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UNIVERSITÉ PIERRE ET MARIE CURIE
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THÈSE DE DOCTORAT

Discipline : Mathématiques

pour obtenir le grade de

Docteur en Sciences de l'Université Pierre et Marie Curie

présentée par

Eugénie POULON

Etude qualitative d'éventuelles singularités dans les équations de Navier-Stokes tridimensionnelles pour un fluide visqueux.

Soutenue le 26 juin 2015 devant le jury composé de Mmes et MM. :

Jean-Yves CHEMIN	Directeur
Claire DAVID	Examinatrice
Isabelle GALLAGHER	Examinatrice
Patrick GÉRARD	Rapporteur
Fabrice PLANCHON	Rapporteur
Franck SUEUR	Examinateur

"Ce à quoi l'un s'était failli, l'autre y est arrivé et ce qui était inconnu à un siècle, le siècle suivant l'a éclairci, et les sciences et les arts ne se jettent pas en moule mais se forment et figurent en les maniant et polissant à plusieurs fois [...] Ce que ma force ne peut découvrir, je ne laisse pas de le sonder et essayer et, en retastant et pétrissant cette nouvelle matière, la remuant et l'eschaufant, j'ouvre à celui qui me suit quelque facilité et la lui rends plus souple et plus maniable. Autant en fera le second au tiers qui est cause que la difficulté ne me doit pas désespérer, ni aussi peu mon impuissance..."

Montaigne, Les Essais, Livre II, Chapitre XII.

Etude qualitative d'éventuelles singularités pour les équations de Navier-Stokes tridimensionnelles pour un fluide visqueux.

Eugénie Poulon

À ma maman,

Étude qualitative d'éventuelles singularités pour les équations de Navier-Stokes tridimensionnelles pour un fluide visqueux.

Résumé. Nous nous intéressons dans cette thèse aux équations de Navier-Stokes pour un fluide visqueux incompressible. Dans la première partie, nous étudions le cas d'un fluide homogène. Rappelons que la grande question de la régularité globale en dimension 3 est plus ouverte que jamais : on ne sait pas si la solution de l'équation correspondant à un état initial suffisamment régulier mais arbitrairement loin du repos, va perdurer indéfiniment dans cet état (*régularité globale*) ou exploser en temps fini (*singularité*). Une façon d'aborder le problème est de supposer cette éventuelle rupture de régularité et d'envisager les différents *scénarii* possibles. Après un rapide survol de la structure propre aux équations de Navier-Stokes et des résultats connus à ce jour (**chapitre** 1), nous nous intéressons (**chapitre** 2) à l'existence locale (en temps) de solutions dans des espaces de Sobolev qui ne sont pas invariants d'échelle. Partant d'une donnée initiale qui produit une singularité, on prouve l'existence d'une constante optimale qui minore le temps de vie de la solution. Cette constante, donnée par la méthode rudimentaire du point fixe, fournit ainsi un bon ordre de grandeur sur le temps de vie maximal de la solution. Au **chapitre** 3, nous poursuivons les investigations sur le comportement de telles solutions explosives à la lumière de la méthode des éléments critiques.
Dans le seconde partie de la thèse, nous sommes intéressés à un modèle plus réaliste du point de vue de la physique, celui d'un fluide incompressible à densité variable. Ceci est modélisé par les équations de Navier-Stokes incompressible et inhomogènes. Nous avons étudié le caractère globalement bien posé de ces équations dans la situation d'un fluide évoluant dans un tore de dimension 3, avec des données initiales appartenant à des espaces critiques et sans hypothèse de petitesse sur la densité.

Mots-clés. équations de Navier-Stokes, incompressibilité, explosion, singularité, invariance d'échelle, théorie des profils, concentration-compacité, élément critique, espace de Sobolev, espace de Besov.

Description of potential singularities in the Navier-Stokes equations for a viscous fluid in dimension 3.

Abstract. This thesis is concerned with incompressible Navier-Stokes equations for a viscous fluid. In the first part, we study the case of an homogeneous fluid. Let us recall that the big question of the global regularity in dimension 3 is still open : we do not know if the solution associated with a data smooth enough and far from the immobile stage will last over time (*global regularity*) or on the contrary will stop living in finite time and blow up (*singularity*). The goal of this thesis is to study this regularity break. One way to deal withthis question is to assume that such a phenomenon occurs and to study differents scenarii. The **chapter** 1 is devoted to a recollection of well-known results. In **chapter** 2, we are interesting in the local (in time) existence of a solution in some Sobolev spaces which are not invariant under the natural sclaing of Navier-Stokes. Starting with a data generating a singularity, we can prove there exists an optimal lower boundary of the lifespan of such a solution. In ths way, the lower boundary provided by the elementary procedure of fixed-point, gives the correct order of magnitude. Then, we keep on investigations about the behaviour of regular solution near the blow up, thanks to the method of critical elements (**chapter** 3).
In the second part, we are concerned with a more relevant model, from a physics point of view : the inhomogeneous Navier-Stokes system. We deal with the global wellposeness of such a model for a inhomogeneous fluid, evolving on a torus in dimension 3, with critical data and without smallness assumption on the density.

Keywords. Navier-Stokes equations, incompressible fluid, blow-up, singularities, scaling invariance, profile theory, concentration-compactness, critical element, Sobolev space, Besov space.

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Introduction générale

1 Mécanique des fluides et turbulence

1.1 D'Euler à Navier-Stokes

L'équation de Navier-Stokes constitue un modèle mathématique de l'écoulement des fluides. Elle s'inscrit dans le vaste domaine de la mécanique des fluides qui a pour vocation de décrire les écoulements de type fluide (vent, eau, gaz) et d'expliquer la résistance d'un corps en mouvement dans un fluide.Nombreux sont les phénomènes que nous observons au quotidien et qui nous interrogent : pourquoi un avion ne tombe-t-il pas sous son propre poids ; un pont ne s'effondre-t-il pas sous l'action d'un vent violent ? Pourquoi cette étonnante danse d'une feuille tombant d'un arbre ? Comment expliquer la formation des tourbillons atmosphériques que nous admirons sur les cartes météorologiques ? Par ailleurs, il apparaît alors évident que la compréhension de tous ces phénomènes naturels permettrait d'exploiter les fluides à des fins très pratiques. Citons par exemple le pompage des puits de pétrole, l'adduction d'eau, les forces de frottements de l'eau sur un nageur, de l'air sur un cycliste.

Une façon d'expliquer ces types d'écoulements consiste à les modéliser, c'est-à-dire à les mettre en équation. Les premiers travaux remontent au 18ème siècle avec D. Bernouilli, qui analyse la conservation de l'énergie des fluides visqueux. Mais c'est avec J. d'Alembert et L. Euler que naissent les équations fondamentales de la mécanique des fluides, dites équations d'Euler (1748). Celles-ci répondent au problème mis à prix par l'Académie des Sciences de Berlin : *"déterminer la théorie de la résistance que souffrent les corps solides dans leur mouvement, en passant par un fluide, tant par rapport à la figure et aux divers degrés de vitesse des corps qu'à la densité et aux divers degrés de compression du fluide"*. Si l'on doit beaucoup aux travaux de d'Alembert concernant la dynamique des fluides, ils sont cependant incomplets car ils ne tiennent pas compte du rôle essentiel joué par la pression pour rendre compte du caractère incompressible du fluide. C'est finalement L. Euler qui sera le premier à établir en 1755, un système complet modélisant un fluide parfait(sans frottement interne) incompressible :

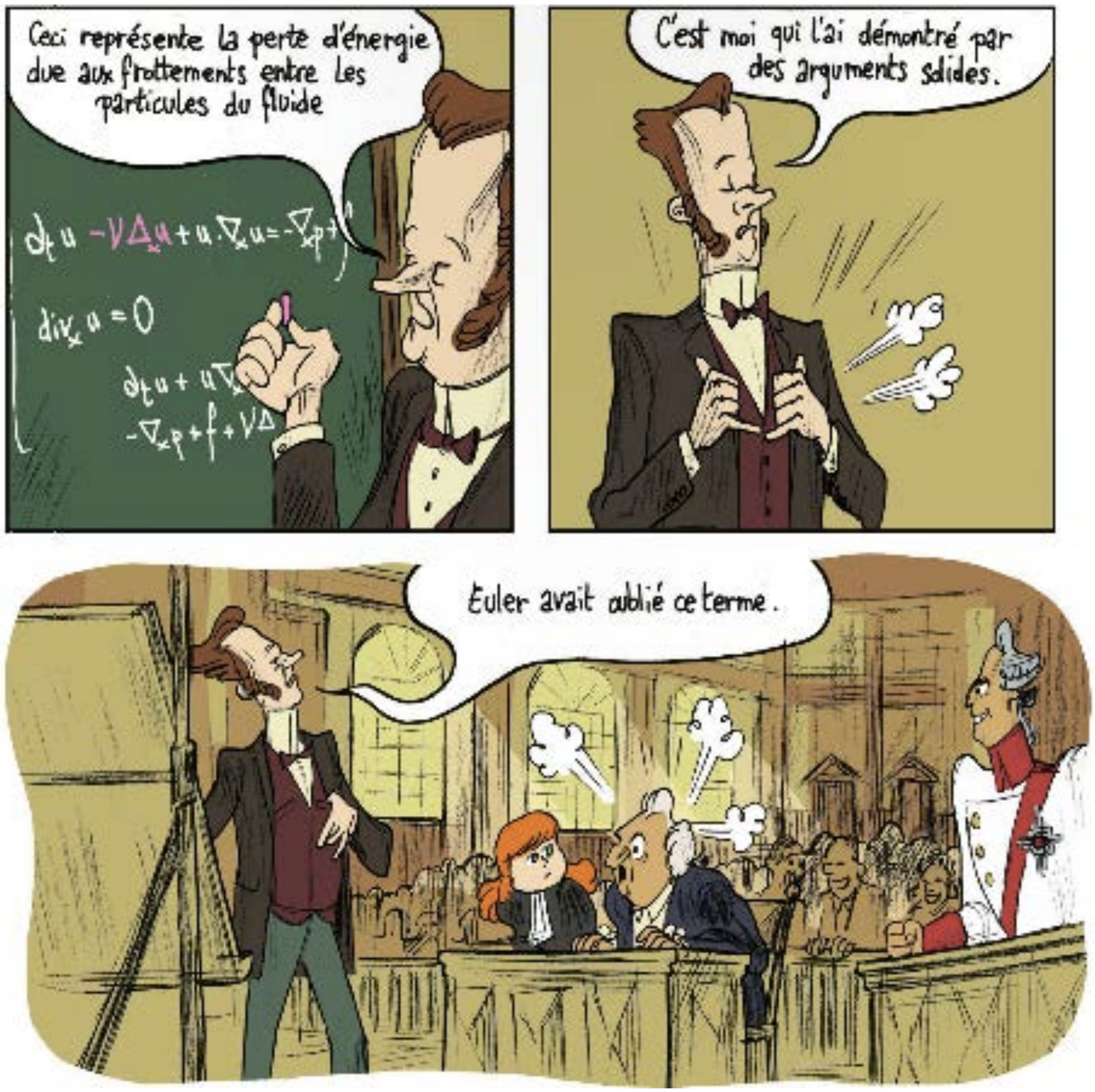
$$\left\{\begin{array}{rcl} \partial_t u + u\cdot\nabla u & = & -\nabla p \\ \operatorname{div} u & = & 0. \\ u_{|t=0} & = & u_0. \end{array}\right. \tag{1}$$

Ici $u(t,x)$ désigne le champ de vitesse d'un fluide à l'instant t et à la position x, soumis à une pression $p(t,x)$: en ce sens, on parle de formulation *eulérienne* et non *lagrangienne* car on suit non pas les particules du fluide mais leur vitesse à chaque instant et en chaque point de l'espace. Les équations d'Euler constituent les premières équations aux dérivées partielles de physique mathématique. Il est assez piquant de remarquer qu'aujourd'hui encore, elles hantent l'esprit de bon nombre de mathématiciens. Néanmoins, d'Alembert met en évidence un écueil : un corps soumis aux équations d'Euler qui serait plongé dans un fluide en mouvement ne rencontrerait aucune résistance de la part de celui-ci... Ainsi, si l'on calcule à partir des équations d'Euler la pression exercée par l'air sur les ailes d'un avion, on trouve 0, ce qui signifierait que l'avion tombe... Ou encore, une barque naviguant sur l'eau avance sans jamais se voir opposer une quelconque résistance de la part de l'eau. C'est ce qu'on appelle le *« paradoxe de d'Alembert »*, qu'il formule ainsi : *« Il me semble que la théorie, développée avec toute la rigueur possible, donne, au moins dans plusieurs cas, une résistance nulle, paradoxe singulier que je laisse les Géomètres futurs résoudre. »*

C'est au 19ème siècle avec C. Navier que cet écueil est dissipé : l'ingénieur des ponts (à qui l'on doit entre autres le pont d'Argenteuil sur la Seine) a l'intuition qu'il manque à l'équation d'Euler un élément crucial : un fluide en mouvement dissipe de l'énergie sous forme de chaleur, résultant du frottement microscopique des particules du fluide. C'est cette perte d'énergie qui est responsable de la résistance d'un corps plongé dans un fluide en mouvement. C'est ainsi que C. Navier et G. Stokes proposent en 1845 le modèle, dit de Navier-Stokes, pour décrire l'évolution d'un fluide visqueux :

$$\left\{\begin{array}{ccc} \partial_t u + u \cdot \nabla u - \nu\,\Delta u & = & -\nabla p \\ \operatorname{div} u & = & 0 \\ u_{|t=0} & = & u_0. \end{array}\right. \tag{2}$$

Le paramètre $\nu > 0$ est appelée la *viscosité* du fluide : il mesure la différence entre un fluide parfait ($\nu = 0$) et un fluide visqueux ($\nu > 0$), c'est-à-dire entre les équations d'Euler et celles de Navier-Stokes.



Notons une question naturelle : lorsque la viscosité tend vers 0, les solutions des équations de Navier-Stokes convergent-elles vers celles des équations d'Euler ? Cela fait l'objet de nombreux travaux, notamment dans le cas de domaines à bords ;ce passage à la limite constitue le point de départ de l'étude d'une équation mal posée, dite équation de Prandtl.
Revenons à l'équation de Navier-Stokes. Le lecteur aura noté la présence de deux termes qui se concurrencent l'un l'autre : le premier $u \cdot \nabla u$, dit terme convectif, est non linéaire, générateur d'instabilités et tenu pour coupable de l'écoulement *turbulent* d'un fluide dans certaines situations. C'est par exemple le cas en météorologie et en océanographie : la présence du terme non linéaire rend l' évolution du fluide à un instant $t + dt$ très sensible à de petites variations à l'instant t. En d'autres termes, il est indispensable de connaître en détail l'écoulement à l'instant t pour prédire sa situation à l'instant $t + dt$. Ceci explique pourquoi les météorologues ne peuvent prédire le beau temps à plus de quelques jours ; l'état initial n'est jamais suffisamment bien connu pour pouvoir prédire avec certitude à long terme. Le second terme, $\nu\,\Delta u$, dit terme visqueux, a pour effet au contraire de lisser l'écoulement et donc de le revêtir d'un aspect *laminaire*. On appelle nombre de Reynolds de l'écoulement Re, le

rapport entre ces deux termes :

$$Re \stackrel{\text{def}}{=} \frac{|u \cdot \nabla u|}{|\nu \, \Delta u|} \approx \frac{U\, L}{\nu},$$

où L et U sont respectivement la longueur et vitesse caractéristique de l'écoulement. L'observation expérimentale suggère l'existence d'un nombre de Reynolds critique Re^* (ce nombre critique dépend évidemment du domaine dans lequel évolue le fluide, ainsi que de la condition initiale). Il est alors convenu d'appeler écoulement laminaire tout écoulement dont le nombre de Reynolds satisfait $Re < Re^*$. On parle d'écoulement turbulent lorsque $Re > Re^*$. Donnons un exemple numérique (cf l'article de Raoul Robert, *CNRS UMR* 5582*, Institut Fourier, St Martin d'Heres*) : disons que le nombre de Reynolds critique pour l'air est d'environ $Re^* = 100$; pour un écoulement météorologique, le nombre de Reynolds est alors d'environ $Re = 10^{12}$, celui d'un écoulement autour d'une automobile, de l'ordre de $Re = 10^7$; ce sont des écoulements pleinement turbulents.

1.2 De la résolution des équations de Navier-Stokes

L'idée naturelle qui vient à l'esprit à la vue des équations de Navier-Stokes est d'en chercher des solutions explicites. Entreprise vaine et vite abandonnée par les mathématiciens contemporains de C. Navier et G. Stokes. La stratégie mise en place pour pallier cet échec de solutions explicites consiste à en chercher des solutions approchées. En effet, d'un point de vue numérique, s'il est impossible de rentrer l'équation exacte dans un ordinateur, il est cependant tout à fait possible de lui en donner une approximation (cela passe par une discrétisation de l'équation considérée). La solution obtenue est donc une solution approchée. Reste à savoir son degré de proximité de la solution exacte. Pour ce faire, il est indispensable d'étudier au préalable le caractère *bien posé au sens d'Hadamard* de l'équation considérée. Cela signifie que l'équation doit satisfaire à trois conditions. Elles se définissent chacune en ces termes : existence, unicité, stabilité :
- l'existence : supposons l'état du fluide connu à l'instant initial, alors il existe une solution à l'équation aux instants ultérieurs, coïncidant avec l'état initial à $t = 0$;
- l'unicité : il n'existe qu'une seule solution qui coïncide avec l'état initial à $t = 0$;
- la stabilité : si deux solutions sont proches à un instant t, elles doivent le rester à un instant $t + dt$.

Désormais, toute étude d'une équation aux dérivées partielles commencera par la quête de réponses à ces trois questions.

2 Liste des travaux présentés dans la thèse

Les chapitres de ce manuscrit sont composés des travaux suivants :

– Chapitre 2 : article soumis
– Chapitre 3 : article en voie de soumission, prépublication.
– Chapitre 4 : article en voie de soumission, prépublication.

* Les dessins sont issus de la bande-dessinée *L'équation du Millénaire*, éditée par la Fondation Sciences Mathématiques de Paris. Nous remercions les auteurs et éditeurs pour leur autorisation à faire figurer quelques extraits de la bande-dessinée dans la thèse.

Première partie

Équations de Navier-Stokes pour un fluide homogène et incompressible

Chapitre 1

Le problème de Cauchy pour les équations de Navier-Stokes incompressible

L'objet de cette première partie de la thèse consiste en l'étude des équations de Navier-Stokes pour un fluide évoluant dans l'espace $\mathbb{R}^3$ tout entier, présentant les caractéristiques suivantes :
- homogénéité : la densité ρ du fluide est constante en la variable d'espace ; on la normalise à $\rho = 1$,
- incompressibilité : le volume du fluide ne peut ni augmenter, ni diminuer,
- viscosité que l'on choisit égale à 1, afin d'alléger les notations.

Noter que l'hypothèse d'homogénéité du fluide associée à celle d'incompressibilité implique que la densité du fluide étudié est constante à tout instant et en tout point de l'espace. Dans tout ce qui va suivre, le fluide considéré vivra dans l'espace tout entier ; nous n'écrirons donc plus $\mathbb{R}^3$. Plus précisément, nous nous intéressons au système suivant, où u est un champ de vitesse de $\mathbb{R}^3$ et p, la pression du fluide. Ce sont les deux inconnues du problème : elles dépendent du temps $t > 0$ et de la position $x \in \mathbb{R}^3$.

$$\left\{\begin{array}{ccc} \partial_t u + u \cdot \nabla u - \Delta u & = & -\nabla p \\ \operatorname{div} u & = & 0 \\ u_{|t=0} & = & u_0. \end{array}\right. \tag{1.1}$$

Tout l'enjeu de ce problème de Cauchy est de déterminer les conditions qu'il faut imposer au champ de vitesse initial u_0 pour qu'il existe une solution (u, p).
Explicitons les notations :

$$\operatorname{div} u \stackrel{\text{def}}{=} \sum_{j=1}^{3} \partial_j\, u^j, \quad u \cdot \nabla \stackrel{\text{def}}{=} \sum_{j=1}^{3} u^j\, \partial_j, \quad \text{et} \quad \Delta \stackrel{\text{def}}{=} \sum_{j=1}^{3} \partial_j^2.$$

Remarque 1.1. La condition de divergence nulle (qui traduit l'hypothèse d'incompressibilité du fluide) joue un rôle majeur dans l'étude du système de Navier-Stokes. Par exemple, elle implique que si le champ de vitesse u est assez régulier, on a la propriété suivante :

$$u \cdot \nabla u \;=\; \operatorname{div}(u \otimes u), \quad \text{où} \quad \operatorname{div}(u \otimes u)^j \;\stackrel{\text{def}}{=}\; \sum_{j=1}^{3} \partial_k (u^j\, u^k).$$

Ainsi, le système d'équations de Navier-Stokes s'écrit aussi

$$\left\{\begin{array}{ccc} \partial_t u + \operatorname{div}(u \otimes u) - \Delta u & = & -\nabla p \\ \operatorname{div} u & = & 0 \\ u_{|t=0} & = & u_0. \end{array}\right. \tag{1.2}$$

Remarque 1.2. En appliquant l'opérateur de divergence à l'équation de Navier-Stokes et utilisant l'hypothèse d'incompressibilité, on montre que la pression p est en fait fonction de la vitesse u du fluide, ce n'est donc pas une inconnue du problème. En effet, on obtient

$$\operatorname{div}(u \cdot \nabla u) = -\Delta p,$$

ce qui conduit (grâce à la formule de Leibniz) à

$$p = (-\Delta)^{-1} \sum_{i,j=1}^{3} \partial_i \, \partial_j (u^i \, u^j).$$

1.1 Conservation d'énergie et invariance d'échelle

L'équation de Navier-Stokes présente deux propriétés fondamentales, qui conditionnent beaucoup de résultats remarquables sur cette équation.

1.1.1 Estimation d'énergie

La première propriété est celle de **la conservation d'énergie** L^2. Au moins formellement, si l'on suppose u solution assez régulière de (1.1), alors on a

$$\frac{1}{2}||u(t)||_{L^2}^2 + \int_0^t ||\nabla u(t')||_{L^2}^2 \, dt' = \frac{1}{2}||u_0||_{L^2}^2. \tag{1.3}$$

Démonstration. En effet, on effectue le produit scalaire L^2 de u avec l'équation de Navier-Stokes, ce qui conduit à

$$\frac{1}{2}\frac{d}{dt}||u(t)||_{L^2}^2 + ||\nabla u(t)||_{L^2}^2 = -\int_{R^3} \big(u \cdot \nabla u(t) | u(t)\big)_{L^2} - \int_{R^3} \big(\nabla p(t) | u(u)\big)_{L^2}.$$

On remarque ensuite que la condition de divergence nulle sur le champ u implique en particulier

$$\big(u \cdot \nabla u | u\big)_{L^2} = 0 = \big(\nabla p | u\big)_{L^2}.$$

Par ailleurs, il est clair (par intégration par partie et en supposant la décroissance à l'infini vers 0 de u, ce qui est licite par densité des fonctions régulières à support compact dans L^2) que l'on a

$$\big(-\Delta u | u\big)_{L^2} = ||\nabla u(t)||_{L^2}^2.$$

Finalement, on obtient

$$\frac{1}{2}\frac{d}{dt}||u(t)||_{L^2}^2 + ||\nabla u(t)||_{L^2}^2 = 0,$$

ce qui conduit au résultat après intégration en temps. □

Ceci signifie que l'énergie du système $||u(t)||_{L^2}$ est une fonction décroissante du temps, contrôlée par l'énergie du système à l'état initial. Par ailleurs, il est intéressant de noter dès à présent l'effet régularisant suivant : dès qu'on choisit une donnée initiale u_0 dans l'espace d'énergie L^2, la solution découlant d'une telle donnée est alors "régularisée" au sens où son gradient appartient aussi à l'espace L^2.

1.1.2 Invariance d'échelle

La seconde propriété est celle de **l'invariance d'échelle** de l'équation de Navier-Stokes (NS). Comme beaucoup d'équations aux dérivées partielles, Navier-Stokes présente la propriété intéressante et non moins cruciale d'avoir une échelle : si u est solution de l'équation de NS sur $[0,T]\times\mathbb{R}^3$, pour la donnée initiale u_0, alors pour tout $\lambda>0$, le champ de vecteur remis à l'échelle

$$u_\lambda(t,x)\stackrel{\text{def}}{=}\lambda\, u(\lambda^2 t,\lambda x)$$

est solution de l'équation de NS sur l'intervalle $[0,\lambda^{-2}T]\times\mathbb{R}^3$, pour la donnée initiale

$$u_{0,\lambda}=\lambda\, u_0(\lambda x).$$

De cette propriété, découle une définition naturelle, celle d'espace invariant d'échelle.

Définition 1.3. *Un espace de Banach X est invariant d'échelle (ou critique) si sa norme est elle-même invariante par la transformation $u\mapsto u_\lambda$, e.g*

$$||u_\lambda||_X=||u||_X$$

Donnons quelques exemples en dimension d d'espaces critiques pour NS. De simples calculs montrent que les espaces ci-dessous sont invariants d'échelle.

$$L^\infty(\mathbb{R}^+,L^d(\mathbb{R}^d)),\quad L^q(\mathbb{R}^+,L^r(\mathbb{R}^d)),\ \text{avec}\ \frac{2}{q}+\frac{d}{r}=1,\quad L^q(\mathbb{R}^+,\dot H^s(\mathbb{R}^d)),\ \text{avec}\ \frac{2}{q}-s=1-\frac{d}{2}\cdot$$

Nous donnons désormais une chaine d'espaces critiques en dimension 3. Nous recroiserons cette succession d'espaces invariants d'échelle quelques lignes plus bas.

$$\dot H^{\frac12}\hookrightarrow L^3\hookrightarrow \dot B^{-1+\frac{3}{p}}_{p,\infty}\ \text{avec}\ 3\leqslant p<\infty\hookrightarrow \mathcal{BMO}^{-1}\hookrightarrow \dot B^{-1}_{\infty,\infty}.$$

Cette notion d'échelle est cruciale comme cela apparaitra par la suite, car c'est un outil clé pour démontrer le caractère bien posé des équations de Navier-Stokes dans certains espaces. Cela nous amène à définir la notion d'équation sous-critique, critique et sur-critique. Pour ce faire, on commence par rechercher la quantité *a priori* conservée par l'équation de Navier-Stokes. Elle s'obtient en effectuant une estimation d'énergie L^2. Cette quantité cherchée est la suivante :

$$L^\infty(\mathbb{R}^+,L^2(\mathbb{R}^d))\cap L^2(\mathbb{R}^+,\dot H^1(\mathbb{R}^d)). \tag{1.4}$$

Puis on compare la régularité donnée par la quantité conservée (1.4) et celle donnée par l'invariance d'échelle. Lorsque la régularité de la quantité conservée est plus grande que celle donnée par l'invariance d'échelle, on parle d'équation sous-critique (dans ce cas, le caractère bien posé de l'équation est très facile à démontrer).

Lorsque les régularités sont les mêmes (e.g les normes conservées sont invariantes d'échelle), on parle d'équation critique (c'est le cas des équations de Navier-Stokes bidimensionnelles).

Enfin, lorsque la régularité de la quantité conservée est plus petite que celle donnée par l'invariance d'échelle, on parle d'équation sur-critique. Des équations de ce type constituent un défi pour les mathématiciens. C'est par exemple le cas des équations de Navier-Stokes d-dimensionnelles (avec $d\geqslant 3$), puisque un espace invariant d'échelle est donné par

$$L^\infty(\mathbb{R}^+,L^d(\mathbb{R}^d))\cap L^2(\mathbb{R}^+,\dot H^{\frac{d}{2}}(\mathbb{R}^d)),$$

dont la régularité est bien supérieure à celle donnée par (1.4).

1.2 Différents types de solutions

1.2.1 Solutions faibles et solutions à la Leray

Les travaux de J. Leray sur la mécanique des fluides ont largement contribué à l'essor de l'analyse mathématique des équations aux dérivées partielles. C'est notamment dans son célèbre article fondateur et pionnier [40], paru dans *Acta Mathematica* en 1934 que J. Leray a introduit la notion de solution faible pour les équations de Navier-Stokes incompressibles à une époque où le concept même de solution faible de Sobolev (introduite à l'origine pour des problèmes linéaires) et les distributions de L. Schwartz sont encore inconnues.

Définition 1.4. *On dit qu'un champ de vecteurs u à composantes dans l'espace $L^2_{loc}([0,T]\times \mathbb{R}^d)$ est une solution faible de NS, si pour tout champ de vecteurs ψ dans $\mathcal{C}^\infty([0,T]\times \mathbb{R}^d)$ à support compact en espace, de divergence nulle, on a pour tout $t \leqslant T$*

$$\begin{aligned} \int_{\mathbb{R}^3} u\cdot \psi(t,x)\,dx = &\int_{\mathbb{R}^3} u_0(x)\cdot\psi(0,x)\,dx \\ &+ \int_0^t \int_{\mathbb{R}^3} (u\cdot \Delta\,\psi\, + u\otimes u\,:\,\nabla\psi + u\cdot\partial_t\psi)\,dx\,dt'. \end{aligned} \tag{1.5}$$

Cette définition (ou encore formulation variationnelle des équations de Navier-Stokes) ignore les deux propriétés caractéristiques des équations de NS, ce qui laisse supposer son insuffisance à prouver l'existence de solution. C'est de ce souci d'utiliser la conservation d'énergie que J. Leray définit la notion de solution turbulente, dont le nom souligne la très faible régularité a priori de la solution.

Définition 1.5. *On dit qu'un champ de vecteurs u appartenant à l'espace*

$$L^\infty([0,T],L^2(\mathbb{R}^d))\cap L^2([0,T],\dot H^1(\mathbb{R}^d))$$

est une solution turbulente de NS, (ou "solution à la Leray") associée à une donnée initiale $u_0 \in L^2(\mathbb{R}^d)$ si u est une solution faible et qu'elle satisfait pour tout $t\leqslant T$

$$\frac{1}{2}||u(t)||^2_{L^2} + \int_0^t\, ||\nabla u(t)||^2_{L^2} \leqslant \frac{1}{2}||u_0||^2_{L^2}. \tag{1.6}$$

Le théorème suivant (1934), que l'on doit aussi à J. Leray marque un tournant décisif dans l'étude du problème de Cauchy pour les équations de Navier-Stokes. Dans son article fondateur [40], il démontre l'existence globale en temps de solutions faibles pour le système de Navier-Stokes.

Théorème 1.6. *Soit $u_0\in L^2(\mathbb{R}^d)$ un champ de vecteurs à divergence nulle. Alors il existe une solution turbulente globale, e.g, définie pour tout temps $T\geqslant 0$.*

Remarque 1.7. Ce résultat est fondamental à plusieurs titres. D'une part parce que c'est un résultat d'existence globale de solutions aux équations de Navier-Stokes, sans condition particulière sur la donnée initiale u_0, si ce n'est d'être dans l'espace d'énergie L^2. D'autre part, la méthode de résolution élaborée par J. Leray est novatrice et féconde car pouvant être adaptée à bon nombre de cas.

Nous donnons ici les grandes lignes de la preuve : c'est une méthode par compacité, qui utilise de façon cruciale la conservation de l'énergie. Nous renvoyons le lecteur à l'article original [40].
• La première étape consiste à résoudre globalement un système approché de NS. Plus précisément, on régularise le terme convectif $u\cdot\nabla u$ par convolution, de telle sorte que le système original 1.1 de NS est approché par un système pour lequel on démontre facilement l'existence d'une suite de solutions régulières et globales.
• Ensuite, on établit des estimations a priori sur la suite de solutions approchées. Des arguments de

compacité viennent compléter la preuve.
• Il s'agit alors de passer à la limite dans le système approché. Si les termes linéaires ne posent pas de problème, il n'en va pas de même pour les termes non linéaires. C'est d'ailleurs précisément ce genre de problème qui surgit dès lors qu'on a une équation aux dérivées partielles non linéaire.
• Une fois cette difficulté surmontée, il ne reste plus qu'à montrer que la solution limite est une solution faible, satisfaisant l'inégalité d'énergie.

Remarque 1.8. Si la question de l'existence globale de solutions turbulentes est résolue par J. Leray, celle de l'unicité n'est pas si claire. J. Leray prouve l'unicité de solutions turbulentes en dimension 2 d'espace, mais en dimension 3, cela reste à ce jour une question ouverte majeure. Nous y reviendrons dans le chapitre suivant.

Unicité fort-faible

Si l'unicité en dimension 3 reste en suspens, il y a cependant un résultat partiel d'unicité, dit unicité fort-faible. Voici ce dont il s'agit.
Soit u et v deux solutions turbulentes associées à une même donnée initiale $u_0 \in L^2$. Il est naturel de considérer la différence $w \stackrel{\text{def}}{=} u - v$ et de chercher l'équation satisfaite par w. De simples calculs montrent

$$\partial_t w + w \cdot \nabla w \; - \Delta w + \; u \cdot \nabla w \; + w \cdot \nabla u \;\; = \;\; -\nabla q,$$

où q désigne la différence des deux pressions associées à chacune des solutions turbulentes u et v.
Il s'ensuit (au moins formellement), en vertu de la condition de divergence nulle sur w

$$\frac{1}{2}||w(t)||_{L^2}^2 + \frac{1}{2}||u(t)||_{L^2}^2 + \int_0^t \; ||\nabla u(t)||_{L^2}^2 = \frac{1}{2}||w_0||_{L^2}^2 \; + \Big| \int_0^t (w \cdot \nabla u | w)_{L^2} \Big| dt'. \tag{1.7}$$

Par des estimations standards et la condition de divergence nulle, on obtient :

$$\begin{aligned} \frac{1}{2}||w(t)||_{L^2}^2 + \int_0^t \; ||\nabla w(t')||_{L^2}^2 &\leqslant \frac{1}{2}||w_0||_{L^2}^2 \; + \int_0^t \; ||w(t')||_{L^2} \, ||\nabla w(t')||_{L^2} \, ||u(t')||_{L^\infty} dt' \\ &\leqslant \frac{1}{2}||w_0||_{L^2}^2 \; + \frac{1}{2} \int_0^t \; ||\nabla w(t')||_{L^2}^2 + C \int_0^t \; ||w(t')||_{L^2}^2 \, ||u(t')||_{L^\infty}^2 dt'. \end{aligned} \tag{1.8}$$

On en déduit alors grâce au lemme de Gronwall :

$$||w(t)||_{L^2}^2 + \int_0^t \; ||\nabla w(t')||_{L^2}^2 \leqslant ||w_0||_{L^2}^2 \; \exp(C \int_0^t \; ||u(t')||_{L^\infty}^2 dt). \tag{1.9}$$

Comme par hypothèse $w_0 = 0$, on a bien $w(t) = 0$, sous réserve que la solution turbulente u appartienne à l'espace $L^2(\mathbb{R}^+, L^\infty)$, ce qui n'est pas le cas puisque d'après le théorème de J. Leray une solution turbulente u appartient à l'espace $L^2(\mathbb{R}^+, \dot{H}^1)$. Ainsi, si l'on ajoute une hypothèse plus *forte* à une solution faible, alors toutes les solutions faibles coïncident avec celle-là. C'est ce qu'on appelle l'unicité fort-faible.

De ce résultat d'unicité fort-faible, surgit une idée nouvelle : une façon d'aborder le problème redoutable de l'unicité en dimension 3 consiste à exiger davantage de régularité sur la donnée initiale. On a alors bon espoir de démontrer l'unicité, mais le revers de la médaille est que l'on perd le caractère global de la solution turbulente si l'on ne fait pas une hypothèse supplémentaire de petitesse sur la donnée initiale. C'est ce qu'a démontré J. Leray en 1934 dans [40] où il définit le concept *solutions semi-régulières*.

Théorème 1.9. *Soit u_0 un champ à divergence nulle dans L^2 tel que ∇u_0 appartienne aussi à L^2. Alors il existe un temps $T > 0$ et une unique solution $u = NS(u_0)$ dans l'espace $\mathcal{C}^0([0,T], \dot{H}^1) \cap L^2([0,T], \dot{H}^2)$. De plus, il existe une constante $c_1 > 0$ tel que si la condition de petitesse suivante est satisfaite*

$$||u_0||_{L^2} \, ||\nabla u_0||_{L^2} \leqslant c_1,$$

alors T peut être choisi égal à $+\infty$.

Le lecteur attentif aura noté que la quantité sur laquelle porte la condition de petitesse est invariante sous la transformation $u \mapsto u_\lambda$. Le théorème de J. Leray marque un tournant dans la compréhension et la progression du caractère globalement bien posé des équations de Navier-Stokes. Le rôle joué par l'invariance d'échelle apparaît alors clairement : en considérant des données initiales dans des espaces invariants d'échelle, on démontre l'existence et l'unicité locale en temps des solutions. Pour gagner le caractère global des solutions, il faut exiger une condition de petitesse, elle-même invariante d'échelle : ceci prouve alors que le problème est globalement bien posé au sens d'Hadamard. Nombreux sont les mathématiciens à s'être engouffrés dans la brèche ouverte par les travaux pionniers de J. Leray. Le premier travail célèbre qui a mis en oeuvre cette idée est le théorème de H. Fujita et T. Kato [21] en 1964. Ils démontrent l'existence et l'unicité locale en temps des solutions fortes tridimensionnelles dans l'espace "critique" $\dot{H}^{\frac{1}{2}}$. En l'absence de force extérieure et lorsque le fluide se trouve à l'instant initial dans un état proche de celui du repos, la solution forte est alors globale en temps.
L'outil clé de la démonstration est un argument de contraction (ou encore appelé théorème de point fixe de Picard), dont nous rappelons l'énoncé ci-dessous.

Lemme 1.10. *Soit X un espace de Banach, $\mathcal{B}$ une application bilinéaire continue de $X \times X$ dans X, et $\alpha > 0$ tels que*

$$\alpha < \frac{1}{4\,||\mathcal{B}||} \quad \textit{avec} \quad ||\mathcal{B}|| \stackrel{\text{def}}{=} \sup_{||u||,||v|| \leqslant 1} ||\mathcal{B}(u,v)||. \tag{1.10}$$

Alors pour tout x_0 dans la boule ouverte $\mathcal{B}(0,\alpha)$ dans E, il existe une unique solution x dans la boule ouverte $\mathcal{B}(0, 2\,\alpha)$, tel que

$$x = x_0 + \mathcal{B}(u,v).$$

Afin d'appliquer ce lemme dans le contexte d'espaces invariants d'échelle, il convient d'écrire la solution de l'équation de Navier-Stoke sous forme intégrale (encore appelée formule de Duhamel)

$$u(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-t')\Delta}\, \mathbb{P}(\operatorname{div}(u \otimes u)\, dt' = e^{t\Delta} u_0 + \mathcal{B}(u,u)(t), \tag{1.11}$$

où $\mathbb{P}$ désigne le projecteur de Leray sur les champs à divergence nulle défini par

$$\mathbb{P} \stackrel{\text{def}}{=} Id - \nabla \Delta^{-1} \operatorname{div}.$$

Remarque 1.11. Tous les théorèmes d'existence locale de solutions (et globale à donnée petite dans un espace critique) qui vont suivre se démontrent en utilisant le théorème de point fixe énoncé ci-dessus. Nous commençons par le plus petit espace critique, $\dot{H}^{\frac{1}{2}}$, et nous irons jusqu'au plus grand espace possible invariant d'échelle en dimension 3. Si ces résultats marquent une avancée majeure dans l'étude des équations de Navier-Stokes, il n'en reste pas moins qu'ils n'utilisent pas vraiment la structure de l'équation (la conservation d'énergie, par exemple), contrairement au théorème d'existence de solutions turbulentes de J. Leray.

Remarque 1.12. Désormais, nous appellerons solution forte, toute solution construite par un argument de point fixe, par opposition à solution faible, construite à partir de la conservation de l'énergie et par un argument de compacité.

1.2.2 Théorème de H. Fujita, T. Kato et ses conséquences

Les énoncés de cette section (sauf mention contraire) ne seront pas démontrés. Nombreuses sont les références pour ce théorème qui marque une grande avancée dans la compréhension des équations de Navier-Stokes. Nous renvoyons le lecteur à l'article original de H. Fujita et T. Kato [21], ainsi qu'au chapitre 5 de [1] et au chapitre 15 de [39].

Théorème 1.13. *Soit $u_0 \in \dot{H}^{\frac{1}{2}}$ un champ de vecteurs à divergence nulle. Il existe un unique temps de vie maximal $T_*(u_0)$ et pour tout $T < T_*(u_0)$, une unique solution de Navier-Stokes dans l'espace*

$$C([0,T], \dot{H}^{\frac{1}{2}}) \cap L^2([0,T], \dot{H}^{\frac{3}{2}}).$$

De plus, on a les propriétés suivantes

– *Si $T_*(u_0) < \infty$, alors*

$$\lim_{T \to T_*(u_0)} \int_0^T \|u(t')\|_{\dot{H}^1}^4 \, dt' = +\infty. \tag{1.12}$$

– *Il existe une constante $c > 0$ telle que*

$$\|u_0\|_{\dot{H}^{\frac{1}{2}}} \leqslant c \Longrightarrow T_*(u_0) = +\infty.$$

Enfin, les solutions sont stables au sens suivant : soient u et v deux NS-solutions, alors

$$\|u(t) - v(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \int_0^t \|u(t') - v(t')\|_{\dot{H}^{\frac{3}{2}}}^2 \, dt' \leqslant \|u_0 - v_0\|_{\dot{H}^{\frac{1}{2}}}^2 \exp\left(C_0 \int_0^t (\|u(t')\|_{\dot{H}^1}^4 + \|v(t')\|_{\dot{H}^1}^4) dt' \right).$$

Remarque 1.14. Ce résultat est remarquable car c'est le premier, après le théorème de J. Leray, qui démontre le caractère localement bien posé des équations de Navier-Stokes en dimension 3 (et globalement bien posé à donnée petite). La preuve de ce célèbre théorème utilise la méthode de point fixe de Picard dont nous avons rappelé la procédure itérative. Notons qu'un sous-produit immédiat est *la théorie des solutions petites* : si la donnée initiale est petite, la solution déduite d'une telle donnée reste petite au sens suivant

$$\|u_0\|_{\dot{H}^{\frac{1}{2}}} \leqslant c \Longrightarrow \|u(t)\|_{\dot{H}^{\frac{1}{2}}} \leqslant 2\,c, \quad \text{pour tout temps} \quad t. \tag{1.13}$$

En fait, on a un peu mieux que cela, comme le montre la proposition ci-dessous.

Proposition 1.15. *Si la donnée initiale u_0 est dans la boule ouverte de centre 0 et de rayon c dans l'espace $\dot{H}^{\frac{1}{2}}$, alors la fonction*

$$t \mapsto \|NS(u_0)(t)\|_{\dot{H}^{\frac{1}{2}}}$$

est décroissante.

Quelques mots à propos de la preuve du théorème de H. Fujita T. Kato. Comme annoncé en préambule de cette partie, la démonstration d'origine utilise le théorème de point fixe de Picard. Il s'agit donc de montrer que l'application bilinéaire $\mathcal{B}$ est continue de $L_T^4(\dot{H}^1) \times L_T^4(\dot{H}^1)$ dans $L_T^4(\dot{H}^1)$. Cela repose sur des lois de produit dans les espaces de Sobolev et les inclusions de Sobolev.

Étudions une propriété qualitative des solutions globales de Navier-Stokes en temps grand, sans hypothèse de taille sur la donnée initiale. Celle-ci utilise à la fois la théorie de Fujita-Kato et la régularité en temps grand des "solutions à la Leray". Plus précisément, I. Gallagher, D. Iftimie, et F. Planchon ont démontré dans [23] que toute solution globale de NS, associée à une donnée initiale $u_0 \in \dot{H}^{\frac{1}{2}}$, tend vers 0 en temps grand, et ceci quelle que soit la taille de la donnée initiale.

Bien sûr, dans le cas d'une petite donnée initiale, le résultat est clair puisqu'alors on montre sans trop de difficulté que le comportement asymptotique de la solution de NS est dicté par le flot linéaire de la chaleur, qui tend évidemment vers 0 quand t tend vers $+\infty$. Le fait remarquable est que cela se produit également lorsque la donnée initiale est grande. Voici le théorème.

Théorème 1.16. *Soit u_0 un champ à divergence nulle dans $\dot{H}^{\frac{1}{2}}$ tel que la solution de Navier-Stokes $NS(u_0)$ donnée par le théorème 1.13 est globale. Alors,*

$$\lim_{t\to+\infty} \|NS(u_0)(t)\|_{\dot{H}^{\frac{1}{2}}} = 0 \quad et \quad \int_0^{+\infty} \|NS(u_0)(t)\|^4_{\dot{H}^1} < \infty.$$

Nous donnons ici la preuve de ce théorème, tant les idées mises en oeuvre sont simples mais redoutables d'efficacité.

Démonstration. La démonstration du théorème repose sur deux observations.
(1) si la donnée initiale est suffisamment petite, la solution est globale et par décroissance de la solution en norme $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, on obtient le résultat.
(2) si la donnée initiale u_0 appartient à $L^2(\mathbb{R}^3)$, alors la solution est une solution turbulente (au sens de Leray) donc appartient à l'espace $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3))$. En particulier, par interpolation, la solution appartient à l'espace $L^4(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$. Ainsi, pour tout $\varepsilon > 0$, il existe un temps $T > 0$ tel que $\|u(T)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leqslant \varepsilon$. on est alors ramené à la théorie des solutions petites, ce qui termine la preuve dans le cas où la donnée initiale appartient à $L^2(\mathbb{R}^3)$.
– Dans le cas général où la donnée initiale n'est ni petite ni dans $L^2(\mathbb{R}^3)$, on utilise les deux observations précédentes, en découpant la donnée u_0 en hautes et basses fréquences :
Soit $\rho > 0$ fixé, tel que :

$$u_0 = u_{0,h} + u_{0,l} \quad \text{avec} \quad u_{0,l} = \mathcal{F}^{-1}\left(1_{B(0,\rho)}(\xi)\widehat{u}_0(\xi)\right) \quad \text{et} \quad u_{0,h} \in L^2(\mathbb{R}^3).$$

L'intérêt d'un tel découpage est le suivant : dans l'espace $\dot{H}^{\frac{1}{2}}$, c'est la partie basses fréquences qui empêche la donnée u_0 d'appartenir à l'espace $L^2(\mathbb{R}^3)$, nous privant alors du résultat de la seconde observation. L'idée pour y remédier est de retirer à la donnée u_0, la partie basses fréquences, $u_{0,l}$, qu'on prendra petite en norme $\dot{H}^{\frac{1}{2}}$ de façon à lui appliquer la théorie de solutions petites. Ainsi, l'autre partie, (e.g la partie hautes fréquences $u_{0,h}$) tombe dans $L^2(\mathbb{R}^3)$, par Plancherel Parseval. On peut alors lui appliquer la seconde observation.
Pour la partie basses fréquences : il s'agit de fixer le seuil ρ à partir duquel on coupe en fréquences, de façon à rendre $u_{0,l}$ petite en norme $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$.
Soit $\varepsilon > 0$ fixé arbitrairement petit, on choisit ρ tel que :

$$\|u_{0,l}\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leqslant \min\left\{c, \frac{\varepsilon}{2}\right\}.$$

Soit u_l la solution de (NS) pour la donnée initiale $u_{0,l}$. Alors,

$$\forall t \in \mathbb{R}^+, \|u_l(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leqslant \frac{\varepsilon}{2}. \tag{1.14}$$

Quant à la partie hautes fréquences de u_0, on utilise la deuxième observation. On pose $u_h = u - u_l$. Ainsi définie, u_h est solution de l'équation de (NS) perturbée suivante

$$\left\{\begin{array}{ccc} \partial_t u_h + u_h.\nabla u_h - \Delta u_h + u_h \cdot \nabla u_l + u_l.\nabla u_h & = & -\nabla p \\ \text{div } u_h & = & 0 \\ u_{h|t=0} & = & u_{0,h}. \end{array}\right. \tag{1.15}$$

Nous n'avons pas d'hypothèse de petitesse sur la donnée initiale $u_{0,h}$. En revanche, nous avons montré qu'elle est dans $L^2(\mathbb{R}^3)$ et nous avons donc envie d'utiliser le résultat de la seconde observation. Il faut donc montrer que la solution u_h appartient à l'espace $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3))$.
On procède à une estimation d'énergie L^2 sur u_h en prenant le produit scalaire L^2 de l'équation *(1.15)*

avec u_h.
La condition de divergence nulle implique que les termes $(u_l.\nabla u_h|u_h)_{L^2(\mathbb{R}^3)}$ et $-(\nabla p|u_h)_{L^2(\mathbb{R}^d)}$ sont nuls ; l'estimation d'énergie L^2 après intégration en temps s'écrit donc

$$\frac{1}{2}\|u_h(t)\|^2_{L^2(\mathbb{R}^3)} \;+\; \int_0^t \|\nabla u_h(s)\|^2_{L^2(\mathbb{R}^3)} \;= \frac{1}{2}\|u_{h,0}\|^2_{L^2(\mathbb{R}^3)} \;-\int_0^t (u_h.\nabla u_l|u_h)_{L^2(\mathbb{R}^3)}dt'.$$

Il s'ensuit que

$$\frac{1}{2}\|u_h(t)\|^2_{L^2(\mathbb{R}^3)} \;+\; \int_0^t \|\nabla u_h(t')\|^2_{L^2(\mathbb{R}^3)}dt' \;\leqslant\; \frac{1}{2}\|u_{0,h}\|^2_{L^2(\mathbb{R}^3)} \;+\left|\int_0^t (u_h.\nabla u_l|u_h)_{L^2(\mathbb{R}^3)}dt'\right|.$$

Comme u_h et u_l sont à divergence nulle, $u_h\cdot\nabla u_l = \mathrm{div}(u_h\otimes u_l)$, puis par intégration par parties

$$\begin{aligned}\left|\int_0^t (u_h.\nabla u_l|u_h)_{L^2(\mathbb{R}^3)}dt'\right| &= \left|\int_0^t\int_{\mathbb{R}^3}(u_h\cdot\nabla u_l)\,.u_h\,dx\,dt'\right|\\ &= \left|\int_0^t\int_{\mathbb{R}^3}(u_h\otimes u_l)\,.\nabla u_h\,dx\,dt'\right|.\end{aligned} \tag{1.16}$$

Les inégalités de Cauchy-Schwarz et d'Hölder impliquent

$$\begin{aligned}\left|\int_0^t (u_h.\nabla u_l|u_h)_{L^2(\mathbb{R}^3)}dt'\right| &\leqslant \int_0^t \|u_h\otimes u_l\|_{L^2(\mathbb{R}^3)}\|\nabla u_h\|_{L^2(\mathbb{R}^3)}\,dt'\\ &\leqslant \int_0^t \|u_h\|_{L^6(\mathbb{R}^3)}\|u_l\|_{L^3(\mathbb{R}^3)}\|\nabla u_h\|_{L^2(\mathbb{R}^3)}\,dt'.\end{aligned} \tag{1.17}$$

Enfin, par inclusion de Sobolev, on en déduit que

$$\left|\int_0^t (u_h.\nabla u_l|u_h)_{L^2(\mathbb{R}^3)}dt'\right| \leqslant C\int_0^t \|u_h\|^2_{\dot H^1(\mathbb{R}^3)}\|u_l\|_{\dot H^{\frac12}(\mathbb{R}^3)}\,dt'.$$

L'inégalité d'énergie s'écrit donc

$$\|u_h(t)\|^2_{L^2(\mathbb{R}^3)} \;+\; 2\int_0^t \|\nabla u_h(t')\|^2_{L^2(\mathbb{R}^3)}dt' \;\leqslant\; \|u_{0,h}\|^2_{L^2(\mathbb{R}^3)} \;+C\varepsilon\int_0^t \|u_h(t')\|^2_{\dot H^1(\mathbb{R}^3)}\,dt'.$$

En choisissant ε assez petit ($C\varepsilon\leqslant 1$), on obtient :

$$\|u_h(t)\|^2_{L^2(\mathbb{R}^3)} \;+\; \int_0^t \|\nabla u_h(t')\|^2_{L^2(\mathbb{R}^3)}dt' \;\leqslant\; \|u_{0,h}\|^2_{L^2(\mathbb{R}^3)}.$$

Ceci prouve que u_h appartient à l'espace $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3))\cap L^2(\mathbb{R}^+,\dot H^1(\mathbb{R}^3))$. D'où par interpolation, u_h appartient à l'espace $L^4(\mathbb{R}^+,\dot H^{\frac12}(\mathbb{R}^3))$. Ainsi, il existe un temps $t_\varepsilon>0$ tel que $\|u_h(t_\varepsilon)\|_{\dot H^{\frac12}(\mathbb{R}^3)}\leqslant\frac{\varepsilon}{2}$. On a donc, $\|u(t_\varepsilon)\|_{\dot H^{\frac12}(\mathbb{R}^3)}\leqslant\varepsilon$. Enfin, la solution u étant globale par hypothèse, le résultat en découle par la théorie des solutions petites. □

Remarque 1.17. Le lecteur aura noté un point important : ce théorème utilise de façon cruciale la structure de l'équation de Navier-Stokes, puisqu'on a recours à la conservation de l'énergie L^2.

Corollaire 1.18. *On appelle $\mathcal{G}$ l'ensemble des données initiales $u_0\in\dot H^{\frac12}$ telles que les solutions données par le théorème 1.13 sont globales. Alors l'ensemble $\mathcal{G}$ est un ouvert connexe de l'espace $\dot H^{\frac12}$.*

Remarque 1.19. Noter que la connexité d'un tel ensemble est aisée à démontrer, en vertu du théorème 1.16. En effet, comme toutes les solutions appartenant à $\mathcal{G}$ sont globales et donc tendent vers 0 en norme $\dot H^{\frac12}$ en temps grand. Elle sont donc toutes reliées à 0.

1.2.3 Théorèmes de T. Kato, de F. Weissler et de M. Cannone-Y. Meyer-F. Planchon

Le théorème de H. Fujita et T. Kato a ensuite été étendu par T. Kato [31] au cas de données initiales dans l'espace plus large L^3, qui est bien un espace invariant d'échelle en dimension 3. Voici l'énoncé.

Théorème 1.20. *Soit $u_0 \in L^3$ un champ de vecteurs à divergence nulle et $3 \leqslant p < +\infty$. Il existe un unique temps de vie maximal $T_*(u_0)$ et pour tout $T < T_*(u_0)$, une unique solution de Navier-Stokes dans l'espace K_T où*

$$K_T \stackrel{\text{def}}{=} \Big\{ C(]0,T], L^3) \ |\|u\|_{K_T} \stackrel{\text{def}}{=} \sup_{0<t<T} t^{\frac{1}{2}(1-\frac{3}{p})} \|u\|_{L^p} < \infty \Big\}.$$

De plus, il existe une constante $c > 0$ telle que

$$\|u_0\|_{L^3} \leqslant c \Longrightarrow T_*(u_0) = +\infty.$$

Remarque 1.21. La preuve de ce théorème repose à nouveau sur le théorème de point fixe de Picard, dans un espace bien choisi. Noter que l'idée naturelle d'utiliser un argument de contraction dans l'espace $L^\infty(]0,T], L^3)$ ne marche pas ici. En effet, F. Oru a prouvé dans [45] que la forme bilinéaire $\mathcal{B}$ n'était pas continue de $L^\infty(]0,T], L^3) \times L^\infty(]0,T], L^3)$ dans $L^\infty(]0,T], L^3)$. Néanmoins, l'effet régularisant du noyau de la chaleur permet de contourner cette difficulté : puisque nous pouvons alors définir les espaces K_T, dits espaces de Kato, dans lesquels la méthode de point fixe s'applique. La preuve du théorème 1.20 repose sur les deux lemmes suivants. Nous renvoyons le lecteur à nouveau au livre [1] pour les preuves.

Lemme 1.22. *Il existe une constante $C > 0$ telle que pour toute donnée initiale $u_0 \in L^3$, alors (avec les notations du Théorème 1.20)*

$$\text{pour tout} \quad T > 0, \quad \|e^{t\Delta} u_0\|_{K_T} \leqslant C \|u_0\|_{L^3}, \tag{1.18}$$

$$\lim_{T \to 0} \|e^{t\Delta} u_0\|_{K_T} = 0. \tag{1.19}$$

Lemme 1.23. *Il existe une constante $C > 0$ telle que*

$$\text{pour tout} \quad T > 0, \quad \|\mathcal{B}(u,u)\|_{K_T} \leqslant C \|u\|_{K_T} \|u\|_{K_T}. \tag{1.20}$$

On continue d'étendre ce type d'énoncé d'existence locale de solution (et globale à donnée petite) à des espaces de plus en plus gros. Notons que l'on a la chaîne d'inclusions critiques suivante :

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)_{3\leqslant p<\infty} \hookrightarrow \mathcal{BMO}^{-1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3).$$

Remarque 1.24. Il est important de noter ici que ce qui différencie l'espace de Sobolev $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ de l'espace de Besov $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)_{3\leqslant p<\infty}$ (et même plus généralement avec $1 \leqslant p < \infty$) , ce sont les fonctions homogènes de degré -1, typiquement la fonction $x \mapsto \dfrac{1}{|x|}$. On peut démontrer que cette fonction n'appartient à aucun espace de Lebesgue $L^p(\mathbb{R}^3)$, ni aucun espace de Sobolev . En revanche, elle appartient à des espaces plus exotiques, comme l'espace de Besov critique $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)_{1\leqslant p<\infty}$. Cela nous amène à définir les solutions auto-similaires pour les équations de Navier-Stokes.

Definition 1.25. *On dit que u est une solution auto-similaire pour les équations de Navier-Stokes si pour tout $\lambda > 0$, on a*

$$u(t,x) \stackrel{\text{def}}{=} \lambda\, u(\lambda^2\, t, \lambda\, x).$$

En d'autres termes si,

$$u(t,x) = \frac{1}{\sqrt{t}}\, U\Big(\frac{x}{\sqrt{t}}\Big), \quad \text{où} \quad U = (U_1, U_2, U_3) \quad \textit{est un champ de vecteurs à divergence nulle.}$$

Un calcul simple montre que si u est solution de Navier-Stokes, alors U est solution de l'équation suivante

$$-\Delta U - \frac{U}{2} - \frac{1}{2}\, x\cdot \nabla U + U\cdot \nabla U = -\nabla q, \quad \text{où q est une fonction scalaire} \quad \text{et} \quad \operatorname{div} U = 0. \tag{1.21}$$

On en déduit alors que si u est une solution auto-similaire pour les équations de Navier-Stokes et si $\lim_{t\to 0} u(t,x) = u_0(x)$ existe alors nécessairement la donnée initiale u_0 est homogène de degré -1. Ceci résulte du fait que

$$u_0(x) = u(0,x) = \lambda\, u(0, \lambda x) = \lambda\, u_0(\lambda x). \tag{1.22}$$

Ainsi, pour espérer attraper des solutions auto-similaires pour les équations de Navier-Stokes, tout le jeu est de construire des espaces contenant les fonctions homogènes de degré -1. L'idée du théorème de M. Cannone-Y. Meyer-F. Planchon [11] est la suivante. Dans un tel espace, si la donnée initiale u_0 est homogène de degré -1 et à divergence nulle, alors il existe une unique solution auto-similaire pour les équations de Navier-Stokes, continue de $[0, +\infty[$ sur cet espace.

Ici, il est donc naturel de considérer l'espace de Besov critique $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ avec $3 \leqslant p < \infty$. La méthode de point fixe fonctionne à nouveau dans ce cadre-là, généralisant ainsi la méthode au cas des espaces de Besov d'indice de régularité négatif. Nous renvoyons le lecteur aux articles de M. Cannone [10] (qui traite le cas $3 < p \leqslant 6$) et de F. Planchon [47] (pour touts les $p \geqslant 3$). Nous en donnons ici une version légèrement différente (voir [14]).

Théorème 1.26. *Soit $u_0 \in \dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ un champ de vecteurs à divergence nulle et $3 \leqslant p < \infty$. Il existe une unique solution de Navier-Stokes dans l'espace E_T où*

$$E_T \stackrel{\text{def}}{=} \tilde{L}^\infty([0,T], \dot{B}^{-1+\frac{3}{p}}_{p,\infty}) \cap \tilde{L}^1([0,T], \dot{B}^{1+\frac{3}{p}}_{p,\infty}),$$

et les espaces $\tilde{L}^\rho([0,T], \dot{B}^s_{p,r})$, dits espaces de Chemin-Lerner, sont définis par

$$\|u\|_{\tilde{L}^\rho\left([0,T], \dot{B}^s_{p,r}\right)} \stackrel{\text{def}}{=} \Big\| 2^{js}\, \|\dot{\Delta}_j u\|_{L^\rho\left([0,T], L^p\right)} \Big\|_{\ell^r(\mathbb{Z})}.$$

De plus, il existe une constante $c > 0$ telle que

$$\|u_0\|_{\dot{B}^{-1+\frac{3}{p}}_{p,\infty}} \leqslant c \Longrightarrow T_*(u_0) = +\infty.$$

Enfin, si de plus la donnée initiale u_0 est homogène de degré -1, alors la solution est auto-similaire.

Remarque 1.27. Il est légitime de s'interroger sur l'importance de la condition de petitesse de la donnée initiale pour espérer gagner le caractère global en temps des solutions. En effet, jusqu'à présent, nous avons mis en évidence le fait suivant : seules les données initiales suffisamment petites sont aptes à générer des solutions globales. Le théorème de M. Cannone-Y. Meyer-F. Planchon éclaire cette question. Il est en effet possible de construire des solutions fortes globales en temps, à données grandes en norme $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ ou $L^3(\mathbb{R}^3)$, mais qui deviennent petites en norme de Besov $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$. Il suffit pour cela de faire fortement osciller la donnée initiale, comme l'illustre l'exemple suivant

$$u_0^\varepsilon \stackrel{\text{def}}{=} \frac{1}{\varepsilon^\alpha} \sin\Big(\frac{x_3}{\varepsilon}\Big) (-\partial_2\phi(x), \partial_1\phi(x), 0), \quad \text{où} \quad 0 < \alpha < 1 \quad \text{et} \quad \phi \in \mathcal{S}(\mathbb{R}^3; \mathbb{R}). \tag{1.23}$$

On peut vérifier que cette donnée initiale est de taille 1 en norme $L^3(\mathbb{R}^3)$ et très petite en norme de Besov si ε est petit et $p > 3$.

1.2.4 Théorème de H. Koch et D. Tataru

Plus récemment [37], H. Koch et D. Tataru ont obtenu l'existence d'une unique solution globale en temps pour l'équation de Navier-Stokes pour des données initiales dans un espace encore plus gros : l'espace $\mathcal{BMO}^{-1}$.

Définition 1.28. *On appelle $\mathcal{BMO}^{-1}$ l'espace des distributions tempérées u telles que*

$$\|u\|^2_{\mathcal{BMO}^{-1}} \stackrel{\text{def}}{=} \sup_{t\geqslant 0} t\,\|e^{t\Delta}u_0\|^2_{L^\infty} + \sup_{x\in\mathbb{R}^3, R>0} \frac{1}{R^3} \int_{P(x,R)} |(e^{t\Delta}u_0)(t,y)|^2\,dy\,dt,$$

avec $P(x,R)$ est l'ensemble $[0,R^2] \times B(x,R)$, avec $B(x,R)$ la boule ouverte de centre x et de rayon R. Disons quelques mots à propos de cette norme : la façon la plus naturelle de donner un sens au terme convectif $u\cdot\nabla u$ consiste à l'écrire sous la forme $\nabla(u\otimes u)$ et de demander que u soit L^2_{loc}. Maintenant, l'insensibilité des équations de Navier-Stokes aux changements d'échelle et aux translations conduit à considérer une norme L^2_{loc} dilatée et translatée, qui prend la forme suivante

$$\sup_{x\in\mathbb{R}^3, R>0} \Big(\frac{1}{R^3} \int_{P(x,R)} |u(t,y)|^2\,dy\,dt\Big)^{\frac{1}{2}}.$$

Parce qu'on a mis les hypothèses minimum pour donner un sens au terme non linéaire de l'équation, cette norme apparaît comme optimale, ce qui justifie (intuitivement) que l'espace $\mathcal{BMO}^{-1}$ est en fait optimal pour la mise en oeuvre de la procédure itérative de Picard.
Moralement l'espace $\mathcal{BMO}^{-1}$ est l'espace de distributions qui sont des dérivées de fonctions appartenant à $\mathcal{BMO}$, où

$$\|f\|_{\mathcal{BMO}} \stackrel{\text{def}}{=} \sup_B \frac{1}{|B|} \int_B |f(x) - \bar{f}_B|\,dx, \quad \text{avec} \quad \bar{f}_B = \frac{1}{|B|} \int_B f(x)\,dx,$$

le supremum est pris sur toutes les boules ouvertes de $\mathbb{R}^3$.

Théorème 1.29. *Il existe une constante c telle que si la donnée initiale u_0 est à divergence nulle, appartient à l'espace $\mathcal{BMO}^{-1}$ et satisfait $\|u_0\|_{\mathcal{BMO}^{-1}} \leqslant c$, alors il existe une unique solution globale de Navier-Stokes dans l'espace X défini par*

$$\|u\|_X \stackrel{\text{def}}{=} \sup_{t\geqslant 0} t^{\frac{1}{2}}\,\|u\|_{L^\infty} + \sup_{x\in\mathbb{R}^3, R>0} \frac{1}{R^3} \int_{P(x,R)} |u(t,y)|^2\,dy\,dt.$$

De plus, la solution satisfait $\|u\|_X \leqslant 2\,\|u_0\|_{\mathcal{BMO}^{-1}}$.

Nous terminons cette récollection de résultats par une question qu'il est naturel de se poser, sachant qu'on a noté le rôle primordial joué par l'invariance d'échelle des espaces considérés dans la méthode itérative de Picard : y-a-t-il un espace limite au-delà duquel cette méthode cesse de fonctionner ? La réponse, positive, a été donnée par Y. Meyer [44]. L'espace $\dot{B}^{-1}_{\infty,\infty}$ est l'espace limite. C'est l'objet du lemme ci-dessous.

Lemme 1.30. *Soit B un espace de Banach continûment inclus dans $\mathcal{S}(\mathbb{R}^d)$ et invariant par translation et dilatation au sens suivant*

$$\forall f \in B, \quad \forall \lambda > 0, \quad \forall a \in R^d, \quad \|f\|_X = \lambda \, \|f(\lambda x - a)\|_X,$$

Alors, B est continûment inclus dans $\dot{B}^{-1}_{\infty,\infty}$, où

$$\|f\|_{\dot{B}^{-1}_{\infty,\infty}} \stackrel{\text{def}}{=} \sup_{t \geqslant 0} t^{\frac{1}{2}} \, \|e^{t\Delta} u_0\|_{L^\infty}.$$

Remarque 1.31. Il a été démontré que dans le dernier espace invariant d'échelle $\dot{B}^{-1}_{\infty,\infty}$, les équations de Navier-Stokes sont mal posées (voir [7] et [28]). L'espace $\mathcal{BMO}^{-1}$ est ainsi le meilleur espace connu dans lequel la méthode de point fixe s'applique et donc dans lequel on a existence et unicité locale en temps de solutions de NS (globale à donnée petite).

1.3 Quelques critères d'explosion

Nous avons abordé dans la section précédente le problème de l'existence et l'unicité locale ou globale de solutions pour le système de Navier-Stokes incompressible. Les travaux révolutionnaires de J. Leray [40] ont montré l'existence de solutions globales turbulentes, i.e très peu régulières. Le problème de leur régularité et de leur unicité peut ainsi être énoncé en ces termes : partant d'une donnée initiale régulière (pas de changement de vitesse trop brusque, par exemple), la solution qui en découle, hérite-t-elle de cette régularité *ad vitam eternam* (on parle alors de régularité globale) ou au contraire, cesse-t-elle d'être régulière à un instant T_* fini (on parle alors de singularité, ou encore de solution explosive, comme nous le verrons plus loin) ? Cette question est toujours d'actualité, et les enjeux sont élevés puisque ce problème dit problème de la régularité globale en dimension 3, fait partie des problèmes du Millénaire, mis à prix par la fondation Clay.

Nombreuses ont été les tentatives d'approche du problème, mais la question continue de hanter l'esprit de bon nombre de mathématiciens. J. Leray a démontré l'existence et l'unicité globale de solutions turbulentes en dimension 2. En revanche, en dimension 3, si l'existence globale est acquise, la question de l'unicité reste un problème ouvert.

Comme nous l'avons noté dans la section précédente, une stratégie payante pour attaquer cette question d'unicité en dimension 3, consiste à choisir des données initiales dans des espaces invariants d'échelle. Cela garantira l'existence et l'unicité locale en temps de solutions. Sous réserve d'ajouter une condition de petitesse sur la donnée initiale (condition qui, bien sûr, aura le bon goût d'être invariante d'échelle), alors le problème devient globalement bien posé. En d'autres termes, les équations de Navier-Stokes tridimensionnelles sont globalement bien posées lorsque l'état initial est proche de son état de repos. En dehors de ce sentier battu par de nombreux mathématiciens (de H. Fujita- T. Kato à H. Koch-D. Tataru, par exemple), les pistes de réflexion ne sont pas claires.

Ainsi, faute de réponse, il faut choisir son camp : régularité globale versus solution singulière. J. Leray avait conjecturé une rupture de régularité (ce qui justifie le concept de solutions turbulentes), fournissant même *des critères d'explosion*. Voilà ce qu'il en est : il existe une constante $C > 0$ telle que

$$\text{Si} \quad T_* < \infty, \quad \text{alors} \quad \forall t < T_*, \quad \|\nabla u(t)\|_{L^2} \geqslant \frac{C}{(T_* - t)^{\frac{1}{4}}}. \tag{1.24}$$

$$\text{Si} \quad T_* < \infty, \quad \text{alors} \quad \forall\, 3 < q < \infty, \quad \forall\, t < T_*, \quad \|u(t)\|_{L^q} \geqslant \frac{C_q}{(T_* - t)^{\frac{1}{2}(1-\frac{3}{q})}}. \tag{1.25}$$

De ces critères d'explosion de J. Leray, il en découle immédiatement les critères suivants, aujourd'hui communément appelés *critères de Serrin.*

$$\text{Si} \quad T_* < \infty, \quad \text{alors} \quad \lim_{T \to T_*(u_0)} \int_0^T \|u(t)\|_{L^q}^p \, dt = +\infty, \quad \text{avec} \quad \frac{2}{p} + \frac{3}{q} = 1 \quad \text{avec} \quad 3 < q \leqslant +\infty.$$

Le lecteur aura noté *"le rôle important de l'homogénéité des formules"* (sic). J. Leray souligne que les *"les équations aux dimensions permettent de prévoir a priori toutes les inégalités que nous écrivons"*(sic). Convaincu de la rupture de régularité des solutions de Navier-Stokes, J. Leray propose une méthode pour construire des singularités, aujourd'hui appelées "*profils d'explosion auto-similaire*". Ce sont des solutions de Navier-Stokes de la forme

$$u(t,x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{T_* - t}}\, U\Big(\frac{x}{\sqrt{T_* - t}}\Big), \quad \text{où} \quad U \quad \text{est un profil appartenant à} \quad L^2 \cap \dot{H}^1.$$

Un simple calcul montre, d'une part, que u satisfait les critères d'explosion énoncés ci-dessus (encore une fois, par argument d'échelle) et, d'autre part, que si u est solution de Navier-Stokes alors le profil U est solution de l'équation

$$-\Delta U + U + x\cdot \nabla U + U\cdot \nabla U = -\nabla q, \quad \text{où q est une fonction scalaire.} \tag{1.26}$$

J. Leray n'a *"malheureusement pas réussi à faire l'étude de ce système, laissant donc en suspens cette question de savoir si des singularités peuvent ou non se présenter"*. Une réponse négative à la conjecture de J.Leray a été apportée en 1996 par J. Nečas, M. Ružička, and V. Šverák dans [38], où ils démontrent que l'équation sur le profil U n'admet pas d'autre solution que la solution nulle, dans $L^3(\mathbb{R}^3)$. Ceci met un terme à toute tentative de construction de solutions singulières par cette méthode.

Dans la continuité du travail des trois auteurs précédents, apparaît alors une question naturelle : existe-t-il des singularités qui restent bornées en norme $L^3(\mathbb{R}^3)$? C'est au remarquable travail [20] de L. Escauriaza, G. Seregin et V. Šverák que l'on doit la réponse. Ils démontrent que toute solution faible de Leray-Hopf qui reste bornée en norme $L^3(\mathbb{R}^3)$ ne peut pas développer de singularités en temps fini, généralisant ainsi le résultat de [38].

Théorème 1.32 (ESS)**.** *Soit u_0 un champ de vecteurs de divergence nulle dans $L^3(\mathbb{R}^3)$, et u la solution associée (on parle de solution de Leray-Hopf).*

$$\textit{Si} \quad T_*(u_0) < \infty, \quad \textit{alors} \quad \limsup_{t \to T_*(u_0)} \|u(t)\|_{L^3} = +\infty.$$

Noter que ce théorème fondamental correspond au cas limite ($q = 3$) du critère d'explosion de Serrin. La preuve de ce théorème est difficile. Elle repose sur le concept de solutions faibles à la Cafarelli-Kohn-Nirenberg [9], sur des arguments de "zoom" autour de la singularité éventuelle, et l'unicité rétrograde.

Ce résultat a été réexaminé par par C. Kenig et G. Koch [32], pour des données dans l'espace $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, puis par I. Gallagher, G. Koch et F. Planchon [24] dans le contexte des solutions fortes à données dans l'espace plus grand $L^3(\mathbb{R}^3)$. Voici l'énoncé du théorème de C. Kenig et G. Koch.

Théorème 1.33 ([32])**.** *Soit u_0 un champ de vecteurs de divergence nulle dans $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, et u la solution (forte au sens de T. Kato) de Navier-Stokes associée.*

$$\textit{Si} \quad T_*(u_0) < \infty, \quad \textit{alors} \quad \lim_{t \to T_*(u_0)} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} = +\infty.$$

Détaillons la méthode adoptée dans les deux situations énoncées. Le point de vue est différent de celui de [20], puisqu'il met en oeuvre la méthode dite de "concentration-compacité" combinée à un théorème de rigidité (l'unicité rétrograde). Cette méthode (encore appelée *méthode des éléments critiques*) a été développée par C. Kenig et F. Merle pour traiter les équations dispersives critiques et hyperboliques comme l'équation de Schrödinger non linéaire à énergie critique [33]. Soulignons que l'outil clé de la démonstration de [24] est la théorie des profils, introduite par P. Gérard [26] pour étudier le défaut de compacité des inclusions de Sobolev. Disons quelques mots à propos de cette théorie que nous emploierons très largement par la suite.

La motivation originelle de la théorie des profils était la description du défaut de compacité dans les inclusions de Sobolev. Nous renvoyons le lecteur aux travaux pionniers de P.-L. Lions [41], [42] et ceux de H. Brezis et J.-M. Coron [8]. Ici dans [24] et dans les travaux qui vont suivre, c'est le théorème de P. Gérard [26] qui est adopté. Ce résultat remarquable donne, après un certain nombre d'extractions, la structure d'une suite bornée dans l'espace de Sobolev $\dot{H}^s$, avec $0 < s < \frac{3}{2}\cdot$ Plus précisément, le défaut de compacité de l'inclusion de Sobolev $\dot{H}^s \hookrightarrow L^p$ est décrit en termes d'une somme de profils orthogonaux, dilatés par des échelles, translatés par des coeurs, à un terme de reste petit dans L^p près. Ce théorème a été généralisé par la suite à de nombreuses autres situations. Citons par exemple le travail de H. Bahouri, A. Cohen et G. Koch [2], qui généralisent le résultat au cas des inclusions critiques. La théorie des profils s'avère un outil redoutable d'efficacité dans l'étude des problèmes d'évolution, comme par exemple l'étude haute fréquence des solutions d'énergie finie pour l'équation des ondes quintique dans $\mathbb{R}^3$, par H. Bahouri et P. Gérard [4], ou encore pour l'équation critique de Schrödinger non linéaire en dimension 2 (voir [43]) Concernant les équations de Navier-Stokes, c'est I. Gallagher qui a développé avec succès la décomposition en profils pour des solutions de Navier-Stokes à donnée dans $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ dans [22]. L'idée est simple : d'une suite de données initiales bornées dans l'espace critique $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, qui admet donc une décomposition en profils, I. Gallagher montre que les équations de Navier-Stokes propagent cette décomposition à la solution elle-même. Il est intéressant de souligner le caractère linéaire de ce résultat qui contraste avec la nature fortement non linéaire de l'équation de Navier-Stokes. Ce genre de résultat sera généralisé par la suite au cas des espaces de Besov critiques en dimension 3 (voir [24]).

Revenons à la question de l'existence de singularités qui restent bornées dans des espaces critiques. La méthode des éléments critiques ouvre une voie royale à l'étude de l'explosion d'éventuelles singularités. Nous avons mentionné ci-dessus le résultat [24]. Les mêmes auteurs ont récemment [25] généralisé leur résultat au cas de l'espace de Besov critique $\dot{B}_{p,q}^{-1+\frac{3}{p}}$, avec $3 < p, q < \infty$. Plus précisément,

Théorème 1.34 ([25])**.** *Soient $3 < p, q < \infty$ et u_0 un champ de vecteurs de vecteurs à divergence nulle dans $\dot{B}_{p,q}^{-1+\frac{3}{p}}$. Soit $u = NS(u_0)$ l'unique solution de Navier-Stokes à temps de vie maximal $T_*(u_0)$. Alors*

$$\text{Si} \quad T_*(u_0) < \infty, \quad \text{alors} \quad \limsup_{t \to T_*(u_0)} \|u(t)\|_{\dot{B}_{p,q}^{-1+\frac{3}{p}}} = +\infty.$$

Mentionnons aussi le résultat de J.-Y. Chemin et F. Planchon [19] qui prouve le même théorème dans le cas où $3 < p < \infty$, $q < 2p'$ et avec une hypothèse supplémentaire sur la régularité de la donnée initiale. Signalons aussi le résultat de NC. Phuc [46] qui démontre le résultat dans le cadre des espaces de Lorentz $L^{3,q}$ avec q fini.

Il est intéressant de dire quelques mots sur le cas limite, exclu par les deux théorèmes ci-dessus : $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}$. La question naturelle à se poser est de savoir si le théorème reste valide dans ce cas limite. A notre connaissance, la question est toujours ouverte. En fait, si le théorème est vrai dans le cas limite $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}$, *a* *fortiori*, il est vrai dans l'espace plus petit $\dot{B}^{\frac{1}{2}}_{2,\infty}$, en vertu de l'inclusion en dimension 3 : $\dot{B}^{\frac{1}{2}}_{2,\infty} \hookrightarrow \dot{B}^{-1+\frac{3}{p}}_{p,\infty}$. Ainsi, cela prouverait qu'il n'existe pas de solution explosive, bornée dans l'espace $\dot{B}^{\frac{1}{2}}_{2,\infty}$. Cette interrogation fait écho à l'une des situations rencontrées dans cette thèse. En effet, sous l'hypothèse de l'existence de solutions explosives continues à valeurs H^s nous construisons, au chapitre 3, des solutions explosives, bornées dans l'espace $\dot{B}^{\frac{1}{2}}_{2,\infty}$. Pour terminer, nous mentionnons l'article récent de H. Jia et V. Šverák [30], dans lequel il est prouvé que toute donnée initiale homogène de degré -1 conduit à une solution globale, elle-aussi homogène de degré -1. Malheureusement, la question de l'unicité d'une telle solution n'est pas résolue.

Chapitre 2

About the behaviour of regular Navier-Stokes solutions near the blow up

Abstract: In this paper, we present some results about the blow up of regular solutions to the homogeneous incompressible Navier-Stokes system, in the case of data in the Sobolev space $\dot{H}^s(\mathbb{R}^3)$, where $\dfrac{1}{2} < s < \dfrac{3}{2}\cdot$ Firstly, we will introduce the notion of minimal blow up Navier-Stokes solutions and show that the set of such solutions is not only nonempty but also compact in a certain sense. Secondly, we will state an uniform blow up rate for minimal Navier-Stokes solutions. The key tool is profile theory as established by P. Gérard [26].

2.1 Introduction

We consider the Navier-Stokes system for incompressible fluids evolving in the whole space $\mathbb{R}^3$. Denoting by u the velocity, a vector field in $\mathbb{R}^3$, by p in $\mathbb{R}$ the pressure function, the Cauchy problem for the homogeneous incompressible Navier-Stokes system is given by

$$\left\{\begin{array}{ccc} \partial_t u + u\cdot\nabla u - \Delta u & = & -\nabla p \\ \operatorname{div} u & = & 0 \\ u_{|t=0} & = & u_0. \end{array}\right. \tag{2.1}$$

Throughout this paper, we will adopt the useful notation $NS(u_0)$ to denote the maximal solution of the Navier-Stokes system, associated with the initial data u_0.

Definition 2.1. *Let s in $\mathbb{R}$. The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$ is the space of tempered distributions u over $\mathbb{R}^3$, the Fourier transform of which belongs to $L^1_{loc}(\mathbb{R}^3)$ and satisfies*

$$\|u\|_{\dot{H}^s} \stackrel{\text{def}}{=} \left(\int_{\mathbb{R}^3} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}} < \infty.$$

It is known that $\dot{H}^s(\mathbb{R}^3)$ is an Hilbert space if and only if $s < \frac{3}{2}$. We will denote by $(\cdot|\cdot)_{\dot{H}^s(\mathbb{R}^3)}$, the scalar product in $\dot{H}^s(\mathbb{R}^3)$. From now on, for the sake of simplicity, it will be an implicit understanding that all computations will be done in the whole space $\mathbb{R}^3$.

Before stating the results we prove in this paper, we recall two fundamental properties of the incompressible Navier-Stokes system. The first one is the conservation of the L^2 energy. Formally, let us take the L^2 scalar product with the velocity u in the equation. We get

$$\frac{1}{2}\frac{d}{dt}||u(t)||_{L^2}^2 + ||\nabla u(t)||_{L^2}^2 = -\int_{R^3} \big(u\cdot \nabla u(t)|u(t)\big)_{L^2} - \int_{R^3} \big(\nabla p(t)|u(u)\big)_{L^2}. \tag{2.2}$$

Thanks to the divergence free condition, obvious integration by parts implies that, for any vector field a

$$\big(u\cdot \nabla a|a\big)_{L^2} = 0 = \big(\nabla p|a\big)_{L^2}. \tag{2.3}$$

This gives

$$\frac{1}{2}\frac{d}{dt}||u(t)||_{L^2}^2 + ||\nabla u(t)||_{L^2}^2 = 0. \tag{2.4}$$

The second property of the system is the scaling invariance. Let us define the above operator:

$$\begin{gathered}\forall \alpha \in \mathbb{R}^+,\ \forall \lambda \in \mathbb{R}^+_*,\, \forall x_0 \in \mathbb{R}^3,\quad \Lambda^\alpha_{\lambda,x_0}\, u(t,x) \overset{\text{def}}{=} \frac{1}{\lambda^\alpha} u\Big(\frac{t}{\lambda^2}, \frac{x-x_0}{\lambda}\Big).\\ \text{If}\ \ \alpha = 1,\ \ \text{we note}\ \ \Lambda^1_{\lambda,x_0} = \Lambda_{\lambda,x_0}.\end{gathered} \tag{2.5}$$

It is easy to see that if u is smooth solution of Navier-Stokes system on $[0,T]\times\mathbb{R}^3$ with pressure p associated with the initial data u_0, then, for any positive λ, the vector field and the pressure

$$u_\lambda \overset{\text{def}}{=} \Lambda_{\lambda,x_0}\, u \quad \text{and} \quad p_\lambda \overset{\text{def}}{=} \Lambda^2_{\lambda,x_0}\, p$$

is a solution of Navier-Stokes system on the interval $[0,\lambda^2 T]\times\mathbb{R}^3$, associated with the initial data

$$u_{0,\lambda} = \ \Lambda_{\lambda,x_0}\, u_0.$$

This leads to the definition of scaling invariant space, which is a key notion to investigate local and global well-posedness issues for Navier-Stokes system.

Definition 2.2. *A Banach space X is said to be scaling invariant, if its norm is invariant under the scaling transformation defined by $u \mapsto u_\lambda$*

$$||u_\lambda||_X = ||u||_X$$

The first main result on incompressible Navier-Stokes system is due to J. Leray, who proved (see [40]) in 1934 that being given initial data in the energy space L^2, the associated NS-solutions, called weak solutions, exist globally in time. The key ingredient of the proof is the L^2-energy conservation (2.4). Moreover, such solutions are unique in 2-D; but the uniqueness in 3-D is still an open problem. One way to adress this question of unique solvability in 3-D is to demand smoother initial data. In this case, we definitely get a unique solution, but the other side of coin is that the problem is only locally well-posed (and becomes globally well-posed under a scaling invariant smallness assumption on the initial data). J. Leray stated such a theorem of existence of solutions, which he called semi-regular solutions.

Theorem 2.3. *Let an initial data u_0 be a divergence free vector field in L^2 such that ∇u_0 belongs to L^2. Then, there exists a positive time T, and a unique solution $NS(u_0)$ in $\mathcal{C}^0([0,T],\dot H^1)\,\cap\, L^2([0,T],\dot H^2)$. Moreover, a constant c_1 exists such that if $||u_0||_{L^2}\,||\nabla u_0||_{L^2} \leqslant c_1$, then T can be chosen equal to ∞.*

The reader will have noticed that the quantity $||u_0||_{L^2}\,||\nabla u_0||_{L^2}$ is scaling invariant under the operator Λ_{λ,x_0}. Actually, that is the starting point of many frameworks concerning the global existence in time of solutions under a scaling invariant smallness assumption on the data. The celebrated first one was introduced in 1964, by H. Fujita and T. Kato. These authors stated a similar result as J. Leray, but they demanded less regularity on the data. Indeed, they proved that for any initial data in $\dot{H}^{\frac{1}{2}}$, there exists a positive time T and there exists a unique solution $NS(u_0)$ belonging to $\mathcal{C}^0([0,T],\dot{H}^{\frac{1}{2}})\cap L^2([0,T],\dot{H}^{\frac{3}{2}})$. Moreover, if $\|u_0\|_{\dot{H}^{\frac{1}{2}}}$ is small enough, then the solution is global in time. This theorem can be proved by a fixed-point argument and the key ingredient of the proof is that the Sobolev space $\dot{H}^{\frac{1}{2}}$ is invariant under the operator Λ_{λ,x_0}. In other words, the Sobolev space $\dot{H}^{\frac{1}{2}}$ has exactly the same scaling as Navier-Stokes equation. We refer the reader to [1], [21] or [39] for more details of the proof. But in this paper, we are not interested in the particular kind of space. On the contrary, we work with initial data belonging to homogeneous Sobolev spaces, $\dot{H}^s$ with $\dfrac{1}{2}<s<\dfrac{3}{2}$, which means that we are above the natural scaling of the equation. The first thing to do is to provide an existence theorem of Navier-Stokes solutions with data in such Sobolev spaces $\dot{H}^s$. The Cauchy problem is known to be locally well-posed; it can be proved by a fixed-point procedure in an adequate function space (we refer the reader to the book [39], from page 146 to 148, of P-G. Lemarié-Rieusset).

We shall constantly be using the following simplified notations:

$$L^\infty_T(\dot{H}^s)\stackrel{\text{def}}{=}L^\infty([0,T],\dot{H}^s)\quad\text{and}\quad L^2_T(\dot{H}^{s+1})\stackrel{\text{def}}{=}L^2([0,T],\dot{H}^{s+1}).$$

Let us define the relevant function space we shall be working with in the sequel:

$$X^s_T\stackrel{\text{def}}{=}L^\infty_T(\dot{H}^s)\,\cap\,L^2_T(\dot{H}^{s+1}),\quad\text{equipped with}\quad\|u\|^2_{X^s_T}\stackrel{\text{def}}{=}\|u\|^2_{L^\infty_T(\dot{H}^s)}+\|u\|^2_{L^2_T(\dot{H}^{s+1})}.$$

Theorem 2.4. *Let u_0 be in $\dot{H}^s$, with $\dfrac{1}{2}<s<\dfrac{3}{2}$. Then there exists a time T and there exists a unique solution $NS(u_0)$ such that $NS(u_0)$ belongs to $L^\infty_T(\dot{H}^s)\cap L^2_T(\dot{H}^{s+1})$.*
Moreover, let $T_*(u_0)$ be the maximal time of existence of such a solution. Then, there exists a positive constant c such that

$$T_*(u_0)\,\|u_0\|^{\sigma_s}_{\dot{H}^s}\geqslant c,\quad\textit{with}\quad\sigma_s\stackrel{\text{def}}{=}\frac{1}{\frac{1}{2}(s-\frac{1}{2})}\cdot\tag{2.6}$$

Remark 2.5. *As a by-product of the proof of Picard's Theorem, we get actually for free the following property: if the initial data is small enough (in the sense of there exists a positive constant c_0, such that $T\,\|u_0\|^{\sigma_s}_{\dot{H}^s}\leqslant c_0$), then a unique Navier-Stokes solution associated with it exists (locally in time, until the blow up time given by the relation (2.6)) and satisfies the following linear control*

$$\forall\,0\leqslant T\leqslant\frac{c_0}{\|u_0\|^{\sigma_s}_{\dot{H}^s}},\quad\|NS(u_0)(t,\cdot)\|_{X^s_T}\leqslant 2\,\|u_0\|_{\dot{H}^s}.\tag{2.7}$$

Formula (2.6) invites us to consider the lower boundary, denoted by $A^{\sigma_s}_c$, of the lifespan of such a solution

$$A^{\sigma_s}_c\stackrel{\text{def}}{=}\inf\Big\{\,T_*(u_0)\|u_0\|^{\sigma_s}_{\dot{H}^s}\;\big|\;u_0\in\dot{H}^s\;;\;T_*(u_0)<\infty\Big\}.$$

Obviously, $A^{\sigma_s}_c$ exists and is a positive real number and we always have the formula

$$T_*(u_0)\|u_0\|^{\sigma_s}_{\dot{H}^s}\geqslant A^{\sigma_s}_c.\tag{2.8}$$

Throughout this paper, we made the assumption of blow up, which is still an open problem. More precisely, we claim the following hypothesis.

Hypothesis $\mathcal{H}$: for any $\frac{1}{2} < s < \frac{3}{2}$, a divergence-free vector field u_0 exists in $\dot{H}^s$ such that the lifespan $T_*(u_0)$ is finite.

Let $\mathcal{B}_\rho$ be the open ball in $\dot{H}^s$ defined by $\mathcal{B}_\rho = \{u_0 \in \dot{H}^s \; / \; \|u_0\|_{\dot{H}^s} < \rho\}$. Let $T_* > 0$ be a positive real number. We define a critical radius by the following formula

$$\rho_s(T_*) \stackrel{\text{def}}{=} \frac{A_c}{T_*^{\frac{1}{\sigma_s}}}.$$

Defined in this way and thanks to (2.8), we get an another definition of the critical radius

$$\rho_s(T_*) = \sup\{\ \rho > 0 \mid \|u_0\|_{\dot{H}^s} < \rho \implies T_*(u_0) > T_*\}.$$

Thanks to this definition, we define the notion of minimal blow up solution for the Navier-Stokes system.

Definition 2.6. *(minimal blow up solution)*
We say that $u = NS(u_0)$ is a minimal blow up solution if u_0 satisfies the two following assumptions:

$$\|u_0\|_{\dot{H}^s} = \rho_s(T_*) \quad \textit{and} \quad T_*(u_0) = T_*.$$

Therefore, $u = NS(u_0)$ is a minimal blow up solution if and only if $A_c^{\sigma_s}$ is reached: $T_*(u_0)\|u_0\|_{\dot{H}^s}^{\sigma_s} = A_c^{\sigma_s}$.

Question: If $\rho_s(T_*)$ is finite, do some minimal blow up solutions exist ?

We will prove a stronger result: the set of initial data generating minimal blow up solutions, denoted by $\mathcal{M}_s(T_*)$, is not only a nonempty subset of $\dot{H}^s$ (which, in particular, gives the positive answer to the question) but also compact in a sense which is given in Theorem 2.7. We define the set $\mathcal{M}_s(T_*)$ as follows

$$\mathcal{M}_s(T_*) \stackrel{\text{def}}{=} \Big\{u_0 \in \dot{H}^s \mid T_*(u_0) = T_* \quad \text{and} \quad \|u_0\|_{\dot{H}^s} = \rho_s(T_*)\Big\}.$$

Theorem 2.7. *Assuming hypothesis $\mathcal{H}$. For any finite time T_*, the set $\mathcal{M}_s(T_*)$ is non empty and compact, up to translations. This means that for any sequence $(u_{0,n})_{n\in\mathbb{N}}$ of points in the set $\mathcal{M}_s(T_*)$, a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $(\mathbb{R}^3)^{\mathbb{N}}$ and a function V in $\mathcal{M}_s(T_*)$ exist such that, up to an extraction*

$$\lim_{n\to+\infty} ||u_{0,n}(\cdot + x_n) - V||_{\dot{H}^s} = 0.$$

The second result of this paper states that the blow up rate of a minimal blow up solution can be uniformely controlled since we get a priori bound of these minimal blow up solutions.

Theorem 2.8. *(Control of minimal blow up solutions)*
Assuming $\mathcal{H}$, there exists a nondecreasing function $F_s : [0, A_c^{\sigma_s}[\to \mathbb{R}^+$ with $\lim_{r\to A_c^{\sigma_s}} F_s(r) = +\infty$ such that for any divergence free vector field u_0 in $\dot{H}^s$, generating minimal blow up solution (it means $T_*(u_0)\|u_0\|_{\dot{H}^s}^{\sigma_s} = A_c^{\sigma_s}$), we have the following control on the minimal blow up solution $NS(u_0)$

$$\forall T < T_*(u_0),\ \|NS(u_0)\|_{X_T^s} \leqslant \|u_0\|_{\dot{H}^s} F_s(T^{\frac{1}{\sigma_s}} \|u_0\|_{\dot{H}^s}).$$

Remark 2.9. Let us point out that the quantity $T^{\frac{1}{\sigma_s}} \|u_0\|_{\dot{H}^s}$ is scaling invariant; which is obviously necessary.

The two previous theorems are the analogue of results, proved in the case of the Sobolev space $\dot{H}^{\frac{1}{2}}$. We shall not recall all the statements existing in the literature concerning the regularity of Navier-Stokes solutions in critical spaces, such as $\dot{H}^{\frac{1}{2}}$. We refer for instance the reader to [21] and to the article of C. Kenig et G. Koch [32], where the authors prove that NS-solutions which remain bounded in the space $\dot{H}^{\frac{1}{2}}$ do not become singular in finite time. Concerning Theorem 2.7, we were largely inspired by the article of W. Rusin and V. Šverák [49], in which the authors set up the key concept of minimal blow-up for data in Sobolev space $\dot{H}^{\frac{1}{2}}$. Firstly, they defined a critical radius $\rho_{\frac{1}{2}}$

$$\rho_{\frac{1}{2}} = \sup\{\rho > 0 \quad ; \quad \|u_0\|_{\dot{H}^{\frac{1}{2}}} < \rho \implies T_*(u_0) = +\infty\}.$$

Then, they introduced a subset $\mathcal{M}$ of $\dot{H}^{\frac{1}{2}}$, which describes the set of minimal-norm singularities (we speak about minimal norm in the sense of $\|u_0\|_{\dot{H}^{\frac{1}{2}}}$ is equal to the critical radius $\rho_{\frac{1}{2}}$)

$$\mathcal{M} = \{u_0 \in \dot{H}^{\frac{1}{2}} \quad ; \quad T_*(u_0) < +\infty \quad \text{and} \quad \|u_0\|_{\dot{H}^{\frac{1}{2}}} = \rho_{\frac{1}{2}}\}.$$

Thanks to these definitions, W. Rusin and V. Šverák proved that if there exist elements in the space $\dot{H}^{\frac{1}{2}}$ which develop singularities in finite time (we assume that blow-up occurs), then some of these elements are of minimal $\dot{H}^{\frac{1}{2}}$-norm (and thus, the set $\mathcal{M}$ is nonempty) and compact up to translations and dilations. It means that for any sequence $(u_{0,n})_{n\in\mathbb{N}}$ of points in the set $\mathcal{M}$, a sequence $(\lambda_n, x_n)_{n\in\mathbb{N}}$ and a function φ in $\mathcal{M}$ exist such that, up to an extraction, we have

$$\lim_{n\to+\infty} ||u_{0,n} - \Lambda_{\lambda_n,x_n}\varphi||_{\dot{H}^{\frac{1}{2}}} = 0.$$

Let us point out that I. Gallagher, G. Koch and F. Planchon generalize in [24] the result of W. Rusin and V. Šverák to critical Lebesgue and Besov spaces, such as L^3.

Concerning Theorem 2.8, our main source of inspiration is a result established by I. Gallagher in [22]. Given an initial data u_0 in the open ball $\mathcal{B}_{\rho_{\frac{1}{2}}}$. Then, by definition of $\rho_{\frac{1}{2}}$, $NS(u_0)$ is a global solution and thus belongs to the space $L^4(\mathbb{R}_+, \dot{H}^1)$, thanks to the important paper [23] of I. Gallagher, D. Iftimie and F. Planchon. In this way, the blow up in the $E_{\mathbb{R}_+} = L^\infty(\mathbb{R}_+, \dot{H}^{\frac{1}{2}}) \cap L^2(\mathbb{R}_+, \dot{H}^{\frac{3}{2}})$-norm does not occur. Even better: I. Gallagher proved in [22] the a priori control of the Navier-Stokes solution with data in the open ball $\mathcal{B}_{\rho_{\frac{1}{2}}}$ in the sense of there exists a nondecreasing function F defined from $[0, \rho_{\frac{1}{2}}[$ to $\mathbb{R}^+$ such that for any divergence free vector field u_0 in the open ball $\mathcal{B}_{\rho_{\frac{1}{2}}}$, we have

$$\|NS(u_0)\|_{E_{\mathbb{R}_+}} \leqslant F(\|u_0\|_{\dot{H}^{\frac{1}{2}}}).$$

Notation. We shall denote by C a constant which does not depend on the various parameters appearing in this paper, and which may change from line to line. We shall also denote sometimes $x \lesssim y$ to mean there exists an absolute constant $C > 0$ such that $x \leqslant C\, y$.

The paper is organized in the following way:
In section 2, we recall the fundamental tool of this paper : profile decomposition of a bounded sequence in $\dot{H}^s$. Then, we give the proof of the compactness of minimal blow up solutions set (Theorem 2.7) and control of of such solutions (Theorem 2.8). These two results are based on the crucial Theorem 2.12 about the lifespan of a Navier-Stokes solution associated with a bounded sequence of $\dot{H}^s$.
Section 3 is devoted to the proof of Theorem 2.12, thanks to a regularization process. Firstly, we will see that it is an immediate consequence of Lemma 2.14, which gives the structure of a Navier-Stokes solution associated with a bounded sequence of data in $\dot{H}^s$. Secondly, we will provide some helpful tools in order to prove Lemma 2.14.

In section 4, we prove Lemma 2.14, the result on which all others are based on. This section is the most technical part of the paper. It relies on classical product and paraproduct estimates, which are collected in Appendix A and B.

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2.2 Profiles theory and applications

This section is devoted to the proof of Theorems 2.7 and 2.8. Following I. Gallagher [22], W. Rusin and V. Šverák [49], C. Kenig and G. Koch [32] and I. Gallagher, G. Koch, F. Planchon [24], we shall use profile decomposition theory. The original motivation of this theory was the desciption of the default of compactness in Sobolev embeddings (see for instance the pionneering works of P.-L. Lions in [41], [42] and H. Brezis, J.-M. Coron in [8]. Here, we will use the theorem of P. Gérard [26], which gives, up to extractions, the structure of a bounded sequence of $\dot{H}^s$, with s between 0 and $\frac{3}{2}\cdot$ More precisely, the default of compactness in the critical Sobolev embedding $\dot{H}^s \subset L^p$ is described in terms of a sum of rescaled and translated orthogonal profiles, up to a small term in L^p. That was generalized to other Sobolev spaces $\dot{H}^{s,p}(\mathbb{R}^d)$ with $0 < s < \frac{d}{p}$ by S. Jaffard in [29], to Besov spaces by G. Koch in [36] and to general critical embeddings by H. Bahouri, A. Cohen and G. Koch in [2]. Let us notice the recent work [5] of H. Bahouri, M. Majdoud and N. Masmoudi concerning the lack of compactness of the Sobolev embedding of $H^1(\mathbb{R}^2)$ in the critical Orlicz space $\mathcal{L}(\mathbb{R}^2)$. Then profile decomposition techniques have been applied in many works of evolution problems such as the high frequency study of finite energy solutions to quintic wave equations on $\mathbb{R}^3$, by H. Bahouri and P. Gérard [4]. C. Kenig and F. Merle investigated in [33] the blow up property for the energy critical focusing non linear wave equation. Profile techniques turned out to be also a relevant tool in the study of Schrödinger equations. Notice this kind of decomposition was stated and developped, independently from [26], by F. Merle and L. Vega [43] for L^2-solutions of the critical non linear Schrödinger in $2D$, in the continuation of the work of J. Bourgain [6]. Then, S. Keraani revisited in [35] the work of H. Bahouri and P. Gérard [4] in the context of energy critical non linear Schrödinger equations. C. Kenig and F. Merle investigated in [34] the global well-posedness, scattering and blow up matter for such solutions in the focusing and radial case. We mention the work of I. Gallagher [22] for a relevant utilisation of profile theory in the context of Navier-Stokes equations.

Remark 2.10. Using notation (2.5), we can prove easily that the L^p (as well as $\dot{H}^s$)-norm is conserved under the transformation $u \mapsto \Lambda^{\frac{3}{p}}_{\lambda,x_0} u$. It means $\|\Lambda^{\frac{3}{p}}_{\lambda,x_0} u\| = \|u\|$.

Theorem 2.11. *Let $(u_{0,n})_{n\in\mathbb{N}}$ be a bounded sequence in $\dot{H}^s$. Then, up to an extraction:*
- There exists a sequence of vectors fields, called profiles $(V^j)_{j\in\mathbb{N}}$ in $\dot{H}^s$.
- There exists a sequence of scales and cores $(\lambda_{n,j}, x_{n,j})_{n,j\in\mathbb{N}}$, such that, up to an extraction

$$\forall J \geqslant 0,\ u_{0,n}(x) = \sum_{j=0}^{J} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x) \quad \textit{with} \quad \lim_{J\to+\infty} \limsup_{n\to+\infty} \|\psi_n^J\|_{L^p} = 0, \quad \textit{and} \quad p = \frac{6}{3-2s}\cdot$$

Where $(\lambda_{n,j}, x_{n,j})_{n\in\mathbb{N}, j\in\mathbb{N}^*}$ are sequences of $(\mathbb{R}_+^* \times \mathbb{R}^3)^{\mathbb{N}}$ with the following orthogonality property: for every integers (j,k) such that $j \neq k$, we have

$$\textit{either} \lim_{n\to+\infty} \Big(\frac{\lambda_{n,j}}{\lambda_{n,k}} + \frac{\lambda_{n,k}}{\lambda_{n,j}}\Big) = +\infty \quad \textit{or} \quad \lambda_{n,j} = \lambda_{n,k} \quad \textit{and} \quad \lim_{n\to+\infty} \frac{|x_{n,j} - x_{n,k}|}{\lambda_{n,j}} = +\infty.$$

Moreover, for any J in $\mathbb{N}$, we have the following orthogonality property

$$\|u_{0,n}\|^2_{\dot{H}^s} = \sum_{j=0}^{J} \|V^j\|^2_{\dot{H}^s} + \|\psi_n^J\|^2_{\dot{H}^s} + \circ(1), \quad \textit{when} \quad n \to +\infty. \tag{2.9}$$

A first application of this, is Theorem 2.12 about the lifespan of a NS-solution associated with bounded data in $\dot{H}^s$. The proof of it will be given in section 3.

Theorem 2.12. *Let $(u_{0,n})$ be a bounded sequence of initial data in $\dot{H}^s$ such that its profiles decomposition is given by*

$$u_{0,n}(x) = \sum_{j=0}^{J} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x) \quad \textit{with} \quad \lim_{J\to+\infty} \limsup_{n\to+\infty} \|\psi_n^J\|_{L^p} = 0.$$

Let us define $\mathcal{J}_1$ as the subset of indices j in $\mathbb{N}$, such that the profile V^j is non-zero and such that the associated scale $\lambda_{n,j}$ is identically equal to 1.

$$\textit{If } \mathcal{J}_1 = \emptyset, \textit{ then } \liminf_{n\to+\infty} T_*(u_{0,n}) = +\infty.$$
$$\textit{If } \mathcal{J}_1 \neq \emptyset, \textit{ then } \liminf_{n\to+\infty} T_*(u_{0,n}) \geqslant \inf_{j\in\mathcal{J}_1} T_*(V^j).$$

Remark 2.13. Let us point out some facts. Firstly, if $T_*(V^j) = +\infty$ for any j, then $\liminf_{n\to+\infty} T_*(u_{0,n}) = +\infty$. Secondly, in the case where $\mathcal{J}_1$ is non empty, the quantity $\inf_{j\in\mathcal{J}_1} T_*(V^j)$ exists and obviously, if $|\mathcal{J}_1|$ is finite, we get immediately that $\inf_{j\in\mathcal{J}_1} T_*(V^j) = \min_{j\in\mathcal{J}_1} T_*(V^j)$. In the case where $|\mathcal{J}_1|$ is infinite, we get the same conclusion. Indeed, by virtue of (2.9), the serie $\sum_{j\geqslant 0} \|V^j\|^2_{\dot{H}^s}$ is summable (*a fortiori* if we consider in the summation integers belonging to $\mathcal{J}_1$), and thus $\lim_{j\to+\infty} \|V^j\|_{\dot{H}^s} = 0$. Thanks to Inequality (2.6), we deduce that $\lim_{j\to+\infty} T_*(V^j) = +\infty$ and thus

$$\inf_{j\in\mathcal{J}_1} T_*(V^j) > 0 \quad \text{and} \quad \inf_{j\in\mathcal{J}_1} T_*(V^j) = \min_{j\in\mathcal{J}_1} T_*(V^j).$$

This result gives us an important information: whenever a sequence of initial data which satisfies profiles hypothesis (it means a bounded sequence in $\dot{H}^s$), we get an information on the lifespan of the NS-solution associated with such a sequence of initial data: it mainly depends on the lifespan of profiles with a constant scale. Note that the orthogonality property on scales and cores in Theorem 2.11 implies either the scales are different (in the sense that $\lim_{n\to+\infty} \Big(\frac{\lambda_{n,j}}{\lambda_{n,k}} + \frac{\lambda_{n,k}}{\lambda_{n,j}}\Big) = +\infty$) or the scales are the same ($\lambda_{n,j} = \lambda_{n,k}$), equal to a constant, and the cores go away from one another, in the sense that $\lim_{n\to+\infty} \frac{|x_{n,j} - x_{n,k}|}{\lambda_{n,j}} = +\infty$. In the last case where scales are equal to a constant, we shall assume that it is one, up to rescaling profiles by a fixed constant.
Theorem 2.12 has a key role in the proof of the compactness Theorem 2.7: the set $\mathcal{M}_s(T_*)$, recalled below, is non empty and compact, up to translations.

$$\mathcal{M}_s(T_*) := \Big\{ u_0 \in \dot{H}^s \mid T_*(u_0) = T_* \quad \text{and} \quad \|u_0\|_{\dot{H}^s} = \rho_s(T_*) \Big\}.$$

2.2.1 Compactness result on minimal blow up solutions

Proof. By definition of $A_c^{\sigma_s}$, we consider a minimizing sequence $(u_{0,n})_{n\geqslant 0}$ such that

$$\lim_{n\to+\infty} T_*(u_{0,n})\,\|u_{0,n}\|_{\dot H^s}^{\sigma_s} = A_c^{\sigma_s}.$$

Up to a rescaling process, we can assume that the minimizing sequence $(u_{0,n})_{n\geqslant 0}$ satisfies

$$\lim_{n\to+\infty} \|u_{0,n}\|_{\dot H^s} = \rho_s(T_*) \quad \text{and} \quad T_*(u_{0,n}) = T_*. \tag{2.10}$$

Indeed, consider the sequence $(v_{0,n})_{n\geqslant 0}$ defined as

$$v_{0,n}(x) \overset{\text{def}}{=} \Big(\frac{T_*(u_{0,n})}{T_*}\Big)^{\frac12} u_{0,n}\Big(\Big(\frac{T_*(u_{0,n})}{T_*}\Big)^{\frac12} x\Big).$$

The reader notices that the Navier-Stokes solution associated with such a sequence $(v_{0,n})$ has a lifespan equal to T_*. As $\|v_{0,n}\|_{\dot H^s}^{\sigma_s} = \Big(\frac{T_*(u_{0,n})}{T_*}\Big)\|u_{0,n}\|_{\dot H^s}^{\sigma_s}$, it seems clear now we can assume (2.10), by virtue of definition of $\rho_s(T_*)$. As defined, $(u_{0,n})_{n\geqslant 0}$ is a sequence of points of the set $\mathcal{M}_s(T_*)$; it is a bounded sequence in $\dot H^s$ and thus we can apply Theorem 2.11. Taking limit when $n\to+\infty$ in (2.9), we get

$$\forall J\geqslant 0, \quad \rho_s^2(T_*) \geqslant \sum_{j=0}^{J} \|V^j\|_{\dot H^s}^2.$$

Let us assume that there are two non-zero profiles at least. Then we should have

$$\forall j\in\{0,\cdots,J\},\ \|V^j\|_{\dot H^s}^2 < \rho_s^2(T_*).$$

By definiton of $\rho_s(T_*)$, it means all profiles V^j generate solutions whose lifespan satisfies

$$T_*(V^j) > T_*,\ \forall j\in\{0,\cdots,J\} \tag{2.11}$$

As $T_*(u_{0,n}) = T_* < \infty$ for any $n\in\mathbb{N}$, Theorem 2.12 implies that $\mathcal{J}_1 \neq \emptyset$: there exists at least one profile with constant scale. Moreover, thanks to Remark 2.13, we have $\inf\limits_{j\in\mathcal{J}_1} T_*(V^j) = \min\limits_{j\in\mathcal{J}_1} T_*(V^j)$. Combining this with Relation (2.11) implies that

$$\widetilde{T} \overset{\text{def}}{=} \inf_{j\in\mathcal{J}_1} T_*(V^j) > T_*.$$

By hypothesis on $(u_{0,n})_{n\in\mathbb{N}}$ and thanks to Theorem 2.12, we get a contradiction, since we have

$$\liminf_{n\to+\infty} T_*(u_{0,n}) = T_* \geqslant \widetilde{T} > T_*.$$

It means there exists an integer j_0 such that the profile, V^{j_0} has a lifespan which satisfies $T_*^{j_0} \leqslant T_*$. In particular, by definition of $\rho_s(T_*)$, it implies that $\|V^{j_0}\|_{\dot H^s}^2 \geqslant \rho_s^2(T_*)$. And, thanks to the orthogonal property of the $\dot H^s$-norm (2.9), we deduce the equality

$$\|V^{j_0}\|_{\dot H^s}^2 = \rho_s^2(T_*).$$

Now, we have just to check that $T_* = T_*^{j_0}$. We have already proved a first inequality: $T_*^{j_0} \leqslant T_*$. The other way is given by (2.8): we have always the following relation: $T_*^{j_0}\|V^{j_0}\|_{\dot H^s}^{\sigma_s} \geqslant A_c^{\sigma_s}$. Thanks to the result $\|V^{j_0}\|_{\dot H^s}^{\sigma_s} = \rho_s^{\sigma_s}(T_*) = \dfrac{A_c^{\sigma_s}}{T_*}$, we get the second inequality: $T_*^{j_0} \geqslant T_*$. Thus, the set $\mathcal{M}_s(T_*)$ is non empty and thus, there exists some minimal Navier-Stokes solutions. The compactness of the set $\mathcal{M}_s(T_*)$ is a consequence of the above work. Thanks to (2.9) and $\|V^{j_0}\|_{\dot{H}^s} = \rho_s(T_*)$, we infer that

$$\forall j \neq j_0, \;\; V^j = 0 \quad \text{and} \quad \lim_{n\to+\infty} \|\psi_n^J\|^2_{\dot{H}^s} = 0.$$

The above assumption implies in particular that $j_0 \in \mathcal{J}_1$. Indeed, if $j_0 \notin \mathcal{J}_1$, then $\mathcal{J}_1 = \emptyset$ and thus we should have $T_* = +\infty$, which is absurd. As a result, there exists a unique integer $j_0 \in \mathcal{J}_1$, such that

$$u_{0,n}(x) = V^{j_0}(x - x_{n,j_0}) + \psi_n^J(x).$$

The property $\lim_{n\to+\infty} \|\psi_n^l\|^2_{\dot{H}^s} = 0$ implies $\lim_{n\to+\infty} \|u_{0,n}(\cdot + x_{j_0,n}) - V^{j_0}\|_{\dot{H}^s} = 0.$ $\square$

2.2.2 Control a priori of minimal blow up solutions

Proof. Let us consider a critical element $u = NS(u_0) : T_*^{\frac{1}{\sigma_s}}(u_0)\|u_0\|_{\dot{H}^s} = A_c$. By virtue of a rescaling, we can asume that $\|u_0\|_{\dot{H}^s} = 1$ and thus $T_*^{\frac{1}{\sigma_s}}(u_0) = A_c$. Let us introduce the following set

$$\mathcal{N}_T^s \stackrel{\text{def}}{=} \Big\{ \|NS(u_0)\|_{X_T} \;\Big|\; u_0 \text{ in } \dot{H}^s \text{ such that } \|u_0\|_{\dot{H}^s} = 1 \quad \text{and} \quad T < A_c^{\sigma_s} \Big\}.$$

Theorem 2.7 claims that the set $\mathcal{N}_T^s$ is nonempty. The aim is to prove that $\sup \mathcal{N}_T^s$ is finite for any T. If not, a sequence $(u_{0,n})_{n\geqslant 0}$ in $\dot{H}^s$ exists, such that for any $T < T_*(u_{0,n})$, we have

$$\|u_{0,n}\|_{\dot{H}^s} = 1, \quad T_*(u_{0,n}) = A_c^{\sigma_s} \quad \text{and} \quad \lim_{n\to+\infty} \|NS(u_{0,n})\|_{X_T} = \infty. \tag{2.12}$$

By hypothesis, the sequence $(u_{0,n})_{n\geqslant 0}$ belongs to the set $\mathcal{M}_s(T_*)$. Therefore, there exists a sequence of cores $(x_n)_{n\in\mathbb{N}}$ and a function V in $\mathcal{M}_s(T_*)$ such that, up to an extraction:

$$\lim_{n\to+\infty} ||u_{0,n}(\cdot + x_n) - V||_{\dot{H}^s} = 0. \tag{2.13}$$

We can prove easily that, for any $T < T_*(V)$:

$$NS(u_{0,n}(\cdot + x_n)) = NS(V) + R_n \text{ with } \lim_{n\to+\infty} \|R_n\|_{X_T} = 0 \tag{2.14}$$

Indeed, we define

$$R_{0,n} \stackrel{\text{def}}{=} u_{0,n}(\cdot + x_n) - V.$$

Because of (2.13), the sequence $(R_{0,n})_{n\geqslant 0}$ converges to 0 in $\dot{H}^s$-norm, for n large enough. Moreover, the error term R_n satifies the following perturbed Navier-Stokes system

$$\left\{ \begin{array}{rcl} \partial_t R_n + R_n \cdot \nabla R_n - \Delta R_n + R_n \cdot \nabla NS(V) + NS(V) \cdot \nabla R_n & = & -\nabla p \\ \operatorname{div} R_n & = & 0 \\ R_{n|t=0} & = & R_{0,n}. \end{array} \right. \tag{2.15}$$

Applying forthcoming Theorem 2.30, we infer that, for any $T < T_*(V)$ and for n large enough

$$\|NS(u_{0,n}(\cdot + x_n))\|_{X_T} \leqslant \|NS(V)\|_{X_T} + \circ(1).$$

As $\|NS(u_{0,n}(\cdot + x_n))\|_{X_T} = \|NS(u_{0,n})\|_{X_T}$, we take the limit when $n \to +\infty$ in the above inequality and thus we get a contradiction with the assumption. $\square$

2.3 Profile decomposition on Navier-Stokes solutions with bounded sequence

All the previous results are based on Theorem 2.12. In this section, we prove this theorem, which relies on Lemma 2.14. This last one gives the structure of the Navier-Stokes solution associated with an initial data which has a profile decomposition. In others words, we wonder if, given the profile decomposition of a sequence of data, we get a similar decomposition on the Navier-Stokes solution itself. Lemma 2.14 gives a positive answer.
Let us recall to the reader that this question has already been studied by I. Gallagher in [22] in the case of initial data in the Sobolev space $\dot{H}^{\frac{1}{2}}$ and the same author with G. Koch, F. Planchon [24] in others critical spaces (e.g scaled invariant under the Navier-Stokes transformation). In our case, the difficulty is that the homogeneous Sobolev space $\dot{H}^s$ is not a scale invariant space under the natural scaling of the Navier-Stokes equation. To overcome this issue, the method consists in cutting off frequencies of profiles [4] (such profiles will have the useful property to belong to any $\dot{H}^s$, for any s). In particular, profiles scaled by 0 (resp. ∞) will tend to 0 in some Sobolev spaces (more precisely in $\dot{H}^{s_1}$ with $s_1 < s$), (resp. $\dot{H}^{s_2}$ with $s_2 > s$) and therefore, will not perturb the profile decomposition of the NS-solution.

2.3.1 Structure Lemma

Let $(u_{0,n})_{n\geqslant 0}$ be a bounded sequence of initial data in $\dot{H}^s$. Thanks to Theorem 2.11, $(u_{0,n})_{n\geqslant 0}$ can be written as follows, up to an extraction

$$u_{0,n}(x) = \sum_{j=0}^{J} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x).$$

By virtue of orthogonality of scales and cores given by Theorem 2.11, we sort profiles according to their scales

$$u_{0,n}(x) = \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} V^j(x - x_{n,j}) + \sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x) \tag{2.16}$$

where for any $j\in\mathcal{J}_1$, for any $n\in\mathbb{N}$, $\lambda_{n,j}\equiv 1$.

We claim we have the following structure lemma of the Navier-Stokes solutions, which proof will be provided in section 4. This lemma highlights the specific role of profiles with constant-scales.

Lemma 2.14. *(Profile decomposition of the Navier-Stokes solution)*
Let $(u_{0,n})_{n\geqslant 0}$ be a bounded sequence of initial data in $\dot{H}^s$ which profile decomposition is given by

$$u_{0,n}(x) = \sum_{j=0}^{J} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x).$$

Then, the error term R_n^J defined by $R_n^J \stackrel{\text{def}}{=} NS(u_{0,n}) - U_n^{\text{app,J}}$ is a solution of the below perturbed Navier-Stokes equation

$$\begin{cases} \partial_t R_n^J + R_n^J.\nabla R_n^J - \Delta R_n^J + R_n^J\cdot\nabla U_n^{\text{app,J}} + U_n^{\text{app,J}}.\nabla R_n^J & = & -F_n^J - \nabla p_n^J \\ \text{div } R_n^J & = & 0 \\ R^J_{n|t=0} & = & 0. \end{cases} \tag{2.17}$$

where F_n^J is a forcing term which will be explicitely detailed in (2.26) and

$$U_n^{\mathrm{app,J}}(t,x) \stackrel{\mathrm{def}}{=} \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,x-x_{n,j}) \;+\; e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big).$$

Moreover, the lifespan τ_n^J of the error term R_n^J satisifies

$$\forall \varepsilon>0,\ \exists J\geqslant 0\ \ \exists n_J\geqslant 0\ \forall n\geqslant n_J,\quad \tau_n^J \geqslant \inf_{j\in J_1} T_*(V^j)-\varepsilon.$$

Proof of Theorem 2.12. Clearly, Theorem 2.12 is an immediate consequence of Lemma 2.14. Assume Lemma 2.14 is proved. On the one hand, if there is no non zero profile with constant scale (e.g $\mathcal{J}_1=\emptyset$), the "profile decomposition" of the solution in Lemma 2.14 implies that $\liminf\limits_{n\to+\infty} T_*(u_{0,n})=+\infty$. On the other hand, if $\mathcal{J}_1\neq\emptyset$, the lifespan of sequence $NS(u_{0,n})$ is given by the lifespan of profiles, scaled by the constant 1 and $T_*(u_{0,n})\geqslant \inf\limits_{j\in\mathcal{J}_1} T_*(V^j)$. This ends up the proof of Theorem 2.12.

2.3.2 Tool box

In this subsection, we recall some basic facts about homogeneous Besov spaces and we prove some properties we need to the proof of Lemma 2.14. We refer the reader to [1], from page 63, for a detailed presentation of the theory and analysis of homogeneous Besov spaces.

Definition 2.15. *Let s be in $\mathbb{R}$, (p,r) in $[1,+\infty]^2$ and u in $\mathcal{S}'$. A tempered distribution u is an element of the Besov space $\dot{B}^s_{p,r}$ if u satifies*

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{\mathrm{def}}{=} \Big(\sum_{j\in\mathbb{Z}} 2^{jrs}\ ||\dot{\Delta}_j u||^r_{L^p}\Big)^{\frac{1}{r}}<\infty,$$

where $\dot{\Delta}_j$ is a frequencies localization operator (called Littlewood-Paley operator), defined by

$$\dot{\Delta}_j u(\xi) \stackrel{\mathrm{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{u}(\xi)),$$

with $\varphi\in\mathcal{D}([\frac{1}{2},2])$, such that $\sum\limits_{j\in\mathbb{Z}}\varphi(2^{-j}t)=1$, for any $t>0$.

Remark 2.16. We have the embedding $\dot{H}^s\subset\dot{B}^s_{2,2}$. These spaces coincide if $s<\frac{3}{2}$.

The first thing we have to notice is the following: given a bounded sequence of data in $\dot{H}^s$ (thus we get a profile decomposition of this sequence), Theorem 2.11 implies that the term $\psi_n^J(x)$, (which is bounded in $\dot{H}^s$), satisfies:

$$\lim_{J\to+\infty}\limsup_{n\to+\infty}\|\psi_n^J\|_{L^p}=0.$$

In fact, thanks to an interpolation argument, we can prove that the remaining term ψ_n^J tends to 0 in certain Besov spaces. That is the point in the following proposition.

Proposition 2.17. *For any $0<\theta<1$, let p_θ be a positive real number given by the interpolation relation*

$$\frac{1}{p_\theta}=\frac{\theta}{p}+\frac{1-\theta}{2}\quad \textit{with}\quad \frac{1}{p}=\frac{1}{2}-\frac{s}{3}.$$

Then, under the same hypothesis of Theorem 2.11, we have:

$$\lim_{J\to+\infty}\limsup_{n\to+\infty}\|\psi_n^J\|_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}=0\cdot$$

Proof. Interpolation inequality in the Lebesgue spaces and multiplication by the factor $2^{js(1-\theta)}$ give

$$2^{js(1-\theta)}\|\dot{\Delta}_j\psi_n^J\|_{L^{p_\theta}}\leqslant\|\dot{\Delta}_j\psi_n^J\|^\theta_{L^p}\ \left(2^{js}\|\dot{\Delta}_j\psi_n^J\|_{L^2}\right)^{1-\theta}.$$

Applying Hölder's inequality in the above expression, we get

$$\|\psi_n^J\|_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}\leqslant\|\psi_n^J\|^\theta_{\dot{B}^0_{p,p}}\ \|\psi_n^J\|^{1-\theta}_{\dot{B}^s_{2,2}}.$$

Because p is greater than 2, L^p is continuously included in $\dot{B}^0_{p,p}$. Remark 2.16 leads to

$$\|\psi_n^J\|_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}\leqslant\|\psi_n^J\|^\theta_{L^p}\|\psi_n^J\|^{1-\theta}_{\dot{H}^s}. \tag{2.18}$$

By virtue of Theorem 2.11, we get the result. □

Let us come back to the profile decomposition of the sequence $(u_{0,n})_{n\geqslant 0}$ and introduce some notations. Let $\eta>0$ be the parameter of rough cutting off frequencies. We define by $u_\eta(x)$ and $u_{^c\eta}(x)$ the elements which Fourier transform is given by

$$\widehat{u_\eta}(\xi)=\widehat{u}(\xi)1_{\{\frac{1}{\eta}\leqslant|\xi|\leqslant\eta\}}\quad\text{and}\quad\widehat{u_{^c\eta}}(\xi)=\widehat{u}(\xi)\big(1-1_{\{\frac{1}{\eta}\leqslant|\xi|\leqslant\eta\}}\big). \tag{2.19}$$

From the profiles decomposition (2.16), we infer, thanks to the orthogonality property of scales, that among profiles V^j such that j belongs to ${}^c\mathcal{J}_1$, there are profiles with small scales ($j\in\mathcal{J}_0$) and large scales ($j\in\mathcal{J}_\infty$). These profiles are cut (according to the parameter η), with respect to notations (2.19) and we get

$$\begin{aligned}u_{0,n}(x)&=\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}V^j(x-x_{n,j})+\sum_{\substack{j\in\mathcal{J}_0\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}V^j_\eta(x)+\sum_{\substack{j\in\mathcal{J}_\infty\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}V^j_\eta(x)+\psi^J_{n,\eta}(x)\\ \text{where}\quad\psi^J_{n,\eta}(x)&\stackrel{\text{def}}{=}\sum_{\substack{j\in\mathcal{J}_1^c\equiv\mathcal{J}_0\cup\mathcal{J}_\infty\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}V^j_{^c\eta}(x)\ +\ \psi^J_n(x),\end{aligned} \tag{2.20}$$

with for any $j\in\mathcal{J}_0$, $\lim\limits_{n\to+\infty}\lambda_{n,j}=0$ and for any $j\in\mathcal{J}_\infty$, $\lim\limits_{n\to+\infty}\lambda_{n,j}=+\infty$.

Firstly, we check the remaining term $\psi^J_{n,\eta}$ is still small in $\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}$-norm, in the following sense. That is the point of the proposition below.

Proposition 2.18. *Let $0<\theta<1$. Under the interpolation relation $\dfrac{1}{p_\theta}=\dfrac{\theta}{p}+\dfrac{1-\theta}{2}$, we have*

$$\lim_{J\to+\infty}\lim_{\eta\to+\infty}\limsup_{n\to+\infty}\|\psi^J_{n,\eta}\|_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}=0.$$

Proof. Let $0 < \theta < 1$. By definition of $\psi^J_{n,\eta}$ and thanks to $(a+b)^2 \lesssim a^2+b^2$, we have

$$\|\psi^J_{n,\eta}\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}} \lesssim \Big\| \sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_{c\eta}(x)\Big\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}} + \|\psi^J_n\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}. \tag{2.21}$$

The embedding $\dot{H}^s \subset \dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}$ and the orthogonality of scales and cores imply

$$\begin{aligned}\|\psi^J_{n,\eta}\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}} &\lesssim \Big\| \sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_{c\eta}(x)\Big\|^2_{\dot{H}^s} + \|\psi^J_n\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}\\ &\lesssim \Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \|\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_{c\eta}(x)\|^2_{\dot{H}^s} + \circ(1)\Big) + \|\psi^J_n\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}.\end{aligned} \tag{2.22}$$

By scaling invariance of the norm $\dot{H}^s$ under the transformation $u \mapsto \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} u$, we get

$$\|\psi^J_{n,\eta}\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}} \lesssim \Big(\sum_{j=0}^{\infty} \|V^j_{c\eta}(x)\|^2_{\dot{H}^s} + \circ(1)\Big) + \|\psi^J_n\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}, \text{ when } \quad n\to+\infty.$$

For any $j \geqslant 0$, the term $\|V^j_{c\eta}(x)\|^2_{\dot{H}^s}$ tends to 0 for η large enough, by Lebesgue Theorem. Therefore, applying Lebesgue Theorem once again, we infer that $\lim\limits_{\eta\to+\infty} \sum\limits_{j=0}^{\infty} \|V^j_{c\eta}(x)\|^2_{\dot{H}^s} = 0$. As a result, we take in first the upper limit of $\|\psi^J_{n,\eta}\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}$, when $n\to+\infty$. Then, we take the limit for $\eta\to+\infty$ and at the last, for $J\to+\infty$. Thanks to Proposition 2.17, Proposition 2.18 is proved. □

As it was already mentionned previously, the point of such rough cutting off in frequencies is that profiles which are supported in the annulus $1_{\{\frac{1}{\eta}\leqslant|\xi|\leqslant\eta\}}$, belong to the Sobolev spaces $\dot{H}^s$, for any $s>0$. In particular, we can look at such profiles in the Sobolev spaces such as $\dot{H}^{s_1}$ with $s_1 < s$ and $\dot{H}^{s_2}$ with $s_2 > s$. That is the point in the following proposition: according to the size of the scale (either small j in $\mathcal{J}_0$ or large j in $\mathcal{J}_\infty$), profiles, trapped in the annulus, behave themselves as "remaining terms", seen from the point of view of solving Navier-Stokes.

Proposition 2.19.

For any $\eta>0,\ s_1<s,$ *and* $j\in\mathcal{J}_0,$ *e.g* $\lim\limits_{n\to+\infty}\lambda_{n,j}=0,$ *then* $\lim\limits_{n\to+\infty}\Big\|\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_\eta(x)\Big\|_{\dot{H}^{s_1}} = 0.$

For any $\eta>0,\ s_2>s,$ *and* $j\in\mathcal{J}_\infty,$ *e.g* $\lim\limits_{n\to+\infty}\lambda_{n,j}=+\infty,$ *then* $\lim\limits_{n\to+\infty}\Big\|\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_\eta(x)\Big\|_{\dot{H}^{s_2}} = 0.$

Proof. Let $s_1<s$. Let $j\in\mathcal{J}_0$ and $\eta>0$. Definition of $\dot{H}^{s_1}$-norm and a variable change yield

$$\begin{aligned}\Big\|\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_\eta(x)\Big\|^2_{\dot{H}^{s_1}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^{2s_1} \Big|{\lambda_{n,j}}^{3(1-\frac{1}{p})} \widehat{V^j_\eta}(\lambda_{n,j}\xi)\Big|^2 d\xi\\ &= \lambda_{n,j}^{2(s-s_1)} \int_{\mathbb{R}^3} |\xi|^{2s_1}\, |\widehat{V^j_\eta}(\xi)|^2 d\xi.\end{aligned} \tag{2.23}$$

Let us introduce the factor $|\xi|$. The hypothesis of the ring implies that

$$\begin{aligned}\left\|\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_\eta(x)\right\|^2_{\dot H^{s_1}} &= {\lambda_{n,j}}^{2(s-s_1)} \int_{\mathbb{R}^3} |\xi|^{2s}\, |\widehat{V^j_\eta}(\xi)|^2 \frac{1}{|\xi|^{2(s-s_1)}} d\xi \\ &\leqslant (\eta\ \lambda_{n,j})^{2(s-s_1)} \|V^j\|^2_{\dot H^s}.\end{aligned} \tag{2.24}$$

As $\lambda_{n,j}$ tends to 0; this proves the first part of the proposition. The second part relies on similar arguments and thus the proof is omitted. □

2.4 Proof of structure Lemma

Given a bounded sequence $(u_{0,n})$ in $\dot H^s$ which profile decomposition is given by Theorem 2.11, we search sequences associated solutions $NS(u_{0,n})$, under the form of

$$NS(u_{0,n}) = U_n^{\mathrm{app,J}} + R_n^J, \quad \text{where}$$

$$U_n^{\mathrm{app,J}} \stackrel{\mathrm{def}}{=} \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot - x_{n,j}) + e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big), \tag{2.25}$$

Note that if $\mathcal{J}_1=\emptyset$, the the approximation term $U_n^{\mathrm{app,J}}$ is reduced to the linear part

$$U_n^{\mathrm{app,J}} = e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big).$$

Plugging this decomposition into the Navier-Stokes equation leads to the following perturbed equation on the error term R_n^J

$$\left\{\begin{array}{ccc} \partial_t R_n^J + R_n^J.\nabla R_n^J - \Delta R_n^J + R_n^J\cdot\nabla U_n^{\mathrm{app,J}} + U_n^{\mathrm{app,J}}.\nabla R_n^J & = & -F_n^J - \nabla p_n^J \\ \mathrm{div}\ R_n^J & = & 0 \\ R^J_{n|t=0} & = & 0. \end{array}\right. \tag{2.26}$$

where the forcing term F_n^J is given by $F_n^J = \sum\limits_{\ell=1}^{4} F_n^{J,\ell}$, with

$$\begin{aligned}
F_n^{J,1} &= \sum_{0\leqslant j,k\leqslant \mathcal{J}_1; j\neq k} NS(V^j)(t,\cdot - x_{n,j})\cdot\nabla NS(V^k)(t,\cdot - x_{n,k}),\\
F_n^{J,2} &= e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big)\cdot\nabla\Big(e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big)\Big),\\
F_n^{J,3} &= e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big)\cdot\nabla\Big(\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot - x_{n,j})\Big),\\
F_n^{J,4} &= \Big(\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot - x_{n,j})\Big)\cdot\nabla\Big(e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big)\Big).
\end{aligned} \tag{2.27}$$

Let us admit for a while the two following propositions.

Proposition 2.20. *With notations (2.37), the sequence* $U_n^{\text{app,J}}$ *is bounded in the space* X_T^s*, uniformly in* J,

$$\|U_n^{\text{app,J}}\|_{X_T^s} < \infty, \quad \forall\, T < \widetilde{T} \stackrel{\text{def}}{=} \inf_{j\in\mathcal{J}_1} T_*(V^j).$$

Once again, we use the convention that $\inf\limits_{j\in\mathcal{J}_1} T_*(V^j) = +\infty$ if $\mathcal{J}_1$ is empty. Let us admit for a while the following proposition.

Proposition 2.21.

$$\lim_{J\to+\infty} \limsup_{n\to+\infty} \|F_n^J\|^2_{L^2_T(\dot H^{s-1})} = 0.$$

Completion of the proof of Lemma 2.14. Let $\varepsilon_0 > 0$. Let T_0 be the time defined by

$$T_0 \stackrel{\text{def}}{=} \sup\{0 < T < \widetilde{T} \mid \|R_n^J(t)\|^2_{L^\infty_T(\dot H^s)} \leqslant \varepsilon_0\}.$$

Therefore, for any $T < T_0 \leqslant \widetilde{T}$, Theorem 2.30 implies

$$\|R_n^J\|^2_{X_T^s} \lesssim \|F_n^J\|^2_{L^2_T(\dot H^{s-1})} \exp\Big(\varepsilon_0^{\frac{2}{2s-1}}\, \widetilde{T} + \widetilde{T}^{s-\frac12}\, \|U_n^{\text{app,J}}\|^2_{X_T^s} + \widetilde{T}\, \|U_n^{\text{app,J}}\|^{\frac{4}{2s-1}}_{L^\infty_T(\dot H^s)}\Big). \tag{2.28}$$

Combining Propositions 2.20 and 2.21, Lemma 2.14 is proved. Therefore, to complete the proof, we shall prove the two above propositions.

Proof of Proposition 2.20. By definition of $U_n^{\text{app,J}}$ and vertue of $(a+b)^2 \leqslant 2\left(a^2+b^2\right)$, we have

$$\|U_n^{\text{app,J}}\|^2_{X_T^s} \leqslant 2\Big(\Big\|\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot - x_{n,j})\Big\|^2_{X_T^s} + \Big\|e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac3p}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big)\Big\|^2_{X_T^s}\Big). \tag{2.29}$$

Let us focus for a moment on the heat term $e^{t\Delta}\Big(\sum\limits_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac3p}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big)$. It is well-known that an $\dot H^s$-energy estimate on the heat equation implies that $\|e^{t\Delta}u\|^2_{X_T^s} \leqslant \|u_0\|^2_{\dot H^s}$, for any u solution associated with data u_0 in $\dot H^s$. As a result, we get

$$\Big\|e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac3p}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big)\Big\|^2_{X_T^s} \leqslant \Big\|\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac3p}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big\|^2_{\dot H^s}.$$

Therefore, profile decomposition yields, up to triangular and Young's inequalities

$$\begin{aligned}\|e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac3p}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big)\|^2_{X_T^s} &\leqslant \Big\|u_{0,n} - \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} V^j(\cdot - x_{n,j})\Big\|^2_{\dot H^s}\\ &\leqslant 2\,\|u_{0,n}\|^2_{\dot H^s} + 2\,\Big\|\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} V^j(\cdot - x_{n,j})\Big\|^2_{\dot H^s}.\end{aligned}$$

Let us admit for a while the following statement

$$\Big\|\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} V^j(\cdot - x_{n,j})\Big\|^2_{\dot H^s} = \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} \|V^j\|^2_{\dot H^s} + \circ(1), \quad \text{when} \quad n\to+\infty. \tag{2.30}$$

Thanks to the orthogonality relation (2.9), the term $\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}\|V^j\|^2_{\dot H^s}$ satisfies $\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}\|V^j\|^2_{\dot H^s}\leqslant\|u_{0,n}\|^2_{\dot H^s}+\circ(1)$, for n large enough. As a result,

$$\Big\|e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}V^j(x)+\psi_n^J(x)\Big)\Big\|^2_{X^s_T}\lesssim\|u_{0,n}\|^2_{\dot H^s}+\circ(1),\quad\text{when}\quad n\to+\infty.\tag{2.31}$$

Now, let us come back to (2.29). Thanks to the previous estimate (2.31), we infer that

$$\|U_n^{\mathrm{app,J}}\|^2_{X^s_T}\lesssim\Big\|\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}NS(V^j)(t,\cdot-x_{n,j})\Big\|^2_{X^s_T}+\|u_{0,n}\|^2_{\dot H^s}+\circ(1),\quad\text{when}\quad n\to+\infty.$$

We admit for a while the following statement, for any $T<\widetilde T\overset{\text{def}}{=}\inf_{j\in\mathcal{J}_1}T_*(V^j)$ and $\eta>0$.

$$\Big\|\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}NS(V^j)(t,\cdot-x_{n,j})\Big\|^2_{X^s_T}\leqslant\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}\|NS(V^j)(t,\cdot)\|^2_{X^s_T}+\circ(1),\quad\text{when}\quad n\to+\infty.\tag{2.32}$$

Therefore, we have for any $T<\widetilde T\overset{\text{def}}{=}\inf_{j\in\mathcal{J}_1}T_*(V^j)$

$$\|U_n^{\mathrm{app,J}}\|^2_{X^s_T}\leqslant C\Big(\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}\|NS(V^j)\|^2_{X^s_T}+\|u_{0,n}\|^2_{\dot H^s}+\circ(1)\Big),\quad\text{when}\quad n\to+\infty.$$

As $NS(V^j)$ solves NS-equation with initial data V^j belonging to $\dot H^s$ and since the time T is far away from the blow up time, we infer that each term in the right-hand side is bounded, uniformly in J. Now let us prove (2.30). Clearly we have, thanks to the translations invariance of the $\dot H^s$-norm

$$\begin{aligned}\Big\|\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}V^j(\cdot-x_{n,j})\Big\|^2_{\dot H^s}&=\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}\|V^j(\cdot-x_{n,j})\|^2_{\dot H^s}+2\sum_{\substack{(j,k)\in\mathcal{J}_1\times\mathcal{J}_1\\ j\neq k}}\Big(V^j(\cdot-x_{n,j})\,\big|\,V^k(\cdot-x_{n,k})\Big)_{\dot H^s}\\&=\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}\|V^j\|^2_{\dot H^s}+2\sum_{\substack{(j,k)\in\mathcal{J}_1\times\mathcal{J}_1\\ j\neq k}}\Big(|D|^sV^j(\cdot-x_{n,j})\,\big|\,|D|^sV^k(\cdot-x_{n,k})\Big)_{L^2},\end{aligned}$$

where $|D|=\sqrt{-\Delta}$. The orthogonality of cores (e.g. $\lim_{n\to\infty}|x_{n,j}-x_{n,k}|=+\infty$) implies in particular that the term $|D|^s\,V^k(x+(x_{n,j}-x_{n,k}))$ weakly converges towards 0 in L^2 and thus (notice that $|D|^s\,V^j_\eta(x)$ belongs to L^2, by hypothesis)

$$\forall (j,k)\in\mathcal{J}_1\times\mathcal{J}_1,\quad\lim_{n\to\infty}\int_{\mathbb{R}^3}|D|^s\,V^j(x)\,|D|^s\,V^k(x+(x_{n,j}-x_{n,k}))\,dx=0,$$

which ends up the proof of statement (2.30). Concerning statement (2.32), the proof is similar. Let $\varepsilon>0$. As for any $T\leqslant\widetilde T-\varepsilon$, $NS(V^j_\eta)$ belongs to the space $X^s_T\overset{\text{def}}{=}\mathcal{C}_T(\dot H^s)\cap L^2_T(\dot H^{s+1})$. In particular, the map $t\in[0,\widetilde T-\varepsilon]\mapsto NS(V^j)(t,\cdot)$ belongs to $\dot H^s$. Previous computations hold and, by virtue of translation invariance of the $\dot H^s$-norm, we get for any $t<\widetilde T$,

$$\begin{aligned}\Big\|\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}NS(V^j)(t,\cdot-x_{n,j})\Big\|^2_{\dot H^s}&=\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}\|NS(V^j)(t,\cdot)\|^2_{\dot H^s}\\&\quad+2\sum_{\substack{(j,k)\in\mathcal{J}_1\times\mathcal{J}_1\\ j\neq k}}\Big(|D|^s\,NS(V^j)(t,\cdot-x_{n,j})\,\big|\,|D|^s\,NS(V^k)(t,\cdot-x_{n,k})\Big)_{L^2}.\end{aligned}\tag{2.33}$$

Then, for any t in $[0, \widetilde{T}-\varepsilon]$, we get

$$\Big\|\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot - x_{n,j})\Big\|^2_{L^\infty_T(\dot H^s)} \leqslant \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} \|NS(V^j)(t,\cdot)\|^2_{L^\infty_T(\dot H^s)} + 2\sum_{\substack{(j,k)\in\mathcal{J}_1\times\mathcal{J}_1\\ j\neq k}} \Gamma^{s,j,k}_{\varepsilon,n},$$

where $\Gamma^{s,j,k}_{\varepsilon,n}$ is defined by

$$\begin{aligned}\Gamma^{s,j,k}_{\varepsilon,n} &\stackrel{\text{def}}{=} \sup_{t\in[0,\widetilde{T}-\varepsilon]} \Big(|D|^s\, NS(V^j)(t,\cdot - x_{n,j})\,\big|\, |D|^s\, NS(V^k)(t,\cdot - x_{n,k})\Big)_{L^2}\\ &= \sup_{t\in[0,\widetilde{T}-\varepsilon]} \int_{\mathbb{R}^3} |D|^s\, NS(V^j)(t,\cdot)\, |D|^s\, NS(V^k)(t,\cdot + (x_{n,j}-x_{n,k}))\,dx.\end{aligned} \tag{2.34}$$

The map $\psi : t\in[0,\widetilde{T}-\varepsilon] \mapsto |D|^s\, NS(V^j)(t,\cdot)\, |D|^s\, NS(V^k)(t,\cdot+(x_{n,j}-x_{n,k}))$ is continuous on the compact $[0,\widetilde{T}-\varepsilon]$, with value in $L^1(\mathbb{R}^3)$. Thus, $\psi([0,\widetilde{T}-\varepsilon])$ is precompact in the Lebesgue space $L^1(\mathbb{R}^3)$ and thus can be covered by a finite open ball with an arbitrarily radius $\alpha>0$. Let α be a positive radius. There exists an integer N, such that for any $t\in[0,\widetilde{T}-\varepsilon]$, $\psi(t)$ belongs to $\displaystyle\bigcup_{l=1}^{N}\mathcal{B}(\psi(t_l),\alpha)$. Thus, for any t belonging to the compact $[0,\widetilde{T}-\varepsilon]$, there exists a time t_l such that

$$\|\psi(t)\|_{L^1(\mathbb{R}^3)} \leqslant \alpha + \|\psi(t_l)\|_{L^1(\mathbb{R}^3)}. \tag{2.35}$$

By virtue of the simple fact $\displaystyle\int f \leqslant \int |f|$, we infer that

$$\begin{aligned}\Gamma^{s,j,k}_{\varepsilon,n} &\leqslant \alpha + \|\psi(t_l)\|_{L^1(\mathbb{R}^3)}\\ &= \alpha + \int_{\mathbb{R}^3}\Big||D|^s\, NS(V^j)(t_l,\cdot)\, |D|^s\, NS(V^k)(t_l,\cdot+(x_{n,j}-x_{n,k}))\Big|\,dx.\end{aligned}$$

Now, in order to conclude, we notice that Lebesgue theorem combining with the orthogonality property of cores imply that the right-hand-side tends to 0, when n tends to $+\infty$ (since we can choose α arbitrarily small) and thus, we get

$$\forall (j,k)\in\mathcal{J}_1\times\mathcal{J}_1,\quad \lim_{n\to\infty}\sup_{t\in[0,\widetilde{T}-\varepsilon]} \Big(|D|^s\, NS(V^j)(t,\cdot - x_{n,j})\,\big|\,|D|^s\, NS(V^k)(t,\cdot - x_{n,k})\Big)_{L^2} = 0.$$

Therefore, we have proved for any $T<\widetilde{T}$,

$$\Big\|\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot - x_{n,j})\Big\|^2_{L^\infty_T(\dot H^s)} \leqslant \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} \|NS(V^j)(t,\cdot)\|^2_{L^\infty_T(\dot H^s)} + \circ(1),\quad \text{when}\quad n\to+\infty.$$

Concerning the $L^2_T(\dot H^{s+1})$-norm, we write estimate (2.33) in $\dot H^{s+1}$-norm. Then, the $L^2_T(\dot H^{s+1})$-norm of crossed terms tends to 0, thanks to Lebesgue theorem and orthogonality of cores. Details are left to the reader. Finally, we get (2.32)

$$\Big\|\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot - x_{n,j})\Big\|^2_{X^s_T} \leqslant \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} \|NS(V^j)(t,\cdot)\|^2_{X^s_T} + \circ(1),\quad \text{when}\quad n\to+\infty.$$

In order to complete the proof of Proposition 2.20, we have to prove that the term $\displaystyle\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} \|NS(V^j)(t,\cdot)\|^2_{X^s_T}$ is bounded, *uniformly* in J. This will result from Remark 2.5 and the orthogonality of $\dot H^s$-norm (2.9) in profile theorem. Indeed, by virtue of profile decomposition of the bounded sequence $(u_{0,n})_{n\geqslant 0}$ in the Sobolev space $\dot{H}^s$, we know that $\displaystyle\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} \|V^j\|^2_{\dot{H}^s}$ is bounded. It means that

$$\forall\varepsilon>0,\ \exists\mathcal{J}_1^*\subset\mathcal{J}_1,\ \text{ with } |\mathcal{J}_1^*|<\infty\ \forall\ j\in\mathcal{J}_1\setminus\mathcal{J}_1^*,\ \|V^j\|_{\dot{H}^s}\leqslant\varepsilon.$$

By virtue of Remark 2.5, we infer that for any j belonging to $\mathcal{J}_1\backslash\mathcal{J}_1^*$, the Navier-Stokes solutions $NS(V^j)$ associated with such profiles V^j satisfy $\|NS(V^j)(t,\cdot)\|_{X^s_T}\leqslant 2\,\|V^j\|_{\dot{H}^s}$. Therefore, we infer that

$$\begin{aligned}\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}\|NS(V^j)(t,\cdot)\|^2_{X^s_T}&\leqslant\sum_{\substack{j\in\mathcal{J}_1^*\\ j\leqslant J}}\|NS(V^j)(t,\cdot)\|^2_{X^s_T}+4\sum_{\substack{j\in\mathcal{J}_1\backslash\mathcal{J}_1^*\\ j\leqslant J}}\|V^j\|^2_{\dot{H}^s}\\&\leqslant\sum_{\substack{j\in\mathcal{J}_1^*\\ j\leqslant J}}\|NS(V^j)(t,\cdot)\|^2_{X^s_T}+4\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}\|V^j\|^2_{\dot{H}^s}\\&\leqslant\sum_{\substack{j\in\mathcal{J}_1^*\\ j\leqslant J}}\|NS(V^j)(t,\cdot)\|^2_{X^s_T}+4\limsup_{n\to+\infty}\|u_{0,n}\|^2_{\dot{H}^s}.\end{aligned}\tag{2.36}$$

As we are not so close to the blow up time (since $T<\inf_{j\in\mathcal{J}_1}T_*(V^j)$), the term $\displaystyle\sum_{\substack{j\in\mathcal{J}_1^*\\ j\leqslant J}}\|NS(V^j)(t,\cdot)\|^2_{X^s_T}$ is bounded, uniformly in J (since $\mathcal{J}_1^*$ is a finite set and depends only on the sequence of profiles V^j). Thus, the proof of Proposition 2.20 is complete.

Proof of Proposition 2.21. In order to prove the smallness result on the forcing term, we shall need to use the regularization process mentionned in the tool box of the previous section. Let us recall that we get an approximation of the Navier-Stokes solution associated with such a data, under the form of

$$NS(u_{0,n})=U_n^{\mathrm{app,J}}+R_n^J,\quad\text{where}$$

$$U_n^{\mathrm{app,J}}(t,\cdot)\stackrel{\mathrm{def}}{=}\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}NS(V^j)(t,\cdot-x_{n,j})+e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}V^j(\cdot)+\psi_n^J(\cdot)\Big),\tag{2.37}$$

As already mentionned in (2.20), profiles are sorted with respect of the size of scales. Moreover, we cut off frequencies of profiles with small and big scales and therefore, decomposition (2.37) can be rewritten as follows

$$U_n^{\mathrm{app,J}}(t,\cdot)\stackrel{\mathrm{def}}{=}\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}NS(V^j)(t,\cdot-x_{n,j})+e^{t\Delta}\Big(U^0_{n,\eta}+U^\infty_{n,\eta}+\psi^J_{n,\eta}(\cdot)\Big),\tag{2.38}$$

$$\begin{aligned}&\text{with}\quad U^0_{n,\eta}\stackrel{\mathrm{def}}{=}\sum_{\substack{j\in\mathcal{J}_0\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}V^j_\eta\quad;\quad U^\infty_{n,\eta}\stackrel{\mathrm{def}}{=}\sum_{\substack{j\in\mathcal{J}_\infty\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}V^j_\eta\\&\text{and}\quad\psi^J_{n,\eta}\stackrel{\mathrm{def}}{=}\sum_{\substack{j\in\mathcal{J}_0\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}V^j_{c\eta}+\sum_{\substack{j\in\mathcal{J}_\infty\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}V^j_{c\eta}+\psi^J_n.\end{aligned}\tag{2.39}$$

Let us point out that the main point is that, by virtue of Proposition 2.19, the terms $U^0_{n,\eta}$ and $U^\infty_{n,\eta}$ are small in the sense that, for any $\delta>0$, for any $\eta>0$, $\lim_{n\to+\infty}||U^0_{n,\eta}||_{\dot{H}^{s-\delta}}=0$ and $\lim_{n\to+\infty}||U^\infty_{n,\eta}||_{\dot{H}^{s+\delta}}=0$.

We recall a basic property due to divergence free condition: for any vector field u, smooth enough and divergence-free,

$$u \cdot \nabla v = \operatorname{div}(u \otimes v)\cdot \tag{2.40}$$

The property (2.40) provides us another expression of the exterior force term F_n^J

$$F_n^J = I_{n,\eta}^{J,1} + I_{n,\eta}^{J,2} + I_n^{J,3}. \tag{2.41}$$

where

$$\begin{aligned}
I_{n,\eta}^{J,1} &= \operatorname{div}\Big(\Big(2\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot-x_{n,j}) + e^{t\Delta}\Big(U_{n,\eta}^0 + U_{n,\eta}^\infty \,+\, \psi_{n,\eta}^J\Big)\Big)\otimes\, e^{t\Delta}\Big(U_{n,\eta}^0 + U_{n,\eta}^\infty\Big)\Big).\\
I_{n,\eta}^{J,2} &= \operatorname{div}\Big(\Big(2\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot-x_{n,j}) + e^{t\Delta}\Big(U_{n,\eta}^0 + U_{n,\eta}^\infty \,+\, \psi_{n,\eta}^J\Big)\Big)\,\otimes\, e^{t\Delta}\psi_{n,\eta}^J\Big).\\
I_n^{J,3} &= F_n^{J,1} = \sum_{\substack{0\leqslant j,k\leqslant J\,;j\neq k\\ (j,k)\in\mathcal{J}_1^2}} NS(V^j(t,\cdot-x_{n,j}))\cdot\nabla NS(V^k)(t,\cdot-x_{n,k}).
\end{aligned} \tag{2.42}$$

Concerning $I_{n,\eta}^{J,1}$, we apply (2.50) of Proposition 2.28, for any $\delta>0$, such that $\frac{1}{2}<s-\delta$ and $s+\delta<\frac{3}{2}$,

$$\begin{aligned}
\|I_{n,\eta}^{J,1}\|_{L_T^2(\dot H^{s-1})} \leqslant C\,T^{\frac{1}{2}(s-\frac{1}{2})}\,\Big(T^{\frac{-\delta}{2}}||U_{n,\eta}^0||_{\dot H^{s-\delta}} \,+\, T^{\frac{\delta}{2}}||U_{n,\eta}^\infty||_{\dot H^{s+\delta}}\Big)&\\
\times\Big\|2\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot-x_{n,j}) + e^{t\Delta}\Big(U_{n,\eta}^0+U_{n,\eta}^\infty+\psi_{n,\eta}^J\Big)\Big\|_{X_T^s}&\\
\leqslant C\,T^{\frac{1}{2}(s-\frac{1}{2})}\,\Big(T^{\frac{-\delta}{2}}||U_{n,\eta}^0||_{\dot H^{s-\delta}} \,+\, T^{\frac{\delta}{2}}||U_{n,\eta}^\infty||_{\dot H^{s+\delta}}\Big)&\\
\times\Big(2\,\|U_n^{\mathrm{app,J}}\|_{X_T^s} + \|e^{t\Delta}\Big(U_{n,\eta}^0+U_{n,\eta}^\infty+\psi_{n,\eta}^J\Big)\Big\|_{X_T^s}\Big)\,.&
\end{aligned} \tag{2.43}$$

From (2.31), we infer that

$$\begin{aligned}
\|I_{n,\eta}^{J,1}\|_{L_T^2(\dot H^{s-1})} \leqslant C\,T^{\frac{1}{2}(s-\frac{1}{2})}\,\Big(T^{\frac{-\delta}{2}}||U_{n,\eta}^0||_{\dot H^{s-\delta}} \,+\, T^{\frac{\delta}{2}}||U_{n,\eta}^\infty||_{\dot H^{s+\delta}}\Big)&\\
\times\Big(\|U_n^{\mathrm{app,J}}\|_{X_T^s} + \|u_{0,n}\|_{\dot H^s} + \circ(1)\Big)\,,\quad \text{when}\quad n\to+\infty.&
\end{aligned}$$

Propositions 2.19 and 2.20 implies that $I_{n,\eta}^{J,1}$ tends to 0 when n tends to infinity

$$\forall\varepsilon>0,\ \exists\tilde{n_1}(\varepsilon,J,\eta),\ \forall n\geqslant\tilde{n_1}(\varepsilon,J,\eta),\ \Big\|I_{n,\eta}^{J,1}\Big\|_{L_T^2(\dot H^{s-1})}\leqslant\varepsilon.$$

Concerning $I_{n,\eta}^{J,2}$, we apply the estimate (2.49) of Proposition 2.28

$$\begin{aligned}
\|I_{n,\eta}^{J,2}\|_{L_T^2(\dot H^{s-1})} \leqslant C\,T^{\frac{1}{2}(s-\frac{1}{2})}\,||\psi_{n,\eta}^J||_{\dot B^{s(1-\theta)}_{p_\theta,p_\theta}}&\\
\times\Big\|2\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot-x_{n,j}) + e^{t\Delta}\Big(U_{n,\eta}^0+U_{n,\eta}^\infty+\psi_{n,\eta}^J\Big)\Big\|_{X_T^s}&\\
\leqslant C\,T^{\frac{1}{2}(s-\frac{1}{2})}\,||\psi_{n,\eta}^J||_{\dot B^{s(1-\theta)}_{p_\theta,p_\theta}}\,\Big(\|U_n^{\mathrm{app,J}}\|_{X_T^s} + \|u_{0,n}\|_{\dot H^s} + \circ(1)\Big)\,,\quad \text{when}\quad n\to+\infty.&
\end{aligned} \tag{2.44}$$

Thanks to Proposition 2.18, we infer

$$\forall \varepsilon > 0,\ \exists \tilde{J}(\varepsilon),\ \forall J \geqslant \tilde{J}(\varepsilon),\ \exists \tilde{\eta}(J),\ \exists \tilde{n_2}(J), \forall \eta \geqslant \tilde{\eta}(J),\ \forall n \geqslant \tilde{n_2}(J),\ \|I^{J,2}_{n,\eta}\|_{L^2_T(\dot{H}^{s-1})} \leqslant \varepsilon.$$

Concerning $I^{J,3}_{n,\eta}$, the argument relies on the approximation Lemma 2.29 applied with with $\sigma = \frac{s}{2} + \frac{3}{4}$, which proof is given in Appendix A. For the sake of simplicity, we note:

$$\Phi^j = NS(V^j) \ \text{ and } \ \Phi^k = NS(V^k).$$

Remark 2.22. As Φ^j and Φ^k belong to the space $L^\infty_T(\dot{H}^s) \cap L^2_T(\dot{H}^{s+1})$, an interpolation argument implies they belong to the space $L^4_T(\dot{H}^{\frac{s}{2}+\frac{3}{4}})$. Indeed, we have

$$\|u\|_{\dot{H}^{\frac{s}{2}+\frac{3}{4}}} \leqslant \|u\|^{\frac{1}{2}(s+\frac{1}{2})}_{\dot{H}^s} \|u\|^{\frac{1}{2}(\frac{3}{2}-s)}_{\dot{H}^{s+1}}.$$

Then, by integrating in time, we deduce that

$$\|u\|^4_{L^4_T(\dot{H}^{\frac{s}{2}+\frac{3}{4}})} \leqslant T^{s-\frac{1}{2}} \|u\|^{2(s+\frac{1}{2})}_{L^\infty_T(\dot{H}^s)} \|u\|^{3-2s}_{L^2_T(\dot{H}^{s+1})}.$$

Thanks to the divergence-free condition, we have $\|\Phi^j.\nabla\Phi^k\|^2_{\dot{H}^{s-1}} = \|\Phi^j \otimes \Phi^k\|^2_{\dot{H}^s}$ and thus

$$\begin{aligned}\|\Phi^j.\nabla\Phi^k\|^2_{L^2_T(\dot{H}^{s-1})} &= \int_0^T \|\Phi^j \otimes \Phi^k\|^2_{\dot{H}^s}\\ &\leqslant \int_0^T \|(\Phi^j - \Phi^j_\varepsilon) \otimes \Phi^k\|^2_{\dot{H}^s} + \int_0^T \|\Phi^j_\varepsilon \otimes (\Phi^k - \Phi^k_\varepsilon)\|^2_{\dot{H}^s} + \int_0^T \|\Phi^j_\varepsilon \otimes \Phi^k_\varepsilon\|^2_{\dot{H}^s}.\end{aligned}$$

As $\dfrac{s}{2} + \dfrac{3}{4} < \dfrac{3}{2}$, a product rule in Sobolev spaces implies

$$\|u\, v\|_{\dot{H}^s} \leqslant C(s)\, \|u\|_{\dot{H}^{\frac{s}{2}+\frac{3}{4}}} \|v\|_{\dot{H}^{\frac{s}{2}+\frac{3}{4}}}. \tag{2.45}$$

Therefore, we infer that :

$$\begin{aligned}\|\Phi^j.\nabla\Phi^k\|^2_{L^2_T(\dot{H}^{s-1})} \lesssim &\int_0^T \|(\Phi^j - \Phi^j_\varepsilon)\|^2_{\dot{H}^{\frac{s}{2}+\frac{3}{4}}} \|\Phi^k\|^2_{\dot{H}^{\frac{s}{2}+\frac{3}{4}}} + \int_0^T \|\Phi^j_\varepsilon\|^2_{\dot{H}^{\frac{s}{2}+\frac{3}{4}}} \|\Phi^k - \Phi^k_\varepsilon\|^2_{\dot{H}^{\frac{s}{2}+\frac{3}{4}}}\\ &+ \int_0^T \|\Phi^j_\varepsilon \otimes \Phi^k_\varepsilon\|^2_{\dot{H}^s}.\end{aligned}$$

Finally, Cauchy-Schwarz inequality and approximation Lemma 2.29 yield

$$\|\Phi^j.\nabla\Phi^k\|^2_{L^2_T(\dot{H}^{s-1})} \lesssim \varepsilon^2 \|\Phi^k\|^2_{L^4_T(\dot{H}^{\frac{s}{2}+\frac{3}{4}})} + \varepsilon^2 \|\Phi^j\|^2_{L^4_T(\dot{H}^{\frac{s}{2}+\frac{3}{4}})} + \|\Phi^j_\varepsilon \otimes \Phi^k_\varepsilon\|^2_{L^2_T(\dot{H}^s)}.$$

To conclude, we have to prove that $\|\Phi^j_\varepsilon \otimes \Phi^k_\varepsilon\|^2_{L^2_T(\dot{H}^s)}$ tends to 0, for ε small enough. This will come from the orthogonality of cores. By definition, Φ^j_ε (resp. Φ^k_ε) is an approximation of Φ^j (resp. Φ^k). Because of translations by cores, we define $\Phi^{j,n}_\varepsilon(t, x - x_{n,j})$ (resp. $\Phi^{k,n}_\varepsilon(t, x - x_{n,k})$) as an approximation of $\Phi^j(t, x - x_{n,j})$ (resp. $\Phi^k(t, x - x_{n,k})$). As $\Phi^{j,n}_\varepsilon$ and $\Phi^{k,n}_\varepsilon$ are compactly supported and concentrated around $x_{n,j}$ and $x_{n,k}$, the divergence of cores ($\lim\limits_{n\to+\infty} |x_{n,j} - x_{n,k}| = +\infty$) implies they are supported by disjointed compacts. Therefore, the term $\|\Phi^j_\varepsilon \otimes \Phi^k_\varepsilon\|^2_{L^2_T(\dot{H}^s)}$ converges towards 0, for n large enough. In other words, we have

$$\forall \varepsilon > 0,\ \exists \tilde{n}(\varepsilon),\ \forall n \geqslant \tilde{n}(\varepsilon),\ \Big\| NS(V^j(t, \cdot - x_{n,j}))\cdot \nabla NS(V^k)(t, \cdot - x_{n,k})\Big\|_{L^2_T(\dot{H}^{s-1})} \leqslant \frac{\varepsilon}{|\mathcal{J}_1|}.$$

Therefore, we infer that, $\forall \varepsilon > 0,\ \exists \tilde{n}(\varepsilon),\ \forall n \geqslant \tilde{n}(\varepsilon)$,

$$\Big\| I^{J,3}_{n,\eta} \Big\|_{L^2_T(\dot{H}^{s-1})} = \Big\| \sum_{\substack{0 \leqslant j,k \leqslant J\ ; j \neq k\\ (j,k) \in \mathcal{J}_1^2}} NS(V^j(t, \cdot - x_{n,j}))\cdot \nabla NS(V^k)(t, \cdot - x_{n,k}) \Big\|_{L^2_T(\dot{H}^{s-1})} \leqslant \varepsilon. \tag{2.46}$$

This concludes the proof of Proposition 2.21.

2.5 Appendix A. Product and paraproduct estimates

In this section, we give some typical product estimates, in which splitting frequency allows for a much finer control of the product. The main tool is the homogeneous paradifferential calculus. For a detailed presentation of it, we refer the reader to [1], page 85. We recall two fundamental statements (see for instance Theorem 2.47 and 2.52 in [1]) about continuity of the homogeneous paraproduct operator T, and the remainder operator R. We shall constantly be using these two theorems in the sequel.

Theorem 2.23. *There exists a constant C such that for any real number s and any (p,r) in $[1,\infty]^2$, we have for any (u,v) in $L^\infty \times \dot{B}^s_{p,r}$,*

$$\|T_u v\|_{\dot{B}^s_{p,r}} \leqslant C^{1+|s|} \|u\|_{L^\infty} \|v\|_{\dot{B}^s_{p,r}}$$

Moreover, for any (s,t) in $\mathbb{R}\times]-\infty,0[$, (p,r_1,r_2) in $[1,\infty]^3$, and (u,v) in $\dot{B}^t_{\infty,r_1} \times \dot{B}^s_{p,r_2}$, we have

$$\|T_u v\|_{\dot{B}^{s+t}_{p,r}} \leqslant \frac{C^{1+|s+t|}}{-t} \|u\|_{\dot{B}^t_{\infty,r_1}} \|v\|_{\dot{B}^s_{p,r_2}} \quad \text{with} \quad \frac{1}{r} \stackrel{\text{def}}{=} \min\left\{1, \frac{1}{r_1} + \frac{1}{r_2}\right\}.$$

Theorem 2.24. *A constant C exists which satisfies the following properties.*
Let (s_1,s_2) be in $\mathbb{R}^2$ and (p_1,p_2,r_1,r_2) in $[1,\infty]^4$. Let us assume that

$$\frac{1}{p} \stackrel{\text{def}}{=} \frac{1}{p_1} + \frac{1}{p_2} \leqslant 1 \quad \text{and} \quad \frac{1}{r} \stackrel{\text{def}}{=} \frac{1}{r_1} + \frac{1}{r_2} \leqslant 1.$$

If s_1+s_2 is positive, then we have for any (u,v) in $\dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$,

$$\|R(u,v)\|_{\dot{B}^{s_1+s_2}_{p,r}} \leqslant \frac{C^{1+|s_1+s_2|}}{s_1+s_2} \|u\|_{\dot{B}^{s_1}_{p_1,r_1}} \|v\|_{\dot{B}^{s_2}_{p_2,r_2}}.$$

A lot of results of continuity may be deduced from the two above theorems. For instance, we can state the lemma below.

Lemma 2.25. *(Product rule in $\dot{H}^s$)*
Let u and v be two functions in $\dot{H}^s$ with $-\frac{3}{2} < s < \frac{3}{2}$, then

$$\|uv\|_{\dot{H}^s} \leqslant C(s)\left(\|u\|_{\dot{H}^s} \|v\|_{\dot{H}^{\frac{3}{2}}} + \|u\|_{\dot{H}^{\frac{3}{2}}} \|v\|_{\dot{H}^s}\right) \quad \text{and} \quad \|uv\|_{\dot{H}^s} \leqslant C(s) \|u\|_{L^\infty \cap \dot{H}^{\frac{3}{2}}} \|v\|_{\dot{H}^s}.$$

Proof. We have to estimate a product in Sobolev space thus, we shall use the paradifferential calculus. In particular, thanks to the Bony's paraproduct decomposition, we get

$$u\,v = T_u v + R(u,v) + T_v u.$$

The term $R(v,u)$ can be estimated in $\dot{H}^s$-norm easily, thanks to Theorem 2.24,

$$\|R(u,v)\|_{\dot{B}^{s+\frac{3}{2}}_{1,1}} \leqslant C\|u\|_{\dot{H}^s} \|v\|_{\dot{H}^{\frac{3}{2}}}.$$

Therefore, thanks to embeddings $\dot{B}^{s+\frac{3}{2}}_{1,1} \hookrightarrow \dot{B}^s_{2,1} \hookrightarrow \dot{B}^s_{2,2}$ and Remark 2.16, we infer that

$$\|R(u,v)\|_{\dot{H}^s} \leqslant C\|u\|_{\dot{H}^s}\|v\|_{\dot{H}^{\frac{3}{2}}}. \tag{2.47}$$

Concerning $T_v u$ and $T_u v$, we use once again estimates of Theorem 2.23, which gives

$$\|T_u v\|_{\dot{H}^s} \leqslant C(s)\, \|u\|_{\dot{B}^{s-\frac{3}{2}}_{\infty,\infty}} \|v\|_{\dot{B}^{\frac{3}{2}}_{2,2}}.$$

Because $s - \frac{3}{2}$ is negative, Bernstein's inequality and the classical embedding $\ell^2(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$ give

$$\dot{B}^s_{2,2} \hookrightarrow \dot{B}^{s-\frac{3}{2}}_{\infty,2} \hookrightarrow \dot{B}^{s-\frac{3}{2}}_{\infty,\infty}.$$

Therefore, we deduce that

$$\|T_u v\|_{\dot{H}^s} \leqslant C(s)\, \|u\|_{\dot{H}^s} \|v\|_{\dot{H}^{\frac{3}{2}}}.$$

Permuting the roles of u and v and using (2.47) gives the first part of the result. The second part of Lemma 2.25 is easy. By virtue of Theorem 2.23, we have

$$\|T_u v\|_{\dot{H}^s} \leqslant C(s)\, \|u\|_{L^\infty} \|v\|_{\dot{H}^s} \quad \text{and} \quad \|T_v u\|_{\dot{H}^s} \leqslant C(s)\, \|v\|_{\dot{H}^s} \|u\|_{\dot{H}^{\frac{3}{2}}}.$$

Moreover, it seems clear that, due to Theorem 2.24

$$\|R(u,v)\|_{\dot{B}^{s+\frac{3}{2}}_{1,1}} \leqslant C\|v\|_{\dot{H}^s} \|u\|_{\dot{H}^{\frac{3}{2}}}.$$

This leads to the proof of $\|uv\|_{\dot{H}^s} \leqslant C(s)\, \|u\|_{L^\infty \cap \dot{H}^{\frac{3}{2}}} \|v\|_{\dot{H}^s}$. □

Remark 2.26. Let us point out an interpolation inequality: by definition of s we have

$$\|u\|_{\dot{H}^{\frac{3}{2}}} \leqslant C\|u\|^{s-\frac{1}{2}}_{\dot{H}^s} \|u\|^{\frac{3}{2}-s}_{\dot{H}^{s+1}}. \tag{2.48}$$

Therefore, combining this with Lemma 2.25, we get the result following which will be a frequent use later on.

Corollary 2.27. *Let u and v be in $\dot{H}^s$ with $\frac{1}{2} < s < \frac{3}{2}$, then*

$$\|uv\|_{\dot{H}^s} \leqslant C(s)\left(\|u\|_{\dot{H}^s} \|v\|^{s-\frac{1}{2}}_{\dot{H}^s} \|v\|^{\frac{3}{2}-s}_{\dot{H}^{s+1}} + \|u\|^{s-\frac{1}{2}}_{\dot{H}^s} \|u\|^{\frac{3}{2}-s}_{\dot{H}^{s+1}} \|v\|_{\dot{H}^s}\right).$$

Proposition 2.28. *Let $0 < \theta < 1$.*
Under the interpolation relation $\dfrac{1}{p_\theta} = \dfrac{\theta}{p} + \dfrac{1-\theta}{2}$ with $\dfrac{1}{p} = \dfrac{1}{2} - \dfrac{s}{3}$,

$$\|u \otimes e^{t\Delta} r_0\|_{L^2_T(\dot{H}^s)} \leqslant C\, T^{\frac{1}{2}(s-\frac{1}{2})} \|u\|_{X^s_T} ||r_0||_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}. \tag{2.49}$$

$$\text{For any } \frac{1}{2} < \alpha < \frac{3}{2},\ \|u \otimes e^{t\Delta} r_0\|_{L^2_T(\dot{H}^s)} \leqslant C\, T^{\frac{1}{2}(\alpha-\frac{1}{2})} \|u\|_{X^s_T} ||r_0||_{\dot{H}^\alpha}. \tag{2.50}$$

Proof. Let us start by proving the first inequality. Bony's paraproduct decomposition implies

$$u \otimes e^{t\Delta} r_0 = T_{e^{t\Delta} r_0} u + R(e^{t\Delta} r_0, u) + T_u(e^{t\Delta} r_0).$$

The first two terms can be estimated in $\dot{H}^s$-norm easily. Thanks to Theorem 2.23, we have

$$\|T_{e^{t\Delta} r_0}(u)\|_{\dot{H}^s = \dot{B}^s_{2,2}} \leqslant C\|e^{t\Delta} r_0\|_{\dot{B}^{s-\frac{3}{2}}_{\infty,\infty}} \|u\|_{\dot{B}^{\frac{3}{2}}_{2,2}}.$$

By virtue of Theorem 2.24, we also have

$$\|R(e^{t\Delta}r_0,u)\|_{\dot H^s=\dot B^s_{2,2}} \leqslant C\|e^{t\Delta}r_0\|_{\dot B^{s-\frac32}_{\infty,\infty}}\|u\|_{\dot B^{\frac32}_{2,2}}.$$

Let us recall that $\dfrac{1}{p_\theta}$ is defined by $\dfrac{1}{p_\theta}=\dfrac{\theta}{p}+\dfrac{1-\theta}{2}$, for any θ in $]0,1[$ and with $\frac1p=\frac12-\frac s3$.
A classical result due to Bernstein's inequality gives the following embedding $\dot B^{s(1-\theta)}_{p_\theta,\infty}\hookrightarrow \dot B^{s-\frac32}_{\infty,\infty}$.
Therefore we infer that

$$\|R(e^{t\Delta}r_0,u)\|_{L^2_T(\dot H^s)}+\|T_{e^{t\Delta}r_0}(u)\|_{L^2_T(\dot H^s)}\lesssim \|e^{t\Delta}r_0\|_{L^\infty_T(\dot B^{s(1-\theta)}_{p_\theta,\infty})}\|u\|_{L^2_T(\dot H^{\frac32})}.$$

On the one hand, thanks to the hypothesis $\frac12<s<\frac32$, we recover the Navier-Stokes solution u in X^s_T-norm by an interpolation argument. As $\|u\|_{\dot H^{\frac32}}\lesssim \|u\|^{s-\frac12}_{\dot H^s}\ \|u\|^{\frac32-s}_{\dot H^{s+1}}$, we get

$$\begin{aligned}\|u\|^2_{L^2_T(\dot H^{\frac32})}=\int_0^T\|u\|^{2s-1}_{\dot H^s}\|u\|^{3-2s}_{\dot H^{s+1}}dt&\lesssim \|u\|^{2s-1}_{L^\infty_T(\dot H^s)}\ T^{s-\frac12}\ \|u\|^{3-2s}_{L^2_T(\dot H^{s+1})}\\&\lesssim T^{s-\frac12}\ \|u\|^2_{X^s_T}.\end{aligned}\tag{2.51}$$

On the other hand, the simple embedding $\ell^{p_\theta}(\mathbb Z)\subset\ell^\infty(\mathbb Z)$ implies

$$\|e^{t\Delta}r_0\|_{L^\infty_T(\dot B^{s(1-\theta)}_{p_\theta,\infty})}\leqslant \|r_0\|_{\dot B^{s(1-\theta)}_{p_\theta,\infty}}\leqslant \|r_0\|_{\dot B^{s(1-\theta)}_{p_\theta,p_\theta}}.$$

Finally, we have proved the proposition for the first two terms

$$\|R(e^{t\Delta}r_0,u)\|_{L^2_T(\dot H^s)}+\|T_{e^{t\Delta}r_0}(u)\|_{L^2_T(\dot H^s)}\lesssim T^{\frac12(s-\frac12)}\ \|r_0\|_{\dot B^{s(1-\theta)}_{p_\theta,p_\theta}}\|u\|_{X^s_T}.$$

The last term $T_u(e^{t\Delta}r_0)$ is more delicate. Note that, here, as we work locally in time, low frequencies do not play a major role, unlike high frequencies. As a result, we have to handle low and high frequencies separately. It is natural to split them according to their size: either the frequencies are low (in the sense that $\sqrt T 2^j\leqslant C$) or the frequenties are high (in the sense that $\sqrt T2^j\geqslant C$).
Firstly, let us observe that

$$\|T_u(e^{t\Delta}r_0)\|_{L^2_T(\dot H^s)}=\|T_u(e^{t\Delta}r_0)\|_{L^2_T(\dot B^s_{2,2})}=\Big(2^{js}\|\dot\Delta_jT_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}\Big)_{\ell^2(\mathbb Z)}.$$

We split, according to low and high frequencies

$$\begin{aligned}\|T_u(e^{t\Delta}r_0)\|_{L^2_T(\dot H^s)}\leqslant&\Big(2^{js}\|\dot\Delta_jT_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}1_{\{\sqrt T2^j\leqslant C\}}\Big)_{\ell^2(\mathbb Z)}\\&+\Big(2^{js}\|\dot\Delta_jT_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}1_{\{\sqrt T2^j\geqslant C\}}\Big)_{\ell^2(\mathbb Z)}.\end{aligned}\tag{2.52}$$

A classical result in Littlewood Paley theory gives the following identity

$$\dot\Delta_jT_u(e^{t\Delta}r_0)=\sum_{|j-j'|\leqslant4}\dot S_{j'-1}u\ \dot\Delta_{j'}(e^{t\Delta}r_0).$$

Therefore, Hölder's inequality yields

$$\|\dot\Delta_jT_u(e^{t\Delta}r_0)\|_{L^2}\leqslant\sum_{|j-j'|\leqslant4}\|\dot S_{j'-1}u\|_{L^{q_\theta}}\|\dot\Delta_{j'}(e^{t\Delta}r_0)\|_{L^{p_\theta}}\quad\text{with}\quad \frac12=\frac{1}{p_\theta}+\frac{1}{q_\theta}.$$

In particular, Bernstein's inequality implies

$$\begin{aligned}\|\dot S_{j'-1}u\|_{L^{q_\theta}} &\leqslant \sum_{j''=-\infty}^{j'-2}\|\dot\Delta_{j''}u\|_{L^{q_\theta}}\\ &\lesssim \sum_{j''=-\infty}^{j'-2} 2^{3j''(\frac12-\frac1{q_\theta})}\|\dot\Delta_{j''}u\|_{L^2} = \sum_{j''=-\infty}^{j'-2} 2^{j''(\frac{3}{p_\theta}-s)}2^{j''s}\|\dot\Delta_{j''}u\|_{L^2}.\end{aligned}$$

Applying Young's inequality, we infer there exists for any t, a sequence $(c_j(t))_{j\in\mathbb{Z}}$ belonging to the sphere of $\ell^{q_\theta}(\mathbb{Z})$, such that

$$\|\dot S_{j'-1}u\|_{L^{q_\theta}} \leqslant C\, c_{j'}(t)\; 2^{j'(\frac{3}{p_\theta}-s)}\; \|u(t)\|_{\dot B^s_{2,q_\theta}}.$$

As $q_\theta \geqslant 2$, $\ell^2(\mathbb{Z})$ is included in $\ell^{q_\theta}(\mathbb{Z})$, which implies that

$$\|\dot S_{j'-1}u\|_{L^{q_\theta}} \leqslant C\, c_{j'}(t)\; 2^{j'(\frac{3}{p_\theta}-s)}\; \|u(t)\|_{\dot B^s_{2,2}}.$$

Therefore, we have

$$\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2} \lesssim \sum_{|j-j'|\leqslant 4} c_{j'}(t)2^{j'(\frac{3}{p_\theta}-2s+\theta\, s)}\|u(t)\|_{\dot B^s_{2,2}}\; 2^{sj'(1-\theta)}\|\dot\Delta_{j'}(e^{t\Delta}r_0)\|_{L^{p_\theta}}.$$

As j and j' are equivalent, we can write

$$\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2} \lesssim c_j(t)2^{j(\frac{3}{p_\theta}-2s+\theta\, s)}\|u(t)\|_{\dot B^s_{2,2}}\; 2^{js(1-\theta)}\|\dot\Delta_j(e^{t\Delta}r_0)\|_{L^{p_\theta}}. \tag{2.53}$$

On the other hand, we have (see for instance Lemma 2.4 of [1])

$$\|\dot\Delta_{j'}(e^{t\Delta}r_0)\|_{L^{p_\theta}} \lesssim e^{-t2^{2j'}}\|\dot\Delta_{j'}r_0\|_{L^{p_\theta}}. \tag{2.54}$$

As $e^{-t2^{2j'}} \leqslant 1$, integration in time yields

$$\|\dot\Delta_{j'}(e^{t\Delta}r_0)\|_{L^{p_\theta}_T(L^{p_\theta})} \lesssim T^{\frac{1}{p_\theta}}\|\dot\Delta_{j'}r_0\|_{L^{p_\theta}}.$$

Above result combining with Hölder's inequality in time imply

$$2^{js}\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)} \lesssim 2^{j(\frac{3}{p_\theta}-s+\theta\, s)}\|u(t)\|_{L^\infty_T(\dot H^s)}\|c_j(t)\|_{L^{q_\theta}_T}\; T^{\frac{1}{p_\theta}}\; 2^{js(1-\theta)}\; \|\dot\Delta_j r_0\|_{L^{p_\theta}}.$$

Therefore, as far as the low frequencies are concerned ($\sqrt{T}2^j \leqslant C$), we have

$$\begin{aligned}2^{js}\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}1_{\{\sqrt{T}2^j\leqslant C\}} &\lesssim T^{-\frac12(\frac{3}{p_\theta}-s+\theta\, s)}\|u(t)\|_{L^\infty_T(\dot H^s)}\|c_j(t)\|_{L^{q_\theta}_T}\\ &\quad\times T^{\frac{1}{p_\theta}}\; 2^{js(1-\theta)}\; \|\dot\Delta_j r_0\|_{L^{p_\theta}}.\end{aligned}$$

Applying Hölder's inequality for the $\ell^2(\mathbb{Z})$-norm, we have

$$\begin{aligned}\Big(2^{js}\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}1_{\{\sqrt{T}2^j\leqslant C\}}\Big)_{\ell^2(\mathbb{Z})} &\leqslant T^{-\frac12(\frac{3}{p_\theta}-s+\theta\, s)+\frac{1}{p_\theta}}\;\|u(t)\|_{L^\infty_T(\dot H^s)}\\ &\quad\times\Big(\|c_j(t)\|_{L^{q_\theta}_T}\Big)_{\ell^{q_\theta}(\mathbb{Z})}\|r_0\|_{\dot B^{s(1-\theta)}_{p_\theta,p_\theta}}.\end{aligned}$$

Clearly, we have $\left(\|c_j(t)\|_{L_T^{q_\theta}}\right)_{\ell^{q_\theta}(\mathbb{Z})} \leqslant T^{\frac{1}{q_\theta}}$. Besides, we have

$$-\frac{1}{2}\left(\frac{3}{p_\theta} - s + \theta\, s\right) + \frac{1}{p_\theta} + \frac{1}{q_\theta} = -\frac{1}{2}\left(\frac{3}{p_\theta} - s + \theta\, s - 1\right) = -\frac{1}{2}\left(\frac{3\theta}{p} + \frac{3(1-\theta)}{2} - s + \theta\, s - 1\right)$$
$$= -\frac{1}{2}\left(\frac{3\theta}{2} - \theta\, s + \frac{3(1-\theta)}{2} - s + \theta\, s - 1\right) = \frac{1}{2}\left(s - \frac{1}{2}\right).$$

As a result, we infer that

$$\left(2^{js}\|\dot{\Delta}_j T_u(e^{t\Delta} r_0)\|_{L_T^2(L^2)} 1_{\{\sqrt{T}2^j \leqslant C\}}\right)_{\ell^2(\mathbb{Z})} \leqslant T^{\frac{1}{2}(s-\frac{1}{2})}\ \|u(t)\|_{L_T^\infty(\dot{H}^s)} \|r_0\|_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}.$$

This completes the proof in the case of low frequencies. For the high frequencies, we need to use the smoothing effect of the heat flow. Thanks to (2.54), we infer

$$\|\dot{\Delta}_{j'}(e^{t\Delta} r_0)\|_{L_T^{p_\theta}(L^{p_\theta})} \lesssim 2^{\frac{-2j'}{p_\theta}} \|\dot{\Delta}_{j'} r_0\|_{L^{p_\theta}}. \tag{2.55}$$

We write an estimate for $2^{js}\|\dot{\Delta}_j T_u(e^{t\Delta} r_0)\|_{L_T^2(L^2)} 1_{\{\sqrt{T}2^j \geqslant C\}}$. We come back to (2.53), we integrate in time, applying Hölder's inequality

$$\begin{aligned} 2^{js}\|\dot{\Delta}_j T_u(e^{t\Delta} r_0)\|_{L_T^2(L^2)} \lesssim{}& 2^{j(\frac{1}{p_\theta} - s + \theta\, s)}\ \|u(t)\|_{L_T^\infty(\dot{H}^s)}\ \|c_j(t)\|_{L_T^{q_\theta}} \\ &\times 2^{j\left((1-\theta)s + \frac{2}{p_\theta}\right)}\ \|\dot{\Delta}_j(e^{t\Delta} r_0)\|_{L_T^{p_\theta}(L^{p_\theta})}. \end{aligned} \tag{2.56}$$

High frequencies hypothesis implies

$$\begin{aligned} 2^{js}\|\dot{\Delta}_j T_u(e^{t\Delta} r_0)\|_{L_T^2(L^2)} 1_{\{\sqrt{T}2^j \geqslant C\}} \lesssim{}& T^{-\frac{1}{2}(\frac{1}{p_\theta} - s + \theta\, s)} \|u(t)\|_{L_T^\infty(\dot{H}^s)} \|c_j(t)\|_{L_T^{q_\theta}} \\ &\times 2^{j((1-\theta)s + \frac{2}{p_\theta})}\ \|\dot{\Delta}_j(e^{t\Delta} r_0)\|_{L_T^{p_\theta}(L^{p_\theta})}. \end{aligned}$$

Thanks to (2.55), we infer

$$\begin{aligned} 2^{js}\|\dot{\Delta}_j T_u(e^{t\Delta} r_0)\|_{L_T^2(L^2)} 1_{\{\sqrt{T}2^j \geqslant C\}} \lesssim{}& T^{-\frac{1}{2}(\frac{1}{p_\theta} - s + \theta\, s)} \|u(t)\|_{L_T^\infty(\dot{H}^s)} \|c_j(t)\|_{L_T^{q_\theta}} \\ &\times 2^{j((1-\theta)s)}\ \|\dot{\Delta}_j r_0\|_{L^{p_\theta}}. \end{aligned} \tag{2.57}$$

Once again, we apply Hölder's inequality for the $\ell^2(\mathbb{Z})$-norm and we have

$$\left(2^{js}\|\dot{\Delta}_j T_u(e^{t\Delta} r_0)\|_{L_T^2(L^2)} 1_{\{\sqrt{T}2^j \geqslant C\}}\right)_{\ell^2(\mathbb{Z})} \lesssim T^{-\frac{1}{2}(\frac{1}{p_\theta} - s + \theta\, s) + \frac{1}{q_\theta}} \|u(t)\|_{L_T^\infty(\dot{H}^s)} \|r_0\|_{L_T^{p_\theta}(\dot{B}^{(1-\theta)s}_{p_\theta,p_\theta})}.$$

Then, the simple computation $-\frac{1}{2}\Big(\frac{1}{p_\theta} - s + \theta\, s\Big) + \frac{1}{q_\theta} = \frac{1}{2}\Big(s - \frac{1}{2}\Big)$ implies

$$\left(2^{j}\|\dot{\Delta}_j T_u(e^{t\Delta} r_0)\|_{L_T^2(L^2)} 1_{\{\sqrt{T}2^j \geqslant C\}}\right)_{\ell^2(\mathbb{Z})} \lesssim T^{\frac{1}{2}(s-\frac{1}{2})}\ \|u(t)\|_{L_T^\infty(\dot{H}^s)} \|r_0\|_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}.$$

This ends up the proof for the case of high frequencies and therefore the first inequality of the proposition is proved.

Now, let us prove the second inequality, which proof is very close to the previous one. We gives only outlines. Thanks to Bony's decomposition, we have

$$u \otimes e^{t\Delta} r_0 = T_{e^{t\Delta} r_0} u + R(e^{t\Delta} r_0, u) + T_u(e^{t\Delta} r_0).$$

The first two terms can be estimated in $\dot{H}^s$-norm easily, thanks to mapping of paraproduct in the Besov spaces (cf Theorem 2.23)

$$\|T_{e^{t\Delta} r_0} u\|_{\dot{B}^s_{2,2}} \leqslant C \|e^{t\Delta} r_0\|_{\dot{B}^{\alpha-\frac{3}{2}}_{\infty,\infty}} \|u\|_{\dot{B}^{s+\frac{3}{2}-\alpha}_{2,2}}.$$

On the one hand, Bernstein's Lemma and obvious embedding $\ell^2(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$ ensure that

$$\dot{B}^\alpha_{2,2} \hookrightarrow \dot{B}^{\alpha-\frac{3}{2}}_{\infty,2} \hookrightarrow \dot{B}^{\alpha-\frac{3}{2}}_{\infty,\infty} \quad \text{and thus} \quad \|e^{t\Delta} r_0\|_{\dot{B}^{\alpha-\frac{3}{2}}_{\infty,\infty}} \lesssim \|e^{t\Delta} r_0\|_{\dot{B}^\alpha_{2,2}}.$$

On the other hand, as $s \leqslant s + \frac{3}{2} - \alpha \leqslant s + 1$, u belongs to $\dot{B}^{s+\frac{3}{2}-\alpha}_{2,2}$. Interpolation argument yields

$$\|u\|_{\dot{B}^{s+\frac{3}{2}-\alpha}_{2,2}} \leqslant C \|u\|_{\dot{H}^{s+\frac{3}{2}-\alpha}} \leqslant \|u\|^{\alpha-\frac{1}{2}}_{\dot{H}^s} \|u\|^{\frac{3}{2}-\alpha}_{\dot{H}^{s+1}}.$$

By integration in time and thanks to Hölder's inequality, we have

$$\begin{aligned} \|u\|^2_{L^2_T(\dot{H}^{s+\frac{3}{2}-\alpha})} &\leqslant \int_0^T \|u(t,\cdot)\|^{2\alpha-1}_{\dot{H}^s} \|u(t,\cdot)\|^{3-2\alpha}_{\dot{H}^{s+1}} \, dt \\ &\leqslant T^{\alpha-\frac{1}{2}} \|u\|^{2\alpha-1}_{L^\infty_T(\dot{H}^s)} \|u\|^{3-2\alpha}_{L^2_T(\dot{H}^{s+1})}. \end{aligned}$$

Finally, we get

$$\|u\|_{L^2_T(\dot{H}^{s+\frac{3}{2}-\alpha})} \leqslant T^{\frac{1}{2}(\alpha-\frac{1}{2})} \|u\|_{X^s_T}.$$

Therefore, we deduce an estimate of the term $\|T_{e^{t\Delta} r_0}(u)\|_{L^2_T(\dot{H}^s)}$ and $\|R(e^{t\Delta} r_0, u)\|_{L^2_T(\dot{H}^s)}$.

$$\|T_{e^{t\Delta} r_0} u\|_{L^2_T(\dot{H}^s)} \leqslant T^{\frac{1}{2}(\alpha-\frac{1}{2})} \|u\|_{X^s_T} \|r_0\|_{\dot{H}^\alpha}.$$

$$\|R(e^{t\Delta} r_0, u)\|_{L^2_T(\dot{H}^s)} \leqslant T^{\frac{1}{2}(\alpha-\frac{1}{2})} \|u\|_{X^s_T} \|r_0\|_{\dot{H}^\alpha}.$$

Now, in order to estimate the last term $\|T_u(e^{t\Delta} r_0)\|_{L^2_T(\dot{H}^s)}$, we shall need splitting, according low and high frequencies (e.g $\sqrt{T}\, 2^j \leqslant 1$ or $\sqrt{T}\, 2^j \geqslant 1$). That is exactly the same computations as in the proof of the first inequality of the proposition

$$\|\dot{\Delta}_j T_u(e^{t\Delta} r_0)\|_{L^2} \lesssim \sum_{|j-j'|\leqslant 4} \|\dot{S}_{j'-1} u\|_{L^p} \|\dot{\Delta}_{j'}(e^{t\Delta} r_0)\|_{L^{\frac{3}{s}}}.$$

Thanks to the property $\|\dot{S}_{j'-1} u\|_{L^p} \lesssim \|u\|_{L^p}$ and the equivalence between j and j', we get

$$\|\dot{\Delta}_j T_u(e^{t\Delta} r_0)\|_{L^2} \lesssim \|u\|_{L^p} \|\dot{\Delta}_j(e^{t\Delta} r_0)\|_{L^{\frac{3}{s}}}.$$

By virtue of Sobolev embedding and integration in time

$$2^{js} \|\dot{\Delta}_j T_u(e^{t\Delta} r_0)\|_{L^2_T(L^2)} \leqslant 2^{js} \|u\|_{L^\infty_T(\dot{H}^s)} \|\dot{\Delta}_j(e^{t\Delta} r_0)\|_{L^2_T(L^{\frac{3}{s}})}.$$

Concerning low frequencies (e.g $\sqrt{T}\,2^j \leqslant 1$), we combine (2.54) with the rough boundary $e^{-t2^{2j}} \leqslant 1$ and we get

$$\begin{aligned}2^{js}\,\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}\,1_{\{\sqrt{T}2^j\leqslant C\}} &\lesssim 2^{js}\,\|u\|_{L^\infty_T(\dot H^s)}\,\|\dot\Delta_j(r_0)\|_{L^2_T(L^{\frac{3}{s}})}\\ &\lesssim 2^{js}\,\|u\|_{L^\infty_T(\dot H^s)}\,2^{-j(\alpha+s-\frac{3}{2})}\,2^{j(\alpha+s-\frac{3}{2})}\,\|\dot\Delta_j(r_0)\|_{L^2_T(L^{\frac{3}{s}})}\\ &\lesssim 2^{-j(\alpha-\frac{3}{2})}\,\|u\|_{L^\infty_T(\dot H^s)}\,2^{j(\alpha+s-\frac{3}{2})}\,T^{\frac12}\,\|\dot\Delta_j(r_0)\|_{L^{\frac{3}{s}}}.\end{aligned}$$

Hypothesis of low frequencies implies

$$\Big(2^{js}\,\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}\,1_{\{\sqrt{T}2^j\leqslant C\}}\Big)_{\ell^2(\mathbb{Z})} \lesssim T^{\frac12(\alpha-\frac12)}\,\|u\|_{L^\infty_T(\dot H^s)}\,\|r_0\|_{\dot B^{\alpha+s-\frac32}_{\frac{3}{s},2}}. \tag{2.58}$$

As far as high frequencies are concerned (e.g $\sqrt{T}\,2^j \geqslant 1$), (2.54) combining with the integration of the term $e^{-t2^{2j}}$ on $[0,T]$, gives

$$\begin{aligned}2^{js}\,\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}\,1_{\{\sqrt{T}2^j\geqslant C\}} &\lesssim 2^{js}\,\|u\|_{L^\infty_T(\dot H^s)}\,2^{-j}\|\dot\Delta_j(r_0)\|_{L^{\frac{3}{s}}}\\ &\lesssim 2^{j(s-1)}\,\|u\|_{L^\infty_T(\dot H^s)}\,2^{-j(\alpha+s-\frac{3}{2})}\,2^{j(\alpha+s-\frac{3}{2})}\,\|\dot\Delta_j(r_0)\|_{L^{\frac{3}{s}}}\\ &\lesssim 2^{-j(\alpha-\frac{1}{2})}\,\|u\|_{L^\infty_T(\dot H^s)}\,2^{j(\alpha+s-\frac{3}{2})}\,\|\dot\Delta_j(r_0)\|_{L^{\frac{3}{s}}}.\end{aligned}$$

Hypothesis of high frequencies gives

$$\Big(2^{js}\,\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}\,1_{\{\sqrt{T}2^j\geqslant C\}}\Big)_{\ell^2(\mathbb{Z})} \lesssim T^{\frac12(\alpha-\frac12)}\,\|u\|_{L^\infty_T(\dot H^s)}\,\|r_0\|_{\dot B^{\alpha+s-\frac32}_{\frac{3}{s},2}}. \tag{2.59}$$

Combining (2.58) and (2.59) with the fact that $\dot B^\alpha_{2,2}$ is embedded in $\dot B^{\alpha+s-\frac32}_{\frac{3}{s},2}$, we get finally

$$\|T_u(e^{t\Delta}r_0)\|_{L^2_T(\dot H^s)} \lesssim T^{\frac12(\alpha-\frac12)}\,\|u\|_{L^\infty_T(\dot H^s)}\,\|r_0\|_{\dot H^\alpha}. \tag{2.60}$$

This completes the proof of the second inequality of the proposition. Now, let us state an approximation lemma. □

Lemma 2.29. *Let $0<\sigma<\frac32$ and $\varepsilon>0$. Let a be an element of $L^4_T(\dot H^\sigma)$. Then, there exists a constant $C>0$, there exists a family of compactly supported functions (in space variables), a_ε, which satisfies for any positive T*

$$\lim_{\varepsilon\to0}\|a-a_\varepsilon\|_{L^4_T(\dot H^\sigma)} = 0 \quad \textit{and} \tag{2.61}$$

$$\|a_\varepsilon\|_{L^4_T(\dot H^\sigma)} \leqslant C\,\|a\|_{L^4_T(\dot H^\sigma)}. \tag{2.62}$$

Proof. Let us introduce the approximation function a_ε defined by

$$a_\varepsilon = \chi(\varepsilon\cdot)a,$$

where χ is the usual fonction of $\mathcal{D}(\mathbb{R}^3)$ with value 1 near 0.
Let us start by proving (2.62). Due to product rule in Sobolev spaces recalled in Lemma 2.25, we have

$$\begin{aligned}\|a_\varepsilon\|_{\dot H^\sigma} &\leqslant C(\sigma)\,\|\chi(\varepsilon\cdot)\|_{L^\infty\cap\dot H^{\frac32}}\,\|a\|_{\dot H^\sigma}\\ &\leqslant C(\sigma)\,\|\chi\|_{L^\infty\cap\dot H^{\frac32}}\,\|a\|_{\dot H^\sigma}\\ &\leqslant C(\sigma)\,\|a\|_{\dot H^\sigma}\end{aligned} \tag{2.63}$$

Now we prove (2.61). In order to apply Lebesgue Theorem, we have to prove there exists a positive constant C, such that

$$\lim_{\varepsilon\to 0}\|a_\varepsilon - a\|_{\dot H^\sigma} = 0 \quad \text{and} \quad \|a_\varepsilon - a\|_{\dot H^\sigma} \leqslant C. \tag{2.64}$$

Let us notice that $\|a_\varepsilon - a\|_{\dot H^\sigma}$ is bounded, thanks to (2.63). Concerning the proof of $\lim_{\varepsilon\to 0}\|a_\varepsilon - a\|_{\dot H^\sigma} = 0$, one way is to approach the function a by an truncated element A_η which Fourier transform is defined by $\widehat{A_\eta}(\xi) = \widehat{a}(\xi)1_{\{\eta\leqslant|\xi|\leqslant\frac{1}{\eta}\}}$. In this way, by virtue of Lebesgue Theorem, it seems clear that

$$\lim_{\eta\to 0}\|A_\eta - a\|_{\dot H^\sigma} = 0. \tag{2.65}$$

Therefore, we have

$$\begin{aligned}\|a_\varepsilon - a\|_{\dot H^\sigma} &= \|(1-\chi(\varepsilon\,\cdot))\,a\|_{\dot H^\sigma}\\ &\leqslant \|(1-\chi(\varepsilon\,\cdot))\,(a-A_\eta)\,\|_{\dot H^\sigma} + \|(1-\chi(\varepsilon\,\cdot))\,A_\eta\|_{\dot H^\sigma}.\end{aligned}$$

By virtue of Lemma 2.25, we have

$$\begin{aligned}\|a_\varepsilon - a\|_{\dot H^\sigma} &\leqslant \|1-\chi(\varepsilon\,\cdot)\|_{\dot H^{\frac32}\cap L^\infty}\,\|a-A_\eta\|_{\dot H^\sigma} + \|(1-\chi(\varepsilon\,\cdot))\,A_\eta\|_{\dot H^\sigma}\\ &\leqslant \left(1+\|\chi\|_{\dot H^{\frac32}\cap L^\infty}\right)\|a-A_\eta\|_{\dot H^\sigma} + \|(1-\chi(\varepsilon\,\cdot))\,A_\eta\|_{\dot H^\sigma}.\end{aligned}$$

Now, we have just to prove that $\lim_{\varepsilon\to 0}\|(1-\chi(\varepsilon\,\cdot))\,A_\eta\|_{\dot H^\sigma} = 0$. This comes from an interpolation argument. For any $0<\sigma<\frac32$ and $\sigma<s<\dfrac{3}{2}$,

$$\begin{aligned}\|(1-\chi(\varepsilon\,\cdot))\,A_\eta\|_{\dot H^\sigma} &\leqslant \|(1-\chi(\varepsilon\,\cdot))\,A_\eta\|_{L^2}^{1-\frac{\sigma}{s}}\,\|(1-\chi(\varepsilon\,\cdot))\,A_\eta\|_{\dot H^s}^{\frac{\sigma}{s}}\\ &\leqslant \|(1-\chi(\varepsilon\,\cdot))\,A_\eta\|_{L^2}^{1-\frac{\sigma}{s}}\left(1+\|\chi\|_{\dot H^{\frac32}\cap L^\infty}\right)\|A_\eta\|_{\dot H^s}^{\frac{\sigma}{s}}.\end{aligned}$$

To conclude, we have just to notice that the term $\|(1-\chi(\varepsilon\,\cdot))\,A_\eta\|_{L^2}^{1-\frac{\sigma}{s}}$ tends to 0 for ε small enough, by virtue of Lebesgue Theorem. The other term is obviously bounded, since A_η belongs to any Sobolev spaces, for any $\varepsilon>0$, thanks to truncature process. □

2.6 Appendix B.

In this appendix, we prove a general theorem about an estimate in the X^s_T-space of a solution of a perturbed Navier-Stokes system. The method is standard: the first step consists in establishing an $\dot H^s$-energy estimate. Then, some computations on scalar-product terms lead to an inequality on which we can apply Gronwall's lemma. In particular, we apply this theorem to prove that the map $u_0 \mapsto T_*(u_0)$ is a lower semi-continous function on $\dot H^s$.

Theorem 2.30. *Let q be an element belonging to the space X^s_T, defined by for any $T<\widetilde T(q)=\widetilde T$*

$$\|q\|^2_{X^s_T} \overset{\text{def}}{=} \|q\|^2_{L^\infty_T(\dot H^s)} + \|q\|^2_{L^2_T(\dot H^{s+1})}.$$

Let r be a solution of the following perturbed Navier-Stokes system

$$\left\{\begin{array}{rcl}\partial_t r + r.\nabla r - \Delta r + r\cdot\nabla q + q.\nabla r &=& -f-\nabla p\\ \operatorname{div}\, r &=& 0\\ r_{|t=0} &=& r_0.\end{array}\right.$$

Let $\varepsilon_0 > 0$. Let T_0 be the time defined by

$$T_0 \stackrel{\text{def}}{=} \sup\{0 < T < \widetilde{T}(q) \mid \|r(t)\|^2_{L^\infty_T(\dot H^s)} \leqslant \varepsilon_0\}.$$

Then, for any $t \leqslant T_0$, we have

$$\|r\|^2_{X^s_T} \lesssim \Big(\|r_0\|^2_{\dot H^s} + \|f\|^2_{L^2_T(\dot H^{s-1})}\Big) \exp\Big(\varepsilon_0^{\frac{2}{2s-1}}\, \widetilde{T} + \widetilde{T}^{s-\frac12}\, \|q\|^2_{X^s_T} + \widetilde{T}\, \|q\|^{\frac{4}{2s-1}}_{L^\infty_T(\dot H^s)}\Big).$$

Proof. A $\dot H^s$ scalar-product, an integration in time and triangular inequality yield

$$\begin{aligned}\|r\|^2_{\tilde X^s_T} &\stackrel{\text{def}}{=} \|r\|^2_{\dot H^s} + 2\int_0^t \|r(t')\|^2_{\dot H^{s+1}} dt' \\ &\leqslant \|r_0\|^2_{\dot H^s} + 2\int_0^t \Big|\big((r\cdot\nabla r)\mid r\big)_{\dot H^s}\Big|\, dt' + 2\int_0^t \Big|\big((q\cdot\nabla r)\mid r\big)_{\dot H^s}\Big|\, dt' \\ &+ 2\int_0^t \Big|\big((r\cdot\nabla q)\mid r\big)_{\dot H^s}\Big|\, dt' + 2\int_0^t \Big|\big(f\mid r\big)_{\dot H^s}\Big|\, dt'.\end{aligned} \tag{2.66}$$

We assess each term in the right-hand side; the divergence-free condition implies

$$\begin{aligned}\Big|\big((r\cdot\nabla r)\mid r\big)_{\dot H^s}\Big| &\leqslant \|r\cdot\nabla r\|_{\dot H^{s-1}}\, \|r\|_{\dot H^{s+1}} \\ &\leqslant \|r\otimes r\|_{\dot H^s}\, \|r\|_{\dot H^{s+1}}.\end{aligned}$$

Thanks to Corollary 2.27, we infer that

$$\Big|\big((r\cdot\nabla r)\mid r\big)_{\dot H^s}\Big| \leqslant C(s)\, \|r\|^{s+\frac12}_{\dot H^s}\, \|r\|^{\frac52-s}_{\dot H^{s+1}}.$$

Then, integrating in time and applying Young's inequality $\big(ab \leqslant \dfrac{a^p}{p} + \dfrac{b^{p'}}{p'}$, with $\dfrac1p + \dfrac1{p'} = 1\big)$ yield

$$\begin{aligned}\int_0^t \Big|\big((r\cdot\nabla r)\mid r\big)_{\dot H^s}\Big|\, dt' &\leqslant C(s)\int_0^t \|r\|^{s+\frac12}_{\dot H^s}\, \|r\|^{\frac52-s}_{\dot H^{s+1}} dt' \\ &\leqslant C(s)\int_0^t \|r\|^{2\,\frac{2s+1}{2s-1}}_{\dot H^s}\, dt' + \frac{1}{12}\int_0^t \|r\|^2_{\dot H^{s+1}} dt'.\end{aligned} \tag{2.67}$$

Now we have to estimate $\displaystyle\int_0^t \Big|\big((r\cdot\nabla q)\mid r\big)_{\dot H^s}\Big|\, dt'$ and $\displaystyle\int_0^t \Big|\big((q\cdot\nabla r)\mid r\big)_{\dot H^s}\Big|\, dt'$. Actually, thanks to the divergence-free condition, it is exactly the same estimate and we get

$$\begin{aligned}\Big|\big((r\cdot\nabla q)\mid r\big)_{\dot H^s}\Big| &\leqslant \|r\cdot\nabla q\|_{\dot H^{s-1}}\, \|r\|_{\dot H^{s+1}} \\ &\leqslant \|r\otimes q\|_{\dot H^s}\, \|r\|_{\dot H^{s+1}}.\end{aligned}$$

Once again, Corollary 2.27 gives

$$\begin{aligned}\int_0^t \Big|\big((r\cdot\nabla q)\mid r\big)_{\dot H^s}\Big|\, dt' &\leqslant C(s)\int_0^t \|r\|_{\dot H^s}\|q\|^{s-\frac12}_{\dot H^s}\, \|q\|^{\frac32-s}_{\dot H^{s+1}}\, \|r\|_{\dot H^{s+1}} dt' \\ &+ C(s)\int_0^t \|q\|_{\dot H^s}\, \|r\|^{s-\frac12}_{\dot H^s}\, \|r\|^{\frac52-s}_{\dot H^{s+1}} dt'.\end{aligned}$$

Young's inequality implies

$$\begin{aligned}\int_0^t \Big|\big((r\cdot\nabla q)\mid r\big)_{\dot H^s}\Big|\, dt' &\leqslant C(s)\int_0^t \|r\|^2_{\dot H^s}\|q\|^{2s-1}_{\dot H^s}\, \|q\|^{3-2s}_{\dot H^{s+1}}\, dt' \\ &+ C(s)\int_0^t \|q\|^{\frac{4}{2s-1}}_{\dot H^s}\, \|r\|^2_{\dot H^s} dt' + \frac{2}{12}\int_0^t \|r\|^2_{\dot H^{s+1}} dt'.\end{aligned} \tag{2.68}$$

Same arguments give an estimate of exterior force term

$$\begin{aligned}\int_0^t |(f|\, r)_{\dot H^s}|dt' &\leqslant \int_0^t \|f\|_{\dot H^{s-1}}\, \|r\|_{\dot H^{s+1}}\, dt'\\ &\leqslant C \int_0^t \|f\|^2_{\dot H^{s-1}} dt' + \frac{1}{12}\int_0^t \|r\|^2_{\dot H^{s+1}} dt'.\end{aligned} \tag{2.69}$$

Combining Inequalities (2.66), (2.67), (2.68) and (2.69), we get

$$\begin{aligned}\|r\|^2_{\tilde X^s_T} \leqslant{}& \|r_0\|^2_{\dot H^s} + C \int_0^t \|f\|^2_{\dot H^{s-1}} dt' + 2\int_0^t \frac{6}{12}\|r\|^2_{\dot H^{s+1}} dt'\\ &+ C(s) \int_0^t \|r\|^2_{\dot H^s}\Big(\|r\|^{\frac{4}{2s-1}}_{\dot H^s} + \|q\|^{2s-1}_{\dot H^s}\, \|q\|^{3-2s}_{\dot H^{s+1}} + \|q\|^{\frac{4}{2s-1}}_{\dot H^s}\Big)dt'.\end{aligned} \tag{2.70}$$

Let us introduce the time T_0 defined by

$$T_0 \stackrel{\text{def}}{=} \sup\{0 < T < \widetilde T \mid \|r(t)\|^2_{L^\infty_T(\dot H^s)} \leqslant \varepsilon_0\}.$$

Therefore, for any $t \leqslant T_0$, we have

$$\begin{aligned}\|r\|^2_{X^s_T} &\stackrel{\text{def}}{=} \|r\|^2_{\dot H^s} + \int_0^t \|r(t')\|^2_{\dot H^{s+1}} dt'\\ &\lesssim \|r_0\|^2_{\dot H^s} + \int_0^t \|f\|^2_{\dot H^{s-1}} dt' + \int_0^t \|r\|^2_{\dot H^s}\Big(\varepsilon_0^{\frac{2}{2s-1}} + \|q\|^{2s-1}_{\dot H^s}\, \|q\|^{3-2s}_{\dot H^{s+1}} + \|q\|^{\frac{4}{2s-1}}_{\dot H^s}\Big).\end{aligned}$$

Thanks to Gronwall's lemma, we infer that for any $T < T_0 \leqslant \widetilde T$

$$\|r\|^2_{X^s_T} \lesssim \Big(\|r_0\|^2_{\dot H^s} + \|f\|^2_{L^2_T(\dot H^{s-1})}\Big) \exp\Big(\varepsilon_0^{\frac{2}{2s-1}}\, \widetilde T + \widetilde T^{s-\frac12}\, \|q\|^2_{X^s_T} + \widetilde T\, \|q\|^{\frac{4}{2s-1}}_{L^\infty_T(\dot H^s)}\Big).$$

This concludes the proof Theorem 2.30. $\square$

Proposition below is well-known and can be seen as a consequence of Theorem 2.30. We perturb a data by a small term and we are interested in the consequence on the lifespan of the Navier-Stokes solution associated with such a perturbed data. The lifespan of the perturbed Navier-Stokes solution can not decrease too much, compared to the lifespan of the non-perturbed one. More precisely, we have the following proposition.

Proposition 2.31. *The map $u_0 \mapsto T_*(u_0)$ is a lower semi-continous function on $\dot H^s$*

$$\textit{e.g.}\quad \forall \varepsilon > 0, \exists \alpha > 0,\ \forall v_0 \textit{ in } \dot H^s \textit{ such that } \|v_0\|_{\dot H^s} < \alpha,\ \textit{then } T_*(u_0+v_0) \geqslant T_*(u_0) - \varepsilon.$$

Moreover, (under notations of Theorem 2.30), a constant $C > 0$ exists such that for any $T \leqslant T_*(u_0) - \varepsilon$

$$\begin{aligned}\|NS(u_0+v_0) - NS(u_0)\|^2_{X^s_T} &\leqslant C\, \|v_0\|^2_{\dot H^s}\\ &\times \exp\Big(\varepsilon_0^{\frac{2}{2s-1}}\, T + T^{s-\frac12}\, \|NS(u_0)\|^2_{X^s_T} + T\, \|NS(u_0)\|^{\frac{4}{2s-1}}_{L^\infty_T(\dot H^s)}\Big).\end{aligned} \tag{2.71}$$

Proof. Let u_0 and v_0 be two elements in $\dot H^s$. We operate a small perturbation of the data u_0 by v_0 (the aim is to quantify this smallness condition) and we want to prove that the lifespan of the perturbed Navier-Stokes solution $NS(u_0+v_0)$ can not be much less than the lifespan of $NS(u_0)$. The process is standard. We introduce an error term R defined by

$$R(t,x) = NS(u_0+v_0) - NS(u_0).$$

Classical computations imply that R is solution of the following perturbed Navier-Stokes system

$$\left\{\begin{array}{ccc} \partial_t R + R\cdot\nabla R - \Delta R + R\cdot\nabla NS(u_0) + NS(u_0)\cdot\nabla R & = & -\nabla p \\ \operatorname{div} R & = & 0 \\ R_{|t=0} & = & v_0. \end{array}\right. \tag{2.72}$$

Let $\varepsilon_0 > 0$. Let us introduce the time T_0 defined by

$$T_0 := \sup\big\{0 < T < T_*(u_0) \mid \|R(t)\|^2_{L^\infty_T(\dot H^s)} \leqslant \varepsilon_0\big\}.$$

Thanks to Theorem 2.30, we infer that for any $T \leqslant T_0$

$$\|R\|^2_{X^s_T} \leqslant C\,\|v_0\|^2_{\dot H^s}\, \exp\Big(\varepsilon_0^{\frac{2}{2s-1}}\, T + T^{s-\frac12}\, \|NS(u_0)\|^2_{X^s_T}\, + T\, \|NS(u_0)\|^{\frac{4}{2s-1}}_{L^\infty_T(\dot H^s)}\Big). \tag{2.73}$$

The above expression gives the smallness condition on $\|v_0\|_{\dot H^s}$. Indeed, suppose that v_0 satifies

$$C\,\|v_0\|^2_{\dot H^s}\, \exp\Big(\varepsilon_0^{\frac{2}{2s-1}}\, T + T^{s-\frac12}\, \|NS(u_0)\|^2_{X^s_T}\, + T\, \|NS(u_0)\|^{\frac{4}{2s-1}}_{L^\infty_T(\dot H^s)}\Big) \leqslant \varepsilon_0. \tag{2.74}$$

Therefore, the error term R, keeps on living until the time $T_*(u_0) - \varepsilon$, for any $\varepsilon > 0$. This concludes the proof of the proposition. $\square$

Chapitre 3

About the possibility of minimal blow up for Navier-Stokes solutions with data in $\dot{H}^s$

Abstract: Considering initial data in $\dot{H}^s$, with $\frac{1}{2} < s < \frac{3}{2}$, this paper is devoted to the study of possible blowing-up Navier-Stokes solutions such that $(T_*(u_0) - t)^{\frac{1}{2}(s-\frac{1}{2})} \|u\|_{\dot{H}^s}$ is bounded. Our result is in the spirit of the tremendous works of L. Escauriaza, G. Seregin, and V. Šverák [20] and I. Gallagher, G. Koch, F. Planchon [24], where they proved there is no blowing-up solution which remain bounded in $L^3(\mathbb{R}^3)$. The main idea is that if such blowing-up solutions exist, they satisfy critical properties.

3.1 Introduction and statement of main result

We consider the Navier-Stokes system for incompressible viscous fluids evolving in the whole space $\mathbb{R}^3$. Denoting by u the velocity, a vector field in $\mathbb{R}^3$, by p in $\mathbb{R}$ the pressure function, the Cauchy problem for the homogeneous incompressible Navier-Stokes system is given by

$$\left\{\begin{array}{ccc} \partial_t u + u \cdot \nabla u - \Delta u & = & -\nabla p \\ \operatorname{div} u & = & 0 \\ u_{|t=0} & = & u_0. \end{array}\right. \tag{3.1}$$

We recall a crucial property of the Navier-Stokes equation: the scaling invariance. Let us define the operator

$$\begin{gathered} \forall \alpha \in \mathbb{R}^+,\ \forall \lambda \in \mathbb{R}^+_*,\ \forall x_0 \in \mathbb{R}^3, \quad \Lambda^\alpha_{\lambda,x_0} u(t,x) \stackrel{\text{def}}{=} \frac{1}{\lambda^\alpha} u\Big(\frac{t}{\lambda^2}, \frac{x - x_0}{\lambda}\Big). \\ \text{If } \ \alpha = 1, \ \text{ we note } \ \Lambda^1_{\lambda,x_0} = \Lambda_{\lambda,x_0}. \end{gathered} \tag{3.2}$$

Clearly, if u is a smooth solution of Navier-Stokes system on $[0,T] \times \mathbb{R}^3$ with pressure p associated with the initial data u_0, then, for any positive λ, the vector field and the pressure

$$u_\lambda \stackrel{\text{def}}{=} \Lambda_{\lambda,x_0} u \quad \text{and} \quad p_\lambda \stackrel{\text{def}}{=} \Lambda^2_{\lambda,x_0} p$$

is a solution of Navier-Stokes system on the interval $[0, \lambda^2 T] \times \mathbb{R}^3$, associated with the initial data

$$u_{0,\lambda} = \ \Lambda_{\lambda,x_0} u_0.$$

This leads to the definition of scaling invariant space.

Definition 3.1. *A Banach space X is said to be scaling invariant (or also critical), if its norm is invariant under the scaling transformation defined by $u \mapsto u_\lambda$*

$$||u_\lambda||_X = ||u||_X.$$

Let us give some exemples of critical spaces in dimension 3

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)_{3\leqslant p<\infty} \hookrightarrow \mathcal{BMO}^{-1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3).$$

Our work takes place in functional spaces which are above the natural scaling of Navier-Stokes equations. More precisely, our statements will take place in some Sobolev and Besov spaces, with a regularity index s such that $\dfrac{1}{2} < s < \dfrac{3}{2}\cdot$

Notations. We shall constantly be using the following simplified notations:

$$L^\infty_T(\dot{H}^s) \stackrel{\text{def}}{=} L^\infty([0,T],\dot{H}^s) \quad \text{and} \quad L^2_T(\dot{H}^{s+1}) \stackrel{\text{def}}{=} L^2([0,T],\dot{H}^{s+1}),$$

and the relevant function space we shall be working with in the sequel is

$$X^s_T \stackrel{\text{def}}{=} L^\infty_T(\dot{H}^s) \cap L^2_T(\dot{H}^{s+1}), \quad \text{endowed with the norm} \quad \|u\|^2_{X^s_T} \stackrel{\text{def}}{=} \|u\|^2_{L^\infty_T(\dot{H}^s)} + \|u\|^2_{L^2_T(\dot{H}^{s+1})}.$$

Let us start by recalling the local existence theorem for data in the Sobolev space $\dot{H}^s$.

Theorem 3.2. *Let u_0 be in $\dot{H}^s$, with $\dfrac{1}{2} < s < \dfrac{3}{2}\cdot$ Then there exists a time T and there exists a unique solution $NS(u_0)$ such that $NS(u_0)$ belongs to $L^\infty_T(\dot{H}^s) \cap L^2_T(\dot{H}^{s+1})$.*
Moreover, denoting by $T_*(u_0)$ the maximal time of existence of such a solution, there exists a positive constant c such that

$$T_*(u_0)\ \|u_0\|^{\sigma_s}_{\dot{H}^s} \geqslant c, \quad \text{with} \quad \sigma_s \stackrel{\text{def}}{=} \frac{1}{\frac{1}{2}(s-\frac{1}{2})}\cdot \tag{3.3}$$

Remark 3.3. Throughout this paper, we will adopt the useful notation $NS(u_0)$ to mean the maximal solution of the Navier-Stokes system, associated with the initial data u_0. Notice that our whole work relies on the hypothesis there exists some blowing up NS-solutions, e.g some NS-solutions with a finite lifespan $T_*(u_0)$. This is still an open question.

Remark 3.4. We point out that the infimum of the quantity $T_*(u_0)\ \|u_0\|^{\sigma_s}_{\dot{H}^s}$ exists and is positive (because of the constant c). It has been proved in [48] that there exists some intial data which reach this infimum and that the set of such data is compact, up to dilations and translations.

Remark 3.5. Theorem 3.2 implies there exists a constant $c > 0$, such that

$$(T_*(u_0) - t)\ \|NS(u_0)(t)\|^{\sigma_s}_{\dot{H}^s} \geqslant c, \tag{3.4}$$

and thus we get in particular the blow up of the $\dot{H}^s$-norm

$$\lim_{t\to T_*(u_0)} \|NS(u_0)(t)\|^{\sigma_s}_{\dot{H}^s} = +\infty.$$

Our motivation here is to wonder if there exist some Navier-Stokes solutions which stop living in finite time (e.g $T_*(u_0) < \infty$) and which blow up at a minimal rate, namely: there exists a positive constant M such that $(T_*(u_0) - t)\ \|NS(u_0)\|^{\sigma_s}_{\dot{H}^s} \leqslant M$. In others terms,

Question: Does there exist some blowing up NS-solutions such that $\sup\limits_{t<T_*(u_0)} (T_*(u_0)-t)\,\|NS(u_0)\|^{\sigma_s}_{\dot H^s} \leqslant M$ *?*
If yes, what do they look like ?

We assume an affirmative answer and we search to characterize such solutions.

Hypothesis $\mathcal{H}$: There exist some blowing up NS-solutions such that $\sup\limits_{t<T_*(u_0)} (T_*(u_0)-t)\,\|NS(u_0)\|^{\sigma_s}_{\dot H^s} \leqslant M.$

Notice that a very close question to this one is to prove that

$$\text{If} \quad T_*(u_0)<\infty, \quad \text{does} \quad \limsup_{t\to T_*(u_0)} (T_*(u_0)-t)\,\|NS(u_0)(t)\|^{\sigma_s}_{\dot H^s} = +\infty \quad ?$$

We underline that this question about blowing-up Navier-Stokes solutions has been highly developed in the context of critical spaces, namely $\dot H^{\frac12}(\mathbb{R}^3)$ and $L^3(\mathbb{R}^3)$. Indeed, L. Escauriaza, G. Seregin and V. Šverák showed in the fundamental work [20] that any "Leray-Hopf" weak solution which remains bounded in $L^3(\mathbb{R}^3)$ can not develop a singularity in finite time. Alternatively, it means that

$$\text{If } T_*(u_0)<+\infty, \quad \text{then} \limsup_{t\to T_*(u_0)} \|NS(u_0)(t)\|_{L^3} = +\infty. \tag{3.5}$$

I. Gallagher, G. Koch and F. Planchon revisited the above criteria in the context of mild Navier-Stokes solutions. They proved in [24] that strong solutions which remain bounded in $L^3(\mathbb{R}^3)$, do not become singular in finite time. To perform it, they develop an alternative viewpoint : the method of "critical elements" (or "concentration-compactness"), which was introduced by C. Kenig and F. Merle to treat critical dispersive equations. Recently, same authors extend the method in [25] to prove the same result in the case of the critical Besov space $\dot B^{-1+\frac3p}_{p,q}(\mathbb{R}^3)$, with $3<p,q<\infty$. Notice the work of J.-Y.Chemin and F. Planchon in [19], which gives the same answer in the case of the Besov space $\dot B^{-1+\frac3p}_{p,q}(\mathbb{R}^3)$, with $3<p<\infty,\ q<3$ and with an additional regularity assumption on the data. To conclude the non-exhaustive list of blow up results, we mention the work of C. Kenig and G. Koch who carried out in [32] such a program of critical elements for solutions in the simpler case $\dot H^{\frac12}(\mathbb{R}^3)$. More precisely, they proved for any data u_0 belonging to the smaller critical space $\dot H^{\frac12}(\mathbb{R}^3)$,

$$\text{If } T_*(u_0)<+\infty, \quad \text{then} \lim_{t\to T_*(u_0)} \|NS(u_0)(t)\|_{\dot H^{\frac12}} = +\infty. \tag{3.6}$$

In our case (remind : we consider Sobolev spaces $\dot H^s(\mathbb{R}^3)$ with $\dfrac12<s<\dfrac32$ which are non-invariant under the natural scaling of Navier-Stokes equations), we can not expect to prove our result in the same way, because of the scaling. Indeed, a similar proof leads us to define the critical quantity $M_c^{\sigma_s}$

$$M_c^{\sigma_s} = \sup\{A>0,\ \sup_{t<T_*(u_0)} (T_*(u_0)-t)\,\|NS(u_0)\|^{\sigma_s}_{\dot H^s} \leqslant A \quad \Rightarrow T_*(u_0)=+\infty\}.$$

But unfortunately, such a point of view makes no sense, owing to the meaning of $(T_*(u_0)-t)$ when $T_*(u_0)=+\infty$. We have to proceed in another way and it may be removed by defining the new object $M_c^{\sigma_s}$

$$M_c^{\sigma_s} \stackrel{\text{def}}{=} \inf_{\substack{u_0\in\dot H^s\\ T_*(u_0)<\infty}} \big\{\limsup_{t\to T_*(u_0)} (T_*(u_0)-t)\,\|NS(u_0)(t)\|^{\sigma_s}_{\dot H^s}\big\}.$$

Clearly, (3.4) implies that $M_c^{\sigma_s}$ exists and is positive. As we have decided to work under hypothesis $\mathcal{H}$, *a fortiori*, this implies that $M_c^{\sigma_s}$ is finite. The definition below is the key notion of critical solution in this context.

Definition 3.6. *(Sup-critical solution)*
Let u_0 be an element in $\dot H^s$. We say that $u = NS(u_0)$ is a sup-critical solution if $NS(u_0)$ satisfies the two following assumptions:

$$T_*(u_0) < \infty \quad \text{and} \quad \limsup_{t\to T_*(u_0)} (T_*(u_0) - t)\, \|NS(u_0)(t)\|_{\dot H^s}^{\sigma_s} = M_c^{\sigma_s}.$$

A natural question is to know if such elements exist. The statement given below gives an affirmative answer and provides a general procedure to build some sup-critical solutions. Our main result follows.

Theorem 3.7. *(Key Theorem)*
Let us assume that there exists u_0 in $\dot H^s$ and M in $\mathbb{R}_*^+$ such that

$$T_*(u_0) < \infty \quad \text{and} \quad \sup_{t<T_*(u_0)} (T_*(u_0) - t)\, \|NS(u_0)(t)\|_{\dot H^s}^{\sigma_s} \leqslant M.$$

Then, there exists $\Phi_0 \in \dot H^s \cap \dot B^{\frac12}_{2,\infty}$ such that $\Phi \stackrel{\text{def}}{=} NS(\Phi_0)$ is a sup-critical solution, blowing up at time 1, such that

$$\sup_{\tau<1} (1-\tau)\, \|NS(\Phi_0)(\tau)\|_{\dot H^s}^{\sigma_s} = \limsup_{\tau\to 1} (1-\tau)\, \|NS(\Phi_0)(\tau)\|_{\dot H^s}^{\sigma_s} = M_c^{\sigma_s}. \tag{3.7}$$

In addition, there exists a positive constant C such that

$$\text{for any} \quad \tau < 1, \quad \|NS(\Phi_0)(\tau)\|_{\dot B^{\frac12}_{2,\infty}} \leqslant C, \tag{3.8}$$

where the Besov norm (for regularity index $0 < \alpha < 1$) is defined by

$$\|u\|_{\dot B^{\alpha}_{2,\infty}} \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}^d} \frac{\|u(\cdot - x) - u\|_{L^2}}{|x|^\alpha}.$$

We postpone the proof of (3.7) of the key Theorem 3.7 to the next section. The proof of (3.8) will be given in Section 5. Let us underline that the proof of result (3.8) reminds a result proved in the paper [25] of I. Gallagher, G. Koch and F. Planchon, where it has been shown that for data in the space $\dot B^{-1+\frac3p}_{p,q}$, with $3 < p, q < \infty$, assuming that the NS-solutions associated with such data are locally bounded in time until the blow up time (we assume the blow up occurs), then the NS-solutions can be written as the sum of a part that behaves like the linear part and a remaining term which is bounded in particular in the Besov space $\dot B^{\frac12}_{2,\infty}$. Our result, proved in the non-critical case $\dot H^s$, with $s > \dfrac12$ is somewhat the analogue of the result of [25] established in the critical case $\dot B^{-1+\frac3p}_{p,q}$.
We stress here on the crucial role played by the third index ∞ in the Besov space $\dot B^{-1+\frac3p}_{p,\infty}$. We refer once again the reader to the question raised and solved by the paper of I. Gallagher, G. Koch and F. Planchon in [25], where they prove that for any initial data in the critical Besov space $\dot B^{-1+\frac3p}_{p,q}$, with $3 < p, q < \infty$, the NS-solution, (the lifespan of which is assumed finite) becomes unbounded at the blow-up time. We may wonder if the result holds in the limit case $q = \infty$. As far as the author is aware, the answer is still open. Actually, if it holds, *a fortiori* it holds in the smaller space $\dot B^{\frac12}_{2,\infty}$, by virtue of the embedding $\dot B^{\frac12}_{2,\infty} \hookrightarrow \dot B^{-1+\frac3p}_{p,\infty}$. In others terms, it would mean there is no blowing-up solution, bounded in the critical space $\dot B^{\frac12}_{2,\infty}$. This is related to the concern of our paper since we build some blowing-up solutions bounded in this critical space, under the assumption of blow up at minimal rate. We mention the very interesting work of H. Jia and V. Šverák [30], where they prove that (-1)-homogeneous initial data generate global self-similar solutions. Unfortunately, the uniqueness of such solutions is not guaranteed.

3.2 Existence of sup-critical solutions

The goal of this section is to give a partial proof of key Theorem 3.7. It relies on the two lemmas below.

Lemma 3.8. *(Existence of sup-critical solutions in $\dot{H}^s$)*
Let $(v_{0,n})_{n\in\mathbb{N}}$ be a bounded sequence in $\dot{H}^s$ such that

$$\tau^*(v_{0,n}) = 1 \quad \text{and} \quad \text{for any } \tau < 1, \quad (1-\tau)\,\|NS(v_{0,n})(\tau,\cdot)\|_{\dot{H}^s}^{\sigma_s} \leqslant\; M_c^{\sigma_s} + \varepsilon_n, \tag{3.9}$$

where $\tau^*(v_{0,n})$ stands for the lifespan of $NS(v_{0,n})$ and ε_n is a generic sequence which tends to 0 when n goes to $+\infty$.
Then, there exists Ψ_0 in $\dot{H}^s$ such that $\Psi \stackrel{\text{def}}{=} NS(\Psi_0)$ is a sup-critical solution blowing up at time 1 and satisfies

$$\sup_{\tau<1}(1-\tau)\,\|NS(\Psi_0)(\tau)\|_{\dot{H}^s}^{\sigma_s} \;=\; \limsup_{\tau\to 1}(1-\tau)\,\|NS(\Psi_0)(\tau)\|_{\dot{H}^s}^{\sigma_s} \;=\; M_c^{\sigma_s}. \tag{3.10}$$

Moreover, the initial data of such element is a weak limit of the sequence $(v_{0,n})$ translated, e.g

$$\exists\ (x_{0,n})_{n\geqslant 0}, \quad v_{0,n}(\cdot + x_{0,n}) \rightharpoonup_{n\to+\infty} \Psi_0. \tag{3.11}$$

The proof of Lemma 3.8 will be the purpose of Section 4. It relies essentially on a scaling argument and profile theory, which will be introduced in the next Section 3.

Lemma 3.9. *(Fluctuation estimates)*
Let $u = NS(u_0)$ be a NS-solution associated with a data $u_0 \in \dot{H}^s$, with $\frac{1}{2} < s < \frac{3}{2}$, such that

$$\sup_{t<T_*(u_0)} (T_*(u_0)-t)^{\frac{1}{\sigma_s}}\,\|NS(u_0)(t)\|_{\dot{H}^s} \leqslant M.$$

Then, the following estimates on the fluctuation part $B(u,u)(t) \stackrel{\text{def}}{=} u - e^{t\,\Delta}u_0$ yield

$$\text{for any} \quad s < s' < 2s - \frac{1}{2}, \quad (T_*(u_0)-t)^{\frac{1}{\sigma_{s'}}}\,\|B(u,u)(t)\|_{\dot{H}^{s'}} \leqslant F_{s'}(M^2) \tag{3.12}$$

Moreover, for the critical case $= \frac{1}{2}$, we have

$$\|B(u,u)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \leqslant C\,M^2. \tag{3.13}$$

The proof of this lemma is postponed to Section 8. It merely stems from product laws in Besov spaces, interpolation inequalities and from judicious splitting into low and high frequencies in the following sense

$$(T_*-t)2^{2j} \leqslant 1 \quad \text{and} \quad (T_*-t)2^{2j} \geqslant 1.$$

Remark 3.10. Let us point out that estimates of Lemma 3.9 do not hold if $0 < \alpha < \frac{1}{2}$, because of low frequencies. Indeed, arguments similar to the ones used in the proof of Lemma 3.9 lead only to the following estimate

$$\|B(u,u)(t)\|_{\dot{B}^{\alpha}_{2,\infty}} \leqslant C\,M^2\,T_*(u_0)^{\frac{1}{2}(\alpha-\frac{1}{2})}.$$

Partial proof of Key Theorem 3.7
In all this text, we denote by $(\varepsilon_n)_{n\in\mathbb{N}}$ a non increasing sequence, which tends towards 0, when n tend to $+\infty$.
• Step 1 : Existence of sup-critical elements in $\dot{H}^s$, with $\frac{1}{2}<s<\frac{3}{2}$.
Let us consider the sequence $(M_c+\varepsilon_n)_{n\geqslant 0}$. By definition of M_c, there exists a sequence $(u_{0,n})$ belonging to $\dot{H}^s$, with a finite lifespan $T_*(u_{0,n})$, such that for any $t<T_*(u_{0,n})$:

$$\limsup_{t\to T_*(u_0)}\left(T_*(u_{0,n})-t\right)\|NS(u_{0,n})\|_{\dot{H}^s}^{\sigma_s}\leqslant M_c^{\sigma_s}+\varepsilon_n.$$

By definition of lim sup, there exists a nondecreasing sequence of time t_n, converging towards $T_*(u_0)$, such that

$$\forall t\geqslant t_n,\ \left(T_*(u_{0,n})-t\right)\|NS(u_{0,n})(t,x)\|_{\dot{H}^s}^{\sigma_s}\leqslant M_c^{\sigma_s}+\varepsilon_n. \tag{3.14}$$

By rescaling, we consider the sequence

$$v_{0,n}(y)=\left(T_*(u_{0,n})-t_n\right)^{\frac{1}{2}}NS(u_{0,n})\big(t_n,\left(T_*(u_{0,n})-t_n\right)^{\frac{1}{2}}y\big).$$

and we have

$$\|v_{0,n}\|_{\dot{H}^s}^{\sigma_s}=\left(T_*(u_{0,n})-t_n\right)\|NS(u_{0,n})(t_n)\|_{\dot{H}^s}^{\sigma_s}. \tag{3.15}$$

By virtue of (3.14), the sequence $(v_{0,n})_{n\geqslant 1}$ is bounded $\big($by $M_c^{\sigma_s}+\varepsilon_0\big)$ in the space $\dot{H}^s$. Moreover, such a sequence generates a Navier-Stokes solution, which keeps on living until the time $\tau^*=1$ and satisfies

$$NS(v_{0,n})(\tau,y)=\left(T_*(u_{0,n})-t_n\right)^{\frac{1}{2}}NS(u_{0,n})\big(t_n+\tau\left(T_*(u_{0,n})-t_n\right),\left(T_*(u_{0,n})-t_n\right)^{\frac{1}{2}}y\big). \tag{3.16}$$

We introduce $\widetilde{t_n}=t_n+\tau\left(T_*(u_{0,n})-t_n\right)$. Notice that, because of scaling, an easy computation yields

$$(1-\tau)\|NS(v_{0,n})(\tau)\|_{\dot{H}^s}^{\sigma_s}=\left(T_*(u_{0,n})-\widetilde{t_n}\right)\|NS(u_{0,n})(\widetilde{t_n})\|_{\dot{H}^s}^{\sigma_s}. \tag{3.17}$$

As $\widetilde{t_n}\geqslant t_n$ for any n (by definition of $\widetilde{t_n}$) we combine (3.17) with (3.14) and we get, for any $\tau\in[0,1[$,

$$(1-\tau)\|NS(v_{0,n})(\tau,x)\|_{\dot{H}^s}^{\sigma_s}\leqslant M_c^{\sigma_s}+\varepsilon_n.$$

The sequence $(v_{0,n})$ satisfies the hypothesis of Lemma 3.8. Applying it, we build a sup-critical solution $\Phi=NS(\Psi_0)$ in $\dot{H}^s$ which blows up at time 1, e.g

$$\limsup_{\tau\to 1}(1-\tau)\|NS(\Psi_0)(\tau)\|_{\dot{H}^s}^{\sigma_s}=M_c^{\sigma_s}.$$

This proves the first part of the statement of Theorem 3.7.

• Step 2 : Existence of sup-critical elements in $\dot{H}^s\cap\dot{B}^{\frac{1}{2}}_{2,\infty}\cap\dot{H}^{s'}$, with s and s' such that $s<s'<2s-\frac{1}{2}$.
This will be proved in Section 6. Notice that proving that $NS(\Psi_0)$ is bounded in the Besov space $\dot{B}^{\frac{1}{2}}_{2,\infty}$ is equivalent to prove that Ψ_0 belongs to $\dot{B}^{\frac{1}{2}}_{2,\infty}$, since, by virtue of Lemma 3.9, the fluctuation part is bounded in $\dot{B}^{\frac{1}{2}}_{2,\infty}$ and we have obviously

$$\|NS(\Psi_0)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}}\leqslant\|NS(\Psi_0)(t)-e^{t\Delta}\Psi_0\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}}+\|e^{t\Delta}\Psi_0\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}}.$$

The paper is structured as follows. In Section 3, we recall the main tools of this paper. Essentially, it deals with the profile theory of P. Gérard [26] and a structure lemma concerning a NS-solution

associated with a sequence which satisfies the hypothesis of profile theory. We also recall some basics facts on Besov spaces.
In Section 4, we are going to establish the proof of crucial Lemma 3.8, which provides the proof of the first part of Theorem 3.7: there exists some sup-critical elements in $\dot{H}^s$. The second part of the proof is postponed in Section 6, where we build some sup-critical elements not only in $\dot{H}^s$, but also in others spaces, such as $\dot{B}^{\frac{1}{2}}_{2,\infty}$ and $\dot{B}^{s'}_{2,\infty}$, with $s < s' < 2s - \frac{1}{2}\cdot$ To carry out this, we need some estimates on the fluctuation part of the solution, which will be provided in Section 5.
Then in Section 7, we give a similar sup-inf critical criteria. It turns out that among sup-critical solutions, there exists some of them which are sup-inf-critical in the sense of they reach the biggest infimum limit. Section 8 is devoted to the proof of Lemma 3.12, which gives the structure of a Navier-Stokes solution associated with a bounded sequence of data in $\dot{H}^s$. We recall to the reader that such structure result has been partially proved in [48], except for the orthogonality property of Navier-Stokes solution in $\dot{H}^s$-norm. As a result, we give the proof of such a property, after reminding the ideas of the complete proof.

3.3 Profile theory and Tool Box

We recall the fundamental result due to P. Gérard : the profile decomposition of a bounded sequence in the Sobolev space $\dot{H}^s$. The original motivation of this theory was the desciption, up to extractions, of the defect of compactness in Sobolev embeddings (see for instance the pionneering works of P.-L. Lions in [41], [42] and H. Brezis, J.-M. Coron in [8]. Here, we will use the theorem of P. Gérard [26], which gives, up to extractions, the structure of a bounded sequence of $\dot{H}^s$, with s between 0 and $\frac{3}{2}\cdot$
More precisely, the defect of compactness in the critical Sobolev embedding $\dot{H}^s \subset L^p$ is described in terms of a sum of rescaled and translated orthogonal profiles, up to a small term in L^p. For more details about the history of the profile theory, we refer the reader to the paper [48].

Theorem 3.11. *(Profile Theorem [26])*
Let $(u_{0,n})_{n\in\mathbb{N}}$ be a bounded sequence in $\dot{H}^s$. Then, up to an extraction:
- There exists a sequence vectors fields, called profiles $(\varphi^j)_{j\in\mathbb{N}}$ in $\dot{H}^s$.
- There exists a sequence of scales and cores $(\lambda_{n,j}, x_{n,j})_{n,j\in\mathbb{N}}$, such that, up to an extraction

$$\forall J \geqslant 0,\ u_{0,n}(x) = \sum_{j=0}^{J} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j(x) + \psi_n^J(x) \quad \text{with} \quad \lim_{J\to+\infty} \limsup_{n\to+\infty} \|\psi_n^J\|_{L^p} = 0, \quad \text{and} \quad p = \frac{6}{3-2s}\cdot$$

Where, $(\lambda_{n,j}, x_{n,j})_{n\in\mathbb{N},j\in\mathbb{N}^*}$ are sequences of $(\mathbb{R}^*_+ \times \mathbb{R}^3)^{\mathbb{N}}$ with the following orthogonality property: for every integers (j,k) such that $j \neq k$, we have

$$\text{either} \lim_{n\to+\infty} \Big(\frac{\lambda_{n,j}}{\lambda_{n,k}} + \frac{\lambda_{n,k}}{\lambda_{n,j}}\Big) = +\infty \quad \text{or} \quad \lambda_{n,j} = \lambda_{n,k} \quad \text{and} \quad \lim_{n\to+\infty} \frac{|x_{n,j} - x_{n,k}|}{\lambda_{n,j}} = +\infty.$$

Moreover, for any $J \in \mathbb{N}$, we have the following orthogonality property

$$\|u_{0,n}\|^2_{\dot{H}^s} = \sum_{j=0}^{J} \|\varphi^j\|^2_{\dot{H}^s} + \|\psi_n^J\|^2_{\dot{H}^s} + \circ(1), \quad \text{when} \quad n \to +\infty. \tag{3.18}$$

Let us recall a structure Lemma, based on the crucial profils theorem of P. Gérard (see [26]). Let $(u_{0,n})$ be a bounded sequence in the Sobolev space $\dot{H}^s$, the profile decomposition of which is given by

$$u_{0,n}(x) = \sum_{j\in J} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j(x) + \psi_n^J(x),$$

with the appropriate properties on the error term ψ_n^J. By virtue of orthogonality of scales and cores given by Theorem 3.11, we sort profiles according to their scales

$$u_{0,n}(x) = \sum_{\substack{j \in \mathcal{J}_1 \\ j \leqslant J}} \varphi^j(x - x_{n,j}) + \sum_{\substack{j \in \mathcal{J}_1^c \\ j \leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j}, x_{n,j}} \varphi^j(x) + \psi_n^J(x) \tag{3.19}$$

where for any $j \in \mathcal{J}_1$, for any $n \in \mathbb{N}$, $\lambda_{n,j} \equiv 1$.
Under these notations, we claim we have the following structure lemma of the Navier-Stokes solutions, the proof of which will be provided in Section 8.

Lemma 3.12. *(Profile decomposition of a sequence of Navier-Stokes solutions)*
Let $(u_{0,n})_{n \geqslant 0}$ be a bounded sequence of initial data in $\dot{H}^s$, the profile decomposition of which is given by

$$u_{0,n}(x) = \sum_{j=0}^{J} \Lambda^{\frac{3}{p}}_{\lambda_{n,j}, x_{n,j}} \varphi^j(x) + \psi_n^J(x).$$

Then, $\liminf_{n \geqslant 0} T_*(u_{0,n}) \geqslant \widetilde{T} \stackrel{\text{def}}{=} \inf_{j \in \mathcal{J}_1} T_*(\varphi^j)$ and for any $t < T_*(u_{0,n})$, we have

$$NS(u_{0,n})(t,x) = \sum_{j \in \mathcal{J}_1} NS(\varphi^j)(t, x - x_{n,j}) \; + \; e^{t\Delta}\Big(\sum_{\substack{j \in \mathcal{J}_1^c \\ j \leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j}, x_{n,j}} \varphi^j(x) + \psi_n^J(x) \Big) \; + \; R_n^J(t,x) \tag{3.20}$$

where the remaining term R_n^J satisfies for any $T < \tilde{T}$, $\lim_{J \to +\infty} \lim_{n \to +\infty} \|R_n^J\|_{X_T^s} = 0$.
Moreover, we have the orthogonality property on the $\dot{H}^s$-norm for any $t < \tilde{T}$

$$\|NS(u_{0,n})(t)\|^2_{\dot{H}^s} = \sum_{j \in \mathcal{J}_1} \|NS(\varphi^j)(t)\|^2_{\dot{H}^s} \; + \; \Big\| e^{t\Delta}\Big(\sum_{\substack{j \in \mathcal{J}_1^c \\ j \leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j}, x_{n,j}} \varphi^j + \psi_n^J \Big) \Big\|^2_{\dot{H}^s} + \gamma_n^J(t). \tag{3.21}$$

with $\lim_{J \to +\infty} \limsup_{n \to +\infty} \sup_{t' < t} |\gamma_n^J(t')| = 0$.

For the convenience of the reader, we recall the usual definition of Besov spaces. We refer the reader to [1], from page 63, for a detailed presentation of the theory and analysis of homogeneous Besov spaces.

Definition 3.13. *Let s be in $\mathbb{R}$, (p,r) in $[1, +\infty]^2$ and u in $\mathcal{S}'$. A tempered distribution u is an element of the Besov space $\dot{B}^s_{p,r}$ if u satisfies $\lim_{j \to \infty} ||\dot{S}_j u||_{L^\infty} = 0$ and*

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \Big(\sum_{j \in \mathbb{Z}} 2^{jrs} \; ||\dot{\Delta}_j u||^r_{L^p} \Big)^{\frac{1}{r}} < \infty,$$

where $\dot{\Delta}_j$ is a frequencies localization operator (called Littlewood-Paley operator), defined by

$$\dot{\Delta}_j u(\xi) \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{u}(\xi)),$$

with $\varphi \in \mathcal{D}([\frac{1}{2}, 2])$, such that $\sum_{j \in \mathbb{Z}} \varphi(2^{-j}t) = 1$, for any $t > 0$.

Remark 3.14. Notice that the characterization of Besov spaces with positive indices in terms of finite differences is equivalent to the above definition (cf [1]). In the case where the regularity index is between 0 and 1, one has the following property. Let s be in $]0,1[$ and (p,r) in $[1,\infty]^2$. A constant C exists such that, for any $u \in \mathcal{S}'$,

$$C^{-1}\,\|u\|_{\dot{B}^s_{p,r}} \leqslant \Big\|\frac{\|u(\cdot-y)-u\|_{L^p}}{|y|^s}\Big\|_{L^r(\mathbb{R}^d;\frac{dy}{|y|^d})} \leqslant C\,\|u\|_{\dot{B}^s_{p,r}}. \tag{3.22}$$

Remark 3.15. Notice that $\dot{H}^s \subset \dot{B}^s_{2,2}$ and both spaces coincide if $s < \frac{3}{2}\cdot$

We recall an interpolation property in Besov spaces, which will be useful in the sequel.

Proposition 3.16. *A constant C exists which satisifes the following property. If s_1 and s_2 are real numbers such that $s_1 < s_2$ and $\theta \in]0,1[$, then we have for any $p \in [1,+\infty]$*

$$\|u\|_{\dot{B}^{\theta\, s_1+(1-\theta)\, s_2}_{p,1}} \;\leqslant\; C(s_1,s_2,\theta)\,\|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}}\;\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}.$$

3.4 Application of profile theory to sup-critical solutions

This section is devoted to the proof of Lemma 3.8. The statement given below is actually a bit stronger and clearly entails Lemma 3.8. We shall prove the following proposition.

Proposition 3.17. *Let $(v_{0,n})_{n\in\mathbb{N}}$ be a bounded sequence in $\dot{H}^s$ such that*

$$\tau^*(v_{0,n}) = 1 \quad \text{and} \quad \text{for any } \tau < 1, \quad (1-\tau)\,\|NS(v_{0,n})(\tau,)\|^{\sigma_s}_{\dot{H}^s} \leqslant\; M_c^{\sigma_s} + \varepsilon_n,$$

where $\tau^*(v_{0,n})$ stands for the lifespan of $NS(v_{0,n})$ and ε_n is a generic sequence which tends to 0 when n goes to $+\infty$.
Then, up to extractions, we get the statements below
• *the profile decomposition of such a sequence of data has a unique profile φ^{j_0} with constant scale such that $NS(\varphi^{j_0})$ is a sup-critical solution which blows up at time 1, e.g*

$$\limsup_{\tau\to 1}(1-\tau)\,\|NS(\varphi^{j_0})(\tau)\|^{\sigma_s}_{\dot{H}^s} \;=\; M_c^{\sigma_s}. \tag{3.23}$$

• *"The limsup is actually a sup"*

$$\sup_{\tau<1}(1-\tau)\,\|NS(\varphi^{j_0})(\tau)\|^{\sigma_s}_{\dot{H}^s} \;=\; M_c^{\sigma_s}. \tag{3.24}$$

Proof. Let $(v_{0,n})_{n\geqslant 1}$ be a bounded sequence in $\dot{H}^s$, satisfiying the assumptions of Proposition 3.17. Therefore, $(v_{0,n})_{n\geqslant 1}$ has the profile decomposition below

$$v_{0,n}(x) = \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} \varphi^j(x-x_{n,j}) + \sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j(x) + \psi_n^J(x). \tag{3.25}$$

We denote by $\tau^*_{j_0} \stackrel{\text{def}}{=} \inf_{j\in\mathcal{J}_1} T_*(\varphi^j)$.

• Step 1: we start by proving by a contradiction argument that $\tau^*_{j_0} = 1$.

We have already known by virtue of Lemma 3.12, that $\tau^*_{j_0} \leqslant 1$. Assuming that $\tau^*_{j_0} < 1$, we expect a contradiction. Moreover, orthogonal Estimate (3.21) can be bounded from below by

$$\|NS(v_{0,n})(\tau)\|^2_{\dot{H}^s} \geqslant \|NS(\varphi^{j_0})(\tau)\|^2_{\dot{H}^s} - |\gamma^J_n(\tau)|. \tag{3.26}$$

On the one hand, it seems clear by assumption that for any $\tau < \tau^*_{j_0}$, we have

$$(1-\tau^*_{j_0})^{\frac{2}{\sigma_s}} \leqslant (1-\tau)^{\frac{2}{\sigma_s}}.$$

On the other hand, hypothesis on $NS(v_{0,n})$ yields

$$(1-\tau)^{\frac{2}{\sigma_s}}\, \|NS(v_{0,n})(\tau)\|^2_{\dot{H}^s} \leqslant\ M_c^2 + \varepsilon_n.$$

Therefore, from the above remarks, we get

$$\|NS(v_{0,n})(\tau)\|^2_{\dot{H}^s} \leqslant\ \frac{M_c^2+\varepsilon_n}{(1-\tau^*_{j_0})^{\frac{2}{\sigma_s}}}\cdot \tag{3.27}$$

Combining the above estimate with (3.26), we finally get, after multiplication by the factor $(\tau^*_{j_0}-\tau)^{\frac{2}{\sigma_s}}$,

$$\frac{M_c^2+\varepsilon_n}{(1-\tau^*_{j_0})^{\frac{2}{\sigma_s}}}\ (\tau^*_{j_0}-\tau)^{\frac{2}{\sigma_s}} \geqslant (\tau^*_{j_0}-\tau)^{\frac{2}{\sigma_s}}\ \|NS(\varphi^{j_0})(\tau)\|^2_{\dot{H}^s} - (\tau^*_{j_0}-\tau)^{\frac{2}{\sigma_s}}\ |\gamma^J_n(\tau)|. \tag{3.28}$$

Notice that $(\tau^*_{j_0}-\tau)^{\frac{2}{\sigma_s}}$ is always less than 1, which allows us to get rid of it in front of the remaining term $|\gamma^J_n(\tau)|$. In addition, applying (3.4) and hypothesis on the sequence ε_n, one has

$$\frac{M_c^2+\varepsilon_0}{(1-\tau^*_{j_0})^{\frac{2}{\sigma_s}}}\ (\tau^*_{j_0}-\tau)^{\frac{2}{\sigma_s}} \geqslant c\,-\,|\gamma^J_n(\tau)|.$$

Let us choose $\tau=\tau_c$ such that $\tau_c<\tau^*_{j_0}$ and $\dfrac{M_c^2+\varepsilon_0}{(1-\tau^*_{j_0})^{\frac{2}{\sigma_s}}}\ (\tau^*_{j_0}-\tau_c)^{\frac{2}{\sigma_s}} = \dfrac{c}{4}\cdot$ Then, we take J and n large enough such that $|\gamma^J_n(\tau_c)| \leqslant \dfrac{c}{2}\cdot$ Therefore, we get a contradiction, which proves that $\tau^*_{j_0}=1$.

• Step 2: we prove here that $NS(\varphi^{j_0})$ is a sup-critical solution in $\dot{H}^s$.

Let us come back to Inequality (3.26), which we multiply by the factor $(1-\tau)^{\frac{2}{\sigma_s}}$. As we have shown that $\tau^*_{j_0}=1$, hypothesis on $NS(v_{0,n})$ implies that for any $\tau<1$,

$$M_c^2+\varepsilon_n \geqslant (1-\tau)^{\frac{2}{\sigma_s}}\ \|NS(\varphi^{j_0})(\tau)\|^2_{\dot{H}^s}\ -\ |\gamma^J_n(\tau)|. \tag{3.29}$$

Our aim is to prove that the particular profile φ^{j_0} generates a sup-critical solution. If not, it means that

$$\exists \alpha_0>0, \forall \varepsilon>0,\ \exists \tau_\varepsilon,\ \text{such that } 0<(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\ <\varepsilon \quad\text{and}\quad (1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\ \|NS(u_{0,n})(\tau_\varepsilon)\|^2_{\dot{H}^s} \geqslant M_c^2+\alpha_0.$$

Taking the above inequality at time τ_ε, one has

$$M_c^2+\varepsilon_n \geqslant\ M_c^2+\alpha_0-\ |\gamma^J_n(\tau_\varepsilon)|.$$

Moreover, assumption on the remaining term γ_n^J implies that

$$\forall \eta > 0,\ \exists \widetilde{J}(\eta) \in \mathbb{N},\ \exists N_\eta \in \mathbb{N} \text{ such that } \forall J \geqslant \widetilde{J}(\eta),\ \forall n \geqslant N_\eta,\ |\gamma_n^J(\tau_\varepsilon)| \leqslant \eta.$$

Let $\eta > 0$. For any $J \geqslant \widetilde{J}(\eta)$ and for any $n \geqslant N_\eta$, we get at time τ_ε,

$$M_c^2 \geqslant M_c^2 + \alpha_0 \,-\, \eta.$$

Now, choosing η small enough (namely $\eta = \dfrac{\alpha_0}{2}$) we get a contradiction which proves that $NS(\varphi^{j_0})$ is a sup-critical solution. This concludes the proof of step 2 and thus the point (3.23) is proved.

• Step 3: let us prove the point (3.24) of Proposition 3.17. The proof is a straightforward adaptation of the previous one. We shall use that $NS(\varphi^{j_0})$ is a sup-critical solution:

$$\limsup_{\tau\to 1}(1-\tau)\,\|NS(\varphi^{j_0})(\tau)\|_{\dot H^s}^{\sigma_s} \,=\, M_c^{\sigma_s}.$$

As we always have $\sup\limits_{\tau<1}(1-\tau)\,\|NS(\varphi^{j_0})(\tau)\|_{\dot H^s}^{\sigma_s} \geqslant \limsup\limits_{\tau\to 1}(1-\tau)\,\|NS(\varphi^{j_0})(\tau)\|_{\dot H^s}^{\sigma_s}$, we get a first inequality : $\sup\limits_{\tau<1}(1-\tau)\,\|NS(\varphi^{j_0})(\tau)\|_{\dot H^s}^{\sigma_s} \geqslant M_c^{\sigma_s}$.
According to the previous computations, we have, for any $\tau < 1$,

$$M_c^2 + \varepsilon_n \geqslant (1-\tau)^{\frac{2}{\sigma_s}}\,\|NS(\varphi^{j_0})(\tau)\|_{\dot H^s}^2 \,-\, |\gamma_n^J(\tau)|.$$

Hypothesis on the remaining term $|\gamma_n^J|$ implies that $\sup\limits_{\tau<1}(1-\tau)\,\|NS(\varphi^{j_0})(\tau)\|_{\dot H^s}^{\sigma_s} \leqslant M_c^{\sigma_s}$, which provides the second desired inequality. This ends up the proof of (3.24).

Let us recall some notation and add a few words about profiles with constant scale. Thanks to Lemma 3.12 and obvious bounds from below we get for any $\tau < \tau_{j_0}^* \stackrel{\text{def}}{=} \inf\limits_{j\in\mathcal{J}_1} T_*(\varphi^j) = 1$

$$\|NS(v_{0,n})(\tau)\|_{\dot H^s}^2 \geqslant \sum_{j\in\mathcal{J}_1}\|NS(\varphi^j)(\tau)\|_{\dot H^s}^2 - |\gamma_n^J(\tau)|. \tag{3.30}$$

Among profiles with a scale equal to 1 (e.g $j \in \mathcal{J}_1$), we distinguish profiles with a lifespan equal to $\tau_{j_0}^* = 1$ and profiles with a lifespan τ_j^* strictly greater than 1. In other words, we consider the set

$$\tilde{\mathcal{J}}_1 \stackrel{\text{def}}{=} \{j \in \mathcal{J}_1 \mid \tau_j^* = 1\}.$$

Therefore, for any $\tau < 1$,

$$\begin{aligned}\|NS(v_{0,n})(\tau)\|_{\dot H^s}^2 \geqslant \|NS(\varphi^{j_0})(\tau)\|_{\dot H^s}^2 &+ \sum_{j\in\tilde{\mathcal{J}}_1,\, j\neq j_0}\|NS(\varphi^j)(\tau)\|_{\dot H^s}^2\\ &+ \sum_{j\in\mathcal{J}_1\setminus\tilde{\mathcal{J}}_1}\|NS(\varphi^j)(\tau)\|_{\dot H^s}^2 - |\gamma_n^J(\tau)|,\end{aligned}$$

which be bounded from below once again by

$$\|NS(v_{0,n})(\tau)\|_{\dot H^s}^2 \geqslant \|NS(\varphi^{j_0})(\tau)\|_{\dot H^s}^2 + \sum_{j\in\tilde{\mathcal{J}}_1,\, j\neq j_0}\|NS(\varphi^j)(\tau)\|_{\dot H^s}^2 - |\gamma_n^J(\tau)|, \tag{3.31}$$

since obviously the term $\displaystyle\sum_{j\in\mathcal{J}_1\setminus\tilde{\mathcal{J}}_1} \|NS(\varphi^j)(\tau)\|^2_{\dot H^s}$ is positive.

• Step 4: in order to complete the proof of Lemma 3.8, we have to prove that there exists a unique profile with a lifespan $\tau^*_{j_0}=1$, namely $|\tilde{\mathcal{J}}_1|=1$. Once again, we assume that there exists at least two profiles in $\tilde{\mathcal{J}}_1$. We expect a contradiction. Arguments of the proof are similar to the ones used in the step 2. We shall use the fact $(1-\tau)^{\frac{2}{\sigma_s}}\,\|NS(\varphi^j)(\tau)\|^2_{\dot H^s}$ can not be small as we want, by virtue of (3.4). Indeed, let us come back to Inequality (3.31). We have already proved that φ^{j_0} generates a sup-critical solution, blowing up at time 1. It means that for any $\varepsilon>0$, there exists a time τ_ε such that

$$0<(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}<\varepsilon \quad\text{and}\quad M_c^2-\varepsilon\leqslant(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\,\|NS(\varphi^{j_0})(\tau_\varepsilon)\|^2_{\dot H^s}\leqslant M_c^2+\varepsilon.$$

Therefore, Inequality (3.31) becomes at time τ_ε

$$M_c^2+\varepsilon_n\geqslant M_c^2-\varepsilon+\sum_{j\in\tilde{\mathcal{J}}_1,j\neq j_0}(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\,\|NS(\varphi^j)(\tau_\varepsilon)\|^2_{\dot H^s}-|\gamma_n^J(\tau_\varepsilon)|. \tag{3.32}$$

By virtue of (3.4), there exists a universal constant $c>0$ such that for any $j\in\tilde{\mathcal{J}}_1$ and $j\neq j_0$

$$(1-\tau)^{\frac{2}{\sigma_s}}\,\|NS(\varphi^j)(\tau)\|^2_{\dot H^s}\geqslant c^2. \tag{3.33}$$

As a result, taking the limit for n and J large enough, we infer that (still under the hypothesis $|\tilde{\mathcal{J}}_1|>1$)

$$M_c^2\geqslant M_c^2-\varepsilon+(|\tilde{\mathcal{J}}_1|-1)\,c^2-\eta. \tag{3.34}$$

Choosing ε small enough, we get a contradiction and as a consequence, $|\tilde{\mathcal{J}}_1|=1$. It means there exists a unique profile generating a sub-critical solution, blowing up at time 1. This completes the proof of Proposition 3.17, and thus the proof of Lemma 3.8. □

3.5 Fluctuation estimates in Besov spaces

This section is devoted to the proof of Lemma 3.9. We shall prove some estimates on the fluctuation part which is given by the bilinear form

$$B(u,u)(t)\stackrel{\text{def}}{=}NS(u_0)(t)-e^{t\Delta}u_0=u-e^{t\Delta}u_0.$$

We distinguish the case $\dot B^{\frac12}_{2,\infty}$ from the case $\dot B^{s'}_{2,\infty}$, even if the ideas of the proves are similar: we cut-off according low and high frequencies in the following sense:

$$(T_*-t)2^{2j}\leqslant1\quad\text{and}\quad(T_*-t)2^{2j}\geqslant1.$$

Concerning high frequencies, we shall use the regularization effet of the Laplacian. Let us start by proving the critical part of Lemma 3.9.

Lemma 3.18. *Let $\frac12<s<\frac32$ and $u_0\in\dot H^s$. There exists a positive constant C_s such that*

$$\textit{If}\quad T_*(u_0)<\infty\quad\textit{and}\quad M_u\stackrel{\text{def}}{=}\sup_{t<T_*(u_0)}(T_*(u_0)-t)\,\|NS(u_0)(t)\|^{\sigma_s}_{\dot H^s}<\infty,$$

then, we have

$$\|u(t)-e^{t\Delta}u_0\|_{\dot B^{\frac12}_{2,\infty}}<C_s\,M_u^2.$$

Proof. Duhamel formula gives

$$u(t) - e^{t\Delta}u_0 \stackrel{\text{def}}{=} B(u,u)(t) = -\int_0^t e^{(t-t')\Delta}\,\mathbb{P}(\operatorname{div}(u\otimes u)\,dt'. \tag{3.35}$$

By virtue of classsical estimates on the heat term (see for instance Lemma 2.4 in [1]), we have

$$\|\Delta_j e^{t\Delta}\,a\|_{L^2} \leqslant C\,e^{-ct\,2^{2j}}\,\|\Delta_j a\|_{L^2}. \tag{3.36}$$

Therefore, the fluctuation part becomes

$$\begin{aligned}\|\Delta_j B(u,u)(t)\|_{L^2} &\lesssim \int_0^t e^{-c(t-t')\,2^{2j}}\,2^j\,\|\Delta_j(u\otimes u)(t')\|_{L^2}\,dt'\\ &\lesssim \int_0^t e^{-c(t-t')\,2^{2j}}\,2^j\,2^{-j(2s-\frac{3}{2})}\,\|u\otimes u(t')\|_{\dot{B}^{2s-\frac{3}{2}}_{2,\infty}}\,dt'.\end{aligned} \tag{3.37}$$

We infer thus, thanks to the product laws in Sobolev spaces

$$2^{\frac{j}{2}}\,\|\Delta_j B(u,u)(t)\|_{L^2} \lesssim \int_0^t e^{-c(t-t')\,2^{2j}}\,2^{j(3-2s)}\,\|u(t')\|^2_{\dot{H}^s}\,dt'. \tag{3.38}$$

By hypothesis, we have supposed that

$$M_u^2 \stackrel{\text{def}}{=} \sup_{t<T_*(u_0)} (T_*(u_0)-t)\,\|NS(u_0)(t)\|^{\sigma_s}_{\dot{H}^s} < \infty.$$

As a result,

$$\begin{aligned}2^{\frac{j}{2}}\,\|\Delta_j B(u,u)(t)\|_{L^2} \leqslant C_s &\int_0^t e^{-c(t-t')\,2^{2j}}\,2^{j(3-2s)}\,\frac{M_u^2}{(T_*(u_0)-t')^{\frac{2}{\sigma_s}}}\\ &= \int_0^t 1_{\{(T_*(u_0)-t')2^{2j}\leqslant 1\}}\,e^{-c(t-t')\,2^{2j}}\,2^{j(3-2s)}\,\frac{M_u^2}{(T_*(u_0)-t')^{\frac{2}{\sigma_s}}}\,dt'\\ &\quad+\int_0^t 1_{\{(T_*(u_0)-t')2^{2j}\geqslant 1\}}\,e^{-c(t-t')\,2^{2j}}\,2^{j(3-2s)}\,\frac{M_u^2}{(T_*(u_0)-t')^{\frac{2}{\sigma_s}}}\,dt'.\end{aligned} \tag{3.39}$$

We apply Young inequality: in the first integral, we consider $L^\infty \star L^1$, whereas in the second one, we consider $L^1 \star L^\infty$ in order to use the regularization effect of the Laplacian.

$$2^{\frac{j}{2}}\,\|\Delta_j B(u,u)(t)\|_{L^2} \leqslant C_s\,M_u^2 \int_{T_*(u_0)-2^{-2j}}^{T_*(u_0)} \frac{2^{j(3-2s)}\,dt'}{(T_*(u_0)-t')^{\frac{2}{\sigma_s}}} + C_s\,M_u^2 \int_0^t e^{-c(t-t')\,2^{2j}}\,2^{j(3-2s)}\,2^{2j(s-\frac{1}{2})}\,dt'. \tag{3.40}$$

We recall that $\dfrac{2}{\sigma_s} \stackrel{\text{def}}{=} s-\dfrac{1}{2}$ and $s-\dfrac{1}{2}<1$. As a result,

$$2^{\frac{j}{2}}\,\|\Delta_j B(u,u)(t)\|_{L^2} \leqslant C_s\,M_u^2\left(2^{j(2s-3)}\,2^{j(3-2s)} + \frac{1}{2^{2j}}\,2^{j(3-2s)}\,2^{2j(s-\frac{1}{2})}\right) \lesssim C_s\,M_u^2. \tag{3.41}$$

This concludes the proof on the fluctuation estimate in the critical case. □

The statement given below is a bit more general than the one of Lemma 3.9, which stems from an interpolation argument (the same as given at the end of the proof of Theorem 3.7).

Lemma 3.19. *Let* $\frac{1}{2} < s < \frac{3}{2}$ *and* $u_0 \in \dot{H}^s$. *There exists a positive constant* C_s *such that*

$$\text{If} \quad T_*(u_0) < \infty \quad \text{and} \quad M_u \stackrel{\text{def}}{=} \sup_{t<T_*(u_0)} (T_*(u_0)-t)\,\|NS(u_0)(t)\|_{\dot{H}^s}^{\sigma_s} < \infty,$$

then, we have for any $s < s' < 2s - \frac{1}{2}$

$$(T_*(u_0)-t)^{\frac{1}{2}(s'-\frac{1}{2})}\,\|u(t)-e^{t\Delta}u_0\|_{\dot{B}^{s'}_{2,\infty}} < \infty.$$

Proof. Same arguments as above yield

$$\|\Delta_j B(u,u)(t)\|_{L^2} \lesssim \int_0^t e^{-c(t-t')\,2^{2j}}\,2^j\,2^{-j(2s-\frac{3}{2})}\,\|u\otimes u(t')\|_{\dot{B}^{2s-\frac{3}{2}}_{2,\infty}}\,dt'. \tag{3.42}$$

Product laws in Sobolev spaces and hypothesis on u imply

$$\begin{aligned}2^{js'}\,\|\Delta_j B(u,u)(t)\|_{L^2} &\lesssim \int_0^t e^{-c(t-t')\,2^{2j}}\,2^{j(\frac{5}{2}-2s+s')}\,\|u(t')\|_{\dot{H}^s}^2\,dt'\\ &\lesssim \int_0^t e^{-c(t-t')\,2^{2j}}\,2^{j(\frac{5}{2}-2s+s')}\,\frac{C}{(T_*(u_0)-t')^{s-\frac{1}{2}}}.\end{aligned} \tag{3.43}$$

We split (the same cut-off as before) according low and high frequencies. Concerning high frequencies, since $T_*(u_0)-t \leqslant T_*(u_0)-t'$, we get

$$\begin{aligned}2^{js'}\,\|\Delta_j B(u,u)(t)\,1_{\{(T_*-t)2^{2j}\geqslant 1\}}\|_{L^2} &\lesssim \int_0^t e^{-c(t-t')\,2^{2j}}\,2^{j(\frac{5}{2}-2s+s')}\,\frac{C}{(T_*(u_0)-t)^{s-\frac{1}{2}}}\,dt'\\ &\lesssim 2^{j(\frac{1}{2}-2s+s')}\,\frac{C}{(T_*(u_0)-t)^{s-\frac{1}{2}}}.\end{aligned} \tag{3.44}$$

Choosing s' such that $\frac{1}{2}-2s+s' < 0$, we get

$$2^{js'}\,\|\Delta_j B(u,u)(t)\,1_{\{(T_*-t)2^{2j}\geqslant 1\}}\|_{L^2} \lesssim C\,\frac{(T_*(u_0)-t)^{\frac{1}{2}(-\frac{1}{2}+2s-s')}}{(T_*(u_0)-t)^{s-\frac{1}{2}}} = C\,(T_*(u_0)-t)^{-\frac{1}{2}(s'-\frac{1}{2})},$$

which yields the desired estimate, as far as high frequencies are concerned.
Concerning low frequencies, let us come back to the very beginning.

$$\begin{aligned}2^{js'}\,\|\Delta_j B(u,u)(t)\,1_{\{(T_*(u_0)-t)2^{2j}\leqslant 1\}}\|_{L^2} &\lesssim 2^{j(s'-s)}\,2^{js}\,\|\Delta_j B(u,u)(t)\|_{L^2}\\ &\lesssim 2^{j(s'-s)}\,\|u(t)-e^{t\Delta}u_0\|_{\dot{B}^s_{2,\infty}}.\end{aligned} \tag{3.45}$$

As $\|u(t)-e^{t\Delta}u_0\|_{\dot{B}^s_{2,\infty}} \leqslant \frac{C}{(T_*(u_0)-t)^{\frac{1}{2}(s-\frac{1}{2})}}$, we infer that

$$2^{js'}\,\|\Delta_j B(u,u)(t)\,1_{\{(T_*(u_0)-t)2^{2j}\leqslant 1\}}\,\|_{L^2} \lesssim 2^{j(s'-s)}\,\frac{C}{(T_*(u_0)-t)^{\frac{1}{2}(s-\frac{1}{2})}}.$$

Hypothesis of low frequencies implies

$$2^{js'}\,\|\Delta_j B(u,u)(t)\,1_{\{(T_*(u_0)-t)2^{2j}\leqslant 1\}}\,\|_{L^2} \lesssim \frac{C}{(T_*(u_0)-t)^{\frac{1}{2}(s-\frac{1}{2})+\frac{1}{2}(s'-s)}} = \frac{C}{(T_*(u_0)-t)^{\frac{1}{2}(s'-\frac{1}{2})}}.$$

which ends up the proof for low frequency part. The proof of Lemma 3.19 is thus complete. $\square$

3.6 Existence of sup-critical solutions bounded in $\dot{B}^{\frac{1}{2}}_{2,\infty}$

This section is devoted to complete the proof of Theorem 3.7, namely the part concerning the $\dot{B}^{\frac{1}{2}}_{2,\infty}$-norm of the sup-critical solutions. We have already built some sup-critical elements in the space $\dot{H}^s$. It turns out that, starting from this statement, we shall prove that data generating a sup-critical element are not only in $\dot{H}^s$, but also in some others spaces such as $\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{B}^{s'}_{2,\infty}$, with s' satisfiying the condition given below, which stems from the proof of Lemma 3.9.
The statement given below is actually a bit stronger than the one we want to prove, since we are going to catch some sup-critical solutions not only in $\dot{B}^{\frac{1}{2}}_{2,\infty}$ (as claimed by Theorem 3.7) but also in $\dot{B}^{s'}_{2,\infty}$. The main idea to get such information on the regularity is to focus on the fluctuation part which is more regular than the solution itself. Notice that, in all this section, we use regularity index s' satisfying

$$s < s' < 2s - \frac{1}{2}\cdot$$

Theorem 3.20. *There exists a data $\Phi_0 \in \dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^s \cap \dot{B}^{s'}_{2,\infty}$, such that $T_*(\Phi_0) < \infty$ and*

$$\sup_{t<T_*(\Phi_0)} (T_*(\Phi_0)-t)\,\|NS(\Phi_0)(t)\|^{\sigma_s}_{\dot{H}^s} \;=\; \limsup_{t\to T_*(\Phi_0)} (T_*(\Phi_0)-t)\,\|NS(\Phi_0)\|^{\sigma_s}_{\dot{H}^s} = M_c^{\sigma_s},$$

$$\text{and for any} \quad t < T_*(\Phi_0), \quad \|NS(\Phi_0)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} < \infty.$$

Proof. The idea of the proof is to start with the existence of sup-critical elements in $\dot{H}^s$. Indeed, we have proved previously that there exists data $\Psi_0 \in \dot{H}^s$, such that $\Psi \stackrel{\text{def}}{=} NS(\Psi_0)$ is sup-critical. Therefore, by definition of lim sup, there exists a sequence $t_n \nearrow T_*(\Psi_0)$ such that

$$\lim_{n\to+\infty} (T_*(\Psi_0)-t_n)\,\|NS(\Psi_0)(t_n)\|^{\sigma_s}_{\dot{H}^s} = M_c^{\sigma_s}.$$

Let us introduce as before the rescaled sequence

$$v_{0,n}(y) = \big(T_*(\Psi_0)-t_n\big)^{\frac{1}{2}}\, NS(\Psi_0)(t_n, \big(T_*(\Psi_0)-t_n\big)^{\frac{1}{2}} y).$$

Such a sequence generates a solution which keeps on living until the time 1 and satisfies

$$\|v_{0,n}\|^{\sigma_s}_{\dot{H}^s} = \big(T_*(\Psi_0)-t_n\big)\,\|NS(\Psi_{0,n})(t_n)\|^{\sigma_s}_{\dot{H}^s}. \tag{3.46}$$

In the sake of simplicity, we note

$$\tau_n \stackrel{\text{def}}{=} T_*(\Psi_0) - t_n.$$

Previous computations imply that $(v_{0,n})$ is a bounded sequence of $\dot{H}^s$. Now, inspired by the idea of Y. Meyer (fluctuation-tendancy method, [44]), we decomposed the sequence $(v_{0,n})$ into

$$v_{0,n}(y) \stackrel{\text{def}}{=} v_{0,n}(y) - \tau_n^{\frac{1}{2}}\, e^{t_n\Delta}\Psi_0(\tau_n^{\frac{1}{2}}\, y) \;+\; \tau_n^{\frac{1}{2}}\, e^{t_n\Delta}\Psi_0(\tau_n^{\frac{1}{2}}\, y), \tag{3.47}$$

where we have

$$v_{0,n}(y) \stackrel{\text{def}}{=} \tau_n^{\frac{1}{2}}\, NS(\Psi_0)(t_n, \tau_n^{\frac{1}{2}}\, y)$$

It follows

$$v_{0,n}(y) \stackrel{\text{def}}{=} \tau_n^{\frac{1}{2}} \underbrace{\Big(NS(\Psi_0)(t_n,\cdot) - e^{t_n\Delta}\Psi_0\Big)}_{B(\Psi,\Psi)(t_n)=\text{fluctuation part}}(\tau_n^{\frac{1}{2}}\, y) \;+\; \tau_n^{\frac{1}{2}} \underbrace{e^{t_n\Delta}\Psi_0}_{\text{tendancy part}} (\tau_n^{\frac{1}{2}}\, y). \tag{3.48}$$

Lemma 3.21. *The rescaled fluctuation part* $\phi_n \stackrel{\text{def}}{=} \tau_n^{\frac{1}{2}} B(\Psi,\Psi)(t_n, \tau_n^{\frac{1}{2}}\,\cdot\,)$ *is bounded in* $\dot H^s \cap \dot B^{\frac12}_{2,\infty} \cap \dot B^{s'}_{2,\infty}$.

Proof. Indeed, concerning the $\dot B^{\frac12}_{2,\infty}$-norm, we use firstly the scaling invariance of this norm and then we apply Lemma 3.9, which gives

$$\sup_n \|\phi_n\|_{\dot B^{\frac12}_{2,\infty}} = \sup_n \|NS(\Psi_0)(t_n,\cdot\,) - e^{t_n\Delta}\Psi_0\|_{\dot B^{\frac12}_{2,\infty}} < \infty. \tag{3.49}$$

Concerning the $\dot H^s$-norm, we apply successively the following arguments : scaling, triangular inequality and the fact that $NS(\Psi_0)$ is a sup-critical element in $\dot H^s$.

$$\begin{aligned} \|\phi_n\|^{\sigma_s}_{\dot H^s} &= \tau_n \|NS(\Psi_0)(t_n,\cdot\,) - e^{t_n\Delta}\Psi_0\|^{\sigma_s}_{\dot H^s} \\ &\lesssim \tau_n \|NS(\Psi_0)(t_n,\cdot\,)\|^{\sigma_s}_{\dot H^s} + \tau_n \|e^{t_n\Delta}\Psi_0\|^{\sigma_s}_{\dot H^s} \\ &\lesssim \Big(M_c + \frac{1}{n}\Big)^{\sigma_s} + \tau_n \|\Psi_0\|^{\sigma_s}_{\dot H^s} < \infty. \end{aligned} \tag{3.50}$$

Therefore, $\sup_n \|\phi_n\|^{\sigma_s}_{\dot H^s} < \infty$.
Concerning the $\dot B^{s'}_{2,\infty}$-norm, scaling argument combining with Lemma 3.9 yields

$$\|\phi_n\|^{\sigma_{s'}}_{\dot B^{s'}_{2,\infty}} = \tau_n \|NS(\Psi_0)(t_n,\cdot\,) - e^{t_n\Delta}\Psi_0\|^{\sigma_{s'}}_{\dot B^{s'}_{2,\infty}}. \tag{3.51}$$

This concludes the proof of this Lemma 3.21. □

By virtue of profile theory, we perform a profile decomposition of the sequence ϕ_n in the Sobolev space $\dot H^s$. But in this decomposition, there is only left profiles with constant scale, as Lemma below will prove it. The idea is clear. As ϕ_n is bounded in the Besov space $\dot H^s \cap \dot B^{\frac12}_{2,\infty}$, big scales vanish. Likewise, the fact that ϕ_n is bounded in the Besov space $\dot H^s \cap \dot B^{s'}_{2,\infty}$ implies that small scales vanish. That is the point in the Lemma below.

Lemma 3.22. • *If* $(f_n)_{n\in\mathbb{N}}$ *is a bounded sequence in* $\dot B^{\frac12}_{2,\infty} \cap \dot H^s$ *and if* $\limsup_{n\to+\infty} \|f_n\|_{\dot B^s_{2,\infty}} = L > 0$, *then there is no big scales in the profile decomposition of the sequence* f_n *in* $\dot H^s$.
• *If* $(f_n)_{n\in\mathbb{N}}$ *is a bounded sequence in* $\dot B^{s'}_{2,\infty} \cap \dot H^s$, *with* $s' > s > \frac12$ *and if* $\limsup_{n\to+\infty} \|f_n\|_{\dot B^s_{2,\infty}} = L > 0$, *then there is no small scales in the profile decomposition of the sequence* f_n *in* $\dot H^s$.

Proof. We only proof the first part of the Lemma. The other one is similar. If $\limsup_{n\to+\infty} \|f_n\|_{\dot B^s_{2,\infty}} = L > 0$, it means there exists an extraction $\varphi(n)$ such that $\|f_{\varphi(n)}\|_{\dot B^s_{2,\infty}} \geqslant \frac{L}{2}\cdot$ Otherwise, for any subsequence of (f_n), we would have

$$\|f_{\varphi(n)}\|_{\dot B^s_{2,\infty}} < \frac{L}{2} \quad \text{and thus,} \quad \lim_{n\to+\infty} \|f_{\varphi(n)}\|_{\dot B^s_{2,\infty}} \leqslant \frac{L}{2}\cdot$$

As a result, we would have $\limsup_{n\to+\infty} \|f_n\|_{\dot B^s_{2,\infty}} \leqslant \frac{L}{2} < L$, which is wrong by hypothesis. Moreover, by definition of the Besov norm, we can find a sequence $(k_n)_{n\in\mathbb{Z}}$, such that

$$\lim_{n\to+\infty} 2^{k_n s} \|\Delta_{k_n} f_{\varphi(n)}\|_{L^2} = \|f_{\varphi(n)}\|_{\dot B^s_{2,\infty}}. \tag{3.52}$$

Therefore, $\lim_{n\to+\infty} 2^{k_n s}\|\Delta_{k_n} f_{\varphi(n)}\|_{L^2} \geqslant \frac{L}{2}$.

Let us introduce the scale $\lambda_n \stackrel{\text{def}}{=} 2^{-k_n}$. As (up to extraction) $2^{k_n s}\|\Delta_{k_n} f_{\varphi(n)}\|_{L^2} \geqslant \frac{L}{2}$, then one has

$$2^{k_n(s-\frac{1}{2})}\ \|f_{\varphi(n)}\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \geqslant \frac{L}{2}.$$

Hence, the infimum limit of the sequence k_n is not $-\infty$, otherwise, the term $2^{k_n(s-\frac{1}{2})}$ would tend to 0 and thus $L=0$ (since the sequence $\|f_{\varphi(n)}\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}}$ is bounded by hypothesis), which is false by hypothesis. Therefore, $\lambda_n \nrightarrow +\infty$: big scales are excluded from the profile decomposition of the sequence f_n. This concludes the proof of Lemma 3.22. $\square$

Continuation of the proof of Theorem 3.20.

Let us come back to the proof of sup-critical element in the Besov space $\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{B}^{s'}_{2,\infty}$. First, we check that ϕ_n satisfies hypothesis of Lemma 3.22. As it was already checked previously, ϕ_n is bounded in $\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{H}^s\cap \dot{B}^{s'}_{2,\infty}$. Concerning assumption $\limsup_{n\to+\infty}\|\phi_n\|_{\dot{B}^s_{2,\infty}} > 0$, by a scaling argument, one has

$$\begin{aligned}\|\phi_n\|^{\sigma_s}_{\dot{B}^s_{2,\infty}} &= \tau_n\,\|NS(\Psi_0)(t_n,\cdot) - e^{t_n\Delta}\Psi_0\|^{\sigma_s}_{\dot{B}^s_{2,\infty}} = (T_*(\Psi_0)-t_n)\|NS(\Psi_0)(t_n,\cdot) - e^{t_n\Delta}\Psi_0\|^{\sigma_s}_{\dot{B}^s_{2,\infty}}\\ &\geqslant (T_*(\Psi_0)-t_n)\|NS(\Psi_0)(t_n,\cdot)\|^{\sigma_s}_{\dot{B}^s_{2,\infty}} - (T_*(\Psi_0)-t_n)\|\Psi_0\|^{\sigma_s}_{\dot{H}^s}.\end{aligned} \tag{3.53}$$

Obviously, the term $(T_*(\Psi_0)-t_n)\|\Psi_0\|^{\sigma_s}_{\dot{H}^s}$ tends to 0 when n goes to $+\infty$. By virtue of (3.4) and [39], there exists a constant $c>0$ such that $(T_*(\Psi_0)-t_n)\|NS(\Psi_0)(t_n,\cdot)\|^{\sigma_s}_{\dot{B}^s_{2,\infty}} \geqslant c$. Therefore,

$$\limsup_{n\to+\infty}\|\phi_n\|_{\dot{B}^s_{2,\infty}} > 0$$

and thus profile decomposition of ϕ_n in the space $\dot{H}^s$ is reduced to (with notations of Theorem 3.11)

$$\phi_n = \sum_{j\geqslant 0}^{J} V^j(\cdot - x_{n,j})\ +\ r_n^J. \tag{3.54}$$

Moreover, as the sequence ϕ_n is bounded in $\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{B}^{s'}_{2,\infty}$, profiles V^j belong also to $\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{B}^{s'}_{2,\infty}$. That's the crucial point in the proof. Indeed, each profile V^j can be seen as a translated (by $x_{n,j}$) weak limit of the sequence ϕ_n. As a result, we get immediately

$$\|V^j\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \leqslant \liminf_{n\to+\infty}\|\phi_n\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} < \infty \quad \text{and} \quad \|V^j\|_{\dot{B}^{s'}_{2,\infty}} \leqslant \liminf_{n\to+\infty}\|\phi_n\|_{\dot{B}^{s'}_{2,\infty}} < \infty.$$

Let us come back to the sequence $(v_{0,n})$ defined by

$$v_{0,n} \stackrel{\text{def}}{=} \phi_n\ +\ \tau_n^{\frac{1}{2}}\ e^{t_n\Delta}\Psi_0(\tau_n^{\frac{1}{2}}\,\cdot).$$

As it has been already underlined previously, the term $\gamma_n \stackrel{\text{def}}{=} \tau_n^{\frac{1}{2}}\ e^{t_n\Delta}\Psi_0(\tau_n^{\frac{1}{2}}\,\cdot)$ tends to 0 in $\dot{H}^s$-norm (and thus in L^p-norm, by Sobolev embedding) since

$$\|\tau_n^{\frac{1}{2}}\ e^{t_n\Delta}\Psi_0(\tau_n^{\frac{1}{2}}\,\cdot)\|^{\sigma_s}_{\dot{H}^s} = \tau_n\,\|\ e^{t_n\Delta}\Psi_0\|^{\sigma_s}_{\dot{H}^s} \leqslant \tau_n\,\|\ \Psi_0\|^{\sigma_s}_{\dot{H}^s}. \tag{3.55}$$

Combining the profile decomposition of (ϕ_n) with the definition of $(v_{0,n})$, we finally get

$$v_{0,n} = \sum_{j\geqslant 0}^{J} V^j(\cdot - x_{n,j}) \,+\, r_n^J \,+\, \gamma_n,$$

with $\lim\limits_{J\to+\infty}\limsup\limits_{n\to+\infty}\|r_n^J\|_{L^p}=0$ and $\lim\limits_{n\to+\infty}\|\gamma_n\|_{L^p}=0$. By virtue of Lemma 3.12, one has for any $\tau<1$

$$NS(v_{0,n})(\tau) = \sum_{j\geqslant 0}^{J} NS(V^j)(\tau, \cdot - x_{n,j}) + e^{\tau\Delta}(r_n^J+\gamma_n) + R_n^J(\tau).$$

By definition of the sequence $(v_{0,n})$, $NS(v_{0,n})$ is given by

$$NS(v_{0,n})(\tau,\cdot) = \big(T_*(\Psi_0)-t_n\big)^{\frac12}\, NS(\Psi_0)\big(t_n+\tau\,(T_*(\Psi_0)-t_n)\,,\,(T_*(\Psi_0)-t_n)^{\frac12}\,\cdot\big).$$

Once again, we denote $\widetilde{t_n} = t_n + \tau\,\big(T_*(\Psi_0)-t_n\big)$ and one has

$$(1-\tau)\,\|NS(v_{0,n})(\tau,\cdot)\|_{\dot H^s}^{\sigma_s} = \big(T_*(\Psi_0)-\widetilde{t_n}\big)\,\|NS(\Psi_0)(\widetilde{t_n},\cdot)\|_{\dot H^s}^{\sigma_s}.$$

As $\widetilde{t_n}\geqslant t_n$ for any n, we get

$$(1-\tau)\|NS(v_{0,n})(\tau)\|_{\dot H^s}^{\sigma_s} = (T_*(\Psi_0)-\widetilde{t_n})\|NS(\Psi_0)(\widetilde{t_n})\|_{\dot H^s}^{\sigma_s} \leqslant M_c^{\sigma_s} + \frac{2}{n}\cdot$$

Hence, Proposition 3.17 implies there exists a unique profile Φ_0 in $\dot B^{\frac12}_{2,\infty}\,\cap\,\dot H^s\,\cap\dot B^{s'}_{2,\infty}$ such that the NS-solution generated by this profile is a sup-critical solution. As Φ_0 belongs to $\dot B^{\frac12}_{2,\infty}$, Lemma 3.9 implies that $NS(\Phi_0)$ is bounded in the same space. This ends up the proof of Theorem 3.20.
Hence, we claim that the proof of Theorem 3.7 is over. Indeed, this stems from an interpolation argument. By virtue of Proposition 3.16, we have for any $s<s_1<s'$

$$\|\Phi_0\|_{\dot H^{s_1}} \leqslant \|\Phi_0\|_{\dot B^{s_1}_{2,1}} \leqslant \|\Phi_0\|^{\theta}_{\dot B^{s}_{2,\infty}}\,\|\Phi_0\|^{1-\theta}_{\dot B^{s'}_{2,\infty}} \leqslant \|\Phi_0\|^{\theta}_{\dot H^{s}}\,\|\Phi_0\|^{1-\theta}_{\dot B^{s'}_{2,\infty}}\cdot \tag{3.56}$$

This concludes the proof of Theorem 3.7. □

3.7 Another notion of critical solution

In this section, we wonder if among sup-critical solutions, we can find some of them which reach the biggest infimum limit of the quantity $(T_*(u_0)-t)\,\|NS(u_0)(t)\|_{\dot H^s}^{\sigma_s}$. We define the following set $\mathcal{E}_c$ by

$$\begin{aligned}\mathcal{E}_c \stackrel{\text{def}}{=} \Big\{ & u_0\in \dot B^{\frac12}_{2,\infty}\cap\dot H^s\cap\dot B^{s'}_{2,\infty}\quad\text{such that } T_*(u_0)<\infty\;;\\ & \sup_{t<T_*(u_0)}\,(T_*(u_0)-t)\,\|NS(u_0)(t)\|_{\dot H^s}^{\sigma_s} = \limsup_{t\to T_*(u_0)}(T_*(u_0)-t)\,\|NS(u_0)(t)\|_{\dot H^s}^{\sigma_s} = M_c^{\sigma_s}\;;\\ & \text{for any}\quad t<T_*(u_0),\quad \|NS(u_0)(t)\|_{\dot B^{\frac12}_{2,\infty}}<\infty \quad\text{and}\quad \sup_{t<T_*(u_0)}\,(T_*(u_0)-t)\;\|NS(u_0)(t)\|^{\sigma_{s'}}_{\dot B^{s'}_{2,\infty}}<\infty\Big\}.\end{aligned}$$

Let us introduce the following quantity $m_c^{\sigma_s}$

$$m_c^{\sigma_s} \stackrel{\text{def}}{=} \sup_{u_0\,\in\,\mathcal{E}_c}\big\{\liminf_{t\to T_*(u_0)}(T_*(u_0)-t)\,\|NS(u_0)(t)\|_{\dot H^s}^{\sigma_s}\big\}.$$

Definition 3.23. *(sup-inf-critical solution)*
A solution $u = NS(u_0)$ is said to be a sup-inf-critical solution if u_0 belongs to $\mathcal{E}_c$ and

$$\liminf_{t \to T_*(u_0)} (T_*(u_0) - t) \|NS(u_0)(t)\|_{\dot{H}^s}^{\sigma_s} = m_c^{\sigma_s}. \tag{3.57}$$

Notice we need to look for such elements among sup-critical solutions, otherwise the definition of $m_c^{\sigma_s}$ would be meaningless. We claim that there exist such elements.

Lemma 3.24. *There exists some elements belonging to $\mathcal{E}_c$, which are sup-inf-critical.*

Proof. By definition of $m_c^{\sigma_s}$, we can find a sequence $(u_{0,n}) \in \dot{H}^s$ and a sequence $t_n \nearrow T_*(u_{0,n}) \equiv T_*$ (we can assume this, up to a rescaling) such that

$$m_c - \varepsilon_n \leqslant (T_* - t_n)^{\frac{1}{\sigma_s}} \|NS(u_{0,n})(t_n)\|_{\dot{H}^s} \leqslant m_c + \varepsilon_n \tag{3.58}$$

and

$$\text{For any} \quad t \geqslant t_n, \quad m_c - \varepsilon_n \leqslant (T_* - t)^{\frac{1}{\sigma_s}} \|NS(u_{0,n})(t)\|_{\dot{H}^s}. \tag{3.59}$$

Assume in addition that the sequence $(u_{0,n})$ belongs to the set $\mathcal{E}_c$. As a consequence, we have

$$\text{For any} \quad t \geqslant t_n \quad , m_c - \varepsilon_n \leqslant (T_* - t)^{\frac{1}{\sigma_s}} \|NS(u_{0,n})(t)\|_{\dot{H}^s} \leqslant M_c + \varepsilon_n. \tag{3.60}$$

Considering the rescaled sequence

$$v_{0,n}(y) = (T_* - t_n)^{\frac{1}{2}} NS(u_{0,n})(t_n, (T_* - t_n)^{\frac{1}{2}} y).$$

Hence, $v_{0,n}$ satisfies properties below by scaling argument

$$\begin{aligned} \|v_{0,n}\|_{\dot{H}^s}^{\sigma_s} = (T_* - t_n) \|NS(u_{0,n})(t_n)\|_{\dot{H}^s}^{\sigma_s}, \quad \|v_{0,n}\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} = \|NS(u_{0,n})(t_n)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \\ \text{and} \quad \|v_{0,n}\|_{\dot{B}^{s'}_{2,\infty}}^{\sigma_{s'}} = (T_* - t_n) \|NS(u_{0,n})(t_n)\|_{\dot{B}^{s'}_{2,\infty}}^{\sigma_{s'}}. \end{aligned} \tag{3.61}$$

Combining (3.58) with the fact that $(u_{0,n})$ belongs to $\mathcal{E}_c$, we infer that the sequence $(v_{0,n})_{n \geqslant 1}$ is bounded in $\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^s \cap \dot{B}^{s'}_{2,\infty}$. Moreover, concerning the Navier-Stokes solution generated by such a data $NS(v_{0,n})$, we know that it keeps on living until the time $\tau^* = 1$ and satisfies once again (with $\widetilde{t_n} = t_n + \tau\,(T_* - t_n)$)

$$(1 - \tau)^{\frac{1}{\sigma_s}} \|NS(v_{0,n})(\tau)\|_{\dot{H}^s} = (T_* - \widetilde{t_n})^{\frac{1}{\sigma_s}} \|NS(u_{0,n})(\widetilde{t_n})\|_{\dot{H}^s}. \tag{3.62}$$

As $\widetilde{t_n} \geqslant t_n$ for any n, we infer that for any $\tau < 1$

$$(1 - \tau)^{\frac{1}{\sigma_s}} \|NS(v_{0,n})(\tau)\|_{\dot{H}^s} \geqslant m_c - \varepsilon_n.$$

Let us sum up information we have on the sequence $v_{0,n}$. Firstly, the lifespan of the Navier-Stokes associated with the sequence $v_{0,n}$ is equal to 1. Then,

$$\limsup_{\tau \to 1} (1 - \tau)^{\frac{1}{\sigma_s}} \|NS(v_{0,n})(\tau)\|_{\dot{H}^s} = \limsup_{\widetilde{t_n} \to T_*} (T_* - \widetilde{t_n})^{\frac{1}{\sigma_s}} \|NS(u_{0,n})(\widetilde{t_n})\|_{\dot{H}^s},$$

which implies, thanks to (3.60) and definition of M_c , that for any $\tau < 1$,

$$\limsup_{\tau \to 1} (1 - \tau)^{\frac{1}{\sigma_s}} \|NS(v_{0,n})(\tau)\|_{\dot{H}^s} = M_c \quad \text{and} \quad \|NS(v_{0,n})(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} = \|NS(u_{0,n})(\widetilde{t_n})\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} < \infty.$$

In addition,

$$(1-\tau)^{\frac{1}{\sigma_{s'}}}\ \|NS(v_{0,n})(\tau)\|_{\dot B^{s'}_{2,\infty}} = (T_*-\widetilde{t_n})^{\frac{1}{\sigma_{s'}}}\ \|NS(u_{0,n})(\widetilde{t_n})\|_{\dot B^{s'}_{2,\infty}} < \infty. \tag{3.63}$$

To summerize, from the minimizing sequence $(u_{0,n})$ of the set $\mathcal{E}_c$, we build another sequence $(v_{0,n})$ (the rescaled sequence of $(u_{0,n})$) which also belongs to the set $\mathcal{E}_c$. Moreover, as the sequence $(v_{0,n})$ is bounded in the spaces $\dot B^{\frac12}_{2,\infty}\cap \dot H^s\cap \dot B^{s'}_{2,\infty}$ and satisfies $\limsup\limits_{n\to+\infty}\ \|v_{0,n}\|_{\dot B^{s}_{2,\infty}}<\infty$, Lemma 3.22 implies that profile decomposition in $\dot H^s$ of such a sequence is reduced, up to extractions, to a sum of translated profiles and a remaining term (under notations of Theorem 3.11)

$$v_{0,n}=\sum_{j\in\mathcal{J}_1}\varphi^j(\cdot-x_{n,j})+\psi_n^J.$$

By virtue of Theorem 3.12, combining with Proposition 3.17, we infer there exists only one profile φ^{j_0} which blows up at time 1 and such that

$$NS(v_{0,n})(\tau,\cdot)=NS(\varphi^{j_0})(\tau,\cdot-x_{n,j_0})\ +\sum_{\substack{j\in\mathcal{J}_1,j\neq j_0\\ \tau_*^j>1}}NS(\varphi^j)(\cdot-x_{n,j})+\ e^{\tau\Delta}\psi_n^J(\cdot)\ +\ R_n^J(\tau,\cdot). \tag{3.64}$$

By orthogonality, we have

$$\|NS(v_{0,n})(\tau)\|^2_{\dot H^s}\geqslant \|NS(\varphi^{j_0})(\tau)\|^2_{\dot H^s}\ +\sum_{\substack{j\in\mathcal{J}_1,j\neq j_0\\ \tau_*^j>1}}\|NS(\varphi^j)(\tau)\|^2_{\dot H^s}\ +\ +\|e^{\tau\Delta}\psi_n^J\|^2_{\dot H^s}\ +|\gamma_n^J(\tau)|. \tag{3.65}$$

We want to prove that $\liminf\limits_{\tau\to1}\ (1-\tau)^{\frac{1}{\sigma_s}}\|NS(\varphi^{j_0})(\tau)\|_{\dot H^s}\geqslant m_c$. By definition of m_c, this will imply that $\liminf\limits_{\tau\to1}\ (1-\tau)^{\frac{1}{\sigma_s}}\|NS(\varphi^{j_0})(\tau)\|_{\dot H^s}=m_c$. Let us assume that is not the case. Therefore,

$$\exists\alpha_0>0,\forall\varepsilon>0,\ \exists\tau_\varepsilon,\ \text{ such that }\ 0<(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\ <\varepsilon \text{ and } (1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\|NS(u_{0,n})(\tau_\varepsilon)\|^2_{\dot H^s}\leqslant m_c^2-\alpha_0.$$

From (3.65), we deduce that

$$\begin{aligned}(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\|NS(v_{0,n})(\tau_\varepsilon)\|^2_{\dot H^s}&=(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\|NS(\varphi^{j_0})(\tau_\varepsilon)\|^2_{\dot H^s}\ +(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\Big\{\sum_{\substack{j\in\mathcal{J}_1,j\neq j_0\\ \tau_*^j>1}}\|NS(\varphi^j)(\tau_\varepsilon)\|^2_{\dot H^s}\\ &\quad+\ \|e^{\tau_\varepsilon\Delta}\psi_n^J\|^2_{\dot H^s}\ +\ |\gamma_n^J(\tau_\varepsilon)|\Big\}.\end{aligned}$$

By hypothesis, $(1-\tau_\varepsilon)^{\frac{1}{\sigma_s}}\|NS(v_{0,n})(\tau_\varepsilon)\|_{\dot H^s}\geqslant m_c-\varepsilon_n$, and $1-\tau_\varepsilon\leqslant 1$. Hence, we get

$$(m_c-\varepsilon_n)^2\leqslant\ m_c^2-\alpha_0\ +\ (1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\Big\{\sum_{\substack{j\in\mathcal{J}_1,j\neq j_0\\ \tau_*^j>1}}\sup_{\tau\in[0,1]}\|NS(\varphi^j)(\tau)\|^2_{\dot H^s}\ +\ \|\psi_n^J\|^2_{\dot H^s}\Big\}\ +|\gamma_n^J(\tau_\varepsilon)|. \tag{3.66}$$

On the one hand, as profiles φ^j have a lifespan $\tau_*^j>1$, the quantity $\sup\limits_{\tau\in[0,1]}\ \|NS(\varphi^j)(\tau)\|^2_{\dot H^s}$ is finite. On the other hand, by virtue of profile decomposition of the sequence $(v_{0,n})$, we have obviously that $\|\psi_n^J\|^2_{\dot H^s}\leqslant\|v_{0,n}\|^2_{\dot H^s}$. As we have proved that $(v_{0,n})$ is an element of the set $\mathcal{E}_c$, we get in particular that $\sup\limits_{\tau<1}(1-\tau)^{\frac{1}{\sigma_s}}\|NS(v_{0,n})(\tau)\|_{\dot H^s}=M_c$, which leads to (at $\tau=0$) $\|v_{0,n}\|_{\dot H^s}\leqslant M_c$. Finally, for all τ_ε,

$$(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\Big\{\sum_{\substack{j\in\mathcal{J}_1,j\neq j_0\\ \tau_*^j>1}}\sup_{\tau\in[0,1]}\|NS(\varphi^j)(\tau)\|^2_{\dot H^s}\ +\ \|\psi_n^J\|^2_{\dot H^s}\Big\}\leqslant\frac{\alpha_0}{4},$$

we get

$$(m_c - \varepsilon_n)^2 \leqslant \; m_c^2 - \alpha_0 \; + \; \frac{\alpha_0}{4} + |\gamma_n^J(\tau_\varepsilon)|. \tag{3.67}$$

Now, by assumption of γ_n^J, we take the limit for n and J large enough, and we get

$$m_c^2 \leqslant m_c^2 \; - \; \frac{3\,\alpha_0}{4} \; + \; \frac{\alpha_0}{4}, \tag{3.68}$$

which is obviously absurd. Thus, we have proved that

$$\liminf_{\tau \to 1} \; (1-\tau)^{\frac{1}{\sigma_s}} \, \|NS(\varphi^{j_0})(\tau)\|_{\dot{H}^s} = m_c.$$

This conclude the proof of Lemma 3.24. □

3.8 Structure Lemma for Navier-Stokes solutions with bounded data

This section is devoted to the proof of Lemma 3.12. The idea is to show that the profile decomposition of a bounded sequence of data provides a similar decomposition on the sequence of NS-solutions associated with such data. The main point is that profiles with constant scale play the key role.
Let $(v_{0,n})_{n\geqslant 0}$ be a bounded sequence of initial data in $\dot{H}^s$. Thanks to Theorem 3.11, $(v_{0,n})_{n\geqslant 0}$ can be written as follows, up to an extraction

$$v_{0,n}(x) = \sum_{j=0}^{J} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j(x) + \psi_n^J(x),$$

which can be written as follows

$$v_{0,n}(x) = \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} \varphi^j(x - x_{n,j}) + \sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j(x) + \psi_n^J(x). \tag{3.69}$$

Let $\eta > 0$ be the parameter of rough cutting off frequencies. We define by w_η and w_{c_η} the elements, the Fourier transform of which is given by

$$\widehat{w_\eta}(\xi) = \widehat{w}(\xi) 1_{\{\frac{1}{\eta}\leqslant|\xi|\leqslant\eta\}} \quad \text{and} \quad \widehat{w_{c_\eta}}(\xi) = \widehat{w}(\xi)\big(1 - 1_{\{\frac{1}{\eta}\leqslant|\xi|\leqslant\eta\}}\big). \tag{3.70}$$

After rough cutting off frequencies with respect to the notations (3.70) and sorting profiles supported in the annulus $1_{\{\frac{1}{\eta}\leqslant|\xi|\leqslant\eta\}}$ according to their scale (thanks to the orthogonality property of scales and cores, given by Theorem 3.11). We get the following profile decomposition

$$\begin{aligned} v_{0,n}(x) = \sum_{j\in\mathcal{J}_1} \varphi^j(x - x_{n,j}) + \sum_{j\in\mathcal{J}_0} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j_\eta(x) + \sum_{j\in\mathcal{J}_\infty} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j_\eta(x) + \psi^J_{n,\eta}(x) \\ \text{where} \quad \psi^J_{n,\eta}(x) \stackrel{\text{def}}{=} \sum_{\substack{j\in\mathcal{J}_1^c\equiv\mathcal{J}_0\cup\mathcal{J}_\infty\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_{c_\eta}(x) \; + \; \psi^J_n(x), \end{aligned} \tag{3.71}$$

for any j in $\mathcal{J}_1 \subset J$, $\lambda_{n,j} = 1$, for any j in $\mathcal{J}_0$, $\lim\limits_{n\to+\infty} \lambda_{n,j} = 0$ and for any j in $\mathcal{J}_\infty$, $\lim\limits_{n\to+\infty} \lambda_{n,j} = +\infty$.
As mentionned in the introduction, the whole Lemma 3.12 has been already proved in [48], except for the orthogonality property of the Navier-stokes solution associated with such a sequence of initial data.

Therefore, we refer the reader to [48] for details of the proof and here, we focus on the "Pythagore property". Let us recall the notations

$$U^0_{n,\eta} \stackrel{\text{def}}{=} \sum_{j\in\mathcal{J}_0} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j_\eta \quad \text{and} \quad U^\infty_{n,\eta} \stackrel{\text{def}}{=} \sum_{j\in\mathcal{J}_\infty} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j_\eta.$$

We recall some properties on profiles with small and large scale and remaining term. We refer the reader to [48] to the proof of the two propositions below.

Proposition 3.25.

For any $s_1 < s$, *for any* $\eta > 0$, *for any* $j \in \mathcal{J}_0$, *(e.g* $\lim_{n\to+\infty} \lambda_{n,j} = 0$*), then* $\lim_{n\to+\infty} \|U^0_{n,\eta}\|_{\dot H^{s_1}} = 0.$

For any $s_2 > s$, *for any* $\eta > 0$, *for any* $j \in \mathcal{J}_\infty$, *(e.g* $\lim_{n\to+\infty} \lambda_{n,j} = +\infty$*), then* $\lim_{n\to+\infty} \|U^\infty_{n,\eta}\|_{\dot H^{s_2}} = 0.$

Concerning the remaining term, we can show it tends towards 0, thanks to Lebesgue Theorem.

Proposition 3.26.

$$\lim_{J\to+\infty} \lim_{\eta\to+\infty} \limsup_{n\to+\infty} \|\psi^J_{n,\eta}\|_{L^p} = 0.$$

Continuation of Proof of Lemma 3.12. By virtue of (3.20) in Lemma 3.12, it seems clear that for any $t < \tilde T$

$$\begin{aligned}
\|NS(v_{0,n})(t,\cdot)\|^2_{\dot H^s} = \Big\| \sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j})\Big\|^2_{\dot H^s} &+ \Big\| e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi^J_n\Big)\Big\|^2_{\dot H^s} \\
&+ \|R^J_n(t,\cdot)\|^2_{\dot H^s} + 2\Big(\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j}) \mid e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi^J_n\Big)\Big)_{\dot H^s} \\
&+ 2\Big(\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j}) \mid R^J_n\Big)_{\dot H^s} + 2\Big(e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi^J_n\Big) \;\Big|\; R^J_n\Big)_{\dot H^s}.
\end{aligned}$$

Therefore, proving (3.21) is equivalent to prove Propositions 3.27 and 3.28 below. Both of them essentially stem from the orthogonality of cores and a compactness argument.

Proposition 3.27. *Let* $\varepsilon > 0$*. Then, for any* $t \in [0, \tilde T - \varepsilon]$,

$$\Big\| \sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j})\Big\|^2_{\dot H^s} = \sum_{j\in\mathcal{J}_1} \|NS(\varphi^j)(t,\cdot)\|^2_{\dot H^s} + \gamma_{n,\varepsilon}(t), \tag{3.72}$$

with $\lim_{n\to+\infty} \sup_{t\in[0,\tilde T-\varepsilon]} |\gamma_{n,\varepsilon}(t)| = 0.$

Proof. Once again, we developp the square of $\dot H^s$-norm and we get for any $t < \tilde T$

$$\begin{aligned}
\Big\| \sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j})\Big\|^2_{\dot H^s} &= \sum_{j\in\mathcal{J}_1} \|NS(\varphi^j)(t,\cdot - x_{n,j})\|^2_{\dot H^s} \\
&\quad + 2\sum_{\substack{(j,k)\in\mathcal{J}_1\times\mathcal{J}_1\\ j\neq k}} \Big(\Lambda^s NS(\varphi^j)(t,\cdot - x_{n,j}) \mid \Lambda^s NS(\varphi^k)(t,\cdot - x_{n,k})\Big)_{L^2},
\end{aligned}$$

where $\Lambda = \sqrt{-\Delta}$. Let $\varepsilon > 0$. Then, for any t in $[0, \widetilde{T} - \varepsilon]$, we get

$$\Big\| \sum_{j \in \mathcal{J}_1} NS(\varphi^j)(t, \cdot - x_{n,j}) \Big\|^2_{\dot{H}^s} = \sum_{j \in \mathcal{J}_1} \| NS(\varphi^j)(t, \cdot) \|^2_{\dot{H}^s} + 2 \sum_{\substack{(j,k) \in \mathcal{J}_1 \times \mathcal{J}_1 \\ j \neq k}} \Gamma^{s,j,k}_{\varepsilon,n},$$

where $\Gamma^{s,j,k}_{\varepsilon,n} \stackrel{\text{def}}{=} \Big(\Lambda^s\, NS(\varphi^j)(t, \cdot - x_{n,j}) \mid \Lambda^s\, NS(\varphi^k)(t, \cdot - x_{n,k}) \Big)_{L^2}.$

We denote by

$$K^J_\varepsilon \stackrel{\text{def}}{=} \bigcup_{j \in J} \Lambda^s\, NS(\varphi^j)([0, \widetilde{T} - \varepsilon]).$$

By virtue of the continuity of the map $t \in [0, \widetilde{T} - \varepsilon] \mapsto \Lambda^s\, NS(\varphi^j)(t, \cdot) \in L^2$, we deduce that K^J_ε is compact (and thus precompact) in L^2. It means that it can be covered by a finite open ball with an arbitrarily radius $\alpha > 0$. Let α be a positive radius. There exists an integer N_α, and there exists $(\theta_\ell)_{1 \leqslant \ell \leqslant N_\alpha}$ some elements of $\mathcal{D}(\mathbb{R}^3)$, such that

$$K^J_\varepsilon \subset \bigcup_{\ell=1}^{N_\alpha} B(\theta_\ell, \alpha). \tag{3.73}$$

Let us come back to the proof of Proposition 3.27. Thanks to the previous remark, we approach each profil $\Lambda^s\, NS(\varphi^j)(t, \cdot)$ (resp. $\Lambda^s\, NS(\varphi^k)(t, \cdot)$) by a smooth function: e.g. there exists a integer $\ell \in \{1, \cdots N_\alpha\}$ and there exists a function $\theta_{\ell(j,t)}$ (resp. $\theta_{\ell(k,t)}$) in $\mathcal{D}(\mathbb{R}^3)$ and we get

$$\begin{aligned}
\Gamma^{s,j,k}_{\varepsilon,n} &= \Big(\Lambda^s\, NS(\varphi^j)(t, \cdot - x_{n,j}) - \theta_{\ell(j,t)}(\cdot - x_{n,j}) \mid \Lambda^s\, NS(\varphi^k)(t, \cdot - x_{n,k}) - \theta_{\ell(k,t)}(\cdot - x_{n,k}) \Big)_{L^2} \\
&+ \Big(\Lambda^s\, NS(\varphi^j)(t, \cdot - x_{n,j}) - \theta_{\ell(j,t)}(\cdot - x_{n,j}) \mid \theta_{\ell(k,t)}(\cdot - x_{n,k}) \Big)_{L^2} \\
&+ \Big(\theta_{\ell(j,t)}(\cdot - x_{n,j}) \mid \Lambda^s\, NS(\varphi^k)(t, \cdot - x_{n,k}) - \theta_{\ell(k,t)}(\cdot - x_{n,k}) \Big)_{L^2} \\
&+ \Big(\theta_{\ell(j,t)}(\cdot - x_{n,j}) \mid \theta_{\ell(k,t)}(\cdot - x_{n,k}) \Big)_{L^2}.
\end{aligned} \tag{3.74}$$

The three first terms in the right-hand side of the above estimate tend uniformly (in time) to 0, by virtue of Cauchy-Schwarz and the translation-invariance of the $\dot{H}^s$-norm (we just perform the estimate for the first term, the others are similar). For any $t \in [0, \tilde{T} - \varepsilon]$

$$\begin{aligned}
&\Big(\Lambda^s\, NS(\varphi^j)(t, \cdot - x_{n,j}) - \theta_{\ell(j,t)}(\cdot - x_{n,j}) \mid \Lambda^s \left(NS(\varphi^k)(t, \cdot - x_{n,k}) - \theta_{\ell(k,t)}(\cdot - x_{n,k}) \right) \Big)_{L^2} \\
&\qquad \leqslant \| \Lambda^s\, NS(\varphi^j)(t) - \theta_{\ell(j,t)} \|_{L^2}\ \| \Lambda^s\, NS(\varphi^k)(t) - \theta_{\ell(k,t)} \|_{L^2} \\
&\qquad \leqslant \alpha^2.
\end{aligned} \tag{3.75}$$

Therefore, for any $\alpha > 0$, we have

$$\sup_{t \in [0, \tilde{T} - \varepsilon]} \Big(\Lambda^s\, NS(\varphi^j)(t, \cdot - x_{n,j}) - \theta_{\ell(j,t)}(\cdot - x_{n,j}) \mid \Lambda^s\, NS(\varphi^k)(t, \cdot - x_{n,k}) - \theta_{\ell(k,t)}(\cdot - x_{n,k}) \Big)_{L^2} \leqslant \alpha^2. \tag{3.76}$$

For the last term $\Big(\theta_{\ell(j,t)}(\cdot - x_{n,j}) \mid \theta_{\ell(k,t)}(\cdot - x_{n,k}) \Big)_{L^2}$, we have

$$\Big(\theta_{\ell(j,t)}(\cdot - x_{n,j}) \mid \theta_{\ell(k,t)}(\cdot - x_{n,k}) \Big)_{L^2} = \int_{\mathbb{R}^3} \theta_{\ell(j,t)}(x)\, \theta_{\ell(k,t)}(x + x_{n,j} - x_{n,k})\, dx.$$

It follows immediately that the above term tends to 0, when n tend to $+\infty$, by virtue of Lebesgue theorem combining with the orthogonality property of cores (e.g. $\lim_{n\to\infty} |x_{n,j} - x_{n,k}| = +\infty$). Finally, we have proved that $\Gamma^{s,j,k}_{\varepsilon,n}$ tends towards 0 when n tends to $+\infty$, uniformly in time. This concludes the proof of Proposition 3.27.

□

Concerning the crossed-terms in the profile decomposition, we have to prove they are also negligable, uniformly in time. That is the point in the following proposition.

Proposition 3.28. *Let $\varepsilon > 0$, We denote by*

$$I_n(t,\cdot) \stackrel{\text{def}}{=} \Big(\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j}) \mid e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi^J_n\Big)\Big)_{\dot H^s},$$

then, one has

$$\lim_{J\to+\infty}\lim_{n\to+\infty}\sup_{t\in[0,\tilde T-\varepsilon]} I_n(t,\cdot) = 0, \tag{3.77}$$

$$\lim_{J\to+\infty}\lim_{n\to+\infty}\sup_{t\in[0,\tilde T-\varepsilon]} \Big(\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j}) \mid R^J_n(t)\Big)_{\dot H^s} = 0, \tag{3.78}$$

$$\lim_{J\to+\infty}\lim_{n\to+\infty}\sup_{t\in[0,\tilde T-\varepsilon]} \Big(e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j(x) + \psi^J_n\Big) \mid R^J_n(t)\Big)_{\dot H^s} = 0. \tag{3.79}$$

Proof. Let us start by proving (3.77). We shall use once again an approximation argument. Let us define

$$\Lambda^J_\varepsilon \stackrel{\text{def}}{=} \bigcup_{j\in J} NS(\varphi^j)([0,\widetilde T - \varepsilon]).$$

By virtue of the continuity of the map $t\in[0,\widetilde T-\varepsilon]\mapsto NS(\varphi^j)(t,\cdot)\in \dot H^s$, we deduce that Λ^J_ε is compact (and thus precompact) in $\dot H^s$. It means that it can be covered by a finite open ball with an arbitrarily radius $\beta > 0$. Let β be a positive radius. There exists an integer N_β, and there exists $(\chi_\ell)_{1\leqslant\ell\leqslant N_\beta}$ some elements of $\mathcal{D}(\mathbb{R}^3)$, such that

$$\Lambda^J_\varepsilon \subset \bigcup_{\ell=1}^{N_\beta} B(\chi_\ell,\beta). \tag{3.80}$$

Let us come back to the proof of (3.77). Same arguments as previously imply there exists an integer $\ell\in\{1\cdots N_\beta\}$ and a smooth function $\chi_{\ell(t,j)}$ in $\mathcal{D}(\mathbb{R}^3)$ such that

$$\begin{aligned} I_n(t,\cdot) &\stackrel{\text{def}}{=} \Big(\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j}) \mid e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi^J_n\Big)\Big)_{\dot H^s}\\ &= \Big(\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j}) - \chi_{\ell(t,j)}(\cdot - x_{n,j}) \mid e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi^J_n\Big)\Big)_{\dot H^s}\\ &\quad+ \Big(\sum_{j\in\mathcal{J}_1} \chi_{\ell(t,j)}(\cdot - x_{n,j}) \mid e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi^J_n\Big)\Big)_{\dot H^s}. \end{aligned} \tag{3.81}$$

As $\|e^{t\Delta}\Big(\sum\limits_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j+\psi_n^J\Big)\|_{\dot H^s}\leqslant\|v_{0,n}\|_{\dot H^s}$, we infer that

$$I_n(t,\cdot)\leqslant|\mathcal{J}_1|\,\beta\,\|v_{0,n}\|_{\dot H^s}+\Big(\sum_{j\in\mathcal{J}_1}\chi_{\ell(t,j)}(\cdot-x_{n,j})\mid e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j+\psi_n^J\Big)\Big)_{\dot H^s}\tag{3.82}$$

Concerning the second part of above inequality, we shall use the splitting with respect to the parameter of cut off η. We refer the reader to the beginning of this section for notations.

$$\Big(\sum_{j\in\mathcal{J}_1}\chi_{\ell(t,j)}(\cdot-x_{n,j})\mid e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j+\psi_n^J\Big)\Big)_{\dot H^s}=I^1_{n,\eta}(t,\cdot)+I^2_{n,\eta}(t,\cdot)+I^3_{n,\eta}(t,\cdot),$$

where $\displaystyle I^1_{n,\eta}(t,\cdot)=\sum_{j\in\mathcal{J}_1}\Big(\chi_{\ell(t,j)}(\cdot-x_{n,j})\mid e^{t\Delta}U^0_{n,\eta}\Big)_{\dot H^s}$; $\displaystyle I^2_{n,\eta}(t,\cdot)=\sum_{j\in\mathcal{J}_1}\Big(\chi_{\ell(t,j)}(\cdot-x_{n,j})\mid e^{t\Delta}U^\infty_{n,\eta}\Big)_{\dot H^s}$

and $\displaystyle I^3_{n,\eta}(t,\cdot)=\sum_{j\in\mathcal{J}_1}\Big(\chi_{\ell(t,j)}(\cdot-x_{n,j})\mid e^{t\Delta}\psi^J_{n,\eta}\Big)_{\dot H^s}$.

Let us start with $I^1_{n,\eta}(t,\cdot)$. One has

$$\begin{aligned}|I^1_{n,\eta}(t,\cdot)|&\leqslant|\mathcal{J}_1|\,\|\chi_{\ell(t,j)}\|_{\dot H^{2s-s_1}}\,\|e^{t\Delta}U^0_{n,\eta}\|_{\dot H^{s_1}}\\&\leqslant|\mathcal{J}_1|\,\|\chi_{\ell(t,j)}\|_{\dot H^{2s-s_1}}\,\|U^0_{n,\eta}\|_{\dot H^{s_1}}.\end{aligned}$$

Proposition 3.25 (for η and $j\in\mathcal{J}_1$ fixed) implies thus $\displaystyle\lim_{n\to+\infty}\sup_{t\in[0,\tilde T-\varepsilon]}|I^1_{n,\eta}(t,\cdot)|=0$.

Concerning profiles with large scales, the proof is similar and we get for any $t\in[0,\tilde T-\varepsilon]$

$$|I^2_{n,\eta}(t,\cdot)|\leqslant|\mathcal{J}_1|\,\|\chi_{\ell(t,j)}\|_{\dot H^{2s-s_2}}\,\|U^\infty_{n,\eta}(x)\|_{\dot H^{s_2}}.\tag{3.83}$$

Once again, Proposition 3.25 implies the result: $\displaystyle\lim_{n\to+\infty}\sup_{t\in[0,\tilde T-\varepsilon]}|I^2_{n,\eta}(t,\cdot)|=0$.

Concerning the last term $I^3_{n,\eta}$, Hölder inequality with $\dfrac{1}{p}+\dfrac{1}{p'}=1$ yields

$$\begin{aligned}|I^3_{n,\eta}(t,\cdot)|&\leqslant\big|\Big(\Lambda^{2s}\chi_{\ell(t,j)}\mid e^{t\Delta}\psi^J_{n,\eta}(\cdot+x_{n,j})\Big)_{L^2}\big|\\&\leqslant\|\Lambda^{2s}\chi_{\ell(t,j)}\|_{L^{p'}}\,\|e^{t\Delta}\psi^J_{n,\eta}(\cdot+x_{n,j})\|_{L^p}).\end{aligned}$$

By translation invariance of the L^p-norm and estimate on the heat equation, we get

$$|I^3_{n,\eta}(t,\cdot)|\leqslant\|\Lambda^{2s}\chi_{\ell(t,j)}\|_{L^{p'}}\,\|\psi^J_{n,\eta}\|_{L^p}.\tag{3.84}$$

Obviously the term $\|\psi^J_{n,\eta}\|_{\dot H^s}$ is bounded by profiles hypothesis and the term $\|\Lambda^{2s}\chi_{\ell(t,j)}\|_{L^{p'}}$ is bounded too, since the function χ is as regular as we need. By virtue of Proposition 3.26, the term $\|\psi^J_{n,\eta}\|_{L^p}$ is small in the sense of for any $\varepsilon>0$, there exists an integer $N_0\in\mathbb{N}$, such that for any $n\geqslant N_0$, there exists $\tilde\eta>0$ and $\tilde J\geqslant0$, such that for any $\eta\geqslant\tilde\eta$ and for any $J\geqslant\tilde J$, we have $\|\psi^J_{n,\eta}\|_{L^p}\leqslant\varepsilon$. As a result, we get

$$\lim_{J\to+\infty}\lim_{\eta\to+\infty}\lim_{n\to+\infty}\sup_{t\in[0,\tilde T-\varepsilon]}|I^3_{n,\eta}(t,\cdot)|=0.$$

This ends up the proof of estimate (3.77).

Concerning the proof of (3.78) and (3.79), the proof is very close in both cases and relies on the fact that the error term R_n^J tends towards 0 in the $L^\infty_{\tilde{T}}(\dot{H}^s)$-norm. For any $t \in [0, \tilde{T}-\varepsilon]$, we have

$$\begin{aligned} \big|\big(\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j}) \mid R_n^J\big)_{\dot{H}^s}\big| &\leqslant \sum_{j\in\mathcal{J}_1} \big|\big(NS(\varphi^j)(t,\cdot) \mid R_n^J(t,\cdot + x_{n,j})\big)_{\dot{H}^s}\big| \\ &\leqslant |\mathcal{J}_1|\, \|NS(\varphi^j)(t,\cdot)\|_{L^\infty_{\tilde{T}}(\dot{H}^s)}\, \|R_n^J(t,\cdot)\|_{L^\infty_{\tilde{T}}(\dot{H}^s)}. \end{aligned} \tag{3.85}$$

Obviously, the term $\|NS(\varphi^j)(t,\cdot)\|_{L^\infty_{\tilde{T}}(\dot{H}^s)}$ is bounded since $t \in [0, \tilde{T}-\varepsilon]$. As a result, Lemma 3.12 implies that

$$\lim_{J\to+\infty} \lim_{n\to+\infty} \sup_{t\in[0,\tilde{T}-\varepsilon]} \big|\, \big(\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j}) \mid R_n^J\big)_{\dot{H}^s} \big| = 0.$$

As far as estimate (3.79) is concerned, the idea is the same. For any $t \in [0, \tilde{T}-\varepsilon]$,

$$\begin{aligned} \Big|\Big(e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi_n^J\Big) \mid R_n^J\Big)_{\dot{H}^s}\Big| &\leqslant \Big|\Big(e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi_n^J\Big) \mid R_n^J\Big)_{\dot{H}^s}\Big| \\ &\leqslant \Big\|e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi_n^J\Big)\Big\|_{L^\infty_{\tilde{T}}(\dot{H}^s)} \|R_n^J\|_{L^\infty_{\tilde{T}-\varepsilon}(\dot{H}^s)} \\ &\leqslant \|U^0_{n,\eta} + U^\infty_{n,\eta} + \psi^J_{n,\eta}\|_{\dot{H}^s}\, \|R_n^J\|_{L^\infty_{\tilde{T}-\varepsilon}(\dot{H}^s)}. \end{aligned} \tag{3.86}$$

Thanks to profile decomposition (3.71), we get

$$\|U^0_{n,\eta} + U^\infty_{n,\eta} + \psi^J_{n,\eta}\|^2_{\dot{H}^s} \leqslant \|v_{0,n}\|^2_{\dot{H}^s} + \circ(1). \tag{3.87}$$

Thus, finally we get

$$\Big|\Big(e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi_n^J\Big) \mid R_n^J\Big)_{\dot{H}^s}\Big| \leqslant C\left(\|v_{0,n}\|^2_{\dot{H}^s} + \circ(1)\right) \|R_n^J\|_{L^\infty_{\tilde{T}-\varepsilon}(\dot{H}^s)}. \tag{3.88}$$

We end up the proof as before, thanks to the hypothesis on R_n^J. This completes the proof of Proposition 3.28 and thus Lemma 3.12. $\square$

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Deuxième partie

Équations de Navier-Stokes pour un fluide inhomogène et incompressible

Chapitre 4

Wellposedness for density-dependent incompressible viscous fluids on the torus $\mathbb{T}^3$

Abstract: We investigate the local wellposedness of incompressible inhomogeneous Navier-Stokes equations on the Torus $\mathbb{T}^3$, with initial data in the critical Besov spaces. Under some smallness assumption on the velocity in the critical space $B^{\frac{1}{2}}_{2,1}(\mathbb{T}^3)$, the global-in-time existence of the solution is proved. The initial density is required to belong to $B^{\frac{3}{2}}_{2,1}(\mathbb{T}^3)$ but not supposed to be small.

4.1 Introduction and mains statements

Incompressible flows are often modeled by the incompressible homogeneous Navier-Stokes system (4.1), e.g the density of the fluid is supposed to be a constant

$$\left\{\begin{array}{ccc} \partial_t v + v\cdot\nabla v - \Delta v & = & -\nabla p \\ \operatorname{div} v & = & 0 \\ v_{|t=0} & = & v_0. \end{array}\right. \tag{4.1}$$

However, this model is sometimes far away from the physical situation. Concerning models of blood and rivers, even if the fluid is incompressible, its density can not be considered constant, owing to the complexity of the structure of the flow. As a result, a model which takes into account such constraints, has to be considered. That is the so-called Inhomogeneous Navier-Stokes system, given by

$$\left\{\begin{array}{ccc} \partial_t\rho + \operatorname{div}(\rho u) & = & 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) - \Delta u + \nabla\Pi & = & 0 \\ \operatorname{div} u & = & 0 \\ (\rho, u)_{|t=0} & = & (\rho_0, u_0). \end{array}\right. \tag{4.2}$$

which is equivalent to the system below, by virtue of the transport equation

$$\left\{\begin{array}{ccc} \partial_t\rho + u\cdot\nabla\rho & = & 0 \\ \rho(\partial_t u + u\cdot\nabla u) - \Delta u + \nabla\Pi & = & 0 \\ \operatorname{div} u & = & 0 \\ (\rho, u)_{|t=0} & = & (\rho_0, u_0), \end{array}\right. \tag{4.3}$$

where $\rho = \rho(t,x) \in \mathbb{R}^+$ stands for the density and $u = u(t,x) \in \mathbb{T}^3$ for the velocity field. The term $\nabla\Pi$ (namely the gradient of the pressure) may be seen as the Lagrangian multiplier associated with the constraint $\operatorname{div} u = 0$. The initial data (ρ_0, u_0) are prescribed. Notice, we choose the viscosity of the fluid equal to 1, in a sake of simplicity.

Let us recall some well-known results about the two above systems (homogeneous versus inhomogeneous). In the homogeneous case, the celebrated theorem of J. Leray [15] proves the global existence of weak solutions with finite energy in any space dimension. The uniqueness is garanteed in dimension 2, whereas in dimension 3, this is still an open question. In deal with this issue, H. Fujita and T. Kato [10] built some global strong solutions in the context of scaling invariance spaces, namely spaces which have the same scaling as the system (4.1). Such spaces are said to be critical, in the sense that their norm is invariant for any $\lambda > 0$ under the transformation

$$v_0(x) \mapsto \lambda\, v_0(\lambda x) \quad \text{and} \quad v(t,x) \mapsto \lambda\, v(\lambda^2 t, \lambda x).$$

The point is that such solutions are unique in this framework. In the inhomogeneous case, Leray's approach is still relevant for the system (4.2). Indeed, if the initial density ρ_0 is non negative and belongs to L^∞ and if $\sqrt{\rho_0}\, u_0$ belongs to L^2, then there exists some global weak solutions (ρ, u) with finite energy. However, the question of uniqueness has not been solved, even in dimension 2. We refer the reader to the paper of A. Kazhikhov [12], J. Simon [18] for the existence of global weak solutions. The unique resolvability of (4.2) is first established by the works of O. Ladyzenskaja and V. Solonnikov [13] in the case of a bounded domain Ω with homogeneous Dirichlet condition for the velocity u. As one has already mentionned previously, the approach initiated by H. Fujita and T. Kato is particulary efficient in the scaling invariance framework to face the uniqueness problem. A natural question is to wonder if such an approach is relevant for incompressible inhomogeneous fluids. If one believes so, scaling considerations should help us to find an adaptated functional framework. Firstly, one can check that (4.3) is invariant under the scaling transformation (for any $\lambda > 0$)

$$(\rho_0, u_0)(x) \mapsto (\rho_0, \lambda\, u_0)(\lambda x) \quad \text{and} \quad (\rho, u, \Pi)(t,x) \mapsto (\rho,\ \lambda\, u,\ \lambda^2\, \Pi)(\lambda^2 t, \lambda x).$$

That is an easy exercice to check that $\dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3) \times \dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)$ is scaling invariant under this transformation, in dimension 3, e.g

$$\|\rho_0(\lambda x)\|_{\dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)} = \|\rho_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)} \quad \text{and} \quad \|\lambda u_0(\lambda x)\|_{\dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)} = \|u_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)}.$$

Secondly, as the system (4.3) degenerates if ρ vanishes or becomes unbounded, we further assume that the density is away from zero ($\rho_0^{\pm 1} \in L^\infty$). Denoting

$$\frac{1}{\rho_0} \stackrel{\text{def}}{=} 1 + a_0 \quad \text{and} \quad \frac{1}{\rho} \stackrel{\text{def}}{=} 1 + a,$$

the incompressible inhomogeneous Navier-Stokes system (4.3) can be rewritten as

$$\left\{\begin{array}{ccc} \partial_t a\, + u\cdot \nabla a & = & 0 \\ \partial_t u + u\cdot\nabla u + (1+a)\,(\nabla\Pi - \Delta u) & = & 0 \\ \operatorname{div} u & = & 0 \\ (a,u)_{|t=0} & = & (a_0, u_0), \end{array}\right. \tag{4.4}$$

The question of unique solvability of the above system (4.4) has been adressed by many authors. Let us highlight the work of R. Danchin [6], who studied the unique solvability of (4.4) with constant viscosity coefficient and in scaling invariant (e.g critical) Besov spaces in the whole space $\mathbb{R}^N$. This generalized the celebrated results by H. Fujita and T. Kato, devoted to the classical homogeneous Navier-Stokes system (4.1). Indeed, R. Danchin proved in [6] (under the assumption the density is close to a constant) a local well-posedness for large initial velocity and a global well-posedness for initial velocity small with respect to the viscosity. More precisely, he proved that if the initial data (a_0, u_0) belongs to $\dot{B}^{\frac{N}{2}}_{2,\infty}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \times \dot{B}^{\frac{N}{2}-1}_{2,1}(\mathbb{R}^N)$, with a_0 small enough in $\dot{B}^{\frac{N}{2}}_{2,\infty}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then the system (4.4) has a unique local-in-time solution. In addition, assuming the velocity u_0 is also small enough in the space $\dot{B}^{\frac{N}{2}-1}_{2,1}(\mathbb{R}^N)$, the solution is global.

Our main motivation in this paper is to investigate the local and global wellposedness of the incompressible inhomogeneous Navier-Stokes system, in the case of critical Besov spaces and on the torus $\mathbb{T}^3$. The aim is to get rid of the smallness condition on the density, and just keeping the smallness one on the initial velocity. We point out that such a result has been already proved in the whole space $\mathbb{R}^3$. We refer the reader to the paper [4] of H. Abidi, G. Gui and P. Zhang. The main difference between their work and ours is that, on the torus, we have to be careful, owing to the average of the velocity u, which is not preserved, contrary to the case of classical Navier-Stokes system (4.1). As a consequence, a lot of "classical results" such as Gagliardo-Nirenberg inequalities and Sobolev embeddings, have to take into account the average of the velocity u. We will collect them in section 2. Let us give some remarks about this.

Notation In the sequel, we shall denote by

$$\bar{m} \stackrel{\text{def}}{=} \int_{\mathbb{T}^3} m(x)\, dx, \quad \text{where} \quad |\mathbb{T}^3| = 1.$$

Remark 4.1. It is clear that $\bar{\rho} = \bar{\rho}_0$. Indeed, an integration on the mass conservation equation combining with the fact $\int_{\mathbb{T}^3} u\cdot \nabla\rho = 0$ gives

$$\int_{\mathbb{T}^3} \rho(t,x)\, dx \;=\; \int_{\mathbb{T}^3} \rho_0(x)\, dx.$$

Notice that by virtue of the divergence free condition on the velocity u, the average of any function of ρ is preserved. In particular, the average of a is conserved.

Remark 4.2. An integration on the momentum equation of the system (4.2) (the terms $\int_{\mathbb{T}^3} \operatorname{div}(\rho u \otimes u)$, $\int_{\mathbb{T}^3} \Delta u$ and $\int_{\mathbb{T}^3} \nabla\Pi$ are nul) implies

$$\int_{\mathbb{T}^3} (\rho\, u)(t,x)\, dx = \int_{\mathbb{T}^3} \rho_0\, u_0(x)\, dx.$$

Remark 4.3. Notice that $\rho - \bar{\rho}$ is also solution of the transport equation. Thus, if we take the L^2 inner product of this mass conservation equation with $\rho - \bar{\rho}$ itself, we get the energy conservation of the quantity $\|\rho - \bar{\rho}\|_{L^2}$, because of divergence-free condition of u. Therefore we have :

$$\|\rho - \bar{\rho}\|_{L^2} = \|\rho_0 - \bar{\rho}_0\|_{L^2}.$$

In this paper, our main Theorem can be stated as follows

Theorem 4.4 (Main theorem)**.** *Let $a_0 \in B^{\frac{3}{2}}_{2,1}$, $u_0 \in B^{\frac{1}{2}}_{2,1}$, such that*

$$\operatorname{div} u_0 = 0 \quad ; \quad 1 + a_0 \geqslant b \quad \textit{for some positive constant } b \quad \textit{and} \quad \int_{\mathbb{T}^3} \frac{1}{1+a_0(x)}\, u_0(x)\, dx = 0. \qquad (4.5)$$

Then there exists a positive time T_* such that the system (4.4) has a unique local-in-time solution : for any $T < T_*$,

$$(a, u, \Pi) \;\in\; \mathcal{C}([0,T], B^{\frac{3}{2}}_{2,1}) \,\times\, \mathcal{C}([0,T], B^{\frac{1}{2}}_{2,1}) \,\cap\, L^1([0,T], B^{\frac{5}{2}}_{2,1}) \times\, L^1([0,T], B^{\frac{1}{2}}_{2,1}).$$

In addition, there exists a constant c (depending on $\|a_0\|_{B^{\frac{3}{2}}_{2,1}}$) such that

$$\text{if} \quad \|u_0\|_{B^{\frac{1}{2}}_{2,1}} \leqslant c, \quad \text{then} \quad T_* = +\infty.$$

Our main Theorem 4.4 relies on two Theorems, given below. Indeed, we will face the question of local wellposedness and global wellposedness in a different way. The first one deals with the local wellposed issue: until a small time, we may control the velocity u in some functional Besov spaces, by the initial data u_0. It can stated as follows

Theorem 4.5 (Local-wellposedness theorem)**.** *Let $a_0 \in B^{\frac{3}{2}}_{2,1}$, $u_0 \in B^{\frac{1}{2}}_{2,1}$, such that*

$$\operatorname{div} u_0 = 0 \quad ; \quad 1 + a_0 \geqslant b \quad \text{for some positive constant } b. \tag{4.6}$$

Then there exists a positive time T_* such that the system (4.4) has a unique local-in-time solution : for any $T < T_*$,

$$(a, u, \Pi) \in \mathcal{C}([0,T], B^{\frac{3}{2}}_{2,1}) \times \mathcal{C}([0,T], B^{\frac{1}{2}}_{2,1}) \cap L^1([0,T], B^{\frac{5}{2}}_{2,1}) \times L^1([0,T], B^{\frac{1}{2}}_{2,1}).$$

In addition, there exists a small constant c depending on $\|a_0\|_{B^{\frac{3}{2}}_{2,1}}$ such that if

$$\|u_0\|_{B^{\frac{1}{2}}_{2,1}} \leqslant c,$$

therefore, $T_* \geqslant 1$ and one has for any $T < T_*$,

$$\textit{Density estimate:} \quad \|a\|_{L^\infty_T(B^{\frac{3}{2}}_{2,1})} \leqslant \|a_0\|_{B^{\frac{3}{2}}_{2,1}} \exp\Big(C \|u\|_{L^1_T(B^{\frac{5}{2}}_{2,1})}\Big). \tag{4.7}$$

$$\textit{Velocity estimate:} \quad \|u\|_{L^\infty_T(B^{\frac{1}{2}}_{2,1})} + \|u\|_{L^1_T(B^{\frac{5}{2}}_{2,1})} + \|\nabla\Pi\|_{L^1_T(B^{\frac{1}{2}}_{2,1})} \leqslant C \|u_0\|_{B^{\frac{1}{2}}_{2,1}}. \tag{4.8}$$

Remark 4.6. The difficulty, as mentionned previously, is that the density a is not supposed to be small. To overcome this issue, we split the density $1 + a$ into

$$1 + a = (1 + S_m a) + (a - S_m a), \quad \text{where} \quad S_m a \stackrel{\text{def}}{=} \sum_{j \leqslant m-1} \Delta_j a.$$

The first part is then regular enough, the second part can be made small enough, for some large enough integer m: we fix m in the sequel such that $\|a - S_m a\|_{B^{\frac{3}{2}}_{2,1}} \leqslant c$.

The local wellposedness Theorem 4.5 is an immediate consequence of Lemma below, which will be useful in the sequel.

Lemma 4.7. *Let $T > 0$ be a fixed finite time. For any $t \in [0, T]$, the velocity estimate is given by*

$$\|u\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})} + \|u\|_{L^1_t(B^{\frac{5}{2}}_{2,1})} + \|\nabla\Pi\|_{L^1_t(B^{\frac{1}{2}}_{2,1})} \leqslant C \|u_0\|_{B^{\frac{1}{2}}_{2,1}} + \int_0^t (\|\nabla u(t')\|_{L^\infty} + W(t'))\|u(t')\|_{B^{\frac{1}{2}}_{2,1}} \, dt', \tag{4.9}$$

where

$$W(t') \stackrel{\text{def}}{=} 2^{2m} \|a\|^2_{L^\infty_{t'}(L^\infty)} + 2^{8m} \|a\|^4_{L^\infty_{t'}(L^2)} \big(1 + \|u\|^4_{L^\infty_{t'}(B^{\frac{1}{2}}_{2,1})} + \|a\|^4_{L^\infty_{t'}(L^\infty)}\big).$$

Two above results will provide us the local and uniqueness existence of a solution (a, u). Concerning the global aspect to this solution, we shall use an energy method, which can be achieved by virtue of Theorem 4.8 below.

Theorem 4.8 (Global wellposedness Theorem)**.** *Given the initial data (ρ_0, u_0) and two positive constants m and M such that*

$$u_0 \in H^2(\mathbb{T}^3), \quad 0 < m \leqslant \rho_0(x) \leqslant M, \quad \text{and} \quad \int_{\mathbb{T}^3} \rho_0\, u_0 = 0. \tag{4.10}$$

There exists a constant $\varepsilon_0 > 0$ (depending on m and M) such that if u_0 satisfies the smallness condition $\|u_0\|_{H^2} \leqslant \varepsilon_0$ then, the system (4.3) has a (unique) global solution (ρ, u) which satisfies for any $(t, x) \in [0, +\infty[\times\mathbb{T}^3$

$$\begin{aligned}
&0 < m \leqslant \rho(t,x) \leqslant M,\\
&B_0(t) \leqslant \|\sqrt{\rho_0}u_0\|_{L^2}^2,\\
&B_1(t) \leqslant C\,\|\nabla u_0\|_{L^2}^2,\\
&B_2(t) \leqslant C\left(1 + \|u_0\|_{H^2}^4\right)\|u_0\|_{H^2}^2\ \exp\left(\|u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2\right)
\end{aligned} \tag{4.11}$$

where $B_0(t)$, $B_1(t)$ and $B_2(t)$ are defined by

$$B_0(T) \stackrel{\text{def}}{=} \sup_{t\in[0,T]} \|\sqrt{\rho}u(t)\|_{L^2}^2 + \int_0^T \int_{\mathbb{T}^3} |\nabla u(t,x)|^2\, dx\, dt. \tag{4.12}$$

$$B_1(T) \stackrel{\text{def}}{=} \sup_{t\in[0,T]} \|\nabla u(t)\|_{L^2}^2 + \int_0^T \left(\|\sqrt{\rho}\,\partial_t u(t)\|_{L^2}^2 + \|\nabla\Pi(t)\|_{L^2}^2\right)dt + \frac{1}{8}\int_0^T \|\nabla^2 u(t)\|_{L^2}^2\, dt, \tag{4.13}$$

$$\begin{aligned}
B_2(T) \stackrel{\text{def}}{=} &\sup_{t\in[0,T]} \Big(\frac{1}{2}\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla\Pi(t)\|_{L^2}^2 + \frac{m}{3}\,\|\partial_t u(t)\|_{L^2}^2\Big)\\
&+ \frac{1}{4}\int_0^T \|\nabla\partial_t u(t)\|_{L^2}^2\, dt + \frac{1}{2}\int_0^T \|\nabla^2 u(t)\|_{L^6}^2 dt + \int_0^T \|\nabla^2\Pi(t)\|_{L^6}^2\, dt.
\end{aligned} \tag{4.14}$$

Remark 4.9. We shall prove the existence and global part by an energy method. We underline the very weak assumption (bounded from above and below) on the density we need. We refer the reader to [17] for the uniqueness proof.

Guideline of the proof and organisation of the paper.
Firstly, we prove the local existence and uniqueness of a solution, under hypothesis of Theorem 4.5. Then, we underline that, provided $\|u_0\|_{B^{\frac{1}{2}}_{2,1}}$ is small enough, the lifespan $T^*(u_0)$ of the local solution associated with this data should be greater than 1. This is due to scaling argument. In addition, velocity estimate (4.8) implies

$$\exists t_1 \in [0,1[\quad \text{such that} \quad u(t_1) \in H^2 \quad \text{and} \quad \|u(t_1)\|_{H^2} \leqslant C\,\|u_0\|_{B^{\frac{1}{2}}_{2,1}}. \tag{4.15}$$

This stems from an interpolation argument, provided $T^*(u_0) > 1$. Indeed, assume we have proved there exists an unique solution u such that

$$u \in L^\infty([0,T], B^{\frac{1}{2}}_{2,1}) \cap L^1([0,T], B^{\frac{5}{2}}_{2,1}),$$

and thus, u belongs to $L^{\frac{4}{3}}([0,T], H^2)$, which provide the existence of the small time t_1, such that (4.15) is satisfied.

From this point, the strategy to deal with the global property of our system takes another direction than the strategy set up in [4]. Indeed, we shall prove that, considering $u(t_1)$ as an initial data in H^2, which is small enough (since $\|u_0\|_{B^{\frac{1}{2}}_{2,1}}$ is supposed to be so) and thanks to Theorem 4.8 below, there exists a global solution (the uniqueness is non necessary for what we need in the sequel).
Then, it remains to be seen that such a solution has the relevant regularity, namely the regularity required by Theorem 4.5. In others words, it is crucial to prove the propagation of the regularity of the density function a, from which we infer the regularity of the velocity, thanks to Lemma 4.7. To sum up, we will prove the existence of a global solution with the relevant regularity : this proves the uniqueness of such a solution.

The paper is structured as follows. In Section 2, we collect some basic facts on Littlewood Paley theory, Besov spaces and we will give the classical inequalities (well-known in the whole space $\mathbb{R}^3$), in the case of the torus $\mathbb{T}^3$. In addition, we will stress on the important role of the average u.
Section 3 is devoted to the proof of the main Theorem 4.4. Section 4 deals with the local wellposedness issue of the main theorem : we will prove Theorem 4.5. Section 5 provides the global wellposedness aspect of the main theorem, which will stem from the proof of Theorem 4.8. Let us mention we will only give in both two cases the a priori estimates. It means we skip the standard procedure of Friedrich's regularization. The point is that we deal with uniform estimates, in which we use a standard compactness argument.

4.2 Tool box concerning estimates on the Torus $\mathbb{T}^3$

Proposition 4.10. *(Poincaré-Wirtinger inequality)*
Let u be in $H^1(\mathbb{T}^3)$ and mean free. Then we have :

$$\|u\|_{L^2(\mathbb{T}^3)} \leqslant \|\nabla u\|_{L^2(\mathbb{T}^3)}.$$

In particular, the $\dot{H}^1(\mathbb{T}^3)$ and $H^1(\mathbb{T}^3)$-norms are equivalent, when $\bar{u}$ is mean free.

An obvious consequence of the Poincaré-Wirtinger inequality is the corollary below.

Corollary 4.11. *Let u be in $H^1(\mathbb{T}^3)$. Then we have :*

$$\|u-\bar{u}\|_{L^2(\mathbb{T}^3)} \leqslant \|\nabla u\|_{L^2(\mathbb{T}^3)}.$$

Proposition 4.12. *(Gagliardo-Nirenberg inequality)*

$$\text{In the whole space} \quad \mathbb{R}^3: \quad \|u\|_{L^p} \leqslant \|u\|_{L^2}^{\frac{3}{p}-\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{3}{2}-\frac{3}{p}}, \quad \text{with} \quad 2 \leqslant p \leqslant 6.$$

$$\text{On the torus} \quad \mathbb{T}^3: \quad \|u-\bar{u}\|_{L^p} \leqslant \|u\|_{L^2}^{\frac{3}{p}-\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{3}{2}-\frac{3}{p}}, \quad \text{with} \quad 2 \leqslant p \leqslant 6.$$

In particular, for $p=6$, we find the Sobolev embeddings on the torus :

$$\|u-\bar{u}\|_{L^6(\mathbb{T}^3)} \leqslant C\,\|\nabla u\|_{L^2(\mathbb{T}^3)} \quad \text{instead of} \quad \|u\|_{L^6(\mathbb{R}^3)} \leqslant C\,\|\nabla u\|_{L^2(\mathbb{R}^3)}.$$

The following Lemma is fundamental in this paper. It highlights the crucial role played by the average of the velocity. Because the framework of our work is the torus, we will need several times in the next, to have an estimate on the average. Actually, it provides a general method to compute the average of a quantity we are intesresting in. We will call it *the average method* in the sequel.

Lemma 4.13. *Assuming that* $|\mathbb{T}^3| = 1$ *and* $\int_{\mathbb{T}^3} \rho_0\, u_0 = 0$, *we have :*

$$|\bar{u}(t)| \leqslant \frac{\|\rho_0 - \bar{\rho_0}\|_{L^2}}{|\bar{\rho_0}|} \|\nabla u(t)\|_{L^2}.$$

Proof. Let us consider the integral below and developp it

$$\int_{\mathbb{T}^3} (\rho - \bar{\rho})(t,x)(u - \bar{u})(t,x)\, dx = \int_{\mathbb{T}^3} \rho(t,x)\, u(t,x) - 2\bar{\rho}(t)\, \bar{u}(t) \; + \bar{\rho}(t)\, \bar{u}(t).$$

Thanks to (4.1) and (4.2), we have

$$\begin{aligned} \bar{u}(t) &= -\frac{1}{\bar{\rho}(t)} \int_{\mathbb{T}^3} (\rho - \bar{\rho})(t,x)(u - \bar{u})(t,x)\, dx \\ &= -\frac{1}{\bar{\rho_0}} \int_{\mathbb{T}^3} (\rho - \bar{\rho})(t,x)(u - \bar{u})(t) \\ |\bar{u}(t)| &\leqslant \frac{1}{|\bar{\rho_0}|} \|(\rho - \bar{\rho})(t)\|_{L^2} \|(u - \bar{u})(t)\|_{L^2}. \end{aligned} \tag{4.16}$$

Applying (4.3), we have

$$|\bar{u}(t)| \leqslant \frac{1}{|\bar{\rho_0}|} \|\rho_0 - \bar{\rho_0}\|_{L^2} \|(u - \bar{u})(t)\|_{L^2}. \tag{4.17}$$

Thanks to Poincaré-Wirtinger, we get :

$$|\bar{u}(t)| \leqslant \frac{\|\rho_0 - \bar{\rho_0}\|_{L^2}}{|\bar{\rho_0}|} \|\nabla u(t)\|_{L^2}. \tag{4.18}$$

□

Proposition 4.14. *Assuming that* $|\mathbb{T}^3| = 1$ *and* $\int_{\mathbb{T}^3} \rho_0\, u_0 = 0$, *therefore* $\|u(t)\|_{L^6} \leqslant C(\rho_0)\, \|\nabla u(t)\|_{L^2}$.

Proof.

$$\begin{aligned} \|u(t)\|_{L^6}^2 &\leqslant \|(u - \bar{u})(t)\|_{L^6}^2 + |\bar{u}(t)|^2 \\ &\leqslant C\, \|\nabla u(t)\|_{L^2}^2 + \frac{\|\rho_0 - \bar{\rho_0}\|_{L^2}^2}{\bar{\rho_0}^2} \|\nabla u(t)\|_{L^2}^2 \\ &\leqslant C(\rho_0)\, \|\nabla u(t)\|_{L^2}^2. \end{aligned} \tag{4.19}$$

□

Proposition 4.15. *If* $|\mathbb{T}^3| = 1$ *and* $\int_{\mathbb{T}^3} \rho_0\, u_0 = 0$, *then* $\|u(t)\|_{L^3} \leqslant C(\rho_0)\, \|\nabla u(t)\|_{L^2}$.

Proof. Arguments are similar as before. We introduce the average of u and we apply succesively Gagliardo-Nirenberg and Poincaré-Wirtinger inequalities

$$\begin{aligned} \|u(t)\|_{L^3} &\leqslant \|(u - \bar{u})(t)\|_{L^3} + |\bar{u}(t)| \\ &\leqslant \|(u - \bar{u})(t)\|_{L^2}^{\frac{1}{2}} \|\nabla(u - \bar{u})(t)\|_{L^2}^{\frac{1}{2}} + |\bar{u}(t)| \\ &\leqslant \|\nabla u(t)\|_{L^2}^{\frac{1}{2}} \|\nabla u(t)\|_{L^2}^{\frac{1}{2}} \; + |\bar{u}(t)| \\ &\leqslant \|\nabla u(t)\|_{L^2} \; + |\bar{u}(t)|. \end{aligned} \tag{4.20}$$

Concerning the term $|\bar{u}(t)|$, same computations as in Lemma 4.13 yield

$$\begin{aligned}\|u(t)\|_{L^3} &\leqslant \|\nabla u(t)\|_{L^2} + \frac{1}{|\bar{\rho_0}|}\,\|\rho_0-\bar{\rho_0}\|_{L^2}\,\|(u-\bar{u})(t)\|_{L^2}\\ &\leqslant C(\rho_0)\,\|\nabla u(t)\|_{L^2}.\end{aligned} \tag{4.21}$$

□

4.3 Proof of the main Theorem

Assuming we have proved Theorems 4.5 and 4.8, we can prove the main Theorem. Firstly, notice that Theorem 4.5 implies

$$\exists t_1 \in [0,T],\ u(t_1)\in H^2\cap B^{\frac{1}{2}}_{2,1}, \quad \text{and} \quad \|u(t_1)\|_{H^2} \leqslant \|u_0\|_{B^{\frac{1}{2}}_{2,1}}. \tag{4.22}$$

Moreover, we have a fundamental information on $a(t_1)$:

$$a(t_1)\in B^{\frac{3}{2}}_{2,1}\cap L^\infty. \tag{4.23}$$

Let us underline that we have, by virtue of Remark 4.2,

$$\int_{\mathbb{T}^3}\frac{1}{1+a(t_1)}\,u(t_1)=\int_{\mathbb{T}^3}\frac{1}{1+a_0}\,u_0=0. \tag{4.24}$$

As a consequence, Theorem 4.8 implies there exists a global solution (ρ,w) of the system (4.2) associated with data

$$(\rho,w)_{t=0}\overset{\text{def}}{=}\Big(\frac{1}{1+a(t_1)},u(t_1)\Big).$$

First of all, we adopt the classical point of view : from the solution (ρ,w) of the system (4.2), we define the solution (a_w,w) of the system (4.4), given by

$$\rho\overset{\text{def}}{=}\frac{1}{1+a_w}.$$

Therefore, it follows that the solution (a_w,w) is associated with the data $(a(t_1),u(t_1))$, which belongs to $B^{\frac{3}{2}}_{2,1}\cap L^\infty\times H^2$.
The goal is to prove the uniqueness of such a solution, which will come from the following regularity

$$\forall\, T\geqslant 0,\quad (a_w,w)\in\mathcal{C}([0,T],B^{\frac{3}{2}}_{2,1})\,\times\,\mathcal{C}([0,T],B^{\frac{1}{2}}_{2,1})\,\cap\,L^1([0,T],B^{\frac{5}{2}}_{2,1}).$$

Proving such a regularity on the density function and the velocity field provides us the uniqueness by virtue of local wellposedness Theorem 4.5. The point is the propagation of the regularity of a_w.

4.3.1 Propagation of the regularity of the density

Proposition 4.16. *Let $T>0$ be a time fixed. Then, $\forall t\in[0,T]$,* $\quad a_w(t)\in B^{\frac{3}{2}}_{2,1}$.

Proof. Applying the frequencies localization operator Δ_q on the transport equation, we get

$$\partial_t \Delta_q a_w + w\cdot \nabla \Delta_q a_w = -\left[\Delta_q, w\cdot \nabla\right] a_w. \tag{4.25}$$

Taking the L^2-inner product with $\Delta_q a$, the divergence-free condition implies that

$$\frac{1}{2}\frac{d}{dt}\|\Delta_q a_w\|_{L^2}^2 \leqslant \|\Delta_q a_w\|_{L^2}\, \|\left[\Delta_q, w\cdot \nabla\right] a_w\|_{L^2}. \tag{4.26}$$

By virtue of Gronwall's Lemma 4.19 (given in the appendix), we infer that

$$2^{\frac{3q}{2}}\,\|\Delta_q a_w(t)\|_{L^2} \leqslant 2^{\frac{3q}{2}}\,\|\Delta_q a(t_1)\|_{L^2} + 2^{\frac{3q}{2}}\int_0^t \|\left[\Delta_q, w\cdot \nabla\right] a_w\|_{L^2}\, dt'. \tag{4.27}$$

Therefore, by some classical estimate of the commutator (see Lemma 2.100 in [5]), we get

$$\|a_w(t)\|_{B^{\frac{3}{2}}_{2,1}} \leqslant \|a(t_1)\|_{B^{\frac{3}{2}}_{2,1}} + C\int_0^t (\|a_w(t')\|_{B^{\frac{3}{2}}_{2,1}}\,\|\nabla w(t')\|_{L^\infty} + \|\nabla a_w(t')\|_{L^3}\,\|\nabla w(t')\|_{B^{\frac{1}{2}}_{6,1}})dt'. \tag{4.28}$$

From the following embedding $B^{\frac{3}{2}}_{2,1} \hookrightarrow B^1_{3,1}$ which holds in dimension 3, Gronwall Lemma yields

$$\|a_w(t)\|_{B^{\frac{3}{2}}_{2,1}} \leqslant \|a(t_1)\|_{B^{\frac{3}{2}}_{2,1}}\ \exp\big(C\int_0^t (\|\nabla w(t')\|_{L^\infty} + \|\nabla w(t')\|_{B^{\frac{1}{2}}_{6,1}}\big))dt'. \tag{4.29}$$

It remains to be checked that $\int_0^t \|\nabla w(t')\|_{L^\infty} dt'$ and $\int_0^t \|\nabla w(t')\|_{B^{\frac{1}{2}}_{6,1}}\, dt'$ exist for any time. This stems from energy method applying on w, thanks to Theorem 4.8. Concerning the term $\int_0^t \|\nabla w(t')\|_{L^\infty} dt'$, an interpolation argument gives rise to

$$\begin{aligned}\int_0^t \|\nabla w(t')\|_{L^\infty} dt' &\leqslant \int_0^t \|\nabla w(t')\|_{L^2}^{\frac{1}{4}}\,\|\nabla^2 w(t')\|_{L^6}^{\frac{3}{4}} dt'\\ &\leqslant \frac{1}{4}\int_0^t \|\nabla w(t')\|_{L^2} dt' + \frac{3}{4}\int_0^t \|\nabla^2 w(t')\|_{L^6} dt',\end{aligned} \tag{4.30}$$

and thanks to Hölder's inequality, we get

$$\int_0^t \|\nabla w(t')\|_{L^\infty} \leqslant C\, t^{\frac{1}{2}}\,\big(\|\nabla w(t')\|_{L^2_t(L^2)} + \|\nabla^2 w(t')\|_{L^2_t(L^6)}\big). \tag{4.31}$$

By virtue of Theorem 4.8, $\|\nabla w\|_{L^2_t(L^2)} \leqslant C\,\|u(t_1)\|_{L^2}$ and $\|\nabla^2 w\|_{L^2_t(L^6)} \leqslant C\,\|u(t_1)\|_{H^2}$, therefore,

$$\int_0^t \|\nabla w(t')\|_{L^\infty} dt' \leqslant C\, t^{\frac{1}{2}}\,\|u(t_1)\|_{H^2}. \tag{4.32}$$

Concerning the term $\int_0^t \|\nabla w(t')\|_{B^{\frac{1}{2}}_{6,1}}\, dt'$, arguments are similar to the others ones and lead us to

$$\begin{aligned}\int_0^t \|\nabla w(t')\|_{B^{\frac{1}{2}}_{6,1}} dt' &\leqslant \int_0^t \|w(t')\|_{B^{-1}_{6,\infty}}^{\frac{1}{2}}\,\|\ w(t')\|_{B^2_{6,\infty}}^{\frac{1}{2}} dt'\\ &\leqslant \frac{1}{2}\int_0^t \|w(t')\|_{B^{-1}_{6,\infty}} dt' + \frac{1}{2}\int_0^t \|w(t')\|_{B^2_{6,\infty}} dt'.\end{aligned} \tag{4.33}$$

Notice we have the following embeddings

$$L^2 \hookrightarrow B^{-1}_{6,\infty} \quad \text{and} \quad L^6 \hookrightarrow B^0_{6,\infty}, \tag{4.34}$$

from which we infer that (thanks to Thereom 4.8)

$$\begin{aligned}\int_0^t \|\nabla w\|_{B^{\frac12}_{6,1}} &\leqslant \frac12 \int_0^t \|w\|_{L^2} + \frac12 \int_0^t \|\nabla^2 w\|_{L^6} \\ &\leqslant \frac{t}{2} \|w\|_{L^\infty_t(L^2)} + \frac12 t^{\frac12} \|\nabla^2 w\|_{L^2_t(L^6)} \\ &\leqslant \frac12 t \|u(t_1)\|_{L^2} + \frac12 t^{\frac12} \|u(t_1)\|_{H^2}.\end{aligned} \tag{4.35}$$

Choosing t small enough such that $t \leqslant t^{\frac12}$, we get

$$\int_0^t \|\nabla w(t')\|_{B^{\frac12}_{6,1}} dt' \leqslant C\, t^{\frac12} \|u(t_1)\|_{H^2}.$$

This yields to the desired estimate

$$\|a_w(t)\|_{B^{\frac32}_{2,1}} \leqslant \|a(t_1)\|_{B^{\frac32}_{2,1}} \exp\big(C\, t^{\frac12} \|u(t_1)\|_{H^2}\big). \tag{4.36}$$

This concludes the proof on the propagation of the regularity on the density function.

4.3.2 Regularity of the velocity field

Holding the regularity on the density, we are allowed to apply Lemma 4.7, which gives rise to the following estimate, available, for any $t \in [0, T]$, where T is a fixed finite time.

$$\|w\|_{L^\infty_t(B^{\frac12}_{2,1})} + \|w\|_{L^1_t(B^{\frac52}_{2,1})} + \|\nabla \Pi\|_{L^1_t(B^{\frac12}_{2,1})} \leqslant C\, \|u(t_1)\|_{B^{\frac12}_{2,1}} + C \int_0^t \big(\|\nabla w(t')\|_{L^\infty} + W(t')\big) \|w(t')\|_{B^{\frac12}_{2,1}} dt', \tag{4.37}$$

where

$$W(t') \overset{\text{def}}{=} 2^{2m} \|a_w\|^2_{L^\infty_{t'}(L^\infty)} + 2^{8m} \|a_w\|^4_{L^\infty_{t'}(L^2)} \big(1 + \|w\|^4_{L^\infty_{t'}(B^{\frac12}_{2,1})} + \|a_w\|^4_{L^\infty_{t'}(L^\infty)}\big).$$

We deduce from this estimate, by Gronwall Lemma,

$$\|w\|_{L^\infty_t(B^{\frac12}_{2,1})} + \|w\|_{L^1_t(B^{\frac52}_{2,1})} + \|\nabla \Pi\|_{L^1_t(B^{\frac12}_{2,1})} \leqslant C\, \|u(t_1)\|_{B^{\frac12}_{2,1}} \exp\big(\|\nabla w\|_{L^1_t(L^\infty)} + t\, W(t)\big). \tag{4.38}$$

Concerning the term $W(t)$, on the one hand, by the transport equation, we get immediately

$$\|a_w(t,\cdot)\|_{L^2} = \|a_w(0,\cdot)\|_{L^2},$$

which is bounded by $\|a_w(0)\|_{B^{\frac32}_{2,1}}$, since spaces are inhomogeneous. One the other hand, by an interpolation argument, one has

$$\begin{aligned}\|w\|_{B^{\frac12}_{2,1}} &\leqslant \|w\|^{\frac12}_{B^0_{2,\infty}} \|w\|^{\frac12}_{B^1_{2,\infty}} \\ &\leqslant \|w\|^{\frac12}_{L^2} \|w\|^{\frac12}_{H^1}.\end{aligned}$$

It follows that, by virtue of Theorem 4.8,

$$\begin{aligned}\|w\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})} &\leqslant \|w\|^{\frac{1}{2}}_{L^\infty_t(L^2)}\,\|w\|^{\frac{1}{2}}_{L^\infty_t(H^1)}\\ &\leqslant \|u(t_1)\|^{\frac{1}{2}}_{L^2}\,(\|u(t_1)\|^{\frac{1}{2}}_{L^2}+\|\nabla u(t_1)\|^{\frac{1}{2}}_{L^2}).\end{aligned} \tag{4.39}$$

It results from these simple computations that the factor $W(t)$ is bounded by

$$\forall t\in[0,T],\ W(t)\leqslant C\,\|u(t_1)\|_{H^2}.$$

As it has been already noticed, the term $\|\nabla w\|_{L^1_t(L^\infty)}$ satisfies

$$\int_0^t \|\nabla w(t')\|_{L^\infty}dt' \leqslant C\,t^{\frac{1}{2}}\,\|u(t_1)\|_{H^2}. \tag{4.40}$$

It results from all of this, that for any $t\in[0,T]$, we have

$$\|w\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})}+\|w\|_{L^1_t(B^{\frac{5}{2}}_{2,1})}+\|\nabla\Pi\|_{L^1_t(B^{\frac{1}{2}}_{2,1})}\leqslant C\,\|u(t_1)\|_{B^{\frac{1}{2}}_{2,1}}\exp\big(C\,t^{\frac{1}{2}}\,\|u(t_1)\|_{H^2}\big). \tag{4.41}$$

Combining with the estimate on the density function (4.36), we get $t\in[0,T]$, for a fixed time $T>0$

$$\|a_w\|_{L^\infty_t(B^{\frac{3}{2}}_{2,1})}+\|w\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})}+\|w\|_{L^1_t(B^{\frac{5}{2}}_{2,1})}+\|\nabla\Pi\|_{L^1_t(B^{\frac{1}{2}}_{2,1})}\leqslant C\,\|u(t_1)\|_{B^{\frac{1}{2}}_{2,1}}\exp\big(C\,t^{\frac{1}{2}}\,\|u(t_1)\|_{H^2}\big). \tag{4.42}$$

This ends up the proof of Theorem 4.4.

□

4.4 Proof of the local wellposedness part of the main theorem

This section is devoted to the proof of Theorem 4.5. We give only the proof of the existence part of the theorem, since the uniqueness part has been already proved in [3]. We only mention the start point of the uniqueness proof.

4.4.1 Existence part

The existence proof can be achieved by a regularization process (e.g Fridriech method). The idea is classical : we build smooth approximate solutions, perform uniform estimates on them. A compactness argument leads us to the proof of the existence of a solution of 4.4. We skip this part and provide some a priori estimates for smooth enough solution (a,u).

Let us start by proving the estimate (4.7) on the density. Applying the frequencies localization operator Δ_q on the transport equation, we get

$$\partial_t\Delta_q a+u\cdot\nabla\Delta_q a=-\,[\Delta_q,u\cdot\nabla]\,a.$$

Taking the L^2-inner product with $\Delta_q a$, the divergence-free condition implies that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta_q a\|^2_{L^2} &= -\big([\Delta_q,u\cdot\nabla]\,a\,|\,\Delta_q a\big)_{L^2}\\ &\leqslant \|\Delta_q a\|_{L^2}\,\|\,[\Delta_q,u\cdot\nabla]\,a\|_{L^2}.\end{aligned}$$

By virtue of Gronwall's Lemma 4.19 (given in the appendix), we infer that

$$2^{\frac{3q}{2}}\,\|\Delta_q a\|_{L^2} \leqslant 2^{\frac{3q}{2}}\,\|\Delta_q a_0\|_{L^2} + 2^{\frac{3q}{2}}\int_0^t \|\,[\Delta_q, u\cdot\nabla]\,a\|_{L^2}\,dt'.$$

A classical commutator estimate (see for instance Lemma 2.100 in [5]) shows there exists a sequence (c_q) belonging to $\ell^1(\mathbb{Z})$ such that

$$2^{\frac{3q}{2}}\,\|\,[\Delta_q, u\cdot\nabla]\,a\|_{L^2} \leqslant c_q\,\|a\|_{B^{\frac{3}{2}}_{2,1}}\,\|u\|_{B^{\frac{5}{2}}_{2,1}}\,,$$

and therefore,

$$2^{\frac{3q}{2}}\int_0^t \|\,[\Delta_q, u\cdot\nabla]\,a\|_{L^2}\,dt' \leqslant \sup_t\, c_q(t)\int_0^t \|a(t')\|_{B^{\frac{3}{2}}_{2,1}}\,\|u(t')\|_{B^{\frac{5}{2}}_{2,1}}\,dt'.$$

By summing on $q\in\mathbb{Z}$, we get

$$\|a\|_{B^{\frac{3}{2}}_{2,1}} \leqslant \|a_0\|_{B^{\frac{3}{2}}_{2,1}} + C\int_0^t \|a(t')\|_{B^{\frac{3}{2}}_{2,1}}\,\|u(t')\|_{B^{\frac{5}{2}}_{2,1}}\,dt'.$$

The classical Gronwall's Lemma yields the proof of (4.7).

Let us prove estimate (4.8) on the velocity. Actually, we prove Lemma 4.7, which is a bit more general than we want to get.

Proof of Lemma 4.7.
We may rewrite the system (4.4), after decomposing $(1+a)$ into $(1+S_m a)+(a-S_m a)$.

$$\partial_t u + u\cdot\nabla u - (1+S_m a)\,\Delta u + (1+S_m a)\,\nabla\Pi = (a-S_m a)(\Delta u - \nabla\Pi) \tag{4.43}$$

Notice that $(1+S_m a)\,\nabla\Pi = \nabla\big((1+S_m a)\,\Pi\big) - \Pi\,\nabla S_m a$, which implies

$$\partial_t u + u\cdot\nabla u - (1+S_m a)\,\Delta u + \nabla\big((1+S_m a)\,\Pi\big) = (a-S_m a)(\Delta u - \nabla\Pi) + \Pi\,\nabla S_m a.$$

Let us introduce the notation $E_m \stackrel{\text{def}}{=} (a-S_m a)(\Delta u-\nabla\Pi)$. We reduce the problem to the system below

$$\left\{\begin{array}{ccc} \partial_t u + u\cdot\nabla u - (1+S_m a)\,\Delta u + \nabla\big((1+S_m a)\,\Pi\big) & = & E_m + \Pi\,\nabla S_m a.\\ \operatorname{div} u & = & 0\\ (a,u)_{|t=0} & = & (a_0,u_0), \end{array}\right. \tag{4.44}$$

Step 1: Frequency localization.
Applying the operator Δ_q in (4.44), we localize the velocity in a ring, with a size 2^q, and we get

$$\partial_t\Delta_q u + \Delta_q(u\cdot\nabla u) - \Delta_q\big((1+S_m a)\,\Delta u\big) + \Delta_q\big(\nabla\big((1+S_m a)\,\Pi\big)\big) = \Delta_q E_m + \Delta_q(\Pi\,\nabla S_m a).$$

By definition of the commutator $\Delta_q(u\cdot\nabla u) \stackrel{\text{def}}{=} u\cdot\nabla\Delta_q u + [\Delta_q, u\cdot\nabla]\,u$, this gives

$$\begin{aligned}\partial_t\Delta_q u + u\cdot\nabla\Delta_q u - \Delta_q\big((1+S_m a)\,\Delta u\big) + \Delta_q\big(\nabla\big((1+S_m a)\,\Pi\big)\big) = &-[\Delta_q, u\cdot\nabla]\,u + \Delta_q E_m\\ &+ \Delta_q(\Pi\,\nabla S_m a).\end{aligned}$$

In particular, a simple computation gives

$$-\Delta_q\big((1+S_m a)\,\Delta u\big) = -\operatorname{div}\big((1+S_m a)\,\Delta_q\nabla u\big) - \operatorname{div}\big([\Delta_q, S_m a]\,\nabla u\big) + \Delta_q\big(\nabla S_m a\,\nabla u\big).$$

As a consequence, we get

$$\begin{aligned}\partial_t \Delta_q u + u\cdot\nabla\Delta_q u - \operatorname{div}\Big((1+S_m a)\,\Delta_q \nabla u\, + \Delta_q\Big(\nabla\big((1+S_m a)\,\Pi\big)\Big) = -\,[\Delta_q, u\cdot\nabla]\,u\, + \Delta_q E_m \\ + \Delta_q(\Pi\,\nabla S_m a) + \operatorname{div}([\Delta_q, S_m a]\,\nabla u) - \Delta_q(\nabla S_m a\,\nabla u).\end{aligned} \tag{4.45}$$

Let us take the L^2 inner product with $\Delta_q u$ in the above equation (4.45). Because of the divergence free condition, we have

$$\big(u\cdot\nabla\Delta_q u\,|\,\Delta_q u\big)_{L^2} = 0 \quad\text{and}\quad \big(\Delta_q(\nabla((1+S_m a)\,\Pi))\,|\,\Delta_q u\big)_{L^2} = 0.$$

As a result,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta_q u\|_{L^2}^2 + \int_{\mathbb{T}^3}(1+S_m a)\,|\Delta_q\nabla u|^2\,dx \leqslant \|\Delta_q u\|_{L^2}\Big(\|\,[\Delta_q, u\cdot\nabla]\,u\|_{L^2} + \|\Delta_q E_m\|_{L^2} + \|\Delta_q(\Pi\,\nabla S_m a)\|_{L^2} \\ + 2^q\,\|\,[\Delta_q, S_m a]\,\nabla u\|_{L^2} + \|\Delta_q(\nabla S_m a\,\nabla u)\|_{L^2}\Big)\end{aligned}$$

Let us point that $1+S_m a = 1 + a + S_m a - a$. As we assume that $S_m a - a$ is small enough in norm $L^\infty_t(B^{\frac{3}{2}}_{2,1})$, it follows that

$$1 + S_m a \geqslant \frac{b}{2},$$

which along with Lemma 4.21, ensures that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta_q u\|_{L^2}^2 + \frac{b}{2}\;2^{2q}\,\|\Delta_q u\|_{L^2}^2 \leqslant \|\Delta_q u\|_{L^2}\Big(\|\,[\Delta_q, u\cdot\nabla]\,u\|_{L^2} + \|\Delta_q E_m\|_{L^2} + \|\Delta_q(\Pi\,\nabla S_m a)\|_{L^2} \\ + 2^q\,\|\,[\Delta_q, S_m a]\,\nabla u\|_{L^2} + \|\Delta_q(\nabla S_m a\,\nabla u)\|_{L^2}\Big).\end{aligned}$$

Applying a Gronwall's argument, we get

$$\begin{aligned}\frac{d}{dt}\|\Delta_q u\|_{L^2} + \frac{b}{2}\;2^{2q}\,\|\Delta_q u\|_{L^2} \leqslant \|\,[\Delta_q, u\cdot\nabla]\,u\|_{L^2} + \|\Delta_q E_m\|_{L^2} + \|\Delta_q(\Pi\,\nabla S_m a)\|_{L^2} \\ + 2^q\,\|\,[\Delta_q, S_m a]\,\nabla u\|_{L^2} + \|\Delta_q(\nabla S_m a\,\nabla u)\|_{L^2}.\end{aligned}$$

An integration in time yields

$$\begin{aligned}2^{\frac{q}{2}}\,\|\Delta_q u\|_{L^2} + C\,b\,2^{\frac{5q}{2}}\int_0^t \|\Delta_q u\|_{L^2}\,dt' \leqslant\; 2^{\frac{q}{2}}\,\|\Delta_q u_0\|_{L^2} + \int_0^t 2^{\frac{q}{2}}\,\|\,[\Delta_q, u\cdot\nabla]\,u\|_{L^2}dt' \\ + \int_0^t 2^{\frac{q}{2}}\,\|\Delta_q E_m\|_{L^2}dt' + \int_0^t 2^{\frac{q}{2}}\,\|\Delta_q(\Pi\,\nabla S_m a)\|_{L^2}\,dt' \\ + \int_0^t 2^{\frac{3q}{2}}\,\|\,[\Delta_q, S_m a]\,\nabla u\|_{L^2}dt' + \int_0^t 2^{\frac{q}{2}}\,\|\Delta_q(\nabla S_m a\,\nabla u)\|_{L^2}dt'.\end{aligned}$$

Taking the supremum in time and then summing on $q\in\mathbb{Z}$ provides us the norm $\|u\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})}$ and thus

$$\begin{aligned}\|u\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})} + C\,b\,\|u\|_{L^1_t(B^{\frac{5}{2}}_{2,1})} \leqslant\; \|u_0\|_{B^{\frac{1}{2}}_{2,1}} + \|E_m\|_{L^1_t(B^{\frac{1}{2}}_{2,1})} + \|\Pi\,\nabla S_m a\|_{L^1_t(B^{\frac{1}{2}}_{2,1})} \\ + \sum_{q\in\mathbb{Z}} 2^{\frac{q}{2}}\,\|\,[\Delta_q, u\cdot\nabla]\,u\|_{L^1_t(L^2)} \\ + \sum_{q\in\mathbb{Z}} 2^{\frac{3q}{2}}\,\|\,[\Delta_q, S_m a]\,\nabla u\|_{L^1_t(L^2)} + \|\nabla S_m a\,\nabla u\|_{L^1_t(B^{\frac{1}{2}}_{2,1})}.\end{aligned} \tag{4.46}$$

Step 2: Estimate of each term in the right-hand-side of the above inequality.

⋆ Estimate of $\|E_m\|_{L^1_t(B^{\frac12}_{2,1})}$
Product laws in Besov spaces (cf Lemma 4.20 in Appendix) yield

$$\begin{aligned}\|E_m\|_{L^1_t(B^{\frac12}_{2,1})} &\leqslant C\,\|a-S_m a\|_{L^\infty_t(B^{\frac32}_{2,1})}\,\|\Delta u-\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})}\\ &\leqslant C\,\|a-S_m a\|_{L^\infty_t(B^{\frac32}_{2,1})}\Big(\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})}+\|u\|_{L^1_t(B^{\frac52}_{2,1})}\Big).\end{aligned}\tag{4.47}$$

⋆ Estimate of $\|\Pi\,\nabla S_m a\|_{L^1_t(B^{\frac12}_{2,1})}$.
Concerning the pressure term, as it is defined up to a constant, we can assume it is mean free. Same remark holds for the term $\|\nabla S_m a\|_{B^1_{2,2}}$, since obviously the term $\nabla S_m a$ is mean free. In this way, the norms $\|\cdot\|_{B^1_{2,2}}$ and $\|\cdot\|_{\dot B^1_{2,2}}$ are equivalent. By virtue of paradifferential calculus in inhomogeneous Besov norm, we get

$$\begin{aligned}\|\Pi\,\nabla S_m a\|_{B^{\frac12}_{2,1}} &\leqslant C\,\|\Pi\,\|_{B^1_{2,2}}\,\|\nabla S_m a\|_{B^1_{2,2}}\\ &\leqslant C\,\|\Pi\,\|_{\dot B^1_{2,2}}\,\|\nabla S_m a\|_{\dot B^1_{2,2}}\\ &= C\,\|\nabla\Pi\,\|_{L^2}\,\|\nabla S_m a\|_{\dot H^1},\end{aligned}\tag{4.48}$$

which leads to

$$\|\Pi\,\nabla S_m a\|_{L^1_t(B^{\frac12}_{2,1})}\leqslant C\,\|\nabla\Pi\,\|_{L^1_t(L^2)}\,\|\nabla S_m a\|_{L^\infty_t(\dot H^1)}.$$

⋆ Estimate of $\|\nabla S_m a\,\nabla u\|_{L^1_t(B^{\frac12}_{2,1})}$. Above arguments still provide

$$\begin{aligned}\|\nabla S_m a\,\nabla u\|_{B^{\frac12}_{2,1}} &\leqslant C\,\|\nabla S_m a\|_{B^1_{2,2}}\,\|\nabla u\|_{B^1_{2,2}}\\ &\leqslant C\,\|\nabla u\|_{\dot H^1}\,\|\nabla S_m a\|_{\dot H^1}.\end{aligned}\tag{4.49}$$

Therefore, we deduce that

$$\|\nabla S_m a\,\nabla u\|_{L^1_t(B^{\frac12}_{2,1})}\leqslant C\,\|u\|_{L^1_t(\dot H^2)}\,\|\nabla S_m a\|_{L^\infty_t(\dot H^1)}.$$

⋆ Estimate of $\sum_{q\in\mathbb{Z}}2^{\frac q2}\,\|\,[\Delta_q,u\cdot\nabla]\,u\|_{L^1_t(L^2)}$. By virtue of commutator estimate, we infer that

$$\|\,[\Delta_q,u\cdot\nabla]\,u\|_{L^2}\leqslant C\,d_q\,2^{-\frac q2}\,\|\nabla u\|_{B^{\frac32}_{2,1}}\,\|u\|_{B^{\frac12}_{2,1}}.$$

Therefore, we deduce that

$$\sum_{q\in\mathbb{Z}}2^{\frac q2}\,\|\,[\Delta_q,u\cdot\nabla]\,u\|_{L^1_t(L^2)}\leqslant C\,\int_0^t\|\nabla u(t')\|_{L^\infty}\,\|u(t')\|_{B^{\frac12}_{2,1}}\,dt'.$$

⋆ Estimate of $\sum_{q\in\mathbb{Z}}2^{\frac{3q}{2}}\,\|\,[\Delta_q,S_m a]\,\nabla u\|_{L^1_t(L^2)}$. We can prove the estimate below (see Lemma 4.21)

$$\sum_{q\in\mathbb{Z}}2^{\frac{3q}{2}}\,\|\,[\Delta_q,S_m a]\,\nabla u\|_{L^1_t(L^2)}\leqslant\;C\,2^m\,\|a\|_{L^\infty_t(L^\infty)}\,\|u\|_{L^1_t(B^{\frac32}_{2,1})}\;+\;2^{2m}\,\|a\|_{L^\infty_t(L^2)}\,\|u\|_{L^1_t(\dot H^2)}.\tag{4.50}$$

Plugging all the above estimates in (4.46), we finally get

$$\begin{aligned}\|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + C\,b\,\|u\|_{L^1_t(B^{\frac52}_{2,1})} \leqslant\ & \|u_0\|_{B^{\frac12}_{2,1}} + \|a - S_m a\|_{L^\infty_t(B^{\frac32}_{2,1})}\Big(\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} + \|u\|_{L^1_t(B^{\frac52}_{2,1})}\Big)\\ &+ \int_0^t \|\nabla u(t')\|_{L^\infty}\,\|u(t')\|_{B^{\frac12}_{2,1}}\,dt' \,+\, 2^m\,\|a\|_{L^\infty_t(L^\infty)}\,\|u\|_{L^1_t(B^{\frac32}_{2,1})}\\ &+ 2^{2m+1}\,\|a\|_{L^\infty_t(L^2)}\Big(\|u\|_{L^1_t(\dot H^2)} + \|\nabla\Pi\|_{L^1_t(L^2)}\Big),\end{aligned} \tag{4.51}$$

where we have used

$$\|\nabla S_m a\|_{L^\infty_t(\dot H^1)} = \|\nabla^2 S_m a\|_{L^\infty_t(L^2)} \leqslant 2^{2m}\,\|a\|_{L^\infty_t(L^2)}.$$

Step 3: Estimate of $\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})}$.

We take the divergence operator in (4.43) and thus

$$\mathrm{div}\big((1+S_m a)\nabla\Pi\big) = -\,\mathrm{div}(u\cdot\nabla u) + \Delta u\cdot\nabla S_m a\, + \mathrm{div}\Big((S_m a - a)(\nabla\Pi - \Delta u)\Big).$$

Applying the operator Δ_q and taking the L^2 inner product with $\Delta_q\Pi$ yield

$$\begin{aligned}\big(\Delta_q\Big((1+S_m a)\nabla\Pi\Big)|\Delta_q\Pi\big)_{L^2} &= \big(\Delta_q(u\cdot\nabla u)|\Delta_q\nabla\Pi\big)_{L^2} + \big(\Delta_q(\Delta u\cdot\nabla S_m a)|\Delta_q\Pi\big)_{L^2}\\ &\quad+ \big(\Delta_q((S_m a - a)\nabla\Pi)|\Delta_q\nabla\Pi\big)_{L^2} - \big(\Delta_q((S_m a - a)\Delta u)|\Delta_q\nabla\Pi\big)_{L^2}.\end{aligned}$$

In particular, the left-hand-side can be rewritten and bounded from below as follows

$$\begin{aligned}\big(\Delta_q\Big((1+S_m a)\nabla\Pi\Big)|\Delta_q\nabla\Pi\big)_{L^2} &= \big(\Delta_q\nabla\Pi|\Delta_q\nabla\Pi\big)_{L^2} + \big([\Delta_q, S_m a]\,\nabla\Pi|\Delta_q\nabla\Pi\big)_{L^2}\\ &\quad+ \big(S_m a\,\Delta_q\nabla\Pi|\Delta_q\nabla\Pi\big)_{L^2}\\ &= \big((1+S_m a)\,\Delta_q\nabla\Pi|\Delta_q\nabla\Pi\big)_{L^2} + \big([\Delta_q, S_m a]\,\nabla\Pi|\Delta_q\nabla\Pi\big)_{L^2}\\ &\geqslant b\,\|\Delta_q\nabla\Pi\|^2_{L^2} + \big([\Delta_q, S_m a]\,\nabla\Pi|\Delta_q\nabla\Pi\big)_{L^2}.\end{aligned}$$

It follows

$$\begin{aligned}b\,\|\Delta_q\nabla\Pi\|^2_{L^2} \leqslant\ & \|\Delta_q\nabla\Pi\|_{L^2}\Big(\|\Delta_q(u\cdot\nabla u)\|_{L^2} + \|\Delta_q((S_m a - a)\nabla\Pi)\|_{L^2}\\ &+ \|\Delta_q((S_m a - a)\Delta u)\|_{L^2} + \|\,[\Delta_q, S_m a]\,\nabla\Pi\|_{L^2}\Big)\\ &+ \|\Delta_q\Pi\|_{L^2}\,\|\Delta_q(\Delta u\cdot\nabla S_m a)\|_{L^2}.\end{aligned} \tag{4.52}$$

In particular, Lemma 4.21 provides the inequality below

$$\|\Delta_q\Pi\|_{L^2} \lesssim 2^{-q}\,\|\Delta_q\nabla\Pi\|_{L^2},$$

which gives rise to

$$\begin{aligned}b\,\|\Delta_q\nabla\Pi\|_{L^2} \leqslant\ & \|\Delta_q(u\cdot\nabla u)\|_{L^2} + \|\Delta_q((S_m a - a)\nabla\Pi)\|_{L^2}\, + \|\Delta_q((S_m a - a)\Delta u)\|_{L^2}\\ &+ \|\,[\Delta_q, S_m a]\,\nabla\Pi\|_{L^2}\, + 2^{-q}\,\|\Delta_q(\Delta u\cdot\nabla S_m a)\|_{L^2}.\end{aligned} \tag{4.53}$$

Multiplying by $2^{\frac q2}$ and summing on $q\in\mathbb{Z}$, we have

$$\begin{aligned}b\,\|\nabla\Pi\|_{B^{\frac32}_{2,1}} \lesssim\ & \|u\cdot\nabla u\|_{B^{\frac12}_{2,1}} + \|(S_m a - a)\nabla\Pi\|_{B^{\frac12}_{2,1}} + \|(S_m a - a)\Delta u\|_{B^{\frac12}_{2,1}}\\ &+ \|\Delta u\cdot\nabla S_m a\|_{B^{-\frac12}_{2,1}} + \sum_{q\in\mathbb{Z}} 2^{\frac q2}\|\,[\Delta_q, S_m a]\,\nabla\Pi\|_{L^2}.\end{aligned}$$

Notice that

$$\|\Delta u\cdot \nabla S_m a\|_{B^{-\frac12}_{2,1}} \leqslant C\,\|\nabla S_m a\|_{\dot H^1}\,\|\Delta u\|_{L^2}.$$

On the one hand, product laws in Besov spaces (cf Lemma 4.20) give

$$\|u\cdot \nabla u\|_{B^{\frac12}_{2,1}} \leqslant \|u\|_{B^{\frac12}_{2,1}}\,\|\nabla u\|_{L^\infty}.$$

$$\|(S_m a - a)\Delta u\|_{B^{\frac12}_{2,1}} \leqslant C\,\|(S_m a - a)\|_{B^{\frac32}_{2,1}}\,\|\Delta u\|_{B^{\frac12}_{2,1}}.$$

$$\|(S_m a - a)\nabla \Pi\|_{B^{\frac12}_{2,1}} \leqslant C\,\|(S_m a - a)\|_{B^{\frac32}_{2,1}}\,\|\nabla \Pi\|_{B^{\frac12}_{2,1}}.$$

On the other hand, a classical commutator estimate yields

$$\sum_{q\in\mathbb{Z}} 2^{\frac q2}\|\,[\Delta_q, S_m a]\,\nabla\Pi\|_{L^2} \leqslant C\,\|\nabla S_m a\|_{\dot H^1}\,\|\nabla\Pi\|_{L^2}.$$

As a result, previous estimates imply

$$\begin{aligned} b\,\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} \lesssim \int_0^t \|u(t')\|_{B^{\frac12}_{2,1}}\,\|\nabla u(t')\|_{L^\infty}\,dt' + \|(S_m a - a)\|_{L^\infty_t(B^{\frac32}_{2,1})}\,\|\Delta u\|_{L^1_t(B^{\frac12}_{2,1})} \\ + \|(S_m a - a)\|_{L^\infty_t(B^{\frac32}_{2,1})}\,\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} + \|\nabla S_m a\|_{L^\infty_t(\dot H^1)}\Big(\|\nabla\Pi\|_{L^1_t(L^2)} + \|\Delta u\|_{L^1_t(L^2)}\Big). \end{aligned} \tag{4.54}$$

The smallness condition on $\|(S_m a - a)\|_{L^\infty_t(B^{\frac32}_{2,1})}$ allows to write

$$\begin{aligned} \frac b2\,\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} \lesssim \int_0^t \|u(t')\|_{B^{\frac12}_{2,1}}\,\|\nabla u(t')\|_{L^\infty}\,dt' + \|(S_m a - a)\|_{L^\infty_t(B^{\frac32}_{2,1})}\,\|u\|_{L^1_t(B^{\frac52}_{2,1})} \\ + \|\nabla S_m a\|_{L^\infty_t(\dot H^1)}\Big(\|\nabla\Pi\|_{L^1_t(L^2)} + \|\Delta u\|_{L^1_t(L^2)}\Big). \end{aligned}$$

Obviously, $\|\nabla S_m a\|_{L^\infty_t(\dot H^1)} \leqslant C\,2^{2m}\,\|a\|_{L^\infty_t(L^2)}$. Therefore,

$$\begin{aligned} \frac b2\,\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} \lesssim \int_0^t \|u(t')\|_{B^{\frac12}_{2,1}}\,\|\nabla u(t')\|_{L^\infty}\,dt' + \|(S_m a - a)\|_{L^\infty_t(B^{\frac32}_{2,1})}\,\|u\|_{L^1_t(B^{\frac52}_{2,1})} \\ + 2^{2m}\,\|a\|_{L^\infty_t(L^2)}\Big(\|\nabla\Pi\|_{L^1_t(L^2)} + \|u\|_{L^1_t(\dot H^2)}\Big). \end{aligned} \tag{4.55}$$

This ends up the estimate on the pressure term in $L^1_t(B^{\frac12}_{2,1})$-norm. It is left with estimate the pressure term in the $L^1_t(L^2)$-norm, in order to get rid of it in the above estimate, and thus, it is likely to applying with success Gronwall Lemma in the estimate of the velocity term.

Step 4: Estimate of $\|\nabla\Pi\|_{L^1_t(L^2)}$.
Once again, we take the divergence in the momentum equation, and the $\dot H^{-1}$-norm, so that we get

$$\|\operatorname{div}\big((1+S_m a)\nabla\Pi\big)\|_{\dot H^{-1}} \leqslant \|\operatorname{div}(u\cdot\nabla u)\|_{\dot H^{-1}} + \|\Delta u\cdot\nabla S_m a\|_{\dot H^{-1}} + \|\operatorname{div}\Big((S_m a - a)(\nabla\Pi - \Delta u)\Big)\|_{\dot H^{-1}}.$$

We recall that the smallness condition implies that $(1+S_m a)\geqslant \dfrac b2$ and thus

$$b\,\|\nabla\Pi\|_{L^2} \leqslant C\,\|(1+S_m a)\nabla\Pi\|_{L^2} \leqslant C\,\|u\cdot\nabla u\|_{L^2} + \|\Delta u\cdot\nabla S_m a\|_{\dot H^{-1}} + \|(S_m a - a)(\nabla\Pi - \Delta u)\|_{L^2}.$$

Thanks to the smallness condition and product law, we have

$$\frac{b}{2}\|\nabla\Pi\|_{L^2} \lesssim \|u\|_{L^3}\,\|\nabla u\|_{L^6} + \|\Delta u\cdot\nabla S_m a\|_{\dot H^{-1}} + \|(S_m a - a)\Delta u\|_{L^2}. \tag{4.56}$$

On the one hand, Gagliardo-Nirenberg inequality (notice that average of ∇u is nul) yields

$$\frac{b}{2}\|\nabla\Pi\|_{L^2} \lesssim \|u\|_{L^3}\,\|\nabla^2 u\|_{L^2} + \|\Delta u\cdot\nabla S_m a\|_{\dot H^{-1}} + \|a\|_{L^\infty}\|\Delta u\|_{L^2}.$$

On the other hand, we prove easily thanks to the divergence free condition that

$$\|\Delta u\cdot\nabla S_m a\|_{\dot H^{-1}} \leqslant C\,\|a\|_{L^\infty}\,\|\Delta u\|_{L^2}.$$

Despite the fact that average of u is not nul, we have $\|u\|_{L^3} \leqslant C(\rho_0)\,\|u\|_{B^{\frac12}_{2,1}}$. Hence, one has

$$\frac{b}{2}\|\nabla\Pi\|_{L^1_t(L^2)} \lesssim \big(\|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + 2\|a\|_{L^\infty_t(L^\infty)}\big)\|u\|_{L^1_t(\dot H^2)}. \tag{4.57}$$

Plugging (4.57) in the estimate (4.55), we finally get an estimate of the pressure, in which the right-hand side is independent of the pressure: we got rid of the term $\|\nabla\Pi\|_{L^2}$. Indeed, (4.55) becomes

$$\begin{aligned}\frac{b}{2}\,\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} \lesssim \int_0^t \|u\|_{B^{\frac12}_{2,1}}\,\|\nabla u\|_{L^\infty}\,dt' + \|(S_m a - a)\|_{L^\infty_t(B^{\frac32}_{2,1})}\,\|u\|_{L^1_t(B^{\frac52}_{2,1})}\\ + 2^{2m}\,\|a\|_{L^\infty_t(L^2)}\,\|u\|_{L^1_t(\dot H^2)}\Big(1 + \|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + \|a\|_{L^\infty_t(L^\infty)}\Big).\end{aligned} \tag{4.58}$$

Plugging (4.57) in the estimate (4.51), we also get

$$\begin{aligned}\|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + C\,b\,\|u\|_{L^1_t(B^{\frac52}_{2,1})} \leqslant\ & \|u_0\|_{B^{\frac12}_{2,1}} + \|a - S_m a\|_{L^\infty_t(B^{\frac32}_{2,1})}\Big(\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} + \|u\|_{L^1_t(B^{\frac52}_{2,1})}\Big)\\ &+ \int_0^t \|\nabla u(t')\|_{L^\infty}\,\|u(t')\|_{B^{\frac12}_{2,1}}\,dt' + 2^m\,\|a\|_{L^\infty_t(L^\infty)}\,\|u\|_{L^1_t(B^{\frac32}_{2,1})}\\ &+ 2^{2m+1}\,\|a\|_{L^\infty_t(L^2)}\,\|u\|_{L^1_t(\dot H^2)}\Big(1 + \|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + 2\|a\|_{L^\infty_t(L^\infty)}\Big),\end{aligned} \tag{4.59}$$

Suuming (4.59) with (4.58) and using obvious estimates on the transport equation below

$$\|a\|_{L^\infty_t(L^\infty)} \leqslant \|a_0\|_{L^\infty} \quad \text{and} \quad \|a\|_{L^\infty_t(L^2)} \leqslant \|a_0\|_{L^2},$$

leads to

$$\begin{aligned}\|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + C\,b\,\|u\|_{L^1_t(B^{\frac52}_{2,1})} + \frac{b}{2}\,\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} \lesssim\ & \|u_0\|_{B^{\frac12}_{2,1}}\\ &+ \|a - S_m a\|_{L^\infty_t(B^{\frac32}_{2,1})}\,\big(\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} + \|u\|_{L^1_t(B^{\frac52}_{2,1})}\big)\\ &+ \int_0^t \|\nabla u(t')\|_{L^\infty}\,\|u(t')\|_{B^{\frac12}_{2,1}}\,dt' + 2^m\,\|a_0\|_{L^\infty}\,\|u\|_{L^1_t(B^{\frac32}_{2,1})}\\ &+ 2^{2m+1}\,\|a_0\|_{L^2}\,\|u\|_{L^1_t(\dot H^2)}\,\big(1 + \|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + \|a_0\|_{L^\infty}\big)\\ &+ \|u\|_{L^1_t(\dot H^2)}\big(\|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + 2\,\|a_0\|_{L^\infty}\big).\end{aligned}$$

Once again, the smallness condition simplifies the above estimate

$$\begin{aligned}\|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + C\,b\,\|u\|_{L^1_t(B^{\frac52}_{2,1})} + \frac{b}{2}\,\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} &\lesssim \|u_0\|_{B^{\frac12}_{2,1}} + 2\int_0^t \|u(t')\|_{B^{\frac52}_{2,1}}\,\|u(t')\|_{B^{\frac12}_{2,1}}\,dt'\\ &+ (1+2^{2m+1}\,\|a_0\|_{L^2})\,\|u\|_{L^1_t(\dot H^2)}\,(1+\|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + \|a_0\|_{L^\infty})\\ &+ 2^m\,\|a_0\|_{L^\infty}\,\|u\|_{L^1_t(B^{\frac32}_{2,1})}.\end{aligned} \tag{4.60}$$

Let us recall somme interpolation properties. The following inequalities hold on the torus:

$$\|u\|_{\dot H^2} \leqslant C\,\|u\|^{\frac14}_{B^{\frac12}_{2,1}}\,\|u\|^{\frac34}_{B^{\frac52}_{2,1}} \quad \text{and} \quad \|u\|_{B^{\frac32}_{2,1}} \leqslant C\,\|u\|^{\frac12}_{B^{\frac12}_{2,1}}\,\|u\|^{\frac12}_{B^{\frac52}_{2,1}}.$$

They are due the product laws in Besov spaces (cf Lemma 4.20). For instance, the first one stems from

$$\|u\|_{\dot H^2} = \|\nabla u\|_{\dot H^1} \leqslant \|\nabla u\|_{H^1} \leqslant \|\nabla u\|_{B^1_{2,1}} \leqslant C\,\|\nabla u\|^{\frac14}_{B^{-\frac12}_{2,1}}\,\|\nabla u\|^{\frac34}_{B^{\frac32}_{2,1}}.$$

Obviously, by integration in time and thanks to Hölder's inequality, we have

$$\|u\|_{L^1_t(\dot H^2)} \leqslant C\,\|u\|^{\frac14}_{L^1_t(B^{\frac12}_{2,1})}\,\|u\|^{\frac34}_{L^1_t(B^{\frac52}_{2,1})} \quad \text{and} \quad \|u\|_{L^1_t(B^{\frac32}_{2,1})} \leqslant C\,\|u\|^{\frac12}_{L^1_t(B^{\frac12}_{2,1})}\,\|u\|^{\frac12}_{L^1_t(B^{\frac52}_{2,1})}.$$

By virtue of Young's inequalities

$$xy \leqslant \frac{x^4}{4} + \frac{3\,y^{\frac43}}{4} \quad \text{and} \quad xy \leqslant \frac{x^2}{2} + \frac{y^2}{2},$$

Estimate (4.60) becomes

$$\begin{aligned}\|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + C\,b\,\|u\|_{L^1_t(B^{\frac52}_{2,1})} + \frac{b}{2}\,\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} &\lesssim \|u_0\|_{B^{\frac12}_{2,1}} + 2\int_0^t \|\nabla u(t')\|_{L^\infty}\,\|u(t')\|_{B^{\frac12}_{2,1}}\,dt'\\ &+ (1+2^{8m}\,\|a_0\|^4_{L^2})\,\|u\|_{L^1_t(B^{\frac12}_{2,1})}\,(1+\|u\|^4_{L^\infty_t(B^{\frac12}_{2,1})} + \|a_0\|^4_{L^\infty}) + \frac{b}{4}\,\|u\|_{L^1_t(B^{\frac52}_{2,1})}\\ &+ 2^{2m}\,\|a_0\|^2_{L^\infty}\,\|u\|_{L^1_t(B^{\frac12}_{2,1})} + \frac{b}{4}\|u\|_{L^1_t(B^{\frac52}_{2,1})}.\end{aligned}$$

which can be simplified by

$$\begin{aligned}\|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + C\,\frac{b}{2}\,\|u\|_{L^1_t(B^{\frac52}_{2,1})} + \frac{b}{2}\,\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} &\lesssim \|u_0\|_{B^{\frac12}_{2,1}} + 2\int_0^t \|\nabla u(t')\|_{L^\infty}\,\|u(t')\|_{B^{\frac12}_{2,1}}\,dt'\\ &+ (1+2^{8m}\,\|a_0\|^4_{L^2})\,\|u\|_{L^1_t(B^{\frac12}_{2,1})}\,(1+\|u\|^4_{L^\infty_t(B^{\frac12}_{2,1})} + \|a_0\|^4_{L^\infty})\\ &+ 2^{2m}\,\|a_0\|^2_{L^\infty}\,\|u\|_{L^1_t(B^{\frac12}_{2,1})}.\end{aligned}$$

This concludes the proof of Lemma 4.7.
Continuation of the proof of existence part of Theorem 4.5. This stems from the obvious fact : $B^{\frac32}_{2,1} \hookrightarrow L^\infty$ and thus

$$\|\nabla u\|_{L^\infty} \leqslant \|\nabla u\|_{B^{\frac32}_{2,1}}.$$

Therefore, we get

$$\|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + C\,b\,\|u\|_{L^1_t(B^{\frac52}_{2,1})} + \frac{b}{2}\,\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} \lesssim \|u_0\|_{B^{\frac12}_{2,1}} + 2\int_0^t \|u(t')\|_{B^{\frac52}_{2,1}}\,\|u(t')\|_{B^{\frac12}_{2,1}}\,dt' \\ + \,(1+\,2^{8m}\,\|a_0\|^4_{L^2})\,\|u\|_{L^1_t(B^{\frac12}_{2,1})}\,(1+\,\|u\|^4_{L^\infty_t(B^{\frac12}_{2,1})} + \|a_0\|^4_{L^\infty}) \\ + \,2^{2m}\,\|a_0\|^2_{L^\infty}\,\|u\|_{L^1_t(B^{\frac12}_{2,1})}.$$

As a result, we get

$$\|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + C\,\frac{b}{2}\,\|u\|_{L^1_t(B^{\frac52}_{2,1})} + \frac{b}{2}\,\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} \lesssim \|u_0\|_{B^{\frac12}_{2,1}} + 2\int_0^t \|u(t')\|_{B^{\frac52}_{2,1}}\,\|u(t')\|_{B^{\frac12}_{2,1}}\,dt' \\ + \int_0^t \|u(t')\|_{B^{\frac12}_{2,1}}\Big(2^{2m}\,\|a_0\|^2_{L^\infty} + \,(1+\,2^{8m}\,\|a_0\|^4_{L^2})\,(1+\,\|u\|^4_{L^\infty_t(B^{\frac12}_{2,1})} + \|a_0\|^4_{L^\infty})\Big)\,dt'. \tag{4.61}$$

Let $\varepsilon_0 > 0$. Let us introduce the time T_0 such that

$$T_0 \stackrel{\text{def}}{=} \sup\{0 \leqslant t \leqslant T^* \mid \|u(t)\|_{B^{\frac12}_{2,1}} \leqslant \varepsilon_0\}.$$

Hence, for any $t \leqslant T_0$, we have

$$\|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + C\,\frac{b}{2}\,\|u\|_{L^1_t(B^{\frac52}_{2,1})} + \frac{b}{2}\,\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} \lesssim \|u_0\|_{B^{\frac12}_{2,1}} + 2\,\varepsilon_0\int_0^t \|u(t')\|_{B^{\frac52}_{2,1}}\,dt' \\ + \int_0^t \|u(t')\|_{B^{\frac12}_{2,1}}\Big(2^{2m}\,\|a_0\|^2_{L^\infty} + \,(1+\,2^{8m}\,\|a_0\|^4_{L^2})\,(1+\,\varepsilon_0^4 + \|a_0\|^4_{L^\infty})\Big)\,dt'.$$

Choosing ε_0 small enough, namely $\varepsilon_0 \leqslant \dfrac{C\,b}{4}$, Gronwall lemma implies that for any $t \leqslant T_0$,

$$\|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + \frac{C\,b}{4}\,\|u\|_{L^1_t(B^{\frac52}_{2,1})} + \frac{b}{2}\,\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} \lesssim \|u_0\|_{B^{\frac12}_{2,1}} \\ \times \exp\big((T_0\,(2^{2m}\,\|a_0\|^2_{L^\infty} + \,(1+\,2^{8m}\,\|a_0\|^4_{L^2})\,(1+\,(\frac{b}{4})^4 + \|a_0\|^4_{L^\infty})\big). \tag{4.62}$$

As a result, we get the a priori estima on the velocity

$$\text{For any} \quad t \leqslant T_0, \quad \|u\|_{L^\infty_t(B^{\frac12}_{2,1})} + \|u\|_{L^1_t(B^{\frac52}_{2,1})} + \frac{b}{2}\,\|\nabla\Pi\|_{L^1_t(B^{\frac12}_{2,1})} \leqslant C\,\|u_0\|_{B^{\frac12}_{2,1}}. \tag{4.63}$$

This concludes the proof of (4.8) : until the (small) time T_0, the solution is controlled by initial data, up to a multiplicative constant. This ends up the proof of the local-existence part of Theorem 4.5.

4.4.2 Uniqueness part

The uniqueness part has been already done in [3]. We refer the reader to it for more details. Let us recall some details. Let $(a_1, u_1, \nabla\Pi_1)$ and $(a_2, u_2, \nabla\Pi_2)$ be two solutions of the system (4.4), satisfying the smallness hypothesis $\|a - S_m a\|_{B^{\frac32}_{2,1}} \leqslant c$ and such that

$$(a_i, u_i, \nabla\Pi_i) \in \mathcal{C}([0,T], B^{\frac32}_{2,1}) \times \mathcal{C}([0,T], B^{\frac12}_{2,1}) \cap L^1([0,T], B^{\frac52}_{2,1}) \times L^1([0,T], B^{\frac12}_{2,1})\cdot \tag{4.64}$$

We define as one expects

$$(\delta a, \delta u, \nabla \delta \Pi) \stackrel{\text{def}}{=} (a_2 - a_1, u_2 - u_1, \nabla \Pi_2 - \nabla \Pi_1),$$

so that $(\delta a, \delta u, \nabla \delta \Pi)$ solves the following system

$$\left\{ \begin{array}{ccc} \partial_t \delta a + u_2 \cdot \nabla \delta a & = & -\delta u \cdot \nabla a_1 \\ \partial_t \delta u + u_2 \cdot \nabla \delta u - (1+a_2)\,(-\nabla \delta \Pi - +\Delta \delta u) & = & -\delta u \cdot \nabla u_1 + \delta a (\Delta u_1 - \nabla \Pi_1) \\ \operatorname{div} \delta u & = & 0 \\ (\delta a, \delta u)_{|t=0} & = & (0,0). \end{array} \right. \tag{4.65}$$

We prove that such solution of this system satisifies

$$(\delta a, \delta u, \nabla \delta \Pi) \in, \mathcal{C}([0,T], B^{\frac{3}{2}}_{2,1}) \times \mathcal{C}([0,T], B^{-\frac{1}{2}}_{2,1}) \cap L^1([0,T], B^{\frac{3}{2}}_{2,1}) \times L^1([0,T], B^{-\frac{1}{2}}_{2,1}) \cdot \tag{4.66}$$

Remark 4.17. Notice that, owing to the presence of a transport equation, we loose one derivative in the estimate involving δa.

4.5 Proof of the global wellposedness part of the main theorem

This section is devoted to the proof of Theorem 4.8, which provides the global property of the main Theorem 4.4.

$$\left\{ \begin{array}{ccc} \partial_t \rho + u \cdot \nabla \rho & = & 0 \\ \rho(\partial_t u + u \cdot \nabla u) - \Delta u + \nabla \Pi & = & 0 \\ \operatorname{div} u & = & 0 \\ (\rho, u)_{|t=0} & = & (\rho_0, u_0). \end{array} \right. \tag{4.67}$$

In a sake of simplicity, we skip the regularisation process (Friedrich methods) and we only present the a priori estimates for smooth enough solution (ρ, u), which provide the existence part of Theorem 4.8. Concerning the uniqueness part, we refer the reader to the paper of M. Paicu, P. Zhang and Z. Zhang (see [17]). We underline that Lagragian coordinates are necessary to prove the uniqueness, owing to the very low regularity hypothesis on the density(which is only supposed to be bounded from above and from below). Let us proceed firstly to an L^2-energy estimate, which leads to the result on B_0. Then we will get estimate on B_1, thanks to an H^1-energy estimate.

- Proof of (4.12). Taking the L^2 inner product of momentum equation with u in the system (4.67), we get :

$$\big(\,\rho(\partial_t u + u \cdot \nabla u) \,|\, u \,\big)_{L^2} - \big(\,\Delta u \,|\, u\,\big)_{L^2} + 0 = 0.$$

We check that $\big(\,\rho(\partial_t u + u \cdot \nabla u) \,|\, u \,\big)_{L^2} = \dfrac{1}{2}\dfrac{d}{dt}\|\sqrt{\rho} u\|^2_{L^2}$.
This stems from the computations below

$$\begin{aligned} \big(\,\rho(\partial_t u + u \cdot \nabla u) \,|\, u \,\big)_{L^2} &= \frac{1}{2}\int_{\mathbb{T}^3} \rho\, \partial_t |u|^2\, dx + \frac{1}{2}\int_{\mathbb{T}^3} \rho\; u \cdot \nabla |u|^2\, dx \\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} \rho\, |u|^2 - \frac{1}{2}\int_{\mathbb{T}^3} \partial_t \rho\, |u|^2\, dx + \frac{1}{2}\int_{\mathbb{T}^3} \rho\; u \cdot \nabla |u|^2\, dx. \end{aligned}$$

However, $\displaystyle\int_{\mathbb{T}^3} \rho\, u \cdot \nabla |u|^2 = -\int_{\mathbb{T}^3} (u \cdot \nabla \rho) |u|^2$. Therefore, the transport equation yields

$$\begin{aligned} \big(\,\rho(\partial_t u + u \cdot \nabla u), u \,\big)_{L^2} &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} \rho\, |u|^2 - \frac{1}{2}\int_{\mathbb{T}^3} (\partial_t \rho + u \cdot \nabla \rho)\, |u|^2 \\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} \rho\, |u|^2. \end{aligned}$$

Finally, an integration in time provides the desired estimate

$$\frac{1}{2}\|\sqrt{\rho}u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(t')\|_{L^2}^2\,dt' = \frac{1}{2}\|\sqrt{\rho_0}u_0\|_{L^2}^2. \tag{4.68}$$

This concludes the proof of (4.12). Now let us proceed to the proof of (4.13).

• Proof of (4.13). The idea is the same as the previous one : we take the L^2 inner product of momentum equation with $\partial_t u$ in the system (4.67), we get :

$$\big(\sqrt{\rho}\,\partial_t u\,|\,\sqrt{\rho}\partial_t u\big)_{L^2} + \big(\sqrt{\rho}u\cdot\nabla u\,|\,\sqrt{\rho}\partial_t u\big)_{L^2} + \frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 = 0,$$

which leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\sqrt{\rho}\,\partial_t u(t)\|_{L^2}^2 &\leqslant \|\sqrt{\rho}u\cdot\nabla u(t)\|_{L^2}\,\|\sqrt{\rho}\partial_t u(t)\|_{L^2}\\ &\leqslant \|\sqrt{\rho}u(t)\|_{L^6}\,\|\nabla u(t)\|_{L^3}\,\|\sqrt{\rho}\partial_t u(t)\|_{L^2}\end{aligned} \tag{4.69}$$

Applying Proposition 4.14 on the term $\|u(t)\|_{L^6}$ and Proposition 4.12 on the term $\|\nabla u(t)\|_{L^3}$ gives rise to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\sqrt{\rho}\,\partial_t u(t)\|_{L^2}^2 &\leqslant C(\rho_0)\,\|\nabla u(t)\|_{L^2}\,\|\nabla u(t)\|_{L^2}^{\frac{1}{2}}\,\|\nabla^2 u(t)\|_{L^2}^{\frac{1}{2}}\,\|\sqrt{\rho}\partial_t u(t)\|_{L^2}\\ &\leqslant C(\rho_0)\,\|\nabla u(t)\|_{L^2}^{\frac{3}{2}}\,\|\nabla^2 u(t)\|_{L^2}^{\frac{1}{2}}\,\|\sqrt{\rho}\partial_t u(t)\|_{L^2}.\end{aligned} \tag{4.70}$$

Then, Young inequality yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \frac{1}{2}\,\|\sqrt{\rho}\,\partial_t u(t)\|_{L^2}^2 \leqslant \frac{1}{2}\,C(\rho_0)\,\|\nabla u(t)\|_{L^2}^3\,\|\nabla^2 u(t)\|_{L^2}. \tag{4.71}$$

We have to estimate the term $\|\nabla^2 u\|_{L^2}$. Applying the L^2-norm in the momentum equation, we get

$$\|\nabla^2 u(t)\|_{L^2} + \|\nabla\Pi(t)\|_{L^2} \leqslant \|\rho(t)\|_{L^\infty}^{\frac{1}{2}}\Big(\|\sqrt{\rho}\partial_t u(t)\|_{L^2} + \|\sqrt{\rho}u(t)\|_{L^6}\|\nabla u\|_{L^3}\Big).$$

Once again, by virtue of Proposition 4.14 and Gagliardo-Nirenberg inequality, one has

$$\|\nabla^2 u\|_{L^2} + \|\nabla\Pi\|_{L^2} \leqslant C(\rho_0)\,\Big(\|\sqrt{\rho}\partial_t u(t)\|_{L^2} + \|\nabla u(t)\|_{L^2}\,\|\nabla u(t)\|_{L^2}^{\frac{1}{2}}\,\|\nabla^2 u(t)\|_{L^2}^{\frac{1}{2}}\Big).$$

Young inequality implies

$$\frac{1}{2}\|\nabla^2 u(t)\|_{L^2} + \|\nabla\Pi\|_{L^2} \leqslant C(\rho_0)\,\|\sqrt{\rho}\partial_t u(t)\|_{L^2} + \frac{1}{2}\,\|\nabla u(t)\|_{L^2}^3. \tag{4.72}$$

Plugging Inequality (4.72) in (4.71) and applying Young inequality gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \frac{1}{2}\|\sqrt{\rho}\,\partial_t u(t)\|_{L^2}^2 \leqslant C(\rho_0)\,\|\nabla u(t)\|_{L^2}^6 + \frac{1}{4}\|\sqrt{\rho}\,\partial_t u(t)\|_{L^2}^2. \tag{4.73}$$

As a result, we have :

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \frac{1}{4}\|\sqrt{\rho}\,\partial_t u(t)\|_{L^2}^2 \leqslant C(\rho_0)\,\|\nabla u(t)\|_{L^2}^6. \tag{4.74}$$

We sum (4.74) and (4.72) and we get :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \frac{1}{4}\|\sqrt{\rho}\,\partial_t u(t)\|_{L^2}^2 + \|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla\Pi(t)\|_{L^2}^2 &\leqslant C(\rho_0)\,\big(\|\nabla u(t)\|_{L^2}^6\\ &+ \frac{1}{8}\,\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^6\big).\end{aligned}$$

Finally, we have by integration in time

$$\frac{1}{2}\,\|\nabla u(t)\|_{L^2}^2 \;+\; \int_0^t \Big(\frac{1}{8}\,\|\sqrt{\rho}\,\partial_t u(t')\|_{L^2}^2 + \;\|\nabla^2 u(t')\|_{L^2}^2 + \|\nabla \Pi(t')\|_{L^2}^2\Big)\,dt' \;\leqslant\; \frac{1}{2}\,\|\nabla u_0\|_{L^2}^2 \\ + \;C(\rho_0)\int_0^t \|\nabla u(t')\|_{L^2}^6\,dt'. \tag{4.75}$$

Let us focus for a while on the term $\displaystyle\int_0^t \|\nabla u(t')\|_{L^2}^6\,dt'$. It seems clear that

$$\int_0^t \|\nabla u(t')\|_{L^2}^6 dt' \leqslant \;\|\nabla u\|_{L_t^\infty(L^2)}^4 \int_0^t \|\nabla u(t')\|_{L^2}^2 dt',$$

which leads to, by virtue of (4.13) and définition of B_1

$$\int_0^t \|\nabla u(t')\|_{L^2}^6 dt' \leqslant \;\|u_0\|_{L^2}^2\, B_1^2(t).$$

Finally, we get

$$B_1(t) \leqslant \frac{1}{2}\,\|\nabla u_0\|_{L^2}^2 \;+\; C(\rho_0)\,\|u_0\|_{L^2}^2\, B_1^2(t).$$

As long as the smallness condition on u_0 is satisfied, we obtain Estimate (4.13), which conclude the proof of this estimate.

• Proof of (4.14). Firstly, we derive the momentum equations, with respect to the time t. Then, we take the L^2 inner product with $\partial_t u$.

The derivated momentum equation is given by the following formula :

$$\big(\,\rho\,\partial_{tt}u\,|\,\partial_t u\,\big)_{L^2} - \big(\,\Delta\partial_t u\,|\,\partial_t u\,\big)_{L^2} = -\big(\,\partial_t\rho\;(\partial_t u + u\cdot\nabla u)\,|\,\partial_t u\,\big)_{L^2} - \big(\,\rho\,\partial_t u\cdot\nabla u\,|\,\partial_t u\,\big)_{L^2} \\ -\big(\,\rho\,u\cdot\nabla\partial_t u\,|\,\partial_t u\,\big)_{L^2}.$$

By hypothesis on the density, the left-hand side can be bounded from below by :

$$\frac{m}{2}\,\frac{d}{dt}\big(\,\|\partial_t u\|_{L^2}^2\big) + \;\|\nabla\partial_t u\|_{L^2}^2 \leqslant \frac{m}{2}\,\|\partial_t u\|_{L^2}^2 - \big(\,\rho\;\partial_t u\cdot\nabla u\,|\,\partial_t u\,\big)_{L^2} - \big(\,\rho\;u\cdot\nabla\partial_t u\,|\,\partial_t u\,\big)_{L^2} \\ -\big(\;\partial_t\rho\;\partial_t u\,|\,\partial_t u\,\big)_{L^2} - \big(\;\partial_t\rho\;u\cdot\nabla u)\,|\,\partial_t u\,\big)_{L^2}.$$

Let us point out that $\big(\,\rho\;u\cdot\nabla\partial_t u\,|\,\partial_t u\,\big)_{L^2}$ is in fact nul, by virtue of the divergence free condition. Taking the modulus, applying triangular inequality and finally, using the mass equation on the density:

$$\begin{aligned}\frac{m}{2}\,\frac{d}{dt}\big(\,\|\partial_t u\|_{L^2}^2\big) + \;\|\nabla\partial_t u\|_{L^2}^2 &\leqslant \frac{m}{2}\,\|\partial_t u\|_{L^2}^2 \;+\; \int_{\mathbb{T}^3}\Big|\,\rho\,(\partial_t u\cdot\nabla u)\,\partial_t u\Big|\,dx \\ &+\Big|\big(\;\mathrm{div}(\rho\,u)\;|\,(\partial_t u)^2\,\big)_{L^2}\Big| + \Big|\big(\;\mathrm{div}(\rho\,u)\,u\cdot\nabla u)\,|\,\partial_t u\,\big)_{L^2}\Big| \\ &\leqslant \sum_{k=1}^{6}\;I_k(t),\end{aligned} \tag{4.76}$$

with

$$
\begin{aligned}
I_1(t) &= \frac{m}{2}\,\|\partial_t u\|_{L^2}^2\,dx,\\
I_2(t) &= \int_{\mathbb{T}^3}\Big|\rho\ (\partial_t u\cdot\nabla u)\,\partial_t u\Big|\,dx,\\
I_3(t) &= 2\int_{\mathbb{T}^3}\Big|\rho\ u\,\nabla(\partial_t u)\,\partial_t u\Big|\,dx,\\
I_4(t) &= \int_{\mathbb{T}^3}\Big|\,\rho\,((u\cdot\nabla u)\cdot\nabla u)\cdot\ \partial_t u\Big|\,dx,\\
I_5(t) &= \int_{\mathbb{T}^3}\Big|\,\rho\,((u\,\otimes u):\nabla^2)u\cdot\partial_t u\Big|\,dx,\\
I_6(t) &= \int_{\mathbb{T}^3}\Big|\rho\ (u\cdot\nabla u)\ \cdot(u\cdot\nabla(\partial_t u))\Big|\,dx.
\end{aligned}
\tag{4.77}
$$

As far as $I_2(t)$ is concerned, firstly we apply Hölder's inequality and we get

$$
\begin{aligned}
I_2(t) &= \int_{\mathbb{T}^3}|\rho\ (\partial_t u\cdot\nabla u)\,\partial_t u|\,dx\\
&\leqslant M\ \|\partial_t u(t)\|_{L^2}\,\|\partial_t u(t)\|_{L^6}\,\|\nabla u(t)\|_{L^3}.
\end{aligned}
\tag{4.78}
$$

Once again, classical Sobolev embedding can not be applied directly to the term $\|\partial_t u(t)\|_{L^6}$. We shall consider the term $\overline{\partial_t u(t)}$ and adapt Lemma 4.13. Firstly, notice that $\int_{\mathbb{T}^3}\rho(t,x)\,\partial_t u(t,x)\,dx = 0$, due to an integration of the momentum equation in (4.67)). Hence, *the average method* gives rise to the following computation

$$
\int_{\mathbb{T}^3}(\rho(t,x)-\bar\rho(t))\,(\partial_t u(t,x)-\overline{\partial_t u(t)})\,dx = \int_{\mathbb{T}^3}\rho(t,x)\,\partial_t u(t,x)\ dx\ -\ \bar\rho(t)\,\overline{\partial_t u(t)}.
$$

By virtue of remarks 4.1 and 4.3, one has

$$
|\,\overline{\partial_t u(t)}|\ \leqslant\ \frac{1}{\bar{\rho_0}}\,\|\rho_0-\bar{\rho_0}\|_{L^2}\,\|\partial_t u(t)-\overline{\partial_t u(t)}\|_{L^2},
$$

which gives, thanks to Poincaré-Wirtinger

$$
|\,\overline{\partial_t u(t)}|\ \leqslant\ \frac{1}{\bar{\rho_0}}\,\|\rho_0-\bar{\rho_0}\|_{L^2}\,\|\nabla\partial_t u(t)\|_{L^2}.
$$

Therefore, we deduce from the above computation that

$$
\|\partial_t u(t)\|_{L^6}\leqslant\|\partial_t u(t)-\overline{\partial_t u(t)}\|_{L^6}+|\,\overline{\partial_t u(t)}|\,\leqslant C(\rho_0)\,\|\nabla\partial_t u(t)\|_{L^2}.
$$

Thanks to Gagliardo-Nirenberg and Young inequalities, we infer that

$$
\begin{aligned}
I_2(t) &\leqslant C(\rho_0)\,\|\partial_t u(t)\|_{L^2}\,\|\nabla\partial_t u(t)\|_{L^2}\,\|\nabla u(t)\|_{L^2}^{\frac12}\|\nabla^2 u(t)\|_{L^2}^{\frac12}\\
&\leqslant C(\rho_0)\ \|\partial_t u(t)\|_{L^2}^2\,\|\nabla u(t)\|_{L^2}\,\|\nabla^2 u(t)\|_{L^2}+\frac14\|\nabla\partial_t u(t)\|_{L^2}^2\\
&\leqslant C(\rho_0)\ \|\partial_t u(t)\|_{L^2}^2\left(\|\nabla u(t)\|_{L^2}^2\ +\ \|\nabla^2 u(t)\|_{L^2}^2\right)+\frac14\|\nabla\partial_t u(t)\|_{L^2}^2.
\end{aligned}
\tag{4.79}
$$

Concerning estimate of $I_3(t)$, we get

$$
\begin{aligned}
I_3(t) &= \int_{\mathbb{T}^3}\Big|\rho\ u\,\nabla(\partial_t u(t))\,\partial_t u(t)\Big|\,dx\\
&\leqslant M\ \|u\,\partial_t u\|_{L^2}\,\|\nabla\partial_t u\|_{L^2}\\
&\leqslant M\ \|u\|_{L^3}\,\|\partial_t u\|_{L^6}\,\|\nabla\partial_t u\|_{L^2}.
\end{aligned}
\tag{4.80}
$$

Applying *the average method* for $\|\partial_t u(t)\|_{L^6}$ and $\|u(t)\|_{L^3}$, we infer that

$$\begin{aligned} I_3(t) &\leqslant C(\rho_0)\, \|u(t)\|_{L^3}\, \|\nabla \partial_t u(t)\|_{L^2}^2 \\ &\leqslant C(\rho_0)\ \|\nabla u(t)\|_{L^2}\, \|\nabla \partial_t u(t)\|_{L^2}^2. \end{aligned} \tag{4.81}$$

Concerning $I_4(t), I_5(t)$, and $I_6(t)$, previous computations hold (applying Proposition 4.14 and Young inequality) :

$$\begin{aligned} I_4(t) &= \int_{\mathbb{T}^3} \Big| \rho\, ((u\cdot\nabla u)\cdot\nabla u)\cdot\, \partial_t u \Big|\, dx \\ &\leqslant M\ \|u(t)\|_{L^6}\, \|\nabla u(t)\|_{L^6}^2\, \|\partial_t u(t)\|_{L^2} \\ &\leqslant C(\rho_0)\ \|\nabla u(t)\|_{L^2}\, \|\nabla^2 u(t)\|_{L^2}^2\, \|\partial_t u(t)\|_{L^2} \\ &\leqslant \frac{1}{4}\, C(\rho_0)\, \Big(\|\nabla u(t)\|_{L^2}^2\ +\ \|\nabla^2 u(t)\|_{L^2}^2\Big)\, \Big(\|\nabla^2 u(t)\|_{L^2}^2\ +\ \|\partial_t u(t)\|_{L^2}^2\Big). \end{aligned} \tag{4.82}$$

$$\begin{aligned} I_5(t) &= \int_{\mathbb{T}^3} \Big| \rho\, ((u\,\otimes u) : \nabla^2) u\cdot\partial_t u\Big|\ :, dx \\ &\leqslant M\ \|u^2\, \partial_t u(t)\|_{L^2}\, \|\nabla^2 u(t)\|_{L^2} \\ &\leqslant C(\rho_0)\ \|u(t)\|_{L^6}^2\, \|\nabla \partial_t u(t)\|_{L^2} \|\nabla^2 u(t)\|_{L^2} \\ &\leqslant C(\rho_0)\ \|\nabla u(t)\|_{L^2}^2\, \|\nabla \partial_t u(t)\|_{L^2} \|\nabla^2 u\|_{L^2} \\ &\leqslant C(\rho_0)\ \|\nabla u(t)\|_{L^2}^4\, \|\nabla^2 u(t)\|_{L^2}^2 + \frac{1}{4} \|\nabla \partial_t u(t)\|_{L^2}^2. \end{aligned} \tag{4.83}$$

Similar computation holds for the last term $I_6(t)$.

$$\begin{aligned} I_6(t) &= \int_{\mathbb{T}^3} \Big|\rho\ (u\cdot\nabla u)\ \cdot (u\cdot\nabla(\partial_t u))\Big|\, dx \\ &\leqslant M\ \|u^2\, \nabla u(t)\|_{L^2}\, \|\nabla \partial_t u(t)\|_{L^2} \\ &\leqslant C(\rho_0)\ \|\nabla u(t)\|_{L^2}^4\, \|\nabla^2 u(t)\|_{L^2}^2 + \frac{1}{4} \|\nabla \partial_t u(t)\|_{L^2}^2. \end{aligned} \tag{4.84}$$

Let us keep on the proof. Plugging these above estimates into the (4.76) gives rise to

$$\begin{aligned} \frac{m}{2}\, \frac{d}{dt}(\, \|\partial_t u\|_{L^2}^2) +\ \|\nabla \partial_t u(t)\|_{L^2}^2 \leqslant\ & \frac{m}{2}\, \|\partial_t u(t)\|_{L^2}^2\ + C(\rho_0)\ \|\partial_t u(t)\|_{L^2}^2\, \Big(\|\nabla u(t)\|_{L^2}^2\ +\ \|\nabla^2 u(t)\|_{L^2}^2\Big) \\ & + \frac{1}{4} \|\nabla \partial_t u(t)\|_{L^2}^2\ + C(\rho_0)\, \|\nabla u(t)\|_{L^2}\, \|\nabla \partial_t u(t)\|_{L^2}^2 \\ & + C(\rho_0)\, \Big(\|\nabla u(t)\|_{L^2}^2\ +\ \|\nabla^2 u(t)\|_{L^2}^2\Big)\, \Big(\|\nabla^2 u(t)\|_{L^2}^2\ +\ \|\partial_t u(t)\|_{L^2}^2\Big) \\ & + 2\, C(\rho_0)\ \|\nabla u(t)\|_{L^2}^4\, \|\nabla^2 u(t)\|_{L^2}^2 + \frac{1}{2} \|\nabla \partial_t u(t)\|_{L^2}^2, \end{aligned} \tag{4.85}$$

so that

$$\begin{aligned} \frac{m}{2}\, \frac{d}{dt}(\, \|\partial_t u(t)\|_{L^2}^2) +\ \frac{1}{4} \|\nabla \partial_t u(t)\|_{L^2}^2 \leqslant\ & \frac{m}{2}\, \|\partial_t u(t)\|_{L^2}^2\ +\ C(\rho_0)\, \|\nabla u(t)\|_{L^2}\, \|\nabla \partial_t u(t)\|_{L^2}^2 \\ & + 2\, C(\rho_0)\ \|\nabla u(t)\|_{L^2}^4\, \|\nabla^2 u(t)\|_{L^2}^2 \\ & + C(\rho_0)\, \Big(\|\nabla u(t)\|_{L^2}^2\ +\ \|\nabla^2 u(t)\|_{L^2}^2\Big)\, \Big(\|\nabla^2 u(t)\|_{L^2}^2\ +\ \|\partial_t u(t)\|_{L^2}^2\Big). \end{aligned}$$

By integration in time, we have :

$$\begin{aligned}\frac{m}{2}\,\|\partial_t u(t)\|_{L^2}^2+\frac{1}{2}\int_0^t\|\nabla\partial_t u(t')\|_{L^2}^2\,dt' \leqslant \|u_0\|_{H^2}^2 &+ \frac{m}{2}\int_0^t\|\partial_t u(t')\|_{L^2}^2\,dt'\\ &+ C(\rho_0)\int_0^t\|\nabla u(t')\|_{L^2}\,\|\nabla\partial_t u(t')\|_{L^2}^2\,dt'\\ &+ C(\rho_0)\int_0^t\|\nabla u(t')\|_{L^2}^4\,\|\nabla^2 u(t')\|_{L^2}^2\,dt'\\ &+ C(\rho_0)\int_0^t\Big(\|\nabla u(t')\|_{L^2}^2+\|\nabla^2 u(t')\|_{L^2}^2\Big)\Big(\|\nabla^2 u(t')\|_{L^2}^2+\|\partial_t u(t')\|_{L^2}^2\Big)\,dt'.\end{aligned}\tag{4.86}$$

Concerning the term $\int_0^t\|\nabla u(t')\|_{L^2}\,\|\nabla\partial_t u(t')\|_{L^2}^2\,dt'$

$$\int_0^t\|\nabla u(t')\|_{L^2}\,\|\nabla\partial_t u(t')\|_{L^2}^2\,dt' \leqslant \|\nabla u\|_{L^\infty_T(L^2)}\int_0^t\|\nabla\partial_t u(t')\|_{L^2}^2\,dt',$$

which becomes, by virtue of Theorem 4.8,

$$\int_0^t\|\nabla u(t')\|_{L^2}\,\|\nabla\partial_t u(t')\|_{L^2}^2\,dt' \leqslant C\,\|\nabla u_0\|_{L^2}\int_0^t\|\nabla\partial_t u(t')\|_{L^2}^2\,dt'.$$

Same argument combining with Theorem 4.8 gives rise to

$$\int_0^t\|\nabla u(t')\|_{L^2}^4\,\|\nabla^2 u(t')\|_{L^2}^2\,dt' \leqslant C\,\|\nabla u_0\|_{L^2}^6.$$

As a result, Inequation (4.86) can be rewritten as follows (providing we choose $\|\nabla u_0\|_{L^2}$ small enough)

$$\begin{aligned}\frac{m}{2}\,\|\partial_t u(t)\|_{L^2}^2+\frac{1}{3}\int_0^t\|\nabla\partial_t u(t')\|_{L^2}^2\,dt' &\lesssim \|u_0\|_{H^2}^2+\frac{m}{2}\,\|\nabla u_0\|_{L^2}^2+\|\nabla u_0\|_{L^2}^6\\ &+\int_0^t\Big(\|\nabla u(t')\|_{L^2}^2+\|\nabla^2 u(t')\|_{L^2}^2\Big)\Big(\|\nabla^2 u(t')\|_{L^2}^2+\|\partial_t u(t')\|_{L^2}^2\Big)\,dt'.\end{aligned}\tag{4.87}$$

Moreover, the momentum equation given by

$$-\Delta u+\nabla\Pi = -\rho\,(\partial_t u+u\cdot\nabla u),$$

which along with the classical estimates on the Stokes system, ensures that

$$\begin{aligned}\|\nabla^2 u(t)\|_{L^2}+\|\nabla\Pi(t)\|_{L^2} &\leqslant C\left(\|\partial_t u(t)\|_{L^2}+\|\nabla u(t)\|_{L^2}^{\frac{3}{2}}\|\nabla^2 u(t)\|_{L^2}^{\frac{1}{2}}\right)\\ &\lesssim \|\partial_t u(t)\|_{L^2}+\|\nabla u(t)\|_{L^2}^3+\frac{1}{2}\|\nabla^2 u(t)\|_{L^2}.\end{aligned}$$

So that, we get

$$\frac{1}{2}\,\|\nabla^2 u(t)\|_{L^2}+\|\nabla\Pi(t)\|_{L^2} \lesssim \|\partial_t u(t)\|_{L^2}+\|\nabla u(t)\|_{L^2}^3.\tag{4.88}$$

By virtue of Theorem 4.8, we obtain

$$\sup_{t\in[0,T]}\Big(\frac{1}{2}\,\|\nabla^2 u(t)\|_{L^2}^2+\|\nabla\Pi(t)\|_{L^2}^2\Big) \lesssim \sup_{t\in[0,T]}\big(\,\|\partial_t u(t)\|_{L^2}^2\big)+\|\nabla u_0\|_{L^2}^6.\tag{4.89}$$

Remark 4.18. Let us point out that searching an estimate of $\|u\|_{L^2_T(H^3)}$ is a natural idea here since the initial velocity u_0 belongs to the space H^2. But actually, it is not relevant. Indeed, to perform it, we shall use the theory of Stokes problems. We shall begin derivating the momentum equation with respect to the space, and then, we shall take the L^2 norm. But, such an approach is doomed to fail, because requires an estimate on $\sup\limits_{t\in[0,T]} \|\nabla\rho\|_{L^\infty}$, which is not our case here, since the density function only belongs to $L^\infty([0,T]\times\mathbb{T}^3)$.

Once again, the momentum equation gives

$$-\Delta u + \nabla\Pi = -\rho(\partial_t u + u\cdot\nabla u).$$

We take the L^6-norm and use the fact that $\|u\cdot\nabla u(t)\|_{L^6} \leqslant C\,\|\nabla(u\cdot\nabla u(t))\|_{L^2}$ since $\overline{u\cdot\nabla u} = 0$.

$$\begin{aligned}\|\nabla^2 u(t)\|_{L^6} + \|\nabla^2 p(t)\|_{L^6} &\leqslant \|\rho(\partial_t u + u\cdot\nabla u)\|_{L^6}\\ &\leqslant C(\rho_0)\Big(\|\nabla\partial_t u(t)\|_{L^2} + \|(\nabla u(t))^2\|_{L^2} + \|u(t)(\nabla^2 u(t))\|_{L^2}\Big)\\ &\leqslant C(\rho_0)\Big(\|\nabla\partial_t u(t)\|_{L^2} + \|\nabla u(t)\|_{L^3}\,\|\nabla^2 u(t)\|_{L^2} + \|u(t)\|_{L^3}\,\|\nabla^2 u(t)\|_{L^6}\Big)\end{aligned}$$

Applying Proposition 4.15 to the term $\|\nabla u(t)\|_{L^3}$, we get

$$\|\nabla^2 u(t)\|_{L^6} + \|\nabla^2 p(t)\|_{L^6} \lesssim \|\nabla\partial_t u(t)\|_{L^2} + \|\nabla u(t)\|_{L^2}^{\frac12}\,\|\nabla^2 u(t)\|_{L^2}^{\frac32} + \|\nabla u(t)\|_{L^2}\,\|\nabla^2 u(t)\|_{L^6}.$$

By integration in time :

$$\begin{aligned}\int_0^t \|\nabla^2 u(t')\|_{L^6}^2\,dt' + \int_0^t \|\nabla^2 p(t')\|_{L^6}^2\,dt' &\lesssim \int_0^t \|\nabla\partial_t u(t')\|_{L^2}^2\,dt' + \int_0^t \|\nabla u(t')\|_{L^2}\,\|\nabla^2 u(t')\|_{L^2}^3\,dt'\\ &\quad + \|\nabla u\|_{L^\infty_T(L^2)}^2 \int_0^t \|\nabla^2 u(t')\|_{L^6}^2\,dt'.\end{aligned}$$

On the one hand, Theorem 4.8 provides $\|\nabla u\|_{L^\infty_T(L^2)}^2 \lesssim \|\nabla u_0\|_{L^2}^2$, which implies that

$$\|\nabla u\|_{L^\infty_T(L^2)}^2 \int_0^t \|\nabla^2 u(t')\|_{L^6}^2\,dt' \leqslant \|\nabla u_0\|_{L^2}^2 \int_0^t \|\nabla^2 u(t')\|_{L^6}^2\,dt'.$$

On the other hand, applying Estimates (4.12) and (4.13) of Theorem 4.8, to the term

$$\int_0^t \|\nabla u(t')\|_{L^2}\,\|\nabla^2 u(t')\|_{L^2}^3\,dt',$$

leads to

$$\begin{aligned}\int_0^t \|\nabla u(t')\|_{L^2}\,\|\nabla^2 u(t')\|_{L^2}^3\,dt' &= \int_0^t \|\nabla u(t')\|_{L^2}\,\|\nabla^2 u(t')\|_{L^2}\,\|\nabla^2 u(t')\|_{L^2}^2\,dt'\\ &\leqslant \sup_{t\in[0,T]}\big(\|\nabla^2 u(t)\|_{L^2}^2\big)\Big(\int_0^t \|\nabla u(t')\|_{L^2}^2\,dt'\Big)^{\frac12}\Big(\int_0^t \|\nabla^2 u(t')\|_{L^2}^2\,dt'\Big)^{\frac12}\\ &\lesssim \|u_0\|_{L^2}\,\|\nabla u_0\|_{L^2} \sup_{t\in[0,T]}\big(\|\nabla^2 u(t)\|_{L^2}^2\big)\end{aligned}$$

As a result, if $\|\nabla u_0\|_{L^2}$ is small enough, we have :

$$\begin{aligned}\frac{\mu}{2}\int_0^t \|\nabla^2 u(t')\|_{L^6}^2\,dt' + \int_0^t \|\nabla^2\Pi(t')\|_{L^6}^2\,dt' &\lesssim \|u_0\|_{L^2}\,\|\nabla u_0\|_{L^2} \sup_{t\in[0,T]}\big(\|\nabla^2 u(t)\|_{L^2}^2\big)\\ &\quad + \int_0^t \|\nabla\partial_t u(t')\|_{L^2}^2\,dt'.\end{aligned} \tag{4.90}$$

Summing (4.90) with (4.89) and (4.87), we recognize $B_2(T)$ and we get

$$\begin{aligned} B_2(T) \lesssim \frac{m}{2}\, \|\nabla u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^6}^6 \ + \|u_0\|_{H^2}^2 \\ + \frac{1}{4}\int_0^t \Big(\|\nabla u(t')\|_{L^2}^2 \ + \|\nabla^2 u(t')\|_{L^2}^2 \Big)\, dt' \Big(\sup_{t\in[0,T]} \|\nabla^2 u(t)\|_{L^2}^2 + \sup_{t\in[0,T]} \|\partial_t u(t)\|_{L^2}^2 \Big) \\ + \int_0^t \|\nabla \partial_t u(t')\|_{L^2}^2\, dt' \ + \|u_0\|_{L^2}\, \|\nabla u_0\|_{L^2} \sup_{t\in[0,T]} (\, \|\nabla^2 u(t)\|_{L^2}^2) \end{aligned}$$

The smallness condition on $\|u_0\|_{L^2}\, \|\nabla u_0\|_{L^2}$ implies

$$\begin{aligned} B_2(T) \lesssim \frac{m}{2}\, \|\nabla u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^6 + \|u_0\|_{H^2}^2 \\ + \frac{1}{4}\int_0^t \Big(\|\nabla u(t')\|_{L^2}^2 \ + \|\nabla^2 u(t')\|_{L^2}^2 \Big) dt' \Big(\sup_{t\in[0,T]} \|\nabla^2 u(t)\|_{L^2}^2 + \sup_{t\in[0,T]} \|\partial_t u(t)\|_{L^2}^2 \Big). \end{aligned}$$

Now, we apply Gronwall lemma, and we have :

$$B_2(T) \lesssim \Big(\frac{m}{2}\, \|\nabla u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^6 \ + \|u_0\|_{H^2}^2 \Big) \ \exp \Big(\int_0^t \|\nabla u(t')\|_{L^2}^2\, dt' \ + \|\nabla^2 u(t')\|_{L^2}^2\, dt' \Big). \tag{4.91}$$

Once again Theorem 4.8 gives the expected estimate in the exponential term. Finally, we get

$$B_2(T) \lesssim (1 + \|u_0\|_{H^2}^4\,)\, \|u_0\|_{H^2}^2 \ \exp \Big(\|u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2 \Big). \tag{4.92}$$

This concludes the proof of 4.14. Up to the regularization procedure of Friedrich, we have proved the global existence of solution of 4.67, with data (ρ_0, u_0) satisfiying hypothesis of Theorem 4.8.

4.6 Appendix

Lemma 4.19. *(Gronwall's Lemma)*
Let f and g be two positive functions satisfying $\frac{1}{2}\frac{d}{dt} f^2(t) \leqslant f(t)\, g(t)$. Then, we have

$$f(t) \leqslant f(0) + \int_0^t g(t')dt'.$$

Proof. We introduce the function $H(t) \stackrel{\text{def}}{=} 2\displaystyle\int_0^t f(t')\, g(t')\, dt'$. As defined, we get immediately

$$H'(t) = 2\, f(t)\, g(t) \quad \text{and} \quad f^2(t) - f^2(0) \leqslant H(t). \tag{4.93}$$

This implies that for any $\varepsilon > 0$,

$$f(t) \leqslant \sqrt{H(t) + f^2(0) + \varepsilon^2}.$$

Moreover, we have in particular $H'(t) \leqslant 2\sqrt{H(t) + f^2(0) + \varepsilon^2}\; g(t)$ and thus

$$\frac{d}{dt}\sqrt{H(t) + f^2(0) + \varepsilon^2} \leqslant g(t).$$

By integration in time, we have

$$\sqrt{H(t) + f^2(0) + \varepsilon^2} \leqslant \sqrt{H(0) + f^2(0) + \varepsilon^2} + \int_0^t g(t')\, dt'.$$

Finally, we have for any $\varepsilon > 0$,

$$f(t) \leqslant \sqrt{f^2(0)+\varepsilon^2} + \int_0^t g(t')\,dt',$$

which proves the result. □

Lemma 4.20. *The following properties hold*

1. *Sobolev embedding: if $p_1 \leqslant p_2$ and $r_1 \leqslant r_2$, then*

$$B^s_{p_1,r_1} \hookrightarrow B^{s-N(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}.$$

2. *Product laws in Besov spaces: let $1 \leqslant r, p, p_1, p_2 \leqslant +\infty$.*
If $s_1, s_2 < \frac{N}{p}$ and $s_1 + s_2 + N\ \min(0, 1-\frac{2}{p}) > 0$, then

$$\|uv\|_{B^{s_1+s_2-\frac{N}{p}}_{p,r}} \leqslant C\,\|u\|_{B^{s_1}_{p,r}}\,\|v\|_{B^{s_2}_{p,\infty}}.$$

3. *Another product law: if $|s| < \frac{N}{p}$, then*

$$\|uv\|_{B^s_{p,r}} \leqslant C\,\|u\|_{B^s_{p,r}}\,\|v\|_{B^{\frac{N}{p}}_{p,\infty}\cap L^\infty}.$$

4. *Algebric properties: for $s > 0$, $B^{\frac{N}{p}}_{p,\infty} \cap L^\infty$ is an algebra. Moreover, for any $p \in [1, +\infty$, then*

$$B^{\frac{N}{p}}_{p,1} \hookrightarrow B^{\frac{N}{p}}_{p,\infty} \cap L^\infty.$$

Lemma 4.21. *Let $\mathcal{C}$ a ring of $\mathbb{R}^3$. A constant C exists so that for any positive real number λ, any non-negative integer k, the following hold*

$$\text{If} \quad \text{Supp}\ \widehat{u} \subset \lambda\,\mathcal{C}, \quad \text{then} \quad C^{-1-k}\,\lambda^k\,\|u\|_{L^a} \;\leqslant\; \sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^a} \;\leqslant\; C^{1+k}\,\lambda^k\,\|u\|_{L^a}.$$

Lemma 4.22.

$$\sum_{q\in\mathbb{Z}} 2^{\frac{3q}{2}}\,\big\|[\Delta_q, S_m a]\ \nabla u\big\|_{L^1_t(L^2)} \leqslant C\Big(2^m\,\|a\|_{L^\infty_t(L^\infty)}\,\|u\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} \;+\; 2^{2m}\,\|a\|_{L^\infty_t(L^2)}\,|u\|_{L^1_t(\dot H^2)}\Big) \tag{4.94}$$

Proof. By virtue of Bony's decomposition, the commutator may be decomposed into

$$\begin{aligned}
[\Delta_q, S_m a] &= \Delta_q(S_m a\,\nabla u) - S_m a \Delta_q \nabla u\\
&= \Delta_q(T_{S_m a}\nabla u) + \Delta_q(T_{\nabla u} S_m a) + \Delta_q R(S_m a,\,\nabla u)\\
&\quad - T_{S_m a}\Delta_q\nabla u - T_{\Delta_q\nabla u} S_m a - R(S_m a\,,\,\Delta_q\nabla u)\\
&= [\Delta_q, T_{S_m a}]\,\nabla u \;+\; \Delta_q(T_{\nabla u}S_m a) + \;\Delta_q R(S_m a,\,\nabla u) \;-\; T'_{\Delta_q\nabla u}S_m a,
\end{aligned} \tag{4.95}$$

where $T'_a b \overset{\text{def}}{=} T_a b \,+\, R(a,b)$. Let us analyse each term in the right-hand-side. Firstly, we decompose the first commutator term into

$$\begin{aligned}
[\Delta_q, T_{S_m a}]\,\nabla u &= \Delta_q(T_{S_m a}\,\nabla u) - T_{S_m a}\Delta_q\nabla u\\
&= \Delta_q\Big(\sum_{|q-q'|\leqslant 4} S_{q'-1}S_m a\,\Delta_{q'}\nabla u\Big) - \sum_{|q-q'|\leqslant 4} S_{q'-1}S_m a\,\Delta_{q'}\Delta_q\nabla u\\
&= \sum_{|q-q'|\leqslant 4}\big[\Delta_q, S_{q'-1}S_m a\,\big]\,\Delta_{q'}\nabla u.
\end{aligned} \tag{4.96}$$

Now, let us focus on the commutator term $[\Delta_q, S_{q'-1}S_m a\,]\,\Delta_{q'}\nabla u$. We shall use definiton of Littlewood-Paley theory.

$$\begin{aligned}
[\Delta_q, S_{q'-1}S_m a\,]\,\Delta_{q'}\nabla u &= \Delta_q\Big(S_{q'-1}S_m a\,\Delta_{q'}\nabla u\Big) - S_{q'-1}S_m a\,\Delta_q\Delta_{q'}\nabla u\\
&= \varphi(2^{-q}\,|D|)\,S_{q'-1}S_m a\,\Delta_{q'}\nabla u - S_{q'-1}S_m a\,\varphi(2^{-q}\,|D|)\,\Delta_{q'}\nabla u.
\end{aligned} \tag{4.97}$$

In particular, writting $h \overset{\text{def}}{=} \mathcal{F}^{-1}\varphi(|\cdot|)$, we get

$$\begin{aligned}
\varphi(2^{-q}\,|D|)\,S_{q'-1}S_m a\,\Delta_{q'}\nabla u(x) &\overset{\text{def}}{=} \int_{\mathbb{T}^3} 2^{qd}\,h(2^q y)\,S_{q'-1}S_m a(x-y)\,\Delta_{q'}\nabla u(x-y)\,dy\\
&= \int_{\mathbb{T}^3} h(z)\,S_{q'-1}S_m a(x-2^{-q}z)\,\Delta_{q'}\nabla u(x-2^{-q}z)\,dz.
\end{aligned} \tag{4.98}$$

Likewise, we have

$$\begin{aligned}
S_{q'-1}S_m a\,\varphi(2^{-q}\,|D|)\,\Delta_{q'}\nabla u(x) &\overset{\text{def}}{=} S_{q'-1}S_m a\,(x)\,\int_{\mathbb{T}^3} 2^{qd}\,h(2^q y)\,\Delta_{q'}\nabla u(x-y)\,dy\\
&= \int_{\mathbb{T}^3} S_{q'-1}S_m a\,(x)\;h(z)\,\Delta_{q'}\nabla u(x-2^{-q}z)\,dz.
\end{aligned} \tag{4.99}$$

Therefore, applying the first-order Taylor's formula, we get, for any $x\in\mathbb{T}^3$,

$$\begin{aligned}
[\Delta_q, S_{q'-1}S_m a\,]\,\Delta_{q'}\nabla u(x) &= \int_{\mathbb{T}^3} h(z)\,\left[S_{q'-1}S_m a(x-2^{-q}z)\,-S_{q'-1}S_m a\,(x)\right]\,\Delta_{q'}\nabla u(x-2^{-q}z)\,dz\\
&= -\int_{\mathbb{T}^3}\int_0^1 h(z)\,2^{-q}z\cdot\nabla S_{q'-1}S_m a(x-2^{-q}\,z\,t)\,\Delta_{q'}\nabla u(x-2^{-q}z)\,dz\,dt\\
&= -2^{-q}\int_{\mathbb{T}^3}\int_0^1 2^{qd}\,h(2^q y)\,y\cdot\nabla S_{q'-1}S_m a(x-y\,t)\,\Delta_{q'}\nabla u(x-y)\,dz\,dt.
\end{aligned} \tag{4.100}$$

Therefore, we infer that, for any $x\in\mathbb{T}^3$,

$$\|\,[\Delta_q, S_{q'-1}S_m a\,]\,\Delta_{q'}\nabla u\|_{L^2} \leqslant \|\nabla S_{q'-1}S_m a\|_{L^\infty}\,2^{-q}\left\|\int_{\mathbb{T}^3} 2^{qd}\,(2^q y)\,h(2^q y)\,\Delta_{q'}\nabla u(\cdot\,-y)\,dz\right\|_{L^2}.$$

Applying Young's inequality ($L^1 * L^2 = L^2$), we infer that

$$\|\,[\Delta_q, S_{q'-1}S_m a\,]\,\Delta_{q'}\nabla u\|_{L^2} \leqslant C\,\|\nabla S_{q'-1}S_m a\|_{L^\infty}\,2^{-q}\,\|\Delta_{q'}\nabla u\|_{L^2}. \tag{4.101}$$

Obviously, we have

$$\|\nabla S_{q'-1}S_m a\|_{L^\infty} \leqslant \|\nabla S_m a\|_{L^\infty} \leqslant 2^m\,\|a\|_{L^\infty}.$$

Finally, we get

$$\|\,[\Delta_q, S_{q'-1}S_m a\,]\,\Delta_{q'}\nabla u\|_{L^2} \leqslant C\,2^{-q}\,2^m\,\|a\|_{L^\infty}\,\|\Delta_{q'}\nabla u\|_{L^2},$$

and thus,

$$\|\,[\Delta_q, T_{S_m a}]\,\nabla u\|_{L^2} \leqslant C\sum_{|q-q'|\leqslant 4} 2^{-q}\,2^m\,\|a\|_{L^\infty}\,\|\Delta_{q'}\nabla u\|_{L^2}. \tag{4.102}$$

As a consequence, we have

$$\begin{aligned}
2^{\frac{3q}{2}}\,\|\,[\Delta_q, T_{S_m a}]\,\nabla u\|_{L^2} &\leqslant C\sum_{|q-q'|\leqslant 4} 2^{\frac{3q}{2}}\,2^{-q}\,2^m\,\|a\|_{L^\infty}\,2^{\frac{-q'}{2}}\,2^{\frac{q'}{2}}\,\|\Delta_{q'}\nabla u\|_{L^2}\\
&\leqslant C\,2^m\,\|a\|_{L^\infty}\sum_{|q-q'|\leqslant 4} 2^{\frac{q-q'}{2}}\,2^{\frac{q'}{2}}\,\|\Delta_{q'}\nabla u\|_{L^2}.
\end{aligned} \tag{4.103}$$

By definition of the Besov norm, there exists a serie $(c_{q'})_{q\in\mathbb{Z}}$ belonging to $\ell^1(\mathbb{Z})$ such that

$$2^{\frac{q'}{2}}\left\|\Delta_{q'}\nabla u\right\|_{L^2}\leqslant C\,c_{q'}\left\|\nabla u\right\|_{B^{\frac12}_{2,1}}.$$

And thus,

$$2^{\frac{3q}{2}}\left\|\left[\Delta_q,T_{S_m a}\right]\nabla u\right\|_{L^2}\leqslant C\,2^m\left\|a\right\|_{L^\infty}\left\|\nabla u\right\|_{B^{\frac12}_{2,1}}\sum_{|q-q'|\leqslant 4}2^{\frac{q-q'}{2}}c_{q'}. \tag{4.104}$$

We notice, by virtue of Young's inequality, that the term $\displaystyle\sum_{|q-q'|\leqslant 4}2^{\frac{q-q'}{2}}c_{q'}$ belongs to $\ell^1(\mathbb{Z})$. Indeed, let us define $d_q\stackrel{\text{def}}{=}\displaystyle\sum_{|q-q'|\leqslant 4}2^{\frac{q-q'}{2}}c_{q'}$. Thanks to Young's inequality, we get

$$\left\|d_q\right\|_{\ell^1(\mathbb{Z})}\leqslant\left\|c_q\right\|_{\ell^1(\mathbb{Z})}\times\sum_{-4\leqslant k\leqslant 4}2^{\frac{k}{2}}\leqslant C.$$

Finally, we get

$$\begin{aligned}\sum_{q\in\mathbb{Z}}2^{\frac{3q}{2}}\left\|\left[\Delta_q,T_{S_m a}\right]\nabla u\right\|_{L^2}&\leqslant C\,2^m\left\|a\right\|_{L^\infty}\left\|\nabla u\right\|_{B^{\frac12}_{2,1}}\sum_{q\in\mathbb{Z}}d_q\\&\leqslant C\,2^m\left\|a\right\|_{L^\infty}\left\|\nabla u\right\|_{B^{\frac12}_{2,1}}.\end{aligned} \tag{4.105}$$

By integration in time, we infer that

$$\sum_{q\in\mathbb{Z}}2^{\frac{3q}{2}}\left\|\left[\Delta_q,T_{S_m a}\right]\nabla u\right\|_{L^1_t(L^2)}\leqslant C\,2^m\left\|a\right\|_{L^\infty_t(L^\infty)}\left\|\nabla u\right\|_{L^1_t(B^{\frac12}_{2,1})}. \tag{4.106}$$

This gives the first term in the Lemma. The second term will stem from remainder terms in the Bony's decomposition. More precisely, concerning the term $\displaystyle\sum_{q\in\mathbb{Z}}2^{\frac{3q}{2}}\,\Delta_q T_{\nabla u}S_m a\|_{L^1_t(L^2)}$, we have by definition

$$\sum_{q\in\mathbb{Z}}2^{\frac{3q}{2}}\left\|\Delta_q T_{\nabla u}S_m a\right\|_{L^1_t(L^2)}\stackrel{\text{def}}{=}\left\|T_{\nabla u}S_m a\right\|_{B^{\frac32}_{2,1}}.$$

By virtue of Theorem 2.82 in the book [5], we have

$$\left\|T_{\nabla u}S_m a\right\|_{B^{\frac32}_{2,1}}\leqslant C\left\|\nabla u\right\|_{B^{-\frac12}_{\infty,2}}\left\|S_m a\right\|_{B^2_{2,2}}. \tag{4.107}$$

Moreover, Bernstein result implies the following embedding $B^1_{2,2}\hookrightarrow B^{-\frac12}_{\infty,2}$. Therefore, we have

$$\left\|T_{\nabla u}S_m a\right\|_{B^{\frac32}_{2,1}}\leqslant C\left\|\nabla u\right\|_{B^1_{2,2}\equiv H^1}\left\|S_m a\right\|_{B^2_{2,2}}. \tag{4.108}$$

Applying Poincaré-Wirtinger to $\|\nabla u\|_{H^1}$, (since the average of ∇u is nul), we infer that the norms $\|\nabla u\|_{H^1}$ and $\|\nabla u\|_{\dot H^1}$ are equivalent and thus

$$\left\|T_{\nabla u}S_m a\right\|_{B^{\frac32}_{2,1}}\leqslant C\left\|u\right\|_{\dot H^2}\left\|S_m a\right\|_{B^2_{2,2}}. \tag{4.109}$$

On the other hand, it seems obvious that $\|S_m a\|_{B^2_{2,2}}\leqslant\|S_m a\|_{\dot B^2_{2,2}}\leqslant\|S_m a\|_{\dot H^2}$. As a result,

$$\left\|T_{\nabla u}S_m a\right\|_{B^{\frac32}_{2,1}}\leqslant C\,2^{2m}\left\|u\right\|_{\dot H^2}\left\|S_m a\right\|_{\dot H^2}. \tag{4.110}$$

Finally, by integration in time and by definition of $S_m a$, we get

$$\|T_{\nabla u} S_m a\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} \overset{\text{def}}{=} \sum_{q\in\mathbb{Z}} 2^{\frac{3q}{2}} \, \|\Delta_q T_{\nabla u} S_m a\|_{L^1_t(L^2)} \leqslant C\, 2^{2m} \, \|u\|_{L^1_t(\dot H^2)} \, \|a\|_{L^\infty_t(L^2)}. \tag{4.111}$$

The estimate on the term $\sum_{q\in\mathbb{Z}} 2^{\frac{3q}{2}} \, \|\Delta_q R(S_m a,\, \nabla u)\|_{L^1_t(L^2)}$ is close to the previous one, by virtue of Theorem page 2.85 in [5]. We recall it below.

Remind: If s_1 and s_2 are two real numbers, such that $s_1 + s_2 > 0$, then

$$\|R(u,v)\|_{B^{s_1+s_2}_{p,r}} \leqslant C(s_1,s_2)\, \|u\|_{B^{s_1}_{p_1,r_1}} \, \|v\|_{B^{s_2}_{p_2,r_2}}, \qquad \frac{1}{p} \overset{\text{def}}{=} \frac{1}{p_1} + \frac{1}{p_2} \quad \textit{and} \quad \frac{1}{r} \overset{\text{def}}{=} \frac{1}{r_1} + \frac{1}{r_2}.$$

Therefore, we have

$$\sum_{q\in\mathbb{Z}} 2^{\frac{3q}{2}} \, \|\Delta_q R(S_m a,\, \nabla u)\|_{L^1_t(L^2)} \overset{\text{def}}{=} \|R(S_m a,\, \nabla u)\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} \leqslant C\, 2^{2m} \, \|u\|_{L^1_t(\dot H^2)} \, \|a\|_{L^\infty_t(L^2)}. \tag{4.112}$$

Concerning the last term, $\sum_{q\in\mathbb{Z}} 2^{\frac{3q}{2}} \, \|T'_{\Delta_q \nabla u} S_m a\|_{L^1_t(L^2)}$, we write the definition. Indeed,

$$T'_{\Delta_q \nabla u} S_m a \overset{\text{def}}{=} \sum_{q' \geqslant q-2} S_{q'+2} \Delta_q \nabla u \, \Delta_{q'} S_m a.$$

Therefore, we get

$$\begin{aligned}
\|T'_{\Delta_q \nabla u} S_m a\|_{L^2} &\leqslant C \sum_{q' \geqslant q-2} \|\Delta_q \nabla u\|_{L^\infty} \, \|\Delta_{q'} S_m a\|_{L^2} \\
2^{\frac{3q}{2}} \, \|T'_{\Delta_q \nabla u} S_m a\|_{L^2} &\leqslant C\, 2^{\frac{3q}{2}} \sum_{q' \geqslant q-2} 2^{\frac{q}{2}}\, 2^{-\frac{q}{2}} \, \|\Delta_q \nabla u\|_{L^\infty} \, 2^{-2q'}\, 2^{2q'} \, \|\Delta_{q'} S_m a\|_{L^2} \\
&\leqslant C \sum_{q' \geqslant q-2} 2^{2(q-q')}\, 2^{-\frac{q}{2}} \, \|\Delta_q \nabla u\|_{L^\infty} \, 2^{2q'} \, \|\Delta_{q'} S_m a\|_{L^2}
\end{aligned} \tag{4.113}$$

By definition of the Besov norm, there exists a sequence $c_{q'}$ belonging to $\ell^2(\mathbb{Z})$ such that

$$2^{2q'} \, \|\Delta_{q'} S_m a\|_{L^2} \leqslant C\, c_{q'} \, \|S_m a\|_{B^2_{2,2}}.$$

As a result, by summation on q, we infer that

$$\sum_{q\in\mathbb{Z}} 2^{\frac{3q}{2}} \, \|T'_{\Delta_q \nabla u} S_m a\|_{L^2} \leqslant C \Big(\sum_{q\in\mathbb{Z}} 2^{-\frac{q}{2}} \, \|\Delta_q \nabla u\|_{L^\infty} \, d_q \Big) \, \|S_m a\|_{B^2_{2,2}}, \tag{4.114}$$

where the sequence d_q stems from convolution product: $d_q \overset{\text{def}}{=} \sum_{q' \geqslant q-2} 2^{2(q-q')} \, c_{q'}$. As defined, it is clear that, by virtue of Young's inequality, $\|d_q\|_{\ell^2(\mathbb{Z})} \leqslant C$. Finally, Cauchy-Schwarz inequality yields

$$\sum_{q\in\mathbb{Z}} 2^{\frac{3q}{2}} \, \|T'_{\Delta_q \nabla u} S_m a\|_{L^2} \leqslant C \, \|\nabla u\|_{B^{-\frac{1}{2}}_{\infty,2}} \, \|S_m a\|_{B^2_{2,2}}, \tag{4.115}$$

Once again, the Bernstein's embedding $B^1_{2,2} \hookrightarrow B^{-\frac{1}{2}}_{\infty,2}$, combining with an integration in time gives

$$\sum_{q\in\mathbb{Z}} 2^{\frac{3q}{2}} \, \|T'_{\Delta_q \nabla u} S_m a\|_{L^1_t(L^2)} \leqslant C \, \|S_m a\|_{L^\infty_t(B^2_{2,2})} \, \|\nabla u\|_{L^1_t(B^1_{2,2})} \tag{4.116}$$

Therefore,

$$\sum_{q\in\mathbb{Z}} 2^{\frac{3q}{2}} \, \|T^{'}_{\Delta_q \nabla u} S_m a\|_{L^1_t(L^2)} \leqslant \, C \, 2^{2m} \|a\|_{L^\infty_t(L^2)} \, \|u\|_{L^1_t(\dot{H}^2)} \tag{4.117}$$

Conclusion Summing estimates (4.106), (4.111), (4.112), and (4.117) completes the proof of the Lemma. $\square$

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