Around rationality of algebraic cycles

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Around Rationality of Algebraic Cycles

by

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Abstract

Let F be a field and X, Y some F-varieties. In this dissertation, we are interested in knowing if the class $y \in \mathrm{CH}(Y_{F(X)})$ of an algebraic cycle defined over the function field $F(X)$ is actually defined over the base field, i.e belongs to the image of the pull-back homomorphism $\mathrm{CH}(Y) \to \mathrm{CH}(Y_{F(X)})$. We study this issue in different contexts, the variety X varying among classes of varieties such as quadrics or projective homogeneous varieties.

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This Phd dissertation is dedicated to my father and my mother.

Contents

Chapter I

Introduction

Let Y be a variety over a field F, let X be a geometrically integral variety over F and let us denote its function field as $F(X)$. For any integer $m \geq 0$, consider the following commutative diagram given by change of field homomorphisms for Chow groups CH^m of codimension m classes of algebraic cycles

$$\begin{tikzcd} \mathrm{CH}^m(Y) \arrow[r] \arrow[d] & \mathrm{CH}^m(Y_{F(X)}) \arrow[d] \\ \mathrm{CH}^m(\overline{Y}) \arrow[r, hook] & \mathrm{CH}^m(Y_{\overline{F}(X)}) \end{tikzcd},$$

where we write $\overline{Y} := Y_{\overline{F}}$ with $\overline{F}$ an algebraic closure of F.

An element $\overline{y}$ of $\mathrm{CH}(\overline{Y})$ is $F(X)$-*rational* if its image $\overline{y}_{\overline{F}(X)}$ under $\mathrm{CH}(\overline{Y}) \to \mathrm{CH}(Y_{\overline{F}(X)})$ is in the image of $\mathrm{CH}(Y_{F(X)}) \to \mathrm{CH}(Y_{\overline{F}(X)})$. An element $\overline{y}$ of $\mathrm{CH}(\overline{Y})$ is simply called *rational* if it is in the image of $\mathrm{CH}(Y) \to \mathrm{CH}(\overline{Y})$, denoted by $\overline{\mathrm{CH}}(Y)$. Note that since $\overline{F}$ is algebraically closed, the bottom homomorphism $\mathrm{CH}(\overline{Y}) \to \mathrm{CH}(Y_{\overline{F}(X)})$ is injective by the specialization arguments.

The general question is the following

Question I.0.1. *When is an $F(X)$-rational element y in $\mathrm{CH}^m(\overline{Y})$ actually rational?*

In the aftermath of the previous question, one has the strongest following one.

Question I.0.2. *When is the change of field homomorphism $\mathrm{CH}^m(Y) \to \mathrm{CH}^m(Y_{F(X)})$ surjective?*

Example I.0.3. The change of field homomorphism $\mathrm{CH}(Y) \to \mathrm{CH}(Y_E)$ associated with a purely transcendental extension E/F is surjective, in any codimension (this follows from the continuity property of Chow groups and the Homotopy Invariance). Therefore, for any rational F-variety X, the homomorphism $\mathrm{CH}(Y) \to \mathrm{CH}(Y_{F(X)})$ is surjective, in any codimension. In particular, this applies to X an isotropic quadric over F, see [7].

For any integer p, one can ask the same questions with the Chow group $\mathrm{Ch} = \mathrm{CH}/p\,\mathrm{CH}$ instead of the integral Chow group CH. The choice of the coefficients depends on what variety X is considered.

These questions originally come from the work of Alexander Vishik on the Kaplansky's Conjecture (see [48, Theorem 5.1]). In this paper, A. Vishik showed that for $r \geq 3$, there exists a field F of u-invariant $2^r + 1$ (the u-invariant of a field F is the maximal dimension of an anisotropic quadratic form over F). To prove this result, he needed the following so-called *Main Tool Lemma* (proved in [49]), where Ch is the Chow group modulo 2 (see Theorem III.2.1 for a more complete statement).

Theorem I.0.4. (Vishik) *Let Q be a smooth projective quadric over a field F and let Y be a smooth quasi-projective F-variety. If* ***char(F)=0*** *then any $F(Q)$-rational element $\overline{y} \in \mathrm{Ch}^i(\overline{Y})$ with $i < dim(Q)/2$ is rational.*

Actually, A. Vishik used the contrapositive statement of the MTL for quadrics to find some $F(Q)$-irrational algebraic cycles.

In his proof, A. Vishik states the problem in terms of Chow groups then lifts elements of the Chow groups to elements of the algebraic cobordism groups, which requires him to work with the assumption $\mathrm{char}(F) = 0$ (since the algebraic cobordism theory relies on resolution of singularities). It is a natural idea to try to find a proof which stays at the level of Chow groups. In [23], Nikita Karpenko extends the above early version of the MTL for quadrics to any field of characteristic different from 2 (modulo 2-torsion). He was able to do so because his method uses *Steenrod operations of cohomological type on Chow groups* instead of symmetric operations in the algebraic cobordism theory.

In Chapter III, we extend the method introduced by N. Karpenko to get a complete version of the MTL on quadrics, modulo 2-torsion (the complete version allows one to consider algebraic cycles with a larger codimension, see Sections III.2 and III.3). Since the proof of [48, Theorem 5.1] only deals with torsion-free Chow groups, our versions modulo 2-torsion element can also be used here.

In another paper ([51]), A. Vishik adressed the question treated in Theorem I.0.3 but for integral Chow groups CH. He got a similar result assuming that the quadric Q has a projective line defined over its function field. Since his method uses symmetric operations, he needed once again the assumption $\mathrm{char}(F) = 0$.

Using a similar method but Steenrod operations in place of symmetric operations, we were able to get statements valid in any characteristic different from 2 (modulo 2-torsion). This is the topic of the second part of Chapter III (Sections III.4 and III.5).

There are also existing versions of the MTL for varieties X different from quadrics. For a smooth variety X with a *special correspondence* in the sense of Markus Rost, Kirill Zainoulline proved in [55, Theorem 1.3] a version of the MTL over a field of characteristic 0. In Chapter VI (Section VI.3), we use his method to get an improved version when X is a quadric with a special correspondence.

For such a smooth variety X with a special correspondence, the result [26, Theorem SC.1] by N. Karpenko and Alexander Merkurjev constitutes another version, which allows one to consider algebraic cycles with a larger codimension. We slightly extend this result at the very end of Chapter VI. In the same article, the authors also provided a version of the MTL for X the *norm variety* of a symbol $s \in H^{n+1}(F, \mu_p^{\otimes n})$ ([26, Theorem 4.3]).

In Chapter IV, we prove a version of the MTL for X a projective homogeneous varieties under a linear algebraic group of type F_4 or E_8 over a field of arbitrary characteristic (see Theorem IV.0.1). The proof notably involves a motivic decomposition of the Chow motive of X with the help of a Rost motive $\mathcal{R}$ and the Riemann-Roch Theorem without denominators. Over a field of characteristic 0, this version is, for the type F_4, contained in the aforementioned result [55, Theorem 1.3] (modulo torsion) and it is, for both types, contained in the aforementioned result [26, Theorem 4.3].

In Chapter V, we prove a version of the MTL for X a principal homogeneous space for $\mathbf{SL_1}(A)$, with A a central simple algebra of prime degree (see Theorem V.0.1). The proof mainly relies on a result of I. Panin about the Grothendieck ring of such a variety. In fact, this version implies the previous one on exceptional projective homogeneous varieties.

Finally, in Chapter VI (Sections VI.1 to VI.3), we are interested in a conjecture ([48, Conjecture 3.13]) of A. Vishik. This conjecture would be a version of the MTL for quadrics which brings into play certain algebraic cycles on associated Grassmannians. First, we compare the rationality of this cycles on Grassmannians with the rationality of certain special correspondences on product of quadrics. Then we use this to broach the conjecture.

In this dissertation, the word *scheme* means a separated scheme of finite type over a field and a *variety* is an integral scheme. Basic material is introduced in Chapter II.

Chapter II

Basic material

II.1 Definition and basic properties of Chow groups

In this section, we define the Chow groups, which are the main object of this work. We also describe some basic properties of Chow groups. Further properties are given is the next section.

The Rost complex

We follow the construction given in [7, §49] (see also [42]). For a field L, we write $K_*(L)$ for its *Milnor ring* (see Appendix A). Let X be scheme over a field F. For $x \in X$, we write $F(x)$ for its residue field and $\dim(x)$ for the dimension of the closure $\overline{\{x\}}$. We recall that $x' \in X$ is a *specialization* of x if $x' \in \overline{\{x\}}$.

Let $x, x' \in X$ such that x' is a specialization of x with $\dim(x') = \dim(x) - 1$. Then the local ring $\mathcal{O}_{\overline{\{x\}},x'}$ is a 1-dimensional excellent domain with quotient field $F(x)$ and residue field $F(x')$.

Moreover, it is known that in this situation (i.e more generally when one has a 1-dimensional excellent domain at one's disposal) the integral closure $\tilde{\mathcal{O}}_{\overline{\{x\}},x'}$ in $F(x)$ is semi-local, 1-dimensional and finite as a $\mathcal{O}_{\overline{\{x\}},x'}$-algebra. Let us index by $1, \ldots, n$ the maximal ideals of $\tilde{\mathcal{O}}_{\overline{\{x\}},x'}$. Then for each $i = 1, \ldots, n$, the corresponding localization is a discrete valuation ring with valuation v_i and we denote by F_i the associated residue field. For each $i = 1, \ldots, n$, the field F_i is a finite extension of $F(x')$. Thus, for any such point $x, x' \in X$, one can define the homomorphism

$$\delta_{x,x'} : K_*(F(x)) \to K_{*-1}(F(x'))$$

by the formula

$$\delta_{x,x'} = \sum_{i=1}^{n} c_{F_i/F(x')} \circ \delta_{v_i},$$

where $\delta_{v_i} : K_*(F(x)) \to K_{*-1}(F_i)$ is the residue homomorphism associated with the discrete valuation v_i on $F(x)$ and $c_{F_i/F(x')} : K_*(F_i) \to K_*(F(x'))$ is the norm homomorphism associated with the extension $F_i/F(x')$ (see Appendix A).

Now, for every pair of points $x, x' \in X$, one can consider the homomorphism

$$\delta_x^{x'} : K_*(F(x)) \to K_{*-1}(F(x'))$$

defined by $\delta_x^{x'} = \delta_{x,x'}$ if x' is a specialization of x with $\dim(x') = \dim(x) - 1$ and $\delta_x^{x'} = 0$ otherwise. Furthermore, by [7, Lemma 49.1], for each $x \in X$ and $\alpha \in K_*(F(x))$, the residue $\delta_x^{x'}(\alpha)$ is nontrivial only for finitely many points $x' \in X$. It follows that there is a well-defined endomorphism d_X of the direct sum

$$C(X) := \coprod_{x \in X} K_*(F(x))$$

given by $\delta_x^{x'}$ on the (x, x')-component. The group $C(X)$ is graded by the dimension: for any integer $k \geq 0$, we set

$$C_k(X) := \coprod_{x \in X_{(k)}} K_*(F(x))$$

where $X_{(k)}$ the set of point of X of dimension k, and one can extend this grading by setting $C_k(X) = 0$ for $k < 0$. Note that the endomorphism d_X is of degree -1 with respect to this grading. For any integer n, we also set

$$C_{k,n}(X) := \coprod_{x \in X_{(k)}} K_{k+n}(F(x)).$$

Note that the graded group $C_{*,n}(X)$ is invariant under d_X.

By [7, Proposition 49.30], the endomorphism d_X is such that $d_X^2 = 0$, that is to say $(C_*(X), d_X)$ is a complex, called *the Rost complex* of the scheme X.

Definition of Chow groups

For any integers k and n, let us denote by $A_k(X, K_n)$ the homology group of the sequence

$$C_{k+1,n}(X) \to C_{k,n}(X) \to C_{k-1,n}(X)$$

given by the differential d_X. In other words, the group $A_k(X, K_n)$ is the k-th homology group of the complex $C_{*,n}(X)$.

Definition II.1.1. The group

$$\mathrm{CH}_k(X) := A_k(X, K_{-k})$$

is called the *Chow group of dimension k classes of cycles on X.*

Remark II.1.2. Since for any $x \in X$, one has $K_{-1}(F(x)) = 0$ and $K_0(F(x)) = \mathbb{Z}$, it follows from the very definition that $\mathrm{CH}_k(X)$ is a quotient of the free group

$$\mathrm{Z}_k(X) := \coprod_{x \in X_{(k)}} \mathbb{Z}$$

called the *group of algebraic cycles of dimension k on X*. We denote by $[x]$ the element of $\mathrm{Z}_k(X)$ associated with $x \in X_{(k)}$. Such an algebraic cycle is called *prime cycle*. Hence, any element in $\mathrm{Z}_k(X)$ is a finite formal sum with coefficients in $\mathbb{Z}$ of prime cycles of dimension k. Identifying x with its closure in X, any element in $\mathrm{Z}_k(X)$ can also be considered as a finite formal sum of cycles $[Z]$ associated with closed subvarieties $Z \subset X$ of dimension k. We will use the same notation for the class in $\mathrm{CH}_k(X)$ and all along this dissertation we will simply called an element of a Chow group an *algebraic cycle* or even a *cycle*.

Example II.1.3. Assume that X is of dimension d. Then the group $\mathrm{CH}_d(X) = \mathrm{Z}_d(X)$ is free with basis the classes of the generic points, or equivalently – the irreducible components – of X.

We will use the following notion of Chow group as well.

Definition II.1.4. For X equidimensional of dimension d, the group

$$\mathrm{CH}^k(X) := A_{d-k}(X, K_{k-d})$$

is called the *Chow group of codimension k classes of cycles on X*.

Functorial properties

Push-forward. Let $f : X \to Y$ be a morphism of F-schemes. We define a homomorphism

$$f_* : C_*(X) \to C_*(Y)$$

as follows. Let $x \in X$ and $y \in Y$. If $y = f(x)$ and the extension $F(x)/F(y)$ is finite, we set

$$f_{*y}^{\ x} := c_{F(x)/F(y)} : K_*(F(x)) \to K_*(F(y))$$

(where $c_{F(x)/F(y)}$ is the norm homomorphism associated with the extension $F(x)/F(y)$, see Appendix A), otherwise, we set $f_{*y}^{\ x} = 0$. The homomorphism f_* is of degree zero with respect to the grading by dimension on C_*.

Furthermore, if f is a proper morphism then f_* is a morphism of complexes (see [7, Proposition 49.9]). Therefore, for any integer n, the homomorphism f_* induces some homomorphsims between the homology groups of the respective complexes $C_{*,n}$. In particular, f_* gives rise to a homomorphism at the level of Chow groups

$$f_* : \mathrm{CH}_k(X) \to \mathrm{CH}_k(Y),$$

called the *push-forward* of f. For $g : Y \to Z$ another proper morphism, one has $(g \circ f)_* = g_* \circ f_*$ (this follows from the transitivity of the norm homomorphism).

Example II.1.5. Let X be a complete scheme over F. The pushforward $\deg : \mathrm{CH}_0(X) \to \mathrm{CH}_0(\mathrm{Spec}(F)) = \mathbb{Z}$ of the structure morphism $X \to \mathrm{Spec}(F)$ is called the *degree homomorphism* because for any closed point $x \in X$, one has $\deg([x]) = [F(x) : F]$.

Pull-back. Let $g : Y \to X$ be a flat morphism of F-schemes. One says that g is of *relative dimension* d if for every $x \in X$ in the image of g and for every generic point y of $g^{-1}(\overline{\{x\}})$, one has $\dim(y) = \dim(x) + d$.

Let $g : Y \to X$ be a flat morphism of F-schemes of relative dimension d. For every $x \in X$, we denote by Y_x the fiber scheme $Y \times_X \mathrm{Spec}(F(x))$ over $F(x)$ and we identify its underlying topological space with a subspace of Y. For any generic point y of Y_x, the local ring $\mathcal{O}_{Y_x,y}$ is noetherian 0-dimensional, hence is artinian and consequently has finite length $l\left(\mathcal{O}_{Y_x,y}\right)$. Then one defines a homomorphism

$$g^* : C_*(X) \to C_{*+d}(Y)$$

as follows. Let $x \in X$ and $y \in Y$. If $g(y) = x$ and y is a generic point of Y_x, we set

$$g^{*x}_{\ y} := l\left(\mathcal{O}_{Y_x,y}\right) \cdot r_{F(y)/F(x)} : K_*(F(x)) \to K_*(F(y)),$$

where $r_{F(y)/F(x)}$ is the restriction homomorphism (see Appendix A), otherwise, we set $g^{*x}_{\ y} = 0$.

As in the case of the push-forward, for any integer k, the homomorphism g^* induces a homomorphism at the level of Chow groups

$$g^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+d}(Y),$$

called the *pull-back* of g. If X and Y are equidimensional then the latter pull-back can be rewritten as

$$g^* : \mathrm{CH}^{n-k}(X) \to \mathrm{CH}^{n-k}(Y),$$

with $n = \dim(X)$. For $h : Z \to Y$ another morphism of constant relative dimension, one has $(g \circ h)^* = h^* \circ g^*$.

Example II.1.6. Let X be a scheme over F and let L/F be an extension. The projection $X_L \to X$ is flat of relative dimension 0. The associated pull-back

$$\mathrm{CH}_k(X) \to \mathrm{CH}_k(X_L)$$

is called the *change of field homomorphism*. This homomorphism has the central place in this work.

We conclude this subsection by stating a proposition which mixes push-forwards and pull-backs. Consider a fiber product diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of F-schemes. The following functorial property comes from [7, Proposition 49.20].

Proposition II.1.7. *Assume that g and g' are flat morphisms of constant relative dimension d and that f and f' are proper morphisms. Then the diagram*

$$\begin{tikzcd} \mathrm{CH}_*(X) \arrow[r, "g'^*"] \arrow[d, "f_*"'] & \mathrm{CH}_{*+d}(X') \arrow[d, "f'_*"] \\ \mathrm{CH}_*(Y) \arrow[r, "g^*"] & \mathrm{CH}_{*+d}(Y') \end{tikzcd}$$

is commutative.

Products

Let X and Y be two schemes over F. By [7, §52.C], for any integers k, l, i, j, there is a well defined pairing

$$A_k(X, K_i) \otimes A_l(Y, K_j) \to A_{k+l}(X \times Y, K_{i+j}).$$

Assume that $Y = X$ and that X is smooth of dimension d. The diagonal morphism $\Delta : X \to X \times X$ is a regular closed embedding of codimension d. By taking the particular case $i = -k$ and $j = -l$ in the above pairing and by combining it with the pull-back associated with Δ, one get a new pairing

$$\mathrm{CH}_k(X) \otimes \mathrm{CH}_l(X) \to \mathrm{CH}_{k+l-d}(X),$$

which endows $\mathrm{CH}_*(X)$ with a commutative ring structure, with neutral element the class $[X] \in \mathrm{CH}_d(X)$.

II.2 Further properties of Chow groups

In this section, we introduce some properties of Chow groups we will use all along this dissertation. All facts provided are taken from the book [7, Chapters IX and X] by R. Elman, N. Karpenko and A. Merkurjev.

Let F be a field. The first property below shows that pull-backs commute with the product on the Chow ring CH_* (see [7, Proposition 56.8]).

Proposition II.2.1. *Let $f : X \to Y$ be a morphism of smooth F-varieties. Then one has*

$$f^*(\alpha \cdot \beta) = f^*(\alpha) \cdot f^*(\beta)$$

for every $\alpha, \beta \in \mathrm{CH}(Y)$ and $f^*([Y]) = [X]$.

The Projection Formula below (see [7, Proposition 56.9]) will be extensively use in this dissertation, especially in Chapter VI.

Proposition II.2.2. *Let $f : X \to Y$ be a proper morphism of smooth F-varieties. Then one has*

$$f_*(\alpha \cdot f^*(\beta)) = f_*(\alpha) \cdot \beta$$

for every $\alpha \in \mathrm{CH}(X)$ and $\beta \in \mathrm{CH}(Y)$.

One also has the following analog of the projection formula (see [7, Proposition 56.11])

Proposition II.2.3. *Let $f : X \to Y$ be a morphism of equidimensional smooth F-varieties. Then one has*

$$f_*(f^*(\beta)) = f_*([X]) \cdot \beta$$

for every $\beta \in \mathrm{CH}(Y)$.

The proposition below constitutes the first step of proofs of the main results in Chapter III.

Proposition II.2.4. ([7, Corollary 57.11]) *For every variety X of dimension n and scheme Y over F, the pull-back homomorphism $\mathrm{CH}_*(X \times Y) \to \mathrm{CH}_{*-n}(Y_{F(X)})$ is surjective.*

The following statement is a slightly altered version of the result [7, Lemma 88.5] (see also the proof of [26, Proposition 2.8]). This propostion constitutes the basis of proofs of the results in Chapter IV and V.

Proposition II.2.5 (Karpenko, Merkurjev)**.** *Let X be a smooth variety over a field F and Y an equidimensional F-variety. Given an integer k such that for any nonnegative integer i and any point $y \in Y$ of codimension i the change of field homomorphism*

$$\mathrm{CH}^{k-i}(X) \longrightarrow \mathrm{CH}^{k-i}(X_{F(y)})$$

is surjective, the change of field homomorphism

$$\mathrm{CH}^{k}(Y) \longrightarrow \mathrm{CH}^{k}(Y_{F(X)})$$

is also surjective.

Note that this statement remains true for any prime p when one considers the group Ch with $\mathbb{Z}/p\mathbb{Z}$-coefficients instead of CH.

II.3 Steenrod operations on Chow groups modulo 2

For a smooth scheme X over a field F of characteristic different from 2, P. Brosnan constructed in [4, §10] a certain homomorphism

$$S_X : \mathrm{Ch}(X) \to \mathrm{Ch}(X),$$

where Ch is the Chow group modulo 2, called the *total Steenrod operation on X of cohomological type*. For any integer $j \geq 0$, we denote by

$$S_X^j : \mathrm{Ch}^*(X) \to \mathrm{Ch}^{*+j}(X)$$

the *jth Steenrod operation on X of cohomological type* (or simply by S^j if there is no ambiguity). We refer to the book [7, §61] or to [4, §10] for an introduction to the subject. In this subsection, we just present some basic properties of Steenrod operations of cohomological type we will need in Chapter III.

The following proposition shows that Steenrod operations of cohomological type commute with pull-back homomorphisms.

Theorem II.3.1. ([7, Theorem 61.9]) *Let $f : X \to Y$ be a morphism of smooth schemes. Then the diagram*

$$\begin{array}{ccc} \mathrm{Ch}(X) & \xrightarrow{S_X} & \mathrm{Ch}(X) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{Ch}(Y) & \xrightarrow{S_Y} & \mathrm{Ch}(Y) \end{array},$$

is commutative

The interaction between Steenrod operations of cohomological type and push-forward is more complicated and is described with the following statement. For a vector bundle E over a scheme, we abuse notation and write $c(E)$ for both the total Chern class with value in CH and its modulo 2 reduction (Chern classes are defined in Appendix B).

Proposition II.3.2. ([7, Proposition 61.10]) *Let $f : X \to Y$ be a smooth projective morphism of smooth schemes. Then*

$$S_Y \circ f_* = f_* \circ c(-T_f) \circ S_X,$$

where T_f is the relative tangent bundle of f.

Now, let us recall the basic values taken by S^j.

Theorem II.3.3. ([7, Theorem 61.13]) *Let X be a smooth scheme. Then for any algebraic cycle $\alpha \in \mathrm{Ch}^k(X)$, one has*

$$S^j(\alpha) = \begin{cases} \alpha & if\ j = 0 \\ \alpha^2 & if\ j = k \\ 0 & if\ j < 0\ or\ j > k. \end{cases}$$

The following theorem says that the total Steenrod operation of cohomological type on a product of smooth schemes is just the product of the total Steenrod operations on the respective schemes.

Theorem II.3.4. ([7, Theorem 61.14]) *Let X and Y be two smooth schemes. Then one has*

$$S_{X\times Y} = S_X \times S_Y.$$

One easily deduce the following Cartan formula from the previous theorem.

Corollary II.3.5. ([7, Corollary 61.15]) *Let X be a smooth variety. Then for any j and for any $\alpha, \beta \in \mathrm{Ch}(X)$, one has*

$$S^j(\alpha \cdot \beta) = \sum_{k+i=j} S^k(\alpha) \cdot S^i(\beta).$$

In the following proposition, whose the statement and the proof are very close to [27, Lemma 3.1], we focus on how the Steenrod operations interact with the composition of correspondences (correspondences are defined in Appendix C). This will be useful during the proof of the main result of Section III.4.

Let X_1, X_2, X_3 be smooth schemes over F (of characteristic $\neq 2$), and assume that X_2 is complete (so the push-forward associated with the projection $X_1 \times X_2 \times X_3 \longrightarrow X_1 \times X_3$ is well defined).

Proposition II.3.6. *For any correspondence $\alpha \in \mathrm{Ch}(X_1 \times X_2)$ and for any correspondence $\beta \in \mathrm{Ch}(X_2 \times X_3)$, one has*

(i) $S_{X_1\times X_3}(\beta \circ \alpha) = (S_{X_2\times X_3}(\beta) \cdot c(-T_{X_2})) \circ S_{X_1\times X_2}(\alpha)$;

(ii) $S_{X_1\times X_3}(\beta \circ \alpha) = S_{X_2\times X_3}(\beta) \circ (S_{X_1\times X_2}(\alpha) \cdot c(-T_{X_2}))$,

where T_{X_2} is the tangent bundle of X_2 and c is the total Chern class.

Proof. For any $i, j \in \{1, 2, 3\}$ such that $i < j$, let us write p_{ij} for the projection

$$X_1 \times X_2 \times X_3 \longrightarrow X_i \times X_j.$$

We recall that the composition rule of correspondences (described in Appendix C) is

$$\beta \circ \alpha = p_{13*}\left({p_{12}}^*(\alpha) \cdot {p_{23}}^*(\beta)\right).$$

Therefore, by Proposition II.3.2 applied to p_{13}, we get

$$S_{X_1\times X_3}(\beta \circ \alpha) = p_{13*}\left(S_{X_1\times X_2\times X_3}\left({p_{12}}^*(\alpha) \cdot {p_{23}}^*(\beta)\right) \cdot {p_{12}}^*\left([X_1] \times c(-T_{X_2})\right)\right),$$

and since S commutes with products and pull-backs, we get

$$S_{X_1\times X_3}(\beta \circ \alpha) = p_{13*}\left({p_{12}}^*\left(S_{X_1\times X_2}(\alpha)\right) \cdot {p_{23}}^*\left(S_{X_2\times X_3}(\beta)\right) \cdot ([X_1] \times c(-T_{X_2}) \times [X_3])\right),$$

this gives, on the one hand

$$S_{X_1\times X_3}(\beta \circ \alpha) = p_{13*}\left({p_{12}}^*\left(S_{X_1\times X_2}(\alpha)\right) \cdot {p_{23}}^*\left(S_{X_2\times X_3}(\beta) \cdot c(-T_{X_2})\right)\right),$$

thus (i) is proved, and on the other hand, this gives

$$S_{X_1\times X_3}(\beta \circ \alpha) = p_{13*}\left({p_{12}}^*\left(S_{X_1\times X_2}(\alpha) \cdot c(-T_{X_2})\right) \cdot {p_{23}}^*\left(S_{X_2\times X_3}(\beta)\right)\right),$$

thus (ii) is proved. □

II.4 Grothendieck rings

Definitions and properties

All facts provided here can be found in [11, §15].

Let X be a smooth scheme. The *Grothendieck group* $K(X)$ of X is the abelian group given by generators, the isomorphism classes $[E]$ of vector bundles E over X, modulo the relation

$$[E] = [E'] + [E'']$$

whenever one has an exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of vector bundles over X.

The tensor product on vector bundles over X induces a ring structure on $K(X)$ given by $[E] \cdot [E'] = [E \otimes E']$ for any vector bundles E and E' over X.

For any morphism of smooth schemes $f : X \to Y$ there is an induced pull-back homomorphism

$$f^* : K(Y) \to K(X)$$

taking $[E]$ to $[f^*E]$. For example, if X is a smooth scheme over a field F then for any extension L/F one can consider the associated change of field homomorphism $K(X) \to K(X_L)$. Hence, K is a contravariant functor from the category of smooth schemes to the category of commutative rings.

If f is a proper morphism, there exists also an induced push-forward homomorphism $f_* : K(X) \to K(Y)$.

If X is irreducible, there is a ring epimorphism

$$\mathrm{rk} : K(X) \to \mathbb{Z}$$

taking the class of a vector bundle over X to its rank. This epimorphism is called the *rank homomorphism* and has kernel the term $\tau^1(X)$ of the topological filtration on $K(X)$ defined in the next subsection.

Note that since X is smooth, one can define the Grothendieck ring $K(X)$ in exactly the same way but using coherent sheaves $\mathcal{F}$ on X instead of vector bundles E over X.

Filtrations on the Grothendieck ring

In this subsection, we introduce two particular filtrations on the Grothendieck ring $K(X)$ of a smooth variety X over a field F. Material presented here can be found in [22, §2].

On the one hand, the term of codimension i of the *topological filtration* on $K(X)$ is given by

$$\tau^i(X) = \langle [\mathcal{O}_Z] \,|\, Z \hookrightarrow X \text{ and } \operatorname{codim}(Z) \geq i \rangle,$$

where $[\mathcal{O}_Z]$ is the class in $K(X)$ of the structure sheaf of a closed subvariety Z of X.

On the other hand, the term of codimension i of the γ-*filtration* on $K(X)$ is given by

$$\gamma^i(X) = \langle c^K_{n_1}(a_1) \cdots c^K_{n_m}(a_m) \,|\, n_1 + \cdots + n_m \geq i \text{ and } a_1, \ldots, a_m \in K(X) \rangle,$$

where the endomorphism c^K_n is the n-th Chern class with values in K.

As in the case of Chern classes with values in CH, Chern classes with values in K commute with push-forwards (see Appendix B), i.e for any morphism $f : X \to Y$ of smooth varieties over F and any $a \in K(Y)$, one has the identity $c^K_n(f^*(a)) = f^*(c^K_n(a))$ in $K(X)$.

For any i, one has $\gamma^i(X) \subset \tau^i(X)$ and one even has $\gamma^i(X) = \tau^i(X)$ for $i \leq 2$. We write $\gamma^{i/i+1}(X)$ and $\tau^{i/i+1}(X)$ for the respective quotients.

We denote by pr^i the canonical surjection

$$\begin{array}{ccc} \mathrm{CH}^i(X) & \twoheadrightarrow & \tau^{i/i+1}(X) \\ [Z] & \longmapsto & [\mathcal{O}_Z] \end{array}.$$

By the Riemann-Roch Theorem without denominators the i-th Chern class induces an homomorphism in the opposite way $c_i : \tau^{i/i+1}(X) \to \mathrm{CH}^i(X)$ such that the composition $c_i \circ pr$ is the multiplication by $(-1)^{i-1}(i-1)!$ (see [11, Example 15.3.6]).

Furthermore, for any smooth F-variety X, Chern classes with different values are connected by the following commutative diagram of maps

$$\begin{tikzcd} K(X) \arrow[r, "c_i^{\mathrm{CH}}"] \arrow[d, "c_i^K"'] & \mathrm{CH}^i(X) \arrow[d, "pr_i"] \\ \gamma^{i/i+1}(X) \arrow[r] & \tau^{i/i+1}(X) \end{tikzcd} \tag{II.4.1}$$

(see [22, Lemma 2.16]).

Remark II.4.2. Note that for any prime p, one can also consider the γ-filtration γ_p and the topological filtration τ_p on the ring $K(X)/pK(X)$ by replacing $K(X)$ by $K(X)/pK(X)$ in the previous definitions. In particular, one get that for any $0 \leq i \leq p$, the map $pr^i_p : \mathrm{Ch}^i(X) \twoheadrightarrow \tau_p^{i/i+1}(X)$, where Ch is the Chow group modulo p, is an isomorphism.

Brown-Gersten-Quillen spectral sequence

For any smooth variety X and any $i \geq 1$, the epimorphism pr^i coincides with the edge homomorphism of the spectral Brown-Gersten-Quillen structure $E_2^{i,-i}(X) \Rightarrow K(X)$ (see [41, §7]), that is to say

$$pr^i : \mathrm{CH}^i(X) \simeq E_2^{i,-i}(X) \twoheadrightarrow \cdots \twoheadrightarrow E_{i+1}^{i,-i}(X) = \tau^{i/i+1}(X).$$

In particular, for any prime p, the map pr_p^{p+1} is the compostion of the surjections

$$q_r : E_r^{p+1,-p-1}(X) \pmod p \twoheadrightarrow \frac{E_r^{p+1,-p-1}(X)}{\mathrm{Im}(\delta_r)} \pmod p,$$

for r from 2 to $p+1$, where δ_r is the differential starting at $E_r^{p-r+1,-p+r-2}(X)$.

Moreover, by the result [31, Theorem 3.4] of A. Merkurjev on Adam's operations, every prime divisor l of the order of δ_r is such that $l-1$ divides $r-1$. Therefore, for any $r \leq p-1$, the differential δ_r is of prime to p order and this implies that q_r is an isomorphism. Consequently, for any smooth variety X and any prime p, one has

$$\mathrm{Ch}^{p+1}(X) \simeq E_p^{p+1,-p-1}(X) \pmod p. \tag{II.4.3}$$

Chapter III

Quadrics

In this chapter, we prove some results comparing rationality of algebraic cycles over the function field of a smooth projective quadric and over the base field. First, we deal Chow groups modulo 2 and then with integral Chow groups. Our work is largely inspired by the work of A. Vishik on this topic. We recall that the word *scheme* means a separated scheme of finite type over a field and a *variety* is an integral scheme.

Remark III.0.1. The case of *affine norm quadrics* can be treated as follows. Let U be a non-degenerate anisotropic affine norm quadric given by the equation $\pi = c$ for π a Pfister form over a field F and $c \in F^\times$. By a result of M. Rost, one has $\mathrm{CH}^i(U) = 0$ for any $i > 0$, where CH is the integral Chow group (see [21, Theorem A.4] for a proof). M. Rost used this in [44] in the proof of the *Rost Nilpotence Theorem* for Chow motives of smooth projective quadrics. Combining this with Proposition II.2.5, one get that for any equidimensional F-variety Y, the change of field homomorphism

$$\mathrm{CH}(Y) \to \mathrm{CH}(Y_{F(U)}),$$

is surjective in codimension $< \dim(U)$. It is also surjective in codimension $\dim(U)$ for a given Y provided that for each generic point ζ of Y, the variety U remains anisotropic over $F(\zeta)$.

Remark III.0.2. Let Q_π be an r-fold Pfister quadric over a field F of characteristic $\neq 2$. The combination of [27, Theorem 8.1] with the motivic decomposition result [44, Proposition 19] and Proposition II.2.5 gives that for any equidimensional F-variety Y the change of field homomorphism

$$\mathrm{CH}(Y) \to \mathrm{CH}(Y_{F(Q_\pi)})$$

is surjective in codimension $< \dim(Q_\pi)/2 = 2^{r-1} - 1$. Note that the proof of [27, Theorem 8.1] uses the computation of Chow groups of affine quadrics mentionned in the previous remark. One also has the equivalent result for the norm quadric associated with $\pi \perp \langle -c \rangle$.

III.1 Decomposition on Chow groups of projective quadrics

The main purpose of this section is to introduce the notion of *coordinates* for a cycle $x \in \mathrm{CH}(Q \times Y)$, where Q is a smooth projective quadric over F and Y is a smooth F-variety. This notion will be useful during the proofs of the results of this chapter.

Let Q be a smooth projective quadric over F of dimension n given by a quadratic form φ (see [7, §22]), and let us set $i_0(Q) := i_0(\varphi)$, where $i_0(\varphi)$ is the Witt index of φ, i.e the dimension of a maximal totally isotropic subspace of the form φ.

For $i = 0, ..., n$, let us denote as $h^i \in \mathrm{CH}^i(Q)$ the ith power of the hyperplane section class (note that for any i, the cycle h^i is defined over the base field). For $i < i_0(Q)$, let us denote as $l_i \in \mathrm{CH}_i(Q)$ the class of an i-dimensional totally isotropic subspace of $\mathbb{P}(V)$, where V is the underlying vector space of φ. For $i \leq \lfloor n/2 \rfloor$, we still write $l_i \in \mathrm{CH}_i(\overline{Q})$ for the class of an i-dimensional totally isotropic subspace of $\mathbb{P}(V_{\overline{F}})$, (if $i < i_0(Q)$, the cycle $l_i \in \mathrm{CH}_i(\overline{Q})$ is the image of $l_i \in \mathrm{CH}_i(Q)$ under the change of field homomorphism $\mathrm{CH}(Q) \to \mathrm{CH}(\overline{Q})$). Let us notice that for $i < \lfloor n/2 \rfloor$, the cycle l_i (in $\mathrm{CH}_i(\overline{Q})$ or in $\mathrm{CH}_i(Q)$ if $i < i_0(Q)$) is canonical by [7, Proposition 68.2] (in case of even n, there are two classes of $n/2$-dimensional totally isotropic subspaces and we fix one of the two).

Moreover, we recall that the total Chow group $\mathrm{CH}(\overline{Q})$ is free with basis $\{h^i, l_i | i \in [0, [\frac{n}{2}]]\}$ and that the following multiplication rule holds in the ring $\mathrm{CH}(Q)$:

$$h \cdot l_i = l_{i-1} \text{ for any } i \in [1, [\frac{n}{2}]].$$

(see [7, Proposition 68.1]). Finally, we recall also that

$$h^i = 2l_{n-i} \text{ for any } i > [\frac{n}{2}].$$

Let x be an element of $\mathrm{CH}^r(Q \times Y)$. We write pr for the projection $Q \times Y \to Y$. For every $i = 0, ..., i_0(Q) - 1$, we have the following homomorphisms

$$\begin{array}{rcl} \mathrm{CH}^r(Q \times Y) & \longrightarrow & \mathrm{CH}^{r-i}(Y) \\ x & \longmapsto & pr_*(l_i \cdot x) =: x^i \end{array},$$

and

$$\begin{array}{rcl} \mathrm{CH}^r(Q \times Y) & \longrightarrow & \mathrm{CH}^{r-n+i}(Y) \\ x & \longmapsto & pr_*(h^i \cdot x) =: x_i \end{array}.$$

Definition III.1.1. The cycle $x^i \in \mathrm{CH}^{r-i}(Y)$ is called the *coordinate of x on h^i* while $x_i \in \mathrm{CH}^{r-n+i}(Y)$ is called the *coordinate of x on l_i*.

Note that if $r < n/2$, for any $i = 0, ..., i_0(Q) - 1$, one has $x_i = 0$ by dimensional reasons.

Remark III.1.2. For any nonnegative integer $k < i_0(Q)$, let us set $x(k) := x - \sum_{i=0}^{k} h^i \times x^i - \sum_{i=0}^{k} l_i \times x_i$. Note that for any $i = 0, ..., k$, the coordinate of $x(k)$ on h^i (as well as its coordinate on l_i) is 0. The writing

$$x = x(k) + \sum_{i=0}^{k} h^i \times x^i + \sum_{i=0}^{k} l_i \times x_i$$

is called a *decomposition* of x.

Assume now that $r < i_0(Q)$ and $r \leq k$. Then, by [7, Theorem 66.2], one can write

$$x(k) = \sum_{i=0}^{r} h^i \times w^i$$

with some $w^i \in \mathrm{CH}^{r-i}(Y)$. Since, for any $i = 0, ..., r$, the cycle w^i coincides with the coordinate $x(k)^i$ of $x(k)$ on h^i, we get that $x(k) = 0$.

Recall that one says that the quadric Q is *completely split* if $i_0(Q)$ is maximal, i.e $i_0(Q) = \lfloor n/2 \rfloor + 1$ (this terminology is consistent with the fact that for such a quadric the semisimple group $\mathbf{SO}(Q)$ is split, see [29, §25]).

Remark III.1.3. Assume that $Y = Q$, $r = n$, and that $k < \lfloor n/2 \rfloor$ (what is the case if the quadric Q is not completely split). Let x be an element of $\mathrm{CH}^n(Q \times Q)$. Since, for $i = 0, ..., k$, the group $\mathrm{CH}^{n-i}(Q)$ is free with basis $\{l_i\}$ (because $i < \lfloor n/2 \rfloor$), one can uniquely write

$$x = x(k) + \sum_{i=0}^{k} b_i \cdot h^i \times l_i + \sum_{i=0}^{k} l_i \times x_i,$$

with some $b_i \in \mathbb{Z}$.

Note that everything in this section holds for Chow groups Ch modulo 2 in place of the integral Chow groups CH.

III.2 Rationality on Chow groups modulo 2 - Main result

In this section, we deal with Question I.0.1 in the context of smooth projective quadrics and for Ch is the Chow group modulo 2.

Let Y be a smooth variety over a field F of characteristic different from 2. We recall that for any $j \geq 0$, the map

$$S^j : \mathrm{Ch}^*(Y) \to \mathrm{Ch}^{*+j}(Y)$$

denotes the *j-th Steenrod operation on Y of cohomological type* (see Section II.3).

The complete version of the Main Tool Lemma by A. Vishik is the following (see [49, Theorem 3.1(1)]).

Theorem III.2.1 (Vishik)**.** *Let Y be a smooth quasi-projective variety over a field F of characteristic 0 and let Q be a smooth projective quadric. Then for any $F(Q)$-rational element $\overline{y} \in \mathrm{Ch}^m(\overline{Y})$, with $m < dim(Q)/2 + j$, the element $S^j(\overline{y})$ is rational.*

This can be visualized with the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Ch}^{m+j}(Y) \arrow[d] & \mathrm{Ch}^m(Y) \arrow[l, "S^j"'] \arrow[r] \arrow[d] & \mathrm{Ch}^m(Y_{F(Q)}) \arrow[d] \\
\mathrm{Ch}^{m+j}(\overline{Y}) & \mathrm{Ch}^m(\overline{Y}) \arrow[l, "S^j"] \arrow[r, hook] & \mathrm{Ch}^m(Y_{\overline{F}(Q)})
\end{tikzcd}$$

This technical result plays the crucial role in the construction by A. Vishik of fields with u-invariant $2^r + 1$, for $r \geq 3$ (in fact, he used the contrapositive statement to show that certain irrational cycles are $F(Q)$-irrational, see [48, Theorem 5.1]).

A. Vishik's proof of the Main Tool Lemma uses *symmetric operations in the algebraic cobordism theory* constructed in [52], which requires to work with fields of characteristic 0 since it relies on resolution of singularities.

However, N. Karpenko introduced a method in [23] which works in any characteristic different from 2 because it only uses Steenrod operations on Chow groups modulo 2 and no symmetric operations in the algebraic cobordism theory (Chow theory does not rely on resolution of singularities). Namely, N. Karpenko proved the case $j = 0$ of the following theorem and we generalized his method to get a statement similar to Theorem III.2.1 (see [8, Theorem 1.1]).

Theorem III.2.2. *Let Y be a smooth variety over a field F of characteristic different from 2 and let Q be a smooth projective quadric. Then for any $F(Q)$-rational element $\overline{y} \in \mathrm{Ch}^m(\overline{Y})$, with $m < dim(Q)/2 + j$, the element $S^j(\overline{y})$ is is the sum of a rational element and the class modulo 2 of an integral element of exponent 2.*

The version of A. Vishik remains stronger in the sense that his use of symmetric operations in the algebraic cobordism theory allowed him to get rid of the exponent 2 element appearing in our conclusion.

Nevertheless, since the proof of [48, Theorem 5.1] only deals with torsion-free Chow groups, our versions with exponent 2 element can also be used here.

Moreover, the only use of the Steenrod operations allows one to get rid of the assumption of quasi-projectivity for Y (A. Vishik needed that assumption because the algebraic cobordism theory is defined on the category of smooth quasi-projective schemes over a field of characteristc 0, see [30]).

Note that in the case $j = 0$, A. Vishik provided an example [49, Statement 3.7] proving that, without further assumptions on Q and Y, the bound $\dim(Q)/2$ in the previous statements is sharp.

If one imposes Y to be complete and $\dim(Y) \leq \dim(Q) - i_1(Q)$ then there is the interesting result [34, Theorem 3.1] of N. Karpenko and A. Merkurjev stating that any closed point of $Y_{F(Q)}$ of odd degree actually exists over the base field F.

Most of material needed for the proof of Theorem III.2.2 below is taken from the book [7] by R. Elman, N. Karpenko and A. Merkurjev. In the next section, we prove some other technical results around rationality of algebraic cycles using the same methods (Proposition III.3.1 and Theorem III.3.6). Those results are weaker versions of some proved by A. Vishik in [49] (Proposition 3.3(2) and Theorem 3.1(2)) over fields of characteristic 0.

Proof of Theorem III.2.2

We use material and notation introduced in Section III.1 and we denote by n the dimension of the quadric Q.

We assume that $0 \leq j \leq m$ (otherwise we get $S^j(\overline{y}) = 0$, see Theorem II.3.3). Let $\overline{y}$ be an $F(Q)$-rational element of $\mathrm{Ch}^m(\overline{Y})$. Since the quadric Q is isotropic over $\overline{F}$, the homomorphism $\mathrm{CH}(\overline{Y}) \to \mathrm{CH}(Y_{\overline{F}(Q)})$ is surjective and is consequently an isomorphism (see Chapter I). The element $\overline{y} \in \mathrm{Ch}^m(\overline{Y})$ being $F(Q)$-rational, there exists $y \in \mathrm{Ch}^m(Y_{F(Q)})$ mapped to $\overline{y}$ under the homomorphism

$$\mathrm{Ch}^m(Y_{F(Q)}) \to \mathrm{Ch}^m(Y_{\overline{F}(Q)}) \xrightarrow{\sim} \mathrm{Ch}^m(\overline{Y}).$$

Let us fix an element $x \in \mathrm{Ch}^m(Q \times Y)$ mapped to y under the surjection (see Proposition II.2.4)

$$\mathrm{Ch}^m(Q \times Y) \twoheadrightarrow \mathrm{Ch}^m(Y_{F(Q)}).$$

Since over $\overline{F}$ the quadric Q becomes completely split, by Remark III.1.2, the image $\overline{x} \in \mathrm{Ch}^m(\overline{Q} \times \overline{Y})$ of x decomposes as

$$\overline{x} = h^0 \times x^0 + \cdots + h^{[\frac{n}{2}]} \times x^{[\frac{n}{2}]} + l_{[\frac{n}{2}]} \times x_{[\frac{n}{2}]} + \cdots + l_{[\frac{n}{2}]-j} \times x_{[\frac{n}{2}]-j} \tag{III.2.3}$$

where $x^i \in \mathrm{Ch}^{m-i}(\overline{Y})$ is the coordinate of $\overline{x}$ on h^i and $x_i \in \mathrm{Ch}^{m-n+i}(\overline{Y})$ is the coordinate of $\overline{x}$ on l_i (see Definition III.1.1). Note that, by [49, Lemma 3.2], one has

$$x^0 = \overline{y}.$$

For every $i = 0, ..., m$, let s^i be the image in $\mathrm{CH}^{m+i}(\overline{Q} \times \overline{Y})$ of an element in $\mathrm{CH}^{m+i}(Q \times Y)$ representing $S^i(x) \in \mathrm{Ch}^{m+i}(Q \times Y)$. We also set $s^i := 0$ for $i > m$.

The integer n can be uniquely written in the form $n = 2^t - 1 + s$, where t is a non-negative integer and $0 \leq s < 2^t$. Let us denote $2^t - 1$ as d. Since $d \leq n$, we can fix a smooth subquadric P of Q of dimension d; we write in for the imbedding

$$(P \hookrightarrow Q) \times \mathrm{id}_Y : P \times Y \hookrightarrow Q \times Y.$$

Lemma III.2.4. *For any integer r, one has*

$$S^r pr_* in^* x = \sum_{i=0}^{r} pr_*(c_i(-T_P) \cdot in^* S^{r-i}(x)) \quad \text{in } \mathrm{Ch}^{r+m-d}(Y),$$

where T_P is the tangent bundle of P, c_i are the Chern classes, and pr is the projection $P \times Y \to Y$.

Proof. Since $pr : P \times Y \to Y$ is a smooth projective morphism between smooth schemes, for any integer r one has,

$$S^r \circ pr_* = \sum_{i=0}^{r} pr_*(c_i(-T_{pr}) \cdot S^{r-i})$$

by Proposition II.3.2, where T_{pr} is the relative tangent bundle of pr over $P \times Y$ (so here $T_{pr} = T_P$). Finally, since $in : P \times Y \hookrightarrow Q \times Y$ is a morphism between smooth schemes, the Steenrod operations of cohomological type commute with in^* (see Theorem II.3.1) and we are done. □

We apply Lemma III.2.4 taking $r = d + j$. Since $pr_* in^* x \in \mathrm{Ch}^{m-d}(Y)$ and $m - d < d + j$ (indeed, $m - d < n/2 + j - d$ by assumption, and $n/2 < 2d$ thanks to our choice of d), we have $S^{d+j} pr_* in^* x = 0$.

Hence, we have by Lemma III.2.4,

$$\sum_{i=0}^{d+j} pr_*(c_i(-T_P) \cdot in^* S^{d+j-i}(x)) = 0 \quad \text{in } \mathrm{Ch}^{m+j}(Y).$$

In addition, for any $i = 0, ..., d$, by [7, Lemma 78.1] we have $c_i(-T_P) = \binom{d+i+1}{i} \cdot h^i$, where $h^i \in \mathrm{Ch}^i(P)$ is the ith power of the hyperplane section class, and where the binomial coefficient is considered modulo 2. Furthermore, for any $i = 0, ..., d$, the binomial coefficient $\binom{d+i+1}{i}$ is odd (because d is a power of 2 minus 1, see [7, Lemma 78.6]). Note also that for $i > d$, we have $c_i(-T_P) = 0$ since P is of dimension d. Thus, we get

$$\sum_{i=0}^{d} pr_*(h^i \cdot in^* S^{d+j-i}(x)) = 0 \quad \text{in } \mathrm{Ch}^{m+j}(Y).$$

Therefore, the element

$$\sum_{i=0}^{d} pr_*(h^i \cdot in^* s^{d+j-i}) \in \mathrm{CH}^{m+j}(\overline{Y})$$

is twice a rational element.

Furthermore, for any $i = 0, ..., d$, we have

$$pr_*(h^i \cdot in^* s^{d+j-i}) = pr_*(in_*(h^i \cdot in^* s^{d+j-i}))$$

(the first pr is the projection $P \times Y \to Y$ while the second pr is the projection $Q \times Y \to Y$). Since in is a proper morphism between smooth schemes, we have by the projection formula (see Proposition II.2.2),

$$in_*(h^i \cdot in^* s^{d+j-i}) = in_*(h^i) \cdot s^{d+j-i} = h^{n-d+i} \cdot s^{d+j-i}$$

and we finally get

$$pr_*(h^i \cdot in^* s^{d+j-i}) = pr_*(h^{n-d+i} \cdot s^{d+j-i}).$$

Hence, we get that the element

$$\sum_{i=0}^{d} pr_*(h^{n-d+i} \cdot s^{d+j-i}) \in \mathrm{CH}^{m+j}(\overline{Y})$$

is twice a rational element.

We would like to compute the sum obtained modulo 4. Since $s^{d+j-i} = 0$ if $d+j-i > m$, the ith summand is 0 for any $i < d+j-m$ ($(j-m) \leq 0$ by assumption). Otherwise – if $i \geq d+j-m$ – the factor h^{n-d+i} is divisible by 2 (indeed, we have $h^{n-d+i} = 2l_{d-i}$ because $n-d+i \geq n+j-m > n/2$) and in order to compute the ith summand modulo 4 it suffices to compute s^{d+j-i} modulo 2, that is, to compute $S^{d+j-i}(\overline{x})$.

According to the decomposition (III.2.3), we have

$$S^{d+j-i}(\overline{x}) = \sum_{k=0}^{[\frac{n}{2}]} S^{d+j-i}(h^k \times x^k) + \sum_{k=0}^{j} S^{d+j-i}(l_{[\frac{n}{2}]-k} \times x_{[\frac{n}{2}]-k}).$$

And we set

$$A_i := \sum_{k=0}^{[\frac{n}{2}]} S^{d+j-i}(h^k \times x^k) \text{ and } B_i := \sum_{k=0}^{j} S^{d+j-i}(l_{[\frac{n}{2}]-k} \times x_{[\frac{n}{2}]-k}).$$

For any $k = 0, ..., [\frac{n}{2}]$, we have by Theorem II.3.4,

$$S^{d+j-i}(h^k \times x^k) = \sum_{l=0}^{d+j-i} S^{d+j-i-l}(h^k) \times S^l(x^k).$$

Moreover, for any $l = 0, ..., d+j-i$, we have by [7, Corollary 78.5],

$$S^{d+j-i-l}(h^k) = \binom{k}{d+j-i-l} h^{d+j+k-i-l}.$$

Thus, choosing an integral representative $\varepsilon_{k,l} \in \mathrm{CH}^{m-k+l}(\overline{Y})$ of $S^l(x^k)$ (we choose $\varepsilon_{k,l} = 0$ if $l > m-k$), we get that the element

$$\sum_{k=0}^{[\frac{n}{2}]} \sum_{l=0}^{d+j-i} \binom{k}{d+j-i-l} (h^{d+j+k-i-l} \times \varepsilon_{k,l}) \in \mathrm{CH}^{m+d+j-i}(\overline{Q} \times \overline{Y})$$

is an integral representative of A_i.

Therefore, for any $i \geq d+j-m$, choosing an integral representative $\tilde{B}_i$ of B_i, there exists $\gamma_i \in \mathrm{CH}^{m+d+j-i}(\overline{Q} \times \overline{Y})$ such that

$$s^{d+j-i} = \sum_{k=0}^{[\frac{n}{2}]} \sum_{l=0}^{d+j-i} \binom{k}{d+j-i-l} (h^{d+j+k-i-l} \times \varepsilon_{k,l}) + \tilde{B}_i + 2\gamma_i.$$

Hence, according to the multiplication rules in the ring $\mathrm{CH}(\overline{Q})$ described in Section III.1, for any $i \geq d+j-m$, we have

$$h^{n-d+i} \cdot s^{d+j-i} = 2 \sum_{k=0}^{[\frac{n}{2}]} \sum_{l=0}^{d+j-i} \binom{k}{d+j-i-l} (l_{l-j-k} \times \varepsilon_{k,l}) + h^{n-d+i} \cdot \tilde{B}_i + 4 l_{d-i} \cdot \gamma_i.$$

If $k \leq d-i$, one has $j+k \leq d+j-i$, and for any $0 \leq l \leq d+j-i$, we have by dimensional reasons,

$$pr_*(l_{l-j-k} \times \varepsilon_{k,l}) = \begin{cases} \varepsilon_{k,l} & \text{if } l = j+k \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise $k > d-i$, and $pr_*(l_{l-j-k} \times \varepsilon_{k,l}) = 0$ for any $0 \leq l \leq d+j-i$. Moreover, for $k > d-i$, one has $j+k > j+d-i \geq m > m-k$, therefore $\varepsilon_{k,j+k} = 0$.

Thus we deduce the identity

$$pr_* \left(2 \sum_{k=0}^{[\frac{n}{2}]} \sum_{l=0}^{d+j-i} \binom{k}{d+j-i-l} (l_{l-j-k} \times \varepsilon_{k,l}) \right) = 2 \sum_{k=0}^{[\frac{n}{2}]} \binom{k}{d-i-k} \varepsilon_{k,j+k}.$$

Then,

$$\sum_{i=d+j-m}^{d} pr_* \left(2 \sum_{k=0}^{[\frac{n}{2}]} \sum_{l=0}^{d+j-i} \binom{k}{d+j-i-l} (l_{l-j-k} \times \varepsilon_{k,l}) \right)$$
$$= 2 \sum_{i=d+j-m}^{d} \sum_{k=0}^{[\frac{n}{2}]} \binom{k}{d-i-k} \varepsilon_{k,j+k}.$$

In the latest expression, for every $k = 0, ..., [\frac{m-j}{2}]$, the total coefficient at $\varepsilon_{k,j+k}$ is

$$2 \sum_{i=d+j-m}^{d} \binom{k}{d-i-k} = 2 \sum_{i=d-2k}^{d-k} \binom{k}{d-i-k} = 2 \sum_{s=0}^{k} \binom{k}{s} = 2^{k+1},$$

which is divisible by 4 for $k \geq 1$.

Therefore, the cycle $\sum_{i=d+j-m}^{d} pr_*(h^{n-d+i} \cdot s^{d+j-i}) \in \mathrm{CH}^{m+j}(\overline{Y})$ is congruent modulo 4 to

$$2\varepsilon_{0,j} + \sum_{i=d+j-m}^{d} pr_*(h^{n-d+i} \cdot \tilde{B}_i).$$

Thus, the cycle $2\varepsilon_{0,j} + \sum_{i=d+j-m}^{d} pr_*(h^{n-d+i} \cdot \tilde{B}_i)$ is congruent modulo 4 to twice a rational element.

Finally, the following lemma will lead to the conclusion.

Lemma III.2.5. *For any $d + j - m \leq i \leq d$, one can choose an integral representative $\tilde{B}_i$ of B_i so that*

$$pr_*(h^{n-d+i} \cdot \tilde{B}_i) = 0.$$

Proof. We recall that $B_i := \sum_{k=0}^{j} S^{d+j-i}(l_{[\frac{n}{2}]-k} \times x_{[\frac{n}{2}]-k})$. For any $k = 0, ..., j$, we have

$$S^{d+j-i}(l_{[\frac{n}{2}]-k} \times x_{[\frac{n}{2}]-k}) = \sum_{l=0}^{d+j-i} S^{d+j-i-l}(l_{[\frac{n}{2}]-k}) \times S^{l}(x_{[\frac{n}{2}]-k}).$$

And for any $l = 0, ..., d + j - i$, we have by [7, Corollary 78.5],

$$S^{d+j-i-l}(l_{[\frac{n}{2}]-k}) = \binom{n+1-[\frac{n}{2}]+k}{d+j-i-l} l_{[\frac{n}{2}]-k-d-j+i+l}.$$

Thus, choosing an integral representative $\delta_{k,l} \in \mathrm{CH}^{m-k+l}(\overline{Y})$ of $S^{l}(x_{[\frac{n}{2}]-k})$ (we choose $\delta_{k,l} = 0$ if $l > m + [\frac{n}{2}] - k - n$), we get that the element

$$\sum_{k=0}^{j} \sum_{l=0}^{d+j-i} \binom{n+1-[\frac{n}{2}]+k}{d+j-i-l} (l_{[\frac{n}{2}]-k-d-j+i+l} \times \delta_{k,l}) \in \mathrm{CH}^{m+d+j-i}(\overline{Q} \times \overline{Y})$$

is an integral representative of B_i. Let us denote it $\tilde{B}_i$.

Hence, we have

$$h^{n-d+i} \cdot \tilde{B}_i = \sum_{k=0}^{j} \sum_{l=0}^{d+j-i} \binom{n+1-[\frac{n}{2}]+k}{d+j-i-l} (l_{[\frac{n}{2}]-k-n-j+l} \times \delta_{k,l}).$$

Moreover, we have

$$pr_*(l_{[\frac{n}{2}]-k-n-j+l} \times \delta_{k,l}) \neq 0 \Longrightarrow l = j + k + n - [\frac{n}{2}].$$

Furthermore, for any $0 \leq k \leq j$, we have $d+j-i \leq m < j+\frac{n}{2} \leq j+n-[\frac{n}{2}] \leq j+k+n-[\frac{n}{2}]$. Thus, for any $0 \leq l \leq d+j-i$ and for any $0 \leq k \leq j$, we have $pr_*(l_{[\frac{n}{2}]-k-n-j+l} \times \delta_{k,l}) = 0$. It follows that $pr_*(h^{n-d+i} \cdot \tilde{B}_i) = 0$ and we are done. □

We deduce from Lemma III.2.5 that the cycle $2\varepsilon_{0,j} \in \mathrm{CH}^{m+j}(\overline{Y})$ is congruent modulo 4 to twice a rational cycle. Therefore, there exist a cycle $\gamma \in \mathrm{CH}^{m+j}(\overline{Y})$ and a rational cycle $\alpha \in \mathrm{CH}^{m+j}(\overline{Y})$ so that

$$2\varepsilon_{0,j} = 2\alpha + 4\gamma,$$

hence, there exists an exponent 2 element $\delta \in \mathrm{CH}^{m+j}(\overline{Y})$ so that

$$\varepsilon_{0,j} = \alpha + 2\gamma + \delta.$$

Finally, since $\varepsilon_{0,j}$ is an integral representative of $S^j(x^0)$ and $x^0 = \overline{y}$, we get that $S^j(\overline{y})$ is the sum of a rational element and the class modulo 2 of an integral element of exponent 2. We are done with the proof of Theorem III.2.2.

III.3 Rationality on Chow groups modulo 2 - Other results

In this section, we continue to use notation introduced in the previous section and we prove some results which deal with the limit case of Theorem III.2.2. Those results have already been proved by A. Vishik over fields of characteristic 0 (see [49, Proposition 3.3(2) and Theorem 3.1(2)] respectively).

Proposition III.3.1. *Assume that $m = [\frac{n+1}{2}] + j$. Let $x \in \mathrm{Ch}^m(Q \times Y)$ be some element, and let x^i, x_i be the coordinates of $\overline{x}$ (as in decomposition (III.2.3)). Then the element*

$$S^j(x^0) + x^0 \cdot x_{[\frac{n}{2}]}$$

differs from a rational element by the class of an exponent 2 element of $\mathrm{CH}^{m+j}(\overline{Y})$.

Proof. The image $\overline{x} \in \mathrm{Ch}^m(\overline{Q} \times \overline{Y})$ of x decomposes as in (III.2.3). Let $\mathbf{x} \in \mathrm{CH}^m(Q \times Y)$ be an integral representative of x. The image $\overline{\mathbf{x}} \in \mathrm{CH}^m(\overline{Q} \times \overline{Y})$ decomposes as

$$\overline{\mathbf{x}} = h^0 \times \mathbf{x}^0 + \cdots + h^{[\frac{n}{2}]} \times \mathbf{x}^{[\frac{n}{2}]} + l_{[\frac{n}{2}]} \times \mathbf{x}_{[\frac{n}{2}]} + \cdots + l_{[\frac{n}{2}]-j} \times \mathbf{x}_{[\frac{n}{2}]-j}$$

where the elements $\mathbf{x}^i \in \mathrm{CH}^i(\overline{Y})$ (resp. $\mathbf{x}_i \in \mathrm{CH}^{m-n+i}(\overline{Y})$) are some integral representatives of the elements x^i (resp. x_i) appearing in (III.2.3).

For every $i = 0, ..., m-1$, let s^i be the image in $\mathrm{CH}^{m+i}(\overline{Q}\times\overline{Y})$ of an element in $\mathrm{CH}^{m+i}(Q\times Y)$ representing $S^i(x) \in \mathrm{Ch}^{m+i}(Q\times Y)$. We also set $s^i := 0$ for $i > m$. Finally, we set $s^0 := \overline{\mathbf{x}}$ and $s^m := (s^0)^2$ (because $S^m(x) = x^2$, see Theorem II.3.3). Therefore, for any nonnegative integer i, s^i is the image in $\mathrm{CH}^{m+i}(\overline{Q} \times \overline{Y})$ of an integral representative of $S^i(x)$.

The integer n can be uniquely written in the form $n = 2^t - 1 + s$, where t is a non-negative integer and $0 \leq s < 2^t$. Let us denote $2^t - 1$ as d.

We would like to use again Lemma III.2.4 to get that the sum

$$\sum_{i=d+j-m}^{d} pr_*(h^{n-d+i} \cdot s^{d+j-i}) \in \mathrm{CH}^{m+j}(\overline{Y}) \tag{III.3.2}$$

is twice a rational element. To do this, it suffices to check that $m - d < d + j$. Then the same reasoning as the one used during the proof of Theorem III.2.2 gives us the desired result. We have $m - d = [\frac{n+1}{2}] + j - d = d + j + ([\frac{n+1}{2}] - 2d)$, and since our choice of d and the assumption $n > 0$, one can easily check that $2d > [\frac{n+1}{2}]$. Thus we do get that the sum (III.3.2) is twice a rational element. We would like to compute that sum modulo 4.

For any $i \geq d+j-m$, the factor s^{d+j-i} present in the ith summand is congruent modulo 2 to $S^{d+j-i}(\overline{x})$, which is represented by $\tilde{A}_i + \tilde{B}_i$, where

$$\tilde{A}_i := \sum_{k=0}^{[\frac{n}{2}]} \sum_{l=0}^{d+j-i} \binom{k}{d+j-i-l} (h^{d+j+k-i-l} \times \varepsilon_{k,l})$$

and

$$\tilde{B}_i := \sum_{k=0}^{j} \sum_{l=0}^{d+j-i} \binom{n+1-[\frac{n}{2}]+k}{d+j-i-l} (l_{[\frac{n}{2}]-k-d-j+i+l} \times \delta_{k,l})$$

where $\varepsilon_{k,l} \in \mathrm{CH}^{m-k+l}(\overline{Y})$ (resp. $\delta_{k,l} \in \mathrm{CH}^{m-k+l}(\overline{Y})$) is an integral representative of $S^l(x^k)$ (resp. of $S^l(x_{[\frac{n}{2}]-k})$), and we choose $\varepsilon_{k,l} = 0$ if $l > m-k$ (resp. $\delta_{k,l} = 0$ if $l > m+[\frac{n}{2}]-k-n$). Finally, in the case of even $m - j$, we choose $\varepsilon_{\frac{m-j}{2},\frac{m+j}{2}} = (\mathbf{x}^{\frac{m-j}{2}})^2$.

Furthermore, for any $i \geq d + j - m$, we have

$$h^{n-d+i} \cdot \tilde{B}_i = \sum_{k=0}^{j} \sum_{l=0}^{d+j-i} \binom{n+1-[\frac{n}{2}]+k}{d+j-i-l} (l_{[\frac{n}{2}]-k-n-j+l} \times \delta_{k,l}).$$

And we have

$$pr_*(l_{[\frac{n}{2}]-k-n-j+l} \times \delta_{k,l}) \neq 0 \Longrightarrow l = j + k + n - [\frac{n}{2}].$$

On the one hand, for any $i > d+j-m$, we have $d+j-i < m = n-[\frac{n}{2}]+j \leq j+k+n-[\frac{n}{2}]$. Hence, for any $0 \leq l \leq d + j - i$ and for any $0 \leq k \leq j$, we have $pr_*(l_{[\frac{n}{2}]-k-n-j+l} \times \delta_{k,l}) = 0$. Then, for any $i > d + j - m$, we get that $pr_*(h^{n-d+i} \cdot \tilde{B}_i) = 0$.

On the other hand, for $i = d + j - m$, we have $d + j - i = j + n - [n/2]$ and

$$l = j + k + n - [\frac{n}{2}] \Longleftrightarrow k = 0 \text{ and } l = d + j - i.$$

Thus, we have

$$pr_*(h^{n+j-m} \cdot \tilde{B}_{d+j-m}) = \delta_{0,m}.$$

Since $m > m + [n/2] - n$, one has $\delta_{0,m} = 0$.

Therefore, for any $i \geq d + j - m$, we have

$$pr_*(h^{n-d+i} \cdot \tilde{B}_i) = 0.$$

Then, for any $i > d + j - m$, the cycle h^{n-d+i} is divisible by 2. Hence, according to the multiplication rules in the ring $\mathrm{CH}(\overline{Q})$ described in Section III.1 and by doing the same computations as those done during the proof of Theorem III.2.2, for any $i > d + j - m$, we get the congruence

$$pr_*(h^{n-d+i} \cdot s^{d+j-i}) \equiv 2 \sum_{k=0}^{[\frac{n}{2}]} \binom{k}{d-i-k} \varepsilon_{k,j+k} \pmod 4.$$

Moreover, since $d - i - k \leq k$ if and only if $k \leq [\frac{m-j}{2}]$, for any $i > d + j - m$, we have the congruence

$$pr_*(h^{n-d+i} \cdot s^{d+j-i}) \equiv 2 \sum_{k=0}^{[\frac{m-j}{2}]} \binom{k}{d-i-k} \varepsilon_{k,j+k} \pmod 4. \tag{III.3.3}$$

Now, we would like to study the $(d + j - m)$th summand, that is to say the cycle $pr_*(h^{n+j-m} \cdot s^m)$ modulo 4. That is the purpose of the following lemma.

Lemma III.3.4. *One has*

$$pr_*(h^{n+j-m} \cdot s^m) \equiv \begin{cases} 2\varepsilon_{\frac{m-j}{2},\frac{m+j}{2}} + 2\mathbf{x}^0 \cdot \mathbf{x}_{[\frac{n}{2}]} \pmod 4 & \textit{if } m-j \textit{ is even} \\ 2\mathbf{x}^0 \cdot \mathbf{x}_{[\frac{n}{2}]} \pmod 4 & \textit{if } m-j \textit{ is odd.} \end{cases}$$

Proof. We recall that $s^m = (\overline{\mathbf{x}})^2$. Thus, we have

$$h^{n+j-m} \cdot s^m = h^{n+j-m} \cdot (A + B + C)$$

where

$$A := \sum_{0 \leq i,l \leq [\frac{n}{2}]} h^{i+l} \times (\mathbf{x}^i \cdot \mathbf{x}^l),$$

$$B := \sum_{0 \leq i,l \leq j} (l_{[\frac{n}{2}]-i} \cdot l_{[\frac{n}{2}]-l}) \times (\mathbf{x}_{[\frac{n}{2}]-i} \cdot \mathbf{x}_{[\frac{n}{2}]-l})$$

and

$$C := 2\sum_{i=0}^{[\frac{n}{2}]} h^i \times \mathbf{x}^i \cdot \sum_{l=0}^{j} l_{[\frac{n}{2}]-l} \times \mathbf{x}_{[\frac{n}{2}]-l}.$$

First of all, we have

$$h^{n+j-m} \cdot A = \sum_{0\leq i,l\leq[\frac{n}{2}]} h^{n+j-m+i+l} \times (\mathbf{x}^i \cdot \mathbf{x}^l).$$

Now we have $m = [\frac{n+1}{2}] + j$, so $n + j - m + i + l = [\frac{n}{2}] + i + l$. Thus, if $i \geq 1$ or $l \geq 1$, we have $n + j - m + i + l > [\frac{n}{2}]$, and in this case we have $h^{n+j-m+i+l} = 2l_{m-i-l-j}$. Therefore, the cycle $h^{n+j-m} \cdot A$ is equal to

$$h^{n+j-m} \times (\mathbf{x}^0)^2 + 4 \sum_{\substack{1\leq i,l\leq[\frac{n}{2}]\\ i\neq l}} l_{m-i-l-j} \times (\mathbf{x}^i \cdot \mathbf{x}^l) + 2\sum_{i=1}^{[\frac{n}{2}]} l_{m-j-2i} \times (\mathbf{x}^i)^2.$$

Then, since $n \geq 1$, we have $n + j - m \neq n$. It follows that $pr_*(h^{n+j-m} \times (\mathbf{x}^0)^2) = 0$.

Furthermore, we have

$$pr_*(\sum_{i=1}^{[\frac{n}{2}]} l_{m-j-2i} \times (\mathbf{x}^i)^2) = \begin{cases} (\mathbf{x}^{\frac{m-j}{2}})^2 & \text{if } m-j \text{ is even} \\ 0 & \text{if } m-j \text{ is odd.} \end{cases}$$

Therefore, $pr_*(h^{n+j-m} \cdot A)$ is congruent modulo 4 to $2\varepsilon_{\frac{m-j}{2},\frac{m+j}{2}}$ if $m - j$ is even, and to 0 if $m - j$ is odd.

Then, by dimensional reasons, we have $l_{[\frac{n}{2}]-i} \cdot l_{[\frac{n}{2}]-l} = 0$ if $i \geq 1$ or if $l \geq 1$. Hence, we have $B = (l_{[\frac{n}{2}]} \cdot l_{[\frac{n}{2}]}) \times (\mathbf{x}_{[\frac{n}{2}]})^2$. It follows that

$$h^{n+j-m} \cdot B = (l_0 \cdot l_{[\frac{n}{2}]}) \times (\mathbf{x}_{[\frac{n}{2}]})^2$$

and $l_0 \cdot l_{[\frac{n}{2}]} = 0$ by dimensional reasons. Therefore, we get that $h^{n+j-m} \cdot B = 0$.

Finally, we have

$$h^{n+j-m} \cdot C = 2\sum_{i=0}^{[\frac{n}{2}]} h^{n+j-m+i} \times \mathbf{x}^i \cdot \sum_{l=0}^{j} l_{[\frac{n}{2}]-l} \times \mathbf{x}_{[\frac{n}{2}]-l}.$$

Now for any $i \geq 1$, we have $n + j - m + i > [\frac{n}{2}]$, and in this case the cycle $h^{n+j-m+i}$ is divisible by 2. Thus, the element $h^{n+j-m} \cdot C$ is congruent modulo 4 to

$$2\sum_{l=0}^{j} (h^{[\frac{n}{2}]} \cdot l_{[\frac{n}{2}]-l}) \times (\mathbf{x}^0 \cdot \mathbf{x}_{[\frac{n}{2}]-l}),$$

and, by dimensional reasons, in the latest sum, each summand is 0 except the one corresponding to $l = 0$. Therefore, the cycle $h^{n+j-m} \cdot C$ is congruent modulo 4 to $2l_0 \times (\mathbf{x}^0 \cdot \mathbf{x}_{[\frac{n}{2}]})$. It follows that $pr_*(h^{n+j-m} \cdot C)$ is congruent modulo 4 to $2\mathbf{x}^0 \cdot \mathbf{x}_{[\frac{n}{2}]}$. We are done. □

By the congruence (III.3.3) and Lemma III.3.4, we deduce that the cycle

$$\sum_{i=d+j-m}^{d} pr_*(h^{n-d+i} \cdot s^{d+j-i})$$

is congruent modulo 4 to

$$2 \sum_{i=d+j-m}^{d} \sum_{k=0}^{[\frac{m-j}{2}]} \binom{k}{d-i-k} \varepsilon_{k,j+k} + 2\mathbf{x}^0 \cdot \mathbf{x}_{[\frac{n}{2}]}.$$

It follows that the cycle

$$2 \sum_{i=d+j-m}^{d} \sum_{k=0}^{[\frac{m-j}{2}]} \binom{k}{d-i-k} \varepsilon_{k,j+k} + 2\mathbf{x}^0 \cdot \mathbf{x}_{[\frac{n}{2}]}$$

is congruent modulo 4 to twice a rational element $\alpha \in \mathrm{CH}^{m+j}(\overline{Y})$. Then, we finish as in the proof of Theorem III.2.2. For every $k = 0, ..., [(m-j)/2]$, the total coefficient at $\varepsilon_{k,j+k}$ is 2^{k+1}, which is divisible by 4 for $k \geq 1$. Therefore, there exists a cycle $\gamma \in \mathrm{CH}^{m+j}(\overline{Y})$ such that

$$2\varepsilon_{0,j} + 2\mathbf{x}^0 \cdot \mathbf{x}_{[\frac{n}{2}]} = 2\alpha + 4\gamma,$$

hence, there exists an exponent 2 element $\delta \in \mathrm{CH}^{m+j}(\overline{Y})$ so that

$$\varepsilon_{0,j} + \mathbf{x}^0 \cdot \mathbf{x}_{[\frac{n}{2}]} = \alpha + 2\gamma + \delta.$$

Finally, since $\varepsilon_{0,j}$ is an integral representative of $S^j(x^0)$ and $\mathbf{x}^0$ (resp. $\mathbf{x}_{[\frac{n}{2}]}$) is an integral representative of x^0 (resp. of $x_{[\frac{n}{2}]}$), we get that $S^j(x^0) + x^0 \cdot x_{[\frac{n}{2}]}$ differs from a rational element by the class of an exponent 2 element of $\mathrm{CH}^{m+j}(\overline{Y})$. We are done with the proof of Proposition III.3.1. □

Remark III.3.5. In the case of $j = 0$, and if we make the extra assumption that the image of x under the composition

$$\mathrm{Ch}^m(Q \times Y) \to \mathrm{Ch}^m(Q_{F(Y)}) \to \mathrm{Ch}^m(Q_{\overline{F}(Y)}) \to \mathrm{Ch}^m(\overline{Q})$$

(the last passage is given by the inverse of the change of field isomorphism) is rational, then we get the stronger result that the cycle x^0 differs from a rational element by the class of an exponent 2 element of $\mathrm{CH}^m(\overline{Y})$. That is the subject of [23, Proposition 4.1].

Finally, the following theorem is a consequence of Proposition III.3.1.

Theorem III.3.6. *Assume that* $m = [\frac{n+1}{2}] + j$. *Let* $\overline{y}$ *be an* $F(Q)$*-rational element of* $\mathrm{Ch}^m(\overline{Y})$. *Then there exists a rational element* $z \in \mathrm{Ch}^j(\overline{Y})$ *such that* $S^j(\overline{y}) + \overline{y} \cdot z$ *is the sum of a rational element and the class modulo* 2 *of an integral element of exponent* 2.

Proof. The element $\overline{y}$ being $F(Q)$-rational, there exists $x \in \mathrm{Ch}^m(Q \times Y)$ mapped to $\overline{y}_{\overline{F}(Q)}$ under the composition

$$\mathrm{Ch}^m(Q \times Y) \to \mathrm{Ch}^m(Y_{F(Q)}) \to \mathrm{Ch}^m(Y_{\overline{F}(Q)}).$$

Moreover, the image $\overline{x} \in \mathrm{Ch}^m(\overline{Q} \times \overline{Y})$ of x decomposes as in (III.2.3). Thus, by Proposition III.3.1, the cycle $S^j(\overline{y}) + \overline{y} \cdot x_{[\frac{n}{2}]}$ is the sum of a rational element and the class of an element of exponent 2.

Finally, we have by Proposition II.1.7,

$$\overline{(pr)_*(x \cdot h^{[\frac{n}{2}]})} = pr_*(\overline{x} \cdot h^{[\frac{n}{2}]}) = x_{[\frac{n}{2}]} \in \overline{\mathrm{Ch}^j}(Y)$$

and we are done. □

III.4 Rationality on integral Chow groups - Main version

In this section we continue to use notation introduced in the previous sections and we deal with Question I.0.1 still in the context of smooth projective quadrics but for integral Chow groups CH.

In the aftermath of the Main Tool Lemma, A. Vishik adressed similar questions for integral Chow groups CH instead of Chow groups modulo 2. Namely, he proved the following integral version of the Main Tool Lemma (see [51, Theorem 3.1]).

Theorem III.4.1 (Vishik)**.** *Let Y be a smooth quasi-projective variety over a field F of characteristic 0 and let Q be a smooth projective quadric with $i_1(Q) > 1$. Then any $F(Q)$-rational element $\overline{y} \in \mathrm{CH}^m(\overline{Y})$, with $m < dim(Q)/2$, is rational.*

In the above statement, the assumption that the first Witt index $i_1(Q)$ of Q is strictly greater than 1 means that Q has a projective line defined over the generic point of Q (such quadrics are quite widespread).

Once again, the use of symmetric operations in the algebraic cobordism theory forced A. Vishik to work with a smooth quasi-projective variety Y over a field of characteristic 0.

However, we proved a similar result using only Chow theory itself, which allows one to get a valid statement in any characteristic different from 2 (since Chow theory does not rely on resolution of singularities) and to get rid of the assumption of quasi-projectivity for Y (see [9, Theorem 3.1]).

Theorem III.4.2. *Let Y be a smooth variety over a field F of characteristic different from 2 and let Q be a smooth projective quadric with $i_1(Q) > 1$. Then any $F(Q)$-rational element $\overline{y} \in \mathrm{CH}^m(\overline{Y})$, with $m < dim(Q)/2$, is the sum of a rational and an exponent 2 element.*

Once again, the version of A. Vishik remains stronger in the sense that his use of symmetric operations in the algebraic cobordism theory allowed him to get rid of the exponent 2 element appearing in our conclusion.

The main idea of the proof of Theorem III.4.2 (inspired by the proof of Theorem III.4.1) is as follows. First of all, we consider the $F(Q)$-rational element $\overline{y} \in \mathrm{CH}^m(\overline{Y})$ as the coordinate on h^0 of a rational cycles $\overline{x} \in \overline{\mathrm{CH}}^m(Q \times Y)$, and we use $\overline{x}$ mod 2, the 1-primordial cycle in $\overline{\mathrm{Ch}}(Q \times Q)$ and the Steenrod operations on Chow groups modulo 2 to form "special cycles". Then we choose carefully some integral representatives of these special cycles and we obtain $\overline{y}$ as a specific linear combination of rational cycles (modulo 2-torsion). Most of material needed for the proof can be found Chapter XIII and Chapter XV of the book [7].

Remark III.4.3. Let Q be a smooth projective quadric over F of positive dimension (in that case, Q is geometrically integral) given by a quadratic form φ. Since for isotropic Q, any $F(Q)$-rational element (in any codimension) is rational, one can make the assumption

that the quadric Q is anisotropic in order to prove Theorem III.4.2. In particular, Q is not completely split and one can consider the first Witt index $i_1(\varphi)$ of φ, which we simply denote as i_1.

Proof of Theorem III.4.2

We denote by n the dimension of Q.

The statement being trivial for negative m, we may assume that $m \geq 0$ in the proof. Let $\overline{y}$ be an $F(Q)$-rational element of $\mathrm{CH}^m(\overline{Y})$. Since over $\overline{F}$ the quadric Q becomes completely split and $m < n/2$, one can fix an element $x \in \mathrm{CH}^m(Q \times Y)$, as in the beginning of the proof of Theorem III.2.2, such that the image $\overline{x} \in \mathrm{CH}^m(\overline{Q} \times \overline{Y})$ of x decomposes as

$$\overline{x} = \sum_{i=0}^{m} h^i \times x^i, \tag{III.4.4}$$

where $x^i \in \mathrm{CH}^{m-i}(\overline{Y})$ is the coordinate of $\overline{x}$ on h^i and $x^0 = \overline{y}$ (see Remark III.1.2 with $r = k = m$).

Let $\pi \in \overline{\mathrm{Ch}}_{n+i_1-1}(Q^2)$ be the 1-primordial cycle (see [7, Definition 73.16] and paragraph right after [7, Theorem 73.26]). Since $i_1 > 1$, by [7, Proposition 83.2], we get that the cycle $(h^0 \times h^{i_1-1}) \cdot \pi \in \overline{\mathrm{Ch}}_n(Q^2)$ decomposes as

$$(h^0 \times h^{i_1-1}) \cdot \pi = \sum_{p=0}^{r} \varepsilon_p (h^{2p} \times l_{2p}) + \sum_{p=0}^{r} \varepsilon_p (l_{2p+i_1-1} \times h^{2p+i_1-1}), \tag{III.4.5}$$

where $\varepsilon_p \in \{0, 1\}$, $\varepsilon_0 = 1$, and $r = \lfloor \frac{d-i_1+1}{2} \rfloor$ with $d = \lfloor \frac{n}{2} \rfloor$. Thus, one can choose a rational integral representative $\overline{\gamma} \in \overline{\mathrm{CH}}_n(Q^2)$ of $(h^0 \times h^{i_1-1}) \cdot \pi$ such that $\overline{\gamma}$ decomposes as

$$\overline{\gamma} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_i (h^i \times l_i) + \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \beta_i (l_i \times h^i) + \delta (l_{\lfloor \frac{n}{2} \rfloor} \times l_{\lfloor \frac{n}{2} \rfloor}), \tag{III.4.6}$$

with some integers α_i, β_i and δ, where α_i is even for all odd i and α_0 is odd.

The element $\overline{\gamma}$ being rational, there exists $\gamma \in \mathrm{CH}_n(Q^2)$ mapped to $\overline{\gamma}$ under the change of field homomorphism $\mathrm{CH}_n(Q^2) \to \mathrm{CH}_n(\overline{Q}^2)$. The cycles γ and $\overline{\gamma}$ are considered here as correspondences of degree 0 (correspondences are defined in Appendix C).

Lemma III.4.7. *For any $i = 0, ..., m$, one can choose a rational integral representative $s^i \in \mathrm{CH}^{m+i}(\overline{Q} \times \overline{Y})$ of $S^i((\overline{x} \bmod 2) \circ (\overline{\gamma} \bmod 2))$ such that*

(i) *for any $0 \leq j \leq m$, $2s^{i,j}$ is rational , where $s^{i,j} \in \mathrm{CH}^{m+i-j}(\overline{Y})$ is the coordinate of s^i on h^j;*

(ii) *for any odd $0 \leq j \leq m$, $s^{i,j}$ is rational.*

Proof. First of all, since $m < n/2$, for any $j = 0, ..., m$, one has $h^{n-j} = 2l_j$. Therefore, for any rational cycle $s \in \mathrm{CH}(\overline{Q} \times \overline{Y})$, the element $2pr_*(l_j \cdot s)$ (where pr is the projection $Q \times Y \to Y$) is rational and (i) is proved.

Assume now that j is odd. By Proposition II.3.6(i), for any $i = 0, ..., m$, one has

$$S^i((\overline{x} \bmod 2) \circ (\overline{\gamma} \bmod 2)) = \sum_{k=0}^{m} \sum_{t=0}^{m} (S^t(\overline{x} \bmod 2) \cdot c_{i-k-t}(-T_Q)) \circ S^k(\overline{\gamma} \bmod 2). \quad \text{(III.4.8)}$$

For every $k = 0, ..., m$, let $\tilde{a}^k \in \mathrm{CH}^{n+k}(\overline{Q} \times \overline{Q})$ be a rational integral representative of $S^k(\overline{\gamma} \bmod 2) \in \mathrm{Ch}^{n+k}(\overline{Q} \times \overline{Q})$. We write $\tilde{a}^{k,j} \in \mathrm{CH}^{n+k-j}(\overline{Q})$ for the coordinate of $\tilde{a}^k$ on h^j. For every $k = 0, ..., m$ and every $t = 0, ..., m$, we choose a rational integral representative $d_{k,t} \in \mathrm{CH}^{m+i-k}(\overline{Q} \times \overline{Y})$ of $S^t(\overline{x} \bmod 2) \cdot c_{i-k-t}(-T_Q) \in \mathrm{Ch}^{m+i-k}(\overline{Q} \times \overline{Y})$. Thus, by the equation (III.4.8), the cycle

$$s^i := \sum_{k=0}^{m} \sum_{t=0}^{m} d_{k,t} \circ \tilde{a}^k \in \mathrm{CH}^{m+i}(\overline{Q} \times \overline{Y})$$

is a rational integral representative of $S^i((\overline{x} \bmod 2) \circ (\overline{\gamma} \bmod 2))$.

Moreover, for any $0 \leq k \leq m$, one has by (III.4.5)

$$S^k(\overline{\gamma} \bmod 2) = \sum_{p=0}^{r} \varepsilon_p S^k(h^{2p} \times l_{2p}) + \sum_{p=0}^{r} \varepsilon_p S^k(l_{i_1-1+2p} \times h^{i_1-1+2p}).$$

Therefore, for any $0 \leq k \leq m$, denoting as $a^{k,j} \in \mathrm{Ch}^{n+k-j}(\overline{Q})$ the coordinate of $S^k(\overline{\gamma} \bmod 2)$ on h^j, we have

$$a^{k,j} = \sum_{(p,t) \in \mathcal{E}_{k,j}} \varepsilon_p \binom{2p}{t} S^{k-t}(l_{2p}),$$

where $\mathcal{E}_{k,j} = \{(p,t) \in [\![0,r]\!] \times [\![0,k]\!] \mid 2p + t = j\}$.

Furthermore, since j is odd, for any $(p,t) \in \mathcal{E}_{k,j}$, the binomial coefficient $\binom{2p}{t}$ is even. Therefore, for any $0 \leq k \leq m$, we have $a^{k,j} = 0$ and, consequently, the cycle $\tilde{a}^{k,j} \in \mathrm{CH}^{n+k-j}(\overline{Q})$ is divisible by 2. Since $j - k < n/2$, the group $\mathrm{CH}^{n+k-j}(\overline{Q})$ is generated by l_{j-k} and $2l_{j-k} = h^{n+k-j}$ (see Section III.1). Hence, for any $0 \leq k \leq m$, the cycle $\tilde{a}^{k,j}$ is rational.

According to the composition rules of correspondences described in Appendix C, we have the identity

$$h^j \times s^{i,j} = \sum_{k=0}^{m} \sum_{t=0}^{m} d_{k,t} \circ (h^j \times \tilde{a}^{k,j}) = \sum_{k=0}^{m} \sum_{t=0}^{m} h^j \times pr_*(\tilde{a}^{k,j} \cdot d_{k,t}).$$

Therefore, since for any $0 \leq k \leq m$ and for any $0 \leq t \leq m$, the cycles $\tilde{a}^{k,j}$ and $d_{k,t}$ are rational, we get that $s^{i,j}$ is rational and (ii) is proved. □

Furthermore, we fix a smooth subquadric P of Q of dimension m; we write in for the imbedding

$$(P \hookrightarrow Q) \times id_Y : P \times Y \hookrightarrow Q \times Y.$$

Then, considering x as a correspondence, we set

$$z := in^*(x \circ \gamma) \in \mathrm{CH}^m(P \times Y).$$

In view of decompositions (III.4.4) and (III.4.6), we get that the image $\overline{z} \in \mathrm{CH}^m(\overline{P} \times \overline{Y})$ of z can be written as

$$\overline{z} = \sum_{i=0}^{m} \alpha_i \cdot h^i \times x^i$$

(we recall that the integer α_i is even for all odd i and that α_0 is odd). For every $i = 0, ..., m$, we set $z^i := \alpha_i \cdot x^i \in \mathrm{CH}^{m-i}(\overline{Y})$. Note that since $x^0 = \overline{y}$, the cycle z^0 is an odd multiple of $\overline{y}$.

Note also that since the Steenrod operations of cohomological type commute with in^* (see Theorem II.3.1), for every $i = 0, ..., m$, the cycle $in^*(s^i) \in \mathrm{CH}^{m+i}(\overline{P} \times \overline{Y})$ (with s^i as in Lemma III.4.7) is a rational integral representative of $S^i(\overline{z} \bmod 2) \in \mathrm{Ch}^{m+i}(\overline{P} \times \overline{Y})$.

Lemma III.4.9. *For any $\lfloor (m+1)/2 \rfloor \leq m' \leq m$, the cycle*

$$\sum_{i=0}^{m'} \binom{m'+i+1}{i} s^{m'-i,m'-i} \in \mathrm{CH}^m(\overline{Y})$$

is the sum of a rational element $\overline{\delta_{m'}}$ and an exponent 2 element.

Proof. For any $\lfloor (m+1)/2 \rfloor \leq m' \leq m$, we can fix a smooth subquadric P' of P of dimension m'; we write $in_{m'}$ for the imbedding

$$(P' \hookrightarrow P) \times id_Y : P' \times Y \hookrightarrow P \times Y.$$

By Lemma III.2.4, one has

$$S^{m'} pr_{m'*} in_{m'}{}^*(z \bmod 2) = \sum_{i=0}^{m'} pr_{m'*}(c_i(-T_{P'}) \cdot in_{m'}{}^* S^{m'-i}(z \bmod 2)) \quad \text{in } \mathrm{Ch}^m(Y)$$

(where $T_{P'}$ is the tangent bundle of P', c_i are the Chern classes, and $pr_{m'}$ is the projection $P' \times Y \to Y$).

If $m' \geq \lfloor (m+1)/2 \rfloor + 1$, since $pr_{m'*} in_{m'}{}^*(z \bmod 2) \in \mathrm{Ch}^{m-m'}(Y)$ and $m - m' < m'$, we have $S^{m'} pr_{m'*} in_{m'}{}^*(z \bmod 2) = 0$. Therefore, we get

$$\sum_{i=0}^{m'} pr_{m'*}(c_i(-T_{P'}) \cdot in_{m'}{}^* S^{m'-i}(z \bmod 2)) = 0 \quad \text{in } \mathrm{Ch}^m(Y).$$

Furthermore, by [7, Lemma 78.1], for any $i = 0, ..., m'$, one has $c_i(-T_{P'}) \equiv \binom{m'+i+1}{i} h^i \pmod 2$. By combining the congruence for Chern classes with the observation just prior to the statement of the lemma, we deduce that

$$\sum_{i=0}^{m'} \binom{m'+i+1}{i} pr_{m'*}(h^i \cdot in_{m'}{}^* in^*(s^{m'-i}))$$

is twice a rational element $\overline{\delta_{m'}} \in \mathrm{CH}^m(\overline{Y})$. Since, by the projection formula (Proposition II.2.2), for any $i = 0, ..., m'$, one has $pr_{m'*}(h^i \cdot in_{m'}{}^* in^*(s^{m'-i})) = pr_*(h^{n-m'+i} \cdot s^{m'-i}) = 2s^{m'-i,m'-i}$, we are done with the case $m' \geq \lfloor (m+1)/2 \rfloor + 1$.

If $m' = \lfloor (m+1)/2 \rfloor$ and m is odd, we still have $m - m' < m'$ and we can do the same reasoning as in the first case. If $m' = \lfloor (m+1)/2 \rfloor$ and m is even, we have $m - m' = m' = m/2$, and in this case, we have

$$S^{m/2} pr_{m/2_*} in_{m/2}{}^*(z \bmod 2) = (pr_{m/2_*} in_{m/2}{}^*(z \bmod 2))^2.$$

Therefore, by the same reasoning as in the first case, there exists $\delta_{m/2} \in \mathrm{CH}^m(Y)$ such that

$$2\sum_{i=0}^{m/2} \binom{\frac{m}{2}+i+1}{i} s^{\frac{m}{2}-i,\frac{m}{2}-i} = 2\overline{\delta_{m/2}} + (pr_{m/2_*} in_{m/2}{}^*(\overline{z}))^2.$$

Moreover, we have

$$(pr_{m/2_*} in_{m/2}{}^*(\overline{z}))^2 = (2z^{\frac{m}{2}})^2 = 2 \cdot (2z^{\frac{m}{2}2}),$$

and since for any $i = 0, ..., m$, the cycle $2z^i = pr_{m*}(h^{m-i} \cdot \overline{z})$ is rational, the cycle

$$2z^{\frac{m}{2}2} = pr_{m*}(\overline{z}^2) - 4 \sum_{\substack{0 \leq i \leq m \\ i \neq \frac{m}{2}}} z^i \cdot z^{m-i}$$

is rational also and we are done with the proof of Lemma III.4.9. □

Lemma III.4.10. *For any $j = 0, ..., m$, one can choose an integral representative $v^j \in \mathrm{CH}^m(\overline{Y})$ of $S^j(z^j \bmod 2)$ such that*

(i) *the cycle $2v^j$ is rational and v^0 is an odd multiple of $\overline{y}$;*

(ii) *the cycle v^j is rational for odd j;*

(iii) *for any $k = 0, ..., m$, one has $s^{k,k} = \sum_{j=0}^{k} a_j^k v^j$, where a_j^k is the binomial coefficient $\binom{j}{k-j}$.*

Proof. We induct on j. For $j = 0$, one has $2z^0 = pr_{m*}(h^m \cdot \overline{z})$. Hence the element $2z^0$ is rational, and since the cycle z^0 is an odd multiple of $\overline{y}$, we choose $v^0 := z^0$. For $j = 1$, one has

$$S^1((\overline{x} \bmod 2)\circ(\overline{\gamma} \bmod 2)) = \sum_{i=0}^{m} h^i \times S^1(z^i \bmod 2) + \sum_{i=0}^{m} i \cdot h^{i+1} \times (z^i \bmod 2) \in \mathrm{Ch}^{m+1}(\overline{Q} \times \overline{Y}).$$

In the latter expression, the coordinate on h^1, whose $s^{1,1}$ is an integral representative, is $S^1(z^1 \bmod 2)$. Since, by Lemma III.4.7(ii), the cycle $s^{1,1}$ is rational, we choose $v^1 := s^{1,1}$. Assume that the representatives $v^0, v^1, ..., v^{j-1}$ are already built.

One has

$$S^j((\overline{x} \bmod 2) \circ (\overline{\gamma} \bmod 2)) = \sum_{k=0}^{j} \sum_{i=0}^{m} S^k(h^i) \times S^{j-k}(z^i \bmod 2) \in \mathrm{Ch}^{m+j}(\overline{Q} \times \overline{Y}).$$

In the latter expression, the coordinate on h^j, whose $s^{j,j}$ is an integral representative, is

$$a^j_j \cdot S^j(z^j \bmod 2) + a^j_{j-1} \cdot S^{j-1}(z^{j-1} \bmod 2) + \cdots + a^j_0 \cdot S^0(z^0 \bmod 2),$$

where $a^j_i = \binom{i}{j-i}$ for any $0 \le i \le j$. Therefore, the cycle

$$v^j := s^{j,j} - (a^j_{j-1} \cdot v^{j-1} + \cdots + a^j_0 \cdot v^0)$$

is an integral representative of $S^j(z^j \bmod 2)$. Moreover, the element

$$2s^{j,j} = 2(v^j + a^j_{j-1} \cdot v^{j-1} + \cdots + a^j_0 \cdot v^0)$$

is rational by Lemma III.4.7(i). By the induction hypothesis, we get that the cycle $2v^j$ is rational. Furthermore, if j is odd, then the cycle $s^{j,j}$ is rational by Lemma III.4.7(ii), and for any even $0 \le i \le j$, the binomial coefficient a^j_i is even. Therefore, by the induction hypothesis, we get that the cycle v^j is rational. We are done with the proof of Lemma III.4.10. □

Finally, the following lemma will lead to the conclusion. Denote by $\eta(X)$ the power series $\sum_{i\ge 0} \eta_i \cdot X^i$ in variable X, where $\eta_l = (-1)^l \binom{2l+1}{l}$.

Lemma III.4.11. *For any polynomial $f \in \mathbb{Z}[X]$ of degree $\le \lfloor m/2 \rfloor$, the linear combination*

$$\sum_{j=0}^{m} g_{m-j} \cdot v^j$$

is the sum of a rational element and an exponent 2 element, where $g(X) = \sum_l g_l \cdot X^l$ is the power series $f(X) \cdot \eta(X)$.

Proof. Let $f = \sum f_k \cdot X^k \in \mathbb{Z}[X]$ be some polynomial of degree $\leq \lfloor m/2 \rfloor$. Consider the element

$$\varepsilon := \sum_{m'=\lfloor \frac{m+1}{2} \rfloor}^{m} f_{m-m'} \cdot \delta_{m'} \in \mathrm{CH}^m(Y),$$

with $\delta_{m'}$ as in Lemma III.4.9. Then, we have

$$2\overline{\varepsilon} = 2 \sum_{m'=\lfloor \frac{m+1}{2} \rfloor}^{m} f_{m-m'} \sum_{i=0}^{m'} \binom{m'+i+1}{i} s^{m'-i,m'-i}.$$

Furthermore, by Lemma III.4.10(iii), for any $k = 0, ..., m$, one has $s^{k,k} = \sum_{j=0}^{k} a_j^k v^j$. Hence, we get the identity

$$2\overline{\varepsilon} = 2 \sum_{m'=\lfloor \frac{m+1}{2} \rfloor}^{m} f_{m-m'} \sum_{j=0}^{m'} \left(\sum_{l=0}^{m'-j} \binom{m'+l+1}{l} \binom{j}{m'-l-j} \right) v^j,$$

and the latter identity can be rewritten as

$$2\overline{\varepsilon} = 2 \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{j=0}^{m} f_i \cdot c_{i,j} \cdot v^j,$$

where $c_{i,j} := \sum_{l=0}^{m-i-j} \binom{m-i+l+1}{l} \binom{j}{m-i-j-l}$. If $m-i-j < 0$, then we have $c_{i,j} = \eta_{m-i-j} = 0$. Otherwise – if $m-i-j \geq 0$ – we set $k := m-i-j$, and we have

$$c_{i,j} \equiv \sum_{l=0}^{k} \binom{-k-j-2}{l} \binom{j}{k-l} \pmod 2,$$

which is congruent modulo 2 to $\binom{-k-2}{k}$ by the Chu-Vandermonde Identity (see [1, Corollary 2.2.3]). Therefore, since $\binom{-k-2}{k} \equiv \binom{2k+1}{k} \pmod 2$, we get that, for any $i = 0, ..., \lfloor m/2 \rfloor$ and for any $j = 0, ..., m$,

$$c_{i,j} \equiv \eta_{m-i-j} \pmod 2.$$

Thus, since by Lemma III.4.10(i), for any $j = 0, ...m$, the cycle $2v^j$ is rational, we get that there exists an element $\delta \in \mathrm{CH}^m(Y)$ such that

$$2\overline{\delta} = 2 \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{j=0}^{m} f_i \cdot \eta_{m-i-j} \cdot v^j = 2 \sum_{j=0}^{m} g_{m-j} \cdot v^j,$$

where $g(X) = \sum_l g_l \cdot X^l$ is the power series $f(X) \cdot \eta(X)$. Hence, there exists an exponent 2 element $\lambda \in \mathrm{CH}^m(\overline{Y})$ such that

$$\sum_{j=0}^{m} g_{m-j} \cdot v^j = \overline{\delta} + \lambda,$$

and we are done. $\square$

We finish now the proof of Theorem III.4.2. By [51, Lemma 3.13], there exists a polynomial $f \in \mathbb{Z}[X]$ of degree $\leq \lfloor m/2 \rfloor$ such that the power series $g(X) := f(X) \cdot \eta(X)$ has an odd coefficient g_m at X^m and even coefficients g_{m-j} (with even j) at smaller monomials of the same parity. Applying Lemma III.4.11 to this polynomial f, we get that there exists an exponent 2 element $\lambda \in \mathrm{CH}^m(\overline{Y})$ such that the cycle

$$\sum_{j=0}^{m} g_{m-j} \cdot v^j - \lambda$$

is rational. Since for any $j = 1, ..., m$, the cycle $2v^j$ is rational and v^j is rational for all odd j, the product $g_{m-j} \cdot v^j$, with $j \geq 1$, is always rational. Therefore, we get that the cycle

$$g_m \cdot v^0 - \lambda$$

is rational. Furthermore, since g_m is odd, the cycle $2v^0$ is rational and $v^0 = \alpha_0 \cdot \overline{y}$, where α_0 is odd, there exist an integer k and an element $\delta \in \mathrm{CH}^m(Y)$ such that $g_m \cdot v^0 = \overline{y} + 2k\overline{y} + \overline{\delta}$. Finally, note that the cycle $2\overline{y}$ is rational since it is equal to $pr_*(h^n \cdot \overline{x})$.

This concludes the proof of Theorem III.4.2.

III.5 Rationality on integral Chow groups - A stronger version

In this section, we continue to use notation introduced in the previous section. The following result is stronger than Theorem III.4.2 although its statement is less eloquent.

Let K/F be an extension and X be an F-variety. In the following proof, an element $x \in \mathrm{CH}^*(X_K)$ is called *rational* if it is in the image of the change of field homomorphism $\mathrm{CH}^*(X) \to \mathrm{CH}^*(X_K)$.

In the same way as before, the following theorem is a generalization of [51, Proposition 3.7] to any field of characteristic different from 2 (although, putting aside characteristic, Theorem III.5.1 is still weaker than the original version in the sense that an exponent 2 element appears in the conclusion).

Theorem III.5.1. *Assume that $m < n/2$ and $i_1 > 1$, and let E/F be an extension such that $i_0(Q_E) > m$. Then, for any $y \in \mathrm{CH}^m(Y_{F(Q)})$ there exists $\delta \in \mathrm{CH}^m(Y)$ and an exponent 2 element $\lambda \in \mathrm{CH}^m(Y_{E(Q)})$ such that $y_{E(Q)} = \delta_{E(Q)} + \lambda$.*

Proof. We proceed the same way as in the proof of Theorem III.4.2.

Let us fix an element $x \in \mathrm{CH}^m(Q \times Y)$ mapped to y under the surjection

$$\mathrm{CH}^m(Q \times Y) \twoheadrightarrow \mathrm{CH}^m(Y_{F(Q)}).$$

Since $i_0(Q_E) > m$, by Remark III.1.2 (applied with $r = k = m$), the image $x_{E(Q)} \in \mathrm{CH}^m(Q_{E(Q)} \times Y_{E(Q)})$ of x decomposes as

$$x_{E(Q)} = \sum_{i=0}^{m} h^i \times x^i$$

where $x^i \in \mathrm{CH}^{m-i}(Y_{E(Q)})$ is the coordinate of $x_{E(Q)}$ on h^i.

The image of x under the composition

$$\mathrm{CH}^m(Q \times Y) \to \mathrm{CH}^m(Q_E \times Y_E) \to \mathrm{CH}^m(Y_{E(Q)})$$

is x^0. Therefore, by the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{CH}^m(Q_E \times Y_E) & \longrightarrow & \mathrm{CH}^m(Y_{E(Q)}) \\ \uparrow & & \uparrow \\ \mathrm{CH}^m(Q \times Y) & \longrightarrow & \mathrm{CH}^m(Y_{F(Q)}) \end{array}$$

we get that $x^0 = y_{E(Q)}$ and we want to prove that there exists $\delta \in \mathrm{CH}^m(Y)$ and an exponent 2 element $\lambda \in \mathrm{CH}^m(Y_{E(Q)})$ such that $x^0 = \delta_{E(Q)} + \lambda$.

Let $\pi \in \mathrm{Ch}_{n+i_1-1}(Q^2)$ be an element mapped to the 1-primordial cycle under the homomorphism $\mathrm{Ch}^*(Q) \to \mathrm{Ch}^*(\overline{Q})$. By [7, Proposition 83.2], there is no cycle of type $h^j \times l_j$ with odd j appearing in the decomposition of $(h^0 \times h^{i_1-1}) \cdot \pi_{\overline{E}(Q)} \in \mathrm{Ch}_n(Q^2_{\overline{E}(Q)})$ (and the cycle $h^0 \times l_0$ appears).

Moreover, since the coefficients near the cycles contained in the decomposition of $(h^0 \times h^{i_1-1}) \cdot \pi_{E(Q)} \in \mathrm{Ch}_n(Q^2_{E(Q)})$ given by Remark III.1.3 (with $k = m$) do not change when going over $\overline{E}(Q)$, the cycle $(h^0 \times h^{i_1-1}) \cdot \pi_{E(Q)}$ can be uniquely written as a linear combination of cycles of type $h^j \times l_j$ with even $j \in [\![0, m]\!]$ (and the coefficient near $h^0 \times l_0$ is 1) , of cycles of type $l_j \times h^j$ (where $j \in [\![0, m]\!]$), and of a cycle $\rho \in \mathrm{Ch}^n(Q^2_{E(Q)})$ whose coordinate on h^j (as well as coordinate on l_j) is 0 for $j \in [\![0, m]\!]$.

Thus, fixing a rational integral representative $\gamma_{E(Q)} \in \mathrm{CH}_n(Q^2_{E(Q)})$ of $(h^0 \times h^{i_1-1}) \cdot \pi_{E(Q)}$, we get that the integral coefficient α_j near the cycle $h^j \times l_j$ contained in the decomposition of $\gamma_{E(Q)}$ (given by Remark III.1.3, with $k = m$), is even for all odd j, and that α_0 is odd.

Let $\gamma \in \mathrm{CH}_n(Q^2)$ mapped to $\gamma_{E(Q)}$ under the homomorphism $\mathrm{CH}_n(Q^2) \to \mathrm{CH}_n(Q^2_{E(Q)})$. We have the following lemma, whose the statement and the proof are very close to Lemma III.4.7.

Lemma III.5.2. *For any $i = 0, ..., m$, one can choose a rational integral representative $s^i \in \mathrm{CH}^{m+i}(Q_{E(Q)} \times Y_{E(Q)})$ of $S^i((x_{E(Q)} \bmod 2) \circ (\gamma_{E(Q)} \bmod 2))$ such that*

(i) *for any $0 \le j \le m$, $2s^{i,j}$ is rational , where $s^{i,j}$ is the coordinate of s^i on h^j;*

(ii) *for any odd* $0 \leq j \leq m$, $s^{i,j}$ *is rational.*

Proof. We use same notation as those introduced during the proof of Lemma III.4.7. One can prove (i) exactly as the same way as Lemma III.4.7(i). We need the following proposition to prove (ii).

Proposition III.5.3. *Let* X *be a smooth* F*-variety and let* ρ *be an element of* $\mathrm{Ch}(Q \times X)$ *such that for any* $j = 0, ..., r$, *its coordinate* ρ^j *on* h^j *is* 0. *Then, for any integer* k *and for any* $j = 0, ..., r$, *the coordinate of* $S^k(\rho)$ *on* h^j *is* 0.

Proof. We induct on k. For $k = 0$, one has $S^0 = \mathrm{Id}$. Assume that the statement is true till the rank k and let $j \in [\![0, r]\!]$. By the Cartan formula (Corollary II.3.5), one has

$$S^{k+1}(l_j \cdot \rho) = l_j \cdot S^{k+1}(\rho) + \sum_{i=1}^{k+1} S^i(l_j) \cdot S^{k+1-i}(\rho).$$

Since for any $i = 1, ..., k+1$, the cycle $S^i(l_j)$ is a multiple of l_{j-i} (see [7, Corollary 78.5]), by the induction hypothesis, we get

$$pr_*(l_j \cdot S^{k+1}(\rho)) = pr_*(S^{k+1}(l_j \cdot \rho)).$$

Furthermore, by Proposition II.3.2, one has

$$S^{k+1} \circ pr_*(l_j \cdot \rho) = \sum_{i=0}^{k+1} pr_*(c_{k+1-i}(-T_Q) \cdot S^i(l_j \cdot \rho)),$$

and since $pr_*(l_j \cdot \rho) = 0$, we deduce that

$$pr_*(l_j \cdot S^{k+1}(\rho)) = \sum_{i=0}^{k} a_i,$$

where $a_i = pr_*(c_{k+1-i}(-T_Q) \cdot S^i(l_j \cdot \rho))$. We are going to prove that for any $i = 0, ..., k$, one has $a_i = 0$. Let i be an integer in $[\![0, k]\!]$. Since by [7, Lemma 78.1], the cycle $c_{k+1-i}(-T_Q)$ is a multiple of h^{k+1-i}, it suffices to show that $pr_*(h^{k+1-i} \cdot S^i(l_j \cdot \rho)) = 0$.

By the Cartan Formula and [7, Corollary 78.5], the cycle $pr_*(h^{k+1-i} \cdot S^i(l_j \cdot \rho))$ is a linear combination of cycles of type $pr_*(h^{k+1-i} \cdot l_{j-t} \cdot S^{i-t}(\rho))$, where $t \in [\![0, i]\!]$. Since for any $t = 0, ..., i$, one has $h^{k+1-i} \cdot l_{j-t} = l_{j-t-(k+1-i)}$, we are done by the induction hypothesis. □

We finish now the proof of Lemma III.5.2. Assume that j is odd. Since by Proposition III.5.3, for any $k = 0, ..., m$, the coordinate of $S^k(\rho)$ on h^j is 0, the only fact that we have to explain here to prove (ii) (i.e what is new compared to the proof of Lemma III.4.7) is why the corresponding cycle $\tilde{a}^{k,j} \in \mathrm{CH}^{n+k-j}(Q_{E(Q)})$ is rational.

For the same reasons as in the proof of Lemma III.4.7, the cycle $\tilde{a}^{k,j} \in \mathrm{CH}^{n+k-j}(Q_{E(Q)})$ is divisible by 2. Moreover, since one has $j - k \leq m < i_0(Q_E)$, the cycle l_{j-k} is defined

over E and it is consequently defined over $E(Q)$. Furthermore, since $j-k \leq m < n/2$, the group $\mathrm{CH}^{n+k-j}(Q_{E(Q)})$ is free with basis $\{l_{j-k}\}$ (as well as the group $\mathrm{CH}^{n+k-j}(Q_{\overline{E(Q)}})$) and therefore the restriction homomorphism

$$\mathrm{CH}^{n+k-j}(Q_{E(Q)}) \longrightarrow \mathrm{CH}^{n+k-j}(Q_{\overline{E(Q)}})$$

is injective (it is even an isomorphism). Since $2l_{j-k} = h^{n+k-j}$, we deduce that any cycle of $\mathrm{CH}^{n+k-j}(Q_{E(Q)})$ divisible by 2 is rational. Thus, for any $0 \leq k \leq m$, the cycle $\tilde{a}^{k,j}$ is rational and we finish as in the proof of Lemma III.4.7. $\square$

Now, one can finish the proof of Theorem III.5.1 exactly the same way as the proof of Theorem III.4.2 replacing $\overline{F}$ by $E(Q)$. $\square$

Chapter IV

Exceptional projective homogeneous varieties

The purpose of this chapter is to prove the following theorem dealing with rationality of algebraic cycles over function field of some exceptional projective homogeneous varieties (see [10, Theorem 1.1]). This theorem gives an answer to Question I.0.2 in the context of those exceptional projective homogeneous varieties. We refer to [3] for an introduction to linear algebraic groups.

Theorem IV.0.1. *Let G be a linear algebraic group of type F_4 or E_8 over a field F and let X be a projective homogeneous G-variety. For any equidimensional variety Y, the change of field homomorphism*

$$\mathrm{Ch}(Y) \to \mathrm{Ch}(Y_{F(X)}),$$

where Ch *is the Chow group modulo p, with $p = 3$ when G is of type F_4 and $p = 5$ when G is of type E_8, is surjective in codimension $< p + 1$.*

It is also surjective in codimension $p+1$ for a given Y provided that $1 \notin \deg \mathrm{Ch}_0(X_{F(\zeta)})$ for each generic point $\zeta \in Y$.

In this chapter, a linear algebraic group G over a field F is a twisted form ${}_\xi G_0$ by mean of a cocycle $\xi \in H^1(F, G_0)$ (see Appendix D), where G_0 is a split linear algebraic group of the same type as G (one says that G is of *inner type*). A *projective homogeneous G-variety* X is a twisted form ${}_\xi(G_0/P)$ of G_0/P for P is a parabolic subgroup of G_0. The proof of Theorem IV.0.1 is given in Section IV.4.

The above statement is to put in relation with the result [26, Theorem 4.3] by N. Karpenko and A. Merkurjev, where *generic splitting varieties* have been considered. In characteristic 0, Theorem IV.0.1 is contained in [26, Theorem 4.3]. In an earlier paper (see [55, Corollary 1.4]), K. Zainoulline proved the first conclusion of Theorem IV.0.1 (modulo torsion) in characteristic 0 if G is of type F_4. Our result is valid in any characteristic.

The method of proof is basically the method used to prove [26, Theorem 4.3] combined with a motivic decomposition result for *generically split* projective homogeneous varieties

due to V. Petrov, N. Semenov and K. Zainoulline (see [40, Theorem 5.17]) and involving the *Rost motive*. This is described in Section IV.2.

In Section IV.3, we give the definition of the J-invariant and we present some properties about Chow groups of the Rost motive of groups of strongly inner type (e.g F_4 and E_8) with maximal J-invariant. Those properties make the method particularly suitable for groups of type F_4 and E_8.

The method also relies on a linkage between the γ-filtration on the Grothendieck ring of projective homogeneous varieties and Chow groups, in the spirit of [14].

In the aftermath of Theorem IV.0.1, we get the following statement dealing with integral Chow groups (see [26, Theorem 4.5]).

Corollary IV.0.2. *We use notation introduced in Theorem IV.0.1. If $p \in \deg \mathrm{CH}_0(X)$ (for example, if F is p-special), then for any equidimensional variety Y, the change of field homomorphism*

$$\mathrm{CH}(Y) \to \mathrm{CH}(Y_{F(X)})$$

is surjective in codimension $< p+1$.

It is also surjective in codimension $p+1$ for a given Y provided that $1 \notin \deg \mathrm{Ch}_0(X_{F(\zeta)})$ for each generic point $\zeta \in Y$.

Remark IV.0.3. Our method of proof for Theorem IV.0.1 works for groups of type G_2 as well (with $p = 2$). However, the case of G_2 can be treated in a more elementary way if $\mathrm{char}(F) \neq 2$.

Indeed, it is known that to each group G of type G_2 one can associate a 3-fold Pfister quadratic form π such that, denoting by X_π the associated Pfister quadric, the variety X has a rational point over $F(X_\pi)$ and vice versa (see [46, Theorem 9]). Thus, for any equidimensional variety Y, the right and the bottom maps in the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}(Y) \arrow[r] \arrow[d] & \mathrm{CH}(Y_{F(X)}) \arrow[d] \\
\mathrm{CH}(Y_{F(X_\pi)}) \arrow[r] & \mathrm{CH}(Y_{F(X_\pi \times X)})
\end{tikzcd}$$

are isomorphisms. Furthermore, by Remark III.0.2, the change of field homomorphism $\mathrm{CH}(Y) \to \mathrm{CH}(Y_{F(X_\pi)})$ is surjective in codimension < 3.

IV.1 Filtrations on Grothendieck ring of projective homogeneous varieties

In this section, we prove two statements concerning filtrations on Grothendieck ring of certain class of projective homogeneous varieties. Those propositions play a crucial role in the proof of Theorem IV.0.1 (see Section IV.4). We use notation introduced in Section II.4.

Coincidence of filtrations

We have seen in Section II.4 that for any any smooth variety X over a field F and for any integer $i \geq 0$, the term $\gamma^i(X)$ of codimension i of the γ-filtration on the Grothendieck ring $K(X)$ is contained in the the term $\tau^i(X)$ of codimension i of the topological filtration.

The following proposition provides us a way to get the existence of a variety X for which the two filtrations actually coincide when dealing with a certain class of projective homogeneous varieties. The method of proof is largely inspired by the proof of [24, Theorem 6.4 (2)] by N. Karpenko and A. Merkurjev.

Proposition IV.1.1. *Let G_0 be a split connected semisimple linear algebraic group over a field F and let B be a Borel subgroup of G_0. There exist an extension E/F and a cocycle $\xi \in H^1(E, G_0)$ such that the topological filtration and the γ-filtration on $K({}_\xi(G_0/B))$ coincide.*

Proof. Let n be an integer such that $G_0 \subset \mathbf{GL}_n$ and let us set $S := \mathbf{GL}_n$ and $E := F(S/G_0)$. We denote by $\mathbf{T}$ the E-variety $S \times_{S/G_0} \mathrm{Spec}(E)$ given by the generic fiber of the projection $S \to S/G_0$. Note that since $\mathbf{T}$ is clearly a G_0-torsor over E, there exists a cocycle $\xi \in H^1(E, G_0)$ such that the smooth projective variety $X := \mathbf{T}/B_E$ is isomorphic to ${}_\xi(G_0/B)$. We claim that the Chow ring $\mathrm{CH}(X)$ is generated by Chern classes.

Indeed, the morphism $h : X \to S/B$ induced by the canonical G_0-equivariant morphism $\mathbf{T} \to S$ being a localization, the associated pull-back

$$h^* : \mathrm{CH}(S/B) \longrightarrow \mathrm{CH}(X)$$

is surjective. Furthermore, the ring $\mathrm{CH}(S/B)$ itself is generated by Chern classes: by [24, §6,7] there exists a morphism

$$\mathbb{S}(T^*) \longrightarrow \mathrm{CH}(S/B), \tag{IV.1.2}$$

(where $\mathbb{S}(T^*)$ is the symmetric algebra of the group of characters T^* of a split maximal torus $T \subset B$) with its image generated by Chern classes. Moreover, the morphism (IV.1.2) is surjective by [24, Proposition 6.2]. Since h^* is surjective and Chern classes commute with pull-backs, the claim is proved.

We show now that the two filtrations on $K(X)$ coincide by induction on codimension. Let $i \geq 0$ and assume that $\tau^{i+1}(X) = \gamma^{i+1}(X)$. Since for any $j \geq 0$, one has $\gamma^j(X) \subset \tau^j(X)$, the induction hypothesis implies that

$$\gamma^{i/i+1}(X) \subset \tau^{i/i+1}(X).$$

Thus, the ring $\mathrm{CH}(X)$ being generated by Chern classes, one has $\gamma^{i/i+1}(X) = \tau^{i/i+1}(X)$ by (II.4.1). Therefore one has $\tau^i(X) = \gamma^i(X)$ and the proposition is proved. □

Note that this result remains true when one consider a *special* parabolic subgroup P, i.e a parabolic subgroup P for which $H^1(F, P)$ is trivial, instead of B (see [24]).

γ-filtration on Borel variety of strongly inner groups

We say that a linear algebraic group G over a field F is of *strongly inner type* if it is a twisted form ${}_\xi G_0$ by mean of a cocycle $\xi \in H^1(F, G_0)$ with G_0 a split simply-connected group of the same type as G.

Remark IV.1.3. Assume that G_0 is simply-connected (e.g F_4 and E_8) and let B be a Borel subgroup of G_0. Consider an extension E/F and a cocycle $\xi \in H^1(E, G_0)$. By the result [38, Theorem 2.2.(2)] of I. Panin, the change of field homomorphism

$$K({}_\xi(G_0/B)_E) \to K({}_\xi(G_0/B)_{\overline{E}}) \simeq K(G_0/B),$$

for Grothendieck ring of Borel varieties of strongly inner groups, with $\overline{E}$ an algebraic closure of E, is an isomorphism.

Therefore, since the γ-filtration is defined in terms of Chern classes and the latter commute with pull-backs, the terms of the γ-filtration on $K({}_\xi(G_0/B)_E)$ do not depend nor on the extension E/F neither on the choice of $\xi \in H^1(E, G_0)$.

Chow groups and topological filtration

Now, we prove a result which will be used in Section IV.4 to get the second conclusion of Theorem IV.0.1.

We recall that for any smooth variety X over a field F, for any prime p, and for any $i < p+1$, the canonical surjection $pr_p^i : \mathrm{Ch}^i(X) \twoheadrightarrow \tau_p^{i/i+1}(X)$ is an isomorphism by the Riemann-Roch Theorem without denominators (see Remark II.4.2). The following proposition extends this fact to $i = p+1$ provided that X is a projective homogeneous variety under a certain class of linear algebraic groups (containing the groups of type F_4 and E_8) and $p > 2$.

Proposition IV.1.4. *Let X be a projective homogeneous variety under a semisimple adjoint algebraic group G of strongly inner type. For any prime $p > 2$ the canonical surjection*

$$\mathrm{Ch}^{p+1}(X) \twoheadrightarrow \tau_p^{p+1/p+2}(X),$$

is injective.

Proof. First of all, we have seen in Section II.4 that the surjection pr_p^{p+1} coincides with the composite $q_{p+1} \circ q_p$

$$E_p^{p+1,-p-1}(X) \pmod p \twoheadrightarrow E_{p+1}^{p+1,-p-1}(X) \pmod p \twoheadrightarrow \frac{E_{p+1}^{p+1,-p-1}(X)}{\mathrm{Im}(\delta_{p+1})} \pmod p.$$

Furthermore, since any prime divisor l of the order of δ_{p+1} is such that $l-1$ divides p and $p > 2$, the differential δ_{p+1} is of prime to p order. It follows that q_{p+1} is an isomorphism. Therefore, we have shown that pr_p^{p+1} is injective if and only if q_p is an isomorphism.

Now let us consider the following inclusions given by the Brown-Gersten-Quillen structure

$$E_\infty^{1,-2}(X) \subset \cdots \subset E_3^{1,-2}(X) \subset E_2^{1,-2}(X).$$

By the very definition of the Brown-Gersten-Quillen spectral sequence, one has $E_\infty^{1,-2}(X) = E_2^{1,-2}(X)$ if and only if for any $r \geq 2$ the differential starting from $E_r^{1,-2}(X)$ is zero. In particular, the equality $E_\infty^{1,-2}(X) = E_2^{1,-2}(X)$ implies that the differential δ_p (starting from $E_p^{1,-2}(X)$) is zero and consequently that q_p is an isomorphism. Therefore, the following lemma completes the proof of the proposition.

Lemma IV.1.5. *Let G be a semisimple adjoint algebraic group of strongly inner type. For any projective homogeneous G-variety X, the inclusion $E_\infty^{1,-2}(X) \subset E_2^{1,-2}(X)$ given by the Brown-Gersten-Quillen spectral sequence is an equality.*

Proof. On the one hand, by the very definition, the group $E_\infty^{1,-2}(X)$ is the first quotient $K_1^{(1/2)}(X)$ of the topological filtration on $K_1(X)$. On the other hand, one has $E_2^{1,-2}(X) = A^1(X, K_2)$ (for any integers p and q, one has $E_2^{p,q}(X) = A^p(X, K_{-q})$).

First, we claim that the natural map

$$A^0(X, K_1) \otimes \mathrm{CH}^1(X) \to A^1(X, K_2) \tag{IV.1.6}$$

is an isomorphism. Indeed, since G_{sep} has only trivial Tits algebras (because it is adjoint and simply-connected), by [33, Theorem], one has

$$A^1(X, K_2) \simeq A^1(X_{\mathrm{sep}}, K_2)^\Gamma,$$

where Γ is the absolute Galois group of F. Moreover, since the variety X_{sep} is cellular, by [33, Proposition 1], one has

$$A^1(X_{\mathrm{sep}}, K_2) \simeq K_1 F_{\mathrm{sep}} \otimes \mathrm{CH}^1(X_{\mathrm{sep}}).$$

Note that since X is smooth, the Picard group $\mathrm{Pic}(X_{\mathrm{sep}})$ is identified with $\mathrm{CH}^1(X_{\mathrm{sep}})$. Furthermore, since G_{sep} has only trivial Tits algebras, the group $\mathrm{Pic}(X_{\mathrm{sep}})$ is rational by [36, Proposition 2.3]. Therefore one has $\mathrm{CH}^1(X) \simeq \mathrm{CH}^1(X_{\mathrm{sep}})$ and since $(K_1 F_{\mathrm{sep}})^\Gamma = K_1 F = A^0(X, K_1)$, one has $A^0(X, K_1) \otimes \mathrm{CH}^1(X) \simeq A^1(X, K_2)$ and the claim is proved.

Now, it is known that $\mathrm{CH}^1(X_{\mathrm{sep}})$ is a free abelian group of finite rank (see [45, §2] for example). Let us denote by φ the isomorphism

$$(F^\times)^{\oplus k} \longrightarrow A^1(X, K_2)$$

such that for any $a \in (F^\times)^{\oplus k}$ the element $\varphi(a)$ corresponds by (IV.1.6) to $\sum_{i=0}^k \pi_i(a) \otimes e_i$ in $A^0(X, K_1) \otimes \mathrm{CH}^1(X)$, where $(e_i)_{1 \leq i \leq k}$ is a basis of $\mathrm{CH}^1(X)$ and $\pi_i : (F^\times)^{\oplus k} \to F^\times$ is the standard projection.

Then it suffices to find a homomorphism $\psi : (F^\times)^{\oplus k} \to K_1^{(1/2)}(X)$ such that the diagram

$$\begin{tikzcd} K_1^{(1/2)}(X) \arrow[rr, hook] & & A^1(X, K_2) \\ & (F^\times)^{\oplus k} \arrow[ul, "\psi"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

is commutative to get the conclusion (as in [20, §4]). The homomorphism ψ defined as follow is suitable (and ψ is necessarily defined this way). For every $i = 0, \ldots, k$, let $j_i : Z_i \subset X$ be a subvariety of codimension 1 such that $[Z_i] = e_i$ in $\mathrm{CH}^1(X)$ and let p_i be the structure morphism $Z_i \to \mathrm{Spec}(F)$. Then we set $\psi = \sum_{i=1}^k \psi_i$, with

$$\psi_i : \ (F^\times)^{\oplus k} \xrightarrow{\pi_i} F^\times \xrightarrow{{p_i}^*} K_1(Z_i) \xrightarrow{{j_i}^*} K_1^1(X) \longrightarrow K_1^{1/2}(X).$$

□

This concludes the proof of Proposition IV.1.4. □

IV.2 Generically split projective homogeneous varieties

In this section, we present a motivic decomposition result due to V. Petrov, N. Semenov and K. Zainoulline (see [40, Theorem 5.17]) and we introduce in a more general context the basis of the method we will use in Section IV.4 to prove Theorem IV.0.1.

Let X be a projective homogeneous variety under an algebraic group G over a field F. The variety X is said to be *generically split* if the group G splits over the generic point of X (e.g any projective homogeneous variety X under a group G of type F_4 or E_8 which has no splitting extension of degree coprime to 3 or 5 respectively).

Assume furthermore that G is semisimple, then such a generically split G-variety X presents the interest that for any prime p, its Chow motive $\mathcal{M}(X, \mathbb{Z}/p\mathbb{Z})$ with coefficients in $\mathbb{Z}/p\mathbb{Z}$ decomposes as a sum of twists of an indecomposable motive $\mathcal{R}_p(G)$, called *Rost motive*, by mean of the following theorem.

Theorem IV.2.1 (Petrov, Semenov, Zainoulline)**.** *Let G be a semisimple linear algebraic group over a field F and let p be a prime. Then for any generically split projective homogeneous variety X under G one has the motivic decomposition*

$$\mathcal{M}(X, \mathbb{Z}/p\mathbb{Z}) \simeq \bigoplus_{i \geq 0} \mathcal{R}_p(G)(i)^{\oplus a_i},$$

where $\Sigma_{i\geq 0} a_i t^i = P(\mathrm{CH}(\overline{X}), t)/P(\mathrm{CH}(\overline{\mathcal{R}_p(G)}), t)$, with $P(-, t)$ the Poincaré polynomial.

It follows immediatly from Theorem IV.2.1 that for any integer k and any extension L/F, one has the following decomposition concerning Chow groups

$$\mathrm{Ch}^k(X_L) \simeq \bigoplus_{i\geq 0} \mathrm{Ch}^{k-i}(\mathcal{R}_p(G)_L)^{\oplus a_i}. \tag{IV.2.2}$$

If the group G is of strongly inner type (e.g F_4 and E_8) and has no splitting field of degree coprime to p, the indecomposable motive $\mathcal{R}_p(G)$ coincides with a *generalized Rost motive* and

$$\overline{\mathcal{R}_p(G)} \simeq \bigoplus_{i=0}^{p-1} \mathbb{Z}/p\mathbb{Z}\,(i(p+1))$$

(see [54, (5.4-5.5)]).

Remark IV.2.3. The Poincaré polynomial $P(\mathrm{CH}(\overline{\mathcal{R}_p(G)}), t)$ only depends on the J-invariant modulo p of G defined in the next section. Moreover, the Poincaré polynomial $P(\mathrm{CH}(\overline{X}), t)$ can be computed thanks to the Solomon's Theorem (see [45, §2.5]) when one knows the parabolic subgroup P determining X. Note that this allows one to easily compute the coefficient a_i's of the decomposition (IV.2.2) when X is a twisted form $\mathfrak{B}$ of G_0/B, with B a Borel subgroup of the split group G_0 of the same type as G. Beside, except for the following proposition, we will only apply decomposition (IV.2.2) to $\mathfrak{B}$ in the sequel. Note that since the group G splits over any extension E/F over which $\mathfrak{B}$ admits a rational point, the projective homogeneous variety $\mathfrak{B}$ is in particular generically split.

The following statement, which is obtained by combining the decomposition (IV.2.2) with Proposition II.2.5, constitutes the first step in our way to prove Theorem IV.0.1.

Proposition IV.2.4. *Let G be a semisimple linear algebraic group over a field F. Let p be a prime and $\mathcal{R}_p(G)$ the associated Rost motive of G. If for any extension L/F, the change of field*

$$\mathrm{Ch}(\mathcal{R}_p(G)) \longrightarrow \mathrm{Ch}(\mathcal{R}_p(G)_L)$$

is surjective in codimension $< k$ then for any equidimensional variety Y and for any generically split projective homogeneous G-variety X, the change of field

$$\mathrm{Ch}(Y) \to \mathrm{Ch}(Y_{F(X)})$$

is surjective in codimension $< k$.

IV.3 J-invariant

The notion of J-invariant, at first, of an orthogonal group, has been introduced by A. Vishik in [50], and he notably used it to get his result on the u-invariant of a field (see [48]).

In the aftermath of the works of A. Vishik, V. Petrov, N. Semenov and K. Zainoulline have generalized in [40] the notion of J-invariant to an arbitrary semisimple algebraic group. In this section, we recall some material about the J-invariant of a semisimple algebraic group and then we study some connections between the J-invariant and Chow groups of the Rost motive $\mathcal{R}_p(G)$ introduced in the previous section.

Let G_0 be a split semisimple linear algebraic group over a field F and let B be a Borel subgroup of G_0. Let $G = {}_\xi G_0$ be a twisted form of G_0 given by a cocycle $\xi \in H^1(F, G_0)$ and let $\mathfrak{B} = {}_\xi(G_0/B)$ be the associated Borel variety.

Definition and properties

Most of material and facts described here can be found in [40, §4].

Let us fix a prime p and write Ch for the Chow ring with coefficients in $\mathbb{Z}/p\mathbb{Z}$. We consider the ring homomorphism given by the composite of pullbacks

$$\mathrm{Ch}(\mathfrak{B}) \longrightarrow \mathrm{Ch}(\overline{\mathfrak{B}}) \xrightarrow{\pi} \mathrm{Ch}(\overline{G}) \ .$$

The map π is surjective by [18, p.21]. Moreover, an explicit description of the ring $\mathrm{Ch}(\overline{G})$ is known for all types of G and all torsion primes p of G (see [18, Definition 3]). Namely, by [19, Theorem 3], there exists an integer $r \geq 1$ such that one has

$$\mathrm{Ch}^*(\overline{G}) = \mathbb{Z}/p\mathbb{Z}[x_1, \dots x_r]/(x_1^{p^{k_1}}, \dots, x_r^{p^{k_r}}),$$

where, for every $i = 1, \dots, r$, the variable x_i is of codimension the coprimary part d_i of the ith *p-exceptional degree* of G_0 while the integer k_i is the respective p-primary power. In the case where a prime p is not a torsion prime of G one has $\mathrm{Ch}^*(\overline{G}) = \mathbb{Z}/p\mathbb{Z}$.

We give now the definition of the J-invariant of G in the case where $\mathrm{Ch}^*(\overline{G})$ has only one generator (if G is of type F_4 or E_8 for example) although this definition can easily be generalized for arbitrary r (see [40, Definition 4.6]).

Definition IV.3.1. Let p be a tosion prime of G such that $r = 1$. The *J-invariant $J_p(G)$ of G modulo p* is the smallest non-negative integer j such that $x_1^{p^j} \in \pi(\overline{\mathrm{Ch}}(\mathfrak{B}))$.

It follows immediatly from the definition that for any extension E/F, one has $J_p(G_E) \leq J_p(G)$.

For arbitrary r, the J-invariant $J_p(G)$ of G consists of a r-tuple of integers $(j_1, \dots, j_r)$ with $j_i \leq k_i$ for any $1 \leq i \leq r$. We will need the following fact about the J-invariant (see [40, Corollary 6.7]).

Property IV.3.2. One has the equivalence

(i) The J-invariant $J_p(G)$ of G modulo p is trivial, i.e $(j_1, \dots, j_r) = (0, \dots, 0)$;

(ii) G splits over a finite extension of prime to p degree;

(iii) $\mathcal{R}_p(G) = \mathbb{Z}/p\mathbb{Z}$ (Tate motive).

Groups of strongly inner type with maximal J-invariant

In this subsection, we assume furthermore that G is simple of strongly inner type.

For any torsion prime p of G, we say that the J-invariant $J_p(G) = (j_1, \dots, j_r)$ modulo p of G is *maximal* if for every $i = 1, \dots, r$, one has $j_i = k_i$.

In this subsection, we present some properties about Chow groups of the Rost motive of simple linear algebraic groups of strongly inner type (e.g F_4 and E_8) with maximal J-invariant modulo some torsion prime. In the next section, we will combine those properties with the method described in Proposition IV.2.4 to prove Theorem IV.0.1.

Lemma IV.3.3. *Let G be a simple linear algebraic group of strongly inner type such that its J-invariant $J_p(G)$ is maximal. Then one has*

(i) $p = 3$ *or* 5;

(ii) $\mathrm{Ch}^2(\mathcal{R}_p(G)) \simeq \mathbb{Z}/p\mathbb{Z}$ *and* $\mathrm{Ch}^3(\mathcal{R}_p(G)) = 0$.

Proof. Since $J_p(G)$ is maximal, by [16, Example 5.3], the cocycle $\xi \in H^1(F, G_0)$ corresponds to a *generic* G_0-torsor in the sense of [16] (see also Appendix D). Thus, by [14, Proposition 3.2] and [13, pp. 31, 133], one has $\mathrm{Tors}_p\mathrm{CH}^2(\mathfrak{B}) \neq 0$ (we need the assumption strongly inner to use material from [14, §3]). The conclusion is given by [14, Proposition 5.4]. □

Lemma IV.3.4. *Let G be a simple linear algebraic group of strongly inner type such that its J-invariant $J_p(G)$ is maximal and let L/F be an extension such that $J_p(G_L) = J_p(G)$. Then one has*

(i) $\mathrm{Ch}^2(\mathcal{R}_p(G)_L) \simeq \mathbb{Z}/p\mathbb{Z}$ *and* $\mathrm{Ch}^3(\mathcal{R}_p(G)_L) = 0$;

(ii) *the change of field* $\mathrm{Ch}^2(\mathfrak{B}) \to \mathrm{Ch}^2(\mathfrak{B}_L)$ *is an isomorphism.*

Proof. Since $J_p(G_L)$ is maximal then by Lemma IV.3.3 one has $\mathrm{Ch}^2(\mathcal{R}_p(G_L)) \simeq \mathbb{Z}/p\mathbb{Z}$ and $\mathrm{Ch}^3(\mathcal{R}_p(G_L)) = 0$. Moreover, since $J_p(G_L) = J_p(G)$, one has $\mathcal{R}_p(G_L) \simeq \mathcal{R}_p(G)_L$ (see [40, Proposition 5.18 (i)]) and (i) is proved.

We show now that the change of field $\mathrm{Ch}^2(\mathfrak{B}) \to \mathrm{Ch}^2(\mathfrak{B}_L)$ is an isomorphism. We use material and notation introduced in Sections II.4 and IV.1 about filtrations on Grothendieck rings. Since $J_p(G) = J_p(G_L)$ is maximal, the cocycles ξ and ξ_L correspond to generic G_0-torsors and one consequently has $\gamma^3(\mathfrak{B}) = \tau^3(\mathfrak{B})$ and $\gamma^3(\mathfrak{B}_L) = \tau^3(\mathfrak{B}_L)$ (see [14, Theorem 3.1(ii)]). In particular, it follows that

$$\gamma_p^{2/3}(\mathfrak{B}) = \tau_p^{2/3}(\mathfrak{B}) \quad \text{and} \quad \gamma_p^{2/3}(\mathfrak{B}_L) = \tau_p^{2/3}(\mathfrak{B}_L).$$

Therefore, since $2 < p + 1$, the homomorphism $\mathrm{Ch}^2(\mathfrak{B}) \to \mathrm{Ch}^2(\mathfrak{B}_L)$ coincides with

$$\mathrm{Ch}^2(\mathfrak{B}) \simeq \gamma_p^{2/3}(\mathfrak{B}) \to \gamma_p^{2/3}(\mathfrak{B}_L) \simeq \mathrm{Ch}^2(\mathfrak{B}_L)$$

and the center arrow is an isomorphism by Remark IV.1.3. □

Lemma IV.3.5. *In this statement, one has p=5. Let G be a simple linear algebraic group of strongly inner type such that its J-invariant $J_5(G)$ is maximal and let L/F be an extension such that $J_5(G_L) = J_5(G)$. Then one has*

$$\mathrm{Ch}^4(\mathcal{R}_5(G)_L) = 0 \quad \textit{and} \quad \mathrm{Ch}^5(\mathcal{R}_5(G)_L) = 0.$$

Proof. Since $J_5(G_L) = J_5(G)$ one has $\mathcal{R}_5(G)_L = \mathcal{R}_5(G_L)$ and it suffices to prove that $\mathrm{Ch}^4(\mathcal{R}_5(G)) = \mathrm{Ch}^5(\mathcal{R}_5(G)) = 0$.

By Proposition IV.1.1 there exist an extension E/F and a cocycle $\xi' \in H^1(E, G_0)$ such that the topological filtration and the γ-filtration on $K(\mathfrak{B}')$, with $\mathfrak{B}' = {}_{\xi'}(G_0/B)$, coincide. Let us set $G' = {}_{\xi'}G_0$.

We claim that $J_5(G') \neq (0, \dots, 0)$. Indeed, assume that $J_5(G') = (0, \dots, 0)$. In that case, one has $\mathcal{R}_5(G') = \mathbb{Z}/5\mathbb{Z}$ (Tate motive) and the isomorphism (IV.2.2) gives that $\mathrm{Ch}^2(\mathfrak{B}') \simeq \mathbb{Z}/5\mathbb{Z}^{\oplus a_2}$. Since $2 < p+1$, it implies that $\gamma_5^{2/3}(\mathfrak{B}') \simeq \mathbb{Z}/5\mathbb{Z}^{\oplus a_2}$, and consecutively $\gamma_5^{2/3}(\mathfrak{B}) \simeq \mathbb{Z}/5\mathbb{Z}^{\oplus a_2}$ by Remark IV.1.3. However, we have $\gamma_5^{2/3}(\mathfrak{B}) = \tau_5^{2/3}(\mathfrak{B})$ (because $\gamma^3(\mathfrak{B}) = \tau^3(\mathfrak{B})$ since $\xi \in H^1(F, G_0)$ is generic). Thus, we have $\mathrm{Ch}^2(\mathfrak{B}) \simeq \mathbb{Z}/5\mathbb{Z}^{\oplus a_2}$ which contradicts $\mathrm{Ch}^2(\mathcal{R}_5(G)) \simeq \mathbb{Z}/5\mathbb{Z}$ and the claim is proved (we recall that for any $i < 6 = p+1$, one has $\tau_5^{i/i+1}(X) \simeq \mathrm{Ch}^i(X)$).

We now compute the groups $\gamma_5^{i/i+1}(\mathfrak{B}')$ for $i = 3, 4, 5$. We recall that one has $K(\mathfrak{B}') \simeq K(G_0/B)$ by Remark IV.1.3. Furthermore, the description of the free group $K(G_0/B)$ in terms of generators does not depend on the characteristic of the base field (see [5, Lemma 13.3(4)]). Thus, in order to compute the groups $\gamma_5^{i/i+1}(\mathfrak{B}')$ for $i = 3, 4, 5$, since $J_5(G') \neq (0, \dots, 0)$, one can use the following theorem (adapted from [26, Theorem RM.10] to our situation)

Theorem IV.3.6 (Karpenko, Merkurjev)**.** *Let H be a semisimple linear algebraic group over a field of characteristic 0 and let p be a torsion prime of H. If $J_p(H) \neq (0, \dots, 0)$ then*

$$\mathrm{Ch}^j(\mathcal{R}_p(H)) \simeq \begin{cases} \mathbb{Z}/p\mathbb{Z} & \textit{if } j = 0 \textit{ or } j = k(p+1) - p + 1,\ 1 \leq k \leq p-1 \\ 0 & \textit{otherwise}, \end{cases}$$

which combined with (IV.2.2) gives that

$$\gamma_5^{i/i+1}(\mathfrak{B}') \simeq \mathrm{Ch}^i(\mathfrak{B}') \simeq \mathbb{Z}/5\mathbb{Z}^{\oplus(a_{i-2}+a_i)} \quad \text{for } i = 3, 4, 5$$

(where the first isomorphism is due to $i < p+1$). Therefore, we get

$$\gamma_5^{i/i+1}(\mathfrak{B}) \simeq \mathbb{Z}/5\mathbb{Z}^{\oplus(a_{i-2}+a_i)} \quad \text{for } i = 3, 4, 5.$$

Thus, since $\tau_5^{3/4}(\mathfrak{B}) \simeq \mathrm{Ch}^3(\mathfrak{B})$, the isomorphism (IV.2.2) for $k = 3$ gives that $\tau_5^{3/4}(\mathfrak{B}) = \gamma_5^{3/4}(\mathfrak{B})$. Since the γ-filtration is contained in the topological one, we get

$$\tau_5^4(\mathfrak{B}) = \gamma_5^4(\mathfrak{B}),$$

which implies the existence of an exact sequence

$$0 \to (\tau_5^5(\mathfrak{B})/\gamma_5^5(\mathfrak{B})) \to \gamma_5^{4/5}(\mathfrak{B}) \to \tau_5^{4/5}(\mathfrak{B}) \to 0.$$

Thus, since $\tau_5^{4/5}(\mathfrak{B}) \simeq \mathrm{Ch}^4(\mathfrak{B})$, by applying the isomorphism (IV.2.2) for $k = 4$, we get a surjection

$$\mathbb{Z}/5\mathbb{Z}^{\oplus(a_2+a_4)} \twoheadrightarrow \mathrm{Ch}^4(\mathcal{R}_5(G)) \oplus \mathbb{Z}/5\mathbb{Z}^{\oplus(a_2+a_4)},$$

which implies that $\mathrm{Ch}^4(\mathcal{R}_5(G)) = 0$.

We prove that $\mathrm{Ch}^5(\mathcal{R}_5(G)) = 0$ by proceeding in exactly the same way. □

IV.4 Proof of the result

In this section, we prove Theorem IV.0.1.

Remark IV.4.1. Let G be a semisimple linear algebraic group over a field F and let X be a projective homogeneous G-variety. The F-variety X is *A-trivial* in the sense of [26, Definition 2.3] (see [26, Example 2.5]), i.e for any extension L/F with $X(L) \neq \emptyset$, the degree homomorphism $\deg : \mathrm{CH}_0(X_L) \to \mathbb{Z}$ is an isomorphism.

Since by [26, Lemma 2.9], any A-trivial variety X with $1 \in \deg \mathrm{Ch}_0(X)$ is such that for any equidimensional variety Y the change of field homomorphism $\mathrm{Ch}(Y) \to \mathrm{Ch}(Y_{F(X)})$ is an isomorphism (in any codimension, with Ch the Chow group modulo p, for any prime p), one can assume that $1 \notin \deg \mathrm{Ch}_0(X)$ in order to prove Theorem IV.0.1.

Now, we know from [40, Table 4.13] that if G is of type F_4 or E_8 then the J-invariant $J_p(G)$ of G is equal to (0) or (1) and in the latter case, the J-invariant modulo p is maximal (with $p = 3$ if G is of type F_4 and $p = 5$ if G is of type E_8). However, the assumption $J_p(G) = (0)$ is equivalent to the existence of a splitting field K/F of G of degree coprime to p (see Property IV.3.2). In that case one has $\mathrm{Ch}_0(X) \simeq \mathrm{Ch}_0(X_K)$ and consequently $1 \in \deg \mathrm{Ch}_0(X)$. Thus, under the assumption $1 \notin \deg \mathrm{Ch}_0(X)$, one necessarly has $J_p(G) = (1)$ and that is why we can assume $J_p(G)$ nontrivial, i.e maximal, in the sequel.

Since for G with nontrivial $J_p(G)$ the prime p must divide the degree of any finite splitting extension, every projective homogeneous variety under a group of type F_4 or E_8 with nontrivial $J_p(G)$ ($p = 3$ for the type F_4 and $p = 5$ for the type E_8) is generically split by [40, Example 3.6]. Then, by Proposition IV.2.4, the first conclusion of Theorem IV.0.1 is a direct consequence of the following proposition.

Proposition IV.4.2. *Let G be a linear algebraic group of type F_4 or E_8 over a field F such that $J_p(G)$ is nontrivial, with $p = 3$ if G is of type F_4 and $p = 5$ if G is of type E_8. Then, for any extension L/F, the change of field*

$$\mathrm{Ch}(\mathcal{R}_p(G)) \longrightarrow \mathrm{Ch}(\mathcal{R}_p(G)_L), \tag{IV.4.3}$$

where $\mathcal{R}_p(G)$ is the associated Rost motive, is surjective in codimension $< p + 1$.

Proof. First of all, the homomorphism (IV.4.3) is clearly surjective in codimension 0 since one has $\mathrm{Ch}^0(\mathcal{R}_p(G)_L) = \mathbb{Z}/p\mathbb{Z}$ for any extension L/F. Then, $\mathrm{Ch}^1(\overline{\mathfrak{B}})$ is identified with the Picard group $\mathrm{Pic}(\overline{\mathfrak{B}})$ and is rational (for the same reason as in the proof of Lemma IV.1.5). Furthermore, one can compute the coefficients a_i's in the decomposition (IV.2.2): we get $a_0 = 1$ and $a_1 = \mathrm{rank}(G) = \mathrm{rank}(\mathrm{CH}^1(\overline{\mathfrak{B}}))$ (see Remark IV.2.3). Thus, the isomorphism (IV.2.2) implies that $\mathrm{Ch}^1(\mathcal{R}_p(G)_L) = 0$ for any extension L/F. Therefore, we have already shown that the homomorphism (IV.4.3) is surjective in codimension 0 and 1.

Now we show that it is surjective in codimension 2 and 3 (which proves the proposition for G of type F_4). Since $J_p(G)$ is maximal, one has $\mathrm{Ch}^2(\mathcal{R}_p(G)) \simeq \mathbb{Z}/p\mathbb{Z}$ and $\mathrm{Ch}^3(\mathcal{R}_p(G)) = 0$ by Lemma IV.3.3. Moreover, since $J_p(G_L) \leq J_p(G)$ for any extension L/F, one has $J_p(G_L) = (0)$ or $J_p(G_L) = J_p(G)$ (i.e is maximal).

If $J_p(G_L) = J_p(G)$ then one has $\mathrm{Ch}^2(\mathcal{R}_p(G)_L) \simeq \mathbb{Z}/p\mathbb{Z}$ and $\mathrm{Ch}^3(\mathcal{R}_p(G)_L) = 0$ by Lemma IV.3.4(i) and the homomorphism (IV.4.3) is clearly surjective in codimension 3. Thanks to the decomposition (IV.2.2) and Lemma IV.3.4(ii), we see that it is also surjective in codimension 2.

If $J_p(G_L) = (0)$ then on the one hand one has $\mathcal{R}_p(G_L) = \mathbb{Z}/p\mathbb{Z}$ and on the other hand the motivic decomposition given in [40, Proposition 5.18 (i)] implies the following decomposition on Chow groups for any integer k

$$\mathrm{Ch}^k(\mathcal{R}_p(G)_L) \simeq \bigoplus_{i=0}^{p-1} \mathrm{Ch}^{k-i(p+1)}(\mathcal{R}_p(G_L)). \tag{IV.4.4}$$

In particular, one has $\mathrm{Ch}^k(\mathcal{R}_p(G)_L) = 0$ for $k = 2$ or 3 and the conclusion follows.

For G of type E_8, we now prove that $\mathrm{Ch}(\mathcal{R}_5(G)) \longrightarrow \mathrm{Ch}(\mathcal{R}_5(G)_L)$ is surjective in codimension 4 and 5 by showing that one has $\mathrm{Ch}^4(\mathcal{R}_5(G)_L) = \mathrm{Ch}^5(\mathcal{R}_5(G)_L) = 0$ for any extension L/F. By Lemma IV.3.5, this is true when $J_p(G_L) = J_p(G)$. Moreover, if $J_p(G_L) = 0$ then one has $(\mathcal{R}_5(G_L) = \mathbb{Z}/5\mathbb{Z}$ and the isomorphism (IV.4.4) implies that $\mathrm{Ch}^4(\mathcal{R}_5(G)_L) = \mathrm{Ch}^5(\mathcal{R}_5(G)_L) = 0$. This completes the proof of Proposition IV.4.2. □

Finally, using the same notation as in the statement of Theorem IV.0.1, we want to prove the second conclusion of Theorem IV.0.1. Since for any generic point ζ of Y, one has

$$1 \notin \deg \mathrm{Ch}_0(X_{F(\zeta)}) \Rightarrow J_p(G_{F(\zeta)}) = (1),$$

by Proposition IV.2.4 and in view of what has already been done, it is sufficient to prove the following lemma to get the second conclusion.

Lemma IV.4.5. *Let G be a linear algebraic group of type F_4 or E_8 over a field F such that $J_p(G)$ is nontrivial, with $p = 3$ if G is of type F_4 and $p = 5$ if G is of type E_8. Then one has*

$$\mathrm{Ch}^{p+1}(\mathcal{R}_p(G)) = 0.$$

Proof. Thanks to Proposition IV.1.4, one can prove the lemma by proceeding in exactly the same way Lemma IV.3.5 has been proved. □

This concludes the proof of Theorem IV.0.1.

Chapter V

Principal homogeneous space for $\mathbf{SL_1}(A)$

Let A be a central simple algebra over a field F and let $\mathrm{Nrd} : A^\times \to F^\times$ be the reduced norm homomorphism . We recall that the homomorphism $F^\times \to H^1(F, \mathbf{SL_1}(A))$, associating to $c \in F^\times$ the $\mathbf{SL_1}(A)$-torsor X_c given by the equation $\mathrm{Nrd} = c$, is surjective (with kernel $\mathrm{Nrd}(A^\times)$) – see [15, Proposition 2.7.3] for instance.

The main purpose of this chapter is to prove the following theorem dealing with rationality of algebraic cycles over function field of $\mathbf{SL_1}(A)$-torsors.

Theorem V.0.1. *Let A be a central simple algebra of prime degree p over a field F and let X be a $\mathbf{SL_1}(A)$-torsor. Then*

(i) *for any equidimensional F-variety Y, the change of field homomorphism*

$$\mathrm{CH}(Y) \to \mathrm{CH}(Y_{F(X)}),$$

where CH *is the integral Chow group, is surjective in codimension $< p+1$.*

(ii) *it is also surjective in codimension $p+1$ for a given Y provided that the variety $X_{F(\zeta)}$ does not have any closed point of prime to p degree for each generic point $\zeta \in Y$.*

Note that the previous statement is a version of Theorem IV.0.1 for principal homogeneous space for $\mathbf{SL_1}(A)$ (it gives an answer to Question I.0.2 in this particular context). Besides we describe in Section V.3 how Theorem IV.0.1 is related to Theorem V.0.1.

The method of proof mainly relies on Proposition II.2.5. We need to introduce some new material before giving the proof.

V.1 Preliminaries

Chow groups of principal homogeneous spaces for $\mathbf{SL_1}(A)$

Let X be a $\mathbf{SL_1}(A)$-torsor and let p be a prime. One has $K(X) = \mathbb{Z}$ by the result [39, Theorem A] of I. Panin and consequently, for $i \geq 1$, the term $\tau^i(X)$ of the topological filtration on $K(X)$ is equal to zero. Therefore, for any $1 \leq i \leq p$, one has $\mathrm{Ch}^i(X) = 0$, with Ch the Chow group modulo p (see Remark II.4.2).

Moreover, by the result [47, Theorem 2.7] of A. Suslin, one has $\mathrm{CH}^i(\mathbf{SL_p}) = 0$ for any $i \geq 1$. Hence, for A of degree p (then there exists a splitting field of A of degree p), it follows by transfert argument that $p \cdot \mathrm{CH}^i(X) = 0$ for any $i \geq 1$. Therefore, for X a $\mathbf{SL_1}(A)$-torsor, with A of prime degree p, one has

$$\mathrm{CH}^i(X) = 0 \quad \text{for any} \quad 1 \leq i \leq p. \tag{V.1.1}$$

Note that, by Proposition II.2.5, this gives Theorem V.0.1(i) already.

Brown-Gersten-Quillen spectral sequence

We recall that for any smooth variety X and any $i \geq 1$, the epimorphism pr^i coincides with the edge homomorphism of the spectral Brown-Gersten-Quillen structure $E_2^{i,-i}(X) \Rightarrow K(X)$ (see [41, §7]), that is to say

$$pr^i : \mathrm{CH}^i(X) \simeq E_2^{i,-i}(X) \twoheadrightarrow \cdots \twoheadrightarrow E_{i+1}^{i,-i}(X) = \tau^{i/i+1}(X).$$

Assume that X is a $\mathbf{SL_1}(A)$-torsor, with A of prime degree p. Then it follows from (V.1.1) that $E_{i-1}^{i,-i}(X) = 0$ for $3 \leq i \leq p$. Consequently, one has $A^1(X, K_2) = E_p^{1,-2}(X)$.

Moreover, by the result [31, Theorem 3.4] of A. Merkurjev, for any smooth variety X, every prime divisor l of the order of the differential δ_r ending in $E_r^{p+1,-p-1}(X)$ is such that $l-1$ divides $r-1$. Therefore, for any prime p and $2 \leq r \leq p-1$, the differential δ_r is of prime to p order. Assume furthermore that X is a $\mathbf{SL_1}(A)$-torsor, with A of prime degree p. Since $p \cdot \mathrm{CH}^{p+1}(X) = 0$, one deduce that, for $2 \leq r \leq p-1$, the differential δ_r is trivial. Consequently, one has $\mathrm{CH}^{p+1}(X) = E_p^{p+1,-p-1}(X)$.

Therefore, for X a $\mathbf{SL_1}(A)$-torsor, with A of prime degree p, the differential δ_p in the BGQ-structure is a homomorphism

$$\delta : A^1(X, K_2) \to \mathrm{CH}^{p+1}(X).$$

Remark V.1.2. Let X be a principal homogeneous space for a semisimple group G. By [13, Part II, Example 4.3.3 and Corollary 5.4], one has $E_2^{0,-1}(X) = A^0(X, K_1) = F^\times$ and the composition $F^\times = K_1(F) \to K_1(X) \to A^0(X, K_1)$ of the pullback of the structural morphism with the inclusions

$$K_1^{(0/1)}(X) = E_\infty^{0,-1}(X) \subset \cdots \subset E_3^{0,-1}(X) \subset E_2^{0,-1}(X)$$

given by the BGQ spectral sequence, is the identity. Therefore, for any $i \geq 2$, the differential starting from $E_i^{0,-1}(X)$ is zero, i.e for any $i \geq 2$, one has

$$E_i^{i,-i}(X) = \tau^{i/i+1}(X).$$

In particular, for X a $\mathbf{SL_1}(A)$-torsor, with A of prime degree p, one has $E_{p+1}^{p+1,-p-1}(X) = 0$, i.e the differential $\delta : A^1(X, K_2) \to \mathrm{CH}^{p+1}(X)$ is surjective.

On the group $A^1(X, K_2)$

The proof in the next section will use the work of A. Merkurjev on the *Rost invariant* of simply connected algebraic groups (see [13, Part II]). Let X be a $\mathbf{SL_1}(A)$-torsor over F. The group $A^1(X_{F(X)}, K_2)$ is infinite cyclic with generator q and isomorphic to $A^1(\mathbf{SL_n}, K_2)$ under restriction (where $n = \deg(A)$). Furthermore, the restriction map $r : A^1(X, K_2) \to A^1(X_{F(X)}, K_2)$ is injective with finite cokernel of order the same order as the element $R_{\mathbf{SL_1}(A)}(X)$, where

$$R_{\mathbf{SL_1}(A)} : H^1(F, \mathbf{SL_1}(A)) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2))$$

is the Rost invariant of $\mathbf{SL_1}(A)$ (see [13, Theorem 9.10]). Moreover, the homomorphism $R_{\mathbf{SL_1}(A)}$ is of order $\exp(A)$ by [13, Theorem 11.5].

If $\mathrm{char}(F) = l$ is prime then the modulo l component $H^3(F, \mathbb{Z}/l\mathbb{Z}(2))$ of the Galois cohomology group $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ is the group $H_l^3(F)$ defined by K. Kato in [28] by means of logarithmic differential forms.

V.2 Proof of the result

In this subsection, we prove the result of this chapter.

Theorem V.2.1. *Let A be a central simple algebra of prime degree p over a field F and let X be a $\mathbf{SL_1}(A)$-torsor. Then*

(i) *for any equidimensional F-variety Y, the change of field homomorphism*

$$\mathrm{CH}(Y) \to \mathrm{CH}(Y_{F(X)}),$$

where CH *is the integral Chow group, is surjective in codimension $< p+1$.*

(ii) *it is also surjective in codimension $p+1$ for a given Y provided that the variety $X_{F(\zeta)}$ does not have any closed point of prime to p degree for each generic point $\zeta \in Y$.*

Proof. We use notation and material introduced in the previous section. One can assume that X does not have any rational point over F (or equivalently X does not have any closed point of prime to p degree, by the result [2, Theorem 3.3] of J. Black), if else there is nothing to prove. Note that in this situation, the central simple algebra A is necessarily a division

algebra. We recall that conclusion (i) has already been proved. According to Proposition II.2.5, it suffices to show that $\mathrm{CH}^{p+1}(X_{F(\zeta)}) = 0$ for each generic point $\zeta \in Y$ to get conclusion (ii). Since $X_{F(\zeta)}$ does not have any closed point of prime to p degree, it is enough to prove that $\mathrm{CH}^{p+1}(X) = 0$.

Assume on the contrary that $\mathrm{CH}^{p+1}(X) \neq 0$. Then $\delta : A^1(X, K_2) \to \mathrm{CH}^{p+1}(X)$ is nonzero (since δ is surjective by Remark V.1.2), i.e $E_{p+1}^{1,-2}(X)$ is strictly included in $E_p^{1,-2}(X) = A^1(X, K_2)$. We claim that this implies that, by denoting as q_X the generator of $A^1(X, K_2)$, one has $r(q_X) = q$. Indeed, otherwise one has $r(q_X) = p \cdot q$ by the previous subsection. Consecutively, by denoting as c the corestriction morphism $A^1(\mathbf{SL_p}, K_2) \to A^1(X, K_2)$, for any $i \geq 2$, one has $c(E_i^{1,-2}(\mathbf{SL_p})) = c(A^1(\mathbf{SL_p}, K_2)) = A^1(X, K_2)$ (where the first identity is due to $\mathrm{CH}^i(\mathbf{SL_p}) = 0$ for any $i \geq 2$). In particular, one has $E_p^{1,-2}(X) = c(E_{p+1}^{1,-2}(\mathbf{SL_p})) \subset E_{p+1}^{1,-2}(X)$, which is a contradiction.

Therefore, we have shown that under the assumption $\mathrm{CH}^{p+1}(X) \neq 0$, the generator q of $A^1(X_{F(X)}, K_2)$ is rational. Then it follows that the generator g of $\mathrm{CH}^{p+1}(X_{F(X)})$ is also rational.

However, since $A_{F(X)}$ is a still a division algebra, by [25, Theorem 7.2 and Theorem 8.2], the cycle g^{p-1} in $\mathrm{CH}_0(\mathbf{SL_1}(A_{F(X)}))$ is nonzero and the latter group is cyclic of order p generated by the class of the identity of $\mathbf{SL_1}(A_{F(X)})$. Thus, the degree of the rational cycle g^{p-1} is prime to p.

It follows that X has a closed point of prime to p degree, which is a contradiction.

The Theorem is proved. □

Remark V.2.2. The end of the above proof shows in particular that for a division algebra A of prime degree p over a field F, the kernel of the Rost invariant $R_{\mathbf{SL_1}(A)}$ is trivial. This is already contained in the result [35, Theorem 12.2] of A. Merkurjev and A. Suslin under the assumption $\mathrm{char}(F) \neq p$. Indeed, let $\xi \in H^1(F, \mathbf{SL_1}(A))$ and let X be the associated $\mathbf{SL_1}(A)$-torsor. Assume that $R_{\mathbf{SL_1}(A)}(\xi)$ is trivial. It follows then that the generator of $A^1(X_{F(X)}, K_2)$ is rational (see Section V.1). As we have seen in the above proof, this implies that X has a rational point over F, i.e the cocycle ξ is trivial.

Note also that for a division algebra A of prime degree p over a field F, the Rost invariant $R_{\mathbf{SL_1}(A)}$ coincides, up to sign, with the normalized invariant given by the cup product $[A] \cup (c) \in H^3(F, \mathbb{Z}/p\mathbb{Z}(2))$ for any class $c\,\mathrm{Nrd}(A^\times)$, where $[A]$ is the class of the algebra A in the Brauer group $\mathrm{Br}(F)$, see [13, §11].

V.3 Link with Chapter IV - Exceptional projective homogeneous varieties

In this section, we describe how Theorem V.0.1 implies a similar version of it for projective homogeneous varieties under a group of type F_4 or E_8. Namely, we give an alternative proof of Theorem V.3.1 below (see Theorem IV.0.1). The following proof requires the characteristic

of the base field to be different from p, with $p = 3$ when G is of type F_4 and $p = 5$ when G is of type E_8, although the original result Theorem IV.0.1 is valid for arbitrary characterisitic.

Let X be a nonsplit $\mathbf{SL_1}(A)$-torsor over a field F, with A a division algebra of prime degree p. There exists a smooth compactification $\tilde{X}$ of X such that the Chow motive $\mathcal{M}(\tilde{X}, \mathbb{Z}/p\mathbb{Z})$ decomposes as a direct sum $\mathcal{R}_p \oplus N$, where $\mathcal{R}_p$ is the indecomposable Rost motive associated with the symbol $[A] \cup (c) \in H^3(F, \mathbb{Z}/p\mathbb{Z}(2))$, with $c \in F^\times \backslash \mathrm{Nrd}(A^\times)$ giving X, see [25, Theorem 1.1]. Note that the projective variety $\tilde{X}$ is a *norm variety* of s.

Theorem V.3.1. ([10, Theorem 1.1]) *Let G be a linear algebraic group of type F_4 or E_8 over a field F of characteristic different from p, with $p = 3$ when G is of type F_4 and $p = 5$ when G is of type E_8, and let X' be a projective homogeneous G-variety. For any equidimensional variety Y, the change of field homomorphism*

$$\mathrm{Ch}(Y) \to \mathrm{Ch}(Y_{F(X')}),$$

where Ch *is the Chow group modulo p, is surjective in codimension $< p + 1$.*

It is also surjective in codimension $p + 1$ for a given Y provided that $1 \notin \deg \mathrm{Ch}_0(X'_{F(\zeta)})$ for each generic point $\zeta \in Y$.

Proof. Since the F-variety X' is A-trivial in the sense of [26, Definition 2.3], one can assume that G has no splitting field of degree coprime to p. Indeed, otherwise $1 \in \deg \mathrm{Ch}_0(X')$ by corestricition and this implies that $\mathrm{Ch}(Y) \to \mathrm{Ch}(Y_{F(X')})$ is an isomorphism in any codimension by A-triviality, see [26, Lemma 2.9].

Let us now write $G = {}_\xi G_0$ for a nontrivial cocycle $\xi \in H^1(F, G_0)$, with G_0 a split group of the same type as G. Then the motive $\mathcal{R}_p(G)$ living on the Chow motive (with coefficients in $\mathbb{Z}/p\mathbb{Z}$) of X' given in [40, Theorem 5.17] is the Rost motive of the symbol $R_{G_0,p}(\xi) = [A] \cup (c) \in H^3(F, \mathbb{Z}/p\mathbb{Z}(2))$, where $R_{G_0,p}$ is the the modulo p component of the Rost invariant R_{G_0}, A is a division algebra of degree p and $c \in F^\times \backslash \mathrm{Nrd}(A^\times)$ – see [37, §4] and [12, §14] (here the assumption $\mathrm{char}(F) \neq p$ is needed).

Let us denote as X the nonsplit $\mathbf{SL_1}(A)$-torsor over F associated with c and as $\tilde{X}$ its smooth compactification. We claim that X' has a closed point of prime to p degree over $F(\tilde{X})$ and vice versa.

Indeed, since $\tilde{X}$ is a norm variety for $[A] \cup (c)$, the motive $\mathcal{R}_p(G)$ decomposes as a sum of Tate motives over $F(\tilde{X})$. Therefore, the group $G_{F(\tilde{X})}$ is split by an extension of degree coprime to p and it follows that X' has a closed point of prime to p degree over $F(\tilde{X})$ (this is more generally true for any extension L/F over which $\tilde{X}$ has a closed point of prime to p degree). Moreover, the motive $\mathcal{R}_p(G)$ decomposes as a sum of Tate motives over $F(X')$ because G is split by $F(X')$. Consequently, $\tilde{X}$ has a closed point of prime to p degree over $F(X')$.

It follows then (note that $\tilde{X}$ is A-trivial by [26, Example 5.7]) that the right and the bottom homomorphisms in the commutative square

$$\begin{tikzcd} \mathrm{Ch}(Y) \arrow[r] \arrow[d] & \mathrm{Ch}(Y_{F(X')}) \arrow[d] \\ \mathrm{Ch}(Y_{F(\tilde{X})}) \arrow[r] & \mathrm{Ch}(Y_{F(\tilde{X}\times X')}) \end{tikzcd}$$

are isomorphisms. Since $F(\tilde{X}) = F(X)$, Theorem V.3.1 is now a direct consequence of Theorem V.0.1. □

The following was pointed out to me by Philippe Gille.

Remark V.3.2. Let G_0 a split group of type E_8 over a 5-special field F (i.e F has no proper extension of degree coprime to 5) of characteristic $\neq 5$. The above proof gives rise to a new argument for the triviality of the kernel of the Rost invariant modulo 5

$$H^1(F, G_0) \to H^3(F, \mathbb{Z}/5\mathbb{Z}(2)).$$

This result is originally due to Vladimir Chernousov (under the assumption $\mathrm{char}(F) \neq 2$, 3, 5, see [6, Theorem]).

Indeed, since F is 5-special, for any nontrivial cocycle $\xi \in H^1(F, G_0)$, the group ${}_\xi G_0$ has no splitting field of degree coprime to 5. Then, as we have seen in the proof, there is a division algebra A of degree 5 such that $R_{G_0,5}(\xi)$ is equal to a symbol $[A] \cup (c)$ associated with a nonsplit $\mathbf{SL_1}(A)$-torsor X. The injectivity of $R_{G_0,5}$ follows now from Remark V.2.2.

Chapter VI

Special correspondences

Let p be prime and let X be a geometrically irreducible variety of dimension $p^n - 1$ over a field F of characteristic $\neq p$. We refer to [7, §62] or to Appendix C for an introduction to correspondences. In this chapter, an antisymmetric correspondence $\sigma \in \mathrm{CH}^b(X \times X)$, where $b = (p^n - 1)/(p - 1)$, is said to be *special* if its image $H \in \mathrm{CH}^b(X_{F(X)})$ under the pull-back associated with the morphism $\mathrm{Spec}(F(X)) \times X \to X \times X$ is such that

(i) $\sigma_{F(X)} = 1 \times H - H \times 1$ in $\mathrm{CH}^b(X^2_{F(X)})$;

(ii) $\deg(H^{p-1})$ is not divisible by p.

This notion has initially been introduced by M. Rost in [43].

In the first part of this chapter (section VI.1 to VI.3), we are interested in a conjecture (see Conjecture VI.1.1 below) due to Alexander Vishik. That conjecture deals with rationality of algebraic cycles over function field of quadrics and it can be related to some special correspondences on quadrics. In the second part, we prove the case of equality of a theorem ([26, Theorem SC.1]) due to Nikita Karpenko and Alexander Merkurjev and involving special correspondences on A-trivial varieties.

VI.1 A conjecture of A. Vishik

Let Y be a smooth quasi-projective variety over a field F, let Q be a smooth projective quadric of dimension n over F and let us denote its function field as $F(Q)$. For any integer $m \geq 0$, we recall that one can consider the following commutative diagram given by change of field homomorphisms for Chow groups modulo 2 of codimension m classes of algebraic cycles

$$\begin{tikzcd} \mathrm{Ch}^m(Y) \arrow[r] \arrow[d] & \mathrm{Ch}^m(Y_{F(Q)}) \arrow[d] \\ \mathrm{Ch}^m(\overline{Y}) \arrow[r, hook] & \mathrm{Ch}^m(Y_{\overline{F}(Q)}) \end{tikzcd} ,$$

where we write $\overline{Y} := Y_{\overline{F}}$, with $\overline{F}$ an algebraic closure of F.

We recall that an element $\overline{y}$ of $\mathrm{Ch}(\overline{Y})$ is $F(Q)$-*rational* if its image $\overline{y}_{\overline{F}(Q)}$ under $\mathrm{Ch}(\overline{Y}) \to \mathrm{Ch}(Y_{\overline{F}(Q)})$ is in the image of $\mathrm{Ch}(Y_{F(Q)}) \to \mathrm{Ch}(Y_{\overline{F}(Q)})$. Since $\overline{F}$ is algebraically closed, the bottom homomorphism $\mathrm{Ch}(\overline{Y}) \to \mathrm{Ch}(Y_{\overline{F}(Q)})$ is injective by the specialization arguments.

Furthermore, for any $I \subset \{0, \ldots, [n/2]\}$, let us denote the associated partial flag variety as $G(I)$ (in particular, for any $i \in \{0, \ldots, [n/2]\}$, the variety $G(i)$ is the Grassmannian of i-dimensional totally isotropic subspaces) and for $J \subset I$ we write π with subindex I with J underlined inside it for the natural projection $G(I) \to G(J)$. In particular, for any $i \in \{0, \ldots, [n/2]\}$, one can consider

$$Q \xleftarrow[\pi_{(\underline{0},i)}]{} G(0,i) \xrightarrow[\pi_{(0,\underline{i})}]{} G(i),$$

and we set

$$\mathbf{z}_i := \pi_{(0,\underline{i})_*} \circ \pi^*_{(\underline{0},i)}(l_0) \in \mathrm{CH}^{n-i}(G(i)_{F(Q)}),$$

where $l_0 \in \mathrm{CH}_0(Q_{F(Q)})$ is the class of a closed point $\mathbf{x} \in Q_{F(Q)}$ of degree 1.

In the aftermath of its use of the Main Tool Lemma to refute the Kaplansky's conjecture, A. Vishik stated the following conjecture (see [48, Conjecture 3.13])

Conjecture VI.1.1 (Vishik)**.** *Let Y be a smooth quasi-projective variety over a field F with char$(F) = 0$, let Q be a smooth projective F-quadric of dimension n and let $i \in \{0, \ldots, [n/2]\}$. If the cycle $\mathbf{z}_i$* (mod 2) *is rational then any $F(Q)$-rational element $y \in \mathrm{Ch}^m(\overline{Y})$, with $m \leq n - i$, is rational.*

Note that A. Vishik proved in [48, Proposition 2.5] that the rationality of $\mathbf{z}_i$ implies the rationality of $\mathbf{z}_j$ for any $j > i$.

This conjecture is known for the extremal values $i = 0$ (if $\mathbf{z}_0 = l_0$ is rational, i.e if Q is isotropic, then one has $\mathrm{CH}^m(Y) \simeq \mathrm{CH}^m(Y_{F(Q)})$ for any m) and $i = [n/2]$ (see [48, Proposition 3.12]). It is claimed to be known also for $i = 1$ in [48, end of section 3.2] but a proof is not given (we get the case $i = 1$ by combining Proposition VI.2.8 with Proposition VI.3.2). Moreover, if one considers Conjecture VI.1.1 with $\underline{\mathrm{Ch}} := \underline{\mathrm{CH}}$ (mod 2), where $\underline{\mathrm{CH}} = \mathrm{CH}$ (modulo 2-torsion), instead of Ch, then the case $i = 1$ is given by the result [55, Theorem 1.3] of K. Zainoulline.

In the second section of this chapter, we link the rationality of the cycles $\mathbf{z}_i$ with the rationality of certain special correspondences ρ_i of Q. In the third section, we give a partial answer to Conjecture VI.1.1 (in the case the first Witt index i_1 of Q is sufficiently large) which go in the direction of a positive general answer (see Proposition VI.3.1).

VI.2 Rationality of special correspondences on quadrics

Let Q be a smooth projective anisotropic quadric of dimension n over a field F. Let us denote $[n/2]$ as d. Let $\rho = 1 \times l_0 + l_0 \times 1 \in \mathrm{CH}^n(Q^2_{F(Q)})$ be the so-called *Rost correspondence* on Q (in the sense of [7, §80]). Note that if ρ is rational then $\rho - h^n \times 1$ is a special correspondence as defined in the introduction of this chapter, however we choose to work with ρ for convenience with computations (note that, since Q is anisotropic, the rationality of ρ would imply that $n+1$ is a power of 2 by [7, Corollary 80.8]).

For every $i = 0, \dots, d$, we set $t_i := 1 \times h \times \cdots \times h^{i-1} \times l_0 \in \mathrm{CH}^{n+i(i-1)/2}(Q^{i+1}_{F(Q)})$, where h is the hyperplane section class (always rational), and for any $\sigma \in S_{i+1}$ we also denote by $\sigma : Q^{i+1} \to Q^{i+1}$ the associated isomorphism. Then we set

$$\rho_i := \Sigma_{\sigma \in S_{i+1}} \sigma_*(t_i) \in \mathrm{CH}^{n+i(i-1)/2}(Q^{i+1}_{F(Q)}).$$

We called ρ_i the *i-th Rost correspondence of Q*. One has $\rho_0 = l_0 = \mathbf{z}_0$ and $\rho_1 = \rho$. Note that the rationality of ρ_i implies the rationality of ρ_j for any $j > i$.

The purpose of this section is to prove the two propositons below. We will need the following lemma, which can easily be deduced from [48, Propositon 2.1 and Lemma 2.6] and its proofs.

For any positive integers $k \leq d$ and j such that $j + k \leq n$, we set

$$W^k_j := \pi_{(0,\underline{k})_*} \circ \pi^*_{(\underline{0},k)}(h^{j+k}) \in \mathrm{CH}^j(G(k)).$$

Lemma VI.2.1 (Vishik)**.** *For any $0 \leq k < d$, one has*

(i) $\pi_{(0,\underline{k})_*} \circ \pi^*_{(\underline{0},k)}(h^k) = [G(k)]$;

(ii) $\pi_{(0,\underline{k},\underline{k+1})_*} \circ \pi^*_{(\underline{0},k,k+1)}(h^k) = [G(k,k+1)]$;

(iii)
$$\pi^*_{(\underline{k},k+1)}(\mathbf{z}_k) = c_1(\mathcal{O}(1)) \cdot \pi^*_{(k,\underline{k+1})}(\mathbf{z}_{k+1}),$$
where $\mathcal{O}(1)$ is the standard sheaf on the projective bundle $G(k,k+1) = \mathbb{P}_{G(k)}(\tilde{\mathcal{T}}_{k+1})$, with $\tilde{\mathcal{T}}_{k+1}$ the vector bundle dual to the tautological bundle $\mathcal{T}_{k+1}$ on $G(k+1)$.

(iv)
$$c_1(\mathcal{O}(1)) \cdot \pi^*_{(k,\underline{k+1})}(W^{k+1}_j) = \pi^*_{(\underline{k},k+1)}(W^k_{j+1}) - \pi^*_{(k,\underline{k+1})}(W^{k+1}_{j+1}).$$

In the following statement, the fact that if ρ is rational then $\mathbf{z}_1$ is also rational has already been shown by A. Vishik in the proof of [53, Theorem 4.4].

Proposition VI.2.2. *Let $i \in \{0, \dots, d\}$. If ρ_i is rational then $\mathbf{z}_i$ is also rational.*

Proof. In order to simplify the notation in the proof, for any $0 \leq i \leq d$ we denote $\pi_{(\underline{0},i)}$ as f_i and $\pi_{(0,\underline{i})}$ as g_i.

Since the conclusion is obvious for $i = 0$, we assume $i \geq 1$ in the proof. Let $\Theta \subset Q \times Q \times G(0,1)$ be the subvariety $\{((y,z),(y,l))|\, z \in l\}$ and let

$$\theta := [\Theta] \in \mathrm{CH}_{\dim(G(0,1))+1}(Q \times Q \times G(0,1))$$

be its class as an algebraic cycle.

View as a correspondence, the cycle θ defines an homomorphism $\theta_* : \mathrm{CH}^*(Q \times Q) \to \mathrm{CH}^{*-1}(G(0,1))$ and one easily checks that for any $(\alpha, \beta) \in \{1, h, \dots, h^{i-1}, l_0\}^2$, with $\alpha \neq \beta$, one has

$$\theta_*(\alpha \times \beta) = \begin{cases} {g_1}^*(\mathbf{z}_1) & \text{if } \alpha \times \beta = 1 \times l_0 \\ [G(0,1)] & \text{if } \alpha \times \beta = 1 \times h \\ 0 \text{ or } {f_1}^*(l_0) & \text{if else} \end{cases}$$

Thus one has the following identity in $\mathrm{CH}(Q^{i-1} \times G(0,1)_{F(Q)})$,

$$\begin{aligned}\big((\mathrm{Id}_{{Q^{i-1}}_*} \times \theta_*)(\rho_i)\big) \cdot \big([Q^{i-1}] \times {f_1}^*(h)\big) &= \Sigma_{\sigma \in S_{i-1}} \sigma_*(h \times h^2 \times \cdots \times h^{i-1}) \times ({g_1}^*(\mathbf{z}_1) \cdot {f_1}^*(h)) \\ &\quad + \Sigma_{\sigma \in S_{i-1}} \sigma_*(l_0 \times h^2 \times \cdots \times h^{i-1}) \times {f_1}^*(h).\end{aligned}$$

Then, by applying the homomorphism $\mathrm{Id}_{{Q^{i-1}}_*} \times g_{1*}$ to the previous identity and by combining the fact that $g_{1*} \circ {f_1}^*(h) = [G(1)]$ (see Lemma VI.2.1(i)) with the projection formula (see Proposition II.2.2), one get that in $\mathrm{CH}(Q^{i-1} \times G(1)_{F(Q)})$ the cycle

$$\Sigma_{\sigma \in S_{i-1}} \sigma_*(h \times h^2 \times \cdots \times h^{i-1}) \times \mathbf{z}_1 + \Sigma_{\sigma \in S_{i-1}} \sigma_*(l_0 \times h^2 \times \cdots \times h^{i-1}) \times [G(1)] \qquad \text{(VI.2.3)}$$

is rational. Note that this gives the conclusion if $i = 1$ so one can assume $i \geq 2$ in the sequel.

Let us denote the sum (VI.2.3) as $s = u + v$. We write in for the imbedding $G(0,1) \hookrightarrow Q \times G(1)$ and we set $t = \mathrm{Id}_{{Q^{i-2}}_*} \times r$ where $r : \mathrm{CH}(Q \times G(1)) \to \mathrm{CH}(G(2))$, is defined by

$$r(\alpha) = \pi_{(1,\underline{2})_*}\left(\left(\pi^*_{(1,\underline{2})}(W_1^2)\right) \cdot \left(\pi_{(0,\underline{1},\underline{2})_*} \circ \pi^*_{(\underline{0},\underline{1},2)} \circ in^*(\alpha)\right)\right)$$

for any $\alpha \in \mathrm{CH}(Q \times G(1))$ (we intentionally write r this way for convenience with computations).

We claim that one has $t(v)$=0. Note that since s is rational and t commutes with change of field homomorphisms, this would imply that the cycle $t(u)$ is rational. We prove the claim. For any $2 \leq k \leq i-1$, one has

$$\pi_{(0,\underline{1},\underline{2})_*} \circ \pi^*_{(\underline{0},\underline{1},2)} \circ in^*(h^k \times [G(1)]) = 0$$

for dimensional reasons. Furthermore, one has

$$\pi_{(0,\underline{1},\underline{2})_*} \circ \pi^*_{(\underline{0},\underline{1},2)} \circ in^*(l_0 \times [G(1)]) = \pi^*_{(\underline{1},2)}(\mathbf{z}_1)$$

and by the projection formula it follows that

$$r(l_0 \times [G(1)]) = W_1^2 \cdot \pi_{(1,\underline{2})_*}\left(\pi^*_{(\underline{1},2)}(\mathbf{z}_1)\right).$$

Since $\pi_{(1,\underline{2})_*}\left(\pi^*_{(\underline{1},2)}(\mathbf{z}_1)\right) = 0$ by dimensional reasons, we get that $t(v) = 0$.

Now we would like to compute the rational cycle $t(u)$. First of all, for any integer $2 \leq k \leq i-1$, one has $\pi_{(0,\underline{1},\underline{2})_*} \circ \pi^*_{(\underline{0},\underline{1},2)} \circ in^*(h^k \times \mathbf{z}_1) = 0$ by dimensional reasons. Moreover, one has

$$\pi^*_{(\underline{0},\underline{1},2)} \circ in^*(h \times \mathbf{z}_1) = \pi^*_{(\underline{0},1,2)}(h) \cdot \pi^*_{(0,\underline{1},\underline{2})}\left(\pi^*_{(\underline{1},2)}(\mathbf{z}_1)\right).$$

By Lemma VI.2.1(ii), (iii) and the projection formula, it follows that

$$\pi_{(0,\underline{1},\underline{2})_*} \circ \pi^*_{(\underline{0},\underline{1},2)} \circ in^*(h^k \times \mathbf{z}_1) = c_1(\mathcal{O}(1)) \cdot \pi^*_{(1,\underline{2})}(\mathbf{z}_2). \tag{VI.2.4}$$

Therefore, using Lemma VI.2.1(iv) (with $k = j = 1$) and once again the projection formula, from (VI.2.4) one get

$$r(h \times \mathbf{z}_1) = \mathbf{z}_2 \cdot \pi_{(1,\underline{2})_*} \circ \pi^*_{(\underline{1},2)}(W_2^1). \tag{VI.2.5}$$

Moreover, by [48, Proposition 2.1], one has $W_2^1 = c_2(-\mathcal{T}_1)$, with $\mathcal{T}_1$ the tautological bundle over $G(1)$. Hence, since the bundle $\pi^*_{(\underline{1},2)}(\mathcal{T}_1)$ is naturally identified with the tautological line bundle $\mathcal{O}(-1)$ over $G(1,2) = \mathbb{P}_{G(1)}(\tilde{\mathcal{T}}_2)$, one get

$$\pi_{(1,\underline{2})_*} \circ \pi^*_{(\underline{1},2)}(W_2^1) = \pi_{(1,\underline{2})_*}(c_2(-\mathcal{O}(-1))) = c_0(-\tilde{\mathcal{T}}_2) = [G(2)] \tag{VI.2.6}$$

and it follows from (VI.2.5) that

$$t(u) = \Sigma_{\sigma \in S_{i-2}} \sigma_*(h^2 \times h^3 \times \cdots \times h^{i-1}) \times \mathbf{z}_2.$$

Now, for every $k = 2, \ldots, i$, we set

$$u_k := \Sigma_{\sigma \in S_{i-k}} \sigma_*(h^k \times h^{k+1} \times \cdots \times h^{i-1}) \times \mathbf{z}_k \in \mathrm{CH}(Q^{i-k} \times G(k)_{F(Q)}).$$

Since the cycle $u_2 = t(u)$ is rational and one has the identity $u_i = \mathbf{z}_i$, it suffices to show that, for any $2 \leq k \leq i-1$, the rationality of u_k implies the rationality of u_{k+1}, to conclude.

Let $2 \leq k \leq i-1$ and assume that the cycle u_k is rational. We set $t_k = \mathrm{Id}_{Q^{i-k-1}{}_*} \times r$ where $r : \mathrm{CH}(Q \times G(k)) \to \mathrm{CH}(G(k+1))$, is defined by

$$r(\alpha) = \pi_{(k,\underline{k+1})_*}\left(\left(\pi^*_{(k,\underline{k+1})}(W_k^{k+1})\right) \cdot \left(\pi_{(0,\underline{k},\underline{k+1})_*} \circ \pi^*_{(\underline{0},\underline{k},k+1)} \circ in^*(\alpha)\right)\right)$$

for any $\alpha \in \mathrm{CH}(Q \times G(k))$ (for $k = 1$, t_k is the homomorphism t that we used).

We claim that $t_k(u_k) = u_{k+1}$. Note that this would give us the conclusion since t_k commutes with change of field homomorphisms. To prove this we reproduce what has been done for the computation of $t(u)$

First of all, for any $k+1 \leq j \leq i-1$, one has $\pi_{(0,\underline{k},\underline{k+1})_*} \circ \pi^*_{(\underline{0},\underline{k},k+1)} \circ in^*(h^j \times \mathbf{z}_k) = 0$ by dimensional reasons. Thus, one get

$$t_k(u_k) = t_k\left(\Sigma_{\sigma \in S_{i-k-1}} \sigma_*(h^{k+1} \times \cdots \times h^{i-1}) \times h^k \times \mathbf{z}_k\right).$$

Moreover, one has

$$\pi^*_{(\underline{0},\underline{k},k+1)} \circ in^*(h^k \times \mathbf{z}_k) = \pi^*_{(\underline{0},k,k+1)}(h^k) \cdot \pi^*_{(0,\underline{k},\underline{k+1})}\left(\pi^*_{(\underline{k},k+1)}(\mathbf{z}_k)\right).$$

Hence, by Lemma VI.2.1(ii), (iii) and the projection formula, it follows that

$$\pi_{(0,\underline{k},\underline{k+1})_*} \circ \pi^*_{(\underline{0},\underline{k},k+1)} \circ in^*(h^k \times \mathbf{z}_k) = c_1(\mathcal{O}(1)) \cdot \pi^*_{(k,\underline{k+1})}(\mathbf{z}_{k+1}). \tag{VI.2.7}$$

Then using once again the projection formula and Lemma VI.2.1(iv), from (VI.2.7) one deduce the identity

$$t_k(u_k) = u_{k+1} \cdot \left([Q^{i-k-1}] \times \left(\pi_{(k,\underline{k+1})_*} \circ \pi^*_{(\underline{k},k+1)}(W^k_{k+1})\right)\right).$$

Finally, as in (VI.2.6), one has

$$\pi_{(k,\underline{k+1})_*} \circ \pi^*_{(\underline{k},k+1)}(W^k_{k+1}) = [G(k+1)]$$

and the proposition is proved. □

We simply write $\mathbf{z}$ for $\mathbf{z}_1$.

Proposition VI.2.8. *ρ is rational if and only if $\mathbf{z}$ is rational.*

Proof. By Proposition VI.2.2, it remains to show that the rationality of $\mathbf{z}$ implies the rationality of ρ. We denote $\pi_{(\underline{0},1)}$ as f, $\pi_{(0,\underline{1})}$ as g and $G(1)$ as G.

Let $\Theta' \subset G \times Q \times Q$ be the subvariety $\{(l, y_1, y_2) | \, y_1, y_2 \in l\}$ and let $\theta' := [\Theta'] \in \mathrm{CH}_{\dim(G)+2}(G \times Q \times Q)$ be its class as an algebraic cycle. As a correspondence, the cycle θ' defines an homomorphism

$$\theta'_* : \mathrm{CH}^{n-1}(G) \to \mathrm{CH}^n(Q \times Q).$$

Let us denote by Δ th class of the diagonal $\{(y,y)\} \subset Q \times Q$. We want to show that one has the identity $\theta'_*(\mathbf{z}) + 1 \times l_0 + l_0 \times 1 = \Delta_{F(Q)}$ (this gives us the conclusion). To do so, viewing the previous cycles as correspondences, one has to prove that the homomorphism

$$(\theta'_*(\mathbf{z}) + 1 \times l_0 + l_0 \times 1)_* : \mathrm{CH}^*(Q_{F(Q)}) \to \mathrm{CH}^*(Q_{F(Q)})$$

is the identity. Moreover, one easily checks that for any $\alpha \in \mathrm{CH}^k(Q_{F(Q)})$, one has

$$(1 \times l_0 + l_0 \times 1)_*(\alpha) = \begin{cases} \alpha & \text{if } k = 0 \text{ or } k = n \\ 0 & \text{if else} \end{cases}$$

Therefore, one has to show that for any $\alpha \in \mathrm{CH}^k(Q_{F(Q)})$, one has

$$(\theta'_*(\mathbf{z}))_*(\alpha) = \begin{cases} \alpha & \text{if } 0 < k < n \\ 0 & \text{if else} \end{cases}$$

Note that since the correspondence $\theta'_*(\mathbf{z})$ is symmetric, it is sufficient (see [7, §68]) to show that

$$(\theta'_*(\mathbf{z}))_*(h^k) = \begin{cases} h^k & \text{if } 0 < k < d \\ 0 & \text{if } k = 0 \end{cases}$$

To do so, let us first find an explicit formula for $(\theta'_*(\mathbf{z}))_*$. Consider the following commutative diagram given by projections

$$\begin{tikzcd} G \times Q \times Q \arrow[r, "p_{G\times\underline{Q}\times\underline{Q}}"'] \arrow[d, "p_{\underline{G}\times Q\times\underline{Q}}"'] & Q \times Q \arrow[d, "p_{Q\times\underline{Q}}"] \\ G \times Q \arrow[r, "p_{G\times\underline{Q}}"'] & Q \end{tikzcd} .$$

By the very definition, one has $\theta'_*(\mathbf{z}) = p_{G\times\underline{Q}\times\underline{Q}_*}(\mathbf{z} \times [Q] \times [Q] \cdot \theta')$. Hence it follows by the projection formula that for any $\alpha \in \mathrm{CH}^k(Q_{F(Q)})$, one has

$$(\theta'_*(\mathbf{z}))_*(\alpha) = p_{Q\times\underline{Q}_*} \circ p_{G\times\underline{Q}\times\underline{Q}_*}(\mathbf{z} \times \alpha \times [Q] \cdot \theta'),$$

that is to say, by the commutativity of the above diagram,

$$(\theta'_*(\mathbf{z}))_*(\alpha) = p_{G\times\underline{Q}_*} \circ p_{\underline{G}\times Q\times\underline{Q}_*}(\mathbf{z} \times \alpha \times [Q] \cdot \theta').$$

Moreover, by denoting $\gamma \in \mathrm{CH}^{n-1}(G \times Q)$ the class of the subvariety $\{(l, y) | y \in l\}$, one has the identity $\theta' = \gamma \times [Q] \cdot p^*_{\underline{G}\times Q\times\underline{Q}}(\gamma)$. Consequently, using the projection formula, one get

$$(\theta'_*(\mathbf{z}))_*(\alpha) = p_{G\times\underline{Q}_*}\left(\gamma \cdot \left(p_{\underline{G}\times Q_*}((\mathbf{z} \times \alpha) \cdot \gamma) \times [Q]\right)\right),$$

that is to say, using again the projection formula,

$$(\theta'_*(\mathbf{z}))_*(\alpha) = p_{G\times\underline{Q}_*}\left(\gamma \cdot (\mathbf{z} \times [Q]) \cdot \left(p_{\underline{G}\times Q_*}\left(p^*_{G\times\underline{Q}}(\alpha) \cdot \gamma\right) \times [Q]\right)\right). \tag{VI.2.9}$$

Furthermore, since $\gamma = in_*([G(0,1)])$ (with $in : G(0,1) \hookrightarrow G \times Q$), by the analog Proposition II.2.3 of the projection formula , one has

$$p^*_{G\times\underline{Q}}(\alpha) \cdot \gamma = in_* \circ in^*\left(p^*_{G\times\underline{Q}}(\alpha)\right)$$

and by the very definition of the morphisms f and g, it follows that

$$p_{\underline{G}\times Q_*}\left(p^*_{G\times\underline{Q}}(\alpha) \cdot \gamma\right) = g_* \circ f^*(\alpha).$$

In the same way, one has $\gamma \cdot (\mathbf{z} \times [Q]) = in_* \circ g^*(\mathbf{z})$. Consequently, one deduce from (VI.2.9) the following identity

$$(\theta'_*(\mathbf{z}))_* (\alpha) = p_{G\times \underline{Q}_*} \left((in_* \circ g^*(\mathbf{z})) \cdot \left(p^*_{\underline{G}\times Q}(g_* \circ f^*(\alpha)) \right) \right),$$

that is to say, applying the projection formula to in, followed by Proposition II.2.1,

$$(\theta'_*(\mathbf{z}))_* (\alpha) = f_* \circ g^* (\mathbf{z} \cdot g_* \circ f^*(\alpha)). \tag{VI.2.10}$$

We will use this formula for computations.

First of all, one has $(\theta'_*(\mathbf{z}))_* (h^0) = 0$ since $g_* \circ f^*(h^0) \in \mathrm{CH}^{-1}(G)$. Let us show now that $(\theta'_*(\mathbf{z}))_* (h) = h$. Since $g_* \circ f^*(h) = [G]$ by Lemma VI.2.1(i), one has

$$(\theta'_*(\mathbf{z}))_* (h) = f_* \circ g^*(\mathbf{z}) = p_{G\times \underline{Q}_*} ([X]), \tag{VI.2.11}$$

where X is the subvariety $\{(l, y)|\mathbf{x}, y \in l\} \subset G \times Q$. Let us denote by H the hyperplane section of Q determined by the orthogonal complementary of the vectorial line associated with the point $\mathbf{x}$. Then one has $p_{G\times \underline{Q}}(X) = H$ and the projection $p_{G\times \underline{Q}}$ maps isomorphically the open $\{(l, y)|\mathbf{x}, y \in l; y \neq \mathbf{x}\}$ of X to the open $H\backslash\{\mathbf{x}\}$ of H. Therefore, one has $p_{G\times \underline{Q}_*} ([X]) = [H] = h$, that is to say

$$(\theta'_*(\mathbf{z}))_* (h) = h.$$

Now let $0 \leq k \leq d - 1$. By (VI.2.10), one has

$$(\theta'_*(\mathbf{z}))_* (h^{k+1}) = f_* \left((g^*(\mathbf{z})) \cdot \left(g^* \circ g_* \circ f^*(h^{k+1}) \right) \right),$$

and it follows from Lemma VI.2.1(iv) that

$$(\theta'_*(\mathbf{z}))_* (h^{k+1}) = f_* \left(g^*(\mathbf{z}) \cdot f^*(h^k) \right) - f_* \left(c_1 (\mathcal{O}(1)) \cdot (g^*(\mathbf{z})) \cdot \left(g^* \circ g_* \circ f^*(h^k) \right) \right).$$

By the projection formula and (VI.2.11), the first summand of the right side of the previous identity is equal to $h^k \cdot ((\theta'_*(\mathbf{z}))_* (h))$, that is to say h^{k+1}.

Therefore, it only remains to prove that the second summand of the right side of the previous identity is equal to zero to conclude. Using successively the fact that $c_1 (\mathcal{O}(1)) \cdot (g^*(\mathbf{z})) = f^*(l_0)$ (see Lemma VI.2.1(iii)) and the projection formula, one get that this second summand can be rewritten as

$$-l_0 \cdot \left(f_* \circ g^* \circ g_* \circ f^*(h^k) \right)$$

and this is equal to zero for dimensional reasons. The proposition is proved. □

We do not know if for $i \geq 2$ the rationality of $\mathbf{z}_i$ implies the rationality of ρ_i.

VI.3 A partial answer to the conjecture

Let Q be a smooth projective anisotropic quadric of dimension n over a field F of characteristic 0. We need that assumption on the characteristic of the base field because we will use the algebraic cobordism theory and the latter relies on resolution of singularities. Let us denote by i_1 the first Witt index of Q. The purpose of this section is to prove the following proposition.

Proposition VI.3.1. *Let $0 \leq i \leq i_1$. If ρ_i is rational then for any smooth quasi-projective variety Y and for any integer $m \leq n - i$, the change of field homomorphism*

$$\mathrm{CH}^m(Y) \to \mathrm{CH}^m(Y_{F(Q)})$$

is surjective.

Note that, if the rationality of ρ_i is equivalent to the rationality of $\mathbf{z}_i$, then the previous proposition gives an answer to the Conjecture VI.1.1 in the case the quadric Q has an $(i-1)$-dimensional subspace defined over the generic point of Q.

Proposition VI.3.1 is actually a consequence of the following statement, which is in fact the case $i = 1$ of Proposition VI.3.1 (note that by the very definition of i_1, one always has $i_1 \geq 1$).

Proposition VI.3.2. *If $\rho = 1 \times l_0 + l_0 \times 1$ is rational then for any smooth quasi-projective variety Y and for any integer $m \leq n - 1$, the change of field homomorphism*

$$\mathrm{CH}^m(Y) \to \mathrm{CH}^m(Y_{F(Q)})$$

is surjective.

The proof of Proposition VI.3.2, which is largely inspired by the proof of [55, Theorem 1.3] by K. Zainoulline, is given in the last subsection (we need to introduce some material about algebraic cobordism before giving the proof). For the moment, let us explain how Proposition VI.3.2 implies Proposition VI.3.1.

Proof of Proposition VI.3.1. Since the result is known for $i = 0$ and $i = 1$, one can assume that $i \geq 2$ in the proof. Let $Q' \subset Q$ be a subquadric of codimension $i - 1$. One can consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^*(Y) & \longrightarrow & \mathrm{CH}^*(Y_{F(Q)}) \\ \downarrow & & \downarrow \\ \mathrm{CH}^*(Y_{F(Q')}) & \longrightarrow & \mathrm{CH}^*(Y_{F(Q\times Q')}) \end{array} . \tag{VI.3.3}$$

Since $i_1 > i - 1$, the quadric Q' has a rational point over $F(Q)$ and it follows that the right homomorphism is an isomorphism. Since Q has a rational point over $F(Q')$ as well, the bottom homomorphism is also an isomorphism.

Hence, one has to show that the homomorphism $\mathrm{CH}^m(Y) \to \mathrm{CH}^m(Y_{F(Q')})$ is surjective. Moreover, since $m \leq n - i = \dim(Q') - 1$, it is sufficient by Proposition VI.3.2 to prove that the cycle

$$\rho' = 1 \times l'_0 + l'_0 \times 1 \in \mathrm{CH}_0({Q'}^2_{F(Q')}) \simeq \mathrm{CH}_0({Q'}^2_{F(Q)})$$

is rational.

Let us denote by $\pi \in \overline{\mathrm{CH}}^{n-i_1+1}(Q^2)$ a rational integral representative of the 1-primordial cycle in $\overline{\mathrm{Ch}}^{n-i_1+1}(Q^2)$ (see [7, Definition 73.16] and paragraph right after [7, Theorem 73.26]). Even if it means adding a rational cycle to π, one can assume that π decomposes as

$$\pi = 1 \times l_{i_1-1} + l_{i_1-1} \times 1 + \sum_{j=i_1}^{d-i_1+1} a_j \left(h^j \times l_{j+i_1-1} + l_{j+i_1-1} \times h^j\right),$$

where for every $j = i_1, \ldots, d - i_1 + 1$, the coefficient a_j is an integer (the fact that one can choose to make the previous sum start from $j = i_1$ is due to [7, Proposition 73.27]).

Furthermore, let us denote by $\sigma \in S_{i+1}$ the cyclic permutation which sends k to $k+1$ for $1 \leq k \leq i$ and $i+1$ to 1. We also denote by $\sigma : Q^{i+1} \to Q^{i+1}$ the associated isomorphism. Then, viewing the algebraic cycles at work as correspondences, we recursively define the following sequence of rational cycles

$$\begin{cases} \varepsilon_0 := \rho_i \\ \varepsilon_k := \sigma_* \left(\varepsilon_{k-1} \circ \left((1 \times h^{i_1-i+k-1}) \cdot \pi\right)\right) \quad \text{for } k = 1, \ldots, i-1 \end{cases}$$

One easily checks that, in $\overline{\mathrm{CH}}^n(Q^{i+1})$, one has the identity

$$\varepsilon_{i-1} = \rho \times 1 \times 1 \times \cdots \times 1 + \delta_{i,i_1} \left((1 \times h^{i-1} + h^{i-1} \times 1) \times l_{i-1} \times 1 \times \cdots \times 1\right),$$

where δ is the Kronecker symbol. We write Δ for the diagonal morphism

$$\begin{array}{ccc} Q^2 & \longrightarrow & Q^{i+1} \\ (x,y) & \longmapsto & (x,y,\ldots,y) \end{array}.$$

By applying Δ^* to the previous equation, one get that the cycle

$$\gamma := 1 \times l_0 + l_0 \times 1 + \delta_{i,i_1} \left(1 \times l_0 + h^{i-1} \times l_{i-1}\right) \in \mathrm{CH}^n(\overline{Q^2})$$

is rational. Then, by denoting in the inclusion $Q'^2 \hookrightarrow Q^2$, one has the identity

$$in^* \left(\gamma \circ \pi\right) = l'_{i_1-i} \times 1 + \delta_{i,i_1} \left(1 \times l'_{i_1-i}\right).$$

If $i = i_1$ then the previous identity gives us directly the conclusion. If else – i.e $i < i_1$ – it follows from the previous identity that the cycle $(l'_{i_1-i} \times 1) \cdot (h^{i_1-i}) = l'_0 \times 1$ is rational and consecutively that ρ' is rational. The proposition is proved. $\square$

Symmetric operations in algebraic cobordisms

In this subsection, we briefly recall some properties of *symmetric operations in algebraic cobordisms*.

We recall that for any smooth variety X over a field F of characteristic 0, M. Levine and F. Morel have defined in [30] the ring of *algebraic cobordisms* $\Omega^*(X)$ with a natural surjective map $pr : \Omega^*(X) \twoheadrightarrow \mathrm{CH}^*(X)$. An element $\beta \in \Omega^*(X)$ is a finite sum of classes $[v]$, where the morphism $v : Y \to X$ belongs to a certain class of morphims containing the class of smooth projective morphisms. In particular, for any smooth projective F-variety U of dimension n, the class $[U \to \mathrm{Spec}(F)]$ of the structure morphism is an element of $\mathbb{L}_n \subset \mathbb{L} = \Omega^*(\mathrm{Spec}(F))$, with $\mathbb{L}$ the Lazard ring. Moreover, one has

$$\mathrm{Ker}\,(pr : \Omega^*(X) \twoheadrightarrow \mathrm{CH}^*(X)) = \mathbb{L}_{>0} \cdot \Omega^*(X).$$

For our purpose (the proof of Proposition VI.3.2), one can keep in mind the following commutative diagram, for any extension K/F,

$$\begin{tikzcd} \Omega^*(X) \arrow[r] \arrow[d, "pr"'] & \Omega^*(X_K) \arrow[d, "pr"] \\ \mathrm{CH}^*(X) \arrow[r] & \mathrm{CH}^*(X_K) \end{tikzcd} .$$

Then A. Vishik constructed in [53] and [52] some cohomological operations

$$\Phi^{t^r} : \Omega^d(X) \to \Omega^{2d+r}(X),$$

for $r \geq 0$, called *symmetric operations*. For $r \geq 0$, we write $\phi^{t^r} = pr \circ \Phi^{t^r}$. For $r > 0$, the operation ϕ^{t^r} is additive. For any $q(t) = \sum_{i\geq 0} q_i t^i \in \mathrm{CH}^*(X)\,[[t]]$, we set $\phi^{q(t)} := \sum_{i\geq 0} q_i \phi^{t^i}$.

For a vector bundle E over X, we write $c(E)(t)$ for the total Chern polynomial $\prod_{i\geq 0}(t + \lambda_i)$, where the $\lambda_i \in \mathrm{CH}^1(X)$ are the *roots* of E.

The following properties of symmetric operations, which have been proved by A. Vishik (see [52, Propositions 3.4 and Proposition 3.15] and [51, Proposition 2.4(3)]), will be useful during the proof of Proposition VI.3.2.

The following proposition describes how ϕ interacts with pull-backs and push-forwards.

Proposition VI.3.4 (Vishik)**.** *Let $f : X \to Y$ be a morphism of smooth projective varieties. For any $r \geq 0$, one has*

(i) $f^* \circ \phi^{t^r}([w]) = \phi^{t^r} \circ f^*([w])$;

(ii) *If f is a regular embedding then*

$$\phi^{t^r}\left(f_*([1_X]) \cdot [w]\right) = \phi^{t^r \cdot f_*\left(c(N_f)(t)\right)}([w]),$$

where N_f is the normal bundle of f.

For any smooth projective F-variety X and any integer j, we denote by S^j_{LN} the Landweber-Novikov operation associated with the integer j. We recall that one has the following commutative diagram

$$\begin{tikzcd}
\Omega^*(X) \arrow[r, "S^j_{LN}"] \arrow[d, "pr"'] & \Omega^{*+j}(X) \arrow[d, "pr"] \\
\mathrm{CH}^*(X) \arrow[d] & \mathrm{CH}^{*+j}(X) \arrow[d] \\
\mathrm{Ch}^*(X) \arrow[r, "S^j"'] & \mathrm{Ch}^{*+j}(X)
\end{tikzcd}$$

where S^j is the j-th Steenrod operation of cohomological type on X (we recall that both S^0_{LN} and S^0 are the identity).

Furthermore, the formula below links in a particular case the symmetric operations by A. Vishik with the Chow trace of the Landweber-Novikov operations (see [52, Proposition 3.15]).

Let U be a smooth projective F-variety of positive dimension n and let us denote as $[U] \in \mathbb{L}$ the class of its structure morphism. We write $\eta(U)$ for the Rost invariant $\frac{-\deg(c_n(-T_U))}{2} \in \mathbb{Z}$ of U, where T_U is the tangent bundle of U (the fact that $\eta(U)$ is an integer can be found in [32]).

Proposition VI.3.5 (Vishik)**.** *Let U be a smooth projective F-variety of positive dimension n and let $\beta \in \Omega^i(X)$. For any $j > \max(i-2n\,;0)$, one has*

$$\phi^{t^{j-i+2n}}\left([U]\cdot\beta\right) = (-1)^{j-i+2n}\eta(U)\cdot\left(pr\circ S^j_{LN}(\beta)\right).$$

Proof of Proposition VI.3.2

In this subsection, we prove Proposition VI.3.2. For an element β in CH^* or Ω^* and an extension L/F, we write β_L for the image of β under the associated change of field homomorphism. In the proof, we also denote by ρ an element of $\mathrm{CH}^n(Q^2)$ mapped to $1\times l_0 + l_0\times 1 \in \mathrm{CH}^n(Q^2_{F(Q)})$ under the change of field homomorphism

Since $2 \in \deg\left(\mathrm{CH}_0(Q)\right)$ (because $h^n = 2l_0$), it suffices by [26, Corollary 2.10] to prove the surjectivity at the level of Chow groups modulo 2, i.e one has to show that for any integer $m \leq n-1$, the change of field homomorphism

$$\mathrm{Ch}^m(Y) \to \mathrm{Ch}^m(Y_{F(Q)})$$

is surjective.

Let $y \in \mathrm{Ch}^m(Y_{F(Q)})$ and let us fix an element $x \in \mathrm{Ch}^m(Q \times Y)$ mapped to y under the surjection (see Proposition II.2.4)

$$\mathrm{Ch}^m(Q \times Y) \twoheadrightarrow \mathrm{Ch}^m(Y_{F(Q)}).$$

We set $x' := x \circ (\rho \pmod 2) \in \mathrm{Ch}^m(Q \times Y)$. Note that since ρ is a correspondence of multiplicity 1, the cycle x' is also mapped to y under the previous surjection (see [26, Lemma 2.1(1) and Lemma 2.6]). Furthermore, one has

$$x'_{F(Q)} = x_{F(Q)} \circ (1 \times l_0) + x_{F(Q)} \circ (l_0 \times 1) = 1 \times y, \tag{VI.3.6}$$

the term $x_{F(Q)} \circ (1 \times l_0)$ being equal to zero since $m \leq n-1$ We will use this identity at th very end of the proof.

Since the cycle $1 \times l_0 + l_0 \times 1 \in \mathrm{CH}^n(Q^2_{F(Q)})$ is rational, there exists an element $\alpha \in \mathbb{L}_{>0} \cdot \Omega(Q^2_{F(Q)})$ such that, by denoting the class of $g : pt \hookrightarrow Q_{F(Q)}$ in $\Omega_0(Q_{F(Q)})$ as x_0, the element

$$1 \times x_0 + x_0 \times 1 + \alpha$$

is rational (with $1 = [\mathrm{Id}_Q]$). Now let us fix an element $w \in \Omega^m(Q \times Y)$ mapped to x' under $\Omega^*(Q \times Y) \twoheadrightarrow \mathrm{Ch}^*(Q \times Y)$ and let us consider the following rational element

$$v := (1 \times x_0 + x_0 \times 1 + \alpha)_*(w_{F(Q)}) \in \Omega^m(Q \times Y_{F(Q)}).$$

The element v decomposes as

$$v = 1 \times v_n + x_0 \times v_0 + \alpha_*(w_{F(Q)}), \tag{VI.3.7}$$

where $v_j \in \Omega^{m-n+j}(Y_{F(Q)})$ and $pr(v_n) \pmod 2 = y$ (basic computations for correspondences are the same for both Chow theory and algebraic cobordism theory).

The following lemma constitutes the next step of the proof.

Lemma VI.3.8. *One has*

$$y = \phi^{t^{2n-m}} \circ p_{Y*}(v) \; - \; \phi^{t^{2n-m}}(v_0) \; - \; \phi^{t^{2n-m}} \circ p_{Y*}\left(\alpha_*(w_{F(Q)})\right) \pmod 2,$$

where p_Y is the projection $Q \times Y \to Y$.

Proof. We apply $\phi^{t^{2n-m}} \circ p_{Y*}$ to the decomposition (VI.3.7) of v. First, we deal with the first summand of the right side of the equation. By the projection formula, one has

$$\phi^{t^{2n-m}} \circ p_{Y*}(1 \times v_n) = \phi^{t^{2n-m}}\left(q_*(1) \cdot v_n\right),$$

where $q : Q \to \mathrm{Spec}(F)$ is the structure morphism. Since one has $q_*(1) = [Q] \in \mathbb{L}_n$, one get from Proposition VI.3.5 that

$$\phi^{t^{2n-m}} \circ p_{Y*}(1 \times v_n) = (-1)^m \eta(Q) \cdot pr(v_n) \quad \text{in } \mathrm{CH}^m(Y_{F(Q)}).$$

Consequently, as $\eta(Q) \pmod 2 = 1 \in \mathbb{Z}/2\mathbb{Z}$ since the quadric Q is anisotropic (see [43, Theorem 9.9]), one has

$$\phi^{t^{2n-m}} \circ p_{Y*}(1 \times v_n) \pmod 2 = y\ .$$

Secondly, we deal with the second summand of of the right side of the equation. Once again by the projection formula, one has

$$\phi^{t^{2n-m}} \circ p_{Y*}(x_0 \times v_0) = \phi^{t^{2n-m}}(q_*(x_0) \cdot v_0) = \phi^{t^{2n-m}}(v_0)$$

and the lemma is proved. □

The following lemma constitutes the second step of the proof.

Lemma VI.3.9. *One has*

$$\phi^{t^{2n-m}}(v_0) = p_{Y*} \circ \phi^{t^{n-m}}(v) \ - \ p_{Y*} \circ \phi^{t^{n-m}}\left(\alpha_*(w_{F(Q)})\right).$$

Proof. We apply $p_{Y*} \circ \phi^{t^{n-m}}$ to the decomposition (VI.3.7) of v. Let us start with the first summand of the right side of the equation. The projection formula and Proposition VI.3.4(i) imply the following string of identities

$$\begin{aligned} p_{Y*} \circ \phi^{t^{n-m}}(1 \times v_n) &= p_{Y*} \circ \phi^{t^{n-m}} \circ (p_Y^*(v_n) \\ &= p_{Y*} \circ p_Y^* \circ \phi^{t^{n-m}}(v_n) \\ &= \phi^{t^{n-m}}(v_n) \cdot p_{Y*}(1), \end{aligned}$$

and $p_{Y*}(1) = 0$ for dimensional reasons.

We deal now with the second summand of the right side of equation (VI.3.7). Let $f : \mathrm{Spec}(F(Q)) \times Y \hookrightarrow Q \times Y$ be the regular embedding $g \times \mathrm{Id}_Y$. Since $x_0 \times v_0 = f_*(1 \times 1) \cdot (1 \times v_0)$, it follows from Proposition VI.3.4(ii) that

$$\phi^{t^{n-m}}(x_0 \times v_0) = \phi^{t^{n-m} \cdot f_*\left(c(N_f)(t)\right)}(1 \times v_0). \tag{VI.3.10}$$

Moreover, one has $N_g = g^*(T_Q)$, with T_Q the tangent bundle of Q (see [17, Corollary 17.12.3] or [7, Proposition 104.12]). Hence, by [7, Proposition 54.5] (see also Appendix B), one has $g_*(c(N_g)(t)) = c(T_Q)(t) \cdot g_*(1) = c(T_Q)(t) \cdot l_0$. Since $c(T_Q)(t) = (t-h)^{n+1} \cdot (t-2h)^{-1}$ (see [50, Proposition 6.1] for example), one get from (VI.3.10) that

$$\phi^{t^{n-m}}(x_0 \times v_0) = (l_0 \times 1) \cdot \phi^{t^{2n-m}}(1 \times v_0) = l_0 \times \phi^{t^{2n-m}}(v_0),$$

where the last identity is due to Proposition VI.3.4(i). Consequently, one has $p_{Y*} \circ \phi^{t^{n-m}}(x_0 \times v_0) = \phi^{t^{2n-m}}(v_0)$ and the lemma is proved. □

One get from the two previous lemmas that $y = \phi^{t^{2n-m}} \circ p_{Y*}(v) - p_{Y*} \circ \phi^{t^{n-m}}(v) \pmod 2 + z$, with $z := p_{Y*} \circ \phi^{t^{n-m}}\left(\alpha_*(w_{F(Q)})\right) - \phi^{t^{2n-m}} \circ p_{Y*}\left(\alpha_*(w_{F(Q)})\right) \pmod 2$. Therefore, the conclusion is given by the following lemma.

Lemma VI.3.11. *In* $\mathrm{Ch}^m(Y_{F(Q)})$*, one has* $z = 0$*.*

Proof. Since ϕ^{t^r} is additive for $r > 0$, one can assume there exist an integer $d > 0$ and some elements $[U] \in \mathbb{L}_d$ and $\beta \in \Omega^{n+d}(Q^2_{F(Q)})$ such that $\alpha = [U] \cdot \beta$. By $\mathbb{L}$-linearity, it follows that $\alpha_*(w_{F(Q)}) = [U] \cdot \beta_*(w_{F(Q)})$. Consequently, by Proposition VI.3.5, one has

$$\phi^{t^{n-m}}\left(\alpha_*(w_{F(Q)})\right) = (-1)^{n-m}\eta(U) \cdot \left(pr \circ S^{n-d}_{LN}\left(\beta_*(w_{F(Q)})\right)\right) \text{ in } \mathrm{CH}^m(Y_{F(Q)}),$$

and therefore, in $\mathrm{Ch}^m(Y_{F(Q)})$, one has the following identity

$$p_{Y_*} \circ \phi^{t^{n-m}}\left(\alpha_*(w_{F(Q)})\right) \pmod 2 = \eta(U) \cdot p_{Y_*} \circ S^{n-d}\left(pr\left(\beta_*(w_{F(Q)})\right) \pmod 2\right). \quad \text{(VI.3.12)}$$

Moreover, since again by $\mathbb{L}$-linearity one has $p_{Y_*}\left(\alpha_*(w_{F(Q)})\right) = [U] \cdot p_{Y_*}\left(\beta_*(w_{F(Q)})\right)$, one also has the following identity

$$\phi^{t^{2n-m}} \circ p_{Y_*}\left(\alpha_*(w_{F(Q)})\right) \pmod 2 = \eta(U) \cdot S^{n-d} \circ p_{Y_*}\left(pr\left(\beta_*(w_{F(Q)})\right) \pmod 2\right), \quad \text{(VI.3.13)}$$

and by combining (VI.3.12), (VI.3.13) and Proposition II.3.2, one get that the cycle z can be rewritten as

$$z = \eta(U) \cdot p_{Y_*}\left(\sum_{0<i\leq n-d} c_i(-T_Q) \cdot S^{n-d-i}\left(pr\left(\beta_*(w_{F(Q)})\right) \pmod 2\right)\right).$$

Since $c_i(-T_Q) = h^i \in \mathrm{Ch}^i(Q)$ (see [7, Corollary 80.11 and Lemma 78.1]), by denoting the cycle $pr(\beta) \pmod 2 \in \mathrm{Ch}^{n+d}(Q^2_{F(Q)})$ as β', the latter equation can be rewritten as

$$z = \eta(U) \cdot p_{Y_*}\left(\sum_{0<i\leq n-d} h^i \cdot S^{n-d-i}(x'_{F(Q)} \circ \beta')\right). \quad \text{(VI.3.14)}$$

We claim that, for every $i = 1, \ldots, n-d$, the summand $z_i := p_{Y_*}\left(h^i \cdot S^{n-d-i}(x'_{F(Q)} \circ \beta')\right)$ of (VI.3.14) is equal to zero. Indeed, one deduce from (VI.3.6) that $x'_{F(Q)} \circ \beta' = p_{\underline{Q}\times Q_*}(\beta') \times y$. Then by the Cartan formula (see Corollary II.3.5), it follows that for every $i = 1, \ldots, n-d$, one has

$$z_i = \sum_{j=0}^{n-d-i} z_{i,j} \quad \text{with} \quad z_{i,j} := p_{Y_*}\left(\left(h^i \cdot S^j \circ p_{\underline{Q}\times Q_*}(\beta')\right) \times S^{n-d-i-j}(y)\right).$$

If $j \neq n-d-i$ then one has $z_{i,j} = 0$ for dimensional reasons. Otherwise, $h^i \cdot S^{n-d-i} \circ p_{\underline{Q}\times Q_*}(\beta')$ is a 0-cycles and $z_{i,n-d-i} = 0$ since $\deg\left(h^i \cdot S^{n-d-i} \circ p_{\underline{Q}\times Q_*}(\beta')\right) = 0$ by [43, Lemma 9.3]. □

Proposition VI.3.2 is now completely proved.

VI.4 Special correspondences on A-trivial varieties

In this section, we state the result [26, Theorem SC.1] (see Theorem VI.4.1 below) due to Nikita Karpenko and Alexander Merkurjev and then we give an extension, namely we deal with the case of equality of this theorem.

Let p be a prime and let X be a smooth complete geometrically irreducible variety over a field F of characteristic $\neq p$. We denote by Ch the Chow group modulo p. First, we recall that X is said to be *A-trivial for $\mathbb{Z}/p\mathbb{Z}$* if for any extension L/F such that $X(L) \neq \emptyset$, the degree homomorphism $\deg : \mathrm{Ch}_0(X_L) \to \mathbb{Z}/p\mathbb{Z}$ is an isomorphism (for example, any smooth projective quadric is A-trivial for $\mathbb{Z}/2\mathbb{Z}$). We say that X and X' are *equivalent* if for any extension L/F, one has $1 \in \deg \mathrm{Ch}_0(X_L)$ if and only if $1 \in \deg \mathrm{Ch}_0(X'_L)$. This is an equivalence relation.

For any integer i, we write S^i for the cohomological Steenrod operation on Ch which increases the codimension by i (as in [26]). This indexing differs from that of [4]. In this indexing, one has $S^i = 0$ if i is not divisible by $p-1$. We recall that b refers to $(p^n-1)/(p-1)$.

Theorem VI.4.1 (Karpenko, Merkurjev)**.** *Let X be an A-trivial F-variety for $\mathbb{Z}/p\mathbb{Z}$ equivalent to an A-trivial F-variety of dimension p^n-1 possessing a special correspondence. Then for any smooth irreducible F-variety Y, any $m, s \in \mathbb{Z}$ with $s > (m-b)(p-1)$, and any $y \in \mathrm{Ch}^m(Y_{F(X)})$, the element $S^s(y) \in \mathrm{Ch}^{m+s}(Y_{F(X)})$ is rational up to the class modulo p of an exponent p element of $\mathrm{CH}^{m+s}(Y_{F(X)})$.*

The proof of the following proposition is widely inspired by the proof of the previous statement and is for this reason to be read along with the proof of [26, Theorem SC.1].

Proposition VI.4.2. *Let X be an A-trivial F-variety for $\mathbb{Z}/p\mathbb{Z}$ equivalent to an A-trivial F-variety of dimension p^n-1 possessing a special correspondence. Then for any smooth irreducible F-variety Y, any $m \in \mathbb{Z}$, and any $y \in \mathrm{Ch}^m(Y_{F(X)})$, there exists a polynomial P of degree $\leq p-1$, with rational coefficients in $\mathrm{Ch}(Y_{F(X)})$, such that the element $S^s(y)+P(y) \in \mathrm{Ch}^{m+s}(Y_{F(X)})$, with $s = (m-b)(p-1)$, is rational up to the class modulo p of an exponent p element of $\mathrm{CH}^{m+s}(Y_{F(X)})$.*

Proof. We make the assumption that $\dim(Y) > 0$ (otherwise, the conclusion is immediate). Since the conclusion is given by [26, Lemma 2.9] if $1 \in \deg \mathrm{Ch}_0(X)$, one can assume that $\deg \mathrm{CH}_0(X) \subset p\mathbb{Z}$. Furthermore, using a commutative diagram similar to (VI.3.3), one can also assume that X itself is of dimension $d := p^n - 1$ and possesses a special correspondence $\sigma \in \mathrm{CH}^b(X \times X)$. As in the introduction of this chapter, we write $H \in \mathrm{CH}^b(X_{F(X)})$ for the image of σ under the corresponding pull-back.

As in the proof of [26, Theorem SC.1], one can find an element $x \in \mathrm{Ch}^m(X \times Y)$ such that x decomposes over $F(X)$ as

$$x_{F(X)} = 1 \times y + H \times x_1 + \cdots + H^{p-1} \times x_{p-1} \tag{VI.4.3}$$

with some $x_1, ..., x_{p-1} \in \mathrm{Ch}(Y_{F(X)})$.

Let us fix an integral representative $\tilde{x} \in \mathrm{CH}^m(X \times Y)$ of $x \in \mathrm{Ch}^m(X \times Y)$ and for each integer $k \geq 0$, an integral representative $S^k_\sigma \in \mathrm{CH}^{b+k}(X \times X)$ of $S^k(\sigma) \in \mathrm{Ch}^{b+k}(X \times X)$ (we choose σ for S^0_σ) and an integral representative $S^k_x \in \mathrm{CH}^{m+k}(X \times Y)$ of $S^k(x) \in \mathrm{Ch}^{m+k}(X \times Y)$. By combining the reasoning done in the beginning of [26, Proposition SC.12] with the fact that $S^{d+s}(x) = x^p$, one get that there exist some cycles $b_k \in \mathrm{Ch}^k(X)$ such that the element

$$\sum_{\substack{i+j+k+l_1+\cdots+l_{p-1}=d+s \\ i>0;\, j,k,l_1,\ldots,l_{p-1}\geq 0}} p_{Y*}\left(b_j \cdot \left(p_{X,\underline{X}*}\left(b_i \cdot S^{l_1}_\sigma \cdot \ldots \cdot S^{l_{p-1}}_\sigma\right)\right) \cdot S^k_x\right) + p_{Y*}(\tilde{x}^p) \in \mathrm{CH}(Y),$$

is divisible by p (in the previous sum, $b_k\cdot$ always stands for $(b_k \times 1)\cdot$). Let us denote this sum as $A + p_{Y*}(\tilde{x}^p)$.

Furthermore, one knows from the proof of [26, Proposition SC.12] that there exist a cycle $\beta \in \mathrm{CH}(Y_{F(X)})$, a cycle $\gamma \in \mathrm{CH}(Y)$, and a prime to p integer q such that

$$A_{F(X)} = p^2\beta + p\gamma_{F(X)} + q\deg(b_d)S^s_y,$$

where $S^s_y \in \mathrm{CH}^{m+s}(Y_{F(X)})$ is an integral representative of $S^s(y) \in \mathrm{Ch}^{m+s}(Y_{F(X)})$. Therefore, even if it means modifying β and γ, one can write

$$q\deg(b_d)S^s_y + p_{Y*}(\tilde{x}^p_{F(X)}) = p^2\beta + p\gamma_{F(X)}. \tag{VI.4.4}$$

Moreover, according to the decompositon (VI.4), there exists a cycle $\alpha \in \mathrm{CH}^m(X \times Y_{F(X)})$ such that

$$\tilde{x}_{F(X)} = 1 \times \tilde{y} + H \times \tilde{x_1} + \cdots + H^{p-1} \times \tilde{x}_{p-1} + p\alpha,$$

where the cycles $\tilde{y}, \tilde{x_1}, ..., \tilde{x}_{p-1}$ are some integral representatives of the cycles $y, x_1, ..., x_{p-1}$. Hence, one has

$$\tilde{x}^p_{F(X)} = \sum_{k=0}^{p} \binom{p}{k} (p\alpha)^{p-k} \cdot \left(1 \times \tilde{y} + H \times \tilde{x_1} + \cdots + H^{p-1} \times \tilde{x}_{p-1}\right)^k.$$

In the lattest expression, each summand, except the one corresponding to $k = p$, is divisible by p^2. Thus, even if it means modifying β, we deduce from the equation (VI.4.4) the following identity

$$q\deg(b_d)S^s_y + p_{Y*}\left(\left(1 \times \tilde{y} + H \times \tilde{x_1} + \cdots + H^{p-1} \times \tilde{x}_{p-1}\right)^p\right) = p^2\beta + p\gamma_{F(X)}. \tag{VI.4.5}$$

Furthermore, by the multinomial Theorem, the cycle $p_{Y*}\left(\left(\sum_{0\leq i\leq p-1} H^i \times \tilde{x}_i\right)^p\right)$ (with $x_0 = y$) is equal to

$$\sum_{\substack{k_0+k_1+\cdots+k_{p-1}=p \\ k_1+2k_2+\cdots+(p-1)k_{p-1}=p-1}} \binom{p}{k_0, k_1, \ldots, k_{p-1}} \tilde{y}^{k_0} \cdot \tilde{x_1}^{k_1} \cdots \tilde{x}^{k_{p-1}}_{p-1}.$$

Since for any $i = 0, \ldots, p-1$, one has $k_i < p$, each multinomial coefficient appearing in the previous sum is a multiple of p. Thus, that sum can be rewritten as

$$p \sum_{k=1}^{p-1} a_k \cdot \tilde{y}^k,$$

where

$$a_k := \sum_{\substack{k_1+k_2+\cdots+k_{p-1}=p-k \\ k_1+2k_2+\cdots+(p-1)k_{p-1}=p-1}} \binom{p-1}{k, k_1, \ldots, k_{p-1}} \tilde{x_1}^{k_1} \cdot \tilde{x_2}^{k_2} \cdots \tilde{x}_{p-1}^{k_{p-1}} \in \mathrm{CH}(Y_{F(X)}).$$

Therefore, since the integer $\deg(b_d)$ is divisible by p (by assumption) but not by p^2 (see proof of [26, Proposition SC.12], this results from [43, Theorem 9.9] by M. Rost), we deduce from the equation (VI.4.5) that the element

$$S^s(y) + \sum_{k=1}^{p-1} a_k \cdot y^k \tag{VI.4.6}$$

is rational up to the class modulo p of an exponent p element of $\mathrm{CH}^{m+s}(Y_{F(X)})$ (for $k = 1, ..., p-1$, we still write a_k for the class in $\mathrm{Ch}(Y_{F(X)})$, and we replace the coefficient in $(\mathbb{Z}/p\mathbb{Z})^\star$ near $S^s(y)$ by 1). From now on, we work with Chow groups modulo p. For any $k = 1, ..., p-1$, one has

$$\begin{aligned}
a_k &= \binom{p-1}{k-1} k^{-1} p_{Y*}\left(\left(H \times x_1 + H^2 \times x_2 + ... + H^{p-1} \times x_{p-1}\right)^{p-k}\right) \\
&= (-1)^{k-1} k^{-1} p_{Y*}\left(\left(x_{F(X)} - 1 \times y\right)^{p-k}\right) \\
&= (-1)^{k-1} k^{-1} \sum_{i=0}^{k} \binom{p-k}{i} (-1)^i p_{Y*}\left(x_{F(X)}^{p-k-i} \cdot (1 \times y^i)\right) \\
&= (-1)^{k-1} k^{-1} \sum_{i=0}^{k} \binom{p-k}{i} (-1)^i y^i \cdot p_{Y*}(x_{F(X)}^{p-k-i}).
\end{aligned}$$

Therefore, setting for every $j = 1, ..., p-1$,

$$e_j := \left(\sum_{l=1}^{j} l^{-1} \binom{p-l}{j-l}\right) (-1)^{j-1} p_{Y*}(x^{p-j}) \in \mathrm{Ch}(Y),$$

one get that

$$\sum_{k=1}^{p-1} a_k \cdot y^k = P(y), \tag{VI.4.7}$$

where P is the polynomial in variable Z with coefficients in $\mathrm{Ch}(Y_{F(X)})$ such that $P(Z) = \sum_{j=1}^{p-1} e_{j\,F(X)} \cdot Z^j$ (there is no coefficient e_p because $p_{Y*}(1 \times 1) = 0$). One get the desired result by combining (VI.4.6) and (VI.4.7). □

Appendix A

Milnor K-Theory

All material presented here can be found in [7, §100]

Graded Milnor ring. Let F be a field and let us denote by T_* the tensor ring of the multiplicative group $F^\times$. We write I for the ideal of T_* generated by the tensors $a \otimes b$ with $a + b = 1$ and we set

$$K_*(F) = T_*/I.$$

This graded ring is called the *Milnor ring*. The class of a tensor $a_1 \otimes \cdots \otimes a_n$ in $K_n(F)$ is denoted by $\{a_1, \ldots, a_n\}$ and is called a *symbol*. One has $K_n(F) = 0$ for $n < 0$, $K_0(F) = \mathbb{Z}$ and $K_1(F) = F^\times$. For $n \geq 2$,, $K_n(F)$ is generated by the symbol $\{a_1, \ldots, a_n\}$ with $a_i \in F^\times$ subject to the multilinearity relation and the Steinberg relation.

Note that for any extension L/F, one has a natural restriction homomorphism $r_{L/F} : K_*(F) \to K_*(L)$ associating to any symbol $\{a_1, \ldots, a_n\}_F$ the symbol $\{a_1, \ldots, a_n\}_L$.

Residue homomorphsim. Let L be a field with discrete valuation $v : L^\times \to \mathbb{Z}$ and residue field F. For any $n \geq 0$, the residue homomorphism

$$\delta_v : K_{n+1}(L) \to K_n(F)$$

is uniquely determined by the condition: if $a_0, a_1, \ldots a_n \in L^\times$ are such that $v(a_i) = 0$ for each $i = 1, \ldots, n$, then $\delta_v(\{a_0, a_1, \ldots, a_n\}) = v(a_0)\{\overline{a_1}, \ldots, \overline{a_n}\}$, where $\overline{a_i} \in F^\times$ is the residue of a_i.

Milnor's Theorem. Let X be an integral scheme over F. For any regular point $x \in X$ of codimension 1, the local ring $\mathcal{O}_{X,x}$ is a discrete valuation ring with quotient field $F(X)$ and residue field $F(x)$. We write $\delta_x : K_{*+1}(F(X)) \to K_*(F(x))$ for the associated residue homomorphism.

Milnor's Theorem describes the Milnor K-groups of the function field $F(\mathbb{A}^1_F) = F(t)$ of the affine line, by means, the following sequence

$$0 \longrightarrow K_{n+1}(F) \xrightarrow{r_{F(t)/F}} K_{n+1}(F(t)) \xrightarrow{(\delta_x)} \coprod_{x \in \mathbb{A}^1_{(0)}} K_n(F(x)) \longrightarrow 0$$

is split exact.

Norm homomorphism. For any finite extension L/F and $n \geq 0$, we describe how is constructed the associated norm homomorphism

$$c_{L/F} : K_n(L) \to K_n(F).$$

We assume first that the extension L/F is simple. Then L can be identified with the residue field of $F(y)$ of a closed point $y \in \mathbb{A}^1_F$. Now let $\alpha \in K_n(L) = K_n(F(y))$. By Milnor's Theorem, there exists $\beta \in K_{n+1}(F(t))$ such that $\delta_x(\beta) = \alpha$ if $x = y$ and $\delta_x(\beta) = 0$ otherwise.

Let v be the discrete valuation of the field $F(\mathbb{P}^1_F) = F(\mathbb{A}^1_F) = F(t)$ associatd with the infinite point of the projective line $\mathbb{P}^1_F$. We set

$$c_{L/F}(\alpha) = \delta_v(\beta).$$

In the genreal case, we choose an arbitrary sequence of simple field extensions

$$F \subset F_1 \subset \cdots \subset F_n = L$$

and the composite

$$c_{L/F} := c_{F_1/F} \circ \cdots \circ c_{F_n/F_{n-1}}$$

is well defined.

Appendix B

Chern classes

Let X be a scheme and let $p : E \to X$ be a vector bundle of rank $r > 0$. The associated pull-back homomorphism

$$p^* : A_*(X, K_*) \to A_{*+r}(E, K_{*-r})$$

is an isomorphism (see [7, Corollary 52.14]). Let us denote by $s : X \to E$ the zero section. The composite

$$e(E) := (p^*)^{-1} \circ s_* : A_*(X, K_*) \to A_{*-r}(X, K_{*+r})$$

is called the *Euler class* of E.

We write $q : \mathbb{P}(E) \to X$ for the projective bundle associated with p and L for the tautological line bundle over $\mathbb{P}(E)$. We set $e = e(L)$. The following statement is known as the Projective Bundle Theorem (see [7, Theorem 53.10]).

Theorem B.0.1. *The homomorphism*

$$\coprod_{i=1}^{r} e^{r-i} \circ q^* : \coprod_{i=1}^{r} A_{*-i+1}(X, K_{*+1-i}) \to A_*(\mathbb{P}(E), K_*)$$

is an isomorphism

Let $\alpha \in A_*(X, K_*)$. Since $-e^r \circ q^*(\alpha) \in A_{*-1}(\mathbb{P}(E), K_{*+1})$, by the previous theorem, there exist $\alpha_i \in A_{*-i}(X, K_{*+i})$ such that

$$-e^r \circ q^*(\alpha) = \sum_{i=1}^{r} (-1)^i e^{r-i} \circ q^*(\alpha_i).$$

Therefore, one get some homomorphisms

$$\begin{array}{cccc} c_i(E) & A_*(X, K_*) & \to & A_{*-i}(X, K_{*+i}) \\ & \alpha & \mapsto & \alpha_i \end{array}$$

for $i = 1, \ldots, r$. Moreover, we set $c_0(E) = \mathrm{id}$ and $c_i(E) = 0$ for $i < 0$ or $i > r$. Those homomorphisms are called the *Chern classes* of E and

$$c(E) := c_0(E) + c_1(E) + \cdots c_r(E)$$

is the *total Chern class* of E.

When X is a smooth variety, the homomorphism $c_i(E) : \mathrm{CH}^*(X) \to \mathrm{CH}^{*+i}(X)$ is just the multiplication by an element of $\mathrm{CH}^i(X)$, which is still denoted by $c_i(E)$.

The following statement describes the functorial behaviour of Chern classes.

Proposition B.0.2. ([7, Proposition 54.5]) *Let $f : Y \to X$ be a morphism and E a vector bundle over X. Set $E' = f^*(E)$. Then*

(i) *If f is proper then $c(E) \circ f_* = f_* \circ c(E')$.*

(ii) *If f is flat then $f^* \circ c(E) = c(E') \circ f^*$.*

Appendix C

Correspondences on Chow groups

In this appendix, we only present the category of correspondences (used in Chapters III and VI). Note that the *category of Chow motive* is built from this category (see [7, §64]) All material presented here is taken from [7, §62, §63].

Let Λ be a commutative ring and CH the Chow group with coefficients in Λ. Let X and Y be smooth complete schemes over a field F. Let $X_1, \ldots, X_n$ be the irreducible components of X of dimension $d_1, \ldots, d_n$, respectively. For every $i \in \mathbb{Z}$, we set

$$\mathrm{Corr}_i(X, Y) = \coprod_{k=1}^{n} \mathrm{CH}_{i+d_k}(X_k \times Y).$$

An element $\alpha \in \mathrm{Corr}_i(X, Y)$ is called a *correspondence between X and Y of degree i with coefficients in* Λ. There is an associative composition between correspondences. Namely, if Z is another smooth complete scheme, one can consider the associative composition

$$\mathrm{Corr}_i(Y, Z) \times \mathrm{Corr}_j(X, Y) \to \mathrm{Corr}_{i+j}(X, Z) \tag{C.0.1}$$

given by

$$\beta \circ \alpha = p_{\underline{X},Y,\underline{Z}*}\left(p^*_{\underline{X},\underline{Y},Z}(\alpha) \cdot p^*_{X,\underline{Y},\underline{Z}}(\beta)\right)$$

for any $(\beta, \alpha) \in \mathrm{Corr}_i(Y, Z) \times \mathrm{Corr}_j(X, Y)$, where the underlined schemes in indices determined the associated projection. In Chapters III and VI, for $\alpha \in \mathrm{Corr}_i(X, Y)$, we have denoted by α_* and α^* the composition by α on the left and on the right, respectively.

The category $\mathrm{CR}_*(F)$ of *correspondences* (with coefficients in Λ) over F is defined as follows: objects of $\mathrm{CR}_*(F)$ are smooth complete schemes over F and a morphism between two objects X and Y is an element of the graded group

$$\coprod_{k \in \mathbb{Z}} \mathrm{Corr}_k(X, Y).$$

Composition of morphisms is given by (C.0.1). The identity morphism of X is the class $[\Gamma_{1_X}] \in \mathrm{CH}(X^2)$ of the graph of the identity morphism 1_X. The direct sum in $\mathrm{CR}_*(F)$

is given by the disjoint union of schemes and the composition (C.0.1) being bilinear, the category $\mathrm{CR}_*(F)$ is additive. A smooth complete scheme X, view as an object of $\mathrm{CR}_*(F)$, is denoted by $\mathcal{M}(X)$ and is called a *Chow motive*.

Appendix D

Torsors of algebraic groups

The purpose of this appendix is to recall the notion of a *generic* torsor. All material presented here can be found in [29, Chapter VII], [13, Part II, §3] and [24, §6].

Let G be a linear algebraic group over a field F. We recall that an F-variety Y is called a *G-torsor* (or a *principal homogeneous space* under G) if G acts (on the right) simply transitively on Y. The set of isomorphism classes of G-torsors is in bijection with the set $H^1(F, G)$ of classes of 1-cocycles (sometimes simply referred as 1-cocycles in this dissertation).

Let n be an integer such that $G \subset \mathbf{GL}_n$. Let us set $S := \mathbf{GL}_n$ and $X := S/G$ (X is a *classifying variety* of G). Since for any field extension E/F one has $H^1(E, S) = 1$, the set $H^1(E, G)$ can be identified with the orbit space of the action of $S(E)$ on $X(E)$:

$$H^1(E, G) = X(E)/S(E).$$

The G-torsor Y over the function field $F(X)$ corresponding to the generic point of X is called a *generic torsor*.

One can show that if G is split and Y is a generic torsor with corresponding 1-cocycle $\xi \in H^1(F(X), G)$, the Chow ring $\mathrm{CH}({}_\xi(G/B))$, where B is a Borel subgroup of G, is generated by Chern classes (as in the proof of Proposition IV.1.1). That is the reason why in Chapter IV we call a 1-cocycle ξ *generic* if the associated Chow ring $\mathrm{CH}({}_\xi(G/B))$ is generated by Chern classes.

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