = (x_a \vee x_b \vee x_c)$, of F in the proof of Theorem 4.37.

Given an instance F of MONOTONE-NAE3SAT with k variables and ℓ clauses, we construct an oriented graph H such that F is not-all-equal satisfiable if and only if H has a simple 2-dipath colouring using two colours.

Beginning with a pair of vertices, t and f, construct H as follows.

- For each variable x_i of F $(1 \leq i \leq k)$
  - add the vertices x_i, x_i', and x_i'', and
  - add the arcs necessary to form the 2$-$dipath (tx_if) and the directed 3$-$cycle $(x_ix_i'x_i'')$.
- For each clause $e_j = (x_a \vee x_b \vee x_c)$ of F $(1 \leq j \leq \ell)$
  - construct the oriented graph given in Figure 4.6, and
  - add the arcs necessary to form the directed cycles $(x_a^{e_j}x_a'x_a'')$, $(x_b^{e_j}x_b'x_b'')$, $(x_c^{e_j}x_c'x_c'')$, $(tt_a^{e_j}t_c^{e_j})$, and $(ff_a^{e_j}f_c^{e_j})$.

This completes the construction of H. See Figures 4.5 and 4.6. Note that this construction can be obtained in polynomial time. If H has a simple 2$-$dipath colouring of H, g, using two colours, we make the following observations:

Observation 4.30. *For all clauses* $e_j = (x_a \vee x_b \vee x_c)$, $g(t) = g(t_a^{e_j}) = g(t_c^{e_j})$ *and* $g(f) = g(f_a^{e_j}) = g(f_c^{e_j})$.

Notice that in each of these cases the vertices form a directed 3−cycle. Since only two colours are being used in the simple 2−dipath colouring, each vertex in a directed 3−cycle must receive the same colour.

Observation 4.31. $g(t) \neq g(f)$.

If $g(t) = g(f)$, then all other vertices of H must receive this same colour. This violates that at least two colours are used in simple 2−dipath colouring.

Observation 4.32. *For all variables* x_i *and all clauses* e_j *such that* x_i *is contained in* e_j, $g(x_i) = g(x_i^{e_j}) = g(x_i') = g(x_i'')$.

Notice that in each of these cases the vertices form a directed 3−cycle. Since only 2 colours are being used in the simple 2−dipath colouring, each vertex in a directed 3−cycle must receive the same colour.

Observation 4.33. *For all variables* x_i *and all clauses* e_j *such that* x_i *is contained in* e_j, $g(x_i) \neq g(\overline{x}_i^{e_j}) = g(y_i) = g(z_i)$ *(when* $\overline{x}_i^{e_j}, y_i$, *and* z_i *exist).*

Since $g(x_i) = g(x_i^{e_j})$ (see Observation 4.32) , if $g(x_i) = g(\overline{x}_i^{e_j})$, then $g(t) = g(f)$. Since $\overline{x}_i^{e_j} y_i z_i$ is a directed 3−cycle, each of these vertices must be assigned the same colour.

Observation 4.34. *For all clauses,* $e_j = (x_a \vee x_b \vee x_c)$, $g(x_a)$, $g(x_b)$ *and* $g(x_c)$ *are not all equal.*

If $g(x_a)$, $g(x_b)$ and $g(x_c)$ are all equal, then the colours appearing on the 2−dipath $z_a x_b z_c$ violate the second condition of a simple 2−dipath colouring.

We use these observations to prove the following results relating the 2−colourability of H and the satisfiability of F.

Proposition 4.35. *If* H *has a simple* 2−*dipath colouring, then* F *is not-all-equal satisfiable.*

Proof. If there exists a simple 2−dipath colouring, g, using two colours, then we may use g to make the following truth assignments to the variables of F.

- If $g(x_i) = g(t)$, then assign x_i to be TRUE, otherwise assign x_i to be FALSE $(1 \leq i \leq k)$.

By Observation 4.34 each clause contains at least one TRUE variable, but does not contain three TRUE variables. Therefore if H has a simple 2−dipath colouring using two colours, then F has the required type of satisfying truth assignment. □

Proposition 4.36. *If* F *is not-all-equal satisfiable, then* H *has a simple* 2−*dipath colouring using two colours.*

Proof. Consider a not-all-equal satisfying assignment of the variables of F. We construct a simple 2−dipath colouring of H, g, using two colours.

- For all clauses e_j, let $g(t) = g(t_a^{e_j}) = g(t_c^{e_j}) = 1$ and $g(f) = g(f_a^{e_j}) = g(f_c^{e_j}) = 2$.
- For all x_i such that x_i is FALSE, let $g(x_i) = g(x_i') = g(x_i'') = g(x_i^{e_j}) = 2$ and $g(\overline{x}_i^{e_j}) = g(y_i^{e_j}) = g(z_i^{e_j}) = 1$, where e_j is a clause containing x_i $(1 \leq i \leq k$ and $1 \leq j \leq \ell)$.

- For all x_i such that x_i is TRUE, let $g(x_i) = g(x'_i) = g(x''_i) = g(x_i^{e_j}) = 1$ and $g(\overline{x}_i^{e_j}) = g(y_i^{e_j}) = g(z_i^{e_j}) = 2$, where e_j is a clause containing x_i ($1 \leq i \leq k$ and $1 \leq j \leq \ell$).

To show that g is a simple $2-$dipath colouring, we show at there is no $2-$dipath in H, uvw, such that $g(u) = g(w) \neq g(v)$.

Consider first the set of vertices $T \subset V(H)$ such that $g(v) = 1$ for all $v \in T$. For a contradiction assume there exists $u, v, w \in V(G)$ such that uvw is a $2-$dipath, where $u, w \in T$ and $v \notin T$. It suffices to consider $2-$dipaths in H where a pair of adjacent vertices in a $2-$dipath is not contained in the same directed $3-$cycle. Further, we may also discount those $2-$dipaths that have an end contained in the set

$$\{f\} \cup \{f_a^{e_j}, f_c^{e_j} | e_j = (x_a \vee x_b \vee x_c), 1 \leq j \leq \ell\},$$

as elements of this set are not elements of T.

- $uvw \neq x_a^{e_j} f_a^{e_j} \overline{x_a}^{e_j}$: if $x_a^{e_j} \in T$, then $\overline{x_a}^{e_j} \notin T$.
- $uvw \neq x_c^{e_j} f_c^{e_j} \overline{x_c}^{e_j}$: if $x_c^{e_j} \in T$, then $\overline{x_c}^{e_j} \notin T$.
- $uvw \neq \overline{x_a}^{e_j} z_a x_b^{e_j}$: if $\overline{x_a}^{e_j} \in T$, then $z_a \in T$.
- $uvw \neq x_b^{e_j} z_c \overline{x_c}^{e_j}$: if $\overline{x_c}^{e_j} \in T$, then $z_c \in T$.
- $uvw \neq z_a^{e_j} x_b^{e_j} z_c^{e_j}$: since F is not-all-equal satisfied, it cannot be that both $z_a^{e_j}$ and $z_c^{e_j}$ are in T when $x_b^{e_j} \in F$.

Therefore no such $2-$dipath uvw exists where $x, w \in T$ but $v \notin F$.

Consider now the set of vertices $F \subset V(H)$ such that $g(v) = 2$. For a contradiction assume there exists $u, v, w \in V(G)$ such that uvw is a $2-$dipath where $u, w \in F$ and $v \notin F$. It suffices to consider $2-$dipaths in H where a pair of adjacent vertices in a $2-$dipath is not contained in the same directed $3-$cycle. Further, we may also discount those $2-$dipaths that have an end contained in the set

$$\{t\} \cup \{t_a^{e_j}, t_c^{e_j} | e_j = (x_a \vee x_b \vee x_c), 1 \leq j \leq \ell\},$$

as elements of this set are not elements of F.

- $uvw \neq \overline{x_a}^{e_j} t_a^{e_j} x_a^{e_j}$: if $x_a^{e_j} \in F$, then $\overline{x_a}^{e_j} \notin F$.
- $uvw \neq \overline{x_c}^{e_j} t_c^{e_j} x_c^{e_j}$: if $x_c^{e_j} \in F$, then $\overline{x_c}^{e_j} \notin F$.
- $uvw \neq \overline{x_a}^{e_j} z_a x_b^{e_j}$: if $\overline{x_a}^{e_j} \in F$, then $z_a \in F$.
- $uvw \neq x_b^{e_j} z_c \overline{x_c}^{e_j}$: if $\overline{x_c}^{e_j} \in F$, then $z_c \in F$.
- $uvw \neq z_a^{e_j} x_b^{e_j} z_c^{e_j}$: since F is not-all-equal satisfied, it cannot be that both $x_b^{e_j}$ and $z_c^{e_j}$ are in F when $x_b^{e_j} \in T$.

Therefore there is no $2-$dipath uvw such that $x, w \in F$ and $v \notin T$, and so g is a $2-$dipath colouring of H.

□

Theorem 4.37. *SIMPLE $2-$DIPATH $2-$COLOURING is NP-complete*

Proof. Our reduction is from MONOTONE-NAE3SAT. Given an instance, F of MONOTONE-NAE3SAT we construct H, as described above (see Figures 4.5 and 4.6). We note H can be constructed in polynomial time. Since MONOTONE-NAE3SAT is NP-complete and since F is not-all-equal satisfiable if and only if H has a simple $2-$dipath colouring we have directly that SIMPLE $2-$DIPATH $2-$COLOURING is NP-complete. □

4.4 Conclusions and Future Directions

Our work here on simple colourings of planar graphs provides a possible avenue of attack for improving the upper bound on the oriented chromatic number for the family of planar graphs; rather than considering the entire family of planar graphs, we may restrict our attention to those which are complete-convex. Many questions still remain on the subject of complete-convex planar graphs. There is no known method of constructing such graphs, and it is unknown when a planar graph may be oriented to be complete-convex. Work on these two questions may provide further structure inherent to these graphs, which may in turn aid in finding a new universal target for this family of graphs.

That the problem of simple $2-$dipath colouring using just two colours is NP-complete suggests further structural results concerning simple $2-$dipath colouring may be difficult. It may be possible that for some particular families of graphs, the problem of simple $2-$dipath colouring may be easier to study. A good candidate for a family of such graphs would be oriented $2-$trees. Additionally, given the interest in the oriented chromatic number of planar graphs and the relationship between the simple chromatic number and chromatic number of such graphs, this family would be a priority in the study of the simple $2-$dipath chromatic number.

It is possible that simple $2-$dipath colourings may be generalised to simple $k-$dipath colourings using the same methods as in Chapter 3. However, given that there is no universal target for oriented graphs with simple $2-$dipath chromatic number m, the homomorphism model used in Chapter 3 will not be able to provide a universal target for simple $k-$dipath colourings.

Chapter 5
Vertex Colourings of $k-$edge-coloured Graphs

In Chapter 5 we examine colourings of $(0, k)-$mixed graphs. More often called $k-$edge-coloured graphs, these graphs arise from ordinary graphs by assigning an edge type (colour) to each of the edges. Here we consider vertex colourings of these graphs. We find a lower bound for the chromatic number of the family $2-$edge-coloured graphs with maximum degree 3 by considering a new colouring parameter for these graphs. We find an upper bound for these graphs by constructing targets for graphs in this family. Finally, we consider vertex colourings of $k-$edge-coloured graphs that allow, in some cases, adjacent vertices to receive the same colour. We find that these colourings, called simple colourings, provide an avenue of attack to improve the upper bound on the chromatic number of families of $k-$edge-coloured graphs.

5.1 Background and Preliminaries

When restricted to $(j, k) = (0, k)$, the definitions for homomorphism and colouring given in Chapter 1 give the following.

Definition 5.1. *Let (G, Σ) and (H, Π) be $k-$edge-coloured graphs. We say (G, Σ) admits a homomorphism to (H, Π), denoted $(G, \Sigma) \to (H, \Pi)$, if there exists $\phi : V(G) \to V(H)$ such that, for all $1 \leq i \leq k$, if $uv \in \Sigma_i$, then $\phi(u)\phi(v) \in \Pi_i$. We call ϕ a homomorphism and we write $\phi : (G, \Sigma) \to (H, \Pi)$.*

Definition 5.2. *Let G be a $k-$edge-coloured graph. The chromatic number of G, denoted $\chi_k(G)$, is the least integer m such that there exists a $k-$edge-coloured graph H such that $|V(H)| = m$ and a homomorphism $\phi : G \to H$. We call ϕ an $m-$colouring of G, or a colouring of G using m colours. If $\mathcal{F}$ is a family of k-edge-coloured graphs, then we define $\chi_k(\mathcal{F})$ to be the least m such that for all $F \in \mathcal{F}$, $\chi_k(F) \leq m$.*

Recall the vertex labelling definition for colouring of $k-$edge-coloured graphs.

Definition 5.3. *If (G, Σ) is a $k-$edge-coloured graph and $c : V(G) \to \{1, 2, 3, \ldots, m\}$, then c is an $m-$colouring of G provided the following conditions are met:*

- *for all $uv \in E(G)$, $c(u) \neq c(v)$, and*
- *for all $1 \leq i \leq k$, $uv \in \Sigma_i$, and $xy \in E(G)$, if $c(u) = c(x)$ and $c(v) = c(y)$, then $xy \in \Sigma_i$.*

When restricted to the case $k = 1$, our definitions for homomorphism and colouring match exactly the usual definitions for graphs. When restricted to the case $k = 2$, we arrive at the following definitions.

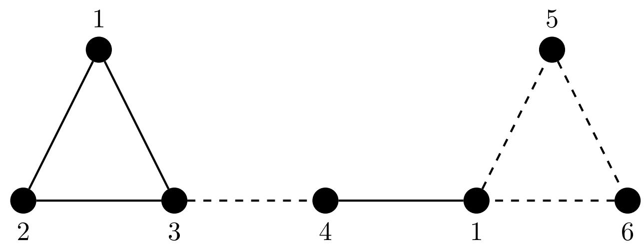


Figure 5.1: A 2−edge-coloured graph coloured with 6 colours.

Definition 5.4. *A* 2−edge-coloured graph *is a simple graph, G, together with a function $\Sigma : E(G) \to \{1,2\}$. For all $e \in E(G)$, if $e \in \Sigma_1$, then we say e is a* red *edge, otherwise it is a* blue *edge.*

In our diagrams we denote red edges by solid lines and blue edges by dashed lines.

Definition 5.5. *Let (G, Σ) be a 2−edge-coloured graph. A* colouring of (G, Σ) using m colours *or, alternatively an $m-$*colouring of G*, is a function $c : V(G) \to \{1, 2, 3, \ldots, m\}$ such that*

- *if $uv \in E(G)$, then $c(u) \neq c(v)$, and*
- *if $uv, xy \in E(G)$ such that $c(u) = c(x)$ and $c(v) = c(y)$, then either $uv, xy \in \Sigma_1$ or $uv, xy \in \Sigma_2$.*

Consider the 2-edge-coloured graph G in Figure 5.1. The simple graph underlying G has chromatic number 3; however 6 colours are needed for a colouring of the 2−edge-coloured graph. Though colour 1 may be used again on the 3−cycle formed from blue edges, the colour 2 may not be used on this 3−cycle, as there is already a red edge between a vertex coloured 1 and a vertex coloured 2. The 2−edge-coloured graph in Figure 5.2, H, is a target in a homomorphism that is equivalent to this colouring; that is, $G \to H$.

As with oriented colouring, the second requirement of a colouring of a 2−edge-coloured graph gives rise to a local requirement in any colouring of a 2−edge-coloured graph. We see if G contains a path uvw where $uv \in \Sigma_1$ and $vw \in \Sigma_2$, then it must be u and w receive different colours. This condition, that vertices at the end of such a path receive different colours, is necessary but not sufficient for a colouring of a 2−edge-coloured graph. Therefore we can use colourings that satisfy this condition to find a lower bound for the chromatic number. With this in mind we define the term *alternating* 2−*path.*

Definition 5.6. *If G is a 2−edge-coloured graph, and $u, v, w \in V(G)$ such that $uv \in \Sigma(uv) \neq \Sigma(vw)$, then the path uvw is an* alternating 2−path*.*

Colourings of oriented graphs that assign to each vertex a colour such that pairs of vertices at directed distance at most 2 receive different colours (2−dipath colourings) are useful when constructing lower bounds for the oriented chromatic number.(See Chapter 3). We consider an analogous colouring parameter for 2−edge-coloured graphs.

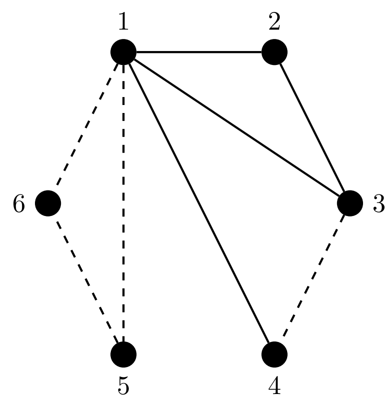


Figure 5.2: A homomorphic image of the 2−edge-coloured graph in Figure 5.1.

Definition 5.7. *For a* 2−*edge-coloured graph,* (G, Σ)*, we define an* alternating 2−path colouring using m colours*, or an* alternating 2−path m−colouring*, to be a function* $c : V(G) \to \{1, 2, 3, \ldots, m\}$ *such that*

- *if* $uv \in E(G)$*, then* $c(u) \neq c(v)$*, and*
- *if* $uv, vw \in E(G)$ *is an alternating* 2−*path, then* $c(u) \neq c(w)$*.*

Definition 5.8. *The* alternating 2−path chromatic number *of* (G, Σ)*, denoted by* $\chi_2^a(G, \Sigma)$ *is the least* m *such that* (G, Σ) *has an alternating* 2−*path colouring using* m *colours. For a family of graphs,* $\mathcal{F}$*, define* $\chi_2^a(\mathcal{F})$ *to be the least* m *such that for all* $(G, \Sigma) \in \mathcal{F}$*,* $\chi_2^a((G, \Sigma)) \leq m$*.*

Since every m−colouring is also an alternating 2−path colouring, we get immediately that for all (G, Σ), $\chi_2^a(G, \Sigma) \leq \chi_s(G, \Sigma)$.

5.2 Vertex Colourings of 2−edge-coloured Graphs with $\Delta \leq 3$

We begin by showing each 2−edge-coloured cubic graph has an alternating 2−path 8−colouring. To do so, we first define a graph akin to the square of an oriented graph. Let (G, Σ) be a 2−edge-coloured graph. Define $G^\star$, a simple graph, with

- $V(G^\star) = V(G)$ and
- $E(G^\star) = E(G) \cup \{uv | u$ and v are the ends of an alternating 2−path in G$\}$.

In this graph a pair of vertices are adjacent if and only if they are adjacent in G or they are at the ends of an alternating 2−path in G. As such, a proper colouring of $G^\star$ is also an alternating 2−path colouring of G and so $\chi(G^\star) = \chi_2^a(G)$.

Lemma 5.1. *If* (G, Σ) *is a* 2−*edge-coloured graph with* $\Delta \leq 3$*, then either* $G^\star$ *contains a vertex with degree at most* 6*, or* $G^\star$ *has a proper* 8*-colouring.*

Proof. Let (G, Σ) be a 2−edge-coloured graph, and let $G^\star$ be defined as above. To prove our lemma, we show the average degree in $G^\star$ is no more than 7. Each edge of G contributes exactly one edge to $G^\star$. Since each vertex of G is the centre vertex of at most two alternating 2−paths, each vertex of G contributes at most two edges to $G^\star$. Therefore

$$|E(G^\star)| \leq |E(G)| + 2|V(G)| = \frac{3|(V(G))|}{2} + 2|V(G)| = \frac{7|V(G)|}{2}$$

Since $G^\star$ has at most $\frac{7|(V(G))|}{2}$ edges it is either 7−regular and so, by Brooks' Theorem is 8−colourable, or $G^\star$ has a vertex of degree at most 6. □

Consider the 2−edge-coloured graph G given in Figure 5.3. In this case $G^\star$ is K_8, and so G is an example of a 2−edge-coloured subcubic graph that requires 8 colours in an alternating 2−path colouring. Recall that a graph is *subcubic* if it has maximum degree 3, but is not 3−regular. To show 8 colours suffice for 2−edge-coloured cubic graphs, we first show 8 colours suffice for a 2−edge-coloured subcubic graphs.

Lemma 5.2. *If* (G, Σ) *is a connected* 2−*edge-coloured subcubic graph, then* $\chi_2^a(G, \Sigma) \leq 8$.

Proof. We proceed by induction on n, the number of vertices of G. Note our statement is trivially true for all $n \leq 8$. Let u be a vertex of degree 2 in G, with neighbours x and y. Consider an alternating 2−path colouring, c, of $G \setminus \{u\}$. Since x and y are of degree at most 2 in $G \setminus \{u\}$, c can be constructed such that $c(x) \neq c(y)$, as each of x and y only need to disagree in colour from at most 6 other vertices in a colouring of $G \setminus \{u\}$ and 8 colours are available. This colouring can be extended to one of G since there are only 6 vertices from which u needs to disagree in colour and 8 available colours. □

Using Lemmas 5.1 and 5.2, we give a proof of the main result regarding alternating 2−path colourings of cubic graphs.

Theorem 5.3. *Every* 2−*edge-coloured cubic graph has an alternating* 2−*path colouring using no more than* 8 *colours.*

Proof. Figure 5.3 gives an example of a 2−edge-coloured cubic graph that requires 8 colours. This shows $\chi_2^a(\mathcal{F}) \geq 8$ for, $\mathcal{F}$, the family of 2−edge-coloured cubic graphs.

Let G be a 2−edge-coloured cubic graph. We may assume $G^\star$ is not 7−regular, as otherwise we have directly that G has an alternating 2−path colouring using no more than 8 colours. By Lemma 5.1, there exists a vertex v such that v has degree less than 7 in $G^\star$. We may assume $d_G(v) = 3$, as otherwise we have directly that $d_{G^\star}(v) \leq 6$. Let a, b, d be the neighbours of v in G. We proceed by cases.

Case I: All of va, vb, vd *are red.* By Lemma 5.2, $G \setminus \{v\}$ has an alternating 2−path colouring, c, using 8 colours. Since v must disagree in colour with no more than 6 vertices, and we have a palette of 8 colours, c can be extended, as adding v creates no new alternating 2−paths between the neighbours of v.

Case II: Exactly one of va, vb, vd *is red.* Without loss of generality, assume $va \in \Sigma_1$ and $vb, vd \in \Sigma_2$. By Lemma 5.2, $G \setminus \{v\}$ can be coloured using 8 colours. Further, in this colouring there are at least two choices for each of a, b, d. Therefore a colouring, c, of $G \setminus \{v\}$ exists where $c(a) \neq c(b)$ and $c(a) \neq c(d)$. This colouring can be extended to one of G, since v must disagree in colour with no more than 6 vertices, as v has degree at most 6 in $G^\star$. □

The example given in Figure 5.3 is the only 2−edge-cubic graph known, at the time of writing to require 8 colours in an alternating 2−path colouring.

We turn now to the task of bounding the chromatic number of the family of 2−edge-coloured graphs with maximum degree 3. We begin by observing that since each vertex

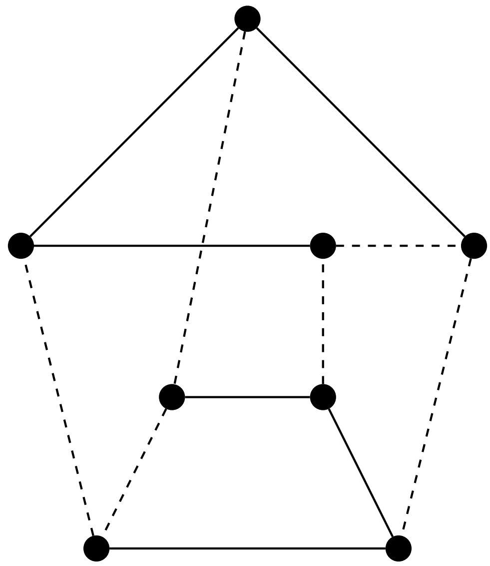


Figure 5.3: A $2-$edge-coloured cubic graph that requires 8 colours in an alternating $2-$path colouring.

in the graph in Figure 5.3 requires its own colour in an alternating $2-$path colouring, then this $2-$edge-coloured graph also requires 8 colours in a proper colouring. This gives directly that $\chi_2 \geq 8$ for this family. To give an upper bound for this parameter, we will make use of the following property.

Property 5.4. *A $2-$edge-coloured complete graph (G, Σ) has property $P_{i,j}$ if for every subset $X \subset V(G)$ of size i and for every sequence $(z_1, z_2, \ldots, z_i)$, $z_k \in \{1, 2\}$ $(1 \leq k \leq i)$, there exist j distinct vertices in $V(G) \setminus X$, $y_1, y_2, \ldots, y_j$, such that for all $1 \leq \ell \leq j$, $x_i y_\ell \in E(G)$ and $\Sigma(x_i y_\ell) = z_i$.*

Variants of this property have appeared in previous work on $2-$edge-coloured graphs and also in work on oriented graphs. A nice survey on graphs with property $P_{1,n}$ is given by Bonato (here called $n-$existentially closed) [5]. Here, it is shown that such graphs can be found among the Payley graphs. Sopena and Vignal use the existence of an oriented version of this property to give an upper bound on the oriented chromatic number of cubic graphs [53].

To show no more than 11 colours are required for a colouring of a $2-$edge-coloured cubic graph, we exhibit a $2-$edge-coloured graph on 9 vertices with property $P_{2,1}$ and then show, with a few exceptions, every subcubic $2-$edge-coloured graph admits a homomorphism to this graph. Using this fact, we can then, for any $2-$edge-coloured subcubic graph, find a $2-$edge-coloured target with 11 vertices.

Let (H, Σ) be the $2-$edge coloured graph formed from the complete graph on 9 vertices where the red edges are those shown in Figure 5.4. Observe that $H[\Sigma_1] = H[\Sigma_2] = C_3 \Box C_3$. And so the subgraph induced by the red edges is isomorphic to the one induced by the blue edges, each of these subgraphs is edge transitive, and H is vertex transitive.

The $2-$edge-coloured graph H exhibits the following properties.

Property 5.5. *For every edge xy of H there exists:*

1. *a single vertex z such that $\Sigma(xz) = \Sigma(yz) = \Sigma(xy)$,*

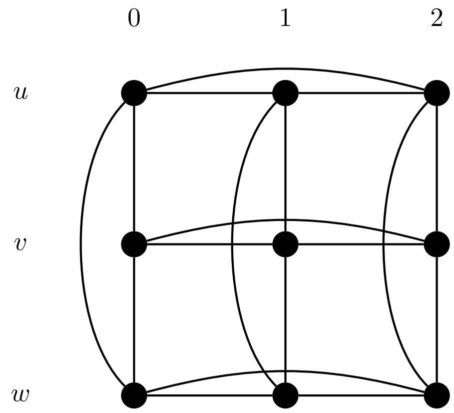


Figure 5.4: The red edges of H, a $2-$edge-coloured graph with property $P_{2,1}$. The vertices are labelled by their row and column index.

2. *a pair of vertices z_1 and z_2 such that $\Sigma(xz_1) = \Sigma(xz_2) = 1$ and $\Sigma(z_1y) = \Sigma(z_2y) = 2$, and*

3. *a pair of vertices z_1 and z_2 such that $\Sigma(xz_1) = \Sigma(xz_2) = \Sigma(z_1y) = \Sigma(z_2y) \neq \Sigma(xy)$.*

We observe these properties by considering the neighbourhood of the vertex set $\{u_0, v_0\}$. In [37] the authors use these properties of H to show H is a universal target for the family of $2-$edge-coloured outerplanar graphs and in fact all $2-$edge-coloured $2-$trees.

Corollary 5.6 (Montejano et al. [37])**.** *The $2-$edge-coloured graph H has property $P_{2,1}$.*

That H has property $P_{2,1}$ is not enough to show that each $2-$edge-coloured cubic graph admits a homomorphism to H. Consider the pair of $2-$edge-coloured graphs, A_1 and A_2, in Figure 5.5. Call this set of graphs $\mathcal{A}$. Neither of these graphs admits a homomorphism to H. To see this, observe that in any colouring of A_1 or A_2, a_1 and a_2 must receive different colours. By Property 5.5.3 this means a_3 and a_4 receive the same colour. However this is a contradiction, as they are at the ends of an alternating $2-$path.

It turns out, however, that these subcubic graphs are the lone obstructions to subcubic homomorphism to H. In order to show subcubic graphs that contain neither A_1 nor A_2 admit a homomorphism to H we require the following property of A_1 and A_2.

Property 5.7. *Let A' be a graph produced from a graph in $\mathcal{A}$ by changing the colour of any edge and then subdividing this edge. Let x be the new vertex created by this process. There exists a pair of homomorphisms $c_1, c_2 : A' \to H$ such that $c_1(d) = c_2(d)$ but $c_1(x) \neq c_2(x)$.*

Figure 5.6 gives all possible graphs formed by subdividing an edge of A_2, as above. Table 5.1 gives explicit colourings that verify the property. Similar colourings may also be obtained for the possible graphs obtained by subdividing an edge of A_1, as above.

Theorem 5.8. *Every connected 2-edge-coloured subcubic graph with no subgraph isomorphic to a graph in $\mathcal{A}$ admits a homomorphism to H.*

Proof. Let (G, Σ) be a minimum counter-example with respect to number of vertices and, subject to that, with respect to the number of edges. Since G is the smallest counter-example and is subcubic, there exists a vertex z with neighbours x and y such that $G\backslash\{z\}$

Figure 5.5: A_1 and A_2.

B_1	$\mathbf{c_1}$	$\mathbf{c_2}$
$\mathbf{a_1}$	v_0	v_0
$\mathbf{a_2}$	u_2	u_2
$\mathbf{a_3}$	v_0	v_0
$\mathbf{a_4}$	w_1	w_1
$\mathbf{d}$	u_0	u_0
$\mathbf{x}$	v_1	v_2

B_2	$\mathbf{c_1}$	$\mathbf{c_2}$
$\mathbf{a_1}$	u_2	u_2
$\mathbf{a_2}$	v_0	v_0
$\mathbf{a_3}$	v_0	v_0
$\mathbf{a_4}$	w_1	w_1
$\mathbf{d}$	u_0	u_0
$\mathbf{x}$	v_1	v_2

B_3	$\mathbf{c_1}$	$\mathbf{c_2}$
$\mathbf{a_1}$	u_2	u_2
$\mathbf{a_2}$	w_1	w_1
$\mathbf{a_3}$	v_0	v_0
$\mathbf{a_4}$	w_1	w_1
$\mathbf{d}$	u_0	u_0
$\mathbf{x}$	w_0	w_2

B_4	$\mathbf{c_1}$	$\mathbf{c_2}$
$\mathbf{a_1}$	w_1	w_1
$\mathbf{a_2}$	u_2	u_2
$\mathbf{a_3}$	v_0	v_0
$\mathbf{a_4}$	w_1	w_1
$\mathbf{d}$	u_0	u_0
$\mathbf{x}$	w_0	w_2

B_5	$\mathbf{c_1}$	$\mathbf{c_2}$
$\mathbf{a_1}$	u_2	u_2
$\mathbf{a_2}$	u_2	u_2
$\mathbf{a_3}$	v_0	v_0
$\mathbf{a_4}$	w_1	w_1
$\mathbf{d}$	u_0	u_0
$\mathbf{x}$	u_0	u_1

B_6	$\mathbf{c_1}$	$\mathbf{c_2}$
$\mathbf{a_1}$	w_2	u_2
$\mathbf{a_2}$	u_1	u_1
$\mathbf{a_3}$	v_0	w_0
$\mathbf{a_4}$	v_0	w_0
$\mathbf{d}$	u_0	u_0
$\mathbf{x}$	w_0	v_0

B_7	$\mathbf{c_1}$	$\mathbf{c_2}$
$\mathbf{a_1}$	u_0	u_0
$\mathbf{a_2}$	u_2	w_2
$\mathbf{a_3}$	w_1	v_1
$\mathbf{a_4}$	w_1	v_1
$\mathbf{d}$	u_0	u_0
$\mathbf{x}$	v_2	w_2

Table 5.1: Homomorphisms to H of the graphs in Figure 5.6.

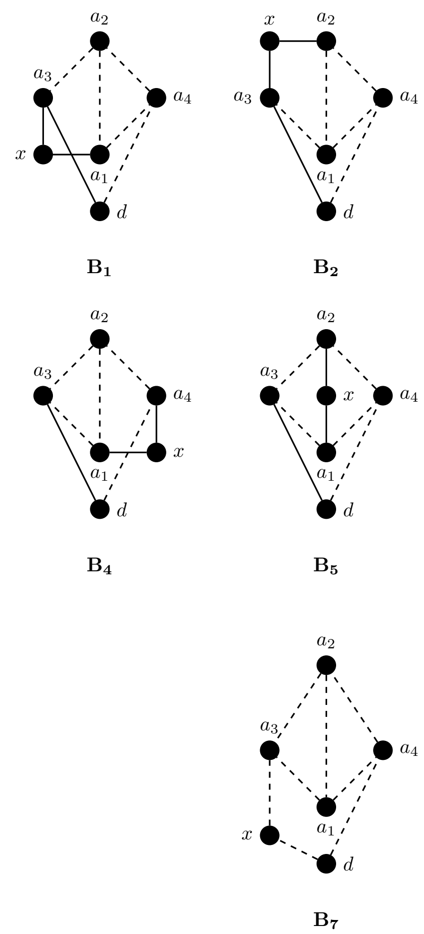


Figure 5.6: Possibilities for subdividing an edge in A_2, as described in Property 5.7.

admits a homomorphism to H. Further, it must be, by Property 5.5, that $xz \in \Sigma_1$, $yz \in \Sigma_2$ and in every homomorphism $\phi : G \setminus \{z\} \to H$ we have $\phi(x) = \phi(y)$. Since H is vertex transitive, we may also assume G has no cut-edge. Further, since $\Delta(G) \leq 3$, G has no cut-vertex, as the existence of a cut-vertex implies the existence of a cut-edge.

Let x_1 and x_2 (respectively y_1 and y_2) be the neighbours of x (respectively y) in $G \setminus \{z\}$. It must be x_1x and x_2x (respectively y_1y and y_2y) have the same colour, as otherwise, by Property 5.5 there would exist a homomorphism $\phi : G \setminus \{z\} \to H$ such that $\phi(x) \neq \phi(y)$.

We proceed by considering the existence and colour of an edge x_1x_2. By Properties 5.5.2 and 5.5.3, we may assume xx_1 is a red edge and xx_2 is a red edge.

Case I: There is a blue edge between x_1 and x_2: If there is a blue edge between x_1 and x_2, then by Property 5.5.2 any colouring $G \setminus \{z, x\}$ can be extended to one of $G \setminus \{z\}$ in two ways. This is a contradiction.

Case II: There is no edge between x_1 and x_2: If there is no edge between x_1 and x_2 we can obtain a homomorphism $\phi : G \setminus \{z\} \to H$ such that $\phi(x) \neq \phi(y)$ by

- removing x from $G \setminus \{z\}$, and
- adding a blue edge x_1x_2.

If this new graph has no subgraph isomorphic to a graph in $\mathcal{A}$, then since this new graph has fewer vertices than G it admits a homomorphism, δ, to H. By Property 5.5.3, δ can be extended to be a homomorphism from $G \setminus \{z\}$ to H in two ways. This implies that there exists a homomorphism $\phi : G \setminus \{z\} \to H$ such that $\phi(x) \neq \phi(y)$. This is a contradiction.

If this new graph does contain a subgraph isomorphic to a graph in $\mathcal{A}$, then it must be replacing the red edges xx_1 and xx_2 with the blue edge x_1x_2 yields a block isomorphic to a graph in $\mathcal{A}$. However, by Property 5.7, any colouring of $G \setminus \{z, x\}$ can be extended to one of $G \setminus \{z\}$ such that x and y receive different colours. This is a contradiction.

Case III: There is a red edge between x_1 and x_2: Assume there is a red edge between x_1 and x_2. Let s_1 and s_2 be neighbours of x_1 and x_2, respectively, in $G \setminus \{z, x\}$.

If $s_1 = s_2$, then s_1 is a cut vertex in $G \setminus \{z\}$. As such, any colouring of $G \setminus \{z, x, x_1, x_2, s_1\}$ can be extended to one of $G \setminus \{z\}$ such that x and y receive different colours. To see this, note there are three possible colours for s_1 in such an extension.

Otherwise, we require the following claim.

Claim 11. *There exists a colouring of $G \setminus \{z, x, x_1, x_2\}$ with H such that s_1 and s_2 receive different colours.*

Consider the graph G' formed from G by removing z, x, x_1, x_2, adding new vertex t, a red edge s_1t and a blue edge s_2t. If this graph contains no subgraph from $\mathcal{A}$, then it admits a homomorphism to H such that s_1 and s_2 receive different colours. If this graph does contain a subgraph from $\mathcal{A}$, then it must be that the path from s_1 to s_2 through t corresponds to the path between a_3 and a_4 through d in a copy of either A_1 or A_2. We show that this implies there is only a single vertex of degree at most 2 in $G \setminus \{z\}$. This is a contradiction as both x and y have degree at most 2.

Consider the graph induced by the vertices corresponding to a_1, a_2, a_3 and a_4 when the vertex t is added. This graph is $K_4 \setminus \{a_1a_2\}$. However, s_1 and s_2 (the vertices corresponding to a_1 and a_2) are adjacent to x_1 and x_2, respectively. Further, by assumption, x_1 and x_2 are adjacent and are also adjacent to x. As G has maximum degree 3, there can be no other vertices in $G \setminus \{x, z\}$, as z is not a cut vertex. Therefore $G \setminus \{z\}$ has only a single vertex of degree 2, a contradiction as each of x and y are of degree 2.

Consider a colouring of $G \setminus \{z, x, x_1, x_2\}$ with H in which s_1 and s_2 receive different colours. Figure 5.7 shows all possibilities for the edges in the subgraph induced by

Γ_1	$\mathbf{c_1}$	$\mathbf{c_2}$
$\mathbf{x}$	w_1	w_2
$\mathbf{x_1}$	v_1	v_2
$\mathbf{x_2}$	u_1	u_2
$\mathbf{s_1}$	v_0	v_0
$\mathbf{s_2}$	u_0	u_0

Γ_2	$\mathbf{c_1}$	$\mathbf{c_2}$
$\mathbf{x}$	w_2	u_0
$\mathbf{x_1}$	v_2	v_0
$\mathbf{x_2}$	u_2	w_0
$\mathbf{s_1}$	v_1	v_1
$\mathbf{s_2}$	u_0	u_0

Γ_3	$\mathbf{c_1}$	$\mathbf{c_2}$
$\mathbf{x}$	w_0	u_1
$\mathbf{x_1}$	w_2	v_1
$\mathbf{x_2}$	w_1	w_1
$\mathbf{s_1}$	v_0	w_0
$\mathbf{s_2}$	u_0	u_0

Γ_4	$\mathbf{c_1}$	$\mathbf{c_2}$
$\mathbf{x}$	w_0	u_2
$\mathbf{x_1}$	w_2	w_2
$\mathbf{x_2}$	w_1	v_2
$\mathbf{s_1}$	v_1	v_1
$\mathbf{s_2}$	u_0	u_0

Γ_5	$\mathbf{c_1}$	$\mathbf{c_2}$
$\mathbf{x}$	u_0	v_2
$\mathbf{x_1}$	u_2	w_2
$\mathbf{x_2}$	u_1	v_2
$\mathbf{s_1}$	v_1	v_1
$\mathbf{s_2}$	u_0	w_2

Γ_6	$\mathbf{c_1}$	$\mathbf{c_2}$
$\mathbf{x}$	u_0	w_1
$\mathbf{x_1}$	u_2	w_2
$\mathbf{x_2}$	u_1	w_0
$\mathbf{s_1}$	v_0	v_0
$\mathbf{s_2}$	u_0	u_0

Table 5.2: Homomorphisms to H of the graphs in Figure 5.7.

x, x_1, x_2, s_1 and s_2, up to symmetry. For each graph in Figure 5.7, Table 5.2 gives a pair of colourings that each give the same colour to s_1 and s_2 but the two colourings give a different colour for x. This contradicts that every homomorphism from $G \setminus \{z\}$ gives the same colours to both x and y.

Therefore, every $2-$edge-coloured connected subcubic graph that does not contain a subgraph from $\mathcal{A}$ admits a homomorphism to H. $\square$

Theorem 5.9. *Every connected* $2-$*edge-coloured graph with maximum degree* 3 *admits a homomorphism to a* $2-$*edge-coloured graph on* 11 *vertices.*

Proof. Let G be a connected $2-$edge-coloured graph with maximum degree 3.

Case I: G contains no subgraph isomorphic to a graph in $\mathcal{A}$: Let uv be an edge of G. Form G' by deleting uv . By Theorem 5.8, G' admits a homomorphism, ϕ, to H. By colouring each vertex other than u and v as prescribed by ϕ and giving u colour 0 and v colour $0'$ we have an 11-colouring of G.

Case II: G contains a subgraph isomorphic to a graph in $\mathcal{A}$: From each of these subgraphs remove the vertex a_2. What remains admits a homomorphism to H. By colouring all removed vertices from copies of A_1 with colour 0^+ and those from copies of A_2 with colour 0^-, we obtain an 11-colouring of G. $\square$

Both the result and method of proof of this result are similar in flavour to the result on oriented cubic graphs presented in Chapter 2. In both results we consider a target graph with convenient adjacency properties, consider a family of subgraphs that do not admit a homomorphism to this target, and then construct a homomorphism to a modified target depending on the existence, or lack thereof, of these subgraphs in the input graph. In Chapter 2 this method is also used to construct colourings of oriented graphs with maximum degree 4. It is likely that this technique can be useful in constructing colourings of $(j, k)-$mixed graphs with bounded degree, provided target graphs with appropriate adjacency properties exist. For example, the graph formed from H using the Tromp construction has property $P_{3,1}$ [37].

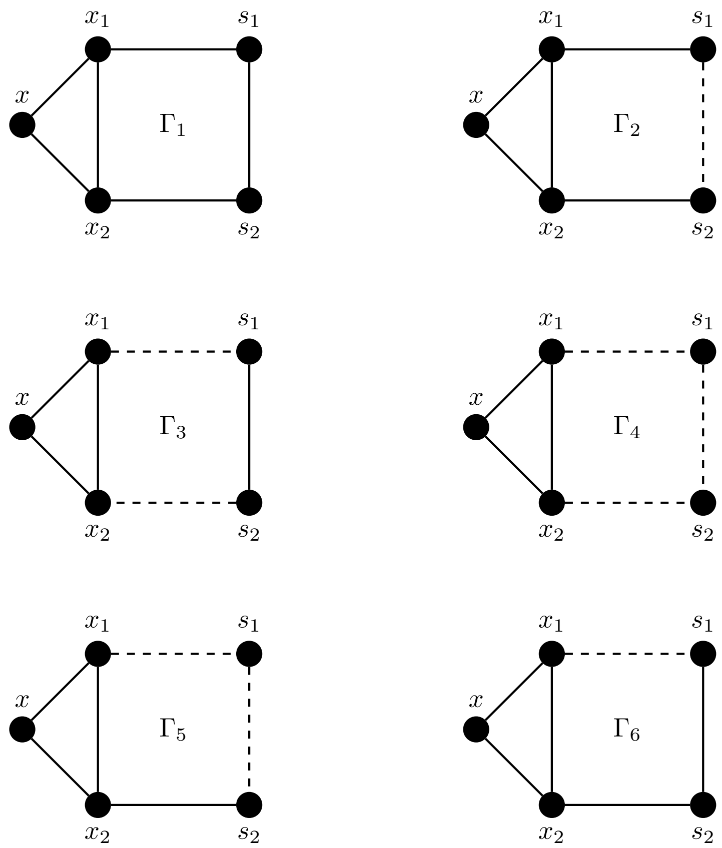


Figure 5.7: Possibilities in *Case III* if $s_1 \neq s_2$

5.3 Simple Colourings of $k-$edge-coloured Graphs

Definition 5.9. *Let (G, Σ) and (H, Π) be $k-$edge-coloured graphs. We say G* admits a simple homomorphism to *H, denoted $G \to_s H$, if there exists a mapping $\phi : V(G) \to V(H)$, such that the following conditions are satisfied:*

1. *there exist $u, v \in V(G)$ such that $\phi(u) \neq \phi(v)$, and*
2. *for all $1 \leq i \leq k$ if $uv \in \Sigma_i$, then $\phi(u)\phi(v) \in \Pi_i$ or $\phi(u) = \phi(v)$.*

If $|V(H)| = m$, we call ϕ a simple colouring of G using m colours *or* a simple $m-$colouring of *G.*

Alternatively we may view simple homomorphisms of $k-$edge-coloured graphs as homomorphisms where the target graph has a loop of each edge colour and the range of the vertex mapping consists of at least two vertices.

Definition 5.10. *If G is a $k-$edge-coloured graph and c is a simple colouring of G, then a proper subgraph, H, of G is called* monochromatic *if for all $u, v \in V(G)$, $c(u) = c(v)$.*

Definition 5.11. *For a $k-$edge-coloured graph G the* simple chromatic number of *G is the least m such that there exists a simple $m-$colouring of G. We denote this value as $\chi_k^s(G)$. If $\mathcal{F}$ is a family of $k-$edge-coloured graphs we define $\chi_k^s(\mathcal{F})$ as the smallest m such that for all $F \in \mathcal{F}$, $\chi_k^s(F) \leq m$. In the event no such m exists we say $\chi_k^s(\mathcal{F}) = \infty$.*

Simple colourings were first introduced by Smolíková [49] in her Ph.D thesis. She studied simple colourings of oriented graphs. Here we show her results and methods can be adapted for $2-$edge-coloured graphs.

5.3.1 Simple Colourings of $2-$edge-coloured Graphs

Those $k-$edge-coloured graphs with simple chromatic number equal to two are easily characterised.

Proposition 5.10. *A $k-$edge-coloured graph G has $\chi_k^s(G) = 2$ if and only if G has a monochromatic edge cut.*

This follows by observing that, up to the colour of the edges, there is a single target for simple $2-$colouring. This target is a single edge with loops of both colours at either end.

Proposition 5.11. *There exist no $2-$edge-coloured graph with simple chromatic number equal to* 3.

This follows by observing that, up to the the colour of the edges, there are exactly two $2-$edge-coloured complete graphs on 3 vertices, one in which all of the edges have the same colour, and one in which exactly two of the edges agree in colour. Each of these graphs has simple chromatic number 2. And so any $2-$edge-coloured graph that admits surjective simple homomorphism to either of these graphs must satisfy the hypothesis of Proposition 5.10.

Definition 5.12. *Let (G, Σ) be a $2-$edge-coloured graph and let $u, v, w \in V(G)$ such that $uv, vw \in E(G)$. We say v is* between *u and w if uvw is an alternating $2-$path. We say $C \subset V(G)$ is* convex *if for any pair $u, w \in C$ there is no $v \in V(G) \setminus C$ such that v is between u and w. Let $N \subseteq V(G)$. The* convex hull *of N is the minimum convex set of vertices of G that has N as a subset. We denote this set $conv(N)$.*

Note that if N is a convex set, then $conv(N) = N$.

Proposition 5.12. *Let c be a simple colouring of a $2-$edge-coloured graph G and consider $N \subseteq V(G)$ such that for all $u \in N$, $c(u) = i$. For all $x \in conv(N)$, it must be $c(x) = i$.*

Proof. Consider a vertex $x \in conv(N)$ and let N' be the largest subset of $conv(N)$ such that $x \notin N'$ and for all $y, z \in N'$ such that if there is a vertex $w \neq x$ between y and z, then $w \in N'$. We proceed by induction on the cardinality of N'. If $|N'| = 2$, then, since N' is largest, x is between the two vertices in N' and so $c(x) = i$.

Assume now that $|N'| = k > 2$. Since N' is largest, there exists a pair of vertices $y, z \in N'$ such that x is between y and z. If c is a simple colouring of G, then by induction $c(y) = c(z) = c(v)$ for all $v \in N'$. Since x is between y and z, it must also be $c(x) = c(u)$. □

In the remainder of this chapter we explore families of graphs, $\mathcal{F}$, for which $\chi_2(\mathcal{F}) = \chi_2^s(\mathcal{F})$.

Definition 5.13. *A family of $2-$edge-coloured graphs, $\mathcal{F}$ is* optimally simply colourable *if $\chi_2(\mathcal{F}) = \chi_2^s(\mathcal{F})$.*

We begin our study of such families by considering the family of $2-$edge-coloured planar graphs, $\mathcal{P}$, as the long-standing upper bound of $\chi_2(\mathcal{P}) \leq 80$ [2] is of particular interest.

Theorem 5.13. *The family $\mathcal{P}$ of planar $2-$edge-coloured graphs is optimally simply colourable.*

Proof. As every $m-$colouring is also a simple $m-$colouring, we have directly that

$$\chi_2^s(\mathcal{P}) \leq \chi_2(\mathcal{P}).$$

Let $m = \chi_2^s(\mathcal{P})$ and let $(P, \Sigma) \in \mathcal{P}$ have at least 3 vertices. We show there is an $m-$colouring of P. Since P was chosen arbitrarily, this implies $\chi_2(\mathcal{P}) \leq m = \chi_2^s(\mathcal{P})$.

Let $(P^\star, \Sigma^\star)$ be a triangulation of (P, Σ) such that $\Sigma_1 \subseteq \Sigma_1^\star$ and $\Sigma_2 \subseteq \Sigma_2^\star$. Denote by $C(P^\star)$ the set of all simple m-colourings of $(P^\star, \Sigma^\star)$ that contain a monochromatic edge. For every $c \in C(P^\star)$, $P^\star$ has a triangular face F_c that is not monochromatic under c, but contains a monochromatic edge, as otherwise we have directly that P has an $m-$colouring. Denote the vertices of such a face by x_c, y_c, z_c. We note that, with respect to the colours of the edges, there are four possibilities for the edges in this face (up to relabelling of x_c, y_c, z_c). We will refer to these four possibilities as follows:

1. Type A: $x_cy_c, y_cz_c, z_cx_c \in \Sigma_1$
2. Type B: $x_cy_c, z_cx_c \in \Sigma_1$ and $y_cz_c \in \Sigma_2$
3. Type C: $z_cy_c \in \Sigma_1$ and $x_cy_c, x_cz_c \in \Sigma_2$
4. Type D: $x_cy_c, y_cz_c, z_cx_c \in \Sigma_2$

Notice that by reversing the roles of red and blue edges in Types A and B we obtain types D and C, respectively.

We define a new $2-$edge-coloured graph (R, Π) that has $(P^\star, \Sigma^\star)$ as a subgraph and show any simple $m-$colouring of R when restricted to the edges in $P^\star$ is an $m-$colouring. As such, it must be P has an $m-$colouring. We do this by showing no colouring in $C(P^\star)$ can be extended to one of R.

Construct R from $P^\star$ as follows: (See Figure 5.8)

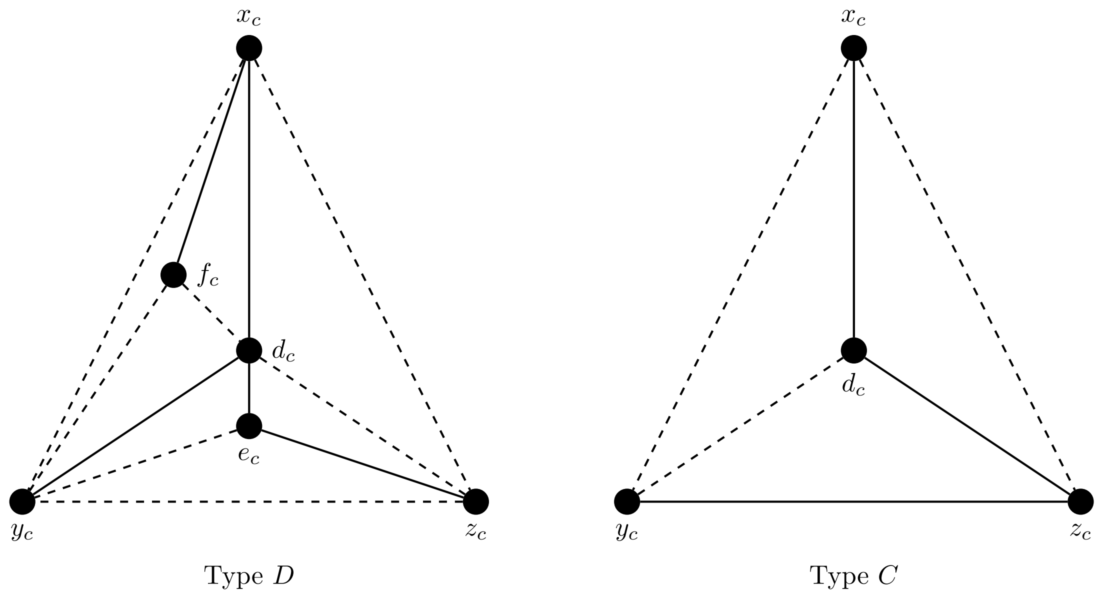


Figure 5.8: Construction for Types D and C

- For each F_c of Type A add vertices d_c, e_c, f_c together with blue edges x_cf_c, x_cd_c, y_cd_c, d_ce_c,z_ce_c and red edges y_cf_c, y_ce_c,z_cd_c,d_cf_c.
- For each F_c of Type B add vertex d_c together with blue edges x_cd_c, z_cd_c and red edge y_cd_c.
- For each F_c of Type C add vertex d_c together with red edges x_cd_c, z_cd_c and blue edge y_cd_c.
- For each F_c of Type D add vertices d_c, e_c, f_c together with red edges x_cf_c, x_cd_c, y_cd_c, d_ce_c,z_ce_c and blue edges y_cf_c, y_ce_c,z_cd_c,d_cf_c.

Notice R is a planar 2−edge-coloured graph (See Figure 5.8), and so has a simple m-colouring. Let c_r be such a colouring. By construction, for a given triangular face F_c, and for any chosen pair from the set $\{x_c, y_c, z_c\}$, the unchosen vertex is part of the convex hull formed of the chosen pair. As such, in any simple $m-$colouring all 3 vertices either receive the same colour or receive distinct colours. Therefore c_r does not extend c for any $c \in C(P^\star)$. Therefore when restricted to $P^\star$, c_r is an $m-$colouring. As $P \in \mathcal{P}$ was chosen arbitrarily, this gives $\chi_2(\mathcal{P}) \leq m = \chi_2^s(\mathcal{P})$. □

For a fixed integer $m > 1$, let us consider the smallest 2−edge-coloured planar graph with simple chromatic number m. Call this 2−edge-coloured graph H and let c be a simple m-colouring of H. Let uv be an edge of H and consider the result of contracting this edge. If the convex hull of u and v contains only u and v, then contracting this edge yields no parallel edges with different colours. This smaller 2−edge-coloured graph must have a simple $(m-1)-$colouring. Such a colouring can be extended to one of H by giving both u and v the same colour. As such, it must be that the convex hull of u and v contains at least one other vertex. This idea can be extended to any connected subgraph of H. That is to say, if H is a smallest 2−edge-coloured graph with simple chromatic number

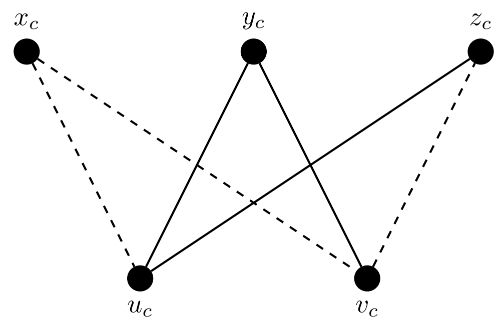


Figure 5.9: The edges added to form T' in Theorem 5.14.

m, it must be that the convex hull of any connected set of vertices is the entire vertex set of H. If it were not, then we could identify the convex hull into a single vertex, and either obtain a smaller graph with simple chromatic number m, or colour H with fewer colours. This reasoning suggests a family of graphs to examine for trying to improve $\chi_2(\mathcal{P}) \leq 80$. We need only consider the graphs so that the convex hull of any connected subgraph is the entire graph.

We turn now to another family of optimally simply colourable graphs, partial $p-$trees.

Theorem 5.14. *For any $p \geq 3$, the family, $\mathcal{T}_p$, of $2-$edge-coloured partial $p-$trees is optimally simply colourable.*

Proof. Let $p \geq 3$. We have directly that $\chi_2^s(\mathcal{T}_p) \leq \chi_2(\mathcal{T}_p)$, as any homomorphism is also a simple homomorphism. To prove our claim it suffices to show $\chi_2(\mathcal{T}_p) \leq \chi_2^s(\mathcal{T}_p)$.

Since $\chi_s(\mathcal{T}_p)$ is bounded [1], it must also be $\chi_2^s(\mathcal{T}_p)$ is bounded. Let $\chi_2^s(\mathcal{T}_p) = m$ and consider $T \in \mathcal{T}_p$ such that $\chi_2^s(T) = m$. We may assume T is a $p-$tree. Let $\mathcal{C}$ be the set of simple $m-$colourings of T such that T has a monochromatic edge.

As T is a $p-$tree it is constructed with a sequence of cliques of order $p+1$ $V_1, V_2, \dots, V_\ell$. For every colouring $c \in \mathcal{C}$ there must exist some clique V_i such that V_i has a monochromatic edge, but contains vertices of two different colours. Let $x_c \in V_i$ and $y_c \in V_i$ be the ends of this monochromatic edge, and let $z_c \in V_i$ be coloured differently than x_c and y_c under c.

Consider the following partial $p-$tree, T', constructed from T. (See Figure 5.9)

$$V(T') = V(T) \cup \bigcup_{c \in \mathcal{C}} \{v_c, w_c\},$$

$$E(T') = E(T) \cup \{x_c v_c, x_c w_c, y_c v_c, y_c w_c, z_c v_c, z_c w_c\},$$

$$\Sigma_2^{T'} = \Sigma_2^T \cup \{x_c v_c, x_c w_c, z_c w_c\}.$$

Consider an $m-$colouring of T', c'. Since for all $c \in \mathcal{C}$, $z_c, v_c, w_c \in conv(\{x_c, y_c\})$, if x_c and y_c receive the same colour under c', then z_c, v_c and w_c must also receive this same colour. Therefore c' does not extend any colouring in $\mathcal{C}$. This implies that when restricted to the vertices of T c' is an $m-$colouring of T. Therefore $\chi_2(\mathcal{T}_p) \leq \chi_2^s(\mathcal{T}_p)$ and so $\chi_2(\mathcal{T}_p) = \chi_2^s(\mathcal{T}_p)$. ☐

The families of $2-$edge-coloured planar graphs and $2-$edge-coloured $p-$trees each have bounded chromatic number and so have bounded simple chromatic number. Here we examine a family in which both of these parameters are unbounded.

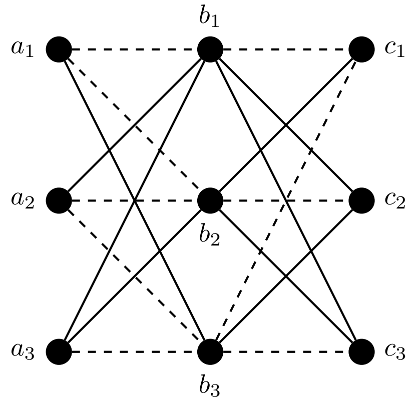


Figure 5.10: Construction of G_m for $m = 3$

Let $\mathcal{B}$ be the family of 2−edge-coloured bipartite graphs. We show, by way of construction, $\chi_k^s(\mathcal{B}) = \infty$.

Consider the complete bipartite graph, G_m ($m \geq 3$), with vertex set

$$V(G_m) = \{a_1, a_2, \ldots, a_m\} \cup \{b_1, b_2, \ldots, b_m\} \cup \{c_1, c_2, \ldots, c_m\}, m \geq 3$$

and edge set

$$E(G_m) = \{a_i b_j | 1 \leq i, j \leq m\} \cup \{c_i b_j | 1 \leq i, j \leq m\}.$$

We form a 2−edge-coloured graph (G_m, Σ) by partitioning $E(G_m)$ as follows:

- $a_i b_j \in \Sigma_1$ for all $i > j$ and for all $i < j$ where i and j have the same parity,
- $c_i b_j \in \Sigma_1$ for all $i > j$ and for all $i < j$ where i and j have the different parity, and
- all other edges are placed in Σ_2.

Proposition 5.15. $\chi_2^s(G_m) \geq m$

Proof. We proceed by induction on m, noting by inspection (see Figure 5.10) that for $m = 3$ in any simple colouring of G each of the vertices in $B = \{b_1, b_2, b_3\}$ must all receive distinct colours. We show for all $m > 3$ that the convex hull of any pair of vertices in B consists all of the vertices of G.

Case I: conv$(\{b_i, b_j\}) = V(G)$*, where* $1 < i < j < m$. By induction we have

$$\{a_1, a_2, \ldots, a_{m-1}\} \cup \{b_1, b_2, \ldots, b_{m-1}\} \cup \{c_1, c_2, \ldots, c_{m-1}\} \subseteq conv(\{b_i, b_j\}).$$

Since exactly one of $a_1 b_m$ and $a_2 b_m$ is red, $b_m \in conv(\{a_1, a_2\}) \subseteq conv(\{b_i, b_j\})$. Therefore $b_m \in conv(\{b_i, b_j\})$. Further, since $a_m b_m$ and $c_m b_m$ are blue and $a_m b_{m-2}$ and $c_m b_{m-1}$ are red, we have $\{a_m, c_m\} \subseteq conv(\{b_{m-1}, b_m\}) \subseteq conv(\{b_i, b_j\})$. Therefore $conv(\{b_i, b_j\}) = V(G_m)$.

Case II: conv$(\{b_i, b_m\}) = V(G)$*, where* $i \equiv m \pmod 2$. Since $a_m b_m$ is blue and $a_m b_i$ is red, $a_m \in conv(\{b_i, b_j\})$. Since $a_{m-1} b_m$ is blue and $a_{m-1} b_i$ is red we notice that since $i \equiv m \pmod 2$, it must be that for $i < m - 1$, $a_{m-1} \in conv(\{b_i, b_j\})$. Finally, since $a_m b_{m-2}$ is red and $a_{m-1}b_{m-1}$ is blue, $b_{m-1} \in conv(\{b_i, b_m\})$. By Case I, it must be $V(G) = conv(\{b_i, b_m\})$.

Case III $conv(\{b_i, b_n\}) = V(G)$, *where* $i \not\equiv m \pmod 2$. Proceed as in Case II swapping vertices in C for vertices in A and red for blue, where required.

Therefore, for every pair $\{b_i, b_j\}$ $(1 \leq i \neq j \leq m)$ the convex hull of $\{b_i, b_j\}$ is all vertices in G. As such, in any simple $2-$edge-coloured colouring of G the vertices in B must receive pairwise distinct colours. This gives directly that $\chi_2^s(G) \geq m$. □

Corollary 5.16. $\chi_2^s(\mathcal{B}) = \infty$.

Corollary 5.17. $\chi_2(\mathcal{B}) = \infty$.

Finally, we consider a family of $2-$edge-coloured graphs that is not optimally simply colourable, connected cubic graphs.

Theorem 5.18. *Every connected* $2-$*edge-coloured cubic graph requires at most* 4 *colours in any simple colouring.*

Proof. Let (G, Σ) be a $2-$edge-coloured cubic graph that has simple chromatic number at least 5. We may also assume G has at least 5 vertices. Since G does not have a simple $2-$colouring, for any subset of the vertices $N \subset V(G)$, the edge cut formed from those edges with exactly one end in N contains both red and blue edges. Consider now the subgraph induced by the set of red (respectively blue) edges. Call this graph G_1 (respectively G_2). Each of these graphs must be connected, as otherwise there is an edge cut consisting solely of either red or blue edges, implying G has a simple $2-$colouring.

If G_1 contains a cycle it must be a Hamilton cycle in G, as otherwise G would contain a vertex that has all of its incident edges red since G is cubic. However, if this is the case, then G_2 is a matching and so is not connected. Therefore each of G_1 and G_2 is connected, acyclic, spanning, and has maximum degree 2; they are Hamilton paths. Since G is cubic, a vertex of degree 2 in G_1 is a vertex of degree 1 in G_2, and vice versa. Since G_2 has 2 vertices of degree 1, G_1 has two vertices of degree 2. However, since G_1 has no vertex of degree 3, and has at least 5 vertices, it cannot have exactly two vertices of degree 2 and two vertices of degree 1. This contradicts that G has at least 5 vertices. □

Corollary 5.19. *The family of* $2-$*edge-coloured cubic graphs is not optimally simply colourable.*

The proof of the previous theorem allows us to classify the simple chromatic number of $2-$edge-coloured connected cubic graphs.

Theorem 5.20. *If* (G, Σ) *is a connected* $2-$*edge-coloured graph, then* $\chi_2^s(G) = 2$ *unless* (G, Σ) *is the complete graph on* 4 *vertices in which* $G[\Sigma_1]$ *and* $G[\Sigma_2]$ *each are a path on* 4 *vertices. In this case* $\chi_2^s(G) = 4$.

Proof. Let (G, Σ) be a connected $2-$edge-coloured graph. As observed in Theorem 5.18, if either of $G_1 = G[\Sigma_1]$ or $G_2 = G[\Sigma_2]$ is not connected, then $\chi_2^s(G) = 2$. Assume now G_1 and G_2 are connected. Following the proof of Theorem 5.18, we see each of G_1 and G_2 are Hamilton paths and a vertex of degree 2 in G_1 is a vertex of degree at most 1 in G_2, and vice versa. Therefore G has at most 4 vertices. Since G is cubic it must be the complete graph on 4 vertices. Therefore (G, Σ) is the complete graph on 4 vertices in which $G[\Sigma_1]$ and $G[\Sigma_2]$ each are a path on 4 vertices. □

5.4 Conclusions and Future Directions

Throughout our work on the chromatic number of 2−edge-coloured graphs with maximum degree 3, we have been careful to note that all of the graphs we are considering are connected. Unlike proper colourings of graphs, we cannot just consider the component which requires the most colours. For an easy example, consider the disjoint union of two copies of K_3, one with all red edges and the other with all blue edges. We see no obvious way to adapt the method of proof of Theorem 5.9 to work with non-connected graphs, as the proof may construct different targets for each of the components. As such, the question of the chromatic number of the entire family of 2−edge-coloured cubic graphs remains open. Obviously, this number is at least 8, but it is unknown if it is equal to the chromatic number for the family of connected 2−edge-coloured cubic graphs. It is also an open question whether the bound of 11 for the family of connected cubic graphs can be improved. If this number can be improved, it may be there exists a 2−edge-coloured graph on fewer than 11 vertices to which each 2−edge-coloured cubic graph admits a homomorphism. As mentioned previously, when considering families of 2−edge-coloured graphs with bounded maximum degree, the technique applied in Theorem 5.9 can be useful in constructing colourings. For example for the family of connected 2−edge-coloured graphs with maximum degree at most 4, a target graph with Property $P_{3,1}$ may be of use. The Tromp construction (see [37]) applied the target graph used in Theorem 5.2 yields a 2−edge-coloured graph with Property $P_{3,1}$. It is possible this graph may be a universal target for the family of connected 2−edge-coloured graphs that have maximum degree 4, but are not 4−regular.

In our study of simple colourings of 2−edge-coloured graphs, the families we considered are the same families considered by Smolíková [49] in her Ph.D. thesis. In [2] Alon and Marshall note, when referring to the similarity of their results on k−edge-coloured graphs to those of Raspaud and Sopena on oriented graphs, though similar methods were utilised they see no way to deduce their set of results for k−edge-coloured graphs from the results on oriented graphs and vice versa. Here we make the same observation for the results of Smolíková [49]. In [42], Nešetřil and Raspaud unify the results of Alon and Marshall, and Raspaud and Sopena, by finding a similar result for (j, k)−mixed graphs. It is possible a similar generalisation exists when considering simple colourings of (j, k)−mixed graphs.

Chapter 6
Incidence Colourings and Oriented Incidence Colourings

Incidence colouring arose in 1993 when Brualdi and Massey first defined the incidence chromatic number of a graph (then called the incidence colouring number) [9]. In this paper they gave upper and lower bounds for the incidence chromatic number based on maximum degree. These authors used their results as a method to improve a bound for the strong chromatic index of bipartite graphs. Since then, bounds for the incidence chromatic number have been investigated for a variety of families of graphs, including planar graphs, $k-$trees, $k-$regular graphs, toroidal grids and $k-$degenerate graphs ([15], [55], [54], [57]). In this chapter we find a new characterisation of the incidence chromatic number using systems of distinct representatives and also introduce a directed version of this parameter.

6.1 Introduction and Preliminaries

Definition 6.1. *Let G be a simple undirected graph. For every $u \in V(G)$ and $e = uv \in E(G)$ we call the pair $(u, e) = (u, uv)$ an* incidence*. Let $\mathcal{I}_G$ denote the set of incidences of G. A pair of distinct incidences (v, e) and (w, f) are* adjacent *if $v = w$, $e = f$, $vw = e$, or $vw = f$. (See Figure 6.1).*

Definition 6.2. *An* incidence colouring of G using k colours *is a mapping, $c : \mathcal{I}_G \to \{1, 2, 3, \ldots, k\}$ such that for adjacent incidences $(v, e), (w, f) \in \mathcal{I}_\mathcal{G}$, $c((v, e)) \neq c((w, f))$. The* incidence chromatic number of G*, denoted $\chi_i(G)$, is the least integer k such that G has an incidence colouring using k colours. For a family of graphs, $\mathcal{F}$, the* incidence chromatic number of $\mathcal{F}$*, denoted $\chi_i(\mathcal{F})$ is the least k such that for all $F \in \mathcal{F}$, $\chi_i(F) \leq k$.*

In describing explicit incidence colourings we will drop the extra pair of parentheses. That is, we denote $c((v, e))$ as $c(v, e)$.

In their introduction to incidence colouring, Brualdi and Massey make the following contributions.

Proposition 6.1 (Brualdi and Massey [9]). *Let G be a simple graph.*

- $\chi_i(G) \leq |V(G)|$.
- *If G is a complete graph, then $\chi_i(G) = |V(G)|$.*
- *If G is a tree, then $\chi_i(G) \leq \Delta(G) + 1$.*
- *If G is a path, then $\chi_i(G) \leq 3$.*
- *If G is a cycle, then $\chi_i(G) \leq 4$.*

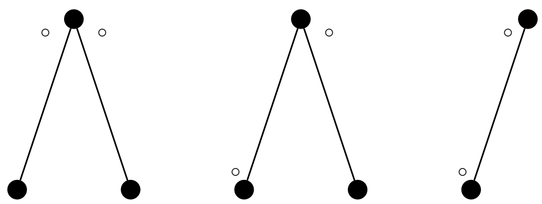


Figure 6.1: Incidences defined to be adjacent.

Based on their initial observations, Brualdi and Massey conjectured that for any graph, G, $\chi_i(G) \leq \Delta(G) + 2$. This was shown to be false by Guiduli [24] by considering the family of Paley graphs. Guiduli gives the following upper bound.

Proposition 6.2 (Guiduli [24])**.** *If G is graph, then $\chi_i(G) \leq \Delta(G) + 20 \log \Delta(G) + 84$.*

As with other colouring parameters, homomorphism is useful in establishing upper bounds. For incidence colourings, however, we require injective homomorphisms.

Definition 6.3. *Let G and H be graphs. We say G* admits an injective homomorphism *to H if there exists a homomorphism $\phi : G \to H$ such that for all $u \in V(G)$ and every pair of edges $ux, uy \in E(G)$, $\phi(x) \neq \phi(y)$.*

Theorem 6.3. *If G and H are simple graphs such that G admits an injective homomorphism to H, then $\chi_i(G) \leq \chi_i(H)$.*

Proof. Let G and H be simple graphs such that $\phi : G \to H$ is an injective homomorphism. Let c be an incidence colouring of H using k colours. Consider the mapping $c' : \mathcal{I}_G \to \{1, 2, 3, \ldots, k\}$ given by $c'(u, uv) = c(\phi(u), \phi(u)\phi(v))$. Consider a pair of edges $uv, vw \in E(G)$. Since ϕ is injective, we note $\phi(u) \neq \phi(w)$. If c' is not an incidence colouring, then one of the following must be true.

- $c'(u, uv) = c'(v, uv)$: If this is true, then $c(\phi(u), \phi(u)\phi(v)) = c(\phi(v), \phi(u)\phi(v))$. However this would contradict that c is an incidence colouring of H.

- $c'(u, uv) = c'(v, vw)$: If this is true, then $c(\phi(u), \phi(u)\phi(v)) = c(\phi(v), \phi(v)\phi(w))$. However this would contradict that c is an incidence colouring of H, as since $\phi(u) \neq \phi(w)$ the incidences $(\phi(u), \phi(u)\phi(v))$ and $(\phi(v), \phi(v)\phi(w))$ are adjacent in H.

- $c'(v, uv) = c'(w, vw)$: If this is true, then $c(\phi(v), \phi(u)\phi(v)) = c(\phi(w), \phi(v)\phi(w))$. However this would contradict that c is an incidence colouring of H, as since $\phi(u) \neq \phi(w)$ the incidences $(\phi(v), \phi(u)\phi(v))$ and $(\phi(w), \phi(v)\phi(w))$ are adjacent in H.

Therefore c' is an incidence colouring of G using no more than $\chi_i(H)$ colours. $\square$

6.2 Incidence Chromatic Number as a System of Sets

Let G be simple graph and c be an incidence colouring of G using k colours. For each vertex u, let

$$A_u = \{c(u, e) | (u, e) \in \mathcal{I}_G\}.$$

If c is a surjection, then we observe

$$\bigcup_{u \in V(G)} A_u = \{1, 2, 3, \ldots, k\}.$$

Observe the following properties of these sets.

Property 6.4. *For all $e = uv$, $A_u \setminus A_v \neq \emptyset$.*

Consider the incidence (u, uv). The colour appearing at this incidence must be an element of A_u, and since c is an incidence colouring, this colour cannot appear as an element of A_v.

Property 6.5. *For all $u \in V(G)$ the collection of sets $\{A_u \setminus A_v | v \in N(u)\}$ has a system of distinct representatives.*

For each $A_u \setminus A_v$ we select $c(u, uv)$ as the representative element.

Though these sets and their properties arise from a particular incidence colouring of G, we can use such a system of sets to define incidence colouring.

Theorem 6.6. *If $A = \{A_u | \; u \in V(G)\}$ is a collection of sets such that*

- *$|A_u| = d(u)$, and*
- *for all $u \in V(G)$ the collection of sets $B_u = \{A_u \setminus A_v | \; v \in N(u)\}$ has a system of distinct representatives,*

then G has an incidence colouring using exactly $\left|\bigcup_{u \in V(G)} A_u\right|$ colours.

Proof. Let G be a graph and let $A = \{A_u | \; u \in V(G)\}$ be a collection of sets that satisfy the hypotheses. For every collection of sets $B_u = \{A_u \setminus A_v | \; v \in N(u)\}$, let b_{uv} be the representative of the set $A_u \setminus A_v$ in the system of distinct representatives of B_u. We claim a colouring, c, that assigns the colour b_{uv} to the incidence (u, uv) is an incidence colouring of G.

Consider the pair of edges $uv, vw \in E(G)$. If c is not an incidence colouring then one of the following must be true.

- $b_{uv} = b_{vu}$: Since $b_{uv} \in A_u \setminus A_v$ and $b_{vu} \in A_v$, it must be $b_{uv} \neq b_{vu}$;
- $b_{uv} = b_{vw}$: Since $b_{uv} \in A_u \setminus A_v$ and $b_{vw} \in A_v$, it must be $b_{uv} \neq b_{vw}$;
- $b_{vu} = b_{vw}$: Since b_{vu} and b_{vw} are each representatives in a system of distinct representatives of the collection of sets B_v it must be $b_{vw} \neq b_{vu}$.

Therefore c is an incidence colouring of G. Further since for every vertex u, $|A_u| = d(u)$, it must be every element of A_u appears as a colour on some incidence (u, e). Therefore c uses exactly $\left|\bigcup_{u \in V(G)} A_u\right|$ colours. □

Using this theorem we find an alternate characterisation for the incidence chromatic number of a graph G.

Definition 6.4. *Let G be a graph. The* incidence chromatic number *of G, denoted $\chi_i(G)$, is the cardinality of the smallest set U such that there exist subsets $A_u \subset U$ for all $u \in V(G)$ so the following properties hold.*

- *For all $u \in V(G), |A_u| = d(u)$, and*
- *for all $u \in V(G)$, the collection of sets $\{A_u \setminus A_v\}$ has a system of distinct representatives.*

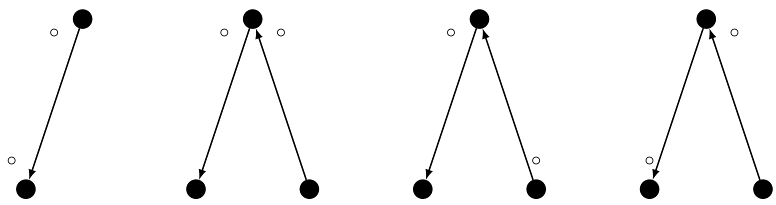


Figure 6.2: Oriented incidences defined to be adjacent.

6.3 Oriented Incidence Colouring

We now consider adapting the spirit of incidence colouring to directed graphs.

Definition 6.5. *For a digraph, G, we define* incidences *of two types:*

- *an ordered pair (u, uv), where $uv \in E(G)$; and*
- *an ordered pair (xy, y), where $xy \in E(G)$.*

Let $\mathcal{I}_G$ denote the set of incidences of G, a digraph. Consider the incidences $(u, uv), (uv, v), (x, xy), (x$
$\mathcal{I}_G$. *We define* adjacency *as follows (see Figure 6.2).*

- *If $v = x$, then*
  - *(uv, v) is adjacent to (x, xy),*
  - *(u, uv) is adjacent to (x, xy), and*
  - *(uv, v) is adjacent to (xy, y).*
- *If $uv = xy$, then (u, uv) is adjacent to (xy, y).*

Definition 6.6. *An* oriented incidence colouring *of G assigns to each incidence of G a colour such that adjacent incidences receive different colours. That is, an oriented incidence colouring of G with k colours is a function $c\colon \mathcal{I}_G \to \{1, 2, \dots, k\}$ such that if $\alpha, \beta \in \mathcal{I}_G$ are adjacent incidences, then $c(\alpha) \neq c(\beta)$.*

As with incidence colouring, in describing explicit oriented incidence colourings we will drop the extra pair of parentheses. That is, we denote $c((x, xy))$ as $c(x, xy)$.

Definition 6.7. *For a digraph G we define the* oriented incidence chromatic number *to be the least k such that G has an oriented incidence colouring using k colours. We denote this value as $\overrightarrow{\chi_i}(G)$. If $\mathcal{F}$ is a family of digraphs we define $\overrightarrow{\chi_i}(\mathcal{F})$ to be the least k such that for all $F \in \mathcal{F}$, $\overrightarrow{\chi_i}(F) \leq k$.*

Figures 6.3, 6.4, 6.5, and6.6 give examples of oriented incidence colourings of some digraphs with few vertices. Notice that in Figure 6.3, we see a pair of incidences at the same vertex receiving the same colour.

In this section our main goal is to study the relationship between oriented incidence colouring and digraph homomorphism. Using this relationship we find a connection between the oriented incidence chromatic number of a digraph and the chromatic number

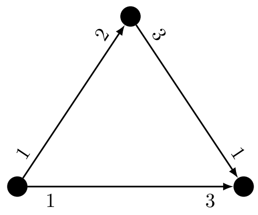


Figure 6.3: An oriented incidence 3−colouring of the transitive triple.

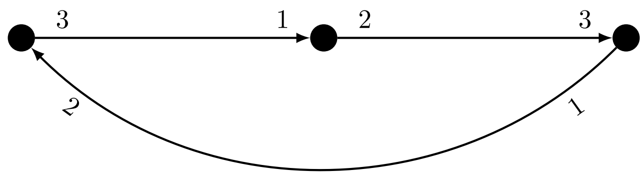


Figure 6.4: An oriented incidence 3−colouring of the directed cycle on 3 vertices

of its underlying simple graph. Subsequently, we find upper and lower bounds for the oriented chromatic number of complete symmetric digraphs.

The study of 2−dipath colourings of oriented graphs in the thesis of Sherk [58] contains a result that provides an upper bound on the oriented chromatic number as a function of the 2−dipath chromatic number (see Chapter 3). We consider the possibility of a result relating the oriented chromatic number and the oriented incidence chromatic number. This idea is explored in Section 6.3.5.

We begin by finding the oriented incidence chromatic number of the family of orientations of stars. By Figure 6.5 we see at least 3 colours are required to colour every oriented star. We show that 3 colours always suffice.

Proposition 6.7. *If G is an orientation of a star, then* $\overrightarrow{\chi_i}(G) \leq 3$.

Proof. Let S_k be an oriented star on $k+1$ vertices. Let u be the centre vertex of S_k, A be the set of out-neighbours of u and B be the set of in-neighbours of u. Consider a function, $c : \mathcal{I}_{S_k} \to \{1, 2, 3\}$ defined as follows. For all $a \in A$ and all $b \in B$ let

- $c(u, ua) = 3$,
- $c(ua, a) = 1$,

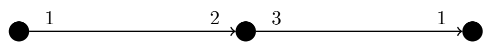


Figure 6.5: An oriented incidence 3−colouring of the directed path on 3 vertices

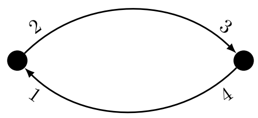


Figure 6.6: An oriented incidence 4−colouring of the 2−cycle

- $c(bu, u) = 2$, and
- $c(b, bu) = 1$.

It is easy to observe that c is an oriented incidence colouring of S_k.

□

We begin our study of the oriented incidence chromatic number by relating the oriented incidence chromatic number of an oriented graph to the incidence chromatic number of the underlying simple graph. To do so, we observe that the set of incidences of an oriented graph is exactly equal to the set of incidences of the underlying graph, as defined in Definition 6.5, and that any incidences adjacent in the oriented sense are also adjacent in the undirected sense. From this it follows directly that:

Proposition 6.8. *If G is an oriented graph, then $\chi_i(U(G)) \geq \overrightarrow{\chi_i}(G)$.*

By Theorem 6.1, we see that if T is a tournament on k vertices, then $\overrightarrow{\chi_i}(T) \leq k$. We improve this bound in Section 6.3.4 by observing tournaments are subgraphs of symmetric complete graphs.

6.3.1 A Homomorphism Model for Oriented Incidence Colouring

Consider a homomorphism that maps an orientation of a star to P_2. We can obtain the oriented incidence colouring of S_k exhibited in Proposition 6.7 by lifting back the oriented incidence colouring of P_2 given in Figure 6.5 to the incidences of S_k. This idea leads us to the following general result relating oriented incidence colouring and digraph homomorphism.

Theorem 6.9. *If G and H are digraphs such that $G \to H$, then $\overrightarrow{\chi_i}(G) \leq \overrightarrow{\chi_i}(H)$.*

Proof. Let G and H be digraphs, let f be an oriented incidence colouring of H using $\overrightarrow{\chi_i}(H)$ colours, and let ϕ be a homomorphism from G to H. Construct c, an oriented incidence colouring of G, as follows. For all $uv \in E(G)$:

- let $c(u, uv) = f\left(\phi(u), \phi(u)\phi(v)\right)$ and,
- let $c(uv, v) = f\left(\phi(u)\phi(v), \phi(v)\right)$.

If c is not an oriented incidence colouring of G, then one of the following must occur:
Case I: There exist $x, y \in V(G)$ and $xy \in E(G)$ such that $c(x, xy) = c(xy, y)$. However this would imply $f(\phi(x), \phi(x)\phi(y)) = f(\phi(x)\phi(y), \phi(y))$, a contradiction as f is an oriented incidence colouring of H.

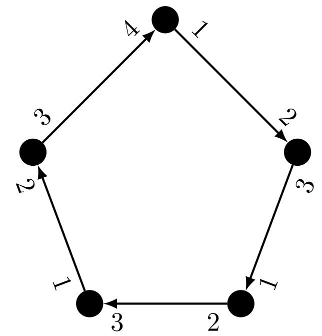


Figure 6.7: An oriented incidence 4−colouring of the directed cycle on 5 vertices

Case II: There exist $x, y, z \in V(G)$ and $xy, yz \in E(G)$ such that $c(y, yz) = c(xy, y)$. However this would imply $f(\phi(y), \phi(y)\phi(z)) = f(\phi(x)\phi(y), \phi(y))$, a contradiction.

Case III: There exist $x, y, z \in V(G)$ and $xy, yz \in E(G)$ such that $c(x, xy) = c(y, yz)$. However this would imply $f(\phi(x), \phi(x)\phi(y)) = f(\phi(y), \phi(y)\phi(z))$, a contradiction.

Case IV: There exist $w, x, y \in V(G)$ and $xy, wx \in E(G)$ such that $c(xy, y) = c(wx, x)$. However this would imply $f(\phi(x)\phi(y), \phi(y)) = f((\phi(w)\phi(x), \phi(x))$, a contradiction.

Therefore c is an oriented incidence colouring of G using at most $\overrightarrow{\chi_i}(H)$ colours. □

Corollary 6.10. *If G is an oriented graph, then $\overrightarrow{\chi_i}(G) \leq \chi_o(G)$.*

Proof. If G is an oriented graph such that $\chi_o(G) = m$, then there exists T, a tournament on m vertices such that $G \to T$. By Proposition 6.8 and Theorem 6.1 $\overrightarrow{\chi_i}(T) \leq m$. And so by Theorem 6.9, $\overrightarrow{\chi_i}(G) \leq m$. □

Corollary 6.11. *Let G be an oriented graph.*

- *If $U(G)$ is a path, then $\overrightarrow{\chi_i}(G) \leq 3$.*
- *If $U(G)$ is a tree, then $\overrightarrow{\chi_i}(G) \leq 3$.*
- *If $U(G)$ is a cycle, then $\overrightarrow{\chi_i}(G) \leq 4$.*
- *If $U(G)$ is a complete graph, then $\overrightarrow{\chi_i}(G) \leq |V(G)|$.*

All of these results follow directly from bounds for the oriented chromatic number for these families of oriented graphs. The only case that requires further comment is the case where G is a directed 5−cycle. By inspection we can see only 4 colours are required for an oriented incidence colouring (see Figure 6.7), even though 5 colours are required for an oriented colouring.

Corollary 6.12. *If $\mathcal{F}$ is a family of oriented graphs with bounded oriented chromatic number, then $\mathcal{F}$ also has bounded oriented incidence chromatic number.*

Theorem 6.9 provides a direct link between the oriented incidence chromatic number of an oriented graph and the oriented chromatic number of the same oriented graph. However, by considering the family of bipartite graphs, we are led to a relationship between the oriented incidence chromatic number of an oriented graph and the chromatic number of its underlying simple graph.

n	1	2	3	4	5	6	7
$\overrightarrow{\chi_i}(\overrightarrow{K_n})$	0	4	4	5	5	6	6

Table 6.1: The oriented incidence chromatic numbers of $\overrightarrow{K_n}$ for $1 \leq n \leq 7$.

Proposition 6.13. *If B is an oriented bipartite graph, then $\overrightarrow{\chi_i}(B) \leq 4$.*

Proof. Let $B = [X, Y]$ be an oriented bipartite graph. Partition the arcs into two sets, those that have their head in X and those that have their head in Y. Denote by xy an arc that has its head in Y and by $y'x'$ $(x, y, x', y' \in V(B))$ an arc that has its head in X. Consider a function $c : \mathcal{I}_B \to \{1, 2, 3, 4\}$ such that

- $c(x, xy) = 1$,
- $c(xy, y) = 2$,
- $c(y'x', x') = 3$, and
- $c(y', y'x') = 4$.

It is easily observed that c is an oriented incidence colouring of B. □

The technique applied here suggests a method for constructing oriented incidence colouring using a proper vertex colouring of the underlying graph. Observe that if $\chi(U(G)) \leq k$, then G admits a homomorphism to the digraph formed from K_k by replacing each edge with a pair of oppositely oriented arcs. We call this graph the symmetric tournament on k vertices and denote it by $\overleftrightarrow{K}_k$. In Proposition 6.13 we are noticing every orientation of a bipartite graph admits a homomorphism to $\overleftrightarrow{K}_2$

Theorem 6.14. *If G is digraph, then $\overrightarrow{\chi_i}(G) \leq \overrightarrow{\chi_i}(\overleftrightarrow{K}_{\chi(U(G))})$.*

Proof. Let G be a digraph and assume $\chi(U(G)) \leq k$. Since G admits a homomorphism to $\overleftrightarrow{K}_k$, by Theorem 6.9 it must be that $\overrightarrow{\chi_i}(G) \leq \overrightarrow{\chi_i}(\overrightarrow{K_k})$. □

Given that homomorphism to the symmetric complete graph is useful in finding an upper bound for the oriented incidence chromatic number, we consider the problem of finding the oriented incidence chromatic number of a symmetric complete graph.

Table 6.1 gives the oriented incidence chromatic number of $\overrightarrow{K_k}$, for $0 \leq k \leq 7$. These values were found by computer search. Figures 6.8 and 6.9 give oriented incidence colourings of $\overrightarrow{K_3}$ and $\overrightarrow{K_6}$, respectively, using the fewest possible number of colours. Though this table tempts us into making a conjecture about the oriented incidence chromatic number of a symmetric complete digraphs, we resist this temptation, as later we show this conjecture would be false. After developing some further tools in Section 6.3.2 and Section 6.3.3, we return to the question of the oriented incidence chromatic number of symmetric complete digraphs.

6.3.2 Constructions and Decompositions

Here we examine oriented incidence colourings of digraph decompositions and products. We begin with an upper bound for digraphs that can be realised as the union of digraphs.

Proposition 6.15. *If G is a digraph such that $G = G_1 \cup G_2$ where $V(G_1) \cap V(G_2) = \emptyset$, then*

$$\overrightarrow{\chi_i}(G) = \max\{\overrightarrow{\chi_i}(G_1), \overrightarrow{\chi_i}(G_2)\}.$$

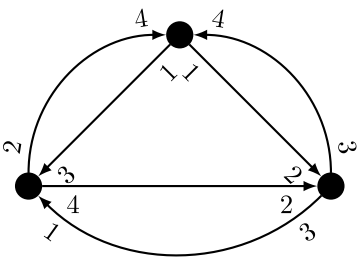


Figure 6.8: An oriented incidence $4-$colouring of the symmetric complete graph on 3 vertices, $\overrightarrow{K_3}$.

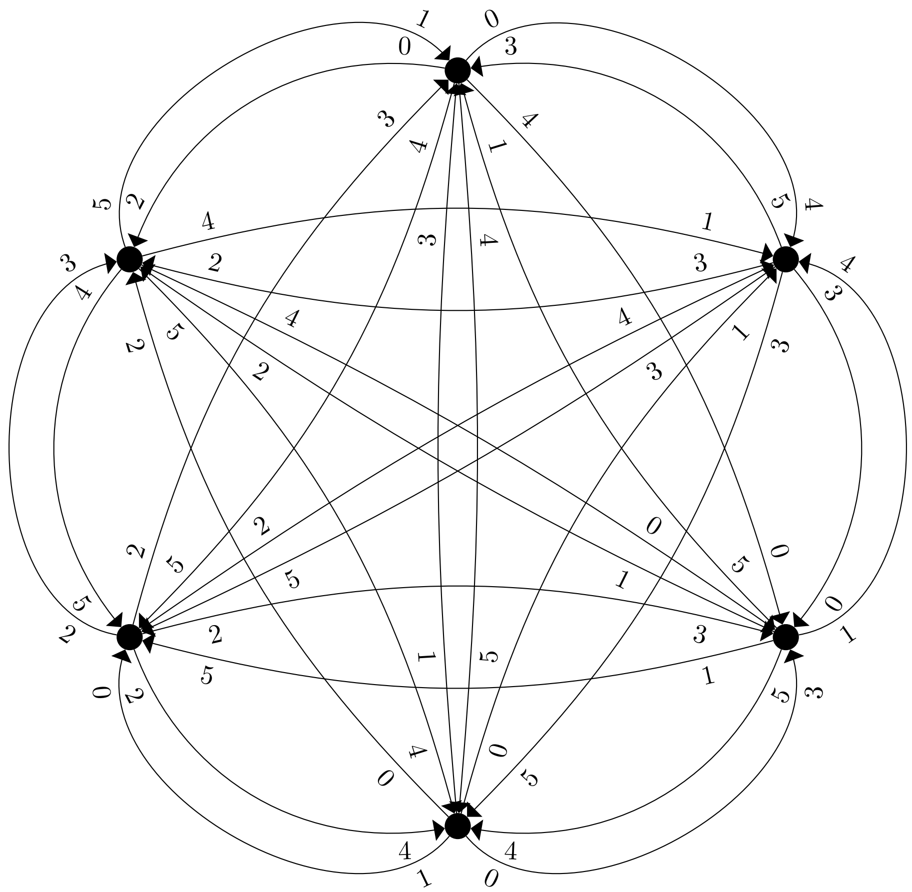


Figure 6.9: An oriented incidence $6-$colouring of the symmetric complete graph on 6 vertices, $\overrightarrow{K_6}$.

Proposition 6.16. *If G is a digraph such that $G = G_1 \cup G_2$ where $V(G_1) \subseteq V(G_2)$ and $E(G_1) \cap E(G_2) = \emptyset$, then*
$$\overrightarrow{\chi_i}(G) \leq \overrightarrow{\chi_i}(G_1) + \overrightarrow{\chi_i}(G_2).$$

Proof. Let
$$\phi_1 : V(G_1) \to \{1, 2, 3, \ldots, \overrightarrow{\chi_i}(G_1)\}$$
and let
$$\phi_2 : V(G_2) \to \{-1, -2, -3, \ldots, -\overrightarrow{\chi_i}(G_2)\}$$
be oriented incidence colourings of G_1 and G_2, respectively. Define ϕ as follows.

- $\phi(u, uv) = \phi_\ell(u, uv)$, and $\phi(uv, v) = \phi_\ell(uv, v)$ for all $uv \in E(G_\ell)$ ($\ell \in \{1, 2\}$).

We show by contradiction that ϕ is an oriented incidence colouring of G. If ϕ is not an oriented incidence colouring of G, then one of the following must be true.

Case I: There exist $x, y \in V(G)$ and $xy \in E_\ell(G)$ ($\ell \in \{1, 2\}$) such that $c(x, xy) = c(xy, y)$. However this would imply $\phi_\ell(x, xy) = \phi_\ell(xy, y)$. This contradicts that ϕ_ℓ is an oriented incidence colouring of G_ℓ.

The remainder of the cases follow similarly to the proof of Theorem 6.9. □

Recall the *arboricity* of a graph, G, is the smallest number of forests needed to cover $E(G)$. The *in-star arboricity (respectively, out-star)* of a digraph, G, is the smallest number of in-stars (respectively, out-stars) needed to cover $E(G)$.

Corollary 6.17. *If G is a directed graph, then $\overrightarrow{\chi_i}(G) \leq 3 \cdot arb(U(G))$, where $arb(U(G))$ denotes the arboricity of $U(G)$.*

Proof. Consider a decomposition of $U(G)$ into forests. When oriented, each of these forests requires at most 3 colours, regardless of the orientation of G. Colouring each forest with a unique set of 3 colours yields an oriented incidence colouring using at most $3 \cdot arb(U(G))$ colours. □

Corollary 6.18. *If G is a digraph, then $\overrightarrow{\chi_i}(G) \leq 2 \cdot \min\{arb_{in}(G), arb_{out}(G)\}$, where $arb_{in}(H)$ and $arb_{out}(H)$ denote the in-star and out-star arboricity of G, respectively.*

We consider now a graph operation that arises in the study of oriented colourings and oriented cliques (for an example see [47]). Let G and H be digraphs on disjoint vertex sets. We define the digraph $G \star H$ as follows.

- $V(G \star H) = V(G) \cup V(H) \cup \{z\}$, and
- $E(G \star H) = E(G) \cup E(H) \cup \{uz | u \in V(G)\} \cup \{zv | v \in V(H)\}$.

Theorem 6.19. *Let G and H be digraphs and let $k = \max\{\overrightarrow{\chi_i}(G), \overrightarrow{\chi_i}(H)\}$. Then $k \leq \overrightarrow{\chi_i}(G \star H) \leq k + 2$.*

Proof. Let c_G be an oriented incidence colouring of G using the colours $\{1, 2, 3, \ldots, k\}$ and let c_H be an oriented incidence colouring of H using the colours
$$\{3, 4, \ldots, k + 1, k + 2\}.$$

Construct an oriented incidence colouring, c, of $G \star H$, using the colours $\{1, 2, \ldots, k + 2\}$, as follows.

- $c(u, uv) = c_G(u, uv)$ and $c(uv, v) = c_G(uv, v)$, for all $uv \in E(G)$,

- $c(u, uv) = c_H(u, uv)$ and $c(uv, v) = c_H(uv, v)$, for all $uv \in E(H)$,
- $c(u, uz) = k + 1$ and $c(uz, z) = k + 2$, for all $u \in V(G)$, and
- $c(z, zv) = 1$ and $c(zv, v) = 2$, for all $v \in V(H)$.

□

The upper bound in Theorem 6.19 is not always achieved with equality. The oriented graph in Figure 6.11 is $P_2 \star P_2$. The directed path on 3 vertices can be coloured using 3 colours, but $P_2 \star P_2$ requires only 4 colours, not the 5 given by the upper bound in Theorem 6.19.

Finally we consider the oriented incidence chromatic number of the join of digraphs. Let G and H be digraphs. The join of G and H, denoted $G + H$, is the digraph with

- $V(G + H) = V(G) \cup V(H)$, and
- $E(G + H) = E(G) \cup E(H) \cup \{u_G v_H | u_G \in V(G), v_H \in V(H)\} \cup \{u_H v_G | u_H \in V(H), v_G \in V(G)\}$.

Informally, the join of digraphs is the disjoint union of the digraphs together with all possible arcs between vertices of different digraphs. We give a pair of bounds for the oriented incidence chromatic number of the join of a pair of digraphs.

Theorem 6.20. *If G and H are digraphs, then*

$$\vec{\chi_i}(G + H) \leq \max\{\vec{\chi_i}(G), \vec{\chi_i}(H)\} + 4.$$

This follows directly from Theorem 6.14, and Propositions 6.13, 6.15 and 6.16.

6.3.3 Oriented Incidence Colourings as a System of Sets

Let c be an oriented incidence colouring of a digraph G. For a vertex u, let

$$A_u = \bigcup_{uv \in E(G)} c(u, uv)$$

and let

$$B_u = \bigcup_{vu \in E(G)} c(vu, u).$$

Informally A_u is the set of colours assigned to incidences of the type (u, uv) and B_u is the set of colours assigned to incidences of the type (vu, u).

Property 6.21. *For all vertices v, $A_v \cap B_v = \emptyset$.*

No colour can appear on an incidence of the form (uv, v) and one of the form (v, vw).

Property 6.22. *For all vertices v that have an out-neighbour, A_v is non-empty.*

Property 6.23. *For all vertices v that have an in-neighbour, B_v is non-empty.*

Property 6.24. *For all arcs uv, $A_u \setminus A_v \neq \emptyset$ and $B_v \setminus B_u \neq \emptyset$.*

For every arc uv it must be $c(u, uv) \in A_u \setminus A_v$.

Property 6.25. *For all arcs uv, if $A_u \setminus A_v = B_v \setminus B_u$, then $|A_u \setminus A_v| \neq 1$.*

If $A_u \setminus A_v = B_v \setminus B_u$ and $|A_u \setminus A_v| = 1$, then it would imply $c(u, uv) = c(uv, v)$.

As with our new characterisation for incidence colouring using systems of distinct representatives, existence of sets satisfying these properties is enough to construct an oriented incidence colouring.

Theorem 6.26. *Let G be a digraph with n vertices. The* oriented incidence chromatic number of G *is the least k such that there exist sets*

$$A_{u_1}, A_{u_2}, \ldots, A_{u_n} \subseteq \{1, 2, 3, \ldots, k\}$$

and sets

$$B_{u_1}, B_{u_2}, \ldots, B_{u_n} \subseteq \{1, 2, 3, \ldots, k\}$$

such that the following hold.

1. *For all vertices v, $A_v \cap B_v = \emptyset$.*
2. *For all vertices v that have an out-neighbour, A_v is non-empty.*
3. *For all vertices v that have an in-neighbour, B_v is non-empty.*
4. *For all arcs uv, $A_u \setminus A_v \neq \emptyset$ and $B_v \setminus B_u \neq \emptyset$.*
5. *For all arcs uv, if $A_u \setminus A_v = B_v \setminus B_u$, then $|A_u \setminus A_v| \neq 1$.*

Proof. Assume there exist sets

$$A_{u_1}, A_{u_2}, \ldots, A_{u_n} \subseteq \{1, 2, 3, \ldots, k\}$$

and sets

$$B_{u_1}, B_{u_2}, \ldots, B_{u_n} \subseteq \{1, 2, 3, \ldots, k\}$$

that satisfy the hypotheses. Construct an oriented incidence colouring c by assigning to each incidence (u, uv) a colour from the set $A_u \setminus A_v$ and to each incidence (uv, v) a colour from the set $B_v \setminus B_u$ such that $c(u, uv) \neq c(uv, v)$. □

Corollary 6.27. *If c is an oriented incidence colouring of $\overrightarrow{K}_n$, then the collection of sets $A_{u_1}, A_{u_2}, \ldots, A_{u_n}$ form an antichain in the Boolean lattice of subsets of $\{1, 2, 3, \ldots, \overrightarrow{\chi_i}(\overrightarrow{K}_n)\}$.*

Corollary 6.28. *If c is an oriented incidence colouring of $\overrightarrow{K}_n$, then the collection of sets $B_{u_1}, B_{u_2}, \ldots, B_{u_n}$ form an antichain in the Boolean lattice of subsets of $\{1, 2, 3, \ldots, \overrightarrow{\chi_i}(\overrightarrow{K}_n)\}$.*

We use these results in Section 6.3.4 to find both upper and lower bounds for the oriented incidence chromatic number of a symmetric complete digraph.

6.3.4 Symmetric Complete Digraphs

To find a lower bound for $\overrightarrow{\chi_i}(\overrightarrow{K}_n)$ we first observe that for every pair of sets A_u, A_v (as defined in Section 6.3.3), it must be $A_u \neq A_v$.

Theorem 6.29. *The complete symmetric digraph on n vertices has oriented incidence chromatic number at least $\lceil \log_2(n) \rceil$.*

Proof. Let c be an oriented incidence colouring using k colours and let A_{v_i} $(1 \leq i \leq n)$ be the set of colours appearing on an incidence of the form $(v_i, v_i v_j)$. By Theorem 6.26, for every $1 \leq i < j \leq n$ it must be $A_{v_i} \neq A_{v_j}$. Since each $A_{v_j} \subseteq \{1, 2, 3, \ldots k\}$, it must be $k \geq \log_2(n)$. □

To find an upper bound on the oriented incidence chromatic number of a symmetric complete digraph we first recall the classic result of Sperner.

Theorem 6.30 (Sperner's Theorem)**.** *The size of a largest antichain in the lattice of subsets of* $\{1, 2, 3, \ldots, k\}$ *is*

$$\binom{k}{\lfloor k/2 \rfloor}.$$

Theorem 6.31. *If* k *is the smallest integer such that* $\binom{k}{\lfloor k/2 \rfloor} \geq n$*, then*

$$k \leq \overrightarrow{\chi_i}(\overrightarrow{K}_n) \leq 2k.$$

Proof. Let c be an oriented incidence colouring of $\overrightarrow{K}_n$ using $\overrightarrow{\chi_i}(\overrightarrow{K}_n)$ colours. Consider the collection of sets $A_{u_1}, A_{u_2}, \ldots, A_{u_n}$, where A_{u_i} is the set of colours that appear on incidences of the type $(u_i, u_i u_j)$. By Corollary 6.27, this collection of sets forms an antichain of length n. This implies directly that $\overrightarrow{\chi_i}(\overrightarrow{K}_n) \geq k$.

Let k be the smallest integer such that $\binom{k}{\lfloor k/2 \rfloor} \geq n$. Let

$$A_{u_1}, A_{u_2}, \ldots, A_{u_n} \subset \{1, 2, 3, \ldots k\}$$

be pairwise distinct sets of size $\binom{k}{\lfloor k/2 \rfloor}$ and let

$$B_{u_1}, B_{u_2}, \ldots, B_{u_n} \subset \{1', 2', 3', \ldots, k'\}$$

be pairwise distinct sets of size $\binom{k}{\lfloor k/2 \rfloor}$. By Sperner's Theorem, these sets satisfy the hypothesis of Theorem 6.26 and so there exists an oriented incidence colouring of $\overrightarrow{K}_n$ using $2k$ colours. $\square$

Using Theorem 6.31 we find an upper bound for $\overrightarrow{\chi_i}(\overrightarrow{K}_n)$ as a function of $\log_2(n)$. To do so we require the following observations.

Observation 6.32. *For all* $n \geq 9$*,*

$$\binom{\lceil c \cdot \log_2(n) \rceil}{\lceil (c/2) \cdot \log_2(n)) \rceil} \geq n,$$

where $c = 1 + \log_n(\log_2(n))$*.*

Observation 6.33.

$$\lim_{n \to \infty} \log_n(\log_2(n)) = 0.$$

Combining these observations with the statement of Theorem 6.31 gives the following.

Lemma 6.34. *For all* $n \geq 2$*,*

$$\log_2(n) \leq \overrightarrow{\chi_i}(\overrightarrow{K}_n) \leq (2 + o(1)) \log_2(n).$$

We use this result to find the following upper bound for $\overrightarrow{\chi_i}(\overrightarrow{K}_n)$, where n is a central binomial coefficient.

Theorem 6.35. *For all* $n = \binom{2k}{k}$*,*

$$\overrightarrow{\chi_i}(\overrightarrow{K}_n) \leq (1 + o(1)) \log_2(n).$$

Proof. Let $n = \binom{2k}{k}$. We show the existence of an oriented incidence colouring of $\overrightarrow{K}_n$, with vertex set $\{v_1, v_2, \ldots, v_n\}$, using no more than $(1 + o(1)) \log_2(n)$ colours.

Let $A_1, A_2, \ldots, A_n$ be a collection of subsets of $\{1, 2, 3, \ldots, 2k\}$ that satisfy the following properties.

- For all $i \neq j$ $(1 \leq i \leq j \leq n)$, $A_i \not\subset A_j$ and $A_i \not\subset A_j$, and
- for all i $(1 \leq i \leq n)$, $|A_i| = k$.

For all i $(1 \leq i \leq n)$ let $B_i = \overline{A_i}$. Construct a colouring, c, as follows. For all $i, j \in \{1, 2, 3, \ldots, n\}$ such that $|A_i \setminus A_j| > 1$, assign to incidence (u_i, u_iu_j) any element of $A_i \setminus A_j$ and to incidence (u_j, u_ju_i) any element of $B_j \setminus B_i$ such that $c(u_i, u_iu_j) \neq c(u_j, u_ju_i)$. Observe that by Theorem 6.3.3 the colouring constructed thus far does not assign identical colours to any adjacent incidences. At this point we observe for every set A_i that there exist k^2 sets A_j such that $|A_i \setminus A_j| = 1$. And so the symmetric graph, S, induced by those arcs xy where the incidences $(x, xy), (xy, y), (y, yx)$ and (yx, x) remain uncoloured is a regular symmetric digraph where each vertex has in-degree k^2 and out-degree k^2. The simple graph underlying S is k^2-colourable and so, by Theorem 6.9 and Lemma 6.34,

$$\overrightarrow{\chi_i}(S) \leq \overrightarrow{\chi_i}(\overrightarrow{K}_{k^2}) \leq (2 + o(1)) \log_2(k^2) = (4 + o(1)) \log_2(k).$$

We can complete c to be an oriented incidence colouring of $\overrightarrow{K}_n$ using at most an additional $(4 + o(1)) \log_2(k)$ colours. The total number of colours used by c is at most $2k + (4 + o(1)) \log_2(k)$. Since $\log_2(n) \leq 2k$ and

$$\lim_{n \to \infty} \frac{(4 + o(1)) \log_2(k)}{\log_2(n)} = 0,$$

we observe:

$$2k + (4 + o(1)) \log_2(k) \leq (1 + o(1)) \log_2(n).$$

Therefore,

$$\overrightarrow{\chi_i}(\overrightarrow{K}_n) \leq (1 + o(1)) \log_2(n).$$

□

To extend this result for values of n that are not central binomial coefficients we require the following observations.

Observation 6.36. *For all $k > 1$,*

$$\log\left(\binom{2(k+1)}{k+1}\right) - \log\left(\binom{2k}{k}\right) \leq 2.$$

Observation 6.37. *For all $n \geq 2$, where $\binom{2(k-1)}{k-1} < n < \binom{2k}{k}$*

$$\log(n) + 2 > \log\left(\binom{2k}{k}\right).$$

Combining these observations with Theorem 6.29 gives the following statements.

Theorem 6.38. *For all $n \geq 2$*

$$\log_2(n) \leq \overrightarrow{\chi_i}(\overrightarrow{K}_n) \leq (1 + o(1)) \log_2(n) + 2.$$

Corollary 6.39. *If G is a digraph, then $\overrightarrow{\chi_i}(G) \leq (1 + o(1)) \log_2(\chi(G)) + 2$.*

Corollary 6.40. *If T is a tournament on n vertices, then*

$$\log(n) \leq \overrightarrow{\chi_i}(T) \leq (1 + o(1)) \log_2(n) + 2.$$

Corollary 6.41. *If T is a transitive tournament on n vertices, then*

$$\frac{1}{2} \log_2(n) \leq \overrightarrow{\chi_i}(T) \leq (1 + o(1)) \log_2(n) + 2.$$

The lower bound here comes by observing that the arcs of any symmetric complete digraph on n vertices may be partitioned in a pair of transitive tournaments on n vertices.

We note the upper bounds given in Theorem 6.38 are not the best possible. Continued work on this bound by Pascal Ochem in [16] gives the following bounds.

Theorem 6.42. *If $n \geq 8$, then*

$$\log_2(n) + \tfrac{1}{2} \log_2(\log_2(n)) \leq \overrightarrow{\chi_i}(\overrightarrow{K_n}) \leq \log_2(n) + \tfrac{3}{2} \log_2(\log_2(n)) + 2.$$

6.3.5 Graphs with small Oriented Incidence Chromatic Number

In her Masters thesis [58] (more recently published as [34]), Sherk explores the relationship between oriented graph homomorphism and 2−dipath colouring. One of the main results of this work is to define a family of oriented graphs, G_k ($k > 1$), with the property that an oriented graph H has a 2−dipath colouring using k colours if and only if H admits a homomorphism to G_k. See Chapter 3 for a more thorough discussion of this result. Here we consider the possibility of a similarly-styled result for the oriented incidence chromatic number. For the case $\overrightarrow{\chi_i}(G) = 2$, a fairly straightforward characterisation exists.

Theorem 6.43. *Let G be a digraph with at least one arc, then $\overrightarrow{\chi_i}(G) = 2$ if and only if G admits a homomorphism to $\overrightarrow{P_1}$.*

To find a characterisation for those digraphs for which 3 colours suffice, consider the oriented graphs given in Figure 6.10. Observe that $H_1 \to H_2$.

Theorem 6.44. *For any digraph, G, $\overrightarrow{\chi_i}(G) \leq 3$ if and only if G admits a homomorphism to H_2.*

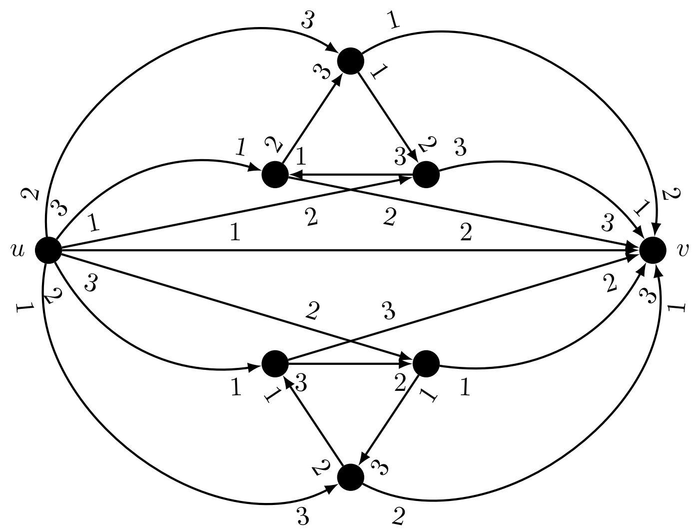
Proof. Let G be a digraph. If G admits a homomorphism to H_2, then by Theorem 6.9 we have directly $\overrightarrow{\chi_i}(G) \leq 3$, as H_2 is a subgraph of H_1 and $\overrightarrow{\chi_i}(H_1) = 3$. To show $\overrightarrow{\chi_i}(G) \leq 3$ implies homomorphism to H_2, we show $\overrightarrow{\chi_i}(G) \leq 3$ implies homomorphism to H_1.

Let g be an oriented incidence colouring of G that uses at most 3 colours. Construct the mapping $f : V(G) \to V(H_1)$ as follows.

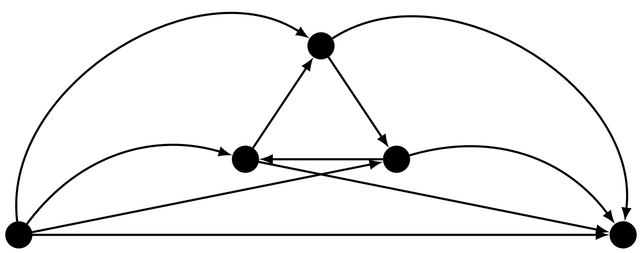
- For all $s \in V(G)$ such that $d^-(s) = 0$, let $f(s) = u$.
- For all $t \in V(G)$ such that $d^+(t) = 0$, let $f(t) = v$.
- For all $x \in V(G)$ such that there exist $wx, xy \in E(G)$, let $f(x)$ be the unique vertex $h \in H_1 \setminus \{u, v\}$ such that for all $h_1h \in E(H_1)$, $c(h_1h, h) = g(wx, x)$ and for all $hh_2 \in E(H_1)$; let $c(h, hh_2) = g(x, xy)$.

It can be checked $f : G \to H_1$. Since $H_1 \to H_2$ we have $G \to H_2$.

□



H_1



H_2

Figure 6.10: The oriented graphs, H_1 and H_2, used in the proof of Theorem 6.44.

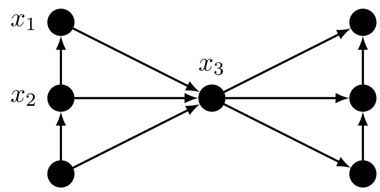


Figure 6.11: An outerplanar graph that requires 4 colours in an oriented incidence colouring.

Corollary 6.45. *If G is an oriented graph with $\overrightarrow{\chi_i}(G) \leq 3$, then $\chi_o(G) \leq 5$.*

We note here this bound is tight – H_1 has oriented incidence chromatic number 3 and oriented chromatic number 5.

These results allow us to find the oriented incidence chromatic number of oriented outerplanar graphs.

Corollary 6.46. *The family of oriented outerplanar graphs has oriented incidence chromatic number* 4.

Proof. Consider the oriented graph, G, shown in Figure 6.11. It is outerplanar and so its underlying simple graph has chromatic number at most 3. By Theorem 6.14 and Table 6.1, $\overrightarrow{\chi_i}(G) \leq 4$.

To show $\overrightarrow{\chi_i}(G) = 4$ we show G does not admit a homomorphism to H_2. Consider the vertices labelled x_1, x_2, x_3 in Figure 6.11. Each of these vertices has positive in- and out-degree, and so if G admits a homomorphism to H_2, then these three vertices must map to the directed $3-$cycle. However these vertices form a transitive triple. Therefore G does not admit a homomorphism to H_2. This gives that the family of oriented outerplanar graphs has oriented incidence chromatic number at least 4. $\square$

Given there is an oriented graph that is a universal target for all digraphs that have oriented incidence chromatic number at most 3 it is natural to wonder if there is an oriented graph that is a universal target for all digraphs that have oriented chromatic number at most k, for each k. This turns out not to be the case.

Theorem 6.47. *For all $k > 3$, there is no finite oriented graph G such that every oriented graph with oriented incidence chromatic number no more than k admits a homomorphism to G.*

Proof. Consider the family of oriented bipartite graphs. Every oriented bipartite graph has an oriented incidence colouring using at most 4 colours, but the oriented chromatic number of the family of oriented bipartite graphs is unbounded [50]. This implies there is no finite oriented graph that is a universal target for the family of oriented bipartite graphs. $\square$

Definition 6.8. *Let G be a digraph. Define the* directed line graph of G, *denoted $\overrightarrow{L}(G)$, to be the digraph with the following vertex and arc sets.*

- $V(\overrightarrow{L}(G)) = \{x_{uv} | uv \in E(G)\}$, *and*
- $E(\overrightarrow{L}(G)) = \{x_{uv}x_{vw} | uv, vw \in E(G)\}$.

Informally, the directed line graph of a digraph G has as its vertex set the arc set of G and has an arc e_1e_2 whenever the head of e_1 is incident with the tail of e_2. We note the directed line graph has been used in the study of the oriented chromatic index [43].

Using the directed line graph we build a homomorphism model of oriented incidence colouring. To do this we first define a digraph that will be the target for a homomorphism from a directed line graph. We call this graph Γ_k ($k > 1$), and define it as follows.

- $V(\Gamma_k) = \{(a, b)|a, b \in \{1, 2, \dots k\}, a \neq b\}$, and
- $E(\Gamma_k) = \{((a, b)(c, d))\,|a \neq c, b \neq c, d\}$.

Theorem 6.48. *$\overrightarrow{\chi_i}(G) \leq k$ if and only if $\overrightarrow{L}(G) \to \Gamma_k$.*

Proof. Let G be a digraph and assume $\overrightarrow{\chi_i}(G) \leq k$. Let c be an oriented incidence colouring of G using at most k colours. Using c we construct a homomorphism $\overrightarrow{L}(G) \to \Gamma_k$. We first define a map between the vertices of $\overrightarrow{L}(G)$ and the vertices of Γ_k and then show it is a homomorphism. Let $\phi : V(\overrightarrow{L}(G)) \to V(\Gamma_k)$ be defined by

- $\phi(x_{uv}) = (c(u, uv), c(uv, v))$.

Since c uses at most k colours and is an oriented incidence colouring, we see $(c(u, uv), c(uv, v)) \in V(\Gamma_k)$. To show ϕ is a homomorphism consider the image of an arc $x_{uv}x_{vw}$ under ϕ.

$$\phi(x_{uv})\phi(x_{vw}) = ((c(u, uv), c(uv, v)), (c(v, vw), c(vw, w)))$$

Since c is an oriented incidence colouring, it must be $c(u, uv) \neq c(v, vw)$, $c(uv, v) \neq c(v, vw)$, and $c(uv, v) \neq c(vw, w)$. Therefore

$$((c(u, uv), c(uv, v)), c(v, vw), c(vw, w))$$

is an arc of Γ_k and so ϕ is a homomorphism. Therefore if $\overrightarrow{\chi_i}(G) \leq k$, then $\overrightarrow{L}(G) \to \Gamma_k$.

To prove the opposite direction, let $\phi : \overrightarrow{L}(G) \to \Gamma_k$. Using this homomorphism construct an oriented incidence colouring as follows.

- If $\phi(x_{uv}) = (a, b)$, then let $c(u, uv) = a$ and $c(uv, v) = b$.

If c is not an oriented incidence colouring, then there must be a pair of adjacent incidences of G that receive the same colour.

Case I: $c(u, uv) = c(uv, v)$. Assume $\phi(x_{uv}) = (a, b)$. Since $a \neq b$, it must be $c(u, uv) \neq c(uv, v)$.

Case II: $c(u, uv) = c(v, vw)$. Assume $\phi(x_{uv}) = (a, b)$ and $\phi(x_{vw}) = (c, d)$. Since ϕ is a homomorphism, the image of the arc $x_{uv}x_{vw}$ is an arc of Γ_k. Therefore $a \neq c$ and so it must be $c(u, uv) \neq c(v, vw)$.

Case III: $c(uv, v) = c(v, vw)$. Assume $\phi(x_{uv}) = (a, b)$ and $\phi(x_{vw}) = (c, d)$. Since ϕ is a homomorphism, the image of the arc $x_{uv}x_{vw}$ is an arc of Γ_k. Therefore $b \neq c$ and so it must be $c(uv, v) \neq c(v, vw)$.

Case IV: $c(uv, v) = c(vw, w)$. Assume $\phi(x_{uv}) = (a, b)$ and $\phi(x_{vw}) = (c, d)$. Since ϕ is a homomorphism, the image of the arc $x_{uv}x_{vw}$ is an arc of Γ_k. Therefore $b \neq d$ and so it must be $c(uv, v) \neq c(vw, w)$.

Since no pair of adjacent incidences receive the same colour, c is an oriented incidence colouring of G using at most k colours. □

For the case $k = 3$, we observe Γ_k is the disjoint union of two directed $3-$cycles.

Proposition 6.49. *If G is a digraph, then $\overrightarrow{\chi_i}(G) \leq 3$ if and only if $\overrightarrow{L}(G) \to C_3$, where C_3 is the directed cycle on 3 vertices.*

6.4 Conclusions and Future Directions

In the study of oriented incidence colourings of digraphs, many open problems and areas of enquiry remain. One open area is the construction of universal targets for digraphs with given oriented incidence chromatic number. For digraphs with oriented incidence chromatic number 3 there is a complete characterisation. Digraphs with oriented incidence chromatic number at most 3 are necessarily oriented graphs, and the universal target for this family is an oriented graph. For digraphs with oriented incidence chromatic number at least 4, observe that such digraphs may contain $2-$cycles. And so any universal target for this family of digraphs necessarily contains $2-$cycles.

The definition of oriented colouring enforces that if there is an arc with its tail coloured i and its head coloured j, then there is no arc with its tail coloured j and its head coloured i. To enforce this constraint with respect to the colours of the incidences would not drastically change the analysis given above. Undoubtedly this extra constraint would increase the oriented incidence chromatic number, but the methods used above may still be utilized. The homomorphism model utilizing the chromatic number would still exist, however the logarithmic upper bound for a colouring of a symmetric complete graph may not hold.

Glossary of Colouring Parameters

$\chi_{j,k}$	$(j,k)-$chromatic number	9
χ_o	oriented chromatic number	12
χ_2	chromatic number (of a $2-$edge coloured graph)	11
χ_k	chromatic number (of a $k-$edge coloured graph)	11
χ_{2d}	$2-$dipath chromatic number	34
χ_{kd}	$k-$dipath chromatic number	34
χ_s	simple chromatic number (of an oriented graph)	46
χ_{2s}	simple $2-$dipath chromatic number	53
χ_2^a	alternating $2-$path chromatic number (or a $2-$edge coloured graph)	62
χ_2^s	simple chromatic number (of a $2-$edge coloured graph)	71
χ_k^s	simple chromatic number (of a $k-$edge coloured graph)	71
χ_i	incidence chromatic number	78
$\overrightarrow{\chi_i}$	oriented incidence chromatic number	81

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