Random trees, fires and non-crossing partitions

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Summary

Trees play a fundamental role in various areas of mathematics such as combinatorics, graph theory, population genetics, and theoretical computer science. In the first part of this thesis, we consider fires on large random trees. We define dynamics where, as time goes by, fires start randomly on a tree and propagate through the neighboring flammable edges, whereas in the meantime, some edges become fireproof and stop the propagation of subsequent fires. We study the effect of such dynamics, which is intimately related to the geometry of the underlying tree, and consider two different models of trees: Cayley trees and random recursive trees. The techniques used are related to the procedure of cutting-down a tree (in which edges are successively removed from the tree) and fragmentation theory.

In the second part, we use trees (in particular Galton–Watson trees) as tools to study large random non-crossing configurations of the unit disk. Such configurations consist of a graph formed by non-intersecting diagonals of a regular polygon. We consider two variants: first when the connected components of the graph are pairwise disjoint polygons (the configuration is then called a non-crossing partition), and then when this graph is a tree (it is called a non-crossing tree). These objects have been studied from the perspective of combinatorics and probability but in the past, research has focused on the uniform distribution. We generalize the latter using Boltzmann sampling. In both models, we observe a universality phenomenon: all large non-crossing partitions for which the distribution of the size of a typical polygon belongs to the domain of attraction of a stable law resemble the same random object, namely a stable lamination. A similar result holds for non-crossing trees, for which we construct a novel universal limit that we call a stable triangulation.

Zusammenfassung

Bäume spielen eine grundlegende Rolle in verschiedenen Bereichen der Mathematik, wie beispielsweise in der Kombinatorik, Graphentheorie, Populationsgenetik oder theoretischen Informatik. Große zufällige Bäume liegen im Kern dieser Dissertation. Im ersten Teil betrachten wir eine Feuerdynamik: mit fortlaufender Zeit entfachen Brände zufällig auf einem Baum und verbreiten sich über die benachbarten brennbaren Kanten; währenddessen werden einige Kanten feuerfest und stoppen die Ausbreitung der nachfolgenden Brände. Wir untersuchen das Verhalten dieser Dynamik, die eng mit der Geometrie der darunterliegenden Graphenstruktur verbunden ist. Wir betrachten zwei verschiedene Modelle von Bäumen: Cayley-Bäume und zufällige rekursive Bäume. Die verwendeten Techniken stehen im Zusammenhang mit den Verfahren von Vernichtung von Bäumen (hierbei werden Kanten sukzessive von dem Baum entfernt) und Fragmentierungstheorie.

Im zweiten Teil verwenden wir Bäume (insbesondere Galton–Watson Bäume) als Werkzeuge, um große zufällige kreuzungsfreie Konfigurationen des Kreises zu untersuchen. Letztere wurden bereits in der Kombinatorik studiert und bestehen aus Diagonalen eines regelmäßigen Polygons, die sich nicht überschneiden. Wir betrachten zwei Fälle: zum einen seien die Zusammenhangskomponenten des Graphen, die durch diese Diagonalen gebildet werden, paarweise disjunkte Polygone (dann wird die Konfiguration als kreuzungsfreie Partition bezeichnet); zum anderen sei dieser Graph ein Baum (und er wird ein kreuzungsfreier Baum genannt). In der Vergangenheit wurden probabilistische Versionen dieser Modelle nur bezüglich der Gleichverteilung untersucht. Wir verallgemeinern diese Modelle mit einem sogenannten „Boltzmann sampling". In beiden Fällen beobachten wir ein Universalitätsphänomen. Alle großen kreuzungsfreien Partitionen (bzw. kreuzungsfreie Bäume), für die die Verteilung der Größe eines typischen Polygons (bzw. der Grad eines typischen Knotens) im Anziehungsbereich einer stabilen Verteilung liegt, sind sich ähnlich: sie entsprechen entweder einer stabilen Laminierung für kreuzungsfreie Partitionen oder einer stabilen Triangulierung für kreuzungsfreie Bäume.

Acknowledgement

I will keep this section short and sober, as I am not good at writing (this might not be the best way to start your thesis).

My first words naturally go to Jean Bertoin who proposed me to set trees on fire with him and who guided me during these three years. His rigor, clarity and confidence also really helped me. I have been impressed by his availability and I thank him for all these mathematics discussions and his scientific advises. He was always here when I had a question, and also when I did not have any for too long. *Merci Jean, ce fut un privilège !*

Maybe as naturally as the first words, the next ones should be addressed to Igor Kortchemski who opened me the door of the beautiful[1] world of laminations of the disk. It is a real pleasure to work with him, in an office, at a conference, or in a bar. I hope we will continue in the future. *Merci Igor pour ça, et pour tout le reste aussi !*

I thank particularly Nicolas Curien and Thomas Duquesne for reading and commenting this thesis and for their help for the next stage. I also have had the pleasure to meet Valentin Féray and Ashkan Nikeghbali here in Zürich and I am honoured to have them participating to my jury. *Merci à vous tous !*

I must also acknowledge and thank all the secretaries of the mathematics institute which, everyday, make life easier.

I will end with a thought, without listing all the names, to my colleagues in Zürich, those who left before me, disseminated throughout the world, and those who are still here, as well as a few, but important, friends.

[1] If you are not convinced, please see pages 46 & 48.

Contents

Part I

Introduction

1

Outline and background

This thesis is divided into two parts. We first consider random fire dynamics on random graphs, based on the articles [81] and [82]. Then we focus on random non-crossing configurations of the unit disk, based on a joint work with Igor Kortchemski [72]. In this chapter, we first describe informally these models and we then give some required background. The new results developed in this work are stated in the next two chapters.

1.1 Outline

1.1.1 Random fires on trees

Given a finite connected graph and a parameter $p \in [0,1]$, we consider the following random dynamics: initially every edge is flammable, then successively, in a uniform random order, each edge is either fireproof with probability $1-p$ or set on fire with probability p; in the latter case, the edge burns, sets on fire its flammable neighbors and the fire propagates instantly in the graph, only stopped by fireproof edges. An edge which has burnt because of the propagation of fire is not subject to the dynamics thereafter. The dynamics continue until all edges are either burnt or fireproof. A vertex is called fireproof if all its adjacent edges are fireproof and called burnt otherwise. We discard the fireproof edges with at least one burnt extremity and thus get two families of subgraphs: one consists of fireproof subgraphs and the other of burnt subgraphs; see Figure 1.1 for an illustration. We study the asymptotic behavior of the size of these two families of subgraphs and of their connected components as the size n of the original graph tends to infinity and the parameter $p = p(n)$ tends to 0.

Fires on a graph find applications in statistical physics and in the study of epidemics propagating in a network. If the graph models a network, then the fires may be thought of as infections, and burnt and fireproof vertices respectively as infected and immune nodes. We stress that in the present model, infected nodes do not recover, we talk about a Susceptible-Infected-Removed epidemic, as opposed to Susceptible-Infected-Susceptible epidemics described by usual fire forest models in which the infected nodes recover and may be infected again later, see e.g. Drossel & Schwabl [42]. Another fire model in which burnt components are removed was studied by Ráth [101] and Ráth & Tóth [102] who considered an Erdős–Renyi graph in which edges appear at unit rate and the vertices are set on fire at small rate; when a vertex is set on fire, the whole connected component which contains it burns and is removed from the graph (edges and vertices). These are in some sense dual dynamics of the present ones: all edges are present at the beginning but fireproof edges act as barriers that stop the propagation of fires.

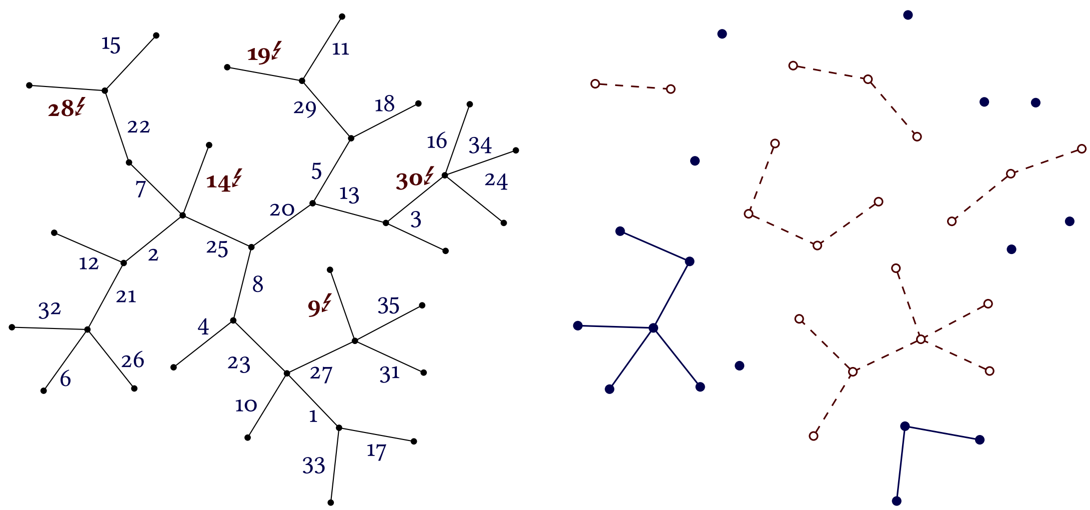


Figure 1.1: Given a tree with a numbering of its edges on the left, if the edges set on fire are the 9th, 13th, 19th, 28th and 30th, we get the two forests on the right where the burnt components are drawn with dotted lines and the fireproof ones with plain lines.

We shall focus on the case where the underlying graph is a tree, i.e. contains no loop, or equivalently, for which the removal of any edge disconnects it into two connected components. Furthermore, we study this dynamics on random trees. We first consider in Chapter 4 a Cayley tree of size n, i.e. chosen uniformly at random among the finite set of (labelled) trees with n vertices; this dynamics was indeed originally defined and studied in this context by Bertoin [16]. We then turn our attention to uniform random recursive trees of size n in Chapter 5. Such trees are constructed recursively as follows: we start with a single vertex 1, and then successively, for every $k = 2, \dots, n$, the vertex k is added to the tree, attached by an edge to a vertex chosen uniformly at random among the $k-1$ already present. We shall see that the behavior in the two cases are different since the geometry of the trees are different (see below for a first indication). However Cayley trees and recursive trees are the two (non-trivial) trees which fulfill the *splitting property*, which is a crucial feature in our study: consider t_n a random Cayley tree or uniform random recursive tree of size n and remove a uniform random edge, then the two resulting subtrees are, conditional on their size, say, k and $n-k$, independent and distributed as t_k and t_{n-k} respectively.

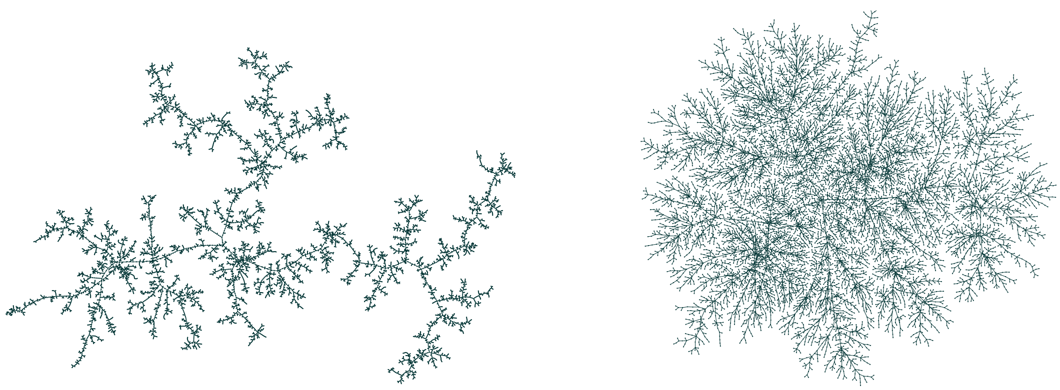


Figure 1.2: Samples of a Cayley tree and a random recursive tree of size 10 000.

The first result is a phase transition phenomenon. Consider the fire dynamics with parameter p_n on a Cayley tree with n vertices. Intuitively, if p_n is large, most of the tree is burnt at the end of the dynamics, whereas if p_n is small, most of it is fireproof. Bertoin [16] proved that a phase transition indeed occurs, and that the critical regime — for which the proportion of burnt and fireproof vertices has a non-trivial limit — is for p_n comparable to $1/\sqrt{n}$. We obtain the same result for random recursive trees, where the critical regime is at $\ln n/n$. Note that $\ln n/n \ll 1/\sqrt{n}$, which can be intuitively understood by the fact that in a large Cayley tree, vertices tend to have a small degree, whereas in a large random recursive tree, some vertices have a large degree, which helps the fire to propagate. Further, we obtain in both cases a limit theorem for the joint sizes of the burnt connected components, rescaled by n, in the critical regime. For Cayley trees, the latter can be described by logging Aldous' Brownian continuum random tree [3, 4, 5] at the atoms of a certain point process on its skeleton, in the spirit of Aldous & Pitman [8], whereas the limit is simpler for random recursive trees. In the subcritical regime, when $p_n \gg 1/\sqrt{n}$ for Cayley trees and $p_n \gg \ln n/n$ for random recursive trees, we know that the number of fireproof vertices is small compared to n; we obtain in both cases the correct normalizing factor for convergence in distribution to an explicit non-trivial limit. Consider finally the supercritical regime, when $p_n \ll 1/\sqrt{n}$

for Cayley trees and $p_n \ll \ln n/n$ for random recursive trees. The number of burnt vertices is now small compared to n and we obtain for Cayley trees the correct normalizing factor for convergence in distribution to an explicit non-trivial limit. Moreover, for Cayley trees, Bertoin [16] proved that with a probability tending to 1 as $n \to \infty$, there exists a "giant" fireproof connected component, with size $n - o(n)$. We prove the same result for random recursive trees and also estimate the size of the largest fireproof connected component in the two other regimes.

This fire dynamics on trees is closely related to the problem of isolating nodes, first introduced by Meir & Moon [85]. In this model, one is given a tree t_n of size n and k vertices (chosen randomly or deterministically), say, $u_1, \dots, u_k$; then the edges of t_n are successively removed in a uniform random order, and at each step, if a connected component newly created does not contain any of the k selected vertices, it is immediately discarded. This random dynamics eventually ends when the graph is reduced to the k selected singletons, we say that the k vertices have been isolated. The main interest in [85] and several subsequential papers concerns the behavior of the (random) number of steps $X(t_n; u_1, \dots, u_k)$ of this algorithm, as $n \to \infty$ and the number of selected vertices k is fixed. We shall see that $X(t_n; u_1, \dots, u_k)$ is related to the fire dynamics on t_n. Indeed, if one sees fireproof edges as being removed from the tree, then a vertex is fireproof if and only if it is isolated.

Meir & Moon [85] studied the first two moments of $X(t_n; U_1)$ when t_n is a Cayley tree of size n and U_1 is chosen uniformly at random in t_n. Janson [63] and Panholzer [94] then obtained a limit theorem for the latter in a more general setting (conditioned Galton–Watson trees and simply generated trees respectively). Finally, for Cayley trees, Addario-Berry, Broutin & Holmgren [2] as well as Bertoin [16] obtained a limit theorem for $X(t_n; U_1, \dots, U_k)$ as $n \to \infty$ for every $k \geq 1$ fixed, where $U_1, \dots, U_k$ are independent uniform random vertices of t_n.

When t_n is a random recursive tree, rooted at the vertex 1, Meir & Moon [86] estimated the first two moments of $X(t_n; 1)$; a limit theorem for the latter was first derived by Drmota *et al.* [41] and recovered by Iksanov & Möhle [62] using probabilistic argument. Then Kuba & Panholzer [74] obtained a limit theorem for $X(t_n; U_1, \dots, U_k)$ when $U_1, \dots, U_k$ are either the first k vertices $\{1, \dots, k\}$, or the last k vertices $\{n - k + 1, \dots, n\}$ or independent uniform random vertices of t_n. Finally Bertoin [19] recovered and gave a multidimensional extension of these results.

As a last remark, let us mention that in both cases the proofs of Bertoin [16, 19] rely on the so-called cut-tree associated with t_n. The latter records the genealogy of the fragmentation obtained by removing the edges of t_n one after the others in a uniform random order (see Chapter 2 below for a formal definition). In [16, 19], limit theorems for the cut-tree are obtained, which yields the above results on the number of cuts, but it will also enable us to derive strong results on the fire dynamics. The denomination "cut-tree" has been used recently by several authors: Bertoin & Miermont [20] as well as Dieuleveut [38] considered the cut-tree of large Galton–Watson trees and Broutin & Wang [24, 25] that of **p**-trees. However, we stress that the definition of the cut-tree differs slightly in these works, depending on the context.

1.1.2 Random non-crossing partitions

A partition of the set of integers $[n] = \{1, \dots, n\}$ is a collection of (pairwise) disjoint subsets — called *blocks* — whose union is $[n]$. As defined by Kreweras [73], such a partition is said to be non-crossing if it fulfills the following property: for every quadruple $1 \leq i < j < k < \ell \leq n$, if i and k belong to the same block and j and ℓ belong to the same block as well, then i, j, k, and ℓ all belong to the same block. We denote by $\mathbb{NC}_n$ the set of all non-crossing partitions of $[n]$.

In Chapter 6, we consider for each integer n a random element of $\mathbb{NC}_n$ and study the behavior of this sequence as $n \to \infty$. The case of the uniform distribution on the finite set $\mathbb{NC}_n$ has been considered by Arizmendi & Vargas [10], Ortmann [91], as well as Curien & Kortchemski [31]. We consider more generally *simply generated* non-crossing partitions, using a Boltzmann sampling: each non-crossing partition in $\mathbb{NC}_n$ is given a weight, we then sample one proportionally to its weight. The main tool relies on a bijection between non-crossing partitions and rooted plane trees, defined by Dershowitz & Zaks [35]. If the partition is a simply generated non-crossing partition of $[n]$, then the associated tree is a simply generated tree with $n + 1$ vertices. Such random trees were introduced by Meir & Moon [87] and studied by Janson [64].

We first consider statistics on large random non-crossing partitions; as examples, we establish limit theorems for the size of the block containing 1 and that of a block chosen uniformly at random, as well as for the number of blocks with size in a given set of integers. We give two applications of our results. First, for any set $A \subset \mathbb{N}$ fixed, we give an asymptotic formula as $n \to \infty$ for the number of non-crossing partitions of $[n]$ with blocks of size only belonging to a A; for n fixed, exact formulas for the latter are known only when $A = k\mathbb{N}$ (Edelman [46]) and when $A = \{k\}$ for a given $k \geq 2$ (Arizmendi & Vargas [10]). Second, in free probability, it is known that a compactly supported probability measure on $\mathbb{R}$, say, μ, is characterized by the sequence of its free cumulants. Assuming that all the free cumulants of μ are non-negative (and that μ is not a Dirac mass), we give an expression of the right edge of its support, more explicit than that obtained by Ortmann [91] under the same assumption.

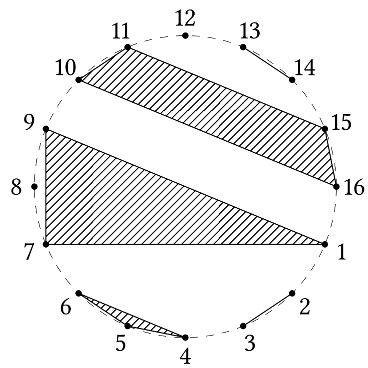


Figure 1.3: The partition $\{\{1,7,9\},\{2,3\},\{4,5,6\},\{8\},\{10,11,15,16\},\{12\},\{13,14\}\}$.

Following Curien & Kortchemski [31], we then adopt a geometrical point of view. Indeed, any partition of $[n]$ can be visualized in the unit disk of the complex plane as follows: each integer $k \in [n]$ is placed on the complex number $\exp(-2i\pi k/n)$ and each block is represented by the convex polygon spanned by its elements. A partition is non-crossing if and only if these polygons do not cross, i.e. if their convex hulls are pairwise disjoint; see Figure 1.3 for an example. Given a non-crossing partition, we consider the closed subset of the unit disk formed by the vertices and edges of these polygons, viewed as line segments in the plane. For every integer n, we consider a random element of $\mathbb{NC}_n$ and the associated closed set of the disk; we then look for a convergence in distribution of this sequence of random sets. In [31], such a convergence was obtained for the uniform distribution on $\mathbb{NC}_n$, and the limit is the Brownian triangulation introduced by Aldous [6] and studied by Le Gall & Paulin [77]. We shall see that this object is universal, in the sense that it appears as the limit of any simply generated non-crossing partitions for which the distribution of the size of a typical block has finite variance. When this distribution has a heavy tail, we obtain at the limit a stable lamination introduced by Kortchemski [71].

In Chapter 7, we consider another random non-crossing configuration of the disk. A non-crossing tree is a tree drawn in the unit disk having as vertices the n-th roots of unity for some $n \in \mathbb{N}$, and whose edges are straight line segments and do not cross. A non-crossing tree can be mapped to a plane tree, rooted at the complex number 1, however this mapping is not one-to-one: several non-crossing trees have the same planar structure.

We consider the following canonical embedding of a planar tree into the disk: given a planar tree with n vertices, we list these vertices in lexicographical (also called depth-first search) order from 0 to $n-1$, then the k-th vertex is sent to the complex $\exp(-2i\pi k/n)$ to form a non-crossing tree, see Figure 1.5 for an example. This defines a bijection between

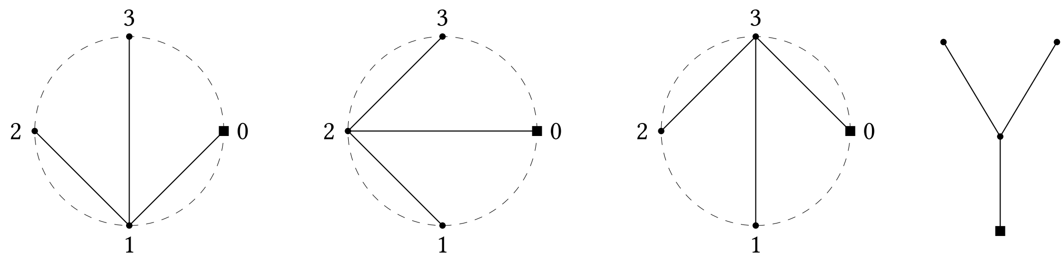


Figure 1.4: Three non-crossing trees with the same planar structure.

plane trees and non-crossing trees which "always turn to the right" in the sense that if the vertices $\exp(-2i\pi k/n)$ and $\exp(-2i\pi\ell/n)$ are linked by a chord and if $\exp(-2i\pi k/n)$ is closer to 1 than $\exp(-2i\pi\ell/n)$ for the graph distance, then $k < \ell$.

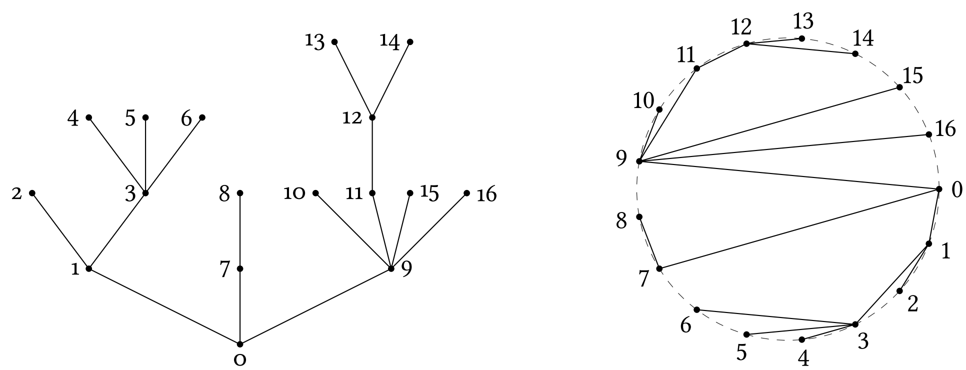


Figure 1.5: A plane tree and its embedding in the disk.

As for partitions, we view non-crossing trees as closed subsets of the unit disk; for every integer n, we consider the embedding of a critical Galton–Watson tree conditioned to have n vertices and look for a limit in distribution as $n \to \infty$. When the offspring distribution has finite variance, we recover the Brownian triangulation at the limit. When this distribution has a heavy tail, we obtain at the limit a new object, that we call stable triangulation, which is informally obtained by "filling-in" a stable lamination. We give two constructions of this random set and compute its Hausdorff dimension.

The rest of this chapter is devoted to some background on random trees, which are needed to state precisely our results. The latter are postponed to the next two chapters. We first introduce simply generated trees and Galton–Watson trees, which are a simple model of random trees and which will play a crucial role in Chapters 6 and 7; they will also be used in Chapter 4. We then recall the concept of real trees, and present several Polish topologies on sets of trees. Finally, we discuss a particular and central example of random real tree: the Brownian continuum random tree.

1.2 Galton–Watson trees and simply generated trees

1.2.1 Definitions

Plane trees We follow the formalism of Neveu [89]. Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of all positive integers, set $\mathbb{N}^0 = \{\varnothing\}$ and consider the set of labels

$$\mathbb{U} = \bigcup_{n \geq 0} \mathbb{N}^n.$$

For $u = (u_1, \dots, u_n) \in \mathbb{U}$, we denote by $|u| = n$ the length of u; if $n \geq 1$, we define $pr(u) = (u_1, \dots, u_{n-1})$ and for $i \geq 1$, we let $ui = (u_1, \dots, u_n, i)$; more generally, for $v = (v_1, \dots, v_m) \in \mathbb{U}$, we let $uv = (u_1, \dots, u_n, v_1, \dots, v_m) \in \mathbb{U}$ be the concatenation of u and v. We endow $\mathbb{U}$ with the lexicographical order: given $v, w \in \mathbb{U}$, let $z \in \mathbb{U}$ be their longest common prefix, that is $v = z(v_1, \dots, v_n)$, $w = z(w_1, \dots, w_m)$ and $v_1 \neq w_1$, then $v \prec w$ if $v_1 < w_1$.

Definition 1.1. A plane tree is a nonempty, finite subset $\tau \subset \mathbb{U}$ such that:

(i) $\varnothing \in \tau$;

(ii) if $u \in \tau$ with $|u| \geq 1$, then $pr(u) \in \tau$;

(iii) if $u \in \tau$, then there exists an integer $k_u \geq 0$ such that $ui \in \tau$ if and only if $1 \leq i \leq k_u$.

We will view each vertex u of a tree τ as an individual of a population for which τ is the genealogical tree. The vertex $\varnothing$ is called the root of the tree and for every $u \in \tau$, k_u is the number of children of u (if $k_u = 0$, then u is called a leaf, otherwise, u is called an internal vertex), $|u|$ is its generation, $pr(u)$ is its parent and more generally, the vertices $u, pr(u), pr \circ pr(u), \dots, pr^{|u|}(u) = \varnothing$ are its ancestors.

We denote by $\mathbb{T}$ the set of plane trees and for each integer n, by $\mathbb{T}_n$ the set of plane trees with n edges, or equivalently $n + 1$ vertices.

Galton–Watson trees Let μ be a probability measure on $\mathbb{Z}_+$ which satisfies $\mu(0) > 0$ and with expectation $m := \sum_{k=0}^{\infty} k\mu(k) \leq 1$. We shall always further assume that $\mu(0) + \mu(1) < 1$ to avoid trivial cases. We define the law of a Galton–Watson tree with offspring distribution μ as the unique probability measure GW^{μ} on $\mathbb{T}$ satisfying the following conditions:

(i) $\mathrm{GW}^\mu(k_\varnothing = j) = \mu(j)$ for every $j \geq 0$;

(ii) For every $j \geq 1$ such that $\mu(j) > 0$, conditional on the event $\{k_\varnothing = j\}$, the subtrees that stem from the children of the root $\{u \in \mathbb{U}; 1u \in \tau\}, \dots, \{u \in \mathbb{U}; ju \in \tau\}$ are independent and distributed as GW^μ.

The following explicit formula for the law GW^μ is originally due to Otter [93]:

$$\mathrm{GW}^\mu(\tau) = \prod_{u\in\tau} \mu(k_u). \tag{1.1}$$

The denomination comes from the fact that if we denote by Z_n the number of vertices at generation n for each $n \geq 0$, then under GW^μ, the sequence $(Z_n; n \geq 0)$ is a Galton–Watson process issued from 1. The assumption $m \leq 1$ is equivalent to $\sum_{n=0}^\infty Z_n < \infty$, GW^μ-almost surely. We shall focus on the case where $m = 1$, for which the expectation of $\sum_{n=0}^\infty Z_n$ under GW^μ is infinite; such an offspring distribution (and then such a tree) is called *critical*.

Let us give two examples of a distribution μ on $\mathbb{Z}_+$ which give rise to remarkable critical Galton–Watson trees. We denote by GW^μ_n the law on $\mathbb{T}_n$ of a Galton–Watson tree with offspring distribution μ conditioned to have $n + 1$ vertices, providing that this conditioning makes sense.

Example 1.2. (i) When μ is the geometric distribution with parameter 1/2, the law GW^μ_n is the uniform distribution on $\mathbb{T}_n$. Indeed, for every $\tau \in \mathbb{T}_n$, since each vertex, except the root, has a unique parent, we have $\sum_{u\in\tau} k_u = n$, whence

$$\mathrm{GW}^\mu(\tau) = \prod_{u\in\tau} \mu(k_u) = \prod_{u\in\tau} 2^{-(k_u+1)} = 2^{-(2n+1)}.$$

Then $\mathrm{GW}^\mu_n(\tau) = 2^{-(2n+1)}\mathrm{GW}^\mu(\mathbb{T}_n)^{-1}$ for every $\tau \in \mathbb{T}_n$, which does not depend on the choice of τ.

(ii) Let μ be the Poisson distribution with parameter 1. Sample a tree according to GW^μ_n, view it as a non-planar (or unordered) tree and assign labels from 1 to $n + 1$ to the vertices uniformly at random. Then the tree that we obtain is a uniform rooted Cayley tree with $n + 1$ vertices. Indeed, in this case, for every $\tau \in \mathbb{T}_n$,

$$\mathrm{GW}^\mu(\tau) = \prod_{u\in\tau} \frac{\mathrm{e}^{-1}}{k_u!} = \mathrm{e}^{-(n+1)} \prod_{u\in\tau} \frac{1}{k_u!}.$$

Then there are $(n + 1)! \prod_{u\in\tau} \frac{1}{k_u!}$ different ways to make τ a labelled rooted unordered tree. It follows that the probability that a tree sampled according to GW^μ with a uniform random labelling of its vertices is a particular labelled rooted unordered tree of size $n + 1$, say, t_n, is $\mathrm{e}^{-(n+1)}/(n + 1)!$, which does not depend on the choice of t_n. Note that the parameter of the Poisson law is irrelevant.

Simply generated trees More general distributions of random plane trees are that of simply generated trees, which were introduced by Meir & Moon [87]. We generalize the law of a Galton–Watson tree (1.1) as follows. Given a sequence $w = (w(k); k \geq 0)$ of nonnegative real numbers (thought of as elementary weights), with every $\tau \in \mathbb{T}$ we associate a weight

$$\Omega^w(\tau) = \prod_{u \in \tau} w(k_u);$$

we define then for every integer n a partition function

$$Z_n^w = \sum_{T \in \mathbb{T}_n} \Omega^w(T).$$

Implicitly, we shall always restrict our attention to the values of n for which $Z_n^w > 0$. In this case, for every $\tau \in \mathbb{T}_n$, we set

$$\mathbb{Q}_n^w(\tau) = \frac{\Omega^w(\tau)}{Z_n^w}. \tag{1.2}$$

A random tree of $\mathbb{T}_n$ sampled according to $\mathbb{Q}_n^w$ is called a simply generated tree. Observe that if w is a probability measure on $\mathbb{Z}_+$ with expectation at most 1, then $\mathbb{Q}_n^w = \mathrm{GW}_n^\mu$ for every integer n.

The following simple remark, which will be useful in Chapter 6, is originally due to Kennedy [69], see also Janson [64, Chapters 3 & 4]. We say that two sequences $w = (w(k); k \geq 0)$ and $v = (v(k); k \geq 0)$ are equivalent when there exist $a, b > 0$ such that $v(k) = ab^k w(k)$ for every $k \geq 0$. In this case, we compute for every $\tau \in \mathbb{T}_n$

$$\Omega^v(\tau) = \prod_{u \in \tau} ab^{k_u} w(k_u) = a^{n+1} b^n \Omega^w(\tau),$$

from which it follows that

$$\mathbb{Q}_n^v = \mathbb{Q}_n^w \qquad \text{for every} \quad n \geq 1.$$

We see that we do not change the law of the simply generated tree when we change the sequence of weights for an equivalent sequence in the above sense. Therefore, if w admits a probability measure on $\mathbb{Z}_+$ with mean 1 in its equivalence class, then we reduce the study of a simply generated tree to that of a conditioned critical Galton–Watson tree. It is possible to define a probability equivalent to w if and only if the generating series $z \mapsto \sum_{k=0}^\infty w(k) z^k$ has a non-zero radius of convergence, say, ρ; then every $b \in (0, \rho)$ defines a probability measure $\mu = (ab^k w(k); k \geq 0)$, with the correct normalizing factor $a > 0$. Moreover, μ has mean 1 if and only if $\sum_{k=0}^\infty kab^k w(k) = \sum_{k=0}^\infty ab^k w(k) = 1$. Janson [64, Lemma 3.1] observed that the function

$$\Psi : t \mapsto \frac{\sum_{k=0}^\infty k w(k) t^k}{\sum_{k=0}^\infty w(k) t^k}$$

is finite, null at 0, continuous and strictly increasing on $[0, \rho)$. Therefore every value in the interval $(0, \Psi(\rho-))$ is the expectation of exactly one probability measure equivalent to w and w is equivalent to a (unique) critical probability measure if and only if

$$\rho := \left(\limsup_{k\to\infty} w(k)^{1/k}\right)^{-1} > 0 \qquad \text{and} \qquad \lim_{t\uparrow\rho} \Psi(t) \geq 1.$$

Moreover, one can control the variance and tail distribution of this probability measure, see [64, Section 4].

Example 1.3. Let A be a subset of $\mathbb{Z}_+$ containing 0 and at least one other integer. Set $w_A(k) = 1$ if $k \in A$ and $w_A(k) = 0$ otherwise. Observe that $\mathbb{Q}_n^{w_A}$ is the uniform distribution on the set of trees of size $n+1$ for which each vertex has a number of children in A. Then there exists a critical probability measure π_A equivalent to w_A, defined by

$$\pi_A(k) = \frac{\xi_A^k}{\sum_{j\in A} \xi_A^j} \mathbb{I}_{k\in A} \qquad (k \geq 0),$$

where $\xi_A > 0$ satisfies

$$\sum_{j\in A} \xi_A^j = \sum_{j\in A} j\xi_A^j.$$

In particular, for $A = N\mathbb{Z}_+$ for a fixed $N \geq 1$, we have

$$\pi_{N\mathbb{Z}_+}(k) = \frac{N}{(1+N)^{1+k/N}} \mathbb{I}_{k\in N\mathbb{Z}_+} \qquad (k \geq 0).$$

Note that for $N = 1$, we obtain the geometric distribution $\pi_{\mathbb{Z}_+} = (2^{-(k+1)}; k \geq 0)$, which recovers Example 1.2 (i).

1.2.2 Coding planar trees by a discrete paths

We fix for the whole subsection a tree $\tau \in \mathbb{T}_n$ and we let $\varnothing = u(0) \prec u(1) \prec \cdots \prec u(n)$ be its vertices, listed in lexicographical order. We describe three bijections between τ and discrete paths. Recall that k_u denotes the number of children of $u \in \tau$ and $|u|$ its generation.

The Łukasiewicz path. Define $W = (W_j; 0 \leq j \leq n+1)$ by $W_0 = 0$ and for every $0 \leq j \leq n$,

$$W_{j+1} = W_j + k_{u(j)} - 1.$$

One easily checks that $W_j \geq 0$ for every $0 \leq j \leq n$ but $W_{n+1} = -1$. Observe that $W_{j+1} - W_j \geq -1$ for every $0 \leq j \leq n$, with equality if and only if $u(j)$ is a leaf of τ. We shall think of such a

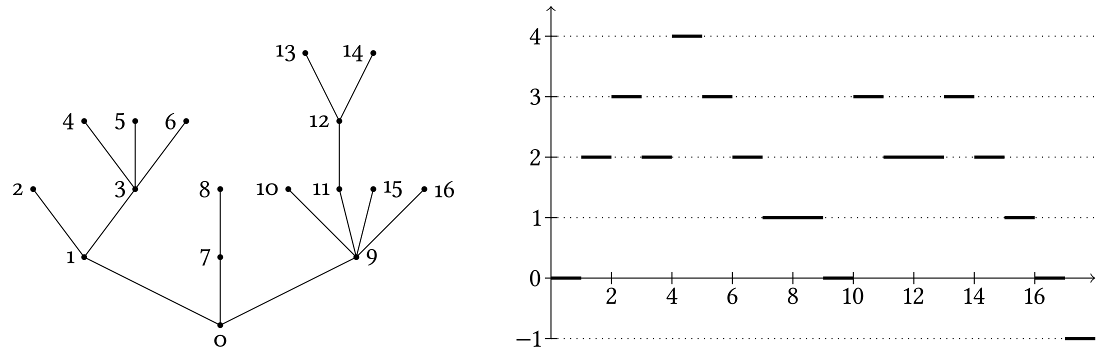


Figure 1.6: A plane tree and its Łukasiewicz path.

path as the step function on $[0, n]$ given by $s \mapsto W_{\lfloor s \rfloor}$. The next proposition describes how to reconstruct a plane tree from its Łukasiewicz path.

Proposition 1.4. *Let W be the Łukasiewicz path of τ. Fix $0 \le j \le n-1$ such that $k := W_{j+1} - W_j + 1 \ge 1$. Let $s_1, \dots, s_k \in \{1, \dots, n\}$ be defined by*

$$s_i = \inf\{\ell \ge j+1 : W_\ell = W_{j+1} - (i-1)\} \qquad \textit{for} \quad 1 \le i \le k.$$

Note that $s_1 = j+1$. The vertices $u(s_1), u(s_2), \dots, u(s_k)$ are the children of $u(j)$ listed in lexicographical order.

As a consequence, with the notations of the proposition, for every $j \in \{1, \dots, n\}$, the parent of $u(j)$ is $u(\ell)$ where $\ell = \sup\{m < j : W_m \le W_j\}$; furthermore $u(j)$ is the $W_{\ell+1} - W_j + 1$-st child of $u(\ell)$. We can then describe the ancestors of a vertex $u(j)$ by considering the set

$$\left\{ \ell \in \{0, \dots, j\} : W_\ell = \inf_{\ell \le m \le j} W_m \right\}.$$

The latter contains $|u(j)| + 1$ elements, say, $0 = a_0 < a_1 < \dots < a_{|u(j)|} = j$, and for every $\ell \in \{0, \dots, |u(j)| - 1\}$, $u(a_\ell)$ is the parent of $u(a_{\ell+1})$.

The Łukasiewicz path associated with a Galton–Watson tree with offspring distribution μ is a quite simple object. Indeed, define a probability measure on $\{-1, 0, 1, \dots\}$ by $\hat{\mu}(k) = \mu(k+1)$ for every $k \ge -1$; then if we sample a tree according to GW^μ, the associated Łukasiewicz path is distributed as a random walk on $\mathbb{Z}$ starting from 0 with step distribution $\hat{\mu}$ and stopped at the first hitting time of -1 (see e.g. Le Gall & Le Jan [75]). This allows to deduce properties of large Galton–Watson trees from that of random walks. However, if some properties of a tree can be easily obtained from its Łukasiewicz path, like the largest degree (which is given the highest jump plus one), some others are not, like the number of vertices at a given height. The latter is better expressed by the height process that we now describe.

The height process. Following Le Gall & Le Jan [75], we define a process $H = (H_j; 0 \leq j \leq n)$ by

$$H_j = |u(j)| \qquad \text{for every} \quad j \in \{0, \ldots, n\}.$$

As opposed to the Łukasiewicz path, we shall think of H as a continuous function on $[0, n]$, obtained by linear interpolation: $s \mapsto (1-\{s\})H_{\lfloor s \rfloor} + \{s\}H_{\lfloor s \rfloor + 1}$, where $\{x\} = x - \lfloor x \rfloor$. Similarly to the Łukasiewicz path, one can recover the tree from its height process.

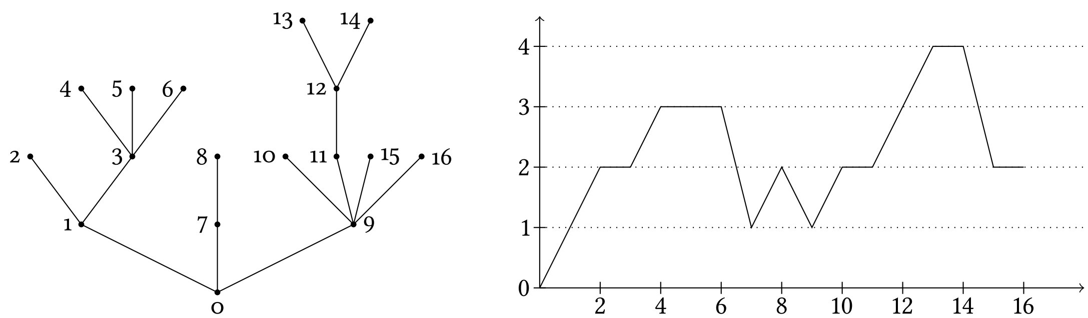


Figure 1.7: A plane tree and its height process.

Proposition 1.5 ([75]). *Let H be the height process of τ. Fix $0 \leq j \leq n-1$ such that $H_{j+1} > H_j$ and consider the set*

$$\left\{ \ell \geq j+1 : H_\ell = H_{j+1} = \min_{j+1 \leq m \leq \ell} H_m \right\}.$$

Denote by $k \geq 1$ its cardinal and by $j+1 = s_1 < \cdots < s_k$ its elements. Then the vertices $u(s_1), u(s_2), \ldots, u(s_k)$ are the children of $u(j)$ listed in lexicographical order.

The height process associated with a Galton–Watson tree is not Markovian in general. However, it is closely related to the Łukasiewicz path. Indeed, from the previous discussion, for every $j \in \{0, \ldots, n\}$, we have

$$H_j = \mathrm{Card} \left\{ \ell \in \{0, \ldots, j-1\} : W_\ell = \inf_{\ell \leq m \leq j} W_m \right\}, \tag{1.3}$$

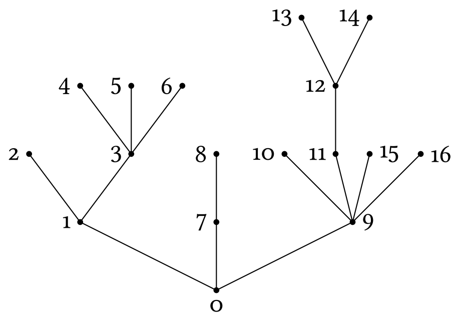
since the set on the right describes the ancestors of $u(j)$, including the root and excluding $u(j)$ itself.

We will use the Łukasiewicz path and the height process to define non-crossing partitions and non-crossing trees in Chapters 6 & 7. Let us briefly mention a third coding of a plane tree by a path, since the latter is close to the procedure that we adopt in the next section for real trees.

The contour process. Define the contour sequence $(c_0, c_1, \dots, c_{2n})$ of τ as follows: $c_0 = \varnothing$ and for each $i \in \{0, \dots, 2n-1\}$, c_{i+1} is either the first child of c_i which does not appear in the sequence $(c_0, \dots, c_i)$, or the parent of c_i if all its children already appear in this sequence. We then define the contour process $C = (C_j; 0 \le j \le 2n)$ by

$$C_j = |c_j| \qquad \text{for every} \quad j \in \{0, \dots, 2n\}.$$

Again, we shall think of C as a continuous function on $[0, 2n]$, obtained by linear interpolation. Note that $C_j \ge 0$ for every $0 \le j \le 2n$ and $C_0 = C_{2n} = 0$.



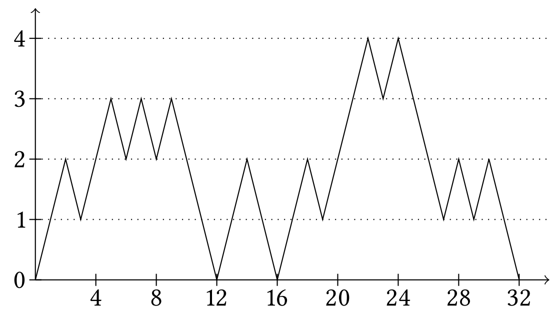


Figure 1.8: A plane tree and its contour process.

One can define this process in terms of the height process which shows the bijection between a tree and its contour process. Let us only give an intuitive idea of how to recover a tree from its contour process as a similar procedure will be used below for real trees. Consider the following equivalence relation on $\{0, \dots, 2n\}$: $i \sim j$ when $C_i = C_j = \min_{i \wedge j \le \ell \le i \vee j} C_\ell$. There are $n+1$ equivalence classes; we merge two points (i, C_i) and (j, C_j) whenever $i \sim j$ to form the vertices of the tree, the line segments of C then merge to form the edges.

1.3 Gromov–Hausdorff–Prokhorov topology and real trees

1.3.1 Real trees

Definition 1.6. A real tree is a compact metric space $(\mathcal{T}, d)$ which satisfies the following two properties:

(i) For every $x, y \in \mathcal{T}$, there is a unique isometric map $\varphi_{x,y}$ from $[0, d(x, y)]$ into $\mathcal{T}$ such that $\varphi_{x,y}(0) = x$ and $\varphi_{x,y}(d(x, y)) = y$.

(ii) For every $x, y \in \mathcal{T}$ and every continuous injective map f from $[0, 1]$ into $\mathcal{T}$ such that $f(0) = x$ and $f(1) = y$, we have $f([0, 1]) = \varphi_{x,y}([0, d(x, y)])$.

Some authors suppose the space complete instead of compact; however in this work we will only consider compact real trees so we include the compactness in the definition. Real trees can also be defined as the path-connected compact (or complete) metric spaces $(\mathcal{T}, d)$ which fulfill the so-called four points inequality: for all $x_1, \ldots, x_4 \in \mathcal{T}$,

$$d(x_1, x_2) + d(x_3, x_4) \leq \max\{d(x_1, x_3) + d(x_2, x_4), d(x_1, x_4) + d(x_2, x_3)\};$$

see e.g. Evans [47] or Dress, Moulton and Terhalle [39].

The range of the mapping $\varphi_{x,y}$ above is the geodesic between x and y in $\mathcal{T}$ and is denoted by $[\![x, y]\!]$. We will say that a real tree $(\mathcal{T}, d)$ is rooted if there is a distinguished element $\rho \in \mathcal{T}$ called the root. In this case, similarly to the discrete setting, we can interpret $\mathcal{T}$ as a genealogical tree: for every $x, y \in \mathcal{T}$, we say that x is an ancestor of y when $x \in [\![\rho, y]\!]$; this defines a partial order on $\mathcal{T}$. The degree of a point $x \in \mathcal{T}$ is the (possibly infinite) number of connected components of the open set $\mathcal{T} \setminus \{x\}$. A point with degree one is called a leaf; we denote by $\mathrm{Lf}(\mathcal{T})$ the set of leaves of $\mathcal{T}$ and by $\mathrm{Sk}(\mathcal{T}) := \mathcal{T} \setminus \mathrm{Lf}(\mathcal{T})$ its skeleton. Note that the closure of the skeleton is the whole tree.

An easy way to construct a real tree (and a similar procedure shall appear later for laminations) is from an "excursion-type" function. Let $g : [0, \infty) \to [0, \infty)$ be a continuous and compactly supported function such that $g(0) = 0$. We define a pseudo-distance on $[0, \infty)$ by setting

$$d_g(s, t) = g(s) + g(t) - 2 \min_{r \in [s \wedge t, s \vee t]} g(r)$$

for every $s, t \in [0, \infty)$. Define then an equivalence relation on $[0, \infty)$ by setting $s \overset{g}{\sim} t$ if and only if $d_g(s, t) = 0$ or, equivalently, $g(s) = g(t) = \min_{[s \wedge t, s \vee t]} g$. Consider the quotient space

$$\mathcal{T}_g = [0, \infty) / \overset{g}{\sim}$$

equipped with the distance induced by d_g; we keep the notation d_g for simplicity. Denote by $p_g : [0, \infty) \to \mathcal{T}_g$ the canonical projection. Since g is continuous then p_g is continuous as well from $[0, \infty)$ equipped with the Euclidean distance to $(\mathcal{T}_g, d_g)$. In particular, the latter is a compact and connected metric space.

Theorem 1.7 ([45]). *The metric space $(\mathcal{T}_g, d_g)$ is a real tree, which can be naturally rooted at $\rho = p_g(0)$. Conversely, any rooted real tree can be represented in such a form $\mathcal{T}_g$.*

Observe that any finite plane tree can be seen as a (compact) metric space, when endowed with the graph distance. Furthermore, it can be turned into a real tree, by replacing each edge by a line segment of unit length; the associated function g is the linear interpolation of the contour process. See Haas & Miermont [58, Section 3.2.1] for a detailed discussion on "turning discrete trees into real trees".

In addition to the structure of metric space, we will need to consider measures on trees. When needed, we will equip the tree $(\mathcal{T}, d)$ with a Borel finite "mass" measure μ. When $\mathcal{T} = \mathcal{T}_g$, we consider the push-forward of the Lebesgue measure Leb on the support of g by p_g: $\mu = \mu_g = p_g \star \mathrm{Leb} := \mathrm{Leb}(p_g^{-1}(\cdot))$. In the sequel, we shall always consider functions g supported by $[0,1]$, so that μ_g is a probability measure. A triple $(\mathcal{T}, d, \mu)$ is a particular case of compact metric measured space. In order to consider random such objects and their weak convergence, we next recall how to endow them with Polish topologies.

1.3.2 Gromov–Hausdorff–Prokhorov topology

Let (E, δ) be a Polish space and denote by $C(E)$ and $K(E)$ respectively the sets of closed sets and of compact sets of E. We recall the Hausdorff distance on $K(E)$:

$$\delta_{\mathrm{H}}^E(A, B) = \inf\{\varepsilon > 0 : A \subset B^\varepsilon \text{ and } B \subset A^\varepsilon\} \quad \text{for every} \quad A, B \in K(E),$$

where $A^\varepsilon = \{x \in E : \delta(x, A) < \varepsilon\}$ is the ε-enlargement of A. Recall that if (E, δ) is compact, then so is $(K(E), \delta_{\mathrm{H}}^E)$, see e.g. Burago, Burago & Ivanov [26, Theorem 7.3.8]. Let $\mathcal{M}_1(E)$ denote the set of all Borel probability measures on E. Recall also the definition of the Prokhorov metric: for every $\mu, \nu \in \mathcal{M}_1(E)$,

$$\delta_{\mathrm{P}}^E(\mu, \nu) = \inf\{\varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \text{ for any } A \in C(E)\}.$$

It is well-known (see e.g. Billingsley [21]) that $(\mathcal{M}_1(E), \delta_{\mathrm{P}}^E)$ is a Polish space, and that the topology generated by δ_{P}^E is that of weak convergence.

In order to compare two compact metric spaces, we will use the Gromov–Hausdorff distance, which has been introduced by Gromov (see e.g. [57]): for every compact metric spaces (X, d) and (X', d'), we set

$$d_{\mathrm{GH}}(X, X') = \inf\{\delta_{\mathrm{H}}^E(\phi(X), \phi'(X'))\}$$

where the infimum is taken over all possible choices of a metric space (E, δ) and isometric embeddings $\phi : X \to E$ and $\phi' : X' \to E$. Similarly, if X and X' are equipped with Borel probability measures μ and μ' respectively, we consider their Gromov–Prokhorov distance: set

$$d_{\mathrm{GP}}(X, X') = \inf\{\delta_{\mathrm{P}}^E(\phi \star \mu, \phi' \star \mu')\}$$

where again the infimum is taken over all possible choices of a metric space (E, δ) and all isometric embeddings $\phi : X \to E$ and $\phi' : X' \to E$, and $\phi \star \mu$, $\phi' \star \mu'$ denote the push-forward of μ, μ' by ϕ, ϕ' respectively. Finally, we can consider both the metric and the measure with the Gromov–Hausdorff–Prokhorov distance:

$$d_{\mathrm{GHP}}(X, X') = \inf\{\delta_{\mathrm{H}}^{E}(\phi(X), \phi'(X')) \vee \delta_{\mathrm{P}}^{E}(\phi \star \mu, \phi' \star \mu')\},$$

where the infimum is as above.

For every $a > 0$ and every compact metric measured spaces (X, d, μ) and (X', d', μ'), we denote by aX the space (X, ad, μ) and similarly aX' the space (X', ad', μ'). Observe that $d_{\mathrm{GH}}(aX, aX') = ad_{\mathrm{GH}}(X, X')$ but in general $d_{\mathrm{GP}}(aX, aX')$ differs from $ad_{\mathrm{GP}}(X, X')$ and $d_{\mathrm{GHP}}(aX, aX')$ from $ad_{\mathrm{GHP}}(X, X')$ as only the distances are scaled and not the measures. Nonetheless, when τ is a discrete tree and $\mathfrak{T}$ is the real tree obtained by replacing the edges of τ by line segments of unit length, we have by the following bound:

$$d_{\mathrm{GHP}}(a\tau, a\mathfrak{T}) \leq a \qquad \text{for every} \quad a > 0. \tag{1.4}$$

Note that the three functions d_{GH}, d_{GP} and d_{GHP} are only pseudo-distances; two compact metric spaces (X, d) and (X', d') are called equivalent if there is an isometry that maps X onto X'; (X, d, μ) and (X', d', μ') are called equivalent if in addition the push-forward of μ by the isometry is μ'. We shall always implicitly identify two equivalent spaces. We denote by $\mathscr{M}$ the set of equivalence classes of compact metric spaces, and by $\mathscr{M}_{\mathrm{w}}$ the set of equivalence classes of compact metric measured spaces.

Theorem 1.8 ([49], [56], [88])**.** *The spaces $(\mathscr{M}, d_{\mathrm{GH}})$, $(\mathscr{M}_{\mathrm{w}}, d_{\mathrm{GP}})$ and $(\mathscr{M}_{\mathrm{w}}, d_{\mathrm{GHP}})$ are separable and complete metric spaces.*

Denote next by $\mathscr{T}$ the set of all equivalence classes of real trees and by $\mathscr{T}_{\mathrm{w}}$ the set of equivalence classes of measured real trees.

Theorem 1.9 ([48], [49])**.** *The space $\mathscr{T}$ is a closed subspace of $(\mathscr{M}, d_{\mathrm{GH}})$; the space $\mathscr{T}_{\mathrm{w}}$ is a closed subspace of both $(\mathscr{M}_{\mathrm{w}}, d_{\mathrm{GP}})$ and $(\mathscr{M}_{\mathrm{w}}, d_{\mathrm{GHP}})$.*

Remark 1.10. We will be considering pointed spaces, i.e. with one or several distinguished elements (such as the root of a real tree). The distances can then be adapted to complete and separable metrics on pointed spaces. As an example, if (X, d, μ) and (X', d', μ') are a metric spaces and $x_1, \dots, x_k \in X$ and $x'_1, \dots, x'_k \in X'$ are k distinguished elements, we define the k-pointed Gromov–Hausdorff–Prokhorov distance between X and X' as the infimum of the quantities

$$\delta_{\mathrm{H}}^{E}(\phi(X), \phi'(X')) \vee \delta_{\mathrm{P}}^{E}(\phi \star \mu, \phi' \star \mu') \vee \max_{1 \leq i \leq k} \delta(\phi(x_i), \phi'(x'_i))$$

over all possible choices of a metric space (E, δ) and all isometric embeddings $\phi : X \to E$ and $\phi' : X' \to E$.

Let us give a simple characterization of the convergence in the sense of the (pointed) Gromov–Prokhorov topology, see e.g. Löhr [78]. For each $n \in \mathbb{N} \cup \{\infty\}$, consider a k-pointed compact measured metric space $(X_n, \{x_{n,1}, \dots, x_{n,k}\}, d_n, \mu_n)$, set $\chi_n(i) = x_{n,i}$ for each $i = 1, \dots, k$ and let $(\chi_n(i); i \geq k+1)$ be i.i.d. random variables sampled according to μ_n. By Gromov's reconstruction theorem [57, Chapter $3\frac{3}{2}$], the distribution of $(d_n(\chi_n(i), \chi_n(j)); i, j \geq 1)$ characterizes X_n.

Proposition 1.11 ([78]). *The convergence $X_n \to X_\infty$ as $n \to \infty$ for the k-pointed Gromov–Prokhorov topology is equivalent to the convergence in distribution of the matrices*

$$(d_n(\chi_n(i), \chi_n(j)); 1 \leq i, j \leq \ell) \quad \xrightarrow[n\to\infty]{(d)} \quad (d_\infty(\chi_\infty(i), \chi_\infty(j)); 1 \leq i, j \leq \ell)$$

for every integer $\ell \geq 1$ fixed.

Let us end this section with a few words on random real trees. In view of Theorem 1.7, one is tempted to define a random real tree as the tree $\mathcal{T}_g$ coded by a random function g. The next lemma due to Duquesne & Le Gall shows that this procedure indeed defines a random variable, i.e. that the map $g \mapsto \mathcal{T}_g$ is measurable.

Lemma 1.12 ([45]). *Let $g, g' : [0, \infty) \to [0, \infty)$ be two continuous functions with compact support such that $g(0) = g'(0) = 0$. Then,*

$$d_{\mathrm{GH}}(\mathcal{T}_g, \mathcal{T}_{g'}) \leq 2\|g - g'\|,$$

where $\|\cdot\| : g \mapsto \sup_{x\in[0,\infty)} |g(x)|$ is the uniform norm.

The proof in [45] easily follows from the representation of the Gromov–Hausdorff distance in terms of distortions, see e.g. Burago, Burago & Ivanov [26] for this alternative representation. Note finally that if $g_n \to g$ for the uniform topology, then $\mathcal{T}_{g_n} \to \mathcal{T}_g$ for the Gromov–Hausdorff–Prokhorov topology.

1.4 The Brownian tree

Let $B^{\mathrm{ex}} = (B^{\mathrm{ex}}(s); 0 \leq s \leq 1)$ be the standard Brownian excursion with duration 1. A possible definition (see Section 3.2 below for a more detailed construction) is the following: denote by

$B = (B(s); s \geq 0)$ a standard Brownian motion and consider $g_1 := \sup\{s < 1; B(s) = 0\}$ and $d_1 := \inf\{s > 1; B(s) = 0\}$. Note that $d_1 > g_1$ almost surely, we then set

$$B^{\text{ex}}(s) = \frac{1}{\sqrt{d_1 - g_1}} |B(g_1 + (d_1 - g_1)s)| \qquad \text{for} \quad s \in [0, 1].$$

The process B^{ex} takes values in the Polish space $\mathcal{C}([0,1], \mathbb{R})$ of real-valued continuous functions on $[0,1]$ equipped with the uniform distance. Theorem 1.7 and Lemma 1.12 justify the next definition.

Definition 1.13. The Brownian continuum random tree (CRT for short) is the random real tree $\mathcal{T}_{2B^{\text{ex}}}$ coded by twice the standard Brownian excursion.

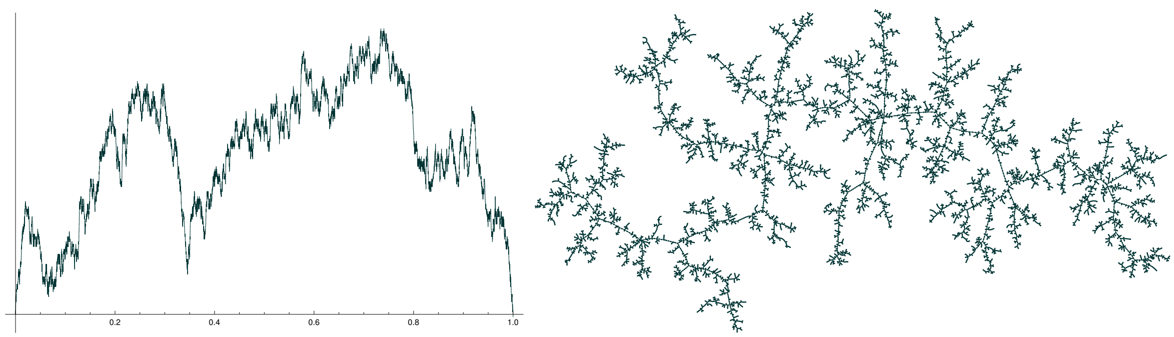


Figure 1.9: Simulations of B^{ex} and $\mathcal{T}_{2B^{\text{ex}}}$.

Aldous [3, 4, 5] introduced the CRT, first as a compact subset of the space $\ell^1(\mathbb{R})$ of real-valued summable sequences, and then as above. As proved by Theorem 23 in [5], the CRT arises naturally as limit of large Galton–Watson trees. Recall the notation $C(\tau)$ for the contour process of a plane tree τ and GW_n^μ for the law of a Galton–Watson tree with offspring distribution μ conditioned to have n edges.

Theorem 1.14 ([5]). *Let μ be a critical probability measure on $\mathbb{Z}_+$ with variance $\sigma^2 \in (0, \infty)$. For every $n \geq 1$ for which GW_n^μ is well defined, sample a tree τ_n according to GW_n^μ. Then the convergence in distribution*

$$\left(\frac{\sigma}{\sqrt{n}} C_{2ns}(\tau_n); s \in [0,1] \right) \quad \xrightarrow[n\to\infty]{(d)} \quad (2B^{\text{ex}}(s); s \in [0,1])$$

holds in $\mathcal{C}([0,1], \mathbb{R})$.

It then follows from Lemma 1.12 that, under the above assumptions,

$$\frac{\sigma}{\sqrt{n}} \tau_n \quad \xrightarrow[n\to\infty]{\text{(d)}} \quad \mathcal{T}_{2B^{\text{ex}}} \tag{1.5}$$

for the Gromov–Hausdorff–Prokhorov topology, where τ_n is equipped with the graph distance and the uniform probability measure on the set of vertices. Here we implicitly approximate the discrete tree τ_n by the real tree $\mathcal{T}_n$ obtained by replacing the edges by line segments of unit length. This is plainly justified since, according to (1.4), the Gromov–Hausdorff–Prokhorov distance between these rescaled trees is bounded from above by $\sigma/\sqrt{n}$. We will use the convergence (1.5) in Chapter 4 when μ is the Poisson distribution with parameter 1 (recall from Example 1.2 that τ_n is then a uniform rooted Cayley tree). We shall recall in Chapter 3 an important extension of Theorem 1.14 due to Duquesne [43].

Markov branching trees and the CRT Aldous [5] observed that the mass measure μ on the CRT is diffuse and supported by the set of leaves: $\mu(\mathrm{Sk}(\mathcal{T}_{2B^{\mathrm{ex}}})) = 0$. The CRT indeed also appears as the limit for the Gromov–Hausdorff–Prokhorov topology of trees equipped with the uniform probability distribution on leaves. As an example, Haas & Miermont [59] studied the behavior of *Markov branching trees*. The latter is a sequence of random trees $(T_n; n \geq 1)$ where T_n has n leaves, and which satisfy the following Markov branching property: conditional on the event that the root of T_n has k children-trees with $\ell_1, \dots, \ell_k$ leaves respectively, the tree T_n is distributed as that obtained by gluing on a common root k independent trees distributed respectively as $T_{\ell_1}, \dots, T_{\ell_k}$. The distributions of such trees T_n are entirely characterized by the probabilities of the previous events and under some assumption on these splitting probabilities, Haas & Miermont [59] (see also Rizzolo [103]) obtained the convergence of the trees T_n, properly rescaled and equipped with the uniform probability distribution on leaves to so-called fragmentation trees, that they introduced previously in [58].

In the next Chapter, we recall the definition of the cut-tree $\mathrm{Cut}(t)$ of a tree t, and we will see that if $(t_n; n \geq 1)$ is a sequence of Cayley trees of size n respectively, then the sequence $(\mathrm{Cut}(t_n); n \geq 1)$ fulfills the Markov branching property (which is a direct consequence of the splitting property previously described) as well as the assumptions of [59] so that $\mathrm{Cut}(t_n)$ converges to a fragmentation tree which is, in this case, the Brownian CRT. When each tree t_n is a random recursive tree of size n, the sequence $(\mathrm{Cut}(t_n); n \geq 1)$ fulfills the Markov branching property but not the technical assumption of [59]; the convergence follows from other considerations and indeed, the limit — the interval $[0, 1]$ — is not a fragmentation tree in the sense of [58].

2

Fires on large random trees

In this chapter, we study the model of random fire dynamics presented in Section 1.1 on general random trees, based on the work [82]. For every integer $n \geq 1$, we consider a random tree T_n with n labelled (for convenience) vertices, say, $[n] = \{1, \dots, n\}$ and a fire rate $p_n \in [0, 1]$. In the first section below, we state and prove a rigorous phase transition phenomenon, and express the critical regime — for which the proportion of burnt and fireproof vertices of T_n has a non-trivial limit — in terms of the law of T_n. We then state and prove the joint convergence of the sizes of the burnt subtrees of T_n in the critical regime. In Sections 2.2 and 2.3, we then present the main results developed in Chapters 4 and 5 respectively, where this dynamics is studied in more details for Cayley trees and random recursive trees.

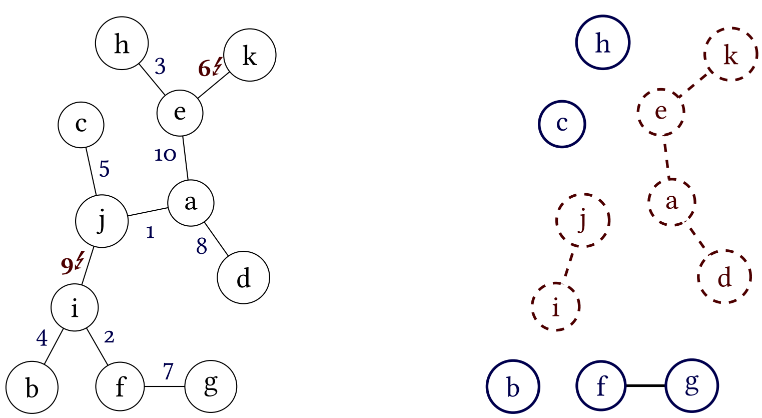


Figure 2.1: Given a tree and an enumeration of its edges on the left, if the edges set on fire are the 6th and the 9th, we get the two forests on the right where the burnt components are drawn with dotted lines and the fireproof ones with plain lines.

2.1 General results

Let us first recall the definition of the cut-tree introduced by Bertoin [16]. We associate with T_n a random rooted binary tree $\mathrm{Cut}(\mathrm{T}_n)$ with n leaves which records the genealogy induced by the fragmentation of T_n: each vertex of $\mathrm{Cut}(\mathrm{T}_n)$ corresponds to a subset (or block) of $[n]$, the root of $\mathrm{Cut}(\mathrm{T}_n)$ is the entire set $[n]$ and its leaves are the singletons $\{1\}, \dots, \{n\}$. We remove successively the edges of T_n in a uniform random order; at each step, a subtree of T_n with set of vertices, say, V, splits into two subtrees with sets of vertices, say, V' and V'' respectively; in $\mathrm{Cut}(\mathrm{T}_n)$, V' and V'' are the two offsprings of V. Notice that, by construction, the set of leaves of the subtree of $\mathrm{Cut}(\mathrm{T}_n)$ that stems from some block coincides with this block. See Figure 2.2 below for an illustration.

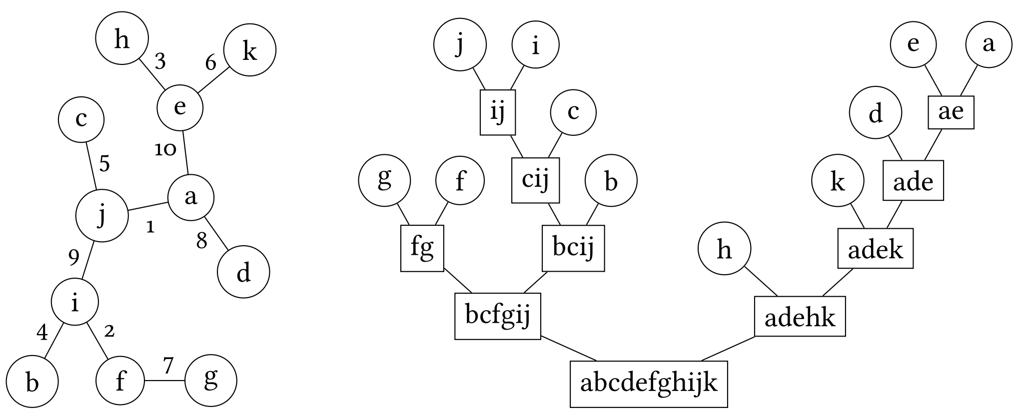


Figure 2.2: A tree with the order of cuts on the left and the corresponding cut-tree on the right. The order of the children in the cut-tree is irrelevant and has been chosen for aesthetic reasons only.

As alluded in Chapter 1, the cut-tree is an interesting tool to study the isolation of nodes in a tree. As an example, sample k vertices uniformly at random with replacement in the tree T_n, say, $U_1, \dots, U_k$. Then independently, start the fragmentation of the tree as described above but in addition, each time the removal of an edge creates a connected component which does not contain any of the k selected vertices, discard immediately the latter component. This dynamics stop when the tree is reduced to the k selected singletons. It should be plain that the subtrees which are not discarded correspond to the blocks of the tree $\mathrm{Cut}(\mathrm{T}_n)$ reduced to its branches from its root to the k leaves $\{U_1\}, \dots, \{U_k\}$. As a consequence, the number of steps of this isolation procedure is given by the number of internal nodes of this reduced tree or, equivalently, its length minus the number of distinct leaves plus one; in the case $k = 1$, the latter is simply the height of a uniform random leaf in $\mathrm{Cut}(\mathrm{T}_n)$.

We endow $\mathrm{Cut}(\mathrm{T}_n)$ with a mark process φ_n: at each generation, each internal (i.e. non-singleton) block is marked independently of the others with probability p_n provided that none of its ancestors has been marked, and not marked otherwise. This is equivalent to the following two-steps procedure: mark first every internal block independently with probability p_n, then along each branch from the root to a leaf, keep only the closest mark to the root and erase the other marks. Throughout this work, "marks on $\mathrm{Cut}(\mathrm{T}_n)$" shall always refer to the marks induced by φ_n.

Consider the fire dynamics on T_n; if we remove each edge as soon as it is fireproof, then, when an edge is set on fire, it immediately burns the whole subtree which contains it. Observe that the only information which is lost with this point of view is the geometry of the fireproof forest. We can couple this dynamics on T_n and the cut-tree $\mathrm{Cut}(\mathrm{T}_n)$ endowed with the marks induced by φ_n in such a way that the marked blocks of $\mathrm{Cut}(\mathrm{T}_n)$ correspond to the burnt subtrees of T_n and the leaves of $\mathrm{Cut}(\mathrm{T}_n)$ which do not possess a marked ancestor correspond to the fireproof vertices of T_n, see Figure 2.3 for an illustration. We implicitly assume in the sequel that the fire dynamics on T_n and the pair $(\mathrm{Cut}(\mathrm{T}_n), \varphi_n)$ are coupled in this way.

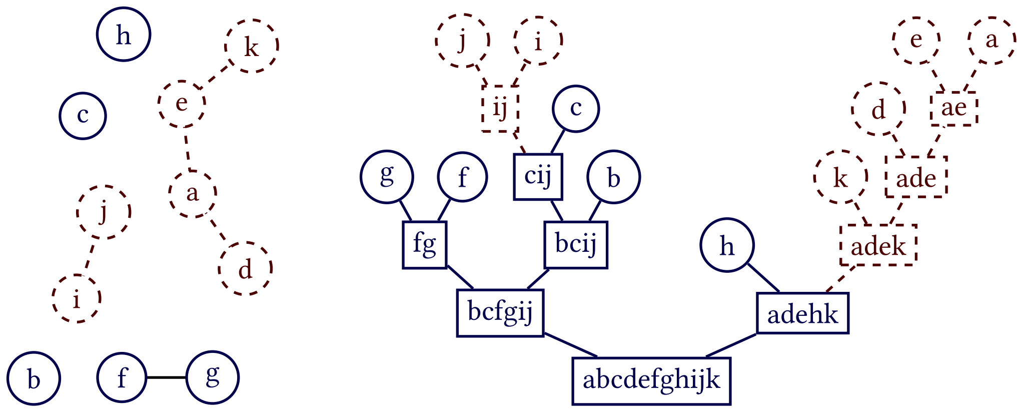


Figure 2.3: The forest after the dynamics and the corresponding marked cut-tree. The burnt components are drawn with dotted lines and the fireproof ones with plain lines

Fix $k \in \mathbb{N}$ and let $R_{n,k}$ be the tree $\mathrm{Cut}(\mathrm{T}_n)$ reduced to its branches from its root to k leaves chosen uniformly at random with replacement; denote by $L_{n,k}$ the length of $R_{n,k}$. Let $r : \mathbb{N} \to \mathbb{R}$ be some function such that $\lim_{n\to\infty} r(n) = \infty$. We introduce the following hypothesis:

$$r(n)^{-1} L_{n,k} \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad L_k, \tag{H_k}$$

where L_k is a non-negative and finite random variable. The arguments developed by Bertoin [16, 17] allow us to derive the following result.

Proposition 2.1. *Denote by I_n the number of fireproof vertices in T_n.*

(i) If (H_1) holds and $\lim_{n\to\infty} r(n)p_n = 0$, then $n^{-1}I_n$ converges in probability to 1.

(ii) If (H_1) holds and $\lim_{n\to\infty} r(n)p_n = \infty$, then for every $\varepsilon > 0$, we have $\limsup_{n\to\infty} \mathbb{P}(I_n > \varepsilon n) \leq \varepsilon^{-1}\mathbb{P}(L_1 = 0)$. In particular, if $L_1 > 0$ almost surely, then $n^{-1}I_n$ converges in probability to 0.

(iii) If (H_k) holds for every $k \in \mathbb{N}$ and $\lim_{n\to\infty} r(n)p_n = c$ with $c \in (0,\infty)$ fixed, then we have the convergence in distribution

$$n^{-1}I_n \quad \underset{n\to\infty}{\overset{(d)}{\longrightarrow}} \quad D(c),$$

where the law of $D(c)$ is characterized by its entire moments: for every $k \geq 1$,

$$\mathbb{E}[D(c)^k] = \mathbb{E}[\exp(-cL_k)]. \tag{2.1}$$

Note that $D(c) \not\equiv 0$ and also $D(c) \equiv 1$ if and only if $L_k \equiv 0$.

Proof. Fix $k \in \mathbb{N}$; the variable $n^{-1}I_n$ represents the proportion of fireproof vertices in T_n, therefore its k-th moment is the probability that k vertices of T_n chosen uniformly at random with replacement and independently of the dynamics are fireproof. Using the coupling with $\mathrm{Cut}(\mathrm{T}_n)$, the latter is the probability that there is no atom of the point process φ_n on $R_{n,k}$. We thus have

$$\mathbb{E}[(n^{-1}I_n)^k] = \mathbb{E}[(1-p_n)^{X_{n,k}}],$$

where $X_{n,k}$ denotes the number of internal nodes of $R_{n,k}$. Observe that $L_{n,k} - X_{n,k}$ is equal to the number of distinct leaves of $R_{n,k}$ minus one, which is bounded by $k-1$, then (H_k) yields

$$\lim_{n\to\infty} \mathbb{E}[(n^{-1}I_n)^k] = \mathbb{E}[\exp(-cL_k)] \quad \text{when} \quad r(n)p_n \to c \in [0,\infty),$$

as well as

$$\limsup_{n\to\infty} \mathbb{E}[(n^{-1}I_n)^k] \leq \mathbb{P}(L_k = 0) \quad \text{when} \quad r(n)p_n \to \infty,$$

and the three assertions follow. □

The next proposition offers a reciprocal to Proposition 2.1, which shows that (H_k), $k \in \mathbb{N}$, form a necessary and sufficient condition for the critical case (iii). Observe that, if $D(c)$ is defined as in (2.1) for every $c \in (0,\infty)$, then $\lim_{c\to 0+} D(c) = 1$ in probability.

Proposition 2.2. *Suppose that for every $c \in (0,\infty)$, if $\lim_{n\to\infty} r(n)p_n = c$, then $n^{-1}I_n$ converges in distribution as $n \to \infty$ to a limit $D(c)$ which satisfies and $\lim_{c\to 0+} D(c) = 1$ in probability. Then (H_k) is fulfilled for every $k \in \mathbb{N}$ and the Laplace transform of L_k is given by (2.1).*

Proof. Fix $k \in \mathbb{N}$. For every $c \in (0, \infty)$, the convergences $n^{-1} I_n \to D(c)$ in distribution and $r(n)p_n \to c$ imply that

$$\mathbb{E}[D(c)^k] = \lim_{n\to\infty} \mathbb{E}[(n^{-1} I_n)^k] = \lim_{n\to\infty} \mathbb{E}[(1-p_n)^{L_{n,k}}] = \lim_{n\to\infty} \mathbb{E}[\exp(-cr(n)^{-1} L_{n,k})].$$

Moreover, the assumption $\lim_{c\to 0+} D(c) = 1$ in probability implies that $\mathbb{E}[D(c)^k]$ converges to 1 as $c \to 0+$. We conclude from a classical theorem, see e.g. Feller [50, Theorem XIII.1.2] that the function $c \mapsto \mathbb{E}[D(c)^k]$ is the Laplace transform of some random variable $L_k \geq 0$ and $r(n)^{-1} L_{n,k}$ converges in distribution to L_k. □

For the critical case $p_n \sim c/r(n)$, under a stronger assumption, we obtain the joint convergence in distribution of I_n and the sizes of the burnt components, all rescaled by n. To this end, let $(\mathcal{T}, d, \rho, \mu)$ be a random (compact) rooted real tree, equipped with it mass-measure μ (recall the definitions in Section 1.3). For each integer k, denote by $\mathcal{R}(\mathcal{T}, k)$ the tree $\mathcal{T}$ spanned by its root and k i.i.d. elements chosen according to μ:

$$\mathcal{R}(\mathcal{T}, k) = \bigcup_{i=1}^{k} [\![\rho, \chi_i]\!] \quad \text{where} \quad (\chi_1, \ldots, \chi_k) \text{ are i.i.d. with law } \mu. \tag{2.2}$$

Observe that $R_{n,k} = \mathcal{R}(\mathrm{Cut}(\mathrm{T}_n), k)$ if $\mathrm{Cut}(\mathrm{T}_n)$ is viewed as a real tree, where edges are replaced by line segments of unit length, rooted at $[n]$ and equipped with the uniform distribution on leaves. Let $r : \mathbb{N} \to \mathbb{R}$ be some function such that $\lim_{n\to\infty} r(n) = \infty$. We introduce the following hypothesis:

$$r(n)^{-1} R_{n,k} \xrightarrow[n\to\infty]{\text{(d)}} \mathcal{R}(\mathcal{T}, k). \tag{H'_k}$$

Recall from Proposition 1.11 that the fact that all the (H'_k), $k \in \mathbb{N}$, hold is equivalent to the convergence of $r(n)^{-1}\mathrm{Cut}(\mathrm{T}_n)$ to $\mathcal{T}$ for the pointed Gromov–Prokhorov topology. Indeed, the distance is a continuous function and the matrix of the distances as defined in Proposition 1.11 determines entirely the reduced tree. It will be however easier to work with the reduced trees.

The distance on $\mathcal{T}$ induces an extra length-measure ℓ, which is the unique σ-finite measure assigning measure $d(x, y)$ to the geodesic path between $[\![x, y]\!]$. We define on $\mathcal{T}$ a point process Φ_c analogous to φ_n on $\mathrm{Cut}(\mathrm{T}_n)$: first sample a Poisson point process with intensity $c\ell(\cdot)$, then, along each branch from the root to a leaf, keep only the closest mark to the root (if any) and erase the other marks. The process Φ_c induces a partition of $\mathcal{T}$ in which two elements $x, y \in \mathcal{T}$ are connected if and only if there is no atom on the geodesic with extremities x and y. Denote by $\#(\mathcal{T}, \Phi_c)$ the sequence of the μ-mass of each connected component of $\mathcal{T}$ after logging at the atoms of Φ_c, the root-component first, and the next in non-increasing order.

Recall that I_n denotes the number of fireproof vertices in T_n and let $b^*_{n,1} \geq b^*_{n,2} \geq \cdots \geq 0$ be the sizes of the burnt subtrees, ranked in non-increasing order.

Proposition 2.3. *If* (H'_k) *holds for every* $k \in \mathbb{N}$*, then in the regime* $p_n \sim c/r(n)$*, the convergence*

$$n^{-1}(I_n, b^*_{n,1}, b^*_{n,2}, \ldots) \quad \xrightarrow[n\to\infty]{(d)} \quad \#(\mathcal{T}, \Phi_c)$$

holds in distribution for the ℓ^1 *topology.*

Remark that (H'_k) implies (H_k) where L_k is the total length of $\mathcal{R}(\mathcal{T}, k)$; further the first element of $\#(\mathcal{T}, \Phi_c)$ is the variable $D(c)$ of Proposition 2.1 and we can interpret identity (2.1) using $\mathcal{T}$. Sample $U_1, \ldots, U_k \in \mathcal{T}$ independently according to μ and denote by $\mathcal{R}(\mathcal{T}, k)$ the associated reduced tree. Denote also by C_c the root-component of $\mathcal{T}$ after logging at the atoms of Φ_c. Then $D(c) = \mu(C_c)$ and

$$\mathbb{E}[D(c)^k] = \mathbb{E}[\mu(C_c)^k] = \mathbb{P}(U_1, \ldots, U_k \in C_c).$$

Since $U_1, \ldots, U_k$ belong to C_c if and only if there is no atom of the Poisson random measure on $\mathcal{R}(\mathcal{T}, k)$, we also have

$$\mathbb{P}(U_1, \ldots, U_k \in C_c) = \mathbb{E}[\exp(-c\ell(\mathcal{R}(\mathcal{T}, k)))] = \mathbb{E}[\exp(-cL_k)].$$

The proof of Proposition 2.3 is essentially that of Lemma 2 in [81] which considers the case where T_n is a Cayley tree of size n, for which, as we will see, (H'_k) holds for every $k \in \mathbb{N}$ with $r(n) = \sqrt{n}$ and where $\mathcal{T}$ is the Brownian CRT defined in Section 1.4. The arguments follow closely Section 2.3 of Aldous & Pitman [8] who considered the logging of the CRT at all the atoms of a Poisson point process with intensity $c\ell$ (and not only Φ_c).

Proof. Consider the cut-tree $\mathrm{Cut}(\mathrm{T}_n)$ with the marks induced by φ_n and log it at the midpoint of each edge connecting a marked block to its parent. Let $\#(\mathrm{Cut}(\mathrm{T}_n), \varphi_n)$ be the vector whose entries count the number of leaves of each component, with the root-component first, and the next in the non-increasing order of their size. From the coupling introduced at the beginning of this section, we have

$$\#(\mathrm{Cut}(\mathrm{T}_n), \varphi_n) = (I_n, b^*_{n,1}, b^*_{n,2}, \ldots). \tag{2.3}$$

For each $k \geq 1$, we denote by $\#\mathcal{R}_n(k, \varphi_n)$ the vector whose entries count the number of leaves of each tree in the forest obtained by logging the reduced tree $R_{n,k}$ at the marks induced by φ_n, the root-component first, and the next in non-increasing order. Denote by $\#\mathcal{R}(k, \Phi_c)$

the similar quantity for the reduced tree $\mathcal{R}(\mathcal{T}, k)$ logged at the marks induced by Φ_c, where we count the number of vertices χ_i defined as in (2.2). Then it is easy to extend (H'_k) to the convergence of the reduced trees endowed with the point process of marks and it follows that

$$\#\mathcal{R}_n(k, \varphi_n) \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad \#\mathcal{R}(k, \Phi_c)$$

for every $k \in \mathbb{N}$. The law of large numbers entails that

$$k^{-1}\#\mathcal{R}(k, \Phi_c) \quad \xrightarrow[k\to\infty]{} \quad \#(\mathcal{T}, \Phi_c).$$

We can then build a sequence $k_n \to \infty$ sufficiently slowly as $n \to \infty$, so that

$$k_n^{-1}\#\mathcal{R}_n(k_n, \varphi_n) \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad \#(\mathcal{T}, \Phi_c),$$

from which we conclude that

$$n^{-1}\#(\mathrm{Cut}(\mathrm{T}_n), \varphi_n) \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad \#(\mathcal{T}, \Phi_c);$$

see e.g. Lemma 11 of Aldous & Pitman [8]. Appealing (2.3), this last convergence is the claim of Proposition 2.3. □

2.2 Fires on large Cayley trees

In Chapter 4, we study this fire dynamics in more details in the case where each T_n is a Cayley tree of size n. Bertoin [16] proved that the assumptions (H'_k) hold for all $k \geq 1$, where

$$r(n) = \sqrt{n} \quad \text{and} \quad \mathcal{T} \text{ is the Brownian CRT.}$$

This follows from the work of Haas & Miermont [59] discussed in Section 1.4. Indeed, the splitting property asserts that the removal of a uniform random edge of Cayley tree produces two subtrees which are, conditional on their size, independent Cayley trees. It follows that the sequence $(\mathrm{Cut}(\mathrm{T}_n); n \geq 1)$ fulfills the Markov branching property defined in Section 1.4. Bertoin [16, Lemma 1] proves then the technical assumption of Haas and Miermont [59] so that their limit theorem applies. Therefore Proposition 2.1 holds (see [16, Theorem 1]) with $r(n) = \sqrt{n}$ and the limit $D(c)$ in the critical regime is given by the mass of the root-component of the CRT after logging at the atoms of Φ_c or, equivalently, of an entire Poisson point process with rate c per unit length (this does not affect the root-component); the distribution of the latter is given explicitly by Aldous & Pitman [8]:

$$\mathbb{P}(D(c) \in \mathrm{d}x) = \frac{c}{\sqrt{2\pi x(1-x)^3}} \exp\left(-\frac{c^2 x}{2(1-x)}\right)\mathrm{d}x, \quad 0 < x < 1. \tag{2.4}$$

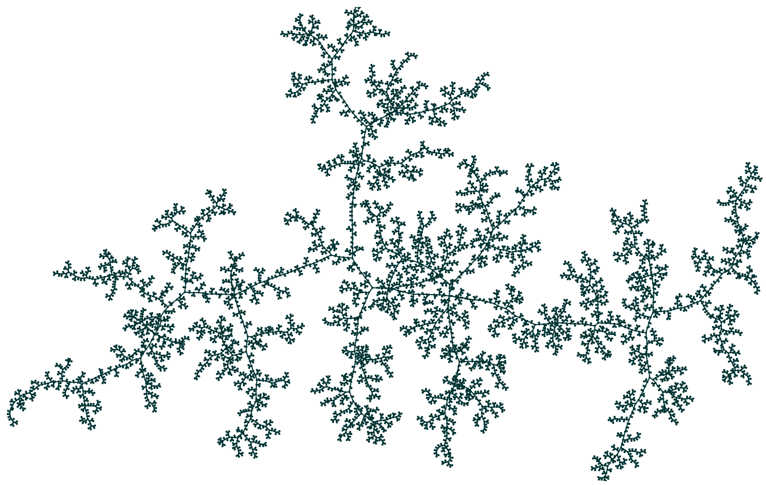


Figure 2.4: A sample of $\mathrm{Cut}(\mathrm{T}_{5\,000})$.

Asymptotic size of the burnt subtrees in the critical regime Our first result gives a representation of the limit $\#(\mathcal{T}, \Phi_c)$ of Proposition 2.3 in terms of the jumps made by a certain conditioned stable subordinator. More precisely, we consider $(\sigma(t); t \geq 0)$ the first-passage time process of a linear Brownian motion and $J_1 \geq J_2 \geq \cdots \geq 0$ the ranked sizes of its jumps made during the time interval $[0, 1]$. One can make sense of the conditional distribution of the sequence $(J_i)_{i \geq 1}$ given $\sigma(1) = z$ in the set $\ell^1(\mathbb{R})$ of real-valued summable sequences.

Theorem 2.4. *In the regime $p_n \sim c/\sqrt{n}$, we have for all continuous and bounded maps $f : (0, 1) \to \mathbb{R}$ and $F : \ell^1(\mathbb{R}) \to \mathbb{R}$:*

$$\lim_{n\to\infty} \mathbb{E}\left[f\left(\frac{I_n}{n}\right) F\left(\frac{b^*_{n,1}}{n}, \ldots, \frac{b^*_{n,\kappa_n}}{n}\right)\right] = \int_0^1 f(x) \mathbb{E}\left[F\left(\frac{(1-x)J_1}{\sigma(1)}, \frac{(1-x)J_2}{\sigma(1)}, \ldots\right) \,\middle|\, \sigma(1) = \frac{1-x}{c^2x^2}\right] \mathbb{P}(D(c) \in \mathrm{d}x),$$

where $\mathbb{P}(D(c) \in \mathrm{d}x)$ is defined in (2.4).

In order to establish this result, we approximate the CRT marked by Φ_c by Galton–Watson trees with offspring distribution Poisson with parameter 1, conditioned to have size n, and endowed with a point process similar to φ_n on $\mathrm{Cut}(\mathrm{T}_n)$. Recall indeed from Theorem 1.14 that such Galton–Watson trees converge in distribution, as $n \to \infty$ to the CRT; the joint convergence of the trees and the marks is similar to Proposition 2.3. Using both the properties of Galton–Watson trees and the Poisson distribution, we make explicit calculations on these discrete trees and pass to the limit as $n \to \infty$.

Asymptotic proportion of fireproof vertices in the subcritical regime It follows from Proposition 2.1 that, taking $p_n \gg 1/\sqrt{n}$, we have $I_n/n \to 0$ in probability as $n \to \infty$. We establish in this regime a non-trivial convergence of I_n.

Theorem 2.5. *In the regime $p_n \gg 1/\sqrt{n}$, we have*

$$p_n^2 I_n \quad \xrightarrow[n\to\infty]{(d)} \quad Z^2,$$

where Z is a standard Gaussian random variable.

We consider the time when the first fire occurs: at this instant, if we remove the edges previously fireproof, then we get a decomposition of the tree into a forest, we next pick an edge uniformly at random in this forest and burn the subtree which contains it. Note that the number of subtrees is geometrically distributed, with parameter p_n, and so is of order $1/p_n \ll \sqrt{n}$. The joint distribution of the sizes of these subtrees is known and Aldous & Pitman [8] have shown that there is one giant component — with size of order n — and the other components have size at most of order $1/p_n^2$. Since the subtree which is set on fire is chosen at random proportionally to its number of edges, the giant component burns with high probability; we are left with the forest of small trees and the dynamics continue independently on each. Observe that the largest such subtrees are now critical, we can then estimate the number of fireproof vertices in each at the end of the dynamics thanks to Proposition 2.1. Informally, $p_n^2 I_n$ converges in distribution to a random variable of the form $\sum_{k=1}^{\infty} D(c_k)$, where each c_k is random, and, conditional on the sequence $(c_k)_{k\geq 1}$, the variables $D(c_k)$ are independent and distributed as in (2.4). The distribution of the c_k's can be made explicit in order to conclude that $\sum_{k=1}^{\infty} D(c_k)$ is distributed as the square of a standard Gaussian random variable.

Asymptotic proportion of burnt vertices in the supercritical regime We finally consider the number $B_n = n - I_n$ of burnt vertices. As previously, from Proposition 2.1, we have $B_n/n \to 0$ in probability as $n \to \infty$ when $p_n \ll 1/\sqrt{n}$. We establish in this regime a non-trivial convergence of B_n.

Theorem 2.6. *In the regime $p_n \ll 1/\sqrt{n}$, we have*

$$(np_n)^{-2} B_n \quad \xrightarrow[n\to\infty]{(d)} \quad Z^{-2},$$

where Z is a standard Gaussian random variable.

We again consider the dynamics at the instant of the first fire, however now we have fireproof a number of order $1/p_n \gg 1/\sqrt{n}$ of edges and there is no giant component anymore. We show by induction that for every integer $j \geq 1$, the vector of the rescaled sizes of the first j burnt subtrees $(np_n)^{-2}(b_{n,1}, \dots, b_{n,j})$ converges in distribution as $n \to \infty$ to an explicit limit, say $(X_1, \dots, X_j)$. We then argue that for every $\varepsilon > 0$, one can fix $j_0 \geq 1$ such that for every n large enough, $(np_n)^{-2} \sum_{j=j_0}^{\infty} b_{n,j} \leq \varepsilon$ with a probability greater than $1 - \varepsilon$. We conclude that the sequence $((np_n)^{-2} b_{n,j})_{j\geq 1}$ converges in distribution to $(X_j)_{j\geq 1}$ for the ℓ^1 topology. It follows in particular that $(np_n)^{-2} B_n$ converges in distribution to $\sum_{j=1}^{\infty} X_j$ and we show that the latter is distributed as Z^{-2}.

2.3 Fires on large random recursive trees

We then study the case of random recursive trees in Chapter 5. In this setting, Bertoin [19] proved that the assumptions (H'_k) hold for all $k \geq 1$, where

$$r(n) = n/\ln n \quad \text{ and } \quad \mathcal{T} \text{ is the interval } [0,1],$$

equipped with the Euclidean distance and Lebesgue measure. It follows that Proposition 2.3 holds with $r(n) = n/\ln n$ and, moreover,

$$n^{-1}(I_n, b^*_{n,1}, b^*_{n,2}, \dots) \xrightarrow[n\to\infty]{(\mathrm{d})} (\mathbf{e}_c \wedge 1, 1 - (\mathbf{e}_c \wedge 1), 0, 0, \dots), \tag{2.5}$$

where $\mathbf{e}_c$ is an exponential random variable with rate c.

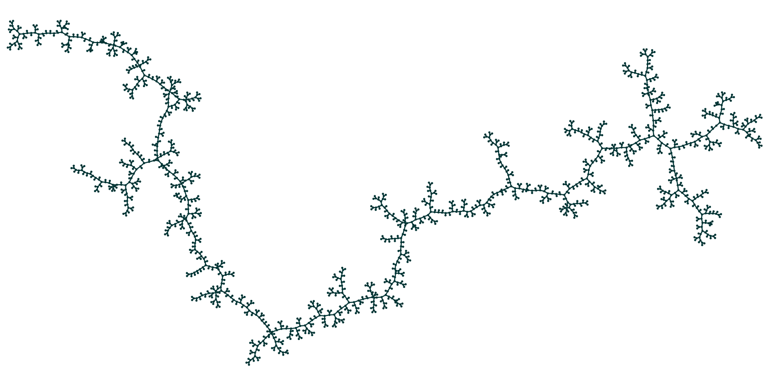


Figure 2.5: A sample of $\mathrm{Cut}(\mathrm{T}_{1\,000})$.

Density of fireproof vertices Our first result provides a finer limit theorem for the number of fireproof vertices in the subcritical regime $p_n \gg \ln n/n$.

Theorem 2.7. *In the regime $p_n \gg \ln n/n$, we have*

$$\frac{p_n}{\ln(1/p_n)} I_n \quad \xrightarrow[n\to\infty]{(d)} \quad \mathbf{e}_1,$$

where $\mathbf{e}_1$ is an exponential random variable with rate 1.

The proof relies on fine properties of the cut-tree of large random recursive trees, and in particular a decomposition into a trunk and bushes due to Bertoin [19]. As for Cayley trees, we consider fireproof edges of T_n as being deleted, but here, we first focus on the connected component which contains the root: as soon as a subtree of T_n gets disconnected from the root, we freeze it and consider the dynamics on it later. We show that with high probability, the root burns and the whole connected component which burns with it has size of order n. We then consider the frozen subtrees which were disconnected from the root. First, we are able to prove a limit theorem for the total size, say, S_n, of this forest:

$$\frac{p_n}{\ln(1/p_n)} S_n \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad \mathbf{e}_1.$$

This is essentially due to Iksanov and Möhle [62] who proved that the sizes of the subtrees disconnected from the root of T_n are given by the jumps of a certain increasing random walk. Then we prove that the proportion of fireproof vertices in this forest converges to 1 in probability. Recall from Section 2.1 that the proportion of fireproof vertices in a tree τ is distributed as $(1-p_n)^X$ where X is the height of a uniform random leaf of $\mathrm{Cut}(\tau)$. We show that such a quantity converges to 1 in probability uniformly over all the subtrees τ of T_n disconnected from the root when the latter burns; the claim then follows.

Connectivity properties of the fireproof forest We then focus on the connectivity of the fireproof forest and in particular, on the size of the largest fireproof connected component. In the case of Cayley trees, Bertoin [16] proved that the latter is of order n in the supercritical regime, but is small compared to n in the critical regime. For random recursive trees, we prove that with high probability as $n \to \infty$,

- In the supercritical regime, there exists a giant fireproof component of size $n - o(n)$;
- In the subcritical regime, the largest fireproof component has size of order p_n^{-1};
- In the critical regime, the largest fireproof component has size of order $p_n^{-1} \asymp n/\ln n$ if the root burns and $n - o(n)$ if it is fireproof.

These three results follow from the following one.

Theorem 2.8. *Let $c \in [0, \infty)$ and p_n such that $\lim_{n\to\infty} np_n / \ln n = c$. Let also X_n be a uniform random vertex in $[n]$ independent of T_n and the fire dynamics. Then the probability that X_n and 1 belong to the same fireproof subtree converges towards e^{-c} as $n \to \infty$.*

The proof relies on a so-called *spinal decomposition* of T_n. In order to have X_n and the root in the same fireproof subtree, all the edges along the path from the root to X_n must be fireproof; this occurs with probability $\mathbb{E}[(1 - p_n)^{h(X_n)}]$ where $h(X_n)$ is the height of X_n and it is well-known that $h(X_n) \sim \ln n \ll 1/p_n$ in probability so the previous expectation converges to 1 as $n \to \infty$. Further, if we remove all the edges of this path, then each of the $h(X_n) + 1$ subtrees contains one of the ancestors of X_n (including itself) and we also require that each such ancestor is fireproof for the dynamics restricted to the subtree which contains it. We describe the joint distribution of the sizes of these subtrees and prove that, conditional on their sizes, they are independent uniform random recursive trees. Since the root of each is the ancestor of X_n, we can then control the probability of the latter being fireproof.

On the sequence of burnt subtrees We finally study the sizes of the burnt subtrees, in order of appearance, in the critical regime $p_n \sim c \ln /n$. Note indeed from (2.5) that at most one is of order n, and the sum of all the others is negligible compared to n. One can show that the "giant" burnt component exists if and only if the root burns, and is the burnt component containing the root in this case.

For every $i \in \mathbb{N}$, denote by $b_{n,i}$ the size of the i-th burnt subtree of T_n. Let also $\gamma_0 = 0$ and $(\gamma_j - \gamma_{j-1})_{j\geq 1}$ be a sequence of i.i.d. exponential random variables with rate c and conditional on $(\gamma_j)_{j\geq 1}$, let $(Z_j)_{j\geq 1}$ be a sequence of independent random variables, where Z_j is distributed as an exponential random variable with rate γ_j conditioned to be smaller than 1.

Theorem 2.9. *For every $j \geq 1$, the probability that the root burns with the j-th fire converges to*

$$\mathbb{E}\left[\mathrm{e}^{-\gamma_j} \prod_{i=1}^{j-1} (1 - \mathrm{e}^{-\gamma_i})\right]$$

as $n \to \infty$. Moreover, on this event, for every $k \geq j + 1$, the vector

$$\left(\frac{\ln b_{n,1}}{\ln n}, \dots, \frac{\ln b_{n,j-1}}{\ln n}, \frac{b_{n,j}}{n}, \frac{\ln b_{n,j+1}}{\ln n}, \dots, \frac{\ln b_{n,k}}{\ln n}\right)$$

converges in distribution towards

$$(Z_1, \dots, Z_{j-1}, \mathrm{e}^{-\gamma_j}, Z_{j+1}, \dots, Z_k).$$

The proof consists of four main steps. We first consider the case of the root: we view the fire dynamics as a dynamical percolation in continuous-time (using exponential waiting times) where each fireproof edge is deleted and each burnt component is discarded. Then the results of Bertoin [18] allow us to derive for every $j \geq 1$ the probability that the root burns with the j-th fire, and the size of its burnt component.

In a second step, we investigate the size of the first burnt subtree, conditional on the event that it does not contain the root. Consider the first edge e which is set on fire. Since the root does not burn with e, there exists a fireproof edge on the path from the root to e; call $Z_{n,1}$ the closest extremity of such an edge to e and $T_{n,1}$ the subtree of T_n that stems from $Z_{n,1}$. Then $b_{n,1}$ is the size of the first burnt subtree of $T_{n,1}$ and the latter contains its root. Observe that conditionally given its size, $T_{n,1}$ is a random recursive tree. We first estimate the size of $T_{n,1}$ and then the size $b_{n,1}$ of its burnt component containing its root.

In the third step, we extend the results of the second one to the first two burnt components, conditional on the event that none of them contains the root. We prove that the paths between the root of T_n and the first two edges which are set on fire become disjoint close to the root so that the dynamics on each are essentially independent: in particular the variables $Z_{n,1}$ and $Z_{n,2}$ (the latter plays the same role as $Z_{n,1}$ for the second fire) become independent at the limit and we obtain the joint convergence of $b_{n,1}$ and $b_{n,2}$. We conclude by induction that the estimate holds for the sizes of the first k burnt subtrees, conditional on the event that the root does not burn with any of the first k fires.

The last step is a simple remark: the fact that the root burns does not affect the previous reasoning, so the estimate for the size of the burnt components before that the root burns holds also for the burnt components which come after that the root has burnt.

3

Random non-crossing configurations

A geodesic lamination of the closed unit disk $\overline{\mathbb{D}}$ — we will write simply "lamination" — is a closed subset of $\overline{\mathbb{D}}$ which can be written as the union of a collection of chords with extremities on the unit circle $\mathbb{S}^1$ which do not intersect in the open disk $\mathbb{D}$. In this chapter, we view non-crossing partitions and trees as laminations. We first explain in Section 3.1 how to code them by discrete paths, namely a Łukasievicz path and the associated height process. We then consider continuous analogous objects: after presenting some continuous-time processes related to stable Lévy processes in Section 3.2, we recall in Section 3.3 the definition of the Brownian triangulation and the stable laminations and we define the stable triangulations. We present in Sections 3.4 and 3.5 the results developed in Chapters 6 and 7 respectively. This is based on a joint work with Igor Kortchemski. Finally, in Section 3.6, we give some perspectives and open questions.

3.1 Non-crossing partitions and non-crossing trees

Fix a plane tree τ with, say, $n+1$ vertices $\varnothing = u(0) \prec u(1) \prec \cdots \prec u(n)$ listed in lexicographical order. We define a partition $P(\tau)$ of the set $[n] = \{1, \dots, n\}$ by considering that $i, j \in [n]$ belong to the same block of $P(\tau)$ when $u(i)$ and $u(j)$ have the same parent in τ. Then such a partition is non-crossing in the sense defined in Chapter 1; this in fact defines a bijection between non-crossing partition of $[n]$ and plane trees with $n+1$ vertices, see Dershowitz & Zaks [35] as well as Section 6.2 below. Recall that a non-crossing partition of $[n]$ can be seen as a graph on the set of vertices $\{\exp(-2i\pi k/n); k \in [n]\}$ and whose connected components, the blocks of the partition, are non-intersecting polygons. The blocks of $P(\tau)$ are in bijection with the internal vertices of τ and for each block, the edges of the polygon join two consecutive children of the corresponding internal vertex of τ, where the last and the first are consecutive by convention.

We also define similarly the embedding $\Gamma(\tau)$ of τ into the unit disk as the graph on the set of vertices $\{\exp(-2i\pi k/(n+1)); k \in \{0, \dots, n\}\}$ in which $\exp(-2i\pi k/(n+1))$ and $\exp(-2i\pi \ell/(n+1))$ are joined by a chord whenever $u(k)$ and $u(\ell)$ are linked by an edge in τ; in this case $u(k \wedge \ell)$ is the parent of $u(k \vee \ell)$.

Since a plane tree is coded by its Łukasiewicz path and its height process (recall Section 1.2), we can then code a non-crossing partition and a tree embedded in the disk by such paths as well.

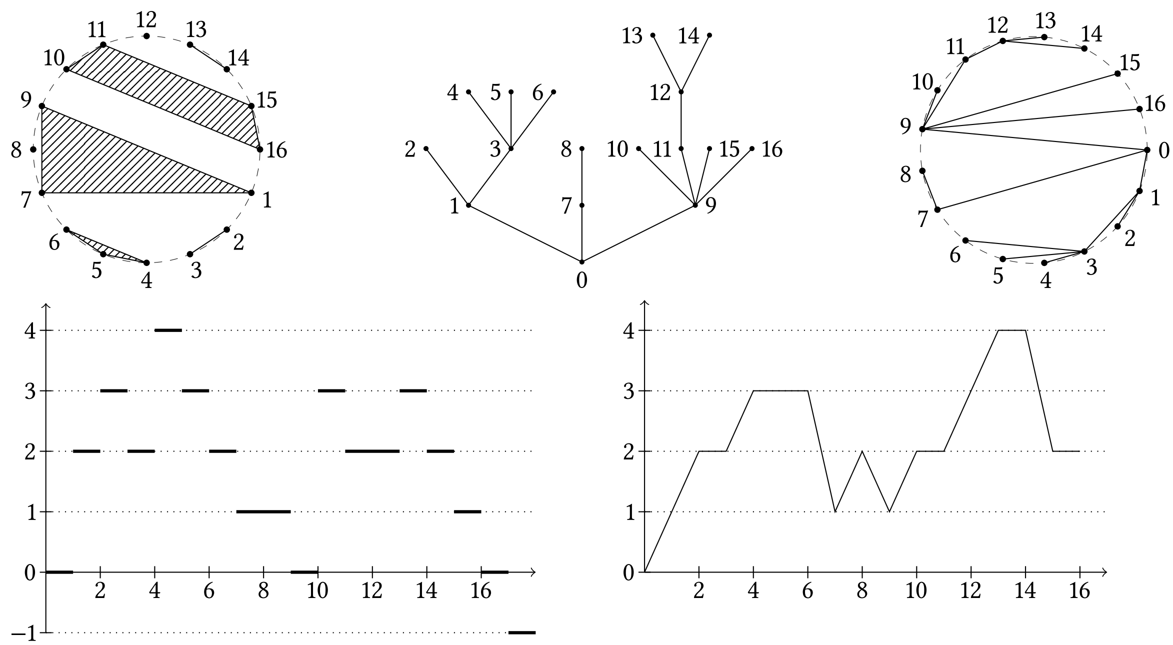


Figure 3.1: The partition $\{\{1,7,9\},\{2,3\},\{4,5,6\},\{8\},\{10,11,15,16\},\{12\},\{13,14\}\}$, the associated plane tree, the embedding in the disk of the latter and its Łukasiewicz path and height process.

The Łukasiewicz path Fix $n \in \mathbb{N}$ and $W = (W_j; 0 \le j \le n+1)$ a path such that $W_0 = 0$, for every $0 \le j \le n$, $W_{j+1} - W_j \ge -1$ with the condition that $W_j \ge 0$ for every $0 \le j \le n$ and $W_n = -1$. Define for every $0 \le j \le n-1$

$$k_j = W_{j+1} - W_j + 1.$$

If $k_j \geq 1$, then for every $1 \leq \ell \leq k_j$, let

$$s^j_\ell = \inf\{m \geq j+1 : W_m = W_{j+1} - (\ell - 1)\}$$

and then set $s^j_{k_j+1} = s^j_1 = j+1$. Finally define

$$\mathbf{P}(W) = \bigcup_{j:k_j\geq 1} \bigcup_{\ell=1}^{k_j} \left[\exp\left(-2\mathrm{i}\pi \frac{s^j_\ell}{n} \right), \exp\left(-2\mathrm{i}\pi \frac{s^j_{\ell+1}}{n} \right) \right], \tag{3.1}$$

and

$$\mathbf{C}(W) = \bigcup_{j:k_j\geq 1} \bigcup_{\ell=1}^{k_j} \left[\exp\left(-2\mathrm{i}\pi \frac{j}{n+1} \right), \exp\left(-2\mathrm{i}\pi \frac{s^j_{\ell}}{n+1} \right) \right]. \tag{3.2}$$

Then $\mathbf{P}(W)$ is a non-crossing partition and $\mathbf{C}(W)$ is a non-crossing tree. Moreover, if W is the Łukasiewicz path of τ, then k_j is the number of children of $u(j)$ and, as we have seen in Proposition 1.4, $s^j_1, \dots, s^j_{k_j}$ are the indices of these children listed in lexicographical order; it follows that $\mathbf{P}(W)$ is $P(\tau)$ the non-crossing partition associated with τ and $\mathbf{C}(W)$ is its embedding $\Gamma(\tau)$ in the unit disk.

The height process Recall from Section 1.2 that the times of the form s^j_ℓ are easily defined from the height process. Indeed, consider $H = (H_j; 0 \leq j \leq n)$ such that $H_0 = 0$ and $H_{j+1} - H_j \leq 1$ for every $0 \leq j \leq n-1$ with the condition that $H_j \geq 0$ for every $0 \leq j \leq n$. Next, for every $0 \leq j \leq n-1$, set $k_j = 0$ if $H_{j+1} \leq H_j$ and otherwise consider the set

$$\left\{ s^j_1, \dots, s^j_{k_j} \right\} = \left\{ \ell \geq j+1 : H_\ell = H_{j+1} = \min_{j+1 \leq m \leq \ell} H_m \right\}.$$

As shown by Proposition 1.5 if H is the height process of τ, then for every $0 \leq j \leq n-1$, k_j is the number of children of $u(j)$ and $j+1 = s^j_1 < \cdots < s^j_{k_j}$ are the indices of these children listed in lexicographical order. Finally, we define

$$\mathbf{P}(H) = \bigcup_{j:k_j\geq 1} \bigcup_{\ell=1}^{k_j} \left[\exp\left(-2\mathrm{i}\pi \frac{s^j_\ell}{n} \right), \exp\left(-2\mathrm{i}\pi \frac{s^j_{\ell+1}}{n} \right) \right], \tag{3.3}$$

and

$$\mathbf{C}(H) = \bigcup_{j:k_j\geq 1} \bigcup_{\ell=1}^{k_j} \left[\exp\left(-2\mathrm{i}\pi \frac{j}{n+1} \right), \exp\left(-2\mathrm{i}\pi \frac{s^j_{\ell}}{n+1} \right) \right]. \tag{3.4}$$

The sets $\mathbf{P}(H)$ and $\mathbf{C}(H)$ are then respectively the non-crossing partition $P(\tau)$ associated with τ and its embedding $\Gamma(\tau)$ in the unit disk.

3.2 Stable Lévy processes and excursions

We next define paths which are the analogs in continuous-time of the Łukasiewicz path and the discrete height process. We recall some definitions, properties and constructions with no proof and refer the interesting reader to Bertoin [14] and Duquesne [43] for more details.

Spectrally positive stable Lévy process Fix $\alpha \in (1, 2]$ and consider a random process $X_\alpha = (X_\alpha(s); s \geq 0)$ with paths in the set $\mathcal{D}([0, \infty), \mathbb{R})$ of càdlàg functions endowed with the Skorokhod topology (see e.g. Billingsley [21] for details on this space), which has independent and stationary increments, no negative jump and such that

$$\mathbb{E}[\exp(-\lambda X_\alpha(t))] = \exp(t\lambda^\alpha) \qquad \text{for every} \quad t, \lambda > 0.$$

Such a process is called a (strictly) stable spectrally positive Lévy process of index α. An important feature of X_α is the *scaling property*: for every $c > 0$,

$$(c^{-1/\alpha} X_\alpha(ct); t \geq 0) = (X_\alpha(t); t \geq 0) \qquad \text{in distribution.}$$

The process X_α is continuous for $\alpha = 2$, and indeed $X_2/\sqrt{2}$ is the standard Brownian motion, whereas the set of discontinuities of X_α is dense in $[0, \infty)$ for every $\alpha \in (1, 2)$; we shall thereby treat the two cases separately in the next section.

Normalized excursion Let $\underline{X}_\alpha$ be the infimum process of X_α, defined by

$$\underline{X}_\alpha(t) = \inf\{X_\alpha(s); s \in [0, t]\} \qquad \text{for every} \quad t \geq 0.$$

Observe that $\underline{X}_\alpha$ is continuous since X_α has no negative jump. The process $X_\alpha - \underline{X}_\alpha$ is a strong Markov process; we may, and do, choose $-\underline{X}_\alpha$ as its local time at 0. By Itō's excursion theory for Markov processes, the excursions of $X_\alpha - \underline{X}_\alpha$ away from 0 are distributed according to a Poisson random measure whose intensity is given by the Itō excursion measure N_α. We denote by $\zeta(\epsilon) = \sup\{t > 0 : \epsilon(t) > 0\} \in (0, \infty)$ the duration of an excursion ϵ and we let $N_\alpha^{(v)}$ be a regular version of the probability law $N_\alpha(\cdot \mid \zeta = v)$, i.e. the measure such that for every positive and continuous functional F,

$$N_\alpha(F) = \int_0^\infty N_\alpha(\zeta \in \mathrm{d}v) N_\alpha^{(v)}(F).$$

Such a law can be obtained by scaling: for any fixed $t > 0$, the process

$$\left((v/\zeta)^{1/\alpha} \epsilon(\zeta s/v); s \in [0, v]\right) \qquad \text{under} \quad N_\alpha(\cdot \mid \zeta > t) = \frac{N_\alpha(\cdot, \zeta > t)}{N_\alpha(\zeta > t)}$$

is distributed as $N_\alpha^{(v)}$. We denote by X_α^{ex} the normalized excursion of X_α, which is a random variable taking values in $\mathcal{D}([0,1],\mathbb{R})$ with law $N_\alpha^{(1)}$. We refer to Chaumont [28] for interesting constructions of processes with law $N_\alpha^{(v)}$ by path transformations. As an example, let us recall the following construction of $N_\alpha^{(1)}$ already described in Section 1.4 in the Brownian case. Consider the excursion straddling 1: let

$$\zeta_1 = d_1 - g_1 \quad \text{where} \quad \begin{cases} g_1 = \sup\{t \leq 1 : X_\alpha(t) = \underline{X}_\alpha(t)\}, \\ d_1 = \inf\{t > 1 : X_\alpha(t) = \underline{X}_\alpha(t)\}, \end{cases} \tag{3.5}$$

and define a process $X_\alpha^* = (X_\alpha^*(t); t \in [0,1])$ by

$$X_\alpha^*(t) = \zeta_1^{-1/\alpha}(X_\alpha(g_1 + \zeta_1 t) - X_\alpha(g_1)) \qquad \text{for every} \quad t \in [0,1].$$

Then X_α^* is distributed as X_α^{ex}. Note that $X_\alpha^{\text{ex}}(0) = X_\alpha^{\text{ex}}(1) = 0$ and $X_\alpha^{\text{ex}}(s) > 0$ for every $s \in (0,1)$.

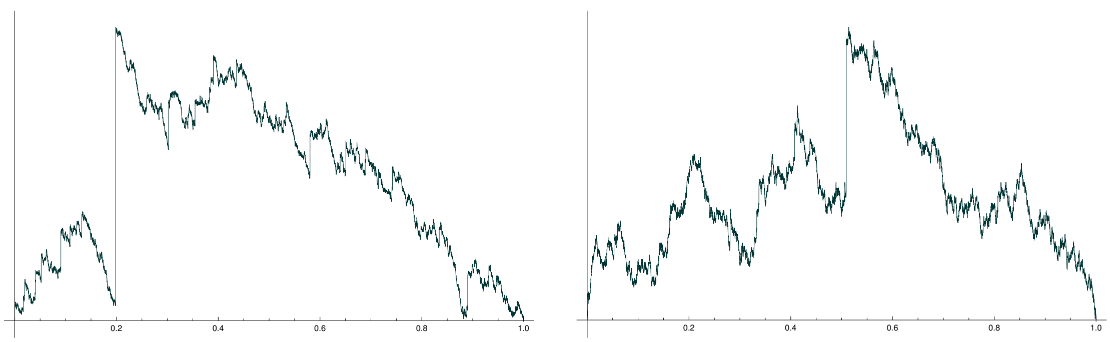


Figure 3.2: Simulations of X_α^{ex}, respectively for α equals 1.2 and 1.6.

The height process The height process associated with a spectrally positive Lévy process was introduced by Le Gall & Le Jan [75] and studied by Duquesne [43] and Duquesne & Le Gall [44] in the context of so-called continuum random trees. We do not enter into details here. Let us define the height process associated with X_α by

$$H_\alpha(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{I}_{\{X_\alpha(s) \leq \inf_{[s,t]} X_\alpha + \varepsilon\}} \mathrm{d}s \qquad \text{for every} \quad t \geq 0,$$

where the convergence holds in probability. The process $(H_\alpha(t); t \geq 0)$ admits a continuous modification that we now consider.

If X_α is the continuous analog of a Łukasiewicz path, then H_α is that of the associated discrete height process. Indeed, fix $t > 0$, consider the time-reversed process defined by

$$Y_\alpha(s) = X_\alpha(t) - X_\alpha((t-s)-) \qquad \text{for} \quad s \in [0,t]$$

and the supremum process of the latter

$$\overline{Y}_\alpha(s) = \sup\{Y_\alpha(u); u \in [0, s]\} \qquad \text{for} \quad s \in [0, t].$$

Then the (normalized) local time of $\overline{Y}_\alpha - Y_\alpha$ at 0 is distributed as $H_\alpha(t)$ so that, loosely speaking, $H_\alpha(t)$ measures the size of the set $\{s \in [0, t]; Y_\alpha(s) = \overline{Y}_\alpha(s)\}$ or, equivalently, that of $\{s \in [0, t]; X_\alpha(s) = \inf_{s \le r \le t} X_\alpha(r)\}$. This corresponds intuitively to the discrete relation (1.3) between a Łukasiewicz path and the corresponding discrete height process.

Next, one can deduce from the scaling property of X_α that H_α satisfies

$$(c^{-1+1/\alpha} H_\alpha(ct); t \ge 0) = (H_\alpha(t); t \ge 0) \qquad \text{in distribution}$$

for every $c > 0$. As for X_α, we can then define the normalized excursion $(H_\alpha^{\mathrm{ex}}(t); t \in [0, 1])$. As above, H_α^{ex} can be obtained by path transformation and in fact, the pair $(X_\alpha^{\mathrm{ex}}, H_\alpha^{\mathrm{ex}})$ can be constructed as follows: let g_1, d_1 and ζ_1 be as in (3.5), then the process

$$\Big(\Big(\zeta_1^{-1/\alpha}(X_\alpha(g_1 + \zeta_1 t) - X_\alpha(g_1)), \zeta_1^{-1+1/\alpha} H_\alpha(g_1 + \zeta_1 t)\Big); t \in [0, 1]\Big)$$

is distributed as

$$\Big((X_\alpha^{\mathrm{ex}}(t), H_\alpha^{\mathrm{ex}}(t)); t \in [0, 1]\Big).$$

Again, we have $H_\alpha^{\mathrm{ex}}(0) = H_\alpha^{\mathrm{ex}}(1) = 0$ and $H_\alpha^{\mathrm{ex}}(s) > 0$ for every $s \in (0, 1)$.

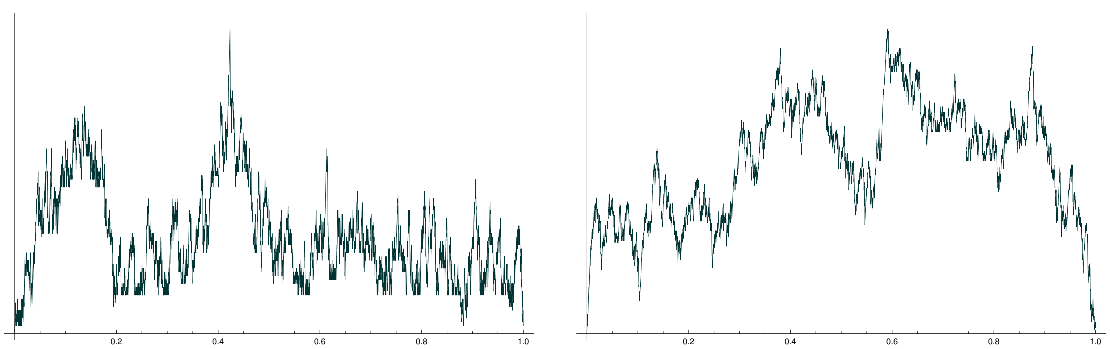


Figure 3.3: Simulations of H_α^{ex}, respectively for α equals 1.2 and 1.6.

Duquesne's Theorem We finally extend Theorem 1.14 which shows that the normalized excursion and height process of a stable Lévy process are the limits of the Łukasievicz path and height process of large Galton–Watson trees. Let μ be a probability measure on $\mathbb{Z}_+$ and $\alpha \in (1, 2]$. Recall that μ is said to be critical if its expectation is 1. We say that it is in the domain of attraction of a stable law of index α if $\alpha = 2$ and the variance of μ is finite and non-zero, or if the tail distribution is given by $\sum_{k=j}^{\infty} \mu(k) = j^{-\alpha} L(j)$, where L is a slowly varying function at infinity, meaning that for every $a > 0$, we have $\lim_{x\to\infty} L(ax)/L(x) = 1$.

Recall from Section 1.2 the notations $W(\tau)$, $H(\tau)$ and $C(\tau)$ for the Łukasievicz path, the height process and the contour process of a discrete tree τ. Recall that W is viewed as a step function, whereas H and C are seen as continuous functions. Finally, recall the notation GW_n^{μ} for the law of a Galton–Watson tree with offspring distribution μ conditioned to have n edges.

Theorem 3.1 ([43]). *Let $\alpha \in (1, 2]$ and μ a critical probability measure on $\mathbb{Z}_+$ in the domain of attraction of a stable law of index α. For every $n \geq 1$ for which GW_n^{μ} is well defined, sample τ_n according to GW_n^{μ}. Then there exists a sequence $(B_n)_{n\geq 1}$ of positive constants satisfying $\lim_{n\to\infty} B_n = \lim_{n\to\infty} n/B_n = \infty$, such that the triplet*

$$\left(\frac{1}{B_n} W_{\lfloor ns \rfloor}(\tau_n), \frac{B_n}{n} H_{ns}(\tau_n), \frac{B_n}{n} C_{2ns}(\tau_n); s \in [0,1]\right)$$

converges in distribution towards

$$(X_\alpha^{\mathrm{ex}}(s), H_\alpha^{\mathrm{ex}}(s), H_\alpha^{\mathrm{ex}}(s); s \in [0,1])$$

in the space $\mathcal{D}([0,1],\mathbb{R}) \otimes \mathcal{C}([0,1],\mathbb{R}) \otimes \mathcal{C}([0,1],\mathbb{R})$.

The value of B_n can be made explicit, although rather complicated and we will not need it, let us note that it is of order $n^{1/\alpha}$, and indeed the sequence $(n^{-1/\alpha} B_n)_{n\geq 1}$ is slowly varying (Kortchemski [70, Theorem 1.10]). In the case $\alpha = 2$, the processes X_2^{ex} and H_2^{ex} coincide — both with $\sqrt{2}$ times the standard Brownian excursion — and the above convergence was proved by Marckert & Mokkadem [79] under the stronger assumption of a finite exponential moment for μ. As for Theorem 1.14, Theorem 3.1 implies the convergence for the Gromov–Hausdorff–Prokhorov topology of $(n^{-1}B_n)\tau_n$ towards the random real tree $\mathcal{T}_{H_\alpha^{\mathrm{ex}}}$; the latter is called the α-stable Lévy tree.

3.3 Stable laminations

We next define random laminations of the disk from the excursions paths previously defined. We first start with the Brownian case for which we refer to Aldous [6] and Le Gall & Paulin [77]. We then consider the α-stable case, with $\alpha \in (1, 2)$, we recall the definition of the stable lamination of Kortchemski [71] and define a new object: the stable triangulation.

3.3.1 The Brownian triangulation

Let $e = X_2^{\mathrm{ex}}$ be $\sqrt{2}$ times the standard Brownian excursion. Recall the equivalence relation defined in Section 1.3: for every $s, t \in [0,1]$, we set $s \overset{e}{\sim} t$ when $e(s) = e(t) = \min_{[s\wedge t, s\vee t]} e$. We then define a subset of $\overline{\mathbb{D}}$ by

$$\mathbf{L}(e) := \bigcup_{s \overset{e}{\sim} t} \left[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}\right]. \tag{3.6}$$

Using the fact that, almost surely, e is continuous and its local minima are distinct, one can prove the following result.

Proposition 3.2 (Aldous [6] - Le Gall & Paulin [77])**.** *Almost surely,* $\mathbf{L}(e)$ *is a geodesic lamination of* $\overline{\mathbb{D}}$*. Furthermore, it is maximal for the inclusion relation among geodesic laminations of* $\overline{\mathbb{D}}$*.*

Let us briefly sketch the arguments, as they extend to more general laminations. First, to prove that the chords above are non-crossing, one checks that if $s < t$ and $u < v$ are such that $s \overset{e}{\sim} t$ and $u \overset{e}{\sim} v$, then either the intervals (s,t) and (u,v) are disjoint, or one is contained in the other. Note that uniqueness of local minima is used here since a quadruple $0 \leq s < t < u < v \leq 1$ such that $s \overset{e}{\sim} t \overset{e}{\sim} u \overset{e}{\sim} v$ would define the two crossing chords $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi u}]$ and $[\mathrm{e}^{-2\mathrm{i}\pi t}, \mathrm{e}^{-2\mathrm{i}\pi v}]$. Next, the fact that $\mathbf{L}(e)$ is closed is due to the fact that the equivalence relation $\overset{e}{\sim}$ is closed, in the sense that its graph is a closed subset of $[0,1]^2$. Finally, for the maximality property, one checks that for every $s < t$, either $s \overset{e}{\sim} t$, or there exist $u \in (s,t)$ and $v \in [0,1] \setminus [s,t]$ such that $u \overset{e}{\sim} v$; it follows that each chord in the disk either belongs to $\mathbf{L}(e)$ or intersects a chord of the latter so we cannot build a lamination containing strictly $\mathbf{L}(e)$.

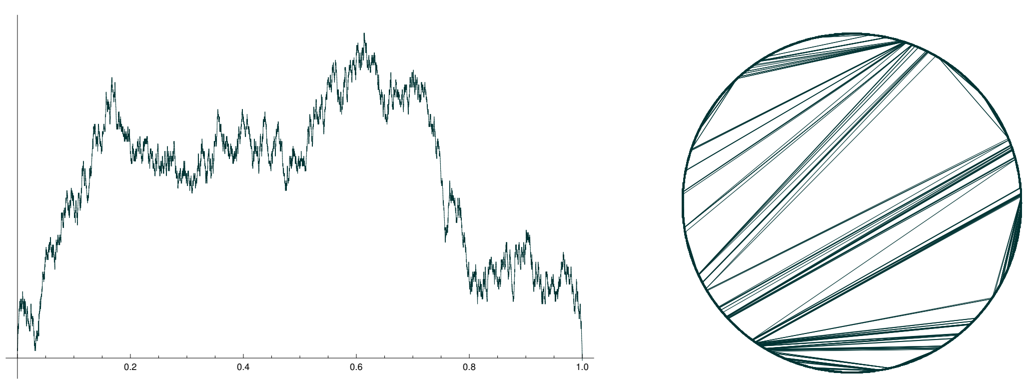


Figure 3.4: Simulations of e and $\mathbf{L}(e)$.

Observe that $s \overset{e}{\sim} s$ for every $s \in [0,1]$ so $\mathbb{S}^1 \subset \mathbf{L}(e)$. Also, since $\mathbf{L}(e)$ is maximal, its faces, i.e. the connected components of the open set $\overline{\mathbb{D}} \setminus \mathbf{L}(e)$, are open triangles whose vertices belong to $\mathbb{S}^1$. Aldous [6] indeed defined $\mathbf{L}(e)$ as

$$\mathbf{L}(e) = \overline{\mathbb{D}} \setminus \bigcup_{\substack{s<t<u \\ s \overset{e}{\sim} t \overset{e}{\sim} u}} \Delta(s,t,u),$$

where $\Delta(s,t,u)$ denotes the open triangle with vertices at $\mathrm{e}^{-2i\pi s}$, $\mathrm{e}^{-2i\pi t}$ and $\mathrm{e}^{-2i\pi u}$. Note that $s < t < u$ and $s \overset{e}{\sim} t \overset{e}{\sim} u$ means that t is a time of local minimum of e. Making use of the (almost sure) density of such times in $[0,1]$, this gives a triangulation "with zero area" [6]. The set $\mathbf{L}(e)$ is called the *Brownian triangulation*.

3.3.2 Laminations coded by a function with no negative jumps

Fix $\alpha \in (1,2)$ and consider X_α^{ex} the normalized excursion of the α-stable Lévy process. Recall the definition of a non-crossing partition (3.1) and a non-crossing tree (3.2) from a Łukasievicz path. We construct laminations from X_α^{ex} in a similar way. We define two relations (not equivalence relations in general) on $[0,1]$ using X_α^{ex}: for every $0 \le s < t \le 1$, we set

$$s \simeq^{X_\alpha^{\mathrm{ex}}} t \quad \text{if} \quad t = \inf\{u > s : X_\alpha^{\mathrm{ex}}(u) \le X_\alpha^{\mathrm{ex}}(s-)\},$$

and

$$s \cong^{X_\alpha^{\mathrm{ex}}} t \quad \text{if} \quad X_\alpha^{\mathrm{ex}}(s-) \le X_\alpha^{\mathrm{ex}}(t) \quad \text{and} \quad X_\alpha^{\mathrm{ex}}(t) = \inf_{[s,t]} X_\alpha^{\mathrm{ex}};$$

then for $0 \le t < s \le 1$, we set $s \simeq^{X_\alpha^{\mathrm{ex}}} t$ if $t \simeq^{X_\alpha^{\mathrm{ex}}} s$, and we agree that $s \simeq^{X_\alpha^{\mathrm{ex}}} s$ for every $s \in [0,1]$. We do the same operation for $\cong^{X_\alpha^{\mathrm{ex}}}$. We finally define two subsets of $\overline{\mathbb{D}}$ by

$$L(X_\alpha^{\mathrm{ex}}) := \bigcup_{s \simeq^{X_\alpha^{\mathrm{ex}}} t} \left[\mathrm{e}^{-2i\pi s}, \mathrm{e}^{-2i\pi t}\right] \quad \text{and} \quad \widehat{L}(X_\alpha^{\mathrm{ex}}) := \bigcup_{s \cong^{X_\alpha^{\mathrm{ex}}} t} \left[\mathrm{e}^{-2i\pi s}, \mathrm{e}^{-2i\pi t}\right]. \tag{3.7}$$

Note that for every $s,t \in [0,1]$, if $s \simeq^{X_\alpha^{\mathrm{ex}}} t$, then $s \cong^{X_\alpha^{\mathrm{ex}}} t$ and so $\mathbb{S}^1 \subset L(X_\alpha^{\mathrm{ex}}) \subset \widehat{L}(X_\alpha^{\mathrm{ex}})$.

The set $L(X_\alpha^{\mathrm{ex}})$ was defined and studied by Kortchemski [71]; he proved that it is a geodesic lamination of $\overline{\mathbb{D}}$, called the *α-stable lamination*. It is far from being maximal since all its faces are bounded by infinitely many chords. We will show in Chapter 7 that $\widehat{L}(X_\alpha^{\mathrm{ex}})$ is geodesic lamination of $\overline{\mathbb{D}}$ as well, which is, moreover, maximal for the inclusion relation among geodesic laminations of $\overline{\mathbb{D}}$. We call it the *α-stable triangulation*. Informally, $\widehat{L}(X_\alpha^{\mathrm{ex}})$ is obtained from $L(X_\alpha^{\mathrm{ex}})$ by "filling the faces", i.e. inside each face of $L(X_\alpha^{\mathrm{ex}})$, we join by a chord each vertex of this face to the closest one to the complex number 1 in $\mathbb{S}^1$ in clockwise order (and then we take the closure of the set thus obtained), thus making it a triangulation.

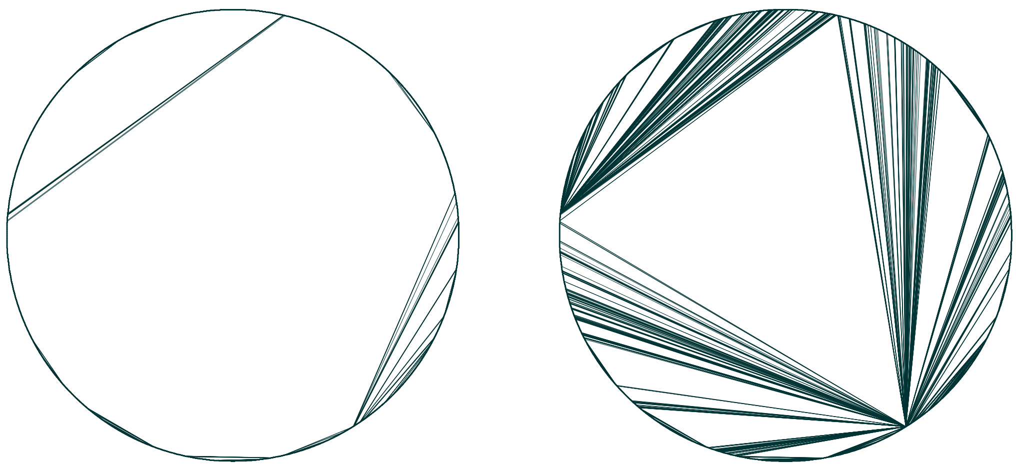


Figure 3.5: Simulations of $L(X_\alpha^{\mathrm{ex}})$ and $\widehat{L}(X_\alpha^{\mathrm{ex}})$ for $\alpha = 1.1$.

3.3.3 Laminations coded by a continuous function

We provide an alternative construction of stable laminations and triangulations, closer in spirit to that of the Brownian triangulation and the discrete relations (3.3) and (3.4); for the stable lamination, this can be found in Section 4 of [71]. The relevant excursion function here is that of the height process.

Let us first work in a general setting: we consider a continuous function $g : [0,1] \to [0,\infty)$ such that $g(0) = g(1) = 0$ and we construct two laminations from g similarly to the Brownian triangulation. Recall the equivalence relation on $[0,1]$ given by $s \overset{g}{\sim} t$ if and only if $g(s) = g(t) = \min_{[s\wedge t, s\vee t]} g$. Recall that if the local minima of g are not distinct, then mimicking the definition (3.6) of the Brownian triangulation does not give a lamination. In order to circumvent this problem, we define two other relations (again, not equivalence relations in general) on $[0,1]$. For every $s \in [0,1]$, let $\mathrm{cl}_g(s)$ be the equivalence class of s for $\overset{g}{\sim}$. For every $s, t \in [0,1]$, we set

(i) $s \approx^g t$ when $s \overset{g}{\sim} t$ and at least one the following conditions holds: $g(r) > g(s)$ for every $r \in (s \wedge t, s \vee t)$, or $(s \wedge t, s \vee t) = (\min \mathrm{cl}_g(s), \max \mathrm{cl}_g(s))$;

(ii) $s \cong^g t$ when $s \overset{g}{\sim} t$ and at least one the following conditions holds: $g(r) > g(s)$ for every $r \in (s \wedge t, s \vee t)$, or $s \wedge t = \min \mathrm{cl}_g(s)$.

The arguments of Proposition 3.2 can be adapted to obtain the following result.

Proposition 3.3. *For any continuous function $g : [0,1] \to [0,\infty)$ such that $g(0) = g(1) = 0$,* *the sets*

$$\mathbf{L}(g) := \bigcup_{s \approx^g t} \left[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}\right] \qquad \textit{and} \qquad \widehat{\mathbf{L}}(g) := \bigcup_{s \cong^g t} \left[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}\right]$$

are geodesic laminations of $\overline{\mathbb{D}}$ *and, further,* $\widehat{\mathbf{L}}(g)$ *is maximal for the inclusion relation among geodesic laminations of* $\overline{\mathbb{D}}$*.*

We call $\mathbf{L}(g)$ and $\widehat{\mathbf{L}}(g)$ respectively the lamination and the triangulation coded by the function g. Intuitively, $\mathbf{L}(g)$ is defined by drawing a chord between two points $\mathrm{e}^{-2\mathrm{i}\pi s}$ and $\mathrm{e}^{-2\mathrm{i}\pi t}$ on the circle if they correspond to two consecutive times $s \overset{g}{\sim} t$, where the smallest and largest such times are consecutive; for $\widehat{\mathbf{L}}(g)$ we draw the same chords but we also join for each equivalence class, every element to the smallest one.

The set $\mathbf{L}(g)$ can be seen as the dual graph of the real tree $\mathcal{T}_g = [0, \infty)/ \overset{g}{\sim}$ defined in Section 1.3. Indeed, each vertex of $\mathcal{T}_g$ corresponds to an equivalence class cl_g; loosely speaking, $\mathcal{T}_g$ is obtained by merging all the vertices of a face of $\mathbf{L}(g)$ together. See Le Gall & Paulin [77, Section 2] and references therein for more precise relations between geodesic laminations and real trees.

Note that $\mathbb{S}^1 \subset \mathbf{L}(g) \subset \widehat{\mathbf{L}}(g)$ for every g; moreover, if the local minima of g are distinct (this can be slightly weakened, see Curien & Le Gall [33, Proposition 2.5]), then for every $s, t \in [0, 1]$, one has $s \overset{g}{\sim} t$ if and only if $s \approx^g t$ if and only if $s \cong^g t$. In particular, if $g = e$, we recover the Brownian triangulation $\mathbf{L}(e) = \widehat{\mathbf{L}}(e)$. Kortchemski [71] proved that for every $\alpha \in (1, 2)$, almost surely, the relation $\approx^{H_\alpha^{\mathrm{ex}}}$ coincides with $\simeq^{X_\alpha^{\mathrm{ex}}}$; we will prove in Chapter 7 that, almost surely, the relation $\cong^{H_\alpha^{\mathrm{ex}}}$ coincides with $\cong^{X_\alpha^{\mathrm{ex}}}$. It follows that

$$\mathbf{L}(H_\alpha^{\mathrm{ex}}) = L(X_\alpha^{\mathrm{ex}}) \qquad \text{and} \qquad \widehat{\mathbf{L}}(H_\alpha^{\mathrm{ex}}) = \widehat{L}(X_\alpha^{\mathrm{ex}}) \qquad \text{a.s.}$$

for every $\alpha \in (1, 2)$.

3.4 Large non-crossing partitions

In Chapter 6, we study large random non-crossing partitions. For every integer n, we sample a non-crossing partition of $[n]$ (recall that this set is denoted by $\mathbb{NC}_n$) using Boltzmann weights, similarly to simply generated trees defined in Section 1.2. Given a sequence of non-negative real numbers $w = (w(i); i \geq 1)$, with every partition $P \in \mathbb{NC}_n$, we associate a weight

$$\Omega^w(P) = \prod_{B \text{ block of } P} w(\text{size of } B).$$

Then, for every $P \in \mathbb{NC}_n$, set

$$\mathbb{P}_n^w(P) = \frac{\Omega^w(P)}{\sum_{Q \in \mathbb{NC}_n} \Omega^w(Q)}.$$

Implicitly, we shall always restrict our attention to those values of n for which we have $\sum_{Q \in \mathbb{NC}_n} \Omega^w(Q) > 0$. A random non-crossing partition of $[n]$ sampled according to $\mathbb{P}_n^w$ is called a *simply generated non-crossing partition.*

Such a distribution recovers the uniform law on $\mathbb{NC}_n$, which corresponds to $\mathbb{P}_n^w$ when $w(i) = 1$ for every $i \geq 1$. More generally, if A is a non-empty subset of $\mathbb{N}$, and $w_A(i) = 1$ if $i \in A$ and $w_A(i) = 0$ otherwise, then $\mathbb{P}_n^{w_A}$ corresponds to the uniform distribution on the subset $\mathbb{NC}_n^A \subset \mathbb{NC}_n$ formed by the partitions with all block sizes belonging to A (provided that they exist) and which we call A-constrained non-crossing partitions. In the case where $A = k\mathbb{N}$ for some $k \geq 1$ fixed, the elements of $\mathbb{NC}_n^A$ are called k-divisible, see Edelman [46] and Arizmendi & Vargas [10]; when $A = \{k\}$ for some $k \geq 1$ fixed, they are called k-equal [10].

Bijections with plane trees The main tool is to reduce the study of simply generated non-crossing partitions to that of simply generated trees. We introduce several bijections with trees and recover, by geometric considerations, the bijection of Dershowitz & Zaks [35] presented earlier in Section 3.1 (but also another bijection due to Prodinger [100] and the Kreweras complement [73]). If the partition is sampled according to $\mathbb{P}_n^w$, the associated plane tree is distributed as a simply generated tree with law $\mathbb{Q}_n^w$ given by (1.2), where we set $w(0) = 1$. Recall from Section 1.2 that if a sequence $\pi = (\pi(i), i \geq 0)$ is equivalent to $w = (w(i), i \geq 0)$ in the sense that there exist $a, b > 0$ such that $\pi(i) = ab^i w(i)$ for every $i \geq 0$, then $\mathbb{Q}_n^\pi = \mathbb{Q}_n^w$ and so $\mathbb{P}_n^\pi = \mathbb{P}_n^w$ for every $n \geq 1$.

Statistics on large non-crossing partitions Arizmendi [9] computed the expected number of blocks of given size for uniform random k-divisible or k-equal non-crossing partitions of $[n]$ and Ortmann [91] showed that the distribution of a uniform random block in a uniform non-crossing partition of $[n]$ converges to a geometric random variable of parameter $1/2$ as $n \to \infty$. Recall from Section 1.2 that if $\limsup_{k \to \infty} w(k)^{1/k} < \infty$, then there exist probabilities on $\mathbb{Z}_+$ equivalent to w in the above sense and the latter are characterized by their mean; we denote by π the unique probability equivalent to w with mean 1 if such a measure exists, and that with the largest mean otherwise.

Theorem 3.4. *For each $n \geq 1$, sample P_n according to $\mathbb{P}_n^w$. As $n \to \infty$,* *(i) For every $k \geq 1$, the probability that the block containing 1 in P_n has size k converges towards $k\pi(k)$.*

(ii) If $\pi(0) < 1$, then for every $k \geq 1$, the probability that a block chosen uniformly at random in P_n has size k converges towards $\pi(k)/(1-\pi(0))$.

(iii) For every $A \subset \mathbb{N}$, the number of blocks of P_n whose size belongs to A, rescaled by n, converges in probability, and its expectation converges as well, towards $\pi(A)$.

We express these quantities in terms of the simply generated tree associated with P_n, which enables us to use several results obtained by Janson [64] in this context.

Two applications The bijection with trees allows us also to count A-constrained non-crossing partitions. For n fixed, only the cases of k-divisible and k-equal non-crossing partitions are known: Edelman [46] and Arizmendi & Vargas [10] respectively computed

$$\#\mathbb{NC}_{kn}^{\{k\}} = \frac{1}{(k-1)n+1}\binom{kn}{n} \qquad \text{and} \qquad \#\mathbb{NC}_{kn}^{k\mathbb{Z}_+} = \frac{1}{kn+1}\binom{(k+1)n}{n}.$$

For every $A \subset \mathbb{N}$, $A \neq \{1\}$, we obtain an asymptotic formula for the cardinal $\#\mathbb{NC}_n^A$ of the form

$$\#\mathbb{NC}_n^A \quad \underset{n\to\infty}{\sim} \quad c\left(\frac{1}{\xi_A}\left(1+\sum_{k\in A}\xi_A^k\right)\right)^n n^{-3/2},$$

where $n \to \infty$ in such a way that n is divisible by $\gcd A$. The constant c depends on A and can be written explicitly, the number ξ_A is that defined in Example 1.3.

The second application concerns free probability. Indeed, fix a probability measure μ on $\mathbb{R}$ with compact support, different from a Dirac mass. It is known that μ is characterized by the real sequence $(\kappa_n(\mu); n \geq 1)$ of its free cumulants. The latter are the coefficients in the series expansion of the R-transform R_μ of μ, which is the analog in free probability of the Laplace transform, see e.g. Bercovici & Voiculescu [12].

Theorem 3.5. *Suppose that all the free cumulants of μ are nonnegative. Let s_μ be the maximum of the support of μ and $\rho = (\limsup_{n\to\infty} \kappa_n(\mu)^{1/n})^{-1}$. Then*

$$s_\mu \quad = \quad \frac{1}{\xi} + R_\mu(\xi),$$

where $\xi \in (0, \rho]$ is the unique solution of $R'_\mu(\xi) = \xi^{-2}$ if such a number exists, and $\xi = \rho$ otherwise.

This gives a more explicit formula (see Example 6.14 for calculations with several laws μ) than that obtained by Ortmann [91], which reads, under the same condition,

$$\log(s_\mu) = \sup\left\{\frac{1}{m_1(p)}\sum_{n\in L} p_n \log\left(\frac{\kappa_n(\mu)}{p_n}\right) - \frac{\theta(m_1(p))}{m_1(p)};\ p \in \mathcal{M}_1(L)\right\},$$

where $L = \{n \geq 1; \kappa_n(\mu) \neq 0\}$, $\theta(x) = \log(x-1) - x\log(x-1/x)$, $\mathcal{M}_1(L)$ is the set of probability measures p on L and $m_1(p)$ is the mean of p.

The proof relies on an asymptotic result due to Janson [64] for the partition function in the definition of $\mathbb{P}_n^w$:

$$Z_n^w = \sum_{P\in\mathbb{NC}_n} \prod_{B \text{ block of } P} w(\text{size of } B).$$

Indeed, the moments of μ and its free cumulants are related as follows (see Speicher [105]):

$$\int_{\mathbb{R}} t^n \mu(\mathrm{d}t) = \sum_{P\in\mathbb{NC}_n} \prod_{B \text{ block of } P} \kappa_{\text{size of } B}(\mu),$$

and we know that $s_\mu = \limsup_{n\to\infty}(\int t^n\mu(\mathrm{d}t))^{1/n}$. Assuming that $\kappa_i(\mu) \geq 0$ for every $i \geq 1$, we may therefore interpret these free cumulants as weights $(w(i); i \geq 1)$.

Large random laminations In a second part, we view non-crossing partitions as laminations of the disk, as described in Section 3.1. We assume that w is equivalent to a probability measure μ which is critical and belongs to the domain of attraction of a stable law of index $\alpha \in (1,2]$.

Theorem 3.6. *If P_n is a random non-crossing partition sampled according to $\mathbb{P}_n^w = \mathbb{P}_n^\mu$, for every integer $n \geq 1$ such that $\mathbb{P}_n^w$ is well defined, then the convergence*

$$P_n \quad \xrightarrow[n\to\infty]{(d)} \quad \mathbf{L}_\alpha \tag{3.8}$$

holds in distribution for the Hausdorff distance on the space of compact subsets of $\overline{\mathbb{D}}$. For $\alpha = 2$, the set $\mathbf{L}_2$ is the Brownian triangulation, while $\mathbf{L}_\alpha$ is the α-stable lamination for $\alpha \in (1,2)$.

Observe that the assumptions above hold with $\alpha = 2$ for the uniform distribution on A-constrained non-crossing partitions of $[n]$ since this law is given by $\mathbb{P}_n^{w_A}$ where $w_A = \mathbb{I}_{A\cup\{0\}}$; the equivalent probability distribution defined in Example 1.3 is then critical and with finite variance. This extends a previous result of Curien & Kortchemski [31] who proved (3.8) in the case of the uniform distribution on $\mathbb{NC}_n$, given by $\mathbb{P}_n^\mu$ where $\mu(k) = 2^{-(k+1)}$ for $k \geq 0$.

To prove (3.8), we associate with each P_n its Łukasiewicz path $W^{(n)}$ as explained in Section 3.1; under the assumption above, Theorem 3.1 applies and $W^{(n)}$ converges in distribution, after renormalization, to X_α^{ex}. Using Skorokhod's representation theorem, we may suppose that this convergence holds almost surely and we prove that for any ω fixed in the probability space for which the convergence of $W^{(n)}(\omega)$ holds, we have $P_n(\omega) \to L(X_\alpha^{\mathrm{ex}}(\omega))$ as $n \to \infty$. Since the disk $\overline{\mathbb{D}}$ is compact, then as recalled in Section 1.3, the sequence $(P_n(\omega); n \geq 1)$ takes values in a compact space so we only need to check that $L(X_\alpha^{\mathrm{ex}}(\omega))$ is its only accumulation point.

3.5 Large non-crossing trees

In Chapter 7, we consider the embedding $\Gamma(\tau)$ of a plane tree τ in the disk described in Section 3.1. As the statistics of $\Gamma(\tau)$ are exactly the same as that of τ, we focus on the geometric point of view. Fix $\alpha \in (1, 2]$ and a critical probability measure μ in the domain of attraction of a stable law of index α.

Theorem 3.7. *For every integer $n \geq 1$ such that this conditioning makes sense, sample a Galton–Watson tree τ_n with offspring distribution μ, conditioned to have n vertices. Then the convergence*

$$\Gamma(\tau_n) \quad \underset{n\to\infty}{\overset{(d)}{\longrightarrow}} \quad \widehat{\mathbf{L}}_\alpha \tag{3.9}$$

holds for the Hausdorff distance on the space of all compact subsets of $\overline{\mathbb{D}}$. For $\alpha = 2$, the set $\widehat{\mathbf{L}}_2 = \mathbf{L}_2$ is the Brownian triangulation, while $\widehat{\mathbf{L}}_\alpha$ is the α-stable triangulation for $\alpha \in (1, 2)$.

The proof is similar to that of (3.8): we assume the convergence of the associated Łukasiewicz path $W^{(n)}$ to X_α^{ex} and reduce the problem to a deterministic statement; then we show that $\widehat{L}(X_\alpha^{\mathrm{ex}})$ is the only accumulation point of the sequence $(\Gamma(\tau_n); n \geq 1)$.

We also study the properties of the random set $\widehat{\mathbf{L}}_\alpha$ for $\alpha \in (1, 2)$. We show that it is almost surely a geodesic triangulation of the disk and we prove the identity $\widehat{L}(X_\alpha^{\mathrm{ex}}) = \widehat{\mathbf{L}}(H_\alpha^{\mathrm{ex}})$. Finally, we compute two Hausdorff dimensions: that of the whole set $\widehat{\mathbf{L}}_\alpha \subset \overline{\mathbb{D}}$ and that of the set $\widehat{\mathbf{A}}_\alpha \subset \mathbb{S}^1$ of all end-points of chords in $\widehat{\mathbf{L}}_\alpha$. In the case $\alpha = 2$, it is known, see Aldous [6] as well as Le Gall & Paulin [77] for a detailed proof, that almost surely,

$$\dim(\widehat{\mathbf{A}}_2) = \frac{1}{2} \qquad \text{and} \qquad \dim(\widehat{\mathbf{L}}_2) = \frac{3}{2}.$$

We prove more generally that for every $\alpha \in (1, 2)$, almost surely,

$$\dim(\widehat{\mathbf{A}}_\alpha) = \frac{1}{\alpha} \qquad \text{and} \qquad \dim(\widehat{\mathbf{L}}_\alpha) = 1 + \frac{1}{\alpha}.$$

Intuitively, the set $\widehat{\mathbf{L}}_\alpha$ is obtained from $\mathbf{L}_\alpha$ by adding in each face of the latter a line segment linking each vertex to the closest one to the complex 1 in clockwise order. Kortchemski [71] computed for each face V of $\mathbf{L}_\alpha$ the Hausdorff dimension of the set $\overline{V} \cap \mathbb{S}^1$ of its vertices, which is given by $1/\alpha$. Then, informally, the restriction of $\widehat{\mathbf{A}}_\alpha$ to $\overline{V}$ is $\overline{V} \cap \mathbb{S}^1$ and the Hausdorff dimension of the restriction of $\widehat{\mathbf{L}}_\alpha$ to $\overline{V}$ is given by $1 + \dim(\overline{V} \cap \mathbb{S}^1) = 1 + 1/\alpha$. Since $\widehat{\mathbf{A}}_\alpha$ and $\widehat{\mathbf{L}}_\alpha$ are the union over the countable set of faces of these restrictions, the result follows.

3.6 Some perspectives and sub-sequential work

In Chapters 6 and 7, we study the behavior of the non-crossing partition associated with a large Galton–Watson tree and its embedding in the disk. Another non-crossing configuration of the disk associated with a tree is that formed by the union of the sides of the convex polygon spanned by the n-th roots of unity for some integer n and of a collection of diagonals that may intersect only at their endpoints; such a set is called a *dissection*. Curien & Kortchemski [31] proved the convergence in distribution of large uniform random dissections towards the Brownian triangulation. Kortchemski [71] then extended this result by considering laminations chosen according to Boltzmann weights which are critical and in the domain of attraction of a stable law of index $\alpha \in (1, 2]$; he proved in this setting the convergence in distribution towards the α-stable lamination. In both cases, the proof relies as here on a bijection with critical Galton–Watson trees conditioned on their size, where "size" refers to the number of leaves in [31, 71], as opposed to the total number of vertices here.

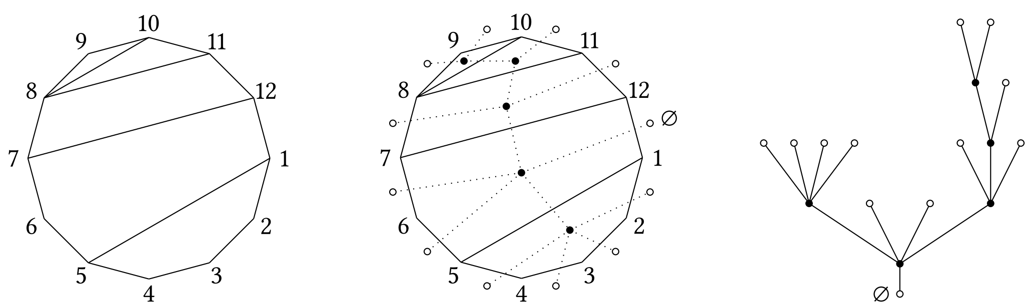


Figure 3.6: A dissection of size 12 and the associated dual plane tree, with 11 leaves.

One could also consider other sequences of random trees indexed by their size. As conditioned Galton–Watson trees fulfill the Markov branching property described in Section 1.4, it is then natural try to generalize the results presented here and that of [31, 71] to non-crossing partitions and laminations associated with Markov branching trees, as well as their embedding in the disk, as proposed (for dissections) by Curien [30, Open question 3]. Note that Duquesne's Theorem does not extend to Markov branching trees: the behavior of the discrete associated paths (for a suitable planar embedding of such trees) as the size of the tree tends to infinity is not known. The convergence of such trees, viewed as metric spaces, by Haas & Miermont [59] and Rizzolo [103] follows from other considerations and so would that of the associated non-crossing configurations of the disk.

Finally, in Chapter 7, we consider random non-crossing trees built from plane trees: we first sample a plane tree at random and then we embed it in the disk. One can also directly sample a non-crossing tree (without the property of "always turning to the right" explained in Chapter 1). Curien & Kortchemski [31] considered uniform random non-crossing trees with n vertices and proved their convergence in distribution as $n \to \infty$ to the Brownian triangulation. One may then ask for a generalization of this result when the non-crossing tree is sampled according to critical Boltzmann weights in the domain of attraction of a stable law of index $\alpha \in (1, 2]$. Surprisingly, it seems that for every $\alpha \in (1, 2]$, such a sequence converges in distribution to the Brownian triangulation.

Part II

Fires on large random trees

4

Fires on large Cayley trees

In this chapter, we study the random dynamics described in Chapter 2 where the trees considered are uniform Cayley tree with n vertices and we prove the results discussed in Section 2.2. This is based on the article [81].

4.1 Introduction and main results

Recall the fire dynamics on random trees studied in Chapter 2. Throughout this chapter, the trees T_n considered are uniform Cayley tree of size n, i.e. picked uniformly at random amongst the n^{n-2} different trees on a set of n labelled vertices, say, $[n] = \{1, \dots, n\}$. For this model, the system exhibits a phase transition as it is shown by Bertoin [16]. Theorem 1 in [16] is stated in the case where the probability to set on fire a given edge is $p_n \sim cn^{-\alpha}$ with $c, \alpha > 0$ but extends verbatim as follows: denote by I_n and B_n respectively the total number of fireproof and burnt vertices of T_n; then, as $n \to \infty$,

(i) If $n^{1/2}p_n \to \infty$, then $n^{-1}I_n \to 0$ in probability;

(ii) If $n^{1/2}p_n \to 0$, then $n^{-1}B_n \to 0$ in probability;

(iii) If $n^{1/2}p_n \to c$ for some $c > 0$, then $n^{-1}I_n \to D(c)$ in distribution where

$$\mathbb{P}(D(c) \in \mathrm{d}x) = \frac{c}{\sqrt{2\pi x(1-x)^3}} \exp\left(-\frac{c^2 x}{2(1-x)}\right)\mathrm{d}x, \quad 0 < x < 1. \tag{4.1}$$

To obtain this result, Bertoin proves that the sequence T_n satisfies the assumptions (H_k), page 27, with $r(n) = \sqrt{n}$ and an explicit limit L_k for every $k \geq 1$, see Lemma 1 in [16]; the claim then follows from Proposition 2.1 here. The aim of this chapter is to improve these three convergences. For the first two regimes, we prove a convergence in distribution to a non-trivial limit under an appropriate scaling of I_n and B_n respectively, see the statements below. For the critical regime, the proof of Lemma 1 in [16] shows in fact that the sequence T_n satisfies the assumptions (H_k'), page 29, for every $k \geq 1$ where the limiting tree $\mathcal{T}$ is the Brownian CRT; we can then apply Proposition 2.3 and give an explicit expression of the limit. The precise statement requires some notations and is postponed to Section 4.3, see Theorem 4.6 there. We next state our main result concerning the subcritical regime.

Theorem 4.1. *Suppose that* $\lim_{n\to\infty} n^{1/2}p_n = \infty$*. Then*

$$p_n^2 I_n \xrightarrow[n\to\infty]{(d)} Z^2,$$

where Z is a standard Gaussian random variable.

Consider then the supercritical regime $p_n \ll n^{-1/2}$; as we are interested in the asymptotic behavior of B_n we assume that $p_n \gg n^{-1}$ so that the probability that no fire occurs is $(1-p_n)^{n-1} \to 0$.

Theorem 4.2. *Suppose that* $\lim_{n\to\infty} n^{1/2}p_n = 0$ *and* $\lim_{n\to\infty} np_n = \infty$*. Then*

$$(np_n)^{-2} B_n \xrightarrow[n\to\infty]{(d)} Z^{-2},$$

where Z is a standard Gaussian random variable.

Remark 4.3. Let Z be a standard Gaussian random variable. One can check from (4.1) that

$$D(c) = \frac{Z^2}{c^2 + Z^2} \qquad \text{in distribution}$$

from which it follows that

$$c^2 D(c) \xrightarrow[c\to\infty]{\text{(d)}} Z^2 \qquad \text{and} \qquad c^{-2}(1 - D(c)) \xrightarrow[c\to 0]{\text{(d)}} Z^{-2}. \tag{4.2}$$

Very informally, if we write $I_n(p_n)$ for the number of fireproof vertices of T_n when the probability to set on fire a given edge is p_n, and similarly $B_n(p_n)$, then (iii) above shows that for every $c \in (0, \infty)$ fixed,

$$I_n(cn^{-1/2}) \approx nD(c) \quad \text{and} \quad B_n(cn^{-1/2}) \approx n(1 - D(c)).$$

From (4.2), one is tempted to write more generally for $p_n \gg n^{-1/2}$,

$$I_n(p_n) \approx nD(n^{1/2}p_n) \approx p_n^{-2} Z^2,$$

and for $p_n \ll n^{-1/2}$,

$$B_n(p_n) \approx n(1 - D(n^{1/2}p_n)) \approx (np_n)^2 Z^{-2}.$$

However, it does not seem clear to the author how to prove respectively Theorem 4.1 and Theorem 4.2 from this sketch. Indeed the argument in [16] does not enable one to deal with the sub or supercritical regime and the proofs given here are different from that of (i), (ii) and (iii) above.

The rest of this chapter is organized as follows. Relying on Pitman [98] and Chaumont & Uribe Bravo [29], we briefly discuss in Section 4.2 the existence of a conditional distribution for the sequence of the ranked sizes of the jumps made during the time interval $[0, 1]$ by a certain subordinator, say, σ, conditionally given the value of the latter at time 1. We also prove the continuity of this conditional distribution in the terminal value $\sigma(1)$, which will be used to derive our first result.

We then focus on the critical regime in Section 4.3. We consider the cut-tree associated with the Cayley tree T_n and the point process φ_n on the latter related to the fire dynamics as described in Section 2.1. Proposition 2.3 yields the joint convergence of the number of fireproof vertices and the sizes of the burnt connected components to the masses of the components of the CRT logged at the atoms of a point process which is a slight modified version of that studied by Aldous & Pitman [8]. Using a second approximation of the CRT with finite trees, we further express this limit as a mixture of the jumps of the previous subordinator σ conditioned on the value of the latter at time 1, with a mixing law $D(c)$ defined by (4.1).

We prove Theorem 4.1 in Section 4.4. For this, we shall see that, with high probability, the remaining forest after the first fire has a total size of order p_n^{-2} and so have its largest trees. Note that the dynamics then continue on each subtree independently. Informally, the smallest ones do not contribute much and may be neglected, while the dynamics on the largest subtrees are now critical. A slight generalization of (iii) then yields an asymptotic result for the number of fireproof vertices in each subtree and so for the total number of fireproof vertices.

Finally, we prove Theorem 4.2 in Section 4.5. Consider the sequence of the sizes of the burnt subtrees, ranked in order of appearance, and all rescaled by a factor $(np_n)^{-2}$. We prove that the latter converges in distribution for the ℓ^1 topology, from which Theorem 4.2 follows readily. To this end, we first show for every integer j the joint convergence for the size of the j first burnt subtrees; then we show that, taking j large enough, the next trees are arbitrary small.

4.2 Preliminaries on subordinators and bridges

Let $(\sigma(t); t \geq 0)$ be the first-passage time process of a linear Brownian motion: σ is a stable subordinator of index $1/2$ such that

$$\mathbb{E}[\exp(-q\sigma(t))] = \exp(-t\sqrt{2q}) \qquad \text{for any} \quad t, q \geq 0. \tag{4.3}$$

Let $J_1 \geq J_2 \geq \cdots \geq 0$ be the ranked sizes of its jumps made during the time interval $[0,1]$. We need to make sense of the conditional distribution of the sequence $(J_i)_{i\geq 1}$ conditionally given the null event $\{\sigma(1) = z\}$ in the set $\ell^1(\mathbb{R})$ of real-valued summable sequences.

From the Lévy–Itō decomposition, we know that the process σ is right-continuous, non-decreasing and increases only by jumps — we say that σ is a pure jump process — and that the pairs (t, x) induced by the times and sizes of the jumps are distributed as the atoms of a Poisson random measure on $[0,1] \times (0,\infty)$ with intensity $(2\pi x^3)^{-1/2}\mathrm{d}t\mathrm{d}x$. Denote by $(P_i)_{i\geq 1}$ a size-biased permutation of the sequence $(J_i/\sum_k J_k)_{i\geq 1}$. Pitman [98] gives an inductive construction of a regular conditional distribution for $(P_i)_{i\geq 1}$ given $\{\sum_k J_k = z\}$ for arbitrary $z > 0$. The latter determines the conditional distribution of $(J_i/\sum_k J_k)_{i\geq 1}$ given $\{\sum_k J_k = z\}$ called Poisson–Kingman distribution. Descriptions of finite-dimensional distributions can be found in Perman [96] or in Pitman & Yor [99]. Our purpose here is to check that these distributions depend continuously on the variable $z \in (0,\infty)$.

Proposition 4.4. *The conditional distribution of the ranked jump-sizes $(J_i)_{i\geq 1}$ conditionally given $\{\sigma(1) = z\}$ is continuous in z.*

Proof. In the recent work of Chaumont & Uribe Bravo [29], sufficient conditions on the distribution of a Markov process $(X_t; t \geq 0)$ in a quite general metric space are given in order to make sense of a conditioned version of $(X_s; 0 \leq s \leq t)$ given $\{X_0 = x \text{ and } X_t = y\}$. The latter is called Markovian bridge from x to y of length t and its law is denoted by $\mathbf{P}^t_{x,y}$. The process σ fulfills the framework of their Theorem 1 and Corollary 1, it follows that the bridge laws $\mathbf{P}^1_{0,z}$ are well defined and continuous in z for the Skorohod topology. Thanks to Skorohod's representation Theorem, the claim thus reduces to the deterministic result below. □

Let $f, f_1, f_2, \ldots$ be functions defined from $[0,1]$ to $[0,\infty)$ which are non-decreasing, right-continuous and null at 0. Denote by $j_1 \geq j_2 \geq \cdots \geq 0$ the ranked sizes of the jumps of f and respectively, $j_1^{(n)} \geq j_2^{(n)} \geq \cdots \geq 0$ that of f_n for every $n \geq 1$.

Lemma 4.5. *Suppose that f_n converges to f for the Skorohod topology. Then*

(i) *For any integer* N, $(j_1^{(n)}, \dots, j_N^{(n)})$ *converges to* $(j_1, \dots, j_N)$ *in* $\mathbb{R}^N$.

(ii) *If* f *is a pure jump function, then* $(j_k^{(n)})_{k\geq 1}$ *converges to* $(j_k)_{k\geq 1}$ *in* $\ell^1(\mathbb{R})$.

Proof. For the first claim, suppose first that f has infinitely many jumps. We may, and do, assume that N is such that $j_N > j_{N+1}$. For any t, denote by $\Delta f(t) := f(t) - f(t-)$ the size of the jump made by f at time t and similarly Δf_n for every $n \geq 1$. Upon changing the time scale using a sequence of increasing homeomorphisms from $[0, 1]$ onto itself which converges uniformly to the identity, we may assume that f_n converges to f uniformly. This does not affect the jump-sizes of f_n. Then $\Delta f_n(t)$ converges to $\Delta f(t)$ for every t and $(j_1, \dots, j_N)$ are limits of N jumps of f_n. Moreover, these jumps are $(j_1^{(n)}, \dots, j_N^{(n)})$ for n large enough since, for any $\varepsilon \in (0, j_N - j_{N+1})$, for any n large enough, as f_n converges to f uniformly, it admits no other jump larger than $j_{N+1} + \varepsilon/2 < j_N - \varepsilon/2$. If f has only finitely many jumps, say, N, this reasoning yields the convergences $(j_1^{(n)}, \dots, j_N^{(n)}) \to (j_1, \dots, j_N)$ and $j_k^{(n)} \to 0$ for any $k \geq N + 1$.

For the second claim, we write for any integer N fixed,

$$\sum_{k=1}^{\infty} |j_k^{(n)} - j_k| \leq \sum_{k=1}^{N} |j_k^{(n)} - j_k| + \sum_{k=N+1}^{\infty} j_k + \sum_{k=N+1}^{\infty} j_k^{(n)}.$$

As $n \to \infty$, the first term tends to 0 from (i). Let $\varepsilon > 0$ and fix N such that $\sum_{k=N+1}^{\infty} j_k < \varepsilon$. Since f is a pure jump function, we have $f(1) = \sum_{k=1}^{\infty} j_k$ and so $\sum_{k=1}^{N} j_k \geq f(1) - \varepsilon$. Finally, since $\lim_{n\to\infty} f_n(1) = f(1)$, we conclude that $\sum_{k=N+1}^{\infty} j_k^{(n)} \leq f_n(1) - \sum_{k=1}^{N} j_k^{(n)} \leq 2\varepsilon$ for n large enough. □

4.3 Asymptotic size of the burnt subtrees in the critical regime

Fix $c \in (0, \infty)$ and consider the critical regime $p_n \sim cn^{-1/2}$ of the fire dynamics on T_n. Let κ_n be the number of burnt subtrees, $b_{n,1}, \dots, b_{n,\kappa_n}$ their respective size, listed in order of appearance, and finally $b_{n,1}^* \geq \cdots \geq b_{n,\kappa_n}^*$ a non-increasing rearrangement of the latter. We can now state the main result of this section.

Theorem 4.6. *For all continuous and bounded maps* $f : (0, 1) \to \mathbb{R}$ *and* $F : \ell^1(\mathbb{R}) \to \mathbb{R}$, *we* *have*

$$\lim_{n\to\infty} \mathbb{E}\left[f\left(\frac{I_n}{n}\right)F\left(\frac{b^*_{n,1}}{n},\ldots,\frac{b^*_{n,\kappa_n}}{n}\right)\right]$$
$$= \int_0^1 f(x)\mathbb{E}\left[F\left(\frac{(1-x)J_1}{\sigma(1)},\frac{(1-x)J_2}{\sigma(1)},\ldots\right)\,\middle|\, \sigma(1)=\frac{1-x}{c^2x^2}\right]\mathbb{P}(D(c)\in \mathrm{d}x),$$

where σ is a subordinator distributed as (4.3) and $\mathbb{P}(D(c)\in \mathrm{d}x)$ is defined in (4.1).

Note that, taking $F\equiv 1$, this recovers the result (iii) in the beginning of this chapter; moreover, since $\sum_i J_i = \sigma(1)$, it strengthens (iii) by giving the decomposition of the burnt forest conditionally given its total size.

The proof is divided in two parts. As discussed in Chapter 2, we view the fire dynamics on T_n as a mark process on the associated cut-tree $\mathrm{Cut}(\mathrm{T}_n)$, which translates the vector $n^{-1}(I_n, b^*_{n,1},\ldots,b^*_{n,\kappa_n})$ into the proportion of leaves of the trees in the forest obtained by logging $\mathrm{Cut}(\mathrm{T}_n)$ at the marks. We prove that the marked tree $\mathrm{Cut}(\mathrm{T}_n)$, properly rescaled, converges to the Brownian CRT endowed with a certain point process; it follows that the previous vector converges to the masses of the trees in the forest obtained by logging the CRT at the atoms of the point process. We then study the distribution of the latter. As direct computations with the CRT seem rather complicated, we approximate the marked CRT by a Galton–Watson tree with Poisson(1) offspring distribution conditioned to have n vertices and endowed with a similar mark process as the cut-tree $\mathrm{Cut}(\mathrm{T}_n)$. We refer to Chapter 1 for prerequisites about the CRT and convergence of conditioned Galton–Watson trees, as well as Aldous & Pitman [8] for its logging by a Poisson point process.

4.3.1 Cut-tree, fires and mark a process

Recall the definition of the cut-tree $\mathrm{Cut}(\mathrm{T}_n)$ associated with T_n from Section 2.1. As described there, we define on $\mathrm{Cut}(\mathrm{T}_n)$ a point process φ_n by the following two-steps procedure: mark first every internal block independently with probability p_n, then along each branch from the root to a leaf, keep only the closest mark to the root and erase the other marks. We have seen that the fire dynamics on T_n can be coupled with the cut-tree endowed with the marks induced by φ_n in such a way that the marked blocks of $\mathrm{Cut}(\mathrm{T}_n)$ correspond to the burnt subtrees of T_n and the leaves of $\mathrm{Cut}(\mathrm{T}_n)$ which do not possess a marked ancestor correspond to the fireproof vertices of T_n, see Figure 4.1 for an illustration. It follows that the entries of the vector $(I_n, b^*_{n,1},\ldots,b^*_{n,\kappa_n})$ count the number of leaves of each tree in the forest obtained by logging $\mathrm{Cut}(\mathrm{T}_n)$ at the marks of φ_n, the root-component first, and the next in non-increasing order. We implicitly assume in the sequel that the fire dynamics on T_n and the pair $(\mathrm{Cut}(\mathrm{T}_n), \varphi_n)$ are coupled in this way.

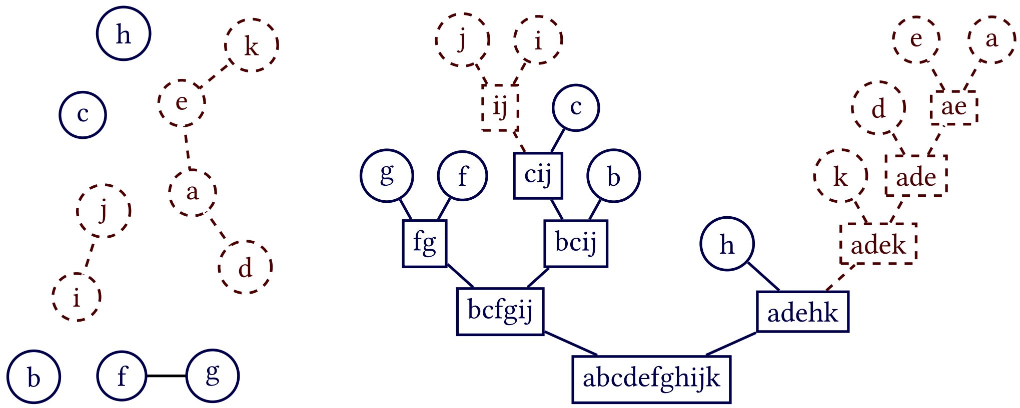


Figure 4.1: The forest after the dynamics and the corresponding marked cut-tree.

Let $\mathcal{T}$ be a rooted Brownian CRT, μ its uniform probability mass measure on leaves and ℓ its length measure. Recall the definition of the process Φ_c on the skeleton of $\mathcal{T}$ defined in Section 2.1: first sample a Poisson point process with intensity $c\ell(\cdot)$, then, along each branch from the root to a leaf, keep only the closest mark to the root (if any) and erase the other marks. Denote by $\#(\mathcal{T}, \Phi_c)$ the sequence of the μ-mass of each tree in the forest obtained by logging $\mathcal{T}$ at the atoms of Φ_c, the root-component first, and the next in non-increasing order. As explained in Section 2.2, the proof of Lemma 1 of Bertoin [16] shows that the assumption (H'_k) is fulfilled for every $k \geq 1$ with $r(n) = \sqrt{n}$: if we denote by $\mathcal{R}(\mathcal{T}, k)$ the tree $\mathcal{T}$ spanned by its root and k i.i.d. elements chosen according to μ and similarly $R_{n,k}$ the tree $\mathrm{Cut}(\mathrm{T}_n)$ reduced to k independent uniform random leaves, then

$$\frac{1}{\sqrt{n}} R_{n,k} \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad \mathcal{R}(\mathcal{T}, k),$$

for every $k \geq 1$ fixed. Proposition 2.3 then reads

$$n^{-1}(I_n, b^*_{n,1}, \ldots, b^*_{n,\kappa_n}) \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad \#(\mathcal{T}, \Phi_c). \tag{4.4}$$

To complete the proof of Theorem 4.6, we need to identify the limiting distribution $\#(\mathcal{T}, \Phi_c)$. As direct computations with the CRT seem rather complicated, we use a second discrete approximation of the latter. Denote by T_n a Galton–Watson tree with Poisson(1) offspring distribution conditioned to have n vertices and where labels are assigned to the vertices uniformly at random. As discussed in Example 1.2, T_n is distributed as a uniform rooted Cayley tree with n vertices; moreover, we have seen in Section 1.4 that rescaled by a factor $n^{-1/2}$, it converges to the Brownian CRT. We endow T_n with the mark process ψ_n defined in two steps: first mark every vertex of T_n independently with probability p_n, then along each branch from the root to a leaf, keep only the closest mark to the root and erase the others. Adapting Proposition 2.3 to T_n and the uniform probability on vertices, we get

$$n^{-1}(C_{n,0}, C^*_{n,1}, \ldots, C^*_{n,M_n}) \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad \#(\mathcal{T}, \Phi_c). \tag{4.5}$$

where $C_{n,0}$ denotes the size of the connected component of T_n that contains the root, M_n the number of marks of ψ_n and $C^*_{n,1} \geq \cdots \geq C^*_{n,M_n}$ the respective sizes of the other connected components, listed in non-increasing order. We now study the asymptotic behavior of this vector in order to show that the right-hand side above is the limit in Theorem 4.6.

4.3.2 Asymptotic behavior of the size of the burnt blocks

Using the same notation as in (4.5), we have the following limit theorem for T_n.

Proposition 4.7. *For all continuous and bounded maps $f\colon (0,1) \to \mathbb{R}$ and $F\colon \ell^1(\mathbb{R}) \to \mathbb{R}$, we have*

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left[f\left(\frac{C_{n,0}}{n}\right)F\left(\frac{C^*_{n,1}}{n}, \ldots, \frac{C^*_{n,M_n}}{n}\right)\right] \\ = \int_0^1 f(x)\mathbb{E}\left[F\left(\frac{(1-x)J_1}{\sigma(1)}, \frac{(1-x)J_2}{\sigma(1)}, \ldots\right) \,\middle|\, \sigma(1) = \frac{1-x}{c^2x^2}\right]\mathbb{P}(D(c) \in \mathrm{d}x),\end{aligned}$$

where σ is a subordinator distributed as (4.3) and $\mathbb{P}(D(c) \in \mathrm{d}x)$ is defined in (4.1).

Before proving this result, notice first that Theorem 4.6 is a direct consequence of Proposition 4.7 and the convergences (4.4) and (4.5).

Proof of Theorem 4.6. Let $f\colon (0,1) \to \mathbb{R}$ and $F\colon \ell^1(\mathbb{R}) \to \mathbb{R}$ be two continuous and bounded maps. From (4.4) and (4.5), the sequences

$$\mathbb{E}\left[f\left(\frac{I_n}{n}\right)F\left(\frac{b^*_{n,1}}{n}, \ldots, \frac{b^*_{n,\kappa_n}}{n}\right)\right] \qquad \text{and} \qquad \mathbb{E}\left[f\left(\frac{C_{n,0}}{n}\right)F\left(\frac{C^*_{n,1}}{n}, \ldots, \frac{C^*_{n,M_n}}{n}\right)\right]$$

both converge to the same limit as $n \to \infty$ and Proposition 4.7 gives the expression of the latter, which is the one claimed in Theorem 4.6. $\square$

It remains to prove Proposition 4.7. For any positive real number a, we define the Borel distribution with parameter a, which is the law of the size of a Galton–Watson tree with Poisson(a) offspring distribution:

$$\mathbb{P}(\text{Borel}(a) = n) = \frac{1}{n!}\mathrm{e}^{-na}(na)^{n-1}, \qquad n \geq 1.$$

We also define for any integer k, the Borel-Tanner distribution with parameter k as the sum of k i.i.d. Borel(1) variables:

$$\mathbb{P}(\text{Borel-Tanner}(k) = n) = \frac{k}{(n-k)!}\mathrm{e}^{-n}n^{n-k-1}, \qquad n \geq k.$$

Borel and Borel-Tanner distributions appear in our context as the sizes of the connected components of T_n.

Lemma 4.8. *For any integers k, ℓ with $k + \ell \leq n$, conditional on the event $\{C_{n,0} = k, M_n = \ell\}$, the vector $(C^*_{n,1}, \dots, C^*_{n,\ell})$ is distributed as a non-increasing rearrangement of ℓ i.i.d.* Borel(1) *random variables conditioned to have sum $n - k$.*

Proof. We explicitly write the condition for the size of the tree. Let T be a Galton–Watson tree with Poisson(1) offspring distribution; we endow it with the same mark process ψ_n. Denote by $\tilde{M}_n$ the number of marks, $\tilde{C}_{n,0}$ the size of the root-component and, conditional on $\{\tilde{M}_n = \ell\}$, let $\tilde{C}^*_{n,1} \geq \dots \geq \tilde{C}^*_{n,\ell}$ be the ranked sizes of the other components. Note that on the event $\{\tilde{M}_n = \ell\}$, we have $|\mathrm{T}| = \tilde{C}_{n,0} + \tilde{C}^*_{n,1} + \dots + \tilde{C}^*_{n,\ell}$.

Condition on the event $\{\tilde{M}_n = \ell\}$; it is known that the subtrees of T generated by the ℓ atoms of the point process are independent Galton–Watson trees with Poisson(1) offspring distribution, independent of $\tilde{C}_{n,0}$. Hence, on the event $\{\tilde{M}_n = \ell\}$, $\tilde{C}^*_{n,1}, \dots, \tilde{C}^*_{n,\ell}$ are i.i.d. Borel(1) random variables, listed in non-increasing order and independent of $\tilde{C}_{n,0}$. Further, on the event $\{|\mathrm{T}| = n, \tilde{M}_n = \ell, \tilde{C}_{n,0} = k\}$, $\tilde{C}^*_{n,1}, \dots, \tilde{C}^*_{n,\ell}$ are conditioned to have sum $n - k$. □

The Borel(1) distribution belongs to the domain of attraction of a stable law of index 1/2. A consequence tailored for our need is the following: let $(\beta_i)_{i\geq 1}$ be i.i.d. Borel(1) random variables, and for any $k \geq 1$, denote by $\beta^*_1 \geq \dots \geq \beta^*_k$ the order statistics of the first k elements of the latter. Let also σ be a subordinator distributed as (4.3) and $J_1 \geq J_2 \geq \dots \geq 0$ the ranked sizes of its jumps made during the time interval $[0, 1]$.

Lemma 4.9. *Let $\lambda, \nu > 0$ and two sequences of integers k_n and a_n such that $n^{-1/2}k_n \to \lambda$ and $n^{-1}a_n \to \nu$ as $n \to \infty$. Then the convergence in distribution*

$$\left(\left(\frac{1}{n}\sum_{i=1}^{\lfloor\sqrt{n}t\rfloor\wedge k_n}\beta_i; t \geq 0\right) \middle| \sum_{i=1}^{k_n}\beta_i = a_n\right) \xrightarrow[n\to\infty]{(d)} \left((\sigma(t\wedge\lambda), t \geq 0) \,\middle|\, \sigma(\lambda) = \nu\right)$$

holds for the Skorohod topology. As a consequence, the convergence in distribution of the ranked jumps

$$\left(\left(\frac{\beta_1^*}{n},\dots,\frac{\beta_{k_n}^*}{n}\right)\,\middle|\,\sum_{i=1}^{k_n}\beta_i=a_n\right)\quad\xrightarrow[n\to\infty]{(d)}\quad\left(\frac{\nu J_1}{\sigma(1)},\frac{\nu J_2}{\sigma(1)},\cdots\,\middle|\,\sigma(1)=\frac{\nu}{\lambda^2}\right)$$

holds for the ℓ^1 topology.

Proof. The first convergence is the result stated in Lemma 11 of Aldous & Pitman [7]. The second then follows from the continuity obtained in Lemma 4.5. □

We apply this convergence to the random sequences M_n and $n-C_{n,0}$ instead of k_n and a_n. They fulfill the assumptions of Lemma 4.9 as it is shown in the following Lemma that we prove in the next subsection.

Lemma 4.10. *Let $D(c)$ be a random variable distributed as (4.1). Then*

$$\left(\frac{C_{n,0}}{n},\frac{M_n}{\sqrt{n}}\right)\quad\xrightarrow[n\to\infty]{(d)}\quad(D(c),cD(c)).$$

In order to go from deterministic sequences to random sequences, we also use the following elementary result (see Carathéodory [27], Part Four, Chapter I). Let $\mathbb{X}$ and $\mathbb{Y}$ be metric spaces and $f,f_1,f_2,\dots$ be functions defined from $\mathbb{X}$ to $\mathbb{Y}$. We say that f_n *converges continuously* to f if for any $x,x_1,x_2,\dots\in\mathbb{X}$ such that $\lim_{n\to\infty}x_n=x$ in $\mathbb{X}$, we have $\lim_{n\to\infty}f_n(x_n)=f(x)$ in $\mathbb{Y}$. Then f_n converges continuously to f if and only if f is continuous and f_n converges to f uniformly on compact sets.

Proof of Proposition 4.7. Let $f\colon(0,1)\to\mathbb{R}$ and $F\colon\ell^1(\mathbb{R})\to\mathbb{R}$ be two continuous and bounded maps. With the notations of Lemma 4.9, define for any $(u,v)\in(0,1)\times(0,\infty)$

$$\Upsilon_n(u,v):=f(u)\mathbb{E}\left[F\left(\frac{\beta_1^*}{n},\dots,\frac{\beta^*_{\lfloor\sqrt{n}v\rfloor}}{n}\right)\,\middle|\,\sum_{i=1}^{\lfloor\sqrt{n}v\rfloor}\beta_i=n-\lfloor nu\rfloor\right],$$

and

$$\Upsilon(u,v):=f(u)\mathbb{E}\left[F\left(\frac{(1-u)J_1}{\sigma(1)},\frac{(1-u)J_2}{\sigma(1)},\dots\right)\,\middle|\,\sigma(1)=\frac{1-u}{v^2}\right].$$

Then Lemma 4.9 states that $\Upsilon_n(u_n,v_n)\to\Upsilon(u,v)$ whenever $(u_n,v_n)\to(u,v)$. On the one hand, $\mathbb{E}[\Upsilon(D(c),cD(c))]$ is the limit claimed in Proposition 4.7 and, from Lemma 4.8, the $C_{n,i}^*$'s are, conditionally given $C_{n,0}$ and M_n, distributed as ranked i.i.d. Borel(1) random variables conditioned to have sum $n-C_{n,0}$. Then we also have

$$\mathbb{E}\left[\Upsilon_n\left(\frac{C_{n,0}}{n},\frac{M_n}{\sqrt{n}}\right)\right]=\mathbb{E}\left[f\left(\frac{C_{n,0}}{n}\right)F\left(\frac{C_{n,1}^*}{n},\dots,\frac{C_{n,M_n}^*}{n}\right)\right].$$

On the other hand, from the discussion above, Υ is continuous (which is also a consequence of Proposition 4.4) and $\Upsilon_n \to \Upsilon$ uniformly on compact sets. Let us bound from above

$$\left|\mathbb{E}\left[\Upsilon_n\left(\frac{C_{n,0}}{n}, \frac{M_n}{\sqrt{n}}\right)\right] - \mathbb{E}\left[\Upsilon(D(c), cD(c))\right]\right|$$

by

$$\mathbb{E}\left[\left|\Upsilon_n\left(\frac{C_{n,0}}{n}, \frac{M_n}{\sqrt{n}}\right) - \Upsilon\left(\frac{C_{n,0}}{n}, \frac{M_n}{\sqrt{n}}\right)\right|\right] + \left|\mathbb{E}\left[\Upsilon\left(\frac{C_{n,0}}{n}, \frac{M_n}{\sqrt{n}}\right)\right] - \mathbb{E}\left[\Upsilon(D(c), cD(c))\right]\right|.$$

From Lemma 4.10, since Υ is continuous and bounded, the second term tends to 0. Moreover $\Upsilon, \Upsilon_1, \Upsilon_2, \dots$ are uniformly bounded, say by $C > 0$, therefore the first term is bounded from above by

$$\sup_{x \in K}\left|\Upsilon_n(x) - \Upsilon(x)\right| + 2C\mathbb{P}\left(\left(\frac{C_{n,0}}{n}, \frac{M_n}{\sqrt{n}}\right) \notin K\right),$$

for any compact K. The first term of the latter converges to 0 for any K and the second can be made arbitrary small as the sequence is tight. □

4.3.3 Asymptotic behavior of the number of burnt blocks

We finally prove Lemma 4.10 which completes the proof of Proposition 4.7 and thereby that of Theorem 4.6. We use the two following observations: the convergence of the first marginal $n^{-1}C_{n,0}$ holds and the conditional distribution of M_n given $C_{n,0}$ is known explicitly.

Lemma 4.11. *We have $n^{-1}C_{n,0} \to D(c)$ in distribution as $n \to \infty$.*

Proof. Denote by $\mu(\xi_0)$ the mass of the root-component of the CRT after logging at the atoms of a Poisson point process with rate c per unit length (here keeping only the closest atoms to the root does not matter). Then (4.5) yields $\lim_{n\to\infty} n^{-1}C_{n,0} = \mu(\xi_0)$ in distribution. The claim follows from the identity $\mu(\xi_0) = D(c)$ in distribution stated in Corollary 5 of Aldous & Pitman [8] since $\mu(\xi_0)$ here is $Y_1^*(c)$ there. □

Lemma 4.12. *For any $n \geq 2$, the pair $(C_{n,0}, M_n)$ is distributed as follows: for any integers k, ℓ such that $k + \ell \leq n$,*

$$\mathbb{P}(C_{n,0} = k, M_n = \ell) = \frac{n!(k(1-p_n))^{k-1}(kp_n)^{\ell}(n-k)^{n-k-\ell-1}}{n^{n-1}k!(\ell-1)!(n-k-\ell)!}.$$

Then, on the event $\{C_{n,0} = k\}$, M_n is distributed as $X_n + 1$ where X_n is a binomial random variable with parameters $n - k - 1$ and $(kp_n)/(n - k + kp_n)$.

Proof. For the first claim, as in the proof of Lemma 4.8, we explicitly write the condition on the size of the tree and work with a Galton–Watson tree with Poisson(1) offspring distribution T:

$$\mathbb{P}(C_{n,0} = k, M_n = \ell \mid |\mathrm{T}| = n) = \frac{\mathbb{P}(C_{n,0} = k)\mathbb{P}(M_n = \ell \mid C_{n,0} = k)\mathbb{P}(|\mathrm{T}| = n \mid C_{n,0} = k, M_n = \ell)}{\mathbb{P}(|\mathrm{T}| = n)}.$$

We know that $|\mathrm{T}|$ is Borel(1) distributed. Moreover, the root-component of T is a Galton–Watson tree with Poisson($1-p_n$) offspring distribution, so that $C_{n,0}$ is Borel($1-p_n$) distributed. Next, on the event $\{C_{n,0} = k\}$, M_n is the sum of k i.i.d. Poisson(p_n) random variables, so is Poisson(kp_n) distributed. Finally, from Lemma 4.8, on $\{C_{n,0} = k, M_n = \ell\}$, $|\mathrm{T}| - k$ is the sum of ℓ i.i.d. Borel(1) random variables, i.e. is Borel-Tanner(ℓ) distributed. Putting the pieces together gives the first claim. For the second claim (with the implicit condition $|\mathrm{T}| = n$), we then directly compute

$$\begin{aligned}\mathbb{P}(M_n = \ell \mid C_{n,0} = k) &= \mathbb{P}(C_{n,0} = k, M_n = \ell)\left(\sum_{j=1}^{n-k} \mathbb{P}(C_{n,0} = k, M_n = j)\right)^{-1} \\ &= \frac{(n-k-1)!}{(\ell-1)!(n-k-\ell)!}\left(\frac{kp_n}{n-k+kp_n}\right)^{\ell-1}\left(\frac{n-k}{n-k+kp_n}\right)^{n-k-\ell} \\ &= \mathbb{P}(X_n = \ell - 1),\end{aligned}$$

where X_n is the desired binomial random variable. □

We can now prove Lemma 4.10.

Proof of Lemma 4.10. We aim to show that for any $s, t \geq 0$,

$$\lim_{n\to\infty} \mathbb{E}\left[\exp\left(-s\frac{C_{n,0}}{n} - t\frac{M_n}{\sqrt{n}}\right)\right] = \mathbb{E}[\exp(-(s+ct)D(c))].$$

From Lemma 4.11, $n^{-1}C_{n,0} \to D(c)$ in distribution, it is thus sufficient to show

$$\limsup_{n\to\infty}\left|\mathbb{E}\left[\exp\left(-s\frac{C_{n,0}}{n} - t\frac{M_n}{\sqrt{n}}\right)\right] - \mathbb{E}\left[\exp\left(-(s+ct)\frac{C_{n,0}}{n}\right)\right]\right| = 0.$$

Let $\varepsilon > 0$ and fix $\delta > 0$ such that for any n large enough, $\mathbb{P}(C_{n,0} > (1-\delta)n) \leq \varepsilon$. We then reduce to show the above convergence on the event $\{C_{n,0} \leq (1-\delta)n\}$. Using Lemma 4.12, we compute for any $1 \leq k \leq n-1$ and any $t \geq 0$,

$$\mathbb{E}\left[\mathrm{e}^{-tM_n} \mid C_{n,0} = k\right] = \mathbb{E}\left[\mathrm{e}^{-t(X_n+1)}\right] = \mathrm{e}^{-t}\left(1 - \frac{kp_n(1-\mathrm{e}^{-t})}{n-k+kp_n}\right)^{n-k-1}.$$

Conditioning first on the value of $C_{n,0}$ and then averaging, we obtain

$$\mathbb{E}\left[\exp\left(-s\frac{C_{n,0}}{n}-t\frac{M_n}{\sqrt{n}}\right)\mathbb{I}_{\{C_{n,0}\leq(1-\delta)n\}}\right] \\ =\sum_{k=1}^{\lfloor(1-\delta)n\rfloor}\mathbb{P}(C_{n,0}=k)\exp\left(-\frac{sk}{n}\right)\exp\left(-\frac{t}{\sqrt{n}}\right)\left(1-\frac{kp_n(1-\mathrm{e}^{-t/\sqrt{n}})}{n-k+kp_n}\right)^{n-k-1}.$$

Remark that, uniformly for $k\leq\lfloor(1-\delta)n\rfloor$,

$$\frac{kp_n(1-\mathrm{e}^{-t/\sqrt{n}})}{n-k+kp_n}=\frac{1}{n-k}\left(\frac{kct}{n}+o(1)\right)\qquad\text{as } n\to\infty.$$

As a consequence, as $n\to\infty$,

$$\exp\left(-\frac{t}{\sqrt{n}}\right)\left(1-\frac{kp_n(1-\mathrm{e}^{-t/\sqrt{n}})}{n-k+kp_n}\right)^{n-k-1}=\exp\left(-\frac{kct}{n}\right)(1+o(1)),$$

uniformly for $k\leq\lfloor(1-\delta)n\rfloor$. Finally, the difference

$$\mathbb{E}\left[\exp\left(-s\frac{C_{n,0}}{n}-t\frac{M_n}{\sqrt{n}}\right)\mathbb{I}_{\{C_{n,0}\leq(1-\delta)n\}}\right]-\mathbb{E}\left[\exp\left(-(s+ct)\frac{C_{n,0}}{n}\right)\mathbb{I}_{\{C_{n,0}\leq(1-\delta)n\}}\right]$$

tends to 0 as $n\to\infty$, which completes the proof. □

4.4 Asymptotic proportion of fireproof vertices in the subcritical regime

We now consider the subcritical regime $p_n\gg n^{-1/2}$ of the dynamics on T_n. We prove Theorem 4.1:

$$p_n^2 I_n \xrightarrow[n\to\infty]{(\mathrm{d})} Z^2, \tag{4.6}$$

where Z is a standard Gaussian random variable, as well as the following result on the size of the largest fireproof component.

Proposition 4.13. *For any $\varepsilon>0$, with a probability converging to 1 as $n\to\infty$, there exists at least one fireproof subtree larger than $n^{-\varepsilon}p_n^{-2}$ but none larger than εp_n^{-2}.*

Let us sketch our approach to establish (4.6). We let the dynamics evolve until an edge is set on fire for the first time, denoting this random time by $\zeta_n\in\mathbb{N}\cup\{\infty\}$. The event $\{\zeta_n=\infty\}$

corresponds to the case where the whole tree is fireproof at the end. Conditional on $\{\zeta_n = k\}$ with $k \in \mathbb{N}$, if we delete the $k-1$ first fireproof edges, we get a decomposition of T_n into a forest of k trees. Then we set on fire an edge of this forest uniformly at random and burn the whole subtree that contains the latter. The burnt subtree is therefore picked at random with a probability proportional to its number of edges. We then study the dynamics which continue independently on each of the $k-1$ other subtrees.

Let σ be a subordinator distributed as (4.3) and $J_1 \geq J_2 \geq \cdots \geq 0$ the sizes of its jumps made during the time interval $[0,1]$. Let also $\mathbf{e}$ be an exponential random variable with parameter 1 independent of σ. We shall see that $p_n\zeta_n$ converges to $\mathbf{e}$ in distribution and that the sequence given by the sizes of the non-burnt subtrees at time ζ_n, ranked in non-increasing order and rescaled by a factor p_n^2, converges in distribution to $(\mathbf{e}^2 J_k)_{k\geq 1}$ in ℓ^1. Conditionally given $(\mathbf{e}^2 J_k)_{k\geq 1}$, we define a sequence $(X_k(\mathbf{e}))_{k\geq 1}$ of independent random variables sampled according to $\mu_{\mathbf{e}^2 J_k}$ respectively, where for every $x > 0$, μ_x is the probability measure given by

$$\mu_x(\mathrm{d}y) = \left(\frac{x^3}{2\pi y(x-y)^3}\right)^{1/2} \exp\left(-\frac{xy}{2(x-y)}\right)\mathrm{d}y, \qquad 0 < y < x. \tag{4.7}$$

Note that if X is distributed as μ_x, then $x^{-1}X$ is distributed as $D(x^{1/2})$, defined in (4.1). Indeed, μ_x is the limit of the number of fireproof vertices in a subtree of asymptotic size xp_n^{-2} (see Lemma 4.15 for a precise statement). Informally, summing over all subtrees, since the dynamics on each are independent, we get

$$p_n^2 I_n \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad \sum_{k=1}^{\infty} X_k(\mathbf{e}). \tag{4.8}$$

Finally, (4.6) follows from the identity

$$\sum_{k=1}^{\infty} X_k(\mathbf{e}) = Z^2 \qquad \text{in distribution.} \tag{4.9}$$

To derive this, note that, conditionally given $\mathbf{e}$, the sequence $(\mathbf{e}^2 J_k)_{k\geq 1}$ is distributed as the ranked atoms of a Poisson random measure on $(0,\infty)$ with intensity $\mathbf{e}(2\pi x^3)^{-1/2}\mathrm{d}x$. Further, conditionally given $\mathbf{e}$, the sequence $(\mathbf{e}^2 J_k, X_k(\mathbf{e}))_{k\geq 1}$ is distributed as the atoms of a Poisson random measure on $(0,\infty)^2$ with intensity $\mathbf{e}(2\pi x^3)^{-1/2}\mathrm{d}x\mu_x(\mathrm{d}y)$, ranked in the non-increasing order of the first coordinate. Therefore, conditioning first on $\mathbf{e}$, using Laplace formula and then averaging, we have for any $q > 0$,

$$\mathbb{E}\left[\exp\left(-q\sum_{k=1}^{\infty} X_k(\mathbf{e})\right)\right] = \int_0^{\infty} \exp\left(-\int_{(0,\infty)^2} (1-\mathrm{e}^{-qy})\frac{t}{\sqrt{2\pi x^3}}\mathrm{d}x\mu_x(\mathrm{d}y)\right)\mathrm{e}^{-t}\mathrm{d}t.$$

Using the definition of μ_x and the change of variables $(x, y) \mapsto (y(x-y)^{-1/2}, y)$, we see that the right-hand side is equal to

$$\int_0^\infty \exp\left(- t - t \int_0^\infty (1 - \mathrm{e}^{-qy}) \mathrm{e}^{-y/2} \frac{\mathrm{d}y}{\sqrt{2\pi y^3}} \right) \mathrm{d}t.$$

We write

$$(1 - \mathrm{e}^{-qy}) \mathrm{e}^{-y/2} = \left(1 - \exp\left(- \frac{2q+1}{2} y \right)\right) - \left(1 - \exp\left(- \frac{y}{2} \right)\right);$$

since

$$\int_0^\infty \left(1 - \exp\left(- \frac{z^2 y}{2} \right)\right) \frac{\mathrm{d}y}{\sqrt{2\pi y^3}} = z \qquad \text{for any} \quad z > 0,$$

we finally obtain for any $q > 0$,

$$\mathbb{E}\left[\exp\left(- q \sum_{k=1}^\infty X_k(\mathbf{e}) \right) \right] = \int_0^\infty \exp\left(- t\sqrt{2q+1} \right) \mathrm{d}t = \frac{1}{\sqrt{2q+1}} = \mathbb{E}[\exp(-qZ^2)].$$

In the rest of this section, we first prove the convergence of the sequence of the sizes of the non-burnt trees after the first fire. We then establish (4.8) which, by (4.9), proves Theorem 4.1. Finally, we prove Proposition 4.13.

4.4.1 Configuration at the instant of the first fire

Recall that we denote by ζ_n the first instant where an edge is set on fire during the dynamics on T_n. Then ζ_n is a truncated geometric random variable:

$$\begin{aligned} &\mathbb{P}(\zeta_n = \infty) = (1 - p_n)^{n-1} \to 0 \text{ as } n \to \infty, \\ \text{and} \quad &\mathbb{P}(\zeta_n = k) = p_n(1 - p_n)^{k-1} \quad \text{for every } k \in \{1, \dots, n-1\}. \end{aligned}$$

So $p_n \zeta_n$ converges in distribution to an exponential random variable with rate 1. For each integer $k \le n$, we denote by $\mathrm{T}_{n,1}, \dots, \mathrm{T}_{n,k}$ the forest obtained by deleting $k-1$ edges of T_n uniformly at random, where the labelling is made uniformly at random, and $|\mathrm{T}_{n,1}|^* \ge \cdots \ge |\mathrm{T}_{n,k}|^*$ a non-increasing rearrangement of their sizes. We know from Lemma 5 of Bertoin [16] (see also Pavlov [95] or Pitman [97]) that the sizes of these k subtrees are distributed as i.i.d. Borel(1) random variables conditioned to have sum n: for any $n_1, \dots, n_k \ge 1$ such that $n_1 + \cdots + n_k = n$,

$$\mathbb{P}(|\mathrm{T}_{n,1}| = n_1, \dots, |\mathrm{T}_{n,k}| = n_k) = \frac{(n-k)!}{kn^{n-k-1}} \prod_{j=1}^k \frac{n_j^{n_j - 1}}{n_j!}. \tag{4.10}$$

Moreover, conditionally on the partition of $\{1, \dots, n\}$ induced by the k subsets of vertices of these subtrees, the $\mathrm{T}_{n,i}$'s are independent uniform Cayley trees on their respective set of vertices (recall the splitting property).

Notice that on the event $\{\zeta_n \geq k\}$, the forest obtained by deleting the first $k-1$ fireproof edges is distributed as $\mathrm{T}_{n,1}, \dots, \mathrm{T}_{n,k}$ and, further, ζ_n is independent of the latter. The forest at the instant of the first fire is then distributed as follows: we first sample a (truncated) geometric variable ζ_n and then independently the uniform forest $\mathrm{T}_{n,1}, \dots, \mathrm{T}_{n,\zeta_n}$. Recall that the Borel(1) distribution belongs to the domain of attraction of a stable law of index $1/2$ so that, taking a number of order p_n^{-1} of i.i.d. such random variables, the sum is typically of order p_n^{-2}. Then, loosely speaking, conditioning this sum to be abnormally large, here of order n, essentially amounts to conditioning one single variable to be large, the others being almost unaffected. This feature is formalized in the next proposition.

Proposition 4.14. *The convergence in distribution*

$$p_n^2(n - |\mathrm{T}_{n,1}|^*, |\mathrm{T}_{n,2}|^*, \dots, |\mathrm{T}_{n,\zeta_n}|^*) \quad \xrightarrow[n\to\infty]{(d)} \quad \mathbf{e}^2(\sigma(1), J_1, J_2, \dots)$$

holds for the ℓ^1 topology.

Proof. The proof is similar to that of Proposition 4.7. Aldous & Pitman [8, Equation (34)] provide the convergence in distribution

$$k_n^{-2}(n - |\mathrm{T}_{n,1}|^*, |\mathrm{T}_{n,2}|^*, \dots, |\mathrm{T}_{n,k_n}|^*) \quad \xrightarrow[n\to\infty]{\text{(d)}} \quad (\sigma(1), J_1, J_2, \dots)$$

for any deterministic sequence $k_n = o(n^{1/2})$. Let $f : \ell^1(\mathbb{R}) \to \mathbb{R}$ be a continuous and bounded function and set for any $x > 0$

$$F_n(x) := \mathbb{E}\left[f\left(p_n^2\left(n - |\mathrm{T}_{n,1}|^*, |\mathrm{T}_{n,2}|^*, \dots, |\mathrm{T}_{n,\lfloor xp_n^{-1}\rfloor}|^*\right)\right)\right],$$

and

$$F(x) := \mathbb{E}[f(x^2(\sigma(1), J_1, J_2, \dots))].$$

The previous convergence yields $\lim_{n\to\infty} F_n(x_n) = F(x)$ whenever $\lim_{n\to\infty} x_n = x$. Using Skorohod's representation Theorem, we may suppose $\lim_{n\to\infty} p_n\zeta_n = \mathbf{e}$ almost surely. Since ζ_n is independent of the $\mathrm{T}_{n,i}$'s, we have $\lim_{n\to\infty} F_n(p_n\zeta_n) = F(\mathbf{e})$ almost surely and the claim follows from Lebesgue's Theorem. □

Recall that the first fire burns one subtree in the forest $\mathrm{T}_{n,1}, \dots, \mathrm{T}_{n,\zeta_n}$ and that the latter is chosen at random with a probability proportional to its size minus one. Therefore, with high probability, this burnt subtree has a size of order n and the forest that we obtain by discarding this tree and the edges previously fireproof has a total size of order $p_n^{-2} = o(n)$. We already recover the convergence $n^{-1}I_n \to 0$ of Bertoin [16]. The fire dynamics then continue independently on each tree of this forest and the total number of fireproof vertices is the sum of the number of fireproof vertices in each component.

4.4.2 Total number of fireproof vertices

We now study the dynamics on the remaining forest after the first fire. We know from Proposition 4.14 that with high probability, the largest trees have size of order p_n^{-2} so that they are now critical for the dynamics which continue on each with parameter $p_n = (p_n^{-2})^{-1/2}$. To see this, we slightly generalize the convergence (iii) at the beginning of this Chapter. Let $(\mathrm{T}'_n)_{n\geq 1}$ be a sequence of Cayley trees with size $|\mathrm{T}'_n| \sim ap_n^{-2}$ as $n \to \infty$ for some $a > 0$ and denote by I'_n the number of fireproof vertices of T'_n.

Lemma 4.15. *The law of $p_n^2 I'_n$ converges weakly to the distribution μ_a defined by (4.7).*

Proof. The proof of Theorem 1 in [16] shows that $|\mathrm{T}'_n|^{-1}I'_n$, the proportion of fireproof vertices in T'_n, converges in distribution to $D(a^{1/2})$, as defined in (4.1). Since $p_n^2|\mathrm{T}'_n| \to a$, we get $p_n^2 I'_n \to aD(a^{1/2})$ in distribution. One easily checks that the latter is distributed according to μ_a. □

Using Proposition 4.14 and Lemma 4.15, we can now prove the convergence (4.8) and so, Theorem 4.1.

Proof of Theorem 4.1. Conditionally given ζ_n, we write $(\mathrm{T}'_{n,1}, \dots, \mathrm{T}'_{n,\zeta_n})$ for the trees obtained by deleting the first $\zeta_n - 1$ fireproof edges, listed so that $\mathrm{T}'_{n,1}$ is the tree burnt at time ζ_n and $|\mathrm{T}'_{n,2}| \geq \cdots \geq |\mathrm{T}'_{n,\zeta_n}|$. Note that

$$I_n = \sum_{k=2}^{\zeta_n} \mathrm{Card}\{i \in \mathrm{T}'_{n,k} : i \text{ is fireproof}\}.$$

From Proposition 4.14, we have

$$p_n^2(n - |\mathrm{T}'_{n,1}|, |\mathrm{T}'_{n,2}|, \dots, |\mathrm{T}'_{n,\zeta_n}|) \quad \underset{n\to\infty}{\overset{(\mathrm{d})}{\longrightarrow}} \quad \mathbf{e}^2(\sigma(1), J_1, J_2, \dots).$$

Therefore, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ and then $n_0 \in \mathbb{N}$ such that

$$\mathbb{P}\left(\sum_{k=N}^{\infty} \mathbf{e}^2 J_k > \varepsilon\right) < \varepsilon, \quad \text{and for any } n \geq n_0, \quad \mathbb{P}\left(\sum_{k=N+1}^{\zeta_n} p_n^2|\mathrm{T}'_{n,k}| > \varepsilon\right) < \varepsilon.$$

Recall that, conditionally given $(\mathbf{e}^2 J_k)_{k\geq 1}$, $(X_k(\mathbf{e}))_{k\geq 1}$ is a sequence of independent random variables sampled according to $\mu_{\mathbf{e}^2 J_k}$ respectively, where for every $x > 0$, μ_x is the probability measure on $(0, x)$ given by (4.7). In particular, $X_k(\mathbf{e}) \leq \mathbf{e}^2 J_k$ for every $k \geq 1$; we also have $\mathrm{Card}\{i \in \mathrm{T}'_{n,k} : i \text{ is fireproof}\} \leq |\mathrm{T}'_{n,k}|$. Then

$$\mathbb{P}\left(\sum_{k=N}^{\infty} X_k(\mathbf{e}) > \varepsilon \right) < \varepsilon,$$

and for any $n \geq n_0$,

$$\mathbb{P}\left(\sum_{k=N+1}^{\zeta_n} p_n^2 \mathrm{Card}\{i \in \mathrm{T}'_{n,k} : i \text{ is fireproof}\} > \varepsilon \right) < \varepsilon.$$

Conditional on the partition of $\{1, \dots, n\}$ induced by the subsets of vertices of the subtrees, the $\mathrm{T}'_{n,k}$'s are independent uniform Cayley trees on their respective set of vertices. Proposition 4.14 and Lemma 4.15 thus yield

$$\sum_{k=2}^{N} p_n^2 \mathrm{Card}\{i \in \mathrm{T}'_{n,k} : i \text{ is fireproof}\} \quad \underset{n\to\infty}{\overset{(\mathrm{d})}{\longrightarrow}} \quad \sum_{k=1}^{N-1} X_k(\mathbf{e}).$$

Since the rests are arbitrary small with high probability, we get

$$\sum_{k=2}^{\zeta_n} p_n^2 \mathrm{Card}\{i \in \mathrm{T}'_{n,k} : i \text{ is fireproof}\} \quad \underset{n\to\infty}{\overset{(\mathrm{d})}{\longrightarrow}} \quad \sum_{k=1}^{\infty} X_k(\mathbf{e}).$$

The above convergence is (4.8), Theorem 4.1 then follows from (4.9). □

Combined with the results of Bertoin [16], Proposition 4.14 and Lemma 4.15 also entail Proposition 4.13 about the size of the largest fireproof connected component.

Proof of Proposition 4.13. Fix $\varepsilon > 0$ and $\delta \in (0, 1/2)$. Let σ be a subordinator distributed as (4.3) and $\chi \in (0, \varepsilon)$ such that the probability that σ admits no jump larger than χ during the time interval $[0, 1]$ is less than δ. Consider the subtrees of T_n larger than χp_n^{-2} when an edge is set on fire for the first time. From Proposition 4.14, we know that the number of such trees converges to the number of jumps larger than χ made by σ before time 1. The latter is almost surely finite and non-zero with a probability greater than $1 - \delta$. Now from Lemma 4.15, these subtrees are critical and thus for each, from Corollary 1 and Proposition 1 of Bertoin [16], the probability that there exists a fireproof component larger than εp_n^{-2} tends to 0 and the probability that there exists at least one larger than $n^{-\varepsilon} p_n^{-2}$ tends to 1. Therefore for any n large enough, on the one hand there exists in T_n a fireproof subtree larger than $n^{-\varepsilon} p_n^{-2}$ and on the other hand there exists none larger than εp_n^{-2}, both with a probability at least $1 - 2\delta$. The claim follows since δ is arbitrary. □

4.5 Asymptotic proportion of burnt vertices in the supercritical regime

We finally consider the supercritical regime $n^{-1} \ll p_n \ll n^{-1/2}$ and prove Theorem 4.2:

$$(np_n)^{-2} B_n \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad Z^{-2}, \tag{4.11}$$

where Z is a standard Gaussian random variable. Recall that $b_{n,1}, \dots, b_{n,\kappa_n}$ denote the sizes of the burnt subtrees, listed in order of appearance. Let $(\mathbf{e}_i)_{i\geq 1}$ be a sequence of independent exponential random variables with parameter 1 and for each $i \geq 1$, denote by $\gamma_i \coloneqq \mathbf{e}_1 + \cdots + \mathbf{e}_i$. Let also $(Z_i)_{i\geq 1}$ be a sequence of i.i.d. standard Gaussian random variables, independent of $(\gamma_i)_{i\geq 1}$. We shall prove the following result.

Theorem 4.16. *The convergence in distribution*

$$(np_n)^{-2}(b_{n,1}, \dots, b_{n,\kappa_n}) \quad \xrightarrow[n\to\infty]{(d)} \quad (\gamma_1^{-2} Z_1^2, \gamma_2^{-2} Z_2^2, \dots)$$

holds for the ℓ^1 topology.

Theorem 4.2 follows as a corollary.

Proof of Theorem 4.2. As a consequence of Theorem 4.16, we have the convergence of the sums:

$$(np_n)^{-2} B_n \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad \sum_{i=1}^{\infty} \gamma_i^{-2} Z_i^2.$$

Note that the sequence $(\gamma_i)_{i\geq 1}$ is distributed as the atoms of a Poisson random measure on $(0, \infty)$ with intensity $\mathrm{d}x$, it follows readily that the sequence $(\gamma_i^{-2} Z_i^2)_{i\geq 1}$ is distributed as the atoms of a Poisson random measure on $(0, \infty)$ with intensity $(2\pi x^3)^{-1/2}\mathrm{d}x$. The above limit is thus distributed as $\sigma(1)$ where σ is the subordinator defined by (4.3); then (4.11) finally follows from the well-known identity $\sigma(1) = Z^{-2}$ in distribution. $\square$

In order to prove Theorem 4.16, we first show the joint convergence of the first j coordinates for any $j \geq 1$, and then that, taking j large enough, the other coordinates are arbitrary small with high probability. We conclude in the same manner as in the proof of Theorem 4.1.

4.5.1 Asymptotic size of the first burnt subtrees

We first prove the convergence of the size of the first burnt subtree $b_{n,1}$. As in the preceding section, we let the dynamics evolve until an edge is set on fire for the first time, denoting this random time by ζ_n. The size of the tree that burns at this instant is distributed as one among ζ_n i.i.d. Borel(1) random variables conditioned to have sum n, chosen proportionally to its value minus 1. As we have seen, $p_n\zeta_n$ converges in distribution to an exponential random variable with parameter 1, thus ζ_n is typically of order p_n^{-1} and the sum of ζ_n i.i.d. Borel(1) random variables is of order p_n^{-2}. In the previous section, we considered $p_n^{-2} = o(n)$ and we have seen in Proposition 4.14 that conditioning these random variables to have sum n essentially amounts to conditioning one to be of order n. The behavior is notoriously different when $n = o(p_n^{-2})$. As an example, Pavlov [95, Theorem 3] gives a limit theorem for the size of the largest subtree when one removes $k_n - 1$ edges uniformly at random, with $n = o(k_n^2)$.

Lemma 4.17. *As $n \to \infty$, $(np_n)^{-2} b_{n,1}$ converges in distribution to $\mathbf{e}^{-2}Z^2$ where Z and $\mathbf{e}$ are independent, respectively standard Gaussian and exponential with parameter 1 distributed.*

Proof. We work throughout the proof conditionally on $\{\zeta_n = k_n\}$ with $k_n \sim cp_n^{-1}$, $c > 0$ arbitrary, and we show the convergence in distribution $(np_n)^{-2} b_{n,1} \to c^{-2}Z^2$. The general claim then follows as in the proof of Proposition 4.14. For any $\lambda \geq 0$, we write

$$\begin{aligned}
&\mathbb{E}\Big[\mathrm{e}^{-\lambda(np_n)^{-2}b_{n,1}} \;\Big|\; \zeta_n = k_n\Big] \\
&= \sum_{m=0}^{\infty} \mathrm{e}^{-\lambda(np_n)^{-2}m}\,\mathbb{P}(b_{n,1} = m \mid \zeta_n = k_n) \\
&= \int_0^{\infty} \mathrm{e}^{-\lambda(np_n)^{-2}\lfloor x\rfloor}\,\mathbb{P}(b_{n,1} = \lfloor x\rfloor \mid \zeta_n = k_n)\mathrm{d}x \\
&= \int_0^{\infty} \mathrm{e}^{-\lambda(np_n)^{-2}\lfloor x(np_n)^2\rfloor}(np_n)^2\,\mathbb{P}(b_{n,1} = \lfloor x(np_n)^2\rfloor \mid \zeta_n = k_n)\mathrm{d}x.
\end{aligned}$$

We show the pointwise convergence of the densities

$$\lim_{n\to\infty} (np_n)^2\,\mathbb{P}(b_{n,1} = \lfloor x(np_n)^2\rfloor \mid \zeta_n = k_n) = \frac{c}{\sqrt{2\pi x}}\exp\Big(-\frac{c^2 x}{2}\Big),$$

then Scheffé's Lemma implies that this convergence also holds in L^1, which allows us to pass to the limit in the above integral:

$$\lim_{n\to\infty} \mathbb{E}\Big[\mathrm{e}^{-\lambda(np_n)^{-2}b_{n,1}} \;\Big|\; \zeta_n = k_n\Big] = \int_0^{\infty} \mathrm{e}^{-\lambda x}\frac{c}{\sqrt{2\pi x}}\exp\Big(-\frac{c^2 x}{2}\Big)\mathrm{d}x = \mathbb{E}\Big[\mathrm{e}^{-\lambda c^{-2}Z^2}\Big].$$

Recall from (4.10) the distribution of k_n i.i.d. Borel(1) random variables conditioned to have sum n: for any integers $n_1, \dots, n_{k_n} \geq 1$ such that $n_1 + \cdots + n_{k_n} = n$,

$$\mathbb{P}(\beta_{n,1} = n_1, \dots, \beta_{n,k_n} = n_{k_n}) = \frac{(n-k_n)!}{k_n n^{n-k_n-1}} \prod_{j=1}^{k_n} \frac{n_j^{n_j-1}}{n_j!}.$$

In particular, the $\beta_{n,j}$'s are identically distributed and for any $m_n \in \{1, \dots, n-k_n+1\}$, summing over all the $n_2, \dots, n_{k_n} \geq 1$ such that $n_2 + \cdots + n_{k_n} = n - m_n$,

$$\mathbb{P}(\beta_{n,1} = m_n) = \frac{(n-k_n)!}{k_n n^{n-k_n-1}} \frac{m_n^{m_n-1}}{m_n!} \frac{(k_n-1)(n-m_n)^{n-m_n-k_n}}{(n-m_n-k_n+1)!}.$$

Recall that on the event $\{\zeta_n = k_n\}$, $b_{n,1}$ is distributed as one of the variables $\beta_{n,1}, \dots, \beta_{n,k_n}$ chosen proportionally to its value minus 1. Therefore, for any $m_n \in \{1, \dots, n-k_n+1\}$, we have

$$\begin{aligned}
\mathbb{P}(b_{n,1} = m_n \mid \zeta_n = k_n) &= \sum_{j=1}^{k_n} \mathbb{P}(b_{n,1} = \beta_{n,j} \mid \beta_{n,j} = m_n)\mathbb{P}(\beta_{n,j} = m_n) \\
&= k_n \frac{m_n - 1}{n - k_n} \mathbb{P}(\beta_{n,1} = m_n) \\
&= (m_n - 1)(k_n - 1) \frac{(n-k_n-1)!}{n^{n-k_n-1}} \frac{m_n^{m_n-1}}{m_n!} \frac{(n-m_n)^{n-m_n-k_n}}{(n-m_n-k_n+1)!}.
\end{aligned}$$

Suppose that $m_n, k_n \to \infty$ as $n \to \infty$ with $m_n, k_n = o(n)$, then Stirling's formula yields

$$\begin{aligned}
&\mathbb{P}(b_{n,1} = m_n \mid \zeta_n = k_n) \\
&\qquad = \frac{1}{\sqrt{2\pi}} \frac{k_n}{n\sqrt{m_n}} \exp\left(- \frac{k_n^2 m_n}{2n^2} + O\left(\frac{k_n^3 m_n}{n^3}\right) + O\left(\frac{(k_n m_n)^2}{n^3}\right)\right)(1 + o(1)).
\end{aligned}$$

For any $x, c > 0$, taking $m_n = \lfloor x(np_n)^2 \rfloor$ and $k_n \sim c p_n^{-1}$, we obtain

$$\begin{aligned}
(np_n)^2 \mathbb{P}(b_n = \lfloor x(np_n)^2 \rfloor \mid \zeta_n = k_n) &= \frac{c}{\sqrt{2\pi x}} \exp\left(- \frac{c^2 x}{2} + O\left(\frac{1}{np_n}\right) + O(np_n^2)\right)(1 + o(1)) \\
&= \frac{c}{\sqrt{2\pi x}} \exp\left(- \frac{c^2 x}{2}\right)(1 + o(1)),
\end{aligned}$$

and the proof is now complete. □

More generally, for any integer $j \geq 1$, denote by $\zeta_{n,j}$ the time of the j-th fire, so that $b_{n,j}$ denotes the size of the subtree burnt at time $\zeta_{n,j}$.

Proposition 4.18. *For any $j \geq 1$, the convergence in distribution*

$$(np_n)^{-2}(b_{n,1}, \dots, b_{n,j}) \quad \xrightarrow[n\to\infty]{(d)} \quad \left(\gamma_1^{-2} Z_1^2, \dots, \gamma_j^{-2} Z_j^2\right)$$

holds in $\mathbb{R}^j$.

Proof. We prove the claim for $j = 2$ for simplicity of notation, the general case follows by induction in the same manner. Notice first that the times at which the first j fires appear jointly converge:

$$p_n(\zeta_{n,1}, \dots, \zeta_{n,j}) \quad \xrightarrow[n\to\infty]{\text{(d)}} \quad (\gamma_1, \dots, \gamma_j). \tag{4.12}$$

Indeed, conditionally given the size of the first burnt subtree $b_{n,1} = m$ and the number of edges previously fireproof $\zeta_{n,1} - 1 = k - 1$, after the first fire, it remains a forest containing $(n-1) - (k-1) - (m-1) = n - m - k + 1$ edges and the time $\zeta_{n,2} - \zeta_{n,1}$ we wait for the second fire after the first one is again a truncated geometric random variable which takes value

$$\begin{aligned} &\infty \text{ with probability } (1-p_n)^{n-m-k+1}, \\ \text{and} \quad &\ell \text{ with probability } p_n(1-p_n)^{\ell-1}, \text{ for any } \ell \in \{1, \dots, n-m-k+1\}. \end{aligned}$$

Since $b_{n,1} + \zeta_{n,1} = o(n)$ in probability, we see that $p_n(\zeta_{n,2} - \zeta_{n,1})$, conditionally given $b_{n,1}$ and $\zeta_{n,1}$, converges in distribution to an exponential random variable with parameter 1. This yields (4.12) in the case $j = 2$.

The same idea gives the claim of Proposition 4.18. The remaining forest after the first fire is, conditionally given $b_{n,1}$ and $\zeta_{n,1}$, uniformly distributed amongst the forests with $\zeta_{n,1} - 1$ trees and $n - b_{n,1}$ vertices. Therefore, conditionally given $b_{n,1}$ and $\zeta_{n,2}$, $b_{n,2}$ is distributed as the size of a tree chosen at random with probability proportional to its number of edges in a forest consisting of $\zeta_{n,2} - 1$ trees with total size $n - b_{n,1} \sim n$. Then the proof of Lemma 4.17 shows that such a random variable, rescaled by a factor $(np_n)^{-2}$, converges in distribution to $\gamma_2^{-2} Z_2^2$. This yields

$$(np_n)^{-2}\left(b_{n,1}, b_{n,2}\right) \quad \xrightarrow[n\to\infty]{\text{(d)}} \quad \left(\gamma_1^{-2} Z_1^2, \gamma_2^{-2} Z_2^2\right),$$

and the proof is complete after an induction on j. □

4.5.2 Asymptotic size of all burnt subtrees

To strengthen the convergence from finite dimensional vectors to the ℓ^1 convergence, we need to bound the remainders. This is done in the following lemma, the last ingredient for the proof of Theorem 4.2.

Lemma 4.19. *For any $\varepsilon > 0$, we have*

$$\lim_{j_0\to\infty} \limsup_{n\to\infty} \mathbb{P}\left((np_n)^{-2} \sum_{j=j_0}^{\infty} b_{n,j} > \varepsilon \right) = 0.$$

In order to prove this result, we consider a slightly different sequence of random subtrees of T_n, which can be coupled with the sequence of burnt subtrees and for which the study is easier. Precisely, consider the following random dynamics on T_n: we remove successively the edges in a uniform random order and at each step, we mark one subtree at random proportionally to its number of edges. We stress that in this procedure, the subtrees are not burnt, which implies that the edges of a marked subtree can be removed afterward and that a subtree of a marked one may be marked as well. For each $k \in \{1, \dots, n-2\}$, we denote by $b'_{n,k}$ the size of the subtree which is marked when k edges have been removed.

Lemma 4.20. *There exists a numerical constant $C > 0$ such that for any $a > 0$, we have*

$$\limsup_{n\to\infty} n^{-2} p_n^{-1} \sum_{k=\lfloor a p_n^{-1} \rfloor}^{\lfloor n-u_n \rfloor} \mathbb{E}[b'_{n,k}] \le \frac{C}{a},$$

where $u_n = n\exp(-\sqrt{np_n})$ for every integer n.

The role of the sequences u_n and ap_n^{-1} shall appear in the proofs of Lemma 4.20 and Lemma 4.19; note that since $\lim_{n\to\infty} np_n = \infty$, we have

$$\lim_{n\to\infty} p_n u_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} (np_n)^{-1} \ln\left(\frac{n-u_n}{u_n}\right) = 0. \tag{4.13}$$

The first convergence shows that the sum in Lemma 4.20 is not empty for n large enough.

Proof. Fix $k \le n-2$ and let $(\beta_{n,1}, \dots, \beta_{n,k})$ be a k-tuple formed by i.i.d. Borel(1) random variables conditioned to have sum n. As we have seen, $b'_{n,k}$ can be viewed as one the $\beta_{n,i}$'s picked at random with probability proportional to its value minus one and hence,

$$\mathbb{E}[b'_{n,k}] = \mathbb{E}\left[\sum_{i=1}^{k} \beta_{n,i} \frac{\beta_{n,i}-1}{n-k} \right] = \frac{n}{n-k} \mathbb{E}\left[\sum_{i=1}^{k} (\beta_{n,i}-1) \frac{\beta_{n,i}}{n} \right].$$

Bertoin [15, Section 3.1] provides an upper bound for the expectation on the right-hand side. Precisely, Proposition 1 in [15], together with Lemma 5 and equation (2) there, shows that there exists a numerical constant $K > 0$ such that for every integers $1 \le k \le n$, we have

$$\mathbb{E}\left[\sum_{i=1}^{k} (\beta_{n,i}-1) \frac{\beta_{n,i}}{n} \right] \le K \left(\frac{n}{k}\right)^2.$$

Hence for every n,

$$\sum_{k=\lfloor ap_n^{-1}\rfloor}^{\lfloor n-u_n\rfloor} \mathbb{E}[b'_{n,k}] \le Kn^3 \sum_{k=\lfloor ap_n^{-1}\rfloor}^{\lfloor n-u_n\rfloor} \frac{1}{k^2(n-k)}.$$

Comparing sums and integrals, we have on the one hand,

$$\sum_{k=\lfloor ap_n^{-1}\rfloor}^{\lfloor 3n/4\rfloor} \frac{1}{k^2(n-k)} \le \frac{4}{n} \sum_{k=\lfloor ap_n^{-1}\rfloor}^{\lfloor 3n/4\rfloor} \frac{1}{k^2} = \frac{4}{a} n^{-1} p_n(1+o(1)),$$

and on the other hand,

$$\sum_{k=\lceil 3n/4\rceil}^{\lfloor n-u_n\rfloor} \frac{1}{k^2(n-k)} \le n^{-2}\left[\ln(x) - \ln(n-x) - \frac{n}{x}\right]_{3n/4}^{n-u_n} = n^{-2}\ln\left(\frac{n-u_n}{u_n}\right)(1+o(1)).$$

Summing the two terms and appealing (4.13), we obtain

$$\sum_{k=\lfloor ap_n^{-1}\rfloor}^{\lfloor n-u_n\rfloor} \mathbb{E}[b'_{n,k}] \le \frac{4K}{a} n^2 p_n(1+o(1)),$$

and the claim follows. □

We have a natural coupling between burnt and marked subtrees which enables us to deduce Lemma 4.19 from Lemma 4.20: for each $k \in \{1, \dots, n-2\}$, we toss a coin which gives Head with probability p_n; the first burnt subtree, say, $\mathrm{T}_{n,1}$, is distributed as the first marked subtree, say, $\mathrm{T}'_{n,k}$, for which the outcome is Head. Then, the second burnt subtree $\mathrm{T}_{n,2}$ is distributed as the next marked subtree for which the outcome is Head and which is not contained in $\mathrm{T}'_{n,k}$, and so on. In the next proof, we implicitly assume that the marked and burnt subtrees are indeed coupled.

Proof of Lemma 4.19. Fix $\varepsilon, \delta > 0$ and $a > \delta^{-1}$. Since $p_n\zeta_{n,j} \to \gamma_j$ in distribution as $n \to \infty$ for any $j \ge 1$ and $j^{-1}\gamma_j \to 1$ in probability as $j \to \infty$ from the law of large numbers, we may, and do, fix $j_0 \ge 1$ and further $n_0 \ge 1$ such that for any $n \ge n_0$, we have

$$\mathbb{P}(\zeta_{n,j_0} > ap_n^{-1}) \ge 1 - \delta.$$

For any $j \ge 1$, denote by $\theta_{n,j} - 1$ the number of edges that have been removed in the marking procedure when we mark the subtree corresponding to the burnt subtree $b_{n,j}$. We have

$$\sum_{j \ge j_0} b_{n,j} \le \sum_{k=1}^{n-2} b'_{n,k} \mathbb{I}_{\{\eta_k = 1\}} \mathbb{I}_{\{k \ge \theta_{n,j_0}\}},$$

where $\eta_k = 1$ if and only if the outcome of the coin which is tossed at the k-th step is Head. Further, since $\zeta_{n,1} = \theta_{n,1}$ and $\zeta_{n,j} \leq \theta_{n,j}$ for every $j \geq 2$, we see that

$$\mathbb{P}\left((np_n)^{-2}\sum_{j=j_0}^{\infty} b_{n,j} > \varepsilon \,\middle|\, \zeta_{n,j_0} > ap_n^{-1}\right) \leq \mathbb{P}\left((np_n)^{-2}\sum_{k=\lfloor ap_n^{-1}\rfloor}^{n-2} b'_{n,k}\mathbb{I}_{\{\eta_k=1\}} > \varepsilon\right).$$

Recall that $\lim_{n\to\infty} p_n u_n = 0$, which implies that the probability that no tree is marked after the $\lfloor n-u_n\rfloor$-th step is $(1-p_n)^{\lceil u_n\rceil-2} \geq 1-\delta$ for any n large enough. Finally,

$$\mathbb{P}\left((np_n)^{-2}\sum_{k=\lfloor ap_n^{-1}\rfloor}^{\lfloor n-u_n\rfloor} b'_{n,k}\mathbb{I}_{\{\eta_k=1\}} > \varepsilon\right) \leq \varepsilon^{-1}(np_n)^{-2}\sum_{k=\lfloor ap_n^{-1}\rfloor}^{\lfloor n-u_n\rfloor} \mathbb{E}[b'_{n,k}]p_n,$$

so, thanks to Lemma 4.20,

$$\limsup_{n\to\infty}\mathbb{P}\left((np_n)^{-2}\sum_{k=\lfloor ap_n^{-1}\rfloor}^{\lfloor n-u_n\rfloor} b'_{n,k}\mathbb{I}_{\{\eta_k=1\}} > \varepsilon\right) \leq \varepsilon^{-1}\frac{C}{a}.$$

We conclude that

$$\limsup_{n\to\infty}\mathbb{P}\left((np_n)^{-2}\sum_{j=j_0}^{\infty} b_{n,j} > \varepsilon\right) \leq \varepsilon^{-1}C\delta + 2\delta,$$

and the claim follows since δ is arbitrary. □

Using the same reasoning as in the proof of Theorem 4.1, Theorem 4.16 follows readily from Proposition 4.18 and Lemma 4.19.

Proof of Theorem 4.16. For every $n, j \geq 1$, we write the sequence $((np_n)^{-2}b_{n,k}; k \geq 1)$ as $S_n(j) + R_n(j)$ where

$$S_n(j) = (np_n)^{-2}(b_{n,1}, b_{n,2}, \ldots, b_{n,j}, 0, 0, \ldots),$$

and

$$R_n(j) = (np_n)^{-2}(0, \ldots, 0, b_{n,j+1}, b_{n,j+2}, \ldots);$$

and similarly, $(\gamma_i^{-2}Z_i^2; i \geq 1) = S(j) + R(j)$, with

$$S(j) = (\gamma_1^{-2}Z_1^2, \gamma_2^{-2}Z_2^2, \ldots, \gamma_j^{-2}Z_j^2, 0, 0, \ldots),$$

and

$$R(j) = (0, \ldots, 0, \gamma_{j+1}^{-2}Z_{j+1}^2, \gamma_{j+2}^{-2}Z_{j+2}^2, \ldots).$$

From Proposition 4.18, for any $j \geq 1$, $\lim_{n\to\infty} S_n(j) = S(j)$ in distribution. Further, for any $\varepsilon > 0$, since the sequence $(\gamma_i^{-2} Z_i^2; i \geq 1)$ is summable, and thanks to Lemma 4.19, there exists $j_0 \geq 1$ and then $n_0 \geq 1$ such that

$$\mathbb{P}(\|R(j_0)\| > \varepsilon) < \varepsilon, \quad \text{and for every } n \geq n_0, \quad \mathbb{P}(\|R_n(j_0)\| > \varepsilon) < \varepsilon.$$

which completes the proof. □

5

Fires on large random recursive trees

In this chapter, we study the same fire dynamics as in Chapter 4, described in Chapter 2, but now on a uniform random recursive tree with n vertices, and we prove the results discussed in Section 2.3. This work is based on the article [82].

5.1 Introduction

A tree on the set of vertices $[n] := \{1, \ldots, n\}$ is called recursive if, when rooted at 1, the sequence of vertices along any branch from the root to a leaf is increasing; see Figure 5.1 for an illustration. There are $(n-1)!$ such trees and we pick one of them uniformly at random, that we simply call *random recursive tree*, and denote by T_n. A random recursive tree on $[n]$ can be inductively constructed by the following algorithm: we start with the singleton $\{1\}$, then for every $i = 2, \ldots, n$, the vertex i is added to the current tree by an edge $\{u_i, i\}$, where u_i is chosen uniformly at random in $\{1, \ldots, i-1\}$ and independently of the previous edges.

We shall see that random recursive trees fulfill the assumptions of Proposition 2.1 (and Proposition 2.3 as well), which reads as follows: denote by I_n the number of fireproof vertices in T_n, then, as $n \to \infty$,

(i) If $np_n / \ln n \to 0$, then $n^{-1} I_n \to 1$ in probability;

(ii) If $np_n / \ln n \to \infty$, then $n^{-1} I_n \to 0$ in probability;

(iii) If $np_n / \ln n \to c \in (0, \infty)$, then $n^{-1} I_n \to \mathbf{e}_c \wedge 1$ in distribution, where $\mathbf{e}_c$ is exponentially distributed with rate c.

Using precise information about the cut-tree of random recursive trees, in particular a coupling with a certain random walk due to Iksanov & Möhle [62], we improve the con-

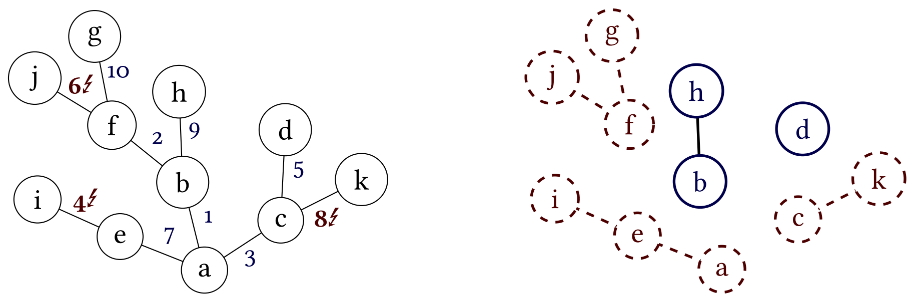


Figure 5.1: Given a recursive tree and an enumeration of its edges on the left, if the edges set on fire are the 4th, the 6th and the 8th, we get the two forests on the right where the burnt components are drawn with dotted lines and the fireproof ones with plain lines.

vergence (ii) and obtain a convergence in distribution to a non-trivial limit for I_n in the subcritical regime, specifically:

$$\frac{p_n}{\ln(1/p_n)} I_n \quad \xrightarrow[n\to\infty]{(d)} \quad \mathbf{e}_1 \qquad \text{when} \quad p_n \gg \ln n/n,$$

where $\mathbf{e}_1$ is an exponential random variable with rate 1.

We further study the connectivity in the fireproof forest. We prove that with high probability as $n \to \infty$, in the supercritical regime, there exists a giant fireproof component of size $n - o(n)$, in the subcritical regime, the largest fireproof component has size of order p_n^{-1}, and finally for the critical regime, the largest fireproof component has size of order $p_n^{-1} \asymp n/\ln n$ if the root burns and $n - o(n)$ if the root is fireproof.

In the last part, we study the sizes of the burnt subtrees, in order of appearance, in the critical regime. The one which contains the root (if the root burns) has size of order n, while for the others, the logarithm of their size, rescaled by $\ln n$ converge in distribution to limits strictly smaller than 1.

This work leaves open the question of the total number of burnt vertices in the supercritical regime: as above for the subcritical one, one would ask for a convergence in distribution of $n - I_n$, rescaled by some sequence which is negligible compared to n (recall Theorem 4.2 for Cayley trees).

The rest of this chapter is organized as follows: in Section 5.2, we first recall the relations between the marked cut-tree and the fire dynamics presented in Section 2.1 and apply these results to random recursive trees. After recalling some known results in Section 5.2.2, in particular a coupling of Iksanov & Möhle [62], we prove the scaling convergence in distribution of the total number of fireproof vertices in the subcritical regime in Section 5.3. Section 5.4 is devoted to the existence of giant fireproof components in the three regimes.

Finally, our main results about the sizes of the burnt subtrees in the critical regime are stated and proved in Section 5.5.

5.2 Cut-tree and fires

5.2.1 The cut-tree of a random recursive tree and fires

Recall from Section 2.1 the definitions of the cut-tree $\mathrm{Cut}(\mathrm{T}_n)$ associated with T_n as well as the mark process φ_n on the latter describing the fire dynamics on T_n in the sense that the marked blocks of $\mathrm{Cut}(\mathrm{T}_n)$ correspond to the burnt subtrees of T_n and the leaves of $\mathrm{Cut}(\mathrm{T}_n)$ which do not possess a marked ancestor correspond to the fireproof vertices of T_n.

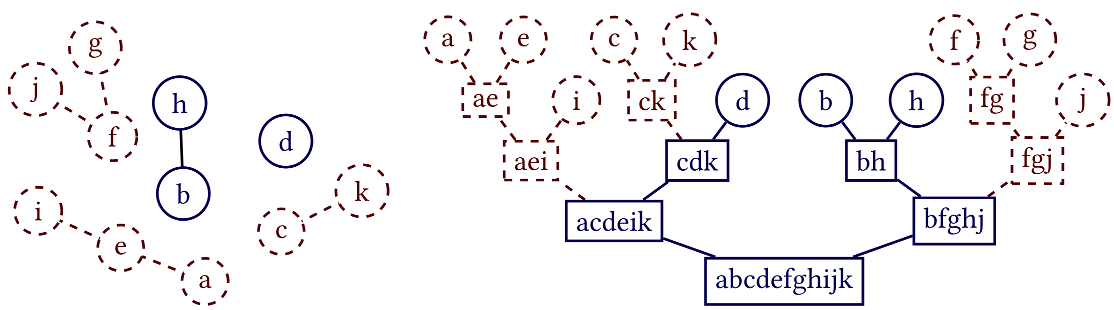


Figure 5.2: The forests after the dynamics on the recursive tree on the left and the corresponding cut-tree on the right: the thick blocks are the ones connected to the root, the dotted ones, those disconnected from the root. The leaves in the root-component of the cut-tree are the fireproof vertices, the other components are the burnt subtrees.

Kuba & Panholzer [74] and Bertoin [19] provide the assumptions (H_k) and (H'_k) for every $k \in \mathbb{N}$ respectively. Indeed, for each $k \in \mathbb{N}$, denote by $R_{n,k}$ the smallest connected subset of $\mathrm{Cut}(\mathrm{T}_n)$ which contains its root and k leaves chosen uniformly at random with replacement; denote by $L_{n,k}$ the length of $R_{n,k}$. Then Kuba & Panholzer [74, Theorem 3] proved

$$\frac{\ln n}{n} L_{n,k} \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad \beta_k \qquad \text{for every} \quad k \in \mathbb{N}, \tag{5.1}$$

where β_k is a beta$(k, 1)$ random variable, i.e. with law $kx^{k-1}\mathrm{d}x$ on $[0, 1]$. Their result is stated in terms of the number of cuts needed to isolate k uniform random vertices; the latter is indeed given by $L_{n,k}$ as explained in Section 2.1. Further, Bertoin [19] obtained the convergence of the entire cut-tree; precisely, the proof of Theorem 1 in [19] shows the following.

Lemma 5.1 ([19]). *Consider* $\mathrm{Cut}(\mathrm{T}_n)$ *equipped with the metric* d_n *given by the graph distance rescaled by a factor* $\ln n/n$*, and the uniform probability measure on the* n *leaves* μ_n*. Then the convergence*

$$\Big(\mathrm{Cut}(\mathrm{T}_n), \{[n], \{1\}\}, d_n, \mu_n\Big) \underset{n\to\infty}{\longrightarrow} \Big([0,1], \{0,1\}, |\cdot|, \mathrm{Leb}\Big)$$

holds in probability for the two-pointed Gromov–Hausdorff–Prokhorov topology, where $|\cdot|$ *and* Leb *refer respectively to the Euclidean distance and Lebesgue measure.*

As in the previous chapters, we define on $[0,1]$ a point process Φ_c analogous to φ_n on $\mathrm{Cut}(\mathrm{T}_n)$: first sample a Poisson point process with intensity c per unit length, then, along each branch from the root to a leaf, keep only the closest mark to the root (if any) and erase the other marks. The process Φ_c thus reduces to the point process with at most one mark, given by the smallest atom of a Poisson random measure on $[0,1]$ with intensity $c\mathrm{d}x$ if it exists, and no mark otherwise. Proposition 2.1 and Proposition 2.3 then read as follows for recursive trees.

Corollary 5.2. *For every integer* n*, denote by* I_n *the number of fireproof vertices in* T_n *and by* $b^*_{n,1} \geq b^*_{n,2} \geq \cdots \geq 0$ *the sizes of the burnt subtrees, ranked in non-increasing order. Then, as* $n \to \infty$*,*

(i) If $np_n/\ln n \to 0$*, then* $n^{-1}I_n \to 1$ *in probability;*

(ii) If $np_n/\ln n \to \infty$*, then* $n^{-1}I_n \to 0$ *in probability;*

(iii) If $np_n/\ln n \to c \in (0,\infty)$*, then*

$$n^{-1}(I_n, b^*_{n,1}, b^*_{n,2}, \dots) \underset{n\to\infty}{\overset{(d)}{\longrightarrow}} (\mathbf{e}_c \wedge 1, 1-(\mathbf{e}_c \wedge 1), 0, 0, \dots),$$

where $\mathbf{e}_c$ *is an exponential random variable with rate* c*.*

Proof. Recall from the proof of Proposition 2.1 the identity

$$\mathbb{E}[(n^{-1}I_n)^k] = \mathbb{E}[(1-p_n)^{X_{n,k}}] = \mathbb{E}[(1-p_n)^{L_{n,k}}](1+o(1)),$$

where $X_{n,k}$ denotes the number of internal nodes of $R_{n,k}$, so $L_{n,k} - X_{n,k}$ is equal to the number of distinct leaves of $R_{n,k}$ minus one, which is bounded by $k-1$. It then follows from (5.1) that $n^{-1}I_n$ converges in probability to 1 when $p_n \ll \ln n/n$, to 0 when $p_n \gg \ln n/n$ and to $\mathbf{e}_c \wedge 1$ when $p_n \sim c\ln n/n$, since, for every $k \geq 1$, we have

$$\mathbb{E}\Big[\mathrm{e}^{-c\beta_k}\Big] = \int_0^1 \mathrm{e}^{-cx}kx^{k-1}\mathrm{d}x = \mathrm{e}^{-c} + \int_0^1 c\mathrm{e}^{-cx}x^k\mathrm{d}x = \mathbb{E}\Big[(\mathbf{e}_c \wedge 1)^k\Big].$$

Further, using Lemma 5.1, we may apply Proposition 2.3 where $\mathcal{T} = [0,1]$ so the limit $\#(\mathcal{T}, \Phi_c)$ is the sequence whose entries are the Lebesgue measure of each connected component of $[0,1]$ after logging at the atoms of Φ_c, the root-component first, and the next in non-increasing order, i.e. $\#(\mathcal{T}, \Phi_c) = (\mathbf{e}_c \wedge 1, 1 - (\mathbf{e}_c \wedge 1), 0, 0, \dots)$. □

Consider the critical regime of the fire dynamics on T_n, that is $p_n \sim c \ln n/n$ for some fixed $c \in (0, \infty)$. We identify the macroscopic burnt component appearing in Corollary 5.2 (iii) above as the one containing the root.

Proposition 5.3. *Denote by b_n^0 the size of the burnt subtree which contains the root of T_n if the latter is burnt, and 0 otherwise, and A_n the event that the root of T_n burns. Then in the critical regime $p_n \sim c \ln n/n$, we have*

$$(\mathbb{I}_{A_n}, n^{-1} I_n, n^{-1} b_n^0) \quad \xrightarrow[n\to\infty]{(d)} \quad (\mathbb{I}_{\mathbf{e}_c < 1}, \mathbf{e}_c \wedge 1, 1 - (\mathbf{e}_c \wedge 1)),$$

where $\mathbf{e}_c$ is an exponential random variable with rate c.

We see that the probability that the root burns converges to $1 - \mathrm{e}^{-c}$. Further, from this result and Corollary 5.2 (iii), we see that in the critical regime, with high probability, when the root burns, its burnt component is the macroscopic one, and when it does not burn, then there is no macroscopic burnt component. Finally, the density of fireproof vertices converges to 1 in probability if we condition the root to be fireproof, and it converges in distribution to an exponential random variable with rate c conditioned to be smaller than 1 if we condition the root to burn.

Proof. On the path of $\mathrm{Cut}(\mathrm{T}_n)$ from $[n]$ to $\{1\}$, there is at most one mark, at a height given by a geometric random variable with parameter $p_n \sim c \ln n/n$ if the latter is smaller than the height of $\{1\}$, and no mark otherwise. Furthermore, b_n^0 is equal to 0 if there is no such mark and is given by the number of leaves of the subtree of $\mathrm{Cut}(\mathrm{T}_n)$ that stems from this marked block otherwise. Thanks to Lemma 5.1, this path and the mark converge in distribution to the interval $[0,1]$ with a mark at distance $\mathbf{e}_c$ from 0 if $\mathbf{e}_c < 1$, and no mark otherwise. Hence

$$(\mathbb{I}_{A_n}, n^{-1} b_n^0) \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad (\mathbb{I}_{\mathbf{e}_c < 1}, 1 - (\mathbf{e}_c \wedge 1)).$$

Moreover, we already know from Corollary 5.2 (iii) that $n^{-1} I_n$ converges in distribution to $\mathbf{e}'_c \wedge 1$, where $\mathbf{e}'_c$ is exponentially distributed with rate c. Notice that we have $I_n \leq n - b_n^0$, so $\mathbf{e}'_c \wedge 1 \leq \mathbf{e}_c \wedge 1$ almost surely. Since they have the same law, we conclude that $\mathbf{e}'_c \wedge 1 = \mathbf{e}_c \wedge 1$ almost surely and the claim follows. □

In order to obtain more precise results on the fire dynamics on T_n, we need more information about $\mathrm{Cut}(\mathrm{T}_n)$. We next recall some known results about the cut-tree of large random recursive trees, due to Meir & Moon [86], Iksanov & Möhle [62] and Bertoin [19], and introduce the notation we shall use subsequently.

5.2.2 More about the cut-tree of a random recursive tree

Let $\zeta(n)$ be the length of the path in $\mathrm{Cut}(\mathrm{T}_n)$ from its root $[n]$ to the leaf $\{1\}$. Set $C_{n,0} := [n]$ and for each $i = 1, \dots, \zeta(n)$, let $C_{n,i}$ and $C'_{n,i}$ be the two offsprings of $C_{n,i-1}$, with the convention that $1 \in C_{n,i}$; finally, denote by $\mathrm{T}_{n,i}$ and $\mathrm{T}'_{n,i}$ the subtrees of T_n restricted to $C_{n,i}$ and $C'_{n,i}$ respectively. Note that for every $i \in \{1, \dots, \zeta(n)\}$, the collection $\{C'_{n,1}, \dots, C'_{n,i}, C_{n,i}\}$ forms a partition of $[n]$. The next lemma shows that the law of $\mathrm{Cut}(\mathrm{T}_n)$ is essentially determined by that of the nested sequence $[n] = C_{n,0} \supset C_{n,1} \supset \cdots \supset C_{n,\zeta(n)} = \{1\}$.

Indeed, random recursive trees fulfill a certain consistency relation called splitting property or randomness preservation property. We extend the definition of a recursive tree to a tree on a totally ordered set of vertices which is rooted at the smallest element and such that the sequence of vertices along any branch from the root to a leaf is increasing. There is a canonical way to transform such a tree with size, say, k, to a recursive tree on $[k]$ by relabelling the vertices.

Lemma 5.4. *Fix $i \in \{1, \dots, \zeta(n)\}$. Conditional on the sets $C'_{n,1}, \dots, C'_{n,i}$ and $C_{n,i}$, the subtrees $\mathrm{T}'_{n,1}, \dots, \mathrm{T}'_{n,i}$ and $\mathrm{T}_{n,i}$ are independent random recursive trees on their respective set of vertices. Furthermore, conditional on these sets, the subtrees of $\mathrm{Cut}(\mathrm{T}_n)$ that stem from these blocks are independent and distributed as cut-trees of random recursive trees on these sets.*

Proof. The first statement should be plain from the inductive construction of random recursive trees described in the introduction. The second follows since, in addition, if we restrict the fragmentation of T_n described earlier to one of its subtree, the edges of this subtree are indeed removed in a uniform random order and this fragmentation is independent of the rest of T_n. □

As a consequence, we only need to focus on the size of the $C_{n,i}$'s. Our main tool relies on a coupling due originally to Iksanov & Möhle [62] that connects the latter with a certain random walk. Let us introduce a random variable ξ with distribution

$$\mathbb{P}(\xi = k) = \frac{1}{k(k+1)}, \qquad k \geq 1,$$

then a random walk

$$S_j = \xi_1 + \cdots + \xi_j, \qquad j \geq 1,$$

where $(\xi_i; i \geq 1)$ are i.i.d. copies of ξ, and finally the last-passage time

$$\lambda(n) = \max\{j \geq 1 : S_j < n\}.$$

A weaker form of the result in [62], which is sufficient for our purpose, is the following.

Lemma 5.5. *One can construct on the same probability space a random recursive tree of size n and its cut-tree, together with a version of the random walk S such that $\zeta(n) \geq \lambda(n)$ and*

$$(|C'_{n,1}|, \dots, |C'_{n,\lambda(n)}|, |C_{n,\lambda(n)}|) = (\xi_1, \dots, \xi_{\lambda(n)}, n - S_{\lambda(n)}). \tag{5.2}$$

From now on, we assume that the recursive tree T_n and its cut-tree $\mathrm{Cut}(\mathrm{T}_n)$ are indeed coupled with the random walk S. This coupling enables us to deduce properties of $\mathrm{Cut}(\mathrm{T}_n)$ from that of S; we shall need the following results.

Lemma 5.6. *The random walk S fulfills the following properties.*

(i) Weak law of large numbers:

$$\frac{1}{k \ln k} S_k \quad \underset{k\to\infty}{\longrightarrow} \quad 1 \quad \textit{in probability.}$$

(ii) The last-passage time satisfies

$$\frac{\ln n}{n} \lambda(n) \quad \underset{n\to\infty}{\longrightarrow} \quad 1 \quad \textit{in probability.}$$

(iii) The undershoot satisfies

$$\frac{\ln n}{n} (n - S_{\lambda(n)}) \quad \underset{n\to\infty}{\longrightarrow} \quad 0 \quad \textit{in probability.}$$

(iv) The random point measure

$$\sum_{i=1}^{\lambda(n)} \delta_{\frac{\ln n}{n}\xi_i}(\mathrm{d}x)$$

converges in distribution on the space of locally finite measures on $(0, \infty]$ endowed with the vague topology towards to a Poisson random measure with intensity $x^{-2}\mathrm{d}x$.

Proof. The first assertion can be checked using generating functions; a standard limit theorem for random walk with step distribution in the domain of attraction of a stable law entails moreover the weak convergence of $k^{-1}S_k - \ln k$ to the so-called continuous Luria-Delbrück distribution, see for instance Geluk & de Haan [54]. Iksanov & Möhle [62], provide finer limit theorems for the last-passage time as well as the undershoot, see respectively Proposition 2 and Lemma 2 there. Finally, the last assertion is the claim of Lemma 1 (ii) of Bertoin [19] and follows readily from the distribution of ξ and Theorem 16.16 of Kallenberg [68]. □

Note from Lemma 5.5 that

$$\lambda(n) \leq \zeta(n) \leq \lambda(n) + |C_{n,\lambda(n)}| = \lambda(n) + n - S_{\lambda(n)}.$$

Lemma 5.6 (ii) and 5.6 (iii) thus entail

$$\lim_{n\to\infty} \frac{\ln n}{n}\zeta(n) = 1 \qquad \text{in probability.}$$

This result was obtained earlier by Meir & Moon [86] who proved

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{\ln n}{n}\zeta(n)\right] = \lim_{n\to\infty} \mathbb{E}\left[\left(\frac{\ln n}{n}\zeta(n)\right)^2\right] = 1.$$

We will use this stronger result in Section 5.4 below.

5.3 Density of fireproof vertices

We prove in this section a non-trivial limit in distribution for the number I_n of fireproof vertices in T_n, under an appropriate scaling, in the subcritical regime. We begin with a lemma.

Lemma 5.7. *Consider the subcritical regime $1 \gg p_n \gg \ln n/n$. Then, as $n \to \infty$, the root of T_n burns with high probability, and the size of its burnt component, rescaled by n, converges to 1 in probability.*

Proof. Consider the path from $[n]$ to $\{1\}$ in $\mathrm{Cut}(\mathrm{T}_n)$. It contains at most one mark, whose height $\sigma(n)$ is distributed as $g_n \wedge \zeta(n)$ where g_n is a geometric random variable with parameter p_n independent of $\zeta(n)$. Recall from Lemma 5.6 that $\zeta(n) \geq \lambda(n) \sim n/\ln n$ in probability, so this mark exists with high probability and, moreover,

$$p_n\sigma(n) \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad \mathbf{e}_1.$$

In particular, we have $\sigma(n) \leq \lambda(n)$ with high probability. On this event, observe thanks to Lemma 5.5 that the size of the burnt component which contains the root is given by

$$|C_{n,\sigma(n)}| = n - S_{\sigma(n)}.$$

It follows from Lemma 5.6 (i) and a standard argument (cf. Theorem 15.17 in Kallenberg [68]) that for every $y \geq 0$,

$$\sup_{x\in[0,y]} \left| \frac{p_n}{\ln(1/p_n)} S_{\lfloor x/p_n \rfloor} - x \right| \quad \underset{n\to\infty}{\longrightarrow} \quad 0 \quad \text{in probability,}$$

and we conclude that

$$\frac{p_n}{\ln(1/p_n)} S_{\sigma(n)} \quad \overset{\text{(d)}}{\underset{n\to\infty}{\longrightarrow}} \quad \mathbf{e}_1. \tag{5.3}$$

Note that $p_n/\ln(1/p_n) \gg 1/n$ when $p_n \gg \ln n/n$, therefore

$$n^{-1}|C_{n,\sigma(n)}| \quad \underset{n\to\infty}{\longrightarrow} \quad 1 \qquad \text{in probability,}$$

and the proof is complete. □

Theorem 5.8. *In the subcritical regime* $1 \gg p_n \gg \ln n/n$, *we have the convergence*

$$\frac{p_n}{\ln(1/p_n)} I_n \quad \overset{(d)}{\underset{n\to\infty}{\longrightarrow}} \quad \mathbf{e}_1,$$

where $\mathbf{e}_1$ *is an exponential random variable with rate* 1.

As we noted, $p_n/\ln(1/p_n) \gg 1/n$ when $p_n \gg \ln n/n$, so this result recovers Corollary 5.2 (ii). Observe also that in the critical regime $p'_n \sim c \ln n/n$, we have $p'_n/\ln(1/p'_n) \sim c/n$; it then follows from Corollary 5.2 (iii) that in this case

$$\frac{p'_n}{\ln(1/p'_n)} I_n \quad \overset{\text{(d)}}{\underset{n\to\infty}{\longrightarrow}} \quad c(\mathbf{e}_c \wedge 1) = \mathbf{e}_1 \wedge c,$$

and the right-hand side further converges to $\mathbf{e}_1$ as $c \to \infty$. The same phenomenon was observed in Remark 4.3 for Cayley trees, where in both regimes, critical and subcritical, one should rescale I_n by a factor p_n^2 and the limit for the critical case converges to that of the subcritical case when $c \to \infty$.

Proof. The proof borrows ideas from [19, Section 3]. Recall that I_n is the number of leaves in the component of $\mathrm{Cut}(\mathrm{T}_n)$ which contains the root $[n]$ after logging at the atoms of the process φ_n. According to the proof of Lemma 5.7, with high probability, there exists a mark on the path from $[n]$ to $\{1\}$ in $\mathrm{Cut}(\mathrm{T}_n)$, at height $\sigma(n)$. Observe that all the other marks of φ_n are contained in the subtrees of $\mathrm{Cut}(\mathrm{T}_n)$ that stem from the blocks $C'_{n,1}, \dots, C'_{n,\sigma(n)}$. Moreover, appealing to Lemma 5.5, (5.3) reads

$$\frac{p_n}{\ln(1/p_n)} \sum_{i=1}^{\sigma(n)} |C'_{n,i}| \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad \mathbf{e}_1.$$

It only remains to show that the proportion of leaves in all these subtrees which belong to the root-component of $\mathrm{Cut}(\mathrm{T}_n)$ converges to 1 in probability. Recall from Lemma 5.4 that, conditional on the sets $C'_{n,1}, \dots, C'_{n,\sigma(n)}$, the subtrees of $\mathrm{Cut}(\mathrm{T}_n)$ that stems from these blocks are independent and distributed respectively as the cut-tree of a random recursive tree on $C'_{n,i}$. As in the proof of Proposition 2.1, we show that the probability that a leaf chosen uniformly at random in these subtrees belongs to the root-component converges to 1. This probability latter is bounded from below by

$$\mathbb{E}\left[(1-p_n)^{\max\{\mathrm{Depth}(\mathrm{Cut}(\mathrm{T}'_{n,i}));1\le i\le\sigma(n)\}}\right],$$

where $\mathrm{Depth}(T)$ denotes the maximal distance in the tree T from the root to a leaf. The proof then boils down to the convergence

$$p_n \max\{\mathrm{Depth}(\mathrm{Cut}(\mathrm{T}'_{n,i})); 1 \le i \le \sigma(n)\} \quad \underset{n\to\infty}{\longrightarrow} \quad 0$$

in probability. Bertoin [19, Proposition 1] proves a similar statement, in the case where $p_n = \ln n/n$ and the maximum is up to $\lambda(n)$. We closely follow the arguments in [19]. Fix $\varepsilon > 0$; from Lemma 5.4, since $\mathrm{Depth}(\mathrm{Cut}(T)) \le |T|$, for every $m \in \mathbb{N}$ and $a > 0$,

$$\mathbb{P}(p_n \max\{\mathrm{Depth}(\mathrm{Cut}(\mathrm{T}'_{n,i})); 1 \le i \le \sigma(n)\} > \varepsilon)$$

is bounded from above by

$$m \sup_{k\le a/p_n} \mathbb{P}(p_n\mathrm{Depth}(\mathrm{Cut}(\mathrm{T}_k)) > \varepsilon) + \mathbb{P}(N(\varepsilon,n) > m) + \mathbb{P}(N(a,n) \ge 1),$$

where $N(z,n) = \mathrm{Card}\{i = 1, \dots, \sigma(n) : |\mathrm{T}'_{n,i}| > z/p_n\}$. On the one hand, from (5.2) and the distribution of ξ, conditionally given $\sigma(n)$ with $\sigma(n) \le \lambda(n)$, $N(z,n)$ is binomial distributed with parameters $\sigma(n)$ and $\lceil z/p_n \rceil^{-1}$; as a consequence, for any $\delta > 0$, we may fix m and a sufficiently large so that

$$\limsup_{n\to\infty} \mathbb{P}(N(\varepsilon,n) > m) + \mathbb{P}(N(a,n) \ge 1) \le \delta.$$

On the other hand, from (5.2), we have

$$\mathrm{Depth}(\mathrm{Cut}(\mathrm{T}_k)) \le \lambda(k) + \max\{\xi_i; 1 \le i \le \lambda(k)\} + (k - S_{\lambda(k)})$$

which, rescaled by a factor p_n, converges in probability to 0 uniformly for $k \le a/p_n$ thanks to Lemma 5.6. This concludes the proof. □

Remark 5.9. Let C_n be the root-component of $\mathrm{Cut}(\mathrm{T}_n)$ after performing a Bernoulli bond percolation, in which each edge is removed with probability p_n; we endow it with the graph distance d_n and the measure ν_n which assigns mass 1 to each leaf. Adapting Section 3 of Bertoin [19], the proofs of Proposition 5.3 and Theorem 5.8 here respectively entail the following weak convergences for the pointed Gromov–Hausdorff–Prokhorov topology:

$$\left(C_n, p_n d_n, \frac{p_n}{\ln(1/p_n)}\nu_n\right) \xrightarrow[n\to\infty]{(\mathrm{d})} \Big([0, \mathbf{e}_1], |\cdot|, \mathrm{Leb}\Big), \quad \text{when } p_n \gg \ln n/n,$$

and

$$\left(C_n, \frac{\ln n}{n} d_n, \frac{1}{n}\nu_n\right) \xrightarrow[n\to\infty]{(\mathrm{d})} \Big([0, \mathbf{e}_c \wedge 1], |\cdot|, \mathrm{Leb}\Big), \quad \text{when } p_n \sim c \ln n/n,$$

where in both cases, $|\cdot|$ and Leb refer respectively to the Euclidean distance and Lebesgue measure, and the intervals are pointed at 0. The same arguments also yield

$$\left(C_n, \frac{\ln n}{n} d_n, \frac{1}{n}\nu_n\right) \xrightarrow[n\to\infty]{(\mathrm{d})} \Big([0, 1], |\cdot|, \mathrm{Leb}\Big), \quad \text{when } p_n \ll c \ln n/n.$$

5.4 Connectivity properties of the fireproof forest

We next focus on the fireproof forest. As in Section 4 of Bertoin [16] for Cayley trees, we first find an asymptotic estimate for the probability that the root and a uniform random vertex belong to the same fireproof subtree, in both the critical and supercritical cases. We then deduce estimates on the size of the largest fireproof component in all of the three regimes.

Theorem 5.10. *Let $c \in [0, \infty)$ and p_n such that $\lim_{n\to\infty} np_n/\ln n = c$. Let also X_n be a uniform random vertex in $[n]$ independent of T_n and the fire dynamics. Then the probability that X_n and 1 belong to the same fireproof subtree converges towards e^{-c} as $n \to \infty$.*

Let us postpone the proof of Theorem 5.10 to first state some consequences in terms of the existence of giant fireproof components. Denote by $f^*_{n,1}$ the size of the largest fireproof subtree of T_n.

Corollary 5.11. *In the supercritical regime $p_n \ll \ln n/n$, we have the convergence*

$$n^{-1} f^*_{n,1} \underset{n\to\infty}{\longrightarrow} 1 \quad \textit{in probability.}$$

This further yields the following result for the subcritical regime.

Corollary 5.12. *In the subcritical regime $1 \gg p_n \gg \ln n/n$, the sequence $(p_n f^*_{n,1}; n \geq 1)$ is tight.*

Finally, in the critical regime, the behavior resembles that of sub or supercritical, according to the final state of the root. Fix $c > 0$.

Corollary 5.13. *Consider the critical regime $p_n \sim c \ln n/n$. We distinguish two cases:*

(i) On the event that the root burns, the sequence $((\ln n)n^{-1} f^*_{n,1}; n \geq 1)$ is tight.

(ii) On the event that the root is fireproof, $n^{-1} f^*_{n,1}$ converges to 1 in probability.

Proof of Theorem 5.10. We use a so-called *spinal decomposition*: fix $X \in [n]$ and denote by $h(X) = d(X, 1)$ the height of X in T_n. Let $V_0, \ldots, V_{h(X)}$ be the vertices on the oriented branch from 1 to X: $V_0 = 1$, $V_{h(X)} = X$ and for each $i = 1, \ldots, h(X)$, V_{i-1} is the parent of V_i. Removing all the edges $\{V_i, V_{i+1}\}$ disconnects T_n into $h(X) + 1$ subtrees denoted by $T_0, \ldots, T_{h(X)}$ where T_i contains V_i for every $i = 0, \ldots, h(X)$. Clearly, $V_0 = 1$ and $V_{h(X)} = X$ belong to the same fireproof connected component if and only if all the V_i's are fireproof, i.e. when all the edges $\{V_i, V_{i+1}\}$ are fireproof and each V_i is fireproof for the dynamics restricted to the tree T_i.

Using the inductive construction of random recursive trees described in the introduction, one sees that, when removing the edge $\{V_0, V_1\}$, the two subtrees we obtain are, conditional on their set of vertices, independent random recursive trees. The one containing V_0 is T_0. Removing the edge $\{V_1, V_2\}$ in the other subtree, we obtain similarly that T_1 is, conditional on its set of vertices and that of T_0, a random recursive tree independent of T_0. We conclude by induction that conditional on their set of vertices, the T_i's are independent random recursive trees rooted at V_i respectively.

Recall that for every $k \geq 1$, $\zeta(k)$ denotes the height of the leaf $\{1\}$ in the cut-tree $\mathrm{Cut}(\mathrm{T}_k)$ of a random recursive tree of size k. We have seen that the root of T_k is fireproof with probability $\mathbb{E}[(1-p_n)^{\zeta(k)}]$. Thus, from the discussion above, the probability that X and 1 belong to the same fireproof connected component is given by

$$\mathbb{E}\left[\exp\left(\ln(1-p_n)\left(h(X) + \sum_{i=0}^{h(X)} \zeta_i(|T_i|)\right)\right)\right], \tag{5.4}$$

where $(\zeta_i(k); k \geq 1)_{i\geq 0}$ is a sequence of i.i.d. copies of $(\zeta(k); k \geq 1)$. We prove that if X_n is uniformly distributed on $[n]$, then

$$\frac{\ln n}{n}\left(h(X_n) + \sum_{i=0}^{h(X_n)} \zeta_i(|T_i|)\right) \underset{n\to\infty}{\longrightarrow} 1 \qquad \text{in probability,} \tag{5.5}$$

which yields Theorem 5.10. It is well-known that $h(X_n) \sim \ln n$ in probability as $n \to \infty$ so we only need to consider the sum in (5.5). Let us first discuss the distribution of the $|T_i|$'s.

Let $S_n(0)$ be a random variable uniformly distributed on $[n]$. Then, for every $i \geq 1$, conditionally given $\overline{S}_n(i-1) := S_n(0) + \cdots + S_n(i-1)$, let $S_n(i)$ be uniformly distributed on $[n - \overline{S}_n(i-1)]$ if $\overline{S}_n(i-1) < n$ and set $S_n(i) = 0$ otherwise. Let $\kappa_n := \inf\{i \geq 0 : \overline{S}_n(i) = n\}$; note that $S_n(i) = 0$ if and only if $i > \kappa_n$ and that $S_n(0) + \cdots + S_n(\kappa_n) = n$. We call the sequence $\mathrm{S}_n := (S_n(0), \dots, S_n(\kappa_n))$ a *discrete stick-breaking process*. Denote finally by $\tilde{S}_n$ a size-biased pick from S_n. Then $\tilde{S}_n$ is uniformly distributed on $[n]$ (see Lemma 5.14 below) and for every measurable and non-negative functions f and g, we have

$$\mathbb{E}\left[g\left(\sum_{i=0}^{\kappa_n} f(S_n(i))\right)\right] = \mathbb{E}\left[g\left(\sum_{i=0}^{\kappa_n} n\frac{f(S_n(i))}{S_n(i)}\mathbb{P}(\tilde{S}_n = S_n(i) \mid \mathrm{S}_n)\right)\right] = \mathbb{E}\left[g\left(n\frac{f(\tilde{S}_n)}{\tilde{S}_n}\right)\right] \tag{5.6}$$

The stick breaking-process appears in a random recursive tree in two ways: *vertically* and *horizontally*. Indeed, if we discard the root of T_{n+1} and its adjacent edges, then the sequence formed by the sizes of the resulting subtrees, ranked in increasing order of their root is distributed as S_n. In particular, the one containing the leaf $n+1$ has size $\tilde{S}_n$ so is uniformly distributed on $[n]$. Further, if $n+1$ is not the root of this subtree, we can iterate the procedure of removing the root and discarding all the components but the one containing $n+1$. Conditionally given the size s_i of the component containing $n+1$ at the i-th step, its size at the $i+1$-st step is uniformly distributed on $[s_i - 1]$, thus defining a stick-breaking process. We continue until the component containing $n+1$ is reduced to the singleton $\{n+1\}$; this takes $h(n+1) = \kappa_n + 1$ steps.

Let X_n be the parent of $n+1$ in T_{n+1}; then X_n is distributed as a uniform random vertex of T_n. Moreover, we just saw that $h(X_n) = h(n+1) - 1$ is distributed as κ_n and, further, the sequence $|T_0|, \dots, |T_{h(X_n)}|$ previously defined is distributed as S_n. Theorem 5.10 will thereby follow from the convergence

$$\frac{\ln n}{n}\sum_{i=0}^{\kappa_n} \zeta_i(S_n(i)) \underset{n\to\infty}{\longrightarrow} 1 \tag{5.7}$$

in probability. We prove the convergence of the first and second moments. Let us define $f_1(\ell) := \mathbb{E}[\zeta(\ell)]$ and $f_2(\ell) := \mathbb{E}[\zeta(\ell)^2]$ for every $\ell \geq 1$. We already mentioned that Meir &

Moon [86] proved that, as $\ell \to \infty$,

$$f_1(\ell) = \frac{\ell}{\ln \ell}(1 + o(1)) \qquad \text{and} \qquad f_2(\ell) = \left(\frac{\ell}{\ln \ell}\right)^2 (1 + o(1)). \tag{5.8}$$

Conditioning first on S_n and then averaging, we have

$$\mathbb{E}\left[\sum_{i=0}^{\kappa_n} \zeta_i(S_n(i))\right] = \mathbb{E}\left[\sum_{i=0}^{\kappa_n} f_1(S_n(i))\right],$$

and, using the conditional independence of the ζ_i's,

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=0}^{\kappa_n} \zeta_i(S_n(i))\right)^2\right] &= \mathbb{E}\left[\sum_{i=0}^{\kappa_n} \zeta_i(S_n(i))^2\right] + \mathbb{E}\left[\sum_{i\neq j} \zeta_i(S_n(i))\zeta_j(S_n(j))\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{\kappa_n} f_2(S_n(i))\right] + \mathbb{E}\left[\sum_{i\neq j} f_1(S_n(i))f_1(S_n(j))\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{\kappa_n} f_2(S_n(i))\right] + \mathbb{E}\left[\left(\sum_{i=0}^{\kappa_n} f_1(S_n(i))\right)^2\right] - \mathbb{E}\left[\sum_{i=0}^{\kappa_n} f_1(S_n(i))^2\right].
\end{aligned}$$

We finally compute these four expectations appealing to (5.6), (5.8) and Lemma 5.14: as $n \to \infty$,

$$\mathbb{E}\left[\sum_{i=0}^{\kappa_n} f_1(S_n(i))\right] = n\mathbb{E}\left[\frac{f_1(\tilde{S}_n)}{\tilde{S}_n}\right] = \sum_{\ell=1}^{n} \frac{f_1(\ell)}{\ell} \sim \sum_{\ell=2}^{n} \frac{1}{\ln \ell} \sim \frac{n}{\ln n};$$

similarly

$$\mathbb{E}\left[\sum_{i=0}^{\kappa_n} f_2(S_n(i))\right] = n\mathbb{E}\left[\frac{f_2(\tilde{S}_n)}{\tilde{S}_n}\right] = \sum_{\ell=1}^{n} \frac{f_2(\ell)}{\ell} \sim \sum_{\ell=2}^{n} \frac{\ell}{(\ln \ell)^2} \sim \frac{n^2}{2(\ln n)^2};$$

and

$$\mathbb{E}\left[\left(\sum_{i=0}^{\kappa_n} f_1(S_n(i))\right)^2\right] = n^2\mathbb{E}\left[\left(\frac{f_1(\tilde{S}_n)}{\tilde{S}_n}\right)^2\right] = n\sum_{\ell=1}^{n} \left(\frac{f_1(\ell)}{\ell}\right)^2 \sim n\sum_{\ell=2}^{n} \frac{1}{(\ln \ell)^2} \sim \frac{n^2}{(\ln n)^2};$$

finally

$$\mathbb{E}\left[\sum_{i=0}^{\kappa_n} f_1(S_n(i))^2\right] = n\mathbb{E}\left[\frac{f_1(\tilde{S}_n)^2}{\tilde{S}_n}\right] = \sum_{\ell=1}^{n} \frac{f_1(\ell)^2}{\ell} \sim \sum_{\ell=2}^{n} \frac{\ell}{(\ln \ell)^2} \sim \frac{n^2}{2(\ln n)^2}.$$

Thus, the first two moments of $n^{-1}\ln n \sum_{i=0}^{\kappa_n} \zeta_i(S_n(i))$ converge to 1, which implies (5.7) (the convergence even holds in L^2). $\square$

In the course of the proof we used the following lemma.

Lemma 5.14. *A size-biased pick $\tilde{S}_n$ from a discrete stick-breaking process* S_n *is uniformly distributed on* $[n]$.

Proof. As we have seen, S_n is distributed as the sizes of the subtrees of T_{n+1} after removing the root and its adjacent edges. Furthermore, as there are $n!$ recursive trees of size $n+1$, the latter are in bijection with permutations of $[n]$. Indeed, there is an explicit bijection between uniform random recursive trees of size $n+1$ and uniform random permutation of $[n]$, via the chinese restaurant process, see e.g. Goldschmidt and Martin [55], in which the vertex-sets of the subtrees of T_{n+1} after removing the root and its adjacent edges are exactly the cycles of the permutation. Then, for every $k \in [n]$,

$$\mathbb{P}(\tilde{S}_n = k) = \sum_{i \geq 0} \frac{k}{n} \mathbb{P}(S_n(i) = k) = \frac{k}{n} \mathbb{E}[\mathrm{Card}\{i \geq 0 : S_n(i) = k\}],$$

and $\mathrm{Card}\{i \geq 0 : S_n(i) = k\}$ is distributed as the number of cycles of length k in a uniform random permutation of $[n]$. Given k distinct elements of $[n]$ fixed, there are $(k-1)!(n-k)!$ permutations of $[n]$ for which they form a cycle, so they form a cycle of a uniform random permutation with probability $(k-1)!(n-k)!/n!$. Summing over all the k-tuples of $[n]$, we obtain

$$\mathbb{E}[\mathrm{Card}\{i \geq 0 : S_n(i) = k\}] = \binom{n}{k} \frac{(n-k)!(k-1)!}{n!} = \frac{1}{k},$$

and the proof is complete. □

We end this section with the proof of the three corollaries of Theorem 5.10.

Proof of Corollary 5.11. Let $f^*_{n,1} \geq f^*_{n,2} \geq \cdots \geq 0$ be the sizes of the fireproof subtrees of T_n, ranked in non-increasing order. Let also X_n and X'_n be two independent uniform random vertices in $[n]$, independent of the fire dynamics. Since $\sum_{i \geq 1} f^*_{n,i} \leq n$, we have

$$\begin{aligned}
\mathbb{E}[n^{-1} f^*_{n,1}] &\geq \mathbb{E}\left[n^{-2} \sum_{i \geq 1} (f^*_{n,i})^2 \right] \\
&= \mathbb{P}(X_n \text{ and } X'_n \text{ belong to the same fireproof component}) \\
&\geq \mathbb{P}(X_n, X'_n \text{ and } 1 \text{ belong to the same fireproof component}) \\
&\geq 2\mathbb{P}(X_n \text{ and } 1 \text{ belong to the same fireproof component}) - 1,
\end{aligned}$$

and the latter converges to 1 as $n \to \infty$ from Theorem 5.10. We conclude that $n^{-1} f^*_{n,1}$ converges to 1 in probability. □

Proof of Corollary 5.12. With the notations of the proof of Theorem 5.8, the root burns with high probability, so we implicitly condition on this event, and the number of edges fireproof in the root-component is given by $\sigma(n)$ which, rescaled by a factor p_n, converges in distribution towards $\mathbf{e}_1$. Then the argument of Lemma 5.6 (iv) entails the joint convergence in distribution of the pair

$$\left(p_n\sigma(n), \sum_{i=1}^{\sigma(n)} \delta_{p_n|\mathrm{T}'_{n,i}|}(\mathrm{d}x)\right)$$

towards $\mathbf{e}_1$ and a Poisson random measure with intensity $\mathbf{e}_1 x^{-2}\mathrm{d}x$ on $(0,\infty)$. In particular, for every $\varepsilon \in (0,1)$, there exist two constants, say, m and M, for which

$$\mathbb{P}\left(m \le \max_{1\le i\le\sigma(n)} p_n|\mathrm{T}'_{n,i}| \le M\right) > 1-\varepsilon,$$

for every n large enough. Observe that

$$p_n \frac{M/p_n}{\ln(M/p_n)} \quad \underset{n\to\infty}{\longrightarrow} \quad 0,$$

and therefore a subtree which satisfies $m \le p_n|\mathrm{T}'_{n,i}| \le M$ is supercritical. It then follows from Theorem 5.10 that such a subtree contains a fireproof component larger than $(1-\varepsilon)m/p_n$ (and smaller that M/p_n) with high probability and the proof is complete. □

Proof of Corollary 5.13. For the first statement, on the event that the root burns, the number $\sigma(n)$ of edges fireproof in the root-component, rescaled by a factor $\ln n/n$ converges in distribution towards an exponential random variable with rate c conditioned to be smaller than 1. The rest of the proof follows verbatim from that of Corollary 5.12 above. For the second statement, we already proved in Proposition 5.3 that the probability that the root of T_n is fireproof converges to e^{-c} as $n\to\infty$. Thus, on this event, the probability that the root and an independent uniform random vertex belong to the same fireproof subtree converges to 1 and the claim follows from the proof of Corollary 5.11. □

5.5 On the sequence of burnt subtrees

In the last section, we investigate the behavior of the burnt subtrees, in the critical regime $p_n \sim c\ln n/n$. Let $\gamma_0 = 0$ and $(\gamma_j - \gamma_{j-1})_{j\ge1}$ be a sequence of i.i.d. exponential random variables with rate c and conditional on $(\gamma_j)_{j\ge1}$, let $(Z_j)_{j\ge1}$ be a sequence of independent random variables, where Z_j is distributed as an exponential random variable with rate γ_j conditioned to be smaller than 1. For every $i \in \mathbb{N}$, denote by $\theta_{n,i}$ the time at which the i-th fire occurs and by $b_{n,i}$ the size of the corresponding burnt subtree of T_n.

Theorem 5.15. *Consider the critical regime $p_n \sim c \ln n/n$. We have for every $i \in \mathbb{N}$,*

$$\frac{\ln n}{n}(\theta_{n,1}, \dots, \theta_{n,i}) \quad \xrightarrow[n\to\infty]{(d)} \quad (\gamma_1, \dots, \gamma_i).$$

Furthermore, for every $j \in \mathbb{N}$, the probability that the root burns with the j-th fire converges to

$$\mathbb{E}\left[\mathrm{e}^{-\gamma_j} \prod_{i=1}^{j-1}(1 - \mathrm{e}^{-\gamma_i})\right]$$

as $n \to \infty$. Finally, on this event, for every $k \geq j+1$, the vector

$$\left(\frac{\ln b_{n,1}}{\ln n}, \dots, \frac{\ln b_{n,j-1}}{\ln n}, \frac{b_{n,j}}{n}, \frac{\ln b_{n,j+1}}{\ln n}, \dots, \frac{\ln b_{n,k}}{\ln n}\right)$$

converges in distribution towards

$$(Z_1, \dots, Z_{j-1}, \mathrm{e}^{-\gamma_j}, Z_{j+1}, \dots, Z_k).$$

We can compute the expectation above by setting

$$\mathbb{E}\left[\mathrm{e}^{-\gamma_j} \prod_{i=1}^{j-1}(1 - \mathrm{e}^{-\gamma_i})\right] = \rho_{j-1} - \rho_j,$$

where for every $j \geq 0$,

$$\begin{aligned}
\rho_j &= \mathbb{E}\left[\prod_{i=1}^{j}(1 - \mathrm{e}^{-\gamma_i})\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{j}(-1)^i \sum_{1\leq \ell_1 < \cdots < \ell_i \leq j} \mathrm{e}^{-\gamma_{\ell_1}} \cdots \mathrm{e}^{-\gamma_{\ell_i}}\right] \\
&= \sum_{i=0}^{j}(-1)^i \sum_{1\leq \ell_1 < \cdots < \ell_i \leq j} \mathbb{E}[\mathrm{e}^{-i\gamma_{\ell_1}} \mathrm{e}^{-(i-1)(\gamma_{\ell_2}-\gamma_{\ell_1})} \cdots \mathrm{e}^{-(\gamma_{\ell_i}-\gamma_{\ell_{i-1}})}] \\
&= \sum_{i=0}^{j}(-1)^i \sum_{1\leq \ell_1 < \cdots < \ell_i \leq j} \left(\frac{c}{c+i}\right)^{\ell_1}\left(\frac{c}{c+i-1}\right)^{\ell_2-\ell_1} \cdots \left(\frac{c}{c+1}\right)^{\ell_i-\ell_{i-1}}.
\end{aligned}$$

Before tackling the proof of Theorem 5.15, let us give a consequence of it.

Corollary 5.16. *We have in the critical regime $p_n \sim c \ln n/n$*

$$\lim_{k\to\infty} \lim_{n\to\infty} \mathbb{P}(\textit{the root of } \mathrm{T}_n \textit{ is not burnt after } k \textit{ fires}) = \lim_{n\to\infty} \mathbb{P}(\textit{the root of } \mathrm{T}_n \textit{ is fireproof}),$$

and both are equal to e^{-c}.

In particular, we see that for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\liminf_{n\to\infty} \mathbb{P}(\text{the root of } \mathrm{T}_n \text{ is fireproof} \mid \text{it is not burnt after } k \text{ fires}) > 1-\varepsilon.$$

In words, we can find k large enough but independent of n, such that if the root of T_n has not burnt after the first k fires, then with high probability, it will be fireproof at the end of the dynamics.

Proof. We already proved in Proposition 5.3 that the right-hand side is equal e^{-c}. To prove that the left-hand side is equal to e^{-c} as well, observe from Theorem 5.15 that for every $k \in \mathbb{N}$, the probability that the root of T_n is not burnt after k fires converges as $n \to \infty$ towards $\mathbb{E}[\prod_{i=1}^{k}(1-\mathrm{e}^{-\gamma_i})]$, which in turn converges as $k \to \infty$ towards

$$\mathbb{E}\left[\prod_{i=1}^{\infty}(1-\mathrm{e}^{-\gamma_i})\right] = \mathbb{P}(\varepsilon_i \leq \gamma_i \text{ for every } i \geq 1),$$

where $(\varepsilon_i)_{i\geq 1}$ is a sequence of i.i.d. standard exponential random variables which is independent of the sequence $(\gamma_i)_{i\geq 1}$. For each $k \geq 1$, define $\tilde{\gamma}_k \coloneqq \gamma_k$ if $\gamma_i \geq \varepsilon_i$ for every $i \leq k$ and $\tilde{\gamma}_k \coloneqq \partial$, some cemetery state, otherwise. Then $\tilde{\gamma}$ is a homogeneous killed Markov chain with (defective) transition kernel given by

$$Q(x, x+\mathrm{d}y) = (1-\mathrm{e}^{-(x+y)})c\mathrm{e}^{-cy}\mathrm{d}y, \qquad x, y \in \mathbb{R}.$$

We define for every $x \in \mathbb{R}$, $\rho(x)$ the probability, starting from x, that $\tilde{\gamma}$ lives for ever, i.e. never hits the absorbing state ∂. Note that $\rho(\infty) = 1$ and $\rho(0) = \mathbb{E}[\prod_{i=1}^{\infty}(1-\mathrm{e}^{-\gamma_i})]$. Using the Markov property at the first step, we have for every $x \in \mathbb{R}$,

$$\rho(x) = \int_0^\infty (1-\mathrm{e}^{-(x+y)})c\mathrm{e}^{-cy}\rho(x+y)\mathrm{d}y.$$

Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \mathrm{e}^{-cx}\rho(x)$, then f satisfies the integral equation

$$f(x) = \int_x^\infty c(1-\mathrm{e}^{-z})f(z)\mathrm{d}z, \qquad x \in \mathbb{R},$$

with $f(\infty) = 0$. Taking the derivative of both sides, we obtain

$$f'(x) = -c(1-\mathrm{e}^{-x})f(x), \qquad x \in \mathbb{R},$$

and then

$$\ln\left(\frac{f(x)}{f(0)}\right) = \int_0^x -c(1-\mathrm{e}^{-z})\mathrm{d}z = -(cx - c + c\mathrm{e}^{-x}), \qquad x \in \mathbb{R}.$$

We conclude that

$$\rho(0) = f(0) = f(x)\exp(cx - c + c\mathrm{e}^{-x}) = \mathrm{e}^{-c}\rho(x)\exp(c\mathrm{e}^{-x}), \qquad x \in \mathbb{R}.$$

Letting $x \to \infty$, we obtain $\rho(0) = \mathrm{e}^{-c}$ and more generally $\rho(x) = \exp(-c\mathrm{e}^{-x})$ for every $x \in \mathbb{R}$, which is the distribution function of a Gumbel random variable evaluated at $x - \ln c$. This completes the proof. □

In order to prove Theorem 5.15, we shall need the following three results on random recursive trees. Let E_n be the set of all edges of T_n which are not adjacent to the root. Fix $k \in \mathbb{N}$ and consider a simple random sample $(e_{n,1}, \dots, e_{n,k})$ of k edges from E_n. For each $i = 1, \dots, k$, denote by $v_{n,i}$ and $v'_{n,i}$ the two extremities of $e_{n,i}$ with the convention that $v_{n,i}$ is the closest one to the root. Finally, denote by $\rho_{n,k} := \max_{1 \le i,j \le k} h(v_{n,i} \wedge v_{n,j})$, where $a \wedge b$ denotes the last common ancestor of a and b in T_n and $h(a)$ the height of a in T_n. Then in the complement of the ball centered at the root and of radius $\rho_{n,k}$, the paths from 1 to $v_{n,i}$, $i = 1, \dots, k$ are disjoint.

Lemma 5.17. *For every $i \in \mathbb{N}$ fixed, we have*

$$\frac{1}{\ln n} h(v_{n,i}) \underset{n\to\infty}{\longrightarrow} 1 \quad \textit{in probability.}$$

Proof. We may replace $v_{n,1}$ by $v'_{n,1}$, which is uniformly distributed on the set of vertices of T_n with height at least two. It is known that the degree of the root of T_n rescaled by $\ln n$ converges to 1 in probability so, as $n \to \infty$, $v'_{n,1}$ is close (in the sense of total variation) to a uniform random vertex on $[n]$. It follows that

$$\frac{1}{\ln n} \ln v_{n,1} \underset{n\to\infty}{\longrightarrow} 1 \quad \text{in probability.}$$

Finally, in a random recursive tree, the label ℓ of a vertex and its height $h(\ell)$ are related as follows:

$$\frac{1}{\ln \ell} h(\ell) \underset{\ell\to\infty}{\longrightarrow} 1 \quad \text{in probability,} \tag{5.9}$$

which concludes the proof. □

Lemma 5.18. *For every $k \in \mathbb{N}$ fixed, $\rho_{n,k} = O(1)$ in probability as $n \to \infty$.*

Proof. It suffices to consider $k = 2$; moreover, we may approximate $v_{n,1}$ and $v_{n,2}$ by a two independent uniform random vertices in $[n]$, say, u_n and v_n. We have $h(u_n \wedge v_n) \ge 1$ if and only if u_n and v_n belong to the same tree-component in the forest, say, $(T_n^1, \dots, T_n^\kappa)$ obtained by removing the root and all its adjacent edges from T_n. We already noticed that this vector forms a discrete stick-breaking process on $[n-1]$. Let $\Upsilon_1 = 0$ and $\Upsilon_n := \sum_{j=1}^{\kappa} |T_n^j|^2$ for $n \geq 1$; then, by decomposing with respect to $|T_n^1|$, since the latter is uniformly distributed on $[n-1]$, we obtain

$$\mathbb{E}[\Upsilon_n] = \sum_{\ell=1}^{n-1} \frac{1}{n-1}(\ell^2 + \mathbb{E}[\Upsilon_{n-\ell}]) = (n-1) + \mathbb{E}[\Upsilon_{n-1}] = \sum_{\ell=1}^{n}(\ell-1) = \frac{n(n-1)}{2}.$$

Finally,

$$\mathbb{P}(h(u_n \wedge v_n) \geq 1) = \frac{\mathbb{E}[\Upsilon_n]}{n^2} = \frac{n-1}{2n} \quad \underset{n\to\infty}{\longrightarrow} \quad \frac{1}{2}.$$

We can iterate with the same reasoning: $h(u_n \wedge v_n) \geq \ell$ if and only if u_n and v_n belong to the same tree-component in the forest obtained by removing all the vertices at distance at most ℓ from the root and their adjacent edges. Then as previously,

$$\mathbb{P}(h(u_n \wedge v_n) \geq \ell + 1 \mid h(u_n \wedge v_n) \geq \ell) \quad \underset{n\to\infty}{\longrightarrow} \quad \frac{1}{2}.$$

Therefore $h(u_n \wedge v_n)$ converges weakly to the geometric distribution with parameter $1/2$. It follows that for every $k \in \mathbb{N}$ fixed, the sequence $(\rho_{n,k})_{n\in\mathbb{N}}$ is bounded in probability. $\square$

Lemma 5.19. *For each $v \in [n]$, denote by $\mathrm{T}_n(v)$ the subtree of T_n that stems from v. Then for any sequence of integers $1 \ll v_n \ll n$, we have*

$$\frac{v_n}{n}|\mathrm{T}_n(v_n)| \quad \underset{n\to\infty}{\overset{(d)}{\longrightarrow}} \quad \mathbf{e}_1,$$

where $\mathbf{e}_1$ is an exponential random variable with rate 1.

Proof. We interpret the size of $\mathrm{T}_n(v)$ in terms of a Pólya urn. Recall the iterative construction of random recursive trees described in the introduction. At the v-th step, we add the vertex v to the current tree; then cut the edge which connects v to its parent to obtain a forest with two components with sets of vertices $\{v\}$ and $[v-1]$ respectively. Next, the vertices $v+1, \ldots, n$ are added to this forest independently one after the others, and the parent of each is uniformly chosen in the system. Considering the two connected components, we see that their sizes evolve indeed as an urn with initial configuration of 1 red ball and $v-1$ black balls and for which at each step, a ball is picked uniformly at random and then put back in the urn, along with one new ball of the same color. Then $|\mathrm{T}_n(v)| - 1$ is equal to the number of "red" outcomes after $n-v$ trials and this is known to have the beta-binomial distribution with parameters $(n-v, 1, v-1)$, i.e.

$$\mathbb{P}(|\mathrm{T}_n(v)| = \ell + 1) = (v-1)\frac{(n-v)!}{(n-v-\ell)!}\frac{(n-\ell-2)!}{(n-1)!}, \quad \ell = 0, \ldots, n-v.$$

Using Stirling formula, we compute for every $x \in (0, \infty)$ and $1 \ll v_n \ll n$,

$$\mathbb{P}(|\mathrm{T}_n(v_n)| = \lfloor xn/v_n \rfloor) = \frac{v_n}{n}\mathrm{e}^{-x}(1 + o(1)),$$

and the claim follows from this local convergence. □

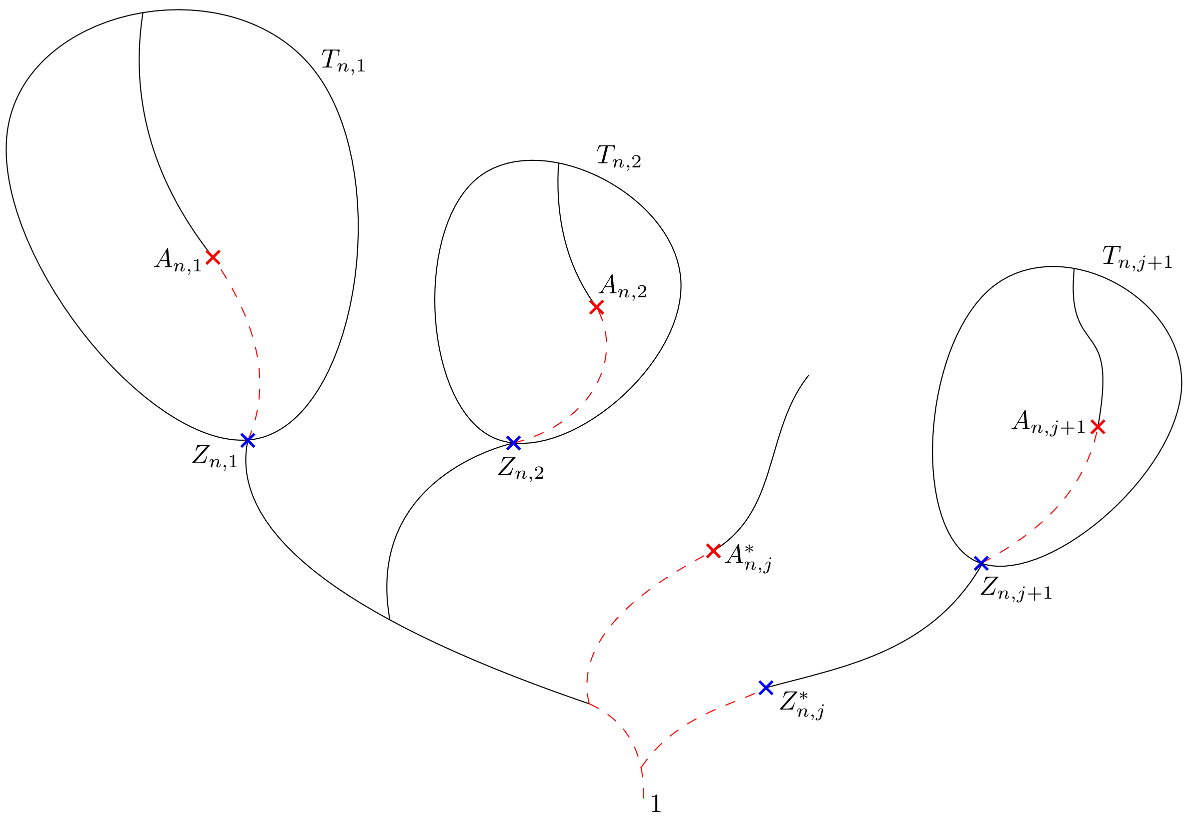


Figure 5.3: Illustration of the proof of Theorem 5.15.

The proof of Theorem 5.15 consists of four main steps, which are indicated by "Step 1", …, "Step 4". We first consider the case of the root: we view the fire dynamics as a dynamical percolation in continuous-time where each fireproof edge is deleted and each burnt component is discarded. Then the results of Bertoin [18] allow us to derive the size of the root-cluster at the instant of the first fire, which gives us the probability that this cluster burns with the first fire. If it does not, then we discard a small cluster (this is proven in step 2) and then continue the percolation on the remaining part until the second fire; again we know the size of the root-cluster at this instant. By induction, we obtain for every $j \geq 1$ the probability that the root burns with the j-th fire, and the size of its burnt component.

In a second step, we investigate the size of the first burnt subtree, conditionally given that it does not contain the root. We denote by $A_{n,1}$ the closest extremity to the root of the first edge which is set on fire. Since the root does not burn at this instant, there exists at least one fireproof edge on the path from the root to $A_{n,1}$; consider all the vertices on this path which are adjacent to such a fireproof edge and let $Z_{n,1}$ be the closest one to $A_{n,1}$. Finally, let $T_{n,1}$ be the subtree of T_n that stems from $Z_{n,1}$. Then $b_{n,1}$ is the size of the first burnt subtree of $T_{n,1}$ and the latter contains $A_{n,1}$ and its root $Z_{n,1}$. We estimate the size of $T_{n,1}$; further, we know the number of fireproof edges in $T_{n,1}$ before the first fire and, thanks to the first step (recall that, conditionally given its size, $T_{n,1}$ is a random recursive tree), we know the size of the root-component which burns with the first fire.

In the third step, we extend the results of the second one to the first two burnt components, conditionally given that none of them contains the root. The reasoning is the same as for the second step; we prove that the paths between the root of T_n and the starting points of the first two fires, respectively $A_{n,1}$ and $A_{n,2}$, become disjoint close to the root so that the dynamics on each are essentially independent: in particular the variables $Z_{n,1}$ and $Z_{n,2}$ (the latter plays the same role as $Z_{n,1}$ for the second fire) become independent at the limit. We conclude by induction that the estimate holds for the sizes of the first k burnt subtrees, conditionally given that the root does not burn with any of the first k fires.

The last step is a simple remark: the fact that the root burns does not affect the previous reasoning. Indeed, if the root burns with, say, the j-th fire, then there exists a vertex $Z_{n,j}^*$ between the root and the starting point $A_{n,j+1}$ of the $j+1$-first fire where the j-th fire stops. Therefore the estimate for the size of the burnt components before that the root burns holds also for the burnt components which come after that the root has burnt.

Proof of Theorem 5.15. Step 1: Let us first consider the root-component. It will be more convenient to work in a continuous-time setting. We attach to each edge e of T_n two independent exponential random variables, say, $\varepsilon^{(f)}(e)$ and $\varepsilon^{(b)}(e)$, with parameter $(1-p_n)/\ln n$ and $p_n/\ln n$ respectively. They should be thought of as the time at which the edge e becomes fireproof or is set on fire respectively; if the edge is already burnt because of the propagation of a prior fire, we do not do anything; likewise, if an edge e is fireproof, we do not set it on fire at the time $\varepsilon^{(b)}(e) > \varepsilon^{(f)}(e)$. Then the time $\tau(n)$ corresponding to the first fire is given by

$$\tau(n) = \inf\{\varepsilon^{(b)}(e) : \varepsilon^{(b)}(e) < \varepsilon^{(f)}(e)\}.$$

By the properties of exponential distribution, the variable $\inf_e \varepsilon^{(b)}(e)$ is exponentially distributed with parameter $(n-1)p_n/\ln n \to c$ as $n \to \infty$. Denote by $\hat{e}$ the edge of T_n such that $\varepsilon^{(b)}(\hat{e}) = \inf_e \varepsilon^{(b)}(e)$. Then $\varepsilon^{(f)}(\hat{e})$ is exponentially distributed with parameter $(1-p_n)/\ln n$

and so $\varepsilon^{(b)}(\hat{e}) < \varepsilon^{(f)}(\hat{e})$ with high probability. We conclude that $\tau(n) = \inf_e \varepsilon^{(b)}(e)$ with high probability and the latter converges in distribution to γ_1. It then follows from Corollary 2 of Bertoin [18] that the size of the root-component at the instant $\tau(n)$, rescaled by a factor n^{-1}, converges to $\mathrm{e}^{-\gamma_1}$ as $n \to \infty$. We conclude that the probability that the root of T_n burns with the first fire converges to $\mathbb{E}[\mathrm{e}^{-\gamma_1}]$ and, conditional on this event, the size of the corresponding burnt component rescaled by a factor n^{-1} converges to $\mathrm{e}^{-\gamma_1}$ in distribution.

If the root does not burn with the first fire, then we shall prove in the next step that the size $b_{n,1}$ of the first burnt component is negligible compared to n with high probability. The previous reasoning then shows that the time of the second fire converges in distribution to γ_2, the probability that the root of T_n burns at the second fire converges to $\mathbb{E}[(1-\mathrm{e}^{-\gamma_1})\mathrm{e}^{-\gamma_2}]$ and, on this event, the size of the corresponding burnt component rescaled by a factor n^{-1} converges to $\mathrm{e}^{-\gamma_2}$ in distribution. Again, if the root does not burn with the second fire, then the second burnt component is negligible compared to n with high probability and the general claim follows by induction.

Step 2: For the rest of this proof, we condition on the event that the root of T_n burns with the j-th fire with $j \geq 1$ fixed. We first prove the convergence of the logarithm of the sizes of the first $j-1$ burnt subtrees. Observe that the $j-1$ first edges which are set on fire are distributed as a simple random sample of $j-1$ edges from the set E_n of edges of T_n not adjacent to the root. Lemma 5.17 then entails that

$$\frac{1}{\ln n}(h(A_{n,1}), \dots, h(A_{n,j-1})) \underset{n\to\infty}{\longrightarrow} (1, \dots, 1) \quad \text{in probability.}$$

Consider first the first burnt subtree. The number $\theta_{n,1}$ of edges fireproof when the first edge is set on fire follows the geometric distribution with parameter $p_n \sim c \ln n/n$, truncated at $n-1$, so

$$\frac{\ln n}{n}\theta_{n,1} \xrightarrow[n\to\infty]{(\mathrm{d})} \gamma_1. \tag{5.10}$$

Conditioning the root not to burn with the first fire amounts to conditioning the path from 1 to $A_{n,1}$ to contain at least one of the $\theta_{n,1}$ first fireproof edges. Since, conditionally given $\theta_{n,1}$, these edges are distributed as a simple random sample from the $n-2$ edges of T_n different from the first one which is set on fire, then for every $x \in (0,1)$, the probability that $d(A_{n,1}, Z_{n,1})$ is smaller than $x \ln n$, conditionally given that it is smaller than $h(A_{n,1})$ is given by

$$\mathbb{E}\left[\frac{1-(1-\frac{\lfloor x\ln n\rfloor}{n-2})\cdots(1-\frac{\lfloor x\ln n\rfloor}{n-\theta_{n,1}-1})}{1-(1-\frac{h(A_{n,1})}{n-2})\cdots(1-\frac{h(A_{n,1})}{n-\theta_{n,1}-1})}\right] \sim \mathbb{E}\left[\frac{1-(1-\frac{x\ln n}{n})^{\theta_{n,1}}}{1-(1-\frac{\ln n}{n})^{\theta_{n,1}}}\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[\frac{1-\exp(-x\gamma_1)}{1-\exp(-\gamma_1)}\right].$$

We conclude that
$$\left(\frac{\ln n}{n}\theta_{n,1}, \frac{1}{\ln n} d(A_{n,1}, Z_{n,1})\right) \xrightarrow[n\to\infty]{(d)} (\gamma_1, Z_1).$$
Appealing to (5.9), we further have
$$\frac{1}{\ln n} \ln Z_{n,1} \xrightarrow[n\to\infty]{(d)} 1 - Z_1,$$
jointly with the convergence (5.10). Note that with the notation of Lemma 5.19, we have $T_{n,1} = \mathrm{T}_n(Z_{n,1})$. Since $1 \ll Z_{n,1} \ll n$ in probability, we obtain
$$\frac{1}{\ln n} \ln |T_{n,1}| \xrightarrow[n\to\infty]{(d)} Z_1,$$
again jointly with (5.10). Consider finally the fire dynamics on $T_{n,1}$ and denote by $N_{n,1}$ the number of edges which are fireproof in this tree before the first fire. Note that conditionally given $N_{n,1}$, these edges are distributed as a simple random sample of $N_{n,1}$ edges from the complement in $T_{n,1}$ of the path from its root $Z_{n,1}$ to $A_{n,1}$ and that the fire burns this path. Conditionally given $\theta_{n,1}$, $d(A_{n,1}, Z_{n,1})$ and $|T_{n,1}|$, the variable $N_{n,1}$ has a hypergeometric distribution: it is the number of edges picked amongst the $|T_{n,1}| - 1 - d(A_{n,1}, Z_{n,1}) \sim |T_{n,1}|$ "admissible" edges from the $n-1$ edges of T_n after $\theta_{n,1}$ draws without replacement. Since $\theta_{n,1} = o(n)$ in probability, conditionally given $\theta_{n,1}$, $d(A_{n,1}, Z_{n,1})$ and $|T_{n,1}|$, the variable $N_{n,1}$ is close (in total variation) to a binomial variable with parameter $\theta_{n,1}$ and $n^{-1}|T_{n,1}|$. It is easy to check that a binomial random variable with parameters, say, n and $p(n)$, rescaled by a factor $(np(n))^{-1}$, converges in probability to 1; it follows from the previous convergences that
$$\frac{\ln |T_{n,1}|}{|T_{n,1}|} N_{n,1} = \frac{\ln n}{n}\theta_{n,1} \frac{\ln |T_{n,1}|}{\ln n} \frac{n}{\theta_{n,1}|T_{n,1}|} N_{n,1} \xrightarrow[n\to\infty]{(d)} \gamma_1 Z_1.$$
We see that in $T_{n,1}$, we fireproof $N_{n,1} \approx \gamma_1 Z_1 |T_{n,1}| / \ln |T_{n,1}|$ edges before setting the root on fire. From the first step of the proof, the size $b_{n,1}$ of the first burnt component (which, by construction, contains the root of $T_{n,1}$) is comparable to $|T_{n,1}|$; in particular,
$$\frac{1}{\ln n} \ln b_{n,1} \xrightarrow[n\to\infty]{(d)} Z_1.$$

Step 3: Consider next the first two fires and, again, condition the root not to be burnt after the second fire; in particular, there exists at least one fireproof edge on the path from 1 to $A_{n,1}$ and on that from 1 to $A_{n,2}$. Thanks to Lemma 5.18, we have $h(A_{n,1} \wedge A_{n,2}) = o(\ln n)$ in probability. Then with high probability, $Z_{n,1}$ and $Z_{n,2}$ are located outside the ball centered at 1 and of radius $h(A_{n,1} \wedge A_{n,2})$ where the two paths from 1 to $A_{n,1}$ and to $A_{n,2}$ are disjoint so where the location of the fireproof edges are independent (conditionally given $\theta_{n,1}$ and $\theta_{n,2}$). With the same reasoning as for the first fire, we obtain

$$\left(\frac{\ln n}{n}(\theta_{n,1},\theta_{n,2}),\frac{1}{\ln n}\Big(d(A_{n,1},Z_{n,1}),d(A_{n,2},Z_{n,2})\Big)\right) \quad \xrightarrow[n\to\infty]{\text{(d)}} \quad \Big((\gamma_1,\gamma_2),(Z_1,Z_2)\Big).$$

Then

$$\frac{1}{\ln n}(\ln Z_{n,1},\ln Z_{n,2}) \quad \xrightarrow[n\to\infty]{\text{(d)}} \quad (1-Z_1,1-Z_2),$$

and

$$\frac{1}{\ln n}(\ln|T_{n,1}|,\ln|T_{n,1}|) \quad \xrightarrow[n\to\infty]{\text{(d)}} \quad (Z_1,Z_2).$$

Finally,

$$\frac{1}{\ln n}(\ln b_{n,1},\ln b_{n,2}) \quad \xrightarrow[n\to\infty]{\text{(d)}} \quad (Z_1,Z_2).$$

We conclude by induction that

$$\frac{1}{\ln n}(\ln b_{n,1},\ldots,\ln b_{n,j-1}) \quad \xrightarrow[n\to\infty]{\text{(d)}} \quad (Z_1,\ldots,Z_{j-1}).$$

Step 4: Consider next the $j+1$-st fire. The number of flammable edges after the j-th fire is given by $q_{n,j} := n-(1+\theta_{n,j}+(b_{n,1}-1)+\cdots+(b_{n,j}-1))$ which, rescaled by a factor n^{-1}, converges in distribution towards $1-\mathrm{e}^{-\gamma_j}$. As previously, $\theta_{n,j+1}-\theta_{n,j}$ is then distributed as a geometric random variable with parameter $p_n \sim c\ln n/n$ and truncated at $q_{n,j} \gg n/\ln n$. It follows that

$$\frac{\ln n}{n}(\theta_{n,1},\ldots,\theta_{n,j+1}) \quad \xrightarrow[n\to\infty]{\text{(d)}} \quad (\gamma_1,\ldots,\gamma_{j+1}).$$

Moreover since the root has burnt, there is at least one fireproof edge on the path from 1 to $A_{n,j+1}$, the starting point of the $j+1$-st fire. All the previous work then applies and we obtain the convergence

$$\frac{1}{\ln n}(\ln b_{n,1},\ldots,\ln b_{n,j-1},\ln b_{n,j+1}) \quad \xrightarrow[n\to\infty]{\text{(d)}} \quad (Z_1,\ldots,Z_{j-1},Z_{j+1}).$$

The general claim follows by a last induction. □

Part III

Random non-crossing configurations

6

Simply generated non-crossing partitions

In this chapter, we study the behavior of large random non-crossing partition, as presented in Chapter 3. This work is based on the article [72] in collaboration with Igor Kortchemski.

6.1 Introduction

We are interested in the structure of non-crossing partitions. The latter were introduced by Kreweras [73], and quickly became a standard object in combinatorics. They have also appeared in many different other contexts, such as low-dimensional topology, geometric group theory and free probability (see e.g. the survey [84] and the references therein). In this work, we study combinatorial and geometric aspects of large *random* non-crossing partitions.

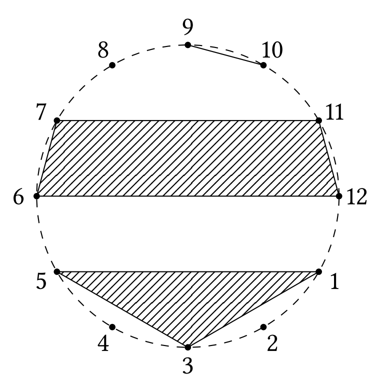


Figure 6.1: The non-crossing partition $\{\{1,3,5\},\{2\},\{4\},\{6,7,11,12\},\{8\},\{9,10\}\}$ of $[12]$.

Recall that a partition of $[n] := \{1,2,\ldots,n\}$ is a collection of (pairwise) disjoint subsets, called blocks, whose union is $[n]$. A non-crossing partition of $[n]$ is a partition of the vertices of a regular n-gon (labelled by the set $[n]$ in clockwise order) with the property that the convex hulls of its blocks are pairwise disjoint (see Figure 6.1 for an example).

Large discrete combinatorial structures. There are many ways to study discrete structures. Given a finite combinatorial class $\mathtt{A}_n$ of objects of "size" n, a first step is often to calculate as explicitly as possible its cardinal $\#\mathtt{A}_n$, using for instance bijective arguments or generating functions. For non-crossing partitions, it is well-known that they are enumerated by Catalan numbers. It is also often of interest to enumerate elements of $\mathtt{A}_n$ satisfying constraints. For instance, the number of non-crossing partitions of $[n]$ with given block sizes [73], or the total number of blocks [46] have been studied. Edelman [46] also introduced and enumerated k-divisible non-crossing partitions (where all blocks must have size divisible k), which have also been studied by Arizmendi & Vargas [10] in connection with free probability. Arizmendi & Vargas also studied k-equal non-crossing partitions (where all blocks must have size exactly k).

In probabilistic combinatorics, one is interested in the properties of a *typical* element of $\mathtt{A}_n$. In other words, one studies statistics of a random element $\mathtt{a}_n$ of $\mathtt{A}_n$ chosen uniformly at random. Graph theoretical properties of different uniform plane non-crossing structures obtained from a regular polygon have been considered in the past years. For example, [37, 53, 40, 31] study the maximal degree in random triangulations, [13, 31] obtain concentration bounds for the maximal degree in random dissections, and [80, 36, 31] are interested in the structure of non-crossing trees. However, uniform non-crossing partitions have attracted less attention. Arizmendi [9] finds the expected number of blocks of given size for non-crossing partitions of $[n]$ with certain constraints on the block sizes, Ortmann [91] shows that the distribution of a uniform random block in a uniform non-crossing partition P_n of $[n]$ converges to a geometric random variable of parameter $1/2$ as $n \to \infty$ and limit theorems concerning the length of the longest chord of P_n are obtained in [31].

It is also of interest to sample an element $\mathtt{a}_n$ of $\mathtt{A}_n$ according to a probability distribution different from the uniform law; one then studies the impact of this change on the asymptotic behavior of $\mathtt{a}_n$ as $n \to \infty$. Certain families of probability distributions lead to the same asymptotic properties, and are said to belong the same universality class. However, the structure of $\mathtt{a}_n$ may drastically be impacted. To the best of our knowledge, only uniform non-crossing partitions have yet been studied in [10, 91, 31].

Finally, another direction is to study distributional limits of $\mathtt{a}_n$. Indeed, if it is possible to see the elements of the combinatorial class under consideration as elements of a same metric space, it makes sense to study the convergence in distribution of the sequence of random variables $(\mathtt{a}_n)_{n\geq 1}$ in this metric space. In the case of uniform non-crossing partitions, this approach has been followed in [31] by seeing them as compact subsets of the unit disk; we extend the result obtained there to simply generated non-crossing partitions.

Simply generated non-crossing partitions. In this work, we propose to sample non-crossing partitions at random according to a Boltzmann-type distribution, which depends on a sequence of weights. For every integer $n \geq 1$, denote by $\mathbb{NC}_n$ the set of all non-crossing partitions of $[n]$; given a sequence of non-negative real numbers $w = (w(i); i \geq 1)$, with every partition $P \in \mathbb{NC}_n$, we associate a weight $\Omega^w(P)$:

$$\Omega^w(P) = \prod_{B \text{ block of } P} w(\text{size of } B).$$

Then, for every $P \in \mathbb{NC}_n$, set

$$\mathbb{P}_n^w(P) = \frac{\Omega^w(P)}{\sum_{Q \in \mathbb{NC}_n} \Omega^w(Q)}.$$

Implicitly, we shall always restrict our attention to those values of n for which we have $\sum_{P \in \mathbb{NC}_n} \Omega^w(P) > 0$. A random non-crossing partition of $[n]$ sampled according to $\mathbb{P}_n^w$ is called a *simply generated non-crossing partition*. We chose this terminology because of the similarity with the model of simply generated trees, introduced by Meir & Moon [87] and whose definition we recall in Section 6.2.2 below. We were also inspired by recent work on scaling limits of Boltzmann-type random graphs [76, 71].

We point out that, taking $w(i) = 1$ for every $i \geq 1$, $\mathbb{P}_n^w$ is the uniform distribution on $\mathbb{NC}_n$; more generally, if A is a non-empty subset of $\mathbb{N} = \{1, 2, 3, \ldots\}$, and $w_A(i) = 1$ if $i \in A$ and $w_A(i) = 0$ if $i \notin A$, then $\mathbb{P}_n^{w_A}$ is the uniform distribution on the subset of $\mathbb{NC}_n$ formed by partitions with all block sizes belonging to A (provided that they exist), and which we call A-constrained non-crossing partitions. In particular, by taking $A = \{k\}$ one gets uniform k-equal non-crossing partitions, and by taking $A = k\mathbb{N}$ one gets uniform k-divisible non-crossing partitions.

Bijections between non-crossing partitions and plane trees. Our main tools to study simply generated non-crossing partitions are bijections with plane trees. We explain here the main ideas, and refer to Section 6.2.1 for details. With a non-crossing partition, we start by associating a (two-type) dual tree, as depicted in Figure 6.2.

We choose an appropriate root for this two-type tree, and then apply a recent bijection due to Janson & Stefánsson [67]; this yields a bijection $\mathcal{B}^\circ$ between $\mathbb{NC}_n$ and plane trees with $n + 1$ vertices. We mention here that this bijection was directly defined by Dershowitz

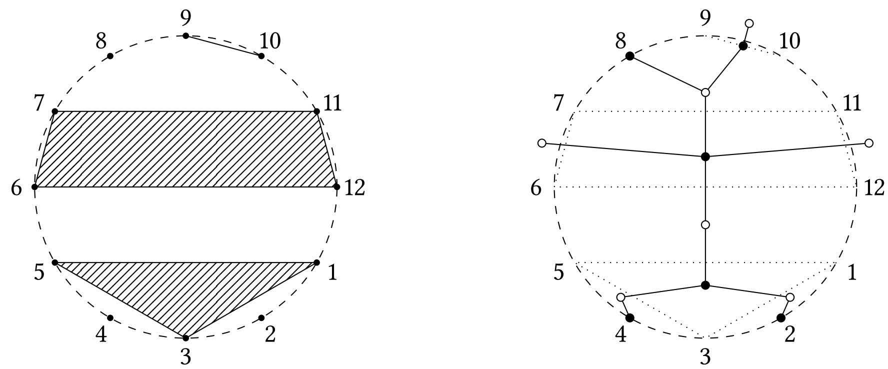


Figure 6.2: The (non-crossing) partition $\{\{1,3,5\},\{2\},\{4\},\{6,7,11,12\},\{8\},\{9,10\}\}$ and its dual tree.

& Zaks [35] without using the dual two-type tree. It turns out that other known bijections between non-crossing partitions and plane trees, such as Prodinger's bijection [100] and the Kreweras complement [73], can be obtained by choosing to distinguishing another root in the dual two-type tree (again see Section 6.2.1 below for details). Our contribution is therefore to unify previously known bijections between non-crossing partitions and plane trees by showing that they all amount to doing certain operations on the dual tree of a non-crossing partition, and to use them to study *random* non-crossing partitions.

It turns out that the dual tree of a simply generated non-crossing partition is a two-type simply generated tree (Proposition 6.7). A crucial feature of the bijection $\mathcal{B}^\circ$ is that it maps simply generated non-crossing partitions into simply generated trees in such a way that blocks of size k are in correspondence with vertices with outdegree k (Proposition 6.6). This allows to reformulate questions on simply generated non-crossing partitions involving block sizes in terms of simply generated trees involving outdegrees. The point is that the study of simply generated trees is a well-paved road. In particular, this allows us to show that if P_n is a simply generated non-crossing plane partition of $[n]$, then, under certain conditions, the size of a block chosen uniformly at random in P_n converges in distribution as $n \to \infty$ to an explicit probability distribution depending on the weights. We also obtain, for a certain family of weights, asymptotic normality of the block sizes and limit theorems for the sizes of the largest blocks. We specify here some of these results for A-constrained non-crossing partitions, and refer to Section 6.3.4 for more general statements and further applications.

Theorem 6.1. *Let A be a non-empty subset of $\mathbb{N}$ with $A \neq \{1\}$, and let P_n^A be a random non-crossing partition chosen uniformly at random among all those with block sizes belonging to A*

(provided that they exist). Let π_A be the probability measure on $\mathbb{Z}_+ = \{0,1,2,\ldots\}$ defined by

$$\pi_A(k) = \frac{\xi_A^k}{1+\sum_{i\in A}\xi_A^i}\mathbb{I}_{k\in\{0\}\cup A},$$

where $\xi_A > 0$ is such that

$$1+\sum_{i\in A}\xi_A^i = \sum_{i\in A} i\cdot\xi_A^i.$$

(i) Let $S_1(P_n^A)$ be the size of the block containing 1 in P_n^A. Then for every $k \geq 1$, we have $\mathbb{P}(S_1(P_n^A) = k) \to k\pi_A(k)$ as $n \to \infty$.

(ii) Let B_n be a block chosen uniformly at random in P_n^A. Then for every $k \geq 1$, we have $\mathbb{P}(|B_n| = k) \to \pi_A(k)/(1-\pi_A(0))$ as $n \to \infty$.

(iii) Let C be a non-empty subset of $\mathbb{N}$ and denote by $\zeta_C(P_n^A)$ the number of blocks of P_n^A whose size belongs to C. As $n \to \infty$, the convergence $\zeta_C(P_n^A)/n \to \pi_A(C)$ holds in probability and, in addition, $\mathbb{E}[\zeta_C(P_n^A)]/n \to \pi_A(C)$.

In the particular case of uniform k-divisible non-crossing partitions, Theorem 6.1 (ii) and (iii) have been obtained by Ortmann [91, Section 2.3]. Also, Arizmendi [9] obtained by combinatorial means closed formulas for the expected number of blocks of given size in k-divisible non-crossing partitions.

Applications in free probability. An additional motivation for introducing simply generated non-crossing partitions comes from free probability. Indeed, the partition function

$$Z_n^w := \sum_{P\in\mathbb{NC}_n}\prod_{B \text{ block of } P} w(\text{size of } B)$$

expresses the moments of a measure in terms of its free cumulants. More precisely, if μ is a probability measure on $\mathbb{R}$ with compact support, its Cauchy transform

$$G_\mu(z) = \int_{\mathbb{R}}\frac{\mu(\mathrm{d}t)}{z-t}, \qquad z\in\mathbb{C}\setminus\operatorname{supp}\mu$$

is analytic and locally invertible on a neighbourhood of ∞; its inverse K_μ is meromorphic around zero, with a simple pole of residue 1 (see e.g. [12, Section 5]). One can then write

$$R_\mu(z) = K_\mu(z) - \frac{1}{z} = \sum_{n=0}^{\infty}\kappa_n(\mu)z^n.$$

The analytic function R_μ is called the R-transform of μ, and uniquely defines μ. In addition, the coefficients $(\kappa_n(\mu); n \geq 1)$ are called the free cumulants of μ. The importance of R-transforms stems in the fact that they linearize free additive convolution and characterize weak convergence of probability measures, see Bercovici & Voiculescu [12]. The following relation between the moments of μ and its free cumulants is a well-known fact, that goes up to Speicher [105]. Let μ be a compactly supported probability measure on $\mathbb{R}$. Then, for every $n \geq 1$,

$$\int_{\mathbb{R}} t^n \mu(\mathrm{d}t) = \sum_{P \in \mathbb{NC}_n} \prod_{B \text{ block of } P} \kappa_{\mathrm{size}(B)}(\mu). \tag{6.1}$$

In other words, the n-th moment of μ is the partition function of simply generated non-crossing partitions on $[n]$ with weights $w(i) = \kappa_i(\mu)$ given by the free cumulants of μ. Using the bijection $\mathcal{B}^\circ$, we establish the following result.

Theorem 6.2. *Let μ be a compactly supported probability measure on $\mathbb{R}$, different from a Dirac mass, and such that all its free cumulants $(\kappa_n(\mu); n \geq 1)$ are nonnegative. Let s_μ be the maximum of its support. Set*

$$\rho = \left(\limsup_{n \to \infty} \kappa_n(\mu)^{1/n} \right)^{-1} \qquad \text{and} \qquad \nu = 1 + \lim_{t \uparrow \rho} \frac{t^2 R'_\mu(t) - 1}{t R_\mu(t) + 1}.$$

If $\nu \geq 1$, there exists a unique number ξ in $(0, \rho]$ such that $R'_\mu(\xi) = 1/\xi^2$, and, in addition,

$$s_\mu \quad = \quad \begin{cases} \frac{1}{\xi} + R_\mu(\xi) & \text{if } \nu \geq 1, \\ \frac{1}{\rho} + R_\mu(\rho) & \text{if } \nu < 1. \end{cases}$$

See Section 6.3.3 for examples. This gives a more explicit formula that the one obtained by Ortmann [91, Theorem 5.4], which reads

$$\log(s_\mu) = \sup \left\{ \frac{1}{m_1(p)} \sum_{n \in L} p_n \log\left(\frac{\kappa_n(\mu)}{p_n} \right) - \frac{\theta(m_1(p))}{m_1(p)};\ p \in \mathfrak{M}_1^1(L) \right\},$$

where $L = \{n \geq 1; \kappa_n(\mu) \neq 0\}$, $\theta(x) = \log(x-1) - x\log(x - 1/x)$, $\mathfrak{M}_1^1(L)$ is the set of probability measures $p = (p_n; n \in \mathbb{N})$ on $\mathbb{N}$ with $p(L^c) = 0$ and $m_1(p)$ is the mean of p.

Non-crossing partitions seen as compact subsets of the unit disk. Finally, if P_n is a simply generated non-crossing partition of $[n]$, we study the distributional limits of P_n, seen as compact subset of the unit disk by identifying each integer $l \in [n]$ with the complex number $\mathrm{e}^{-2\mathrm{i}\pi l/n}$. This route was followed in [31], where it was shown that as $n \to \infty$, a uniform non-crossing partition of $[n]$ converges in distribution to Aldous' Brownian triangulation of the disk [6], in the space of all compact subsets of the unit disk equipped with the Hausdorff metric, and where the Brownian triangulation is a random compact subset of the unit disk constructed from the Brownian excursion. We show more generally that a whole family of simply generated non-crossing partitions of $[n]$ (including uniform A-constrained non-crossing partitions) converges in distribution to the Brownian triangulation, and show that other families converge in distribution to the stable lamination, which is another random compact subset of the unit disk introduced in [71]. We refer to Section 6.4 for details and precise statements.

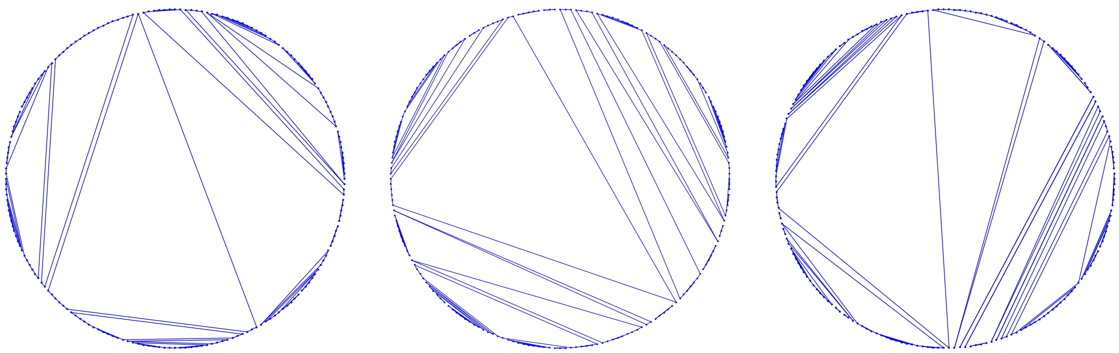


Figure 6.3: Simulations of random non-crossing partitions of $[200]$ chosen uniformly at random among all those having respectively only block sizes that are multiples of 5, block sizes that are odd and block sizes that are prime numbers.

This has in particular applications concerning the length of the longest chord of P_n. By definition, the (angular) length of a chord $[e^{-2i\pi s}, e^{-2i\pi t}]$ with $0 \leq s \leq t \leq 1$ is $\min(t-s, 1-t+s)$. Denote by $\mathtt{C}(P_n)$ the length of the longest chord of P_n. In the case of A-constrained non-crossing partitions, we prove in particular the following result.

Theorem 6.3. *Let A is a non-empty subset of $\mathbb{N}$ with $A \neq \{1\}$, and let P_n^A be a random non-crossing partition chosen uniformly at random among all those with block sizes belonging to A (provided that they exist). Then, as $n \to \infty$, $\mathtt{C}(P_n^A)$ converges in distribution to a random variable with distribution*

$$\frac{1}{\pi} \frac{3x-1}{x^2(1-x)^2\sqrt{1-2x}} \mathbb{I}_{\frac{1}{3} \leq x \leq \frac{1}{2}} \mathrm{d}x.$$

It is remarkable that the limiting distribution in Theorem 6.3 does not depend on A (it seems that this is *not* the case for the largest block area, see Section 6.5).

This bears some similarity with [31], but we emphasize that this is not a simple adaptation of the arguments of [31]. Indeed, roughly speaking, [31] manages to code uniform non-crossing partitions of $[n]$ by a dual-type uniform plane tree. In the more general case of simply generated non-crossing partitions, the dual tree is a more complicated two-type tree and the Janson–Stefánsson bijection is needed.

6.2 Bijections between non-crossing partitions and plane trees

We denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk of the complex plane, by $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle and by $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{S}^1$ the closed unit disk. For every $x, y \in \mathbb{S}^1$, we write $[x, y]$ for the line segment between x and y in $\overline{\mathbb{D}}$, with the convention $[x, x] = \{x\}$. A geodesic lamination L of $\overline{\mathbb{D}}$ is a closed subset of $\overline{\mathbb{D}}$ which can be written as the union of a collection of non-crossing such chords, i.e. which do not intersect in $\mathbb{D}$. In this Chapter, by lamination we will always mean geodesic lamination of $\overline{\mathbb{D}}$.

We view a partition of $[n]$ as a closed subset of $\overline{\mathbb{D}}$ by identifying each integer $l \in [n]$ with the complex number $\mathrm{e}^{-2\mathrm{i}\pi l/n}$ and by drawing a chord $[\mathrm{e}^{-2\mathrm{i}\pi l/n}, \mathrm{e}^{-2\mathrm{i}\pi l'/n}]$ whenever $l, l' \in [n]$ are two consecutive elements of the same block of the partition, where the smallest and the largest element of a block are consecutive by convention. The partition is non-crossing if and only if these chords do not cross; we implicitly identify a non-crossing partition with the associated lamination throughout this Chapter.

Let $\mathbb{T}$ be the set of all finite plane trees (recall the definition from Chapter 1), and $\mathbb{T}_n$ be the set of all plane trees with n edges, or equivalently $n + 1$ vertices. We construct two bijections between $\mathbb{NC}_n$ and $\mathbb{T}_n$. The study of a (random) non-crossing partition then reduces to that of the associated (random) plane tree.

6.2.1 Non-crossing partitions and plane trees

We define the (planar, but non-rooted) dual tree $T(P)$ of a non-crossing partition P of $[n]$ as follows: we place a black vertex inside each block of the partition and a white vertex inside each other face, then we join two vertices if the corresponding faces share a common edge; here we shall view the singletons as self-loops and the blocks of size two with one double edge. See Figure 6.4 for an illustration. Observe that the graph thus obtained is a indeed a planar tree (meaning that there is an order among all edges adjacent to a same vertex, up to cyclic permutations), with $n+1$ vertices, and that the latter is bipartite: each edge connects two vertices of different colours.

In order to fully recover the partition from the tree (and therefore obtain a bijection), we need to assign a root by distinguishing a corner of $T(P)$ (a corner of a vertex in a planar tree is a sector around this vertex delimited by two consecutive edges), thus making it a plane tree. We will do so in two different ways, which will give rise to two different bijections. First, $T^{\circ}(P)$ is the tree $T(P)$ rooted at the corner of the white vertex that lies in the face containing the vertices 1 and n, and that has the black vertex in the block containing 1 as its first child; $T^{\bullet}(P)$ is the tree $T(P)$ rooted at the corner of the black vertex in the block containing n and that has the white vertex that lies in the face containing the vertices 1 and n as its first child.

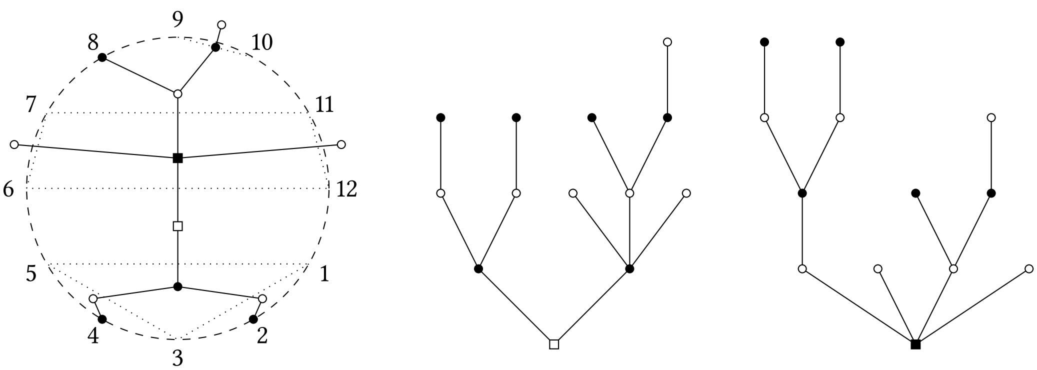


Figure 6.4: The dual tree $P(T)$ and the two rooted trees $T^{\circ}(P)$ and $T^{\bullet}(P)$ associated with the partition $P = \{\{1,3,5\},\{2\},\{4\},\{6,7,11,12\},\{8\},\{9,10\}\}$.

The trees $T^{\circ}(P)$ and $T^{\bullet}(P)$ are two-type plane trees: vertices at even generation are coloured in one colour and vertices at odd generation are coloured in another colour. We apply to each a bijection due to Janson & Stefánsson [67, Section 3] which maps such a tree into a one-type tree, that we now describe. This bijection enjoys useful probabilistic features, see Corollary 6.8 below.

We denote by T a plane tree and by $\mathcal{G}(T)$ its image by this bijection; T and $\mathcal{G}(T)$ have the same vertices but the edges are different. If $T=\{\varnothing\}$ is a singleton, then set $\mathcal{G}(T)=\{\varnothing\}$; otherwise, for every vertex $u \in T$ at even generation with $k_u \geq 1$ children, do the following: first, if $u \neq \varnothing$, draw an edge between its parent $pr(u)$ and its first child $u1$, then draw edges between its consecutive children $u1$ and $u2$, $u2$ and $u3$, ..., $u(k_u-1)$ and uk_u, and finally draw an edge between uk_u and u; if u is a leaf of T, then this procedure reduces to drawing an edge between u and $pr(u)$. We root $\mathcal{G}(T)$ at the first child of the root of T. One can check that $\mathcal{G}(T)$ thus defined is indeed a plane tree, and that the mapping is invertible. Also observe that every vertex at even generation in T is mapped to a leaf of $\mathcal{G}(T)$, and every vertex at odd generation with $k \geq 0$ children in T is mapped to a vertex with $k+1$ children in $\mathcal{G}(T)$.

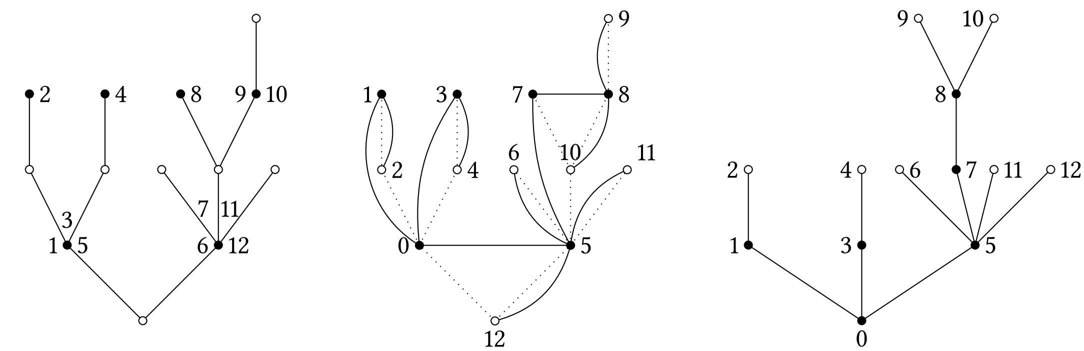


Figure 6.5: The tree T° associated with the partition from Figure 6.2, with its black corners indexed according to the contour sequence, and its image $\mathcal{T}^\circ$ by the Janson–Stefánsson bijection, with its vertices indexed in lexicographical order.

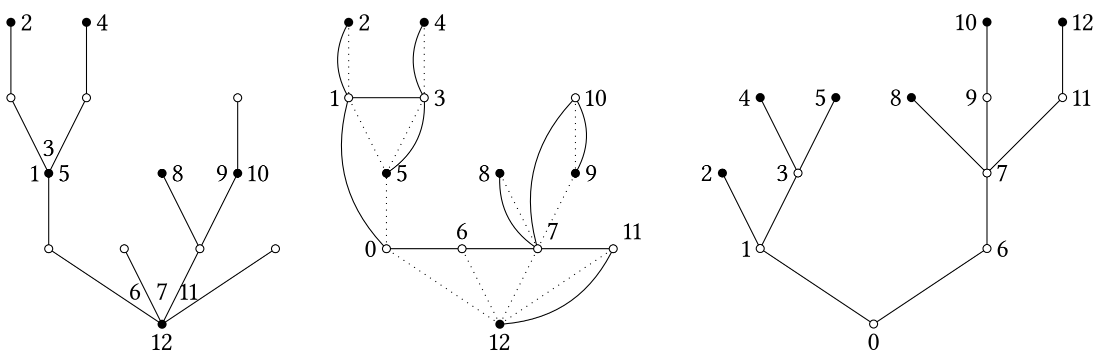


Figure 6.6: The tree $T^\bullet$ associated with the partition from Figure 6.2, black corners indexed according to the contour sequence, and its image $\mathcal{T}^\bullet$ by the Janson–Stefánsson bijection, with its vertices indexed in lexicographical order.

We let

$$\mathcal{T}^\circ(P) \coloneqq \mathcal{G}(T^\circ(P)) \qquad \text{and} \qquad \mathcal{T}^\bullet(P) \coloneqq \mathcal{G}(T^\bullet(P))$$

be the (one-type) trees associated with $T^\circ(P)$ and $T^\bullet(P)$ respectively and now explain how to reconstruct the non-crossing partition P from the trees $\mathcal{T}^\circ(P)$ and $\mathcal{T}^\bullet(P)$.

To this end, we introduce the notion of *twig*. If T is a tree and $u, v \in T$, denote by $[\![u, v]\!]$ the shortest path between u and v in T. A twig of T is a set of the form $[\![u, v]\!]$, where u is an ancestor of v and such that all the vertices of $]\!]u, v]\!]$ are the last child of their parent; we agree that $[\![u, u]\!]$ is a twig for every vertex u. Now, if $\tau \in \mathbb{T}_n$ is a tree, let $\varnothing = u(0) \prec u(1) \prec \cdots \prec u(n)$ be its vertices listed in lexicographical order. We define two partitions $P_\circ(\tau)$ and $P_\bullet(\tau)$ of $[n]$ as follows:

- $i, j \in [n]$ belong to the same block of $P_\circ(\tau)$ when $u(i)$ and $u(j)$ have the same parent in τ;
- $i, j \in [n]$ belong to the same block of $P_\bullet(\tau)$ when $u(i)$ and $u(j)$ belong to a same twig.

It is an easy exercise to check that for every $\tau \in \mathbb{T}$, $P_\circ(\tau)$ and $P_\bullet(\tau)$ are indeed partitions which, further, are non-crossing. As illustrated by Figure 6.5 and Figure 6.6, we have the following result.

Proposition 6.4. *For every non-crossing partition P we have*

$$P = P_\circ(\mathcal{T}^\circ(P)) = P_\bullet(\mathcal{T}^\bullet(P)).$$

Proof. Fix a non-crossing partition P of $[n]$. Let us first prove the first equality. Define the contour sequence $(u_0, u_1, \ldots, u_{2n})$ of the tree $T^\circ(P)$ as follows: $u_0 = \varnothing$ and for each $i \in \{0, \ldots, 2n-1\}$, u_{i+1} is either the first child of u_i which does not appear in the sequence $(u_0, \ldots, u_i)$, or the parent of u_i if all its children already appear in this sequence. Recall that a corner of a vertex $v \in T^\circ(P)$ is a sector around v delimited by two consecutive edges. We index from 1 to n the corners of the black vertices of $T^\circ(P)$, following the contour sequence. By construction of $T^\circ(P)$, we recover P from these corners: for each black vertex of $T^\circ(P)$, the indices of its corners, listed in clockwise order, form a block of P. Now assign labels to the vertices of $\mathcal{T}^\circ(P)$ as follows. By definition of the bijection $\mathcal{G}$, each edge of $\mathcal{T}^\circ(P)$ starts from one of these corners, we then label its other extremity by the label of the corner. The root of $\mathcal{T}^\circ(P)$ is not labelled, we assign it the label 0; the labels thus obtained correspond to the lexicographical order in $\mathcal{T}^\circ(P)$ and the first identity follows.

For the second equality, define similarly the contour sequence of $T^\bullet(P)$, but starting from the first child of the root, and label the black corners as before. We then label the vertices of $T^\bullet(P)$ as follows: the label of every black vertex is the largest label of its adjacent corners, and then assign the remaining labels of its adjacent corners in decreasing order to its children, starting from the last one. Observe that the root of $T^\bullet(P)$ has as many children as corners, and all the other black vertices have one child less than the number of corners. Thus all the vertices of $T^\bullet(P)$ have labels, except the first child of the root which we label 0. We recover P from $T^\bullet(P)$ as follows: for each black vertex of $T^\bullet(P)$, its label, together with the labels of its children, form a block of P (and one does not take into account the label 0). As the vertex set of $T^\bullet(P)$ and of $\mathcal{T}^\bullet(P)$ is the same, we also get a labeling of the vertices of

$\mathcal{T}^\bullet(P)$. Again, by definition of the $\mathcal{G}$, these labels correspond to the lexicographical order in $\mathcal{T}^\bullet(P)$ and the second identity follows. □

Observe from the previous results that the plane trees $\mathcal{T}^\circ(P)$ and $\mathcal{T}^\bullet(P)$ are in bijection. Let us describe a direct operation on trees which maps $\mathcal{T}^\circ(P)$ onto $\mathcal{T}^\bullet(P)$. Starting from a tree $\tau \in \mathbb{T}$, we construct a tree $\mathcal{B}(\tau)$ on the same vertex-set by defining edges (called "new" edges in the sequel) as follows: first, we link any two consecutive children in τ; second, we link every vertex v which is the first child of its parent to its youngest ancestor u such that $[\![u, pr(v)]\!]$ is a twig in τ (in this case observe that either u is the root of τ, or v is not the last child of u in $\mathcal{B}(\tau)$).

We leave it as an exercise to check that this mapping preserves the lexicographical order.

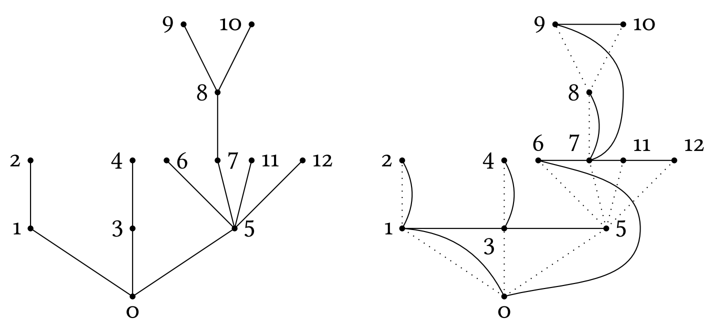


Figure 6.7: The transformation $\tau \mapsto \mathcal{B}(\tau)$.

Proposition 6.5. *For every non-crossing partition P we have*

$$\mathcal{B}(\mathcal{T}^\circ(P)) = \mathcal{T}^\bullet(P).$$

Proof. Fix a non-crossing partition P. Thanks to Proposition 6.4, it is equivalent to show that

$$P_\bullet(\mathcal{B}(\mathcal{T}^\circ(P))) = P,$$

and we set by $P' = P_\bullet(\mathcal{B}(\mathcal{T}^\circ(P)))$ to simplify notation.

Suppose first that $i, j \geq 1$ lie in the same block of P. We shall show that i and j belong to the same block of P'. The two corresponding vertices, say, $u(i)$ and $u(j)$ have the same parent in $\mathcal{T}^\circ(P)$. Without loss of generality, assume that $u(i) \prec u(j)$ are consecutive children in $\mathcal{T}^\circ(P)$. It suffices to check that, in $\mathcal{B}(\mathcal{T}^\circ(P))$, $u(j)$ is the last child of $u(i)$. This simply follows from the fact $\mathcal{B}$ preserves the lexicographical order and that the children of $u(i)$ in $\mathcal{B}(\mathcal{T}^\circ(P))$, $u(j)$ excluded, are descendants of $u(i)$ in $\mathcal{T}^\circ(P)$.

Conversely, suppose that $i, j \geq 1$ lie in the same block of P'. Without loss of generality, we may assume that, in $\mathcal{B}(\mathcal{T}^\circ(P))$, $u(j)$ is the last child of $u(i)$. We argue by contradiction and assume that, in $\mathcal{T}^\circ(P)$, $u(i)$ and $u(j)$ are not siblings. We saw that in this case, by definition of $\mathcal{B}$, either $u(i)$ is the root, or $u(j)$ is not the last child of $u(i)$ in $\mathcal{B}(\mathcal{T}^\circ(P))$. Both of these cases are excluded. Therefore i and j belong to the same block of P. □

We already mentioned in the Introduction that the bijection $\tau \leftrightarrow P_\circ(\tau)$ was defined by Dershowitz and Zaks [35]; the bijection $\tau \leftrightarrow P_\bullet(\tau)$ was defined by Prodinger [100] and further used in combinatorics, see e.g. Yano and Yoshida [107] and in (free) probability, see Ortmann [91]. Roughly speaking, here we unify these two bijections by seeing that they amount (up to the Janson–Stefánsson bijection) to choosing different distinguished corners in the dual two-type planar tree. In this spirit, if P is a non-crossing partition, let us also mention that its Kreweras complement $K(P)$ is just obtained by re-rooting $T(P)$ at a new corner; more precisely, the mappings $(T^\bullet)^{-1} \circ T^\circ$ and $(\mathcal{T}^\bullet)^{-1} \circ \mathcal{T}^\circ$ coincide and both correspond to K.

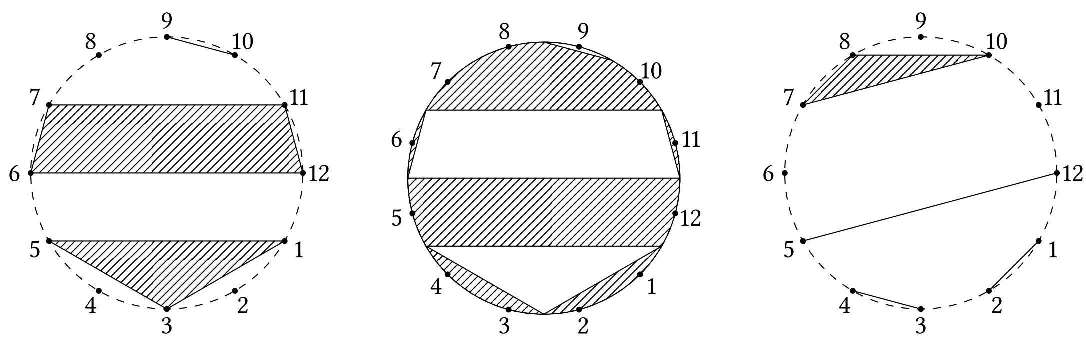


Figure 6.8: The Kreweras complement of $\{\{1,3,5\},\{2\},\{4\},\{6,7,11,12\},\{8\},\{9,10\}\}$ is $\{\{1,2\},\{3,4\},\{5,12\},\{6\},\{7,8,10\},\{11\}\}$.

The Kreweras complement can be formally defined as follows. If we denote by $\mathbb{NC}(A)$ the set of non-crossing partitions on a finite subset $A \subset \mathbb{N}$, then we have canonical isomorphisms $\mathbb{NC}_n := \mathbb{NC}(\{1,2,\ldots,n\}) \cong \mathbb{NC}(\{1,3,\ldots,2n-1\}) \cong \mathbb{NC}(\{2,4,\ldots,2n\})$. Given two non-crossing partitions $P \in \mathbb{NC}(\{1,3,\ldots,2n-1\})$ and $P' \in \mathbb{NC}(\{2,4,\ldots,2n\})$, one constructs a (possibly crossing) partition $P \cup P'$ of $\{1,2,\ldots,2n\}$. The Kreweras complement of a non-crossing partition $P \in \mathbb{NC}_n \cong \mathbb{NC}(\{1,3,\ldots,2n-1\})$ is then given by

$$K(P) = \max\{P' \in \mathbb{NC}_n \cong \mathbb{NC}(\{2,4,\ldots,2n\}) : P \cup P' \in \mathbb{NC}_{2n}\},$$

where the maximum refers to the partial order of reverse refinement: $P_1 \preceq P_2$ when every block of P_1 is contained in a block of P_2.

The Kreweras complementation can be visualized as follows: consider the representation of $P \in \mathbb{NC}_n$ in the unit disk as in Figure 6.2; invert the colors and rotate the vertices of the regular n-gon by an angle $-\pi/n$; then the blocks of $K(P)$ are given by the vertices lying in the same "coloured" component. See Figure 6.8 for an illustration.

6.2.2 Simply generated non-crossing partitions and simply generated trees

An important feature of the bijection $\mathcal{B}^\circ : P \mapsto \mathcal{T}^\circ(P)$ is that it transforms simply generated non-crossing partitions into simply generated trees, which were introduced by Meir & Moon [87] and whose definition we now recall.

Given a sequence $w = (w(i); i \geq 0)$ of nonnegative real numbers, with every $\tau \in \mathbb{T}$, associate a weight $\Omega^w(\tau)$:

$$\Omega^w(\tau) = \prod_{u \in \tau} w(k_u).$$

Then, for every $\tau \in \mathbb{T}_n$, set

$$\mathbb{Q}_n^w(\tau) = \frac{\Omega^w(\tau)}{\sum_{T \in \mathbb{T}_n} \Omega^w(T)}.$$

Again, we always restrict our attention to those values of n for which $\sum_{T \in \mathbb{T}_n} \Omega^w(T) > 0$. A random tree of $\mathbb{T}_n$ sampled according to $\mathbb{Q}_n^w$ is called a simply generated tree. A particular case of such trees on which we shall focus in Section 6.4 is when the sequence of weights w defines a probability measure on $\mathbb{Z}_+$ with mean 1 (see the discussion in Section 6.3.1 below). In this case, $\mathbb{Q}_n^w$ is the law of a Galton–Watson tree with critical offspring distribution w conditioned to have n edges.

Proposition 6.6. *Let $(w(i); i \geq 1)$ be any sequence of nonnegative real numbers. Set $w(0) = 1$. Then, for every $P \in \mathbb{NC}_n$,*

$$\mathbb{P}_n^w(P) = \mathbb{Q}_n^w(\mathcal{T}^\circ(P)).$$

In other words, the bijection $\mathcal{B}^\circ$ transforms simply generated non-crossing partitions into simply generated trees.

Proof. By Proposition 6.4, we have $P = P_\circ(\mathcal{T}^\circ(P))$. In particular, blocks of size $k \geq 1$ in P are in bijection with vertices with out-degree k in $\mathcal{T}^\circ(P)$. The claim immediately follows. □

It is also possible to give an explicit description of the law of T° under $\mathbb{P}_n^w$, which turns out to be a two-type simply generated tree. We denote by $\mathbb{T}^{(\mathsf{e},\mathsf{o})}$ the set of finite two-type trees: for every $\tau \in \mathbb{T}^{(\mathsf{e},\mathsf{o})}$, we denote by $\mathsf{e}(\tau)$ and $\mathsf{o}(\tau)$ the set of vertices respectively at even and odd generation in τ. Given two sequences of weights w^{e} and w°, we define the weight of tree $\tau \in \mathbb{T}^{(\mathsf{e},\mathsf{o})}$ by

$$\Omega^{(w^{\mathsf{e}},w^{\circ})}(\tau) = \prod_{u\in\mathsf{e}(\tau)} w^{\mathsf{e}}(k_u) \prod_{u\in\mathsf{o}(\tau)} w^{\circ}(k_u).$$

and we define for every $\tau \in \mathbb{T}_n^{(\mathsf{e},\mathsf{o})}$ the set of two-type trees with n edges,

$$\mathbb{Q}_n^{(w^{\mathsf{e}},w^{\circ})}(\tau) = \frac{\Omega^{(w^{\mathsf{e}},w^{\circ})}(\tau)}{\sum_{T\in\mathbb{T}_n^{(\mathsf{e},\mathsf{o})}} \Omega^{(w^{\mathsf{e}},w^{\circ})}(T)},$$

where, again, we implicitly restrict ourselves to the values of n for which $\sum_{T\in\mathbb{T}_n^{(\mathsf{e},\mathsf{o})}} \Omega^{(w^{\mathsf{e}},w^{\circ})}(T) > 0$. A random tree sampled according to $\mathbb{Q}_n^{(w^{\mathsf{e}},w^{\circ})}$ is called a two-type simply generated tree.

Proposition 6.7. *Let $w = (w(i), i \geq 1)$ be a sequence of nonnegative real numbers and $c > 0$ be a positive real number. For every $i \geq 0$, set $w^{o}(i) = w(i+1)$ and $w^{e}(i) = c^{-(i+1)}$. Then, for every $P \in \mathbb{NC}_n$,*

$$\mathbb{P}_n^{w}(P) = \mathbb{Q}_n^{(w^{e},w^{o})}(T^{\circ}(P)).$$

Proof. Fix $P \in \mathbb{NC}_n$; by construction of $T^{\circ}(P)$ (recall the proof of Proposition 6.4), the vertices at odd generation in $T^{\circ}(P)$ are in bijection with the blocks of P and the degree of each corresponds to the size of the associated block. Consequently, we have on the one hand

$$\prod_{u\in\mathsf{o}(T^{\circ}(P))} w^{\circ}(k_u) = \prod_{u\in\mathsf{o}(T^{\circ}(P))} w(k_u+1) = \prod_{B \text{ block of } P} w(\text{size of } B) = \Omega^{w}(P);$$

on the other hand, since $T^{\circ}(P) \in \mathbb{T}_n^{(\mathsf{e},\mathsf{o})}$,

$$\prod_{u\in\mathsf{e}(T^{\circ}(P))} w^{\mathsf{e}}(k_u) = \prod_{u\in\mathsf{e}(T^{\circ}(P))} c^{-(k_u+1)} = c^{-\sum_{u\in\mathsf{e}(T^{\circ}(P))}(k_u+1)} = c^{-(n+1)}.$$

This last term only depends on n and not on P and the claim follows. □

Recall that $\mathcal{G}$ denotes the Janson–Stefánsson bijection. Then, combining Propositions 6.6 and 6.7, we obtain the following result.

Corollary 6.8. *Let $w = (w(i), i \geq 1)$ be any sequence of nonnegative real numbers and $c > 0$ be a positive real number. Set $w(0) = 1$ and for every $i \geq 0$, define $w^{o}(i) = w(i+1)$ and $w^{e}(i) = c^{-(i+1)}$. Then, for every $T \in \mathbb{T}_n^{(e,o)}$, we have*

$$\mathbb{Q}_n^{w^{e},w^{o}}(T) = \mathbb{Q}_n^{w}(\mathcal{G}(T)).$$

In other words, the Janson–Stefánsson bijection transforms a certain class of two-type simply generated trees into one-type simply generated trees. A similar result implicitly appears in their work [67, Appendix A] in the particular case of Galton–Watson trees, where w^{e} and w° are probability distribution on $\{0, 1, \dots, \}$ and moreover w^{e} is a geometric distribution.

6.3 Applications

In this section, we use simply generated trees to study combinatorial properties of simply generated non-crossing partitions. Indeed, as suggested by Proposition 6.6, it is possible to reformulate questions concerning random non-crossing partitions in terms of random trees, which are more familiar grounds.

6.3.1 Asymptotics of simply generated trees

Following Janson [64], here we describe all the possible regimes arising in the asymptotic behavior of simply generated trees. All the following discussion appears in [64], but we reproduce it here for the reader's convenience in view of future use and refer to the latter reference for details and proofs.

Let $(w(i); i \geq 0)$ be a sequence of nonnegative real numbers with $w(0) > 0$ and $w(k) > 0$ for some $k \geq 2$ (and keeping in mind that we will take $w(0) = 1$ in view of Proposition 6.6). Set

$$\Phi(z) = \sum_{k=0}^{\infty} w(k) z^k, \quad \Psi(z) = \frac{z\Phi'(z)}{\Phi(z)} = \frac{\sum_{k=0}^{\infty} k w(k) z^k}{\sum_{k=0}^{\infty} w(k) z^k}, \quad \rho = \left(\limsup_{k \to \infty} w(k)^{1/k} \right)^{-1}.$$

If $\rho = 0$, set $\nu = 0$ and otherwise

$$\nu = \lim_{t \uparrow \rho} \Psi(t).$$

We now define a number $\xi \geq 0$ according to the value of ν.

- If $\nu \geq 1$, then ξ is the unique number in $(0, \rho]$ such that $\Psi(\xi) = 1$.
- If $\nu < 1$, then we set $\xi = \rho$.

In both cases, we have $0 < \Phi(\xi) < \infty$, and we set

$$\pi(k) = \frac{w(k)\xi^k}{\Phi(\xi)}, \qquad k \geq 0,$$

so that π is a probability distribution with expectation $\min(\nu, 1)$ and variance $\xi\Psi'(\xi) \leq \infty$.

We say that another sequence of weights $\widetilde{w} = (\widetilde{w}(i); i \geq 0)$ is equivalent to w when there exist $a, b > 0$ such that $\widetilde{w}(i) = ab^i w(i)$ for every $i \geq 0$. In this case, one can check that $\Omega^{\widetilde{w}}(\tau) = a^{n+1}b^n\Omega^w(\tau)$ for every $\tau \in \mathbb{T}_n$ so that $\mathbb{Q}_n^{\widetilde{w}} = \mathbb{Q}_n^w$ and $\mathbb{P}_n^{\widetilde{w}} = \mathbb{P}_n^w$ for every $n \geq 1$. We see that w is equivalent to a probability distribution if and only if $\rho > 0$. In addition, when $\rho > 0$, π defined as above is the unique probability distribution with mean 1 equivalent to w, if such distribution exists; if no such distribution exists, then π is the probability distribution equivalent to w that has the maximal mean.

Example 6.9. Let A is a non-empty subset of $\{1, 2, 3, \ldots\}$ with $A \neq \{1\}$. Set $w_A(0) = 1$, $w_A(k) = 1$ if $k \in A$ and $w_A(k) = 0$ if $k \notin A$. Then the equivalent probability measure π_A is defined by

$$\pi_A(k) = \frac{\xi_A^k}{1 + \sum_{i\in A}\xi_A^i}\mathbb{I}_{k\in\{0\}\cup A},$$

where $\xi_A > 0$ is such that

$$1 + \sum_{i\in A}\xi_A^i = \sum_{i\in A} i \cdot \xi_A^i.$$

For example, for fixed $n \geq 1$, we have

$$\pi_{n\mathbb{N}}(k) = \frac{n}{(1+n)^{1+k/n}}\mathbb{I}_{k\in n\mathbb{Z}_+} \qquad (k \geq 0).$$

In particular, $\pi_{\mathbb{Z}_+}(k) = 1/2^{k+1}$ for every $k \geq 0$. Also,

$$\pi_{(2\mathbb{Z}_+ + 1)}(k) = \frac{1-z^2}{1+z-z^2} \cdot z^k \cdot \mathbb{I}_{k=0 \text{ or } k \text{ odd}} \qquad (k \geq 0),$$

where z is the unique root of $1 - 2z^2 - 2z^3 + z^4 = 0$ in $[0, 1]$.

Now let T_n be a random element of $\mathbb{T}_n$ sampled according to $\mathbb{Q}_n^w$.

Theorem 6.10 (Janson [64], Theorems 7.10 and 7.11)**.** *Fix $k \geq 0$.*

(i) We have $\mathbb{P}(k_\varnothing(T_n) = k) \to k\pi(k)$ as $n \to \infty$;

(ii) Let $N_k(T_n)$ be the number of vertices with outdegree k in T_n. Then $N_k(T_n)/n$ converges in probability to π_k as $n \to \infty$.

Theorem 6.11 (Janson [64], Theorem 18.6)**.** *If $\rho > 0$, we have*

$$\frac{1}{n}\log\sum_{T\in\mathbb{T}_n}\Omega^w(T) \quad \underset{n\to\infty}{\longrightarrow} \quad \log(\Phi(\xi)/\xi).$$

6.3.2 Applications in the enumeration of non-crossing partitions with prescribed block sizes

By Proposition 6.4, counting non-crossing partitions of $[n]$ with conditions on the number of blocks of given sizes reduces to counting plane trees of $\mathbb{T}_n$ with conditions on the number of vertices with given outdegrees, which is a well-paved road (see e.g. [106, Section 5.3]). Since our main interest lies in probabilistic aspects of non-crossing partitions, we shall only give one such example of application. Let A be a non-empty subset of $\{1, 2, 3, \dots\}$ with $A \neq \{1\}$, and denote by $\mathbb{NC}_n^A$ the set of all non-crossing partitions of $[n]$ with blocks of size only belonging to A. Recall the definition of ξ_A from Example 6.9.

Proposition 6.12. *Set* $\Phi(z) = 1 + \sum_{k \in A} z^k$. *Then*

$$\#\mathbb{NC}_n^A \quad \underset{n\to\infty}{\sim} \quad \gcd(A) \cdot \sqrt{\frac{\Phi(\xi_A)}{2\pi\Phi''(\xi_A)}} \cdot \left(\frac{\Phi(\xi_A)}{\xi_A}\right)^n \cdot n^{-3/2},$$

where $n \to \infty$ *in such a way that* n *is divisible by* $\gcd(A)$.

Setting $\overline{A} = \{0\} \cup A$, observe that $\#\mathbb{NC}_n^A = \#\mathbb{T}_n^{\overline{A}}$ by Proposition 6.4. But, by [52, Proposition I.5.], the generating function $T^{\overline{A}}(z) = \sum_{n \geq 1} \#\mathbb{T}_n^{\overline{A}} \cdot z^n$ satisfies the implicit equation $T^{\overline{A}}(z) = z\Phi(T^{\overline{A}}(z))$. Proposition 6.12 then immediately follows from [52, TheoremVII.2 and Rem. VI.17].

Let us mention that explicit expressions for $\#\mathbb{NC}_n^A$ for n fixed are known for two particular choices of A. Edelman [46] has found an explicit formula for $\#\mathbb{NC}_{kn}^{k\mathbb{Z}_+}$ (i.e. for k-divisible non-crossing partitions) and Arizmendi & Vargas [10] have found the explicit expression of $\#\mathbb{NC}_{kn}^{\{k\}}$ (i.e. for k equal non-crossing partitions):

$$\#\mathbb{NC}_{kn}^{\{k\}} = \frac{1}{(k-1)n+1}\binom{kn}{n} \qquad \text{and} \qquad \#\mathbb{NC}_{kn}^{k\mathbb{Z}_+} = \frac{1}{kn+1}\binom{(k+1)n}{n}.$$

6.3.3 Applications in free probability

Recall from the Introduction the definition of the R-transform R_μ of a compactly supported probability measure μ on the real line, and that it is related to its associated free cumulants $(\kappa_i(\mu); i \geq 0)$ by the formula

$$R_\mu(z) = \sum_{n=0}^{\infty} \kappa_n(\mu) z^n.$$

Theorem 6.13. *Let μ be a compactly supported probability measure on $\mathbb{R}$ different from a Dirac mass. Assume that its free cumulants $(\kappa_i(\mu); i \geq 1)$ are all nonnegative. Set*

$$\rho = \left(\limsup_{n\to\infty} \kappa_n(\mu)^{1/n}\right)^{-1} \qquad \text{and} \qquad \nu = 1 + \lim_{t\uparrow\rho} \frac{t^2 R'_\mu(t) - 1}{tR_\mu(t) + 1}.$$

(i) If $\nu \geq 1$, there exists a unique number ξ in $(0, \rho]$ such that $R'_\mu(\xi) = 1/\xi^2$ and

$$\frac{1}{n} \cdot \log \int_{\mathbb{R}} t^n \mu(\mathrm{d}t) \quad \underset{n\to\infty}{\longrightarrow} \quad \log\left(\frac{1}{\xi} + R_\mu(\xi)\right).$$

(ii) If $\nu < 1$, we have

$$\frac{1}{n} \cdot \log \int_{\mathbb{R}} t^n \mu(\mathrm{d}t) \quad \underset{n\to\infty}{\longrightarrow} \quad \log\left(\frac{1}{\rho} + R_\mu(\rho)\right).$$

Note that the equality $R'_\mu(\xi) = 1/\xi^2$ is equivalent to $K'_\mu(\xi) = 0$, where we recall that K_μ denotes the inverse of the Cauchy transform of μ.

Proof. First note that $\rho > 0$, as R_μ is analytic on a neighbourhood of the origin. We then apply the results of Section 6.3.1 with weights w defined by $w(0) = 1$ and $w(i) = \kappa_i(\mu)$ for $i \geq 1$. The fact that μ is different from a Dirac mass guaranties that $w(k) > 0$ for some $k \geq 2$. Observe that

$$\Phi(z) = 1 + zR_\mu(z) = zK_\mu(z) \qquad \text{and} \qquad \Psi(z) = 1 + \frac{z^2R'(z) - 1}{zR(z) + 1}.$$

In particular, $\Psi(z) = 1$ if and only if $R'_\mu(z) = 1/z^2$. The claim then follows by combining (6.1) with Theorem 6.11. □

See Example 6.14 below for an example where $\nu < 1$. If μ is the uniform measure on $[0, 1]$, its free cumulants are not all nonnegative, as $R_\mu(z) = 1/(1 - e^{-z}) - 1/z$. See also [11] for information concerning Taylor series of the R-transform of measures which are not compactly supported.

Let s_μ be the maximum of the support of a compactly supported probability measure μ on $\mathbb{R}$. It is well known and simple to check that

$$\log(s_\mu) = \limsup_{n\to\infty} \frac{1}{n} \log \int_{\mathbb{R}} t^n \mu(\mathrm{d}t).$$

Hence, taking into account (6.1), we immediately get Theorem 6.2 from Theorem 6.13.

Example 6.14. (i) If $\mu(\mathrm{d}x) = 1/(\pi\sqrt{1-x^2})\mathbb{I}_{|x|\leq 1}\mathrm{d}x$ is the arcsine law (which is also the free additive convolution $\lambda \boxplus \lambda$ with $\lambda = (\delta_{-1/2} + \delta_{1/2})/2)$), one has $\rho = \infty$, $\nu = 0$, so that $R_\mu(z) = (\sqrt{1+z^2} - 1)/z$, and one recovers that $s_\mu = 1/\infty + R(\infty) = 1$.

(ii) If μ is the free convolution of a free Poisson law of parameter 1 and the uniform distribution on $[-1,1]$, then $R_\mu(z) = \coth(z) - z^{-1} + (1-z)^{-1}$, $\rho = 1$, $\nu = \infty$ so that

$$s_\mu = \coth(z_*) + \frac{1}{1-z_*} \simeq 4.16, \quad \text{where} \quad \operatorname{csch}(z_*)(1-z_*)^2 = 1 \text{ with } z_* \in (0,1).$$

This gives a simpler expression that the one of [91, Example 6.2], which involves solutions of two implicit equations.

(iii) If μ is such that $R_\mu(z) = \frac{1}{z} - \pi\cot(\pi z)$ (this corresponds to the Lévy area corresponding to the free Brownian bridge introduced in [92]), then

$$s_\mu = \frac{2 - \sqrt{2 - \pi^2 z_*^2}}{z_*} \simeq 3.94, \quad \text{where} \quad \frac{\sin(\pi z_*)}{\pi z_*} = \frac{\sqrt{2}}{2} \text{ with } z_* \in (0,1).$$

This gives a simpler expression that the one of [91, Proposition 5.12],

(iv) As noted by Ortmann [91, Section 6.1], if λ is a finite compactly supported measure on $\mathbb{R}$ and $\alpha \in \mathbb{R}$, by [12] or [61, Theorem 3.3.6], there exists a compactly supported probability measure μ such that

$$R_\mu(z) = \alpha + \int \frac{z}{1-xz}\lambda(\mathrm{d}x),$$

and all the cumulants of μ are nonnegative, so that Theorem 6.13 and Theorem 6.2 apply to the corresponding normalized probability measure. This actually corresponds to the class of so-called freely infinitely divisible measures.

In particular, if $\lambda(\mathrm{d}x) = c(1-x)^\alpha \mathbb{I}_{0\leq x\leq 1}\mathrm{d}x$ with $c > 0, \alpha > 1$, then μ is such that $R_\mu(z) = \int_{\mathbb{R}} \frac{z}{1-xz}\lambda(\mathrm{d}x)$ and

$$\kappa_n(\mu) = c\frac{\Gamma(1+\alpha)\cdot\Gamma(n-1)}{\Gamma(n+\alpha)}\mathbb{I}_{\{n\geq 2\}}, \quad \rho = 1, \quad \nu = \frac{(2\alpha-1)c}{(\alpha-1)(\alpha+c)}.$$

Note that $\kappa_n(\mu) \sim c\Gamma(1+\alpha)\cdot n^{-1-\alpha}$ as $n \to \infty$ and that $\nu = 1$ if and only if $c = \alpha - 1$. For example, for $\alpha = 2$ and $c = 1/2$, we have $\nu = 3/5 < 1$ and $s_\mu = 1 + R_\mu(1) = 5/4$.

6.3.4 Distribution of the block sizes in random non-crossing partitions

We are now interested in the distribution of block sizes in large simply generated non-crossing partitions. We fix a sequence of nonnegative weights $w = (w(i); i \geq 1)$ such that

$w(k) > 0$ for some $k \geq 2$. Set $w(0) = 1$, and let P_n be a random non-crossing partition with law $\mathbb{P}_n^w$. Denote by π the probability distribution equivalent to the weights w in the sense of Section 6.3.1. Finally, set $T_n = \mathcal{T}^\circ(P_n)$, so that by Proposition 6.6, T_n is a simply generated tree with $n+1$ vertices with law $\mathbb{Q}_n^w$.

Blocks of given size. If P is a non-crossing partition and A is a non-empty subset of $\mathbb{N}$, we let $\zeta_A(P)$ be the number of blocks of P whose size belongs to A. In particular, notice that $\zeta_{\mathbb{N}}(P)$ is the total number of blocks of P.

Theorem 6.15. *(i) Let $S_1(P_n)$ be the size of the block containing* 1 *in P_n. Then, for every $k \geq 1$, $\mathbb{P}(S_1(P_n) = k) \to k\pi(k)$ as $n \to \infty$.*

(ii) Let B_n be a block chosen uniformly at random in P_n. Assume that $\pi(0) < 1$. Then, for every $k \geq 1$, $\mathbb{P}(|B_n| = k) \to \pi(k)/(1-\pi(0))$ as $n \to \infty$.

(iii) Let A be a non-empty subset of $\mathbb{N}$. As $n \to \infty$, the convergence $\zeta_A(P_n)/n \to \pi(A)$ holds in probability and, in addition, $\mathbb{E}[\zeta_A(P_n)]/n \to \pi(A)$.

In particular, the total number of blocks of P_n is of order $(1-\pi(0))n$ when $\pi(0) < 1$.

Proof. For the first assertion, simply note that $S_1(P_n) = k_\varnothing(T_n)$, and the claim immediately follows from Theorem 6.10 (i). For the second one, if T is a tree, denote by $N_k(T)$ the number of vertices of T with outdegree k. Note that B_n has the law of the outdegree of an internal (i.e. not a leaf) vertex of T_n chosen uniformly at random. As a consequence,

$$\mathbb{P}(|B_n| = k) = \mathbb{E}\left[\frac{N_k(T_n)}{n - N_0(T_n)}\right].$$

By Theorem 6.10 (ii), $N_k(T_n)/(n - N_0(T_n))$ converges in probability to $\pi(k)/(1-\pi(0))$ as $k \to \infty$, and is clearly bounded by 1. The second first assertion then follows from the dominated convergence theorem. For the last assertion, observe that $\zeta_A(P_n) = N_A(T_n)$, where $N_A(T_n)$ denotes the number of vertices of T_n with outdegree in A. Then, fix $K \geq 1$, and to simplify notation, set $A_K = A \cap [K]$, so that by Theorem 6.10 (ii), the convergence $\zeta_{A_K}(P_n)/n \to \pi(A_K)$ holds in probability as $n \to \infty$. Since $|\zeta_A(P_n) - \zeta_{A_K}(P_n)| \leq n/K$, the quantity $|\zeta_A(P_n)/n - \zeta_{A_K}(P_n)/n|$ can be made arbitrarily small by choosing K sufficiently large. It follows that $\zeta_A(P_n)/n \to \pi(A)$ in probability as $n \to \infty$, and the last claim readily by the dominated convergence theorem. □

In the case $\pi(0) = 1$ (which corresponds to $\rho = 0$), Theorem 6.15 (i) tells us that the convergence $S_1(P_n) \to \infty$ holds in probability as $n \to \infty$, but the asymptotic behavior of

$|B_n|$ and the total number of blocks of P_n remains unclear. Unfortunately, it seems that one cannot say anything more in full generality. Indeed:

(i) If $w(k) = k!^\alpha$ with $\alpha > 1$, by [66, Remark 2.9], with probability tending to one as $n \to \infty$, the root of T_n has n children which are all leaves. Therefore, as $n \to \infty$, $\mathbb{P}(S_1(P_n) = n) \to 1$, $\mathbb{P}(|B_n| = n) \to 1$ and $\mathbb{P}(\zeta_{\mathbb{N}}(P_n) = 1) \to 1$.

(ii) If $w(k) = k!$, by [66, Theorem 2.4], with probability tending to one as $n \to \infty$, the root of T_n has $n - U_n$ children which are all leaves, except U_n of them (which have only one vertex grafted on them), and U_n converges in distribution to X, a Poisson random variable of parameter 1, as $n \to \infty$. Therefore, as $n \to \infty$, $n - S_1(P_n) \to X$ in distribution, $\mathbb{P}(|B_n| = 1) \to \mathbb{E}[X/(X+1)] = 1/e$, $\mathbb{P}(|B_n| = S_1(P_n)) \to 1 - 1/e$ and $\zeta_{\mathbb{N}}(P_n) \to X + 1$ in distribution.

(iii) If $w(k) = k!^\alpha$ with $0 < \alpha < 1$ and $1/\alpha \notin \mathbb{N}$ for simplicity, by [66, Theorem 2.5], as $n \to \infty$, $k_\varnothing(T_n)/n \to 1$ in probability, for every $1 \le i \le \lfloor 1/\alpha \rfloor$, $N_i(T_n)/n^{1-i\alpha} \to i!^\alpha$ in probability and, with probability tending to one as $n \to \infty$, $N_i(T_n) = 0$ for every $i > \lfloor 1/\alpha \rfloor$. Therefore, as $n \to \infty$, $S_1(P_n)/n \to 1$ in probability. Also, noting that

$$\mathbb{P}(|B_n| = k) = \mathbb{E}\left[\frac{N_k(T_n)}{\sum_{i\ge 1} N_i(T_n)}\right], \qquad \zeta_{\mathbb{N}}(P_n) = \sum_{i\ge 1} N_i(T_n),$$

we get that and $\mathbb{P}(|B_n| = 1) \to 1$ and $\zeta_{\mathbb{N}}(P_n)/n^{1-\alpha} \to 1$ in probability.

In addition, [64, Example 19.39] gives an example where $\rho = 0$ and $k_\varnothing(T_n)/n \to 0$ in probability.

Asymptotic normality of the block sizes. Theorem 6.15 (ii) shows that a law of large numbers holds for $\zeta_A(P_n)$. Under some additional regularity assumptions on the weights, it is possible to obtain a central limit theorem. Specifically, assume that w is equivalent (in the sense of Section 6.3.1) to a probability distribution π which is critical (meaning that its mean is equal to 1) and has finite positive variance σ^2. In this case, the following result holds.

Theorem 6.16. *Fix an integer $k \ge 1$, and let $A_1, \ldots, A_k$ be non-empty subsets of $\mathbb{N}$. Then there exists a centered Gaussian vector $(X_{A_1}, \ldots, X_{A_k})$ such that the convergence*

$$\left(\frac{\zeta_{A_1}(P_n) - \pi(A_1)n}{\sqrt{n}}, \ldots, \frac{\zeta_{A_k}(P_n) - \pi(A_k)n}{\sqrt{n}}\right) \xrightarrow[n\to\infty]{(d)} (X_{A_1}, \ldots, X_{A_k})$$

holds in distribution. In addition we have

$$\mathbb{E}\left[X_{A_i}^2\right] = \pi(A_i)(1 - \pi(A_i)) - \frac{1}{\sigma^2}\sum_{r\in A_i}(r-1)^2\pi(r)$$

for $1 \leq i \leq k$ *and*

$$\mathrm{Cov}(X_{A_i}, X_{A_j}) = -\pi(A_i)\pi(A_j) - \frac{1}{\sigma^2} \sum_{r \in A_i} (r-1)^2 \pi(r) \cdot \sum_{s \in A_j} (s-1)^2 \pi(s)$$

if $1 \leq i \neq j \leq k$ *are such that* $A_i \cap A_j = \emptyset$.

This result is just a translation of the corresponding known result for conditioned Galton–Watson trees: recalling that $T_n = \mathcal{T}^{\circ}(P_n)$, let $N_A(T_n)$ denote the number of vertices of T_n with outdegree in A, then

$$(\zeta_{A_1}(P_n), \ldots, \zeta_{A_k}(P_n)) = (N_{A_1}(T_n), \ldots, N_{A_k}(T_n)),$$

and Theorem 6.16 then follows from [65, Example 2.2] (in this reference, the results are stated when $\#A_i = 1$ for every i, but it is a simple matter to see that they still hold).

Large deviations for the empirical block size distribution. Denote by $\mathcal{M}_n$ the law of the size of a block of P_n, chosen uniformly at random among all possible blocks, so that $\mathcal{M}_n$ is a random probability measure on $\mathbb{N}$. Dembo, Mörters & Sheffield [34, Theorem 2.2] establish a large deviation principle for the empirical outdegree distribution in Galton–Watson trees. Therefore, we believe that an analogue large deviation principle holds for $\mathcal{M}_n$ (at least when the weights are equivalent to a critical probability distribution having a finite exponential moment), which would in particular extend a result of Ortmann [91, Theorem 1.1], who established such a large deviation principle in the case of uniformly distributed k-divisible non-crossing partitions. The point is that Ortmann uses the bijection $P \leftrightarrow \mathcal{T}^{\bullet}(P_n)$, but we believe that it is simpler to use the bijection $P \leftrightarrow \mathcal{T}^{\circ}(P)$ since $\mathcal{T}^{\circ}(P_n)$ is a simply generated tree, but in general not $\mathcal{T}^{\bullet}(P_n)$. However, we have not worked out the details.

Largest blocks. Depending on the weights, Janson [64, Section 9 and 19] obtains general results concerning the largest outdegrees of simply generated trees. Since the sequence of outdegrees of vertices of T_n that are not leaves, listed in non increasing order, is equal to the sequence of sizes of blocks of P_n, listed in non increasing order, one gets estimates on the sizes of the largest blocks of P_n. We do not enter details, and refer to [64] for precise statements.

Local behavior. Theorem 6.15 (i) describes the distributional limit of the size of the block of P_n containing 1; it is also possible to describe the behavior of the blocks at "finite distance" of the latter. Indeed, as we have seen in Section 6.2.2, when P_n is sampled according to $\mathbb{P}_n^w$, then its two-type dual tree $T_n^\circ = T^\circ(P_n)$ is distributed according to $\mathbb{Q}_n^{(w^{\mathsf{e}},w^{\mathsf{o}})}$ where $w^{\mathsf{o}}(i) = w(i+1)$ and $w^{\mathsf{e}}(i) = 1$ for every $i \geq 0$. In this case, for every tree $\tau \in \mathbb{T}^{(\mathsf{e},\mathsf{o})}$ we have

$$\Omega^{(w^{\mathsf{e}},w^{\mathsf{o}})}(\tau) = \prod_{u \in \mathsf{e}(\tau)} w^{\mathsf{e}}(k_u) \prod_{u \in \mathsf{o}(\tau)} w^{\mathsf{o}}(k_u) = \prod_{u \in \mathsf{o}(\tau)} w(\deg(u)),$$

and Björnberg & Stefánsson [22, Theorem 3.1] have obtained a limit theorem for the measure $\mathbb{Q}_n^{(w^{\mathsf{e}},w^{\mathsf{o}})}$ on $\mathbb{T}_n^{(\mathsf{e},\mathsf{o})}$ as $n \to \infty$, in the local topology. Loosely speaking, the dual tree T_n° converges locally to a limiting infinite two-type tree which can be explicitly constructed, and which is in a certain sense a two-type Galton–Watson tree conditioned to survive. We do not enter details as we will not use this and refer to [22] for precise statements and proofs.

6.4 Non-crossing partitions as compact subsets of the unit disk

We investigate in this section the asymptotic behavior, as $n \to \infty$, of a non-crossing partition sampled according to $\mathbb{P}_n^\mu$ and viewed as an element of the space of all compact subsets of the unit disk equipped with the Hausdorff distance.

Main assumptions. We restrict ourselves to the case where $\mu = (\mu(k), k \geq 0)$ defines a critical probability measure, i.e. $\sum_{k=0}^\infty \mu(k) = \sum_{k=0}^\infty k\mu(k) = 1$. Recall from Section 6.3.1 that any sequence of weights $(w(k), k \geq 0)$ such that

$$\rho = \left(\limsup_{k \to \infty} w(k)^{1/k} \right)^{-1} > 0 \qquad \text{and} \qquad \lim_{t \uparrow \rho} \frac{\sum_{k=0}^\infty k w(k) t^k}{\sum_{k=0}^\infty w(k) t^k} \geq 1$$

is equivalent to such a measure μ and then $\mathbb{P}_n^\mu = \mathbb{P}_n^w$ for every $n \geq 1$. We shall in addition assume that μ belongs to the domain of attraction of a stable law of index $\alpha \in (1, 2]$, i.e. either it has finite variance: $\sum_{k=0}^\infty k^2 \mu(k) < \infty$ (in the case $\alpha = 2$), or $\sum_{k=j}^\infty \mu(k) = j^{-\alpha} L(j)$, where L is a slowly varying function at infinity. Without further notice, we always assume that $\mu(0) + \mu(1) < 1$ to discard degenerate cases.

In this section, we shall establish the following result.

Theorem 6.17. *Fix $\alpha \in (1, 2]$. There exists a random compact subset of the unit disk $\mathbf{L}_\alpha$ such that for every critical offspring distribution μ belonging to the domain of attraction of a stable law of index α, if P_n is a random non-crossing partition sampled according to $\mathbb{P}_n^\mu$, for every integer $n \geq 1$ such that $\mathbb{P}_n^\mu$ is well defined, the convergence*

$$P_n \quad \xrightarrow[n \to \infty]{(d)} \quad \mathbf{L}_\alpha$$

holds in distribution for the Hausdorff distance on the space of all compact subsets of $\overline{\mathbb{D}}$.

The random compact set $\mathbf{L}_\alpha$ is a geodesic lamination; for $\alpha = 2$, the set $\mathbf{L}_2$ is Aldous' Brownian triangulation of the disk [6], while $\mathbf{L}_\alpha$ is the α-stable lamination introduced in [71] for $\alpha \in (1, 2)$. Observe that Theorem 6.17 applies for uniform A-constrained non-crossing partitions of $[n]$ when $A \neq \{1\}$, since this law is $\mathbb{P}_n^{w_A}$ where $w_A(k) = 1$ if $k \in A$ and $w_A(k) = 0$ otherwise; the equivalent probability distribution defined in Example 6.9 is then critical and with finite variance and thus corresponds to $\alpha = 2$.

Before explaining the construction of $\mathbf{L}_\alpha$, we mention an interesting corollary. Recall from the Introduction the notation $\mathtt{C}(P_n)$ for the (angular) length of the longest chord.

Corollary 6.18. *Fix $\alpha \in (1, 2]$. There exists a random variable $\mathtt{C}_\alpha$ such that for every critical offspring distribution μ belonging to the domain of attraction of a stable law of index α, if P_n is a random non-crossing partition sampled according to $\mathbb{P}_n^\mu$, for every integer $n \geq 1$ such that $\mathbb{P}_n^\mu$ is well defined, the convergence*

$$\mathtt{C}(P_n) \quad \xrightarrow[n\to\infty]{(d)} \quad \mathtt{C}_\alpha$$

holds in distribution.

This immediately follows from Theorem 6.17, since the functional "longest chord" is continuous on the set of laminations. Aldous [6] (see also [37]) showed that the law of $\mathtt{C}_2$ has the following explicit distribution:

$$\frac{1}{\pi} \frac{3x-1}{x^2(1-x)^2\sqrt{1-2x}} \mathbb{I}_{\frac{1}{3} \leq x \leq \frac{1}{2}} \mathrm{d}x.$$

See Shi [104] for a study of the longest chord of stable laminations. As before, observe that Theorem 6.3 follows from Corollary 6.18, which applies with $\alpha = 2$ for uniform A-constrained non-crossing partitions of $[n]$ when $A \neq \{1\}$.

Techniques. We briefly comment on the main techniques involved in the proof of Theorem 6.17. Since it is simple to recover P_n from its dual two-type tree $T^\circ(P_n)$, it seems natural to study scaling limits of $T^\circ(P_n)$. However, this is not the road we take: we rather code P_n by the associated one-type tree $\mathcal{T}^\circ(P_n)$, which, as we have earlier seen, has the law of a Galton–Watson tree with offspring distribution μ conditioned to have $n+1$ vertices, and is therefore simpler to study. We then follow the route of [71]: we code $\mathcal{T}^\circ(P_n)$ via a discrete walk; the latter converges in distribution to a continuous-time process, we then define $\mathbf{L}_\alpha$ from this limit path and we show that it is indeed the limit of the discrete non-crossing partitions.

In [71], it is shown that certain random dissections of $[n]$ (a dissection of a polygon with n edges is a collection of non-crossing diagonals) are shown to converge to the stable lamination, by using the fact that their dual trees are Galton–Watson trees conditioned to have a fixed number of leaves. Our arguments are similar to that of [71, Section 2 and 3], but the devil is in the details since the objects under consideration and their coding by trees are different: first, vertices with outdegree 1 are forbidden in [71], and second a vertex with outdegree k in [71] corresponds to $k+1$ chords in the associated discrete lamination, whereas in our case a vertex with outdegree k corresponds to k chords in the associated non-crossing partition. In particular, the proofs of [71, Section 2 and 3] do not carry out with mild modifications, and for this reason we give a complete proof of Theorem 6.17.

From now on, we fix $\alpha \in (1,2]$, a critical offspring distribution μ belonging to the domain of attraction of a stable law of index α, and we let P_n be a random non-crossing partition sampled according to $\mathbb{P}_n^{\mu}$, for every integer $n \geq 1$ such that $\mathbb{P}_n^{\mu}$ is well defined.

6.4.1 Non-crossing partitions and paths

We have seen in Section 3.1 that a non-crossing partition P can be coded by the Łukasiewicz path of an associated plane tree. Observe that the tree there is $\mathcal{T}^{\circ}(P)$. We briefly recall the construction. Fix $n \in \mathbb{N}$ and $W = (W_j; 0 \leq j \leq n+1)$ a path such that $W_0 = 0$, for every $0 \leq j \leq n$, $W_{j+1} - W_j \geq -1$ with the condition that $W_j \geq 0$ for every $0 \leq j \leq n$ and $W_n = -1$. Define for every $0 \leq j \leq n-1$

$$k_j = W_{j+1} - W_j + 1.$$

If $k_j \geq 1$, then for every $1 \leq \ell \leq k_j$, let

$$s_\ell^j = \inf\{m \geq j+1 : W_m = W_{j+1} - (\ell - 1)\}$$

and then set $s_{k_j+1}^j = s_1^j = j+1$. Finally define

$$\mathbf{P}(W) = \bigcup_{j:k_j \geq 1} \bigcup_{\ell=1}^{k_j} \left[\exp\left(-2\mathrm{i}\pi \frac{s_\ell^j}{n}\right), \exp\left(-2\mathrm{i}\pi \frac{s_{\ell+1}^j}{n}\right) \right]. \tag{6.2}$$

Let us briefly explain what this means: if W is the Łukasiewicz path of a tree τ with its vertices labelled $\varnothing = u(0) \prec u(1) \prec \cdots \prec u(n)$ in lexicographical order, then (recall Proposition 1.4) k_j is the number of children of $u(j)$, and $s_1^j < \cdots < s_{k_j}^j$ are the indices of its children. Recall from Section 6.2.1 that from a tree τ, we can define a non-crossing partition $P_\circ(\tau)$ by joining two consecutive children in τ (where the first and the last ones are consecutive by convention);

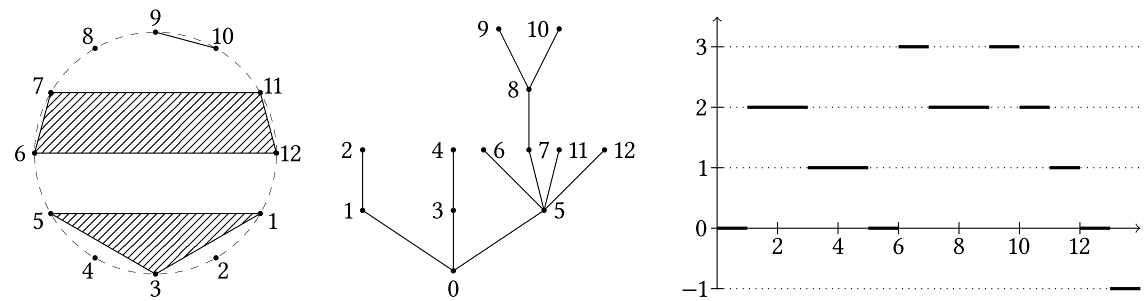


Figure 6.9: The partition $P = \{\{1,3,5\},\{2\},\{4\},\{6,7,11,12\},\{8\},\{9,10\}\}$, the tree $\mathcal{T}^\circ(P)$ and its Łukasiewicz path.

this is exactly what is done in (6.2). Recall also from Section 6.2.1 the construction of the plane tree $\mathcal{T}^\circ(P)$ from a non-crossing partition P.

Proposition 6.19. *For every non-crossing partition P, we have*

$$P = \mathbf{P}(W(\mathcal{T}^\circ(P))).$$

Proof. To simplify notation, let $\mathcal{W}^\circ$ denote the Łukasiewicz path of $\mathcal{T}^\circ(P)$ and let n be its length. First, note that $\mathbf{P}(\mathcal{W}^\circ)$ is a partition of $[n]$: with the notation used in (6.2), the blocks are given by the sets $\{s_1^j, \dots, s_{k_j}^j\}$ for the j's such that $k_j \geq 1$. To show that it is non-crossing, fix $j, j' \in \{0, \dots, n-1\}$ with $k_j, k_{j'} \geq 1$ and fix $\ell \in \{1, \dots, k_j+1\}$ and $\ell' \in \{1, \dots, k_{j'}+1\}$ with $(j,\ell) \neq (j',\ell')$; one checks that the intervals $(s_\ell^j, s_{\ell+1}^j)$ and $(s_{\ell'}^{j'}, s_{\ell'+1}^{j'})$ either are disjoint or one is included in the other so that the chords

$$\left[\exp\left(-2i\pi\frac{s_\ell^j}{n}\right), \exp\left(-2i\pi\frac{s_{\ell+1}^j}{n}\right)\right] \quad \text{and} \quad \left[\exp\left(-2i\pi\frac{s_{\ell'}^{j'}}{n}\right), \exp\left(-2i\pi\frac{s_{\ell'+1}^{j'}}{n}\right)\right]$$

do not cross. Further, as explained above, by construction, the chords of $\mathbf{P}(\mathcal{W}^\circ)$ are chords between consecutive children of $\mathcal{T}^\circ(P)$. The equality $P = \mathbf{P}(\mathcal{W}^\circ)$ then simply follows from the fact that, by construction and Proposition 6.4, $i, j \in [n]$ belong to the same block of P if and only if $u(i)$ and $u(j)$ have the same parent in $\mathcal{T}^\circ(P)$. □

As previously explained, we will prove the convergence, when $n \to \infty$, of a random non-crossing partition P_n of $[n]$ sampled according to $\mathbb{P}_n^\mu$, by looking at the scaling limit of the Łukasiewicz path of the conditioned Galton–Watson tree $\mathcal{T}^\circ(P_n)$. As we have seen in Section 3.2, the latter is known to be the normalized excursion of a spectrally positive strictly α-stable Lévy process X_α^{ex} which we next recall. The main advantage of this approach is that $\mathcal{T}^\circ(P_n)$ is a (conditioned) one-type Galton–Watson tree, whereas the dual tree $T^\circ(P_n)$ of P_n is a (conditioned) two-type Galton–Watson tree. We mention here that [1] uses a "modified" Łukasiewicz path to study a two-type Galton–Watson tree; actually this path is just the Łukasiewicz path of the one-type tree associated with the two-type tree by the Janson–Stefánsson bijection.

6.4.2 Convergence to the stable excursion

Fix $\alpha \in (1,2]$ and consider a strictly stable spectrally positive Lévy process of index α: X_α is a random process with paths in the set $\mathcal{D}([0,\infty),\mathbb{R})$ of càdlàg functions endowed with the Skorokhod J_1 topology (see e.g. Billingsley [21] for details on this space) which has independent and stationary increments, no negative jumps and such that $\mathbb{E}\left[\exp(-\lambda X_\alpha(t))\right] = \exp(t\lambda^\alpha)$ for every $t, \lambda > 0$. Using excursion theory, it is then possible to define X_α^{ex}, the normalized excursion of X_α, which is a random variable with values in $\mathcal{D}([0,1],\mathbb{R})$, such that $X_\alpha^{\mathrm{ex}}(0) = X_\alpha^{\mathrm{ex}}(1) = 0$ and, almost surely, $X_\alpha^{\mathrm{ex}}(t) > 0$ for every $t \in (0,1)$. We do not enter into details, see Section 3.2 and references therein for background.

An important point is that X_α^{ex} is continuous for $\alpha = 2$, and indeed $X_2^{\mathrm{ex}}/\sqrt{2}$ is the standard Brownian excursion, whereas the set of discontinuities of X_α^{ex} is dense in $[0,1]$ for every $\alpha \in (1,2)$; we shall treat the two cases separately. Duquesne [43, Proposition 4.3 and proof of Theorem 3.1] provides the following limit theorem which is the steppingstone of our results in this section.

Theorem 6.20 (Duquesne [43]). *Fix $\alpha \in (1,2]$ and let $(\mu(k), k \geq 0)$ be a critical probability measure in the domain of attraction of a stable law of index α. For every integer n such that $\mathbb{Q}_n^\mu$ is well defined, sample τ_n according to $\mathbb{Q}_n^\mu$. Then there exists a sequence $(B_n)_{n\geq 1}$ of positive constants converging to ∞ such that the convergence*

$$\left(\frac{1}{B_n} W_{\lfloor ns \rfloor}(\tau_n); s \in [0,1]\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (X_\alpha^{\mathrm{ex}}(s), s \in [0,1])$$

holds in distribution for the Skorokhod topology on $\mathcal{D}([0,1],\mathbb{R})$.

Recall that if we sample P_n according to $\mathbb{P}_n^\mu$, then the plane tree $\mathcal{T}^\circ(P_n)$ is distributed according to $\mathbb{Q}_n^\mu$. Thus, denoting by $W^{(n)} = W(\mathcal{T}^\circ(P_n))$ the Łukasiewicz path of $\mathcal{T}^\circ(P_n)$, the convergence

$$\left(\frac{1}{B_n} W^{(n)}_{\lfloor ns \rfloor}; s \in [0,1]\right) \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad (X_\alpha^{\mathrm{ex}}(s), s \in [0,1]) \tag{6.3}$$

holds in distribution for the Skorokhod topology on $\mathcal{D}([0,1],\mathbb{R})$.

We next define continuous laminations by replacing the Łukasiewicz path by X_α^{ex} and mimicking the definition (6.2). We prove, using (6.3), that they are the limit of P_n as $n \to \infty$. We first consider the case $\alpha = 2$ as a warm-up before treating the more involved the case $\alpha \in (1, 2)$.

6.4.3 The Brownian case

Let $e = X_2^{\mathrm{ex}}$; we define an equivalence relation $\overset{e}{\sim}$ on $[0,1]$ as follows: for every $s, t \in [0,1]$, we set $s \overset{e}{\sim} t$ when $e(s \wedge t) = e(s \vee t) = \min_{[s\wedge t, s\vee t]} e$. We then define a subset of $\overline{\mathbb{D}}$ by

$$\mathbf{L}(e) \coloneqq \bigcup_{s \overset{e}{\sim} t} \left[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}\right]. \tag{6.4}$$

Using the fact that, almost surely, e is continuous and its local minima are distinct, one can prove (see Aldous [6] and Le Gall & Paulin [77]) that almost surely, $\mathbf{L}(e)$ is a geodesic lamination of $\overline{\mathbb{D}}$ and that, furthermore, it is maximal for the inclusion relation among geodesic laminations of $\overline{\mathbb{D}}$. Observe that $s \overset{e}{\sim} s$ for every $s \in [0,1]$ so $\mathbb{S}^1 \subset \mathbf{L}(e)$. Also, since $\mathbf{L}(e)$ is maximal, its faces, i.e. the connected components of $\overline{\mathbb{D}} \setminus \mathbf{L}(e)$, are open triangles whose vertices belong to $\mathbb{S}^1$; $\mathbf{L}(e)$ is called the Brownian triangulation and corresponds to $\mathbf{L}_2$ in Theorem 6.17.

Proof of Theorem 6.17 for $\alpha = 2$. Using Skorokhod's representation theorem, we assume that the convergence (6.3) holds almost surely with $\alpha = 2$; we then fix ω in the probability space such that this convergence holds for ω. Since the space of compact subsets of $\overline{\mathbb{D}}$ equipped with the Hausdorff distance is compact, we have the convergence, along a subsequence (which depends on ω), of P_n to a limit P_∞, and it only remains to show that $P_\infty = \mathbf{L}(e)$. Observe first that, since the space of geodesic laminations of $\overline{\mathbb{D}}$ is closed, P_∞ is a lamination. Then, by maximality of $\mathbf{L}(e)$, it suffices to prove that $\mathbf{L}(e) \subset P_\infty$ to obtain the equality of these two sets.

Fix $\varepsilon > 0$ and $0 \le s < t \le 1$ such that $s \overset{e}{\sim} t$. Using the convergence (6.3) and the properties of the Brownian excursion (namely that times of local minima are almost surely dense in $[0,1]$), we can find integers $j_n, \ell_n \in \{1, \dots, n-1\}$ such that every n large enough, we have

$$|n^{-1} j_n - s| < \varepsilon \qquad \text{and} \qquad |n^{-1} \ell_n - t| < \varepsilon,$$

as well as

$$W_{j_n}^{(n)} > W_{j_n - 1}^{(n)} \qquad \text{and} \qquad \ell_n = \min\{m > j_n : W_m^{(n)} < W_{j_n}^{(n)}\}.$$

In other words, $u(j_n)$ and $u(\ell_n)$ are consecutive children of $u(j_n - 1)$ in $\mathcal{T}^\circ(P_n)$. By Proposition 6.19, the last two properties yield

$$\left[\exp\left(-2i\pi\frac{j_n}{n}\right), \exp\left(-2i\pi\frac{\ell_n}{n}\right)\right] \subset P_n.$$

Thus, for every n large enough, the chord $[e^{-2i\pi s}, e^{-2i\pi t}]$ lies within distance 2ε from P_n. Letting $n \to \infty$, along a subsequence, we obtain that $[e^{-2i\pi s}, e^{-2i\pi t}]$ lies within distance 2ε from P_∞. As ε is arbitrary, we have $[e^{-2i\pi s}, e^{-2i\pi t}] \subset P_\infty$, hence $\mathbf{L}(e) \subset P_\infty$ and the proof is complete. □

6.4.4 The stable case

We follow the presentation of [71]. Fix $\alpha \in (1,2)$ and consider X_α^{ex} the normalized excursion of the α-stable Lévy process. For every $t \in (0,1]$, we denote by $\Delta X_\alpha^{\text{ex}}(t) = X_\alpha^{\text{ex}}(t) - X_\alpha^{\text{ex}}(t-) \geq 0$ its jump at t, and we set $\Delta X_\alpha^{\text{ex}}(0) = X_\alpha^{\text{ex}}(0-) = 0$. We recall from [71, Proposition 2.10] that X_α^{ex} fulfills the following four properties with probability one:

(H1) For every $0 \leq s < t \leq 1$, there exists at most one value $r \in (s,t)$ such that $X_\alpha^{\text{ex}}(r) = \inf_{[s,t]} X_\alpha^{\text{ex}}$;

(H2) For every $t \in (0,1)$ such that $\Delta X_\alpha^{\text{ex}}(t) > 0$, we have $\inf_{[t,t+\varepsilon]} X_\alpha^{\text{ex}} < X_\alpha^{\text{ex}}(t)$ for every $0 < \varepsilon \leq 1 - t$;

(H3) For every $t \in (0,1)$ such that $\Delta X_\alpha^{\text{ex}}(t) > 0$, we have $\inf_{[t-\varepsilon,t]} X_\alpha^{\text{ex}} < X_\alpha^{\text{ex}}(t-)$ for every $0 < \varepsilon \leq t$;

(H4) For every $t \in (0,1)$ such that X_α^{ex} attains a local minimum at t (which implies $\Delta X_\alpha^{\text{ex}}(t) = 0$), if $s = \sup\{u \in [0,t] : X_\alpha^{\text{ex}}(u) < X_\alpha^{\text{ex}}(t)\}$, then $\Delta X_\alpha^{\text{ex}}(s) > 0$ and $X_\alpha^{\text{ex}}(s-) < X_\alpha^{\text{ex}}(t) < X_\alpha^{\text{ex}}(s)$.

We will always implicitly discard the null-set for which at least one of these properties does not hold. We next define a relation (not equivalence relation in general) on $[0,1]$ as follows: for every $0 \leq s < t \leq 1$, we set

$$s \simeq^{X_\alpha^{\text{ex}}} t \quad \text{if} \quad t = \inf\{u > s : X_\alpha^{\text{ex}}(u) \leq X_\alpha^{\text{ex}}(s-)\},$$

and then for $0 \leq t < s \leq 1$, we set $s \simeq^{X_\alpha^{\text{ex}}} t$ if $t \simeq^{X_\alpha^{\text{ex}}} s$, and finally we agree that $s \simeq^{X_\alpha^{\text{ex}}} s$ for every $s \in [0,1]$. We next define the following subset of $\overline{\mathbb{D}}$:

$$\mathbf{L}_\alpha := \bigcup_{s \simeq^{X_\alpha^{\text{ex}}} t} \left[e^{-2i\pi s}, e^{-2i\pi t}\right]. \tag{6.5}$$

Observe that $\mathbb{S}^1 \subset \mathbf{L}_\alpha$. Using the above properties, it is proved in [71, Proposition 2.9] that $\mathbf{L}_\alpha$ is a geodesic lamination of $\overline{\mathbb{D}}$, called the α-stable lamination. The latter is not maximal: each face is bounded by infinitely many chords (the intersection of the closure of each face and the unit disk has indeed a non-trivial Hausdorff dimension in the plane).

We next prove Theorem 6.17; as in the case $\alpha = 2$, we assume using Skorokhod's representation theorem that (6.3) holds almost surely and we work with ω fixed in the probability space such that this convergence (as well as the properties (H1) to (H4)) holds for ω. To simplify notation, we set

$$X^{(n)}(s) = \frac{1}{B_n} W^{(n)}_{\lfloor ns \rfloor} \qquad \text{for every } s \in [0,1].$$

Along a subsequence (which depends on ω), we have the convergence of P_n to a limit P_∞, which is a lamination. It only remains to prove the identity $P_\infty = \mathbf{L}_\alpha$. To do so, we shall prove the inclusions $\mathbf{L}_\alpha \subset P_\infty$ and $P_\infty \subset \mathbf{L}_\alpha$ in two separate lemmas.

Lemma 6.21. *We have* $\mathbf{L}_\alpha \subset P_\infty$.

Proof. Notice that if $s < t$ and $s \simeq^{X^{\mathrm{ex}}_\alpha} t$, then $X^{\mathrm{ex}}_\alpha(t) = X^{\mathrm{ex}}_\alpha(s-)$ and $X^{\mathrm{ex}}_\alpha(r) > X^{\mathrm{ex}}_\alpha(s-)$ for every $r \in (s,t)$, hence $s \simeq^{X^{\mathrm{ex}}_\alpha} t$ if and only if one the following cases holds:

(i) $\Delta X^{\mathrm{ex}}_\alpha(s) > 0$ and $t = \inf\{u > s : X^{\mathrm{ex}}_\alpha(u) = X^{\mathrm{ex}}_\alpha(s-)\}$, we write $(s,t) \in \mathcal{E}_1(X^{\mathrm{ex}}_\alpha)$;

(ii) $\Delta X^{\mathrm{ex}}_\alpha(s) = 0$, $X^{\mathrm{ex}}_\alpha(s) = X^{\mathrm{ex}}_\alpha(t)$ and $X^{\mathrm{ex}}_\alpha(r) > X^{\mathrm{ex}}_\alpha(s)$ for every $r \in (s,t)$, we write $(s,t) \in \mathcal{E}_2(X^{\mathrm{ex}}_\alpha)$.

Using the observation ([71, Proposition 2.14]) that, almost surely, for every pair $(s,t) \in \mathcal{E}_2(X^{\mathrm{ex}}_\alpha)$ and every $\varepsilon \in (0, (t-s)/2)$, there exist $s' \in [s, s+\varepsilon]$ and $t' \in [t-\varepsilon, t]$ with $(s',t') \in \mathcal{E}_1(X^{\mathrm{ex}}_\alpha)$, one can prove ([71, Proposition 2.15]) that almost surely

$$\mathbf{L}_\alpha = \overline{\bigcup_{(s,t)\in\mathcal{E}_1(X^{\mathrm{ex}}_\alpha)} [\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}]}. \tag{6.6}$$

The proof thus reduces to showing that, for any $0 \le u < v \le 1$ such that $\Delta X^{\mathrm{ex}}_\alpha(u) > 0$ and $v = \inf\{w \ge u : X^{\mathrm{ex}}_\alpha(w) = X^{\mathrm{ex}}_\alpha(u-)\}$ fixed, we have $[\mathrm{e}^{-2\mathrm{i}\pi u}, \mathrm{e}^{-2\mathrm{i}\pi v}] \subset P_\infty$. Further, as in the case $\alpha = 2$, it is sufficient to find sequences $u_n \to u$ and $v_n \to v$ as $n \to \infty$ such that for every n large enough, $[\mathrm{e}^{-2\mathrm{i}\pi u_n}, \mathrm{e}^{-2\mathrm{i}\pi v_n}] \subset P_n$. Informally, the main difference with [71] is that we choose different sequences u_n, v_n: with the notation used in (6.2), we shall take the pair (u_n, v_n) of the form $n^{-1}(s^j_1, s^j_{k_j})$ for a certain j.

More precisely, fix $\varepsilon > 0$ and observe that, since v cannot be a time of local minimum of X^{ex}_α by (H4), then

$$\inf_{[v-\varepsilon, v+\varepsilon]} X^{\mathrm{ex}}_\alpha < X^{\mathrm{ex}}_\alpha(v) = X^{\mathrm{ex}}_\alpha(u-) < \inf_{[u, v-\varepsilon]} X^{\mathrm{ex}}_\alpha.$$

Using the convergence (6.3), we can then find a sequence $(u_n)_{n\geq 1}$ such that for every n sufficiently large, we have

$$u_n \in (u-\varepsilon, u+\varepsilon) \cap n^{-1}\mathbb{N} \qquad \text{and} \qquad \inf_{[v-\varepsilon,v+\varepsilon]} X^{(n)} < X^{(n)}(u_n-) < \inf_{[u_n,v-\varepsilon]} X^{(n)}.$$

Define then $v_n \coloneqq \inf\{r \geq u_n : X^{(n)}(r) = X^{(n)}(u_n-)\}$ and observe that $v_n \in (v-\varepsilon, v+\varepsilon) \cap n^{-1}\mathbb{N}$. Moreover, as $B_n X^n(u_n) = W^{(n)}_{nu_n}$ and $B_n X^n(u_n-) = W^{(n)}_{nu_n-1}$, we have $W^{(n)}_{nu_n-1} \leq W^{(n)}_{nu_n}$ and

$$nv_n = \inf\{l \geq nu_n : W^{(n)}_l = W^{(n)}_{nu_n} - (W^{(n)}_{nu_n} - W^{(n)}_{nu_n-1})\}.$$

We conclude from Proposition 6.19 that

$$\left[\mathrm{e}^{-2\mathrm{i}\pi u_n}, \mathrm{e}^{-2\mathrm{i}\pi v_n}\right] \subset P_n$$

for every n large enough and the proof is complete. □

Finally, we end the proof of Theorem 6.17 with the converse inclusion.

Lemma 6.22. *We have $P_\infty \subset \mathbf{L}_\alpha$.*

Proof. Recall that P_∞ is the limit of P_n along a subsequence, say, $(n_k)_{k\geq 1}$. Let us rewrite (6.2), combined with Proposition 6.19, as

$$P_{n_k} = \bigcup_{(u,v)\in\mathcal{E}_{(n_k)}} \left[\mathrm{e}^{-2\mathrm{i}\pi u}, \mathrm{e}^{-2\mathrm{i}\pi v}\right],$$

where $\mathcal{E}_{(n_k)}$ is a symmetric finite subset of $[0,1]^2$. Upon extracting a further subsequence, we may, and do, assume that $\mathcal{E}_{(n_k)}$ converges in the Hausdorff sense as $k \to \infty$ to a symmetric closed subset $\mathcal{E}_\infty$ of $[0,1]^2$. One then checks that

$$P_\infty = \bigcup_{(u,v)\in\mathcal{E}_\infty} \left[\mathrm{e}^{-2\mathrm{i}\pi u}, \mathrm{e}^{-2\mathrm{i}\pi v}\right].$$

It only remains to prove that every pair $(u,v) \in \mathcal{E}_\infty$ satisfies $u \simeq^{X^{\mathrm{ex}}_\alpha} v$. Fix $(u,v) \in \mathcal{E}_\infty$ with $u < v$; we aim to show that $v = \inf\{r > u : X^{\mathrm{ex}}_\alpha(r) \leq X^{\mathrm{ex}}_\alpha(u-)\}$.

For every integer $j \in \{1,\dots,n\}$ and let $p(j)$ be the index of the parent of vertex labelled j in $\mathcal{T}^\circ(P_n)$: $p(j) = \sup\{m < j : W^{(n)}_m \leq W^{(n)}_j\}$. Observe then that $[\mathrm{e}^{-2\mathrm{i}\pi j_n/n}, \mathrm{e}^{-2\mathrm{i}\pi \ell_n/n}] \subset P_n$ when $p(j_n) = p(\ell_n)$ and, either $\ell_n = \inf\{m \geq j_n : W^{(n)}_m = W^{(n)}_{j_n} - 1\}$, or $j_n = p(j_n) + 1$ and $\ell_n = \inf\{m \geq j_n : W^{(n)}_m = W^{(n)}_{p(j_n)}\}$.

By definition, (u, v) is the limit as $k \to \infty$ of elements (u_{n_k}, v_{n_k}) in $\mathcal{E}_{(n_k)}$. Upon extracting a subsequence, we may, and do, suppose that either each pair $(j_{n_k}, l_{n_k}) = (n_k u_{n_k}, n_k v_{n_k})$ fulfills the first condition above, or they all fulfill the second one. We first focus on the first case. We therefore suppose that we can find integers $j_{n_k} < l_{n_k}$ in $\{1, \dots, n_k\}$ such that $(u, v) = \lim_{k\to\infty} n_k^{-1}(j_{n_k}, l_{n_k})$ and

$$l_{n_k} = \inf\{m \geq j_{n_k} : W_m^{(n_k)} = W_{j_{n_k}}^{(n_k)} - 1\} \qquad \text{for every} \quad k \geq 1.$$

We see that

$$X^{(n_k)}(r) \geq X^{(n_k)}\left(\frac{j_{n_k}}{n_k}\right) = X^{(n_k)}\left(\frac{l_{n_k} - 1}{n_k}\right) \qquad \text{for every} \quad r \in \left[\frac{j_{n_k}}{n_k}, \frac{l_{n_k} - 1}{n_k}\right], \tag{6.7}$$

which yields, together with the functional convergence $X^{(n)} \to X_\alpha^{\text{ex}}$,

$$X_\alpha^{\text{ex}}(r) \geq X_\alpha^{\text{ex}}(v-) \qquad \text{for every} \quad r \in (u, v). \tag{6.8}$$

By (H3), we must have $\Delta X_\alpha^{\text{ex}}(v) = 0$ and so $X^{(n_k)}(n_k^{-1}(l_{n_k} - 1)) \to X_\alpha^{\text{ex}}(v)$ as $k \to \infty$. On the other hand, the only possible accumulation points of $X^{(n_k)}(n_k^{-1} j_{n_k})$ are $X_\alpha^{\text{ex}}(u-)$ and $X_\alpha^{\text{ex}}(u)$.

We consider two cases. Suppose first that $\Delta X_\alpha^{\text{ex}}(u) = 0$; then $X^{(n_k)}(n_k^{-1} j_{n_k}) \to X_\alpha^{\text{ex}}(u)$ as $k \to \infty$ and it follows from (6.7) that $X_\alpha^{\text{ex}}(u) = X_\alpha^{\text{ex}}(v)$. This further implies that $X_\alpha^{\text{ex}}(u) < X_\alpha^{\text{ex}}(r)$ for every $r \in (u, v)$, otherwise it would contradict either (H1) or (H4), depending on whether X_α^{ex} admits a local minimum at u or not. We conclude that in this case, we have $u \simeq^{X_\alpha^{\text{ex}}} v$.

Suppose now that $\Delta X_\alpha^{\text{ex}}(u) > 0$; then, by (H2), for every $\varepsilon > 0$, there exists $r \in (u, u + \varepsilon)$ such that $X_\alpha^{\text{ex}}(r) < X_\alpha^{\text{ex}}(u)$. Consequently, we must have $X^{(n_k)}(n_k^{-1} j_{n_k}) \to X_\alpha^{\text{ex}}(u-)$ as $k \to \infty$, otherwise (6.7) would give $X_\alpha^{\text{ex}}(u) = X_\alpha^{\text{ex}}(v) = X_\alpha^{\text{ex}}(v-)$ and we would get a contradiction with (6.8). We thus have $X_\alpha^{\text{ex}}(u-) = X_\alpha^{\text{ex}}(v) \leq X_\alpha^{\text{ex}}(r)$ for every $r \in (u, v)$; moreover the latter inequality is strict since an element $r \in (u, v)$ such that $X_\alpha^{\text{ex}}(r) = X_\alpha^{\text{ex}}(u-)$ is the time of a local minimum of X_α^{ex} and this contradicts (H4). We see again that $u \simeq^{X_\alpha^{\text{ex}}} v$.

In the second case when each pair (j_{n_k}, l_{n_k}) satisfies $j_{n_k} = p(j_{n_k}) + 1$ and $l_{n_k} = \inf\{m \geq j_{n_k} : W_m^{(n)} = W_{p(j_{n_k})}^{(n)}\}$, the very same arguments apply, which completes the proof. □

6.5 Extensions

If P_n is a simply generated non-crossing partition generated using a sequence of weights w, a natural question is to ask how behaves the largest block area of P_n. In this direction, if P is a non-crossing partition, we propose to study $P^\circledast$, which is by definition the union of the convex hulls of the blocks of P (see Figure 6.10 for an example).

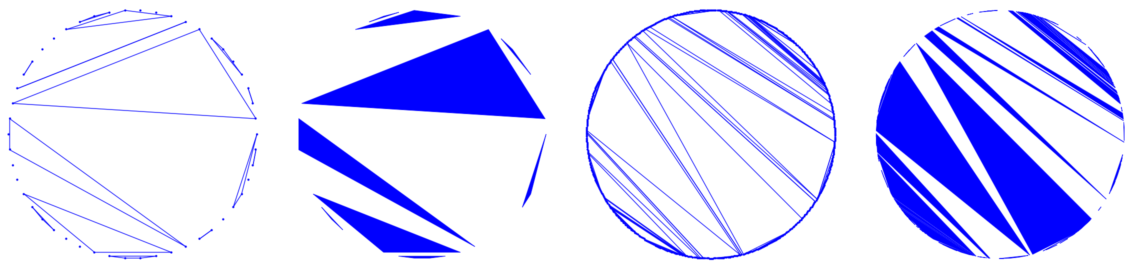


Figure 6.10: From left to right: $P_{50}, P^{\bullet}_{50}, P_{500}, P^{\bullet}_{500}$, where P_{50} (resp. P_{500}) is a uniform non-crossing partition of [50] (resp. [500]).

Question 6.23. Assume that the weights w are equivalent to a critical probability distribution which has finite variance. Is it true that $P^{\bullet}_n$ converges in distribution as $n \to \infty$ to a random compact subset of the unit disk?

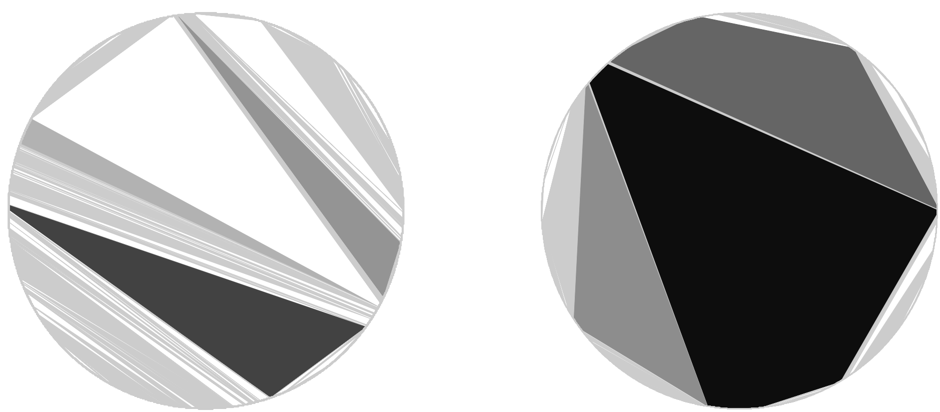


Figure 6.11: A simulation of $P^{\bullet}_{20000}$ for respectively $\alpha = 2$ and $\alpha = 1.3$, where the largest faces are the darkest ones.

If the answer was positive, the limiting object would be obtained from the Brownian triangulation by "filling-in" some triangles, and this would imply that the largest block area of P_n converges in distribution to the area of the largest "filled-in face" of the distributional limit.

In the case of A-constrained uniform plane partitions, numerical simulations based on the calculation of the total area of $P^{\bullet}_n$ indicate that this limiting distribution should depend on the weights A (note that in the particular case $A \subset \{1, 2\}$ it is clear that $(P_n, P^{\bullet}_n) \to (\mathbf{L}_2, \mathbf{L}_2)$ in distribution as $n \to \infty$).

When the weights w are equivalent to a critical probability distribution that belongs to the domain of attraction of a stable law of index $\alpha \in (1, 2)$, it is not difficult to adapt the arguments of the previous section to check that

$$\left(P_n, \overline{\overline{\mathbb{D}} \setminus P_n^\circ}\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (\mathbf{L}_\alpha, \mathbf{L}_\alpha),$$

meaning that the faces of P_n cover in the limit the whole disk (see Figure 6.11 for an illustration). In particular, in this case, the largest block area of P_n converges in distribution to the largest area face of $\mathbf{L}_\alpha$.

7

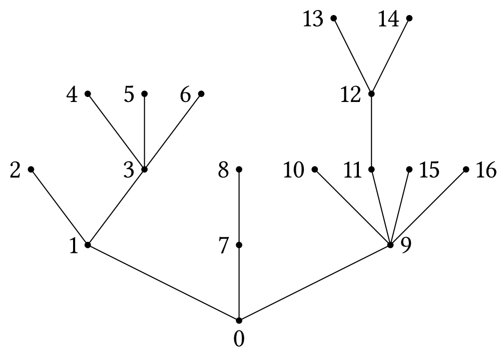
Non-crossing trees and stable triangulations

In this chapter, we study the embedding into the unit disk of large Galton–Watson trees as presented in Chapter 3. This work is currently being prepared for publication separately.

7.1 Introduction and main results

The behavior of large random trees has motivated a lot of work and a usual way to consider a convergence in distribution of trees is to view them as compact metric spaces, equipped with the Gromov–Hausdorff distance (recall the definition in Chapter 1), see e.g. Aldous [5] and Duquesne [43] for Galton–Watson trees, Haas & Miermont [59] and Rizzolo [103] for sequences of trees satisfying a Markov branching branching property, Haas & Stephenson [60] for k-ary trees and Broutin & Sulzbach [23] who consider the dual tree of the recursive triangulation of the disk studied by Curien & Le Gall [33].

Here, we view trees as a closed subsets of the closed unit disk $\overline{\mathbb{D}}$ of the complex plane, equipped with the Hausdorff distance. Fix a plane tree τ with n vertices, that we denote by $\varnothing = u(0) \prec u(1) \prec \cdots \prec u(n-1)$ in lexicographical order. We define a closed subset $\Gamma(\tau)$ of $\overline{\mathbb{D}}$ by considering the n-th roots of unity $\{e^{-2i\pi k/n}; 0 \le k \le n-1\}$ and by drawing a chord $[e^{-2i\pi k/n}, e^{-2i\pi \ell/n}]$ whenever the vertices $u(k)$ and $u(\ell)$ are linked by an edge in τ. See Figure 7.1 for an illustration. As we have seen in Chapter 6, the main benefit of this point of view is that the set of closed subsets of $\overline{\mathbb{D}}$ is compact for the Hausdorff distance. Therefore, given a sequence of trees, we know a priori that it admits sub-sequential limits so in order to prove its convergence, it suffices to show that it admits a unique accumulation point, and to identify the latter if possible.



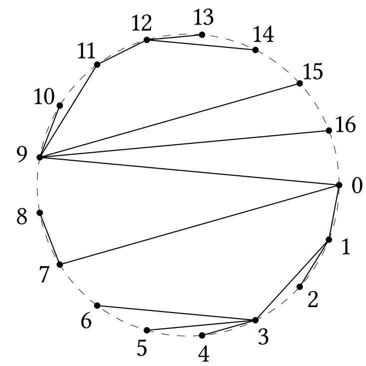


Figure 7.1: A plane tree and its embedding in the disk.

Such a set $\Gamma(\tau)$ is a particular example of non-crossing trees which are more generally trees drawn in the disk with the n-th roots of unity as vertices and whose edges do not cross. Here, the embedding of a plane tree has the property that each vertex lies after its parent in clockwise order. Therefore, and as opposed to the previous chapter, we do not consider statistics on this embedding such as the distribution of the degree of the root, the number of leaves, number of vertices with a given degree, the largest degree or the height of a typical vertex (all these have been studied for uniform random non-crossing trees by Noy [90], Flajojet & Noy [51] and Deutsch & Noy [36]); indeed the latter are exactly the same as for the plane version. We thus focus here on the geometrical point of view by considering the embedding of a plane tree in the disk as a lamination. Let us mention that Marckert & Panholzer [80] and then Curien & Kortchemski [31] considered the reversed question and studied the properties of the random plane tree induced by a uniform random non-crossing tree.

Main assumptions. Let μ be a critical probability measure on $\mathbb{Z}_+$: $\mu(0) > 0$ and $\sum_{k=0}^{\infty} k\mu(k) = 1$. We consider for each integer $n \geq 1$ a Galton–Watson tree with offspring distribution μ conditioned to have n vertices (recall the definition from Section 1.2); we shall always implicitly restrict ourselves to the values of n for which this conditioning is well defined. We assume in addition that μ belongs to the domain of attraction of a stable law of index $\alpha \in (1,2]$, i.e. either it has finite variance: $\sum_{k=0}^{\infty} k^2\mu(k) < \infty$ (in the case $\alpha = 2$), or $\sum_{k=j}^{\infty} \mu(k) = j^{-\alpha}L(j)$, where L is a slowly varying function at infinity.

Theorem 7.1. *Fix $\alpha \in (1,2]$. There exists a random compact subset of the unit disk $\widehat{\mathbf{L}}_\alpha$ such that for every critical offspring distribution μ belonging to the domain of attraction of a stable* *law of index α, if τ_n is a Galton–Watson tree with offspring distribution μ conditioned to have n vertices, the convergence*

$$\Gamma(\tau_n) \xrightarrow[n\to\infty]{(d)} \widehat{\mathbf{L}}_\alpha$$

holds in distribution for the Hausdorff distance on the space of all compact subsets of $\overline{\mathbb{D}}$.

Consider $\alpha \in (1,2]$ and the above set $\widehat{\mathbf{L}}_\alpha$. We will prove that, almost surely, the latter is a geodesic lamination of $\overline{\mathbb{D}}$, which further is maximal for the inclusion relation on laminations, so $\widehat{\mathbf{L}}_\alpha$ is a triangulation (each face is an open triangle with vertices on the circle) that we call the α-stable triangulation. In the case $\alpha = 2$, $\widehat{\mathbf{L}}_2 = \mathbf{L}_2$ is Aldous' Brownian triangulation [6]. For $\alpha \in (1,2)$, $\widehat{\mathbf{L}}_\alpha$ strictly contains the α-stable lamination $\mathbf{L}_\alpha$ of Kortchemski [71].

Denote by $\widehat{\mathbf{A}}_\alpha$ the set of all end-points of chords in $\widehat{\mathbf{L}}_\alpha$. In the case $\alpha = 2$, it is known, see Aldous [6] as well as Le Gall & Paulin [77] for a detailed proof, that almost surely,

$$\dim(\widehat{\mathbf{A}}_2) = \frac{1}{2} \qquad \text{and} \qquad \dim(\widehat{\mathbf{L}}_2) = \frac{3}{2}.$$

We shall prove more generally that for every $\alpha \in (1,2)$, almost surely,

$$\dim(\widehat{\mathbf{A}}_\alpha) = \frac{1}{\alpha} \qquad \text{and} \qquad \dim(\widehat{\mathbf{L}}_\alpha) = 1 + \frac{1}{\alpha}.$$

Curien & Kortchemski [31, Theorem 3.1] proved the convergence in distribution to $\widehat{\mathbf{L}}_2$ of a uniform random non-crossing tree of size n as $n \to \infty$. Our non-crossing trees are certainly not uniformly chosen since each vertex lies after (in clockwise order) its parent. Nevertheless, taking $\mu = (2^{-(k+1)}, k \geq 0)$, $\Gamma(\tau_n)$ in Theorem 7.1 is uniformly distributed on the set of non-crossing trees of size n satisfying this property. Finally, since the law of $\widehat{\mathbf{L}}_2$ is invariant under reflection about the real axis, then the same result holds for non-crossing trees for which each vertex lies before its parent.

Remark 7.2. Recall from Section 6.2 the map $\mathcal{T}^\circ$ which associates with a non-crossing partition P_n of $\{1,\dots,n\}$ a plane tree with $n+1$ vertices. Let μ be a critical offspring distribution belonging to the domain of attraction of a stable law of index $\alpha \in (1,2]$; sample a simply generated non-crossing partition P_n according to $\mathbb{P}_n^\mu$ and let $\tau_{n+1} = \mathcal{T}^\circ(P_n)$. Then both Theorems 6.17 and 7.1 hold and moreover, it shall be plain from the proof that the convergences hold jointly. One could also consider the tree $\tau'_{n+1} = \mathcal{T}^\bullet(P_n)$ which is not a conditioned Galton–Watson in general. However it can be proved by the same arguments as that of Theorem 6.17 that its embedding into the disk converges to $\mathbf{L}_\alpha$ and in fact, we have the joint convergence

$$\Big(P_n, \Gamma(\mathcal{T}^\circ(P_n)), \Gamma(\mathcal{T}^\bullet(P_n))\Big) \xrightarrow[n\to\infty]{(\mathrm{d})} \Big(\mathbf{L}_\alpha, \widehat{\mathbf{L}}_\alpha, \mathbf{L}_\alpha\Big).$$

Note as we pointed out at the beginning of this Chapter the usefulness of viewing plane trees as laminations of the disk since no scaling limit is known for $\mathcal{T}^\bullet(P_n)$ viewed as a compact metric space; we conjecture nonetheless that the latter (exists and) is the so-called α-stable Lévy tree, which is the limit of $\mathcal{T}^\circ(P_n)$ as well (recall Theorem 3.1). This shows that there is no clear relation between convergence for the Gromov–Hausdorff topology and the convergence considered here.

The proof of Theorem 7.1 and the construction of $\widehat{\mathbf{L}}_\alpha$ closely follow Section 6.4: we define $\Gamma(\tau_n)$ from the Łukasievicz path of τ_n; the latter converges in distribution to the continuous-time excursion X_α^{ex}, we then define $\widehat{\mathbf{L}}_\alpha$ from this limit path and we show that it is indeed the limit of $\Gamma(\tau_n)$.

7.2 The discrete setting

Recall from Section 3.1 that the embedding $\Gamma(\tau)$ into the disk of plane tree τ can be easily deduced from its Łukasiewicz path. Fix $n \in \mathbb{N}$ and $W = (W_j; 0 \le j \le n)$ a path such that $W_0 = 0$, $W_{j+1} - W_j \ge -1$ for every $0 \le j \le n-1$, with the condition that $W_j \ge 0$ for every $0 \le j \le n-1$ and $W_n = -1$. Define for every $0 \le j \le n-2$,

$$k_j = W_{j+1} - W_j + 1.$$

If $k_j \ge 1$, then for every $1 \le \ell \le k_j$, let

$$s_\ell^j = \inf\{m \ge j+1 : W_m = W_{j+1} - (\ell - 1)\}$$

and then set $s_{k_j+1}^j = s_1^j = j+1$. Finally define

$$\mathbf{C}(W) = \bigcup_{j:k_j \ge 1} \bigcup_{\ell=1}^{k_j} \left[\exp\left(-2\mathrm{i}\pi \frac{j}{n}\right), \exp\left(-2\mathrm{i}\pi \frac{s_\ell^j}{n}\right)\right]. \tag{7.1}$$

As we have seen in Proposition 1.4, if W is the Łukasiewicz path of a plane tree τ, then k_j is the number of children of its j-th vertex in lexicographical order and $s_1^j < \cdots < s_{k_j}^j$ are the indices of these children. The next proposition should be then plain.

Proposition 7.3. *For every plane tree τ we have*

$$\Gamma(\tau) = \mathbf{C}(W(\tau)).$$

As in the previous chapter, we will prove the convergence, when $n \to \infty$, of the embedding in the disk of random trees of size n by looking at the scaling limit of their Łukasiewicz paths. We have seen that the latter is the normalized excursion of a spectrally positive strictly α-stable Lévy process X_α^{ex} as we next recall.

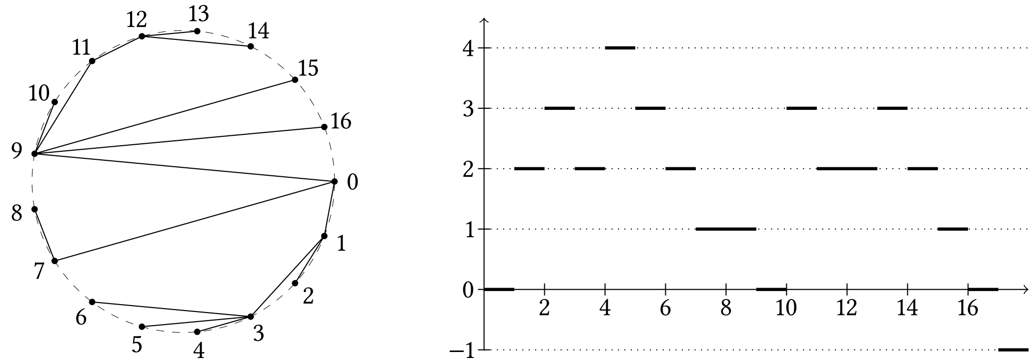


Figure 7.2: A non-crossing tree and the associated Łukasiewicz path.

7.3 Large non-crossing trees

Fix $\alpha \in (1,2]$ and consider X_α a strictly stable spectrally positive Lévy process of index α. As explained in Section 3.2, one can make sense of X_α^{ex}, the normalized excursion of X_α, which is a random variable with values in the Skorokhod space $\mathcal{D}([0,1],\mathbb{R})$, such that $X_\alpha^{\mathrm{ex}}(0) = X_\alpha^{\mathrm{ex}}(1) = 0$ and, almost surely, $X_\alpha^{\mathrm{ex}}(t) > 0$ for every $t \in (0,1)$. Recall also that, almost surely, X_α^{ex} is continuous for $\alpha = 2$, and indeed $X_2^{\mathrm{ex}}/\sqrt{2}$ is the standard Brownian excursion. We finally recall Duquesne's limit theorem.

Theorem 7.4 ([43]). *Fix $\alpha \in (1,2]$ and let μ be a critical probability measure in the domain of attraction of a stable law of index α. Then there exists a sequence $(B_n)_{n\geq 1}$ of positive constants converging to ∞ such that if $W^{(n)}$ denotes the Łukasiewicz path of a Galton–Watson tree with offspring distribution μ conditioned to have n vertices, the convergence*

$$\left(\frac{1}{B_n} W^{(n)}_{\lfloor ns \rfloor}; s \in [0,1]\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (X_\alpha^{\mathrm{ex}}(s), s \in [0,1]) \tag{7.2}$$

holds in distribution for the Skorokhod topology on $\mathcal{D}([0,1],\mathbb{R})$.

We next define laminations using X_α^{ex} similarly to Proposition 7.3 and prove, using (7.2), that they are the limit of $\Gamma(\tau_n)$ as $n \to \infty$. We first consider the case $\alpha = 2$ and postpone the more involved case $\alpha \in (1,2)$ to the next two sections. The constructions and proofs closely follow that of Section 6.4.

7.3.1 The Brownian case

Let $e = X_2^{\mathrm{ex}}$; recall the definition of the Brownian triangulation:

$$\mathbf{L}(e) := \bigcup_{s \overset{e}{\sim} t} \left[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}\right], \tag{7.3}$$

where $s \overset{e}{\sim} t$ when $e(s) = e(t) = \min_{[s \wedge t, s \vee t]} e$. Recall from Aldous [6] and Le Gall & Paulin [77] that, almost surely, $\mathbf{L}(e)$ is a geodesic lamination of $\overline{\mathbb{D}}$, which furthermore, is maximal for the inclusion relation among geodesic laminations of $\overline{\mathbb{D}}$.

We next prove Theorem 7.1 for $\alpha = 2$. Let μ be a critical probability measure in the domain of attraction of a stable law with index 2 and for each n, let τ_n be a Galton–Watson tree with offspring distribution μ conditioned to have n vertices. We show the convergence

$$\Gamma(\tau_n) \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad \mathbf{L}(e)$$

for the Hausdorff distance on the space of all compact subsets of $\overline{\mathbb{D}}$.

Proof of Theorem 7.1 for $\alpha = 2$. The proof is close to that of Theorem 6.17. Using Skorokhod's representation theorem, we assume that the convergence (7.2) holds almost surely with $\alpha = 2$; we then fix ω in the probability space such that this convergence holds for ω. Since the space of compact subsets of $\overline{\mathbb{D}}$ equipped with the Hausdorff distance is compact, we have the convergence, along a subsequence (which depends on ω), of $\Gamma(\tau_n)$ to a limit C_∞. Moreover, since the space of geodesic laminations of $\overline{\mathbb{D}}$ is closed, C_∞ is a lamination. Finally, by maximality of $\mathbf{L}(e)$, it suffices to prove that $\mathbf{L}(e) \subset C_\infty$ to obtain the equality of the two sets.

Fix $\varepsilon > 0$ and $0 \le s < t \le 1$ such that $s \overset{e}{\sim} t$. Using the convergence (7.2) and the properties of the Brownian excursion (namely that times of local minima are almost surely dense in $[0, 1]$), we can find integers $j_n, \ell_n \in \{1, \dots, n-1\}$ such that every n large enough, we have

$$|n^{-1} j_n - s| < \varepsilon \qquad \text{and} \qquad |n^{-1} \ell_n - t| < \varepsilon;$$

as well as

$$W_{j_n}^{(n)} > W_{j_n - 1}^{(n)} \qquad \text{and} \qquad \ell_n = \min\{m > j_n : W_m^{(n)} < W_{j_n}^{(n)}\}.$$

In other words, $u(j_n)$ and $u(\ell_n)$ are respectively the first and second children of $u(j_n - 1)$ in τ_n. By (7.1) and Proposition 7.3, the last two properties yield

$$\left[\exp\left(-2\mathrm{i}\pi \frac{j_n - 1}{n}\right), \exp\left(-2\mathrm{i}\pi \frac{\ell_n}{n}\right)\right] \subset \Gamma(\tau_n).$$

Thus, for every n large enough, the chord $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}]$ lies within distance 2ε from $\Gamma(\tau_n)$. Letting $n \to \infty$, along a subsequence, we obtain that $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}]$ lies within distance 2ε from C_∞. As ε is arbitrary, we have $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}] \subset C_\infty$, hence $\mathbf{L}(e) \subset C_\infty$ and the proof is complete. □

7.3.2 The stable triangulation

Fix $\alpha \in (1,2)$ and consider X_α^{ex} the normalized excursion of the α-stable Lévy process. For every $t \in (0,1]$, we denote by $\Delta X_\alpha^{\mathrm{ex}}(t) = X_\alpha^{\mathrm{ex}}(t) - X_\alpha^{\mathrm{ex}}(t-) \geq 0$ its jump at t, and we set $\Delta X_\alpha^{\mathrm{ex}}(0) = X_\alpha^{\mathrm{ex}}(0-) = 0$. We recall that X_α^{ex} fulfills the following four properties with probability one [71, Proposition 2.10]:

(H1) For every $0 \leq s < t \leq 1$, there exists at most one value $r \in (s,t)$ such that $X_\alpha^{\mathrm{ex}}(r) = \inf_{[s,t]} X_\alpha^{\mathrm{ex}}$.

(H2) For every $t \in (0,1)$ such that $\Delta X_\alpha^{\mathrm{ex}}(t) > 0$, we have $\inf_{[t,t+\varepsilon]} X_\alpha^{\mathrm{ex}} < X_\alpha^{\mathrm{ex}}(t)$ for every $0 < \varepsilon \leq 1-t$;

(H3) For every $t \in (0,1)$ such that $\Delta X_\alpha^{\mathrm{ex}}(t) > 0$, we have $\inf_{[t-\varepsilon,t]} X_\alpha^{\mathrm{ex}} < X_\alpha^{\mathrm{ex}}(t-)$ for every $0 < \varepsilon \leq t$;

(H4) For every $t \in (0,1)$ such that X_α^{ex} attains a local minimum at t (which implies $\Delta X_\alpha^{\mathrm{ex}}(t) = 0$), if $s = \sup\{u \in [0,t] : X_\alpha^{\mathrm{ex}}(u) < X_\alpha^{\mathrm{ex}}(t)\}$, then $\Delta X_\alpha^{\mathrm{ex}}(s) > 0$ and $X_\alpha^{\mathrm{ex}}(s-) < X_\alpha^{\mathrm{ex}}(t) < X_\alpha^{\mathrm{ex}}(s)$.

We will always implicitly discard the null-set for which at least one of these properties does not hold. We next define a relation (not equivalence relation in general) on $[0,1]$ using X_α^{ex}: for every $0 \leq s < t \leq 1$, we set

$$s \cong^{X_\alpha^{\mathrm{ex}}} t \qquad \text{if} \qquad X_\alpha^{\mathrm{ex}}(s-) \leq X_\alpha^{\mathrm{ex}}(t) \quad \text{and} \quad X_\alpha^{\mathrm{ex}}(t) = \inf_{[s,t]} X_\alpha^{\mathrm{ex}};$$

then for $0 \leq t < s \leq 1$, we set $s \cong^{X_\alpha^{\mathrm{ex}}} t$ if $t \cong^{X_\alpha^{\mathrm{ex}}} s$, and finally we set $s \cong^{X_\alpha^{\mathrm{ex}}} s$ for every $s \in [0,1]$. We then define the following subset of $\overline{\mathbb{D}}$:

$$\widehat{L}(X_\alpha^{\mathrm{ex}}) := \bigcup_{s \cong^{X_\alpha^{\mathrm{ex}}} t} \left[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}\right]. \tag{7.4}$$

Proposition 7.5. *For every $\alpha \in (1,2)$, $\widehat{L}(X_\alpha^{\mathrm{ex}})$ is a geodesic lamination of $\overline{\mathbb{D}}$. Moreover, it is maximal for the inclusion relation among geodesic laminations of $\overline{\mathbb{D}}$.*

Recall the definition of the stable lamination:

$$L(X_\alpha^{\mathrm{ex}}) := \bigcup_{s \simeq^{X_\alpha^{\mathrm{ex}}} t} \left[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}\right], \tag{7.5}$$

where

$$s \simeq^{X_\alpha^{\mathrm{ex}}} t \qquad \text{if} \qquad t = \inf\{u > s : X_\alpha^{\mathrm{ex}}(u) \leq X_\alpha^{\mathrm{ex}}(s-)\}.$$

Note that for every $s, t \in [0,1]$, if $s \simeq^{X_\alpha^{\mathrm{ex}}} t$, then $s \cong^{X_\alpha^{\mathrm{ex}}} t$; in particular $\mathbb{S}^1 \subset L(X_\alpha^{\mathrm{ex}}) \subset \widehat{L}(X_\alpha^{\mathrm{ex}})$. Informally, the set $\widehat{L}(X_\alpha^{\mathrm{ex}})$ is obtained from $L(X_\alpha^{\mathrm{ex}})$ by "filling the faces", i.e. inside each face of $L(X_\alpha^{\mathrm{ex}})$, we join by a chord each vertex of this face to the closest one to the complex number 1 in $\mathbb{S}^1$ in clockwise order (and then take the closure of this set), thus making it a triangulation.

Proof. Kortchemski [71, Proposition 2.9] proved that $L(X_\alpha^{\mathrm{ex}})$ is a lamination; we follow the same idea to prove that $\widehat{L}(X_\alpha^{\mathrm{ex}})$ is a lamination as well. First note that the chords in (7.4) are non-crossing: there exists no 4-tuple $0 \leq s < s' < t < t' \leq 1$ such that $s \cong^{X_\alpha^{\mathrm{ex}}} t$ and $s' \cong^{X_\alpha^{\mathrm{ex}}} t'$. Indeed, suppose that there exist such pairs, then $X_\alpha^{\mathrm{ex}}(s-) \leq X_\alpha^{\mathrm{ex}}(t) = \inf_{[s,t]} X_\alpha^{\mathrm{ex}}$ and $X_\alpha^{\mathrm{ex}}(s'-) \leq X_\alpha^{\mathrm{ex}}(t') = \inf_{[s',t']} X_\alpha^{\mathrm{ex}}$; in particular, by (H3),

$$X_\alpha^{\mathrm{ex}}(s') = X_\alpha^{\mathrm{ex}}(t) = X_\alpha^{\mathrm{ex}}(t') = \inf_{[s,t']} X_\alpha^{\mathrm{ex}} \qquad \text{and} \qquad \Delta X_\alpha^{\mathrm{ex}}(s') = \Delta X_\alpha^{\mathrm{ex}}(t) = 0.$$

Therefore X_α^{ex} attains the same local minimum at s' and t, which contradicts (H1).

To show that $\widehat{L}(X_\alpha^{\mathrm{ex}})$ is closed, it is enough to consider two sequences $(s_n)_{n\geq1}$ and $(t_n)_{n\geq1}$ such that $0 \leq s_n < t_n \leq 1$, $s_n \cong^{X_\alpha^{\mathrm{ex}}} t_n$ and (s_n, t_n) converges to some (s,t) with $s < t$, and to verify that $s \cong^{X_\alpha^{\mathrm{ex}}} t$. First, for every $r \in (s_n, t_n)$, we have $X_\alpha^{\mathrm{ex}}(r) \geq X_\alpha^{\mathrm{ex}}(t_n)$ so, letting $n \to \infty$, $X_\alpha^{\mathrm{ex}}(r) \geq X_\alpha^{\mathrm{ex}}(t-)$ for every $r \in (s,t)$. Together with (H3), this implies that $\Delta X_\alpha^{\mathrm{ex}}(t) = 0$ so as $n \to \infty$, $X_\alpha^{\mathrm{ex}}(t_n) \to X_\alpha^{\mathrm{ex}}(t) = X_\alpha^{\mathrm{ex}}(t-) \leq X_\alpha^{\mathrm{ex}}(r)$ for every $r \in (s,t)$. It only remains to show that $X_\alpha^{\mathrm{ex}}(s-) \leq X_\alpha^{\mathrm{ex}}(t)$. If $\Delta X_\alpha^{\mathrm{ex}}(s) = 0$, then $X_\alpha^{\mathrm{ex}}(s_n)$ converges to $X_\alpha^{\mathrm{ex}}(s)$ as $n \to \infty$ and the claim follows since $X_\alpha^{\mathrm{ex}}(s_n-) \leq X_\alpha^{\mathrm{ex}}(t_n)$ for every n. Suppose next that $\Delta X_\alpha^{\mathrm{ex}}(s) > 0$ and so $s > 0$. By (H2) and right-continuity, there exists $0 < \delta < (s+t)/2$ such that

$$\inf_{[s,s+\delta)} X_\alpha^{\mathrm{ex}} > \inf_{[s+\delta,(s+t)/2]} X_\alpha^{\mathrm{ex}}.$$

Moreover, by (H3), the infimum on the right-hand side is achieved at some point, say, r_0, of the interval $[s+\delta, (s+t)/2]$. In particular, if $s < s_n$ for infinitely many n, then we can find infinitely many values of n for which $s < s_n < s+\delta \leq r_0 < t_n$, which implies $X_\alpha^{\mathrm{ex}}(r_0) < X_\alpha^{\mathrm{ex}}(s_n-)$, which in turn contradicts $s_n \cong^{X_\alpha^{\mathrm{ex}}} t_n$. We conclude that $s_n \leq s$ for every n sufficiently large and thus $X_\alpha^{\mathrm{ex}}(s_n-) \to X_\alpha^{\mathrm{ex}}(s-)$ as $n \to \infty$. Again, the claim follows from $X_\alpha^{\mathrm{ex}}(s_n-) \leq X_\alpha^{\mathrm{ex}}(t_n)$ for every n.

Finally, we show that $\widehat{L}(X_\alpha^{\mathrm{ex}})$ is a maximal lamination. We argue by contradiction that for every $x, y \in \mathbb{S}^1$ with $x \neq y$, the open chord $(x,y) := [x,y] \setminus \{x,y\}$ must intersect $\widehat{L}(X_\alpha^{\mathrm{ex}})$, otherwise $\widehat{L}(X_\alpha^{\mathrm{ex}}) \cup [x,y]$ would be a bigger lamination. Fix $0 \leq u < v \leq 1$ and suppose that $(\mathrm{e}^{-2\mathrm{i}\pi u}, \mathrm{e}^{-2\mathrm{i}\pi v}) \cap \widehat{L}(X_\alpha^{\mathrm{ex}}) = \varnothing$. Suppose first that $X_\alpha^{\mathrm{ex}}(u-) > X_\alpha^{\mathrm{ex}}(v)$; set $z =$

$(X_\alpha^{\mathrm{ex}}(u-)+X_\alpha^{\mathrm{ex}}(v))/2$ and define $s=\sup\{r<u : X_\alpha^{\mathrm{ex}}(r)\leq z\}$ and $t=\inf\{r>u : X_\alpha^{\mathrm{ex}}(r)\leq z\}$. Then $s<u<t<v$ and $s \cong^{X_\alpha^{\mathrm{ex}}} t$ which is a contradiction, therefore we must have $X_\alpha^{\mathrm{ex}}(u-)\leq X_\alpha^{\mathrm{ex}}(v)$. Suppose next that there exists $r\in(u,v)$ such that $X_\alpha^{\mathrm{ex}}(r)<X_\alpha^{\mathrm{ex}}(v)$; by a similar argument, we can find $s\in(u,v)$ and $t>v$ with $s \cong^{X_\alpha^{\mathrm{ex}}} t$. Thus, we must have $X_\alpha^{\mathrm{ex}}(u-)\leq X_\alpha^{\mathrm{ex}}(v)\leq X_\alpha^{\mathrm{ex}}(r)$ for every $r\in(u,v)$, so $u \cong^{X_\alpha^{\mathrm{ex}}} v$. We see that in any case, $(\mathrm{e}^{-2\mathrm{i}\pi u},\mathrm{e}^{-2\mathrm{i}\pi v})\cap \widehat{L}(X_\alpha^{\mathrm{ex}})\neq\varnothing$. □

7.3.3 Convergence to the stable triangulation

Using the convergence (7.2), we now prove Theorem 7.1 in the stable case. Fix $\alpha\in(1,2)$ and μ a critical probability measure belonging to the domain of attraction of a stable law of index α. For each n, let τ_n be a Galton–Watson tree with offspring distribution μ conditioned to have n vertices. We show the convergence

$$\Gamma(\tau_n) \quad \xrightarrow[n\to\infty]{(\mathrm{d})} \quad L(X_\alpha^{\mathrm{ex}})$$

for the Hausdorff distance on the space of all compact subsets of $\overline{\mathbb{D}}$.

Proof of Theorem 7.1 for $\alpha\in(1,2)$**.** Following the proof of the case $\alpha=2$, we assume using Skorokhod's representation theorem that (7.2) holds almost surely and we implicitly work with ω fixed in the probability space such that this convergence (as well as the properties (H1) to (H4)) holds for ω. We set

$$X^{(n)}(s)=B_n^{-1}W^{(n)}_{\lfloor ns\rfloor} \qquad \text{for every} \quad s\in[0,1].$$

Along a subsequence (which depends on ω), we have the convergence of $\Gamma(\tau_n)$ to a limit C_∞ which is a lamination. It only remains to prove the identity $C_\infty=\widehat{L}(X_\alpha^{\mathrm{ex}})$. Since the latter is maximal, we only need to prove that $C_\infty\subset\widehat{L}(X_\alpha^{\mathrm{ex}})$. The arguments are close to that of Kortchemski [71, Sections 2 and 3]; we give the main ideas and leave some details to the reader.

As for the proof of Lemma 6.21, we start by writing $\widehat{L}(X_\alpha^{\mathrm{ex}})$ as the closure of a union of certain chords. Indeed, for every $0\leq s<t\leq 1$, $s \cong^{X_\alpha^{\mathrm{ex}}} t$ when $X_\alpha^{\mathrm{ex}}(s-)\leq X_\alpha^{\mathrm{ex}}(t)$ and $X_\alpha^{\mathrm{ex}}(t)=\inf_{[s,t]}X_\alpha^{\mathrm{ex}}$ so if and only if one the following cases holds:

(i) $\Delta X_\alpha^{\mathrm{ex}}(s)>0$ and $X_\alpha^{\mathrm{ex}}(s-)\leq X_\alpha^{\mathrm{ex}}(t)<X_\alpha^{\mathrm{ex}}(r)$ for every $r\in(s,t)$, we write $(s,t)\in\widehat{\mathcal{E}}_1(X_\alpha^{\mathrm{ex}})$;

(ii) $\Delta X_\alpha^{\mathrm{ex}}(s)>0$, $X_\alpha^{\mathrm{ex}}(s-)\leq X_\alpha^{\mathrm{ex}}(t)=\inf_{[s,t]}X_\alpha^{\mathrm{ex}}$ and there exists $r\in(s,t)$ such that $X_\alpha^{\mathrm{ex}}(r)=X_\alpha^{\mathrm{ex}}(t)$, we write $(s,t)\in\widehat{\mathcal{E}}_2(X_\alpha^{\mathrm{ex}})$;

(iii) $\Delta X_\alpha^{\mathrm{ex}}(s) = 0$ and $X_\alpha^{\mathrm{ex}}(s) = X_\alpha^{\mathrm{ex}}(t) = \inf_{[s,t]} X_\alpha^{\mathrm{ex}}$, we write $(s,t) \in \widehat{\mathcal{E}}_3(X_\alpha^{\mathrm{ex}})$.

Then, almost surely for any pair $(s,t) \in \widehat{\mathcal{E}}_2(X_\alpha^{\mathrm{ex}})$, the element $r \in (s,t)$ such that $X_\alpha^{\mathrm{ex}}(r) = X_\alpha^{\mathrm{ex}}(t)$ is a time of a local minimum of X_α^{ex} by (H3), so it is unique by (H1) and t is not a time of a local minimum of X_α^{ex}; moreover, by (H4), we have $X_\alpha^{\mathrm{ex}}(t) = X_\alpha^{\mathrm{ex}}(r) > X_\alpha^{\mathrm{ex}}(s-)$; finally $\Delta X_\alpha^{\mathrm{ex}}(t) = 0$ by (H3). We conclude that for any $\varepsilon > 0$, there exists $u \in (t, t+\varepsilon)$ such that $X_\alpha^{\mathrm{ex}}(s-) < X_\alpha^{\mathrm{ex}}(u) < X_\alpha^{\mathrm{ex}}(t) = \inf_{[s,t]} X_\alpha^{\mathrm{ex}}$ and so there exists $t' \in (t, t+\varepsilon)$ such that $(s,t') \in \widehat{\mathcal{E}}_1(X_\alpha^{\mathrm{ex}})$.

Similarly, almost surely for any pair $(s,t) \in \widehat{\mathcal{E}}_3(X_\alpha^{\mathrm{ex}})$, we must have $X_\alpha^{\mathrm{ex}}(s) = X_\alpha^{\mathrm{ex}}(t) < X_\alpha^{\mathrm{ex}}(r)$ for every $r \in (s,t)$ otherwise we have a contradiction either with (H1) if s is a time of local minimum of X_α^{ex} or with (H4) if it is not. But then $s \simeq^{X_\alpha^{\mathrm{ex}}} t$ and it follows from [71, Proposition 2.14] that there exist $s' \in [s, s+\varepsilon]$ and $t' \in [t-\varepsilon, t]$ such that $\Delta X_\alpha^{\mathrm{ex}}(s') > 0$ and $t' = \inf\{u \geq s' : X_\alpha^{\mathrm{ex}}(u) = X_\alpha^{\mathrm{ex}}(s'-)\}$ and so $(s',t') \in \widehat{\mathcal{E}}_1(X_\alpha^{\mathrm{ex}})$.

We conclude as in [71, Proposition 2.15] that almost surely

$$\widehat{L}(X_\alpha^{\mathrm{ex}}) = \overline{\bigcup_{(s,t) \in \widehat{\mathcal{E}}_1(X_\alpha^{\mathrm{ex}})} [\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}]}, \tag{7.6}$$

and, as for Lemma 6.21, the proof reduces to showing that, for any $0 \leq s < t \leq 1$ fixed such that $\Delta X_\alpha^{\mathrm{ex}}(s) > 0$ and $X_\alpha^{\mathrm{ex}}(s-) \leq X_\alpha^{\mathrm{ex}}(t) < X_\alpha^{\mathrm{ex}}(r)$ for every $r \in (s,t)$, we can find two sequences $s_n \to s$ and $t_n \to t$ as $n \to \infty$ such that for every n large enough, $[\mathrm{e}^{-2\mathrm{i}\pi s_n}, \mathrm{e}^{-2\mathrm{i}\pi t_n}] \subset \Gamma(\tau_n)$. Informally, with the notation used in (7.1), we shall take the pair (s_n, t_n) of the form $n^{-1}(j, s_l^j)$ for a certain j with $k_j \geq 1$ and a certain $l \leq k_j$.

Consider first the case $X_\alpha^{\mathrm{ex}}(s-) = X_\alpha^{\mathrm{ex}}(t)$. Then $(s,t) \in \mathcal{E}_1(X_\alpha^{\mathrm{ex}})$ and we have seen in the proof of Lemma 6.21 that for any $\varepsilon > 0$ we can find a sequence $(s_n)_{n \geq 1}$ such that for every n sufficiently large, we have

$$s_n \in (s-\varepsilon, s+\varepsilon) \cap n^{-1}\mathbb{N} \qquad \text{and} \qquad \inf_{[t-\varepsilon, t+\varepsilon]} X^{(n)} < X^{(n)}(s_n-) < \inf_{[s_n, t-\varepsilon]} X^{(n)}.$$

Then setting $t_n \coloneqq \inf\{r \geq s_n : X^{(n)}(r) = X^{(n)}(s_n-)\}$, we have $t_n \in (t-\varepsilon, t+\varepsilon) \cap n^{-1}\mathbb{N}$ and

$$nt_n = \inf\{l \geq ns_n : W_l^{(n)} = W_{ns_n}^{(n)} - (W_{ns_n}^{(n)} - W_{ns_n-1}^{(n)})\}.$$

Informally, in view of (7.1), nt_n is the index of the last child of $ns_n - 1$. Using Proposition 7.3, we thus get

$$\left[\exp\left(-2\mathrm{i}\pi \frac{ns_n - 1}{n}\right), \exp\left(-2\mathrm{i}\pi \frac{nt_n}{n}\right)\right] \subset \Gamma(\tau_n)$$

for every n large enough.

Suppose now that $X_\alpha^{\mathrm{ex}}(s-) < X_\alpha^{\mathrm{ex}}(t) < X_\alpha^{\mathrm{ex}}(r)$ for every $r \in (s,t)$. Then for $\varepsilon > 0$ small enough, we can find s'_n such that for every n sufficiently large, we have

$$s'_n \in (s-\varepsilon, s+\varepsilon) \cap n^{-1}\mathbb{N} \qquad \text{and} \qquad X^{(n)}(s'_n-) < \inf_{[t-\varepsilon,t+\varepsilon]} X^{(n)} < \inf_{[s'_n,t-\varepsilon]} X^{(n)}.$$

Set $z_n = (\inf_{[t-\varepsilon,t+\varepsilon]} X^{(n)} + \inf_{[s'_n,t-\varepsilon]} X^{(n)})/2$, observe that $X^{(n)}(s'_n-) < z_n < X^{(n)}(s'_n)$ and then set $t'_n \coloneqq \inf\{r \geq s'_n : X^{(n)}(r) \leq z_n\}$. We have $t'_n \in (t-\varepsilon, t+\varepsilon) \cap n^{-1}\mathbb{N}$ and

$$nt'_n = \inf\{l \geq ns'_n : W_l^{(n)} = W_{ns'_n}^{(n)} - (W_{ns'_n}^{(n)} - B_n z_n)\},$$

with $W_{ns'_n}^{(n)} - B_n z_n \in \{1, \dots, W_{ns'_n}^{(n)} - W_{ns'_n-1}^{(n)} - 1\}$. Informally, in view of (7.1), nt'_n is the index of a child of $ns'_n - 1$ (and neither its first or its last). We conclude by Proposition 7.3 that

$$\left[\exp\left(-2\mathrm{i}\pi\frac{ns'_n-1}{n}\right), \exp\left(-2\mathrm{i}\pi\frac{nt'_n}{n}\right)\right] \subset \Gamma(\tau_n)$$

for every n large enough and the proof is now complete. □

7.4 Hausdorff dimension

We denote by $\dim(K)$ the Hausdorff dimension of a subset K of $\mathbb{C}$, see e.g. Mattila [83] for background. Fix $\alpha \in (1,2)$ and consider X_α^{ex} the normalized excursion of the α-stable Lévy process; Kortchemski [71, Theorem 5.1] proves the following result.

Theorem 7.6 ([71])**.** *Let $A(X_\alpha^{\mathrm{ex}})$ be the set of all end-points of chords in $L(X_\alpha^{\mathrm{ex}})$. Almost surely*

$$\dim(A(X_\alpha^{\mathrm{ex}})) = 1 - \frac{1}{\alpha} \qquad \textit{and} \qquad \dim(L(X_\alpha^{\mathrm{ex}})) = 2 - \frac{1}{\alpha}, \tag{7.7}$$

and for every face V of $L(X_\alpha^{\mathrm{ex}})$,

$$\dim(\overline{V} \cap \mathbb{S}^1) = \frac{1}{\alpha}.$$

Recall that for $\alpha = 2$, $X_2^{\mathrm{ex}} = e$ is $\sqrt{2}$ times the standard Brownian excursion; (7.7) extends verbatim to the Brownian triangulation $\mathbf{L}(e)$, see Aldous [6] as well as Le Gall & Paulin [77] for a detailed proof. The aim of this section is to extend Theorem 7.6 to the stable triangulation.

Theorem 7.7. *Let $\widehat{A}(X_\alpha^{\mathrm{ex}})$ be the set of all end-points of chords in $\widehat{L}(X_\alpha^{\mathrm{ex}})$. Almost surely*

$$\dim(\widehat{A}(X_\alpha^{\mathrm{ex}})) = \frac{1}{\alpha} \qquad \textit{and} \qquad \dim(\widehat{L}(X_\alpha^{\mathrm{ex}})) = 1 + \frac{1}{\alpha}. \tag{7.8}$$

Remark 7.8. We see that the dimensions of the sets in (7.7) and (7.8) have the same limit as $\alpha \uparrow 2$ and indeed, for $\alpha = 2$, the stable lamination and triangulation coincide (both with the Brownian triangulation). On the other hand, we also see that

$$(\dim(L(X_\alpha^{\mathrm{ex}})), \dim(\widehat{L}(X_\alpha^{\mathrm{ex}}))) \to (1, 2) \quad \text{as} \quad \alpha \downarrow 1.$$

Informally, as $\alpha \downarrow 1$, the process X_α^{ex} converges towards the deterministic function $f : [0,1] \to \mathbb{R}$ defined by $f(0) = 0$ and $f(x) = 1 - x$ for every $x \in (0,1]$. Observe that f is not càdlàg, we refer to Curien & Kortchemski [32, Theorem 3.6] for a precise statement and proof (essentially, we have the convergence in $\mathcal{D}([0,1], \mathbb{R})$ of the time-reversed processes to the time-reversal of f which is càdlàg). If we try then to define $L(f)$ and $\widehat{L}(f)$ mimicking (7.4) and (7.5), we obtain $L(f) = \mathbb{S}^1$ and $\widehat{L}(f) = \overline{\mathbb{D}}$.

We recall from [71, Proposition 3.10] that the faces of $L(X_\alpha^{\mathrm{ex}})$ are in one-to-one correspondence with the jump times of X_α^{ex} (observe that the latter set is countable since X_α^{ex} is càdlàg). For every $s, t \in (0,1)$, let $\mathbb{H}(s,t)$ be the open half-plane bounded by the line containing $\mathrm{e}^{-2\mathrm{i}\pi s}$ and $\mathrm{e}^{-2\mathrm{i}\pi t}$, which does not contain the complex number 1. Then for every jump time s of X_α^{ex}, letting $t = \inf\{u > s : X_\alpha^{\mathrm{ex}}(u) = X_\alpha^{\mathrm{ex}}(s-)\}$, the face V of $L(X_\alpha^{\mathrm{ex}})$ associated with s is the unique one contained in $\mathbb{H}(s,t)$ whose boundary contains the chord $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}]$. Moreover, one has

$$\overline{V} \cap \mathbb{S}^1 = \{r \in [s,t] : X_\alpha^{\mathrm{ex}}(r) = \inf_{[s,r]} X_\alpha^{\mathrm{ex}}\}, \tag{7.9}$$

where we identify the interval $[0,1)$ with the circle $\mathbb{S}^1$ via the mapping $t \mapsto \mathrm{e}^{-2\mathrm{i}\pi t}$ to ease notation.

Proof of Theorem 7.7. Fix a face V of $L(X_\alpha^{\mathrm{ex}})$, let s be the jump-time of X_α^{ex} associated with V and $t = \inf\{u > s : X_\alpha^{\mathrm{ex}}(u) = X_\alpha^{\mathrm{ex}}(s-)\}$. Observe from (H2), (H3) and (H4) that all the times $r \in [s,1]$ such that $r \overset{X_\alpha^{\mathrm{ex}}}{\simeq} s$ belong to the interval $[s,t]$. Then (7.9) reads $\overline{V} \cap \mathbb{S}^1 = \{r \in [s,1] : s \overset{X_\alpha^{\mathrm{ex}}}{\simeq} r\}$. Notice also that all the chords of $\widehat{L}(X_\alpha^{\mathrm{ex}})$ who lie in $\overline{V}$ are either of the form $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi r}]$ for $r \in \overline{V} \cap \mathbb{S}^1$, or belong to the boundary ∂V, and so to $L(X_\alpha^{\mathrm{ex}})$. Therefore, if we denote by $\widehat{L}_V(X_\alpha^{\mathrm{ex}})$ the triangulation $\widehat{L}(X_\alpha^{\mathrm{ex}})$ restricted to $\overline{V}$ and by $\widehat{A}_V(X_\alpha^{\mathrm{ex}})$ the set of all end-points of the latter, we have

$$\widehat{A}_V(X_\alpha^{\mathrm{ex}}) = \overline{V} \cap \mathbb{S}^1.$$

Since $\widehat{A}(X_\alpha^{\mathrm{ex}}) = \bigcup_V \widehat{A}_V(X_\alpha^{\mathrm{ex}})$, where the union runs over the countable set of faces of $L(X_\alpha^{\mathrm{ex}})$, we conclude from Theorem 7.6 that

$$\dim(\widehat{A}(X_\alpha^{\mathrm{ex}})) = \sup\{\dim(\widehat{A}_V(X_\alpha^{\mathrm{ex}})); V \text{ face of } L(X_\alpha^{\mathrm{ex}})\} = \dim(\overline{V} \cap \mathbb{S}^1) = \frac{1}{\alpha}.$$

Similarly, we have

$$\dim(\widehat{L}(X_\alpha^{\text{ex}})) = \sup\{\dim(\widehat{L}_V(X_\alpha^{\text{ex}})); V \text{ face of } L(X_\alpha^{\text{ex}})\}$$

so it only remains to show that for any given face V of $L(X_\alpha^{\text{ex}})$, we have

$$\dim(\widehat{L}_V(X_\alpha^{\text{ex}})) = 1 + \dim(\widehat{A}_V(X_\alpha^{\text{ex}})) = 1 + \frac{1}{\alpha}.$$

We prove the two inequalities by adapting the argument of Le Gall & Paulin [77, Proposition 2.3]. It is actually sufficient to only consider the subset $\widetilde{L}_V(X_\alpha^{\text{ex}}) \subset \widehat{L}_V(X_\alpha^{\text{ex}})$ which is the union of the chords $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi r}]$ for $r \in \overline{V} \cap \mathbb{S}^1 = \widehat{A}_V(X_\alpha^{\text{ex}})$. Indeed as we remarked previously, the remaining chords belong to $L(X_\alpha^{\text{ex}})$ which, by (7.7), has Hausdorff dimension $2 - \frac{1}{\alpha} < 1 + \frac{1}{\alpha}$ for every $\alpha \in (1, 2)$.

We first show that $\dim(\widetilde{L}_V(X_\alpha^{\text{ex}})) \geq 1 + \dim(\widehat{A}_V(X_\alpha^{\text{ex}}))$. Fix $0 < \gamma < \dim(\widehat{A}_V(X_\alpha^{\text{ex}}))$; thanks to Frostman's lemma [83, Theorem 8.8], there exists a non-trivial finite Borel measure ν supported on $\widehat{A}_V(X_\alpha^{\text{ex}})$ such that

$$\nu(B(x, r)) \leq r^\gamma$$

for every $x \in \mathbb{C}$ and every $r > 0$, where $B(x, r)$ is the Euclidean ball centered at x and of radius r. Next, for every $x \in \widehat{A}_V(X_\alpha^{\text{ex}})$, denote by λ_x the one-dimensional Hausdorff measure on the chord joining x to $\mathrm{e}^{-2\mathrm{i}\pi s}$. We define a finite Borel measure Λ on $\mathbb{C}$, supported by $\widetilde{L}_V(X_\alpha^{\text{ex}})$, by setting for every Borel set B

$$\Lambda(B) = \int \nu(\mathrm{d}x)\lambda_x(B).$$

Fix $0 < R < 1$ such that $\Lambda(B(0, R)) > 0$; let $z_0 \in B(0, R) \cap \widetilde{L}_V(X_\alpha^{\text{ex}})$ and then $x_0 \in \widehat{A}_V(X_\alpha^{\text{ex}})$ such that the chord $[x_0, \mathrm{e}^{-2\mathrm{i}\pi s}]$ contains z_0. Fix $\varepsilon \in (0, 1]$; every $x \in \widehat{A}_V(X_\alpha^{\text{ex}})$ such that the chord $[x, \mathrm{e}^{-2\mathrm{i}\pi s}]$ intersects the ball $B(z_0, \varepsilon)$ must satisfy $|x - x_0| \leq C\varepsilon$, where the constant C only depends on R. We conclude that

$$\Lambda(B(z_0, \varepsilon)) = \int_{|x - x_0| \leq C\varepsilon} \nu(\mathrm{d}x)\lambda_x(B(z_0, \varepsilon)) \leq C'\varepsilon^{1+\gamma},$$

where the constant C' does not depend on ε nor z_0. Appealing again to Frostman's lemma, we obtain $\dim(\widetilde{L}_V(X_\alpha^{\text{ex}})) \geq 1 + \gamma$, whence, as $\gamma < \dim(\widehat{A}_V(X_\alpha^{\text{ex}}))$ is arbitrary, $\dim(\widetilde{L}_V(X_\alpha^{\text{ex}})) \geq 1 + \dim(\widehat{A}_V(X_\alpha^{\text{ex}}))$.

It remains to show the converse inequality. We denote respectively by $\underline{\dim}_M(K)$ and $\overline{\dim}_M(K)$ the lower and upper Minkowski dimensions of a subset K of $\mathbb{C}$ (see e.g. Mattila [83, Chapter 5]); recall that for every $K \subset \overline{\mathbb{D}}$, we have $\dim(K) \leq \underline{\dim}_M(K) \leq \overline{\dim}_M(K) \leq 2$.

Observe from the proof of Theorem 5.1 of Kortchemski [71] (in particular, Proposition 5.3 there) that we have $\dim(\widehat{A}_V(X_\alpha^{\mathrm{ex}})) = \overline{\dim}_M(\widehat{A}_V(X_\alpha^{\mathrm{ex}}))$. Fix $\beta > \dim(\widehat{A}_V(X_\alpha^{\mathrm{ex}})) = \underline{\dim}_M(\widehat{A}_V(X_\alpha^{\mathrm{ex}}))$; then there exists a sequence $(\varepsilon_k; k \ge 1)$ decreasing to 0 such that for every $k \ge 1$, there exists a positive integer $M(\varepsilon_k) \le \varepsilon_k^{-\beta}$ and $M(\varepsilon_k)$ disjoint subarcs of $\mathbb{S}^1$ with length less than ε_k and which cover $\widehat{A}_V(X_\alpha^{\mathrm{ex}})$. It follows that the two-dimensional Lebesgue measure of the ε_k-enlargement of $\widetilde{L}_V(X_\alpha^{\mathrm{ex}})$ is bounded above by $C\varepsilon_k^{1-\beta}$, where the constant C does not depend on k. We conclude from [83, page 79] that $\dim(\widetilde{L}_V(X_\alpha^{\mathrm{ex}})) \le \overline{\dim}_M(\widetilde{L}_V(X_\alpha^{\mathrm{ex}})) \le 1+\beta$ for every $\beta > \dim(\widehat{A}_V(X_\alpha^{\mathrm{ex}}))$, which completes the proof. □

7.5 Stable triangulation and height process

We end our study of stable laminations and triangulations by an alternative construction of these objects, presented in Section 3.3. Recall indeed that from any continuous function $g : [0,1] \to [0,\infty)$ such that $g(0) = g(1)$, we can construct a lamination $\mathbf{L}(g)$ and a triangulation $\widehat{\mathbf{L}}(g)$ as follows. For every $s \in [0,1]$, let $\mathrm{cl}_g(s)$ be the equivalence class of s for the relation $s \overset{g}{\sim} t$ if and only if $g(s) = g(t) = \min_{[s\wedge t, s\vee t]} g$. Then we define two relations (not equivalence relations in general) on $[0,1]$ by

(i) $s \approx^g t$ when $s \overset{g}{\sim} t$ and at least one the following conditions holds: $g(r) > g(s)$ for every $r \in (s\wedge t, s\vee t)$, or $(s\wedge t, s\vee t) = (\min \mathrm{cl}_g(s), \max \mathrm{cl}_g(s))$;

(ii) $s \approxeq^g t$ when $s \overset{g}{\sim} t$ and at least one the following conditions holds: $g(r) > g(s)$ for every $r \in (s\wedge t, s\vee t)$, or $s\wedge t = \min \mathrm{cl}_g(s)$.

We have claimed in Proposition 3.3 that the sets

$$\mathbf{L}(g) \coloneqq \bigcup_{s \approx^g t} \left[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}\right] \qquad \text{and} \qquad \widehat{\mathbf{L}}(g) \coloneqq \bigcup_{s \approxeq^g t} \left[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}\right] \tag{7.10}$$

are geodesic laminations of $\overline{\mathbb{D}}$ and, further, that $\widehat{\mathbf{L}}(g)$ is maximal for the inclusion relation.

Recall next the normalized excursion H_α^{ex} of the height process associated with X_α, which is, according to Theorem 3.1 the scaling limit of the height process of large Galton–Watson trees with critical offspring distribution on the domain of attraction of a stable law with index α. The process H_α^{ex} has continuous paths and satisfies $H_\alpha^{\mathrm{ex}}(s) \ge 0$ for every $s \in [0,1]$ and $H_\alpha^{\mathrm{ex}}(0) = H_\alpha^{\mathrm{ex}}(1) = 0$ so we can define $\mathbf{L}(H_\alpha^{\mathrm{ex}})$ and $\widehat{\mathbf{L}}(H_\alpha^{\mathrm{ex}})$ by (7.10).

Theorem 7.9. *For every $\alpha \in (1,2)$, almost surely, the relation $\approx^{H_\alpha^{\mathrm{ex}}}$ coincides with $\simeq^{X_\alpha^{\mathrm{ex}}}$ and the relation $\approxeq^{H_\alpha^{\mathrm{ex}}}$ coincides with $\cong^{X_\alpha^{\mathrm{ex}}}$. In particular,*

$$\mathbf{L}(H_\alpha^{\mathrm{ex}}) = L(X_\alpha^{\mathrm{ex}}) \qquad \textit{and} \qquad \widehat{\mathbf{L}}(H_\alpha^{\mathrm{ex}}) = \widehat{L}(X_\alpha^{\mathrm{ex}}) \qquad \textit{a.s.}$$

We shall need the following relations between H_α^{ex} and X_α^{ex}; we refer to Kortchemski [71, Section 4] for their proof.

Lemma 7.10 ([71]). *For every $\alpha \in (1,2)$, almost surely, the following holds:*

(i) If $s \in (0,1)$ is a jump time of X_α^{ex} and $t = \inf\{u > s : X_\alpha^{\mathrm{ex}}(u) = X_\alpha^{\mathrm{ex}}(s-)\} < 1$, then $H_\alpha^{\mathrm{ex}}(u) \geq H_\alpha^{\mathrm{ex}}(s)$ for every $u \in [s,t]$, with equality if and only if $X_\alpha^{\mathrm{ex}}(u) = \inf_{[s,u]} X_\alpha^{\mathrm{ex}}$. Moreover, for every $\varepsilon > 0$, $\inf_{[s-\varepsilon,s]} H_\alpha^{\mathrm{ex}} < H_\alpha^{\mathrm{ex}}(s)$ and $\inf_{[s,t+\varepsilon]} H_\alpha^{\mathrm{ex}} < H_\alpha^{\mathrm{ex}}(s)$.

(ii) For every $0 \leq s < t \leq 1$ such that $H_\alpha^{\mathrm{ex}}(s) = H_\alpha^{\mathrm{ex}}(t) < H_\alpha^{\mathrm{ex}}(r)$ for every $r \in (s,t)$ and for every $\varepsilon > 0$ sufficiently small, there exist $s' \in [s, s+\varepsilon]$ and $t' \in [t-\varepsilon, t]$ such that H_α^{ex} does not attain a local minimum neither at s' nor at t' and, further, $H_\alpha^{\mathrm{ex}}(s') = H_\alpha^{\mathrm{ex}}(t') = \inf_{[s',t']} H_\alpha^{\mathrm{ex}}$ and there exists $r \in (s',t')$ such that $H_\alpha^{\mathrm{ex}}(r) = H_\alpha^{\mathrm{ex}}(s')$.

(iii) For every $r \in [0,1]$ such that $\mathrm{Card}(\mathrm{cl}_{H_\alpha^{\mathrm{ex}}}(r)) \geq 3$, there exists a jump time s of X_α^{ex} such that $s \in \mathrm{cl}_{H_\alpha^{\mathrm{ex}}}(r)$; conversely, if s is a jump time of X_α^{ex}, then $s = \min \mathrm{cl}_{H_\alpha^{\mathrm{ex}}}(s)$ and $\inf\{u > s : X_\alpha^{\mathrm{ex}}(u) = X_\alpha^{\mathrm{ex}}(s-)\} = \max \mathrm{cl}_{H_\alpha^{\mathrm{ex}}}(s)$.

Proof of Theorem 7.9. The equivalence between $\approx^{H_\alpha^{\mathrm{ex}}}$ and $\simeq^{X_\alpha^{\mathrm{ex}}}$ is the claim of Theorem 4.5 in [71]; we prove that between $\approxeq^{H_\alpha^{\mathrm{ex}}}$ and $\cong^{X_\alpha^{\mathrm{ex}}}$. Notice first that we showed during the proof of Proposition 7.5 that the relation $\cong^{X_\alpha^{\mathrm{ex}}}$ is closed, in the sense that its graph is a closed subset of $[0,1]^2$. It is easily shown that the same holds for $\approxeq^{H_\alpha^{\mathrm{ex}}}$ and we leave the details to the reader. Moreover, with the notation used in (7.6), any pair $0 \leq s < t \leq 1$ such that $s \cong^{X_\alpha^{\mathrm{ex}}} t$ can be approximated by a sequence $(s_n, t_n) \in \widehat{\mathcal{E}}_1(X_\alpha^{\mathrm{ex}})$. Thus, to show that for every $s,t \in [0,1]$, the relation $s \cong^{X_\alpha^{\mathrm{ex}}} t$ implies $s \approxeq^{H_\alpha^{\mathrm{ex}}} t$, it is sufficient to consider $(s,t) \in \widehat{\mathcal{E}}_1(X_\alpha^{\mathrm{ex}})$.

Fix such a pair: $\Delta X_\alpha^{\mathrm{ex}}(s) > 0$ and $X_\alpha^{\mathrm{ex}}(s-) \leq X_\alpha^{\mathrm{ex}}(t) < X_\alpha^{\mathrm{ex}}(r)$ for every $r \in (s,t)$. It follows from Lemma 7.10 (i) that $H_\alpha^{\mathrm{ex}}(s) = H_\alpha^{\mathrm{ex}}(t) \leq H_\alpha^{\mathrm{ex}}(r)$ for every $r \in (s,t)$. Moreover for every $\varepsilon \in (0,s)$, $\inf_{[s-\varepsilon,s]} H_\alpha^{\mathrm{ex}} < H_\alpha^{\mathrm{ex}}(s)$ so $s = \min \mathrm{cl}_{H_\alpha^{\mathrm{ex}}}(s)$. We conclude that $s \approxeq^{H_\alpha^{\mathrm{ex}}} t$.

For the converse implication, fix $0 \leq s < t \leq 1$ such that $s \approxeq^{H_\alpha^{\mathrm{ex}}} t$, i.e. $H_\alpha^{\mathrm{ex}}(s) = H_\alpha^{\mathrm{ex}}(t) = \inf_{[s,t]} H_\alpha^{\mathrm{ex}}$ and, either $H_\alpha^{\mathrm{ex}}(r) > H_\alpha^{\mathrm{ex}}(s)$ for every $r \in (s,t)$, or $s = \min \mathrm{cl}_{H_\alpha^{\mathrm{ex}}}(s)$. If (s,t) is of the first kind: $H_\alpha^{\mathrm{ex}}(s) = H_\alpha^{\mathrm{ex}}(t)$ and $H_\alpha^{\mathrm{ex}}(r) > H_\alpha^{\mathrm{ex}}(s)$ for every $r \in (s,t)$, then, from Lemma 7.10 (ii), it can be approximated by pairs of the second kind; as before we therefore only focus on this case. Consequently, suppose that $s = \min \mathrm{cl}_{H_\alpha^{\mathrm{ex}}}(s)$ and there exists $r \in (s,t)$ such that $H_\alpha^{\mathrm{ex}}(r) = H_\alpha^{\mathrm{ex}}(s) = H_\alpha^{\mathrm{ex}}(t) = \inf_{[s,t]} H_\alpha^{\mathrm{ex}}$. From Lemma 7.10 (iii) we have that $\Delta X_\alpha^{\mathrm{ex}}(s) > 0$ and $t \leq \max \mathrm{cl}_{H_\alpha^{\mathrm{ex}}}(s) = \inf\{u > s : X_\alpha^{\mathrm{ex}}(u) = X_\alpha^{\mathrm{ex}}(s-)\}$ so, in particular, $X_\alpha^{\mathrm{ex}}(r) \geq X_\alpha^{\mathrm{ex}}(s-)$ for every $r \in [s,t]$. Finally, since $H_\alpha^{\mathrm{ex}}(t) = H_\alpha^{\mathrm{ex}}(s)$, it follows from Lemma 7.10 (i) that $X_\alpha^{\mathrm{ex}}(t) = \inf_{[s,t]} X_\alpha^{\mathrm{ex}}$. We conclude that $s \cong^{X_\alpha^{\mathrm{ex}}} t$. $\square$

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