Elliptical systems related to superconductor model

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Université Paris-Est

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THÈSE

pour obtenir le grade de

Docteur de l'Université Paris-Est

Spécialité : **Mathématiques**

préparée au laboratoire **LAMA : Laboratoire d'Analyse et de Mathématiques Appliquées**

dans le cadre de l'École Doctorale **MSTIC : Mathématiques et Sciences et Technologies de l'Information et de la Communication**

présentée et soutenue publiquement
par

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le 5 Décembre 2014

Titre:
Systèmes Elliptiques Issus de la Modélisation des Supraconducteurs

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Remerciements

Je voudrais en premier lieu remercier **Yuxin Ge** et **Étienne Sandier** pour avoir dirigé ma thèse et m'avoir soumis une série de problèmes très intéressants qui m'ont amené à apprendre de nombreuses techniques mathématiques fort utiles. Ils m'ont témoigné leur confiance et m'ont soutenu tout au long de cette thèse. Leurs remarques, toujours simples et claires, m'ont beaucoup aidé à trouver une solution aux questions posées.

Je suis de même très reconnaissant envers **Petru Mironescu** et **Robert Jerrard** pour avoir rapporté cette thèse. Je remercie également les autre membres du jury qui ont accepté avec gentillesse de juger ce travail: **Fabrice Bethuel**, **Sylvia Serfaty** et **Ahmad El Soufi**.

Sans le professeur **Xingbin Pan** de l'Université normale de la Chine de l'Est, je n'aurais pas eu l'idée de venir faire mes études en France. Qu'il soit ici tout spécialement remercié pour ce conseil et ses constants encouragements.

Durant toute ma thèse, j'ai bénéficié de conditions de travail très avantageuses au **Laboratoire d'Analyse et de Mathématiques Appliquées**. Je voudrais à cette occasion remercier tous les membres du Laboratoire et en particulier **Sébastien Cartier**, **Mickaël Dos Santos**, **Anais Delgado** et **Lingmin Liao**.

Je voudrais remercier du fond du coeur mes camarades thésards de l'UPEC et d'ailleurs, qui m'ont accompagnée pendant ces années: **Antoine**, **Cecilia**, **Cosmin** , **David**, **Eduardo**, **Francesco**, **Harry**, **Houda**, **Jean-Maxime**, **Jérémy**, **Johann**, **Khaled**, **Marwa**, **Rana**, **Salwa**,**Victor**, **Zeina**, ... Un remerciement particulier s'adresse à **Laurent** et **Rémy**, pour leur gentillesse, les discussions partagées et leur soutien et encouragements constants. Les thésards chinois en France m'ont également beaucoup aidé, aussi bien en mathématiques que pour la vie quotidienne: **Deng Wen**, **Liao Xian**, **Lu Yong**, **Xia Bo**, **Xu Haiyan**, **Xu Liping**, **Yang Xiaochuan**, **Zhang Xin**, **Zhao Lei**, ...

Mes parents, mon frère et l'ensemble de ma famille ont su montrer leur soutien constant pendant toute la durée des mes études et je leur en suis infiniment reconnaissant.

Enfin, je tiens à exprimer ma profonde affection à ma femme **Li Zhenzhen**, qui m'a toujours soutenu.

Contents

Chapter 1

Introduction

Ce travail porte sur des équations aux dérivées partielles issues de la physique mathématique, plus particulièrement sur celles régissant la supraconductivité. Ainsi, la majorité du travail concerne le modèle de Ginzburg-Landau, qui est un modèle macroscopique de supraconducteurs de type-II. Ce travail est divisé en deux parties principales:

- La première partie se focalise sur l'analyse des vortex du modèle de Ginzburg-Landau en deux dimensions pour les supraconducteurs de type-II, modèle conduisant à une estimation de la variation du nombre de vortex et à l'optimalité du réseau d'Abrikosov parmi les réseaux de Bravais. Nous avons également étudié certains modèles de stuctures des matériaux comme ceux de Lennard-Jones et de Thomas-Fermi.

- La seconde partie est consacrée à la fonctionnelle de Ginzburg-Landau en dimension n. Deux résultats principaux sont obtenus. L'un porte sur l'énergie renormalisée pour les minimiseurs de la fonctionnelle de Ginzburg-Landau. L'autre concerne les limites des solutions de l'équation de Ginzburg-Landau. Ces deux résultats sont fortement reliés aux applications n-harmoniques.

1.1 Analyse des vortex

Pour un supraconducteur de type-II refroidi en deçà de la température critique, les vortex apparaissent quand le champs magnétique extérieur est supérieur à une première valeur dite critique. Le physicien russe Abrikosov a prédit l'apparition de réseaux de vortex parfaitement triangulaires, désormais appelés réseaux d'Abrikosov,à partir du modèle de Ginzburg-Landau en 1950. Celui-ci, destiné au départ à décrire les phénomènes de supraconductivité, a conduità de nombreux travaux en physique théorique. En revanche, il n'y avait pas réellement de preuve mathématique rigoureuse pour la transition de phase se produisant à la première valeur critique ni pour l'émergence des réseaux d'Abrikosov. Depuis 1990, de nombreux mathématiciens se sont intéressés au modèle de Ginzburg-Landau, dont par exemple Berger, Baumann, Chapman, Du, Schatzman, Phillips, etc([32][49][11][21]). Parmi eux, Bethuel-Brezis-Helein[13] ont fait un travail remarquable sur le modèle de Ginzburg-Landau sans champ magnétique, sous la contrainte d'un nombre fixé de vortex dans la limite où les vortex deviennent des points. Ensuite, Bethuel et Rivière ont étudié le modèle avec jauge et une autre condition au bord ([15], [14]).

Concernant le modèle avec champ magnétique et avec un nombre de vortex devenant infini dans la limite où les vortex deviennent des points, c'est en particulier grÃćce aux outils dévelopés par Sandier[62] et Jerrard[48], que l'on commenceà pouvoir résoudre ce problème. Etienne Sandier et Sylvia Serfaty ont beaucoup écrit, individuellement ou ensemble, sur le modèle complet([63]). Notre travail sur les analyses des vortex est fortement liéà leur démarche. Nous obtenons les résultats suivants :

1.1.1 Variation du Nombre de Vortex

Dans [82], nous étudions l'évolution du nombre de vortex dans un modèle de Ginzburg-Landau périodique. Il est conjecturé que le nombre de vortex pour un minimiseur de la fonctionnelle de Ginzburg-Landau varie par pas d'une unité lorsque le champ magnétique augmente. Par contre, il y a très peu de preuves mathématiques rigoureuses de cette conjecture. Nous avons étudié ce problème dans un cas particulier. Nous montrons que pour le modèle de Ginzburg-Landau doublement périodique, quand la cellule de périodicité dégénère en un segment, le nombre de vortex pour un minimiseur de la fonctionnelle de Ginzburg-Landau augmente un par un en fonction du champ magnétique appliqué. Ce travail s'appuie sur [9]. Nous réduisons le problème à celui de l'étude des minimiseurs de l'énergie de Ginzburg-Landau renormalisée. Utilisant la Γ-convergence, nous parvenons à connecter le modèle en deux dimensions avec un modèle en une dimension. Nous montrons que pour l'énergie renormalisée en une dimension, le nombre de vortex du minimiseur augmente un par un en fonction du champ magnétique appliqué. Autrement dit, l'énergie renormalisée en deux dimensions et le modèle de Ginzburg-Landau périodique ont pour limite une énergie renormalisée en une dimension quand la hauteur de la cellule tend vers 0.

1.1.2 Optimalité des Réseaux d'Abrikosov

Nous avons plusieurs résultats pour ce problème. Dans [81], nous montrons que le réseau d'Abrikosov, modulo les rotations, est un minimiseur unique pour l'énergie renormalisée de Ginzburg-Landau parmi tous les réseaux de Bravais à densité fixée. Ceci décrit un supraconducteur dans un champ extérieur égal à $H_{c_1} + C$, pour lequel le réseau de vortex est dilué: dans ce cas les vortex interagissent par un potentiel de Bessel au lieu du potentiel log. Adaptant les méthodes de [64], nous pouvons réécrire l'énergie renormalisée grâce à une formule explicite utilisant les fonctions θ de Jacobi. Ensuite, le résultat de Montgomery [54] sur les fonctions θ de Jacobi peut alors être appliqué pour obtenir notre résultat. En collaboration avec Laurent Bétermin, nous étudions l'optimalité des réseaux d'Abrikosov dans les modèles de Lennard-Jones et de Thomas-Fermi. Nous montrons dans [12] que le minimiseur de l'énergie par particule pour l'interaction de Lennard-Jones parmi les réseaux de Bravais est le réseau hexagonal pour de forte densité de particules, mais que cela est faux pour une densité suffisamment faible. Nous montrons également des résultats sur le minimiseur sans contrainte sur la densité. Dans cet article, nous prouvons également que le minimiseur de l'énergie par particule dans le modèle de Thomas-Fermi dans le plan parmi les réseaux de Bravais avec densité fixée est aussi le réseau hexagonal, et ceci fournit une autre preuve de l'optimalité des réseaux d'Abrikosov parmi les réseaux de Bravais dans le modèle de Ginzburg-Landau.

1.2 Fonctionnelle de type Ginzburg-Landau en dimensions supérieures

Il y a beaucoup de travaux de recherche concernant le système de Ginzburg-Landau en deux dimensions. Dans cette partie, nous étudions une fonctionnelle de type Ginzburg-Landau en n dimensions.

Dans [13], Bethuel-Brézis-Hélein, en dimension deux, définissent une énergie renormalisée $\mathbf{W}$ pour des applications harmoniques à valeurs dans $\mathbb{S}^1$ ayant un nombre fini de singularités, et

1. Donnent une formule explicite de $\mathbf{W}$;
2. Montrent que l'énergie de minimiseurs de l'énergie de Ginzburg-Landau a un développement asymptotique

$$\mathbf{E}_\varepsilon(u_\varepsilon) = \pi d|\ln \varepsilon| + \mathbf{W}(a_1, \cdots, a_d) + O_\varepsilon(1) \qquad (*)$$

   où $\mathbf{W}$ est l'énergie renormalisée citée précédemment.

Pour la dimension n, l'asymptotique des minimiseurs de l'énergie de Ginzburg-Landau est étudiée par Han and Li [42] qui démontrent que, comme en 2D les minimiseurs convergent en dehors d'un nombre fini de points $a_1, \cdots, a_d$ vers une application $n-$ harmonique ayant une singularité de degré 1 en chaque point.

Par ailleurs, Hardt-Lin-Wang[45] définissent une énergie renormalisée pour les applications $n-$ harmoniques avec un nombre fini de singularités de degré 1, de $\Omega \subset \mathbb{R}^n$ à valeurs dans $\mathbb{S}^{n-1}$. Ils démontrent également que les applications $p-$ harmoniques minimisantes convergent quand $p \nearrow n$ vers une telle application $n-$ harmonique qui minimise l'énergie renormalisée.

En collaboration avec Y.X. Ge et E. Sandier [40], nous montrons des résultats suivants:

1. L'application $n-$ harmonique limite des minimiseurs de Ginzburg-Landau de Han-Li [42] minimise l'énergie renormalisée.
2. Nous avons l'équivalent du développement asymptotique $(*)$ pour les minimiseurs de Ginzburg-Landau en dimension n.
3. Nous étudions la limite de points critiques non nécessairement minimisants pour l'énergie de Ginzburg-Landau en dimension n, et montrons un équivalent de la "vanishing gradient property" de Bethuel-Brézis-Hélein [13] dans ce cadre. Contrairement au cas bidimensionnel, ou au cas des minimiseurs en dimension n, des singularités d'énergie finie ne peuvent pas être exclues a priori dans notre étude.
4. Nous montrons également l'existence de telles suites de points critiques non minimisants pour l'énergie de Ginzburg-Landau en dimension trois.

Chapter 2

Variations of the Vortex Number

In this chapter, we study the variations of the number of vortices contained in the minimizer of a two-dimensional Ginzburg-Landau functional describing a Type-II superconductor in the London limit, with periodic conditions on the boundary of the sample. We prove that, under the assumption that the sample is rectangular of area 1 with height far smaller than its length, the number of vortices contained in the minimizer of the periodic Ginzburg-Landau functional jumps by unit step as the applied magnetic field increases. We convert the problem into the study of corresponding renormalized energy. By using the Γ-convergence we reduce the two dimensional renormalized energy to one dimensional one. Then the result is obtained by analyzing the one dimensional model.

2.1 Introduction

In this chapter, we focus on the two-dimensional periodic Ginzburg-Landau model for type-II superconductivity. We study the variations of the vortex number contained in the minimizer of a Ginzburg-Landau energy with periodic boundary conditions in the London limit. We firstly study the related renormalized energy W, and get that if the height of the lattice is small enough, then we get the variations of the number of points contained in the minimizer of W. By using Theorem 2.1.2 of [9], we have the same result about the Ginzburg-Landau energy.

2.1.1 Background

Since the discovery of superconductivity in 1911 by Dutch physicist Kamerlingh Onnes, many scientists from various subjects such as physics, material, mathematics and so on have being abstracted by this magical phenomenon. There are a lot of excellent sources for introduction to superconductivity. For example [73], [29], [61], [11] and [49]. For the Type-II superconductor, when the applied magnetic field is above a certain value which is called the first critical applied magnetic field, vortices would appear. The number of vortices varies with the applied magnetic field.

An interesting question arising here is “How does the number of the vortices vary? ” It grows one by one, or, for example, from 3 to 5 directly. The problem is complicated due to the fact that the sample has boundaries if we consider common sample. Thus we assume that the superconductor is large, and we are far from the boundary. In this case, the physically relevant variables are in some sense periodic. We will study the question in the periodic case. Periodic solutions of the Ginzburg-Landau energy were firstly studied by A. Abrikosov in [2]. And since then, it has been being studied by many mathematicians and physicists. For example the existence ([7],[57]), regularity of solutions ([32]), and numeric analysis ([31]).

In the framework of periodic case, we solve this problem in a special case: when the height of the lattice is small, the number of vortices jumps one by one as the applied magnetic field grows. Our work will based on the work of [9] which gave a fairly unified description of the vortices of minimizers of the periodic Ginzburg-Landau energy in the London limit.

2.1.2 The main result

The Model.
Denote L be a parallelogram generated by two vectors $(\vec{u}, \vec{v})$, and $\mathcal{L}$ be the group of translations generated by $(\vec{u}, \vec{v})$. We say a function $f(\mathbf{x})$ is periodic with respect to $(\vec{u}, \vec{v})$, if

$$f(\mathbf{x} + k\vec{u} + m\vec{v}) = f(\mathbf{x}), \ \forall k, m \in \mathbb{Z}, \forall \mathbf{x} \in \mathbb{R}^2.$$

Here the function $f(\mathbf{x})$ can be real, complex or vector valued.

We need the following function spaces

$$H^m_{loc}(\mathbb{R}^2) = \left\{u : \mathbb{R}^2 \to \mathbb{C} \;\middle|\; \Re(u), \Im(u) \in H^m(\mathcal{D}) \text{ for all bounded } \mathcal{D} \subset \mathbb{R}^2\right\},$$

$$\mathbf{H}^m_{loc}(\mathbb{R}^2) = \left\{\mathbf{A} = (A_1, A_2) : \mathbb{R}^2 \to \mathbb{R}^2 \;\middle|\; A_1, A_2 \in H^m(\mathcal{D}) \text{ for all bounded } \mathcal{D} \subset \mathbb{R}^2\right\}.$$

Definition 2.1.1 (Gauge Equivalent). *We say that two configurations $(u, \mathbf{A})$ and $(v, \mathbf{B})$ are gauge equivalent if there exists a (smooth) function $f : \mathbb{R}^2 \to \mathbb{R}$, such that*

$$v = ue^{if}, \quad \mathbf{B} = \mathbf{A} + \nabla f$$

The transformation from $(u, \mathbf{A})$ to $(v, \mathbf{B})$ is called a gauge transformation.

Definition 2.1.2 (Periodic Space). *We define the space $\mathcal{H}_{per}$ to be the set of all $(u, \mathbf{A}) \in H^1_{loc}(\mathbb{R}^2) \times \mathbf{H}^1_{loc}(\mathbb{R}^2)$ such that for any $k, m \in \mathbb{Z}$, the configuration $(u(\cdot + k\vec{u} + m\vec{v}), \mathbf{A}(\cdot + k\vec{u} + m\vec{v}))$ is gauge equivalent to $(u, \mathbf{A})$.*

For $(u, \mathbf{A}) \in \mathcal{H}_{per}$, an $\varepsilon > 0$ and an applied magnetic field $h_{ex}(\varepsilon)$, we define the periodic Ginzburg-Landau energy as follows

$$GL_\varepsilon(u, \mathbf{A}) = \frac{1}{2} \int_L |\nabla_{\mathbf{A}} u|^2 + |\mathrm{curl}\mathbf{A} - h_{ex}|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2.$$

Here $\nabla_{\mathbf{A}} u :\equiv \nabla u - i\mathbf{A}u$, u is called "order parameter " in physics, which indicates the local state of the material(superconducting phase or normal phase), $\mathbf{A}$ is the vector potential of the magnetic field, curl$\mathbf{A}$ is the induced magnetic field, h_{ex} is the applied magnetic field, and ε is the inverse of the Ginzburg-Landau parameter κ. In $\mathcal{H}_{per}$, $\nabla_{\mathbf{A}} u$ and curl$\mathbf{A}$ are periodic with respect to $(\vec{u}, \vec{v})$. The energy is invariant under the Gauge transformation. For this periodic model, when the area of L equals to 1, we have the following two propositions and one theorem in [9](See Proposition 2.1, Proposition 2.2 and Theorem 1 of [9]).

Proposition 2.1.1. *The minimum of $GL_\varepsilon(u, \mathbf{A})$ over $\mathcal{H}_{per}$ is achieved.*

Proposition 2.1.2. *Given any $(u, \mathbf{A}) \in \mathcal{H}_{per}$, then*

$$\frac{1}{2\pi}\int_L \mathrm{curl}\mathbf{A} \in \mathbb{Z}$$

Moreover, if $(u_1, \mathbf{A}_1)$ minimizes the Ginzburg-Landau energy with parameters ε and $h_{ex} = h_1$, and if $(u_2, \mathbf{A}_2)$ minimizes the Ginzburg-Landau energy with parameters ε and $h_{ex} = h_2 > h_1$, then $n_2 \geq n_1$, where for $i = 1, 2$,

$$n_i = \frac{1}{2\pi}\int_L curl\mathbf{A}_i.$$

Remark 1. *For each $\varepsilon > 0$, the number of vortices contained in the minimizer of $GL_\varepsilon(u, \mathbf{A})$ in one lattice cell does not decrease as the applied magnetic field increases. Then there exist a well defined value $H_{C_1}(\varepsilon)$ such that the minimizer of $GL_\varepsilon(u, \mathbf{A})$ with parameters ε and h_{ex} satisfies $n = 0$ if $h_{ex} < H_{C_1}(\varepsilon)$, and $n > 0$ if $h_{ex} \geq H_{C_1}(\varepsilon)$. We call this value "the first critical applied magnetic field ". The author in [8] proved that the regime of the $H_{C_1}(\varepsilon)$ is $\frac{1}{2}|\log\varepsilon|$. We define $\Delta_{ex} :\equiv h_{ex} - \frac{1}{2}|\log\varepsilon|$.*

Remark 2. *Note that when $n = 0$, the minimizers are gauge equivalent to Meissner solution* $(1, 0)$. *In physics, it means that the material is in the superconducting state.*

Theorem ([9])**.** *Let $(u_\varepsilon, \mathbf{A}_\varepsilon)$ be any minimizer of GL_ε, $h_\varepsilon = \mathrm{curl}\mathbf{A}_\varepsilon$, and*

$$n_\varepsilon = \frac{1}{2\pi}\int_L \mathrm{curl}\mathbf{A}_\varepsilon.$$

Then the following behaviors of $h_\varepsilon, n_\varepsilon$ holds, according to the applied field h_{ex}.

- *If $1 \ll \Delta_{ex} \ll 1/\varepsilon^2$, then as $\varepsilon \to 0$,*

$$\frac{h_\varepsilon}{2\pi n_\varepsilon} \to 1 \; in \; W^{1,p}(L)\,(\forall p < 2), and \, n_\varepsilon \approx \frac{\Delta_{ex}}{2\pi}.$$

- *If $|\Delta_{ex}|$ is bounded independently of ε, then so are $\|h_\varepsilon\|_{W^{1,p}}, \forall p < 2$ and n_ε. If $\{\varepsilon\}$ is a subsequence such that $\{h_\varepsilon\}_\varepsilon$ converges to h_* and $\{\Delta_{ex}\}_\varepsilon$ converges to a value Δ^*_{ex}, then $n_\varepsilon \to n_* \in \mathbb{N}$, and in particular $n_\varepsilon = n_*$ for small enough ε, then there are n_* distinct points $\{a_i\}_{i=1}^{n_*}$ in L such that*

$$-\Delta h_* + h_* = 2\pi\sum_{i=1}^{n_*}\delta_{a_i}.$$

Moreover, for $\boldsymbol{p} = (p_1, \cdots, p_n) \in \mathcal{P}$, where $\mathcal{P}$ is the family of sets of finite points. Let

$$W(\boldsymbol{p}, \Delta^*_{ex}) = \lim_{\rho\to 0}\Big(\pi n\log\rho + \frac{1}{2}\int\limits_{L\setminus\cup_i B(p_i,\rho)} |\nabla h_{\boldsymbol{p}}|^2 + h_{\boldsymbol{p}}^2\Big) + n(\gamma - 2\pi\Delta^*_{ex}),$$

where $h_{\boldsymbol{p}}$ is the unique L-periodic solution of $-\Delta h_{\boldsymbol{p}} + h_{\boldsymbol{p}} = 2\pi\sum_{i=1}^{n}\delta_{p_i}$. Then $(a_1, \cdots, a_{n_*})$ minimizes W over $\mathcal{P}$.

The number γ is defined in [13, 63] as

$$\gamma = \lim_{R\to+\infty} -\pi \log R + \frac{1}{2}\int_{B(0,R)} |\nabla u_0|^2 + \frac{(1-|u_0|^2)^2}{2},$$

where u_0 is the unique solution of $-\Delta u_0 = u_0(1-|u_0|^2)$ in $\mathbb{R}^2$ of the form $u_0(r,\theta) = f(r)e^{i\theta}$, with $f : \mathbb{R}_+ \to \mathbb{R}_+$.

- *There exists a possibly negative $\Delta_1 \in \mathbb{R}$ s.t. if $\Delta_{ex} < \Delta_1$ and ε is small enough, then $n_* = 0$. In this case $(u_\varepsilon, \mathbf{A}_\varepsilon)$ is gauge equivalent to the Meissner solution$(1,0)$.*

Remark 3. *From Proposition 2.1.2 and the Theorem above, we could get that the number of the points contained in the minimizer of the renormalized energy W would not decrease as Δ^*_{ex} increases. We give a short proof as follows.*

Proposition 2.1.3. *The number of the points contained in the minimizer of the renormalized energy W defined over $\mathcal{P}$ would not decrease as Δ^*_{ex} increases.*

Proof. The proof is easy. Let $(\mathbf{p}_1, \Delta_1)$ (respectively $(\mathbf{p}_2, \Delta_2)$) minimizes the renormalized energy W with parameter Δ_1 (respectively Δ_2 and $\Delta_1 < \Delta_2$). Then we have

$$W(\mathbf{p}_1, \Delta_1) \leq W(\mathbf{p}_2, \Delta_1), \tag{2.1}$$
$$W(\mathbf{p}_1, \Delta_2) \geq W(\mathbf{p}_2, \Delta_2). \tag{2.2}$$

then apply (2.1) − (2.2), we have

$$2\pi n_1(\Delta_2 - \Delta_1) \leq 2\pi n_2(\Delta_2 - \Delta_1).$$

Then we have

$$n_1 \leq n_2$$

□

In the left of the paper, we only consider a special case of the periodic model. We consider the small rectangles, i.e. $\vec{u}\perp\vec{v}$. Let $L = [-\frac{1}{2}, \frac{1}{2}] \times [0, l]$. Note that now the area of the lattice cell is l, rather than 1. However the existence of the renormalized energy W is still true (see the proof in section 6 of [9]). It would be interesting to verify the regime of the $H_{C_1}(\varepsilon)$ if the area of the lattice cell is l. One can refer [50] for the regime of the $H_{C_1}(\varepsilon)$ for a domain with size tending to 0 or infinity as ε tends to 0. Here we redefine $\Delta_{ex} :\equiv h_{ex} - H_{C_1}(\varepsilon)$.

We are interested in the variation of vortex number contained in the minimizer of periodic Ginzburg-Landau energy. We have a result in a special case.

Theorem 2.1.1. *Let L be the lattice cell defined above. For every $N \in \mathbb{N}$, there exists l_N and if $\varepsilon \ll l \leq l_N \ll 1$, then there exists an exactly increasing sequence of N values*

$$\Delta_1 + o(1) < \Delta_2 + o(1) < \cdots < \Delta_N + o(1),$$

where Δ_i does not depend on ε and l, such that at $h_{ex} = H_n = H_{C_1}(\varepsilon) + \frac{\Delta_n}{l} + o(\frac{1}{l})$, $1 \leq n \leq N$, the number of vortices of the minimizer of GL_ε jumps from n to $n+1$.

The remainder of this chapter is organized as follows. Before studying the periodic Ginzburg-Landau energy, we first study the corresponding renormalized energy W. In Section 2.2.1, we prove that in dimension one, the number of the points contained in the minimizer of $\mathbf{W}_{1D}$ jumps one by one. In Section 2.2.2, we prove Γ-convergence of l^2E to F, where E is the main part of W and F is the main part of $\mathbf{W}_{1D}$, and both of them will be defined later. Then in Section 2.3, we give the proof of Theorem 2.1.1.

2.2 The Renormalized Energy

In this section, we study the renormalized energy W. First, we study the properties of a energy $\mathbf{W}_{1D}$ in one dimension. The motivation is in the case of two-dimension, when the height l of the lattice cell converges to 0, intuitively, the vortices would form lines, thus the two dimensional model would degenerate to a one dimensional model. Second, we prove the Γ-convergence of $l^2 \cdot E$ in two-dimension to F in one-dimension, where E is the main part of W and F is the main part of $\mathbf{W}_{1D}$, and they will be defined later. In the proof, we use ball growth method which was introduced independently in [48] and [62] to get the lower bound. And then by using a similar method in [65] we get the upper bound. Combine the upper bound and lower one together, we finish the proof of Γ-convergence.

2.2.1 The Case of One Dimension

In this subsection, we consider the case of one dimension. Denote K be $[-a/2, a/2]$ in $\mathbb{R}^1$ for any $a > 0$. Denote still by $\mathcal{P}$ the family of sets of finite points over K and for $\mathbf{p} = \{p_1, ..., p_n\} \in \mathcal{P}$ where $n \in \mathbb{N}$, we define an energy

$$\mathbf{W}_{1D}(\mathbf{p}, \Delta_{ex}) = \frac{1}{2}\int_K |\nabla h_{\mathbf{p}}|^2 + h_{\mathbf{p}}^2 \,\mathrm{d}x - 2\pi n \Delta_{ex}$$

where $h_{\mathbf{p}}$ is the unique K-periodic solution of

$$-\Delta h_{\mathbf{p}} + h_{\mathbf{p}} = 2\pi \sum_{i=1}^{n} \delta_{p_i}.$$

Theorem 2.2.1. *There exists an increasing sequence of critical values $\{\Delta_k\}_{k\in\mathbb{N}}$, such that at each Δ_k, the number of points contained in the minimizer of $\mathbf{W}_{1D}$ jumps from k to $k+1$. That means the number of points contained in the minimizer of $\mathbf{W}_{1D}$ jumps one by one as the Δ_{ex} grows.*

First, let us consider the case of one point in K, i.e.

$$\begin{cases} -h_0''(x) + h_0(x) &= 2\pi\delta_0 \qquad \text{in } K \\ h_0(0^+) &= h_0(0^-) \\ h_0(a/2) &= h_0(-a/2) \\ h_0'(a/2) &= h_0'(-a/2) \end{cases}$$

The solution has the form of

$$h_0(x) = \begin{cases} c_1 \mathrm{e}^x + c_2 \mathrm{e}^{-x}, & x \geq 0; \\ c_3 \mathrm{e}^x + c_4 \mathrm{e}^{-x}, & x < 0. \end{cases} \tag{2.1}$$

We can determine these constants by using the equation and boundary conditions(periodic boundary conditions).

$$\begin{cases} (c_3 - c_4) - (c_1 - c_2) &= 2\pi \\ c_1 + c_2 &= c_3 + c_4 \\ c_1 \mathrm{e}^{a/2} + c_2 \mathrm{e}^{-a/2} &= c_3 \mathrm{e}^{-a/2} + c_4 \mathrm{e}^{a/2} \\ c_1 \mathrm{e}^{a/2} - c_2 \mathrm{e}^{-a/2} &= c_3 \mathrm{e}^{-a/2} - c_4 \mathrm{e}^{a/2}. \end{cases} \tag{2.2}$$

Then we have the solution

$$\begin{cases} c_1 = \dfrac{\pi}{\mathrm{e^a} - 1} \\ c_2 = \dfrac{\pi \mathrm{e^a}}{\mathrm{e^a} - 1} \\ c_3 = \dfrac{\pi \mathrm{e^a}}{\mathrm{e^a} - 1} \\ c_4 = \dfrac{\pi}{\mathrm{e^a} - 1}. \end{cases} \tag{2.3}$$

For arbitrary $p \in K$, $h_p(x) = h_0(x - p)$ due to the periodicity. And what's more, in the case of only one point in K, all the solutions have the same energy.

$$\begin{aligned} \mathbf{W}_{1D}(\{0\}, \Delta_{ex}) &= \frac{1}{2} \int_K |\nabla h_0|^2 + h_0^2 \, \mathrm{d}x - 2\pi\Delta_{ex} \\ &= \frac{\mathrm{e}^a + 1}{\mathrm{e}^a - 1}\pi^2 - 2\pi\Delta_{ex}. \end{aligned}$$

Second, we consider the case of two points in K. From the linearity, we can easily get that for points $\mathbf{p} = \{p_1, p_2\}$, the solution $h_{\mathbf{p}} = h_{p_1} + h_{p_2}$.

Now we need to adjust the locations of these two points to minimize $\mathbf{W}_{1D}(\mathbf{p}, \Delta_{ex})$. Due to the periodicity, we can fix one point at 0 firstly, then adjust the other one, i.e. consider $\mathbf{p} = \{0, s\}, 0 < s \leq a/2$. Then

$$\begin{aligned} \mathbf{W}_{1D}(\mathbf{p}, \Delta_{ex}) &= \frac{1}{2} \int_K |\nabla h_{\mathbf{p}}|^2 + h_{\mathbf{p}}^2 \, \mathrm{d}x - 2 \cdot 2\pi\Delta_{ex} \\ &= 2 \cdot \frac{\mathrm{e}^a + 1}{\mathrm{e}^a - 1}\pi^2 - 2 \cdot 2\pi\Delta_{ex} + f_2(0, s) \end{aligned}$$

where $f_2(0, s) = \int_K \nabla h_0 \cdot \nabla h_s + h_0 \cdot h_s$.

Divide K into four parts

$$\begin{aligned} f_2(0, s) &= \int_{-a/2}^{s-a/2} + \int_{s-a/2}^{0} + \int_0^s + \int_s^{a/2} \\ &= \int_{-a/2}^{s-a/2} \frac{2\pi^2 \mathrm{e}^a}{(\mathrm{e}^a - 1)^2} \mathrm{e}^{2x-s+a} + \frac{2\pi^2 \mathrm{e}^a}{(\mathrm{e}^a - 1)^2} \mathrm{e}^{-(2x-s+a)} \\ &\quad + \int_{s-a/2}^{0} \frac{2\pi^2 \mathrm{e}^{2a}}{(\mathrm{e}^a - 1)^2} \mathrm{e}^{2x-s} + \frac{2\pi^2}{(\mathrm{e}^a - 1)^2} \mathrm{e}^{-(2x-s)} \\ &\quad + \int_0^s \frac{2\pi^2 \mathrm{e}^a}{(\mathrm{e}^a - 1)^2} \mathrm{e}^{2x-s} + \frac{2\pi^2 \mathrm{e}^a}{(\mathrm{e} - 1)^2} \mathrm{e}^{-(2x-s)} \end{aligned}$$

$$
\begin{aligned}
& + \int_s^{a/2} \frac{2\pi^2}{(\mathrm{e}^a - 1)^2}\mathrm{e}^{2x-s} + \frac{2\pi^2\mathrm{e}^{2a}}{(\mathrm{e}^a - 1)^2}\mathrm{e}^{-(2x-s)} \\
= \; & \frac{2\pi^2\mathrm{e}^a}{(\mathrm{e}^a - 1)^2}(\mathrm{e}^s - \mathrm{e}^{-s}) \\
& + \frac{\pi^2\mathrm{e}^{2a}}{(\mathrm{e}^a - 1)^2}(\mathrm{e}^{-s} - \mathrm{e}^{s-a}) + \frac{\pi^2}{(\mathrm{e}^a - 1)^2}(\mathrm{e}^{a-s} - \mathrm{e}^s) \\
& + \frac{2\pi^2\mathrm{e}^a}{(\mathrm{e}^a - 1)^2}(\mathrm{e}^s - \mathrm{e}^{-s}) \\
& + \frac{\pi^2\mathrm{e}^{2a}}{(\mathrm{e}^a - 1)^2}(\mathrm{e}^{-s} - \mathrm{e}^{s-a}) + \frac{\pi^2}{(\mathrm{e}^a - 1)^2}(\mathrm{e}^{a-s} - \mathrm{e}^s) \\
= \; & 4\frac{\pi^2\mathrm{e}^a}{(\mathrm{e}^a - 1)^2}(\mathrm{e}^s - \mathrm{e}^{-s}) + 2\left(\frac{\pi^2\mathrm{e}^{2a}}{(\mathrm{e}^a - 1)^2}(\mathrm{e}^{-s} - \mathrm{e}^{s-a}) + \frac{\pi^2}{(\mathrm{e}^a - 1)^2}(\mathrm{e}^{a-s} - \mathrm{e}^s)\right) \\
= \; & \frac{2\pi^2}{\mathrm{e}^a - 1}(\mathrm{e}^s + \mathrm{e}^{a-s}).
\end{aligned}
$$

The derivative of $f_2(0, s)$ with respect to s is

$$
\frac{\mathrm{d}f_2(0, s)}{\mathrm{d}s} = \frac{2\pi^2}{\mathrm{e}^a - 1}(\mathrm{e}^s - \mathrm{e}^{a-s}).
$$

And we can also get that f_2 is a strictly convex function by taking the second derivative of $f_2(0, s)$. Then $s = a/2$ minimizes $f_2(0, s)$, i.e. minimizes $\mathbf{W}_{1D}(\mathbf{p}, \Delta_{ex})$.

We now know that in the case of two points $\mathbf{p} = \{p_1, p_2\}$, when $|p_1 - p_2| = a/2$, the minimum of $\mathbf{W}_{1D}$ is achieved,

$$
\min_{|\mathbf{p}|=2} \mathbf{W}_{1D}(\mathbf{p}, \Delta_{ex}) = 2 \cdot \frac{\mathrm{e}^a + 1}{\mathrm{e}^a - 1}\pi^2 - 2 \cdot 2\pi\Delta_{ex} + \frac{4 \cdot \pi^2 \cdot \mathrm{e}^{a/2}}{\mathrm{e}^a - 1}.
$$

While

$$
\min_{|\mathbf{p}|=1} \mathbf{W}_{1D}(\mathbf{p}, \Delta_{ex}) = \frac{\mathrm{e}^a + 1}{\mathrm{e}^a - 1}\pi^2 - 2\pi\Delta_{ex}
$$

if we want $\min_{|\mathbf{p}|=2} \mathbf{W}_{1D}(\mathbf{p}, \Delta_{ex}) \leq \min_{|\mathbf{p}|=1} \mathbf{W}_{1D}(\mathbf{p}, \Delta_{ex})$, we need

$$
2\pi\Delta_{ex} \geq \frac{\mathrm{e}^a + 1}{\mathrm{e}^a - 1}\pi^2 + \frac{4 \cdot \pi^2 \cdot \mathrm{e}^{a/2}}{\mathrm{e}^a - 1}
$$

At last, we consider the general case. In fact, we can get some clue from the case of three points. Let $\mathbf{p_3} = \{p_1, p_2, p_3\}$, and $\mathrm{dist}(p_1, p_2) = s_1$, $\mathrm{dist}(p_2, p_3) = s_2$, $\mathrm{dist}(p_1, p_3) = s_3 = s_1 + s_2$. Then the energy

$$
\mathbf{W}_{1D}(\mathbf{p_3}, \Delta_{ex}) = 3 \cdot \frac{\mathrm{e}^a + 1}{\mathrm{e}^a - 1}\pi^2 - 3 \cdot 2\pi\Delta_{ex} + f_3(\mathbf{p_3})
$$

where $f_3(\mathbf{p_3}) = f_2(s_1) + f_2(s_2) + f_2(s_1 + s_2)$.

We need to minimize f_3, which is in fact a function of s_1, s_2. Denote it as $f_3(s_1, s_2)$. The minimizer of f_3 satisfies the equations as follows:

$$
\begin{cases}
\dfrac{\partial f_3(s_1)}{\partial s_1} = 0 \\[2ex]
\dfrac{\partial f_3(s_2)}{\partial s_2} = 0
\end{cases}
\tag{2.4}
$$

i.e.

$$\begin{cases} e^{s_1} - e^{a-s_1} + e^{s_1+s_2} - e^{a-(s_1+s_2)} = 0 \\ e^{s_2} - e^{a-s_2} + e^{s_1+s_2} - e^{a-(s_1+s_2)} = 0. \end{cases} \tag{2.5}$$

The solution is

$$s_1 = s_2 = \frac{a}{3},$$

then

$$\min_{|\mathbf{p_3}|=3} f_3(\mathbf{p_3}) = \frac{2\cdot\pi^2}{e^a - 1}\cdot 3\cdot(e^{a/3} + e^{2a/3}).$$

Lemma 2.2.1. *For arbitrary $n \in \mathbb{N}$, if there are n points in K, denote them as $\mathbf{p_n} = \{p_1, p_2, \cdots, p_n\}$, and* $\mathrm{dist}(p_i, p_{i+1}) = s_i$ *for $i = 1, \cdots, n-1$, then the minimum of $f_n(\mathbf{p_n})$ is reached if and only if $s_1 = s_2 = \cdots = s_{n-1} = \frac{a}{n}$, and*

$$\min_{|\mathbf{p_n}|=n} f_n(\mathbf{p_n}) = \frac{2\pi^2}{e^a - 1}\cdot n\cdot(e^{a/n} + e^{2a/n} + \cdots + e^{(n-1)a/n}) = \frac{2\cdot\pi^2}{e^a - 1}\cdot n\cdot\frac{e^a - e^{a/n}}{e^{a/n} - 1}.$$

Proof. In fact, f_n is a function of $s_1, \cdots, s_{n-1}$. Denote is as $f_n(s_1, \cdots, s_{n-1})$. We write f_n as a sum of functions of $f_2(\cdot)$, and the variable are $s_1, s_2, \cdots, s_{n-1}$, i.e.

$$\begin{aligned} f_n \quad = \quad & f_2(s_1) + f_2(s_1+s_2) + \cdots + f_2(s_1+s_2+\cdots+s_{n-1}) \\ & + f_2(s_2) + f_2(s_2+s_3) + \cdots + f_2(s_2+s_3+\cdots+s_{n-1}) \\ & \vdots \\ & + f_2(s_{n-1}). \end{aligned}$$

It is not difficult to certify that $s_1 = s_2 = \cdots = s_{n-1} = \frac{a}{n}$ is a critical point, i.e. at the point $(\frac{a}{n}, \frac{a}{n}, \cdots, \frac{a}{n}) \in \mathbb{R}^{n-1}$, we have

$$\begin{cases} \dfrac{\partial f_n}{\partial s_1} = 0 \\ \dfrac{\partial f_n}{\partial s_2} = 0 \\ \qquad\vdots \\ \dfrac{\partial f_n}{\partial s_{n-1}} = 0 \end{cases}$$

Since f_n is a strictly convex function (it is the sum of convex functions $f_2(\cdot)$, and there are $f_2(s_i)$ in the sum) in $\mathbb{R}^{n-1}$, it is the unique minimizer of f_n. We substitute the value of s_i into f_n, and then get its expression as above. □

Proof of Theorem 2.2.1. For $x > 0$, define function

$$g(x) = x\cdot\frac{e^a - e^{a/x}}{e^{a/x} - 1},$$

then $\min f_n = \frac{2\cdot\pi^2}{e^a-1}\cdot g(n)$ and because

$$g''(x) > 0,$$

thus it is a strictly convex function.

Now we prove that

$$g(x) \quad \text{is strictly convex} \Longrightarrow \Delta_1 < \Delta_2 < \Delta_3 < \cdots .$$

i.e. the number of vortex jumps one by one.

We prove the theorem by contradiction. If the number of points contained in the minimizer does not jump one by one, then there would exist $m_1 < m_2 < m_3$ and Δ_{ex} such that

$$\min_{|\mathbf{p}|=m_3} \mathbf{W}_{1D}(\mathbf{p}, \Delta_{ex}) \leq \min_{|\mathbf{p}|=m_1} \mathbf{W}_{1D}(\mathbf{p}, \Delta_{ex}) \leq \min_{|\mathbf{p}|=m_2} \mathbf{W}_{1D}(\mathbf{p}, \Delta_{ex})$$

Note that here

$$\mathbf{W}_{1D}(\mathbf{p}_m, \Delta_{ex}) = m \cdot \frac{\mathrm{e}^a + 1}{\mathrm{e}^a - 1}\pi^2 - m \cdot 2\pi\Delta_{ex} + f_m.$$

From the inequality above, we get

$$\min_{|\mathbf{p}|=m_2} \mathbf{W}_{1D}(\mathbf{p}, \Delta_{ex}) - \min_{|\mathbf{p}|=m_1} \mathbf{W}_{1D}(\mathbf{p}, \Delta_{ex}) \geq 0, \tag{2.6}$$

$$\min_{|\mathbf{p}|=m_2} \mathbf{W}_{1D}(\mathbf{p}, \Delta_{ex}) - \min_{|\mathbf{p}|=m_3} \mathbf{W}_{1D}(\mathbf{p}, \Delta_{ex}) \geq 0. \tag{2.7}$$

then apply $(m_3 - m_2) \times (2.6) + (m_2 - m_1) \times (2.7)$, we obtain

$$\min_{|\mathbf{p}|=m_2} f_{m_2}(\mathbf{p}) \geq \frac{m_2 - m_1}{m_3 - m_1} \min_{|\mathbf{p}|=m_3} f_{m_3}(\mathbf{p}) + \frac{m_3 - m_2}{m_3 - m_1} \min_{|\mathbf{p}|=m_1} f_{m_1}(\mathbf{p}).$$

This contradicts the fact that $g(x)$ is strictly convex. That finishes the proof. □

2.2.2 Γ−Convergence

Recall the definition of $L = [-\frac{1}{2}, \frac{1}{2}] \times [0, l]$, and the two dimensional Ginzburg-Landau renormalized energy

$$W(\mathbf{p}, \Delta_{ex}) = \lim_{\rho \to 0} \left(\pi n \log \rho + \frac{1}{2} \int\limits_{L \setminus \cup_i B(p_i, \rho)} |\nabla H_l|^2 + H_l^2 \right) + n(\gamma - 2\pi\Delta_{ex}^*),$$

where $\mathbf{p} = \{p_i\}_{1 \leq i \leq n}$ for $n \in \mathbb{N}$ are any n points in L, and H_l is the unique L-periodic solution of

$$\begin{cases} -\Delta H_l + H_l = 2\pi \sum\limits_{i}^{n} \delta_{p_i} \quad \text{in } L \\ \text{periodic boundary conditions on } \partial L. \end{cases} \tag{2.8}$$

Denote S be the square $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$.

We want to relate this two dimensional renormalized energy $W(\mathbf{p}, \Delta_{ex})$ with the one dimensional energy $\mathbf{W}_{1D}(\mathbf{p}, \Delta_{ex})$, because we have already had the result on the variation of vortex number in one dimension. For the solution $h_\mathbf{p}$ in the case of one dimension, we can expend it in the direction of Y-axis to get a two dimensional function over the square S, i.e. $h_\mathbf{p}(x, y) \equiv h_\mathbf{p}(x, 0)$, where $h_\mathbf{p}(x, 0)$ is the solution of the one dimensional equation, thus $h_\mathbf{p}(x, y)$ satisfies

$$\begin{cases} -\Delta h_\mathbf{p}(x, y) + h_\mathbf{p}(x, y) = 2\pi \sum\limits_{i=1}^{n} \delta_{Lp_i} \text{ in } S \\ \text{periodic boundary conditions on } \partial S \end{cases} \tag{2.9}$$

where Lp_i is the line in $\mathbb{R}^2$ passing p_i and parallel with Y-axis, and $\delta_{Lp_i} \in (C^{0,\alpha}(S))^*$ for any $\alpha \in [0,1)$ such that

$$\langle \delta_{Lp_i}, f \rangle :\equiv \int_{Lp_i} f \mathrm{d}s \text{ for any } f \in C^{0,\alpha}(S). \tag{2.10}$$

For the convenience of notation, we write h as $h_{\mathbf{p}}$ if there is no confusion.

By using the periodic boundary condition, we rewrite the function (2.8) as follows, it is the same as (2.8) because of the periodicity and uniqueness of the solution.

$$\begin{cases} -\Delta H_l + H_l = 2\pi \sum_{j=1}^{1/l} \sum_{i=1}^{n} \delta_{p_{i,j}} \quad \text{in } S \\ \text{periodic boundary conditions on } \partial S \end{cases} \tag{2.11}$$

where the set of points $\{p_{i,j}\}$ with $1 \le i \le n, 1 \le j \le \frac{1}{l}$ comes from the periodic extension of $\{p_i\}$ in L to the square S. For any i fixed the set of points $\{p_{i,j}\}_{j=1}^{\frac{1}{l}}$ lie on the same line L_{p_i} (in fact $p_i = p_{i,\frac{1}{2l}+1}$). It is not difficult to prove that $l\sum_{j=1}^{\frac{1}{l}} \sum_{i=1}^{n} \delta_{p_i} \overset{l\to 0}{\longrightarrow} \sum_{i=1}^{n} \delta_{Lp_i}$ in $(C^{0,\alpha}(S))^*, \forall \alpha \in [0,1)$. Then by the compact embedding of $W^{1,q}(S)$ into $C^{0,\beta}(S)$ for any $q > 2$, we extract a convergent subsequence in $W^{-1,p}(S)$ for any $p < 2$. Then we have

$$lH_l \to h \text{ in } W^{1,p}(S), p < 2$$

We define two related energy E and F as follows

$$\begin{aligned} E &= \lim_{\rho\to 0}\left(\pi\frac{n}{l}\log\rho + \frac{1}{2}\int_{S\setminus\cup_{i,j}B(p_{i,j},\rho)} |\nabla H_l|^2 + |H_l|^2\right) \\ &= \frac{1}{l}\lim_{\rho\to 0}\left(\pi n\log\rho + \frac{1}{2}\int_{L\setminus\cup_{i=1}^{n}B_i(p_i,\rho)} |\nabla H_l|^2 + |H_l|^2\right) \end{aligned} \tag{2.12}$$

$$F = \frac{1}{2}\int_S |\nabla h|^2 + h^2 \mathrm{d}x\mathrm{d}y$$

Theorem 2.2.2 (Γ convergence of l^2E)**.** *For the energies E, F defined above, we have*

$$l^2E \overset{\Gamma}{\longrightarrow} F \text{ as } l \to 0.$$

More precisely,

- *If $\sum_{i=1}^{n} \delta_{Lp_i} \overset{l\to 0}{\longrightarrow} \mu$ in $(C^{0,\alpha}(S))^*, \forall\alpha \in [0,1)$, where $\mu = \sum_{i=1}^{n}\delta_{L_i}$ and independent of l, then we have*

$$l\sum_{j=1}^{\frac{1}{l}}\sum_{i=1}^{n}\delta_{p_{i,j}} \overset{l\to 0}{\longrightarrow} \mu \text{ in } (C^{0,\alpha}(S))^*, \forall\alpha \in [0,1),$$

and

$$\liminf_{l\to 0} l^2E(H_l) \ge F(h).$$

- *For every measure $\mu_L = \sum_{i=1}^{n} \delta_{L_i}$ in $(C^{0,\alpha}(S))^*$, where δ_{L_i} is defined as (2.10), L_i is a line which is parallel with Y-axis, then there exists a sequence of distribution*

$$\tilde{\mu}_l := l\mu_l \to \mu_L \text{ as } l \to 0,$$

where μ_l is in the form of $\mu_l = \sum_{j=1}^{\frac{1}{l}} \sum_{i=1}^{n} \delta_{p_{i,j}}$, such that

$$\limsup_{l\to 0} l^2 E(H_l) \le F(h).$$

Lower bound

Ball growth method is a technical method to calculate energy on annuli which was introduced independently in [48] and [62]. In this subsection, we use the frame of [63, Chaper 4]. By using this method, we can merge two tangent or overlapped balls into a single ball that contains the original balls, and the radius of the new ball is equal to the sum of the radii of the original balls. We will write $r(B)$ for the radius of a ball B, $r(\mathfrak{B})$ for the sum of the radii of the balls in the collection of balls $\mathfrak{B}$, and $\mathfrak{B} \cap U$ for the collection $\{B \cap U | B \in \mathfrak{B}\}$.

Lemma 2.2.2 (Ball growth)**.** *Let $\mathfrak{B}_0$ be a finite collection of disjoint closed balls. There exists a family $\{\mathfrak{B}(t)\}_{t\in\mathbb{R}^+}$ of collections of disjoint closed balls such that $\mathfrak{B}(0) = \mathfrak{B}_0$ and*
1. For every $s \ge t \ge 0$,

$$\bigcup_{B\in\mathfrak{B}(t)} B \subset \bigcup_{B\in\mathfrak{B}(s)} B$$

2. There exists a finite set $T \subset \mathbb{R}^+$ such that if $[t_0, t_1] \subset \mathbb{R}^+ \setminus T$, then $\mathfrak{B}(t_1) = e^{t_1 - t_0}\mathfrak{B}(t_0)$.
3. For every $t \in \mathbb{R}^+$, $r(\mathfrak{B}(t)) = e^t r(\mathfrak{B}(0))$.

Refer to Theorem 4.2 of [63] for the proof.

Lemma 2.2.3 (Merging)**.** . *Assume B_1 and B_2 are two closed balls in $\mathbb{R}^n$ such that $B_1 \cap B_2 \neq \emptyset$, then there is a ball B such that $r(B) = r(B_1) + r(B_2)$ and $B_1 \cup B_2 \subset B$.*

Refer to Lemma 4.1 of [63] for the proof.

Notation: We see function $\mathcal{F}(x, r) : \mathbb{R}^2 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be defined also for collections of balls. We write

$$\mathcal{F}(B) = \mathcal{F}(B(x, r)) := \mathcal{F}(x, r)$$

and

$$\mathcal{F}(\mathfrak{B}) = \sum_{B\in\mathfrak{B}} \mathcal{F}(B).$$

Here we say that $\mathcal{F}$ is monotonic if $\mathcal{F}$ is continuous with respect to r and for any families of disjoint closed balls $\mathfrak{B}_1$, $\mathfrak{B}_2$ such that $\bigcup_{B\in\mathfrak{B}_1} B \subset \bigcup_{B\in\mathfrak{B}_2} B$

$$\mathcal{F}(\mathfrak{B}_1) \le \mathcal{F}(\mathfrak{B}_2).$$

This implies that $\mathcal{F}$ is non-decreasing with respect to r.

Proposition 2.2.1. *Let function $\mathcal{F}(x,r) : \mathbb{R}^2 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be monotonic in the above sense. Let $\mathfrak{B}_0$ be a finite collection of disjoint closed balls and by applying the ball growth method to $\mathfrak{B}_0$ we can get $\mathfrak{B}(t)$. Then for every $s \geq 0$,*

$$\mathcal{F}(\mathfrak{B}(s)) - \mathcal{F}(\mathfrak{B}_0) \geq \int_{t=0}^{s} \sum_{B(x,r)\in\mathfrak{B}(t)} r\frac{\partial\mathcal{F}}{\partial r}(x,t)\mathrm{d}t,$$

and for every $B \in \mathfrak{B}(s)$, we have

$$\mathcal{F}(B) - \mathcal{F}(\mathfrak{B}_0 \cap B) \geq \int_{t=0}^{s} \sum_{B(x,r)\in\mathfrak{B}(t)\cap B} r\frac{\partial\mathcal{F}}{\partial r}(x,t)\mathrm{d}t.$$

Refer Proposition 4.1 of [63] for the proof.

If we denote the collection of original balls $\{B_i(p_i,\rho) \subset L\}_{i=1}^n$ as $\mathfrak{B}(\rho)$, then by using the ball growth method we can get a new collection of balls $\mathfrak{B}(l^2)$, i.e. $0 \leq t \leq T = \log\frac{l^2}{\rho}$.

Let

$$E_{l^2} = \frac{1}{l}\left(\pi n \log l^2 + \frac{1}{2}\int_{L\setminus\cup_{B\in\mathfrak{B}(l^2)}B} |\nabla H_l|^2 + |H_l|^2\right),$$

then

$$\lim_{l\to 0} l^2 E = \lim_{l\to 0}\left(lE_{l^2} + \lim_{\rho\to 0}\left(l\pi n\log\frac{\rho}{l^2} + l\frac{1}{2}\int_{\cup_{B\in\mathfrak{B}(l^2)}B\setminus\cup_{B\in\mathfrak{B}(\rho)}B} |\nabla H_l|^2 + |H_l|^2\right)\right).$$

Let $B(t) \in \mathfrak{B}(t)$, and $B(t) \cap \mathfrak{B}(0) = \mathfrak{B}(\rho) = \{B_1, \cdots, B_m\}$, it means that these m balls grow to be one ball $B(t)$ at time t.

From the function, we find for $\rho \leq r \leq l^2$, i.e. $0 \leq t \leq T = \log\frac{l^2}{\rho}$ that

$$-\int_{\partial B(t)} \frac{\partial H_l}{\partial \nu} = -\int_{B(t)} \Delta H_l = 2\pi m - \int_{B(t)} H_l.$$

By using the Cauchy-Schwartz inequality, we get

$$\left(2\pi m - \int_{B(t)} H_l\right)^2 \leq 2\pi r\int_{\partial B(t)} |\frac{\partial H_l}{\partial \nu}|^2 \leq 2\pi r\int_{\partial B(t)} |\nabla H_l|^2.$$

Since $m^2 \geq m, m \in \mathbb{N}$, we could get

$$4\pi^2 m - C\int_{B(t)} H_l \leq 2\pi r\int_{\partial B(t)} |\nabla H_l|^2.$$

We already have $lH_l \to h$ in $W^{1,p}, \forall p < 2$, where h is bounded, so we have

$$lH_l \to h \text{ in } L^2.$$

For l small enough, we find

$$\int_{B(t)} (lH_l)^2 \leq \int_{B(l^2)} (lH_l)^2 \leq C\int_{B(l^2)} h^2 \leq Cl^4,$$

where C depends only on l and is independent of ρ.

Therefore we could get

$$\begin{aligned}
|\int_{B(t)} H_l| &\leq (\int_{B(t)} 1)^{\frac{1}{2}} (\int_{B(t)} H_l^2)^{\frac{1}{2}} \\
&\leq Clr.
\end{aligned}$$

Now for $\mathcal{F}(t) = \int_{\cup_{B\in\mathfrak{B}(r(t))}B} |\nabla H_l|^2 + |H_l|^2$ using the ball growth method, where $r(t) = \rho e^t, 0 \leq t \leq T = \log \frac{l^2}{\rho}$, we find

$$\begin{aligned}
\int_{\cup_{B\in\mathfrak{B}(l^2)}B \setminus \cup_{B\in\mathfrak{B}(\rho)}B} |\nabla H_l|^2 + |H_l|^2 &= \int_0^T r(t) \int_{\cup_{B(t)\in\mathfrak{B}(t)}\partial B(t)} |\nabla H_l|^2 + |H_l|^2 \mathrm{d}t \\
&\geq \int_0^T 2\pi n - Cr\mathrm{d}t \\
&\geq 2\pi n T - C(l^2 - \rho) \\
&\geq 2\pi n \log \frac{l^2}{\rho} - C.
\end{aligned}$$

So we get

$$\pi n \log \frac{\rho}{l^2} + \frac{1}{2} \int_{\cup_{B\in\mathfrak{B}(l^2)}B \setminus \cup_{B\in\mathfrak{B}(\rho)}B} |\nabla H_l|^2 + |H_l|^2 \geq C.$$

Combine the results above, we can get a lower bound of $l^2 E$

$$\begin{aligned}
\lim_{l\to 0} l^2 E(H_l) &= \lim_{l\to 0} \left(lE_{l^2} + \lim_{\rho\to 0} \left(l\pi n \log \frac{\rho}{l^2} + l\frac{1}{2} \int_{\cup_{B\in\mathfrak{B}(l^2)}B \setminus \cup_{B\in\mathfrak{B}(\rho)}B} |\nabla H_l|^2 + |H_l|^2 \right)\right) \\
&\geq \lim_{l\to 0} (lE_{l^2} - lC) \\
&= \lim_{l\to 0} \left(l\pi n \log l^2 + \frac{1}{2} \int_{S \setminus \cup_{B\in\mathfrak{B}(l^2)}B} |l\nabla H_l|^2 + |lH_l|^2 - l \cdot C \right) \\
&\geq F(h) \text{ by Fatou's lemma.}
\end{aligned}$$

This finishes the proof of the lower bound.

Upper bound

In this subsection, we prove the upper bound in the Γ convergence by using a similar method in [65]. For every distribution $\mu_L = \sum_{i=1}^n \delta_{L_i}$ in $(C^{0,\alpha}(S))^*$, where δ_{L_i} is Radon measure with support on line which is parallel with Y-axis, in lattice $L = [-\frac{1}{2}, \frac{1}{2}] \times [0, l]$, we choose one point on each line L_i. Note that if two lines coincide we could choose two different points on the same line. Denote these points as $\{p_i\}_{i=1}^n$, and by using the periodic boundary condition, we could get a set of points in square S, written as $\{p_{i,j}\}$ with $1 \leq i \leq n, 1 \leq j \leq \frac{1}{l}$ and for any i the set of points $\{p_{i,j}\}_{j=1}^{1/l}$ lie on the same line L_{p_i} (in fact $p_i = p_{i,\frac{1}{2l}+1}$). In fact, we can rewrite the function (2.8) as follows, it is the same as (2.8) because of the periodicity and uniqueness of the solution.

$$\begin{cases} -\Delta H_l + H_l = 2\pi \sum_{j=1}^{1/l} \sum_{i=1}^{n} \delta_{p_{i,j}} \quad \text{in } S \\ \qquad \text{periodic boundary conditions} \end{cases} \tag{2.13}$$

Let $G(x, y)$ be the Green function satisfies

$$\begin{cases} -\Delta_x G(x, y) + G(x, y) = 2\pi\delta_y(x) \quad \text{in } S \\ \qquad \text{periodic boundary conditions} \end{cases} \tag{2.14}$$

$G(x, y)$ has some properties as follows

- $G(x, y) = G(y, x)$;
- $G(x, y) = G(x - y, 0)$;
- $G(x, y) = -\log|x - y| + g(x, y)$, where $g(x, y)$ is C^1 and bounded in the diagonal of $S \times S$.

Then we have

$$H_l(x) = \sum_{i,j} G(x, p_{i,j})$$

We rewrite it as follows for convenience

$$H_l(x) = \sum_{i=1}^{\frac{n}{l}} G(x, p_i).$$

Then

$$E = \lim_{\rho\to 0}\left(\pi\frac{n}{l}\log\rho + \frac{1}{2}\int_{S\setminus\cup_{i=1}^{\frac{n}{l}} B(p_i,\rho)} |\nabla H_l|^2 + |H_l|^2\right).$$

For the second term on the right hand side, we have

$$\begin{aligned}
&\frac{1}{2}\int_{S\setminus\cup_{i=1}^{\frac{n}{l}} B(p_i,\rho)} |\nabla H_l|^2 + |H_l|^2 \\
= \;& -\frac{1}{2}\int_{\cup_i \partial B(p_i,\rho)} \frac{\partial H_l}{\partial \nu} H_l + \underbrace{\frac{1}{2}\int_{S\setminus\cup_i B(p_i,\rho)} -\Delta H_l \cdot H_l + |H_l|^2}_{=0 \text{ by the equation}} \\
= \;& -\frac{1}{2}\int_{\cup_i \partial B(p_i,\rho)} \frac{\partial \sum_j G(x, p_j)}{\partial \nu} \sum_j G(x, p_j) \\
= \;& -\frac{1}{2}\sum_i \int_{\partial B(p_i,\rho)} \frac{\partial G(x, p_i)}{\partial \nu} \sum_{j\neq i} G(x, p_j) - \frac{1}{2}\sum_i \int_{\partial B(p_i,\rho)} \frac{\partial G(x, p_i)}{\partial \nu} G(x, p_i) \\
& -\frac{1}{2}\sum_i \int_{\partial B(p_i,\rho)} \sum_{j\neq i} \frac{\partial G(x, p_j)}{\partial \nu} \sum_k G(x, p_k) \\
= \;& \pi \sum_{i\neq j} G(p_i, p_j) - \pi \sum_i \log\rho + \pi \sum_i g(p_i, p_i) + o(1).
\end{aligned}$$

The last equality is because we have

$$G(x, p_j) = -\log|x - p_j| + g(x, p_j).$$

Thus when we take the limit of $\rho \to 0$, we can get

$$E = \pi \sum_{i \neq j} G(p_i, p_j) + \sum_i g(p_i, p_i)$$

also we can rewrite it as

$$E = \pi \iint_{S\times S\setminus\Gamma} G(x, y) \mathrm{d}\mu_l \mathrm{d}\mu_l + \int_S g(x, x) \mathrm{d}\mu_l$$

where Γ is the diagonal of $S \times S$, $\mu_l = \sum_i \delta_{p_i}$.

If denote $\tilde{\mu}_l := l\mu_l$, then we have $\tilde{\mu}_l := l\mu_l \to \mu_L$ in $(C^{0,\alpha})^*$, $\forall\alpha \in [0, 1)$.

Now multiply l^2 to E we have

$$l^2 E = \pi \iint_{S\times S\setminus\Gamma} G(x, y) \mathrm{d}\tilde{\mu}_l \mathrm{d}\tilde{\mu}_l + l \int_S g(x, x) \mathrm{d}\tilde{\mu}_l$$

the second term at the right hand side is converge to 0 as $l \to 0$, for g is bounded in a neighborhood of the diagonal.

Next we will study the first term. Let $M > 0$, Γ_M is a neighborhood of the diagonal where $G(x, y)$ is greater than M, and Γ'_{2M} is another neighborhood of the diagonal such that $G(x, y)$ is less than $2M$ outside of it. What's more, we can construct the two neighborhoods such that $\Gamma'_{2M} \subset \Gamma_M$. Γ_M and Γ'_{2M} satisfy

- $G(x, y) \geq M \quad$ when $(x, y) \in \Gamma_M$;
- $G(x, y) \leq 2M \quad$ when $(x, y) \notin \Gamma'_{2M}$;
- $\Gamma'_{2M} \subsetneq \Gamma_M$.

Let $G_M(x, y) = \min(2M, G(x, y))$ in Γ_M, then we have

$$\begin{aligned}\pi \iint_{S\times S\setminus\Gamma} G(x, y)\mathrm{d}\tilde{\mu}_l \mathrm{d}\tilde{\mu}_l =& \pi \iint_{S\times S} G_M(x, y)\mathrm{d}\tilde{\mu}_l \mathrm{d}\tilde{\mu}_l \\ &+ \pi \iint_{\Gamma_M\setminus\Gamma} G(x, y) - G_M(x, y)\mathrm{d}\tilde{\mu}_l \mathrm{d}\tilde{\mu}_l - \pi nl \cdot 2M\end{aligned}$$

where the term $\pi nl \cdot 2M$ comes from the diagonal.

If we first take the limit of $l \to 0$, and then $M \to \infty$ we can use the weak convergence of the measure, and connect the two dimensional energy E with the one dimensional energy F.

Now we prove that

$$\pi \iint_{\Gamma_M\setminus\Gamma} G(x, y) - G_M(x, y)\mathrm{d}\tilde{\mu}_l \mathrm{d}\tilde{\mu}_l \to 0 \quad \text{as} \quad \text{firstly} \quad l \to 0, \quad \text{secondly } M \to \infty.$$

Note that $G(x, y) = -\log|x - y| + g(x, y)$ where g is bounded near the diagonal and $G_M(x, y)$ is bounded by $2M$, so we only need to prove

$$\pi \iint_{\Gamma_M \setminus \Gamma} - \log|x-y| \mathrm{d}\tilde{\mu}_l \mathrm{d}\tilde{\mu}_l \to 0 \quad \text{as} \quad \text{firstly} \quad l \to 0, \quad \text{secondly } M \to \infty.$$

Denote $I_{i,l}$ be the line segment of length l with p_i its midpoint, and parallel with the $Y - axis$.

For $i \neq j$, precisely only one of the following statement about $I_{i,l}$ and $I_{j,l}$ holds

α :either
$I_{i,l}$ and $I_{j,l}$ touch each other;

β : or else
$I_{i,l}$ and $I_{j,l}$ are separated.

The number of the pairs of segments in the first case is at most $\frac{n}{l} \cdot 2n$, and

$$\sum_{i,j \in \alpha} \iint_{I_{i,l} \times I_{j,l}} - \log|x-y| \mathrm{d}l\delta_{p_i} \mathrm{d}l\delta_{p_j} \leq -2\frac{n^2}{l} \cdot l^2 \log(\theta l)$$

where $\theta = \min\{\theta_{i,j} : |p_i - p_j| = \theta_{i,j} l, \ \forall i, j \quad \text{and} \quad i \neq j\}$.

In the second case, if l is small enough, we could get for $\forall (x, y) \in I_{i,l} \times I_{j,l}$

$$|p_i - p_j| > \frac{1}{2}|x - y|$$

so

$$-\log|p_i - p_j| < -\log|x - y| + \log 2$$

and then

$$\iint_{I_{i,l} \times I_{j,l}} - \log|x-y| \, \mathrm{d}l\delta_{p_i} \mathrm{d}l\delta_{p_j} = -l^2 \log|p_i - p_j| < \iint_{I_{i,l} \times I_{j,l}} - \log|x-y| + \log 2 \, \mathrm{d}\mu_L \mathrm{d}\mu_L$$

Sum all pairs $(i, j) \in \beta$, we could get

$$\sum_{i,j \in \beta} \iint_{I_{i,l} \times I_{j,l}} - \log|x-y| \, \mathrm{d}l\delta_{p_i} \mathrm{d}l\delta_{p_j} < \iint_{\Gamma_M \setminus \Gamma} - \log|x-y| + \log 2 \, \mathrm{d}\mu_L \mathrm{d}\mu_L$$

Sum up the two case, and we could get what we want

$$\pi \iint_{\Gamma_M \setminus \Gamma} G(x, y) - G_M(x, y) \mathrm{d}\tilde{\mu}_l \mathrm{d}\tilde{\mu}_l \to 0 \quad \text{as} \quad \text{firstly} \quad l \to 0, \quad \text{secondly } M \to \infty.$$

While the term

$$\pi \iint_{S \times S} G_M(x, y) \mathrm{d}\tilde{\mu}_l \mathrm{d}\tilde{\mu}_l \leq \pi \iint_{S \times S} G(x, y) \mathrm{d}\mu_L \mathrm{d}\mu_L$$

The right hand side is exactly the energy F. In fact the measure $\mu_L \in H^{-1}$, because by using the Trace operator, we can define $\langle \mu_L, f \rangle$ for any $f \in H_0^1$. Then let

$$u(x) = \int G(x, y) \mathrm{d}\mu_L(y)$$

by using Theorem 1 of [19] we get

$$\|u(x)\|_{H^1} = 2\pi \iint G(x,y)\mathrm{d}\mu_L \mathrm{d}\mu_L.$$

That finishes the proof of the upper bound.

2.3 Proof of Theorem 2.1.1

We have proven that $l^2 E \xrightarrow{\Gamma} F$ as $l \to 0$. From Theorem 2.2.1, we know that the number of points contained in the minimizer of $W_{1D}(\mathbf{p}, \Delta_{ex})$ jumps one by one as the Δ_{ex} grows. So for every $N \in \mathbb{N}$, there exists l_N, such that for any $l < l_N$, the number of vortices contained in the minimizer of $W(\mathbf{p}, \Delta_{ex})$ jumps one by one from 1 to N as the Δ_{ex} grows. For fixed l, we have the lattice $L = [-\frac{1}{2}, \frac{1}{2}] \times [0, l]$. If the Theorem 2.1.1 is not true. Then we could find a sequence $\{\varepsilon\}$, such that there exists a value $\Delta_m(\varepsilon)$ which is bounded independent of ε and at $h_{ex} = H_{C_1}(\varepsilon) + \Delta_m(\varepsilon) + o(1)$, the number of vortices contained in the minimizer of the Ginzburg-Landau energy GL_ε jumps from m to $m+k$, for some $m, k \in \mathbb{N}$ and $k \geq 2$. When we take the limit $\varepsilon \to 0$, from Theorem 2.1.2, we have that the minimizers $(u_\varepsilon.\mathbf{A}_\varepsilon)$ of GL_ε satisfies $h_\varepsilon = \mathrm{curl}\mathbf{A}_\varepsilon$ converge to h_* and $n_\varepsilon = \frac{1}{2\pi}\int_L \mathrm{curl}\mathbf{A}$ converge to n_*. And there are n_* distinct points $\{a_i\}_{i=1}^{n_*}$ in L which minimizes $W(\mathbf{p}, \Delta_{ex})$ over $\mathcal{P}$. So at $\Delta_m = \lim_{\varepsilon \to 0} \Delta_m(\varepsilon)$, the number of points of the minimizer $W(\mathbf{p}, \Delta_{ex})$ would jumps from m to $m+k$. This contradicts our conclusion about the renormalized energy. This finishes the proof.

Chapter 3

Optimality of Abrikosov Lattice in a Periodic Ginzburg-Landau Model

In this chapter, we study the configuration of vortices which minimizes a renormalized energy related to Ginzburg-Landau model. Among all the Bravais lattices, we prove that the triangular lattice minimizes this renormalized energy.

3.1 Introduction

For type-II superconductors, A. Abrikosov [2] predicted that the triangular lattice, now called "Abrikosov lattice", would appear. There are some rigorous mathematical results related to this phenomenon, for example [4],[5],[13],[64]. In [64], E. Sandier and S. Serfaty have proven that the vortices of minimizers of the Ginzburg-Landau energy, blown-up at a suitable scale, converges to minimizers of a "Coulombian Renormalized Energy ", and in the periodic case, the triangular lattice minimizes this renormalized energy. In this paper, we consider another renormalized energy for a periodic Ginzburg-Landau energy introduced in [9] and prove that the triangular lattice is the unique minimizer of this renormalized energy among all the Bravais lattices. One can refer to [22] for a related work in one dimension.

Let $\mathcal{L} = \{\mathbb{Z}\vec{u} \oplus \mathbb{Z}\vec{v} \mid \det(\vec{u}, \vec{v}) = 1\}$. For $\Lambda \in \mathcal{L}$, we define $L = \mathbb{R}^2/\Lambda$, hence $|L| = 1$. We introduce the renormalized energy W which is defined in [9] over $\mathcal{L}$ as follows

$$W(n, \Lambda) = \lim_{\varepsilon \to 0} \left(\pi n \log \varepsilon + \frac{1}{2} \int_{L \setminus \cup_{i=1}^{n} B(p_i, \varepsilon)} |\nabla h|^2 + h^2 \right),$$

where $\{p_i\}_{i=1}^n$ are n points in L, and h satisfies

$$\begin{cases} -\Delta h + h = 2\pi \sum_{i=1}^{n} \delta_{p_i} \quad \text{in} \quad L \\ \text{periodic boundary conditions.} \end{cases} \tag{3.1}$$

In fact, this energy is a renormalized energy for the Ginzburg-Landau energy in the periodic setting. In the case of $n = 1$, i. e. among the Bravais lattices, we prove

Theorem 3.1.1. *The triangular lattice, modulo rotations, is the unique minimizer of W among all Bravais lattices.*

In the proof of this theorem, we use a technique which has already been used in [64] to rewrite the renormalized energy W in an explicit formula related to Jacobi Theta Function, then by applying a result of H.L.Montgomery[54], we complete the proof.

3.2 Proof of Theorem 3.1.1

We follow the idea of [64] to rewrite the renormalized energy W in an explicit formula. When $n = 1$,

$$W(\Lambda) = \lim_{\varepsilon\to 0}\left(\pi \log \varepsilon + \frac{1}{2}\int_{L\setminus B(0,\varepsilon)} |\nabla h|^2 + h^2\right),$$

where h satisfies

$$\begin{cases} -\Delta h + h = 2\pi\delta_0 \quad \text{in} \quad L \\ \text{periodic boundary conditions.} \end{cases} \tag{3.1}$$

Lemma 3.2.1. *For any $\Lambda \in \mathcal{L}$, we have*

$$W(\Lambda) = \pi \lim_{x\to 0}(h(x) + \log|x|).$$

Proof. We have

$$\pi \log \varepsilon + \frac{1}{2}\int_{L\setminus B(0,\varepsilon)} |\nabla h|^2 + h^2 = \pi \log \varepsilon - \frac{1}{2}\int_{\partial B(0,\varepsilon)} \frac{\partial h}{\partial \nu}\cdot h$$

where ν is the outer-pointing unit normal vector with respect to the corresponding boundary. In fact, $h(x) = -\log|x| + g(x)$, where $g(x)$ is C^1 near origin. So

$$\frac{\partial h}{\partial \nu}\Big|_{\partial B(0,\varepsilon)} = -\frac{1}{\varepsilon} + \frac{\partial g}{\partial \nu}\Big|_{\partial B(0,\varepsilon)}.$$

Therefore,

$$W(\Lambda) = \lim_{x\to 0}(\pi \log|x| + \pi h(x) + O(|x|\cdot \log|x|)) = \pi \lim_{x\to 0}(h(x) + \log|x|).$$

$\square$

Next we prove an important lemma by following the same method in [64].

Lemma 3.2.2. *There exists a constant $C_0 \in \mathbb{R}$, such that for any $\Lambda \in \mathcal{L}$, we have*

$$W(\Lambda) = C_0 + \pi \lim_{x\to 0}\left(\zeta_{\Lambda^*}(x) - \int_{\mathbb{R}^2} \frac{2\pi}{1 + 4\pi^2|y|^{2+x}}\mathrm{d}y\right),$$

where Λ^* is the dual lattice of Λ, i.e. the set of vectors q such that $q \cdot p \in \mathbb{Z}$ for every $p \in \Lambda$, and $\zeta_{\Lambda^*}(x) = \sum_{p\in\Lambda^*} \frac{2\pi}{1+4\pi^2|p|^{2+x}}$.

Proof. We already have

$$W(\Lambda) = \pi \lim_{x\to 0}(h(x) + \log|x|).$$

We introduce the Green function $G(x) \in L^2(\mathbb{R}^2)$ which is the solution of $-\Delta G + G = 2\pi\delta_0$ in $\mathbb{R}^2$, and by the periodic boundary conditions, we can consider the function $h(x)$ as a function in $\mathbb{R}^2$, i.e. the solution of

$$-\Delta h_\Lambda + h_\Lambda = 2\pi \sum_{p\in\Lambda} \delta_p.$$

Then we can write

$$h_\Lambda(x) + \log|x| = G(x) + \log|x| + u_\Lambda(x),$$

where $u_\Lambda(x) = h_\Lambda(x) - G(x)$ and it depends on lattice Λ. It is well known that $h_\Lambda(x) + \log|x|$, $G(x) + \log|x|$, $u_\Lambda(x)$ are C^1 near 0. Note that $G(x) + \log|x|$ is independent of lattice Λ, so

$$W(\Lambda) = \pi \lim_{x\to 0}(h_\Lambda(x) + \log|x|) = C_0 + \pi \cdot u_\Lambda(0),$$

where $C_0 = \lim_{x\to 0} G(x) + \log|x|$.

Denote by $\varphi(x) = (2\pi)^{-1}\mathrm{e}^{-|x|^2/2}$ the Gaussian distribution in $\mathbb{R}^2$ and $\varphi_n(x) = n^2\varphi(nx)$ for any $n \in \mathbb{N}$, so $\{\varphi_n(x)\}_n$ is an approximation of the Dirac mass. Since $u_\Lambda(x)$ is C^1 near 0, we have

$$u_\Lambda(0) = \lim_{n\to\infty} w(n,\Lambda),$$

where

$$w(n,\Lambda) = \int_{\mathbb{R}^2} \varphi_n(x)u_\Lambda(x)\mathrm{d}x = \int_{\mathbb{R}^2} \hat{\varphi}_n(\xi)\hat{u}_\Lambda(\xi)\mathrm{d}\xi.$$

We know that $\hat{\varphi}_n(\xi) = \mathrm{e}^{-2\pi^2|\xi|^2/n^2}$, and $\hat{u}_\Lambda(\xi) = \hat{h}(\xi) - \hat{G}(\xi)$, where $\hat{h}(\xi) = \frac{2\pi\sum_{p\in\Lambda^*}\delta_p(\xi)}{4\pi^2|\xi|^2+1}$ (2π comes form the fact that $|L| = 1$) and $\hat{G}(\xi) = \frac{2\pi}{4\pi^2|\xi|^2+1}$. Hence

$$w(n,\Lambda) = 2\pi\left(\sum_{p\in\Lambda^*}\frac{\mathrm{e}^{-2\pi^2|p|^2/n^2}}{4\pi^2|p|^2+1} - \int_{\mathbb{R}^2}\frac{\mathrm{e}^{-2\pi^2|y|^2/n^2}}{4\pi^2|y|^2+1}\mathrm{d}y\right).$$

We claim that

$$\lim_{n\to\infty} w(n,\Lambda) = \lim_{x\to 0^+} v(x,\Lambda),$$

where $v(x,\Lambda) = 2\pi\left(\sum_{p\in\Lambda^*}\frac{1}{4\pi^2|p|^{2+x}+1} - \int_{\mathbb{R}^2}\frac{1}{4\pi^2|y|^{2+x}+1}\mathrm{d}y\right)$, $x > 0$.

In fact, for any $p \in \Lambda^*$, denote by K_p the Voronoi cell centered at p, i.e. the region in $\mathbb{R}^2$ consisting of all the points closer to p than to any other points in Λ^*. Note that K_p is periodic due to the periodicity of lattice Λ^* and $|K_p| = 1$. Denote by $\mathbf{1}_{K_p}$ the characteristic function with respect to K_p, then we have

$$w(n,\Lambda) = 2\pi\int_{\mathbb{R}^2}\sum_{p\in\Lambda^*}\mathbf{1}_{K_p}\cdot\left(\frac{\mathrm{e}^{-2\pi^2|p|^2/n^2}}{4\pi^2|p|^2+1} - \frac{\mathrm{e}^{-2\pi^2|y|^2/n^2}}{4\pi^2|y|^2+1}\right)\mathrm{d}y.$$

By applying the mean value theorem to $\frac{\mathrm{e}^{-2\pi^2|p|^2/n^2}}{4\pi^2|p|^2+1} - \frac{\mathrm{e}^{-2\pi^2|y|^2/n^2}}{4\pi^2|y|^2+1}$, we get a bound for the integrand function

$$\Big|\sum_{p\in\Lambda^*}\mathbf{1}_{K_p}\cdot\left(\frac{\mathrm{e}^{-2\pi^2|p|^2/n^2}}{4\pi^2|p|^2+1} - \frac{\mathrm{e}^{-2\pi^2|y|^2/n^2}}{4\pi^2|y|^2+1}\right)\Big| \le C\frac{1}{|y|^3+1},$$

where the constant C is independent of n. The function at the right hand side is an integrable function over the whole plane. The LebesgueâĂŹs dominated convergence theorem implies that

$$\lim_{n\to\infty} w(n,\Lambda) = 2\pi\int_{\mathbb{R}^2}\sum_{p\in\Lambda^*}\mathbf{1}_{K_p}\cdot\left(\frac{1}{4\pi^2|p|^2+1} - \frac{1}{4\pi^2|y|^2+1}\right)\mathrm{d}y.$$

Similarly, we have

$$\lim_{x\to 0^+} v(x,\Lambda) = 2\pi \int_{\mathbb{R}^2} \sum_{p\in\Lambda^*} \mathbf{1}_{K_p} \cdot \left(\frac{1}{4\pi^2|p|^2+1} - \frac{1}{4\pi^2|y|^2+1} \right) \mathrm{d}y.$$

By combining the results above, we prove the lemma.

□

Now we consider the term

$$\zeta_{\Lambda^*}(x) = \sum_{p\in\Lambda^*} \frac{2\pi}{4\pi^2|p|^{2+x}+1}.$$

Let $\zeta^0_{\Lambda^*}(x) = \sum_{p\in\Lambda^*\setminus\{0\}} \frac{2\pi}{4\pi^2|p|^{2+x}}$, we can split $\zeta_{\Lambda^*}(x)$ as follows,

$$\begin{aligned} \zeta_{\Lambda^*}(x) &= 2\pi + \zeta^0_{\Lambda^*}(x) - 2\pi \sum_{p\in\Lambda^*\setminus\{0\}} \frac{1}{4\pi^2|p|^{2+x}\cdot(4\pi^2|p|^{2+x}+1)} \\ &= 2\pi + \zeta^0_{\Lambda^*}(x) - 2\pi \sum_{p\in\Lambda^*\setminus\{0\}} \frac{1}{4\pi^2|p|^{2}\cdot(4\pi^2|p|^{2}+1)} + o(1). \end{aligned}$$

Note here $o(1)$ means $o(1)\to 0$ as $x\to 0$ for any fixed $\Lambda\in\mathcal{L}$, but the convergence is not uniform w.r.t. Λ.

We will consider $\zeta^0_{\Lambda^*}(x) - 2\pi\sum_{p\in\Lambda^*\setminus\{0\}} \frac{1}{4\pi^2|p|^2\cdot(4\pi^2|p|^2+1)}$ together.

If $4\pi^2|p|^2 > 1$, we can have a series expansion of the second term. We can do this at least in a neighborhood of the triangular lattice, because the length of the edge is $\sqrt{2/\sqrt{3}} > 1$.

$$\sum_{p\in\Lambda^*\setminus\{0\}} \frac{1}{4\pi^2|p|^2\cdot(4\pi^2|p|^2+1)} = \sum_{p\in\Lambda^*\setminus\{0\}} \frac{1}{(4\pi^2|p|^2)^2\cdot(1+(4\pi^2|p|^2)^{-1})} = \sum_{p\in\Lambda^*\setminus\{0\}} \sum_{n=2}^{\infty} \frac{(-1)^n}{(4\pi^2|p|^2)^n}.$$

Since the summation $\sum_{p\in\Lambda^*\setminus\{0\}}\sum_{n=2}^{\infty}\frac{(-1)^n}{(4\pi^2|p|^2)^n}$ converges absolutely, we can change the order of the summation.

$$\sum_{p\in\Lambda^*\setminus\{0\}} \sum_{n=2}^{\infty} \frac{(-1)^n}{(4\pi^2|p|^2)^n} = \sum_{n=2}^{\infty} \sum_{p\in\Lambda^*\setminus\{0\}} \frac{(-1)^n}{(4\pi^2|p|^2)^n}$$

We write $\sum_{n=2}^{\infty}\sum_{p\in\Lambda^*\setminus\{0\}}\frac{(-1)^n}{(4\pi^2|p|^2)^n} = \sum_{n=2}^{\infty}(-1)^n g_{n,\Lambda^*}$ for convenience, where $g_{n,\Lambda^*} = \sum_{p\in\Lambda^*\setminus\{0\}}\frac{1}{(4\pi^2|p|^2)^n}$. Let $s = 1+\frac{x}{2}$, $x>0$, then by using a result in [54], we have

$$\frac{1}{2\pi}\cdot 4\pi^2\cdot\zeta^0_{\Lambda^*}(x)\cdot 2^s\cdot\Gamma(s)\cdot(2\pi)^{-s} = \frac{1}{s-1} - \frac{1}{s} + \int_1^{+\infty} (\theta_{\Lambda^*}(\alpha)-1)(\alpha^s+\alpha^{1-s})\frac{\mathrm{d}\alpha}{\alpha}$$

where $\theta_{\Lambda^*}(\alpha) = \sum_{p\in\Lambda^*} \mathrm{e}^{-\pi\alpha|p|^2}$.
Similarly, we have

$$(4\pi^2)^n\cdot g_{n,\Lambda^*}(x)\cdot 2^n\cdot\Gamma(n)\cdot(2\pi)^{-n} = \frac{1}{n-1} - \frac{1}{n} + \int_1^{+\infty} (\theta_{\Lambda^*}(\alpha)-1)(\alpha^n+\alpha^{1-n})\frac{\mathrm{d}\alpha}{\alpha}.$$

Therefore, we have

$$\begin{aligned}
\zeta_{\Lambda^*}(x) &= 2\pi + \zeta^0_{\Lambda^*}(x) - 2\pi \sum_{p\in\Lambda^*\backslash\{0\}} \frac{1}{4\pi^2|p|^2\cdot(4\pi^2|p|^2+1)} + o(1) \\
&= 2\pi + \frac{\pi^{s-1}}{2\Gamma(s)}(\frac{1}{s-1}-\frac{1}{s}) + \sum_{n=2}^{\infty} 2\pi\frac{(-1)^{n-1}}{(4\pi)^n\Gamma(n)}(\frac{1}{n-1}-\frac{1}{n}) \\
&\quad +2\pi\int_1^{+\infty}(\theta_{\Lambda^*}(\alpha)-1)\cdot\frac{\pi^{s-1}}{4\pi\Gamma(s)}\cdot(\alpha^s+\alpha^{1-s})\frac{\mathrm{d}\alpha}{\alpha} \\
&\quad +\sum_{n=2}^{\infty}2\pi\int_1^{+\infty}(\theta_{\Lambda^*}(\alpha)-1)\frac{(-1)^{n-1}}{(4\pi)^n\Gamma(n)}(\alpha^n+\alpha^{1-n})\frac{\mathrm{d}\alpha}{\alpha} + o(1) \\
&= 2\pi + f(x) + c_0 + 2\pi\int_1^{+\infty}(\theta_{\Lambda^*}(\alpha)-1)\cdot I(x,\alpha)\frac{\mathrm{d}\alpha}{\alpha} + o(1)
\end{aligned}$$

where $f(x) = \frac{\pi^{s-1}}{2\Gamma(s)}(\frac{1}{s-1}-\frac{1}{s})$, $c_0 = \sum_{n=2}^{\infty} 2\pi\frac{(-1)^{n-1}}{(4\pi)^n\Gamma(n)}(\frac{1}{n-1}-\frac{1}{n})$ and $I(x,\alpha) = \frac{\pi^{s-1}}{4\pi\Gamma(s)}\cdot(\alpha^s+\alpha^{1-s}) + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(4\pi)^n\Gamma(n)}(\alpha^n+\alpha^{1-n})$.
For any α fixed, we have

$$\begin{aligned}
& I(x,\alpha) \\
=\ & \left(\frac{\pi^{s-1}}{4\pi\Gamma(s)}\cdot\alpha^s + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(4\pi)^n\Gamma(n)}\alpha^n\right) + \left(\frac{\pi^{s-1}}{4\pi\Gamma(s)}\alpha^{1-s} + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(4\pi)^n\Gamma(n)}\alpha^{1-n}\right) \\
=\ & \frac{\alpha}{4\pi}\left(\frac{(\pi\alpha)^{s-1}}{\Gamma(s)} + \mathrm{e}^{-\frac{\alpha}{4\pi}} - 1\right) + \frac{1}{4\pi}\left(\frac{\pi^{s-1}}{\Gamma(s)}\alpha^{1-s} + \mathrm{e}^{-\frac{1}{4\pi\alpha}} - 1\right)
\end{aligned}$$

$\Gamma(s)$ is convex in $[1,2]$, and $\Gamma(1) = \Gamma(2) = 1$, so for $s \in [1,2]$, $\Gamma(s) \leq 1$, while $(\pi\alpha)^{s-1} \geq 1$, for $\alpha \geq 1$, $s \in [1,2]$. Hence

$$\frac{(\pi\alpha)^{s-1}}{\Gamma(s)} - 1 \geq 0.$$

Similarly, we have $\frac{\pi^{s-1}}{\Gamma(s)}\alpha^{1-s} \geq \alpha^{1-s}$, and the fact that $1-\mathrm{e}^{-\frac{1}{4\pi\alpha}} < \frac{1}{4\pi\alpha}$ implies that $\frac{\pi^{s-1}}{\Gamma(s)}\alpha^{1-s} + \mathrm{e}^{-\frac{1}{4\pi\alpha}} - 1 > 0$ for $\alpha \geq 1$, $s \in [1,2]$.
By combining the results above, we have $I(x,\alpha) > 0$ for $\alpha \geq 1$, $s \in [1,2]$.

Next we will prove that

$$\zeta_{\Lambda^*}(x) = 2\pi + f(x) + c_0 + 2\pi\int_1^{+\infty}(\theta_{\Lambda^*}(\alpha)-1)I(x,\alpha)\frac{\mathrm{d}\alpha}{\alpha} + o(1)$$

is true not just for lattices in a neighborhood of triangular lattice but for all Bravais lattices with area 1. We claim that both

$$f_1(\Lambda) = \zeta^0_{\Lambda^*}(x) - 2\pi\sum_{p\in\Lambda^*\backslash\{0\}}\frac{1}{4\pi^2|p|^2\cdot(4\pi^2|p|^2+1)}$$

and

$$f_2(\Lambda) = f(x) + c_0 + 2\pi\int_1^{+\infty}(\theta_{\Lambda^*}(\alpha)-1)I(x,\alpha)\frac{\mathrm{d}\alpha}{\alpha}$$

are analytic w.r.t. lattice. It means that if we denote by $\vec{u} = (a,0), a > 0$, $\vec{v} = (b,c) = (b,1/a)$ the vectors which generate the lattice Λ^*, the two functions are analytic w.r.t.

$\vec{u}, \vec{v}$, i.e. a, b. If $p = m\vec{u} + n\vec{v} = (ma + nb, nc)$, then $|p|^2 = (ma + nb)^2 + n^2c^2$. For $\sum_{p\in\Lambda^*\setminus\{0\}} \frac{1}{4\pi^2|p|^2\cdot(4\pi^2|p|^2+1)}$, at $(a_0, b_0, c_0), a_0 > 0$, we have

$$\begin{aligned}
&\sum_{p\in\Lambda^*\setminus\{0\}} \frac{1}{4\pi^2|p|^2\cdot(4\pi^2|p|^2+1)} \\
= &\sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{4\pi^2[(ma+nb)^2+n^2c^2]\cdot[4\pi^2((ma+nb)^2+n^2c^2)+1)]} \\
= &\sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{4\pi^2[(ma_0+nb_0)^2+n^2c_0^2+R(a-a_0,b-b_0,c-c_0)]} \cdot \frac{1}{4\pi^2[(ma_0+nb_0)^2+n^2c_0^2+R(a-a_0,b-b_0,c-c_0)]+1} \\
= &\sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{[4\pi^2(m^2a_0^2+2a_0b_0mn+n^2(b_0^2+c_0^2))]\cdot[4\pi^2(m^2a_0^2+2a_0b_0mn+n^2(b_0^2+c_0^2))+1]} \\
&\cdot \frac{1}{1+\frac{R(a-a_0,b-b_0,c-c_0)}{m^2a_0^2+2a_0b_0mn+n^2(b_0^2+c_0^2)}} \cdot \frac{1}{1+\frac{4\pi^2R(a-a_0,b-b_0,c-c_0)}{4\pi^2(m^2a_0^2+2a_0b_0mn+n^2(b_0^2+c_0^2))+1}}
\end{aligned}$$

We obtain a series expansion of the formula above by expanding the function $\frac{1}{1+x}$ at 0 and rearranging the terms since that the coefficients converge absolutely. Take a function composition with $c = 1/a$, we obtain that

$$\sum_{p\in\Lambda^*\setminus\{0\}} \frac{1}{4\pi^2|p|^2\cdot(4\pi^2|p|^2+1)}$$

is analytic w.r.t. lattice.

Similarly, the function $\zeta^0_{\Lambda^*}(x)$ is analytic w.r.t. lattice.

For the function $f_2(\Lambda)$, $f(x) + c_0$ is independent of lattice, so we only need to prove that $2\pi\int_1^{+\infty}(\theta_{\Lambda^*}(\alpha) - 1)I(x,\alpha)\frac{\mathrm{d}\alpha}{\alpha}$ is analytic w.r.t. lattice. The series is a positive series, it converges absolutely. The function $\theta_{\Lambda^*}(\alpha) - 1$ is a positive series and converges absolutely for any α, and each term in the series is analytic, so we rewrite the function $2\pi\int_1^{+\infty}(\theta_{\Lambda^*}(\alpha) - 1)I(x,\alpha)\frac{\mathrm{d}\alpha}{\alpha}$ in the form of series w.r.t. lattice. Therefore, the function $f(x) + c_0 + 2\pi\int_1^{+\infty}(\theta_{\Lambda^*}(\alpha) - 1)I(x,\alpha)\frac{\mathrm{d}\alpha}{\alpha}$ is analytic w.r.t. lattice.

Now we know that the functions $f_1(\Lambda)$ and $f_2(\Lambda)$ are analytic, and $f_1 = f_2$ in a neighborhood of triangular lattice, so $f_1 \equiv f_2$ for all lattices with fixed area 1.

We use a result due to Montgomery,

Theorem 3.2.1 ([54]). *For any $\alpha > 0$,*

$$\theta_f(\alpha) \geq \theta_h(\alpha),$$

where $f(\mathbf{u}) = f(u_1, u_2) = au_1^2 + bu_1u_2 + cu_2^2$ be a positive definite binary quadratic form with real coefficient and discriminant $b^2-4ac = -1$, and $h(\mathbf{u}) = \frac{1}{\sqrt{3}}(u_1^2+u_1u_2+u_2^2)$. If there is an $\alpha > 0$ such that $\theta_f(\alpha) = \theta_h(\alpha)$, then f and h are equivalent forms and $\theta_f(\alpha) \equiv \theta_h(\alpha)$.

From the theorem above, we know that the minimum of the Jacobi Theta function θ over $\mathcal{L}$ (recall that $\mathcal{L}$ is the set of all Bravais lattices with area 1) is uniquely achieved by Λ_0^*, $\Lambda_0 = \sqrt{\frac{2}{\sqrt{3}}}(\mathbb{Z}(1,0)\oplus\mathbb{Z}(1/2, \sqrt{3}/2))$. Denote by Λ a Bravais lattice, then apply Lebesgue's dominated convergence theorem, we have

$$W(\Lambda) - W(\Lambda_0) = \pi\lim_{x\to0}(\zeta_{\Lambda^*}(x) - \zeta_{\Lambda_0^*}(x)) = \pi\lim_{x\to0}2\pi\int_1^{+\infty}(\theta_{\Lambda^*} - \theta_{\Lambda_0^*})I(x,\alpha)\frac{\mathrm{d}\alpha}{\alpha}$$

$$= \quad 2\pi^2 \int_1^{+\infty} (\theta_{\Lambda^*} - \theta_{\Lambda_0^*}) I(0, \alpha) \, \frac{\mathrm{d}\alpha}{\alpha}$$

By using Theorem 1 of [54] and the fact that $I(0, \alpha) > 0$, we have $W(\Lambda) \geq W(\Lambda_0)$ for all lattice $\Lambda \in \mathcal{L}$, and the equality holds if and only if $\Lambda = \Lambda_0$. Therefore the triangular lattice is the unique minimizer of energy $W(\Lambda)$.

Chapter 4

The Lennard-Jones Model and Thomas-Fermi Model

This is joint work with Laurent Bétermin. It was accepted by the journal *Communications in Contemporary Mathematics* and published online

Communications in Contemporary Mathematics, 2014, doi: 10.1142/S0219199714500497.

Original title:

Minimization of energy per particle among Bravais lattices in $\mathbb{R}^2$: Lennard-Jones and Thomas-Fermi cases.

In this chaper, we prove that the minimizer of Lennard-Jones energy per particle among Bravais lattices is a triangular lattice, i.e. composed of equilateral triangles, in $\mathbb{R}^2$ for large density of points, while it is false for sufficiently small density. We show some characterization results for the global minimizer of this energy and finally we also prove that the minimizer of the Thomas-Fermi energy per particle in $\mathbb{R}^2$ among Bravais lattices with fixed density is triangular.

4.1 Introduction

Understanding the structure of matter at low temperature has been a challenge for many years. In this case, one of the simplest models is to consider identical points as particles interacting in a Lennard-Jones potential. This model is deterministic, therefore we do not consider either entropy nor other quantum effects. The problem is to find the configuration of the points which minimize the total interaction energy, called the Lennard-Jones energy. Radin, in [39], studied this problem in one dimension and showed that, in the case of infinite points, the minimizer is periodic. His method is not adaptable in higher dimensions and he studied, in [46, 58] the case of short range interactions and proved the first result of crystallization in two dimensions for a hard-sphere model. In the meantime, Ventevogel and Nijboer gave in [75, 76, 77] more general results in one dimension for Lennard-Jones energy per particle. Indeed, they showed that a unique lattice of the form

$a_0\mathbb{N}$ minimizes the Lennard-Jones energy and that all lattices $a\mathbb{N}$ with $a \leq a_0$ minimize this energy when the density of points $\rho = a^{-1}$ is fixed. Our paper gives some results in the spirit of the latter paper.

After a numerical investigation of Yedder, Blanc, Le Bris, in [18], about the minimization of the Lennard-Jones and the Thomas-Fermi energy in $\mathbb{R}^2$, it seemed that the triangular lattice, also called “hexagonal lattice" – which is composed of equilateral triangles – is the minimum configuration for Lennard-Jones energy among any lattices and for Thomas-Fermi energy with nuclei density fixed. Some time after, Theil, in [72], gave the first proof of crystallization in two dimensions for a “Lennard-Jones like" potential, with a minimum less than one but very close to one and long range interaction. He showed that the global minimizer of the total energy is triangular. His method was adapted by E and Li, in [34], for a three-body potential with long range interactions in order to obtain a honeycomb lattice as global minimizer – see also the works of Mainini, Piovano and Stefanelli in [51, 52] about the crystallization in square and honeycomb lattices for three-body potentials with short range interactions – and by Theil and Flatley in three dimensions in [38].

Furthermore Montgomery, in [54], proved that the triangular lattice is the unique minimizer of theta functions among Bravais lattices with fixed density and hence the unique minimizer of the Epstein zeta function, thanks to the link between these two functions. As the Lennard-Jones potential is a linear sum of Epstein zeta functions, it is natural to study the problem of minimization of the Lennard-Jones energy among Bravais lattices with and without fixed density. However, there are few results about minimization in the general case of periodic systems. For example, Cohn and Kumar described in [24] a method and a conjecture for completely monotonic functions. It is interesting to observe that this kind of problem is connected with the theory of spherical design due to Delsarte, Goethals and Seidel in [30] and linked to the layers of a lattice, among others, by Venkov and Bachoc in [74, 10] and by Coulangeon et al. in [26, 28, 27].

In this paper, our main results are :

Theorem:

- *Let $V_{LJ}(r) = r^{-12} - 2r^{-6}$ be the Lennard-Jones potential, then the minimizer of the energy $E_{LJ}(L) = \displaystyle\sum_{x \in L\backslash\{0\}} V_{LJ}(\|x\|)$ among all Bravais lattices of $\mathbb{R}^2$ with fixed density sufficiently large is triangular and unique, up to rotation.*

- *A minimizer of E_{LJ} among all Bravais lattices with fixed density sufficiently small cannot be triangular.*

- *Let $W_{TF} : \mathbb{R}_+^* \to \mathbb{R}$ be the solution of $-\Delta h + \pi h = \delta_0$ which goes to 0 at infinity, then the minimizer of the Thomas-Fermi energy $E_{TF}(L) = \displaystyle\sum_{x \in L\backslash\{0\}} W_{TF}(\|x\|)$ among all Bravais lattices of $\mathbb{R}^2$ with density fixed is triangular and unique, up to rotation.*

This paper is structured as follows : in Section 2, we introduce the notations; in Section 3, we show that the minimizer of the Lennard-Jones energy per particle among Bravais lattices with fixed density, if the density is sufficiently large, it is triangular and unique. Moreover we give numerical results and a conjecture for the minimization with density fixed and we have arguments in order to explain why the global minimizer, among Bravais lattices without fixed density, is triangular; in Section 4, we use proof of Blanc in [16] to find a lower bound for the interparticle distance of the global minimizer, and finally in Section 5 we study the same kind of problem for the Thomas-Fermi model only when the density is fixed and we prove that the triangular lattice is the unique minimizer of the Thomas-Fermi energy per particle in $\mathbb{R}^2$.

4.2 Preliminaries

A Bravais lattice (also called a "simple lattice") of $\mathbb{R}^2$ is given by $L = \mathbb{Z}u \oplus \mathbb{Z}v$ where (u, v) is a basis of $\mathbb{R}^2$. By Engel's theorem (see [35]), we can choose u and v so that $\|u\| \leq \|v\|$ and $(\widehat{u,v}) \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ in order to obtain the unicity of the lattice, up to a rotation. We note $|L| = \|u \wedge v\| = \|u\|\|v\| \left|\sin(\widehat{u,v})\right|$ the area of L which is in fact the area of the lattice primitive cell and $L^* := L\backslash\{0\}$. The positive definite quadratic form associated with the Bravais lattice L is, for $(m, n) \in \mathbb{Z}^2$,

$$Q_L(m, n) = \|mu + nv\|^2 = \|u\|^2 m^2 + \|v\|^2 n^2 + 2\|u\|\|v\| \cos(\widehat{u,v})mn.$$

For a positive definite quadratic form $q(m, n) = am^2 + bmn + cn^2$, we define its discriminant $D = 4ac - b^2 \geq 0$. Hence for Q_L, we obtain :

$$D = 4\|u\|^2\|v\|^2 - 4\|u\|^2\|v\|^2 \cos^2(\widehat{u,v}) = 4\|u\|^2\|v\|^2 \sin^2(\widehat{u,v}) = 4|L|^2.$$

In this paper, the term "lattice" will mean a "Bravais lattice", and we define, for $s > 2$, the Epstein zeta function of the lattice L by

$$\zeta_L(s) := \sum_{x \in L^*} \frac{1}{\|x\|^s} = \sum_{(m,n) \neq (0,0)} \frac{1}{Q_L(m, n)^{s/2}}.$$

Let $\Lambda_A = \sqrt{\frac{2A}{\sqrt{3}}}\left[\mathbb{Z}(1, 0) \oplus \mathbb{Z}(1/2, \sqrt{3}/2)\right]$ be the triangular lattice of area A, also called the hexagonal lattice. Its length is the norm of its vector u, i.e. the minimum distance strictly positive of Λ_A, $\|u\| = \sqrt{2A/\sqrt{3}}$. We notice, for any $s > 2$, that

$$\zeta_{\Lambda_A}(s) = \frac{\zeta_{\Lambda_1}(s)}{A^{s/2}} \tag{4.1}$$

and this relation of scaling is true for any lattice L of area A.
We recall the result of Montgomery about theta functions :

Theorem 4.2.1. (Montgomery, [54]) *For any real number $\alpha > 0$ and a Bravais lattice L, let*

$$\theta_L(\alpha) := \Theta_L(i\alpha) = \sum_{m,n \in \mathbb{Z}} e^{-2\pi\alpha Q_L(m,n)},$$

where Θ_L is the Jacobi theta function of the lattice L defined for $Im(z) > 0$. Then, for any $\alpha > 0$, Λ_A is the unique minimizer of $L \to \theta_L(\alpha)$ among lattices of area A, up to rotation.

Remark 4. *The same kind of results were obtained by Nonnenmacher and Voros in [56]. The previous theorem implies that the triangular lattice is the unique minimizer, up to rotation, of $L \mapsto \zeta_L(s)$ among lattices with density fixed for any $s > 2$ which is also proved by Rankin (in [59]).*

We consider the classical Lennard-Jones potential

$$V_{LJ}(r) = \frac{1}{r^{12}} - \frac{2}{r^6}$$

whose minimum is obtained at $r = 1$, and for $L = \mathbb{Z}u \oplus \mathbb{Z}v$ a Bravais lattice of $\mathbb{R}^2$, we let

$$E_{LJ}(L) := \sum_{x \in L^*} V_{LJ}(\|x\|) = \zeta_L(12) - 2\zeta_L(6)$$

be the Lennard-Jones energy of lattice L. By (4.1) this energy among lattices of area A can be viewed as energy $L \mapsto E_{LJ}(\sqrt{A}L)$ over lattices of area 1 and we parametrize L with its length $\|u\|$ and $\|v\|$ by

$$Q_L(m,n) = \|u\|^2 m^2 + \|v\|^2 n^2 + 2mn\sqrt{\|u\|^2\|v\|^2 - 1}.$$

It follows that we can write Lennard-Jones energy among lattices of area A as

$$(\|u\|, \|v\|) \mapsto \sum_{(m,n) \neq (0,0)} V_{LJ}\left(\sqrt{A}\sqrt{\|u\|^2 m^2 + \|v\|^2 n^2 + 2mn\sqrt{\|u\|^2\|v\|^2 - 1}}\right). \tag{4.2}$$

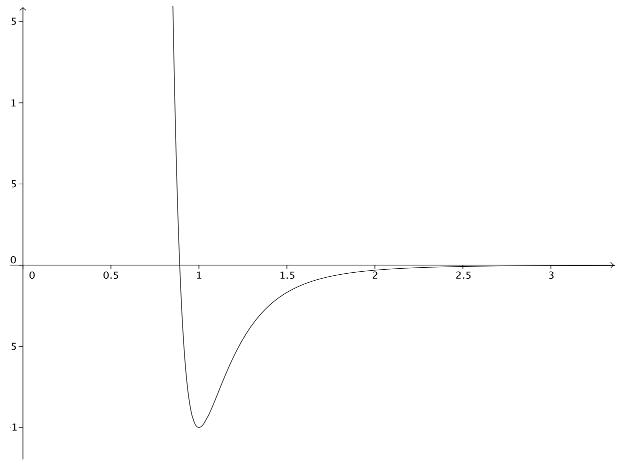


Fig. 1: *Graph of the Lennard-Jones potential V_{LJ}*

The aim of this paper is to study the following two minimization problems, up to rotation :

(P_A) : Find the minimizer of E_{LJ} among lattices L with fixed $|L| = A$;

(P) : Find the minimizer of E_{LJ} among lattices.

Proposition 4.2.1. *The minimum of E_{LJ} among lattices is achieved.*

Proof. We parametrize a lattice L by $x = \|u\|$, $y = \|v\|$ and $\theta = (\widehat{u,v})$, therefore

$$f(x,y,\theta) := E_{LJ}(L)$$

$$= \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(x^2m^2 + y^2n^2 + 2xymn\cos\theta)^6} - \frac{2}{(x^2m^2 + y^2n^2 + 2xymn\cos\theta)^3} \right).$$

First case : minimization without fixed area. If L is the solution of (P) then x and y cannot be too small, otherwise the energy is too large and a proof of a lower bound for x is given in Section 4. Moreover $y \leq 1$ because if $y > 1$ then a contraction of the line $\mathbb{R}v$ gives smaller energy. Therefore we have $x, y \in [m, M]$ and $\theta \in [\pi/3, \pi/2]$. The function $(x, y, \theta) \mapsto f(x, y, \theta)$ is continuous on $[m, M] \times [m, M] \times [\pi/3, \pi/2]$ hence its minimum is achieved.
Second case : minimization with fixed area. We can parametrize L with only two variables x and y – as in (4.2) – such that when $x \to 0$ then $y \to +\infty$. As L should be a Bravais lattice, it is clear that the minimum of f is achieved. $\square$

4.3 Minimization among lattices with fixed area

4.3.1 A sufficient condition

Our idea is to write E_{LJ} in terms of θ_L and to use Theorem 4.2.1 in order to find a sufficient condition for the minimality of the triangular lattice among Bravais lattices with a fixed area.

Theorem 4.3.1. *If $A^3 \leq \dfrac{\pi^3}{120}$, then Λ_A is the unique solution of (P_A).*

Proof. As it is explained in [54] or [71], we can write the Epstein zeta function in terms of a theta function. Indeed, we have the following identity, where the discriminant of Q_L is $D = 1$:

$$\text{for } \mathrm{Re}(s) > 1, \zeta_L(2s)\Gamma(s)(2\pi)^{-s} = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty (\theta_L(\alpha) - 1)(\alpha^s + \alpha^{1-s})\frac{d\alpha}{\alpha}. \quad (4.1)$$

Thus, for $|L| = A$, we write $E_{LJ}(L) = \zeta_L(12) - 2\zeta_L(6)$ as an integral $\displaystyle\int_1^{+\infty} g_A(\alpha)\left(\theta_L\left(\frac{\alpha}{2A}\right) - 1\right)\frac{d\alpha}{\alpha}$, up to a constant independent of L and we find A so that $g_A(\alpha) \geq 0$ for any $\alpha \geq 1$. As Λ_A is the unique minimizer of $\theta_L(\alpha)$ for any $\alpha > 0$, we have for any L such that $|L| = A$:

$$E_{LJ}(L) - E_{LJ}(\Lambda_A) = \int_1^{+\infty} \left(\theta_L\left(\frac{\alpha}{2A}\right) - \theta_{\Lambda_A}\left(\frac{\alpha}{2A}\right)\right) g_A(\alpha)\frac{d\alpha}{\alpha} \geq 0$$

and Λ_A is the unique solution of (P_A).
In fact (4.1) it is the classic “Riemann’s trick" and here we will briefly recall its proof : as

$$\Gamma(s)(2\pi)^{-s}Q_L(m,n)^{-s} = \int_0^\infty t^{s-1}e^{-t}(2\pi)^{-s}Q_L(m,n)^{-s}dt$$

for $\mathrm{Re}(s) > 1$, and by putting $t = 2\pi Q_L(m,n)y$, we obtain

$$\Gamma(s)(2\pi)^{-s}Q_L(m,n)^{-s} = \int_0^\infty e^{-2\pi y Q_L(m,n)}y^{s-1}dy.$$

Summing over $(m,n) \neq (0,0)$ and using the identity $\theta_L(1/\alpha) = \alpha\theta_L(\alpha)$ for any $\alpha > 0$, proved by Montgomery in [54], we obtain

$$\begin{aligned}
\Gamma(s)(2\pi)^{-s}\zeta_L(2s) &= \int_0^\infty (\theta_L(y)-1)y^{s-1}dy = \int_0^1 (\theta_L(y)-1)y^{s-1}dy + \int_1^\infty (\theta_L(y)-1)y^{s-1}dy \\
&= \int_1^\infty (\theta_L(1/\alpha)-1)\alpha^{-1-s}d\alpha + \int_1^\infty (\theta_L(\alpha)-1)\alpha^{s-1}d\alpha \\
&= \int_1^\infty (\alpha\theta_L(\alpha)-1)\alpha^{-1-s}d\alpha + \int_1^\infty (\theta_L(\alpha)-1)\alpha^{s-1}d\alpha \\
&= \int_1^\infty \theta_L(\alpha)\alpha^{-s}d\alpha - \int_1^\infty \alpha^{-1-s}d\alpha + \int_1^\infty (\theta_L(\alpha)-1)\alpha^{s-1}d\alpha \\
&= \int_1^\infty (\theta_L(\alpha)-1)\alpha^{-s}d\alpha + \int_1^\infty (\theta_L(\alpha)-1)\alpha^{s-1}d\alpha + \int_1^\infty \alpha^{-s}d\alpha - \int_1^\infty \alpha^{-1-s}d\alpha \\
&= \int_1^\infty (\theta_L(\alpha)-1)\alpha^{-s}d\alpha + \int_1^\infty (\theta_L(\alpha)-1)\alpha^{s-1}d\alpha + \frac{1}{s-1} - \frac{1}{s} \\
&= \int_1^\infty (\theta_L(\alpha)-1)(\alpha^s + \alpha^{1-s})\frac{d\alpha}{\alpha} + \frac{1}{s-1} - \frac{1}{s}.
\end{aligned}$$

Now if $|L| = A$, by the equality $D = (2A)^2$ there are two identities :

$$(2\pi)^{-6}(2A)^6\Gamma(6)\zeta_L(12) = \frac{1}{5} - \frac{1}{6} + \int_1^{+\infty} \left(\theta_L\left(\frac{\alpha}{2A}\right) - 1\right)(\alpha^6 + \alpha^{1-6})\frac{d\alpha}{\alpha}$$

$$(2\pi)^{-3}(2A)^3\Gamma(3)\zeta_L(6) = \frac{1}{2} - \frac{1}{3} + \int_1^{+\infty} \left(\theta_L\left(\frac{\alpha}{2A}\right) - 1\right)(\alpha^3 + \alpha^{1-3})\frac{d\alpha}{\alpha}$$

and we find

$$\zeta_L(12) = \frac{(2\pi)^6}{30(2A)^6 5!} + \int_1^{+\infty} \left(\theta_L\left(\frac{\alpha}{2A}\right) - 1\right)\frac{(2\pi)^6}{(2A)^6 5!}(\alpha^6 + \alpha^{-5})\frac{d\alpha}{\alpha}$$

$$\zeta_L(6) = \frac{(2\pi)^3}{6(2A)^3 2!} + \int_1^{+\infty} \left(\theta_L\left(\frac{\alpha}{2A}\right) - 1\right)\frac{(2\pi)^3}{(2A)^3 2!}(\alpha^3 + \alpha^{-2})\frac{d\alpha}{\alpha}.$$

Therefore, for any L of area A,

$$E_{LJ}(L) = C_A + \frac{\pi^3}{A^3}\int_1^{+\infty} \left(\theta_L\left(\frac{\alpha}{2A}\right) - 1\right)g_A(\alpha)\frac{d\alpha}{\alpha}$$

where $g_A(\alpha) := \dfrac{\pi^3}{A^3 5!}(\alpha^6 + \alpha^{-5}) - (\alpha^3 + \alpha^{-2})$, and C_A is a constant depending on A but independent of L. Now we want to prove that if $\pi^3 \geq 120A^3$ then $g_A(\alpha) \geq 0$ for any $\alpha \geq 1$. First, we remark that

$$g_A(1) \geq 0 \iff \frac{\pi^3}{A^3 5!} - 1 \geq 0 \iff \pi^3 \geq 120A^3.$$

Secondly, we compute $g_A'(\alpha) = \dfrac{\pi^3}{A^3 5!}(6\alpha^5 - 5\alpha^{-6}) - (3\alpha^2 - 2\alpha^{-3})$, and if $\pi^3 \geq 120A^3$ then

$$g_A'(1) = \frac{\pi^3}{A^3 5!} - 1 \geq 0.$$

Finally, we compute $g_A''(\alpha) = \dfrac{\pi^3}{A^3 5!}(30\alpha^4 + 30\alpha^{-7}) - (6\alpha + 6\alpha^{-4})$. As $\dfrac{\pi^3}{A^3 5!} \geq 1$ and $\alpha \geq 1$,

$$\frac{\pi^3}{A^3 5!}(30\alpha^4+30\alpha^{-7})-(6\alpha+6\alpha^{-4}) \geq 30\alpha^4+30\alpha^{-7}-6\alpha-6\alpha^{-4} \geq 24\alpha+30\alpha^{-7}-6\alpha^{-4} \geq 0.$$

Thus, we have shown that, for any A so that $\pi^3 \geq 120A^3$, $g_A''(\alpha) \geq 0$ for any $\alpha \geq 1$, $g_A'(1) \geq 0$ and $g_A(1) \geq 0$. Hence $g_A(\alpha) \geq 0$ for any $\alpha \geq 1$ if $\pi^3 \geq 120A^3$. □

Remark 5. *We have* $\left(\dfrac{\pi^3}{120}\right)^{1/3} \approx 0.63693$, *hence for* $A \leq 0.63692$, Λ_A *is the unique solution of* (P_A).

Remark 6. *We prove below (see Proposition 4.3.1) that when A is sufficiently large then* Λ_A *is no longer a solution of* (P_A). *However, our bound* $\pi^3 \geq 120A^3$ *is likely not to be optimal. If it were, by the Proposition 4.4.2 and its remark, then the triangular lattice is not the solution to* (P).

This result explains that the behavior of the potential is important for the interaction between the first neighbors because in this case the reverse power part r^{-12} is the strongest interaction. This method can be adapted to any potential of the form $V(r) = \dfrac{K_1}{r^n} - \dfrac{K_2}{r^p}$ with $n > p > 2$ to obtain similar results in two dimensions.

Remark 7. ***The three-dimensional case*** *is an open problem. Indeed, there is no result related to the minimization of theta and Epstein functions among Bravais lattices of* $\mathbb{R}^3$ *with fixed volume. Sarnak and Strömbergsson recalled in [66] that Ennola had shown in [36] the local minimality of the face centered cubic lattice for* $\zeta_L(s)$ *and for any* $s > 0$. *They also prove that the face centered cubic lattice cannot be the minimizer of* $\zeta_L(s)$ *for all* $s > 0$. *Hence the problem of minimization of Lennard-Jones energy among lattices of* $\mathbb{R}^3$, *and of course in higher dimensions, seems to be very difficult.*

4.3.2 A necessary condition

Proposition 4.3.1. Λ_A *is a solution of* (P_A) *if and only if* $A \leq \inf_{\substack{|L|=1\\ L\neq\Lambda_1}} \left(\dfrac{\zeta_L(12)-\zeta_{\Lambda_1}(12)}{2(\zeta_L(6)-\zeta_{\Lambda_1}(6))}\right)^{1/3}$.
Hence if A is sufficiently large, Λ_A *is not a solution of* (P_A).

Proof. We have the following equivalences

$$\begin{aligned}
&E_{LJ}(\Lambda_A) \leq E_{LJ}(L) \text{ for any } L \text{ such that } |L| = A\\
\iff\ & \zeta_{\Lambda_A}(12) - 2\zeta_{\Lambda_A}(6) \leq \zeta_L(12) - 2\zeta_L(6) \text{ for any } L \text{ such that } |L| = A\\
\iff\ & 2(\zeta_L(6) - \zeta_{\Lambda_A}(6)) \leq \zeta_L(12) - \zeta_{\Lambda_A}(12) \text{ for any } L \text{ such that } |L| = A\\
\iff\ & \frac{2(\zeta_L(6)-\zeta_{\Lambda_1}(6))}{A^3} \leq \frac{\zeta_L(12)-\zeta_{\Lambda_1}(12)}{A^6} \text{ for any } L \text{ such that } |L| = 1
\end{aligned}$$

by the scaling property (4.1). We recall that $\zeta_L(6) > \zeta_{\Lambda_1}(6)$ for any L of area A so that $L \neq \Lambda_1$, as a consequence of Theorem 4.2.1 and the Riemann's trick (4.1). Then we obtain

$$E_{LJ}(\Lambda_A) \leq E_{LJ}(L) \text{ for any } L \text{ such that } |L| = A$$

$$\iff A \leq \inf_{\substack{|L|=1 \\ L\neq\Lambda_1}} \left(\frac{\zeta_L(12) - \zeta_{\Lambda_1}(12)}{2(\zeta_L(6) - \zeta_{\Lambda_1}(6))}\right)^{1/3}.$$

□

It is difficult to study the minimum of function $L \mapsto \left(\frac{\zeta_L(12) - \zeta_{\Lambda_1}(12)}{2(\zeta_L(6) - \zeta_{\Lambda_1}(6))}\right)^{1/3}$ among lattices $L \neq \Lambda_1$ such that $|L| = 1$. However, we can numerically look for a lower bound. This function can be parametrized with two variables – here the lengths $\|u\|$ and $\|v\|$ of the lattice L as in (4.2) – and we can plot the level sets of it. We notice that the large differences between the values of the function only give a domain where the function is minimum.

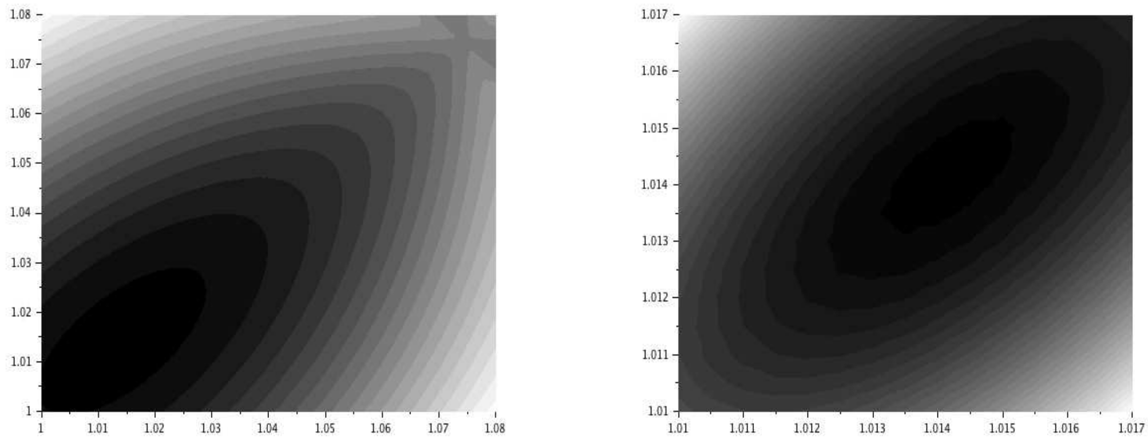


Fig. 2 : *Level sets of* $(\|u\|, \|v\|) \mapsto \left(\frac{\zeta_L(12) - \zeta_{\Lambda_1}(12)}{2(\zeta_L(6) - \zeta_{\Lambda_1}(6))}\right)^{1/3}$
(black = minimum, white = maximum)

Indeed, its minimum seems to be around lattice L of area 1 such that $\|u\| = \|v\| = 1.014$ and for this one, we have $\left(\frac{\zeta_L(12) - \zeta_{\Lambda_1}(12)}{2(\zeta_L(6) - \zeta_{\Lambda_1}(6))}\right)^{1/3} \approx 1.1378475$, hence numerically the minimum of this function is between 1.13 and 1.14.

Actually Fig. 3 gives the Lennard-Jones energy – viewed as a function of two variables $\|u\|$ and $\|v\|$ over the lattices of area one (see (4.2)) – for $(\|u\|, \|v\|) \in [1, 1.08]^2$. The triangular lattice Λ_1 corresponds to the point $\left(\sqrt{2/\sqrt{3}}, \sqrt{2/\sqrt{3}}\right) \approx (1.075, 1.075)$ and the square lattice $\mathbb{Z}^2$ corresponds to the point $(1, 1)$. In fact it is clear that the point associated with the triangular lattice is a critical point of this energy, because the triangular lattice is the unique minimizer of Epstein zeta function among lattices of area A. Moreover we can prove that the square lattice is also a critical point, by using an other parametrization as $(\|u\|, \theta)$. We numerically obtain :

- For $A = 1$, Λ_1 seems to be its minimizer and $\mathbb{Z}^2$ is a local maximizer.
- For $A = 1.13$, Λ_1 seems to be its minimizer but $\mathbb{Z}^2$ seems to be not a local maximizer.

- For $A = 1.14$, $\mathbb{Z}^2$ seems to be its minimizer because we estimate $E_{LJ}(\sqrt{1.14}\Lambda_1) \approx -4.435$ is larger than $E_{LJ}(\sqrt{1.14}\mathbb{Z}^2) \approx -4.437$

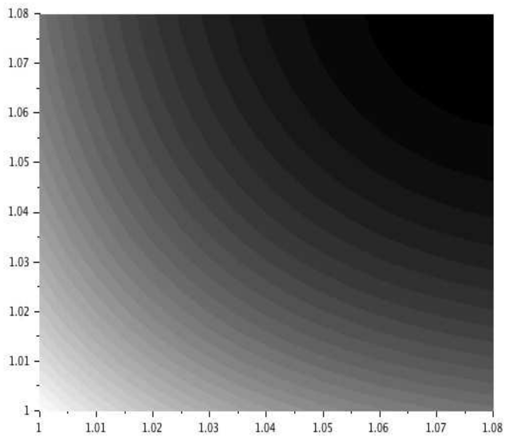
- For $A = 1.16$, $\mathbb{Z}^2$ seems to be its minimizer.

- For $A = 1.2$, $\mathbb{Z}^2$ seems to be its minimizer and Λ_1 is a local maximizer.

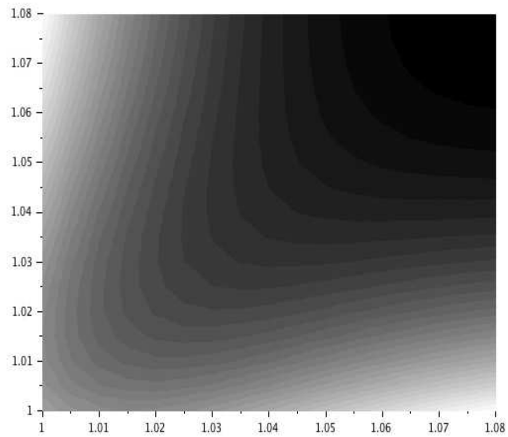
- For $A = 2$ (and more), $\mathbb{Z}^2$ seems to be its minimizer and Λ_1 is a local maximizer.

Hence, we can write the following conjecture based on our numerical study of $L \mapsto E_{LJ}(\sqrt{A}L)$ among all lattices with area 1 :

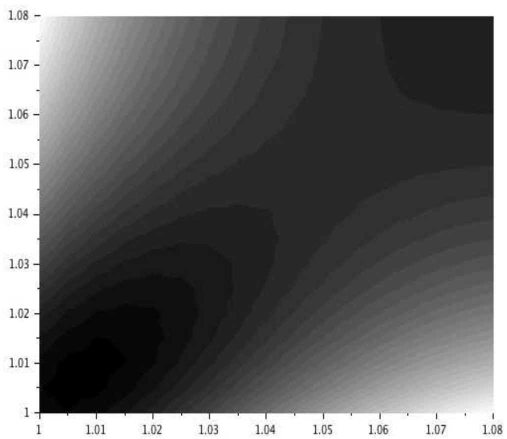
Conjecture : *If A is sufficiently large, the square lattice is the unique solution of (P_A).*



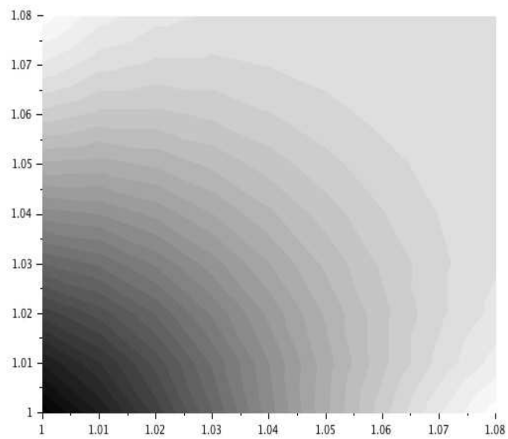
A=1



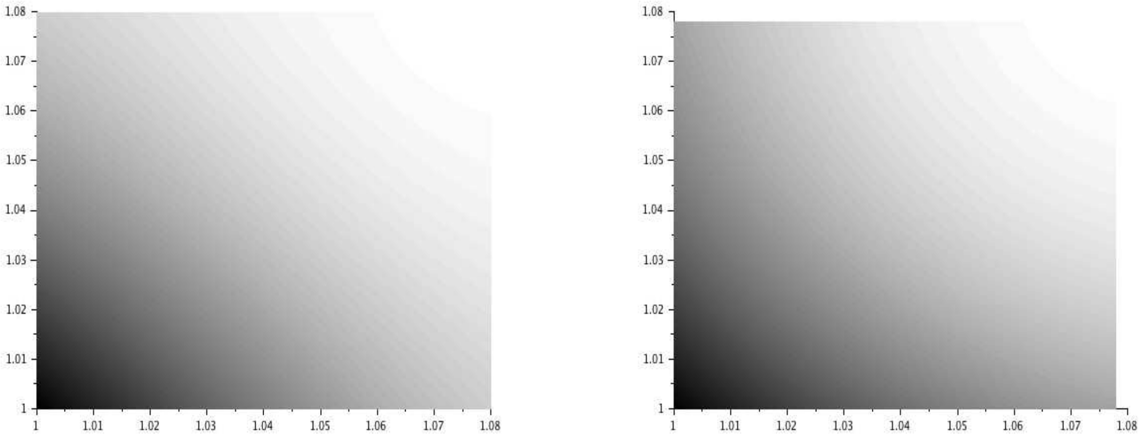
A=1.13



A=1.14



A=1.16



A=1.2 A=2

Fig. 3 : *Level sets of* $(\|u\|,\|v\|) \mapsto E_{LJ}(\sqrt{A}L)$ *for some interesting values of A*
(black = minimum , white = maximum)

4.4 Global minimization of E_{LJ} among lattices

Now we study the problem (P). We give high properties for the global minimizer among lattices and some indications of its shape.

4.4.1 Characterization of the global minimizer

Proposition 4.4.1. *If* $L_0 = \mathbb{Z}u \oplus \mathbb{Z}v$ *is a solution of* (P) *then*
i) $E_{LJ}(L_0) = -\zeta_{L_0}(6) = -\zeta_{L_0}(12) < 0$,
ii) $\|u\| < 1$ *and* $\|v\| \leq 1$,
iii) $\zeta_{L_0}(6) = \max\{\zeta_L(6); L \text{ such that } \zeta_L(12) \leq \zeta_L(6)\}$.

Proof. *i)* We consider the function $f(r) = E_{LJ}(rL_0) = r^{-12}\zeta_{L_0}(12) - 2r^{-6}\zeta_{L_0}(6)$. As L_0 is a global minimizer of E_{LJ}, $r = 1$ is the critical point of f and $f'(r) = -12r^{-13}\zeta_{L_0}(12) + 12r^{-7}\zeta_{L_0}(6)$, hence

$$f'(1) = 0 \iff \zeta_{L_0}(12) = \zeta_{L_0}(6)$$

and $E_{LJ}(L_0) = \zeta_{L_0}(12) - 2\zeta_{L_0}(6) = -\zeta_{L_0}(6) = -\zeta_{L_0}(12)$.

ii) As $\zeta_{L_0}(12) = \zeta_{L_0}(6)$, it is clear that $\|u\| < 1$ because if $r > 1$ then $r^{-12} < r^{-6}$. If $\|v\| > 1$, a little contraction of $\mathbb{R}v$ yields a new lattice L_1 such that $E_{LJ}(L_1) < E_{LJ}(L_0)$ because some of the distances of the lattice decrease while $\|u\|$ is constant, therefore the energy decreases.
iii) $-\zeta_{L_0}(6) = E_{LJ}(L_0) \leq E_{LJ}(L) \iff \zeta_L(6) - \zeta_{L_0}(6) \leq \zeta_L(12) - \zeta_L(6)$ and if L is a lattice such that $\zeta_L(12) \leq \zeta_L(6)$, we get $\zeta_L(6) \leq \zeta_{L_0}(6)$. □

Corollary 1. *The triangular lattice of length* 1 *cannot be the solution of* (P) *though the minimum of the potential* V_{LJ} *is achieved for* $r = 1$.

Proposition 4.4.2. *The minimizer of E_{LJ} among triangular lattices is Λ_{A_0} such that*

$$A_0 = \left(\frac{\zeta_{\Lambda_1}(12)}{\zeta_{\Lambda_1}(6)}\right)^{1/3}.$$

Proof. As in the above proof, we define the function $f(r) = E_{LJ}(r\Lambda_1)$ and we compute its first derivative $f'(r) = -12r^{-13}\zeta_{\Lambda_1}(12) + 12r^{-7}\zeta_{\Lambda_1}(6)$. It follows that :

$$f'(r) \geq 0 \iff r \geq \left(\frac{\zeta_{\Lambda_1}(12)}{\zeta_{\Lambda_1}(6)}\right)^{1/6} =: r_0$$

hence $\Lambda_{A_0} = r_0\Lambda_1$, with $A_0 = r_0^2 = \left(\frac{\zeta_{\Lambda_1}(12)}{\zeta_{\Lambda_1}(6)}\right)^{1/3}$, is the minimizer of E_{LJ} among all triangular lattices. □

Remark 8. *We compute $A_0 \approx 0.84912$, therefore the length of this lattice is $\|u\| \approx 0.99019$. Moreover we notice that $E_{LJ}(\Lambda_{A_0}) = -\zeta_{\Lambda_{A_0}}(6) \approx -6.76425$ (it will be useful for the next part).*
Because $A_0 > 0.63692$, Theorem 4.3.1 is not sufficient to prove that Λ_{A_0} is the solution of (P) but a numerical investigation of $L \mapsto E_{LJ}(\sqrt{A_0}L)$ among all lattices of area 1 seems to indicate that the solution of (P_{A_0}) is triangular and unique.

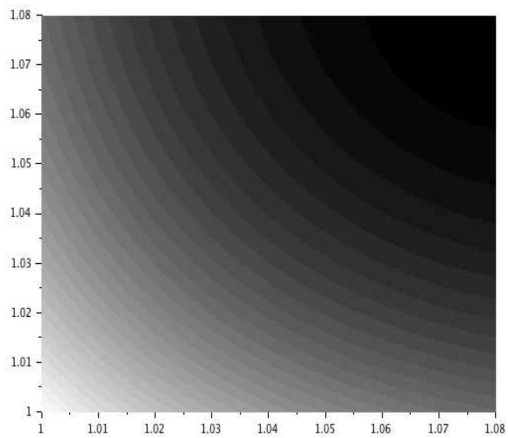


***Fig. 4 :** Level sets of $(\|u\|, \|v\|) \mapsto E(\sqrt{A_0}L)$*
(black = minimum, white = maximum)

Moreover it is not difficult to prove numerically that Λ_{A_0} is a local minimizer among all lattices. Hence we can write the following conjecture for this problem :

***Conjecture :** The triangular lattice Λ_{A_0} is the unique solution of (P).*

4.4.2 Minimum length of the global minimizer

Because our method does not show that the triangular lattice of area A_0 is the global minimizer of the Lennard-Jones energy among lattices, we use Blanc's proof, from [16], in order to find a lower bound for the minimal distance in the globally minimizing lattice.

His result was for the Lennard-Jones interaction of N points in $\mathbb{R}^2$ and $\mathbb{R}^3$. Xue in [80] and Schachinger, Addis, Bomze and Schoen in [3] improved this. We use Blanc's method because it is well suited to our problem.

Proposition 4.4.3. *If $L_0 = \mathbb{Z}u \oplus \mathbb{Z}v$ is a solution of (P), then the minimal distance is greater than an explicit constant c. Furthermore, we have $c > 0.74035$.*

Proof. In [16], Blanc proved that

$$E_{LJ}(L_0) \geq V_{LJ}(\|u\|) - 23 + \frac{1}{\|u\|^{12}} \sum_{k\geq 2} \frac{16k+8}{k^{12}} - \frac{1}{\|u\|^6} \sum_{k\geq 2} \frac{32k+16}{k^6}.$$

As we have $E_{LJ}(L_0) \leq E_{LJ}(\Lambda_{A_0}) = -\zeta_{\Lambda_{A_0}}(6)$ we obtain

$$23 - \zeta_{\Lambda_{A_0}}(6) \geq \frac{P+1}{\|u\|^{12}} - \frac{Q+2}{\|u\|^6}.$$

with $P := \sum_{k\geq 2} \frac{16k+8}{k^{12}}$ and $Q := \sum_{k\geq 2} \frac{32k+16}{k^6}$.

Now, setting $t = \|u\|^{-6}$, we have $(P+1)t^2 - (Q+2)t - 23 + \zeta_{\Lambda_{A_0}}(6) \leq 0$ which implies

$$t \leq \frac{Q+2+\sqrt{(Q+2)^2 + 4(23-\zeta_{\Lambda_{A_0}}(6))(P+1)}}{2(P+1)}$$

and we obtain

$$\|u\| \geq \left(\frac{2(P+1)}{Q+2+\sqrt{(Q+2)^2 + 4\left(23-\zeta_{\Lambda_{A_0}}(6)\right)(P+1)}} \right)^{1/6} =: c.$$

Since $P \approx 0.00988$, $Q \approx 1.45918$ and $\zeta_{\Lambda_{A_0}}(6) \approx 6.76425$ we get $c > 0.74035$. $\square$

Remark 9. *As we think that Λ_{A_0} is the unique solution of (P), this lower bound is the best that we can find with this method. Moreover, this bound and the second point of Proposition 4.4.1 imply that $0.47468 < |L_0| < 1$.*

4.5 The Thomas-Fermi model in $\mathbb{R}^2$

In Thomas-Fermi's model for interactions in a solid, we consider N nuclei at positions $X_N = (x_1, ..., x_N)$, with for any $1 \leq i \leq N$, $x_i \in \mathbb{R}^2$, associated with N electrons with total density $\rho \geq 0$. Then the Thomas-Fermi energy is given by

$$\begin{aligned} E^{TF}(\rho, X_N) = &\int_{\mathbb{R}^2} \rho^2(x)dx - \frac{1}{2} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\|x-y\|\rho(x)\rho(y)dxdy \\ &+ \sum_{j=1}^N \int_{\mathbb{R}^2} \log\|x-x_j\|\rho(x)dx - \frac{1}{2}\sum_{j\neq k} \log\|x_j - x_k\|. \end{aligned}$$

To introduce this kind of model property in quantum chemistry, refer to [20]. Because the system is neutral, the number of electrons is exactly N and we study the minimization problem $I_N^{TF} = \inf_{X_N}\{E^{TF}(X_N)\}$ where

$$E^{TF}(X_N) := \inf_{\rho}\left\{E^{TF}(\rho, X_N), \rho \geq 0, \rho \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2), \int_{\mathbb{R}^2}\rho = N\right\}.$$

By the Euler-Lagrange equations for this minimization problem, we find – as it is explained in Section 2 of [18] and Section 4 of [17] – that the minimizer $\bar{\rho}$ is the solution of

$$-\Delta\bar{\rho} + \pi\bar{\rho} = \pi\sum_{j=1}^N \delta_{x_j}.$$

It is known that the fundamental solution of the modified Helmholtz equation $-\Delta h + h = \delta_0$ – also called "screened Poisson equation" – which goes to 0 at infinity, is the radial modified Bessel function of the second kind, also called the Yukawa potential, defined in [41] and [78], by

$$K_0(\|x\|) = \int_0^{+\infty} e^{-\|x\|\cosh t}dt.$$

Therefore we obtain $\bar{\rho}(x) = \pi\sum_{j=1}^N W_{TF}(\|x-x_j\|)$ where $W_{TF}(\|x\|) = \frac{1}{2}K_0(\sqrt{\pi}\|x\|)$ and finally

$$E^{TF}(X_N) = \sum_{i\neq j} W_{TF}(\|x_i - x_j\|) + NC$$

where C is a constant independent of N and X_N. Now, if we consider that the nuclei are in lattice L, we can study, by taking the mean value of the total energy, the following energy per point

$$E_{TF}(L) = \sum_{x\in L^*} W_{TF}(\|x\|).$$

A simple idea enables us to use theta functions and we have the following result :

Theorem 4.5.1. *Λ_A is the unique minimizer of E_{TF} among all lattices of fixed area A.*

Proof. This problem is equivalent to finding the minimizer of $\sum_{x\in L^*} K_0(\|x\|)$ among lattices with a fixed area. We put $y = \frac{1}{2}\|x\|e^t$ for $x \neq 0$ in the integral formula for $K_0(\|x\|)$:

$$\begin{aligned}K_0(\|x\|) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-\|x\|\cosh t}dt &= \frac{1}{2}\int_0^{+\infty} e^{-\|x\|\cosh(\ln(2y/\|x\|))}\frac{dy}{y}\\ &= \frac{1}{2}\int_0^{+\infty} e^{-y-\frac{\|x\|^2}{4y}}\frac{dy}{y}\\ &= \frac{1}{2}\int_0^{+\infty} e^{-\frac{\|x\|^2}{4y}}e^{-y}\frac{dy}{y}.\end{aligned}$$

Now, for any $y > 0$ and any lattice L of area A, we obtain $\sum_{x\in L^*} e^{-\frac{\|x\|^2}{4y}} = \theta_L\left(\frac{1}{8\pi y}\right) - 1$. Hence, by Montgomery's theorem, the triangular lattice Λ_A minimizes $\theta_L(\alpha)$ for any $\alpha > 0$, and it is the unique minimizer of $L \mapsto \theta_L(\alpha)$ among all Bravais lattices with a fixed area A. Therefore, for any $y > 0$, Λ_A is the unique minimizer of the energy $E_y(L) := \sum_{x\in L^*} e^{-\frac{\|x\|^2}{4y}}$ among lattices with a fixed area A. Now it is clear, because $E_y(\Lambda_A) \leq E_y(L)$ for any $y > 0$ and for any lattice L with area A, that

$$\frac{1}{2}\int_0^{+\infty} E_y(\Lambda_A)e^{-y}\frac{dy}{y} \leq \frac{1}{2}\int_0^{+\infty} E_y(L)e^{-y}\frac{dy}{y}.$$

Hence, for any L of a fixed area A : $E_{TF}(\Lambda_A) = \sum_{x\in\Lambda_A^*} W_{TF}(\|x\|) \leq \sum_{x\in L^*} W_{TF}(\|x\|) = E_{TF}(L)$.

□

Remark 10. *The Yukawa potential appears in many vortex interaction models, as the α-model in fluid mechanics and in superconductivity (see for example [1] and [68]). Indeed, the second author recently studied, in [81], Ginzburg-Landau's model for the interactions between vortices in superconductors. He proved, by using a more general method – that it can certainly be used for other potentials – the same result was obtained for minimality of the triangular lattice among all lattices with fixed density. The use of results from Number Theory in Ginzburg-Landau's models for vortices can also be seen in [64].*

Remark 11. *The potential W_{TF} decreases. We notice that $W_{TF}(\sqrt{.})$ is completely monotonic on $\mathbb{R}_+^*$, i.e. $(-1)^n(W_{TF}(\sqrt{.}))^{(n)}(r)$ is positive for any $n \geq 0$ and any $r > 0$(see Corollary 1 of [53]). It is explained in [24], by using Bernstein's Theorem (see Theorem 12b of [79]) about the following representation of a completely monotonic function f*

$$f(r) = \int_0^{+\infty} e^{-rt}d\alpha(t)$$

where α is a non decreasing function, and Montgomery's Theorem 4.2.1 for theta functions, that the triangular lattice is the unique minimizer among lattices of $E_f(L) := \sum_{x\in L^*} f(\|x\|^2)$, provided we have the correct assumptions of convergence, for instance $f(r) = O(r^{-1-\eta})$ at infinity for some $\eta > 0$. This is another proof of our theorem.

Chapter 5

Limits of Solutions to n-dimensional Ginzburg-Landau Equations

This is joint work with Yuxin Ge and Etienne Sandier.

5.1 Introduction

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $g : \partial\Omega \to \mathbb{S}^{n-1}$ is a smooth prescribed map, and $d = \deg(g, \partial\Omega, \mathbb{S}^{n-1})$ is the degree of g. We consider the functional

$$\mathbf{E}_\varepsilon(u, \Omega) = \int_\Omega \left[\frac{|\nabla u|^n}{n} + \frac{1}{4\varepsilon^n} \left(1 - |u|^2\right)^2 \right] \mathrm{d}x \tag{5.1}$$

for $u \in W_g^{1,n}(\Omega, \mathbb{R}^n) = \left\{ w \in W^{1,n}(\Omega, \mathbb{R}^n) : w|_{\partial\Omega} = g \right\}$. It is easy to see that $W_g^{1,n}(\Omega, \mathbb{R}^n)$ is not empty.

In the case of $n = 2$, the functional defined above is the classical Ginzburg-Landau functional. A minimizer $u_\varepsilon \in W_g^{1,2}\left(\Omega, \mathbb{R}^2\right)$ of $\mathbf{E}_\varepsilon(u, \Omega)$ satisfies the so called Ginzburg-Landau system

$$\begin{cases} -\Delta u_\varepsilon = \frac{1}{\varepsilon^2}\left(1 - |u_\varepsilon|^2\right) u_\varepsilon & \text{in } \Omega \\ \quad u_\varepsilon = g & \text{on } \partial\Omega\,. \end{cases} \tag{5.2}$$

Bethuel, Brézis, Hélein did notable contribution on this model, and eventually their work leaded to the publication of the book [13]. In [13], they proved, among the others, the following theorems.

Theorem (BBH1). *Assume that Ω is star-shaped, and that $d \neq 0$, then there exists a subsequence of $\varepsilon_k \to 0$, exactly $|d|$ distinct points $a_1, a_2, \cdots, a_{|d|}$, and a harmonic map $u_* \in \mathbf{C}^\infty(\Omega \setminus \{a_1, a_2, \cdots, a_{|d|}\})$ with boundary value g such that*

$$u_{\varepsilon_n} \to u_* \quad in \quad \mathbf{C}^k_{loc}(\Omega \setminus \cup_i \{a_i\}) \quad for \quad \forall k \quad and\ in \quad \mathbf{C}^{1,\alpha}_{loc}(\bar{\Omega} \setminus \cup_i \{a_i\}) \quad for \quad \forall \alpha < 1.$$

In addition, each singularity has degree $sign(d)$.

For non-star shaped domain, see [69] for references.

Also, in their paper, they introduced a renormalized energy which is defined on configurations of points. For any given configuration $b = (b_1, b_2, \cdots, b_{|d|})$ of distinct points in Ω, the renormalized energy

$$W(b,d,g) := -\pi \sum_{i\neq j} \ln\left|b_i - b_j\right| + \frac{1}{2}\int_{\partial\Omega} \Phi(g\times g_\tau) - \pi \sum_{i=1}^{|d|} R(b_i)$$

where Φ is the solution of the linear Neumann problem

$$\begin{cases} \Delta\Phi = 2\pi \sum_{i=1}^{|d|} \delta_{b_i} & \text{in } \Omega, \\ \dfrac{\partial\Phi}{\partial\nu} = g\times g_\tau & \text{on } \partial\Omega \end{cases} \tag{5.3}$$

where ν is the unit outward normal to $\partial\Omega$, τ is a unit tangent vector to $\partial\Omega$ and

$$R(x) = \Phi(x) - \sum_{i=1}^{|d|} \ln|x - b_i|\,.$$

For this renormalized energy, they proved

Theorem (BBH2). *Let $\cup_i \{a_i\}$ be the limit singular points of Theorem(BBH1), then the configuration $\cup_i \{a_i\}$ minimizes $W(b,d,g)$.*

Near the singularity, they had a vanishing gradient property

Theorem (BBH3). *Near each singularity a_j,*

$$u_*(z) = \frac{z-a_j}{\left|z-a_j\right|} e^{iH_j(z)}, \tag{5.4}$$

where H_j is a real harmonic function such that

$$H_j(z) = H_j(a_j) + O(\left|z-a_j\right|^2), \text{ as } z \to a_j. \tag{5.5}$$

In other words,

$$\nabla H_j(a_j) = 0. \tag{5.6}$$

As for the case of $n \geq 3$, the infimum of the Ginzburg-Landau type functional $\mathbf{E}_\varepsilon(u,\Omega)$ is attained. We have the higher dimension analogue of (5.2) for any minimizer u_ε of $\mathbf{E}_\varepsilon(u,\Omega)$

$$\begin{cases} -div\left(|\nabla u_\varepsilon|^{n-2}\nabla u_\varepsilon\right) &= \frac{1}{\varepsilon^n}\left(1-|u_\varepsilon|^2\right)u_\varepsilon & \text{in } \Omega \\ u_\varepsilon &= g & \text{on } \partial\Omega\,. \end{cases} \tag{5.7}$$

We are interested in the case of $d \neq 0$, therefore we assume throughout the rest of the paper, for notational convenience, that $d = \deg(g,\partial\Omega,\mathbb{S}^{n-1}) > 0$. For convenience, we define a constant

$$\kappa_n = \frac{1}{n}(n-1)^{\frac{n}{2}}\omega_n$$

where $\omega_n = |\mathbb{S}^{n-1}|$.

In [70], Strzelecki proved minimizers $u_\varepsilon \in W_g^{1,n}(\Omega,\mathbb{R}^n)$ of the n-dimensional functional $\mathbf{E}_\varepsilon(u,\Omega)$, which satisfy the Dirichlet boundary condition $u_\varepsilon = g$ with zero topological degree, converge in $W^{1,n}(\Omega)$ and $C^\alpha_{loc}(\Omega)$ for any $\alpha < 1$ - upon passing to a subsequence

$\varepsilon_k \to 0$ - to some minimizing n-harmonic map. For the case $d = \deg g \neq 0$, Hong [47] established a weak convergence away from the singularities for a sequence of selected minimizers. In [42], Han and Li proved, among other things, the corresponding results on the singular limits of minimizers of n-dimensional Ginzburg-Landau type functional. This can be regarded as the higher dimensional analogue of Theorem(BBH1).

Theorem (HL). *Assume $d \neq 0$, $n \geq 3$. For any sequence $\varepsilon_k \to 0$, let $\{u_k\} \subset W_g^{1,n}(\Omega, \mathbb{R}^n)$ be the corresponding sequence of minimizer for $\mathbf{E}_{\varepsilon_k}$. Then there exists a subsequence $\{u_{k'}\}$, a collection of $|d|$ distinct points $\{a_1, a_2, \cdots, a_{|d|}\} \subset \Omega$, and an n-harmonic map $u_* : \Omega \setminus \cup_i \{a_i\} \to \mathbb{S}^{n-1}$ such that*

$$u_{k'} \to u_* \quad \textit{strongly in} \quad \mathbf{W}^{1,n}_{loc}(\bar{\Omega} \setminus \cup_i \{a_i\}; \mathbb{R}^n), \tag{5.8}$$

$$u_{k'} \to u_* \quad \textit{in} \quad \mathbf{C}^0_{loc}(\bar{\Omega} \setminus \cup_i \{a_i\}; \mathbb{R}^n), \tag{5.9}$$

$$u_{k'} \to u_* \quad \textit{strongly in} \quad \mathbf{W}^{1,p}(\Omega; \mathbb{R}^n) \textit{ for all } 1 \leq p < n. \tag{5.10}$$

Furthermore, $\deg(u_*, \partial B_\sigma, \mathbb{S}^{n-1}) = sign(d)$ *for all* $1 \leq j \leq |d|$ *and* $\sigma > 0$ *small enough.*

In our paper, we shall prove an analogue of Theorem(BBH2), i.e. the existence of renormalized energy for the higher dimensional Ginzburg-Landau functional. We shall prove that it coincides with the renormalized energy for n-harmonic map introduced by Hardt, Lin and Wang in the paper [45]. In [45], for an arbitrary subset A of Ω consisting of d distinct points $\{a_1, a_2, \cdots, a_d\}$ that are separated from each other and from the boundary $\partial\Omega$ by at least $2\sigma_0$, they introduced a renormalized energy W_g defined on the configuration of singular points of n-harmonic maps in Ω as follows. For $0 < \delta < \sigma_0$, let

$$\Omega_{A,\delta} = \Omega \setminus \cup_{i=1}^d B_\delta(a_i).$$

Define

$$E_{A,\delta}(w) := \int_{\Omega_{A,\delta}} \frac{|\nabla w|^n}{n} \, \mathrm{d}x$$

where w is in the family

$$\mathcal{W}_{A,\delta} = \left\{ w \in W^{1,n}(\Omega_{A,\delta}; \mathbb{S}^{n-1}) : w|\partial\Omega = g, \deg(w, \partial B_\delta(a_i)) = 1 \text{ for all } i \right\}.$$

Suppose that $w_{A,\delta}$ minimizes $E_{A,\delta}$.

Then

$$E_{A,\delta}(w_{A,\delta}) - d\kappa_n |\ln \delta|$$

is increasing with respect to δ and bounded from below for any $\delta > 0$.

So

$$W_g(a_1, a_2, \cdots, a_d) := \lim_{\delta \to 0} (E_{A,\delta}(w_{A,\delta}) - d\kappa_n |\ln \delta|) \tag{5.11}$$

makes sense.

For this renormalized energy W_g of n-harmonic maps, we will prove that **it is** the renormalized energy for n-dimensional Ginzburg-Landau type functional $\mathbf{E}_\varepsilon(u, \Omega)$. When there is no possibility of confusion, we will write W as W_g. We have the following result.

Theorem 5.1.1. *Let $a = \{a_i\}_{i=1}^d$ be the limit singular points as in Theorem (HL), then*

$$\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) = d\kappa_n |\ln \varepsilon| + W_g(a) + d\gamma + o(1) \, as \, \varepsilon \to 0,$$

where γ is a constant which will be defined in Section 5.2.1 later, and the configuration $\{a_i\}_{i=1}^d$ minimizes W_g.

The proof of this theorem can be divided into the following two lemmas.

Lemma 5.1.1. *Let $\bar{a} = \{\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_d\}$ be any configuration of d points in Ω. Then for every $\varepsilon > 0$ small enough, we have*

$$\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \le d\kappa_n |\ln\varepsilon| + W_g(\bar{a}) + d\gamma + o(1).$$

Lemma 5.1.2. *Let $a = \{a_1, a_2, \cdots, a_d\}$ be the singular points of u_* as in theorem(HL), then there is an integer $N = N(g, \Omega)$ such that for every $k > N$,*

$$\mathbf{E}_{\varepsilon_k}(u_{\varepsilon_k}, \Omega) \ge d\kappa_n |\ln\varepsilon_k| + W_g(a) + d\gamma - o(1).$$

Most of the results above deal only with the sequences of energy-minimizers. In our paper, we shall study the limits of solutions to Ginzburg-Landau equations in n-dimensions.

Suppose u_ε is a critical point of $\mathbf{E}_\varepsilon(u, \Omega)$, which satisfies (5.7)

$$\begin{cases} -\mathrm{div}\left(|\nabla u_\varepsilon|^{n-2}\nabla u_\varepsilon\right) = \frac{1}{\varepsilon^n}\left(1-|u_\varepsilon|^2\right)u_\varepsilon & \text{in } \Omega\,, \\ u_\varepsilon = g & \text{on } \partial\Omega\,. \end{cases}$$

and has an energy upper bound

$$\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \le d\kappa_n |\ln\varepsilon| + M \tag{5.12}$$

where M is a constant. We have

Theorem 5.1.2. *Suppose that $\{u_{\varepsilon_k}\}$, $\varepsilon_k \searrow 0$ is a sequence of critical points of $\mathbf{E}_{\varepsilon_k}$ and satisfies the upper bound condition (5.12). Then there exists a subsequence $\{u_{\varepsilon_{k'}}\}$, a collection of exactly d distinct points $\{a_1, a_2, \cdots, a_d\} \subset \Omega$, a finite subset S of Ω, and an n-harmonic map $u_0 : \Omega_0 := \Omega \setminus (\{a_1, a_2, \cdots, a_d\} \cup S) \to \mathbb{S}^{n-1}$, such that*

$$u_{\varepsilon_{k'}} \to u_0 \quad in \quad \mathbf{W}^{1,n}_{loc}(\Omega_0, \mathbb{R}^n).$$

Furthermore, $\deg(u_0, \partial B_\sigma(a_j), \mathbb{S}^{n-1}) = 1$, *for $1 \le j \le d$ and $\sigma > 0$ small.*

For a sequence of functions which satisfy only the upper bound condition (5.12), R. Jerrard proved in [48] that there exists a weakly convergence subsequence. In our work, we prove the strong convergence, but this depends on the fact that the sequence of functions solve the Ginzburg-Landau equations.

We also give a higher dimensional “vanishing gradient property" analogous to the one in [13]. Let $u : \Omega_0 \to \mathbb{S}^{n-1}$ be an n-harmonic map. We say u is a stationary n-harmonic map if its stress tensor is divergence free in Ω_0, that is

$$\sum_i \partial_i T_{i,j} = 0$$

where

$$T_{i,j} := |\nabla u|^{n-2}\langle \partial_i u, \partial_j u\rangle - \frac{1}{n}|\nabla u|^n \delta_{i,j}$$

and satisfies

$$\oint_{\partial B_\rho} \sum_i T_{i,j}\nu_i = 0 \tag{5.13}$$

for $\partial B_\rho \subset \Omega_0$, $\nu = (\nu_1, \cdots, \nu_n)$.

Proposition 5.1.1. *Assume* $u : \Omega_0 \subset \mathbb{R}^n \to \mathbb{S}^{n-1}$ *is a stationary* $n-$*harmonic map where* $\Omega_0 := \Omega \setminus (\{a_1, \cdots, a_d\} \cup S)$ *in the above sense, and* $\deg(u, a_i) = 1$. *Assume around each singular point* a_i*, one has the asymptotic expansion*

$$u(x) = e^{B(x)} \frac{x - a_i}{|x - a_i|}$$

where $B(x) \in so(n)$ *is antisymmetric matrix satisfying* $B(0) = 0$ *such that* B *is* C^1 *in a neighborhood of* x*. Then*

$$\sum_{k=1}^{n} \partial_k B(0) e_k = 0$$

where $(e_1, \cdots, e_n)$ *is the canonical basis in* $\mathbb{R}^n$*. Moreover, we can write*

$$u(x) = \frac{x - a_i}{|x - a_i|} + \frac{Q(x - a_i)}{|x - a_i|} + O(|x - a_i|^2)$$

where $Q(x)$ *is some harmonic polynomial of degree 2. In particular, when* $n = 2$*, we have* $B(x) = O(|x - a_i|^2)$.

At the end of the paper, we give an example of non-minimizing sequence of critical points.

Theorem 5.1.3. *In three dimensions, there exists a domain* Ω *and a boundary value* g*, such that a sequence of critical values* u_ε *of the 3-dimensional Ginzburg-Landau type functional* $\mathbf{E}_\varepsilon(u, \Omega)$ *satisfies the upper bound condition (5.12) and is not the minimizer.*

5.2 Renormalized Energy

In this section, we study the renormalized energy for minimizers of n-dimensional Ginzburg-Landau type functional. We show that it coincides with the renormalized energy for n-harmonic maps.

5.2.1 Estimates when $\Omega = B_R$ and $g(x) = g_0 = \frac{x}{|x|}$

In this subsection, we introduce some quantities similar with [13]. The quantities in this section will play an important role in the proof of Lemma 5.1.1 and Lemma 5.1.2. For convenience, denote

$$\begin{aligned} \mathbf{E}_{\varepsilon,R} &= \mathbf{E}_\varepsilon(u, B_R), \\ \mathbf{I}(\varepsilon, R) &= \min_{u \in W_{R,g_0}} \mathbf{E}_{\varepsilon,R}. \end{aligned}$$

where $W_{R,g_0} = \{v \mid v \in W^{1,n}(B_R, \mathbb{R}^n), u|_{\partial B_R} = g_0 = \frac{x}{|x|}\}$. By scaling it is easy to see that

$$\mathbf{I}(\varepsilon, R) = \mathbf{I}(1, R/\varepsilon) = \mathbf{I}(\varepsilon/R, 1).$$

Denote $\mathbf{I}(t) = \mathbf{I}(t, 1)$ for notational convenience. Let u_t be the minimizer of $\mathbf{I}(1, \frac{1}{t})$.

Lemma 5.2.1. *For* $0 < t_1 < t_2 < 1$*, we have*

$$\mathbf{I}(t_1) + \kappa_n \ln(t_1) \leq \mathbf{I}(t_2) + \kappa_n \ln(t_2)$$

i.e. the function $t \to \mathbf{I}(t) + \kappa_n \ln(t)$ *is increasing.*

Proof. Set

$$v(x) = \begin{cases} u_{t_2} & |x| < \frac{1}{t_2}, \\ \frac{x}{|x|} & \frac{1}{t_2} \le |x| \le \frac{1}{t_1}. \end{cases} \tag{5.1}$$

Then by the definition of $\mathbf{I}(t)$, we have

$$\begin{aligned} \mathbf{I}(t_1) &= \mathbf{I}(1, t_1^{-1}) \le \mathbf{E}_{1,t_1^{-1}}(v) \\ &= \mathbf{I}(t_2) + \int_{B_{t_1^{-1}} \setminus B_{t_2^{-1}}} \frac{\left|\nabla\left(\frac{x}{|x|}\right)\right|^n}{n} \,\mathrm{d}\,x = \mathbf{I}(t_2) + \int_{t_2^{-1}}^{t_1^{-1}} \frac{(n-1)^{\frac{n}{2}}}{n \cdot r} \cdot \omega_n \,\mathrm{d}\,r \\ &= \mathbf{I}(t_2) + \frac{(n-1)^{\frac{n}{2}}}{n} \omega_n \cdot \ln \frac{t_2}{t_1} \end{aligned} \tag{5.2}$$

This implies the conclusion. □

By using Theorem 1.1 of [48], we have a lower bound of $\mathbf{I}(t) + \kappa_n \ln(t)$.

Theorem. *If $u \in W_g^{1,n}(\Omega, \mathbb{R}^n)$, then*

$$\mathbf{E}_\varepsilon(u, \Omega) \ge d\kappa_n |\ln(\varepsilon)| + C(\Omega, g).$$

This leads to the definition of constant

$$\gamma := \lim_{t \to 0} \mathbf{I}(t) + \kappa_n \ln(t).$$

If we apply a rotation θ on the boundary value g_0, i.e. if the boundary value is $\theta \circ g_0$, we can get the same constant γ, that is because rotation on u does not change the energy.

5.2.2 Proof of Lemma 5.1.1

In this subsection, we prove Lemma 5.1.1. We will construct a comparison map which is in $W_g^{1,n}(\Omega, \mathbb{R}^n)$ to get the upper bound. For the construction, we need an important lemma (Lemma 9.1 in [45]) on the behavior of $w_{A,\delta}$ near the singularities. In [45], they proved that near a_i, one of the singularities, $w_{A,\delta}$ will be, at each sufficiently small scale, close to some rotation. We will state the lemma without proof. Let $\bar{a} = \{\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_d\}$ be any configuration of d points of Ω that are separated from each other and from the boundary $\partial\Omega$ by at least $2\sigma_0$. Then

Lemma 5.2.2 ([45])**.** *For any $\mu > 0$, there exists a positive $\tau_0 < \sigma_0$ so that if $\delta > 0$, $s \in [4\delta, \tau_0]$, and $i \in \{1, \ldots, d\}$, then*

$$\|w_{\bar{a},\delta}(\bar{a}_i + s(\cdot)) - \theta_{\delta,s,i}\|_{C^1(B \setminus B_{1/2})} < \mu/3d$$

for some orthogonal rotation $\theta_{\delta,s,i}$ of $\mathbb{R}^n$.

Now we prove Lemma 5.1.1. For any $\mu > 0$, from (5.11), we may choose a positive constant $\delta_0 < \tau_0$ such that

$$E_{\bar{a},\delta}(w_{\bar{a},\delta}) \le W_g(\bar{a}) + d\kappa_n |\ln \delta| + \mu/3 \tag{5.3}$$

whenever $\delta < \delta_0$. We now fix such a $\delta < \delta_0/4$, and fix an $s \in [4\delta, \tau_0]$ as in Lemma 5.2.2, then we have the comparison map $u_0(x) \in W_g^{1,n}(\Omega, \mathbb{R}^n)$ defined by

$$u_0(x) = \begin{cases} w_{\bar{a},\delta} & \text{if } x \in \Omega_{\bar{a},\delta} \setminus \cup_{i=1}^d B_s(\bar{a}_i), \\ v_i(x) & \text{if } x \in B_s(\bar{a}_i) \setminus B_{s/2}(\bar{a}_i), \\ u_{s/2,g_i} & \text{if } x \in B_{s/2}(\bar{a}_i). \end{cases} \tag{5.4}$$

where $g_i(x) = \theta_{\delta,s,i}(\frac{x-\bar{a}_i}{|x-\bar{a}_i|})$ is defined on $\partial B_{s/2}(\bar{a}_i)$, and $u_{s/2,g_i}$ is the corresponding minimizer of $\mathbf{E}_{\varepsilon,s/2}$ in Section 5.2.1. The map $v_i(x)$ is the interpolation map

$$v_i(x) = \frac{v_i^*}{|v_i^*|}$$

where

$$v_i^*(x) = (2 - 2\,|x-\bar{a}_i|\,/s)\theta_{\delta,s,i}\left(\frac{x-\bar{a}_i}{|x-\bar{a}_i|}\right) + (2\,|x-\bar{a}_i|\,/s - 1)w_{\bar{a},\delta},$$

and from Lemma 5.2.2(also see section 7 of [43]), it satisfies the energy estimate

$$\int_{B_s(\bar{a}_i)\setminus B_{s/2}(\bar{a}_i)} \frac{|\nabla v_i|^n}{n}\,\mathrm{d}\,x \le \kappa_n \ln 2 + \mu/3d. \tag{5.5}$$

We need an energy lower bound estimate of $\mathbb{S}^{n-1}$ valued function on annulus.

Lemma 5.2.3 (Annulus estimate)**.** *If* $0 < r < s < \infty$, $u \in W^{1,n}(B_s(a) \setminus B_r(a), \mathbb{S}^{n-1})$, *and* $\deg u|\partial B(\rho) = 1$ *for almost all* $\rho \in (r,s)$, *then*

$$\int_{B_s(a)\setminus B_r(a)} \frac{|\nabla u|^n}{n}\,\mathrm{d}\,x \ge \kappa_n \ln\frac{s}{r}.$$

Proof.

$$\begin{aligned} \int_{B_s(a)\setminus B_r(a)} \frac{|\nabla u|^n}{n}\,\mathrm{d}\,x &\ge \int_r^s \int_{\partial B_\rho} |\nabla_{\tan} u|^n /n\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}\,\rho \\ &\ge \int_r^s \frac{1}{n\rho}\left[\int_{\mathbb{S}^{n-1}} |\nabla_{\tan} u|^{n-1}\,\mathrm{d}\mathcal{H}^{n-1}\right]^{\frac{n}{n-1}} \omega_n^{1-\frac{n}{n-1}}\,\mathrm{d}\,\rho \\ &\ge \int_r^s \frac{1}{n\rho}\left|(n-1)^{\frac{n-1}{2}} \int_{\mathbb{S}^{n-1}} \mathrm{Jac}\,u\,\mathrm{d}\mathcal{H}^{n-1}\right|^{\frac{n}{n-1}} \omega_n^{1-\frac{n}{n-1}}\,\mathrm{d}\rho \\ &\ge \kappa_n \ln\frac{s}{r}. \end{aligned} \tag{5.6}$$

The third inequality comes from the inequality of arithmetic and geometric means. ∎

Lemma 5.2.3 gives the lower bound on annulus,

$$\int_{\cup(B_s(a_i)\setminus B_\delta(a_i))} \frac{|\nabla u_0|^n}{n}\,\mathrm{d}\,x \ge d\kappa_n \ln\frac{s}{\delta}.$$

Combining this with (5.3) we have the estimate

$$\begin{aligned} \mathbf{E}_{\bar{a},s}(u_0) &= \int_{\Omega_{\bar{a},s}} \frac{|\nabla u_0|^n}{n}\,\mathrm{d}\,x \\ &= \int_{\Omega_{\bar{a},\delta}} \frac{|\nabla u_0|^n}{n}\,\mathrm{d}\,x - \int_{\cup(B_s(a_i)\setminus B_\delta(a_i))} \frac{|\nabla u_0|^n}{n}\,\mathrm{d}\,x \\ &\le W_g(\bar{a}) + d\kappa_n\,|\ln s| + \mu/3. \end{aligned} \tag{5.7}$$

In the balls $B_{s/2}(\bar{a}_i)$, there exists a constant ε_0 such that

$$\mathbf{E}_\varepsilon(u_{s/2,g_i}, B_{s/2}) = \mathbf{I}(\varepsilon, s/2) = \mathbf{I}(2\varepsilon/s, 1) \le \gamma + \kappa_n \left|\ln(2\varepsilon/s)\right| + \mu/3d \tag{5.8}$$

whenever $\varepsilon < \varepsilon_0$.

Combining (5.5),(5.7) and (5.8) we have the desired upper bound

$$\begin{aligned}\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) &\le \mathbf{E}_\varepsilon(u_0, \Omega)\\ &= \mathbf{E}_{\bar{a},s}(u_0) + \sum_{i=1}^{d} \int_{B_s(\bar{a}_i)\setminus B_{s/2}(\bar{a}_i)} \frac{|\nabla v_i|^n}{n} \,\mathrm{d}\,x + \sum_{i=1}^{d} \mathbf{E}_\varepsilon(u_{s/2,g_i}, B_{s/2}) \\ &\le W_g(\bar{a}) + d\kappa_n \left|\ln \varepsilon\right| + d\gamma + \mu.\end{aligned} \tag{5.9}$$

That finishes the proof of Lemma 5.1.1.

5.2.3 Proof of Lemma 5.1.2

In this section, we will give a proof of Lemma 5.1.2. Let $a = \{a_1, a_2, \cdots, a_d\}$ be the singularities of u_* in Theorem(HL). Suppose these points are separated from each other and from the boundary by at least $2\sigma_0$. On the one hand, from the convergence of the $u_\varepsilon \to u_*$, we can have a lower bound $\mathbf{E}_{a,\rho}(w_{a,\delta}) - o(1)$ of the functional $\mathbf{E}_\varepsilon$ away from these singularities. On the other hand, we need to prove that, near a_i, one of the singularities, for ε small enough,

$$\mathbf{E}_\varepsilon(u_\varepsilon, B_\rho(a_i)) \ge \mathbf{I}(\varepsilon, \rho) + o(1). \tag{5.10}$$

In order to prove (5.10), we need to prove an important lemma which is similar with Lemma 5.2.2. It says that, near each singularities, at sufficiently small scale, the minimizer is close to some rotation on sphere. Define $u_{i,r}(x) := u_*(a_i + rx)$ in B_1. Then we have

Lemma 5.2.4. *For any $\mu > 0$, there exists a sequence $\sigma_k \to 0$, and an integer $N(\mu)$, such that if $k > N$, then*

$$\|u_{i,\sigma_k}(\cdot) - \theta_i\|_{W^{1,n}(\mathbb{S}^{n-1}, \mathbb{R}^n)} < \mu \tag{5.11}$$

for some orthogonal rotation θ_i of $\mathbb{R}^n$.

Proof. If the lemma were false, then for any sequence $\sigma_k \to 0$,

$$u_{i,\sigma_k}(x) \nrightarrow \theta \frac{x}{|x|} \text{ in } W^{1,n}(\mathbb{S}^{n-1}, \mathbb{R}^n)$$

for any orthogonal rotation θ .

Then there exists $\delta, \bar{\sigma} > 0$, such that for $\sigma < \bar{\sigma}$, all $i = 1, 2, \cdots, d$, we have

$$\int_{B_{\bar{\sigma}(a_i)}\setminus B_\sigma(a_i)} \frac{|\nabla u_*|^n}{n} \,\mathrm{d}\,x \ge (\kappa_n + \delta) \ln \frac{\bar{\sigma}}{\sigma}. \tag{5.12}$$

In the balls $B_\sigma(a_i)$, by using a conclusion on the lower bound of $\mathbf{E}_\varepsilon(u_\varepsilon, B_\sigma(a_i))$ (see theorem 1.2 of [48]), we have

$$\mathbf{E}_\varepsilon(u_\varepsilon, B_\sigma(a_i)) \ge \kappa_n \ln \frac{\sigma}{\varepsilon} - C(n) \tag{5.13}$$

where $C(n)$ is a constant which depends only on n.

Then for ε small enough, say $\varepsilon < \varepsilon_1$, (5.12) and (5.13) imply

$$\mathbf{E}_\varepsilon(u_\varepsilon, \cup B_{\bar{\sigma}}(a_i)) \geq d\kappa_n \ln \frac{\bar{\sigma}}{\varepsilon} + \delta \ln \frac{\bar{\sigma}}{\sigma} - C(n). \tag{5.14}$$

With fixed $\bar{\sigma}$, the lower bound contradicts the upper bound (5.9) in Section 5.2.2 as $\sigma \to 0$. This completes the proof of the lemma. □

For any $\mu > 0$, from the definition of W_g, there exists $\tau_0 < \sigma_0$, such that for any $\delta < \tau_0$, we have

$$\mathbf{E}_{a,\delta}(w_{a,\delta}, \Omega_{a,\delta}) \geq W_g(a) + d\kappa_n |\ln \delta| - \mu/6. \tag{5.15}$$

From Lemma 5.2.4, choose $\sigma_k < \tau_0/2$ small enough, such that

$$||u_{i,\sigma_k}(\cdot) - \theta_i||_{W^{1,n}(\mathbb{S}^{n-1},\mathbb{R}^n)} < \mu/6d. \tag{5.16}$$

Now fix σ_k, and let $\rho = \frac{\sigma_k}{3}$, in ball $B_{4\rho}(a_i)$ define a function

$$\tilde{u}_\varepsilon(x) = \min\{(1 + f(|x-a_i|))\,|u_\varepsilon|\,, 1\} \cdot \frac{u_\varepsilon}{|u_\varepsilon|}$$

where

$$f(r) = \begin{cases} 0 & \text{if } |x-a_i| \leq \rho, \\ \left(\frac{|x-a_i|}{\rho} - 1\right)\delta_0 & \text{if } \rho < |x-a_i| \leq 2\rho, \\ \delta_0 & \text{if } 2\rho < |x-a_i| \leq 4\rho. \end{cases} \tag{5.17}$$

We will choose δ_0 later.

Recall firstly from Theorem (HL) that

$$u_\varepsilon \to u_* \quad \text{in} \quad \mathbf{C}^0_{loc}(\bar{\Omega} \setminus \cup_i \{a_i\}\,; \mathbb{R}^n).$$

Then there exists an $\varepsilon_1(\sigma_k, \delta_0)$, such that

$$|u_\varepsilon| \geq \frac{1}{1+\delta_0} \text{ on } B_{4\rho} \setminus B_{2\rho}$$

whenever $\varepsilon < \varepsilon_1$. It is clear that $\tilde{u}_\varepsilon = \frac{u_\varepsilon}{|u_\varepsilon|}$ on $B_{4\rho} \setminus B_{2\rho}$.

Recall that ρ is fixed, and we have $|\tilde{u}_\varepsilon| \geq |u_\varepsilon|$ in $B_{4\rho}(a_i)$, thus we can choose δ_0 small enough, and $\varepsilon < \varepsilon_2(\sigma_k, \delta_0, \mu)$ such that

$$\mathbf{E}_\varepsilon(u_\varepsilon, B_{4\rho}(a_i)) \geq \mathbf{E}_\varepsilon(\tilde{u}_\varepsilon, B_{4\rho}(a_i)) - \mu/6d. \tag{5.18}$$

To prove Lemma 5.1.2, we construct another function

$$\bar{u}_\varepsilon = \begin{cases} \tilde{u}_\varepsilon & \text{if } |x-a_i| \leq 2\rho, \\ \bar{v}_i & \text{if } 2\rho < |x-a_i| \leq 3\rho, \\ \bar{w}_i & \text{if } 3\rho < |x-a_i| \leq 4\rho. \end{cases} \tag{5.19}$$

where both $\bar{v}_i$ and $\bar{w}_i$ are interpolation maps defined by

$$\bar{v}_i = \frac{v_i}{|v_i|}$$

where $v_i = (3 - |x - a_i|/\rho)\tilde{u}_\varepsilon + (|x - a_i|/\rho - 2)u_*$. And

$$\bar{w}_i = \frac{w_i}{|w_i|}$$

where $w_i = (4 - |x - a_i|/\rho) \cdot u_*(3\rho \cdot \frac{x-a_i}{|x-a_i|}) + (|x - a_i|/\rho - 3) \cdot \theta_i(\frac{x-a_i}{|x-a_i|})$.

The fact of the convergence of u_ε to u_* implies that

$$\mathbf{E}_\varepsilon(\tilde{u}_\varepsilon, B_{3\rho}(a_i) \setminus B_{2\rho}(a_i)) \geq \mathbf{E}_\varepsilon(\bar{u}_\varepsilon, B_{3\rho}(a_i) \setminus B_{2\rho}(a_i)) - \mu/6d. \tag{5.20}$$

From the Lemma 5.2.4, we have

$$\begin{aligned} \mathbf{E}_\varepsilon(\tilde{u}_\varepsilon, B_{4\rho}(a_i) \setminus B_{3\rho}(a_i)) &\geq \kappa_n \ln \frac{4}{3} \\ &= \mathbf{E}_\varepsilon(\theta_i(\frac{x-a_i}{|x-a_i|}), B_{4\rho}(a_i) \setminus B_{3\rho}(a_i)) \\ &\geq \mathbf{E}_\varepsilon(\bar{u}_\varepsilon(a_i), B_{4\rho}(a_i) \setminus B_{3\rho}(a_i)) - \mu/6d. \end{aligned} \tag{5.21}$$

Combining (5.18), (5.20) and (5.21) gives

$$\mathbf{E}_\varepsilon(u_\varepsilon, B_{4\rho}(a_i)) \geq \mathbf{E}_\varepsilon(\bar{u}_\varepsilon, B_{4\rho}(a_i)) - \mu/2d \geq \mathbf{I}(\varepsilon, 4\rho) - \mu/2d. \tag{5.22}$$

Now we choose $\delta = 4\rho$. From the convergence of u_ε in the domain $\Omega_{a,\delta}$, choose ε small enough, such that

$$\begin{aligned} \mathbf{E}_\varepsilon(u_\varepsilon, \Omega_{a,\delta}) &\geq \mathbf{E}_\varepsilon(u_*, \Omega_{a,\delta}) - \mu/6 \\ &\geq \mathbf{E}_\varepsilon(w_{a,\delta}, \Omega_{a,\delta}) - \mu/6 \\ &\geq W_g(a) + d\kappa_n |\ln \delta| - \mu/3. \end{aligned} \tag{5.23}$$

Combining (5.22) and (5.23), for ε small enough, we have

$$\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \geq d\mathbf{I}(\varepsilon, 4\rho) + W_g(a) + d\kappa_n |\ln \delta| - 5\mu/6 \geq W_g(a) + d\kappa_n |\ln \varepsilon| + d\gamma - \mu. \tag{5.24}$$

This completes the proof of Lemma 5.1.2.

5.3 Limits of Solutions to Ginzburg-Landau equations

In this section, we start to study a sequence of critical points of $n-$dimensional Ginzburg-Landau type functional which have proper upper bounds. Suppose u_ε is a critical point of $\mathbf{E}_\varepsilon(u, \Omega)$, which satisfies (5.7)

$$\begin{cases} -\operatorname{div}\left(|\nabla u_\varepsilon|^{n-2} \nabla u_\varepsilon\right) = \frac{1}{\varepsilon^n}\left(1 - |u_\varepsilon|^2\right) u_\varepsilon & \text{in } \Omega\,, \\ u_\varepsilon = g & \text{on } \partial\Omega\,. \end{cases}$$

and has an energy upper bound

$$\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \leq d\kappa_n |\ln \varepsilon| + M \tag{5.1}$$

where M is a constant.

Note that these points are not necessarily the minimizers of the functional. We study the compactness and other properties of this sequence of critical points.

5.3.1 The Divergence Free Stress-Energy Tensor

Pohozaev identity plays a crucial role in [13] and [69] in two dimensions, and in [42] in n dimensions. There are two methods to derive the Pohozaev identity. The first method is to multiply both sides of the equations by appropriate multipliers. And the other one is by using the fact that the stress-energy tensor is divergence free. These two methods use only the equations. While the proof of [42] in n dimensions depends on the minimality of the functions. In our paper, we will use the divergence free of stress tensor to get the Pohozaev type identity. To do this, we need to discuss the regularity.

Maximum Principle

Lemma 5.3.1. *Let $u_\varepsilon \in W_g^{1,n}(\Omega, \mathbb{R}^n)$ be a solution of equations (5.7), then we have $|u_\varepsilon| \leq 1$.*

Proof. Let $\mathcal{M} = \{x \,|\, |u_\varepsilon(x)| \geq 1\}$, then on $\mathcal{M}$, we have

$$\begin{aligned}
\frac{1}{2}\mathrm{div}\left(|\nabla u_\varepsilon|^{n-2}\nabla |u_\varepsilon|^2\right) &= \left\langle \mathrm{div}\left(|\nabla u_\varepsilon|^{n-2}\nabla u_\varepsilon\right), u_\varepsilon\right\rangle + |\nabla u_\varepsilon|^n \\
&= \left\langle \frac{1}{\varepsilon^n}\left(|u_\varepsilon|^2 - 1\right) u_\varepsilon, u_\varepsilon\right\rangle + |\nabla u_\varepsilon|^n \\
&= \frac{1}{\varepsilon^n}\left(|u_\varepsilon|^2 - 1\right)|u_\varepsilon|^2 + |\nabla u_\varepsilon|^n \\
&\geq 0.
\end{aligned} \tag{5.2}$$

Therefore

$$\begin{aligned}
&\int_{\mathcal{M}} -\mathrm{div}\left(|\nabla u_\varepsilon|^{n-2}\nabla |u_\varepsilon|^2\right)\frac{\left(|u_\varepsilon|^2 - 1\right)}{|u_\varepsilon|^2} \\
&= \int_{\mathcal{M}} |\nabla u_\varepsilon|^{n-2}\left(\nabla |u_\varepsilon|^2\right)^2 \cdot \frac{1}{|u_\varepsilon|^4} \\
&\leq 0.
\end{aligned} \tag{5.3}$$

Thus either

- $|\mathcal{M}| = 0 \Rightarrow |u_\varepsilon| \leq 1$;

  or

- $|\nabla u_\varepsilon| = 0$ on $\mathcal{M} \Rightarrow |u_\varepsilon| = 1$ on $\mathcal{M}$.

That finishes the proof. □

An Auxiliary Problem

In this part, we shall discuss the regularity of the solutions, and prove

Lemma 5.3.2. *Let $u_\varepsilon \in W_g^{1,n}(\Omega, \mathbb{R}^n)$ be a solution of equations (5.7), then*

$$|\nabla u_\varepsilon|^{n-2}\frac{\partial u_\varepsilon}{\partial x_l} \in \mathbf{H}^1_{loc}(\Omega)$$

for all $l = 1, \cdots, n$.

Now we prove the Lemma 5.3.2. We follow the same method in the proof of Lemma 2.2 in [6].

For simplicity, in this part we suppose that $l = 1$. We consider the auxiliary problem below. In ball $B(a,r) \subset \Omega$, consider the energy functional

$$\mathbf{F}_\delta(w) := \frac{1}{n}\int_{B(a,r)} (|\nabla w|^2 + \delta^2)^{n/2}\,\mathrm{d}x - \int_{B(a,r)} f(u_\varepsilon)w\,\mathrm{d}x$$

for $w \in W^{1,n}_{u_\varepsilon}(B(a,r),\mathbf{R}^n) = \left\{w \in W^{1,n}\left(B(a,r),\mathbf{R}^n\right), w = u_\varepsilon \text{ on } \partial B(a,r)\right\}$.

The corresponding Euler-Lagrange equation of functional $\mathbf{F}_\delta$ is

$$\begin{cases} L_\delta w_\delta = f(u_\varepsilon) = \dfrac{1}{\varepsilon^n}\left(1 - |u_\varepsilon|^2\right)u_\varepsilon & \text{in } B(a,r) \\ \quad w_\delta = u_\varepsilon & \text{on } \partial B(a,r) \end{cases} \tag{5.4}$$

where

$$L_\delta w_\delta := -\mathrm{div}\left((|\nabla w_\delta|^2 + \delta^2)^{\frac{n-2}{2}}\nabla w_\delta\right).$$

Note that $f(u_\varepsilon) \in W^{1,n} \cap L^\infty$ if ε is fixed.

Differentiate the equation above(here we assume $w_\delta \in W^{2,n}$, if not we can use the difference quotients to get the same conclusion), then we have

$$\frac{\partial}{\partial x_1}L_\delta w_\delta = \frac{\partial}{\partial x_1}(f(u_\varepsilon)).$$

The corresponding variational equality is

$$\begin{aligned}&\int_{B(a,r)} (|\nabla w_\delta|^2 + \delta^2)^{\frac{n-2}{2}}\left\langle \nabla\frac{\partial w_\delta}{\partial x_1}, \nabla v\right\rangle + (n-2)\int_{B(a,r)} \langle\nabla w_\delta, \nabla v\rangle\,(|\nabla w_\delta|^2 + \delta^2)^{\frac{n-4}{2}}\left\langle\nabla w_\delta, \nabla\frac{\partial w_\delta}{\partial x_1}\right\rangle \\ &= \int_{B(a,r)} \frac{\partial}{\partial x_1}\left(\frac{1}{\varepsilon^n}(1 - |u_\varepsilon|^2)u_\varepsilon\right)\cdot v\end{aligned} \tag{5.5}$$

for $v \in W^{1,n}_0(B(a,r))$.

Choose $\varphi = \psi^2$, where

$$\psi = \begin{cases} 1, & \text{in } B(a,\frac{r}{2}) \\ 0, & \text{on } \partial B(a,r). \end{cases} \tag{5.6}$$

and $|\nabla\psi| \leq \dfrac{C}{r}$.

Let $v = \varphi\dfrac{\partial w_\delta}{\partial x_1}$, then $\nabla v = \varphi\cdot\nabla\dfrac{\partial w_\delta}{\partial x_1} + \dfrac{\partial w_\delta}{\partial x_1}\cdot(\nabla\varphi)^T$. Then we have

$$\begin{aligned}&\int_{B(a,r)} (|\nabla w_\delta|^2 + \delta^2)^{\frac{n-2}{2}}\left\langle\nabla\frac{\partial w_\delta}{\partial x_1}, \nabla v\right\rangle \\ &= \int_{B(a,r)} \varphi\left|\nabla\frac{\partial w_\delta}{\partial x_1}\right|^2 (|\nabla w_\delta|^2 + \delta^2)^{\frac{n-2}{2}} + (|\nabla w_\delta|^2 + \delta^2)^{\frac{n-2}{2}}\left\langle\nabla\frac{\partial w_\delta}{\partial x_1}, \frac{\partial w_\delta}{\partial x_1}\cdot(\nabla\varphi)^T\right\rangle.\end{aligned} \tag{5.7}$$

By using Young's Inequality, we can estimate the second term on the right hand side,

$$\begin{aligned}&\left|\int_{B(a,r)} (|\nabla w_\delta|^2+\delta^2)^{\frac{n-2}{2}} \left\langle \nabla \frac{\partial w_\delta}{\partial x_1}, \frac{\partial w_\delta}{\partial x_1}\cdot (\nabla\varphi)^T \right\rangle\right| \\ \leq&\theta \int_{B(a,r)} \varphi \left|\nabla \frac{\partial w_\delta}{\partial x_1}\right|^2 (|\nabla w_\delta|^2+\delta^2)^{\frac{n-2}{2}} + C(\theta, r)\int_{B(a,r)} (|\nabla w_\delta|^2+\delta^2)^{\frac{n}{2}}.\end{aligned} \tag{5.8}$$

While

$$\begin{aligned}&(n-2)\int_{B(a,r)} \langle \nabla w_\delta, \nabla v\rangle (|\nabla w_\delta|^2+\delta^2)^{\frac{n-4}{2}} \left\langle \nabla w_\delta, \nabla\frac{\partial w_\delta}{\partial x_1}\right\rangle \\ =&(n-2)\int_{B(a,r)} \varphi \left\langle \nabla w_\delta, \nabla\frac{\partial w_\delta}{\partial x_1}\right\rangle^2 (|\nabla w_\delta|^2+\delta^2)^{\frac{n-4}{2}} \\ &+2(n-2)\int_{B(a,r)} \left\langle \nabla w_\delta, \frac{\partial w_\delta}{\partial x_1}\cdot(\nabla\psi)^T\right\rangle (|\nabla w_\delta|^2+\delta^2)^{\frac{n-4}{2}} \left\langle \nabla w_\delta, \psi\cdot\nabla\frac{\partial w_\delta}{\partial x_1}\right\rangle.\end{aligned} \tag{5.9}$$

We can estimate the second term on the right hand side by using Young's Inequality again.

$$\begin{aligned}&\left|(n-2)\int_{B(a,r)} \left\langle \nabla w_\delta, \frac{\partial w_\delta}{\partial x_1}\cdot(\nabla\psi)^T\right\rangle (|\nabla w_\delta|^2+\delta^2)^{\frac{n-4}{2}} \left\langle \nabla w_\delta, \psi\cdot\nabla\frac{\partial w_\delta}{\partial x_1}\right\rangle\right| \\ \leq&\theta \int_{B(a,r)} \varphi \left|\nabla \frac{\partial w_\delta}{\partial x_1}\right|^2 (|\nabla w_\delta|^2+\delta^2)^{\frac{n-2}{2}} + C(\theta, r)\int_{B(a,r)} (|\nabla w_\delta|^2+\delta^2)^{\frac{n}{2}}.\end{aligned} \tag{5.10}$$

Combine the inequalities above, we have

$$\begin{aligned}&\int_{B(a,r/2)} \left|\nabla \frac{\partial w_\delta}{\partial x_1}\right|^2 (|\nabla w_\delta|^2+\delta^2)^{\frac{n-2}{2}} \\ \leq&C(r,\varepsilon)\int_{B(a,r)} \left(1+|\nabla w_\delta|^n + \left|\frac{\partial u_\varepsilon}{\partial x_1}\right| \left|\frac{\partial w_\delta}{\partial x_1}\right|\right)\end{aligned} \tag{5.11}$$

Therefore, $(|\nabla w_\delta|^2+\delta^2)^{\frac{n-2}{4}}\cdot\dfrac{\partial w_\delta}{\partial x_1}$ is bounded in $H^1_{loc}(\Omega)$. Recall that $\mathbf{F}_\delta$ is coercive, we have w_δ is bounded in $W^{1,n}_{u_\varepsilon}(B(a,r),\mathbf{R}^n)$. Then

$$\begin{aligned} w_\delta &\rightharpoonup w_0 \quad \text{in } W^{1,n}; \\ w_\delta &\to w_0 \quad \text{in } L^s, \forall 1\leq s<\infty.\end{aligned} \tag{5.12}$$

From the lower semi-continuity, we have

$$\int_{B(a,r)} |\nabla w_0|^n \leq \liminf_{\delta\to 0}\int_{B(a,r)} |\nabla w_\delta|^n.$$

And from the convergence of w_δ, we have

$$\int_{B(a,r)} f(u_\varepsilon)\cdot w_0 = \lim_{\delta\to 0}\int_{B(a,r)} f(u_\varepsilon)\cdot w_\delta.$$

Thus w_0 is a minimizer of $\mathbf{F}_0$. From the uniqueness of minimizer of $\mathbf{F}_0$(in fact, $\mathbf{F}_0$ is convex), we have $w_0 = u_\varepsilon$. By the minimality of w_0, we have

$$\lim_{\delta\to 0}\int_{B(a,r)} |\nabla w_\delta|^n \le \int_{B(a,r)} |\nabla u_\varepsilon|^n .$$

Hence

$$w_\delta \to u_\varepsilon \text{ in } W^{1,n}.$$

Then

$$(|\nabla w_\delta|^2+\delta^2)^{\frac{n-2}{4}}\cdot\frac{\partial w_\delta}{\partial x_1} \to |\nabla u_\varepsilon|^{\frac{n-2}{2}}\frac{\partial u_\varepsilon}{\partial x_1} \quad \text{in } L^2.$$

Therefore

$$|\nabla u_\varepsilon|^{\frac{n-2}{2}}\frac{\partial u_\varepsilon}{\partial x_1} \in H^1_{loc}(\Omega).$$

From [33], we know that $\nabla u_\varepsilon \in L^\infty$, then we have

$$|\nabla u_\varepsilon|^{n-2}\frac{\partial u_\varepsilon}{\partial x_1} \in H^1_{loc}(\Omega).$$

Divergence Free of Stress-Energy Tensor

The stress-energy tensor associated to a critical point u_ε of the Ginzburg-Landau functional is

$$T_{i,j} = |\nabla u_\varepsilon|^{n-2}\left\langle \partial_i u_\varepsilon, \partial_j u_\varepsilon\right\rangle - (\frac{1}{n}|\nabla u_\varepsilon|^n + \frac{1}{4\varepsilon^n}(1-|u_\varepsilon|^2)^2)\delta_{i,j}.$$

Then

$$\begin{aligned}
\mathrm{div} T_{\cdot,j} &= \left\langle \mathrm{div}(|\nabla u_\varepsilon|^{n-2}\nabla u_\varepsilon), \partial_j u_\varepsilon\right\rangle + \left\langle |\nabla u_\varepsilon|^{n-2}\partial_i u_\varepsilon, \partial_i\partial_j u_\varepsilon\right\rangle \\
&\quad - \left\langle |\nabla u_\varepsilon|^{n-2}\partial_i u_\varepsilon, \partial_i\partial_j u_\varepsilon\right\rangle + \left\langle \frac{1}{\varepsilon^n}(1-|u_\varepsilon|^2)u_\varepsilon, \partial_j u_\varepsilon\right\rangle \\
&= \left\langle \mathrm{div}(|\nabla u_\varepsilon|^{n-2}\nabla u_\varepsilon), \partial_j u_\varepsilon\right\rangle + \left\langle \frac{1}{\varepsilon^n}(1-|u_\varepsilon|^2)u_\varepsilon, \partial_j u_\varepsilon\right\rangle \\
&= 0.
\end{aligned} \tag{5.13}$$

Lemma 5.3.3. *For any vector $X = (x_1, \cdots, x_n)$, we have*

$$\int_{\partial\Omega}\sum_{i,j} x_j\nu_i T_{i,j} = \int_\Omega \sum_{i,j}(\partial_i x_j)T_{i,j}.$$

Proof. By using the Divergence theorem directly, we have

$$\begin{aligned}
\int_{\partial\Omega}\sum_{i,j} x_j\nu_i T_{i,j} &= \int_\Omega \sum_j \mathrm{div}(T_{\cdot,j}x_j) \\
&= \int_\Omega \sum_j (\mathrm{div} T_{\cdot,j})x_j + \sum_{i,j} T_{i,j}\partial_i(x_j) \\
&= \int_\Omega \sum_{i,j}(\partial_i x_j)T_{i,j}.
\end{aligned} \tag{5.14}$$

□

5.3.2 Covering of Bad Sets

We call the sets where $|u_\varepsilon|$ is near 0 “Bad Sets". In this part, we cover these “Bad Sets" by a finite collection of small balls. This covering also provide a finite singular set of the limit map.

Pohozaev Inequality

Pohozaev Inequality plays a crucial rule to cover the bad sets. We use the divergence free of the stress-energy tensor to prove this inequality. On the boundary ∂D of a domain D, let ν be the outward pointing unit normal to ∂D, and $\tau^k, k = 1, \cdots, n-1$ be the orthogonal unit tangent vectors to ∂D, then for every vector field $X = (x_1, \cdots, x_n)$,

Lemma 5.3.4. *We have* $\sum_{i,j=1}^n x_j \nu_i T_{i,j} = X_\nu T_{\nu,\nu} + \sum_{k=1}^{n-1} X_{\tau^k} T_{\nu,\tau^k}$.

Proof. For fixed i, j, by expanding the notations, we have

$$\begin{aligned}
&\sum_{s=1}^n x_s\nu_s\nu_i T_{i,j}\nu_j + \sum_{k=1}^{n-1}\sum_{s=1}^n x_s\tau_s^k\nu_i T_{i,j}\tau_j^k \\
&= x_j\nu_j^2\nu_i T_{i,j} + \sum_{k=1}^{n-1} x_j(\tau_j^k)^2\nu_i T_{i,j} + \sum_{k=1}^{n-1}\sum_{s\neq j}(x_s\nu_s\nu_j\nu_i T_{i,j} + \sum_{k=1}^{n-1} x_s\tau_s^k\tau_j^k\nu_i T_{i,j}) \\
&= x_j\nu_i T_{i,j}.
\end{aligned} \tag{5.15}$$

The last equality comes from the fact that $(\nu_s, \tau_s^1, \cdots, \tau_s^{n-1}) \perp (\nu_j, \tau_j^1, \cdots, \tau_j^{n-1})$ if $s \neq j$. □

In a bounded strictly star-shaped domain, by taking the particular choice of the vector field X, we have the following Pohozaev Inequality

Proposition 5.3.1. *Let $D \subset \mathbb{R}^n$ be a bounded strictly star-shaped domain with respect to $x_0 \in D$, such that $(x - x_0)\cdot\nu \geq \alpha\mathrm{diam}(D)$ for all $x \in \partial\Omega$, and u_ε is solution of equation (5.7). Then there exists a constant C depending only on n, α, such that*

$$\begin{aligned}
&\int_D \frac{1}{4\varepsilon^n}\left(1 - |u_\varepsilon|^2\right)^2 + \alpha\mathrm{diam}(D)\int_{\partial D} |\nabla u_\varepsilon|^{n-2}\,|\partial_\nu u_\varepsilon|^2 \\
&\leq C(n,\alpha)\mathrm{diam}(D)\int_{\partial D} |\nabla u_\varepsilon|^{n-2}\,|\partial_\tau u_\varepsilon|^2 + \frac{1}{4\varepsilon^n}\left(1 - |u_\varepsilon|^2\right)^2.
\end{aligned} \tag{5.16}$$

Proof. We take a particular choice of $X(x) = x - x_0$, then $\partial_i(X_j) = \delta_{i,j}$. From Lemma 5.3.3 and Lemma 5.3.4, we have

$$\begin{aligned}
\int_{\partial D} X_\nu T_{\nu,\nu} + \sum_{k=1}^{n-1} X_{\tau^k} T_{\nu,\tau^k} &= \int_{\partial D}\sum_{i,j} X_j\nu_i T_{i,j} \\
&= \int_D \sum_{i,j}(\partial_i x_j)T_{i,j} = \int_D \sum_i T_{i,i} \\
&= \int_D -\frac{n}{4\varepsilon^n}\left(1 - |u_\varepsilon|^2\right)^2.
\end{aligned} \tag{5.17}$$

Also on the boundary $\partial\Omega$, we have

$$\begin{aligned}
&\int_{\partial D} X_\nu T_{\nu,\nu} + \sum_{k=1}^{n-1} X_{\tau^k} T_{\nu,\tau^k} \\
&= \int_{\partial D} X\cdot\nu\left(|\nabla u_\varepsilon|^{n-2}\,|\partial_\nu u_\varepsilon|^2 - \frac{1}{n}\,|\nabla u_\varepsilon|^n - \frac{1}{4\varepsilon^n}\left(1-|u_\varepsilon|^2\right)^2\right) \\
&+ \int_{\partial D}\sum_{k=1}^{n-1} X\cdot\tau_{\tau^k}\left(|\nabla u_\varepsilon|^{n-2}\,\langle\partial_\nu u_\varepsilon, \partial_{\tau^k} u_\varepsilon\rangle\right)
\end{aligned} \tag{5.18}$$

Combine (5.17) and (5.18) above, and by using Cauchy-Schwarz inequality and Young inequality, we have

$$\begin{aligned}
\int_D \frac{n}{4\varepsilon^n}\left(1-|u_\varepsilon|^2\right)^2 \le \operatorname{diam}(D)&\int_{\partial D}\frac{1}{n}\,|\nabla u_\varepsilon|^{n-2}\,|\partial_\tau u_\varepsilon|^2 + \frac{1}{4\varepsilon^n}\left(1-|u_\varepsilon|^2\right)^2 \\
&- \alpha\operatorname{diam}(D)\int_{\partial D}\frac{n-1}{n}\,|\nabla u_\varepsilon|^{n-2}\,|\partial_\nu u_\varepsilon|^2 \\
&+ \frac{\alpha}{2}\operatorname{diam}(D)\int_{\partial D}|\nabla u_\varepsilon|^{n-2}\,|\partial_\nu u_\varepsilon|^2 \\
&+ C(n,\alpha)\operatorname{diam}(D)\int_{\partial D}|\nabla u_\varepsilon|^{n-2}\,|\partial_\tau u_\varepsilon|^2\,.
\end{aligned} \tag{5.19}$$

This implies the conclusion of the proposition. □

Covering of Bad Sets

Next we follow the methods of [69] and [42] to cover the bad sets. For any $x_0 \in \Omega$, $\rho > 0$, we introduce

$$f(x_0,\rho) = \rho\int_{\partial B_\rho(x_0)\bigcap\Omega}\frac{|\nabla u_\varepsilon|^n}{n} + \frac{1}{4\varepsilon^n}\left(1-|u_\varepsilon|^2\right)^2.$$

The following is related to the “Courant-Lebesgue lemma"; see Lemma 2.3 in [69] and Lemma3.5 in [42].

Lemma 5.3.5. *(i). If we have an upper bound for the energy of critical points u_ε, i.e.*

$$\mathbf{E}_\varepsilon(u_\varepsilon,\Omega) \le d\kappa_n\,|\ln\varepsilon| + M,$$

then for any point $x_0 \in \Omega$, and $0 < \varepsilon \le \mathrm{e}^{-1}$, we have

$$\inf_{\varepsilon^{1/2}\le\rho\le\varepsilon^{1/4}} f(x_0,\rho) \le \frac{4\mathbf{E}_\varepsilon(u_\varepsilon,\Omega)}{|\ln\varepsilon|} \le C_1$$

and

$$\inf_{5\varepsilon^{1/4}\le\rho\le5\varepsilon^{1/8}} f(x_0,\rho) \le \frac{8\mathbf{E}_\varepsilon(u_\varepsilon,\Omega)}{|\ln\varepsilon|} \le 2C_1.$$

(ii). There are constants γ and ε_0 depending on Ω, g, such that for $0 < \varepsilon < \varepsilon_0$,

$$\inf_{B_\rho\bigcap\Omega}|u_\varepsilon| \ge 1/2.$$

whenever $\varepsilon^{1/2} \le \rho \le \varepsilon^{1/4}$ and $f(x_0,\rho) \le \gamma$.

Proof. (i). For $0 < \varepsilon \leq \mathrm{e}^{-1}$, we have

$$
\begin{aligned}
\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) &\geq \mathbf{E}_\varepsilon(u_\varepsilon, \Omega \cap B_{\varepsilon^{1/4}}(x_0)) \\
&\geq \int_{\varepsilon^{1/2}}^{\varepsilon^{1/4}} f(x_0, \rho) \frac{1}{\rho} \,\mathrm{d}\rho \\
&\geq \frac{1}{4} |\ln \varepsilon| \inf_{\varepsilon^{1/2} \leq \rho \leq \varepsilon^{1/4}} f(x_0, \rho).
\end{aligned}
\tag{5.20}
$$

Thus we have the first part of (i) if we have an upper bound of the energy. The second part of (i) follows the same idea.
(ii). From the regularity result in Proposition 3.3 of [42] for solutions of equations (5.7), we have the Hölder continuity of u_ε

$$
[u_\varepsilon]_{C^\alpha(\bar{\Omega})} \leq C_2 \varepsilon^{-\alpha}.
$$

Choose $0 < \rho < \rho_0(\Omega)$ small, s.t. $D = B_\rho(x_0) \bigcap \Omega$ is strongly star-shaped w.r.t. $y_0 \in D$ and $(x - y_0) \cdot \nu \geq \frac{1}{4}\rho$ for $\forall x \in \partial D$. If there is a $y \in D$ such that $|u_\varepsilon(y)| \leq \frac{1}{2}$, then

$$
|u_\varepsilon(x)| \leq \frac{3}{4} \quad \text{for } |x - y| \leq \frac{\varepsilon}{(4C_2)^{\frac{1}{\alpha}}}.
$$

and therefore

$$
\begin{aligned}
&\int_D \frac{1}{4\varepsilon^n} \left(1 - |u_\varepsilon|^2\right)^2 \\
&\geq \int_{D \bigcap B_{\frac{\varepsilon}{(4C_2)^{1/\alpha}}}(y)} \frac{1}{4\varepsilon^n} \left(1 - |u_\varepsilon|^2\right)^2 \\
&\geq C_3 > 0.
\end{aligned}
\tag{5.21}
$$

However, by the Pohozaev inequality,

$$
\begin{aligned}
&\int_D \frac{1}{4\varepsilon^n} \left(1 - |u_\varepsilon|^2\right)^2 + \frac{\rho}{4} \int_{\partial D} |\nabla u_\varepsilon|^{n-2} |\partial_\nu u_\varepsilon|^2 \\
&\leq C(n) \left[f(x_0, \rho) + \rho \int_{B_\rho(x_0) \bigcap \partial\Omega} \frac{|\nabla u_\varepsilon|^{n-2} |\partial_\tau g|^2}{n} \right].
\end{aligned}
\tag{5.22}
$$

Recall that g is smooth, thus we have

$$
\begin{aligned}
&\int_D \frac{1}{4\varepsilon^n} \left(1 - |u_\varepsilon|^2\right)^2 + \frac{\rho}{4} \int_{\partial D} |\nabla u_\varepsilon|^{n-2} |\partial_\nu u_\varepsilon|^2 - C(n, g)\rho \int_{B_\rho(x_0) \bigcap \partial\Omega} \frac{|\partial_\nu u_\varepsilon|^{n-2}}{n} \\
&\leq C(n) \left[f(x_0, \rho) + \rho \int_{B_\rho(x_0) \bigcap \partial\Omega} \frac{|\partial_\tau g|^n}{n} \right] \\
&\leq C(n) \left(f(x_0, \rho) + C(g)\rho^n \right).
\end{aligned}
\tag{5.23}
$$

While

$$
\frac{1}{4} \int_{\partial D} |\nabla u_\varepsilon|^{n-2} |\partial_\nu u_\varepsilon|^2 - C(n, g) \int_{B_\rho(x_0) \bigcap \partial\Omega} \frac{|\partial_\nu u_\varepsilon|^{n-2}}{n} \geq -C_4(n, g)
$$

uniformly, therefore we can choose ρ small enough such that $\rho \cdot C_4 \leq \frac{C_3}{2}$.

Then (5.23) contradicts (5.21) if we choose γ and ρ small enough.

□

For $0 < \varepsilon \le \varepsilon_0$ and critical point u_ε of energy E_ε, denote

$$S_\varepsilon := \left\{ x \in \Omega : |u_\varepsilon| < \frac{1}{2} \right\}.$$

Let $\{B_{\varepsilon^{1/4}}(x)\}_{x \in S_\varepsilon}$ be a covering of S_ε. Then by lemma 5.3.5, we have for $x \in S_\varepsilon$, there exits a $\rho_0 \in (\varepsilon^{1/2}, \varepsilon^{1/4})$, such that

$$\gamma \le f(x, \rho_0) \le C_1.$$

By Vitali's covering lemma, we can find a finite collection of disjoint balls $B_{\varepsilon^{\frac{1}{4}}}(x_i)$, $x_i \in S_\varepsilon, 1 \le i \le I_\varepsilon$, such that

$$(\Omega \cap \cup_{x \in S_\varepsilon} B_{\varepsilon^{1/4}}(x)) \subset \cup_{1 \le i \le I_\varepsilon} B_{5\varepsilon^{1/4}}(x_i). \tag{5.24}$$

Thus we have an uniform upper bound for the number of the bad balls $B_{\varepsilon^{1/4}}(x_i)$.

$$I_\varepsilon \le \sum_{i=1}^{I_\varepsilon} \frac{4\mathbf{E}_\varepsilon}{\gamma |\ln \varepsilon|} \le \frac{C_1}{\gamma} \le I_0.$$

We refine the initial choice. For $1 \le i \le I_\varepsilon$, choose $\rho_i \in \left[5\varepsilon^{1/4}, 5\varepsilon^{1/8}\right]$, such that

$$f(x_i, \rho_i) \le 2C_1$$

and let $D_i = \Omega \cap B_{\rho_i}(x_i)$. From the Pohozaev inequality, we have

Lemma 5.3.6. *There exists a constant* $C_4 = C_4(\Omega, g) > 0$ *such that*

$$\int_{D_i} \frac{1}{4\varepsilon^n} \left(1 - |u_\varepsilon|^2\right)^2 \le C_4$$

uniformly for $0 < \varepsilon \le \varepsilon_0$, $1 \le i \le I_\varepsilon$.

Lemma 5.3.7. *There exists a number* $J_0 = J_0(\Omega, g) \in \mathbb{N}$ *such that for any disjoint collection of balls* $B_{\varepsilon/5}(x_j), x_j \in S_\varepsilon, 1 \le j \le J_\varepsilon$, *there holds* $J_\varepsilon \le J_0$.

Proof. From the definition of D_i and (5.24),

$$\left(\Omega \cap \bigcup_j B_{\varepsilon/5}(x_j) \right) \subset \bigcup_{1 \le i \le I_\varepsilon} D_i.$$

Then by (5.21),

$$J_\varepsilon C_3 \le \sum_j \int_{B_{\varepsilon/5}(x_j)} \frac{1}{4\varepsilon^n} \left(1 - |u_\varepsilon|^2\right)^2 \le \sum_i \int_{D_i} \frac{1}{4\varepsilon^n} \left(1 - |u_\varepsilon|^2\right)^2 \le C_4 I_0.$$

□

Now consider the covering $(B_{\varepsilon/5}(x))_{x \in S_\varepsilon}$. By Vitali's covering lemma again, we can find a disjoint collection of balls $B_{\varepsilon/5}(x_j), x_j \in S_\varepsilon, 1 \le j \le J_\varepsilon$, such that

$$S_\varepsilon \subset \bigcup_j B_\varepsilon(x_j).$$

And moreover, by Lemma 5.3.7, $J_\varepsilon \le J_0$ independent of ε.

For any $\sigma > 0$, denote

$$\Omega_\varepsilon^\sigma = \Omega \setminus \cup_j B_\sigma(x_j^\varepsilon).$$

We have the following estimate

Proposition 5.3.2. *There exist $C = C(\Omega, g) > 0$, such that for any $\sigma > 0$,*

$$\mathbf{E}(u_\varepsilon; \Omega_\varepsilon^\sigma) \leq d\kappa_n |\ln \sigma| + C$$

uniformly for $0 < \varepsilon < \varepsilon_0$.

To prove Proposition 5.3.2, we need an estimate on the annulus, which is stated in Lemma 3.9 of [42].

Lemma 5.3.8 (Lemma 3.9 of [42])**.** *Fix an $R_1 > 0$, and any $x_0 \in \Omega$. Let $\varepsilon < R_0 < R \leq R_1$ and suppose $u \in W_g^{1,n}(\Omega; \mathbb{R}^n)$ satisfies $|u| \leq 1$ in Ω, and $|u| \geq \dfrac{1}{2}$ in $A_{R,R_0} = \Omega \cap (B_R(x_0) \setminus B_{R_0}(x_0))$ and the estimates*

$$\int_{\Omega \cap B_{\varepsilon^{1/4}}(x_0)} \frac{1}{\varepsilon^n} \left(1 - |u_\varepsilon|^2\right)^2 \leq K,$$

as well as

$$\mathbf{E}_\varepsilon(u) \leq K |\ln \varepsilon| + K.$$

Then there holds

$$\int_{A_{R,R_0}} |\nabla u|^n \, \mathrm{d}x \geq \left|\hat{d}\right|^{\frac{n}{n-1}} (n-1)^{n/2} \left|\mathbb{S}^{n-1}\right| \ln \frac{R}{R_0} - C(n, \hat{d}, \Omega, g), \tag{5.25}$$

where $\hat{d}$ is the degree of u, restricted to $\partial(\Omega \cap B_R(x_0))$.

Proof. The proof is as the same as in [42]. □

Proof of Proposition 5.3.2 We have already an upper bound for the energy of the critical points u_ε, i.e.

$$\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \leq d\kappa_n |\ln \varepsilon| + M$$

By applying the Lemma 5.3.8 and a ball grow method in the proof of Proposition 3.3 of [69], we have a lower bound of the energy.

$$\mathbf{E}(u_\varepsilon, \Omega_\varepsilon^\sigma) \geq d\kappa_n \ln \frac{\sigma}{\varepsilon} - C(n, d, \Omega, g).$$

The proposition follows immediately from the upper bound and lower bound above. □

For any sequence $\varepsilon_k \to 0$, consider the corresponding sequence of critical points $u_k = u_{\varepsilon_k}$. By Lemma 5.3.7, we have a bounded number of sequences of centers $\{x_j^k\}$, $1 \leq j \leq J_k \leq J_0$, of “bad balls". Passing to a subsequence, we have $\hat{J}$ independent of ε_k, such that

$$x_j^k \to x_j \in \bar{\Omega} \text{ as } k \to +\infty \text{ for each } j = 1, 2, \cdots, \hat{J}. \tag{5.26}$$

Note that here x_j may be the same, however, we can choose a collection of distinct points $\{a_j\}_{j=1}^J$ in $\{x_j\}$.

Now we give more discussion on the number J, and prove that $J = d$. Let $\sigma_0 = \frac{1}{6} \min_{i \neq j} \left|a_i - a_j\right|$, then for $\rho < \sigma_0$, there exists ε_0 and j' such that $\left|x_{j'}^k - a_j\right| \leq \rho$ if $\varepsilon_k < \varepsilon_0$. We have

1. If $d_j = \deg(u_\varepsilon, B(a_j, 2\rho)) = 0$, then in $B(x^k_{j'}, \varepsilon_k^{1/4}) \subset B(a_j, 2\rho)$, by using Lemma 5.3.5, we have

$$\mathbf{E}(u_{\varepsilon_k}, B(a_j, 2\rho)) \geq \mathbf{E}(u_{\varepsilon_k}, B(x^k_{j'}, \varepsilon_k^{1/4})) \geq \frac{\gamma}{4} |\ln \varepsilon| . \tag{5.27}$$

2. If $d_j = \deg(u_\varepsilon, B(a_j, 2\rho)) \neq 0$, then by using Lemma 5.3.8 and Proposition 3.3 of [69], we have

$$\mathbf{E}(u_{\varepsilon_k}, B(a_j, \sigma_0) \setminus \cup_{j'} B(x_{j'}, 5\varepsilon)) \geq \left|d_j\right| |\ln \varepsilon| + (\left|d_j\right|^{\frac{n}{n-1}} - \left|d_j\right|) |\ln \rho| - C(n, d_j, \Omega, g, \sigma_0). \tag{5.28}$$

While we have $\sum_j d_j = d$ and the upper bound condition of (5.1), therefore

1. There is no point a_j such that $d_j = \deg(u_\varepsilon, B(a_j, 2\rho)) = 0$ if $\varepsilon < \varepsilon_0$.
2. There are only d points a_j such that $d_j = \deg(u_\varepsilon, B(a_j, 2\rho)) \neq 0$, and $d_j = 1$.

Now we can say we choose the limit collection of distinct points $\{a_j\}_{j=1}^d$.

5.3.3 $\varepsilon-$ Regularity

We rewrite

$$u_\varepsilon = \rho_\varepsilon \cdot \theta_\varepsilon,$$

where $\rho_\varepsilon = |u_\varepsilon|$ and $\theta_\varepsilon = \dfrac{u_\varepsilon}{|u_\varepsilon|}$. In this part, we prove $\varepsilon-$regularity of θ_ε.

Denote

$$\Omega_\sigma := \Omega \setminus \cup_j B_\sigma(a_j),$$

thus we have $|u_\varepsilon| \geq \dfrac{1}{2}$ in Ω_σ if ε is small enough.

Then

$$\nabla u_\varepsilon = \rho_\varepsilon \nabla \theta_\varepsilon + \theta_\varepsilon \cdot (\nabla \rho_\varepsilon)^\top .$$

By substituting u_ε in the function, we have

$$-\mathrm{div}(|\nabla u_\varepsilon|^{n-2} (\rho_\varepsilon \nabla \theta_\varepsilon + \theta_\varepsilon \nabla \rho_\varepsilon^\top)) = \frac{1}{\varepsilon^n}(1 - |u_\varepsilon|^2) u_\varepsilon,$$

i.e.

$$-\mathrm{div}(|\nabla u_\varepsilon|^{n-2} \nabla \rho_\varepsilon) \theta_\varepsilon - 2 |\nabla u_\varepsilon|^{n-2} \nabla \theta_\varepsilon \cdot \nabla \rho_\varepsilon^\top - \mathrm{div}(| \nabla u_\varepsilon |^{n-2} \nabla \theta_\varepsilon) \rho_\varepsilon = \frac{1}{\varepsilon^n}(1 - \rho_\varepsilon^2) u_\varepsilon$$

Then multiply both sides by θ_ε, we have

$$-\mathrm{div}(|\nabla u_\varepsilon|^{n-2} \nabla \rho_\varepsilon) - 0 - \mathrm{div}(|\nabla u_\varepsilon|^{n-2} \nabla \theta_\varepsilon) \rho_\varepsilon \cdot \theta_\varepsilon = \frac{1}{\varepsilon^n}(1 - \rho_\varepsilon^2) \rho_\varepsilon.$$

Therefore, $-\mathrm{div}(|\nabla u_\varepsilon|^{n-2} \nabla \theta_\varepsilon) \theta_\varepsilon = |\nabla u_\varepsilon|^{n-2} |\nabla \theta_\varepsilon|^2$ (that is because of the fact $\theta_\varepsilon \cdot \nabla \theta_\varepsilon = 0$) implies that

$$-\mathrm{div}(|\nabla u_\varepsilon|^{n-2} \nabla \rho_\varepsilon) + |\nabla u_\varepsilon|^{n-2} |\nabla \theta_\varepsilon|^2 \cdot \rho_\varepsilon = \frac{1}{\varepsilon^n}(1 - \rho_\varepsilon^2) \rho_\varepsilon.$$

Recall that $\theta_\varepsilon = \dfrac{u_\varepsilon}{|u_\varepsilon|}$, thus $\nabla\theta_\varepsilon = -\dfrac{u_\varepsilon}{|u_\varepsilon|^2}\cdot\nabla|u_\varepsilon| + \dfrac{\nabla u_\varepsilon}{\rho_\varepsilon}$, then

$$
\begin{aligned}
&-\operatorname{div}(|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2\nabla\theta_\varepsilon)\\
=&\operatorname{div}[|\nabla u_\varepsilon|^{n-2}(u_\varepsilon\cdot(\nabla\rho_\varepsilon)^\top - \rho_\varepsilon\cdot\nabla u_\varepsilon)]\\
=&\operatorname{div}(|\nabla u_\varepsilon|^{n-2}\nabla\rho_\varepsilon)\cdot u_\varepsilon + |\nabla u_\varepsilon|^{n-2}\cdot\nabla u_\varepsilon\cdot(\nabla\rho_\varepsilon)^\top - \operatorname{div}(|\nabla u_\varepsilon|^{n-2}\nabla u_\varepsilon)\rho_\varepsilon - |\nabla u_\varepsilon|^{n-2}\nabla u_\varepsilon(\nabla\rho_\varepsilon)^\top\\
=&\mid\nabla u_\varepsilon\mid^{n-2}\rho_\varepsilon^2\mid\nabla\theta_\varepsilon\mid^2\theta_\varepsilon - \frac{1}{\varepsilon^n}(1-\rho_\varepsilon^2)\rho_\varepsilon^2\theta_\varepsilon + \frac{1}{\varepsilon^n}(1-\rho_\varepsilon^2)\rho_\varepsilon^2\theta_\varepsilon\\
=&|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\theta_\varepsilon.
\end{aligned}
\tag{5.29}
$$

So we have

$$
\begin{cases}
-\operatorname{div}(\mid\nabla u_\varepsilon\mid^{n-2}\nabla\rho_\varepsilon) + \mid\nabla u_\varepsilon\mid^{n-2}\mid\nabla\theta_\varepsilon\mid^2\rho_\varepsilon = \dfrac{1}{\varepsilon^n}(1-\rho_\varepsilon^2)\rho_\varepsilon, & (5.30)\\
-\operatorname{div}(\mid\nabla u_\varepsilon\mid^{n-2}\rho_\varepsilon^2\nabla\theta_\varepsilon) - \mid\nabla u_\varepsilon\mid^{n-2}\rho_\varepsilon^2\mid\nabla\theta_\varepsilon\mid^2\theta_\varepsilon = 0. & (5.31)
\end{cases}
$$

Lemma 5.3.9. *For any $K \subset\subset \Omega\setminus\{a_1,\cdots,a_d\}$, we have*

(a). $\rho_\varepsilon \longrightarrow 1$ uniformly in K, as $\varepsilon \longrightarrow 0$;

(b). $\displaystyle\frac{1}{\varepsilon^n}\int_K\left(1-\mid u_\varepsilon\mid^2\right)^2 + \int_K\mid\nabla u_\varepsilon\mid^{n-2}|\nabla\rho_\varepsilon|^2 \longrightarrow 0$, *as $\varepsilon \longrightarrow 0$;*

(c). $\dfrac{(1-|u_\varepsilon|^2)u_\varepsilon}{\varepsilon^n} \in L^1(K)$ *with the norm bounded independent of ε.*

Proof.

(a).

We claim that $\rho_\varepsilon \longrightarrow 1$ uniformly in K, as $\varepsilon \longrightarrow 0$.
If it is not true, then there exist $\delta > 0$, $\varepsilon_i \longrightarrow 0$ and $\{y_i\} \in K$, s.t. $\mid u_{\varepsilon_i}(y_i)\mid \le 1-\delta$. In $B_{\varepsilon_i^{1/4}}(y_i)$, there exists $\rho_i \in [\varepsilon^{1/2}, \varepsilon^{1/4}]$, such that

$$
f(y_i,\rho_i) \le C\frac{E(u_{\varepsilon_i}, B_{\varepsilon_i^{1/4}}(y_i)\setminus B_{\varepsilon_i^{1/2}}(y_i))}{\mid\ln\varepsilon_i\mid} \le \frac{C(n,\Omega,g,\sigma)}{\mid\ln\varepsilon_i\mid}.
$$

By applying Pohozaev's inequality, we have

$$
\int_{B_{\rho_i(x_i)}}\frac{1}{\varepsilon_i^n}(1-\mid\rho\mid^2)^2 \le C(n)f(y_i,\rho_i) \le \frac{C(n,\Omega,g,\sigma)}{\mid\ln\varepsilon_i\mid}
\tag{5.32}
$$

while From the regularity result in Proposition 3.3 of [42] for solutions of equations (5.7) again, we have the Hölder continuous of u_ε

$$
[u_\varepsilon]_{C^\alpha(\bar\Omega)} \le C(n)\varepsilon^{-\alpha},
$$

then $\left|u_{\varepsilon_i}(x)\right| \le 1-\dfrac{\delta}{2}$ for $\mid x-y_i\mid \le \dfrac{\delta^{1/\alpha}\varepsilon_i}{(2C(n))^{1/\alpha}}$, therefore

$$
\int_{B_{\frac{\delta\varepsilon_i}{2C(n)}}}\frac{(1-\mid\rho\mid^2)^2}{\varepsilon_i^n} \ge C(n)\delta^2,
$$

this contradicts with the upper bound (5.32).

(b).

Because K is compact, we can find a finite collection of open set $K \subset \cup_{i=1}^{I_k} B_{r_i}(b_i) \subset \cup_{i=1}^{I_k} B_{2r_i}(b_i) \subset \Omega_\sigma$. For any i fixed, in $A_{r_i}(b_i) := B_{2r_i}(b_i) \backslash B_{r_i}(b_i)$, we have the estimate

$$\left| \int_{A_{r_i}(b_i)} |\nabla u_\varepsilon|^{n-2} \nabla \rho_\varepsilon (1-\rho_\varepsilon) \right| \le \left(\int_{A_{r_i}(b_i)} |\nabla u_\varepsilon|^n \right)^{\frac{n-1}{n}} \left(\int_{A_{r_i}(b_i)} (1-\rho_\varepsilon)^n \right)^{\frac{1}{n}} \le C(\sigma)\varepsilon \tag{5.33}$$

Then Fubini theorem implies that there exists $\bar{r}_i(\varepsilon) \ge r_i$, s.t.

$$\int_{\partial B_{\bar{r}_i(\varepsilon)}(b_i)} | \nabla u_\varepsilon |^{n-2} | \nabla \rho_\varepsilon || 1-\rho_\varepsilon | \le \frac{C(\sigma)\varepsilon}{r_i}.$$

Multiply both sides of (5.30) by $(1-\rho_\varepsilon)$ and integrate over $B_{\bar{r}_i(\varepsilon)}(\varepsilon)$, then we have

$$-\int_{\partial B_{\bar{r}_i(\varepsilon)}(b_i)} | \nabla u_\varepsilon |^{n-2} \nabla \rho_\varepsilon \nu (1-\rho_\varepsilon) - \int_{B_{\bar{r}_i(\varepsilon)}} | \nabla u_\varepsilon |^{n-2} | \nabla \rho_\varepsilon |^2 + \int_{B_{\bar{r}_i(\varepsilon)}} | \nabla u_\varepsilon |^{n-2} | \nabla \theta_\varepsilon |^2 \rho_\varepsilon (1-\rho_\varepsilon)$$

$$= \frac{1}{\varepsilon^n} \int_{B_{\bar{r}_i}(\varepsilon)} (1-\rho_\varepsilon^2)\rho_\varepsilon(1-\rho_\varepsilon),$$

thus

$$\begin{aligned}
&\frac{1}{\varepsilon^n} \int_{B_{\bar{r}_i}(\varepsilon)} (1-\rho_\varepsilon^2)\rho_\varepsilon(1-\rho_\varepsilon) + \int_{B_{\bar{r}_i(\varepsilon)}} | \nabla u_\varepsilon |^{n-2} | \nabla \rho_\varepsilon |^2 \\
&= -\int_{\partial B_{\bar{r}_i}(\varepsilon)} | \nabla u_\varepsilon |^{n-2} \nabla \rho_\varepsilon \nu (1-\rho_\varepsilon) + \int_{B_{\bar{r}_i(\varepsilon)}} | \nabla u_\varepsilon |^{n-2} | \nabla \theta_\varepsilon |^2 \rho_\varepsilon (1-\rho_\varepsilon) \\
&\le C(K)\varepsilon + \int_{B_{\bar{r}_i(\varepsilon)}} | \nabla u_\varepsilon |^{n-2} | \nabla \theta_\varepsilon |^2 \rho_\varepsilon (1-\rho_\varepsilon) \\
&\le C(K)\varepsilon + \max_K | 1-\rho_\varepsilon | \cdot \int_{B_{\bar{r}_i(\varepsilon)}} | \nabla u_\varepsilon |^n \longrightarrow 0 \quad \text{as } \varepsilon \to 0.
\end{aligned} \tag{5.34}$$

therefore

$$\int_{B_{r_i}(b_i)} \frac{1}{\varepsilon^n}(1-\rho_\varepsilon^2)\rho_\varepsilon(1-\rho_\varepsilon) + \int_{B_{r_i}(b_i)} | \nabla u_\varepsilon |^{n-2} |\nabla \rho_\varepsilon|^2 \longrightarrow 0 \quad \text{as}\, \varepsilon \to 0.$$

By the covering of K,

$$\int_K \frac{1}{\varepsilon^n}(1-\rho_\varepsilon^2)\rho_\varepsilon(1-\rho_\varepsilon) + \int_K | \nabla u_\varepsilon |^{n-2} |\nabla \rho_\varepsilon|^2 \longrightarrow 0 \quad \text{as}\, \varepsilon \to 0.$$

Then the conclusion follows immediately.

(c).

In fact, we only need to prove that $\frac{(1-|u_\varepsilon|^2)|u_\varepsilon|^2}{\varepsilon^n} \in L^1(K)$ with norm bounded independent of ε. We use the same method in the proof of (b) above. We take the same covering balls, and by Fubini theorem and Hölder inequality we choose $\bar{r}_i(\varepsilon) \in [r_i, 2r_i]$, such that

$$\int_{\partial B_{\bar{r}_i(\varepsilon)}(b_i)} | \nabla u_\varepsilon |^{n-2} \nabla u_\varepsilon \cdot \nu \cdot u_\varepsilon \le C(\sigma).$$

Multiply both sides of Equation (5.7) with u_ε, in each ball B_{r_i} we have

$$\int_{B_{r_i}(b_i)} \frac{(1-|u_\varepsilon|^2)\,|u_\varepsilon|^2}{\varepsilon^n} \le -\int_{\partial B_{\tilde{r}_i(\varepsilon)}(b_i)} |\nabla u_\varepsilon|^{n-2}\,\nabla u_\varepsilon \cdot \nu \cdot u_\varepsilon + \int_{B_{\tilde{r}_i(\varepsilon)}(b_i)} |\nabla u_\varepsilon|^n\,. \quad (5.35)$$

This implies that $\frac{(1-|u_\varepsilon|^2)|u_\varepsilon|^2}{\varepsilon^n} \in L^1(K)$.

□

Now, we prove the weak convergence of a subsequence of $\{u_{\varepsilon_k}\}$. Recall that $\sigma_0 = \frac{1}{6}\min_{i\neq j}\left|a_i - a_j\right|$, then for fixed $\sigma < \sigma_0$, there exists ε_0, such that $\Omega_\sigma \subset \Omega_\varepsilon^{\sigma/2}$ if $\varepsilon < \varepsilon_0$. Then Propositoin 5.3.2 implies that

$$\begin{aligned} \int_{\Omega_\sigma} \frac{\left|\nabla u_{\varepsilon_k}\right|^n}{n} &\le E_{\varepsilon_k}(\varepsilon_k, \Omega_\sigma) \\ &\le E_{\varepsilon_k}(\varepsilon_k, \Omega_\varepsilon^{\sigma/2}) \\ &\le d\kappa_n\,|\ln\sigma| + C(n, d, \Omega, g) \end{aligned} \quad (5.36)$$

Then $\{u_{\varepsilon_k}\}$ is bounded in $W^{1,n}(\Omega_\sigma, \mathbb{R}^n)$. By a diagonal process, we find a subsequence, still denoted by u_{ε_k} , such that

$$u_{\varepsilon_k} \rightharpoonup u_0 \text{weakly in } W^{1,n}_{loc}(\Omega \setminus \{a_1, \cdots a_d\}, \mathbb{R}^n).$$

From the Euler-Lagrange equation 5.7 of u_{ε_k}, we have

$$\operatorname{div}(\left|\nabla u_{\varepsilon_k}\right|^{n-2} \nabla u_{\varepsilon_k}) \wedge u_{\varepsilon_k} = 0 \text{ weakly.}$$

By using the conclusion of [44] and (c) of Lemma 5.3.9, we have $u_{\varepsilon_k} \to u_0$ strongly in $W^{1,p}_{loc}(\Omega \setminus \{a_1, \cdots a_d\}, \mathbb{R}^n)$ for $p < n$. Then by passing to the limit, we have

$$\operatorname{div}(|\nabla u_0|^{n-2}\,\nabla u_0 \wedge u_0) = 0 \text{ weakly.}$$

In fact, by applying a similar argument of the Lemma 2.2 in [23], a map $u \in W^{1,n}(\Omega, \mathbb{S}^{n-1})$ is a $n-$harmonic map if and only if it satisfies the equation above. Similar arguments can also be found in [60] and [67]. While Lemma 5.3.9 implies that $|u_0| = 1$, i.e. $u_0 \in W^{1,n}(\Omega \setminus \{a_1, \cdots a_d\}, \mathbb{S}^{n-1})$, thus u_0 is a $n-$harmonic map. Here, we give a short argument. The fact

$$\operatorname{div}(|\nabla u_0|^{n-2}\,\nabla u_0 \wedge u_0) = 0 \text{ weakly}$$

implies that

$$\operatorname{div}(|\nabla u_0|^{n-2}\,\nabla u_0) = \lambda(x) u_0.$$

Then multiplies both sides by $\phi(x)u_0$ where $\phi(x)$ is a test function, and we get $\lambda = |\nabla u_0|^n$, which means that u_0 is a $n-$harmonic map.

Later we need some properties of Hardy space $\mathcal{H}^1$ and the space $BMO(\mathbb{R}^n)$. We shall use the following famous theorem of Fefferman and Stein in [37].

Theorem 5.3.1. *$\mathcal{H}^1(\mathbb{R}^n)^* = BMO(\mathbb{R}^n)$. In, particular, the integral $\int_{\mathbb{R}^n} f \cdot g$ is well defined for $f \in \mathcal{H}^1(\mathbb{R}^n) \bigcap C^\infty$ and $g \in BMO(\mathbb{R}^n)$, and it can be extend to any $f \in \mathcal{H}^1(\mathbb{R}^n)$, and there is a constant $C = C(n)$ such that*

$$\left|\int_{\mathbb{R}^n} f \cdot g\right| \le C\,\|f\|_{\mathcal{H}^1}\,\|g\|_{BMO}\,.$$

In our paper, functions are defined on Ω. When we say a function $f \in \mathcal{H}^1(\Omega)$, we mean that in each $U \subset\subset \Omega$, f agrees with a function in $\mathcal{H}^1(\mathbb{R}^n)$. And we define

$$\|f\|_{\mathcal{H}^1(U)} = \inf\{\|g\|_{\mathcal{H}^1(\mathbb{R}^n)} : f|_U = g|_U\}.$$

From (5.31), we have

$$\begin{aligned}
&-\operatorname{div}(|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2\nabla\theta_\varepsilon)\\
=&|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\theta_\varepsilon\\
=&|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\theta_\varepsilon\\
=&|u_\varepsilon|^{n-2}\rho_\varepsilon^2(\sum_{i,j}\partial_i\theta_\varepsilon^j\partial_i\theta_\varepsilon^j\theta_\varepsilon^k)\\
=&|u_\varepsilon|^{n-2}\rho_\varepsilon^2[\sum_{i,j}\partial_i\theta_\varepsilon^j(\partial_i\theta_\varepsilon^j\theta_\varepsilon^k-\partial_i\theta_\varepsilon^k\theta_\varepsilon^j)]
\end{aligned} \tag{5.37}$$

Next, we shall prove that $|u_\varepsilon|^{n-2}\rho_\varepsilon^2[\sum_{i,j}\partial_i\theta_\varepsilon^j(\partial_i\theta_\varepsilon^j\theta_\varepsilon^k-\partial_i\theta_\varepsilon^k\theta_\varepsilon^j)]$ is in $\mathcal{H}^1(\Omega)$.

Let $B_j = \nabla\theta_\varepsilon^j$, and $E_j = |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2(\nabla\theta_\varepsilon^j\cdot\theta_\varepsilon^k-\nabla\theta_\varepsilon^k\cdot\theta_\varepsilon^j)$. Then $E_j \in L^{\frac{n}{n-1}}$, $B_j \in L^n$. We have $\mathrm{curl}B_j = 0$, because of $\mathrm{curl}\nabla = 0$. And $\operatorname{div}E_j = 0$, in fact, for any $\Omega' \subset \Omega \setminus \{a_1, \cdots, a_d\}$, and $\phi \in W_0^{1,n}(\Omega', \mathbb{R})$, we have

$$\begin{aligned}
&\int_{\Omega'}\operatorname{div}E_j\cdot\phi\\
=&-\int_{\Omega'}|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2(\nabla\theta_\varepsilon^j\cdot\theta_\varepsilon^k)\cdot\nabla\phi+\int_{\Omega'}|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2(\nabla\theta_\varepsilon^k\cdot\theta_\varepsilon^j)\cdot\nabla\phi\\
=&-\int_{\Omega'}|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\theta_\varepsilon^j\cdot\phi\cdot\theta_\varepsilon^k+\int_{\Omega'}|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\theta_\varepsilon^k\cdot\phi\cdot\theta_\varepsilon^j\\
=&0.
\end{aligned} \tag{5.38}$$

The second equality follows from the equation (5.31).

From a conclusion in [25], we have $E_j \cdot B_j \in \mathcal{H}^1$, and

$$\left\|E_j\cdot B_j\right\|_{\mathcal{H}^1} \le C(n)\left\|E_j\right\|_{L^{\frac{n}{n-1}}}\left\|B_j\right\|_{L^n}.$$

Therefore

$$|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\theta_\varepsilon \in \mathcal{H}^1$$

and

$$\begin{aligned}
&\left\||\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\theta_\varepsilon\right\|_{\mathcal{H}^1}\\
\le&C(n)\|\nabla\theta\|_{L^n}\cdot\left\||\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2\cdot(\nabla\theta_\varepsilon\cdot\theta_\varepsilon^k-\nabla\theta_\varepsilon^k\cdot\theta_\varepsilon)\right\|_{L^{\frac{n}{n-1}}}\\
\le&C(n)\int|\nabla u_\varepsilon|^n\\
=&C(n)\|\nabla u_\varepsilon\|_{L^n}^n
\end{aligned} \tag{5.39}$$

Recall that $|u_\varepsilon| > 1/2$ in Ω_σ. In Ω_σ, we have the following lemma

Lemma 5.3.10. *There exist $\delta, \tau \in (0,1)$ depending only on n, $\varepsilon_0 \in (0,1)$ depending only on n and σ, so that if $\varepsilon < \varepsilon_0$, and $B_r(x) \subset\subset \Omega_\sigma$, then*

$$e(x,r,\varepsilon) := \int_{B_r(x)} |\nabla\theta_\varepsilon|^n \le 2^n \int_{B_r(x)} |\nabla u_\varepsilon|^n \le \delta$$

implies $e(x,\tau r,\varepsilon) \le \frac{1}{2} e(x,r,\varepsilon)$.

Proof. If the conclusion is not true, then for any $\tau \in (0, \frac{1}{8})$ fixed, there exist $B_{r_i}(x_i) \subset\subset \Omega_\sigma$ and $\varepsilon_i \searrow 0$ s.t.

$$\int_{B_{r_i}(x_i)} \left|\nabla\theta_{\varepsilon_i}\right|^n = \lambda_i^n \searrow 0,$$

but

$$e(x_i, \tau r_i, \varepsilon_i) \ge \frac{1}{2}\lambda_i^n.$$

From Lemma 5.3.9, for any $K \subset\subset \Omega \setminus \{a_1, ...a_d\}$, we have $\int_K |\nabla u_\varepsilon|^{n-2} |\nabla\rho_\varepsilon|^2 \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ and $\rho_\varepsilon \longrightarrow 1$ uniformly in K, as $\varepsilon \longrightarrow 0$. We can choose a sub-sequence $\{\varepsilon_{k_i}\}$ of $\{\varepsilon_i\}$, for the convenience of notations we still denote $\{\varepsilon_{k_i}\}$ as $\{\varepsilon_i\}$, s.t.

$$\begin{aligned}
&\int_{B_{r_i}(x_i)} \left|\nabla u_{\varepsilon_i}\right|^n \\
\le &\int_{B_{r_i}(x_i)} (|\nabla\rho_\varepsilon|^2 + |\rho_\varepsilon \cdot \nabla\theta_\varepsilon|^2)^{\frac{n}{2}} \\
\le &\lambda_i^n + o(\lambda_i^n)
\end{aligned} \tag{5.40}$$

Define

$$U_i(z) = \lambda_i^{-1} u_{\varepsilon_i}(x_i + r_i z),$$

and

$$V_i(z) = \lambda_i^{-1}(\theta_{\varepsilon_i}(x_i + r_i z) - \bar{\theta}_{\varepsilon_i, x_i, r_i}),$$

where

$$\bar{\theta}_{\varepsilon_i, x_i, r_i} = \fint_{B_{r_i}(x_i)} \theta_{\varepsilon_i}(x)\,\mathrm{d}x$$

Then we have

$$\int_{B_1} |\nabla V_i(z)|^n = 1,$$

and Poincare Inequality implies that

$$\int_{B_1} |V_i|^n \le C(n).$$

And also from the condition we have

$$\int_{B_\tau} |\nabla V_i|^n \ge \frac{1}{2}.$$

From the embedding theorem, we have

$$V_i \to V_0 \quad \text{in} \quad L^n(B_1, R^n)$$

$$\nabla V_i \rightharpoonup \nabla V_0 \quad \text{in} \quad L^n(B_1, R^{nn})$$

for some $V_0 \in W^{1,n}(B_1, R^n)$.

We claim that $V_k \longrightarrow V_0$ in $W^{1,n}(B_1, R^n)$. In fact, let $\xi(x) \in C_0^1(B_{1/2}, [0, 1])$, and satisfy

$$\xi(z) = \begin{cases} 1 & \text{in} \quad B_{1/4}; \\ 0 & \text{on} \quad \partial B_{1/2}. \end{cases} \tag{5.41}$$

and $|\nabla\xi| \leq 5$.

For $a, b \in \mathbb{R}^n$ and any $p \geq 2$, we have

$$|a-b|^p \leq 2^{p-2}(|a|^{p-2} + |b|^{p-2})\,|a-b|^2 \leq 2^{p-1}(|a|^{p-2}\,a - |b|^{p-2}\,b)\cdot(a-b). \tag{5.42}$$

Thus we have

$$\begin{aligned} &\int_{B_1} \xi\,|\nabla V_k - \nabla V_l|^n \\ \leq& C(n)\int_{B_1} [|\nabla V_k|^{n-2}\,\nabla V_k - |\nabla V_l|^{n-2}\,\nabla V_l]\cdot\nabla[V_k - V_l]\cdot\xi \\ =& C(n)\int_{B_1} (|\nabla V_k|^{n-2}\,\nabla V_k - |\nabla V_l|^{n-2}\,\nabla V_l)\cdot[\nabla((V_k - V_l)\xi) - (V_k - V_l)\nabla\xi] \end{aligned} \tag{5.43}$$

Let $\psi = (V_k - V_l)\xi$, $\phi(y) = \psi(\dfrac{y - x_k}{r_k})$. We claim that $\psi, \phi \in BMO$. In fact we only need to prove that $w = \xi V_k \in BMO$, because $||\phi||_{BMO} = ||\psi||_{BMO}$.

$$\begin{aligned} &⨍_{B_r(z)} \left|w - \bar{w}_{z,r}\right| \\ \leq& C(n) r^{1-n}\int_{B_r(z)} |\nabla w| \\ \leq& C(n)(\int_{B_r(z)} |\nabla w|^n)^{1/n} \\ \leq& C(n)(\int_{B_r(z)} |\nabla V_k|^n + \int_{B_r(z)} |V_k|^n)^{1/n} \\ \leq& C(n). \end{aligned} \tag{5.44}$$

Thus

$$
\begin{aligned}
&\left|\int_{B_1} |\nabla V_k|^{n-2}\, \nabla V_k \cdot \nabla((V_k - V_l)\xi)\, \mathrm{d}z\right| \\
\leq &\left|\int_{B_1} |\nabla U_k|^{n-2}\, \rho^2_{\varepsilon_k}(x_k + r_k z) \nabla V_k \cdot \nabla((V_k - V_l)\xi)\right| + \left|\int_{B_1} (|\nabla V_k|^{n-2} - |\nabla U_k|^{n-2}) \nabla V_k \cdot \nabla((V_k - V_l)\xi)\right| \\
+ &\left|\int_{B_1} |\nabla U_k|^{n-2}\, (1 - \rho^2_{\varepsilon_k}(x_k + r_k z)) \nabla V_k \cdot \nabla((V_k - V_l)\xi)\right| \\
\leq &\left|\int_{B_{r_k}(x_k)} \frac{1}{\lambda_k^{n-1}} \left|\nabla u_{\varepsilon_k}\right|^{n-2} \rho^2_{\varepsilon_k} \nabla\theta_{\varepsilon_k} \nabla\phi\right| + o(1) \\
\leq &\frac{1}{\lambda_k^{n-1}} C(n)\, \|f\|_{\mathcal{H}^1} \cdot \|\phi\|_{BMO} + o(1) \\
\leq &C(n)\, \|\phi\|_{BMO} \cdot \lambda_k \longrightarrow 0.
\end{aligned}
\tag{5.45}
$$

where $f = |\nabla u_{\varepsilon_k}|^{n-2}\, \rho^2_{\varepsilon_k}\, |\nabla\theta_{\varepsilon_k}|^2\, \theta_{\varepsilon_k}$.

Similarly, we have

$$\int_{B_1} |\nabla V_l|^{n-2}\, \nabla V_l \cdot \nabla((V_k - V_l)\xi) \leq C(n)\, \|\phi\|_{BMO} \cdot \lambda_l \longrightarrow 0.$$

Also we have

$$\int_{B_1} (|\nabla V_k|^{n-2}\, \nabla V_k - |\nabla V_l|^{n-2}\, \nabla V_l) \cdot (V_k - V_l)\nabla\xi \leq C(n)\, \|V_k - V_l\|_{L^n} \longrightarrow 0.$$

These estimates imply that $V_k \longrightarrow V_0$ in $W^{1,n}(B_{1/4}, R^n)$.

This implies that $\int_{B_{1/4}} |\nabla V_0|^n \leq 1$, $\int_{B_{1/4}} |V_0|^n \leq C(n)$.

For any $\phi \in C^1_0(B_{1/4}, R^n)$,

$$\left|\int_{B_{1/4}} |\nabla V_0|^{n-2} \cdot \nabla V_0 \cdot \nabla\phi\right| = \left|\lim_{k\longrightarrow\infty} \int_{B_{1/4}} |\nabla V_k|^{n-2} \cdot \nabla V_k \cdot \nabla\phi\right| \leq C(n)\, \|\phi\|_{BMO} \cdot \lambda_k \longrightarrow 0.$$

Thus

$$\int_{B_{1/4}} |\nabla V_0|^{n-2}\, \nabla V_0 \nabla\phi = 0.$$

V_0 is n-harmonic in $B_{1/4}$. For the n-harmonic map, we have the theorem as follows, which is Theorem 2.4 in [55],

Theorem 5.3.2. *If $u \in W^{1,n}(B_r(x), \mathbb{R}^n)$ is n-harmonic, then $u \in C^{1,\alpha}(B_r(x), \mathbb{R}^n)$ for some $\alpha \in (0,1)$, and for some constant $C(n)$,*

$$\sup_{B_{r/2}(x)} |\nabla u|^n \leq C(n) \fint_{B_r(x)} |\nabla u|^n\, .$$

Then by using the theorem above, we have

$$\sup_{B_{1/8}} |\nabla V_0|^n \leq C(n) \fint_{B_{1/4}} |\nabla V_0|^n \leq C(n) \frac{4^n}{\omega_n}.$$

Therefore, for $0 < \tau < \frac{1}{8}$,

$$\int_{B_\tau} |\nabla V_0|^n \leq C \cdot 4^n \tau^n \leq \frac{1}{4}$$

for τ chosen small enough. It contradicts with the condition of the lower bound.

□

Lemma 5.3.11. *In $B(x, 2r) \subset\subset \Omega_\sigma$, if $e(x, 2r, \varepsilon) \leq \delta$ where δ is the constant in Lemma 5.3.10, then θ_ε is C^α for some $\alpha \in [0, 1]$, and $\|\theta_\varepsilon\|_{C^\alpha} \leq C(n)$ independent of ε.*

Proof. We follow the proof of [55].

Take $\varepsilon \leq \varepsilon_0$. If $B_{2\rho}(x) \subset\subset \Omega \setminus \{a_1, ...a_d\}$, and $e(x, 2\rho) \leq \delta$, then for $\forall y \in B_\rho(x)$, $r \in (0, \rho)$, we have

$$e(y, r) \leq e(y, \rho) \leq e(x, 2\rho) \leq \delta.$$

From the lemma, we have for some $\tau \in [0, 1]$, and all $r \in (0, \rho)$,

$$e(y, \tau r) \leq \frac{1}{2} e(y, r).$$

for $\forall r \in (0, \rho)$ fixed, there exists k, s.t. $r \in [\tau^k \rho, \tau^{k-1}\rho]$, then

$$e(y, r) \leq e(y, \tau^{k-1}\rho) \leq 2^{-k+1} e(y, \rho) \leq 2\varepsilon_0 \left(\frac{r}{\rho}\right)^\beta,$$

where $\beta_\varepsilon = \log_\tau 1/2 > 0$.

Morrey's lemma 3.2.5 implies that $\|\theta_\varepsilon\|_{C^{\beta/n}} \leq C(n, \tau, \rho)$ independent of ε.

□

5.3.4 Proof of Theorem 5.1.2

Proof of Theorem 5.1.2

Step 1.

Assume $K \subset\subset \Omega$. Denote

$$S = \bigcap_{r>0} \left\{ x \in \bar{\Omega} \setminus \{a_1, a_2, \cdots, a_d\} \,\Big|\, \liminf_{\varepsilon \longrightarrow 0} \int_{B_r(x)} |\nabla u_\varepsilon|^n > \delta \right\} \tag{5.46}$$

where δ is the constant in Lemma 5.3.10. Then for any $x \in K$, by the definition of Ω_0 and S, there exists r, s.t.

$$\liminf_{\varepsilon_k} \int_{B_r(x)} |\nabla u_k|^n < \delta.$$

Thus there is an $\bar{\varepsilon} < \varepsilon_0$ (recall that ε_0 is the value in Lemma 5.3.10), and a subsequence $u_{\varepsilon_{k'}}$, s.t.

$$\left\|\theta_{\varepsilon_{k'}}\right\|_{C^\alpha} \leq C(n) \quad \text{independent of} \quad \varepsilon,$$

for $\varepsilon_{k'} < \varepsilon_0$.

Therefore Arzela-Ascoli theorem implies that there is a subsequence $\{\theta_{\varepsilon_{k''}}\}$, s.t.

$$\theta_{\varepsilon_{k''}} \to \theta_0 \quad \text{uniformly in} \quad \mathbf{C}^0(B_r),$$

where $\theta_0 \in \mathbf{C}^{0,\alpha}(B_r)$. In fact θ_0 is u_0 in the theorem.

Step 2.

By abuse of notation, we write θ_k for $\theta_{\varepsilon_{k''}}$. Let

$$\eta(x) = \begin{cases} 1 & \text{in} \quad B_{r/2}; \\ 0 & \text{on} \quad \partial B_r, \end{cases} \tag{5.47}$$

and $|\nabla \eta| \leq \frac{5}{r}$. Then we have

$$\begin{aligned} &\int_{B_r(x)} [|\nabla u_k|^{n-2} \rho_k^2 \nabla \theta_k - |\nabla u_l|^{n-2} \rho_l^2 \nabla \theta_l] \cdot \nabla[(\theta_k - \theta_l) \cdot \eta] \\ =&\int_{B_r(x)} [|\nabla u_k|^{n-2} \rho_k^2 |\nabla \theta_k|^2 \theta_k - |\nabla u_l|^{n-2} \rho_l^2 |\nabla \theta_l|^2 \theta_l] \cdot [(\theta_k - \theta_l) \cdot \eta] \\ \leq& \max_{B_r(x)} |\theta_k - \theta_l| \int_{B_r(x)} (|\nabla u_k|^n + |\nabla u_l|^n) \\ \to& 0. \end{aligned} \tag{5.48}$$

By the equation (5.31) and Holder inequality, we have

$$\begin{aligned} &\int_{B_{r/2}(x)} |\nabla \theta_k - \nabla \theta_l|^n \\ \leq& C(n) \int_{B_r(x)} [|\nabla \theta_k|^{n-2} \nabla \theta_k - |\nabla \theta_l|^{n-2} \nabla \theta_l] \cdot [\eta \cdot \nabla(\theta_k - \theta_l)] \\ \leq& C(n) \int_{B_r(x)} [|\nabla u_k|^{n-2} \nabla \theta_k - |\nabla u_l|^{n-2} \nabla \theta_l] \cdot [\nabla(\eta \cdot (\theta_k - \theta_l)) - \nabla \eta \cdot (\theta_k - \theta_l)] + o(1) \\ \to& 0. \end{aligned} \tag{5.49}$$

Therefore we have the strong convergence

$$\theta_k \to \theta_0 \quad \text{in} \quad \mathbf{W}^{1,n}(B_r).$$

Then by the convergence of ρ_k, we get the convergence of u_k in $\mathbf{W}^{1,n}(B_r)$. By the finite covering theorem, we get the convergence

$$u_k \to u_0 \quad \text{in} \quad \mathbf{W}^{1,n}(K, \mathbb{S}^{n-1}).$$

Step 3.

Let $\rho_0 = \min_{j \neq k} \left\{ \frac{1}{6} \left| a_j - a_k \right| \right\}$. By using Proposition 5.3.2 and the strong convergence of u_ε, for $\rho_1 < \rho_0$, we have

$$\mathbf{E}(u_0; \Omega_{\rho_1}) \leq d\kappa_n |\ln \rho_1| + C(n, g). \tag{5.50}$$

On the other hand, by using (5.25) of Lemma 5.3.8, we have

$$\mathbf{E}(u_0; \Omega_{\rho_1}) \geq \sum_j \left| \hat{d}_j \right|^{\frac{n}{n-1}} \kappa_n |\ln \rho_1| - C(n, \hat{d}, \Omega, g, \rho_0), \tag{5.51}$$

where $\hat{d}_j$ is the degree of u_0, restricted to $\partial(\Omega \cap B_{\rho_1}(a_j))$.

Therefore, (5.50) and (5.51) imply that for $a \in \{a_1, a_2, \cdots, a_d\}$, if $d_a = \deg|_{\partial B_\rho(a)} u_0 \neq 0$ where $\rho < \rho_0$, then $d_a = 1$. This implies that there are exactly d singularities of degree one.

Step 4.

In this step, we prove the set S is finite. If this is not true, then we have infinitely many points in S. By Proposition 5.3.2, there are only finitely many points in Ω_ρ for $\rho \leq \rho_0$. Therefore, there exists a sequence of points in S which converges to some singularity a_i. Then there exists $\rho_2 \leq \rho_0$, such that in Ω_{ρ_2}, there are M points of S denoted $\{b_j\}_{j=1}^M$ and $M \cdot \delta > 2(C(n,g) + C(n, \hat{d}, \Omega, g, \rho_0))$ where $C(n,g)$ and $C(n, \hat{d}, \Omega, g, \rho_0))$ are the constants in (5.50) and (5.51).

For u_0, we have a lower bound

$$\mathbf{E}(u_0; \Omega_{\rho_2} \setminus \cup_{j=1}^M B_r(b_j)) \geq d\kappa_n |\ln \rho_2| - 5/4C(n, \hat{d}, \Omega, g, \rho_0), \tag{5.52}$$

if we choose r small enough. Then by the strong convergence of u_ε, there is an ε_{ρ_2}, such that

$$\mathbf{E}(u_\varepsilon; \Omega_{\rho_2} \setminus \cup_{j=1}^M B_r(b_j)) \geq d\kappa_n |\ln \rho_2| - 6/4C(n, \hat{d}, \Omega, g, \rho_0) \tag{5.53}$$

if $\varepsilon < \varepsilon_{\rho_2}$.

By the definition of S, we have

$$\begin{aligned} \mathbf{E}(u_\varepsilon; \Omega_{\rho_2}) &\geq d\kappa_n |\ln \rho_2| - 6/4C(n, \hat{d}, \Omega, g, \rho_0) + M \cdot \delta \\ &\geq d\kappa_n |\ln \rho_2| + 2C(n,g) \end{aligned} \tag{5.54}$$

for ε small enough. This contradicts the upper bound of u_ε proved in Proposition 5.3.2.

Step 5.

$$a_i \in \Omega, \forall i.$$

From all the proof before, we only have the information that $a_i \in \bar{\Omega}$. In this step, we shall exclude the possibility that $a_i \in \partial\Omega$ for some i. For convenience, suppose $a_1 \in \partial\Omega$. Then we enlarge a little the domain Ω, as done in [13]. Fix a smooth, bounded and simply connected domain $\hat{\Omega}$ such that $\bar{\Omega} \subset\subset \hat{\Omega}$. Also fix an arbitrarily smooth map $\hat{g} : \hat{\Omega} \setminus \Omega \to \mathbb{S}^{n-1}$, such that $\hat{g} = g$ on $\partial\Omega$. Then we extend the map u_0 to a larger domain $\hat{\Omega}$ such that $u_0 = \hat{g}$ on $\hat{\Omega} \setminus \Omega$. We have a higher dimensional analogue of Lemma VI.1 in [13].

Lemma 5.3.12. *Let $a \in \partial\Omega$. For every map u that belongs to $W^{1,n}(\overline{B_R(a)} \setminus \{a\}; \mathbb{S}^{n-1})$ such that $u = \hat{g}$ in $(\hat{\Omega} \setminus \Omega) \cap B_R(a)$ and* $\deg(u, \partial B_R(a)) = 1$. *We have*

$$\frac{1}{n} \int_{B_R(a) \setminus B_\rho(a)} |\nabla u|^n \geq 2^{1/(n-1)} \kappa_n |\ln \rho| - C,$$

where C depends only on $\hat{g}$ and R.

We postpone the proof of Lemma 5.3.12.

Proof of Step 5. In $\hat{\Omega}_\rho = \hat{\Omega} \setminus \cup_{i=1}^n B_\rho(a_i)$, we also have an upper bound of the energy

$$\mathbf{E}(u_0; \hat{\Omega}_\rho) \leq d\kappa_n |\ln \rho| + C(n, \hat{\Omega}, \hat{g}),$$

while from the Lemma 5.3.12 and Lemma 5.3.8, if a_1 is on the boundary, we have a lower bound of the energy

$$\mathbf{E}(u_0; \hat{\Omega}_\rho) \geq d\kappa_n |\ln \rho| + (2^{1/(n-1)} - 1)\kappa_n |\ln \rho| - C(n, \hat{\Omega}, \hat{g}).$$

This contradicts the upper bound above.

Proof of Lemma 5.3.12.

By a conformal change of variables, we may assume that locally, Ω is the half-space $\{x \in \mathbb{R}^n | x_n > 0\}$, and that $a = 0$. In this transformation, $B_R(a) \setminus B_\rho(a)$ is transformed into a domain $B_{R'}(0) \setminus B_{\rho'}(0)$ with $R' \simeq R$ and $\rho' \simeq \rho$. Thus in $B_r(0)$ with $\rho \le r \le R$, we have

$$\begin{aligned}
1 &= \frac{1}{\omega_n} \int_{\partial B_r} \det(\nabla_{\tan} u) \, d\mathcal{H}^{n-1} \\
&\le \frac{1}{\omega_n} \int_{(\partial B_r)^+} (n-1)^{-\frac{n-1}{2}} |\nabla_{\tan} u|^{n-1} \, d\mathcal{H}^{n-1} + \frac{1}{\omega_n} \int_{(\partial B_r)^-} (n-1)^{-\frac{n-1}{2}} |\nabla_{\tan} \hat{g}|^{n-1} \, d\mathcal{H}^{n-1} \\
&\le \frac{1}{\omega_n} \left(\int_{(\partial B_r)^+} (n-1)^{-\frac{n}{2}} |\nabla_{\tan} u|^n \, d\mathcal{H}^{n-1} \right)^{\frac{n-1}{n}} \left(\int_{(\partial B_r)^+} 1 \, d\mathcal{H}^{n-1} \right)^{1/n} + O(r^{n-1}) \\
&\le \left(\frac{1}{\omega_n} \int_{(\partial B_r)^+} (n-1)^{-\frac{n}{2}} |\nabla_{\tan} u|^n \, d\mathcal{H}^{n-1} \right)^{\frac{n-1}{n}} \left(\frac{1}{2} \right)^{1/n} r^{\frac{n-1}{n}} + O(r^{n-1})
\end{aligned} \tag{5.55}$$

Then we have

$$2^{\frac{1}{n-1}} \kappa_n \frac{1}{r} - O(r^n) \le \frac{1}{n} \int_{(\partial B_r)^+} |\nabla u|^n \le \frac{1}{n} \int_{\partial B_r} |\nabla u|^n .$$

Integral this inequality over $[\rho, R]$, we finished the proof of the lemma.

□

5.3.5 The Divergence Free Condition

Let $u : \Omega_0 \to \mathbb{S}^{n-1}$ be an n-harmonic map. We say u is a stationary n-harmonic map if its stress tensor is divergence free in Ω_0, that is

$$\sum_i \partial_i T_{i,j} = 0$$

where

$$T_{i,j} := |\nabla u|^{n-2} \langle \partial_i u, \partial_j u \rangle - \frac{1}{n} |\nabla u|^n \delta_{i,j}$$

and satisfies

$$\oint_{\partial B_\rho} \sum_i T_{i,j} \nu_i = 0 \tag{5.56}$$

for $\partial B_\rho \subset \Omega_0$, $\nu = (\nu_1, \cdots, \nu_n)$. We claim that the $n-$harmonic map u_0 in Theorem 5.1.2 is stationary.

In fact, from (5.13) the divergence free of the energy tensor for $T_{i,j}(u_\varepsilon)$, we have for any ball in Ω,

$$\int_B \mathrm{div} T_{\cdot,j}(u_\varepsilon) = \int_{\partial B} \sum_i \nu_i \cdot T_{i,j}(u_\varepsilon) = 0.$$

Then on any annulus $B_R(y) \setminus B_r(y) \subset \Omega_0$, we have

$$\int_{B_R \setminus B_r} \sum_i \frac{x_i - y_i}{|x - y|} \cdot T_{i,j}(u_\varepsilon) = 0.$$

Then let $\varepsilon \to 0$, by using Lemma 5.3.9 and Theorem 5.1.2, we have

$$\int_{B_R \setminus B_r} \sum_i \frac{x_i - y_i}{|x-y|} \cdot T_{i,j}(u_0) = 0.$$

Therefore, for almost every ρ, we get

$$\oint_{\partial B_\rho} \sum_i T_{i,j} \nu_i = 0 \tag{5.57}$$

This gives the sense of divergence free condition around each singularity. We will understand now such condition.

Proposition 5.3.3. *Assume $u : \Omega_0 \subset \mathbb{R}^n \to \mathbb{S}^{n-1}$ is a stationary $n-$harmonic map where $\Omega_0 := \Omega \setminus (\{a_1, \cdots, a_d\} \cup S)$ in the above sense, and* $\deg(u, a_i) = 1$. *Assume that around each singular point a_i, one has the asymptotic expansion*

$$u(x) = \mathrm{e}^{B(x-a_i)} \frac{x - a_i}{|x - a_i|}$$

where $B(x) \in so(n)$ is antisymmetric matrix satisfying $B(0) = 0$ such that B is C^1 in a neighborhood of x. Then

$$\sum_{k=1}^{n} \partial_k B(0) e_k = 0 \tag{5.58}$$

where $(e_1, \cdots, e_n)$ is the canonical basis in $\mathbb{R}^n$. Moreover, we can write

$$u(x) = \frac{x - a_i}{|x - a_i|} + \frac{Q(x - a_i)}{|x - a_i|} + O(|x - a_i|^2)$$

where $Q(x)$ is some harmonic polynomial of degree 2. In particular, when n = 2, we have $B(x) = O(|x - a_i|^2)$.

Proof. Without loss of generality, we assume $a_i = 0$. We have

$$\partial_j u(x) = \mathrm{e}^{B(x)} \partial_j B(x) \frac{x}{|x|} + \mathrm{e}^{B(x)} \left(\frac{e_j}{|x|} - \frac{\langle e_j, \nu \rangle \nu}{|x|}\right)$$

and

$$\partial_\nu u(x) = \mathrm{e}^{B(x)} \partial_\nu B \cdot \nu.$$

Therefore, we can write

$$|\nabla u|^2 = \frac{n-1}{|x|^2} + 2 \sum_{j=1}^{n} \langle \left(\frac{e_j}{|x|} - \frac{\langle e_j, \nu \rangle \nu}{|x|}\right), \partial_j B(0) \cdot \nu \rangle + O(1),$$

and

$$\begin{aligned} |\nabla u|^n &= (|\nabla u|^2)^{n/2} \\ &= \left(\frac{n-1}{|x|^2}\right)^{n/2} \left(1 + \frac{n \cdot r}{n-1} \sum_{j=1}^{n} \langle \left(\frac{e_j}{|x|} - \frac{\langle e_j, \nu \rangle \nu}{|x|}\right), \partial_j B(0) \cdot \nu \rangle \right) + O(|x|^{2-n}). \end{aligned} \tag{5.59}$$

Recall the divergence free condition around the singularity a_i for all index j

$$0 = \oint_{\partial B(0,r)} |\nabla u|^{n-2} \langle \partial_\nu u, \partial_j u\rangle - \frac{1}{n} |\nabla u|^n \langle \nu, e_j\rangle,$$

which implies

$$0 = \oint_{\partial B(0,r)} \langle \partial_\nu B(0)\cdot \nu, (e_j - \langle e_j, \nu\rangle\nu)\rangle - \sum_{k=1}^{n} \langle (e_k - \langle e_k, \nu\rangle\nu), \partial_k B(0)\cdot\nu\rangle\langle \nu, e_j\rangle + O(r).$$

Now using the fact that the matrices $\partial_k B(0)$ and $\partial_\nu B_0$ are antisymmetric and the symmetry of the integrand, we have

$$\begin{aligned}
&\oint_{\partial B(0,r)} \langle \partial_\nu B(0)\cdot \nu, (e_j - \langle e_j, \nu\rangle\nu)\rangle - \sum_{k=1}^{n} \langle (e_k - \langle e_k, \nu\rangle\nu), \partial_k B(0)\cdot\nu\rangle\langle \nu, e_j\rangle \\
=&\oint_{\partial B(0,r)} \langle \partial_\nu B(0)\cdot \nu, e_j\rangle - \sum_{k=1}^{n} \langle e_k, \partial_k B(0)\cdot\nu\rangle\langle \nu, e_j\rangle \\
=&\oint_{\partial B(0,r)} \langle \partial_\nu B(0)\cdot \nu, e_j\rangle + \sum_{k=1}^{n} \langle \partial_k B(0)\cdot e_k, \nu\rangle\langle \nu, e_j\rangle \qquad (5.60)\\
=&\oint_{\partial B(0,r)} \sum_{k=1}^{n} \frac{x_k^2}{|x|^2}\langle \partial_k B(0)\cdot e_k, e_j\rangle + \frac{x_j^2}{|x|^2}\langle \partial_k B(0)\cdot e_k, e_j\rangle \\
=&2\oint_{\partial B(0,r)} \sum_{k=1}^{n} \frac{x_1^2}{|x|^2}\langle \partial_k B(0)\cdot e_k, e_j\rangle.
\end{aligned}$$

The third equality comes from the fact that

$$\oint_{\partial B(0,r)} \sum_{k\neq l} \frac{x_k\cdot x_l}{|x|^2}\langle \partial_k B(0)\cdot e_l, e_j\rangle + \sum_{l\neq j} \frac{x_j\cdot x_l}{|x|^2}\langle \sum_k \partial_k B(0)\cdot e_k, e_j\rangle = 0. \qquad (5.61)$$

Therefore, we get

$$\sum_{k=1}^{n} \partial_k B(0)\cdot e_k = 0.$$

Then we make the expansion

$$u(x) = \frac{x}{|x|} + |x|\,\partial_\nu B(0)\cdot\nu + O(|x|^2).$$

By using the above condition

$$\Delta\, |x|^2\, \partial_\nu B(0)\cdot\nu = 0.$$

□

Remark 12. *When $n = 2$, we write*

$$B(x) = \begin{pmatrix} 0 & \alpha(x) \\ -\alpha(x) & 0 \end{pmatrix} \qquad (5.62)$$

The above condition (5.58) is equivalent to $\nabla\alpha(0) = 0$.

5.3.6 Construction of non-minimizing sequence of critical points

In this part, we prove Theorem 5.1.3. Let $n = 3$ and $x = (x', x_3)$ with $x' \in \mathbb{R}^2$. We consider $\Omega = \{x \in \mathbb{R}^3 | |x'| \leq 1, |x_3| \leq L\} \bigcup \{x \in \mathbb{R}^3 | |x - (0,0,L)| \leq 1, x_3 \geq L\} \bigcup \{x \in \mathbb{R}^3 | |x - (0,0,-L)| \leq 1, x_3 \leq -L\}$ for some large $L > 0$ to be fixed later. We define a boundary map $g : \partial\Omega \to \mathbb{S}^2$ of degree one as follows:
On the set $\{x \in \mathbb{R}^3 | |x - (0,0,L)| = 1, x_3 \geq L\}$,

$$g(x) = \frac{x - (0,0,L)}{|x - (0,0,L)|};$$

On the set $\{x \in \mathbb{R}^3 | |x - (0,0,-L)| = 1, x_3 \leq -L\}$,

$$g(x) = \frac{x - (0,0,-L)}{|x - (0,0,-L)|};$$

On the set $\{x \in \mathbb{R}^3 | |x'| = 1, 1 < x_3 < L - 1\}$,

$$g(x) = \sqrt{\frac{1}{1+h^2}}(x', -h),$$

where $h > 0$ to be fixed later.

Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be a symmetry map by $S(x', x_3) = (x', -x_3)$ and R_θ the rotation of angle equal to θ in x_1x_2 plane. We can extend it to be a C^1 by piece and equivariant map, that is , $g \circ S = S \circ g$. We define

$$W^{1,3}_g(\Omega, \mathbb{S}^2) := \{u : \Omega \to \mathbb{S}^2 | u \in W^{1,3}, u(x) \in \mathbb{S}^2 a.e., u|_{\partial\Omega} = g\}$$

and a closed subspace $\bar{W}(\Omega, \mathbb{S}^2)$ of $W^{1,3}_g(\Omega, \mathbb{S}^2)$,

$$\bar{W}(\Omega, \mathbb{S}^2) := \{u \in W^{1,3}_g(\Omega, \mathbb{S}^2), u \circ S = S \circ u, u \circ R_\theta = R_\theta \circ u, \forall\theta\}.$$

Similarly, we consider $W^{1,3}_g(\Omega, \mathbb{R}^3)$ and $\bar{W}(\Omega, \mathbb{R}^3)$.

Let $\mathbf{D}$ be the unit disc, A be the part of the sphere $\{|x| = 1, x_3 \geq -\frac{h}{\sqrt{1+h^2}}\}$, and $B = \mathbb{S}^2 \backslash A$. Let E_ε be the Ginzburg-Landau functional. We define two constants

$$a := \min\{\frac{1}{3}\int_{\mathbf{D}} |\nabla u|^3 \,|\, u : \mathbf{D} \to A, u|_{\partial\mathbf{D}} = g_A\}, \tag{5.63}$$

$$b := \min\{\frac{1}{3}\int_{\mathbf{D}} |\nabla u|^3 \,|\, u : \mathbf{D} \to B, u|_{\partial\mathbf{D}} = g_B\}, \tag{5.64}$$

where $g_A(x) = g_B(x) = (\frac{x}{\sqrt{1+h^2}}, -\frac{h}{\sqrt{1+h^2}})$ on the unit circle.

We claim that

$$\min_{W^{1,3}_g(\Omega,\mathbb{R}^3)} E_\varepsilon \leq \frac{2^{3/2}}{3} \cdot 4\pi |\ln\varepsilon| + L \cdot (a+b) + O(1)$$

and

$$\min_{\bar{W}(\Omega,\mathbb{R}^3)} E_\varepsilon \geq \frac{2^{3/2}}{3} \cdot 4\pi |\ln\varepsilon| + 2L \cdot a + O(1).$$

In fact, for the first inequality above, we put the blow up point at the point $(0,0,L)$ to get the result. We construct a map in $W^{1,3}_g(\Omega, \mathbb{R}^3)$. In the ball $B_\varepsilon((0,0,L))$, we define

$u(x) = \frac{1}{\varepsilon}(x - (0, 0, L))$. On the slice $\{x_3 = Constant, x_3 \in (1, L-1)\}$, u is a map from the disc to B whose 3-energy is close to b and on the slice $\{x_3 = Constant, x_3 \in (-L+1, -1)\}$, u is a map from the disc to A whose 3-energy is close to a. Then

$$\min_{W_g^{1,3}(\Omega,\mathbb{R}^3)} E_\varepsilon \le \frac{2^{3/2}}{3} \cdot 4\pi\, |\ln \varepsilon| + L \cdot (a + b) + O(1)$$

The second lower bound of the energy comes from the equivariant setting and the fact that u_ε converge strongly to 3-harmonic map far from the singularity. From the equivariant setting and symmetry of the functions in $\bar{W}(\Omega, \mathbb{R}^3)$, we get that the singularity of the critical point u_ε is at 0. And also we can construct a function in $\bar{W}(\Omega, \mathbb{R}^3)$, and its energy satisfies the upper bound condition (5.12). Thus u_ε is a sequence of critical points of the functional $E_\varepsilon(u)$ and satisfy the upper bound condition (5.12). By using Lemma 5.3.8 we have a lower bound of the energy near the singularity

$$E_\varepsilon(u_\varepsilon, B_R(0)) \ge \frac{2^{3/2}}{3} \cdot 4\pi\, |\ln \varepsilon| - C(R),$$

here note that the constant $C(R)$ depends only on R if $B_R \cap \Omega = \emptyset$.

Theorem 5.1.2 implies that $u_\varepsilon \to u_0$ in $W^{1,n}$. On the slice $\{x_3 = Constant, x_3 \in (1, L-1)\}$ and the slice $\{x_3 = Constant, x_3 \in (-L+1, -1)\}$, u_0 is a map from the disc to A whose 3-energy is greater than a. Then we have the energy

$$\min_{\bar{W}(\Omega,\mathbb{R}^3)} E_\varepsilon \ge \frac{2^{3/2}}{3} \cdot 4\pi\, |\ln \varepsilon| + 2L \cdot a + O(1).$$

If h is large enough, B is almost flat, then we can choose an almost constant map such that $b < a$. In fact, let $\theta = \arccos \frac{h}{\sqrt{1+h^2}}$, then we define a map $u : \mathbf{D} \to B$ as follows

$$\begin{aligned} u: \quad \mathbf{D} &\to B \\ (x, y) &\to \begin{pmatrix} \sin(\theta\sqrt{x^2+y^2})\frac{x}{\sqrt{x^2+y^2}} \\ \sin(\theta\sqrt{x^2+y^2})\frac{y}{\sqrt{x^2+y^2}} \\ \cos(\theta\sqrt{x^2+y^2}) \end{pmatrix} \end{aligned} \tag{5.65}$$

The 3-energy of this map is $O(\theta^3)$.

Therefore

$$\min_{W_g^{1,3}(\Omega,\mathbb{R}^3)} E_\varepsilon < \min_{\bar{W}(\Omega,\mathbb{R}^3)} E_\varepsilon$$

if L large enough.

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