Irregular Perturbations and Rough Differential Systems

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Perturbations irrégulières et systèmes différentiels rugueux

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Résumé

Ce travail, à la frontière de l'analyse et des probabilités, s'intéresse à l'étude de systèmes différentiels a priori mal posés. Nous cherchons, grâce à des techniques issues de la théorie des chemins rugueux et de l'étude trajectorielle des processus stochastiques, à donner un sens à de tels systèmes puis à les résoudre, tout en montrant que les notions proposées ici étendent bien les notions classiques de solutions. Cette thèse comporte trois chapitres. Le premier traite des systèmes différentiels ordinaires perturbés additivement par des processus irréguliers. Le deuxième concerne l'équation de transport linéaire perturbée multiplicativement pas des chemins rugueux ; enfin, le dernier chapitre s'intéresse à une équation de la chaleur non linéaire perturbée par un bruit blanc espace-temps, l'équation de quantisation stochastique Φ^4 en dimension 3.

Après un chapitre introductif sur les théories mises en oeuvre, cette thèse se décompose en trois grands chapitres, chacun avec son propre thème.

Premier chapitre Dans ce premier chapitre, nous cherchons à résoudre des équations différentielles ordinaires perturbées par des processus irréguliers, ainsi qu'aux propriétés de régularisations que de telles perturbations induisent sur le système. Lorsque le champs de vecteur b est seulement höldérien, il est bien connu que l'équation $\dot{y} = b(y)$ n'admet pas une unique solution. Néanmoins, Davie [20] mais aussi Flandoli et al. [30] montrent que lorsque w suit la loi du mouvement brownien, et avec des hypothèses de régularité minimale sur b l'équation

$$\mathrm{d}x_t = b(x_t)\mathrm{d}t + \mathrm{d}w_t,$$

admet une unique solution, dans un sens à préciser.

L'objectif de ce chapitre est d'étendre ce type de résultat à une classe plus générale de perturbations additives, qui contient notamment les mouvements brownien fractionnaires mais aussi les processus de Poisson α-stables fortement non dégénérés. Pour ce faire, nous montrons qu'il est possible de donner un sens à l'équation translatée

$$\theta_t = \theta_0 + \int_0^t b(\theta_u + w_u)\mathrm{d}u$$

pour une large classe de champs de vecteurs qui peut notamment contenir des distributions. Sous les bonnes hypothèses sur b nous montrons d'une part l'existence et l'unicité des solutions, d'autre part leur régularité par rapport aux paramètres de l'équation. Enfin nous nous intéressons au cas où w suit la loi du mouvement brownien fractionnaire B^H, et nous montrons qu'il existe une unique solution (au sens trajectoires par trajectoires défini par Davie dans [20]) à l'équation précédente lorsque b appartient à l'espace de Besov-Hölder $B^{\alpha+1}_{\infty,\infty}$ avec $\alpha > -\frac{1}{1H}$.

Deuxième chapitre Dans ce chapitre nous nous intéressons à l'équation de transport conduite par un champ de vitesse b faiblement régulier et perturbée multiplicativement par un chemin rugueux $\boldsymbol{X} = (X, \mathbb{X}^2)$, pour des conditions initiales u_0 bornées.

$$\partial_t u + b.\nabla u + \nabla u.\mathrm{d}\boldsymbol{X}_t = 0.$$

Lorsque le champ de vecteur b a une faible régularité, l'équation est a priori mal posée, puisque $\nabla u.\mathrm{d}\boldsymbol{X}_t$ n'est pas défini, même au sens faible. Nous interpréterons l'équation sous sa forme

intégrale

$$u_t(\varphi) - u_0(\varphi) = \int_0^t u_q(b_q.\nabla\varphi + (\operatorname{div} b_q)\varphi)\mathrm{d}q + \int_0^t u_q(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q$$

où $u(\psi) = \langle u, \psi\rangle$ désigne le crochet de dualité de u comme distribution. Lorsque X est un mouvement brownien, et en interprétant $\int_0^\cdot u_q(\nabla\varphi).\mathrm{d}\boldsymbol{X}_t$ comme l'intégrale de Stratonovitch, Flandoli et al. [30] mais aussi Beck et al. [6] montrent qu'il y a unicité des solutions pour des champs de vecteur b seulement höldériens d'indice $\varepsilon > 0$. Néanmoins, dans le cas de perturbations pour lesquelles le calcul stochastique classique n'est pas disponible, la méthode adoptée dans les travaux sus-cités ne fonctionne pas aussi bien.

Pour surmonter cette difficulté, nous nous intéressons à des chemins rugueux. Il est néanmoins nécessaire dans ce cas de définir une nouvelle notion de solutions.

Le premier travail de ce chapitre est donc de donner une notion raisonnable de solutions, que nous appellerons les solutions faiblement (ou fortement) contrôlées par le processus X. Dans un second temps, nous montrons que de telles solutions faiblement contrôlées existent pour des chemins rugueux d'indice $1/2 \geqslant \gamma > \frac{1}{3}$ et des b à croissance linéaire. Dans un troisième temps, grâce aux résultats du Chapitre 1 sur les flots d'équations différentielles perturbées, nous montrons qu'il existe une unique solution pour l'équation de transport rugueuse lorsque $\boldsymbol{X}$ a de bonnes propriétés. Ces résultats s'appliquent directement au mouvement brownien fractionnaire. Nous montrons dans cette partie qu'il est notamment possible de considérer des champs de vecteurs aléatoires.

Troisième chapitre Dans ce chapitre nous nous intéressons à l'équation de quantisation stochastique Φ^4 en dimension trois. Cette équation fait partie d'une classe d'équations différentielles partielles stochastiques qui sont des perturbations non linéaires de l'équation de la chaleur stochastique, *i.e.*

$$\partial_t u - \Delta u = f(u) + \xi$$

où ξ est un bruit blanc gaussien. Dans notre cas, la perturbation qui nous intéresse est $f(x) = x^3$ et la dimension est $d = 3$. Le bruit blanc étant une distribution, et malgré le caractère régularisant de l'opérateur de la chaleur, cette équation est mal posée, le terme u^3 n'ayant pas de sens classique. Dans le cas de la dimension $d = 2$, Da Prato et Debussche [17] ont montré comment interpréter cette équation de façon à ce qu'elle soit bien posée. Ils ont notamment mis en évidence la nécessité de renormaliser l'équation pour que cette dernière ait un sens.

Dans le cas de la dimension $d = 3$, Hairer dans son article [50] montre, grâce aux structures de régularité basées entre autre sur la théorie des ondelettes, comment donner un sens à cette équation de manière à ce qu'elle soit bien posée. Nous nous proposons de donner une preuve alternative de ce résultat à l'aide de la théorie développée par Gubinelli et al. dans [44]. Nous nous appuyons sur les paraproduits de Bony et les distributions para-contrôlées, préférant un travail dans l'espace de Fourier et l'utilisation des blocs de Paley-Littlewood à un travail en espace direct et l'utilisation d'ondelettes.

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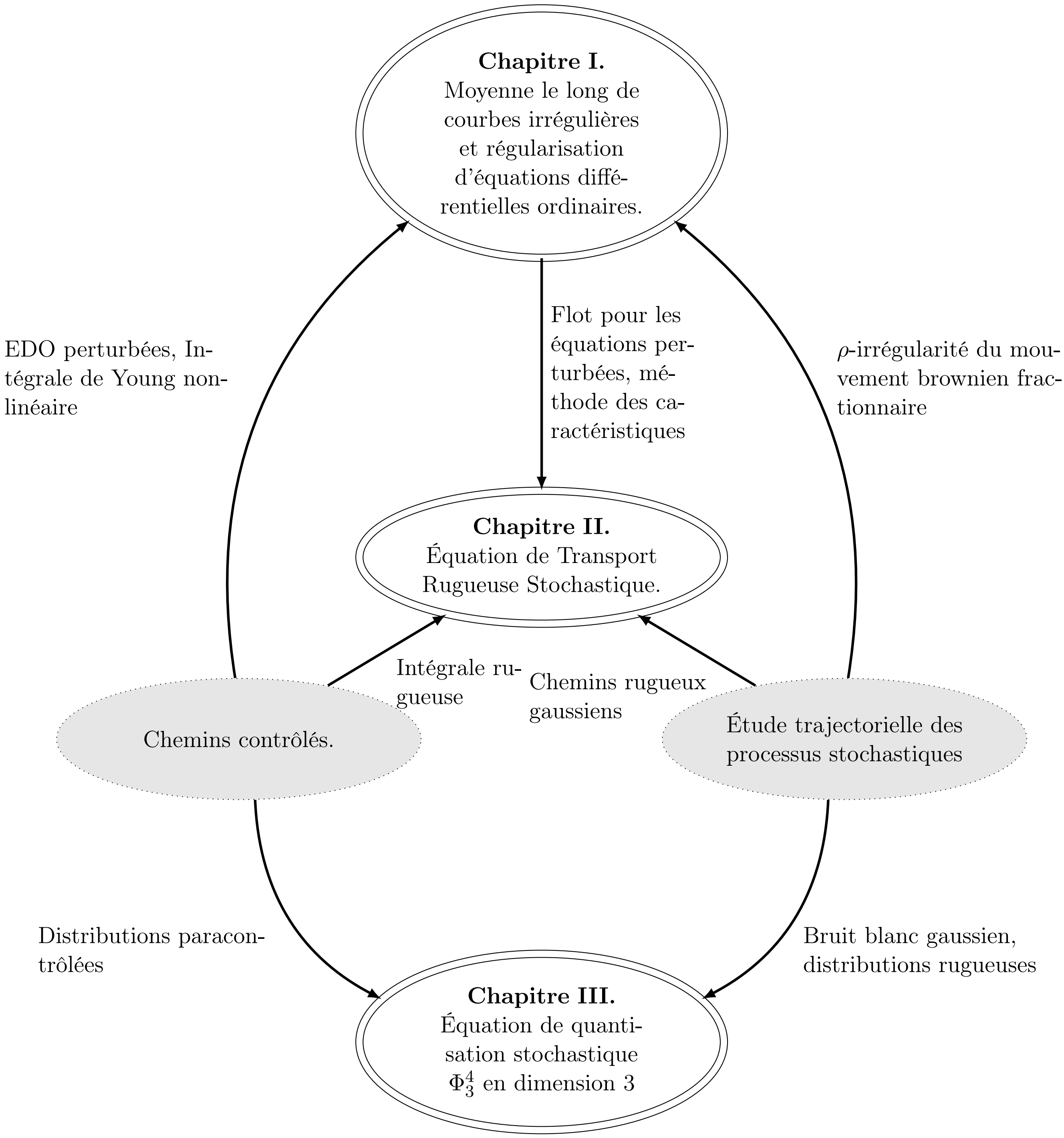
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Notations générales

- Tout au long de cette thèse, $d \in \mathbb{N}$ sera utilisé pour la dimension de l'espace $\mathbb{R}^d$. Lorsque la norme sur cet espace n'est pas spécifiée, on considèrera toujours la norme euclidienne usuelle.

- Lorsque (E, d_E) et (F, d_F) sont des espaces métriques, on note $C(E; F)$ l'ensemble des fonctions continues de E dans F. Lorsque il n'y a pas d'ambiguïté, on notera $C(E)$ ou même C.

- Si $\gamma \in (0, 1]$, $T \in \mathbb{R}_+$, et $(F, \|.\|_F)$ est un espace de Banach, on note

$$\mathcal{C}^\gamma([0,T];F) = \left\{ f : [0,T] \to \mathbb{R}^d : [\![f]\!]_\gamma := \sup \frac{\|f_t - f_s\|_F}{|t-s|^\gamma} < +\infty \right\}$$

  l'ensemble des fonctions höldériennes de $[0, T]$ dans F. Cet espace muni d'une des deux normes équivalentes suivantes

$$\|f\| = \|f\|_\gamma + \|f\|_{\infty,[0,T]} \quad \text{ou} \quad \|f\| = \|f\|_\gamma + |f_0|$$

  est alors complet. On notera indifférement l'une ou l'autre de ces deux normes $\|f\|_{\gamma,[0,T];F}$.

- L'ensemble des fonctions m-fois différentiables de $\mathbb{R}^d$ dans $\mathbb{R}^{d'}$ est noté $C^m(\mathbb{R}^d, \mathbb{R}^{d'})$. L'espace des fonctions C^∞ à support compact de $\mathbb{R}^d$ dans $\mathbb{R}^{d'}$ est noté $C_c^\infty(\mathbb{R}^d, \mathbb{R}^{d'})$.

- La différentielle sur $\mathbb{R}^d$ est notée D, et si $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ est un multi-indice et $|\alpha| = \alpha_1 + \cdots + \alpha_d$, on note

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_d} x_d}.$$

  De plus, lorsque $f \in C^1(\mathbb{R}^d, \mathbb{R})$, on note $\nabla f = (\partial_1 f, \dots, \partial_d f)$ le gradient de f, et $\nabla^2 f = (\partial_i \partial_j f)_{1 \leqslant i,j \leqslant d}$ la Hessienne de f. Lorsque $f \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ le déterminant de la jacobienne de f est noté $\operatorname{Jac} f := \det((\partial_i f^j)_{1 \leqslant i,j \leqslant d})$.

- Soit $d \in \mathbb{N}$, on note $\mathcal{S}(\mathbb{R}^d)$ l'espace des fonctions de Schwartz et $\mathcal{S}'(\mathbb{R}^d)$ l'ensemble des distributions tempérées. On notera indifféremment $\mathcal{F}f$ ou $\hat{f}$ la transformée de Fourier d'un élément $f \in \mathcal{S}'(\mathbb{R}^d)$.

- Une fonction f de $\mathbb{R}^d$ dans $\mathbb{R}^{d'}$ est lipschitzienne si

$$\|f\|_{\mathrm{Lip}} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} < +\infty$$

  Une fonction est localement lispchitzienne si sa restriction à tout compact est lipschitzienne.

- Une fonction f de $\mathbb{R}^d$ dans $\mathbb{R}^{d'}$ est à croissance linéaire si

$$\|f\|_{\mathrm{Lin}} := \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{1 + |x|} < +\infty.$$

- Soit $[s,t] \subset \mathbb{R}^d$. L'ensemble $\mathcal{P} \subset [s,t]^n$ est une partition de $[s,t]$ si

$$\mathcal{P} = \{(t_0, \dots, t_n) : s = t_0 < t_i < \dots < t_n = t\}.$$

  On note $|\mathcal{P}| = \max_{0 \leqslant i \leqslant n-1} |t_{i+1} - t_i|$.

- Soit $f : [0,T] \to F$ un Banach. On notera de façon indifférenciée

$$f_t - f_s = \delta f_{s,t} = f_{s,t}.$$

- Soit $a, b \in \mathbb{R}$. On note $a \lesssim b$ s'il existe une constante $C > 0$ indépendante de a et b telle que $a \leqslant Cb$. On note $a \sim b$ quand $a \lesssim b$ et $b \lesssim a$. De plus pour $c > 0$ $a \lesssim_c b$ indique que la constante C dépend de c.

Chapitre 1

Introduction

1.1 Moyennes le long de courbes irrégulières et régularisation d'équations différentielles ordinaires

Dans ce chapitre nous étudions les propriétés de régularisation de certains processus irréguliers sur l'équation différentielle perturbée suivante.

$$x_t = x_0 + \int_0^t b(q, x_q)\mathrm{d}q + w_t. \tag{1.1}$$

Il est bien connu que lorsque $w = 0$, et lorsque b est seulement bornée, cette équation n'est pas bien posée. Cependant dans [20], Davie montre que si w suit la loi du mouvement brownien, et que $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ est bornée et mesurable, l'équation précédente admet une unique solution pour presque toutes les trajectoires w. Ce résultat peut être interprété comme un résultat de régularisation par le mouvement brownien.

La régularisation par le bruit dans le cadre du mouvement brownien est un sujet assez bien compris. On pourra consulter par exemple Veretennikov [73], Krylov et Röckner [57], Flandoli, Gubinelli et Priola [30], Flandoli et Da Prato [18], Zhang [76]. Tous ces travaux s'appuientt essentiellement sur le calcul d'Itô. A l'aide du calcul de Malliavin, Meyer-Brandis et Proske [64] arrivent à des conclusions similaires. Le travail récent de Delarue et Diel [23] traite le problème d'un champ de vecteur irrégulier, aléatoire, donné par la solution d'une équation de Burgers stochastique. La notion de solution est cependant faible au sens probabiliste, et liée à un problème de martingale mal posé au sens classique. La contribution de Davie à ce sujet est un peu différente dans la mesure où le résulte concerne l'équation différentielle ordinaire et non pas l'équation différentielle stochastique associée. En effet, l'existence et l'unicité des solutions est étudiée dans l'espace des fonctions continues et non pas dans l'espace des processus adaptés. Cette différence a été souligné par Flandoli dans [28] qui appelle ce type de solutions les solutions trajectoires par trajectoires. De ce point de vu, le travail de Davie est uniquement analytique, et un des objectifs du chapitre est de caractériser les propriétés de régularisations des perturbations, continues ou non, aléatoire ou non sur la dynamique définie par un champ de vecteurs b irrégulier.

La régularisation par des processus réguliers est un phénomène qui se produit aussi dans le cas d'équations aux dérivées partielles déterministes, par exemple pour l'équation de Kortweg-de-Vries [42, 3] et pour les équations de Navier-Stokes et d'Euler [4]. Les techniques d'intégration de Young utilisées dans ce travail sont d'ailleurs essentiellement les mêmes que celles employées dans [42] et s'inspirent de la théorie des chemins rugueux [60, 61, 34]. Récemment, un article de Hu et Le utilise des techniques similaires à celles présentées dans cette thèse, et ce pour des problèmes proches liés aux dynamiques décrites pas des champs de vecteurs höldériens.

Dans deux articles récents Chouk et Gubinelli [14, 15] analysent le phénomène de régularisation dans le contexte des équations aux dérivées partielles dispersives modulées par un signal irrégulier. En particulier, ils considèrent des équations de la forme

$$\frac{\partial}{\partial t}\varphi = A\varphi \frac{\mathrm{d}}{\mathrm{d}t} w + \mathcal{N}(\varphi)$$

où w est une fonction continue arbitraire, A un opérateur linéaire non borné, comme l'opérateur de Schrödinger $i\partial^2$ où l'opérateur d'Airy ∂^3, qui agit sur l'ensemble des fonctions périodiques sur le tore, et où $\mathcal{N}$ est une non linéarité polynomiale. Notre travail et celui de Chouk et Gubinelli sont reliés par les outils utilisés pour analyser les effets de la régularisation par w et notamment *l'opérateur de moyennisation* T_t^w défini pour toute fonction f mesurable $f : \mathbb{R}^d \to \mathbb{R}^d$ par

$$T_t^w f(x) = \int_0^t f(x + w_u)\mathrm{d}u.$$

L'opérateur de moyennisation peut être interprété formellement comme la convolution entre le temps local de w et la fonction f. Pour plus de détails on pourra consulter l'Annexe B.

L'observation principale de Davie [20] est que si $b : \mathbb{R}^d \to \mathbb{R}^d$ est une fonction bornée, alors pour presque toute trajectoire brownienne $w : [0, T] \to \mathbb{R}^d$ et pour tout $0 \leqslant t \leqslant T$ la fonction $x \to T_t^w b(x)$ est quasi lipschitzienne, *i.e.*

$$|T_t^w b(x) - T_t^w b(y)| \lesssim |x - y| \log_+^{1/2}(1/|x - y|). \tag{1.2}$$

Ce gain de quasiment un degré de régularité est l'ingrédient essentiel pour prouver l'unicité de l'équation (1.1) dans le cas de champs de vecteurs b bornés.

Avant de nous intéresser plus particulièrement à l'opérateur de moyennisation et aux équations différentielles ordianires perturbées, nous allons introduire les espaces de fonctions mis en jeu dans cette étude.

1.1.1 Préliminaires

Nous allons nous intéresser aux propriétés de l'opérateur de moyennisation pour des fonctions ou des distributions dont les transformées de Fourier possèdent de bonnes propriétés d'intégrabilité : les espaces de Besov-Hölder. Pour une théorie générale de ces espaces, on pourra consulter le livre de Triebel [72]. Dans ce chapitre nous n'avons besoin que de peu d'informations sur ces espaces de Besov-Hölder. Pour une présentation plus en profondeur de ces espaces, nous renvoyons à la partie 1.3.1 de cette introduction.

Pour obtenir les propriétés qui nous intéressent l'idée est de décomposer les éléments de $\mathcal{S}'$ en somme de fonctions dont les transformées de Fourier sont localisées. Pour cela nous introduisons deux fonctions de classe C^∞ de $\mathbb{R}^d$ dans $\mathbb{R}$ à support compact et à symétrie radiale, θ et χ telles que

1. Le support de χ est contenu dans une boule B et le support de θ est inclue dans un anneau $\mathcal{A} = \{\xi \in \mathbb{R}^d : r \leqslant |\xi| \leqslant R\}$.

2. Pour tout $k \in \mathbb{R}^d$, $\chi(\xi) + \sum_{j \geqslant 0} \theta(2^{-j}\xi) = 1$.

3. Pour $i, j \in \mathbb{N}$ avec $|i - j| > 1$, $\operatorname{supp} \theta(2^{-j}.) \cap \operatorname{supp} \theta(2^{-i}.) = \emptyset$ et pour $i \geqslant 1$ $\operatorname{supp} \chi \cap \operatorname{supp}(\theta(2^{-i}.)) = \emptyset$.

Pour ce qui est de l'existence de telles fonctions, on pourra consulter [5], Proposition 2.10. On définit alors les blocs de Paley-Littlewood pour tout $f \in \mathcal{S}'(\mathbb{R}^d)$ par

$$\Delta_{-1} f = \mathcal{F}^{-1}(\chi \mathcal{F} f) \quad \text{et pour} \quad i \geqslant 0 \quad \Delta_i f = \mathcal{F}^{-1}(\theta(2^{-i}.)\mathcal{F} f).$$

Il est utile de noter que les blocs de Paley-Littlewood d'une distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ sont localisés dans l'espace de Fourier. Ainsi pour tout $i \geqslant -1$ le bloc $\Delta_i f$ est une fonction $C^\infty(\mathbb{R}^d)$ et de plus appartient à tous les espaces $L^p(\mathbb{R}^d)$ pour $1 \leqslant p \leqslant +\infty$.

Les blocs de Paley-Littlewood nous permettent de définir les espaces de Besov-Hölder comme suit.

Définition 1.1.1. Pour $\alpha \in \mathbb{R}$ on définit l'espace de Besov-Hölder $\mathcal{C}^\alpha(\mathbb{R}^d, \mathbb{R}^m)$ par

$$\mathcal{C}^\alpha(\mathbb{R}^d, \mathbb{R}^m) = \{f \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^m) : \|u\|_\alpha = \sup_{j \geqslant -1} (2^{j\alpha} \|\Delta_j u\|_{\infty, \mathbb{R}^d}) < +\infty\}$$

Lorsque $\alpha \in (0,1)$, l'espace de Besov-Hölder s'identifie à l'espace des fonctions höldériennes bornées de $\mathbb{R}^d$ dans $\mathbb{R}^m$, la norme $\|.\|_\alpha$ étant équivalente à la norme usuelle sur cet espace. De plus, quand $\alpha > 0$, les éléments de l'espace $\mathcal{C}^\alpha$ sont des fonctions continues bornées et non plus des distributions.

Ces espaces ne suffisent pas cependant pour étudier T^w en tant qu'opérateur. Si on se réfère au résultat de Davie (équation (1.2)), la moyennisation d'une fonction bornée f contre w, *i.e.* la fonction $T^w f$ ne peut pas appartenir à un espace $\mathcal{C}^\alpha$, le module de continuité n'étant pas de la bonne forme. Afin de surmonter cette difficulté, nous introduisons alors les espaces de Hölder avec poids.

Définition 1.1.2. Soit $\psi : \mathbb{R} \to \mathbb{R}$ une fonction positive, croissante et continue telle que pour tout $x \in \mathbb{R}_+$ et tout $c > 0$ $\psi(cx) \leqslant C_{c,\psi} \psi(x)$ où $C_{c,\psi}$ est une constante qui dépend de ψ et de c mais pas de x. Dans la suite la fonction ψ sera appelée poids.

Soit $0 < \alpha \leqslant 1$ et $\mathbb{N} \in n$. L'espace de Hölder avec poids $\mathcal{C}^{\alpha,\psi}$ est défini par

$$\mathcal{C}^{n+\alpha,\psi} = \left\{ f : \mathbb{R}^d \to \mathbb{R}^m : [\![f]\!]_{n+\alpha,\psi} := \sup_{x \neq y} \frac{|D^n f(x) - D^n f(y)|}{|x-y|^\alpha \psi(|x| + |y|)} < +\infty \right\}$$

où $D^n f$ est la différentielle n-ième de f. On définit alors la norme $\|f\|_{n+\alpha,\psi} = [\![f]\!]_{\alpha,\psi} + \sum_{k=0}^n |D^k f(0)|$ qui rend l'espace $\mathcal{C}^{n+\alpha,\psi}$ complet.

Pour différentes raisons, ces espaces ne suffisent toujours pas, lorsque w est un processus stochastique, à définir T^w comme un opérateur presque sûrement borné. Il sera possible dans ces espaces d'étudier uniquement les marginales finies dimensionnelles de l'opérateur $T^w f$. Afin de caractériser T^w en tant qu'opérateur aléatoire presque sûrement borné, il est nécessaire d'introduire les espaces de Fourier-Lebesgue suivants.

Définition 1.1.3. Soit $\alpha \in \mathbb{R}^d$, on définit l'espace de Fourier Lebesgue d'indice α, de la manière suivante

$$\mathcal{F}L^\alpha(\mathbb{R}^d, \mathbb{R}^m) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^m) : N_\alpha(f) := \int_{\mathbb{R}^d} (1 + |\xi|)^\alpha |\hat{f}(\xi)| \mathrm{d}\xi < +\infty \right\}.$$

Muni de la norme N_α, l'espace $\mathcal{F}L^\alpha(\mathbb{R}^d, \mathbb{R}^m)$ est un espace complet. De plus lorsque $\alpha > 0$ les éléments de $\mathcal{F}L^\alpha(\mathbb{R}^d, \mathbb{R}^m)$ sont des fonctions continues qui tendent vers 0 lorsque $|x| \to +\infty$.

Finalement les espaces précédents sont reliés par les inclusions suivantes.

Proposition 1.1.4. *Soit $\alpha' < \alpha$ et ψ un poids. Alors les inclusions continues suivantes sont vérifiées*

$$\mathcal{F}L^\alpha \subset \mathcal{C}^\alpha \subset \mathcal{C}^{\alpha'}.$$

De plus si $\alpha > 0$

$$\mathcal{C}^\alpha \subset \mathcal{C}^{\alpha,\psi} \subset C(\mathbb{R}^d, \mathbb{R}^m).$$

1.1.2 Moyennisation et ρ-irrégularité

Dans ce chapitre, nous analysons donc le comportement de l'opérateur de moyennisation T^w en tant qu'opérateur sur les espaces de Besov-Hölder ou de Fourier-Lebesgue, et ce pour tous les $\alpha \in \mathbb{R}$. Nous allons considérer une classe de perturbations w qui inclut le mouvement brownien fractionnaire d'indice de Hurst $H \in (0, 1)$, qui est l'unique processus d–dimensionnel gaussien centré $(B^H_t)_t$ avec pour fonction de covariance

$$K_H(s,t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} + |t - s|^{2H}),$$

pour tout $s, t \in \mathbb{R}_+$. On renvoie à l'annexe A pour des propriétés basiques d'un tel processus.

Comme application des propriétés de moyennisation de w étudiées à travers les propriétés de T^w nous obtiendrons différents résultats d'existence et d'unicité pour l'équation (1.1), ainsi que des propriétés sur le flot d'une telle equation et ce pour des champs de vecteurs irréguliers et éventuellement des distributions.

Le mouvement brownien fractionnaire n'est pas choisi au hasard. Ce processus est assez simple, et un certain nombre de résultats de régularisation sont déjà connus, bien que le calcul d'Itô ou les techniques de Davie ne puissent pas s'appliquer. On pourra par exemple consulter Nualart et Ouknine [67, 68]. La possibilité de faire varier l'exposant de Hurst H permet aussi de considérer différentes échelles d'irrégularités. De plus les propriétés d'invariance en loi du mouvement brownien fractionnaire vont nous permettre d'étudier effectivement l'opérateur de moyennisation T^{B^H}.

Ainsi, si on s'intéresse aux propriétés de moyennisation du mouvement brownien fractionnaire, on obtient le résultat suivant

Théorème 1.1.5. *Soit $H \in (0, 1)$ et $\rho < 1/2H$. Il esiste $\gamma > 1/2$ et un poids ψ tel que pour tout $\alpha + \rho > 0$ et pour tout $f \in \mathcal{C}^\alpha(\mathbb{R}^d, \mathbb{R})$, il existe un sous ensemble $\mathcal{N}_{f,\gamma,\alpha} \subset C([0, T], \mathbb{R}^d)$, qui dépend de f et de γ, de mesure 0 relativement à la loi du mouvement fractionnaire de dimension d et d'indice de Hurst H tel que pour tout $w \notin \mathcal{N}_{f,\gamma,\alpha}$, on a*

$$\|T^w_t f - T^w_s f\|_{\mathcal{C}^{\alpha+\rho,\psi}} \lesssim_w \|f\|_{\mathcal{C}^\alpha} |t - s|^\gamma.$$

Si on laisse un instant de côté la régularité temporelle dans le résultat précédent, on constate que l'opérateur de moyennisation permet de gagner quasiment $1/2H$ dérivées en espace. Le résultat du Théorème 1.1.5 n'est néanmoins pas satisfaisant puisqu'il ne permet pas de montrer que l'operateur de moyennisation est presque sûrement borné de $\mathcal{C}^\alpha$ dans $\mathcal{C}^{\alpha+\rho,\psi}$. La difficulté vient du fait que l'ensemble de mesure nulle dépend de f. L'analyse suivante nous donne la clé pour pallier à cette difficulté. Si $f : \mathbb{R}^d \to \mathbb{R}^m$ est assez régulière, on a pour tout $0 \leqslant s < t \leqslant T$

$$(T_t^w f - T_s^w f)(x) = \int_s^t f(w_u + x)\mathrm{d}u = \int_{\mathbb{R}^d} \hat{f}(\xi)e^{i\xi.x}\left(\int_s^t e^{i\xi.w_u}\mathrm{d}u\right)\mathrm{d}\xi$$

Ainsi, si $f \in \mathcal{F}L^\alpha$ et $\gamma \in [0,1]$, on a

$$|(T_t^w f - T_s^w f)(x)| \leqslant \int_{\mathbb{R}^d} |\hat{f}(\xi)|(1+|\xi|)^\alpha \mathrm{d}\xi \left(\sup_{\xi\in\mathbb{R}^d}\sup_{0\leqslant s<t\leqslant T}(1+|\xi|)^{-\alpha}\frac{\left|\int_s^t e^{i\xi.w_u}\mathrm{d}u\right|}{|t-s|^\gamma}\right)|t-s|^\gamma.$$

Ainsi $T_t^w - T_s^w$ est borné en tant qu'opérateur de $\mathcal{F}L^\alpha$ dans $\mathcal{F}L^{\alpha+\rho}$ si la quantité

$$\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} := \sup_{\xi\in\mathbb{R}^d}\sup_{0\leqslant s<t\leqslant T}(1+|\xi|)^\rho\frac{|\Phi_t^w(\xi)-\Phi_s^w(\xi)|}{|t-s|^\gamma} < +\infty$$

où

$$\Phi_t^w(\xi) = \int_0^t e^{i\xi.w_u}\mathrm{d}u,$$

et de plus

$$\|T_t^w - T_s^w\|_{\mathcal{L}(\mathcal{F}L^\alpha;\mathcal{F}L^{\alpha+\rho})} \lesssim \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}|t-s|^\gamma.$$

Grâce à cette observation, l'étude de l'opérateur de moyennisation T^w sur $\mathcal{F}L^\alpha$ se réduit à l'étude de la fonctionnelle de moyennisation Φ^w, transformée de Fourier de la mesure d'occupation. Pour des généralités sur les mesures d'occupation et leurs densité pour les fonctions déterministes et aléatoires, on pourra consulter l'article de synthèse de Geman et Horowitz [37]. Cette remarque nous incite à introduire la notion suivante.

Définition 1.1.6. Soit $\rho > 0$ et $\gamma \in (0,1]$. Une fonction $w \in L^\infty([0,T];\mathbb{R}^d)$ est dite (ρ,γ)–irrégulière si

$$\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} < +\infty.$$

De plus w est ρ–irrégulière s'il existe $\gamma > 1/2$ tel que w est (ρ,γ)–irrégulière.

L'observation précédente peut alors se comprendre comme suit :

Proposition 1.1.7. *Soit $\rho > 0$, $\gamma \in (0,1]$, et $w \in L^\infty([0,T];\mathbb{R}^d)$ une fonction (ρ,γ)–irrégulière. L'opérateur de moyennisation T^w est borné de $\mathcal{F}L^\alpha$ dans $\mathcal{F}L^{\alpha+\rho}$ et de plus, pour tous $s \leqslant t \in [0,T]$*

$$\|T_t^w - T_s^w\|_{\mathcal{L}(\mathcal{F}L^\alpha,\mathcal{F}L^{\alpha+\rho})} \leqslant \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}|t-s|^\gamma.$$

La valeur $1/2$ pour γ ne semble pas avoir de signification particulière, mais grâce à des techniques d'intégration de Young cette régularité temporelle est suffisante pour traiter l'essentiel des résultats de régularisation de l'équation (1.1) présentés dans ce travail. Le théorème suivant montre de plus qu'il est possible de trouver un grand nombre de fonctions ρ-irrégulières.

Théorème 1.1.8. *Soit $H \in (0,1)$ et $\rho < 1/2H$. Alors le mouvement brownien fractionnaire d–dimensionnel B^H est presque sûrement ρ–irrégulier. De plus pour $\gamma > 1/2$ comme dans la Définition 1.1.6 il existe $\lambda > 0$ tel que*

$$\mathbb{E}[\exp(\lambda \|\Phi^{B^H}\|^2_{\mathcal{W}_T^{\rho,\gamma}})] < +\infty.$$

Le mouvement brownien fractionnaire n'est évidemment pas le seul processus stochastique à posséder cette propriété. Les processus de Lévy α–stables non dégénérés sont aussi des processus ρ–irréguliers. On obtient alors le théorème suivant dont on trouvera une preuve dans l'Annexe B.

Théorème 1.1.9. *Soit $\alpha \in (0,2]$ et X un processus de Lévy α–stable non dégénéré de dimension d. Presque sûrement pour tout $\rho < \frac{\alpha}{2}$ le processus X est ρ–irrégulier.*

Outre la définition de l'opérateur de moyennisation, la notion de (ρ,γ)-irrégularité semble prometteuse pour l'étude d'un certains nombre de problèmes reliés aux propriétés trajectorielles des processus. On renvoie à l'Annexe B pour plus de résultats sur cette notion.

1.1.3 Principaux résultats du chapitre

Les résultats exposés sur les propriétés de moyennisation par le mouvement brownien fractionnaire ou les processus ρ-irréguliers vont finalement nous permettre d'étudier l'équation suivante.

$$y_u = x + \int_0^t b(y_u)\mathrm{d}u + w_t. \tag{1.3}$$

Deux situations un peu différentes se présentent.

1. Lorsque $b \in \mathcal{C}^\alpha$, ou $b \in \mathcal{F}L^\alpha$ pour un certain $\alpha > 0$. Dans ce cas l'équation a un sens naturel, puisque b est une fonction continue bornée. Nous pouvons donc nous interroger sur l'existence et la régularité du flot d'une telle équation.

2. Lorsque $b \in \mathcal{C}^\alpha$, ou $b \in \mathcal{F}L^\alpha$ pour $\alpha \leqslant 0$. Dans ce cas la définition même de l'équation pose problème. Nous allons, grâce aux propriétés de l'opérateur de moyennisation, donner un sens raisonnable à cette équation, et étudier dans ce cadre les propriétés d'existence et d'unicité des solutions.

L'analyse de ces deux situations repose sur une seule et même technique qui provient de l'heuristique suivante. Pour $0 < \varepsilon, \theta \leqslant 1$, $f \in \mathcal{C}^\varepsilon$ et $\theta \in \mathcal{C}^\delta([0,T])$ et $\mathcal{P}$ une partition de $[0,T]$, on a

$$\begin{aligned}
\left|\int_0^T f(\theta_u + w_u)\mathrm{d}u - \sum_i \int_{t_{i+1}}^{t_i} f(\theta_{t_i} + w_u)\mathrm{d}u\right| &= \left|\int_0^T f(\theta_u + w_u)\mathrm{d}u - \sum_i T^w_{t_i,t_{i+1}} f(\theta_{t_i})\right| \\
&\leqslant \sum_i \left|\int_{t_{i+1}}^{t_i} f(\theta_u - w_u) - f(\theta_{t_i} + w_u)\mathrm{d}u\right| \\
&\leqslant \|f\|_\varepsilon \sum_i \int_{t_i}^{t_{i+1}} |\theta_{t_i} - \theta_u|^\varepsilon \mathrm{d}u \\
&\leqslant \|f\|_\varepsilon \|\theta\|_\delta^\varepsilon |\mathcal{P}|^{\varepsilon\delta+1}.
\end{aligned}$$

Ainsi, pour des f et θ régulières, les sommes de Riemann $\sum_i T^w_{t_i,t_{i+1}} f(\theta_{t_i})$ convergent vers $\int f(\theta_u + w_u)\mathrm{d}u$. Il est alors naturel de se demander sous quelles hypothèses sur les fonctions f, w et θ les sommes de Riemann $\sum_i T^w_{t_i,t_{i+1}} f(\theta_{t_i})$ convergent. Cette question est à mettre en perspective avec la théorie de l'intégrale de Young [75] et plus généralement la théorie des chemins rugueux [60, 61, 34, 41].

En adaptant les techniques qui permettent de définir l'intégrale de Young, on peut alors montrer le théorème suivant :

Théorème 1.1.10. *Soit ψ un poids, $0 < \gamma, \delta, \eta \leqslant 1$ tels que $\delta\eta + \gamma > 1$ et $\theta \in \mathcal{C}^\delta([0,T])$. Si $T^w f \in \mathcal{C}^\gamma([0,T]; \mathcal{C}^{\eta,\psi}(\mathbb{R}^d, \mathbb{R}^d))$ alors il existe $I : [0,T]^2 \to \mathbb{R}^d$ telle que pour tout $0 \leqslant s \leqslant t \leqslant T$,*

$$I_{s,t}(\theta, T^w f) = \lim_{\substack{\mathcal{P}\text{ partition de}[s,t] \\ |\mathcal{P}| \to 0}} \sum_{[t_i,t_{i+1}]\subset \mathcal{P}} T^w_{t_i,t_{i+1}} f(\theta_{t_i}).$$

De plus pour tout $0 \leqslant s \leqslant u \leqslant t \leqslant T$ $I_{s,t}(\theta, T^w f) = I_{s,u}(\theta, T^w f) + I_{u,t}(\theta, T^w f)$ et de plus

$$|I_{s,t}(\theta, T^w f) - T^w_{s,t} f(\theta_s)| \lesssim |t-s|^{\gamma+\eta\delta} \|\theta\|^\eta_\delta \psi(\|\theta\|_\infty) \|T^w f\|_{\mathcal{C}^\gamma([0,T];\mathcal{C}^{\eta,\psi}(\mathbb{R}^d,\mathbb{R}^d))} \tag{1.4}$$

où

$$\|T^w f\|_{\mathcal{C}^\gamma([0,T];\mathcal{C}^{\eta,\psi}(\mathbb{R}^d,\mathbb{R}^d))} = \sup_{0\leqslant s<t\leqslant T} \sup_{x\neq y\in\mathbb{R}^d} \frac{|T^w_{s,t} f(x) - T^w_{s,t} f(y)|}{|t-s|^\gamma |x-y|^\eta \psi(|x|+|y|)}.$$

Comme la fonctionnelle I est la limite de sommes de Riemann, elle peut être interprétée comme l'intégrale de θ le long de l'opérateur de moyennisation $T^w f$. Pour cette raison on notera

$$\int_s^t T^w_{\mathrm{d}u} f(\theta_u) := I_{s,t}(\theta, T^w f).$$

Cette définition étendue de l'intégrale, et en particulier la borne (1.4), est la pierre angulaire de l'étude de l'équation (1.3). On comprend désormais l'intérêt des processus ρ–irréguliers dans ce contexte. En effet, grâce à la Proposition 1.1.7, lorsque w est un processus ρ–irregulier, $\alpha > -\rho$ et $f \in \mathcal{F}L^{\alpha+1}$ alors $T^w f$ vérifie exactement les hypothèses nécessaires à la définition de l'intégrale de Young non linéaire. De plus, dans le cas $\alpha + 1 > 0$, toute fonction $b \in \mathcal{F}L^{\alpha+1}$ est continue bornée. Il est bien connu que dans ce cas l'équation (1.3) admet des solutions continues. Ainsi, si y est une solution de cette équation, $\theta = y - w$ est une solution de l'équation associée

$$\theta_u = x + \int_0^t b(\theta_u + w_u)\mathrm{d}u,$$

et comme b est bornée, θ est lipschitzienne. Ainsi, θ est aussi solution de l'équation

$$\theta_u = x + \int_0^t T^w_{\mathrm{d}u} b(\theta_u).$$

Grâce à la borne (1.4), il est alors possible de montrer le théorème suivant.

Théorème 1.1.11. *Soit $\rho > 0$, $\alpha > -\rho$ tel que $\alpha + 3/2 > 0$. Soit w une fonction ρ–irrégulière et $b \in \mathcal{F}L^{\alpha+3/2}$. Alors il existe une unique fonction continue y solution de*

$$y_t = x + \int_0^t b(y_u)\mathrm{d}u + w_u.$$

De plus le flot $x \to y$ est localement lipschitzien uniformément en temps.

Ce théorème s'applique évidemment au cas du mouvement brownien fractionnaire. Cependant grâce à la même analyse, et en utilisant le Théorème 1.1.5 il est possible de relâcher quelque peu les conditions de régularité du champ de vecteurs.

Théorème 1.1.12. *Soit $H \in (0,1)$, $\alpha > -1/2H$ tel que $\alpha + 1 > 0$. Pour tout $b \in \mathcal{C}^{\alpha+1}$ et tout $x \in \mathbb{R}^d$ il existe un ensemble $\mathcal{N}_{b,x} \subset C([0,T], \mathbb{R}^d)$ qui est de mesure nulle par rapport à la loi du mouvement brownien fractionnaire d–dimensionnel d'indice de Hurst H et tel que pour tout $w \notin \mathcal{N}_{b,x}$ il existe une unique solution continue $y \in C([0,T]; \mathbb{R}^d)$ à l'équation différentielle (1.3).*

Dans le cas $\alpha < 0$ l'équation n'est pas bien définie, puisque l'évaluation d'une distribution le long d'une courbe n'est pas possible en général. Cependant, si on se restreint à un sous-espace adéquat des fonctions continues, il est possible de montrer que cette procédure a bien un sens. Les éléments de ce sous-espace sont en fait des perturbations additives de w au sens de la définition suivante.

Définition 1.1.13. L'espace $\mathcal{D}^w_\gamma$ des chemins (w, γ)-controlés est l'espace

$$\mathcal{D}^w_\gamma = \{y \in C([0,T]; \mathbb{R}^d) : y - w \in \mathcal{C}^\gamma([0,T]; \mathbb{R}^d)\}.$$

Pour les chemins contrôlés il est alors possible de définir l'intégrale d'une distribution b le long de tels chemins, au sens du théorème suivant.

Théorème 1.1.14. *Soit $b \in \mathcal{S}'(\mathbb{R}^d; \mathbb{R}^d)$ et ψ un poids, et on suppose que $\|T^w b\|_{\mathcal{C}^\gamma([0,T];\mathcal{C}^{0,\psi})} < +\infty$. Soit $k \in \mathcal{S}(\mathbb{R}^d)$ une fonction positive, avec $k(0) = 1$ et on pose $k_\varepsilon = \frac{1}{\varepsilon} k\left(\frac{\cdot}{\varepsilon}\right)$. Alors pour tout $x \in \mathcal{D}^w_\gamma$,*

$$\int_0^t b(x_s) \mathrm{d}s := \lim_{\varepsilon \to 0} \int_0^t (k_\varepsilon * b)(x_u) \mathrm{d}u$$

existe uniformément pour $t \in [0,T]$, est indépendant de k et étend la définition standard de l'intégrale pour des b réguliers. De plus, la fonction $t \to \int_0^t b(x_s)\mathrm{d}s$ est höldérienne d'indice γ.

Le théorème précédent nous permet de donner un sens naturel à l'intégrale $\int_0^t b(x_s)\mathrm{d}s$. Ainsi on dira que $y \in \mathcal{D}^w_\gamma$ est une solution de l'équation (1.3) si

$$y_t - w_t = x + \int_0^t b(y_s)\mathrm{d}s.$$

pour tout $t \in [0,T]$. Ici encore, la borne (1.4) permet de montrer l'analogue des Théorèmes 1.1.11 et 1.1.12.

Théorème 1.1.15. *Soit $\rho > 0$, $\alpha > -\rho$. Soit w une fonction ρ–irrégulière. Soit $b \in \mathcal{F}L^{\alpha+2}$. Alors il existe une unique fonction contrôlée $y \in \mathcal{D}^w_\gamma$ solution de*

$$y_t = x + \int_0^t b(y_u)\mathrm{d}u + w_u.$$

De plus le flot de l'équation $x \to y$ est localement lipschitzienne uniformément en temps.

Théorème 1.1.16. *Soit $H \in (0,1)$, $\alpha > -1/2H$. Pour tout $b \in \mathcal{C}^{\alpha+1}$ et tout $x \in \mathbb{R}^d$ il existe un un ensemble $\mathcal{N}_{b,x} \subset C([0,T],\mathbb{R}^d)$ qui est de mesure nulle par rapport à la loi du mouvement brownien fractionnaire d'indice de Hurst H et tel que pour tout $w \notin \mathcal{N}_{b,x}$, il existe une unique solution contrôlée $y \in \mathcal{D}^w_\gamma$ à l'équation différentielle* (1.3).

Finalement, quand w est distribuée selon la loi du mouvement brownien fractionnaire, il est possible d'appliquer ces résultats. On a alors le théorème suivant.

Théorème 1.1.17. *Soit $H \in (0,1)$. Soit $\alpha > -1/2H$. Soit $b \in \mathcal{C}^{\alpha+2}$ (respectivement $\mathcal{C}^{\alpha+3/2}$ et $\alpha+3/2 > 0$). Il existe un ensemble mesurable de l'ensemble des fonctions continues de $[0,T]$ dans $\mathbb{R}^d$, $\mathcal{N}_b \subset C([0,T],\mathbb{R}^d)$ qui est de mesure nulle pour la loi du mouvement brownien fractionnaire d'indice H, et tel que pour tout $w \notin \mathcal{N}_b$ et pour tout $x \in \mathbb{R}^d$, il existe une unique solution $y \in \mathcal{D}^w_\gamma$ (respectivement $\alpha+3/2 > 0$ et $y \in C([0,T],\mathbb{R}^d)$) à l'équation* (1.3)*. De plus le flot correspondant $t,x \to y_t$ est localement lipschitzien en espace uniformément en temps. Enfin, l'ensemble $\mathcal{N}_b$ peut être choisi comme étant le même pour tout $b \in \mathcal{F}L^{\alpha+2}$ (respectivement $\alpha+3/2 > 0$ et $b \in \mathcal{F}L^{\alpha+3/2}$).*

Lorsque l'on considère l'équation différentielle (1.3) comme une équation différentielle stochastique au sens fort, et que w suit la loi du mouvement brownien fractionnaire, il est possible de choisir des $b \in \mathcal{F}L^{\alpha+3/2}$ (ou $b \in \mathcal{F}L^{\alpha+2}$) aléatoires et l'unicité persiste. Ceci est une des principales raisons de l'introduction des espaces $\mathcal{F}L$. Dans les espaces de Besov-Hölder ce résultat semble pour l'instant être hors de portée.

Les deux figures suivantes regroupent l'ensemble des nos résultats sur la régularisation par le bruit des équations différentielles ordinaires. La première (Figure 1.1) concerne les perturbations ρ-irrégulières, la suivante (Figure 1.2) s'intéresse plus spécifiquement au mouvement brownien fractionnaire.

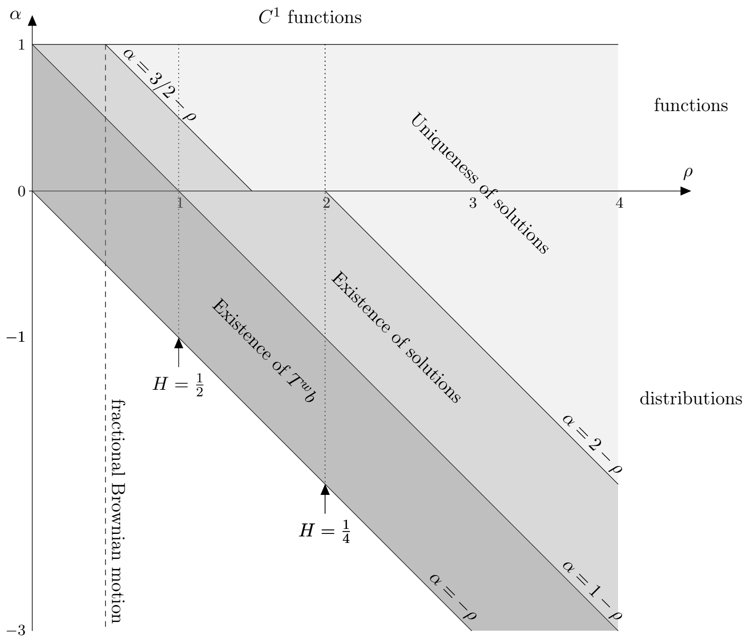

FIGURE 1.1 – Averaging ODE for ρ-irregular perturbations

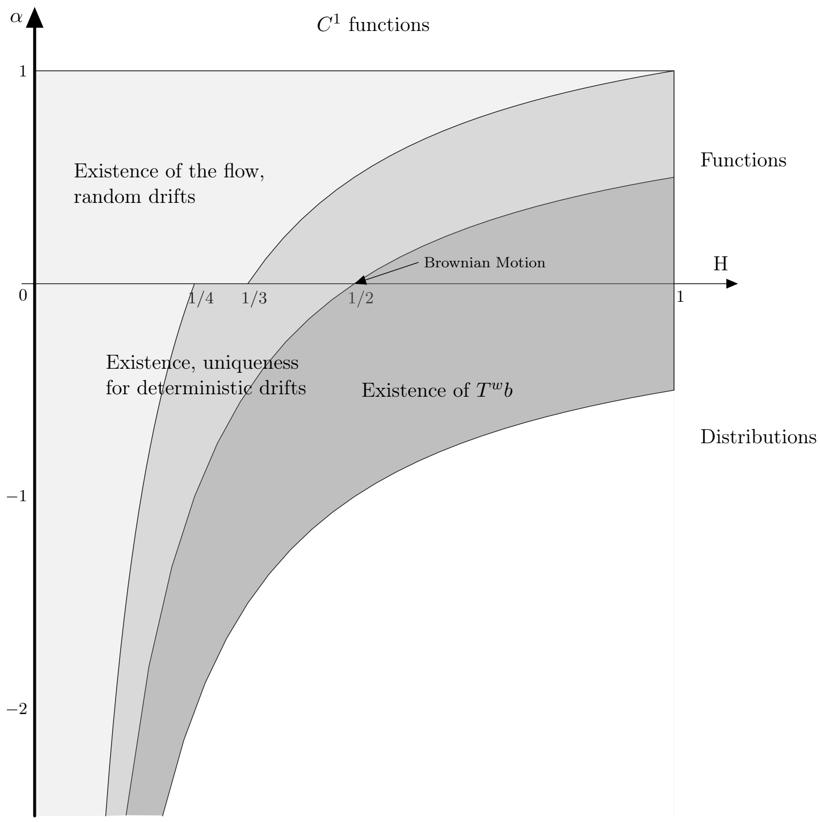

FIGURE 1.2 – Averaging ODE for fractional Brownian motion

1.2 Equation de transport linéaire rugueuse.

Dans ce chapitre nous nous intéressons à l'étude de l'équation de transport linéaire suivante

$$\frac{\partial}{\partial t}u(t,x) + b(t,x).\nabla u(t,x) + \nabla u(t,x).\frac{\partial}{\partial t}X(t) = 0, \qquad u(0,x) = u_0(x) \tag{1.5}$$

où b est un champ de vecteurs suffisamment régulier et $X : [0,T] \to \mathbb{R}^d$ est à valeur dans $\mathbb{R}^d$; on supposera toujours que $X_0 = 0$. Nous cherchons des solutions pour lesquelles u_0 est une fonction bornée ; ainsi l'équation devra être comprise au sens faible en espace.

Lorsque $X = 0$, Di Perna et Lyons [27] ont montré l'unicité d'une solution faible dans L^∞ lorsque le champ de vecteurs $b \in L^1([0,T]; W^{1,1}_{\text{loc}}(\mathbb{R}^d))$, avec, en outre, des conditions de croissance au plus linéaire. Beaucoup de résultat ont suivi ce premier travail. On pourra consulter une synthèse partielle des résultats existants dans [2].

Sous des conditions plus faibles pour le champ de vecteurs b, il est bien connu que l'équation est mal posée. Cependant, quand on considère l'unicité forte au sens stochastique, l'ajout d'une perturbation brownienne à l'équation permet d'affaiblir les hypothèses sur le champ de vecteurs b. Lorsque $b \in C([0,T]; \mathcal{C}^\alpha_b(\mathbb{R}^d))$ et $\operatorname{div} b \in L^p([0,T] \times \mathbb{R}^d), p \geqslant 2$, il existe une unique solution faible, comme l'ont montré Flandoli, Gubinelli et Priola [30, 31]. Dans ce cas la perturbation doit être comprise au sens de l'intégrale de Stratonovitch contre le mouvement brownien. La preuve de leurs résultats est basée sur l'effet de régularisation qu'induit le mouvement brownien sur le flot de l'équation caractéristique

$$\Phi_t(x) = x + \int_0^t b_q(\Phi_q(x))\mathrm{d}q + B_t.$$

et repose en grande partie sur les outils du calcul stochastique.

Lorsque l'on considère un processus qui n'est pas le mouvement brownien, le calcul stochastique n'est pas forcément disponible. Cependant nous avons montré au Chapitre 2 que lorsque la perturbation était un mouvement brownien fractionnaire par exemple, le même phénomène de régularisation par le bruit avait lieu. On peut noter que dans un article récent, Beck, Flandoli, Gubinelli et Maurelli [6] montrent des effets de régularisation du mouvement brownien sur l'équation de transport pour des champs de vecteurs plus généraux et sans utiliser le flot de l'équation caractéristique. Cependant, leur méthode dépend encore fortement de la formule d'Ito pour les semi-martingales et ne peut pas être appliquée dans le cas du mouvement brownien fractionnaire.

Pour utiliser les résultats du Chapitre 2 il convient tout de même de définir l'équation lorsque X n'est pas différentiable et que le calcul stochastique n'est pas disponible pour X.

Une première idée serait d'approcher le signal X par une suite de signaux $(X^\varepsilon)_\varepsilon$ réguliers, et de construire une solution de l'équation approchée u^ε. Ensuite, en utilisant des bornes a priori, il faudrait faire converger la solution u^ε vers une fonction u qui serait alors définie comme étant la solution de l'équation. Cette idée est utilisée par Lyons, Perthame et Souganidis [59] dans un cadre plus général mais pour un processus X unidimensionnel, et dans un autre contexte par Caruana, Friz et Oberhauser [11]. Cependant, dans ces deux approches l'équation est remplacée par un procédé d'approximation, et l'objet limite ne vérifie pas d'équation a priori.

Lorsque l'on s'intéresse à la régularisation par le bruit, cette manière de procéder ne semble pas appropriée. En effet, quand on approche le processus X par un processus régulier X^ε, les propriétés de régularisation disparaissent. L'existence de solutions est évidement plus facile, mais l'unicité semble alors hors de portée. Après passage à la limite, il est probable que l'on puisse retrouver les propriétés de régularisation, mais dans ce cas c'est le sens de l'équation qui pose problème.

Pour donner un sens à l'équation, puis afin d'étudier les propriétés d'existence et d'unicité de la solution, ainsi que les effets de régularisation par le bruit, nous nous inspirons d'un travail récent de Gubinelli, Tindel et Toricella [41]. Dans ce travail, les auteurs utilisent la notion de chemins contrôlés introduite par Gubinelli dans [41], et définissent des solutions de viscosité contrôlées pour des équations aux dérivées partielles non linéaires perturbées par un chemin rugueux d'ordre 2.

Pour donner un sens à l'équation (1.5) nous avons donc besoin d'un chemin rugueux géométrique $\boldsymbol{X}$, et nous allons donner une notion de solutions faiblement contrôlées par $\boldsymbol{X}$. Nous montrerons que cette nouvelle notion est une extension de la notion de solution faible au sens classique du terme. Puis nous montrerons que, sous des conditions raisonnables sur le champ de vecteurs b, ces solutions sont uniques. Enfin nous étudierons les effets de régularisation par le bruit. Comme cette théorie est uniquement analytique, il sera de plus possible de considérer des champs de vecteurs aléatoires dans les effets de la régularisation.

1.2.1 Chemins rugueux contrôlés

Un des principaux ingrédients de l'étude proposée dans ce chapitre repose sur la théorie des chemins rugueux contrôlées. Cette théorie permet de décrire l'effet d'un signal irrégulier sur des systèmes différentiels non linéaires. La théorie des chemins rugueux a d'abord été développée par Lyons et ses co-auteurs ; on pourra consulter par exemple [60, 61] et le livre de Friz et Victoir [34]. À la théorie de Lyons, nous allons préférer la théorie des chemins contrôlés introduite par Gubinelli [41], dont on pourra trouver une présentation appréciable dans [32].

Lorsque le processus X n'est pas à variations finies, il n'est pas possible de définir directement une intégrale contre la dérivée, même faible, de X. La théorie des chemins contrôlés permet de surmonter cette difficulté et donne un contexte général dans lequel il est possible de définir une intégrale contre des processus irréguliers.

Il convient dans un premier temps d'expliciter la notion de processus - ou chemins - irréguliers. Comme le but est d'étudier les équations lorsque le terme X suit notamment la loi d'un mouvement Brownien fractionnaire, et que les trajectoires du mouvement brownien fractionnaire sont höldériennes, nous choisirons d'étudier l'intégrale rugueuse dans ce cas. On définit donc les espaces suivants.

Définition 1.2.1. Soit $T > 0$, $\gamma \in (0, 1]$, on définit l'espace des fonctions höldériennes sur $[0, T]$ à valeur dans un espace de Banach $(F, \|.\|_F)$ comme

$$\mathcal{C}^\gamma([0,T],F) = \left\{ f : [0,T] \to \mathbb{R}^d : [\![f]\!]_{\gamma,[0,T]} := \sup_{s \neq t \in [0,T]} \frac{\|f_t - f_s\|_F}{|t-s|^\gamma} < +\infty \right\}.$$

De plus $[\![.]\!]_\gamma$ définit une semi-norme sur $\mathcal{C}^\gamma([0, T], \mathbb{R}^d)$, et

$$\|f\|_{\gamma,[0,T]} := |f_0| + [\![f]\!]_{\gamma,[0,T]}$$

définit une norme sur l'espace $\mathcal{C}^\gamma([0, T], \mathbb{R}^d)$, qui, muni de cette norme, est un espace complet.

Remarque 1.2.2. Si on définit $\|f\|_{\infty,[0,T]} = \sup_{t\in[0,T]} \|f_t\|_F$, on a évidemment $\|f\|_{\gamma,[0,T]} \leqslant \|f\|_{\infty,[0,T]} + [\![f]\!]_{\gamma,[0,T]}$. De plus, pour tout $t \in [0, T]$

$$|f_t| \leqslant |f_t - f_0| + |f_0| \leqslant T^\gamma [\![f]\!]_{\gamma,[0,T]} + |f_0| \leqslant (1 + T^\gamma)\|f\|_{\gamma,[0,T]}.$$

Ainsi $\|f\|_{\infty,[0,T]} + [\![f]\!]_{\gamma,[0,T]}$ et $\|f\|_{\gamma,[0,T]}$ définissent deux normes équivalentes. Dans la suite nous utiliserons allègrement l'une ou l'autre de ces deux normes, et nous noterons de façon indifférenciée $\|.\|_{\gamma,[0,T]}$. De plus si la notation n'est pas ambigue, nous ne noterons que $\|.\|_\gamma$.

Enfin, nous pouvons remarquer deux faits intéressants. Lorsque $\gamma = 1$, les fonctions concernées sont des fonctions lipschitziennes. De plus, lorsque $\gamma > 1$, la propriété suivante nous renseigne sur la forme de telles fonctions.

Proposition 1.2.3. *Soit $\mu > 0$ et $f : [0, T] \to F$. On suppose de plus qu'il existe une constante $C > 0$ telle que pour tout $s \neq t \in [0, T]$,*

$$|f_t - f_s| \leqslant C|t - s|^\mu.$$

Alors pour tout $t \in [0, T]$ $f_t = f_0$.

La théorie des chemins rugueux, pour des raisons expliquées plus bas, s'intéresse aussi à des fonctions höldériennes du carré $[0,T]^2$ à valeurs dans un Banach F, qui sont définies de la manière suivante :

Définition 1.2.4. Soit $T > 0$ et $\mu > 0$ et $(F, \|\|_F)$ un espace de Banach. On définit l'ensemble des fonctions höldériennes d'indice μ du carré $[0,T]^2$ dans F de la façon suivante

$$\mathcal{C}_2^\mu([0,T]^2; F) = \left\{ f : [0,T]^2 \to F : \|f\|_{\mu,[0,T]^2} := \sup_{s \neq t \in [0,T]} \frac{|f_{s,t}|}{|t-s|^\mu} \right\}.$$

Muni de la norme $\|.\|_{\mu,[0,T]^2}$, l'espace $\mathcal{C}_2^\mu([0,T]^2; F)$ est un espace complet.

Remarque 1.2.5. Contrairement aux fonctions höldériennes sur le segment, $\|.\|_{\mu,[0,T]^2}$ définit bien une norme sur $\mathcal{C}_2^\mu([0,T]^2)$. De plus si $\mu > 1$, les fonctions dans $\mathcal{C}_2^\mu([0,T]^2)$ ne sont pas forcément constantes.

La proposition suivante est l'analogue de la Proposition 1.2.3 dans le cas des fonctions höldériennes sur le carré.

Proposition 1.2.6. *Soit $f \in \mathcal{C}_2^\mu([0,T]^2)$ avec $\mu > 1$ et telle que $f_{s,t} - f_{u,t} - f_{s,u} = 0$ alors $f = 0$.*

La propriété suivante est une généralisation des Propositions 1.2.3 et 1.2.6. Cette proposition donne l'existence d'une application linéaire, dite application de la couturière (Sewing Map) et est nécessaire notamment pour construire l'intégrale rugueuse. Comme constaté dans la Proposition 1.2.6, dans le cas de fonctions höldériennes définie sur le carré $[0,T]^2$, il est nécessaire de s'intéresser aux propriétés de l'application du cube $[0,T]^3$ dans F définie par $f : [0,T]^2 \to F$, $(s,u,t) \to f_{s,t} - f_{s,u} - f_{u,t}$. Nous allons dans en premier temps formaliser cette intuition par la définition suivante.

Définition 1.2.7. Soit $\mu > 0$ et $(F, \|.\|_F)$ un espace de Banach. Une fonction $h : [0,T]^3 \to F$ est dite höldériennes d'indice μ si

$$\|f\|_{\mu,[0,T]^3} := \sup_{0 \leq s < u < t \leq T} \frac{\|f_{s,u,t}\|_F}{|t-s|^\mu} < +\infty$$

On note alors $h \in \mathcal{C}_3^\mu([0,T]^3, F)$.

Nous ne nous intéressons qu'à des fonctions définies sur le carré et le cube. On définit alors l'espace des fonctions du carré dans F qui s'étendent en des fonctions höldériennes du cube dans F de la manière suivante.

Définition 1.2.8. Soit $\mu > 0$ et $(F, \|.\|_F)$ un espace de Banach. On définit l'espace

$$\Lambda\mathcal{C}_3^\mu := \left\{ f : [0,T]^2 \to F \;:\; h : (s,u,t) \to f_{s,t} - f_{s,u} - f_{u,t} \in \mathcal{C}_3^\mu([0,T]^3, F) \right\}$$

que l'on munit de $\|f\|_{\Lambda\mathcal{C}_3^\mu} := \|h\|_{\mu,[0,T]^3}$. L'espace $\Lambda\mathcal{C}_3^\mu$ munit de la distance engendrée par la semi norme $\|.\|_{\Lambda\mathcal{C}_3^\mu}$ est alors complet.

Proposition 1.2.9 (Gubinelli, Proposition 1 [41]). *Soit $\mu > 1$, $(F, \|.\|_F)$ un espace de Banach et $f \in \Lambda\mathcal{C}_3^\mu$. Alors il existe un unique $\Lambda f \in \mathcal{C}_2^\mu([0,T]^2, F)$ tel que*

$$\Lambda f_{s,t} - \Lambda f_{s,u} - \Lambda f_{u,t} = f_{s,t} - f_{s,u} - f_{u,t}.$$

De plus

$$\|\Lambda f\|_{\mathcal{C}_2^\mu([0,T]^2)} \lesssim \|f\|_{\Lambda\mathcal{C}_3^\mu}.$$

L'application Λ est donc linéaire et continue de $\Lambda\mathcal{C}_3^\mu$ dans $\mathcal{C}_2^\mu([0,T], F)$.

La première quantité que nous souhaiterions définir dans le cadre d'une théorie de l'intégration est l'intégrale du processus contre lui même. L'idée de la théorie des chemins rugueux est de présupposer l'existence d'une telle quantité, puis de construire une théorie de l'intégration contre ce processus étendu, le processus lui même et son intégrale itérée. La définition suivante décrit donc le cadre dans lequel nous allons travailler.

Définition 1.2.10. Soit $1/3 < \gamma \leqslant 1/2$. Le couple $\boldsymbol{X} = (X, \mathbb{X})$ est un chemin rugueux d'ordre γ si $X \in \mathcal{C}^\gamma([0,T]; \mathbb{R}^d)$, $\mathbb{X} \in \mathcal{C}^{2\gamma}([0,T]^2; \mathcal{M}_d(\mathbb{R}^d))$ et si pour tout $0 \leqslant t \leqslant u \leqslant s \leqslant T$

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = (X_u - X_s) \otimes (X_t - X_u) = ((X_u^i - X_s^i)(X_t^j - X_u^j))_{0 \leqslant i,j \leqslant d}.$$

De plus, on définit $\|\boldsymbol{X}\|_{\mathcal{R}^\gamma} = \|X\|_\gamma + \|\mathbb{X}\|_{2\gamma}$ et pour deux chemins rugueux d'ordre γ, $\boldsymbol{X}$ et $\boldsymbol{Y}$, on définit la distance

$$d_{\mathcal{R}^\gamma}(\boldsymbol{X}, \boldsymbol{Y}) = \|\boldsymbol{X} - \boldsymbol{Y}\|_{\mathcal{R}^\gamma} = \|X - Y\|_{\gamma,[0,T]} + \|\mathbb{X} - \mathbb{Y}\|_{2\gamma,[0,T]^2}.$$

Il est alors possible d'identifier $\mathbb{X}$ avec l'intégrale itérée de X, ce qui donne formellement

$$\mathbb{X}_{s,t} = \int_s^t (X_r - X_s) \otimes \mathrm{d}X_r.$$

Cette égalité est à comprendre dans le sens suivant : le terme de gauche est la définition pour le terme de droite et non l'inverse. Quand X est un processus régulier, c'est à dire $X \in C^1([0,T])$, il est toujours possible de définir le relèvement naturel de X de la façon suivante

$$\boldsymbol{X} = (X, \mathbb{X}) \,\text{avec}\, \mathbb{X} = \int_s^t (X_r - X_s) \otimes \dot{X}_r \mathrm{d}r.$$

De plus, afin d'approximer les chemins rugueux par des chemins réguliers, nous définissons maintenant l'espace des chemins rugueux géométriques, comme étant la fermeture des processus réguliers par la distance des chemins rugueux $d_{\mathcal{R}^\gamma}$.

Définition 1.2.11. Soit $1/3 < \gamma \leqslant 1/2$, un chemin rugueux d'ordre γ $\boldsymbol{X}$ est dit géométrique, et on note $\boldsymbol{X} \in \mathcal{R}^\gamma([0,T])$ s'il existe une suite $X^\varepsilon \in C^1([0,T])$ telle que

$$\|\boldsymbol{X} - \boldsymbol{X}^\varepsilon\|_{\mathcal{R}^\gamma} \to_{\varepsilon \to 0} 0,$$

où $\boldsymbol{X}^\varepsilon$ est l'extension naturelle de X^ε en tant que chemin rugueux d'ordre γ.

Dans toute la suite, nous ne considérerons que des chemins rugueux géométriques. Pour une étude générale de la différence entre chemins rugueux et chemin rugueux géométriques, on pourra consulter le travail de Hairer et Kelly [51].

On peut maintenant se demander quels sont les processus stochastiques qu'il est possible de relever en des chemins rugueux géométriques. Le théorème suivant nous assure que le mouvement brownien fractionnaire possède une telle propriété. Ce résultat peut être trouvé dans [34, 32]. Il est intéressant de noter que le premier résultat de relèvement pour le mouvement brownien fractionnaire est dû à Coutin et Qian [16].

Théorème 1.2.12. *Soit* $H \in (1/3, 1/2]$, $(\Omega, \mathcal{F}, \mathbb{P})$ *un espace de probabilité et* B^H *un mouvement brownien fractionnaire d-dimensionnel d'indice de Hurst* H *défini sur* $(\Omega, \mathcal{F}, \mathbb{P})$. *Presque surement,* B^H *peut être relevé en un chemin rugueux* $\boldsymbol{B}^H = (B^H, \mathbb{B}^H)$ *d'ordre* γ *pour tout* $\gamma \in (1/3, H)$. *De plus, pour tout* $1 \leqslant p < +\infty$,

$$\mathbb{E}[\|\boldsymbol{B}^H\|^p_{\mathcal{R}^\gamma([0,T])}] < +\infty$$

et de plus il existe une approximation lisse $B^{\varepsilon,H}$ *de* B^H *telle que, presque surement,* $\|\boldsymbol{B}^{H,\varepsilon} - \boldsymbol{B}^H\|_{\mathcal{R}^\gamma} \to 0$ *et pour tout* $1 \leqslant p < +\infty$

$$\mathbb{E}[\|\boldsymbol{B}^{H,\varepsilon} - \boldsymbol{B}^H\|^p_{R^\gamma}] \to 0$$

où $\boldsymbol{B}^{H,\varepsilon}$ *est le relèvement naturel de* $B^{H,\varepsilon}$ *en un chemin d'indice* γ.

Nous allons maintenant nous intéresser à la définition de l'intégrale contre un chemin rugueux. à l'instar du calcul stochastique, où il n'est possible d'intégrer que des processus progressivement mesurables, nous devons décrire l'espace des intégrandes. Comme l'intégrale de X contre $\mathrm{d}X$ est déjà définie par la définition même du chemin rugueux, l'idée est de ne considérer que des fonctions qui, au premier ordre, ressemblent localement à X. De telles fonctions sont appelées contrôlées par X et sont définies comme suit.

Définition 1.2.13. Soit $1/3 < \gamma \leqslant 1/2$ et $X \in \mathcal{C}^\gamma([0,T], \mathbb{R})$. Une fonction $y \in \mathcal{C}^\gamma([0,T], \mathbb{R})$ est dite γ-contrôlée par X et on note $y \in \mathcal{D}^\gamma_X([0,T]; \mathbb{R}^d)$ s'il existe $y' \in \mathcal{C}^\gamma([0,T], \mathcal{L}(\mathbb{R}^d; \mathbb{R}))$ et $y^\# \in \mathcal{C}^{2\gamma}_2([0,T], \mathbb{R})$ telles que pour tout $0 \leqslant s \leqslant t \leqslant T$

$$y_t - y_s = y'_s(X_t - X_s) + y^\#_{s,t}.$$

De plus on munit cet espace de la distance

$$\|x - y\|_{\mathcal{D}^\gamma_X([0,T],\mathbb{R})} = d_{\mathcal{D}^\gamma_X([0,T],\mathbb{R})}(x, y) = \|x - y\|_{\gamma,[0,T]} + \|x' - y'\|_{\gamma,[0,T]} + \|x^\# - y^\#\|_{\gamma,[0,T]^2}.$$

Remarque 1.2.14. La définition précédente s'étend directement aux fonctions höldériennes à valeur dans $\mathbb{R}^m$. On dira qu'une fonction $y = (y^1, \dots, y^m) : [0,T] \to \mathbb{R}^m$ est contrôlée par X si chacune de ses coordonnées y^i est contrôlée par X. De plus dans ce cas on note $y' = ((y^1)', \dots, y^{m'})$ et $y^\# = (y^{1,\#}, \dots, y^{m,\#})$.

Muni de cet ensemble de fonctions, il est alors possible de définir l'intégrale rugueuse en utilisant l'application de la couturière.

Théorème 1.2.15. *Soit $1/3 < \gamma \leqslant 1/2$, $\boldsymbol{X} \in \mathcal{R}^\gamma$ et $y \in \mathcal{D}_X^\gamma([0,T],\mathbb{R})$. Pour tout $0 \leqslant s \leqslant t \leqslant T$, la limite des sommes de Riemann existe*

$$\int_s^t y_r \mathrm{d}\boldsymbol{X}_r := \lim_{\substack{\mathcal{P}\,\text{partition de}[s,t] \\ |\mathcal{P}| \to 0}} \sum_{[t_i,t_{i+1}]\in\mathcal{P}} y_{t_i}(X_{t_{i+1}} - X_{t_i}) + y'_{t_i}\mathbb{X}_{t_i,t_{i+1}}$$

et ne dépend pas du choix de la partition. De plus

$$\left| \int_s^t y_r \mathrm{d}\boldsymbol{X}_r - y_s(X_t - X_s) + y'_s \mathbb{X}_{s,t} \right| \lesssim |t-s|^{3\gamma} \|\boldsymbol{X}\|_{\mathcal{R}^\gamma} \|y\|_{\mathcal{D}_X^\gamma([0,T],\mathbb{R})}.$$

et l'application de $\mathcal{D}_X^\gamma$ dans $\mathcal{D}_X^\gamma$ donnée par

$$y \to \int_0^\cdot y_r \mathrm{d}\boldsymbol{X}_r$$

est linéaire et continue et de plus

$$\left\| \int_0^\cdot y_r \mathrm{d}\boldsymbol{X}_r \right\|_{\mathcal{D}_X^\gamma} \lesssim (1 + \|\boldsymbol{X}\|_{\mathcal{R}^\gamma}) \|y\|_{\mathcal{D}_X^\gamma}.$$

La démonstration de ce théorème repose sur l'application de la couturière, c'est à dire la Proposition 1.2.9.

Démonstration. Soit $0 \leqslant s \leqslant u \leqslant t \leqslant T$. On note $f_{s,t} = y_s(X_t - X_s) + y'_s\mathbb{X}_{s,t}$. Grâce aux propriétés de $\boldsymbol{X}$, et au fait que y est contrôlé par X,

$$\begin{aligned} f^{\boldsymbol{X}}_{s,t}(y) - f^{\boldsymbol{X}}_{s,u}(y) - f^{\boldsymbol{X}}_{u,t}(y) &= y'_s(X_u - X_s) \otimes (X_t - X_u) - (y'_u - y'_s)\mathbb{X}_{u,t} - (y_u - y_s)(X_t - X_u) \\ &= -y^{\#}_{s,u}(X_t - X_u) - (y'_u - y'_s)\mathbb{X}_{u,t}. \end{aligned}$$

Mais

$$|y^{\#}_{s,u}(X_t - X_u)| \leqslant \|y^{\#}\|_{2\gamma,[0,T]} \|X\|_{\gamma,[0,T]} |s-u|^{2\gamma} |t-u|^\gamma$$

et

$$|(y'_u - y'_s)\mathbb{X}_{u,t}| \leqslant \|y'\|_\gamma \|\mathbb{X}\|_{2\gamma} |u-s|^{2\gamma} |t-u|^\gamma.$$

De plus $3\gamma > 1$, ainsi $f^{\boldsymbol{X}}(y) \in \Lambda\mathcal{C}_3^\mu$ et grâce à la Proposition 1.2.9, il existe un unique $\Lambda f^{\boldsymbol{X}}(y) \in \mathcal{C}_2^{3\gamma}([0,T]^2,\mathbb{R}^d)$ tel que

$$\|\Lambda f^{\boldsymbol{X}}(y)\|_{3\gamma,[0,T]^2} \lesssim \|\boldsymbol{X}\|_{\mathcal{R}^\gamma} \|y\|_{\mathcal{D}_X^\gamma}$$

et

$$\Lambda f^{\boldsymbol{X}}_{s,t}(y) - \Lambda f^{\boldsymbol{X}}_{s,u}(y) - \Lambda f^{\boldsymbol{X}}_{t,u}(y) = f^{\boldsymbol{X}}_{s,t}(y) - f^{\boldsymbol{X}}_{s,u}(y) - f^{\boldsymbol{X}}_{u,t}(y).$$

De plus, $y \to f^{\boldsymbol{X}}(y)$ est linéaire en y, et donc $y \to \Lambda f^{\boldsymbol{X}}(y)$ est linéaire en y. Pour tout $y \in \mathcal{D}_X^\gamma([0,T],\mathbb{R})$ et tout $0 \leqslant s \leqslant t \leqslant T$, on définit alors

$$I_{s,t}(y,\boldsymbol{X}) = \Lambda f^{\boldsymbol{X}}_{s,t}(y) + f^{\boldsymbol{X}}_{s,t}(y) = \Lambda f^{\boldsymbol{X}}_{s,t}(y) + y_s(X_t - X_s) + y'_s\mathbb{X}_{s,t}.$$

D'après la définition de Λ et de f, on a directement $I(y,\boldsymbol{X}) \in \mathcal{D}_X^\gamma([0,T],\mathbb{R}^d)$ et

$$I(y,\boldsymbol{X})' = y \qquad I_{s,t}(y,\boldsymbol{X})^{\#} = y'_s\mathbb{X}_{s,t} + \Lambda f^{\boldsymbol{X}}_{s,t}(y).$$

Donc

$$\|I(y,\boldsymbol{X})\|_{\mathcal{D}^\gamma([0,T],\mathbb{R}^d)} \lesssim (1+\|\boldsymbol{X}\|_{\mathcal{R}^\gamma})\|y\|_{\mathcal{D}_X^\gamma}$$

et par définition de Λ,

$$|I(y,\boldsymbol{X}) - y_s(X_t - X_s) - y'_s\mathbb{X}_{s,t}| \lesssim |t-s|^{3\gamma}\|\boldsymbol{X}\|_{\mathcal{R}^\gamma}\|y\|_{\mathcal{D}_X^\gamma}.$$

La fonction I possède une partie des caractéristiques d'une intégrale. Il reste alors à montrer que cette fonction est bien la limite des sommes de Riemann. Mais on a grâce à la définition de f et de I,

$$\begin{aligned} I_{s,t}(y,\boldsymbol{X}) - I_{s,u}(y,\boldsymbol{X}) - I_{u,t}(y,\boldsymbol{X}) &= \Lambda f^{\boldsymbol{X}}_{s,t}(y) - \Lambda f^{\boldsymbol{X}}_{s,u}(y) - \Lambda f^{\boldsymbol{X}}_{u,t}(y) \\ &\quad + f^{\boldsymbol{X}}_{s,t}(y) - f^{\boldsymbol{X}}_{s,u}(y) - f^{\boldsymbol{X}}_{u,t}(y) \\ &= 0. \end{aligned}$$

Ainsi, la relation de Chasles est bien vérifiée pour $I(y,\boldsymbol{X})$. Alors, pour toute partition $\mathcal{P}$ de $[0,T]$, on a

$$\begin{aligned} \left| I_{s,t}(y,\boldsymbol{X}) - \sum_{[t_i,t_{i+1}]\subset\mathcal{P}} y_{t_i}(X_{t_{i+1}} - X_{t_i}) + y'_{t_i}\mathbb{X}_{t_i,t_{i+1}} \right| &= \left| \sum_{[t_i,t_{i+1}]\subset\mathcal{P}} I_{t_i,t_{i+1}}(y,\boldsymbol{X}) - f^{\boldsymbol{X}}_{t_i,t_{i+1}}(y) \right| \\ &\leqslant \sum_{[t_i,t_{i+1}]\subset\mathcal{P}} |\Lambda f^{\boldsymbol{X}}_{t_i,t_{i+1}}(y)| \\ &\lesssim |\mathcal{P}|^{3\gamma-1}\|\boldsymbol{X}\|_{\mathcal{R}^\gamma}\|y\|_{\mathcal{D}_X^\gamma}. \end{aligned}$$

Ainsi, toute somme de Riemann converge vers I. Il reste donc à définir

$$\int_s^t y_r \mathrm{d}\boldsymbol{X}_r := I_{s,t}(y,\boldsymbol{X}),$$

ce qui finit la démonstration. □

Remarque 1.2.16. On peut remarquer que quand $X \in C^1([0,T])$, du fait de la convergence des sommes de Riemann, l'intégrale rugueuse contre le relèvement naturel $\boldsymbol{X} = (X,\mathbb{X})$ de X et l'intégrale de Riemann coïncident sur l'ensemble des chemins contrôlés par X.

1.2.2 Principaux résultats du chapitre

Définition et existence des solutions

Nous avons désormais tous les outils pour énoncer les principaux résultats du chapitre. Nous cherchons donc à étudier l'équation de transport linéaire suivante

$$\partial_t u + b.\nabla u + \nabla u.\mathrm{d}X = 0 \qquad u_0 \in L^\infty(\mathbb{R}^d) \tag{1.6}$$

lorsque b est une fonction irrégulière en espace et X un processus lui aussi irrégulier, mais en temps. Avant de traiter d'un quelconque effet de régularisation dû au terme $\nabla u.\mathrm{d}X$, il est évidemment nécessaire de définir ce terme. De plus, puisque $u_0 \in L^\infty(\mathbb{R}^d)$, cette équation doit s'interpréter au sens faible. Ainsi, on cherche les fonctions $u \in L^\infty([0,T] \times \mathbb{R}^d)$ telles que pour tout $\varphi \in C_c^\infty(\mathbb{R}^d)$ l'équation suivante est vérifiée.

$$u_t(\varphi) - u_s(\varphi) = \int_s^t u_q((\operatorname{div} b_q)\varphi + b_q.\nabla\varphi) + \int_s^t u_q(\nabla\varphi).\mathrm{d}X_r \qquad u_0 \in L^\infty(\mathbb{R}^d). \tag{1.7}$$

où $u_t(\varphi) = \langle u_t, \varphi \rangle = \int_{\mathbb{R}^d} u_t(x)\varphi(x)\mathrm{d}x$ désigne le crochet de dualité de u_t sur φ.

Le terme $\int_s^t u_q(\nabla\varphi).\mathrm{d}X_r$ n'est toujours pas bien défini sous cette forme, mais la théorie des chemins rugueux contrôlés exposée plus haut nous indique une façon de définir cette intégrale. Pour cela, il est nécessaire que $t \to u_t(\nabla\varphi)$ soit contrôlée pour X. Cette contrainte nous donne une indication sur la façon de définir la notion de solutions qui nous intéresse dans ce cas.

Définition 1.2.17. Soit $1/3 < \gamma \leqslant 1/2$ et $\boldsymbol{X} = (X, \mathbb{X}) \in \mathcal{R}^\gamma$ un chemin rugueux géométrique d'ordre γ. Soit $b \in L^\infty([0,T]; \operatorname{Lin}(\mathbb{R}^d))$ telle que $\operatorname{div} b \in L^\infty([0,T] \times \mathbb{R}^d)$ et $u_0 \in L^\infty(\mathbb{R}^d)$. Une solution faiblement contrôlée de l'équation de transport rugueuse (3.9) avec pour condition initiale u_0 est une fonction $u \in L^\infty([0,T] \times \mathbb{R}^d)$ telle que

1. Pour toute fonction $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$ la fonction $q \to u_q(\varphi)$ est contrôlée par X.

2. Pour toute fonction $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$, l'équation suivante est vérifiée

$$u_t(\varphi) - u_s(\varphi) = \int_s^t u_q((\operatorname{div} b_q)\varphi + b_q.\nabla\varphi) + \int_s^t u_q(\nabla\varphi).\mathrm{d}\boldsymbol{X}_r$$

   où l'intégrale $\int_s^t u_q(\nabla\varphi).\mathrm{d}\boldsymbol{X}_r$ est comprise au sens de l'intégrale rugueuse construite au Théorème 1.2.15.

Une fois cette définition donnée, il reste à traiter un certain nombres de difficultés. Dans un premier temps, il est nécessaire de montrer qu'il existe des solutions faiblement contrôlées, et que cette notion étend, en un sens à préciser, la notion de solutions faibles dans le sens classique. Il convient aussi de montrer que sous des conditions raisonnables, l'équation de transport rugueuse possède une unique solution.

Toute l'analyse de cette notion de solutions repose sur l'équation caractéristique associée à l'équation de transport rugueuse. En effet lorsque b, u_0 et X sont lisses, l'unique solution u de l'équation de transport (1.5) est de la forme

$$u_t(x) = u_0(\Phi_t^{-1}(x)) \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d$$

où Φ est le flot de l'équation différentielle ordinaire suivante :

$$\Phi_t(x) = x + \int_0^t b_q(\Phi_q(x))\mathrm{d}q + X_t.$$

Dans ce cas, un changement de variable permet de donner une forme plus exploitable pour la fonction $t \to u_t(\varphi)$:

$$u_t(\varphi) = \int_{\mathbb{R}^d} u(\Phi_t^{-1}(x))\varphi(x)\mathrm{d}x = \int_{\mathbb{R}^d} u_0(x)\varphi(\Phi_t(x))|\operatorname{Jac}(\Phi_t(x))|\mathrm{d}x.$$

On rappelle ici que $Jac(\Phi_t(x))$ désigne le déterminant de la Jacobienne de Φ_t. Ainsi, pour montrer que la fonction $u(\varphi)$ est contrôlée par X, il suffit de montrer que la fonction

$$t \to \varphi(\Phi_t(x))|\operatorname{Jac}(\Phi_t(x))|$$

est contrôlée par X, et que sa norme dans $\mathcal{D}_X^\gamma$ possède de bonnes conditions d'intégrabilité en espace. Cette analyse de la fonction $t \to \varphi(\Phi_t(x))|\operatorname{Jac}(\Phi_t(x))|$ ne dépend pas de l'existence de solutions faiblement contrôlées ni d'ailleurs de l'existence du chemin rugueux relatif au processus X. Ainsi la proposition suivante est vérifiée.

Proposition 1.2.18. *Soit $1/3 < \gamma \leqslant 1/2$ et $X \in \mathcal{C}^\gamma([0,T],\mathbb{R}^d)$. Soit $b \in L^\infty([0,T];\operatorname{Lin}(\mathbb{R}^d))$ telle que $\operatorname{div} b \in L^\infty([0,T]\times\mathbb{R}^d)$. Ici l'espace Lin désigne l'espace des fonctions à croissance au plus linéaire. On pourra consulter la définition 3.2.10. On suppose de plus que le flot Φ de l'équation*

$$\Phi_t(x) = x + \int_0^t b_q(\Phi_q(x))\mathrm{d}q + X_t$$

est bien défini. Pour tout $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$t \to \varphi(\Phi_t(x))|\operatorname{Jac}(\Phi_t(x))|$$

est bien définie et est contrôlée par X. De plus, pour tout $N \geqslant 0$ il existe une constante $C_N > 0$ qui dépend de $\|b\|_{L^\infty([0,T];\operatorname{Lin}(\mathbb{R}^d))}$, $\|\operatorname{div} b\|_{L^\infty([0,T]\times\mathbb{R}^d)}$,$T$ et de φ telle que

$$\|\varphi(\Phi_.(x))|\operatorname{Jac}(\Phi_.(x))|\|_{\mathcal{D}_X^\gamma} \lesssim C_N(1+\|X\|_\gamma)^{1+1/\gamma}(1+|x|)^{-N}.$$

Grâce à cette propriété sur l'évaluation des fonctions test le long du flot, il est possible de montrer que lorsque X est régulier, les solutions faibles de l'équation de transport sont aussi des solutions faiblement contrôlées, et qu'ainsi la notion de solutions faiblement contrôlées étend la notion de solutions faibles. De plus, grâce à un argument de compacité, il est aussi possible, en approchant à la fois le champ de vecteur et le chemin rugueux par des fonctions lisses, de montrer qu'il existe bien des solutions faiblement contrôlées.

Théorème 1.2.19. *Soit $1/3 < \gamma \leqslant 1/2$ et $\boldsymbol{X} \in \mathcal{R}^\gamma$ un chemin rugueux géométrique d'ordre γ. Soit $b \in L^\infty([0,T];\operatorname{Lin}(\mathbb{R}^d))$ telle que $\operatorname{div} b \in L^\infty([0,T]\times\mathbb{R}^d)$ et $u_0 \in L^\infty(\mathbb{R}^d)$. Il existe une solution faiblement contrôlée à l'équation (3.9) de condition initiale u_0.*

Unicité

Afin de montrer l'unicité des solutions faiblement contrôlées pour l'équation de transport linéaire, nous utilisons un argument de dualité qui repose sur l'analyse heuristique suivante. Tout d'abord, comme l'équation est linéaire, pour montrer l'unicité il suffit de considérer le cas $u_0 = 0$. Ensuite, pour $\varphi \in C_c^\infty(\mathbb{R}^d)$ et $t_0 \in [0,T]$ si ρ vérifie l'équation suivante, dite équation de continuité rétrograde,

$$\partial_t\rho_t(x) + b_t(x).\nabla\rho_t(x) - (\operatorname{div} b_t)(x)\rho_t(x) + \nabla\rho_t(x).\dot{X}_t = 0 \qquad \rho_{t_0}(x) = \varphi(x),$$

le calcul suivant est alors vérifié

$$\begin{aligned}\partial_t(u_t(\rho_t)) &= \partial_t u_t(\rho_t) + u_t(\partial_t \rho_t)\\ &= -\int_{\mathbb{R}^d}(b_t(x)+\dot{X}_t).\nabla u_t(x)\rho_t(x)\mathrm{d}x\\ &\quad -\int_{\mathbb{R}^d} u_t(x)b_t(x).\nabla\rho_t(x) - (\operatorname{div} b_t)(x).\nabla\rho_t(x) + \nabla\rho_t(x).\dot{X}_t\mathrm{d}x\\ &= 0\end{aligned}$$

Enfin, il suffit de remarquer que

$$u_{t_0}(\varphi) = u_{t_0}(\rho_{t_0}) - u_0(\rho_0) = 0.$$

Ainsi, s'il est possible de résoudre l'équation de continuité pour une classe de fonction φ suffisantes, on aura bien $u_{t_0} = 0$ et ce pour tout $t_0 \in [0, T]$.

L'adaptation de cet argument heuristique au cadre de l'équation de transport rugueuse nécessite quelques aménagements. En effet, pour utiliser l'heuristique précédente, il est nécessaire que les solutions de l'équation de continuité rétrograde soient suffisamment régulières, en temps comme en espace. Pour la régularité temporelle, il est alors nécessaire de montrer qu'il existe une solution *fortement* contrôlée à l'équation de continuité rugueuse. Il est néanmoins illusoire d'espérer qu'un telle solution sera assez régulière en espace pour pouvoir appliquer la formulation faible de l'équation de transport rugueuse. Nous allons alors procéder par régularisation. En fait, la proposition suivante est vérifiée.

Proposition 1.2.20. *Soit $1/3 < \gamma \leqslant 1/2$ et $\mathbf{X} \in \mathcal{R}^\gamma$ un chemin rugueux géométrique d'ordre γ. Soit $b \in L^\infty([0,T]; \mathrm{Lin}(\mathbb{R}^d))$ telle que $\operatorname{div} b \in L^\infty([0,T]\times\mathbb{R}^d)$ et u une solution faiblement contrôlée de l'équation de transport rugueuse avec $u_0 = 0$. Soit $\tilde{b} \in L^\infty([0,T]; C_c^\infty(\mathbb{R}^d))$ et $\tilde{\Phi}$ le flot de l'équation*

$$\tilde{\Phi}_t(x) = x + \int_0^t \tilde{b}_q(\tilde{\Phi}_q(x))\mathrm{d}q + X_t.$$

Pour $\varphi \in C_c^\infty(\mathbb{R}^d)$ et $t_0 \in [0,T]$ on définit alors pour tout $t \leqslant t_0$

$$\tilde{\rho}_t(x) = \varphi(\tilde{\Phi}_{t_0-t}(x)) \exp\left(\int_t^{t_0} (\operatorname{div}\tilde{b}_q)(\Phi_{q-t}(x))\mathrm{d}q\right).$$

Alors pour tout $t \in [0, t_0]$, $\tilde{\rho}_t \in C_c^\infty(\mathbb{R}^d)$ et de plus

$$u_{t_0}(\varphi) = \int_0^{t_0} u_q((b_q - \tilde{b}_q).\nabla\tilde{\rho}_q + \operatorname{div}(b_q - \tilde{b}_q)\tilde{\rho}_q)\mathrm{d}q.$$

Cette proposition est la pierre angulaire des deux théorèmes d'unicité suivants. On approche le champ de vecteur b par un champ régulier $\tilde{b}$ et on obtient :

Théorème 1.2.21. *Soit $1/3 < \gamma \leqslant 1/2$ et $\boldsymbol{X} \in \mathcal{R}^\gamma$ un chemin rugueux géométrique d'ordre γ. Soit $b \in L^\infty([0,T]; L^\infty(\mathbb{R}^d) \cap \mathrm{Lip}(\mathbb{R}^d))$ telle que $\operatorname{div} b \in L^\infty([0,T]\times\mathbb{R}^d)$. Alors il existe une unique solution faiblement contrôlée à l'équation de transport rugueuse avec condition initiale $u_0 \in L^\infty(\mathbb{R}^d)$.*

Comme mentionné plus haut, l'intérêt d'une telle théorie est de considérer des processus irréguliers avec des propriétés de régularisation. C'est typiquement ce genre de processus que nous avons étudié au chapitre 2. Si on considère alors que $\boldsymbol{X} = (X, \mathbb{X})$ est un chemin rugueux, et que X est ρ-irregulier, on a le résultat suivant.

Théorème 1.2.22. *Soit $1/3 < \gamma \leqslant 1/2$ et $0 < \rho$ et $-\rho < \alpha$ tel que $\alpha + 3/2 > 0$. Soit $\boldsymbol{X} = (X, \mathbb{X}) \in \mathcal{R}^\gamma$ un chemin rugueux géométrique d'ordre γ, et de plus X est ρ-irregulier. Soit $b \in \mathcal{F}L^{\alpha+3/2}$ avec $\operatorname{div} b \in \mathcal{F}L^{\alpha+3/2}$. Alors il existe une unique solution faiblement contrôlée à l'équation de transport rugueuse avec condition initiale $u_0 \in L^\infty(\mathbb{R}^d)$.*

Processus stochastiques

Les notions précédentes permettent de définir les solutions de façon trajectorielle. Néanmoins, dans le cadre stochastique, il est nécessaire d'étendre la notion de solution pour prendre en compte l'espace de probabilité. On a alors la définition suivante

Définition 1.2.23. Soit $1/3 < \gamma < H \leqslant 1/2$, $(\Omega, \mathcal{F}, \mathbb{P})$ un espace de probabilité, $\boldsymbol{B}^H = (B^H, \mathbb{B}^H) \in \mathcal{R}^\gamma$ le chemin rugueux géométrique d'ordre γ associé au mouvement brownien fractionnaire B^H de paramètre de Hurst H tel que défini au Théorème 1.2.12. Soit $b \in L^\infty(\Omega \times [0,T]; \operatorname{Lin}(\mathbb{R}^d))$ telle que $\operatorname{div} b \in L^\infty(\Omega \times [0,T] \times \mathbb{R}^d)$ et $u_0 \in L^\infty(\Omega \times \mathbb{R}^d)$. Une solution stochastiquement faiblement contrôlée de l'équation de transport rugueuse (3.9) avec pour condition initiale u_0 est une fonction $u \in L^\infty(\Omega \times [0,T] \times \mathbb{R}^d)$ telle que presque sûrement

1. Pour toute fonction $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$ la fonction $q \to u_q(\varphi)$ est contrôlée par B^H.
2. Pour toute fonction $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$, presque sûrement et pour presque tout $t \in [0,T]$ l'équation suivante est vérifiée
$$u_t(\varphi) - u_s(\varphi) = \int_s^t u_q((\operatorname{div} b_q)\varphi + b_q.\nabla\varphi) + \int_s^t u_q(\nabla\varphi).\mathrm{d}\boldsymbol{B}_r^H$$
où l'intégrale $\int_s^t u_q(\nabla\varphi).\mathrm{d}\boldsymbol{B}_r^H$ est comprise au sens de l'intégrale rugueuse construite au Théorème 1.2.15.

Grâce aux résultats du Chapitre 2, on obtient alors le théorème suivant.

Théorème 1.2.24. *Soit $1/3 < \gamma < H \leqslant 1/2$, $(\Omega, \mathcal{F}, \mathbb{P})$ un espace de probabilité, $\boldsymbol{B}^H = (B^H, \mathbb{B}^H) \in \mathcal{R}^\gamma$ le chemin rugueux géométrique d'ordre γ associé au mouvement brownien fractionnaire B^H de paramètre de Hurst H tel que défini au Théorème 1.2.12. Alors*

1. *Pour $b \in L^\infty(\Omega \times [0,T]; \operatorname{Lin}(\mathbb{R}^d))$ telle que $\operatorname{div} b \in L^\infty(\Omega \times [0,T] \times \mathbb{R}^d)$ et $u_0 \in L^\infty(\Omega \times \mathbb{R}^d)$ il existe une solution stochastiquement faiblement contrôlée.*
2. *Pour $b \in L^\infty(\Omega \times [0,T]; \operatorname{Lip}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ telle que $\operatorname{div} b \in L^\infty(\Omega \times [0,T] \times \mathbb{R}^d)$ et $u_0 \in L^\infty(\Omega \times \mathbb{R}^d)$, il existe une unique solution stochastiquement faiblement contrôlée.*
3. *Pour $-1/2H < \alpha$ et $\alpha + 3/2 > 0$ et pour $b \in L^\infty(\Omega; \mathcal{F}L^{\alpha+3/2})$ telle que $\operatorname{div} b \in L^\infty(\Omega; \mathcal{F}L^{\alpha+3/2})$ et $u_0 \in L^\infty(\Omega \times \mathbb{R}^d)$, il existe une unique solution stochastiquement faiblement contrôlée.*

4. *Pour $\alpha > -1/2H$ et $\alpha + 1 > 0$ et $b \in \mathcal{C}_b^{\alpha+1}$ telle que* $\operatorname{div} b \in \mathcal{C}_b^{\alpha+1}$ *et $u_0 \in L^\infty(\Omega \times \mathbb{R}^d)$, il existe une unique solution stochastiquement faiblement contrôlée.*

Remarque 1.2.25. Le point 3 du théorème précédent permet de considérer des champs de vecteur b et des conditions initiales u_0 aléatoires, ce que des méthodes utilisant le calcul stochastique ne semblent pas pouvoir obtenir.

Dans le point 4 Du théorème précédent, pour $H = 1/2$, on retrouve quasiment les conditions d'unicité énoncées dans [30].

1.3 Distributions paracontrôlées et équation de quantisation stochastique Φ^4 en dimension 3.

Nous consacrons cette troisième partie à un travail effectué en collaboration avec Khalil Chouk. Nous étudions l'équation de quantisation stochastique Φ^4 en dimension trois. Plus précisément, nous considérons le problème de Cauchy suivant

$$\begin{cases} \partial_t u = \Delta_{\mathbb{T}^3} u - u^3 + \xi \\ u(0,x) = u_0(x) \end{cases} \tag{1.8}$$

où $\Delta_{\mathbb{T}^3}$ est le laplacien sur le tore tridimensionnel $\mathbb{T}^3$, u_0 est une condition initiale à choisir dans un espace convenable et ξ est un bruit blanc gaussien espace temps.

Formellement, la mesure invariante de l'équation (1.8) est la mesure sur les distributions de Schwarz associée à la théorie quantique des champs Φ^4 en dimension trois. Cette équation a été introduite dans le but d'étudier cette mesure invariante par Parisi et Wu [69]. On pourra consulter le livre de Glimm et Jaffe [40] pour de plus amples détails sur la théorie quantique des champs. Depuis les années 1980 ce problème est étudié en physique théorique, on pourra consulter par exemple [8], [55], et [54], ainsi que les références associées.

Cette équation est aussi reliée au comportement du modèle d'Ising en dimension 3 soumis à sa dynamique de Glauber au voisinage de la température critique. En dimension 1, Bertini, Presutti, Rüdiger et Saade [9] ont effectivement montré que cette équation décrivait cette dynamique pour le modèle d'Ising avec une interaction de type Kac. Dans [38] Giacomin, Lebowitz et Marra donnent des arguments heuristiques pour étendre ce résultat en plus grande dimension. Plus récemment, Weber et Mourrat [65] ont montré que ce résultat était vrai en dimension 2.

Il convient de noter que pour les dimensions plus grandes que 2, le bruit blanc espace temps est une distribution assez irrégulière. Malgré la présence de l'opérateur de la chaleur, les potentielles solutions resteront des distributions. Ainsi, le terme u^3 n'est pas défini a priori. Une grosse partie du chapitre sera dédiée à surmonter cette difficulté, afin de donner une formulation raisonnable de l'équation. Pour ce faire, nous utiliserons la théorie des distributions paracontrôlées introduite par Gubinelli, Imkeller et Perkowski [44].

L'étude mathématique de cette équation a néanmoins connue des avancés considérables ces dernières années. Jona-Lasinio et Mitter [55] ont donné une formulation faible de l'équation en utilisant une transformation de Girsanov. Albeverio et Röckner ont pour leur part étudié ce problème grâce aux formes de Dirichlet. On pourra consulter [1] pour une synthèse des résultats utilisant les formes de Dirichlet afin de traiter ce genre d'équations.

En dimension 2, Da Prato et Debussche [17] ont donné une formulation forte du problème en utilisant les produits de Wick et des techniques de renormalisation. Les techniques que nous présentons dans ce chapitre s'inspirent en partie de ce travail.

Dans un article récent, grâce à sa théorie des structures de régularités, Hairer [50] donne notamment une solution à l'équation (1.8) comme solution d'un problème de point fixe. Comme la théorie des distributions paracontrôlées, la théorie des structures de régularité est une généralisation de la théorie des chemins rugueux contrôlés. Hairer obtient son résultat en généralisant la notion de fonctions localement höldériennes. Il lui est alors possible de travailler dans un espace abstrait, dans lequel les solutions sont construites comme solutions d'un problème de point fixe. Il lui reste alors à projeter la solution abstraite dans l'espace des distributions grace à un théorème de reconstruction (Théorème 3.10 de [50]). Cette approche n'est pas limitée au seul problème de l'équation étudiée ici, et s'applique à un grand nombre d'équations singulières, comme l'équation de KPZ, le modèle d'Anderson parabolique [50], l'équation de la chaleur stochastique [25] mais aussi à l'équation de Navier Stokes en trois dimensions conduite par un bruit blanc espace temps [77]. Pour une introduction aux structures de régularité, on pourra consulter [32, 48, 49].

Nous nous proposons d'appliquer une théorie alternative à la théorie de structures de régularité pour résoudre cette équation par un point fixe. Cette théorie, développée par Gubinelli, Imkeller et Perkowski [44] est basée sur la décomposition de Paley-Littlewood et les paraproduits de Bony (voir [10] et [5]). Bien que moins générale que la théorie des structures de régularité, la théorie des distributions paracontrôlées a un certain nombre d'avantages. Elle permet, comme nous le verrons dans la suite, de considérer des objets définis globalement, et s'appuie sur une littérature déjà bien établie. Cette théorie est elle aussi capable de traiter un certain nombre d'équations singulières et notamment toutes celles traitées plus haut, on pourra consulter [77, 35, 44, 43].

1.3.1 Espaces de Besov et Paraproduits de Bony

Bloc de Paley-Littlewood et espaces de Besov

Dans cette section nous donnons quelques éléments de calcul dans les espaces de Besov, de décomposition de Paley-Littlewood et de paraproduits de Bony. Comme spécifié précédemment, on pourra consulter [5] pour plus de détails. Dans l'étude de l'équation (1.8), nous allons tirer parti de la décomposition de Paley-Littlewood pour des éléments de $\mathcal{S}'$. L'idée est de décomposer les éléments de $\mathcal{S}'$ en somme de fonctions dont les transformées de Fourier sont localisées. Pour cela nous introduisons deux fonctions de classe C^∞ de $\mathbb{R}^d$ dans $\mathbb{R}$ à support compact et à symétrie radiale, θ et χ telles que

1. Le support de χ est contenu dans une boule B et le support de θ est inclue dans un anneau $\mathcal{A} = \{k \in \mathbb{R}^d : r \leqslant |k| \leqslant R\}$.
2. Pour tout $k \in \mathbb{R}^d$, $\chi(k) + \sum_{j \geqslant 0} \theta(2^{-j}k) = 1$.
3. Pour $i, j \in \mathbb{N}$ avec $|i-j| > 1$, $\operatorname{supp}\theta(2^{-j}.) \cap \operatorname{supp}\theta(2^{-i}.) = \emptyset$ et pour $i \geqslant 1$ $\operatorname{supp}\chi \cap \operatorname{supp}(\theta(2^{-i}.)) = \emptyset$.

Pour l'existence de telles fonctions, on pourra consulter [5], Proposition 2.10. On définit alors les blocs de Paley-Littlewood pour tout $f \in \mathcal{S}'(\mathbb{R}^d)$ par

$$\Delta_{-1}f = \mathcal{F}^{-1}(\chi\mathcal{F}f)\,\text{et pour}\, i \geqslant 0 \Delta_i f = \mathcal{F}^{-1}(\theta(2^{-i}.)\mathcal{F}f).$$

Il convient de noter que les blocs de Paley-Littlewood d'une distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ sont localisés dans l'espace de Fourier. Ainsi pour tout $i \geqslant -1$ le bloc $\Delta_i f$ est une fonction $C^\infty(\mathbb{R}^d)$ et de plus appartient à tous les espaces $L^p(\mathbb{R}^d)$ pour $1 \leqslant p \leqslant +\infty$.

Munis de ces blocs de Paley-Littlewood, nous pouvons maintenant introduire les espaces de Besov.

Définition 1.3.1. Pour $1 \leqslant p, q \leqslant +\infty$ et $\alpha \in \mathbb{R}$ on définit l'espace de Besov $B^\alpha_{p,q}$ par

$$B^\alpha_{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{B^\alpha_{p,q}} = \left(\sum_{j \geqslant -1} 2^{j\alpha q} \|\Delta_j u\|^q_{L^p(\mathbb{R}^d)} \right)^{1/q} < +\infty \right\}.$$

avec la modification habituelle pour $q = +\infty$.

Bien que la norme $\|.\|_{B^\alpha_{p,q}}$ dépende du choix des fonctions θ et χ, la topologie induite sur l'espace $B^\alpha_{p,q}$, elle, n'en dépend pas et la norme induite par un autre choix de fonctions est une norme équivalente. De plus l'espace $B^\alpha_{p,q}$ muni de sa norme $\|.\|_{B^\alpha_{p,q}}$ est un espace de Banach. Dans la suite on s'interessera particulièrement à l'espace de Besov-Hölder d'indice α, $\mathcal{C}^\alpha := B^\alpha_{\infty,\infty}$. Dans ce cas on notera la norme simplement par $\|.\|_\alpha$. Lorsque $\alpha \in (0,1)$ cet espace n'est autre que l'espace des fonctions höldériennes bornées sur $\mathbb{R}^d$.

La façon dont les blocs de Paley-Littlewood sont construits garantit en fait que la somme desdits blocs est l'identité sur l'ensemble des distributions tempérées $\mathcal{S}'$. On a en fait le résultat suivant.

Proposition 1.3.2 (Proposition 2.12 dans [5])**.** *Soit $f \in \mathcal{S}'$, alors*

$$\pi_n f = \sum_{i=-1}^{n} \Delta_i f \to^{\mathcal{S}'} f.$$

Démonstration. Pour tout $\varphi \in \mathcal{S}(\mathbb{R}^d)$, on a $\langle \pi_n f - f, \varphi \rangle = \langle f, \pi_n \varphi - \varphi \rangle$. Ainsi, il suffit de montrer que $\pi_n \varphi \to^{\mathcal{S}} \varphi$. De plus, comme la transformée de Fourier sur $\mathcal{S}$ est un automorphisme, il suffit de montrer que $\widehat{\pi_n \varphi} \to^{\mathcal{S}} \hat{\varphi}$, ce qui est immédiat d'après la définition des blocs de Paley-Littlewood. $\square$

L'étude des propriétés des fonctions dans les espaces de Besov est facilité par les propriétés des blocs de Paley-Littlewood. La transformée de Fourier de ces derniers étant une fonctions C^∞ à support compact, plusieurs inclusions entre espaces sont immédiates. L'inégalité suivante nous permet d'en déduire un certain nombre.

Proposition 1.3.3 (Lemma 2.1 dans [5])**.** *Soit B une boule. Il existe une constante $C > 0$ telle que pour tout $p \leqslant p' \in [1, +\infty]$, pour toute fonction $f \in L^p$ et pour tout $n \geqslant 0$*

$$\operatorname{supp} \hat{f} \subset \lambda B \Longrightarrow \sup_{|\alpha|=n} \|\partial^\alpha u\|_{L^{p'}} \leqslant C^{k+1} \lambda^{n+d\left(\frac{1}{p}-\frac{1}{p'}\right)} \|u\|_{L^p},$$

Démonstration. En utilisant une dilatation, il suffit de montrer ce résultat pour $\lambda = 1$. Soit ϕ une fonction C^∞ à support compact égale à 1 au voisinage de B. Comme $\hat{f} = \phi \hat{f}$, on a

$$\partial^\alpha f = \partial^\alpha (g * f) \operatorname{avec} g = \mathcal{F}^{-1}(\phi).$$

L'inégalité de Young donne alors

$$\|\partial^\alpha f\|_{L^{p'}} \leqslant \|\partial^\alpha g\|_{L^r}\|f\|_{L^p} \quad \text{avec} \quad \frac{1}{r} = 1 + \frac{1}{p'} - \frac{1}{p}$$

Et de plus

$$\|\partial^\alpha g\|_{L^r} \leqslant \|\partial^\alpha g\|_{L^\infty} + \|\partial^\alpha g\|_{L^1} \leqslant C\|(1+|.|^2)^d \partial^\alpha g\|_{L^\infty}$$

avec $C = 1 + \int_{\mathbb{R}^d}(1+|x|^2)^{-d}\mathrm{d}x$. La dernière inégalité donne alors

$$\|\partial^\alpha g\|_{L^r} \leqslant C\|(\mathrm{id} - \Delta_{\mathbb{R}^d})^d((.)^\alpha g)\|_{L^1} \leqslant C^{k+1}.$$

ce qui conclut la démonstration. □

Si on applique la proposition précédente aux blocs de Paley-Littlewood, avec $\lambda = 2^{-i}$, $p' = +\infty$, on a le résultat suivant, qui sera très utile dans la suite lorsque nous travaillerons avec le bruit blanc.

Proposition 1.3.4. *Soit $\alpha \in \mathbb{R}$ et $p, q \in [1, +\infty]$ alors*

$$B^\alpha_{p,q} \subset \mathcal{C}^{\alpha - \frac{d}{p}}$$

et

$$\|f\|_{\alpha - d/p} \lesssim \|f\|_{B^\alpha_{p,q}}.$$

Nous finissons cette partie en montrant que les espaces définis plus haut sont pertinents lorsque l'on travaille avec le bruit blanc. En ce qui concerne la définition du bruit blanc comme une mesure gaussienne sur l'ensemble des distributions tempérées $\mathcal{S}'$ on pourra consulter [53].

Lemme 1.3.5. *Soit ξ un bruit blanc sur le tore $\mathbb{T}^d$. Alors presque sûrement, pour tout $\varepsilon > 0$ $\xi \in \mathcal{C}^{-d/2-\varepsilon}(\mathbb{T}^d)$.*

Démonstration. On rappelle que dans $L^2(\Omega, \mathcal{S}'(\mathbb{T}^d))$ le bruit blanc vérifie l'égalité

$$\xi = \sum_{k \in \mathbb{Z}^d} \hat{\xi}(k) e_k$$

où $(e_k)_k$ est la base de Fourier de $L^2(\mathbb{T}^d)$ et $(\hat{\xi}(k))_{k \in \mathbb{Z}^d}$ une famille i.i.d. de variables gaussiennes centrées réduites. Ainsi, pour tout $x \in \mathbb{T}^d$, et pour tout $i \geqslant -1$,

$$\mathbb{E}[|\Delta_i \xi(x)|^2] = \sum_{|k| \sim 2^i} \theta(2^{-i}k) \lesssim 2^{id}.$$

En utilisant l'hypercontractivité gaussienne [53], on obtient

$$\mathbb{E}[\|\Delta_i \xi\|^p_{L^p(\mathbb{T}^d)}] = \int_{\mathbb{T}^d} \mathbb{E}[|\Delta_i \xi(x)|^2]^{p/2} \mathrm{d}x \lesssim 2^{ipd/2},$$

et finalement

$$\mathbb{E}[\|\xi\|^p_{B^{-d/2-\varepsilon/2}_{p,p}}] < +\infty.$$

Pour $d/p < \varepsilon/2$, en utilisant la Proposition 1.3.4, on obtient alors

$$\mathbb{E}[\|\xi\|^p_{-d/2-\varepsilon}] < +\infty$$

ce qui est le résultat attendu. □

Paraproduits de Bony

Nous introduisons ici l'étude des paraproduits commencée par Bony dans [10]. Dans l'équation (1.8), il est nécessaire, et c'est là une des difficulté de toute étude de cette équation, de définir le terme u^3. Étant donné que u sera à valeur dans $\mathcal{S}'$, la difficulté revient ici à « multiplier » deux distributions. C'est exactement ce à quoi s'intéresse la théorie de Bony.

L'identité de la Proposition 1.3.2 nous permet d'écrire que pour toute distribution $f \in \mathcal{S}'$, on a

$$\sum_{i \geqslant -1} \Delta_i f = f.$$

Si on considère maintenant $f, g \in \mathcal{S}'$, on peut alors écrire formellement

$$fg = \sum_{i,j \geqslant -1} \Delta_i f \Delta_j g = \pi_<(f,g) + \pi_0(f,g) + \pi_>(f,g) \tag{1.9}$$

où

$$\pi_<(f,g) = \sum_{i \geqslant -1, j > i+1} \Delta_i(f)\Delta_j(g) = \pi_>(g,f) \text{ et } \pi_0(f,g) = \sum_{|i-j| \leqslant 1} \Delta_i(f)\Delta_j(g).$$

La décomposition précédente n'a évidemment pas de sens pour toutes les distributions f et g. Néanmoins, dans le cadre des espaces de Besov-Hölder et sous des conditions assez naturelles, le calcul précédent a bien un sens.

Théorème 1.3.6 (Théorème 2.47 et 2.52 dans [5]). *Soit $f \in L^\infty$ et $g \in \mathcal{C}^\alpha$, $\alpha \in \mathbb{R}$. Le terme $\pi_<(f,g)$ est bien défini et*

$$\|\pi_<(f,g)\|_\alpha \lesssim_\alpha \|f\|_\infty \|g\|_\alpha.$$

Si de plus $\alpha < 0$, $\beta \in \mathbb{R}$ et $f \in \mathcal{C}^\beta$,

$$\|\pi_>(f,g)\|_{\alpha+\beta} \lesssim_{\alpha,\beta} \|f\|_\beta \|g\|_\alpha.$$

Finalement si $\alpha < 0$ et $\alpha + \beta > 0$,

$$\|\pi_0(f,g)\|_\alpha \lesssim \|f\|_\beta \|g\|_\alpha.$$

Enfin, pour définir la produit u^3 il faut considérer une succession de décomposition de la forme (1.9). En effet, lorsqu'on « multiplie » trois distributions f, g et h, on a formellement

$$fgh = (\pi_<(f,g) + \pi_0(f,g) + \pi_>(f,g))h.$$

Dans le bon cadre les termes de la forme $\pi_<(\cdot, h)$ et $\pi_>(\cdot, h)$ seront toujours bien définis. Cependant les termes de la forme

$$\pi_0(\pi_<(f,g), h)$$

ne le sont pas, et on voudra les comparer à des quantités de la forme $f\pi_0(g,h)$ qui elles seront bien définies.

Dans un contexte un peu différent, les auteurs de [44] ont obtenu le théorème suivant qui permet de commuter les paraproduits.

Proposition 1.3.7 (Gubinelli, Imkeller, Perkowski [44]). *Soit $\alpha, \beta, \gamma \in \mathbb{R}$ tels que $0 < \alpha < 1$, $\alpha + \beta + \gamma > 0$ et $\beta + \gamma < 0$. On suppose de plus que $f \in \mathcal{C}^\alpha$, $g \in \mathcal{C}^\beta$ et $h \in \mathcal{C}^\gamma$. Alors la quantité*

$$R(f, g, h) = \pi_0(\pi_<(f, g), h) - f\pi_0(g, h)$$

est bien définie, et de plus

$$\|R(f, g, h)\|_{\alpha+\beta+\gamma} \lesssim \|f\|_\alpha \|g\|_\beta \|h\|_\gamma.$$

L'équation (1.8) est une équation de la chaleur perturbée par un terme non linéaire et un bruit blanc. Ainsi, sous sa forme de Duhamel (ou forme intégrée, ou forme Mild) cette équation fait intervenir le noyau de la chaleur $(P_t)_{t \geqslant 0} = (e^{t\Delta})_{t \geqslant 0}$. Il est donc naturel d'étudier la façon dont le noyau de la chaleur agit sur les objets définis précédemment.

Lemme 1.3.8. *Soit $\alpha \in \mathbb{R}$ et $\varepsilon, \eta \geqslant 0$. Pour tout $f \in \mathcal{C}^\alpha$ l'inégalité suivante est satisfaite.*

$$\|P_t f\|_{\alpha+2\eta} \lesssim t^{-\eta} \|f\|_\alpha; \|(P_{t-s} - 1)f\|_{\alpha-2\varepsilon} \lesssim |t - s|^\varepsilon \|f\|_\alpha.$$

De plus, si $\alpha < 1$ et $\beta \in \mathbb{R}$, pour tout $g \in \mathcal{C}^\beta$,

$$\|P_t \pi_<(f, g) - \pi_<(f, P_t g)\|_{\alpha+\beta+2\eta} \lesssim t^{-\eta} \|f\|_\alpha \|g\|_\beta.$$

La commutation entre le paraproduit et le noyau de la chaleur semble assez peu connue, et nous renvoyons à la fin du Chapitre 4 pour une preuve de ce lemme.

1.3.2 Principaux résultats du chapitre

Dans ce troisième et dernier chapitre de cette thèse, nous nous intéresserons donc au problème de Cauchy suivant

$$\begin{cases} \partial_t u = \Delta_{\mathbb{T}^3} u - u^3 + \xi \\ u(0, x) = u_0(x) \end{cases}$$

ou ξ est un bruit blanc espace temps, et où, pour simplifier les notations de cette présentation, nous prendrons $u_0 = 0$. Sous la forme de Duhamel (forme Mild) l'équation devient alors

$$u = X + I(u^3) \tag{1.10}$$

où $(P_t)_{t \geqslant 0}$ est le semi groupe de la chaleur, $X = \int_0^t \mathrm{d}s P_{t-s} \xi_s$ est le processus d'Orsntein-Uhlenbeck considéré ici comme étant stationnaire et $I(f)_t = -\int_0^t \mathrm{d}s P_{t-s} f_s$. Plus précisément, on a

$$\mathbb{E}[|\hat{X}_s(k_1)\hat{X}_t(k_2)|^2] = \delta_{k_1+k_2=0} \frac{e^{-|k_1|^2|t-s|}}{|k_1|^2}$$

avec $\hat{X}(0) = 0$. Grâce aux résultats de la partie précédente, on constate que presque sûrement $X \in \mathcal{C}_T^{-1/2-\delta} := C([0, T]; \mathcal{C}^{-1/2-\delta})$ pour tout $\delta > 0$. Ainsi, si u résout l'équation (1.10), u ne peut pas être a priori plus régulier que X. Malgré les résultats sur le paraproduit, définir u^3 n'est alors pas possible, et une approche différente est nécessaire.

La première idée, que l'on pourrait qualifier de naïve, est de régulariser le bruit ξ en

$$\xi^\varepsilon = \sum_{k \in \mathbb{Z}^3} f(\varepsilon k) \hat{\xi}(k) e_k,$$

où f est une fonction positive, radiale, lisse et à support compact telle que $f(0) = 1$, et $(e_k)_k$ est la base de Fourier de $L^2(\mathbb{T}^3)$. Dans ce cas, il existe une solution, locale en temps, u^ε à l'équation approchée $u^\varepsilon = \Delta u^\varepsilon - (u^\varepsilon)^3 + \xi^\varepsilon$, soit avec les notations précédentes

$$u^\varepsilon = X^\varepsilon + I((u^\varepsilon)^3).$$

La stratégie serait alors de trouver des bornes a priori pour la fonction u^ε, puis de montrer qu'il est possible de passer à la limite lorsque $\varepsilon \to 0$. Malheureusement, en l'état, cette méthode ne peut pas aboutir. Afin d'estimer $(u^\varepsilon)^3$, il est nécessaire d'estimer $I((u^\varepsilon)^3)$, et au vu de l'équation vérifiée par u^ε, $(X^\varepsilon)^2$ et $(X^\varepsilon)^3$. Or un calcul presque immédiat donne

$$\begin{aligned}
\mathbb{E}[|X_t^\varepsilon|^2] &= \sum_{k\in\mathbb{Z}^3} \sum_{k_1+k_2=k} f(\varepsilon k_1) f(\varepsilon k_2) \frac{1}{|k_1|^2} \delta_{k_1+k_2=0} \\
&= \sum_{k\in\mathbb{Z}^3} \frac{f(\varepsilon k_1)^2}{|k_1|^2} \\
&\sim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \frac{f(x)}{(1+|x|)^2} \mathrm{d}x.
\end{aligned}$$

Obtenir une limite non triviale pour u^ε semble alors illusoire. Néanmoins, et nous le montrerons dans la suite, bien que $(X^\varepsilon)^2$ ne converge pas il est possible de trouver une limite non triviale au processus suivant,

$$(X^\varepsilon)^2 - \mathbb{E}[(X^\varepsilon)^2],$$

uniformément en temps et presque sûrement dans $\mathcal{C}_T^{-1-\delta}$ pour tout $\delta > 0$.

Il est alors tentant de remplacer l'équation vérifiée par u^ε par l'équation suivante

$$\partial_t u^\varepsilon = \Delta u^\varepsilon - ((u^\varepsilon)^3 - C_\varepsilon u^\varepsilon) + \xi^\varepsilon,$$

où C_ε sera choisi convenablement pour que $(X^\varepsilon)^2$ converge. Le même genre de phénomènes apparaissant avec d'autres fonctionnelles de X^ε, il sera nécessaire de choisir convenablement la constante C_ε afin que la solution u^ε converge elle aussi uniformément en t et presque sûrement dans un certain espace de Besov-Hölder. Introduire une telle constante revient dans un certain sens, à renormaliser la définition du cube $(u^\varepsilon)^3$ en $(u^\varepsilon)^{\diamond 3}$. On notera alors que la fonction u^ε vérifie l'équation suivante

$$u^\varepsilon = X^\varepsilon + I((u^\varepsilon)^{\diamond 3})$$

et que la limite u vérifie l'équation

$$u = X + I(u^{\diamond 3}).$$

Le produit renormalisé $\diamond$ ne désigne pas forcément le produit de Wick, et en ce sens la méthode présentée ici diffère de celle de Da Prato et Debussche [17].

La stratégie de preuve est donc la suivante, et est à rapprocher des stratégies développées dans la théorie de chemins rugueux. Dans un premier temps, nous allons montrer que sous de bonne hypothèses sur Y et sur certaines fonctionnelles de Y (les termes d'aires de Y), il est possible d'utiliser une méthode de point fixe pour résoudre l'équation

$$u = \mathbb{Y} + I(u^{\diamond 3})$$

où la notation $\mathbb{Y}$ désigne non seulement la donnée de Y mais aussi de ses termes d'aire. Dans un deuxième temps, nous allons montrer qu'il est possible de construire une telle distribution rugueuse à l'aide de X^ε, disons $\mathbb{X}^\varepsilon$, et qu'il existe une distribution rugueuse $\mathbb{X}$ de telle sorte que $\mathbb{X}^\varepsilon \to \mathbb{X}$ presque sûrement.

Méthode semi-perturbative et distributions paracontrôlées.

Afin de prendre en compte les problèmes de renormalisation, on s'intéresse donc désormais non plus à une seule équation, mais à une classe d'équations $\partial_t u = \Delta u - u^3 + Cu + \Xi$, où Ξ est une distribution a priori régulière, mais qui nous intéressera en tant que distribution dans $\mathcal{C}_T^{-5/2-\delta}$. Ce qui donne, si on choisit la constante $C = 3a + 9b$,

$$u = Y + I(u^3) - I((3a+9b)u) = Y + \Phi$$

avec $\Phi = u - Y$. Grâce aux propriétés du noyau de la chaleur, $\Phi \in \mathcal{C}_T^{1/2-\delta}$. De plus Φ satisfait l'équation suivante

$$\Phi = I(Y^3 - 3aY) + 3I(\Phi(Y^2 - a) - 3b(Y + \Phi)) + 3I(\Phi^2 Y) + I(\Phi^3) \tag{1.11}$$

A ce stade, nous constatons qu'il est nécessaire pour la définition de l'équation que les termes

$$I(Y^3 - 3aY) \quad \text{et} \quad (Y^2 - a)$$

soient bien définis. Cependant, même sous cette hypothèse, les deuxièmes et troisièmes termes du membre de droite de l'équation ne sont pas correctement définis. Le paraproduit introduit dans la section précédente va nous permettre de pousser l'analyse un peu plus loin.

Si on décompose l'équation précédente à l'aide des paraproduits, et qu'on note $B_<(f,g) = I(\pi_<(f,g))$, la décomposition suivante apparaît

$$\Phi = I(Y^3 - 3aY) + 3B_<(\Phi, Y^2 - a) + \mathcal{C}_T^{3/2-\delta}.$$

Ainsi, si Φ est solution de l'équation, Φ possède forcément une telle décomposition en termes de régularité croissante ($I(Y^3 - 3aY) \in \mathcal{C}^{1/2-\delta}$, $B_<(\Phi, Y^2 - a) \in \mathcal{C}_T^{1-\delta}$ et le reste est dans $\mathcal{C}_T^{3/2-\delta}$). Nous allons donc à ce stade faire une supposition sur la forme des solutions de l'équation.On suppose qu'il existe des $\Phi' \in \mathcal{C}_T^{1/2-\delta}$ et $\Phi^\# \in \mathcal{C}_T^{3/2-\delta}$ telles que

$$\Phi = I(Y^3 - 3aY) + B_<(\Phi', Y^2 - a) + \Phi^\#.$$

Grâce à cette ansatz il est possible de continuer l'analyse de l'équation (1.11). Il s'avère alors que les termes dans le membre de droite de l'équation (1.11) qui ne sont pas définis sont

$$I(\pi_0(\Phi, Y^2 - a)) \, \text{et} \, I(\pi_0(\Phi^2, X)).$$

Mais il est possible d'exprimer ces termes en fonction des quantités bien définies suivantes

$$\pi_0(I(Y^3 - 3a), Y), \quad \pi_0(I(Y^2 - a), Y^2 - a) - b \quad \text{et} \quad \pi_0(I(Y^3 - 3a), Y^2 - a) - 3bY.$$

Ainsi, à l'instar de $I(Y^3 - 3aY)$ et $(Y^2 - a)$ il faut définir un certain nombre d'autres quantités pour que l'équation ait un sens, et ce malgré l'ansatz. De plus, lorsque nous voudrons appliquer cette décomposition à des régularisation du bruit blanc, il sera nécessaire d'introduire une fonction auxiliaire déterministe φ de telle sorte que les quantités introduites précédemment convergent dans des espaces de distributions en espace et des espaces de fonctions continues en temps. La définition suivante spécifie les « termes d'aires » qui doivent être définis de façon à ce que l'équation (1.11) ait un sens.

Définition 1.3.9. Soit $T > 0$, $\nu > \rho > 0$, on note $\bar{C}_T^{\nu,\rho}$ la fermeture de l'espace $C^\infty([0,T],\mathbb{R})$ par la semi norme

$$\|\varphi\|_{\nu,\rho} = \sup_{t\in[0,T]} t^\nu|\varphi_t| + \sup_{s\leqslant t\in[0,T],s\neq t} \frac{s^\nu|\varphi_t-\varphi_s|}{|t-s|^\rho}.$$

De plus on note $\mathcal{C}_T^{\gamma,\nu} = \mathcal{C}^\gamma([0,T];\mathcal{C}^\nu(\mathbb{R}^d))$.

Pour $K = (\delta,\delta',\nu,\rho)$ et $0 < 4\delta' < \delta$ on définit l'espace

$$\mathcal{W}_{T,K} = \mathcal{C}_T^{\delta',-1/2-\delta} \times \mathcal{C}_T^{\delta',-1-\delta} \times \mathcal{C}_T^{\delta',1/2-\delta} \times \mathcal{C}_T^{\delta',-\delta} \times \mathcal{C}_T^{\delta',-\delta} \times \mathcal{C}_T^{\delta',-1/2-\delta} \times \bar{C}_T^{\nu,\rho}$$

que l'on munit de sa topologie produit et qui en fait un espace métrique complet. Pour $(Y,\varphi) \in C([0,T];C(\mathbb{T}^3)) \times C^\infty([0,T];\mathbb{R})$ et $(a,b) \in \mathbb{R}^d$ on définit $R^\varphi_{a,b}Y \in \mathcal{W}_{T,K}$ par

$$\begin{aligned} R^\varphi_{a,b}Y = (Y, Y^2-a, I(Y^3-3a), \pi_0(I(Y^3-3a),Y), \pi_0(I(Y^2-a),Y^2-a)-b-\varphi \\ , \pi_0(I(Y^3-3a),Y^2-a)-3bY-3\varphi Y,\varphi). \end{aligned}$$

L'ensemble des distributions rugueuses $\mathcal{X}_{T,K}$ est alors défini comme la fermeture dans $\mathcal{W}_{T,K}$ de

$$\{R^\varphi_{a,b}Y, (Y,\varphi) \in C([0,T];C(\mathbb{T}^3)) \times C^\infty([0,T];\mathbb{R}), (a,b) \in \mathbb{R}^2\}.$$

Dans la suite nous dirons que Y s'étend en une distribution rugueuse s'il existe $\mathbb{Y} \in \mathcal{X}_{T,K}$ telle que la première composante de $\mathbb{Y}$ soit Y.

Il est enfin possible de définir la notion de produit renormalisé telle qu'évoquée plus haut.

Notation 1.3.10. Soit $\mathbb{Y} \in \mathcal{X}_{T,K}$, on note alors les composantes de $\mathbb{Y}$

$$\mathbb{Y} = (Y, Y^{\diamond 2}, I(Y^{\diamond 3}), \pi_{0,\diamond}(I(Y^{\diamond 3}),Y), \pi_{0,\diamond}(I(Y^{\diamond 2}),Y^{\diamond 2})-\varphi, \pi_{0,\diamond}(I(Y^{\diamond 3}),Y^{\diamond 2})-3\varphi Y,\varphi).$$

Grâce à cette définition de distributions rugueuses, il est alors possible de formaliser l'ansatz faite pour avancer dans l'analyse de l'équation.

Définition 1.3.11. Soit $\mathbb{Y} \in \mathcal{X}_{T,K}$. On définit l'espace $\mathcal{D}^\delta_{\mathbb{Y},T}$ des distributions paracontrôlées par $\mathbb{Y}$ par

$$\mathcal{D}^\delta_{\mathbb{Y},T} = \{(\Phi,\Phi') \in (\mathcal{C}_T^{1/2-\delta})^3 : \Phi^\# = \Phi - I(Y^{\diamond 3}) - B_<(\Phi', Y^{\diamond 2}) \in \mathcal{C}_T^{3/2-\delta}\}.$$

L'espace $\mathcal{D}^\delta_{\mathbb{Y},T}$ est un espace affine muni de la semi-norme

$$\|\Phi\|_{\mathcal{D}^\delta_{\mathbb{Y},T}} = \|\Phi\|_{\mathcal{C}_T^{1/2-\delta}} + \|\Phi'\|_{\mathcal{C}_T^{1/2-\delta}} + \|\Phi^\#\|_{\mathcal{C}_T^{3/2-\delta}}.$$

ainsi que de la distance associée $d_{\mathcal{D}^\delta_{\mathbb{Y},T}}(\Phi_1,\Phi_2) = \|\Phi_1-\Phi_2\|_{\mathcal{D}^\delta_{\mathbb{Y},T}}$, ce qui munit $\mathcal{D}^\delta_{\mathbb{Y},T}$ d'une structure d'espace métrique complet.

La définition de tels espaces permet alors de formuler l'équation pour $\mathbb{Y} \in \mathcal{X}_{T,K}$ et $\Phi \in \mathcal{D}^{\delta}_{\mathbb{Y},T}$

$$\Phi = I(Y^{\diamond 3}) + 3I(\Phi^2 Y) + 3I(\Phi \diamond Y^{\diamond 2}) + I(\Phi^3).$$

Le terme $I(\Phi \diamond Y^{\diamond 2})$ n'est pas le produit usuel, mais désigne le produit compatible avec la renormalisation et les composantes de $\mathbb{Y}$.

Remarque 1.3.12. Les conditions analytiques diffèrent légèrement de celles du chapitre 4, et ce car nous considérons ici seulement $u_0 = 0$.

Dans ces espaces précédemment introduits, il est alors possible de résoudre l'équation par un point fixe de Banach pour un temps petit. Nous donnons ici la version sans condition initiale, et renvoyons au chapitre 4 pour la version générale du théorème.

Théorème 1.3.13. *Soit $F : C(\mathbb{R}^+; C^\infty(\mathbb{T}^3)) \times \mathbb{R}^2 \to C(\mathbb{R}^+, C^1)$ le flot de l'équation*

$$\begin{cases} \partial_t u = \Delta_{\mathbb{T}^3} u - (u^3 - (3a+9b)u) + \Xi, & 0 \leqslant t < T_C(\Xi, \varphi, (a,b)) \\ \partial_t u = 0 \,\text{pour}\, t \geqslant T_C(u_0, \Xi, \varphi, (a,b)) \end{cases}$$

où $\Xi \in C(\mathbb{R}^+; C^0(\mathbb{T}^3))$ et $T_C(\Xi) > 0$ est un temps pour lequel la solution u satisfait l'équation. Soit maintenant $z \in (1/2, 1/3)$, alors il existe $\tilde{T}_C : \mathcal{X}^{\delta}_{K,T} \to \mathbb{R}^+$ une fonction semi-continue inférieurement et $\tilde{F} : \mathcal{X}^{\delta}_{K,T} \to C(\mathbb{R}^d, \mathcal{C}^{-z}(\mathbb{T}^3))$ qui est continue en $\mathbb{Y} \in \mathcal{X}^{\delta}_{K,T}$ et tels que $(\tilde{T}_C, \tilde{F})$ étend (T, F) dans le sens suivant. Pour tout $(\Xi, \varphi, (a,b)) \in C(\mathbb{R}^+; C^\infty(\mathbb{T}^3)) \times \mathbb{R}^2$ avec $Y = -I(\Xi)$, on a

$$T_C(\Xi, \varphi, (a,b)) \geqslant \tilde{T}_C(R^{\varphi}_{a,b} Y)$$

$$F(\Xi, \varphi, (a,b))(t) = \tilde{F}(R^{\varphi}_{a,b} Y)(t) \text{pour tout } t < \tilde{T}_C(R^{\varphi}_{a,b} Y).$$

Il est désormais possible de résoudre l'équation (1.8) quand on considère le bruit blanc gaussien. Puisque le flot F défini précédemment est continu, il suffit de montrer qu'il existe des constantes a_ε et b_ε et une fonction φ_ε de telle sorte $R^{\varphi_\varepsilon}_{a_\varepsilon, b_\varepsilon} X^\varepsilon$ converge vers un élément $\mathbb{X} \in \mathcal{X}_{T,K}$.

Théorème 1.3.14. *Soit $X = -I(\xi)$ le processus d'Ornstein Uhlenbeck associé à ξ le bruit blanc gaussien espace temps sur le tore $\mathbb{T}^3$. Alors il existe deux constantes C_1^ε et C_2^ε, une fonction $\varphi^\varepsilon \in C^\infty(\mathbb{R}_+; \mathbb{R})$ telle que $R^{\varphi^\varepsilon}_{C_1^\varepsilon, C_2^\varepsilon} X^\varepsilon$ converge en probabilité dans $\mathcal{X}_{T,K}$ vers un processus $\mathbb{X} \in \mathcal{X}_{T,K}$. De plus la première composante de $\mathbb{X}$ est X.*

Ainsi les deux résultats précédents mis bout à bout donnent le théorème de convergence suivant, utile pour définir les solutions de l'équation (1.8).

Corollaire 1.3.15. *Soit u^ε la solution de l'équation*

$$\begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon - ((u^\varepsilon)^3 - (3C_1^\varepsilon + 9C_2^\varepsilon) u^\varepsilon) + \xi^\varepsilon; t \in [0, T^\varepsilon) \\ \partial_t u^\varepsilon = 0; t \geqslant T^\varepsilon \end{cases}$$

avec ξ^ε une régularisation par convolution du bruit blanc gaussien espace temps sur le tore $\mathbb{T}^3$ et T^ε le temps d'existence de u^ε. Alors u^ε converge en probabilité dans $C(\mathbb{R}^+; \mathcal{C}^{-z})$ vers $u = \tilde{F}(\mathbb{X})$ pour tout $z \in (1/3, 1/2)$.

Chapitre 2

Moyenne le long de courbes irrégulières et régularisation d'équations différentielles ordinaires

Résumé

Nous étudions à l'équation différentielle ordinaire $\mathrm{d}x_t = b(t, x(t))\mathrm{d}t + \mathrm{d}w_t$ où w est une fonction continue et b un champs de vecteur irrégulier Nous quantifions les propriétés de régularisation de perturbations w arbitraires sur l'existence et unicité de solutions de cette équation. Pour cela nous introduisons la notion de (ρ, γ)-irregularité et montrons quel joue un rôle fondamentale dans des phénomènes de régularisation par le bruit.

Lorsque w est distribué suivant la loi du mouvement brownien fractionnaire de paramètre de Hurst $H \in (0, 1)$, nous montrons que presque surement l'équation différentielle admet une unique solution lorsque b est dans l'espace de Besov-Hölder $\in B^{\alpha+1}_{\infty,\infty}$, avec $\alpha > -1/2H$. Il est intéressant de montrer que lorsque $1 + \alpha < 0$ le champs de vecteurs b est une distribution, nous fournissons un cadre naturelles pour des solutions dans ce cas.

Résumé

We consider the ordinary differential equation (ODE) $\mathrm{d}x_t = b(t, x_t)\mathrm{d}t + \mathrm{d}w_t$ where w is a continuous driving function and b is a time-dependent vector field which possibly is only a distribution in the space variable. We quantify the regularizing properties of an arbitrary continuous path w on the existence and uniqueness of solutions to this equation. In this context we introduce the notion of *ρ-irregularity* and show that it plays a key role in some regularization by noise phenomenon. When w is sampled according to the law of the fractional Brownian motion of Hurst index $H \in (0, 1)$, we prove that almost surely the ODE admits a solution for all b in the Besov-Hölder space $B^{\alpha+1}_{\infty,\infty}$ with $\alpha > -1/2H$. We also investigate the regularity of the flow for various values of α. Note that when $\alpha < 0$ the vectorfield b is only a distribution, nonetheless there exists a natural notion of solution for which the above results apply.

Sommaire

2.1 Introduction

In [20] A. M. Davie showed that the integral equation

$$x_t = x_0 + \int_0^t b(s, x_s)\mathrm{d}s + w_t, \qquad t \in [0, T] \tag{2.1}$$

with $x, w \in C([0, T]; \mathbb{R}^d)$ and $b : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ bounded and measurable has a unique continuous solution for almost every path w sampled from the law of the d-dimensional Brownian motion. This result can be interpreted as a phenomenon of regularisation by noise, in the sense that it is well known that the same equation without w can show non-uniqueness.

Regularisation by noise in the case of stochastic differential equations (SDEs) driven by Brownian motion is nowadays a well understood subject: see for example Veretennikov, Krylov and Roeckner [64], Flandoli, Gubinelli and Priola [30], Zhang, Flandoli and Da Prato [18]. All these work are essentially based of the use of Itô calculus to highlight the regularising properties of Brownian paths. Meyer-Brandis and Proske [64] use Malliavin calculus to derive similar conclusions. Davie's contribution [20] is more subtle in the sense that it is a result for an ordinary differential equation (ODE) and not for the related SDE, i.e. the existence and uniqueness of solutions is studied in the space of continuous paths and not in the more common probabilistic framework of continuous adapted processes on a given filtered probability space. This has been clearly pointed out by Flandoli [28] which called these more general solutions *path-by-path*. In this respect Davie's contribution is purely analytical and one of the aim of the present work is to *analytically* caracterize the regularisation effect for general continuous perturbation w (whether random or not) to the evolution dictated by an irregular vectorfield.

Regularisation by "fast" or "dispersive" motions is an interesting phenomenon which appears also in some deterministic PDE situations, for example for Korteweg-de-Vries equation [42, 3] and for fast-rotating Euler and Navier-Stokes equations [4]. In particular the technique of Young integration we employ in the present work is essentially the same used in the paper [42] to study the periodic Korteweg-de-Vries equation and take inspiration in the theory of rough paths [61, 41, 34].

In a recent paper [14, 15] Chouk and Gubinelli analyse the regularisation phenomenon in the context of non-linear dispersive PDEs modulated by an irregular signal. In particular they considered equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = A\varphi_t\frac{\mathrm{d}w_t}{\mathrm{d}t} + \mathcal{N}(\varphi_t), \qquad t \geq 0 \tag{2.2}$$

where $w : \mathbb{R}_+ \to \mathbb{R}$ is an arbitrary continuous function, A is an unbounded linear operator (like the Schrödinger operator $i\partial^2$ or the Airy operator ∂^3 acting on periodic or non-periodic functions) and $\mathcal{N}$ some local polynomial non-linearity with possibly derivative terms. The unifying theme of this last study and the present one is the fact that the regularising properties of $w \in C([0,T];\mathbb{R}^d)$ are analysed in terms of the *averaging operator* T_t^w defined as

$$T_t^w f(x) = \int_0^t f(x + w_r)\mathrm{d}r, \qquad x \in \mathbb{R}^d \tag{2.3}$$

for any measurable functions $f : \mathbb{R}^d \to \mathbb{R}$. Characterising the mapping properties of T^w for various kind of perturbations w seems very interesting and not straightforward. Mapping properties of T^w for deterministic smooth curves w are, for reasons not related to the regularisation by noise phenomenon, an interesting subject in analysis: we have in mind, for example the work of Tao and Wright [71] on L^p improving bounds for averages along curves (we thanks F. Flandoli and V. M. Tortorelli for having pointed us the existence of these results).

The averaging operator can be seen as the convolution against the *occupation measure* L_t^w of the path w defined as

$$L_t^w(\mathrm{d}y) = \int_0^t \delta_{w_u}(\mathrm{d}y).$$

Indeed, for continuous b, the following computation holds

$$T_t^w b(x) = \int_0^t b(x + w_u)\mathrm{d}u = \int_0^t \mathrm{d}u \int_{\mathbb{R}^d} b(x - y)\delta_{w_u}(\mathrm{d}y) = (b * L_t^w)(x).$$

The basic observation contained in Davie's paper [20] is that if $b : \mathbb{R}^d \to \mathbb{R}^d$ is a given bounded function then for almost every d-dimensional Brownian path $w : [0,T] \to \mathbb{R}^d$ and for all $0 \leqslant t \leqslant T$ the function $x \mapsto T_t^w b(x)$ has almost Lipschitz regularity (its modulus of continuity is of the type $|x| \log^{1/2}(1/|x|)$). Morally this is a gain of almost 1 degree of the regularity and one of the key step to prove uniqueness of the ODE (2.1) for bounded measurable drift b.

In this paper we analyse the behaviour of the averaging operator T^w in the scale of Hölder-Besov spaces $\mathcal{C}^\alpha = \mathcal{C}^\alpha(\mathbb{R}^d, \mathbb{R}^n) = B^\alpha_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^n)$ for arbitrary regularity $\alpha \in \mathbb{R}$. We consider a class of perturbations w given by the sample paths of the d-dimensional fractional Brownian motion (fBm) of Hurst index $H \in (0,1)$, that is the unique centered Gaussian process $(B_t^H)_{t\geqslant 0}$ with values on $\mathbb{R}^d$ and covariance function

$$\mathbb{E}[B_t^H B_s^H] = c_H(|t+s|^{2H} - |t|^{2H} - |s|^{2H})$$

for all $t, s \geqslant 0$.

As an application of the averaging properties we obtain various existence and uniqueness results for solutions of the ODE (2.1) and relative flow properties for distributional vectorfield b.

The choice of fBm has the advantage of being a simple process for which many other results about existence and uniqueness of associated SDE are available [67, 68]. More interestingly, the approach based on Itô calculus, used in most of the papers on the regularisation effect for Brownian motion, does not easily extend to the fBm case, nor the explicit computations of Davie [20]. The freedom in the choice of the Hurst parameter gives us the possibility to explore the effect of different degrees of irregularity of the perturbation on the regularisation phenomenon and the quasi-invariance of the law of the fBm will allow us to study the effect of perturbations on the the averaging properties of the paths.

Returning to the averaging behavior of fBm paths we obtain the following result

Theorem 2.1.1. *Take $H \in (0,1)$ and $\rho < 1/2H$ and $\alpha > 0$. Then there exists $\gamma > 1/2$ such that for all $f \in \mathcal{C}^\alpha(\mathbb{R}^d;\mathbb{R})$ there exists a Borel set $\mathcal{N}_{f,\gamma} \subseteq C([0,1],\mathbb{R}^d)$ (which depends on f,γ) of zero measure with respect to the law of the d-dimensional fractional Brownian motion (fBm) of Hurst index H such that for all $w \notin \mathcal{N}_{f,\gamma}$ we have*

$$\|T_t^w f - T_s^w f\|_{\mathcal{C}^{\alpha+\rho,\psi}} \lesssim_w \|f\|_{\mathcal{C}^\alpha} |t-s|^\gamma$$

for all $0 \leqslant s < t \leqslant 1$.

In this statement the weighted space $\mathcal{C}^{\alpha,\psi}$ is a subspace of the space of local Besov-Hölder functions with given grow at infinity described by the weight ψ.

Letting for a moment aside the time regularity, this result shows that the averaging against fBm paths gains almost $1/2H$ derivatives in the space variable. Unfortunately the result stated in Theorem 2.1.1 is not very satisfying since one would really like to have the almost sure boundedness of $T_t^w : \mathcal{C}^\alpha \to \mathcal{C}^{\alpha+\rho,\psi}$. The difficulty is, of course, the fact that the exceptional set $\mathcal{N}_f$ of Theorem 2.1.1 depends itself on the function f. Using the Littewood–Paley decomposition of Holder–Besov distributions and the scaling of the fractional Brownian motion, the problem of finding a version of T^w which is almost surely continuous can be related to the following conjecture:

Conjecture 2.1.2. *Let $(B_t^H)_{t\geqslant 0}$ be a d-dimensional fBm of Hurst index $H \in (0,1)$. Let $K : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that*

$$|K(x)| \lesssim (1+|x|)^{-N}, \qquad \int_{\mathbb{R}^d} K(x)\mathrm{d}x = 0$$

where $N > d$ can be chosen arbitrarily large. Then

$$\mathbb{E}(\|T_{0,t}^{B^H} K\|_{L^1(\mathbb{R}^d)}^p) = \mathbb{E}\left[\left(\int_{\mathbb{R}^d}\left|\int_0^t K(x+B_s^H)\mathrm{d}s\right|\mathrm{d}x\right)^p\right] \lesssim t^{p/2}$$

for $t \to +\infty$.

If the function K has a bounded support the estimation is true as an easy consequence of our results, however currently we are unable to prove or disprove this conjecture.

On the positive side if we replace $\mathcal{C}^\alpha$ by the Fourier–Lebesgue spaces $\mathcal{F}L^{\alpha,p}$ defined as

$$\mathcal{F}L^{\alpha,p}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : N_{\alpha,p}(f)^p = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^p (1+|\xi|)^{\alpha p}\mathrm{d}\xi < \infty\}$$

and let $\mathcal{F}L^{\alpha} = \mathcal{F}L^{\alpha,1}$ then it is easy to see that for $0 \leqslant \gamma \leqslant 1$ and $\rho \in \mathbb{R}$:

$$\|T_t^w - T_s^w\|_{\mathcal{F}L^{\alpha} \to \mathcal{F}L^{\alpha+\rho}} = \sup_{f \in \mathcal{F}L^{\alpha}} \frac{N_{\alpha+\rho}(T_t^w f - T_s^w f)}{N_{\alpha}(f)} \lesssim \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-s|^{\gamma}$$

where $\Phi_t^w(a) = \int_0^t e^{i\langle a, w_r\rangle} \mathrm{d}r = e^{-i\langle a,x\rangle} T_t^w(e^{i\langle a,\cdot\rangle})(x)$ and where we introduced the norm

$$\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} = \sup_{a \in \mathbb{R}^d} \sup_{0 \leqslant s < t \leqslant T} \langle a\rangle^{\rho} \frac{|\Phi_t^w(a) - \Phi_s^w(a)|}{|s-t|^{\gamma}}.$$

This observation reduces the question of the boundedness of T^w to that of the decay of the Fourier transform $a \mapsto \Phi_t^w(a)$ of the *occupation measure* of w (for generalities about occupation measures and densities for deterministic and random functions see for example the review of Geman and Horowitz [37]). This suggests to introduce the following novel notion of "irregularity" of the perturbation w :

Definition 2.1.3. Let $\rho > 0$ and $\gamma > 0$. We say that a function $w \in C([0,T];\mathbb{R}^d)$ is (ρ,γ)-irregular if

$$\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} < +\infty.$$

Moreover we say that w is ρ-irregular if there exists $\gamma > 1/2$ such that w is (ρ,γ)-irregular.

The time regularity of this Fourier transform, measured by the Hölder exponent γ, will also be crucial in our analysis. The notion of ρ-irregularity is also relevant to the boundedness of T^w in other functional spaces, for example we easily see that for all $\alpha \in \mathbb{R}$:

$$\|T_t^w f - T_s^w f\|_{H^{\alpha+\rho}(\mathbb{R}^d)} \leqslant \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-s|^{\gamma} \|f\|_{H^{\alpha}(\mathbb{R}^d)}$$

where $H^{\alpha}(\mathbb{R}^d) = \mathcal{F}L^{\alpha,2}$ are the usual Sobolev spaces on $\mathbb{R}^d$ and in general similar inequalities holds in Fourier–Lebesgue spaces $\mathcal{F}L^{\alpha,p}$ of arbitrary integrability $p \in [1,+\infty]$. However the notion of ρ-irregularity seems not enough to control the boundedness of the averaging operator in Besov spaces.

The limiting value $1/2$ for γ seems not have any special meaning, as far as the occupation measure is concerned, however if $\gamma > 1/2$ we are able to develop a quite simple integration theory for the averaging operator using Young integral techniques and quite surprisingly it turns out that this is sufficient for the purpose of this paper. Indeed a proof similar to that of Theorem 2.1.1 gives the existence of (plenty of) perturbations w which are ρ-irregular :

Theorem 2.1.4. *Let $(B_t^H)_{t \geqslant 0}$ be a fractional Brownian motion of Hurst index $H \in (0,1)$ then for any $\rho < 1/2H$ there exist $\gamma > 1/2$ so that with probability one the sample paths of B^H are (ρ,γ)-irregular.*

In particular there exist continuous paths which are ρ-irregular for arbitrarily large ρ and thus paths which deliver an arbitrary degree of regularisation. Using well known properties of support of the law of the fractional Brownian motion $\boxed{\mathtt{r}}$eference it is also possible to show that there exists ρ-irregular trajectories which are arbitrarily close in the supremum norm to any smooth path.

As a direct corollary of Theorem 2.1.4 we have the boundedness of T^w in the Lebesgue–Fourier spaces $\mathcal{F}L^{\alpha}$:

Corollary 2.1.5. *Let $H \in (0,1)$ and $\rho < 1/2H$. Then almost surely wrt the law of the fBm of Hurst index H we have that for all $0 \leqslant s \leqslant t \leqslant T$ the averaging operator T^w is bounded from $\mathcal{F}L^\alpha$ to $\mathcal{F}L^{\alpha+\rho}$ and satisfy*

$$\|T_t^w - T_s^w\|_{\mathcal{L}(\mathcal{F}L^\alpha, \mathcal{F}L^{\alpha+\rho})} \leqslant C_{w,\gamma,\rho}|t-s|^\gamma$$

for some constant $C_{w,\gamma,\rho}$ which depends only on w, γ, ρ. This means that

$$T^w \in C^\gamma\left([0,T]; \mathcal{L}\left(\mathcal{F}L^\alpha, \mathcal{F}L^{\alpha+\rho}\right)\right).$$

One of the contributions of our work is the observation that the regularity of the occupation measure of w seems to play a major role in the understanding of the regularising properties of w in a non-linear context and it would be desirable to understand more deeply the link of the notion of ρ-irregularity with the path-wise properties of w, for example linking them to the notion of true roughness appearing in the literature on densities for differential equations driven by rough paths [33].

It would also be interesting to study more deeply the notion of irregularity for "generic" continuous paths (for example in the class of Hölder continuous paths). Indeed, set aside the classic contribution of Geman and Horowitz [37] mentioned above, the authors are not aware of any systematic study of occupation measures of random processes from the point of view of their action on spaces of functions or distributions, topic which seems central to our analysis.

An open problem is, for example, understanding what happens if we replace w with a regularised version w^ε or with a perturbed version. In this respect we conjecture that if w is (ρ,γ)-irregular then for any smooth function $\varphi \in C([0,1];\mathbb{R}^d)$ the perturbed path $w^\varphi = w + \varphi$ is still (ρ,γ)-irregular. In relation to this last problem we have obtained the following general result:

Theorem 2.1.6. *Let $\rho \in \mathbb{R}$ and $\varphi \in \mathcal{C}^\beta([0,T];\mathbb{R}^d)$ with $1/2 \leqslant \beta < 1$. Then if w is ρ-irregular the path $w^\varphi = w + \varphi$ is $(\rho - 1/2\beta)$-irregular. Moreover for $\gamma > 1/2$ we have*

$$\|T^{w+\varphi} f\|_{C^\gamma\left([0,T];\mathcal{C}^{\alpha+\rho-1/2\beta}\right)} \lesssim_{T,\beta,\gamma} \|T^w f\|_{\mathcal{C}^{\gamma,\psi}([0,T];\mathcal{C}^{\alpha+\rho})} \|\varphi\|_{\mathcal{C}^\beta}$$

and in particular if $T^w \in \mathcal{C}^\gamma\left([0,T]; \mathcal{L}\left(\mathcal{C}^\alpha; \mathcal{C}^{\alpha+\rho,\psi}\right)\right)$ then $T^{w+\varphi} \in \mathcal{C}^\gamma\left([0,T]; \mathcal{L}\left(\mathcal{C}^\alpha; \mathcal{C}^{\alpha+\rho-1/2\beta}\right)\right)$.

In particular the irregularity property is preserved at the price of a loss at least $1/2$ in regularity (which happens when β is close to 1).

If w is sampled according to the law of a fBm and if the perturbation φ is adapted to the natural filtration of w then it is possible to exploit the quasi-invariance of the fBm measure wrt adapted shifts to prove the irregularity of the perturbed path without any loss on the irregularity exponent:

Theorem 2.1.7. *Let B^H be a fBm of Hurst index $H \in (0,1)$ and let $\Phi : [0,T] \to \mathbb{R}^d$ be an Hölder continuous process which is adapted to the natural filtration of W. Then, for all $\rho < 1/2H$ almost surely the process $W^\Phi = W + \Phi$ is ρ-irregular and for any $f \in \mathcal{C}^\alpha$*

$$\|T^{W+\Phi} f\|_{C^\gamma\left([0,T];\mathcal{C}^{\alpha+\rho,\psi}\right)} < +\infty$$

almost surely.

The disadvantage of this result is that the exceptional set where the irregularity property fails depends a-priori on Φ and this poses problems in applications to path-wise results valid for a large class of perturbations (for example smooth and adapted Φ).

One of our aims is to apply these results on the averaging properties of paths w and of its perturbations to the study of existence and uniqueness of solutions the ODE (2.1) for distributional b. Two main situation will be considered

1. $b \in \mathcal{C}^\alpha$ (or $b \in \mathcal{F}L^\alpha$) for some $\alpha > 0$. In this case b will be a bounded continuous function and the ODE (2.1) has a natural meaning and allows for continuous solution, we will then consider the related uniqueness problem and the existence of a Lipshitz flow.

2. $b \in \mathcal{C}^\alpha$ (or $b \in \mathcal{F}L^\alpha$) for some $\alpha < 0$. In this case even the appropriate meaning to give to the ODE (2.1) is not clear and we will investigate this problem and the related well-posedness and continuity issues.

In the case $\alpha \geqslant 0$ we have the following results:

Theorem 2.1.8. *Let $b \in \mathcal{C}(\mathbb{R}^d)$ and assume that $\|T^w b\|_{C^\gamma([0,T];\mathcal{C}^{3/2,\psi})} < +\infty$. Then for any $x_0 \in \mathbb{R}^d$ there exists a unique continuous solution $x \in C([0,T];\mathbb{R}^d)$ of the ODE (2.1) and the flow map $x_0 \mapsto x_t$ of the equation is locally Lipshitz continuous in space uniformly in $t \in [0,T]$.*

Theorem 2.1.9. *Let $b \in \mathcal{C}^\alpha$ and assume that $\alpha > 1 - 1/2H$. Then for any $x_0 \in \mathbb{R}^d$ there exists a measurable set of perturbations $\mathcal{N}_{b,x_0} \subseteq C([0,1];\mathbb{R}^d)$ which is of zero measure with respect to the law of the fBm with index $H \in (0,1)$ and such that, for all $w \notin \mathcal{N}_{b,x_0}$ there exists a unique continuous solution $x \in C([0,1];\mathbb{R}^d)$ of the ODE (2.1).*

As we already remarked, in the case where $b \in \mathcal{C}^\alpha$ for $\alpha < 0$, the ODE (2.1) is not well defined since in general the evaluation of the distribution b along a continuous curve is not possible. However if we take into account a particular class of continuous paths we can show that this coupling has a meaning. A suitable class of continuous functions is given by a space of paths which are perturbations of w:

Definition 2.1.10. The space $\mathcal{Q}^w_\gamma$ of (w,γ)-controlled paths is the space

$$\mathcal{Q}^w_\gamma = \{x \in C([0,1];\mathbb{R}^d) \ : (x - w) \in C^\gamma([0,1];\mathbb{R}^d)\}.$$

Then for *controlled paths* we can prove the following result.

Theorem 2.1.11. *Let $b \in \mathcal{S}'(\mathbb{R}^d;\mathbb{R}^d)$ and assume that $\|T^w b\|_{\mathcal{C}^\gamma\left([0,T];\mathcal{C}^{0,\psi}\right)} < +\infty$. Let $\rho \in \mathcal{S}(\mathbb{R}^d)$ be a positive function with $\rho(0) = 1$ and let $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(x/\varepsilon)$. Then, for all $x \in \mathcal{Q}^w_\gamma$,*

$$\lim_{\varepsilon \to 0} \int_0^t (\rho_\varepsilon * b)(x_s)\mathrm{d}s =: \int_0^t b(x_s)\mathrm{d}s \tag{2.4}$$

exists uniformly in $t \in [0,T]$, is independent of ρ and extends the usual definition of the r.h.s. for smooth b. Moreover the function $t \mapsto \int_0^t b(x_s)\mathrm{d}s$ is Hölder continuous of exponent γ.

Theorem 2.1.11 allows to give a natural meaning to $\int_0^t b(x_s)\mathrm{d}s$ for all $x \in \mathcal{Q}_\gamma^w$ and from this we can say that $x \in \mathcal{Q}_\gamma^w$ is a solutions of the ODE (2.1) if

$$x_t - w_t = \int_0^t b(x_s)\mathrm{d}s$$

for all $t \in [0,1]$. That is the ODE has a meaning not in the space of all continuous functions, as it was when b is a function, but in the more restricted space of functions which can be seen as "not too irregular" additive modifications of w. In this context we have natural generalisations of the Theorems 2.1.8 and 2.1.9 provided we restrict the space of allowed functions to $\mathcal{Q}_\gamma^w$:

Theorem 2.1.12. *Assume that* $\|T^w b\|_{C^\gamma([0,T];\mathcal{C}^{3/2,\psi})} < +\infty$. *Then for any* $x_0 \in \mathbb{R}^d$ *there exists a unique continuous solution* $x \in \mathcal{Q}_\gamma^w$ *of the ODE* (2.1) *and the flow map* $x_0 \mapsto x_t$ *of the equation is Lipshitz continuous uniformly in* $t \in [0,T]$.

Theorem 2.1.13. *Let* $b \in \mathcal{C}^{\alpha+1}$ *and assume that* $\alpha > -1/2H$. *Then for any* $x_0 \in \mathbb{R}^d$ *there exists a measurable set of perturbations* $\mathcal{N}_{b,x_0} \subseteq C([0,1];\mathbb{R}^d)$ *which is of zero measure with respect to the law of the fBm with index* $H \in (0,1)$ *and such that, for all* $w \notin \mathcal{N}_{b,x_0}$ *there exists a unique continuous solution* $x \in \mathcal{Q}_\gamma^w$ *of the ODE* (2.1).

Note that Theorem 2.1.12 and 2.1.13 are applicable also when $\alpha \geqslant 0$. In this case existence of solutions is simply a result of a compactness argument in $C([0,1];\mathbb{R}^d)$ and given a continuous solution it belongs necessarily to $\mathcal{Q}_\gamma^w$ so, in this case, Theorem 2.1.12 and 2.1.13 are natural generalisations of Theorems 2.1.8 and 2.1.9.

When w is sampled according to the law of the fBm with Hurst parameter H Theorem 2.1.12 give the following corollary

Theorem 2.1.14. *Fix* $H \in (0,1)$ *and assume that* $b \in \mathcal{C}^{\alpha+3/2}$ *for some* $\alpha > -1/2H$. *Then there exists a measurable set of perturbations* $\mathcal{N}_\lfloor \subseteq C([0,1];\mathbb{R}^d)$ *which is of zero measure with respect to the law of the fBm with index* $H \in (0,1)$ *and such that, for all* $w \notin \mathcal{N}_\lfloor$ *and for all* $x_0 \in \mathbb{R}^d$ *there exists a unique continuous solution* $x \in \mathcal{Q}_\gamma^w$ *of the ODE* (2.1) *and the corresponding flow map* $\Phi_t : x_0 \mapsto x_t$ *is locally Lipshitz. Moreover the exceptional set* $\mathcal{N}_b$ *can be chosen to be the same for all* $b \in \mathcal{F}L^\alpha$.

An interesting consequence of Theorem 2.1.14 is the fact that if one consider the ODE (2.1) as a strong SDE (that is an equation for stochastic processes adapted to the filtration generated by the process w) and if w has the law of the fBm of index H then we can allow general random $b \in \mathcal{F}L^\alpha$ and still retain uniqueness under the regularity conditions of the theorem. This was one of our main motivation to introduce the scale of Fourier-Lebesgue regularities $(\mathcal{F}L^\alpha)_\alpha$. Similar results for the Besov scale $(\mathcal{C}^\alpha)_\alpha$ are not known since we are not able to prove the corresponding mapping properties for the averaging operator T^w. Note that even in the case of the Brownian motion this was an open problem [30] since the standard approach using stochastic calculus cannot be applied in this case. Allowing random b could open the way to the study of a general class of stochastic transport equations where the drift itself depends on the solution.

The key to obtain these results (the existence part when $\alpha < 0$ and the uniqueness part for $\alpha \geqslant 0$ or $\alpha < 0$) lies in the fact that in all cases the ODE (2.1) is equivalent to an equation of Young type (YE) of the form

$$\theta_t = \theta_0 + \int_0^t X_{\mathrm{d}s}(\theta_s) \tag{2.5}$$

where here $X : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ plays the röle of a time-varying, integrated, vector field and $\theta_t = x_t - w_t$ is the perturbation which by the hypothesis $x \in \mathcal{Q}^w_\gamma$ belongs to $C^\gamma([0,1];\mathbb{R}^d)$. The integral operation featuring in (2.5) has to be understood as a natural non-linear generalisation of the Young intergal [75] defined as limit of Riemman sums:

$$\int_0^t X_{\mathrm{d}s}(\theta_s) = \lim_{|\Pi|\to 0} \sum X_{t_i,t_{i+1}}(\theta_{t_i})$$

where $X_{s,t}(x) = X_t(x) - X_s(x)$. In the case of the ODE (2.1) the integrated vector field X corresponds to the average of the original vector field b given by $X_t(x) = T^w_{0,t}b(x)$ for all $t \in [0,1]$ and $x \in \mathbb{R}^d$. Young differential equations of the type (2.5) are used also in [14, 14] to study the regularisation phenomenon for some non-linear dispersive equations. The theory of such equations is very similar to the theory for standard Young-type equation but for the sake of the reader we rederive here the main results in our slightly non standard setting.

This paper is then divided naturally into two parts: in the first we study the non-linear Young integral and the YE (2.5) and derive the results announced above about existence and uniqueness for the ODE (2.1). In the second we analyse the averaging properties of fBm sample paths and apply the results to the study of the regularisation phenomenon for eq. (2.1) driven by fBm paths.

2.2 Notations

Several functions spaces are involved in the rest of the article. In this section we define those spaces, and specify some notations.

Let $\psi, \varphi \in \mathcal{D}$ be nonnegative radial functions such that

1. The support of ψ is contained in a ball and the support of φ is contained in an annulus;
2. $\psi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^d$;
3. $\mathrm{supp}(\psi) \cap \mathrm{supp}(\varphi(2^{-j}\cdot)) = \emptyset$ for $i \geq 1$ and if $|i-j| > 1$, then $\mathrm{supp}(\varphi(2^{-i}\cdot)) \cap \mathrm{supp}(\varphi(2^{-j}.)) = \emptyset$.

For the existence of ψ and φ see [5]. The Littlewood-Paley blocks are now defined as

$$\Delta_{-1}u = \mathcal{F}^{-1}(\psi \mathcal{F}u) \quad \text{and} \quad \text{for} j \geq 0, \quad \Delta_j u = \mathcal{F}(\varphi(2^{-j}.)\mathcal{F}u).$$

The $\Delta_j u$ are smooth function with Fourier transform with compact support. We define the Hölder-Besov space $\mathcal{C}^\alpha$ by

$$\mathcal{C}^\alpha(\mathbb{R}^d, \mathbb{R}^n) = B^\alpha_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^n) : \|u\|_\alpha = \|(2^{j\alpha}\|\Delta_j u\|_\infty)_j\|_\infty < \infty \right\}$$

While the norm $\|.\|_\alpha$ depends on the choice of ψ and φ, the space $\mathcal{C}^\alpha$ does not and each choice of ψ, φ correspond to an equivalent semi-norm on $\mathcal{C}^\alpha$. If $\alpha \in \mathbb{R}_+ - \mathbb{N}$, then the space $\mathcal{C}^\alpha$ is the space of $[\alpha]$ times differentiable functions , whose partial derivatives up to orderer $[\alpha]$ are bounded, and whose partial derivatives of orderer $[\alpha]$ are $(\alpha - [\alpha])$-Hölder continuous. Note that we have the following continuous embedding, for $\alpha' \leqslant \alpha$ then $\mathcal{C}^\alpha \subset \mathcal{C}^{\alpha'}$ and $\|u\|_{\alpha'} \lesssim \|u\|_\alpha$.

When $f \in C([0,T], \mathcal{C}^\alpha)$, we denote abusively $\|f\|_\alpha = \sup_{t\in I} \|u(t,.)\|_\alpha$. When $\alpha > 0$, the space $\mathcal{C}^\alpha = B^\alpha_{\infty,\infty}$ is the space of bounded Hölder continuous functions, indeed, for $m \in \mathbb{N}\backslash\{0\}$ and $m-1 \leqslant \alpha < m$, when we define $[\![f]\!]_\nu = \sup_{x\neq y} |f(x)-f(y)|/|x-y|^\nu$ for $\nu \in (0,1]$ and

$$\mathcal{C}^\alpha = \{f : \mathbb{R}^d \to \mathbb{R}^d : \|f\|_{\infty,\mathbb{R}^d} + [\![D^{m-1}f]\!]_{m-\nu} < +\infty\}.$$

Furthermore $\|.\|_\alpha$ and $\|f\|_\infty + [\![f]\!]_{m-\alpha}$ are equivalent norms. We will indifferently use either one or the other. We will also need some localised Hölder spaces described as follows:

Definition 2.2.1. Let $\nu \in [0,1)$. A weight is a continuous non-decreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for $c > 0$, there exists a constant $C_{c,\psi} > 0$ such that

$$\psi(Cx) \leqslant C_{c,\psi}\psi(x).$$

A ν-weight is a weight such that

$$x^{-(1-\nu)}\psi(x) \xrightarrow[x\to+\infty]{} 0.$$

Hence, the weighted Hölder spaces are defined in that case as

Definition 2.2.2. Let ψ a weight and $\nu \in (0,1]$ and V and W two Banach spaces. The ψ-weighted Hölder space of index ν is the space $\mathcal{C}^{\nu,\psi}(V,W)$ defined by

$$\mathcal{C}^{\nu,\psi}(V,W) = \left\{ f : \mathcal{C}^\nu_{\mathrm{loc}} : [\![f]\!]_{\nu,\psi} = \sup_{x\neq y\in V} \frac{|f(x)-f(y)|_W}{|x-y|_V \psi(|x|_V + |y|_V)} < +\infty \right\},$$

with the usual extension when $\nu > 1$.

To simplify the notation, we introduce also the following spaces related to time dependent nonlinear mappings between Banach spaces V and W.

Definition 2.2.3. Let $0 < \gamma, \nu \leqslant 1$ and ψ a weight. Let $I = [0,T]$ and V and W two Banach spaces. For all $n \in \mathbb{N}$ any $G : I \times V \to W$ we define

$$[\![G]\!]_{\gamma,\nu,\psi} = \sup_{s\neq t}\sup_{x\neq y} \frac{|G_{s,t}(x) - G_{s,t}(y)|}{|t-s|^\gamma |x-y|^\nu \psi(|x|+|y|)}$$

and

$$\|G\|_{\gamma,n+\nu,\psi} = [\![D^n G]\!]_{\gamma,\nu,\psi} + \sum_{k=0}^n \sup_{s\neq t} \frac{|D^n G_{s,t}(0)|}{|t-s|^\gamma}$$

$$C^{\gamma,n+\nu,\psi}(I,V,W) = \left\{ G \in L^\infty\left(I; \mathcal{C}^{\nu,\psi}(V,W)\right) : \|G\|_{\gamma,n+\nu,\psi} < \infty \right\}.$$

When $V = W$ we write $C^{\gamma,\nu,\psi}(I,V,V) = C^{\gamma,\nu,\psi}(I,V)$. Furthermore, when it is not ambiguous we only use $\mathcal{C}^{\gamma,\nu,\psi}$. When $\psi = 1$ and there is no ambiguity, we only write $\mathcal{C}^{\gamma,\nu}$.

As stated in the introduction, in order to have estimates for the averaging operator T^w which will not depend on the functions f, we introduce the following Fourier–Lebesgue spaces

Definition 2.2.4. Let $\alpha \in \mathbb{R}$ and

$$N_{\alpha,p}(f) = \int_{\mathbb{R}^d} |\hat{f}(\omega, \xi)|^p (1 + |\xi|)^{p\alpha} \mathrm{d}\xi$$

and $\mathcal{F}L^{\alpha,p}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : N_\alpha(f) < \infty\}$. Then $N_{\alpha,p}$ is a norm on $\mathcal{F}L^{\alpha,p}(\mathbb{R} \times \mathbb{R}^d)$. When $p = 1$ we only write $\mathcal{F}L^\alpha$ and N_α.

When $\alpha \geqslant 0$ and $f \in \mathcal{F}L^\alpha$ implies that $\hat{f}$ is in L_1 and f is bounded continuous function. Furthermore if $\alpha \geq 1$, $f \in \mathcal{F}L^\alpha$ is globally Lipschitz continuous in the second variable. Furthermore for $\alpha \in (0, 1)$, $f \in \mathcal{F}L^\alpha$ is globally Hölder continuous in the second variable. Note that if $\alpha < 0$ the vectorfields are only distributions.

Remark 2.2.5. An easy computation gives $\mathcal{F}L^\alpha \subset \mathcal{C}^\alpha$ for all $\alpha \in \mathbb{R}$, and for $\alpha > 0$ and ψ a weight, $\mathcal{C}^\alpha \subset \mathcal{C}^{\alpha,\psi}$.

It is natural to make some approximations in $\mathcal{C}^\alpha$ and in $\mathcal{F}L^\alpha$. Thought the quantity $\sum_i \Delta_i f$ does not converge in $\mathcal{C}^\alpha$, it converges in all $\mathcal{C}^{\alpha'}$ with $\alpha' < \alpha$, which gives the following lemma :

Lemma 2.2.6. *The sequence* $(\pi_{\leqslant N} u)_{N \geq -1} = (\sum_{j \leqslant N} \Delta_j u)_N$ *converges to* u *in* $\mathcal{C}^{\alpha'}$ *for all* $\alpha' < \alpha$. *Furthermore, for all* α, $\pi_{\leqslant N} f \xrightarrow{\mathcal{F}L^\alpha} f$ *for* $f \in \mathcal{F}L^\alpha$.

Finally if $G : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ we let $G_{s,t}(x) = G_t(x) - G_s(x)$.

2.3 The non-linear Young integral and Young-type equations

As already said, we intend to study the ODE (2.1) where $w : \mathbb{R}_+ \to \mathbb{R}^d$ is a continuous function (with $w_0 = 0$) and $b : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a (time-dependent, distributional) vectorfield. We think w as a very rough function whose oscillations dominate in small time scales the effects of the integrated vectorfield b. In this situation the function x behaves at small scales very much like w and the effects of b are seen only via a average over these fast oscillations. All this will cooks up some regularisation effect which will allow to prove existence and uniqueness even when the vectorfield b does not enjoys sufficient space regularity.

To highlight the effect of the translations induced by w on the flow of b let us introduce the change of variables $\theta_t = x_t - w_t$ so that the above equation now reads:

$$\theta_t = \theta_0 + \int_0^t b(w_s + \theta_s) \mathrm{d}s.$$

If we believe that w oscillate faster than θ then it seems reasonable to approximate the integral in the r.h.s. by a sum over a partition $t_0 = 0, \ldots, t_n = t$ of $[0, t]$ where we have fixed the θ parameter at the initial time of each segment:

$$\int_0^t b(s, w_s + \theta_s) \mathrm{d}s \simeq \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} b(s, w_s + \theta_{t_i}) \mathrm{d}s = \sum_{i=0}^{n-1} (T^w_{t_i, t_{i+1}} b)(\theta_{t_i}). \tag{2.6}$$

where $T^w_{s,t} = T^w_t - T^w_s$.

Under appropriate conditions the expression on the right hand side of eq.(2.6) will have a well defined limit as the size of the partition goes to zero and it defines a kind of integral which we naturally denote by

$$\int_0^t (T^w_{\mathrm{d}s} b)(\theta_s) = \lim \sum_{i=0}^{n-1} (T^w_{t_i,t_{i+1}} b)(\theta_{t_i}).$$

and will enable us to set up an alternative formulation of the above ODE as an integral equation involving the time-dependent integrated vectorfield $G_t = T^w_t b$ which is an averaged version of b. The integral appearing in this equation is a kind of non-linear Young integral [75]. Existence and uniqueness of solutions for equations involving Young integrals are by now standard [61, 41, 34] and easily extended to this context as shown below. In particular the equation

$$\theta_t = \theta_0 + \int_0^t G_{\mathrm{d}s}(\theta_s)$$

will have a solution $\theta \in C^\gamma([0,T], \mathbb{R}^d)$ (the space of γ-Hölder continuous functions on $\mathbb{R}^d$) provided $(x,t) \mapsto G_t(x)$ is a γ-Hölder function of time, locally Lipshitz in space with $\gamma > 1/2$, that is

$$|G_{s,t}(x) - G_{s,t}(y)| \lesssim |x-y||t-s|^\gamma \psi(|x|+|y|)$$

for all $x, y \in \mathbb{R}^d$ and $0 \leqslant s \leqslant t \leqslant 1$. Note that some space regularity is already needed to have existence (to be compared with the classical setup where bounded vectorfields are sufficient to existence).

A strategy to prove uniqueness is to consider the difference between a solutions x and a solution x' of a similar equations

$$x'_t = x_0 + \int_0^t G'_{\mathrm{d}u}(x'_u).$$

It is the necessary to estimate the difference

$$(x_t - x'_t) - (x_s - x'_s) = \int_s^t G_{\mathrm{d}u}(x_u) - G'_{\mathrm{d}u}(x'_u)$$

To deal with such an estimates, we will need an averaged translation operator $\tau_f G_{s,t}(z) = \int_s^t G_{\mathrm{d}u}(f_u + z)$ in order to have an equation on $x - x'$.

In order for these estimates to be useful we need a way to link the regularity of the original vectorfield b with its averaged version $T^w_{s,t} b$ along an arbitrary continuous path w.

Theorem 2.3.1. *Assume that for $\alpha \in \mathbb{R}$, $f \to T^w f$ is defined on the whole space $\mathcal{F}L^{\alpha+\nu}$ for all $\nu \geqslant 0$. Assume also that there exists $\gamma > 1/2$ such that for all $\nu > 0$, there exists a ν-weight ψ such that T^w maps $\mathcal{C}^{\alpha+\nu}$ into $\mathcal{C}^{\gamma,\nu,\psi}$. Then there exists a solution $\theta(x_0) \in \mathcal{C}^\gamma([0,T], \mathbb{R}^d)$ to the Young-type equation*

$$\theta_t(x_0) = x_0 + \int_0^t (T^w_{\mathrm{d}s} b)(\theta_s(x_0))$$

for any $b \in \mathcal{C}^{\alpha+\nu}$ for ν such that $\gamma(1+\nu) > 1$. If $b \in \mathcal{C}^{\alpha+2}$ (or $\alpha + 3/2 > 0$ and $b \in \mathcal{C}^{\alpha+3/2}$) this is the unique γ-Hölder solution to this equation, and for all $t \in [0,T]$, the flow map $x_0 \to \theta_t(x_0)$ is well defined and locally Lipschitz continuous, uniformly in time.

Remark 2.3.2. To prove such a theorem, we need the two hypothesis about T^w. The first one is that this application is well defined. This will follow either from the definition of the application (when $\alpha \geq 0$) or from section 2.4. The second one is to prove that T^w maps $\mathcal{C}^\alpha$ into $\mathcal{C}^{\gamma,\nu,\psi}$. We also need a theory of integration for vectorfields in $\mathcal{C}^{\gamma,\nu,\psi}$. In the next section we will build such a theory.

When $\alpha \geqslant -1$ the vectorfield $b \in \mathcal{C}^{\alpha+1}$ is continuous and the solutions are simply solutions to the classical ODE

$$\theta_t = \theta_0 + \int_0^t b(u, w_u + \theta_u)\mathrm{d}u$$

In the case that $\alpha < -1$ the vectorfield b is a distribution and the previous ODE does not make sense. In that situation the natural meaning of these solution is the following. Let $b_n = \pi_{\leqslant n} b$ for $b \in \mathcal{C}^\alpha$ then

$$\int_0^t b_n(u, w_u + \theta_u)\mathrm{d}u = \int_0^t (T^w_{\mathrm{d}s} b_n)(\theta_s) \to \int_0^t (T^w_{\mathrm{d}s} b)(\theta_s)$$

by continuity of the Young integral and of the averaging with respect to the norm of $\mathcal{F}L^{\alpha+1}$. Then θ solves the equation

$$\theta_t = \theta_0 + \lim_n \int_0^t b_n(u, w_u + \theta_u)\mathrm{d}u$$

where the r.h.s. is well defined for any $\theta \in \mathcal{C}^\gamma([0,T], \mathbb{R}^d)$. At this point we can identify

$$\int_0^t b(u, w_u + \theta_u)\mathrm{d}u = \lim_n \int_0^t b_n(u, w_u + \theta_u)\mathrm{d}u$$

and give meaning to the ODE with a distributional drift b.

Remark 2.3.3. When the vector field b is in $\mathcal{F}L^\alpha$, the limiting procedure does not depend of the choice of the sequence. That is the principal reason of the introduction of that spaces.

One of the aims of this paper is to show that the above program can be carried on successfully in the case of w given by a sample path of a fractional Brownian motion B^H of Hurst parameter $H \in (0,1)$.

2.3.1 Definition of the Young integral

We define now the Young integral [75, 61, 41] for non linear operators.

Theorem 2.3.4. *Let $\gamma, \rho, \nu > 0$ with $\gamma + \nu\rho > 1$, a ν-weight ψ, and V and W two Banach spaces and I a finite interval on $\mathbb{R}$. Let $G \in \mathcal{C}^{\gamma,\nu,\psi}(I, V, W)$ and $f \in \mathcal{C}^\rho(I, V)$. Let $s, t \in I$ with $s \leqslant t$. Then the following limit exists and is independent of the partition*

$$\int_s^t G_{\mathrm{d}u}(f_u) := \lim_{\substack{\Pi \text{partition of} [s,t] \\ |\Pi| \to 0}} \sum_i G_{t_i, t_{i+1}}(f_{t_i})$$

Furthermore

1. *For all $s \leqslant u \leqslant t$ with $s, u, t \in I$ we have*

$$\int_s^t G_{\mathrm{d}r}(f_r) = \int_s^u G_{\mathrm{d}r}(f_r) + \int_u^t G_{\mathrm{d}r}(f_r).$$

2.
$$\left| \int_s^t G_{\mathrm{d}r}(f_r) - G_{s,t}(f_s) \right|_W \leqslant C_{\gamma,\rho,\nu} [\![G]\!]_{\gamma,\nu,\psi} [\![f]\!]_{\rho,I}^{\nu} |t-s|^{\gamma+\nu\rho} \psi(\|f\|_{\infty,I}).$$

3. *For all $s \leqslant t \in I$ and $R > 0$, the map $(f, G) \mapsto \int_s^t G_{\mathrm{d}r}(f_r)$ is continuous as a function of $\left(\left\{g \in \mathcal{C}^{\rho}(I,V), \|g\|_{\gamma,[s,t]} \leqslant R\right\}, \|.\|_{\infty,[s,t]}\right) \times (C^{\gamma,\nu,\psi}(I,V,W), \|.\|_{\gamma,\nu,\psi})$ onto W.*

Proof. Let $s, t \in I$ with $s \leqslant t$ be fixed until the end of the proof. Suppose first that G is differentiable (in time) and $G' \in C^{\gamma,\nu,\psi}(I,V,W)$ and $G \in \mathcal{C}^{\gamma,\nu,\psi}$. For simplicity, in all the proof we write $\|G\|$ and $[\![G]\!]$ instead of $\|G\|_{\gamma,\nu,\psi}$ and $[\![G]\!]_{\gamma,\nu,\psi}$. Then we define for $s \leqslant t$

$$\int_s^t G_{\mathrm{d}u}(f_u) := \int_s^t G'_u(f_u)\mathrm{d}u := I_{s,t}(f,G)$$

and also define $J_{s,t}(f,G) := I_{s,t}(f,G) - G_{s,t}(f_s)$. For $u \in [s,t]$ we have

$$J_{s,t}(f,G) = J_{s,u}(f,G) + J_{u,t}(f,G) + G_{u,t}(f_u) - \delta G_{u,t}(f_s)$$

hence, for $n \geq 1$, $i \in \{0, \ldots, 2^n\}$ and $t_i^n = s + (t-s)i2^{-n}$,

$$J_{s,t}(f,G) = \sum_{i=0}^{2^n-1} J_{t_i^n,t_{i+1}^n}(f,G) + \sum_{k=1}^{n} \sum_{i=1}^{2^k-1} (G_{t_{2i-1}^k,t_{2i}^k}(f_{t_{2i-1}^k}) - G_{t_{2i-1}^k,t_{2i}^k}(f_{t_{2(i-1)}^k}))$$

But, as G is smooth, the following computation holds

$$\begin{aligned}
|J_{t_i^n,t_{i+1}^n}(f,G)|_W &\leqslant \int_{t_i^n}^{t_{i+1}^n} |G'_u(f_u) - G'_u(f_{t_i^n})|_W \mathrm{d}u \\
&\leqslant \int_{t_i^n}^{t_{i+1}^n} [\![G'_u]\!]_{\nu,\psi} |f_u - f_{t_i^n}|_V^{\nu} \psi(|f_u| + |f_{t_i^n}|)\mathrm{d}u \\
&\leqslant 2\|[\![G']\!]_{\nu,\psi}\|_{\infty} \int_{t_i^n}^{t_{i+1}^n} |f|_{\rho}^{\nu} |u - t_i^n|^{\nu\rho} \psi(\|f\|_{\infty,[s,t]})\mathrm{d}u \\
&\lesssim 2^{-(1+\nu\rho)n}
\end{aligned}$$

hence
$$\left| \sum_{i=0}^{2^n-1} J_{t_i^n,t_{i+1}^n}(f,G) \right| \lesssim 2^{-n\nu\rho} \underset{n\to\infty}{\to} 0$$

and then
$$|J_{s,t}(f,G)| \leqslant \sum_{k=1}^{\infty} \sum_{i=1}^{2^k-1} |G_{t_{2i-1}^k,t_{2i}^k}(f_{t_{2i-1}^k}) - G_{t_{2i-1}^k,t_{2i}^k}(f_{t_{2(i-1)}^k})|_W.$$

For $k \geq 1$ and $i \in \{1, \dots, 2^k - 1\}$, we have

$$\begin{aligned} |G_{t^k_{2i-1}, t^k_{2i}}(f_{t^k_{2i-1}}) - G_{t^k_{2i-1}, t^k_{2i}}(f_{t^k_{2(i-1)}})| &\leqslant [\![G]\!] |t^k_{2i-1} - t^k_{2i}|^\gamma |f_{t^k_{2i-1}} - f_{t^k_{2i-2}}|^\nu \psi(|f_{t^k_{2i-1}}| + |f_{t^k_{2i-2}}|) \\ &\lesssim [\![G]\!] [\![f]\!]^\nu_{\rho,[s,t]} \psi(\|f\|_{\infty,[s,t]}) |t-s|^{\gamma+\nu\rho} 2^{-(\gamma+\nu\rho)k} \end{aligned}$$

Hence, the following bound holds

$$\begin{aligned} &\sum_{k=1}^{\infty} \sum_{i=1}^{2^k-1} |G_{t^k_{2i-1}, t^k_{2i}}(f_{t^k_{2i-1}}) - G_{t^k_{2i-1}, t^k_{2i}}(f_{t^k_{2(i-1)}})|_W \\ &\qquad \lesssim [\![G]\!] [\![f]\!]^\nu_{\rho,[s,t]} \psi(\|f\|_{\infty,[s,t]}) |t-s|^{\nu\rho+\gamma} \sum_{k=1}^{\infty} \sum_{i=1}^{2^k-1} 2^{-(\gamma+\nu\rho)k} \\ &\qquad \lesssim \frac{2^{-(\nu\rho+\gamma-1)}}{1 - 2^{-(\nu\rho+\gamma-1)}} [\![G]\!] [\![f]\!]^\nu_{\rho,[s,t]} \psi(\|f\|_{\infty,[s,t]}) |t-s|^{\nu\rho+\gamma}. \end{aligned}$$

The result is proved for smooth G. Let us now $G \in C^{\gamma,\nu,\psi}(I, V, W)$ and f as wanted. Let G^n smooth as above such that $G^n_{s,t}(f_s) \to G_{s,t}(f_s)$ as $n \to \infty$; for all $\gamma' < \gamma$ $\lim_{n\to\infty} \|G - G^n\|_{\gamma',\nu,\psi} = 0$ and for all $n \geq 0$, $\|G^n\|_{\gamma,\nu,\psi} \leqslant \|G\|_{\gamma,\nu,\psi}$. As $I_{s,t}$ is linear in the second variable, we have, for $\gamma' < \gamma$

$$\begin{aligned} |J_{s,t}(f, G^n) - J_{s,t}(f, G^{n+m})|_W &= |J_{s,t}(f, G^n - G^{n+m})|_W \\ &\lesssim [\![G^n - G^{n+m}]\!]_{\gamma',\nu} \\ &\underset{n\to\infty}{\to} 0 \end{aligned}$$

The sequence $(J_{s,t}(f, G^n))_n$ is Cauchy in W which is a Banach space. Let say it converges to $J_{s,t}(f, G)$. Furthermore, the sequence $G^{nn}_{s,t}(f_s)$ converges obviously to $G_{s,t}(f_s)$. Since the following holds, $I_{s,t}(f, G^n) = J_{s,t}(f, G^n) + G^n_{s,t}(f_s)$ the sequence $(I_{s,t}(f, G^n))_n$ converges to a limit called $I_{s,t}(f, G)$. Furthermore,

$$\begin{aligned} |J_{s,t}(f, G^n)|_W &\lesssim_{\gamma,\rho} [\![G_n]\!] [\![f]\!]_{\rho,[s,t]} \psi(\|f\|_\infty) |t-s|^{\gamma+\nu\rho} \\ &\lesssim [\![G]\!] [\![f]\!]_\rho \psi(\|f\|_\infty) |t-s|^{\gamma+\nu\rho} \end{aligned}$$

and so does $|I_{s,t}(f, G) - G_{s,t}(f_s)|_W$. The Chasles property and the triangular inequality are obvious with the definition of I. Moreover since $I(f, G)$ is linear in G it is easy to see that the definition does not depend on the particular sequence G^n.

Let us show that $I_{s,t}(f, G)$ is the limit of Riemann sum. Let $\Pi = \{s = t_0 < t_1 < \ldots < t_n = t\}$ a partition of $[s, t]$. Let

$$S_\Pi = \sum_{k=0}^{n-1} G_{t_i, t_{i+1}}(f_{t_{i+1}})$$

the Riemann sum corresponding to this partition. As $G_{t_i,t_{i+1}}(f_{t_{i+1}}) = I_{t_i,t_{i+1}}(f, G) - J_{t_i,t_{i+1}}(f, G)$ the following equality holds

$$S_\Pi - I_{s,t}(f, G) = -\sum_{i=0}^{n-1} J_{t_i,t_{i+1}}(f, G)$$

Hence

$$\begin{aligned}
|S_\Pi - I_{s,t}(f,G)|_W &\leqslant \sum_{i=0}^{n-1} |J_{t_i,t_{i+1}}(f,G)|_W \\
&\lesssim \sum_{i=0}^{n-1} |t_{i+1}-t_i|^{\gamma+\nu\rho} \\
&\lesssim |\Pi|^{\gamma+\nu\rho-1} \to_{|\Pi|\to 0} 0
\end{aligned}$$

It remains to show the continuity of the map $(f,G) \mapsto I(f,G)$. Take f, f', G, G' and assume for simplicity that $G(0) = G'(0) = 0$ then

$$I_{s,t}(f,G) - I_{s,t}(f',G') = [I_{s,t}(f,G) - I_{s,t}(f',G)] + I_{s,t}(f', G-G')$$

and

$$\begin{aligned}
|I_{s,t}(f', G-G')|_W &\leqslant |(G-G')_{s,t}(f_s) - (G-G')_{s,t}(0)|_W \\
&\quad + |(G-G')_{s,t}(0)| + |J_{s,t}(f, G-G')|_W \\
&\lesssim \|G-G'\| |t-s|^\gamma (|f_s|^\gamma \psi(|f_s|) + 1 + |t-s|^{\nu\rho} [\![f]\!]_\rho^\nu \psi(\|f\|_\infty)) \\
&\lesssim \|G-G'\| |t-s|^\gamma (\|f\|_\infty^\nu \psi(\|f\|_\infty) + 1 + |t-s|^{\nu\rho} [\![f]\!]_\rho^\nu \psi(\|f\|_\infty)). \\
&\lesssim \|G-G'\| |t-s|^\gamma (1 + \|f\|_\rho^\nu \psi(\|f\|_\rho)).
\end{aligned}$$

Furthermore

$$I_{s,t}(f,G) - I_{s,t}(f',G) = G_{s,t}(f_s) - G_{s,t}(f'_s) + J_{s,t}(f,G) - J_{s,t}(f',G).$$

We have also

$$\begin{aligned}
|I_{s,t}(f,G) - I_{s,t}(f',G)|_W &\lesssim \|G\| \|f-f'\|_\infty^\nu \psi(\|f\|_\infty + \|f'\|_\infty) |t-s|^\gamma \\
&\quad + ([\![f]\!]_\rho^\nu \psi(\|f\|_\infty) + [\![f']\!]_\rho^\nu \psi(\|f'\|_\infty)) \|G\| |t-s|^{\nu\rho+\gamma}. \\
&\lesssim \|G\| \|f-f'\|_\infty^\nu \psi(\|f\|_\rho + \|f'\|_\rho) |t-s|^\gamma \\
&\quad + (\|f\|_\rho^\nu \psi(\|f\|_\rho) + \|f'\|_\rho^\nu \psi(\|f'\|_\rho)) \|G\| |t-s|^{\nu\rho+\gamma}.
\end{aligned}$$

By partitioning the interval $[s,t]$ in subintervals $[t_i, t_{i+1}]$ of size 2^{-n} and summing up the contributions according to these bounds we obtain an improved estimate

$$\begin{aligned}
|I_{s,t}(f,G) - I_{s,t}(f',G)|_W &\lesssim \|G\| \|f-f'\|_\infty^\nu \psi(\|f\|_\rho + \|f'\|_\rho) 2^{(1-\gamma)n} |t-s|^\gamma \\
&\quad + \|G\| (\|f\|_\rho^\nu \psi(\|f\|_\rho) + \|f'\|_\rho^\nu \psi(\|f'\|_\rho)) (2^{-n}|t-s|)^{\nu\rho+\gamma} 2^n
\end{aligned}$$

Taking n large enough so that

$$2^{-\nu\rho n} \leqslant \frac{\|f-f'\|_\infty^\nu \psi(\|f\|_\rho + \|f'\|_\rho)}{(\|f\|_\rho^\nu \psi(\|f\|_\rho) + \|f'\|_\rho^\nu \psi(\|f'\|_\rho)) |t-s|^{\nu\rho}} \leqslant 2^{-\nu\rho(n-1)}$$

we have

$$|I_{s,t}(f,G) - I_{s,t}(f',G)|_W \lesssim \|G\| \|f-f'\|_\infty^\nu \psi(\|f\|_\rho + \|f'\|_\rho) 2^{(1-\gamma)n} |t-s|^\gamma$$

which means that it is possible to choose n such that

$$\begin{aligned}
&|I_{s,t}(f,G) - I_{s,t}(f',G)|_W \\
&\lesssim \|G\| \|f-f'\|_\infty^\nu \psi(\|f\|_\rho + \|f'\|_\rho)|t-s|^\gamma \left(\frac{(\|f\|_\rho^\nu \psi(\|f\|_\rho) + \|f'\|_\rho^\nu \psi(\|f'\|_\rho))|t-s|^{\nu\rho}}{\|f-f'\|_\infty^\nu \psi(\|f\|_\rho + \|f'\|_\rho)} \right)^{\frac{1-\gamma}{\nu\rho}} \\
&\lesssim \|G\| \{\|f-f'\|_\infty^\nu \psi(\|f\|_\rho + \|f'\|_\rho)\}^{(\gamma+\nu\rho-1)/\nu\rho} |t-s| (\|f\|_\rho^\nu \psi(\|f\|_\rho) + \|f'\|_\rho^\nu \psi(\|f'\|_\rho))^{\frac{1-\gamma}{\nu\rho}}
\end{aligned}$$

and this allows us to infer the continuity of $I(f,G)$. □

Remark 2.3.5. It is easy to construct a suitable sequence $(G^n)_{n\geq 1}$. Let $h : \mathbb{R} \to \mathbb{R}$ be a compactly supported, smooth positive function with integral 1. Define $h_n(t) = nh(nt)$ and define for all $v \in V$ and all $t \in \mathbb{R}$

$$G_t^n(v) = \int_{\mathbb{R}} h_n(t-s) G_s(v) \mathrm{d}s = \int_{\mathbb{R}} h_n(s) G_{t-s}(v) \mathrm{d}s$$

Then G^n is as wanted. Indeed,

$$\begin{aligned}
&|G_{s,t}^n(v) - G_{s,t}^n(w)|_W \\
&\quad \leqslant \int_{\mathbb{R}} h_n(r) |(G_{t-r} - G_{s-r})(v) - (G_{t-r} - G_{s-r})(w)|_W \mathrm{d}r \\
&\quad \leqslant \int_{\mathbb{R}} h_n(r) |t-s|^\gamma |v-w|_V^\nu \psi(|v|+|w|) \mathrm{d}r \\
&\quad \leqslant \|G\|_{\gamma,\nu,\psi} |t-s|^\gamma |v-w|_V^\nu \psi(|v|+|w|)
\end{aligned}$$

which proves that $G^n \in C^{\gamma,\nu,\psi}(\mathbb{R},V,W))$ and that $\|G^n\|_{\gamma,\nu,\psi} \leqslant \|G\|_{\gamma,\nu,\psi}$. Furthermore G^n is differentiable and $(G^n)' \in C^{\gamma,\nu,\psi}(\mathbb{R},V,W)$. As we can chose h_n to be a good kernel, all the properties required on G^n are satisfied.

Definition 2.3.6. The limit functional I defined in the last theorem is obviously an integral and then we will refer to it as $\int_s^t G_{du}(f_u)$.

Remark 2.3.7. Let $g \in \mathcal{C}^\gamma(I,V')$ and $f \in \mathcal{C}^\rho(I,V)$ with $\gamma + \rho > 1$, where V and V' are (finite–dimensional) Banach spaces. Let $W = V \otimes V'$ and for all $x \in V$, $G_t(x) = x \otimes g_t$. Then $G \in \mathcal{C}^{\gamma,1}(I,V,W)$ and the above integral is the standard Young integral.

Remark 2.3.8. The bound in Theorem 2.3.4 is

$$\left| \int_s^t G_{\mathrm{d}u}(f_u) - G_{s,t}(f_s) \right| \lesssim [\![G]\!]_{\gamma,\nu,\psi} |t-s|^{\gamma+\rho\nu} [\![f]\!]_\gamma^\nu \psi(\|f\|_\infty).$$

But as $[\![f]\!]_\gamma \leqslant \|f\|_\gamma$ and $\|f\|_\infty \leqslant (1+|I|)\|f\|_\gamma$ and ψ is a weight, we also have this other useful bound

$$\left| \int_s^t G_{\mathrm{d}u}(f_u) - G_{s,t}(f_s) \right| \lesssim [\![G]\!]_{\gamma,\nu,\psi} |t-s|^{\gamma+\rho\nu} \|f\|_\gamma^\nu \psi(\|f\|_\gamma)$$

where the new constant depends on the length of the interval $|I|$ and ψ. In the following, we will exploit these three bounds indifferently and without further notice.

We intend to solve differential equations driven by such G. Thanks to the definition of the integral and the bound in Theorem 2.3.4, we are able to define the equation, prove the existence of solutions and give an a priori bound on the norm of the solutions. Here we will use the notion of ν-weight, in order to control the growth of the norm.

Theorem 2.3.9. *Let $\gamma > \frac{1}{2}$, $\nu \in [0,1)$ such that $\gamma(1+\nu) > 1$ and ψ a ν-weight as in Definition 2.2.1. Let $G \in \mathcal{C}^{\gamma,\nu,\psi}([0,T],\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. There exists a solution to the non-linear Young differential equation*

$$\theta_t = \theta_0 + \int_0^t G_{\mathrm{d}u}(\theta_u).$$

Furthermore, there exists two universal constants K_1 and K_2 depending on γ, ν, ψ and T such that

$$\|\theta\|_{\infty,[0,T]} \leqslant K_1(1+\|G\|_{\gamma,\nu,\psi})^{K_2\|G\|^{1/\nu\gamma}_{\gamma,\nu,\psi}}(|\theta_0|+1)$$

and

$$\|\theta\|_{\gamma,[0,T]} \leqslant \tilde{K}_1\|G\|^{1/\nu\gamma}(1+\|G\|_{\gamma,\nu,\psi})^{\tilde{K}_2\|G\|^{1/\nu\gamma}_{\gamma,\nu,\psi}}(1+|\theta_0|).$$

Proof. Let us first deal with the existence of the solutions. Let $t_0 \in I = [0,T]$, $K > 0$ and $0 < S \leqslant T$ to be specify later. Let $J = [t_0, (t_0+S)\wedge T]$ and let us define for all $x \in V$,

$$\mathcal{C}_{t_0,x} = \{\theta \in \mathcal{C}^\gamma(J) : \theta_{t_0} = x, \|\theta\|_{\gamma,J} \leqslant K\}$$

and

$$\Phi_{t_0,x} : \begin{array}{ccc} \mathcal{C}^\gamma(J,V) & \to & C^\gamma(J,V) \\ \theta & \to & x + \int_{t_0}^{\cdot} G_{\mathrm{d}u}(\theta_u) \end{array}$$

By Theorem 2.3.4 the map $\Phi_{t_0,x}$ is well defined. Furthermore we always have

$$|\theta_s| \leqslant |\theta_{t_0}| + T^\nu [\![\theta]\!]_{\gamma,J} \lesssim_T \|\theta\|_{\gamma,J}$$

Hence for $s < t \in J$ we have

$$\begin{aligned} |\delta(\Phi_{t_0,x}(\theta))_{s,t}| &\leqslant \left|\int_s^t G_{\mathrm{d}u}(\theta_u) - G_{s,t}(\theta_s)\right| + |G_{s,t}(\theta_s) - G_{s,t}(0)| + |G_{s,t}(0)| \\ &\lesssim \|G\|_{\gamma,\nu,\psi}|t-s|^\gamma(S^{\nu\gamma}\|\theta\|^\nu_{\gamma,J}\psi(\|\theta\|_{\gamma,J}) + |\theta_s|^\nu\psi(|\theta_s|)+1) \\ &\lesssim \|G\|_{\gamma,\nu,\psi}|t-s|^\gamma(S^{\nu\gamma}\|\theta\|^\nu_{\gamma,J}\psi(\|\theta\|_{\gamma,J}) + |\theta_{t_0}|^\nu\psi(|\theta_0|)+1) \end{aligned} \tag{2.7}$$

Now take $\theta \in \mathcal{C}_{t_0,x}$,

$$[\![\Phi_{t_0,x}(\theta)]\!]_{\gamma,J} \lesssim \|G\|_\gamma(S^{\nu\gamma}\|\theta\|^\nu_{\gamma,J}\psi(\|\theta\|_{\gamma,J}) + |x|^\nu\psi(|x|)+1)$$

But since $\nu < 1$ and ψ i a ν-weight,, there exists a constant $C_{\nu,\psi}$ such that $\|\theta\|^\nu_{\gamma,J}\psi(\|\theta\|_{\gamma,J}) \leqslant C_{\nu,\psi}(1+|x|)$. Hence, there is a universal constant $C > 0$ such that

$$\|\Phi_{t_0,x}(\theta)\|_{\gamma,J} = [\![\Phi_{t_0,x}(\theta)]\!]_{\gamma,J} + |x| \leqslant |x| + C\|G\|_\gamma(S^{\nu\gamma}\|\theta\|_{\gamma,J} + |x|^\nu\psi(|x|)+1).$$

For S such that $C\|G\|_\gamma S^{\nu\gamma} < 1/2$, and for $K \geqslant 2\{|x| + C(|x|^\nu\psi(|x|)+1)\}$, we have

$$\|\Phi_{t_0,x}(\theta)\|_{\gamma,J} \leqslant K.$$

Then $\Phi_{t_0,x}(\theta) \in \mathcal{C}_{t_0,x}$, moreover by the property of the Young integral the map $\Phi_{t_0,x}$ is continuous on $\mathcal{C}_{t_0,x}$ for the norm $\|.\|_{\infty,[t_0,(t_0+T)\wedge 1]}$. By its definition $\mathcal{C}_{t_0,x}$ is immediately a closed convex set of $C^0(J)$. Let us show that $\Phi_{t_0,x}(\mathcal{C}_{t_0,x})$ is relatively compact in $\mathcal{C}^0$. It is obviously equicontinuous as $\|\Phi_{t_0,x}(\theta)\|_\gamma \leqslant K$ and relatively bounded as $|\Phi_{t_0,x}(\theta)_t| \leqslant |x| + K^\nu \psi(K)(t-t_0)^\gamma$. Hence by Ascoli theorem $\Phi_{t_0,x}(\mathcal{C}_{t_0,x})$ is relatively compact. Thanks to Leray-Schauder-Tychonoff fixed point theorem, there exists $\theta^{t_0,x}$ such that $\theta^{t_0,x} = \Phi_{t_0,x}(\theta^{t_0,x}) = x + \int_{t_0}^t G_{\mathrm{d}u}(\theta_u^{t_0,x})$. We then construct by induction a solution on the whole interval. For n such that $nS \leqslant T$ let $\theta^0 = \theta^{0,x_0}$ and $\theta^n = \theta^{nS,\theta_S^{n-1}}$. Let us define $\theta_t = \theta_t^n$ if $t \in [nS,(n+1)S]$. By an immediate induction, θ is solution of the equation $\theta_t = x_0 + \int_0^t G_{\mathrm{d}u}(\theta_u)$ and then is obviously in $\mathcal{C}^\gamma$.

We have all the tools to bound the norm of a solution of the equation. Again take t_0 and S to be specify lated, and θ a solution of the non-linear Young differential equation. And take $J = [t_0,(t_0+S)\wedge T]$. We have

$$\begin{aligned}|\delta\theta_{s,t}| &\leqslant \left|\int_s^t G_{\mathrm{d}u}(\theta_u) - G_{s,t}(\theta_s)\right| + |G_{s,t}(\theta_s) - G_{s,t}(\theta_{t_0})| + |G_{s,t}(\theta_{t_0}) - G_{s,t}(0)| + |G_{s,t}(0)| \\ &\lesssim |t-s|^\gamma \|G\|_{\gamma,\nu,\psi}(S^{\gamma\nu}[\![\theta]\!]_{\gamma,J}^\nu \psi(\|\theta\|_{\gamma,J}) + |\theta_{t_0}|^\nu \psi(|\theta_{t_0}|) + 1)\end{aligned}$$

hence

$$[\![\theta]\!]_{\gamma,[t_0,t_0+S]} \lesssim \|G\|_{\gamma,\nu,\psi}(S^{\gamma\nu}\|\theta\|_{\gamma,J}^\nu \psi(\|\theta\|_{\gamma,J}) + |\theta_{t_0}|^\nu \psi(|\theta_{t_0}|) + 1).$$

Let S such that $C\|G\|_{\gamma,\nu,\psi}S^{\gamma,\nu} \leqslant 1$, and we have

$$\|\theta\|_{\gamma,[t_0,t_0+S]} \lesssim \|\theta\|_{\gamma,J}^\nu \psi(\|\theta\|_{\gamma,J}) + |\theta_{t_0}| + C\|G\|_{\gamma,\nu,\psi}|\theta_{t_0}|^\nu \psi(|\theta_{t_0}|) + C\|G\|_{\gamma,\nu}$$

As $x \to x^\nu \psi(x)$ is sub linear (as before), there exists a constant depending on ν and ψ such that

$$\|\theta\|_{\gamma,[t_0,t_0+S]} \lesssim_{\nu,\psi} 1 + |\theta_{t_0}| + \|G\|_{\gamma,\nu,\psi}|\theta_{t_0}|^\nu \psi(|\theta_{t_0}|) + \|G\|_{\gamma,\nu,\psi}.$$

There also exists a constant C such that

$$|\theta_{t_0}|^\nu \psi(|\theta_{t_0}|) \leqslant C + |\theta_{t_0}|$$

and

$$\|\theta\|_{\gamma,[t_0,t_0+S]} \leqslant (1 + \|G\|_{\gamma,\nu,\psi})|\theta_{t_0}| + C(\|G\|_{\gamma,\nu,\psi} + 1). \tag{2.8}$$

From this we deduce

$$|\theta_t| \leqslant |\theta_t - \theta_{t_0}| + |\theta_{t_0}| \lesssim_T \|\theta\|_{\gamma,J} + |\theta_{t_0}| \lesssim (1 + \|G\|_{\gamma,\nu,\psi})|\theta_{t_0}| + \|G\|_{\gamma,\nu,\psi} + 1$$

and then

$$\|\theta\|_{\infty,J} \lesssim (1 + \|G\|_{\gamma,\nu,\psi})|\theta_{t_0}| + \|G\|_{\gamma,\nu,\psi} + 1.$$

Now let n such that $S = T/n$ and $1/2 \leqslant C\|G\|T^{\nu\gamma}n^{-\nu\gamma} \leqslant 1$ hence $n \geqslant T(C\|G\|)^{1/\nu\gamma}$ and we have for $J_i = [iT/n,(i+1)T/n]$

$$\|\theta\|_{\infty,J_i} \lesssim (1 + \|G\|_{\gamma,\nu,\psi})\|\theta\|_{\infty,J_{i-1}} + \|G\|_{\gamma,\nu,\psi} + 1$$

and

$$\|\theta\|_{\infty,J_0} \lesssim (1 + \|G\|_{\gamma,\nu,\psi})|\theta_0| + \|G\|_{\gamma,\nu,\psi} + 1.$$

Hence

$$\|\theta\|_{\infty,J_i} \lesssim C^i(1+\|G\|_{\gamma,\nu,\psi})^i((1+\|G\|_{\gamma,\nu,\psi})|\theta_0| + \|G\|_{\gamma,\nu,\psi} + 1)$$

and finally

$$\|\theta\|_{\infty,[0,T]} \leqslant K_1(1+\|G\|_{\gamma,\nu,\psi})^{K\|G\|^{1/\nu\gamma}_{\gamma,\nu,\psi}}(|\theta_{t_0}| + 1)$$

where K_1 and K_2 are two universal constants depending on ν, γ, ψ and T. From the equation (2.8), we can deduce, with the same induction argument, that

$$\|\theta\|_{\gamma,[0,T]} \leqslant |\theta_0| + C\|G\|^{1/\nu\gamma}T((1+\|G\|_{\gamma,\nu,\psi})\|\theta\|_{\infty,[0,T]} + \|G\|_{\gamma,\nu,\psi} + 1)$$

and the result follows. □

Remark 2.3.10. The bounds on the solutions of the differential equation allows us to get ride of the ν-weight ψ. Indeed, we have, for a solution θ of the non-linear Young equation we have

$$\|\theta\|_{\infty,[0,T]} \leqslant K_1(1+\|G\|_{\gamma,\nu,\psi})^{K_2\|G\|^{1/\nu\gamma}_{\gamma,\nu,\psi}}(|\theta_{t_0}| + 1).$$

Then for $R > 0$, and $\theta_0 \in B(0,R)$ $\|\theta\|_{\infty,[0,T]} \leqslant K_1(1+\|G\|_{\gamma,\nu,\psi})^{K\|G\|^{1/\nu\gamma}_{\gamma,\nu,\psi}}(R+1)$. Hence, it is enough to consider the localised norm of G

$$\|G\|^R_{\gamma,\nu} = \sup_{t\neq s\in I}\ \sup_{x\neq y\in B(0,R^{G,R})} \frac{|G_{s,t}(x) - G_{s,t}(y)|}{|x-y|^\nu|t-s|^\gamma} + \sup_{s\neq t}\frac{|G_{s,t}(0)|}{|t-s|^\gamma}$$

where

$$R^{G,R} = K_1(1+\|G\|_{\gamma,\nu,\psi})^{K\|G\|^{1/\nu\gamma}_{\gamma,\nu,\psi}}(R+1).$$

From now we will ourself on bounded G, namely $G \in \mathcal{C}^{\gamma,\nu}$, and we will extend the results to $\mathcal{C}^{\gamma,\nu,\psi}$ thanks to the previous remark.

2.3.2 Uniqueness of solutions

Comparison Principle

From now, thanks to remark 2.3.10 we can restrict the study of the properties of the solutions, their uniqueness and their regularity w.r.t. the parameters for bounded G. Hence, we define the following space

$$\mathcal{C}^{\gamma,n+\nu}_b = \left\{ G \in \mathcal{C}^{\gamma,\nu} : \|G\|^b_{\gamma,n+\nu} := [\![D^nG]\!]_{\gamma,\nu} + \sum_{k=0}^n \sup_{s\neq t\in[0,T]}\ \sup_{x\in\mathbb{R}^d} |D^kG_{s,t}(x)|/|t-s|^\gamma < +\infty \right\}.$$

As there will be no ambiguity in the following, we will usually avoid to mention explicitly the b in the norm on that space. Those spaces are nicer than the whole space $\mathcal{C}^{\gamma,\nu}$ as there are natural embeddings:

Lemma 2.3.11. *Let $0 < \gamma' < \gamma \leqslant 1$, $0 \leqslant \nu' \leqslant \nu$ and $G \in \mathcal{C}^{\gamma,\nu}_b$.*

$$\|G\|^b_{\gamma,\nu'} \lesssim \|G\|^b_{\gamma,\nu}.$$

Proof. Let $x,y \in \mathbb{R}^d$ $s,t \in [0,T]$. For $0 \leqslant \nu \leqslant \nu' \leqslant 1$, we have

$$\begin{array}{rcl} \dfrac{|G_{s,t}(x)-G_{s,t}(y)|}{|x-y|^{\nu'}} & \leqslant & \left(\dfrac{|G_{s,t}(x)-G_{s,t}(y)|}{|x-y|^{\nu}}\right)^{\nu'/\nu} |G_{s,t}(x)-G_{s,t}(y)|^{1-\nu'/\nu} \\ & \leqslant & 2^{1-\nu'}|t-s|^{\gamma}[\![G]\!]_{\gamma,\nu}^{\nu'/\nu}(\|G\|_{\gamma,\nu}^{b})^{1-\nu'/\nu} \\ & \lesssim & 2^{1-\nu'/\nu}|t-s|^{\gamma}\|G\|_{\gamma,\nu}^{b} \end{array}$$

and the following bound holds

$$\|G\|_{\gamma,\nu'}^{b} = (1+2^{1-\nu'/\nu})\|G\|_{\gamma,\nu}^{b}.$$

Furthermore, we also have

$$\begin{array}{rcl} |G_{s,t}(x)-G_{s,t}(y)| & \leqslant & \displaystyle\int_0^1 \mathrm{d}r|DG_{s,t}(r(x-y)+y)||x-y| \\ & \leqslant & \|DG\|_{\gamma,1+\nu}^{b}|x-y||t-s|^{\gamma} \end{array}$$

and

$$\|G\|_{\gamma,1}^{b} \leqslant 2\|DG\|_{\gamma,1+\nu}^{b}.$$

The general result follows by an easy induction. □

Remark 2.3.12. These embeddings allows us to state a result for the existence of the solutions when $G \in \mathcal{C}_b^{\gamma,1}$ with $\gamma > \frac{1}{2}$. Indeed, as for all $\nu < 1$, $G \in \mathcal{C}_b^{\gamma,\nu}$ and $\|G\|_{\gamma,\nu}^{b} \lesssim \|G\|_{\gamma,1}^{b}$, there exists a solution θ and the non-linear Young differential equation. Furthermore, for all $\nu < 1$, there exists a constant K_2 such that

$$\|\theta\|_{\infty} \lesssim (1+\|G\|_{\gamma,\nu}^{b})^{K_2(\|G\|_{\gamma,\nu}^{b})^{1/\nu\gamma}}(|\theta_0|+1) \lesssim (1+\|G\|_{\gamma,1}^{b})^{K_2(\|G\|_{\gamma,1}^{b})^{1/\nu\gamma}}(|\theta_0|+1).$$

In fact, a deeper look at the proof of Theorem 2.3.9, allows us to get rid of the ν, and state that there exists a constant K depending on T and γ such that

$$\|\theta\|_{\infty} \lesssim (1+\|G\|_{\gamma,1}^{b})^{K(\|G\|_{\gamma,1}^{b})^{1/\gamma}}(|\theta_0|+1),$$

and a similar bound holds for $\|\theta\|_{\gamma}$.

In order to study the properties of the solutions of the non-linear Young differential equation, we intend to compare two solutions θ^1 and θ^2. In the classical case (when G is differentiable in time), we would have

$$\begin{array}{rcl} \theta_t^1-\theta_t^2 & = & (\theta_0^1-\theta_0^2)+\displaystyle\int_0^t G'_u(\theta_u^1)-G'_u(\theta_u^2)\mathrm{d}u \\ & = & (\theta_0^1-\theta_0^2)+\displaystyle\int_0^t G'_u(\theta_u^1-\theta_u^2+\theta_u^2)-G'_u(\theta_u^2)\mathrm{d}u \\ & = & (\theta_0^1-\theta_0^2)+\displaystyle\int_0^t (\tau_{\theta^2}G)'_u(\theta_u^1-\theta_u^2)\mathrm{d}u-\int_0^t G'_u(\theta_u^2)\mathrm{d}u \end{array}$$

where $(\tau_{\theta^2}G)_t(x) = \int_0^t G'_u(\theta^2_u + x)\mathrm{d}u$. Hence $\theta^1 - \theta^2$ solve a differential equation, but with a translated and averaged function $(\tau_{\theta^2}G)$ and a second member. In order to prove some properties on the solutions, we then have to study this differential equation.

In the case of the Young differential equation, this strategy will be very profitable, we then have to define the averaged translation and to study some of its properties. Hence, we define the natural action of the additive group of $\mathcal{C}^\rho$ paths on the integrated vectorfields $C \in \mathcal{C}^{\gamma,\nu}_b$.

Definition 2.3.13. Let $\gamma, \nu, \rho \in [0,1]$ such that $\gamma + \rho\nu > 1$, $G \in \mathcal{C}^{\gamma,\nu}_b$ and $f \in \mathcal{C}^\rho$. We define the average translation of G by f, and we write $\tau_f G$ the following quantity

$$\tau_f G : (t,x) \to \int_0^t G_{\mathrm{d}u}(f_u + x).$$

Due to the requirements of Young integration, the estimations for the translated integrated vector-field $\tau_f G$ show a loss of regularity quantified by the next lemma.

Lemma 2.3.14. *For $\gamma + \nu\rho > 1$ and $\gamma + \eta\rho > 1$, $f \in \mathcal{C}^\rho$ and $G \in \mathcal{C}^{\gamma,\nu+\eta}_b$ we have $\tau_f G \in \mathcal{C}^{\gamma,\nu}$ and*

$$\|\tau_f G\|_{\gamma,\nu} \lesssim \|G\|_{\gamma,\nu+\eta}(1 + \|f\|^\eta_\rho)$$

Proof. Suppose first that $\eta + \nu \leqslant 1$ Let $x, y \in V$ and define $\tilde{G}(z) = G(x+z) - G(y+z)$. There is two bounds for the increments of $\tilde{G}$.

$$|\tilde{G}_{s,t}(z_1) - \tilde{G}_{s,t}(z_2)| \lesssim [\![G]\!]_{\gamma,\nu+\eta}|t-s|^\gamma|x-y|^{\nu+\eta}$$

and

$$|\tilde{G}_{s,t}(z_1) - \tilde{G}_{s,t}(z_2)| \lesssim [\![G]\!]_{\gamma,\nu+\eta}|t-s|^\gamma|z_1-z_2|^{\nu+\eta}.$$

Hence, by interpolating these to inequalities

$$|\tilde{G}_{s,t}(z_1) - \tilde{G}_{s,t}(z_2)| \lesssim [\![G]\!]_{\gamma,\nu+\eta}|t-s|^\gamma|x-y|^\nu|z_1-z_2|^\eta.$$

When $2 \geqslant \eta + \nu > 1$, we have

$$\begin{aligned} |\tilde{G}_{s,t}(z_1) - \tilde{G}_{s,t}(z_2)| &= \left|\int_0^1 \mathrm{d}r\{DG_{s,t}(r(x-y) + z_1 + y) - DG_{s,t}(r(x-y) + z_2 + y)\}.x - y\right| \\ &\lesssim [\![G]\!]_{\gamma,\nu+\eta}|t-s|^\gamma|x-y||z_1-z_2|^{\nu+\eta-1} \end{aligned}$$

and

$$|\tilde{G}_{s,t}(z_1) - \tilde{G}_{s,t}(z_2)| \lesssim [\![G]\!]_{\gamma,\nu+\eta}|t-s|^\gamma|x-y|^{\nu+\eta-1}|z_1-z_2|$$

and again

$$|\tilde{G}_{s,t}(z_1) - \tilde{G}_{s,t}(z_2)| \lesssim [\![G]\!]_{\gamma,\nu+\eta}|t-s|^\gamma|x-y|^\nu|z_1-z_2|^\eta.$$

In the two cases, we have

$$[\![\tilde{G}]\!]_{\gamma,\eta} \lesssim [\![G]\!]_{\gamma,\nu+\eta}|t-s|^\gamma|x-y|^\nu.$$

Hence

$$\begin{aligned} \tau_f G_{s,t}(x) - \tau_f G_{s,t}(y) &= \int_s^t G_{\mathrm{d}u}(f_u + x) - G_{\mathrm{d}u}(f_u + y) \\ &\leqslant \int_s^t \tilde{G}_{\mathrm{d}u}(f_u) - \tilde{G}_{s,t}(f_s) + \tilde{G}_{s,t}(f_s) \end{aligned}$$

Hence,

$$|\tau_f G_{s,t}(x) - \tau_f G_{s,t}(y)| \lesssim [\![\tilde{G}]\!]_{\gamma,\eta}(\|f\|_\gamma^\eta + 1) + |\tilde{G}_{s,t}(f_s)|$$

and as $|\tilde{G}_{s,t}(f_s)| \leqslant 2|t-s|^\gamma \|G\|_{\gamma,\nu+\eta}$,

$$[\![\tau_f G]\!]_{s,t} \leqslant \|G\|_{\gamma,\nu+\eta}(\|f\|_\gamma^\eta + 1).$$

Furthermore

$$|(\tau_f G)_{s,t}(0)| \leqslant \left|\int_s^t G_{\mathrm{d}u}(f_u) - G_{s,t}(f_s)\right| + |G_{s,t}(f_s)| \leqslant \|G\|_{\gamma,\eta}|t-s|^\gamma(\|f\|_\gamma^\eta + 1)$$

and by the embedding of Lemma 2.3.11, the result follows. □

The averaged translation is a suitable tool to control the difference of two non-linear Young integrals, as soon as we have enough regularity to estimate the integral. The following lemma state the estimation for generic functions.

Lemma 2.3.15. *Let $\gamma, \nu, \nu', \rho \in [0,1]$ such that $\gamma + \rho\nu > 1$ and $\gamma + \rho\nu' > 1$. Let $f^1, f^2 \in \mathcal{C}^\rho$, $G \in \mathcal{C}_b^{\gamma,\nu}$, and suppose that $\tau_{f^2} G \in \mathcal{C}_b^{\gamma,\nu'}$. Then*

$$\left\|\int_0^\cdot G_{\mathrm{d}u}(f_u^1) - G_{\mathrm{d}u}(f_u^2)\right\|_{\infty,[0,T]} \lesssim \|\tau_{f^2}G\|_{\gamma,\nu'} T^\gamma ([\![f^1 - f^2]\!]_\rho^{\nu'} + \|f^1 - f^2\|_\infty^{\nu'})$$

and

$$\left[\!\!\left[\int_0^\cdot G_{\mathrm{d}u}(f_u^1) - G_{\mathrm{d}u}(f_u^2)\right]\!\!\right]_{\gamma,[0,T]} \lesssim \|\tau_{f^2}G\|_{\gamma,\nu'}([\![f^1 - f^2]\!]_\gamma^{\nu'} + \|f^1 - f^2\|_\infty^{\nu'}).$$

Furthermore when $1 \geqslant \eta > 0$ such that $\rho\eta + \gamma > 1$ and $G \in \mathcal{C}^{\gamma,\nu'+\eta}$

$$\left\|\int_0^\cdot G_{\mathrm{d}u}(f_u^1) - G_{\mathrm{d}u}(f_u^2)\right\|_{\infty,[0,T]} \lesssim T^\gamma \|G\|_{\gamma,\nu'+\eta}(1 + \|f^2\|_\rho^\eta)([\![f^1 - f^2]\!]_\rho^{\nu'} + \|f^1 - f^2\|_\infty^{\nu'}).$$

Proof. It is an direct application of the definition of the averaged translation. Let $s,t \in [0,T]$, by definition we have $\int_s^t G_{\mathrm{d}u}(f_u^2) = \tau_{f^2}G_{s,t}(0)$. Hence

$$\begin{aligned}
\left|\int_s^t G_{\mathrm{d}u}(f_u^1) - \int_s^t G_{\mathrm{d}u}(f_u^2)\right| &\leqslant \left|\int_s^t \tau_{f^2}G_{\mathrm{d}u}(f_u^1 - f_u^2) - \tau_{f^2}G_{s,t}(f_s^1 - f_s^2)\right| \\
&\quad + |\tau_{f^2}G_{s,t}(f_s^1 - f_s^2) - \tau_{f^2}G_{s,t}(0)| \\
&\lesssim [\![\tau_{f^2}G]\!]_{\gamma,\nu'}|t-s|^\gamma([\![f^1 - f^2]\!]_\rho |t-s|^{\nu'\gamma} + |f_s^1 - f_s^2|^{\nu'})
\end{aligned}$$

Hence,

$$\left\|\int_0^\cdot G_{\mathrm{d}u}(f_u^1) - G_{\mathrm{d}u}(f_u^2)\right\|_{\infty,[0,T]} \lesssim [\![\tau_{f^2}G]\!]_{\gamma,\nu'} T^\gamma [\|f^1 - f^2]\!]_\rho^{\nu'} + \|f^1 - f^2\|_\infty^{\nu'}]$$

and

$$\left[\!\!\left[\int_0^\cdot G_{\mathrm{d}u}(f_u^1) - G_{\mathrm{d}u}(f_u^2)\right]\!\!\right]_{\gamma,[0,T]} \lesssim [\![\tau_{f^2}G]\!]_{\gamma,\nu'}([\![f^1 - f^2]\!]_\rho^{\nu'} + \|f^1 - f^2\|_\infty^{\nu'}).$$

For the second part of the lemma, we use the bound of Lemma 2.3.14. □

We now ready to prove a comparison principle between two solutions. In order to keep a high degree of generality, we do not use the estimation of the Lemma 2.3.14, but prefer to state a general assumption for the regularity of the averaged translation of the first vector field.

Theorem 2.3.16. *Let $\gamma > \frac{1}{2}$, $\nu \in [0,1]$ such that $\gamma(1+\nu) > 1$. Let $G^1, G^2 \in \mathcal{C}_b^{\gamma,\nu}$, θ^1 (respectively θ^2) a solution of the non-linear Young differential equation driven by G^1 (respectively by G^2). Suppose that $\tau_{\theta^2} G^1 \in \mathcal{C}^{\gamma,1}$. Then*

$$\|\theta^1 - \theta^2\|_{\infty,[0,T]} \leqslant c_1 e^{c_2 \|\tau_{\theta^2} G^1\|_{\gamma,1}^{1/\gamma}} (\|\theta^2\|_\gamma^\nu + 1)(|\theta_0^1 - \theta_0^2| + \|G^1 - G^2\|_{\gamma,\nu}).$$

Proof. Let $t_0 \in [0,T]$, $S > 0$ and define $J = [t_0, (t_0+S) \wedge 1]$. For $s \leqslant t \in J$ we have

$$\begin{aligned}
\delta(\theta^1 - \theta^2)_{s,t} \quad = \quad & \int_s^t \tau_{\theta^2} G^1_{\mathrm{d}u}(\theta^1_u - \theta^2_u) - \tau_{\theta^2} G^1_{s,t}(\theta^1_s - \theta^2_s) \\
& + \tau_{\theta^2} G^1_{s,t}(\theta^1_s - \theta^2_s) - \tau_{\theta^2} G^1_{s,t}(0) \\
& + \int_s^t (G^1 - G^2)_{\mathrm{d}u}(\theta^2_u) - (G^1 - G^2)_{s,t}(\theta^2_s) \\
& + (G^1 - G^2)_{s,t}(\theta^2_s) - (G^1 - G^2)_{s,t}(0) \\
& + (G^1 - G^2)_{s,t}(0)
\end{aligned}$$

hence,

$$\begin{aligned}
|\delta(\theta^1 - \theta^2)_{s,t}| \quad \lesssim \quad & \|\tau_{\theta^2} G^1\|_{\gamma,1} |t-s|^\gamma (S^\gamma [\![\theta^1 - \theta^2]\!]_\gamma + |\theta^1_s - \theta^2_s|) \\
& + \|G^1 - G^2\|_{\gamma,\nu} |t-s|^\gamma (S^{\gamma\nu} [\![\theta^2]\!]_\gamma^\nu + |\theta^2_s|^\nu + 1).
\end{aligned}$$

When C_1 is the universal constant in the previous inequality and for S small enough such that $\frac{1}{4} \leqslant C_1 \|\tau_{\theta^2} G^1\|_{\gamma,1} S^\gamma \leqslant \frac{1}{2}$, there exists an other constant C_2 such that

$$|\delta(\theta^1 - \theta^2)_{s,t}| \leqslant \frac{1}{2} |t-s|^\gamma ([\![\theta^1 - \theta^2]\!]_\gamma + S^{-\gamma} |\theta^1_s - \theta^2_s|) + C_2 \|G^1 - G^2\|_{\gamma,\nu} |t-s|^\gamma (S^{\gamma\nu} \|\theta^2\|_\gamma^\nu + 1).$$

Hence

$$[\![\theta^2 - \theta^2]\!]_\gamma \leqslant S^{-\gamma} \|\theta^1 - \theta^2\|_{\infty,J} + C_3 \|G^1 - G^2\|_{\gamma,\nu} (\|\theta^2\|_\gamma^\nu + 1)$$

and

$$\|\theta^1 - \theta^2\|_{\infty,J} \quad \leqslant \quad \frac{1}{2} (\|\theta^1 - \theta^2\|_{\infty,J} + |\theta^1_s - \theta^2_s|) + C_4 \|G^1 - G^2\|_{\gamma,\nu} S^\gamma (\|\theta^2\|_\gamma^\nu + 1).$$

Finally

$$\|\theta^1 - \theta^2\|_{\infty,J} \leqslant 2 |\theta^1_s - \theta^2_s| + C_5 \|G^1 - G^2\|_{\gamma,\nu} S^\gamma (\|\theta^2\|_\gamma^\nu + 1).$$

By the same gluing argument as in Therorem 2.3.9, we have

$$\|\theta^1 - \theta^2\|_\infty \lesssim 2^{1/S} (|\theta^1_0 - \theta^2_0| + C_5 \|G^1 - G^2\|_{\gamma,\nu} S^\gamma (\|\theta^2\|_\gamma^\nu + 1)).$$

Remind that $\frac{1}{4} \leqslant C_1 \|\tau_{\theta^2} G^1\|_{\gamma,1} S^\gamma \leqslant \frac{1}{2}$, and there exists two universal constants (depending on γ, ν, T) c_1 and c_2 such that

$$\|\theta^1 - \theta^2\|_\infty \leqslant c_1 e^{c_2 \|\tau_{\theta^2} G^1\|_{\gamma,1}^{1/\gamma}} (|\theta^1_0 - \theta^2_0| + \|G^1 - G^2\|_{\gamma,\nu} (\|\theta^2\|_\gamma^\nu + 1))$$

which ends the proof. □

Uniqueness of solutions

We prove here the uniqueness of the solutions when G is regular enough using the comparison principle given in Theorem 2.3.16. In order to use the comparison principle, we ask that the vector field is regular enough (in space) and as stated in Lemma 2.3.14, we are able to estimate the averaged translation if we accept a additional loss of space regularity. Furthermore, as we will use uniqueness results in different context, especially in situation when we will have a priori regularity properties for the solutions, we will give a pretty general theorem of existence, Theorem 2.3.17, and specialise it in the two following corollaries.

Theorem 2.3.17. *Let $\gamma > 1/2$, $\nu \in [0,1]$ such that $\gamma(1+\nu) > 1$ and $G \in \mathcal{C}_b^{\gamma,\nu}$. Suppose that there exists a sequence $(G^\varepsilon)_\varepsilon \in \mathcal{C}_b^{\gamma,\nu}$ such that*

i. For all $\gamma' < \gamma$ and all, $\|G - G^\varepsilon\|_{\gamma',\nu} \to 0$.

ii. For all $\varepsilon > 0$ and $\|G^\varepsilon\|_{\gamma,\nu} \leqslant \|G\|_{\gamma,\nu}$.

iii. For all $\varepsilon > 0$ and all $x \in \mathbb{R}^d$ there exists a unique solution θ^ε for the equation

$$\theta_t^\varepsilon(x) = x + \int_0^t G_{\mathrm{d}u}^\varepsilon(\theta_u^\varepsilon(x))\mathrm{d}u.$$

iv. For all $\varepsilon > 0$, $\tau_{\theta^\varepsilon} G \in \mathcal{C}_b^{\gamma,1}$ and

$$\sup_{\varepsilon>0} \|\tau_{\theta^\varepsilon} G\|_{\gamma,1} < +\infty.$$

The solution of the non-linear Young equation driven by G is unique.

Proof. This theorem is a direct consequence of the principle of comparison of Theorem 2.3.16. Let $x \in \mathbb{R}^d$ an initial condition, and let θ a solution of the non-linear Young differential equation with initial condition x. Furthermore let G^ε and θ^ε as in the hypothesis of the theorem. Take $\frac{1}{2} < \gamma' < \gamma$ such that $\gamma(1+\nu) > 1$. Remark that we can apply the comparison principle to θ and θ^ε with γ' instead of γ, and as $\|G^\varepsilon\|_{\gamma',\nu} \leqslant \|G\|_{\gamma',\nu} \lesssim \|G\|_{\gamma,\nu}$, we have

$$\begin{aligned}
\|\theta^\varepsilon\|_{\gamma'} &\lesssim \|G^\varepsilon\|_{\gamma',\nu}^{1/\nu\gamma'}(1 + \|G^\varepsilon\|_{\gamma',\nu})^{\tilde{K}_2\|G^\varepsilon\|_{\gamma',\nu}^{1/\nu\gamma'}}(1+|x|). \\
&\lesssim \|G\|_{\gamma,\nu}^{1/\nu\gamma'}(1 + \|G\|_{\gamma,\nu})^{\tilde{K}_2\|G\|_{\gamma,\nu}^{1/\nu\gamma'}}(1+|x|).
\end{aligned}$$

But also

$$\begin{aligned}
\|\theta(x) - \theta^\varepsilon(x)\|_\infty &\lesssim e^{c_2\|\tau_{\theta^\varepsilon}G\|_{\gamma',1}^{1/\gamma}}\|G^1 - G^\varepsilon\|_{\gamma',\nu}(\|\theta^\varepsilon\|_{\gamma'}^\nu + 1) \\
&\lesssim \|G\|_{\gamma,\nu}^{1/\nu\gamma'}(1 + \|G\|_{\gamma,\nu})^{\tilde{K}_2\|G\|_{\gamma,\nu}^{1/\nu\gamma'}}(1+|x|)e^{c_2\|\tau_{\theta^\varepsilon}G\|_{\gamma,1}^{1/\gamma}}\|G^1 - G^\varepsilon\|_{\gamma',\nu}.
\end{aligned}$$

As $\sup_{\varepsilon>0}\|\tau_{\theta^\varepsilon}G\|_{\gamma,1} < +\infty$, we have $\|\theta(x) - \theta^\varepsilon(x)\|_\infty \to_{\varepsilon\to 0} 0$. As θ^ε is unique, and since this convergence holds true for every function $\theta(x)$ solution of the equation, the solution is unique. □

We can now use the averaged translation operator to establish uniqueness in the case where we have a priori informations of the regularities of the solutions θ^ε.

Corollary 2.3.18. *Let $\gamma > 1/2$, $\delta > 0$ and $G \in \mathcal{C}_b^{\gamma,1+\eta}$. Suppose that there exists a sequence $(G^\varepsilon)_\varepsilon \in \mathcal{C}_b^{\gamma,1+\eta}$ such that*

i. For all $\gamma' < \gamma$ and all, $\|G - G^\varepsilon\|_{\gamma',1} \to 0$.

ii. For all $\varepsilon > 0$ and $\|G^\varepsilon\|_{\gamma,1+\eta} \leqslant \|G\|_{\gamma,1+\eta}$.

iii. For all $\varepsilon > 0$ and all $x \in \mathbb{R}^d$ there exists a unique solution θ^ε for the equation

$$\theta_t^\varepsilon(x) = x + \int_0^t G_{\mathrm{d}u}^\varepsilon(\theta_u^\varepsilon(x))\mathrm{d}u.$$

iv. There exists $\rho > 0$ such that $\eta\rho + \gamma > 0$ and for which for all $\varepsilon > 0$, $\theta^\varepsilon(x) \in \mathcal{C}^\rho$ and $\sup_{\varepsilon>0} \|\theta^\varepsilon(x)\|_\rho < +\infty$.

Then solution of the non-linear Young equation driven by G is unique.

Furthermore, when the function $x \to \sup_{\varepsilon>0} \|\theta^\varepsilon(x)\|_\rho$ is locally bounded in time, the flow $(t,x) \to \theta_t(x)$ of the equation is locally Lipschitz continuous in space, uniformly in time and

$$\|\theta(x) - \theta(y)\|_\infty \lesssim \exp(C(1 + \log(1 + \|G\|_{\gamma,1+\eta}) + (\sup_{\varepsilon>0} \|\theta^\varepsilon(y)\|_\rho)^\eta)\|G\|_{\gamma,1+\delta}^{1/\gamma})|x-y|(|y|+1).$$

Proof. Condition *i.*, *ii.* and *iii.* are the same of those of Theorem 2.3.17. We only have to prove that the point *iv.* of Theorem 2.3.17 is satisfied. But thanks to Lemma 2.3.14, we know that $\tau_{\theta^\varepsilon} G \in \mathcal{C}_b^{\gamma,1}$. Furthermore

$$\|\tau_{\theta^\varepsilon} G\|_{\gamma,1} \lesssim \|G\|_{\gamma,1+\eta}(\|\theta^\varepsilon\|_\rho^\eta + 1) \lesssim \|G\|_{\gamma,1+\eta}((\sup_{\varepsilon>0} \|\theta^\varepsilon\|_\rho)^\eta + 1),$$

and the uniqueness follows by Theorem 2.3.17. Furthermore, for $y \in \mathbb{R}^d$ since $\|G^\varepsilon\|_{\gamma,1+\eta} \leqslant \|G\|_{\gamma,1+\eta}$, we have

$$\|\tau_{\theta^\varepsilon(y)} G^\varepsilon\|_{\gamma,1} \lesssim \|G^\varepsilon\|_{\gamma,1+\eta}((\sup_{\varepsilon>0} \|\theta^\varepsilon\|_\rho)^\eta + 1) \lesssim \|G\|_{\gamma,1+\eta}((\sup_{\varepsilon>0} \|\theta^\varepsilon\|_\rho)^\eta + 1).$$

Since $\sup_{\varepsilon>0} \|G^\varepsilon\|_{\gamma,1} \lesssim \|G\|_{\gamma,1+\eta}$ we have, thanks to the a priori bounds for the solutions of Theorem 2.3.9, and Remark 2.3.12, that

$$\|\theta^\varepsilon(y)\|_\gamma \lesssim \|\theta\|_\infty \lesssim (1 + \|G\|_{\gamma,1+\eta})^{K(\|G\|_{\gamma,1+\eta})^{1/\gamma}}(|y|+1),$$

which implies

$$\|\theta^\varepsilon(x) - \theta^\varepsilon(y)\|_\infty \lesssim \exp(C(1 + \log(1 + \|G\|_{\gamma,1+\eta}) + (\sup_{\varepsilon>0} \|\theta^\varepsilon(y)\|_\rho)^\eta)\|G\|_{\gamma,1+\delta}^{1/\gamma})|x-y|(|y|+1),$$

the conclusion the easily follows when we let ε goes to zero. $\square$

Remark 2.3.19. Suppose furthermore that for all $t \in [0,T]$, $x \to \theta_t^\varepsilon(x)$ is differentiable in space. Then

$$\sup_{\varepsilon>0} |D\theta_t^\varepsilon(x)| \lesssim \exp(C(1 + \log(1 + \|G\|_{\gamma,1+\eta}) + (\sup_{\varepsilon>0} \|\theta^\varepsilon(x)\|_\rho)^\eta)\|G\|_{\gamma,1+\delta}^{1/\gamma})(|x|+1)$$

Finally, we state the more general results of uniqueness, where all the needed informations are the regularity of G.

Corollary 2.3.20. *Let $\gamma > \frac{1}{2}$, $\nu \in [0,1]$ such that $\gamma(1+\nu) > 0$ and suppose that $G \in \mathcal{C}_b^{\gamma,\nu+1}$, then there exists a unique solution $\theta(x)$ for the non-linear Young equation with initial condition x. Furthermore θ is locally Lipschitz continuous in space uniformly in time.*

Proof. We only have to check the conditions of Corollary 2.3.18 with $\nu = \eta$ and $\gamma = \rho$. Let $G^\varepsilon \in \mathcal{C}^{\gamma,1+\nu}$ such that the time derivative $(G^\varepsilon)'$ exists and lies in $\mathcal{C}_b^{\gamma,1+\nu}$, and such that $\|G^\varepsilon\|_{\gamma,1+\nu} \leqslant \|G\|_{\gamma,1+\nu}$ and for all $\gamma' < \gamma$, $\|G - G^\varepsilon\|_{\gamma',1+\nu} \to 0$. In that case, $\theta^\varepsilon(x)$ is the solution of

$$\theta^\varepsilon(x) = x + \int_0^t (G_r^\varepsilon)'(\theta_r^\varepsilon(x))\mathrm{d}r = x + \int_0^t G_{\mathrm{d}r}^\varepsilon(\theta_r^\varepsilon(x)).$$

As $G^\varepsilon \in \mathcal{C}_b^{\gamma,1+\nu}$, θ^ε is unique and furthermore θ^ε is differentiable in space, and the differential is the solution of the following equation

$$D\theta^\varepsilon(x) = \mathrm{id} + \int_0^t D(G_r^\varepsilon)'(\theta_r^\varepsilon(x))\mathrm{d}r.$$

Thanks to Remark 2.3.12,

$$\|\theta^\varepsilon(x)\|_\infty + \|\theta^\varepsilon(x)\|_\gamma \lesssim (1 + \|G\|_{\gamma,1}^b)^{K(\|G\|_{\gamma,1}^b)^{1/\gamma}}(|x|+1),$$

Hence $x \to \sup_\varepsilon \|\theta^\varepsilon(x)\|_\gamma$ is locally bounded in space. All the conditions of the Corollary 2.3.18 are fulfilled, and the result follows. □

2.3.3 Localisation of unbounded vectorfields.

In order to give a complete survey of the question, we need to go back to the weighted spaces $\mathcal{C}^{\gamma,\nu,\psi}$ and to state the following theorems in that case.

Let $\gamma > 1/2$, $\nu < 1$ and $\gamma(1+\nu) > 1$ and $\psi > 0$ a ν-weight. Let $r > 0$ and $r = K_1(1 + \|G\|_{\gamma,1}^b)^{K_2(\|G\|_{\gamma,1}^b)^{1/\gamma}}(r+1)$, where K_1 and K_2 are define as in Theorem 2.3.9 and depend on ψ. As we intend to use the averaged translation operator, and since any solution lies in balls of radius R, we need to localize G on balls B of center 0 and of radius $2R$. We then let $G_{|R} \in \mathcal{C}_b^{\gamma,\nu}([0,T],B)$ the restriction of G on $[0,T]\times B$. We have of course $\|G_{|R}\|_{\gamma,\nu}^b \lesssim \psi(2R)\|\tilde{G}\|_{\gamma,\nu,\psi}$. Furthermore, as all the arguments hold locally, as they do all the estimations for $x,y \in B(0,r)$ in the previous section.

When $\nu = 1$, it is necessary to have the existence and a bound for the solution in order to localised. As this holds only for $\nu < 1$, the good hypothesis is that there exists $\nu < 1$, $\tilde{\psi}$ a ν-weight such that $G \in \mathcal{C}^{\gamma,\nu,\tilde{\psi}} \cap \mathcal{C}^{\gamma,1,\psi}$. In that case, we are again able to localised and to use the result of the previous section. The following theorem holds:

Theorem 2.3.21. *Let $\gamma > \frac{1}{2}$, $1 \geqslant \nu > 0$ with $\gamma(1+\nu) > 1$ and ψ a weight. Let $1 < \nu' \leqslant \nu$ with $\nu' < 1$ such that $\gamma(1+\nu') > 1$, and ψ' a ν'-weight. Let $G \in \mathcal{C}^{\gamma,\nu',\psi'} \cap \mathcal{C}^{\gamma,\nu,\psi}$. For all $x \in \mathbb{R}^d$ there exists a solution $\theta(x) \in \mathcal{C}^\gamma([0,T])$ to the equation*

$$\theta_t(x) = x + \int_0^t G_{\mathrm{d}u}(\theta_u).$$

Furthermore, there exists K_1 and K_2 two constants depending on γ, ν' and ψ' such that

$$\|\theta\|_{\gamma,[0,T]} \leqslant K_1 \|G\|^{1/\nu'\gamma} (1 + \|G\|_{\gamma,\nu',\psi'})^{K_2 \|G\|^{1/\nu'\gamma}_{\gamma,\nu',\psi'}} (1 + |x|).$$

Let $r > 0$, $R = K_1 \|G\|^{1/\nu'\gamma} (1 + \|G\|_{\gamma,\nu',\psi'})^{K_2 \|G\|^{1/\nu'\gamma}_{\gamma,\nu',\psi'}} (1 + |r|)$ and $B = B(0, 2R)$. Let us take $\tilde{G} \in \mathcal{C}^{\gamma,\nu',\psi'} \cap \mathcal{C}^{\gamma,\nu,\psi}$ such that $\|\tilde{G}\|_{\gamma,\nu,\psi'} \leqslant \|G\|_{\gamma,\nu,\psi'}$, suppose furthermore that for $y \in B(0, r)$, $\tau_{\tilde{\theta}(y)} G \in \mathcal{C}^{\gamma,1,\psi}$ where $\tilde{\theta}$ is the solution of the following equation

$$\tilde{\theta}_t(y) = y + \int_0^t \tilde{G}_{\mathrm{d}u}(\tilde{\theta}_u).$$

Then

$$\|\theta(x) - \tilde{\theta}(y)\|_\infty \lesssim \varphi(R) e^{c_2 \varphi(R) \|\tau_{\tilde{\theta}(y)} G\|^{1/\gamma}_{\gamma,1,\psi}} (|x - y| + \|G - \tilde{G}\|_{\gamma,\nu,\psi})$$

where $\varphi(R) = (R + 1)\psi(R)$.

2.4 Averaging of paths.

We turn now to the study of the averaging operator T^w proper. One of our main results is a proof that fBm paths are ρ-irregular for any $\rho < 1/2H$ and as a consequence that the averaging operator T^w is bounded from the Fourier–Lebesgue space $\mathcal{F}L^\alpha$ to $C^\gamma \mathcal{F}L^{\alpha+\rho}$ for any $\alpha \in \mathbb{R}$ and for almost every fBm path w. This result was one of our main reasons to look at the scale of Lebesgue–Fourier spaces.

For the scale of Besov spaces $(\mathcal{C}^\alpha)_\alpha$ we were unable to prove similar results and we limited ourselves to study the averaged vector-fields $T^w f$ for fixed $f \in \mathcal{C}^\alpha$.

In this section we will first study the almost-sure irregularity of fBm paths. This study proceeds in two steps: first we use well known chaining arguments (essentially going back to Kolmogorov lemma in the form given to it by Garsia–Rodemich and Rumsey) to go from supremum norm to "integral" norms more suitable to probabilistic estimates and then use Hoeffding inequality to prove these estimates.

The use of Hoeffding replaces what in Davie's paper [20] are explicit and painful computations on Brownian motions (relying on the Markov property) and what in other works (e.g. in [30]) is achieved via stochastic calculus (and thus martingale properties). In the fBm context neither technique is applicable and explicit computations using Gaussian tools, while possible are quite cumbersome and moreover we were unable to use them to obtain the exponential square integrability we show here to be valid. So we think that our observation that discrete martingale techniques like Hoeffding inequality are useful in the fBm context is one of the interesting points of our research.

2.4.1 Chaining lemmas

To see the average properties of the fractional Brownian path, we will need some chaining lemmas, to infer global estimates from point-wise ones.

Lemma 2.4.1. *Let X from I^2 to $\mathbb{R}^d$ such that for all $s \leqslant u \leqslant t$*

$$|X_{s,t}| \leqslant |X_{s,u}| + |X_{u,t}| \,\text{and}\, X_{s,s} = 0$$

And let us define

$$R_\mu(X) = \sum_{n\in\mathbb{N}} \sum_{k=0}^{2^n-1} 2^{-2n} \exp(\mu 2^n |X_{k2^{-n},(k+1)2^{-n}}|^2)$$

then there exists a constant $K>0$ such that for all $s \leqslant t$,

$$\exp(\mu |X_{s,t}|^2/|t-s|) \lesssim |t-s|^{-K} R_{\mu K}(X)$$

Proof. Let $0 \leqslant s < t \leqslant 1$, $n \in \mathbb{N}$ the largest $n' \in \mathbb{N}$ such that $2^{-(n'+1)} \leqslant t-s \leqslant 2^{-n'}$. By definition of n there exists l such that $l/2^n \leqslant s < t \leqslant (l+1)/2^n$. We can find some sequences $(s_k)_{k\geq 1}$ and $(t_k)_{k\geq 1}$ such that (s_k) decreases, (t_k) increases, $s_1 = t_1 = (2l+1)/2^{n+1}$, $\lim_{k\to\infty} s_k = s$, $\lim_{k\to\infty} t_k = t$, $s_{k+1} - s_k \leqslant 2^{n+k+1}$, $t_{k+1} - t_k \leqslant 2^{n+k+1}$ and $2^{n+k} t_k \in \mathbb{Z}$ and $2^{n+k} s_k \in \mathbb{Z}$. Hence $[s,t) = \cup_{k\geq 1}[s_{k+1}, s_k) \cup \cup_{k\geq 1}[t_k, t_{k+1})$ and thanks to the definition of the sequences, the following inequalities hold for s_k, but also for t_k.

First, if $s_{k+1} = s_k$, $\sqrt{\mu}|X_{s_{k+1},s_k}| = 0$. Now, if $s_{k+1} < s_k$ then there exists $l_k \in \{0,\ldots,n+k\}$ such that $s_{k+1} = (2l_k - 1)/2^{n+k+1}$ and $s_k = l_k/2^{n+k}$. Hence

$$\begin{aligned}
\sqrt{\mu}|X_{s_{k+1},s_k}| &= 2^{-(n+k+1)/2} \log(2^{(n+k+1)} 2^{-(n+k+1)} \exp(\mu 2^{k+n+1} |X_{s_{k+1},s_k}|^2))^{1/2} \\
&\lesssim 2^{-(n+k)/2} \{(n+k) + \log(R_\mu(X))\}^{1/2}.
\end{aligned}$$

But $2^{-(n+1)} \leqslant |t-s| \leqslant 2^{-n}$, hence

$$\sqrt{\mu}|X_{s_{k+1},s_k}| \lesssim |t-s|^{1/2} 2^{-k/2} \{k + \log_- |t-s| + \log(R_\mu(X))\}^{1/2}.$$

Thanks to the definition of $(s_k)_k$ and $(t_k)_k$, we have

$$\begin{aligned}
\sqrt{\mu}|X_{s,t}| &\leqslant \sum_{k\geqslant 1} \sqrt{\mu}|X_{s_{k+1},s_k}| + \sqrt{\mu}|X_{t_k,t_{k+1}}| \\
&\lesssim |t-s|^{1/2} \{1 + \log_- |t-s| + \log(R_\mu(X))\}^{1/2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\exp(\mu |X_{s,t}|^2/|t-s|) &\lesssim \exp(K \log(1/|t-s|) + K \log R_\mu(X)) \\
&\lesssim |t-s|^{-K} R_{\mu K}(X)
\end{aligned}$$

as by Jensen inequality $R_\mu(X)^K \leqslant R_{\mu K}(X)$. □

In the following, to approach a point of $\mathbb{R}^d$ we will use a similar argument. Namely we will use the graph $(2^{-m}\mathbb{Z})^d$ as a good approximation of $\mathbb{R}^d$. Hence we need to have an approximation of the biggest error we can make using such an approximation. It is well known that for all d and all $m \in \mathbb{N}$, $\sup_{x\in\mathbb{R}^d} \inf_{y\in(2^{-m}\mathbb{Z})^d} |x-y| = \sqrt{d}/2^{m+1}$.

Lemma 2.4.2. *Let X from $\mathbb{R}^d$ to $\mathbb{R}^d$ and g such that $g \geqslant 1$, $\sup_{|\zeta-\zeta'|\leqslant\sqrt{d}/2} g(\zeta')/g(\zeta) < \infty$ and with $\|g^{-1}\|_{L^1(\mathbb{R}^d)} < +\infty$. Suppose furthermore that the following quantity is finite*

$$C_X := \sup_{\substack{m\in\mathbb{N} \\ \zeta : 2 \geqslant g(\zeta)/2^m \geqslant 1/2 \\ |\zeta-\zeta'| \leqslant \sqrt{d}/2}} |X(\zeta) - X(\zeta')|/(2^m|\zeta-\zeta'|) < +\infty.$$

Let

$$S_\mu(X) = \sum_{n\in\mathbb{N}} \sum_{\zeta\in(2^{-n}\mathbb{Z})^d} 2^{-(d+1)n} g(\zeta')^{-1} \exp(\mu|X(\zeta')|^2).$$

There exists a constant $C \geqslant 1$ such that

$$\exp(\mu|X(\zeta)|^2) \lesssim g(\zeta)^{-C} \exp(\mu K C_X^2) S_{\mu C}(X).$$

Proof. Let $\zeta \in \mathbb{R}^d$ and m such that $g(\zeta) \sim 2^m$. Let $\zeta' \in (2^{-m}\mathbb{Z})^d$ such that $|\zeta - \zeta'| \leqslant 2^{-m}\sqrt{d}/2$ then

$$|X(\zeta) - X(\zeta')| \leqslant C_X 2^m |\zeta - \zeta'| \lesssim C_X.$$

Furthermore, the hypothesis on g gives us that $\log(g(\zeta')) \lesssim 1 + \log(g(\zeta))$. Hence

$$\begin{aligned}
\sqrt{\mu}|X(\zeta)| &\leqslant \sqrt{\mu}|X(\zeta) - X(\zeta')| + \sqrt{\mu}|X(\zeta')| \\
&\leqslant \sqrt{\mu}C_X + \{\log(2^m 2^{-m} g(\zeta') g(\zeta')^{-1} \exp(\mu|X(\zeta')|^2))\}^{1/2} \\
&\lesssim \sqrt{\mu}C_X + \{m + \log(g(\zeta')) + \log(S_\mu(X))\}^{1/2} \\
&\lesssim \sqrt{\mu}C_X + \{1 + \log(g(\zeta)) + \log(S_\mu(X))\}^{1/2}.
\end{aligned}$$

Finally we have

$$\begin{aligned}
\exp(\mu|X(\zeta)|^2) &\leqslant \exp(K\{\mu C_X^2 + 1 + \log(g(\zeta)) + \log S_\mu(X)\}) \\
&\lesssim g(\zeta)^K \exp(\mu C C_X^2) S_{\mu C}(X).
\end{aligned}$$

□

We can think at g as $g(\zeta) = (1 + |\zeta|)^{d+1}$.

Lemma 2.4.3. *For all $\beta \in \mathbb{R}$ and all $R > 0$ there exists a constant $C(\beta, R)$ such that for all $\zeta' \in B(\zeta, R)$*

$$|(1 + |\zeta|)^\beta - (1 + |\zeta'|)^\beta| \leqslant C(\beta, R)(1 + |\zeta|)^{\beta-1}|\zeta - \zeta'|$$

Proof. Let us suppose first that $|\zeta'| \geqslant |\zeta|$ by the choice of ζ' we have $0 \leqslant \frac{|\zeta'|-|\zeta|}{1+|\zeta|} \leqslant R$. Then

$$\begin{aligned}
|(1 + |\zeta|)^\beta - (1 + |\zeta'|)^\beta| &= (1 + |\zeta|)^\beta \left| \left(1 + \frac{|\zeta'| - |\zeta|}{1 + |\zeta|}\right)^\beta - 1 \right| \\
&\leqslant (1 + |\zeta|)^\beta \sup_{x\in[1,R]} |f'_\beta(x)| \frac{||\zeta'| - |\zeta||}{1 + |\zeta|} \\
&\leqslant \sup_{x\in[0,R]} |f'_\beta(x)| (1 + |\zeta|)^{\beta-1} |\zeta - \zeta'|
\end{aligned}$$

Where the function f_β is define from $[0, R]$ to $\mathbb{R}$ by $f_\beta(x) = (1 + x)^\beta$ We have

$$|f'_\beta(x)| = |\beta(1 + x)^{\beta-1}| \leqslant |\beta|((1 + R)^{\beta-1} \vee 1).$$

If $|\zeta| > |\zeta'|$, the same computation gives

$$|(1 + |\zeta|)^\beta - (1 + |\zeta'|)^\beta| \lesssim_{R,\beta} (1 + |\zeta'|)^{\beta-1}|\zeta - \zeta'|.$$

When $\beta - 1 \geqslant 0$, the result follows. Suppose now that $\alpha b - 1 < 0$, we have to prove that $(1+|\zeta|) \lesssim (1+|\zeta'|)$. When $|\zeta| \leqslant 2R$, then

$$(1+|\zeta'|)/(1+|\zeta|) \quad \geqslant \quad 1/(1+2R).$$

When $|\zeta| > 2R$

$$(1+|\zeta'|)/(1+|\zeta|) \geqslant (1+|\zeta| - |\zeta'-\zeta|)/(1+|\zeta|) \geqslant 1 - |\zeta'-\zeta|/(1+|\zeta|) \geqslant 1/2$$

and the result follows. □

2.4.2 Application of the chaining lemmas, control of the averaging along curves.

The last lemmas allows us to control the average of a function (or a distribution) along the curve w. Indeed, to estimate on the quantity $\int_s^t f_u(x+w_u)\mathrm{d}u$ it will be enough to have a control on simpler quantities. We will apply those lemmas in two similar situation, namely when $f \in \mathcal{C}^\alpha$ and when $f \in \mathcal{F}L^\alpha$. In that case, we will see that it is enough to control Φ^w.

Averaging property of the occupation measure.

Recall that we already defined $\Phi_t^w(\xi) = \int_0^t e^{i\langle \xi, w_r\rangle}\mathrm{d}r$ and

$$\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} = \sup_{\xi\in\mathbb{R}^d}\sup_{0\leqslant s<t\leqslant T}(1+|\xi|)^\rho\frac{|\Phi_t^w(\xi)-\Phi_s^w(\xi)|}{|s-t|^\gamma}.$$

Lemma 2.4.4. *For all $-\beta < \alpha$ there exists a constant $a > 0$ and $\gamma > 0$ such that for all $\lambda > 0$,*

$$|\Phi_t^w(\xi) - \Phi_s^w(\xi)| \lesssim |t-s|^\gamma(1+|\xi|)^{-\alpha'}(1+\log^{1/2}(e^{a\mu\|w\|_\infty}K_\alpha^w(\lambda)))$$

where

$$K_\alpha^w(\lambda) = \sum_{\substack{n,m\in\mathbb{N}\\ 0\leqslant k\leqslant 2^n-1\\ \xi'\in(2^{-m}\mathbb{Z})^d}} 2^{-2n+(d+1)m}(1+|\xi'|)^{-(d+1)}\exp(\lambda 2^n(1+|\xi'|)^{2\beta}|\Phi^w_{k2^{-n},(k+1)2^{-n}}(\xi')|^2).$$

Proof. We apply the Lemmas 2.4.1 and 2.4.2 to

$$X_{s,t}(\xi) = (1+|\xi|)^\beta|\Phi_{s,t}^w(\xi)|/|t-s|^{1/2}$$

thanks to Lemma 2.4.3 and the definition of $\Phi_{s,t}^w$, for all $\xi\in\mathbb{R}^d$, and all $\xi'\in B\left(\xi, \sqrt{d}/2\right)$

$$\begin{aligned}|X_{s,t}(\xi) - X_{s,t}(\xi')| &\leqslant |(1+|\xi|)^\beta - (1+|\xi'|)^\beta||\Phi_{s,t}^w(\xi')|/|t-s|^{1/2}\\ &\quad +(1+|\xi|)^\beta|\Phi_{s,t}^w(\xi)-\Phi_{s,t}^w(\xi')|/|t-s|^{1/2}\\ &\lesssim (1+|\xi|)^\beta|\xi-\xi'|(1+\|w\|_\infty)|t-s|^{1/2}\end{aligned}$$

Here we take $\zeta = \xi$, $g(\zeta) = (1+|\zeta|)^{\beta+Cd}$ such that $\beta + Cd \geqslant d+1$. With those choices, $X_{s,t}$ and g verify the hypothesis of lemma 2.4.2, furthermore $C_X \lesssim (1+\|w\|_\infty)$

$$\begin{aligned}\exp(\mu|X_{s,t}(\xi)|^2) &= \exp(\mu(1+|\xi|)^{2\beta}|\Phi^w_{s,t}(\xi)|^2/|t-s|) \\ &\lesssim (1+|\xi|)^{C(d+1)} S_{C\mu}(X_{s,t}) \exp(\mu C\|w\|_\infty^2)\end{aligned}$$

Now, let apply Lemma 2.4.1 to $(1+|\xi|)^\beta \Phi^w_{s,t}(\xi)$, then

$$\exp(\mu|X_{s,t}(\xi)|^2) \lesssim |t-s|^{-K} R_{K\mu}((1+|\xi|)^\beta \Phi^w_{\cdot}(\xi))$$

But

$$\begin{aligned}R_{K\mu}((1+|\xi|)^\beta \Phi_{\cdot}(\xi)) &= \sum_{n\in\mathbb{N}} \sum_{k=0}^{2^n-1} 2^{-2n} \exp(\mu K 2^n |\Phi^w_{k2^{-n},(k+1)2^{-n}}(\xi)|^2 (1+|\xi|)^{2\beta}) \\ &\lesssim (1+|\xi|)^{C(d+1)} \sum_{n\in\mathbb{N}} \sum_{k=0}^{2^n-1} 2^{-2n} S_{\mu CK}(X_{k2^{-n},(k+1)2^{-n}}) \exp(\mu CK\|w\|_\infty^2) \\ &\lesssim (1+|\xi|)^{C(d+1)} \exp(\lambda\|w\|_\infty^2) K^w_\beta(\lambda).\end{aligned}$$

When we take the logarithm, we have

$$|\Phi^w_{s,t}(\xi)| \lesssim \mu^{-1/2}|t-s|^{1/2}(1+|\xi|)^{-\beta}(1+\log_-|t-s| + \log(1+|\xi|) + \log(\exp(\lambda\|w\|_\infty^2)K^w_\beta(a\mu))).$$

Hence, for all $\varepsilon_1, \varepsilon_2 > 0$, we have

$$|\Phi^w_{s,t}(\xi)| \lesssim_{\varepsilon_1,\varepsilon_2} \mu^{-1/2}|t-s|^{1/2-\varepsilon_1}(1+|\xi|)^{-\beta+\varepsilon_2}(1+\log(\exp(\lambda\|w\|_\infty^2)K^w_\beta(a\mu))).$$

Furthermore, by interpolating with the trivial estimates $|\Phi_{s,t}(\xi)| \leqslant |t-s|$, for all $-\beta < \alpha$, there exists $\gamma > 1/2$, and a constant $a > 0$ such that

$$|\Phi^w_{s,t}(\xi)| \lesssim |t-s|^\gamma (1+|\xi|)^\alpha (1+\log^{1/2}(\exp(a\mu\|w\|_\infty^2)K^w_\beta(a\mu))).$$

□

Averaging of Besov functions along paths

In this section we analyse the averaging effect of paths on functions belonging to the scale of Besov spaces $(\mathcal{C}^\alpha)_\alpha$. The computation are mostly the same as for the operator Φ. Nevertheless as we are unable to give an estimate for the whole space $\mathcal{C}^\alpha$, the chaining arguments now depends on the chosen function f.

Lemma 2.4.5. *For all $-\beta < \alpha$ there exists $\gamma > 1/2$ such that for all $f \in \mathcal{S}'(\mathbb{R}^d)$, all $\lambda > 0$ and all i,*

$$|T^w_{s,t}(\Delta_i f)(x)| \lesssim_\lambda 2^{\alpha i}\|\Delta_i f\|_\infty |t-s|^\gamma (1+\log^{1/2}(1+|x|) + \log^{1/2}(K^w_{f,\beta}(\lambda)))$$

where

$$K^w_{f,\beta}(\lambda) = \sum_{\substack{n,m\in\mathbb{N} \\ i \geqslant -1 \\ 0 \leqslant k \leqslant 2^n-1 \\ x' \in (2^{-m}\mathbb{Z})^d}} 2^{-c_\beta(m+n+i)}(1+|x'|)^{-(d+1)} \exp(\lambda 2^{n+2i\beta}|T^w_{k/2^n,(k+1)/2^n}(\Delta_i f)(x')|^2/\|\Delta_i f\|_\infty^2)$$

and c_β is a constant depending only of β and d such that the sum without the exponential is finite.

Proof. The proof is very similar to the proof of Lemma 2.4.4. We will apply the Lemmas 2.4.1 and 2.4.2 to

$$X^i_{s,t}(x) = 2^{i\alpha}|T^w_{s,t}(\Delta_i f)(x)|/(\|\Delta_i f\|_\infty |t-s|^{1/2})$$

with the convention that $X^i = 0$ when $\Delta_i f = 0$. We have, thank to the definition of T^w,

$$|T^w_{s,t}(\Delta_i f)(x)| \leqslant \|\Delta_i f\|_\infty |t-s| \tag{2.9}$$

Furthermore, as the Fourier transform of $\Delta_i f$ is compactly supported in an annulus, we have the trivial estimate

$$\begin{aligned} |X^i_{s,t}(x) - X^i_{s,t}(x')| &\lesssim 2^{i\beta}\|\nabla\Delta_i f\|_\infty |t-s|^{1/2}|x-x'|/\|\Delta_i f\|_\infty \\ &\lesssim 2^{i(\beta+1)}|x-x'|. \end{aligned}$$

Let us take $g_i(x) = 2^{(a+\beta)i}(1+|x|)^{d+1}$ with $a \geqslant 1$ and $a+\beta \geqslant c_\beta$. Hence, $C_{X^i_{s,t}} \lesssim 1$. By Lemma 2.4.2, there exists $b, c > 1$ such that

$$\exp(\mu|X^i_{s,t}(x)|^2) \lesssim 2^{b(a+\beta)i}(1+|x|)^{b(d+1)} S_{\mu b}(X^i_{s,t}).$$

Now, thanks to Lemma 2.4.1, there exists a', b', c', d' such that

$$\exp(\mu|X^i_{s,t}(x)|^2) \leqslant 2^{a'i}(1+|x|)^{b'}|t-s|^{-c'} K^w_{f,\beta}(d'\mu).$$

Hence by taking the logarithm, and by losing a small power of time and on i, we have

$$|T^w_{s,t}\Delta_i f(x)| \lesssim 2^{(-\beta+\varepsilon_1)i}\|\Delta_i f\|_\infty |t-s|^{1/2-\varepsilon_2}(1+\log^{1/2}(1+|x|)+\log^{1/2}(K^w_{f,\beta}(d'\mu))) \tag{2.10}$$

Now we interpolate (2.10) and (2.9) and there exists for all $\alpha > -\beta$, there exists $\rho > 0$ and $\gamma > 1/2$

$$|T^w_{s,t}(\Delta_i f)(x)| \lesssim 2^{\alpha i}\|\Delta_i f\|_\infty |t-s|^\gamma (1+\log^{1/2}(1+|x|)+\log(K^w_{f,\beta}(d'\mu)))^{1/2}$$

□

The operator T^w

We are now able to define the function $T^w f$ for all $f \in \mathcal{C}^\alpha$ (respectively $\mathcal{F}L^\alpha$) for all $\alpha > -\beta$, as soon as there exists $\lambda > 0$ small enough such that $K^w_{f,\beta}(\lambda)$ (respectively $K^w_\beta(\lambda)$) is finite. As we already mentioned it remains an open problem to study the boundedness of T^w as an operator with range in Besov spaces so we restrict ourselves to study the image of $T^w f$ for fixed f and with w in the support of the fBm law without any attempt to obtain estimates which are uniform in f. On the contrary, for $\mathcal{F}L^\alpha$, the estimate on Φ^w are good enough to define T^w as an operator on the whole space.

Definition 2.4.6. Let $\beta \in \mathbb{R}$, $\alpha > -\beta$ and let $f \in \mathcal{C}^\alpha$. We define

$$T^w_{s,t} f(x) = \sum_{i \geqslant -1} T^w_{s,t}(\Delta_i f)(x) = \lim_{N\to\infty} T^w_{s,t}(\pi_{\leqslant N} f)(x)$$

and

$$T^w_{s,t} g(x) = \lim_{\substack{h \,\in\, \mathcal{F}L^{0\vee\alpha} \\ h \xrightarrow{\mathcal{F}L^\alpha} g}} T^w_{s,t} h(x)$$

As these object are defined by some limiting procedures, it is not straightforward that they exist. Furthermore, for the consistency of the definition, we must show that when $f \in \mathcal{F}L^\alpha$ these two limiting procedures gives the same object, and that the limits does not depends of the choice of sequence (Δ_i). This is the purpose of the following theorem.

Theorem 2.4.7. *Let $\beta \in \mathbb{R}$ and let $\alpha > -\beta$.*

i. Suppose that there exists λ_0 such that $K^w_\beta(\lambda_0) < +\infty$. Then for all $g \in \mathcal{F}L^\alpha$, $T^w g$ exists and does not depends of the choice of the sequence. Furthermore, for all $\lambda \leqslant \lambda_0$ we have

$$|T^w_{s,t} g(x)| \lesssim |t-s|^\gamma N_\alpha(g)(1 + \log^{1/2} K^w_\beta(\lambda)).$$

Hence, $(T^w_{s,t})_{0\leqslant s\leqslant t\leqslant 1}$ is well define as a family of operators on $\mathcal{F}L^\alpha$.

ii. For $f \in \mathcal{C}^\alpha$ suppose that there exists λ_0 such that $K^w_{f,\beta}(\lambda_0) < +\infty$. Then $T^w f$ exists and the following bound holds

$$|T^w f_{s,t}(x)| \lesssim_\lambda |t-s|^\gamma \|f\|_\alpha (1 + \log^{1/2}(1+|x|) + \log^{1/2} K^w_{f,\beta}(\lambda)).$$

Furthermore let us suppose that for $g \in \mathcal{C}^\alpha$, $K^w_{f,\beta}(\lambda_0)$ is also finite, then for all $\lambda \leqslant \lambda_0/2$

$$|T^w_{s,t}(f-g)(x)| \lesssim |t-s|^\gamma \|f-g\|_\alpha (1 + \log^{1/2}(1+|x|) + (K^w_{f,\beta}(\lambda) + K^w_{g,\beta}(\lambda)))$$

iii. These two limiting procedures are compatible when $f \in \mathcal{F}L^\alpha$.

Proof. The proof is quite straightforward when $g \in \mathcal{F}L^\alpha$. Indeed, for h^1 and h^2 in $\mathcal{F}L^0 \cap \mathcal{F}L^\alpha$, we have

$$T^w_{s,t}(h_1 - h_2)(x) = \int_{\mathbb{R}^d} \mathrm{d}\xi (\hat{h}_1 - \hat{h}_2)(\xi) \exp(i\xi.x) \Phi^w_{s,t}(\xi)$$

hence

$$|T^w_{s,t}(h_1 - h_2)(x)| \lesssim N_\alpha(h_1 - h_2)|t-s|^\gamma (1 + \log^{1/2}(K^w_\beta(\lambda_0)))$$

and the result of (i) follows.

Let us prove (ii). For $f \in \mathcal{C}^\alpha$ let us show as the quantity $T^w_{s,t}(\pi_{\leqslant N} f)(x)$ converges when $N \to +\infty$. Indeed, thanks to Lemma 2.4.5, for $\varepsilon > 0$ such that $-\beta < \alpha - \varepsilon < \alpha$, we have

$$\begin{aligned} |T^w_{s,t}(\pi_{\leqslant N} f)(x) - T^w_{s,t}(\pi_{\leqslant N+M} f)(x)| &\lesssim \sum_{i=N+1}^{N+M} |T^w_{s,t}(\Delta_i f)(x)| \\ &\lesssim C_{s,t,x}(\log^{1/2} K^w_{f,\beta}(\lambda_0)) \|f\|_\alpha 2^{-\varepsilon N}. \end{aligned}$$

Hence, $\left(T^w_{s,t}(\pi_{\leqslant N} f)(x)\right)$ is Cauchy, and then the limit $T^w f$ exists. Furthermore we have the straightforward bound for all $\lambda < \lambda_0$

$$|T^w_{s,t}(\pi_{\leqslant N} f)(x)| \lesssim |t-s|^\gamma \|f\|_\alpha (1 + \log^{1/2}(1+|x|) + \log^{1/2}(K^w_{f,\beta}(\lambda_0)))$$

and the same bound holds for $N \to +\infty$.

For f and g we have

$$|T^w_{s,t}(f-g)(x)| \lesssim |t-s|^\gamma \|f-g\|_\alpha (1 + \log^{1/2}(1+|x|) + \log^{1/2}(K^w_{f-g,\beta}(\lambda_0))),$$

but thanks to the definition of the constants, $K^w_{f-g,\beta}(\lambda) \lesssim K^w_{f,\beta}(2\lambda) + K^w_{g,\beta}(2\lambda)$, and the result follows.

For (iii) let us consider $f \in \mathcal{F}L^\alpha$, we have

$$\begin{aligned} |T^w_{s,t}(\pi_{\leqslant N}f)(x) - T^w_{s,t}(\pi_{\leqslant N+M}f)(x)| &= |T^w_{s,t}(\pi_{\leqslant N}f - \pi_{\leqslant N+M}f)(x)| \\ &\lesssim C_{s,t,x} N_\alpha(\pi_{\leqslant N}f - \pi_{\leqslant N+M}f) \log^{1/2} K^w_\beta(\lambda_0) \end{aligned}$$

and as the sequence $(\pi_{\leqslant N}f)_N$ converges in $\mathcal{F}L^\alpha$ the limit also exists, and of course it is the same. Furthermore, for two functions in $\mathcal{F}L^{\alpha\vee 0}$,

$$|T^w_{s,t}(\pi_{\leqslant N}f)(x) - T^w_{s,t}(\pi_{\leqslant N+M}f)(x)| \lesssim_{s,t,x} N_\alpha(f-g)$$

and the limiting procedure in the $\mathcal{F}L^\alpha$ case is correct. □

Remark 2.4.8. The definition of $T^w f$ given above seems to depend on the choice of the Littlewood-Paley decomposition $(\Delta_i)_i$. It is indeed the fact. When we will consider w being a stochastic process, this will lead us to a choice of a version of this averaging process defined almost surely. In fact, if $(\widetilde{\Delta}_i)_i$ is an other sequence of Littlewood–Paley operators, and $\tilde{K}_i$ is the associated integral kernels, we have

$$\begin{aligned} |K_j| * \log(1+|.|)(x) &\leqslant \log(1+|x|) + \int_{\mathbb{R}^d} \mathrm{d}y |\tilde{K}_j(y)| |\log(1+|x-y|) - \log(1+|x|)| \\ &\lesssim \log(1+|x|) + 2^{-j} \int_{\mathbb{R}^d} |2^j y| 2^{jd} \tilde{K}(2^j y) \mathrm{d}y \\ &\lesssim 1 + \log(1+|x|) \end{aligned}$$

Hence, for all $\alpha > -\beta$ and for all $\varepsilon > 0$ small enough, we have

$$\begin{aligned} |(T^w_{s,t}\tilde{\Delta}_j f)(x)| &\leqslant \sum_{i\sim j} |(\tilde{\Delta}_j T^w \Delta_i f)(x)| \\ &\lesssim 2^{-j\varepsilon} \|f\|_\alpha |t-s|^\gamma \sum_{i\sim j} \int_{\mathbb{R}^d} \tilde{K}_j * (1 + \log(1+|.|) + K^w_{f,\beta}(\lambda_0))(x) \\ &\lesssim 2^{-j\varepsilon} \|f\|_\alpha (1 + \log(1+|x|) + K^w_{f,\beta}(\lambda_0)) \end{aligned}$$

which gives the convergence of $T^w \tilde{\pi}_{\leqslant N} f$ to a limit we called $T^{w,\tilde{\Delta}} f$, and with the same stochastic constant $K^w_{f,\beta}(\lambda_0)$.

In order to apply the results the section related to the Young integral, it is necessary to have a better understanding of the space regularity of an average function. Thanks to the property of the operator T^w, as soon as we ask f is regular, this regularity will holds. Furthermore the definition of T^w allows us to differentiate it whenever f is regular enough, and the constant are finite. Namely we have the following propositions.

Proposition 2.4.9. *Let $\nu \in [0,1]$, $\alpha > -\beta$, and $f \in \mathcal{F}L^{\alpha+\nu}$ (respectively in $\mathcal{C}^{\alpha+\nu}$). Furthermore we suppose that there exists $\lambda_0 > 0$ such that $K^w_\beta(\lambda_0) < \infty$ (respectively $K^w_{\nabla f,\beta}(\lambda_0) < +\infty$). Then $T^w f \in \mathcal{C}^{\gamma,\nu}_b$ (respectively $T^w f \in \mathcal{C}^{\gamma,\nu,\psi}$ where $\psi(r) = 1 + \log^{1/2}(1+r)$ and the following bounds hold*

$$|T^w_{s,t} f(x) - T^w_{s,t} f(y)| \lesssim N_{\alpha+\nu}(b) |t-s|^\gamma |x-y|^\nu (1 + \log^{1/2} K^w_\beta(\lambda_0))$$

(respectively)

$$|T^w_{s,t}f(x) - T^w_{s,t}f(y)| \lesssim |t-s|^\gamma |x-y|^\nu \|f\|_{\alpha+\nu}(\psi(|x|+|y|) + \log^{1/2} K^w_{\nabla f,\beta}(\lambda_0) + \log^{1/2} K^w_{f,\beta}(\lambda_0))$$

Proof. For $f \in \mathcal{C}^{\alpha+\nu}$, and for all $\beta < \alpha' < \alpha$

$$\begin{aligned}
&|T^w \Delta_i f(x) - T^w \Delta_i f(y)| \\
&= \left| \int_0^1 \nabla T^w \Delta_i f(r(x-y)+y).(x-y)\mathrm{d}r \right| \\
&\leqslant |x-y| \sup_{r\in[0,1]} |T^w \Delta_i \nabla f(r(x-y)+y)| \\
&\lesssim 2^{i\alpha} \|\Delta_i \nabla f\|_\infty |x-y||t-s|^\gamma (1 + \sup_{r\in[0,1]} \log^{1/2}(1+|r(x-y)+y|) + K^w_{\nabla f,\beta}(\lambda_0)) \\
&\lesssim 2^{i(\alpha+1)} \|\Delta_i f\|_\infty |x-y||t-s|^\gamma (1 + \log^{1/2}(1+|x|+|y|) + K^w_{\nabla f,\beta}(\lambda_0))
\end{aligned}$$

Furthermore, we also have

$$|T^w \Delta_i f(x) - T^w \Delta_i f(y)| \lesssim 2^{i\alpha} \|\Delta_i f\|_\infty |t-s|^\gamma (1 + \log^{1/2}(1+|x|) + \log^{1/2}(1+|y|) + K^w_{f,\beta}(\lambda_0)),$$

and by interpolation we have the bound for $T^w \Delta_i f$. The argument of Theorem 2.4.7 gives us the result. A similar argument holds when $f \in \mathcal{F}L^\alpha$. ☐

The next proposition shows that the definition of the averaging operator T is compatible with the space differential in the Hölder spaces.

Proposition 2.4.10. *Let $r \in \mathbb{N}^d$, $|r| = r_1 + \ldots . r_d$, and $\alpha > -\beta$. Suppose furthermore that for $f \in \mathcal{C}^{\alpha+|r|}$ there exists λ_0 such that $K^w_{D^{|r|}f,\lambda_0}(\lambda_0) < +\infty$ (respectively there exists $\lambda_0 > 0$ suc that $K^w_\beta(\lambda) < +\infty$. Then the derivative $\partial^r T^w f$ is well defined and we have $\partial^r T^w f = T^w \partial^r f$.*

Proof. First, let us take $f \in \mathcal{F}L^{\alpha+r}$. For $N \geqslant 0$, the projection $\pi_{\leqslant N}$ is a convolution operator, hence

$$\partial^r T^w(\pi_{\leqslant N} f)(x) = T^w \pi_{\leqslant N}(\partial^r f)(x).$$

But for $f \in \mathcal{F}L^{\alpha+|r|}$, $\pi_{\leqslant N}(\partial^r f) \to^{\mathcal{F}L^\alpha} \partial^r f$, which gives the result for $f \in \mathcal{F}L^\alpha$. Now take $-\beta < \alpha' < \alpha$. We know that for all $f \in \mathcal{C}^{\alpha+r}$, $\pi_{\leqslant N}\partial^r f \to^{\mathcal{C}^{\alpha'}} \partial^r f$, and the result follows for $f \in \mathcal{C}^\alpha$. ☐

2.5 Averages along Fractional Brownian paths.

2.5.1 Fractional Brownian motion case.

The results of Lemmas 2.4.4 and 2.4.5 show that in order to control the irregularity constant of fBm paths it is enough to prove that the there exists $\lambda > 0$ and $\alpha \in \mathbb{R}$ such that the random variable $e^{\lambda \|B^H\|^2_\infty} K^{B^H}_\alpha(\lambda)$ is almost surely finite when W is a continuous random path with the law of the fBm. Then we only have to consider the following two quantities :

$$\exp(\lambda \|W\|^2_\infty)$$

$$\exp(\lambda(1+|\xi|)^\alpha |\Phi^W_{s,t}(\xi)|^2/|t-s|).$$

If the expectation of those quantities are bounded independently of s, t, x, ω, ξ then the expectations of $K_f^{B^H}$ and $e^{\lambda \|B^H\|_\infty} K_\alpha^{B^H}(\lambda)$ are finite and the variable are finite almost surely. For $\exp(\lambda \|B^H\|_\infty^2)$ it is an application of the well known theorem due to Fernique

Theorem 2.5.1. *Let X be a Gaussian process on a Banach space $(\mathcal{B}, \|.\|)$. Then there exists a constant $\mu > 0$ such that*

$$\mathbb{E}[\exp(\mu \|X\|^2)] < +\infty$$

Remark 2.5.2. This holds for the fractional Brownian motion of Hurst parameter $H \in (0,1)$ and for the Banach spaces $\left(C^{H-\varepsilon}([0,1], \mathbb{R}^d), \|.\|_{0,H-\varepsilon}\right)$ and for $\left(C([0,1], \mathbb{R}^d), \|.\|_{\infty,[0,1]}\right)$.

To control the square exponential integrability of $\Phi_{s,t}^{B^H}(\xi)$ we devised a novel technique based on an elementary application of Hoeffding inequality for discrete martingale increments. This bypasses the explicit Gaussian computations or the computations based on Malliavin calculus usual in the studies involving the fBm. The following theorem then gives general estimates for additive functionals of the fBm of the form

$$\int_s^t f_u(B_u^H) \mathrm{d}u$$

where $f : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ is a measurable and bounded function. Note that the following theorem suggest in general that such functionals have the same Gaussian deviation behavior of Brownian martingales.

Theorem 2.5.3. *Let $B^H = (B^{H,(1)}, \ldots, B^{H,(d)})$ a d-dimensional fractional Brownian motion of Hurst parameter $H \in (0,1)$ and let f be a function bounded by 1 and such that*

$$C_f := \sup_u \int_0^\infty |P_{t^{2H}} f_u|_\infty \mathrm{d}t < +\infty$$

where P is the heat kernel on $\mathbb{R}^d$. Then for $\mu > 0$ small enough independent of f we have

$$\sup_{t \neq s} \mathbb{E}\left[\exp\left(\mu \left|\int_s^t f_u(B_u^H) \mathrm{d}u\right|^2 / (|t-s| C_f)\right)\right] < +\infty.$$

Proof. The fBm B^H can be represented as a stochastic integral over a d-dimensional standard Brownian motion $W = (W^{(1)}, \cdots, W^{(d)})$ defined on the whole $\mathbb{R}$ (with $W_0 = 0$):

$$B_u^{H,(i)} = \int_{-\infty}^u (K(u,r) - K(0,r)) \mathrm{d}W_r^{(i)},$$

where $K(u,r) = (u-r)_+^{H-1/2} / \Gamma(H+1/2)$. We let $(\mathcal{F}_t)_{t \in \mathbb{R}}$ be the natural filtration of $(W_t)_{t \in \mathbb{R}}$. For $v \leqslant u$ we have the decomposition

$$\begin{aligned} B_u^{H,(i)} &= \int_{-\infty}^u (K(u,r) - K(0,r)) \mathrm{d}W_r^{(i)} \\ &= \int_v^u K(u,r) \mathrm{d}W_r^{(i)} + \int_{-\infty}^v (K(u,r) - K(0,r)) \mathrm{d}W_r^{(i)} \\ &= W_{u,v}^{1,(i)} + W_{u,v}^{2,(i)} \end{aligned}$$

where the random variable $W^{1,(i)}_{u,v}$ is independent of $\mathcal{F}_v$ and $W^{2,(i)}_{u,v}$ is $\mathcal{F}_v$ measurable. We define $W^j = (W^{j,(1)}, \ldots, W^{j,(d)})$ for $j \in \{1,2\}$ and in the following we will note abusively

$$\mathrm{Var}(W^j) = \mathrm{Var}(W^{j,(1)}).$$

Now we have two cases we have to consider. Suppose first that $(t-s)/C_f \leqslant 1$. Then

$$\left|\int_s^t f_u(B^H_u)\mathrm{d}u\right| \leqslant |t-s|^2 \leqslant |t-s|C_f$$

and the result follows in that case. Suppose now that $|t-s|C_f^{-1} \geqslant 1$. Let $N \in \mathbb{N}$ to be specify later. For $n \in \{0, \ldots, N\}$, let us define $t_n = s + (t-s)n/N$ and

$$Z_n \quad = \quad \mathbb{E}\left[\int_s^t f_u(B^H_u)\mathrm{d}u|\mathcal{F}_{t_{n+1}}\right] - \mathbb{E}\left[\int_s^t f_u(B^H_u)\mathrm{d}u|\mathcal{F}_{t_n}\right].$$

Thanks to the previous decomposition of the fractional Brownian motion, we are able to bound Z_n and to apply Hoeffding's Lemma to the sum of the martingale increments $(Z_n)_{1\leqslant n\leqslant N}$. Let $S_N = \sum_{n=0}^{N-1} Z_n$, then

$$\int_s^t f_u(B^H_u)\mathrm{d}u = S_N + \mathbb{E}\left[\int_s^t f_u(B^H_u)\mathrm{d}u\middle|\mathcal{F}_s\right]. \tag{2.11}$$

Let us first estimate the conditional expectation in eq. (2.11) : for all $u \geqslant 0$ we get

$$\begin{aligned}\left|\mathbb{E}\left[\int_s^t f_u(B^H_u)\mathrm{d}u\middle|\mathcal{F}_s\right]\right| &= \left|\int_s^t P_{Var(W^1_{u,s})} f_u(W^2_{u,s})\mathrm{d}u\right| \\ &\leqslant \int_s^t |P_{\mathrm{Var}(W^1_{u,s})} f_u|_\infty \mathrm{d}u \leqslant C_f < +\infty\end{aligned}$$

since $\mathrm{Var}(W^1_{u,s}) = C(u-s)^{2H}$. But we also have the trivial bound

$$\left|\mathbb{E}\left[\int_s^t f_u(B^H_u)\mathrm{d}u\middle|\mathcal{F}_s\right]\right| \leqslant |t-s|$$

hence

$$\left|\mathbb{E}\left[\int_s^t f_u(B^H_u)\mathrm{d}u\middle|\mathcal{F}_s\right]\right| \leqslant |t-s|^{1/2} C_f^{1/2}. \tag{2.12}$$

Next we bound Z_n by decomposing it into three pieces which are easier to estimate. Let

$$\begin{aligned} U_n &= \int_{t_n}^t \mathbb{E}[f_u(B^H_u)|\mathcal{F}_{t_n}]\mathrm{d}u = \int_{t_n}^t \mathbb{E}[f_u(W^1_{u,t_n} + W^2_{u,t_n})|\mathcal{F}_{t_n}]\mathrm{d}u \\ &= \int_{t_n}^t P_{\mathrm{Var}(W^1_{u,t_n})} f_u(W^2_{u,t_n})\mathrm{d}u \\ &= \int_{t_n}^{t_{n+1}} P_{\mathrm{Var}(W^1_{u,t_n})} f_u(W^2_{u,t_n})\mathrm{d}u + \int_{t_{n+1}}^t P_{\mathrm{Var}(W^1_{u,t_n})} f_u(W^2_{u,t_n})\mathrm{d}u. \end{aligned}$$

Hence

$$Z_n = \int_{t_n}^{t_{n+1}} f_u(B_u^H)\mathrm{d}u + U_{n+1} - U_n,$$

moreover

$$|U_n| \leqslant \int_{t_n}^{t} |P_{\mathrm{Var}(W^1_{u,t_n})} f_u(W^2_{u,t_n})|\mathrm{d}u \leqslant \int_{t_n}^{t} |P_{\mathrm{Var}(W^1_{u,t_n})} f_u|_\infty \mathrm{d}u \leqslant C_f < +\infty$$

and of course $|\int_{t_n}^{t_{n+1}} f_u(B_u^H)\mathrm{d}u| \leqslant (t-s)/N$ which implies that $|Z_n| \lesssim (t-s)/N + C_f$. By the standard Hoeffding inequality we obtain

$$\mathbb{P}(|S_N| > \lambda) \lesssim \exp(-2\lambda^2/((t-s)N^{-1/2} + N^{1/2}C_f)^2)$$

Hence for $0 \leqslant \nu < 1$, we have

$$\mathbb{E}[\exp(2\nu|S_N|^2/((t-s)N^{-1/2} + N^{1/2}C_f)^2)] \lesssim \nu/(1-\nu) + 1$$

Now, we can chose $N = [1 + |t-s|/C_f]$, hence

$$((t-s)N^{-1/2} + N^{1/2}C_f) \lesssim |t-s|C_f,$$

and thanks to (2.12) we have

$$\mathbb{E}\left[\exp\left(\mu\left|\int_s^t f_u(B_u^H)\mathrm{d}u\right|^2/(|t-s|C_f)\right)\right] \lesssim \mathbb{E}[\exp(C\mu|S_N|^2/(|t-s|C_f))] \lesssim_\mu 1$$

□

As an immediate corollary we have the wanted result for the ρ-irregularity constant for the fractional Brownian motion.

Corollary 2.5.4. *For λ small enough,*

$$\mathbb{E}\exp(\lambda(1+|\xi|)^{1/H}|\Phi^{B^H}_{s,t}(\xi)|^2/|t-s|) \leqslant C < +\infty$$

uniformly in ξ, t, s.

Proof. When $\xi \leqslant 1$, we have

$$\exp(\lambda(1+|\xi|)^{1/H}|\Phi^{B^H}_{s,t}(\xi)|^2/|t-s|) \leqslant \exp(\lambda 2^{1/H}).$$

For $|\xi| \geqslant 1$, we have $(1+|\xi|)^{1/H} \lesssim |\xi|^{1/H}$. But we also have

$$\begin{aligned} |P_{t^{2H}} f(x)| &= |\mathbb{E}[\exp(i\xi(x + B_t^H))]| \\ &= \exp(-|\xi|^2 t^{2H}/2), \end{aligned}$$

hence

$$C_{f_\xi} \sim |\xi|^{-1/H}.$$

Finally there exists a constant $C > 0$ such that

$$\mathbb{E}\exp(\lambda(1+|\xi|)^{1/H}|\Phi^{B^H}_{s,t}(\xi)|^2/|t-s|) \leqslant \mathbb{E}\exp(C\lambda|\Phi^{B^H}_{s,t}(\xi)|^2/(|t-s|C_{f_\xi}))$$

and for λ small enough, thanks to Theorem 2.5.3 the RHS is bounded by a constant independent of ξ, s, t. □

We are now in condition to prove Theorem 2.1.4 (ρ-irregularity of the fBm paths for all $\rho < 1/2H$).

Proof. (of Theorem 2.1.4) By Lemma 2.4.4 we have

$$\|\Phi^{B^H}\|_{\mathcal{W}_1^{\rho,\gamma}} = \sup_{\xi \in \mathbb{R}^d} \sup_{0 \leqslant s < t \leqslant 1} (1+|\xi|)^\rho \frac{|\Phi_t^{B^H}(\xi) - \Phi_s^{B^H}(\xi)|}{|s-t|^\gamma} \lesssim 1 + \log^{1/2}(e^{\lambda\|B^H\|_\infty} K_{1/2H}^{B^H}(\lambda)).$$

Moreover by Theorem 2.5.1 the quantity $e^{\lambda\|B^H\|_\infty^2}$ is almost surely finite and by Corollary 2.5.4 we readily have that

$$\mathbb{E}[K_{1/2H}^{B^H}(\lambda)] < +\infty$$

as soon as λ is small enough. □

Remark 2.5.5. A byproduct of the proof of Theorem 2.1.4 is that the irregularity constant $\|\Phi^{B^H}\|_{\mathcal{W}_1^{\rho,\gamma}}$ is exponentially square integrable, as easily shown: for small $\lambda > 0$ and all $\gamma > 1/2$ and $\rho < 1/2H$ we have

$$\mathbb{E}\left[e^{\lambda\|\Phi^W\|^2_{\mathcal{W}_1^{\rho,\gamma}}}\right] < +\infty.$$

When we consider the spaces $\mathcal{C}^\alpha$ instead of $\mathcal{F}L^\alpha$,the ρ-irregularity of the fractional Brownian path is not enough. Nevertheless Theorem 2.5.3 allows us to give the correct bound for $T^{B^H}\Delta_i f$ in order to define $T^{B^H} f$ for any $f \in \mathcal{C}^\alpha$.

Corollary 2.5.6. *Let B^H a d-dimensional fractional Brownian motion of Hurst parameter $H \in (0,1)$. There exists $\lambda > 0$ such that for $f \in C([0,T]; \mathcal{S}'(\mathbb{R}^d))$, all $i \geqslant -1$*

$$\sup_{x \in \mathbb{R}^d, i \geqslant -1, s \neq t} \mathbb{E}[\exp(\lambda 2^{i/H} |T^w(\Delta_i f)(x)|^2 / (|t-s| \|\Delta_i f\|_\infty^2))] < +\infty$$

Proof. The function $g_u = \Delta_i f_u / \|\Delta_i f\|_\infty$ is bounded by 1. Furthermore, for $i \geqslant 0$, $\operatorname{supp} \hat{g}_u \subset 2^i \mathcal{A}$ where $\mathcal{A}$ is an annulus. Hence by the Lemma 2.4 page 54 of [5], there exists a constant $c > 0$ independent of g such that

$$\|P_{t^{2H}} g_u\|_\infty \lesssim \exp(-ct^{2H} 2^{2i})$$

hence $C_g \lesssim 2^{-i/H}$ and the result follows immediately by applying Theorem 2.5.3 to g.

When $i = -1$, we have

$$\left|\int_s^t \Delta_{-1} f(B_u^H) \mathrm{d}u\right|^2 \leqslant 2^{1/H} |t-s| \|\Delta_{-1} f\|_\infty^2$$

and the result follows. □

The result is exactly the wanted hypothesis of Theorem 2.4.7, but also Propositions 2.4.9 and 2.4.10, depending on the regularity of f. Hence, the averaging operator T^{B^H}, or its finite dimensional marginals depending of the space, is well defined and has the right range for applying results of the Section 2.3. As this operator is defined almost surely, almost surely (depending of b when $b \in \mathcal{C}^\alpha$) the following equation has a solution

$$\theta_u = \theta_0 + \int_0^t T_{\mathrm{d}u}^W b(\theta_u)$$

when $b \in \mathcal{C}^{\alpha}$, and Theorem 2.3.9 which guarantees this existence. Furthermore, when $b \in \mathcal{C}^{\alpha+2}$, $\alpha > -1/2H$ (or $\mathcal{C}^{\alpha+\gamma+1}$) there is uniqueness and the flow is Lipschitz-continuous, thanks to the Corollary 2.3.18. As the set where $T^w b$ is not defined does not depend on b when $b \in \mathcal{F}L^{\alpha}$, those results does not depend on b. The uniqueness for $b \in \mathcal{C}^{\alpha+1}$ or $b \in \mathcal{F}L^{\alpha+1}$ is a more probabilistic argument and is the subject of the following section.

Remark 2.5.7. If the regularity of b is not enough to guarantee uniqueness by the above arguments the solution constructed via Theorem 2.3.9 lacks, a priori, measurability with respect to W. If a measurable solution is needed the fix–point argument of Theorem 2.3.9 has to be repeated in a space of random processes, for example in $L^p(\Omega, C^{\gamma}([0,1];\mathbb{R}^d))$.

2.5.2 Averaging for absolutely continuous perturbations of the fBm

In this section, we analyse the properties of the averaging operator along a path of the form $B^H + \theta^n$ where θ^n is a solution to the approximate equation $\mathrm{d}\theta^n = T^{B^H}_{\mathrm{d}u} b^n(\theta^n)$. In order to do so we will use a version of the Girsanov theorem for fractional Brownian motion. The results holds of course for the both considered functions spaces $\mathcal{C}^{\alpha+1}$ and $\mathcal{F}L^{\alpha+1}$. We will only give the proof for $b \in \mathcal{C}^{\alpha+1}$, and give some comments for the case $\mathcal{F}L^{\alpha+1}$. Let $(b^n)_{n\geq 1}$ be a sequence of smooth vector fields such that $\|b^n\|_{\alpha} \leqslant C$ uniformly in $n \geq 1$. By a standard fixed point argument, it is well known that the following equation

$$X^n_t = x_0 + \int_0^t b^n_s(X^n_s)ds + B^H_t \tag{2.13}$$

has an adapted solution X^n (to the standard filtration of the fractional Brownian motion).

Here we analyse the averaging constant of X_n and we prove that it satisfy the requirements of Theorem 2.3.17 implying uniqueness of the limit ODE for $b \in \mathcal{C}^{\alpha+1}$ and convergence of X^n to this unique solution. Furthermore, if we consider, as in Section 2.3.2, the averaged translation by $\theta^n = X^n - B^H$, we only have to check the hypothesis of Theorem 2.3.17. Indeed, if θ is solution of

$$\theta_t = \theta_0 + \int_0^t T^{B^H}_{\mathrm{d}u} b(\theta_u)$$

and θ^n is solution of

$$\theta^n_t = \theta_0 + \int_0^t T^{B^H}_{\mathrm{d}u} b^n(\theta^n_u)$$

then $\theta^n + B^H_u = X^n$ and by the averaged translation by θ^n, as $\tau_{\theta^n} T^{B^H} b = T^{X_n} b$. Then $\theta - \theta^n$ is the solution of the following Young equation

$$(\theta - \theta^n)_t = \int_0^t T^{X^n}_{\mathrm{d}u} b(\theta_u - \theta^n_u) + \int_0^t T^{B^H}_{\mathrm{d}u} b^n(\theta^n_u)$$

These considerations are the motivation to introduce the comparison principle based on averaged translations in the proof of uniqueness in Section 2.3.2.

Below we will take advantage of the absolute continuity of the law of X^n w.r.t. the law of the fBm B^H to transfer the averaging properties of the fBM to the stochastic process X^n. This approach is an extension of an observation of Davie [20] to the fBm context.

A drawback of this approach is that the exceptional set will necessarily depend on the initial point x_0 and on the vectorfield b. This prevents to easily apply the uniqueness result to the case of random b and to the analysis of the flow of the ODE.

The computation of the Radon-Nikodym derivative between the law of X^n and the law of B^H will result in a Girsanov transform. For technical reasons we will do this transformation only on a sub interval $[0, T_{Gir}] \subset [0, T]$. For b^n regular enough, and as X^n is regular enough, according to Nualart and Ouknine [67], there exist a Brownian motion W adapted to the filtration associated with B^H and a probability $\mathbb{P}_n$ such that the process $(X_t^n)_{t\in[0,T_{Gir}]}$ is a fractional Brownian motion of Hurst parameter H, where

$$\frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}} = \exp\left(-\int_0^{T_{Gir}} H_t^n.\mathrm{d}W_t - \frac{1}{2}\int_0^{T_{Gir}} |H_t^n|^2 \mathrm{d}t\right)$$

and where for $H \geq \frac{1}{2}$

$$H_t^n = \frac{t^{H-\frac{1}{2}}}{\Gamma(\frac{3}{2}-H)}\left(t^{1-2H}b^n(t, X_t^n) + \left(H - \frac{1}{2}\right)\int_0^t \frac{t^{\frac{1}{2}-H}b_t^n(X_t^n) - s^{\frac{1}{2}-H}b_s^n(X_s^n)}{(t-s)^{H+\frac{1}{2}}}\mathrm{d}s\right)$$

and for $H < \frac{1}{2}$

$$H_t^n = \frac{t^{H-\frac{1}{2}}}{\Gamma(\frac{1}{2}-H)}\int_0^t (s(t-s))^{\frac{1}{2}-H} b_s^n(X_s^n)\mathrm{d}s$$

Thanks to that Girsanov transform, the almost sure bound for $T_b^{B^H}$ can be used to estimate $T_b^{X_n}$ since $\mathbb{P}_n$ and $\mathbb{P}$ are equivalent.

Lemma 2.5.8. *Let $0 \geqslant \alpha > -1/2H$. There exists a constant $\lambda > 0$ and a constant C_λ independent of n such that for all $b \in \mathcal{C}^{H-\varepsilon}\left([0,1]; \mathcal{C}^{\alpha+1}\right) \cap \mathcal{C}^{\alpha+1}\left(\mathbb{R}^d, \mathcal{C}^{H-1/2+\varepsilon}([0,T])\right)$,*

$$\mathbb{E}[K_{b,1/2H}^{X_n}(\lambda)] \leqslant C_\lambda$$

Until the end of the section, we will only consider $K_{b,1/2H}^{X_n}$. For simplicity we only write it as $K_b^{X_n}$.

Proof. Let $K > 0$. By using the notation above

$$\begin{aligned}
\mathbb{E}[K_b^{X_n}(\lambda)]^2 &= \mathbb{E}_{\mathbb{P}_n}\left[K_b^{X_n}(\lambda)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_n}\right]^2 \\
&\leqslant \mathbb{E}_{\mathbb{P}_n}[K_b^{X_n}(\lambda)^2]\mathbb{E}_{\mathbb{P}_n}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_n}\right)^2\right] \\
&\lesssim \mathbb{E}[K_b^{B^H}(2\lambda)]\mathbb{E}_{\mathbb{P}_n}\left[\exp\left(2\int_0^T H_t^n \mathrm{d}W_t + \int_0^T H_t^n \mathrm{d}t\right)\right]
\end{aligned}$$

Where we have used that under $\mathbb{P}_n$, X^n is a fractional Brownian motion of same Hurst parameter H. If ρ is small enough the first term is finite by the above results. To prove the lemma, it is sufficient to prove that

$$\mathbb{E}_{\mathbb{P}_n}\left[\exp\left(2\int_0^T H_t^n \mathrm{d}W_t + \int_0^T |H_t^n|^2 \mathrm{d}t\right)\right]$$

is bounded by a constant independent of n. As W is a Brownian motion, it is enough to bound

$$\mathbb{E}_{\mathbb{P}_n}\left[\exp\left(\int_0^T |H_t^n|^2 \mathrm{d}t\right)\right]$$

The arguments are quite different depending whether $H > 1/2$ or $H < 1/2$. First suppose that $H < \frac{1}{2}$.

$$\begin{aligned}
|H_t^n|^2 &= \left|K_H^{-1}\left(\int_0^\cdot b_s^n(X_s^n)\mathrm{d}s\right)(t)\right|^2 \\
&= \left|\frac{t^{H-\frac{1}{2}}}{\Gamma(\frac{1}{2}-H)}\int_0^t (s(t-s))^{\frac{1}{2}-H} b_s^n(X_s^n)\mathrm{d}s\right|^2 \\
&= \left|\frac{-\left(\frac{1}{2}-H\right)t^{H-\frac{1}{2}}}{\Gamma(\frac{1}{2}-H)}\int_0^t (s(t-s))^{-(\frac{1}{2}+H)}(t-2s)\underbrace{\int_0^s b_u^n(X_u^n)\mathrm{d}u}_{(T_s^{X^n}b^n)(0)}\mathrm{d}s\right|^2 \\
&\lesssim t^{2H}\int_0^t (s(t-s))^{-(1+2H)}|t-2s||(T_s^{X^n}b^n)(0)|^2\mathrm{d}s \\
&\lesssim \|b^n\|_{\mathcal{C}^\alpha}^2(1+\log^{1/2}(K_b^{X_n}(\lambda)))^2 t^{2H}\int_0^t (s(t-s))^{-(1+2H)}|t-2s|s^{2\gamma}\mathrm{d}s \\
&\lesssim \|b^n\|_{\mathcal{C}^\alpha}^2(1+\log(K_b^{X_n}(\lambda)))t^{2(\gamma-H)}\int_0^1 (u(1-u))^{-(1+2H)}|1-2u|u^{2\gamma}\mathrm{d}s \\
&\leqslant C(b,H,\gamma,\lambda)(1+\log(K_b^{X_n}(\lambda)))
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_n}\left[\exp\left(\int_0^T |H_t^n|^2\mathrm{d}t\right)\right] &\lesssim \mathbb{E}_{\mathbb{P}_n}[K_b^{X_n}(C\lambda)] \\
&\lesssim \mathbb{E}[K_b^{B^H}(C\lambda)]
\end{aligned}$$

For λ small enough, this quantity is bounded, and the Lemma is proved in this case.

For $H \geqslant \frac{1}{2}$, b^n is $(1+\alpha)$-Hölder continuous and $\|b^n\|_{\mathcal{C}^{\alpha+1}} \lesssim \|b\|_{\mathcal{C}^{\alpha+1}}$. Furthermore, since $|(b^n(0,0))_n|$ is uniformly bounded we have

$$\begin{aligned}
|H_t^n| &\lesssim \left|t^{H-1/2}\left(t^{1-2H}b^n(t,X_t^n) + \int_0^t \frac{t^{1/2-H}b_t^n(X_t^n) - s^{1/2-H}b_s^n(X_s^n)}{(t-s)^{H+1/2}}\mathrm{d}s\right)\right| \\
&\lesssim t^{1/2-H}(\|b^n\|_{\alpha+1} + \|b_\cdot^n(0)\|_\infty)(|X_t^n - X_0^n|^{1+\alpha}+1) \\
&\quad + t^{H-1/2}\int_0^t (t-s)^{-(H+1/2)}|(t^{1/2-H} - s^{1/2-H})(b_t^n(X_t^n) + b_s^n(X_s^n))|\mathrm{d}s \\
&\quad + t^{H-1/2}\int_0^t |(t-s)^{-(H+1/2)}(t^{1/2-H} + s^{1/2-H})(b_t^n(X_t^n) - b_s^n(X_s^n))|\mathrm{d}s
\end{aligned}$$

The first term is bounded by $(\|b^n\|_\alpha + \|b^n_\cdot(0)\|_\infty)(\|X^n\|_{H-\varepsilon}^{1+\alpha} + 1)t^{1/2-H}$ and is integrable. The second is bounded by

$$t^{H-1/2}\int_0^t \mathrm{d}s(t-s)^{-(H+1/2)}|(t^{1/2-H}-s^{1/2-H})(|b^n_t(X^n_t)-b^n(t,0)|+|b^n_s(X^n_s)-b^n_t(0)|+2\|b_\cdot(0)\|_\infty)|$$

$$\lesssim t^{H-1/2+1-H-1/2+1/2-H}\int_0^1 (1-u)^{-H-1/2}(1-u^{1/2-H})(\|b^n\|_\alpha + \|b^n_\cdot(0)\|_\infty)\|X^n\|_\infty^{\alpha+1}$$
$$\lesssim_{x_0} t^{1/2-H}((\|b^n\|_\alpha + \|b^n_\cdot(0)\|_\infty)(\|X^n\|^{\alpha+1}_{H-\varepsilon}+1) + \|b_\cdot(X^n_s)\|_{H-1/2+\varepsilon}|t-s|^{H-1/2+\varepsilon})$$

The third term is bounded by

$$t^{H-\frac{1}{2}}\int_0^t \left|(t-s)^{-(H+\frac{1}{2})}s^{\frac{1}{2}-H}(|b^n_t(X^n_t)-b^n_t(X^n_s)|+|b^n_t(X^n_s)-b^n_s(X^n_s)|)\right|\mathrm{d}s$$
$$\lesssim (\|b^n\|_\alpha + \|b(.,0)\|_{H-1/2+\varepsilon})(\|X^n\|^{1+\alpha}_{H-\varepsilon}+1)t^{H-\frac{1}{2}}$$
$$\times \int_0^t (t-s)^{-(H+\frac{1}{2})}s^{\frac{1}{2}-H}\left(|t-s|^{H-\frac{1}{2}+\varepsilon}+|t-s|^{(\alpha+1)(H-\varepsilon)}\right)\mathrm{d}s$$
$$\lesssim (\|b^n\|_\alpha + \|b^n_\cdot(0)\|_{H-1/2+\varepsilon} + \|b^n_\cdot(0)\|_\infty)(\|X^n\|^{1+\alpha}_{H-\varepsilon}+1)t^\varepsilon \int_0^1 (1-u)^{-1+\varepsilon}u^{-H+1/2}\mathrm{d}u$$

Finally, we choose b^n such that $(\|b^n\|_\alpha + \|b^n_\cdot(0)\|_{H-1/2+\varepsilon} + \|b^n_\cdot(0)\|_\infty) \lesssim_b 1$

$$|H^n_t| \lesssim C_{b,x_0}(\|X^n\|^{1+\alpha}_{H-\varepsilon}+1)t^\varepsilon$$

But under $\mathbb{P}_n$, X^n is a fractional Brownian motion of Hurst parameter H. Thanks to Fernique theorem, $\|X^n\|^{1+\alpha}_{H-\varepsilon}$ is exponentially integrable in $\mathbb{P}_n$ and the result follows. □

This bound in L^1 is not enough to use the Theorem 2.3.17, as we need an almost surely, uniformly in n bound for $\|T^{X^n}b\|_{\mathcal{C}^{\gamma,1,\psi}}$. Nevertheless, using the results of section 2.4, we already know that

$$\exp\left(C\|T^{X^n}b\|_{\mathcal{C}^{\gamma,1,\psi}}\right) \lesssim 1 + K_b^{X^n}(\lambda)$$

We have all the tools to prove the following theorem

Theorem 2.5.9. *Assume that $b \in \mathcal{C}^{\alpha+1}$. There then there exists $\lambda > 0$ and sequence of smooth vectorfields $(b^n)_n$ such that $b^n \to b$ in $\mathcal{C}^{\alpha'}$ for all $\alpha' < \alpha$ and almost surely*

$$K_b^{X_n}(\lambda)\|b-b^n\|_{\alpha'+1} \to 0$$

which implies uniqueness of the Young equation for b by Theorem 2.3.17.

Proof. By the previous result we have that the L^1 norm of $K_b^{X_n}(\lambda)$ is uniformly bounded in n. Moreover consider b^n such that $b^n = \rho_n * b$ where $\rho_n(x) = \frac{1}{2}n^d\exp(-n|x|)$. Then b^n is smooth, $b^n \to b$ in $\mathcal{C}^{1+\alpha'}$ for all $\alpha' < \alpha$ by the dominated convergence theorem and there exists a subsequence which will still denote with b^n such that $\|b-b^n\|_{\alpha'+1} \lesssim n^{-2}$. On this subsequence (which depends on b) consider the random variable

$$D = \sum_{n\geq 1} K_b^{X_n}(\lambda)\|b-b^n\|_{\alpha'+1}$$

then

$$\mathbb{E}D = \sum_{n\geq 1} \mathbb{E}[K_b^{X_n}(\lambda)] \|b - b^n\|_{\alpha'+1} \lesssim \sum_{n\geq 1} n^{-2} \lesssim 1$$

so that almost surely $D < \infty$ which implies that $K_b^{X_n}(\lambda)\|b - b^n\|_{\alpha'+1} \to 0$ □

Note that this argument give an exceptional set of zero measure which a priori depends on b (and on the sequence $(b^n)_n$) and of x_0. As remarked previously, this fact prevents straightforward extension of the uniqueness results in $\mathcal{C}^\alpha$ to random b. Furthermore, it also prevent to consider the regularity of the flow of the equation by path-wise methods.

Chapitre 3

Équation de transport linéaire rugueuse

Résumé

Nous nous intéressons à l'équation de transport linéaire

$$\frac{\partial}{\partial t}u(t,x) + b(t,x)\cdot\nabla u(t,x) + \nabla u(t,x)\cdot\frac{\partial}{\partial t}X(t) = 0, \qquad u(0,x) = u_0(x)$$

où b est un champs de vecteur assez irrégulier et X est une perturbation höldérienne. En utilisant la théorie des chemins rugueux contrôlés nous donnons un sens à la formulation faible de l'équation et la résolvons pour des champs de vecteurs réguliers. Lorsque X est un mouvement brownien fractionnaire nous montrons un phénomène de régularisation par le bruit, qui permet de considérer des champs de vecteurs plus irréguliers.

Abstract

We study the linear transport equation

$$\frac{\partial}{\partial t}u(t,x) + b(t,x)\cdot\nabla u(t,x) + \nabla u(t,x)\cdot\frac{\partial}{\partial t}X(t) = 0, \qquad u(0,x) = u_0(x)$$

where b is a vectorfield of limited regularity and X a vector-valued Hölder continuous driving term. Using the theory of controlled rough paths we give a meaning to the weak formulation of the PDE and solve that equation for smooth vectorfields b. In the case of the fractional Brownian motion a phenomenon of regularization by noise is displayed.

Sommaire

3.1 Introduction

In this work we study the driven linear transport equation

$$\frac{\partial}{\partial t}u(t,x) + b(t,x)\cdot\nabla u(t,x) + \nabla u(t,x)\cdot\frac{\partial}{\partial t}X(t) = 0, \qquad u(0,x) = u_0(x) \tag{3.1}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is a sufficiently smooth vectorfield and $X : [0,T] \to \mathbb{R}^d$ a vector-valued driving term (which we always take such that $X_0 = 0$). We consider solutions u and initial conditions u_0 which are only bounded functions of space so that the transport equation has to be understood in the weak sense with respect to the spatial variable.

When $X = 0$, Di Perna and Lions [27] showed that when $b \in L^1([0,T]; W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d))$ with linear growth condition and $\operatorname{div} b \in L^1([0,T] \times \mathbb{R}^d)$, a unique L^∞ weak solution exists. Furthermore a wide range of results follows this one, a partial survey can be found in [2].

Under weaker condition on regularity of the vectorfield b, the equation is known to be ill-posed. Despite of that, Flandoli, Gubinelli and Priola [30, 31] showed that when $b \in C([0,T]; \mathcal{C}^\alpha_b(\mathbb{R}^d))$ and $\operatorname{div} b \in L^p([0,T] \times \mathbb{R}^d)$ for $p \geqslant 2$, adding a Brownian perturbation X allows to maintain uniqueness of L^∞ solutions which are strong in the probabilistic sense. In that case the driving term in eq. (3.1) has to be understood as a Stratonovitch integral against the Brownian motion X. Their result is based on the regularization effect of the Brownian perturbation on the flow of the characteristic equation

$$\Phi_t(x) = x + \int_0^t b_q(\Phi_q(x))\mathrm{d}q + X_t, \qquad x \in \mathbb{R}^d, t \geqslant 0. \tag{3.2}$$

The key estimates on this regularization effect depends quite heavily on stochastic calculus techniques and so breaks down easily if we try to apply the same approach to more general perturbations X, for example a fractional Brownian motion of given Hurst parameter. Note that in a recent paper, Beck, Flandoli Gubinelli and Maurelli [6] proved directly regularization properties of the Brownian motion on the transport equation with more general vector fields without relying on the flow of caracteristics, but they still have to rely on stochastic calculus tools, in particular the Itô formula for Brownian semi–martingales.

While stochastic calculus is not available for the fractional Brownian motion (fBm), in a recent paper, Catellier and Gubinelli [13] proved that, at the level of the characteristic equation (3.2), the same phenomenon of regularization by noise appears for arbitrary Hurst parameter $H \in (0, 1)$ of the fractional Brownian motion X. In particular, the phenomenon of regularization gets stronger as H gets smaller.

In the perspective of this last result it is then interesting to investigate the regularization by noise phenomenon at the level of the transport equation (3.1). In order to do so we need first to give an appropriate meaning to the transport equation with non-differentiable driver X.

In a more general setting, Lions, Perthame and Souganidis [59] use the entropy solutions and the kinetic formulation of scalar conservation laws to overcome the difficulty given by the rough drivers. On that setting the authors use a one-dimensional irregular path only. For a multidimensional noise, Caruana, Friz and Oberhauser [11] use Lyons' theory of rough paths [60] and the notion of viscosity solutions. These two approaches are based on a common idea. The irregular signal X is approximaded by a family $(X^\varepsilon)_{\varepsilon>0}$ of smooth functions, and a solution u^ε of the approximate equation constructed. Then, using suitable a priori bounds, the authors show that the solutions converge to a function u which is then *defined* to be the solution of the equation. In these approaches the equation is replaced by a limiting procedure and no attempts are made to investigate the equation satisfied by the limiting object.

This way of proceeding is not very useful in the context of the regularization by noise phenomenon we would like to study. Indeed, as soon as X is replaced by a more regular signal X^ε the regularization phenomenon is lost and there is no hope that the approximate equations have unique solutions. So while the existence problem is easier, the uniqueness gets out of reach. In the limit where the regularization is removed one expects to regain uniqueness but then it is the meaning of the equation which is not clear.

In order to have a well–defined setting in which to discuss the existence, uniqueness and regularization effect for transport equations we will follow the recent work of Gubinelli, Tindel and Toricella [46] where they use the theory of controlled rough paths introduced by Gubinelli in [41] to define controlled viscosity solutions of fully non–linear PDEs with driving signals given by a general (step–2) rough path, thus filling the conceptual gap left behind in the approach of Caruana, Friz and Oberhauser [11]. Their approach shows the versatility of the controlled approach to deal with various problems in rough PDE theory.

In order to do so we need a geometric rough path $\boldsymbol{X}$ and we will define a certain class of solutions which are controlled, in a weak sense, by the rough path $\boldsymbol{X}$. Finally, we will show that this notion of solution is an extension of the classical notion of solutions, which enjoys uniqueness in a natural class of vectorfields b and that the regularization properties of X can be used to extend the class of vectorfields for which uniqueness holds, proving for the first time the regularization by noise result for this class of rough PDEs.

Let us remark that Gubinelli and Jara [45] have already used the controlled path approach in a more probabilistic setting in order to prove regularization by noise results for the Kardar–Parisi–Zhang equation.

Denoting with $u_t(\varphi) = \langle u(t,\cdot), \varphi(\cdot)\rangle$ the pairing of u with a smooth test function φ (depending only on the space variable) we can reformulate the PDE as the infinite set of integral equations

$$u_t(\varphi) = u_0(\varphi) + \int_0^t u_s(\operatorname{div}(b_s\varphi))\mathrm{d}s + \int_0^t u_s(\nabla\varphi)\cdot \mathrm{d}\boldsymbol{X}_s \tag{3.3}$$

for all $t \geqslant 0$ and test functions φ in a suitable class. The last integral in the r.h.s. will be understood as a rough integral for the *controlled path* $s \mapsto u_s(\nabla\varphi)$ w.r.t. the rough path $\boldsymbol{X}$.

In the controlled path theory, the idea is to ask the integrand to "look like" the driving term, at least at a first order level. We will ask the pairing of the solutions against test functions to have this property, and this will lead to the following definition.

Definition 3.1.1 (Definition 3.3.1 below)**.** Let $X \in \mathcal{C}^\gamma([0,T])$ and $b \in L^\infty([0,T];\mathrm{Lin}(\mathbb{R}^d))$ with $\operatorname{div} b \in L^\infty([0,T]\times\mathbb{R}^d)$. Here *Lin* denote the space of functions with linear growth, as in definition 3.2.10 below. A function $u \in L^\infty([0,T]\times\mathbb{R}^d)$ is a Weak controlled solution to equation (3.3) with initial condition $u_0 \in L^\infty(\mathbb{R}^d)$ and driving term X if for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ the pairing $u(\varphi)$ is controlled by X in the sense of Definition 3.2.16 and if furthermore Equation (3.3) is fulfilled, where the term $\int_0^t u_s(\nabla\varphi)\cdot\mathrm{d}X_s$ is understood as the controlled rough integral of $u(\nabla\varphi)$ against the rough path $\boldsymbol{X}$.

As it will be shown, this definition is an extension of the classical notion of weak solutions, in the case where X is smooth. Besides we will show that the same phenomenon of regularization by noise appears in the case of the fractional Brownian motion, and we will mostly retrieve the results of Flandoli, Gubinelli and Priola but for a larger class of noises. Moreover as our theory is completely deterministic we will be able to handle random vectorfield b.

This paper is structured as follows. Section 3.2 is devoted to preliminary results: we recall some notation for the involved function spaces and we give a short overview of the theory of controlled rough paths. Then we recall some of the results of [13] for the regularization by non Brownian noise, and finally we give some standard results about flows and the characteristic method. Section 3.3 is devoted to the definition of weak controlled solutions and to the proof of their existence in a general case. Finally in Section 3.4, thanks to a duality method we prove uniqueness for equation (3.3).

3.2 Preliminaries

In this section we gather some material needed below, in particular we define some relevant functional spaces and then we give a short introduction to the theory of controlled rough paths and the controlled integral. For the reader's sake we explicit also some easy estimates of the flow of perturbed ordinary differential equations.

3.2.1 Useful functional spaces

We will use the following function spaces to measure the regularity of the objects we consider.

Notation 3.2.1. In all the following, the notation $D^n f$ is for the n–th Differential of a function f. When $f : \mathbb{R}^d - \mathbb{R}$, $\nabla f = (\partial_1 f, \ldots, \partial_d f)$ is the gradient, and $\nabla^2 f = (\partial_i \partial j f)_{1\leqslant i,j\leqslant d}$ is the Hessien of f. Furthermore when $f : \mathbb{R}^d \to \mathbb{R}^d$ we denote by $\operatorname{Jac} f = \det(Df)$ the Jacobien determinant of f.

Definition 3.2.2. Let (E, d_E) a complete metric linear space and $(F, \|\|_F)$ a Banach space. For $n \in \mathbb{N}$ and $0 < \alpha \leqslant 1$ we define the space of α-Hölder-continuous functions from E to F by

$$\mathcal{C}^{n+\alpha} = \mathcal{C}^{n+\alpha}(E, F) = \left\{ f \in C^n(E, F) : [\![f]\!]_{n+\alpha} = \sup_{x\neq y} \frac{\|D^n f(x) - D^n f(y)\|_F}{d_E(x, y)^\alpha} < +\infty \right\}.$$

The quantity $[\![f]\!]_{n+\alpha}$ is only a semi-norm for the space $\mathcal{C}^{n+\alpha}$. For $x_0 \in E$ the following quantity is a norm such that the space $\mathcal{C}^\alpha$ is complete

$$\|f\|_{x_0,n+\alpha;F} = [\![f]\!]_{n+\alpha} + \sum_{k=0}^{n} \|D^k f(x_0)\|_F.$$

When $x_0 = 0$ we only write $\|f\|_{n+\alpha}$.

Remark 3.2.3. The space $\mathcal{C}^1(E, F)$ is the space of Lipschitz continuous functions from E to F. To avoid confusion with the space of continuously differentiable functions, we will write this space $\operatorname{Lip}(E, F)$.

Notation 3.2.4. When it is not specified, F is always assumed to be the space $\mathbb{R}^d$ and $|.|_F = |.|$ the usual Euclidean norm.

Definition 3.2.5. For E and F as before and $n < \alpha \leqslant n+1$, $f \in \mathcal{C}_b^\alpha(E, F)$ if $f \in \mathcal{C}^\alpha(\mathbb{R}^d)$ and if for all $k \in \{0, \ldots, n\}$ $\|D^k f\|_\infty := \sup_{x\in E} |D^k f(y)| < +\infty$. On this space we consider the norm

$$\|f\|_{\mathcal{C}_b^\alpha} = [\![f]\!]_\alpha + \sum_{k=0}^{n} \|D^k f\|_\infty.$$

Remark 3.2.6. We always have $\|f\|_\alpha \leqslant \|f\|_{\mathcal{C}_b^\alpha}$. When E is bounded (*i.e.* $\sup_{x\neq y} d_E(x, y) < +\infty$) we also have $\|f\|_{\mathcal{C}_b^\alpha} \lesssim \|f\|_\alpha$. In that case, especially when $E = [0, T]$ and $d_E(s, t) = |t - s|$, we will identify the two norms.

Remark 3.2.7. There is a way to extend the space $\mathcal{C}_b^\alpha$ to nonpositive value of α. This is via the Besov spaces $B^\alpha_{\infty,\infty}$, as it can be found in [5] or [72]. When $\alpha < 0$, the results of part 3.2.3 about the flow of the caracteristic equation are still true, see [13]. Nevertheless, the definition of the transport equation for such irregular vectorfields seems to be tricky in that case. The method of Chouk and Gubinelli [14] and [15] does not apply and another definition has to be found. We postpone the analysis of this situation to a further publication.

Definition 3.2.8. Let $\nu \in (0, 1]$, (E, d_E) a complete metric space and $(F, \|\|_F)$ a Banach space. We define the space of ν-Hölder-continuous functions from E^2 to F by

$$\mathcal{C}_2^\nu(E^2, F) = \left\{ f \in C(E^2, F) : f(x, x) = 0 \,\text{and}\, \|f\|_\nu = \sup_{x\neq y} \frac{|f(x, y)|_F}{d_E(x, y)^\nu} < +\infty \right\}.$$

Unlike the case of the space $\mathcal{C}^{\alpha}(E,F)$, $\|.\|_{\nu}$ is a norm on $\mathcal{C}_2^{\nu}$. Finally, we introduce some notations for the usual L^p spaces with image in Banach spaces.

Definition 3.2.9. Let $p \geqslant 1$ and F a Banach space and $T > 0$. We define

$$L^p([0,T],F) = \left\{ b : [0,T] \to F : \|b\|_{p;F} := \left(\int_0^T \|b_u\|_F^p \mathrm{d}u \right)^{1/p} < +\infty \right\}$$

with the usual modification for $p = +\infty$.

In order to have existence of global weak solution for the transport equation in the classical case, the vectorfield must have at most linear growth in the space variable. In order to quantify that, let us define a space of function with linear growth.

Definition 3.2.10. Let $d \geqslant 1$, the space of functions with linear growth is defined as follows

$$\mathrm{Lin}(F) = \left\{ b \,\text{measurable from}\, \mathbb{R}^d \,\text{to}\, \mathbb{R}^d : \|b\|_{\mathrm{lin}} = \left\| \frac{f(.)}{1 + |.|} \right\|_{\infty} < +\infty \right\}.$$

Furthermore $(\mathrm{Lin}(F), \|\|_{\mathrm{Lin}})$ is a Banach space.

Finally let us give three useful notations for the following.

Notation 3.2.11. For a function u of $[0,T]$, we define $\delta u_{s,t} = u_t - u_s$ the increment of u.

Notation 3.2.12. We denote with $u(\varphi) = \langle u(\cdot), \varphi(\cdot) \rangle$ the pairing of u with a smooth test function φ.

Notation 3.2.13. Let $a, b \in \mathbb{R}$. The write $a \lesssim b$ if there exists a constant $C > 0$ independent of a and b such that $a \leqslant Cb$. When $a \lesssim b$ and $b \lesssim a$ we write $a \sim b$. Furthermore, the notation $a \lesssim_c b$ specifies that the constant $C > 0$ depends on c.

3.2.2 Rough path theory in a nutshell

Rough path theory is a way to describe the effects of irregular signals on certain non–linear systems. It has been first developed by Lyons and his coauthors, see for example [60, 62] and the book by Friz and Victoir [34]. In order to use this theory to define integrals againts irregular signals we will use the notion of *controlled paths* developed by Gubinelli in [41]. An enjoyable exposition of this theory can also be found in [32]. When the path is not of finite variation, there is not enough informations to define an integral against its (weak)-derivative. The theory of controlled rough paths overcomes this problem and gives a general setting for the theory of integration against irregular paths.

One of the first quantities we would like to define is the integral of the path against itself. The idea of rough path integration is to presuppose the existence of a first order iterated integral, and to construct a theory of integration related to that enhanced path (the path and its iterated integral). This idea leads us to the following definition

Definition 3.2.14. Let $1/3 < \gamma \leqslant 1/2$. The pair $\boldsymbol{X} = (X, \mathbb{X}^2)$ is a rough path of order γ if $X \in \mathcal{C}^\gamma([0,T], \mathbb{R}^d)$, $\mathbb{X} \in \mathcal{C}_2^{2\gamma}([0,T]^2; \mathcal{M}_d(\mathbb{R}))$ and for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = (X_u - X_s) \otimes (X_t - X_u) = ((X_u^i - X_s^i)(X_t^j - X_u^j))_{0 \leqslant i,j \leqslant d}.$$

Furthermore, we define $\|\boldsymbol{X}\|_{\mathcal{R}^\gamma} = \|X\|_\gamma + \|\mathbb{X}\|_{2\gamma}$ and for two γ-rough paths $\boldsymbol{X}$ and $\boldsymbol{Y}$ we define

$$\|\boldsymbol{X} - \boldsymbol{Y}\|_{\mathcal{R}^\gamma} = d_{\mathcal{R}^\gamma}(\boldsymbol{X}, \boldsymbol{Y}) = \|X - Y\|_\gamma + \|\mathbb{X} - \mathbb{Y}\|_{2\gamma}.$$

The term $\mathbb{X}$ can be understood as the iterated integral of X against itself. Formally we have

$$\mathbb{X}_{s,t} = \int_s^t (X_r - X_s) \otimes \mathrm{d}X_r.$$

In fact, in the last equality, the left hand side is a definition for the right hand side. On the other hand, when X is a smooth path, for example $X \in C^1([0,T])$, we can always define a natural lift $\boldsymbol{X}$ to X by

$$\boldsymbol{X} = (X, \mathbb{X}) \,\text{where}\, \mathbb{X}_{s,t} = \int_s^t (X_r - X_s)\dot{X}_r \mathrm{d}r.$$

In order to approximate irregular rough paths by smooth paths, we define the space of geometric rough paths as the closure of smooth rough paths for the rough path distance. This leads to the following definition

Definition 3.2.15. Let $1/3 < \gamma \leqslant 1/2$, a γ-rough path $\boldsymbol{X}$ is a geometric rough path, and we write $\boldsymbol{X} \in \mathcal{R}^\gamma$ if there exists a sequence $X^\varepsilon \in C^1([0,T]; \mathbb{R}^d)$ such that

$$\|\boldsymbol{X} - \boldsymbol{X}^\varepsilon\|_{\mathcal{R}^\gamma} \to_{\varepsilon \to 0} 0$$

where $\boldsymbol{X}^\varepsilon$ is the natural lift of X^ε to a γ-rough path.

In all the following we will consider only geometric rough paths. A general discussion about rough paths and geometric rough paths can be found in [51].

As in the stochastic calculus setting, where we can integrate progressively measurable processes only, one has to give a structure to the paths we can integrate. In fact, as the integral of X against $\mathrm{d}X$ is already defined by the definition of the rough path $\boldsymbol{X}$, the idea is to consider functions which locally up to the first order look like X. Such functions are called controlled by X and they are defined in the following definition.

Definition 3.2.16. Let $1/3 < \gamma \leqslant 1/2$ and $X \in \mathcal{C}^\gamma([0,T])$. A function $y \in \mathcal{C}^\gamma([0,T])$ is γ-controlled by X, and we write $y \in \mathcal{D}_X^\gamma([0,T])$ if

$$y_t - y_s = y'_s(X_t - X_s) + y^\#_{s,t}$$

with $y' \in \mathcal{C}^\gamma$ and $y^\# \in \mathcal{C}_2^{2\gamma}$. Furthermore, we define the controlled norm of y by

$$\|y\|_{\mathcal{D}_X^\gamma} = \|y\|_\gamma + \|y'\|_\gamma + \|y^\#\|_{2\gamma}.$$

When there is no ambiguity, we will omit the γ and say that y is controlled by X.

The space of controlled paths has a rich algebraic structure. In particular, it is stable by products. Indeed the following estimate holds.

Lemma 3.2.17 (Gubinelli [41]). *Let $a, b \in \mathcal{D}_X^\gamma$ and $\tilde{a}, \tilde{b} \in \mathcal{D}_Y^\gamma$. Then $ab \in \mathcal{D}_X^\gamma$ and $\tilde{a}\tilde{b} \in \mathcal{D}_Y^\gamma$ and*

$$\|ab - \tilde{a}\tilde{b}\|_{\mathcal{C}^\gamma} \leqslant \|a - \tilde{a}\|_\gamma (\|b\|_{\mathcal{D}_X^\gamma} + \|\tilde{b}\|_{\mathcal{D}_Y^\gamma}) + \|b - \tilde{b}\|_\gamma (\|a\|_{\mathcal{D}_X^\gamma} + \|\tilde{a}\|_{\mathcal{D}_Y^\gamma})$$

$$\begin{aligned}\|(ab)' - (\tilde{a}\tilde{b})'\|_\gamma &\leqslant (\|a\|_{\mathcal{D}_X^\gamma} + \|\tilde{a}\|_{\mathcal{D}_Y^\gamma} + \|b\|_{\mathcal{D}_X^\gamma} + \|\tilde{b}\|_{\mathcal{D}_Y^\gamma}) \\ &\quad \times (\|a - \tilde{a}\|_\gamma + \|a' - \tilde{a}'\|_\gamma + \|b - \tilde{b}\|_\gamma + \|b' - \tilde{b}'\|_\gamma)\end{aligned}$$

and

$$\begin{aligned}\|(ab)^\# - (\tilde{a}\tilde{b})^\#\|_{\mathcal{C}^{2\gamma}} &\leqslant (\|b\|_{\mathcal{D}_X^\gamma} + \|\tilde{b}\|_{\mathcal{D}_Y^\gamma})(\|a - \tilde{a}\|_\gamma + \|a^\# - \tilde{a}^\#\|_\gamma) \\ &\quad + (\|a\|_{\mathcal{D}_X^\gamma} + \|\tilde{a}\|_{\mathcal{D}_Y^\gamma})(\|b - \tilde{b}\|_\gamma + \|b^\# - \tilde{b}^\#\|_\gamma)\end{aligned}$$

Proof. We have

$$\begin{aligned}\delta(ab)_{s,t} &= \delta a_{s,t} b_s + a_s \delta b_{s,t} + \delta a_{s,t} \delta b_{s,t} \\ &= a'_s X_{s,t} b_s + a_s b'_{s,t} X_{s,t} + a^\#_{s,t} b_s + a_s b^\#_{s,t} + \delta a_{s,t} \delta b_{s,t}\end{aligned}$$

Hence $(ab)' = a'b + ab'$ and $(ab)^\# = a^\#_{s,t} b_s + a_s b^\#_{s,t} + \delta a_{s,t} \delta b_{s,t}$ and the same equality holds for $\tilde{a}$ and $\tilde{b}$. Finally, we have

$$\delta(ab - \tilde{a}\tilde{b})_{s,t} = \delta(a - \tilde{a})_{s,t} b_t + (a_s - \tilde{a}_s)\delta\tilde{b}_{s,t} + \delta\tilde{a}_{s,t}(b_t - \tilde{b}_t) + a_s \delta(b - \tilde{b})_{s,t}$$

which leads to the first inequality. We also have

$$\|(ab)' - (\tilde{a}\tilde{b})'\|_\gamma \leqslant \|a'b - \tilde{a}'\tilde{b}\|_\gamma + \|ab' - \tilde{a}\tilde{b}'\|_\gamma$$

and with the first inequality it gives the second one. Finally

$$\begin{aligned}(ab)^\#_{s,t} - (\tilde{a}\tilde{b})^\#_{s,t} &= (a^\#_{s,t} - \tilde{a}^\#_{s,t}) b_s + \tilde{a}_{s,t}(b_s - \tilde{b}_s) + (a_s - \tilde{a}_s) b^\#_{s,t} \\ &\quad + \tilde{a}_s (b^\#_{s,t} - \tilde{b}^\#_{s,t}) + \delta(a - \tilde{a})_{s,t} \delta b_{s,t} + \delta\tilde{a}_{s,t} \delta(b - \tilde{b})_{s,t}\end{aligned}$$

and the last inequality follows. □

Whenever a path is uniformly locally controlled by X, it is globally controlled by X as stated in the following lemma.

Lemma 3.2.18. *Suppose that $a : [0, T] \to \mathbb{R}^d$ is uniformly locally controlled by X, that is there exists a' and $a^\#$ such that for all $s, t \in [0, T]$*

$$a_t - a_s = a'_s (X_t - X_s) + a^\#_{s,t}$$

and there exists $\varepsilon > 0$ such that for all $s, t \in [0, T]$ with $|s - t| \leqslant \varepsilon$ we have

$$[\![a]\!]_{\mathcal{D}_{X,\varepsilon}^\gamma} := \sup_{s \neq t, |t-s| < \varepsilon} |a_t - a_s| / |t - s|^\gamma + |a'_t - a'_s| / |t - s|^\gamma + |a^\#_{s,t}| / |t - s|^{2\gamma} < +\infty$$

Then $a \in \mathcal{D}_X^\gamma$ and we have

$$\|a\|_{\mathcal{D}_X^\gamma} \lesssim (T/\varepsilon)^{1-\gamma} [\![a]\!]_{\mathcal{D}_{X,\varepsilon}^\gamma} (1 + \|X\|_\gamma) + |a_0| + |a'_0|.$$

Proof. As for all $0 \leqslant s, t \leqslant T$, $a_t - a_s = a'_s(X_t - X_s) + a^{\#}_{s,t}$, we have for $s \leqslant u \leqslant t$

$$a^{\#}_{s,t} = a^{\#}_{s,u} + a^{\#}_{u,t} + \delta a'_{s,u}\delta X_{u,t}$$

hence for $s = t_0 \leqslant \cdots \leqslant t_n = t$ with $t_{i+1} - t_i \leqslant \varepsilon$, we have

$$\delta a_{s,t} = \sum_{i=0}^{n-1} \delta a_{t_i,t_{i+1}} \text{ and } \delta a'_{s,t} = \sum_{i=0}^{n-1} \delta a'_{t_i,t_{i+1}}$$

and

$$a^{\#}_{s,t} = \sum_{i=0}^{n-1} (a^{\#}_{t_i,t_{i+1}} + \delta a'_{t_i,t_{i+1}} \delta X_{t_i,t_n}).$$

Hence

$$|\delta a_{s,t}| \leqslant [\![a]\!]_{\mathcal{D}^\gamma_{X,\varepsilon}} \sum_{i=0}^{n-1} (t_{i+1} - t_i)^\gamma \lesssim \|a\|_{\mathcal{D}^\gamma_{X,\varepsilon}} \sum_{i=0}^{n-1} (t_{i+1} - t_i)^\gamma \lesssim \|a\|_{\mathcal{D}^\gamma_{X,\varepsilon}} (T/\varepsilon)^{1-\gamma} |t - s|^\gamma$$

and the same holds for $\delta a'_{s,t}$. Finally,

$$|a^{\#}_{s,t}| \leqslant \sum_{i=0}^{n-1} (|a^{\#}_{t_i,t_{i+1}}| + |\delta a'_{t_i,t_{i+1}} \delta X_{t_i,t_n}|) \leqslant [\![a]\!]_{\mathcal{D}^\gamma_{X,\varepsilon}} |t - s|^\gamma (1 + \|X\|_\gamma) \sum_{i=0}^{n-1} |t_{i+1} - t_i|^\gamma$$

and the result follows. □

Remark 3.2.19. The same proof shows that when a is locally Hölder continuous, we have

$$[\![a]\!]_\gamma \lesssim (T/\varepsilon) \sup_{t \neq s, |t-s| \leqslant \varepsilon} |a_t - a_s| / |t - s|^\gamma.$$

The definition of controlled paths and the definition of rough paths allow us to construct a controlled rough integral as the limit of the Riemann sum. This construction and the properties are a byproduct of the existence of the *sewing map* (Proposition 1 in [41]).

Theorem 3.2.20 (Controlled Rough Integral, Gubinelli [41])**.** *Let* $1/3 < \gamma \leqslant 1/2$*,* $\boldsymbol{X} \in \mathcal{R}^\gamma([0,T])$ *and let* $a \in \mathcal{D}^\gamma_X([0,T])$*. For all* $0 \leqslant s \leqslant t \leqslant T$*, the following limit of Riemann sums exists*

$$\int_s^t a_r \mathrm{d}\boldsymbol{X}_r := \lim_{\substack{\mathcal{P} \text{ partition of } [s,t] \\ |\mathcal{P}| \to 0}} \sum_{[t_i,t_{i+1}] \in \mathcal{P}} (a_{t_i} \delta X_{t_i,t_{i+1}} + a'_{t_i} \mathbb{X}_{t_i,t_{i+1}})$$

and does not depends on the partition. Furthermore,

$$\left| \int_s^t a_r \mathrm{d}\boldsymbol{X}_r - a_s \delta X_{s,t} - a'_s \mathbb{X}_{s,t} \right| \lesssim |t - s|^{3\gamma} \|X\|_{\mathcal{R}^\gamma} \|a\|_{\mathcal{D}^\gamma_X}$$

and the map from $\mathcal{D}^\gamma_X$ *to* $\mathcal{D}^\gamma_X$ *given by*

$$a \to \int_0^\cdot a_r \mathrm{d}\boldsymbol{X}_r$$

is linear and continuous and we have

$$\left\| \int_0^\cdot a_r \mathrm{d}\boldsymbol{X}_r \right\|_{\mathcal{D}^\gamma_X} \lesssim (1 + \|\boldsymbol{X}\|_{\mathcal{R}^\gamma}) \|X\|_{D^\gamma_X}.$$

Let us finally give the equivalent for a function $[0, T]^2 \to F$ of the classical result which says that when a continuous function f from $[0, T]$ to F is such that there exists $\varepsilon > 0$ with $|f_t - f_s| \lesssim |t - s|^{1+\varepsilon}$ then $f \equiv 0$.

Proposition 3.2.21 (Gubinelli [41])**.** *Let $\mu > 1$. Let $h \in \mathcal{C}_2^\mu$ such that for all $0 \leqslant s < u < t \leqslant T$*

$$h_{s,t} - h_{u,t} - h_{s,u} = 0$$

then $h \equiv 0$.

Finally in order for this theory to be useful, we need to be able to lift to the space of rough path the class of signals X we consider. The following theorem gives a whole set of stochastic processes with such a property. It can be found in [32] and [34]. Note that the first result for the lift of the fractional Brownian motion is due to Coutin and Qian [16]

Theorem 3.2.22. *Let $H \in (1/3, 1/2]$ and B^H a d-dimensional fractional Brownian motion of Hurst parameter H. For any $1/3 < \gamma < H$, almost surely B^H can be lifted as a γ-rough path $\boldsymbol{B}^H = (B^H, \mathbb{B}^H)$. Furthermore for every $+\infty > p \geqslant 1$,*

$$\mathbb{E}[\|\boldsymbol{B}^H\|^p_{\mathcal{R}^\gamma([0,T])}] < +\infty$$

and there exists a smooth measurable approximation $B^{H,\varepsilon}$ of B^H such that $\boldsymbol{B}^{H,\varepsilon} \to^{\mathcal{R}_\gamma} \boldsymbol{B}^H$ almost surely and for all $1 \leqslant p < +\infty$

$$\mathbb{E}[\|\boldsymbol{B}^{H,\varepsilon} - \boldsymbol{B}^H\|^p_{\mathcal{R}^\gamma([0,T])}] \to 0,$$

where $\boldsymbol{B}^{H,\varepsilon} = (B^{H,\varepsilon}, \mathbb{B}^{H,\varepsilon})$ with $\mathbb{B}^{H,\varepsilon}_{s,t} = \int_s^t (B^{H,\varepsilon}_r - B^{H,\varepsilon}_s)\dot{B}^{H,\varepsilon}_r \mathrm{d}r$.

3.2.3 Irregular paths and the fractional Brownian motion

The method of characteristic relies on the regularity of the flow of the characteristic ODE associated with a transport equation. In order to use the properties of regularization of the fractional Brownian motion and other (stochastic) processes, let us recall some results of the first chapter 2. We use the oscillations of specific processes to regularize ordinary differential systems. The following definition describes the typical function for which a regularization will occur in perturbed differential systems.

Definition 3.2.23. Let $X : [0, T] \to \mathbb{R}^d$ and $\rho > 0$. We say that the function X is ρ-irregular if there exists $\gamma > 1/2$ such that

$$\|\phi^X\|_{\mathcal{W}_T^{\rho,\gamma}} := \sup_{s \neq t \in [0,T]} \sup_{\xi \in \mathbb{R}^d} (1 + |\xi|)^\rho |t - s|^{-\gamma} \left| \int_s^t e^{i\xi X_q} \mathrm{d}q \right| < +\infty.$$

This definition is not empty: almost every path of the fractional Brownian is ρ-irregular. Furthermore, nondegenerate α-stable Lévy processes also have this property.

Theorem 3.2.24. *Let B^H be a d-dimensional fractional Brownian motion of Hurst parameter $H \in (0, 1)$, then for all $\rho < 1/2H$ almost surely B^H is ρ-irregular. Furthermore, for all $\gamma > 1/2$, there exists $\lambda > 0$ such that*

$$\mathbb{E}[\exp(\lambda \|\phi^{B^H}\|^2_{\mathcal{W}_T^{\rho,\gamma}})] < +\infty.$$

The regularization properties of ρ-irregular paths will occur in Fourier–Lebesgue spaces. The oscillations of the function ϕ^X are enough to regularize the differential system in that setting.

Definition 3.2.25. The space of Fourier-Lebesgue function of order $\alpha > 0$ is defined as

$$\mathcal{F}L^\alpha = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) | \|f\|_{\mathcal{F}L^\alpha} := \int_{\mathbb{R}^d} \mathrm{d}k |\hat{f}(k)|(1+|k|)^\alpha < +\infty \right\}$$

Remark 3.2.26. Note that for $\alpha > 0$, $\mathcal{F}L^\alpha \subset \mathcal{C}_b^\alpha$ the space of bounded Hölder continuous functions of index α.

When the vector-field b lies in a Fourier Lebesgue space and X is ρ-irregular, one has a regularization property for the perturbed differential equation. When b is regular enough, the flow has good spatial regularity and furthermore there are good ways of approaching this flow by a regularized one.

Theorem 3.2.27. *Let $\rho > 0$, let X a ρ-irregular path. Let $\alpha > -\rho$ and $\alpha + 3/2 > 0$. Let $b \in \mathcal{F}L^{\alpha+3/2}$. Then for all $x \in \mathbb{R}^d$ there exists a unique solution $\Phi(x)$ to the equation*

$$\Phi_t(x) = x + \int_0^t b(\Phi_q(x))\mathrm{d}q + X_t.$$

Furthermore, $\theta_t(x) = \Phi_t(x) - X_t$ is Lipschitz continuous in time. The function θ is also locally Lipschitz in space, uniformly in time. Moreover, for any mollification b^ε of b we write Φ^ε the flow of the approximate equation

$$\Phi_t^\varepsilon(x) = x + \int_0^t b^\varepsilon(\Phi_q^\varepsilon(x))\mathrm{d}q + X_t$$

the following bound holds

$$\|\Phi^\varepsilon(x) - \Phi(x)\|_\infty \leqslant K(|x|)\|b - b^\varepsilon\|_{\mathcal{F}L^{\alpha+3/2}}$$

where the constant K is increasing in $|x|$ and independent of ε. The approximate flow Φ^ε is also differentiable in space and

$$\sup_{\varepsilon>0} \sup_{t\in[0,T]} |D\Phi_t^\varepsilon(x)| \leqslant K(|x|) < +\infty.$$

Even though the flow of perturbed differential equation does not have the same regularization properties as ρ-irregular path, one can look at their averaging properties. This is the purpose of the following Lemma, and it will be really important in the following.

Proposition 3.2.28. *Let $f \in C^1(\mathbb{R}^d) \cap \mathcal{F}L^{\alpha+3/2}$, b, b^ε and Φ^ε as in the previous theorem. For all $t \in [0,T]$ let us define the function*

$$F_t^\varepsilon(x) = \int_0^t f(\Phi_q^\varepsilon(x))\mathrm{d}q.$$

Then F_t^ε is differentiable in space and there exists a constant $K(|x|)$ at most with linear growth in $|x|$ and independent of $\varepsilon > 0$ such that

$$\sup_{t\in[0,T]} |DF_t^\varepsilon(x)| \leqslant K(|x|)\|f\|_{\mathcal{F}L^{\alpha+3/2}}.$$

In the case of the fractional Brownian motion, the previous results are not optimal in term of the regularity of the vectorfields involved in the equation. Indeed, using a Girsanov transform, one can show that the fractional Brownian motion has regularization properties for ODEs when b lies in a bigger space of function/distribution. The price for that extension is the loss of the global character of the averaging property for the fractional Brownian motion, as the exceptional set here depends on the function.

Theorem 3.2.29. *Let $H \in (0,1)$, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and B^H a d–dimensional fractional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\alpha > -1/2H$ and $\alpha + 1 > 0$. Let $b \in \mathcal{C}_b^{\alpha+1}$. Then for all $x \in \mathbb{R}^d$, there exists $\mathcal{N}_{b,x} \subset \mathcal{F}$ such that $\mathbb{P}(\mathcal{N}_{b,x}) = 0$ such that for all $\omega \notin \mathcal{N}_{b,x}$ there exists a unique solution of the equation*

$$\Phi_t(\omega, x) = x + \int_0^t b(\Phi_q(\omega, x))\mathrm{d}q + B_t^H(\omega).$$

Furthermore for any mollification b^ε of b and for Φ^ε the flow of the approximate equation, for any $x \in \mathbb{R}^d$

$$\mathbb{E}[\|\Phi^\varepsilon(x) - \Phi(x)\|_{\infty,[0,T]}^2] \to_{\varepsilon\to 0} 0$$

and there exists a constant $K(|x|)$ at most with linear growth in $|x|$ such that

$$\sup_{\varepsilon>0} \mathbb{E}[\sup_{t\in[0,T]} |D\Phi_t^\varepsilon(x)|^2] \leqslant K(|x|)^2 < +\infty.$$

The same kind of regularization properties for the flow occur in the context.

Proposition 3.2.30. *Let $f \in C^1(\mathbb{R}^d) \cap \mathcal{C}_b^{\alpha+1}$ with $\alpha + 1 > 0$ and $\alpha > -1/2H$, b, b^ε and Φ^ε as in the previous Theorem.*

For all $t \in [0,T]$ let us define the function

$$F_t^\varepsilon(x) = \int_0^t f(\Phi_q^\varepsilon(x))\mathrm{d}q.$$

Then F_t^ε is differentiable in space and there exists a constant $K(|x|)$ at most with linear growth in $|x|$ and independent of $\varepsilon > 0$ such that

$$\mathbb{E}[(\sup_{t\in[0,T]} |DF_t^\varepsilon(x)|)^2] \leqslant K(|x|)^2 \|f\|_{\mathcal{C}_b^\alpha}^2.$$

3.2.4 Standard theory of flow for additive rough ODE

Our approach for the study of Rough Transport Equations comes mostly from the method of characteristics. This method requires lots of informations about the regularity of the flow of the characteristic differential equation linked to the transport equation.

As we intend to give existence results for quite general vectorfields, there will not be any good notion for the flow. All along this paper, we will proceed by regularization of the vectorfield b and the perturbation X. Then we will need a priori estimates for such flows, as well as space and time regularity for that object.

The following Theorem is standard, but we recall the form of the Jacobian determinant of the flow. All along this subsection, the perturbation X will be a path lying in $\mathcal{C}^\gamma$ and $\gamma \in (0,1]$.

Theorem 3.2.31. *Let $b \in L^\infty([0,T]; \mathrm{Lip}(\mathbb{R}^d) \cap C^1(\mathbb{R}^d))$ and $X \in \mathcal{C}^\gamma([0,T])$. The equation*

$$\left\{ \begin{array}{l} \mathrm{d}y_t = b_t(y_t) + \mathrm{d}X_t \\ y_0 = x \end{array} \right. , \tag{3.4}$$

understood in its integral form

$$y_t = x + \int_0^t b_s(y_s)\mathrm{d}s + X_t - X_0$$

has a unique solution $\Phi(x) \in \mathcal{C}^\gamma([0,T])$. Furthermore, in that case, for all $t \in [0,T]$, $x \to \Phi_t(x)$ is differentiable and its derivative $D\Phi_t$ is the unique solution of the linear ordinary differential equation

$$D\Phi_t(x) = \mathrm{id} + \int_0^t Db_s(\Phi_s(x)).D\Phi_s(x)\mathrm{d}s.$$

In that case, we have

$$\mathrm{Jac}(\Phi_t(x)) = \exp\left(\int_0^t \mathrm{div}\, b_s(\Phi_s(x))\mathrm{d}s\right).$$

The first assertions are quite standard. The proof of the last assertion relies on the so called Liouville Lemma, where we take $A_t = Db_t(\Phi_t(x))$.

Lemma 3.2.32 (Liouville)**.** *Let $A \in C([0,T]; \mathcal{M}_d(\mathbb{R}))$ and let B the unique solution in the space $C([0,T]; \mathcal{M}_d(\mathbb{R}))$ of the equation*

$$B_t = B_0 + \int_0^t A_s B_s \mathrm{d}s.$$

Then

$$\det B_t = \det(B_0) \exp\left(\int_0^t \mathrm{tr}(A_s)\mathrm{d}s\right).$$

Proof. Thanks to the equation, $B \in C^1([0,T], \mathcal{M}_d)$ and as the determinant is also a C^1 function from $\mathcal{M}_d(\mathbb{R})$ to $\mathbb{R}$, we have for all $t \in [0,T]$

$$\begin{aligned} (\det B_t)' &= \mathrm{tr}({}^t\mathrm{adj}(B_t).B_t') \\ &= \mathrm{tr}({}^t\mathrm{adj}(B_t)A_t B_t) \\ &= \mathrm{tr}(A_t^t\,\mathrm{adj}(B_t)B_t) \\ &= \mathrm{tr}(A_t \det(B_t) I_d) \\ &= \det(B_t)\,\mathrm{tr}(A_t). \end{aligned}$$

where $\mathrm{adj}(M)$ is the adjugate matrix of M. The result follows easily. □

In the classical case of the transport equation, the existence of a solution is granted when the vectorfield b has spatial linear growth. The reminder of this section presents a brief study of flows when b has linear growth. All these a priori bounds will be useful when we use the characteristic method to solve the rough transport equation.

Lemma 3.2.33. *Let $b \in L^\infty([0,T], \mathrm{Lin}(\mathbb{R}^d))$ such that the flow Φ of the equation (3.4) exists for all $t \in [0,T]$. There exists a constant $K(T, \|b\|_{\infty;\mathrm{Lin}}) > 0$ such that for all $x \in \mathbb{R}^d$, $\Phi(x) \in \mathcal{C}^\gamma([0,T])$ and*

$$\|\Phi(x)\|_\gamma \leqslant K(1+|x|)(1+\|X\|_\gamma).$$

Proof. Let $x \in \mathbb{R}^d$ and $0 \leqslant s < t \leqslant T$

$$\begin{aligned}
\frac{|\Phi(x)_t - \Phi(x)_s|}{|t-s|^\gamma} &\leqslant \|X\|_\gamma + \frac{1}{|t-s|^\gamma}\int_s^t |b_r(\Phi_r(x))|\mathrm{d}r \\
&\lesssim \|X\|_\gamma + \|b\|_{\infty;\mathrm{Lin}}\left(T^{1-\gamma}(1+|\Phi_s(x)|) + \int_s^t \frac{|\Phi_r(x) - \Phi_s(x)|}{|r-s|^\gamma}\mathrm{d}r\right).
\end{aligned}$$

By Gronwall's lemma,

$$\frac{|\Phi(x)_t - \Phi(x)_s|}{|t-s|^\gamma} \leqslant (\|X\|_\gamma + T^{1-\gamma}(1+|\Phi_s(x)|)\|b\|_{\infty;\mathrm{Lin}})\exp(T\|b\|_{\infty;\mathrm{Lin}}),$$

hence

$$[\![\Phi(x)]\!]_\gamma \leqslant (\|X\|_\gamma + T^{1-\gamma}(1+\|\Phi(x)\|_\infty)\|b\|_{\infty;\mathrm{Lin}})\exp(T\|b\|_{\infty;\mathrm{Lin}}).$$

It remains to estimate the supremum of $\Phi(x)$. We have

$$|\Phi(x)_t| \leqslant |x| + \|b\|_{\infty;\mathrm{Lin}}\left(T + \int_0^t |\Phi(x)_r|\mathrm{d}r\right) + \|X\|_\infty,$$

which gives, again by Gronwall's lemma

$$\|\Phi(x)\|_\infty \leqslant (|x| + T\|b\|_{\infty;\mathrm{Lin}} + \|X\|_\infty)\exp(T\|b\|_{\infty;\mathrm{Lin}}).$$

The result follows after recalling that $\|X\|_\infty \lesssim \|X\|_\gamma$. □

Remark 3.2.34. The constant K can be chosen as $K(T, \|b\|_{\infty;\mathrm{Lin}}) = K_T g(T\|b\|_{\infty;\mathrm{Lin}})$ with $g(x) = ((x^2+x)e^x + x + 1)e^x$.

As in the standard theory of transport equations, the solutions will involve the inverse of the flow of equation (3.4). The following lemma gives informations about the equation verified by this inverse.

Proposition 3.2.35. *Let b and X such that that the equations (3.5) and (3.6) below have unique solution (for all $t_0 \in [0,T]$):*

$$\Phi_t(x) = x + \int_0^t b_q(\Phi_u(x))\mathrm{d}u + X_t \quad t \in [0,T]. \tag{3.5}$$

and

$$\psi_t^{t_0}(y) = y - \int_0^t b_{t_0-q}(\psi_u^{t_0}(y))\mathrm{d}u - (X_{t_0} - X_{t_0-t}), \quad t \in [0,t_0] \tag{3.6}$$

Then for all $t_0 \in [0,T]$, $\Phi_{t_0}^{-1}(x)$ exists, and $\Phi_{t_0}^{-1}(x) = \psi_{t_0}(x)$ where ψ is the unique solution of the equation. Furthermore, Φ^{-1} also verifies the following equation:

$$\Phi_t^{-1}(x) = x - \int_0^t b_{T-u}(\Phi_u^{-1}(x))\mathrm{d}u - (X_T - X_{T-t}), \quad t \in [0,T].$$

Proof. We have

$$\begin{aligned}
\Phi_{t_0-t}(x) &= x + \int_0^{t_0-t} b_u(\Phi_u(x))\mathrm{d}u + X_{t_0-t} \\
&= x + \int_t^{t_0} b_{t_0-u}(\Phi_{t_0-u}(x))\mathrm{d}u + X_{t_0-t} \\
&= \Phi_{t_0}(x) - \int_0^t b_{t_0-u}(\Phi_{t_0-u}(x))\mathrm{d}u - (X_{t_0} - X_{t_0-t}) \\
&\quad + x + X_{t_0} + \int_0^{t_0} b_{t_0-u}(\Phi_u(x))\mathrm{d}u - \Phi_{t_0}(x) \\
&= \Phi_{t_0}(x) - \int_0^t b_{t_0-u}(\Phi_{t_0-u}(x))\mathrm{d}u - (X_{t_0} - X_{t_0-t}).
\end{aligned}$$

Hence, since the solution of equation (3.6) is unique, $\Phi_{t_0-t}(x) = \psi_t^{t_0}(\Phi_{t_0}(x))$, and we have $\psi_{t_0}^{t_0}(\Phi_{t_0}(x)) = x$ which is the wanted result. As equation (3.6) is of same form as equation (3.5), we also have $\Phi_{t_0} \circ \psi_{t_0}^{t_0} = \mathrm{id}$. Hence $\Phi_{t_0}^{-1}$ exists and we have $\Phi_{t_0}^{-1} = \psi_{t_0}^{t_0}$. Furthermore for $s, t \in [0, T]$

$$\Phi_{s+t} \circ \Phi_s^{-1} \circ \Phi_t^{-1} = \Phi_t \circ \Phi_s \circ \Phi_s^{-1} \circ \Phi_t^{-1} = \mathrm{id}.$$

Hence $\Phi_s^{-1} \circ \Phi_t^{-1} = \Phi_{s+t}^{-1} = \Phi_t^{-1} \circ \Phi_s^{-1}$ and Φ^{-1} has the semi-group property. Furthermore,

$$\begin{aligned}
x - \Phi_t^{-1}(x) &= \Phi_T(\Phi_T^{-1}(x)) - \Phi_{T-t}(\Phi_T^{-1}(x)) \\
&= \int_{T-t}^T b_r(\Phi_r(\Phi_T^{-1}(x)))\mathrm{d}r + X_T - X_{T-t} \\
&= \int_{T-t}^T b_r(\Phi_{T-r}^{-1}(x))\mathrm{d}r + X_T - X_{T-t} \\
&= \int_0^t b_{T-u}(\Phi_u^{-1}(x))\mathrm{d}u + X_T - X_{T-t},
\end{aligned}$$

and the result follows. □

As in the characteristic method (see Appendix 3.2.5), in the following, we will strongly use the spatial regularity of the flow. When the vector-field b is regular enough, the following result gives a bound for the norm of the spatial derivative.

Proposition 3.2.36. *Let $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $N \in \mathbb{N}$, and $b \in L^\infty([0, T]; C_b^N(\mathbb{R}^d))$. Let $X \in \mathcal{C}^\gamma([0, T])$ and Φ the flow of the equation*

$$\Phi_t(x) = x + \int_0^t b_r(\Phi_r(x))\mathrm{d}r + X_t.$$

For all $k \in \{1, \dots N\}$ and all $t \in [0, T]$, the k-th spatial derivative $D^k\Phi_t$ exists, and there exists a constant C_k depending on T $\|X\|_{\mathcal{D}^\gamma}$ and $\|D^k b\|_{\infty; L^\infty}$ for all $k \in \{0, \dots, n\}$ such that

$$|D^k\Phi_t(x)| \leqslant C_k.$$

Proof. The fact that $\Phi_t \in C^N(\mathbb{R}^d)$ is standard. Let us recall the equation verified by the flow

$$D\Phi_t(x) = \mathrm{id} + \int_0^t Db_r(\Phi_r(x)).D\Phi_r(x)\mathrm{d}r.$$

Hence, the proof of the proposition is quite straightforward. We have to use the equation verified by $D^n\Phi(x)$, where all the other derivatives of Φ and all the derivatives of b up to order n appear. We then prove by induction the wanted bound for $|D^n\Phi_t(x)|$, using the bound for the smaller derivatives of Φ and the Gronwall lemma. $\square$

Remark 3.2.37. The proof also gives that C_k has exponential growth with respect to $\|Db\|_{\infty;L^\infty}$ but polynomial growth wrt the other quantities.

Finally, the heart of the following section will be a smooth approximation of the characteristic equation of the transport equation by an approximate one. Indeed, when $X \notin C^1([0,T])$ and $b \notin C^1(\mathbb{R}^d)$, the above results about the differentiability of the flow can not be used. The following result gives the regularity of the flow regarding the perturbation X.

Lemma 3.2.38. *Let $b \in L^\infty([0,T]; C_b^1(\mathbb{R}^d))$ and $X, Y \in \mathcal{C}^\gamma$. Then*

$$\|\Phi^X(x) - \Phi^Y(x)\|_{\gamma,[0,T]} \leqslant C(T, \|Db\|_\infty)\|X - Y\|_{\gamma,[0,T]}$$

where C is independent of x and nondecreasing in T and $\|Db\|_\infty$. Furthermore, when $b \in L^\infty([0,T]; C_b^2(\mathbb{R}^d))$, we also have

$$\|D\Phi^X(x) - D\Phi^Y(x)\|_{\mathrm{Lip}} \leqslant C(T, \|Db\|_\infty, \|D^2 b\|_\infty)\|X - Y\|_\gamma.$$

Proof. First, we give an estimate of the difference in supremum norm.

$$\begin{aligned} |\Phi_t^Y(x) - \Phi_t^X(x)| &\leqslant |X_t - Y_t| + \int_0^t |b_q(\Phi_q^Y(x)) - b_q(\Phi_q^X(x))|\mathrm{d}q \\ &\leqslant t^\gamma\|X - Y\|_\gamma + \|Db\|_\infty \int_0^t |\Phi_q^Y(x) - \Phi_q^X(x)|\mathrm{d}q \end{aligned}$$

By Gronwall's lemma,

$$|\Phi_t^Y(x) - \Phi_t^X(x)| \leqslant T^\gamma\|X - Y\|_\gamma e^{T\|Db\|_\infty}.$$

Furthermore, we have

$$|\delta(\Phi^X - \Phi^Y)_{s,t}(x)| \leqslant |\delta(X - Y)_{s,t}| + \|Db\|_\infty \int_s^t |(\Phi^X - \Phi^Y)_{s,q}(x)| + |(\Phi^X - \Phi^Y)_s(x)|\mathrm{d}q.$$

Again by Gronwall's lemma, we get

$$|\delta(\Phi^X - \Phi^Y)_{s,t}(x)| \lesssim_{T,\|Db\|_\infty} |t - s|^\gamma\|X - Y\|_\gamma,$$

which is the first part of the result. Furthermore, as

$$D\Phi_t^X(x) = \mathrm{id} + \int_0^t Db_q(\Phi_q^X(x)).D\Phi_q^X(x)\mathrm{d}q,$$

we have

$$\|D\Phi^X(x)\|_\infty \leqslant e^{T\|Db\|_\infty}$$

and

$$\begin{aligned}
|D(\Phi^X - \Phi^Y)_t(x)| &\leqslant \left|\int_0^t (Db_q(\Phi_q^X(x)) - Db(\Phi_q^Y(x))).D\Phi_t^X(x)\mathrm{d}q\right| \\
&\quad + \left|\int_0^t Db(\Phi_q^Y(x)).D(\Phi^X - \Phi^Y)_q(x)\mathrm{d}q\right| \\
&\lesssim \|D^2 b\|\|\Phi^X(x) - \Phi^Y(x)\|_\infty e^{T\|Db\|_\infty} + \|Db\|_\infty \int_0^t |D(\Phi^X - \Phi^Y)_q(x)|\mathrm{d}q.
\end{aligned}$$

Hence

$$\|D(\Phi^X - \Phi^Y)_\cdot(x)\|_\infty \lesssim_{T,\|Db\|_\infty,\|D^2 b\|_\infty} \|X - Y\|_\gamma.$$

Finally,

$$|\delta D(\Phi^X - \Phi^Y)_{s,t}(x)| \lesssim \|X - Y\|_\gamma |t - s| + \|Db\|_\infty \int_s^t |\delta D(\Phi^X - \Phi^Y)_{s,q}(x)|\mathrm{d}q,$$

and the result follows by Gronwall's Lemma. □

3.2.5 The method of characteristics

We give a proof of the usual characteristic method, and a proof of weak uniqueness thanks to the dual equation.

Theorem 3.2.39. *Let $b \in L^\infty([0,T]; C_b^1(\mathbb{R}^d))$, $c \in L^\infty([0,T], C_b^1(\mathbb{R}^d))$ and $X \in C^1([0,T])$. Let Φ the flow of the equation*

$$\Phi_t(x) = x + \int_0^t b(\Phi_u(x))\mathrm{d}u + X_t.$$

Let $u_0 \in C^1(\mathbb{R}^d)$. There exists a unique strong solution u to equation (3.7) with initial condition u_0.

$$\partial_t u + (b + \dot{X}).\nabla u + cu = 0. \tag{3.7}$$

Furthermore the solution has the explicit form

$$u_t(x) = u_0(\Phi_t^{-1}(x)) \exp\left(-\int_0^t c_{t-q}(\Phi_q^{-1}(x))\mathrm{d}q\right).$$

Proof. We first prove the existence of such a solution. As $b \in \mathrm{Lip}(\mathbb{R}^d) \cap C^1(\mathbb{R}^d)$, the flow Φ and its inverse are well-defined. Let us define

$$u_t(x) = u_0(\Phi_t^{-1}(x)) \exp\left(-\int_0^t c_{t-q}(\Phi_q^{-1}(x))\mathrm{d}q\right).$$

Then $u_t(\Phi_t(x)) = u_0(x) \exp\left(-\int_0^t c_q(\Phi_q(x))\mathrm{d}q\right)$ and

$$\partial_t(u_t(\Phi_t(x))) = -u_0(x)c_t(\Phi_t(x)) \exp\left(-\int_0^t c_{t-q}(\Phi_q^{-1}(x))\mathrm{d}q\right) = -c_t(\Phi_t(x))u_t(\Phi_t(x)).$$

On the other hand, as $b, c \in L^\infty([0,T], C^1(\mathbb{R}^d))$, $\Phi^{-1} \in C^1([0,T]\times\mathbb{R}^d)$ and $u \in C^1([0,T]\times\mathbb{R}^d)$. Hence

$$\begin{aligned}\partial_t(u_t(\Phi_t(x))) &= \partial_t u_t(\Phi_t(x)) + \nabla u_t(\Phi_t(x)).\Phi_t'(x)\\ &= \partial_t u_t(\Phi_t(x)) + \nabla u_t(\Phi_t(x)).(b(\Phi_t(x)) + \dot{X}_t).\end{aligned}$$

and therefore

$$\partial_t u_t(\Phi_t(x)) + \nabla u_t(\Phi_t(x)).(b_t(\Phi_t(x)) + \dot{X}_t) = -c_t(\Phi_t(x))u_t(\Phi_t(x)).$$

We take $x = \Phi_t^{-1}(y)$, and we have

$$u_t(y) + b_t(y).\nabla u_t(y) + c_t(y)u_t(y) + \nabla u_t(y)\dot{X}_t = 0.$$

Hence, u is a solution of equation (3.7) with initial condition u_0.

For the uniqueness, let φ a solution of equation (3.7) with $\varphi_0 = u_0$ and let $v_t(x) = \varphi_t(\Phi_t(x))$. Then

$$\partial_t(v_t(x)) = -c_t(\Phi_t(x))v_t(x).$$

We thus have

$$\varphi_t(\Phi_t(x)) = v_t(x) = v_0(x)\exp\left(-\int_0^t c_q(\Phi_q(x))\mathrm{d}q\right) = u_0(x)\exp\left(-\int_0^t c_q(\Phi_q(x))\mathrm{d}q\right),$$

and then $\varphi_t(x) = u_t(x)$ which ends the proof. □

Remark 3.2.40. For $c = 0$, we have the transport equation. When $c = \operatorname{div} b$, we have the continuity equation, the dual equation of the transport equation.

Remark 3.2.41. Thanks to the use of Corollary 3.3.6, we see that when b and c are as in the previous theorem, and $u_0 \in C_c^\infty(\mathbb{R}^d)$, and $u \in C^1(\mathbb{R}^d)$ is the unique solution of equation (3.7), then for every multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ and all $t \in [0,T]$,

$$x \mapsto x^\alpha \nabla u_t(x) \quad \text{and} \quad x \mapsto x.\nabla u_t(x)$$

are in $L^1(\mathbb{R}^d)$ where $x^\alpha = x_1^{\alpha_1}\dots x_d^{\alpha_d}$.

We can now prove the existence and the uniqueness of L^∞ weak solutions.

Theorem 3.2.42. *Let $b \in L^\infty([0,T]; C_b^1(\mathbb{R}^d))$, $\operatorname{div} b, c \in L^\infty([0,T], C_b^1(\mathbb{R}^d))$ and $X \in C^1([0,T])$. Let Φ the flow of the equation*

$$\Phi_t(x) = x + \int_0^t b(\Phi_u(x))\mathrm{d}u + X_t.$$

Let $u_0 \in L^\infty(\mathbb{R}^d)$. Then there exists a unique weak solution $u \in L^\infty([0,T]\times\mathbb{R}^d)$ of equation (3.8) with initial condition u_0

$$\partial_t u + (b + \dot{X}).\nabla u + cu = 0, \tag{3.8}$$

furthermore for almost every t and x,

$$u_t(x) = u_0(\Phi_t^{-1}(x))\exp\left(-\int_0^t c_{T-q}(\Phi_q^{-1}(x))\mathrm{d}q\right).$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$, and let us define

$$\varphi_t^{t_0}(x) = \varphi(\Phi_{t_0-t}(x)) \exp\left(\int_t^{t_0} (\operatorname{div} b_q - c_q)(\Phi_{q-t}(x))\mathrm{d}q\right).$$

Hence, φ^{t_0} is the unique strong solution of the equation

$$\partial_t u + (b + \dot{X}).\nabla u + (\operatorname{div} b - c)u = 0$$

such that $\varphi_{t_0}^{t_0}(x) = \varphi(x)$. Furthermore, thanks to the previous remark and since $\varphi \in C_c^\infty$, all the following computations are allowed. Let us define as before

$$u_t(x) = u_0(\Phi_t^{-1}(x)) \exp\left(-\int_0^t c_{T-q}(\Phi_q^{-1}(x))\mathrm{d}q\right).$$

We have

$$\begin{aligned}
u_{t_0}(\varphi) &= \int_{\mathbb{R}^d} u_0(\Phi_{t_0}^{-1}(x)) \exp\left(-\int_0^{t_0} c_{t_0-q}(\Phi_q^{-1}(x))\mathrm{d}q\right)\varphi(x) \\
&= \int_{\mathbb{R}^d} u_0(x)\varphi_0^{t_0}(x)\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \mathrm{d}x u_0(x)\int_0^{t_0} \partial\varphi_q^{t_0}(x)\mathrm{d}q + u_0(\varphi) \\
&= -\int_{\mathbb{R}^d} \mathrm{d}x u_0(x)\int_0^{t_0} (b_q(x) + \dot{X}_q).\nabla\varphi_q(x) + (\operatorname{div} b_q(x) - c_q(x))\varphi_q(x)\mathrm{d}q + u_0(\varphi) \\
&= \int_0^{t_0} \mathrm{d}q \int_{\mathbb{R}^d} \mathrm{d}x u_q((b_q + \dot{X}_q).\nabla\varphi_q + (\operatorname{div} b_q - c_q)\varphi) + u_0(\varphi).
\end{aligned}$$

Hence, by definition, u is a weak solution of Equation (3.8). Furthermore, for a weak solution v with initial condition 0, by testing against φ we have

$$\begin{aligned}
v_{t_0}(\varphi) &= v_{t_0}(\varphi_{t_0}) - v_0(\varphi_0) \\
&= \int_0^{t_0} \mathrm{d}q v_q((b_q + \dot{X}_q).\nabla\varphi_{t_0} + (\operatorname{div} b_q - c_q)\varphi_{t_0}) + v_q(\varphi_q) - v_0(\varphi_q) \\
&= 0.
\end{aligned}$$

And this is true for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, hence $u_{t_0}(x) = 0$ for almost all x. As the equation is linear, the result is proved. $\square$

3.3 Rough Transport Equation, existence of solutions.

In order to deal with the multiplicative perturbation of the classical transport equation we need a new notion of solution. When X is a Brownian motion, one can use, as in Flandoli et al. [30] the Stratonovitch integral to deal with the multiplicative term. As here, we intend to work with processes such as fractional Brownian motion, which are neither a martingale nor a Markov processes, there is no way to use classical stochastic calculus.

In order to replace the stochastic integral we will use controlled rough paths, as presented in [41] but also in [32]. The way we will require the solution to be weakly controlled by the process X is to be linked with the way Gubinelli et al. in [46] define controlled viscosity solution of non linear PDEs. In the following we will focus on rough paths $\boldsymbol{X} \in \mathcal{R}^\gamma$ with $1/3 < \gamma \leqslant 1/2$, since in that case the controlled rough integrals are quite easy to define. When $1 \geqslant \gamma > 1/2$, all the computations are easy as we can consider usual Young integrals.

3.3.1 Weak Controlled solutions, smooth case

We can now focus ourselves on the Rough Transport Equation (RTE). Namely for $b \in L^\infty([0,T];\mathrm{Lin}(\mathbb{R}^d))$ and $\boldsymbol{X} \in \mathcal{D}^\gamma([0,T])$, we want to solve the following Cauchy problem

$$\begin{cases} \partial_t u_t(x) + b_t(x).\nabla u_t(x) + \nabla u_t(x).\mathrm{d}\boldsymbol{X}_t = 0 \\ u_0 \in L^\infty(\mathbb{R}^d) \end{cases}. \tag{3.9}$$

To deal with the term $\nabla u_t(x).\mathrm{d}\boldsymbol{X}_t$ which is a priori ill-defined even in the weak sense, we need to introduce a new notion of solution for equation (3.9).

Definition 3.3.1. Let $u_0 \in L^\infty(\mathbb{R}^d)$, $1/3 < \gamma \leqslant 1/2$ and $\boldsymbol{X} = (X, \mathbb{X}) \in \mathcal{R}^\gamma$.We say that $u \in L^\infty([0,T] \times \mathbb{R}^d)$ is a weak controlled solution (WCS) of equation (3.9) with initial condition u_0 if for all $\varphi \in C_c^\infty(\mathbb{R}^d)$,

1. we have $u(\varphi) \in \mathcal{D}_X^\gamma([0,T])$;
2. for all $t \in [0,T]$,

$$u_t(\varphi) = u_0(\varphi) + \int_0^t u_s(b_s.\nabla\varphi + \mathrm{div}(b_s)\varphi)\mathrm{d}s + \int_0^t u_s(\nabla\varphi)\mathrm{d}\boldsymbol{X}_s$$

   where the quantity $\int_0^t u_s(\nabla\varphi)\mathrm{d}\boldsymbol{X}_s$ is understood as the controlled rough integral of $u(\nabla\varphi)$ against $\mathrm{d}\boldsymbol{X}$.

Remark 3.3.2. Since this definition is true for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, when u is a weak Controlled solution to equation (3.9) and $\varphi \in C_c^\infty(\mathbb{R}^d)$, for all $\alpha \in \{1, \ldots, d\}^d$ with $|\alpha| = 1$, we have

$$\delta u_{r,t}(\partial^\alpha\varphi) = \int_r^t u_s(b_s.\nabla(\partial^\alpha\varphi) + \mathrm{div}(b_s)(\partial^\alpha\varphi))\mathrm{d}s + \int_r^t u_s(\nabla(\partial^\alpha\varphi))\mathrm{d}\boldsymbol{X}_s$$

and the derivative of $u(\nabla\varphi)$ as a controlled path is $u(\nabla^2\varphi)$. Hence as $\boldsymbol{X}$ is geometric, an alternative formulation for Definition 3.3.1 can be:

1. For all $\varphi \in C_c^\infty(\mathbb{R}^d)$, $u_\cdot(\nabla\varphi) \in \mathcal{D}_X^\gamma([0,T])$, with $u_\cdot(\varphi)' = u_\cdot(\nabla\varphi)$.
2. There exists a remainder $R^\varphi : [0,T]^2 \to \mathbb{R}$ such that $|R_{s,t}(\varphi)| \lesssim |t-s|^{3\gamma}$ and for all $0 \leqslant s \leqslant t \leqslant T$ the following equation is verified

$$\delta u_{s,t}(\varphi) = \int_s^t u_q(b_q.\nabla\varphi + \mathrm{div}(b_q)\varphi)\mathrm{d}q + u_s(\nabla\varphi)X_{s,t} + \frac{1}{2}u_s(\nabla^2\varphi)X_{s,t}^{\otimes 2} + R_{s,t}(\varphi).$$

Thanks to the definition of the rough integral (see Theorem 3.2.20 above) these two definitions are equivalent. Hence, in the following we will either use one or the other.

However from this second formulation it is clear that the solution u depends only on the first level X of the rough path $\boldsymbol{X}$ due to the fact that only the symmetric part of the area $\mathbb{X}_{s,t}$ is needed to compute the rough integral on the r.h.s. if $u(\nabla\varphi)' = u(\nabla^2\varphi)$ and that this symmetic part is canonical for geometric rough paths and given by $X_{s,t}^{\otimes 2}/2$.

Whereas this notion of solution is quite different from the classical one, when X is smooth, the rough integral $\int_0^t u_s(\nabla\varphi)\mathrm{d}\boldsymbol{X}_s$ is equal to the classical integral $\int_0^t u_s(\nabla\varphi)\dot{X}_s\mathrm{d}s$. In that case, we have to show that the two notions of weak solutions coincide. This is the purpose of the next theorem. Using the usual weak solutions (see Section 3.2.5), the unique weak solution can be written as $u_t(x) = u_0(\Phi_t^{-1}(x))$. In order to prove that this solution is a controlled weak solution, it will be necessary, thanks to a change of variable, to prove that $t \to \varphi(\Phi_t(x))$ is controlled by X and has some good integrability properties. This will be the purpose of the end of the subsection.

Theorem 3.3.3. *Let $b \in L^\infty([0,T];\mathrm{Lip}(\mathbb{R}^d) \cap C^1(\mathbb{R}^d))$ with $\operatorname{div} b \in L^\infty([0,T];C_b^1(\mathbb{R}^d))$, $u_0 \in L^\infty(\mathbb{R}^d)$ and $X \in C^1([0,T])$. Let $1/3 < \gamma \leqslant 1/2$ and $\boldsymbol{X}$ be the natural lift of X into the space $\mathcal{R}^\gamma$. Then there exists a unique weak controlled solution u to equation (3.9) with initial condition u_0. For almost every $t \in [0,T]$ and almost every $x \in \mathbb{R}^d$ we have*

$$u_t(x) = u_0(\Phi_t^{-1}(x))$$

where Φ is the flow of equation (3.4).

Here, as X is smooth, equation (3.9) can be understood in the weak sense as the classical transport equation

$$\begin{cases} \partial_t u_t(x) + b_t(x).\nabla u_t(x) + \nabla u_t(x).\dot{X}_t = 0 \\ u_0 \in L^\infty(\mathbb{R}^d) \end{cases} . \tag{3.10}$$

Thanks to the hypothesis, and thanks to the usual characteristic method, the function $(t,x) \to u_0(\Phi_t^{-1}(x))$ is the unique weak solution of equation (3.10). In order to show that this solution is a weak controlled solution, the following Lemmas are needed. The whole method relies on a Taylor expansion of the function $t \to \varphi(\Phi_t(x))$.

Lemma 3.3.4. *Let $b \in L^\infty([0,T],\mathrm{Lin}(\mathbb{R}^d))$ and $X \in \mathcal{C}^\gamma(\mathbb{R}^d)$ such that the flow Φ and its inverse Φ^{-1} exist. There exists $\varepsilon(T, \|b\|_{\infty;\mathrm{Lin}}, \|X\|_\gamma) > 0$, such that for all $0 \leqslant s < t \leqslant T$ with $|t-s| < \varepsilon$, all $r \in [0,1]$, all test functions $\psi \in C_c^\infty(\mathbb{R}^d,\mathbb{R}^m)$ and all $N \in \mathbb{N}$, there exists a constant $C_{N,\psi} > 0$ such that*

$$\psi(r\Phi_t(x) + (1-r)\Phi_s(x))(1+|x|)^N \leqslant C_{N,\psi}.$$

Proof. Let $\varepsilon > 0$ to be specified later and $s,t \in [0,T]$ with $|t-s| < \varepsilon$. Thanks to Proposition 3.2.35, where we set $y = \Phi_s(x)$

$$\begin{aligned} |\psi(r\Phi_t(x) + (1-r)\Phi_s(x))|(1+|x|)^N &= |\psi(r\Phi_t(\Phi_s^{-1}(y)) + (1-r)y)|(1+|\Phi_s^{-1}(y)|)^N \\ &\lesssim (1+|y|)^N|\psi(r(\Phi_t(\Phi_s^{-1}(y)) - y) + y)|. \end{aligned}$$

Furthermore, applying Lemma 3.2.33,

$$\begin{aligned} |\Phi_t(\Phi_s^{-1}(y)) - y| &= |\Phi_{t-s}(y) - y| \\ &\leqslant |t-s|^\gamma\|\Phi(y)\|_\gamma \\ &\leqslant K\varepsilon^\gamma(1+|y|)(1+\|X\|_\gamma). \end{aligned}$$

If $|y| \leqslant 1$, then for $\varepsilon \leqslant (2K(1+\|X\|_\gamma))^{-1/\gamma}$,

$$|\Phi_t(\Phi_s^{-1}(y)) - y| \leqslant 1$$

and $r\Phi_t(\Phi_s^{-1}(y)) + (1-r)y \in B(0,2)$, so that

$$\mathbb{1}_{B(0,1)}(|y|)|\psi(r\Phi_t(\Phi_s^{-1}(y)) + (1-r)y)|(1+|y|)^N \lesssim \sup_{B(0,2)} |\psi(z)|.$$

When $|y| > 1$ and $\varepsilon \leqslant 2^{-2/\gamma}(K(1+\|X\|_\gamma))^{-1/\gamma}$, we have

$$|\Phi_t(\Phi_s^{-1}(y)) - y| \leqslant |y|/2$$

and $|r\Phi_t(\Phi_s^{-1}(y)) + (1-r)y| \geqslant |y|/2$. Now let us take $\tilde{C}_\psi > 0$ such that $|\psi(z)| \leqslant \tilde{C}_\psi/(1+2|z|)^N$. We have

$$\begin{aligned} \mathbb{1}_{{}^cB(0,1)}(|y|)|\psi(r(\Phi_t(\Phi_s^{-1}(y)) - y) + y)| &\lesssim (1 + 2|r\Phi_t(\Phi_s^{-1}(y)) + (1-r)y|)^{-N} \\ &\lesssim (1+|y|)^{-N}. \end{aligned}$$

Since Φ^{-1} satisfies the same type of equation as Φ, thanks to Lemma 3.2.33, for ε as before,

$$(1+|x|) \leqslant 1 + |\Phi_s^{-1}(y) - y| + |y| \leqslant 2(1+|y|)$$

and finally

$$|\psi(r(\Phi_t(x) - \Phi_s(x)) + \Phi_s(x))| \lesssim (1+|x|)^{-N}$$

which ends the proof. □

Remark 3.3.5. We can choose $\varepsilon = 2^{-2/\gamma}(K(1+\|X\|_\gamma))^{-1/\gamma}$ where K is the constant of Lemma 3.2.33.

An immediate corollary gives an estimate of the growth of the function $\varphi(\Phi_t(.))$.

Corollary 3.3.6. *Let b and Φ as in the previous lemma and let $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$. Then for all $t \in [0,T]$ and all $N \in \mathbb{N}$, $x \to (1+|x|)^N \varphi(\Phi_t(x)) \in L^\infty(\mathbb{R}^d)$.*

Proof. Let $0 = t_0 < t_1 < \cdots < t_{n+1} = t$ such that $|t_{i+1} - t_i| < \varepsilon$ where $\varepsilon > 0$ is chosen as in the previous lemma and n is chosen as small as possible. Hence

$$|(1+|x|)^N \varphi(\Phi_t(x))| \leqslant (1+|x|)^N \left(|\varphi(x)| + \sum_{i=0}^n |\varphi(\Phi_{t_{i+1}}(x)) - \varphi(\Phi_{t_i}(x))| \right)$$

$$\leqslant (1+|x|)^N \left(\sum_{i=0}^n \int_0^1 \mathrm{d}r |D\varphi(r(\Phi_{t_{i+1}}(x) - \Phi_{t_i}(x)) + \Phi_{t_i}(x))||\Phi_{t_{i+1}}(x) - \Phi_{t_i}(x)| + |\varphi(x)| \right)$$

$$\lesssim (n+1) \lesssim \frac{1}{\varepsilon}.$$

□

Finally, thanks to Lemma 3.2.18, we are able to give some estimates for the $\mathcal{C}^\gamma$ and $\mathcal{D}_X^\gamma$ norms of $\varphi(\Phi_.(x))$.

Corollary 3.3.7. *Let b as in Lemma 3.3.4, and $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$, then $\varphi(\Phi_.(x)) \in \mathcal{D}_X^\gamma([0,T])$. Furthermore for all $N \geqslant 0$, we have*

$$\|\varphi(\Phi_.(x))\|_\gamma \lesssim (1 + \|X\|_\gamma)^{1+1/\gamma}(1+|x|)^{-N}$$

and the implicit constant on the right hand side is nondecreasing in all the parameters.

Proof. Let $\varepsilon > 0$ as in Lemma 3.3.4 and $|t-s| \leqslant \varepsilon$ and $N > 0$. If C denotes the constant of Lemma 3.3.4, we have

$$\begin{aligned}
|\varphi(\Phi_t(x)) - \varphi(\Phi_s(x))| &= \left| \int_s^t \mathrm{d}r D\varphi(r(\Phi_t(x) - \Phi_s(x)) + \Phi_s(x)).(\Phi_t(x) - \Phi_s(x)) \right| \\
&\leqslant C\|\Phi_.(x)\|_\gamma (1+|x|)^{-(N+1)}|t-s|^\gamma. \\
&\leqslant CK(1+\|X\|_\gamma)(1+|x|)^{-N}|t-s|^\gamma
\end{aligned}$$

Hence, thanks to Lemma 3.2.18, we have

$$\|\varphi(\Phi_.(x))\|_\gamma \leqslant \frac{T}{\varepsilon} C_{N+1}(1+\|X\|_\gamma)K(1+|x|)^{-N}.$$

Thanks to Remark 3.3.5 we can choose $\varepsilon = 2^{-2/\gamma}K^{-1/\gamma}$, we finally have

$$\|\varphi(\Phi_.(x))\|_\gamma \leqslant TC_{N+1}K^{1+1/\gamma}(1+\|X\|_\gamma)^{1+1/\gamma}(1+|x|)^{-N}.$$

To end the proof, we just have to remember that K and C_{N+1} are nondecreasing in the parameters. □

Lemma 3.3.8. *Let b and X as in Lemma 3.3.4 and with $\operatorname{div} b \in L^\infty(\mathbb{R}^d)$. For all $\varphi \in C_c^\infty(\mathbb{R}^d)$, $u(\nabla\varphi) \in \mathcal{D}_X^\gamma$ is controlled by X and there exists a constant $C_{\mathcal{D}}(\varphi)$ which is nondecreasing in $\|b\|_{\mathrm{Lin}}, \|\operatorname{div} b\|_\infty$ and T such that*

$$\|u(\nabla\varphi)\|_{\mathcal{D}_X^\gamma} \leqslant C_{\mathcal{D}}(\varphi)\|u_0\|_\infty(1+\|X\|_\gamma)^{1+1/\gamma}.$$

Proof. We first rewrite $u(\varphi)$ in a more suitable way. We use Theorem 3.2.31 and Lemma 3.2.35 and we have

$$\begin{aligned}
u_t(\nabla\varphi) &= \int_{\mathbb{R}^d} u_t(x)\nabla\varphi(x)\mathrm{d}x \\
&= \int_{\mathbb{R}^d} u_0(\Phi_t^{-1}(x))\nabla\varphi(x)\mathrm{d}x \\
&= \int_{\mathbb{R}^d} u_0(x)\nabla\varphi(\Phi_t(x))|\operatorname{Jac}(\Phi_t(x))|\mathrm{d}x.
\end{aligned}$$

Thanks to Lemma 3.2.17, we know that when a, b are controlled by X, the product ab is also controlled by X and furthermore $\|ab\|_{\mathcal{D}_X^\gamma} \lesssim \|a\|_{\mathcal{D}_X^\gamma}\|b\|_{\mathcal{D}_X^\gamma}$. Hence, in order to prove that $u(\nabla\varphi)$ is controlled it is enough to prove that for all $x \in \mathbb{R}^d$, $t \mapsto \nabla\varphi(\Phi_t(x))$ and $t \mapsto |\operatorname{Jac}(\Phi_t(x))|$ are controlled, with good estimates in x for the controlled norms of those two functions. Since everything is smooth here, we can apply Theorem 3.2.31 and we have

$$|\operatorname{Jac}(\Phi_t(x))| = \exp\left(\int_0^t (\operatorname{div} b)(\Phi_q(x))\mathrm{d}q\right).$$

Hence

$$|\operatorname{Jac}(\Phi_t(x))| \leqslant \exp(T\|\operatorname{div} b\|_\infty).$$

Furthermore

$$||\operatorname{Jac}(\Phi_t(x))| - |\operatorname{Jac}(\Phi_s(x))|| \leqslant |t-s|\|\operatorname{div} b\|_\infty \exp(T\|\operatorname{div} b\|_\infty),$$

hence $|\operatorname{Jac}(\Phi_\cdot(x))| \in \mathcal{D}_X^\gamma$, its derivative is zero and its recaller is itself, and we have

$$\|\operatorname{Jac}(\Phi_\cdot(x))\|_{\mathcal{D}_X^\gamma} \lesssim (1 + \|\operatorname{div} b\|_\infty) \exp(T\|\operatorname{div} b\|_\infty).$$

We need a bit more work to handle $\nabla\varphi(\Phi_\cdot(x))$. Thanks to Corollaries 3.3.6 and 3.3.7, for all $N \geqslant 0$ we already have

$$\|\nabla\varphi(\Phi_\cdot(x))\|_\gamma + \|\nabla\varphi(\Phi_\cdot(x))\|_\infty \lesssim (1 + \|X\|_\gamma)^{1+1/\gamma}(1+|x|)^{-N}.$$

Moreover

$$\begin{aligned}
&\nabla\varphi(\Phi_t(x)) - \nabla\varphi(\Phi_s(x)) \\
&\quad= D^2\varphi(\Phi_s(x)).(X_t - X_s) \\
&\qquad + \int_0^1 \mathrm{d}r D^2\varphi(r(\Phi_t(x) - \Phi_s(x)) + \Phi_s(x)) - D^2\varphi(\Phi_s(x)).(X_t - X_s) \\
&\qquad + \int_0^1 \mathrm{d}r D^2\varphi(r(\Phi_t(x) - \Phi_s(x)) + \Phi_s(x)). \int_s^t b_q(\Phi_q(x))\mathrm{d}q \\
&\quad= \nabla\varphi(\Phi_\cdot(x))'_s X_{s,t} + \nabla\varphi(\Phi_\cdot(x))^\#_{s,t}.
\end{aligned}$$

Thanks to Corollary 3.3.7,

$$\|D^2\varphi(\Phi_\cdot(x))\|_\gamma \lesssim (1 + \|X\|_\gamma)^{1+1/\gamma}(1+|x|)^{-N}.$$

Next, thanks to Lemma 3.2.18, it is enough to control the local norm of $(s,t) \mapsto \nabla\varphi(\Phi_\cdot(x))^\#_{s,t}$ when we choose ε as in Lemma 3.3.4. By Lemma 3.2.33, we already know that,

$$\left|\int_s^t b_q(\Phi_q(x))\mathrm{d}q\right| \lesssim K|t-s|(1 + \|b\|_{\mathrm{Lin}})(1+|x|)(1+\|X\|_\gamma),$$

hence it is enough to bound $\int_0^1 \mathrm{d}r D^2\varphi(r(\Phi_t(x) - \Phi_s(x)) + \Phi_s(x))$. But thanks to Lemma 3.3.4, this quantity is bounded by $(1+|x|)^{-(N+1)}$. Furthermore

$$\begin{aligned}
&\int_0^1 \mathrm{d}r (D^2\varphi(r(\Phi_t(x) - \Phi_s(x)) + \Phi_s(x)) - D^2\varphi(\Phi_s(x))).(X_t - X_s) \\
&= \int_0^1 \mathrm{d}r \int_0^1 \mathrm{d}q D^3\varphi(rq(\Phi_t(x) - \Phi_s(x)) + \Phi_s(x)).(\Phi_t(x) - \Phi_s(x)).(X_t - X_s)
\end{aligned}$$

and again thanks to Lemmas 3.2.33 and 3.3.4 we have

$$\sup_{x\in\mathbb{R}^d} (1+|x|)^{-N} \sup_{s\neq t, |t-s|\leqslant\varepsilon} |\nabla\varphi(\Phi_\cdot(x))^\#_{s,t}|/|t-s| < +\infty.$$

Furthermore, since $\varepsilon \sim ((1+\|X\|_\gamma)K)^{-1/\gamma}$ thanks to Remark 3.3.5, thanks to Lemma 3.2.18 we have

$$\|\nabla\varphi(\Phi_\cdot(x))\|_{\mathcal{D}_X^\gamma} \lesssim (1+|x|)^{-N}(1+\|X\|_\gamma)^{1+1/\gamma}.$$

As all the previous constants are nondecreasing with respect to $\|b\|_{\mathrm{Lin}}$, so is the implicit constant in this last inequality. Hence, when we put this inequality and the inequality for $|\operatorname{Jac}(\Phi(x))|$ together, we have

$$\|\nabla\varphi(\Phi_\cdot(x))\operatorname{Jac}(\Phi_\cdot(x))\|_{\mathcal{D}_X^\gamma} \leqslant C_{N,T,\varphi}(\|b\|_{\mathrm{Lin}}, \|\operatorname{div} b\|_\infty)(1+|x|)^{-N}(1+\|X\|_\gamma)^{1+1/\gamma} \tag{3.11}$$

where the constant C is nondecreasing in T, $\|b\|_{\mathrm{Lin}}$ and $\|\operatorname{div} b\|_\infty$. Finally as

$$u_t(\nabla\varphi) = \int_{\mathbb{R}^d} u_0(x)\nabla\varphi(\Phi_t(x))\operatorname{Jac}(\Phi_t(x))\mathrm{d}x,$$

the function $u(\nabla\varphi)$ is controlled by X and

$$\|\nabla\varphi(\Phi_\cdot(x))\operatorname{Jac}(\Phi_\cdot(x))\|_{\mathcal{D}_X^\gamma} \leqslant C(T, \|X\|_\gamma, \|b\|_{\mathrm{Lin}}, \|\operatorname{div} b\|_\infty, \varphi, D\varphi, D^2\varphi, D^3\varphi)\|u_0\|_\infty.$$

where the constant C is the constant of Equation (3.11) with $N = d+1$. □

Remark 3.3.9. The last proof shows gives that for all $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $u(\psi) \in \mathcal{C}^\gamma(\mathbb{R}^d)$ with the bound

$$\|u(\psi)\|_\gamma \leqslant C(T, \|u_0\|_\infty, \|b\|_{\infty,\mathrm{Lin}}, \psi)(1+\|X\|_\gamma)^{1+1/\gamma}.$$

Hence, for all $t \in [0,T]$, $u_t(\psi)$ has a meaning as a continuous function of t.

The last proof, and particularly Equation (3.11) gives us the following corollary

Corollary 3.3.10. *Let $\varphi_0 \in C_c^\infty(\mathbb{R}^d)$. For all $x \in \mathbb{R}^d$ the function $t \to \varphi_0(\Phi_t(x))e^{\int_0^t \operatorname{div} b(\Phi_q(x))\mathrm{d}q}$ is controlled by X. Furthermore $x \to \|\varphi_0(\Phi_\cdot(x))\|_{\mathcal{D}_X^\gamma}$ is decreasing faster than any polynomial and*

$$\|\|\varphi_0(\Phi_\cdot(x))\|_{\mathcal{D}_X^\gamma([0,T])}(1+|x|)^{-N}\|_{L^\infty(\mathbb{R}^d)} \leqslant C(T, \|b\|_{\infty,\mathrm{Lin}}, \varphi_0, D\varphi_0, D^2\varphi_0)(1+\|X\|_\gamma)^{1+1/\gamma}.$$

Remark 3.3.11. In fact, to prove that $u(\nabla\varphi_0)$ is controlled by X we do not need $\varphi_0 \in C_c^\infty(\mathbb{R}^d)$, but only $\varphi_0 \in C^3(\mathbb{R}^d)$ with $x \to (1+|x|)^{d+2}(|\nabla\varphi_0(x)| + |D^2\varphi(x)| + |D^3\varphi(x)|) \in L^\infty(\mathbb{R}^d)$.

When we test $(t,x) \mapsto u_0(\Phi_t^{-1}(x))$ against smooth compactly supported function φ, thanks to a change of variable, proving that $u(\varphi)$ is controlled by X is equivalent proving that $t \mapsto \varphi(\Phi_t(x))$ is controlled by X and has good integrability properties, which is what the previous lemmas have just achieved. We are now able to prove Theorem 3.3.3.

Proof of Theorem 3.3.3. Let us first consider the weak solution u of equation (3.10) (see Section 3.2.5). We have shown that for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, $u(\varphi) \in \mathcal{C}^\gamma([0,T])$ and $u(\nabla\varphi) \in \mathcal{D}_X^\gamma([0,T])$, hence $\int_0^\cdot u_s(\nabla\varphi)\mathrm{d}\mathbb{X}_s$ is well defined as a rough integral. Furthermore, for $t \in [0,T]$

$$\begin{aligned} u_t(\operatorname{div}(b_t\varphi)) &= \int_{\mathbb{R}^d} \mathrm{d}x u_t(x)(b_t(x).\nabla\varphi(x) + \varphi(x)\operatorname{div} b_t(x))\mathrm{d}x \\ &= \int_{\mathbb{R}^d} u_0(x)(b_t(\Phi_t(x)).\nabla\varphi(\Phi_t(x)) + \varphi(\Phi_t(x))\operatorname{div} b_t(\Phi_t(x)))|\operatorname{Jac}(\Phi_t(x))|\mathrm{d}x \end{aligned}$$

and as $|\operatorname{Jac}(\Phi_t(x))| \leqslant e^{\|\operatorname{div} b\|_\infty T}$ and $|\operatorname{Jac}(\Phi_t^{-1}(x))| \leqslant e^{\|\operatorname{div} b\|_\infty T}$

$$\begin{aligned}
|u_t(\operatorname{div}(b_t\varphi))| &\leqslant \|u_0\|_\infty e^{\|\operatorname{div} b\|_\infty T}\int_{\mathbb{R}^d}|(b_t(\Phi_t(x)).\nabla\varphi(\Phi_t(x))+\varphi(\Phi_t(x))\operatorname{div} b_t(\Phi_t(x)))|\mathrm{d}x\\
&\leqslant \|u_0\|_\infty e^{2\|\operatorname{div} b\|_\infty T}\int_{\mathbb{R}^d}\mathrm{d}x|b_t(x).\nabla\varphi(x)+\varphi(x)\operatorname{div} b_t(x)|\mathrm{d}x\\
&\lesssim_{\varphi,\nabla\varphi} \|u_0\|_\infty\|b\|_{\infty;\mathrm{Lin}}e^{2\|\operatorname{div} b\|_\infty T}(\|b\|_{\infty,\mathrm{Lin}}+\|\operatorname{div} b\|_\infty)
\end{aligned}$$

Hence $u(\operatorname{div}(b.\nabla\varphi)) \in L^\infty([0,T])$ and $\int_0^t \mathrm{d}s u_s(b_s.\nabla\varphi)$ is well-defined for all $t\in[0,T]$. As u is a weak solution of (3.10), for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ and all $f \in C_c^\infty([0,T])$, the smooth functions with compact support in $[0,T]$,

$$u_0(\varphi)f_0 - u_T(f_T) + \int_0^T\left(u_t(\varphi) - \int_0^t u_s(b_s.\nabla\varphi + \operatorname{div}(b_s)\varphi)\mathrm{d}s - \int_0^t u_s(\nabla\varphi)\dot{X}_s\mathrm{d}s\right)\dot{f}_t\mathrm{d}t = 0$$

and since $\int_0^t u_s(\nabla\varphi)\dot{X}_s\mathrm{d}s = \int_0^t u_s(\nabla\varphi)\mathrm{d}\boldsymbol{X}_s$, thanks to Lemma 3.3.8, we have

$$\int_0^T\left(u_t(\varphi) - u_0(\varphi) - \int_0^t u_s(\operatorname{div}(b_s\varphi))\mathrm{d}s - \int_0^t u_s(\nabla\varphi)\mathrm{d}\boldsymbol{X}_s\right)\dot{f}_t\mathrm{d}t = 0.$$

But $t \to u_t(\varphi) - u_0(\varphi) - \int_0^t u_s(\operatorname{div}(b_s\varphi))\mathrm{d}s - \int_0^t u_s(\nabla\varphi)\mathrm{d}\boldsymbol{X}_s$ is a continuous function, thanks to Remark 3.3.9, Lemma 3.3.8 and the previous computation. Hence, for every $t\in[0,T]$,

$$u_t(\varphi) - u_0(\varphi) - \int_0^t u_s(\operatorname{div}(b_s\varphi))\mathrm{d}s - \int_0^t u_s(\nabla\varphi)\mathrm{d}\boldsymbol{X}_s = 0$$

and u is a WCS of equation (3.9).

We will show that it is unique. Let v a WCS to eq. (3.10). Then, we have for all $f \in C_c^\infty([0,T])$ and all $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\int_0^T v_t(\varphi)\dot{f}_t\mathrm{d}t = \int_0^T\left(v_0(\varphi) + \int_0^t v_s(\operatorname{div}(b_s\varphi))\mathrm{d}s + \int_0^t v_s(\nabla f)\dot{X}_s\mathrm{d}s\right)\dot{f}_t\mathrm{d}t$$

hence

$$-v(\partial_t f\otimes\varphi) - v_0(f_0\otimes\varphi) + v(\operatorname{div}\{(b+\dot{X})(f\otimes\varphi)\}) = 0.$$

As this equation is linear, and thanks to the density of the linear span of $C_c^\infty([0,T])\otimes C_c^\infty(\mathbb{R}^d)$ into $C_c^\infty([0,T]\times\mathbb{R}^d)$, for all $\psi \in C_c^\infty([0,T]\times\mathbb{R}^d)$,

$$-v(\partial_t\psi) - v_0(\psi(0,.)) + v(\operatorname{div}\{(b+\dot{X})\psi\}) = 0$$

Hence, v is a weak solution of equation (3.10). By Section 3.2.5, $v = u$ and uniqueness holds for the weak controlled solutions in the smooth case. □

3.3.2 Existence of a weak controlled solution

We now have all the tools to deal with the Rough Transport Equation. We are looking for weak controlled solutions. Here we will no longer suppose that X is smooth. As usual in such a setting, we will approximate X and b in a smooth way, and use the a priori bounds of the previous part to obtain compactness. In the case where X is a stochastic process, in order to have qsolution which are measurable in ω, the result will be a bit more subtle and stated in the next subsection. Furthermore, we give the entire proof in the deterministic case as it is quite straighforward relying on a standard compactness argument.

Theorem 3.3.12. *Let $\frac{1}{3} < \gamma \leqslant 1/2$, $b \in L^\infty([0,T];\mathrm{Lin}(\mathbb{R}^d))$ with $\operatorname{div} b \in L^\infty([0,T];L^\infty(\mathbb{R}^d))$ and let $\boldsymbol{X} \in \mathcal{R}^\gamma([0,T])$. Then there exists a weak controlled solution $u \in L^\infty([0,T]\times\mathbb{R}^d)$ to equation (3.9).*

Proof. As $\boldsymbol{X} \in \mathcal{R}^\gamma([0,T])$, there exists a sequence $(\boldsymbol{X}^\varepsilon)_\varepsilon = (X^\varepsilon, \mathbb{X}^\varepsilon)_\varepsilon \in \mathcal{R}^\gamma([0,T])$ with $\mathbb{X}^\varepsilon_{s,t} = \int_s^t X^\varepsilon_s - X^\varepsilon_r \mathrm{d}X^\varepsilon_r$ such that $X^\varepsilon \in C^1(\mathbb{R}^d)$ and $\|\boldsymbol{X} - \boldsymbol{X}^\varepsilon\|_{\mathcal{R}^\gamma} \to 0$. We can also approximate b by b^ε such that $b^\varepsilon \in L^\infty([0,T];\mathrm{Lip}(\mathbb{R}^d)\cap C^1(\mathbb{R}^d))$ and $\operatorname{div} b^\varepsilon \in L^\infty([0,T];C^1_b(\mathbb{R}^d))$. Let us consider the weak Rough solution u^ε of the equation $\partial_t u^\varepsilon + b^\varepsilon.\nabla u^\varepsilon + \dot{X}^\varepsilon_t = 0$ with $u^\varepsilon_0 = u_0$. Thanks to Theorem 3.3.3, we know that for all $t \in [0,T]$ and all $\varphi \in C^\infty_c(\mathbb{R}^d)$

$$u^\varepsilon_t(\varphi) = u_0(\varphi) + \int_0^t u^\varepsilon_s(b^\varepsilon_s.\nabla\varphi)\mathrm{d}s + \int_0^t u^\varepsilon_s(\nabla\varphi)\mathrm{d}\mathbb{X}^\varepsilon_s. \tag{3.12}$$

The strategy here is to extract a subsequence such that each term from the previous equality converges. First let us show that there exists $u \in L^\infty([0,T]\times\mathbb{R}^d)$ and a subsequence ε_n such that

$$u^{\varepsilon_n} \xrightarrow{w-*L^1} u.$$

This is nearly straightforward since $u^\varepsilon_t(x) = u_0((\Phi^\varepsilon)^{-1}_t(x))$ and $\|u^\varepsilon\|_{L^\infty} \leqslant \|u_0\|_{L^\infty}$. Hence $(u^\varepsilon)_\varepsilon$ is relatively compact for the weak-star topology of L^1, and we take a subsequence (ε_n) and $u \in L^\infty([0,T]\times\mathbb{R}^d)$ such that $u^{\varepsilon_n} \xrightarrow{w-*L^1} u$. Furthermore, for all $\varphi \in C^\infty_c(\mathbb{R}^d)$, we have

$$u^{\varepsilon_n}(\varphi) \xrightarrow{w-*L^1([0,T])} u(\varphi)$$

But thanks to Remark 3.3.9, we know that $u^\varepsilon(\varphi) \in \mathcal{C}^\gamma([0,T])$ and that

$$\begin{aligned}
\|u^\varepsilon(\varphi)\|_\gamma &\leqslant C(T,\|u_0\|_\infty,\|b^\varepsilon\|_{\infty,\mathrm{Lin}},\varphi)(1+\|X^\varepsilon\|_\gamma)^{1+1/\gamma} \\
&\lesssim C(T,\|u_0\|_\infty,\|b\|_{\infty,\mathrm{Lin}},\varphi)(1+\|X\|_\gamma)^{1+1/\gamma} \\
&< +\infty.
\end{aligned}$$

We can apply Arzelà-Ascoli to $u^\varepsilon(\varphi)$, and there exists $l(u,\varphi) \in C([0,T])$ such that, for another subsequence $(\tilde{\varepsilon}_n)$ of (ε_n), $(u^{\tilde{\varepsilon}_n}(\varphi))$ converges uniformly to $l(u,\varphi)$. Hence we have $u(\varphi) = l(u,\varphi)$ and $u(\varphi) \in \mathcal{C}^\gamma([0,T])$. The same strategy works for $\int_0^t u^\varepsilon_s(\operatorname{div}(b^\varepsilon_s\varphi))\mathrm{d}s$ up to extraction of an other subsequence. Hence, for all $\varphi \in C^\infty_c(\mathbb{R}^d)$, there exists another subsequence, let us denote it again by (ε_n) such that $\left(\int_0^\cdot u^{\varepsilon_n}_s(\operatorname{div}(b^{\varepsilon_n}_s.\nabla\varphi))\mathrm{d}s\right)$ converges uniformly to $\int_0^\cdot u_s(b_s.\nabla\varphi)\mathrm{d}s$. Thanks to Lemma 3.3.8, as $\sup_{\varepsilon>0}\|X^\varepsilon\|_{0,\gamma} + \|b^\varepsilon\|_{\infty,\mathrm{Lin}} + \|\operatorname{div} b^\varepsilon\|_\infty < +\infty$ we have

$$\sup_{\varepsilon>0}\|u^\varepsilon_s(\nabla\varphi)\|_{\mathcal{D}^\gamma_{X^\varepsilon}} < +\infty.$$

Hence, $u^\varepsilon(\nabla\varphi)$ is bounded in the space $\mathcal{D}^\gamma_{X^\varepsilon}$, uniformly in ε. It is possible to apply the Arzelà-Ascoli theorem to $u^\varepsilon(\nabla\varphi)$, $(u^\varepsilon(\nabla\varphi))'$ and $(u^\varepsilon(\nabla\varphi))^\#$, and there exists $u(\nabla\varphi) \in \mathcal{D}^\gamma_X$ such that

$$u^{\varepsilon_n}(\nabla\varphi) \xrightarrow{\mathcal{D}^\gamma} u(\nabla\varphi).$$

Furthermore, thanks to the definition of the rough integral (see Theorem 3.2.20 above) and the comparison between controlled paths (see Lemma 3.2.17), we know that

$$\int_0^\cdot u_s^{\varepsilon_n}(\nabla\varphi)\mathrm{d}\boldsymbol{X}^{\varepsilon_n} \xrightarrow{\mathcal{C}^\gamma} \int_0^\cdot u_s(\nabla\varphi)\mathrm{d}\boldsymbol{X}_s.$$

Hence the last term in equation (3.12) converges. Finally we have shown that $u \in L^\infty([0,T]\times\mathbb{R}^d)$ is such that for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, $u(\nabla\varphi)$ is controlled by X and for all $t \in [0,T]$,

$$u_t(\varphi) = u_0(\varphi) + \int_0^t u_s(b_s.\nabla\varphi)\mathrm{d}s + \int_0^t u_s(\nabla\varphi)\mathrm{d}\boldsymbol{X}_s$$

and u is a weak controlled solution to equation (3.9). □

3.3.3 Existence of a measurable weak controlled solution

The proof of the previous theorem strongly relies on Arzelà-Ascoli theorem and compactness in continuous functions. Nevertheless, since we are working with pathwise arguments, these extractions do not guarantee the limit to be measurable in ω. The following definition is a refinement of Definition 3.3.1 in the case we are working with a rough path lift of a stochastic process. The subsequent theorem proves that such solutions exist in that case.

Definition 3.3.13. Let $1/3 < \gamma \leqslant 1/2$, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and X a continuous stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ such that almost surely X lifts to a Rough Path $\boldsymbol{X} \in \mathcal{R}^\gamma([0,T])$ in a measurable way. Let $b \in L^\infty(\Omega \times [0,T]; \mathrm{Lin}(\mathbb{R}^d))$ and $u_0 \in L^\infty(\Omega \times \mathbb{R}^d)$.

In that setting, a (Stochastic) weak controlled solution of the Rough Transport Equation with initial condition u_0 is a function $u \in L^\infty(\Omega \times [0,T] \times \mathbb{R}^d)$ such that almost surely, $u(\varphi) \in \mathcal{D}_X^\gamma([0,T])$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ and almost surely, for almost all $t \in [0,T]$,

$$u_t(\varphi) = u_0(\varphi) + \int_0^t u_q(\mathrm{div}(b_q\varphi))\mathrm{d}q + \int_0^t u_q(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q,$$

where that last integral is understood as the controlled rough integral of $u(\nabla\varphi)$ against $\mathrm{d}\boldsymbol{X}$.

Theorem 3.3.14. *Let b, u_0 and X as in the last definition. We also assume that for a smooth measurable approximation $\boldsymbol{X}^\varepsilon$ of $\boldsymbol{X}$ and all $1/3 < \gamma \leqslant 1/2$ and all $+\infty > p \geqslant 1$, $\mathbb{E}[\|\boldsymbol{X} - \boldsymbol{X}^\varepsilon\|_{\mathcal{R}^\gamma}^p] \to 0$.*

Then there exists a Stochastic weak controlled solution for the Rough Transport Equation with initial condition u_0.

The proof is quite similar to the one in the deterministic case. The only - but significant - difference is that we can no longer apply naïvely the Arzelà-Ascoli theorem. As before we will use a weak-$*$-compactness theorem to identify a limit. In order to find a Hölder continuous version of this limit, we will use a sequence of partitions (here the dyadic numbers) and Riemann sums. The end of the proof will be devoted to prove the convergence of each term to the wanted quantities.

Proof. Let $(\boldsymbol{X}^\varepsilon)$ the smooth approximation of $\boldsymbol{X}$. Let $k : \mathbb{R}^d \to \mathbb{R}^d$ a smooth mollifier, *i.e.* $k \in C_c^\infty(\mathbb{R}^d)$, $k = 1$ for $|x| \leqslant 1$ and $k = 0$ for $|x| > 2$, and $k_\varepsilon(x) = \frac{1}{\varepsilon^d} k\left(\frac{x}{\varepsilon}\right)$. Let $b^\varepsilon = k_\varepsilon * b$, hence $b^\varepsilon \in L^\infty(\Omega \times [0,T]; C^\infty(\mathbb{R}^d) \cap \mathrm{Lin}(\mathbb{R}^d))$ and for all $Y \in L^1(\Omega)$

$$\mathbb{E}[\|b^\varepsilon - b\|_{L^\infty([0,T];\mathrm{Lin}(\mathbb{R}^d))} Y] \to 0$$

and $\|b^\varepsilon\|_{L^\infty(\Omega\times[0,T];\mathrm{Lin}(\mathbb{R}^d))} \leqslant \|b\|_{L^\infty(\Omega\times[0,T];\mathrm{Lin}(\mathbb{R}^d))}$. Furthermore $\operatorname{div} b^\varepsilon \rightharpoonup^{L^\infty(\Omega\times[0,T]\times\mathbb{R}^d)} \operatorname{div} b$, *i.e.* for all $f \in L^1(\Omega \times [0,T] \times \mathbb{R}^d)$,

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}^d} \operatorname{div} b_t^\varepsilon(x) f_t(x) \mathrm{d}x\mathrm{d}t\right] \to \mathbb{E}\left[\int_{[0,T]\times\mathbb{R}^d} \operatorname{div} b_t(x) f_t(x) \mathrm{d}x\mathrm{d}t\right]$$

and $\|\operatorname{div} b^\varepsilon\|_{L^\infty(\Omega\times[0,T]\times\mathbb{R}^d)} \lesssim \|\operatorname{div} b\|_{L^\infty(\Omega\times[0,T]\times\mathbb{R}^d)}$.

Let Φ^ε the flow of the approximate equation $\Phi_t^\varepsilon(x) = x + \int_0^t b_q^\varepsilon(\Phi_q^\varepsilon(x))\mathrm{d}q + X_t^\varepsilon$ and $(\Phi^\varepsilon)^{-1}$ its inverse. We know, thanks to Theorem 3.3.3, that $u_t^\varepsilon(x) = u_0((\Phi^\varepsilon)_t^{-1}(x))$ is a weak controlled solution of the approximate Rough Transport Equation with initial condition u_0, *i.e.* for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, $t \to u_t^\varepsilon(\varphi)$ is controlled by X^ε almost surely, and for all $s < t \in [0,T]$,

$$u_t^\varepsilon(\varphi) - u_s^\varepsilon(\varphi) = \int_s^t u_q^\varepsilon(\operatorname{div}(b_q^\varepsilon \varphi))\mathrm{d}q + \int_s^t u_q^\varepsilon(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q^\varepsilon$$

First, as $u_0 \in L^\infty(\Omega \times \mathbb{R}^d)$,

$$\|u^\varepsilon\|_{L^\infty(\Omega\times[0,T]\times\mathbb{R}^d)} \leqslant \|u_0\|_\infty.$$

Hence there exists a subsequence abusively denoted again by (u^ε), and $u \in L^\infty(\Omega \times [0,T] \times \mathbb{R}^d)$ such that for all $Y \in L^1(\Omega)$, $f \in L^1([0,T])$, $\varphi \in L^1(\mathbb{R}^d)$

$$\mathbb{E}\left[\int_0^T \mathrm{d}t \int_{\mathbb{R}^d} u_t^\varepsilon(x)\varphi(x)\mathrm{d}x f_t \mathrm{d}t Y\right] \to \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} u_t(x)\varphi(x)\mathrm{d}x f_t \mathrm{d}t Y\right].$$

Let us find a weakly continuous version of the limit u. Let us define for $n \geqslant 0$ the points of the dyadic partition of $[0,T]$ by $t_i^n = iT2^{-n}$. We denote by $\Pi_n = \{t_i^n : i \in \{0,\dots,2^n\}\}$ and by $\Pi = \cup_{n\geqslant 1}\Pi_n$. As Π is countable and for all $t \in \Pi$

$$\|u_t^\varepsilon\|_{L^\infty(\Omega\times\mathbb{R}^d)} \leqslant \|u_0\|_\infty < +\infty,$$

there exists a subsequence of (u^ε), denoted again by (u^ε) such that for all $t \in \Pi$ there exists $\tilde{u}_t \in L^\infty(\Omega\times\mathbb{R}^d)$ such that $u_t^\varepsilon \rightharpoonup \tilde{u}_t$ weakly in $L^\infty(\Omega\times\mathbb{R}^d)$. Furthermore, for all $t \in \Pi$, $\|\tilde{u}_t\|_{L^\infty(\Omega\times\mathbb{R}^d)} \leqslant \|u_0\|_{L^\infty(\Omega\times\mathbb{R}^d)}$.

Furthermore, let us recall that thanks to Lemma 3.3.8, $u^\varepsilon(\varphi) \in \mathcal{D}_{X^\varepsilon}^\gamma$ almost surely and since $\|b^\varepsilon\|_{L^\infty(\Omega\times[0,T];\mathrm{Lin}(\mathbb{R}^d))} \leqslant \|b\|_{L^\infty(\Omega\times[0,T];\mathrm{Lin}(\mathbb{R}^d))}$ and $\|\boldsymbol{X}^\varepsilon\|_{\mathcal{R}^\gamma} \leqslant \|\boldsymbol{X}\|_{\mathcal{R}^\gamma}$

$$\|u^\varepsilon(\varphi)\|_{\mathcal{D}_{X^\varepsilon}^\gamma} \lesssim K_{\varphi,b,u_0}(1 + \|\boldsymbol{X}\|_{\mathcal{R}^\gamma})^{1+1/\gamma}.$$

We have, for all $s,t \in \Pi$, all $\varphi \in C_c^\infty(\mathbb{R}^d)$, since $\tilde{u}_t(\varphi) - \tilde{u}_s(\varphi) \in L^\infty(\Omega)$, for all $p \geqslant 1$,

$$\mathbb{E}[(\delta u_{s,t}^\varepsilon(\varphi))(\delta\tilde{u}_{s,t}(\varphi))^{2p-1}] \to \mathbb{E}[(\delta\tilde{u}_{s,t}(\varphi))^{2p}].$$

But, by Hölder's inequality,

$$\begin{aligned}\mathbb{E}[\delta u^{\varepsilon_n}_{s,t}(\varphi)(\delta\tilde{u}_{s,t}(\varphi))^{2p-1}] &\leqslant \mathbb{E}[(\delta\tilde{u}_{s,t}(\varphi))^{2p}]^{1-1/2p}\mathbb{E}[(\delta u^{\varepsilon_n}_{s,t}(\varphi))^{2p}]^{1/2p} \\ &\lesssim_p \mathbb{E}[(\delta\tilde{u}_{s,t}(\varphi))^{2p}]^{2p/(2p-1)}|t-s|^{\gamma}\end{aligned}$$

Hence

$$\mathbb{E}[(\delta\tilde{u}_{s,t}(\varphi))^{2p}] \lesssim_p K_{\varphi,\|b\|,\|u_0\|}|t-s|^{2\gamma p}. \tag{3.13}$$

Hence, for $t\in[0,T]$, $(t_k)_k,(\tilde{t}_k)_k\in\Pi^{\mathbb{N}}$ such that $t_k\to t$ and $\tilde{t}_k\to t$, $(\tilde{u}_{t_k}(\varphi)-\tilde{u}_{\tilde{t}_k}(\varphi))_k$ converges to zero in $L^p(\Omega)$. Furthermore $(\tilde{u}_{t_k}(\varphi))_k$ is Cauchy in $L^{2p}(\Omega)$. We call its limit $\tilde{u}_t(\varphi)$. Note that it is independent of the sequence in Π. Since $\|\tilde{u}_t(\varphi)\|_{L^\infty(\Omega)}\leqslant\|u_0\|_{L^\infty(\Omega\times\mathbb{R}^d)}\|\varphi\|_{L^1(\mathbb{R}^d)}$, by the dominated convergence theorem, the bound in Equation (3.13) holds for all $s,t\in[0,T]$. Hence, thanks to the usual Kolmogorov continuity Theorem, almost surely for all $\gamma'<\gamma$, $\tilde{u}(\varphi)\in\mathcal{C}^{\gamma'}([0,T])$ and for all $\mathbb{E}[\|\tilde{u}(\varphi)\|^p_{\mathcal{C}^\gamma}]<+\infty$ for $1\leqslant p<+\infty$.

Let us take $f\in C^\infty_c([0,T])$, $Y\in L^q(\Omega)$ for $1<q\leqslant+\infty$ and, $\varphi\in C^\infty_c(\mathbb{R}^d)$. As for every point $s\in\Pi$, $\tilde{u}_s(\varphi)=u_s(\varphi)$, and thanks to Riemann sum estimates, where we define

$$S^\varepsilon_n(\varphi,f)=2^{-n}\sum_{i=0}^{2^n-1}f_{t_i}u^\varepsilon_{t^n_i}(\varphi)\,\text{and}\,S_n(\varphi,f)=2^{-n}\sum_{i=0}^{2^n-1}f_{t_i}u_{t^n_i}(\varphi)$$

we have

$$\begin{aligned}\left|\mathbb{E}\left[\int_0^T(u^\varepsilon_t(\varphi)-\tilde{u}_t(\varphi))f_t\mathrm{d}tY\right]\right| &\leqslant \mathbb{E}\left[\left|\int_0^T f_tu^\varepsilon_t(\varphi)\mathrm{d}t-S^\varepsilon_n(\varphi,f)\right||Y|\right] \\ &\quad+|\mathbb{E}[(S^\varepsilon_n(\varphi,f)-S_n(\varphi,f))Y]| \\ &\quad+\mathbb{E}\left[\left|\int_0^T f_tu_t(\varphi)\mathrm{d}t-S_n(\varphi,f)\right||Y|\right] \\ &\lesssim \|f'\|_\infty\|u_0\|_{L^\infty(\Omega\times\mathbb{R}^d)}\mathbb{E}[|Y|]2^{-n} \\ &\quad+\|f\|_\infty 2^{-n}\sum_{i=0}^{2^n-1}|\mathbb{E}[u^\varepsilon_{t^n_i}(\varphi)-u^\varepsilon_{t^n_i}(\varphi)Y]| \\ &\quad+\|f\|_\infty 2^{-\gamma n}\mathbb{E}[(\sup_\varepsilon\|u^\varepsilon(\varphi)\|_\gamma+\|\tilde{u}\|_\gamma)|Y|].\end{aligned} \tag{3.14}$$

Letting ε go to zero and n go to infinity, we have

$$\mathbb{E}\left[\int_0^T f_tu_t(\varphi)\mathrm{d}tY\right]=\mathbb{E}\left[\int_0^T f_t\tilde{u}_t(\varphi)\mathrm{d}tY\right].$$

hence almost surely and for almost all $t\in[0,T]$ $u_t(\varphi)=\tilde{u}_t(\varphi)$. Since it is also true for all $\varphi\in C^\infty_c(\mathbb{R}^d)$, we have shown that there exists a version $\tilde{u}$ of the weak limit u such that $\tilde{u}\in L^\infty(\Omega\times[0,T]\times\mathbb{R}^d)$ and

$$u(\varphi)\in L^p(\Omega;\mathcal{C}^\gamma([0,T])).$$

From now on we will consider only this version and denote it by u.

Thanks to the hypothesis on b and b^ε, we know that for all $\varphi\in C^\infty_c(\mathbb{R}^d)$, $\mathrm{div}(b^\varepsilon\varphi)\to\mathrm{div}(b\varphi)$ strongly in $L^1([0,T]\times\mathbb{R}^d)$. Hence, for all $t\in[0,T]$

$$\int_0^t u^\varepsilon_q(\mathrm{div}(b^\varepsilon_q\varphi))\mathrm{d}q=\int_0^T\mathrm{d}q\int_{\mathbb{R}^d}\mathrm{d}xu^\varepsilon_q(x)\,\mathrm{div}(b^\varepsilon_q\varphi)\mathbb{1}_{[0,t]}(q)\to\int_0^t u_q(\mathrm{div}(b_q\varphi))\mathrm{d}q,$$

where the convergence is weak in $L^\infty(\Omega)$. Furthermore, since the following bound holds

$$\|u_q^\varepsilon(\operatorname{div}(b_q^\varepsilon\varphi))\|_{L^\infty(\Omega\times[0,T])} \leqslant \|u_0\|_{(\Omega\times\mathbb{R}^d)}\|\operatorname{div} b\varphi\|_{L^\infty(\Omega\times[0,T];L^1(\mathbb{R}^d))},$$

by the dominated convergence theorem we have

$$\int_0^t u_q^\varepsilon(\operatorname{div}(b_q^\varepsilon\varphi))\mathrm{d}q \rightharpoonup^{L^\infty(\Omega\times[0,T])} \int_0^t u_q(\operatorname{div}(b_q\varphi))\mathrm{d}q.$$

In order to prove that u is a weak controlled solution of the Rough Transport Equation, it remains to show that the last term $\int_0^\cdot u_q^\varepsilon(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q^\varepsilon$ converges weakly in $L^\infty(\Omega;\mathcal{D}'([0,T]))$ where $\mathcal{D}'$ is the set of distributions on $[0,T]$ to $\int_0^\cdot u_q(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q$. In order to do that, it is necessary to show that $u(\nabla\varphi)$ is controlled by X. Thanks to the last construction, for all $m\in\mathbb{N}$ and all $\varphi\in C_c^\infty(\mathbb{R}^d,\mathbb{R}^m)$ and all $t\in\Pi$, $u_t^\varepsilon(\varphi)$ converges to $u_t(\varphi)$ weakly in $L^\infty(\Omega)$ and furthermore $t\to u_t(\varphi)$ is almost surely in $\mathcal{C}^{\gamma'}([0,T])$ for all $\gamma'<\gamma$, and $\|u(\varphi)\|_{\mathcal{C}^{\gamma'}}\in L^p(\Omega)$ for all $1\leqslant p<+\infty$. Hence, this holds for $\nabla\varphi$ and $\nabla^2\varphi$ when $\varphi\in C_c^\infty(\mathbb{R}^d)$. Furthermore, thanks to Lemma 3.3.8, $u^\varepsilon(\nabla\varphi)' = u^\varepsilon(\nabla^2\varphi)$.

For all $s\leqslant t\in\Pi$ and all $Y\in L^q(\Omega)$ for $q>1$, we have $X_{s,t}Y\in L^1(\Omega)$, and the following computation holds

$$\begin{aligned}|\mathbb{E}[u_s^\varepsilon(\nabla\varphi)X_{s,t}^\varepsilon Y]-\mathbb{E}[u_s(\nabla\varphi)X_{s,t}Y]| &\leqslant |\mathbb{E}[u_s^\varepsilon(\nabla\varphi)X_{s,t}Y]-\mathbb{E}[u_s(\nabla\varphi)X_{s,t}Y]|\\ &\quad+\mathbb{E}[|u_s^\varepsilon(\nabla\varphi)||X_{s,t}^\varepsilon-X_{s,t}||Y|]\\ &\lesssim |\mathbb{E}[u_s^\varepsilon(\nabla\varphi)X_{s,t}Y]-\mathbb{E}[u_s(\nabla\varphi)X_{s,t}Y]|\\ &\quad+\mathbb{E}[|X_{s,t}^\varepsilon-X_{s,t}|^p]^{1/p}\mathbb{E}[|Y|^q]^{1/q}.\end{aligned}$$

Since $\mathbb{E}[\|\boldsymbol{X}^\varepsilon-\boldsymbol{X}\|_{\mathcal{R}^{\gamma'}}^p]\to_{\varepsilon\to0}0$ and $|u_s^\varepsilon(\nabla\varphi)|\leqslant\|u_0\|_{L^\infty(\Omega\times\mathbb{R}^d)}\|\nabla\varphi\|_{L^1(\mathbb{R}^d)}$, for all $1\leqslant p<+\infty$ and all $s,t\in\Pi$, $u_s^\varepsilon(\varphi)X_{s,t}^\varepsilon$ converge weakly in $L^p(\Omega)$ to $u_s(\varphi)X_{s,t}$. The same computation holds for $u_s^\varepsilon(\nabla^2\varphi)X_{s,t}^\varepsilon$ and $u_s(\nabla^2\varphi)X_{s,t}$ and $u_s^\varepsilon(\nabla^2\varphi)\mathbb{X}_{s,t}^\varepsilon$ and $u_s(\nabla^2\varphi)\mathbb{X}_{s,t}$.

Furthermore, for all $s,t\in\Pi$

$$r_{s,t}^\varepsilon(\nabla\varphi)=u^\varepsilon(\nabla\varphi)_t-u^\varepsilon(\nabla\varphi)_s-u_s^\varepsilon(\nabla^2\varphi)X_{s,t}^\varepsilon.$$

Hence $r_{s,t}^\varepsilon(\nabla\varphi)\rightharpoonup r_{s,t}(\nabla\varphi)$ weakly in L^p for all $1\leqslant p<+\infty$ and we have for all $Y\in L^q(\Omega)$,

$$\mathbb{E}[(r_{s,t}(\nabla\varphi)-(u(\nabla\varphi)_t-u(\nabla\varphi)_s)-u_s(\nabla^2\varphi)X_{s,t})Y]=0.$$

Hence, almost surely for all $s,t\in\Pi$,

$$u(\nabla\varphi)_t-u(\nabla\varphi)_s=u_s(\nabla\varphi)'X_{s,t}+r_{s,t}(\nabla\varphi).$$

thanks to the same limiting procedure, the previous equation is true for all $s,t\in[0,T]$. Furthermore, by the same computation, we also have that for all $s,t\in[0,T]$.

$$\mathbb{E}[|r_{s,t}|^{2p}]\lesssim_p|t-s|^{4\gamma p}.$$

Therefore, by Kolmogorov's continuity theorem, $r_{s,t}\in\mathcal{C}^{2\gamma'}$. Hence, almost surely $u(\nabla\varphi)$ is γ'-controlled by X and $\|u(\nabla\varphi)\|_{\mathcal{D}_X^{\gamma'}}\in L^p(\Omega)$ for all $1\leqslant p<+\infty$.

For all $n \geqslant 0$, $k \geqslant n$ and all $t \in \cap_{k\geqslant n}\Pi_k$, there exists $i_t^k \in \{0,\dots,2^k\}$ such that $t = t^k_{i^k_t}$. Then we define

$$S_k^\varepsilon(\nabla\varphi, t) = \sum_{i=0}^{i_t^k-1}[u^\varepsilon_{t_i^k}(\nabla\varphi)\delta X^\varepsilon_{t_i^k,t_{i+1}^k} + u^\varepsilon_{t_i^k}(\nabla^2\varphi)\mathbb{X}^\varepsilon_{t_i^k,t_{i+1}^k}]$$

and $S_k(\nabla\varphi,t)$ as the same quantity for u and $\boldsymbol{X}$. Thanks to the definitions of u, we know that $S_k^\varepsilon(\nabla\varphi,r)$ converges weakly in all $L^p(\Omega)$ for $1\leqslant p<+\infty$ to $S_k(\nabla\varphi,r)$. Furthermore, thanks to the estimates for the rough integrals, and since $\|u^\varepsilon(\nabla\varphi)\|_{\mathcal{D}^{\gamma'}_{X^\varepsilon}} \lesssim (1+\|\boldsymbol{X}\|_{\mathcal{R}^{\gamma'}})^{1+1/\gamma}$, we have

$$\begin{aligned}\left|\int_0^t u_q^\varepsilon(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q^\varepsilon - S_k^\varepsilon(\nabla\varphi,t)\right| &\lesssim \|u^\varepsilon_\cdot(\nabla\varphi)\|_{\mathcal{D}^{\gamma'}_{X^\varepsilon}}\|\boldsymbol{X}^\varepsilon\|_{\mathcal{R}^{\gamma'}}2^{-(3\gamma'-1)}\\ &\lesssim (1+\|\boldsymbol{X}\|_{\mathcal{R}^{\gamma'}})^{2+1/\gamma'}2^{-(3\gamma'-1)}\end{aligned}$$

and

$$\left|\int_0^t u_q(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q - S_k(\nabla\varphi,t)\right| \lesssim \|u_\cdot(\nabla\varphi)\|_{\mathcal{D}^{\gamma'}_X}\|\boldsymbol{X}\|_{\mathcal{R}^{\gamma'}}2^{-(3\gamma'-1)}.$$

Hence, for all $1<q\leqslant+\infty$, and $Y\in L^q(\Omega)$,

$$\begin{aligned}&\left|\mathbb{E}\left[\left(\int_0^t u_q^\varepsilon(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q^\varepsilon - \int_0^t u_q(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q\right)Y\right]\right|^p\\ &\leqslant \mathbb{E}\left[\left|\int_0^t u_q^\varepsilon(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q^\varepsilon - S_k^\varepsilon(\nabla\varphi,t)\right|^p\right]\|Y\|^p_{L^q(\Omega)} + |\mathbb{E}[(S_k^\varepsilon(\nabla\varphi,t)-S_k(\nabla\varphi,t))Y]|^p\\ &\quad+\mathbb{E}\left[\left|\int_0^t u_q(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q^\varepsilon - S_k(\nabla\varphi,t)\right|^p\right]\|Y\|^p_{L^q(\Omega)}.\end{aligned}$$

As $\varepsilon\to0$ and $k\to+\infty$, the right hand side of the previous inequality goes to zero, and for all $t\in\Pi$, $\int_0^t u_q^\varepsilon(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q^\varepsilon$ converges weakly to $\int_0^t u_q(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q$ in $L^p(\Omega)$ for all $1\leqslant p<+\infty$.

Furthermore, as for all $t\in[0,T]$

$$\left|\int_0^t u_q^\varepsilon(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q^\varepsilon\right| \lesssim \left(1+\|u^\varepsilon(\nabla\varphi)\|_{\mathcal{D}^{\gamma'}_{X^\varepsilon}}\right)((1+\|\boldsymbol{X}^\varepsilon\|_{\mathcal{R}^{\gamma'}})) \lesssim (1+\|\boldsymbol{X}\|_{\mathcal{R}^{\gamma'}})^{2+1/\gamma'}$$

and the same kind of bound holds for $\left|\int_0^t u_q(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q\right|$. By a similar computation to (3.14), for all $1\leqslant p<+\infty$, all $Y\in L^q(\Omega)$ and all $f\in C_c^\infty([0,T])$,

$$\mathbb{E}\left[\int_0^T f_t\int_0^t u_q^\varepsilon(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q^\varepsilon\mathrm{d}tY\right] \to \mathbb{E}\left[\int_0^T f_t\int_0^t u_q(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q\mathrm{d}tY\right].$$

Hence all terms of the approximate equation converge and we have, for all test functions f and Y,

$$\mathbb{E}\left[\int_0^T\left(u_t(\varphi)-u_0(\varphi)-\int_0^t u_q(\operatorname{div}(b_q\varphi))\mathrm{d}q - \int_0^t u_q(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q\right)f_t\mathrm{d}tY\right]=0.$$

Almost surely and for almost all $t\in[0,T]$,

$$u_t(\varphi) = u_0(\varphi) + \int_0^t u_q(\operatorname{div}(b_q\varphi))\mathrm{d}q + \int_0^t u_q(\nabla\varphi)\mathrm{d}\boldsymbol{X}_q.$$

Hence, $u\in L^\infty(\Omega\times[0,T]\times\mathbb{R}^d)$ is a weak controlled solution of the Rough Transport Equation driven by $\boldsymbol{X}$. $\square$

3.4 Uniqueness of weak controlled solutions

In order to prove the uniqueness of weak controlled solutions, we will use a duality argument. Indeed, we will suppose that everything is smooth, and that ψ is a strong solution of the Continuity Equation

$$\partial_t \psi + \operatorname{div}(b)\psi + b/\nabla\psi + \nabla\psi.\dot{X} = 0 \tag{3.15}$$

with $\psi_t \in C_c^\infty(\mathbb{R}^d)$ for all $t \in [0,T]$, we have for any weak solution of the rough transport equation, using the Leibniz rule on $u_t(\psi_t) = \langle u_t, \psi_t\rangle$,

$$\begin{aligned} \partial_t(u_t(\psi_t)) &= \partial_t u_t(\psi_t) + u_t(\partial_t(\psi_t)) \\ &= u_t(\operatorname{div}(b_t\psi_t)) + u_t(\nabla\psi_t)\dot{X}_t - u_t(\operatorname{div}(b\psi_t) - \nabla\psi_t\dot{X}_t) \\ &= 0. \end{aligned}$$

Hence $u_t(\psi_t) = u_0(\psi_0)$. As the equation is linear, it is enough to prove uniqueness when $u_0 = 0$, then $u_t(\psi_t) = 0$. The trick here is to solve equation (3.15) backward, such that for any fixed $\varphi \in C_c^\infty(\mathbb{R}^d)$ and any $t \in [0,T]$, there exists a solution of the equation (3.15) such that $\psi_t = \varphi$.

Of course such a strategy comes at a price. When X is not smooth, equation (3.15) does not make sense. Hence we will need to give a suitable notion of strong solution for the Rough Continuity Equation

$$\psi_t(x) - \psi_s(x) + \int_s^t \operatorname{div}(b_q\psi_q)(x)\mathrm{d}q + \int_s^t \nabla\psi_q(x)\mathrm{d}\mathbf{X}_q = 0. \tag{3.16}$$

As the test functions for the weak controlled solution of the Rough Transport Equation need to be in $C_c^\infty(\mathbb{R}^d)$, we need to construct smooth compactly supported solution for the Rough Continuity Equation. Since we want to consider b with poor regularity, this will not be possible. We will proceed by regularization of b, and we will need some convergence lemmas.

3.4.1 Existence of strong solutions for the rough continuity equation

We have to define what are strong solutions for the Rough Transport or Continuity equation. Following the definition of weak controlled solutions, we give the following definition. Note here that in all the following part we will not allow linear growth for the vectorfield. This is only a technical issue, since the regularization argument for the flow works only if we have bounded vectorfields.

Definition 3.4.1. Let $b, c \in L^\infty([0,T], L^\infty(\mathbb{R}^d))$, $\frac{1}{3} < \gamma \leqslant 1$ and $\boldsymbol{X} = (X, \mathbb{X}) \in \mathcal{R}^\gamma([0,T])$ a γ-rough path. A strong controlled solution with initial condition u_0 of the Rough Equation

$$\partial_t u_t(x) + b_t.\nabla u_t(x) + c_t(x)u_t(x) + \nabla u_t(x).\mathrm{d}\boldsymbol{X}_t = 0, \quad u(0,x) = u_0(x),$$

is a function $u \in \mathcal{C}^\gamma([0,T]; C^1(\mathbb{R}^d)) \cap L^\infty([0,T]\times\mathbb{R}^d)$ such that

1. For all $x \in \mathbb{R}^d$, the function $t \to \nabla u_t(x)$ is controlled by X.
2. For all $x \in \mathbb{R}^d$ and all $s \leqslant t \in [0,T]$, the following equation is satisfied

$$u_t(x) - u_s(x) + \int_s^t b_q(x).\nabla u_q(x) + c_q(x)u_q(x)\mathrm{d}q + \int_s^t \nabla u_q(x)\mathrm{d}\boldsymbol{X}_q = 0$$

   where the last integral is the rough integral of $\nabla u_\cdot(x)$ against $\mathrm{d}\boldsymbol{X}$.

As for the weak controlled solutions, when $u_t \in C_b^2$, we can replace condition 2 by the following one.

- There exists a function $R^u(x) : [0,T]^2 \to \mathbb{R}$ such that $R^u(x) \in \mathcal{C}_2^{3\gamma}([0,T])$ and

$$u_t(x) - u_s(x) + \int_s^t (b_q(x).\nabla u_q(x) + c_q(x)u_q(x))\mathrm{d}q + \nabla u_s(x).X_{s,t} + \frac{1}{2}\nabla^2 u_s(x)X_{s,t}^{\otimes 2} + R_{s,t}^u(x) = 0$$

In order to prove the existence of strong controlled solution (SCS) for the Rough Continuity Equation (RCE) we will proceed as for the weak solutions. Namely we will approximate X and show that classical solutions (see Subsection 3.2.5) are controlled solutions for the approximate equation, and then remove the approximation. Nevertheless, in order to have an explicit form, we cannot use a compactness argument anymore, and we will have to control all the objects in order to make them converge. As we focus on regular b only, standard argument will be used.

The three following lemmas are quite similar to Lemmas 3.3.4, Corollary 3.3.7 and Lemma 3.3.8, but since we are working with bounded function b we will give an easier proof.

Lemma 3.4.2. *Let $b \in L^\infty([0,T]; L^\infty(\mathbb{R}^d))$ and $X \in \mathcal{C}^\gamma(\mathbb{R}^d)$ such that the flow Φ of the equation*

$$\Phi_t(x) = x + \int_0^t b_q(\Phi_q(x))\mathrm{d}q + X_t.$$

Let $\varphi \in C_c(\mathbb{R}^d)$ a continuous function with compact support. Then for all $s,t \in [0,T]$ and all $r \in [0,1]$

$$x \to \varphi(r(\Phi_t(x) - \Phi_s(x)) + \Phi_s(x)) \,\text{has compact support}\,.$$

Furthermore if $\operatorname{supp}\varphi \subset B(0, R_\varphi)$*,then* $\operatorname{supp}\varphi(r(\Phi_t(.) - \Phi_s(.)) + \Phi_s(.)) \subset B(0,R)$ *where* $R = R_\varphi + 2T^\gamma(\|b\|_\infty + \|X\|_\gamma)$.

Proof. As $b \in L^\infty([0,T], L^\infty(\mathbb{R}^d))$, we have for all $x \in \mathbb{R}^d$ and all $s < t \in [0,T]$

$$|\Phi_t(x) - x| \leqslant T\|b\|_\infty + T^\gamma\|X\|_\gamma.$$

But, thanks to the equation, we also have

$$|\Phi_t(x) - \Phi_s(x)| \leqslant T\|b\|_\infty + T^\gamma\|X\|_\gamma.$$

Hence for all $r \in [0,1]$, $s,t \in [0,T]$,

$$r(\Phi_t(x) - \Phi_s(x)) + \Phi_s(x) \in B(x, 2(T\|b\|_\infty + T^\gamma\|X\|_\gamma)).$$

Let $R_\varphi > 0$ such that $\operatorname{supp}\varphi \subset B_f(0,R_\varphi)$, for all $x \in \mathbb{R}^d$ such that $|x| > R_\varphi + 2(T\|b\|_\infty + T^\gamma\|X\|_\gamma)$,

$$B_f(x, 2(T\|b\|_\infty + T^\gamma\|X\|_\gamma)) \cap B_f(0, 2(T\|b\|_\infty + T^\gamma\|X\|_\gamma)) = \emptyset.$$

And

$$\operatorname{supp} x \to \varphi(r(\Phi_t(x) - \Phi_s(x)) + \Phi_s(x)) \subset B_f(0, R_\varphi + 2(T\|b\|_\infty + T^\gamma\|X\|_\gamma)).$$

which ends the proof. □

Lemma 3.4.3. *Let b and X as in Lemma 3.4.2. Let $\varphi \in C^1_c(\mathbb{R}^d, \mathbb{R}^m)$ a continuous differentiable function with compact support. Then $t \to \varphi(\Phi_t(x)) \in \mathcal{C}^\gamma([0,T])$, and furthermore*

$$\|\varphi(\Phi_.(x))\|_\gamma \lesssim_T (\|b\|_\infty + \|X\|_\gamma)\|D\varphi\|_\infty \mathbb{1}_{B_f(0,R)}(x)$$

where $R = R_\varphi + 2(T\|b\|_\infty + T^\gamma \|X\|_\gamma)$ and R_φ is such that $\operatorname{supp}\varphi \subset B_f(0, R_\varphi)$.

Proof. As b is bounded, we have $\|\Phi\|_\gamma \lesssim_T \|b\|_\infty + \|X\|_\gamma$. As usual we consider the increments, and with the previous lemma we have

$$\begin{aligned} |\varphi(\Phi_t(x)) - \varphi(\Phi_s(x))| &\leqslant \int_0^1 \mathrm{d}r |D\varphi(r(\Phi_t(x) - \Phi_s(x)) + \Phi_s(x))||\Phi_t(x) - \Phi_s(x)| \\ &\lesssim_T (\|b\|_\infty + \|X\|_\gamma)\|D\varphi\|_\infty \mathbb{1}_{B_f(0,R)}(x). \end{aligned}$$

□

Those two previous lemmas guarantee that the function $t \to \varphi(\Phi_t(x))$ is controlled by X and furthermore give a estimate on the controlled norm. Indeed, the following lemma holds.

Lemma 3.4.4. *Let b and X as in Lemma 3.4.2. Let $\varphi \in C^2_c(\mathbb{R}^d, \mathbb{R}^m)$, then $t \to \varphi(\Phi_t(x)) \in \mathcal{D}^\gamma_X(\mathbb{R}^d)$, and furthermore*

$$\varphi(\Phi_.(x))' = D\varphi(\Phi_.(x))$$

and for R as in the previous lemma, we have

$$\|\varphi(\Phi_.(x))\|_{\mathcal{D}^\gamma_X([0,T])} \leqslant C(1 + \|X\|_\gamma)\mathbb{1}_{B(0,R)}(x).$$

The proof is quite similar to the one where b has linear growth. Nevertheless, as it is easier, we present it.

Proof. We already know that $\varphi(\Phi_.(x)) \in \mathcal{C}^\gamma([0,T])$. Let us compute the increments,

$$\begin{aligned} \varphi(\Phi_t(x)) - \varphi(\Phi_s(x)) &= D\varphi(\Phi_s(x)).X_{s,t} \\ &\quad + \int_0^1 \mathrm{d}r \{D\varphi(r(\Phi_s(x) - \Phi_s(x)) + \Phi_s(x)) - D\varphi(\Phi_s(x))\}.X_{s,t} \\ &\quad + \int_0^1 \mathrm{d}r D\varphi(r(\Phi_s(x) - \Phi_s(x)) + \Phi_s(x)). \int_s^t b_q(\Phi_q(x))\mathrm{d}q. \end{aligned}$$

Thanks to the previous lemma, we also know that $D\varphi(\Phi_.(x)) \in \mathcal{C}^\gamma$. Furthermore, thanks to Lemma 3.4.2, we have

$$\left| \int_0^1 \mathrm{d}r D\varphi(r(\Phi_s(x) - \Phi_s(x)) + \Phi_s(x)). \int_s^t b_q(\Phi_q(x))\mathrm{d}q. \right| \leqslant \|D\varphi\|_\infty \mathbb{1}_{B_f(0,R)}(x)|t-s|\|b\|_\infty.$$

where $R = R_\varphi + 2T^\gamma(\|b\|_\infty + \|X\|_\gamma)$. And finally

$$\left| \int_0^1 \mathrm{d}r \{D\varphi(r(\Phi_s(x) - \Phi_s(x)) + \Phi_s(x)) - D\varphi(\Phi_s(x))\}.X_{s,t} \right|$$

$$= \left| \int_0^1 \int_0^1 \mathrm{d}r\mathrm{d}q \{ D^2\varphi(rq(\Phi_s(x) - \Phi_s(x)) + \Phi_s(x)).(\Phi_s(x) - \Phi_s(x)).X_{s,t} \right|$$

$$\leqslant \|D^2\varphi\|_\infty \|X\|_\gamma T^{1-\gamma}(\|b\|_\infty + \|X\|_\gamma) \mathbb{1}_{B_f(0,R)}(x)|t-s|^{2\gamma}.$$

Putting all together, we have the wanted result

$$\|\varphi(\Phi_\cdot(x))\|_{\mathcal{D}_X^\gamma} \leqslant C(T, \|X\|_\gamma, \|b\|_\infty, \|\varphi\|_\infty, \|D\varphi\|_\infty, \|D^2\varphi\|_\infty) \mathbb{1}_{B_f(0,R)}(x),$$

where the constant C is nondecreasing in all the parameters. and so is R. $\square$

In order to prove that there exist strong controlled solutions to equation (3.16), we will approximate the rough path X. Hence, we need some regularity for the controlled norm of the potential solutions of the equations. The following lemma gives us the regularity w.r.t. the rough path norm of $\boldsymbol{X}$ and $\boldsymbol{Y}$ of the controlled test functions.

Lemma 3.4.5. *Let $b \in L^\infty([0,T]; C_b^1(\mathbb{R}^d))$, $X, Y \in \mathcal{C}^\gamma$ and Φ^X, Φ^Y the associated flows. Let $\varphi \in C_c^3(\mathbb{R}^d)$. Then*

$$\|\varphi(\Phi_\cdot^X(x))' - \varphi(\Phi_\cdot^Y(x))'\|_\gamma \leqslant C\|X-Y\|_\gamma (1 + \|Y\|_\gamma)(1 + \|X\|_\gamma) \mathbb{1}_{B_f(0,R)}(x)$$

and

$$\|\varphi(\Phi_\cdot^X(x))^\# - \varphi(\Phi_\cdot^Y(x))^\#\|_{2\gamma} \leqslant C\|X-Y\|_\gamma (1 + \|Y\|_\gamma)(1 + \|X\|_\gamma) \mathbb{1}_{B_f(0,R)}(x),$$

where the two constants $C = C_T(\|b\|_\infty, \|Db\|_\infty, \|\varphi\|_\infty, \|D\varphi\|_\infty, \|D^2\varphi\|_\infty, \|D^3\varphi\|_\infty)$ and $R = R_T(\|b\|_\infty, \|X\|_\gamma, \|Y\|_\gamma)$ are nondecreasing in all the parameters.

Proof. We already know that the two functions $\varphi(\Phi_\cdot^X(x))$ and $\varphi(\Phi_\cdot^Y(x))$ are controlled by X and by Y respectively and that

$$\varphi(\Phi_\cdot^X(x))' = D\varphi(\Phi_\cdot^X(x))$$

and

$$\begin{aligned} \varphi(\Phi_\cdot^X(x))^\# &= \int_0^1 \mathrm{d}r D\varphi(r\delta(\Phi_{s,t}^X)(x) + \Phi_s^X(x)). \int_s^t b_q(\Phi_q^X(x))\mathrm{d}q \\ &+ \int_0^1 \int_0^1 \mathrm{d}r\mathrm{d}q D^2\varphi(rq\delta(\Phi_{s,t}^X)(x) + \Phi_s^X(x)).\delta(\Phi_{s,t}^X)(x).X_{s,t} \end{aligned}$$

and the same holds for $\varphi(\Phi_\cdot^Y(x))$ when we replace X by Y. Hence

$$
\begin{aligned}
&\delta(D\varphi(\Phi^X_\cdot(x)) - D\varphi(\Phi^X_\cdot(x)))_{s,t}\\
&= \int_0^1 \mathrm{d}r D^2\varphi(r\delta(\Phi^X_{s,t})(x) + \Phi^X_s(x)).\delta(\Phi^X_{s,t})(x)\\
&\quad - \int_0^1 \mathrm{d}r D^2\varphi(r\delta(\Phi^Y_{s,t})(x) + \Phi^Y_s(x)).\delta(\Phi^Y_{s,t})(x)\\
&= \int_0^1 \mathrm{d}r D^2\varphi(r\delta(\Phi^X_{s,t})(x) + \Phi^X_s(x)).\delta(\Phi^X - \Phi^Y)_{s,t}(x)\\
&\quad + \int_0^1 (D^2\varphi(r\delta(\Phi^X_{s,t})(x) + \Phi^X_s(x)) - D^2\varphi(r\delta(\Phi^Y_{s,t})(x) + \Phi^Y_s(x))).\delta(\Phi^Y_{s,t})(x)\\
&= \int_0^1 \mathrm{d}r D^2\varphi(r\delta(\Phi^X_{s,t})(x) + \Phi^X_s(x)).\delta(\Phi^X - \Phi^Y)_{s,t}(x)\\
&\quad + \int_0^1\int_0^1 \mathrm{d}r\mathrm{d}q D^3\varphi(q(r\delta(\Phi^X - \Phi^Y)_{s,t} + \Phi^X_s(x) - \Phi^Y_s(x)) + r\delta\Phi^Y_{s,t}(x) + \Phi^Y_s(x))\\
&\qquad\qquad .(r\delta(\Phi^X - \Phi^Y)_{s,t} + \Phi^X_s(x) - \Phi^Y_s(x)).\delta(\Phi^Y_{s,t})(x)\\
&= A_1 + A_2
\end{aligned}
$$

Following the proof of Lemma 3.4.2 and with the estimations of Lemma 3.2.38, there exists $R > 0$ nondecreasing in all the parameters such that

$$
|A_1| \leqslant C(T, \|D^3\varphi\|_\infty, \|D^2\varphi\|_\infty, \|Db\|_\infty, \|b\|_\infty)\|X - Y\|_\gamma \mathbb{1}_{B_f(0,R)}(x)|t - s|,
$$

where C is nondecreasing with respect to the parameters. The same kind of bound holds for A_2 and we have

$$
|A_2| \leqslant C|t - s|^{2\gamma}\|X - Y\|_\gamma \mathbb{1}_{B_f(0,R)}(x)\|X - Y\|_\gamma(1 + \|Y\|_\gamma).
$$

Let us turn now to the remainder. We decompose it into five terms:

$$
\varphi(\Phi^X_\cdot(x))^{\#}_{s,t} - \varphi(\Phi^Y_\cdot(x))^{\#}_{s,t} = B_1 + B_2 + B_3 + B_4 + B_5,
$$

where

$$
B_1 = \int_0^1 \mathrm{d}r D\varphi(r\delta(\Phi^X_{s,t})(x) + \Phi^X_s(x)).\int_s^t b_q(\Phi^X_q(x)) - b_q(\Phi^Y_q(x))\mathrm{d}q,
$$

$$
B_2 = \int_0^1 \mathrm{d}\,\mathrm{rD}\,\varphi(r\delta(\Phi^X_{s,t})(x) + \Phi^X_s(x)) - D\varphi(r\delta(\Phi^Y_{s,t})(x) + \Phi^Y_s(x)).\int_s^t b_q(\Phi^Y_q(x))\mathrm{d}q,
$$

$$
B_3 = \int_0^1\int_0^1 \mathrm{d}r\mathrm{d}q D^2\varphi(rq\delta(\Phi^X_{s,t})(x) + \Phi^X_s(x)).\delta(\Phi^X_{s,t})(x).(X - Y)_{s,t},
$$

$$
B_4 = \int_0^1\int_0^1 \mathrm{d}r\mathrm{d}q D^2\varphi(rq\delta(\Phi^X_{s,t})(x) + \Phi^X_s(x)).\delta(\Phi^X - \Phi^Y)_{s,t}(x).Y_{s,t},
$$

$$
B_5 = \int_0^1\int_0^1 \mathrm{d}r\mathrm{d}q D^2\varphi(rq\delta(\Phi^X_{s,t})(x) + \Phi^X_s(x)) - D^2\varphi(rq\delta(\Phi^Y_{s,t})(x) + \Phi^Y_s(x)).\delta(\Phi^Y_{s,t})(x).Y_{s,t}.
$$

The analysis of those five terms being exactly the same as the analysis of the terms A_1 and A_2, the result follows easily. $\square$

Lemma 3.4.6. *Let $b \in C^1_b(\mathbb{R}^d)$, $X, Y \in \mathcal{C}^\gamma([0,T])$ and Φ^X and Φ^Y the associated flows. Let $c \in C^1_b(\mathbb{R}^d)$ and*

$$K^X_t(x) = \exp\left(\int_0^t c_q(\Phi^X_q(x))\mathrm{d}q\right), K^Y_t(x) = \exp\left(\int_0^t c_q(\Phi^Y_q(x))\mathrm{d}q\right).$$

Then

$$\|K^X(x) - K^Y(x)\|_{\text{Lip}} \leqslant C(T, \|Db\|_\infty, \|c\|, \|\nabla c\|_\infty)\|X - Y\|_\gamma.$$

Furthermore, if $b \in C^2_b(\mathbb{R}^d)$ and $c \in C^2_b(\mathbb{R}^d)$, we have

$$\|\nabla K^X(x) - \nabla K^Y(x)\|_{\text{Lip}} \leqslant C(T, \|Db\|_\infty, \|D^2 b\|_\infty, \|c\|, \|\nabla c\|_\infty, \|D^2 c\|_\infty)\|X - Y\|_\gamma.$$

Finally the two constants are nondecreasing with respect to all the parameters.

Proof. As $K^X_0(x) = K^X_0(y) = 1$, we only have to control the increments of the difference. Let $x \in \mathbb{R}^d$, $s, t \in [0,T]$. Thanks to Lemma 3.2.38, we have

$$\begin{aligned}
|\delta(K^X - K^Y)_{s,t}(x)| &\leqslant |K^X_s(x) - K^Y_s(x)| \left|\exp\left(\int_s^t c_q(\Phi^X_q(x))\mathrm{d}q\right) - 1\right| \\
&\quad + \left|\exp\left(\int_s^t c_q(\Phi^X_q(x))\mathrm{d}q\right) - \exp\left(\int_s^t c_q(\Phi^Y_q(x))\mathrm{d}q\right)\right| K^X_s(x) \\
&\leqslant e^{T\|c\|_\infty}\left|\int_0^s c_q(\Phi^X_q(x))\mathrm{d}q - \int_0^s c_q(\Phi^Y_q(x))\mathrm{d}q\right| |t-s|\|c\|_\infty \\
&\quad + e^{T\|c\|_\infty}\left|\int_s^t c_q(\Phi^X_q(x))\mathrm{d}q - \int_s^t c_q(\Phi^Y_q(x))\mathrm{d}q\right| T\|c\|_\infty \\
&\lesssim C(\|Db\|_\infty, \|c\|_\infty, \|\nabla c\|_\infty, T)|t-s|\|X - Y\|_\gamma.
\end{aligned}$$

Furthermore, as

$$\nabla K^X_t(x) = \int_s^t \nabla c_q(\Phi^X_q(x)).D\Phi^X_q(x)\mathrm{d}q K^X_t(x),$$

we only have to prove the bound for $L^X_t(x) = \int_0^t \nabla c_q(\Phi^X_q(x)).D\Phi^X_q(x)\mathrm{d}q$. We have, again thanks to Lemma 3.2.38,

$$\begin{aligned}
|\delta(L^X - L^Y)_{s,t}(x)| &\leqslant \left|\int_s^t \nabla c_q(\Phi^X_q(x)).(D\Phi^X_q(x) - D\Phi^Y_q(x))\mathrm{d}q\right| \\
&\quad + \left|\int_s^t (\nabla c_q(\Phi^X_q(x)) - \nabla c_q(\Phi^Y_q(x))).D\Phi^Y_q(x)\mathrm{d}q\right| \\
&\lesssim |t-s|\|\nabla c\|_\infty\|X - Y\|_\gamma + |t-s|\|D^2 c\|_\infty\|X - Y\|_\gamma,
\end{aligned}$$

where the constant depends on T, $\|Db\|_\infty$, $\|D^2 b\|_\infty$, $\|c\|_\infty$ and $\|\nabla c\|_\infty$, which proves the result. □

We have gathered all the tools to prove the existence of strong controlled solutions if the initial condition and the the two functions b and c are regular enough.

Theorem 3.4.7. *Let $b \in L^\infty([0,T];C_b^2(\mathbb{R}^d))$, $c \in L^\infty([0,T];C_b^2(\mathbb{R}^d))$, $1/3 < \gamma \leqslant 1/2$, $\boldsymbol{X} \in \mathcal{R}^\gamma$ and $\varphi_0 \in C_c^3(\mathbb{R}^d)$. Let Φ the flow of the equation*

$$\Phi_t(x) = x + \int_0^t b_q(\Phi_s(x)) + X_t$$

and Φ^{-1} its inverse. The function

$$(t,x) \to \psi_t(x) = \varphi_0(\Phi_t^{-1}(x)) \exp\left(-\int_0^t c_{t-q}(\Phi_q^{-1}(x))\mathrm{d}q\right)$$

is a strong controlled solution to equation (3.16). Furthermore the function $x \to \|\nabla\psi_\cdot(x)\|_{\mathcal{D}_X^\gamma}$ is compactly supported.

Proof. Let us take a smooth approximation $\boldsymbol{X}^\eta = (X^\eta, \mathbb{X}^\eta)$ of $\mathbb{X}$ such that $\boldsymbol{X}^\eta \to \boldsymbol{X}$. By the characteristic method (see Subsection 3.2.5) we already know that

$$(t,x) \mapsto \psi_t^\eta(x) = \varphi_0((\Phi^\eta)_t^{-1}(x)) \exp\left(-\int_0^t c_{t-q}((\Phi^\eta)_q^{-1}(x))\mathrm{d}q\right)$$

satisfies the following equation

$$\psi_t^\eta(x) - \psi_s^\eta(x) + \int_s^t b_q(x).\nabla\psi_q^\eta(x)\mathrm{d}q + \int_s^t c_q(x)\psi_q(x)\mathrm{d}q + \int_s^t \nabla\psi_q^\eta(x).\dot{X}_q^\eta\mathrm{d}q = 0. \tag{3.17}$$

Let us define $\varphi_t^\eta(x) = \varphi_0((\Phi^\eta)_t^{-1}(x))$ and $K_t^\eta(x) = \exp\left(-\int_0^t c_{t-q}((\Phi^\eta)_q^{-1}(x))\mathrm{d}q\right)$. Then

$$\begin{aligned}
\nabla\psi_t^\eta(x) &= \nabla\varphi_t^\eta(x)K_t^\eta(x) + \nabla K_t^\eta(x)\varphi_t^\eta(x) \\
&= \nabla\varphi_0((\Phi^\eta)_t^{-1}(x)).D(\Phi^\eta)_t^{-1}(x)K_t^\eta(x) - \psi_t^\eta(x)\int_0^t \nabla c_{t-q}((\Phi^\eta)_q^{-1}(x)).(\Phi^\eta)_q^{-1}(x)\mathrm{d}q,
\end{aligned}$$

and since $(\Phi^\eta)^{-1}$ satisfies equation

$$(\Phi^\eta)_t^{-1}(x) = x - \int_0^t b_{t-q}((\Phi^\eta)_q^{-1}(x))\mathrm{d}q - X_t,$$

the function $t \mapsto \nabla\varphi_0((\Phi^\eta)_t^{-1}(x))$ is controlled by X^η, thanks to Lemma 3.4.4. Furthermore,

$$t \mapsto D(\Phi^\eta)_t^{-1}(x) \in \mathrm{Lip}([0,T])$$

and this function is controlled by X^η, with $(D(\Phi^\eta)_\cdot^{-1}(x))' = 0$. As $c \in C_b^1(x)$, $t \to K_t^\eta(x) \in \mathrm{Lip}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $t \mapsto \int_0^t \nabla c_{t-q}((\Phi^\eta)_q^{-1}(x)).D(\Phi^\eta)_q^{-1}(x)\mathrm{d}q \in \mathrm{Lip}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. The same arguments hold for X, and $t \mapsto \psi_t(x)$. Hence, $\psi_\cdot^\eta(x)$ and $\psi_\cdot(x)$ are controlled respectively by X^η and X. Furthermore, thanks to Lemmas 3.4.5 and 3.2.38:

$$\begin{aligned}
&\|\nabla\varphi_\cdot^\eta(x) - \nabla\varphi_\cdot(x)\|_\gamma \\
&\quad\leqslant \|\nabla\varphi_0((\Phi^\eta)_\cdot^{-1}(x)) - \nabla\varphi_0(\Phi_\cdot^{-1}(x))\|_\gamma(\|D(\Phi^\eta)_t^{-1}(x)\|_\gamma + \|D\Phi_t^{-1}(x)\|_\gamma) \\
&\qquad+(\|\nabla\varphi_0((\Phi^\eta)_\cdot^{-1}(x))\|_{\mathcal{C}^\gamma} + \|\nabla\varphi_0(\Phi_\cdot^{-1}(x))\|_{\mathcal{C}^\gamma})\|D(\Phi^\eta)_t^{-1}(x) - D\Phi_t^{-1}(x)\|_\gamma \\
&\quad\lesssim (1 + \|X\|_\gamma + \|X^\eta\|)^2\|X - X^\eta\|_\gamma \mathbb{1}_{B(0,R^\eta)}(x),
\end{aligned}$$

Where R^η is nondecreasing w.r.t. $\|X\|_\gamma$ and $\|X^\eta\|_\gamma$. As $(D(\Phi^\eta)^{-1}_\cdot(x))' = 0$ and $(D\Phi^{-1}_\cdot(x))' = 0$, we also have, thanks to Lemma 3.2.17

$$\|\nabla\varphi^\eta_\cdot(x)' - \nabla\varphi_\cdot(x)'\|_\gamma \lesssim \mathbb{1}_{B_f(0,R^\eta)}(x)\|X - X^\gamma\|(1 + \|X\|_\gamma + \|X^\eta\|_\gamma)^2$$

and

$$\|\varphi^\eta_\cdot(x)^\# - \varphi_\cdot(x)^\#\|_\gamma \lesssim \mathbb{1}_{B_f(0,R^\eta)}(x)\|X - X^\gamma\|(1 + \|X\|_\gamma + \|X^\eta\|_\gamma)^2,$$

where the radius R^η is nondecreasing with respect to $\|X\|$ and $\|X^\eta\|$. Furthermore, since $K^\eta(x), K(x) \in \mathrm{Lip}([0,T])$ and $\nabla K^\eta(x), \nabla K(x) \in \mathrm{Lip}([0,T])$, and thanks to Lemmas 3.4.6 and 3.2.17, and since $\|X^\eta\|_\gamma \lesssim \|X\|_\gamma$, there exists $R > 0$ depending on $\|X\|_\gamma, \|Db\|_\infty$ and T and a constant C depending on b, c, φ_0, T and $\|X\|_\gamma$ such that

$$\|\psi^\eta_\cdot(x) - \psi_\cdot(x)\|_\gamma + \|\psi^\eta_\cdot(x)' - \psi_\cdot(x)'\|_\gamma + \|\psi^\eta_\cdot(x)^\# - \psi_\cdot(x)^\#\|_{2\gamma} \leqslant C\|X - X^\eta\|_\gamma \mathbb{1}_{B_f(0,R)}(x).$$

Furthermore, $X \to^{\mathcal{C}^\gamma} X^\eta$, hence, by the definition of the rough integral and the comparison between controlled path (see Theorem 3.2.20 and Lemma 3.2.17), for all $s,t \in [0,T]$,

$$\int_s^t \nabla\psi^\eta_q(x).\dot{X}^\eta_q \mathrm{d}q \to_{\eta\to 0} \nabla\psi_s(x).X_{s,t} + \frac{1}{2}\nabla^2\psi_s(x).X^{\otimes 2}_{s,t} + R^\psi_{s,t}(x)$$

and

$$|R^\psi_{s,t}(x)| \lesssim |t-s|^{3\gamma}\mathbb{1}_{B_f(0,R)}(x).$$

It remains to show that the other terms of Equation (3.17) converge to the right quantities. But, thanks to Lemma 3.2.38, we know that $(\Phi^\eta)^{-1}_t(x) \to \Phi^{-1}_t(x)$ and $D(\Phi^\eta)^{-1}_t(x) \to D\Phi^{-1}_t(x)$, hence, as φ_0, $\nabla\varphi_0$, c and ∇c are continuous,

$$\psi^\eta_t(x) \to \psi_t(x) \text{ and } \nabla\psi^\eta_t(x) \to \nabla\psi_t(x).$$

Furthermore

$$|b_q(x).\nabla\psi^\eta_t(x) + c_q(x)\psi^\eta_q(x)| \lesssim (1 + \|X^\eta\|)\mathbb{1}_{B_f(0,R)}(x),$$

hence

$$\int_s^t b_q(x).\nabla\psi^\eta_t(x) + c_q(x)\psi^\eta_q(x)\mathrm{d}q \to \int_s^t b_q(x).\nabla\psi_t(x) + c_q(x)\psi_q(x)\mathrm{d}q$$

and

$$\left|\int_s^t b_q(x).\nabla\varphi_q(x) + c_q(x)\psi_q(x)\mathrm{d}q\right| \lesssim |t-s|\mathbb{1}_{B_f(0,R)}(x).$$

Finally all the quantities converge and we have

$$\psi_t(x) - \psi_s(x) + \int_s^t b_q(x).\nabla\psi_q(x) + c_q(x)\psi_q(x)\mathrm{d}q + \nabla\psi_s(x).X_{s,t} + \frac{1}{2}\nabla\psi_s(x).X^{\otimes 2}_{s,t} + R^\psi_{s,t}(x) = 0$$

with

$$\psi_t(x) = \varphi_0(\Phi^{-1}_t(x))\exp\left(-\int_0^t c_{t-q}(\Phi^{-1}_q(x))\mathrm{d}q\right),$$

which ends the proof. □

If $c = 0$ in the last theorem, we have the existence of a strong controlled solution for the Rough Transport Equation. When $c = \operatorname{div} b$, it is a solution for the Rough Continuity Equation. This result gives us the good dynamic to solve the Rough Transport Equation by a duality argument. Indeed, as stated before, we will be able to test weak controlled solution against good test functions, *i.e.* the solution of the Rough Continuity Equation with an approximate vector.

3.4.2 Uniqueness for the rough transport equation

As the Transport Equation is linear, it is enough to prove uniqueness when $u_0 = 0$. We would like to use the standard duality argument to prove that in that case the only solution is zero. As we need to test the weak controlled solution against smooth compactly supported functions, it is not possible to do it directly. The idea is to approximate the vectorfield b with a smooth one, and hence to show that the error we make by such a trick goes to zero when the regularization goes to zero.

3.4.3 The fundamental Lemma

Lemma 3.4.8 (Fundamental Lemma)**.** *Let $1/3 < \gamma \leqslant 1/2$, $\boldsymbol{X} = (X, \mathbb{X}) \in \mathcal{R}^\gamma$ be a geometric rough path, $b \in L^\infty([0,T]; \mathrm{Lin}(\mathbb{R}^d))$ be two vector spaces such that $\tilde{b} \in L^\infty([0,T]; C_c^\infty(\mathbb{R}^d))$ with $\operatorname{div} b \in L^\infty([0,T]; L^\infty(\mathbb{R}^d))$. Let $\tilde{\Phi}$ the flow associated to X and $\tilde{b}$, and let $\tilde{R}$ the radius of the ball of Lemma 3.4.2 associated to $\tilde{b}$ and X.*

Let $u \in L^\infty([0,T] \times \mathbb{R}^d)$ a weak controlled solution of the Rough Transport Equation with $u_0 = 0$, and

$$\tilde{G}_q^{t_0}(x) = \int_q^{t_0} (\operatorname{div} \tilde{b}_r)(\tilde{\Phi}_{r-q}(x)) \mathrm{d}r.$$

Then

$$|u_{t_0}(\varphi_0)| \lesssim \|u\|_\infty \sup_{q \in [0,t_0]} \int_{B_f(0,\tilde{R})} \mathrm{d}x [|\operatorname{div}(b - \tilde{b})_q| + |(b - \tilde{b})_q(x)|](|D\tilde{\Phi}_{t_0 - q}(x)| + |\nabla \widetilde{G}_q^{t_0}(x)|).$$

Proof. Let $\varphi_0 \in C_c^\infty$ and $\tilde{b} \in C_b^\infty$. For all $t \in [0,T]$ we define,

$$\tilde{\psi}_t : x \to \tilde{\psi}_t(x) = \varphi_0(\tilde{\Phi}_t^{-1}(x)) \exp\left[-\int_0^t (\operatorname{div} \tilde{b})(\tilde{\Phi}_q^{-1}(x)) \mathrm{d}q\right] \in C_c^\infty(\mathbb{R}^d).$$

By Theorem 3.4.7, $\tilde{\psi}$ is a strong controlled solution of the Rough Continuity Equation.

Furthermore, thanks to the definition of $\tilde{R}$, for all $n \in \mathbb{N}$

$$t \to D^n \tilde{\psi}_t(x) \in \mathcal{D}_X^\gamma([0,T])$$

and

$$\|D^n \tilde{\psi}_\cdot(x)\|_{\mathcal{D}_X^\gamma} \lesssim_n \mathbb{1}_{B_f(0,\tilde{R})}(x).$$

Since $\nabla \tilde{\psi}_\cdot(x) \in \mathcal{D}^\gamma([0,T])$ and the function $x \to \|\nabla \tilde{\psi}_\cdot(x)\|_{\mathcal{D}^\gamma([0,T])}$ is compactly supported, we have

$$\begin{aligned}
u_t(\tilde{\psi}_t) - u_s(\tilde{\psi}_s) &= (u_t - u_s)(\tilde{\psi}_s) + u_s(\tilde{\psi}_t - \tilde{\psi}) + (u_t - u_s)(\tilde{\psi}_t - \tilde{\psi}_s) \\
&= \int_s^t u_q(b.\nabla \tilde{\psi}_s + (\operatorname{div} b)\tilde{\psi}_s) \mathrm{d}q + u_s(\nabla \tilde{\psi}_s).X_{s,t} + \frac{1}{2} u_s(\nabla^2 \tilde{\psi}_s).X_{s,t}^{\otimes 2} \\
&\quad - \int_s^t u_s(\tilde{b}.\nabla \tilde{\psi}_q + (\operatorname{div} \tilde{b})\tilde{\psi}_s) \mathrm{d}q - u_s(\nabla \tilde{\psi}_s).X_{s,t} + \frac{1}{2} u_s(\nabla^2 \tilde{\psi}_s).X_{s,t}^{\otimes 2} \\
&\quad + \int_s^t u_q(b.(\nabla \tilde{\psi}_t - \nabla \tilde{\psi}_s) + (\operatorname{div} b)(\tilde{\psi}_t - \tilde{\psi}_s)) \mathrm{d}q + u_s(\nabla \tilde{\psi}_t - \nabla \tilde{\psi}_s).X_{s,t} \\
&\quad + R_{s,t},
\end{aligned}$$

where $|R_{s,t}| \lesssim |t-s|^{3\gamma}$.

But
$$u_s(\nabla\tilde{\psi}_t - \nabla\tilde{\psi}_s) = -u_s(\nabla^2\tilde{\psi}_s).X_{s,t} + \tilde{R}_{s,t}$$
and all the rough terms cancelled. Finally we have

$$\begin{aligned} u_t(\tilde{\psi}_t) - u_s(\tilde{\psi}_s) &= \int_s^t u_q((b-\tilde{b}).\nabla\tilde{\psi}_q)\mathrm{d}q + \int_s^t u_q(\operatorname{div}(b-\tilde{b})\tilde{\psi}_q)\mathrm{d}q \\ &\quad + \int_s^t u_q(b.(\nabla\tilde{\psi}_t - \nabla\tilde{\psi}_q))\mathrm{d}q + \int_s^t u_q((\operatorname{div} b)(\tilde{\psi}_t - \tilde{\psi}_q))\mathrm{d}q \\ &\quad + \int_s^t (u_q - u_s)(\tilde{b}.\nabla\tilde{\psi}_q + (\operatorname{div}\tilde{b})\tilde{\psi}_q)\mathrm{d}q + \tilde{R}_{s,t}. \end{aligned}$$

Furthermore
$$\left|\int_s^t u_q(b.(\nabla\tilde{\varphi}_t - \nabla\tilde{\psi}_q))\mathrm{d}q\right| \lesssim_{\varepsilon,T,\varphi,b,\|X\|_\gamma} \|u\|_\infty \int_s^t |t-q|^\gamma \mathrm{d}q \lesssim |t-s|^{1+\gamma},$$

$$\left|\int_s^t u_q(\operatorname{div} b(\tilde{\psi}_t - \tilde{\psi}_q))\mathrm{d}q\right| \lesssim |t-s|^{1+\gamma},$$
and since $\tilde{b}.\nabla\tilde{\psi}_q + (\operatorname{div}\tilde{b})\tilde{\psi}_q \in C_c^\infty(\mathbb{R}^d)$ and u is a WCS
$$|(u_q - u_s)(\tilde{b}.\nabla\tilde{\psi}_q + (\operatorname{div}\tilde{b})\tilde{\psi}_q)| \lesssim |q-s|^\gamma.$$

Hence, we have the following decomposition
$$u_t(\tilde{\psi}_t) - u_s(\tilde{\psi}_s) = \int_s^t u_q((b-\tilde{b}).\nabla\tilde{\psi}_q + \operatorname{div}(b-\tilde{b})\tilde{\psi}_q)\mathrm{d}q + \tilde{R}_{s,t}$$
where $|\tilde{R}_{s,t}| \lesssim |t-s|^{3\gamma}$. But, thanks to the last equation,
$$\delta\tilde{R}_{r,s,t} = 0$$
and the Lemma 3.2.21 gives
$$\tilde{R}_{s,t} = 0.$$
Finally, we have

$$\begin{aligned} u_t(\tilde{\psi}_t) - u_s(\tilde{\psi}_s) &= \int_s^t u_q((b-\tilde{b}).\nabla\tilde{\psi}_q + \operatorname{div}(b-\tilde{b})\tilde{\psi}_q)\mathrm{d}q \\ &= \int_s^t u_q(\operatorname{div}[(b_q - \tilde{b}_q)\tilde{\psi}_q])\mathrm{d}q. \end{aligned} \tag{3.18}$$

As Equation (3.9) is linear, in order to prove the uniqueness of the solutions, it is enough to prove it when $u_0 = 0$. As stated above, we need backward solutions of the Continuity Equation. When $\tilde{b}$ is bounded smooth, for $\varphi_0 \in C_c^\infty(\mathbb{R}^d)$, $x \to \varphi_0(x) = \varphi_0(\tilde{\Phi}_{t_0}(x)) \exp\left(\int_0^{t_0}(\operatorname{div}\tilde{b}_q)(\tilde{\Phi}_q(x))\mathrm{d}q\right)$ is smooth, compactly supported and $\tilde{\psi}$ is a solution of the Rough Continuity Equation. Furthermore $\tilde{\psi}_{t_0}(x) = \varphi_0(x)$, and

$$\tilde{\psi}_t(x) = \varphi_0(\tilde{\Phi}_{t_0-t}(x)) \exp\left(\int_t^{t_0} (\operatorname{div} \tilde{b}_q)(\tilde{\Phi}_{q-t}(x)) \mathrm{d}q\right).$$

Hence, we can choose $t = t_0$ in (3.18) and $s = 0$, and we have

$$u_{t_0}(\varphi_0) = \int_0^{t_0} u_q(\operatorname{div}[(b_q - \tilde{b}_q)\tilde{\psi}_q]) \mathrm{d}q.$$

We can split the right hand side into three parts

$$A_1 = \int_0^{t_0} u_q(\operatorname{div}(b - \tilde{b})_q \tilde{\psi}_q) \mathrm{d}q,$$

$$A_2 = \int_0^t \mathrm{d}q \int_{\mathbb{R}^d} \mathrm{d}x u_q(x)(b - \tilde{b})_q(x).\nabla\varphi_0(\tilde{\Phi}_{t-q}(x)).D\tilde{\Phi}_{t-q}(x) \exp\left(\int_q^t (\operatorname{div} \tilde{b}_r)(\tilde{\Phi}_{r-q}(x)) \mathrm{d}q\right) \mathrm{d}q,$$

and

$$A_3 = \int_0^t \mathrm{d}q \int_{\mathbb{R}^d} \mathrm{d}x u_q(x)\tilde{\psi}_q(x)(b - \tilde{b})_q(x) \int_q^t \nabla(\operatorname{div} \tilde{b}_r)(\tilde{\Phi}_{r-q}(x)).D\tilde{\Phi}_{r-q}(x) \mathrm{d}q.$$

Let us recall that $|\tilde{\psi}_q(x)| \leqslant \|\varphi_0\|_\infty \exp(T\|\operatorname{div}\tilde{b}\|_\infty)\mathbb{1}_{B_f(0,\tilde{R})}(x)$ where $\tilde{R}$ is nondecreasing in T, $\|X\|_\gamma$ and $\|\tilde{b}\|_\infty$. Hence

$$|A_1| \leqslant \|u\|_\infty T \sup_{q\in[0,T]} \int_{\mathbb{R}^d} \mathrm{d}x |\operatorname{div} b_q - \operatorname{div}\tilde{b}_q| \mathbb{1}_{B_f(0,\tilde{R})}(x).$$

For A_2 we will use the same trick: as $|\nabla\varphi_0(\tilde{\Phi}_{t-q}(x))| \leqslant \|\nabla\varphi_0\|_\infty \mathbb{1}_{B_f(0,\tilde{R})}(x)$, we have

$$|A_2| \lesssim \sup_{q\in[0,T]} \int_{\mathbb{R}^d} \mathrm{d}x |b_q(x) - \tilde{b}_q(x)| \mathbb{1}_{B_f(0,\tilde{R})}(x) |D\tilde{\Phi}_{t-q}(x)|.$$

The same holds for A_3, and we have

$$|A_3| \lesssim \sup_{q\in[0,T]} \int_{\mathbb{R}^d} \mathrm{d}x |b_q(x) - \tilde{b}_q(x)| \mathbb{1}_{B_f(0,\tilde{R})}(x) \left| \int_q^t \nabla(\operatorname{div}\tilde{b}_r)(\tilde{\Phi}_{r-q}(x)).D\tilde{\Phi}_{r-q}(x) \mathrm{d}r \right|$$

Once put together, this gives the wanted result. □

Remark 3.4.9. In fact, the proof gives us a decomposition of $u_q(\varphi_0)$ as the follows:

$$\tilde{\psi}_t(x) = \varphi_0(\tilde{\Phi}_{t_0-t}(x)) \exp\left(\int_t^{t_0} (\operatorname{div} \tilde{b}_q)(\tilde{\Phi}_{q-t}(x)) \mathrm{d}q\right)$$

and every weak controlled solution of the Rough Transport Equation verifies

$$u_{t_0}(\varphi_0) = \int_0^{t_0} u_q(\operatorname{div}[(b_q - \tilde{b}_q)\tilde{\psi}_q]) \mathrm{d}q.$$

3.4.4 Strong uniqueness

In the case of the fractional Brownian motion, a phenomenon of regularization by noise will occur. But even without any regularization, we have the following theorem.

Theorem 3.4.10. *Let* $b \in L^\infty([0,T]; L^\infty(\mathbb{R}^d) \cap C^1(\mathbb{R}^d))$, $\delta > 0$ *and* $\operatorname{div} b \in L^\infty([0,T] \times \mathbb{R}^d)$, $1/3 < \gamma \leqslant 1/2$ *and* $\boldsymbol{X} \in \mathcal{R}^\gamma([0,T])$. *There exists a unique weak controlled solution with* $u_0 \in L^\infty(\mathbb{R}^d)$ *to the Rough Transport Equation.*

Proof. As $b \in L^\infty([0,T]; \operatorname{Lin}(\mathbb{R}^d))$, and since $\operatorname{div} b \in L^\infty([0,T] \times \mathbb{R}^d)$ and $\boldsymbol{X} \in \mathcal{R}^\gamma$, there exists a weak controlled solution. Furthermore, b is locally Lipschitz continuous in the second variable and b has linear growth in the second variable. It is well-known that there exists a unique solution Φ to the equation

$$\Phi_t(x) = x + \int_0^t b_q(\Phi_q(x)) \mathrm{d}q + X_q.$$

Furthermore, Φ is differentiable in space, and its spatial derivative satisfies the following equation

$$D\Phi_t(x) = \mathrm{id} + \int_0^t Db_q(\Phi_q(x)).D\Phi_q(x) \mathrm{d}q.$$

Furthermore, for for $r > 0$ and $x \leqslant r$

$$|D\Phi_t(x)| \lesssim \sup_{q \in [0,T]} e^{\sup_{y \in B(r + \|X\|_\infty + T\|b\|_\infty)} |Db_q(y)|}.$$

In order to prove uniqueness, with the notations of Remark 3.4.9, we only have to check that there exists a sequence $b^\varepsilon \in C^\infty(\mathbb{R}^d)$ such that

$$|u_{t_0}(\varphi)| = \left| \int_0^{t_0} u_q(\operatorname{div}[(b_q - b^\varepsilon_q)\psi^\varepsilon_q]) \mathrm{d}q \right| \to_{\varepsilon \to 0} 0. \tag{3.19}$$

Let $k \in C^\infty_c(\mathbb{R}^d, \mathbb{R})$ a mollifier,*i.e.* $k \geqslant 0$ such that $\int_{\mathbb{R}^d} k(x)\mathrm{d}x = 1$ and $k(x) = k(-x)$. Let $k_\varepsilon(x) = \frac{1}{\varepsilon^d} k(x/\varepsilon)$ and $b^\varepsilon = k_\varepsilon * b$. Hence, $b^\varepsilon \in C^\infty_b(\mathbb{R}^d)$ with $\|\operatorname{div} b^\varepsilon\|_\infty \leqslant \|\operatorname{div} b\|_\infty$. Let us recall, thanks to Lemma 3.3.8 and since $\varphi \in C^\infty$ and $b^\varepsilon \in C^\infty_b(\mathbb{R}^d) \cap \operatorname{Lin}(\mathbb{R}^d)$, for all $t \in [0,T]$, $x \to \varphi(\Phi^\varepsilon_t(x)) \in C^\infty_c(\mathbb{R}^d)$ and since $\|b^\varepsilon\|_\infty \lesssim \|b\|_\infty$, for all $N \in \mathbb{N}$, there exists $R > 0$ independent of ε such that for all $t \in [0,T]$,

$$\operatorname{supp} \varphi_0(\Phi^\varepsilon_t(.))| \subset B(0,R).$$

Furthermore, as $b^\varepsilon \in C^\infty$, for all $t \in [0,T]$, $\Phi^\varepsilon_t \in C^1(\mathbb{R}^d)$ and for $x \leqslant R$

$$\sup_{\varepsilon > 0} \sup_{t \in [0,T]} |D\Phi^\varepsilon_q(x)| \leqslant \sup_{q \in [0,T]} e^{\sup_{y \in B(R + \|X\|_\infty + T\|b\|_\infty)} |Db_q(y)|}.$$

Furthermore, as $b \in L^\infty([0,T]; C^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$, by localization

$$\sup_{\varepsilon > 0} \varepsilon^{-1} \sup_{q \in [0,T], x \in B(0,R)} |b_q(x) - b^\varepsilon_q(x)| < +\infty.$$

In order to prove the theorem when $\operatorname{div} b \in L^\infty(\mathbb{R}^d)$, we need to use an approximation argument. As all the function are localized in a ball of radius R, let $\eta > 0$ and let $\theta \in C^\infty([0,T] \times \mathbb{R}^d)$ such that $\|(\theta - u)\mathbb{1}_{B(0,R)}\|_{L^1([0,T]\times\mathbb{R}^d)} < \eta$. We have

$$\begin{aligned}\int_0^{t_0} \mathrm{d}q \left| \int_{B(0,R)} \mathrm{d}x \theta_q(x) \operatorname{div}((b_q - b_q^e)\psi_q^\varepsilon)(x) \right| &= \int_0^{t_0} \mathrm{d}q \left| \int_{B(0,R)} \mathrm{d}x \nabla\theta_q(x).(b_q - b_q^e)(x)\psi_q^\varepsilon(x) \right| \\ &\lesssim \|b - b^\varepsilon\|_{L^\infty([0,T]\times B(0,R))} \|\nabla\theta\|_{L^\infty([0,T]\times B(0,R))}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\int_0^{t_0} \mathrm{d}q \left| \int_{B(0,R)} \mathrm{d}x (u_q(x) - \theta_q(x)) \operatorname{div}((b(x) - b^\varepsilon(x))\psi_q^\varepsilon(x)) \right| \\ &\qquad\qquad \lesssim \|u - \theta\|_{L^\infty([0,T]\times B(0,R))} \|\operatorname{div}((b_q - b_q^e)\psi_q^\varepsilon)\|_{L^\infty([0,T]\times B(0,R))}.\end{aligned}$$

But

$$\mathbb{1}_{B(0,R)}(x) \operatorname{div}((b_q - b_q^e)\psi_q^\varepsilon)(x) \leqslant e^{T\|\operatorname{div} b\|_{L^\infty([0,T]\times\mathbb{R}^d)}} (A_1^\varepsilon(x) + A_2^\varepsilon(x) + A_3^\varepsilon(x)),$$

where

$$A_1^\varepsilon = |(\operatorname{div} b_q - \operatorname{div} b_q^\varepsilon)(x)||\varphi(\Phi_{t_0-q}^\varepsilon(x))|\mathbb{1}_{B(0,R)}(x) \lesssim \|\operatorname{div} b_q\|_{L^\infty([0,T]\times\mathbb{R}^d)},$$

$$A_2^\varepsilon = |b_q(x) - b_q^\varepsilon(x)||\nabla\varphi(\Phi_{t_0-q}^\varepsilon(x))||D\Phi_{t_0-q}^\varepsilon(x)|\mathbb{1}_{B(0,R)}(x) \lesssim \sup_{q\in[0,T]} \|b_q \mathbb{1}_{B(0,R)}\|_{L^\infty(\mathbb{R}^d)}$$

and

$$\begin{aligned}A_3^\varepsilon &= |b_q(x) - b_q^\varepsilon(x)||\varphi(\Phi_{t_0-q}^\varepsilon(x))| \int_q^{t_0} \nabla \operatorname{div} b^\varepsilon(\Phi_{r-q}^\varepsilon(x))|D\Phi_{r-q}^\varepsilon(x)|\mathrm{d}r \mathbb{1}_{B(0,R)}(x) \\ &\lesssim \sup_{q,r\in[0,T]} \|(b_q - b_q^\varepsilon)\mathbb{1}_{B(0,R)} \nabla \operatorname{div} b^\varepsilon(\Phi_r^\varepsilon(.))\| \\ &\lesssim 1,\end{aligned}$$

since $\|(b_q - b_q^\varepsilon)\mathbb{1}_{B(0,R)}\|_\infty \lesssim \varepsilon$ and $\|\nabla \operatorname{div} b^\varepsilon(\Phi_r^\varepsilon(.))\mathbb{1}_{B(0,R)}\| \lesssim 1/\varepsilon$. Hence

$$\int_0^{t_0} \mathrm{d}q \left| \int_{B(0,R)} \mathrm{d}x (u_q(x) - \theta_q(x)) \operatorname{div}((b(x) - b^\varepsilon(x))\psi_q^\varepsilon(x)) \right| \lesssim \eta.$$

Finally thanks to Remark 3.4.9,

$$\begin{aligned}|u_{t_0}(\varphi)| &\leqslant \int_0^{t_0} \mathrm{d}q \left| \int_{B(0,R)} \mathrm{d}x (u_q(x) - \theta_q(x)) \operatorname{div}((b(x) - b^\varepsilon(x))\psi_q^\varepsilon(x)) \right| \\ &\quad + \int_0^{t_0} \mathrm{d}q \left| \int_{B(0,R)} \mathrm{d}x \theta_q(x) \operatorname{div}((b_q - b_q^e)\psi_q^\varepsilon)(x) \right| \\ &\lesssim \eta + \|b - b^\varepsilon\|_{L^\infty([0,T]\times B(0,R))} \|\nabla\theta\|_{L^\infty([0,T]\times B(0,R))}.\end{aligned}$$

Now ε goes to zero, and $u_{t_0}(\varphi) = 0$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, which is exactly the wanted result. □

Remark 3.4.11. If $b \in \mathrm{Lin}(\mathbb{R}^d)$ the last argument does not work. Indeed, we were not able to prove that in that case the the functions $x \mapsto \varphi(\Phi_t^\varepsilon(x))$ is compactly supported. In order to fix that, one solution might be to localize the function b before mollifying it. Let $\theta \in C_c^\infty(\mathbb{R}^d)$ such that $\theta(x) = 1$ for $|x| \leqslant 1$ and $\theta(x) = 0$ for $|x| \geqslant 2$. For $r > 0$ let us define $\theta_r(x) = \theta(x/r)$ and

$$b_q^{\varepsilon,r}(x) = k_\varepsilon * (b_q \theta_r).$$

In order to complete the duality argument in Lemma 3.4.8 we would have to test u against $\psi^{\varepsilon,r}\theta_r$. In that procedure other remainder terms should appear.

Corollary 3.4.12. *With the hypothesis of the previous theorem, for $u_0 \in C_b^3(\mathbb{R}^d)$ there exists a unique strong controlled solution u to the Rough Transport Equation. Furthermore, for all $(t,x) \in [0,T] \times \mathbb{R}^d$, $u_t(x) = u_0(\Phi_t^{-1}(x))$.*

Proof. Thanks to Theorem 3.4.7, for $u_0 \in C_b^3(\mathbb{R}^d)$ we know that $(t,x) \to u_0(\Phi_t^{-1}(x))$ is a strong controlled solution, and then a weak controlled solution. But the previous theorem guarantees that there is only one weak controlled solution. □

3.4.5 Regularization by noise

In the paper by Catellier and Gubinelli [13], as we presented in Section 3.2.3, if the process X is ρ-irregular, a phenomenon of regularization occurs. Indeed, for less regular vectorfields b the flow of the equation exists, and furthermore its averaging properties are nice. We will give two different results: for a general ρ-irregular path, and for the fractional Brownian motion.

General $\rho-$irregular paths

Theorem 3.4.13. *Let $\frac{1}{3} < \gamma \leqslant 1$, $\boldsymbol{X} = (X, \mathbb{X}) \in \mathcal{R}^\gamma([0,T])$, $\rho > 0$ such that X is ρ-irregular. Let $\alpha > -\rho$ such that $\alpha + 3/2 > 0$ and $b \in \mathcal{F}L^{\alpha+3/2}(\mathbb{R}^d)$ and $\operatorname{div} b \in \mathcal{F}L^{\alpha+3/2}$. Let $u_0 \in L^\infty(\mathbb{R}^d)$. There exists a unique weak controlled solution to the Rough Transport Equation with initial condition u_0.*

Proof. Thanks to Theorem 3.2.27, the flow Φ of the equation

$$\Phi_t(x) = x + \int_0^t b(\Phi_q(x))\mathrm{d}q + X_t$$

exists. Furthermore, we know that for a mollification b^ε of b such that $\|b - b^\varepsilon\|_{\mathcal{F}L^{\alpha+3/2}} \to 0$,

$$\sup_{\varepsilon>0} \sup_{t\in[0,T], x\in\mathbb{R}^d} |D\Phi_q^\varepsilon(x)| < +\infty.$$

As $\alpha + 3/2 > 0$,

$$|b(x) - b(y)| \lesssim |x-y|^{\alpha+3/2} \|b\|_{\mathcal{F}L^{\alpha+3/2}}.$$

Hence $b \in \mathrm{Lin}(\mathbb{R}^d)$, $\operatorname{div} b \in L^\infty(\mathbb{R}^d)$ and there exists a weak controlled solution of the Rough Transport Equation. Finally for all $t \in [0,T]$, thanks to Proposition 3.2.30 the function

$$(G^\varepsilon)_q^{t_0} : x \to \int_q^{t_0} (\operatorname{div} b^\varepsilon)(\Phi_{r-q}^\varepsilon(x))\mathrm{d}q$$

is differentiable and
$$\sup_{\varepsilon>0}\sup_{q\leqslant t_0\in[0,T]}(G^\varepsilon)^{t_0}_q(|x|)\lesssim K(|x|)N_{\alpha+1}(\operatorname{div} b).$$

Thanks to Lemma 3.4.8, it is enough to prove that

$$A_\varepsilon=\sup_{q\in[0,T]}\int_{\mathbb{R}^d}\mathbb{1}_{B_f(0,R^\varepsilon)}(x)|b(x)-b^\varepsilon(x)|(|D\Phi^\varepsilon_{t_0-q}(x)|+|\nabla(G^\varepsilon)^{t_0}_q(x)|)\to 0$$

and
$$B_\varepsilon=\sup_{q\in[0,T]}\int_{\mathbb{R}^d}\mathbb{1}_{B_f(0,R^\varepsilon)}(x)|\operatorname{div} b(x)-\operatorname{div} b^\varepsilon(x)|(|D\Phi^\varepsilon_{t_0-q}(x)|+|\nabla(G^\varepsilon)^{t_0}_q(x)|).$$

Since $\alpha+3/2>0$, $b\in C^0_b([0,T])$, $R^\varepsilon\lesssim R$ where R^ε and R are the radii defined in Lemma 3.4.2, and $\|b-b^\varepsilon\|_\infty\to 0$. Furthermore, thanks to Theorem 3.2.27, $(|D\Phi^\varepsilon_{t_0-q}(x)|+|\nabla(G^\varepsilon)^{t_0}_q(x)|)\lesssim K(|x|)$ uniformely in ε. Hence $A_\varepsilon\to 0$. Furthermore, $\operatorname{div} b^\varepsilon=k_\varepsilon*(\operatorname{div} b)$ and since $\alpha+3/2>0$, $\operatorname{div} b^\varepsilon\to^{L^\infty}\operatorname{div} b^\varepsilon$, and then the result follows. □

As the proof of the last theorem is pathwise, one can prove the following corollary where b and u_0 are random and X a generic continuous and ρ-irregular stochastic process which lifts into $\mathcal{R}^\gamma$ and with good approximation properties in $L^p(\Omega)$.

Corollary 3.4.14. *Let $\rho>0$, $\alpha>-\rho$ such that $3/2+\alpha>0$ and $1/3<\gamma\leqslant 1.2$. Let $(\Omega,\mathcal{F},\mathbb{P})$ a probability space, X a continuous stochastic process on $(\Omega,\mathcal{F},\mathbb{P})$ such that X is almost surely ρ-irregular and almost surely X lifts in a measurable way to $\boldsymbol{X}\in\mathcal{R}^\gamma$. Suppose furthermore that for any smooth approximation $\boldsymbol{X}^\varepsilon$ of $\boldsymbol{X}$ and any $1\leqslant p<+\infty$, $\mathbb{E}[\|\boldsymbol{X}^\varepsilon-\boldsymbol{X}\|^p_{\mathcal{R}^\gamma}]\to 0$.*

Let $b\in L^\infty(\Omega\times\mathbb{R}^d)$, $\operatorname{div} b\in L^\infty(\Omega\times\mathbb{R}^d)$ such that almost surely $b(\omega,.)\in\mathcal{F}L^{\alpha+3/2}$ and $\operatorname{div} b\in\mathcal{F}L^{\alpha+3/2}$. Let $u_0\in L^\infty(\Omega\times\mathbb{R}^d)$.

There exists a unique Stochastic weak controlled solution $u\in L^\infty(\Omega\times[0,T]\times\mathbb{R}^d)$ of the Rough Transport Equation with initial condition u_0.

Proof. The existence of such a solution is proved in Theorem 3.3.14. For the uniqueness result, remark that the lift $\boldsymbol{X}$ of X and the functions b and u_0 satisfy the conditions of the last theorem almost surely. Hence, the solution is unique. □

Remark 3.4.15. For $H\in(0,1)$, $-\alpha<\rho<1/2H$ with $\alpha+3/2>0$, the fractional Brownian motion B^H satisfies the hypothesis of the last theorem. The last result allows us to take random vectorfields.

Fractional Brownian motion, the hölder continuous case

The last proofs, and in particular the proof of the Lemma 3.4.8 relies on the existence of a flow for the characteristic equation associated to the vectorfield b and the path X. In order to prove uniqueness, a uniform bound on the differential of the flow for regularized vectorfields $\tilde{b}$ is also needed. In [13], the authors use a Girsanov transform for the fractional Brownian motion in order to extend the space of vectorfields, namely from $\mathcal{F}L^\alpha$ to $\mathcal{C}^\beta$, the space of Hölder continuous function.

Theorem 3.4.16. *Let $H \in (1/3, 1/2]$ and $\alpha > -1/2H$ with $\alpha+1 > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and B^H a d-dimensional fractional Brownian motion of Hurst parameter H associated to the probability space and $\boldsymbol{B}^H$ its natural lift. Let $b \in \mathcal{C}_b^{\alpha+1}(\mathbb{R}^d)$ with $\operatorname{div} b \in \mathcal{C}_b^{\alpha+1}(\mathbb{R}^d)$, and $u_0 \in L^\infty(\mathbb{R}^d)$. There exists a unique Stochastic weak controlled solution u of the Stochastic Rough Transport Equation driven by $\boldsymbol{B}^H$ with initial condition u_0.*

Proof. The existence holds thanks to Theorem 3.3.14.

Let $-1/2H < \alpha' < \alpha$ such that $\alpha' + 1 > 0$. Let b^ε a mollification of b. We know that $\|b - b^\varepsilon\|_\infty \to 0$. Let $\varphi_0 \in C_c^\infty(\mathbb{R}^d)$ a test function and the radius R^ε associated to the flow Φ^ε and φ_0. As $R^\varepsilon = R_{\varphi_0} + 2T\|b^\varepsilon\|_\infty + \|B^H\|_\infty \leqslant R = R_{\varphi_0} + 2T\|b\|_\infty + \|B^H\|_\infty$. Hence, when $u_0 = 0$, we have, thanks to the Lemma 3.4.8,

$$u_0(\varphi_0)| \lesssim (\|b - b^\varepsilon\|_\infty + \|\operatorname{div} b - \operatorname{div} b^\varepsilon\|_\infty) \\ \times \int_{\mathbb{R}^d} \mathrm{d}x \mathbb{1}_{B_f(0,R)}(x) \sup_{q\in[0,t_0]} (|D\Phi^\varepsilon_{t_0-q}(x)| + |\nabla (G^\varepsilon)^{t_0}_q(x)|). \quad (3.20)$$

Furthermore, thanks to Theorem 3.2.29 and Proposition 3.2.30,

$$\mathbb{E}[\mathbb{1}_{B_f(0,R)}(x) \sup_{q\in[0,t_0]} (|D\Phi^\varepsilon_{t_0-q}(x)| + |\nabla (G^\varepsilon)^{t_0}_q(x)|)] \lesssim \mathbb{E}[\mathbb{1}_{B_f(0,R)}(x)]^{1/2} K(|x|).$$

But

$$\mathbb{E}[\mathbb{1}_{B_f(0,R)}(x)] = \mathbb{P}(R_{\varphi_0} + T\|b\|_\infty + \|B^H\|_\infty \geqslant |x|)$$

and thanks to Fernique's theorem [19], $\exp(2\|B^H\|_\infty) \in L^1(\Omega)$ for all $p \geqslant 1$, so that by the inquelaity of Markov

$$\mathbb{E}[\mathbb{1}_{B_f(0,R)}(x)] \lesssim \exp(-2|x|).$$

Hence

$$\mathbb{E}[\mathbb{1}_{B_f(0,R)}(x) \sup_{q\in[0,t_0]} (|D\Phi^\varepsilon_{t_0-q}(x)| + |\nabla (G^\varepsilon)^{t_0}_q(x)|)] \leqslant \exp(-|x|) K(|x|)$$

and

$$\int \mathbb{1}_{B_f(0,R)}(x) \sup_{q\in[0,t_0]} (|D\Phi^\varepsilon_{t_0-q}(x)| + |\nabla (G^\varepsilon)^{t_0}_q(x)|) \mathrm{d}x \in L^1(\Omega)$$

and is finite almost surely. By letting ε go to zero in (3.20), almost surely $|u_0(\varphi_0)| = 0$, which ends the proof. ☐

Remark 3.4.17. In the Brownian case $H = 1/2$, we have $b \in \mathcal{C}^\varepsilon(\mathbb{R}^d)$ and $\operatorname{div} b \in \mathcal{C}^\varepsilon(\mathbb{R}^d)$ for all $\varepsilon > 0$, which is almost the optimal regularity under uwhivh uniqueness is known in [30]. Furthermore, as proved in Friz and Hairer [32] and Gubinelli [41], when we choose the lift as the Stratonovitch lift, the stratonovitch integral and the rough integral coincides. Hence, in the Brownian case, the notion of solution are quite similar.

Chapitre 4

Distributions paracontrôlées et équation de quantisation stochastique Φ^4 en dimensions 3

Résumé

La théorie des distributions paracontrôlées à été introduite récemment par Gubinelli, Imkeller et Perkowski [44]. Nous utilisons cette théorie pour montrer l'existence et l'unicité de solutions locales en temps à l'équation de quantisation stochastique renormalisée Φ_3^4 sur le tore de dimensions trois. Cette méthode est une alternative à la théorie des structures de régularités développé par Hairer[50].

Abstract

We prove the existence and uniqueness of a local solution to the periodic renormalized Φ_3^4 model of stochastic quantisation using the method of controlled distributions introduced recently by Imkeller, Gubinelli and Perkowski [44].

Sommaire

4.1 Introduction

We study here the following Cauchy problem

$$\begin{cases} \partial_t u = \Delta_{\mathbb{T}^3} u - u^3 + \xi \\ u(0,x) = u^0(x) \quad x \in \mathbb{T}^3 \end{cases} \tag{4.1}$$

where ξ is a space-time with noise such that $\int_{\mathbb{T}^3} \xi(x)\mathrm{d}x = 0$ i.e. it is a centered Gaussian space-time distribution such that

$$\mathbb{E}[\xi(s,x)\xi(t,y)] = \delta(t-s)\delta(x-y)$$

and $u : \mathbb{R}_+ \times \mathbb{T}^3 \to \mathbb{R}$ is a space-time distribution which is continuous in time. We write this equation in its mild formulation

$$u = P_t u^0 - \int_0^t P_{t-s}(u_s)^3 \mathrm{d}s + X_t \tag{4.2}$$

where $P_t = e^{t\Delta}$ is the Heat flow and $X_t = \int_0^t P_{t-s}\xi_s \mathrm{d}s$ is a the solution of the linear equation :

$$\partial_t X_t = \Delta_{\mathbb{T}^3} X_t + \xi; \quad X_0 = 0. \tag{4.3}$$

Moreover X is a Gaussian process and as we see below $X \in \mathcal{C}([0,T], \mathcal{C}^{-1/2-\varepsilon}(\mathbb{T}^3))$ for every $\varepsilon > 0$ where $\mathcal{C}^\alpha = B^\alpha_{\infty,\infty}$ is the Besov-Hölder space. The main difficulty of the equation (4.1) comes from the fact that for any fixed time t the space regularity of the solution $u(t,x)$ cannot be better than the one of X_t. If we measure spatial regularity in the scale of Hölder spaces $\mathcal{C}^\alpha$ we should expect that $u(t,x) \in \mathcal{C}^\alpha(\mathbb{T}^3)$ for any $\alpha < -1/2$ but not better. In particular the term u^3 is not well defined. A natural approach to give a well defined meaning to the equation would consist in regularizing the noise in $\xi^\varepsilon = \xi \star \rho^\varepsilon$ with $\rho^\varepsilon = \varepsilon^{-3}\rho(\frac{\cdot}{\varepsilon})$ a smooth kernel and taking the limit of the solution u^ε of the approximate equation

$$\partial_t u^\varepsilon = \Delta u^\varepsilon - (u^\varepsilon)^3 + \xi^\varepsilon. \tag{4.4}$$

Since the non-linear term diverges when ε goes to zero, an a priori estimate for the wanted solution is hard to find. To overcome this problem we have to focus on the following modified equation

$$\partial_t u^\varepsilon = \Delta u^\varepsilon - ((u^\varepsilon)^3 - C_\varepsilon u^\varepsilon) + \xi^\varepsilon \tag{4.5}$$

where $C_\varepsilon > 0$ is a renormalization constant which diverges when ε goes to 0. We will show that we have to take $C_\varepsilon \sim \frac{a}{\varepsilon} + b\log(\varepsilon) + c$ to obtain a non trivial limit for $(u^\varepsilon)^2 - C_\varepsilon$.

Therefore this paper aims at giving a meaning of the equation (4.2) and at obtaining a (local in time) solution. The method developed here uses some ideas of [47] where the author deals with the KPZ equation. More precisely we use the partial series expansion of the solution to define the reminder term using the notion of paracontrolled distributions introduced in [44]. A solution of this equation has already been constructed in the remarkable paper of Hairer [50] where the author shows the convergence of the solution of the mollified equation (4.5).

The stochastic quantization problem has been studied since the eighties in theoretical physics (see for example [55] and [54] In [8] and the references about it in [50]).

From a mathematical point of view, several articles deals with the 2-dimensional case. Weak probabilistic solutions where find by Jona-Lasinio and and Mitter in [55] and [54]. Some other probabilistic results are obtain thanks to non perturbative methods by Bertini, Jona-Lasinio and Parrinello in [8]. In [17] Da Prato and Debussche found a strong (in the probabilistic sense) formulation for this 2d problem.

In a recent paper, Hairer [50] gives a fixed point solution to the 3-dimensional case thanks to his theory of regularity structures. Like the theory of paracontrolled distributions, Hairer's theory of regularity structures is a generalization of rough path theory. Hairer gets his result by giving a generalization of the notion of pointwise Hölder regularity. With this extended notion, it is possible to work on a more abstract space where the solutions are constructed thanks to a fixed point argument, and then project the abstract solution into a space of distributions via a reconstruction map. The regularity structures approach is quite general and can treat more singular models.

In the approach of the paracontrolled distribution developed in [44] by Gubinelli, Imkeller and Perkowski, on the other hand, it is the notion of controlled path which is generalized. This allows us to give a reasonable notion of product of distributions. Since all the problems treated by the theory of paracontrolled distributions can be solved by using the theory of regularity structures, asking whether or not the opposite is true is a legitimate (and reasonable) question. The following theorems are a piece of the answer.

We will proceed in two steps. In an analytic part we will extend the flow of the regular equation,

$$\partial u_t = \Delta u_t - u^3 + 3au + 9bu + \xi$$

with $(a,b)\in\mathbb{R}^2$ and $\xi\in C([0,T],C^0(\mathbb{T}^3))$ to the situation of more irregular driving noise ξ. More precisely we will prove that the solution u is a continuous function of $(u^0,R^{\varphi}_{a,b}\mathbf{X})$ with

$$\begin{aligned}R^{\varphi}_{a,b}\mathbf{X} =&(X,X^2-a,I(X^3-3aX),\pi_0(I(X^3-3aX),X),\\ &\pi_0(I(X^2-a),(X^2-a))-b-\varphi,\pi_0(I(X^3-3aX),(X^2-a))-3bX-3\varphi X,\varphi)\end{aligned} \tag{4.6}$$

where $X_t=\int_0^t P_{t-s}\xi\mathrm{d}s$, $\pi_0(.,.)$ denotes the reminder term of the paraproduct decomposition given in (4.2.3) and $I(f)_t=\int_0^t P_{t-s}f\mathrm{d}s$. This extension is given in the following theorem.

Theorem 4.1.1. *Let $F:\mathcal{C}^1(\mathbb{T}^3)\times C(\mathbb{R}^+,\mathcal{C}^0(\mathbb{T}^3))\times\mathbb{R}\times\mathbb{R}\to C(\mathbb{R}^+,\mathcal{C}^1(\mathbb{T}^3))$ the flow of the equation*

$$\begin{cases}\partial_t u_t=\Delta u_t-u_t^3+3au_t+9bu_t+\xi_t, & t\in[0,T_C(u^0,X,(a,b))[\\ \partial_t u_t=0, & t\geq T_C(u^0,X,(a,b))\\ u(0,x)=u^0(x)\in\mathcal{C}^1(\mathbb{T}^3)\end{cases}$$

where $\xi\in C(\mathbb{R}^+,\mathcal{C}^0(\mathbb{T}^3))$ and $T_C(u^0,\xi,(a,b))$ is a time such that the the equation holds for $t\leq T_C$. Now let $z\in(1/2,2/3)$, then there exists a Polish space $\mathcal{X}$, called the space of rough distribution, $\tilde{T}_C:\mathcal{C}^{-z}\times\mathcal{X}\to\mathbb{R}^+$ a lower semi-continuous function and $\tilde{F}:\mathcal{C}^{-z}\times\mathcal{X}\to C(\mathbb{R}^+,C^{-z}(\mathbb{T}^3))$ continuous in $(u^0,\mathbb{X})\in C^{-z}(\mathbb{T}^3)\times\mathcal{X}$ such that $(\tilde{F},\tilde{T})$ extends (F,T) in the following sense :

$$T_C(u^0,\xi,(a,b))\geq\tilde{T}_C(u^0,R^{\varphi}_{a,b}\mathbf{X})$$

and

$$F(u^0,\xi,a,b)(t)=\tilde{F}(u^0,R^{\varphi}_{a,b}\mathbf{X})(t),\ \text{for all } t\leq\tilde{T}_C(u^0,R^{\varphi}_{a,b}\mathbf{X})$$

for all $(u^0,\xi,\varphi)\in\mathcal{C}^1(\mathbb{T}^3)\times C(\mathbb{R}^+,\mathcal{C}^0(\mathbb{T}^3))\times C^\infty([0,T])$, $(a,b)\in\mathbb{R}^2$ with $X_t=\int_0^t\mathrm{d}sP_{t-s}\xi$ and where $R^{\varphi}_{a,b}$ is given in the equation (4.6).

In a second part we obtain a probabilistic estimate for the stationary Ornstein Uhlenbeck (O.U.) process which is the solution if the linear equation (4.3) and this allows us to construct the rough distribution in this case.

Theorem 4.1.2. *Let X be the stationary (O.U.) process and X^ε a space mollification of X. There exists two constants $C_1^\varepsilon,C_2^\varepsilon\to^{\varepsilon\to0}+\infty$ and a function $\varphi^\varepsilon\in C^\infty(\mathbb{R}^+)$ such that $R^{\varphi^\varepsilon}_{C_1^\varepsilon,C_2^\varepsilon}\mathbf{X}^\varepsilon$ converge in $L^p(\Omega,\mathcal{X})$ to some $\mathbb{X}\in\mathcal{X}$. Furthermore the first component of $\mathbb{X}$ is X.*

In the setting, the Corollary below follows immediately.

Corollary 4.1.3. *Let ξ a space time white noise, and ξ^ε is a space mollification of ξ such that :*

$$\xi^\varepsilon=\sum_{k\neq0}f(\varepsilon k)\hat{\xi}(k)e_k$$

with f a smooth radial function with compact support satisfying $f(0)=0$, let X the stationary (O.U.) process associated to ξ, $\mathbb{X}$ the element of $\mathcal{X}$ given in the Theorem (4.1.2) and $u^0\in\mathcal{C}^{-z}$ for $z\in(1/2,2/3)$ then if u^ε is the solution of the mollified equation :

$$\begin{cases}\partial_t u_t^\varepsilon=\Delta u_t^\varepsilon-(u_t^\varepsilon)^3+3C_1^\varepsilon u_t+9C_2^\varepsilon u_t+\xi_t^\varepsilon, & t\in[0,T^\varepsilon[\\ \partial_t u_t=0, & t\geq T^\varepsilon\\ u(0,x)=(u^0)^\varepsilon(x)\end{cases}$$

We have the following convergence :

$$\lim_{\varepsilon} u^{\varepsilon} = \tilde{F}(u^0, \mathbb{X})$$

where the limit is understood in the probability sense in the space $C(\mathbb{R}^+, \mathcal{C}^{-z})$.

The proofs of those two theorems are almost independent, but we need the existence and the properties of the rough distribution, specified in the Definition 4.2.9, to prove the first theorem.

Plan of the paper. It is the aim of Section 4.2 to introduce the notion spaces of paracontrolled distributions where the renormalized equation will be solved. In Section 4.3 we prove that for a small time the application associated to the renormalized equation is a contraction, which, by a fixed point argument, gives the existence and uniqueness of the solution, but also the continuity with respect to the rough distribution and the initial condition. The last Section 4.4 is devoted to the existence of the rough distribution for the (O.U.) process.

4.2 Paracontrolled distributions

4.2.1 Besov spaces and paradifferential calculus

The results given in this Subsection can be found in [5] and [44]. Let us start by recalling the definition of Besov spaces via the Littelwood-Paley projectors.

Let $\chi, \theta \in \mathcal{D}$ be nonnegative radial functions such that

1. The support of χ is contained in a ball and the support of θ is contained in an annulus;
2. $\chi(\xi) + \sum_{j\geq 0} \theta(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^d$;
3. $\mathrm{supp}(\chi) \cap \mathrm{supp}(\theta(2^{-j}.)) = \emptyset$ for $i \geq 1$ and $\mathrm{supp}(\theta(2^{-j}.)) \cap \mathrm{supp}(\theta(2^{-j}.)) = \emptyset$ when $|i-j| > 1$.

For the existence of χ and θ see [5], Proposition 2.10. The Littlewood-Paley blocks are defined as

$$\Delta_{-1} u = \mathcal{F}^{-1}(\chi \mathcal{F} u) \text{ and for } j \geq 0, \Delta_j u = \mathcal{F}^{-1}(\theta(2^{-j}.)\mathcal{F} u).$$

We define the Besov space of distribution by

$$B^{\alpha}_{p,q} = \left\{ u \in S'(\mathbb{R}^d); \quad ||u||^q_{B^{\alpha}_{p,q}} = \sum_{j\geq -1} 2^{jq\alpha} ||\Delta_j u||^q_{L^p} < +\infty \right\}.$$

In the sequel we will deal with the special case of $\mathcal{C}^{\alpha} := B^{\alpha}_{\infty,\infty}$ and write $||u||_{\alpha} = ||u||_{B^{\alpha}_{\infty,\infty}}$. We hold the following result for the convergence of localized series in the Besov spaces, which will prove itself useful.

Proposition 4.2.1. *Let* $(p, q, s) \in [1, +\infty]^2 \times \mathbb{R}$, *$B$ a ball in* $\mathbb{R}^d$ *and* $(u_j)_{j\geq -1}$ *a sequence of functions such that* $\mathrm{supp}(u_j)$ *is contained in* $2^j B$ *moreover we assume that*

$$\Xi_{p,q,s} = \left|\left|(2^{js}||u_j||_{L^p})_{j\geq -1}\right|\right|_{l^q} < +\infty$$

then $u = \sum_{j\geq -1} u_j \in B^s_{p,q}$ *and* $||u||_{B^s_{p,q}} \lesssim \Xi_{p,q,s}$.

The trick to manipulate stochastic objects is to deal with Besov spaces with finite indexes and then go back to space $\mathcal{C}^\alpha$. For that we have the following useful Besov embedding.

Proposition 4.2.2. *Let $1 \le p_1 \le p_2 \le +\infty$ and $1 \le q_1 \le q_2 \le +\infty$. For all $s \in \mathbb{R}$ the space $B^s_{p_1,q_1}$ is continuously embedded in $B^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,q_2}$, in particular we have $||u||_{\alpha-\frac{d}{p}} \lesssim ||u||_{B^\alpha_{p,p}}$.*

Taking $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$ we can formally decompose the product as

$$fg = \pi_<(f,g) + \pi_0(f,g) + \pi_>(f,g)$$

with

$$\pi_<(f,g) = \pi_>(g,f) = \sum_{j\ge -1}\sum_{i<j-1} \Delta_i f \Delta_j g; \quad \pi_0(f,g) = \sum_{j\ge -1}\sum_{|i-j|\le 1} \Delta_i f \Delta_j g.$$

With these notations the following results hold.

Proposition 4.2.3. *Let $\alpha, \beta \in \mathbb{R}$*

- $||\pi_<(f,g)||_\beta \lesssim ||f||_\infty ||g||_\beta$ *for $f \in L^\infty$ and $g \in \mathcal{C}^\beta$*
- $||\pi_>(f,g)||_{\alpha+\beta} \lesssim ||f||_\alpha ||g||_\beta$ *for $\beta < 0$, $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$*
- $||\pi_0(f,g)||_{\alpha+\beta} \lesssim ||f||_\alpha ||g||_\beta$ *for $\alpha + \beta > 0$ and $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$*

On of the key result of [44] is a commutation result for the operator $\pi_<$ and π_0.

Proposition 4.2.4. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha < 1$, $\alpha + \beta + \gamma > 0$ and $\beta + \gamma < 0$ then*

$$R(f,x,y) = \pi_0(\pi_<(f,x),y) - f\pi_0(x,y)$$

is well-defined when $f \in \mathcal{C}^\alpha$, $x \in \mathcal{C}^\beta$ and $y \in \mathcal{C}^\gamma$ and more precisely

$$||R(f,x,y)||_{\alpha+\beta+\gamma} \lesssim ||f||_\alpha ||x||_\beta ||y||_\gamma$$

We finish this Section by describing the action of the Heat flow on the Besov spaces and a commutation property with the paraproduct . See the appendix for a proof.

Lemma 4.2.5. *Let $\theta \ge 0$ and $\alpha \in \mathbb{R}$ then the following inequality holds*

$$||P_t f||_{\alpha+2\theta} \lesssim \frac{1}{t^\theta}||f||_\alpha, \quad ||(P_{t-s}-1)f||_{\alpha-2\varepsilon} \lesssim |t-s|^\varepsilon ||f||_\alpha$$

for $f \in \mathcal{C}^\alpha$. Moreover if $\alpha < 1$ and $\beta \in \mathbb{R}$ we have

$$||P_t \pi_<(f,g) - \pi_<(f, P_t g)||_{\alpha+\beta+2\theta} \lesssim \frac{1}{t^\theta}||f||_\alpha ||g||_\beta$$

for all $g \in \mathcal{C}^\beta$.

In the following, we will extensively use some space-time function spaces. Let us introduce the notation

Notation 4.2.6.

$$C_T^\beta = C([0,T],\mathcal{C}^\beta)$$

For $f \in C_T^\beta$ we introduce the norm

$$||f||_\beta = \sup_{t\in[0,T]} ||f_t||_{\mathcal{C}^\beta} = \sup_{t\in[0,T]} ||f_t||_\beta$$

and by

$$\mathcal{C}_T^{\alpha,\beta} := C^\alpha([0,T],\mathcal{C}^\beta(\mathbb{T}^3)).$$

Furthermore, we endow this space with the following distance

$$d_{\alpha,\beta}(f,g) = \sup_{t\neq s\in[0,T]} \frac{\|(f-g)_t-(f-g)_s\|_\beta}{|t-s|^\alpha} + \sup_{t\in[0,T]} ||f_t-g_t||_\beta.$$

4.2.2 Renormalized equation and rough distribution

Let us focus on the mild formulation of the equation (4.1)

$$u = \Psi + X + I(u^3) = X + \Phi \tag{4.7}$$

where we remind the notation $I(f)(t) = -\int_0^t P_{t-s}f_s\mathrm{d}s$, $X = -I(\xi)$ and $\Psi_t = P_t u^0$ for $u^0 \in \mathcal{C}^{-z}(\mathbb{T}^3)$. We can see that a solution u must have at least the same regularity as X. Yet thanks to the definition of I, as $\xi \in C([0,T],\mathcal{C}^{-5/2-\varepsilon})$, for all $\varepsilon > 0$, we have $X \in C([0,T],\mathcal{C}^{-1/2-\varepsilon})$. But in that case the non-linear term u^3 is not well-defined, as there is no universal notion for the product of distributions. A first idea is to proceed by regularization of X, such that products of the regularized quantities are well-defined, and then try to pass to the limit. Let us recall that the stationary O.U process is defined by the fact that $(\hat{X}_t(k))_{t\in\mathbb{R},k\in\mathbb{Z}^3}$ is a centered Gaussian process with covariance function given by

$$\mathbb{E}\left[\hat{X}_t(k)\hat{X}_s(k')\right] = \delta_{k+k'=0}\frac{e^{-|k|^2|t-s|}}{|k|^2}$$

and $\hat{X}_t(0) = 0$. Let $X_t^\varepsilon = \int_0^t P_{t-s}\xi^\varepsilon\mathrm{d}s$ more precisely $\hat{\xi}^\varepsilon = f(\varepsilon k)\hat{\xi}(k)$ where f is a smooth radial function with bounded support such that $f(0) = 1$. Then we have the following approximated equation

$$\Phi^\varepsilon = \Psi^\varepsilon + I((X^\varepsilon)^3) + 3I((\Phi^\varepsilon)^2X^\varepsilon) + 3I(\Phi^\varepsilon(X^\varepsilon)^2) + I((\Phi^\varepsilon)^3)$$

for $\Phi^\varepsilon = I((u^\varepsilon)^3) + \Psi^\varepsilon$ which is well-posed. Then an easy computation gives for $(X^\varepsilon)^2$

$$\begin{aligned}\mathbb{E}\left[(X_t^\varepsilon)^2\right] &= \sum_{k\in\mathbb{Z}^3}\sum_{k_1+k_2=k} f(\varepsilon k_1)f(\varepsilon k_2)\frac{1}{|k_1|^2}\delta_{k_1+k_2=0}\\ &= \sum_{k\in\mathbb{Z}^3}\frac{|f(\varepsilon k)|^2}{|k|^2} \sim_0 \frac{1}{\varepsilon}\int_{\mathbb{R}^3} f(x)(1+|x|)^{-2}\mathrm{d}x\end{aligned}$$

and there is no hope to obtain a finite limit for this term when ε goes to zero. This difficulty has to be solved by subtracting to the original equation these problematic contributions. In order to do so consistently we will introduce a renormalized product. Formally we would like to define

$$X^{\diamond 2} = X^2 - \mathbb{E}[X^2]$$

and show that it is well-defined and that $X^{\diamond 2} \in \mathcal{C}_T^{-1-\delta}$ for $\delta > 0$. More precisely we will defined

$$(X^\varepsilon)^{\diamond 2} = (X^\varepsilon)^2 - \mathbb{E}[(X^\varepsilon)^2]$$

and we will show that it converges to some finite limit. The same phenomenon happens for X^3 and other terms, and we have to renormalize them too. This is the meaning of Theorem 4.1.2. We remind the notation of that theorem

Notation 4.2.7. Let C_1^ε and C_2^ε two positives constants (to be specified later). We denote by

$$\begin{aligned}
&(X^\varepsilon)^{\diamond 2} := (X^\varepsilon)^2 - C_1^\varepsilon \\
&I((X^\varepsilon)^{\diamond 3}) := I((X^\varepsilon)^3 - 3C_1^\varepsilon X^\varepsilon) \\
&\pi_{0\diamond}(I((X^\varepsilon)^{\diamond 2}), (X^\varepsilon)^{\diamond 2}) = \pi_0(I((X^\varepsilon)^{\diamond 2})(X^\varepsilon)^{\diamond 2}) - C_2^\varepsilon \\
&\pi_{0\diamond}(I((X^\varepsilon)^{\diamond 3}), (X^\varepsilon)^{\diamond 2}) = \pi_0(I((X^\varepsilon)^{\diamond 3})(X^\varepsilon)^{\diamond 2}) - 3C_2^\varepsilon X^\varepsilon.
\end{aligned}$$

Remark 4.2.8. In all the sequel the symbol $\diamond$ does not stand for the usual Wick product, also it looks like it, but for renormalized product, where we have subtracted only the diverging quantity in the expression of the stochastic processes. It can be seen as a product between the usual one and the Wick one. When in Section 4.4 we use the usual Wick product (see [53] for its definition and its properties) we use the usual notation :: .

To include such considerations and notations in the approximated equation, we need to add a renormalized term

$$\begin{aligned}
\Phi^\varepsilon &= \Psi^\varepsilon + I((X^\varepsilon)^3) + 3I((\Phi^\varepsilon)^2 X^\varepsilon) + 3I(\Phi^\varepsilon (X^\varepsilon)^2) + I((\Phi^\varepsilon)^3) - C^\varepsilon I(\Phi^\varepsilon + X^\varepsilon) \\
&= \Psi^\varepsilon + I((X^\varepsilon)^3 - 3C_1^\varepsilon X^\varepsilon) + 3I((\Phi^\varepsilon)^2 X^\varepsilon) + 3I(\Phi^\varepsilon((X^\varepsilon)^2 - C_1^\varepsilon) - 3C_2^\varepsilon(\Phi^\varepsilon + X^\varepsilon)) + I((\Phi^\varepsilon)^3)
\end{aligned}$$

with $C^\varepsilon = 3(C_1^\varepsilon - 3C_2^\varepsilon)$. Then the approximated equation is given by

$$\Phi^\varepsilon = \Psi^\varepsilon + I((X^\varepsilon)^{\diamond 3}) + 3I((\Phi^\varepsilon)^2 X^\varepsilon) + 3I(\Phi^\varepsilon \diamond (X^\varepsilon)^{\diamond 2}) + I((\Phi^\varepsilon)^3) \tag{4.8}$$

where

$$I(\Phi^\varepsilon \diamond (X^\varepsilon)^{\diamond 2}) := 3I(\Phi^\varepsilon((X^\varepsilon)^2 - C_1^\varepsilon)) + 9C_2^\varepsilon I(\Phi^\varepsilon + X^\varepsilon)$$

Then our goal is obtain a uniform bound for the solution Φ^ε. For that we proceed in two steps

1. In a first analytic step we build an abstract fix point equation which allows us to extend continuously the flow of the regular equation given by

$$\begin{cases} \Phi = I(X^3 - 3aX) + 3I(\Phi^2 X) + 3\left\{I(X^2 - a)) - 3bI(\Phi + X)\right\} + I(\Phi^3) + \Psi \\ (X, \Psi, (a,b)) \in \mathcal{C}_T^1(\mathbb{T}^3) \times \mathcal{C}_T^1(\mathbb{T}^3)) \times \mathbb{R}^2 \end{cases} \tag{4.9}$$

   to a space $\mathcal{X}$ of a more rough signal X which satisfies some algebraic and analytic assumptions (see (4.2.9) for the exact definition of $\mathcal{X}$).

2. In a second probabilistic step we show that the stationary (O.U) process can be enhanced in a canonical way in an element $\mathbb{X}$ of $\mathcal{X}$.

We will give the exact definition of the space $\mathcal{X}$

Definition 4.2.9. Let $T > 1$, $\nu, \rho > 0$. We denote by $\overline{C}_T^{\nu,\rho}$ the closure of the set of smooth functions $C^\infty([0,T],\mathbb{R})$ by the semi-norm :

$$||\varphi||_{\nu,\rho} = \sup_{t\in[0,T]} t^\nu|\varphi_t| + \sup_{t,s\in[0,T];s\neq t} \frac{s^\nu|\varphi_t - \varphi_s|}{|t-s|^\rho}.$$

For $0 < 4\delta' < \delta$ we define the normed space $\mathcal{W}_{T,K}$

$$\mathcal{W}_{T,K} = C_T^{\delta',-1/2-\delta} \times C_T^{\delta',-1-\delta} \times C_T^{\delta',1/2-\delta} \times C_T^{\delta',-\delta} \times C_T^{\delta',-\delta} \times C_T^{\delta',-1/2-\delta} \times \overline{C}_T^{\nu,\rho}$$

with $K = (\delta, \delta', \nu, \rho)$ equipped with the product topology . For $(X, \varphi) \in C([0,T], C(\mathbb{T}^3)) \times C^\infty([0,T])$, and $(a,b) \in \mathbb{R}^2$ we define $R^\varphi_{a,b}\mathbf{X} \in \mathcal{W}_{T,K}$ by

$$\begin{aligned} R^\varphi_{a,b}\mathbf{X} =& (X, X^2 - a, I(X^3 - 3aX), \pi_0(I(X^3 - 3aX), X), \\ &\pi_0(I(X^2 - a), (X^2 - a)) - b - \varphi, \pi_0(I(X^3 - 3aX), (X^2 - a)) - 3bX - 3\varphi X, \varphi). \end{aligned}$$

The space of the rough distribution $\mathcal{X}_{T,K}$ is defined as the closure of the set

$$\left\{R^\varphi_{a,b}\mathbf{X}, \quad (X,\varphi) \in C([0,T], C(\mathbb{T}^3)) \times C^\infty([0,T]), (a,b) \in \mathbb{R}^2\right\}$$

in $\mathcal{W}_{T,K}$. For $\mathbb{X} \in \mathcal{X}$ we denote its components by

$$\mathbb{X} = (X, X^{\diamond 2}, I(X^{\diamond 3}), \pi_{0\diamond}(I(X^{\diamond 3}), X), \pi_{0\diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi^X, \pi_{0\diamond}(I(X^{\diamond 3}), X^{\diamond 2}) - 3\varphi^X X, \varphi^X).$$

For two rough distributions $\mathbb{X} \in \mathcal{X}_{T,K}$ and $\mathbb{Y} \in \mathcal{X}_{T,K}$ we introduce the distance :

$$\begin{aligned} \mathbf{d_{T,K}}(\mathbb{Y}, \mathbb{X}) =& d_{\delta',-1/2-\delta}(Y, X) + d_{\delta',-1-\delta}(Y^{\diamond 2}, X^{\diamond 2}) + d_{\delta',1/2-\delta}(I(Y^{\diamond 3}), I(X^{\diamond 3})) \\ &+ d_{\delta',-1/2-\delta}(\pi_0(I(Y^{\diamond 3}), Y), \pi_0(I(X^{\diamond 3}), X)) \\ &+ d_{\delta',-1-\delta}(\pi_{0\diamond}(I(Y^{\diamond 2}), Y^{\diamond 2}) - \varphi^Y, \pi_{0\diamond}(I(X^{\diamond 3}), \diamond X^2) - \varphi^X) \\ &+ d_{\delta',-1-\delta}(\pi_{0\diamond}(I(Y^{\diamond 3})Y^{\diamond 2}) - 3\varphi^Y Y, \pi_{0\diamond}(I(X^{\diamond 3})\diamond, X^2) - 3\varphi^X X) \\ &+ \|\varphi^Y - \varphi^X\|_{\nu,\rho}. \end{aligned} \tag{4.10}$$

with $K = (\delta, \delta', \rho, \nu) \in [0,1]^4$ and we denote by $||\mathbb{X}||_{T,K} = \mathbf{d}_{T,K}(\mathbb{X}, 0)$.

Remark 4.2.10. As we see in the Section (4.4), the term $\pi_0(I((X^\varepsilon)^2) - C_1^\varepsilon, (X^\varepsilon)^2 - C_1^\varepsilon) - C_2^\varepsilon$ where X^ε is a mollification of the O.U process does not converge in the space $\mathcal{C}_T^{\delta',-\delta}$. On the other hand it converge in a explosive norm and more precisely there exit a function $\varphi^\varepsilon \in C^\infty([0,T])$ such that $\varphi^\varepsilon \to^{\varepsilon\to 0} \varphi$ in $\overline{C}_T^{\nu,\rho}$ and $\pi_0(I((X^\varepsilon)^2) - C_1^\varepsilon, (X^\varepsilon)^2 - C_1^\varepsilon) - C_2^\varepsilon - \varphi^\varepsilon$ converge in $\mathcal{C}_T^{\delta',-\delta}$ for all $0 < \delta' < \delta/4$.

For $\mathbb{X} \in \mathcal{X}$ we can obviously construct $I(X^{\diamond 2}) \diamond X^{\diamond 2}$ using the Bony paraproduct in the following way

$$I(X^{\diamond 2}) \diamond X^{\diamond 2} = \pi_<(I(X^{\diamond 2}), X^{\diamond 2}) + \pi_>(I(X^{\diamond 2}), X^{\diamond 2}) + \pi_{0\diamond}(I(X^{\diamond 2}), X^{\diamond 2})$$

and a similar definition for $I(X^{\diamond 3}) \diamond X^{\diamond 2}$. In the sequel we might abusively denote $\mathbb{X}$ by X if there is no confusion, and the rough path terminology we denote the other components of $\mathbb{X}$ by the area components of $\mathbb{X}$.

4.2.3 Paracontrolled distributions and fixed point equation

The aim of this Section is to define a suitable space in which it is possible to formulate an fixed point for the eventual limit of the mollified solution, to be more precise let $\mathbb{X} \in \mathcal{X}$ then we know that there exist $X^\varepsilon \in \mathcal{C}^1_T(\mathbb{T}^3)$, $a^\varepsilon, b^\varepsilon \in \mathbb{R}$ and $\varphi^\varepsilon \in C^\infty([0,T])$ such that $\lim_{\varepsilon\to 0} R^{\varphi^\varepsilon}_{a^\varepsilon,b^\varepsilon}\mathbf{X}^\varepsilon = \mathbb{X}$. Let us focus more intently on the regular equation given by :

$$\Phi^\varepsilon = I((X^\varepsilon)^3 - 3a^\varepsilon X^\varepsilon) + 3\left\{I(\Phi^\varepsilon((X^\varepsilon)^2 - a^\varepsilon)) - 3b^\varepsilon I(X^\varepsilon + \Phi^\varepsilon)\right\} + 3I((\Phi^\varepsilon)^2 X^\varepsilon) + I((\Phi^\varepsilon)^3)$$

where we have omitted temporarily the dependence on the initial condition. If we assume that Φ^ε converge to some Φ in $\mathcal{C}^{1/2-\delta}$ we see that the regularity of $\mathbb{X}$ is not sufficient to define $I(\Phi^2 X) := \lim_{\varepsilon\to 0} I((\Phi^\varepsilon)^2 X^\varepsilon)$ and $I(\Phi \diamond X^{\diamond 2}) := \lim_{\varepsilon\to 0} I(\Phi^\varepsilon((X^\varepsilon)^2 - a^\varepsilon)) + 3b^\varepsilon I(X^\varepsilon + \Phi^\varepsilon)$. To bypass this problem we remark that

$$\Phi^\varepsilon = I((X^\varepsilon)^3 - 3a^\varepsilon X^\varepsilon) + 3I(\pi_<(\Phi^\varepsilon, (X^\varepsilon)^2 - a^\varepsilon)) + (\Phi^\varepsilon)^\sharp$$

then if we impose the convergence of $(\Phi^\varepsilon)^\sharp$ to some $\Phi^\sharp$ in $\mathcal{C}^{3/2-\delta}_T$ we see that the limit Φ should satisfy the following relation

$$\Phi = I(X^{\diamond 3}) + 3I(\pi_<(\Phi, X^{\diamond 2})) + \Phi^\sharp.$$

This is the missing ingredient which allows to construct the quantity $I(\Phi^2 X)$ and $I(\Phi \diamond X^{\diamond 2})$ and to solve the equation

$$\Phi = I(X^{\diamond 3}) + 3I(\Phi^2 X) + 3I(\Phi \diamond X^{\diamond 2}) + \Phi^\sharp \tag{4.11}$$

Notation 4.2.11. Let us introduce some useful notations for the sequel

$$B_>(f,g) = I(\pi_>(f,g)), \quad B_0(f,g) = I(\pi_0(f,g)) \quad \text{and} \quad B_<(f,g) = I(\pi_<(f,g)).$$

As we observed in the beginning of this Section to deal with the difficulty of defining the products of distributions, we use the notion of controlled distribution introduced in [44].

Definition 4.2.12. Let $\mathbb{X} \in \mathcal{X}$ and $z \in (1/2, 2/3)$. We say that $\Phi \in \mathcal{C}^{1/2-z}_T$ is controlled by $\mathbb{X}$ if

$$\Phi = I(X^{\diamond 3}) + B_<(\Phi', X^{\diamond 2}) + \Phi^\sharp$$

such that

$$||\Phi^\sharp||_{\star,1,L,T} = \sup_{t\in[0,T]} \left(t^{\frac{1+\delta+z}{2}} ||\Phi^\sharp_t||_{1+\delta} + t^{1/4+\frac{\gamma+z}{2}} ||\Phi^\sharp_t||_{1/2+\gamma} + t^{\frac{\kappa+z}{2}} ||\Phi^\sharp_t||_\kappa \right)$$
$$+ \sup_{(s,t)\in[0,T]^2} s^{\frac{z+a}{2}} \frac{||\Phi^\sharp_t - \Phi^\sharp_s||_{a-2b}}{|t-s|^b} < +\infty$$

and

$$||\Phi'||_{\star,2,L,T} = \sup_{(s,t)\in[0,T]^2} s^{\frac{z+c}{2}} \frac{||\Phi'_t - \Phi'_s||_{c-2d}}{|t-s|^d} + \sup_{t\in[0,T]} t^{\frac{\eta+z}{2}} ||\Phi'_t||_\eta < +\infty$$

with $L := (\delta, \gamma, \kappa, a, b, c, d, \eta) \in [0,1]^8$, $z \in (1/2, 2/3)$ and $2d \le c$, $2b \le a$. Let us denote by $\mathcal{D}^L_{T,\mathbb{X}}$ the space of controlled distributions, endowed with the following metric

$$d_{L,T}(\Phi_1, \Phi_2) = ||\Phi'_1 - \Phi'_2||_{\star,T} + ||\Phi^\sharp_1 - \Phi^\sharp_2||_{\star,T}$$

for $\Phi_1, \Phi_2 \in \mathcal{D}^L_{\mathbb{X}}$ and the quantity

$$||\Phi||_{\star,T,L} = ||\Phi_1||_{\mathcal{D}^L_{T,X}} = d_{L,T}(\Phi_1, I(X^{\diamond 3})).$$

In the following we will omit L when its choice is clear.

We notice that the distance and the metric introduced in this last definition do not depend on $\mathbb{X}$. More generally for $\Phi \in \mathcal{D}^L_{T_1,X}$ and $\Psi \in \mathcal{D}^G_{T_2,Y}$ we denote by $d_{\min(L,G),\min(T_1,T_2)}(\Phi, \Psi)$ the same quantity. We claim that if $\Phi \in \mathcal{D}^L_X$ for a suitable choice of L then we are able to define $I(\Phi \diamond X^{\diamond 2})$ and $I(\Phi^2 X)$ modulo the use of $\mathbb{X}$.

Let us decompose the end of this Section into two parts, namely we show that $I(\Phi \diamond X^{\diamond 2})$ and $I(\Phi^2 X)$ are well-defined when Φ is a controlled distribution. We also have to prove that when Φ is a controlled distribution, $\Psi + I(X^{\diamond 3}) + 3I(\Phi^2 X) + 3I(\Phi \diamond X^{\diamond 2}) + I(\Phi^3)$ is also a controlled distribution. All those verifications being made, the only remaining point will be to show that we can apply a fixed point argument to find a solution to the renormalized equation. This is the aim of Section 4.3.

4.2.4 Decomposition of $I(\Phi^2 X)$

Let $\mathbb{X} \in \mathcal{X}$ and $\Phi \in \mathcal{D}^L_{\mathbb{X},T}$, a quick computation gives :

$$I(\Phi^2 X) = I(I(X^{\diamond 3})^2 X) + I((\theta^\sharp)^2 X) + 2I(\theta^\sharp I(X^{\diamond 3}) X)$$

with

$$\theta^\sharp = B_<(\Phi', X^{\diamond 2}) + \Phi^\sharp.$$

Using the fact that $\Phi \in \mathcal{D}^L_{\mathbb{X},T}$ and that $I(X^{\diamond 3}) \in \mathcal{C}^{1/2-\delta}_T$ we can see that the two terms $I((\theta^\sharp)^2 X)$ and $I(\theta^\sharp I(X^{\diamond 3}) X)$ are well defined. Let us focus on the term $I(X^{\diamond 3})^2 X$ which, at this stage, is not well understood, then a paraproduct decomposition of this term give us that

$$\begin{aligned} I(X^{\diamond 3})^2 X &= 2\pi_0(\pi_<(I(X^{\diamond 3}), I(X^{\diamond 3})), X) + \pi_0(\pi_0(I(X^{\diamond 3}), I(X^{\diamond 3})), X) \\ &\quad + \pi_<(I(X^{\diamond 3})^2, X) + \pi_>(I(X^{\diamond 3})^2, X) \end{aligned}$$

We remark that only the first term of this expansion is not well understood and to overcome this problem we use the Proposition (4.2.4), indeed we know that

$$R(I(X^{\diamond 3}), I(X^{\diamond 3}), X) = \pi_0(\pi_<(I(X^{\diamond 3}), I(X^{\diamond 3})), X) - I(X^{\diamond 3})\pi_0(I(X^{\diamond 3}), X)$$

is well defined and it lies in the space $\mathcal{C}^{1/2-3\delta}_T$ due to the fact that $\mathbb{X} \in \mathcal{X}$

Remark 4.2.13. We remark that the "extension" of the term $I(\Phi^2 X)$ is a functional of "$(\Phi, \mathbb{X}) \in \mathcal{D}^L_{\mathbb{X},T} \times \mathcal{X}$" and then we use sometimes the notation $I(\Phi^2 \mathbb{X})[\Phi, \mathbb{X}]$ to underline this fact.

Proposition 4.2.14. *Let $z \in (1/2, 2/3)$, $\Phi \in \mathcal{D}^L_{\mathbb{X}}$, and assume that $\mathbb{X} \in \mathcal{X}$. Then the quantity $I(\Phi^2 X)[\Phi, \mathbb{X}]$ is well-defined via the following expansion*

$$I(\Phi^2 X)[\Phi, \mathbb{X}] := I(I(X^{\diamond} 3)^2 X) + I((\theta^\sharp)^2 X) + 2I(\theta^\sharp I(X^{\diamond 3}) X)$$

with

$$\theta^\sharp = B_<(\Phi', X^{\diamond 2}) + \Phi^\sharp$$

and

$$I(X^{\diamond 3})^2 X := \pi_0(I(X^{\diamond 3}), I(X^{\diamond 3}))X + 2\pi_<(\pi_<(I(X^{\diamond 3}), I(X^{\diamond 3})), X) + 2\pi_>(\pi_<(I(X^{\diamond 3}), I(X^{\diamond 3})), X) + 2I(X^{\diamond 3})\pi_0(I(X^{\diamond 3}), X) + 2R(I(X^{\diamond 3}), I(X^{\diamond 3}), X) \tag{4.12}$$

where

$$R(I(X^{\diamond 3}), I(X^{\diamond 3}), X) = \pi_0(\pi_<(I(X^{\diamond 3}), I(X^{\diamond 3})), X) - I(X^{\diamond 3})\pi_0(I(X^{\diamond 3}), X))$$

is well-defined by the Proposition 4.2.4. And there exists a choice of L such that the following bound holds

$$||I(\Phi^2 X)[\Phi, \mathbb{X}]||_{\star,1,T} \lesssim T^\theta \left(||\Phi||_{\mathcal{D}^L_X} + 1\right)^2 \left(1 + ||\mathbb{X}||_{T,\nu,\rho,\delta,\delta'}\right)^3$$

for $\theta > 0$ and $\delta, \delta', \rho, \nu > 0$ small enough depending on L and z. Moreover if $X \in \mathcal{C}^1_T(\mathbb{T}^3)$ then

$$I(\Phi^2 X)[\Phi, R^\varphi_{a,b}\mathbf{X}] = I(\Phi^2 X)$$

Proof. By a simple computation it is easy to see that

$$\begin{aligned} ||B_<(\Phi', X^{\diamond 2})(t)||_\kappa &\lesssim \int_0^t \mathrm{d}s (t-s)^{-(\kappa+1+r)/2} ||\Phi'_s||_\kappa ||X^{\diamond 2}_s||_{-1-r} \\ &\lesssim_{r,\kappa} T^{1/2-r/2-\kappa/2-z/2} ||\Phi'||_{\star,2,T} ||X^{\diamond 2}||_{-1-r} \end{aligned}$$

for $r, \kappa > 0$ small enough and $1/2 < z < 2/3$. A similar computation gives

$$\begin{aligned} ||B_<(\Phi', X^{\diamond 2})(t)||_{1/2+\gamma} &\lesssim \int_0^t \mathrm{d}s (t-s)^{-(3/2+\gamma+r)/2} ||\Phi'_s||_\kappa ||X^{\diamond 2}_s||_{-1-r} \\ &\lesssim_{\kappa,r,z} ||\Phi'||_{\star,2,T} ||X^{\diamond 2}||_{-1-r} \int_0^t \mathrm{d}s (t-s)^{-(3/2+\gamma+r)/2} s^{-(\kappa+z)/2} \\ &\lesssim t^{1/4-(\gamma+\kappa+z+r)/2} ||\Phi'||_{\star,2,L,T} ||X^{\diamond 2}||_{-1-r} \end{aligned}$$

for $\gamma, r, \kappa > 0$ small enough. Using this bound we can deduce that

$$\begin{aligned} ||I((\theta^\sharp)^2 X)(t)||_{1+\delta} &\lesssim \int_0^t \mathrm{d}s (t-s)^{-(3/2+\delta+\beta)/2} ||(\theta^\sharp_s)^2 X_s||_{-1/2-\beta} \\ &\lesssim_{\beta,\delta} \int_0^t \mathrm{d}s (t-s)^{-(3/2+\delta+\beta)/2} ||\theta^\sharp_s||_\kappa ||\theta^\sharp_s||_{1/2+\gamma} ||X_s||_{-1/2-\beta} \\ &\lesssim_{L,z} ||\Phi||^2_{\star,L,T} (||X^{\diamond 2}||_{-1-r} + ||X||_{-1/2-\beta} + 1)^2 \\ &\times \int_0^t \mathrm{d}s (t-s)^{-(3/2+\delta+\beta)} s^{-(1/2+\kappa+\gamma+2z)/2} \\ &\lesssim_{L,z} t^{-(\delta+\kappa+\gamma+\beta+2z)} ||\Phi||^2_{\star,L,T} (||X^{\diamond 2}||_{-1-r} + ||X||_{-1/2-\beta} + 1)^2 \end{aligned}$$

for $\gamma, \beta, \delta > 0$ small enough and $2/3 > z > 1/2$. Hence we obtain that

$$\sup_{t \in [0,T]} t^{(1+\delta+z)/2} ||I((\theta^\sharp)^2 X)(t)||_{1+\delta} \lesssim_L T^{\theta_1} ||\Phi||^2_{\star,L,T} (||X^{\diamond 2}||_{-1-r} + ||X||_{-1/2-\beta} + 1)^2$$

for some $\theta_1 > 0$ depending on L and z. The same type of computation gives

$$\sup_{t\in[0,T]} t^{(\kappa+z)/2}||||I((\theta^\sharp)^2X)(t)||_\kappa \lesssim_{L,z} T^{\theta_2}||\Phi||^2_{\star,L,T}(||X^{\diamond 2}||_{-1-r}+||X||_{-1/2-\beta}+1)^2$$

and

$$\sup_{t\in[0,T]} t^{(1/2+\gamma+z)/2}I((\theta^\sharp)^2X)(t)||_{1/2+\gamma} \lesssim_{L,z} T^{\theta_3}||\Phi||^2_{\star,T}(||X^{\diamond 2}||_{-1-r}+||X||_{-1/2-\beta}+1)^2$$

with θ_2 and θ_3 two non negative constants depending only on L and z. To complete our study for this term, we have also

$$||((\theta^\sharp)^2X)(t)-I((\theta^\sharp)^2X)(s)||_{a-2b} \lesssim I^1_{st}+I^2_{st}$$

with

$$I^1_{st}=\left|\left|\int_0^s \mathrm{d}u(P_{t-u}-P_{s-u})(\theta^\sharp_u)^2X_u\right|\right|_{a-2b}, \quad I^2_{st}=\left|\left|\int_s^t \mathrm{d}uP_{t-u}(\theta^\sharp_u)^2X_u\right|\right|_{a-2b}.$$

Let us begin by bounding I^1 :

$$\begin{aligned}
I^1_{st} &\lesssim (t-s)^b\int_0^s \mathrm{d}u||P_{s-u}(\theta^\sharp_u)^2X_u||_a \\
&\lesssim (t-s)^b\int_0^t \mathrm{d}u(s-u)^{-(1/2+a+\beta)}||(\theta^\sharp_u)^2X_u||_{-1/2-\beta} \\
&\lesssim T^{\theta_4}|t-s|^b||\Phi||^2_{\star,T}(||X^{\diamond 2}||_{-1-r}+||X||_{-1/2-\beta}+1)^2
\end{aligned}$$

with $\theta_4 > 0$ depending on L and z. Let us focus on the bound for I^2,

$$\begin{aligned}
I^2_{st} &\lesssim \int_s^t (t-u)^{-(1/2+a-2b+\beta)/2}||(\theta^\sharp_u)^2X_u||_{-1/2-\beta} \\
&\lesssim_{L,z} ||\Phi||^2_{\star,T}(||X^{\diamond 2}||_{-1-r}+||X||_{-1/2-\beta}+1)^2\int_s^t \mathrm{d}u(t-u)^{-(1/2+a-2b+\beta)/2}u^{-(1/2+\kappa+\gamma+2z)/2}
\end{aligned}$$

and

$$\begin{aligned}
&\int_s^t \mathrm{d}u(t-u)^{-(1/2+a-2b+\beta)/2}u^{-(1/2+\kappa+\gamma+2z)/2} \\
&= (t-s)^{3/4-(a-2b+\beta)}\int_0^1 \mathrm{d}x(1-x)^{-(1/2+a-2b+\beta)}(s+x(t-s))^{-(1/2+\kappa+\gamma+2z)/2} \\
&\lesssim_{l,\kappa,\gamma,a,b} (t-s)^{l-(a-2b+\beta)/2}s^{1/2-z+(\kappa+\gamma)/2}\int_0^1 \mathrm{d}x(1-x)^{-(1/2+a-2b+\beta)/2}x^{-3/4+l}.
\end{aligned}$$

Then using the fact $z<1$ and choosing $l,\kappa,\gamma,b>0$ small enough we can deduce that

$$\int_s^t \mathrm{d}u(t-u)^{-(1/2+a-2b+\beta)/2}u^{-(1/2+\kappa+\gamma+2z)/2} \lesssim_L T^{\theta_5}(t-s)^bs^{-(z+a)/2}$$

with $\theta_5>0$. This gives the needed bound for I_2. Finally we have

$$\sup_{(s,t)\in[0,T]} s^{(z+a)/2}\frac{||I((\theta^\sharp)^2X)(t)-I((\theta^\sharp)^2X)(s)||_{a-2b}}{|t-s|^b} \lesssim T^{\theta_5}||\Phi||^2_{\star,T}(||X^{\diamond 2}||_{-1-r}+||X||_{-1/2-\beta}+1)^2$$

hence

$$||I((\theta^\sharp)^2X)||_{\star,1,T}\lesssim_L T^\theta||\Phi||^2_{\star,T}(||X^{\diamond 2}||_{-1-r}+||X||_{-1/2-\beta}+1)^2.$$

The bound for $||I(\theta^\sharp I(X^3)X)||_{\star,1,T}$ can be obtained by a similar way and then, according to the hypothesis given on the area $I(X^{\diamond 3})X$ and the decomposition of $I(I(X^{\diamond 3})^2X)$, we obtain easily from the Proposition 4.2.4 and the Proposition 4.2.3 that

$$\|I(I(X^{\diamond 3}))^2\diamond X\|_{\star,1,T}\lesssim T^\theta(1+\|\pi_0(I(X^{\diamond 3}),X)\|_{\delta',-1/2-\rho}+\|I(X^{\diamond 3})\|_{\delta',1/2-\rho}+\|X\|_{\delta',-1/2-\rho})^3$$

for $3\rho<\delta'$ small enough, which gives the wanted result. $\square$

4.2.5 Decomposition of $I(\Phi\diamond X^{\diamond 2})$

Let us apply the controlled structure to the mollified equation. As in that case the equation is well-posed, we have

$$\tilde\Gamma(\Phi^\varepsilon)=\Phi^\varepsilon \text{ where } \tilde\Gamma(\Phi^\varepsilon)=I((X^\varepsilon)^{\diamond 3})+3I(\Phi^\varepsilon(X^\varepsilon)^{\diamond 2})+3I((\Phi^\varepsilon)^2X^\varepsilon)+I((\Phi^\varepsilon)^3)$$

with of course Φ^ε controlled by X^ε

$$\Phi^\varepsilon=I((X^\varepsilon)^{\diamond 3})+B_<((\Phi^\varepsilon)',(X^\varepsilon)^{\diamond 2})+(\Phi^\varepsilon)^\sharp.$$

By a direct computation we also have

$$\begin{aligned}I(\Phi^\varepsilon(X^\varepsilon)^{\diamond 2})=&B_<(\Phi^\varepsilon,(X^\varepsilon)^{\diamond 2})+B_0(I((X^\varepsilon)^{\diamond 3}),(X^\varepsilon)^{\diamond 2})+B_0(B_<((\Phi^\varepsilon)',(X^\varepsilon)^{\diamond 2}),(X^\varepsilon)^{\diamond 2})\\&+B_0((\Phi^\varepsilon)^\sharp,(X^\varepsilon)^{\diamond 2})+B_>(\Phi^\varepsilon,(X^\varepsilon)^{\diamond 2}).\end{aligned}$$

Indeed, thanks to the Bony paraproduct, the first and the last terms in the r.h.s are well defined. The only problem is to define $B_0(.)$. By an analysis of the regularity, the structure of controlled distribution for $\tilde\Gamma^\varepsilon(\Phi^\varepsilon)$ appears, and we have $\tilde\Gamma^\varepsilon(\Phi^\varepsilon)'=3\Phi^\varepsilon$ hence $(\Phi^\varepsilon)'=3\Phi^\varepsilon$. Furthermore, $B_0(I((X^\varepsilon)^{\diamond 3}),(X^\varepsilon)^{\diamond 2})$ does not converge, and we need to renormalize it by subtracting $3C_2^\varepsilon I(X^\varepsilon)$. We have to deal with the (ill-defined) diagonal term.

$$\begin{aligned}X^{\varepsilon,\diamond}(\Phi')(t)&=B_0(B_<((\Phi^\varepsilon)',(X^\varepsilon)^{\diamond 2}),(X^\varepsilon)^{\diamond 2})(t)\\&=\int_0^t\mathrm{d}sP_{t-s}\pi_0(\int_0^s\mathrm{d}\sigma P_{s-\sigma}\pi_<((\Phi^\varepsilon)'_\sigma,(X^\varepsilon)^{\diamond 2}_\sigma),(X^\varepsilon)^{\diamond 2}_s)\end{aligned}$$

Thanks to the properties of the paraproduct, we decompose this term in the following way

$$\begin{aligned}(X^\varepsilon)^{\varepsilon,\diamond}((\Phi^\varepsilon)')(t)=&\int_0^t\mathrm{d}sP_{t-s}(\Phi^\varepsilon)'_s\pi_0(I((X^\varepsilon)^{\diamond 2})(s),(X^\varepsilon)^{\diamond 2}_s)\\&+\int_0^t\mathrm{d}sP_{t-s}\int_0^s\mathrm{d}\sigma((\Phi^\varepsilon)'_\sigma-(\Phi^\varepsilon)'_s)\pi_0((X^\varepsilon)^{\diamond 2}_s,P_{s-\sigma}(X^\varepsilon)^{\diamond 2}_\sigma)\\&+\int_0^t\mathrm{d}sP_{t-s}\int_0^s\mathrm{d}\sigma\pi_0(R^1_{s-\sigma}((\Phi^\varepsilon)'_\sigma,(X^\varepsilon)^{\diamond 2}_\sigma),(X^\varepsilon)^{\diamond 2}_s)\\&+\int_0^t\mathrm{d}sP_{t-s}\int_0^s R^2((\Phi^\varepsilon)'_\sigma,P_{s-\sigma}(X^\varepsilon)^{\diamond 2}_\sigma,(X^\varepsilon)^{\diamond 2}_s)\\\equiv&\sum_{i=1}^4(X^\varepsilon)^{\diamond,i}(t)\end{aligned}$$

with

$$R^1_{s-\sigma}(f,g) = P_{s-\sigma}\pi_<(f,g) - \pi_<(f,P_{s-\sigma}g), \quad R^2(f,g,h) = \pi_0(\pi_<(f,g),h) - f\pi_0(g,h)$$

Here again, to have a convergent quantity we need to renormalize $(X^\varepsilon)^{\diamond,1}$ by $C_2^\varepsilon \int_0^t P_{t-s}(\Phi^\varepsilon)'_s = C_2^\varepsilon I(\Phi^\varepsilon)$. Hence, the approximated equation must be

$$\tilde{\Gamma}(\Phi^\varepsilon) = \Phi^\varepsilon$$

with

$$\Gamma^\varepsilon(\Phi^\varepsilon) = \tilde{\Gamma}^\varepsilon(\Phi^\varepsilon) + 9C_2^\varepsilon(\Phi^\varepsilon + X^\varepsilon).$$

The same computation holds for the renormalized equation, and we have

$$I(\Phi \diamond X^{\diamond 2}) = B_<(\Phi, X^{\diamond 2}) + B_{0\diamond}(I(X^{\diamond 3}), X^{\diamond 2}) + B_{0,\diamond}(B_<(\Phi', X^{\diamond 2}), X^{\diamond 2}) + B_0(\Phi^\sharp, X^{\diamond 2}) + B_>(\Phi, X^{\diamond 2}).$$

Indeed, thanks to the Bony paraproduct, the first and the last terms in the r.h.s are well-defined. The only problem is to define $B_0(.)$. The term in $\Phi^\sharp X^{\diamond 2}$ is also well-defined as $\Phi^\sharp \in \mathcal{C}_T^{1+\delta}$. The term $B_{0\diamond}(I(X^{\diamond 3}), X^{\diamond 2})$ is also well-defined by Definition 4.2.9. So we only have to deal with the diagonal term

$$X^\diamond(\Phi')(t) = B_{0,\diamond}(B_<(\Phi', X^{\diamond 2}), X^{\diamond 2})(t) = \int_0^t \mathrm{d}s P_{t-s}\pi_{0,\diamond}\left(\int_0^s \mathrm{d}\sigma P_{s-\sigma}\pi_<(\Phi'_\sigma, X^{\diamond 2}_\sigma), X^{\diamond 2}_s\right)$$

Thanks to the properties of the paraproduct, we decompose this term in the following way

$$\begin{aligned} X^\diamond(\Phi')(t) &= \int_0^t \mathrm{d}s P_{t-s}\Phi'_s\pi_{0,\diamond}(I(X^{\diamond 2})_s, X^{\diamond 2}_s) + \int_0^t \mathrm{d}s P_{t-s}\int_0^s \mathrm{d}\sigma(\Phi'_\sigma - \Phi'_s)\pi_0(X^{\diamond 2}_s, P_{s-\sigma}X^{\diamond 2}_\sigma) \\ &+ \int_0^t \mathrm{d}s P_{t-s}\int_0^s \mathrm{d}\sigma\pi_0(R^1_{s-\sigma}(\Phi'_\sigma, X^{\diamond 2}_\sigma), X^{\diamond 2}_s) + \int_0^t \mathrm{d}s P_{t-s}\int_0^s R^2(\Phi'_\sigma, P_{s-\sigma}X^{\diamond 2}_\sigma, X^{\diamond 2}_s) \\ &\equiv \sum_{i=1}^4 X^{\diamond,i}(t) \end{aligned}$$

where

$$R^1_{s-\sigma}(f,g) = P_{s-\sigma}\pi_<(f,g) - \pi_<(f,P_{s-\sigma}g), \quad R^2(f,g,h) = \pi_0(\pi_<(f,g),h) - f\pi_0(g,h)$$

and f,g,h are distributions lying in the suitable Besov spaces for R^1 and R^2 to be defined. Before starting to bound the term $X^\diamond$, let us give a useful lemma to deal with the explosive Hölder type norm.

Lemma 4.2.15. *Let f a space time distribution such that $\sup_{t\in[0,T]} t^{(r+z)/2}||f_t||_r < +\infty$ then the following bound holds*

$$\sup_{s,t\in[0,T]} \frac{||I(f)(t) - I(f)(s)||_{a-2b}}{|t-s|^b} \lesssim_{b,a,z,r} T^\theta \sup_{t\in[0,T]} t^{(r+z)/2}||f_t||_r$$

with $a+z<2$, $z+r<2$, $a-r<2$, $0<a,b<1$ and $\theta>0$ is a constant depending only on a,r,b,z.

Proof. By a simple computation we have

$$I(f)(t) - I(f)(s) = I^1_{st} + I^2_{st}$$

with

$$I^1_{st} = (P_{t-s} - 1)\int_0^s \mathrm{d}u P_{s-u} f_u \text{ and } I^2_{st} = \int_s^t \mathrm{d}u P_{t-u} f_u.$$

Using the lemma (4.2.5) the following bound holds

$$||I^1_{st}||_{a-2b} \lesssim |t-s|^b \int_0^t \mathrm{d}u (t-u)^{-(a-r)/2} u^{-(r+z)/2} \sup_{t\in[0,T]} t^{(r+z)/2} ||f_t||_r < +\infty.$$

To handle the second term we use Hölder inequality,

$$||I^2_{st}||_{a-2b} \lesssim |t-s|^b \left(\int_s^t \mathrm{d}u (t-u)^{\frac{-(a-2b-r)}{2(1-b)}} u^{-\frac{(z+r)}{2(1-b)}} \right)^{1-b} \sup_{t\in[0,T} t^{(r+z)/2} ||f_t||_r < +\infty$$

which ends the proof. □

The following proposition gives us the regularity for our terms.

Proposition 4.2.16. *Assume that* $\mathbb{X} \in \mathcal{X}$ *there exists a choice of* L *such that for all* $z \in (1/2, 2/3)$ *the following bound holds*

$$||X^\diamond(\Phi')||_{\star,1,T} \lesssim T^\theta (1 + ||\mathbb{X}||_{T,K})^2 ||\Phi'||_{\star,2,T}$$

where $K \in [0,1]^4, \theta > 0$ *are two small parameters depending only on* L *and* z*.*

Proof. We begin by estimate the first term of the expansion (4.2.5)

$$\begin{aligned}
||X^{\diamond,1}(\Phi')(t)||_{1+\delta} \lesssim & \int_0^t \mathrm{d}s (t-s)^{-(1+\delta+\eta/2)/2} ||\Phi'_s \pi_{0,\diamond}(I(X^{\diamond 2})(s), X^{\diamond 2}_s) - \varphi^X_s||_{-\eta/2} \\
& + (\sup_{\sigma\in[0,T]} \sigma^\nu |\varphi^X_\sigma|) \int_0^t \mathrm{d}s (t-s)^{-(1+\delta-\eta)/2} ||\Phi'_s||_\eta \\
& \lesssim ||\Phi'||_{\star,2,T} (I(X^{\diamond 2}), X^{\diamond 2}) - \varphi^X||_{-\eta/2} + \sup_{\sigma\in[0,T]} \sigma^\nu |\varphi^X_\sigma| + 1) \\
& \times \int_0^t \mathrm{d}s (t-s)^{-(1+\delta+\eta/2)/2} s^{-(\eta+\nu+z)/2} \\
& \lesssim_{\beta,L} T^{\theta_1} ||\Phi'||_{\star,2,T} (||\pi_{0\diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi^X||_{-\eta/2} + \sup_{\sigma\in[0,T]} \sigma^\nu |\varphi^X_\sigma| + 1)
\end{aligned}$$

for $\nu, \eta, \delta > 0$ small enough and with $\theta_1 > 0$ depending on L. Hence

$$\sup_{t\in[0,T]} t^{(1+\delta+z)/2} ||X^{\diamond,1}(\Phi')(t)||_{1+\delta} \lesssim_{L,z} T^{\theta_1} (||\pi_{0\diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi^X||_{-\eta/2} + \sup_{\sigma\in[0,T]} \sigma^\nu |\varphi^X_\sigma| + 1)$$

Let us focus on the second term. We have

$$\begin{aligned}||X^{\diamond,2}(\Phi')(t)||_{1+\delta} &\lesssim \int_0^t \mathrm{d}s(t-s)^{-(1+\delta-\beta)/2}\int_0^s \mathrm{d}\sigma||(\Phi'_\sigma-\Phi'_s)\pi_0(P_{s-\sigma}X^{\diamond 2}_\sigma, X^{\diamond 2}_s)||_\beta \\ &\lesssim_{\beta,\rho} \int_0^t \mathrm{d}s(t-s)^{-(1+\delta-\beta)/2}\int_0^s \mathrm{d}\sigma(s-\sigma)^{-(2+\rho)/2}||\Phi'_\sigma-\Phi'_s||_{c-2d}||X^{\diamond 2}_s||^2_{-1-\rho} \\ &\lesssim_{L,\beta,\rho} ||\Phi'||_{\star,2,T}||X^{\diamond 2}||^2_{\mathcal{C}_T^{-1-\rho}}\int_0^t \mathrm{d}s(t-s)^{-(1+\delta-\beta)/2}\int_0^s \mathrm{d}\sigma(s-\sigma)^{-1-\rho/2+d}\sigma^{-(c+z)/2} \\ &\lesssim_{L,\beta,\rho} ||\Phi'||_{\star,2,T}||X^{\diamond 2}||^2_{\mathcal{C}_T^{-1-\rho}}\int_0^t \mathrm{d}s(t-s)^{-(1+\delta-\beta)/2}s^{-(\rho+c-2d+z)/2} \\ &\lesssim_{L,\beta,\rho} T^{\theta_2}||\Phi'||_{\star,2,T}||X^{\diamond 2}||^2_{\mathcal{C}_T^{-1-\rho}}\end{aligned}$$

for $\beta=\min(c-2d,\rho)\geq 0$ and all $c,d,\rho>0$ small enough, $z<1$ and $\theta_2>0$ is a constant depending only on L and z. Using the Lemma 4.2.5 we see

$$||R^1_{s-\sigma}(\Phi'_\sigma, X^{\diamond 2}_\sigma)||_{1+2\beta} \lesssim (s-\sigma)^{-(2+3\beta-\eta)/2}||\Phi'_\sigma||_\eta||X^{\diamond 2}_\sigma||_{-1-\beta}$$

for all $\beta>0$, $\beta<\eta/3$ small enough. By a straightforward computation we have

$$\begin{aligned}||X^{\diamond,3}(\Phi')(t)||_{1+\delta} &\lesssim \int_0^t \mathrm{d}s(t-s)^{-(1+\delta-\beta)/2}\int_0^s \mathrm{d}\sigma||\pi_0(R^1_{s-\sigma}(\Phi'_\sigma, X^{\diamond 2}_\sigma), X^{\diamond 2}_s)||_\beta \\ &\lesssim \int_0^t \mathrm{d}s(t-s)^{-(1+\delta-\beta)/2}\int_0^s \mathrm{d}\sigma||R^1_{s-\sigma}(\Phi'_\sigma, X^{\diamond 2}_\sigma)||_{1+2\beta}||X^{\diamond 2}_s||_{-1-\beta} \\ &\lesssim ||X^{\diamond 2}||^2_{\mathcal{C}_T^{-1-\beta}}||\Phi'||_{\star,2,T}\int_0^t \mathrm{d}s(t-s)^{-(1+\delta-\beta)/2}\int_0^s \mathrm{d}\sigma(s-\sigma)^{-(2+3\beta-\kappa)/2}\sigma^{-(\eta+z)/2} \\ &\lesssim ||X^{\diamond 2}||^2_{\mathcal{C}_T^{-1-\beta}}||\Phi'||_{\star,2,T}\int_0^t \mathrm{d}s(t-s)^{-(1+\delta-\beta)/2}s^{-(3\beta-\kappa+\eta+z)/2} \\ &\lesssim T^{\theta_3}||X^{\diamond 2}||^2_{\mathcal{C}_T^{-1-\beta}}||\Phi'||_{\star,2,T}\end{aligned}$$

where $\theta_3>0$ is a constant depending on L and z, $0<\beta<\eta/3$ small enough and $z<1$. To treat the last term it is sufficient to use the commutation result given in the Proposition (4.2.4), indeed we have

$$||R^2(\Phi'_\sigma, P_{s-\sigma}X^{\diamond 2}_\sigma, X^{\diamond 2}_s)||_{\eta-3\beta} \lesssim_{\eta,\beta} s^{-(\eta+z)/2}(s-\sigma)^{-(2-\beta)/2}||X^{\diamond 2}||^2_{\mathcal{C}_T^{-1-\beta}}||\Phi'||_{\star,2,T}$$

for $0<\beta<\eta/3$ small enough and then

$$\begin{aligned}||X^{\diamond,4}(\Phi')(t)||_{1+\delta} &\lesssim_{\eta,\beta} ||X^{\diamond 2}||^2_{\mathcal{C}_T^{-1-\beta}}||\Phi'||_{\star,2,T}\int_0^t \mathrm{d}s(t-s)^{-(1+\delta-\eta+3\beta)/2}\int_0^s \mathrm{d}\sigma s^{-(\eta+z)/2}(s-\sigma)^{-(2-\beta)/2} \\ &\lesssim_{\eta,\beta} ||X^{\diamond 2}||^2_{\mathcal{C}_T^{-1-\beta}}||\Phi'||_{\star,2,T}\int_0^t \mathrm{d}s(t-s)^{-(1+\delta-\eta+3\beta)/2}s^{-(\eta+z+\beta)/2} \\ &\lesssim T^{\theta_4}||X^{\diamond 2}||^2_{\mathcal{C}_T^{-1-\beta}}||\Phi'||_{\star,2,T}\end{aligned}$$

for $\theta_4 > 0$ depending on L and $z < 1$ and $\beta, \eta, \delta > 0$ small enough. Binding all these bounds together we conclude that

$$\sup_{t\in[0,T]} t^{(1+\delta+z)/2}||X^\diamond(\Phi')(t)||_{1+\delta} \lesssim_{L,z} T^\theta(1+||X^{\diamond 2}||_{\mathcal{C}_T^{-1-\rho}} + ||\pi_{0\diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi^X||_{\mathcal{C}_T^{-1-\rho}} + \sup_{t\in[0,T]} t^\nu|\varphi_t^X|)^2||\Phi'||_{\star,2,T}$$

for $\theta > 0$ depending on L and z. The same arguments gives

$$\sup_{t\in[0,T]} t^{(1/2+\gamma+z)/2}||X^\diamond(\Phi')(t)||_{1/2+\gamma} \lesssim_{L,z} T^\theta(1+||X^{\diamond 2}||_{\mathcal{C}_T^{-1-\rho}} + ||\pi_{0\diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi^X||_{\mathcal{C}_T^{-1-\rho}} + \sup_{t\in[0,T]} t^\nu|\varphi_t^X|)^2||\Phi'||_{\star,2,T}$$

and

$$\sup_{t\in[0,T]} t^{(\kappa+z)/2}||X^\diamond(\Phi')(t)||_{\kappa} \lesssim_{L,z} T^\theta(1+||X^{\diamond 2}||_{\mathcal{C}_T^{-1-\rho}} + ||\pi_{0\diamond}(I(X^{\diamond 2}, X^{\diamond 2}) - \varphi^X||_{\mathcal{C}_T^{-1-\rho}} + \sup_{t\in[0,T]} t^\nu|\varphi_t^X|)^2||\Phi'||_{\star,2,T}$$

To obtain the needed bound we still need to estimate the following quantity

$$\sup_{(s,t)\in[0,T]^2} s^{\frac{z+a}{2}} \frac{||X^\diamond(\Phi')(t) - X^\diamond(\Phi')(s)||_{a-2b}}{|t-s|^b}.$$

To deal with is we use the fact that $X^{\diamond,i}(\Phi') = I(f^i)$ with

$$f^1(s) = \Phi'_s\pi_{0,\diamond}(I(X^{\diamond 2})(s), X_s^{\diamond 2}), \quad f^2(s) = \int_0^s \mathrm{d}\sigma(\Phi'_\sigma - \Phi'_s)\pi_0(X_s^{\diamond 2}, P_{s-\sigma}X_\sigma^{\diamond 2})$$

and

$$f^3(s) = \int_0^s \mathrm{d}\sigma\pi_0(R^1_{s-\sigma}(\Phi'_\sigma, X_\sigma^{\diamond 2}), X_s^{\diamond 2}), \quad f^4(s) = \int_0^s R^2(\Phi'_\sigma, P_{s-\sigma}X_\sigma^{\diamond 2}, X_s^{\diamond 2}).$$

By a easy computation we have

$$||f^1(t)||_{\eta/2} \lesssim_\eta s^{-(\eta+z)/2}||\Phi'||_{\star,2,T}(1+||\pi_{0\diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi^X||_{-\eta/4} + \sup_{t\in[0,T]} t^\nu|\varphi_t^X|)^2$$

$$||f^2(s)||_{-d} \lesssim ||\Phi'||_{\star,2,T}||X^{\diamond 2}||^2_{-1-d/4} \times \int_0^s \mathrm{d}\sigma(s-\sigma)^{-1+d/2}\sigma^{-(c+z)/2} \lesssim_{z,c,d} s^{d/2-(c+z)/2}||\Phi'||_{\star,2,T}||X^{\diamond 2}||_{-1-d/4}$$

$$||f^3(s)||_{2\eta/3} \lesssim ||\Phi'||_{\star,2,T}||X^{\diamond 2}||^2_{-1-\eta/9} \times \int_0^s \mathrm{d}s(s-\sigma)^{-1+\eta/9}s^{-(\eta+z)/2} \lesssim s^{-(11\eta+9z)/2}||\Phi'||_{\star,2,T}||X^{\diamond 2}||^2_{-1-\eta/9}$$

with $\nu > 0$ depending only on L, and a similar bound for f^4 which allows us to conclude by the Lemma (4.2.15) that we have

$$\sup_{(s,t)\in[0,T]^2} s^{\frac{z+a}{2}} \frac{||X^\diamond(\Phi')(t) - X^\diamond(\Phi')(s)||_{a-2b}}{|t-s|^b} \lesssim T^\theta ||\Phi'||_{\star,2,T} ||X^{\diamond 2}||^2_{-1-\rho}$$

for some $\rho > 0$, $\theta > 0$ and $\eta, c, d > 0$ small enough and $z \in (1/2, 2/3)$. □

We are now able to give the meaning of $I(\Phi \diamond X^{\diamond 2})$ for a $\Phi \in \mathcal{D}^L_{\mathbb{X}}$.

Corollary 4.2.17. *Assume that $\mathbb{X} \in \mathcal{X}$ and let $\Phi \in \mathcal{D}^L_{\mathbb{X}}$ then for $z \in (1/2, 2/3)$ and for a suitable choice of L the term $I(\Phi \diamond X^{\diamond 2})[\Phi, \mathbb{X}]$ is defined via the following expansion*

$$I(\Phi \diamond X^{\diamond 2})[\Phi, \mathbb{X}] := B_<(\Phi, X^{\diamond 2}) + B_>(\Phi, X^{\diamond 2}) + B_{0,\diamond}(I(X^{\diamond 3}), X^{\diamond 2}) + X^\diamond(\Phi') + B_0(\Phi^\sharp, X^{\diamond 2})$$

And we have the following bound

$$||B_0(\Phi^\sharp, X^{\diamond 2})||_{\star,1,T} + ||B_>(\Phi, X^{\diamond 2})||_{\star,1,T} \lesssim T^\theta ||\Phi||_{\star,T} ||X^{\diamond 2}||_{\mathcal{C}_T^{-1-\rho}}$$

for some $\theta, \rho > 0$ being a non-negative constant depending on L and z. Moreover if $a, b \in \mathbb{R}$, $X \in \mathcal{C}^1_T(\mathbb{T}^3)$ and $\varphi \in C^\infty([0,T])$ then we have that

$$I(\Phi \diamond X^{\diamond 2})[\Phi, R^\varphi_{a,b}\mathbf{X}] = I(\Phi(X^2 - a)) + 3bI(X + \Phi)$$

for every $\Phi \in \mathcal{D}_{R^\varphi_{a,b}\mathbf{X}}$.

Proof. We remark that all the term in the definition of $I(\Phi \diamond X^{\diamond 2})$ are well-defined due to the Proposition 4.2.16 and the definition of the paraproduct, and we also notice that

$$\begin{aligned}
||B_0(\Phi^\sharp, X^{\diamond 2})(t)||_{1+\delta} &\lesssim \int_0^t \mathrm{d}s (t-s)^{-(1+\delta/2)/2} ||\Phi^\sharp_s||_{1+\delta} ||X^{\diamond 2}||_{-1-\delta/2} \\
&\lesssim ||\Phi^\sharp||_{\star,1,T} ||X^{\diamond 2}||_{\mathcal{C}_T^{-1-\delta/2}} \int_0^t \mathrm{d}s (t-s)^{-(1+\delta/2)/2} s^{-(1+\delta+z)/2} \\
&\lesssim s^{-(3/2\delta+z)/2} ||\Phi^\sharp||_{\star,1,T} ||X^{\diamond 2}||_{\mathcal{C}_T^{-1-\delta/2}}
\end{aligned}$$

which gives easily

$$\sup_{t\in[0,T]} t^{(1+\delta+z)/2} ||B_0(\Phi^\sharp, X^{\diamond 2})(t)||_{1+\delta} \lesssim T^{1/2-\delta} ||\Phi^\sharp||_{\star,1,T} ||X^{\diamond 2}||_{\mathcal{C}_T^{-1-\delta/2}}$$

for $\delta < 1/2$. By a similar computation we obtain that there exists $\theta > 0$ depending on L and z such that

$$\sup_{t\in[0,T]} t^{(1/2+\gamma+z)/2} ||B_0(\Phi^\sharp, X^{\diamond 2})(t)||_{1/2+\gamma} + \sup_{t\in[0,T]} t^{(\kappa+z)/2} ||B_0(\Phi^\sharp, X^{\diamond 2})(t)||_\kappa \lesssim T^\theta ||\Phi^\sharp||_{\star,1,T} ||X^{\diamond 2}||_{\mathcal{C}_T^{-1-\delta/2}}.$$

To obtain the needed bound for this term we still need to estimate the Hölder type norm for it. We remark that

$$||\pi_0(\Phi^\sharp_s, X^{\diamond 2}_s)||_{\delta/2} \lesssim s^{-(1+\delta+z)/2} ||\Phi^\sharp_s||_{1+\delta} ||X^{\diamond 2}_s||_{-1-\delta/2}$$

and then as usual we decompose the norm in the following way

$$B_0(\Phi^\sharp_t, X^{\diamond 2})(t) - B_0(\Phi^\sharp_s, X^{\diamond 3}_s) = I^1_{st} + I^2_{st}$$

with

$$I^1_{st} = (P_{t-s} - 1)\int_0^t \mathrm{d}u P_{t-u}\pi_0(\Phi^\sharp_u, X^{\diamond 2}_u), \quad I^2_{st} = \int_s^t \mathrm{d}u P_{t-u}\pi_0(\Phi^\sharp_u, X^{\diamond 2}_u).$$

A straightforward computation gives

$$\begin{aligned}||I^1_{st}||_{a-2b} &\lesssim ||\Phi^\sharp||_{\star,1,T}||X^{\diamond 2}||_{\mathcal{C}^{1-\delta/2}_T}|t-s|^b\int_0^t \mathrm{d}u(t-u)^{-(a-\delta/2)/2}u^{-(1+\delta+z)/2}\\ &\lesssim T^{(1-a-\delta/2-z)/2}|t-s|^b||\Phi^\sharp||_{\star,1,T}||X^{\diamond 2}||_{\mathcal{C}^{1-\delta/2}_T}.\end{aligned}$$

For I^2 we use Hölder inequality which gives

$$\begin{aligned}||I^2_{st}||_{a-2b} &\lesssim |t-s|^b||\Phi^\sharp||_{\star,1,T}||X^{\diamond 2}||_{\mathcal{C}^{1-\delta/2}_T}\left(\int_s^t \mathrm{d}u(t-u)^{-\frac{a-2b-\delta/2}{2(1-b)}}u^{-\frac{1+\delta+z}{2(1-b)}}\right)^{1-b}\\ &\lesssim T^{(1-a-\delta/2-z)/2}|t-s|^b||\Phi^\sharp||_{\star,1,T}||X^{\diamond 2}||_{\mathcal{C}^{1-\delta/2}_T}\end{aligned}$$

for $a, \delta > 0$ small enough and $z < 1$. We have obtained that

$$||B_0(\Phi^\sharp, X^{\diamond 2})||_{\star,1,T} \lesssim T^\theta||\Phi^\sharp||_{\star,1,T}||X^{\diamond 2}||_{\mathcal{C}^{-1-\delta/2}_T}$$

for some $\theta > 0$ depending on L and z. The bound for the term $B_>(\Phi, X^{\diamond 2})$ is obtained by a similar argument and this ends the proof. □

Remark 4.2.18. When there are no ambiguity we use the notation $I(\Phi \diamond X^{\diamond 2})$ instead of $I(\Phi \diamond X^{\diamond 2})[\Phi, \mathbb{X}]$.

4.3 Fixed point procedure

Using the analysis of $I(\Phi \diamond X^{\diamond 2})$ and $I(\Phi^2 X)$ developed in the previous Section, we can now show that the equation

$$\Phi = I(X^{\diamond 3}) + 3I(\Phi \diamond X^{\diamond 2}) + 3I(\Phi^2 X) + I(\Phi^3) + \Psi$$

admits a unique solution $\Phi \in \mathcal{D}^L_{\mathbb{X}}$ for a suitable choice of L and $z \in (1/2, 2/3)$ via the fixed point method. We also show that if u^ε is the solution of the regularized equation and Φ^ε is such that $u^\varepsilon = X^\varepsilon + \Phi^\varepsilon$ then $d(\Phi^\varepsilon, \Phi)$ goes to 0 as ε. Hence, by the convergence of X^ε to X we have the convergence of u^ε to $u = \Phi + X$. Let us begin by giving our fixed point result.

Theorem 4.3.1. *Assume that $\mathbb{X} \in \mathcal{X}$ and $u^0 \in \mathcal{C}^{-z}(\mathbb{T}^3)$ with $z \in (1/2, 2/3)$ and L such that the bounds of Propositions 4.2.14 and 4.2.16 are satisfied. Let $\Phi \in \mathcal{D}^L_{\mathbb{X}}$ and $\Psi = Pu^0$ then we define the application $\Gamma : \mathcal{D}^L_{\mathbb{X},T} \to \mathcal{C}^{-z}_T(\mathbb{T}^3)$ by*

$$\Gamma(\Phi) = I(X^{\diamond 3}) + 3I(\Phi \diamond X^{\diamond 2}) + 3I(\Phi^2 X) + I(\Phi^3) + \Psi$$

where $I(\Phi \diamond X^{\diamond 2})$ and $I(\Phi^2 X)$ are given by the Corollary (4.2.17) and the Proposition (4.2.14). Then $\Gamma(\Phi) \in \mathcal{D}^L_{\mathbb{X}}$ for a suitable choice of L and it satisfies the following bound

$$||\Gamma(\Phi)||_{\star,T} \lesssim (T^{\theta}||\Phi||_{\star,L,T} + 1)^3(1 + ||\mathbb{X}||_{T,K} + ||u^0||_{-z})^3. \tag{4.13}$$

Moreover for $\Phi_1, \Phi_2 \in \mathcal{D}^L_{\mathbb{X}}$ the following bound hold

$$d_{T,L}(\Gamma(\Phi_1), \Gamma(\Phi_2)) \lesssim T^{\theta} d_{T,L}(\Phi_1, \Phi_2)(||\Phi_1||_{\star,L,T} + ||\Phi_2||_{\star,L,T} + 1)^2(1 + ||\mathbb{X}||_{T,K} + ||u^0||_{-z})^3 \tag{4.14}$$

for some $\theta > 0$ and $K \in [0,1]^8$ depending on L and z. We can conclude that for this choice of L there exist $T > 0$ and a unique $\Phi \in \mathcal{D}^L_{\mathbb{X},T}$ such that

$$\Phi = \Gamma(\Phi) = I(X^{\diamond 3}) + 3I(\Phi^2 \diamond X^{\diamond 2}) + 3I(\Phi^2 X) + I(\Phi^3) + \Psi. \tag{4.15}$$

Proof. By the the Corollary (4.2.17) and the Proposition (4.2.14) we see that $\Gamma(\Phi)$ has the needed algebraic structure of the controlled distribution more precisely

$$\Gamma(\Phi)' = 3\Phi, \quad \Gamma(\Phi)^{\sharp} = 3B_{>}(\Phi, X^{\diamond 2}) + X^{\diamond}(\Phi') + 3B_0(\Phi^{\sharp}, X^{\diamond 2}) + 3I(\Phi^2 X) + I(\Phi^3) + \Psi$$

and $\Gamma(\Phi) \in \mathcal{C}^{-z}_T$. To show that $\Gamma(\Phi) \in \mathcal{D}^L_{\mathbb{X}}$ and obtain the first bound it remains to estimate $||\Phi||_{\star,2,L,T}$ and $||\Gamma(\Phi)^{\sharp}||_{\star,1,L,T}$. A straightforward computation gives

$$\begin{aligned}
||\Phi_t||_{\eta} &\lesssim ||I(X^{\diamond 3})(t)||_{\eta} + ||B_{<}(\Phi', X^{\diamond 2})(t)||_{\eta} + ||\Phi^{\sharp}_t||_{\eta} \\
&\lesssim ||I(X^{\diamond 3})||_{\eta} + ||\Phi'||_{\star,2,T}||X^{\diamond 2}||_{-1-\eta}\int_0^t \mathrm{d}s(t-s)^{-(1+2\eta/2)/2}s^{-(\eta+z)/2} + t^{-(\kappa+z)}||\Phi^{\sharp}||_{\star,1,T} \\
&\lesssim (||\Phi||_{\star,L,T} + 1)(||X^{\diamond 2}||_{-1-\eta} + ||I(X^{\diamond 3})||_{\eta} + 1)t^{\min(1/2-(3\eta+z)/2, -(\kappa+z)/2)}.
\end{aligned}$$

Then for $0 < \eta < \kappa$ and $\eta < 1/2$ and $z \in (1/2, 2/3)$ small enough we see that

$$\sup_{t\in[0,T]} t^{(\eta+z)/2}||\Phi||_{\eta} \lesssim T^{\kappa-\eta}(||\Phi||_{\star,T} + 1)(||X^{\diamond 2}||_{\mathcal{C}^{-1-\eta}_T} + ||I(X^{\diamond 3})||_{\mathcal{C}^{\eta}_T} + 1).$$

We focus on the explosive Hölder type norm for this term, indeed a quick computation gives

$$\begin{aligned}
||\Phi_t - \Phi_s||_{c-2d} \lesssim\ & ||I(X^{\diamond 3})(t) - I(X^{\diamond 3})(s)||_{c-2d} \\
& + ||B_{<}(\Phi', X^{\diamond 2})(t) - B_{<}(\Phi', X^{\diamond 2})(s)||_{c-2d} + ||\Phi^{\sharp}_t - \Phi^{\sharp}_s||_{c-2d}.
\end{aligned}$$

Let us estimate the first term in the right hand side. Using the regularity for $I(X^{\diamond 3})$ we obtain that for $d > 0$ small enough and $c < 1/2$

$$||I(X^{\diamond 3})(t) - I(X^{\diamond 3})(s)||_{c-2d} \lesssim |t-s|^d ||I(X^{\diamond 3})||_{d,c-2d}.$$

Then we notice that the increment appearing in second term has the following representation

$$B_{<}(\Phi', X^{\diamond 2}) = I(f)$$

with $f = \pi_{<}(\Phi', X^{\diamond 2})$. To treat this term it is sufficient to notice that

$$||f_t||_{-1-\delta} \lesssim ||\Phi'_t||_{\eta}||X^{\diamond 2}||_{-1-\delta} \lesssim t^{-(\eta+z)/2}||\Phi||_{\star,L,T}||X^{\diamond 2}||_{-1-\delta}$$

and then a usual argument gives

$$||B_<(\Phi', X^{\diamond 2})(t) - B_<(\Phi', X^{\diamond 2})(s)||_{c-2d} \lesssim T^\theta |t-s|^d t^{-(c+z)/2} ||\Phi||_{\star,L,T} ||X^{\diamond 2}||_{-1-\delta}$$

for some $\theta > 0$ and $c, \delta > 0$. For the last term we use that

$$||\Phi^\sharp_t - \Phi^\sharp_s||_{c-2d} \lesssim |t-s|^b t^{-(a+z)/2} ||\Phi||_{\star,T} \lesssim T^{b-d+a-c} |t-s|^d t^{-(c+z)/2} ||\Phi||_{\star,L,T}$$

for $c - 2d < a - 2b$, $d < b$ and then $c < a$ which gives :

$$\sup_{s,t\in[0,T]} s^{-(c+z)/2} \frac{||\Phi_t - \Phi_s||_{c-2d}}{|t-s|^d} \lesssim T^\theta (1 + ||I(X^{\diamond 3})||_{d,c-2d} + ||X^{\diamond 2}||_{-1-\delta})(1 + ||\Phi||_{\star,L,T}).$$

Hence the following bound holds

$$||\Gamma(\Phi)'||_{\star,2,L,T} \lesssim T^\theta (1 + ||I(X^{\diamond 3})||_{d,c-2d} + ||X^{\diamond 2}||_{-1-\delta})(1 + ||\Phi||_{\star,T}). \tag{4.16}$$

We need to estimate the remaining term $\Gamma(\Phi)^\sharp$. Due to the propositions (4.2.14),(4.2.16) and the corollary (4.2.17) it only remains to estimate the following terms $I(\Phi^3)$ and Ψ. In fact a simple computation gives

$$||\Psi||_{\star,1,L,T} \lesssim ||u^0||_{-z}.$$

Let us focus to the term $I(\Phi^3)$. We notice that

$$||I(\Phi^3)(t)||_{1+\delta} \lesssim \int_0^t \mathrm{d}s (t-s)^{-(1+\delta-\eta)/2} s^{-3/2(\eta+z)} ||\Phi||^3_{\star,T} (||X^{\diamond 2}||_{-1-\rho} + 1)^3$$

for $\delta, \kappa > 0$ small enough and $z < 2/3$ and we obtain the existence of some $\theta > 0$ such that

$$\sup_{t\in[0,T]} t^{(1+\delta+z)/2} ||I(\Phi^3)(t)||_{1+\delta} \lesssim T^\theta ||\Phi||_{\star,T}.$$

A similar argument gives

$$\sup_{t\in[0,T]} t^{(1/2+\gamma+z)/2} ||I(\Phi^3)(t)||_{1/2+\gamma} + \sup_{t\in[0,T]} t^{(\kappa+z)/2} ||I(\Phi^3)(t)||_\kappa \lesssim T^\theta ||\Phi||^3_{\star,L,T} (1 + ||X^{\diamond 2}||_{-1-\rho})^3.$$

Let us remark that

$$||\Phi^3_t||_\eta \lesssim t^{-3(\eta+z)/2} ||\Phi||^3_{\star,L,T} (1 + ||X^{\diamond 2}||_{-1-\rho})^3$$

and then as usual to deal with the Hölder norms we begin by writing the following decomposition

$$||I(\Phi^3)(t) - I(\Phi^3)(s)||_{c-2d} \lesssim I^1_{st} + I^2_{st}$$

with

$$I^1_{st} = (P_{t-s} - 1) \int_0^s \mathrm{d}u P_{s-u} \Phi^3_u, \text{ and } I^2_{st} = \int_s^t \mathrm{d}u P_{t-u} \Phi^3_u.$$

For I^1 is suffice to observe that

$$\begin{aligned} ||I^1_{st}||_{c-2d} &\lesssim |t-s|^d \int_0^s \mathrm{d}u (s-u)^{-(c-\eta)/2} u^{-3/2(z+\eta)} ||\Phi||^3_{\star,T} (1 + ||X^{\diamond 2}||_{-1-\rho})^3 \\ &\lesssim T^{1-(c-\eta)-3/2(z+\eta)} |t-s|^d ||\Phi||^3_{\star,L,T} (1 + ||X^{\diamond 2}||_{-1-\rho})^3 \end{aligned}$$

for $\eta, c > 0$ small enough, $z < 2/3$. To obtain the second bound we use the Hölder inequality and then

$$
\begin{aligned}
||I^2_{st}||_{c-2d} &\lesssim |t-s|^d \left(\int_s^t \mathrm{d}u ||P_{t-u}\Phi^3_u||^{1/(1-d)}_{c-2d} \right)^{1-d} \\
&\lesssim |t-s|^d \left(\int_s^t \mathrm{d}u (t-u)^{-\frac{c-2d-\eta}{2-2d}} u^{-\frac{3(z+\eta)}{(2-2d)}} \right)^{1-d} ||\Phi||^3_{\star,T}(1+||X^{\diamond 2}||_{-1-\rho})^3 \\
&\lesssim |t-s|^d T^{1-(c-2\eta+3z)/2} ||\Phi||^3_{\star,T}(1+||X^{\diamond 2}||_{-1-\rho})^3
\end{aligned}
$$

for $c, \eta, d > 0$ small enough and $z < 2/3$. We can conclude that there exists $\theta > 0$ such that

$$
\sup_{s,t} s^{(z+c)/2} \frac{||I(\Phi^3)(t) - I(\Phi^3)(s)||_{c-2d}}{|t-s|^d} \lesssim T^\theta ||\Phi||^3_{\star,T}(1+||X^{\diamond 2}||_{-1-\rho})^3
$$

and then we obtain all needed bounds for the remaining term and we can state that

$$
||\Gamma(\Phi)^\sharp||_{\star,2,L,T} \lesssim (T^\theta ||\Phi||_{\star,T}+1)^3(1+||\mathbb{X}||_{T,K}+||u^0||_{-z})^3
$$

for some $K \in [0,1]^4$ depending on L and this gives the first bound (4.13). The second estimate (4.14) is obtained by the same manner.

Due to the bound (4.13) for $T_1 > T > 0$ small enough, there exists $R_T > 0$ such that $B_{R_T} := \left\{ \Phi \in \mathcal{D}^L_{\mathbb{X},T}; ||\Phi||_{\star,T} \leq R_T \right\}$ is invariant by the map Γ. The bound (4.14) tells us that Γ is a contraction on $B_{R_{T_2}}$ for $0 < T_2 < T_1$ small enough. Then by the usual fixed point theorem there exists $\Phi \in \mathcal{D}^L_{\mathbb{X},T_2}$ such that $\Gamma(\Phi) = \Phi$. The uniqueness is obtained by a standard argument. $\square$

A quick adaptation of the last proof gives a better result (see for example [41] and the continuity result theorem). In fact the flow is continuous with respect to the rough distribution $\mathbb{X}$ and with respect to the initial condition ψ (or u^0).

Proposition 4.3.2. *Let $\mathbb{X}$ and $\mathbb{Y}$ two rough distributions such that $||X||_{T,K}, ||Y||_{T,K} \leq R$, $z \in (-2/3, -1/2)$, u^0_X and u^0_Y two initial conditions and $\Phi^X \in \mathcal{D}^L_{T^X,X}$ and $\Phi^Y \in \mathcal{D}^L_{T^Y,Y}$ the two unique solutions of the equations associating to $\mathbb{X}$ and $\mathbb{Y}$, and T_X and T_Y there respective living times. For $T^\star = \inf T_X, T_Y$ the following bound hold*

$$
||\Phi^X - \Phi^Y||_{C([0,T],\mathcal{C}^{-z}(\mathbb{T}^3))} \lesssim d_{T,L}(\Phi^X, \Phi^Y) \lesssim_R \mathbf{d}_{T,K}(\mathbb{X}, \mathbb{Y}) + \|u^0_X - u^0_Y\|_{-z}
$$

for every $T \leq T^\star$, where d is defined in Definition 4.2.12 and $\mathbf{d}$ is defined in Defintion 4.2.9.

Hence, using this result and combining it with the convergence Theorem (4.4.3) , we have this second corollary, where the convergence of the approximated equation is proved.

Corollary 4.3.3. *Let $z \in (1/2, 2/3)$, $u^0 \in \mathcal{C}^{-z}$ and denote u^ε the unique solutions (with life times T^ε) of the equation*

$$
\partial_t u^\varepsilon = \Delta u^\varepsilon - (u^\varepsilon)^3 + C^\varepsilon u^\varepsilon + \xi^\varepsilon
$$

where ξ^ε is a mollification of the space-time white noise ξ and $C^\varepsilon = 3(C^\varepsilon_1 - 3C^\varepsilon_2)$ with (C^ε_1) and C^ε_2 are the constant given by the Definition 4.4.2. Let us introduce $u = X + \Phi$ where Φ is the *local solution with life-time $T > 0$ for the fixed point equation given in the Theorem 4.3.1. Then we have the following convergence result*

$$\mathbb{P}(d_{T^\star,L}(\Phi^\varepsilon,\Phi) > \lambda) \longrightarrow_{\varepsilon\to 0} 0$$

for all $\lambda > 0$ with $T^\star = \inf(T, T^\varepsilon)$ and $\Phi^\varepsilon = u^\varepsilon - X^\varepsilon \in \mathcal{D}^L_{X^\varepsilon,T}$.

4.4 Renormalization and construction of the rough distribution

To end the proof of existence and uniqueness for the renormalized equation, we need to prove that the O.U. process associated to the white noise can be extend to a rough distribution of $\mathcal{X}$. (see Definition 4.2.9). As explained above, to define the appropriate process we proceed by regularization and renormalization. Let us take a *a smooth radial function f with compact support and such that $f(0) = 1$.* We regularize X in the following way

$$X^\varepsilon_t = \sum_{k\neq 0} f(\varepsilon k)\hat{X}_t(k)e_k$$

and then we show that we can choose two divergent constants $C^\varepsilon_1, C^\varepsilon_2 \in \mathbb{R}^+$ and a smooth function φ^ε such that $R^{\varphi^\varepsilon}_{C^\varepsilon_1,C^\varepsilon_2}\mathbf{X}^\varepsilon := \mathbb{X}^\varepsilon$ converge in $\mathcal{X}$. As it has been noticed in the previous Sections, without a renormalization procedure there is no finite limit for such a process.

Notation 4.4.1. Let $k_1, ..., k_n \in \mathbb{Z}^3$ we denote by $k_{1,...,n} = \sum_{i=1}^n k_i$, and for a function f we denote by δf the increment of the function given by $\delta f_{st} = f_t - f_s$

Definition 4.4.2. Let

$$C^\varepsilon_1 = \mathbb{E}\left[(X^\varepsilon)^2\right]$$

and

$$C^\varepsilon_2 = 2\sum_{k_1\neq 0, k_2\neq 0} \frac{|f(\varepsilon k_1)|^2|f(\varepsilon k_2)|^2}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{1,2}|^2)}.$$

Notice that thanks to the definition of the Littlewood-Paley blocs, we can also choose to write C^ε_2 as

$$C^\varepsilon_2 = 2\sum_{|i-j|\leq 1}\sum_{k_1\neq 0, k_2\neq 0}\theta(2^{-i}|k_{1,2}|)\theta(2^{-j}|k_{1,2}|)\frac{f(\varepsilon k_1)f(\varepsilon k_2)}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{1,2}|^2)}.$$

Let us define the following renormalized quantities

$$\begin{aligned}
&(X^\varepsilon)^{\diamond 2} := (X^\varepsilon)^2 - C^\varepsilon_1\\
&I((X^\varepsilon)^{\diamond 3}) := I((X^\varepsilon)^3 - 3C^\varepsilon_1 X^\varepsilon)\\
&\pi_{0\diamond}(I((X^\varepsilon)^{\diamond 2}),(X^\varepsilon)^{\diamond 2}) = \pi_0(I((X^\varepsilon)^{\diamond 2}),(X^\varepsilon)^{\diamond 2}) - C^\varepsilon_2\\
&\pi_{0\diamond}(I((X^\varepsilon)^{\diamond 3}),(X^\varepsilon)^{\diamond 2}) = \pi_0(I((X^\varepsilon)^{\diamond 3}),(X^\varepsilon)^{\diamond 2}) - 3C^\varepsilon_2 X^\varepsilon.
\end{aligned}$$

Then the following theorem holds.

Theorem 4.4.3. *For $T > 0$, there exists a deterministic sequence $\varphi^\varepsilon : [0,T] \to \mathbb{R}$, a deterministic distribution $\varphi : [0,T] \to \mathbb{R}$ such that for all $\delta, \delta', \nu, \rho > 0$ small enough with $\nu > \rho$ we have*

$$\|\varphi\|_{\nu,\rho,T} = \sup_t t^\nu |\varphi_t| + \sup_{0 \le s < t \le T} s^\nu \frac{|\varphi_t - \varphi_s|}{|t-s|^\rho} < +\infty$$

and the sequence φ^ε converges to φ for that norm, that is

$$\|\varphi^\varepsilon - \varphi\|_{1,\star,T} \to 0.$$

Furthermore there exists some stochastic processes

$$X^{\diamond 2} \in \mathcal{C}([0,T], \mathcal{C}^{-1-\delta})$$
$$I(X^{\diamond 3}) \in \mathcal{C}^{\delta'}([0,T], \mathcal{C}^{1/2-\delta-2\delta'})$$
$$\pi_0(I(X^{\diamond 3}), X) \in \mathcal{C}^{\delta'}([0,T], \mathcal{C}^{-\delta-2\delta'})$$
$$\pi_{0\diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi \in \mathcal{C}^{\delta'}([0,T], \mathcal{C}^{-\delta-2\delta'})$$
$$\pi_{0\diamond}(I(X^{\diamond 3}), X^{\diamond 2}) - 3\varphi X \in \mathcal{C}^{\delta'}([0,T], \mathcal{C}^{-1/2-\delta-2\delta'}).$$

Moreover each component of the sequence $\mathbb{X}^\varepsilon$ converges respectively to the corresponding component of the rough distribution $\mathbb{X}$ in the good topology, that is for all $\delta, \delta' > 0$ small enough, and all $p > 1$,

$$X^\varepsilon \to X \in\ L^p(\Omega, \mathcal{C}^{\delta'}([0,T], \mathcal{C}^{-1-\delta-3\delta'-3/2p})) \tag{4.17}$$

$$(X^\varepsilon)^{\diamond 2} \to X^{\diamond 2} \text{ in } L^p(\Omega, \mathcal{C}^{\delta'}([0,T], \mathcal{C}^{-1-\delta-3\delta'-3/2p})) \tag{4.18}$$

$$I((X^\varepsilon)^{\diamond 3}) \to I(X^{\diamond 3}) \text{ in } L^p(\Omega, \mathcal{C}^{\delta'}([0,T], \mathcal{C}^{1/2-\delta-3\delta'-3/2p})) \tag{4.19}$$

$$\pi_0(I((X^\varepsilon)^{\diamond 3}), X^\varepsilon) \to \pi_0(I(X^{\diamond 3}), X) \text{ in } L^p(\Omega, \mathcal{C}^{\delta'}([0,T], \mathcal{C}^{-\delta-3\delta'-3/2p})) \tag{4.20}$$

$$\pi_{0\diamond}(I((X^\varepsilon)^{\diamond 2}), (X^\varepsilon)^{\diamond 2}) - \varphi^\varepsilon \to \pi_{0\diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi \text{ in } L^p(\Omega, \mathcal{C}^{\delta'}([0,T], \mathcal{C}^{-\delta-3\delta'-3/2p})) \tag{4.21}$$

$$\pi_{0\diamond}(I((X^\varepsilon)^{\diamond 3}), (X^\varepsilon)^{\diamond 2}) - 3\varphi^\varepsilon X \to \pi_{0\diamond}(I(X^{\diamond 3}), X^{\diamond 2}) \text{ in } L^p(\Omega, \mathcal{C}^{\delta'}([0,T], \mathcal{C}^{-1/2-\delta-3\delta'-3/2p})) \tag{4.22}$$

Remark 4.4.4. Thanks to the proof below (especially in Subsection 4.4.5 and 4.4.6) we have the following expressions for φ^ε and φ.

$$\varphi^\varepsilon_t = -\sum_{|i-j|\le 1} \sum_{k_1 \ne 0, k_2 \ne 0} \frac{|\theta(2^{-i}|k_{12}|)||\theta(2^{-j}|k_{12}|)||f(\varepsilon k_1) f(\varepsilon k_2)|}{|k_1|^2|k_2|^2(|k_1|^2 + |k_2|^2 + |k_1+k_2|^2)} \exp\left(-t(|k_1|^2 + |k_2|^2 + |k_1+k_2|^2)\right)$$

and

$$\varphi_t = -\sum_{|i-j|\le 1} \sum_{k_1 \ne 0, k_2 \ne 0} \frac{|\theta(2^{-i}|k_{12}|)||\theta(2^{-j}|k_{12}|)|}{|k_1|^2|k_2|^2(|k_1|^2 + |k_2|^2 + |k_1+k_2|^2)} \exp\left(-t(|k_1|^2 + |k_2|^2 + |k_1+k_2|^2)\right).$$

We split the proof of this theorem according to the various components. We start by the convergence of X^ε to X. Then we also give a full proof for $X^{\diamond 2}$. For the other components we only prove the crucial estimates.

4.4.1 Convergence for X

We start by an easy computation for the convergence of X

Proof of (4.17). By a quick computation we have that

$$\delta(X - X^\varepsilon)_{st} = \sum_k (f(\varepsilon k) - 1)\delta \hat{X}_{st}(k) e_k$$

and then

$$\mathbb{E}\left[|\Delta_q \delta(X - X^\varepsilon)_{st}|^2\right] = 2 \sum_{k\neq 0; |k|\sim 2^q} |f(\varepsilon k) - 1|^2 \frac{1 - e^{-|k|^2|t-s|}}{|k|^2} \lesssim_{h,\rho} c(\varepsilon) 2^{q(1+2h+\rho)}|t-s|^h$$

for $h, \rho > 0$ small enough, and $c(\varepsilon) = \sum_{k\neq 0} |k|^{-3-\rho}|f(\varepsilon k) - 1|^2$. The Gaussian Hypercontractivity gives

$$\mathbb{E}\left[||\Delta_q \delta(X - X^\varepsilon)_{st}||^p_{L^p}\right] \lesssim_p \int_{\mathbb{T}^3} \mathbb{E}\left[|\Delta_q \delta(X - X^\varepsilon)_{st}(x)|^2\right]^{p/2} \mathrm{d}x \lesssim_{\rho,h} c(\varepsilon)^p |t-s|^{hp/2} 2^{qp/2(2h+\rho+1)}.$$

for $p > 1$. We obtain that

$$\mathbb{E}\left[||\delta(X - X^\varepsilon)_{st}||^p_{B^{-1/2-\rho-h}_{p,p}}\right] \lesssim c(\varepsilon)^{p/2}|t-s|^{hp/2}$$

Using the Besov embedding (Proposition 4.2.2) we get

$$\mathbb{E}\left[||\delta(X - X^\varepsilon)_{st}||^p_{\mathcal{C}^{-1/2-\rho-h-3/p}}\right] \lesssim c(\varepsilon)^{p/2}|t-s|^{hp/2}$$

and by the standard Garsia-Rodemich-Rumsey Lemma (see [36]) we finally obtain :

$$\mathbb{E}\left[||X - X^\varepsilon||_{\mathcal{C}^{h-\theta}([0,T],\mathcal{C}^{-1/2-h-\rho-3/p})}\right] \lesssim c(\varepsilon)^p$$

for all $h > \theta > 0$, $\rho > 0$ small enough and $p > 1$. Moreover we have $X_0 = X_0^\varepsilon = 0$ and then using the fact that $c(\varepsilon) \to^{\varepsilon\to 0} 0$ we obtain that

$$\lim_{\varepsilon\to 0} ||X^\varepsilon - X||_{L^p(\Omega, \mathcal{C}_T^{\delta', -1/2-\delta-3/p})} = 0$$

for all $0 < \delta' < \delta/3$ and $T > 0$. □

4.4.2 Renormalization for X^2

To prove the theorem for $X^{\diamond 2}$ we first prove the following estimate, and we use the Garsia-Rodemich-Rumsey lemma to conclude.

Proposition 4.4.5. *Let $p > 1$, $\theta > 0$ small enough, then the following bound hold*

$$\sup_\varepsilon \mathbb{E}\left[||\Delta_q \delta(X^\varepsilon)^{\diamond 2}_{st}||^{2p}_{L^{2p}}\right] \lesssim_{p,\theta} |t-s|^{p\theta} 2^{2qp(1+2\theta)}$$

and

$$\mathbb{E}\left[||\Delta_q \left(\delta(X^\varepsilon)^{\diamond 2}_{st} - \delta(X^{\varepsilon'})^{\diamond 2}_{st}\right))||^{2p}_{L^{2p}}\right] \lesssim_{p,\theta} C(\varepsilon,\varepsilon')^p |t-s|^{2p\theta} 2^{2qp(1+\theta)}$$

with $C(\varepsilon, \varepsilon') \to 0$ when $|\varepsilon - \varepsilon'| \to 0$.

Proof. By a straighforward computation we have

$$\mathrm{Var}(\Delta_q((X_t^\varepsilon - X_s^\varepsilon)X_s^\varepsilon)) = \sum_{k,k'\in\mathbb{Z}^3} \theta(2^{-q}k)\theta(2^{-q}k') \sum_{k_{12}=k;k'_{12}=k'} f(\varepsilon k_1)f(\varepsilon k_2)f(\varepsilon k'_1)f(\varepsilon k'_2) \times (I^1_{st}+I^2_{st})e_k e_{-k'} \tag{4.23}$$

where (e_k) denotes the Fourier basis of $L^2(\mathbb{T}^3)$ and

$$I^1_{st} = \mathbb{E}\left[(\hat{X}_t(k_1)-\hat{X}_s(k_1))\overline{(\hat{X}_t(k'_1)-\hat{X}_s(k'_1))}\right]\mathbb{E}\left[\hat{X}_s(k_2)\overline{X}_s(k'_2)\right] = 2\delta_{k_1=k'_1}\delta_{k_2=k'_2}\frac{1-e^{-|k_1|^2|t-s|}}{|k_1|^2|k_2|^2}$$

$$\begin{aligned} I^2_{st} &= \mathbb{E}\left[(\hat{X}_t(k_1)-\hat{X}_s(k_1))\overline{\hat{X}_s(k'_2)}\right]\mathbb{E}\left[\overline{(\hat{X}_t(k'_1)-\hat{X}_s(k'_1))}\hat{X}_s(k_2)\right] \\ &= \delta_{k_1=k'_2}\delta_{k'_1=k_2}\frac{(1-e^{-|k_1|^2|t-s|})(1-e^{-|k_2|^2|t-s|})}{|k_1|^2|k_2|^2}. \end{aligned}$$

Injecting these two identities in the equation (4.23) we obtain that

$$\begin{aligned} \mathrm{Var}(\Delta_q((X_t^\varepsilon - X_s^\varepsilon)X_s^\varepsilon)) &\lesssim \sum_{\substack{|k|\sim 2^q\\ k_{12}=k}} \frac{1-e^{-|k_1|^2|t-s|}}{|k_1|^2|k_2|^2} + \sum_{\substack{|k|\sim 2^q\\ k_{12}=k}} \frac{(1-e^{-|k_1|^2|t-s|})(1-e^{-|k_2|^2|t-s|})}{|k_1|^2|k_2|^2} \\ &\lesssim \sum_{\substack{|k|\sim 2^q\\ k_{12}=k}} \frac{1-e^{-|k_1|^2|t-s|}}{|k_1|^2|k_2|^2}. \end{aligned} \tag{4.24}$$

We have

$$\begin{aligned} \sum_{\substack{|k|\sim 2^q\\ k_{12}=k,|k_1|\le|k_2|}} \frac{1-e^{-|k_1|^2|t-s|}}{|k_1|^2|k_2|^2} &\lesssim |t-s|^\theta \sum_{k\in\mathbb{Z}^3;|k|\sim 2^q,k_{12}=k} |k_1|^{-2+2\theta}|k_2|^{-2} \\ &\lesssim |t-s|^\theta \left\{ \sum_{\substack{|k|\sim 2^q,k_{12}=k,\\ |k_1|\le|k_2|}} |k_1|^{-2+2\theta}|k_2|^{-2} + \sum_{\substack{|k|\sim 2^q,k_{12}=k\\ |k_1|\ge|k_2|}} |k_1|^{-2+2\theta}|k_2|^{-2} \right\} \\ &\lesssim |t-s|^\theta 2^{2q(1+2\theta)} \left(\sum_{k_1} |k_1|^{-3-2\theta} + \sum_{k_1} |k_2|^{-3-4\theta} \right) < +\infty \end{aligned}$$

and then by the Gaussian hypercontractivity we have

$$\mathbb{E}\left[||\Delta_q \delta(X^\varepsilon)^{\diamond 2}_{st}||^{2p}_{L^{2p}}\right] = \int_{\mathbb{T}^3} (Var\left(\delta(X^\varepsilon)^{\diamond 2}_{st}\right)(\xi))^p \mathrm{d}\xi \lesssim |t-s|^{p\theta}2^{2qp(1+2\theta)}.$$

For the second assertion we see that the computation of the beginning gives

$$\mathrm{Var}((\Delta_q((X_t^\varepsilon - X_s^\varepsilon)X_s^\varepsilon) - (X_t^\varepsilon - X_s^\varepsilon)X_s^\varepsilon)) \lesssim |t-s|^\theta 2^{2q(1+3\theta)}C(\varepsilon,\varepsilon')$$

where

$$C(\varepsilon,\varepsilon')=\sum_{k_{12}=k}(|f(\varepsilon k_1)|^2|f(\varepsilon k_2)|^2-|f(\varepsilon' k_1)|^2|f(\varepsilon' k_2)|^2)|k|^{-3-\theta}|k_1|^{-3-2\theta}\to^{|\varepsilon-\varepsilon'|\to 0}0$$

by the dominated convergence theorem. Once again the Gaussian hypercontractivity gives us the needed bound. □

Using the Besov embedding 4.2.2 combined with the standard Garsia-Rodemich-Rumsey lemma (see [36]) the following convergence result holds.

Proposition 4.4.6. *Let $\theta,\delta,\rho>0$ small enough such that $\rho<\theta/2$ and $p>1$ then the following bound hold*

$$\mathbb{E}\left[||(X^{\varepsilon})^{\diamond 2}-(X^{\varepsilon'})^{\diamond 2}||^{2p}_{\mathcal{C}^{\theta/2-\rho}([0,T],\mathcal{C}^{-1-3/(2p)-\delta-2\theta})}\right]\lesssim_{\theta,p,\delta}C(\varepsilon,\varepsilon')^p$$

and due to the fact that $(X^{\varepsilon}_0)^{\diamond 2}=0$ and $(X^{\diamond 2})_0=0$ we see that the sequence $(X^{\varepsilon})^{\diamond 2}$ converges in $L^{2p}(\Omega,C^{\theta/2-\rho}([0,T],\mathcal{C}^{-1-3/(2p)-\delta-3\theta}))$ to random field noted by $X^{\diamond 2}$.

4.4.3 Renormalization for $I(X^3)$

As the computations are quite similar, we only prove the equivalent of the L^2 estimate in proposition (4.4.5). Furthermore we only prove it for a fixed t and not for an increment.

Proof of (4.19). By a simple computation we have that

$$I((X^{\varepsilon}_t)^{\diamond 3})=\sum_{k\in\mathbb{Z}^3}\left(\int_0^t\mathcal{F}((X^{\varepsilon}_s)^{\diamond 3})(k)e^{-|k|^2|t-s|}\mathrm{d}s\right)e_k$$

and then

$$\begin{aligned}\mathbb{E}\left[\left|\Delta_q I((X^{\varepsilon}_t)^{\diamond 3})\right|^2\right]&=6\sum_{\substack{k\in\mathbb{Z}^3\\k_{123}=k}}|\theta(2^{-q}k)|^2\prod_{i=1,..,3}\frac{|f(\varepsilon k_i)|^2}{|k_i|^2}\int_0^t\mathrm{d}s\\&\qquad\times\int_0^s\mathrm{d}\sigma e^{-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\sigma|-|k|^2(|t-s|+|t-\sigma|)}\\&=\sum_k|\theta(2^{-q}k)|^2\Xi^{\varepsilon,1}(k),\end{aligned}$$

where

$$\begin{aligned}\Xi^{\varepsilon,1}(k)&=\sum_{k_{123}=k,k_i\neq 0}\prod_{i=1,..,3}\frac{|f(\varepsilon k_i)|^2}{|k_i|^2}\int_0^t\mathrm{d}s\int_0^s\mathrm{d}\sigma e^{-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\sigma|-|k|^2(|t-s|+|t-\sigma)}\\&\lesssim\sum_{\substack{k_{123}=k\\\max_{i=1,..3}|k_i|=|k_1|}}\frac{1}{|k_1|^2|k_2|^2|k_3|^2}\int_0^t\mathrm{d}s\int_0^s\mathrm{d}\sigma e^{-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\sigma|-|k|^2|t-s|}\\&\lesssim_T\frac{1}{|k|^{2-\rho}}\sum_{\substack{k_{123}=k\\\max_{i=1,..3}|k_i|=|k_1|}}\frac{1}{|k_1|^{4-\rho}|k_2|^2|k_3|^2}\lesssim_T\frac{1}{|k|^{4-4\rho}}(\sum_{k_2}|k_2|^{-3-\rho})^2.\end{aligned}$$

We have used that

$$\int_0^t \mathrm{d}s \int_0^s \mathrm{d}\sigma e^{-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\sigma|-|k|^2(|t-s|+|t-\sigma)} \lesssim_T \frac{1}{|k_1|^{2-\rho}|k|^{2-\rho}} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}\sigma |t-s|^{-1+\rho/2}|s-\sigma|^{-1+\rho/2}$$

for $\rho > 0$ small enough. Using again the Gaussian hypercontractivity we have

$$\mathbb{E}\left[\left|\left|\Delta_q I((X_t^\varepsilon)^{\diamond 3})\right|\right|_{L^{2p}}^{2p}\right] \lesssim 2^{-2pq(1/2-\rho)}$$

and then the Besov embedding gives

$$\sup_{t\in[0,T],\varepsilon} \mathbb{E}\left[||I((X_t^\varepsilon)^{\diamond 3})||_{1/2-\rho-3/p}\right] < +\infty.$$

The same computation gives

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|I((X_t^\varepsilon)^{\diamond 3}) - I((X_t^{\varepsilon'})^{\diamond 3}\|_{1/2-\rho-3/p}^{2p}\right] \to^{|\varepsilon'-\varepsilon|\to 0} 0$$

and this gives the needed convergence. □

4.4.4 Renormalization for $\pi_0(I(X^{\diamond 3}), X)$

Here, we only prove the L^2 estimate for the term $I(X^{\diamond 3})X$ instead of $\pi_0(I(X^{\diamond 3}), X)$ since the computations in the two cases are essentially similar. We remark that in that case we do not need a renormalization.

Proof of (4.20). We have the following representation formula

$$\mathbb{E}\left[\left|\Delta_q\left(I((X^\varepsilon)^{\diamond 3}X^\varepsilon\right)(t)\right|^2\right] = \sum_k |\theta(2^{-q}k)|^2(6I_1^\varepsilon(t)(k) + 18I_2^\varepsilon(t)(k) + 18I_3^\varepsilon(t)(k))$$

with

$$\begin{aligned}
I_1^\varepsilon(t)(k) &= 2\sum_{k_{1234}=k}\prod_{i=1,..,4}\frac{|f(\varepsilon k_i)|^2}{|k_i|^2}\int_0^t \mathrm{d}s\int_0^s \mathrm{d}\sigma e^{-|k_{123}|^2(|t-s|+|t-\sigma|)-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\sigma|}\\
&\lesssim \sum_{\substack{k_{1234}=k\\ \max_{i=1,2,3}|k_i|=|k_1|}}\frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2}\int_0^t \mathrm{d}s\int_0^s \mathrm{d}\sigma e^{-|k_{123}|^2(|t-s|+|t-\sigma|)-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\sigma|}\\
&= I_{11}^\varepsilon(t)(k) + I_{12}^\varepsilon(t)(k)
\end{aligned}$$

and

$$\begin{aligned}
I_{11}^\varepsilon(t)(k) &\lesssim \sum_{\substack{k_{1234}=k\\ \max_{i=1,2,3}|k_i|=|k_1|\\ |k_{123}|\le|k_4|}}\frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2}\int_0^t \mathrm{d}s\int_0^s \mathrm{d}\sigma e^{-|k_{123}|^2(|t-s|+|t-\sigma|)-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\sigma|}\\
&\lesssim \frac{1}{|k|^2}\sum_{k_2,k_3,k_1,\max|k_i|=|k_1|}\frac{1}{|k_1|^{4-\rho}|k_2|^2|k_3|^2|k_{123}|^{2-\rho}}\\
&\lesssim \frac{1}{|k|^2}\sum_{k_1,k_2,k_3}\frac{1}{|k_2|^{3+\rho}|k_3|^{3+\rho}|k_{123}|^{3+\rho}} \lesssim |k|^{-2}
\end{aligned}$$

for $\rho > 0$ small enough. Hence we obtain the needed result for I_1^ε. We can treat the second term by a similar computation, indeed

$$\begin{aligned}
I_{12}^\varepsilon(t)(k) &\lesssim \sum_{\substack{k_{1234}=k\\ \max_{i=1,2,3}|k_i|=|k_1|\\ |k_{123}|\geq|k_4|}} \frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2}\int_0^t \mathrm{d}s\int_0^s \mathrm{d}\sigma e^{-|k_{123}|^2(|t-s|+|t-\sigma|)-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\sigma|}\\
&\lesssim |k|^{-2+\rho}\sum_{k_2,k_3,k_4}|k_2|^{-3-\rho}|k_2|^{-3-\rho}|k_3|^{-3-\rho}\lesssim |k|^{-2+\rho}
\end{aligned}$$

with $\rho > 0$ small enough; this gives the bound for I_1^ε. More precisely we have $I_1^\varepsilon(t)(k) \lesssim |k|^{-2+\rho}$ for $\rho > 0$ small enough. Let us focus on the second term $I_2^\varepsilon(t)(k)$ which is given by

$$\begin{aligned}
I_2^\varepsilon(t)(k) &= \sum_{\substack{k_{12}=k\\ k_3,k_4}}\prod_{i=1}^4\frac{|f(\varepsilon k_i)|^2}{|k_i|^2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}\sigma e^{-(|k_1|^2+|k_2|^2)|s-\sigma|-(|k-k_3|^2+|k_3|^2)|t-s|-(|k_4|^2+|k-k_4|^2)|t-\sigma|}\\
&\lesssim \sum_{\substack{k_{12}=k\\ \max_{i=1,2}|k_i|=|k_1|\\ k_3,k_4}}\frac{1}{|k_1|^{1-\rho}|k_2|^{3+\rho}|k_3|^2|k_4|^2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}\sigma e^{-(|k-k_3|^2+|k_3|^2)|t-s|-(|k_4|^2+|k-k_4|^2)|t-\sigma|}\\
&\lesssim_\rho \frac{1}{|k|^{1-\rho}}\left(\sum_{k_3}\frac{1}{|k_3|^2}\int_0^t \mathrm{d}s e^{-(|k_3|^2+|k-k_3|^2)|t-s|}\right)^2\\
&\lesssim_{\rho,T}\frac{1}{|k|^{1-\rho}}\left(\sum_{k_3\neq 0,k}\frac{1}{|k_3|^2|k-k_3|^{2-\rho}}\right)^2\lesssim_{T,\rho}\frac{1}{|k|^{3-3\rho}}
\end{aligned}$$

and we obtain the bound for I_2^ε. We notice that

$$\begin{aligned}
I_3^\varepsilon(t)(k) &= \sum_{\substack{k_{12}=k\\ k_3,k_4}}\prod_{i=1}^4\frac{|f(\varepsilon k_i)|^2}{|k_i|^2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}\sigma e^{-(|k_1|^2+|k_2|^2)|s-\sigma|-(|k+k_3|^2+|k_3|^2)|t-s|-(|k_4|^2+|k+k_4|^2)|t-\sigma|}\\
&= I_2^\varepsilon(t)(k)
\end{aligned}$$

We have

$$\sup_{t\in[0,T],\varepsilon}\mathbb{E}\left[\left|\Delta_q\left(I((X^\varepsilon)^{\diamond 3}X^\varepsilon\right)(t)\right|^2\right]\lesssim_{\rho,T} 2^{q(1+\rho)}$$

which is the wanted bound. □

4.4.5 Renormalization for $\pi_0(I(X^{\diamond 2}), X^{\diamond 2})$

We only prove the crucial estimate for a renormalization of $\pi_0(I((X^\varepsilon)^{\diamond 2}, (X^\varepsilon)^{\diamond 2}))$. We recall that since all the other terms of the product $I(X^\varepsilon)^{\diamond 2}\diamond (X^\varepsilon)^{\diamond 2}$ are well-defined and converge to a limit with a good regularity, only this term need to be checked.

Proof of (4.21). Let us begin by giving the computation for the first term. Indeed a chaos decomposition gives

$$
\begin{aligned}
-\pi_0(I((X^\varepsilon)^{\diamond 2})(t),(X^\varepsilon_t)^{\diamond 2}) = \\
&\sum_{k\in\mathbb{Z}^3}\sum_{|i-j|\le 1}\sum_{k_{1234}=k}\theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{34}|)\\
&\times\int_0^t \mathrm{d}s e^{-|k_{12}|^2|t-s|} : \hat X^\varepsilon_s(k_1)\hat X^\varepsilon_s(k_2)\hat X^\varepsilon_t(k_3)\hat X^\varepsilon_t(k_4) : e_k\\
&+4\sum_{k\in\mathbb{Z}^3}\sum_{|i-j|\le 1}\sum_{k_{13}=k,k_2}\theta(2^{-i}(|k_{12}|)\theta(2^{-j}|k_{2(-3)}|)|f(\varepsilon k_2)|^2\int_0^t \mathrm{d}s e^{-(|k_{12}|^2+|k_2|^2)|t-s|}|k_2|^{-2} : \hat X^\varepsilon_s(k_1)\hat X^\varepsilon_t(k_3) : e_k\\
&+2\sum_{|i-j|\le 1}\sum_{k_1,k_2}\theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{12}|)|f(\varepsilon k_1)|^2|f(\varepsilon k_2)|^2\frac{1-e^{-(|k_1|^2+|k_2|^2+|k_{12}|^2)t}}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{12}|^2)}
\end{aligned}
$$

where :: denotes the usual Gaussian Wick product. Let us focus on the last term

$$
A^\varepsilon(t)=\sum_{|i-j|\le 1}\sum_{k_1,k_2}|\theta(2^{-i}k_{12})||\theta(2^{-j}k_{12})||f(\varepsilon k_1)|^2||f(\varepsilon k_2)|^2\frac{1-e^{-(|k_1|^2+|k_2|^2+|k_{12}|^2)t}}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{12}|^2)} = C^\varepsilon_2+I^\varepsilon_3(t)
$$

where I^ε_3 is defined below. Moreover is not difficult to see that

$$
\lim_{\varepsilon\to 0} C^\varepsilon_2=\sum_{|i-j|\le 1}\sum_{k_1,k_2}\frac{\theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{12}|)}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{12}|^2)}=+\infty.
$$

To obtain the needed convergence we have to estimate the following term

$$
I^\varepsilon_1(t)=\sum_{k\in\mathbb{Z}^3}\sum_{|i-j|\le 1}\sum_{k_{1234}=k}\theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{34}|)\int_0^t \mathrm{d}s e^{-|k_{12}|^2|t-s|} : \hat X^\varepsilon_s(k_1)\hat X^\varepsilon_s(k_2)\hat X^\varepsilon_t(k_3)\hat X_t(k_4) : e_k
$$

$$
\begin{aligned}
I^\varepsilon_2(t)=\sum_{k\in\mathbb{Z}^3}\sum_{|i-j|\le 1}\sum_{k_{13}=k,k_2}&\theta(2^{-i}(|k_{12}|)\theta(2^{-j}|k_{2(-3)}|)|f(\varepsilon k_2)|^2\\
&\times\int_0^t \mathrm{d}s e^{-(|k_{12}|^2+|k_2|^2)|t-s|}|k_2|^{-2}\hat X^\varepsilon_s(k_1)\hat X^\varepsilon_t(k_3)e_k
\end{aligned}
$$

and

$$
I^\varepsilon_3(t)=\sum_{|i-j|\le 1}\sum_{k_1,k_2}\theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{12}|)\frac{|f(\varepsilon k_1)|^2|f(\varepsilon k_2)|^2 e^{-(|k_1|^2+|k_2|^2+|k_{12}|^2)t}}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{12}|^2)}.
$$

We notice that for the deterministic part we have the following bound

$$
I^\varepsilon_3(t)\lesssim t^{-\rho}\sum_{k_1,k_2,|k_1|\le|k_2|}\frac{1}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{12}|^2)^{1+\rho}}\lesssim t^{-\rho}\sum_{k_1,k_2,|k_1|\le|k_2|}|k_2|^{-4-2\rho}|k_1|^{-2}\lesssim_\rho t^{-\rho}
$$

and then the dominated convergence gives for $\rho>0$

$$
\sup_{t\in[0,T]} t^\rho|I^\varepsilon_3(t)-I_3(t)|\to^{\varepsilon\to 0} 0
$$

with

$$I_3(t) = \sum_{|i-j|\le 1}\sum_{k_1,k_2}\theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{12}|)\frac{e^{-(|k_1|^2+|k_2|^2+|k_{12}|^2)t}}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{12}|^2)}$$

and this gives the bound for the deterministic part. Let us focus on $I_1^\varepsilon(t)$ and $I_2^\varepsilon(t)$. A simple computation gives

$$\begin{aligned}
\mathbb{E}\left[\Delta_q|I_1^\varepsilon(t)|^2\right] &= 2\sum_{k\in\mathbb{Z}^3}\sum_{i\sim j\sim i'\sim j'}\sum_{k_{1234}=k}\theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{34}|)\theta(2^{-i'}|k_{12}|)\theta(2^{-j'}|k_{34}|)\theta(2^{-q}|k|)^2\prod_{l=1}^4\frac{|f(\varepsilon k_l)|^2}{|k_l|^2}\\
&\times\int_0^t\int_0^t \mathrm{d}s\mathrm{d}\sigma e^{-|k_{12}|^2(|t-s|+|t-\sigma|)-(|k_1|^2+|k_2|^2)|s-\sigma|}\\
&+2\sum_{k\in\mathbb{Z}^3}\sum_{i\sim j\sim i'\sim j'}\sum_{k_{1234}=k}\theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{34}|)\theta(2^{-j'}|k_{12}|)\theta(2^{-i'}|k_{34}|)\theta(2^{-q}|k|)^2\prod_{l=1}^4\frac{|f(\varepsilon k_l)|^2}{|k_l|^2}\\
&\times\int_0^t\int_0^t \mathrm{d}s\mathrm{d}\sigma e^{-(|k_{12}|^2+|k_1|^2+|k_2|^2)|t-s|-(|k_{34}|^2+|k_3|^2+|k_4|^2)|t-\sigma|}\\
&+2\sum_{k\in\mathbb{Z}^3}\sum_{i\sim j;i'\sim j'}\sum_{k_{1234}=k}\theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{34}|)\theta(2^{-i'}|k_{14}|)\theta(2^{-j'}|k_{23}|)\theta(2^{-q}|k|)^2\prod_{l=1}^4\frac{|f(\varepsilon k_l)|^2}{|k_l|^2}\\
&\times\int_0^t\int_0^t \mathrm{d}s\mathrm{d}\sigma e^{-(|k_{12}|^2+|k_2|^2)|t-s|-(|k_{14}|^2+|k_4|^2)|t-\sigma|-|k_1|^2|s-\sigma|}\\
&\equiv\sum_{j=1}^3 I_{1,j}^\varepsilon(t).
\end{aligned}$$

Let us begin by treating the first term. As usual by symmetry we have

$$\begin{aligned}
I_{1,1}^\varepsilon(t) &\lesssim \sum_{k\in\mathbb{Z}^3}\sum_{q\lesssim i}\sum_{\substack{k_{1234}=k\\|k_1|\le|k_2|,|k_3|\le|k_4|\\ \max_{l=1,\dots,4}|k_l|=|k_2|}}\theta(2^{-q}|k|)\theta(2^{-i}|k_{12}|)\prod_{i=l}^4|k_l|^{-2}\int_0^t\int_0^t\mathrm{d}s\mathrm{d}\sigma e^{-|k_{12}|^2(|t-s|+|t-\sigma|)-|k_2|^2|s-\sigma|}\\
&+\sum_{k\in\mathbb{Z}^3}\sum_{q\lesssim i}\sum_{\substack{k_{1234}=k\\|k_1|\le|k_2|,|k_3|\le|k_4|\\ \max_{l=1,\dots,4}|k_l|=|k_4|}}\theta(2^{-q}|k|)\theta(2^{-i}|k_{12}|)\prod_{i=l}^4|k_l|^{-2}\int_0^t\int_0^t\mathrm{d}s\mathrm{d}\sigma e^{-|k_{12}|^2(|t-s|+|t-\sigma|)-|k_2|^2|s-\sigma|}\\
&\equiv A_1^\varepsilon(t)+A_2^\varepsilon(t).
\end{aligned}$$

We notice that if $\max_{l=1,\dots,4}|k_l| = |k_1|$ then $|k|\lesssim|k_1|$, then

$$A_1^\varepsilon(t)\lesssim\sum_{k\in\mathbb{Z}^3}|k|^{-1+2\eta}\theta(2^{-q}|k|)\sum_{\substack{k_{1234}=k\\|k_1|\le|k_2|,|k_3|\le|k_4|\\ \max_{l=1,\dots,4}|k_l|=|k_2|}}|k_1|^{-3-\eta/3}|k_3|^{-3-\eta/3}|k_4|^{-3-\eta/3}\sum_{q\lesssim i}2^{-i(2-\eta)}\lesssim t^\eta 2^{3q\eta}$$

where we have used

$$\int_0^t\int_0^t\mathrm{d}s\mathrm{d}\sigma e^{-|k_{12}|^2(|t-s|+|s-\sigma|)-|k_2|^2|s-\sigma|}\lesssim t^\eta\frac{1}{|k_2|^{2-\eta}|k_{12}|^{2-\eta}}.$$

By a similar argument we have

$$A_2^\varepsilon(t) \lesssim \sum_{k\in\mathbb{Z}^3} |k|^{-1+4\eta}\theta(2^{-q}|k|) \sum_{\substack{k_{1234}=k\\ |k_1|\le|k_2|,|k_3|\le|k_4|\\ \max_{l=1,\dots,4}|k_l|=|k_4|}} |k_1|^{-3-\eta}|k_2|^{-3-\eta}|k_3|^{-3-\eta}\sum_{q\lesssim i} 2^{-i(2-\eta)} \lesssim t^\eta 2^{5q\eta}$$

and then $\sup_\varepsilon I_{1,1}(t) \lesssim t^\eta 2^{5q\eta}$. Let us treat the second term $I_{1,2}^\varepsilon(t)$. we have

$$\begin{aligned}
I_{1,2}^\varepsilon(t) &\lesssim \sum_{k\in\mathbb{Z}^3}\sum_{q\lesssim i\sim j} \sum_{\substack{k_{1234}=k\\ |k_1|\le|k_2|,|k_3|\le|k_4|\\ \max_{l=1,\dots,4}|k_l|=|k_2|}} \theta(2^{-q}|k|)\theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{34}|)\prod_{i=l}^4 |k_l|^{-2}\\
&\times \int_0^t\int_0^t \mathrm{d}s\mathrm{d}\sigma e^{-(|k_{12}|^2+|k_1|^2+|k_2|^2)|t-s|-(|k_{34}|^2+|k_3|^2+|k_4|^2)|s-\sigma|}\\
&\lesssim \sum_{k\in\mathbb{Z}^3}|k|^{-1+4\eta}\sum_{q\lesssim i\sim j}\sum_{\substack{k_{1234}=k\\ |k_1|\le|k_2|,|k_3|\le|k_4|\\ \max_{l=1,\dots 4}|k_l|=|k_2|}} \theta(2^{-q}|k|)\theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{34}|)\frac{1}{|k_1|^2|k_2|^{3+3\eta}|k_3|^2|k_4|^2|k_{34}|^{2-\eta}}\\
&\lesssim t^\eta 2^{q(2+4\eta)}\sum_{q\lesssim j}2^{-j(2-\eta)}\sum_l |l|^{-3-\eta} \lesssim t^\eta 2^{5q\eta}.
\end{aligned}$$

We have to treat the last term in the fourth chaos. A similar computation gives

$$\begin{aligned}
I_{1,3}^\varepsilon(t) &\lesssim \sum_{k\in\mathbb{Z}^3}\sum_{q\lesssim i\sim j;q\lesssim i'\sim j'}\sum_{k_{1234}=k}\theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{34}|)\theta(2^{-i'}|k_{14}|)\theta(2^{-j'}|k_{23}|)\theta(2^{-q}|k|)^2\\
&\times\int_0^t\int_0^t \mathrm{d}s\mathrm{d}\sigma e^{-(|k_{12}|^2+|k_2|^2)|t-s|-(|k_{14}|^2+|k_4|^2)|t-\sigma|}\\
&\lesssim \sum_{k\in\mathbb{Z}^3}\sum_{q\lesssim i'\sim j'}\sum_{\substack{k_{1234}=k\\ |k_4|\le|k_2|,|k_1|\le|k_3|}}\theta(2^{-i}|k_{12}|)\theta(2^{-j}|k_{34}|)\theta(2^{-i'}|k_{14}|\theta(2^{-q}|k|)^2\prod_{l=1}^4\frac{1}{|k_l|^2}\\
&\times\int_0^t\int_0^t \mathrm{d}s\mathrm{d}\sigma e^{-|k_2|^2|t-s|-|k_{14}|^2|t-\sigma|}\\
&\lesssim t^\eta 2^{-q(2-\eta)}\sum_{k\in\mathbb{Z}^3}\theta(2^{-q}|k|)\sum_{\substack{k_{1234}=k\\ |k_4|\le|k_2|,|k_1|\le|k_3|}}\frac{1}{|k_1|^2|k_2|^{4-\eta}|k_3|^2|k_4|^2}.
\end{aligned}$$

We still need to bound the sum

$$\sum_{\substack{k_{1234}=k\\ |k_4|\le|k_2|,|k_1|\le|k_3|}}\frac{1}{|k_1|^2|k_2|^{4-\eta}|k_3|^2|k_4|^2}$$

for that we notice that when $|k_3|\le|k_2|$ we can use the bound

$$\frac{1}{|k_1|^2|k_2|^{4-\eta}|k_3|^2|k_4|^2}\lesssim |k|^{-1+4\eta}|k_1|^{-3-\eta}|k_3|^{-3-\eta}|k_4|^{-3-\eta}$$

and in the case $|k_2|\le|k_3|$ we can use that

$$\frac{1}{|k_1|^2|k_2|^{4-\eta}|k_3|^2|k_4|^2}\lesssim |k|^{-1+4\eta}|k_1|^{-2}|k_2|^{-4+\eta}|k_3|^{-1+4\eta}|k_4|^{-2}\lesssim |k|^{-1+4\eta}|k_1|^{-3-\eta}|k_2|^{-3-\eta}|k_4|^{-3-\eta}$$

where we have used that $|k_4| \leq |k_2|$ and then we can conclude that $\sup_\varepsilon I^\varepsilon_{1,3}(t) \lesssim t^\eta 2^{5q\eta}$. This gives the needed bound for the term lying in the chaos of order four; in fact, we have

$$\sup_\varepsilon \mathbb{E}\left[\Delta_q |I^\varepsilon_1(t)|^2\right] \lesssim t^\eta 2^{5q\eta}.$$

Let us focus on the term lying in second chaos.

$$\begin{aligned}
\mathbb{E}\left[|\Delta_q I^\varepsilon_2(t)|^2\right] &= \sum_{k\in\mathbb{Z}^3} \sum_{q\lesssim i\sim j, q\lesssim i'\sim j'} \sum_{k_{13}=k,k_2,k_4} \theta(2^{-i}|k_{12}|)\theta(2^j|k_{2(-3)}|)\theta(2^{-i'}|k_{14}|)\theta(2^{-j'}|k_{4(-3)}|) \\
&\times \prod_{i=1}^4 \frac{|f(\varepsilon k_l)|^2}{|k_l|^2} \int_0^t\int_0^t \mathrm{d}s\mathrm{d}\sigma e^{-(|k_{12}|^2+|k_2|^2)|t-s|-(|k_{14}|^2+|k_4|^2)|t-\sigma|-|k_1|^2|s-\sigma|} \\
&+ \sum_{k\in\mathbb{Z}^3} \sum_{q\lesssim i\sim j, q\lesssim i'\sim j'} \sum_{k_{13}=k,k_2,k_4} \theta(2^{-i}|k_{12}|)\theta(2^j|k_{2(-3)}|)\theta(2^{-i'}|k_{34}|)\theta(2^{-j'}|k_{4(-3)}|) \\
&\times \prod_{i=1}^4 \frac{|f(\varepsilon k_l)|^2}{|k_l|^2} \int_0^t\int_0^t \mathrm{d}s\mathrm{d}\sigma e^{-(|k_{12}|^2+|k_2|^2+|k_1|^2)|t-s|-(|k_{34}|^2+|k_4|^2+|k_3|^2)|t-\sigma|} \\
&\equiv I^\varepsilon_{2,1}(t) + I^\varepsilon_{2,2}(t).
\end{aligned}$$

We treat these two terms separately. In fact, by symmetry, we have

$$\begin{aligned}
I^\varepsilon_{2,1}(t) &\lesssim \sum_{k\in\mathbb{Z}^3} \sum_{q\lesssim i\sim j; q\lesssim i'\sim j'} \sum_{\substack{k_{13}=k \\ k_2,k_4,|k_1|\leq|k_3|}} \theta(2^{-q}|k|)^2\theta(2^{-i}|k_{12}|)\theta(2^j|k_{2(-3)}|)\theta(2^{-i'}|k_{14}|)\theta(2^{-j'}|k_{4(-3)}|) \\
&\times \prod_{i=1}^4 \frac{|f(\varepsilon k_l)|^2}{|k_l|^2} \int_0^t\int_0^t \mathrm{d}s\mathrm{d}\sigma e^{-(|k_{12}|^2+|k_2|^2)|t-s|-(|k_{14}|^2+|k_4|^2)|t-\sigma|} \\
&\lesssim t^\eta \sum_{|k|\sim q} |k|^{-1+\eta} \sum_{q\lesssim i,i'} \theta(2^{-i}|k_{12}|)\theta(2^{-i'}|k_{14}|) \sum_{k_1,k_2,k_4} |k_1|^{-3-\eta}|k_2|^{-3-\eta}|k_3|^{-3-\eta}|k_{12}|^{-1+2\eta}|k_{14}|^{-1+2\eta} \\
&\lesssim 2^{q(2+\eta)} \sum_{q\lesssim i,i'} 2^{-(i+i')(1-2\eta)} \lesssim t^\eta 2^{3q\eta}
\end{aligned}$$

which gives the first bound. The second term has a similar bound, indeed

$$\begin{aligned}
I^\varepsilon_{2,2}(t) &\lesssim \sum_{k\in\mathbb{Z}^3} \sum_{q\lesssim i, q\lesssim i'} \sum_{k_{13}=k,k_2,k_4,|k_1|\leq|k_3|} \theta(2^{-i}|k_{12}|)\theta(2^{-i'}|k_{34}|) \prod_{i=1}^4 \frac{|f(\varepsilon k_l)|^2}{|k_l|^2} \\
&\times \int_0^t\int_0^t \mathrm{d}s\mathrm{d}\sigma e^{-(|k_{12}|^2+|k_2|^2)|t-s|-(|k_{34}|^2+|k_4|^2)|t-\sigma|} \lesssim t^\eta 2^{3q\eta}
\end{aligned}$$

which ends the proof. □

4.4.6 Renormalization for $\pi_0(I(X^{\diamond 3}), X^{\diamond 2})$

Here again we only give the crucial bound, but for $I(X^{\diamond 3}) \diamond X^{\diamond 2}$ instead of $\pi_{0\diamond}(I(X^{\diamond 3}), X^{\diamond 2})$.

Proposition 4.4.7. *For all $T > 0$, $t \in [0, T]$, $\delta, \delta' > 0$ and all $\gg \nu > 0$ small enough, there exists two constants and $C > 0$ depending on T, δ, δ' and ν such that for all $q \geq -1$,*

$$\mathbb{E}[t^{\delta'+\delta}|\Delta_q(I((X^\varepsilon_t)^{\diamond 3})(X^\varepsilon_t)^{\diamond 2} - 3C^\varepsilon_2 X^\varepsilon_t)|^2] \leq Ct^\delta 2^{q(1+\nu)}.$$

Proof. Thanks to a straightforward computation we have

$$-I((X_t^\varepsilon)^{\diamond 3})(X_t^\varepsilon)^{\diamond 2} = I_t^{(1)} + I_t^{(2)} + I_t^{(3)}$$

where

$$I_t^{(1)} = \sum_{k\neq 0} e_k \sum_{\substack{k_{12345}=k \\ k_i \neq 0}} \int_0^t \mathrm{d}s e^{-|k_1+k_2+k_3|^2|t-s|} : \hat{X}_s^\varepsilon(k_1)\hat{X}_s^\varepsilon(k_2)\hat{X}_s^\varepsilon(k_3)\hat{X}_t^\varepsilon(k_4)\hat{X}_t^\varepsilon(k_5) :$$

$$I_t^{(2)} = 6\sum_{k\neq 0} e_k \sum_{\substack{k_3, k_{124}=k \\ k_i \neq 0}} \int_0^t \mathrm{d}s e^{-|k_1+k_2+k_3|^2|t-s|} \frac{e^{-|k_3|^2|t-s|}}{|k_3|^2} f(\varepsilon k_3)^2 : \hat{X}_s^\varepsilon(k_1)\hat{X}_s^\varepsilon(k_2)\hat{X}_t^\varepsilon(k_4) :$$

and

$$I_t^{(3)} = 6\sum_{k\neq 0} e_k \int_0^t \mathrm{d}s \sum_{k_1\neq 0, k_2\neq 0} \frac{f(\varepsilon k_1)^2 f(\varepsilon k_2)^2}{|k_1|^2|k_2|^2} e^{-(|k+k_1+k_2|^2+|k_1|^2+|k_2|^2)|t-s|}\hat{X}_s^\varepsilon(k)$$

Hence,

$$\begin{aligned}-\left(I((X_t^\varepsilon)^{\diamond 3})(X_t^\varepsilon)^{\diamond 2} - 3C_2^\varepsilon X_t^\varepsilon\right) = I((X_t^\varepsilon)^{\diamond 3})(X_t^\varepsilon)^{\diamond 2} I_t^{(3)} \\ + (I_t^{(3)} - \tilde{I}_t^{(3)}) + (\tilde{I}_t^{(3)} - 3\tilde{C}_2^\varepsilon(t)X_t^\varepsilon) + 3(C_2^\varepsilon - \tilde{C}_2^\varepsilon(t))X_t^\varepsilon\end{aligned}$$

where we remind that

$$C_2^\varepsilon = \sum_{k_1\neq 0, k_2\neq 0} \frac{f(\varepsilon k_1)f(\varepsilon k_2)}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_1+k_2|^2)}$$

and where we have defined

$$\tilde{I}_t^{(3)} = 6\sum_{k\neq 0} e_k \int_0^t \mathrm{d}s \sum_{k_1\neq 0, k_2\neq 0} \frac{f(\varepsilon k_1)^2 f(\varepsilon k_2)^2}{|k_1|^2|k_2|^2} e^{-(|k+k_1+k_2|^2+|k_1|^2+|k_2|^2)|t-s|}\hat{X}_t^\varepsilon(k)$$

and

$$\tilde{C}_2^\varepsilon = 2\int_0^t \mathrm{d}s \sum_{k_1\neq 0, k_2\neq 0} \frac{f(\varepsilon k_1)^2 f(\varepsilon k_2)^2}{|k_1|^2|k_2|^2} e^{-(|k_1+k_2|^2+|k_1|^2+|k_2|^2)|t-s|}.$$

Hence for $q \geq -1$,

$$\begin{aligned}\mathbb{E}[|\Delta_q(I((X_t^\varepsilon)^{\diamond 3})(X_t^\varepsilon)^{\diamond 2} - 3C_2^\varepsilon X_t^\varepsilon)|^2] \lesssim \mathbb{E}[|\Delta_q(I_t^{(1)})|^2] + \mathbb{E}[|\Delta_q(I_t^{(2)})|^2] + \mathbb{E}[|\Delta_q(I_t^{(3)} - \tilde{I}_t^{(3)})|^2] \\ + \mathbb{E}[|\Delta_q(\tilde{I}_t^{(3)} - \tilde{C}_2^\varepsilon(t)X_t^\varepsilon)|^2] + |C_2^\varepsilon - \tilde{C}_2^\varepsilon(t)|^2\,\mathbb{E}[|\Delta_q X_t^\varepsilon|^2].\end{aligned}$$

Terms in the first chaos. Let us first deal with the "deterministic" part, here $C_2^\varepsilon - \tilde{C}_2^\varepsilon(t)$. An obvious computation gives for all $\delta' > 0$, $|C_2^\varepsilon - \tilde{C}_2^\varepsilon(t)|^2 \lesssim_{\delta'} 1/t^{\delta'}$. Furthermore, $\mathbb{E}[|\Delta_q X_t^\varepsilon|2] \lesssim 2^q$, hence for all $\delta' > 0$,

$$|C_2^\varepsilon - \tilde{C}_2^\varepsilon(t)|^2\, \mathbb{E}[|\Delta_q X_t^\varepsilon|^2] \lesssim 2^q/t^{\delta'}$$

Let us deal with $\mathbb{E}[|\Delta_q(I_t^{(3)} - \tilde{I}_t^{(3)})|^2]$. For $k \neq 0$ we define

$$a_k(t-s) = \sum_{k_1\neq 0, k_2\neq 0} \frac{f(\varepsilon k_1)^2 f(\varepsilon k_2)^2}{|k_1|^2|k_2|^2} e^{-(|k+k_1+k_2|^2+|k_1|^2+|k_2|^2)|t-s|}$$

such that

$$\begin{aligned}
&\mathbb{E}[|\Delta_q(I_t^{(3)} - \tilde{I}_t^{(3)})|^2] \\
&= \mathbb{E}\left[\left|\int_0^t \sum_k \theta(2^{-q}k) e_k a_k(t-s)(\hat{X}_s^\varepsilon(k) - \hat{X}_t^\varepsilon(k))\right|^2\right] \\
&= \int_{[0,t]^2} \mathrm{d}\bar{s}\mathrm{d}s \sum_{\substack{k\neq 0\\ \bar{k}\neq 0}} e_k e_{\bar{k}} \theta(2^{-q}k)\theta(2^{-q}\bar{k}) \\
&\qquad\qquad \times a_k(t-s) a_{\bar{k}}(t-\bar{s}) \mathbb{E}[(\hat{X}_s^\varepsilon(k) - \hat{X}_t^\varepsilon(k))(\hat{X}_{\bar{s}}^\varepsilon(\bar{k}) - \hat{X}_t^\varepsilon(\bar{k}))]
\end{aligned}$$

But

$$\begin{aligned}
\mathbb{E}[(\hat{X}_s^\varepsilon(k) - \hat{X}_t^\varepsilon(k))(\hat{X}_{\bar{s}}^\varepsilon(\bar{k}) - \hat{X}_t^\varepsilon(\bar{k}))] &= \delta_{k=-\bar{k}} \frac{f(\varepsilon k)^2}{|k|^2}(e^{-|s-\bar{s}||k|^2} - e^{-|t-\bar{s}||k|^2} - e^{-|t-s||k|^2} + 1) \\
&\lesssim \delta_{k=-\bar{k}} \frac{f(\varepsilon k)^2}{|k|^2} |k|^{2\eta}|t-s|^{\eta/2}|t-\bar{s}|^{\eta/2}.
\end{aligned}$$

Hence

$$\mathbb{E}[|\Delta_q(I_t^{(3)} - \tilde{I}_t^{(3)})|^2] \lesssim \sum_{k\neq 0} \theta(2^{-q}k)^2 \frac{f(\varepsilon k)^2}{|k|^{2(1-\eta)}} \left(\int_0^t \mathrm{d}s|t-s|^{\eta/2} a_k(|t-s|)\right)^2$$

and

$$\begin{aligned}
\int_0^t \mathrm{d}s|t-s|^{\eta/2} a_k(|t-s|) &= \sum_{\substack{k_1\neq 0\\ k_2\neq 0}} \int_0^t \mathrm{d}s|t-s|^{\eta/2} e^{-(|k+k_1+k_2|^2+|k_1|^2+|k_2|^2)|t-s|} \frac{f(\varepsilon k_1)^2 f(\varepsilon k_1)^2}{|k_1|^2|k_2|^2} \\
&\lesssim \sum_{\substack{k_1\neq 0\\ k_2\neq 0}} |k_1|^{-3-\eta'}|k_2|^{-3-\eta'} \int_0^t \mathrm{d}s|t-s|^{-1+(\eta/2-\eta')} \lesssim t^{\eta/2-\eta'}
\end{aligned}$$

for $\eta/2 - \eta' > 0$. Then we have

$$\mathbb{E}[|\Delta_q(I_t^{(3)} - \tilde{I}_t^{(3)})|^2] \lesssim 2^{q(1+2\eta)} t^{\eta-2\eta'}.$$

We have furthermore

$$\mathbb{E}[|\Delta_q(\tilde{I}_t^{(3)} - C_2^\varepsilon X_t^\varepsilon)|^2] = \sum_{k\neq 0} \frac{f(\varepsilon k)^2}{|k|^2}\theta(2^{-q}k)^2 b_k(t)^2$$

with

$$b_k(t) = \int_0^t \sum_{\substack{k_1 \neq 0 \\ k_2 \neq 0}} \frac{f(\varepsilon k_1)^2 f(\varepsilon k_2)^2}{|k_1|^2|k_2|^2} e^{-(|k_1|^2+|k_2|^2)|t-s|}\{e^{-|k_1+k_2|^2|t-s|} - e^{-|k_1+k_2+k|^2|t-s|}\}.$$

Using that

$$|e^{-|k_1+k_2+k|^2|t-s|} - e^{-|k_1+k_2|^2|t-s|}| \lesssim |t-s|^\eta |k|^\eta (|k| + \max\{|k_1|,|k_2|\})^\eta$$

we have the following bound

$$b_k(t) \lesssim \int_0^t \sum_{\substack{k_1 \neq 0 \\ k_2 \neq 0}} |k_1|^{-3-\eta'}|k_2|^{-3-\eta''}|k|^\eta(|k| + \max\{|k_1|,|k_2|\})^\eta |t-s|^{-1+(\eta-\eta'/2-\eta''/2)}.$$

We can suppose that $\max\{|k_1|,|k_2|\} = |k_1|$ as the expression is symmetric in k_1, k_2, then if $|k| > |k_1|$,

$$b_k(t) \lesssim t^{(\eta-\eta'/2-\eta''/2)}|k|^{2\eta}$$

for $\eta - \eta'/2 - \eta''/2 > 0$. Furthermore if $|k_1| > |k|$, and $\eta' > \eta$ then

$$b_k(t) \lesssim t^{(\eta-\eta'/2-\eta''/2)}|k|^\eta \sum_{\substack{k_1 \neq 0 \\ k_2 \neq 0}} |k_1|^{-3-(\eta'-\eta)}|k_2|^{-3-\eta''} \lesssim t^{(\eta-\eta'/2-\eta''/2)}|k|^\eta.$$

Hence, there exists $\delta > 0$ and $\nu > 0$ such that

$$\mathbb{E}[|\Delta_q(\tilde{I}_t^{(3)} - 3C_2^\varepsilon X_t^\varepsilon)|^2] \lesssim t^\delta 2^{(1+\nu)q}.$$

Terms in the third chaos. Let us define $c_{k_1,k_2}(t-s) = \sum_{k_3\neq 0} \frac{f(\varepsilon k_3)^2}{|k_3|^2} e^{-(|k_1+k_2+k_3|^2+|k_3|^2)|t-s|}$ such that

$$I_t^{(2)} = 6 \sum_{\substack{k \neq 0, k_i \neq 0 \\ k_{124} = k}} e_k \int_0^t \mathrm{d}s c_{k_1,k_2}(t-s) : \hat{X}_s(k_1)\hat{X}_s(k_2)\hat{X}_t^\varepsilon(k_4) :$$

But for all suitable variables we have

$$\begin{aligned}
&\mathbb{E}[: \hat{X}_s^\varepsilon(k_1)\hat{X}_s^\varepsilon(k_2)\hat{X}_t^\varepsilon(k_4) :: \hat{X}_{\bar{s}}^\varepsilon(\bar{k}_1)\hat{X}_{\bar{s}}^\varepsilon(\bar{k}_2)\hat{X}_t^\varepsilon(\bar{k}_4) :] \\
&= \quad 2\delta_{k_1=-\bar{k}_1}\frac{f(\varepsilon k_1)^2}{|k_1|^2}\delta_{k_2=-\bar{k}_3}\frac{f(\varepsilon k_2)^2}{|k_2|^2}\delta_{k_3=-\bar{k}_3}\frac{f(\varepsilon k_3)^2}{|k_3|^2}e^{-(|k_1|^2+|k_2|^2)|s-\bar{s}|} \\
&\quad +2\delta_{k_1=-\bar{k}_1}\frac{f(\varepsilon k_1)^2}{|k_1|^2}\delta_{k_2=-\bar{k}_3}\frac{f(\varepsilon k_2)^2}{|k_2|^2}\delta_{k_3=-\bar{k}_2}\frac{f(\varepsilon k_3)^2}{|k_3|^2}e^{-|k_1|^2|s-\bar{s}|}e^{-(|k_3|^2)|t-\bar{s}|}e^{-(|k_2|^2)|t-s|} \\
&\quad \times e^{-|k_1|^2|s-\bar{s}|}e^{-(|k_3|^2)|t-\bar{s}|}
\end{aligned}$$

and by an easy computation the following holds $\mathbb{E}[|\Delta_q(I_t^{(2)})|^2] = E_t^{2,1} + E_t^{2,2}$ with

$$E_t^{2,1} = 2\int_0^t \mathrm{d}s \int_0^s \mathrm{d}\overline{s} \sum_{\substack{k, k_i \neq 0 \\ k_{124}=k}} \theta(2^{-q}k)^2 \prod_i \frac{f(\varepsilon k_i)^2}{|k_i|^2} c_{k_1,k_2}(t-s) c_{k_1,k_2}(t-\overline{s}) e^{-(|k_1|^2+|k_2|^2)|s-\overline{s}|}$$

and

$$E_t^{2,1} = 2\int_0^t \mathrm{d}s \int_0^s \mathrm{d}\overline{s} \sum_{\substack{k \neq 0, k_i \neq 0, \\ k_{124}=k}} \theta(2^{-q}k)^2 \prod_i \frac{f(\varepsilon k_i)^2}{|k_i|^2}$$
$$\times c_{k_1,k_2}(t-s) c_{k_1,k_4}(t-\overline{s}) e^{-|k_1|^2|s-\overline{s}|} e^{-|k_4|^2|t-\overline{s}|} e^{-|k_2|^2|t-s|}.$$

In $E_t^{2,1}$, we have a symmetry in k_1, k_2, hence we can assume that $|k_1| \geq |k_2|$. Furthermore, we have $c_{k_1,k_2}(t-s) \lesssim |t-s|^{-\frac{1+\eta}{2}}$ and $c_{k_1,k_2}(t-\overline{s}) \lesssim |s-\overline{s}|^{-\frac{1+\eta}{2}}$ If we assume that $|k_1| \geq |k_4|$ and that $\eta'/2 - \eta > 0$, then

$$\begin{aligned} E_t^{2,1} &\lesssim \int_0^t \mathrm{d}s \int_0^s \mathrm{d}\overline{s} |t-s|^{-\frac{1+\eta}{2}} |s-\overline{s}|^{-1+(\eta'/2-\eta)} \sum_{\substack{k \neq 0, k_i \neq 0, \\ k_{124}=k}} \theta(2^{-q}k)^2 \frac{1}{|k_1|^{3-\eta'}|k_2|^2|k_4|^2} \\ &\lesssim t^\delta \sum_{k\neq 0} \frac{\theta(2^{-q}k)^2}{|k|^{1-\eta''}} \sum_{k_2,k_3} |k_2|^{-3-\frac{\eta''-\eta'}{2}} |k_4|^{-3-\frac{\eta''-\eta'}{2}} \lesssim t^\delta 2^{q(2+\eta'')} \end{aligned}$$

for $\eta'' > \eta'$. When $|k_4| \geq |k_1|$ it is almost the same computation.

In $E_t^{2,2}$, we can assume that $|k_2| \geq |k_4|$, so

$$\begin{aligned} E_t^{2,2} &\lesssim \int_0^t \mathrm{d}s \int_0^t \mathrm{d}\overline{s} \sum_{\substack{k \neq 0, k_i \neq 0, \\ k_{124}=k \\ |k_2| \lesssim |k_4|}} \theta(2^{-q}k)^2 |k_1|^{-3+\eta'} |k_2|^{-3+\eta'} |k_4|^2 |t-s|^{-1+\frac{\eta'-\eta}{2}} |s-\overline{s}|^{-1+\frac{\eta'-\eta}{2}} \\ &\lesssim t^\delta \sum_{\substack{k \neq 0, k_i \neq 0, \\ k_{124}=k}} \theta(2^{-q}k)^2 |k|^{-1+\eta''} |k_1|^{-3+\eta'} |k_2|^{-3+\eta'} |k_4|^2 \max(|k_i|)^{1-\eta''} \lesssim t^\delta 2^{q(1+\eta'')} \end{aligned}$$

that we decompose as in the previous term whether $|k_1| \geq |k_4|$ or $|k_4| \geq |k_1|$.

Terms in the fifth chaos. For all suitable variables, we have

$$\begin{aligned} &\mathbb{E}[: \hat{X}_s^\varepsilon(k_1)\hat{X}_s^\varepsilon(k_2)\hat{X}_s^\varepsilon(k_3)\hat{X}_t^\varepsilon(k_4)\hat{X}_t^\varepsilon(k_5) :: \hat{X}_{\overline{s}}^\varepsilon(\overline{k}_1)\hat{X}_{\overline{s}}^\varepsilon(\overline{k}_2)\hat{X}_{\overline{s}}^\varepsilon(\overline{k}_3)\hat{X}_t^\varepsilon(\overline{k}_4)\hat{X}_t^\varepsilon(\overline{k}_5) :] \\ &= 12 \prod_{i=1}^5 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \delta_{k_i=-\overline{k}_i} e^{-|s-\overline{s}|(|k_1|^2+|k_2|^2+|k_3|^2)} \\ &\quad +72 \prod_{i=1}^5 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \delta_{k_1=-\overline{k}_1} \delta_{k_2=-\overline{k}_2} \delta_{k_3=-\overline{k}_4} \delta_{k_4=-\overline{k}_3} \delta_{k_5=-\overline{k}_5} e^{-|s-\overline{s}|(|k_1|^2+|k_2|^2)-|t-s||k_3|^2-|t-\overline{s}||k_4|^2} + \\ &\quad +36 \prod_{i=1}^5 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \delta_{k_1=-\overline{k}_1} \delta_{k_2=-\overline{k}_4} \delta_{k_3=-\overline{k}_5} \delta_{k_4=-\overline{k}_3} \delta_{k_5=-\overline{k}_2} e^{-|s-\overline{s}||k_1|^2-|t-s|(|k_2|^2+|k_3|^2)-|t-\overline{s}|(|k_4|^2+|k_5|^2)} \end{aligned}$$

Then

$$\mathbb{E}[|\Delta_q I_t^1|^2] = E_t^{1,1} + E_t^{1,2} + E_t^{1,3}$$

with

$$E_t^{1,1} = 12 \int_{[0,t]^2} \mathrm{d}s\mathrm{d}\overline{s}\theta(2^{-q}k)^2 \sum_{\substack{k \\ k_{12345}=k}} \prod_{i=1}^{5} \frac{f(\varepsilon k_i)^2}{|k_i|^2} e^{-|k_{123}|^2|t-s|} e^{-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\overline{s}|}$$

$$E_t^{1,2} = 72 \int_{[0,t]^2} \mathrm{d}s\mathrm{d}\overline{s} \sum_{\substack{k \\ k_{12345}=k}} \theta(2^{-q}k)^2 \prod_{i=1}^{5} \frac{f(\varepsilon k_i)^2}{|k_i|^2}$$
$$\times e^{-(|k_{123}|^2+|k_3|^2)|t-s|} e^{-(|k_{124}|^2+|k_4|^2)|t-\overline{s}|} e^{-|s-\overline{s}|(|k_1|^2+|k_2|^2)}$$

and

$$E_t^{1,3} = 36 \int_0^t \mathrm{d}s \int_0^t \mathrm{d}\overline{s} \sum_{\substack{k\neq 0, k_i \neq 0 \\ k_{12345}=k}} \theta(2^{-q}k)^2 \prod_{i=1}^{5} \frac{f(\varepsilon k_i)^2}{|k_i|^2}$$
$$\times e^{-(|k_{123}|^2+|k_2|^2+|k_3|^2)|t-s|} e^{-(|k_{145}|^2+|k_5|^2+|k_4|^2)|t-\overline{s}|} e^{-|s-\overline{s}||k_1|^2}$$

id est

$$E_t^{1,3} = e^{-(|k_{123}|^2+|k_2|^2+|k_3|^2)|t-s|} e^{-(|k_{145}|^2+|k_5|^2+|k_4|^2)|t-\overline{s}|} e^{-|s-\overline{s}||k_1|^2}$$

Estimation of $E_t^{1,1}$. Let us rewrite it in a form better to handle

$$E_t^{1,1} = 12 \int_{[0,t]^2} \mathrm{d}s\mathrm{d}\overline{s} \sum_{\substack{k, k\neq 0 \\ k_1+k_2+l=k \\ l_1+l_2+l_3=l \\ k_i\neq 0, l_i \neq 0}} \theta(2^{-q}k)^2 \prod_{i=1}^{2} \frac{f(\varepsilon k_i)^2}{|k_i|^2} \prod_{i=1}^{3} \frac{f(\varepsilon l_i)^2}{|l_i|^2} e^{-|l|^2|t-s|} e^{-(|l_1|^2+|l_2|^2+|l_3|^2)|s-\overline{s}|}$$

Thanks to the symmetries of this term, we can always assume that $|k_1| = \max(|k_i|)$ and $l_1 = \max(|l_i|)$.

For $l=0$, we have

$$\int_{[0,t]^2} \mathrm{d}s\mathrm{d}\overline{s} \sum_{\substack{k, k\neq 0 \\ k_1+k_2=k \\ l_1+l_2+l_3=0 \\ k_i\neq 0, l_i \neq 0}} \theta(2^{-q}k)^2 \prod_{i=1}^{2} \frac{f(\varepsilon k_i)^2}{|k_i|^2} \prod_{i=1}^{3} \frac{f(\varepsilon l_i)^2}{|l_i|^2} e^{-(|l_1|^2+|l_2|^2+|l_3|^2)|s-\overline{s}|}$$
$$\lesssim \int_{[0,t]^2} \mathrm{d}s\mathrm{d}\overline{s} \sum_{k\neq 0} \theta(2^{-q}k)^2 |k|^{-1+\eta} \sum_{k_2\neq 0} |k|^{-3-\eta} \sum_{l_2\neq 0, l_3\neq 0} |l_2|^{-4+\eta}|l_3|^{-4+\eta}|s-\overline{s}|^{-1+\eta}$$
$$\lesssim 2^{q(2+\eta)}t.$$

Let us assume that $|l| = \max(|l|, |k_1|)$; as we have the following estimate $|l_1|^{-1} \lesssim |l|^{-1}$, the following bound holds

$$\int_{[0,t]^2} \mathrm{d}s\mathrm{d}\bar{s} \sum_{k \neq 0} \theta(2^{-q}k)^2|k|^{-1+\eta} \sum_{\substack{k_1k_2 \neq 0 \\ l_2, l_3 \neq 0}} (|k_1||k_2|)^{-4+9\eta/2}(|t-s||s-\bar{s}|)^{-1+\eta}|l_2|^{-3-\eta}|l_3|^{-3-\eta}$$
$$\lesssim t^\eta 2^{q(2+\eta)}$$

The case in which $|k_1| = \max(|l|, |k_1|)$ is quite similar, and the conclusion holds for $E_t^{1,1}$.

Estimation of $E_t^{1,2}$. This term is symmetric in k_1, k_2 and in k_3, k_4. Hence, we can assume that $|k_1| \geq |k_2|$ and $|k_3| \geq |k_4|$ First let us assume that $|k_5| = \max\{|k_i|\}$. Then

$$\begin{aligned}
E_t^{1,2} \lesssim & \sum_{\substack{k \\ k_{12345} = k}} \theta(2^{-q}k)^2 \int_0^t \mathrm{d}s \int_0^s \mathrm{d}\bar{s}(|t-s||s-\bar{s}|)^{-1+\eta} \\
& \qquad \times |k_1|^{-4+2\eta}|k_2|^{-2}|k_3|^{-4+2\eta}|k_4|^{-2}|k_5|^{-(1+\eta')}|k|^{-(1-\eta')} \\
& \lesssim t^\eta \sum_k \theta(2^{-q}k)^2|k|^{-(1-\eta')} \sum_{k_{12345} = k} |k_1|^{-7/2+2\eta}|k_2|^{-3-\eta'/2}|k_3|^{-7/2+2\eta}|k_4|^{-3-\eta'/2} \\
& \lesssim t^\eta 2^{(2+\eta')q}
\end{aligned}$$

for η small enough.

Then assume that $\max\{|k_i|\} = |k_1|$

$$\begin{aligned}
E_t^{1,2} \lesssim t^\delta & \sum_{\substack{k \\ k_{12345} = k}} \theta(2^{-q}k)^2|k_1|^{-4+2\eta}|k_2|^{-2}|k_3|^{-3+\eta'}|k_4|^{-3+\eta'}|k_5|^{-2} \\
& \times \int_0^t \mathrm{d}s \int_0^s \mathrm{d}\bar{s}|t-s|^{-1+\eta'}|s-\bar{s}|^{-1+\eta} \\
\lesssim t^{\eta'} & \sum_{\substack{k \\ k_{12345} = k}} \theta(2^{-q}k)^2|k|^{-1+\eta''}|k_2|^{-3-\eta''} \\
& \times |k_3|^{-7/2+(2\eta+\eta''+\eta')/2}|k_4|^{-7/2+(2\eta+\eta''+\eta')/2}|k_5|^{-3-\eta''} \\
\lesssim t^\delta & 2^{(2+\eta')q}
\end{aligned}$$

For $\max\{|k_i|\} = |k_3|$

$$\begin{aligned}
& \lesssim t^\delta \sum_{\substack{k \\ k_{12345} = k}} \theta(2^{-q}k)^2|k_1|^{-4+\eta}|k_2|^{-2}|k_3|^{-4+\eta'}|k_4|^{-2}|k_5|^{-2} \\
& \lesssim t^\delta \sum_{\substack{k \\ k_{12345} = k}} \theta(2^{-q}k)^2|k_1|^{-3+\eta+1/4}|k_2|^{-3+1/4}|k|^{-1+\eta'}|k_4|^{-3+1/4}|k_5|^{-3+1/4} \lesssim t^\delta 2^{(2+\eta')q}
\end{aligned}$$

hence there exists $\delta > 0$ and $\nu > 0$ such that

$$E_t^{1,2} \lesssim t^\delta 2^{(2+\nu)q}.$$

Estimation of $E_t^{1,3}$. Let us deal with this last term. Here the symmetries are in k_2, k_3 and k_4, k_5. Then we can suppose that $|k_2| \geq |k_3| \geq$ and $|k_4| \geq |k_5|$. Furthemore, the role of k_2, k_3 and k_4, k_5 are symmetrical, then we can assume that $|k_1| \geq |k_4|$

$$E_t^{1,3} = \int_{[0,t]^2} \mathrm{d}s\mathrm{d}\overline{s} \sum_{\substack{k \neq 0, k_i \neq 0 \\ k_{12345} = k}} \theta(2^{-q}k)^2 \prod_{i=1}^{5} \frac{f(\varepsilon k_i)^2}{|k_i|^2}$$
$$\times e^{-(|k_{123}|^2+|k_2|^2+|k_3|^2)|t-s|} e^{-(|k_{145}|^2+|k_5|^2+|k_4|^2)|t-\overline{s}|} e^{-|s-\overline{s}||k_1|^2}$$

If $|k_1| = \max(|k_i|)$ then

$$\lesssim \int_{[0,t]^2} \mathrm{d}s\mathrm{d}\overline{s}(|t-s||t-\overline{s}|)^{-1+\eta} \sum_{\substack{k \neq 0, k_i \neq 0 \\ k_{12345} = k}} \theta(2^{-q}k)^2 |k|^{-1+\eta} (|k_2||k_3||k_3||k_4|)^{-7/4+3\eta/4}$$
$$\lesssim 2^{q(2+\eta)} t^\eta$$

If $|k_2| = \max(|k_i|)$ then

$$\lesssim 2 \int_0^t \int_0^s \mathrm{d}s\mathrm{d}\overline{s}(|t-s||s-\overline{s}|)^{-1+\eta} \sum_{\substack{k \neq 0, k_i \neq 0 \\ k_{12345} = k}} \theta(2^{-q}k)^2 |k|^{-1+\eta} (|k_1||k_3||k_3||k_4|)^{-7/4+3\eta/4}$$
$$\lesssim t^\eta 2^{q(2+\eta)}.$$

□

4.A A commutation lemma

We give the proof of the Lemma 4.2.5. This proof is from Gubinelli, Imkeller and Perkowski, and can be found in the first version of [44] and also in [70] Lemmas 5.3.20 and 5.5.7. In fact we give a stronger result, and apply it with $\varphi(k) = \exp(-|k|^2/2)$.

Lemma 4.A.1. *Let $\alpha < 1$ and $\beta \in \mathbb{R}$. Let $\varphi \in \mathcal{S}$, let $u \in C^\alpha$, and $v \in C^\beta$. Then for every $\varepsilon > 0$ and every $\delta \geq -1$ we have*

$$\|\varphi(\varepsilon\mathcal{D})\pi_<(u,v) - \pi_<(u, \varphi(\varepsilon\mathcal{D})v)\|_{\alpha+\beta+\delta} \lesssim \varepsilon^{-\delta} \|u\|_\alpha \|v\|_\beta,$$

where

$$\varphi(\mathcal{D})u = \mathcal{F}^{-1}(\varphi\mathcal{F}u).$$

Proof. We define for $j \geq -1$,

$$S_{j-1}u = \sum_{i=-1}^{j-2} \Delta_i u$$

$$\varphi(\varepsilon\mathcal{D})\pi_<(u,v) - \pi_<(u,\varphi(\varepsilon\mathcal{D})v) = \sum_{j\geq -1} (\varphi(\varepsilon\mathcal{D})(S_{j-1}u\Delta_j v) - S_{j-1}u\Delta_j\varphi(\varepsilon\mathcal{D})v),$$

and every term of this series has a Fourier transform with support in an annulus of the form $2^j\mathcal{A}$. Lemma 2.69 in [5] implies that it suffices to control the L^∞ norm of each term. Let $\psi \in \mathcal{D}$ with support in an annulus be such that $\psi \equiv 1$ on $\mathcal{A}$. We have

$$\begin{aligned}
&\varphi(\varepsilon\mathcal{D})(S_{j-1}u\Delta_j v) - S_{j-1}u\Delta_j\varphi(\varepsilon\mathcal{D})v \\
&\qquad = (\psi(2^{-j}\cdot)\varphi(\varepsilon\cdot))(\mathcal{D})(S_{j-1}u\Delta_j v) - S_{j-1}u(\psi(2^{-j}\cdot)\varphi(\varepsilon\cdot))(\mathcal{D})\Delta_j v \\
&\qquad = [(\psi(2^{-j}\cdot)\varphi(\varepsilon\cdot))(\mathcal{D}), S_{j-1}u]\Delta_j v,
\end{aligned}$$

where $[(\psi(2^{-j}\cdot)\varphi(\varepsilon\cdot))(\mathcal{D}), S_{j-1}u]$ denotes the commutator. In the proof of Lemma 2.97 in [5], it is shown that writing the Fourier multiplier as a convolution operator and applying a first order Taylor expansion and then Young's inequality yields

$$\begin{aligned}
&\|[(\psi(2^{-j}\cdot)\varphi(\varepsilon\cdot))(\mathcal{D}), S_{j-1}u]\Delta_j v\|_{L^\infty} \\
&\qquad \lesssim \sum_{\eta\in\mathbb{N}^d:|\eta|=1} \|x^\eta\mathcal{F}^{-1}(\psi(2^{-j}\cdot)\varphi(\varepsilon\cdot))\|_{L^1}\|\partial^\eta S_{j-1}u\|_{L^\infty}\|\Delta_j v\|_{L^\infty}.
\end{aligned} \tag{4.25}$$

Now $\mathcal{F}^{-1}(f(2^{-j}\cdot)g(\varepsilon\cdot)) = 2^{jd}\mathcal{F}^{-1}(fg(\varepsilon 2^j\cdot))(2^j\cdot)$ for every f, g, and thus we have for every multi-index η of order one

$$\begin{aligned}
&\|x^\eta\mathcal{F}^{-1}(\psi(2^{-j}\cdot)\varphi(\varepsilon\cdot))\|_{L^1} \\
&\qquad \leq 2^{-j}\|\mathcal{F}^{-1}((\partial^\eta\psi)(2^{-j}\cdot)\varphi(\varepsilon\cdot))\|_{L^1} + \varepsilon\|\mathcal{F}^{-1}(\psi(2^{-j}\cdot)\partial^\eta\varphi(\varepsilon\cdot))\|_{L^1} \\
&\qquad = 2^{-j}\|\mathcal{F}^{-1}((\partial^\eta\psi)\varphi(\varepsilon 2^j\cdot))\|_{L^1} + \varepsilon\|\mathcal{F}^{-1}(\psi\partial^\eta\varphi(\varepsilon 2^j\cdot))\|_{L^1} \\
&\qquad \lesssim 2^{-j}\|(1+|\cdot|)^{2d}\mathcal{F}^{-1}((\partial^\eta\psi)\varphi(\varepsilon 2^j\cdot))\|_{L^\infty} + \varepsilon\|(1+|\cdot|)^{2d}\mathcal{F}^{-1}(\psi\partial^\eta\varphi(\varepsilon 2^j\cdot))\|_{L^\infty} \\
&\qquad = 2^{-j}\|\mathcal{F}^{-1}((1-\Delta)^d((\partial^\eta\psi)\varphi(\varepsilon 2^j\cdot)))\|_{L^\infty} + \varepsilon\|\mathcal{F}^{-1}((1-\Delta)^d(\psi\partial^\eta\varphi(\varepsilon 2^j\cdot))\|_{L^\infty} \\
&\qquad \lesssim 2^{-j}\|(1-\Delta)^d((\partial^\eta\psi)\varphi(\varepsilon 2^j\cdot))\|_{L^\infty} + \varepsilon\|(1-\Delta)^d(\psi\partial^\eta\varphi(\varepsilon 2^j\cdot))\|_{L^\infty},
\end{aligned} \tag{4.26}$$

where the last step follows because ψ has compact support. For j satisfying $\varepsilon 2^j \geq 1$ we obtain

$$\|x^\eta\mathcal{F}^{-1}(\varphi(\varepsilon\cdot)\psi(2^{-j}\cdot))\|_{L^1} \lesssim (\varepsilon + 2^{-j})(\varepsilon 2^j)^{2d} \sum_{\eta:|\eta|\leq 2d+1} \|\partial^\eta\varphi(\varepsilon 2^j\cdot)\|_{L^\infty(\mathrm{supp}(\psi))}, \tag{4.27}$$

where we used that ψ and all its partial derivatives are bounded, and where $L^\infty(\mathrm{supp}(\psi))$ means that the supremum is taken over the values of $\partial^\eta\varphi(\varepsilon 2^j\cdot)$ restricted to $\mathrm{supp}(\psi)$. Now φ is a Schwartz function, and therefore it decays faster than any polynomial. Hence, there exists a ball $\mathcal{B}_\delta$ such that for all $x \notin \mathcal{B}_\delta$ and all $|\eta| \leq 2d+1$ we have

$$|\partial^\eta\varphi(x)| \leq |x|^{-2d-1-\delta}. \tag{4.28}$$

Let $j_0 \in \mathbb{N}$ be minimal such that $2^{j_0}\varepsilon\mathcal{A} \cap \mathcal{B}_\delta = \emptyset$ and $\varepsilon 2^{j_0} \geq 1$. Then the combination of (4.25), (4.27), and (4.28) shows for all $j \geq j_0$ that

$$\begin{aligned}
&\|[(\psi(2^{-j}\cdot)\varphi(\varepsilon\cdot))(\mathcal{D}), S_{j-1}u]\Delta_j v\|_{L^\infty} \\
&\qquad \lesssim (\varepsilon + 2^{-j})(\varepsilon 2^j)^{2d} \sum_{\eta:|\eta|\leq 2d+1} \|(\partial^\eta\varphi)(\varepsilon 2^j\cdot)\|_{L^\infty(\operatorname{supp}(\psi))} 2^{j(1-\alpha)}\|u\|_\alpha 2^{-j\beta}\|v\|_\beta \\
&\qquad \lesssim (\varepsilon + 2^{-j})(\varepsilon 2^j)^{2d}(\varepsilon 2^j)^{-2d-1-\delta} 2^{j(1-\alpha-\beta)}\|u\|_\alpha\|v\|_\beta \\
&\qquad \lesssim (1 + (\varepsilon 2^j)^{-1})\varepsilon^{-\delta} 2^{-j(\alpha+\beta+\delta)}\|u\|_\alpha\|v\|_\beta.
\end{aligned}$$

Here we used that $\alpha < 1$ in order to obtain $\|\partial^\eta S_{j-1}u\|_{L^\infty} \lesssim 2^{j(1-\alpha)}\|u\|_{L^\infty}$. Since $\varepsilon 2^j \geq 1$, we have shown the desired estimate for $j \geq j_0$. On the other side Lemma 2.97 in [5] implies for every $j \geq -1$ that

$$\|[\varphi(\varepsilon\mathcal{D}), S_{j-1}u]\Delta_j v\|_{L^\infty} \lesssim \varepsilon \max_{\eta\in\mathbb{N}^d:|\eta|=1} \|\partial^\eta S_{j-1}u\|_{L^\infty}\|\Delta_j v\|_{L^\infty} \lesssim \varepsilon 2^{j(1-\alpha-\beta)}\|u\|_\alpha\|v\|_\beta.$$

Hence, we obtain for $j < j_0$, i.e. for j satisfying $2^j\varepsilon \lesssim 1$, that

$$\|[\varphi(\varepsilon\mathcal{D}), S_{j-1}u]\Delta_j v\|_{L^\infty} \lesssim (\varepsilon 2^j)^{1+\delta}\varepsilon^{-\delta}2^{-j(\alpha+\beta+\delta)}\|u\|_\alpha\|v\|_\beta \lesssim \varepsilon^{-\delta}2^{-j(\alpha+\beta+\delta)}\|u\|_\alpha\|v\|_\beta,$$

where we used that $\delta \geq -1$. This completes the proof. $\square$

Annexe A

Mouvement brownien fractionnaire

We first give a definition of the fractional Brownian motion, and then prove the Fernique theorem and its spatial regularity.

Definition A.0.2. Let $H \in (0,1)$. The fractional Brownian motion (fBm) of Hurst parameter H is a centered gaussian process B^H such that for all $s,t \in \mathbb{R}_+$

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(|t|^{2H} + |t|^{2H} - |t-s|^{2H}).$$

A d-dimensional fBm is a d-dimensional Gaussian process such that $B^H = (B^{H,1}, \dots, B^{H,d})$ where the $B^{H,i}$ are independent fractional Brownian motion of Hurst parameter H.

This process was first introduced by Kolmogorov in 1940 [56] and first studied by Mandelbrot and Van Ness [63]. A broad literature has been devoted to the study of several aspect of this process. Here, we only want to give the first and easiest properties of the fractional Brownian motion.

Proposition A.0.3. *Let $H \in (0,1)$ and B^H a fractional Brownian motion. The following properties holds.*

1. *Self-similarity. For $\lambda > 0$, the following equality holds $(\lambda^{-H} B_{\lambda t}^H)_{t \geqslant 0} =^{\mathcal{L}} (B_t^H)_{t \geqslant 0}$.*
2. *Stationary of the increments. For $s \leqslant t$, the following equality holds $B_t^H - B_s^H =^{\mathcal{L}} B_{t-s}^H \sim \mathcal{N}(0, |t-s|^{2H})$.*

Thanks to these two properties, we can deduce the support of the Law of the fractional Brownian motion. The following proposition holds.

Proposition A.0.4. *Let $H \in (0,1)$, and B^H a fractional Brownian motion of Hurst parameter H. There exists a random variable $L > 0$ such that for all $T \in \mathbb{R}_+$, almost surely, for all $\varepsilon > 0$, $B^H \in \mathcal{C}^{H-\varepsilon}([0,T])$ such that*

$$\mathbb{E}[\exp(L^2)] < +\infty$$

and for all $\varepsilon > 0$, there exists a constant depending on $\varepsilon > 0$ such that

$$\|B^H\|_{\mathcal{C}^{H-\varepsilon}} \leqslant C_\varepsilon T^{2H} \log^{1/2}(1+T) L.$$

Remark A.0.5. We can choose

$$C_\varepsilon = C \sup_{x \geqslant 1} x^{-\varepsilon} \log^{1/2}(x).$$

where C is a universal constant.

Proof. Let B a d-dimensional fBm of Hurst parameter $H \in (0,1)$. Let $0 \leqslant s < t$, and $l \geqslant 0$. We have

$$\mathbb{E}[|B_t^H - B_s^H|^{2l}] = |t-s|^{2Hl}\mathbb{E}[|B_1^H|^{2l}] = |t-s|^{2Hl} 2^l \Gamma(l+d/2)/\Gamma(d/2).$$

For $T \in \mathbb{R}_+$, $n \in \mathbb{N}$ and $k \in \mathbb{N}$ let us define $t_k^n = k2^{-n}$. Let us define

$$K(\lambda) = \sum_{n=0}^{+\infty}\sum_{k=1}^{+\infty} 2^{-n}k^{-2} \exp(\lambda 2^{2nH}|B^H_{t^n_{k+1}} - B^H_{t^n_k}|^2 T^{-2H}).$$

We have

$$\begin{aligned}
\mathbb{E}[K(\lambda)] &= \sum_{n=0}^{+\infty}\sum_{k=1}^{+\infty} 2^{-n}k^{-2} \sum_{l=0}^{+\infty} \frac{\lambda^l 2^{nlH} T^{-2H}}{l!} \mathbb{E}[|B^H_{t^n_{k+1}} - B^H_{t^n_k}|^{2l}] \\
&= \sum_{n=0}^{+\infty}\sum_{k=1}^{+\infty} 2^{-n}k^{-2} \sum_{l=0}^{+\infty} \frac{2^l\lambda^l \Gamma(l+d/2)}{l!\Gamma(d/2)}.
\end{aligned}$$

But we have

$$\begin{aligned}
\frac{2^l\lambda^l\Gamma(l+d/2)}{l!} &\sim (2\lambda)^l (l+d/2)^{l+d/2-1/2} e^{-l-d/2} e^l l^{-l-1/2} \\
&\sim (2\lambda)^l \left(1+\frac{d}{2l}\right)^l \left(l+\frac{d}{2}\right)^{d/2} (l(l+d/2))^{-1/2}
\end{aligned}$$

Hence, for $\lambda < \frac{1}{2}$, $\mathbb{E}[K(\lambda)] = C_{d,\lambda} < +\infty$ and is independent from T.

Now we choose $T > 0$ and we want to evaluate the divergence of the Hölder norm of B^H on the whole interval $[0,T]$. Let $0 \leqslant s < t \leqslant T$. First, we assume that $\frac{t-s}{T} \leqslant 1$. Let $n \geqslant 0$ such that $T2^{-(n+1)} < t-s \leqslant T2^{-n}$. Let $k \in \{0,\dots,2^{n+1}-1\}$ such that. We have

$$s \leqslant Tk2^{-(n+1)} < t \leqslant T(k+1)2^{-(n+1)}$$

We can now construct the two sequences $(t_i)_i = (t^{k_i}_{n+i+1})_i$ and $(s_i) = (s^{k'_i}_{n+i+1})$, with $k_0 = k'_0 = k$, such that $t_i \nearrow t$, $t_{i+1} - t_i \leqslant T2^{-(n+i+2)}$, and $s_i \searrow s$ and $s_i - s_{i+1} \leqslant T2^{-(n+i+2)}$. Hence, as $k_i = 2^{-(n+i+1)}k_i 2^{(n+i+1)} \leqslant t_i 2^{(n+i+1)} \leqslant t2^{(n+i+1)}$ and the same holds for k'_i, we have

$$\begin{aligned}
\lambda^{1/2}|B_t^H - B_s^H| &\lesssim \lambda^{1/2}\sum_{i\geqslant 0} |B^H_{t_{i+1}} - B^H_{t_i}| + |B^H_{s_i} - B^H_{s_{i+1}}| \\
&\lesssim \sum_{i \geqslant 0} T^H 2^{-H(n+i+2)} \log^{1/2}\{2^{4H(n+i+2)} T^{2H} K(\lambda)\} \\
&\lesssim T^H \sum_i 2^{-H(n+i+2)} \{1+n+i+\log_+(T) + \log K\}^{1/2}
\end{aligned}$$

But, as $2^{-n} \lesssim |t-s|$ and $n \lesssim \log(T/|t-s|)$, we have

$$\sqrt{\lambda}|B_t^H - B_s^H| \lesssim |t-s|^H T^H \sum_i 2^{-iH} \left(1 + i + \log \frac{T}{|t-s|} + \log_+ T + \log K(\lambda)\right)^{1/2}.$$

Hence, we have

$$|B_t^H - B_s^H| \lesssim \lambda^{-1/2} |t-s|^H T^H \left(1 + \log^{1/2} \frac{1}{|t-s|} + \log_+^{1/2} T + \log^{1/2} K(\lambda)\right)$$

and by losing a small power of $|t-s|$ we have

$$|B_t^H - B_s^H| \lesssim_\varepsilon \lambda^{-1/2} |t-s|^{H-\varepsilon} T^H \log^{1/2}(1+T)(1 + \log^{1/2} K(\lambda)).$$

which ends the proof. $\square$

Annexe B

Processus (ρ, γ)-irréguliers, quelques résultats

In this Appendix, we investigate a bit further the notion of (ρ, γ)-irregular path introduced in Chapter 2 We first remind some of the defininition and basic properties of (ρ, γ)-irregular paths, and we give useful applications of this notion. Finally we prove that non-degenerate α-stable Levy processes are also ρ-irregular.

B.1 Definition and first properties

Let us remind the definition of (ρ, γ)-irregular paths.

Definition B.1.1. Let $\rho > 0$ and $\gamma > 0$. We say that a function $w : [0, T] \to \mathbb{R}^d$ is (ρ, γ)-irregular if

$$\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} := \sup_{a \in \mathbb{R}^d} \sup_{0 \leqslant s < t \leqslant T} \langle a \rangle^\rho \frac{|\Phi_t^w(a) - \Phi_s^w(a)|}{|s-t|^\gamma} < +\infty$$

where $\Phi_t^w(a) = \int_0^t e^{i\langle a, w_r \rangle} \mathrm{d}r$.

Moreover we say that w is ρ-irregular if there exists $\gamma > 1/2$ such that w is (ρ, γ)-irregular.

Modulation of function toward irregular paths When w is an irregular path, the following straigntforward method will give plenty of application to control the regularity of different transformation of the path. It relies on Young Integration as first developped by Young in [75] and developped for example in Chapter 2.

Definition B.1.2. Let w a E valued (ρ, γ)-irregular path. For all $f \in L^1([0, T]; \mathbb{R})$ we define the modulation of f by Φ^w and we note $\Phi^w f$ the following quantity.

$$(\Phi^w f)_{s,t}(\xi) = \int_s^t e^{i\xi . w_u} f_u \mathrm{d}u.$$

This definition allows us to compute the irregularity of a the modulated path. As before, we define

$$\|\Phi^w f\|_{\mathcal{W}_T^{\rho,\gamma}} = \sup_{a \in \mathbb{R}^d} \sup_{0 \leqslant s < t \leqslant T} (1 + |a|)^{-\rho} \frac{|\Phi_t^w f(a) - \Phi_s^w f(a)|}{|t-s|^\gamma}.$$

Proposition B.1.3. *Suppose that $f \in \mathcal{C}^\nu([0,T];\mathbb{R})$ with $\gamma+\nu>1$ then $\Phi^w f$ is (ρ,γ)-irregular, furthermore*

$$\|\Phi^w f\|_{\mathcal{W}_T^{\rho,\gamma}} \leqslant (1+T^\nu)\|f\|_\nu(1+|\xi|)^{-\rho}\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}$$

Proof. The proof is a direct application of the usual estimates for the Younf Integral. Let us recall that for two functions $f \in \mathcal{C}^\delta([0,T];\mathbb{R})$ and $g \in \mathcal{C}^\nu([0,T];\mathbb{R})$ with $\gamma+\nu>1$ we can define the Young integral of f against $\mathrm{d}g$ as the limit of the Riemann sums $\sum f_{t_i}(g_{t_{i+1}}-g_{t_i})$. In that case, the following bound holds

$$\left|\int_s^t f_s \mathrm{d}g_s - f_s(g_t-g_s)\right| \lesssim |t-s|^{\delta+\nu}\|f\|_\delta\|g\|_\nu.$$

The proposition is straignforward with this bound. Indeed, as $\nu+\gamma>1$, the following computation holds

$$\begin{aligned}
|\Phi^w f_{s,t}(\xi)| &= \left|\int_s^t f_u e^{i\xi.w_u}\mathrm{d}u\right| \\
&= \left|\int_s^t f_u \mathrm{d}\Phi^w_u(\xi)\right| \\
&\leqslant |f_s||t-s|^\gamma(1+|\xi|)^{-\rho}\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} + \|f\|_\nu(1+|\xi|)^{-\rho}\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}|t-s|^{\gamma+\nu} \\
&\leqslant (\|f\|_{\infty,[0,T]}+T^\nu\|f\|_\nu)(1+|\xi|)^{-\rho}\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}|t-s|^\gamma.
\end{aligned}$$

and the result follows. □

Regularity of irregular paths The name for this notion, ρ-irregularity has to be quastionned. Usualy one try to quantifu the regularity of a process. But here the quantity $\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}$ measure indeed the irregularity of the process w in the sens of the following proposition, due to Chouk and Gubinelli [14].

Proposition B.1.4 (Chouk & Gubinelli, Theorem 1.2 in [14])**.** *Let w a δ-Hölder continuous function on $[0,T]$, and $\rho,\gamma>0$ such that $\gamma+\delta>1$ and $\rho>(1-\gamma)/\delta$. Then*

$$\|\Phi^w\|_{\mathcal{W}_T^{\gamma,\rho}} = +\infty.$$

Proof. Let us suppose that w is (ρ,γ)-irregular. Then for all $0\leqslant s<t\leqslant T$ we have

$$e^{ia}(t-s) = \int_s^t e^{ia(1-w_u)}e^{iaw_u}\mathrm{d}u = \Phi^w_{s,t}f(a)$$

with $f_t = e^{ia(1-w_t)}$. Furthermore, for all $0<\eta\leqslant 1$, the Hölder norm of f verified the following bound

$$\|f\|_{\mathcal{C}^{\eta\delta}} \lesssim a^\eta\|w\|^\eta_{\mathcal{C}^\delta}.$$

Now thanks to the hypothesis we can chose a $\eta>0$ such that $1\geqslant\eta>\frac{1-\gamma}{\delta}$ with $\rho>\eta$. We can apply B.1.3, and we have

$$|t-s| = |e^{ia}(t-s)| \lesssim |t-s|^\gamma(1+|a|)^{\eta-\rho}\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}\|w\|^\eta_{\mathcal{C}^\delta}.$$

This is obviously wrong for large $|a|$, and this allows us to conclude that $\|\Phi^w\|_{\mathcal{W}_T^{\gamma,\rho}}=+\infty$. □

In fact, such a proposition gives an upper bound for the regularity of a (ρ,γ)-irregular path. Indeed, if w is (ρ,γ)-irregular and is δ-Hölder continuous then $\delta < (1-\gamma)/\rho$, and this justify the name for such a notion.

When we add an smooth additive perturbation to a δ-hölder continuous function, the regularity of such a function does not change. One can wondered if such a property is still true when dealing with ρ-irregular functions. In fact even when we consider a really smooth perturbation of an irregular path, we are not able to prove that the translated path have still the same regularity. The following proposition is the best we are able to prove concerning the translation of an irregular path.

Proposition B.1.5. *Let $(\rho,\gamma) \in \mathbb{R}_+^2$ and $w : [0,T] \to \mathbb{R}^d$ a ρ-irregular path. Let $\varphi \in \mathrm{Lip}(\mathbb{R}^d)$ then $w+\varphi$ is $(\rho-\delta,\gamma)$- irregular for all $1 \geqslant \delta > 0$ such that $\delta+\gamma > 1$.*

Proof. Let δ as in the hypothesis. Hence $f : u \to e^{ia.\varphi_u}$ is a δ-Hölder continuous function and and we have $\|f\|_\delta \lesssim |a|^\delta \|\varphi\|_{\mathrm{Lip}}$. The result follows by Proposition B.1.3 which gives the result. □

Change of time formula The comportement of the (ρ,γ)-irregularity towards change of time can also be investigate thanks to Proposition B.1.3.

Proposition B.1.6. *Let $w : [0,T] \to \mathbb{R}^d$ a (ρ,γ)-irregular path. Let $\frac{1}{2-\gamma} > \beta > \frac{1}{2}$ then $w^\beta : u \to w_{u^\beta}$ is $(\rho,\gamma\beta)$ irregular and*

$$\|\Phi^{w^\beta}\|_{\mathcal{W}_T^{\rho,\gamma}} \lesssim \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma\beta}}.$$

Proof. We have

$$\begin{aligned}
\Phi_{s,t}^{w^\beta}(\xi) &= \int_s^t \exp(i\xi.w_{u^\beta})\mathrm{d}u \\
&= \frac{1}{\beta}\int_{s^\beta}^{t^\beta} \exp(i\xi.w_v)v^{1/\beta-1}\mathrm{d}v \\
&= (\Phi^w f)_{s^\beta,t^\beta}(\xi)
\end{aligned}$$

where $f_u = \frac{1}{\beta}v^{1/\beta-1}$. Hence, thanks to Proposition B.1.3, as $0 < \frac{1}{\beta}-1 < 1$, $f \in \mathcal{C}^{1/\beta-1}$, and $1/\beta - 1 + \gamma > 1$

$$\begin{aligned}
|\Phi_{s,t}^{w^\varphi}(\xi)| &\leqslant \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t^\beta - s^\beta|^\gamma (1+|\xi|)^{-\rho}(\|f\|_\infty + T^\nu\|f\|_\nu) \\
&\lesssim_T \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-s|^{\gamma\beta}(1+|\xi|)^{-\rho}
\end{aligned}$$

and the result follows. □

B.2 Some applications for (ρ,γ)-irregular paths

This notion of (ρ,γ)-irregular path seems to be really powerful in order to treat a wide range of problem. We give different applications of this notion.

B.2.1 Existence of local time

The notion of (ρ,γ)-irregular path allows us ta specify the existence of the local of the path whenever ρ is big enough.

Lemma B.2.1. *Let $\rho > 0$, $\gamma \in (0,1]$ and $w \in L^\infty([0,T],\mathbb{R}^d)$ a (ρ,γ)–path. If $2\rho > d$ the local time ℓ^w_t of w exists and $\ell_t(x) = \mathcal{F}^{-1}(\Phi^w_t)(x)$ in $L^2(\mathbb{R}^d)$.*

Proof. The occupation measure of the path w is well-defined as $L_t(A) = \int_0^t \mathbb{1}_A(w_u)\mathrm{d}u$. Furthermore, as a distribution we have $\hat{L}_t(\xi) = \Phi^w_t(\xi)$. Moreover, as $w \in L^\infty([0,T];\mathbb{R}^d)$, there exists a compact set $K \subset \mathbb{R}^d$ such that for almost all $u \in [0,T]$ $w_u \in K$. Hence, for any Borel set $A \subseteq \mathbb{R}^d$ we have

$$L_t(A) - L_s(A) = \int_s^t \mathbb{1}_A(w_u)\mathrm{d}u = \int_s^t \mathbb{1}_{A\cap K}(w_u)\mathrm{d}u.$$

By Cauchy–Schwarz inequality and Plancherel formula, we have

$$\begin{aligned} |L_t(A) - L_s(A)| &= \left| \int \mathrm{d}\xi \widehat{\mathbb{1}_{K\cap A}}(\xi)\Phi^w_{s,t}(\xi) \right| \\ &= \|\widehat{\mathbb{1}_{K\cap A}}\|_{L^2}\|\Phi^w_{s,t}\|_{L^2} \\ &\leqslant \|\mathbb{1}_{A\cap K}\|^{1/2}_{L^1}\|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T}|t-s|^\gamma \end{aligned}$$

As $2\rho > d$, $\|\Phi^w_t\|_{L^2} < +\infty$ and this implies that L_t is absolutely continuous toward the Lebesgue measure. Set $\ell_t = \mathrm{d}L_t/\mathrm{d}x \in L^1(\mathbb{R}^d)$ the Radon derivative of L_t, then, by definition, for all $f \in L^\infty(\mathbb{R}^d)$

$$\int_0^t f(w_s)\mathrm{d}s = \int_{\mathbb{R}^d} f(x)\ell^w_t(x)\mathrm{d}x.$$

and

$$\int_{\mathbb{R}^d} e^{i\xi.x}\ell^\omega_t(x)\mathrm{d}x =^{L^2} \int_0^t e^{i\xi.w_u}\mathrm{d}u = \Phi^w_t(\xi)$$

which ends the proof. $\square$

The same kind of argument allows us to state wether or not ℓ^w is continuous in space.

Lemma B.2.2. *For w as in the previous Lemma; the local time ℓ is continuous if $2\rho > 2d$ and is continuously differentiable if $2\rho > 2d+1$.*

Proof. If $2\rho > 2d$ we have that $\|(1+|\xi|^2)^{d/4+\varepsilon/2}\Phi^w_t(\xi)\|_{L^2} < +\infty$ for some small $\varepsilon > 0$ so ℓ_t belongs to $H^{d/2+\varepsilon/2}(\mathbb{R}^d)$ and by Sobolev embedding also to $\mathcal{C}^\delta$ for some small $\delta > 0$. A similar argument give the second statement. $\square$

Some similar computation allows us to state the existence of the intersection local time for to irregular path. We define the intersection measure of w and w_2 as

$$I^{w^1,w^2}_t(A) = \int_0^t\int_0^t \mathbb{1}_A(w^1_u - w^2_v)\mathrm{d}u\mathrm{d}v$$

As before, if we suppose that w^1 and w^2 are bounded measurable, for any Borel set A and for $K \subset \mathbb{R}^d$ a compact set such that $w^1_u - w^2_v \in K$ for all $u, v \in [0,T]$, we have

$$I_t^{w^1,w^2}(A) = \int_{\mathbb{R}^d} \mathrm{d}\xi \widehat{\mathbb{1}_{A\cap K}}(\xi) \Phi_t^{w^1}(\xi)\Phi_t^{w^2}(-\xi).$$

This leads us to the following Lemma about the existence of the intersection local time ℓ^{w^1,w^2}, the Radon derivative of the intersection measure

Lemma B.2.3. *Let us suppose that $w^1, w^2 \in L^\infty([0,T];\mathbb{R}^d)$. Let $\rho_1, \gamma_1 > 0$ and $\rho_2, \gamma_2 > 0$ such that w^1 is (ρ_1,γ_1)–irregular, and w^2 is (ρ_2,γ_2)–irregular. If $\rho_1 + \rho_2 > d$ the intersection local time of w^1 and w^2 exists.*

Proof. As before, we have, as a distribution $\mathcal{F}(I_t^{w^1,w^2}) = \Phi_t^{w^1}(\xi)\Phi_t^{w^2}(-\xi)$. Hence, by the same computation the Radon derivative of the intersection measure exists. □

B.3 Stochastic processes as (ρ,γ)-irregular paths

Let us give here a proof of the (ρ,γ)-irregularity gor the strongly non-degenerate alpha stable Levy processes. Let us first remind what we called an alpha-stable Levy process.

Definition B.3.1. Let $\alpha \in (0,2]$. A strongly non-degenerate d-dimensional α-stable Lévy process X is a Lévy process such that

$$\mathbb{E}[\exp(i\xi.X_t)] = \exp\left(-C\int_{\mathbb{S}^{d-1}} |\xi.\eta|^\alpha \mathrm{d}\nu(\eta)\right)$$

where there exists two constants c_1 and c_2 such that

$$c_1|\xi|^\alpha \leqslant \int_{\mathbb{S}^{d-1}} |\xi.\eta|^\alpha \mathrm{d}\nu(\eta) \leqslant c_2|\xi|^\alpha.$$

Proposition B.3.2. *Let $\alpha \in (0,2]$ and let X a non-degenerate d-dimensional α-stable Lévy process. There exists $\lambda > 0$ small enough, depending only on the law of X such that*

$$\sup_{s\neq t\in[0,T],\xi\in\mathbb{R}^d} \mathbb{E}[\exp(\lambda(1+|\xi|)^\alpha|\Phi_{s,t}^X(\xi)|^2/|t-s|)] < C_\lambda < +\infty$$

where C_λ is independent of s,t,ξ.

Proof. Let ν the measure associated measure on the sphere $\mathbb{S}^{d-1}$. We have

$$\mathbb{E}[\exp(i\xi.X_t)] = \exp\left(-C\int_{\mathbb{S}^{d-1}} |\xi.\eta|^\alpha \mathrm{d}\nu(\eta)\right).$$

As usual let us define

$$\Phi_{s,t}^X(\xi) = \int_s^t e^{i\xi.X_u}\mathrm{d}u.$$

As X is a self-similar with stationary increments process, we have

$$\Phi_{s,t}^X(\xi) =^{\mathcal{L}} (t-s)e^{i\xi.X_s}\int_0^1 e^{i(t-s)^{1/\alpha}\xi.X_u}\mathrm{d}u.$$

in order to prove the (ρ,γ)-irregularity and to have the gaussian tails for the norm, we will control all the moment of Φ^X.

$$\mathbb{E}[|\Phi_1^X(\zeta)|^{2p}] = \mathbb{E}\left[\left|\int_0^1 e^{i\zeta X_u}\mathrm{d}u\right|^{2p}\right] = \int_0^1 \mathrm{d}u_1\ldots\mathrm{d}u_{2p}\mathbb{E}\left[e^{i\zeta.\sum_{j=1}^{2p}\varepsilon_j X_{u_j}}\right]$$

where the $\varepsilon_j = \pm 1$. In order to use the independence of the increment of X, we rearrange the sum such that

$$\mathbb{E}[|\Phi_{0,1}^X(\zeta)|^{2p}] = \sum_{\sigma\in\mathfrak{S}_{2p}}\int_{\Delta^{2p}}\mathrm{d}u_1\ldots\mathrm{d}u_{2p}\mathbb{E}\left[e^{i\zeta.\sum_{j=1}^{2p-1}a_j^\sigma(X_{u_{j+1}}-X_{u_j})}\right] \tag{B.1}$$

where for $\sigma\in\mathfrak{S}_{2p}$, $a_j^\sigma = \sum_{l=0}^j \varepsilon_{\sigma(j)}$ and $\Delta^{2p} = \{(u_1,\ldots,u_{2p}) : 0\leqslant u_1\leqslant\cdots\leqslant u_{2p}\leqslant 1\}$. Hence, as X has independent increments,

$$\begin{aligned}
\mathbb{E}[|\Phi_1^X(\zeta)|^{2p}] &= \sum_{\sigma\in\mathfrak{S}_{2p}}\int_{\Delta^{2p}}\mathrm{d}u_1\ldots\mathrm{d}u_{2p}\prod_{j=1}^{2p-1}\mathbb{E}\left[e^{ia_j^\sigma\zeta.(X_{u_{j+1}}-X_{u_j})}\right]\\
&= \sum_{\sigma\in\mathfrak{S}_{2p}}\int_{\Delta^{2p}}\mathrm{d}u_1\ldots\mathrm{d}u_{2p}\exp\left(-C\sum_{j=1}^{2p}|a_j^\sigma|^\alpha\int_{\mathbb{S}^{d-1}}|\zeta.\eta|^\alpha\mathrm{d}\nu(\eta)(u_{j+1}-u_j)\right).
\end{aligned}$$

We are looking for an upper bound for $\mathbb{E}[|\Phi_1^X(\zeta)|^{2p}]$. As we have in the worst case $|\alpha_{2j}^\sigma| = 0$ and $|a_{2j+1}^\sigma| = 1$, and $\int_{\mathbb{S}^{d-1}}|\zeta.\eta|^\alpha\mathrm{d}\nu(\eta)\geqslant c_1|\zeta|^\alpha$, we have

$$\begin{aligned}
\mathbb{E}[|\Phi_1^X(\zeta)|^{2p}] &\leqslant (2p)!\int_{\Delta^{2p}}\mathrm{d}u_1\ldots\mathrm{d}u_{2p}\exp\left(-\tilde{C}|\zeta|^\alpha\sum_{j=1}^p(u_{2j}-u_{2j-1})\right)\\
&= (2p)!\int_{\Delta^{2p}}\mathrm{d}v_1\ldots\mathrm{d}v_{2p}\exp\left(-\tilde{C}|\zeta|^\alpha\sum_{j=1}^p u_{2j-1}\right)\\
&= \frac{(2p)!}{p!}\left(\int_0^1\mathrm{d}u\int_0^u\mathrm{d}v\exp(-\tilde{C}|\zeta|^\alpha v)\right)^p\\
&\leqslant \frac{(2p)!}{p!}(\tilde{C}|\zeta|^\alpha)^{-p}
\end{aligned}$$

Hence, for $\zeta = (t-s)^{1/\alpha}\xi$, we have

$$\begin{aligned}
\mathbb{E}[|\Phi_{s,t}^X(\xi)|^{2p}] &\leqslant \frac{(2p)!|t-s|^{2p}}{p!}(\tilde{C}|\xi|^\alpha|t-s|)^{-p}\\
&\leqslant \frac{(2p)!|t-s|^p}{p!\tilde{C}^p|\xi|^{\alpha p}}.
\end{aligned}$$

When $|\xi|\leqslant 1$ we also have

$$\mathbb{E}[|\Phi_{s,t}(\xi)|^{2p}] \leqslant |t-s|^{2p} \leqslant \frac{(2p)!}{p!}|t-s|^p$$

hence

$$\mathbb{E}[|\Phi_{s,t}(\xi)|^{2p}] \lesssim \frac{(2p)!}{\tilde{C}^p p!}|t-s|^p(1+|\xi|)^{-\alpha p}$$

Finally we have

$$\mathbb{E}[\exp(\lambda(1+|\xi|)^\alpha|\Phi^X_{s,t}(\xi)|^2/|t-s|)] \leqslant \sum_{p\geqslant 0}\frac{\lambda^{2p}(2p)!}{\tilde{C}^p(p!)^2} < +\infty$$

for λ small enough which depends only on ν. Hence

$$\sup_{s,t,\xi}\mathbb{E}[\exp(\lambda(1+|\xi|^\alpha)|\Phi^X_{s,t}(\xi)|^2/|t-s|)] < +\infty.$$

□

Corollary B.3.3. *Let $\alpha \in (0,2]$ and let X a non-degenerate d-dimensional α-stable Lévy process. Then fro all $\rho < \frac{\alpha}{2}$ X is almost surely (ρ,γ)-irregular for all $\rho < \alpha$ and $\gamma > \frac{1}{2}$.*

Proof. This is an direct corollary of the third part of the Chapter 2, since $\|X\|_{L^\infty([0,T])} < +\infty$ almost surely. □

Remark B.3.4. In fact, in the demonstration of Proposition B.3.2 the independence of the increments is not needed. In Equation (B.1) we do not need an equality but only the inequality. For example if X is a gaussian process, we have

$$\mathbb{E}\left[e^{i\zeta.\sum_{j=1}^{2p-1}a^\sigma_j(X_{u_{j+1}}-X_{u_j})}\right] = \mathbb{E}\left[e^{-|\zeta|^2\operatorname{Var}\left(\sum_{j=1}^{2p-1}|a^\sigma|_j(X_{u_{j+1}}-X_{u_j})\right)/2}\right]$$

Hence, one only need to have

$$\operatorname{Var}\left(\sum_{j=1}^{2p-1}|a^\sigma_j|(X_{u_{j+1}}-X_{u_j})\right) \geqslant C_p\sum_{j=1}^{2p-1}|a^\sigma_j|\operatorname{Var}(X_{u_{j+1}}-X_{u_j}). \tag{B.2}$$

and it will allows to prove the (ρ,γ)-irregularity, but not the gaussian integrability of the quantity of proposition B.3.2. As the last inequality is fulfilled for the fractional Brownian motion, this is a cheap way to prove the ρ-irregularity of fractional Brownian motion paths. The property of Equation (B.2) is called Local non-determinism in Berman sense. It has been introduced by Berman in [7]. A very good survey about this notion can be found in [74].

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Résumé

Ce travail, à la frontière de l'analyse et des probabilités, s'intéresse à l'étude de systèmes différentiels a priori mal posés. Nous cherchons, grâce à des techniques issues de la théorie des chemins rugueux et de l'étude trajectorielle des processus stochastiques, à donner un sens à de tels systèmes puis à les résoudre, tout en montrant que les notions proposées ici étendent bien les notions classiques de solutions. Cette thèse se décompose en trois chapitres. Le premier traite des systèmes différentiels ordinaires perturbés additivement par des processus irréguliers éventuellement stochastiques ainsi que des effets de régularisation de tels processus. Le deuxième chapitre concerne l'équation de transport linéaire perturbée multiplicativement par des chemins rugueux ; enfin, le dernier chapitre s'intéresse à une équation de la chaleur non linéaire perturbée par un bruit blanc espace-temps, l'équation de quantisation stochastique Φ^4 en dimension 3.

Mot clés Integrale de Young, Chemins Contrôlés, Regularization by noise, Mouvement brownien Fractionaire, Equation différentielles stochastiquess, Équation différentielles partielles stochastiques, Chemins rugueux, Paraproduits, Espaces de Besov, Bruit blanc, Equation de quantisation stochastique.

Abstract

In this work we investigate a priori ill-posed differential systems from an analytic and probabilistic point of view. Thanks to technics inspired by the rough path theory and pathwise study of stochastic processes, we want to define those ill-posed systems and then study them. The first chapter of this thesis is related to ordinary differential equations perturbed by some irregular (stochastic) processes and the effects induced by the regularization of such processes. The second chapter deals with the linear transport equation multiplicatively perturbed by a rough path. Finally, in the last chapter we investigate the stochastic quantization equation ϕ^4 in three dimensions.

Keywords Young integral, Controlled Path, Regularization by noise, Fractional Brownian motion, Stochastic differential equation, Partial stochastic differential equation, Rough path, Paraproducts, Besov spaces, White noise, Stochastic quantisation equation.