Contributions to arithmetic geometry in mixed characteristic : lifting covers of curves, non-archimedean geometry and the l-modular Weil representation

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Contributions to arithmetic geometry in mixed characteristic

Lifting covers of curves, non-Archimedean geometry and the ℓ-modular Weil representation

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Introduction en français

Cette thèse traite différents problèmes de géométrie arithmétique. D'une coté on y étudie le problème de relèvement locale de revêtements Galoisiens de courbes, d'une autre la généralisation du groupe metaplectique et de la représentation de Weil au cas d'un anneau quelconque. Ces deux thèmes sont indépendantes par rapport à leur motivation historique et aux techniques nécessaires à leur étude. Cependant, ils partagent plusieurs thématiques telles que la relation entre caractéristique positive et nulle, les techniques non-Archimédiennes et le rôle de la théorie de la ramification et des revêtements finis.

La première question s'insère dans le contexte des problèmes de relèvement. Ceux-ci sont intimement liés au point de vu relatif en géométrie algébrique introduit par Grothendieck au début des années '60. Cela permet, en remplaçant l'étude des objets d'une catégorie par l'étude de ses morphismes, de voir les objets relatifs comme *avatars* de la même structure "absolue" et permet de comparaisons chaque fois qu'on dispose d'un morphisme entre les objets de base. Dans le cas de problèmes de relèvement, le morphisme en question est l'application de réduction d'un anneau à valuation discrète de caractéristique mixte R sur son corps résiduel $\tilde{K}$. Cela induit une correspondance

$$\{Objets/R\} \to \{Objets/\widetilde{K}\}$$

pour nombreux objets en géométrie algébrique. En ce contexte, les problèmes de relèvement demandent de décrire l'image de telle correspondance, c'est-à-dire une caractérisation des objets en caractéristique positive qui proviennent de la caractéristique nulle de façon telle que les propriétés géométriques fondamentales soient conservés. Par exemple, dans le cas des variétés algébriques, le problème de relèvement demande si, à partir d'un schéma intégral, séparé et de type fini X sur $\tilde{K}$, on peut trouver X_R schéma plat, tel que sa fibre spéciale soit égale à X.

Dans la première partie de cette thèse, on y étudie le problème de relèvement de courbes projectives et lisses avec un groupe fini d'automorphismes. On travaille avec un problème analogue, de nature locale, en étudiant les relèvements à la caractéristique zéro d'un groupe fini d'automorphismes de $\tilde{K}[[t]]$. Le résultats originaux principaux sur ce sujet ont été obtenus en suivant deux approches differentes, mais complémentaires qu'on pourrait désigner d' *arithmétique* et *géométrique.*

L'approche arithmétique consiste à travailler explicitement avec les propriétés de certains formes différentielles. On fixe le groupe $G = (\mathbb{Z}/p\mathbb{Z})^n$, pour lequel les travaux de Raynaud ([**?**]), Matignon ([38]) et Pagot ([47]) montrent que l'existence de certains espaces vectoriels de formes différentielles logarithmiques, dits espaces $L_{m+1,n}$, entraine l'existence de relèvements d'actions locaux de G. Dans [47], l'existence de ces espaces est étudié pour certains valeurs de m. On poursuit cet étude, en établissant une formule qui donne une nouvelle relation entre les pôles et les résidus de ces formes différentielles, en permettant de simplifier la stratégie de Pagot et de donner une condition nécessaire à l'existence d'espaces $L_{m+1,2}$ quand $p = 3$. Les calculs impliqués sont purement de nature arithmétique, mais ils présentent une haute complexité computationnelle. En principe, cette stratégie nous permettrait de décider l'existence d'espaces $L_{15,2}$, mais pour aboutir à un résultat il faut envisager une implémentation à l'ordinateur des formules découvertes.

L'approche géométrique adresse le problème de relèvement dans une généralité majeure, en faisant intervenir notions abstraites plus sophistiquées. On considère les actions locales d'un groupe fini quelconque G et la notion d'arbre de Hurwitz, introduite par Henrio dans [31] et partiellement généralisée par Brewis et Wewers dans [14]. L'arbre de Hurwitz est un objet combinatoire associé à une action locale en caractéristique nulle, qui en encode à la fois la géométrie des points fixes rigides et la théorie de la ramification. Il est utilisé pour donner des conditions nécessaires - et, dans le cas où $G = \mathbb{Z}/p\mathbb{Z}$, aussi suffisantes - au relèvement de certaines classes d'actions locales en caractéristique strictement positive. Le résultat principal qu'on montre avec cet approche est une caractérisation de l'arbre de Hurwitz dans le cadre de la géométrie analytique non-Archimédienne. On montre d'abord que l'arbre admet un plongement canonique en tant qu'espace métrique dans le disque unitaire fermé de Berkovich sur K, le corps de fractions de R. Cela nous permet de décrire les données de Hurwitz en utilisant les propriétés analytiques des courbes de Berkovich. En particulier, on décrit les formes différentielles qui entrainent les relèvement des actions de $\mathbb{Z}/p\mathbb{Z}$ en termes d'un fibré un droite sur le disque épointé, décrit explicitement et appelé *faisceau des déformations*. Cela est une première étape vers une généralisation de ces formes différentielles pour G quelconque, problème qui intéresse depuis longtemps les mathématiciens qui travaillent sur ces questions. Finalement on donne une caractérisation de l'arbre de Hurwitz plongé en termes de théories diverses, qui ont été reliées récemment avec les espaces de Berkovich par différents auteurs : la dynamique non-Archimédienne, la géométrie tropicale et les groupes fondamentaux.

Dans la dernière partie de cette thèse, dans un travail commun avec Gianmarco Chinello, on adresse le problème de définir le groupe metaplectique et la représentation de Weil sur un anneau intègre. Ces notions apparaissent dans les travaux d'André Weil [62], où l'auteur construit une représentation projective complexe du groupe symplectique qu'il l'étend ensuite sur certains revêtements fini pour en obtenir une vraie représentation. Cette théorie s'est révélé très fructueuse et riche de connexions avec des théories mathématiques très différentes : formes modulaires, mécanique quantique et analyse harmonique entre autres. L'intérêt récent apporté à la théorie des représentations ℓ-modulaires et avec coefficients dans $\mathbb{Z}_\ell^{nr}$ soulève naturellement le problème de définir une telle représentation dans le cadre de ces théories (voir par exemple l'introduction de [41] où cette question est formulée explicitement). Nous montrons qu'il y a une réponse positive, en construisant plus en général une représentation de Weil à coefficients dans un anneau intègre satisfaisant certaines conditions supplémentaires.

Dans les **chapitres 2 et 3** nous introduisons les outils qui seront exploités dans la suite, c'est-à-dire la géométrie non-Archimédienne *à la Berkovich* et la théorie classique des relèvements d'actions locales. Les résultat contenus dans le reste de la thèse sont à considérer originales sauf mention spécifique du contraire.

Regardons plus en détail ces résultats ainsi que le contexte qui les entoure.

Relèvements de revêtements Galoisiens

Soit $\widetilde{K}$ un corps algébriquement clos de caractéristique $p > 0$ et soit $W(\widetilde{K})$ son anneau de vecteurs de Witt, l'anneau à valuation discrète complet de caractéristique zéro minimal pour la propriété d'avoir $\tilde{K}$ comme corps résiduel. Soit $\widetilde{C}$ une courbe projective et lisse sur $\widetilde{K}$. Des résultats de Grothendieck en théorie des déformations ([29], III 7.3) assurent l'existence d'une courbe relative lisse et projective sur $W(\widetilde{K})$ telle que sa fibre spéciale soit $\widetilde{C}$. On peut demander si cette propriété s'étend aussi aux automorphismes de ces courbes: a-t-on $\mathrm{Aut}(C) = \mathrm{Aut}(\widetilde{C})$? La reponse est. en général, négative : il y a en effet un morphisme $\mathrm{Aut}_{W(\widetilde{K})}(C) \to \mathrm{Aut}_{\widetilde{K}}(\widetilde{C})$ induit par la réduction modulo p, mais il n'est pas une bijection en général. Par exemple, si C est une courbe de genre au moins 2 la borne de Hurwitz nous donne $|\mathrm{Aut}(C)| \leq 84(g-1)$, mais il n'y a pas d'analogue en caractéristique strictement positive. L'exemple du modèle projective de la courbe plane définie par $y^2 = x^p - x$, donné par Roquette ([53]) montre en fait que cette courbe a genre $(p-1)/2$ et groupe d'automorphismes d'ordre $2p(p^2-1)$. Dans d'autres cas, pourtant, les deux groupes d'automorphismes coincident. Par exemple, si C est la quartique de Klein, d'équation $x^3y + y^3z + z^3x$ on a $\mathrm{Aut}(C) = PSL(2,7) = \mathrm{Aut}(\widetilde{C})$.
Il y a beaucoup d'autres exemples et contrexemples à ce problème, de nature si différente qu'il ne semble donc pas pouvoir être traité directement. Il est mieux étudié d'un point de vue locale et on peut montrer qu'une action sur $\widetilde{C}$ se relève en caractéristique zéro si et seulement si elle se relève localement sur l'ensemble des points qui ont stabilisateur non-trivial. Puisque la courbe est lisse, ces actions locaux s'identifient avec des automorphismes de $\widetilde{K}$-algèbres de $\widetilde{K}[[t]]$.

Relèvements locaux

Soit $G \hookrightarrow \mathrm{Aut}(\widetilde{K}[[t]])$ un groupe fini d'automorphismes de $\widetilde{K}$-algèbres. Le problème de relèvement peut être formalisé comme suit:

Question 0.0.1. Existe-t-il une extension R d'anneaux à valuation discrète sur $W(\widetilde{K})$ et un plongement $G \hookrightarrow \mathrm{Aut}(R[[T]])$ tels que le diagramme

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & \mathrm{Aut}_R R[[t]] \arrow[d] \\ & \mathrm{Aut}_{\widetilde{K}} \widetilde{K}[[t]] \end{tikzcd}$$

est commutatif ?

Si la réponse est positive, on dit que l'action locale de G se relève de la caractéristique p à la caractéristique zéro. Cette question est lié à nombre d'autres interrogatifs. On peut demander si, le groupe G fixé, toute action locale de G se relève à la caractéristique zéro. Dans ce cas G est dit *groupe de Oort local*. On doit cette définition au fait que Oort ([45]) a conjecturé que tout groupe cyclique satisfait cette condition. Cette conjecture a été montré par Pop ([50]) and Obus-Wewers ([44]) en 2012. Le problème qui reste ouvert est maintenant une caractérisation des groupes de Oort locaux. Dans [16]) Chinburg, Guralnick et Harbater formulent la conjecture suivante:

Conjecture 0.0.2 (conjecture de Oort forte)**.** Soit G un groupe fini. Alors chaque action locale $(\widetilde{K}[[t]], G)$se relève à la caractéristique zéro si et seulement si une des conditions suivantes est vérifiée :

- le groupe G est cyclique
- le groupe G est dihédral d'ordre $2p^n$
- le groupe G est le groupe alterné A_4 et $p = 2$.

Quand G est un groupe de Oort local, on peut demander de classifier tous les relèvements possibles. Autrement, on peut demander une caractérisation des actions qui se relèvent. Des techniques très différentes ont été utilisés pour répondre à ces questions. Ma contribution concerne principalement l'existence de formes différentielles logarithmiques en caractéristique positive et l'étude des arbres de Hurwitz en géométrie analytique non-Archimédienne.

Espaces $L_{m+1,n}$ et relèvements

Considérons des actions locales du groupe $G = (\mathbb{Z}/p\mathbb{Z})^n$. Si on impose la condition que la distance réciproque des points de ramification géométriques soit constante, l'existence d'un relèvement local pour les actions de $(\mathbb{Z}/p\mathbb{Z})^n$ est équivalent à l'existence d'espaces de formes différentielles définis comme suit.

Definition 0.0.3 (Espaces $L_{m+1,n}$)**.** *Un $\mathbb{F}_p$-espace vectoriel V de formes différentielles meromorphes sur $\mathbb{P}^1_k$ est noté $L_{m+1,n}$ si les propriétés suivantes sont vérifiées pour tout $\omega \in V$:*

1. *$dim(V) = n \in \mathbb{N}$*
2. *La forme $\omega = \frac{df}{f}$, $\exists f \in k(t)$*
3. *La forme ω a un seul zéro d'ordre $m-1$ en $\{\infty\}$.*

Pagot étudie les espaces $L_{m+1,n}$ dans [47] en formulant la conjecture (cfr section 2.3 of [47], *Remarque* 2) qu'un $\mathbb{F}_p$-espace vectoriel $L_{m+1,n}$ existe si et seulement si $p^{n-1}(p-1)|m$, en montrant que cela se vérifie si $n = 2$ et $m + 1 = p$, $m + 1 = 2p$ et $m + 1 = 3p$.
Dans le **chapitre 4** on suppose qu'il existe un espace $L_{m+1,2}$, engendré par deux formes ω_1 et

ω_2. En utilisant la combinatoire des pôles de ω_1 et ω_2 on montre une condition additionnelle sur les résidus qui n'était pas exploité dans [47]. Cela nous permet de simplifier les preuves de [47] et de pousser l'étude de l'existence d'espaces $L_{m+1,2}$ plus loin. En particulier nous montrons le résultat suivant

Theorem 0.0.4. Soient

$$\omega_1 := \frac{u \cdot z^{13}}{\prod_{j=1}^{3}\prod_{i=1}^{5}(1-x_{i,j}z)}dz \quad \text{et} \quad \omega_2 := \frac{v \cdot z^{13}}{\prod_{j=0}^{2}\prod_{i=1}^{5}(1-x_{i,j}z)}dz, \quad u, v \in k$$

deux formes différentielles méromorphes sur $\mathbb{P}^1_k$. Si l'espace $\langle \omega_1, \omega_2 \rangle$ est un $\mathbb{F}_3$-espace $L_{15,2}$, alors $N(\frac{u}{v}) = 0$, où

$$\begin{aligned} N(x) =& x^{84} - x^{83} - x^{81} + x^{80} + x^{79} + x^{78} - x^{77} + x^{76} - x^{73} - x^{72} + x^{71} - x^{70} + x^{69} + \\ & x^{66} + x^{64} - x^{63} + x^{62} - x^{60} - x^{59} + x^{56} + x^{55} + x^{53} - x^{51} + x^{50} - x^{49} + x^{48} - \\ & x^{47} - x^{45} + x^{43} + x^{42} - x^{40} + x^{38} + x^{36} - x^{35} - x^{34} - x^{33} - x^{32} - x^{31} - x^{29} - \\ & x^{28} + x^{27} - x^{26} - x^{25}. \end{aligned}$$

La technique principale de preuve est l'étude arithmétique des pôles de ω_1 et ω_2, formalisé par le biais de certains polynômes s'annulant dans des sous-ensembles de pôles bien choisis. Nous exprimons ensuite les conditions satisfaites par les pôles en termes des coefficients de tels polynômes. Les formules de Newton nous permettent enfin de confronter les conditions ainsi obtenues, en parvenant au résultat final.

Arbres de Hurwitz et disque de Berkovich

Les arbres de Hurwitz sont des objets mathématiques qui classifient les actions locales en caractéristique nulle. Les propriétés des automorphismes d'ordre p du disque p-adique, étudiées par Green-Matignon ([28]) et Raynaud ([52]) ont été encodées par Henrio ([31]) qui a définit, pour tout automorphisme $\sigma \in Aut_R R[[T]]$, un arbre métrique enraciné $\mathcal{H}_\sigma$ avec des données supplémentaires qui encode la position relative des points de ramification et les propriétés de la réduction de σ. Cet arbre est appelé *arbre de Hurwitz*, et les informations supplémentaires *données de Hurwitz*. L'existence et compatibilité de certaines données de Hurwitz est une condition nécessaire et suffisante pour l'existence d'automorphismes qui décrivent un arbre donné.

Brewis et Wewers ([14]) étendent la définition d'arbre de Hurwitz pour décrire les actions locales d'un groupe fini quelconque. Dans leur définition, les données de Hurwitz sont définies par des caractères provenant des représentations de G qui décrivent la théorie de la ramification d'une action locale en caractéristique zéro donnée. Ces données sont appelés *caractère d'Artin* et *caractère de profondeur*. Toutefois, cette généralisation est seulement partielle et la question de définir un arbre de Hurwitz analogue à celui de [31] reste ouverte pour la plupart des groupes finis.

Dans le **chapitre 5**, on caractérise les arbres de Hurwitz comme des objets analytiques non-Archimédiens, au sens de Berkovich. Le résultat central est le suivant :

Theorem 0.0.5. Soit $\mathcal{T}_\Lambda$ l'arbre de Hurwitz associé à une action locale en caractéristique zéro $\Lambda : G \hookrightarrow \mathrm{Aut}_R(R[[T]])$. Alors, il existe un plongement canonique d'espaces métriques

$$\mathcal{T}_\Lambda \hookrightarrow \mathcal{M}(K\{T\})$$

dans le disque fermé de Berkovich $\mathbb{D}(0,1)$.

Ce théorème permet l'identification d'un sommet $v \in V(\mathcal{T}_\Lambda)$ avec un disque fermé $D^\bullet(v)$, et d'un arête $e \in E(\mathcal{T}_\Lambda)$ avec un disque ouvert $D^\circ(e)$. Nous utilisons cette idéntification pour traduire les données de Hurwitz en termes d'espaces analytiques. D'abord on prouve que le caractère de profondeur δ_v de [14] coincide avec l' evaluation de la fonction analytique $\sigma(T) - T$ sur le point qui correspond à v dans l'arbre de Hurwitz plongé. Ensuite, on donne une formulation similaire pour le caractère d'Artin a_e. Pour ce qui concerne les formes différentielles, le résultat principal est le suivant

Theorem 0.0.6. Soit $\Lambda : \mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathrm{Aut}(R[[T]])$ une action locale en caractéristique zéro et soit L_Λ l'ensemble de ses points de ramification géométriques. Il existe un fibré en droites métrisé Ω_Λ sur $\mathbb{D}(0,1) \setminus L_\Lambda$ tel que la bonne donnée de déformation associée à un sommet $v \in V(\mathcal{T}_\Lambda)$ s'identifie avec une section d'un faisceau sur la réduction de Temkin $\widetilde{\Omega_\Lambda, v}$.

La preuve de ce théorème relie sur la description explicite du fibré Ω_Λ, et sur l'application d'un résultat de Chambert-Loir et Ducros sur la réduction des fibrés vectoriels dans les espaces de Berkovich (voir la séction 6 de [15]). L'emploi de la réduction des germes à la Temkin est nécessaire pour décrire les bonnes données de déformation avec des fibrés en droite. Cela n'est pas possible dans la situation classique, où les données de déformation sont définies sur un arbre de droites projectives, où le faisceau des formes différentielles n'est pas localement libre. De plus, la théorie des fibrés en droites métrisés a un caractère fortement combinatoire qui permet de réaliser les calculs de façon explicite. Le faisceau Ω_Λ peut être construit pour tout groupe G et constitue un point de départ pour une généralisation maniable des bonnes données de déformation.

La représentation de Weil et le groupe metaplectique

Dans l'article [18], nous définissons et décrivons explicitement la représentation de Weil sur un domaine d'intégrité quelconque. Cette représentation et le groupe metaplectique ont été introduits par André Weil dans son incontournable article [62], en ayant pour but de éclaircir certains propriétés des fonctions theta présents dans les travaux de Siegel et d'explorer les implications arithmétiques de cette construction. Cette théorie s'est révélés à la fois très profonde et très fertile. Elle est à la base des travaux de Shimura sur les formes modulaires de poids demi-entier et de ceux de Jacquet et Langlands sur les représentations automorphes dans un contexte adélique.

Dans la construction de Weil on considère F un corps local, X un espace vectoriel de dimension finie et $\mathrm{Sp}(W)$ le groupe symplectique sur l'espace $W = X \times X^*$. On note $\mathcal{T}$ le groupe multiplicatif des nombres complexes de valeur absolue unitaire et $\chi : F \to \mathcal{T}$ un caractère lisse non-trivial. En utilisant ce dernier, Weil montre l'existence d'une action sur $\mathrm{Sp}(W)$ du groupe de Heisenberg, définie à multiplication d'un élément de $\mathcal{T}$ près. Avec cela, il construit un revêtement $\mathrm{Mp}(W)$ de $\mathrm{Sp}(W)$ par $\mathcal{T}$, qui se manifeste naturellement avec un morphisme $\mathrm{Mp}(W) \to GL(L^2(X))$, la représentation de Weil. Il montre enfin que $\mathrm{Mp}(W)$ contient un double revêtement de $\mathrm{Sp}(W)$ sur lequel la représentation de Weil peut être restreinte.
Dans le **chapitre 6**, on suppose que F est non-Archimédien de caractéristique $\neq 2$ avec cardinalité de corps résiduel $q = p^e$. On remplace $\mathcal{T}$ par le groupe multiplicatif d'un domaine d'intégrité R tel que $p \in R^\times$, R contient une racine carrée de q et les racines p^n-émes de l'unité pour chaque n, ce qui assure l'existence d'un caractère lisse non-trivial $\chi : F \to R^\times$. Dans cette généralité on est capables de reproduire les résultats de Weil, en montrant l'existence du *groupe métaplectique réduit*, défini comme suit. On construit d'abord le groupe métaplectique $\mathrm{Mp}(W)$ de façon telle à avoir une suite exacte courte

$$1 \longrightarrow R^\times \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1. \tag{$\star$}$$

Ensuite on donne une description d'un sous-groupe de $\mathrm{Mp}(W)$, revêtement à deux feuillets de $\mathrm{Sp}(W)$, qui décrit dans $H^2(\mathrm{Sp}(W), \mu_2(R))$ la classe provenant de la classe de $H^2(\mathrm{Sp}(W), R^\times)$ associée à $\mathrm{Mp}(W)$. Nous formalisons ce résultat de façon suivante.

Theorem 0.0.7. Soit $\mathrm{car}(R) \neq 2$. Il existe un sous-groupe $\mathrm{Mp}_2(W)$ de $\mathrm{Mp}(W)$ tel que la suite exacte courte ($\star$) se restraint en

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}_2(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1, \tag{$\star\star$}$$

suite exacte courte non scindée.

Il y a plusieurs problèmes qui se manifestent en travaillant dans la nouvelle généralité. Pour surmonter ces difficultés nous considérons les mesures de Haar à valeurs dans R et des opérateurs agissant sur les fonctions de Schwarz sur un F-espace vectoriel, à valeurs dans R. Cela remplace les opérateurs unitaires considérés par Weil. En outre, puisqu'on admet R de caractéristique positive, les calculs diffèrent dans plusieurs situations, par exemple lorsqu'il faut calculer explicitement les propriétés de la norme réduite sur les quaternions en relation avec l'arithmétique de R.

Chapter 1

Introduction

The common thread of this work is the comparison between mathematical objects of various kinds (finite Galois covers of curves, logarithmic differential forms, representations of the symplectic group and related entities) when they are defined over different base rings. This is an extremely challenging topic, but also very wide, and we choose to concentrate our attention on two specific issues: the problem of lifting Galois covers of curves to characteristic zero and the problem of defining the Weil representation over rings of any characteristic.

The first question is an example of *lifting problem*. The relative point of view, that consists in replacing the study of properties of objects of a category with the study of properties of its morphisms, allows to consider relative objects as avatars of the same structure, and to compare them whenever there is a morphism between the bases. In the case of lifting problems, the morphism is the surjective reduction from a discrete valuation ring in mixed characteristic R and its residue field $\widetilde{K}$. This yields a correspondence

$$\{Objects/R\} \to \{Objects/\widetilde{K}\}$$

for a wide number of geometrical objects. Given this setting, lifting problems ask what is the image of such a correspondence, or equivalently, what are the objects in positive characteristic that come from objects in characteristic zero. Usually, the assumption that the lifting preserves the geometry of the objects involved is made. For instance, when the objects are algebraic varieties, the lifting problem can be rephrased by asking wether for a given integral separated scheme of finite type over $\tilde{K}$ there exists a flat scheme over R whose special fiber is the initial datum.

In the first part of this thesis, we study the problem of lifting a smooth projective curve together with a finite group of its automorphisms. We work with the analog "local" problem, of studying liftings of a finite group G of automorphisms of $\tilde{K}[[t]]$. The main original results on this topic are obtained with two different, but complementary approaches. The first involves explicit one, we fix the group $G = (\mathbb{Z}/p\mathbb{Z})^n$, and we use the fact (stated by Pagot in [47] using methods coming from the paper [38] by Matignon) that the lifting of a G-action is yielded by the existence of $\mathbb{F}_p$-vector spaces of logarithmic differential forms on $\mathbb{P}^1_{\tilde{K}}$ (i.e. that can be written as $\frac{df}{f}$ for $f \in \tilde{K}(t))$, with a unique zero at ∞. In [47], the existence of such vector spaces is discussed in some cases. We establish a new formula involving the position of poles and the value of residues of such forms, that in the case where $p = 3$ and $n = 2$ permits to simplify the proofs of [47] and to find necessary conditions for lifting in cases that have not been studied yet. The calculations involved are elementary, but they present a high level of computational complexity. In principle, when $p = 3$ and $n = 2$, our approach provides a complete answer, but in order to do this, the use of a computer is necessary.

The second approach address the lifting problem in a wider generality, but requires also more sophisticated theoretical notions. We deal with a local action of any finite group G, and with the two notions of Hurwitz tree existing in the literature: the first introduced by Henrio in [31] for $\mathbb{Z}/p\mathbb{Z}$ and the second by Brewis-Wewers for any finite group. The Hurwitz tree is a combinatorial object which encodes both the geometry of fixed points of an automorphism of finite order and the associated ramification data. It is used to give necessary (and, in the case of $\mathbb{Z}/p\mathbb{Z}$, also sufficient) conditions for lifting actions to characteristic zero. The principal result, that we show with this approach is that the Hurwitz tree is canonically embedded in the Berkovich unit disc over $K := \mathrm{Frac}(R)$. This is the starting point for a study of the Hurwitz tree as a non-Archimedean analytic object. We describe the Hurwitz data using analytic properties of Berkovich curves. In particular, we describe the differential forms yielding liftings of $\mathbb{Z}/p\mathbb{Z}$-actions in terms of a precise metrized line bundle on the pointed unit disc, that we call *sheaf of deformations*. Finally, we give a characterization of the embedded Hurwitz tree in terms of different theories that have been related with Berkovich spaces by recent developments: non-Archimedean dynamics, tropical geometry and fundamental groups.

In the last part of this thesis, in a a joint work with Gianmarco Chinello, we address the problem of defining the metaplectic group and the Weil representation over an integral domain. These notions appear for the first time in the celebrated *Acta* paper of Weil [62], where the author constructed a certain complex projective unitary representation of the symplectic group. This representation has shown many interesting features, bridging between different mathematical theories. The recent interest in ℓ-modular representations, and in representations over $\mathbb{Z}_\ell^{ur}$, raises the question if such a representation could be defined in the framework of these theories (see, for example, the introduction of [41], where this question appears explicitly). We show that this question has a positive answer, by constructing a Weil representation with coefficients in every integral domain satisfying certain assumptions.

In **chapter 2** and in **chapter 3** of the present work, we introduce the tools that will be exploited in the following, namely non-Archimedean analytic geometry in the sense of Berkovich and the techniques used in the study of lifting to characteristic zero local actions of finite groups on curves. In the rest of the thesis, the main ideas stand on their own as original work, but they contribute together to shed new light on some phenomena arising in the study of the local lifting problem and of those number theoretical questions related to the Weil representation.

Let us discuss in a more detailed way the framework where these results are developed.

Lifting Galois covers to characteristic zero

Let $\widetilde{K}$ be an algebraically closed field of characteristic $p > 0$ and let $W(\widetilde{K})$ be its ring of Witt vectors, the minimal complete discrete valuation ring of characteristic 0 which has $\widetilde{K}$ as residue field. Let $\widetilde{C}$ be a smooth projective curve defined over $\widetilde{K}$. Then by results of Grothendieck ([29], III 7.3) there exists a relative smooth projective curve C defined over $W(\widetilde{K})$ such that its special fiber is $\widetilde{C}$. One might ask if this "lifting property" applies also to automorphisms of such curves : is $\mathrm{Aut}(C) = \mathrm{Aut}(\widetilde{C})$? The answer is negative, in general: there is indeed a natural map $\mathrm{Aut}_{W(\widetilde{K})}(C) \to \mathrm{Aut}_{\widetilde{K}}(\widetilde{C})$ induced by reduction modulo the maximal ideal, but it is far from being bijective. For example, when C is of genus ≥ 2 one has the so called Hurwitz bound: $|\mathrm{Aut}(C)| \leq 84(g-1)$ but there is no such analogue in positive characteristic. The example of the projective model of the plane curve defined by equation $y^2 = x^p - x$ was given by Roquette in [53]. This is a curve of genus $(p-1)/2$ and $2p(p^2-1)$ automorphisms. Then it violates the Hurwitz bound whenever $p \geq 5$. In other cases we have a positive answer, like when $p \neq 2, 3, 7$ and the curve is the Klein quartic (of equation $x^3y + y^3z + z^3x$): in this situation $\mathrm{Aut}(C) = PSL(2, 7) = \mathrm{Aut}(\widetilde{C})$.
There exist several other examples of automorphisms of curves that admit liftings, as well as many other counterexamples of automorphisms that do not lift to characteristic zero. This problem is better studied locally, and in fact one can show that, if the action lift locally at every point with nontrivial stabilizer, then there is a lifting of the global action. In this way it is interesting to study automorphisms of $\widetilde{K}$-algebras of $\widetilde{K}[[t]]$ and their liftings.

Local liftings

Let $G \hookrightarrow \mathrm{Aut}(\widetilde{K}[[t]])$ be a finite group of automorphisms of $\widetilde{K}$-algebras. The local lifting problem asks the following question:

Question 1.0.8. Are there an extension R of discrete valued rings over the Witt vector ring $W(\widetilde{K})$ and an immersion $G \hookrightarrow \mathrm{Aut}(R[[T]])$ such that the diagram

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & \mathrm{Aut}_R R[[t]] \arrow[d] \\ & \mathrm{Aut}_{\widetilde{K}} \widetilde{K}[[t]] \end{tikzcd}$$

commutes?

When the answer is positive we say that the local action of G in characteristic p lifts to characteristic zero. This question raises a number of related interrogatives. For example it may be asked wether or not a given group G is a local Oort group (*i.e.* all the local actions of G lift to characteristic zero). Oort ([45]) conjectured that all cyclic groups satisfy this condition, and this has been proved by Pop ([50]) and Obus-Wewers ([44]) in 2012. The Oort conjecture was refined by Chinburg, Guralnick and Harbater ([16]) to the following:

Conjecture 1.0.9 (Strong Oort conjecture)**.** Let G be a finite group. Then every local action $(\widetilde{K}[[t]], G)$ in characteristic p lifts to characteristic zero if and only if G is one of the following :

- a cyclic group
- the dihedral group of order $2p^n$ for some n
- the alternating group A_4.

If G is a local Oort group one may ask to parametrize all possible actions, if it is not one may ask for a characterization of those action coming from characteristic zero and those who are not. Several tools were used to partially answer these questions. Our contributions concern mainly the existence of logarithmic differential forms in positive characteristic and the study of Hurwitz trees in a non-Archimedean analytic context.

In discussing Hurwitz trees we chose to adopt a different viewpoint from the existing literature on this topic. Classically one fixes an action in characteristic p and supposes the existence of a lifting. Then, this lifting is studied either looking for a contradiction or to obtain the deformations providing the lifting explicitly. In this thesis, we do not fix a local action in characteristic p, and we consider Hurwitz trees as objects on their own right. In this way, the role of Hurwitz tree is intimately related to the deformation theory of torsors in characteristic zero, and this perspective permits to study local actions in characteristic zero "in families". As a result of this difference, our definition of Hurwitz tree is more restrictive than the one of [31], [14] or [11]. It corresponds to what is called "Hurwitz tree associated to an automorphism of the p-adic open disc" in the literature.

Vector spaces of logarithmic differential forms in positive characteristic

When the mutual distance between the ramification points is constant, the existence of a local lifting for the action of $(\mathbb{Z}/p\mathbb{Z})^n$ is equivalent to the existence of n-dimensional vector spaces of logarithmic differential forms over the projective line in characteristic p. Pagot studied such spaces in [47], asking the following question

Question 1.0.10. Given a prime number p, and natural numbers $n, m > 0$ such that $(m, p) = 1$ do there exist $\mathbb{F}_p$-vector spaces $L_{m+1,n}$ of logarithmic differential forms on $\mathbb{P}^1_k$ of dimension n such that every $\omega \in L_{m+1,n}$ has m+1 simple poles and a single zero in $\{\infty\}$?

He conjectures (see section 2.3 of [47], *Remarque* 2) that an $\mathbb{F}_p$-vector space $L_{m+1,n}$ exists if and only if $p^{n-1}(p-1)|m$, and proves it for $m+1 = p$, $m+1 = 2p$ and $m+1 = 3p$.
Suppose that a $L_{m+1,2}$ exists, and that it is generated by two logarithmic differential forms ω_1 and ω_2. In **chapter 4** we provide additional algebraic conditions for the residues of ω_i. Mixing these conditions with combinatorial arguments in positive characteristic we can give shorter proofs for the known results and go further in the study of the conjecture, treating the case where $p = 3$ and $m + 1 = 15$. The main tool used is the combinatorics of the poles of the differential forms, incarnated in this case by some polynomials having zeroes in a subset of the

set of poles of ω_i. We express the conditions satisfied by the poles in terms of the coefficients of such polynomials, that are symmetric functions in terms of the poles. In this way we get the necessary conditions for the existence of $\langle \omega_1, \omega_2 \rangle$ as a $L_{m+1,2}$, in terms of the coefficients of ω_1 and ω_2.

Hurwitz trees and the Berkovich pointed unit disc

Hurwitz trees are mathematical objects parametrizing local actions in characteristic zero. The properties of order p automorphisms of the p-adic disc studied by Green-Matignon ([28]) and Raynaud ([52]) has been encoded combinatorially in an article by Henrio ([31]) by associating to such an automorphism $\sigma \in Aut_R R[[T]]$ a rooted metric tree $\mathcal{H}_\sigma$ with additional information, encoding the position of the ramification points and the reduction of the action of σ. The tree is classically called Hurwitz tree, and the information, Hurwitz data. The existence and mutual compatibility of Hurwitz data is a necessary and sufficient condition to the existence of an automorphism giving rise to that Hurwitz tree.
Brewis and Wewers ([14]) extended the definition of Hurwitz tree in order to describe local actions of any finite group G. In their definition, Hurwitz data are defined as characters arising from representations of G, describing the ramification theory of the local action in characteristic zero. They are called the *depth character* and the *Artin character*. Not all Hurwitz data are generalized in this definition, and the question if a Hurwitz tree in the sense of [31] can be defined for any finite group remains open.

In **chapter 5**, we characterize the Hurwitz tree as a non-Archimedean analytic object, in the sense of Berkovich. We start by proving the following result

Theorem 1.0.11. Let $\mathcal{T}_\Lambda$ be the Hurwitz tree associated to the local action of a finite group in characteristic zero $\Lambda : G \hookrightarrow \mathrm{Aut}_R(R[[T]])$. Then there is a metric embedding

$$\mathcal{T}_\Lambda \hookrightarrow \mathcal{M}(K\{T\})$$

of the Hurwitz tree in the Berkovich closed unit disc $\mathbb{D}(0,1)$ such that the image is contained in the set of points fixed by the action on $\mathcal{M}(K\{T\})$.

This theorem permits the identification of a vertex $v \in V(\mathcal{T}_\Lambda)$ with a closed disc $D^\bullet(v)$, and of an edge $e \in E(\mathcal{T}_\Lambda)$ with an open disc $D^\circ(e)$. We exploit this identification to translate Hurwitz data in analytic terms. We first prove that the depth character δ_v of [14] coincides with the evaluation of the analytic function $\sigma(T) - T$ on the point corresponding to v in the embedded Hurwitz tree. We then give a similar formulation for the Artin character a_e. The identification permits to characterize the groups G_v as the stabilizers of closed discs $D^\bullet(v)$, and helps proving the following theorem, that gives an analytic description of good deformation data

Theorem 1.0.12. Let $\Lambda : \mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathrm{Aut}(R[[T]])$ be a local action in characteristic zero, having L_Λ as set of fixed points. Then there exists a metrized line bundle Ω_Λ on $\mathbb{D}(0,1) \setminus L_\Lambda$, such that the good deformation datum associated to a vertex $v \in V(\mathcal{T}_\Lambda)$ is a section of the (Temkin) reduction $\widetilde{\Omega_\Lambda, v}$ at v.

This theorem is proved by constructing explicitly the line bundle Ω_Λ, and applying a result of Chambert-Loir and Ducros on the reduction of vector bundles on Berkovich spaces (see section 6 of [15]). The use of Temkin reduction permits to describe a collection of good deformation data using vector bundles. This is not the case in the classical setting, where the collection of good deformation data is defined over a tree of projective lines, whose sheaf of differential forms is not locally free. Moreover, the theory of metrized vector bundles on Berkovich curves has nice combinatorial features (for instance the Poincaré-Lelong formula and the notion of current are described in a combinatorial way), that permits explicit computable conditions. The sheaf Ω_Λ can be constructed for every group G, and constitutes a starting point for a handful generalization of good deformation data.

Weil representation and metaplectic group

In the paper [18] we define and describe explicitly the Weil representation over any integral domain. This representation has been introduced together with the metaplectic group by André Weil in his *Acta* paper [62] in order to shed light on the results of Siegel on theta functions and to formulate them in an adelic setting. This led to various developments in number theory, for example the work of Shimura on modular forms of half-integral weight and the one of Jacquet and Langlands on automorphic representations of adele groups.
The construction of Weil is as follows: he considers a local field F, a finite dimensional F-vector space X and the symplectic group $\mathrm{Sp}(W)$ over $W = X \times X^*$. He lets $\mathcal{T}$ be the multiplicative group of complex numbers of unitary absolute value and $\chi : F \to \mathcal{T}$ be a non-trivial continuous character. Using χ he shows the existence of an action of $\mathrm{Sp}(W)$ over the Heisenberg group, defined up to multiplication by an element of $\mathcal{T}$. Therefore he constructs a cover $\mathrm{Mp}(W)$ of $\mathrm{Sp}(W)$ such that φ lifts to a complex infinite representation of $\mathrm{Mp}(W)$, the now-so-called *Weil representation*. Finally he shows that $\mathrm{Mp}(W)$ contains properly a double cover of $\mathrm{Sp}(W)$ on which the Weil representation can be restricted.
In **chapter 6**, we suppose that F is non-Archimedean of characteristic $\neq 2$ and residue field of cardinality $q = p^e$. We replace $\mathcal{T}$ by an integral domain R such that $p \in R^\times$, R contains a square root of q and p^n-th roots of unity for every n, to ensure the existence of a nontrivial smooth character $\chi : F \to R^\times$. In this generality we are able to reproduce the results of Weil showing the existence of the *reduced metaplectic group*, defined in the following way. Firstly we construct the metaplectic group $\mathrm{Mp}(W)$ in such a way to have a non-split short exact sequence

$$1 \longrightarrow R^\times \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1. \qquad (\star)$$

Then, we give a description of a minimal subgroup of $\mathrm{Mp}(W)$ which is a non-trivial extension of $\mathrm{Sp}(W)$, in our main theorem:

Theorem 1.0.13. Let $\mathrm{char}(R) \neq 2$. There exists a subgroup $\mathrm{Mp}_2(W)$ of $\mathrm{Mp}(W)$ such that the short exact sequence $(\star)$ restricts to a short exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}_2(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1 \qquad (\star\star)$$

that does not split.

Different kinds of problems occur in the new generality and the paper contains several new ideas. We consider Haar measures with values in R and operators acting over the space of R-valued Schwartz functions over an F-vector space instead of L^2-functions, using Vignéras' approach. Moreover, allowing R to be of positive characteristic makes it necessary to change computations. Despite these changes, the final result on the description of the metaplectic group holds still.

Chapter 2

Non-Archimedean analytic geometry

This chapter is aimed to provide an introduction to non-Archimedean analytic spaces in the sense of Berkovich. We fix k a field, which is complete for a (possibly trivial) non-Archimedean valuation, we write k^o for its valuation ring, k^{oo} for the unique maximal ideal and $\tilde{k}$ for the residue field. We begin by defining a k-analytic space in whole generality and we explain in details how to get such spaces as generic fibers of formal schemes defined over k^o. This description is fundamental for our results, because it permits to study on the generic fiber some features of their formal models. Some of the tools that are introduced here, like the reduction map and boundaries, are ideas that predate Berkovich's theory. These notions are in fact widely used also in Raynaud's formal approach to Tate's theory. Other, like Temkin theory of reduction of analytic germs, are only possible on Berkovich spaces. In both cases we use Berkovich formalism to define these tools, in such a way to prove the results of chapter 5 in a homogeneous setting.

The second part of the chapter is dedicated to the study of k-analytic curves. The topology and the graph theoretical properties of such curves permit to obtain a combinatorial description of some of their features like the genus, the semi-stable reduction and the structure of vector bundles. We detail these techniques, that we apply to the study pointed discs arising from local actions in chapter 5. We concentrate finally on the interplay between positive and zero characteristic, by describing Temkin reduction theory. This is used to relate vector bundles on Berkovich pointed discs with sheaves of differential forms on the special fiber in chapter 5. These theories are proper of Berkovich approach, and are the main motivation for choosing this theory to describe local actions.

We decide to focus the attention on the results that are useful to this thesis. Consequently, many important features of Berkovich spaces are not included in the present chapter. The interested reader can find in the writings of Berkovich ([4] and [5]) the original references, that contain the motivation and the construction of k-analytic spaces. Let us point out also the excellent surveys by Conrad ([20]) and Ducros ([22]).

2.1 Berkovich spaces

To avoid the problem of total disconnectedness (and the consequent lack of interest) of p-adic analytic spaces treated in a *naïve* way, Tate establishes in the late '60s the basis of rigid geometry. In the spirit of scheme theoretic constructions he allows only convergent series on some closed polydiscs in k^n to be admissible functions, with the problem that the spaces obtained in such a way are not topological spaces but they are endowed with a G-topology (namely it is clear what a covering for these spaces is, but there's no such a thing as an open set).
Slightly less than thirty years later Berkovich proposed another approach in which analytic spaces are real topological spaces enjoying good properties (they are locally arc connected, locally compact and Hausdorff). These spaces were originally constructed as spaces of norm for spectral theory purposes, but they can be seen as obtained by adding to a rigid analytic space the points with residue field of infinite degree over k. These "added" points will represent a global point of view of what happens at closed non-Archimedean discs.

2.1.1 Construction and algebraic description

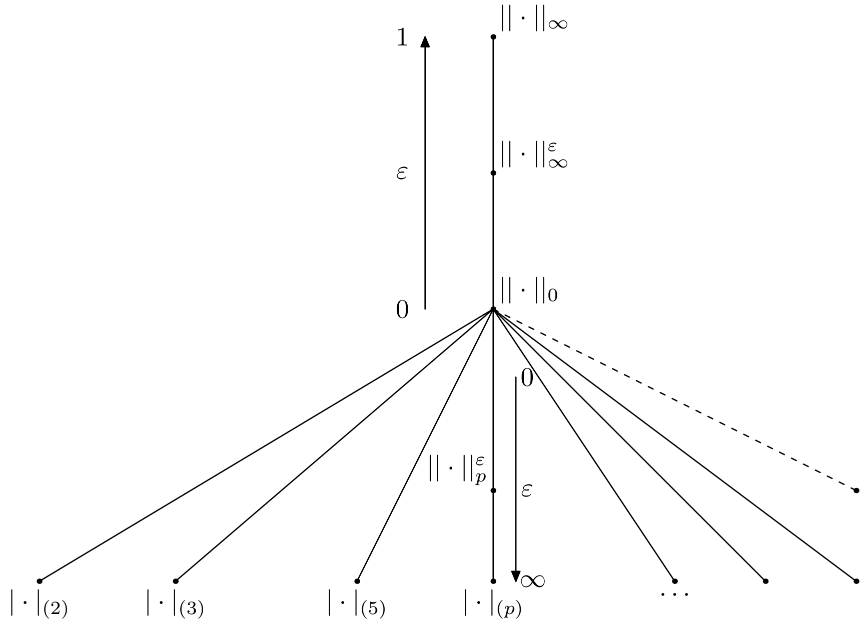
In modern geometric theories, spaces (algebraic, topological, differential or analytical) are firstly defined locally and then properly glued in order to have at the same time a complete and workable description. Berkovich spaces are no exception to this: the local bricks are given by spectra of algebras of analytic functions. Let us describe this construction for a general Banach ring, keeping in mind that the ultimate aim is to apply it to algebras of convergent functions.

Definition 2.1.1. *Let $(\mathcal{A}, \|\cdot\|)$ be a Banach ring. A* ***bounded multiplicative semi-norm*** *over $\mathcal{A}$ is a function $x : \mathcal{A} \to \mathbb{R}$ such that the following properties are satisfied for every $a, b \in \mathcal{A}$:*

- $x(0) = 0$;
- $x(a + b) \leq x(a) + x(b)$;
- $x(a * b) = x(a)x(b)$
- $x(a) \leq \|a\|$.

Definition 2.1.2. *Let $\mathcal{A}$ be a Banach ring. We call* ***analytic spectrum*** *$\mathcal{M}(\mathcal{A})$ of $\mathcal{A}$ the topological space of bounded multiplicative semi-norms over $\mathcal{A}$ endowed with the topology induced by the product topology on $\mathbb{R}^{\mathcal{A}}$.*

Example 2.1.3. *Consider $\mathcal{A} = \mathbb{Z}$ normed with the euclidean absolute value $\|\cdot\|_\infty$. Since $x(n) \leq n = \|n\|_\infty$, every seminorm over $\mathbb{Z}$ is bounded by $\|\cdot\|_\infty$. Moreover Ostrowski's theorem tells us that every nontrivial norm over $\mathbb{Q}$ is either equivalent to an euclidean one or to a p-adic one, therefore $\mathcal{M}(\mathcal{A})$ is made of the norm $\|\cdot\|_\infty^{\varepsilon_\infty}$, $\|\cdot\|_0$ and $\|\cdot\|_p^{\varepsilon_p}$ where $0 < \varepsilon_\infty < 1$ and $0 < \varepsilon_p < \infty$. We have to add the semi-norms which are not norms. Looking at their possible kernels we notice* *that they are all and only the one induced by the trivial norm over $\mathbb{F}_p$ for every p (another way to think at them is to let $\varepsilon_p \to \infty$). The topological picture that results is the following*



Notice that the maps

$$\varepsilon \to \|\cdot\|_p^\varepsilon \text{ and } \varepsilon \to \|\cdot\|_\infty^\varepsilon$$

are homeomorphisms that make each edge in the picture topologically equivalent to a real interval.

Given $x \in \mathcal{M}(\mathcal{A})$ we define its **residue field** $\mathcal{H}(x) = \mathrm{Frac}(\widehat{\mathcal{A}/\ker}(x))$, and we remark that every point $x \in \mathcal{M}(\mathcal{A})$ factorizes (as a character) through $\mathcal{H}(x)$. This remark motivates the following notations: if $x \in \mathcal{M}(\mathcal{A})$ and $f \in \mathcal{A}$ we write $f(x)$ for the image of f inside $\mathcal{H}(x) = \mathrm{Frac}(\widehat{\mathcal{A}/\ker}(x))$ and $|f(x)|$ for $x(f)$.

Affinoid spaces

Analytic spectra are studied in various fashions. Several arithmetical applications can be stated in this way (see for example [49] and [34]). It seems nevertheless proper to warn the reader that it has not been proved if we can make the correspondence $\mathcal{A} \to \mathcal{M}(\mathcal{A})$ into an equivalence of categories between the category of Banach rings and a suitable category of locally ringed spaces. To get this correspondence we have to study a full subcategory of the category of Banach k-algebras, that of **k-affinoid algebras**.

Definition 2.1.4. *Fix a complete non-Archimedean field $(k, |\cdot|)$ and let $\mathbf{r} := (r_1, \dots, r_n) \in \mathbb{R}^n$. We call **$k$-affinoid algebra** any Banach k-algebra $\mathcal{A}$ isomorphic to $k\{\frac{T_1}{r_1}, \dots, \frac{T_n}{r_n}\}/\mathfrak{I}$ where*

$$k\{\frac{T_1}{r_1}, \dots, \frac{T_n}{r_n}\} := \{\sum_{I=(i_1,\dots,i_n)} a_I T^I : |a_I||\mathbf{r}^I| \to 0 \textit{ when } |I| \to \infty, a_I \in k\}$$

is the algebra of power series which converge in the polydisc of dimension n with multi-radius $\mathbf{r} = (r_1, \ldots, r_n)$ *and* $\mathfrak{I}$ *any ideal of this ring. The Banach norm on* $k\{\frac{T_1}{r_1}, \ldots, \frac{T_n}{r_n}\}$ *is given by the* ***Gauss norm*** $\eta_{0,\mathbf{r}} = \|\sum a_I T^I\|_{sup} := \max |a_I||\mathbf{r}^I|$*. If for every i,* $r_i \in |k^\times|$ *(or equivalently 1 can be taken as* r_i *for every i)* $\mathcal{A}$ *is called* ***strictly k-affinoid****.*

It is important to stress this notation as in classical rigid geometry only strictly k-affinoid algebras are considered whereas from Berkovich point of view every polydisc can be taken as convergence domain.
We are now ready to study the class of analytic spaces obtained applying the $\mathcal{M}(\cdot)$ functor to these algebras.

Definition 2.1.5. *A* ***k-affinoid space*** *is any topological space isomorphic to the analytic spectrum of a k-affinoid algebra.*

In ([4], $\mathcal{S}$1.2) Berkovich shows that these spaces, despite their non-Archimedean base field, have nice topological properties.

Fact 2.1.6. A k-affinoid space is a compact, locally arc connected and separated topological space.

Unsing a universal property, we get a definition that permits to locate the subsets of an affinoid space that behave like affinoid spaces.

Definition 2.1.7. *Let* $\mathcal{A}$ *be a k-affinoid algebra and let V be a subset of* $\mathcal{M}(\mathcal{A})$*. We say that V is an affinoid domain in* $\mathcal{M}(\mathcal{A})$ *if there exists a k-affinoid* $\mathcal{A}$*-algebra* $\mathcal{A}_V$ *for which the following two properties hold*

(1) *The image of* $\mathcal{M}(\mathcal{A}_V)$ *in* $\mathcal{M}(\mathcal{A})$ *coincides with V*

(2) *Every morphism of k-affinoid algebras* $\mathcal{A} \to \mathcal{B}$ *such that the image of* $\mathcal{M}(\mathcal{B})$ *is contained in V factorizes uniquely through* $\mathcal{A} \to \mathcal{A}_V$*.*

2.1.2 General k-analytic spaces, G-topology and regular functions

The affinoid spaces are the local bricks that form general k-analytic spaces in the sense of Berkovich. The process of glueing is explained in [5], section 1.2. In the same paper, in section 1.3, the G-topology is defined over a k-analytic space, as well as a structure sheaf for this Grothendieck topology. We briefly review in this section the results that are necessary for our constructions.
Since affinoid spaces are compact, one can not perform glueing in the same way as scheme theoretically, namely by glueing locally ringed spaces. Berkovich construction takes into account this concern and relies on the notion of **quasi-net** and **affinoid atlas**.

Definition 2.1.8. *A* ***quasi-net*** *on a locally separated topological space X is a collection* τ *of compact separated subsets of X such that each* $x \in X$ *has a neighborhood of the form* $\cup V_i$ *for finitely many* $V_i \in \tau$*, with the property that* $x \in \cap V_i$*.*

The notion of quasi-net is defined in such a way to provide a substitute of the notion of open covering in this setting.

Definition 2.1.9. *A k-**affinoid atlas** on a locally separated topological space X is the datum consisting of a quasi-net τ on X such that*

(1) *for all $U, U' \in \tau$, the collection $\{V \in \tau : V \subseteq U \cap U'\}$ is a quasi-net on $U \cap U'$*

(2) *for each $V \in \tau$, there is a k-affinoid algebra $\mathcal{A}_V$ and a homeomorphism $V \cong \mathcal{M}(\mathcal{A}_V)$ such that if $V' \in \tau$ and $V' \subset V$, then V' is a k-affinoid subdomain of $\mathcal{M}(\mathcal{A}_V)$ with coordinate ring $\mathcal{A}_{V'}$.*

*The triple $(X, \mathcal{A}, \tau)$ is called k-**analytic space**. If all $\mathcal{A}_V$ are strictly k-analytic, then this triple is called a **strictly** k-**analytic space**.*

On these spaces is possible to define a Grothendieck topology that plays the role of the one given by admissible coverings in Tate's approach. The construction relies on the idea of an analytic domain.

Definition 2.1.10. *Let $(X, \mathcal{A}, \tau)$ be a k-analytic space and let $U \subset X$ be a subset of X. We say that a family $\{U_i\} \subset \mathcal{P}(U)$ of subsets of U is a G-**covering**[1] of U if $\{U_i\}$ is a quasi-net on U. A k-**analytic domain** is a subset of X which is G-covered by its affinoid subdomains. The Grothendieck topology on X that has for objects the k-analytic subdomains of X and for coverings of such objects, the G-coverings is called **G-topology**.*

Let $\{V_i\}$ be a finite affinoid covering of a k-affinoid space $X = \mathcal{M}(\mathcal{A})$. Berkovich proved in [5], the following results for k-analytic spaces, using techniques from [10].

Theorem 2.1.11 (Tate acyclicity)**.** For every finite Banach $\mathcal{A}$module M, the Čech complex

$$0 \to M \to \prod_i M \otimes_{\mathcal{A}} \mathcal{A}_{V_i} \to \prod_{i,j} M \otimes_{\mathcal{A}} \mathcal{A}_{V_i \cap V_j} \to \dots$$

is exact and admissible.

As a corollary of this theorem one can extend the correspondence $V \mapsto \mathcal{A}_V$ in a sheaf over X_G. More precisely, for every k-analytic domain $V \subset X$ which is G-covered by $\{V_i \to V\}$, one defines the ring $\mathcal{O}_X(V) := \mathrm{Ker}\ (\prod_i \mathcal{A}_{V_i} \to \prod_{i,j} \mathcal{A}_{V_i \cap V_j})$. Thanks to Tate acyclicity's theorem, when V is an affinoid domain one gets $\mathcal{O}_X(V) = \mathcal{A}_V$ and, more in general, $\mathcal{O}_X(V)$ does not depend on the G-covering.

Remark 2.1.12. This definition agrees, in parallelism with the notion of regular function in algebraic geometry, with the one given by $\mathcal{O}_X(V) = \mathrm{Hom}(V, \mathbb{A}_k^{1,an})$. In fact one can show that

[1] Here the G stands for "Grothendieck" as pointed out in [22]. There is no connection with the action of a finite group G appearing in chapter 3 of this thesis.

the sequence of sets

$$0 \to \mathrm{Hom}(V, \mathbb{A}_k^{1,an}) \to \prod_i \mathrm{Hom}(V_i, \mathbb{A}_k^{1,an}) \rightrightarrows \prod_{i,j} \mathrm{Hom}(V_i \cap V_j, \mathbb{A}_k^{1,an})$$

is exact.

If we denote by $|X|$ the site associated to the topology of the analytic spectrum and X_G the one associated to the G-topology, we can canonically construct a morphism of sites

$$X_G \to |X|.$$

In fact one can show that every open subset of X is a union of a locally finite family of k-affinoid domains. As a result, the structure sheaf for the G-topology induces by push-forward a structure sheaf for the Berkovich topology on any k-analytic space.

Fact 2.1.13. Let the valuation over k be nontrivial. Then the law $\mathcal{A} \to \mathcal{M}(\mathcal{A})$ can be extended to a controvariant functor realizing equivalence of categories between k-affinoid algebras and k-affinoid spaces.

Remark 2.1.14. Any spectrum of a general Banach ring can be endowed with a structure of locally ringed space, by taking the uniform limit over rational functions (see [4], Definition 1.5.3). Nevertheless this notion does not coincides with the one just introduced.

By glueing affinoid spaces, one can construct k-analytic spaces that are not affinoid, as shown in the following examples.

Example 2.1.15. *Let $D = \cup_{0<\rho<1}\mathcal{M}(k\{\rho^{-1}T\})$. For $\rho_2 > \rho_1$, the inclusions $\mathcal{M}(k\{\rho_1^{-1}T\}) \hookrightarrow \mathcal{M}(k\{\rho_2^{-1}T\})$ define a k-affinoid atlas of D, which is then a k-analytic space. In the same way one shows that $A = \cup_{\ell<\rho<1}\mathcal{M}(k\{\rho T^{-1}, T\})$ is a k-analytic space. We call **open disc** any k-analytic space isomorphic to D. We call **open annulus of length** ℓ any k-analytic space isomorphic to A.*

Example 2.1.16. *One can construct the projective line $\mathbb{P}_k^{1,an}$ by glueing $\mathcal{M}(k\{T\})$ and $\mathcal{M}(k\{T^{-1}\})$ along $\mathcal{M}(k\{T, T^{-1}\})$. Notice that the glueing of any other pair of strictly k-affinoid closed discs with the same choice of orientation is isomorphic to $\mathbb{P}_k^{1,an}$.*

2.2 Reduction techniques

We are particularly interested in using Berkovich spaces to understand the interplay between positive characteristic and characteristic zero. The natural techniques arising in this context, relate non-Archimedean analytic spaces with schemes in positive characteristic. From a global point of view, this relation is obtained through the use of formal geometry, from a local point of view, with the theory of reduction of analytic germs, developed by Temkin in [57]. In this section we expose both these approaches, and how they allow to read on an analytic space the properties of its reduction. Let us mention that Temkin theory has been refined by the same author in [58] with the use of graded commutative algebra, to include also the case of non strict k-analytic spaces. We will never work in this situation, therefore we chose to expose only the results of [57].

In [51], Raynaud suggested to study rigid spaces in the framework of formal geometry. His intuition led to several applications, and it can be adapted very well to Berkovich spaces. Moreover, it permits to link the analytic theory over k with the algebraic theory over $\tilde{k}$, throughout the notion of **generic fiber** and **special fiber** of a **formal model**. The fact that we can keep track of the reduction of a model on the generic fiber, is a fundamental tool to establish our results on lifting local actions.

Raynaud's approach extends also to Berkovich spaces: to a strictly k-affinoid domain $\mathcal{A}$, we can associate $M_0(\mathcal{A}) := \{x \in \mathcal{M}(\mathcal{A}) \mid [\mathcal{H}(x) : k] < \infty\}$ which is a dense subset of $\mathcal{M}(\mathcal{A})$ in bijection with $\mathrm{Spm}(\mathcal{A})$ ([5], section 1.6). It is a classical result ([5], $\mathcal{S}$1.6.1) that the correspondence $\mathcal{M}(\mathcal{A}) \to M_0(\mathcal{A})$ extends to an equivalence of categories between strictly k-affinoid spaces and rigid k-affinoid spaces with an admissible covering of finite type. With this equivalence it is easy to see that the Grothendieck topology of rigid admissible coverings can be lifted to the G-topology on any strict k-analytic space. This permits to subsume the results of Tate theory in the framework of Berkovich theory, beginning with the fruitful description provided by methods in formal geometry.

2.2.1 The Berkovich generic fiber of a formal scheme

We recall here some definition and results on the relations between formal and analytic geometry, that we will use in the following chapters. Let X be a noetherian scheme over k^o. In algebraic geometry we encounter its special fiber X_s and its generic fiber X_η, which are naturally defined by the cartesian diagrams below:

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\tilde{k}) \arrow[r] & \mathrm{Spec}(k^o) \end{tikzcd} \qquad \begin{tikzcd} X_\eta \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(k^o) \end{tikzcd}$$

In the first diagram $\mathrm{Spec}(\tilde{k}) \hookrightarrow \mathrm{Spec}(k^o)$ is a closed immersion, hence $X_s \hookrightarrow X$ is also a closed immersion. Let then $\widehat{X}$ be the formal completion of X along the ideal sheaf corresponding to X_s in X. It is nowadays classical to consider the generic fiber of $\widehat{X}$ as an analytic space.

Formal models and reduction in Berkovich theory

Let $\mathcal{X} = \mathrm{Spf}(\mathcal{A})$ be a flat formal scheme of finite presentation over k^o. Define $\mathcal{X}_\eta = \mathcal{M}(\mathcal{A}\widehat{\otimes}_{k^o} k)$ and consider the subset $\mathcal{A}^o = \{f \in \mathcal{A} \mid \rho(f) \leq 1\}$. It is a subring of $\mathcal{A}$ having an interesting prime ideal $\mathcal{A}^{oo} = \{f \in \mathcal{A} \mid \rho(f) < 1\}$. We can then consider the $\tilde{k}$-algebra $\widetilde{\mathcal{A}} = \mathcal{A}^o/\mathcal{A}^{oo}$ and the scheme of finite type $\widetilde{\mathcal{X}} = \mathrm{Spec}(\tilde{\mathcal{A}})$.

Definition 2.2.1. *We call $\widetilde{\mathcal{A}}$ the* ***residue algebra*** *of $\mathcal{A}$.*

Whenever $\psi : \mathcal{A} \to \mathcal{B}$ is a continuous homomorphism of commutative Banach algebras, there is an induced morphism $\psi^o : \mathcal{A}^o \to \mathcal{B}^o$ such that $\psi^o(\mathcal{A}^{oo}) \subset \mathcal{B}^{oo}$. Then the morphism $\widetilde{\psi} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is also defined naturally. Recall from 2.1.1 that any point $x \in \mathcal{M}(\mathcal{A})$ can be defined also by a morphism $\chi_x : \mathcal{A} \to \mathcal{H}(x)$. Then there is a homomorphism of $\tilde{k}$ algebras $\widetilde{\chi}_x : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{H}(x)}$.

Definition 2.2.2. *Let $\mathcal{X} = \mathrm{Spf}(A)$ be as above. The* ***reduction map*** *is the set theoretic morphism*

$$\begin{array}{rcl} \tilde{r} : \mathcal{X}_\eta & \to & \widetilde{\mathcal{X}} \\ x & \mapsto & \mathrm{Ker}(\widetilde{\chi}_x). \end{array}$$

Let $\mathcal{A}$ be a strictly k-affinoid algebra. The reduction map enjoys the following properties:

- The reduction map $\tilde{r}$ is anticontinuous: the preimage of an open (resp. closed) subset of $\widetilde{X}$ is a closed (resp. open) subset of $\mathcal{X}_\eta$
- The reduction map $\tilde{r}$ is surjective
- The preimage $\tilde{r}^{-1}(\tilde{x})$ of the generic point of an irreducible component of $\widetilde{X}$ consists of one point $x \in \mathcal{X}_\eta$ and one has $\widetilde{k(\tilde{x})} \cong \widetilde{\mathcal{H}(x)}$ where $\widetilde{k(\tilde{x})} = \mathrm{Frac}(\widetilde{\mathcal{A}}/\tilde{r}(x))$.

More generally, for any $\mathcal{X}$ formal scheme separated and locally of finite presentation over $\mathrm{Spf}(R)$ there is a reduction map $\tilde{r} : \mathcal{X}_\eta \to \mathcal{X}_s$.

Formal blowing-up

The relationship of a formal scheme with its special and generic fiber is investigated throughout the notion of **admissible blowing-up**.

Definition 2.2.3. *Let $\mathcal{X}$ be an admissible formal scheme having $\mathcal{I}$ as ideal of definition. Let $\mathcal{J}$ be a coherent $\mathcal{O}_\mathcal{X}$-module of open ideals. Then there is a map*

$$\varphi : \mathcal{X}' = \varinjlim_{\lambda} \mathrm{Proj}\Big(\bigoplus_{n=0}^{\infty} \mathcal{J}^n \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_\mathcal{X}/\mathcal{I}^{\lambda+1}\Big) \longrightarrow \mathcal{X}$$

which is called formal blowing-up of $\mathcal{J}$ on $\mathcal{X}$.

We have the following properties for a formal blowing-up.

Fact 2.2.4. Let $\varphi : \mathcal{X}' \to \mathcal{X}$ be the formal blowing up of $\mathcal{J}$ on $\mathcal{X}$. Then

- $\mathcal{X}'$ is an admissible formal S-scheme on which the ideal $\mathcal{J}\mathcal{O}'_\mathcal{X}$ is invertible
- φ commutes with basechange
- Whenever $\psi : \mathcal{Z} \to \mathcal{X}$ is a morphism of formal S-schemes such that $\mathcal{J}\mathcal{O}_\mathcal{Z}$ is invertible on $\mathcal{Z}$, then there is a unique S-morphism $\psi' : \mathcal{Z} \to \mathcal{X}'$ such that $\psi = \varphi \circ \psi'$.

By functoriality of the previous constructions, an automorphism of R[[T]] induces also automorphisms over special and generic fibers.

Definition 2.2.5. *If ϕ is an endomorphism of $\mathbb{D}$, we obtain by tensorization a continue function on the generic fiber (ϕ_η) and an endomorphism on the special fiber (ϕ_s).*

2.2.2 Boundaries

A very useful concept when dealing with Berkovich spaces is that of Shilov boundary. This notion, borrowed from the field of functional analysis, permits to define objects that blend the combinatorial aspects of the theory together with the spectral analytic and the valuation theoretical ones.

Definition 2.2.6. *Let $(\mathcal{A}, \|\cdot\|)$ be a Banach k-algebra. The **spectral norm** of an element $f \in \mathcal{A}$ is the real number defined by the formula $\rho(f) = \lim_{n\to\infty} \|f^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|f^n\|^{\frac{1}{n}}$.*

Remark 2.2.7. The equality $\lim_{n\to\infty} \|f^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|f^n\|^{\frac{1}{n}}$ is true by monotony of the sequence $\|f^n\|^{\frac{1}{n}}$, which comes from the submultiplicativity: $\|f^n\| \leq \|f\|^n$ implies $\|f^n\|^{\frac{1}{n}} \leq \|f\|$ and in the same way can be shown the fact that the sequence $\|f^n\|^{\frac{1}{n}}$ is decreasing.

Definition 2.2.8. *Let $(\mathcal{A}, \|\cdot\|)$ be a Banach k-algebra and we set $X = \mathcal{M}(\mathcal{A})$. We say that a closed subset Γ of X is a **functional boundary** of X if it verifies*

$$\forall f \in \mathcal{A}, \sup_{x\in X}\{|f(x)|\} = \sup_{x\in\Gamma}\{|f(x)|\} \in \mathbb{R}.$$

*We call **Shilov boundary**, noted $\partial_S X$, the minimal functional boundary, when it is unique.*

What relates the notion of spectral norm and the notion of boundary is the following (see [4], Theorem 1.3.1):

Fact 2.2.9. Let $\mathcal{A}$ be a Banach ring, $f \in \mathcal{A}$ and $X = \mathcal{M}(\mathcal{A})$. Then

$$\rho(f) = \sup_{x\in X} |f(x)|.$$

Moreover, the following facts about the Shilov boundary of an affinoid space are also proved by Berkovich in [4]:

Fact 2.2.10. Let $X = \mathcal{M}(\mathcal{A})$ be a k-affinoid space.

i) The Shilov Boundary $\partial_S X$ exists and it is finite

ii) For every $x_0 \in \partial_S X$ then for every open neighborhood U of x there is $f \in \mathcal{A}$ and $\varepsilon > 0$ such that $|f(x_0)| = \rho(f)$ and $\{x \in X : |f(x)| > \rho(f) - \varepsilon\} \subset U$

iii) Let $V \subset X$ be an affinoid domain. Then one has $\partial_S X \cap V \subset \partial_S(\mathcal{M}(\mathcal{A}_V)) \subset \partial(X \setminus V) \cup (\partial_S \cap V)$, where ∂ is the boundary in the sense of topological spaces.

Finally, let us mention a result that characterizes the boundary in terms of reduction :

Proposition 2.2.11. Let $\mathcal{X}$ be a flat formal scheme of finite presentation over k^o. Then

i) Let Ξ be the set of generic points of the special fiber $\mathcal{X}_s$. Then $\partial_S \mathcal{X}_\eta = \{\tilde{r}^{-1}(\xi) : \xi \in \Xi\}$.

ii) In the affine case, where $\mathcal{X}_s = \tilde{\mathrm{A}}$, the topological interior $\mathring{\mathcal{X}_\eta}$ coincides with $\tilde{r}^{-1}(\mathrm{Spm}(\tilde{A}))$

The cases of discs and annuli provide the first interesting computations of Shilov boundaries.

Example 2.2.12 (Discs and Annuli)**.**
Let $\mathcal{A} = k\{t\}$. Then the Shilov boundary is the singleton $\{\eta_{0,1}\}$. To prove it, it suffices to remark that the Banach norm over $\mathcal{A}$ is the Gauss norm.
Let now $a \in k^o$ be a nonzero element and consider the Banach k-algebra $\mathcal{A} = k\{t, s\}/(ts - a)$. Then the Shilov boundary is $\{\eta_{0,1}, \eta_{0,|a|}\}$, which is just the Gauss point when $|a| = 1$.

2.2.3 Reduction *à la Temkin*

In this section, we sketch the construction of a functor of reduction of germs of analytic spaces, following the original Temkin article [57]. Let $|k^\times| \neq \{1\}$ and let X be a separated k-analytic space. We explain how to associate to every $x \in X$ a quasi-compact open subset in the Riemann-Zariski space of $\mathcal{H}(x)$ over $\tilde{k}$.

Riemann-Zariski spaces

Let us start with the definition of Riemann-Zariski spaces.

Definition 2.2.13. *Let L/K be a finite type extension of fields. The* ***Riemann-Zariski space*** *of the extension L/K, noted $\mathbf{P}_{L/K}$, is the set of (equivalence classes of) valuations of L that are trivial over K. The* ***affine open subsets*** *of $\mathbf{P}_{L/K}$ are those of the form*

$$\mathbf{P}_{L/K}\{f_1, \ldots, f_n\} := \{v \in \mathbf{P}_{L/K} : f_i \in \mathcal{O}_v \ \forall\, i = 1, \ldots, n\}$$

for a choice of the n-uple $\{f_1, \ldots, f_n\} \in L^n$. The space $\mathbf{P}_{L/K}$ is endowed with the topology having the open affine subsets as a basis.

The space $\mathbf{P}_{L/K}$ is turned into a locally ringed space by defining a sheaf of rings $\mathcal{O}_{\mathbf{P}_{L/K}}$ in such a way that $\mathcal{O}_{\mathbf{P}_{L/K}}(U) = \{f : v(f) = 0 \ \forall\, v \in U\} \subset L$. A vector bundle over $\mathbf{P}_{L/K}$ is then simply a locally free $\mathcal{O}_{\mathbf{P}_{L/K}}$-module of finite type. Its rank as a module is a constant function and is then well defined: a rank one vector bundle is called line bundle.

Vector bundles of rank r are uniquely determined by 1-cocycles with values in $\mathrm{GL}_r(\mathcal{O}_{\mathbf{P}_{L/K}})$; a 1-cocycle on $\mathbf{P}_{L/K}$ with values in $\mathrm{GL}_r(\mathcal{O}_{\mathbf{P}_{L/K}})$ defines the trivial bundle if and only if it is a cobord.

The reduction functor

The category *Germs*, introduced by Berkovich in [5], is the localization of the category of punctual k-analytic spaces with respect to the system of morphisms $(X,x) \to (Y,y)$ that induce an isomorphism between X and an open neighborhood of y in Y. The category $bir_{\tilde{k}}$ has as objects the triples (C,ℓ,f) where C is a connected quasi-compact and quasi-separated topological space, ℓ is a field extension of finite type of $\tilde{k}$ and f is a local homeomorphism from C to $\mathbf{P}_{\ell/\tilde{k}}$.
The reduction is defined through the introduction of two other categories: $Var_{\tilde{k}}$, the category of triples (V,ℓ,η), with V integral scheme of finite type over $\tilde{k}$, $\ell/\tilde{k}$ extension of finite type and $\eta : \mathrm{Spec}(\ell) \to V$ a $\tilde{k}$ morphism having as image the generic point of V. To any such triple, define the topological space $\mathrm{Val}(V)$ as the set of pairs (v,ϕ) with $v \in \mathbf{P}_{\ell/\tilde{k}}$ and $\phi : \mathrm{Spec}(\mathcal{O}_v) \to V$ morphism compatible with η, endowed with the weakest topology that makes the maps $\alpha : \mathrm{Val}(V) \to V$ and $\beta : \mathrm{Val}(V) \to \mathbf{P}_{\ell/\tilde{k}}$ given by $\alpha(v,\phi) = \phi$ and $\beta(v,\phi) = v$ continuous. One can show that β is a local homeomorphism.
The reduction is then constructed for the category Adm_{k^o} of pointed quasi-compact admissible formal schemes over k^o, by defining the correspondence

$$(\mathcal{X},x) \mapsto \mathrm{red}(\mathcal{X},x) = (\mathrm{Val}(V_{\tilde{x}}), \widetilde{\mathcal{H}(x)}, \beta) \in bir_{\tilde{k}},$$

where $V_{\tilde{x}}$ is the closure of $\tilde{x}$ in $\mathcal{X}_s$.

Definition 2.2.14. *The* ***reduction functor*** *is the functor*

$$\begin{array}{ccc} Germs & \to & bir_{\tilde{k}} \\ (X,x) & \mapsto & \widetilde{(X,x)} \end{array}$$

obtained by the localization of the functor $\mathrm{red} : Adm_{k^o} \to bir_{\tilde{k}}$ *with respect to formal admissible blowing-ups.*

The arrow $(X,x) \to \widetilde{(X,x)}$ realizes a bijection between analytic domains in (X,x) and nonempty quasi-compact open subsets of $\widetilde{(X,x)}$.

2.3 Curves

From now on, we concentrate our attention on the case of curves. As in other geometries, these are easier to describe than objects of higher dimension.

Definition 2.3.1. *Let k be a non-Archimedean complete field. A* ***k-analytic curve*** *is a k-analytic space, separated and purely of dimension 1.*

2.3.1 Types of points

Let C be a k-analytic curve and $x \in C$ be a point having $\mathcal{H}(x)$ as residue field. Then, following the properties of $\mathcal{H}(x)$ we have four exclusive different possibilities.

I. The field $\mathcal{H}(x)$ is the completion of an algebraic extension of k; we say in this case that x is of **type 1**;

II. The field $\widetilde{\mathcal{H}(x)}$ is of finite type and transcendence degree 1 over $\tilde{k}$ and $|\mathcal{H}(x)^\times|/|k^\times|$ is finite; we say in this case that x is of **type 2**;

III. The field $\widetilde{\mathcal{H}(x)}$ is a finite extension of $\tilde{k}$ and $|\mathcal{H}(x)^\times|/|k^\times|$ is of rank one; we say in this case that x is of **type 3**;

IV. The field $\mathcal{H}(x)$ is not as in the first case, but it admits an isometric embedding in an immediate extension of $\widehat{k^{alg}}$; we say in this last case that x is of **type 4**.

The affine line

Let $\mathbb{A}_k^{1,an}$ be the analytification of the scheme theoretic affine line $\mathbb{A}_k^1$. It can be defined as a space of multiplicative seminorms as follows:

$$\mathbb{A}_k^{1,an} = \{x : k[T] \to \mathbb{R}_+ \text{ multiplicative seminorm} : x_{|k} = |\cdot|_k\}.$$

As before we write $|f(x)|$ for the value of f at x. For every $a \in k$ and $\rho \in \mathbb{R}_+$ write $\eta_{a,\rho}$ for the element of $\mathbb{A}_k^{1,an}$ defined by

$$|f(\eta_{a,\rho})| = \sup_{b \in D(a,\rho)} |f(b)|.$$

If the field k is algebraically closed there is an explicit description of these points in terms of valuations of the form $\eta_{a,\rho}$.

Proposition 2.3.2. Let k be an algebraically closed field and $x \in \mathbb{A}_k^{1,an}$. Then

I. The point x is of type 1 if and only if $x = \eta_{a,0}$ for some $a \in k$;

II. The point x is of type 2 if and only if $x = \eta_{a,\rho}$ for some $a \in k$ and $0 < \rho \in |k^\times|$;

III. The point x is of type 3 if and only if $x = \eta_{a,\rho}$ for some $a \in k$ and $\rho \notin |k^\times|$;

IV. The point x is of type 4 if and only if $x = \inf_i(\eta_{a_i,\rho_i})$ for a sequence of discs $D(a_i, \rho_i)$ in k such that $D(a_{i+1}, \rho_{i+1}) \subset D(a_i, \rho_i)$ and $\cap_i D(a_i, \rho_i) = \emptyset$. In particular, when k is maximally complete, there are no points of type 4 in $\mathbb{A}_k^{1,an}$.

2.3.2 Semi-stable reduction

In studying covering of curves in characteristic 0 arising as lifting of covering in positive characteristic, one has to check that the special fibers are smooth. Nevertheless, not every model of a k-analytic curve has good reduction on the special fiber. The best one can have is to have semi-stable reduction, after finite extension of k. One interesting feature of k-analytic curves with semi-stable reduction, is that their models over k^o can be parametrized in terms of points of the k-analytic curve itself.

Definition 2.3.3. *Let C be a smooth k-analytic curve. We say that C has* ***good reduction*** *if there exists $\mathfrak{C}$ a formal model of C over $\mathrm{Spf}(k^o)$ such that the special fiber $\mathfrak{C}_s$ is smooth. We say that C has* ***semi-stable reduction*** *if there exists $\mathfrak{C}$ a formal model of C over $\mathrm{Spf}(k^o)$ such that the special fiber $\mathfrak{C}_s$ has at most ordinary double points as singularities. In this case, when $\mathfrak{C}_s$ has only smooth irreducible components, C is said to have* ***simple semi-stable reduction****.*

Let C be a smooth k-analytic curve with semi-stable reduction and $\mathfrak{C}$ a formal model of C with semi-stable special fiber $\mathfrak{C}_s$. Let $\tilde{r} : C \to \mathfrak{C}_s$ be the reduction map. The set $\tilde{r}^{-1}(x)$ is a k-analytic subspace of C, and it is called **formal fiber** of C at x. Using results of Bosch and Lütkebohmert on semi-stable reduction for rigid analytic varieties, Berkovich ([4], Theorem 4.3.1.) proved the following

Proposition 2.3.4. Let $x \in \mathfrak{C}_s$ be a point of the special fiber of C.

i) If x is the generic point of an irreducible component C_i of the special fiber, then there is a unique point in the formal fiber at x, $x_i = \tilde{r}^{-1}(x)$

ii) If x is a smooth closed point of the special fiber of X. Then its formal fiber is an open disc $D \in C$ such that the relative boundary $\bar{D} \setminus D$ is equal to x_i.

iii) If $x \in X_s$ be a singular ordinary double point belonging to the components C_i and C_j. Then its formal fiber is an open annulus A. If $i \neq j$, then the relative boundary $\bar{A} \setminus A$ is equal to $\{x_i, x_j\}$.

To each model $\mathfrak{C}$ of C with semi-stable reduction is associated the **reduction graph** $\Gamma(\mathfrak{C}_s)$ of its special fiber. This is the datum of a vertex for each irreducible component of $\mathfrak{C}_s$ and two vertices are joined by an edge if the associated components meet in an ordinary double point. In particular, when the double point is given by a self intersection it gives rises to a loop in the reduction graph. It is sometimes referred in the literature as **dual graph**. Let $\{v_i\}$ be the set of vertices of $\Gamma(\mathfrak{C}_s)$. We will denote by C_i the irreducible component of $\mathfrak{C}_s$ that corresponds to the vertex v_i in the construction. The following result is classical (proved in [4]):

Theorem 2.3.5. Let C be a smooth k-analytic curve with semi-stable reduction and $\mathfrak{C}$ a formal model of C with semi-stable special fiber $\mathfrak{C}_s$ Then the dual graph $\Gamma(\mathfrak{C}_s)$ is canonically embedded in C.

Proof. Let v_i be a vertex of $\Gamma(\mathfrak{C}_s)$, and let ξ_i be the generic point of the irreducible component C_i. Then the embedding sends v_i to $\tilde{r}^{-1}(\xi_i)$. By point i) of Proposition 2.3.4, this assignment is well defined. Let e be the edge corresponding to an intersection of C_i and C_j in a double point $x \in \mathfrak{C}_s$. Then e is sent to the unique path, in the annulus $\tilde{r}^{-1}(x)$, joining ξ_i and ξ_j. By point iii) of Proposition 2.3.4, this assignment is also well defined. □

The dual graph has also a canonical **metric structure**: let e be an edge of $\Gamma(\mathfrak{C}_s)$. It corresponds to a double point on $\mathfrak{C}_s$. On $\mathfrak{C}$, the local ring at this point is of the form $k^o[[X,Y]]/XY - z$ with $z \in k^o$. One then sets the length of e to be the (k-adic) valuation of z.

2.4 Functions on analytic curves

One of the features that is peculiar of Berkovich spaces with respect to other non-Archimedean analytic theories, is the natural presence of combinatorial structures. This is strictly related to the theory of regular functions and, more in general, sheaves of modules. We introduce in this section the notion of skeleton and the one of vector bundle on a k-analytic curve, pointing out the relations between these two objects.

2.4.1 The skeleton of a curve

Let C be a smooth, proper, connected k-analytic curve. If $\mathfrak{C}$ is any k^o-model with semi-stable reduction for C, there is an associated subset $\mathfrak{S} = \mathfrak{S}_{\mathfrak{C}}$ of C called the **skeleton** of $\mathfrak{C}$. Berkovich proved that the skeleton of $\mathfrak{C}$ is homeomorphic to the dual graph $\Gamma(\mathfrak{C}_s)$. It can therefore inherit a metric structure: we may think at $\mathfrak{S}$ as a finite graph in which each edge is identified to an Euclidean line segment of some length, possibly infinite. By results of the previous section on dual graphs, $\mathfrak{S}$ is a subset of C. Berkovich proved also that $\mathfrak{S}_{\mathfrak{C}}$ is a deformation retract of C.

Definition 2.4.1. *Let X be a k-affinoid space. We say that it is **potentially isomorphic to the unit disc** if there exists a finite separable extension k' of k such that the k'-affinoid space $X \otimes_k k'$ it is isomorphic to a finite sum of copies of the closed unit disc over k'.*

Lemma 2.4.2. A connected k-affinoid domain $V \subset \mathbb{A}_k^{1,an}$ is potentially isomorphic to the unit disc if and only if its Shilov boundary is a singleton, $\partial_S V = \{\xi\}$, such that V coincides with the complement of the unique unbounded connected component of $\mathbb{A}_k^{1,an} \setminus \{\xi\}$.

Definition 2.4.3. *Let $\mathfrak{C}$ be a k^o-curve with simple semi-stable reduction, and let $O(\mathfrak{C})$ be the subset of $\mathfrak{C}_\eta - \partial_S(\mathfrak{C}_\eta)$ whose elements are points having an affinoid neighborhood potentially isomorphic to the unit disc. The **skeleton** of $\mathfrak{C}$ is defined to be the complement $\mathfrak{S}_{\mathfrak{C}} = O(\mathfrak{C})^c$.*

Remark 2.4.4. The definition of the skeleton by Berkovich is given for smooth k-analytic curves admitting distinguished formal coverings with semi-stable reduction in section 4.3 of [4], and generalized for nondegenerate pluri-stable formal schemes in chapter 4 of [6]. The characterization above for models with simple semi-stable reduction is due to Thuillier ([59]) and fits better to our purposes.

In [59], the following properties are shown for the skeleton $\mathfrak{S}_{\mathfrak{C}}$.

Theorem 2.4.5. Let $\mathfrak{S}_{\mathfrak{C}}$ be the skeleton of a k^o-curve with simple semi-stable reduction. Then

- the topological space underlying $\mathfrak{S}_{\mathfrak{C}}$ is a closed subset of $\mathfrak{C}_\eta$ and there exists a retraction $\tau_{\mathfrak{C}} : \mathfrak{C}_\eta \to \mathfrak{S}_{\mathfrak{C}}$;
- for every open subset $\mathfrak{U}$ of $\mathfrak{C}$ then $\mathfrak{U}_\eta$ is a polyhedral domain inside $\mathfrak{S}_{\mathfrak{C}}$ and the embedding $\mathfrak{U}_\eta \hookrightarrow \mathfrak{C}_\eta$ induces an isomorphism $\mathfrak{S}_{\mathfrak{U}} \cong \mathfrak{S}_{\mathfrak{C}} \cap \mathfrak{U}_\eta$.

- For every étale morphism $\mathfrak{C} \to \mathfrak{C}'$ of k^o-curves with simple semi-stable reduction, there is an induced morphism of polyhedra $\mathfrak{S}_{\mathfrak{C}} \to \mathfrak{S}_{\mathfrak{C}'}$ such that the diagram

$$\begin{tikzcd} \mathfrak{C}_\eta \arrow[r, "\tau_{\mathfrak{C}}"] \arrow[d] & \mathfrak{S}_{\mathfrak{C}} \arrow[d] \\ \mathfrak{C}'_\eta \arrow[r, "\tau_{\mathfrak{C}'}"] & \mathfrak{S}_{\mathfrak{C}'} \end{tikzcd}$$

commutes.

Example 2.4.6. *Let $\widehat{\mathbb{A}}^1_R := \mathrm{Spf}(R\{T\})$ be the formal affine line over R. Its generic fiber is already a closed unit disc then its skeleton is just its boundary $S(\widehat{\mathbb{A}}^1_R) = \{\eta_{0,1}\}$.*

Example 2.4.7. *Let us consider the formal annulus $\mathcal{C}_e = \mathrm{Spf}(\frac{R\{S,T\}}{ST-\pi^e})$. Its generic fiber is an annulus of depth e over K having as a boundary the set $\{\eta_{0,1}, \eta_{0,|\pi^e|}\}$. We claim that the skeleton of C is exactly the (unique) segment joining these two points, namely the image of the map*

$$\begin{array}{ccc} [|\pi^e|, 1] & \to & \mathbb{D}_\eta \\ r & \mapsto & \eta_{0,r}. \end{array}$$

Take in fact an affinoid domain potentially isomorphic to the unit disc V and show that it can not contain any point of this segment: the complementary $C - V$ is in fact arcwise connected and it contains 0. We have also $\{\eta_{0,1}\} \subset C - V$, otherwise V would be of radius one, contained in the unit disc and therefore containing 0. Then $C - V$ contains the segment $\eta_{0,|a|} \to \eta_{0,1}$ for every V.
Conversely every other point is contained in such a V. We can in fact write this point as $\eta_{b,\rho}$ with $|\pi^e| \leq |T(b)| \leq 1$ and $\rho < |b|$. Then there exists an $\rho < \varepsilon < |b|$ such that the affinoid domain $\{x : 0 \leq |T(x)| \leq \varepsilon\}$ is a potentially isomorphic to the unit disc neighborhood of $\eta_{b,\rho}$. Hence the claim is proved and $S(\mathcal{C}_e)$ is given by a polytope consisting of a single edge.

2.4.2 Vector bundles

We define and discuss some properties of vector bundles on k-analytic spaces, in particular in the case of curves.

Definition 2.4.8. *Let X be a k-analytic space. A vector bundle on X is a locally free of finite rank sheaf of modules on the ringed site X_G.*

From the discussion of Section 2.1.2, this defines by restriction to the topological site $|X|$ a sheaf of $\mathcal{O}_X$ modules, that may not be locally free. It is so in the case where X is good (i.e. every $x \in X$ has a neighborhood which is an affinoid domain).

Example 2.4.9 (Differential forms)**.** *Modules of differential forms for affinoid algebras can be defined with derivations: whenever $\mathcal{A}$ is an affinoid algebra over K, then $\Omega^1_{\mathcal{A}}$ is the universal object representing the functor $\mathrm{Der}(\mathcal{A}, \cdot)$. This definition extends by glueing to define a vector bundle over any k-analytic space X.*

Metrics on vector bundles and formal metrics

We briefly describe the notion of metric for a vector bundle on a Berkovich curve. A more detailed description is found in section 6.2 of [15], or chapter 7 of [30] which are also our main references. Let X be a k-analytic vector space, and E a vector bundle on X. A continuous metric over E is a continuous map from the total space of E to $\mathbb{R}_+$ which is an ultrametric norm on every fiber. This amounts to the following less concise but more standard definition.

Definition 2.4.10. *A* ***continuous metric*** *over E is the datum, for every U analytic domain in X, of a continuous map*

$$\begin{array}{rcl} \|\cdot\| : \Gamma(U,E) & \to & \mathscr{C}^0(U,\mathbb{R}_+) \\ s & \mapsto & \|s\| \end{array}$$

such that

- *for every analytic domain U of X and sections $f \in \Gamma(U,\mathcal{O}_X)$ and $s \in \Gamma(U,E)$, one has $\|fs\| = |f|\|s\|$;*
- *for every analytic domain U of X and sections $s_1, s_2 \in \Gamma(U,E)$, one has $\|s_1+s_2\| \leq \max(\|s_1\|,\|s_2\|)$;*
- *for every analytic domain U of X, section $s \in \Gamma(U,E)$ and point $x \in U$,*

$$\|s\|(x) = 0 \quad \textit{if and only if} \quad s(x) = 0.$$

An important class of metrics on a given vector bundle, are the formal metric, used in chapter 5 to define the sheaf of deformations. Let E a vector bundle over a k-analytic space X, and let $(\mathcal{X},\mathcal{E})$ be a formal model of the couple (X,E). By this, we mean that $\mathcal{X}$ is a model of X over k^o and $\mathcal{E}$ is a formal vector bundle over $\mathcal{X}$ with an isomorphism $\mathcal{E}_\eta \cong E$. There is a canonical way to associate to $(\mathcal{X},\mathcal{E})$ a continuous metric on E, which goes as follows. For every formal covering $\{\mathcal{U}_i\}$ such that $\mathcal{E}_{|\mathcal{U}_i}$ is triviali, we have that i, $\mathcal{U}_{i,\eta}$ is an analytic domain in X. If $(e_1,\ldots,e_n)$ is a trivialization of $\mathcal{E}$ on $\mathcal{U}_i$, for every section $s \in E(\mathcal{U}_{i,\eta})$ there exists a unique n-uple $(f_1,\ldots,f_n)$ yielding $s = \sum f_i e_i$. The rule $\|s\| = \max(|f_1|,\ldots,|f_n|)$ defines a metric on E, that does not depend on the trivialization. This is called the **formal metric** associated to the model $(\mathcal{X},\mathcal{E})$. When E is a line bundle, this is a smooth metric.

Chapter 3

Local actions on curves

Thorough this chapter, let $\tilde{K}$ be an algebraically closed field of characteristic $p > 0$. For every complete discrete valuation ring R of characteristic zero, having $\tilde{K}$ as residue field, we denote by K its fraction field and by ϖ a uniformizer of R. Such R is necessarily a finite extension of complete DVR of the ring of Witt vectors $W(\tilde{K})$. In this chapter we introduce the local lifting problem for action of finite groups on curves, briefly explaining results and techniques involved, open problems, and related perspectives. We start by discussing the problem of lifting to characteristic zero automorphism of curves, and we relate it with its local analogue, the problem of lifting to characteristic zero automorphisms of $\tilde{K}[[t]]$, the $\tilde{K}$-algebra of formal power series with coefficients in $\tilde{K}$. The main strategy to prove results on liftings such automorphisms is to study the behavior of liftings in characteristic zero. One then studies algebraic and arithmetic properties of automorphisms of the ring $R[[T]]$ and their deformations, with the purpose of obtain a description of their reductions. A particular care is needed in manipulating this objects. For instance, the lifting problem asks to maintain the order of the automorphisms when reduced in characteristic p and this is not always the case. We introduce the main techniques to deal with such problems and the results that could be get in this way. The main object serving this scope is called Hurwitz tree and has two main descriptions. The first one is introduced by Henrio in [31] and deals only with actions of $\mathbb{Z}/p\mathbb{Z}$, the second one is defined by Brewis and Wewers in [14] and is associated to any local action of a finite group, but the results that one can get are less general. The reason is that in the Hurwitz tree for $\mathbb{Z}/p\mathbb{Z}$ there are some differential forms that make possible to have a classification of automorphisms in characteristic 0 with fixed points. These differential form are not available for any G, even if we have a generalization by Bouw and Wewers ([11]), in the very particular case of D_{2p}. Moreover, in the case of $(\mathbb{Z}/p\mathbb{Z})^n$, the situation boils down to the case $\mathbb{Z}/p\mathbb{Z}$ plus certain compatibility conditions, by results of Green and Matignon ([27] and [38]). At the end of the chapter we sketch how to construct a Hurwitz tree for that keeps track of these results. This is used in chapter 4 where we deal with explicit examples of actions of $(\mathbb{Z}/p\mathbb{Z})^2$ that do not lift. The general case is discussed in Chapter 5, where a characterization of Hurwitz trees in terms of non-Archimedean analytic geometry is given.

3.1 Local and global actions on curves

Definition 3.1.1. *Let G be a finite group. A G-**cover** is a finite generically étale morphism $\pi : Y \longrightarrow X$ of (geometrically) connected normal schemes such that $G \cong \mathrm{Aut}(Y/X)$ acts transitively on the (geometric) fibers of π.*

Let C be a smooth projective curve over $\tilde{K}$ and $G \hookrightarrow \mathrm{Aut}(C)$ be a finite group of automorphisms of C. Then the morphism $C \longrightarrow C/G$ is a G-cover. Conversely, any G-cover of smooth projective curves $C \to D$ induces an embedding $G \hookrightarrow \mathrm{Aut}(C)$.
Given C_R be a flat (relative) curve over R such that $C_R \otimes_R \tilde{K} = C$ (such curve is called a **lift** of C to characteristic zero, and its existence is assured by an application of Grothendieck existence theorem, cfr. [29], III 7.3), one would like to know if the automorphisms of the curve C can be "lifted" to C_R as well.

Definition 3.1.2. *Let $\pi : C \longrightarrow C'$ be a G-cover of smooth projective curves defined over $\tilde{K}$. We say that π **lifts to characteristic zero** if there exists $R/W(\tilde{K})$ a finite extension of complete DVR G-cover $\pi_R : C_R \longrightarrow C'_R$ of lifts of C and C' respectively, such that its special fiber is π.*

Let x be a closed point of C. A faithful action of G over C induces an action of the stabilizer G_x at the point x over the completed local ring $\widehat{\mathcal{O}}_{C,x} \cong \tilde{K}[[t]]$. This is analogous to what happens over C_R and justifies the definition of a local action on a curve.

Definition 3.1.3. *A **local action in characteristic** p is a pair $(\tilde{K}[[t]], G)$ where $G \subset \mathrm{Aut}_{\tilde{K}}\tilde{K}[[t]]$ is a finite subgroup of automorphisms. A **local action in characteristic 0** is a pair $(R[[T]], G')$ where $G' \subset \mathrm{Aut}_R R[[T]]$ is a finite subgroup.*

The problem of lifting automorphisms of C to C_R has then a natural local analogue.

Question 3.1.4 (Local lifting problem)**.** Let $(\tilde{K}[[t]], G)$ be a local action in characteristic p. Does there exist a local action in characteristic 0 $(R[[T]], G)$ such that the diagram

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & \mathrm{Aut}_R R[[T]] \arrow[d] \\ & \mathrm{Aut}_{\tilde{K}}\tilde{K}[[t]] \end{tikzcd}$$

where the vertical arrow represents the reduction map, commutes ?

The answer is known to be positive when $(|G|, p) = 1$ by Grothendieck's theory of specialization of the tame fundamental group (see [29] XIII, Corollaire 2.12) and when G is cyclic (as proved by Pop in [50] using results of Obus-Wewers contained in [44]). There are also counterexamples in many other cases: Chinburg, Guralnick and Harbater formulated in [16] the following conjecture

Conjecture 3.1.5. (Strong Local Oort Conjecture) If $\tilde{K}$ is an algebraically closed field of characteristic p and $G \subset \mathrm{Aut}_{\tilde{K}}\tilde{K}[[t]]$ is a finite group, then the local action of G lift to characteristic

0 if and only if G is either a cyclic group, or a dihedral group of order $2p^n$ for some n, or $p = 2$ and G is the alternating group A_4.

In the same paper they also provide counterexamples for all finite groups that are not on the conjecture.

These advances are important for the original problem of lifting covers of curves. In fact it turns out that the local information is sufficient to study the global one.

Fact 3.1.6 (Local-global principle)**.** Let $G \hookrightarrow \mathrm{Aut}(C)$ be a finite group of automorphisms and let $\{x_1, \ldots, x_n\}$ be the set of points where G acts with nontrivial inertia. For each $1 \leq j \leq n$, let G_j be the inertia group of x_j and let $G_j \hookrightarrow \mathrm{Aut}_{\tilde{K}}\tilde{K}[[t_j]]$ be the induced local action. Then the action of G over C lifts to charachteristic zero if and only if the local actions $(G_j, \tilde{K}[[t_j]])$ lift for every j.

This fact allows us to focus for the rest of the chapter on the local lifting problem, without dropping the motivation involving the understanding of global actions and covers of curves.

3.2 Lifting local actions

Let us review some known facts about lifting local actions. We introduce in this section some of the methods that are fundamental to the understanding of Hurwitz trees and their importance.

Lifting via reduction of coverings

In the same way as to faithful G-actions on curves correspond G-covers, one can reformulate the local actions in terms of Galois covers. Given a local action $(\tilde{K}[[t]], G)$ in characterstic $p > 0$ then $\tilde{K}[[t]]^G$ is of the form $\tilde{K}[[z]]$ and we can ask to the cover $\tilde{K}[[t]] \xrightarrow{G} \tilde{K}[[z]]$ to be liftable to a G-cover $R[[T]] \xrightarrow{G} R[[Z]]$.
The point of view of G-covers and the one of G-actions are interchangeable, but one has to be careful, when starting from a local G-cover in characteristic zero, to check that $R[[T]] \xrightarrow{G} R[[T]]^G$ has good reduction. This amounts to check that $R[[T]]^G$ is of the form $R[[Z]]$ for some $Z \in R[[T]]$. This motivates the care to criteria of good reduction that is lavished in the following sections.

3.2.1 Geometric local actions

We describe here how studying models of the unit disc, in the spirit of Section 2.2.1, provide a deeper understanding of the properties of local actions.

The stably marked model

Let $\Lambda = (G, R[[T]])$ be a local action in characteristic zero. This action naturally induces an action on the (formal) spectrum of $R[[T]]$. At first, this does not carry a lot of information at the level of topological spaces. We need indeed to perform a series of blow-up in such a way to study more deeply the properties of Λ.

Lemma 3.2.1. For every $\Lambda = (G, R[[T]])$ local action in characteristic zero, there exists a unique minimal formal model $\mathcal{X}_\Lambda/\mathrm{Spf}(R)$ of the unit disc $\mathbb{D}(0,1)$ such that the following properties are satisfied

- The special fiber $\mathcal{X}_s$ is semistable;
- The group G acts over $\mathcal{X}$ without inertia (i.e. reduces to a proper action);
- Let $L_\Lambda = \{x_1, \ldots, x_m\}$ be the set of ramification points for the action of G on the rigid generic fiber $D(0,1)$. Then the set of specializations $r(L_\Lambda)$ consists of m distinct smooth points of $\mathcal{X}_s$.

Proof. We can suppose $m \neq 1$ (if $m = 1$ then the model is $\mathrm{Spf}(R[[T]])$ itself). Let then $e = \min\{v_R(x_j - x_1) : j = 1, \ldots, m\}$ and consider the admissible blowing up $\mathcal{X}_1 \to \mathrm{Spf}(R[[T]])$, of open ideal $(\varpi^e, T - x_1)$. Its special fiber $\mathcal{X}_{1,s}$ is isomorphic to $\mathbb{P}^1_{\tilde{K}}$ and the specialization map $r_1 : \mathcal{X}_{1,\eta} \to \mathcal{X}_{1,\eta}$ is such that there are at least two distinct points in the set $r_1(L_\Lambda)$. Each element of this set is a closed point in $\mathcal{X}_{1,s}$. If $|r_1(L_\Lambda)| = m$ then we are done, otherwise we can perform a blowing up, with depth $\min\{v_R(x_j - x_i) : r_1(x_j) = r_1(x_i)\}$, of the point corresponding to $r_1^{-1}(r(x_i))$, for every $r_1(x_i)$, to obtain a formal morphism $\mathcal{X}_2 \to \mathcal{X}_1$ with $|r_2(L_\Lambda)| > |r_1(L_\Lambda)|$. Repeating this operation one gets, in a finite number of steps, $\mathcal{X}_\Lambda \to \mathcal{X}_1$ such that $|r(L_\Lambda)| = m$. This model satisfies the conditions above: each specialization is on a smooth distinct point of $\mathcal{X}_s$, which is by construction a tree of projective lines over $\tilde{K}$, hence semi-stable. Let us notice also that, since these blowing up are performed with the minimal depth possible, then the model obtained in this way is minimal. $\square$

Such a model $\mathcal{X}_\Lambda$ is called **stably marked model** of the local action in characteristic zero Λ. By construction, the irreducible components of its special fiber are all projective lines. This fact is important when studying local actions on boundaries (cfr. Section 3.2.2).

Remark 3.2.2. Our definition of stably marked model differs from the one of [12] in the fact that we associate a stably marked model to a local action in characteristic zero rather than to a Galois covering of curves in positive characteristic. This is essentially due to the difference of approaches outlined in the introduction of this chapter.

Example 3.2.3. *Let p be odd, $G = D_{2p}$ and let R be in such a way that $\lambda := \zeta_p - 1 \in R$. Consider the G-cover*

$$\frac{R[[Z]][X,Y]}{(Z - (X + \lambda^p/2)^2, XY^p + X + \lambda^p)} \cong R[[T]] \to R[[Z]].$$

The dihedral group G is then realized as group of automorphism of $R[[T]]$ by the formulas

$$\begin{cases} \sigma(X) = X \\ \sigma(Y) = \zeta_p Y \end{cases} \quad \textit{and} \quad \begin{cases} \tau(X) = -X - \lambda^p \\ \tau(Y) = \frac{1}{Y} \end{cases} .$$

Fixed points for σ if $Y = 0$, $X = -\lambda^p$ and $Z = (\lambda^p/2)^2$.
Fixed points for τ if $Y = 1$, $X = -\lambda^p/2$ and $Z = 0$.

3.2.2 Local actions on boundaries

Let $\Lambda = (R[[T]], G)$ be a local action in characteristic 0. Let ξ be the generic point of an irreducible component of the special fiber of the stably marked model (any such point will be called **boundary point**). We can look at ξ in the stably marked model where, being of codimension 1, is a Weil divisor inside a normal scheme. Thanks to the following classical lemma we can construct a valuation attached to a boundary point.

Lemma 3.2.4. Let A be a Noetherian local domain of dimension 1, $\mathfrak{m}$ its unique maximal ideal.

Then the following are equivalent:

i) A is normal;

ii) Every non-zero ideal is a power of $\mathfrak{m}$;

iii) A is a discrete valuation ring.

Then, if we apply the lemma for the local ring $\mathcal{O}_{\mathcal{X}_\Lambda,\xi}$, which fits in the hypotheses (the stably marked model is normal by definition yielding normality of $\mathcal{O}_{\mathcal{X}_\Lambda,\xi}$), we get a discrete valuation v_ξ on $\mathcal{K} := \mathrm{Frac}(\mathcal{O}_{\mathcal{S}_\sigma,\xi})$. It has the property that there exist $a \in K$, $|a| < 1$ and $\rho \in \mathbb{R}_+$ such that:

- $v_\xi(f) = n > 0$ implies that $f \in \mathcal{K}$ has a zero of order n in the closed disc $D(a, \rho)$.
- $v_\xi(f) = -n < 0$ implies that $f \in \mathcal{K}$ has a pole of order n in $D(a, \rho)$.
- $v_\xi(f) = 0$ implies that $f \in \mathcal{K}$ has neither a zero nor a pole in $D(a, \rho)$.

We define $v_{a,\rho}$ to be a valuation as above: this is well defined because $v_{a,\rho} = v_{b,\rho}$ if and only if $a \in D(b, \rho)$.

Let ξ be as before, such that $v_\xi = v_{a,\rho}$ and let z be a double point of $\mathcal{X}_{\Lambda,s}$.
Let $R[[T]]\{T^{-1}\} = \{\sum_{i\in\mathbb{Z}} a_i T^i : |a_i| \to 0 \text{ if } i \to -\infty\}$. Then we have an isomorphism

$$\mathcal{O}_{\xi,z} := (\widehat{\mathcal{O}_{\mathcal{X}_\Lambda,x}})_\xi \xrightarrow{\sim} R[[\rho^{-1}(T-a)]]\{(T-a)^{-1}\}.$$

The valuation v_ξ extends naturally over the ring $\mathcal{O}_{\xi,z}$ which is complete for v_ξ. Moreover its residue field is isomorphic to $k((t - \bar{a}))$. It is endowed then with a discrete valuation v_z : $k((t - \bar{a})) \to \mathbb{Z}$ which depends only on z.

Definition 3.2.5. *We call* ***Gauss valuation*** *centered in a of radius ρ, the valuation v_ξ. We call* ***residue valuation*** *centered in a of radius ρ the valuation v_z. The couple (v_ξ, v_z) is therefore a valuation of rank two over $\mathcal{O}_{\xi,z}$.*

Notice that the Gauss valuation depends on the closed disc $D(a, \rho)$, as well as the residue valuation depends on the open disc $D(a, \rho)^-$. Namely, for any $a' \in D(a, \rho)$ such that $|a' - a| = \rho$, we have that $v_{a,\rho} = v_{a',\rho}$ but the residue valuations are different.
We have an explicit description of the valuations v_ξ and v_z.

Lemma 3.2.6. Let T be a parameter such that $\mathcal{O}_{\xi,z} \cong R[[T]]\{T^{-1}\}$ and let t be the class of T in $\mathcal{O}_{\xi,z}/\varpi\mathcal{O}_{\xi,z}$. Then $v_\xi : R[[T]]\{T^{-1}\} \to \mathbb{Q}$ is such that $v_\xi(\sum a_i T^i) = \min(v_R(a_i))$ and $v_z : k((t)) \to \mathbb{Z}$ is the t-adic valuation.

Consider the action of Λ on the stably marked model. When ξ and z are as before, the action induced by Λ on $\mathcal{O}_{\xi,z}$ is described by a surjective morphism $Y \to X$, where $X \cong \mathrm{Spec}(R[[T]]\{T^{-1}\})$, that becomes a $G_{\xi,z}$-cover $Y' \to X$, for a subgroup $H \leq G$, when one takes the normalization Y' of Y. This cover corresponds to an extension of integrally closed rings whose extension of fraction fields K_Y/K_X is Galois with Galois group H. One can check, by noticing that the choice of another double point on the same component correspond to a translation by an element of $R^\times$, that the valuation of the different ideal $v_\xi(\mathfrak{D}_{K_Y/K_X})$ does not depend on the choice of z. This number is crucial when studying the reduction of local actions at boundaries for the sake of the following result

3.3 The Hurwitz tree for $\mathbb{Z}/p\mathbb{Z}$

We describe in this section the role of the Hurwitz tree as an object parametrizing local actions in characteristic zero in such a way to keep track of their reduction properties.

Notations: basics on trees

In what follows, the word **tree** denotes a finite oriented rooted tree $\mathcal{T}$. This means that the set $V(\mathcal{T})$ of its vertices is endowed with a partial order relation, denoted by $v \to v'$, in such a way that the following conditions are satisfied:

- there exists a unique $v_0 \in V(\mathcal{T})$ with $v_0 \to v$ for every $v \in V(\mathcal{T})$;
- for every couple $v, v' \in V(\mathcal{T})$ such that $v \to v'$ and such that there is no v'' with $v \to v'' \to v'$ there is an unique edge connecting v and v' ;
- conversely, for each edge of $\mathcal{T}$ joining two vertices $v, v' \in V(\mathcal{T})$ these are such that $v \to v'$ and such that there is no v'' with $v \to v'' \to v'$.

The vertex v_0 is called the **root** of the tree. A vertex v such that there is no $v' \in V(\mathcal{T})$ with $v \to v'$ is called a **leaf**. The couple associated to an edge e of $\mathcal{T}$ is noted $v_e^s \to v_e^t$ and they are respectively called **starting vertex** and **ending vertex** of e. Finally, we set e_v^- for the unique edge having v as ending vertex and $E^+(v) = \{e \in E(\mathcal{T}_\Lambda) : v_e^s = v\}$.

Motivation: the deformation between Kummer and Artin-Schreier

Suppose that R is ramified over $W(\tilde{K})$ with ramification index divided by $p-1$. When $G=\mathbb{Z}/p\mathbb{Z}$ acts on $R[[T]]$ in such a way that the m ramification points have the same valuation the equation of the covering is of the special Kummer type $X^p=(1+T^m)$. In order to deform this equation to one reducing to an Artin-Schreier one, one needs to deform the equation.
Set $X=1+\beta U$ and $S=\beta^{\frac{p}{m}}T$ with $\beta\in R$. Then the equation becomes

$$\begin{aligned}(1+\beta U)^p=(1+\beta^p T^m) &\iff (\beta U)^p+p(\beta U)^{p-1}+\cdots+p\beta U=\beta^p T^m\\ &\iff U^p+\cdots+\frac{p}{\beta^{p-1}}U=T^m.\end{aligned}\tag{3.1}$$

When $v_p(\beta)=\frac{1}{p-1}$ the reduction mod. p of this equation is $u^p-cu=t^m$, with $c\neq 0$, that is, an Artin-Schreier equation of which the original Kummer equation is a lift.

3.3.1 The Hurwitz tree associated to an automorphism of order p

The deformation from Kummer to Artin-Schreier (3.1) describe the lifting of some particular local actions of $\mathbb{Z}/p\mathbb{Z}$. The underlying assumption is in fact that all the ramification points are at the same mutual distance. Not all local actions are of this kind, but one can reduce every action to those by a procedure of “disc shrinking”. The existence of appropriate deformations permitting such a procedure is equivalent to the existence of some differential forms and to the satisfaction of some compatibility conditions. The study of these requirements gives rise to the definition of a Hurwitz tree for an automorphism of order p, as introduced in [31].

Definition 3.3.1 (Hurwitz trees of type $\mathbb{Z}/p\mathbb{Z}$)**.** *Let $\Lambda=(\mathbb{Z}/p\mathbb{Z},R[[T]])$ be a local action in characteristic zero and let $\mathcal{X}_\Lambda$ be the stably marked model of Λ and $L_\Lambda=\{x_1,\dots,x_m\}$ the set of ramification points for the action of G on the rigid generic fiber $D(0,1)$. The Hurwitz tree associated to Λ is the datum of a tree $\mathcal{T}_\Lambda$ endowed with additional data (d,ϵ,m) constructed by enriching the reduction graph of $\mathcal{X}_\Lambda$:*

- *the set of vertices is given by $V(\mathcal{T}_\Lambda)=V(\Gamma_{\mathcal{X}_\Lambda})\cup L_\Lambda$, and the vertex arising from a ramification point x is denoted by v_x ;*
- *the set of edges is given by $E(\mathcal{T}_\Lambda)=E(\Gamma_{\mathcal{X}_\Lambda})\cup L_\Lambda$ and the edge arising from a ramification point x is denoted by e_x ;*
- *whenever an edge $e\in E(\Gamma_{\mathcal{X}_\Lambda})$ has v_e^s and v_e^t as respectively a starting vertex and an ending vertex in the reduction graph, the same property holds still when we see e in $E(\mathcal{T}_\Lambda)$;*
- *we have $v_{e_x}^t=v_x$ and $v_{e_x}^s=v$, the vertex whose corresponding irreducible component of $\mathcal{X}_{\Lambda,s}$ is the one where x specializes.*

The additional data (d,ϵ,m) are given as follows:

- *the datum $d:V(\mathcal{T}_\Lambda)\longrightarrow\mathbb{Q}$ is defined by $d(v)=v_\xi(\mathfrak{D}_v)$, the valuation of the different associated to the irreducible component ξ of $\mathcal{X}_{\Lambda,s}$ corresponding to the vertex v ;*

- *the datum $\epsilon : E(\mathcal{T}_\Lambda) \to \mathbb{N} \cup \infty$ describes a metric on the Hurwitz tree by associating to each edge $e \in E(\Gamma_{\mathcal{X}_\Lambda})$ the thickness of the blowing up giving rise to the double point corresponding to e, and to each edge $e_x \in L_\Lambda$ the value ∞ ;*

- *the datum $m : E(\mathcal{T}_\Lambda) \to \mathbb{Z}$ is defined by $m(e) = |\{x : v_e^t \to v_x\}| - 1$. Morally, $m(e)$ "counts" the number of ramification points that specialize on components obtained by blowing up the irreducible component corresponding to v_e^t.*

Remark 3.3.2. The value of $d(v)$ for every vertex of the Hurwitz tree can be recovered from the single value $d(r_0)$ at the root of the tree, by the formula

$$d(v) = d_0 + (p-1) \cdot \sum_{e \in E^0(v)} m(e)\epsilon(e),$$

where $E^0(v)$ is the set of edges that form the unique path joining r_0 and v.

Remark 3.3.3. In the original definition, the edges of the form e_x are defined to be of length zero were considered. This choice has the advantage to give sense to the quantity $d(v)$ also when v is a leaf. However this feature is irrelevant for our purposes and we decided to set $\epsilon(e_x) = \infty$. In fact, in chapter 5 we show that this metric is more natural when considering Hurwitz trees as non-Archimedean analytic objects.

An Hurwitz tree associated to a local action in characteristic zero satisfies the following properties:

H1. Let $e \in E(\mathcal{T}_\Lambda)$ such that $m(e) \neq 0$. Then $(m(e), p) = 1$;

H2. For each vertex $v \in V(\Gamma_{\mathcal{X}_\Lambda})$ we have

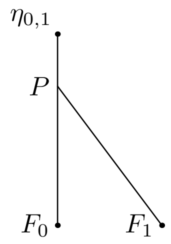
$$\sum_{e \in E^+(v)} (m(e) + 1) - (m(e_v^-) + 1) = 2$$

recalling that e_v^- is the unique edge having v as ending vertex and $E^+(v) = \{e \in E(\mathcal{T}_\Lambda) : v_e^s = v\}$;

H3. For each vertex v which is not a leaf, we have $0 \leq d(v) \leq v_K(p)$;

H4. The vertex v is a leaf of $\Gamma_{\mathcal{X}_\Lambda}$ if and only if $d(v) = v_K(p)$.

Example 3.3.4. *Let ζ_3 be a primitive third root of unity and σ be the automorphism of $\mathbb{Z}_3^{ur}(\zeta_3, \sqrt{3})[[T]]$ given by $T \mapsto \frac{T-3}{T-2}$. It has order 3 and on the special fiber it reduces to the automorphism $\bar{\sigma} : \overline{\mathbb{F}_p}[[t]] \to \overline{\mathbb{F}_p}[[t]]$ given by $t \mapsto \frac{t}{t+1}$. Solving the equation $T = \frac{T-3}{T-2}$ gives two distinct fixed points that we call x_0 and x_1 (this number could be computed also noticing that $v_t(\bar{\sigma}(t)) = 1$). We have $v(x_1 - x_0) = v_p(p)/2$ which gives an Hurwitz tree with four vertices $\{\eta_{0,1}, P = \eta_{0,3^{(-\frac{1}{2})}}, x_0, x_1\}$ and three edges $\{e_1, e_2, e_3\}$ appearing as follows:*



The Hurwitz data attached to each edge are $d_0 = 0$, $m(e_1) = 2; m(e_2) = m(e_3) = 0$ and $h(e_1) = 0; h(e_2) = 1; h(e_3) = -1$. From these is possible to calculate the depth at P: $d(\eta_{0,\frac{\sqrt{3}}{3}}) = 2 = v_K(p)$ which confirms the fact that it is a terminal branch.

3.3.2 Good deformation data

Recall that, for any faithfully flat and locally of finite type commutative group scheme G over a scheme X with the $fppf$ topology, the G-torsors over X are classified by the group $H^1(X, G)$. In positive characteristic, when the group scheme is α_p or μ_p, we have a description of this group in terms of differential forms (see section III.4 of [39] for further details on torsors and their relations with cohomology).

Proposition 3.3.5. Let X be a smooth variety over a perfect field of characteristic $p > 0$. Then

$$H^1(X, \alpha_p) = \{\omega \in H^0(X, \Omega^1_X) : \omega \text{ is locally exact } \}$$

and

$$H^1(X, \mu_p) = \{\omega \in H^0(X, \Omega^1_X) : \omega \text{ is locally logarithmic } \}.$$

For the local action in characteristic zero $\Lambda = (\mathbb{Z}/p\mathbb{Z}, R[[T]])$, recall from 3.2.2 that on every boundary point there is an induced $\mathbb{Z}/p\mathbb{Z}$-covering $\tau : R[[T]]\{T^{-1}\} \to R[[Z]]\{Z^{-1}\}$. It is possible to show that there is a group scheme $\mathcal{G}$ of order p on R such that $R[[T]]\{T^{-1}\} \to R[[Z]]\{Z^{-1}\}$ can be realized as a $\mathcal{G}$-torsor. Then, on the special fiber, we have a torsor $\tau_s : k[[t]] \to k[[z]]$ under $\mathcal{G}_s$. By the classification of group schemes of order p (cfr. [46]) we have only three possibilities for $\mathcal{G}_s$.

Definition 3.3.6. *Let $\tau : R[[T]]\{T^{-1}\} \to R[[Z]]\{Z^{-1}\}$ be a $\mathcal{G}$-torsor over R. Then we say that τ has*

- ***multiplicative reduction**, if τ_s is a μ_p-torsor;*
- ***additive reduction**, if τ_s is an α_p-torsor;*
- ***étale reduction**, if τ_s is a $\mathbb{Z}/p\mathbb{Z}$-torsor.*

In the deformation from Kummer to Artin-Schreier (3.1), one starts with a torsor with multiplicative reduction and ends up with a torsor with étale reduction. When one deals with more general local actions of $\mathbb{Z}/p\mathbb{Z}$ in characteristic zero, it may happen that the induced torsor on some boundaries has additive reduction. One has the following result on the reduction of

μ_p-torsors from K to $\tilde{K}$ (see Proposition 1.6. of [31]), which makes more precise Kato's different criterion in the case where $G = \mathbb{Z}/p\mathbb{Z}$.

Proposition 3.3.7. Let $A = R[[T]]\{T^{-1}\}$, and B a finite A-algebra of degree p which is flat over R and such that $A \cong B^{\mathbb{Z}/p\mathbb{Z}}$. Let us suppose that the special fiber of the torsor $\tau : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is given by a purely inseparable extension of the residue field $\bar{A} \cong \tilde{K}((t))$. Denote by δ the valuation of the different of the extension B/A. Then $0 \leq \delta \leq v_K(p)$ and we have one of the following cases

1. If $\delta = v_K(p)$, then τ has multiplicative reduction: the algebra B is such that
$$B = \frac{A[X]}{X^p - u} \quad , \quad u \in A^\times.$$

2. If $0 < \delta < v_K(p)$, then it is $\delta = v_K(p) - n(p-1)$ for some $0 < n < \frac{v_K(p)}{p-1}$. In this case, τ has additive reduction:
$$B = \frac{A[X]}{\frac{(\varpi^n X+1)^p - 1}{\varpi^{pn}} - u} \quad , \quad u \in A^\times.$$

3. If $\delta = 0$ then τ has étale reduction:
$$B = \frac{A[X]}{X^p - X - u_1 + \varpi^k u_2} \quad , \quad u_1, u_2 \in A^\times \quad \text{and} \quad k > 0.$$

Thanks to Propositions 3.3.5 and 3.3.7, one can attach to each vertex of the Hurwitz tree a differential form that keeps track of the reduction of the torsor associated to boundary action at that vertex.

Assignment 3.3.8. *Let $v \in V(\mathcal{T}_\Lambda)$ be a vertex of the Hurwitz tree associated to a local action in characteristic zero $\Lambda = (\mathbb{Z}/p\mathbb{Z}, R[[T]])$ such that v is not a leaf. Then to v corresponds a torsor* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *given by the local action at the boundary corresponding to v. The differential form ω_v is then defined according to the classification and the notations of Proposition 3.3.7.*

1. *When $d(v) = v_K(p)$, set $\omega_v = \frac{d\bar{u}}{\bar{u}}$;*

2. *When $0 < d(v) < v_K(p)$, set $\omega_v = d\bar{u}$;*

3. *When $d(v) = 0$, $\omega_v = 0$;*

where $\bar{u}$ denotes the reduction of u on $\bar{A}$. Such differential forms are called ***good deformation data****.*

In [31], Henrio proves that Hurwitz trees with good deformation data classify automorphisms of the formal disc with fixed points. He also provides, by means of a "realizability criterion", necessary and sufficient conditions to the lifting of local actions to characteristic zero.

3.4 Hurwitz trees of general type

The possibility to extend the definition of a Hurwitz tree associated to the action of any finite group $G \subset \mathrm{Aut}_R(R[[T]])$ has been considered by several authors.
In [14], Brewis and Wewers propose such a generalization. As the Hurwitz tree associated to a local action of $\mathbb{Z}/p\mathbb{Z}$, their construction describes the geometry of fixed points and the ramification of the local action of G with respect to boundary valuations. To obtain such a description they replace the numbers $m(e), d(v)$, with representation theoretical virtual characters, coming from the ramification theory for analytic curves developed by Huber ([33]). In the case of $\mathbb{Z}/p\mathbb{Z}$ these characters are uniquely determined by their value at a generator and the Hurwitz data boil down to those of [31]. The possibility to extend good deformation data is still open, since the theory of deformation of G-torsors is not understood for the great majority of finite group schemes. Without these differential forms it is possible nevertheless to to get interesting necessary conditions to lifting: the so-called "Hurwitz tree obstruction" defined in section 4.1 of [14] is strictly stronger than the Bertin obstruction (cfr [8]).

3.4.1 Definition of Hurwitz tree

We recall here what is the definition of a Hurwitz tree for a general finite group G following [14]. We first need some preliminaries on the role played by representations in the ramification theory of the local action.

Valuations, characters and class functions

Let $R(G, \mathbb{C})$ be the set of class functions of G (i.e. maps $G \to \mathbb{C}$ which are invariant by conjugation). It contains the set $R^+(G)$ of characters associated to representations of G and we call $R(G)$ the smallest subgroup of $R(G, \mathbb{C})$ containing $R^+(G)$. The elements of $R(G)$ are called **virtual characters**. As explained in [54], $\mathcal{S}14.1$, each $\mathbb{Q}$-valued character (meaning an element of $R(G, \mathbb{Q}) := R(G) \otimes_{\mathbb{Z}} \mathbb{Q}$) is uniquely determined by a class function in $R(G, \mathbb{C})$.
Recall also that the Gauss valuation and the residue valuations on $R[[T]]$ are defined respectively by $\mathrm{val}_{0,1}(\sum a_i T^i) := -\log \eta_{0,1}(\sum a_i T^i) = \min_i(v_R(a_i))$ and $\mathrm{val}_z(f) := \mathrm{ord}_t(f/\varpi^{\mathrm{val}_{0,1}(f)})$. With these valuations we can define the objects that are necessary to generalize Hurwitz data: the depth character and the Artin character.

Definition 3.4.1. *The **depth character** associated to a local action in characteristic zero $\Lambda = (G, R[[T]])$ is the $\mathbb{Q}$-valued character $\delta_\Lambda \in R(G, \mathbb{Q})$ associated to the class function defined by*

$$\delta_\Lambda(\sigma) := \begin{cases} -|G| \cdot \mathrm{val}_{0,1}(\sigma(T) - T) & \text{if } \sigma \neq 1 \\ |G| \sum_{\sigma' \in G \setminus \{1\}} \mathrm{val}_{0,1}(\sigma'(T) - T) & \text{if } \sigma = 1. \end{cases}$$

*The **Artin character** associated to Λ is the element of $R(G)$ associated to the class function* *defined by*

$$a_\Lambda(\sigma) := \begin{cases} -\mathrm{val}_z(\sigma(T) - T) & \text{if } \sigma \neq 1 \\ \sum_{\sigma' \in G\setminus\{1\}} \mathrm{val}_z(\sigma'(T) - T) & \text{if } \sigma = 1. \end{cases}$$

These virtual characters are indeed characters arising from some representation. In fact, if we let u_G to be the character associated to the augmentation representation of G, we have the formula

$$\delta_\Lambda = \sum_{i=1}^{m} |G_i| \cdot (h_i - h_{i-1}) \cdot u^*_{G_i}$$

where the $G = G_{h_0} \supset \cdots \supset G_{h_i} \supset \cdots \supset G_{h_m} = \{1\}$ is the lower ramification filtration of G and h_i are the ramification jumps in this filtration (with $h_0 = 0$). We notice from these relations that the the depth character is intimately related to the ramification of the action of G at the boundary, and in particular with the inertia of this action at the boundary.
We have a similar relation for the Artin character:

$$a_\Lambda = \sum_{x \in L_\Lambda} u^*_{G_x},$$

where G_x is the stabilizer of the ramification point $x \in L_\Lambda$.

Construction

The Hurwitz tree is constructed by induction on the cardinality of the set of ramification points $|L_\Lambda|$. One firstly describes its structure in the case where it consists of a unique edge joining two vertices, and then relates the general situation to this simpler one.

Definition 3.4.2. *Let $\Lambda = (G, R[[T]])$ be a local action of a finite group in characteristic zero. The **Hurwitz tree** associated to Λ is an oriented metric tree $(\mathcal{T}_\Lambda, \epsilon)$ with a datum $([G_v], a_e, \delta_v)$ as follows:*

- *for every vertex $v \in V(\mathcal{T}_\Lambda)$, $[G_v]$ is the conjugacy class of a subgroup of G and $\delta_v : G \to \mathbb{C}$ is a $\mathbb{Q}$-valued character of G;*
- *For every edge $e \in E(\mathcal{T}_\Lambda)$, $a_e : G \to \mathbb{C}$ is a character of G.*

When $|L_\Lambda| = 0$ the Hurwitz tree is empty and there is nothing to say. When $|L_\Lambda| = 1$ one constructs the Hurwitz tree as follows:

- *The tree consists of two vertices $V(\mathcal{T}_\Lambda) = \{r_0, v\}$ and one edge $E(\mathcal{T}_\Lambda) = \{e\}$. The metric is such that $\epsilon(e) = \infty$;*
- *The conjugacy class of a subgroup is the whole group: $G_{r_0} = G_v = G$;*
- *The $\mathbb{Q}$-valued character attached to vertices is $\delta_{r_0} = \delta_v = \delta_\Lambda$, the depth character;*
- *The $\mathbb{Z}$-valued character attached to the edge is $a_e = a_\Lambda$, the Artin character, which is in this case equal to the augmentation character u_G.*

Let now $|L_\Lambda| > 1$ and $x \in L_\Lambda$. Consider inside R the closed disc $D^\bullet$ centered in x of radius $\rho_x = \max\{|x' - x| : x' \in L_\Lambda\}$: it does not depend on x and it is the smallest closed disc containing L_Λ. Since L_Λ is finite, there is a finite number of disjoint residue classes $D_j^\circ \subset D^\bullet$ such that

$$L_{\Lambda,j} := D_j^\circ \cap L_\Lambda \neq \emptyset \quad \text{and} \quad L_\Lambda \subset \cup_j D_j^\circ.$$

The group G fixes $D^\bullet$ and, for every j, a subgroup $G_j \subset G$ fixes D_j°. The action of G_j induces on D_j° is a local action in characteristic zero $\Lambda_j := (G_j, R[[\rho_x^{-1}(T-x)]])$ with $|L_{\Lambda_j}| = |L_{\Lambda,j}| < |L_\Lambda|$. By induction there is a Hurwitz tree $\mathcal{T}_{\Lambda_j}$ for every j, such that $\delta_{\Lambda_j} =$, because it depends only on $D^\bullet$ and not on the residue classes D_j°.
The Hurwitz tree $\mathcal{T}_\Lambda$ is constructed by glueing the trees $\mathcal{T}_{\Lambda_j}$:

- *The vertices are defined by*

  $$V(\mathcal{T}_\Lambda) = \cup_j (V(\mathcal{T}_{\Lambda_j})/\sim) \cup \{r_0\}$$

  where the relation $\sim$ is such that $v \sim v'$ if and only if $v = v'$ or v and v' are both roots of some $\mathcal{T}_{\Lambda_j}$.

- *The edges are defined by $E(\mathcal{T}_\Lambda) = \cup_j E(\mathcal{T}_{\Lambda_j}) \cup \{e_0\}$.*

- *The metric is given by $\epsilon(e_0) = -\log(\rho_x) \cdot |G|$;*

- *The conjugacy class of a subgroup is set to be $G = G_{r_0}$, as well as $G = G_v$ for v any root of $\mathcal{T}_{\Lambda_j}$;*

- *The $\mathbb{Q}$-valued character δ_Λ is attached to r_0, the character δ_{Λ_j} is attached to any v root of $V(\mathcal{T}_{\Lambda_j})$;*

- *The $\mathbb{Z}$-valued character a_Λ is attached to e_0, which is in this case equal to the augmentation character u_G;*

- *For every j, every $v \in V(\mathcal{T}_{\Lambda_j})$ not a root, and $e \in E(\mathcal{T}_{\Lambda_j})$, the Hurwitz data are those induced by $\mathcal{T}_{\Lambda_j}$.*

An Hurwitz tree defined in this way satisfies the following conditions:

H1. For every couple of vertices v, v' such that $v > v'$ we have $G_v \subset G_{v'}$ up to conjugation. When v_0 is the root and v_1 the only successor of v_0 then $G_{v_1} = G_{v_0} = G$.

H2. The group G_b is nontrivial and cyclic for every b in the set of leaves $B \subset V(T)$.

H3. For every edge $e \in E$ we have

$$\delta_{v_e^t} = \delta_{v_e^s} + \epsilon_e \cdot (a_e - u^*_{G_{v_e^t}})$$

H4. Let $b \in B$ be a leaf and let $P_b \cong \mathbb{Z}/p^n\mathbb{Z}$ be the Sylow p-subgroup of G_b (which is unique by H2). Then the character δ_b is given by the class function

$$\delta_b(\sigma^a) = \begin{cases} -\frac{p^{ord_p(a)+1}}{p-1}, & a \in \mathbb{Z} \mid a \not\equiv 0 \bmod p^n; \\ np^n, & \sigma = id. \end{cases}$$

3.4.2 Comparaison with the Hurwitz tree for $\mathbb{Z}/p\mathbb{Z}$

Brewis and Wewers remarked (cfr Remark 3.9. of [14]) that their construction of the Hurwitz tree partially generalizes that of Henrio. In order to make this correspondence clear, we prove their remark and we explicit the relations between the Hurwitz data in the two frameworks.

Proposition 3.4.3. Let $(R[[T]], \mathbb{Z}/p\mathbb{Z})$ be a local action of $\mathbb{Z}/p\mathbb{Z}$. Then the Hurwitz tree of section 3.3.1 is isomorphic, as a rooted metric tree, to the Hurwitz tree of section 3.4.2. Moreover we have the following relations between the Hurwitz data

i) $\epsilon_e = \epsilon(e)$ for every edge e;

ii) $a_e(\sigma) = -m(e) - 1$ for every edge e;

iii) $\delta_v(\sigma) = -\frac{d(v)}{p-1}$ for every vertex v.

Proof. To any closed disc $\mathbb{D} = \mathbb{D}(a, \rho) \subset \mathbb{D}(0, 1)$ one can associate a blowup $\mathcal{X}_{\mathbb{D}} \to \mathrm{Spf}(R\{T\})$, centered in (a, ϖ) whose exceptional divisor is a projective line $\mathbb{P}^1_R$. In the construction of 3.4.2, one associates a disc to each vertex of the Hurwitz tree. The set of all discs obtained in this way gives the sequence of blowups that are necessary to construct the stably marked model associated to σ. Each exceptional divisor corresponds then at the same time to a vertex in both Hurwitz trees. It remains to show that the partial order relation on vertices is respected. To do this, let E_v be the exceptional divisor corresponding to a vertex of the Hurwitz tree of 3.3.1. Then $v < v'$ iff $E_{v'}$ is obtained by a blowing up a point of E_v and this is true if and only if $D_{v'} \subset D_v$, which gives the order relation on the Hurwitz tree of 3.4.2.
Let us now compare Hurwitz data. The thickness of the blowup associated to $\mathbb{D}(a, \rho) \subset \mathbb{D}(0, 1)$ is by construction the thickness of the annulus $\mathbb{D}(0, 1) \setminus \mathbb{D}(a, \rho)$, then we get equality i).
For every $\sigma \in \mathbb{Z}/p\mathbb{Z}$ of order p, one easily calculates $u_G(\sigma) = -1$. Then the Artin character $a_e(\sigma)$ is the opposite of the number of ramification points contained in the residue class associated to e. With the description of 3.3.1, we get the second equality.
For an edge e starting from v and ending in v' we have the formulas

$$d(v') = d(v) + (p-1)(m(e)\epsilon(e)) \quad \text{and} \quad \delta_{v'}(\sigma) = \delta_v(\sigma) + \epsilon_e \cdot (a_e - u_G)(\sigma) = \delta_v(\sigma) + \epsilon_e \cdot (a_e(\sigma) + 1).$$

We then have $\delta_{v'}(\sigma) = \delta_v(\sigma) - m(e)\epsilon(e)$, using the relation just proved between the Artin character and $m(e)$. Notice also that every Hurwitz tree is a sub-Hurwitz tree of one having a root v_0 such that $d(v_0) = 0$. This condition implies that the action at the boundary point corresponding to v_0 has trivial inertia. Therefore, for this Hurwitz tree $\delta_{v_0}(\sigma) = 0$ as well.

Iterating the formulas above then gives for every vertex v:

$$d(v) = (p-1)(\sum_i (m(e_i)\epsilon(e_i))) \quad \text{and} \quad \delta_v(\sigma) = -\sum_i (m(e_i)\epsilon(e_i))$$

where the sum is taken over all the edges joining v_0 with v, hence the third equality holds as well.

□

3.5 The elementary abelian case

Let $G = (\mathbb{Z}/p\mathbb{Z})^n$. It is possible to use results of section 3.3.1 to lift each intermediate $\mathbb{Z}/p\mathbb{Z}$-extension, but there are obstructions to the compatibility of these liftings. In the case of equidistant points, the obstruction can be formulated by saying that the good deformation data giving rise to each lifting form a k-vector space of dimension n.
In [27] the following congruence conditions are given for liftings of actions of $(\mathbb{Z}/p\mathbb{Z})^2$:

Theorem 3.5.1. Let $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and, for $i \in \{1, \ldots, p+1\}$, let G_i be the distinct subgroups of order p of G. Let $\lambda = (G, \tilde{K}[[t]])$ be a local action in characteristic p, inducing $p+1$ $\mathbb{Z}/p\mathbb{Z}$-Galois covers

$$\mathrm{Spec}(\tilde{K}[[t]]^{G_i}) \to \mathrm{Spec}(\tilde{K}[[t]]^G)$$

with conductors $m_i + 1$ and other $p+1$ $\mathbb{Z}/p\mathbb{Z}$-Galois covers

$$\mathrm{Spec}(\tilde{K}[[t]]) \to \mathrm{Spec}(\tilde{K}[[t]]^{G_i})$$

with conductors $m_i' + 1$. Suppose that $m_1 \leq \cdots \leq m_{p+1}$. If there is a lifting to characteristic zero of the local action λ, then only the following two situations can occur:

1. If $m_1 < m_2$, then $m_1 \equiv -1 \mod p$, $m_1' = m_2 p - m_1(p-1)$, $m_i = m_2$ and $m_i' = m_1$ for $i \in \{2, \ldots, p+1\}$
2. If $m_1 = m_2$ then $m_i = m_1 \equiv -1 \mod p$ and $m_i' = m_1$ for $1 \leq i \leq p+1$.

In both cases, the two covers $\mathrm{Spec}(\tilde{K}[[t]]^{G_i}) \to \mathrm{Spec}(\tilde{K}[[t]]^G)$ for $i \in \{1,2\}$ have $\frac{(p-1)(m_1+1)}{p}$ geometric ramification points in common.
Conversely, if $m_1 \equiv -1 \mod p$ and if we suppose that we can lift $\mathrm{Spec}(\tilde{K}[[t]]^{G_i}) \to \mathrm{Spec}(\tilde{K}[[t]]^G)$ to characteristic zero for $i \in \{1,2\}$in such a way that the liftings have $\frac{(p-1)(m_1+1)}{p}$ geometric ramification points in common, then there exist a lift of λ to characteristic zero, given by the normalization of the compositum of the two liftings.

This theorem has some immediate consequences on the Hurwitz trees associated to actions of $(\mathbb{Z}/p\mathbb{Z})^2$. We see more in details in the following chapters how the structure of Hurwitz trees is determined by explicit calculations allowing the existence of local actions in characteristic zero with given properties.

Chapter 4

Explicit calculations

Despite the fact that the local lifting problem has been studied for many years, it is difficult to have explicit examples of actions that lift. The most direct approach to the problem would be that of writing down explicit elements of $\tilde{K}[[t]]$ and try to lift their coefficients. This method is nevertheless not effective at all, when wild ramification occur. More in general, when G is not an Oort group, finding a criterion to decide whether a given action lifts or not is an open problem for every group. We think that it is worthwhile to spend some time on explicit calculation, in order to understand the difficulties that arise in this framework. Combining this approach with the more abstract one of chapter 5, we try to shed new light on different aspects of the local lifting problem.

In the first part of the chapter we focus on the tree structure of an Hurwitz tree, and we get some restrictions on the possible shapes of Hurwitz trees, using only the fact that we deal with actions of finite groups on graphs. In the second part we study actions of $G = (\mathbb{Z}/p\mathbb{Z})^n$ in the case where the ramification points are all at the same mutual distance. In this case the conditions on the existence of liftings reduce to prove that there are $\mathbb{F}_p$ vector spaces of multiplicative good deformation data. Writing down their form, one finds explicit equations in characteristic p, with the poles of those deformation data as unknowns, and their residues as parameters, that have been studied by Pagot in [47]. We find a new condition on these residues, that in the case where $p = 3$ determines them completely. Restricting to the case $p = 3$ and studying the combinatorics of the poles we can reprove the results of Pagot and go further by studying actions with more ramification points.

4.1 Combinatorial rigidity of Hurwitz tree

The Hurwitz tree is a quite complex object, in the sense that it encodes properties coming from different behaviors of coverings in characteristic 0. We claim that we can infer some properties of actions, just by studying the tree structure and knowing the group that acts on every vertex. The main idea is that the Hurwitz subtreees contained in a given Hurwitz tree shall satisfy the same defining properties. With this in mind, we can formulate some necessary conditions that a tree shall satisfy in order to have a chance of giving rise to a Hurwitz tree. We do this by looking at the algebraic structure of the Tate algebra $R\{T\}$. Its reduction is the polynomial algebra $\tilde{K}[t]$, in which automorphisms of finite order are easy to understand. We study in this way the "local lifting problem for closed discs". Then, by studying local actions of open discs on their closed subdiscs, we can find combinatorial properties satisfied by Hurwitz trees.

Let $\mathcal{T}_\Lambda$ be an Hurwitz tree and let $\mathcal{T}'$ be a subtree of $\mathcal{T}_\Lambda$, rooted in a vertex $v \in V(T)$. From the construction by induction of Section 3.4.1, there are Hurwitz data naturally induced by $\mathcal{T}_\Lambda$ on $\mathcal{T}'$. This Hurwitz subtree represents the restriction of the action of G on a disc D_v contained in the unit disc. Following this correspondence, $[G_v]$ is the conjugacy class of the biggest group contained in G that fixes D_v.
We can then decompose an Hurwitz tree into several subtrees carrying actions of their stabilizer. Let us try a systematic approach to this decomposition.
The simplest scenario it is the one with just a single fixed point $b \in B$. In this case the Hurwitz tree has just one edge joining the root to the fixed point and $G = G_b$ is a cyclic group.

4.1.1 Lifting actions to the closed unit disc

Let us begin the study of actions on closed discs, with a description of automorphisms of finite order of the affine line in positive characteristic.

Proposition 4.1.1. Let $\bar{\sigma} \in Aut(\tilde{K}[t])$ be an automorphism of finite order. Then it is determined by one of the following rules:

- $\bar{\sigma}(t) = t$
- $\bar{\sigma}(t) = \zeta_m \cdot t$ with $(m, p) = 1$
- $\bar{\sigma}(t) = t + b$ with $b \in \tilde{K}$,

or by a composition of those.

Proof. Since $\bar{\sigma}$ is surjective we have $\bar{\sigma}(t) = at + b$ with $a \neq 0$. The finiteness of its order gives a further condition over the coefficients :

$$t = \bar{\sigma}^n(t) = a^n t + \sum_{i=0}^{n-1} a^i b$$

gives $a^n = 1$ and we have two options: either $a = 1$ or $a = \zeta_n$ (with n coprime with p). In the first case the equation becomes $t = t + n \cdot b$ giving $n \cdot b = 0$. When $b = 0$ we have identity and when $b \neq 0$ we have $p = n$ (and we fall into the third option). In the second case $a = \zeta_n$ satisfying automatically the condition $\sum_{i=0}^{n-1} a^i = 0$. Therefore $\bar{\sigma}$ is a translation composed with a rotation of order prime to p. □

Remark 4.1.2. The Proposition 4.1.1, tells us that the problem of lifting local actions for the closed unit disc is simpler than the one for open discs. Every rotation can be lifted just by lifting the root of unity. In the case of translations, one may ask if they admit a lifting as an automorphism of order p of $R\{T\}$. If such lifting exists, it does not restrict to an automorphism of $R[[T]]$. This is in fact a behavior that appear only when studying closed discs.

This has some consequences at the level of automorphisms of closed discs in mixed characteristic. The reduction of such actions are automorphisms of finite order of the affine line over $\tilde{K}$. Applying the classification of Proposition 4.1.1 to this reduction, one gets the following description of the behavior of the action on residue classes.

Corollary 4.1.3. Let $\sigma \in \mathrm{Aut}(R\{T\})$ define a local action of $\mathbb{Z}/p^n\mathbb{Z}$ on the closed unit disc. If there exist $a \in \mathfrak{m}$ and an open unit disc $D^\circ(a, 1)$ fixed by σ, then σ reduces to the identity ($\bar{\sigma} = id$ over $\tilde{K}[t]$). In other words a residue class is fixed by σ if and only if all residue classes are fixed by σ.

Proof. Such action can not reduce to a nontrivial roto-translation. In fact the rotation component is trivial because it is of order p^n, and the translation component is trivial because at least one point is fixed. Then the reduction of the whole action is trivial. $\square$

Remark 4.1.4. The corollary guarantees that every open unit subdisc is fixed whenever there is a fixed open disc, but it does not assure the existence of fixed points in $D(0, 1)$. There are, in fact, actions of open discs without fixed points. However, it tells that, whenever a wildly ramified action of a closed disc has fixed points, then it must necessarily reduce to the identity. If we drop the assumption of wild ramification, we need to have two fixed residue classes to assure that $\bar{\sigma}$ is the identity.

Theorem 4.1.5. Let $\Lambda = (G, R[[T]])$ be a local action in characteristic zero, let $\mathcal{T}_\Lambda$ be the associated Hurwitz tree and let $v \in V(\mathcal{T}_\Lambda)$ be a vertex of this tree. Let $G_v \cong \mathbb{Z}/p^n\mathbb{Z}$ for some $n \in \mathbb{N}$. Then, for every couple $e', e'' \in E^+(v)$, one has $G_{v_t^{e'}} = G_{v_t^{e''}}$, and it is either $G_{v_t^{e'}} = \mathbb{Z}/p^n\mathbb{Z}$ or $G_{v_t^{e'}} = \mathbb{Z}/p^{n-1}\mathbb{Z}$.

Proof. Let σ be a generator of G_v. After the classification of Proposition 4.1.1, its reduction $\bar{\sigma}$ is either the identity or a translation. In the first case, the stabilizer of any residue class is G_v itself, in the second one, it is generated by σ^p (in fact such a translation has always order p), which is an element of order p^{n-1}. $\square$

This theorem, saying that one can not get rid all at once of the wildly ramification, has some consequences on the structure of Hurwitz trees. For example, for a local action of $\mathbb{Z}/p^n\mathbb{Z}$, this implies that the edges between the root and a leaf of the Hurwitz tree are at least $n + 1$. The same result could be obtained with classical ramification theory. In fact the higher ramification jumps (which are strictly related to the number of ramification points) in the upper ramification filtration of a $\mathbb{Z}/p^n\mathbb{Z}$ extension are always distinct (see [55], section 4), but this is somehow a simpler argument. We shall mention on this subject, that the control of the distance between the jumps in the ramification filtration plays a major role in the proof of the Oort conjecture both in the article by Pop ([50]) and in the one by Obus and Wewers ([44]). It would be interesting to deduce the properties of these jumps by elementary arguments, similar to those contained in the present section.

4.2 Lifting actions of elementary abelian p-groups

We consider in what follows liftings of local actions of $G = (\mathbb{Z}/p\mathbb{Z})^n$ with $m+1$ ramification points. It is known that such local actions do not lift to characteristic zero in general. Nevertheless, there are examples of actions that lift, and the question of giving a criterium to determine if a local action of an elementary abelian p-group lifts to characteristic zero is still open. The final aim of such a study is to obtain a parametrization (or, at least, a classification) of local actions in characteristic zero and their reduction. This problem has been studied in various fashions by Raynaud in [52], by Green and Matignon in [27] and by Pagot, in [47]. Most of this work has been done assuming that the ramification points of possible liftings have the same relative distance from each other. We also make this assumption, calling such actions **equidistant liftings**. The existence of equidistant liftings is discussed in the present section at first in the whole generality, by exposing known results and proving- new conditions that permit us to go further in this study. In a first instance, we relate the existence of those liftings with the existence of some multiplicative good deformation data - that assure liftings of intermediate $\mathbb{Z}/p\mathbb{Z}$ actions (cfr. section 3.3.2) - together with certain compatibility conditions for these differential forms. Then, we make further assumptions in order to establish additional constraints on the structure of the liftings: first of all, we work with $n = 2$; then, we fix the number of ramification points. We recall the cases studied by Pagot ($m+1 = p$, $m+1 = 2p$ and $m+1 = 3p$) and we are able to go further, finding additional conditions on the residues of multiplicative good deformation data. These conditions permit to simplify the proofs of Pagot's results, and to study explicitly the cases where $m+1 = 4p$ and $m+1 = 5p$. Finally we set $p = 3$. In this case, Green and Matignon showed that there exist liftings with $m+1 = 12$. With the previous results, we are able to find necessary conditions for liftings, when $m+1 = 15$. Such conditions are expressed in terms of a polynomial, that the coefficients of multiplicative good deformation data shall satisfy, if they are mutually compatible.

4.2.1 Lifting intermediate $\mathbb{Z}/p\mathbb{Z}$-extensions

The results of Henrio, recalled in section 3.3.2, and those of Green-Matignon and Raynaud, exposed in section 3.5, permit to relate equidistant liftings of $(\mathbb{Z}/p\mathbb{Z})^n$ with liftings of their $\mathbb{Z}/p\mathbb{Z}$ sub-extensions. Recall that a lifting to characteristic zero of equidistant local actions of the form $(k[[t]], \mathbb{Z}/p\mathbb{Z})$ with conductor m is equivalent to the existence of a **multiplicative good deformation datum**, namely a logarithmic differential form $\omega \in \Omega_{\mathbb{P}^1_k}$, having a unique zero of order $m-1$ in ∞. Let $x_i \in k$ be the poles of ω and let $h_i \in \mathbb{F}_p^{\times}$ be the residue of ω in x_i. Finally, let x be a parameter for $\mathbb{P}^1_k$. Then we can write

$$\omega = \sum_{i=0}^{m} \frac{h_i}{x - x_i} dx = \sum_{i=0}^{m} \frac{h_i x_i}{1 - x_i z} dz$$

as well as

$$\omega = \frac{u}{\prod_{i=0}^{m}(x - x_i)} dx = \frac{u z^{m-1}}{\prod_{i=0}^{m}(1 - x_i z)} dz$$

after change of parameter $z = \frac{1}{x}$. A comparison of the two formulas leads to the following set of conditions that have to be satisfied by poles and residues :

$$\begin{cases} \sum_{i=0}^{m} h_i x_i^k = 0 & \forall\, 1 \leq k \leq m-1 \qquad (*) \\ \prod_{i<j}(x_i - x_j) \neq 0 & \qquad (**) \end{cases}$$

Conversely, any $m+1$-uple of couples $\{(x_i, h_i) \in k \times \mathbb{F}_p^\times\}_{i=0,\dots,m}$ satisfying these equations gives rise to a multiplicative good deformation datum ω. We call such $\{(x_i, h_i)\}$ the **characterizing datum** of ω.
The equations $(*)$ gives rise to a closed subvariety $X \hookrightarrow \mathbb{A}_k^{m+1}$ and the inequalities $(**)$ to an open subvariety $U \hookrightarrow X$. The conditions of existence of good deformation data are then equivalent to the existence of rational points on U. Henrio then states a criterion for the existence of a multiplicative good deformation datum (Proposition 3.16 in [31]), formulated in terms of partitions of the set of residues of ω.

Definition 4.2.1. *Let* $\mathbf{h} = \{h_0, \dots, h_m\}$ *be a* $m+1$*-uple of elements of* $\mathbb{F}_p^\times$ *such that* $\sum h_i = 0$. *A partition* $\mathcal{P}$ *of* $\{0, \dots, m\}$ *is called* $\mathbf{h}$*-adapted, if* $\sum_{j \in J} h_j = 0$ *for every* $J \in \mathcal{P}$.

Using the definition, we can state the criterion in the following way.

Proposition 4.2.2 (Partition condition)**.** If there is a maximal $\mathbf{h}$-adapted partition $\mathcal{P}$ of $\{0, \dots, m\}$ such that $|\mathcal{P}| \leq \left[\frac{m}{p}\right] + 1$, then there is a $m+1$-uple $\{x_0, \dots, x_m\}$ of elements of k and a multiplicative good deformation datum ω, such that $\{(x_i, h_i)\}$ is the characterizing datum of ω.

Sketch of proof. To every partition $\mathcal{P}$ of $\{0, \dots, m\}$, one associates a closed subscheme $X_{\mathcal{P}} \hookrightarrow X$, defined by the equations $X_i - X_j = 0$ for every i, j that are in the same element of $\mathcal{P}$. If P is $\mathbf{h}$-adapted, then $X_{\mathcal{P}} \cong \mathbb{A}_{\tilde{K}}^{|\mathcal{P}|}$.
At this point, one shows that the irreducible components of X have dimension greater or equal than $\left[\frac{m}{p}\right] + 2$. Then, when there exists, as in the assumption, a maximal $\mathbf{h}$-adapted partition $\mathcal{P}$ of $\{0, \dots, m\}$ such that $|\mathcal{P}| \leq \left[\frac{m}{p}\right] + 1$, one shows that $X \setminus X_{\mathcal{P}}$ contains a $\tilde{K}$-rational point of U. The coordinates of this point are exactly the $m+1$-uple $\{x_0, \dots, x_m\}$ which gives rise to ω. □

When the number of poles of ω is a multiple of p this condition becomes quite restrictive:

Proposition 4.2.3. If $m+1 = \lambda p$, then the partition condition is equivalent to ask that, after possibly renumbering of the poles,

$$\begin{cases} h_i = h_0 & \text{if } i \leq p-1 \\ h_i = h_p & \text{if } p \leq i \leq 2p-1 \\ \dots & \\ h_i = h_{(\lambda-1)p} & \text{if } p(\lambda-1) \leq i \leq \lambda p - 1. \end{cases}$$

Proof. Let $\mathcal{P}$ be a maximal $\mathbf{h}$-adapted partition of $\{h_i\}$ with $|\mathcal{P}| \leq \lambda$.
Every set $J \in \mathcal{P}$ is of cardinality at most p. Otherwise, if we let $J = \{h_{i_1}, \dots, h_{i_p}, \dots, \}$, the partition $\mathcal{P}$ is no more maximal, since the set $\{h_{i_1}, h_{i_1} + h_{i_2}, h_{i_1} + h_{i_2} + h_{i_3}, \dots, h_{i_1} + \dots + h_{i_p}\}$ contains necessarily the element 0. Moreover $|J| \geq p$ because $|\mathcal{P}| \leq \lambda$. Hence $|J| = p$ for every $J \in \mathcal{P}$.
Let then $J = \{h_{i_1}, \dots, h_{i_p}\}$, we want to show that $h_{i_j} = h_{i_k}$ for every j and k. To do this, consider the set $\{h_{i_1}, h_{i_1} + h_{i_2}, \dots, h_{i_1} + h_{i_2} + \dots + h_{i_p}\}$. As above, it contains all the elements of $\mathbb{F}_p$ because any repetition would result in a contradiction of the maximality of $\mathcal{P}$. It is the same for the set $\{h_{i_2}, h_{i_1} + h_{i_2}, \dots, h_{i_1} + h_{i_2} + \dots + h_{i_p}\}$, so that $h_{i_1} = h_{i_2}$. With analogous arguments, one shows that $h_{i_1} = h_{i_j}$ for every j. $\square$

Lemma 4.2.4. Let $\{(x_i, h_i)\}$ be a characterizing datum for a multiplicative good deformation datum ω. Then

$$\sum_{i=0}^{m} x_i = 0.$$

Proof. We write $\omega = \frac{uz^{m-1}}{\prod_{i=0}^{m}(1-x_i z)} dz$ and we calculate $u \sum x_i$. Define the following $m+1$-dimensional diagonal matrix and $m+1$-dimensional vector

$$A = \begin{pmatrix} x_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & x_m \end{pmatrix} \quad , \quad H = \begin{pmatrix} h_0 \\ \vdots \\ h_m \end{pmatrix} \quad ,$$

and notice that $u = \operatorname{tr}(A^{\lambda p - 1} H)$.
Now, applying the Cayley-Hamilton theorem and using equations $(*)$, we get:

$$\operatorname{tr}(A) \cdot A^{\lambda p - 1} H = A^{\lambda p} H$$

which gives, taking the traces on both sides, that $(\sum_{i=0}^{m} x_i) \cdot u = \sum_{i=0}^{m} h_i x_i^{\lambda p} = (\sum_{i=0}^{m} h_i x_i^{\lambda})^p = 0$. Since $u \neq 0$, we must have $\sum_{i=0}^{m} x_i = 0$. $\square$

4.2.2 $\mathbb{F}_p$-vector spaces of multiplicative good deformation data

We can use the results of section 4.2.1 to study equidistant lifting of $(\mathbb{Z}/p\mathbb{Z})^n$-actions. In this case, the problem of lifting each intermediate p-extension in a compatible way is equivalent to the existence of n-dimensional $\mathbb{F}_p$-vector spaces of multiplicative good deformation data. We denote such spaces by $L_{m+1,n}$. In [38], examples of these spaces are constructed for $n = 2$ and $p(p-1)|m+1$. Pagot conjectures that this is the only possible case and proves the following (cfr. Section 1.2. of [47]) :

Lemma 4.2.5. Let us assume that there exists a vector space $L_{m+1,n}$. Then $m + 1 = \lambda p^{n-1}$, with $\lambda \in \mathbb{N}$. If $\{\omega_1, \dots, \omega_n\} \in L_{m+1,n}$ is a basis for this vector space, then any pair (ω_i, ω_j) with $i \neq j$ has exactly $\lambda(p-1)^{n-1}$ poles in common.

Let $n = 2$ and suppose to have a $\mathbb{F}_p$-vector spaces of multiplicative good deformation data $L_{m+1,2}$ generated by two forms ω_1 and ω_2. By Lemma 4.2.5, we can partition the set of poles of these forms in such a way that $\omega_1 + j\omega_2$ has its poles in all but the set $X^{(j)} := \{x_1^{(j)}, \dots, x_\lambda^{(j)}\}$ for $j = 0, \dots, p-1$, and that ω_2 has its poles in all but the set $X^{(p)} := \{x_1^{(p)}, \dots, x_\lambda^{(p)}\}$. We can then write

$$\omega_1 := \frac{uz^{m-1}}{\prod_{j=1}^{p}\prod_{i=1}^{\lambda}(1 - x_i^{(j)}z)}dz \quad \text{and} \quad \omega_2 := \frac{vz^{m-1}}{\prod_{j=0}^{p-1}\prod_{i=1}^{\lambda}(1 - x_i^{(j)}z)}dz, \quad \text{for suitable} \quad u, v \in k.$$

Then, we consider the $p+1$ polynomials $P^{(j)}$, for $j \in \{0, \dots, p\}$, defined by

$$\begin{cases} P^{(0)} = \prod_{i=1}^{\lambda}(X - x_i^{(0)}) \\ \dots \\ P^{(p)} = \prod_{i=1}^{\lambda}(X - x_i^{(p)}) \end{cases}$$

which appear also in the article by Pagot (cfr. section 1.3. in [47]).
Our aim is to express the equations $(*)$ in terms of the coefficients of such $P^{(j)}$, which are symmetric functions in the variables given by the poles. We set $p_n(X^{(j)}) = \sum_{i=1}^{\lambda} {x_i^{(j)}}^n$ the n-th symmetric power sum and

$$S_n(X^{(j)}) = \begin{cases} \sum_{i_1 < \dots < i_n} x_{i_1}^{(j)} \dots x_{i_n}^{(j)} & \text{if} \quad 1 \le n \le \lambda \\ 1 & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n > \lambda \end{cases}$$

the n-th elementary symmetric polynomial. The polynomials are defined in such a way to get the following conditions on the $S_i(X^{(j)})$:

Lemma 4.2.6. Let $a = \frac{u}{v}$ and denote by $S_i(t_1, \dots, t_n)$ the i-th elementary symmetric polynomial in the variables $\{t_1, \dots, t_n\}$. Then

$$(a + j)S_i(X^{(j)}) = aS_i(X^{(0)}) + jS_i(X^{(p)})$$

for every $j \in \{0, \dots, p\}$.

Proof. Let us suppose that $j \in \{1, \dots, p-1\}$, otherwise we have a trivial equality. We have

$$\omega_1 = \frac{uP^{(0)}(x)dx}{\prod_{k=0}^{p} P^{(k)}(x)} \quad , \quad \omega_2 = \frac{vP^{(p)}(x)dx}{\prod_{k=0}^{p} P^{(k)}(x)} \quad \text{and} \quad \omega_1 + j\omega_2 = \frac{w_j P^{(j)}(x)}{\prod_{k=0}^{p} P^{(k)}(x)}$$

which implies that $w_j P^{(i)}(x) = uP^{(0)}(x) + jvP^{(p)}(x)$. Since the polynomials $P^{(j)}$ are monic and of the same degree, one gets $w_j = u + jv$ and then the claim. □

Corollary 4.2.7. We have $S_1(X^{(j)}) = S_1(X^{(j')})$ for every $j, j' \in \{0, \dots, p\}$.

Proof. Lemma 4.2.4 gives $\sum_{j=1}^{p} S_1(X^{(j)}) = 0$ when applied to ω_1 and $\sum_{j=0}^{p-1} S_1(X^{(j)}) = 0$ when applied to ω_2. We then have $S_1(X^{(0)}) = S_1(X^{(p)})$. Then, applying Lemma 4.2.6, we find $S_1(X^{(j)}) = S_1(X^{(0)})$ for every j. $\square$

4.2.3 Actions of $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

From now on, we suppose that $n = 2$, and that $p = 3$. The fundamental remark is that, with these assumptions, the characterizing data for ω_1 and ω_2 satisfy the partition condition, contained in Proposition 4.2.2. Some of the results contained in this section stay true when assuming that the partition condition is satisfied, even for $p \neq 3$. Nevertheless, for every $p > 3$, there are examples of multiplicative good deformation data not satisfying the partition condition, as shown in section 1.1 of [47].

We investigate here the algebraic restrictions that the residues shall satisfy in order to appear in the characterizing datum of a multiplicative good deformation datum. The main tool to prove the results of this section is the combinatorics of the poles of the differential forms ω_1 and ω_2. The features of such combinatorics are expressed via the set of variables $X^{(j)} = (x_1^{(j)}, \dots, x_\lambda^{(j)})$, and their symmetric functions $p_n(X^{(j)})$ and $S_n(X^{(j)})$.
Notice that we have chosen the $P^{(j)}$ in such a way that the poles of $X^{(1)}$ have the same residues for ω_1 and ω_2, whence the poles of $X^{(2)}$ have exactly the opposite residues when considered in ω_1 with respect as when considered in ω_2. Then one loses no information in considering only the residues of ω_2 at poles of $X^{(0)}$: let us call $h_i^{(j)}$ the residue of ω_1 in $x_i^{(j)}$ if $j \in \{1, \dots, p\}$ and $h_i^{(0)}$ the residue of ω_2 in $x_i^{(0)}$. They are all element of $\mathbb{F}_p^\times$ and, by Proposition 4.2.3, the cardinality of the set $\{x_i^{(j)} : h_i^{(j)} = 1, j \in \{1, 2, 3\}\}$ is a multiple of 3. If we define $q_n(X^{(j)}) = \sum_{i=1}^{\lambda} h_i^{(j)} {x_i^{(j)}}^n$, then the equations (*) become, for $0 \leq k \leq \lambda p - 2$,

$$\begin{cases} q_k(X^{(1)}) + q_k(X^{(2)}) + q_k(X^{(3)}) = 0 \\ q_k(X^{(0)}) - q_k(X^{(1)}) + q_k(X^{(2)}) = 0 \end{cases} .$$

Definition 4.2.8. *Let S be a set of poles for a differential form $\omega \in L_{m+1,2}$. Then S is said of type $(n_1, \dots, n_{p-1})$ if there are exactly n_i poles in S with residue equal to i for every $i \in \mathbb{F}_p^\times$.*

Example 4.2.9. *The fact that $q_0(X^{(1)}) + q_0(X^{(2)}) + q_0(X^{(3)}) = 0$ and $q_0(X^{(0)}) - q_0(X^{(1)}) + q_0(X^{(2)}) = 0$ implies that there exists at least a j with $q_0(X^{(j)}) = 0$. Hence, $X^{(j)}$ is a set of poles of type (n_1, n_2) with $n_1 \equiv n_2 \bmod 3$.*

The result in Example 4.2.9 has already some consequences in the study of configurations of residues that cannot occur in spaces of good deformation data.

Proposition 4.2.10. Let $\lambda = 5$. Then there are at least two values of $j \in \{0, 1, 2, 3\}$, such that $X^{(j)}$ is not of type $(5, 0)$ or $(0, 5)$.

Proof. With a proper choice of a basis for $L_{15,2}$, we may suppose that $X^{(0)}$ is of type $(4, 1)$ or $(1, 4)$, and that $x_5^{(0)}$ is the pole having residue different from the others. If all $X^{(j)}$ are of type $(5, 0)$ for $j = 1, 2, 3$, then we have $x_5^{(0)} = x_1^{(0)} + x_2^{(0)} + x_3^{(0)} + x_4^{(0)}$, since $S_1(X^{(j)})$ is a constant.

We have then that $p_k(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}) = p_1(x_1^{(0)} + x_2^{(0)} + x_3^{(0)} + x_4^{(0)})^k$ for every k. This leads, by Newton identities to get the relations

$$S_2(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}) = 0$$

and

$$S_3(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)})S_1(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}) = S_4(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}).$$

After possible translation on the set of poles, we may take $S_1(X^{(0)}) = 0$ (or, equivalently, $x_5^{(0)} = 0$). But this would imply $S_4(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}) = 0$, so that $0 \in \{x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}\}$. This is a contradiction, since all the poles are distinct, by condition $(**)$. □

Remark 4.2.11. With the same argument, we can show that Proposition 4.2.10 holds more in general when $\lambda \equiv -1 \bmod 3$, assuming that $X^{(0)}$ is of type $(\lambda - 1, 1)$ when $\lambda > 5$. This leads, among other things, to show that, if $\lambda = 2$, $X^{(j)}$ are of type $(1, 1)$ for every j. This is a simpler proof of the one in [47], Theorem 2.2, second part.

We may suppose without loss of generalty that $X^{(0)}$ is a set of poles of type (n_1, n_2) with $n_1 \equiv n_2 \bmod 3$, (i.e. it is of type $(4, 1)$ or $(1, 4)$). From now on, we make this assumption, and we study the possible types of $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$. When $\lambda = 5$, we can also use a more direct approach to determine the set of residues, in order to get an explicit form for the equation $(*)$.

Theorem 4.2.12. Let $\lambda = 5$. Then the sets of poles $(X^{(j)})$ are of type $(4, 1)$ or $(1, 4)$ for every $j \in \{0, 1, 2, 3\}$. More in general, for any λ, the $(X^{(j)})$ are of type (n_1, n_2) with $n_1 \equiv n_2$ mod. 3.

Proof. To prove this result one can use a generalization of the construction of the proof of Lemma 4.2.6. We start by writing down an identity that is used several times in this proof:

$$\sum_{i=1}^{n} \frac{h_i}{X - x_i} = \frac{\sum_{j=0}^{n-1} \sum_i (-1)^j h_i S_j(x_1, \ldots, \hat{x}_i, \ldots, x_n) X^{n-1-j}}{\prod_i (X - x_i)}.$$

We consider then the set of differential forms $\{\omega_j^*\}_{j=0,\ldots,3}$ defined by $\omega_j^* = \sum_{i=1}^{5} \frac{h_i}{X - x_i^{(j)}} dX$, and the polynomials $Q^{(j)}$ such that $\omega_j^* = \frac{Q^{(j)}(X)}{\prod_i (X - x_i^{(j)})} dX = \frac{Q^{(j)}(X)}{P^{(j)}(X)}\mathrm{dX}$. Using the previous description of differentials with simple poles, plus the relation

$$S_n(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \sum_{k=0}^{n} (-1)^k x_i^k S_{n-k}(x_1, \ldots, x_n),$$

we get

$$Q^{(j)}(X) = \sum_{i=0}^{4} q_i(X^{(j)}) \hat{P}_{4-i}^{(j)}(X),$$

where the “hatted” polynomials are defined by $\hat{P}_n^{(j)}(X) = \sum_{i=0}^{n} S_{n-i}(X^{(j)}) X^i$ for every $n \leq 4$. Notice that the polynomials $Q^{(1)}$ and $Q^{(2)}$ are related to $Q^{(0)}$ and $Q^{(3)}$. In fact, from Lemma 4.2.6, one gets $(a+j)\hat{P}_n^{(j)}(X) = a\hat{P}_n^{(0)}(X) + j\hat{P}_n^{(3)}(X)$. Moreover, for $k \leq 13$ we have $q_k(X^{(j)}) =$

$q_k(X^{(3)}) - jq_k(X^{(0)})$. From this we get

$$(a+j)Q^{(j)}(X) = jQ^{(3)}(X) - ajQ^{(0)}(X) + R(X), \tag{4.1}$$

where $R(X) := a\sum_{i=0}^{4} q_i(X^{(3)})\hat{P}_{4-i}^{(0)}(X) - \sum_{i=0}^{4} q_i(X^{(0)})\hat{P}_{4-i}^{(3)}(X)$, and does not depend on j. From now on, we omit the variable X in the polynomials of the following equations, for the sake of readability.

The relations between $Q^{(j)}$s and $P^{(j)}$s are made explicit by the relations $\omega_2 = \omega_0^* - \omega_1^* + \omega_2^*$ and $\omega_1 = \omega_1^* + \omega_2^* + \omega_3^*$. These yield

$$\begin{cases} Q^{(0)}P^{(1)}P^{(2)} - Q^{(1)}P^{(0)}P^{(2)} + Q^{(2)}P^{(0)}P^{(1)} = v \\ Q^{(1)}P^{(2)}P^{(3)} + Q^{(2)}P^{(1)}P^{(3)} + Q^{(3)}P^{(1)}P^{(2)} = u \end{cases}$$

Now, after Lemma 4.2.6 and Corollary 4.2.7 one has the relations $(a+1)P^{(1)} = aP^{(0)} + P^{(3)}$ and $(a-1)P^{(2)} = aP^{(0)} - P^{(3)}$ that turn the previous conditions into

$$[a^2Q^{(0)} - a(a+1)Q^{(1)} + a(a-1)Q^{(2)}]{P^{(0)}}^2 + [(a-1)Q^{(2)} + (a+1)Q^{(1)}]P^{(0)}P^{(3)} - \\ Q^{(0)}{P^{(3)}}^2 = v(a^2-1)$$

and

$$[-(a+1)Q^{(1)} + (a-1)Q^{(2)} - Q^{(3)}]{P^{(3)}}^2 + [a(a+1)Q^{(1)} + a(a-1)Q^{(2)}]P^{(0)}P^{(3)} + \\ a^2Q^{(3)}{P^{(0)}}^2 = u(a^2-1).$$

We use now the relations between the $Q^{(j)}$, to get the formula

$$aQ^{(3)}{P^{(0)}}^2 - RP^{(0)}P^{(3)} - Q^{(0)}{P^{(3)}}^2 = v(a^2-1).$$

Now, $\deg(R) \leq 3$, yielding $\deg(Q^{(3)}) \leq 3$. But this means, by equation (4.1), that $\deg(Q^{(j)}) \leq 3$ for every j. Then, $\sum_{i=0}^{5} h_i^{(j)}$ vanishes for every j, and the theorem is proved. An analogous argument permits to treat the case where $\lambda > 5$. $\square$

Remark 4.2.13. The *decoupages* $\omega_2 = \omega_0^* - \omega_1^* + \omega_2^*$ and $\omega_1 = \omega_1^* + \omega_2^* + \omega_3^*$ permit already to exclude several possibilities. In fact, ω_1, ω_2 and ω_0^* have no poles at ∞, and this forces the ω_j^* to have the same residue at ∞. By Proposition 4.2.10, one excludes the case where this residue is -1, and by Theorem 4.2.12 the case of residue equal to 1. This is a rather strong result: in this way we have determined uniquely the possible values of residues $h_i^{(j)}$. After possibly renumbering the poles (but still assuming that $X^{(j)}$ is a set of poles that are all not belonging to the same differential form), we may in fact assume without loss of generality that $h_i^{(j)} = 1$ if $2 \leq i \leq 5$, and $h_1^{(j)} = -1$ for every $j \in \{0, 1, 2, 3\}$. This is what permits us to study completely the case where $\lambda = 5$, finding explicitly the poles x_i in this framework.

4.2.4 The case "(4,1)"

Let $p = 3$ and $\lambda = 5$. Theorem 4.2.12 allows us to suppose, without loss of generality, that $h_1^{(j)} = -1$ for every $j \in \{0, 1, 2, 3\}$ and that $h_i^{(j)} = 1$ for $i \in \{2, 3, 4, 5\}$ and $j \in \{0, 1, 2, 3\}$. Recall that we set $p_\ell(X^{(j)}) = {x_1^{(j)}}^\ell + \cdots + {x_5^{(j)}}^\ell$, the ℓ-th power sum symmetric polynomial. Then one can reformulate the set of equations $(*)$ (respectively for ω_1 and ω_2) as

$$\begin{cases} p_\ell(X^{(1)}) + p_\ell(X^{(2)}) + p_\ell(X^{(3)}) = -{x_1^{(1)}}^\ell - {x_1^{(2)}}^\ell - {x_1^{(3)}}^\ell \\ p_\ell(X^{(0)}) - p_\ell(X^{(1)}) + p_\ell(X^{(2)}) = -{x_1^{(0)}}^\ell + {x_1^{(1)}}^\ell - {x_1^{(2)}}^\ell \end{cases} \tag{4.2}$$

Moreover, since these equations are invariant by translation and homothetic transformations, we can suppose that the constant $S_1(X^{(j)})$ vanishes, and that $x_1^{(0)} = 1$. For this change to be admissible, one has only to show that $x_1^{(0)} \neq 0$ when $S_1(X^{(j)}) = 0$. This is true, since $x_1^{(0)} - x_1^{(1)} + x_1^{(2)} = S_1(X^{(j)})$ and the poles are distinct. From (4.2), with $\ell = 1$, one gets the following linear conditions on poles of ω_1:

$$\begin{cases} x_1^{(1)} + x_1^{(2)} + x_1^{(3)} = 0 \\ x_1^{(1)} - x_1^{(2)} = 1 \end{cases} \tag{4.3}$$

which gives $x_1^{(1)} = x_1^{(3)} - 1$ and $x_1^{(2)} = x_1^{(3)} + 1$. Once that one has these relations, it is not hard to find the right hand side of the equations in terms of the sole number $x_1^{(3)}$. To get the same for the left hand side, we apply Newton identities, combined with the relations of Lemma 4.2.6, to reduce the number of variables.

Proposition 4.2.14. There are the following relations between $p_\ell(X^{(j)})_{j=1,2}$, $p_\ell(X^{(0)})$ and $p_\ell(X^{(3)})$:

- $(a+j)p_2(X^{(j)}) = ap_2(X^{(0)}) + jp_2(X^{(3)})$
- $(a+j)^2 p_4(X^{(j)}) = (a+j)(ap_4(X^{(0)}) + jp_4(X^{(3)})) + aj(p_2(X^{(0)}) - p_2(X^{(3)}))^2$
- $(a+j)^2 p_5(X^{(j)}) = (a+j)(ap_5(X^{(0)}) + jp_5(X^{(3)})) - aj((p_2(X^{(0)}) - p_2(X^{(3)}))(S_3(X^{(0)}) - S_3(X^{(3)})))$

Proof. The first equation is proved by observing that $S_1(X^{(j)}) = 0$ entails $p_2(X^{(j)}) = S_2(X^{(j)})$. Let us prove the relation in degree 4: Newton identities give

$$p_4(X^{(j)}) = -S_2(X^{(j)})p_2(X^{(j)}) - S_4(X^{(j)}) \text{ for every } j \in \{0, 1, 2, 3\}.$$

Then, using Lemma 4.2.6 and the relation in degree 2, we can write

$$\begin{aligned}(a+j)^2 p_4(X^{(j)}) &= -(aS_2(X^{(0)}) + jS_2(X^{(3)})(ap_2(X^{(0)}) + jp_2(X^{(3)})) - (a+j)(aS_4(X^{(0)}) + jS_4(X^{(3)})) \\ &= a(a+j)(-S_2(X^{(0)})p_2(X^{(0)}) - S_4(X^{(0)})) + j(a+j)(-S_2(X^{(3)})p_2(X^{(3)}) - S_4(X^{(3)})) + \\ &\quad aj(S_2(X^{(0)}) - S_2(X^{(3)}))(p_2(X^{(0)}) - p_2(X^{(3)})) \\ &= (a+j)(ap_4(X^{(0)}) + jp_4(X^{(3)})) + aj(p_2(X^{(0)}) - p_2(X^{(3)}))^2.\end{aligned}$$

With the same strategy we can compute the relation in the degree 5:

$$\begin{aligned}(a+j)^2 p_5(X^{(j)}) &= (aS_3(X^{(0)})+jS_3(X^{(3)})(ap_2(X^{(0)})+jp_2(X^{(3)}))-(a+j)(aS_5(X^{(0)})+jS_5(X^{(3)}))\\ &= a(a+j)(-S_3(X^{(0)})p_2(X^{(0)})-S_5(X^{(0)}))+j(a+j)(-S_3(X^{(3)})p_2(X^{(3)})-S_5(X^{(3)}))-\\ &aj(S_3(X^{(0)})-S_3(X^{(3)}))(p_2(X^{(0)})-p_2(X^{(3)}))\\ &= (a+j)(ap_5(X^{(0)})+jp_5(X^{(3)}))-aj((p_2(X^{(0)})-p_2(X^{(3)}))(S_3(X^{(0)})-S_3(X^{(3)}))).\end{aligned}$$

□

Proposition 4.2.14 is the key fact to perform our strategy. By expressing $p_\ell(X^{(j)})$ in terms of $p_\ell(X^{(0)})$ and $p_\ell(X^{(3)})$ we obtain the left hand side of equations (4.2) uniquely in terms of power sums in the variables of $X^{(0)}$ and $X^{(3)}$. Since the right hand side is expressed in terms of $x_1^{(3)}$, we can explicit the $x_i^{(j)}$ in terms only of $x_1^{(3)}$. This proceeding, despite its elementary nature, is computationally rather complex. We introduce then the following notations to help us simplify the formulae.
Define $\alpha_i = p_i(X^{(0)})$, $\beta_i = (p_i(X^{(0)}) - p_i(X^{(3)}))$, $\gamma_i = (S_i(X^{(0)}) - S_i(X^{(3)}))$, $\delta_i = p_i(X^{(0)})$ and let $j \in \{1, 2\}$. If one wants to extend the results of Proposition 4.2.14 to homogeneous polynomial of higher degree the following lemma turns out to be useful.

Lemma 4.2.15. For every couple of natural numbers $i, k > 0$ and every $j \in \{1, 2\}$ we have

$$(aS_i(X^{(0)})+jS_i(X^{(3)}))(ap_k(X^{(0)})+jp_k(X^{(3)}))-(a+j)[aS_i(X^{(0)})p_k(X^{(0)})+jS_i(X^{(3)})p_k(X^{(3)})] = -aj\gamma_i\beta_k.$$

Proof.

$$\begin{aligned}&(aS_i(X^{(0)})+jS_i(X^{(3)}))(ap_k(X^{(0)})+jp_k(X^{(3)}))-(a+j)[aS_i(X^{(0)})p_k(X^{(0)})+jS_i(X^{(3)})p_k(X^{(3)})] =\\ &aj(S_i(X^{(0)})p_k(X^{(3)})+S_i(X^{(3)})p_k(X^{(0)})-S_i(X^{(0)})p_k(X^{(0)})-S_i(X^{(3)})p_k(X^{(3)})) =\\ &aj(S_i(X^{(0)})-S_i(X^{(3)}))(p_k(X^{(3)})-p_k(X^{(0)})) = -aj\gamma_i\beta_k.\end{aligned}$$

□

Once the notations introduced, Proposition 4.2.14 transforms formulas (4.1) for $\ell = 2, 4, 5$ into

$$\begin{cases} {x_1^{(1)}}^2 + {x_1^{(2)}}^2 + {x_1^{(3)}}^2 = \frac{a^2}{a^2-1}\beta_2 \\ {x_1^{(1)}}^2 - {x_1^{(2)}}^2 - {x_1^{(0)}}^2 = \alpha_2 - \frac{a}{a^2-1}\beta_2 \end{cases}$$
$$\begin{cases} {x_1^{(1)}}^4 + {x_1^{(2)}}^4 + {x_1^{(3)}}^4 = \frac{a^2}{a^2-1}\beta_4 + \frac{a^2}{(a^2-1)^2}\beta_2^2 \\ {x_1^{(1)}}^4 - {x_1^{(2)}}^4 - {x_1^{(0)}}^4 = \alpha_4 - \frac{a}{a^2-1}\beta_4 + \frac{a^3+a}{(a^2-1)^2}\beta_2^2 \end{cases}$$
$$\begin{cases} {x_1^{(1)}}^5 + {x_1^{(2)}}^5 + {x_1^{(3)}}^5 = \frac{a^2}{a^2-1}\beta_5 - \frac{a^2}{(a^2-1)^2}\beta_2\gamma_3 \\ {x_1^{(1)}}^5 - {x_1^{(2)}}^5 - {x_1^{(0)}}^5 = \alpha_5 - \frac{a}{a^2-1}\beta_5 - \frac{a^3+a}{(a^2-1)^2}\beta_2\gamma_3. \end{cases}$$

The left hand sides can be easily calculated, using equalities 4.3:

$$\begin{aligned}
{x_1^{(1)}}^2 + {x_1^{(2)}}^2 + {x_1^{(3)}}^2 &= -1\\
{x_1^{(1)}}^2 - {x_1^{(2)}}^2 - {x_1^{(0)}}^2 &= -x_1^{(3)} - 1\\
{x_1^{(1)}}^4 + {x_1^{(2)}}^4 + {x_1^{(3)}}^4 &= -1\\
{x_1^{(1)}}^4 - {x_1^{(2)}}^4 - {x_1^{(0)}}^4 &= {x_1^{(3)}}^3 + x_1^{(3)} - 1\\
{x_1^{(1)}}^5 + {x_1^{(2)}}^5 + {x_1^{(3)}}^5 &= x_1^{(3)}(1 - {x_1^{(3)}}^2)\\
{x_1^{(1)}}^5 - {x_1^{(2)}}^5 - {x_1^{(0)}}^5 &= {x_1^{(3)}}^2(1 - {x_1^{(3)}}^2)
\end{aligned}$$

so that the α_i and β_i can be explicited in terms of a, $x_1^{(3)}$ and γ_3:

$$\begin{aligned}
\alpha_2 &= -\frac{1}{a} - 1 - x_1^{(3)}\\
\beta_2 &= \frac{1-a^2}{a^2}\\
\alpha_4 &= {x_1^{(3)}}^3 + x_1^{(3)} - 1 + \frac{a^2+1}{a^3}\\
\beta_4 &= \frac{1-a^4}{a^4}\\
\alpha_5 &= (x_1^{(3)} + \frac{1}{a})(x_1^{(3)} - {x_1^{(3)}}^3) - \frac{\gamma_3}{a}\\
\beta_5 &= \frac{a^2-1}{a^2} x_1^{(3)}(1 - {x_1^{(3)}}^2) - \frac{\gamma_3}{a^2}.
\end{aligned}$$

Remark 4.2.16. Since we are dealing with power sums in characteristic 3, when $3|i$ the computation of α_i and β_i gives a tautological condition and hence we are not interested in it.

Now we look for the values of δ_i and γ_i, in terms of the variables a and $x_1^{(3)}$. In order to do this, we relate the γ_i and δ_i with α_i and β_i, using Newton identities.
We get $\gamma_4 + \beta_4 = \alpha_2\beta_2 + \beta_2^2$, $\delta_4 = -\alpha_4 - \alpha_2^2$, $\alpha_5 = \alpha_2\delta_3 - \delta_5$, and $\gamma_5 + \beta_5 = \delta_3\alpha_2 + (\alpha_2 - \beta_2)\gamma_3$. Then

$$\begin{aligned}
\gamma_4 &= \frac{(a^2-1)(ax_1^{(3)} + 1)}{a^3}\\
\delta_4 &= -\frac{a^3{x_1^{(3)}}^3 + a^3{x_1^{(3)}}^2 - a^2x_1^{(3)} + a + 1}{a^3}.
\end{aligned}$$

Moreover, evaluating the polynomial $P^{(0)}$ in $x_1^{(0)} = 1$, we have that $\delta_5 = S_4(x_2^{(0)}, \ldots, x_5^{(0)}) = \delta_4 - \delta_3 + \delta_2 + 1$. Then on the one hand we have $\delta_5 = -\delta_3 - \frac{a^3({x_1^{(3)}}^3 + {x_1^{(3)}}^2 + x_1^{(3)}) + a^2(-x_1^{(3)}+1) + a + 1}{a^3}$, and on the other hand it is $\delta_5 = \alpha_2\delta_3 - \alpha_5$. The two conditions give

$$\delta_3 = \frac{\gamma_3}{1 + ax_1^{(3)}} + \frac{a^2({x_1^{(3)}}^3 + {x_1^{(3)}}^2 + 1) + a(-x_1^{(3)} + 1) + 1}{a^2} \tag{4.4}$$

$$\delta_5 = -\frac{\gamma_3}{1 + ax_1^{(3)}} + \frac{a^3({x_1^{(3)}}^3 + {x_1^{(3)}}^2 - x_1^{(3)} - 1) + a^2(1 - x_1^{(3)}) + a - 1}{a^3} \tag{4.5}$$

In the same spirit, evaluating $P^{(3)}$ in $x_1^{(3)}$, one finds

$$\delta_5 - \gamma_5 = x_1^{(3)}(\delta_4 - \gamma_4) - {x_1^{(3)}}^2(\delta_3 - \gamma_3) + {x_1^{(3)}}^3(\delta_2 - \gamma_2) + {x_1^{(3)}}^5.$$

This can be used to obtain a linear system in the variables γ_3 and γ_5:

$$\begin{cases} -\gamma_5 = \frac{1+a{x_1^{(3)}}^3}{1+ax_1^{(3)}}\gamma_3 + \frac{a^3(-{x_1^{(3)}}^2 - x_1^{(3)} - 1) + a^2(x_1^{(3)}+1) + a(-{x_1^{(3)}}^3 - x_1^{(3)} + 1) - 1}{a^3} \\ \gamma_5 = \frac{-a^2{x_1^{(3)}}^2 - a + 1}{a(1+ax_1^{(3)})}\gamma_3 + \frac{a^3(-{x_1^{(3)}}^4 - {x_1^{(3)}}^3 - {x_1^{(3)}}^2 + x_1^{(3)} - 1) + a^2(-{x_1^{(3)}}^3 + 1) + a(-{x_1^{(3)}}^3 + x_1^{(3)} + 1) - 1}{a^3} \end{cases}$$

One gets, when $1 + a^2({x_1^{(3)}}^3 - {x_1^{(3)}}^2) \neq 0$, that

$$\begin{cases} \gamma_3 = & -\frac{1+ax_1^{(3)}}{1+a^2({x_1^{(3)}}^3 - {x_1^{(3)}}^2)} \cdot \frac{a^3(-{x_1^{(3)}}^4 - {x_1^{(3)}}^3 + {x_1^{(3)}}^2 + 1) + a^2(-{x_1^{(3)}}^3 + x_1^{(3)} - 1) + a({x_1^{(3)}}^3 + 1) + 1}{a^2} \\ \gamma_5 = & \frac{a^5(-{x_1^{(3)}}^7 - {x_1^{(3)}}^6 - {x_1^{(3)}}^5 + {x_1^{(3)}}^3 - {x_1^{(3)}}^2) + a^4(-{x_1^{(3)}}^6 - {x_1^{(3)}}^4 + {x_1^{(3)}}^3 - {x_1^{(3)}}^2 + 1)}{a^3(1+a^2({x_1^{(3)}}^3 - {x_1^{(3)}}^2))} + \\ & \frac{a^3(-{x_1^{(3)}}^6 - {x_1^{(3)}}^5 + {x_1^{(3)}}^4 - {x_1^{(3)}}^3 + {x_1^{(3)}}^2 - x_1^{(3)}) + a^2(-{x_1^{(3)}}^2 - x_1^{(3)}) + a({x_1^{(3)}}^3 + x_1^{(3)} + 1) + 1}{a^3(1+a^2({x_1^{(3)}}^3 - {x_1^{(3)}}^2))}. \end{cases}$$

Finally, substituting the value of γ_3 in (4.4) and (4.4) we find

$$\begin{cases} \delta_3 = & \frac{a^3({x_1^{(3)}}^6 - {x_1^{(3)}}^4 + {x_1^{(3)}}^3 - {x_1^{(3)}}^2) + a^2({x_1^{(3)}}^2 - 1) + a(-x_1^{(3)} - 1) - {x_1^{(3)}}^3 - x_1^{(3)}}{a(1+a^2({x_1^{(3)}}^3 - {x_1^{(3)}}^2))} \\ \delta_5 = & \frac{a^5({x_1^{(3)}}^6 + {x_1^{(3)}}^4 + {x_1^{(3)}}^2) + a^4({x_1^{(3)}}^4 + {x_1^{(3)}}^3 + 1) + a^3({x_1^{(3)}}^3 + 1) + a^2({x_1^{(3)}}^2 - x_1^{(3)} - 1) - a - 1}{a^3(1+a^2({x_1^{(3)}}^3 - {x_1^{(3)}}^2))}. \end{cases}$$

With these data we can, in principle find explicitly the poles of $X^{(0)}$ and $X^{(3)}$: they are respectively the roots of

$$P^{(0)}(x) = x^5 + \alpha_2 x^3 - \delta_3 x^2 + \delta_4 x - \delta_5$$

and of

$$P^{(3)}(x) = x^5 + (\alpha_2 - \beta_2)x^3 - (\delta_3 - \gamma_3)x^2 + (\delta_4 - \gamma_4)x - (\delta_5 - \gamma_5).$$

Moreover, since we know that $P^{(j)} = \frac{aP^{(0)} + jP^{(3)}}{a+j}$ (cfr. Lemma 4.2.6), this gives also the values of $X^{(1)}$ and $X^{(2)}$. Once the poles expressed in terms of a and $x_1^{(3)}$, one can directly verify if the conditions expressed by equations $(*)$ are satisfied. Nevertheless, this proceeding is of a complex computational nature, and it is not possible to do it directly by hand. To perform completely this strategy, one shall be helped by computational algebra programs.

A necessary condition for a

We give now a necessary polynomial condition that u and v shall satisfy, if a lifting of an equidistant $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$-action with 20 ramification points exists. We get this condition by computing α_7 and β_7 in two different ways. Firstly, in the same spirit of the calculations of the preceding section. Then, using Newton formulas, and the fact that the values of $\alpha_i, \beta_i, \gamma_i, \delta_i$ are known for $i \leq 5$. The same proceeding can be applied to every α_i and β_i for $i \leq 13$, but it is difficult to perform it by hand already for $i = 8$. If no contradiction arises, then the configuration of poles, gives rise to a 2-dimensional vector space of multiplicative good deformation data. Otherwise, there is no such lifting, for $n = 2$, $p = 3$ and $\lambda = 5$. We arrive at the point of verifying a polynomial condition on the value of a, which is completely determined by the values of poles.

With Newton identities one gets

$$p_7(X^{(j)}) = -S_2(X^{(j)})p_5(X^{(j)}) + S_3(X^{(j)})p_4(X^{(j)}) + S_5(X^{(j)})p_2(X^{(j)}),$$

hence, using the same calculations of Proposition 4.2.14,

$$(a+j)^2 p_7(X^{(j)}) = (a+j)(ap_7(X^{(0)}) + jp_7(X^{(3)})) + \frac{aj}{(a+j)^2}(\beta_2(\beta_5 - \gamma_5) + \gamma_4\gamma_3 + \beta_2^2\delta_3 - \frac{a}{a+j}\beta_2^2\gamma_3).$$

The linear system in the variables α_7 and β_7 is then given by

$$\begin{cases} x_1^{(3)}({x_1^{(3)}}^2 - 1) = \frac{a^2}{a^2-1}\beta_7 + \frac{a^2}{(a^2-1)^2}(\beta_2\beta_5 - \beta_2\gamma_5 + \gamma_4\gamma_3 + \beta_2^2\delta_3) + \frac{a^2}{(a^2-1)^3}\beta_2^2\gamma_3 \\ {x_1^{(3)}}^4({x_1^{(3)}}^2 - 1) = \alpha_7 - \frac{a}{a^2-1}\beta_7 + \frac{a^3+a}{(a^2-1)^2}(\beta_2\beta_5 - \beta_2\gamma_5 + \gamma_4\gamma_3 + \beta_2^2\delta_3) - \frac{a^5}{(a^2-1)^3}\beta_2^2\gamma_3. \end{cases}$$

Making the terms explicit, this gives

$$\begin{cases} \beta_7 = & \frac{a^9({x_1^{(3)}}^6+{x_1^{(3)}}^4+{x_1^{(3)}}^2)+a^7({x_1^{(3)}}^7-{x_1^{(3)}}^6+{x_1^{(3)}}^5+{x_1^{(3)}}^4+{x_1^{(3)}}^3-{x_1^{(3)}}^2-x_1^{(3)})+a^6({x_1^{(3)}}^6-{x_1^{(3)}}^4-{x_1^{(3)}}^3-{x_1^{(3)}}^2-x_1^{(3)})}{a^5(a^2-1)(1+a^2({x_1^{(3)}}^3-{x_1^{(3)}}^2))} + \\ & \frac{a^5(-{x_1^{(3)}}^7-{x_1^{(3)}}^5+{x_1^{(3)}}^3-{x_1^{(3)}}^2+x_1^{(3)}+1)+a^4(-{x_1^{(3)}}^6+{x_1^{(3)}}^4-x_1^{(3)}-1)+a^3(-{x_1^{(3)}}^5+{x_1^{(3)}}^4-{x_1^{(3)}}^3-{x_1^{(3)}}^2-x_1^{(3)})}{a^5(a^2-1)(1+a^2({x_1^{(3)}}^3-{x_1^{(3)}}^2))} + \\ & \frac{a^2{x_1^{(3)}}^4+a(-{x_1^{(3)}}^3+x_1^{(3)})+{x_1^{(3)}}^3+x_1^{(3)}}{a^5(a^2-1)(1+a^2({x_1^{(3)}}^3-{x_1^{(3)}}^2))} \\ \alpha_7 = & \frac{a^8({x_1^{(3)}}^9-{x_1^{(3)}}^8-{x_1^{(3)}}^7+{x_1^{(3)}}^6)+a^7({x_1^{(3)}}^7-{x_1^{(3)}}^6+{x_1^{(3)}}^5+{x_1^{(3)}}^2-x_1^{(3)})+a^6(-{x_1^{(3)}}^9+{x_1^{(3)}}^8+{x_1^{(3)}}^7+{x_1^{(3)}}^6+x_1^{(3)}-1)}{a^4(a^2-1)(1+a^2({x_1^{(3)}}^3)-{x_1^{(3)}}^2)} + \\ & \frac{a^5(-{x_1^{(3)}}^7+{x_1^{(3)}}^6-{x_1^{(3)}}^4+{x_1^{(3)}}^3+{x_1^{(3)}}^2-x_1^{(3)}+1)+a^4({x_1^{(3)}}^6-{x_1^{(3)}}^4-{x_1^{(3)}}^3+x_1^{(3)}+1)+a^3(-{x_1^{(3)}}^5+{x_1^{(3)}}^4-{x_1^{(3)}}^2+x_1^{(3)}-1)}{a^4(a^2-1)(1+a^2({x_1^{(3)}}^3)-{x_1^{(3)}}^2)} + \\ & \frac{a^2{x_1^{(3)}}^4+a(-{x_1^{(3)}}^3+x_1^{(3)})+{x_1^{(3)}}^3+x_1^{(3)}}{a^4(a^2-1)(1+a^2({x_1^{(3)}}^3)-{x_1^{(3)}}^2)}. \end{cases}$$

One can also calculate α_7 and β_7 with Newton identities, then getting the formulae

$$\alpha_7 = -\delta_2\alpha_5 + \delta_3\alpha_4 + \delta_5\alpha_2$$

and

$$\alpha_7-\beta_7=-(\delta_2-\gamma_2)(\alpha_5-\beta_5)+(\delta_3-\gamma_3)(\alpha_4-\beta_4)+(\delta_5-\gamma_5)(\alpha_2-\beta_2).$$

From the first equation we get

$$\alpha_7 = \frac{a^6({x_1^{(3)}}^9-{x_1^{(3)}}^8-{x_1^{(3)}}^7+{x_1^{(3)}}^6)+a^5({x_1^{(3)}}^7+{x_1^{(3)}}^5+{x_1^{(3)}}^4+{x_1^{(3)}}^2-x_1^{(3)})+a^4(-{x_1^{(3)}}^6-{x_1^{(3)}}^4-{x_1^{(3)}}^3-{x_1^{(3)}}^2-1)}{a^4(1+a^2(x^3-x^2))}+ \frac{a^3({x_1^{(3)}}^5-{x_1^{(3)}}^4+{x_1^{(3)}}^2-1)+a^2(-{x_1^{(3)}}^4+{x_1^{(3)}}^3+{x_1^{(3)}}^2+1)+a({x_1^{(3)}}^3-x_1^{(3)})-{x_1^{(3)}}^3-x_1^{(3)}-1}{a^4(1+a^2(x^3-x^2))}.$$

By comparing the two expressions for α_7, we find that $x_1^{(3)}$ must be a root of the following polynomial:

$$\begin{aligned}N_1(X)=&(-a^7+a^5)X^6+(-a^7+a^6+a^5-a^4)X^4+(a^6+a^5-a^3-a^2)X^3\\&+(a^6+a^5+a^4+a^2)X^2+(a^6+a^5+a^4-a^3+a^2)X-a^5-a^4+a^3-a^2-1.\end{aligned}$$

From the second equation we get

$$\begin{aligned}\beta_7=&\frac{a^7({x_1^{(3)}}^6+{x_1^{(3)}}^4+{x_1^{(3)}}^2)+a^6(-{x_1^{(3)}}^5-{x_1^{(3)}}^3)+a^5(-{x_1^{(3)}}^7-{x_1^{(3)}}^6-{x_1^{(3)}}^5-{x_1^{(3)}}^4+{x_1^{(3)}}^3-x_1^{(3)}-1)}{a^5(1+a^2(x^3-x^2))}+\\&\frac{a^4(-{x_1^{(3)}}^6-{x_1^{(3)}}^5+{x_1^{(3)}}^3-x_1^{(3)}+1)+a^3({x_1^{(3)}}^6+{x_1^{(3)}}^5+{x_1^{(3)}}^4+1)+a^2(-{x_1^{(3)}}^4-{x_1^{(3)}}^3-{x_1^{(3)}}^2+1)}{a^5(1+a^2(x^3-x^2))}+\\&\frac{a(-{x_1^{(3)}}^4-{x_1^{(3)}}^3+1)+{x_1^{(3)}}^3+x_1^{(3)}+1}{a^5(1+a^2(x^3-x^2))}.\end{aligned}$$

By comparison with the other formulation of β_7, we find that $x_1^{(3)}$ is also a root of the polynomial

$$\begin{aligned}N_2(X)=&(-a^6-a^5+a^4)X^6+(a^8+a^6)X^5+(a^7-a^6+a^5+a^4)X^4+\\&(-a^8-a^7-a^5+a^4)X^3+(a^6+a^3+a^2)X^2+(a^4-a^2+a)X-a^3-a-1.\end{aligned}$$

To avoid a contradiction, N_1 and N_2 must have a root in common. A computation of their resultant with Sage shows that it is a polynomial of degree 84 in the variable a. The result obtained in this way is $N(a):=\mathrm{Res}(N_1,N_2)=a^{84}-a^{83}-a^{81}+a^{80}+a^{79}+a^{78}-a^{77}+a^{76}-a^{73}-a^{72}+a^{71}-a^{70}+a^{69}+a^{66}+a^{64}-a^{63}+a^{62}-a^{60}-a^{59}+a^{56}+a^{55}+a^{53}-a^{51}+a^{50}-a^{49}+a^{48}-a^{47}-a^{45}+a^{43}+a^{42}-a^{40}+a^{38}+a^{36}-a^{35}-a^{34}-a^{33}-a^{32}-a^{31}-a^{29}-a^{28}+a^{27}-a^{26}-a^{25}$. These calculations prove the following result.

Theorem 4.2.17. Let

$$\omega_1:=\frac{uz^{13}}{\prod_{j=1}^3\prod_{i=1}^5(1-x_i^{(j)}z)}dz \quad\text{and}\quad \omega_2:=\frac{vz^{13}}{\prod_{j=0}^2\prod_{i=1}^5(1-x_i^{(j)}z)}dz,\quad\text{with}\quad u,v\in k$$

be two multiplicative good deformation data for an equidistant action of $\mathbb{Z}/3\mathbb{Z}$. If the vector space $\langle\omega_1,\omega_2\rangle$ is a two-dimensional $\mathbb{F}_3$-vector space, then $N(\frac{u}{v})=0$.

We have then, in principle, 59 possible values of $a=\frac{u}{v}$, allowing repetitions. But the value of a is completely determined by the $x_i^{(j)}$, and, more precisely, by the $q_{14}(X^{(j)})$s: in fact $u = q_{14}(X^{(1)}) + q_{14}(X^{(2)}) + q_{14}(X^{(3)})$ and $v = q_{14}(X^{(0)}) - q_{14}(X^{(1)}) + q_{14}(X^{(2)})$. This is a strong condition and it is not likely to be satisfied. Still, due to the complexity of these equations, we are not able to perform a direct computation. To verify the compatibility of the poles and of their power sums, with the equations found in this section, we shall use a computer program like *Sage*. In this way, we would be able either to find the expected contradiction, or to show that there is a lifting, yielding a counterexample in the conjecture of Pagot.

Final remarks

As the computation of this section show, a direct approach to some concrete case of the local lifting problem is possible. Nevertheless, even in the simplest cases, the limits of this approach are evident: the complexity of calculations grows enormously with the number of ramification points and the behavior changes completely when p changes. This phenomenon is observed in other tentatives of giving explicit description of local actions, both in positive and zero characteristic. For example, the recent work of several authors (see for instance [17], [37] and [9]) on finite order elements in the Nottingham group (i.e. the group of automorphisms of $\mathbb{F}_p[[T]]$ of the form $T \mapsto T + T^2 f(T)$), shows that it is a very hard problem even to describe order 4 automorphisms, when $p = 2$.

The main progresses in the general theory of local actions have been made by studying deeper algebraic and arithmetic structures of such actions and their liftings. In the next chapter, we introduce an approach that is based on non-Archimedean analytic geometry, with the purpose to get a better understanding of the general phenomena arising in different problems related to these actions.

Chapter 5

Hurwitz trees in non-Archimedean analytic geometry

We have seen that the Hurwitz tree plays a central role in the study of the local lifting problem. It is nevertheless an abstract object, containing data whose mutual relation is sometimes mysterious. In this chapter we show that Hurwitz trees can defined also as non-Archimedean analytic objects in the spirit of chapter 2. We have already suggested in the previous discussions, that there are similarities between the world of Berkovich spaces and that Hurwitz trees. Now, we make this correspondence explicit, namely we prove that there is a metric embedding of the Hurwitz tree inside the Berkovich closed unit disc. The properties of Hurwitz trees can then be studied in this new context and several generalizations are made.

We realize first the embedding, using the correspondences between the (Berkovich) generic fiber and the (scheme theoretic) special fiber of the formal stably marked model over $\mathrm{Spf}(R)$. Then we give a geometric sense to Hurwitz data, relating them with the evaluation of the analytic function $\sigma(T) - T$, for $\sigma \in G$, on the embedded Hurwitz tree. Finally, for $G = \mathbb{Z}/p\mathbb{Z}$, we relate good deformation data with vector bundles on the Berkovich disc pointed at ramification points, and we study their reduction.

One of the motivations to study the Hurwitz tree as embedded in a K-analytic space is that the theory of Berkovich spaces has been intensively studied in several fashions. In the last years, for instance, there have been pointed out connections to and applications from several domains of mathematics such as tropical geometry, non-Archimedean dynamical systems and graph theory. We conclude the chapter by giving characterizations of the embedded Hurwitz tree in relation to some of these theories, with the hope that a further study of the Hurwitz tree from different perspectives can provide a better understanding of the deformations arising in this way and of the local lifting problem. Much research has been developed around the discovery of common patterns between Berkovich spaces and other mathematical objects, and at a very fast pace. No claim for completeness or complete relevance is made for the choice of the arguments of the last section. We shall mention for example the connexions with model theory (cfr. the work [32] by Hrushovski and Loeser, exposed also by Ducros in his *Séminaire*

Bourbaki [23]) that led to many important applications but are absent in this section, due to our lack of knowledge of the subject.

Since we are now working with discrete valued fields, our notations are different form those of chapter 2. For the reader that leapfrogged chapters 3 and 4, and to avoid confusion, we precise the relationship between the two notations. We write $k = K$ for a discrete valued field, $k^o = R$ for its valuation ring, $k^{oo} = \mathfrak{m} = \{x \in R : |x| < 1\}$ for the maximal ideal and $\tilde{k} = \tilde{K} := R/\mathfrak{m}$ for the residue field, which we suppose to be algebraically closed. Let us also recall the notations we use, when dealing with closed and open discs in this chapter. We set $D(a, r) = \{x \in K : |x - a| \leq r\}$ and $D(a, r)^\circ = \{x \in K : |x - a| < r\}$. We use also open and closed discs in the sense of Berkovich: we set $\mathbb{D}(a, r) = \{x \in \mathbb{A}_K^{1,an} : |(T - a)(x)| \leq r\}$ and $\mathbb{D}(a, r)^\circ = \{x \in \mathbb{A}_K^{1,an} : |(T - a)(x)| < r\}$.

5.1 Automorphisms of open and closed analytic unit discs

The first technical point, when relating the local lifting problem with Berkovich spaces, is in the fact that in the first place we deal with formal power series $K[[T]]$, and in the second with Tate series $K\{T\}$. Let us get a clearer vision of the correspondence between the analytic theory of actions of G on these two K-algebras.

Recall that, for σ automorphism of finite order of the open unit disc, we denote by

$$\widetilde{\sigma} : \mathrm{Spm}(R[[T]] \otimes K) \to \mathrm{Spm}(R[[T]] \otimes K)$$

the induced correspondence of points of the generic fiber, which is in bijective correspondence with elements of $D(0, 1)^\circ$.

Proposition 5.1.1. Let $\sigma \in \mathrm{Aut}_R(R\{T\})$ be an automorphism such that $\sigma^n = \mathrm{id}$ with at least a fixed point $x_0 \in \mathfrak{m}$. Then it induces an automorphism of finite order $\sigma^\circ \in \mathrm{Aut}_R(R[[T]])$. Moreover we have $\widetilde{\sigma^\circ}(D(a, \rho)) = D(\sigma(a), \rho)$ for every $D(a, \rho) \subset \mathbb{D}(0, 1)^\circ$.

Proof. We may suppose $x_0 = 0$. Every disc centered in a fixed point is a fixed disc, then the open unit disc is fixed. This means that σ can be restricted/extended to σ°.

Once this action restricted, we compare $|b - a|$ with $|\sigma(b) - \sigma(a)|$. By a theorem of structure of finite order actions of open disc ([19], Lemma 14, pag.245) we have

$$\sigma^\circ(T) = \zeta T(1 + \alpha_1 T + \alpha_2 T^2 + \dots).$$

Then $\widetilde{\sigma^\circ}(b) - \widetilde{\sigma^\circ}(a) = \zeta((b - a) + \alpha_1(b^2 - a^2) + \alpha_2(b^3 - a^3) + \dots$.

But we have also that $|b^n - a^n| = |b - a||b^{n-1} + ab^{n-2} + \dots + a^{n-1}| < |b - a|$ since $a, b \in \mathfrak{m}$. Therefore $|b - a| = |\sigma(b) - \sigma(a)|$ so that $b \in D(a, \rho)$ if and only if $\sigma(b) \in D(\sigma(a), \rho)$. ☐

The automorphism σ induces functorially an homeomorphism $\Sigma : \mathbb{D}(0, 1) \to \mathbb{D}(0, 1)$ given by $\Sigma(x) = x \circ \sigma$. We can deduce by previous proposition the following easy, but important, fact.

Corollary 5.1.2. A point $\eta_{a,\rho} \in \mathbb{D}(0, 1)$ is fixed for Σ if and only if $\Sigma(\mathbb{D}(a, R)) = \mathbb{D}(a, R)$.

Proof. By Proposition 5.1.1 we have that $\Sigma(\eta_{a,\rho}) = \eta_{\tilde{\sigma}(a),\rho}$ and then the lemma follows. $\square$

When the assumptions of the corollary are satisfied, the restriction of Σ on $\mathbb{D}(a, R)$ induces an homeomorphism of Berkovich discs. We use this fact in the following sections, to relate the Hurwitz tree with dynamical properties of the morphism Σ.

5.2 The Berkovich-Hurwitz tree

We study in this section the Hurwitz trees as non-Archimedean analytic objects. Firstly, we show in which way they can be embedded, as metric trees, in $\mathbb{D}(0, 1)$. Then, we show how the vertices, the edges and the Artin and depth character have an interpretation in terms of natural structures of Berkovich curves. Once proved the embedding, one can introduce the Hurwitz data in a completely formal way. Our aim is to avoid this construction, in favor of an intrinsic one, that we get by showing that Hurwitz data have a precise meaning in terms of analytical objects. This provides a further motivation for our approach and permits to formulate conjectures about the interplay between non-Archimedean analytic geometry and ramification theory.

5.2.1 The embedding

We show that the Hurwitz trees embeds canonically in the Berkovich unit disc. This is the main theorem of the Chapter, and the whole discussion that follows relies on this result.

Theorem 5.2.1. Let $\Lambda = (G, R[[T]])$ be a local action in characteristic zero and let $\mathcal{T}_\Lambda$ be the Hurwitz tree associated to it. Then there is an embedding of topological spaces

$$\iota : \mathcal{T}_\Lambda \hookrightarrow \mathbb{D}(0, 1).$$

Calling B the set of leaves of $\mathcal{T}_\Lambda$ we also have that:

1. the embedding ι induces an isomorphism of metric spaces between $\mathcal{T}_\Lambda - B$ and the skeleton of the Berkovich curve $\mathbb{D}(0, 1) \setminus \iota(B)$;

2. the image $\iota(V \setminus B)$ is the set of formal fibers of boundary points and $\iota(B)$ is the set of fixed rigid points by the action of some $\sigma \in G \setminus \{id\}$;

3. the functions $\sigma(z) - z \in \mathcal{O}_{\mathbb{D}(0,1)}$ are locally constant outside $\iota(\mathcal{T}_\Lambda)$.

Proof. The stably marked model having semi-stable reduction, the dual graph of its special fiber is canonically embedded in $\mathbb{D}(0, 1)$ by Theorem 2.3.5. We shall then discuss only the embedding of terminal edges and vertices. Let v be such a vertex. It is associated to a point of type 1 in $\mathbb{D}(0, 1)^-$, which is of the form $\eta_{a,0}$, with $a \in \mathfrak{m}$ and such that $\tilde{r}(\eta_{a,0})$ is a smooth point on the special fiber, belonging then to a single irreducible component. We denote the generic point of this component by ξ_v. There is a unique path that joins $\eta_{a,0}$ with the point $\tilde{r}^{-1}(\xi_v)$ (which is unique by Proposition 2.3.4). Then the embedding is realized sending the terminal edge ending in v to this path and v to $\eta_{a,0}$. This realizes the embedding for the construction of Henrio. By

Proposition 3.4.3, it is realized also for the construction of Brewis-Wewers. Moreover realizing the embedding in this way proves statement 2, by using Proposition 2.3.4.
Let us prove statement 1. Suppose firstly that $|B| = 1$. In this case we can use the calculation of the skeleton of $\mathcal{C}_e$ in the example 2.4.7: let the unique fixed point be in 0, then we have

$$\mathbb{D}(0,1) \setminus \{0\} = \bigcup_{e \in \mathbb{Z}} \mathcal{C}_e$$

which makes the skeleton of $\mathbb{D}(0,1) \setminus \{0\}$ homeomorphic to $]0,1]$ (the homeomorphism being given by $r \mapsto \eta_{0,r}$). Let now $|B| > 1$. Let $b' \neq b$ be any other leaf and note $\mathbb{D}_{b,b'} = \mathbb{D}(0,1) \setminus \{b, b'\}$. For $\Gamma_K = \mathrm{Gal}(\bar{K}|K)$, we have a canonical surjective map

$$\pi : \mathbb{D}_{b,b'} \otimes \bar{K} \to (\mathbb{D}_{b,b'} \otimes \bar{K})/\Gamma_K \cong \mathbb{D}_{b,b'}.$$

Each point of the form $\eta_{b',r}$ with $r \leq |b' - b|$ is in the skeleton of $\mathbb{D}_{b,b'}$. Moreover, if x is a point of $\mathbb{D}_{b,b'}$, not in the image of $r \mapsto \eta_{b,r}$ or $r \mapsto \eta_{b',r}$, then it has a neighborhood potentially isomorphic to the unit disc. In fact, picking an element in $\pi^{-1}(x)$, it is of the form $\eta_{a,\rho}$ in such a way that $b \notin D(a,\rho)$. Then the affinoid domain $\pi(\mathbb{D}(a, \min\{|b-a|, |b'-a|\}))$ is a neighborhood of x that is potentially isomorphic to the unit disc. This can be seen with Lemma 2.4.2. Repeating this argument a finite number of times, one for every other element of B, we get the wanted isomorphism.
To prove point 3: write $\sigma(T) - T = \pi^n u(T) P(T)$ in the form given by Weierstrass preparation. Then $y(\pi^n u(T))$ is constant for every $y \in Y$ and $P(T) = \prod_{b \in B}(T - b)$, which is constant on any disc $D(a,r)$ such that $r < |b - a|$ for every $b \in B$ (because for every $x, y \in D(a,r)$ and $b \in B$ we have $|x - b| = |y - b|$). After possible basechange, every element of $\mathbb{D}(0,1) \setminus \iota(\mathcal{T}_\Lambda)$ has a neighborhood of this form, by point 2. Then the claim is proved. □

With this theorem and the characterization of points in section 2.3.2, we can perform the following assignment:

Assignment 5.2.2.

- *For each vertex $v \in V$, consider the point $\iota(v)$. It is of the form $\eta_{a,\rho}$ for some closed disc $D(a,\rho)$, $a \in R$ and $0 \leq \rho < 1$. We can then define the assignment of this closed disc as:*

$$\begin{aligned} D^\bullet : V &\longrightarrow \{\textit{Closed subdiscs of } R\} \\ v &\longmapsto D(a,\rho). \end{aligned}$$

- *For each edge $e \in E$, starting in v and ending in v', consider the closed disc $D^\bullet(v)$. There is a unique open disc $D(c,\rho)^-$ having the same radius as $D^\bullet(v)$ and containing $D^\bullet(v')$. We can then define the assignment of this open disc as:*

$$\begin{aligned} D^\circ : E &\longrightarrow \{\textit{Open subdiscs of } R\} \\ e &\longmapsto D(c,\rho)^\circ. \end{aligned}$$

The center c of $D^\circ(e)$ is not well defined, but its residue class $\tilde{c} \in \tilde{K}$ is unique. In the following, we set $\mathrm{cl}_e := \tilde{c}$.

Note that, following this assignment, the root of $\mathcal{T}_\Lambda$ is sent to the unit disc $\mathbb{D}(0,1)$. More in general, for every $v \in V(\mathcal{T}_\Lambda)$, every point $\iota(v)$ is of type 1 if v is a leaf of the Hurwitz tree, and of type 2 otherwise.
After this assignment, the orientation on the Hurwitz tree is such that $v \to v'$ if and only if $D^\bullet(v') \subset D^\bullet(v)$.
We denote by $\mathbb{D}^\bullet(v)$ and $\mathbb{D}^\circ(e)$ the closed and open Berkovich discs with the same center and radius as, respectively, $D^\bullet(v)$ and $D^\circ(e)$.

5.2.2 Translation of Hurwitz data

Let us fix a local action in characteristic zero $(R[[T]], G)$ and we denote by $\mathcal{T}_\Lambda$ its Hurwitz tree. Notice that the group G_v is the stabilizer of the subdisc $D^\bullet(v)$. With the help of Assignment 5.2.2 we can express Artin and depth characters in the language of non-Archimedean analytic geometry.

Proposition 5.2.3. Let $(\mathcal{T}_\Lambda, G_{[v]}, a_e, \delta_v)$ be the Hurwitz tree associated to a local action in characteristic zero $(G, R[[T]])$. Then

$$\frac{\delta_v(\sigma)}{|G_v| - 1} = -\log |(\sigma(T) - T)(x)| = v_\xi(\sigma(T) - T)$$

and

$$a_e(\sigma) = v_z(\sigma(T) - T)$$

for every $\sigma \in G - \{id\}$.

Proof. To each non-terminal vertex v of $\mathcal{T}_\Lambda$ we associate a valuation

$$\mathrm{val}_v : H^0(\mathbb{D}^\bullet(v), \mathcal{O}_Y) \cong R\{T'\} \to \mathbb{Q} \cup \{\infty\}$$

defined by the formula

$$\mathrm{val}_v(\sum a_i T'^i) = \min_i \{v_R(a_i)\}.$$

If e is an edge of $\mathcal{T}_\Lambda$ such that $v_e^s = v$, then the valuation val_v can be extended, with the same definition on $H^0(\mathbb{D}^\bullet(v), \mathcal{O}_Y) \cong R[[T' - a]] \supset R\{T'\}$, where $a \in D^\bullet(v)$. To the edge e we associate the valuation

$$\mathrm{val}_e : H^0(\mathbb{D}^\bullet(e), \mathcal{O}_Y) \to \mathbb{Q} \cup \{\infty\}$$

defined by the formula

$$\mathrm{val}_e(f) = \mathrm{ord}_{t'}(\overline{\lambda f})$$

where $\lambda \in R$ is any element such that $|\lambda| = \pi^{-\mathrm{val}_v(f)}$ and $f \mapsto \bar{f}$ is the reduction map from

$R[[T']]$ to $k[[t']]$. Let $\sigma \neq 1$. By sections 3.2 and 3.3 in [14] we have

$$\delta_v(\sigma) = -|G_v|\mathrm{val}_v(\sigma(T') - T')$$

and

$$a_e(\sigma) = \mathrm{val}_e(\sigma(T') - T').$$

Now, by the characterization of Lemma 3.2.6, val_v and val_e are the restrictions of the valuations v_ξ and v_z, and hence we can conclude. □

Remark 5.2.4. Proposition 5.2.3 can be formulated also in terms of Huber adic spaces. In fact, let $e \in E$ be an edge of the Hurwitz tree, originating from a vertex v, and let $\sigma \in G$. Then, the function $f \to (v_\xi(f), v_z(f))$ is a rank 2 valuation over $K\{T\}$ corresponding to a point P of $\mathrm{Spa}(K\{T\})$. Proposition 5.2.3 says then that $\left(\frac{\delta_v(\sigma)}{|G_v|-1}, a_e(\sigma)\right) = |(\sigma(T) - T)(P)|$, in the framework of adic geometry.
In this perspective, the Artin character is associated to an infinitesimal point next to its starting vertex (it can be visualized as a "direction") more than to an edge of the Hurwitz tree. The correspondence between edges and residue classes representing these directions is explicitly given by $e \mapsto D^\circ(e)$ of Assignment 5.2.2.

The importance of Proposition 5.2.3 is in the fact that the Artin and depth characters contribute to generalize the results of the "equidistant case". As an example, in chapter 4 is shown how explicit calculations in this particular case permit to deduce properties related to the Hurwitz tree of $(\mathbb{Z}/p\mathbb{Z})^n$. Nevertheless, these properties concern only the structure of metric tree and the good deformation data, since the structure of characters is very simple under the equidistant assumption. If one wants to pursue the study in this direction, the Proposition helps to get more information about the structure of characters.

5.3 Analytic good deformation data

In chapter 3, Hurwitz trees for $G = \mathbb{Z}/p\mathbb{Z}$ are endowed with **good deformation data**. In this way, one obtains a parametrization of automorphisms of order p of $R[[T]]$ with fixed points. More in general, to have a notion of good deformation datum for other finite groups that provides such a parametrization, is an important advancement in the understanding of the local lifting problem. Bouw and Wewers, in their paper [11], introduced such a definition for D_{2p}, to show that actions of such dihedral group admit a lifting. An analog strategy is outlined by the same authors in [12], to show liftings of local actions of A_4 for $p = 2$. Thanks to the results of previous sections, we can propose a definition of good deformation datum in the non-Archimedean analytic setting, as a section of the reduction, in the sense of Temkin, of a particular metrized sheaf on $\mathbb{D}(0, 1) \setminus L_\Lambda$, that we construct in this section. We start by recall the definition of good deformation data in the known cases, and we describe the problems that arise when trying to formulate the compatibility conditions between good deformation data on the special fiber of the stable marked model. We give, then a characterization of families of good deformation data in the Berkovich setting. Finally, This result, which is the central one in this section, permits to propose a definition of good deformation datum for any finite group.

5.3.1 Good deformation data and the analytic sheaf of deformations

We have recalled in Section 3.3.2 the notion of good deformation datum for $\mathbb{Z}/p\mathbb{Z}$. In [11] and [13] the authors propose a generalization of this notion for any cyclic-by-p group $G = P \rtimes N$ to show lifting to characteristic zero of actions of $G = D_{2p}$. Let us recall their definition

Definition 5.3.1. *Let $G = P \rtimes N$ be a cyclic by p group with character χ, $f \in \tilde{K}(z)$ and $\omega = f dz \in \Omega^1_{\mathbb{P}^1_{\tilde{K}}}$. We call ω a* ***good deformation datum*** *for G if the following conditions are satisfied:*

$D1$ - The form ω is either logarithmic or exact;

$D2$ - If ω is logarithmic, then it has an unique zero on $\mathbb{P}^1_{\tilde{K}}$ at the point ∞;

$D3$ - There is a faithful action of N on $\Omega^1_{\mathbb{P}^1_{\tilde{K}}}$ such that

$$\sigma.\omega = \chi(\sigma) \cdot \omega$$

when $\sigma \in N$.

When N is trivial, this definition matches the one of Section 3.3.2. In the following we always work with this more general definition.

Let $\Lambda = (G, R[[T]])$ be a local action in characteristic zero for a cyclic-by-p group G. Let $\varphi : \mathcal{X}_\Lambda \longrightarrow S = \mathrm{Spf}(R[[T]])$ be the canonical sequence of formal admissible blowing-ups associated to the stably marked model. The special fiber $\mathcal{S}_{\Lambda,s}$ has normal crossing singularities with a finite set $\{s_i\}$ of singular points. We set $D_{sing} = \sum s_i$ the divisor associated to those singular points and we let $\mathcal{O}_{D_{sing}}$ be the sheaf of ideals over $\mathcal{S}_{\Lambda,s}$ associated to D_{sing}, which is a Weil divisor, but not an effective Cartier divisor. The sheaf $\mathcal{O}_{D_{sing}}$, hence, is not invertible.
A collection of good deformation data arising from the action Λ can be described as a global section of the sheaf $\omega_\Lambda = \mathcal{I} \otimes \Omega_{\mathcal{S}_{\Lambda,s}}$. This sheaf is never locally free, due to the non-smoothness of $\mathcal{S}_{\Lambda,s}$. The best we can obtain is the following proposition.

Proposition 5.3.2. The sheaf ω_Λ is invertible on the regular locus of $\mathcal{S}_{\Lambda,s}$.

Proof. It is sufficient to show the claim for the sheaf $\Omega_{\mathcal{S}_{\Lambda,s}}$, since the support of $\mathcal{I}$ is contained in the regular locus. Let Γ_s be the dual graph of $\mathcal{S}_{\Lambda,s}$ and let $\{v_i\}_{i\in I}$ be its set of vertices. Recall from section 2.3.2 the correspondence $v_i \mapsto S_i$ between vertices of the dual graph and irreducible components of the special fiber, and consider the open covering $\{U_i\}_{i\in I}$ of $\mathcal{S}_{\Lambda,s}$, given in the following way: $U_i = S_{v_i} \setminus \{\infty\}$ if v_i is a leaf of Γ_s and by the amalgamated sum $U_i = \coprod_{v_i \to v_j}(S_{v_j} \setminus \{0\}) \coprod_{v \to v_i}(S_v \setminus \{\infty\})$ otherwise. Now, $U_i \cong \mathbb{A}^1_k$ if v_i is a leaf of Γ_s and $\mathrm{Pic}(\mathbb{A}^1_k)$ is trivial. When v_i is not a leaf, U_i is isomorphic to $\mathrm{Spec}(k[X, Y_j]/(X - a_j)Y_j)$, for appropriate $a_j \in k$. Then, on U_i the sheaf $\Omega_{\mathcal{S}_{\Lambda,s}}$ is generated by the set $\{dx, dy_j\}$ (for every j such that $v_i \to v_j$) with relations $y_j dx - (x-a_j)dy_j$. Hence it is locally free of rank one outside the singular points. □

Remark 5.3.3. The proof of Proposition 5.3.2 is indeed very classical in algebraic geometry. We chose nevertheless to write it down completely in order to have an explicit description of the trivializations of the sheaf ω_Λ. This is crucial in order to investigate the relationship between ω_Λ and sheaves on $\mathbb{D}(0,1)$.

Definition 5.3.4. *For a local action in characteristic zero $\Lambda = (G, R[[T]])$, the* ***sheaf of deformations*** *of Λ is the metrized line bundle Ω_Λ on $\mathbb{D}(0,1) \setminus L_\Lambda$ defined as follows:*

- *the trivializing G-covering is given by $\{U_v\}_{v \in V(\Gamma_s)}$ where*

$$U_v = \mathbb{D}^\bullet(v) \setminus \Big(\bigcup_{v \to w} \mathbb{D}^\bullet(w) \Big),$$

  in particular when v is a leaf of $V(\Gamma_s)$, then U_v is the pointed disc $\mathbb{D}^\bullet(v) \setminus (L_\Lambda \cap \mathbb{D}^\bullet(v))$;

- *the G-covering $\{U_v\}$ is such that all intersections are of the form $U_{vw} \cong \mathcal{M}(\frac{K\{S,T\}}{ST-1})$. The cocycles g_{vw} on U_{vw} are then given by multiplication by the function $-\frac{1}{T^2} = -S^2$;*

- *the metric on Ω_Λ is the formal metric induced by the couple $(\mathcal{S}_\Lambda, \Omega(D_\Lambda))$.*

Remark 5.3.5. The sheaf of deformations is an analogue of the sheaf $\Omega(D)$ of differential forms with possible poles on a prescribed divisor D. In this case, D is the divisor of rigid ramification points, counted with multiplicity one. The pointed open disc is smooth, yielding locally freeness for Ω_Λ. This is one of the advantages of studying the reduction of the sheaf Ω_Λ rather than ω_Λ.

Let $x \in \mathbb{D}(0,1) \setminus L_\Lambda$. Using the techniques of [15] about local reduction of metrized vector bundles, one gets an invertible sheaf $\widetilde{\Omega_{\Lambda,x}}$ on the space $\widetilde{(X,x)}$ in the sense of Temkin's reduction theory (cfr. Section 2.2.3). Recall that, when x is a point of type 2, then $\widetilde{(X,x)} \cong \mathbb{P}^1_{\tilde{K}}$ and that, for every $f \in \tilde{K}[t]$, we defined $\widetilde{(X,x)}\{f\} = \{\nu \in \widetilde{(X,x)} : f \in \mathcal{O}_\nu\}$.

Proposition 5.3.6. Let $x \in \mathbb{D}(0,1) \setminus L_\Lambda$. The sheaf $\widetilde{\Omega_{\Lambda,x}}$ is isomorphic to $\Omega^1_{\mathbb{P}^1_{\tilde{K}}}$ when $x \in V(\Gamma_s)$, and trivial otherwise.

Proof. The sheaf $\widetilde{\Omega_{\Lambda,x}}$ is defined by reduction of cocycles, as outlined in section (6.5.4) of [15]. Let $\mathcal{O}^\circ_{X,x} = \{f \in \mathcal{O}_{X,x} : |f| \le 1\}$, and let (g_{vw}) be a cocycle on (X,x) with coefficients in $GL_n(\mathcal{O}^\circ_{X,x})$ with respect to a covering (X_v). Then, the family $\widetilde{g_{vw}}$ is a cocycle on $\widetilde{(X,x)}$ with coefficients in $GL_n(\mathcal{O}_{\widetilde{(X,x)}})$. Moreover, two cocycles that are cohomologous maintain this property after reduction, just by reducing the cochain.
In our situation, the open covering is the usual affine covering on $\mathrm{Proj}(\tilde{K}[t,s])$, and the cocycles $\widetilde{g_{vw}}$ are given by multiplication by $-1/t^2$, which is exactly the cocycle defining $\Omega^1_{\mathbb{P}^1_{\tilde{K}}}$, since $ds = -\frac{dt}{t^2}$. □

Let $v \in V(\Gamma_s)$. After identification of $\widetilde{(X,v)}$ with the suitable projective line in $\mathcal{S}_{\Lambda,s}$, for the sake of Proposition 5.3.2, we have $\omega_\Lambda \cong \widetilde{\Omega_{\Lambda,x}}$ outside the singular locus of $\mathcal{S}_{\Lambda,s}$. As a consequence, a collection of good deformation data can be described as a collection of sections $\omega_v \in \widetilde{\Omega_{\Lambda,v}}$ when v varies in the set of vertices of the Berkovich-Hurwitz tree that are not leaves, satisfying the following conditions:

$D'1$ - The section ω_v corresponds to a logarithmic differential form if v is a leaf of $\Gamma(\mathcal{S}_{\Lambda,s})$, and to an exact differential form otherwise;

$D'2$ - If ω_v corresponds to a logarithmic differential form, its simple poles are outside the open quasi-compact subset

$$\widetilde{(X,v)}\{\frac{1}{\prod_{e\in E^+(v)}(t-\mathrm{cl}_e)}\},$$

and the unique zero is in the complement of $\widetilde{(X,v)}\{t\}$. If ω_v corresponds to an exact differential form, its zeroes are outside the open quasi-compact subset

$$\widetilde{(X,v)}\{\frac{t^{m+1}}{\prod_{e\in E^+(v)}(t-\mathrm{cl}_e)}\},$$

where $m = |E^+(v)|$;

$D'3$ - In the case of dihedral action, the condition $D3$ of definition 5.3.1 is respected.

Remark 5.3.7. Looking at the definition of good deformation data in Section 3.3.2, we remark that they are constructed as logarithmic (resp. exact) differential forms that are reduction of forms that are already logarithmic (resp. exact) in characteristic zero. This suggests that it is interesting to study the conditions of "logarithmicness" (resp. exactness) of differential forms over R. It turns out that on smooth K-analytic spaces every differential form is locally exact thanks to a version of Poincaré's lemma (see chapter 1 of the book by Berkovich on integration theory, [7]), and then it can be written in logarithmic form on some smaller disc. It would be interesting to use this result in order to discuss the existence of good deformation data.

The reformulation of good deformation data as reductions of sections of metrized vector bundles permits to get a new perspective on conditions to lift local actions to characteristic zero. Given $(G, \tilde{K}[[t]])$ a local action in characteristic p, one can look at the Hurwitz trees of possible liftings (those matching the ramification theory, the number of ramification points, and so on), and study the existence of deformation sheaves whose reduction at vertices of those Hurwitz trees gives the deformation data that one expects.

5.4 Characterizations of the Hurwitz tree

We explicit in this section how the embedded Hurwitz tree can be characterized in relation to different theories, that have been linked to Berkovich spaces by several authors very recently. The hope is to use the techniques arising in this way to get new conditions on the local lifting problem. In the first part, we give an account of how the Berkovich-Hurwitz tree is related to the study of maps $\mathbb{D}(0,1)^- \to \mathbb{D}(0,1)^-$, and their iterated behavior. In the second section, we characterize the Berkovich-Hurwitz tree as the tropicalization of an affine line with respect to a certain embedding. Finally, we discuss how the results developed in this chapter can be related to the theory of tempered covers.

5.4.1 Dynamical properties

The action of G over can be seen also as a finite dynamical system, obtained by iteration of the map $g : \mathbb{D}(0,1)^- \to \mathbb{D}(0,1)^-$, induced by $R[[T]] \to R[[T]]^G$. The ramification locus of the system plays an important role in the description of the system in several works (see [3] for a general introduction to the subject and [24], [25] or [26] for a more specific discussion about the structure of the ramification locus). In this spirit, it seems useful to study the Hurwitz tree in relation with the ramification locus of g.

Proposition 5.4.1. The embedded Hurwitz tree is contained in the branch locus of the G-covering $\mathbb{D}(0,1)^- \to \mathbb{D}(0,1)^-$ induced by the action of G.

Proof. Every point v of the tree as defined above is fixed by Σ, for some $\sigma \in G$. If $v \in B$, then it is so by definition. Otherwise $v = \eta_{a,\rho}$ for some $a \in B$ and $0 < \rho < 1$. By Corollary 5.1.2, $\Sigma(\eta_{a,\rho}) = \eta_{\sigma(a),\rho}$. Being a a fixed point, v turns out to be fixed as well. □

The proof of Proposition 5.4.1 uses the fact that discs containing fixed points are fixed discs. The converse is not true: there are in fact local actions of G without fixed points (a construction of order p^n automorphisms of the open unit disc with no fixed points using Lubin-Tate formal groups can be found in section 3.3.3 of [28]). This entails also that the converse of proposition 5.4.1 is not true in general. An interesting question would be to characterize the branch locus in general, in order to obtain a generalization of the Hurwitz tree. A useful Lemma in this sense, is the following

Lemma 5.4.2. Consider a rigid point $a \in \mathfrak{m}_R$. The value $\rho(a) := |(\sigma(a) - a)|$ is exactly the radius of the smallest disc fixed by Σ and centered in a.

Proof. The homeomorphism Σ fixes the point $\eta_{a,\rho(a)}$ by the formula $\Sigma(\eta_{a,\rho(a)}) = \eta_{\sigma(a),\rho(a)}$.
Let us show the minimality: every disc $\eta_{a,\rho} < \eta_{a,\rho(a)}$ is not fixed, otherwise $\Sigma(\eta_{a,\rho}) = \eta_{a,\rho}$ would imply $\eta_{a(\sigma(z)),\rho} = \eta_{a,\rho}$ and then $|\sigma(a) - a| \leq \rho < \rho(a)$ leading to a contradiction. □

Example 5.4.3. *By Weierstrass preparation theorem, we have $\sigma(T) - T = \pi^n \cdot u(T) \cdot P(T)$ with $u(T)$ unit in $R[[T]]$ and $P(T) \in R[T]$ polynomial of degree m which reduces to $t^m \in \tilde{K}[t]$. Then $|a(\sigma(T) - T)| = p^{-n \cdot v_R(P(a))}$. As a consequence, if σ is an automorphism without fixed points, the polynomial $P(T)$ is a constant and $|a(\sigma(T) - T)| = \pi^{-n}$. Then, the fixed discs containing a are all those of the form $\mathbb{D}(a,\rho)$ with $\rho > \pi^{-n}$. This set may be bigger than the Hurwitz tree, which consist in this case of the singleton $\{\eta_{0,1}\}$.*

Remark 5.4.4. Being its proof purely analytic, Lemma 5.4.2 holds also when σ has infinite order.

5.4.2 The Berkovich-Hurwitz tree as tropicalization

The relationships between tropical geometry and Berkovich spaces have been studied by several authors in the very recent years. In the article [48], Payne shows that analytifications, in the sense of Berkovich, can be described as inverse limits of images of tropicalization maps. Baker, Payne and Rabinoff make in [2] a detailed study of the Berkovich skeleton in the tropical setting. We can describe the Hurwitz tree in the context of tropical geometry thanks to the following result.

Theorem 5.4.5. Let $\Lambda = (G, R[[T]])$ be a local action in characteristic zero, with set of ramification points $L_\Lambda = \{x_1, \ldots, x_m\}$. Then the Berkovich-Hurwitz tree $\iota(\mathcal{T}_\Lambda)$ is isometric to $\mathrm{Trop}(\mathbb{A}^1_{\widehat{\widehat{K}}}, j)$, the tropicalization of the affine line $\mathbb{A}^1_{\widehat{\widehat{K}}}$ associated to the embedding

$$j : x \hookrightarrow (x - x_1, \ldots, x - x_m)$$

in the m-dimensional affine space $\mathbb{A}^m_{\widehat{\widehat{K}}}$.

Proof. Let $|\cdot| := |\cdot|_{\widehat{\widehat{K}}}$ be the non-Archimedean norm on $\widehat{\widehat{K}}$. The tropicalization map $\mathbb{A}^1_{\widehat{\widehat{K}}} \to \mathbb{R}^m$ associates to $x \in \mathbb{A}^1_{\widehat{\widehat{K}}}$, the closure in $\mathbb{R}^m$, of the m-uple

$$\{|x - x_1|, \ldots, |x - x_m|\}.$$

Since all the ramification points are in R, if $|x| \geq 1$, then $\mathrm{Trop}(x) = (|x|, \ldots, |x|)$. At the opposite, one of the coordinates of $\mathrm{Trop}(x)$ is zero if and only if $x \in L_\Lambda$. Let now $x \in D^\circ(0, 1)$. Then the set L_Λ can be split in two parts: the subset

$$L_x = \{x_i \in L_\Lambda : |x - x_i| \leq |x - x_j| \forall 1 \leq j \leq m\}$$

and its complement in L_Λ. For every subset $L_J \subset L_\Lambda$, consider the set $S_J = \{x \in D^\circ(0, 1) : L_x = L_J\}$. It is either the empty set or an annulus of the form $D^\circ(e) \setminus D^\bullet(v_e^t)$ for some edge $e \in E(\mathcal{T}_\Lambda)$. In this second case, the image $\mathrm{Trop}(S_J)$ is isometric to $\iota(e)$. In fact, when x varies in S_J, we have that

$$|x - x_i| = \begin{cases} \text{constant } \forall\, x \in S_J & \text{if } x_i \notin L_J \\ |x - x_j| \;\forall\, x_j \in L_J & \text{if } x_i \in L_J \end{cases}$$

so that the image is homeomorphic to a rational interval. Then, its closure in $\mathbb{R}^m$ is a real interval. It is easy to verify that the length of this interval is equal to the one of e in the Hurwitz tree.

Conversely, every edge of the Hurwitz tree is associated to some L_J: it suffices to take $L_J = L_\Lambda \cap \mathbb{D}^\circ(e)$. $\square$

Remark 5.4.6. The fact that $\mathrm{Trop}(D^\circ(e) \setminus D^\bullet(v_e^t))$ coincides with $\iota(e)$ can be proved also using the relation between the skeleton and the Newton polygon, discussed in section 2.4.1.

Payne shows in [48] that every tropicalization embeds in the Berkovich affine line $\mathbb{A}^{1,an}_K$. One can show that, after this embedding, the Hurwitz tree is not only isometric, but coincides with $\mathbb{D}(0,1) \cap \mathrm{Trop}(\mathbb{A}^1_{\widehat{K}}, j)$.

5.4.3 Covers of Berkovich curves and metric structure of the Hurwitz tree

For an automorphism of the open (resp. closed) unit disc $\sigma : R[[T]] \to R[[T]]$ (resp. $R\{T\} \to R\{T\}$), we can functorially associate, as in Section 5.1, an automorphism $\Sigma : \mathbb{D}^\circ(0,1) \to \mathbb{D}^\circ(0,1)$ (resp. $\mathbb{D}(0,1) \to \mathbb{D}(0,1)$). When $G = \langle\sigma\rangle$, we get a covering $\mathbb{D}^\circ(0,1) \to \mathbb{D}^\circ(0,1)/G$ (resp.$\mathbb{D}(0,1) \to \mathbb{D}(0,1)/G$), that can be investigated using the theory of coverings of k-analytic curves. In this way, the study of the fundamental group of pointed unit discs in the Berkovich setting intersects the study of the Hurwitz tree. Let us explicit this interrelation.
There are different notions of fundamental group for non-Archimedean analytic spaces. The main issue in finding a suitable definition is that finite étale coverings do not induce topological coverings in general. To provide a solution to this problem, a notion of (not necessarily finite) étale morphism and étale fundamental group was given by De Jong [21]. Later, in [1], André modified this notion to provide a more handful definition. His work resulted in the notion of tempered fundamental group, classifying étale coverings (in the sense of De Jong) that become coverings in the topological sense after pull-back by finite étale coverings. The great advantage of this definition is that a tempered covering is always dominated by the universal covering (in the topological sense) of a finite étale covering. In this sense, such a definition is much more connected to our study of finite Galois covers of curves.
The tempered fundamental group is used in several works on non-Archimedean anabelian geometry. In [42], Mochizuki proves that one can reconstruct the graph of the stable reduction of a Mumford curve from the tempered fundamental group of its analytification. In [36], Lepage improves this result by showing that also the metric structure of the skeleton of such analytification can be recovered from the tempered fundamental group, treating also the example of $\mathbb{P}^1_K$ minus a finite number of points (see [36]). In theorem 5.2.1, we proved that the Hurwitz tree coincides with the skeleton of the pointed disc $(\mathbb{P}^1_K \setminus L_\Lambda) \cap \mathbb{D}(0,1)$. As a consequence, if T_1 and T_2 be rooted metric trees that are not isomorphic as metric spaces, then, there exist a finite group G and a local action in characteristic zero $\Lambda = (R[[T]], G)$ such that $T_1 \cong T_\Lambda$ and no other local action of group G has Hurwitz tree isomorphic to T_2. This is related to the problem of realizing abstract trees as Hurwitz trees, in order to parametrize G-covers of the unit disc. More in general, the role of Berkovich-Hurwitz trees in the study of covers, is a problem that we think deserves to be deepened.

Final remarks

In both the examples of this section, we have used exclusively results involving the topological and the metric structure of the Hurwitz tree. Nevertheless, as we have seen, the Berkovich-Hurwitz tree is a richer and deeper object. Further investigations in this direction may shed new light not only on topics related to the main motivation for the definition of the Hurwitz tree, i.e. the local lifting problem, but also on a wide spectrum of topics related to Berkovich spaces more in general. In this way, the Berkovich-Hurwitz tree becomes an object that can be studied in its own right.

Chapter 6

Weil representation and metaplectic groups over an integral domain

This chapter deals with the Weil representation and the metaplectic group. In [62], Weil gives an interpretation of the behavior of theta functions throughout the definition of the metaplectic group with a complex linear representation attached to it, known as the Weil representation. A central tool in his construction is the group $T = \{z \in \mathbb{C} : |z| = 1\}$, in which most computations are developed. We replace T with the multiplicative group of an integral domain R and we construct a Weil representation in this more general context. The scope is to help fitting Weil's theory to give applications in number theoretical questions related to modular representations (see, for example, [41]).

We start by defining objects that are analogue to those in Weil's construction: the groups $\mathcal{B}$, $\mathbb{B}_0(W)$ and the metaplectic group $\mathrm{Mp}(W)$. Then, we prove the main result of this chapter, namely the existence of the **reduced metaplectic group** in this new generality.

The main problems in following Weil's approach are the lack of complex conjugation and complex absolute value. Because of this, Fourier and integration theory shall be adapted in the new context: mainly we consider Haar measures with values in R and operators acting over the space of R-valued Schwartz functions instead of L^2-functions, using Vignéras' approach (section I.2 of [61]). Moreover, allowing R to be of positive characteristic makes it necessary to change some formulas, for example in the proof of Theorem 6.4.1 to include the case where $q^2 = 1$ in R.

The chapter being independent from the rest of the thesis, we decided to completely reset the notations. We hope that this does not generate confusion in the reader.

6.1 Notation and definitions

Let F be a locally compact non-archimedean field of characteristic different from 2. We write $\mathcal{O}_F$ for the ring of integers of F, we fix a uniformizer ϖ of $\mathcal{O}_F$, we denote p the residue characteristic and q the cardinality of the residue field of F. Let R be an integral domain such that $p \in R^\times$.

We assume that there exists a smooth non-trivial character $\chi : F \longrightarrow R^{\times}$, that is a group homomorphism from F to $R^{\times}$ whose kernel is an open subgroup of F. These properties assure the existence of an integer $l = \min\{j \in \mathbb{Z} \,|\, \varpi^j \mathcal{O}_F \subset \ker(\chi)\}$ called the **conductor** of χ.

Quadratic forms

We denote by G any finite dimensional vector space over F.

We recall that a **quadratic form** on G is a continuous map $f : G \to F$ such that $f(ux) = u^2 f(x)$ for every $x \in G$ and $u \in F$ and $(x, y) \longmapsto f(x + y) - f(x) - f(y)$ is F-bilinear. A **character of degree 2** of G is a map $\varphi : G \to R^{\times}$ such that $(x, y) \longmapsto \varphi(x+y)\varphi(x)^{-1}\varphi(y)^{-1}$ is a bicharacter (i.e. a smooth character on each variable) of $G \times G$. We denote by $Q(G)$ the F-vector space of quadratic forms on G, by $X_2(G)$ the group of characters of degree 2 of G endowed with the pointwise multiplication and by $X_1(G)$ the multiplicative group of smooth R-characters of G, that is a subgroup of $X_2(G)$.

We denote by $G^* = \mathrm{Hom}(G, F)$ the dual vector space of G. We write $[x, x^*] = x^*(x) \in F$ and $\langle x, x^* \rangle = \chi([x, x^*]) \in R^{\times}$ for every $x \in G$ and $x^* \in G^*$. We identify $(G^*)^* = G$ by means of $[x^*, x] = [x, x^*]$. We have a group isomorphism

$$\begin{array}{ccc} G^* & \longrightarrow & X_1(G) \\ x^* & \longmapsto & \langle \cdot \, , x^* \rangle. \end{array} \tag{6.1}$$

Indeed if $\langle x, x^* \rangle = 1$ for every $x \in X$ then $[x, x^*] \in \ker(\chi)$ for every $x \in X$ and this implies that $x^* = 0$ since $\ker(\chi) \neq F$. The surjectivity follows by Theorem II.3 of [63] and I.3.9 of [61].

Definition 6.1.1. *Let $\mathcal{B}$ be the bilinear map from $(G \times G^*) \times (G \times G^*)$ to F defined by $\mathcal{B}\big((x_1, x_1^*), (x_2, x_2^*)\big) = [x_1, x_2^*]$ and let $\mathcal{F} = \chi \circ \mathcal{B}$.*

For a F-linear map $\alpha : G \to H$ we denote by $\alpha^* : H^* \to G^*$ its **transpose**. If $H = G^*$ and $\alpha = \alpha^*$ we say that α is **symmetric**. We associate to every quadratic form f on G the symmetric homomorphism $\rho = \rho(f) : G \to G^*$ defined by $\rho(x)(y) = f(x + y) - f(x) - f(y)$ for every $x, y \in G$. Since $\mathrm{char}(F) \neq 2$, the map $f \mapsto \rho(f)$ is an isomorphism from $Q(G)$ to the F-vector space of symmetric homomorphisms from G to G^* with inverse the map sending ρ to the quadratic form $f(x) = [x, \frac{\rho(x)}{2}]$. We say that $f \in Q(G)$ is **non-degenerate** if $\rho(f)$ is an isomorphism and we denote by $Q^{nd}(G)$ the subgroup of $Q(G)$ of non-degenerate quadratic forms on G. We remark that the composition with the character χ gives an injective group homomorphism from $Q(G)$ to $X_2(G)$.

Integration theory

Let dg be a Haar measure on G with values in R (see I.2 of [61]). We denote by $\mathcal{S}(G)$ the R-module of compactly supported locally constant functions on G with values in R. We can write every $\Phi \in \mathcal{S}(G)$ as $\Phi = \sum_{h \in K_1/K_2} x_h \mathbb{1}_{h+K_2}$ where K_1 and K_2 are two compact open subgroups of G, $x_h \in R$, $\mathbb{1}_{h+K_2}$ is the characteristic function of $h+K_2$ and the sum is taken over the finite number of right cosets of K_2 in K_1.

The **Fourier transform** of $\Phi \in \mathcal{S}(G)$ is the function from G^* to R defined by

$$\mathscr{F}\Phi(g^*) = \int_G \Phi(g)\langle g, g^*\rangle dg \tag{6.2}$$

for every $g^* \in G^*$.

For every compact open subgroup K of G let $K_* = \{g^* \in G^* \,|\, \langle k, g^*\rangle = 1 \,\forall\, k \in K\}$ define a subgroup of G^*. Notice that the map $K \mapsto K_*$ is inclusion-reversing.
If L is any $\mathcal{O}_F$-lattice of G and l is the conductor of χ, then $L_* = \{g^* \in G^* \,|\, g^*(L) \subset \varpi_F^l \mathcal{O}_F\}$. Explicitly, if $L = \bigoplus_i \varpi_F^{a_i}\mathcal{O}_F$ (with $a_i \in \mathbb{Z}$ for all i) with respect a fixed basis $(e_1, \ldots, e_N)$ of G, then $L_* = \bigoplus_i \varpi_F^{l-a_i}\mathcal{O}_F$ with respect to the dual basis of $(e_1, \ldots, e_N)$ of G^*. These facts imply that K_* is a compact open subgroup of G^* for every compact open subgroup K of G.

Given a Haar measure dg on G such that $\mathrm{vol}(K', dg) = 1$ we call **dual measure of** dg the Haar measure dg^* on G^* such that $\mathrm{vol}(K'_*, dg^*) = 1$.

The **inverse Fourier transform** of $\Psi \in \mathcal{S}(G^*)$ is the function from G to R defined by

$$\mathscr{F}^{-1}\Psi(g) = \int_{G^*} \Psi(g^*)\langle g, -g^*\rangle dg^* \tag{6.3}$$

for every $g \in G$.

For every $\Psi_1, \Psi_2 \in \mathcal{S}(G^*)$, we denote by $\Psi_1 * \Psi_2 \in \mathcal{S}(G^*)$ the **convolution product** defined by

$$(\Psi_1 * \Psi_2)(x^*) = \int_{G^*} \Psi_1(g^*)\Psi_2(x^* - g^*)dg^*$$

for every $x^* \in G^*$.

Proposition 6.1.2. Formulas (6.2) and (6.3) give an isomorphism of R-algebras from $\mathcal{S}(G)$, endowed with the pointwise product, to $\mathcal{S}(G^*)$, endowed with the convolution product.

Proof. The R-linearity of $\mathscr{F}$ and $\mathscr{F}^{-1}$ is clear from their definitions. Let now K be a compact open subgroup of G and $h \in G$; we have that

$$\mathscr{F}\mathbb{1}_{h+K}(g^*) = \int_G \mathbb{1}_K(g-h)\langle g, g^*\rangle dg = \langle h, g^*\rangle \int_K \langle g, g^*\rangle dg.$$

Moreover we have $\int_K \langle g, g^*\rangle dg = \langle k, g^*\rangle \int_K \langle g, g^*\rangle dg$ for every $k \in K$ and, since R is an integral domain, we obtain that $\mathscr{F}\mathbb{1}_{h+K}(g^*) = \mathrm{vol}(K, dg)\langle h, g^*\rangle \mathbb{1}_{K_*}(g^*)$. Then $\mathscr{F}\Phi \in \mathcal{S}(G^*)$ for every $\Phi \in \mathcal{S}(G)$, since $\mathscr{F}$ is R-linear and Φ is a finite sum of the form $\sum_h x_h \mathbb{1}_{h+K_1}$ with $x_h \in R$ and K_1 a compact open subgroup of G.

Denoting $K_{**} = \{g \in G \,|\, \langle g, g^* \rangle \, \forall g^* \in K_*\}$ we have that

$$\begin{aligned}\mathscr{F}^{-1}\mathscr{F}\mathbb{1}_{h+K}(g) &= \mathrm{vol}(K, dg) \int_{G^*} \langle h, g^* \rangle \mathbb{1}_{K_*}(g^*) \langle g, -g^* \rangle dg^* = \mathrm{vol}(K, dg) \int_{K_*} \langle h - g, g^* \rangle dg^* \\ &= \mathrm{vol}(K, dg)\mathrm{vol}(K_*, dg^*)\mathbb{1}_{h+K_{**}}.\end{aligned}$$

Moreover if $L = \bigoplus_i \varpi_F^{a_i} \mathcal{O}_F$ is an $\mathcal{O}_F$-lattice of G as above then $L_{**} = \bigoplus_i \varpi_F^{l-(l-a_i)} \mathcal{O}_F = L$. Let now L be an $\mathcal{O}_F$-lattice and K be a compact open subgroup of G such that $L \subset K$; we can write $\mathbb{1}_K = \sum_{h \in K/L} I_{h+L}$ and then we obtain

$$\begin{aligned}\mathscr{F}^{-1}\mathscr{F}\mathbb{1}_K &= \mathrm{vol}(K, dg)\mathrm{vol}(K_*, dg^*)\mathbb{1}_{K_{**}} \\ &= \mathrm{vol}(L, dg)\mathrm{vol}(L_*, dg^*) \sum_{h \in K/L} \mathbb{1}_{h+L_{**}} = \mathrm{vol}(L, dg)\mathrm{vol}(L_*, dg^*)\mathbb{1}_K.\end{aligned}$$

This implies that $K = K_{**}$ and $\mathrm{vol}(K, dg)\mathrm{vol}(K_*, dg^*) = 1$ for every compact open subgroup K of G. This proves that $\mathscr{F}$ is an isomorphism whose inverse is $\mathscr{F}^{-1}$.
Finally for every $\Psi_1, \Psi_2 \in \mathcal{S}(G^*)$ we have

$$\begin{aligned}\mathscr{F}^{-1}(\Psi_1 * \Psi_2)(g) &= \int_{G^*} \int_{G^*} \Psi_1(g_1^*)\Psi_2(g_2^* - g_1^*) dg_1^* \langle -g, g_2^* \rangle dg_2^* \\ &= \int_G \Psi_1(g_1^*) \int_G \Psi_2(g_3^*) \langle -g, g_3^* + g_1^* \rangle dg_3^* dg_1^* = \mathscr{F}^{-1}(\Psi_1)(g) \cdot \mathscr{F}^{-1}(\Psi_2)(g)\end{aligned}$$

where we have used the change of variables $g_2^* \longmapsto g_3^* = g_2^* - g_1^*$. □

Definition 6.1.3. *Let G and H be two finite dimensional F-vector spaces and let dx and dy be two Haar measures on G and H. If $\nu : G \longrightarrow H$ is an isomorphism then the* ***module*** *of ν is the constant $|\nu| = \frac{d(\nu x)}{dy}$, which means that we have*

$$\int_H \Phi(y) dy = |\nu| \int_G \Phi(\nu(x)) dx$$

where $\Phi \in \mathcal{S}(H)$. Notice that it is an integer power of q in R.

If dx^* and dy^* are the dual measures on G^* and H^* of dx and dy, then $|\nu| = |\nu^*|$ for every isomorphism $\nu : G \longrightarrow H$. Indeed if K is a compact open subgroup of G then

$$\mathrm{vol}(K, dx) = |\nu|^{-1}\mathrm{vol}(\nu(K), dy) = |\nu|^{-1}\mathrm{vol}(\nu(K)_*, dy^*)^{-1} = |\nu|^{-1}|\nu^*|\mathrm{vol}(\nu^*(\nu(K))_*, dx^*)^{-1}$$

and $\nu^*(\nu(K))_* = \{g^* \in G^* \,|\, \langle \nu(k), \nu^{*-1}(g^*) \rangle = 1 \; \forall\, k \in K\} = K_*$. Then $|\nu| = |\nu^*|$.
Moreover if $G = H$ and $dx = dy$ we have that $|\nu|$ is independent of the choice of the Haar measure dx on G.

The symplectic group

From now on, let X be a finite dimensional F-vector space and let W be the F-vector space $X \times X^*$. We denote by $\mathrm{Sp}(W)$ the group of symplectic automorphisms of W, said to be the **symplectic group** of W, that is the group of automorphisms of W such that

$$\mathcal{B}\big(\sigma(w_1), \sigma(w_2)\big) - \mathcal{B}\big(\sigma(w_2), \sigma(w_1)\big) = \mathcal{B}(w_1, w_2) - \mathcal{B}(w_2, w_1), \tag{6.4}$$

or equivalently, by (6.1), such that $\mathcal{F}\big(\sigma(w_1), \sigma(w_2)\big)\mathcal{F}\big(\sigma(w_2), \sigma(w_1)\big)^{-1} = \mathcal{F}(w_1, w_2)\mathcal{F}(w_2, w_1)^{-1}$.

Proposition 6.1.4. Every group automorphism $\sigma : W \longrightarrow W$ which satisfies (6.4) is F-linear.

Proof. Applying the change of variables $w_1 \mapsto uw_1$ with $u \in F$ in the equality (6.4), we obtain $\mathcal{B}\big(\sigma(uw_1), \sigma(w_2)\big) - \mathcal{B}\big(\sigma(w_2), \sigma(uw_1)\big) = u\big(\mathcal{B}(w_1, w_2) - \mathcal{B}(w_2, w_1)\big)$ and then using (6.4) again we obtain $\mathcal{B}\big(\sigma(uw_1) - u\sigma(w_1), \sigma(w_2)\big) = \mathcal{B}\big(\sigma(w_2), \sigma(uw_1) - u\sigma(w_1)\big)$ for every $w_1, w_2 \in W$. This implies that $\mathcal{B}\big(\sigma(uw_1) - u\sigma(w_1), \sigma(w_2)\big) = 0$ for every $w_2 \in \sigma^{-1}(0 \times X^*)$ and $\mathcal{B}\big(\sigma(w_2), \sigma(uw_1) - u\sigma(w_1)\big) = 0$ for every $w_2 \in \sigma^{-1}(X \times 0)$. Then $\sigma(uw_1) = u\sigma(w_1)$ for every $w_1 \in W$. □

We can write every $\sigma \in \mathrm{Sp}(W)$ as a matrix of the form $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where $\alpha : X \to X$, $\gamma : X \to X^*$, $\beta : X^* \to X$ and $\delta : X^* \to X^*$ are F-linear. The transpose of σ is $\sigma^* = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}$ which is an automorphism of $W^* = X^* \times X$ such that $|\sigma^*| = |\sigma|$. Furthermore if $\xi : X \times X^* \longrightarrow X^* \times X$ is the isomorphism defined by $(x, x^*) \longmapsto (-x^*, x)$ and $\sigma^I = \xi^{-1}\sigma^*\xi = \begin{pmatrix} \delta^* & -\beta^* \\ -\gamma^* & \alpha^* \end{pmatrix}$, then we have $|\sigma| = |\sigma^I|$. With these definitions, an element $\sigma \in \mathrm{Aut}(W)$ is symplectic if and only if $\sigma^I\sigma = 1$ and then the module of every symplectic automorphism is equal to 1.
Moreover we can remark that if $\sigma \in \mathrm{Sp}(W)$ then $\alpha^*\gamma = \gamma^*\alpha : X \longrightarrow X^*$ and $\beta^*\delta = \delta^*\beta : X^* \longrightarrow X$ are symmetric homomorphisms and $\alpha^*\delta - \gamma^*\beta = 1$ and $\delta^*\alpha - \beta^*\gamma = 1$.

We associate to every $\sigma \in \mathrm{Sp}(W)$ the quadratic form defined by

$$f_\sigma(w) = \frac{1}{2}\big(\mathcal{B}(\sigma(w), \sigma(w)) - \mathcal{B}(w, w)\big).$$

It is easy to check that $f_{\sigma_1 \circ \sigma_2} = f_{\sigma_1} \circ \sigma_2 + f_{\sigma_2}$ for every $\sigma_1, \sigma_2 \in \mathrm{Sp}(W)$ and that

$$f_\sigma(w_1 + w_2) - f_\sigma(w_1) - f_\sigma(w_2) = \mathcal{B}(\sigma(w_1), \sigma(w_2)) - \mathcal{B}(w_1, w_2) \tag{6.5}$$

for every $\sigma \in \mathrm{Sp}(W)$ and $w_1, w_2 \in W$.

Symplectic realizations of forms

We introduce some applications, similar to those in 33 of [62], with values in $\mathrm{Sp}(W)$ and we give some relations between them. When comparing our calculations with those of sections 6 and 7 of [62] it shall be remarked that we change most of the definitions because we consider matrices acting on the left rather than on the right, to uniform notation to the contemporary standard. This affects also the formulas that explicit the relations between these applications.

Definition 6.1.5. *We define the following maps.*

- *An injective group homomorphism from* $\mathrm{Aut}(X)$ *to* $\mathrm{Sp}(W)$*:*

$$\begin{aligned} d : \mathrm{Aut}(X) &\longrightarrow \mathrm{Sp}(W) \\ \alpha &\longmapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix}. \end{aligned}$$

- *An injective map from* $\mathrm{Iso}(X^*, X)$ *to* $\mathrm{Sp}(W)$ *where* $\mathrm{Iso}(X^*, X)$ *is the set of isomorphisms from* X^* *to* X*:*

$$\begin{aligned} d' : \mathrm{Iso}(X^*, X) &\longrightarrow \mathrm{Sp}(W) \\ \beta &\longmapsto \begin{pmatrix} 0 & \beta \\ -\beta^{*-1} & 0 \end{pmatrix}. \end{aligned}$$

  We remark that $d'(\beta)^{-1} = d'(-\beta^*)$ *for every* $\beta \in \mathrm{Iso}(X^*, X)$*.*

- *An injective group homomorphism from* $Q(X)$ *to* $\mathrm{Sp}(W)$*:*

$$\begin{aligned} t : Q(X) &\longrightarrow \mathrm{Sp}(W) \\ f &\longmapsto \begin{pmatrix} 1 & 0 \\ -\rho & 1 \end{pmatrix} \end{aligned}$$

  where $\rho = \rho(f)$ *is the symmetric homomorphism associated to* f*.*

- *An injective group homomorphism from* $Q(X^*)$ *to* $\mathrm{Sp}(W)$*:*

$$\begin{aligned} t' : Q(X^*) &\longrightarrow \mathrm{Sp}(W) \\ f' &\longmapsto \begin{pmatrix} 1 & -\rho' \\ 0 & 1 \end{pmatrix} \end{aligned}$$

  where $\rho' = \rho(f')$ *is the symmetric homomorphism associated to* f'*.*

Let G be either X or X^*. If $f \in Q(G)$ and $\alpha \in \mathrm{Aut}(G)$ we write f^α for $f \circ \alpha$.

Proposition 6.1.6.

(i) Let $f \in Q(X)$, $f' \in Q(X^*)$ and $\alpha \in \mathrm{Aut}(X)$. Then $d(\alpha)^{-1} t(f) d(\alpha) = t(f^\alpha)$ and $d(\alpha) t'(f') d(\alpha)^{-1} = t'(f'^{\alpha^*})$.

(ii) Let $\alpha \in \mathrm{Aut}(X)$, $\beta \in \mathrm{Iso}(X^*, X)$. Then $d'(\alpha\beta) = d(\alpha) d'(\beta)$ and $d'(\beta\alpha^{*-1}) = d'(\beta) d(\alpha)$.

Proof.

(i) We have $d(\alpha)^{-1}t(f)d(\alpha) = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\rho & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\alpha^*\rho\alpha & 1 \end{pmatrix}$. It is easy to check that the symmetric homomorphism associated to f^α is $-\alpha^*\rho\alpha$. With similar explicit calculations the second equality can be proven as well.

(ii) We have $d(\alpha)d'(\beta) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix} \begin{pmatrix} 0 & \beta \\ -\beta^{*-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha\beta \\ -\alpha^{*-1}\beta^{*-1} & 0 \end{pmatrix} = d'(\alpha\beta)$, and

$$d'(\beta)d(\alpha) = \begin{pmatrix} 0 & \beta \\ -\beta^{*-1} & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix} = \begin{pmatrix} 0 & \beta\alpha^{*-1} \\ -(\beta\alpha^{*-1})^{*-1} & 0 \end{pmatrix} = d'(\beta\alpha^{*-1}). \qquad \square$$

We have $d(\alpha)d'(\beta)d(\alpha)^{-1} = d'(\alpha \circ \beta \circ \alpha^*)$ so that the group $d(\mathrm{Aut}(X))$ acts on the set $d'(\mathrm{Iso}(X^*, X))$ by conjugacy in $\mathrm{Sp}(W)$.

A set of generators for the symplectic group

Let us provide a description of $\mathrm{Sp}(W)$ by generators and relations. We denote by $\Omega(W)$ the subset of $\mathrm{Sp}(W)$ of elements $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ such that β is an isomorphism. The set $\Omega(W)$ is a set of generators for $\mathrm{Sp}(W)$ (cf. 42 of [62]). The precise statement is as follows.

Proposition 6.1.7. The group $\mathrm{Sp}(W)$ is generated by the elements of $\Omega(W)$ with relations $\sigma\sigma' = \sigma''$ for every $\sigma, \sigma', \sigma'' \in \Omega(W)$ such that the equality $\sigma\sigma' = \sigma''$ holds in $\mathrm{Sp}(W)$.

Weil states also the following fact about the set $\Omega(W)$ (cf. formula (33) of [62]).

Proposition 6.1.8. Every element $\sigma \in \Omega(W)$ can be written as $\sigma = t(f_1)d'(\beta')t(f_2)$ for unique $f_1, f_2 \in Q(X)$ and $\beta' \in \mathrm{Iso}(X^*, X)$.

Remark 6.1.9. Let $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Omega(W)$. Then $\sigma = t(f_1)d'(\beta)t(f_2)$ where f_1 and f_2 are the quadratic forms associated to the symmetric homomorphisms $-\delta\beta^{-1}$ and $-\beta^{-1}\alpha$. In particular we have the formula

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \delta\beta^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & \beta \\ -\beta^{*-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta^{-1}\alpha & 1 \end{pmatrix}.$$

6.2 The metaplectic group

Following Weil's strategy we define the metaplectic group, attached to R and χ, as a central extension of the symplectic group by $R^\times$. To do so, we shall construct the groups $\mathrm{B}_0(W)$ and $\mathbb{B}_0(W)$. In particular, in Theorem 6.2.5 we characterize $\mathbb{B}_0(W)$ as central extension of $\mathrm{B}_0(W)$ by $R^\times$. This characterization permits to define the metaplectic group as fiber product over $\mathrm{B}_0(W)$ of the symplectic group and $\mathbb{B}_0(W)$ and to show that the metaplectic group is a central extension of the symplectic group by $R^\times$.
The main issue related to this group, rather than its formal definition, is to study the maps $\mu : \mathrm{Sp}(W) \to \mathrm{B}_0(W)$ and $\pi_0 : \mathbb{B}_0(W) \to \mathrm{B}_0(W)$, that depend both on R.

6.2.1 The group $B_0(W)$

Let $A(W)$ be the group whose underlying set is $W \times R^\times$ with the multiplication law

$$(w_1, t_1)(w_2, t_2) = (w_1 + w_2, t_1 t_2 \mathcal{F}(w_1, w_2))$$

where $\mathcal{F}$ is as in Definition 6.1.1. Its center is $Z = Z(A(W)) = \{(0, t), t \in R^\times\} \cong R^\times$.

We denote by $B_0(W)$ the subgroup of $\mathrm{Aut}(A(W))$ of group automorphisms of $A(W)$ acting trivially on Z, i.e. $B_0(W) = \{s \in \mathrm{Aut}(A(W)) \mid s_{|Z} = \mathrm{id}_Z\}$.

Proposition 6.2.1. Let $s \in B_0(W)$. Then there exists a unique pair $(\sigma, \varphi) \in \mathrm{Sp}(W) \times X_2(W)$ satisfying the property

$$\varphi(w_1 + w_2)\varphi(w_1)^{-1}\varphi(w_2)^{-1} = \mathcal{F}\big(\sigma(w_1), \sigma(w_2)\big)\mathcal{F}(w_1, w_2)^{-1} \tag{6.1}$$

such that $s(w, t) = (\sigma(w), \varphi(w)t)$ for every $w \in W$ and $t \in R^\times$. Conversely if the pair $(\sigma, \varphi) \in \mathrm{Sp}(W) \times X_2(W)$ satisfies (6.1), then $(w, t) \mapsto (\sigma(w), \varphi(w)t)$ defines an element of $B_0(W)$.

Proof. Let $\eta : A(W) \longrightarrow W$ and $\theta : A(W) \longrightarrow R^\times$ such that $s(w, t) = (\eta(w, t), \theta(w, t))$. For every $w_1, w_2 \in W$ and $t_1, t_2 \in R^\times$ we have

$$\begin{aligned} s((w_1, t_1)(w_2, t_2)) &= \big(\eta(w_1 + w_2, t_1 t_2 \mathcal{F}(w_1, w_2)), \theta(w_1 + w_2, t_1 t_2 \mathcal{F}(w_1, w_2))\big) \\ s(w_1, t_1)s(w_2, t_2) &= \big(\eta(w_1, t_1) + \eta(w_2, t_2), \theta(w_1, t_1)\theta(w_2, t_2)\mathcal{F}(\eta(w_1, t_1), \eta(w_2, t_2))\big). \end{aligned}$$

Since s is a homomorphism then η is so and since $s_{|Z} = \mathrm{id}_Z$ then $\eta(0, t) = 0$ for every $t \in R^\times$. These two facts imply that $\eta(w, t) = \eta(w, 1)$ for every $t \in R^\times$ so that σ, defined by $\sigma(w) = \eta(w, 1)$, is a group endomorphism of W. We have also

$$\theta(w_1 + w_2, t_1 t_2 \mathcal{F}(w_1, w_2)) = \theta(w_1, t_1)\theta(w_2, t_2)\mathcal{F}(\sigma(w_1), \sigma(w_2)). \tag{6.2}$$

Setting $w_2 = 0$ and $t_1 = 1$ and using the fact that $\theta(0, t) = t$ for every $t \in R^\times$ (since $s_{|Z} = \mathrm{id}_Z$) we obtain that $\theta(w_1, t_2) = \theta(w_1, 1)t_2$ for every $w_1 \in W$ and $t_2 \in R^\times$. So, if we set $\varphi(w) = \theta(w, 1)$, we obtain that $s(w, t) = (\sigma(w), \varphi(w)t)$ and (6.2) becomes

$$\varphi(w_1 + w_2)t_1 t_2 \mathcal{F}(w_1, w_2) = \varphi(w_1)t_1\varphi(w_2)t_2\mathcal{F}(\sigma(w_1), \sigma(w_2))$$

that is exactly the condition (6.1). Furthermore, if we take $\sigma' \in \mathrm{End}(W)$ and $\varphi' : W \longrightarrow R^\times$ such that $s^{-1}(w, t) = (\sigma'(w), \varphi'(w)t)$, then $(w, t) = s(s^{-1}(w, t)) = (\sigma(\sigma'(w)), \varphi(\sigma'(w))\varphi'(w)t)$ that implies that σ is a group automorphism of W with $\sigma^{-1} = \sigma'$. Now, the left-hand side of (6.1) is symmetric on w_1 and w_2, so σ verify the symplectic property and by Proposition 6.1.4, $\sigma \in \mathrm{Sp}(W)$. Furthermore the right-hand side of (6.1) is a bicharacter and so φ is a character of degree 2 of W.

For the vice-versa, it is easy to check that $(w, t) \mapsto (\sigma(w), \varphi(w)t)$ is an endomorphism of $A(W)$ thanks to the property (6.1), and that it is invertible with inverse $(w, t) \mapsto (\sigma^{-1}(w), (\varphi(\sigma^{-1}w))^{-1}t)$.

Notice that it acts trivially on Z, so it is an element of $\mathrm{B}_0(W)$. $\square$

From now on, we identify an element $s \in \mathrm{B}_0(W)$ with the corresponding pair (σ, φ) such that $s(w,t) = (\sigma(w), \varphi(w)t)$. If $s_1, s_2 \in B_0(W)$ and (σ_1, φ_1) and (σ_2, φ_2) are their corresponding pairs, then the composition law of $\mathrm{B}_0(W)$ becomes $s_1 \circ s_2 = (\sigma_1, \varphi_1)(\sigma_2, \varphi_2) = (\sigma_1 \circ \sigma_2, \varphi)$ where φ is defined by $\varphi(w) = \varphi_2(w)\varphi_1(\sigma_2(w))$. We observe that the identity element is $(\mathrm{id}, 1)$ and the inverse of (σ, φ) is $(\sigma^{-1}, (\varphi \circ \sigma^{-1})^{-1})$.

The projection $\pi' : \mathrm{B}_0(W) \longrightarrow \mathrm{Sp}(W)$ defined by $\pi'(\sigma, \varphi) = \sigma$ is a group homomorphism whose kernel is $\{(\mathrm{id}, \tau), \tau \in X_1(W)\}$. Furthermore, by (6.5) and (6.1), we have an injective group homomorphism

$$\begin{array}{rcl} \mu : \mathrm{Sp}(W) & \longrightarrow & \mathrm{B}_0(W) \\ \sigma & \longmapsto & (\sigma, \chi \circ f_\sigma) \end{array} \tag{6.3}$$

such that $\pi' \circ \mu$ is the identity of $\mathrm{Sp}(W)$. This means that $\mathrm{B}_0(W)$ is the semidirect product of $\{(\mathrm{id}, \tau), \tau \in X_1(W)\}$ and $\mu(\mathrm{Sp}(W))$ and in particular, by Propositions 6.1.7 and 6.1.8, it is generated by $\mu(t(Q(X)))$, $\mu(d'(\mathrm{Iso}(X^*, X)))$ and $\{(\mathrm{id}, \tau), \tau \in X_1(W)\}$.

Let us define some applications with values in $\mathrm{B}_0(W)$, similar to those in 6 of [62], composing those with values in $\mathrm{Sp}(W)$ with μ. We call them $d_0 = \mu \circ d$, $d_0' = \mu \circ d'$, $t_0 = \mu \circ t$ and $t_0' = \mu \circ t'$.

6.2.2 The group $\mathbb{B}_0(W)$

We define $\mathcal{A}$ as the image of a faithful infinite dimensional representation of $A(W)$ over R and $\mathbb{B}_0(W)$ as its normalizer in $\mathrm{Aut}(\mathcal{S}(X))$. Then we show that in fact $\mathbb{B}_0(W)$ is a central extension of $\mathrm{B}_0(W)$ by $R^\times$.

$\mathcal{A}$ and $\mathbb{B}_0(W)$

For every $w = (v, v^*) \in X \times X^* = W$ and every $t \in R^\times$, we denote by $U(w,t)$ the R-linear operator on $\mathcal{S}(X)$ defined by

$$U(w,t)\Phi : x \mapsto t\Phi(x+v)\langle x, v^*\rangle$$

for every function $\Phi \in \mathcal{S}(X)$. It can be directly verified that $U(w,t)$ lies in $\mathrm{Aut}(\mathcal{S}(X))$ for every $w \in W$ and $t \in R^\times$. With a slight abuse of notation we write $U(w) = U(w,1)$ for every $w \in W$. Let $\mathcal{A} = \{U(w,t) \in \mathrm{Aut}(\mathcal{S}(X)) \mid t \in R^\times, w \in W\}$. It is not hard to see that it is a subgroup of $\mathrm{Aut}(\mathcal{S}(X))$ and that its multiplication law is given by

$$U(w_1, t_1)U(w_2, t_2) = U(w_1 + w_2, t_1 t_2 \mathcal{F}(w_1, w_2)). \tag{6.4}$$

Lemma 6.2.2. The map

$$\begin{array}{rcl} U : A(W) & \longrightarrow & \mathbb{A}(W) \\ (w,t) & \longmapsto & U(w,t). \end{array}$$

is a group isomorphism.

Proof. By (6.4) the map U preserves operations and it is surjective. For injectivity we have to prove that if $t\Phi(x+v)\langle x, v^*\rangle = \Phi(x)$ for every $\Phi \in \mathcal{S}(X)$ and every $x \in X$ then $t = 1$ and $(v, v^*) = (0, 0)$. If we take $x = 0$ and Φ the characteristic function $\mathbb{1}_K$ of any compact open subgroup K of X, we obtain that $t\mathbb{1}_K(v) = 1$ for every K and so $t = 1$ and $v = 0$. Therefore we have that $\langle x, v^*\rangle = 1$ for every $x \in X$ and so $v^* = 0$ by (6.1). □

Remark 6.2.3. The homomorphism U is a representation of $A(W)$ on the R-module $\mathcal{S}(X)$.

The group $\mathrm{B}_0(W)$ acts on $A(W)$ and so on $\mathbb{A}(W)$ via the isomorphism in Lemma 6.2.2. This action is given by

$$\begin{array}{rcl} \mathrm{B}_0(W) \times \mathbb{A}(W) & \longrightarrow & \mathbb{A}(W) \\ ((\sigma, \varphi), U(w, t)) & \longmapsto & U(\sigma(w), t\varphi(w)). \end{array}$$

Moreover, we can identify $\mathrm{B}_0(W)$ with the group of automorphisms of $\mathbb{A}(W)$ acting trivially on the center $Z(\mathbb{A}(W)) = \{t \cdot \mathrm{id}_{\mathcal{S}(X)} \in \mathrm{Aut}(\mathcal{S}(X)) \mid t \in R^\times\} \cong R^\times$.

We denote by $\mathbb{B}_0(W)$ the normalizer of $\mathcal{A}$ in $\mathrm{Aut}(\mathcal{S}(X))$, that is

$$\mathbb{B}_0(W) = \left\{\mathbf{s} \in \mathrm{Aut}(\mathcal{S}(X)) \mid \mathbf{s}\mathbb{A}(W)\mathbf{s}^{-1} = \mathbb{A}(W)\right\}.$$

So, if $\mathbf{s}$ is an element of $\mathbb{B}_0(W)$, conjugation by $\mathbf{s}$, denoted by $\mathrm{conj}(\mathbf{s})$, is an automorphism of $\mathbb{A}(W)$.

Lemma 6.2.4. The map

$$\begin{array}{rcl} \pi_0 : \mathbb{B}_0(W) & \longrightarrow & \mathrm{B}_0(W) \\ \mathbf{s} & \longmapsto & \mathrm{conj}(\mathbf{s}) \end{array}$$

is a group homomorphism

Proof. Clearly $\mathrm{conj}(\mathbf{s})$ is trivial on $Z(\mathbb{A}(W)) = \{t \cdot \mathrm{id}_{\mathcal{S}(X)} \in \mathrm{Aut}(\mathcal{S}(X)) \mid t \in R^\times\}$ and so it lies in $\mathrm{B}_0(W)$. Moreover $\mathrm{conj}(\mathbf{s}_1\mathbf{s}_2) = \mathrm{conj}(\mathbf{s}_1)\mathrm{conj}(\mathbf{s}_2)$ so that π_0 preserves the group operation. □

Theorem 6.2.5. The following sequence is exact:

$$1 \longrightarrow R^\times \longrightarrow \mathbb{B}_0(W) \xrightarrow{\pi_0} B_0(W) \longrightarrow 1$$

where $R^\times$ injects in $\mathrm{B}_0(W)$ by $t \mapsto t \cdot \mathrm{id}_{\mathcal{S}(X)}$.

We prove this theorem in paragraph 6.2.2. Before that, we need to construct, as proposed in 13 of [62], some "liftings" to $\mathbb{B}_0(W)$ of the applications d_0, d_0' and t_0.

Realization of forms on $\mathbb{B}_0(W)$

We fix a Haar measure dx on the finite dimensional F-vector space X with values in R. We denote by dx^* the dual measure of dx on X^* and $dw = dxdx^*$ the product Haar measure on W.

From now on, we suppose that there exists a fixed square root $q^{\frac{1}{2}}$ of q in R. If ν is an isomorphism of F-vector spaces and $|\nu| = q^a$ is its module, we denote $|\nu|^{\frac{1}{2}} = (q^{\frac{1}{2}})^a \in R$.

Definition 6.2.6. *We define the following maps.*

- *A group homomorphism* $\mathbf{d}_0 : \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(\mathcal{S}(X))$ *defined by* $\mathbf{d}_0(\alpha)\Phi = |\alpha|^{-\frac{1}{2}}(\Phi \circ \alpha^{-1})$ *for every* $\alpha \in \mathrm{Aut}(X)$ *and every* $\Phi \in \mathcal{S}(X)$.

- *A map* $\mathbf{d}'_0 : \mathrm{Iso}(X^*, X) \longrightarrow \mathrm{Aut}(\mathcal{S}(X))$ *defined by* $\mathbf{d}'_0(\beta)\Phi = |\beta|^{-\frac{1}{2}}(\mathscr{F}\Phi \circ \beta^{-1})$ *for every* $\beta \in \mathrm{Iso}(X^*, X)$ *and every* $\Phi \in \mathcal{S}(X)$*, where* $\mathscr{F}\Phi$ *is the Fourier transform of* Φ *as in (6.2). We remark that* $\mathbf{d}'_0(\beta)^{-1} = \mathbf{d}'_0(-\beta^*) = |\beta|^{\frac{1}{2}}\mathscr{F}^{-1}(\Phi \circ \beta)$.

- *A group homomorphism* $\mathbf{t}_0 : Q(X) \longrightarrow \mathrm{Aut}(\mathcal{S}(X))$ *defined by* $\mathbf{t}_0(f)\Phi = (\chi \circ f) \cdot \Phi$ *for every* $f \in Q(X)$ *and every* $\Phi \in \mathcal{S}(X)$.

We shall now to prove that they are actually onto $\mathbb{B}_0(W)$ and that they lift in $\mathbb{B}_0(W)$ the applications d_0, d'_0 and t_0.

Proposition 6.2.7. The images of $\mathbf{d}_0$, $\mathbf{d}'_0$ and $\mathbf{t}_0$ are in $\mathbb{B}_0(W)$ and they satisfy

$$\pi_0 \circ \mathbf{d}_0 = d_0 \qquad \pi_0 \circ \mathbf{d}'_0 = d'_0 \qquad \text{and} \qquad \pi_0 \circ \mathbf{t}_0 = t_0.$$

Proof. For every $\alpha \in \mathrm{Aut}(X)$, $\Phi \in \mathcal{S}(X)$, $w = (v, v^*) \in W$ and $x \in X$ we have

$$\begin{aligned} \mathbf{d}_0(\alpha)U(w)\mathbf{d}_0(\alpha)^{-1}\Phi(x) &= \mathbf{d}_0(\alpha)U(w)|\alpha|^{\frac{1}{2}}(\Phi \circ \alpha)(x) = \Phi(\alpha(\alpha^{-1}(x) + u))\langle \alpha^{-1}(x), v^*\rangle \\ &= \Phi(x + \alpha(u))\langle x, \alpha^{*-1}(v^*)\rangle = d_0(\alpha)U(w)\Phi(x). \end{aligned}$$

For every $\beta \in \mathrm{Iso}(X^*, X)$, $\Phi \in \mathcal{S}(X)$, $w = (v, v^*) \in W$ and $x \in X$ we have

$$\begin{aligned} \mathbf{d}'_0(\beta)U(w)\mathbf{d}'_0(\beta)^{-1}\Phi(x) &= \mathbf{d}'_0(\beta)U(w)|\beta|^{\frac{1}{2}}\mathscr{F}^{-1}(\Phi \circ \beta)(x) \\ &= \int_X \left(\int_{X^*} \Phi(\beta(x^*))\langle x_1 + v, -x^*\rangle dx^*\right) \langle x_1, v^*\rangle\langle x_1, \beta^{-1}(x)\rangle dx_1 \\ &= \int_X \left(\int_{X^*} \Phi(\beta(x^*))\langle -v, x^*\rangle\langle x_1, -x^*\rangle dx^*\right) \langle x_1, v^* + \beta^{-1}(x)\rangle dx_1 \\ &= \Phi(\beta(v^* + \beta^{-1}(x)))\langle -v, v^* + \beta^{-1}(x)\rangle \\ &= \Phi(x + \beta(v^*))\langle x, -\beta^{*-1}(v)\rangle\langle v, -v^*\rangle = d'_0(\beta)U(w)\Phi(x). \end{aligned}$$

For every $f \in Q(X)$, $\Phi \in \mathcal{S}(X)$, $w = (v, v^*) \in W$ and $x \in X$ we have

$$\begin{aligned} \mathbf{t}_0(f)U(w)\mathbf{t}_0(f)^{-1}\Phi(x) &= \chi(f(x))\chi(f(x+v))^{-1}\Phi(x+v)\langle x, v^*\rangle \\ &= \chi(f(v))^{-1}\langle x, \rho(v)\rangle^{-1}\Phi(x+v)\langle x, v^*\rangle = t_0(f)U(w)\Phi(x). \end{aligned}$$

These equalities prove at the same time that the images of $\mathbf{d}_0$, $\mathbf{d}'_0$ and $\mathbf{t}_0$ are in $\mathbb{B}_0(W)$ and that they lift in $\mathbb{B}_0(W)$ respectively the applications d_0, d'_0 and t_0. □

Proposition 6.2.7 and the injectivity of d_0 and t_0 entail injectivity for $\mathbf{d}_0$ and $\mathbf{t}_0$. Moreover Propositions 6.1.6 and 6.2.7 say that for every $f \in Q(X)$, $\alpha \in \mathrm{Aut}(X)$ and $\beta \in \mathrm{Iso}(X^*, X)$, the three elements $\mathbf{d}_0(\alpha)^{-1}\mathbf{t}_0(f)\mathbf{d}_0(\alpha)$, $\mathbf{d}'_0(\alpha\circ\beta)$ and $\mathbf{d}'_0(\beta\circ\alpha^{*-1})$ of $\mathbb{B}_0(W)$ differ, respectively from $\mathbf{t}_0(f^\alpha)$, $\mathbf{d}_0(\alpha)\mathbf{d}'_0(\beta)$ and $\mathbf{d}'_0(\beta)\mathbf{d}_0(\alpha)$ just by elements of $R^\times$. A direct calculation gives

$$\mathbf{d}_0(\alpha)^{-1}\mathbf{t}_0(f)\mathbf{d}_0(\alpha) = \mathbf{t}_0(f^\alpha) \quad \mathbf{d}'_0(\alpha\circ\beta) = \mathbf{d}_0(\alpha)\mathbf{d}'_0(\beta) \quad \mathbf{d}'_0(\beta\circ\alpha^{*-1}) = \mathbf{d}'_0(\beta)\mathbf{d}_0(\alpha) \tag{6.5}$$

so that in fact these elements are the identity.

Proof of Theorem 6.2.5

In this paragraph we give a proof of Theorem 6.2.5 that is fundamental for the definition of the metaplectic group.

Firstly we prove that π_0 is surjective: we know that $\mathrm{B}_0(W)$ is generated by $\mu(t(Q(X)))$, $\mu(d'(\mathrm{Iso}(X^*,X)))$ and $\{(\mathrm{id},\tau),\tau \in X_1(W)\}$ so that it is sufficient to prove that every element in these sets is in the image of π_0. By Proposition 6.2.7, this is proved for the sets $\mu(t(Q(X)))$ and $\mu(d'(\mathrm{Iso}(X^*,X)))$. Moreover by (6.1) we have that every character τ of W is of the form $\tau(v,v^*) = \langle a,v^*\rangle\langle v,a^*\rangle$ for suitable $a \in X$ and $a^* \in X^*$. For every $w=(v,v^*)\in W$ and $t\in R^\times$ we have $(1,\tau)U(w,t) = U(w,t\cdot\tau(w)) = U(w,t\langle a,v^*\rangle\langle v,a^*\rangle) = U(a,-a^*)U(w,t)U(-a,a^*,\langle a,-a^*\rangle)$ and so $(\mathrm{id},\tau) = \pi_0(U(a,-a^*))$.

Let us now calculate the kernel of π_0. For $\phi\in\mathcal{S}(X\times X^*)$ we denote by $\mathcal{U}(\phi)$ the operator on $\mathcal{S}(X)$ defined by

$$\mathcal{U}(\phi) = \int_W U(w,\phi(w))dw = \int_W \phi(w)U(w)dw.$$

This means that for every $\Phi\in\mathcal{S}(X)$ and every $x\in X$ we have

$$\mathcal{U}(\phi)\Phi(x) = \int_W \phi(w)(U(w)\Phi)(x)dw = \int_W \phi(v,v^*)\Phi(x+v)\langle x,v^*\rangle dv dv^*$$

where $w=(v,v^*)$. Given $P,Q\in\mathcal{S}(X)$ we denote by $\phi_{P,Q}\in\mathcal{S}(X\times X^*)$ the function defined by

$$\phi_{P,Q}(v,v^*) = \int_X P(v')Q(v'+v)\langle -v',v^*\rangle dv'$$

for every $v\in X, v^*\in X^*$. With this definition we obtain

$$\mathcal{U}(\phi_{P,Q})\Phi(x) = \int_X \Phi(x+v)\int_{X^*}\int_X P(v')Q(v'+v)\langle x-v',v^*\rangle dv'dv^*dv$$

and using Proposition 6.1.2 we have

$$\mathcal{U}(\phi_{P,Q})\Phi(x) = \int_X \Phi(x+v)P(x)Q(x+v)dv = \int_X \Phi(v)Q(v)dvP(x).$$

If we denoted by $[P,Q] = \int_X P(x)Q(x)dx$ for every $P,Q\in\mathcal{S}(X)$ we have $\mathcal{U}(\phi_{P,Q})\Phi = [\Phi,Q]P$. Now, $\mathbf{s}$ is in the kernel of π_0 if and only if it lies in the centralizer of $\mathbb{A}(W)$ in $\mathrm{Aut}(\mathcal{S}(X))$. If this is the case, then $\mathbf{s}$ commutes with $\mathcal{U}(\phi)$ in $\mathrm{End}(\mathcal{S}(X))$ for every $\phi\in\mathcal{S}(X\times X^*)$, i.e. $\mathbf{s}(U(\phi)\Phi) = U(\phi)(\mathbf{s}(\Phi))$. In particular $\mathbf{s}$ commutes with operators of the form $\mathcal{U}(\phi_{P,Q})$ for every

$P, Q \in \mathcal{S}(X)$, that is $[\mathbf{s}\Phi, Q]P = [\Phi, Q]\mathbf{s}P$ for every $\Phi, P, Q \in \mathcal{S}(X)$. If we choose $\Phi = Q = \mathbb{1}_K$ where K is a compact open subgroup of X with $\mathrm{vol}(K, dx) \in R^\times$, we can write

$$\mathbf{s}P = \frac{[\mathbf{s}\Phi, Q]}{[\Phi, Q]}P.$$

In other words $\mathbf{s}$ is of the form $\Phi \mapsto t\Phi$ for a suitable $t \in R$ and t has to be invertible since $\mathbf{s}$ is an automorphism. Hence $\ker(\pi_0) \subseteq \{t \cdot \mathrm{id}_{\mathcal{S}(X)} \in \mathrm{Aut}(\mathcal{S}(X)) \,|\, t \in R^\times\}$. The converse is true because the center of a group is always contained in its centralizer.

Remark 6.2.8. In proving Theorem 6.2.5 the techniques used in [62] could be adapted to show that $\ker(\pi_0) \cong R^\times$, but not to prove surjectivity of π_0.

6.2.3 The metaplectic group

We have just defined in (6.3) and Lemma 6.2.4 the group homomorphisms

$$\begin{array}{rcl} \mu : \mathrm{Sp}(W) & \longrightarrow & \mathrm{B}_0(W) \\ \sigma & \longmapsto & (\sigma, \chi \circ f_\sigma) \end{array} \qquad \text{and} \quad \begin{array}{rcl} \pi_0 : \mathbb{B}_0(W) & \longrightarrow & \mathrm{B}_0(W) \\ \mathbf{s} & \longmapsto & conj(\mathbf{s}). \end{array}$$

The first one is injective, while the second one is surjective with kernel isomorphic to $R^\times$. We remark that the definition of $\mathbb{B}_0(W)$ and these two homomorphisms depend on the choice of the integral domain R and the smooth non-trivial character χ.

Definition 6.2.9. *The **metaplectic group** of W, attached to R and χ, is the subgroup* $\mathrm{Mp}_{R,\chi}(W) = \mathrm{Sp}(W) \times_{\mathrm{B}_0(W)} \mathbb{B}_0(W)$ *of* $\mathrm{Sp}(W) \times \mathbb{B}_0(W)$ *of the pairs* $(\sigma, \mathbf{s})$ *such that* $\mu(\sigma) = \pi_0(\mathbf{s})$.

From now on, we write $\mathrm{Mp}(W)$ instead of $\mathrm{Mp}_{R,\chi}(W)$. We have a group homomorphism

$$\begin{array}{rcl} \pi : \mathrm{Mp}(W) & \longrightarrow & \mathrm{Sp}(W) \\ (\sigma, \mathbf{s}) & \longmapsto & \sigma. \end{array}$$

The morphism π_0 is surjective and surjectivity in the category of groups is preserved under base-change, therefore π is surjective. Moreover an element $(\sigma, \mathbf{s})$ is in the kernel of π if and only if $\mathbf{s}$ is in the kernel of π_0, that is isomorphic to $R^\times$. Thus we obtain:

Theorem 6.2.10. The following sequence is exact:

$$1 \longrightarrow R^\times \longrightarrow \mathrm{Mp}(W) \xrightarrow{\pi} \mathrm{Sp}(W) \longrightarrow 1 \tag{6.6}$$

where $R^\times$ injects in $\mathrm{Mp}(W)$ by $t \mapsto (\mathrm{id}, t \cdot \mathrm{id}_{\mathcal{S}(X)})$.

Since $\mathbb{B}_0(W) = \mathrm{B}_0(W)/R^\times$ and $\mathbb{B}_0(W) \subset \mathrm{Aut}(\mathcal{S}(X))$, we may regard μ as a projective representation of the symplectic group. Then, the metaplectic group is defined in such a way that the map

$$\begin{array}{rcl} \mathrm{Mp}(W) & \longrightarrow & \mathbb{B}_0(W) \\ (\sigma, \mathbf{s}) & \longmapsto & \mathbf{s} \end{array} \tag{6.7}$$

is a faithful representation on the R-module $\mathcal{S}(X)$ that lifts μ.

6.3 The Weil factor

The sequence (6.6) constitutes the object of our study and the rest of the article is devoted to study its properties. Following the idea of Weil, we define in this section a map γ that associates to every non-degenerate quadratic form f on X an invertible element $\gamma(f) \in R^\times$ (cfr. 14 of [62]). This object, that we call **Weil factor**, shows up at the moment of understanding the map π by lifting a description of $\mathrm{Sp}(W)$ by generators and relations. The study of its properties is at the heart of the results in [62]. We prove that similar properties hold for $\gamma(f) \in R^\times$.
The general idea is: we find the relation (6.2) in $\mathrm{B}_0(W)$ and we lift it into $\mathbb{B}_0(W)$ finding an element of $R^\times$ thanks to Theorem 6.2.5. Then we proceed in two directions: on one hand we prove results that are useful to calculate $\gamma(f)$ while on the other we use the Weil factor to lift to $\mathrm{Mp}(W)$ the relations of Proposition 6.1.7.

6.3.1 The Weil factor

Let $f \in Q^{nd}(X)$ be a non-degenerate quadratic form on X and let $\rho \in \mathrm{Iso}(X, X^*)$ be its associated symmetric isomorphism. Explicit calculations in $\mathrm{Sp}(W)$ give the equality

$$d'(\rho^{-1})t(f)d'(-\rho^{-1})t(f) = t(-f)d'(\rho^{-1}). \tag{6.1}$$

Moreover, applying Proposition 6.1.6, (6.1) is equivalent to $\big(t(f)d'(\rho^{-1})\big)^3 = \big(d'(\rho^{-1})t(f)\big)^3 = 1$. It follows from equation (6.1) that

$$d_0'(\rho^{-1})t_0(f)d_0'(-\rho^{-1})t_0(f) = t_0(-f)d_0'(\rho^{-1}). \tag{6.2}$$

We denote $\mathbf{s} = \mathbf{s}(f) = \mathbf{d}_0'(\rho^{-1})\mathbf{t}_0(f)\mathbf{d}_0'(-\rho^{-1})\mathbf{t}_0(f)$ and $\mathbf{s}' = \mathbf{s}'(f) = \mathbf{t}_0(-f)\mathbf{d}_0'(\rho^{-1})$. We have by Proposition 6.2.7 and equation (6.2), $\pi_0(\mathbf{s}) = \pi_0(\mathbf{s}')$. Hence $\mathbf{s}$ and $\mathbf{s}'$ differ by an element of $R^\times$

Definition 6.3.1. *Let $\gamma(f) \in R^\times$ be such that $\mathbf{s} = \gamma(f)\mathbf{s}'$. We call $\gamma(f)$ the Weil factor associated to $f \in Q^{nd}(X)$.*

By formulas (6.5) we have $\gamma(f) = \big(\mathbf{t}_0(f)\mathbf{d}_0'(\rho^{-1})\big)^3 = \big(\mathbf{d}_0'(\rho^{-1})\mathbf{t}_0(f)\big)^3$.

We are now ready to investigate some properties of γ, starting from seeing what changes under the action of $\mathrm{Aut}(X)$.

Proposition 6.3.2. Let $f \in Q^{nd}(X)$.

(i) We have $\gamma(-f) = \gamma(f)^{-1}$.

(ii) For every $\alpha \in \mathrm{Aut}(X)$ we have $\gamma(f^\alpha) = \gamma(f)$.

Proof. Let $f \in Q^{nd}(X)$ be associated to the symmetric isomorphism ρ.

(i) We have $\gamma(-f) = \big(\mathbf{t}_0(-f)\mathbf{d}_0'(-\rho^{-1})\big)^3 = \big(\mathbf{d}_0'(\rho^{-1})\mathbf{t}_0(f)\big)^{-3} = \gamma(f)^{-1}$.

(ii) The symmetric isomorphism associated to f^α is $\alpha^*\rho\alpha$. Then we have

$$\begin{aligned}\gamma(f^\alpha) &= \left(\mathbf{t}_0(f^\alpha)\mathbf{d}'_0(\alpha^{-1}\rho^{-1}\alpha^{*-1})\right)^3 = \left(\mathbf{d}_0(\alpha)^{-1}\mathbf{t}_0(f)\mathbf{d}_0(\alpha)\mathbf{d}_0(\alpha)^{-1}\mathbf{d}'_0(\rho^{-1})\mathbf{d}_0(\alpha)\right)^3 \\ &= \mathbf{d}_0(\alpha)^{-1}\left(\mathbf{t}_0(f)\mathbf{d}'_0(\rho^{-1})\right)^3\mathbf{d}_0(\alpha) = \gamma(f). \qquad \square\end{aligned}$$

Proposition 6.3.2 gives actually a strong result in a particular case: if $-1 \in (F^\times)^2$ and $a^2 = -1$ with $a \in F^\times$ then $x \mapsto ax$ is an automorphism of X. By Proposition 6.3.2 we have $\gamma(f) = \gamma(-f) = \gamma(f)^{-1}$, in other words $\gamma(f)^2 = 1$. This does not hold in general for a local field F without square roots of -1.

Let $f \in Q^{nd}(X)$ be associated to ρ and define $\varphi = \chi \circ f$. Notice that $\varphi(-x) = \varphi(x)$. For every $\Phi \in \mathcal{S}(X)$, we denote by $\Phi * \varphi$ the convolution product defined by

$$(\Phi * \varphi)(x) = \int_X \Phi(x')\varphi(x - x')dx'$$

for every $x \in X$. We have that $\Phi * \varphi \in \mathcal{S}(X)$, indeed

$$\begin{aligned}(\Phi * \varphi)(x) &= \int_X \Phi(x')\varphi(x - x')dx' = \varphi(x)\int_X \Phi(x')\varphi(-x')\langle x, \rho(-x')\rangle dx' \\ &= \varphi(x)\int_X \Phi(x')\varphi(x')\langle x', -\rho(x)\rangle dx' = |\rho|^{-\frac{1}{2}}\mathbf{t}_0(f)\mathbf{d}_0(-\rho^{-1})\mathbf{t}_0(f)\Phi(x)\end{aligned}$$

where we have used that $\varphi(x + y) = \varphi(x)\varphi(y)\langle x, \rho(y)\rangle$ for every $x, y \in X$.

Now we state a proposition that gives a summation formula for $\gamma(f)$ and that allows us to calculate in Theorem 6.4.1 the value of γ for a specific quadratic form over F.

Proposition 6.3.3. Let $f \in Q^{nd}(X)$ be associated to the symmetric isomorphism $\rho \in \mathrm{Iso}(X, X^*)$ and let $\mathbf{s}, \mathbf{s}' \in \mathbb{B}_0(W)$ as in Definition 6.3.1. We set $\varphi = \chi \circ f$.

1. For every $\Phi \in \mathcal{S}(X)$ and for every $x \in X$ we have

$$\mathbf{s}\Phi(x) = |\rho|\mathscr{F}(\Phi * \varphi)(\rho(x)) \quad \text{and} \quad \mathbf{s}'\Phi(x) = |\rho|^{\frac{1}{2}}\mathscr{F}\Phi(\rho(x))\varphi(x)^{-1}.$$

2. For every $\Phi \in \mathcal{S}(X)$ and for every $x^* \in X^*$ we have

$$\mathscr{F}(\Phi * \varphi)(x^*) = \gamma(f)|\rho|^{-\frac{1}{2}}\mathscr{F}\Phi(x^*)\varphi(\rho^{-1}x^*)^{-1}. \tag{6.3}$$

3. There exists a sufficiently large compact open subgroup K_0 of X such that for every compact open subgroup K of X containing K_0 and for every $x^* \in X^*$, the integral $\int_K \varphi(x)\langle x, x^*\rangle dx$ does not depend on K. Moreover we have

$$\int_K \varphi(x)\langle x, x^*\rangle dx = \gamma(f)|\rho|^{-\frac{1}{2}}\varphi(\rho^{-1}x^*)^{-1} \tag{6.4}$$

and we denote $\mathscr{F}\varphi = \int_K \varphi(x)\langle x, x^*\rangle dx$.

4. If K is a sufficiently large compact open subgroup of X, we have

$$\gamma(f) = |\rho|^{\frac{1}{2}} \int_K \chi(f(x))dx. \tag{6.5}$$

Proof.

1. For every $\Phi \in \mathcal{S}(X)$ and every $x \in X$ we have

$$\begin{aligned}
\mathbf{s}\Phi(x) &= \mathbf{d}'_0(\rho^{-1})\mathbf{t}_0(f)\mathbf{d}'_0(-\rho^{-1})\mathbf{t}_0(f)\Phi(x) \\
&= |\rho| \int_X \int_X \Phi(x_1)\varphi(x_1)\langle x_1, -\rho(x_2)\rangle\varphi(x_2)\langle x_2, \rho(x)\rangle dx_1 dx_2 \\
&= |\rho| \int_X \int_X \Phi(x_1)\varphi(-x_1)\langle x_1, -\rho(x_2)\rangle\varphi(x_2)\langle x_2, \rho(x)\rangle dx_1 dx_2 \\
&= |\rho| \int_X \int_X \Phi(x_1)\varphi(x_2 - x_1)\langle x_2, \rho(x)\rangle dx_1 dx_2 = |\rho|\mathscr{F}(\Phi * \varphi)(\rho(x))
\end{aligned}$$

   and $\mathbf{s}'\Phi(x) = \mathbf{t}_0(-f)\mathbf{d}'_0(\rho^{-1})\Phi(x) = \mathbf{t}_0(-f)|\rho|^{\frac{1}{2}}\mathscr{F}(\Phi \circ \rho)(x) = \varphi(x)^{-1}|\rho|^{\frac{1}{2}}\mathscr{F}(\Phi \circ \rho)(x)$.

2. By the equality $\mathbf{s} = \gamma(f)\mathbf{s}'$ we have $|\rho|\mathscr{F}(\Phi * \varphi)(\rho(x)) = \gamma(f)|\rho|^{\frac{1}{2}}\mathscr{F}\Phi(\rho(x))\varphi(x)^{-1}$ and replacing $\rho(x)$ by x^* we obtain the equality (6.3).

3. Taking $\Phi = \mathbb{1}_H$ for a compact open subgroup H of X in formula (6.3), we obtain

$$\int_X (\mathbb{1}_H * \varphi)(x_1)\langle x_1, x^*\rangle dx_1 = \gamma(f)|\rho|^{-\frac{1}{2}}\mathscr{F}\mathbb{1}_H(x^*)\varphi(\rho^{-1}x^*)^{-1}.$$

   We want to calculate the integral in the left hand side. We can take a compact open subgroup K_0 of X large enough to contain both H and the support of $\mathbb{1}_H * \varphi$ obtaining

$$\int_X (\mathbb{1}_H * \varphi)(x_1)\langle x_1, x^*\rangle dx_1 = \int_{K_0} \int_H \varphi_{|K_0}(x_1 - x_2)dx_2\langle x_1, x^*\rangle dx_1.$$

   Now, we can prove that $\varphi_{|K_0}$ is locally constant and that we can change the order of the two integrals, i.e.

$$\begin{aligned}
\int_X (\mathbb{1}_H * \varphi)(x_1)\langle x_1, x^*\rangle dx_1 &= \int_H \int_{K_0} \varphi_{|K_0}(x_1 - x_2)\langle x_1, x^*\rangle dx_1 dx_2 \\
&= \int_H \int_{K_0} \varphi_{|K_0}(x'_1)\langle x'_1 + x_2, x^*\rangle dx'_1 dx_2 \\
&= \mathscr{F}\mathbb{1}_H(x^*) \int_{K_0} \varphi_{|K_0}(x'_1)\langle x'_1, x^*\rangle dx'_1.
\end{aligned}$$

   Since $\mathscr{F}\mathbb{1}_H = \mathrm{vol}(H)\mathbb{1}_{H_*}$ and $\mathrm{vol}(H) \neq 0$, we obtain the equality (6.4) for every $x^* \in H_*$ and every H compact open subgroup of X. Now H_* cover X^*, varying H, and so the equality holds for every $x^* \in X^*$. It is clear that the equality holds also for every compact open subgroup K of X containing K_0.

4. Setting $x^* = 0$ in (6.4) we obtain $\gamma(f) = |\rho|^{\frac{1}{2}} \int_K \varphi(x)dx = |\rho|^{\frac{1}{2}} \int_K \chi(f(x))dx$. $\square$

Remark 6.3.4. The second result in Proposition 6.3.2 is true more generally for every $\alpha' \in \mathrm{Iso}(X', X)$ where X' is a finite dimensional F-vector space. In fact if K' is a compact open subgroup of X' large enough, $f \in Q^{nd}(X)$ and $\alpha \in \mathrm{Iso}(X', X)$ by (6.5) we have

$$\begin{aligned}\gamma(f \circ \alpha) &= |\alpha^*\rho\alpha|^{\frac{1}{2}} \int_{K'} \chi(f(\alpha(x')))dx' = |\rho|^{\frac{1}{2}}|\alpha| \int_{X'} \mathbb{1}_{\alpha(K')}(\alpha(x'))\chi(f(\alpha(x')))dx' \\ &= |\rho|^{\frac{1}{2}} \int_X \mathbb{1}_{\alpha(K')}(x)\chi(f(x))dx' = \gamma(f).\end{aligned}$$

Symplectic generators in $\mathbb{B}_0(W)$

Definition 6.3.5. *Let $\sigma \in \Omega(W)$. By Proposition 6.1.8 we can write $\sigma = t(f_1)d'(\beta)t(f_2)$ for unique $f_1, f_2 \in Q(X)$ and $\beta \in \mathrm{Iso}(X^*, X)$. We define a map $\mathbf{r}_0 : \Omega(W) \to \mathbb{B}_0(W)$ by*

$$\mathbf{r}_0(\sigma) = \mathbf{t}_0(f_1)\mathbf{d}'_0(\beta)\mathbf{t}_0(f_2)$$

for every $\sigma \in \Omega(W)$.

Now we state a theorem that says how an equality $\sigma'' = \sigma\sigma'$ in $\Omega(W)$ lifts to $\mathbb{B}_0(W)$. After a comparison with section 15 of [62] the differences turn out to be the use of Fourier transform for Schwartz functions and previous changes in notations. Finally we have clarified some points and made them explicit.

Theorem 6.3.6. Let $\sigma = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$, $\sigma' = \begin{pmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{pmatrix}$ and $\sigma'' = \begin{pmatrix}\alpha'' & \beta'' \\ \gamma'' & \delta''\end{pmatrix}$ be elements of $\Omega(W)$ such that $\sigma'' = \sigma\sigma'$. Then

$$\mathbf{r}_0(\sigma)\mathbf{r}_0(\sigma') = \gamma(f_0)\mathbf{r}_0(\sigma'')$$

where f_0 is the non-degenerate quadratic form on X associated to the symmetric isomorphism $-\beta^{-1}\beta''\beta'^{-1} : X \to X^*$.

Proof. Since $\mathbf{r}_0(\sigma)\mathbf{r}_0(\sigma')$ and $\mathbf{r}_0(\sigma'')$ have the same image by π_0, we can set $\mathbf{r}_0(\sigma)\mathbf{r}_0(\sigma') = \lambda\mathbf{r}_0(\sigma'')$ where $\lambda \in R^\times$ depends on σ, σ'. By Definition 6.3.5 we have

$$\mathbf{t}_0(f_1)\mathbf{d}'_0(\beta)\mathbf{t}_0(f_2)\mathbf{t}_0(f'_1)\mathbf{d}'_0(\beta')\mathbf{t}_0(f'_2) = \lambda\mathbf{t}_0(f''_1)\mathbf{d}'_0(\beta'')\mathbf{t}_0(f''_2)$$

for suitable $f_1, f_2, f'_1, f'_2, f''_1, f''_2 \in Q(X)$. Setting $f_0 = f_2 + f'_1$, $f_3 = -f_1 + f''_1$ and $f_4 = f''_2 - f'_2$ we obtain

$$\mathbf{d}'_0(\beta)\mathbf{t}_0(f_0)\mathbf{d}'_0(\beta') = \mathbf{d}'_0(\beta)\mathbf{t}_0(f_0)\mathbf{d}'_0(-\beta'^*)^{-1} = \lambda\mathbf{t}_0(f_3)\mathbf{d}'_0(\beta'')\mathbf{t}_0(f_4)$$

where we have used that $\mathbf{d}'_0(\beta')^{-1} = \mathbf{d}'_0(-\beta'^*)$. By Remark 6.1.9 the symmetric homomorphisms associated to f_2 and f'_1 are $\rho_2 = -\beta^{-1}\alpha$ and $\rho'_1 = -\delta'\beta'^{-1}$, hence the symmetric homomorphism associated to f_0 is $\rho_0 = \rho_2 + \rho'_1 = -\beta^{-1}(\alpha\beta' + \beta\delta')\beta'^{-1} = -\beta^{-1}\beta''\beta'^{-1} = -\beta'^{*-1}\beta''^{*}\beta^{*-1}$ that is also an isomorphism.

We set $\varphi_i = \chi \circ f_i$ for $i = 0, 3, 4$. For every $\Phi \in \mathcal{S}(X)$ and $x \in X$ we have

$$\mathbf{d}_0'(\beta)\mathbf{t}_0(f_0)\mathbf{d}_0'(-\beta'^*)^{-1}\Phi(x) = |\beta|^{-\frac{1}{2}}|\beta'|^{\frac{1}{2}}\mathscr{F}(\mathscr{F}^{-1}(\Phi \circ (-\beta'^*)) \cdot \varphi_0)(\beta^{-1}x).$$

By Proposition 6.1.2 the Fourier transform of a pointwise product is the convolution product of the Fourier transforms and then $\mathbf{d}_0'(\beta)\mathbf{t}_0(f_0)\mathbf{d}_0'(\beta')\Phi(x) = |\beta|^{-\frac{1}{2}}|\beta'|^{\frac{1}{2}}\big((\Phi \circ \beta'^*) * \mathscr{F}\varphi_0\big)(\beta^{-1}x)$. Using formula (6.4) we obtain

$$\begin{aligned}
\mathbf{d}_0'(\beta)\mathbf{t}_0(f_0)\mathbf{d}_0'(\beta')\Phi(x) &= \gamma(f_0)|\rho_0|^{-\frac{1}{2}}|\beta|^{-\frac{1}{2}}|\beta'|^{\frac{1}{2}}\big((\Phi \circ \beta'^*) * (\varphi_0 \circ \rho_0^{-1})^{-1}\big)(\beta^{-1}x) \\
&= \gamma(f_0)|\beta''|^{-\frac{1}{2}}|\beta'|\big((\Phi \circ \beta'^*) * (\varphi_0 \circ \rho_0^{-1})^{-1}\big)(\beta^{-1}x) \\
&= \gamma(f_0)|\beta''|^{-\frac{1}{2}}|\beta'|\int_{X^*} \Phi(\beta'^*(x^*))\varphi_0(\beta^*\beta''^{*-1}\beta'^*(x^*) - \beta'\beta''^{-1}(x))^{-1}dx^* \\
&= \gamma(f_0)|\beta''|^{-\frac{1}{2}}\int_X \Phi(x_1)\varphi_0(-\beta'\beta''^{-1}(x) + \beta^*\beta''^{*-1}(x_1))^{-1}dx_1
\end{aligned}$$

where in the last step we have used the change of variables $\beta'^*(x^*) \mapsto x_1$. Furthermore we have

$$\mathbf{t}_0(f_3)\mathbf{d}_0'(\beta'')\mathbf{t}_0(f_4)\Phi(x) = |\beta''|^{-\frac{1}{2}}\int_X \Phi(x_1)\varphi_4(x_1)\varphi_3(x)\langle x_1, \beta''^{-1}x\rangle dx_1$$

and then

$$\gamma(f_0)\int_X \Phi(x_1)\varphi_0(-\beta'\beta''^{-1}(x) + \beta^*\beta''^{*-1}(x_1))^{-1}dx_1 = \lambda\int_X \Phi(x_1)\varphi_4(x_1)\varphi_3(x)\langle x_1, \beta''^{-1}x\rangle dx_1.$$

We observe that the two sides are of the form $c_i\int_X \Phi(x_1)\vartheta_i(x_1, x)dx_1$ for $i = 1, 2$, where $c_i \in R^\times$ and ϑ_i are characters of degree 2 of $X \times X$. Since the equality holds for every $\Phi \in \mathcal{S}(X)$ and every $x \in X$, we obtain that $c_1 = c_2$ and $\vartheta_1 = \vartheta_2$ and so $\gamma(f_0) = \lambda$. $\square$

6.3.2 Metaplectic realizations of forms

Definitions 6.1.5 and 6.2.6 allow us to define some applications from $\mathrm{Aut}(X)$, $\mathrm{Iso}(X^*, X)$ and $Q(X)$ to $\mathrm{Mp}(W)$, similar to those in 34 of [62], that satisfy relations analogous to those of $\mathbf{d}_0$, $\mathbf{d}_0'$ and $\mathbf{t}_0$.

Definition 6.3.7. *Let* $\mathrm{Mp}(W)$ *be as in Definition 6.2.9. We define the following applications.*

- *The injective group homomorphism* $\mathbf{d} : \mathrm{Aut}(X) \longrightarrow \mathrm{Mp}(W)$ *given by* $\mathbf{d}(\alpha) = (d(\alpha), \mathbf{d}_0(\alpha))$ *for every* $\alpha \in \mathrm{Aut}(X)$.

- *The injective map* $\mathbf{d}' : \mathrm{Iso}(X^*, X) \longrightarrow \mathrm{Mp}(W)$ *given by* $\mathbf{d}'(\beta) = (d'(\beta), \mathbf{d}_0'(\beta))$ *for every* $\beta \in \mathrm{Iso}(X^*, X)$.

- *The injective group homomorphism* $\mathbf{t} : Q(X) \longrightarrow \mathrm{Mp}(W)$ *given by* $\mathbf{t}(f) = (t(f), \mathbf{t}_0(f))$ *for every* $f \in Q(X)$.

By Proposition 6.1.6 and by (6.5) we have

$$\mathbf{d}(\alpha)^{-1}\mathbf{t}(f)\mathbf{d}(\alpha) = \mathbf{t}(f^{\alpha}) \tag{6.6}$$

for every $f \in Q(X)$ and $\alpha \in \mathrm{Aut}(X)$. We have also $\mathbf{d}'(\alpha \circ \beta) = \mathbf{d}(\alpha)\mathbf{d}'(\beta)$ and $\mathbf{d}'(\beta \circ \alpha^{*-1}) = \mathbf{d}'(\beta)\mathbf{d}(\alpha)$ for every $\alpha \in \mathrm{Aut}(X)$ and $\beta \in \mathrm{Iso}(X^*, X)$.

As in Definition 6.3.5, we can define a map from $\Omega(W)$ to $\mathrm{Mp}(W)$. By Proposition 6.1.8 every element $\sigma \in \Omega(W)$ can be written uniquely as $\sigma = t(f_1)d'(\beta)t(f_2)$: we define

$$\mathbf{r}(\sigma) = \mathbf{t}(f_1)\mathbf{d}'(\beta)\mathbf{t}(f_2) \tag{6.7}$$

that is equivalent to write $\mathbf{r}(\sigma) = (\sigma, \mathbf{r}_0(\sigma))$.

Let $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\sigma' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$ and $\sigma'' = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix}$ be in $\Omega(W)$ such that $\sigma\sigma' = \sigma''$. By Theorem 6.3.6 we have

$$\mathbf{r}(\sigma)\mathbf{r}(\sigma') = \gamma(f_0)\mathbf{r}(\sigma'') \tag{6.8}$$

where f_0 is the non-degenerate quadratic form on X associated to the symmetric isomorphism $-\beta^{-1}\beta''\beta'^{-1}$.

6.4 Fundamental properties of the Weil factor

In this section we find the possible values of $\gamma(f)$ for every non-degenerate quadratic form f over F. Proposition 6.3.3 gives a summation formula for $\gamma(f)$ and we use it to prove that $\gamma(n) = -1$ where n is the reduced norm of the quaternion division algebra over F. In Theorem 6.4.7 we see that γ is a R-character of the Witt group of F. Moreover we already know by Proposition 6.3.2 that $\gamma(f)^2 = 1$ if F contains a square root of -1 and at the end of this section this is generalized by saying that, for any F, $\gamma(f)$ is a fourth root of unity in R.

For every positive integer m, we denote by q_m the non-degenerate quadratic form $q_m(x) = \sum_{i=1}^{m} x_i^2$ defined on the m-dimensional vector space F^m.

6.4.1 The quaternion division algebra over F

In this paragraph we use some results on quaternion algebras over F ([60]) to prove that if $\mathrm{char}(R) \neq 2$ the map $\gamma : Q^{nd}(X) \longrightarrow R^{\times}$ is non-trivial by means of a concrete example.

By Theorem II.1.1 of [60] we know that there exists a unique quaternion division algebra over F (up to isomorphism) that we denote by A. The reduced norm $n : A \longrightarrow F$ is a non-degenerate quadratic form on the F-vector space underlying A and it induces a surjective group homomorphism $n_{|A^{\times}} : A^{\times} \longrightarrow F^{\times}$. Moreover by Lemma II.1.4 of [60], if v is a discrete valuation of F such that $v(\varpi) = 1$ then $v \circ n$ is a discrete valuation of A; so we can consider the ring of integers $\mathcal{O}_A = \{z \in A \,|\, n(z) \in \mathcal{O}_F\}$ of A and fix a uniformizer ϖ_A of $\mathcal{O}_A$ such that $\varpi_A^2 = \varpi$. The unique prime ideal of $\mathcal{O}_A$ is $\varpi_A \mathcal{O}_A$ and the cardinality of the residue field of A is q^2 where q is the cardinality of the residue field of F. According to Definition 6.1.3, we define the module of $x \in F$ (resp. $z \in A$), denoted by $|x|$ (resp. $|z|_A$), as the module of the multiplication (resp. right multiplication) by x (resp. z). We can easily prove that $|x| = q^{-v(x)}$ and $|z|_A = |n(z)|^2$. We denote by dx and dz the Haar measures on F and A such that $\mathrm{vol}(\mathcal{O}_F, dx) = \mathrm{vol}(\mathcal{O}_A, dz) = 1$.

Theorem 6.4.1. Let A be the quaternion division algebra over F and let $n : A \longrightarrow F$ be the reduced norm of A. Then $\gamma(n) = -1$.

In order to prove this theorem, we start by calculating $|\rho_n|$, where $\rho_n \in \mathrm{Iso}(A, A^*)$ is the symmetric isomorphism associated to the quadratic form n, and then we prove that $\gamma(n)$ does not depend on the choice of the non-trivial character χ.

Lemma 6.4.2. If l is the conductor of χ, then $|\rho_n| = q^{4l-2}$.

Proof. By Definition 6.1.3 with $\Phi = \mathbb{1}_{(\mathcal{O}_A)_*}$, we have $|\rho_n| = \mathrm{vol}(\rho^{-1}((\mathcal{O}_A)_*), dz)^{-1}$. Moreover $\rho_n(z_1)(z_2) = tr(z_1 \bar{z}_2)$ for every $z_1, z_2 \in A$, where $z \mapsto \bar{z}$ is the conjugation of A (cf. page 1 of [60]). Then we have the following equivalences:

$$z \in \rho^{-1}((\mathcal{O}_A)_*) \Longleftrightarrow \langle z, \rho(\mathcal{O}_A) \rangle = 1 \Longleftrightarrow tr(z\mathcal{O}_A) \subset \ker(\chi).$$

We know that $\{z \in A \,|\, tr(z\mathcal{O}_A) \subset \mathcal{O}_F\}$ is a fractional ideal (its inverse is called codifferent ideal), and by Corollary II.1.7 of [60] it is exactly $\varpi_A^{-1}\mathcal{O}_A$. Then $z \in \rho^{-1}((\mathcal{O}_A)_*)$ if and only if $z \in \varpi^l \varpi_A^{-1} \mathcal{O}_A = \varpi_A^{2l-1}\mathcal{O}_A$. Hence $|\rho_n| = q^{4l-2}$. □

Lemma 6.4.3. Let A and n as in Theorem 6.4.1. Then $\gamma(n)$ does not depend on the choice of the non-trivial smooth R-character χ of F.

Proof. We know that every non-trivial smooth R-character of F is of the form $\chi_a : x \mapsto \chi(ax)$ with $a \in F^\times$; in particular the conductor of χ_a is $l - v(a)$ where l is the conductor of χ. Moreover, by (6.5) and Lemma 6.4.2, we have

$$\gamma(n) = |\rho_n|^{\frac{1}{2}} \int_{\varpi_A^{-\lambda}\mathcal{O}_A} \chi(n(z))dz = q^{2l-1} \int_{\varpi_A^{-\lambda}\mathcal{O}_A} \chi(n(z))dz$$

for λ large enough. Now we fix $a \in F^\times$ and we denote by $\gamma_a(n)$ the value of $\gamma(n)$ obtained replacing χ by χ_a. Since n is surjective there exists $z_a \in \varpi^{v(a)}\mathcal{O}_A$ such that $n(z_a) = a$. If we take $\lambda' = \lambda + v(a)$ we obtain

$$\begin{aligned}\gamma_a(n) &= q^{2(l-v(a))-1} \int_{\varpi_A^{-\lambda'}\mathcal{O}_A} \chi_a(n(z))dz = q^{2(l-v(a))-1} \int_{\varpi_A^{-\lambda'}\mathcal{O}_A} \chi(n(z_a z))dz \\ &= q^{2(l-v(a))-1}|z_a|^{-1} \int_{\varpi_A^{-\lambda'+v(a)}\mathcal{O}_A} \chi(n(z))dz = q^{2l-1} \int_{\varpi_A^{-\lambda}\mathcal{O}_A} \chi(n(z))dz = \gamma(n)\end{aligned}$$

which concludes the proof. □

Proof of Theorem 6.4.1. For every $k \geq 0$, we fix a set of representatives $\Xi_{A,k}$ of the classes of $\mathcal{O}_A$ modulo $\varpi_A^k \mathcal{O}_A$. We denote by $\Xi_{A,k}^\times \subset \Xi_{A,k}$ the set of representatives of $\left(\mathcal{O}_A/\varpi_A^k \mathcal{O}_A\right)^\times$ and by $\Xi_{F,k}^\times \subset \Xi_{F,k} \subset n(\Xi_{A,k})$ two sets of representatives of $\left(\mathcal{O}_F/\varpi^k \mathcal{O}_F\right)^\times$ and $\mathcal{O}_F/\varpi^k \mathcal{O}_F$.

By Lemma 6.4.3 we can suppose that the conductor of χ is 1, so that χ is trivial on $\varpi_A \mathcal{O}_A$ but not on $\mathcal{O}_A$. Then, for λ large enough, we have

$$\begin{aligned}\gamma(n) = q \int_{\varpi_A^{-\lambda}\mathcal{O}_A} \chi(n(z))dz &= q^{1+2\lambda} \int_{\mathcal{O}_A} \chi\left(\varpi^{-\lambda} n(z)\right) dz \\ &= q^{1+2\lambda} \sum_{z' \in \Xi_{A,\lambda+1}} \int_{\varpi_A^{\lambda+1}\mathcal{O}_A} \chi\left(\varpi^{-\lambda} n(z'+z)\right) dz \\ &= q^{1+2\lambda} \mathrm{vol}(\varpi_A^{\lambda+1}\mathcal{O}_A, dz) \sum_{z' \in \Xi_{A,\lambda+1}} \chi\left(\varpi^{-\lambda} n(z')\right) \\ &= q^{-1} \sum_{z' \in \Xi_{A,\lambda+1}} \chi\left(\varpi^{-\lambda} n(z')\right).\end{aligned}$$

Since $n : \left(\mathcal{O}_A/\varpi_A^{k+1}\mathcal{O}_A\right)^\times \longrightarrow \left(\mathcal{O}_F/\varpi^{k+1}\mathcal{O}_F\right)^\times$ is surjective and its kernel has cardinality $q^k(q+1)$, we have

$$\sum_{z \in \Xi_{A,k+1}^\times} \chi(\varpi^{-k} n(z)) = q^k(q+1) \sum_{x \in \Xi_{F,k+1}^\times} \chi(\varpi^{-k} x) = -q^k(q+1) \sum_{x \in \Xi_{F,k}} \chi(\varpi^{-k+1} x)$$

that is 0 if $k > 0$ and $-(q+1)$ if $k = 0$. Notice that in the last equality we used that the sum of the values of a non-trivial character over all elements of a finite group is 0.
Then we have that $\sum_{z' \in \Xi_{A,k+1}} \chi\left(\varpi^{-k} n(z')\right) = \sum_{z' \in \Xi_{A,k}} \chi\left(\varpi^{-k+1} n(z')\right)$ for every $k > 0$ and we obtain

$$\gamma(n) = q^{-1} \sum_{z' \in \Xi_{A,1}} \chi(n(z')) = q^{-1}\Big(1 + \sum_{z' \in \Xi_{A,1}^\times} \chi(n(z'))\Big) = q^{-1}(1 - (q+1)) = -1. \qquad \square$$

Remark 6.4.4. The Theorem 6.4.1 corresponds to Proposition 4 of [62]. Weil proves it showing that $\gamma(n)$ is a negative real number of absolute value 1 and hence his proof does not suit in our presentation. Our proof works for every integral domain R verifying our hypotheses but requires F to be non-Archimedean.

6.4.2 The Witt group

In this paragraph we introduce the definition of Witt group of F and we prove that γ defines a R-character of this group.

Let G_1, G_2 be two finite dimensional vector spaces over F and f_1, f_2 be two non-degenerate quadratic forms on G_1 and G_2. We define $f_1 \oplus f_2 \in Q^{nd}(G_1 \times G_2)$ by $(f_1 \oplus f_2)(x_1 \oplus x_2) = f_1(x_1) + f_2(x_2)$ for every $x_1 \in G_1$ and $x_2 \in G_2$.

Remark 6.4.5. If $\rho_1 : G_1 \to G_1^*$ and $\rho_2 : G_2 \to G_2^*$ are the symmetric isomorphisms associated to f_1 and f_2, then $\rho_1 \oplus \rho_2 : G_1 \times G_2 \to (G_1 \times G_2)^*$, defined by $(\rho_1 \oplus \rho_2)(y_1 \oplus y_2) = \rho_1(y_1) \oplus \rho_2(y_2)$ is the symmetric isomorphism associated to $f_1 \oplus f_2$. Indeed, calling this latter $\rho_{1,2}$, we have

$$[x_1 \oplus x_2, (\rho_1 \oplus \rho_2)(y_1 \oplus y_2)] = f_1(x_1 + y_1) - f_1(x_1) - f_1(y_1) + f_2(x_2 + y_2) - f_2(x_2) - f_2(y_2) =$$

$$= (f_1 \oplus f_2)(x_1 \oplus x_2 + y_1 \oplus y_2) - (f_1 \oplus f_2)(x_1 \oplus x_2) - (f_1 \oplus f_2)(y_1 \oplus y_2) = [x_1 \oplus x_2, \rho_{1,2}(y_1 \oplus y_2)].$$

Definition 6.4.6. *We say that $f_1 \in Q^{nd}(G_1)$ and $f_2 \in Q^{nd}(G_2)$ are **equivalent** (and we write $f_1 \sim f_2$) if one can be obtained from the other by adding an hyperbolic quadratic form of dimension* $\max\{\dim(G_1), \dim(G_2)\} - \min\{\dim(G_1), \dim(G_2)\}$ *(see [40]). We call **Witt group** of F the set of equivalence classes of non-degenerate quadratic forms over F endowed with the operation induced by $(f_1, f_2) \longmapsto f_1 \oplus f_2$.*

Theorem 6.4.7. The map $f \mapsto \gamma(f)$ is a R-character of the Witt group of F.

Proof. Let G_1 and G_2 be two finitely dimensional vector spaces over F, $f_1 \in Q^{nd}(G_1)$ and $f_2 \in Q^{nd}(G_2)$. Proposition 6.3.3 gives

$$\gamma(f_1 \oplus f_2) = |\rho_1 \oplus \rho_2|^{\frac{1}{2}} \int_{K_1 \times K_2} \chi((f_1 \oplus f_2)(x_1 \oplus x_2)) dx_1 dx_2$$

for compact open subgroups K_1 and K_2 of G_1 and G_2, both large enough. Now, if we consider $\mathbb{1}_{K_{1,*}} \in \mathcal{S}(G_1^*)$, $\mathbb{1}_{K_{2,*}} \in \mathcal{S}(G_2^*)$ and $\mathbb{1}_{K_{1,*} \times K_{2,*}} \in \mathcal{S}(G_1^* \times G_2^*)$, Definition 6.1.3 gives

$$|\rho_1||\rho_2| \int_{G_1} \mathbb{1}_{K_{1,*}}(\rho_1(x_1)) dx_1 \int_{G_2} \mathbb{1}_{K_{2,*}}(\rho_2(x_2)) dx_2 = \int_{G_1^*} \mathbb{1}_{K_{1,*}}(x_1^*) dx_1^* \int_{G_2^*} (x_2^*) \mathbb{1}_{K_{2,*}} dx_2^* =$$

$$= \int_{G_1^* \times G_2^*} \mathbb{1}_{K_{1,*} \times K_{2,*}}(x_1^* \oplus x_2^*) dx_1^* dx_2^* = |\rho_1 \oplus \rho_2| \int_{G_1 \times G_2} \mathbb{1}_{K_{1,*} \times K_{2,*}}(\rho_1(x_1) \oplus \rho_2(x_2)) dx_1 dx_2$$

and then $|\rho_1||\rho_2| = |\rho_1 \oplus \rho_2|$. Hence we obtain

$$\gamma(f_1 \oplus f_2) = |\rho_1|^{\frac{1}{2}} |\rho_2|^{\frac{1}{2}} \int_{K_1} \chi(f_1(x_1)) dx_1 \int_{K_2} \chi(f_2(x_2)) dx_2 = \gamma(f_1)\gamma(f_2).$$

We shall now to check that γ is equivariant on the equivalence classes of bilinear forms. To see that, recall that $f_1 \sim f_2$ if and only if there exist $n \in \mathbb{N}$ and an hyperbolic quadratic form $h(\mathbf{x}) = \sum x_i x_{i+n}$ of rank $2n$ such that $f_1 = f_2 \oplus h$. After what proven in the first part $\gamma(f_1) = \gamma(f_2)$ if and only if $\gamma(h) = 1$ and since every hyperbolic form is a sum of the rank 2 form $h_2 : (x_1, x_2) \mapsto x_1 x_2$ it's sufficient to show that $\gamma(h_2) = 1$. Now, if we apply the base change $x_1 \mapsto x_1 + x_2$ and $x_2 \mapsto x_1 - x_2$ we obtain $h_2(x_1 + x_2, x_1 - x_2) = (x_1 + x_2)(x_1 - x_2) = x_1^2 - x_2^2$ and Proposition 6.3.2 gives that $\gamma(h_2) = \gamma(q_1 \oplus (-q_1)) = \gamma(q_1)\gamma(q_1)^{-1} = 1$. □

6.4.3 The image of the Weil factor

We exploit some classical results on quadratic forms over F to prove that γ takes values in the group of fourth roots of unity in R.

Definition 6.4.8. *Let G_1, G_2 be two finite dimensional vector spaces over F and f_1, f_2 be two non-degenerate quadratic forms on G_1 and G_2. We say that f_1 and f_2 are* ***isometric*** *if there exists an isomorphism $\vartheta : G_1 \longrightarrow G_2$ such that $f_1(x) = f_2(\vartheta(x))$ for every $x \in G_1$.*

Notice that, by Remark 6.3.4, if f_1 and f_2 are isometric then $\gamma(f_1) = \gamma(f_2)$. We know also that there are only two isometry classes of non-degenerate quadratic forms on a 4-dimensional vector space over F whose discriminant is a square in $F^\times$. One class is represented by the norm n over the quaternion division algebra and the other by $q_2 \oplus -q_2$. Moreover, if $a, b \in F^\times$ and (a, b) is the Hilbert symbol with values in $R^\times$, the quadratic form $x_1^2 - ax_2^2 - bx_3^2 + abx_4^2$ lies in the first class if $(a, b) = -1$ and in the second one if $(a, b) = 1$. Furthermore by Theorems 6.4.7 and 6.4.1 we have that

$$\gamma(x_1^2 - ax_2^2 - bx_3^2 + abx_4^2) = (a, b)\,. \tag{6.1}$$

In particular, for $b = -1$ we apply Theorem 6.4.7 to this formula to get the equalities

$$\gamma(q_1)^2\gamma(-aq_1)^2 = (a, -1) \qquad \text{and} \qquad \gamma(aq_1)^2 = (a, -1)\,\gamma(q_1)^2$$

by Proposition 6.3.2. Since every non-degenerate quadratic form is isometric to $\sum_{i=1}^m a_i x_i^2$ for suitable $m \in \mathbb{N}$ and $a_i \in F^\times$, we have

$$\gamma(f)^2 = \prod_{i=1}^m (a_i, -1)\,\gamma(q_1)^2 = (D(f), -1)\,\gamma(q_1)^{2m} \tag{6.2}$$

where $D(f)$ is the discriminant of f. Notice that, since F is non-archimedean, then -1 is either a square or a norm in $F(\sqrt{-1})$. Therefore $\gamma(q_4) = (-1, -1) = 1$ and it follows that $\gamma(f)^4 = 1$ for every non-degenerate quadratic form f over F as announced.
This is in fact the best possible result whenever -1 is not a square in F. Indeed, in this case, there exists at least an element $a \in F^\times$ such that $(a, -1) = -1$. For such an a, formula (6.1) gives $\gamma(q_1 \oplus -aq_1)^2 = -1$ and then a square root of -1 shall be in the image of γ.

Remark 6.4.9. This result shows also that, whenever -1 is not a square in F and $\mathrm{char}(R) \neq 2$ (in which case $X^4 - 1$ is a separable polynomial) then R contains a primitive fourth root of unity. This fact has an elementary explaination: denote ζ_p an element of order p in $R^\times$ and consider the Gauss sum $\tau = \sum_{i=1}^{p-1} \left(\frac{i}{p}\right) \zeta_p^i \in R$, where $\left(\frac{i}{p}\right)$ is the Legendre symbol. The formula

$$\tau^2 = \left(\frac{-1}{p}\right) p$$

holds thanks to a classical argument that can be found, for example, in 3.3 of [35]. The fact that -1 is not a square in F implies that $\left(\frac{-1}{p}\right) = -1$ and that $q = p^f$ with f odd. Since R contains a square root of q, then there exists an element $x \in R^\times$ such that $x^2 = p$ and $(\tau \cdot \frac{1}{x})^2 = -1$: there is a primitive fourth root of unity in R.

6.5 The reduced metaplectic group

The metaplectic group, associated with R and χ, is an extension of $\mathrm{Sp}(W)$ by $R^\times$ through the short exact sequence (6.6). We want to understand when this sequence does (or does not) split, looking for positive numbers $n \in \mathbb{N}$ yielding the existence of subgroups $\mathrm{Mp}_n(W)$ of $\mathrm{Mp}(W)$ such that $\pi_{|\mathrm{Mp}_n(W)}$ is a finite cyclic cover of $\mathrm{Sp}(W)$ with kernel $\mu_n(R)$. We show that, for F locally compact non-discrete non-archimedean field, it is possible to construct $\mathrm{Mp}_2(W)$. Then we prove that, when $\mathrm{char}(R) \neq 2$, $n = 1$ does not satisfy the condition above, namely that the sequence (6.6) does not split. Finally we show what happens in the simpler case when $\mathrm{char}(R) = 2$.

For a closer perspective we suppose that, for some $n \in \mathbb{N}$, $\mathrm{Mp}_n(W)$ exists and we look at the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \mu_n(R) \arrow[r] \arrow[d] & \mathrm{Mp}_n(W) \arrow[r] \arrow[d] & \mathrm{Sp}(W) \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & R^\times \arrow[r] \arrow[d, ".^n"] & \mathrm{Mp}(W) \arrow[r, "\pi"] \arrow[d, "\psi_n"] & \mathrm{Sp}(W) \arrow[r] & 1 \\
 & R^\times \arrow[r, "id"] & R^\times & &
\end{tikzcd}$$

where $\mu_n(R)$ is the group of n-th roots of unity in R. The existence of a homomorphism $\psi_n : \mathrm{Mp}(W) \longrightarrow R^\times$ such that its restriction on $R^\times$ is the n-th power map implies the existence of the first line in the diagram. Indeed, if such ψ_n exists, let $\mathrm{Mp}_n(W)$ be its kernel; then π induces a surjective homomorphism from $\mathrm{Mp}_n(W)$ to $\mathrm{Sp}(W)$ whose kernel is $\mathrm{Mp}_n(W) \cap R^\times = \mu_n(R)$. Then, as in 43 of [62], the question to address is whether or not there exists $\psi_n : \mathrm{Mp}(W) \to R^\times$ such that $\psi_{n|_{R^\times}}(x) = x^n$ for every $x \in R^\times$.

Lemma 6.5.1. A R-character $\psi_n : \mathrm{Mp}(W) \longrightarrow R^\times$ whose restriction on $R^\times$ is the n-th power map is completely determined by $\widetilde{\psi_n} = \psi_n \circ \mathbf{r} : \Omega(W) \longrightarrow R^\times$ where $\mathbf{r}$ is as in (6.7).

Proof. Let $(\sigma, \mathbf{s}) \in \mathrm{Mp}(W)$. By Proposition 6.1.7 we can write σ as a product $\sigma = \prod_i \sigma_i$ with $\sigma_i \in \Omega(W)$. We set $(\sigma, \mathbf{s}') = \prod_i \mathbf{r}(\sigma_i)$ where $\mathbf{r}$ is as in (6.7). Then, since $\ker(\pi) = R^\times$, we have that $(\sigma, \mathbf{s}) = c(\sigma, \mathbf{s}')$ for a suitable $c \in R^\times$. This implies that the values of ψ_n at $(\sigma, \mathbf{s})$ is $\psi_n(c(\sigma, \mathbf{s}')) = c^n \prod_i \widetilde{\psi_n}(\sigma_i)$. $\square$

By (6.8), the morphism $\widetilde{\psi_n}$ of Lemma 6.5.1 shall verify the condition

$$\widetilde{\psi_n}(\sigma)\widetilde{\psi_n}(\sigma') = \gamma(f_0)^n \widetilde{\psi_n}(\sigma'') \tag{6.1}$$

for every $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\sigma' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$ and $\sigma'' = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix}$ in $\Omega(W)$ satisfying $\sigma'' = \sigma\sigma'$, where f_0 is a non-degenerate quadratic form on X associated to the symmetric isomorphism $-\beta^{-1}\beta''\beta'^{-1}$. Conversely we have:

Lemma 6.5.2. If $\widetilde{\psi_n} : \Omega(W) \longrightarrow R^\times$ satisfies (6.1), then there exists a unique R-character ψ_n of $\mathrm{Mp}(W)$ such that its restriction to $R^\times$ is the n-th power map and $\psi_n \circ \mathbf{r} = \widetilde{\psi_n}$.

Proof. Let $(\sigma, \mathbf{s}) \in \mathrm{Mp}(W)$. By Proposition 6.1.7 we can write σ as a product $\sigma = \prod_i \sigma_i$ with $\sigma_i \in \Omega(W)$ and $(\sigma, \mathbf{s}) = c \prod \mathbf{r}(\sigma_i)$ for a suitable $c \in R^\times$. We define $\psi_n(\sigma, \mathbf{s}) = c^n \prod_i \widetilde{\psi_n}(\sigma_i)$. We have to prove that it is well defined. Let $\sigma = \prod_j \sigma_j$ be another presentation of σ that differs from $\prod_i \sigma_i$ by a single relation $\sigma\sigma' = \sigma''$; by (6.8) we obtain

$$(\sigma, \mathbf{s}) = c \prod_i \mathbf{r}(\sigma_i) = \gamma(f_0) c \prod_j \mathbf{r}(\sigma_j)$$

for a suitable $f_0 \in Q^{nd}(X)$ and by (6.1) we have

$$\psi_n(\sigma, \mathbf{s}) = c^n \prod_i \widetilde{\psi_n}(\sigma_i) = c^n \gamma(f_0)^n \prod_j \widetilde{\psi_n}(\sigma_j) = (c\,\gamma(f_0))^n \prod_j \widetilde{\psi_n}(\sigma_j).$$

Now, since every presentation $\sigma = \prod_k \sigma_k$ with $\sigma_k \in \Omega(W)$ differs from $\prod_i \sigma_i$ by a finite number of relations $\sigma\sigma' = \sigma''$, the definition $\psi_n(\sigma, \mathbf{s}) = c^n \prod_i \widetilde{\psi_n}(\sigma_i)$ makes sense. □

After these results the existence of a character ψ_n, and then of a subgroup $\mathrm{Mp}_n(W)$ of $\mathrm{Mp}(W)$ as above, is equivalent to the existence of $\widetilde{\psi_n} : \Omega(W) \longrightarrow R^\times$ that satisfies (6.1).

First of all we suppose that -1 is a square in F. By Proposition 6.3.2 we have $\gamma(f)^2 = 1$ for every $f \in Q^{nd}(X)$ and so $\widetilde{\psi_2} = 1$ satisfies (6.1) with $n = 2$.

We suppose now that -1 is not a square in F. We fix a basis over the F-vector space X and its dual basis over X^*. By definition of $\Omega(W)$ we have that the determinant $\det(\beta)$ of β with respect to these basis is not zero for every $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Omega(W)$. Moreover, since f_0 is associated to the symmetric isomorphism $-\beta^{-1}\beta''\beta'^{-1}$ we have that the discriminant of f_0 is $D(f_0) = \det(-\beta)^{-1} \cdot \det(-\beta'') \cdot \det(-\beta')^{-1}$. Hence taking $\widetilde{\psi_2}(\sigma) = (\det(-\beta), -1)\,\gamma(q_1)^{2m}$ for every $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Omega(W)$ and using formula (6.2) we obtain the equality (6.1) with $n = 2$.
We have then proved the

Theorem 6.5.3. There exists a subgroup $\mathrm{Mp}_2(W)$ of $\mathrm{Mp}(W)$ that is a cover of $\mathrm{Sp}(W)$ with kernel $\mu_2(R)$. In particular, when $\mathrm{char}(R) \neq 2$, $\mathrm{Mp}_2(W)$ is a 2-cover of $\mathrm{Sp}(W)$.

Now we want to see if this reduction is optimal in the sense that there does not exist any $\mathrm{Mp}_1(W)$ fitting into the diagram. If this is the case, then the group $\mathrm{Mp}_2(W)$ is the minimal subgroup of $\mathrm{Mp}(W)$ which is a central extension of $\mathrm{Sp}(W)$ and therefore is called **reduced metaplectic group**.

Theorem 6.5.4. Let $\mathrm{char}(R) \neq 2$. Then there does not exist a character $\psi : Mp(W) \to R^\times$ such that $\psi_{|R^\times} = id$.

Proof. Let suppose the existence of such ψ. Then there exists a character $\psi' : \mathrm{Mp}(F \times F^*) \to R^\times$ such that $\psi'_{|R^\times} = id$. In fact the extension by triviality

$$\begin{array}{ccc} \iota : \Omega(F \times F^*) & \to & \Omega(W) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & \begin{pmatrix} a & 0 & b & 0 \\ 0 & \mathbb{1}_{n-1} & 0 & \mathbb{1}_{n-1} \\ c & 0 & d & 0 \\ 0 & \mathbb{1}_{n-1} & 0 & \mathbb{1}_{n-1} \end{pmatrix} \end{array}$$

is such that $\sigma'' = \sigma\sigma'$ yields $\iota(\sigma'') = \iota(\sigma)\iota(\sigma')$. Then $\widetilde{\psi}' := \widetilde{\psi} \circ \iota$ satisfies the relation

$$\widetilde{\psi}'(\sigma'') = \gamma(f_0)^{-1}\widetilde{\psi}'(\sigma)\widetilde{\psi}'(\sigma')$$

and Lemma 6.5.2 implies the existence of ψ'. Clearly ψ' takes values 1 on the group of commutators of $\mathrm{Mp}(F \times F^*)$. By (6.6) we have

$$\mathbf{t}\left(\frac{c}{1-a^2}x^2\right)\mathbf{d}\left(a^{-1}\right)\mathbf{t}\left(-\frac{c}{1-a^2}x^2\right)\mathbf{d}\left(a\right) = \mathbf{t}\left(\frac{c}{1-a^2}x^2\right)\mathbf{t}\left(-\frac{ca^2}{1-a^2}x^2\right) = \mathbf{t}\left(cx^2\right)$$

for every $a \notin \{0, 1, -1\}$ in F and every $c \in F$. Then for every quadratic form f on F, $\mathbf{t}(f)$ is a commutator of $\mathrm{Mp}(F \times F^*)$ and so $\psi'(\mathbf{t}(f)) = 1$. By Definition 6.3.1 we obtain the equality

$$\mathbf{d}'(\rho^{-1})\mathbf{t}(f)\mathbf{d}'(-\rho^{-1})\mathbf{t}(f) = \gamma(f)\mathbf{t}(-f)\mathbf{d}'(\rho^{-1})$$

in $\mathrm{Mp}(F \times F^*)$ for every $f \in Q^{nd}(F)$ associated to ρ and applying ψ' we obtain $\gamma(f) = \psi'(\mathbf{d}'(\rho^{-1}))$. So, if we denote by ρ_a the symmetric isomorphism associated to $aq_1 : x \longrightarrow ax^2$ we obtain

$$\gamma\left(aq_1\right) = \psi'(\mathbf{d}'(\rho_a^{-1})) = \psi'(\mathbf{d}(2a))\psi'(\mathbf{d}'(\rho_1^{-1})).$$

Now, since every quadratic form f over F is of the form $f(x) = \sum_{i=1}^m a_i x_i^2$, we can conclude that $\gamma(f) = \prod_{i=1}^m \psi'(\mathbf{d}(2a_i))\psi'(\mathbf{d}'(\rho_1^{-1}))^m$ depends only on m and on the discriminant. But this implies that γ takes the same value on every non-degenerate quadratic form on a 4-dimensional vector space over F with discriminant equal to 1. But this contradicts Theorem 6.4.1. □

We shall remark that, if R has characteristic 2, then necessarily $\gamma(f) = 1$ for every quadratic form f. Then Theorem 6.5.4 is clearly false and the sequence (6.6) splits yielding the existence of $\mathrm{Mp}_1(W) \cong \mathrm{Sp}(W)$.

Relationship with Steinberg theory

The theory of universal central extensions, as exposed for example in [56], permits us to show additional features of the reduced metaplectic group. Since $\mathrm{Sp}(W)$ is a perfect group, it has an universal central extension $\vartheta : \mathcal{U} \to \mathrm{Sp}(W)$ and since the reduced metaplectic group $\mathrm{Mp}_2(W)$ is a central extension of it, there exists a unique map $\varphi : \mathcal{U} \to \mathrm{Mp}_2(W)$ such that $\pi \circ \varphi = \vartheta$.

Moreover, by universal property of $\mathcal{U}$, the image of φ is a central extension of $\mathrm{Sp}(W)$ contained in $\mathrm{Mp}_2(W)$ and in fact φ is surjective by Theorem 6.5.4. Then $\mathrm{Mp}_2(W)$ is contained in every subgroup of $\mathrm{Mp}(W)$ which is a central extension of $\mathrm{Sp}(W)$ and in particular it is contained in all its conjugates: $\mathrm{Mp}_2(W)$ is a normal subgroup of $\mathrm{Mp}(W)$. Moreover the unique map $\varphi' : \mathcal{U} \to \mathrm{Mp}(W)$ given by the universal property factorizes necessarily through φ.
In [56], Steinberg describes the structure of the universal central extension of any Chevalley group by means of generators and relations and he also introduces the so-called Steinberg symbol, which characterizes the kernel of this extension. It has already been noticed (see chapter II of [43]) that the Hilbert symbol enjoys the same properties as the Steinberg symbol. We can actually describe this relationship in our case by studying the behavior of φ on generators of $\mathcal{U}$. As an example, let us fix an identification of F with F^* and make this correspondence explicit in the case of $\mathrm{SL}(2, F)$. Let Λ be the free group generated by the set $\{x(u), y(u) : u \in F\}$. Define $w(u) = x(u)y(-u^{-1})x(u)$ and $h(u) = w(u)w(-1)$. Then we have the following (cfr. section 6 of [56]):

Theorem 6.5.5. Consider the following relations on Λ:

A. $x(u_1 + u_2) = x(u_1)x(u_2)$ and $y(u_1 + u_2) = y(u_1)y(u_2)$;

B. $w(u)x(v)w(-u) = y(-u^{-2}v)$;

C. $h(u_1u_2) = h(u_1)h(u_2)$.

Then A et B are a complete set of relations for the universal central extension $\mathcal{U} \to \mathrm{SL}(2, F)$ and adding C, we obtain a complete set of relations for $\mathrm{SL}(2, F)$.
Moreover if $\pi' : \Lambda/(A, B) \to \Lambda/(A, B, C)$ is the canonical projection, then every element of the form $h(u_1)h(u_2)h(u_1u_2)^{-1} \in \ker \pi'$ coincides with the Steinberg symbol associated to u_1 and u_2.

We remark that condition B implies $x(u) = w(1)^{-1}y(-u)w(1)$ and we can check that the map $\phi : \Lambda/(A, B, C) \to \mathrm{SL}(2, F)$ given by $y(u) \mapsto t(uq_1)$ and $w(1) \mapsto d'(-\frac{1}{2})$ is an isomorphism such that $\phi(x(u)) = d'(\frac{1}{2})t(-uq_1)d'(-\frac{1}{2})$, $\phi(w(u)) = d'(-\frac{u}{2})$ and $\phi(h(u)) = d(u)$. Theorem 6.5.5 assures the existence of a unique map $\varphi : \Lambda/(A, B) \to \mathrm{Mp}(2, F)$ making the following diagram commute

$$\begin{tikzcd}
1 \arrow[r] & \ker \pi' \arrow[r] \arrow[d] & \Lambda/(A, B) \arrow[r, "\pi'"] \arrow[d, "\varphi"] & \Lambda/(A, B, C) \arrow[r] \arrow[d, "\phi"] & 1 \\
1 \arrow[r] & R^\times \arrow[r] & \mathrm{Mp}(2, F) \arrow[r, "\pi"] & \mathrm{SL}(2, F) \arrow[r] & 1.
\end{tikzcd}$$

Let us prove that the image of the Steinberg symbol by φ in $R^\times$ is the Hilbert symbol.
We know that $\mathbf{t}$ and $\mathbf{d}'$ are liftings of t and d' to $\mathrm{Mp}(2, F)$. Then $\varphi(y(u)) = c_1(u)\mathbf{t}(uq_1)$ and $\varphi(w(1)) = c_2\mathbf{d}'(-\frac{1}{2})$ for $c_1(u)$ and c_2 suitable elements in $R^\times$. This gives $\varphi(x(u)) = c_1(-u)\mathbf{d}'(\frac{1}{2})\mathbf{t}(-uq_1)\mathbf{d}'(-\frac{1}{2})$. Now, by relation A and B of Theorem 6.5.5 we have that $c_1(u_1 + u_2) = c_1(u_1)c_1(u_2)$ and $c_1(u_1u_2^2) = c_1(u_1)$ for every $u_1, u_2 \in F$ and then $c_1(u) = 1$ for every $u \in F$. Using relations in section 6.3.2 and the definition of the Weil factor we obtain $\varphi(w(u)) =$

$\gamma(-uq_1)\mathbf{d}'(-\frac{u}{2})$ and then $\varphi(h(u)) = \gamma(q_1 \oplus -uq_1)\mathbf{d}(u)$. So we can calculate the image of the Steinberg symbol: $\varphi(h(u_1)h(u_2)h(u_1u_2)^{-1}) = \gamma(q_1 \oplus -u_1q_1 \oplus -u_2q_1 \oplus u_1u_2q_1) = (u_1, u_2)$ by formula (6.1). This gives another proof of the fact that the Hilbert symbol satisfies all the relations of the Steinberg symbol.

Notice that we have shown in this way that the images of $\mathbf{d}, \mathbf{d}'$ and $\mathbf{t}$ lie in $\mathrm{Mp}_2(2, F)$.

Further directions

We conclude by saying that we can restrict the representation of the metaplectic group given by (6.7) to a representation of the reduced metaplectic group. This is the **Weil representation** defined over R. As pointed out in the introduction, the relevance of having an explicit form for this representation lies in the fact that its understanding has important applications. Considering R in whole generality may help understand more deeply the essential features underlying results like Howe and Shimura correspondences. Let us mention also a more concrete question. Given a morphism of rings $R_1 \to R_2$ and fixed two smooth non-trivial characters $\chi_1 : F \to R_1$ and $\chi_2 : F \to R_2$, it would be interesting to study the relationships between metaplectic groups and the Weil representation respectively over R_1 and R_2.

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