Local regularity of some fractional Brownian fields

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Régularité locale de certains champs browniens fractionnaires

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Résumé

Dans cette thèse, nous examinons les propriétés de régularité locale de certains processus stochastiques multiparamètres définis sur $\mathbb{R}_+^N$, sur une collection d'ensembles, ou encore sur des fonctions de L^2. L'objectif est d'étendre certains outils standard de la théorie des processus stochastiques, en particulier concernant la régularité hölderienne locale, à des ensembles d'indexation qui ne sont pas totalement ordonnés.

Le critère de continuité de Kolmogorov donne classiquement une borne inférieure pour la régularité hölderienne d'un processus stochastique indicé par un sous-ensemble de $\mathbb{R}$ ou $\mathbb{R}^N$. Tirant partie de la structure de treillis des ensembles d'indexations dans la théorie des processus indicés par des ensembles de Ivanoff et Merzbach, nous étendons le critère de Kolmogorov dans ce cadre. Différents accroissements pour les processus indicés par des ensembles sont considérés, et leur sont attachés en conséquence des exposants de Hölder. Pour les processus gaussiens, ces exposants sont, presque surement et uniformément le long des trajectoires, déterministes et calculés en fonction de la loi des accroissements du processus. Ces résultats sont appliqués au mouvement brownien fractionnaire set-indexed, pour lequel la régularité est constante. Afin d'exhiber un processus pour lequel la régularité n'est pas constante, nous utilisons la structure d'espace de Wiener abstrait pour introduire un champ brownien fractionnaire indicé par $(0,1/2]\times L^2(T,m)$, relié à une famille de covariances k_h, $h\in(0,1/2]$. Ce formalisme permet de décrire un grand nombre de processus gaussiens fractionnaires, suivant le choix de l'espace métrique (T,m). Il est montré que la loi des accroissements d'un tel champ est majorée par une fonction des accroissements en chacun des deux paramètres. Les techniques développées pour mesurer la régularité locale s'appliquent alors pour prouver qu'il existe dans ce cadre des processus gaussiens indicés par des ensembles ou par L^2 ayant une régularité prescrite.

La dernière partie est consacrée à l'étude des singularités produites par le processus multiparamètre défini par k_h sur $L^2([0,1]^\nu, \mathrm{d}x)$. Ce processus est une extension naturelle du mouvement brownien fractionnaire et du drap brownien. Au point origine de $\mathbb{R}_+^N$, ce mouvement brownien fractionnaire multiparamètre possède une régularité hölderienne différente de celle observée en tout autre point qui ne soit pas sur les axes. Une loi du logarithme itéré de Chung permet d'observer finement cette différence.

Mots clés: champs aléatoires, processus multiparamètres et set-indexed, régularité hölderienne, mouvement brownien (multi)fractionnaire, mesures gaussiennes, espaces de Wiener abstraits, propriétés trajectorielles.

Abstract

In this thesis, local regularity properties of some multiparameter, set-indexed and eventually L^2-indexed random fields are investigated. The goal is to extend standard tools of the theory of stochastic processes, in particular local Hölder regularity, to indexing collection which are not totally ordered.

The classic Kolmogorov continuity criterion gives a lower estimate of the Hölder regularity of a stochastic process indexed by a subset of $\mathbb{R}$ or $\mathbb{R}^N$. Using the lattice structure of the indexing collections in the theory of set-indexed processes of Ivanoff and Merzbach, Kolmogorov's criterion is extended to this framework. Different increments for set-indexed processes are considered, and several Hölder exponents are defined accordingly. For Gaussian processes, these exponents are, almost surely and uniformly along the sample paths, deterministic and related to the law of the increments of the process. This is applied to the set-indexed fractional Brownian motion, for which the regularity is constant. In order to exhibit a process having a variable regularity, we resorted to structures of Abstract Wiener Spaces, and defined a fractional Brownian field indexed by a product space $(0,1/2] \times L^2(T,m)$, based on a family of positive definite kernels k_h, $h \in (0,1/2]$. This field encompasses a large class of existing multiparameter fractional Brownian processes, which are exhibited by choosing appropriate metric spaces (T,m). It is proven that the law of the increments of such a field is bounded above by a function of the increments in both parameters of the field. Applying the techniques developed to measure the local Hölder regularity, it is proven that this field can lead to a set-indexed, or L^2-indexed, Gaussian process with prescribed local regularity.

The last part is devoted to the study of the singularities induced by the multiparameter process defined by the covariance k_h on $L^2([0,1]^\nu, \mathrm{d}x)$. This process is a natural extension of the fractional Brownian motion and of the Brownian sheet. At the origin 0 of $\mathbb{R}_+^N$, this multiparameter fractional Brownian motion has a different regularity behaviour. A Chung (or *lim inf*) law of the iterated logarithm permits to observe this.

Key words: random fields, multiparameter and set-indexed processes, Hölder regularity, (multi)fractional Brownian motion, Gaussian measures, Abstract Wiener Spaces, sample path properties.

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Introduction 1

Contents

The work presented here intersects several domains in the theory of stochastic processes, and is concerned in particular with multiparameter and set-indexed processes, fractional processes and their sample path properties. In terms that will be explained in this introduction, the main question I tried to answer during my PhD (and that branched in several directions) can be formulated like this: *is it possible to define a fractional Brownian field with prescribed regularity on a general metric space?*

The study of stochastic processes is one of the oldest topic in probability theory, tracing back to the early discovery of Brownian motion in the 19$^{\text{th}}$ century, attributed surprisingly to the botanist BROWN (1828), and then popularized by BACHELIER (1900), EINSTEIN (1905) and SCHMOLUCHOWSKI (1906), before the rigorous mathematical foundation of WIENER [149] in 1923. The diversity of disciplines Brownian motion has appeared in since the beginning is striking: biology, economy, physics, mathematics. One of the reasons for its success is the simplicity in its definition and use, while it accounts for various invariance properties and limit theorems. Brownian motion $\{B_t, t \in \mathbb{R}_+\}$ is the Gaussian process started at 0 with stationary increments:

$$\forall s \geq 0, \quad \{B_{t+s} - B_s, t \in \mathbb{R}_+\} \overset{(d)}{=} \{B_t, t \in \mathbb{R}_+\} \ ,$$

and (statistical) self-similarity of order 1/2:

$$\forall a > 0, \quad \left\{a^{-1/2} B_{at}, t \in \mathbb{R}_+\right\} \overset{(d)}{=} \{B_t, t \in \mathbb{R}_+\} \ .$$

As such, the sample paths of Brownian motion are random objects, whose fractal nature is closely determined by the self-similarity index 1/2. Driven by the observation in turbulent fluid dynamics of a different self-similarity index, KOLMOGOROV [81] defined what he called Wiener spirals, by changing the self-similarity index into any parameter γ between 0 and 1. A few years later, another physical observation led the hydrologist HURST [66] to notice that the Central Limit Theorem failed in the statistical study of the run-offs of the Nile river. To resolve this issue, Hurst proposed a different scaling in n^H (n being the size of his dataset) with $H \neq 1/2$. This phenomenon was later explained with the help of fractional Gaussian noises. The letter H was inherited from Hurst, replacing often γ. Later, MANDELBROT [98] coined the term *fractional* Brownian motion, referring to the fractal nature of this process, and gave a stochastic integral representation:

$$B_t^H = \frac{1}{\Gamma(H+1/2)} \int_{\mathbb{R}} \left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right) \mathrm{d}B_s \,, \quad t \in \mathbb{R}_+ \,,$$

where $x_+ = x \vee 0$. Compared to the Brownian motion ($H = 1/2$), the fractional Brownian motion loses several properties: independence of increments, martingality, Markov property; but on the other hand it provides more flexibility, notably as a modelling tool. Indeed its increments can now express long or short-range dependence, the self-similarity can be chosen in $(0,1)$, etc. Here, we shall focus on the continuity properties of the fractional Brownian motion, whose sample paths are Hölder continuous of order "almost" H, and of its generalizations to more general indexing sets.

In the remainder of this introduction, we present some tools and important results, as well as essential references, that are necessary to understand the next chapters. They concern some elements of the theory of multiparameter, and most of all, set-indexed processes; a presentation of some Gaussian extensions of the Brownian motion, with important results in the regularity theory of the sample paths of random fields; and a brief account of the theory of Gaussian measures, in particular in abstract Wiener spaces. Hopefully, these reminders will permit to make a self-contained presentation of our results, in the last part of this introduction.

1.1 Multiparameter and set-indexed processes

Questions related to the study of stochastic processes with a multidimensional (or even more abstract) parameter set started to appear in the 1940's. It is rather late if we compare it to the development of multivariate and functional analysis at that time, but quite early compared to the rise of modern probability, with the axiomatic of Kolmogorov in the 1930's. It is indeed as natural to study multiparameter processes as is to study functions of several variables. Multiparameter processes allow to model more complex phenomena, and not all physical experiences depend on a single "time" parameter. A typical example is the Poisson equation (that appears in gravitation, electrostatics, etc.), which is a partial differential equation with no notion of time. For that matter, there has been an increasing interest in stochastic partial differential equations over the last thirty years, starting with the pioneering work of WALSH [147]. Perhaps more importantly, generalizing stochastic processes to random fields raises many theoretical questions that were irrelevant in the one-parameter case, one of which is the loss of total order: what does the Markov property become? The martingale property? What does a Brownian motion look like? Does it have a single extension? etc.

Historically, the first appearance of a multiparameter stochastic process is the Lévy Brownian motion, which owes its name to Paul LÉVY [92]. The Brownian sheet followed quickly after, with the work of the statistician KITAGAWA [80]. Both were studied a lot since then, but it seems that the latter received more attention, due perhaps to its many uses in applications. A property that attracted early the attention of probabilists is the Markov property. LÉVY [91] defined the following property: let $\{X_t,\ t \in \mathbb{R}_+^N\}$ be a multiparameter process and put, for any Borel set U in $\mathbb{R}_+^N$, the σ-algebra $\mathscr{F}_U = \sigma(\{X_t,\ t \in U\})$. X is *sharp Markov* with respect to a Borel set $U \in \mathbb{R}_+^N$ if $\mathscr{F}_U$ and $\mathscr{F}_{U^c}$ are conditionally independent with respect to $\mathscr{F}_{\partial U}$, where U^c and ∂U denote respectively the complementary set of U in $\mathbb{R}_+^N$ and its boundary. To the question whether extensions of classical one-parameter Markov processes are sharp Markov, RUSSO [124] gave a positive answer for processes with independent increments, which are sharp Markov with respect to finite unions of rectangleswith sides parallel to the axes. For these processes, DALANG AND WALSH [35] characterised completely the sets for which this Markov property holds. Observing that certain important processes were not sharp Markov for elementary sets, such as the Brownian sheet which is not Markov with respect to a simple triangle with vertices $(0,0)$, $(0,1)$, $(1,0)$ in $\mathbb{R}_+^2$, MCKEAN [102] defined a weaker Markov property known as *germ-Markov*, by substituting $\cap_{\epsilon>0}\sigma\{X_t,\ d(t,U)<\epsilon\}$ for $\mathscr{F}_{\partial U}$. As a consequence of this definition, RUSSO [124] proved that the Brownian sheet is germ-Markov with respect to any open set, and MERZBACH AND NUALART [104] proved that for point processes, the germ-Markov property is equivalent to the sharp Markov property. Note that other definitions for multiparameter Markov properties were given, that we do not discuss here.

Multiparameter martingales were initially studied on discrete subsets of $\mathbb{R}_+^N$ by CAIROLI [26] in the early 1970's. A detailed overview of this theory can be found in [76]. Soon after, WONG AND ZAKAI [150] and CAIROLI AND WALSH [27, 148] introduced martingales in the plane and developed a stochastic calculus for multiparameter processes. As for Markov processes, different notions of "history" yield different type of martingales. See [68] for an survey of these different kinds of history in a non-totally ordered setting.

Alongside were studied processes indexed by subsets of the Euclidean space, primarily as Gaussian processes with CHENTSOV [30], but then from the point of view of empirical processes (see DUDLEY [45] and many other after him). Many questions related to processes indexed by sets of $\mathbb{R}^N$ and empirical statistics can be found in the review made by PYKE [118] on limit theorems for empirical and partial-sum processes. Appearing as limit of these statistical processes, we mention also some works on sample path properties of Gaussian processes indexed by sets due to ALEXANDER [8], and Lévy processes indexed by sets in BASS AND PYKE [19]. Note that among processes indexed by sets, Gaussian noises (as in [45, 8]) will play a particular role in this thesis: on a measure space $(T, \mathscr{T}, m)$, a Gaussian white noise M with control measure m is a centred Gaussian process indexed by sets of $\mathscr{T}$ with the following covariance:

$$\mathbb{E}\left(M(A)\, M(B)\right) = m(A \cap B)\ , \quad A, B \in \mathscr{T}\ . \tag{1.1}$$

These objects will appear in numerous situations, especially in relation with multiparameter processes and the definition of Wiener integrals.

Paragraphs 1.1.1 and 1.1.2 deal with set-indexed processes in the sense of IVANOFF AND MERZBACH [70], which is implicitely the sense given to "set-indexed" from now on. In case a process would be indexed by sets not pertaining to this theory, we would call it a process "indexed by sets".

1.1.1 *The set-indexed framework*

The theory of set-indexed processes of Ivanoff and Merzbach appeared as the will to give a common treatment to two different fields of research: one that focused on processes indexed by sets, such as spatial processes (Poisson and point processes, in particular) and random measures; and another one that studied martingales indexed by directed sets. One of the major constraint that appeared in the construction of this theory was to reconcile, on the one hand, the need to have a collection of sets which is rich enough so as to generate the Borel sets of the underlying metric space; and on the other hand, not letting it be too large, in order to have processes with "good" modifications (càdlàg, continuous, Hölder continuous, etc.). The framework of IVANOFF AND MERZBACH [70] is a synthesis that provides a general structure. Besides, it has many applications in spatial statistics, empirical processes, stochastic geometry, random measures, etc. for which we refer again to the book [70]. Recently, new set-indexed processes were defined: set-indexed Lévy processes [63] and set-indexed fractional [60] and multifractional Brownian motions [121]. These last two processes will be studied extensively in Chapters 2, 3 and 4 of this thesis.

The set-indexed framework was designed for the needs presented above, which sometimes go beyond what will be necessary in this thesis, which is why we drop here some of the assumptions of [70]. The interested reader might have a look there for more details.
Let T be a locally compact complete separable metric and measure space, with metric d and Radon measure m defined on the Borel sets of T.

Definition 1.1 (adapted from [70]). *A nonempty class $\mathscr{A}$ of compact, connected subsets of T is called an* indexing collection *if it satisfies the following:*

1. *$\emptyset \in \mathscr{A}$ and for all $A \in \mathscr{A}$, $A^\circ \neq A$ if $A \notin \{\emptyset, T\}$.*
2. *$\mathscr{A}$ is closed under arbitrary intersections and if $A, B \in \mathscr{A}$ are nonempty, then $A \cap B$ is nonempty. If (A_i) is an increasing sequence in $\mathscr{A}$ then $\overline{\bigcup_i A_i} \in \mathscr{A}$.*
3. *The σ-algebra $\sigma(\mathscr{A})$ generated by $\mathscr{A}$ is the collection $\mathscr{B}$ of all Borel sets of T.*
4. Separability from above*: There exists an increasing sequence of finite subclasses $\mathscr{A}_n = \{\emptyset, A_1^n, \dots, A_{k_n}^n\}$ ($n \in \mathbb{N}, k_n \geq 1$) of $\mathscr{A}$ closed under intersections and a sequence of functions $g_n : \mathscr{A} \to \mathscr{A}_n$ defined by*
$$\forall U \in \mathscr{A}, \quad g_n(U) = \bigcap_{\substack{V \in \mathscr{A}_n \\ V^\circ \supseteq U}} V$$
and such that for each $U \in \mathscr{A}$, $U = \bigcap_{n \in \mathbb{N}} g_n(U)$.

(Note: '$\overline{(\cdot)}$' and'$(\cdot)^\circ$' denote respectively the closure and the interior of a set.)

The g_n's will play the role of dyadic approximation, and are particularly relevant for the study of the regularity of set-indexed processes, as we will see in Chapter 2. This definition implies that an indexing collection cannot be dsicrete, because of the separability from above. We might use several times the notation $\mathscr{A}(u)$ to indicate the set of finite unions of elements of $\mathscr{A}$. Let us present a few examples of indexing collections:

- The first one produces the link with multiparameter processes. Let $T = \mathbb{R}_+^N$ and $\mathscr{A} = \{[0,t],\ t \in T\}$, where $[0,t] = \left\{s \in \mathbb{R}_+^N :\ s \preccurlyeq t\right\}$ denotes the rectangle of all points between 0 and t (for the partial order). Processes indexed by $\mathscr{A}$ are equivalent to those defined on T by the relation:
$$X_t = X([0,t]) \ , \tag{1.2}$$
In the sequel, $\mathscr{A}$ will often be referred to as the class of rectangles. Hence, any multiparameter process can be seen as a set-indexed processes. Put $\mathscr{A}_n = \{[0,t] :\ t_i = k_i 2^{-n},\ k_i = 0, \dots, 2^{2n},\ i = 1, \dots, N\}$. It is easy to verify that $\mathscr{A}$ satisfies all the assumptions in the previous definition.

- It is also possible to consider $T = [-1,1]^N$ or $T = \mathbb{R}^N$ with the indexing collection $\mathscr{A} = \{[0,t],\ t \in T\}$.
- The lower layers $\mathscr{A}$ on $T = [0,1]^N$ are those compact subsets A of T such that $A \supseteq [0,t],\ \forall t \in A$. For the lower layers, $\mathscr{A} = \mathscr{A}(u)$ which gives a hint that this class is very rich. Indeed, we will see that processes such as Brownian motion are not continuous on $\mathscr{A}$.
- Let $T = [0,\tau] \times S_N$, where $\tau > 0$ and S_N is the N-dimensional sphere, and let $\mathscr{A} = \{[0,t] \times U(\vartheta) :\ t \in [0,\tau],\ \vartheta \in [0,\pi]^{N-1} \times [0,2\pi]\}$, with $U(\vartheta) = \{t \in S_N :\ t_i \leq \vartheta_i,\ i = 1,\dots,N\}$ where t_i are the angular coordinates. This collection can be interpreted as the history of the regions $U(\vartheta)$ of the sphere.

Let us present two types of objects of particular importance in the set-indexed theory: the class $\mathscr{C}$ and flows. The first one is an important indexing collection to study the *increments* of set-indexed processes, built from $\mathscr{A}$:

$$\mathscr{C} = \left\{ C = A \setminus \cup_{i=1}^n A_i,\ A, A_1, \dots, A_n \in \mathscr{A},\ n \in \mathbb{N} \right\} .$$

Indeed, if X is an $\mathscr{A}$-indexed process, ΔX is its increment process defined by the inclusion-exclusion process:

$$\Delta X_C = X_A - \sum_{k=1}^{n} \sum_{j_1 < \dots < j_k} (-1)^{k-1} X_{A \cap A_{j_1} \cap \dots \cap A_{j_k}} .$$

This increment process permits to define set-indexed martingales and Markov processes, and we will study the Hölder continuity of processes with respect to these increments. As for $\mathscr{A}(u)$, $\mathscr{C}(u)$ indicates the set of finite unions of elements of $\mathscr{C}$.
The second one is the concept of *flow*. A flow is an increasing and continuous path f from $[0,1]$ to $\mathscr{A}$ or $\mathscr{A}(u)$. It is often useful to find characterizations of set-indexed processes by the properties of their projections on any flow. For instance, the classical martingale characterizations of the Poisson process (due to Watanabe) and of the Brownian motion (due to Lévy) extend to their set-indexed equivalents, and the proofs rely on flows. If X is an $\mathscr{A}$-indexed process and f is a flow, the projection of X on f is defined as $X_t^f = X_{f(t)}$.

1.1.2 *Continuity of set-indexed Markov processes*

In the set-indexed framework, the Markov property appeared in Ivanoff and Merzbach [69] as an extension of the multiparameter sharp Markov property (in fact two other extensions are considered). With a view to constructing set-indexed Markov processes, Balan and Ivanoff [14] introduced $\mathscr{Q}$-Markov processes, $\mathscr{Q}$ referring to a transition system. In this context, Balan [13] extended a result of Bass and Pyke [19], proving that for stochastically continuous Markov processes with independent increments, i.e. Lévy processes, a criterion mixing metric entropy of the collection of sets and decay of the tails of the Lévy measure implies the existence of a right continuous modification of such processes. We recall the following definitions:

Definition 1.2 ($\mathscr{Q}$-Markov process)**.** *Let $\mathscr{Q}$ be a transition system, i.e. a system of transition probabilities $\{Q_{B,B'}(x,\Gamma),\ x \in \mathbb{R}, \Gamma \in \mathscr{B}(\mathbb{R}),\ B, B' \in \mathscr{A}(u)\}$ satisfying a Chapman-Kolmogorov condition, and let $X = \{X_U, U \in \mathscr{A}\}$ be a process with a unique extension on $\mathscr{C}(u)$ and $(\mathscr{F}_U)_{U \in \mathscr{A}}$ its minimal filtration. X is a $\mathscr{Q}$-Markov process if for all $U \subseteq V \in \mathscr{A}(u)$,*

$$\forall B \in \mathscr{B}(\mathbb{R}), \quad \mathbb{P}(\Delta X_V \in B | \mathscr{F}_U) = Q_{U,V}(\Delta X_U, B),$$

where for $U \in \mathscr{A}(u)$, $\mathscr{F}_U = \bigvee_{\substack{V \in \mathscr{A} \\ V \subseteq U}} \mathscr{F}_V$.

Definition 1.3 (Inner limit-Outer continuity)**.** *Let* Σ *be a sub-indexing collection of* $\mathscr{A}(u)$. *A set-indexed function* $x : \mathscr{A} \to \mathbb{R}$ *is said to satisfy the* Σ*-ILOC property (Inner Limit and Outer Continuity) if it admits an extension* Δx *on* Σ *for which, for any* $t \in \mathscr{T}$, *there exist a real number* $\Delta x_{A_t^-}$ *such that:*
$\forall \epsilon > 0$, *there exist* $\delta_t > 0$ *and* $\eta_t > 0$ *such that*

$$\forall V \in \Sigma \text{ with } V \subset A_t \setminus \{t\},\ m(A_t \setminus V) < \delta_t \Rightarrow |\Delta x_V - \Delta x_{A_t^-}| < \epsilon,$$

and

$$\forall W \in \Sigma \text{ with } A_t \subseteq W,\ m(W \setminus A_t) < \eta_t \Rightarrow |\Delta x_W - \Delta x_{A_t}| < \epsilon.$$

In [63], set-indexed Lévy processes are constructed so as to have this property.

We have made some steps towards the proof of existence of ILOC modifications of a class of $\mathscr{Q}$-Markov processes that would correspond to Feller processes in the real-parameter setting, in the sense that they can be approximated by set-indexed pseudo-Poisson processes (see [74, Chapter 17] for approximation of Feller processes by pseudo-Poisson processes). However the proof is incomplete and we chose not to present it here, but we thought it was important to mention that there was a gap in the theory. We also mention a result of BALANÇA [15] on $\mathscr{C}$-Markov processes (another recently defined class of set-indexed Markov processes), who proved the existence of outer-continuous modifications for *multiparameter* $\mathscr{C}$-Feller processes.

1.2 Fractional Brownian motion and its extensions

1.2.1 *Multiparameter and set-indexed extensions*

We gave in the previous paragraph the origins of multiparameter and white noise extensions of Brownian motion. We now present carefully the Lévy Brownian motion and the Brownian sheet, which are the two distinct multiparameter extensions of the Brownian motion and we will see that they can be both expressed as processes indexed by sets of $\mathbb{R}^N$, $N \in \mathbb{N}^*$. Then, we give a similar description of the multiparameter extensions of fractional Brownian motion, with set-indexed representations, but this time there will be three distinct extensions.

The Brownian sheet is a tensorized process, in the sense that along any direction parallel to an axis of $\mathbb{R}^N$, it is a Brownian motion. It is defined as the centred[1] Gaussian process over $\mathbb{R}_+^N$ with covariance:

$$\mathbb{E}(W_s\, W_t) = \prod_{k=1}^{N} (s_k \wedge t_k)\ , \quad s,t \in \mathbb{R}_+^N\ ,$$

where $s_1, \ldots, s_N$ are the coordinates of the point s. The Brownian sheet can be viewed as a set-indexed process via the analogy that we describe now. Let $\mathbb{W}$ be the Gaussian process indexed by Borel sets of $\mathbb{R}_+^N$ with covariance:

$$\mathbb{E}(\mathbb{W}(U)\, \mathbb{W}(V)) = \lambda(U \cap V)\ , \quad U,V \in \mathscr{B}(\mathbb{R}_+^N)\ , \tag{1.3}$$

[1] We point out that all Gaussian processes will be centred, whether it is explicitely stated or not. This is only a matter of convenience, since the addition of a mean function will not be of any trouble.

where λ is the N-dimensional Lebesgue measure. The existence of this process is ensured by the positive definiteness of $(U,V) \mapsto \lambda(U \cap V)$. $\mathbb{W}$ is known as *white noise*[2]. Finally, one can easily verify using (1.3) that $W_t = \mathbb{W}([0,t])$, defined for $t \in \mathbb{R}_+^N$, is a Brownian sheet.

The other multiparameter Brownian motion we mentioned is the Lévy Brownian motion. It is a multiparameter Gaussian process with incremental variance $\|s-t\|$ and stationary increments in $\mathbb{R}^N$. Its covariance then reads:

$$\mathbb{E}(X_s\, X_t) = \frac{1}{2}\left(\|s\| + \|t\| - \|s-t\|\right) \ , \quad s,t \in \mathbb{R}^N \ ,$$

where $\|\cdot\|$ is the Euclidean norm. Perhaps less standard is the representation of Lévy Brownian motion as a set-indexed process. Let us describe the original construction that CHENTSOV [30] provided in the late 50's. Let S_N denote the unit sphere of $\mathbb{R}^N$ ant put $(S, \mathscr{S}, \mu)$ the measure space made of $S = S_N \times (0,\infty)$, $\mathscr{S}$ the Borel sets of S and μ the product measure of the uniform measure on S_N with the Lebesgue measure on $(0,\infty)$. For any $t \in \mathbb{R}^N$, We consider sets in $\mathscr{S}$ of the form:

$$U_t = \{(s,r) \in S : \ r < (s,t)_N\} \ ,$$

where $(\cdot,\cdot)_N$ denotes the Euclidean scalar product of $\mathbb{R}^N$. This set has a nice geometric interpretation in terms of hyperplanes separating 0 and the point t (see [127, p.400] for details). It can be shown that, for some positive constant c_N,

$$\mu(U_t) = c_N \|t\| \ \text{ and } \ \mu(U_t \bigtriangleup U_s) = c_N \|t-s\| \ ,$$

where $U_t \bigtriangleup U_s = (U_t \setminus U_s) \cup (U_s \setminus U_t)$ is the symmetric difference of sets. We renormalize μ so that $\mu(U_t) = \|t\|$. Let M be the Gaussian white noise on $\mathscr{S}$ with control measure μ. Then,

$$\begin{aligned}\mathbb{E}(X_t - X_s)^2 = \mathbb{E}(M(U_t) - M(U_s))^2 &= \mu(U_t \bigtriangleup U_s)\\ &= \|t-s\| \ .\end{aligned}$$

Note that X is indeed a special instance of white noise (1.1) since its covariance reads:

$$\mathbb{E}(X_t\, X_s) = \frac{1}{2}\left(\mu(U_t) + \mu(U_s) - \mu(U_t \bigtriangleup U_s)\right) = \mu(U_t \cap U_s) \ .$$

Fractional processes will be the cental in this thesis. They represent a large class of processes that encloses Brownian motion, and despite the natural apppearance of the latter in many probabilistic phenomena, it can be interesting, from a theoretical or modelling point of view, to relax certain hypothesis such as independence of increments, or simply change the order of self-similarity. For these reasons, the class of fractional Brownian motions offers a wide range of behaviours, in particular on their sample paths (for instance their modulus of continuity involve the Hurst parameter, see Section 1.4.4) but also on their statistical properties (such as long-range dependence). From the theoretical point of view, the loss of several properties[3], such as martingale and Markov property (except for Brownian motion), implies that proofs of results on fractional Brownian motions must rely on different tools than those used for standard Brownian motion. For instance, the law of the iterated logarithm proved by CHUNG in the late 40's used independence of increments[4]. This theorem was extended to fractional Brownian motion

[2] In the literature, white noise often refers to this process on $\mathbb{R}_+^N$, rather than the more general one defined by (1.1), however in this thesis we will encounter different white noises. Note that we consider only Gaussian noises.

[3] A good review of these properties (or lack of properties) can be found in the monograph by NOURDIN [109].

[4] In fact, this result concerned sums of independent random variables, but still holds for Brownian motion, by invariance.

almost half a century later [106], using completely different technique of local nondeterminism and small deviations.

Let us recall briefly that the fractional Brownian motion of order $H \in (0,1]$ is the only H-self-similar Gaussian process with stationary increments, extending the definition given for Brownian motion at the beginning of this thesis. We shall write it B^H, and it has covariance:

$$\mathbb{E}\left(B_t^H\, B_s^H\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)\ , \quad t,s \in \mathbb{R}\ .$$

For $H = 1/2$, notice that this is a Brownian motion.
The Lévy fractional Brownian motion is the fractional process associated to the Lévy Brownian motion, whose covariance is:

$$\mathbb{E}\left(X_t^H X_s^H\right) = \frac{1}{2}\left(\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H}\right)\ .$$

This is positive definite for any $H \in (0,1]$, according to Proposition A.4. Regarding the representation of Lévy Brownian motion ($H = 1/2$) as a process indexed by sets, the Lévy fractional Brownian motion is also a process indexed by the subsets of $\mathscr{S}$:

$$\mathbb{E}\left(X_t^H X_s^H\right) = \frac{1}{2}\left(\mu(U_t)^{2H} + \mu(U_s)^{2H} - \mu(U_s \triangle U_t)^{2H}\right)\ .$$

A lot is known on the sample paths of the Lévy fBm. However, so far as we know, it is not very much used in applications. This contrasts with the class of Brownian sheets, perhaps due to the isotropy of the first one and the tensorized structure of the latter. The fractional Brownian sheet appeared in KAMONT [75]. For a vector $\mathbf{H} = (H_1,\ldots,H_N) \in (0,1]^N$ of Hurst parameters, its covariance reads:

$$\mathbb{E}\left(W_t^{\mathbf{H}}\, W_s^{\mathbf{H}}\right) = \frac{1}{2^N}\prod_{k=1}^{N}\left(|t_k|^{2H_k} + |s_k|^{2H_k} - |t_k - s_k|^{2H_k}\right)$$

Unlike the Brownian sheet ($\mathbf{H} = (1/2,\ldots,1/2)$), we hardly see how the fractional Brownian sheet could be related to a set-indexed process. Yet it is again possible by means of Takenaka measures. Let us present now TAKENAKA's construction [137] of fractional Brownian motion. The underlying measure space is the set Z_N of spheres of $\mathbb{R}^N$ parametrized by their center and radius $(x,r) \in \mathbb{R}^N \times \mathbb{R}_+$, with its Borel σ-algebra $\mathscr{Z}_N$. The indexing sets are given by:

$$V_t = \{(x,r) \in Z :\ \|x\| \le r\} \triangle \{(x,r) \in Z :\ \|x-t\| \le r\}\ ,$$

which can be interpreted as the set of spheres that separate 0 and t (we refer again to [127, pp.402–405]). Then, for any $H < 1/2$, we define the measure $\mu_{N,H}$ on $\mathscr{Z}_N$:

$$\mu_{N,H}(\mathrm{d}x,\mathrm{d}r) = r^{2H-N-1}\mathrm{d}x\mathrm{d}r\ .$$

A correct renormalization of $\mu_{N,H}$ (that is still denoted by $\mu_{N,H}$) gives:

$$\mu_{N,H}(V_s \triangle V_t) = \|t-s\|^{2H}\ ,\ s,t \in \mathbb{R}^N\ .$$

Thus, we obtain another set-indexed representation for the Lévy fractional Brownian motion by defining the white noise on $\mathscr{Z}_N$ with control measure $\mu_{N,H}$. However, we now fix $N = 1$ and another integer $\nu \in \mathbb{N}^*$, and for a Hurst vector $\mathbf{H} = (H_1,\ldots,H_\nu) \in (0,1/2)^\nu$ we tensorize the measure spaces $(Z_1, \mathscr{Z}_1, m_{H_1}), \ldots, (Z_1, \mathscr{Z}_1, m_{H_\nu})$ and denote by $\mu_{\mathbf{H}}$ the tensorized measure. We have, for $t_1, \ldots, t_\nu, s_1, \ldots, s_\nu \in \mathbb{R}$:

$$\begin{aligned}\mu_{\mathbf{H}}\left(V_{t_1} \times \cdots \times V_{t_n} \cap V_{s_1} \times \cdots \times V_{s_n}\right) &= \prod_{k=1}^{\nu} \mu_{1,H_k}(V_{t_k} \cap V_{s_k}) \\ &= \mathbb{E}\left(W_t^{\mathbf{H}}\, W_s^{\mathbf{H}}\right) ,\end{aligned}$$

which shows that $W_t^{\mathbf{H}} = \mathbb{W}^{\mathbf{H}}(V_t)$, where $V_t = V_{t_1} \times \cdots \times V_{t_n}$ and $\mathbb{W}^{\mathbf{H}}$ is the white noise with control measure $\mu_{\mathbf{H}}$.

All these processes can be considered as particular cases of a class of Gaussian processes defined on some measure space $(T, \mathscr{T}, m)$, with covariance k_H:

$$k_H(U,V) = \frac{1}{2}\left(m(U)^H + m(V)^H - m(U \triangle V)^H\right)$$

for $H \in (0,1]$. Such covariances were defined and proved to be positive definite in Herbin and Merzbach [60], and used in the context of set-indexed processes. By extrapolation, we will use them on the space of square integrable functions $L^2(T,m)$, for the reason that $m(U \triangle V) = \|\mathbf{1}_U - \mathbf{1}_V\|_{L^2}^2$, that we also denote by $m\left((\mathbf{1}_U - \mathbf{1}_V)^2\right)^2$. For any $f, g \in L^2(T,m)$, this reads:

$$k_h(f,g) = \frac{1}{2}\left(m(f^2)^{2h} + m(g^2)^{2h} - m\left((f-g)^2\right)^{2h}\right) \tag{1.4}$$

for $h \in (0,1/2]$. We changed the notation from H to h in order to emphasize the fact h does not belong to $(0,1]$. In fact, the set-indexed fractional Brownian motion is originally defined with this latter convention in [60], for it has the additional advantage to look more like the covariance of the usual fractional Brownian motion, with a power $2h$ in the covariance. Yet, in terms of $L^2(T,m)$ norm, the power is to the $4h$.

Finally, there is a new multiparameter process defined easily from this covariance, with covariance on the rectangles of $\mathbb{R}^N$:

$$k_h^{(N)}(s,t) = \frac{1}{2}\left(\lambda([0,s])^{2h} + \lambda([0,t])^{2h} - \lambda([0,s] \triangle [0,t])^{2h}\right) , \tag{1.5}$$

where λ is the Lebesgue measure on $\mathbb{R}^N$. When $N = 1$, this process is a standard fractional brownian motion of parameter h (and $k_h^{(1)}$ is still a covariance for $h > 1/2$). Hence, the process $\boldsymbol{B}^h$ having covariance $k_h^{(N)}$ is a third extension of fractional Brownian motion. Unlike the Lévy fBm and the fractional Brownian sheet, it is defined only for $h \leq 1/2$. Besides, $k_{1/2}^{(N)}$ is the covariance of the Brownian sheet, so that $\boldsymbol{B}^h$ can also be seen as a fractional extension of the Brownian sheet. A thorough comparison of these three processes is carried out in [61].
The multiparameter process $\boldsymbol{B}^h$ will be the object of interest in Chapter 5, and will be referred to as *multiparameter fractional Brownian motion*.

Remark 1.4. *In the sequel, we will often consider multiparameter and set-indexed Gaussian processes as processes indexed by functions of L^2 spaces, for the reasons exposed above. Processes with covariance k_h, without necesarily specifying the underlying measure space, will be called L^2-fractional Brownian motion and denoted by $\boldsymbol{B}^h$. Note that we keep the same notation for the multiparameter and the general $L^2(T,m)$-indexed process. Hopefully, the context will be clear enough for the reader to distinguish them.*

1.2.2 *Fractional Brownian fields*

Besides multiparameter extensions, other directions appeared in the literature to generalize fractional Brownian motion. If one wishes to keep the Gaussian nature of this process, it is for instance possible to weaken the increment stationarity assumption, which can be useful in applications. One may also want to let the Hurst parameter vary along the sample paths (to change the self-similarity and regularity). This led to the definition of the multifractional Brownian motion in the 90's, independently by PELTIER AND LÉVY VÉHEL [116] and BENASSI, JAFFARD AND ROUX [20], a process that behaves locally as a fractional Brownian motion. We will study local behaviours, such as local self-similarity, in Chapter 4.

Another related question is the statistical estimation of the Hurst parameter in models based upon fractional Brownian motion. It was studied in several articles, including recently [25] which gives an exact confidence interval. Given an estimation $\hat{H}$ of the Hurst parameter, we might have to plug this parameter in a model, for instance a functional of some fractional Brownian motion with hidden Hurst parameter H_0. It is therefore important to know how the estimation error influences the law of the model. We refer to the works of JOLIS ET AL. ([73] among a series of related articles) for (multiple) integrals against fractional Brownian motion and their approximation in law as $\hat{H}$ approaches H_0. Ou result of Chapter 3 can be interpreted in that sense.

For a function $H : \mathbb{R}_+ \to (0,1]$, one way to define a *multifractional Brownian motion* is to use Mandelbrot's integral representation, in which the constant parameter H is replaced by the function $H(t)$:

$$B_t^{H(t)} = \frac{1}{\Gamma(H(t)+1/2)} \int_{\mathbb{R}} \left((t-s)_+^{H(t)-1/2} - (-s)_+^{H(t)-1/2} \right) \mathrm{d}B_s \,, \quad t \in \mathbb{R}_+ \,. \tag{1.6}$$

This integral is in the sense of Wiener, with a deterministic integrand belonging to $L^2(\mathbb{R})$. Unless stated otherwise, this will be the case of most of the stochastic integrals encountered here.

The following figure represents a sample path of a multifractional Brownian motion with a sinusoidal Hurst function:

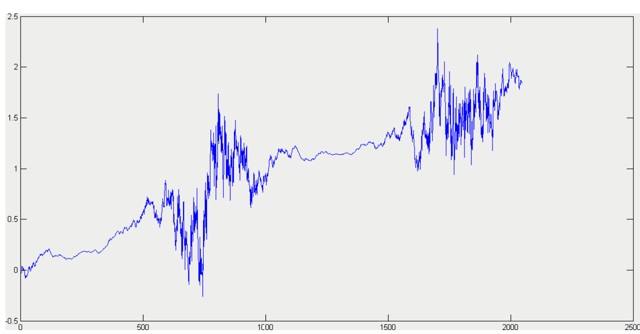

Figure 1.1 – A multifractional Brownian motion with sinusoidal Hurst function

In (1.6), we can even let H vary independently of t. The result is a Gaussian process indexed by the product set $\mathbb{R}_+ \times (0,1]$, called fractional Brownian field in this thesis.

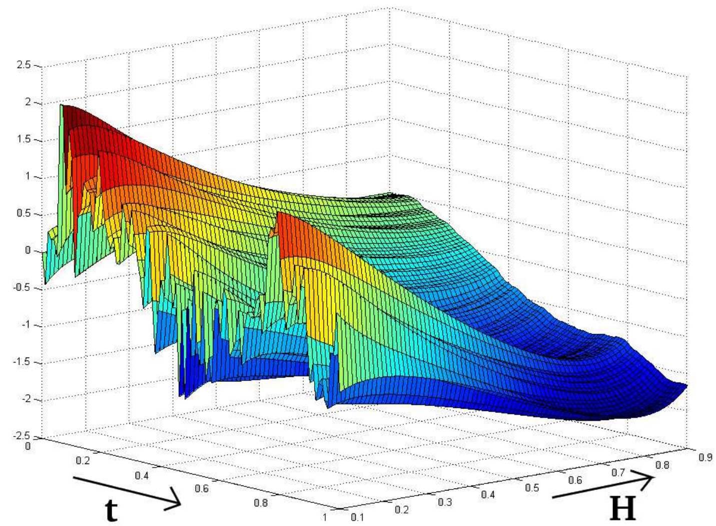

Figure 1.2 – A sample path of a fractional Brownian field

The graph of the above realization of a fractional Brownian field appears to be continuous. Proving the existence of (Hölder) continuous modifications of stochastic processes will be a reccurrent problem in the next chapters.
By means of integral representations, other fractional fields can be defined, especially for the multiparameter extensions of fractional Brownian motion (we refer in particular to HERBIN [57], who studied multifractional Brownian sheets and Lévy multifractional Brownian motions). Instead of the multifractional process, we give the fractional fields:

- the Lévy fractional Brownian field:

$$X_t^h = c_h \int_{\mathbb{R}^N} \left(\|t-u\|^{h-N/2} - \|u\|^{h-N/2} \right) \mathrm{d}\mathbb{W}_u \ , \ t \in \mathbb{R}^N, h \in (0,1] \ ;$$

- the fractional Brownian sheet with varying Hurst parameter:

$$W_t^{\mathbf{h}} = c_{\mathbf{h}} \int_{\mathbb{R}^N} \prod_{n=1}^{N} \left(|t_n - u_n|^{h_n - 1/2} - |u|^{h_n - 1/2} \right) \mathrm{d}\mathbb{W}_u \ , \ t \in \mathbb{R}_+^N, \ \mathbf{h} \in (0,1]^N \ ,$$

where c_h and $c_{\mathbf{h}}$ are normalizing constants. However, the lack of integral representation for the multiparameter fractional Brownian motion means that there is no direct way to define a fractional field extension. This remark is true for any other process with covariance given by k_h on some $L^2(T,m)$, with the notable exceptions of X^h and $W^{\mathbf{h}}$ above.

The term *fractional Brownian field* was used by several authors with different meanings, thus we clarify what we will use it for. Generally, the term *random field* describes a stochastic process whose index set is not $\mathbb{R}_+$, but for instance $\mathbb{R}_+^N$, or any other abstract set T. In this thesis, fractional Brownian field means a process indexed by $(0,1/2] \times L^2(T,m)$, with the additional constraint that the second coordinate is the set of Hurst parameters, and that for any fixed h in $(0,1/2]$, the fractional Brownian field has the law of a L^2-fractional Brownian motion. We have seen in the previous paragraph that the class of L^2-fractional Brownian motion includes various known processes. We summarize this in the following definition:

Definition 1.5 (Fractional Brownian Field). *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $(T, \mathscr{T}, m)$ a measure space. A fractional Brownian field $\boldsymbol{B}$ is a* Gaussian *random field on $(0,1/2] \times L^2(T,m)$ with values in $\mathbb{R}$, such that for each $h \in (0,1/2]$, $\boldsymbol{B}^h = \{\boldsymbol{B}(h,t), t \in L^2(T,m)\}$ is a L^2-fractional Brownian motion.*

To construct a fractional Brownian field, one might think of "sticking" together a family of h-fractional Brownian motions. However, unless these h-fractional Brownian motions are chosen independently, the resulting field might not be Gaussian; and if they were chosen independent, there is no hope to recover interesting regularity properties in the h direction. The existence of fractional Brownian fields is proven in Chapter 3, alongside several regularity properties such as continuity.

The reader might think, perhaps rightly, that we have led so far a laborious work of listing some of the numerous extensions of Brownian motion. Even though we have not detailed yet the regularity property we searched for, and despite the many motivations to look at such processes that we tried to give here and there; if the reader remained unconvinced of the interest of studying fractional Brownian fields, it would certainly be reasonable to leave the last word to Michel Talagrand:

> *"Our motivation for extending results classical for Brownian motion to the (N,d,α) Gaussian process is not the importance of this process, but rather that the case of Brownian motion suffers from an over abundance of special properties; and that moving away from these forces to find proofs that rely upon general principles, and arguably lie at a more fundamental level. Fractional Brownian motion might not be an object of central mathematical importance but abstract principles are."*

M. TALAGRAND [143]

1.3 Gaussian measures and abstract Wiener spaces

1.3.1 *Construction of Gaussian measures in infinite-dimensional spaces*

In Chapters 3, 4 and 5, we will see that Gaussian processes are embedded into infinite-dimensional spaces, considering their laws as Gaussian measures on these spaces, in an attempt to construct new processes and to study their regularity. To explain this approach, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let us a consider a linear mapping F defined on $\mathbb{R}^N$ with values in the space of Gaussian random variables. Assume that for any $x_1, \dots, x_p \in \mathbb{R}^N$, $(F(x_1), \dots, F(x_p))$ is a centred Gaussian random vector with covariance matrix given by the entries $\{(x_i, x_j)\}_{i,j=1\dots p}$ (where $(\cdot,\cdot)$ is the Euclidean scalar product). The most simple example might be to put $(U_1, \dots, U_N)$ a family of i.i.d. standard normal random variables and

$$F_\omega(x) = \sum_{k=1}^{N} U_k(\omega)\,(x, e_k)\ ,$$

where $\{e_k,\ k = 1\dots N\}$ is an orthonormal basis of $\mathbb{R}^N$. In particular, the characteristic function reads:

$$\mathbb{E}\left(e^{iF(x)}\right) = \int_{\mathbb{R}^N} e^{i(x,z)} \mu_N(\mathrm{d}z) = e^{-\frac{1}{2}\|x\|^2},\ \forall x \in \mathbb{R}^N\ , \tag{1.7}$$

where μ_N is the standard Gaussian measure defined for all Borel sets A of $\mathbb{R}^N$ by:

$$\mu_N(A) = \frac{1}{(2\pi)^{N/2}} \int_A e^{-\frac{1}{2}\|z\|^2} \, \mathrm{d}z \ .$$

Let us remark that F is a (linear) Gaussian process indexed by $\mathbb{R}^N$, and that μ_N is absolutely continuous with respect to the Lebesgue measure of $\mathbb{R}^N$. Now, if $\mathbb{R}^N$ is replaced by an infinite-dimensional Hilbert space H, there cannot be a measure satisfying Equation (1.7). Otherwise, since $(x, e_n) \to 0$ as $n \to \infty$ for any $x \in H$, the integral in (1.7) would converge to 0, while $\exp(-\|x\|^2/2)$ would remain constant. Besides, even though we will be able to deal with *standard* Gaussian measures[5], the absolute continuity will not make any sense, since there does not exist an analogue to the Lebesgue measure in infinite dimensions.

Despite these limitations, SEGAL [131] proved the existence on any separable Hilbert space H of a linear mappping F having Gaussian marginals such that:

$$\mathbb{E}(F(x)\, F(y)) = (x, y)_H \ , \ x, y \in H \ .$$

This process is now referred to as the *isonormal process*. The law of F in H fails to be countably additive, as we noticed in the previous paragraph. However, we can define a finitely additive measure $\tilde{\mu}$ on H which satisfies, for any $p \in \mathbb{N}$, any measurable function Φ from $\mathbb{R}^p$ to $\mathbb{R}$, any $x_1, \ldots, x_p \in H$ and any Borel set A of $\mathbb{R}$:

$$\mathbb{P}\Big(\Phi(F(x_1), \ldots, F(x_p)) \in A\Big) = \tilde{\mu}\Big(C(\Phi, x_1, \ldots, x_p, A)\Big) \ ,$$

where $C(\Phi, x_1, \ldots, x_p, A) = \big\{y \in H : \ \Phi((x_1, y)_H, \ldots, (x_p, y)_H) \in A\big\}$. $\tilde{\mu}$ is called a cylinder measure because it measures cylinder sets, that is sets of the form $C = \Pi^{-1}(A)$, where Π is an orthogonal projection of H with finite-dimensional range, and A a Borel set of this range. The pushforward measure of $\tilde{\mu}$ by Π is then a (true) countably additive measure on the σ-algebra of Borel sets of $\Pi(H)$. In fact, letting N be the dimension of $\Pi(H)$ and i be a linear isometry between $\Pi(H)$ and $\mathbb{R}^N$, we have for any cylinder set $C = \Pi^{-1}(A)$:

$$\tilde{\mu}(C) = \mu_N(i(A)) \ .$$

Motivated by the works of WIENER [149], who had overcome the difficulty to define properly the Brownian motion by constructing a cylinder measure on the space of absolutely continuous functions and extending it to the Banach space of continuous functions, GROSS [56] defined abstract Wiener spaces.

Theorem 1.6 (Gross, [56])**.** *Let $(H, \|\cdot\|_H)$ be a real separable Hilbert space. Assume that there exists a seminorm $\|\cdot\|_1$ on H such that for every $\epsilon > 0$ there is a finite-dimensional projection Π_0 such that for any finite-dimensional projection Π which is orthogonal to Π_0, we have:*

$$\tilde{\mu}(\{x \in H : \ \|\Pi x\|_1 > \epsilon\}) < \epsilon \ .$$

The completion of H with respect to this seminorm is denoted by E, and S is the canonical injection from E^* to $H^* \equiv H$. Then, $\tilde{\mu}$ extends to a countably additive measure on the Borel sets of E, satisfying:

$$\int_E e^{i\langle \xi, x\rangle} \, \mu(dx) = e^{-\frac{1}{2}\|S\xi\|_H^2} \ , \tag{1.8}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between E^* and E.

[5] i.e. centred measures with "unit" variance, extending (1.7) in an infinite-dimensional space.

H is densely embedded into E, and by (1.8), the embedding is also continuous. We will say that the triple (H, E, μ) is an *Abstract Wiener Space*. $\|\cdot\|_1$ is a *measurable seminorm*.

Another consequence of this theorem is that there exists a linear operator $\mathscr{I}$ that maps H into $L^2(\mu)$, defined by $\mathscr{I}(S\xi) = \langle \xi, \cdot \rangle$ for any $\xi \in E^*$, and extended by linear isometry ($\{S\xi,\ \xi \in E^*\}$ is dense in H) to the whole space H. Following the terminology of [136], we call this mapping the *Paley-Wiener map*. There is an obvious link between $\mathscr{I}$ and the isonormal process F on H, since considering $\mathscr{I}(x)$ as a random variable on $(E, \mathscr{B}(E), \mu)$, we have:

$$\mathbb{E}^\mu \left(\mathscr{I}(\phi)\ \mathscr{I}(\psi) \right) = (\phi, \psi)_H\ ,\ \ \phi, \psi \in H\ .$$

This association shows that μ is the law of the isonormal process in an abstract Wiener space, and is a *true* (i.e. countably additive) measure. The other important operator appearing here is S. In the theory of Gaussian measures, S is the covariance operator of μ and can be represented as:

$$S\xi = \int_E x\ \langle \xi, x \rangle\ \mu(\mathrm{d}x)\ , \tag{1.9}$$

where this integral is understood in the sense of Bochner. $S\xi$ is an element of H and S is reasonably called covariance operator since:

$$\langle \xi', S\xi \rangle = \int_E \langle \xi, x \rangle\ \langle \xi', x \rangle\ \mu(\mathrm{d}x) = \left(S\xi', S\xi \right)_H^2\ .$$

Example 1.7. *If H is the classical Cameron-Martin space, i.e. the Hilbert space of absolutely continuous functions on* $[0,1]$*,*

$$H = \left\{ f(\cdot) = \int_0^{\cdot} \dot{f}(t)\ dt\ :\ \dot{f} \in L^2([0,1]) \right\}\ ,$$

and if $\|\cdot\|$ is the sup-norm on H, then the completion of H with respect to $\|\cdot\|$ is the space of continuous functions on $[0,1]$ *started at* 0*. The Wiener measure on this space is the law of the Brownian motion.*

Note that in any abstract Wiener space, H is in general called the Cameron-Martin space, or reproducing kernel Hilbert space of μ. Some authors make a difference between these two notions (although they are canonically isomorphic), but we will not need this distinction here. Abstract Wiener spaces turned out to be a good framework for stochastic analysis, and the Cameron-Martin space is the right space to define directional derivatives of functionals of the Wiener process. This permitted to develop a stochastic calculus of variations, now famously referred to as Malliavin calculus.

Remark 1.8. *An alternative construction of infinite-dimensional Gaussian measures, inspired by the theory of distributions, emerged at the same period with the impulse of Russian mathematicians. In this way,* ITO *[67] defined random distributions on the space of infinitely differentiable functions with compact support on $\mathbb{R}$, and so did* GEL'FAND *[51]. Let us mention* YAGLOM *[154], who gave an important spectral representation (that will be mentioned again in Chapter 5),* ROZANOV *[122],* MINLOS *[105], who gave his name to the Bochner-Minlos theorem for nuclear spaces, and the essential book of* GEL'FAND AND VILENKIN *[52], which gathers most of the results known on such processes. This approach echoes the rise of the theory of distributions at that time,* SCHWARTZ *himself contributing [130]. Take the following setup: $\mathscr{L}$ is a nuclear space (for instance the Schwartz space*

$\mathscr{S}(\mathbb{R})$) and C is a positive definite continuous functional on $\mathscr{L}$ such that $C(0) = 1$. Then, the theorem of Minlos tells us that there exists a unique probability measure μ on $\mathscr{L}'$ such that:

$$\int_{\mathscr{L}'} e^{i\langle \xi, x\rangle}\, \mu(dx) = C(\xi)\ .$$

Thus, if $\mathscr{L} = \mathscr{S}(\mathbb{R})$ and $C(\xi) = \exp(-\|\xi\|^2_{L^2(\mathbb{R})}/2)$, μ is the Gaussian measure on the space of tempered distributions. Then, any element ξ of $\mathscr{L}$ can be considered as a random variable on $\mathscr{L}'$ (with the Borel sets given by the topology on this nuclear space). Although we will not be dealing with nuclear spaces, we mention this example as one of the extensions of Bochner's theorem. Another one for Hilbert spaces with Hilbert-Schmidt embedding is used in Chapter 5.

This construction on nuclear spaces has important interactions with theoretical physics, as can testify the works of Nelson [107, 108] on construction of quantum fields, and many others after him. In fact, Euclidean free fields can also be constructed on abstract Wiener spaces, as explained in [136], and this approach was used in recent articles as [6] and [29]. Both reflect the need that appeared in the late 50's to give rigorous mathematical meaning to some objects introduced by theoretical physicists, such as Feynman path integrals.

1.3.2 A general scheme to construct Gaussian processes

A practical tool to construct abstract Wiener spaces is the following theorem, which can be found in [136, p.317].

Theorem 1.9. *Let H and H' be two separable Hilbert spaces and u a linear isometry from H to H'. Assume that an abstract Wiener space (H, E, μ) is given. Then, there exists a separable Banach space $E' \supset H'$ and a linear isometry $\tilde{u} : E \to E'$ whose restriction to H is u and $(H', E', \tilde{u}_*\mu)$ is an abstract Wiener space ($\tilde{u}_*\mu$ denotes the push-forward measure of μ by $\tilde{u}$).*

We will build Gaussian processes upon given covariances, to which we can associate special Hilbert spaces:

Definition 1.10 (Reproducing Kernel Hilbert Space)**.** *Let (T, m) be a separable and complete metric space and R a continuous covariance function on $T \times T$. R determines a unique Hilbert space $H(R)$ satisfying the following properties:*

i) $H(R)$ is a space of functions on $T \to \mathbb{R}$;

ii) for all $t \in T$, $R(\cdot, t) \in H(R)$;

iii) for all $t \in T$, $\forall f \in H(R)$, $(f, R(\cdot, t))_{H(R)} = f(t)$.

We shall use extensively the abbreviation *RKHS* for such spaces. $H(R)$ can be constructed from $\text{Span}\{R(\cdot, t), t \in T\}$, completing this space with respect to the norm given by the scalar product of the previous definition. The continuity of the kernel and the separability of T suffice to prove that $H(R)$ is itself separable [24].

Let R be a covariance kernel on T, and let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space. Provided that there exists one abstract Wiener space (some are presented in the next parapraph), Theorem 1.9 gives the existence of an abstract Wiener space $(H(R), E, \mu)$, and we can associate to this construction the isonormal process F_R on $H(R)$ with law μ on E. For some $s,t\in T$, we see that $h_s=R(\cdot,s)$ and $h_t=R(\cdot,t)$ belong to $H(R)$, and

$$\mathbb{E}(F_R(h_s)\,F_R(h_t)) = \int_E \langle \mathscr{I}(h_s),x\rangle\ \langle \mathscr{I}(h_t),x\rangle\ \mu(\mathrm{d}x) = (h_s,h_t)_{H(R)} \\ = R(s,t)\ ,$$

by the reproducing kernel property (we extended the dual pairing notation $\langle\cdot,\cdot\rangle$ to $\mathscr{I}$ although it is not an element of E^*). Thus, the restriction of the isonormal process given by $X=\{F_R(h_t),\ t\in T\}$ is a centred Gaussian process on T with covariance R, for which we provided a useful embedding. In particular, we see that F_R is a generalized version of X, in the sense of generalized functions. This is the typical procedure we shall adopt in Chapter 3 for the family of covariances $\{k_h, h\in(0,1/2]\}$.

1.3.3 Examples of Wiener spaces

To end this section, let us describe the representation of some Gaussian random fields as abstract Wiener spaces:

- The Brownian motion on $[0,1]$ has covariance $R(s,t)=s\wedge t$, hence its reproducing kernel Hilbert space is generated by the functions $R(\cdot,t)=\int_0^\cdot \mathbf{1}_{[0,t]}(u)\,\mathrm{d}u$, for any $t\in[0,1]$. Thus, it is not difficult to see that $H(R)=H_0^1([0,1])$, the Sobolev space of absolutely continuous functions started at 0, with square integrable weak derivative. We obtain the following representations:

$$R(s,t)=\mathbb{E}(B_tB_s)=\mathbb{E}\big(\boldsymbol{B}(R(\cdot,s))\ \boldsymbol{B}(R(\cdot,t))\big)\ ,$$

  where $\boldsymbol{B}$ is the isonormal process on $H(R)$. The supremum norm $\|\cdot\|_\infty$ is a measurable norm [56], and the completion of $H_0^1([0,1])$ with respect to this norm is the space of continuous functions started at 0, $C_0([0,1])$. The law of the Brownian motion in $C_0([0,1])$ is the (standard) Wiener measure $\mathscr{W}$, and we finally obtain the classical Wiener space $(H_0^1([0,1]),C_0([0,1]),\mathscr{W})$.

  Note that this is the classical Wiener space, but this is not the only way to embed the RKHS of Brownian motion in an abstract Wiener space. See Chapter 5 where we embed it into a larger Hilbert space. We also remark that it is possible to define the classical Wiener space on $\mathbb{R}_+$ instead of $[0,1]$, after a few minor changes (think that the space of continuous functions started at 0, $C_0(\mathbb{R}_+)$, is not a Banach space for the sup-norm).

- The AWS of the fractional Brownian motion on $[0,1]$ was given by DECREUSEFOND AND ÜSTÜNEL [39] in terms of fractional integrals. For $h\in(0,1)$, let R_h denote the covariance of the fBm, and H_h the RKHS of R_h. The reader is referred to [126, p.187], where the meaning of the following integral operator is explained and proved to be an isometric isomorphism from $L^2([0,1])$ onto H_h:

$$K_h = I_{0+}^{2h}x_{0+}^{1/2-h}I_{0+}^{1/2-h}x^{h-1/2}\ (\text{for } h\le 1/2)\ , \tag{1.10}$$

  where I_{0+}^{α} is the right fractional integral of order α (note that a slightly modified operator yields the same result for $h\in(1/2,1)$). This operator has a kernel $K_h(\cdot,\cdot)$ and satisfies the two properties that:

$$R_h(s,t)=\int_0^1 K_h(t,r)\,K_h(s,t)\,\mathrm{d}r$$

and its image is precisely the RKHS H_h of the fBm:

$$H_h = K_h\left(L^2([0,1])\right) \equiv I_{0+}^{h+1/2}\left(L^2([0,1])\right) \ .$$

The last equivalence is an equality of vector spaces only, since the scalar products differ. This Hilbert space can be embedded in $C_0([0,1])$, thus giving the AWS of the fractional Brownian motion. This structure is explained at length in Chapter 3.

- The same way we deduced the RKHS of the Brownian motion gives for the Brownian sheet:

$$H_N(W) = \left\{ \varphi(t) = \int_{[0,t]} f(s)\ \lambda(\mathrm{d}s),\ t \in [0,1]^N; \quad f \in L^2([0,1]^N) \right\} \ .$$

This space can be embedded in the space of continuous functions on $[0,1]^N$ with null values on the axes to get the abstract Wiener space of the Brownian sheet.

- It is interesting to see the difference with the Gaussian (massless) free field, which is another Gaussian process on $\mathbb{R}^N$. For a bounded domain D of $\mathbb{R}^N$, letting $H^1(D)$ denote the completion of the space of smooth functions with respect to the Dirichlet inner product:

$$(f,g)_\nabla = \int_D \nabla f\ \nabla g\ \mathrm{d}\lambda \ ,$$

the Gaussian free field is the random variable $h = \sum_k h_k\ U_k$, where $\{h_k, k \in \mathbb{N}\}$ is a complete orthonormal system of $H^1(D)$ and $\{U_k, k \in \mathbb{N}\}$ a family of i.i.d standard normal random variable. In [133], it is proven that the sum converges almost surely in the space $(-\Delta)^b L^2(D)$, for any $b > (N-2)/4$, and that the embedding is dense and continuous.

1.4 Regularity of stochastic processes

1.4.1 *Classic theory*

In this section, we go over some results concerning the regularity theory of random fields, which gives conditions to derive almost sure boundedness, continuity and Hölder continuity of the sample paths $t \mapsto X_t$ of some process X indexed by a metric space (T,d). This theory is now well established, and we recall first some results on general processes, before focusing on Gaussian processes.

The general setting will be a metric space (T,d) with a property close to compactness. Of course, since we will deal with local results, the properties below extend to σ-compact sets. (T,d) is *totally bounded* if for any $\varepsilon > 0$, it can be covered by a finite number of balls of radius at most ε. The minimal number of balls that are necessary to cover T is called *metric entropy* and is denoted by $N(T,d,\varepsilon)$. The rate at which $N(T,d,\varepsilon)$ goes to 0 is a good indicator of the "complexity" of the index set, and most regularity results are expressed in those terms. Note that a compact set is totally bounded. We will also need the definition of diameter of a set $D(S) = \sup\{d(s,t),\ s,t \in S\}$.

On some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, the properties that we propose to investigate: boundedness, continuity (or absolute continuity, since most of the time, we will be on compact sets); involve quantities that make us look at objects of the form:

$$\Omega_X(F,S) = \{\omega \in \Omega:\ X_t \in F,\ \forall t \in S\} \ ,$$

for some Borel sets F of $\mathbb{R}$ (we assume throughout that processes are $\mathbb{R}$-valued) and some measurable subsets S of T. For these quantities to make sense from a probabilistic point of view, one must either restrict S to be at most countable, or check that $\Omega_X(F,S) \in \mathscr{F}$. In Appendix B, the notion of *separability* for random fields is defined in a very general context. By a famous result of DOOB [42, p.54], all processes indexed by $\mathbb{R}^N$ have a separable modification, which implies that the aforementioned quantities are measurable, as well as $\sup_{t\in S} X_t$ and other related quantities. In Appendix B, this is extended to any $\mathbb{R}$-valued random field indexed by a separable metric space. Hence, it will always be assumed that stochastic processes are separable.

The following general theorem illustrates these ideas:

Theorem 1.11. *Let (T,d) be a totally bounded metric space. For any subset $S \subseteq T$, assume that there is a real number $p \geq 1$ such that:*

$$(\mathbb{E}(X_s - X_t)^p)^{1/p} \leq d(s,t) \ , \ \forall s,t \in S \ .$$

If, in addition, the following entropic condition holds:

$$\int_0^{D(S)} (N(S,d,\varepsilon))^{1/p} \ d\varepsilon < \infty \ ,$$

then X is almost surely bounded and

$$\mathbb{E}\left(\sup_{s,t\in S} |X_s - X_t|\right) \leq 8 \int_0^{D(S)} (N(S,d,\varepsilon))^{1/p} \ d\varepsilon \ .$$

The last inequality even yields almost sure uniform continuity, by a direct application of Borel-Cantelli lemma. Note also that to avoid technicalities, we did not express this result in the full generality of Orlicz spaces (see Theorem 11.1 of [89]).

If (T,d) is the Euclidean space, we obtain with a similar but stronger condition on X, the Kolmogorov's continuity criterion. To this end, let us define the *modulus* of a process X on any metric space:

$$\omega_{X,d}(\delta) = \sup_{s,t\in T : d(s,t)\leq \delta} |X_s - X_t|$$

X is said to be Hölder continuous of order $\alpha > 0$ if:

$$\limsup_{\delta \to 0^+} \ \delta^{-\alpha} \omega_X(\delta) < \infty \text{ a.s.}$$

Theorem 1.12. *Let X be a multiparameter process on $\mathbb{R}^N$ satisfying, for some constants $C, p > 0$ and $q > 0$:*

$$\mathbb{E}\,(X_s - X_t)^p \leq C \ \|s-t\|^{N+q} \ ,$$

for any $s,t \in \mathbb{R}^N$. Then X has a modification which is almost surely α-Hölder continuous, for any $\alpha \in [0, q/p)$.

The proof of this theorem can be found in the book on multiparameter processes by KHOSHNEVISAN [76]. Both proofs rely on the chaining idea, that we exploit in the setting of set-indexed processes in Chapter 2.

1.4.2 *Regularity of Gaussian processes*

In some sense, the structure of $\mathbb{R}^N$ permits to obtain Kolmogorov's continuity criterion[6], while without additional assumption the chaining would fail on any metric space. Hence, the structure of the index set is a limiting factor for general stochastic processes, but the rapid decay of the tails of the Gaussian distribution produces better results. We do not discuss here Gaussian processes with stationary increments, for which precise modulus continuous of continuity were obtained early (see for instance [99]). For general Gaussian processes, the first steps were made by GARSIA, RODEMICH AND RUMSEY [50] with the real-variable lemma for processes on the real line (but this extends to multiparameter processes). Note here that by *modulus of continuity* of a process X on some metric space (T,d), it is generally meant a continuous increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(0) = 0$, such that almost surely:

$$|X_s - X_t| \leq C_\omega \, \psi\left(d(s,t)\right) \ , \ \forall s,t \in T \ ,$$

where C_ω is a random variable which is finite almost surely.

A major step forward was made by DUDLEY [44] who provided a modulus of continuity for any Gaussian process on a totally bounded metric space (T,d), depending on the pseudo-distance generated by the process: $d_X(s,t) = \sqrt{\mathbb{E}(X_s - X_t)^2}$, and depending also on the metric entropy, as follows:

Theorem 1.13 (Dudley's modulus of continuity)**.** *Let X be a Gaussian process indexed by T and assume that (T,d_X) is a totally bounded metric space. Then,*

$$\psi(\delta) = \int_0^\delta \sqrt{\log N(T,d_X,\varepsilon)}\, d\varepsilon$$

is a modulus of continuity for X, i.e. $\forall \delta > 0$, $\omega_{X,d_X}(\delta) \leq C_\omega \psi(\delta)$ almost surely.

To echo the previous paragraph 1.3, it is interesting to notice that Dudley transposed the problem for X indexed by T to the canonical isonormal process indexed by $\{X_t, t \in T\}$ as a subset of the Hilbert space $L^2(\Omega,\mathbb{P})$ (with scalar product given by the covariance of X).

The metric entropy integral *almost* characterizes continuity and boundedness of Gaussian processes. Combining the upper bound of Dudley with Sudakov's minoration gives:

Theorem 1.14. *There are universal constants $k_1, k_2 > 0$ such that for any Gaussian process X indexed by a set T,*

$$k_1 \sup_{\varepsilon>0} \varepsilon \sqrt{\log N(T,d_X,\varepsilon)} \leq \mathbb{E}\left(\sup_{t\in T} X_t\right) \leq k_2 \int_0^{D(T)} \sqrt{\log N(T,d_X,\varepsilon)}\, d\varepsilon \ .$$

However, there are cases when the difference between the two bounds is not tight. This gap in the theory was filled with the conjecture of FERNIQUE [49] using majorizing measures, and proven by TALAGRAND [139] (see also his recent book [144] for a modern treatment of the *generic chaining*).

[6]For the reader familiar with chaining, this can be understood as follows: in the chaining argument, the set of dyadic numbers of order n forms a net, and the structure of $\mathbb{R}^N$ implies that any dyadic number of order n has a bounded, independently of n, number of dyadic neighbours of order n.

1.4.3 Local Hölder regularity

Let us observe the example of a multiparameter process X whose law satisfies the hypothesis of Kolmogorov's criterion, so that we obtain:

$$\limsup_{\delta\to 0^+} \delta^{-\alpha}\omega_X(\delta) < \infty \text{ a.s.}$$

for any $\alpha < q/p$. We retain that the sample paths of X belong almost surely to the space of α-Hölder continuous functions (denoted from now by $\mathscr{C}^\alpha$) for any $\alpha < q/p$, but we did not prove that they were not in $\mathscr{C}^\beta$ for $\beta \geq q/p$. Besides, we obtained a global modulus, while X might have slower variations, locally. This will become clear with our second example: let H be a smooth function from $[0,1]$ to $(0,1)$ and B^H a multifractional Brownian motion on $[0,1]$ with regularity function H. We could prove, as in [57], that the induced distance satisfies $d_H(s,t) \leq |s-t|^h \equiv d_h(s,t)$, where $h = \inf_{[0,1]} H(t)$. Since this implies that

$$N([0,1]^N, d_H, \varepsilon) \leq N([0,1]^N, d_h, \varepsilon) = \varepsilon^{-1/h} ,$$

we obtain from Dudley's modulus:

$$\begin{aligned}\sup_{s,t\in[0,1]:|s-t|\leq\delta} |B^H_s - B^H_t| \leq \omega_{B^H,d_H}(\delta^h) &\leq C_\omega \int_0^{\delta^h} \sqrt{\log N([0,1], d_H, \varepsilon)}\, \mathrm{d}\varepsilon \text{ a.s.}\\ &\leq C_\omega \int_0^{\delta^h} \sqrt{\log N([0,1], d_h, \varepsilon)}\, \mathrm{d}\varepsilon \text{ a.s.}\\ &\leq C_\omega\, \delta^h \sqrt{-\log(\delta^h)} \text{ a.s.}\end{aligned}$$

Hence, we are likely to miss the local behaviour of B^H, on some subsets of $[0,1]$ where it would be more regular than h.

For these reasons, we define Hölder exponents. On a metric space (T,d), the definition of local Hölder continuity can be twofold, since it is no longer equivalent, in a ball of radius ρ, to compare the local oscillations to ρ^α or to $d(s,t)^\rho$ (see the following example). Let f be a mapping from T to $\mathbb{R}$, let $t_0 \in T$ and denote by $B(t_0,\rho)$ the ball centred at t_0 with radius ρ. We define the *pointwise Hölder exponent*:

$$\boldsymbol{\alpha}_f(t_0) = \sup\left\{\alpha :\ \limsup_{\rho\to 0} \sup_{s,t\in B(t_0,\rho)} \frac{|f(s)-f(t)|}{\rho^\alpha} < \infty\right\},$$

and the *local Hölder exponent*:

$$\widetilde{\boldsymbol{\alpha}}_f(t_0) = \sup\left\{\alpha :\ \limsup_{\rho\to 0} \sup_{s,t\in B(t_0,\rho)} \frac{|f(s)-f(t)|}{d(s,t)^\alpha} < \infty\right\}.$$

Each one allows to measure the regularity of the function f. In general, we have

$$\tilde{\boldsymbol{\alpha}}_f \leq \boldsymbol{\alpha}_f ,$$

but the inequality can be strict, as in the following example.

Example 1.15. *Consider the case of the metric space $(\mathbb{R}, |\cdot|)$. Fix $\gamma > 0$ and $\delta > 0$. Let f be a chirp function defined by $x \mapsto |x|^\gamma \sin\frac{1}{|x|^\delta}$. The two Hölder exponents at 0 can be computed and $\widetilde{\alpha}_f(0) = \frac{\gamma}{1+\delta} < \alpha_f(0) = \gamma$. We give in Figure 1.3 the graph of this funtion (blue) with its envelop $x \mapsto \pm|x|^\gamma$ (red). Here, the envelop gives the pointwise regularity.*

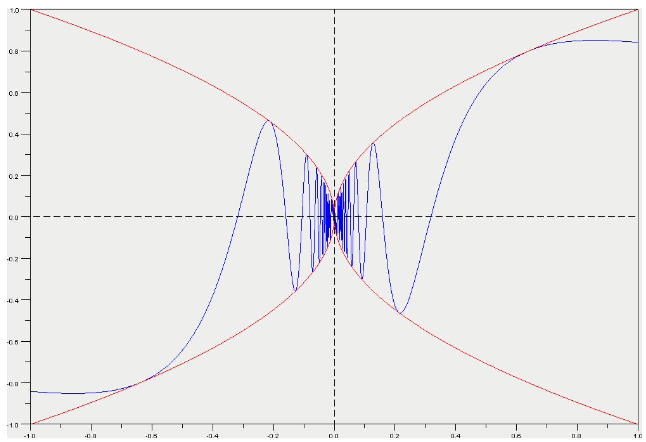

Figure 1.3 – The chirp function

If f is a stochastic process, it is reasonable to expect that these exponents will depend on $\omega \in \Omega$. By the result of Appendix B, we can assume (up to choosing a modification) that $\boldsymbol{\alpha}_f(t_0)$ and $\widetilde{\boldsymbol{\alpha}}_f(t_0)$ are random variables, if (T,d) is separable.

If X is a Gaussian process, we define the *deterministic pointwise Hölder exponent*:

$$\mathbb{\alpha}_X(t_0) = \sup\left\{\alpha;\ \limsup_{\rho\to 0}\ \sup_{s,t\in B(t_0,\rho)} \frac{\mathbb{E}(X_s - X_t)^2}{\rho^{2\alpha}} < \infty\right\} \tag{1.11}$$

and the *deterministic local Hölder exponent*:

$$\widetilde{\mathbb{\alpha}}_X(t_0) = \sup\left\{\alpha;\ \limsup_{\rho\to 0}\ \sup_{s,t\in B(t_0,\rho)} \frac{\mathbb{E}(X_s - X_t)^2}{d(s,t)^{2\alpha}} < \infty\right\}. \tag{1.12}$$

Herbin and Lévy Véhel [59] have shown the following result on uniform (along the sample paths) Hölder regularity theorem:

Theorem 1.16. *Let $\{X_t,\ t\in\mathbb{R}_+^N\}$ be a Gaussian process, and assume that the functions $t\mapsto \liminf_{u\to t}\widetilde{\mathbb{\alpha}}_X(u)$ and $t\mapsto \liminf_{u\to t}\mathbb{\alpha}_X(u)$ are positive. Then, almost surely,*

$$\forall t\in\mathbb{R}_+^N,\ \liminf_{u\to t}\widetilde{\mathbb{\alpha}}_X(u) \le \widetilde{\boldsymbol{\alpha}}_X(t) \le \limsup_{u\to t}\widetilde{\mathbb{\alpha}}_X(u)\ ,$$

and

$$\forall t\in\mathbb{R}_+^N,\ \liminf_{u\to t}\mathbb{\alpha}_X(u) \le \boldsymbol{\alpha}_X(t)\ .$$

In the latter case, even if the upper bound is missing, there is still $\mathbb{P}(\boldsymbol{\alpha}_X(t) = \mathbb{\alpha}_X(t)) = 1$ for any $t\in\mathbb{R}_+^N$.

Note that the last assertion is weaker: permuting the two symbols "a.s." and $\forall$ is not a trivial operation. The definition and study of local Hölder regularity for set-indexed processes is the main topic of Chapter 2.

1.4.4 *Exact modulus of continuity of fractional processes*

We easily deduce from the last theorem that the fractional Brownian motion of parameter $H \in (0,1)$ has Hölder exponents equal to H. Hence H rules the regularity of the fBm, in addition to the other properties we already pointed out. However, this is not enough yet to decide whether the sample paths belong to $\mathscr{C}^H$. The following result, expressed for the N-parameter Lévy fractional Brownian motion, answers negatively due to the logarithm term that appears in. Indeed, there exists a positive constant C_1 such that the exact modulus of continuity of the Lévy fBm is:

$$\limsup_{\delta\to 0+} \sup_{\substack{s,t\in[0,1]^N\\ \|s-t\|<\delta}} \frac{|B^H_s - B^H_t|}{\sqrt{2\delta^{2H}\log(\delta^{-1})}} = C_1 \text{ a.s.}$$

In the unidimensional case $N = 1$, we have $C_1 = 1$. This is Lévy's modulus of continuity of Brownian motion when $H = 1/2$; for $H < 1/2$, this was proven originally in MARCUS [99] while for $H > 1/2$, the first proof of this result we could find is due to KHOSHNEVISAN AND SHI [78] as part of a much more powerful theorem. In the multiparameter case, this is a consequence of the recent work of MEERSCHAERT, WANG AND XIAO [103, Theorem 4.1].

A local version of the previous modulus appeared initially in the works of Khinchine on random walks, and is known as law of the iterated logarithm. It was extended to the Brownian motion, and eventually we have for the N-parameter Lévy fBm of parameter H the existence of a positive constant C_2, such that at any point $t \in \mathbb{R}^N_+$:

$$\lim_{\delta\to 0+} \sup_{\|s\|<\delta} \frac{|B^H_{t+s} - B^H_t|}{\sqrt{2\delta^{2H}\log\log(\delta^{-1})}} = C_2 \text{ a.s.}$$

Hence, the local oscillations are closer to the modulus of H-Hölder continuity than they were in the uniform result. Note that this result can again be found in [99] when $H \leq 1/2$ in the one-dimensional setting (and then $C_2 = 1$), and follows from a theorem of OREY [112] for $H \in (0,1)$. The multiparameter result is again a consequence of [103, Theorem 5.6]. Another form of the law of the iterated logarithm will draw our attention in Chapter 5. It is known as Chung's law of the iterated logarithm, following CHUNG [31], who was the first to consider the lower rate of convergence for random walks (and Brownian motion as a consequence). MONRAD AND ROOTZÉN [106] extended this to the fractional Brownian motion. For the N-parameter fBm, it follows that there exists a constant c_H, such that for any $t \in \mathbb{R}^N_+$,

$$\liminf_{\delta\to 0+} \frac{\sup_{s:\|s\|<\delta} |B^H_{t+s} - B^H_t|}{\delta^H (\log\log(\delta^{-1}))^{-h/N}} = c_H \text{ a.s.}$$

It is interesting to see that c_H is unknown in general, but that bounds exist and depend on the constant of small deviations of the fBm. If $N = 1$ and $H = 1/2$, the constant of the standard Brownian motion is $\pi/\sqrt{8}$. We did not focus on fractional Brownian sheets (nor more generally on anisotropic Gaussian fields), because it would require more care and the results of Chapter 5 are more easily compared to the Lévy fractional Brownian motion. On this topic, we refer to [103] for the latest results and to XIAO [152] for a review.

Along the years, several variations appeared, among which the modulus of non-differentiability (which is still open for the Lévy fBm) and functional laws such as Strassen's, which are particularly noteworthy.

1.5 Main results of this thesis

The topics that are addressed in this thesis follow the work of Herbin and Merzbach on the set-indexed fractional Brownian motion [60, 61, 62]. We extend it to processes whose Hurst parameter is allowed to vary along the sample paths, as it was done for multifractional extensions of multiparameter fractional Brownian processes. To carry out this program, we define in Chapter 2 appropriate tools to measure the regularity of set-indexed processes and prove a Kolmogorov criterion for set-indexed processes and a theorem similar to Theorem 1.16 for set-indexed Gaussian processes. This Chapter was written in collaboration with Erick Herbin and is available as a preprint [64]. In Chapter 3, we construct a L^2-indexed fractional Brownian motion via a representation on abstract Wiener spaces, and study its regularity properties using the results of the previous chapter. This work is also available as a preprint [121]. Chapter 4 gathers several results related to the fractional Brownian field, among which a property of local nondeterminism for the L^2-fractional Brownian motion. Lastly, we establish in Chapter 5 a Chung's law of the iterated logarithm for the multiparameter fractional Brownian motion with covariance given in (1.5), thanks to sompe techniques of analysis on Wiener spaces. These results appear in our last preprint [120].

In the terminology of Ivanoff and Merzbach (Definition 1.1), we recall that $(\mathscr{A}_n)_{n\in\mathbb{N}}$ is a sequence of finite sub-collections of $\mathscr{A}$, and denote k_n the size of $\mathscr{A}_n$ ($k_n = \#\mathscr{A}_n$). In this abstract setting, formulating minimal assumptions is often as important as proving the results themselves. The assumption introduced in Chapter 2 and that we retain in the sequel is formulated as follows:

Assumption ($\mathscr{H}_{\underline{\mathscr{A}}}$). Let $d_{\mathscr{A}}$ be a (pseudo-)distance on $\mathscr{A}$. Let us suppose that for $\underline{\mathscr{A}} = (\mathscr{A}_n)_{n\in\mathbb{N}}$, there exist positive real numbers $q_{\underline{\mathscr{A}}}$ and M_1 such that:

1. For all $n \in \mathbb{N}$,
$$\sup_{U\in\mathscr{A}_n} d_{\mathscr{A}}(U, g_n(U)) \leq M_1\ k_n^{-1/q_{\underline{\mathscr{A}}}}, \tag{H1}$$

2. and the collection $(\mathscr{A}_n)_{n\in\mathbb{N}}$ is minimal in the sense that: setting for all $n \in \mathbb{N}$ and all $U \in \mathscr{A}_n$,
$$\mathscr{V}_n(U) = \{V \in \mathscr{A}_n : V \supsetneq U, d_{\mathscr{A}}(U,V) \leq 3M_1 k_n^{-1/q_{\underline{\mathscr{A}}}}\},$$
the sequence $(N_n)_{n\geq 1}$ defined by $N_n = \max_{U\in\mathscr{A}_n} \#\mathscr{V}_n(U)$ for all $n \geq 1$ satisfies
$$\forall \delta > 0, \quad \sum_{n=1}^{\infty} k_n^{-\delta}\ N_n < \infty. \tag{H2}$$

The hypotheses $(H1)$ and $(H2)$ are thoroughly explained and discussed in Chapter 2. Let us notice that in case of Gaussian processes, the hypothesis $(H2)$ is no longer necessary. The first theorem we obtained extends Kolmogorov's continuity criterion:

Theorem (2.9)**.** *Let $d_{\mathscr{A}}$ be a (pseudo-)distance on the indexing collection $\mathscr{A}$, whose subclasses $\underline{\mathscr{A}} = (\mathscr{A}_n)_{n\in\mathbb{N}}$ satisfy Assumption ($\mathscr{H}_{\underline{\mathscr{A}}}$) with a discretization exponent $q_{\underline{\mathscr{A}}} > 0$. Let $X = \{X_U;\ U \in \mathscr{A}\}$ be a set-indexed process such that:*

$$\forall U,V \in \mathscr{A}, \quad \mathbb{E}(|X_U - X_V|^{\alpha}) \leq K\ d_{\mathscr{A}}(U,V)^{q_{\underline{\mathscr{A}}}+\beta}$$

where K, α and β are positive constants. Then, the sample paths of X are almost surely locally γ-Hölder continuous for all $\gamma \in (0, \frac{\beta}{\alpha})$, i.e. there exist a random variable h^* and a constant $L > 0$ such that almost surely:

$$\forall U, V \in \mathscr{A}, \qquad d_{\mathscr{A}}(U,V) < h^* \Rightarrow |X_U - X_V| \leq L\ d_{\mathscr{A}}(U,V)^{\gamma}.$$

Thus, a process which satisfies these assumptions belongs almost surely to the space of Hölder continuous functions of order γ, when this space is defined for simple increments, i.e. of the form $X_U - X_V$. We shall present results on the regularity with respect to increments on $\mathscr{C}$, as defined in Section 1.1.1. Before doing so, the next theorem improves on Kolmogorov's criterion to provide an almost sure value for the Hölder exponents of Gaussian processes. Let us recall that the deterministic pointwise $\mathbb{a}_X$ and local $\widetilde{\mathbb{a}}_X$ Hölder exponents are defined on a general metric space (see Section 1.4.3), and so they are on $(\mathscr{A}, d_{\mathscr{A}})$.

Theorem (2.35). *Let $X = \{X_U;\ U \in \mathscr{A}\}$ be a set-indexed centered Gaussian process, where $(\mathscr{A}_n)_{n\in\mathbb{N}}$ and $d_{\mathscr{A}}$ satisfy assumption $(H1)$ of $(\mathscr{H}_{\underline{\mathscr{A}}})$. Suppose that the functions $U_0 \mapsto \liminf_{U\to U_0} \widetilde{\mathbb{a}}_X(U)$ and $U_0 \mapsto \liminf_{U\to U_0} \mathbb{a}_X(U)$ are positive over $\mathscr{A}$. Then, with probability one,*

$$\forall U_0 \in \mathscr{A}, \quad \liminf_{U\to U_0} \widetilde{\mathbb{a}}_X(U) \leq \widetilde{\boldsymbol{\alpha}}_X(U_0) \leq \limsup_{U\to U_0} \widetilde{\mathbb{a}}_X(U)$$

and

$$\forall U_0 \in \mathscr{A}, \quad \liminf_{U\to U_0} \mathbb{a}_X(U) \leq \boldsymbol{\alpha}_X(U_0)\ .$$

We already mentioned in this introduction how important the $\mathscr{C}$-increments are in the study of multiparameter and set-indexed processes. Without further assumption, the regularity on $\mathscr{C}$ is likely to be more complicated than on $\mathscr{A}$: for instance, the Brownian motion indexed by rectangles of $\mathbb{R}^2_+$ (which are pinned at 0) is locally bounded and continuous, while on the associated class $\mathscr{C}$ it is almost surely unbounded (see a similar example with proof in [5, p.28]). Hence, we restricted the class $\mathscr{C}$ to increments of the form $U \setminus \cup_{k=1}^{L} U_k$, where L is a fixed integer. This leads to the definition of Hölder exponents on this class $\mathscr{C}^L$, and it is proved that they are in fact independent of the choice of L. Therefore, they are denoted by $\boldsymbol{\alpha}_{X,\mathscr{C}}$ and $\widetilde{\boldsymbol{\alpha}}_{X,\mathscr{C}}$.

Corollary (2.36). *Let $X = \{X_U,\ U \in \mathscr{A}\}$ be a centered Gaussian set-indexed process. If the subcollections $(\mathscr{A}_n)_{n\in\mathbb{N}}$ satisfy assumption $(H1)$ of $(\mathscr{H}_{\underline{\mathscr{A}}})$ and if the deterministic $\mathscr{C}$-Hölder exponents are finite, then for $U_0 \in \mathscr{A}$,*

$$\boldsymbol{\alpha}_{X,\mathscr{C}}(U_0) = \sup\left\{\alpha :\ \limsup_{\rho\to 0} \sup_{\substack{U,V\in B_{d_{\mathscr{A}}}(U_0,\rho)\\ V\subset U}} \frac{\mathbb{E}\left(|\Delta X_{U\setminus V}|^2\right)}{\rho^{\alpha}} < \infty\right\} \text{ a.s.}$$

$$\widetilde{\boldsymbol{\alpha}}_{X,\mathscr{C}}(U_0) = \sup\left\{\alpha :\ \limsup_{\rho\to 0} \sup_{\substack{U,V\in B_{d_{\mathscr{A}}}(U_0,\rho)\\ V\subset U}} \frac{\mathbb{E}\left(|\Delta X_{U\setminus V}|^2\right)}{d_{\mathscr{A}}(U,V)^{\alpha}} < \infty\right\} \text{ a.s.}$$

Finally, the results of this chapter are applied to the set-indexed fractional Brownian motion. All exponents are proved to equal the Hurst parameter H almost surely, and under an additional weak assumption, the pointwise exponent is also uniformly equal to H.

Theorem (2.38). *Let $\mathbf{B}^H$ be a set-indexed fractional Brownian motion on $(T, \mathscr{A}, m)$, $H \in (0, 1/2]$. Assume that the subclasses $(\mathscr{A}_n)_{n\in\mathbb{N}}$ satisfy assumption $(H1)$ of $(\mathscr{H}_{\underline{\mathscr{A}}})$. Then, the local and pointwise*

Hölder exponents of $\mathbf{B}^H$ *at any* $U_0 \in \mathscr{A}$*, defined with respect to the distance* d_m *or any equivalent distance, satisfy*

$$\mathbb{P}\Big(\forall U_0 \in \mathscr{A},\ \widetilde{\boldsymbol{\alpha}}_{\mathbf{B}^H}(U_0) = H\Big) = 1$$

and if the additional Condition (2.22) holds,

$$\mathbb{P}(\forall U_0 \in \mathscr{A},\ \boldsymbol{\alpha}_{\mathbf{B}^H}(U_0) = H) = 1\ .$$

In particular, the multiparameter fBm defined with the covariance (1.5) on rectangles of $[0,1]^N$*, satisfies* $\mathbb{P}\Big(\forall t_0 \in [0,1]^N,\ \boldsymbol{\alpha}_{\mathbf{B}^H}(t_0) = \widetilde{\boldsymbol{\alpha}}_{\mathbf{B}^H}(t_0) = H\Big) = 1$.

Chapter 3 extends the work done in Chapter 2, in the sense that a multifractional field is constructed to which we will apply Theorem 2.38. The fractional Brownian fields that are about to be presented are indexed by some $L^2(T,m)$ spaces, but the restriction to indicator functions implies that they can be seen as set-indexed processes.

First, we sketch the construction of a fractional Brownian field on $[0,1]$. Let $\mathscr{W}$ be the Wiener measure on the space W of continuous functions started at 0. Theorem 3.3 ensures the existence of an operator $\mathscr{K}_h$ which maps H_h (as given in the construction of the AWS of the fractional Brownian motion, Section 1.3.3) into W^*, such that on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, the following formula defines a fractional Brownian field on $(0,1)\times[0,1]$:

$$\forall (h,t) \in (0,1)\times[0,1], \quad \boldsymbol{B}_{h,t} = \int_W \langle \mathscr{K}_h R_h(\cdot,t), w\rangle \ \mathrm{d}\mathbb{B}_w\ ,$$

where $\mathbb{B}$ is a Gaussian white noise on W with control measure $\mathscr{W}$ (see Definition (1.1)). Moreover, $\mathscr{K}_h$ is built so that for each $h \in (0,1)$ fixed, $\{\boldsymbol{B}_{h,t},\ t \in [0,1]\}$ is a fractional Brownian motion. Thus, $\boldsymbol{B}$ is indeed a fractional Brownian field in the sense of Definition 1.5.

Let now $\{k_h(\cdot,\cdot),\ h \in (0,1/2]\}$ be the collection of covariances on $L^2(T,m)$ defined by (1.4). For each h, let $H(k_h)$ be the reproducing kernel Hilbert space attached to k_h, for which there exists an abstract Wiener space $(H(k_h), E_h, \mu_h)$, by the proposition given in Section 1.3.2. For each $h \in (0,1/2]$, there is a linear isometry between $(H_h, W, \mathscr{W}_h)$ eand $(H(k_h), E_h, \mu_h)$, so we deduce in Section 3.2.2 the existence of an operator $\tilde{\mathscr{K}}_h$ from $H(k_h)$ into $E^*_{1/2}$, such that the process defined by:

$$\forall (h,f) \in (0,1/2]\times L^2(T,m), \quad \boldsymbol{B}_{h,f} = \int_E \langle \tilde{\mathscr{K}}_h k_h(\cdot,f), x\rangle \ \mathrm{d}\mathbb{W}(x)\ ,$$

is a fractional Brownian field. It is then proven that this process has increments whose law satisfies:

Theorem (3.15)**.** *Let* $\boldsymbol{B}$ *be a fBf on* $(0,1/2]\times L^2(T,m)$*. For any* $\eta \in (0,1/4)$ *and any compact subset* D *of* L^2*, there exists a constant* $C_{\eta,D} > 0$ *such that for any* $f_1, f_2 \in D$*, and any* $h_1, h_2 \in [\eta, 1/2-\eta]$*,*

$$\mathbb{E}\Big((\boldsymbol{B}_{h_1,f_1} - \boldsymbol{B}_{h_2,f_2})^2\Big) \leq C_{\eta,D}\Big((h_2-h_1)^2 + m\Big((f_1-f_2)^2\Big)^{2(h_1\wedge h_2)}\Big)\ .$$

Similarly, this type of result applies to some functionals of the fractional Brownian field which appear as solutions of stochastic partial differential equations. In Section 3.3.2, we consider an elliptic problem on a domain $U \subset [0,1]^2$:

$$\Delta u = \dot{\mathbb{W}}^{h_1,h_2}\ ,$$

where $\dot{\mathbb{W}}^{h_1,h_2}$ is a fractional Gaussian noise on $[0,1]^2$ which can be "extracted" (more precisions are given in the corresponding section) from a single fractional Brownian field. The solutions are *mild* solutions, and the regularity of the laws when h_1 and h_2 vary in $(0,1/2]$ is proven, and is in adequation with the previous theorem.

These theorems are especially important from the point of view of the regularity of sample paths, as the following proposition suggests. Let us recall that the natural distance on $L^2(T,m)$ is $d_m(f,g) = m\left((f-g)^2\right)^{1/2}$.

Theorem (3.19)**.** *Let $\boldsymbol{B}$ be a fBf indexed on a compact subset I of $(0,1/2]$, and K be a compact subset of $L^2(T,m)$ of d_m-diameter smaller than 1. Let $\iota = \inf I$. If the following Dudley integral converges:*

$$\int_0^1 \sqrt{\log N(K, d_m, \varepsilon^{1/2\iota})}\, d\varepsilon < \infty \ ,$$

then $\boldsymbol{B}$ indexed by $I \times K$ has almost surely continuous sample paths.

In the last part of this chapter, we consider an indexing collection $\mathscr{A}$ on the measurable space (T,m). Let $\boldsymbol{B}$ be a fractional Brownian field on $(0,1/2] \times L^2(T,m)$. Let $\boldsymbol{h}$ be a function indexed by $\mathscr{A}$ with values in $(0,1/2]$, and assume that for any $U \in \mathscr{A}$, $\boldsymbol{h}(U)$ is smaller than its local Hölder regularity at U. Such a function is called regular. The *set-indexed multifractional Brownian motion* (SImBm) with regularity function $\boldsymbol{h}$, denoted by $\{\boldsymbol{B}^{\boldsymbol{h}}_U,\ U \in \mathscr{A}\}$, is defined as follows:

$$\forall U \in \mathscr{A}, \quad \boldsymbol{B}^{\boldsymbol{h}}_U = \boldsymbol{B}_{\boldsymbol{h}(U), \mathbf{1}_U} \ .$$

The right-hand side term is the initial fractional Brownian field valuated at $(h,f) = (\boldsymbol{h}(U), \mathbf{1}_U)$, where $\mathbf{1}_U$ is indeed an element of $L^2(T,m)$. This multifractional process verifies the following regularity property:

Proposition (3.25)**.** *Let $\mathscr{A}$ be an indexing collection satisfying assumption $(H1)$ of $(\mathscr{H}_{\underline{\mathscr{A}}})$. Let $\boldsymbol{B}^{\boldsymbol{h}}$ be a SImBm on $\mathscr{A}$ with a regular function $\boldsymbol{h}$. Then, almost surely,*

$$\forall U_0 \in \mathscr{A}, \quad \widetilde{\boldsymbol{\alpha}}_{\boldsymbol{B}^{\boldsymbol{h}}}(U_0) = \boldsymbol{h}_{U_0} \ \text{ and } \ \boldsymbol{\alpha}_{\boldsymbol{B}^{\boldsymbol{h}}}(U_0) \geq \boldsymbol{h}_{U_0} \ .$$

We did construct a set-indexed multifractional Brownian motion, whose regularity can be prescribed by a function $\boldsymbol{h}$, producing the same effect than for the multifractional Brownian motion on $\mathbb{R}$.

Chapter 4 gathers several independent results concerning the L^2-fractional Brownian motion (h fixed) and results on approximation of the L^2-multifractional Brownian motion by L^2-fractional Brownian motions. A local self-similarity result is also presented.

Firstly, two different couples consisting of an increment stationarity property and a self-similarity property are discussed. The first couple is similar to the multiparameter notions, while the second one accounts for the measure m and prolong the ideas developed in the set-indexed framework [62]. Each of these couples characterizes the L^2-fBm, see Propositions 4.1 and 4.3.

In a second part, it is shown that for any $h \in (0,1/2)$, the L^2-fBm is *locally nondeterministic*, that is:

Proposition (4.4)**.** *Let $h \in (0,1/2)$. There exists a positive constant C_0 such that for all $f \in L^2(T,m)$ and for all $r \leq \|f\|$, the following holds:*

$$Var\left(\boldsymbol{B}^h_f \mid \boldsymbol{B}^h_g, \|f-g\| \geq r\right) = C_0 r^{2h}.$$

This property is wrong for $h = 1/2$, as discussed in Section 4.2. The property of local nondeterminism is a valuable tool for the study of the sample paths of processes which do not have the Markov property. It allows for instance to get estimates for the small ball probabilities of Gaussian processes.

Theorem (4.6)**.** *Let $\mathbf{B}^h$ be a L^2-fractional Brownian motion of parameter $h \in (0,1/2)$ and K a compact set in $L^2(T,m)$. Let d_h denote the distance induced by $\mathbf{B}^h$ on L^2. Assume that there exists a function ψ such that for any $\varepsilon > 0$, $N(K,d_h,\varepsilon) \leq \psi(\varepsilon)$ and $\psi(\varepsilon) \asymp \psi(\varepsilon/2)$. Then, there are some constants $\kappa_1, \kappa_2 > 0$ such that:*

$$\exp\left(-\kappa_2\ \psi(\varepsilon)\right) \leq \mathbb{P}\left(\sup_{f\in K} |\mathbf{B}_f^h| \leq \varepsilon\right) \leq \exp\left(-\kappa_1\ N(K,d_h,\varepsilon)\right)\ ,$$

This result is the first step towards the Chung law of the iterated logarithm of Chapter 5. Finally we provide two results of approximation of a L^2-mBm by L^2-fractional Brownian motions. Similarly to the set-indexed mBm, the L^2-mBm is defined for a regularity function $\mathbf{h} : L^2(T,m) \to (0,1/2]$:

$$\mathbf{B}_f^{\mathbf{h}} = \mathbf{B}_{\mathbf{h}(f),f}\ ,\quad f \in L^2(T,m)\ .$$

The first result states that it is possible to recreates a L^2-multifractional Brownian motion from piecewise L^2-fractional Brownian motions, see Theorem 4.8.

The second approximation result concerns the tangent processes of the L^2-mBm. The idea of a tangent field appeared in FALCONER [47], and we prove that for a fractional Brownian field $\mathbf{B}$ and $\mathbf{h}$ a function from $L^2(T,m)$ to $(0,1/2]$, the following process defined for any $f_0 \in L^2(T,m)$ and any $\rho > 0$ by:

$$Y_f^{f_0,\alpha}(\varrho) = \varrho^{-\alpha}\left(\mathbf{B}_{f_0+\varrho f}^{\mathbf{h}} - \mathbf{B}_{f_0}^{\mathbf{h}}\right)\ ,\quad f \in L^2(T,m)$$

converges when $\varrho \to 0$ to a L^2-fBm:

Theorem (4.10)**.** *Let K be a compact of $L^2(T,m)$. Assume that $\mathbf{h}$ is regular and with values in $[\eta, 1/2-\eta]$ for some $\eta \in (0,1/4)$. Let $\iota = \inf_K \mathbf{h}(f)$ $(\geq \eta)$ and assume further the convergence of Dudley's integral on K:*

$$\int_0^{D(K)} \sqrt{\log N(K,d_m,\varepsilon^{1/2\iota})}\ d\varepsilon < \infty\ .$$

Then, for $\alpha = 2\mathbf{h}(f_0)$, $Y^{f_0,\alpha}(\varrho)$ converges in law on the space of continuous functions $C(K)$ as $\varrho \to 0$, and the limit is a L^2-fractional Brownian motion of parameter $\mathbf{h}(f_0)$.

In Chapter 5, we try to answer a question left open by HERBIN AND XIAO [65] on the behaviour of the multiparameter fractional Brownian motion near 0 (we recall that the multiparameter fBm has covariance $k_h^{(\nu)}$ as given in (1.5)). Local modulus of continuity of this process are nontrivial, which could be guessed by observing the distance induced by the multiparameter fBm on $\mathbb{R}^N$. Indeed, this distance is related to the distance $d_\lambda(s,t) = \lambda([0,s] \triangle [0,t])$ where λ is the Lebesgue measure on $\mathbb{R}^\nu$, which is singular with respect to the Euclidean distance when $\nu \geq 2$ in any neighbourhood of 0. This singularity affects the regularity of the multiparameter fBm: it is shown that Chung's law of the iterated logarithm of this process at 0 is different from any other point away from the axes.

Thanks to Theorem 4.6, we are able to compute a sharp estimate of the probability of small deviations of the multiparameter fBm. This and a spectral representation theorem (Proposition

5.8) allow to obtain a Chung law of the iterated logarithm at 0, with a lower bound given by the following modulus:

$$\Psi_h^{(\ell)}(r) = r^{\nu h}\, \widetilde{\Psi}_h^{(\ell)}(r) = r^{\nu h}\, (\log\log r^{-1})^{-h/\nu} \,,$$

and an upper bound given by the modulus: $\Psi_h^{(u)} = r^{\nu h}\, \widetilde{\Psi}_h^{(u)}$, where we prove the existence of $\widetilde{\Psi}_h^{(u)}$ as a non-decreasing function started at 0, but without obtaining an explicit form. In both cases, $\widetilde{\Psi}_h^{(\ell)}$ and $\widetilde{\Psi}_h^{(u)}$ are negligible compared to $r^{\nu h}$. The supremum of the multiparameter fBm in the neighbourhood of 0 is denoted by $M^h(r) = \sup_{t\in[0,r]^\nu} |\boldsymbol{B}_t^h|,\ r \in [0,1]$. Then we obtained:

Theorem (5.1)**.** *Let $h \in (0,1/2)$ and let M^h, $\Psi_h^{(\ell)}$ and $\Psi_h^{(u)}$ be as above. We have almost surely:*

$$\liminf_{r\to 0^+} \frac{M^h(r)}{r^{\nu h}\widetilde{\Psi}_h^{(\ell)}(r)} \geq \kappa_1^{h/\nu} \quad \textit{and} \quad \liminf_{r\to 0^+} \frac{M^h(r)}{r^{\nu h}\widetilde{\Psi}_h^{(u)}(r)} \leq \kappa_2^{h/\nu} \,,$$

where $\kappa_1 \leq \kappa_2$ are the constants appearing in the small deviations of the L^2-fBm.

As in the article of MONRAD AND ROOTZÉN [106], this theorem is extended into a functional law of the iterated logarithm, thus providing an invariance principle for the rescaled multiparameter fBm. Following the notations of the previous theorem, we define two different scalings given for any $r \in (0,1)$ by:

$$\eta_r^{(h,\ell)}(t) = \frac{\boldsymbol{B}^h(rt)}{r^{\nu h}\sqrt{\log\log(r^{-1})}} \,, \forall t \in [0,1]^\nu$$

and

$$\eta_r^{(h,u)}(t) = \frac{\boldsymbol{B}^h(rt)}{r^{\nu h}\left(\widetilde{\Psi}_h^{(u)}(r)\right)^{-\nu/2h}} \,, \forall t \in [0,1]^\nu \,.$$

We obtain the speed of convergence and the set towards which $\eta^{(h,u)}$ and $\eta^{(h,\ell)}$ converge:

Theorem (5.2)**.** *Let $h \in (0,1/2)$ and let H_h^ν denote the reproducing kernel Hilbert space of $k_h^{(\nu)}$. Let $\varphi \in H_h^\nu$ having norm strictly smaller than* 1*. Then, there exist two positive and finite constants $\gamma^{(\ell)}(\varphi)$ and $\gamma^{(u)}(\varphi)$ such that, almost surely,*

$$\liminf_{r\to 0^+} \widetilde{\Psi}_h^{(\ell)}(r)^{-1-\nu/2h} \sup_{t\in[0,1]^\nu} |\eta_r^{(h,\ell)}(t) - \varphi(t)| \geq \gamma^{(\ell)}(\varphi)$$
$$\liminf_{r\to 0^+} \widetilde{\Psi}_h^{(u)}(r)^{-1-\nu/2h} \sup_{t\in[0,1]^\nu} |\eta_r^{(h,u)}(t) - \varphi(t)| \leq \gamma^{(u)}(\varphi) \,.$$

Note that when $\varphi = 0$, this implies the previous theorem.

Lastly, the Hausdorff dimension of the range of the multiparameter fractional Brownian motion is computed, and we observed that it does not capture the irregularity of this process at 0 (see Proposition 5.17). However, similarly to the chirp function (Example 1.15), the singularity of the multiparameter fBm can also be measured in terms of local Hölder regularity, see Remark 5.18. To conclude, it is worth considering that there might exist a link, for general Gaussian processes, between the local modulus of continuity in Chung's form and a new local Hölder exponent. This is discussed at the end of Chapter 5.

1.6 Principaux résultats de la thèse

Les sujets abordés dans cette thèse succèdent aux travaux communs d'Erick Herbin et Ely Merzbach sur le mouvement brownien fractionnaire set-indexed [60, 61, 62]. Nous poursuivons ce travail et donnons un sens à une extension de ce processus dont le paramètre de Hurst est autorisé à varier le long des trajectoires, de la même manière que des extensions multifractionnaires avaient été construites pour les mouvements browniens fractionnaires multiparamètres. Pour cela, le Chapitre 2 se propose de construire les objets permettant de mesurer la régularité locale des processus set-indexed, qui permettront par la suite de travailler sur des processus set-indexed dont la régularité hölderienne locale varie. Ce chapitre a été écrit en collaboration avec Erick Herbin et fait l'objet d'une pré-publication [64]. Dans le Chapitre 3, nous abordons la construction d'un champ brownien fractionnaire représenté sur les espaces de Wiener et étudions ses propriétés de régularité, à l'aide des résultats du chapitre précédent. Ce chapitre fait l'objet de la pré-publication [121]. Le Chapitre 4 est un recueil de propriétés diverses liées au champ brownien fractionnaire sur L^2, dont une propriété de non-déterminisme local apparaissant dans [121]. Enfin, nous établissons dans le Chapitre 5 une loi du logarithme itéré de Chung pour le mouvement brownien fractionnaire multiparamètre de covariance (1.5), grâce notamment à des techniques d'analyse sur les espaces de Wiener. Ces résultats font l'objet d'une dernière pré-publication, [120].

Dans la terminologie d'Ivanoff et Merzbach (Définition 1.1), on rappelle que $(\mathscr{A}_n)_{n\in\mathbb{N}}$ est une suite de sous-ensembles finis de $\mathscr{A}$, et on note k_n le cardinal de $\mathscr{A}_n$ ($k_n = \#\mathscr{A}_n$). Dans ce cadre abstrait, la formulation des hypothèses minimales est souvent tout aussi importante que la démonstration des résultats eux-mêmes. L'hypothèse introduite au Chapitre 2 et que nous retiendrons pour l'étude des processus set-indexed est la suivante:

Hypothèse ($\mathscr{H}_{\underline{\mathscr{A}}}$)**.** *Soit $d_{\mathscr{A}}$ une (pseudo-)distance sur $\mathscr{A}$. Supposons que pour $\underline{\mathscr{A}} = (\mathscr{A}_n)_{n\in\mathbb{N}}$, il existe des nombres positifs strictement $q_{\underline{\mathscr{A}}}$ et M_1 tels que:*

1. *pour tout $n \in \mathbb{N}$,*
$$\sup_{U\in\mathscr{A}_n} d_{\mathscr{A}}(U, g_n(U)) \leq M_1\, k_n^{-1/q_{\underline{\mathscr{A}}}}, \tag{H1}$$
2. *la collection $(\mathscr{A}_n)_{n\in\mathbb{N}}$ est minimale au sens où: si pour tout $n \in \mathbb{N}$ et tout $U \in \mathscr{A}_n$, on note*
$$\mathscr{V}_n(U) = \{V \in \mathscr{A}_n : V \supsetneq U, d_{\mathscr{A}}(U,V) \leq 3M_1 k_n^{-1/q_{\underline{\mathscr{A}}}}\},$$
alors la suite $(N_n)_{n\geq 1}$ définie par $N_n = \max_{U\in\mathscr{A}_n} \#\mathscr{V}_n(U)$ pour tout $n \geq 1$ vérifie:
$$\forall \delta > 0, \quad \sum_{n=1}^{\infty} k_n^{-\delta}\, N_n < \infty. \tag{H2}$$

Les hypothèses ($H1$) et ($H2$) sont expliquées et discutées en détails dans le Chapitre 2. On retiendra notamment que lorsque les processus sont gaussiens, l'hypothèse ($H2$) n'est plus nécessaire. Le premier théorème obtenu étend le critère de continuité de Kolmogorov:

Théorème (2.9)**.** *Soit $d_{\mathscr{A}}$ une (pseudo-)distance sur la classe d'indexation $\mathscr{A}$, dont les sous-classes $\underline{\mathscr{A}} = (\mathscr{A}_n)_{n\in\mathbb{N}}$ satisfont l'hypothèse ($\mathscr{H}_{\underline{\mathscr{A}}}$) pour l'exposant de discrétisation $q_{\underline{\mathscr{A}}} > 0$. Soit $X = \{X_U;\ U \in \mathscr{A}\}$ un processus set-indexed tel que:*
$$\forall U, V \in \mathscr{A}, \quad \mathbb{E}\left(|X_U - X_V|^{\alpha}\right) \leq K\, d_{\mathscr{A}}(U,V)^{q_{\underline{\mathscr{A}}}+\beta}$$

où K, α et β sont des constantes strictement positives. Alors les trajectoires de X sont presque surement localement γ-Hölder continues pour tout $\gamma \in (0, \frac{\beta}{\alpha})$, c'est-à-dire qu'il existe une variable aléatoire h^* et une constante $L > 0$ telles que presque surement:

$$\forall U, V \in \mathscr{A}, \qquad d_{\mathscr{A}}(U,V) < h^* \Rightarrow |X_U - X_V| \leq L\ d_{\mathscr{A}}(U,V)^{\gamma}.$$

Ainsi, un processus vérifiant ces hypothèses appartient presque surement à l'espace des fonctions Hölder continues d'ordre γ pour les accroissements simples, c'est-à-dire de la forme $X_U - X_V$. Nous présenterons par la suite des résultats sur la régularité par rapport aux accroissements sur $\mathscr{C}$ définis au Paragraphe 1.1.1. Avant cela, le prochain théorème raffine le critère de Kolmogorov pour fournir une valeur presque sure aux exposants de Hölder pour les processus gaussiens. On rappelle que les exposants déterministes ponctuel $\mathbb{\alpha}_X$ et local $\widetilde{\mathbb{\alpha}}_X$ sont définis sur un espace métrique quelconque dans la Section 1.4.3, et le sont donc bien sur $(\mathscr{A}, d_{\mathscr{A}})$.

Théorème (2.35)**.** *Soit $X = \{X_U;\ U \in \mathscr{A}\}$ un processus gaussien set-indexed, avec $(\mathscr{A}_n)_{n\in\mathbb{N}}$ et $d_{\mathscr{A}}$ vérifiant l'hypothèse $(H1)$ de $(\mathscr{H}_{\underline{\mathscr{A}}})$. Supposons ques les fonctions $U_0 \mapsto \liminf_{U\to U_0} \widetilde{\mathbb{\alpha}}_X(U)$ et $U_0 \mapsto \liminf_{U\to U_0} \mathbb{\alpha}_X(U)$ soient strictement positives sur $\mathscr{A}$. Alors avec probabilité 1,*

$$\forall U_0 \in \mathscr{A}, \quad \liminf_{U\to U_0} \widetilde{\mathbb{\alpha}}_X(U) \leq \widetilde{\boldsymbol{\alpha}}_X(U_0) \leq \limsup_{U\to U_0} \widetilde{\mathbb{\alpha}}_X(U)$$

et

$$\forall U_0 \in \mathscr{A}, \quad \liminf_{U\to U_0} \mathbb{\alpha}_X(U) \leq \boldsymbol{\alpha}_X(U_0)\ .$$

Nous avons déjà mentionné dans cette introduction l'importance des $\mathscr{C}$-accroissements pour les processus multiparamètres et set-indexed. Sans rajouter d'hypothèses, la régularité sur $\mathscr{C}$ s'avére rapidement plus compliquée que sur $\mathscr{A}$: considérons par exemple le mouvement brownien indexé par les rectangles (attachés en 0) de $\mathbb{R}^2_+$, puisque celui-ci a de bonnes propriétés de régularité, alors que sur la classe $\mathscr{C}$ associée on peut montrer qu'il est presque surement nonborné (voir l'exemple similaire du mouvement brownien indicé par les lower layers de $[0,1]^2$ dans [5, p.28]). Nous avons donc restreint la classe $\mathscr{C}$ aux accroissements du type $U \setminus \cup_{k=1}^{L} U_k$ où L est un entier fixé. On montre alors que les exposants de Hölder définis sur les accroissements de $\mathscr{C}^L$ ne dépendent pas du choix de L, raison pour laquelle on les note $\boldsymbol{\alpha}_{X,\mathscr{C}}$ et $\widetilde{\boldsymbol{\alpha}}_{X,\mathscr{C}}$.

Corollaire (2.36)**.** *Soit $X = \{X_U,\ U \in \mathscr{A}\}$ un processus gaussien set-indexed. Si les $(\mathscr{A}_n)_{n\in\mathbb{N}}$ vérifient l'hypothèse $(\mathscr{H}_{\underline{\mathscr{A}}})$ et si les exposants $\mathscr{C}$-Hölder déterministes sont finis, alors pour tout $U_0 \in \mathscr{A}$,*

$$\boldsymbol{\alpha}_{X,\mathscr{C}}(U_0) = \sup\left\{\alpha :\ \limsup_{\rho\to 0} \sup_{\substack{U,V\in B_{d_{\mathscr{A}}}(U_0,\rho)\\ V\subset U}} \frac{\mathbb{E}\left(|\Delta X_{U\setminus V}|^2\right)}{\rho^{\alpha}} < \infty\right\} \text{ p.s. },$$

$$\widetilde{\boldsymbol{\alpha}}_{X,\mathscr{C}}(U_0) = \sup\left\{\alpha :\ \limsup_{\rho\to 0} \sup_{\substack{U,V\in B_{d_{\mathscr{A}}}(U_0,\rho)\\ V\subset U}} \frac{\mathbb{E}\left(|\Delta X_{U\setminus V}|^2\right)}{d_{\mathscr{A}}(U,V)^{\alpha}} < \infty\right\} \text{ p.s.}$$

Enfin, les résultats de ce chapitre sont appliqués au mouvement brownien fractionnaire set-indexed. Tous les exposants sont égaux p.s. au paramètre de Hurst H, et sous une hypothèse supplémentaire assez faible, l'exposant ponctuel est même uniformément égal à H.

Théorème (2.38)**.** *Soit* $\mathbf{B}^H$ *un mouvement brownien fractionnaire set-indexed sur* $(T, \mathscr{A}, m)$*,* $H \in (0, 1/2]$*. Supposons que les sous-classes* $(\mathscr{A}_n)_{n\in\mathbb{N}}$ *satisfont l'hypothèse* $(H1)$ *de* $(\mathscr{H}_{\underline{\mathscr{A}}})$*. Alors les exposants de Hölder local et ponctuel de* $\mathbf{B}^H$*, définis par rapport à la distance* d_m*, vérifient en tout point* $U_0 \in \mathscr{A}$*:*

$$\mathbb{P}\Big(\forall U_0 \in \mathscr{A},\ \widetilde{\boldsymbol{\alpha}}_{\mathbf{B}^H}(U_0) = H\Big) = 1$$

et si la condition supplémentaire (2.22) est vérifiée,

$$\mathbb{P}(\forall U_0 \in \mathscr{A},\ \boldsymbol{\alpha}_{\mathbf{B}^H}(U_0) = H) = 1 \ .$$

En particulier, on a pour le fBm multiparamètre, défini par la covariance (1.5) sur les rectangles de $[0,1]^N$*, que* $\mathbb{P}\Big(\forall t_0 \in [0,1]^N,\ \boldsymbol{\alpha}_{\mathbf{B}^H}(t_0) = \widetilde{\boldsymbol{\alpha}}_{\mathbf{B}^H}(t_0) = H\Big) = 1$.

Le Chapitre 3 prolonge le travail du Chapitre 2, au sens où y est construit un processus multifractionnaire auquel le Théorème 2.38 s'applique, à la différence que le paramètre de Hurst est désormais variable. Les champs browniens fractionnaires que nous construisons seront indicés par des espace $L^2(T,m)$, et par restriction pourront être vus comme des processus set-indexed.

Donnons en premier lieu une construction du champ brownien fractionnaire sur $[0,1]$. Si $\mathscr{W}$ est la mesure de Wiener sur l'espace des fonctions continues issues de 0, noté W, le Théorème 3.3 donne l'existence d'un opérateur $\mathscr{K}_h$ agissant sur l'espace de Hilbert H_h (celui de la construction de l'espace de Wiener du mouvement brownien fractionnaire, Section 1.3.3) vers W^* permettant d'écrire que sur l'espace de probabilité $(\Omega, \mathscr{F}, \mathbb{P})$, l'expression suivante définie un champ brownien fractionnaire sur $(0,1) \times [0,1]$:

$$\forall (h,t) \in (0,1) \times [0,1], \quad \boldsymbol{B}_{h,t} = \int_W \langle \mathscr{K}_h R_h(\cdot,t), w\rangle \ \mathrm{d}\mathbb{B}_w \ ,$$

où $\mathbb{B}$ est un bruit blanc sur W de mesure de contrôle $\mathscr{W}$ (voir la Définition (1.1)). De plus, $\mathscr{K}_h$ est construit de sorte que pour $h \in (0,1)$ fixé, $\{\boldsymbol{B}_{h,t},\ t \in [0,1]\}$ est un mouvement brownien fractionnaire. Ainsi, $\boldsymbol{B}$ est bien un champ brownien fractionnaire au sens de la Définition 1.5.

Soit maintenant $\{k_h(\cdot,\cdot),\ h \in (0,1/2]\}$ la famille de covariance sur $L^2(T,m)$ définie par (1.4), avec pour chaque h, un espace de Hilbert à noyau reproduisant $H(k_h)$ auquel est associé un espace de Wiener $(H(k_h), E_h, \mu_h)$ par le procédé décrit dans le Paragraphe 1.3.2. Pour chaque $h \in (0,1/2]$, les espaces de Wiener $(H_h, W, \mathscr{W}_h)$ et $(H(k_h), E_h, \mu_h)$ sont reliés par une isométrie linéaire. On en déduit au Paragraphe 3.2.2 l'existence d'un opérateur $\tilde{\mathscr{K}_h}$ défini sur $H(k_h)$ et à valeurs dans $E^*_{1/2}$, tel que le processus défini par:

$$\forall (h,f) \in (0,1/2] \times L^2(T,m), \quad \boldsymbol{B}_{h,f} = \int_E \langle \tilde{\mathscr{K}_h} k_h(\cdot,f), x\rangle \ \mathrm{d}\mathbb{W}(x) \ ,$$

est un champ brownien fractionnaire. Il est ensuite prouvé que ce processus a des accroissements dont la loi vérifie le théorème suivant:

Théorème (3.15)**.** *Soit* $\boldsymbol{B}$ *un champ brownien fractionnaire sur* $(0,1/2] \times L^2(T,m)$*. Pour tout* $\eta \in (0,1/4)$ *et tout compact* D *de* L^2*, il existe une constante* $C_{\eta,D} > 0$ *telle que pour tous* $f_1, f_2 \in D$*, et tous* $h_1, h_2 \in [\eta, 1/2-\eta]$*,*

$$\mathbb{E}\big((\boldsymbol{B}_{h_1,f_1} - \boldsymbol{B}_{h_2,f_2})^2\big) \leq C_{\eta,D}\Big((h_2-h_1)^2 + m\big((f_1-f_2)^2\big)^{2(h_1\wedge h_2)}\Big) \ .$$

De la même manière, nous montrons comment ce type de résultat s'applique à des fonctionnelles de certains champs browniens fractionnaires apparaissant comme solutions d'équations aux dérivées partielles stochastiques. Dans la Section 3.3.2, nous considérons le problème elliptique suivant, sur un domaine $U \subset [0,1]^2$:

$$\Delta u = \dot{\mathbb{W}}^{h_1,h_2} ,$$

où $\dot{\mathbb{W}}^{h_1,h_2}$ est un bruit gaussien fractionnaire sur $[0,1]^2$ qu'on peut “extraire” (avant plus de précisions dans le paragraphe concerné) d'un seul et même champ brownien fractionnaire. Les solutions de cette équation sont définies au sens *mild*, et la régularité des lois quand les paramètres h_1 et h_2 varient dans $(0,1/2]$ est prouvée, et similaire à celle trouvée au théorème précédent.

Ces théorèmes sont particulièrement importants du point de vue de la régularité des trajectoires, comme nous pouvons le constater avec la proposition suivante. Pour cela, rappelons la distance naturelle sur $L^2(T,m)$, $d_m(f,g) = m\big((f-g)^2\big)^{1/2}$.

Théorème (3.19)**.** *Soit $\boldsymbol{B}$ un champ brownien fractionnaire indicé par un sous-ensemble compact I de $(0,1/2]$, et K un sous-ensemble compact de $L^2(T,m)$ de d_m-diamètre plus petit que 1. Définissons $\iota = \inf I$. Si l'intégrale de Dudley suivante converge:*

$$\int_0^1 \sqrt{\log N(K,d_m,\varepsilon^{1/2\iota})}\, d\varepsilon < \infty ,$$

alors $\boldsymbol{B}$ indicé par $I \times K$ admet une modification ayant des trajectoires presque surement continues.

Considérons désormais la collection set-indexed $\mathscr{A}$ définie sur l'espace mesurable (T,m), et le champ brownien fractionnaire précédent défini sur $(0,1/2] \times L^2(T,m)$. Soit $\boldsymbol{h}$ une fonction indicé par $\mathscr{A}$ et à valeurs dans $(0,1/2]$, dont on suppose qu'elle vérifie en tout point $U \in \mathscr{A}$ que sa valeur en U est plus faible que sa régularité hölderienne en ce point. Une telle fonction est dite régulière. Le *mouvement brownien multifractionnaire set-indexed* (SImBm) de fonction de régularité $\boldsymbol{h}$, noté $\{\boldsymbol{B}^{\boldsymbol{h}}_U,\ U \in \mathscr{A}\}$, est défini comme suit:

$$\forall U \in \mathscr{A}, \quad \boldsymbol{B}^{\boldsymbol{h}}_U = \boldsymbol{B}_{\boldsymbol{h}(U),\mathbf{1}_U} .$$

Le membre de droite est donc le champ brownien fractionnaire initial évalué en $(h,f) = (\boldsymbol{h}(U),\mathbf{1}_U)$, où $\mathbf{1}_U$ est l'indicatrice de $U \in \mathscr{A}$ qui appartient donc bien à $L^2(T,m)$. Ce processus multifractionnaire vérifie la propriété de régularité suivante:

Proposition (3.25)**.** *Soit $\mathscr{A}$ une collection set-indexed vérifiant l'hypothèse $(H1)$ de $(\mathscr{H}_{\underline{\mathscr{A}}})$. Soit $\boldsymbol{h} : \mathscr{A} \to (0,1/2]$ une fonction régulière et $\boldsymbol{B}^{\boldsymbol{h}}$ un SImBm sur $\mathscr{A}$ de fonction de régularité $\boldsymbol{h}$. Alors, presque surement,*

$$\forall U_0 \in \mathscr{A}, \quad \widetilde{\boldsymbol{\alpha}}_{\boldsymbol{B}^{\boldsymbol{h}}}(U_0) = \boldsymbol{h}_{U_0} \ \text{ et } \ \boldsymbol{\alpha}_{\boldsymbol{B}^{\boldsymbol{h}}}(U_0) \geq \boldsymbol{h}_{U_0} .$$

Nous avons donc construit un mouvement brownien multifractionnaire indicé par des ensembles, dont la régularité peut être prescrite par une fonction $\boldsymbol{h}$, au même titre qu'un mouvement brownien multifractionnaire sur $\mathbb{R}$.

Le Chapitre 4 est composé de plusieurs résultats indépendants concernant le L^2-mouvement brownien fractionnaire (h fixé) d'une part, et des résultats d'approximation du L^2-mouvement brownien multifractionnaire par des L^2-mouvements browniens fractionnaires d'autre part. On y trouvera également une propriété d'auto-similarité locale du champ brownien fractionnaire.

Dans un premier temps, deux propriétés de stationarité des accroissements et deux propriétés d'auto-similarité sont discutées. Deux couples stationarité des accroissements–auto-similarité sont formés, le premier étant similaire à la définition multiparamètre, tandis que le deuxième tient compte de la mesure m et permet de poursuivre la discussion entamée dans le cadre set-indexed [62]. Pour chacun de ces couples, un théorème de caractérisation du L^2-mouvement brownien fractionnaire est exprimé (Propositions 4.1 et 4.3).

Dans une deuxième partie, il est prouvé que pour tout $h \in (0,1/2)$, le L^2-mouvement brownien fractionnaire est *localement non-déterministe*, au sens où:

Proposition (4.4)**.** *Pour $h \in (0,1/2)$, il existe une constante strictement positive C_0 telle que pour tout $f \in L^2(T,m)$ et pour tout $r \leq \|f\|$, nous avons l'égalité suivante:*

$$Var\left(\boldsymbol{B}^h_f \mid \boldsymbol{B}^h_g, \|f-g\| \geq r\right) = C_0 r^{2h}.$$

Cette propriété est fausse pour $h = 1/2$, ce qui est discuté dans la Partie 4.2. La propriété de non-déterminisme local est un outils précieux pour l'étude des trajectoires des processus ne possédant pas la propriété de Markov. Elle permet notamment d'obtenir les probabilités de petites déviations pour les processus gaussiens.

Théorème (4.6)**.** *Soit $\boldsymbol{B}^h$ un L^2-mouvement brownien fractionnaire de paramètre $h \in (0,1/2)$ et K un compact de $L^2(T,m)$. Soit d_h la distance induite par $\boldsymbol{B}^h$ sur L^2. Supposons qu'il existe une fonction ψ telle que $\psi(\varepsilon) \asymp \psi(\varepsilon/2)$ au voisinage de 0, et que l'entropie vérifie $N(K,d_h,\varepsilon) \leq \psi(\varepsilon)$ pour tout $\epsilon > 0$. Alors il existe deux constantes $\kappa_1, \kappa_2 > 0$ telles que:*

$$\exp\left(-\kappa_2\ \psi(\varepsilon)\right) \leq \mathbb{P}\left(\sup_{f\in K} |\boldsymbol{B}^h_f| \leq \varepsilon\right) \leq \exp\left(-\kappa_1\ N(K,d_h,\varepsilon)\right)\ .$$

Ce résultat est un premier pas vers la loi du logarithme itéré qui sera présentée par la suite. Enfin, nous exposons des résultats d'approximation du L^2-mBm par des L^2-fBm. Comme dans le cadre set-indexed, le L^2-mBm est défini pour une fonction de régularité $\boldsymbol{h} : L^2(T,m) \to (0,1/2]$ par:

$$\boldsymbol{B}^{\boldsymbol{h}}_f = \boldsymbol{B}_{\boldsymbol{h}(f),f}\ ,\quad f \in L^2(T,m)\ .$$

Le premier résultat consiste à reconstituer un L^2-mouvement brownien multifractionnaire à partir d'un processus qui est un L^2-mouvement brownien fractionnaire par morceaux (voir Théorème 4.8). Le deuxième résultat d'approximation concerne les processus tangents du L^2-mouvement brownien multifractionnaire. Cette notion fut introduite par FALCONER [47], et nous montrons que pour $\boldsymbol{B}$ un champ brownien fractionnaire et $\boldsymbol{h}$ une fonction de $L^2(T,m)$ à valeurs dans $(0,1/2]$, le processus défini en tout point $f_0 \in L^2(T,m)$ et tout $\rho > 0$ par:

$$Y^{f_0,\alpha}_f(\varrho) = \varrho^{-\alpha}\left(\boldsymbol{B}^{\boldsymbol{h}}_{f_0+\varrho f} - \boldsymbol{B}^{\boldsymbol{h}}_{f_0}\right)\ ,\quad f \in L^2(T,m)$$

converge lorsque $\varrho \to 0$ vers un L^2-mouvement brownien fractionnaire.

Théorème (4.10)**.** *Soit K un compact de $L^2(T,m)$. Supposons que $\boldsymbol{h}$ est régulière et à valeurs dans $[\eta, 1/2-\eta]$ pour un certain $\eta \in (0,1/4)$. Soit $\iota = \inf_K \boldsymbol{h}(f)$ $(\geq \eta)$ et supposons la convergence de l'intégrale de Dudley sur K:*

$$\int_0^{D(K)} \sqrt{\log N(K,d_m,\varepsilon^{1/2\iota})}\, d\varepsilon < \infty\ .$$

Alors, pour $\alpha = 2\boldsymbol{h}(f_0)$, $Y^{f_0,\alpha}(\varrho)$ converge en loi dans l'espace des fonctions continues $C(K)$ lorsque $\varrho \to 0$, et la limite est un L^2-mouvement brownien fractionnaire de paramètre $\boldsymbol{h}(f_0)$.

Le Chapitre 5 tente de répondre à une question laissée ouverte par HERBIN AND XIAO [65] sur le comportement du mouvement brownien fractionnaire multiparamètre en 0 (dont on rappelle que la covariance $k_h^{(\nu)}$ est donnée par (1.5)). Les modules de continuité locaux de ce processus sont non triviaux, ce dont on pouvait se douter en observant la distance induite par le processus. En effet, celle-ci est liée à la distance $d_\lambda(s,t) = \lambda([0,s] \triangle [0,t])$ où λ est la mesure de Lebesgue sur $\mathbb{R}^\nu$, qui est singulière par rapport à la distance euclidienne si $\nu \geq 2$. Cette singularité affecte la régularité du fBm multiparamètre: en effet, il est montré que la loi du logarithme itéré de Chung en 0 est différente de celle pour tous les autres points loin des axes.

Grâce au théorème 4.6, on calcule les petites déviations du fBm multiparamètre. Ce résultat et un théorème de représentation spectrale (Proposition 5.8) permettent de calculer une loi du logarithme itéré de type Chung en 0, sous la forme d'une borne inférieure donnée par le module suivant:

$$\Psi_h^{(\ell)}(r) = r^{\nu h}\, \widetilde{\Psi}_h^{(\ell)}(r) = r^{\nu h}\, (\log\log r^{-1})^{-h/\nu}\ ,$$

et d'une borne supérieure avec pour module: $\Psi_h^{(u)} = r^{\nu h}\, \widetilde{\Psi}_h^{(u)}$, où $\widetilde{\Psi}_h^{(u)}$ est une fonction croissante partant de 0 dont on prouve l'existence, sans obtenir d'écriture explicite. Dans les deux cas, $\widetilde{\Psi}_h^{(\ell)}$ et $\widetilde{\Psi}_h^{(u)}$ sont négligeables par rapport au terme $r^{\nu h}$. Le supremum du fBm multiparamètre au voisinage de 0 sera noté $M^h(r) = \sup_{t\in[0,r]^\nu} |\boldsymbol{B}_t^h|$, $r \in [0,1]$. On a alors:

Théorème (5.1)**.** *Soit $h \in (0,1/2)$ et soit M^h, $\Psi_h^{(\ell)}$ et $\Psi_h^{(u)}$ comme définis précédemment. Alors, presque surement:*

$$\liminf_{r\to 0^+} \frac{M^h(r)}{r^{\nu h}\widetilde{\Psi}_h^{(\ell)}(r)} \geq \kappa_1^{h/\nu} \quad et \quad \liminf_{r\to 0^+} \frac{M^h(r)}{r^{\nu h}\widetilde{\Psi}_h^{(u)}(r)} \leq \kappa_2^{h/\nu}\ ,$$

où $\kappa_1 \leq \kappa_2$ sont les constantes apparaissant dans les petites déviations du L^2-fBm.

Comme dans l'article de MONRAD ET ROOTZÉN [106], ce théorème est étendu sous forme de loi fonctionelle, qui donne un principe d'invariance pour le fBm multiparamètre correctement renormalisé. Pour tout $r \in (0,1)$, on définit comme dans le théorème précédent deux renormalisations:

$$\eta_r^{(h,\ell)}(t) = \frac{\boldsymbol{B}^h(rt)}{r^{\nu h}\sqrt{\log\log(r^{-1})}}\ , \forall t \in [0,1]^\nu$$

et

$$\eta_r^{(h,u)}(t) = \frac{\boldsymbol{B}^h(rt)}{r^{\nu h}\left(\widetilde{\Psi}_h^{(u)}(r)\right)^{-\nu/2h}}\ , \forall t \in [0,1]^\nu\ .$$

On obtient alors la vitesse de convergence ainsi que l'ensemble vers lequel convergent les processus $\eta^{(h,u)}$ et $\eta^{(h,\ell)}$.

Théorème (5.2)**.** *Soit $h \in (0,1/2)$ et H_h^ν l'espace de Hilbert à noyau reproduisant de $k_h^{(\nu)}$. Soit $\varphi \in H_h^\nu$ ayant une norme strictement inférieure à 1. Alors, il existe deux constantes strictement positives et finies $\gamma^{(\ell)}(\varphi)$ et $\gamma^{(u)}(\varphi)$ telles que, presque surement,*

$$\liminf_{r\to 0^+} \widetilde{\Psi}_h^{(\ell)}(r)^{-1-\nu/2h} \sup_{t\in[0,1]^\nu} |\eta_r^{(h,\ell)}(t) - \varphi(t)| \geq \gamma^{(\ell)}(\varphi)$$

$$\liminf_{r\to 0^+} \widetilde{\Psi}_h^{(u)}(r)^{-1-\nu/2h} \sup_{t\in[0,1]^\nu} |\eta_r^{(h,u)}(t) - \varphi(t)| \leq \gamma^{(u)}(\varphi)\ .$$

Notons que lorsque $\varphi = 0$, on retrouve le théorème précédent.

La dimension de Hausdorff de l'image du fBm multiparamètre ne capture quant à elle pas l'irrégularité de ce processus en 0, comme l'indique la Proposition 5.17. En revanche, tout comme la fonction chirp (Exemple 1.15), la singularité du fBm multiparamètre se manifeste aussi au niveau de la régularité hölderienne locale (Remarque 5.18). Pour conclure, il semble envisageable que pour les processus gaussiens en général, il existe un lien entre le module de continuité local de Chung et un nouvel exposant de Hölder local. Ceci est discuté à la fin du Chapitre 5.

Hölder regularity of set-indexed processes 2

Contents

2.1 Introduction

Sample path properties of stochastic processes have been deeply studied for a long time, starting with the works of KOLMOGOROV, LÉVY and others on the modulus of continuity and laws of the iterated logarithm of the Brownian motion. Since the late 1960s, these results were extended to general Gaussian processes, while a finer study of the local properties of these sample paths was carried out (we refer to BERMAN [21, 22], DUDLEY [43, 44], OREY AND PRUITT [113], OREY AND TAYLOR [114] and STRASSEN [135], for the early study of Gaussian paths and their rare events). Among the large literature dealing with fine analysis of regularity, Hölder exponents continue to be widely used as a local measure of oscillations (see [16, 18, 87, 94, 146] for examples of recent works in this area). Two different definitions, called local and pointwise Hölder exponents, are usually considered for a stochastic process $\{X_t;\ t \in \mathbb{R}_+\}$, depending whether the increment $X_t - X_s$ is compared with a power $|t-s|^\alpha$ or ρ^α inside a ball $B(t_0,\rho)$ when $\rho \to 0$. As an example, with probability one, the local regularity of fractional Brownian motion $\{B^H;\ t \in \mathbb{R}_+\}$ is constant along the path: the pointwise and local Hölder exponents at any $t \in \mathbb{R}_+$ are equal to the self-similarity index $H \in (0,1)$ (e.g. see [59]).

This field of research is also very active in the multiparameter context and a non-exhaustive list of authors and recent works in this area includes AYACHE [11], DALANG [34], KHOSHNEVISAN [34, 79], LÉVY-VÉHEL [59], XIAO [103, 152, 153]. As an extension to the multiparameter one, the set-indexed context appeared to be the natural framework to describe invariance principles studying convergence of empirical processes (e.g. see [115]). The understanding of set-indexed processes and particularly their regularity is a more complex issue than on points of $\mathbb{R}^N$. The simple continuity property is closely related to the nature of the indexing collection. As an example, Brownian motion indexed by the lower layers of $[0,1]^2$ (i.e. the subsets $A \subseteq [0,1]^2$ such that $[0,t] \subseteq A$ for all $t \in A$) is discontinuous with probability one (we refer to [3] or [70] for the detailed proof). As a matter of fact, necessary and sufficient conditions for the sample path continuity property were investigated, starting with DUDLEY [44] who introduced a sufficient condition on the metric entropy of the indexing set, followed by FERNIQUE [49] who gave a necessary conditions in the specific case of stationary processes on $\mathbb{R}^N$. TALAGRAND gave a definitive answer in terms of majorizing measures [139] (see [5] or [76] for a complete survey and also [7, 8] for a LIL and Lévy's continuity moduli for set-indexed Brownian motion). The question was left open so far concerning the exact Hölder regularity of set-indexed processes.

A formal set-indexed setting has been introduced by IVANOFF AND MERZBACH in order to study standard issues of stochastic processes, such as martingale and Markov properties (see [68, 70]). In this framework, an *indexing collection* $\mathscr{A}$ is a collection of subsets of a measure space (T,m), which is assumed to satisfy certains properties such as stability by intersection of its elements. Section 2.2 of the present chapter uses these properties, instead of conditions on the metric entropy, to derive a Kolmogorov-like criterion for Hölder-continuity of a set-indexed process. The collection of sets $\mathscr{A}$ is endowed with a metric $d_{\mathscr{A}}$ and a nested sequence $\underline{\mathscr{A}} = (\mathscr{A}_n)_{n\in\mathbb{N}}$ of finite subcollections of $\mathscr{A}$ such that each element of $\mathscr{A}$ can be approximated as the decreasing limit (for the inclusion) of its projections on the $\mathscr{A}_n$'s. We consider a supplementary Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$ on $\underline{\mathscr{A}}$ and $d_{\mathscr{A}}$ which impose that: 1) the distance from any $U \in \mathscr{A}$ to $\mathscr{A}_n$ can be related to the cardinal $k_n = \#\mathscr{A}_n$, roughly by $d_{\mathscr{A}}(U,\mathscr{A}_n) = O(k_n^{-1/q_{\underline{\mathscr{A}}}})$, where $q_{\underline{\mathscr{A}}}$ is called the *discretization exponent* of $(\mathscr{A}_n)_{n\in\mathbb{N}}$; and 2) a minimality condition on the class $(\mathscr{A}_n)_{n\in\mathbb{N}}$ that is verified in most cases. This is discussed in Section 2.2, together with the links between our assumption and entropic conditions of previous works. We prove in Theorem 2.9: If $\{X_U;\ U \in \mathscr{A}\}$ is a set-indexed process and α,β,K are positive constants such that $\mathbb{E}[|X_U - X_V|^\alpha] \leq K\ d_{\mathscr{A}}(U,V)^{q_{\underline{\mathscr{A}}}+\beta}$ for all $U,V \in \mathscr{A}$, then for all $\gamma \in (0,\beta/\alpha)$, there exist a random variable h^* and a constant $L>0$ such that almost surely

$$\forall U,V \in \mathscr{A};\quad d_{\mathscr{A}}(U,V) < h^* \Rightarrow |X_U - X_V| \leq L\ d_{\mathscr{A}}(U,V)^\gamma.$$

Alternatively, Hölder-continuity can be based on the usual definition for increments of set-indexed processes. Instead of quantities $X_U - X_V$, the increments of a set-indexed process $\{X_U;\ U \in \mathscr{A}\}$ are defined on the class $\mathscr{C}$ of sets $C = U_0 \setminus \bigcup_{1\leq k\leq n} U_k$ where $U_0, U_1, \ldots, U_n \in \mathscr{A}$ by the *inclusion-exclusion formula*

$$\Delta X_C = X_{U_0} - \sum_{k=1}^{n} \sum_{j_1<\cdots<j_k} (-1)^{k-1} X_{U_0\cap U_{j_1}\cap\cdots\cap U_{j_k}}.$$

This definition extends the notion of rectangular increments for multiparameter processes. For instance, quantities like $\Delta_{[\mathbf{u},\mathbf{v}]}\mathbf{B} = \mathbf{B_v} - \mathbf{B}_{(u_1,v_2)} - \mathbf{B}_{(v_1,u_2)} + \mathbf{B_u}$, where $\mathbf{u} \preccurlyeq \mathbf{v} \in \mathbb{R}_+^2$ and $\mathbf{B}$ is the Brownian sheet, were proved to be useful to derive geometric sample path properties of the process (see e.g. some of the works of DALANG AND WALSH [36]). Let us notice that some processes can satisfy an increment stationarity property with respect to these increments while they do not for quantities $X_U - X_V$. Moreover, this inclusion-exclusion principle is very useful when it comes to martingale and Markov properties. According to this definition, another way to express the Hölder-continuity of X is $|\Delta X_C| \leq L\ m(C)^\gamma$, for $C \in \mathscr{C}$. This question is clarified in Section 2.2.2.

The purpose of Hölder exponents is the (optimal) localization of the Hölder-continuity concept. Following the previous discussion, the first definition for local and pointwise Hölder exponents is based on the comparison between $|X_U - X_V|$ and a power $d_{\mathscr{A}}(U,V)^\alpha$ or ρ^α in a ball $B_{d_{\mathscr{A}}}(U_0,\rho)$ around $U_0 \in \mathscr{A}$ when $\rho \to 0$. Another definition compares $|\Delta X_C|$ for $C = U \setminus \bigcup_{1\leq k\leq n} V_k$ in $\mathscr{C}$ with $d_{\mathscr{A}}(U,U_0) < \rho$ and $d_{\mathscr{A}}(U_0,V_k) < \rho$ for each k, to a power $m(C)^\alpha$ when $\rho \to 0$. As in the real-parameter setting, these two kinds of exponents, precisely defined in Sections 2.3 and 2.3.1, provide a fine knowledge of the local behaviour of the sample paths. In Section 2.4, the different Hölder exponents are linked to the Hölder regularity of projections of the set-indexed process on increasing paths.

The *pointwise continuity* has been introduced in the multiparameter setting in [4] and in the set-indexed setting in [63] as a weak form of continuity. In this definition, the *point mass jumps* are the only kind of discontinuity considered. Without any supplementary condition on the indexing collection, the set-indexed Brownian motion satisfies this property, even on lower layers where it is not continuous. In Section 2.3.2, we define the pointwise continuity Hölder exponent of a pointwise continuous process X by a comparison between $\Delta X_{C_n(t)}$ with a power $m(C_n(t))^\alpha$ when $n \to \infty$, where $(C_n(t))_{n\in\mathbb{N}}$ is a decreasing sequence of elements in $\mathscr{C}$ which converges to $t \in T$.

In the Gaussian case, we prove in Section 2.5 that the different aforementioned Hölder exponents admit almost sure values. Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$ is the key to extend this result from the multiparameter to the set-indexed setting. Moreover these almost sure values can be obtained uniformly on $\mathscr{A}$ for the local exponent. However, this a.s. result cannot be obtained for the pointwise exponent (even for multiparameter processes). Nevertheless, we proved that it holds for the set-indexed fractional Brownian motion (defined in [60]) in Section 2.6, thus improving on a result in the multiparameter case [59]. As this requires some specific extra work, we believe that the uniform almost sure result might not be true for the pointwise exponent of any Gaussian process. Finally, we also applied our results to the set-indexed Ornstein-Uhlenbeck (SIOU) process [17], for which all exponents are almost surely equal to $1/2$ at any set $U \in \mathscr{A}$.

2.2 Hölder continuity of a set-indexed process

In the classical case of real-parameter (or multiparameter) stochastic processes, Kolmogorov's continuity criterion is a useful tool to study sample path Hölder-continuity (e.g. see [76, 59]). In this section, we focus on the definition of a suitable assumption on the indexing collection, that allows to prove an extension of this result to the set-indexed (possibly non-Gaussian) setting.

2.2.1 *Indexing collection for set-indexed processes*

A general framework was introduced by Ivanoff and Merzbach to study martingale and Markov properties of set-indexed processes (we refer to [68, 70] for the details of the theory). The structure of these indexing collections allowed the study of the set-indexed extension of fractional Brownian motion [60], its increment stationarity property [62] and a complete characterization of the class of set-indexed Lévy processes [63].

Let T be a locally compact complete separable metric and measure space, with metric d and Radon measure m defined on the Borel sets of T. All stochastic processes will be indexed by a class $\mathscr{A}$ of compact connected subsets of T.

In the whole chapter, the class of finite unions of sets in any collection $\mathscr{D}$ will be denoted by $\mathscr{D}(u)$. In the terminology of [70], we assume that the *indexing collection* $\mathscr{A}$ satisfies stability and separability conditions in the sense of Ivanoff and Merzbach:

Definition 2.1 (adapted from [70])**.** *A nonempty class $\mathscr{A}$ of compact, connected subsets of T is called an* indexing collection *if it satisfies the following:*

1. *$\emptyset \in \mathscr{A}$ and for all $A \in \mathscr{A}$, $A^\circ \neq A$ if $A \notin \{\emptyset, T\}$.*

2. *$\mathscr{A}$ is closed under arbitrary intersections and if $A, B \in \mathscr{A}$ are nonempty, then $A \cap B$ is nonempty. If (A_i) is an increasing sequence in $\mathscr{A}$ then $\overline{\bigcup_i A_i} \in \mathscr{A}$.*

3. *The σ-algebra $\sigma(\mathscr{A})$ generated by $\mathscr{A}$ is the collection $\mathscr{B}$ of all Borel sets of T.*

4. Separability from above*: There exists an increasing sequence of finite subclasses $\mathscr{A}_n = \{\emptyset, A_1^n, \dots, A_{k_n}^n\}$ ($n \in \mathbb{N}, k_n \geq 1$) of $\mathscr{A}$ closed under intersections and a sequence of functions $g_n : \mathscr{A} \to \mathscr{A}_n$ defined by*
$$\forall U \in \mathscr{A}, \quad g_n(U) = \bigcap_{\substack{V \in \mathscr{A}_n \\ V^\circ \supseteq U}} V$$
and such that for each $U \in \mathscr{A}$, $U = \bigcap_{n \in \mathbb{N}} g_n(U)$.

(Note: '$\overline{(\cdot)}$' and'$(\cdot)^\circ$' denote respectively the closure and the interior of a set.)

Standard examples of indexing collections can be mentionned, such as rectangles $[0, t]$ of $\mathbb{R}^N$, arcs of the circle $\mathbb{S}^2$ or lower layers. Some of them are detailed in Examples 2.5 and 2.6 below.

Distances on sets. In order to study the Hölder-continuity of set-indexed processes, we consider a distance on the indexing collection. Along this chapter, we may sometimes specify the distance on $\mathscr{A}$ that we are using. Among them, the following distances are of special interest:

- The classical Hausdorff metric d_H defined on $\mathscr{K} \setminus \emptyset$, the nonempty compact subsets of T, by
$$\forall U, V \in \mathscr{K} \setminus \emptyset; \quad d_H(U, V) = \inf\{\epsilon > 0 : U \subseteq V^\epsilon \text{ and } V \subseteq U^\epsilon\},$$
where $U^\epsilon = \{x \in T : d(x, U) \leq \epsilon\}$;

- and the pseudo-distance d_m defined by
$$\forall U, V \in \mathscr{A}; \quad d_m(U, V) = m(U \triangle V),$$
where m is the measure on T and $\triangle$ denotes the symetric difference of sets.

Remark 2.2. *In the case of $\mathscr{A} = \left\{[0, t];\ t \in \mathbb{R}_+^N\right\}$, $(s, t) \mapsto d_m([0, s], [0, t])$ induces a distance on $\mathbb{R}_+^N$. This distance can be compared to the classical distances of $\mathbb{R}^N$,*

$$d_1 : (s, t) \mapsto \|t - s\|_1 = \sum_{i=1}^N |t_i - s_i|,$$
$$d_2 : (s, t) \mapsto \|t - s\|_2 = \sum_{i=1}^N (t_i - s_i)^2,$$
$$d_\infty : (s, t) \mapsto \|t - s\|_\infty = \max_{1 \leq i \leq N} |t_i - s_i|.$$

If m is the Lebesgue measure λ on $\mathbb{R}^N$, the distance d_λ is equivalent to d_1, d_2 and d_∞ on any compact of $\mathbb{R}_+^N \setminus \{0\}$.
More precisely, for all $a \prec b$ in $\mathbb{R}_+^N \setminus \{0\}$, there exist two positive constants $m_{a,b}$ and $M_{a,b}$ such that

$$\forall s,t \in [a,b]; \quad m_{a,b}\ d_1(s,t) \leq \lambda([0,s] \triangle [0,t]) \leq M_{a,b}\ d_\infty(s,t).$$

We refer to [58] for a proof of these assertions.

Total boundedness of indexing collections. As discussed in the introduction, the study of continuity of stochastic processes is closely related to the control of the metric entropy of the indexing collection. Following the conditions of Definition 2.1, some additional assumptions on the collection $\mathscr{A}$ are required to guarantee that $(\mathscr{A}, d_{\mathscr{A}})$ is totally bounded (or at least locally totally bounded). We recall that a metric space (T,d) is *totally bounded* if for any $\epsilon > 0$, T can be covered by a finite number of balls of radius smaller than ϵ. The minimal number of such balls is called the *metric entropy* and is denoted by $N(T,d,\epsilon)$.

Before getting to the main assumption on the metric $d_{\mathscr{A}}$ and the finite subclasses $(\mathscr{A}_n)_{n\in\mathbb{N}}$ that approximate $\mathscr{A}$, we notice that the sequence $(k_n)_{n\in\mathbb{N}} = (\#\mathscr{A}_n)_{n\in\mathbb{N}}$ is an increasing sequence that tends to ∞, as $n \to \infty$. This property comes from condition *(4)* in Definition 2.1. We will say that $(\mathscr{A}_n, k_n)_{n\in\mathbb{N}}$ is *admissible* if:

$$\forall \delta > 0, \quad \sum_{n=1}^{\infty} \frac{k_{n+1}}{k_n^{1+\delta}} < \infty. \tag{2.1}$$

This should not appear as a restriction anyhow, because: if $(k_n)_{n\in\mathbb{N}}$ was going to ∞ too slowly, it would suffice to extract a subsequence ; and in the opposite situation, the gap between one scale to the other is too large and can then be filled with additional subclasses.

Assumption ($\mathscr{H}_{\underline{\mathscr{A}}}$). Let $d_{\mathscr{A}}$ be a (pseudo-)distance on $\mathscr{A}$. Let us suppose that for $\underline{\mathscr{A}} = (\mathscr{A}_n)_{n\in\mathbb{N}}$, there exist positive real numbers $q_{\underline{\mathscr{A}}}$ and M_1 such that:

1. For all $n \in \mathbb{N}$,
$$\sup_{U\in\mathscr{A}_n} d_{\mathscr{A}}(U, g_n(U)) \leq M_1\ k_n^{-1/q_{\underline{\mathscr{A}}}}, \tag{H1}$$
2. and the collection $(\mathscr{A}_n)_{n\in\mathbb{N}}$ is minimal in the sense that: setting for all $n \in \mathbb{N}$ and all $U \in \mathscr{A}_n$,
$$\mathscr{V}_n(U) = \{V \in \mathscr{A}_n : V \supsetneq U, d_{\mathscr{A}}(U,V) \leq 3M_1 k_n^{-1/q_{\underline{\mathscr{A}}}}\},$$
the sequence $(N_n)_{n\geq 1}$ defined by $N_n = \max_{U\in\mathscr{A}_n} \#\mathscr{V}_n(U)$ for all $n \geq 1$ satisfies
$$\forall \delta > 0, \quad \sum_{n=1}^{\infty} k_n^{-\delta}\ N_n < \infty. \tag{H2}$$

The real $q_{\underline{\mathscr{A}}}$ is not unique and it depends *a priori* on the distance $d_{\mathscr{A}}$ and the sub-semilattices $\underline{\mathscr{A}} = (\mathscr{A}_n)_{n\in\mathbb{N}}$. Such a real $q_{\underline{\mathscr{A}}}$ is called *discretization exponent* of $(\mathscr{A}_n)_{n\in\mathbb{N}}$. Note that if N_n can be bounded independently of n, then the last assumption is satisfied by admissibility of k_n.

Remark 2.3. *Without loss of generality, the distance $d_{\mathscr{A}}$ can be normalized such that $M_1 = 1$.*

Remark 2.4. *The summability condition* (H2) *of Assumption* $(\mathscr{H}_{\underline{\mathscr{A}}})$ *is close to the notion of entropy with inclusion developped by Dudley [45] in the context of empirical processes. On the contrary to the present work, [45] focused exclusively on the sample path boundedness and continuity in the Brownian case.*

The following example shows that Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$ is satisfied in simple situations.

Example 2.5.
- *In the case of* $\mathscr{A} = \{[0,t];\ t \in [0,1] \subset \mathbb{R}_+\}$, *the subclasses* $\mathscr{A}_n$ $(n \in \mathbb{N})$ *are commonly* $\{[0,k.2^{-n}];\ k = 0,\dots,2^n\}$. *The two (pseudo-)distances* d_H *and* d_λ *(where* λ *is the Lebesgue measure on* $\mathbb{R}$*) on* $\mathscr{A}$ *are equal to*

$$d_{\mathscr{A}} : ([0,s],[0,t]) \mapsto |t-s|,$$

and we have

$$\forall k = 0,\dots,2^n-1; \quad d_{\mathscr{A}}([0,k.2^{-n}],[0,(k+1).2^{-n}]) = 2^{-n}.$$

Then the two conditions of Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$ *both are satisfied for* $q_{\underline{\mathscr{A}}} = 1$.

- *In the case of* $\mathscr{A} = \left\{[0,t];\ t \in [0,1] \subset \mathbb{R}_+^N\right\}$, *the subclasses* $\mathscr{A}_n$ $(n \in \mathbb{N})$ *can be chosen as*

$$\left\{[0,2^{-n}.(l_1,\dots,l_N)];\ 0 \le l_1,\dots,l_N \le 2^n\right\}.$$

Let U *be a set in* $\mathscr{A}$. *The distance (induced by the Lebesgue measure* λ*) between* U *and* $g_n(U)$ *is the volume difference between the two sets. It can be easily bounded from above by the sum of the volumes of the outer faces, minus a negligible residue*

$$\sup_{U\in\mathscr{A}} d_\lambda(U,g_n(U)) = \sup_{U\in\mathscr{A}} \lambda(g_n(U)\setminus U) = N.2^{-n} + o(2^{-n}).$$

Since $k_n = (2^n+1)^N$, *this leads to*

$$d_\lambda(U,g_n(U)) = O(k_n^{-1/q_{\underline{\mathscr{A}}}}),$$

with $q_{\underline{\mathscr{A}}} = N$ *and the other condition of Assumption* $(\mathscr{H}_{\underline{\mathscr{A}}})$ *are satisfied.*

On the contrary to the rectangles case, the following example shows that the collection of *lower layers* of $\mathbb{R}^N$ does not satisfy Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$. We will see later that this result is not surprising in the view of Theorem 2.9, since Brownian motion indexed by lower layers of $[0,1]^2$ does not have a continuous modification, as can be seen for instance in [3, 70].

Example 2.6. *Let* $\mathscr{A}$ *be the collection of lower layers of* $[0,1]^2$, *i.e. the subsets* A *of* $[0,1]^2$ *such that* $\forall t \in A$, $[0,t] \subseteq A$. *For all* $n \in \mathbb{N}$, *let* $\mathscr{A}_n$ *be the collection of finite unions of sets in the dissecting collection of the diadic rectangles of* $[0,1]^2$, *i.e.*

$$\mathscr{A}_n = \left\{ \bigcup_{finite} [0,x] : 2^n x \in \mathbf{Z}^2 \cap (0,2^n]^2 \right\} \cup \{0\} \cup \{\emptyset\}.$$

Then, it can be shown that the cardinal k_n *of* $\mathscr{A}_n$ *satisfies* $k_n \ge 2^{2^n}$ *for all* $n \in \mathbb{N}$. *For all* $U \in \mathscr{A}_n$, *we can see that* $\inf_{V\in\mathscr{A}_n, V\supsetneq U} d_\lambda(U,V) = 2^{-2n}$, *hence there does not exist any* $q_{\underline{\mathscr{A}}}$ *such that* 2^{-2n} *and* $k_n^{-1/q_{\underline{\mathscr{A}}}}$ *are of the same order. Consequently the subclasses* $(\mathscr{A}_n)_{n\in\mathbb{N}}$ *cannot verify Assumption* $(\mathscr{H}_{\underline{\mathscr{A}}})$.

To conclude this section, we emphasize the fact that Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$ implies the total boundedness of $(\mathscr{A}, d_{\mathscr{A}})$. Since

$$\forall n \in \mathbb{N}, \quad d_{\mathscr{A}}(U, g_n(U)) \leq k_n^{-1/q_{\underline{\mathscr{A}}}},$$

$\mathscr{A}_n$ constitutes a $k_n^{-1/q_{\underline{\mathscr{A}}}}$-net for all $n \in \mathbb{N}$, and thus $(\mathscr{A}, d_{\mathscr{A}})$ is totally bounded.

2.2.2 *Kolmogorov's criterion*

As $(\mathscr{A}, d_{\mathscr{A}})$ is not generally totally bounded, for any deterministic function $f : \mathscr{A} \to \mathbf{R}$, we consider the *modulus of continuity* on any totally bounded $\mathscr{B} \subset \mathscr{A}$

$$\omega_{f,\mathscr{B}}(\delta) = \sup_{\substack{U,V \in \mathscr{B} \\ d_{\mathscr{A}}(U,V) \leq \delta}} |f(U) - f(V)|.$$

Recall that the function f is said *Hölder continuous of order* $\alpha > 0$ if for all totally bounded $\mathscr{B} \subset \mathscr{A}$ one of the following equivalent conditions holds (e.g. see [76], Chapter 5)

(i)
$$\limsup_{\delta \to 0} \delta^{-\alpha}.\omega_{f,\mathscr{B}}(\delta) < \infty.$$

(ii) There exists $M > 0$ and $\delta_0 > 0$ such that for all $U, V \in \mathscr{B}$ with $d_{\mathscr{A}}(U, V) < \delta_0$, $|f(U) - f(V)| \leq M.d_{\mathscr{A}}(U, V)^{\alpha}$.

For any general set-indexed Gaussian process, Dudley's Corollary 2.3 in [44] allows to compute a modulus of continuity (giving the same kind of result than following Corollary 2.10). This result holds under certain entropic conditions on the indexing collection, which are different from these of our setting. Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$ and more precisely its second condition allows to prove a continuity criterion in the non-Gaussian case. Although Adler and Taylor [5] emphasize that the Gaussian property is only used through the exponential decay of the tail probability of the process in the proof of the previous results, they do not suggest any Kolmogorov criterion for non-Gaussian processes. The following Theorem 2.9 do so in the general set-indexed framework of Ivanoff and Merzbach, thanks to the discretization exponent.

Definition 2.7. *A (pseudo-)distance $d_{\mathscr{A}}$ on $\mathscr{A}$ is said:*

(i) Outer-continuous *if for any non-increasing sequence $(U_n)_{n\in\mathbb{N}}$ in $\mathscr{A}$ converging to $U = \bigcap_{n\in\mathbb{N}} U_n \in \mathscr{A}$, $d_{\mathscr{A}}(U_n, U)$ tends to 0 as n goes to ∞ ;*

(ii) Contractive *if it is outer-continuous and if for any $U, V, W \in \mathscr{A}$,*

$$d_{\mathscr{A}}(U \cap W, V \cap W) \leq d_{\mathscr{A}}(U, V).$$

Remark 2.8. *The most important metrics in the context of set-indexed processes, d_m and d_H, are contractive.*

Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$ on the subcollections $(\mathscr{A}_n)_{n\in\mathbb{N}}$ and the contractivity of the metric $d_{\mathscr{A}}$ allow to state the first important result of the chapter:

Theorem 2.9. *Let $d_{\mathscr{A}}$ be a contractive (pseudo-)distance on the indexing collection $\mathscr{A}$, whose subclasses $\underline{\mathscr{A}} = (\mathscr{A}_n)_{n\in\mathbb{N}}$ satisfy Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$ with a discretization exponent $q_{\underline{\mathscr{A}}} > 0$. Let $X = \{X_U;\ U \in \mathscr{A}\}$ be a set-indexed process such that*

$$\forall U,V \in \mathscr{A}, \quad \mathbb{E}[|X_U - X_V|^\alpha] \leq K\ d_{\mathscr{A}}(U,V)^{q_{\underline{\mathscr{A}}}+\beta} \tag{2.2}$$

where K, α and β are positive constants.
Then, the sample paths of X are almost surely locally γ-Hölder continuous for all $\gamma \in (0, \frac{\beta}{\alpha})$, i.e. there exist a random variable h^* and a constant $L > 0$ such that almost surely

$$\forall U,V \in \mathscr{A}, \qquad d_{\mathscr{A}}(U,V) < h^* \Rightarrow |X_U - X_V| \leq L\ d_{\mathscr{A}}(U,V)^\gamma.$$

Proof. Let us fix $\gamma \in (0, \frac{\beta}{\alpha})$ and denote $\mathscr{D} = \bigcup_{n\in\mathbb{N}} \mathscr{A}_n$ a countable dense subset of $\mathscr{A}$. First, let $(a_j)_{j\in\mathbb{N}}$ be any sequence of positive real numbers such that $\sum_{j\in\mathbb{N}} a_j < +\infty$, and for $n \in \mathbb{N}$ such that $\sum_{j\geq n} a_j \leq 1$, we have

$$\begin{aligned}
\mathbb{P}\Big(\sup_{U\in\mathscr{D}} |X_U - X_{g_n(U)}| \geq k_{n+1}^{-\gamma/q_{\underline{\mathscr{A}}}}\Big) \\
&\leq \mathbb{P}\Bigg(\exists U \in \mathscr{D},\ \sum_{j=n}^{\infty} |X_{g_{j+1}(U)} - X_{g_j(U)}| \geq k_{n+1}^{-\gamma/q_{\underline{\mathscr{A}}}}\Bigg) \\
&\leq \mathbb{P}\Big(\exists U \in \mathscr{D}, \exists j \geq n, |X_{g_{j+1}(U)} - X_{g_j(U)}| \geq a_j k_{n+1}^{-\gamma/q_{\underline{\mathscr{A}}}}\Big) \qquad (2.3) \\
&\leq \mathbb{P}\Big(\exists j \geq n,\ \exists V \in \mathscr{A}_{j+1},\ |X_V - X_{g_j(V)}| \geq a_j k_{n+1}^{-\gamma/q_{\underline{\mathscr{A}}}}\Big) \\
&\leq \sum_{j=n}^{\infty} \sum_{V\in\mathscr{A}_{j+1}} \mathbb{P}\Big(|X_V - X_{g_j(V)}| \geq a_j k_{n+1}^{-\gamma/q_{\underline{\mathscr{A}}}}\Big).
\end{aligned}$$

Now applying successively Tchebyshev's inequality, (2.2) and Equation (H1) of Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$,

$$\begin{aligned}
\mathbb{P}\Big(\sup_{U\in\mathscr{D}} |X_U - X_{g_n(U)}| \geq k_{n+1}^{-\gamma/q_{\underline{\mathscr{A}}}}\Big) &\leq \sum_{j=n}^{\infty} k_{j+1} a_j^{-\alpha} k_{n+1}^{\alpha\gamma/q_{\underline{\mathscr{A}}}} \sup_{V\in\mathscr{A}_{j+1}} \mathbb{E}\Big[|X_V - X_{g_j(V)}|^\alpha\Big] \\
&\leq K\ k_{n+1}^{\alpha\gamma/q_{\underline{\mathscr{A}}}} \sum_{j=n}^{\infty} a_j^{-\alpha} k_{j+1} \sup_{V\in\mathscr{A}_{j+1}} d_{\mathscr{A}}(V, g_j(V))^{q_{\underline{\mathscr{A}}}+\beta} \\
&\leq K\ k_{n+1}^{\alpha\gamma/q_{\underline{\mathscr{A}}}} \sum_{j=n}^{\infty} a_j^{-\alpha} \frac{k_{j+1}}{k_j} k_j^{-\beta/q_{\underline{\mathscr{A}}}}.
\end{aligned}$$

The admissibility of $(k_n)_{n\in\mathbb{N}}$ implies that for $\delta > 0$, and for n large enough (depending on δ), $k_{n+1}^{\alpha\gamma/q_{\underline{\mathscr{A}}}} \leq (k_n^{\alpha\gamma/q_{\underline{\mathscr{A}}}})^{1+\delta}$, so that:

$$\begin{aligned}
\mathbb{P}\Big(\sup_{U\in\mathscr{D}} |X_U - X_{g_n(U)}| \geq k_{n+1}^{-\gamma/q_{\underline{\mathscr{A}}}}\Big) &\leq K\ k_n^{\delta\alpha\gamma/q_{\underline{\mathscr{A}}}} \sum_{j=n}^{\infty} a_j^{-\alpha} \frac{k_{j+1}}{k_j} k_j^{-\beta/q_{\underline{\mathscr{A}}}} k_n^{\gamma\alpha/q_{\underline{\mathscr{A}}}} \\
&\leq K\ k_n^{\delta\alpha\gamma/q_{\underline{\mathscr{A}}}} \sum_{j=n}^{\infty} a_j^{-\alpha} \frac{k_{j+1}}{k_j} k_j^{-(\beta-\gamma\alpha)/q_{\underline{\mathscr{A}}}}.
\end{aligned}$$

Since $\beta-\alpha.\gamma>0$, $(a_j^\alpha)_{j\in\mathbb{N}}$ can be chosen equal to $(k_j^{-(\beta-\alpha\gamma)/3q_{\underline{\mathscr{A}}}})_{j\in\mathbb{N}}$ (which is indeed summable because k_n is admissible), and then:

$$\mathbb{P}\left(\sup_{U\in\mathscr{D}}|X_U-X_{g_n(U)}|\geq k_{n+1}^{-\gamma/q_{\underline{\mathscr{A}}}}\right)\leq K\ k_n^{\delta\alpha\gamma/q_{\underline{\mathscr{A}}}}\sum_{j=n}^{\infty}\frac{k_{j+1}}{k_j}k_j^{-2(\beta-\gamma\alpha)/3q_{\underline{\mathscr{A}}}},$$

which finally leads to, for $\delta=(\beta-\alpha\gamma)/(6\alpha\gamma)$,

$$\mathbb{P}\left(\sup_{U\in\mathscr{D}}|X_U-X_{g_n(U)}|\geq k_{n+1}^{-\gamma/q_{\underline{\mathscr{A}}}}\right)\leq K\ k_n^{-\delta\alpha\gamma/q_{\underline{\mathscr{A}}}}\sum_{j=n}^{\infty}\frac{k_{j+1}}{k_j}k_j^{-(\beta-\gamma\alpha)/3q_{\underline{\mathscr{A}}}}.$$

Thus, this probability is summable and Borel-Cantelli's theorem implies the existence of $\Omega^*\subset\Omega$ with $\mathbb{P}(\Omega^*)=1$ such that $\forall\omega\in\Omega^*$,

$$\exists n^*(\omega)\in\mathbb{N},\ \forall n\geq n^*,\forall U\in\mathscr{D},\quad |X_U-X_{g_n(U)}|<k_{n+1}^{-\gamma/q_{\underline{\mathscr{A}}}}. \tag{2.4}$$

Now, we develop the same argument for the following probability:

$$\begin{aligned}\mathbb{P}\left(\sup_{U\in\mathscr{A}_n}\sup_{V\in\mathscr{V}_n(U)}|X_U-X_V|\geq k_{n+1}^{-\gamma/q_{\underline{\mathscr{A}}}}\right)&\leq k_n\ N_n\sup_{U\in\mathscr{A}_n}\sup_{V\in\mathscr{V}_n(U)}\mathbb{P}\left(|X_U-X_V|\geq k_{n+1}^{-\gamma/q_{\underline{\mathscr{A}}}}\right)\\&\leq K\ N_n\ k_{n+1}^{\alpha\gamma/q_{\underline{\mathscr{A}}}}\ k_n^{-\beta/q_{\underline{\mathscr{A}}}}\\&\leq K\ N_n\ k_n^{-(\beta-\alpha\gamma)/2q_{\underline{\mathscr{A}}}},\end{aligned}\tag{2.5}$$

where we used δ as in the previous paragraph. This is summable by (H2), hence there exists Ω^{**} a measurable subset of Ω of probability 1 and n^{**} a integer-valued finite random variable such that on Ω^{**}:

$$\forall n\geq n^{**},\quad \sup_{U\in\mathscr{A}_n}\sup_{V\in\mathscr{V}_n(U)}|X_U-X_V|<k_{n+1}^{-\gamma/q_{\underline{\mathscr{A}}}}. \tag{2.6}$$

For any $U,V\in\mathscr{D}$, there is a unique $n\in\mathbb{N}$ such that $k_{n+1}^{-1/q_{\underline{\mathscr{A}}}}\leq d_{\mathscr{A}}(U,V)<k_n^{-1/q_{\underline{\mathscr{A}}}}$. Let $I_n=[k_{n+1}^{-1/q_{\underline{\mathscr{A}}}},k_n^{-1/q_{\underline{\mathscr{A}}}})$. Without any restriction, we assume that $U\subseteq V$. Indeed, if this not the case, we shall consider $X_U-X_V=X_U-X_{U\cap V}+X_{U\cap V}-X_V$, where $d_{\mathscr{A}}(U,U\cap V)\leq d_{\mathscr{A}}(U,V)$ by contractivity. Since this implies that $g_n(V)\in\mathscr{V}_n(g_n(U))$, we will write, on $\Omega^*\cap\Omega^{**}$, for any $n\geq n^*\vee n^{**}$:

$$\begin{aligned}\sup_{\substack{U,V\in\mathscr{D}\\ d_{\mathscr{A}}(U,V)\in I_n}}|X_U-X_V|&\leq\sup_{\substack{U,V\in\mathscr{D}\\ d_{\mathscr{A}}(U,V)\in I_n}}\left(|X_U-X_{g_n(U)}|\ +|X_{g_n(U)}-X_{g_n(V)}|+|X_{g_n(V)}-X_V|\right)\\&\leq 3\ k_{n+1}^{-\gamma/q_{\underline{\mathscr{A}}}}\\&\leq 3\ d_{\mathscr{A}}(U,V)^\gamma,\end{aligned}\tag{2.7}$$

as a consequence of Equations (2.4) and (2.6). Since $\Omega^*\cap\Omega^{**}$ is of probability 1, we have proved that there exist a constant $L>0$ and a random variable h^* such that

$$\forall U,V\in\mathscr{D};\quad d_{\mathscr{A}}(U,V)<h^*\Rightarrow|X_U-X_V|\leq L\ d_{\mathscr{A}}(U,V)^\gamma\quad\text{a.s.} \tag{2.8}$$

In the last part of the proof, we need to extend (2.8) to the whole class $\mathscr{A}$. From the outer-continuity of $d_{\mathscr{A}}$, we can claim:

On Ω^*, for all $\epsilon \in (0,h^*)$, for all U and V in $\mathscr{A}$ with $d_{\mathscr{A}}(U,V) < h^* - \epsilon$, there exists $n_0 > n^*$ such that $d_{\mathscr{A}}(g_n(U), g_m(V)) < h^*$ for all $n \geq n_0$ and $m \geq n_0$. Thus by (2.8),

$$\forall n > n_0, \forall m > n_0; \quad |X_{g_n(U)} - X_{g_m(V)}| \leq L\ d_{\mathscr{A}}(g_n(U), g_m(V))^{\gamma}. \tag{2.9}$$

We define the process $\tilde{X}$ by

- $\forall \omega \notin \Omega^*$, $\forall U \in \mathscr{A}$, $\tilde{X}_U(\omega) = 0$,
- $\forall \omega \in \Omega^*$,
  - $\forall U \in \mathscr{D}$, $\tilde{X}_U(\omega) = X_U(\omega)$
  - $\forall U \in \mathscr{A} \setminus \mathscr{D}$, $\tilde{X}_U(\omega) = \lim_{n\to\infty} X_{g_n(U)}(\omega)$.
    Applying (2.9) with $V = U$, the outer-continuity property of $d_{\mathscr{A}}$ implies that $\left(X_{g_n(U)}(\omega)\right)_{n\in\mathbb{N}}$ is a Cauchy sequence and then converges in $\mathbb{R}$.

The process $\tilde{X}$ satisfies almost surely

$$\forall U, V \in \mathscr{A}; \quad d_{\mathscr{A}}(U,V) < h^* \Rightarrow |\tilde{X}_U - \tilde{X}_V| \leq L.\ d_{\mathscr{A}}(U,V)^{\gamma}.$$

Moreover,

- $\forall U \in \mathscr{D}$, $\tilde{X}_U = X_U$ almost surely.
- $\forall U \in \mathscr{A} \setminus \mathscr{D}$, by construction, $X_{g_n(U)} \xrightarrow{\text{a.s.}} \tilde{X}_U$ as $n \to \infty$.
  Since $\mathbb{E}\left[|X_{g_n(U)} - X_U|^{\alpha}\right]$ converges to 0 when $n \to \infty$, the sequence $\left(X_{g_n(U)}\right)_{n\in\mathbb{N}}$ converges in probability to X_U. Then, there exists a subsequence converging almost surely.
  From these two facts, we get $\tilde{X}_U = X_U$ a.s.

□

As in the multiparameter's case, a simpler statement holds for Gaussian processes (see [76] for a detailed study of the Kolmogorov criterion in the multiparameter frame).

Corollary 2.10. *Let $d_{\mathscr{A}}$ be a (pseudo-)distance on the indexing collection $\mathscr{A}$, whose subclasses $\underline{\mathscr{A}} = (\mathscr{A}_n)_{n\in\mathbb{N}}$ satisfy Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$. Let $X = \{X_U;\ U \in \mathscr{A}\}$ be a centered Gaussian set-indexed process such that*

$$\forall U, V \in \mathscr{A}, \quad \mathbb{E}\left[|X_U - X_V|^2\right] \leq K\ d_{\mathscr{A}}(U,V)^{2\beta}$$

where $K > 0$ and $\beta > 0$.
Then, the sample paths of X are almost surely locally γ-Hölder continuous for all $\gamma \in (0,\beta)$.

Proof. For any $p \in \mathbb{N}^*$, there exists a constant $M_p > 0$ such that for all centered Gaussian random variable Y, we have $\mathbb{E}\left[Y^{2p}\right] = M_p\ (\mathbb{E}\left[Y^2\right])^p$. Then,

$$\forall U, V \in \mathscr{A}; \quad \mathbb{E}\left[|X_U - X_V|^{2p}\right] \leq K\ M_p\ d_{\mathscr{A}}(U,V)^{2p\beta}.$$

For all $\gamma \in (0,\beta)$, there exists $p \in \mathbb{N}^*$ such that $2p\beta > q_{\underline{\mathscr{A}}}$, where $q_{\underline{\mathscr{A}}}$ is the discretization exponent of $(\mathscr{A}_n)_{n\in\mathbb{N}}$. By Theorem 2.9 the result follows. □

Remark 2.11. *The proof of Theorem 2.9 shows that when Condition (H2) is removed from Assumption ($\mathcal{H}_{\underline{\mathcal{A}}}$), the conclusion remains true when the hypothesis (2.2) is strengthened in*

$$\forall U, V \in \mathcal{A}; \quad \mathbb{E}[|X_U - X_V|^\alpha] \leq K\, d_{\mathcal{A}}(U,V)^{2q_{\underline{\mathcal{A}}}+\beta}.$$

The result follows from the simple estimation $N_n \leq k_n$ in Equation (2.5). In that case, the validity of Corollary 2.10 persists, since the integer p can be chosen such that $2p\beta > 2q_{\underline{\mathcal{A}}}$ (instead of $2p\beta > q_{\underline{\mathcal{A}}}$).

As previously mentioned, the Brownian motion indexed by the lower layers of $[0,1]^2$ is discontinuous with probability one (e.g. see Theorem 1.4.5 in [5] or [3, 70]). The previous Theorem 2.9 and Corollary 2.10 do not contradict this fact, since the collection of lower layers of $[0,1]^2$ do not satisfy Assumption ($\mathcal{H}_{\underline{\mathcal{A}}}$) according to Example 2.6 in the specific case of the separating subclasses $(\mathcal{A}_n)_{n\in\mathbb{N}}$ mentioned there. This latter result is improved by the following corollary of Theorem 2.9.

Corollary 2.12. *Any subclasses $(\mathcal{A}_n)_{n\in\mathbb{N}}$ satisfying Condition (4) of Definition 2.1 for the indexing collection of lower layers of $[0,1]^2$ do not satisfy Assumption ($\mathcal{H}_{\underline{\mathcal{A}}}$).*

Following the early work of Dudley, the restriction of the set-indexed Brownian motion to an indexing collection satisfying certain conditions can admit a continuous modification. We refer to [5] for a modern survey of these results. In particular, the set-indexed Brownian motion is a.s. continuous over any Vapnik-Červonenkis class of sets (see Corollary 1.4.10 in [5]), as the collection of rectangles of $\mathbb{R}^N$ is an example.

Remark 2.13. *According to Dudley's Theorem (see Theorem 2.7.1 of Chapter 5 in [76] and also Theorems 1.3.5 and 1.5.4 in [5]), the existence of a continuous modification of a centered Gaussian $\mathcal{A}$-indexed process can be proved if $(\mathcal{A}, d_{\mathcal{A}})$ is totally bounded and $\int_0^1 \sqrt{\log N(\mathcal{A},\epsilon)}\, d\epsilon < +\infty$, where $N(\mathcal{A},\epsilon)$ denotes the entropy function (relative to the distance $d_{\mathcal{A}}$).*

Following the continuity on processes indexed by Vapnik-Červonenkis classes of sets and the role of Assumption ($\mathcal{H}_{\underline{\mathcal{A}}}$) in Theorem 2.9, we emphasize the fact that upper bounds for the entropy function can be obtained in the two cases. Let us define

$$\forall n \in \mathbb{N}, \quad \phi(n) = k_n^{-1/q_{\underline{\mathcal{A}}}}.$$

Let us also define, for $\epsilon \in (0, \frac{1}{2}]$, $n(\epsilon) = \inf\{k : \phi(k) < \epsilon\}$.
From Condition (H1) of Assumption ($\mathcal{H}_{\underline{\mathcal{A}}}$), for all $U \in \mathcal{A}$,

$$d_{\mathcal{A}}(U, g_{n(\epsilon)}(U)) \leq \phi(n(\epsilon)) \leq \epsilon,$$

which implies $N(\mathcal{A},\epsilon) \leq k_{n(\epsilon)}$.
We can see easily that:

$$0 < \epsilon \leq k_{n(\epsilon)}^{-1/q_{\underline{\mathcal{A}}}},$$

which allows to get a bound for the entropy function (relative to the distance $d_{\mathcal{A}}$),

$$N(\mathcal{A},\epsilon) \leq k_{n(\epsilon)} \leq \epsilon^{-q_{\underline{\mathcal{A}}}}. \tag{2.10}$$

In the case of a Vapnik-Červonenkis class $\mathscr{D}$ of sets in a measure space $(E, \mathscr{E}, \nu)$, the entropy function (relative to the distance $\nu(\bullet \triangle \bullet)$) is bounded as:

$$\forall 0 < \varepsilon \leq 1/2, \quad N(\mathscr{D}, \varepsilon) \leq K \varepsilon^{-2v} |\ln \varepsilon|^{v}, \tag{2.11}$$

where K and v are positive constants (e.g. see [5], Theorem 1.4.9).

2.2.3 $\mathscr{C}$-increments

So far, we only considered simple *increments* of X of the form $X_U - X_V$ for $U, V \in \mathscr{A}$ not necessarily ordered. However these quantities do not constitute the natural extension of the one-parameter $X_t - X_s$ ($s, t \in \mathbb{R}_+$) to multiparameter (e.g. [76, 4, 57]) and set-indexed (e.g. [70, 62]) settings, particularly when increment stationarity property is concerned. This section is devoted to usual increments of set-indexed processes, which extend the rectangular increments of multiparameter processes. Let us define, for any given indexing collection $\mathscr{A}$, the collection $\mathscr{C}$ of subsets of T, defined as

$$\mathscr{C} = \{U_0 \setminus \cup_{i=1}^{k} U_i;\ U_0, U_1, \ldots, U_k \in \mathscr{A}, k \in \mathbb{N}\}.$$

This collection is used to index the process ΔX, defined by $\Delta X_C = X_{U_0} - \Delta X_{U_0 \cap \bigcup_{i\geq 1} U_i}$ for $C = U_0 \setminus \cup_{i=1}^{k} U_i$, where $\Delta X_{U_0 \cap \bigcup_{i\geq 1} U_i}$ is given by the *inclusion-exclusion* formula

$$\Delta X_{U_0 \cap \bigcup_{i\geq 1} U_i} = \sum_{i=1}^{k} \sum_{j_1 < \cdots < j_i} (-1)^{i-1} X_{U_0 \cap U_{j_1} \cap \cdots \cap U_{j_i}}. \tag{2.12}$$

The existence of the increment process ΔX indexed by $\mathscr{C}$ requires that for any $C \in \mathscr{C}$, the value ΔX_C does not depend on the representation of C.

Corollary 2.14. *Under the hypotheses of Theorem 2.9 and if the distance $d_{\mathscr{A}}$ on the class $\mathscr{A}$ is assumed to be contractive, for each fixed integer $l \geq 1$, for all $\gamma \in (0, \beta/\alpha)$, there exist a random variable h^{**} and a constant $L > 0$ such that, with probability one,*

$$\forall C = U \setminus \bigcup_{i \leq l} V_i \text{ with } U, V_1, \ldots, V_l \in \mathscr{A},$$
$$\max_{i \leq l} \{m(U \setminus V_i)\} < h^{**} \Rightarrow |\Delta X_C| \leq L\ m(C)^{\gamma}. \tag{2.13}$$

For a proof of this result, see Appendix 2.7.1.

Corollary 2.14, as a result on the class $\mathscr{C}^l = \left\{U \setminus V;\ U \in \mathscr{A}, V \in \mathscr{B}^l\right\}$ where $\mathscr{B}^l = \left\{\bigcup_{i=1}^{l} V_i;\ V_1, \ldots, V_l \in \mathscr{A}\right\}$, does *not* extend to the whole $\mathscr{C} = \bigcup_{l \geq 1} \mathscr{C}^l$, as the following example shows. The next result is an adaptation of an example in [5, 70] to the set-indexed setting. It states that the Brownian motion can be unbounded on $\mathscr{C}$ when $\mathscr{A}$ is the collection of rectangles of $[0,1]^2$.

Proposition 2.15. *Let W be a Brownian motion indexed by the Borelian sets of $[0,1]^2$, i.e. a centered Gaussian process with covariance structure*

$$\mathbb{E}[W_C W_{C'}] = \lambda(C \cap C'), \quad \forall C, C' \in \mathscr{B}([0,1]^2)$$

where λ denotes the Lebesgue measure.
Let $\mathscr{A}$ be the collection of rectangles of $[0,1]^2$. In the sequel, we consider the restriction on the class $\mathscr{C}$, related to $\mathscr{A}$, of the Brownian motion defined above.
Then for all $h>0$, all $M>0$, and for almost all $\omega\in\Omega$, there exist sequences of sets $(C_n(\omega))_{n\in\mathbb{N}}$, $\big(C'_n(\omega)\big)_{n\in\mathbb{N}}$ in $\mathscr{C}$ such that $\lambda(C_n(\omega))\vee\lambda(C'_n(\omega))<h$ and for n big enough,

$$\max\{|W_{C_n(\omega)}(\omega)|,|W_{C'_n(\omega)}(\omega)|\}>\frac{M}{8}.$$

Without any stronger condition than Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$ on the sub-semilattices $(\mathscr{A}_n)_{n\in\mathbb{N}}$, the previous example of set-indexed Brownian motion dismisses a possible definition of the Hölder continuity for stochastic processes of the form:

$$\exists M>0,\exists\delta_0>0:\ \forall C\in\mathscr{C}\text{ with }m(C)<\delta_0,\ |\Delta X_C|\leq M.m(C)^{\alpha}.$$

2.3 Hölder exponents for set-indexed processes

Back to the beginning of Section 2.2.2, localizing the two expressions (i) and (ii) for Hölder-continuity leads to two different notions. Indeed, for the distance $d_{\mathscr{A}}$ on $\mathscr{A}$, if $B_{d_{\mathscr{A}}}(U_0,\rho)$ (or simply $B(U_0,\rho)$ if the context is clear) denotes the open ball centered in $U_0\in\mathscr{A}$ and whose radius is $\rho>0$, we get

(i)$_{loc}$

$$\limsup_{\delta\to 0+}\delta^{-q}\sup_{U,V\in B_{d_{\mathscr{A}}}(U_0,\delta)}|f(U)-f(V)|<\infty.$$

(ii)$_{loc}$ There exist $M>0$ and $\delta_0>0$ such that

$$\forall U,V\in B_{d_{\mathscr{A}}}(U_0,\delta_0),\quad |f(U)-f(V)|\leq M\ d_{\mathscr{A}}(U,V)^q.$$

Although the conditions (i) and (ii) are equivalent, localizing around $U_0\in\mathscr{A}$ only gives (ii)$_{loc}$ $\Rightarrow$(i)$_{loc}$. This leads usually to consider two kinds of Hölder exponent at $U_0\in\mathscr{A}$:

- the pointwise Hölder exponent

$$\boldsymbol{\alpha}_f(U_0)=\sup\left\{\alpha:\ \limsup_{\rho\to 0}\sup_{U,V\in B(U_0,\rho)}\frac{|f(U)-f(V)|}{\rho^{\alpha}}<\infty\right\},\tag{2.14}$$

- and the local Hölder exponent

$$\widetilde{\boldsymbol{\alpha}}_f(U_0)=\sup\left\{\alpha:\ \limsup_{\rho\to 0}\sup_{U,V\in B(U_0,\rho)}\frac{|f(U)-f(V)|}{d_{\mathscr{A}}(U,V)^{\alpha}}<\infty\right\}.\tag{2.15}$$

Each one allows to measure the regularity of the function f. In general, we have

$$\tilde{\boldsymbol{\alpha}}_f\leq\boldsymbol{\alpha}_f\ ,\tag{2.16}$$

but the inequality can be strict.

Remark 2.16. *We can see that condition (i)$_{loc}$ is equivalent to $q < \boldsymbol{\alpha}(U_0)$, and condition (ii)$_{loc}$ is equivalent to $q < \widetilde{\boldsymbol{\alpha}}(U_0)$. Then $\tilde{\boldsymbol{\alpha}}_f \leq \boldsymbol{\alpha}_f$ is another statement for (ii)$_{loc}$ $\Rightarrow$(i)$_{loc}$. Note that the discussion of this whole paragraph is not specific to indexing collection, but can be adapted to any totally bounded metric space.*

Example 2.17. *Consider the case of the metric space $(\mathbb{R}, |.|)$. Fix $\gamma > 0$ and $\delta > 0$. Let f be a chirp function defined by $t \mapsto |t|^\gamma \sin \frac{1}{|t|^\delta}$. The two Hölder exponents at 0 can be computed and $\widetilde{\alpha}_f(0) = \frac{\gamma}{1+\delta} < \alpha_f(0) = \gamma$.*

This example shows that the sole pointwise exponent is not sufficient to describe the irregularity of the function. The local exponent *can see* the oscillations around 0, while the pointwise exponent cannot. These two notions can be applied to study sample path regularity of a stochastic process.

In the case of Gaussian processes (see [59]), we define the *deterministic pointwise Hölder exponent*

$$\mathbb{a}_X(U_0) = \sup \left\{ \sigma;\ \limsup_{\rho \to 0} \sup_{U,V \in B(U_0,\rho)} \frac{\mathbb{E}[X_U - X_V]^2}{\rho^{2\sigma}} < \infty \right\}$$

and the *deterministic local Hölder exponent*

$$\widetilde{\mathbb{a}}_X(U_0) = \sup \left\{ \sigma;\ \limsup_{\rho \to 0} \sup_{U,V \in B(U_0,\rho)} \frac{\mathbb{E}[X_U - X_V]^2}{d_{\mathscr{A}}(U,V)^{2\sigma}} < \infty \right\}.$$

On the space $(\mathbb{R}^N, \|.\|)$, it is shown in [59] that for all $t_0 \in \mathbb{R}_+^N$, the pointwise and local Hölder exponents of X at t_0 satisfy almost surely

$$\boldsymbol{\alpha}_X(t_0) = \mathbb{a}_X(t_0) \quad \text{and} \quad \widetilde{\boldsymbol{\alpha}}_X(t_0) = \widetilde{\mathbb{a}}_X(t_0).$$

In the following sections, several other definitions are studied for Hölder regularity of set-indexed processes. They are connected to the various ways to study the local behaviour of the sample paths of X around a given set $U_0 \in \mathscr{A}$.

2.3.1 Definition of Hölder exponents on $\mathscr{C}^l$

Following expression (2.12) for the definition of the increments of a set-indexed process, we consider alternative definitions for Hölder exponents, where the quantities $X_U - X_V$ are substituted with $\Delta X_{U \setminus V}$.

As stated in Section 2.2.3, it is not wise to consider $\Delta X_{U\setminus V}$ when $U \in \mathscr{A}$ and $V \in \mathscr{A}(u)$ are close to a given $U_0 \in \mathscr{A}$. Indeed, Proposition 2.15 shows that the quantity $|\Delta X_{U\setminus V}|$ can stay far away from 0 when $m(U \setminus V)$ is small, even in the simple case of a Brownian motion indexed by $[0,1]^2$. However, when $U \in \mathscr{A}$ and V is restricted to sets of the form $V = \bigcup_{1\leq i \leq l} V_i$ where l is fixed and $V_1, \ldots, V_l \in \mathscr{A}$, the Hölder regularity can be defined from the study of $\Delta X_{U\setminus V}$.
Fix any integer $l \geq 1$ and set for all $U \in \mathscr{A}$ and $\rho > 0$,

$$\mathscr{B}^l(U,\rho) = \left\{ \bigcup_{1\leq i\leq l} V_i;\ V_1, \ldots, V_l \in \mathscr{A},\ \max_{1\leq i\leq l} d_{\mathscr{A}}(U, V_i) < \rho \right\}.$$

The pointwise and local Hölder $\mathscr{C}^l$-exponents at $U_0 \in \mathscr{A}$ are respectively defined as

$$\boldsymbol{\alpha}_{X,\mathscr{C}^l}(U_0) = \sup \left\{ \alpha : \limsup_{\rho \to 0} \sup_{\substack{U \in B_{d_{\mathscr{A}}}(U_0,\rho) \\ V \in \mathscr{B}^l(U_0,\rho)}} \frac{|\Delta X_{U\setminus V}|}{\rho^\alpha} < \infty \right\},$$

and

$$\widetilde{\boldsymbol{\alpha}}_{X,\mathscr{C}^l}(U_0) = \sup\left\{ \alpha : \limsup_{\rho\to 0} \sup_{\substack{U\in B_{d_{\mathscr{A}}}(U_0,\rho)\\ V\in\mathscr{B}^l(U_0,\rho)}} \frac{|\Delta X_{U\setminus V}|}{d_{\mathscr{A}}(U,V)^{\alpha}} < \infty \right\}.$$

The following result shows that the $\mathscr{C}^l$-exponents do not depend on l and, consequently, they provide a definition of Hölder exponents on the class $\mathscr{C}$. Moreover, these exponents can be compared to the exponents defined by (2.14) and (2.15).

Proposition 2.18. *If $d_{\mathscr{A}}$ is a contractive distance, for any $U_0 \in \mathscr{A}$, the exponents $\boldsymbol{\alpha}_{X,\mathscr{C}^l}(U_0)$ and $\widetilde{\boldsymbol{\alpha}}_{X,\mathscr{C}^l}(U_0)$ do not depend on the integer $l \geq 1$. They are denoted by $\boldsymbol{\alpha}_{X,\mathscr{C}}(U_0)$ and $\widetilde{\boldsymbol{\alpha}}_{X,\mathscr{C}}(U_0)$ respectively.*
Moreover, for all $U_0 \in \mathscr{A}$ and all $\omega \in \Omega$,

$$\boldsymbol{\alpha}_{X,\mathscr{C}}(U_0)(\omega) \geq \boldsymbol{\alpha}_X(U_0)(\omega) \quad \text{and} \quad \widetilde{\boldsymbol{\alpha}}_{X,\mathscr{C}}(U_0)(\omega) \geq \widetilde{\boldsymbol{\alpha}}_X(U_0)(\omega).$$

Proof. We only detail the case of the pointwise exponent. The proof for the local exponent is totally similar.
From the definition of the $\mathscr{C}^l$-exponents, since $l \geq l'$ implies $\mathscr{B}^{l'}(U_0,\rho) \subseteq \mathscr{B}^l(U_0,\rho)$, it is clear that

$$\forall\omega\in\Omega, \forall l \geq l', \quad \boldsymbol{\alpha}_{X,\mathscr{C}^l}(U_0)(\omega) \leq \boldsymbol{\alpha}_{X,\mathscr{C}^{l'}}(U_0)(\omega).$$

For the sake of readability, we prove the converse inequality for $l = 2, l' = 1$ (the other cases are very similar). For any $\rho > 0$, let $U \in B_{d_{\mathscr{A}}}(U_0,\rho)$, and $V = V_1 \cup V_2 \in \mathscr{B}^l(U_0,\rho)$ with $V_1, V_2 \in \mathscr{A}$. From the inclusion-exclusion formula,

$$\begin{aligned} |\Delta X_{U\setminus V}| &= |X_U - X_{U\cap V_1} - X_{U\cap V_2} + X_{U\cap V_1\cap V_2}| \\ &= |\Delta X_{U\setminus V_1} + \Delta X_{U\setminus V_2} - \Delta X_{U\setminus(V_1\cap V_2)}| \\ &\leq |\Delta X_{U\setminus V_1}| + |\Delta X_{U\setminus V_2}| + |\Delta X_{U\setminus(V_1\cap V_2)}|. \end{aligned}$$

We have $d_{\mathscr{A}}(U_0,V_1) \leq \rho$, $d_{\mathscr{A}}(U_0,V_2) \leq \rho$ and

$$\begin{aligned} d_{\mathscr{A}}(U_0, V_1\cap V_2) &\leq d_{\mathscr{A}}(U_0,V_1) + d_{\mathscr{A}}(V_1, V_1\cap V_2) \\ &\leq d_{\mathscr{A}}(U_0,V_1) + d_{\mathscr{A}}(V_1,V_2) \leq 2d_{\mathscr{A}}(U_0,V_1) + d_{\mathscr{A}}(U_0,V_2) \leq 3\rho, \end{aligned}$$

using $d_{\mathscr{A}}(V_1, V_1\cap V_2) \leq d_{\mathscr{A}}(V_1,V_2)$ from the contracting property of $d_{\mathscr{A}}$.
Then, for all $\alpha < \boldsymbol{\alpha}_{X,\mathscr{C}^{l'}}(U_0)(\omega)$,

$$\limsup_{\rho\to 0} \sup_{\substack{U\in B_{d_{\mathscr{A}}}(U_0,\rho)\\ V\in\mathscr{B}^l(U_0,\rho)}} \frac{|\Delta X_{U\setminus V}|}{\rho^{\alpha}} < \infty,$$

which says that $\alpha < \boldsymbol{\alpha}_{X,\mathscr{C}^l}(U_0)(\omega)$. Thus, $\boldsymbol{\alpha}_{X,\mathscr{C}^{l'}}(U_0)(\omega) \leq \boldsymbol{\alpha}_{X,\mathscr{C}^l}(U_0)(\omega)$.
This inequality achieves to prove that $\boldsymbol{\alpha}_{X,\mathscr{C}^l}(U_0)(\omega)$ does not depend on the integer $l \geq 1$.

To prove the second part of the Proposition, it suffices then to prove the inequality for $l = 1$. This is straightforward, since for a fixed $U \in B_{d_{\mathscr{A}}}(U_0,\rho)$,

$$\sup_{V\in\mathscr{B}^1(U_0,\rho)} |\Delta X_{U\setminus V}| \leq \sup_{W\in B_{d_{\mathscr{A}}}(U_0,\rho)} |X_U - X_W|.$$

Hence $\boldsymbol{\alpha}_X(U_0) \leq \boldsymbol{\alpha}_{X,\mathscr{C}^l}(U_0)$. The inequality for the local exponent can be obtained identically, or one can notice that it is a direct consequence of Corollary 2.14.

The converse inequality does not hold in general since quantities $|X_U - X_V|$ cannot be obtained from the increment process ΔX when U, V are not ordered. □

Remark 2.19. *The previous definition of the pointwise Hölder exponent on $\mathscr{C}^l$ is not equivalent to the quantity*

$$\sup\left\{\alpha : \limsup_{\rho\to 0} \sup_{\substack{U\in B_{d_{\mathscr{A}}}(U_0,\rho)\\ V\in\mathscr{B}^l:\ d_{\mathscr{A}}(U_0,V)<\rho}} \frac{|\Delta X_{U\setminus V}|}{\rho^\alpha} < \infty\right\},$$

as the following example shows.
In the particular case of the indexing collection $\mathscr{A}$ equal to the rectangles of $\mathbb{R}^2_+$ and the distance $d_\lambda = \lambda(\bullet \bigtriangleup \bullet)$ induced by the Lebesgue measure λ of $\mathbb{R}^2$, we show that the assertion ($V \in \mathscr{B}^l$: $d_\lambda(U_0, V) < \rho$) is not equivalent to ($V \in \mathscr{B}^l(U_0,\rho)$).
Consider $V_1 = [0;(n^2, n^2 + \frac{1}{n})], V_2 = [0;(n^2 + \frac{1}{n}, n^2)]$ and $U = [0;(n^2 + \frac{1}{n}, n^2 + \frac{1}{n})]$. We have

$$d_\lambda(U, V_1 \cup V_2) = \frac{1}{n^2} \ \textit{ while } \ d_\lambda(U, V_1) = d_\lambda(U, V_2) \approx n.$$

Then, $V_1 \cup V_2 \notin \mathscr{B}^2(U,\rho)$ for small ρ and it is not possible to control the quantity $|X_U - \Delta X_{V_1\cup V_2}|$ using $|X_U - X_{V_1}|$, $|X_U - X_{V_1}|$ and $|X_U - X_{V_1\cap V_2}|$ as was done in the previous proofs.

The notation $\boldsymbol{\alpha}_{X,\mathscr{C}}$ must be considered with care: Proposition 2.15 shows that the Hölder exponents cannot be defined directly by taking the supremum on $U \in \mathscr{A}$ and $V \in \mathscr{A}(u)$ with $d_{\mathscr{A}}(U_0, U) < \rho$ and $d_{\mathscr{A}}(U_0, V) < \rho$ (and then, on the class $\mathscr{C}$). This is the reason why the set V is restricted to be in $\mathscr{B}^l(U,\rho)$.

The arguments of the proof of Proposition 2.18 in the particular case of $l = 1$ leads to: for all ω,

$$\boldsymbol{\alpha}_{X,\mathscr{C}}(U_0)(\omega) \geq \sup\left\{\alpha :\ \limsup_{\rho\to 0} \sup_{\substack{U,V\in B_{d_{\mathscr{A}}}(U_0,\rho)\\ U\subset V}} \frac{|X_U(\omega) - X_V(\omega)|}{\rho^\alpha} < \infty\right\},$$

and

$$\widetilde{\boldsymbol{\alpha}}_{X,\mathscr{C}}(U_0)(\omega) \geq \sup\left\{\alpha :\ \limsup_{\rho\to 0} \sup_{\substack{U,V\in B_{d_{\mathscr{A}}}(U_0,\rho)\\ U\subset V}} \frac{|X_U(\omega) - X_V(\omega)|}{d_{\mathscr{A}}(U,V)^\alpha} < \infty\right\}.$$

The converse inequalities follow from the fact that the set of $U, V \in B_{d_{\mathscr{A}}}(U_0,\rho)$ with $U \subset V$ is included in the set of $U \in B_{d_{\mathscr{A}}}(U_0,\rho)$ and $V \in \mathscr{B}^1(U_0,\rho)$. Then, we can state:

Corollary 2.20. *If $d_{\mathscr{A}}$ is a contractive distance, the pointwise and local Hölder $\mathscr{C}$-exponents at $U_0 \in \mathscr{A}$ are respectively given by*

$$\boldsymbol{\alpha}_{X,\mathscr{C}}(U_0) = \sup\left\{\alpha :\ \limsup_{\rho\to 0} \sup_{\substack{U,V\in B_{d_{\mathscr{A}}}(U_0,\rho)\\ U\subset V}} \frac{|X_U - X_V|}{\rho^\alpha} < \infty\right\},$$

and

$$\widetilde{\boldsymbol{\alpha}}_{X,\mathscr{C}}(U_0) = \sup\left\{\alpha :\ \limsup_{\rho\to 0} \sup_{\substack{U,V\in B_{d_{\mathscr{A}}}(U_0,\rho)\\ U\subset V}} \frac{|X_U - X_V|}{d_{\mathscr{A}}(U,V)^\alpha} < \infty\right\}.$$

2.3.2 *Pointwise continuity*

As previously mentioned, the set-indexed Brownian motion can be not continuous, when the indexing collection is not a Vapnik-Červonenkis class (see [5, 70] for the detailed study).

In [63], a weak form of continuity is considered in the study of set-indexed Poisson process, set-indexed Brownian motion and more generally set-indexed Lévy processes. In particular, the sample paths of the set-indexed Brownian motion are proved to be pointwise continuous as a set-indexed Lévy process with Gaussian increments. Notice that such a property does not require Assumption ($\mathscr{H}_{\underline{\mathscr{A}}}$) on $\mathscr{A}$. We recall the following definitions:

Definition 2.21 ([63])**.** *The* point mass jump *of a set-indexed function $x : \mathscr{A} \to \mathbb{R}$ at $t \in T$ is defined by*

$$J_t(x) = \lim_{n\to\infty} \Delta x_{C_n(t)}, \quad \text{where} \quad C_n(t) = \bigcap_{\substack{C\in\mathscr{C}_n \\ t\in C}} C \tag{2.17}$$

and for each $n \geq 1$, $\mathscr{C}_n$ denotes the collection of subsets $U \setminus V$ with $U \in \mathscr{A}_n$ and $V \in \mathscr{A}_n(u)$.

Definition 2.22 ([63])**.** *A set-indexed function $x : \mathscr{A} \to \mathbb{R}$ is said* pointwise continuous *at $t \in T$ if $J_t(x) = 0$.*

Let us recall that a subset $\mathscr{A}'$ of $\mathscr{A}$ which is closed under arbitrary intersections is called a *lower sub-semilattice* of $\mathscr{A}$. The ordering of a lower sub-semilattice $\mathscr{A}' = \{A_1, A_2, \ldots\}$ is said to be *consistent* if $A_i \subset A_j \Rightarrow i \leq j$. Proceeding inductively, we can show that any lower sub-semilattice admits a consistent ordering, which is not unique in general (see [70]).
If $\{A_1, \ldots, A_n\}$ is a consistent ordering of a finite lower sub-semilattice $\mathscr{A}'$, the set $C_i = A_i \setminus \bigcup_{j\leq i-1} A_j$ is called *the left neighbourhood* of A_i in $\mathscr{A}'$. Since $C_i = A_i \setminus \bigcup_{A\in\mathscr{A}', A\nsubseteq A_i} A$, the definition of the left neighbourhood does not depend on the ordering.

As in the classical Kolmogorov criterion of continuity, the pointwise continuity of a set-indexed process X can be proved from the study of $\mathbb{E}[|\Delta X_{C_n(t)}|^p]$ when n goes to infinity.

Proposition 2.23. *Let $X = \{X_U;\ U \in \mathscr{A}\}$ be a set-indexed process and let $U_{\max}$ be a subset in $\mathscr{A}$ such that $m(U_{\max}) < +\infty$ and assume that there exist $p > 0$, $q > 1$, $N \geq 1$ and $K > 0$ such that for all $t \in U_{\max}$ and all $n \geq N$,*

$$\mathbb{E}\Big[|\Delta X_{C_n(t)}|^p\Big] \leq K\ m(C_n(t))^q\ . \tag{2.18}$$

Then, for any $\gamma \in (0, (q-1)/p)$, there exists an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ and a random variable $n^* \geq 1$ satisfying, with probability one,

$$\forall t \in U_{\max},\ \forall n \geq n^*, \quad |\Delta X_{C_{\varphi(n)}(t)}| \leq m(C_{\varphi(n)}(t))^{\gamma}.$$

Proof. Up to restricting the indexing collection to $\{U \cap U_{max}, U \in \mathscr{A}\}$, we assume in this proof that the indexing collection $\mathscr{A}$ is included in $U_{\max}$.
For all $0 < \gamma < \dfrac{q-1}{p}$, we consider $S_n = \sup\left\{\dfrac{|\Delta X_{C_n(t)}|}{m(C_n(t))^{\gamma}};\ t \in U_{\max}\right\}$, where $C_n(t)$ is defined in (2.17). When t ranges $U_{\max}$, the subset $C_n(t)$ ranges $\mathscr{C}^l(\mathscr{A}_n)$, the collection of the disjoint left-neighbourhoods of $\mathscr{A}_n$. Consequently we can write $S_n = \sup\left\{\dfrac{|\Delta X_C|}{m(C)^{\gamma}};\ C \in \mathscr{C}^l(\mathscr{A}_n)\right\}$.

For any integer $p \geq 1$, we have

$$\begin{aligned}\mathbb{P}(S_n \geq 1) &\leq \sum_{C \in \mathscr{C}^l(\mathscr{A}_n)} \mathbb{P}(|\Delta X_C| \geq m(C)^{\gamma}) \\ &\leq \sum_{C \in \mathscr{C}^l(\mathscr{A}_n)} \frac{\mathbb{E}[|\Delta X_C|^p]}{m(C)^{\gamma p}} \leq K \sum_{C \in \mathscr{C}^l(\mathscr{A}_n)} m(C)^{q-\gamma p} .\end{aligned}$$

Since $q - \gamma p > 1$, we have

$$\begin{aligned}\mathbb{P}(S_n \geq 1) &\leq K \left(\sum_{C \in \mathscr{C}^l(\mathscr{A}_n)} m(C) \right) \sup_{C \in \mathscr{C}^l(\mathscr{A}_n)} \left\{ m(C)^{q-\gamma p-1} \right\} \\ &\leq K\ m(U_{max}) \sup_{C \in \mathscr{C}^l(\mathscr{A}_n)} \left\{ m(C)^{q-\gamma p-1} \right\}\end{aligned}$$

where the fact that the $C \in \mathscr{C}^l(\mathscr{A}_n)$ are disjoint is used. Up to choosing an extraction φ for the sequence $u_n = \sup_{C \in \mathscr{C}^l(\mathscr{A}_n)} \left\{ m(C)^{q-\gamma p-1} \right\}$, we can assume that u_n is summable. Hence the Borel-Cantelli Lemma implies that for $0 < \gamma < (q-1)/p$, $\{S_{\varphi(n)} < 1\}$ happens infinitely often, which gives the result. □

Remark 2.24. *Proposition 2.23 does not require Assumption ($\mathscr{H}_{\underline{\mathscr{A}}}$) for the collection $\{\mathscr{A}_n\}_{n \in \mathbb{N}}$ and the distance d_m.*

From Proposition 2.23, it is natural to define local Hölder regularity of a set-indexed process by a comparison of $\Delta X_{C_n(t)}$ to quantities $m(C_n(t))^{\alpha}$ with $\alpha > 0$, when n is large.

Definition 2.25. *The* pointwise continuity Hölder exponent *at any $t \in T$ is defined by*

$$\boldsymbol{\alpha}_X^{pc}(t) = \sup \left\{ \alpha :\ \limsup_{n \to \infty} \frac{|\Delta X_{C_n(t)}|}{m(C_n(t))^{\alpha}} < \infty \right\}.$$

According to Proposition 2.23, if X is a $\mathscr{A}$-indexed process satisfying hypothesis (2.18), then with probability one, $\boldsymbol{\alpha}_X^{pc}(t) \geq (q-1)/p$ for all $t \in U_{max}$.

Remark 2.26. *As in the continuity criterion (Theorem 2.9 and Corollary 2.10), the proof of Proposition 2.23 can be improved for $\gamma \in (0, (kq-1)/kp)$ for any $k \in \mathbb{N}$, when the process is Gaussian. In that specific case, the upper bound for admissible values of γ is q/p (instead of $(q-1)/p$).*

2.4 Connection with Hölder exponents of projections on flows

In this section, we consider the concept of *flow*, which is a useful tool to reduce characterization or convergence problems to a one-dimensional issue. *Flows* have been used to characterize: strong martingales [70], set-Markov processes [14], set-indexed fractional Brownian motion [62] and set-indexed Lévy processes [63].

Definition 2.27 ([70]). *An* elementary flow *is defined to be a continuous increasing function* $f : [a,b] \subset \mathbf{R}_+ \to \mathscr{A}$*, i.e. such that*

$$\begin{aligned} \forall s,t \in [a,b]; &\quad s<t \Rightarrow f(s) \subseteq f(t) \\ \forall s \in [a,b); &\quad f(s) = \bigcap_{v>s} f(v) \\ \forall s \in (a,b); &\quad f(s) = \overline{\bigcup_{u<s} f(u)}. \end{aligned}$$

A simple flow *is a continuous function* $f : [a,b] \to \mathscr{A}(u)$ *such that there exists a finite sequence* $(t_0, t_1, \dots, t_n)$ *with* $a = t_0 < t_1 < \dots < t_n = b$ *and elementary flows* $f_i : [t_{i-1}, t_i] \to \mathscr{A}$ $(i = 1,\dots,n)$ *such that*

$$\forall s \in [t_{i-1}, t_i]; \quad f(s) = f_i(s) \cup \bigcup_{j=1}^{i-1} f_j(t_j).$$

The set of all simple (resp. elementary) flows is denoted $S(\mathscr{A})$ *(resp.* $S^e(\mathscr{A})$*).*

According to [62], we use the parametrization of flows which allows to preserve the increment stationarity property under projection on flows (it avoids the appearance of a time-change).

Definition 2.28 ([62]). *For any set-indexed process* $X = \{X_U;\ U \in \mathscr{A}\}$ *on the space* $(T, \mathscr{A}, m)$ *and any simple flow* $f : [a,b] \to \mathscr{A}(u)$*, the* m-standard projection of X on f *is defined as the process*

$$X^{f,m} = \left\{ X_t^{f,m} = \Delta X_{f \circ \theta^{-1}(t)};\ t \in \theta([a,b]) \right\},$$

where θ *is the function* $t \mapsto m[f(t)]$ *and* θ^{-1} *its right inverse.*

The importance of flows in the study of set-indexed processes follows the fact that the finite dimensional distributions of an additive $\mathscr{A}$-indexed process X determine and are determined by the finite dimensional distributions of the class $\{X^{f,m},\ f \in S(\mathscr{A})\}$ ([68], Lemma 6).

As the projection of a set-indexed process on any flow is a real-parameter process, its classical Hölder exponents can be considered and compared to the exponents of the set-indexed process. In the sequel, we study how regularity of flows connects the exponents $\boldsymbol{\alpha}_X(U_0)$ (resp. $\widetilde{\boldsymbol{\alpha}}_X(U_0)$) and $\boldsymbol{\alpha}_{X^{f,m}}(t_0)$ (resp. $\widetilde{\boldsymbol{\alpha}}_{X^{f,m}}(t_0)$), when $U_0 \in \mathscr{A}$ and $f \circ \theta^{-1}(t_0) = U_0$.

For any $U_0 \in \mathscr{A}$, let us denote by $S(\mathscr{A}, U_0)$ the subset of $S(\mathscr{A})$ containing all the simple flows $f : \theta^{-1}(I_f) \to \mathscr{A}(u)$ such that there exists $t_0 > 0$ satisfying $f \circ \theta^{-1}(t_0) = U_0$, and where I_f is a closed interval of $\mathbb{R}_+$ containing a ball centered in t_0. Such a t_0 does not depend on the flow f, since $t_0 = m(U_0)$. In the same way, we define $S^e(\mathscr{A}, U_0)$ for elementary flows.

Lemma 2.29. *Let* $f \in S(\mathscr{A}, U_0)$ *and* $\eta > 0$ *such that* $B(t_0, \eta) \subset I_f$*. For all* $t \in B(t_0, \eta)$*,* $f \circ \theta^{-1}(t) \in B_{d_m}^{(u)}(U_0, \eta) = \{A \in \mathscr{A}(u) : m(A \triangle U_0) < \eta\}$.

Proof. $\theta^{-1}(t) = \inf\{x \in I_f : \theta(x) \geq t\}$. As θ is increasing, θ^{-1} is increasing as well. We assume without loss of generality that $t \geq t_0$. Then,

$$\begin{aligned} d_m(f \circ \theta^{-1}(t), U_0) &= m\big(f \circ \theta^{-1}(t) \triangle f \circ \theta^{-1}(t_0)\big) \\ &= m\big(f \circ \theta^{-1}(t) \setminus f \circ \theta^{-1}(t_0)\big) \\ &= m(f \circ \theta^{-1}(t)) - m(f \circ \theta^{-1}(t_0)) \\ &= t - t_0. \end{aligned}$$

□

Using Lemma 2.29, we can compare the Hölder regularity of X and the Hölder regularity of its projections on flows.

Proposition 2.30. *Let $X = \{X_U;\ U \in \mathscr{A}\}$ be a set-indexed process on $(T, \mathscr{A}, m)$, with finite Hölder exponents at $U_0 \in \mathscr{A}$. Then,*

$$\inf_{f \in S^e(\mathscr{A}, U_0)} \boldsymbol{\alpha}_{X^{f,m}}(t_0) = \boldsymbol{\alpha}_{X,\mathscr{C}}(U_0) \geq \boldsymbol{\alpha}_X(U_0) \quad a.s.$$
$$\inf_{f \in S^e(\mathscr{A}, U_0)} \widetilde{\boldsymbol{\alpha}}_{X^{f,m}}(t_0) = \widetilde{\boldsymbol{\alpha}}_{X,\mathscr{C}}(U_0) \geq \widetilde{\boldsymbol{\alpha}}_X(U_0) \quad a.s.$$

where the metric considered on $\mathscr{A}$ is d_m.

Proof. The proof is only given for the pointwise Hölder exponent. The case of the local Hölder exponent is totally similar.
From Proposition 2.18, the inequality $\boldsymbol{\alpha}_{X,\mathscr{C}}(U_0) \geq \boldsymbol{\alpha}_X(U_0)$ for all $\omega \in \Omega$ is already known.
The equality $\inf_{f \in S^e(\mathscr{A}, U_0)} \boldsymbol{\alpha}_{X^{f,m}}(t_0) = \boldsymbol{\alpha}_{X,\mathscr{C}}(U_0)$ follows from Corollary 2.20 and Lemma 2.29. □

The natural question is then to wonder if the previous inequality could be improved in an equality. The answer is generally no, as the following example shows.

Example 2.31. *In this example, we only consider deterministic functions, instead of random processes. Let F be a set-indexed function on $\mathscr{A}$, the usual collection of rectangles of $[0,1]^2$. Let $U_0 \in \mathscr{A}$ and assume that F is α-Hölder continuous in U_0, for some $\alpha \in (0,1)$. We assume without loss of generality that $F(U_0) = 0$.*

Let us divide $\mathscr{A}$ into four quadrants around $U_0 = [0, (x_0, y_0)]$ in the following manner:

$$\mathscr{Q}_1 = \{[0,(x,y)] \in \mathscr{A} : x \leq x_0 \text{ and } y < y_0\},$$
$$\mathscr{Q}_2 = \{[0,(x,y)] \in \mathscr{A} : x \leq x_0 \text{ and } y \geq y_0\},$$
$$\mathscr{Q}_3 = \{[0,(x,y)] \in \mathscr{A} : x > x_0 \text{ and } y \geq y_0\},$$
$$\mathscr{Q}_4 = \{[0,(x,y)] \in \mathscr{A} : x > x_0 \text{ and } y < y_0\}.$$

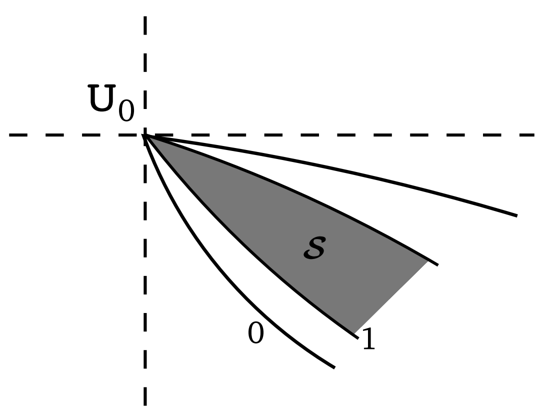

Figure 2.1 – Value of G around U_0

Let us fix $\epsilon > 0$. As F is α-Hölder continuous at U_0, for all $K > 0$, there exists a sequence of sets in $\mathscr{A}$ converging to U_0 and such that

$$\forall n \geq 0, \quad |F(U_n)| = |F(U_n) - F(U_0)| > K\ d_{\mathscr{A}}(U_n, U_0)^{\alpha+\epsilon}.$$

There is at least one of the quadrants in which there are infinitely many sets U_n. Up to a rotation, assume $\mathscr{Q}_4$ is this quadrant. We now assume (without restriction) that a subsequence of (U_n) belongs to a closed subset $\mathscr{S} \subset \mathscr{Q}_4$ (see figure 2.1).

Let G be a smooth function except maybe at U_0, taking its values in $[0,1]$ and such that $G(U_0) = 0$ and $G(U) = 0$ for all $U \in \mathscr{Q}_1 \cup \mathscr{Q}_2 \cup \mathscr{Q}_3$ (see figure 2.1 above), and $G(U) = 1$ for all $U \in \mathscr{S}$. We denote by H the product of F and G.

Up to an extraction that we detailed previously, the sequence $(U_n)_{n\in\mathbb{N}}$ belongs to $\mathscr{S}$. Then,

$$\forall n \geq 0, \quad |H(U_n) - H(U_0)| = |H(U_n)| = |F(U_n)| > K\ d_{\mathscr{A}}(U_n, U_0)^{\alpha+\epsilon}.$$

Thus, if H is β-Hölder continuous at U_0, then the inequality $\beta \leq \alpha$ holds necessarily.

For $\gamma < \alpha$, there exist $\rho > 0$ and $K > 0$ such that

$$\forall U \in B_{d_{\mathscr{A}}}(U_0, \rho), \quad |F(U)| = |F(U) - F(U_0)| \leq K\ d_{\mathscr{A}}(U, U_0)^{\gamma}.$$

Thus,

$$|H(U) - H(U_0)| = |H(U)| \leq G(U).|F(U)| \leq K\ d_{\mathscr{A}}(U, U_0)^{\gamma}.$$

We have built a function H which is α-Hölder continuous. On the other hand, the projection of H on any elementary flow $f \in S^e(\mathscr{A}, U_0)$ is uniformly 0 and consequently, $\inf_{f\in S^e(\mathscr{A},U_0)} \widetilde{\boldsymbol{\alpha}}_{H^{f,m}} = \infty > \alpha$.

2.5 Almost sure values for the Hölder exponents

2.5.1 Separability of stochastic processes

As in the real-parameter case, we prove that the random Hölder exponents of the sample paths have almost sure values when the process is Gaussian: these values are determined in Theorems 2.34 and 2.35.

Defining Hölder exponents by expressions (2.14) and (2.15) leads us to ask whether they are random variables, in order to consider measurable events related to these quantities. This question was first answered by Doob (see [42]) for linear parameter space, see [76] for a contemporary exposition.

Definition 2.32 ([42])**.** *A process $\{X_U, U \in \mathscr{A}\}$ is said* separable *if there exist an at most countable collection $\mathscr{S} \subset \mathscr{A}$ and a null set Λ such that for all closed sets $F \subset \mathbb{R}$ and all open set $\mathscr{O}$ for the topology induced by $d_{\mathscr{A}}$,*

$$\{\omega : X_U(\omega) \in F \text{ for all } U \in \mathscr{O} \cap \mathscr{S}\} \setminus \{\omega : X_U(\omega) \in F \text{ for all } U \in \mathscr{O}\} \subset \Lambda$$

This definition is well suited for set-indexed processes since we have the following:

Theorem 2.33 (from [53, Theorem 2 p.153])**.** *Any stochastic process from a separable metric space with values in a locally compact space admits a separable modification. Hence, if the subcollections $(\mathscr{A}_n)_{n\in\mathbb{N}}$ and the metric $d_{\mathscr{A}}$ satisfy Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$, any $\mathbb{R}$-valued set-indexed stochastic process $X = \{X_U;\ U \in \mathscr{A}\}$ has a separable modification.*

We shall now consider that all our processes are separable. As a consequence, assuming without any restriction on the probability space, variables such as $\sup_{U\in\mathscr{O}} X_U$, for $\mathscr{O}$ an open set of $\mathscr{A}$, are indeed measurable. Hence the random Hölder coefficients aforementioned are random variables.

2.5.2 Uniform results for Gaussian processes

Recall that according to Remark 2.11, Condition (H2) can be removed from Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$ when the process X is Gaussian and therefore in all this section.

Theorem 2.34. *Let $X = \{X_U;\ U \in \mathscr{A}\}$ a set-indexed centered Gaussian process, where $(\mathscr{A}_n)_{n\in\mathbb{N}}$ and $d_{\mathscr{A}}$ satisfy Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$. If the deterministic local Hölder exponent of X at $U_0 \in \mathscr{A}$ is positive and finite, we have*

$$\mathbb{P}\big(\widetilde{\boldsymbol{\alpha}}_X(U_0) = \widetilde{\mathbb{\alpha}}_X(U_0)\big) = 1\ ,$$

and

$$\mathbb{P}\big(\boldsymbol{\alpha}_X(U_0) = \mathbb{\alpha}_X(U_0)\big) = 1.$$

In a similar way to Theorem 3.14 of [59], we can also obtain almost sure results on the exponents $\boldsymbol{\alpha}_X(U_0)$ and $\widetilde{\boldsymbol{\alpha}}_X(U_0)$ uniformly in $U_0 \in \mathscr{A}$.

Theorem 2.35. *Let $X = \{X_U;\ U \in \mathscr{A}\}$ be a set-indexed centered Gaussian process, where $(\mathscr{A}_n)_{n\in\mathbb{N}}$ and $d_{\mathscr{A}}$ satisfy Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$.*
Suppose that the functions $U_0 \mapsto \liminf_{U\to U_0} \widetilde{\mathbb{\alpha}}_X(U)$ and $U_0 \mapsto \liminf_{U\to U_0} \mathbb{\alpha}_X(U)$ are positive over $\mathscr{A}$. Then, with probability one,

$$\forall U_0 \in \mathscr{A},\quad \liminf_{U\to U_0} \widetilde{\mathbb{\alpha}}_X(U) \le \widetilde{\boldsymbol{\alpha}}_X(U_0) \le \limsup_{U\to U_0} \widetilde{\mathbb{\alpha}}_X(U) \tag{2.19}$$

and

$$\forall U_0 \in \mathscr{A},\quad \liminf_{U\to U_0} \mathbb{\alpha}_X(U) \le \boldsymbol{\alpha}_X(U_0). \tag{2.20}$$

The proof of Theorem 2.34 is an adaptation of proofs in [57], but with a conceptual improvement due to the well-suited formulation of Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$, and a technical improvement in Section 2.7.2 that we obtained through Theorem 2.9. The proofs are detailed in Appendix 2.7.2. The proof of Theorem 2.35 is given in Section 2.7.2.

2.5.3 Corollaries for the $\mathscr{C}$-Hölder exponents and the pointwise continuity exponent

Theorem 2.34 can be transposed to the $\mathscr{C}$-Hölder exponent, and the pointwise continuity exponent.

If X is a Gaussian set-indexed process, we define respectively the *deterministic pointwise* and *local $\mathscr{C}$-Hölder exponents* on one hand, for all integer $l \geq 1$,

$$\mathbb{a}_{X,\mathscr{C}^l}(U_0) = \sup\left\{\alpha : \limsup_{\rho\to 0} \sup_{\substack{U\in B_{d_{\mathscr{A}}}(U_0,\rho)\\ V\in\mathscr{B}^l(U_0,\rho)}} \frac{\mathbb{E}[|\Delta X_{U\setminus V}|^2]}{\rho^{2\alpha}} < \infty\right\},$$

$$\widetilde{\mathbb{a}}_{X,\mathscr{C}^l}(U_0) = \sup\left\{\alpha : \limsup_{\rho\to 0} \sup_{\substack{U\in B_{d_{\mathscr{A}}}(U_0,\rho)\\ V\in\mathscr{B}^l(U_0,\rho)}} \frac{\mathbb{E}[|\Delta X_{U\setminus V}|^2]}{d_{\mathscr{A}}(U,V)^{2\alpha}} < \infty\right\}$$

and the *deterministic pointwise continuity exponent* on the other hand,

$$\mathbb{a}_X^{pc}(t_0) = \sup\left\{\alpha : \limsup_{n\to\infty} \frac{\mathbb{E}[|\Delta X_{C_n(t)}|^2]}{m(C_n(t))^{2\alpha}} < \infty\right\}.$$

Similarly to Proposition 2.18, the pointwise and local deterministic exponents do not depend on l. Hence they are denoted respectively by $\mathbb{a}_{X,\mathscr{C}}(U_0)$ and $\widetilde{\mathbb{a}}_{X,\mathscr{C}}(U_0)$.

Corollary 2.36. *Let $X = \{X_U,\ U \in \mathscr{A}\}$ be a centered Gaussian set-indexed process. If the subcollections $(\mathscr{A}_n)_{n\in\mathbb{N}}$ satisfy Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$ and if the deterministic $\mathscr{C}$-Hölder exponents are finite, then for $U_0 \in \mathscr{A}$,*

$$\boldsymbol{\alpha}_{X,\mathscr{C}}(U_0) = \mathbb{a}_{X,\mathscr{C}}(U_0) \text{ a.s.} \quad \text{and} \quad \widetilde{\boldsymbol{\alpha}}_{X,\mathscr{C}}(U_0) = \widetilde{\mathbb{a}}_{X,\mathscr{C}}(U_0) \text{ a.s.}$$

Proof. It suffices to prove the result for $l = 1$, which corresponds to $V \subseteq U$ in the definition of the standard Hölder exponent. Thus one can apply the previous proofs (Sections 2.7.2 and 2.7.2) which are still valid when restricted to $V \subseteq U$. □

Corollary 2.37. *Let $X = \{X_U,\ U \in \mathscr{A}\}$ be a centered Gaussian set-indexed process. If the deterministic exponent of pointwise continuity is finite, then for $t_0 \in T$,*

$$\boldsymbol{\alpha}_X^{pc}(t_0) = \mathbb{a}_X^{pc}(t_0) \quad \text{a.s.}$$

Moreover, for any $U_{max} \in \mathscr{A}$ such that $m(U_{max}) < \infty$,

$$\mathbb{P}\left(\forall t \in U_{max},\ \boldsymbol{\alpha}_X^{pc}(t) \geq \mathbb{a}_X^{pc}(t)\right) = 1.$$

Proof. Fix $t_0 \in T$. Let $\alpha < \mathbb{a}_X^{pc}(t_0)$. The inequality $\alpha < \boldsymbol{\alpha}_X^{pc}(t_0)$ a.s. is a direct consequence of Proposition 2.23. This gives $\boldsymbol{\alpha}_X^{pc}(t_0) \geq \mathbb{a}_X^{pc}(t_0)$ almost surely.

For the converse inequality, denote $\mu = \mathbb{a}_X^{pc}(t_0)$. Then for all $\epsilon > 0$, there exist a subsequence $\left(C_{\varphi(n)}(t_0)\right)_{n\in\mathbb{N}}$ of $(C_n(t_0))_{n\in\mathbb{N}}$ and a constant $c > 0$ such that

$$\forall n \in \mathbb{N}^*, \quad \mathbb{E}\left[|\Delta X_{C_{\varphi(n)}(t_0)}|^2\right] \geq c\ m(C_{\varphi(n)}(t_0))^{2\mu+\epsilon}.$$

For all $n \in \mathbb{N}$, the law of the random variable $\dfrac{\Delta X_{C_{\varphi(n)}(t_0)}}{m(C_{\varphi(n)}(t_0))^{\mu+\epsilon}}$ is $\mathcal{N}(0,\sigma_n^2)$. The previous inequality implies that $\sigma_n \to \infty$ as $n \to \infty$. Then for all $\lambda > 0$, the same computation as in Lemmas 2.48 and 2.49 leads to

$$\begin{aligned}\mathbb{P}\left(\frac{m(C_{\varphi(n)}(t_0))^{\mu+\epsilon}}{\Delta X_{C_{\varphi(n)}(t_0)}} < \lambda\right) &= \mathbb{P}\left(\frac{\Delta X_{C_{\varphi(n)}(t_0)}}{m(C_{\varphi(n)}(t_0))^{\mu+\epsilon}} > \frac{1}{\lambda}\right)\\ &= \int_{|x|>\frac{1}{\lambda}} \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left(-\frac{x^2}{2\sigma_n^2}\right).dx\\ &= \frac{1}{2\pi}\int_{|x|>\frac{1}{\lambda\sigma_n}} \exp\left(-\frac{x^2}{2}\right).dx \overset{n\to+\infty}{\longrightarrow} 1.\end{aligned}$$

Therefore the sequence $\left(\dfrac{m(C_{\varphi(n)}(t_0))^{\mu+\epsilon}}{\Delta X_{C_{\varphi(n)}(t_0)}}\right)_{n\in\mathbb{N}}$ converges to 0 in probability. As a consequence, there exists a subsequence which converges to 0 almost surely. Then for all $\epsilon > 0$, we have almost surely $\boldsymbol{\alpha}_X^{pc}(t_0) \leq \mu + \epsilon$. Taking $\epsilon \in \mathbb{Q}_+$, this yields $\boldsymbol{\alpha}_X^{pc}(t_0) \leq \mathbb{\alpha}_X^{pc}(t_0)$ a.s.

The second equation is a direct consequence of Proposition 2.23. □

2.6 Application: Hölder regularity of the set-indexed fractional Brownian motion and the set-indexed Ornstein-Ülhenbeck process

The various general results proved in Section 2.5 allow to describe the local behaviour of recent set-indexed extensions of two well-known stochastic processes: fractional Brownian motion and Ornstein-Ülhenbeck process.

2.6.1 *Hölder exponents of the SIfBm*

The local regularity of fractional Brownian motion $B^H = \{B_t^H;\ t \in \mathbb{R}_+\}$ is known to be constant a.s. and given by the self-similarity index $H \in (0,1)$. More precisely, the two classical Hölder exponents satisfy, with probability one,

$$\forall t \in \mathbb{R}_+, \quad \boldsymbol{\alpha}_{B^H}(t) = \widetilde{\boldsymbol{\alpha}}_{B^H}(t) = H.$$

In [60, 62], a set-indexed extension for fractional Brownian motion has been defined and studied. A mean-zero Gaussian process $\mathbf{B}^H = \left\{\mathbf{B}_U^H, U \in \mathcal{A}\right\}$ is called a *set-indexed fractional Brownian motion (SIfBm for short)* on $(T, \mathcal{A}, m)$ if

$$\forall U, V \in \mathcal{A}, \quad \mathbb{E}\left[\mathbf{B}_U^H \mathbf{B}_V^H\right] = \frac{1}{2}\left[m(U)^{2H} + m(V)^{2H} - m(U \mathbin{\triangle} V)^{2H}\right], \tag{2.21}$$

where $H \in (0, 1/2]$ is the index of self-similarity of the process.

In [58], the deterministic local Hölder exponent and the almost sure value of the local Hölder exponent have been determined for the particular case of an SIfBm indexed by the collection

$\{[0,t];\ t\in\mathbb{R}_+^N\}\cup\{\emptyset\}$, called the *multiparameter fractional Brownian motion*. If X denotes the $\mathbb{R}_+^N$-indexed process defined by $X_t = \mathbf{B}^H_{[0,t]}$ for all $t\in\mathbb{R}_+^N$, it is proved that for all $t_0\in\mathbb{R}_+^N$, $\widetilde{\alpha}_X(t_0)=H$ and with probability one, for all $t_0\in\mathbb{R}_+^N$, $\widetilde{\boldsymbol{\alpha}}_X(t_0)=H$.

However, the local regularity has not been studied so far, in the general case of an indexing collection which is not reduced to the collection of rectangles of $\mathbb{R}_+^N$. Theorem 2.34, Theorem 2.35 and Corollary 2.37 provide new results for the sample paths of SIfBm.

In Section 2.5, Theorem 2.35 failed to provide a uniform almost sure upper bound for the pointwise Hölder exponent of a general Gaussian set-indexed process. In the specific case of the set-indexed fractional Brownian motion, this result can be improved under some additional requirement. We consider a supplementary condition on the collection $\mathcal{A}$ and the distance d_m: there exists $\eta>0$ such that $\forall U_0\in\mathcal{A}$,

$$\inf_{\rho>0}\sup\left\{\frac{d_m(U,g_n(U))}{\rho};\ n\in\mathbb{N},\ U,g_n(U)\in B_{d_m}(U_0,\rho)\right\}\geq\eta\ . \tag{2.22}$$

Theorem 2.38. *Let $\mathbf{B}^H$ be a set-indexed fractional Brownian motion on $(T,\mathcal{A},m)$, $H\in(0,1/2]$. Assume that the subclasses $(\mathcal{A}_n)_{n\in\mathbb{N}}$ satisfy Assumption $(\mathcal{H}_{\underline{\mathcal{A}}})$.*
Then, the local and pointwise Hölder exponents of $\mathbf{B}^H$ at any $U_0\in\mathcal{A}$, defined with respect to the distance d_m or any equivalent distance, satisfy

$$\mathbb{P}\Big(\forall U_0\in\mathcal{A},\ \widetilde{\boldsymbol{\alpha}}_{\mathbf{B}^H}(U_0)=H\Big)=1$$

and if the additional Condition (2.22) holds,

$$\mathbb{P}(\forall U_0\in\mathcal{A},\ \boldsymbol{\alpha}_{\mathbf{B}^H}(U_0)=H)=1.$$

Consequently, when $\mathcal{A}$ is the collection of rectangles of $\mathbb{R}_+^N$ and $m=\lambda$ is the Lebesgue measure, i.e. $\mathbf{B}^H$ is a multiparameter fractional Brownian motion, we have

$$\mathbb{P}\Big(\forall U_0\in\mathcal{A},\ \boldsymbol{\alpha}_{\mathbf{B}^H}(U_0)=\widetilde{\boldsymbol{\alpha}}_{\mathbf{B}^H}(U_0)=H\Big)=1.$$

Proof. From the definition of the set-indexed fractional Brownian motion, the following expression of the incremental variance,

$$\forall U,V\in\mathcal{A},\quad \mathbb{E}\Big[|\mathbf{B}^H_U-\mathbf{B}^H_V|^2\Big]=m(U\bigtriangleup V)^{2H},$$

directly implies that the deterministic pointwise and local Hölder exponents are equal to H. By Theorem 2.34, the random exponents on an indexing collection satisfying Assumption $(\mathcal{H}_{\underline{\mathcal{A}}})$ are also equal to H.
For the uniform almost sure result on $\mathcal{A}$, according Theorem 2.35, it remains to prove that $\mathbb{P}(\forall U_0\in\mathcal{A},\ \boldsymbol{\alpha}_{\mathbf{B}^H}(U_0)\leq H)=1$. This fact is the object of the following Section 2.6.2.

For the particular case of the multiparameter fractional Brownian motion, it suffices to notice that the collection $\mathcal{A}$ of rectangles of $\mathbb{R}^N$ endowed with the Lebesgue measure λ satisfies Condition (2.22).
Let us recall that for any $U_0\in\mathcal{A}$, $d_\lambda(U_0,g_n(U_0))=N.2^{-n}+o(2^{-n})$. Hence for a given $\rho>0$, choosing the smallest integer n such that $N.2^{-n}\leq\rho/2$ ensures that

$$\frac{d_\lambda(U_0,g_n(U_0))}{\rho}\geq\frac{N.2^{-(n+1)}}{\rho}\geq\frac{1}{8},$$

and that $g_n(U_0)\in B_{d_\lambda}(U_0,\rho)$. □

If the collection $\mathscr{A}$ or the metric d_m do not satisfy the additional requirement (2.22), then the lower bound for the pointwise exponent remains true by Theorem 2.35: $\mathbb{P}(\forall U_0 \in \mathscr{A},\ \boldsymbol{\alpha}_{\mathbf{B}^H}(U_0) \geq H) = 1$.

In [60], it is shown that for all $U, V \in \mathscr{A}$, $\mathbb{E}[|\Delta \mathbf{B}^H_{U \setminus V}|^2] = m(U \setminus V)^{2H}$. This implies that for all $U_0 \in \mathscr{A}$, $\widetilde{\mathbb{\alpha}}_{\mathbf{B}^H,\mathscr{C}}(U_0) = \mathbb{\alpha}_{\mathbf{B}^H,\mathscr{C}}(U_0) = H$, and so by Corollary 2.36:

$$\widetilde{\boldsymbol{\alpha}}_{\mathbf{B}^H,\mathscr{C}}(U_0) = \boldsymbol{\alpha}_{\mathbf{B}^H,\mathscr{C}}(U_0) = H \quad \text{a.s.}$$

The case of the exponent of pointwise continuity needs to determine the behaviour of $\mathbb{E}\left[|\Delta \mathbf{B}^H_C|^2\right]$ when $C \in \mathscr{C}$ (and not only $C = U \setminus V \in \mathscr{C}_0$, with $U, V \in \mathscr{A}$ as previously). In the specific case of an SIfBm with $H = 1/2$, we can state:

Proposition 2.39. *Let $\mathbf{B}$ be a Brownian motion on $\mathscr{A}$. Then, for all $t_0 \in T$,*

$$\boldsymbol{\alpha}^{pc}_{\mathbf{B}}(t_0) = \mathbb{\alpha}^{pc}_{\mathbf{B}}(t_0) = \frac{1}{2} \quad a.s.$$

A uniform lower bound in any $U_{max} \in \mathscr{A}$ such that $m(U_{max}) < \infty$, is given by:

$$\mathbb{P}\left(\forall t_0 \in U_{max}, \quad \boldsymbol{\alpha}^{pc}_{\mathbf{B}}(t_0) \geq \mathbb{\alpha}^{pc}_{\mathbf{B}}(t_0) = \frac{1}{2}\right) = 1.$$

Proof. Since $\mathbb{E}\left[|\Delta \mathbf{B}_C|^2\right] = m(C)$, the result follows from Corollary 2.37. □

This property cannot be extended directly to any SIfBm for which $H < 1/2$, since we do not have $\mathbb{E}\left[|\Delta \mathbf{B}^H_C|^2\right] = m(C)^{2H}$ for all $C \in \mathscr{C}$ (see [60]). However, the results of Proposition 2.39 hold in the specific case of rectangles of $\mathbb{R}^N$, i.e. for the multiparameter fractional Brownian motion (see Remark 2.46).

2.6.2 *Proof of the uniform a.s. pointwise exponent of the SIfBm*

In [2], the isotropic fractional Brownian field is proved to have a uniform pointwise exponent equal to H using techniques such as local times; and in [16], the same result holds for the regular multifractional Brownian motion (mBm), with a proof based on the integral representation of the mBm. This result relies on tools that are not available in the set-indexed framework, although some attempts have been made to introduce set-indexed local times ([71]).

In [16], the following technical lemma is proved for a multifractional Brownian motion. We restrict it to fBm's case:

Lemma 2.40. *Let $B^H = \{B^H_t, t \in \mathbb{R}_+\}$ be a fractional Brownian motion of index $H \in (0,1)$. Let $\epsilon > 0$, $\rho > 0$, $0 \leq s < t$, $n \in \mathbb{N}^*$ and $\delta u = \frac{\rho}{n}$. Then, let $u_0 = s$ and for all $k \in \{0, \ldots, n\}$, $u_{k+1} = u_k + \delta u$. We have the following:*

$$\mathbb{P}\left(\bigcap_{k=1}^{n} \{|B^H_{u_k} - B^H_{u_{k-1}}| < \rho^{H+\epsilon}\}\right) \leq \left(\frac{2}{\sqrt{2\pi}}\right)^n \left(\frac{\rho^{H+\epsilon}}{C.(\delta u)^H}\right)^n,$$

where C is a constant depending only on H.

In the sequel, for $U \subset V \in \mathscr{A}$, we denote by $\mathscr{R}(f, U \to V)$, the range of the elementary flow $f : [0,d] \to \mathscr{A}$ such that $f(0) = U$ and $f(d) = V$, where $d = d_m(U,V)$ (the distance considered here is always $d_m = m(\bullet \triangle \bullet)$). Hence $\mathscr{R}(f, U \to V)$ is a totally ordered subset of $\mathscr{A}$ which forms a continuum. We also denote by $\mathscr{R}_n(f,U)$, the range $\mathscr{R}(f, U \to g_n(U))$. Since the choice of a particular f does not matter, these notations can be used without specifying f, considering that a choice has been made.

Lemma 2.41. *Let $\mathbf{B}^H$ be a SIfBm on $(\mathscr{A}, T, m)$ of index $H \in (0, \frac{1}{2}]$. Let $U \in \mathscr{A}$, $i \in \mathbb{N}$ and $\rho_i = d_m(U, g_i(U))$. Let $\epsilon > 0$, $n \in \mathbb{N}^*$. In any $\mathscr{R}_i(f,U)$, there exist an increasing sequence $(U_j)_{0\le j\le n}$ such that $U_0 = U$, $U_n = g_i(U)$, and $\delta U = d_m(U_{j-1}, U_j) = \frac{\rho_i}{n}$ for all $j \in \{1,\dots,n\}$. Then,*

$$\mathbb{P}\left(\bigcap_{k=1}^{n}\left\{|\mathbf{B}^H_{U_k} - \mathbf{B}^H_{U_{k-1}}| < \rho_i^{H+\epsilon}\right\}\right) \le \left(\frac{2}{\sqrt{2\pi}}\right)^n \left(\frac{\rho_i^{H+\epsilon}}{\sigma}\right)^n$$

where $\sigma = C.(\delta U)^H$ and $C > 0$ only depends on H. Equivalently, there exists a constant $\widetilde{C} > 0$, which only depends on H, such that

$$\mathbb{P}\left(\bigcap_{k=1}^{n}\left\{|\mathbf{B}^H_{U_k} - \mathbf{B}^H_{U_{k-1}}| < \rho_i^{H+\epsilon}\right\}\right) \le \left(\widetilde{C}\; n^H \rho_i^{\epsilon}\right)^n. \tag{2.23}$$

Proof. Let us consider the range $\mathscr{R}_i(f,U)$ of a flow f connecting U to $g_i(U)$. The standard projection of $X = \mathbf{B}^H$ on f is a standard fractional Brownian motion that we denote $X^{f,m} = \left\{X^{f,m}_t, t \in [0,\rho_i]\right\}$. As usual, $\theta = m \circ f$ and in the present situation, $\theta : [0,\rho_i] \to [m(U), m(g_i(U))]$. For $k \in \{0,\dots,n\}$, let $u_k = m(U) + k.\frac{\rho_i}{n}$ and define $U_k = f \circ \theta^{-1}(u_k)$. The U_k's contitute the sequence of the statement and we remark that

$$\mathbb{P}\left(\bigcap_{k=1}^{p}\left\{|X_{U_k} - X_{U_{k-1}}| < \rho_n^{H+\epsilon}\right\}\right) = \mathbb{P}\left(\bigcap_{k=1}^{p}\left\{|X^{f,m}_{u_k} - X^{f,m}_{u_{k-1}}| < \rho_n^{H+\epsilon}\right\}\right).$$

The result follows from Lemma 2.40. □

The following Proposition 2.42 is the key result to prove the uniform almost sure upper bound for the SIfBm.

Proposition 2.42. *Let $\mathbf{B}^H$ be a SIfBm on $(\mathscr{A}, T, m)$ of parameter $H \in (0,1/2]$. We assume that $(\mathscr{A}_n)_{n\in\mathbb{N}}$ endowed with the distance d_m satisfies Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$ and that Condition (2.22) holds. Then, with probability one, for all $\epsilon > 0$, there exists a random variable $h > 0$ such that for all $\rho \le h(\omega)$ and for all $U_0 \in \mathscr{A}$,*

$$\sup_{U,V \in B_{d_{\mathscr{A}}}(U_0,\rho)} \left|\mathbf{B}^H_U - \mathbf{B}^H_V\right| \ge \rho^{H+\epsilon}.$$

Proof. Let us fix $\epsilon > 0$. For all $U \in \mathscr{A}$, let $\rho_{n,U} = d_m(U, g_n(U))$ and $p_{n,U} = \lfloor \rho_{n,U}^{-\epsilon} \rfloor$. For all $N \in \mathbb{N}^*$, we consider the event

$$A_N = \bigcup_{n\ge N} \bigcup_{U \in \mathscr{A}_n} \left\{\forall V, W \in \mathscr{R}_n(f,U), |X_V - X_W| < \rho_{n,U}^{H+\epsilon}\right\}.$$

We have

$$\begin{aligned}\mathbb{P}(A_N) &\le \sum_{n\ge N} \sum_{U\in\mathscr{A}_n} \mathbb{P}\left(\forall V, W \in \mathscr{R}_n(f,U), |X_V - X_W| < \rho_{n,U}^{H+\epsilon}\right) \\ &\le \sum_{n\ge N} \sum_{U\in\mathscr{A}_n} \mathbb{P}\left(\bigcap_{k=1}^{p_{n,U}}\left\{|X_{U_k} - X_{U_{k-1}}| < \rho_{n,U}^{H+\epsilon}\right\}\right),\end{aligned}$$

where $U_0, \dots, U_{p_{n,U}}$ are defined as in Lemma 2.41.

Following equation (2.23) and since $\rho_{n,U} = d_{\mathscr{A}}(U, g_n(U)) \leq k_n^{-1/\underline{q_{\mathscr{A}}}}$, there exist positive constants C_1 and C_2 such that

$$\mathbb{P}\left(\bigcap_{k=1}^{p_{n,U}} \left\{|X_{U_k} - X_{U_{k-1}}| < \rho_{n,U}^{H+\epsilon}\right\}\right) \leq \left(C_1\ \rho_{n,U}^{\epsilon(1-H)}\right)^{\rho_{n,U}^{-\epsilon}-1}$$
$$\leq \left(C_2\ k_n^{-1/\underline{q_{\mathscr{A}}}}\right)^{\epsilon(1-H)(k_n^{\epsilon/\underline{q_{\mathscr{A}}}}-1)}.$$

Going back to the previous equation, we obtain

$$\mathbb{P}(A_N) \leq \sum_{n\geq N} k_n \left(C_2\ k_n^{-1/\underline{q_{\mathscr{A}}}}\right)^{\epsilon(1-H)(k_n^{\epsilon/\underline{q_{\mathscr{A}}}}-1)} = R_N.$$

Since k_n is admissible, we can easily show that $\sum_{N\in\mathbb{N}^*} R_N < \infty$. Hence, Borel-Cantelli Lemma implies the existence of a random variable $N(\omega)$ such that: with probability one, for all $n \geq N(\omega)$ and for all $U \in \mathscr{A}_n$,

$$\exists V, W \in \mathscr{R}_n(f, U); \quad |X_V - X_W| \geq \rho_{n,U}^{H+\epsilon}. \tag{2.24}$$

For $U_0 \in \mathscr{A}$ and $\rho > 0$, Assumption (2.22) gives the existence of $\mathscr{R}_n(f,U) \subset B_{d_{\mathscr{A}}}(U_0, \rho)$, for some $n \geq N(\omega)$ and $U \in \mathscr{A}$ such that $\rho_{n,U} \geq \eta\rho$. Then, there exist $V, W \in \mathscr{A}$ (the same that in (2.24)), such that

$$|X_V - X_W| \geq \rho_{n,U}^{H+\epsilon} \geq (\eta^{H+\epsilon})\ \rho^{H+\epsilon}$$

which concludes the proof. □

As a consequence of Proposition 2.42, with probability one, the random pointwise Hölder exponent of a SIfBm is uniformly smaller than H (and thus, equal to H, by Theorem 2.35), provided that Assumption ($\mathscr{H}_{\underline{\mathscr{A}}}$) and the additional requirement (2.22) hold.

2.6.3 *Hölder exponents of the SIOU process*

Theorems 2.34 and 2.35 can be also applied to derive Hölder exponents of the set-indexed Ornstein-Uhlenbeck (SIOU in short) process, studied in [17]. This process was introduced as an example of set-indexed process satisfying some stationarity and Markov properties.

A mean-zero Gaussian process $Y = \{Y_U;\ U \in \mathscr{A}\}$, where $\mathscr{A}$ is an indexing collection on the measure space (T, m), is called a *stationary set-indexed Ornstein-Uhlenbeck process* if

$$\forall U, V \in \mathscr{A}, \quad \mathbb{E}[Y_U Y_V] = \frac{\sigma^2}{2\gamma} \exp(-\gamma\ m(U \triangle V)),$$

for given positive constants γ and σ.

Fixing $U_0 \in \mathscr{A}$, and for all U, V close to U_0 for the metric d_m, $\mathbb{E}[|Y_U - Y_V|^2] = \dfrac{\sigma^2}{\gamma}(1 - e^{-\gamma\ m(U\triangle V)})$ implies that $\mathbb{E}[|Y_U - Y_V|^2] = \sigma^2 [m(U \triangle V) + o(m(U \triangle V))]$. This leads to $\mathbb{\alpha}_Y(U_0) = \widetilde{\mathbb{\alpha}}_Y(U_0) = 1/2$. Consequently, the following result follows directly from Theorem 2.35.

Proposition 2.43. *Let $Y = \{Y_U;\ U \in \mathscr{A}\}$ be a stationary set-indexed Ornstein-Uhlenbeck process on $(T, \mathscr{A}, m)$. Assume that the subclasses $(\mathscr{A}_n)_{n\in\mathbb{N}}$ of $\mathscr{A}$ satisfy Assumption $(\mathscr{H}_{\underline{\mathscr{A}}})$.*
Then, the pointwise and local Hölder exponents satisfy, with probability one,

$$\forall U_0 \in \mathscr{A}, \quad \widetilde{\boldsymbol{\alpha}}_Y(U_0) = \frac{1}{2} \quad \text{and} \quad \boldsymbol{\alpha}_Y(U_0) \geq \frac{1}{2}$$

and $\forall U_0 \in \mathscr{A}$, $\boldsymbol{\alpha}_Y(U_0) = \frac{1}{2}$ a.s.
As another consequence of the previous remark, the equality holds for the $\mathscr{C}$-Hölder exponents, for all $U_0 \in \mathscr{A}$, almost surely.

As mentioned in the case of the SIfBm, the computation of the exponent of pointwise continuity requires a fine estimation of the variance of the process over $\mathscr{C}$. When $\mathscr{A}$ is the collection of the rectangles of $\mathbb{R}^N_+$, the estimation of $\mathbb{E}[|\Delta Y_{C_n(t)}|^2]$ is easier, as the example of the SIOU process shows.

Lemma 2.44. *Let $\mathscr{A} = \{[0,t] : t \in [0,1]^N\}$ endowed with the usual dissecting class $(\mathscr{A}_n)$ made of the dyadics. Let $t \in (0,1)^N$, $t = (t_1, \dots, t_N)$ and define:*

$$t_j^n = \begin{cases} t_j & \text{if } 2^n t_j \in \mathbb{N} \\ 2^{-n}\lfloor 2^n t_j + 1 \rfloor & \text{otherwise,} \end{cases} \quad \text{and} \quad \tilde{t}_k^n = \begin{cases} 2^{-n}\lfloor 2^n t_k - 1 \rfloor & \text{if } 2^n t_k \in \mathbb{N} \\ 2^{-n}\lfloor 2^n t_k \rfloor & \text{otherwise.} \end{cases}$$

Then,

$$C_n(t) = [0, (t_1^n, \dots, t_N^n)] \setminus \bigcup_{k=1}^{N} [0, (t_1^n, \dots, \tilde{t}_k^n, \dots, t_N^n)].$$

Proof. We recall that $C_n(t)$, the left-neighbourhood of A_t in $\mathscr{A}_n$, is defined as $\bigcap_{\substack{C\in\mathscr{C}_n \\ t\in C}} C$. In the particular case of the rectangles, it corresponds to the expression given in the lemma. □

As usual, let λ be the Lebesgue measure of $\mathbb{R}^N$. A direct consequence of this result is that any Gaussian process X satisfying the assumptions of Corollary 2.10 satisfies, for all $t \in [0,1]^N$ and for all ω,

$$\widetilde{\boldsymbol{\alpha}}_{X,\mathscr{C}}(A_t) \leq \boldsymbol{\alpha}_{X,\mathscr{C}}(A_t) \leq \boldsymbol{\alpha}_X^{pc}(t),$$

with respect to the Lebesgue measure λ and the distance d_λ.
More precise results are available for the SIOU process and the SIfBm.

Proposition 2.45. *Let $Y = \{Y_U : U \in \mathscr{A}\}$ be a SIOU process, where $\mathscr{A}$ refers to the rectangles of $[0,1]^N$ as in the Lemma 2.44. Then, the pointwise continuity of Y with respect to the Lebesgue measure λ of $\mathbb{R}^N$ satisfies*

$$\forall t_0 \in [0,1]^N, \quad \mathbb{P}\left(\boldsymbol{\alpha}_Y^{pc}(t_0) = \mathbb{\alpha}_Y^{pc}(t_0) = \frac{1}{2} \right) = 1,$$

and,

$$\mathbb{P}\left(\forall t_0 \in [0,1]^N, \quad \boldsymbol{\alpha}_Y^{pc}(t_0) \geq \mathbb{\alpha}_Y^{pc}(t_0) = \frac{1}{2} \right) = 1.$$

Proof. For the sake of readability, the proof is written for $N = 2$. Let $t = (t_1, t_2) \in [0,1]^N$. To show there is no difference in the final result, we assume t_1 is dyadic and t_2 is not. Let $k, l \in \mathbb{N}, k < 2^l$ such that $t_1 = k.2^{-l}$. Let $n \in \mathbb{N}, n \geq l$.

First, we notice that, by Lemma 2.44,

$$C_n(t) = \left[0,(t_1, 2^{-n}\lfloor 2^n t_2 + 1\rfloor)\right] \setminus \left\{\left[0,(2^{-n}\lfloor 2^n t_1 - 1\rfloor, 2^{-n}\lfloor 2^n t_2 + 1\rfloor)\right] \cup \left[0,(t_1, 2^{-n}\lfloor 2^n t_2\rfloor)\right]\right\}.$$

Re-writing this for short $C_n(t) = A_n \setminus \left\{B_{1,n} \cup B_{2,n}\right\}$, the inclusion-exclusion formula gives

$$\begin{aligned}\mathbb{E}[|\Delta Y_{C_n(t)}|^2] = \mathbb{E}[&Y_{A_n}^2 + Y_{B_{1,n}}^2 + Y_{B_{2,n}}^2 + Y_{B_{1,n}\cap B_{2,n}}^2 - 2Y_{A_n}Y_{B_{1,n}} - 2Y_{A_n}Y_{B_{2,n}} \\ &+ 2Y_{A_n}Y_{B_{1,n}\cap B_{2,n}} + 2Y_{B_{1,n}}Y_{B_{2,n}} - 2Y_{B_{1,n}}Y_{B_{1,n}\cap B_{2,n}} - 2Y_{B_{2,n}}Y_{B_{1,n}\cap B_{2,n}}].\end{aligned}$$

Combined with the covariance of the SIOU, a second-order Taylor expansion gives:

$$\mathbb{E}[|\Delta Y_{C_n(t)}|^2] = \frac{\sigma^2}{2\gamma}\left(8\gamma.2^{-2n} + 16\gamma^2.2^{-4n}\lfloor 2^n t_1\rfloor.\lfloor 2^n t_2\rfloor + o(2^{-2n})\right).$$

Considering the fact that $\lambda(C_n(t)) = 2^{-2n}$, the previous expansion implies $\widetilde{\alpha}_Y^{pc}(t) = \frac{1}{2}$. Therefore, Corollary 2.37 gives the result. □

Remark 2.46. *With the notations of Proposition 2.45, we can consider the case of the SIfBm $\mathbf{B}^H$ indexed by $\mathcal{A} = \left\{[0,t], t \in \mathbb{R}_+^N\right\} \cup \{\emptyset\}$,*

$$\begin{aligned}\mathbb{E}\left[|\Delta \mathbf{B}^H_{C_n(t)}|^2\right] = &\, m(A_n \setminus B_{1,n})^{2H} + m(A_n \setminus B_{2,n})^{2H} - m(A_n \setminus (B_{1,n} \cap B_{2,n})^{2H} \\ &- m(B_{1,n} \triangle B_{2,n})^{2H} + m(B_{1,n} \setminus B_{2,n})^{2H} + m(B_{2,n} \setminus B_{1,n})^{2H}.\end{aligned}$$

Then, the same development as the previous proof gives $\widetilde{\alpha}^{pc}_{\mathbf{B}^H}(t_0) = H$ for all $t_0 \in [0,1]^N$. Consequently, we can state:

$$\forall t_0 \in [0,1]^N, \quad \mathbb{P}\left(\boldsymbol{\alpha}^{pc}_{\mathbf{B}^H}(t_0) = \widetilde{\alpha}^{pc}_{\mathbf{B}^H}(t_0) = H\right) = 1,$$

and,

$$\mathbb{P}\left(\forall t_0 \in [0,1]^N, \quad \boldsymbol{\alpha}^{pc}_{\mathbf{B}^H}(t_0) \geq \widetilde{\alpha}^{pc}_{\mathbf{B}^H}(t_0) = H\right) = 1.$$

2.7 Proofs of several technical results

2.7.1 Proof of Corollary 2.14

In order to prove Corollary 2.14, we need the following lemma:

Lemma 2.47. *If the distance $d_{\mathscr{A}}$ on the class $\mathscr{A}$ is contracting, then for $U, V_1, V_2 \in \mathscr{A}$,*

$$d_{\mathscr{A}}(U, V_1) \vee d_{\mathscr{A}}(U, V_2) \leq \rho \Rightarrow d_{\mathscr{A}}(U, V_1 \cap V_2) \leq 3\rho.$$

Moreover, for any integer $l \geq 1$ and for all $U, V_1, \ldots, V_l \in \mathscr{A}$,

$$\max_{i \leq l}\{d_{\mathscr{A}}(U, V_i)\} \leq \rho \Rightarrow d_{\mathscr{A}}(U, V_1 \cap \cdots \cap V_l) \leq K(l)\, \rho,$$

for some constant $K(l) > 0$ which only depends on l.

Proof of Lemma 2.47. The proof relies on the triangular inequality and the contracting property of $d_{\mathscr{A}}$. □

Proof of Corollary 2.14. Assuming that g_n can be extended to $\mathscr{A}(u)$ in the following way:

$$\forall V_1, \ldots, V_p \in \mathscr{A}, \quad g_n\left(\bigcup_{i=1}^{p} V_i\right) = \bigcup_{i=1}^{p} g_n(V_i)\,,$$

the following inequality holds:

$$|X_U - \Delta X_{\cup V_i}| \leq |X_{g_{n_0}(U)} - \Delta X_{g_{n_0}(\cup V_i)}| + \sum_{j \geq n_0} |X_{g_{j+1}(U)} - X_{g_j(U)}| + \sum_{j \geq n_0} |\Delta X_{g_{j+1}(\cup V_i)} - \Delta X_{g_j(\cup V_i)}|\,. \tag{2.25}$$

Since for all $V_1, \ldots, V_p \in \mathscr{A}$,

$$\Delta X_{\cup V_i} = \sum_{i=1}^{p} X_{V_i} + \cdots + (-1)^{k-1} \sum_{i_1 < \cdots < i_k} X_{\cap_{i_1 < \cdots < i_k} V_i} + \cdots + (-1)^{p-1} X_{V_1 \cap \cdots \cap V_p},$$

one can express

$$\begin{aligned} |\Delta X_{g_{n+1}(\cup V_i)} - \Delta X_{g_n(\cup V_i)}| \leq & \sum_{i=1}^{p} |X_{g_{n+1}(V_i)} - X_{g_n(V_i)}| + \ldots \\ & + \sum_{i_1 < \cdots < i_k} |X_{g_{n+1}(\cap_{i_1 < \cdots < i_k} V_i)} - X_{g_n(\cap_{i_1 < \cdots < i_k} V_i)}| + \ldots \\ & + |X_{g_{n+1}(\cap_{i=1}^{p} V_i)} - X_{g_n(\cap_{i=1}^{p} V_i)}|. \end{aligned} \tag{2.26}$$

Now assume that $U, V_1, \ldots, V_p \in \mathscr{D}$. When $p \leq l$, the number of terms in the right side of inequality (2.26) is bounded by a constant, independent of the set $V_1, \ldots, V_p \in \mathscr{A}$. Thus, there exists a positive constant $K_2(l)$ such that

$$|\Delta X_{g_{n+1}(\cup V_i)} - \Delta X_{g_n(\cup V_i)}| \leq K_2(l) \sup_{W \in \mathscr{D}} |X_{g_{n+1}(W)} - X_{g_n(W)}|\,. \tag{2.27}$$

Using the same sequence $(a_j)_{j\in\mathbb{N}}$ as in the proof of Theorem 2.9, and the above equation (2.27) in the third inequality below:

$$\begin{aligned}
\mathbf{P}\Big(\sup_{V_1,\dots,V_p\in\mathscr{D}} \sum_{j\geq n_0} |\Delta X_{g_{j+1}(\cup V_i)} - \Delta X_{g_j(\cup V_i)}| \geq K_2(l) k_{n_0+1}^{-\gamma/q_{\underline{\mathscr{A}}}}\Big) \\
&\leq \mathbb{P}\Big(\exists V_1,\dots,V_p \in \mathscr{D}, \exists j\geq n_0,\ |\Delta X_{g_{j+1}(\cup V_i)} - \Delta X_{g_j(\cup V_i)}| \geq a_j K_2(l) k_{n_0+1}^{-\gamma/q_{\underline{\mathscr{A}}}}\Big) \\
&\leq \mathbb{P}\Big(\exists W\in\mathscr{D}, \exists j\geq n_0,\ |X_{g_{j+1}(W)} - X_{g_j(W)}| \geq a_j k_{n_0+1}^{-\gamma/q_{\underline{\mathscr{A}}}}\Big)\ .
\end{aligned}$$

We obtain the same expression (2.3) that we had in the proof of Theorem 2.9, thus the same conclusion holds: if $\max_{i\leq l}\{m(U\setminus V_i)\} \leq k_{n_0}^{-1/q_{\underline{\mathscr{A}}}}$, then almost surely, $k_{n_0}^{-1/q_{\underline{\mathscr{A}}}} \leq h^*$ implies that:

$$\sup_{V_1,\dots,V_p\in\mathscr{D}} \sum_{j\geq n_0} |\Delta X_{g_{j+1}(\cup V_i)} - \Delta X_{g_j(\cup V_i)}| \leq K_2(l)\ k_{n_0+1}^{-\gamma/q_{\underline{\mathscr{A}}}}\ .$$

In the same way, the second term of the upper bound (2.25) is proved to be bounded by $K_4(\gamma, q_{\underline{\mathscr{A}}})\ m(C)^\gamma$, where $K_4(\gamma, q_{\underline{\mathscr{A}}}) > 0$ only depends on γ and $q_{\underline{\mathscr{A}}}$.
The first term of (2.25) can be bounded by a finite sum (whose number of terms only depends on l) of the form $|X_{g_{n_0}(U)} - X_{g_{n_0}(V_{i_1,\dots,i_k})}|$, where $V_{i_1,\dots,i_k} = V_{i_1}\cap\dots\cap V_{i_k}$ for $i_1 < \dots < i_k \leq l$:

$$|X_{g_{n_0}(U)} - \Delta X_{g_{n_0}(\cup V_i)}| \leq \sum_{j=1}^{l} \sum_{i_1<\dots<i_j} |X_{g_{n_0}(U)} - X_{g_{n_0}(V_{i_1,\dots,i_j})}|. \tag{2.28}$$

Finally, if $\max_{i\leq l}\{m(U\setminus V_i)\} \leq k_{n_0}^{-1/q_{\underline{\mathscr{A}}}}$, condition (H1) of Assumption ($\mathscr{H}_{\underline{\mathscr{A}}}$) and Lemma 2.47 imply

$$\begin{aligned}
d_m(g_{n_0}(U), g_{n_0}(V_{i_1,\dots,i_j})) &\leq d_m(g_{n_0}(U), U) + d_m(U, V_{i_1,\dots,i_j}) + d_m(V_{i_1,\dots,i_j}, g_{n_0}(V_{i_1,\dots,i_j})) \\
&\leq K(l)\ \max_{i\leq l}\{m(U\setminus V_i)\} + 2k_{n_0}^{-1/q_{\underline{\mathscr{A}}}} \\
&\leq (K(l)+2)\ k_{n_0}^{-1/q_{\underline{\mathscr{A}}}}\ .
\end{aligned}$$

Hence, Theorem 2.9 implies that when $k_{n_0}^{-1/q_{\underline{\mathscr{A}}}} < (K(l)+2)^{-1}\ h^*$, each term of equation (2.28) is bounded by a quantity proportional to $m(C)^\gamma$. Then, the random variable h^{**} of the statement can be chosen to be $(K(l)+2)^{-1}\ h^*$ and the result follows. □

2.7.2 *Proof of Theorems 2.34 and 2.35*

Lower bound for the pointwise and local Hölder exponents

A lower bound for the local Hölder exponent is directly given by Corollary 2.10.
For all $U_0 \in \mathscr{A}$ and all $0 < \alpha < \widetilde{\mathbb{\alpha}}_X(U_0)$, there exists $\rho_0 > 0$ and $K > 0$ such that

$$\forall U, V \in B_{d_{\mathscr{A}}}(U_0,\rho_0);\quad \mathbb{E}\big[|X_U - X_V|^2\big] \leq K\ d_{\mathscr{A}}(U,V)^{2\alpha}.$$

Therefore, the sample paths of X are almost surely ν-Hölder continuous in $B_{d_{\mathscr{A}}}(U_0,\rho_0)$ for all $\nu\in(0,\alpha)$, which leads to $\alpha \leq \widetilde{\boldsymbol{\alpha}}_X(U_0)$ almost surely. Then we get

$$\mathbb{P}\big(\widetilde{\boldsymbol{\alpha}}_X(U_0) \geq \widetilde{\mathbb{\alpha}}_X(U_0)\big) = 1.$$

By inequality (2.16), any lower bound for the local Hölder exponent is also a lower bound for the pointwise exponent. Moreover it can be improved in the case of strict inequality $0 < \widetilde{\mathbb{\alpha}}_X(U_0) < \mathbb{\alpha}_X(U_0)$.
For any $\epsilon > 0$, there exist $0 < \rho_1 < \rho_0$ and $M > 0$ such that

$$\forall \rho < \rho_1,\ \forall U, V \in B(U_0, \rho); \quad \mathbb{E}\left[\left|\frac{X_U - X_V}{\rho^{\mathbb{\alpha}_X(U_0)-\epsilon}}\right|^2\right] \leq M\ \rho^{\epsilon}.$$

Then setting $\gamma = \mathbb{\alpha}_X(U_0) - \epsilon$, the exponential inequality for the centered Gaussian variable $X_U - X_V$ implies

$$\mathbb{P}\big(|X_U - X_V| \geq \rho^{\gamma}\big) \leq \exp\left(-\frac{1}{2}\ \frac{\rho^{2\gamma}}{\mathbb{E}[|X_U - X_V|^2]}\right) \leq \exp\left(-\frac{1}{2} M \rho^{\epsilon}\right).$$

We consider the particular case $\rho = k_n^{-1/q_{\underline{\mathscr{A}}}} < \rho_1$ for $n \in \mathbb{N}$ large enough. Using the above estimate in the proof of Theorem 2.9 still leads to equation (2.7), where we had that on Ω^*, for all $n \geq n^*$:

$$\sup_{\substack{U,V \in \mathscr{D} \\ d_{\mathscr{A}}(U,V) \leq \rho}} |X_U - X_V| \leq 3\rho^{\gamma}\ .$$

Hence this inequality gives:

$$\sup_{U,V \in B(U_0, k_N^{-1/q_{\underline{\mathscr{A}}}})} |X_U - X_V| \leq C\ k_N^{-\gamma/q_{\underline{\mathscr{A}}}} \quad \text{a.s.}$$

and since the sequence $\left(k_n^{-1/q_{\underline{\mathscr{A}}}}\right)_{n \in \mathbb{N}}$ is decreasing,

$$\limsup_{\rho \to 0} \sup_{U,V \in B(U_0,\rho)} \frac{|X_U - X_V|}{\rho^{\gamma}} < \infty \quad \text{a.s.}$$

Therefore, $\forall \epsilon > 0$, $\boldsymbol{\alpha}_X(U_0) \geq \mathbb{\alpha}_X(U_0) - \epsilon$ almost surely and $\mathbb{P}(\boldsymbol{\alpha}_X(U_0) \geq \mathbb{\alpha}_X(U_0)) = 1$.

Upper bounds for the pointwise and local Hölder exponents

As in [57], upper bounds for the pointwise and local Hölder exponents are given by the following two lemmas. Their proof are totally identical to multiparameter setting.

Lemma 2.48. *Let $X = \{X_U;\ U \in \mathscr{A}\}$ be a centered Gaussian process. Assume that for $U_0 \in \mathscr{A}$, there exists $\mu \in (0,1)$ such that for all $\epsilon > 0$, there exist a sequence $(U_n)_{n \in \mathbb{N}^*}$ of $\mathscr{A}$ converging to U_0, and a constant $c > 0$ such that*

$$\forall n \in \mathbb{N}^*; \quad \mathbb{E}\big[|X_{U_n} - X_{U_0}|^2\big] \geq c\ d_{\mathscr{A}}(U_n, U_0)^{2\mu + \epsilon}.$$

Then, we have almost surely

$$\boldsymbol{\alpha}_X(U_0) \leq \mu.$$

Since the process X has a finite deterministic Hölder exponent, for $\mu = \mathbb{\alpha}_X(U_0)$, one can find a sequence (U_n) as in Lemma 2.48. Hence $\mathbb{P}(\boldsymbol{\alpha}_X(U_0) \leq \mathbb{\alpha}_X(U_0)) = 1$.

Lemma 2.49. *Let $X = \{X_U;\ U \in \mathscr{A}\}$ be a centered Gaussian process. Assume that for $U_0 \in \mathscr{A}$, there exists $\mu \in (0,1)$ such that for all $\epsilon > 0$, there exist two sequences $(U_n)_{n\in\mathbb{N}^*}$ and $(V_n)_{n\in\mathbb{N}^*}$ of $\mathscr{A}$ converging to U_0, and a constant $c > 0$ such that*

$$\forall n \in \mathbb{N}^*; \quad \mathbb{E}\left[|X_{U_n} - X_{V_n}|^2\right] \geq c\, d_{\mathscr{A}}(U_n, V_n)^{2\mu+\epsilon}.$$

Then, we have almost surely

$$\widetilde{\boldsymbol{\alpha}}_X(U_0) \leq \mu.$$

As for the pointwise case, $\mathbb{P}(\widetilde{\boldsymbol{\alpha}}_X(U_0) \leq \widetilde{\mathbb{\alpha}}_X(U_0)) = 1$ follows from Lemma 2.49 with $\mu = \widetilde{\mathbb{\alpha}}_X(U_0)$.

Proof of the uniform almost sure result

This section is devoted to the proof of Theorem 2.35. We only consider the local Hölder exponent. The uniform almost sure lower bound for the pointwise exponent is proved in a similar way.

Starting with the lower bound, from Theorem 2.9, for all $U_0 \in \mathscr{A}$ and all $\epsilon > 0$, there is a modification Y_{U_0} of X which is α-Hölder continuous for all $\alpha \in (0, \widetilde{\mathbb{\alpha}}_X(U_0) - \epsilon)$ on $B_{d_{\mathscr{A}}}(U_0, \rho_0)$.

- In the first step, $\widetilde{\mathbb{\alpha}}_X$ is assumed to be constant over $\mathscr{A}$. Hence the local Hölder exponent of Y_{U_0} satisfies almost surely

$$\forall U \in B_{d_{\mathscr{A}}}(U_0, \rho_0), \quad \widetilde{\boldsymbol{\alpha}}_{Y_{U_0}}(U) \geq \widetilde{\mathbb{\alpha}}_X - \epsilon. \tag{2.29}$$

  The collection $\mathscr{A}$ is totally bounded, so it can be covered by a countable number of balls of radius at most η, for all $\eta > 0$. Let B be one of these balls. For all $U_0 \in \mathscr{A}$, we consider $\rho_0 > 0$ such that (2.29) holds. We have obviously

$$B \subseteq \bigcup_{U_0 \in B} B_{d_{\mathscr{A}}}(U_0, \rho_0).$$

  For each open ball, there exists an integer n such that $B_{d_{\mathscr{A}}}(U_0, \rho_0) \cap \mathscr{A}_n \neq \emptyset$ so that for $V_0 \in B_{d_{\mathscr{A}}}(U_0, \rho_0) \cap \mathscr{A}_n$, there exists an integer m_0 such that $U_0 \in B_{d_{\mathscr{A}}}(V_0, 2^{-m_0}) \subseteq B_{d_{\mathscr{A}}}(U_0, \rho_0)$. Thus

$$B \subseteq \bigcup B_{d_{\mathscr{A}}}(V_0, 2^{-m_0}),$$

  where the union is countable. Each of these balls satisfies

$$\mathbb{P}\left(\forall U \in B_{d_{\mathscr{A}}}(V_0, 2^{-m_0}),\ \widetilde{\boldsymbol{\alpha}}_X(U) \geq \widetilde{\mathbb{\alpha}}_X - \epsilon\right) = 1,$$

  and since $\mathscr{A}$ is a countable union of balls $B_{d_{\mathscr{A}}}(V_0, 2^{-m_0})$, we get

$$\mathbb{P}\left(\forall U \in \mathscr{A},\ \widetilde{\boldsymbol{\alpha}}_X(U) \geq \widetilde{\mathbb{\alpha}}_X - \epsilon\right) = 1.$$

  Taking $\epsilon \in \mathbb{Q}_+^*$, we conclude that

$$\mathbb{P}\left(\forall U \in \mathscr{A},\ \widetilde{\boldsymbol{\alpha}}_X(U) \geq \widetilde{\mathbb{\alpha}}_X\right) = 1. \tag{2.30}$$

- In the general case of a not constant exponent $\widetilde{\mathbb{\alpha}}_X$, for any ball B of radius η previously introduced, we set $\beta = \inf_{U\in B} \widetilde{\mathbb{\alpha}}_X(U) - \epsilon$, $\epsilon > 0$. Then, there exists a constant $C > 0$ such that

$$\forall U, V \in B, \quad \mathbb{E}[|X_U - X_V|^2] \leq C\, d_{\mathscr{A}}(U, V)^{2\beta}.$$

In a similar way as we proved (2.30), we deduce the existence of an event $\Omega^* \subseteq \Omega$ of probability one such that for all $\omega \in \Omega^*$:

$$\forall U \in \mathscr{A},\ \forall n \geq 0,\ \forall \epsilon \in \mathbb{Q}_+^*,$$
$$\forall U_0 \in B_{d_{\mathscr{A}}}(U, 2^{-n}), \quad \widetilde{\boldsymbol{\alpha}}_X(U_0) \geq \inf_{V \in B_{d_{\mathscr{A}}}(U, 2^{-n})} \widetilde{\mathbb{\alpha}}_X(V) - \epsilon.$$

By letting $n \to \infty$, the previous equation leads to

$$\mathbb{P}\left(\forall U_0 \in \mathscr{A}, \quad \widetilde{\boldsymbol{\alpha}}_X(U_0) \geq \liminf_{U \to U_0} \widetilde{\mathbb{\alpha}}_X(U) \right) = 1.$$

In order to prove the converse inequality (which holds only for the local exponent), we adapt a proof in [59]. We first assume that $\widetilde{\mathbb{\alpha}}_X$ is constant on $\mathscr{A}$.
Using the fact that $\mathscr{D} = \bigcup_{n \in \mathbb{N}} \mathscr{A}_n$ is countable, Lemma 2.49 gives

$$\mathbb{P}(\forall U \in \mathscr{D},\ \widetilde{\boldsymbol{\alpha}}_X(U) \leq \widetilde{\mathbb{\alpha}}_X) = 1.$$

Let $\Omega' \in \mathscr{F}$ be the set of ω, such that $\widetilde{\boldsymbol{\alpha}}_X(U) \leq \widetilde{\mathbb{\alpha}}_X$ for all $U \in \mathscr{D}$. Let $U_0 \in \mathscr{A} \setminus \mathscr{D}$. Let $(U^{(i)})_{i \in \mathbb{N}}$ be a sequence in $\mathscr{D}$ converging to U_0. On Ω', $\widetilde{\boldsymbol{\alpha}}_X(U^{(i)}) \leq \widetilde{\mathbb{\alpha}}_X$, for all $i \in \mathbb{N}$. For each fixed $i \in \mathbb{N}$, there exist two sequences $(V_n^{(i)})_{n \in \mathbb{N}}$ and $(W_n^{(i)})_{n \in \mathbb{N}}$ in $\mathscr{A}$ converging to $U^{(i)}$ as $n \to \infty$, and for all $n \in \mathbb{N}$,

$$\lim_{n \to +\infty} \frac{|X_{V_n^{(i)}} - X_{W_n^{(i)}}|}{d_{\mathscr{A}}(V_n^{(i)}, W_n^{(i)})^{\widetilde{\mathbb{\alpha}}_X + \epsilon}} = +\infty.$$

As in [59], we build two other sequences $(V_n)_{n \in \mathbb{N}}$ and $(W_n)_{n \in \mathbb{N}}$ so that $V_n \to U_0$ and $W_n \to U_0$ and

$$\lim_{n \to +\infty} \frac{|X_{V_n} - X_{W_n}|}{d_{\mathscr{A}}(V_n, W_n)^{\widetilde{\mathbb{\alpha}}_X + \epsilon}} = +\infty.$$

This implies the expected inequality for all $U_0 \in \mathscr{A}$.
The general case for $\widetilde{\mathbb{\alpha}}_X$ not constant is proved in the same way as for the lower bound.

The L^2-fractional Brownian field 3

Contents

"Un professeur avait défini un objet abstrait comme un objet qu'on ne peut ni voir ni toucher, et un objet concret comme un objet qu'on peut voir et toucher. Oui, avait acquiescé un élève, mon caleçon, par exemple, est concret tandis que le vôtre est abstrait, parce que je ne peux ni le voir ni le toucher."

L. SCHWARTZ, Un mathématicien aux prises avec le siècle.

3.1 Introduction

In this chapter, we consider generalizations of fractional Brownian motion (fBm) in two directions: *a)* the family of fBm is considered for the different Hurst parameters as a single Gaussian process indexed by $(h,t) \in (0,1) \times \mathbb{R}_+$; *b)* the "time" indexing is replaced by any separable L^2 space. We prove that there exists a Gaussian process indexed by $(0,1/2] \times L^2(T,m)$, with the additional constraint that the variance of its increments is as well behaved as it is on $(0,1) \times \mathbb{R}_+$.

The study of the first generalization originated in the works [116, 20] on what is now known as multifractional Brownian motion (mBm). The mBm can be introduced in a tractable way following the approaches of [41, 12], where a fractional Brownian field (fBf) is primarily defined. Recall that fractional Brownian field is understood in the sense of Definition 1.5. A mBm is then built from a fBf and any given path in the h direction, $\{h(t), t \in \mathbb{R}_+\}$. In [12], the authors use a wavelet series expansion of fBm to construct a fBf, while in [41], the harmonizable integral representation of fBm is used. In both cases, harmonic analysis arguments allow to prove that for any compact subset of $\mathbb{R}_+$, there is a constant $C > 0$ such that for any t in this compact, and any $h, h' \in (0,1)$,

$$\mathbb{E}\left(B_t^h - B_t^{h'}\right)^2 \leq C\,(h-h')^2. \tag{3.1}$$

This inequality is of some importance since it ensures the sample path regularity of the field in h, while the behaviour with respect to the increments in t is already known.

More generally, we will consider processes over $L^2(T, m)$, and an important subclass formed by processes restricted to indicator functions of subsets of T. In particular, multiparameter when $(T, m) = (\mathbb{R}_+^d, \text{Leb.})$, and to a bigger extent set-indexed processes [60, 70], naturally appear and thus motivate generalization *b)*, besides the inherent interest of studying processes over an abstract space. Therefore, our goal will be to construct a fractional Brownian field such that inequality (3.1) holds when t is not in $\mathbb{R}_+$ anymore, but in some L^2 space. We shall write L^2–fBf for any such fractional Brownian field, or simply fBf if the context is clear, and h-fBm when looking at the L^2–fBf with a fixed h. A h-fBm will have the following covariance: for each $h \in (0, 1/2]$,

$$k_h : (f,g) \in L^2 \times L^2 \mapsto \frac{1}{2}\left(m(f^2)^{2h} + m(g^2)^{2h} - m(|f-g|^2)^{2h}\right). \tag{3.2}$$

Note that according to Remark 2.10 of [60], k_h is positive definite. $m(\cdot)$ on L^2 denotes the canonical linear functional associated to m, $\int_T (\cdot)\, \mathrm{d}m$.

This form of covariance is particularly interesting for several reasons: it was thoroughly studied when restricted to indicator functions of some indexing collections, in particular in [60, 65], where it is the covariance of the set-indexed fractional Brownian motion (SIfBm) and of the multiparameter fractional Brownian motion (mpfBm, a particular SIfBm indexed by rectangles of $\mathbb{R}^d$ with Lebesgue measure). Also, this covariance belongs to a larger class of functions on a metric space (S, d), of the form:

$$C(s,t) = 1/2\,(d(t_0,t) + d(t_0,s) - d(t,s))$$

for $s, t \in S$, and an arbitrarily chosen origin $t_0 \in S$. Whenever d is such that $C(s,t)$ is positive definite (see, for instance, [138] for a discussion), we call the resulting Gaussian process a Lévy Brownian motion, after Paul Lévy, who introduced it in the Euclidean setting [92]. Accordingly, the covariances will be said to be of the Lévy type. Let us assume that d is a metric such that replacing it by d^2 in C still yields a covariance. Then, by a result on Bernstein functions (see Appendix A), defining C_H by substituting d with d^{2H} for some $H \in (0,1]$ gives again a covariance, and thus a process referred to as Lévy fractional Brownian motion. From this point of view, $m(|\cdot - \cdot|^2)^{1/2}$ is the L^2 metric, and k_h is of the same form as C_H (with $H = 2h$). Since C_H is positive definite for $H \in (0,1]$, it is coherent that $h \in (0, 1/2]$ only. In the multiparameter setting ([57]), the most studied fractional Brownian processes include the Lévy fBm, with covariance associated to the Euclidean distance: $2R_H(t,s) = \|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H}$; and the fractional Brownian sheet, with covariance $2^d R_H(t,s) = \prod_{i=1}^d \{|t_i|^{2H_i} + |s_i|^{2H_i} - |t_i - s_i|^{2H_i}\}$. Interestingly in this setting, these covariances are not only of the Lévy type, but also of the form (3.2). It is also possible to express the fractional Brownian sheet as a set-indexed Brownian motion, although the constructed product measure depends on H. We recall these facts from Section 1.2.1 of the Introduction, where these ideas are developed. We will explore in section 3.3.2 this construction, with an application to the regularity of solutions of a class of stochastic partial differential equations. It should now be clear that the form of covariance (3.2) encompasses a wide class of processes.

Decreusefond and Üstünel [39] introduced a family of fractional operators on the Wiener space W (i.e. the space of continuous functions on $[0,1]$, started at 0), characterizing for each

$h \in (0,1)$ a Cameron-Martin space H_h. Using these fractional operators, we express the fractional Brownian field as a white noise integral over the Wiener space:

$$\left\{ \int_W \langle \mathscr{K}_h R_h(\cdot,t), w \rangle \, \mathrm{d}\mathbb{B}_w, \ (h,t) \in (0,1) \times [0,1] \right\} ,$$

where $\mathbb{B}$ is the white noise associated to the standard Gaussian measure of W, $\mathscr{K}_h$ is derived from fractional operators appearing in [39], R_h is the covariance of the fBm, and $\langle \cdot, \cdot \rangle$ denotes the usual pairing between W and its topological dual W^*. The advantage of this approach is to allow the transfer of techniques of calculus on the Wiener space to any other linearly isometric space with the same structure. Those spaces, called Abstract Wiener Spaces (AWS), were introduced in his seminal work by Gross [56]. Using the separability and reproducing kernel property of the Cameron-Martin spaces built from the kernels $k_h, h \in (0,1/2]$, we prove the existence of a fractional Brownian field $\boldsymbol{B} = \{\boldsymbol{B}_{h,f},\ h \in (0,1/2], f \in L^2(T,m)\}$ over some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. This is the topic of the second section, where the aforementioned notions are defined.

The third section is devoted to proving that the above L^2-fBf $\boldsymbol{B}$ has good h-increments, as in (3.1). These results rely on Hilbert space analysis and analytic function theory, and are to be found first in Theorem 3.11 for a generalised version of the fBf (in the sense of generalised processes [52]) and then in Theorem 3.15 for the L^2-fBf. Some of the computations are reported in Appendix 3.6.1. As an application of the first Theorem, we look at the $L^2(\Omega)$-continuity of the mild solutions of a class of stochastic partial differential equations (SPDE) with additive and anisotropic fractional noise, when the regularity of the noise changes. We remark that an interest in the continuity with respect to the Hurst parameter of some functionals of the fBm already appeared in the works of Jolis and Viles (see [73] and previous works).

Then, in the fourth section, we use the increment properties of the variance of the L^2–fBf to derive a sufficient condition for almost sure continuity. We express in Theorem 3.19 this condition under the form of a Dudley entropy integral which does not depend on the h coordinate. This is an interesting application of the result of the previous part, since it means that many regularity properties of the fBf can be obtained from the sole observation of the h-fBm, for any fixed h.

We take a closer look at the Hölder regularity of the fBf in the fifth section, when the L^2 indexing collection is restricted to the indicator functions of the rectangles of $\mathbb{R}^d$ (multiparameter processes) or to some indexing collection (in the sense of [70]). This restriction permits to use local Hölder regularity exponents, in the flavour of what was done in Chapter 2. When a regular path $\boldsymbol{h} : L^2 \to (0,1/2]$ is specified, this defines a multifractional Brownian field as $\boldsymbol{B}^{\boldsymbol{h}}_f = \boldsymbol{B}_{\boldsymbol{h}(f),f}$, whose Hölder regularity at each point is proved to equal $\boldsymbol{h}(f)$ almost surely.

3.2 Fractional processes in an abstract Wiener space

Let us start with a few general remarks. $L^2(T,m)$ with its classical dot product $(\cdot,\cdot)_m$ will always be assumed to be separable. This is the case, for example, when T is a locally compact metric space with a countable basis, and m is a Borel measure (cf Chapter IV of [119]).
We recall that it is impossible to construct a "standard" countably additive Gaussian measure on an infinite-dimensional Hilbert space (see, for instance, [86]). By "standard", we mean that every one-dimensional cylindrical projection of this measure is a standard Gaussian measure over $\mathbb{R}$. In particular, describing the law of a Brownian motion indexed over $L^2(T,m)$ in terms of Gaussian measure is not straightforward. However, given a Hilbert space H and a cylindrical measure μ on H, it is possible to embed this Hilbert space in a larger Banach space E such that

μ is countably additive on E, as this will be exposed in the next paragraph. The most natural process obtained from this construction is a Brownian process indexed by H. In order to produce fractional variations of Brownian motion, we will make use of reproducing kernel Hilbert spaces (RKHS), as defined in Definition 1.10.

We recall briefly the construction of GROSS [56] of an Abstract Wiener space on H equipped with its scalar product $(\cdot,\cdot)_H$, and refer to Section 1.3 of the Introduction for more details. Let $\tilde{\mu}$ be the following cylindrical measure: for any cylindrical subset $S \subset H$, i.e. of the form $S = P^{-1}(B)$, where P is an orthogonal projection of H with finite rank equal to n and B is a Borel subset of $P(H)$,

$$\tilde{\mu}(S) = \frac{1}{(2\pi)^{n/2}} \int_{i(B)} e^{-\|x\|^2/2}\, \mathrm{d}x \ ,$$

where i is a linear isometry between $P(H)$ and $\mathbb{R}^n$, $\|\cdot\|$ is the Euclidean norm. The measure $\tilde{\mu}$ is centred Gaussian, but is not countably additive on H when it is infinite-dimensional. The following definition allows to extend $\tilde{\mu}$ to a proper measure on a larger space. A *measurable norm* is a norm $\|\cdot\|_1$ on H such that for any $\varepsilon > 0$, there exists an orthogonal projection P_ε with finite rank such that for any finite-rank projection P which is orthogonal to P_ε, the following holds:

$$\tilde{\mu}\big(\{x \in H : \|Px\|_1 > \varepsilon\}\big) < \varepsilon \ .$$

If such a norm exists, we may call E the completion of H with respect to this norm. Then, $(E, \|\cdot\|_1)$ is a Banach space in which $(H, \|\cdot\|_H)$ is dense and such that the canonical injection is continuous. The same relationship holds between their topological duals E^* and H^* (assimilated to H in the following). The main result in [56] then reads: $\tilde{\mu}$ extends to a countably additive measure μ on all the cylinders of E.

From now on, the image of $x^* \in E^*$ by the canonical injection will be denoted $Sx^* \in H$. A major consequence of Gross's theorem is that there is a measure whose Fourier transform is given by:

$$\forall x^* \in E^*, \quad \int_E e^{i\langle x^*, x\rangle}\, \mathrm{d}\mu(x) = e^{-\frac{1}{2}\|Sx^*\|_H^2} \ , \tag{3.3}$$

or, written in terms of the second moment:

$$\forall x^*, y^* \in E^*, \quad \int_E \langle x^*, x\rangle\, \langle y^*, x\rangle\, \mathrm{d}\mu(x) = (Sx^*, Sy^*)_H \,.$$

The triple (H, E, μ) is an *abstract Wiener space* and H is referred as *Cameron-Martin space* of the process μ. For the sake of completeness, we recall an important theorem already stated in the Introduction that says that new abstract Wiener spaces can be easily constructed from others that already exist.

Theorem 3.1 ([136]). *Let H and H' be two separable Hilbert spaces and F a linear isometry from H to H'. Assume that an AWS (H, E, μ) is given. Then, there exists a Banach space $E' \supset H'$ and a linear isometry $\tilde{F} : E \to E'$ whose restriction to H is F and $(H', E', \tilde{F}_*\mu)$ is an AWS ($\tilde{F}_*\mu$ denotes the push-forward measure of μ by $\tilde{F}$).*

In particular, starting from the AWS of continuous functions on $[0,1]$ with the sup-norm and the Wiener measure, it is possible to construct a large class of AWS. However, this does not mean that all AWS are the same, sicne for a single Hilbert space, there can be an uncountable family of AWS. However, starting from a Banach space and a measure, there is a unique Cameron-Martin space explicitely constructed from it.

Lemma 3.2 ([136]). *1. For any $x^* \in E^*$, there is a unique $g_{x^*} \in H$ such that $(g, g_{x^*})_H = \langle x^*, g\rangle$ for all $g \in H$ and it is realized by the covariance operator $S : x^* \in E^* \mapsto g_{x^*} \in H$, which is continuous, injective and of dense image in H.*

2. *If $x \in E \setminus H$, then $\sup_{\{x^*\in E^*:\|Sx^*\|_H\leq 1\}}\langle x^*, x\rangle = \infty$ and for any $g \in H$, the norm is given by $\|g\|_H = \sup\{\langle x^*, g\rangle,\ x^* \in E^* \text{ and } \|Sx^*\|_H \leq 1\}$.*

3. *There exists a sequence $(x_n^*) \in (E^*)^{\mathbb{N}}$ such that $(Sx_n^*)_{n\in\mathbb{N}}$ is an orthonormal basis of H.*

Our approach in this section will be to identify abstract Wiener spaces related to the covariance functions of the fractional Brownian motion:

$$\forall s,t \in \mathbb{R}_+, \quad R_h(s,t) = \frac{1}{2}\left(|s|^{2h} + |t|^{2h} - |s-t|^{2h}\right) ,$$

$h \in (0,1]$, and operators providing links between these different AWS. Then, each fractional Brownian motion is expressed as an integral in the standard Wiener space, eventually providing a Gaussian field in t and h. The second step is to extend this object to another family of abstract Wiener spaces based on the reproducing kernel Hilbert space (RKHS) of k_h, relying heavily on Theorem 3.1. Finally, we prove the resulting field has a "good" covariance structure, in the sense of (3.1).

3.2.1 R_h in the standard Wiener space

The standard Wiener space on $[0,1]$ is the triple consisting of the Banach space of continuous functions started at 0, denoted by W; the Cameron-Martin space H^1 of absolutely continuous functions started at 0 with square integrable weak derivative; and the Gaussian measure $\mathscr{W}$ on W, characterized by equation (3.3) with appropriate change ($E = W$, $H = H^1$ and $\mu = \mathscr{W}$).

The norm on H^1 is given by $\|g\|_{H^1} = \|\dot{g}\|_{L^2}$, where $g(t) = \int_0^t \dot{g}(s)\,\mathrm{d}s$. Using the Riesz representation theorem on $C([0,1])$, any $x^* \in C([0,1])^*$ can be expressed as:

$$\forall w \in C([0,1]),\ x^*(w) = \int_0^1 w(t)\ \Lambda_{x^*}(\mathrm{d}t) ,$$

with Λ_{x^*} a finite signed Radon measure on $[0,1]$. Besides, this equality yields for the total variation of Λ_{x^*}: $|\Lambda_{x^*}|([0,1]) = \|x^*\|$. Thus, we shall assimilate x^* with Λ_{x^*}, writing:

$$\forall x^* \in W^*, \forall w \in W, \quad \langle x^*, w\rangle = \int_{[0,1]} w(t)\ x^*(\mathrm{d}t). \tag{3.4}$$

We now characterize the existence of a family of fractional Wiener spaces, as described in [39]: for any $h \in (0,1)$, there is a one-to-one operator K_h acting on $L^2([0,1])$ satisfying the following properties:

1) The space $H_h = K_h\left(L^2([0,1])\right)$ is a subspace of W. If we denote $(\cdot,\cdot)_{H_h}$ the scalar product ($\|\cdot\|_{H_h}$ the norm) that makes H_h isometric to L^2, and $\mathscr{W}_h$ the Gaussian measure whose Fourier transform is characterized by $(\cdot,\cdot)_{H_h}$, then $(H_h, W, \mathscr{W}_h)$ is an AWS. For $h = 1/2$: $(H_{1/2}, W, \mathscr{W}_{1/2}) = (H^1, W, \mathscr{W})$.

2) Let K_h^* be the adjoint operator of K_h. When K_h^* is restricted to W^*, the operator $K_h \circ K_h^*$ is the canonical injection from W^* to H_h and it is a kernel operator whose kernel is precisely R_h. As a consequence, we use the same notation for both the operator and the kernel.

3) K_h is a Hilbert Schmidt operator and has a kernel on $[0,1]^2$ which is denoted K_h too, so that for any $g \in L^2([0,1])$, $K_h g(t) = \int_0^1 K_h(t,s)g(s)\,\mathrm{d}s$. We notice that while the kernel R_h is symmetric, K_h is not, as this will become clear.

This operator is given explicitely in Equation (1.10) of the introductory chapter. This is summarized, for all h, by:

$$W^* \xrightarrow{K_h^*} L^2([0,1]) \xrightarrow{K_h} H_h \xrightarrow{R_h^*} W.$$

More details on K_h are given along this section and in Appendix 3.6.1, especially its integral formula ([39, 110]), while the link with fractional integrals is clearly established in [39]. For $t \in [0,1]$, we denote by δ_t the Dirac measure at point t, considered here as an element of W^*.

Theorem 3.3. *Let $h \in (0,1)$ and $H(R_h)$ be the RKHS of R_h. Then, $H(R_h) \subseteq H_h$ and for any $f, g \in H(R_h)$, $(f,g)_{H(R_h)} = (f,g)_{H_h}$. Besides, there is a linear isometry $\tilde{J}_h^*$ from W^* to itself such that for any $\eta, \nu \in W^*$,*

$$\int_W \langle \tilde{J}_h^* \eta, w\rangle \, \langle \tilde{J}_h^* \nu, w\rangle \, d\mathscr{W}(w) = (R_h \eta, R_h \nu)_{H_h} . \tag{3.5}$$

Furthermore, for all $s,t \in [0,1]$,

$$\begin{aligned} R_h(s,t) &= (K_h^*\delta_s, K_h^*\delta_t)_{L^2} \\ &= (R_h\delta_s, R_h\delta_t)_{H_h} \\ &= \int_W \langle \mathscr{K}_h R_h(\cdot,s), w\rangle \, \langle \mathscr{K}_h R_h(\cdot,t), w\rangle \, d\mathscr{W}(w), \end{aligned} \tag{3.6}$$

with $\mathscr{K}_h : H_h \to W^*$ defined by the relationship $\mathscr{K}_h = \tilde{J}_h^* \circ R_h^{-1}$.

Before proving this result, consider the following immediate application. For a white noise $\mathbb{B}$ on W with control measure $\mathscr{W}$ on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, the following formula defines a Gaussian random field on $(0,1)\times[0,1]$:

$$\forall (h,t) \in (0,1)\times[0,1], \quad \boldsymbol{B}_{h,t} = \int_W \langle \mathscr{K}_h R_h(\cdot,t), w\rangle \, \mathrm{d}\mathbb{B}_w . \tag{3.7}$$

Indeed, according to equation (3.6), the mapping $w \mapsto \langle \mathscr{K}_h R_h(\cdot,t), w\rangle$ belongs to $L^2(W,\mathscr{W})$. In addition, the previous theorem shows that for fixed h, this process is a fractional Brownian motion.

Remark 3.4. *A similar two-parameter Gaussian field appeared for the first time in [12, 41], although it was not expressed as an integral over the Wiener space. We present in Corollary 3.5 a different form for (3.7).*

Proof. By the definition of point 2), R_h is the operator $K_h \circ K_h^*$ mapping W^* to H_h, with the kernel R_h of the fractional Brownian motion. The assertion $R_h\delta_t = R_h(t,\cdot)$, for any $t \in [0,1]$, follows. As a consequence, $R_h(\cdot,t) \in H_h$ for all t, which in turns suffices to prove that $H(R_h) \subseteq H_h$, since $H(R_h)$ is the completion of $\mathrm{Span}\{R_h(\cdot,t), t \in [0,1]\}$. For the same reason, proving that $(\cdot,\cdot)_{H(R_h)} = (\cdot,\cdot)_{H_h}$ on $H(R_h)$ amounts to show that for all $s,t \in [0,1]$, $(R_h(s,\cdot), R_h(t,\cdot))_{H(R_h)} = (R_h(s,\cdot), R_h(t,\cdot))_{H_h}$, where the first term in the equality is, by definition, equal to $R_h(s,t)$. Then, we infer from point 2) (of the definition of the adjoint operator) that $K_h^*\eta$ can be evaluated in the following manner:

$$\forall f \in L^2, \quad (f, K_h^*\eta)_{L^2} = \langle \eta, K_h f \rangle = \int_0^1 K_h f(t)\ \eta(\mathrm{d}t)\ .$$

Applied to δ_t and taking into account point 3), this yields:

$$(f, K_h^*\delta_t)_{L^2} = K_h f(t) = \int_0^1 K_h(t,s) f(s)\ \mathrm{d}s$$

Thus $K_h^*\delta_t = K_h(t,\cdot)$ in L^2 and it follows that $K_h \circ K_h^*\delta_t = \int_0^1 K_h(\cdot,r)\ K_h(t,r)\ \mathrm{d}r$ which is also equal to $R_h(t,\cdot)$, as mentioned at the beginning of the proof. According to point 1), $(\cdot,\cdot)_{H_h}$ satisfies, for $f,g \in H_h$, $(f,g)_{H_h} = (K_h^{-1}f, K_h^{-1}g)_{L^2}$. As a consequence,

$$\begin{aligned}(R_h(\cdot,t), R_h(\cdot,s))_{H_h} &= (K_h^*\delta_t, K_h^*\delta_s)_{L^2} \\ &= \int_0^1 K_h(s,r)\ K_h(t,r)\ \mathrm{d}r \\ &= R_h(s,t)\ ,\end{aligned}$$

and the first result follows.

To prove the second point, let us define $J_h = K_h \circ K_{1/2}^{-1}$. From its definition, J_h is an isometric isomorphism from $H_{1/2}$ towards H_h, thus admitting a unique (linear) isometric extension to W. Let $\tilde{J}_h$ be this extension and notice that the image space of $\tilde{J}_h$ is $\overline{H_h}^{\|\cdot\|_1}$, where $\|\cdot\|_1$ is the norm defined by $\|g\|_1 = \|J_h^{-1}(g)\|_W, g \in H_h$. It is clear that $\overline{H_h}^{\|\cdot\|_1} = \overline{H_{1/2}}^{\|\cdot\|_W} = W$, and as a consequence, $(H_h, W, \tilde{J}_{h\,*}\mathscr{W})$ is the image of the standard Wiener space by the isometry J_h. In particular, this identifies $\tilde{J}_{h\,*}\mathscr{W} = \mathscr{W}_h$ (these measures have the same Fourier transform). Let $\tilde{J}_h^* : W^* \to W^*$ be the adjoint operator of $\tilde{J}_h$. Then:

$$\begin{aligned}\int_W \langle \tilde{J}_h^*\eta, w\rangle\ \langle \tilde{J}_h^*\nu, w\rangle\ \mathrm{d}\mathscr{W}(w) &= \int_W \langle \eta, \tilde{J}_h w\rangle\ \langle \nu, \tilde{J}_h w\rangle\ \mathrm{d}\mathscr{W}(w) \\ &= \int_W \langle \eta, w\rangle\ \langle \nu, w\rangle\ \mathrm{d}\left(\tilde{J}_{h\,*}\mathscr{W}\right)(w) \\ &= \int_W \langle \eta, w\rangle\ \langle \nu, w\rangle\ \mathrm{d}\mathscr{W}_h(w) \\ &= (R_h\eta, R_h\nu)_{H_h}\ .\end{aligned}$$

Finally, (3.6) directly follows from (3.5), using the fact that $R_h(\cdot,t) = R_h\delta_t$. □

To end this section, we show that the process defined by (3.7) is equal to the process that appeared in [39]. This is no surprise, since the same operators are involved. However, we include the proof for completeness.

Corollary 3.5. *Let* $\boldsymbol{B}$ *a process as defined in equation (3.7). Then* $\boldsymbol{B}$ *has the same law as the process* $B_1 = \left\{\int_0^1 K_h(t,s)\ dW_s,\ (h,t) \in (0,1)\times[0,1]\right\}$.

Proof. Since both processes are centred Gaussian it suffices to study their covariance. For any $(h,t),(h',s) \in (0,1)\times[0,1]$,

$$C_1\big((h,t),(h',s)\big) = \int_0^1 K_h(t,u)\, K_{h'}(s,u)\, \mathrm{d}u = \big(K_h^*\delta_t, K_{h'}^*\delta_s\big)_{L^2}.$$

The covariance of $\boldsymbol{B}$ is also derived, leading to:

$$C_2\big((h,t),(h',s)\big) = \int_W \langle \tilde{J}_h^*\delta_t, w\rangle \langle \tilde{J}_{h'}^*\delta_s, w\rangle\ \mathscr{W}(\mathrm{d}w)\ ,$$

and then, considering $\tilde{J}_h^*\delta_t \in W^*$ as a measure:

$$\begin{aligned}
C_2\big((h,t),(h',s)\big) &= \int_W \left(\int_0^1\int_0^1 w(u)w(v)\ (\tilde{J}_h^*\delta_t)(\mathrm{d}u)\ (\tilde{J}_{h'}^*\delta_s)(\mathrm{d}v)\right)\mathscr{W}(\mathrm{d}w)\\
&= \int_0^1\int_0^1 \left(\int_W w(u)w(v)\ \mathscr{W}(\mathrm{d}w)\right)(\tilde{J}_h^*\delta_t)(\mathrm{d}u)\ (\tilde{J}_{h'}^*\delta_s)(\mathrm{d}v)\\
&= \int_0^1\int_0^1 R_{1/2}(u,v)\ (\tilde{J}_h^*\delta_t)(\mathrm{d}u)\ (\tilde{J}_{h'}^*\delta_s)(\mathrm{d}v)\\
&= \big(K_{1/2}^*\tilde{J}_h^*\delta_t, K_{1/2}^*\tilde{J}_{h'}^*\delta_s\big)_{L^2}.
\end{aligned}$$

Hence, we are to prove that for any $h \in (0,1)$, $K_{1/2}^*\tilde{J}_h^*$ and K_h^* are equal, as operators from W^* to $L^2[0,1]$. The main ingredient is that when restricted to $H_{1/2}$, $\tilde{J}_h$ is equal to $K_h \circ K_{1/2}^{-1}$. Then, for any $x^* \in W^*, f \in L^2[0,1]$,

$$\begin{aligned}
\big(K_{1/2}^*\tilde{J}_h^* x^*, f\big)_{L^2} &= \langle \tilde{J}_h^* x^*, K_{1/2} f\rangle\\
&= \langle x^*, \tilde{J}_h K_{1/2} f\rangle\\
&= \langle x^*, J_h K_{1/2} f\rangle\\
&= \langle x^*, K_h f\rangle,
\end{aligned}$$

where the third equality holds because $K_{1/2} f \in H_{1/2}$. □

Remark 3.6. *We make a final remark in this section, with the definition of an extension of* $\{\boldsymbol{B}_{h,t},(h,t)\in(0,1)\times[0,1]\}$ *(or B_1) to a generalised process (in the sense of [52]):* $\{\boldsymbol{B}_{h,\xi},(h,\xi)\in(0,1)\times W^*\}$*, where following (3.5),*

$$\boldsymbol{B}_{h,\xi} = \int_W \langle \tilde{J}_h^*\xi, w\rangle\ d\mathbb{B}_w\ . \tag{3.8}$$

3.2.2 *Extended decomposition in an abstract Wiener space*

Equipped with a tractable way of expressing a fractional Gaussian field given the reproducing kernel R_h on $[0,1]$, we now present how to extend this decomposition to any separable $L^2(T,m)$ space with the family of kernels $k_h, h\in(0,1/2]$ introduced in (3.2). The RKHS of k_h is written $H(k_h)$.

For any $h\in(0,1/2]$, $H_h \supset H(R_h)$ and $H(k_h)$ are separable Hilbert spaces, so let us choose a linear isometry between the two latter spaces and extend it to H_h. We call u_h such a linear isometry and write $\mathscr{H}_h = u_h(H_h)$. As a consequence of its definition, the restriction of u_h to $H(R_h)$ is a linear isometry between $H(R_h)$ and $H(k_h)$. Recall that the parameter h is restricted to $(0,1/2]$ because k_h is not positive definite for $h \in (1/2,1]$ (see [60] for a counterexample). In the next section, we will be more specific about the choice of this isometry. u_h is isometrically extended to all of W as in Theorem 3.1, and the extension is denoted by $\tilde{u}_h$. By this very Theorem, it is possible to define an abstract Wiener space which is the image of $(H_h, W, \mathscr{W}_h)$ by u_h, and we denote it $(\mathscr{H}_h, E_h, \mu_h)$, where $E_h = \tilde{u}_h(W)$ and $\mu_h = (\tilde{u}_h)_* \mathscr{W}_h$, the pushforward measure of $\mathscr{W}_h$ by $\tilde{u}_h$. The adjoint operator $\tilde{u}_h^T$ of $\tilde{u}_h$ is the mapping from E_h^* to W^* such that for any $w \in W$, any $w^* \in E_h^*$, $\langle w^*, \tilde{u}_h(w)\rangle = \langle \tilde{u}_h^T(w^*), w\rangle$. Since the space associated to $h = 1/2$ plays a special part, we just drop the h in the notations. Especially, $(H, W, \mathscr{W})$ denotes the standard Wiener space.

Now define $\tilde{\mathscr{K}}_h = (\tilde{u}^T)^{-1} \circ \mathscr{K}_h \circ u_h^{-1}$, the linear operator from $\mathscr{H}_h$ to E^*. For $\phi, \psi \in \mathscr{H}_h$,

$$\begin{aligned}(\phi,\psi)_{\mathscr{H}_h} = (u_h^{-1}\phi, u_h^{-1}\psi)_{H_h} &= \int_W \langle \mathscr{K}_h u_h^{-1}\phi, w\rangle \ \langle \mathscr{K}_h u_h^{-1}\psi, w\rangle \ \mathrm{d}\mathscr{W}(w) \\ &= \int_E \langle \mathscr{K}_h u_h^{-1}\phi, \tilde{u}^{-1}(x)\rangle \ \langle \mathscr{K}_h u_h^{-1}\psi, \tilde{u}^{-1}(x)\rangle \ \mathrm{d}\,(\tilde{u}_*\mathscr{W})(x) \\ &= \int_E \langle \tilde{\mathscr{K}}_h \phi, x\rangle \ \langle \tilde{\mathscr{K}}_h \psi, x\rangle \ \mathrm{d}\mu(x) \ .\end{aligned}$$

When applied to $\phi = k_h(\cdot, f)$ and $\psi = k_h(\cdot, g)$, for $f, g \in L^2(T, m)$, the previous relation reads:

$$k_h(f,g) = \int_E \langle \tilde{\mathscr{K}}_h k_h(\cdot, f), x\rangle \ \langle \tilde{\mathscr{K}}_h k_h(\cdot, g), x\rangle \ \mathrm{d}\mu(x) \ .$$

Definition 3.7 (fractional Brownian field). *Let $(\Omega, \mathscr{F}, \mathbb{P})$ a probability space and $\mathbb{W}$ a white noise on E associated to the measure μ. The following formula defines a Gaussian random field over $(0,1/2] \times L^2(T,m)$:*

$$\forall (h,f) \in (0,1/2] \times L^2(T,m), \quad \boldsymbol{B}_{h,f} = \int_E \langle \tilde{\mathscr{K}}_h k_h(\cdot, f), x\rangle \ \mathrm{d}\mathbb{W}(x) \ .$$

The previous calculus proves that for fixed h, this process has covariance (3.2), so $\boldsymbol{B}$ is a L^2–fBf as defined in the introduction. Noticeably, if $\mathbb{W}^h$ is a white noise of $(\mathscr{H}_h, E_h, \mu_h)$, the process:

$$\left\{ \int_{E_h} \langle \, \mathbf{k}_h(\cdot, f), x\rangle \ \mathrm{d}\mathbb{W}_x^h, \ f \in L^2(T,m) \right\}$$

and $\left\{ \boldsymbol{B}_{h,f}, \ f \in L^2(T,m) \right\}$ have the same law, where $\mathbf{k}_h(\cdot, f) \in E_h^*$ is the pre-image of $k_h(\cdot, f)$ by the canonical injection of $E_h^* \to \mathscr{H}_h$. The same calculus as in the proof of Corollary 3.5, shows that the covariance of the new process is given by:

$$\mathbb{E}\left(\boldsymbol{B}_{h,f} \ \boldsymbol{B}_{h',g}\right) = \left(K_h^{-1} u_h^{-1} k_h(f,.), K_{h'}^{-1} u_{h'}^{-1} k_{h'}(g,.)\right)_{L^2[0,1]} \ . \tag{3.9}$$

Hence, the choice of the family of isometries $\{u_h\}_{h \in (0,1/2]}$ matters.

Let us set up a family of isometries $u = \{u_h\}_{h \in (0,1/2]}$ allowing the fBf to have good increments, as discussed already. Let $\mathbb{D} = \{t_n, n \in \mathbb{N}\}$ be the set of dyadics in $[0,1]$. Let $h \in (0,1/2]$, then from the definition of $H(R_h)$, $\{R_h(\cdot, t), t \in \mathbb{D}\}$ is a linear basis of this space (although the linear independence is proved in the following lemma). We shall establish the existence of a family of functions $\{f_n \in L^2(T,m), n \in \mathbb{N}\}$ such that $\{k_h(\cdot, f_n), n \in \mathbb{N}\}$ is a linear basis for $\mathscr{H}_h$.

Lemma 3.8. *Let $h \in (0,1/2]$. Let $n \in \mathbb{N}^*$ and $(f_0,\dots,f_n) \in L^2(T,m)$ be linearly independent. Then, $(k_h(\cdot,f_0),\dots,k_h(\cdot,f_n))$ is linearly independent in $\mathscr{H}_h$.*

The proof is reported in Appendix 3.6.2. This lemma suggests that we will choose a dense (countable) linear basis of $L^2(T,m)$. This can be done as follows. $L^2(T,m)$ is a separable metric space, hence it admits a countable topological basis[1], so denote by $(O_n)_{n\in\mathbb{N}}$ this basis of open sets for the topology of $L^2(T,m)$. Let us prove inductively the existence of a dense linearly independent family $(f_n)_{n\in\mathbb{N}}$. Let $f_0 \in O_0$ and assume that $(f_0,\dots,f_n)$ already exist. $F_n = \mathrm{Span}\{f_0,\dots,f_n\}$ is a finite-dimensional subspace of an infinite-dimensional space. Therefore F_n is of empty interior and one can pick $f_{n+1} \in O_{n+1} \setminus F_n$. This mechanism provides a linearly independent family, which is dense since its intersection with any O_n is non-empty.

From now on, $\{f_n, n\in\mathbb{N}\}$ denotes such a family, and for any $h \in (0,1/2]$, Lemma 3.8 implies that $\{k_h(\cdot,f_n), n\in\mathbb{N}\}$ is linearly independent in $H(k_h)$. That $\{f_n\}_{n\in\mathbb{N}}$ is dense in L^2 yields that this is actually a spanning family of $H(k_h)$. Let us prove this last statement: let $K \in H(k_h)$, so that there is a sequence of real numbers $(\alpha_n)_{n\in\mathbb{N}}$ and a sequence $(g_n)_{n\in\mathbb{N}}$ of elements of $L^2(m)$, such that $\sum_{n\in\mathbb{N}} \alpha_n k_h(\cdot,g_n)$ is equal to K in $H(k_h)$. Let $\varepsilon > 0$ and for any $n \in \mathbb{N}$, let $\varepsilon_n = \varepsilon\, |\alpha_n|^{-1}\, (n+1)^{-2}$. For any n, there is f_{i_n} in the dense basis of $L^2(m)$ that satisfies $\|g_n - f_{i_n}\|_{L^2}^{2h} < \varepsilon_n$. Let $N \in \mathbb{N}$ such that for all $p \geq N$, $\|K - \sum_{n=0}^{p} \alpha_n k_h(\cdot,g_n)\|_{H(k_h)} < \varepsilon$. Then:

$$\begin{aligned}
\Big\|K - \sum_{n=0}^{p} \alpha_n k_h(\cdot,f_{i_n})\Big\|_{\mathscr{H}_h} &\leq \Big\|K - \sum_{n=0}^{p} \alpha_n k_h(\cdot,g_n)\Big\|_{\mathscr{H}_h} + \Big\|\sum_{n=0}^{p} \alpha_n k_h(\cdot,g_n) - \sum_{n=0}^{p} \alpha_n k_h(\cdot,f_{i_n})\Big\|_{\mathscr{H}_h} \\
&\leq \varepsilon + \sum_{n=0}^{p} |\alpha_n|\; \|k_h(\cdot,g_n) - k_h(\cdot,f_{i_n})\|_{\mathscr{H}_h} \\
&\leq \varepsilon + \sum_{n=0}^{p} |\alpha_n|\; \|g_n - f_{i_n}\|_{L^2}^{2h} \\
&\leq 3\varepsilon\ .
\end{aligned}$$

The same reasoning shows that $\{R_h(\cdot,t_n), n\in\mathbb{N}\}$ is a basis for $H(R_h)$. $\{\underline{R}_h(\cdot,t_n), n\in\mathbb{N}\}$ and $\{\underline{k}_h(\cdot,f_n), n\in\mathbb{N}\}$ stand for the two corresponding orthogonal bases obtained from the Gram-Schmidt process. Then, the linear map $v_h : H(R_h) \to H(k_h)$ can be properly defined:

$$\forall n \in \mathbb{N}, \quad v_h\left(\underline{R}_h(\cdot,t_n)\right) = \frac{\|\underline{R}_h(\cdot,t_n)\|}{\|\underline{k}_h(\cdot,f_n)\|}\ \underline{k}_h(\cdot,f_n)\ . \tag{3.10}$$

This mapping is an isometry. Enlarging the family $\{\underline{R}_h(\cdot,t_n), n\in\mathbb{N}\}$ into a complete orthogonal system of H_h, v_h is extended to an isometry u_h mapping H_h to $u_h(H_h) = \mathscr{H}_h$. In the following v_h and u_h are both denoted by u_h. From now on, $\boldsymbol{B}$ will always refer to a fractional Brownian field built from this particular kind of isometry.

3.3 Variance of the h-increments

This section is devoted to proving that the different fractional Brownian fields we can construct satisfy inequalities of the type (3.1), and in particular the L^2-fBf built with the family of isometries we just introduced.

[1] any separable metric space is second-countable.

3.3.1 The one-dimensional generalised fractional Brownian field

Firstly, the question is answered in the standard framework of the field indexed over $(0,1)\times[0,1]$, and to a larger extent to the corresponding generalised field. Before the main result, we need the following lemma and proposition, whose proofs are reported in Appendix 3.6.1.

Lemma 3.9. *For every $n\in\mathbb{N}$, $\underline{R}_h(\cdot,t_n)$ is a positive linear functional, in the sense that for any nonnegative function $g\in H_h$, $\left(\underline{R}_h(\cdot,t_n),g\right)_{H_h}\geq 0$.*

Let W_+^* denote the set of positive linear functionals over W. As can be seen from equation (3.4), any element of W_+^* can also be considered as a finite nonnegative measure.

Proposition 3.10. *For all $\eta>0$, there exists a constant $M_\eta>0$, such that for all $h_1<h_2\in(\eta,1/2-\eta)$, for all $\xi\in W_+^*$,*

$$\int_0^1\left(K_{h_2}^*\xi(u)-K_{h_1}^*\xi(u)\right)^2\,du\leq M_\eta\left((h_2-h_1)L(h_2-h_1)\right)^2\|\xi\|^2_{H_{h_1}^*}\ ,$$

where for all $x\in(-1,1)$, $L(x)=\log(|x|^{-1})\vee 1$ if $x\neq 0$, and 0 otherwise.

Theorem 3.11. *Let $\boldsymbol{B}_{h,\xi}$ be the generalised fBf defined in (3.8). For all $\eta>0$, there exists a constant $M_\eta>0$, such that for all $h_1<h_2\in(\eta,1/2-\eta)$, for all $\xi\in W^*$,*

$$\mathbb{E}\left(\boldsymbol{B}_{h_1,\xi}-\boldsymbol{B}_{h_2,\xi}\right)^2\leq M_\eta\left((h_2-h_1)L(h_2-h_1)\right)^2\|\xi\|^2_{H_{h_1}^*}\ .$$

Proof. If $\xi\in W^*$, then it also belongs to $H_{h_1}^*$. Note that as an element of H_{h_1}, the image of ξ by R_{h_1} reads:

$$\begin{aligned}R_{h_1}\xi&=\sum_{n=0}^{\infty}\frac{(R_{h_1}\xi,\underline{R}_{h_1}(t_n,\cdot))_{H_{h_1}}}{\|\underline{R}_{h_1}(t_n,\cdot)\|^2_{H_{h_1}}}\,\underline{R}_{h_1}(t_n,\cdot)\\&=\sum_{n=0}^{\infty}\frac{\left\{(R_{h_1}\xi,\underline{R}_{h_1}(t_n,\cdot))_{H_{h_1}}\vee 0\right\}}{\|\underline{R}_{h_1}(t_n,\cdot)\|^2_{H_{h_1}}}\,\underline{R}_{h_1}(t_n,\cdot)-\sum_{n=0}^{\infty}\frac{\left\{-(R_{h_1}\xi,\underline{R}_{h_1}(t_n,\cdot))_{H_{h_1}}\vee 0\right\}}{\|\underline{R}_{h_1}(t_n,\cdot)\|^2_{H_{h_1}}}\,\underline{R}_{h_1}(t_n,\cdot)\\&=R_{h_1}\xi_+-R_{h_1}\xi_-\ .\end{aligned}$$

Since $\|\xi\|^2_{H_{h_1}^*}=\|\xi_+\|^2_{H_{h_1}^*}+\|\xi_-\|^2_{H_{h_1}^*}$, it suffices to notice that:

$$\mathbb{E}\left(\boldsymbol{B}_{h_1,\xi_\pm}-\boldsymbol{B}_{h_2,\xi_\pm}\right)^2=\left\|K_{h_1}^*\xi_\pm-K_{h_2}^*\xi_\pm\right\|^2_{L^2}\ ,$$

and then to apply Proposition 3.10 to ξ_+ and ξ_- (which are positive linear functionals of W^*, according to Lemma 3.9 and the density of H_{h_1} in W). □

While the W^*-norm is rougher than a H_h^*-norm, it suffices in the following application. Indeed, the next proposition follows by taking $\xi=\delta_t\in W_+^*, t\in[0,1]$ because then $\|\delta_t\|_{H_h^*}\leq C_h\|\delta_t\|_{W^*}=C_h$, where C_h is the norm of the canonical injection from W^* to H_h^*.

Corollary 3.12. *The fractional Brownian field on $(0,1/2)\times[0,1]$ has a continuous version.*

Proof. For any $t \in [0,1]$, any $h_1, h_2 \in (0, 1/2]$,

$$\mathbb{E}\left(\boldsymbol{B}_{h_1,t} - \boldsymbol{B}_{h_2,t}\right)^2 = \mathbb{E}\left(\boldsymbol{B}_{h_1,\delta_t} - \boldsymbol{B}_{h_2,\delta_t}\right)^2 .$$

Proposition 3.11 then implies that for any $\eta > 0$, there is a constant M_η such that for all $h_1 < h_2 \in [\eta, 1/2 - \eta]$, and all $t \in [0,1]$,

$$\begin{aligned}\int_0^1 \left(K_{h_1}^* \delta_t(u) - K_{h_2}^* \delta_t(u)\right)^2 \, \mathrm{d}u &\le M_\eta \ (h_2 - h_1)^2 L(h_2 - h_1)^2 \|\delta_t\|_{W^*} \\ &\le M_\eta \ C_{h_1} \ (h_2 - h_1)^2 L(h_2 - h_1)^2 .\end{aligned}$$

It follows that for all $s, t \in [0,1]$,

$$\begin{aligned}\mathbb{E}\left(\boldsymbol{B}_{h_1,s} - \boldsymbol{B}_{h_2,t}\right)^2 &\le 2\mathbb{E}\left(\boldsymbol{B}_{h_1,s} - \boldsymbol{B}_{h_2,s}\right)^2 + 2\mathbb{E}\left(\boldsymbol{B}_{h_2,s} - \boldsymbol{B}_{h_2,t}\right)^2 \\ &\le 2M_\eta \ C_{h_1} \ (h_2 - h_1)^2 L(h_2 - h_1)^2 + 2|t - s|^{2h_2} \\ &\le 2M_\eta \ C_{h_1} \ (h_2 - h_1)^2 L(h_2 - h_1)^2 + 2|t - s|^{2\eta}.\end{aligned}$$

The Kolmogorov continuity theorem allows to conclude that the fBf admits a continuous version on $[\eta, 1/2 - \eta] \times [0,1]$, for any $\eta > 0$. The result is proved on $(0, 1/2) \times [0,1]$. □

Remark 3.13. *Working on K_h for $h \ge 1/2$, we could in fact prove that the fBf has a continuous modification on $(0,1) \times [0,1]$. See for instance [39] where it is proved that $K_h, h \ge 1/2$, has an analytic extension to $(0,1)$.*

3.3.2 *An application to the mild solutions of a family of stochastic partial differential equations*

In this section, we suggest an application of the previous results to the solutions of a class of stochastic partial differential equations with additive fractional noises. Our aim is not to solve them explicitly, but rather to prove the L^2 continuity of the solutions when the regularity of the (anisotropic) noise varies. The exposition is made on $[0,1]^2$, but extends easily to higher dimensions.

It was proved in [28] that the tensor product of two abstract Wiener spaces is an abstract Wiener space. This means that for $h_1, h_2 \in (0,1)$, $(H_{h_1} \bar{\otimes} H_{h_2}, W \bar{\otimes}_\varepsilon W, \mathscr{W}_{h_1} \otimes \mathscr{W}_{h_2})$ is an abstract Wiener space, where $H_{h_1} \bar{\otimes} H_{h_2}$ is the completion of the algebraic tensor product $H_{h_1} \otimes H_{h_2}$ with respect to the norm given by the scalar product: $\forall x_1, x_1' \in H_{h_1}, x_2, x_2' \in H_{h_2}$, $(x_1 \otimes x_2, x_1' \otimes x_2') = (x_1, x_1')_{H_{h_1}} (x_2, x_2')_{H_{h_2}}$; and where $W \bar{\otimes}_\varepsilon W$ is the completion of $W \otimes W$ with respect to the norm given by: $\|x\|_\varepsilon = \sup\{|x_1^* \otimes x_2^*(x)| : \|x_1^*\|_{W^*} = 1, \|x_2^*\|_{W^*} = 1\}$. Note that $W \bar{\otimes}_\varepsilon W$ is in fact $C_0([0,1]^2)$ with the sup-norm topology ([125] gives a detailed account on topological tensor products), the space of continuous functions vanishing on the axes (this is an application of the Stone-Weierstrass theorem). The canonical operator for this new Wiener space is the tensor operator $R_{h_1} \otimes R_{h_2}$ (in fact, its continuous extension, but we keep the same notation for operators). Let $\mathbb{W}^{h_1,h_2}$ be the white noise associated to the tensor Wiener space. Then, for (s,t) and $(s',t') \in$

$[0,1]^2$,

$$\mathbb{E}\Big(\int_{W\bar{\otimes}_\varepsilon W}\langle\delta_s\otimes\delta_t,w\rangle\,\mathrm{d}\mathbb{W}^{h_1,h_2}_w\int_{W\bar{\otimes}_\varepsilon W}\langle\delta_{s'}\otimes\delta_{t'},w\rangle\,\mathrm{d}\mathbb{W}^{h_1,h_2}_w\Big)$$
$$=\big(R_{h_1}\otimes R_{h_2}(\delta_s\otimes\delta_t),R_{h_1}\otimes R_{h_2}(\delta_{s'}\otimes\delta_{t'})\big)_{H_{h_1}\otimes H_{h_2}}$$
$$=\big(R_{h_1}\delta_s,R_{h_1}\delta_{s'}\big)_{H_{h_1}}\ \big(R_{h_2}\delta_t,R_{h_2}\delta_{t'}\big)_{H_{h_2}}$$
$$=R_{h_1}(s,s')\,R_{h_2}(t,t')\ .$$

The last expression is the covariance of an anisotropic (h_1,h_2)-fractional Brownian sheet (see [152]). Thus, we shall also denote by $\{\mathbb{W}^{h_1,h_2}_{s,t},\ (s,t)\in[0,1]^2\}$ this process, and write it as:

$$\left\{\int_{[0,1]^2}K^*_{1/2}\otimes K^*_{1/2}(\delta_s\otimes\delta_t)(u,v)\,\mathrm{d}\mathbb{W}^{h_1,h_2}_{u,v},\ (s,t)\in[0,1]^2\right\}\ . \tag{3.11}$$

Similarly, the process

$$\left\{\int_{[0,1]^2}K^*_{h_1}\otimes K^*_{h_2}(\delta_s\otimes\delta_t)(u,v)\,\mathrm{d}\mathbb{W}_{u,v},\ (s,t)\in[0,1]^2\right\}\ , \tag{3.12}$$

where $\mathbb{W}_{u,v}$ is the standard Brownian sheet of $[0,1]^2$ (corresponding to $h_1=h_2=1/2$), is equal in distribution to the two aforementioned processes. This construction still holds with $\xi=\xi_1\otimes\xi_2$ in the space $\mathrm{Span}\{\delta_s,s\in[0,1]\}\otimes\mathrm{Span}\{\delta_s,s\in[0,1]\}$ and to its completion with respect to the norm $\|K^*_{h_1}\otimes K^*_{h_2}(\xi)\|_{L^2}$, that we denote by V_{h_1,h_2}. This corresponds to the standard construction of the Wiener integral with step functions. The image of V_{h_1,h_2} by $K^*_{h_1}\otimes K^*_{h_2}$ is denoted by $\mathscr{D}_{h_1,h_2}$ and is the space of integrands of the (h_1,h_2)-fractional Brownian sheet. Besides, the processes in (3.11) and (3.12) extended to $\mathscr{D}_{h_1,h_2}$, are equal in law. We note that the space of integrands of the fractional Brownian sheet on $\mathbb{R}^2$ is partly described in [90], so that on $[0,1]^2$, we will consider $\tilde{\mathscr{D}}_{h_1,h_2}$ which consists of square integrable functions ϕ with support in $[0,1]^2$, for which there is an extension $\tilde{\phi}\in L^2(\mathbb{R}^2)$ with the same support and such that:

$$\int_{\mathbb{R}^2}|\mathscr{F}\tilde{\phi}(\lambda_1,\lambda_2)|^2|\lambda_1|^{1-2h_1}|\lambda_2|^{1-2h_2}\,\mathrm{d}\lambda_1\mathrm{d}\lambda_2<\infty\ ,$$

where $\mathscr{F}$ is the Fourier transform. We note that $\tilde{\mathscr{D}}_{h_1,h_2}\subseteq\mathscr{D}_{h_1,h_2}$ and that the equality is not established (in the one-dimensional case, this requires some care, [72]).

This discussion shows that the generalised processes defined by (3.11) and (3.12) extend to V_{h_1,h_2} and are equal, which can be written as:

$$\text{for any }\xi\in V_{h_1,h_2},\quad \dot{\mathbb{W}}^{h_1,h_2}\Big(K^*_{1/2}\otimes K^*_{1/2}(\xi)\Big)\overset{(d)}{=}\dot{\mathbb{W}}\Big(K^*_{h_1}\otimes K^*_{h_2}(\xi)\Big)\ . \tag{3.13}$$

Consider now the following family of elliptic SPDEs with additive noise, on a bounded open domain $U\subset[0,1]^2$ with smooth boundary:

$$\Delta u=\dot{\mathbb{W}}^{h_1,h_2}\quad\text{on }U, \tag{$\mathscr{L}_{h_1,h_2}$}$$

and with the condition that $u=0$ on ∂U. It is assumed that all fractional noises below come from a unique white noise $\mathbb{W}$, i.e. that they can be written as in the right-hand term of (3.13). Let

$\tilde{\mathscr{D}}_{h_1,h_2}(U)$ be the restriction of $\tilde{\mathscr{D}}_{h_1,h_2}$ to functions with support in U. Let G_U be the Green function associated to this problem, which is known to be locally integrable and gives a fundamental solution to the Poisson problem on U, for any $\varphi \in L^2(U)$:

$$\Delta(G_U * \varphi) = \varphi, \quad \text{where} \quad G_U * \varphi(x,y) = \int_U G_U\left((x,y),(s,t)\right)\varphi(s,t)\,\mathrm{d}s\mathrm{d}t\ ,(x,y) \in U\ .$$

This type of equation with the Brownian sheet has already been considered in [111] (with a reflection term), and u is a distributional (or mild, as studied in [33]) solution to this problem if it acts on functions $\phi \in C_c^\infty(U)$ in the following way:

$$\langle u, \Delta\phi\rangle = \int_{[0,1]^2} \phi(x,y)\,\mathrm{d}\mathbb{W}^{h_1,h_2}_{x,y}\ .$$

The last integral is a well-defined Wiener integral, since $C_c^\infty(U) \subset \tilde{\mathscr{D}}_{h_1,h_2}(U)$ when $h_1 \leq 1/2$ and $h_2 \leq 1/2$ (which will be assumed from now). Plugging the fundamental solution into the previous equation yields, for $\varphi \in C_c^\infty(U)$:

$$\langle u, \varphi\rangle = \int_{[0,1]^2} G_U * \varphi(x,y)\,\mathrm{d}\mathbb{W}^{h_1,h_2}_{x,y}\ . \tag{3.14}$$

For this last expression to make sense, we need $G_U * \varphi$ to be in $\tilde{\mathscr{D}}_{h_1,h_2}(U)$. This is the case if $\varphi \in C_c^\infty(U)$, and u can more generally be defined for any φ such that $G_U * \varphi \in \tilde{\mathscr{D}}_{h_1,h_2}(U)$. It follows from the definition of $\mathscr{D}_{h_1,h_2}(U)$ and (3.13) that if $G_U * \varphi \in \tilde{\mathscr{D}}_{h_1,h_2}(U)$, there exists $\xi_\varphi \in V_{h_1,h_2}(\subseteq C_0([0,1]^2)^*)$ such that $K^*_{1/2} \otimes K^*_{1/2}(\xi_\varphi) = G_U * \varphi$. We can now state the following regularity result:

Proposition 3.14. *Let $\eta \in (0,1/4)$. Let (h_1,h_2) and (h'_1,h'_2) be in $(\eta, 1/2-\eta)^2$ such that $h_1 \leq h'_1$ and $h_2 \leq h'_2$, and let $u_{(h_1,h_2)}$ and $u_{(h'_1,h'_2)}$ be the mild solutions to $(\mathscr{L}_{h_1,h_2})$ and $(\mathscr{L}_{h'_1,h'_2})$ respectively. Then, for all φ such that $G_U * \varphi \in \tilde{\mathscr{D}}_{h_1,h_2}(U) \cap \tilde{\mathscr{D}}_{h'_1,h'_2}(U)$,*

$$\begin{aligned}\mathbb{E}\left(u_{(h_1,h_2)}(\varphi) - u_{(h'_1,h'_2)}(\varphi)\right)^2 \leq M_\eta\ &(h_1-h'_1)^2 L(h_1-h'_1)^2 \|\xi_\varphi\|^2_{H^*_{h_1}\bar{\otimes} H^*_{h_2}} \\ &+ M_\eta\ (h_2-h'_2)^2 L(h_2-h'_2)^2 \|\xi_\varphi\|^2_{H^*_{h_1}\bar{\otimes} H^*_{h_2}}\end{aligned}$$

Proof. Recall first that $\tilde{\mathscr{D}}_{h_1,h_2}(U) \cap \tilde{\mathscr{D}}_{h'_1,h'_2}(U)$ is not empty since it contains $C_c^\infty(U)$, hence let φ be such that $G_U * \varphi \in \tilde{\mathscr{D}}_{h_1,h_2}(U) \cap \tilde{\mathscr{D}}_{h'_1,h'_2}(U)$. Let ξ_φ be such that $K^*_{1/2} \otimes K^*_{1/2}(\xi_\varphi) = G_U * \varphi$. According to Equations (3.13) and (3.14), a mild solution of $(\mathscr{L}_{h_1,h_2})$ can be expressed as

$$\langle u_{(h_1,h_2)}, \varphi\rangle = \int_{[0,1]^2} K^*_{h_1} \otimes K^*_{h_2}(\xi_\varphi)(x,y)\,\mathrm{d}\mathbb{W}_{x,y}.$$

Thus, the above expectation is in fact the $L^2([0,1]^2)$-norm of $K^*_{h_1} \otimes K^*_{h_2}(\xi_\varphi) - K^*_{h'_1} \otimes K^*_{h'_2}(\xi_\varphi)$. As ξ_φ may not have a tensorized form, we express it as the limit of elements of the form: $\sum_{k=1}^n \xi_k \otimes \xi'_k \in C_0([0,1])^* \otimes C_0([0,1])^*$. This reads:

$$\begin{aligned}&\left\| K^*_{h_1} \otimes K^*_{h_2}\left(\sum_{k=1}^n \xi_k \otimes \xi'_k\right) - K^*_{h'_1} \otimes K^*_{h'_2}\left(\sum_{k=1}^n \xi_k \otimes \xi'_k\right)\right\|^2_{L^2([0,1]^2)} \\ &\quad \leq 2\left\| K^*_{h_1} \otimes (K^*_{h_2} - K^*_{h'_2})\left(\sum_{k=1}^n \xi_k \otimes \xi'_k\right)\right\|^2_{L^2} + 2\left\| K^*_{h'_2} \otimes (K^*_{h_1} - K^*_{h'_1})\left(\sum_{k=1}^n \xi_k \otimes \xi'_k\right)\right\|^2_{L^2},\end{aligned}$$

and up to an orthogonalisation procedure, we can assume that the $\xi_1,\dots,\xi_n$ are orthogonal in $H^*_{h_1}$ (i.e. that $(K^*_{h_1}\xi_i, K^*_{h_1}\xi_j)_{L^2} = \|\xi_i\|_{H^*_{h_1}}\|\xi_j\|_{H^*_{h_1}}\delta_{ij}$) and that the $\xi'_1,\dots,\xi'_n$ are orthogonal in $H^*_{h'_2}$. Then, the tensor product on $L^2([0,1]^2)$ implies that the first term in the above sum decomposes as:

$$\sum_{k=1}^n \|K^*_{h_1}\xi_k\|^2_{L^2[0,1]}\ \|(K^*_{h_2}-K^*_{h'_2})\xi'_k\|^2_{L^2[0,1]}\ ,$$

which is now smaller than:

$$M_\eta(h_2-h'_2)^2 L(h_2-h'_2)^2\|K_{h_1}(\cdot,\cdot)\|_{L^2([0,1]^2)}\sum_{k=1}^n \|\xi_k\|^2_{H^*_{h_1}}\ \ \|\xi'_k\|^2_{H^*_{h_2}}\ ,$$

using Theorem 3.11. The last sum is exactly $\|\sum_{k=1}^n \xi_k\otimes\xi'_k\|^2_{H^*_{h_1}\bar{\otimes}H^*_{h_2}}$, which is the result of the Proposition for elements of the algebraic tensor product. So by a density argument, this gives the result for ξ_φ. □

3.3.3 *The fractional Brownian field over L^2*

The L^2-fBf, with a proper family of isometries defined in section 3.2.2, is now looked at. A slightly better estimate is attained on the h-increments than on the previous results of this section, due to the different underlying structure of the process. In particular, the result of this section would not permit to obtain the previous estimate on solutions of SPDEs.

Theorem 3.15. *Let $\boldsymbol{B}$ be a fBf on $(0,1/2]\times L^2(T,m)$. For any $\eta\in(0,1/4)$ and any compact subset D of L^2, there exists a constant $C_{\eta,D}>0$ such that for any $f\in D$, and any $h_1,h_2\in[\eta,1/2-\eta]$,*

$$\mathbb{E}\Big((\boldsymbol{B}_{h_1,f}-\boldsymbol{B}_{h_2,f})^2\Big)\le C_{\eta,D}\ (h_2-h_1)^2\ .$$

Proof. This proof is divided into two parts. In the first part, we show that for any $n\in\mathbb{N}$, for any $f\in L^2(m)$, $h\mapsto \underline{k}_h(f,f_n)$ is analytic. This will be needed in the rest of the proof, while in the second part we compute the main estimates. Like $\underline{R}_h$, $\underline{k}_h$ is the Gram-Schmidt transform of k_h : for any $f\in L^2$, $\underline{k}_h(f,f_0)=k_h(f,f_0)$ and $\forall n\ge 1$,

$$\underline{k}_h(f,f_n) = k_h(f,f_n)-\sum_{j=0}^{n-1}\frac{\big(\underline{k}_h(\cdot,f_j),k_h(\cdot,f_n)\big)_{\mathscr{H}_h}}{\|\underline{k}_h(\cdot,f_j)\|^2_{\mathscr{H}_h}}\underline{k}_h(f,f_j) \tag{3.15}$$

$$= k_h(f,f_n)+\sum_{j=0}^{n-1}\left(\sum_{l=j}^{n-1}\alpha_h(f_n,j,l)\right)k_h(f,f_j)\ , \tag{3.16}$$

where the coefficients $\alpha_h(f_n,j,l)$ correspond to the inverse Gram-Schmidt transform. Note that $\alpha_h(f_n,j,l)$ depends on n only through the terms $(\underline{k}_h(\cdot,f_j),k_h(\cdot,f_n))_{\mathscr{H}_h}=\underline{k}_h(f_n,f_j)$, and we define $\alpha_h(g,j,l)$ by an obvious substitution. It is straightforward that for all $f,g\in L^2(m)$, $h\mapsto k_h(f,g)$ is analytic over $h\in(0,1/2)$ (in the sequel we will say, for short, that $k_h(f,g)$ is analytic). Hence, proceding by induction, assume that for all $f,g\in L^2$, $\underline{k}_h(f,f_{n-1})$ and $\alpha_h(g,j,l)$, for $j\le n-2$ and $j\le l\le n-2$, are analytic. We will show that all the terms in (3.15) are analytic. By the preceding remarks, the choice of g is unimportant since, under the induction hypothesis, $\underline{k}_h(g,f_j)$ is analytic. $k_h(f,f_n)$ is analytic, as was previously stated, so it remains to assess the terms in the sum of (3.15). For $j\le n-1$, $(\underline{k}_h(\cdot,f_j),k_h(\cdot,g))_{\mathscr{H}_h}=\underline{k}_h(g,f_j)$, which is analytic by assumption. In particular, this is true for $g=f_n$. Then, decomposing $\underline{k}_h(\cdot,f_j)$ $(j\le n-1)$ as in

(3.16), $\|\underline{k}_h(\cdot,f_j)\|^2$ is a combination of sums and products of $\alpha_h(g,p,l)$ ($g=f_j$, $p\le j-1$) and of $k_h(f_i,f_j)$. Hence, it is analytic. The only term left to conclude this induction proof, is $\alpha(g,n-1,n-1)$. The correspondence with (3.15) indicates that it is equal to $-\underline{k}_h(g,f_{n-1})\,\|\underline{k}_h(\cdot,f_{n-1})\|^{-2}_{\mathcal{H}_h}$. Again, this is analytic by the induction hypothesis and what we just said on $\|\underline{k}_h(\cdot,f_j)\|^2$. All this also holds for R_h and the corresponding quantities.
The analytic property will also be needed for:

$$h'\in(0,1/2]\mapsto\int_0^1 K_h(t,r)K_{h'}(s,r)\,\mathrm{d}r\ ,$$

for any $h\in(0,1/2]$, $s,t\in[0,1]$. In the proof of Lemma 3.1 of [39], the authors show that for any $s,t\in[0,1]$, $H\in(0,1)\mapsto\int_0^1 K_H(t,r)K_H(s,r)\,\mathrm{d}r$ is analytic. A direct adaptation of their proof suffices to show what we want.

Turning to the second part of this proof, let h_1,h_2 be fixed elements in $I_\eta=[\eta,1/2-\eta]$. We recall from (3.9) that:

$$\mathbb{E}\left((\boldsymbol{B}_{h_1,f}-\boldsymbol{B}_{h_2,f})^2\right)=\int_0^1\left(K^*_{h_1}R^{-1}_{h_1}u^{-1}_{h_1}k_{h_1}(f,\cdot)-K^*_{h_2}R^{-1}_{h_2}u^{-1}_{h_2}k_{h_2}(f,\cdot)\right)^2(u)\,\mathrm{d}u\ .$$

From the proof of Lemma 3.9, we recall that for any $n\in\mathbb{N}$, $\underline{K}_h(t_n,\cdot)=K_h^{-1}\underline{R}_h(t_n,\cdot)$ and that $\{\underline{K}_h(t_n,\cdot),n\in\mathbb{N}\}$ is an orthogonal family of L^2. The decomposition of $k_h(f,\cdot)$ in $\mathcal{H}_h$ gives:

$$K_h^{-1}u_h^{-1}k_h(f,\cdot)=\sum_{n=0}^{\infty}\underline{k}_h(f,f_n)\frac{\underline{K}_h(t_n,\cdot)}{\|\underline{R}_h(t_n,\cdot)\|_{H_h}}\ ,$$

where the equality is in $L^2([0,1])$. By definition of $\underline{K}_h$, $\|\underline{K}_h(t_n,\cdot)\|_{L^2}=\|\underline{R}_h(t_n,\cdot)\|_{H_h}$, so we will drop the last norm in the above formula to consider that $\{\underline{K}_h(t_n,\cdot),n\in\mathbb{N}\}$ is an orthonormal family.
Therefore,

$$\mathbb{E}\left(\boldsymbol{B}_{h,f}-\boldsymbol{B}_{h',f}\right)^2=\Big\|\sum_{n=0}^{\infty}\underline{k}_h(f,f_n)\underline{K}_h(t_n,\cdot)-\sum_{n=0}^{\infty}\underline{k}_{h'}(f,f_n)\underline{K}_{h'}(t_n,\cdot)\Big\|^2_{L^2}\ .$$

Let us define, for h,h' in $I_\eta=[\eta,1/2-\eta]$,

$$u_N(h,h')=\Big\|\sum_{n=0}^{N}\underline{k}_h(f,f_n)\underline{K}_h(t_n,\cdot)-\sum_{n=0}^{N}\underline{k}_{h'}(f,f_n)\underline{K}_{h'}(t_n,\cdot)\Big\|^2_{L^2}\ . \tag{3.17}$$

For now, we will assume that this converges uniformly in $h,h'\in I_\eta$ and $f\in D$, as $N\to\infty$. This will be proved in the next paragraph. The limit is denoted by $u(h,h')$ and is the quantity we are interested in. Let us show that $h'\mapsto u_N(h,h')$ is analytic in $h'\in I_\eta$, for any $N\in\mathbb{N}$. For this purpose, we rewrite it as:

$$u_N(h,h')=\sum_{n=0}^{N}\underline{k}_h(f,f_n)^2+\sum_{n=0}^{N}\underline{k}_{h'}(f,f_n)^2-2\sum_{i=0}^{N}\sum_{j=0}^{N}\underline{k}_h(f,f_i)\underline{k}_{h'}(f,f_j)\left(\underline{K}_h(t_i,\cdot),\underline{K}_{h'}(t_j,\cdot)\right)_{L^2}\ .$$

The first term is a constant (N and h are fixed), while according to the first part of this proof, the second term is analytic. The coefficients in the linear decomposition of $\underline{K}_{h'}(t_j,\cdot)$ on $\mathrm{Span}\{K_{h'}(t_l,\cdot),l\le$

$j\}$ are the one obtained in (3.16), making the appropriate adaptation to R_h. They are also analytic, for the reasons mentioned in the first part, and denoted $\beta_{h'}(j)$, by analogy with the α_h's of the first part. Taking into account the analytic terms $\underline{k}_h(f,f_i)\underline{k}_{h'}(f,f_j)$, we write the double sum in $u_N(h,h')$ in the following way (β becomes $\tilde{\beta}$ due to these multiplicative terms):

$$\sum_{i=0}^{N}\sum_{j=0}^{N}\tilde{\beta}_h(i)\tilde{\beta}_{h'}(j)\big(K_h(t_i,\cdot),K_{h'}(t_j,\cdot)\big)_{L^2} \ .$$

It was proven in the first part that the scalar products are analytic, which finishes to prove our assertion that u_N is analytic in the second variable (and so, in the first variable too when the second is fixed). Now, a standard result on analytic functions states that if a function is the uniform limit on a compact of analytic functions, then it is itself analytic, and the sequence of the derivative functions converges uniformly towards the derivative of the limit (see [123, p.214]). So, $u(h,h')$ is analytic (h fixed) and its derivative reads:

$$\begin{aligned} u'(h,h') = \lim_{N\to\infty}\Bigg(2\sum_{n=0}^{N}\underline{k}_{h'}(f,f_n)\underline{k}'_{h'}(f,f_n) - 2\sum_{i=0}^{N}\sum_{j=0}^{N}\underline{k}_h(f,f_i)\underline{k}'_{h'}(f,f_j)\big(\underline{K}_h(t_i,\cdot),\underline{K}_{h'}(t_j,\cdot)\big)_{L^2} \\ - 2\sum_{i=0}^{N}\sum_{j=0}^{N}\underline{k}_h(f,f_i)\underline{k}_{h'}(f,f_j)\big(\underline{K}_h(t_i,\cdot),\underline{K}'_{h'}(t_j,\cdot)\big)_{L^2}\Bigg) , \end{aligned} \tag{3.18}$$

where the limit is uniform. In fact, it is also uniform in h and f, as an adaptation of the proof of Theorem 10.28 of [123] (using Cauchy's estimate) shows. The continuity in the first variable of the partial sums $u_N(h,h')$ follows the same line than for the second variable. The continuity in $f\in D$ of these partial sums is obvious from equation (3.16). As such, a limiting argument implies that $u'(h,h')$ is continuous in both variables and in $f\in D$.
Hence,

$$M_u = \sup_{(h,h')\in I_\eta^2, f\in D} |u'(h,h')| < \infty \ .$$

We also have that $M_u^{(2)} = \sup_{(h,h')\in I_\eta^2, f\in D}|u''(h,h')|$ is finite. For the sake of brevity, we do not develop the proof, which follows by applying the same arguments as we did on the first derivative. Furthermore, we have that $u'(h_1,h_1)=0$. Indeed, the first two terms in (3.18) annihilates when evaluated at h_1, while the last one becomes:

$$\begin{aligned} &\sum_{i\le j}\underline{k}_{h_1}(f,f_i)\underline{k}_{h_1}(f,f_j)\Big(\big(\underline{K}_{h_1}(t_i,\cdot),\underline{K}'_{h_1}(t_j,\cdot)\big)_{L^2}+\big(\underline{K}_{h_1}(t_j,\cdot),\underline{K}'_{h_1}(t_i,\cdot)\big)_{L^2}\Big) \\ &\qquad = \sum_{i\le j}\underline{k}_{h_1}(f,f_i)\underline{k}_{h_1}(f,f_j)\left.\frac{d}{dh}\right|_{h=h_1}\big(\underline{K}_h(t_i,\cdot),\underline{K}_h(t_j,\cdot)\big)_{L^2} \ , \end{aligned}$$

which is zero. Thus, the previous discussion and the mean value theorem applied on the second order Taylor expansion of $u(h_1,h')$ shows that, for $h'\in I_\eta$,

$$\Big\|\sum_{n=0}^{\infty}\underline{k}_{h_1}(f,f_n)\underline{K}_{h_1}(t_n,\cdot) - \sum_{n=0}^{\infty}\underline{k}_{h'}(f,f_n)\underline{K}_{h'}(t_n,\cdot)\Big\|_{L^2}^2 \le M_u^{(2)}\,(h_1-h')^2 \ . \tag{3.19}$$

To conclude the proof, it remains to prove the uniform convergence in (3.17). We first notice that:

$$\sup_{(h,h')\in I_\eta^2, f\in D} |u(h,h') - u_N(h,h')| \le \sup_{(h,h')\in I_\eta^2, f\in D} \Big\| \sum_{n=N+1}^{\infty} \underline{k}_h(f,f_n)\underline{K}_h(t_n,\cdot) - \sum_{n=N+1}^{\infty} \underline{k}_{h'}(f,f_n)\underline{K}_{h'}(t_n,\cdot)\Big\|_{L^2}^2$$
$$\le 2 \sup_{h\in I_\eta, f\in D} \sum_{n=N+1}^{\infty} \underline{k}_h(f,f_n)^2 + 2 \sup_{h'\in I_\eta, f\in D} \sum_{n=N+1}^{\infty} \underline{k}_{h'}(f,f_n)^2 .$$

The initial problem now comes down to the proof that $k_h(f,f)$ is the uniform limit in $h\in I_\eta$ and $f\in D$ of $\sum \underline{k}_h(f_n,f)^2$. Let $\nu>0$. We recall that for any $g\in L^2(T)$, $\|k_h(f,\cdot)-k_h(g,\cdot)\|_{\mathcal{H}_h} = \|f-g\|_{L^2}^{2h}$. It follows from the density of $\{f_n\}_{n\in\mathbb{N}}$ in $L^2(T)$ that for any $f\in L^2$, there is an index $\alpha\in\mathbb{N}$ such that $\|f-f_\alpha\|_{L^2}\le \nu^{1/4\eta}$. In fact, the compactness of D implies that there is an integer N_ν such that D can be covered by balls of radius at most $\nu^{1/4\eta}$ centered in $\{f_{\alpha_j}, j=1\dots N_\nu\}\subset \{f_n\}_{n\in\mathbb{N}}$. Besides, from the construction of $\left\{\underline{k}_h(f_n,\cdot), n\in\mathbb{N}\right\}$, $k_h(f_\alpha,\cdot)\in \text{Span}\left\{\underline{k}_h(f_j,\cdot), j\le\alpha\right\}$. As a consequence of the previous points, if f is in the ball centered in f_{α_j},

$$\sup_{h\in I_\eta} \|k_h(f,\cdot)-k_h(f_{\alpha_j},\cdot)\|_{\mathcal{H}_h}^2 = \sup_{h\in I_\eta}\left(\sum_{n=1}^{\alpha_j}\Big(\underline{k}_h(f,f_n)-\underline{k}_h(f_{\alpha_j},f_n)\Big)^2 + \sum_{n=\alpha_j+1}^{\infty} \underline{k}_h(f,f_n)^2\right)$$
$$= \sup_{h\in I_\eta} \|f-f_{\alpha_j}\|_{L^2}^{4h}$$

and therefore, $\sup_{h\in I_\eta}\sum_{n=\alpha_j+1}^{\infty} \underline{k}_h(f,f_n)^2 \le \sup_{h\in I_\eta}\|f-f_{\alpha_j}\|_{L^2}^{4h}$ which is less than ν. This finally reads: for any $N\ge\alpha=\max_{j=1\dots N_\nu}\alpha_j$,

$$\sup_{h\in I_\eta, f\in D} |k_h(f,f) - \sum_{n=1}^{N} \underline{k}_h(f,f_n)^2| = \sup_{h\in I_\eta, f\in D} \sum_{n=N+1}^{\infty} \underline{k}_h(f,f_n)^2$$
$$\le \nu,$$

so the convergence is uniform and this ends the proof. □

Let d_m denote the distance induced by the $L^2(T,m)$ norm. Expressed in a more general form, the following corollary is obtained from the previous result:

Corollary 3.16. *For all compact subset D of $L^2(T,m)$ of d_m-diameter smaller than 1, for any $\eta\in(0,1/4)$, there exists a constant $C>0$ (depending on D and η) such that, $\forall f,f'\in D$, $\forall h,h'\in[\eta,1/2-\eta]$,*

$$\mathbb{E}\left((\boldsymbol{B}_{h,f}-\boldsymbol{B}_{h',f'})^2\right) \le C\,(h'-h)^2 + 2\,m\left(|f-f'|^2\right)^{2(h\wedge h')} . \tag{3.20}$$

3.4 Continuity of the fractional Brownian field

In this section, we address the following question: under which conditions does the fBf have a continuous modification? As this is often the case, the answer is closely related to metric entropy of the indexing collection. We remark here that speaking of continuous modification of a process requires the process to have a separable modification, in the sense of Doob. This is always the case for multiparameter processes [76], but it is no longer clear when $\mathbb{R}^d$ is replaced by an L^2

space. Theorem 2 of [53, p.153] provides an answer when the process is indexed by a separable metric space with value in a locally compact space, which includes the L^2–fBf.

Following equation (3.20), on a subdomain $D \subset L^2$ of d_m-diameter smaller than 1:

$$\begin{aligned}\mathbb{E}\left((\boldsymbol{B}_{h,f} - \boldsymbol{B}_{h',g})^2\right) &\leq 2C \ \max\left(|h-h'|, m(|f-g|^2)^{1/2}\right)^{4(h\wedge h')} \\ &\leq 2C \ d\left((h,f),(h',g)\right)^{4(h\wedge h')} ,\end{aligned} \tag{3.21}$$

where d is the product distance on $(0,1/2] \times L^2(T,m)$.

Let $\tilde{K}$ be a compact of $L^2(T,m)$ of d_m-diameter less than 1, and $a \in (0,1/2)$. Let $\eta > 0$, $K_a = [a, 1/2-\eta] \times \tilde{K}$ and $C = \left\{\boldsymbol{B}_{h,f}, (h,f) \in K_a\right\}$ be a subspace of $L^2(\Omega)$. To measure the distance between points, let δ be defined by $\delta(\boldsymbol{B}_{\varphi_1}, \boldsymbol{B}_{\varphi_2}) = \sqrt{\mathbb{E}(\boldsymbol{B}_{\varphi_1} - \boldsymbol{B}_{\varphi_2})^2}$, for $\varphi_1, \varphi_2 \in K_a$. For any $\varepsilon < 1$, $N(C,\delta,\varepsilon)$ denotes the metric entropy of C, that is, the smallest number of δ-balls of radius at most ε needed to cover C. We will also make use of the notation $N(\varepsilon)$ when the context is clear, and denote by $H(\varepsilon)$ the log-entropy $\log(N(\varepsilon))$. We give a first result on the modulus of continuity, which is a simple consequence of a famous theorem of Dudley [44] and of inequality (3.21).

Proposition 3.17. *Assume that there exist some $M, \alpha \in \mathbb{R}_+$, such that for all sufficiently small ε, $N(C,\delta,\varepsilon) \leq M\varepsilon^{-\alpha}$. Then, the mapping $x \mapsto x^{2a}\ \sqrt{-\log x}$ is a* uniform modulus of continuity *for $\left\{\boldsymbol{B}_{h,f}, (h,f) \in K_a\right\}$, meaning that there exists a measurable c_ω such that almost surely:*

$$\forall (h,f),(h',g) \in K_a, \quad |\boldsymbol{B}_{h,f} - \boldsymbol{B}_{h',g}| \leq c_\omega \ d\left((h,f),(h',g)\right)^{2a} \ \sqrt{-\log d\left((h,f),(h',g)\right)} \ .$$

In particular, the fBf on K_a is a.s. Hölder-continuous for any $b < 2a$. Such exponential bounds on the entropy appear frequently in statistics, for instance when C is a Vapnick-Cervonenkis class with exponent ν:

$$\forall \varepsilon > 1, \quad N(C,\varepsilon) \leq K\varepsilon^{-2\nu} |\log \varepsilon|^{\nu}.$$

See for instance [5] for a review of these properties. The conditions of the previous Proposition are thus met on a Vapnick-Cervonenkis indexing class, choosing any $\alpha > 2\nu$.

Proof. The elements of Dudley's Theorem are described as follows: let L be the isonormal process over $L^2(\Omega)$, that is, on the same probability space Ω, the centred Gaussian process whose covariance is given by $\mathbb{E}(L(X_1)L(X_2)) = \mathbb{E}(X_1X_2)$, for all $X_1, X_2 \in C$. Thus $\mathbb{E}\left((L(X_1) - L(X_2))^2\right) = \delta(X_1,X_2)$. Using a chaining argument and Borel-Cantelli lemma, Dudley proved that $F(x) = \int_0^x \sqrt{\log N(C,\delta,\varepsilon)}\, \mathrm{d}\varepsilon$ is a modulus of continuity (uniform, and potentially infinite) for the sample paths of L on C. A straightforward calculus shows that under the assumptions on the entropy, $x\sqrt{-\log x} \leq F(x) \leq 2x\sqrt{-\log x}$ for all $x \in (0, e^{-1/2}]$. Hence, $x\sqrt{-\log x}$ is a modulus of continuity of L. Let $G : x \in \mathbb{R}_+ \mapsto x^{2a}$, so that according to (3.21): $\delta(\boldsymbol{B}_{h,f}, \boldsymbol{B}_{h',g}) \leq G(d((h,f),(h',g)))$. Then, $(h,f) \mapsto L(\boldsymbol{B}_{h,f})$ and $(h,f) \mapsto \boldsymbol{B}_{h,f}$ have the same law so there exists a measurable subset $\tilde{\Omega} \subseteq \Omega$ of measure 1, and a measurable c_ω such that for any $\omega \in \tilde{\Omega}$:

$$\begin{aligned}\forall (h,f),(h',g) \in K_a, \quad |\boldsymbol{B}_{h,f} - \boldsymbol{B}_{h',g}| &\leq c_\omega \ F\left(\delta\left((h,f),(h',g)\right)\right) \\ &\leq c_\omega \ F \circ G\left(d\left((h,f),(h',g)\right)\right) .\end{aligned}$$

□

The rest of this section is dedicated to improving this result, in various directions. First we argue that studying entropy conditions for the fBf is essentially the same as studying the entropy of the h-fBm. This is the purpose of Theorem 3.19, preceded by the following technical lemma. Then, in section 3.5.1, we will consider more specific indexing collections for which the regularity results are more precise.

Lemma 3.18. *Let (T_1, d_1) and (T_2, d_2) be two compact metric spaces and denote by d the product distance on $T_1 \times T_2$. The log-entropies on (T_1, d_1), (T_2, d_2) and $(T_1 \times T_2, d)$ are respectively denoted by $H_1(\varepsilon)$, $H_2(\varepsilon)$ and $H(\varepsilon)$. Then, the following lower and upper bounds on H hold (allowing the integral to be infinite):*

$$\frac{1}{2} \int_0^1 \left(\sqrt{H_1(\varepsilon)} + \sqrt{H_2(\varepsilon)} \right) d\varepsilon \leq \int_0^1 \sqrt{H(\varepsilon)}\, d\varepsilon \leq \sqrt{2} \int_0^1 \left(\sqrt{H_1(\varepsilon)} + \sqrt{H_2(\varepsilon)} \right) d\varepsilon. \quad (3.22)$$

Proof. Let $B^i(c, r)$ the open ball of (T_i, d_i) centred at c with radius r, $i = 1, 2$. Let $\varepsilon > 0$ and $\{B^1_j(c^1_j, \varepsilon)\ ,\ 1 \leq j \leq N_1(\varepsilon)\}$ (resp. $\{B^2_j(c^2_j, \varepsilon)\ ,\ 1 \leq j \leq N_2(\varepsilon)\}$) be a ε-covering of T_1 (resp. T_2). First notice that for the product distance d, one has $B^1_i(c^1_i, \varepsilon) \times B^2_j(c^2_j, \varepsilon) = B_d((c^1_i, c^2_j), \varepsilon)$ fol all $(i, j) \in \{1, \ldots, N_1(\varepsilon)\} \times \{1, N_2(\varepsilon)\}$. A first inequality follows:

$$N(T_1 \times T_2, d, \varepsilon) \leq N_1(\varepsilon)\, N_2(\varepsilon)\ .$$

Reciprocically, if $\{B_1(c_1, \varepsilon), \ldots, B_{N(\varepsilon)}(c_{N(\varepsilon)}, \varepsilon)\}$ is a ε-covering of $T_1 \times T_2$, then each c_j rewrites: $c_j = (c^1_j, c^2_j)$ and so $B_j(c_j, \varepsilon) = B^1_j(c^1_j, \varepsilon) \times B^2_j(c^2_j, \varepsilon)$. Then we have:

$$T_i \subseteq \bigcup_{j=1}^{N(\varepsilon)} B^i_j(c^i_j, \varepsilon)\ , \quad i = 1, 2.$$

Hence, $N(\varepsilon) \geq N_1(\varepsilon) \vee N_2(\varepsilon)$. The upper and lower bounds in (3.22) follow. □

Theorem 3.19. *Let $\boldsymbol{B}$ be a fBf indexed on a compact subset I of $(0, 1/2]$, and K be a compact subset of $L^2(T, m)$ of d_m-diameter smaller than 1. Let $\iota = \inf I$. If the following Dudley integral converges:*

$$\int_0^1 \sqrt{\log N(K, d_m^{2\iota}, \varepsilon)}\, d\varepsilon < \infty\ , \quad (3.23)$$

then $\boldsymbol{B}$ indexed by $I \times K$ has almost surely continuous sample paths.

Remark 3.20. *Fernique showed in [49] that for a stationary process indexed on $\mathbb{R}^d$, the convergence of the Dudley integral is a necessary condition (see [89, Chap.13], where the result is derived from a majorizing measure argument combined with Haar measures for processes indexed on a locally compact Abelian group). The extension of this result to increment stationary processes is explained clearly in [100, p.251]. In the case of the h-fBm (increment stationary), the indexing collection is an infinite-dimensional Hilbert space, hence it has no locally compact subgroups of noticeable interest. Whether condition (3.23) is necessary remains open.*

Proof. Remember that δ denotes the canonical pseudo-distance induced by the fBf. We prove that the convergence of the integral (3.23) implies the convergence of this other integral:

$$\int_0^1 \sqrt{\log N(I \times K, \delta, \varepsilon)}\, d\varepsilon\ ,$$

which then implies the result, according to a famous theorem of Dudley [44]. For $h, h' \in I$, and $f, f' \in K$, and since $\iota > 0$, it follows from Equation (3.21) that:

$$\delta\Big((h,f),(h',f')\Big) \leq C\ d\Big((h,f),(h',f')\Big)^{2\iota}\ ,$$

where d is still the product distance of d_m and d_1 (the absolute value distance on $\mathbb{R}$). Note that the previous inequality implies that the balls measured with distance δ are bigger than those measured with $d^{2\iota}$, hence $N(K,\delta,\varepsilon) \leq N(K,d^{2\iota},\varepsilon)$. Since $\int_0^1 \left(\log(N(I,d_1^{2\iota},\varepsilon))\right)^{1/2}\ \mathrm{d}\varepsilon < \infty$, Lemma 3.18 implies that the convergence of the Dudley integral for $d^{2\iota}$ is equivalent to the convergence of the Dudley integral for $d_m^{2\iota}$. Hence the result. □

Examples of indexing classes for which the fBf is a.s. continuous will be discussed in section 3.5. We simply recall that an object as simple as the Brownian motion indexed over the Borel sets of $[0,1]^2$, that is, the centred Gaussian process with covariance:

$$\forall U,V \in \mathscr{B}([0,1]^2), \quad \mathbb{E}(W_U\ W_V) = \lambda(U \cap V)\ ,$$

is almost surely unbounded [5, p.28].

3.5 Applications to the regularity of the multiparameter and set-indexed fractional Brownian fields

In this section, we present the fBf and the h-fBm in the more familiar framework of multiparameter processes, enhancing the fact that these processes are rather different from the Lévy fractional Brownian motion and the fractional Brownian sheet, as well as from their multifractional counterparts ([57]). This study is then extended to set-indexed processes. In both cases, the meeting point will be that the fBf is now considered as a multifractional process, meaning that on the indexing collection $\mathscr{A}$ (to be specified), we have a function $\boldsymbol{h} : \mathscr{A} \to (0,1/2]$, and denote $\boldsymbol{B^h}$ the process indexed over $\mathscr{A}$ defined by $\left\{\boldsymbol{B}_{\boldsymbol{h}(U),U}, U \in \mathscr{A}\right\}$. This framework allows to establish more precise regularity results, such as the measure of local Hölder exponents.

3.5.1 Multiparameter multifractional Brownian motion

For some $d \in \mathbb{N}^*$, let $\mathscr{A} = \{[0,t], t \in [0,1]^d\}$. Let $\boldsymbol{B}$ the fBf on $L^2([0,1]^d, m)$ where m is not necessarily the Lebesgue measure. Then a multiparameter multifractional Brownian motion is a process $\boldsymbol{B^h}$ defined for some function $\boldsymbol{h} : [0,1]^d \to (0,1/2]$ by:

$$\forall t \in [0,1]^d, \quad \boldsymbol{B}_t^{\boldsymbol{h}} = \boldsymbol{B}_{\boldsymbol{h}_t, \mathbf{1}_{[0,t]}}\ .$$

For $\boldsymbol{h}$ a constant function equal to 1/2, this is the usual Brownian sheet of $\mathbb{R}^d$ (when $m = \lambda_d$). For any other constant function, this is neither the fractional Brownian sheet nor the Lévy fractional Brownian motion, but a process called multiparameter fBm (mpfBm) with covariance:

$$\mathbb{E}\left(\boldsymbol{B}_s^{\boldsymbol{h}}\ \boldsymbol{B}_t^{\boldsymbol{h}}\right) = \frac{1}{2}\left(m([0,s])^{2h} + m([0,t])^{2h} - m([0,s] \mathbin{\triangle} [0,t])^{2h}\right)\ ,$$

where $\triangle$ is the symmetric difference between sets. Some of the differences between this process and the aforementioned are discussed in [60]. As stated in the introduction, one can also obtain the Lévy fractional Brownian motion from the fBf, choosing another class $\mathscr{A}$ and a specific measure. Hence, the results of regularity for the Lévy fBm (or its multifractional counterpart, see for instance [57]) follow from the results of the next subsection rather than this one.

In this case and unlike the previous section, the entropy of $\mathscr{A}$ is perfectly known when m is the Lebesgue measure[2], and the Dudley integral is easily seen to be finite. Hence the fBf on $\mathscr{A}$ has a continuous modification, and so does any multiparameter mBm. It is possible to establish precise Hölder regularity coefficients. For easier comparison with prior works on mpfBm, we will not use the distance d_m but a variant defined as:

$$\text{for } s,t \in [0,1]^d, \quad d'_m(s,t) = m([0,s] \triangle [0,t]) \,.$$

Note that $d'_m(s,t) = d_m(\mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]})^2$. We explained in Remark 2.2 that this distance is equivalent to the Euclidean distance when m is the Lebesgue measure and the set of indexing points stays within a compact away from 0.

For a stochastic process X indexed on $\mathscr{A}$, let us define the deterministic pointwise Hölder exponent at $t_0 \in \mathscr{A}$:

$$\mathbb{a}_X(t_0) = \sup\left\{\alpha :\ \limsup_{\rho\to 0} \sup_{s,t\in B_{d'_m}(t_0,\rho)} \frac{\mathbb{E}\left(|X_s - X_t|^2\right)}{\rho^{2\alpha}} < \infty\right\} \,,$$

where $B_{d'_m}(t_0,\rho)$ is the ball of the d'_m distance. Similarly, the deterministic local Hölder exponent is:

$$\widetilde{\mathbb{a}}_X(t_0) = \sup\left\{\alpha :\ \limsup_{\rho\to 0} \sup_{s,t\in B_{d'_m}(t_0,\rho)} \frac{\mathbb{E}\left(|X_s - X_t|^2\right)}{d'_m(s,t)^{2\alpha}} < \infty\right\} .$$

We will compare these exponents to their stochastic analogue, straightforwardly defined getting rid of the expectation in the above definitions. The random coefficients are denoted $\boldsymbol{\alpha}_X(t_0)$ and $\widetilde{\boldsymbol{\alpha}}_X(t_0)$. This extends to continuous (deterministic) functions on $\mathscr{A}$.

As the terminology is commonly accepted in the multifractional literature, a *regular* multiparameter mBm will be a fBf with a function $\boldsymbol{h}$ such that, at each point, the value of the function is smaller than its local and pointwise exponents (ie $\boldsymbol{h}_t \leq \boldsymbol{\alpha}_{\boldsymbol{h}}(t)$).

Proposition 3.21. *Let $\boldsymbol{B}^{\boldsymbol{h}}$ be a regular multiparameter mBm on $[0,1]^d$. Then, for all $t_0 \in [0,1]^d$, both equalities hold almost surely:*

$$\boldsymbol{\alpha}_{\boldsymbol{B}^{\boldsymbol{h}}}(t_0) = \boldsymbol{h}_{t_0} \ \text{ and } \ \widetilde{\boldsymbol{\alpha}}_{\boldsymbol{B}^{\boldsymbol{h}}}(t_0) = \boldsymbol{h}_{t_0} \,.$$

When $t_0 \neq 0$, these equalities still hold true for the exponents defined replacing d'_m with the Euclidean distance. This is another consequence of the equivalence between those distances on a compact away from 0.

Proof. The first step is to evaluate $\mathbb{a}_{\boldsymbol{B}^{\boldsymbol{h}}}(t_0)$ and $\widetilde{\mathbb{a}}_{\boldsymbol{B}^{\boldsymbol{h}}}(t_0)$. Theorem 2.34 then states that a Gaussian process X, indexed by a collection of sets satisfying certain technical assumptions has the following property:

$$\mathbb{P}(\boldsymbol{\alpha}_X(t_0) = \mathbb{a}_X(t_0)) = 1 \ \text{ and } \ \mathbb{P}\left(\widetilde{\boldsymbol{\alpha}}_X(t_0) = \widetilde{\mathbb{a}}_X(t_0)\right) = 1 \,. \tag{3.24}$$

As discussed in Chapter 2, the technical assumptions are satisfied by the class $\mathscr{A}$ of rectangles. A result of that sort actually originated in [59], but we use the one in Chapter 2 to introduce the extended results of the following section on set-indexed processes.

[2]which is assumed until the end of this section.

Let K be a compact of $[0,1]^d$ with d'_m-diameter smaller than 1, whose interior contains t_0. As a consequence of the continuity of $\boldsymbol{h}$, $\boldsymbol{h}(K) \subseteq [\eta, 1/2-\eta]$ for some $\eta > 0$. For all $s,t \in B(\rho)$, the ball centred in t_0 of radius ρ,

$$\begin{aligned}\mathbb{E}\left(\boldsymbol{B}^{\boldsymbol{h}}_t - \boldsymbol{B}^{\boldsymbol{h}}_s\right)^2 &\geq \frac{1}{2}\mathbb{E}\left(\boldsymbol{B}_{\boldsymbol{h}_t,t} - \boldsymbol{B}_{\boldsymbol{h}_t,s}\right)^2 - \mathbb{E}\left(\boldsymbol{B}_{\boldsymbol{h}_t,s} - \boldsymbol{B}_{\boldsymbol{h}_s,s}\right)^2 \\ &\geq \frac{1}{2} d'_m(s,t)^{2\boldsymbol{h}_t} - C_{\eta,K}(\boldsymbol{h}_t - \boldsymbol{h}_s)^2 ,\end{aligned}$$

where we used Theorem 3.15, and this inequality yields that for any $\alpha > \inf_{B(\rho)} \boldsymbol{h}$:

$$\frac{\mathbb{E}\left(\boldsymbol{B}^{\boldsymbol{h}}_t - \boldsymbol{B}^{\boldsymbol{h}}_s\right)^2}{d'_m(s,t)^{2\alpha}} \geq \frac{1}{2} d'_m(s,t)^{2\inf_B \boldsymbol{h} - 2\alpha} - \tilde{C}_{\eta,K}\, d'_m(s,t)^{2\inf_B \boldsymbol{\alpha}_{\boldsymbol{h}} - 2\alpha} .$$

The regularity property of $\boldsymbol{h}$ implies that for ρ sufficiently small, α can be chosen so that $\inf_{B(\rho)} \boldsymbol{\alpha}_{\boldsymbol{h}} \geq \alpha > \inf_{B(\rho)} \boldsymbol{h}$, and the previous inequality diverges as $\rho \to 0$. Hence

$$\mathbb{\alpha}_{\boldsymbol{B}^{\boldsymbol{h}}}(t_0) \leq \lim_{\rho\to 0} \inf_{B(\rho)} \boldsymbol{h} = \boldsymbol{h}(t_0) .$$

The converse inequality follows from the result of Corollary 3.16 and the same reasoning. Thus the deterministic exponents are both equal to $\boldsymbol{h}_{t_0}$, and the property (3.24) leads to the result. □

This result, which holds for all points, almost surely, is greatly strenghtened into paths properties by the following proposition:

Proposition 3.22. *Let $\boldsymbol{B}^{\boldsymbol{h}}$ be a regular multiparameter mBm on $[0,1]^d$. Then, almost surely,*

$$\forall t_0 \in [0,1]^d, \quad \widetilde{\boldsymbol{\alpha}}_{\boldsymbol{B}^{\boldsymbol{h}}}(t_0) = \boldsymbol{h}_{t_0} \text{ and } \boldsymbol{\alpha}_{\boldsymbol{B}^{\boldsymbol{h}}}(t_0) \geq \boldsymbol{h}_{t_0} .$$

Proof. This is a direct application of Theorem 2.35 and of the values of $\mathbb{\alpha}_{\boldsymbol{B}^{\boldsymbol{h}}}$ and $\widetilde{\mathbb{\alpha}}_{\boldsymbol{B}^{\boldsymbol{h}}}$ computed in the proof of the previous proposition. □

In the case of the SIfBm (see Section 2.6) or of the regular mBm ([59]), the previous uniform lower bound of the pointwise exponent is an equality. This provides tangible argument for an improvement of our result, but the question is left open for now.

3.5.2 *Set-Indexed multifractional Brownian motion*

This section is a discussion on a natural extension of the results on multiparameter processes to a wider class of indexing collections. The framework of set-indexed processes of Ivanoff and Merzbach [70], and the results of Chapter 2 provide a definition of Hölder exponents and fine regularity results.

Let T be a locally compact complete separable metric and measure space with metric d and Radon measure m defined on the Borel sets of T. We recall briefly the definition of an indexing collection, as it appeared in Definition 2.1:

Definition 3.23. *A nonempty class $\mathscr{A}$ of compact, connected subsets of T is called an* indexing collection *if it satisfies the following:*

1. *$\emptyset \in \mathscr{A}$, and the interior $A^\circ \neq A$ if $A \neq \emptyset$ or T. In addition, there is an increasing sequence $(B_n)_{n\in\mathbb{N}}$ of union of sets of $\mathscr{A}$ such that $T = \cup_{n=1}^{\infty} B_n^\circ$.*

2. *$\mathscr{A}$ is closed under arbitrary intersections and if $A, B \in \mathscr{A}$ are nonempty, then $A \cap B$ is nonempty. The σ-algebra generated by $\mathscr{A}$ is equal to $\mathscr{B}$, the collection of Borel sets of T.*

3. [Separability from above], *there exists a nested sequence of finite dissecting classes $\mathscr{A}_n$ whose elements approximate sets $A \in \mathscr{A}$ from above (they are bigger for the inclusion), and this approximation is finer as $n \in \mathbb{N}$ increases, until it equals A at the limit.*

The construction of a set-indexed multifractional Brownian motion relies on what was said at the beginning of Section 3.5, and it follows that for all $U \in \mathscr{A}$, $\boldsymbol{B}^{\boldsymbol{h}}_U = \boldsymbol{B}_{\boldsymbol{h}_U, \mathbf{1}_U}$ is a well defined set-indexed process. Its multiple Hölder coefficents are defined as in the multiparameter case (think of point t_0 as a set $[0, t_0] \in \mathscr{A}$ in the previous paragraph), with respect to the distance d'_m, defined for all $U, V \in \mathscr{A}$, by $d'_m(U, V) = m(U \bigtriangleup V)$.

Under entropic assumptions described in Section 2.2, the results on local and pointwise Hölder exponents, as presented for multiparameter processes, also hold for the SImBm:

Proposition 3.24. *Let $\mathscr{A}$ be an indexing collection satisfying Assumption ($\mathscr{H}_{\underline{\mathscr{A}}}$) (of Section 2.2). Let $\boldsymbol{B}^{\boldsymbol{h}}$ be a regular SImBm on $\mathscr{A}$. Then, for all $U_0 \in \mathscr{A}$, both equalities hold almost surely:*

$$\widetilde{\boldsymbol{\alpha}}_{\boldsymbol{B}^h}(U_0) = \boldsymbol{h}_{U_0} \quad \text{and} \quad \boldsymbol{\alpha}_{\boldsymbol{B}^h}(U_0) = \boldsymbol{h}_{U_0} \ .$$

Proposition 3.25. *Let $\mathscr{A}$ be an indexing collection satisfying Assumption ($\mathscr{H}_{\underline{\mathscr{A}}}$). Let $\boldsymbol{B}^{\boldsymbol{h}}$ be a regular SImBm on $\mathscr{A}$. Then, almost surely,*

$$\forall U_0 \in \mathscr{A}, \quad \widetilde{\boldsymbol{\alpha}}_{\boldsymbol{B}^h}(U_0) = \boldsymbol{h}_{U_0} \quad \text{and} \quad \boldsymbol{\alpha}_{\boldsymbol{B}^h}(U_0) \geq \boldsymbol{h}_{U_0} \ .$$

Finally, we note that there is no evidence of another (Gaussian) process with prescribed regularity in a general set-indexed setting, other than the one we defined. In particular, the (multi-)fractional Brownian sheet and Lévy fBm do not have extensions in the set-indexed setting.

3.6 Technical results

This section is an appendix that deals with several technical results that we delayed until the end of this chapter.

3.6.1 A bound for the increments of K_h^* in L^2

This first appendix collects the proofs of the technical results of the beginning of section 3.3. We recall that W_+^* denote the set of positive linear functionals over W, and that for all $x \in (-1,1)$, $L(x) = \log(|x|^{-1}) \vee 1$ if $x \neq 0$, and 0 otherwise.

Proof of Lemma 3.9. For any $t \in \mathbb{D}$, we write $\underline{K}_h(t_n,\cdot) = K_h^{-1}\underline{R}_h(\cdot,t_n)$. $\underline{R}_h$ is the Gram-Schmidt transform of R_h, which implies that $\underline{R}_h(\cdot,t_0) = R_h(\cdot,t_0)$ and $\forall n \geq 1$,

$$\underline{R}_h(\cdot,t_n) = R_h(\cdot,t_n) - \sum_{j=0}^{n-1} \frac{\left(\underline{R}_h(\cdot,t_j), R_h(\cdot,t_n)\right)_{H_h}}{\|\underline{R}_h(\cdot,t_j)\|^2} \underline{R}_h(\cdot,t_j) \; . \tag{3.25}$$

Hence, $\underline{K}_h$ can be written:

$$\begin{aligned}\underline{K}_h(t_n,\cdot) &= K_h(t_n,\cdot) - \sum_{j=0}^{n-1} \frac{\left(\underline{R}_h(\cdot,t_j), R_h(\cdot,t_n)\right)_{H_h}}{\|\underline{R}_h(\cdot,t_j)\|^2} \underline{K}_h(t_j,\cdot) \\ &= K_h(t_n,\cdot) - \sum_{j=0}^{n-1} \frac{\left(\underline{K}_h(t_j,\cdot), K_h(t_n,\cdot)\right)_{L^2}}{\|\underline{K}_h(t_j,\cdot)\|_{L^2}^2} \underline{K}_h(t_j,\cdot) \; ,\end{aligned} \tag{3.26}$$

and this shows that $\{\underline{K}_h(t_n,\cdot), n \in \mathbb{N}\}$ is the Gram-Schmidt orthogonal family of L^2, obtained from $\{K_h(t_n,\cdot), n \in \mathbb{N}\}$. Then for any $g \in L^2$ such that $g \geq 0$, the non-negativeness of $K_h(t,s), \forall t,s \in [0,1]$ (see the closed form (3.27)), implies that $\int_0^1 g(s) K_h(t_n,s)\, ds \geq 0$. Thus, if g is orthogonal to the linear span of $\left\{\underline{K}_h(t_0,\cdot),\dots,\underline{K}_h(t_{n-1},\cdot)\right\}$, it follows from (3.26) that $\int_0^1 g\ \underline{K}_h(t_n,\cdot)$ is non-negative. It is obviously also the case if $g \in \mathrm{Span}\left\{\underline{K}_h(t_0,\cdot),\dots,\underline{K}_h(t_{n-1},\cdot)\right\}$, hence $\left(\underline{K}_h(t_n,\cdot),\ \cdot\ \right)_{L^2}$ is a positive linear functional over $\mathrm{Span}\left\{\underline{K}_h(t_j,\cdot), j \in \mathbb{N}\right\}$. This leads to the following partial result:

$$\text{for any } j \in \mathbb{N}\ , \quad \underline{R}_h(t_j,t_n) = \left(\underline{K}_h(t_n,\cdot), K_h(t_j,\cdot)\right)_{L^2} \geq 0 \; .$$

Now let $g \in H(R_h)$ such that $g \geq 0$. As any element of $H(R_h)$, g can be approximated by a sequence $\{R_h(\cdot,t_{\varphi_j}), j \in \mathbb{N}\}$. By continuity, $\underline{R}_h(t_{\varphi_j},t_n)$ tends to $(\underline{R}(\cdot,t_n),g)$ as j goes to infinity. Since we have seen that the first term is non-negative for any $j \in \mathbb{N}$, this concludes the proof. □

Before the proof of Proposition 3.10, we prove a useful technical lemma:

Lemma 3.26. *For all $h_1 < h_2 \in (0,1/2)$, there exists a constant $\tilde{M}_{h_1} > 0$ such that for all $\xi \in W_+^*$,*

$$\int_0^1 \sup_{h \in [h_1,h_2]} \left(K_h^* \xi(u)\right)^2\ du < \tilde{M}_{h_1}\ \|\xi\|_{H_{h_1}^*}^2 \; .$$

Proof. Recall that $K_h^*\xi \in L^2[0,1]$. In [110, Chap. 5.1.3], for $h < 1/2$, K_h is given by the following formula: $\forall s,t \in [0,1]$,

$$\begin{aligned}K_h(t,s) &= c_h \left(\left(\frac{t(t-s)}{s} \right)^{-(1/2-h)} + (1/2-h)\ s^{1/2-h} \int_s^t u^{h-3/2}(u-s)^{h-1/2}\ \mathrm{d}u \right) \mathbf{1}_{[0,t)}(s) \\ &= c_h \left(C_h(t,s) + (1/2-h) D_h(t,s)\right) ,\end{aligned} \tag{3.27}$$

where $h \mapsto c_h$ is positive and infinitely differentiable. The second term of this sum is uniformly bounded in $s < t \in [0,1]$ and $h \in [h_1, h_2]$, while the first one diverges when t tends to s. Hence for ϵ small enough, $|t-s| < \epsilon$ implies that $K_h(t,s) \leq K_{h_1}(t,s)$ uniformly in $h \in [h_1,h_2]$, since then,

$$\left(\frac{t(t-s)}{s}\right)^{-(1/2-h)} < \left(\frac{t(t-s)}{s}\right)^{-(1/2-h_1)} ,$$

and the rest of $K_h(t,s)$ is negligeable compared to this last expression. Hence for $u \in [0,1]$,

$$\begin{aligned}
\sup_{h\in[h_1,h_2]} \left(K_h^*\xi(u)\right)^2 &= \sup_{h\in[h_1,h_2]} \left(\int_u^1 K_h(t,u)\,\mathrm{d}\xi(t)\right)^2 \\
&= \sup_{h\in[h_1,h_2]} \left(\int_u^{u+\epsilon} K_h(t,u)\,\mathrm{d}\xi(t) + \int_{u+\epsilon}^1 K_h(t,u)\,\mathrm{d}\xi(t)\right)^2 \\
&\leq 2 \sup_{h\in[h_1,h_2]} \left(\int_u^{u+\epsilon} K_\eta(t,u)\,\mathrm{d}\xi(t)\right)^2 + 2 \sup_{h\in[h_1,h_2]} \left(\int_{u+\epsilon}^1 K_h(t,u)\,\mathrm{d}\xi(t)\right)^2 .
\end{aligned}$$

According to the remark that on the set $\{(t,u,h) : |t-u| \geq \epsilon, h \in [h_1,h_2]\}$, $K_h(t,u)$ is uniformly bounded, there is a positive constant (possibly depending on ϵ) M such that $K_h(t,u)/K_{h_1}(t,u) \leq M$. Because ξ is a finite nonnegative Radon measure,

$$\int_{u+\epsilon}^1 K_h(t,u)\,\mathrm{d}\xi(t) \leq M \int_{u+\epsilon}^1 K_{h_1}(t,u)\,\mathrm{d}\xi(t) .$$

It follows that:

$$\sup_{h\in[h_1,h_2]} \left(K_h^*\xi(u)\right)^2 \leq 2\left(\int_u^{u+\epsilon} K_{h_1}(t,u)\,\mathrm{d}\xi(t)\right)^2 + 2M\left(\int_{u+\epsilon}^1 K_{h_1}(t,u)\,\mathrm{d}\xi(t)\right)^2 ,$$

and finally:

$$\begin{aligned}
\int_0^1 \sup_{h\in[h_1,h_2]} \left(K_h^*\xi(u)\right)^2\,\mathrm{d}u &\leq 2\| K_{h_1}^*\xi \|_{L^2}^2 + 2M\| K_{h_1}^*\xi \|_{L^2}^2 \\
&\leq \tilde{M}_{h_1} \|\xi\|_{H_{h_1}^*}^2 .
\end{aligned}$$

A direct consequence of this equation is that the $H_{h_1}^*$-norm is bigger than any other H_h^*-norm, when $h \geq h_1$. □

Proof of Proposition 3.10. We first recall that for any $h \in (0,1/2]$, $\xi \in W^*$, $\int_0^1 \left(K_h^*\xi(t)\right)^2\,\mathrm{d}t < \infty$, as well as the facts that ξ is considered as a nonnegative measure, and that c_h, $C_h(\cdot,\cdot)$ and $D_h(\cdot,\cdot)$ are nonnegative quantities. For the sake of readability, we shall use the symbol $'$ to denote the h-derivation. For all $s,t \in [0,1]$,

$$K_h'(t,s) = c_h' K_h(t,s) + c_h\, C_h'(t,s) + c_h\,(1/2-h) D_h'(t,s) - c_h\, D_h(t,s), \tag{3.28}$$

where

$$\begin{aligned}
C_h'(t,s) &= \log\left(s^{-1}t(t-s)\right) C_h(t,s) , \\
D_h'(t,s) &= (\log s)\, D_h(t,s) + s^{1/2-h}\left(\int_s^t (\log u) u^{h-3/2} \log(u-s)(u-s)^{h-1/2}\,\mathrm{d}u\right) \mathbf{1}_{[0,t]}(s) .
\end{aligned}$$

Each part in the sum of (3.28) will be treated separately, and each but C_h' using the Cauchy-Schwarz inequality. The first part in (3.28) gives:

$$\int_0^1 \left(\int_{h_1}^{h_2} c_h' K_h^* \xi(s)\, \mathrm{d}h \right)^2 \mathrm{d}s \le (h_2 - h_1) \int_0^1 \int_{h_1}^{h_2} \left(c_h'\right)^2 \left(K_h^* \xi(s)\right)^2 \mathrm{d}h\, \mathrm{d}s \, .$$

For $\eta > 0$, the infinite differentiability and boundedness away from 0 of c_h implies that there exists a constant M_η^1 such that for all $h \in [\eta, 1/2 - \eta]$,

$$\int_0^1 \left(c_h' \int_0^1 K_h(t,s)\, \mathrm{d}\xi(t) \right)^2 \mathrm{d}s \le M_\eta^1 \, \|\xi\|_{H_h^*}^2 \, .$$

Now, Lemma 3.26 and Fubini's Theorem imply that:

$$(h_2 - h_1) \int_{h_1}^{h_2} \int_0^1 \left(c_h' \int_0^1 K_h(t,s)\, \mathrm{d}\xi(t) \right)^2 \mathrm{d}s\, \mathrm{d}h \le \tilde{M}_\eta^1 (h_2 - h_1)^2 \|\xi\|_{H_{h_1}^*}^2 \, . \tag{3.29}$$

For the rest of this proof, we might as well consider that c_h is uniformly equal to 1.

Then for $h \in [\eta, 1/2 - \eta]$, we look at the second term in the sum of (3.28). Let $\alpha_s \in (s, 1]$ such that $s^{-1}\alpha_s(\alpha_s - s) = 1$ if $s \le 1/2$ and $\alpha_s = 1$ otherwise. Since $t \mapsto s^{-1}t(t-s)$ is increasing and maps $[s,1]$ to $[0, s^{-1}(1-s)]$, α_s is uniquely defined. Let some $\nu > 0$ such that $h \pm \nu \in (0, 1/2)$. Let us remark that $u \in [1, \infty) \mapsto \log(u) u^{-\nu}$ is bounded between 0 and $(e\ \nu)^{-1}$. Similarly, $u \in (0,1] \mapsto \log(u) u^{\nu}$ is bounded between $-(e\ \nu)^{-1}$ and 0. Thus for $s \in (0,1)$, the map

$$t \in (s,1] \mapsto \log\left(s^{-1}t(t-s)\right) C_{-\nu}(t,s) \mathbf{1}_{\{t > \alpha_s\}} + \log\left(s^{-1}t(t-s)\right) C_\nu(t,s) \mathbf{1}_{\{t \le \alpha_s\}}$$

is uniformly bounded (in s *and* t) by $-(e\ \nu)^{-1}$ and $(e\ \nu)^{-1}$. We note that when $s \ge 1/2$, the first term in the sum is automatically zero. It follows that:

$$\begin{aligned}
\left(\int_{h_1}^{h2} \int_s^1 C_h'(t,s)\, \mathrm{d}\xi(t)\, \mathrm{d}h \right)^2 &= \Bigg(\int_s^{\alpha_s} \left((s^{-1}t(t-s))^{h_2 - h_1} - 1 \right) C_{h_1}(t,s)\, \mathrm{d}\xi(t) \\
&\quad + \int_{h_1}^{h_2} \int_{\alpha_s}^1 \log\left(s^{-1}t(t-s)\right) C_{-\nu}(t,s) C_{h+\nu}(t,s)\, \mathrm{d}\xi(t)\, \mathrm{d}h \Bigg)^2 \\
&\le ((h_2 - h_1) L(h_2 - h_1))^2 \times \\
&\qquad \left(\int_s^{\alpha_s} \frac{(s^{-1}t(t-s))^{h_2 - h_1} - 1}{(h_2 - h_1) L(h_2 - h_1)} C_{h_1}(t,s)\, \mathrm{d}\xi(t) \right)^2 \qquad (3.30) \\
&\quad + 2(e\ \nu)^{-2} (h_2 - h_1) \int_{h_1}^{h_2} \left(\int_{\alpha_s}^1 C_{h+\nu}(t,s)\, \mathrm{d}\xi(t) \right)^2 \mathrm{d}h \, . \qquad (3.31)
\end{aligned}$$

(3.31) can be treated easily with Lemma 3.26, since it suffices to choose $\nu = \eta/2$ so that $h + \nu \in [\eta, 1/2 - \eta/2]$. Thus, for the same reasons as in (3.28), we have that:

$$\begin{aligned}
\int_{h_1}^{h_2} \int_0^1 2(e\ \nu)^{-2} \left(\int_{\alpha_s}^1 C_{h+\eta/2}(t,s)\, \mathrm{d}\xi(t) \right)^2 \mathrm{d}s\, \mathrm{d}h &\le 8(e\ \eta)^{-2}\, \tilde{M}_{h_1}^2 (h_2 - h_1) \|\xi\|_{H_{h_1}^*}^2 \\
&\le \tilde{M}_\eta^2 (h_2 - h_1) \|\xi\|_{H_{h_1}^*}^2 \, .
\end{aligned}$$

(3.30) requires more care since the same method would involve $C_{h-\nu}$ with $h-\nu$ occasionally smaller than h_1. For all $s \in (0,1)$, we define the application:

$$\psi_s(t,h) = \frac{(s^{-1}t(t-s))^h - 1}{hL(h)}$$

on the domain $K_s = \{(t,h):\ s \le t \le \alpha_s,\ 0 < h \le 1-2\eta\}$. Because $s^{-1}t(t-s) \in [0,1]$, it follows that $\psi_s(t,h) \to 0$ as $h \to 0$. Hence ψ_s can be continuously extended to

$$K_s^o = \{(t,h):\ s \le t \le \alpha_s,\ 0 \le h \le 1-2\eta\}\ ,$$

which is compact. It follows from the last two remarks that $\psi_s(t,h)$ is bounded by a constant $\sqrt{M_\eta^3}$ which is independent of s, t, and h. Thus, the term in (3.30) is smaller than $M_\eta^3\left(C^*_{h_1}\xi(s)\right)^2$. This finally yields, for the second term of (3.28):

$$\begin{aligned}\int_0^1\left(\int_{h_1}^{h2}\int_s^1 C_h'(t,s)\,\mathrm{d}\xi(t)\,\mathrm{d}h\right)^2\mathrm{d}s &\le 8(e\ \eta)^{-2}\tilde{M}_\eta^2(h_2-h_1)^2\|\xi\|^2_{H^*_{h_1}}\\ &\quad + M_\eta^3\left((h_2-h_1)L(h_2-h_1)\right)^2\|\xi\|^2_{H^*_{h_1}}\ . \qquad (3.32)\end{aligned}$$

The same technique leads to the following bounds for D_h': first,

$$\begin{aligned}\int_0^1\left(\int_s^1(\log s)D_h(t,s)\,\mathrm{d}\xi(t)\right)^2\mathrm{d}s &= \int_0^1\left(\int_s^1(\log s)s^\nu\ .s^{-\nu}D_h(t,s)\,\mathrm{d}\xi(t)\right)^2\mathrm{d}s\\ &\le \int_0^1(\log s)^2 s^{2\nu}\left(\int_s^1 D_{h+\nu}(t,s)\,\mathrm{d}\xi(t)\right)^2\mathrm{d}s\\ &\le (e\ \nu)^{-2}\int_0^1\left(\int_s^1 D_{h+\nu}(t,s)\,\mathrm{d}\xi(t)\right)^2\mathrm{d}s\ .\end{aligned}$$

Then,

$$\begin{aligned}&\int_0^1\left(s^{1/2-h}\int_s^1\int_s^t(\log u)u^{h-3/2}\log(u-s)(u-s)^{h-1/2}\,\mathrm{d}u\,\mathrm{d}\xi(t)\right)^2\mathrm{d}s\\ &\le \int_0^1\left(s^{1/2-(h-\nu)}s^\nu\int_s^1\int_s^t(\log u)u^\nu u^{h-\nu-3/2}\times\right.\\ &\qquad\left.\log(u-s)(u-s)^\nu(u-s)^{h-\nu-1/2}\,\mathrm{d}u\,\mathrm{d}\xi(t)\right)^2\mathrm{d}s\\ &\le (e\ \nu)^{-2}\int_0^1\left(s^{1/2-(h-\nu)}\int_s^1\int_s^t u^{h-\nu-3/2}(u-s)^{h-\nu-1/2}\,\mathrm{d}u\,\mathrm{d}\xi(t)\right)^2\mathrm{d}s\\ &= (e\ \nu)^{-2}\int_0^1\left(\int_s^1 D_{h-\nu}(t,s)\,\mathrm{d}\xi(t)\right)^2\mathrm{d}s\ .\end{aligned}$$

So that, for $\nu = \eta/2$, the estimates on D_h in the proof of Lemma 3.26 imply again:

$$\int_0^1\left(\int_s^1 D_h'(t,s)\,\mathrm{d}\xi(t)\right)^2\mathrm{d}s \le 16(e\ \eta)^{-2}\tilde{M}_\eta^4\|\xi\|^2_{H^*_{h_1}}\ . \qquad (3.33)$$

All three inequalities (3.29),(3.32) and (3.33), put together with a bound on the last term of (3.28) (which is easily obtained), end the proof. □

3.6.2 *Proof of Lemma 3.8*

Proof. The proof is divided into two cases, depending on whether $h = 1/2$ or not. The case $h = 1/2$ is immediate since $k_{1/2}(f,g) = \int f\, g\; dm$. Then

$$\forall g \in L^2,\ \lambda_1 k_{1/2}(f_1, g) + \cdots + \lambda_n k_{1/2}(f_n, g) = 0 \Rightarrow \lambda_1 f_1 + \cdots + \lambda_n f_n = 0$$

and this yields $\lambda_1 = \cdots = \lambda_n = 0$ because $(f_1, \ldots, f_n)$ was assumed to be linearly independent.

In the remaining of this proof, $h \in (0, 1/2)$. At first we look at the situation when $n = 2$, and the proof is led in two steps, depending on whether $m(f_1^2) = m(f_2^2)$ or not.

Assume first that $m(f_1^2) \neq m(f_2^2)$. Using fractional integration as a linear operator over the indicator functions of the form $\mathbf{1}_{[0,t]}$ straightforwardly implies that for $s \neq t \in [0,1]$, $(R_h(\cdot,t), R_h(\cdot,s))$ is linearly independent. This technique extends to $n \geq 2$ and we shall use it later. Our problem in $L^2(T,m)$ reduces to the aforementionned one via the following trick: let $g \in L^2(T,m)$ be non-zero and orthogonal to f_1 and f_2. Then, for any $\lambda \in \mathbb{R}$:

$$\begin{aligned} k_h(f_1, \lambda g) &= \frac{1}{2}\left(m(f_1^2)^{2h} + \lambda^{4h} m(g^2)^{2h} - |m(f_1^2) - \lambda^2 m(g^2)|^{2h}\right) \\ &= R_h(t, u_\lambda), \end{aligned}$$

where $t = m(f_1^2)$ and $u_\lambda = \lambda^2 m(g^2)$. Let $s = m(f_2^2)$ which is different from t by hypothesis, then the linear independence of $(\lambda \mapsto R_h(t, u_\lambda), \lambda \mapsto R_h(s, u_\lambda))$ implies the linear independence of $(k_h(f_1, .), k_h(f_2, .))$ in $H(k_h)$.

Assume now we are in the case of f_1 and f_2 having the same norm ($\neq 0$) and that $k_h(\cdot, f_1)$ and $k_h(\cdot, f_2)$ satisfy: there is $\lambda \in \mathbb{R}$ such that $k_h(\cdot, f_1) = \lambda k_h(\cdot, f_2)$, ie $\forall g \in L^2(T,m)$,

$$m(f_1^2)^{2h} - \lambda m(f_2^2)^{2h} = (\lambda - 1) m(g^2)^{2h} + m(|f_1 - g|^2)^{2h} - \lambda m(|f_2 - g|^2)^{2h}. \tag{3.34}$$

Applying this equality to $g = f$, λ has to be:

$$\lambda\ k_h(f_1, f_2) = 2m(f_1^2)^{2h}\ ,$$

and identically with $g = f_2$, one obtains:

$$\lambda\ m(f_2^2)^{2h} = \frac{1}{2} k_h(f_1, f_2).$$

Thus $\lambda^2 = 1$. If $\lambda = 1$, this is $m(|f_1 - g|^2)^{2h} = m(|f_2 - g|^2)^{2h}, \forall g \in L^2$, and we deduce that $f_1 = f_2$. Let us prove that $\lambda = -1$ is impossible. Let us consider equation (3.34) applied to any g which is orthogonal to f_1 and f_2 and such that $m(g^2) = m(f_1^2)$:

$$\begin{aligned} 4m(f_1^2)^{2h} &= m(|f_1 - g|^2)^{2h} + m(|f_2 - g|^2)^{2h} \\ &= \left(m(f_1^2) + m(g^2)\right)^{2h} + \left(m(f_2^2) + m(g^2)\right)^{2h} \\ &= 2^{2h+1} m(f_1^2)^{2h}, \end{aligned}$$

which is impossible whenever $h \neq 1/2$.

In a second step, we extend the result for $n \geq 2$: let $f_1, \dots, f_{n+1} \in L^2$ and assume that $k_h(\cdot, f_{n+1})$ is a linear combination of the family $k_h(\cdot, f_1), \dots, k_h(\cdot, f_n)$. The coefficient in this linear combination are denoted (λ_n). Splitting the maps $f_1, \dots, f_n$ into several groups inside which they have the same norm, we index them differently: $f_{1,1}, \dots, f_{1,i_1}, \dots, f_{l,1}, \dots, f_{l,i_l}$ where for all $j \in \{1, \dots, l\}$, and all $p, q \in \{1, \dots, i_j\}$, $m(f_{j,p}^2) = m(f_{j,q}^2)$. Then, let $g \in L^2$ be orthogonal to $\operatorname{span}\{f_1, \dots, f_{n+1}\}$. We already computed that $k_h(f_i, g) = R_h(m(f_i^2), m(g^2))$. The linear combination is expressed, for all $\mu \in \mathbb{R}$, as follows:

$$k_h(f_{n+1}, \mu g) = \sum_{j=1}^{l} \sum_{k=1}^{i_j} \lambda_{j,k} k_h(f_{j,k}, \mu . g),$$

which is better understood in terms of R_h:

$$\begin{aligned} R_h\left(m(f_{n+1}^2), \mu^2 m(g^2)\right) &= \sum_{j=1}^{l} \sum_{k=1}^{i_j} \lambda_{j,k} R_h\left(m(f_{j,k}^2), \mu^2 m(g^2)\right) \\ &= \sum_{j=1}^{l} \left(\sum_{k=1}^{i_j} \lambda_{j,k} \right) R_h\left(m(f_{j,1}^2), \mu^2 m(g^2)\right). \end{aligned}$$

The linear independence for R_h thus commands that $m(f_{n+1}^2)$ be equal to $m(f_{j,1}^2)$ for some $j \in \{1, \dots, l\}$. We will assume, without restriction, that $j = 1$. It is then necessary that $\sum_{k=1}^{i_1} \lambda_{1,k} = 1$ and that for all $j > 1$, $\sum_{k=1}^{i_j} \lambda_{j,k} = 0$. In case $i_1 < n$, an induction on n ends the proof. Otherwise, the situation is that $m(f_1^2) = \cdots = m(f_{n+1}^2)$ and for all $g \in L^2$:

$$m\left((f_{n+1} - g)^2\right)^{2h} = \sum_{i=1}^{n} \lambda_i m\left((f_i - g)^2\right)^{2h}.$$

Because f_{n+1} is linearly independent of $f_1, \dots, f_n$, there exists g orthogonal to every $f_i, i \leq n$ but which is not orthogonal to f_{n+1}. Then, the previous equation reads:

$$\left(m(f_{n+1}^2) + m(g^2) - 2m(g\ f_{n+1})\right)^{2h} = \left(m(f_1^2) + m(g^2)\right)^{2h},$$

which is impossible due to the fact that $m(g\ f_{n+1}) \neq 0$. □

Miscellaneous properties of the L^2-indexed fractional Brownian motion

4

Contents

In this chapter, we give further results on the L^2-fractional Brownian motion (h fixed) and on the L^2-multifractional Brownian motion ($\boldsymbol{h}$ is a function of L^2, see Section 3.5 for a description of these processes). Except for the property of local nondeterminism and the estimate of the small balls which will be used in the next chapter, the results of this section are largely independent of the other chapters. They all have in common the fact that they extend properties of the standard (multi)fractional Brownian motion in a natural way.

In the introductory chapter, it was recalled that the fractional Brownian motion of Hurst parameter $H \in (0,1)$ is the only (up to normalisation of its variance) Gaussian process on $\mathbb{R}$ that has stationary increments and self-similarity of order H. The Lévy fractional Brownian motion of order H, whose covariance is given by:

$$\mathbb{E}\left(X_s^H X_t^H\right) = \frac{1}{2}\left(\|s\|^{2H} + \|t\|^{2H} - \|s-t\|^{2H}\right) , \quad s,t \in \mathbb{R}^N,$$

is self-similar of order H and has a strong increment stationarity property on $\mathbb{R}^N$, i.e. against translations and rotations in $\mathbb{R}^N$:

$$\forall g \in \mathscr{G}(\mathbb{R}^N), \quad \{X_{g(t)} - X_{g(0)},\ t \in \mathbb{R}^N\} \stackrel{(d)}{=} \{X_t,\ t \in \mathbb{R}^N\} ,$$

where $\mathscr{G}(\mathbb{R}^N)$ is the group of rigid motions of $\mathbb{R}^N$. Reciprocically, a Gaussian process having these properties has the above covariance (see the monograph of Samorodnitsky and Taqqu [127, p.393]). There is no such characterization for the fractional Brownian sheet (see the review by Herbin and Merzbach [61]), and a tentative characterization for the set-indexed fBm appeared in [62]. With the notions of self-similarity and increment stationarity introduced in [62], we extend this characterization to the L^2-fBm.

In section 4.2, we establish a sharp estimate on the small balls of the L^2-fBm, in terms of metric entropy (Theorem 4.6). This is a natural extension of a result due to Monrad and

ROOTZÉN [106] for the fBm, and TALAGRAND [142] for the Lévy fBm. While doing so, a local nondeterminism property of this process is proved, similar to the one originally established by PITT [117] in the 1970's.

Then we give various approximation results of the fractional Brownian field by L^2-fractional Brownian motion. If we draw a path on the fractional Brownian field by means of $\boldsymbol{h} : K \to (0,1/2]$ (K a compact of L^2), the result is a L^2-multifractional Brownian motion, which is continuous if $\boldsymbol{h}$ is continuous and if the finiteness of the Dudley integral in Theorem 3.19 is ensured. On a partition $\{K_n\}_{n=1,\dots,N}$ of K, we put on each K_n a L^2-fBm coming from the original fBf by fixing the value of the parameter h close to $\boldsymbol{h}$ on K_n, and we prove that this construction approximates uniformly almost surely the L^2-mBm. In [88], a similar result on the $\mathbb{R}$-indexed fractional Brownian field was used to provide a stochastic integral with respect to the multifractional Brownian motion. Finally we prove another approximation result (which is also known in the multiparameter framework, see [57]), known as local self-similarity. This phenomenon was described as soon as the multifractional Brownian motion was invented (see [116, 20]) and accounts for the loss of self-similarity of this process. Instead, it is possible to define a tangent process at each point t_0 where the mBm is defined, whose self-similarity parameter will be $\boldsymbol{h}(t_0)$, if $\boldsymbol{h}$ (the regularity function of the mBm) is not too irregular. Similarly, we will exhibit the fact that a L^2-mBm locally looks like a L^2-fBm.

The last section of this chapter is devoted to another key feature of set-indexed processes: in [60], it was proved that the projection of the SIfBm on any flow (that is, increasing paths on $\mathscr{A}$) is a time-changed ($\mathbb{R}$-indexed) fractional Brownian motion. We will show that this is still true when h can fluctuate on the indexing collection.
In all this chapter, $\|\cdot\|$ will denote the L^2-norm.

4.1 Stationarity and self-similarity characterisation

We give two characterizations of the L^2-fBm: the first one is very similar to the characterization of the Lévy fBm, while the second one uses a notion of stationarity similar to the one defined for set-indexed processes in [62]. As an exception, we use in this section the parametrization by $H \in (0,1)$ for consistency with the multiparameter framework [1].

We start with the definitions. We will need the following group of transformations on L^2: consider the set $\mathscr{G}$ which is a restriction of the general linear group of L^2 to bounded linear mappings $\varphi : L^2 \to L^2$ such that:

$$\forall f,g \in L^2(T,m), \quad \|f\| = \|g\| \text{ implies that } \|\varphi(f)\| = \|\varphi(g)\| .$$

Let $\varrho : \mathscr{G} \to \mathbb{R}_+$ be the square of the operator norm and notice that for $\varphi \in \mathscr{G}$, $\|\varphi(f)\| = \sqrt{\varrho(\varphi)}\,\|f\|$ and that ϱ is a group morphism.

We will say that a L^2-indexed stochastic process X:

- is H–self-similar, if:

$$\forall a>0, \quad \{a^{-H}X_{af},\ f\in L^2\} \overset{(d)}{=} \{X_f,\ f\in L^2\}\ ; \tag{SS1}$$

- is strongly H–self-similar, if:

$$\forall \varphi \in \mathscr{G}, \quad \{X_{\varphi(f)},\ f\in L^2\} \overset{(d)}{=} \{\varrho(\varphi)^H X_f,\ f \in L^2\}\ ; \tag{SS2}$$

[1] instead of $h \in (0,1/2)$, hence recall that $H = 2h$.

- is increment stationary in the strong sense, if for any ψ a translation or an orthogonal transformation of L^2:

$$\{X_{\psi(f)} - X_{\psi(0)},\ f \in L^2\} \stackrel{(d)}{=} \{X_f - X_0,\ f \in L^2\}\ ; \tag{SI1}$$

- is m–increment stationary, if for any $f_1, \ldots, f_n \in L^2$ and $g_0, g_1, \ldots, g_n \in L^2$ such that $(f_1 - g_0, \ldots, f_n - g_0)$ and $(g_1, \ldots, g_n)$ are isometric:

$$\left(X_{f_1} - X_{g_0}, \ldots, X_{f_n} - X_{g_0}\right) \stackrel{(d)}{=} \left(X_{g_1}, \ldots, X_{g_n}\right)\ . \tag{SI2}$$

One can readily check that the L^2-fractional Brownian motion satisfies all of the above properties. (SS1) and (SI1) are direct analogues of the multiparameter properties presented above. They give a similar characterization:

Proposition 4.1. *Let X be a L^2-indexed Gaussian process which satisfies (SS1) of order $H \in (0,1)$ and (SI1). Then X is a L^2-fBm of Hurst parameter H.*

Proof. This proof follows precisely the one given in [127, p.393], but we write it (with the necessary modifications due to the passage to L^2) for the sake of completeness.
The first step is to prove that X has mean 0. By self-similarity, it is clear that $X_0 = 0$. Let f_0 be a unit vector of L^2, and any $f, g \in L^2$,

$$\begin{aligned}\mathbb{E}\left(X_{f+g} - X_g\right) = \mathbb{E}\left(X_f - X_0\right) = \mathbb{E}(X_f) &= \mathbb{E}\left(X_{\|f\| f_0}\right) \\ &= \|f\|^H\ \mathbb{E}(X_{f_0})\ ,\end{aligned}$$

where the first equality is (SI1) for a translation, the third is (SI1) for an orthogonal transformation mapping f to $\|f\| f_0$, and the last equality is (SS1).
But self-similarity and rotation invariance also yield:

$$\mathbb{E}\left(X_{f+g} - X_g\right) = \left(\|f+g\|^H - \|f\|^H\right)\ \mathbb{E}(X_{f_0})\ .$$

The equality between the last two equations implies that $\mathbb{E}(X_{f_0}) = 0$, and so $\mathbb{E}(X_f) = 0,\ \forall f \in L^2$.
The covariance follows with the same arguments:

$$\mathbb{E}\left(X_f - X_g\right)^2 = \mathbb{E}\left(X_{f-g}\right)^2 = \|f-g\|^{2H}\ \mathbb{E}(X_{f_0})^2\ .$$

The L^2-fBm is called standard if $\mathbb{E}(X_{f_0})^2 = 1$ for any unit vector. □

Properties (SS2) and (SI2) may seem less obvious, but they appeared rather naturally in the context of set-indexed processes. Let us recall a few facts and definitions from [62]. For $\mathcal{A}$ an indexing collection on (T, m), m-stationarity for $\mathcal{C}_0$-increments is defined for an $\mathcal{A}$-indexed process X as: for any $n \in \mathbb{N}$, any $V \in \mathcal{A}$, and any increasing sequences $(U_i)_{1\le i\le n}$ and $(A_i)_{1\le i\le n}$ of elements of $\mathcal{A}$ satisfying $m(U_i \setminus V) = m(A_i)$ for all $1 \le i \le n$,

$$\left(X_{U_1} - X_{U_1 \cap V}, \ldots, X_{U_n} - X_{U_n \cap V}\right) \stackrel{(d)}{=} \left(X_{A_1}, \ldots, X_{A_n}\right)\ . \tag{$m-\mathcal{C}_0$}$$

Property (SI2) generalizes m-stationarity for $\mathcal{C}_0$-increments, since for increasing sequences $(U_i)_{1\le i\le n}$ and $(A_i)_{1\le i\le n}$ of elements of $\mathcal{A}$ satisfying $m(U_i \setminus V) = m(A_i)$, $(\mathbf{1}_{U_1\setminus V}, \ldots, \mathbf{1}_{U_n\setminus V})$ and $(\mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_n})$

are isometric: for any $k \geq j$,

$$\begin{aligned} m(\mathbf{1}_{U_j \setminus V}\ \mathbf{1}_{U_k \setminus V}) &= m\big((U_j \setminus V) \cap (U_k \setminus V)\big) \\ &= m(U_j \setminus V) \\ &= m(A_j) \\ &= m(\mathbf{1}_{A_j}\ \mathbf{1}_{A_k}). \end{aligned}$$

The property (SS2) is also a clear generalization of the self-similarity proposed in [62].

Remark 4.2. *In [62], self-similarity and m-stationary $\mathscr{C}_0$-increments is proved to characterize the law of the projections on elementary flows of a set-indexed Gaussian process (recall the definition of flows 2.27), but this does not extend to simple flows. Since we know that the law of a set-indexed process is characterized by its projections on simple flows (see for instance Lemma 6 of [68], but knowing the law on elementary flows is not sufficient), ($\mathrm{m}-\mathscr{C}_0$) is in fact too weak to characterize the law of the set-indexed fractional Brownian motion.*

Proposition 4.3. *Let X be a L^2-indexed Gaussian process. X is a L^2-fractional Brownian motion of parameter $H \in (0,1)$ if and only if X satisfies (SI2) and (SS2) of order H.*

Proof. We first prove that X is a mean zero process. Let $f_0 \in L^2$ be a unit vector, and for any $f, g \in L^2$ we have:

$$\mathbb{E}\left(X_{f+g} - X_g\right) = \mathbb{E}\left(\varrho(\varphi_1)^H X_{f_0} - \varrho(\varphi_2)^H X_{f_0}\right)$$

where $\varphi_1, \varphi_2 \in \mathscr{G}$ are such that $f+g = \varphi_1(f_0)$ and $g = \varphi_2(f_0)$. We also have, by (SI2), that:

$$\mathbb{E}\left(X_{f+g} - X_g\right) = \mathbb{E}\left(X_f\right) = \varrho(\varphi_3)^H \mathbb{E}(X_{f_0})$$

where $\varphi_3 \in \mathscr{G}$ is such that $f = \varphi_3(f_0)$. We know by definition of ϱ that $\varrho(\varphi_1) = \|f+g\|^2$, $\varrho(\varphi_2) = \|g\|^2$ and $\varrho(\varphi_3) = \|f\|^2$. Hence, the equality between the last two equations implies that:

$$\left(\|f+g\|^{2H} - \|g\|^{2H}\right)\mathbb{E}\left(X_{f_0}\right) = \|f\|^{2H}\ \mathbb{E}\left(X_{f_0}\right)\ .$$

Since this is true for any $f, g \in L^2$, we must have $\mathbb{E}(X_{f_0}) = 0$, and so $\mathbb{E}(X_f) = 0$, $\forall f \in L^2$.

To obtain the covariance, just notice by using (SI2) and (SS2) in the same fashion that:

$$\mathbb{E}\left((X_f - X_g)^2\right) = \|f-g\|^{2H}\ \frac{\mathbb{E}\left(X_{f_0}^2\right)}{\|f_0\|^{2H}} = \|f-g\|^{2H}\ \mathbb{E}\left(X_{f_0}^2\right)\ .$$

Therefore,

$$\begin{aligned} \mathbb{E}\left(X_f\ X_g\right) &= \frac{1}{2}\left(\mathbb{E}(X_f^2) + \mathbb{E}(X_g^2) - \mathbb{E}\left((X_f - X_g)^2\right)\right) \\ &= \frac{1}{2}\mathbb{E}\left(X_{f_0}^2\right)\left(\|f\|^{2H} + \|g\|^{2H} - \|f-g\|^{2H}\right) \\ &\left(= \frac{1}{2}\mathbb{E}\left(X_{f_0}^2\right)\left(m(f^2)^{2h} + m(g^2)^{2h} - m\left((f-g)^2\right)^{2h}\right)\right)\ . \end{aligned}$$

Finally, stationarity implies that $\mathbb{E}\left(X_{f_0}^2\right) = \mathbb{E}\left(X_{g_0}^2\right)$ for any g_0 of norm 1. The process with $\mathbb{E}\left(X_{f_0}^2\right) = 1$ is sometimes called *standard*. In fact, apart from these characterization results, the L^2-fBm we consider are always assumed to be standard. □

4.2 Local nondeterminism and small balls

We study the link between the h-fBm and metric entropy, providing an estimate of the small deviations of this process. In Theorem 4.6, a connection between the small deviations of the h-fBm and metric entropy is expressed, opening the field of the measure of local properties of the fBf in the next chapter, such as Chung laws of the iterated logarithm or measure of Hausdorff dimension of the paths.

Perhaps the most general result on entropy and small ball probabilities over Wiener spaces is due to GOODMAN [54] who showed that, for K_μ the unit ball of the RKHS of μ,

$$\lim_{\varepsilon\to 0} \varepsilon^2 H(K_\mu,\varepsilon) = 0 ,$$

where $H(K_\mu,\varepsilon)$ is the log-entropy computed under the Banach norm (which makes K_μ compact in the Banach space).
KUELBS AND LI [84] considerably refined this equality, establishing a link between the small balls of a Gaussian measure and the metric entropy of K_μ. To state it, let us introduce some notation: as $x \to a$, we shall write $f(x) \asymp g(x)$ if

$$0 < \liminf_{x\to a} f(x)/g(x) \le \limsup_{x\to a} f(x)/g(x) < \infty .$$

Let us assume that there exists a function g such that $g(x^{-1})$ is regularly varying at infinity[2] and:

$$H(K_\mu,\varepsilon) \asymp g(\varepsilon) ,$$

as $\varepsilon \to 0$. Since g is regularly varying, there exists $\beta \ge 0$ and a slowly varying function at infinity[3] J such that $g(\varepsilon) = \varepsilon^{-\beta} J(\varepsilon^{-1})$. If $0 < \beta < 2$ and $B_E(0,\varepsilon)$ denotes the ball of E centred at 0 with radius ε, then:

$$-\log \mu\left(B_E(0,\varepsilon)\right) \asymp \varepsilon^{-2\beta/(2-\beta)} J\left(\varepsilon^{-1}\right)^{2/(2-\beta)} ,$$

which means that if we were able to compute $H(K_\mu,\varepsilon)$, we could obtain the small deviations of the fractional Brownian motion. In [84], it is mentioned that for K_μ the unit ball of the RKHS of the fBm (that we identified in Chapter 3), computing $H(K_\mu,\varepsilon)$ is not trivial (this has not been done, except if $h = 1/2$). Instead, we can recover $H(K_\mu,\varepsilon)$ from the small deviations (which will not be needed), since we know that if B^h is a fBm, then:

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |B^h_t| \le \varepsilon\right) \asymp \varepsilon^{-1/h}, \quad \forall\, 0 < \varepsilon < 1. \tag{4.1}$$

This small deviation result was obtained in [106] with a probabilistic method. In Appendix C, we explain how to obtain this result without probability and using the tools of Kuelbs and Li.

A multiparameter version (for the Lévy fBm) of (4.1) was obtained independently in [132, 142]. We will generalise this result for the h-fBm. The difference between our result and the result of [84] is then discussed in Remark 4.7. The following lemma will be needed. It is interesting in itself, since it establishes that for each $h \in (0,1/2)$, the h-fBm is *strongly locally nondeterministic* (SLND) in the following sense:

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[2] i.e. a function such that there exists $\rho \in \mathbb{R}$ satisfying $\lim_{x\to\infty} g(\lambda/x)/g(1/x) = \lambda^\rho$ for all $\lambda > 0$.
[3] a regularly varying function with $\rho = 0$.

Proposition 4.4. *Let $h \in (0,1/2)$. There exists a positive constant C_0 such that for all $f \in L^2(T,m)$ and for all $r \leq \|f\|$, the following holds:*

$$Var\left(\boldsymbol{B}_f^h \mid \boldsymbol{B}_g^h, \|f-g\| \geq r\right) = C_0 r^{2h}.$$

In order to prove this result, let us state the following lemma which will make up for the lack of Fourier transform in E.

Lemma 4.5. *For $\varphi \in L^2(\mu)$, define:*

$$\mathscr{F}\varphi(x^*) = \int_E \cos\langle x^*, x\rangle \ \varphi(x)\ \mu(\ dx)\ ,\ x^* \in E^*.$$

Then for $x^*, \varphi \in E^*$:

$$\mathscr{F}\varphi(x^*) \neq 0 \Leftrightarrow x^* = \lambda\varphi,\ \textit{for some}\ \lambda \in \mathbb{R}\setminus\{0\}.$$

Proof. First assume that $x^*, \varphi \in E^*$ are linearly independent:

$$\begin{aligned} I = \int_E \cos\langle x^*, x\rangle\ \langle \varphi, x\rangle\ \ \mathrm{d}\mu(x) &= \int_{\mathbb{R}^2} \cos(t_1)\ t_2\ \ \mathrm{d}\mu_\Sigma(t_1,t_2) \\ &= \frac{1}{2\pi\sqrt{\det\Sigma}} \int_{\mathbb{R}^2} \cos(t_1)\ t_2\ \exp\left(-\frac{1}{2}\mathbf{t}^T\Sigma^{-1}\mathbf{t}\right)\ \ \mathrm{d}\lambda(t_1,t_2) \end{aligned}$$

where Σ represents the covariance structure between the Gaussian random variables x^* and φ (defined on the probability space $(E, \mathscr{B}(E), \mu)$). Precisely,

$$\Sigma = \begin{pmatrix} \mathbb{E}^\mu\left(\langle x^*,\cdot\rangle^2\right) & \mathbb{E}^\mu\left(\langle x^*,\cdot\rangle\langle\varphi,\cdot\rangle\right) \\ \mathbb{E}^\mu\left(\langle x^*,\cdot\rangle\langle\varphi,\cdot\rangle\right) & \mathbb{E}^\mu\left(\langle\varphi,\cdot\rangle^2\right) \end{pmatrix}$$

By the linear independence hypothesis on x^* and φ, Σ is not degenerated. Up to renormalization, we can consider that the diagonal in Σ is 1. Let γ be the non-diagonal term. Then I reads:

$$\begin{aligned} I &= \frac{1}{2\pi\sqrt{\det\Sigma}} \int_{\mathbb{R}^2} \cos(t_1)\ t_2\ \exp\left(-\frac{1}{2-2\gamma^2}(t_1^2+t_2^2-\gamma t_1 t_2)\right)\ \ \mathrm{d}\lambda(t_1,t_2) \\ &= \frac{1}{2\pi\sqrt{\det\Sigma}}\left(I_{t_1>0,t_2>0} + I_{t_1>0,t_2<0} + I_{t_1<0,t_2>0} + I_{t_1<0,t_2<0}\right) \\ &= \frac{1}{2\pi\sqrt{\det\Sigma}}\left(I_{t_1>0,t_2>0} + I_{t_1>0,t_2<0} - I_{t_1>0,t_2<0} - I_{t_1>0,t_2>0}\right) \\ &= 0. \end{aligned}$$

The converse gives, up to a multiplicative constant, $I = \int_{\mathbb{R}} t \exp\left(it - t^2/2\right)\ \ \mathrm{d}t > 0$, when $\varphi = x^*$. Thus, whenever x^* and φ are linearly dependent, I is non-zero. □

Proof of Proposition 4.4. The proof for $C_0 \geq 0$ relies essentially on the metric structure of the covariance of fBm, from which follow increment stationarity and scale invariance of the process. As such, the proof is the same as for the Lévy fractional Brownian motion in $\mathbb{R}^d$, as it appeared first in Lemma 7.1 of [117]. In his paper, Pitt used Fourier analysis to obtain $C_0 > 0$. Although this tool is finite-dimensional (because the Lebesgue measure is) and despite the non-existence of a standard infinite-dimensional Fourier transform, Gaussian measures provide a natural extension to an infinite-dimensional framework. Using Pitt's arguments, we obtain that for $\tilde{f} = (r/\|f\|)\, f$:

$$\mathrm{Var}\left(\boldsymbol{B}^h_f \mid \boldsymbol{B}^h_g, \|f-g\| \geq r\right) = \mathrm{Var}\left(\boldsymbol{B}^h_{\tilde{f}} \mid \boldsymbol{B}^h_g, \|\tilde{f}-g\| \geq r = \|\tilde{f}\|\right) = C_0 r^{2h}.$$

If C_0 was to be 0, there would exist a sequence of random variables B_n of the form $B_n = \sum_j a_j(n)\boldsymbol{B}^h_{g_j}$, where $\|g_j - \tilde{f}\| \geq \|\tilde{f}\|$, such that B_n converges to $\boldsymbol{B}^h_{\tilde{f}}$ in $L^2(\mathbb{P})$. Stated differently, the sequence $b^*_n = \sum_j a_j(n)\langle \tilde{\mathscr{K}}_h k_h(g_j,\cdot),\cdot\rangle$ converges to $b^* = \langle \tilde{\mathscr{K}}_h k_h(\tilde{f},\cdot),\cdot\rangle$ in $L^2(\mu)$. Let $\mathscr{F}_0$ be a restriction of the mapping $\mathscr{F}$ (see the previous lemma) defined by:

$$\mathscr{F}_0\varphi(f_2) = \int_E \cos\left(\langle \tilde{\mathscr{K}}_h k_h(f_2,\cdot), x\rangle\right)\ \varphi(x)\ \mu(\,\mathrm{d}x)\ ,\ f_2 \in L^2(T), \varphi \in L^2(\mu).$$

Lemma 4.5 says that for any fixed $f_1 \in L^2(T)$, $\mathscr{F}_0\left(\tilde{\mathscr{K}}_h k_h(f_1,\cdot)\right)(f_2)$ is non-zero only if $f_2 \in L^2(T)$ is such that, for some $\lambda \in \mathbb{R}\setminus\{0\}$, $\tilde{\mathscr{K}}_h k_h(f_2,\cdot) = \lambda \tilde{\mathscr{K}}_h k_h(f_1,\cdot)$. Hence it is non-zero only if $k_h(f_2,\cdot) = \lambda k_h(f_1,\cdot)$. A by-product of the proof of Lemma 3.8 is that this equality can only hold if $f_1 = f_2$. This implies that the support of $\mathscr{F}_0 b^*_n$ is included in $\{g_j\ ,\ j\in\mathbb{N}\}$ which is strictly disjoint from the support of $\mathscr{F}_0 b^*$.

Applying the Cauchy-Schwarz inequality to $|\mathscr{F}_0 b^*_n(f) - \mathscr{F}_0 b^*(f)|$, one proves that for all $f \in L^2(T)$, $\mathscr{F}_0 b^*_n(f) \to \mathscr{F}_0 b^*(f)$ as n tends to infinity. This is a contradiction with the fact that the supports are strictly disjoint. □

Theorem 4.6. *Let $h \in (0,1/2)$, $\boldsymbol{B}^h$ a h-fractional Brownian motion and K a compact set in $L^2(T,m)$. Then, for some constant $\kappa_1 > 0$,*

$$\mathbb{P}\left(\sup_{f\in K}|\boldsymbol{B}^h_f| \leq \varepsilon\right) \leq \exp\left(-\kappa_1\ N(K,d_h,\varepsilon)\right)\ ,$$

and if there exists ψ such that for any $\varepsilon > 0$, $N(K,d_h,\varepsilon) \leq \psi(\varepsilon)$ and $\psi(\varepsilon) \asymp \psi(\varepsilon/2)$, then for some constant $\kappa_2 > 0$,

$$\mathbb{P}\left(\sup_{f\in K}|\boldsymbol{B}^h_f| \leq \varepsilon\right) \geq \exp\left(-\kappa_2\ \psi(\varepsilon)\right)\ .$$

Proof. The lower bound follows from Lemma 2.2 in [142] and is a general result for Gaussian processes. The upper bound is specific to Lévy-type fractional Brownian motions and is a consequence of the SLND property proved above, and of an argument of conditional expectations as described in [106].

Let $\eta > 0$ and $M(\eta) \subset K$ be a finite set of maximal cardinality, in the sense that for any elements, $f \neq g \in M(\eta) \Rightarrow \|f-g\| \geq \eta^{1/2h}$. The cardinal $|M(\eta)|$ is generally referred to as packing number. The elements of $M(\eta)$ are arbitrarily ordered and denoted $f_1,\ldots,f_{|M(\eta)|}$. Then,

$$\mathbb{P}\left(\sup_{f\in K}|\boldsymbol{B}^h_f| \leq \varepsilon\right) \leq \mathbb{P}\left(\sup_{f\in M(\eta)}|\boldsymbol{B}^h_f| \leq \varepsilon\right)\ ,$$

and since the conditional distributions of a Gaussian process are Gaussian, the SLND property of Lemma 4.4 implies that for any $k \in \{2,\ldots,|M(\eta)|\}$:

$$\mathbb{P}\left(|\boldsymbol{B}^h_{f_k}| \leq \varepsilon \,\middle|\, \boldsymbol{B}^h_{f_j}\ ,\ j \leq k-1\right) = \Phi(C_0^{-1}\eta^{-1}\varepsilon)\ ,$$

where Φ is the cumulative distribution function of a standard normal random variable. By repeated conditioning,

$$\mathbb{P}\left(\sup_{f\in M(\eta)} |\boldsymbol{B}^h_f| \leq \varepsilon\right) \leq \left(\Phi(C_0^{-1}\eta^{-1}\varepsilon)\right)^{|M(\eta)|} .$$

As $N(2\varepsilon) \leq |M(\varepsilon)|$, taking $\eta = \varepsilon/2$ in the previous inequality yields:

$$\mathbb{P}\left(\sup_{f\in K} |\boldsymbol{B}^h_f| \leq \varepsilon\right) \leq \exp(-\kappa_1 N(\varepsilon)) ,$$

with $\kappa_1 = -\log\Phi(2C_0^{-1}) > 0$. □

Estimating the small balls of the fBf (i.e. when h is not fixed anymore) seems more complicated. Talagrand's lower bound estimate still holds, leading to: for K compact in $(0,1/2)\times L^2(T,m)$, $\mathbb{P}(\sup_{(h,f)\in K}|\boldsymbol{B}_{h,f}| \leq \varepsilon) \geq \exp(-\kappa_2\, N(K,d_{\boldsymbol{B}},\varepsilon))$. A sharp estimate of this last entropy in terms of the entropy on both coordinates would be required, while for the upper bound, the notion of SLND for the fBf does not seem appropriate: intuitively, the regularity in the h-direction contrasts with the nondeterminism studied above.

Remark 4.7. *Theorem 4.6 is rather different from what is obtained via [84]. Indeed, their result is concerned with the supremum of the process over the elements of the RKHS measured with the Banach norm:*

$$\mathbb{P}\left(\sup_{\varphi\in B_{E_h^*}(0,1)} |\boldsymbol{B}^h_\varphi| \leq \varepsilon\right) = \mathbb{P}\left(\|\boldsymbol{B}^h\|_{E_h} \leq \varepsilon\right) = \mu_h\left(B_{E_h}(0,\varepsilon)\right)$$

and $\mu_h\left(B_{E_h}(0,\varepsilon)\right) = \mathscr{W}_h\left(B_W(0,\varepsilon)\right)$ by isometry. Meanwhile, it comes from (4.1) that:

$$\mathscr{W}_h\left(B_W(0,\varepsilon)\right) = \mathbb{P}\left(\sup_{t\in[0,1]} |\boldsymbol{B}^h_t| \leq \varepsilon\right) \asymp \exp\left\{-\frac{1}{\varepsilon^{1/h}}\right\} ,$$

which in general is different from our bound. This does not contradict the previous Theorem, because of the difference between the Hilbert norm and the Banach norm.

A remark concerning fractional Brownian sheets. It has been said several times (see 1.2.1) that fractional Brownian sheets could be represented as L^2-fractional Brownian motions. This is indeed true, but let us recall that for $\mathbf{H}\in(0,1/2)^N$, and $\mu_{\mathbf{H}}$ the tensorized Takenaka measure of Section 1.2.1, a fractional Brownian sheet on $\mathbb{R}_+^N$ of parameter $\mathbf{H}$ is the following L^2-indexed process:

$$\mathbb{E}\left(W_t^{\mathbf{H}}\, W_s^{\mathbf{H}}\right) = \frac{1}{2}\left(\mu_{\mathbf{H}}(\mathbf{1}_{V_t}) + \mu_{\mathbf{H}}(\mathbf{1}_{V_s}) - \mu_{\mathbf{H}}\left(|\mathbf{1}_{V_t} - \mathbf{1}_{V_s}|\right)\right) .$$

where V_t is the set defined in 1.2.1. Hence it corresponds to $h = 1/2$, so the previous LND result does not apply here, which agrees with the known fact that fractional Brownian sheets do not have the strong LND property.

Our interpretation for $h = 1/2$ comes from the proof of the local nondeterminism (proof of Proposition 4.4 where Lemma 3.8 fails to apply for $h = 1/2$): it happens exactly the same as for the standard fractional Brownian motion on the line when $H = 1$, some linearity in the indexing parameter appears and the process becomes "more deterministic".

4.3 Approximation of L^2-multifractional Brownian motion by L^2-fBm

For an indexing set on which the fractional Brownian field has a continuous modification, we define a process which is a piecewise L^2-fractional Brownian motion based on a single white noise, and that approximates uniformly almost surely a L^2-multifractional Brownian motion. This extends a Theorem of [88] on the approximation of ($\mathbb{R}$-indexed) multifractional Brownian motion. This result turned out to be of practical importance to build a stochastic calculus with respect to multifractional Brownian motion.

4.3.1 *Approximation in the L^2 space*

Let K be a compact subset of $L^2(T,m)$ with nonempty interior, and call a dissecting class any collection $\{K_j\}_{j=1,\dots,J}$, $J \in \mathbb{N}$ which is a partition of K into disjoint sets having nonempty interior. Let $\boldsymbol{B}$ be a fBf and $\boldsymbol{h} : K \to (0,1/2]$ be a continuous mapping. We let

$$\Big\{\{K_j^n\}_{j=1,\dots,n},\ n \in \mathbb{N}\Big\}$$

be a sequence of dissecting classes such that:

$$\lim_{n\to\infty} \max_{j=1,\dots,n} D_m(K_j^n) = 0 \ , \tag{4.2}$$

where D_m denotes the d_m-diameter of subsets of L^2. For each $n \in \mathbb{N}$, each $j = 1,\dots,n$, let us choose g_j^n an element of $\mathring{K}_j^n$. For any $n \in \mathbb{N}^*$, the following process can be defined:

$$\boldsymbol{B}_f^{\boldsymbol{h},n} = \sum_{j=1}^{n} \boldsymbol{B}_{\boldsymbol{h}(g_j^n),f}\ \mathbf{1}_{\{f\in K_j^n\}}\ , \quad \forall t \in K\ .$$

This is a piecewise L^2-fBm, since on each K_j^n, $\boldsymbol{B}^{h,n}$ has constant Hurst parameter. It provides a good approximation of a L^2-mBm with regularity function $\boldsymbol{h}$, in the sense of the following Proposition. We recall that the notation $d_m(f,g)$ refers to $\|f-g\|_{L^2(T,m)}$.

Theorem 4.8. *Let K be a compact subset of $L^2(T,m)$ and let $\boldsymbol{B}$ be a fBf on $(0,1/2]\times K$. Let $\{\{K_j^n\}_{j=1,\dots,n},\ n\in\mathbb{N}\}$ be a sequence of dissecting classes satisfying (4.2). Let $\eta>0$ and assume that $\boldsymbol{h} : K \to [\eta, 1/2-\eta]$ is a continuous function. If $\iota = \inf_K \boldsymbol{h}(f)$ $(\geq \eta)$, and K has finite Dudley integral:*

$$\int_0^\infty \sqrt{\log N(K, d_m^{2\iota}, \varepsilon)}\, d\varepsilon < \infty\ ,$$

then $\boldsymbol{B}^{h,n}$ converges almost surely uniformly on K and in $L^2(\Omega)$ towards the L^2-mBm $\boldsymbol{B}^{\boldsymbol{h}}$.

Proof. By Theorem 3.19, the convergence of Dudley's integral suffices to ensure that there exists Ω' a measurable subset of Ω of probability 1, such that for all $\omega \in \Omega'$, $\mathbf{B}$ has a modification which is continuous on $[\eta,1/2]\times K$, and thus uniformly continuous. Without restriction, we assume that $\boldsymbol{B}$ is this modification. Let $\epsilon > 0$, and for all $\omega \in \Omega'$, there exists δ such that, for any $h, h' \in [\eta, 1/2]$, any $f, g \in K$,

$$\big(|h-h'| \vee d_m(f,g)\big) \leq \delta \Rightarrow |\boldsymbol{B}_{h,f}(\omega) - \boldsymbol{B}_{h',g}(\omega)| \leq \epsilon\ . \tag{4.3}$$

Then, notice that:

$$\sup_{f\in K}|\boldsymbol{B}^{\boldsymbol{h},n}_f - \boldsymbol{B}^{\boldsymbol{h}}_f| = \max_{j=1,\dots,n}\sup_{f\in K^n_j}|\boldsymbol{B}_{\boldsymbol{h}(g^n_j),f} - \boldsymbol{B}_{\boldsymbol{h}(f),f}| \ . \tag{4.4}$$

Let us fix $\omega\in\Omega'$ and δ as above. In the uniform continuity of $\boldsymbol{h}$ over K, let δ' be such that for $f,g\in K$, $d_m(f,g)<\delta'$ implies that $|\boldsymbol{h}(f)-\boldsymbol{h}(g)|\leq\delta$. Then, there is N large enough (depending on ω) such that for any $n\geq N$,

$$\sup_{j=1,\dots,n} D_m(K^n_j)\leq\delta'$$

which now implies that:

$$\sup_{f,g\in K^n_j}|\boldsymbol{h}(f)-\boldsymbol{h}(g)|\leq\delta \ .$$

Finally, for $n\geq N$, Equations (4.3) and (4.4) give:

$$\sup_{f\in K}|\boldsymbol{B}^{\boldsymbol{h},n}_f - \boldsymbol{B}^{\boldsymbol{h}}_f|\leq\epsilon \ .$$

This proves the almost sure uniform convergence:

$$\forall\omega\in\Omega',\quad \lim_{n\to\infty}\sup_{f\in K}|\boldsymbol{B}^{\boldsymbol{h},n}_f(\omega)-\boldsymbol{B}^{\boldsymbol{h}}_f(\omega)|=0 \ .$$

The $L^2(\Omega)$ convergence can be proved like this:

$$\mathbb{E}\left(\boldsymbol{B}^{\boldsymbol{h},n}_f-\boldsymbol{B}^{\boldsymbol{h}}_f\right)^2\leq C_{\eta,K}(\boldsymbol{h}(g^n_j)-\boldsymbol{h}(f))^2$$

where j is determined by $f\in K^j_n$, and we applied the inequality given by Theorem 3.15. We already mentioned that $D_m(K^n_j)\to 0$ as $n\to\infty$, so the fact that $d_m(g^n_j,f)\leq D_m(K^n_j)$ gives the $L^2(\Omega)$ convergence. □

4.3.2 *Set-indexed version*

This section provides a nice application of the previous result to the framework of set-indexed processes. Here we simply describe a natural dissecting class and show how a set-indexed multifractional Brownian motion (SImBm) is approximated by a piecewise SIfBm.

Let $\mathscr{A}$ be an indexing collection on (T,m), endowed with the pseudo-distance d_m. Let $\mathscr{A}'$ be a compact sub-indexing collection of $\mathscr{A}$. We may also work directly with $\mathscr{A}$ if $\{\mathbf{1}_A, A\in\mathscr{A}\}$ is compact in $L^2(T,m)$. We recall the *separability from above* condition, as it appears in Definition 2.1. We assume that there exists an increasing sequence of finite subclasses of $\mathscr{A}$, denoted by $\mathscr{A}_n=\{\emptyset,U^n_1,\dots,U^n_{k_n}\}$ for any $n\in\mathbb{N}$. These classes are closed under intersections and we recall the approximating functions $g_n:\mathscr{A}\to\mathscr{A}_n$ defined by

$$\forall U\in\mathscr{A},\quad g_n(U)=\bigcap_{\substack{V\in\mathscr{A}_n\\ V^\circ\supseteq U}} V \ .$$

The dissecting classes are given by $C^n_k=\{U\in\mathscr{A}':g_n(U)=U^n_k\}$ for $n\in\mathbb{N}$ and $k=1,\dots,k_n$. Condition (4.2) is satisfied if we assume for instance that Assumption (H1) from Chapter 2 is satisfied (this is really stronger than (4.2)). In fact, considering the general definition of IVANOFF AND MERZBACH given in [70], (4.2) is satisfied because it is assumed that $\cap_{n\in\mathbb{N}}g_n(U)=$

$U, \forall U \in \mathscr{A}$. According to Remark 2.13, Assumption (H1) also ensures that the Dudley integral of Theorem 4.8 converges.

In Section 2.6, we have made the link between $\mathscr{A}$-indexed and $L^2(T,m)$-indexed processes. We recall that the first one can be defined from the second, when valuated on indicator functions. Thus we conclude by Theorem 4.8 that if $\boldsymbol{B}$ is a fBf on $L^2(T,m)$, if $\boldsymbol{h}$ is a continuous regularity function on $\mathscr{A}'$ with values in $[\eta, 1/2]$ for some $\eta > 0$, the piecewise SIfBm given by:

$$\boldsymbol{B}_U^{h,n} = \boldsymbol{B}_{\boldsymbol{h}(g_n(U)),U} = \sum_{k=1}^{k_n} \boldsymbol{B}_U^{\boldsymbol{h}(U_k^n)} \mathbf{1}_{C_k^n}(U) \;, \quad U \in \mathscr{A}'$$

converges almost surely uniformly on $\mathscr{A}'$ towards the SImBm $\{\boldsymbol{B}_U^{\boldsymbol{h}}, U \in \mathscr{A}'\}$.

4.3.3 *Local self-similarity*

In the definition of the fractional Brownian field, authorizing the Hurst parameter to vary entails the loss of self-similarity and L^2-increment stationarity. We prove here that, as for the mBm [116, 20] and its multiparameter extensions [57], these properties appear in a weak form, via a tangent process Y. Let $\boldsymbol{h}$ be a continuous regularity function with values in $(0, 1/2]$ and assume that the local and pointwise Hölder exponents of $\boldsymbol{h}$ coincide and that $\boldsymbol{h}(f) < \boldsymbol{\alpha}_{\boldsymbol{h}}(f)$. Here, $\boldsymbol{\alpha}_{\boldsymbol{h}}(f)$ is the exponent of local Hölder regularity with respect to the distance d'_m (i.e. the square of the $L^2(T,m)$ distance, as in Section 3.5). The *irregular* case is probably more difficult since there is no lower bound for the variance of the increments of the fBf, unlike the multiparameter case. Hence we restrict here to $\boldsymbol{h}$ regular. At each $f_0 \in L^2(T,m)$, define for any $\rho > 0$,

$$Y_f^{f_0,\alpha}(\varrho) = \varrho^{-\alpha}\left(\boldsymbol{B}_{f_0+\varrho f}^{\boldsymbol{h}} - \boldsymbol{B}_{f_0}^{\boldsymbol{h}}\right) \;, \quad f \in L^2(T,m) \;.$$

We will be interested in the limit as $\varrho \to 0$.

Lemma 4.9. *Let $\eta > 0$ and $\boldsymbol{h} : L^2 \to [\eta, 1/2 - \eta]$ be smaller than its Hölder regularity (i.e. regular). Then the limit:*

$$\lim_{\varrho \to 0} \mathbb{E}\left(Y_f^{f_0,\alpha}(\varrho) - Y_g^{f_0,\alpha}(\varrho)\right)^2 \;,$$

is finite and positive if and only if $\alpha = 2\boldsymbol{h}(f_0)$. Then, this limit is equal to $\|f - g\|^{4\boldsymbol{h}(f_0)}$, which implies that $\{Y_f^{f_0,\alpha}(\varrho), f \in L^2\}$ converges in finite-dimensional distribution towards a L^2-fBm of parameter $\boldsymbol{h}(f_0)$.

Proof. We first notice that for $f, g \in L^2$,

$$\sqrt{\mathbb{E}\left(\boldsymbol{B}_{\boldsymbol{h}(f),f} - \boldsymbol{B}_{\boldsymbol{h}(f),g}\right)^2} \leq \sqrt{\mathbb{E}\left(\boldsymbol{B}_f^{\boldsymbol{h}} - \boldsymbol{B}_g^{\boldsymbol{h}}\right)^2} + \sqrt{\mathbb{E}\left(\boldsymbol{B}_{\boldsymbol{h}(f),g} - \boldsymbol{B}_{\boldsymbol{h}(g),g}\right)^2} \;.$$

Hence, Theorem 3.15 implies that on a neighbourhood D_0 of f_0,

$$\varrho^{-\alpha+2\boldsymbol{h}(f_0+\varrho f)}\|f-g\|^{2\boldsymbol{h}(f_0+\varrho f)} \leq \sqrt{\mathbb{E}\left(Y_f^{f_0,\alpha}(\varrho) - Y_g^{f_0,\alpha}(\varrho)\right)^2} + \varrho^{-\alpha}\sqrt{C_{\eta,D_0}}|\boldsymbol{h}(f_0+\varrho f) - \boldsymbol{h}(f_0+\varrho g)|$$

Let $\epsilon > 0$ be such that $2\epsilon \leq \alpha_{\boldsymbol{h}}(f_0) - \boldsymbol{h}(f_0)$. If ϱ is small enough, the Hölder continuity of $\boldsymbol{h}$ yields:

$$\begin{aligned}
|\boldsymbol{h}(f_0+\varrho f) - \boldsymbol{h}(f_0+\varrho g)| &\leq d'_m(\varrho f, \varrho g)^{\boldsymbol{\alpha}_h(f_0)-\epsilon} \\
&\leq d'_m(\varrho f, \varrho g)^{\boldsymbol{h}(f_0)+\epsilon} \\
&\leq \varrho^{2\boldsymbol{h}(f_0)+2\epsilon}\|f-g\|^{2\boldsymbol{h}(f_0)+2\epsilon}
\end{aligned}$$

Thus, if we put $\alpha = 2\boldsymbol{h}(f_0)$, it results that:

$$\liminf_{\varrho\to 0} \mathbb{E}\left(Y_f^{f_0,\alpha}(\varrho) - Y_g^{f_0,\alpha}(\varrho)\right)^2 \geq \|f-g\|^{4\boldsymbol{h}(f_0)} .$$

The upper bound is more straightforward, since we can use Corollary 3.16, and the same Hölder estimates on the increments of $\boldsymbol{h}$. Let $\bar{\boldsymbol{h}}_{f_0,D_0}(\varrho) = \inf\{\boldsymbol{h}(f_0+f) : f \in \varrho D_0\}$.

$$\begin{aligned}
\sqrt{\mathbb{E}\left(Y_f^{f_0,\alpha}(\varrho) - Y_g^{f_0,\alpha}(\varrho)\right)^2} &\leq \varrho^{-\alpha}\left(\sqrt{C_{\eta,D_0}}|\boldsymbol{h}(f_0+\varrho f) - \boldsymbol{h}(f_0+\varrho g)| + \|\varrho f - \varrho g\|^{2(\boldsymbol{h}(f_0+\varrho f)\wedge \boldsymbol{h}(f_0+\varrho g))}\right) \\
&\leq \varrho^{-\alpha}\left(\sqrt{C_{\eta,D_0}}\|\varrho f - \varrho g\|^{2\boldsymbol{h}(f_0)+2\epsilon} + \|\varrho f - \varrho g\|^{2\bar{\boldsymbol{h}}_{f_0,D_0}(\varrho)}\right) \\
&\leq \left(1+\varrho^{2\epsilon}\sqrt{C_{\eta,D_0}}\right)\|f-g\|^{2\bar{\boldsymbol{h}}_{f_0,D_0}(\varrho)} , \qquad (4.5)
\end{aligned}$$

where ϵ and ϱ are chosen as above, and in particular, such that $\varrho\ D_m(D_0) \leq 1$ (recall that D_m is the diameter of a set, measured with d_m). The continuity of $\boldsymbol{h}$ implies that $\bar{\boldsymbol{h}}_{f_0,D_0}(\varrho)$ converges to $\boldsymbol{h}(f_0)$ and this finishes the proof. □

Theorem 4.10. *Let K be a compact subset of $L^2(T,m)$. Let $\iota = \inf_K \boldsymbol{h}(f)$ $(\geq \eta)$ and assume the convergence of the Dudley integral on K:*

$$\int_0^\infty \sqrt{\log N(K, d_m^{2\iota}, \varepsilon)}\, d\varepsilon < \infty .$$

Under the conditions of Lemma 4.9 and for $\alpha = 2\boldsymbol{h}(f_0)$, $Y^{f_0,\alpha}(\varrho)$ converges in law in the space of continuous functions $C(K)$ towards a L^2-fBm of parameter $\boldsymbol{h}(f_0)$.

Proof. We will use a general criterion for tightness of random fields (as for instance in KALLENBERG [74, Corollary 14.9]). We shall use the notation

$$\omega_X(K, d_m, \delta) = \sup\left\{|X_f - X_g|,\ f,g \in K, d_m(f,g) \leq \delta\right\}$$

for the modulus of continuity of a process X on K measured with d_m. Then, a sequence of random fields $\{X^n,\ n \in \mathbb{N}\}$ converges in law in $C(K)$ if it converges in finite-dimensional distributions and

$$\lim_{\delta\to 0^+} \limsup_{n\to\infty} \mathbb{E}\left(\omega_{X^n}(K, d_m, \delta) \wedge 1\right) = 0 .$$

The convergence of Dudley integral and Theorem 3.19 allows us to consider an almost sure continuous version of $Y^{f_0,\alpha}(\varrho)$ on K. To make notations simpler, we denote this process $Y(\varrho)$. By Equation (4.5), if ϵ and ϱ are small enough (as in the proof of Lemma 4.9),

$$\begin{aligned}
\omega_{Y(\varrho)}(K, d_m, \delta) &\leq \sup\left\{|Y_f(\varrho) - Y_g(\varrho)|,\ d_{Y(\varrho)}(f,g) \leq (1+\varrho^{2\epsilon}\sqrt{C_{\eta,K}})\ \delta^{2\bar{\boldsymbol{h}}_{f_0,K}(\varrho)}\right\} \\
&\leq \omega_{Y(\varrho)}\left(K, d_{Y(\varrho)}, 2\delta^{2\bar{\boldsymbol{h}}_{f_0,K}(\varrho)}\right) ,
\end{aligned}$$

where we bounded $(1+\varrho^{2\epsilon}\sqrt{C_{\eta,K}})$ by 2 (which is true for ϱ sufficiently close to 0).
From the general theory of Gaussian processes, we know (e.g. refer to [5, Corollary 1.3.4]) that there exists a universal constant $K > 0$ such that:

$$\mathbb{E}\left(\omega_{Y(\varrho)}(K, d_{Y(\varrho)}, \delta)\right) \leq K \int_0^\delta \sqrt{\log N(K, d_{Y(\varrho)}, \varepsilon)}\, \mathrm{d}\varepsilon$$

Hence,

$$\begin{aligned}\mathbb{E}\left(\omega_{Y(\varrho)}(K,d_m,\delta)\wedge 1\right) &\leq \mathbb{E}\left(\omega_{Y(\varrho)}(K,d_{Y(\varrho)},\delta^{2\bar{h}_{f_0,K}(\varrho)})\right)\\ &\leq K\int_0^{2\delta^{2\bar{h}_{f_0,K}(\varrho)}}\sqrt{\log N(K,d_{Y(\varrho)},\varepsilon)}\,\mathrm{d}\varepsilon\\ &\leq K\int_0^{2\delta^{2\bar{h}_{f_0,K}(\varrho)}}\sqrt{\log N\left(K,2d_m^{2\bar{h}_{f_0,K}(\varrho)},\varepsilon\right)}\,\mathrm{d}\varepsilon\\ &\leq 2K\int_0^{\delta^{2\bar{h}_{f_0,K}(\varrho)}}\sqrt{\log N\left(K,d_m^{2\iota},\varepsilon\right)}\,\mathrm{d}\varepsilon\end{aligned}$$

which is a well-defined integral. By continuity of $\boldsymbol{h}$,

$$\limsup_{\varrho\to 0^+}\ \mathbb{E}\left(\omega_{Y(\varrho)}(K,d_m,\delta)\wedge 1\right)\leq 2K\int_0^{\delta^{2\boldsymbol{h}(f_0)}}\sqrt{\log N\left(K,d_m^{2\iota},\varepsilon\right)}\,\mathrm{d}\varepsilon\ ,$$

and the continuity of the integral gives the desired result as $\delta\to 0^+$. □

4.4 Projection on flows of the L^2-multifractional Brownian motion

In this paragraph, the concept of flow (cf Definition 2.27 and [70]) is extended to $L^2(T,m)$. It involves increasing functions with values in L^2, which has to be understood for any function $\Phi:[0,1]\to L^2(T,m)$, as:

$$s\leq t\Rightarrow \Phi(t)-\Phi(s)\geq 0\ m-\text{almost everywhere}.$$

Definition 4.11. *A* flow *is a continuous increasing function from $[0,1]$ to $L^2(T,m)$.*

We see that this definition is even simpler to express on L^2 than it was on an indexing collection. Identically, we have the definition of m–standard projection:

Definition 4.12. *Let Φ be a flow and define the m-standard projection of a process X on $L^2(T,m)$ as:*

$$X^{\Phi}=\left\{X_{\Phi\circ\theta^{-1}(t)}\ ,\ t\in[0,\|\Phi(1)\|]\right\}\ ,$$

where θ^{-1} is the inverse of the continuous and increasing function $\theta: t\mapsto\|\Phi(t)\|$.

Proposition 4.13. *Let $\boldsymbol{B}$ be a fractional Brownian field indexed by $(0,1/2]\times L^2(T,m)$ and $\boldsymbol{h}$: $L^2(t,m)\to(0,1/2]$ a continuous regularity function. For any flow Φ, $\boldsymbol{B}^{\boldsymbol{h},\Phi}$ denotes the m–standard projection along Φ of the L^2-mBm. $\boldsymbol{B}^{\boldsymbol{h},\Phi}$ is a standard multifractional Brownian motion on $I_\Phi=[0,\|\Phi(1)\|]$, with regularity function $\boldsymbol{h}\circ\Phi\circ\theta^{-1}$.*

Proof. Let us choose a flow Φ. In the first place, we consider the process $\boldsymbol{B}^{\Phi}=\{\boldsymbol{B}_{h,\Phi\circ\theta^{-1}(t)},\ (h,t)\in(0,1/2]\times[0,\|\Phi(1)\|]\}$ which is, as we will see, a fractional Brownian field defined on a sub-interval of $\mathbb{R}_+$. Let $s<t\in[0,\|\Phi(1)\|]$,

$$\begin{aligned}\mathbb{E}\left(\boldsymbol{B}^{\Phi}_{h,t}\boldsymbol{B}^{\Phi}_{h,s}\right)&=k_h\left(\Phi\circ\theta^{-1}(t),\Phi\circ\theta^{-1}(s)\right)\\ &=\frac{1}{2}\left(\|\Phi\circ\theta^{-1}(t)\|^{4h}+\|\Phi\circ\theta^{-1}(s)\|^{4h}-\left(\|\Phi\circ\theta^{-1}(t)\|-\|\Phi\circ\theta^{-1}(s)\|\right)^{4h}\right)\\ &=\frac{1}{2}\left(t^{4h}+s^{4h}-(t-s)^{4h}\right)\ ,\end{aligned}$$

where we used the fact that Φ is increasing. Hence for any $h \in (0, 1/2]$, $\boldsymbol{B}^{\Phi}_{h,\cdot}$ is a fractional Brownian motion of parameter $2h$, which confirms our claim that $\boldsymbol{B}^{\Phi}$ is a fBf on I_Φ. If $\boldsymbol{h}$ is a regularity function on L^2, $\boldsymbol{h}' = \boldsymbol{h} \circ \Phi \circ \theta^{-1}$ is a regularity function on I_Φ. Thus, $\boldsymbol{B}_{\boldsymbol{h}'(\cdot),\cdot}$ is a multifractional Brownian motion over I_Φ. □

We proved that $\boldsymbol{B}^{\boldsymbol{h},\Phi}$ is a multifractional Brownian motion in the widest acceptation of this term (as for instance in [88, Definition 1.3] or in our introduction). However, we do not provide evidence that $\boldsymbol{B}^{\boldsymbol{h},\Phi}$ has a more usual moving average or harmonizable representation, or even more generally an integral representation as in [134]. This will be the topic of further investigations, as this would clearly give a strengthened legitimacy to our L^2-fractional Brownian field.

Singularities of the multiparameter fractional Brownian motion 5

Contents

5.1 Introduction

In the 1920's, KHINCHINE introduced for the first time a law of the iterated logarithm for sums of independent and identically distributed random variables. Thereafter, many works extended this result and in particular CHUNG [31] presented a new law of the iterated logarithm for Brownian motion of the *lim inf* type, thus capturing the slowest local oscillations. This law was generalized in numerous ways to Gaussian processes and Gaussian samples. A key step in establishing Chung's laws of the iterated logarithm (that we may hereafter abbreviate as LIL) for Gaussian processes is usually to determine the small ball probabilities, together with good independence properties.

In this chapter, we propose to determine a Chung's law of the iterated logarithm for the multiparameter fractional Brownian motion (multiparameter fBm for short), which is neither increment stationary (see [106, 142, 151] and for a more general theory [103, 96, 97] where this is an hypothesis) nor has an immediate decomposition into independent processes (as for Brownian sheet [141]). We recall that the multiparameter fractional Brownian motion is a centred Gaussian process on $\mathbb{R}^\nu$, $\nu \in \mathbb{N}^*$, with covariance defined for $h \in (0, 1/2]$:

$$k_h^{(\nu)}(s,t) = \frac{1}{2}\left(\lambda([0,s])^{2h} + \lambda([0,t])^{2h} - \lambda([0,s] \triangle [0,t])^{2h}\right) , \tag{5.1}$$

where λ denotes the Lebesgue measure in $\mathbb{R}^\nu$ and $\triangle$ is the symmetric difference of sets. This is a special case of a family of covariance on sets introduced by HERBIN AND MERZBACH [60] to define the set-indexed fractional Brownian motion, and extended to its most general expression in Chapter 3 to functions of $L^2(T,m)$. We also refer to the introduction of this thesis, where this family of covariances is presented in details. Besides, $h = 1/2$ in (5.1) yields the usual Brownian sheet. However, our results hold for $h < 1/2$ and cannot be extended to $h = 1/2$ (see the discussion at the end of Section 4.2).

The (pseudo-)metrics induced by the Lévy fBm X^h and the multiparameter fBm $\boldsymbol{B}^h$, which are defined respectively as $d_{X^h}(s,t) = \sqrt{\mathbb{E}(X^h_s - X^h_t)^2} = \|s-t\|^h$ and $d_{\boldsymbol{B}^h}(s,t) = \lambda([0,s]\triangle[0,t])^h$, are in fact equivalent on a domain of $\mathbb{R}^\nu_+$ that does not approach the axes. Thus, it is expected that these processes will share certain sample path properties, at least away from the axes. This is the purpose of HERBIN AND XIAO [65], where the authors propose a modulus of continuity, laws of the iterated logarithm and the Hausdorff dimension of the level sets of $\boldsymbol{B}^h$. These results coincide with their analogue for the Lévy fractional Brownian motion, but for the law of the iterated logarithm, which is a local result, this is only true away from the axes. The lack of stationarity forbids here to conclude that these modulus are the same whatever the point we choose. We note here that the same happens for the fractional Brownian sheet: its law of the iterated logarithm (not Chung's law) is known away from 0 (cf [103]), but not in the neighbourhood of the origin. If t_0 is not on the axes, the Chung's law of the iterated logarithm given in [65] is:

$$\liminf_{r\to 0^+}\left(\log\log(r^{-1})\right)^{h/\nu} \frac{\sup_{\|t\|\le r} |\boldsymbol{B}^h_{t_0+t} - \boldsymbol{B}^h_{t_0}|}{r^h} = c\ , \quad \text{a.s.}\ ,$$

for some deterministic c that may depend on t_0. Near 0, we will show that the local modulus is in fact of order $r^{\nu h}\,\widetilde{\Psi}_h(r)$, where $\widetilde{\Psi}_h$ is a correcting term (i.e. negligible compared to $r^{\nu h}$). Hence the slowest local oscillations of the multiparameter fractional Brownian motion are of order $r^{\nu h}$, which differs significantly from r^h (as soon as $\nu \ge 2$) and justifies this notion of singularity at the origin. The main new ingredients are a sharp estimate of the small ball probabilities, and a spectral representation in its abstract Wiener space of the multiparameter fBm.

This representation is related to stable measures in Banach spaces: we prove that for H the RKHS of the ν-dimensional Brownian sheet, there exists an abstract Wiener space (H,E,μ) such that for any $h \in (0,1/2)$, there is a strictly stable measure Γ^h on E whose characteristic function is given by $\exp(-1/2\|S\xi\|^{4h}_H)$, $\xi \in E^*$. This measure has a Lévy-Khintchine decomposition, with Lévy measure Δ^h. $\mathscr{I}$ still denotes the Paley-Wiener map, and we let $\mathbb{B}^h$ be the white noise on the Borel sets of E with control measure Δ^h. Then, the multiparameter fractional Brownian motion has the following expression:

$$\boldsymbol{B}^h_t = \int_E \left(1 - e^{i\langle \mathscr{I}(\varphi_t), x\rangle}\right)\, \mathrm{d}\mathbb{B}^h_x\ , \quad t \in [0,1]^\nu\ ,$$

where $\varphi_t(\cdot) = \lambda(\mathbf{1}_{[0,t]}\cdot) \in H$.

We will prove a lower and an upper bound in Chung's LIL. The modulus for the lower bound is given by

$$\Psi^{(\ell)}_h(r) = r^{\nu h}\, \widetilde{\Psi}^{(\ell)}_h(r) = r^{\nu h}\, (\log\log r^{-1})^{-h/\nu}\ ,$$

and the modulus for the upper bound is $\Psi^{(u)}_h = r^{\nu h}\, \widetilde{\Psi}^{(u)}_h$, where $\widetilde{\Psi}^{(u)}_h$ is an increasing function started at 0, whose existence is proven in Section 5.3, and related implicitely to the following decay function of Δ^h:

$$\mathbf{F}(\mathbf{x}) = \sup_{\varphi\in A} \int_{\|x\|_E < \mathbf{x}} \left(1 - \cos\langle \mathscr{I}(\varphi), x\rangle\right)\, \Delta^\alpha(\mathrm{d}x)\ , \tag{5.2}$$

where A is a compact subset of H, that we will define later in this chapter. For every $h \in (0,1/2)$, let us finally define $M^h(r) = \sup_{t\in[0,r]^\nu} |\boldsymbol{B}^h_t|$, $r \in [0,1]$.

Theorem 5.1. *Let $h \in (0,1/2)$ and let M^h, $\Psi_h^{(\ell)}$ and $\Psi_h^{(u)}$ be as above. Then we have almost surely:*

$$\liminf_{r\to 0^+} \frac{M^h(r)}{r^{\nu h}\widetilde{\Psi}_h^{(\ell)}(r)} \geq \kappa_1^{h/\nu} \quad \text{and} \quad \liminf_{r\to 0^+} \frac{M^h(r)}{r^{\nu h}\widetilde{\Psi}_h^{(u)}(r)} \leq \kappa_2^{h/\nu} ,$$

where $\kappa_1 \leq \kappa_2$ are the constants appearing in the small deviations (see Equation (5.3)).

This result is not sharp, as we were unable to give the rate of decay of F. We discuss how this gap could be filled at the end of Section 5.3.

In Strassen [135], while looking for an invariance principle for scaled random walks, the author obtained the fact that the same scaling on a Brownian motion gives a family of processes which is almost surely relatively compact in the unit ball of the Sobolev space H_0^1 of continuous functions started at 0 with square-integrable weak derivative. Functional laws of the iterated logarithm have now been widely studied in the literature: Csáki [32] was the first to get a rate of convergence for certain functions in this unit ball, and this result was extended by de Acosta [38] to scaled random walks, for any function of the unit ball of the RKHS (with radius strictly smaller than 1). Then, we can mention the contributions of Goodman, Grill, Kuelbs, Li, Talagrand, in particular in [55] and [85], where the authors bring a new understanding of the rate of convergence towards the unit sphere in the general frame of Gaussian samples in Banach spaces. Similarly to the standard LIL, the functional result for fractional Brownian motion was also given by Monrad and Rootzén [106]. So for the multiparameter fBm, let us define, for $r \in (0,1)$,

$$\eta_r^{(h,\ell)}(t) = \frac{\boldsymbol{B}^h(rt)}{r^{\nu h}\sqrt{\log\log(r^{-1})}} , \forall t \in [0,1]^\nu$$

and

$$\eta_r^{(h,u)}(t) = \frac{\boldsymbol{B}^h(rt)}{r^{\nu h}\left(\widetilde{\Psi}_h^{(u)}(r)\right)^{-\nu/2h}} , \forall t \in [0,1]^\nu$$

the lower and upper rescaled multiparameter fBm for which we seek an invariance principle.

Theorem 5.2. *Let $h \in (0,1/2)$ and let H_h^ν denote the reproducing kernel Hilbert space of $k_h^{(\nu)}$. Let $\varphi \in H_h^\nu$ having norm strictly smaller than 1. Then, there exist two positive and finite constants $\gamma^{(\ell)}(\varphi)$ and $\gamma^{(u)}(\varphi)$ such that, almost surely,*

$$\liminf_{r\to 0^+} \widetilde{\Psi}_h^{(\ell)}(r)^{-1-\nu/2h} \sup_{t\in[0,1]^\nu} |\eta_r^{(h,\ell)}(t) - \varphi(t)| \geq \gamma^{(\ell)}(\varphi)$$
$$\liminf_{r\to 0^+} \widetilde{\Psi}_h^{(u)}(r)^{-1-\nu/2h} \sup_{t\in[0,1]^\nu} |\eta_r^{(h,u)}(t) - \varphi(t)| \leq \gamma^{(u)}(\varphi) .$$

As usual, taking $f = 0$ yields the standard law of the iterated logarithm.

Observing the techniques of Talagrand [142] and Xiao [151], it appears that for the Lévy fractional Brownian motion and similar processes, Chung's law of the iterated logarithm and the exact Hausdorff measure of the range of these processes could be obtained by means of the same estimates. However, we will prove that the Hausdorff dimension does not grasp the slow oscillations of the multiparameter fBm at the origin. The cause is again the non-increment stationarity.

Outline of this chapter. In section 5.2, we prove preliminary results. The new ideas essentially lie in this section. They concern the small deviations around the origin 0, which are obtained using the local nondeterminism of Chapter 4, and a spectral representation on the Wiener space of this process. We prove Theorem 5.1 in Section 5.3 and Theorem 5.2 in Section 5.4. Hausdorff dimension of the graph is computed in 5.5 and finally, a discussion on possible extensions of this work is included in Section 5.6. The last section gathers a couple of technical results.

5.2 Small balls and spectral representation

5.2.1 *Entropy estimate and small balls*

Let us state the following basic observation on the metric induced by the multiparameter fractional Brownian motion:

Lemma 5.3. *For any $a \in (0,1)$, any $b > a$, there exist $m_{a,b}$ depending on a and b and M_b depending on b only, such that for any $s,t \in [a,b]^\nu$,*

$$m_{a,b}\|s-t\| \leq \lambda([0,s] \triangle [0,t]) \leq M_b\|s-t\|$$

In particular, the upper bound holds even if $s,t \in [0,b]$. However, for any given $\alpha \in (0,1)$, we have that for all $\epsilon > 0$, there exist $s,t \in [0,1]^\nu$ such that $\lambda([0,s] \triangle [0,t]) \leq \varepsilon$ but $\|s-t\| \geq \alpha$.

Proof. The upper and lower bounds on $\lambda([0,s] \triangle [0,t])$ are stated in Lemma 3.1 of [58] (up to equivalence of l^1 and l^∞ distances with the Euclidean distance), except that there, the constant in the upper bound is said to depend also on a. From their proof, it is clear that this is not necessary.
To prove the last statement, let $s_n = (2^{-n}, b, \dots, b) \in [0,b]^\nu$ and $t_n = (b, 2^{-n}, b, \dots, b) \in [0,b]^\nu$. It appears that $\lambda([0,s_n] \triangle [0,t_n]) \to 0$ as $n \to \infty$, while $\|s_n - t_n\|$ increases to $\sqrt{2}b$. □

Concerning notations, we will have to compare several distances, so d_E will denote the Euclidean distance in $\mathbb{R}^\nu$, and for any $h \in (0,1]$, d_h is the following distance:

$$\text{for } s,t \in [0,1]^\nu, \quad d_h(s,t) = \lambda([0,s] \triangle [0,t])^h \ .$$

When $h = 1$, we will prefer the notation d_λ. Note that we will only consider results for $h \leq 1/2$ because of the definition of $\boldsymbol{B}^h$, but d_h is still a distance for $h \in (1/2, 1]$ (but no longer negative definite which prevents the definition of a multiparameter fBm for such values). Accordingly, $B_h(t,r)$ is the ball of d_h-radius r centred at t. If no subscript is written, this will be the Euclidean ball. The notation $\asymp$ between two functions f and g means that near a point a, $f(x) = O(g(x))$ and $g(x) = O(f(x))$. We recall that on a (pre-)compact metric space (T,d), the metric entropy $N(T,d,\varepsilon)$ gives, for any $\varepsilon > 0$, the minimal number of balls of radius ε that are necessary to cover T.

Lemma 5.4. *Let $\nu \in \mathbb{N}$, then the d_λ-metric entropy of $[0,1]^\nu$ is, for ε small enough:*

$$N([0,1]^\nu, d_\lambda, \varepsilon) \asymp \varepsilon^{-\nu} \ .$$

Proof. Let us remark that due to Lemma 5.3, $d_\lambda(s,t) \leq M_1 d_E(s,t)$, for any $s,t \in [0,1]^\nu$. Thus, for any ball one has $B_\lambda(t_0,r) \supseteq B(t_0, M_1^{-1}r)$. We can assert that:

$$N([0,1]^\nu, d_\lambda, \varepsilon) \leq N\left([0,1]^\nu, d_E, M_1^{-1}\varepsilon\right) \asymp \left(M_1^{-1}\varepsilon\right)^{-\nu} \ ,$$

as $\varepsilon \to 0$. Conversely,

$$\begin{aligned} N([0,1]^\nu, d_\lambda, \varepsilon) &\geq N([1/2,1]^\nu, d_\lambda, \varepsilon) \\ &\geq N\left([1/2,1]^\nu, d_E, m_{1/2,1}^{-1}\varepsilon\right) \\ &\asymp \left(2m_{1/2,1}^{-1}\varepsilon\right)^{-\nu}, \end{aligned}$$

so both inequalities give the expected result. □

Another proof of this lemma, geometric and combinatorial, can be found in Appendix 5.7.1. It is more lengthy, but gives more explicit bounds in front of $\varepsilon^{-\nu}$, and an overview of the shape of d_λ-balls.

Proposition 5.5. *For $h < 1/2$, there are constants $\kappa_1 > 0$ and $\kappa_2 > 0$ such that for any fixed $r \in (0,1)$ and any ε small enough (compared to r),*

$$\exp\left\{-\kappa_2 \frac{r^{\nu^2}}{\varepsilon^{\nu/h}}\right\} \leq \mathbb{P}\left(\sup_{t\in[0,r]^\nu} |\boldsymbol{B}_t^h| \leq \varepsilon\right) \leq \exp\left\{-\kappa_1 \frac{r^{\nu^2}}{\varepsilon^{\nu/h}}\right\} \tag{5.3}$$

Proof. First notice the isometry between the metric spaces $([0,1]^\nu, d_h)$ and the subset of $H(k_h)$ defined by $\{k_h(\mathbf{1}_{[0,t]}, \cdot),\ t \in [0,1]^\nu\}$ with the metric induced by the fBm on $L^2([0,1]^\nu, \lambda)$. Hence, we can apply Theorem 4.6, which states that (for $h < 1/2$), for any $\varepsilon > 0$,

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]^\nu} |\boldsymbol{B}_t^h| \leq \varepsilon\right) \asymp N([0,1]^\nu, d_h, \varepsilon) .$$

For any $\varepsilon > 0$, any $t \in [0,1]^\nu$, the ball $B_h(t,\varepsilon)$ is such that $B_h(t,\varepsilon) = B_\lambda(t, \varepsilon^{1/h})$. A direct consequence is that $N([0,1]^\nu, d_h, \varepsilon) = N([0,1]^\nu, d_\lambda, \varepsilon^{1/h})$. Hence it suffices to calculate the d_λ-entropy to obtain the result for any h. Besides, $\boldsymbol{B}^h$ satisfies the subsequent self-similarity property: for any $r > 0$,

$$\left\{\boldsymbol{B}_t^h, t \in [0,1]^\nu\right\} \stackrel{(d)}{=} \left\{r^{-\nu h}\boldsymbol{B}_{rt}^h, t \in [0,1]^\nu\right\} .$$

Therefore, $\mathbb{P}\left(\sup_{t\in[0,r]^\nu} |\boldsymbol{B}_t^h| \leq \varepsilon\right) = \mathbb{P}\left(\sup_{t\in[0,1]^\nu} |\boldsymbol{B}_t^h| \leq r^{-\nu h}\varepsilon\right)$ and so:

$$-\log \mathbb{P}\left(\sup_{t\in[0,r]^\nu} |\boldsymbol{B}_t^h| \leq \varepsilon\right) \asymp N\left([0,1]^\nu, d_\lambda, r^{-\nu}\varepsilon^{1/h}\right) .$$

Lemma 5.4 permits to conclude. □

Remark 5.6. *This is different from the Lévy fBm X^h for which the above* log-probability *is of the order $r^\nu \varepsilon^{-\nu/h}$ (see [142]). In fact, the small deviations of the multiparameter fBm away from the axes are also different of those at 0, and similar to the Lévy fBm. Indeed, if t_0 is not on the axes and r is such that $B(t_0, r) \subset (0,\infty)^\nu$, the equivalence between distances d_λ and d_E yields, as $\varepsilon \to 0$:*

$$\begin{aligned} -\log \mathbb{P}\left(\sup_{t\in B(t_0,r)} |\boldsymbol{B}_t^h| \leq \varepsilon\right) &\asymp N(B(t_0,r), d_E, \varepsilon^{1/h}) \\ &\asymp \left(\frac{r}{\varepsilon^{1/h}}\right)^\nu . \end{aligned}$$

5.2.2 *Integral decomposition of the multiparameter fBm*

The multiparameter fBm does not have independent increments, hence there is no spectral measure in the sense of YAGLOM [154]. Such a situation already appeared for the bifractional Brownian motion in [146], but the difficulty was overcome due to the equivalence of the distances, even near 0. These tricks could not work here. We shall use instead the L^2-increment stationarity in the Wiener space, to produce independent processes.

We recall a few notions about Gaussian measures on Banach spaces and abstract Wiener spaces from the introductory chapter (see also [83, Lemma 2.1]). Let E be a separable Banach space and μ a Gaussian measure on E. Let H be the completion of E^* by the action of the covariance operator S of μ, as defined in (1.9). This permits to define a sequence $\{\xi_n, n \in \mathbb{N}\}$ in E^*, such that $\{S\xi_n, n \in \mathbb{N}\}$ is a complete orthonormal system (CONS) in H. Recall that $\mathscr{I}$ is the Paley-Wiener map, defined as the isometric extension of the map $\xi \in E^* \mapsto \langle \xi, \cdot \rangle$ to a map from H to $L^2(\mu)$. Conversely, it is also possible to start from a separable Hilbert space and to construct an embedding in a larger (Banach space), on which there exists a Gaussian measure whose covariance will be related to the inner product on H. This is the abstract Wiener space approach. In Chapter 3, starting from the reproducing kernel Hilbert space of the multiparameter fBm, denoted H_h, built upon the kernel $k_h(f,g) = 1/2\big(\lambda(f^2)^{2h} + \lambda(g^2)^{2h} - \lambda((f-g)^2)^{2h}\big)$ for $f, g \in L^2([0,1]^\nu, \lambda)$, this led to the representation:

$$\mathbf{W}^h_t = \int_E \langle \mathscr{I}(k_h(\mathbf{1}_{[0,t]}, \cdot)), x\rangle \ \mathrm{d}\mathbb{W}^h_x \ . \tag{5.4}$$

For notational reasons that will become clear in the sequel, we write this process $\mathbf{W}^h$ instead of $\mathbf{B}^h$, but it is indeed the same L^2-fBm as in Chapter 3.

Next we present a Lévy-Khinchine decomposition for negative-definite functions in abstract Wiener spaces. Since this relies on an extension of Bochner's theorem, we will need the following lemma. In general, the embedding between H and E is continuous. We will need it to be Hilbert-Schmidt. The following lemma states that starting from a separable Hilbert space H, it is possible to find E and μ satisfying this property such that (H, E, μ) is an AWS.

Lemma 5.7. *Let H be a separable Hilbert space. There is a separable Hilbert space $(E, \|\cdot\|)$ and a Gaussian measure μ on E such that (H, E, μ) is an abstract Wiener space and the embedding $H \subset E$ is Hilbert-Schmidt.*

Proof. Let us assume that there exists separable Hilbert spaces H_0 and E_0 such that H_0 is densely embedded into E_0 by an operator R which is Hilbert-Schmidt, and that there exists a Gaussian measure μ_0 such that (H_0, E_0, μ_0) is an abstract Wiener space. In that case, R is the covariance operator. Let u be any linear isometry between H_0 and H, and denote by (H, E, μ) the AWS given by $E = \tilde{u}(E_0)$ and $\mu = \tilde{u}_* \mu_0$, where $\tilde{u}$ is the isometric extension of u (see [136, p.317]). Since E_0 is a Hilbert space, E is also a Hilbert space and the operator $R' = \tilde{u} \circ R \circ u^{-1}$ is the natural embedding from H into E, and is of Hilbert-Schmidt type.
The existence of such a (H_0, E_0, μ_0) triple follows either from examples as in sections 6 and 7 of [85], or by the construction of the next paragraph. □

Let us detail the Hilbert space structure of (H, E, μ) when E is a Hilbert space, and explain how this permits to obtain sharper results. Let $\{x_n, n \in \mathbb{N}\}$ be a complete orthonormal system of $(E, (\cdot,\cdot)_E)$. For each n, let λ_n^2 be the variance of $(x_n, \cdot)_E \in E^*$ under μ. Note that $\sum_{n \geq 1} \lambda_n^2 < \infty$, which follows from the fact that:

$$\sum_{n\in\mathbb{N}} \lambda_n^2 = \sum_{n\in\mathbb{N}} \int_E (x_n, x)_E^2 \ \mu(\mathrm{d}x)$$
$$= \int_E \|x\|_E^2 \ \mu(\mathrm{d}x) \ ,$$

and this quantity is finite (we know from FERNIQUE [48] that μ has exponential moments). Then H is given by:

$$H = \left\{ x \in E : \sum_{n=1}^{\infty} \left(\frac{(x, x_n)_E}{\lambda_n} \right)^2 < \infty \right\} \ . \tag{5.5}$$

$\{h_n = \lambda_n x_n, \ n \in \mathbb{N}\}$ defines a CONS of H for the scalar product given by $(x, h_n)_H = \lambda_n^{-1}(x, x_n)_E$, for any $x \in H$, and any $n \in \mathbb{N}$. Then, one can check that H is densely and continuously embedded into E.

This framework being set, we can now address the spectral decomposition itself. For any $\alpha \in (0,1]$, the application $\xi \in E^* \mapsto \|S\xi\|_H^{2\alpha}$ is continuous (because of the inequality $\|\cdot\|_H \le C \ \|\cdot\|_{E^*}$) and negative definite (by an argument on Bernstein functions, see Appendix A). Thus, according to Schoenberg's theorem, $\xi \mapsto \exp(-t\|S\xi\|_H^{2\alpha})$ is positive definite for any $t \in \mathbb{R}_+^*$. It follows from Lemma 5.7 and Sazonov's theorem, according to which a Hilbert-Schmidt map is γ-radonifying[1], that since $\xi \mapsto \exp(-\frac{1}{2}\|S\xi\|_H^{2\alpha})$ is continuous on H, it is the Fourier transform of a measure Γ^α on E, i.e:

$$e^{-\frac{1}{2}\|S\xi\|_H^{2\alpha}} = \int_E e^{i\langle \xi, x\rangle} \ \mathrm{d}\Gamma^\alpha(x) \ .$$

As this will be useful, we already note that the measure Γ^α is a strictly stable and symmetric measure on E of index α, since it satisfies (we denote by $\widehat{\Gamma}^\alpha$ the Fourier transform of Γ^α), for any integer k, and any $\xi \in E^*$:

$$\left(\widehat{\Gamma}^\alpha(\xi)\right)^k = \widehat{\Gamma}^\alpha(k^{1/2\alpha}\xi) \quad \text{and} \quad \widehat{\Gamma}^\alpha(-\xi) = \widehat{\Gamma}^\alpha(\xi) \ .$$

In particular, we see that Γ^α is infinitely divisible.
KUELBS [82] extended the spectral decomposition of α-stable measures on $\mathbb{R}$ (see e.g. [128, pp.77–78]) to the Hilbert space setting. Shortly after this was also extended to Banach spaces, and Theorem 2.4 and Corollary 2.5 of [10] (see also [37]) allow to assert that when $\alpha \in (0,1)$, Γ^α has a Lévy measure Δ^α and can be written:

$$\int_E e^{i\langle \xi, x\rangle} \ \mathrm{d}\Gamma^\alpha(x) = \exp\left\{ \int_E \left(e^{i\langle \xi, x\rangle} - 1 - i\frac{\langle \xi, x\rangle}{1 + \|x\|_E} \right) \Delta^\alpha(\,\mathrm{d}x) \right\} \ ,$$

with Δ^α satisfying $\int_E (1 \wedge \|x\|_E^2) \ \Delta^\alpha(\,\mathrm{d}x) < \infty$ and $\Delta^\alpha(\{0\}) = 0$. That α is strictly smaller than 1 is essential, and this will be assumed implicitly throughout the rest of this chapter. It follows, cancelling the imaginary part (by symmetry of Δ^α), that:

$$\forall \xi \in E^*, \qquad -\|S\xi\|_H^{2\alpha} = 2\int_E (\cos\langle \xi, x\rangle - 1) \ \Delta^\alpha(\,\mathrm{d}x) \tag{5.6}$$

[1] see for instance [155] for Sazonov's theorem, and [23] for its use in a similar context, as well as the references therein.

In the finite dimensional setup, Δ^α is known explicitely and appears in the spectral representation of the Lévy fractional Brownian motion, as in [142]. In fact, Corollary 2.5 of [10] gives a radial decomposition of Δ^α in terms of a finite measure σ^α defined on the Borel sets of the unit ball $\mathscr{S} = \{x \in E : \|x\|_E = 1\}$, such that for any borel set B of E:

$$\Delta^\alpha(B) = \int_0^\infty \frac{\mathrm{d}r}{r^{1+2\alpha}} \int_{\mathscr{S}} \mathbf{1}_B(ry)\, \sigma^\alpha(\,\mathrm{d}y)\,. \tag{5.7}$$

Besides, $\sigma^\alpha(\,\mathrm{d}y) = \Delta^\alpha(\{x \in E : \|x\|_E \geq 1 \text{ and } x/\|x\|_E \in \mathrm{d}y\})$. The previous discussion is summarized in the following proposition.

Proposition 5.8. *Let (H, E, μ) be any abstract Wiener space such that E is a Hilbert space and the embedding $H \subset E$ is Hilbert-Schmidt. Let $\alpha \in (0,1)$. Then there exists a non-trivial Lévy measure Δ^α on E such that Equation (5.6) is satisfied, and that can be radially decomposed as in (5.7).*

In the sequel, H will be specifically the RKHS of the Brownian sheet in $\mathbb{R}^\nu$, that is, the Hilbert space with kernel $\{\lambda(g\cdot), g \in L^2([0,1]^\nu)\}$, where λ is the Lebesgue measure of $\mathbb{R}^\nu$ and for $g \in L^2([0,1]^\nu)$, $\lambda(g\cdot)$ is the mapping:

$$f \in L^2([0,1]^\nu) \mapsto \int_{[0,1]^\nu} f\, g \;\mathrm{d}\lambda.$$

H is a separable Hilbert space and we endow it with E and μ chosen as in Lemma 5.7 to get an AWS. Then Δ_α denotes the Lévy measure discussed in the previous paragraphs. Let $\mathbb{B}^\alpha$ be the (Gaussian) white noise on E with control measure Δ^α, and define the stochastic process $\{\boldsymbol{B}_\xi^{\alpha/2}, \xi \in E^*\}$ by:

$$\boldsymbol{B}_\xi^{\alpha/2} = \int_E \left(1 - e^{i\langle \xi, x\rangle}\right) \;\mathrm{d}\mathbb{B}_x^\alpha\,.$$

The variance of the increments reads ($\overline{(\cdot)}$ denotes complex conjugation):

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\left(\boldsymbol{B}_\xi^{\alpha/2} - \boldsymbol{B}_{\xi'}^{\alpha/2}\right) &= \mathbb{E}\left((\boldsymbol{B}_\xi^{\alpha/2} - \boldsymbol{B}_{\xi'}^{\alpha/2})\overline{(\boldsymbol{B}_\xi^{\alpha/2} - \boldsymbol{B}_{\xi'}^{\alpha/2})}\right)\\
&= \int_E \left(e^{i\langle\xi,x\rangle} - e^{i\langle\xi',x\rangle}\right)\left(e^{-i\langle\xi,x\rangle} - e^{-i\langle\xi',x\rangle}\right)\,\Delta^\alpha(\,\mathrm{d}x)\\
&= 2\int_E (1 - \cos\langle\xi - \xi', x\rangle)\,\Delta^\alpha(\,\mathrm{d}x)\\
&= \|S(\xi - \xi')\|_H^{2\alpha}\,.
\end{aligned}$$

Hence this process has the following covariance:

$$\mathbb{E}\left(\boldsymbol{B}_\xi^{\alpha/2}\,\boldsymbol{B}_{\xi'}^{\alpha/2}\right) = \frac{1}{2}\left(\|S\xi\|_H^{2\alpha} + \|S\xi'\|_H^{2\alpha} - \|S(\xi - \xi')\|_H^{2\alpha}\right)\,.$$

By analogy with the Paley-Wiener map $\mathscr{I}$ that maps H to $L^2(\mu)$, let $\mathscr{I}_\alpha$ be the mapping from E^* to $L^2(\Delta^\alpha)$ such that $\mathscr{I}_\alpha(\xi) = 1 - e^{i\langle\xi,\cdot\rangle}$, and extend it to H using $\mathscr{I}$ by simply putting $\mathscr{I}_\alpha(\varphi) = 1 - e^{i\langle\mathscr{I}(\varphi),\cdot\rangle}$, for any $\varphi \in H$. Similarly to $\mathscr{I}(\varphi)$ in $L^2(\mu)$, $\mathscr{I}_\alpha(\varphi)$ is a well-defined isometry from $(H, \|\cdot\|_H^{2\alpha})$ to $L^2(\Delta^\alpha)$. Thus, $\boldsymbol{B}^\alpha$ is a well-defined process on H.

Example 5.9 (Spectral representation of the multiparameter fBm)**.** *In this context, for any $f,g \in L^2([0,1]^\nu)$, $\lambda(f\cdot)$ and $\lambda(g\cdot)$ are in H, which yields for $h=\alpha/2$:*

$$\mathbb{E}\left(\boldsymbol{B}^h_{\lambda(f\cdot)}\,\boldsymbol{B}^h_{\lambda(g\cdot)}\right) = \frac{1}{2}\left(\lambda(f^2)^{2h} + \lambda(g^2)^{2h} - \lambda((f-g)^2)^{2h}\right) ,$$

and for $f=\mathbf{1}_{[0,s]}$ and $g=\mathbf{1}_{[0,t]}$, this is a multiparameter fBm of index h, as defined in equation (5.1).

Remark 5.10. $\mathbf{W}^h$ *defined by (5.4) on $H(k_h)$ and $\boldsymbol{B}^{\alpha/2}$ on H (when $\alpha=2h$) are different processes: they are not defined on the same spaces, and the first one is a linear application for fixed ω, which is not true for the second. Nevertheless, we have just seen in the previous example that $\{\mathbf{W}^h_{k_h(f,\cdot)},\ f\in L^2([0,1]^\nu)\}$ and $\{\boldsymbol{B}^\alpha_{\lambda(f\cdot)},\ f\in L^2([0,1]^\nu)\}$ are equal in distribution. This implies that they have the same RKHS. In particular, we will be interested only in the multiparameter process, which means that the RKHS is given by*

$$H^\nu_h = \overline{Span\left\{k^{(\nu)}_h(t,\cdot),\ t\in[0,1]^\nu\right\}} ,$$

where $k^{(\nu)}_h(t,\cdot) = k_h(\mathbf{1}_{[0,t]},\mathbf{1}_{[0,\cdot]})$ and the completion is with respect to the scalar product given by:

$$\left(k^{(\nu)}_h(t,\cdot),k^{(\nu)}_h(s,\cdot)\right)_{h,\nu} = k^{(\nu)}_h(t,s) .$$

To conclude this section, we present inequalities on Δ^α that will be useful in the proof of the LIL. These are extensions of the truncation inequalities of LOÈVE [95, p.209]. For any $r\in[0,1]$, let us define the subset $A(r)$ of H:

$$A(r) = \left\{\varphi_{s,t}\ ;\ s,t\in[0,r]^\nu\right\} , \tag{5.8}$$

where $\varphi_{s,t} = \lambda\left(\mathbf{1}_{[0,t]\triangle[0,s]}\cdot\right)$. Note that $A(r)$ is a subset of H.

Lemma 5.11. *For any $a>0$ and $\varphi\in A(1)$, we have:*

$$\int_{\|x\|_E<a} (1-\cos\langle\mathscr{I}(\varphi),x\rangle)\ \Delta^\alpha(dx) \le \|\varphi\|^{2\alpha}_H\ \mathbf{F}(a\Phi) , \tag{5.9}$$

where $\mathbf{F}$ is the function defined in Equation (5.2) with $A=A(1)$, and $\mathbf{F}$ continuously decreases to 0. Besides, there is a constant $C(\alpha)>0$ such that for any $b>0$ and $\varphi\in H$,

$$\int_{\|x\|_E>b} (1-\cos\langle\mathscr{I}(\varphi),x\rangle)\ \Delta^\alpha(dx) \le C(\alpha)b^{-2\alpha} . \tag{5.10}$$

Proof. We start with the first inequality, that we prove by approximation of φ by elements of E^*. Let Φ denote the norm of φ. Let $(\zeta'_n)_{n\in\mathbb{N}} = \{(z'_n,\cdot)_E,\ n\in\mathbb{N}\}$ be a sequence of E^* such that $S\zeta'_n$ belongs to the H-sphere of radius Φ and converges to φ in H. For all n, let ζ_n and z_n be the associated normalized (in E^* and E) vectors and $(\lambda'_n)_{n\in\mathbb{N}}$ be the family of norms in H: $\lambda'_n = \|S\zeta_n\|_H$. Note that if $\varphi\in H\setminus S(E^*)$, $\lambda'_n\to 0$ as $n\to\infty$. By construction, $\lambda'_n = \Phi\|\zeta'_n\|^{-1}_{E^*}>0$. Then, the radial decomposition (5.7) of σ^α yields:

$$\begin{aligned}\int_{\|x\|_E<a} (1-\cos\langle\zeta'_n,x\rangle)\ \Delta^\alpha(\mathrm{d}x) &= \int_0^a \frac{\mathrm{d}r}{r^{1+2\alpha}}\int_{\mathscr{S}}\left(1-\cos\left\{\frac{r\Phi}{\lambda'_n}(z_n,y)_E\right\}\right)\ \sigma^\alpha(\mathrm{d}y)\\ &= \Phi^{2\alpha}\int_0^{a\Phi}\frac{\mathrm{d}u}{u^{1+2\alpha}}\int_{\mathscr{S}}\left(1-\cos\left\{\frac{u}{\lambda'_n}(z_n,y)_E\right\}\right)\ \sigma^\alpha(\mathrm{d}y) ,\end{aligned}$$

where we applied the change of variable $u=\Phi r$. The last integral converges in $L^2(\sigma^\alpha)$, so this reads:

$$\int_{\|x\|_E<a} (1-\cos\langle \mathscr{I}(\varphi),x\rangle)\ \Delta^\alpha(\mathrm{d}x) = \Phi^{2\alpha}\int_0^{a\Phi}\frac{\mathrm{d}u}{u^{1+2\alpha}}\int_{\mathscr{S}}(1-\cos\langle u\mathscr{I}(\varphi/\|\varphi\|_H),y\rangle)\ \sigma^\alpha(\mathrm{d}y)$$
$$\leq \Phi^{2\alpha}\mathbf{F}(a\Phi)\ ,$$

which gives (5.9). Finally, $\mathbf{F}$ decreases continuously to 0 since the mapping:

$$(\varphi,\mathbf{x})\in A(1)\times[0,1]\mapsto \int_{\|x\|_E<\mathbf{x}}(1-\cos\langle \mathscr{I}(\varphi/\|\varphi\|_H),x\rangle)\ \Delta^\alpha(\mathrm{d}x)$$

is continuous on a compact ($A(1)$ is compact as the continuous image of $[0,1]^\nu\times[0,1]^\nu$).

To show (5.10) holds, we use a simple inequality on the cosine function:

$$\int_{\|x\|_E>b}(1-\cos\langle\xi,x\rangle)\ \Delta^\alpha(\ \mathrm{d}x)\leq 2\int_{\|x\|_E>b}\Delta^\alpha(\ \mathrm{d}x)$$
$$\leq 2\int_b^\infty \frac{\mathrm{d}r}{r^{1+2\alpha}}\sigma^\alpha(\mathscr{S})$$
$$\leq \frac{2\sigma(\mathscr{S})}{2-2\alpha}b^{-2\alpha}\ .$$

This concludes the proof of this lemma.

□

5.3 Chung's law of the iterated logarithm

In this section, we prove Theorem 5.1. The abstract Wiener space is the same as in the end of the previous section. We recover the notation h instead of α so that by the previous construction, for any fixed $h\in(0,1/2)$, there exists a measure Δ^{2h} producing the spectral representation of the h-multiparameter fBm. We will simply write Δ for Δ^{2h} in the sequel.

Remark 5.12. *The case $h=1/2$ is special since it corresponds to the Brownian sheet. Its behaviour differs very much from the h-multiparameter fBm, $h<1/2$, although we recall that the 1/2-multiparameter fBm is the Brownian sheet. This difference is due to the loss of the property of local nondeterminism, which the multiparameter fBm possesses when $h<1/2$, which is discussed at the end of Section 4.2. For more information on small deviations and Chung's law of the iterated logarithm of the Brownian sheet, we refer to [141].*

Proof of Theorem 5.1. The proof will be carried out in three steps. In the first, we obtain the lower bound for some constant β_1. In the second and third steps, we follow the scheme proposed in [106], but with the addition of methods related to the infinite dimensional setting described above.

1) Let $\gamma>1$, $r_k=\gamma^{-k}$ and $\beta_1=(\kappa_1/(1+\epsilon))^{h/\nu}$, where κ_1 is the constant in the upper bound of the small deviation probability of $\boldsymbol{B}^h$. The upper bound in the small deviations (5.3) implies:

$$\sum^\infty \mathbb{P}\left(M^h(r_k)\leq\beta_1\Psi_h^{(\ell)}(r_k)\right)\leq\sum^\infty \exp\left\{-\kappa_1\beta_1^{-\nu/h}\log\log(r_k^{-1})\right\}$$
$$\leq\sum^\infty(\log\gamma^k)^{-(1+\epsilon)}<\infty\ ,$$

where the sums start at k large enough (i.e. so that $\beta_1(\log\log\gamma^k)^{-h}$ is small enough, as in Proposition 5.5). Then, the Borel-Cantelli lemma gives:

$$\liminf_{k\to\infty} M^h(r_k)/\Psi_h^{(\ell)}(r_k) \geq \beta_1 \quad \text{a.s.}$$

so for $r_{k+1} < r \leq r_k$:

$$M^h(r)/\Psi_h^{(\ell)}(r) \geq M^h(r_k)/\Psi_h^{(\ell)}(r_{k+1}) \geq \beta_1 \frac{\Psi_h^{(\ell)}(r_k)}{\Psi_h^{(\ell)}(r_{k+1})} \geq (\kappa_1/(1+\epsilon))^{h/\nu}\,\gamma^{-\nu h} \ .$$

This is true for any $\epsilon > 0, \gamma > 1$, hence we get the following lower bound:

$$\mathbb{P}\left(\liminf_{r\to 0} \frac{M^h(r)}{\Psi_h^{(\ell)}(r)} \geq \kappa_1^{h/\nu}\right) = 1 \ . \tag{5.11}$$

2) Now, let κ_2 be the constant in the lower bound of the small balls, and define $\beta_2 = \kappa_2^{h/\nu}$. For some small (fixed) $\eta > 0$, we define the sequence $(\epsilon_k)_{k\in\mathbb{N}^*}$ by:

$$\epsilon_k = \mathbf{F}^{-1}\left((\log k)^{-2h/\nu - 2\eta}\right) \ . \tag{5.12}$$

By Lemma 5.11, $\mathbf{F}$ is a continuous increasing function on any interval $[0, T]$ such that $\mathbf{F}(0) = 0$. Thus, ϵ_k is a well-defined sequence which converges to 0 and satisfies:

$$\frac{(\log k)^{h/\nu}}{\sqrt{-\mathbf{F}(\epsilon_k)\log \mathbf{F}(\epsilon_k)}} \to \infty \qquad \text{as } k \to \infty \ .$$

Let's define another sequence $(r_k)_{k\in\mathbb{N}^*}$ by the following induction:

$$r_1 = 1 \quad \text{and} \quad \forall k \geq 2,\ r_{k+1} = r_k\ \mathbf{F}(\epsilon_k)^{1/(2\nu h)}\ \epsilon_{k+1}^{2/\nu} \ . \tag{5.13}$$

One can now choose $\widetilde{\Psi}_h^{(u)}$ to be any increasing continuous function on $[0,1]$, satisfying the following set of conditions: for any $k \in \mathbb{N}^*$,

$$\widetilde{\Psi}_h^{(u)}(r_k) = (\log k)^{-h/\nu} \ . \tag{5.14}$$

We recall that for a given $\widetilde{\Psi}_h^{(u)}$, chosen as above, $\Psi_h^{(u)}$ is defined by $\Psi_h^{(u)}(r) = r^{\nu h}\,\widetilde{\Psi}_h^{(u)}(r)$, $r \in [0,1]$.

For these parameters, the lower bound in the small deviations of $\boldsymbol{B}^h$ implies:

$$\begin{aligned}\sum^{\infty} \mathbb{P}\left(\sup_{t\in[0,r_k]^\nu} |\boldsymbol{B}_t^h|/\Psi_h^{(u)}(r_k) \leq \beta_2\right) &\geq \sum^{\infty} \exp\left\{-\kappa_2(\beta_2 \widetilde{\Psi}_h^{(u)}(r_k))^{-\nu/h}\right\} \\ &\geq \sum^{\infty} \frac{1}{k} = \infty \ ,\end{aligned} \tag{5.15}$$

where the sums start at k large enough (i.e. so the hypothesis of Proposition 5.5 is satisfied). This is not enough to prove the expected result, because these events are not independent. We will fix this using an idea that appeared in [142] and [106], to create independence by means of increment stationarity. Since this last property is not satisfied by the multiparameter fractional Brownian motion, we shall rely instead on the spectral representation obtained in the previous section.

We recall that $\varphi_t = \lambda(\mathbf{1}_{[0,t]}\cdot), t \in [0,1]^\nu$ is an element of H. For a family of disjoint intervals $\{I_k = (a_k, a_{k+1}],\ k \in \mathbb{N}\}$, where $(a_k)_{k\in\mathbb{N}}$ is an increasing sequence of $\mathbb{R}_+$ such that $a_k \to \infty$ (a_k will be specified later), we define the following processes:

$$\boldsymbol{B}^{h,k}_t = \int_{\|x\|_E \in I_k} \left(1 - e^{i\langle \mathscr{I}(\varphi_t), x\rangle}\right)\ \mathrm{d}\mathbb{B}^{\Delta}_x\ ,\ t \in [0,1]^\nu \tag{5.16}$$

$$\tilde{\boldsymbol{B}}^{h,k}_t = \boldsymbol{B}^h_t - \boldsymbol{B}^{h,k}_t = \int_{\|x\|_E \notin I_k} \left(1 - e^{i\langle \mathscr{I}(\varphi_t), x\rangle}\right)\ \mathrm{d}\mathbb{B}^{\Delta}_x\ ,\ t \in [0,1]^\nu\ . \tag{5.17}$$

Let Σ denote the covariance operator of $\boldsymbol{B}^h$ and Σ_k denote the covariance operator of the $\boldsymbol{B}^{h,k}$. It is clear that $\Sigma - \Sigma_k$ is a positive semi-definite operator. Hence, Anderson's correlation inequality [9] applies and we get, for all $k \in \mathbb{N}$:

$$\mathbb{P}\left(\sup_{t\in[0,r_k]^\nu} |\boldsymbol{B}^{h,k}_t|/\Psi^{(u)}_h(r_k) \le \beta_2\right) \ge \mathbb{P}\left(\sup_{t\in[0,r_k]^\nu} |\boldsymbol{B}^{h}_t|/\Psi^{(u)}_h(r_k) \le \beta_2\right)\ .$$

As a consequence of Equation (5.15), we see that:

$$\sum_{k\ge 1} \mathbb{P}\left(\sup_{t\in[0,r_k]^\nu} |\boldsymbol{B}^{h,k}_t|/\Psi^{(u)}_h(r_k) \le \beta_2\right) = \infty\ .$$

Since the events $\left\{\sup_{t\in[0,r_k]^\nu} |\boldsymbol{B}^{h,k}_t|/\Psi^{(u)}_h(r_k) \le \beta_2\right\},\ k \in \mathbb{N}$, are independent, the reciprocal of Borel-Cantelli lemma yields that almost surely,

$$\liminf_{k\to\infty} \sup_{t\in[0,r_k]^\nu} |\boldsymbol{B}^{h,k}_t|/\Psi^{(u)}_h(r_k) \le \beta_2\ . \tag{5.18}$$

3) For any $k \in \mathbb{N}^*$, let $a_k = r_k^{-\nu/2}\epsilon_k$. Note that (5.13) implies that:

$$a_{k+1} r_k^{\nu/2} = r_{k+1}\ \mathbf{F}(\epsilon_k)^{-1/4h}\ \epsilon_{k+1}^{-1} \ge \mathbf{F}(\epsilon_k)^{-1/4h}\ . \tag{5.19}$$

In particular, $a_{k+1} r_k^{\nu/2}$ goes to infinity. Now, Lemma 5.11 acts on the incremental variance of $\tilde{\boldsymbol{B}}^{h,k}$ as follows: for any $s,t \in [0,r_k]^\nu$, letting $\varphi_{s,t} = \varphi_s - \varphi_t$,

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\left(\tilde{\boldsymbol{B}}^{h,k}_s - \tilde{\boldsymbol{B}}^{h,k}_t\right) &= \mathbb{V}\mathrm{ar}\left(\tilde{\boldsymbol{B}}^{h,k}_{\varphi_{s,t}}\right) \\
&= \int_{\|x\|_E < a_k} \left(1 - \cos\langle \mathscr{I}(\varphi_{s,t}), x\rangle\right)\ \Delta(\mathrm{d}x) + \int_{\|x\|_E \ge a_{k+1}} \left(1 - \cos\langle \mathscr{I}(\varphi_{s,t}), x\rangle\right)\ \Delta(\mathrm{d}x) \\
&\le C\left(\|\varphi_{s,t}\|_H^{4h}\ \mathbf{F}\left(a_k\ \|\varphi_{s,t}\|_H\right) + a_{k+1}^{-4h}\right) \qquad (5.20) \\
&\le C\left(r_k^{2\nu h}\ \mathbf{F}(a_k r_k^{\nu/2}) + a_{k+1}^{-4h}\right) \\
&\le C r_k^{2\nu h}\left(\mathbf{F}(\epsilon_k) + (a_{k+1}\ r_k^{\nu/2})^{-4h}\right)\ ,
\end{aligned}$$

for some positive constant C, where $\|\varphi_{s,t}\|_H^2 = \lambda([0,s] \triangle [0,t]) \le \lambda([0,r_k]^\nu) = r_k^\nu$. Thus, for this choice of r_k and a_k, letting D_k^2 denote this incremental variance, the previous equation and (5.19) give:

$$\begin{aligned}
D_k^2 &= \sup_{s,t\in[0,r_k]^\nu} \mathbb{V}\mathrm{ar}\left(\tilde{\boldsymbol{B}}^{h,k}_s - \tilde{\boldsymbol{B}}^{h,k}_t\right) \\
&\le 2C\ r_k^{2\nu h}\ \mathbf{F}(\epsilon_k)\ ,
\end{aligned}$$

which decreases faster than $\sup_{s,t\in[0,r_k]^\nu} \mathbb{V}\mathrm{ar}\left(\boldsymbol{B}^h_s - \boldsymbol{B}^h_t\right)$ (as $k \to \infty$). By a Gaussian concentration result, we will see that D_k will permit us to obtain an upper bound for the large deviations of $\tilde{\boldsymbol{B}}^{h,k}$. Let $\tilde{d}_{h,k}$ be the distance induced by this process. We have just seen that $\tilde{d}_{h,k} \leq d_h$. Thus, as already noticed (for instance in the proof of Theorem 3.19), the metric entropy of a set computed with $\tilde{d}_h$ is smaller than the one computed with d_h.

$$\int_0^{D_k} \sqrt{\log N([0,r_k]^\nu, \tilde{d}_{h,k}, \varepsilon)}\, \mathrm{d}\varepsilon \leq \int_0^{D_k} \sqrt{\log N([0,r_k]^\nu, d_h, \varepsilon)}\, \mathrm{d}\varepsilon \leq \int_0^{D_k} \sqrt{\log\left(\kappa \frac{r_k^{\nu^2}}{\varepsilon^{\nu/h}}\right)}\, \mathrm{d}\varepsilon\ ,$$

where for some $\kappa > 0$, the upper bound for $N([0,r_k]^\nu, d_h, \varepsilon)$ is due to the link with the small balls of $\boldsymbol{B}^h$ and the estimate given in Proposition 5.5.

$$\int_0^{D_k} \sqrt{\log N([0,r_k]^\nu, \tilde{d}_{h,k}, \varepsilon)}\, \mathrm{d}\varepsilon \leq \sqrt{\frac{\nu}{h}} \kappa^{h/\nu} r_k^{\nu h} \int_0^{\sqrt{2C}\kappa^{-h/\nu}\sqrt{\mathbf{F}(\epsilon_k)}} \sqrt{\log x^{-1}}\, \mathrm{d}x \leq C_1 r_k^{\nu h} \sqrt{-\mathbf{F}(\epsilon_k)\ \log(C_2 \mathbf{F}(\epsilon_k))}\ ,$$

where we made the change of variable $x = \varepsilon\ \kappa^{-h/\nu} r_k^{-\nu h}$, and C_1 and C_2 are given by:

$$C_1 = \sqrt{\frac{C\,\nu}{h}}\ , \qquad C_2 = 2C\kappa^{-2h/\nu}\ .$$

By Talagrand's lemma [142], if $u > u_0(k) := C_1 r_k^{\nu h} \sqrt{-\mathbf{F}(\epsilon_k)\ \log(C_2\mathbf{F}(\epsilon_k))}$,

$$\mathbb{P}\left(\sup_{t\in[0,r_k]^\nu} |\tilde{\boldsymbol{B}}^{h,k}_t| \geq u \right) \leq \exp\left(-\frac{(u-u_0(k))^2}{D_k^2} \right)\ .$$

Let $\epsilon > 0$. In order to replace u by $\epsilon\beta_2\Psi_h^{(u)}(r_k)$, one notices that:

$$\frac{\Psi_h^{(u)}(r_k)}{u_0(k)} = \frac{\widetilde{\Psi}_h^{(u)}(r_k)}{C_1\sqrt{-\mathbf{F}(\epsilon_k)\ \log(C_2\mathbf{F}(\epsilon_k))}}\ ,$$

and this quantity goes to infinity, by definition of ϵ_k in (5.12) and $\widetilde{\Psi}_h^{(u)}$ in (5.14). Thus, the concentration inequality applies replacing u with $\epsilon\beta_2\psi_h(r_k)$ for k big enough, so that:

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,r_k]^\nu} |\tilde{\boldsymbol{B}}^{h,k}_t| \geq \epsilon\beta_2\widetilde{\Psi}_h^{(u)}(r_k) \right) &\leq \exp\left(-\frac{(\epsilon\beta_2\widetilde{\Psi}_h^{(u)}(r_k) - u_0(k))^2}{D_k^2} \right) \\
&\leq \exp\left(-\frac{u_0(k)^2}{D_k^2}\ \left(\epsilon\beta_2\Psi_h^{(u)}(r_k)\ u_0(k)^{-1} - 1\right)^2 \right) \\
&\leq \exp\left(-\frac{C_1\sqrt{-\log(C_2\mathbf{F}(\epsilon_k))}}{2C}\ \left(\epsilon\beta_2\Psi_h^{(u)}(r_k)\ u_0(k)^{-1} - 1\right)^2 \right) \\
&= (C_2\mathbf{F}(\epsilon_k))^{-C_3\left(\epsilon\beta_2\Psi_h^{(u)}(r_k)\ u_0(k)^{-1}-1\right)^2}\ ,
\end{aligned}$$

whose sum is finite, since $\psi_h(r_k)u_0(k)^{-1}$ diverges and $\mathbf{F}(\epsilon_k)$ goes to 0. Hence, applying once again the Borel-Cantelli lemma, we have almost surely,

$$\liminf_{k\to\infty} \sup_{t\in[0,r_k]^\nu} |\tilde{\boldsymbol{B}}_t^{h,k}|/\Psi_h^{(u)}(r_k) \leq \epsilon\beta_2 \ .$$

Therefore, combining this with (5.18), we see that almost surely:

$$\liminf_{k\to\infty} \sup_{t\in[0,r_k]^\nu} |\boldsymbol{B}_t^{h}|/\Psi_h^{(u)}(r_k) \leq (1+\epsilon)\beta_2 \ .$$

Since this is true for any $\epsilon > 0$, we obtain the expected upper bound:

$$\mathbb{P}\left(\liminf_{r\to 0} \frac{M^h(r)}{\Psi_h^{(u)}(r)} \leq \kappa_2^{h/\nu}\right) = 1 \ . \tag{5.21}$$

□

We end this part with a discussion on the consequences of the rate of decay of $\mathbf{F}$. We see that to make this *lim inf* result precise, we would need to find $\widetilde{\Psi}_h^{(u)}$ explicitely, which depends only on the rate of decay of $\mathbf{F}$ near 0. For instance, if we were able to prove that $\mathbf{F}(\mathbf{x}) \leq \mathbf{x}^\gamma$ for some $\gamma > 0$, as $\mathbf{x} \to 0$, then some computations lead to the conclusion that $\widetilde{\Psi}_h^{(u)}(r) = \left(\log\log(r^C)\right)^{-h/\nu}$, where $C = -\nu^{-2}(1+4h/\gamma)$, is a function for which (5.14) holds. Since in that case $\widetilde{\Psi}_h^{(u)}(r) \sim \widetilde{\Psi}_h^{(\ell)}(r)$ as $r \to 0$, we would get

$$\mathbb{P}\left(\liminf_{r\to 0} \frac{M^h(r)}{r^{\nu h}(\log\log(r^{-1}))^{-h/\nu}} \in [\kappa_1^{h/\nu}, \kappa_2^{h/\nu}]\right) = 1 \ .$$

Note that in this situation, a 0—1 law (which is explained in Remark 5.13) implies that the above limit is constant almost surely. A faster rate woul yield the same conlusion, while a slower rate for $\mathbf{F}$ would certainly mean that $\widetilde{\Psi}_h^{(\ell)}$ converges to 0 too quickly.

Remark 5.13. (0 − 1 law of the multiparameter fBm if $\mathbf{F}(\mathbf{x}) \leq \mathbf{x}^\gamma$.) *The following is very similar to the* 0 − 1 *law presented in [96]. Let $\mathscr{F}_n$ be the σ-algebra generated by the process $\sum_{k=n}^\infty \boldsymbol{B}^{h,k}$ and let $\mathscr{F}^\infty = \cap_{n\geq 1}\mathscr{F}_n$ be the tail σ-algebra. According to Kolmogorov's* 0 − 1 *law, any event A in $\mathscr{F}^\infty$ is trivial, i.e. $\mathbb{P}(A) = 0$ or 1. Thus, if the event:*

$$A = \left\{\liminf_{r\to 0} M^h(r)/\Psi_h(r) = constant\right\}$$

belongs to $\mathscr{F}^\infty$, we will have proved the theorem.

Fix $n \in \mathbb{N}^*$. We know from Lemma 5.11 and the first part of Equation (5.20) that

$$\begin{aligned}\sup_{s,t\in[0,1]^\nu} \mathbb{V}ar\left(\sum_{k=1}^{n-1}\boldsymbol{B}_s^{h,k} - \sum_{k=1}^{n-1}\boldsymbol{B}_t^{h,k}\right) &\leq K d_\lambda(s,t)^{2h}\ \mathbf{F}(a_n d_\lambda(s,t)^{1/2}) \\ &\leq K a_n^\gamma\ d_\lambda(s,t)^{2h+\gamma/2} \ .\end{aligned}$$

Thus, Kolmogorov's continuity criterion for multiparameter Gaussian processes implies that $\sum_{k=1}^{n-1}\boldsymbol{B}^{h,k}$ is almost surely (d_λ-)Hölder-continuous of order $h+\gamma/4-\epsilon$, for any $\epsilon \in (0,1)$. For $\epsilon < \gamma/4$, this implies that almost surely:

$$\liminf_{r\to 0} \sup_{t\in[0,r]^\nu} \frac{\left|\sum_{k=1}^{n-1}\boldsymbol{B}_t^{h,k}\right|}{\Psi_h(r)} \leq \liminf_{r\to 0} \frac{\lambda([0,r]^\nu)^{h+\gamma/4-\epsilon}}{r^{\nu h}\tilde{\Psi}_h(r)} = 0 \ .$$

Here $\Psi_h = \Psi_h^{(u)}$, since we have seen that if $\mathbf{F}(\mathbf{x}) \leq \mathbf{x}^\gamma$, we would obtain the same upper and lower modulus. As a consequence, we have almost surely that:

$$\liminf_{r\to 0} M^h(r)/\Psi_h(r) = \liminf_{r\to 0} \sup_{t\in[0,r]^\nu} \Big|\sum_{k=n}^{\infty} \boldsymbol{B}_t^{h,k}\Big|/\Psi_h(r)\ ,$$

which is a $\mathscr{F}_n$-measurable random variable. Hence A is a tail event.

5.4 Functional law of the iterated logarithm

We prove Theorem 5.2. As in the previous part, we also have values for $\gamma^{(\ell)}(\varphi)$ and $\gamma^{(u)}(\varphi)$:

$$\gamma^{(\ell)}(\varphi) = \frac{1}{\sqrt{2}}\kappa_1^{h/\nu}(1-\|\varphi\|_{h,\nu}^2)^{-h/\nu} \ \text{ and } \ \gamma^{(u)}(\varphi) = \frac{1}{\sqrt{2}}\kappa_2^{h/\nu}(1-\|\varphi\|_{h,\nu}^2)^{-h/\nu}\ .$$

Thanks to the preliminary tools of Section 5.2, we have almost all the ingredients to follow the proofs of [38, 106], and the following technical lemma is adapted from these papers. The norm of H_h^ν (see Remark 5.10) is denoted by $\|\cdot\|_{h,\nu}$ and we will also abbreviate $\sup_{t\in[0,1]^\nu}|f(t)| = \|f\|_\infty$.

Lemma 5.14. *For $0<s<r<u<e^{-1}$ and $\varphi \in H_h^\nu$,*

$$\begin{aligned}\left(\log\log r^{-1}\right)^{h/\nu+1/2}\|\eta_r^{(h,\ell)}-\varphi\|_\infty \geq{}& \left(\frac{s\log\log u^{-1}}{u\log\log s^{-1}}\right)^{h\nu}\left(\log\log s^{-1}\right)^{h/\nu+1/2}\|\eta_s^{(h,\ell)}-\varphi\|_\infty\\ &- M_1\left(\log\log u^{-1}\right)^{h/\nu+1/2}\left(\frac{u-s}{u}\right)^h\|\varphi\|_{h,\nu}\\ &-\left(\log\log u^{-1}\right)^{h/\nu+1/2}\sqrt{\left(1-\left(\frac{s}{u}\right)^{2\nu h}\frac{\log\log s^{-1}}{\log\log u^{-1}}\right)}\|\varphi\|_\infty\ ,\end{aligned}$$

where M_1 is the constant in Lemma 5.3 which corresponds to $b=1$.

For the proof of this lemma, one can refer to appendix 5.7.2 .

We recall the following nice proposition from [106] [2], concerning the Gaussian measure of shifted convex sets:

Proposition 5.15. *Let μ be a Gaussian measure on a separable Banach space E. For any convex, symmetric, bounded and measurable subset V of E of positive measure, if φ belongs to the RKHS of μ, then*

$$\lim_{t\to\infty} t^{-2}\left(\log\mu(V+t\varphi)-\log\mu(V)\right) = -\frac{1}{2}\|\varphi\|_\mu^2\ .$$

Proof of Theorem 5.2. This proof is divided into two parts: the first one to give the lower bound on $\gamma(\varphi)$, and the second one for the upper bound.

I) Proof of the lower bound

Let $\epsilon>0$ and γ_1 defined by:

$$\gamma_1 = \left(\frac{\kappa_1}{(1+\epsilon)}\right)^{h/\nu}(1-\|\varphi\|_{h,\nu}^2)^{-h/\nu}\ .$$

[2] It existed before in the literature, in a more general form. See the references therein.

We recall that $\widetilde{\Psi}_h^{(\ell)}(r) = (\log\log r^{-1})^{-h/\nu}$, so that the following events, defined for $k \in \mathbb{N}$ by:

$$A_k = \left\{ \widetilde{\Psi}_h^{(\ell)}(r_k)^{-1-\nu/2h} \|\eta_{r_k}^{(h,\ell)} - \varphi\|_\infty \leq \gamma_1 \right\}$$

for some decreasing sequence r_k (explicited later), will be written:

$$A_k = \left\{ \left\| r_k^{-\nu h} \boldsymbol{B}^h(r_k \cdot) - \sqrt{2\log\log r_k^{-1}}\,\varphi \right\|_\infty \leq \gamma_1 (\log\log r_k^{-1})^{-h/\nu} \right\} .$$

Let $\delta > 0$ and $\delta < \epsilon(1 - \|\varphi\|_{h,\nu}^2)$. By Proposition 5.15, and then by the small deviations of $\boldsymbol{B}^h$, we have for k large enough (depending on δ),

$$\begin{aligned}
\log \mathbb{P}(A_k) &\leq \log \mathbb{P}\left(\sup_{t\in[0,1]^\nu} |\boldsymbol{B}^h(r_k t)| \leq \gamma_1 r_k^{h\nu} (\log\log r_k^{-1})^{-h/\nu} \right) - (\log\log r_k^{-1})(\|\varphi\|_{h,\nu}^2 - \delta) \\
&\leq -(1+\epsilon)(1 - \|\varphi\|_{h,\nu}^2)(\log\log r_k^{-1}) - (\log\log r_k^{-1})(\|\varphi\|_{h,\nu}^2 - \delta) .
\end{aligned}$$

This implies that

$$\mathbb{P}(A_k) \leq \exp\left\{ -\left(1 + \epsilon(1 - \|\varphi\|_{h,\nu}^2) - \delta\right) \log\log r_k^{-1} \right) .$$

Now we put:

$$r_k = \exp\{-k y(k)\} ,$$

where

$$y(k) = \frac{\log\log k}{(\log k)^{h^{-1}+1}} .$$

Since we chose δ appropriately, $\epsilon(1 - \|\varphi\|_{h,\nu}^2) - \delta$ is positive, and

$$\sum^{\infty} \mathbb{P}(A_k) < \infty ,$$

where the sum is over k large enough, according to the previous remarks. Therefore, almost surely,

$$\liminf_{k\to\infty} \left(\log\log r_k^{-1}\right)^{h/\nu+1/2} \sup_{t\in[0,1]^\nu} |\eta_{r_k}(t) - \varphi(t)| \geq \frac{1}{\sqrt{2}}\gamma_1 .$$

To obtain the result for $r \to 0$, we use Lemma 5.14 with $u = r_k$, $s = r_{k+1}$ and r in between. Then

$$\begin{aligned}
\left(\log\log r^{-1}\right)^{h/\nu+1/2} \|\eta_r - \varphi\|_\infty &\geq \left(\frac{r_{k+1} \log\log r_k^{-1}}{r_k \log\log r_{k+1}^{-1}} \right)^{h\nu} \left(\log\log r_{k+1}^{-1}\right)^{h/\nu+1/2} \|\eta_{r_{k+1}} - \varphi\|_\infty && (\star) \\
&\quad - M_1 \left(\log\log r_k^{-1}\right)^{h/\nu+1/2} \left(\frac{r_k - r_{k+1}}{r_k} \right)^h \|\varphi\|_{h,\nu} && (\star\star) \\
&\quad - \left(\log\log r_k^{-1}\right)^{h/\nu+1/2} \sqrt{\left(1 - \left(\frac{r_{k+1}}{r_k}\right)^{2\nu h} \frac{\log\log r_{k+1}^{-1}}{\log\log r_k^{-1}} \right)} \|\varphi\|_\infty . && (\star\star\star)
\end{aligned}$$

Note that by the inequality $e^{-x} \geq 1 - x$, and the decrease of $y(k)$ (for k large),

$$\begin{aligned}
\frac{r_{k+1}}{r_k} &\geq 1 - \{y(k+1)\log(y(k+1)) - y(k)\log(y(k))\} \\
&\geq 1 - y(k+1) .
\end{aligned}$$

Thus, the ratio in Equation (⋆) converges to 1. Likewise, the ratio in (⋆⋆) is smaller than $y(k+1)^h$, so that:

$$\left(\frac{r_k - r_{k+1}}{r_k}\right)^h \left(\log\log r_k^{-1}\right)^{h/\nu+1/2} \leq y(k+1)^h \left(\log(ky(k))\right)^{h/\nu+1/2} \leq \frac{(\log\log(k+1))^h}{(\log(k+1))^{h+1}} (\log k)^{h/\nu+1/2} \left(1 + \frac{\log y(k)}{\log k}\right)^{h/\nu+1/2},$$

which clearly goes to 0. For the last term (⋆ ⋆ ⋆),

$$\begin{aligned}\left(\log\log r_k^{-1}\right)^{2h/\nu+1} &\left(1 - \left(\frac{r_{k+1}}{r_k}\right)^{2\nu h} \frac{\log\log r_{k+1}^{-1}}{\log\log r_k^{-1}}\right) \\ &\leq \left(\log\log r_k^{-1}\right)^{2h/\nu} \Big\{ \log(ky(k)) - \log((k+1)y(k+1)) \\ &\qquad + \log((k+1)y(k+1))\left[1 - (1 - y(k+1))^{2h\nu}\right] \Big\} \\ &\leq \left(\log\log r_k^{-1}\right)^{2h/\nu} \Big\{ \log(ky(k)) - \log((k+1)y(k+1)) \\ &\qquad + 2\nu h\ y(k+1)\ \log((k+1)y(k+1)) \Big\}.\end{aligned}$$

One can show that $\log(ky(k)) - \log((k+1)y(k+1)) \sim -k^{-1}$, thus

$$\left(\log\log r_k^{-1}\right)^{2h/\nu} \{\log(ky(k)) - \log((k+1)y(k+1))\}$$

converges to 0, and so does the remaining term, since:

$$\left(\log\log r_k^{-1}\right)^{2h/\nu}\ y(k+1)\ \log((k+1)y(k+1)) \sim (\log k)^{2h/\nu-1-h^{-1}+1}\ \log\log(k+1)$$

and the sum of the exponents $1 + 2h/\nu - h^{-1} - 1$ is strictly negative ($\nu \geq 1$ and $h < 1/2$).

II) Proof of the upper bound

The proof of Theorem 5.1 and Proposition 5.15 allow to make a quick proof for this bound. Let us denote $\gamma_2 = \kappa_2^{h/\nu}\left(1 - \|\varphi\|_{h,\nu}^2\right)^{-h/\nu}$. Let us put r_k and a_k as in steps 2) and 3) of the proof of the LIL. Again, put $\boldsymbol{B}^{h,k}$ and $\tilde{\boldsymbol{B}}^{h,k}$ the processes defined by (5.16) and (5.17). As in [106], we define the following events, for any $\epsilon > 0$:

$$\begin{aligned} A_k(\epsilon) &= \left\{ \left\| r_k^{-\nu h} \boldsymbol{B}^h(r_k\cdot) - \sqrt{2}\left(\widetilde{\Psi}_h^{(u)}(r_k)\right)^{-\nu/2h} \varphi \right\|_\infty \leq \gamma_2(1+\epsilon)\ \widetilde{\Psi}_h^{(u)}(r_k) \right\} \\ B_k(\epsilon) &= \left\{ \left\| r_k^{-\nu h} \boldsymbol{B}^{h,k}(r_k\cdot) - \sqrt{2}\left(\widetilde{\Psi}_h^{(u)}(r_k)\right)^{-\nu/2h} \varphi \right\|_\infty \leq \gamma_2(1+\epsilon)\ \widetilde{\Psi}_h^{(u)}(r_k) \right\} \\ C_k(\epsilon) &= \left\{ \left\| r_k^{-\nu h} \tilde{\boldsymbol{B}}^{h,k}(r_k\cdot) \right\|_\infty \geq \gamma_2 \epsilon\ \widetilde{\Psi}_h^{(u)}(r_k) \right\}. \end{aligned}$$

This time, we apply Proposition 5.5 and Proposition 5.15 to deduce the existence of a small $\delta > 0$ such that for k large enough, we obtain a lower bound on the probability of the event

$A_k(\epsilon)$:

$$\begin{aligned}\log \mathbb{P}(A_k(\epsilon)) &\geq \log \mathbb{P}\left(\sup_{t\in[0,1]^\nu} |\boldsymbol{B}^h(r_k t)| \leq \gamma_2(1+\epsilon) r_k^{h\nu} \widetilde{\Psi}_h^{(u)}(r_k)\right) - \left(\widetilde{\Psi}_h^{(u)}(r_k)\right)^{-\nu/h} \left(\|\varphi\|_{h,\nu}^2 + \delta\right) \\ &\geq -(1+\epsilon)^{-\nu/h}\left(1 - \|\varphi\|_{h,\nu}^2\right)\left(\widetilde{\Psi}_h^{(u)}(r_k)\right)^{-\nu/h} - \left(\widetilde{\Psi}_h^{(u)}(r_k)\right)^{-\nu/h} \left(\|\varphi\|_{h,\nu}^2 + \delta\right) \\ &\geq -\log k \, \left((1+\epsilon)^{-\nu/h}\left(1 - \|\varphi\|_{h,\nu}^2\right) - \left(\|\varphi\|_{h,\nu}^2 + \delta\right)\right) .\end{aligned}$$

Therefore, choosing δ small enough to ensure that $-(1+\epsilon)^{-\nu/h}\left(1 - \|\varphi\|_{h,\nu}^2\right) - \left(\|\varphi\|_{h,\nu}^2 + \delta\right)$ is greater than -1 implies that:

$$\begin{aligned}\sum_{k=1}^{\infty} \mathbb{P}(A_k(\epsilon)) &\geq \sum_{k=1}^{\infty} k^{-(1+\epsilon)^{-\nu/h}\left(1-\|\varphi\|_{h,\nu}^2\right) - \left(\|\varphi\|_{h,\nu}^2+\delta\right)} \\ &= \infty .\end{aligned}$$

All that remains to notice is that:

$$A_k(\epsilon) \subset B_k(2\epsilon) \,\cup\, C_k(\epsilon) \subset A_k(3\epsilon) \,\cup\, C_k(\epsilon) ,$$

and that the choice of a_k and r_k implies that $\sum \mathbb{P}(C_k(\epsilon)) < \infty$ (as in the proof of Theorem 5.1). The rest follows strictly the proof of [106]. □

As in Remark 5.13, if **F** were proven to have fast decay, the same $0-1$ law that we used for the Chung's law would give the same conclusion, i.e. that there is a constant between $\gamma^{(\ell)}(\varphi)$ and $\gamma^{(u)}(\varphi)$ such that almost surely:

$$\liminf_{r\to 0^+} (\log\log(r^{-1}))^{h/\nu+1/2} \sup_{t\in[0,1]^\nu} |\eta_r^{(h,\ell)}(t) - \varphi(t)| = \gamma(\varphi) .$$

We end this part on laws of the iterated logarithm with a remark concerning the previous result when $\|\varphi\|_{h,\nu} = 1$. This case was studied a lot in the literature, as it yields a different rate of convergence. In fact, part **I)** of the previous proof can be directly adapted to give:

$$\liminf_{r\to 0^+} (\log\log(r^{-1}))^{h/\nu+1/2} \sup_{t\in[0,1]^\nu} |\eta_r(t) - \varphi(t)| = \infty \quad \text{a.s.} ,$$

if $\|\varphi\|_{h,\nu} = 1$. The exact rate when $\|\varphi\|_{h,\nu} = 1$ was computed in many situations and we believe that standard techniques (as in [38, 106]) and our spectral representation and small deviations will permit to compute the exact rate in the functional law of the iterated logarithm on the unit sphere for the multiparameter fBm.

5.5 Hausdorff dimension of the range of the multiparameter fBm

Let us first recall the definition of a Hausdorff measure and of Hausdorff dimension. Following [46], let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and increasing function, which we call *gauge function*. Let the diameter be the set function defined by $\text{diam}(A) = \sup\{\|s-t\| : s,t \in A\}$ for any subset A of $\mathbb{R}^\nu$. For any $\delta > 0$, we call δ–cover of A a family $\{A_i\}$ of subsets of $\mathbb{R}^\nu$ of diameter smaller than δ and such that $A \subseteq \cup_i A_i$. Then for any $A \subset \mathbb{R}^\nu$,

$$\mu_\Phi(A) = \lim_{\delta\to 0} \inf\left\{\sum_i \Phi(\text{diam}(A_i)) : \{A_i\} \text{ is a } \delta-\text{covering of } A\right\}$$

defines the Hausdorff measure on $\mathbb{R}^\nu$ with gauge function Φ.
A case of special interest is when the gauge function is $\Phi(x) = x^s$, and we denote by μ_s the Hausdorff measure. Then the Hausdorff dimension of a set $A \subset \mathbb{R}^\nu$ is:

$$\dim_H(A) = \inf\{s > 0 : \mu_s(A) = 0\} = \sup\{s > 0 : \mu_s(A) = \infty\} \ .$$

In general, it can happen that a set A has Hausdorff dimension s but that $\mu_s(A)$ is either 0 or ∞. So choosing a special gauge function such that $\mu_\Phi(A) \in (0,\infty)$ gives more information on A. A measure having this property will be called an *exact Hausdorff measure* on A. For more information on the measure of random sets associated to stochastic processes, one can refer to the survey paper by TAYLOR [145].

The method to prove Chung's LIL is known to rely on the same estimates than the method to compute the exact Hausdorff measure of the range of a Gaussian process with stationary increments. To make the expostion clearer, we assume that $\mathbf{F}(\mathbf{x}) \leq x^\gamma$, which in the opposite case would require arrangements that do not alter our point. We refer to XIAO [151, Prop. 3.1], which extends the work of TALAGRAND [142, Prop. 4.1] on the Lévy fractional Brownian motion, to formulate the following probabilistic estimate:

Proposition 5.16. *There exists a constant $\delta > 0$ such that for any $r_0 < \delta$:*

$$\mathbb{P}\Big(\exists r,\ r_0^2 \leq r \leq r_0,\ M^h(r) \leq \Psi_h^{(\ell)}(r)\Big) \geq 1 - \exp\Big(-\sqrt{\log(r_0^{-1})}\Big) \ .$$

Proof. Adapting the proof that led to Equation (5.21) gives the result, as in the proof of Proposition 4.1 of [142]. □

Hence, it is tempting to think that, as in the case of the Chung's LIL, this yields a singularity at 0 on the Hausdorff measure. Precisely, if $\boldsymbol{B}^h$ was increment stationary, we would obtain the following Hausdorff measure:

$$\mu_\Phi\left(\boldsymbol{B}^h([0,1]^\nu)\right) \in (0,\infty) \ \ a.s. \ ,$$

where

$$\Phi(x) = x^{1/h} \log\log(x^{-1}) \ .$$

This would in turn imply that the Hausdorff dimension of $\boldsymbol{B}^h([0,1]^\nu)$ is equal to $1/h$. However, this is not the case, as seen in the following (partial, we only compute the Hausdorff dimension) result: in some sense, the Hausdorff measure of the range of the multiparameter fBm on a rectangle containing 0 does not "see" the singularity.

Proposition 5.17. *Let us assume that $\nu \leq hd$. Then,*

$$\dim_H\left(\boldsymbol{B}^h([0,1]^\nu)\right) = \frac{\nu}{h} \ a.s.$$

Proof. The Hausdorff dimension of the image of a set $I \subset [0,1]^\nu$ by the multiparameter fBm has been computed in [58], when I does not intersect the axes. When $\nu \leq hd$, this result says that $\dim_H(\boldsymbol{B}^h(I)) = \nu/h$. Since I is included in $[0,1]^\nu$, this implies that:

$$\dim_H\left(\boldsymbol{B}^h(I)\right) \leq \dim_H\left(\boldsymbol{B}^h([0,1]^\nu)\right) \ ,$$

so it remains to prove the upper bound. To do this, a result of ADLER [1] relates the Hausdorff dimension of the range of an increment stationary Gaussian process to its Hölder regularity.

Since our process is not increment stationary, we cannot use it directly, but instead we rely on Corollary 2.7 of [58] that gives an upper bound for the Hausdorff dimension of the range of a (non-increment stationary) Gaussian process, in terms of the infimum of its deterministic local Hölder exponent (as defined in (1.12)). Thus, it suffices to compute this coefficient at any point of $[0,1]^\nu$.
If t_0 is not on the axes, we already know that $\widetilde{\alpha}_{\boldsymbol{B}^h}(t_0) = h$. Let us prove that the same happens on the axes, and since this is enough to prove it at 0, let $t_0 = 0$. Let $\rho > 0$ and $s,t \in B(0,\rho)$. Lemma 5.3 implies:

$$\frac{\lambda([0,t] \triangle [0,s])^{2h}}{\|t-s\|^{2\alpha}} \leq \frac{\lambda([0,t] \triangle [0,s])^{2h}}{M_1^{2\alpha} \lambda([0,t] \triangle [0,s])^{2\alpha}}$$

where M_1 is the constant of Lemma 5.3. The last expression is bounded on $B(0,\rho)$ whenever $\alpha \leq h$. Therefore, $\widetilde{\alpha}_{\boldsymbol{B}^h}(0) \geq h$. In the opposite situation, let $u \in B(0,\rho/2)$ with positive coordinates and $s_n, t_n \in B(0,\rho)$ such that $s_n = (u_1 + 1/n, u_2, \dots, u_\nu)$ and $t_n = (u_1, u_2 + 1/n, u_3, \dots, u_\nu)$ defined for any $n \in \mathbb{N}$. Then, for n large enough, $s_n, t_n \in B(0,\rho)$ and

$$\frac{\lambda([0,t_n] \triangle [0,s_n])^{2h}}{\|t_n - s_n\|^{2\alpha}} = \frac{n^{-2h}(u_1 u_3 \dots u_\nu + u_2 \dots u_\nu)^{2h}}{2^h n^{-2\alpha}} .$$

This is unbounded as $n \to \infty$ (if $\alpha > h$), which proves that $\widetilde{\alpha}_{\boldsymbol{B}^h}(0) \leq h$.
Now, by Corollary 2.7 of [58],

$$\dim_H \left(\boldsymbol{B}^h([0,1]^\nu) \right) \leq \frac{\nu}{\inf_{t_0 \in [0,1]^\nu} \widetilde{\alpha}_{\boldsymbol{B}^h}(t_0)} = \frac{\nu}{h} ,$$

which completes the proof. □

Remark 5.18. *We have shown in the course of this proof that $\widetilde{\alpha}_{\boldsymbol{B}^h}(0) = h$. Another aspect of the singular behaviour of $\boldsymbol{B}^h$ at 0 lies in the fact that the pointwise Hölder exponent (defined in (1.11)) is different from $\widetilde{\alpha}_{\boldsymbol{B}^h}(0)$, since $\alpha_{\boldsymbol{B}^h}(0) = \nu h$. Indeed,*

$$\sup_{s,t \in B(0,\rho) \cap [0,1]^\nu} \lambda([0,t] \triangle [0,s]) \leq \lambda([0,\rho]^\nu) = \rho^\nu$$

and

$$\sup_{s,t \in B(0,\rho) \cap [0,1]^\nu} \lambda([0,t] \triangle [0,s]) \geq \sup_{t \in B(0,\rho) \cap [0,1]^\nu} \lambda([0,t]) = \frac{\rho^\nu}{\nu^{\nu/2}} ,$$

so that

$$\alpha_{\boldsymbol{B}^h}(0) = \sup \left\{ \alpha > 0 : \lim_{\rho \to 0} \frac{(\rho^\nu)^{2h}}{\rho^{2\alpha}} < \infty \right\} = \nu h .$$

A third Hölder exponent was introduced in [58], specifically to give a lower bound for the local Hausdorff dimension of the range of random fields. The deterministic local subexponent is defined, for a random field X, as:

$$\underline{\alpha}_X(t_0) = \sup \left\{ \alpha > 0 : \lim_{\rho \to 0} \inf_{s,t \in B(t_0,\rho)} \frac{\mathbb{E}(X_s - X_t)^2}{\|s-t\|^{2\alpha}} < \infty \right\} .$$

This exponent also differs from the local Hölder exponent when $t_0 = 0$, and we have for $\nu \geq 2$:

$$\widetilde{\alpha}_{\boldsymbol{B}^h}(0) = h < \nu h = \alpha_{\boldsymbol{B}^h}(0) = \underline{\alpha}_{\boldsymbol{B}^h}(0) .$$

However, all these exponents are equal (for the multiparameter fBm) when t_0 is not on the axes (this is a simple consequence of Lemma 5.3).

5.6 Perspectives

Getting the rate of decay of **F** is probably the best and most obvious way to improve the results of this chapter. However, there are several reasons to believe that this is difficult.

Then, the LIL we obtained holds for a non-increment stationary Gaussian random field. We believe that for certain locally non-deterministic Gaussian processes possessing a form of L^2-increment stationarity, there might be a general link between the local Hölder subexponent and Chung's law of the iterated logarithm. What is missing at this stage is a spectral representation of L^2-increment stationary Gaussian random fields (here we only have it for fractional processes, using stable measures).

With the tools of this chapter, other topics of inquiries for the multiparameter fBm include the multiple points as in [143], some limsup random fractals as in [40] and [101], and more generally [77].

5.7 Technical results

This section is an appendix for this chapter, gathering a couple of independent technical results.

5.7.1 *Metric entropy under d_λ*

Proof of Lemma 5.4. We write the proof for $\nu = 2$. It extends to larger values of ν but becomes much more difficult to write. The idea is to decompose $[0,1]^2$ into layers (disks centred at 0) of width 2ε, and count the number of balls inside each of these layers. So let $\mathscr{B}_{0,1}$ be the ball centred at 0. This ball, the first layer, and all the points we will need are drawn in Figure 5.1. For the first layer, let $O_{1,1} = \big((2\varepsilon)^{1/2}, (2\varepsilon)^{1/2}\big)$ such that $d_1(O, O_{1,1}) = 2\varepsilon$. For layer p, $O_{p,1} = \big((2p\varepsilon)^{1/2}, (2p\varepsilon)^{1/2}\big)$. Layer p consists of a covering of points whose distance from the origin O is between $(2p-1)\varepsilon$ and $(2p+1)\varepsilon$. For each p, n, denote $\mathscr{B}_{p,n}$ the ball centred in $O_{p,n}$ of radius ε. We now work on the first layer and extend the result to other layers by similarity. The balls are symmetric with respect to the axis $y = x$, so we shall only work on the lower side of this axis. Denoting $A_{1,1}$ the extremal point of $\mathscr{B}_{1,1}$ which is at distance 3ε of O, and $B_{1,1}$ its counterpart at distance ε of 0 (see figure), the furthest point from O_1 whose ε-ball contains $A_{1,1}$ is $O_{1,2} = \big((9\varepsilon/2)^{1/2}, (8\varepsilon/9)^{1/2}\big)$. Since, $B_{1,1} = \big((2\varepsilon)^{1/2}, (\varepsilon/2)^{1/2}\big)$, $B_{1,1} \preccurlyeq O_{1,2}$ and so $d_1(B_{1,1}, O_{1,2}) = \varepsilon$. Thus $O_{1,2}$ is a good candidate. Iterating the process, $A_{1,2}$ has the same first coordinate as $O_{1,2}$ and $B_{1,2}$ the same second coordinate. The general scheme is this one:

$$\begin{cases} x_{O_{1,n+1}} y_{O_{1,n+1}} = 2\varepsilon \\ x_{A_{1,n+1}} y_{A_{1,n+1}} = 3\varepsilon \\ x_{O_{1,n+1}} = x_{A_{1,n}} \\ y_{A_{1,n+1}} = y_{O_{1,n+1}} \end{cases}$$

so that for all n, $x_{O_{1,n+1}} = 3\varepsilon / y_{O_{1,n}} = 3/2\ x_{O_{1,n}}$. Finally:

$$O_{1,n+1} = \left(\left(\frac{3}{2}\right)^n (2\varepsilon)^{1/2}, \left(\frac{2}{3}\right)^n (2\varepsilon)^{1/2} \right) ,$$

and denoting n_1 the number of balls within the first layer, ie twice the smallest integer such that $O_{1,n_1} \notin [0,1]^2$, it easily comes that:

$$n_1 = 2E\left[\frac{\log\big((2\varepsilon)^{-1/2}\big)}{\log(3/2)} \right] + 1 ,$$

where $E[\cdot]$ denotes the integer part. This extends to any layer p into:

$$O_{p,n+1} = \left(\left(\frac{2p+1}{2p}\right)^n (2p\varepsilon)^{1/2}, \left(\frac{2p}{2p+1}\right)^n (2p\varepsilon)^{1/2} \right) , \tag{5.22}$$

and thus n_p reads:

$$n_p = 2E\left[\frac{\log\big((2p\varepsilon)^{-1/2}\big)}{\log(1+1/2p)} \right] + 1 . \tag{5.23}$$

Considering the first $O_{p,1}$ not in $[0,1]^2$, there are at most $\mathscr{P}(\varepsilon) = E[(2\varepsilon)^{-1}] + 1$ layers. Unless specified otherwise, we will just write $\mathscr{P}$ for $\mathscr{P}(\varepsilon)$. This family is a covering of $[0,1]^2$,

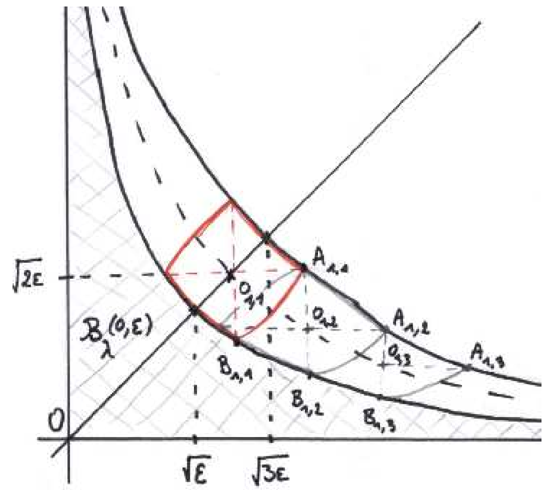

Figure 5.1 – Balls of d_λ

Thus,

$$
\begin{aligned}
N([0,1]^2, d_\lambda, \varepsilon) &\leq \sum_{p=0}^{\mathscr{P}} n_p \\
&= 1 + \sum_{p=1}^{\mathscr{P}} n_p \\
&\leq 1 + \mathscr{P} + 2\sum_{p=1}^{\mathscr{P}} \frac{\log\left((2p\varepsilon)^{-1/2}\right)}{\log(1+1/2p)} \\
&\leq 1 + \mathscr{P} - \sum_{p=1}^{\mathscr{P}} \frac{\log(p/\mathscr{P})}{\log(1+1/2p)} \ . \qquad (5.24)
\end{aligned}
$$

To obtain the lower bound, we use the packing number $D([0,1]^2, d_\lambda, \varepsilon)$, which represents the biggest number of disjoint balls of radius ε on $[0,1]^2$. It is well-known that $D(\varepsilon) \leq N(\varepsilon/2)$. It suffices to prove that the balls centred in $\{\left(O_{p,n+1}\right)_{n\in\{0,\ldots,n_p\}}, p \in \{0,\ldots,\mathscr{P}\}\}$ with radius $\varepsilon/2$ are disjoint. It follows then that:

$$
N(\varepsilon/4) \geq D(\varepsilon/2) \geq 1 + \mathscr{P} - \sum_{p=1}^{\mathscr{P}} \frac{\log(p/\mathscr{P})}{\log(1+1/2p)} \ ,
$$

or after renormalization:

$$
N(\varepsilon) \geq 1 + \mathscr{P}(4\varepsilon) - \sum_{p=1}^{\mathscr{P}(4\varepsilon)} \frac{\log(p/\mathscr{P}(4\varepsilon))}{\log(1+1/2p)} \ . \qquad (5.25)
$$

Hence, it remains to evaluate this sum, for which we rely on the following inequality:

$$
\forall x \in [0,1/2], \quad x - \frac{x^2}{2} \leq \log(1+x) \leq x \ ,
$$

thus obtaining:

$$\sum_{p=1}^{\mathscr{P}}(2p)\log(p/\mathscr{P}) \geq \sum_{p=1}^{\mathscr{P}} \frac{\log(p/\mathscr{P})}{\log(1+1/2p)} \geq \sum_{p=1}^{\mathscr{P}}(2p)\frac{\log(p/\mathscr{P})}{1-(4p)^{-1}}$$
$$\geq \frac{4}{3}\sum_{p=1}^{\mathscr{P}}(2p)\log(p/\mathscr{P}) \ .$$

By convergence of Riemann sums:

$$\frac{1}{\mathscr{P}}\sum_{p=1}^{\mathscr{P}}\frac{p}{\mathscr{P}}\log\left(\frac{p}{\mathscr{P}}\right) = \int_0^1 x\log x \ \ \mathrm{d}x + O(\mathscr{P}^{-1})$$
$$= -\frac{1}{4} + O(\mathscr{P}^{-1}) \ .$$

Finally, equations (5.24) and (5.25) yield:

$$\frac{1}{2}\mathscr{P}(4\varepsilon)^2 + O(\mathscr{P}(4\varepsilon)) \leq N([0,1]^2, d_\lambda, \varepsilon) \leq \frac{2}{3}\mathscr{P}(\varepsilon)^2 + O(\mathscr{P}(\varepsilon)) \ ,$$

and the result follows. □

5.7.2 *Proof of Lemma 5.14*

For the original proof, see Lemma 5.3 of [38]. We make here the necessary modifications.

$$\left(\log\log r^{-1}\right)^{h/\nu+1/2}\|\eta_r^{(h,\ell)} - f\|_\infty = \frac{(\log\log r^{-1})^{h/\nu}}{r^{\nu h}}\left\|\boldsymbol{B}^h(r\cdot) - r^{\nu h}\sqrt{\log\log r^{-1}}f\right\|_\infty$$
$$\geq \frac{(\log\log r^{-1})^{h/\nu}}{r^{\nu h}}\left\|\boldsymbol{B}^h(s\cdot) - r^{\nu h}\sqrt{\log\log r^{-1}}f\left(\frac{s}{r}\cdot\right)\right\|_\infty$$
$$\geq \frac{(\log\log u^{-1})^{h/\nu}}{u^{\nu h}}\left\|\boldsymbol{B}^h(s\cdot) - r^{\nu h}\sqrt{\log\log r^{-1}}f\left(\frac{s}{r}\cdot\right)\right\|_\infty \ .$$

Now choosing $a = s^{\nu h}\sqrt{\log\log s^{-1}}$ and $b = u^{\nu h}\sqrt{\log\log u^{-1}}$,

$$\left\|\boldsymbol{B}^h(s\cdot) - r^{\nu h}\sqrt{\log\log r^{-1}}f\left(\frac{s}{r}\cdot\right)\right\|_\infty \geq \left\|\boldsymbol{B}^h(s\cdot) - af\right\|_\infty - b\left\|f - f\left(\frac{s}{r}\cdot\right)\right\|_\infty - (b-a)\|f\|_\infty$$

and we find a bound for each of the last two terms (the first one is exactly the one given in the Lemma). We need the following inequality for $f \in H_h^\nu$, $s, t \in [0,1]^\nu$:

$$|f(s) - f(t)|^2 \leq M_1\|s-t\|^{2h}\|f\|_{h,\nu}^2 \ ,$$

which follows from approximation of f by linear combinations of simple functions of the form $\lambda(\mathbf{1}_{[0,t_i]}\mathbf{1}_{[0,\cdot]})$ and the upper bound in Lemma 5.3 (where the constant M_1 comes from). Thus,

$$-b\frac{(\log\log u^{-1})^{h/\nu}}{u^{\nu h}}\left\|f - f\left(\frac{s}{r}\cdot\right)\right\|_\infty \geq -M_1\left(\log\log u^{-1}\right)^{h/\nu+1/2}\left(1-\frac{s}{u}\right)^h\|f\|_{h,\nu} \ .$$

For the last term, we use the fact that:

$$b - a \leq \sqrt{u^{2\nu h}\log\log u^{-1} - s^{2\nu h}\log\log s^{-1}} \ ,$$

which ends the proof of this lemma.

Bernstein and negative definite functions

A

Our goal is not to provide a comprehensive list of definitions and results on the topic of Bernstein functions and negative definiteness. For this we refer to [129] (mainly to Chapter 3 and 4), from which the sequel can be deduced. Instead, we give as few definitions as possible to understand the last proposition of this appendix, which I believe gives a good understanding of the range of the Hurst parameter, depending on the nature of the index set.

Let $(G,+)$ be an Abelian group. In concrete situations, G will be the Euclidean space of a Hilbert space, such as $L^2(T,m)$.

Definition A.1. *A function $f : G \to \mathbb{R}$ is* negative definite *if $f(-x) = f(x)$, and*

$$\sum_{j,k=1}^{n} \lambda_j \lambda_k \big(f(x_j) + f(x_k) - f(x_j - x_k)\big) \geq 0 \ ,$$

for any $n \in \mathbb{N}$, any $x_1, \dots, x_n \in G$ and any $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

Below, we give a definition of Bernstein functions that some consider as a characterization, preferring another definition that we omit here.

Definition A.2. *A* Bernstein function *is a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ that admits a Lévy-Khintchine decomposition:*

$$f(x) = a + bx + \int_0^\infty \left(1 - e^{-\lambda x}\right) \mu(d\lambda) \ ,$$

where $a, b \in \mathbb{R}_+$ and μ is a measure on $\mathbb{R}_+$ satisfying $\int_0^\infty (1 \wedge \lambda)\, \mu(d\lambda) < \infty$.

The following lemma of Schöenberg [129, p.36] will provide an essential tool in the proof of Proposition A.4.

Lemma A.3. *$f : G \to \mathbb{R}$ is negative definite if and only if any of the two following equivalent assertions hold:*

1. *$f(0) \geq 0$, $f(-x) = f(x)$ and for any $x_1, \dots, x_n \in G$, and any $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\sum_{j=1}^n \lambda_j = 0$, the following holds:*

$$\sum_{j,k=1}^{n} \lambda_j \lambda_k\, f(x_j - x_k) \leq 0 \ .$$

2. *$f(0) \geq 0$ and $x \mapsto \exp(-tf(x))$ is positive definite for any $t > 0$.*

Proposition A.4. *Let $\alpha \in (0,1]$. Let H be a separable Hilbert space. Then $x \mapsto \|x\|^{2\alpha}$ is negative definite. In other words,*

$$(x,y) \mapsto \|x\|^{2\alpha} + \|y\|^{2\alpha} - \|x-y\|^{2\alpha}$$

is positive definite.

Proof. We assume without restriction that $H = L^2(T,m)$. Indeed, if H is not a L^2 space, we choose a linear isometry between H and a separable $L^2(T,m)$ space.
Let $\lambda_1,\dots,\lambda_n \in \mathbb{R}$ and $x_1,\dots,x_n \in H$. We prove that $x \mapsto \|x\|^2$ is negative definite.

$$\begin{aligned}\sum_{j,k=1}^{n} \lambda_j\lambda_k \left(\|x_j\|^2 + \|x_k\|^2 - \|x_j - x_k\|^2\right) &= -2\sum_{j,k=1}^{n} \lambda_j\lambda_k \left(x_j, x_k\right) \\ &= 2\sum_{j,k=1}^{n} \lambda_j\lambda_k \int_T x_j\ x_k\ \mathrm{d}m \\ &= 2\int_T \sum_{j,k=1}^{n} \lambda_j\lambda_k\ x_j\ x_k\ \mathrm{d}m \\ &= 2\int_T \left(\sum_{j=1}^{n} \lambda_j x_j\right)^2 \mathrm{d}m \geq 0\ .\end{aligned}$$

The authors of [129] remarked at the beginning of Chapter 3 that the function $x \mapsto x^\alpha$ on $\mathbb{R}_+$ is a Bernstein function for $\alpha \in [0,1]$. Hence, this function has a Lévy-Khintchine decomposition, for which a and b can be seen to equal 0. To prove negative definiteness, it more convenient to use Schöenberg's characterisation: let $\lambda_1,\dots,\lambda_n \in \mathbb{R}$ be such that $\sum_{j=1}^{n} \lambda_j = 0$,

$$\begin{aligned}\sum_{j,k=1}^{n} \lambda_j\lambda_k\ \|x_j - x_k\|^{2\alpha} &= \int_0^\infty \sum_{j,k=1}^{n} \lambda_j\lambda_k \left(1 - e^{-\lambda\|x_j - x_k\|^2}\right)\ \mu(\mathrm{d}\lambda) \\ &= -\int_0^\infty \sum_{j,k=1}^{n} \lambda_j\lambda_k e^{-\lambda\|x_j-x_k\|^2}\ \mu(\mathrm{d}\lambda) \leq 0\ ,\end{aligned}$$

since the second assertion of Schöenberg's characterisation ensures that $\sum_{j,k=1}^{n} \lambda_j\lambda_k e^{-\lambda\|x_j-x_k\|^2}$ is non-negative. This ends the proof. □

Separability of stochastic processes B

This topic comes up many times in this thesis, see in particular Section 1.4 of the introduction, so we provide here complete definitions and proofs in a very general setting. We adapt a proof due to Doob [42, p.54]. In this book, only processes indexed by a linear space are considered, while in Gikhman and Skorokhod [53, pp.163–167], processes can be indexed by a separable metric space. Here, we transpose essentially the same proof assuming only that the index set is a topological second countable space. If it is metric, then it is the same than assuming it is separable.

Let $(T, \mathscr{O})$ be a topological space. We assume that this space is *second-countable*, i.e. that there exists a countable subset $\widetilde{\mathscr{O}} \subseteq \mathscr{O}$ such that any open set of $\mathscr{O}$ can be expressed as a union of elements of $\widetilde{\mathscr{O}}$.
A process $\{X_t, t \in T\}$ is *separable* if there exists an at most countable set $S \subset T$ and a null set Λ such that for all closed sets $F \subset \mathbb{R}$ and all open set $O \in \mathscr{O}$,

$$\{\omega : X_s(\omega) \in F \text{ for all } s \in O \cap S\} \setminus \{\omega : X_s(\omega) \in F \text{ for all } s \in O\} \subset \Lambda.$$

This definition is different of the one found in [42], where the space is "linear", in that these authors consider the previous equation only when O is an interval. However when restricted to a vector space, these definitions coincide.

Theorem B.1 (Doob's separability theorem). *Let $(T, \mathscr{O})$ be a second-countable topological space. Any stochastic T-indexed process $X = \{X_t;\ t \in T\}$ has a separable modification.*

It should be noted that this modification has values in $\overline{\mathbb{R}}$, the compactification of $\mathbb{R}$. This does not affect the topology on $\mathbb{R}$, and the probability that any such separable process attains $\pm\infty$ is 0. Hence, this consequence of Doob's separability theorem is commonly forgotten.

Let us prove this theorem. $\mathscr{J}$ is the collection of all closed intervals in $\mathbb{R}$ with rational or infinite endpoints, $\mathring{\mathscr{J}}$ is the analogous of $\mathscr{J}$ with open sets, and $\mathscr{C}$ is the collection of all closed subsets of $\mathbb{R}$. Let also denote $\mathscr{K}$ the sets that are finite unions of open or closed intervals with rational or infinite endpoints.

Lemma B.2. *There exists a countable set $S \subset T$ such that for any fixed $t \in T$, the following is a null set:*

$$N_t = \bigcup_{A \in \mathscr{K}} \{\omega : X_s(\omega) \in A \ for\ all\ s \in S,\ X_t(\omega) \notin A\}.$$

Proof. Temporarily, fix some Borel set $A \subset \mathbb{R}$ and let t_0 be any point in T and

$$\varepsilon_1 = \sup_{t \in T} \mathbb{P}(X_{t_0} \in A,\ X_t \notin A).$$

Having constructed distinct $t_1, t_2, \ldots, \in T$ and $\varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_k$ define

$$\varepsilon_{k+1} = \sup_{t \in T} \mathbb{P}(X_{t_j} \in A \text{ for all } 1 \leq j \leq k,\ X_t \notin A).$$

Clearly, $\varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_{k+1}$. Moreover, one can always choose some $t_{k+1} \in T \setminus \{t_1, \ldots, t_k\}$ such that:

$$\mathbb{P}(X_{t_j} \in A \text{ for all } 1 \leq j \leq k,\ X_{t_{k+1}} \notin A) \geq \frac{\varepsilon_{k+1}}{2}.$$

Thus,

$$\begin{aligned}\sum_{k=2}^{\infty} \varepsilon_k &\leq 2 \sum_{k=2}^{\infty} \mathbb{P}(X_{t_j} \in A \text{ for all } 1 \leq j \leq k-1,\ X_{t_k} \notin A) \\ &= 2\mathbb{P}(X_{t_k} \notin A \text{ for some } k \geq 2) < \infty.\end{aligned}$$

In particular, $\lim_{k \to \infty} \varepsilon_k = 0$. So for any Borel set $A \subset \mathbb{R}$, there exists a countable set T_A such that

$$\sup_{t \in T} \mathbb{P}(X_s \in A \text{ for all } s \in T_A,\ X_t \notin A) = 0.$$

Then define $S = \cup_{A \in \mathscr{K}} T_A$ which is countable. Indeed, $\mathscr{K}$ can be written:

$$\mathscr{K} = \bigcup_{n=1}^{\infty} \left\{ A_1 \cup \cdots \cup A_n :\ A_1, \ldots, A_n \in \mathscr{J} \cup \mathring{\mathscr{J}} \right\}.$$

□

Lemma B.3. *For each $t \in T$,*

$$\bigcup_{A \in \mathscr{C}} \{\omega :\ X_s(\omega) \in A \; for\ all\ s \in S,\ X_t(\omega) \notin A\} \subset N_t.$$

Proof. Any $A \in \mathscr{C}$ can be written as $A = \cap_{n=1}^{\infty} A_n$, where $A_n \in \mathscr{K}$. By the previous lemma, for any such $A \in \mathscr{C}$ and $A_1, A_2, \cdots \in \mathscr{K}$,

$$\begin{aligned}\{\omega \quad : \quad & X_s(\omega) \in A \text{ for all } s \in S,\ X_t(\omega) \notin A_n\} \\ & \subset \{\omega : X_s(\omega) \in A_n \text{ for all } s \in S,\ X_t(\omega) \notin A_n\} \\ & \subset \bigcup_{E \in \mathscr{K}} \{\omega : X_s(\omega) \in E \text{ for all } s \in S,\ X_t(\omega) \notin E\} = N_t.\end{aligned}$$

To conclude, we notice that $A = \cap_{n \geq 1} A_n$ and the result follows. □

Proof of the Theorem. Let $O \in \widetilde{\mathscr{O}}$ and apply Lemma B.3 to the stochastic process $\{X_t;\ t \in O\}$ to conclude the existence of null sets $N_t(O)$ and a countable set $S_O \subset O$ such that

$$\bigcup_{A \in \mathscr{C}} \{\omega :\ X_s(\omega) \in A \text{ for all } s \in S_O,\ X_t(\omega) \notin A\} \subset N_t(O).$$

Since $\widetilde{\mathscr{O}}$ is countable, $S_* = \bigcup_{O\in\widetilde{\mathscr{O}}} S_O$ is countable. Similarly, for all $t \in T$, $\Lambda_t = \bigcup_{O\in\widetilde{\mathscr{O}}} N_t(O)$ is a null set. Define

$$R_O(\omega) = \overline{\{X_s(\omega);\ s \in O \cap S_*\}}.$$

R_O may include the values $\pm\infty$. R_O is closed and nonempty in $\mathbb{R}\cup\{\pm\infty\}$. For any $t \in T$, define the random set

$$R_t = \bigcap_{O\in\widetilde{\mathscr{O}}:\ t\in O} R_O.$$

Clearly, $R_t \subset \mathbb{R}\cup\{\pm\infty\}$ is closed and nonempty, for all $\omega \in \Omega$. Moreover,

$$\text{if } t \in T \text{ and } \omega \notin \Lambda_t, \text{ then } X_t(\omega) \in R_t(\omega). \tag{B.1}$$

Now we can start building the desired modification of X. For all $\omega \in \Omega$ and all $t \in S_*$, define $\widetilde{X}_t(\omega) = X_t(\omega)$. If $t \notin S_*$ and $\omega \notin \Lambda_t$, define also $\widetilde{X}_t(\omega) = X_t(\omega)$. Finally, whenever $t \notin S_*$ and $\omega \in \Lambda_t$, define $\widetilde{X}_t(\omega)$ to be some designated element of $R_t(\omega)$. Since $\mathbb{P}(\Lambda_t) = 0$ for each $t \in T$, $\mathbb{P}(\widetilde{X}_t = X_t) = 1$.

It remains to show that $\widetilde{X}$ is separable. Fix $A \in \mathscr{C}$ and $O \in \widetilde{\mathscr{O}}$, and suppose ω satisfies:

$$\widetilde{X}_s(\omega) \in A \text{ for all } s \in O \cap S_*.$$

If $s \in O \cap S_*$ but $\omega \notin \Lambda_t$, $\widetilde{X}_s(\omega) = X_s(\omega) \in R_O(\omega) \subset R_s(\omega) \subset A$, since A is closed. Similarly, if $s \in O$, but $s \notin S_*$ and $\omega \notin \Lambda_t$, then $\widetilde{X}_s(\omega) = X_s(\omega) \in R_O(\omega) \subset A$, by equation (B.1). Define

$$\Lambda = \bigcup_{s\in S_*} \Lambda_s.$$

Since S_* is countable, Λ is a null set. It is also chosen independently of all $A \in \mathscr{C}$ and $O \in \widetilde{\mathscr{O}}$. Finally we have shown that

$$\left\{\omega :\ \widetilde{X}_s(\omega) \in A,\ \ \forall s \in O \cap S_*\right\} \cap \Lambda^c \subset \left\{\omega :\ \widetilde{X}_s(\omega) \in A,\ \ \forall s \in O\right\},$$

and it is clear that

$$\left\{\omega :\ \widetilde{X}_s(\omega) \in A,\ \ \forall s \in O \cap S_*\right\} \supset \left\{\omega :\ \widetilde{X}_s(\omega) \in A,\ \ \forall s \in O\right\}.$$

The last two inclusions imply that

$$\left\{\omega :\ \widetilde{X}_s(\omega) \in A,\ \ \forall s \in O \cap S_*\right\} \setminus \left\{\omega :\ \widetilde{X}_s(\omega) \in A,\ \ \forall s \in O\right\} \subset \Lambda.$$

We conclude with the fact that any $U \in \mathscr{O}$ can be written $U = \cup_{n=1}^{\infty} O_n$ where $O_n \in \widetilde{\mathscr{O}}, \forall n$. Hence the last equation remains true for U:

$$\left\{\omega :\ \widetilde{X}_s(\omega) \in A,\ \ \forall s \in U \cap S_*\right\} \setminus \left\{\omega :\ \widetilde{X}_s(\omega) \in A,\ \ \forall s \in U\right\} \subset \Lambda\ ,$$

since Λ was built independently of all $O \in \widetilde{\mathscr{O}}$.

This proves the separability of $\widetilde{X}$. □

Remark B.4. *This theorem ensures that random variables such as* $\inf_{t\in O} X_t$, $\sup_{t\in O} X_t$, $\liminf_{t\to t_0} X_t$ *or* $\limsup_{t\to t_0} X_t$ *are indeed measurable. For any open set* $O \in \mathscr{O}$ *and* $\lambda \in \mathbb{R}$*:*

$$\begin{aligned}
\left\{\sup_{t\in O} X_t \leq \lambda\right\} &= \{X_t \leq \lambda, \textit{ for all } t \in O\} \\
&= \{X_t \leq \lambda, \textit{ for all } t \in O\cap S\} \cap \{X_t \leq \lambda, \ \forall t \in O \setminus S\} \\
&= \{X_t \leq \lambda, \textit{ for all } t \in O\cap S\} \setminus \\
&\quad \{\{X_t \leq \lambda, \textit{ for all } t \in O\cap S\} \setminus \{X_t \leq \lambda \textit{ for all } t \in O\}\}
\end{aligned}$$

The last part of the equality is included in a null set according to the theorem, thus assuming without restriction that the probability space is complete, it is also a null set. Hence the subset of Ω *considered above is measurable.*

Application to set-indexed processes. For set-indexed processes of IVANOFF AND MERZBACH [70] and their applications, the necessity of the existence of a separable modification was pointed out in Chapter 2 and in [63]. Given a set $\mathscr{T}$ and $\mathscr{A}$ a collection of subsets of $\mathscr{T}$, we assume that there is a distance $d_{\mathscr{A}}$ on $\mathscr{A}$. Hence it is required that the topology induced by $d_{\mathscr{A}}$ is second-countable. This happens for instance when $(\mathscr{A}, d_{\mathscr{A}})$ is totally bounded, which is always assumed in this thesis.

Analytic calculus of the small deviations of the fractional Brownian motion

C

This paragraph echoes a remark made at the beginning of Section 4.2, where we recall the (analytic) link established by KUELBS AND LI [84] between the small ball values of a Gaussian measure and the metric entropy of its RKHS. These authors note that it seems difficult to compute the metric entropy of the RKHS of the fractional Brownian motion, except for $H = 1/2$, and so to recover the small deviations analytically. Without computing directly the metric entropy of the RKHS, we propose another analytic method for $H > 1/2$. All the material needed here is covered in the article of LI AND LINDE [93] but we try to see it with another perspective.

Lemma 2.2 of [142] gives a general lower bound for the small deviations of a Gaussian process X, of the form:

$$\mathbb{P}\left(\sup_{t\in T}|X_t| \leq \varepsilon\right) \geq \exp\left(-K\, N(T, d_X, \varepsilon)\right)\ , \quad \forall \varepsilon > 0\ ,$$

for some positive constant K, where $N(T, d_X, \varepsilon)$ is the ε-metric entropy of T measured with the distance induced by X, and $N(T, d_X, \varepsilon)$ is assumed to be regularly varying. It is recalled that the upper bound was obtained in [106] using local nondeterminism.
Although Li and Linde claim that this bound is not sharp for many Gaussian processes[1], it is indeed for fractional Brownian motion. Note that this result was proved analytically in [140] for Gaussian measures. Bearing in mind Mandelbrot and Van Ness' integral representation of fractional Brownian motion, the rules of fractional calculus yield that the process B^H, defined for $H > 1/2$ and for a Brownian motion B by:

$$\begin{aligned} B^H_t &= a_H \int_{-\infty}^{0} \left((t-s)^{H-1/2} - (-s)^{H-1/2}\right)\, \mathrm{d}B_s + b_H \int_0^t (t-s)^{H-3/2}\, B_s\, \mathrm{d}s,\ \forall t \in [0,1] \\ &=: Z_H(t) + b_H W_{H+1/2}(t) \end{aligned}$$

is a fractional Brownian motion of parameter H (with the appropriate choice of constants a_H and b_H). We refer to Theorem 6.2 of [93] for a proof of this result, and notice that under this form, $W_{H+1/2}$ is in fact a fractional integral of Brownian motion of order $H - 1/2$.

Hence, Anderson's inequality [9] implies:

$$\mathbb{P}\left(\sup_{t\in[0,1]} |B^H_t| \leq \varepsilon\right) \leq \mathbb{P}\left(\sup_{t\in[0,1]} |b_H W_{H+1/2}(t)| \leq \varepsilon\right)\ .$$

[1] They even indicate that this is the reason that pushed them to produce the results of [93], which are more precise than Talagrand's bound.

The upper bound of the small deviations of $W_{H+1/2}$ is given exactly in Theorem 6.2 of [93]. Nevertheless, we want to emphasize that this does not come from a probabilistic estimate by sketching the proof of this result. Let us denote I_{0+}^{β} the right fractional integral (or Riemann-Liouville fractional integral) of order β. For $H > 0$, let

$$e_n\left(I_{0+}^{H+1/2} : L^2[0,1] \to C[0,1]\right) = \inf\left\{\varepsilon > 0 : \ N\left(I_{0+}^{H+1/2}(K), \|\cdot\|, \varepsilon\right) \le 2^{n-1}\right\}$$

where K is the unit ball of $L^2[0,1]$ and $I_{0+}^{H+1/2}(K)$ is a compact subset of $C[0,1]$ with the sup-norm $\|\cdot\|$, since $H + 1/2 > 1/2$. Next, we simply denote this quantity by e_n. We refer to Proposition 6.1 of [93] where it is shown that $e_n \succcurlyeq n^{-H-1/2}$. A brief calculus suffices to deduce that

$$\log N(I_{0+}^{H+1/2}(K), \|\cdot\|, \varepsilon) \succcurlyeq \varepsilon^{-1/(H+1/2)} \ .$$

For $H \in (0,1)$, it is known (see Section 1.3.3 and [39]) that $I_{0+}^{H+1/2}(K)$ is the same vector space as the RKHS of the fractional Brownian motion of parameter $\beta - 1/2$. Therefore, the link between the entropy of the RKHS and the small balls, as in [84] or Theorem 1.1 iii) of [93], reads here:

$$\begin{aligned} -\log \mathbb{P}\left(\sup_{t\in[0,1]} |B_t^H| \le \varepsilon\right) &\succcurlyeq \varepsilon^{-2(H+1/2)^{-1}/(2-(H+1/2)^{-1})} \\ &\succcurlyeq \varepsilon^{-2/H} \ . \end{aligned}$$

This is the correct upper bound for the small deviations of the H-fBm, and local nondeterminism was not used to derive it.

Bibliography

[1] R. J. Adler, *Hausdorff dimension and Gaussian fields*, Ann. Probab. **5** (1977), no. 1, 145–151.

[2] ______, *The Geometry of Random Fields*, Wiley and sons, 1981.

[3] ______, *An introduction to continuity, extrema and related topics for general gaussian processes*, Lecture Notes- Monographs Series, Institute of mathematical statistics, 1990.

[4] R. J. Adler, D. Monrad, R.H. Scissors, and R. Wilson, *Representations, decompositions and sample function continuity of random fields with independent increments*, Stochastic Process. Appl. **15** (1983), no. 1, 3–30.

[5] R.J. Adler and J.E. Taylor, *Random fields and geometry*, Springer Verlag, 2007.

[6] S. Albeverio and M. W. Yoshida, *$H - C^1$ Maps and elliptic SPDEs with polynomial and exponential perturbations of Nelson's Euclidean free field*, J. Funct. Anal. **196** (2002), no. 2, 265–322.

[7] K. S. Alexander, *Probability Inequalities for Empirical Processes and a Law of the Iterated Logarithm*, Ann. Probab. **12** (1984), no. 4, 1041–1067.

[8] ______, *Sample Moduli for Set-Indexed Gaussian Processes*, Ann. Probab. **14** (1986), no. 2, 598–611.

[9] T. W. Anderson, *The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities*, Proc. Am. Math. Soc. **6** (1955), no. 2, 170–176.

[10] A. Araujo and E. Giné, *On tails and domains of attraction of stable measures in Banach spaces*, Trans. Amer. Math. Soc. **248** (1979), no. 1, 105–119.

[11] A. Ayache, N.-R. Shieh, and Y. Xiao, *Multiparameter multifractional Brownian motion: Local nondeterminism and joint continuity of the local times*, Ann. Inst. Henri Poincaré Probab. Stat. **47** (2011), no. 4, 1029–1054.

[12] A. Ayache and M. S. Taqqu, *Multifractional processes with random exponent*, Publ. Mat. **49** (2005), 459–486.

[13] R. M. Balan, *Set-Markov processes*, Ph.D. thesis, University of Ottawa, 2001.

[14] R. M. Balan and B. G. Ivanoff, *A Markov Property for Set-Indexed Processes*, J. Theoret. Probab. **15** (2002), no. 3, 553–588.

[15] P. Balança, *An increment type set-indexed Markov property*, (2012), arXiv:1207.6568.

[16] P. Balança and E. Herbin, *2-Microlocal Analysis of Martingales and Stochastic Integrals*, Stochastic Process. Appl. **122** (2012), no. 6, 2346–2382.

[17] ______, *A set-indexed Ornstein-Uhlenbeck process*, Electron. Commun. Probab. **17** (2012), no. 39, 1–14.

[18] D. Baraka, T. S. Mountford, and Y. Xiao, *Hölder properties of local times for fractional Brownian motions*, Metrika **69** (2008), no. 2-3, 125–152.

[19] R. F. Bass and R. Pyke, *The existence of set-indexed lévy processes*, Z. Wahrsch. Verw. Gebiete **66** (1984), no. 2, 157–172.

[20] A. Benassi, S. Jaffard, and D. Roux, *Elliptic Gaussian random processes*, Rev. Mat. Iberoamericana **13** (1997), no. 1, 19–90.

[21] S. M. Berman, *Gaussian Processes with Stationary Increments: Local Times and Sample Function Properties*, Ann. Math. Stat. **41** (1970), no. 4, 1260–1272.

[22] ______, *Gaussian sample functions: uniform dimension and hölder conditions nowhere*, Nagoya Math. J. **46** (1972), 63–86.

[23] L. Beznea, A. Cornea, and M. Röckner, *Potential theory of infinite dimensional Lévy processes*, J. Funct. Anal. **261** (2011), no. 10, 2845–2876.

[24] C. Borell, *Gaussian Radon measure on locally convex spaces*, Math. Scand. **38** (1976), 265–284.

[25] J.-C. Breton, I. Nourdin, and G. Peccati, *Exact confidence intervals for the Hurst parameter of a fractional Brownian motion*, Electron. J. Stat. **3** (2009), 416–425.

[26] R. Cairoli, *Un théorème de convergence pour martingales à indices multiples*, CR Acad. Sci. Paris Sér. AB **269** (1969), A587–A589.

[27] R. Cairoli and J. B. Walsh, *Stochastic integrals in the plane*, Acta Math. **134** (1975), no. 1, 111–183.

[28] R. Carmona, *Tensor product of Gaussian measures*, Vector Space Measures and Applications I, Springer, 1978, pp. 96–124.

[29] L. Chen and D. Jakobson, *Gaussian Free Fields and KPZ Relation in $\mathbb{R}^4$*, Ann. Henri Poincaré (2013).

[30] N. N. Chentsov, *Lévy Brownian Motion for Several Parameters and Generalized White Noise*, Theory Probab. Appl. **2** (1957), no. 2, 265–266.

[31] K. L. Chung, *On the maximum partial sums of sequences of independent random variables*, Trans. Amer. Math. Soc. **64** (1948), no. 2, 205–233.

[32] E. Csáki, *A relation between Chung's and Strassen's laws of the iterated logarithm*, Z. Wahrsch. Verw. Gebiete **54** (1980), no. 3, 287–301.

[33] R. C. Dalang, *Extending martingale measure stochastic integral with applications to spatially homogeneous SPDEs*, Electron. J. Probab. **4** (1999), 1–29.

[34] R. C. Dalang, E. Nualart, D. Khoshnevisan, D. Wu, and Y. Xiao, *Critical Brownian sheet does not have double points*, Ann. Probab. **40** (2012), no. 4, 1–32.

[35] R. C. Dalang and J. B. Walsh, *The sharp Markov property of Lévy sheets*, Ann. Probab. (1992), 591–626.

[36] ______, *Geography of the level sets of the Brownian sheet*, Probab Theory Related Fields **96** (1993), no. 2, 153–176.

[37] A. de Acosta, *Stable measures and seminorms*, Ann. Probab. **3** (1975), no. 5, 865–875.

[38] ______, *Small deviations in the functional central limit theorem with applications to functional laws of the iterated logarithm*, Ann. Probab. **11** (1983), no. 1, 78–101.

[39] L. Decreusefond and A.S. Üstünel, *Stochastic Analysis of the Fractional Brownian Motion*, Potential Anal. **411** (1999), 177–214.

[40] P. Deheuvels and D. M. Mason, *Random fractal functional laws of the iterated logarithm*, Studia Scientiarum Mathematicarum Hungarica **34** (1998), no. 1, 89–106.

[41] V. Dobric and F. M. Ojeda, *Fractional Brownian fields, duality, and martingales*, IMS Lecture Notes-Monograph series. High dimensional probability **51** (2006), 77–95.

[42] J.L. Doob, *Stochastic processes*, Wiley, 1953 (Reprinted 1990).

[43] R. M. Dudley, *The sizes of compact subsets of Hilbert space and continuity of Gaussian processes*, J. Funct. Anal. **1** (1967), no. 3, 290–330.

[44] ______, *Sample Functions of the Gaussian Process*, Ann. Probab. **1** (1973), no. 1, 66–103.

[45] ______, *Central limit theorems for empirical measures*, Ann. Probab. **6** (1978), no. 6, 899–929.

[46] K. Falconer, *Fractal geometry: Mathematical Foundations and Applications*, John Wiley & Sons, 2003.

[47] K. J. Falconer, *Tangent Fields and the Local Structure of Random Fields*, J. Theoret. Probab. **15** (2002), no. 3, 731–750.

[48] X. Fernique, *Intégrabilité des vecteurs Gaussiens*, CR Acad. Sci. Paris Sér. AB **270** (1970), 1698–1699.

[49] ______, *Des résultats nouveaux sur les processus gaussiens*, Séminaire de Probabilités IX Université de Strasbourg, Springer, 1975, pp. 318–335.

[50] A. Garsia, E. Rodemich, and H. Rumsey, *A Real Variable Lemma and the Continuity of Paths of Some Gaussian Processes*, Indiana Univ. Math. J. **20** (1970), no. 6, 565–578.

[51] I. M. Gel'fand, *Generalized random processes*, Dok. Akad. Nauk SSSR **100** (1955), no. 5, 853–856, Translated version available in the book *Collected Parpers, Vol. III* of Gel'fand.

[52] I.M. Gel'fand and N.Y. Vilenkin, *Generalized Functions Vol.4: Applications of harmonic analysis*, Academic Press, 1964.

[53] I.I. Gikhman and A.V. Skorokhod, *Introduction to the theory of random processes*, W.B. Saunders Company, 1969.

[54] V. Goodman, *Characteristics of Normal Samples*, Ann. Probab. **16** (1988), no. 3, 1281–1290.

[55] V. Goodman and J. Kuelbs, *Rates of clustering for some Gaussian self-similar processes*, Probab. Theory Related Fields **88** (1991), no. 1, 47–75.

[56] L. Gross, *Abstract Wiener spaces*, Fifth Berkeley symposium on Math. Statist. and Prob., 1967, pp. 31–42.

[57] E. Herbin, *From N-Parameter Fractional Brownian Motions to N-Parameter Multifractional Brownian Motions*, Rocky Mountain J. Math. **36** (2006), no. 4, 1249–1284.

[58] E. Herbin, B. Arras, and G. Barruel, *From almost sure local regularity to almost sure Hausdorff dimension for Gaussian fields*, ESAIM: Probability and Statistics (2013).

[59] E. Herbin and J. Lévy-Véhel, *Stochastic 2-microlocal analysis*, Stochastic Process. Appl. **119** (2009), no. 7, 2277–2311.

[60] E. Herbin and E. Merzbach, *A Set-indexed Fractional Brownian Motion*, J. Theoret. Probab. **19** (2006), no. 2, 337–364.

[61] ______, *The multiparameter fractional Brownian motion*, Math Everywhere, Springer, 2007, pp. 93–101.

[62] ______, *Stationarity and Self-Similarity Characterization of the Set-Indexed Fractional Brownian Motion*, J. Theoret. Probab. **22** (2008), no. 4, 1010–1029.

[63] ______, *The set-indexed Lévy process: Stationarity, Markov and sample paths properties*, Stochastic Process. Appl. **123** (2013), no. 5, 1638–1670.

[64] E. Herbin and A. Richard, *Local Hölder regularity of set-indexed processes*, Preprint (2012), arXiv:1203.0750v2.

[65] E. Herbin and Y. Xiao, *Sample paths properties of the set-indexed fractional Brownian motion*, In preparation (2014).

[66] H. E. Hurst, *Long-term storage capacity of reservoirs*, Trans. Amer. Soc. Civil Engineers **116** (1951), 770–808.

[67] K. Ito, *Stationary random distributions*, Mem. College Sci. Univ. Kyoto Ser. A Math. **28** (1954), no. 3, 209–223.

[68] G. Ivanoff, *Set-Indexed Processes: Distributions and Weak Convergence*, Topics in Spatial Stochastic Processes, Lecture Notes in Mathematics, vol. 1802, Springer Berlin / Heidelberg, 2003, pp. 85–125.

[69] G. Ivanoff and E. Merzbach, *Set-indexed Markov processes*, Stochastic models (Ottawa, ON, 1998), Conference proceedings of the CMS, vol. 26, 2000, pp. 217–232.

[70] ______, *Set-Indexed Martingales*, Chapman & Hall/CRC, 2000.

[71] G. Ivanoff and P. Sawyer, *Local time for processes indexed by a partially ordered set*, Stat. Probab. Lett. **61** (2003), 1–15.

[72] M. Jolis, *On the Wiener integral with respect to the fractional Brownian motion on an interval*, J. Math. Anal. Appl. **330** (2007), no. 2, 1115–1127.

[73] M. Jolis and N. Viles, *Continuity in the Hurst parameter of the law of the symmetric integral with respect to the fractional Brownian motion*, Stochastic Process. Appl. **120** (2010), no. 9, 1651–1679.

[74] O. Kallenberg, *Foundations of modern probability*, Springer, 2002.

[75] A. Kamont, *On the fractional anisotropic Wiener field*, Probab. Math. Stat. **16** (1995), no. 1, 85–98.

[76] D. Khoshnevisan, *Multiparameter Processes: an introduction to random fields*, Springer, 2002.

[77] D. Khoshnevisan, Y. Peres, and Y. Xiao, *Limsup random fractals*, Electron. J. Probab **5** (2000), no. 4, 1–24.

[78] D. Khoshnevisan and Z. Shi, *Fast sets and points for Fractional Brownian Motion*, Séminaire de Probabilités XXXIV, Springer, 2000, pp. 393–416.

[79] D. Khoshnevisan and Y. Xiao, *Lévy processes: Capacity and Hausdorff dimension*, Ann. Probab. **33** (2005), no. 3, 841–878.

[80] T. Kitagawa, *Analysis of variance applied to function space*, Mem. Fac. Sci U. Kyusu **6** (1951), no. 1, 41–53.

[81] A. N. Kolmogorov, *Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum*, CR (Dokl.) Acad. Sci. URSS **26** (1940), 115–118.

[82] J. Kuelbs, *A representation theorem for symmetric stable processes and stable measures on H*, Z. Wahrsch. Verw. Gebiete **26** (1973), no. 4, 259–271.

[83] ———, *A Strong Convergence Theorem for Banach Space Valued Random Variables*, Ann. Probab. **4** (1976), no. 5, 744–771.

[84] J. Kuelbs and W. V. Li, *Metric Entropy and the Small Ball Problem for Gaussian Measures*, J. Funct. Anal. **116** (1993), no. 1, 133–157.

[85] J. Kuelbs, W. V. Li, and M. Talagrand, *Lim Inf Results for Gaussian Samples and Chung's Functional LIL*, Ann. Probab. **22** (1994), 1879–1903.

[86] H.H. Kuo, *Gaussian measures in Banach spaces*, Berlin:Springer-Verlag, 1975.

[87] G. F. Lawler and F. J. Viklund, *Optimal Hölder exponent for the SLE path*, Duke Math. J. **159** (2011), no. 3, 351–383.

[88] J. Lebovits, J. Lévy Véhel, and E. Herbin, *Stochastic integration with respect to multifractional Brownian motion via tangent fractional Brownian motions*, Stochastic Process. Appl. (2013).

[89] M. Ledoux and M. Talagrand, *Probability in Banach Spaces: Isoperimetry and Processes*, A Series of Modern Surveys in Mathematics Series, Springer, 1991.

[90] N. Leonenko, M. D. Ruiz-Medina, and M. S. Taqqu, *Fractional Elliptic, Hyperbolic and Parabolic Random Fields*, Electron. J. Probab. **16** (2011), 1134–1172.

[91] P. Lévy, *Exemples de processus doubles de Markoff*, CR Acad. Sci. Paris **226**, 307–308.

[92] ———, *Processus stochastiques et mouvement brownien*, Gauthier-Villars, Paris (1948).

[93] W. V. Li and W. Linde, *Approximation, metric entropy and small ball estimates for Gaussian measures*, Ann. Probab. **27** (1999), no. 3, 1556–1578.

[94] M. Lifshits, W. Linde, and Z. Shi, *Small deviations of Riemann-Liouville processes in L_q-norms with respect to fractal measures*, Proc. London Math. Soc. **92** (2006), 224–250.

[95] M. Loève, *Probability Theory I*, 4th ed., Springer-Verlag, 1977.

[96] N. Luan and Y. Xiao, *Chung's law of the iterated logarithm for anisotropic Gaussian random fields*, Statistics & Probability Letters **80** (2010), no. 23-24, 1886–1895.

[97] ———, *Spectral Conditions for Strong Local Nondeterminism and Exact Hausdorff Measure of Ranges of Gaussian Random Fields*, Journal of Fourier Analysis and Applications **18** (2011), no. 1, 118–145.

[98] B. B. Mandelbrot and J. W. Van Ness, *Fractional Brownian motions, fractional noises and applications*, SIAM review **10** (1968), no. 4, 422–437.

[99] M. B. Marcus, *Hölder conditions for Gaussian processes with stationary increments*, Trans. Amer. Math. Soc. **134** (1968), no. 1, 29–52.

[100] M. B. Marcus and J. Rosen, *Markov Processes, Gaussian Processes, and Local Times*, Cambridge University Press, Cambridge, 2006.

[101] D. M. Mason and Z. Shi, *Small Deviations for Some Multi-Parameter Gaussian Processes*, J. Theoret. Probab. **14** (2001), no. 1, 213–239.

[102] H. P. McKean, *Brownian Motion with a Several-Dimensional Time*, Theory Probab. Appl. **8** (1963), no. 4, 335–354.

[103] M. M. Meerschaert, W. Wang, and Y. Xiao, *Fernique-type inequalities and moduli of continuity for anisotropic Gaussian random fields*, Trans. Amer. Math. Soc. **365** (2012), no. 2, 1081–1107.

[104] E. Merzbach and D. Nualart, *Markov Properties for Point Processes on the Plane*, Ann. Probab. **18** (1990), no. 1, 342–358.

[105] R. A. Minlos, *Generalized random processes and their extension to a measure*, Selected Translations in mathematical statistics and probability **3** (1959).

[106] D. Monrad and H. Rootzén, *Small values of Gaussian processes and functional laws of the iterated logarithm*, Probab. Theory Related Fields **101** (1995), no. 2, 173–192.

[107] E. Nelson, *Construction of quantum fields from Markoff fields*, J. Funct. Anal. **12** (1973), no. 1, 97–112.

[108] ———, *The free Markoff field*, J. Funct. Anal. **12** (1973), no. 2, 211–227.

[109] I. Nourdin, *Selected aspects of fractional Brownian motion*, Springer, 2013.

[110] D. Nualart, *The Malliavin calculus and related topics*, Springer, 2006.

[111] D. Nualart and S. Tindel, *Quasilinear stochastic elliptic equations with reflection*, Stochastic Process. Appl. **57** (1995), no. 1, 73–82.

[112] S. Orey, *Growth rate of certain Gaussian processes*, Proc. Sixth Berkeley Symp. Math. Statist. Probab, 1972, pp. 443–451.

[113] S. Orey and W. E. Pruitt, *Sample Functions of the N-Parameter Wiener Process*, Ann. Probab. **1** (1973), no. 1, 138–163.

[114] S. Orey and S. J. Taylor, *How Often on a Brownian Path Does the Law of Iterated Logarithm Fail?*, Proc. London Math. Soc. **28** (1974), no. 3, 174–192.

[115] M. Ossiander and R. Pyke, *Lévy's Brownian motion as a set-indexed process and a related central limit theorem*, Stochastic Process. Appl. **21** (1985), no. 1, 133–145.

[116] R. Peltier and J. Lévy Véhel, *Multifractional Brownian motion: definition and preliminary results*, INRIA technical report 2645 (1995).

[117] L. D. Pitt, *Local times for Gaussian vector fields*, Indiana Univ. Math. J. **27** (1978), 309–330.

[118] R. Pyke, *Asymptotic results for empirical and partial-sum processes: A review*, Canad. J. Statist. **12** (1984), no. 4, 241–264.

[119] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol.I: Functional analysis*, Academic Press, 1980.

[120] A. Richard, *Some singular sample path properties of a multiparameter fractional Brownian motion*, Submitted (2014), arXiv:1410.4430.

[121] ______, *A fractional Brownian field indexed by L^2 and a varying Hurst parameter*, To appear in Stochastic Process. Appl. (2015), arXiv:1312.6069.

[122] Y. A. Rozanov, *Spectral Analysis of Abstract Functions*, Theory Probab. Appl. **4** (1959), no. 3, 271–287.

[123] W. Rudin, *Real and complex analysis*, Series in higher mathematics, McGraw-Hill, 1987.

[124] F. Russo, *Etude de la propriété de Markov étroite en relation avec les processus planaires à accroissements indépendants*, Séminaire de probabilités XVIII, 1984, pp. 353–378.

[125] R. A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer Monographs in Mathematics, Springer London, 2002.

[126] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach Science, 1993.

[127] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes*, 1994.

[128] K.-I. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1999.

[129] R.L. Schilling, R. Song, and Z. Vondracek, *Bernstein functions: theory and applications*, vol. 37, Walter de Gruyter, 2012.

[130] L. Schwartz, *La fonction aléatoire du mouvement brownien*, Séminaire Bourbaki **4** (1958), 327–349.

[131] I. E. Segal, *Abstract probability spaces and a theorem of Kolmogoroff*, Am. J. Math. (1954), 721–732.

[132] Q.-M. Shao and D. Wang, *Small ball probabilities of Gaussian fields*, Probab. Theory Related Fields **102** (1995), no. 4, 511–517.

[133] S. Sheffield, *Gaussian free fields for mathematicians*, Probab. Theory Related Fields **139** (2007), no. 3-4, 521–541.

[134] S. Stoev and M. S. Taqqu, *How rich is the class of multifractional Brownian motions?*, Stochastic Process. Appl. **116** (2006), no. 2, 200–221.

[135] V. Strassen, *An invariance principle for the law of the iterated logarithm*, Z. Wahrsch. Verw. Gebiete **3** (1964), no. 3, 211–226.

[136] D. W. Stroock, *Probability Theory: An Analytic View*, 2nd ed., Cambridge University Press, 2010, Available at http://math.mit.edu/~dws/177/prob08.pdf.

[137] S. Takenaka, *Representation of Euclidean random field*, Nagoya Math. J. **105**, 19–31.

[138] S. Takenaka, I. Kubo, and H. Urakawa, *Brownian motion parametrized with metric space of constant curvature*, Nagoya Math. J. **82** (1981), 131–140.

[139] M. Talagrand, *Regularity of gaussian processes*, Acta Math. **159** (1987), no. 1, 99–149.

[140] ———, *New Gaussian estimates for enlarged balls*, Geom. Funct. Anal. **3** (1993), no. 5, 502–526.

[141] ———, *The Small Ball Problem for the Brownian Sheet*, Ann. Probab. **22** (1994), no. 3, 1331–1354.

[142] ———, *Hausdorff Measure of Trajectories of Multiparameter Fractional Brownian Motion*, Ann. Probab. **23** (1995), no. 2, 767–775.

[143] ———, *Multiple points of trajectories of multiparameter fractional Brownian motion*, Probab. Theory Related Fields **112** (1998), no. 4, 545–563.

[144] ———, *Upper and lower bounds for stochastic processes*, Springer, 2014.

[145] S. J. Taylor, *The measure theory of random fractals*, Math. Proc. Cambridge Philos. Soc., vol. 100, Cambridge Univ Press, 1986, pp. 383–406.

[146] C. A. Tudor and Y. Xiao, *Sample path properties of bifractional Brownian motion*, Bernoulli **13** (2007), no. 4, 1023–1052.

[147] J. B. Walsh, *An introduction to stochastic partial differential equations*, École d'Été de Probabilités de Saint Flour XIV - 1984 (P.L. Hennequin, ed.), Lecture Notes in Mathematics, vol. 1180, Springer Berlin Heidelberg, 1986, pp. 265–439.

[148] ———, *Martingales with a multidimensional parameter and stochastic integrals in the plane*, Lectures in Probability and Statistics (Springer Berlin / Heidelberg, ed.), 1986, pp. 329–491.

[149] N. Wiener, *Differential space*, J. Math. Phys. **2** (1923), 131–174.

[150] E. Wong and M. Zakai, *Martingales and stochastic integrals for processes with a multi-dimensional parameter*, Probab. Theory Related Fields **29** (1974), no. 2, 109–122.

[151] Y. Xiao, *Hausdorff measure of the sample paths of Gaussian random fields*, Osaka J. Math. **33** (1996), 895–913.

[152] ———, *Sample path properties of anisotropic Gaussian random fields*, In: A minicourse on stochastic partial differential equations, Springer, 2009, pp. 145–212.

[153] ———, *Uniform modulus of continuity of random fields*, Monatsh. Math. **159** (2010), no. 1-2, 163–184.

[154] A. M. Yaglom, *Some Classes of Random Fields in n -Dimensional Space, Related to Stationary Random Processes*, Theory Probab. Appl. **2** (1957), no. 3, 273–320.

[155] J. A. Yan, *Generalizations of Gross' and Minlos' theorems*, Séminaire de probabilités XXIII, 1989, pp. 395–404.