On the modelling of wireless communication networks using non-poisson point processes

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ON THE MODELLING OF WIRELESS COMMUNICATION NETWOEKS USING NON-POISSON POINT PROCESSES

SUR LA MODÉLISATION LES RÉSEAUX DE COMMUNCATION SANS FIL EN UTILISANT LES PROCESSUS PONCTUELS NON-POISSON

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Summary

Stochastic geometry is a powerful tool to model large wireless networks with high variation of node locations. In this framework, a common assumption is that the node locations form a realization of a Poisson Point Process (PPP). Using available results on the Laplace transform of the Shot Noise processes associating with PPPs, one can obtain closed form expressions of many performance metrics of interest such as the Medium Access Probability (MAP), the Coverage Probability (COP) and the Spatial Density of Throughput (SDT). However, in many wireless network deployments, there is a Carrier Sensing (CS) mechanism to refrain nodes which are too close to each other from transmitting at the same time. In these network, the process of nodes concurrently transmitting at any time does not form a realization of a PPP any more, and this makes the analysis of the network performance a challenging problem.

The aim of this dissertation is to study this problem in two directions. In the first direction, we provide a comprehensive stochastic geometry framework based on Point Processes with exclusion to model the transmitting nodes in different types of wireless networks with CS mechanism. The considered networks are Carrier Sensing Multiple Access (CSMA) networks with perfect CS, Cognitive Radio networks where secondary users use Carrier Sensing to detect primary users, and CSMA networks with imperfect CS mechanism. For the first two cases, we provide approximations of the main network performance metrics, namely the MAP, the COP and the SDT. For the last case, we give analytic bounds on the critical spatial density of nodes where CSMA starts to behave like ALOHA (i.e. the process of concurrent transmitting nodes in the network forms a realization of a PPP). Although this phenomenon has been studied earlier by means of simulations, no analytic result was known to the best of our knowledge.

In the second direction, we go deeper into the problem of studying the distribution of points patterns of the Point Processes associated with the classical Matérn type II and Matérn type III models [Matérn 68]. These are the two models that are used to model CSMA networks with perfect CS. Although these model were introduced long ago and have many applications in many disciplines, the distribution of the points patterns in their associated Point Processes in general and the Laplace transform of the corresponding Shot Noise processes are still open problems. We prove that the probability generating functional of this Point Process, when properly parameterized, is the unique solution of some

systems of differential functional equations. Using these systems of equations, one can get a lower bound and an upper bound on these generating functional. This result can then be applied to the stochastic geometry framework mentioned above to further bridge the gap between analytic mathematical frameworks and practical network deployments.

Sommaire

La géométrie stochastique est un outil puissant pour modéliser des grands réseaux sans fil avec une grande variation de la position des nœuds. Dans ce cadre, une hypothèse courante est que l'emplacement des nœuds forme une réalisation d'un processus ponctuel de Poisson (PPP). En utilisant les résultats disponibles concernant la transformée de Laplace du processus bruit de grenaille associé à des PPPs, on peut obtenir des solutions de forme fermée des métriques de performance de réseau telles que la probabilité d'accès au médium (MAP), la probabilité de couverture (COP) et de la densité spatiale de débit (SDT). Cependant, dans de nombreux déploiements de réseaux sans fil, il y a un mécanisme de détection des porteuses(CS) pour empêcher nœuds qui sont trop proches les uns des autres de transmettre en même temps. Dans ces réseaux, le processus des nœuds qui transmettent simultanément à tout moment ne forme plus une réalisation d'un PPP, ce qui rend l'analyse des performances des réseaux dans ces cas, un problème difficile. L'objectif de cette thèse est d'étudier ce problème dans deux directions. Dans la première direction, nous proposons un cadre complet de la géométrie stochastique qui utilise des processus ponctuels avec exclusion pour modéliser des transmetteurs dans différents types de réseaux sans fil avec un mécanisme de CS. Les réseaux considérés sont les réseaux à accès multiple en cherchant à détecter une porteuse (Carrier Sensing Multiple Access-CSMA) avec un mécanisme de détection (CS) parfait, les réseaux de radio-communications cognitifs où les utilisateurs secondaires utilisent la détection de porteuse pour détecter les utilisateurs principaux et les réseaux CSMA avec un CS imparfait. Pour les deux premiers cas, nous dérivons des approximations des métriques de performances principales de réseau, c'est-à-dire la MAP, la COP et la SDT. Pour le dernier cas, nous donnons des bornes sur la densité spatiale critique des nœuds où CSMA commence à se comporter comme ALOHA (c'est-à-dire le processus de des nœuds qui transmettent simultanément dans le réseau forme une réalisation d'un PPP). Bien que ce phénomène ait été étudié auparavant par simulations, aucun résultat d'analyse n'a été connu à de notre connaissance. Dans la seconde direction, nous étudions la distribution processus ponctuel s associés avec les classique Matérn modèles de type II et type III [Matérn 68]. Ce sont les deux modèles utilisés pour modéliser les réseaux CSMA avec un CS parfait. Bien que ces modèles aient été introduits il y a longtemps et qu'ils aient de nombreuses applications dans de nombreuses disciplines, la distribution de leurs processus ponctuels associés et la transformation de Laplace des

processus bruit de grenaille correspondant est encore un problème ouvert. Nous montrons ici que la fonctionnelle génératrice des probabilités de ces processus ponctuels, lorsqu'elle est correctement paramétrée, est la solution unique de certain système d'équations différentielles. Grace à l'utilisation de ce système d'équations, on peut obtenir une borne inférieure et une borne supérieure de ces fonctionnelle. Ce résultat peut ensuite être appliqué au cadre de la géométrie stochastique mentionnée ci-dessus pour mieux connecter les cadres d'analyse mathématiques et les déploiements de réseaux pratiques.

Preface

Wireless networks are collections of terminals wishing to communicate with each other using radio signals. Nowadays, they become more and more pervasive due to their ability to handle mobility, their flexibility and their low cost. The operation of wireless networks is quite different from that of wired networks in the way that different transmissions interact with each other. In wired networks, if two terminals share the same cable, their signal will be completely destroyed when they transmit at the same time; however, if they do not share the same cable, they can transmit independently without interfering with each other. In wireless networks, all transmissions take place in a common air medium and they interact in a semi-destructive way in the sense that a transmission is successful if and only if (iff) its signal power is strong enough compared to the total interference signal power from other terminals in the network.

In all networks, no matter wired or wireless, there is always a need for a set of rules to coordinate the transmissions in such a way that the resources are employed efficiently. These sets of rules are called Multiple Access Control (MAC) protocols. A good analogy is to think of a communication network as a network of roads in a city, transmissions as cars travelling on the roads and the corresponding MAC protocol as the system of traffic lights which dictates which cars can run and which cars have to stop and wait. Due to the difference alluded to above, the MAC protocols of wireless networks are very different in nature from those of wired networks. For this reason, the modelling and analysis of MAC protocols for wireless networks have attracted a lot of attentions over the last decades, from the 1970's.

In the very first studies [13, 4], the point of view is mostly adopted from the analysis of wired networks. It is assumed that the common air medium is a broadcast medium, so that terminals behave as if they were sharing a common cable. When a terminal transmits, all other terminals can "see" it and if another terminal transmits at the same time, the signals will collide and both transmissions will be lost. Of course, when two terminals are far away, due to the decaying of radio signal power, they may not always "see" each other and hence the above assumption only holds when the considered network is small. One step further to model the interference more accurately is to consider only the strongest interferers. In this viewpoint, a transmission is successful iff there is no strong interferer transmitting at the same time. This gives rise to the notion of *spatial reuse*, i.e. if two terminals are far apart enough, their interference to each other is small and they can transmit at the same time, thus reusing the space resource [14]. Nevertheless, such an approach neglects the fact that interference does add up and sometimes the total interference from a large number of weak interferers can still be strong enough to destroy the tagged signal. Hence, this approach is only applicable for medium scale networks, where the number of weak interferers is small. During the last few years, *Stochastic geometry* has gained its popularity as an effective tool to model the aggregated interference in *large scale wireless networks* quite accurately. By presenting the positions of the terminals as a random collection of points in the plane, which is referred to as a point process in the stochastic geometry literature, the aggregated interference from all other terminals to the tagged terminal can be expressed as a Shot-Noise process at that terminal. The analysis of the network of interest can be carried out using the knowledge about the statistical distribution of this Shot-Noise process [6].

So far, most of the applications of stochastic geometry in wireless network modelling and analysis have been limited by the technical need to use Poisson point processes to model terminals locations. The popularity of Poisson point processes stems from their tractability and from the fact that they can model quite accurately the terminals locations of a popular MAC protocol-the ALOHA protocol. By tractability we mean that the Laplace transforms of the corresponding Shot-Noise processes, which contain all the knowledge about their statistical distribution, can be computed in closed-form. These closed-form expressions can then be used to compute many performance metrics of interest of the networks. On the other hand, not all MAC protocols can be modelled using Poisson point processes. More specifically, the points of a Poisson point process in disjoint areas are distributed independently of each other. This is linked to the independent behaviour of terminals, i.e. each terminal acts on their own without taking any notice of the actions of the others, which is usually found in ALOHA protocols. In other protocols, there is usually some coordination among terminals in order to utilize network resources (time, frequency, energy, etc.) more efficiently. An example is the Carrier Sensing Multiple Access (CSMA) protocol where each terminal has the "listen before talk" type of behaviour. When a terminal has a message to transmit, it first senses the common air medium to see if there is any strong interferer nearby who is transmitting. If there is, it has to refrain from transmitting and wait for a random period before it can start another attempt. Another example where PPPs fail to provide an appropriate model is the modelling of Cognitive Radio Networks, a new wireless communication paradigm which has emerged during the last few years as a response to the problem of bandwidth scarcity. In these networks, there are two classes of terminals: the primary and the secondary, where the primary terminals have higher priority than the secondary ones. Each secondary terminal wishing to transmit has to monitor the common air medium (carrier sensing) and at the presence of a nearby active primary terminal, it has to refrain its own transmission, hence giving up the air medium to the primary terminal.

The first aim of this thesis is to propose appropriate models for the kinds of networks alluded to above, namely the CSMA Networks and the Cognitive Radio

Networks using various MAC schemes. All the results in this directions are systematically presented in Chapter 2 after a brief introduction of the Stochastic Geometry framework in Chapter 1.

The proposed model for CSMA Networks can be any exclusion based model, i.e. PPs where points are in some sense far from each other, but we focus on the Matérn models of type II and type III (see definitions in Sections 2.1, 3.1) due to their close connection to Poisson Point Processes. This modelling is discussed in detail in Section 2.1, where an heuristic analysis of the Matérn type II model case is also provided.

The modelling of Cognitive Radio Networks is the focus of Section 2.2. In these networks, there is a wide range of options for the employed MAC schemes. A common feature of these schemes is that the secondary transmissions should *avoid* the primary transmissions. Within this constraint, any MAC scheme can be used in each class. For example, one can use Time Division Multiple Access to schedule primary terminals transmissions and ALOHA to schedule secondary terminals transmissions; one could also use CSMA for primary terminals and tree based protocols for secondary ones or vice versa, etc. We only consider here three representatives of this class of models:

- the network with a single primary terminal and a population of secondary terminals using the ALOHA MAC protocol;
- the network with a population of primary terminals and a population of secondary terminals both using the ALOHA MAC protocol; and
- the network with a population of primary terminals and a population of secondary terminals both using the CSMA MAC protocol.

The proposed models here are doubly stochastic PPPs [27, Section 5. 2] where the first level of randomness represents the locations of the primary terminals and the second level represents the locations of secondary terminals. These models are presented in the increasing order of their complexity.

Coming back to CSMA, it is worth noting that the carrier sensing in all CSMA Networks is *imperfect* due to the *propagation delay*. More concretely, the propagation delay is the small time it takes for radio signal to travel from a terminal to another one who is monitoring the air medium. If the monitoring terminal decides to start transmitting before this delay, both transmissions will collide and both will be lost. In most cases, there are mechanisms to minimize this effect and to guarantee that the carrier sensing is as nearly perfect as possible. Thus, the exclusion based models, which implicitly assume perfect carrier sensing, are applicable in these cases. Nevertheless, there are instances of CSMA Networks where these mechanisms are not available. For these networks, simulation studies show that if the intensity of terminals is too high, there will be congestion and the terminal locations pattern will break down to that of an ALOHA network, signalling a failure of the carrier sensing mechanism. We quantify this phenomenon by providing an upper bound and a lower bound of the critical intensity where the CSMA type of behaviour starts breaking down to the ALOHA type of behaviour in Section 2.3. The model used in this analysis is, however, closer to the doubly stochastic models used in the analysis of Cognitive Radio Networks than the exclusion based models. In fact, it can be considered as a multilevel-stochastic system, where there are many class of terminal and each level of randomness represents the locations of the terminals in the same class . All terminals in the same class start their transmissions at roughly the same time, i.e. within an interval equals the propagation delay, and only terminals in different classes can sense each other. For this reason, the section containing this analysis is placed right after the section containing the analysis of Cognitive Radio Networks.

In all of the above models, an important feature is the *spatial separation* between terminals caused by the carrier sensing. As we will see, the analysis of these models is not as successful as that of the ALOHA model. The reason is that the probabilistic law of their points locations is not fully understood, or more concretely, the Laplace transform of the corresponding Shot-Noise process is not known. This motivates us to dwell deeper into the problem of studying this probability law, which is characterized by their probability generating functionals, in Chapter 3.

The probability generating functional is an important notion in the theory of point processes. All the knowledge about the distribution of a point process can be systematically deduced from it in the same way as all the knowledge about the distribution of a random variable can be deduced from its probability generating function [11, Section 9. 4] or [27, p. 115]. In particular, the Laplace transform of the Shot-Noise process associated with a point process can be straightforwardly computed from its probability generating functional.

In particular, the non-Poisson Point Processes that we study are the Matérn type II and type III models. By varying their density of points, we prove that their probability generating functionals are the unique solutions of two systems of differential equations.

The structure the second Chapter is as follows. The rigorous constructions of the Matérn models of type II and type III are given in Section 3.1. The main results are then provided in Section 3.2. Section 3.3 contains the discussion of all these results in a special context where the point processes are stationary. This is usually an interesting and important case since most networks are deployed in a statistically even manner.

For the sake of completeness, a short discussion about the theory of point processes is provided in the appendices where all the notions which are used throughout this thesis are briefly introduced.

Chapter 1

Stochastic Geometry and the Modelling of Wireless Networks

In this chapter is a brief introduction of the stochastic geometry framework used in the modelling of wireless networks. We first recall in Section 1.1 some basic notions that characterize the operation of a wireless network such as the path loss, the fading, the Signal to Interference and Noise Ratio, the Multiple Access Control protocols, etc. Then, we demonstrate the modelling of these notions in the context of a large scale wireless network by using notions in Stochastic Geometry in Section 1.2. The materials presented here can be regarded as the background for the next chapter.

1.1 Preliminaries on Wireless Communication

1.1.1 Basic Notions

Consider a simple network which consists of a transmitting terminal (t.t.) T and a receiving terminal (r.t.) R. T wants to send some information to R. This information, when transmitted via the air medium, might be distorted by the ambient noise. When the noise is too large, the signal will be so distorted that R cannot detect the original information. R can only receive a message from T successfully if the signal to noise ratio (SNR) is larger than some threshold T, which depends on the employed coding scheme, where the SNR is the ratio between the received signal power and the ambient noise power [31, Section 3. 1].

Furthermore, the received signal power at R is usually not the same as the transmitted power at T; it is modified by two factors, the *path-loss* and the (multipath) *fading*. The *path-loss* comes from the that the power of a radio

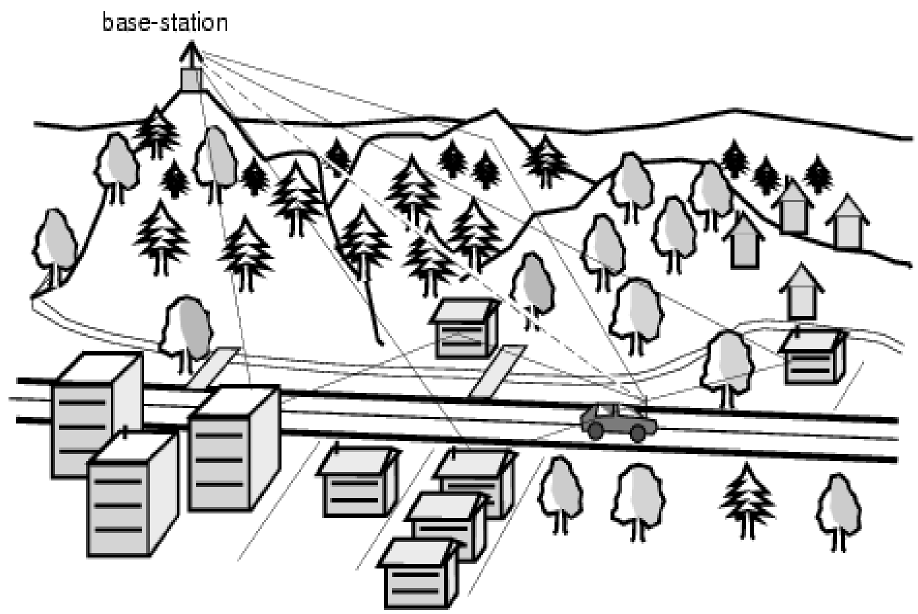

Figure 1.1: An example of multipath fading. The scatterers are the buildings, the landscape, the trees, ect. This is a rich scattering environlent, so the fading will be Rayleigh if there is no direct path from the t.t. (the antena) to the r.t. (the car) and it will be Rician otherwise. Picture taken from http://www.ice.rwth-aachen.de/research/algorithms-projects/entry/detail/rake-receivers/

wave decays as it travels further. This loss of power is measured by the *path-loss function* $l(r)$, with r the distance that the radio wave has to travel. The *fading* represents the effect that a radio wave may be reflected by many objects on its way propagating to the r.t. As all the path-loss functions used in practice are integrable and this often leads to nice analytical results, we assume that this condition holds throughout this thesis. Several copies of the same signal are then received, each following different paths. They may add up in either a *constructive* way or a *destructive* way, thus making the signal power either stronger or weaker. This is formalized by the *random fading* F. Hence, if the distance between T and R is r and T transmits a signal with power P, the received signal power at R is $PFl(r)$. The corresponding SNR is then $\frac{PFl(r)}{N}$ with N the ambient noise power.

EXAMPLE of models for path loss and fading.

- *The log-distance path loss model and its variants [31, p. 18]*: a simple path loss model that is widely used in the literature is the log-distance path loss model where $l_1(r) = (Ar)^{\alpha}$. The parameter α is called the path loss exponent which is assumed to be greater than 2. Note that this is a simplified model that makes no sense when r is small. To take care of this problem, some other variants which give roughly the same value for large r are proposed, for example $l_2(r) = (A\max(r_0, r))^\alpha$ and $l_3(r) = (1+Ar)^\alpha$.

- Fading
  - *The Rayleigh fading [31, p. 36]* is a quite accurate model for fading in rich scattering environment where there is no line of sight between the transmitter and the receiver and there are many independently located objects (scatterers) that attenuate, reflect, refract, and diffract the signal. This is also a quite common assumption in analytical works since the probabilistic law of the random fading is known in a quite simple closed-form. In particular, the random fading here is the square of the magnitude of a complex circular Gaussian random variable, which is an exponential r.v. of some parameter μ. The reason behind the quite unnatural formulation above is that the radio wave at any location is usually viewed as the value of the electric field at that location, which is expressed as a complex value. Thus the fraction of signal coming from many scattered and reflected paths are represented by a large number of independent complex r.v.s. The aggregated signal, which is the sum of such complex r.v.s, is then a circular Gaussian complex r.v. thanks to the Central Limit Theorem. The random fading is then defined as the square of the magnitude of this r.v. since we are only interested in the signal power.
  - *The Rician fading [31, p. 36]* models the fading in the environments where beside a large number of independent scatterers, there is a strong line of sight between the transmitter and the receiver. The random fading in this case is the square of the magnitude of a complex r.v. which consists of two independent components. The first component, which corresponds to the line of sight path, is a complex r.v. of fixed magnitude and uniform phase and the second component is a circular Gaussian complex r.v. corresponding to the scattered paths. The distribution of the random fading in this case is called the Rician distribution.
  - *The Nakagami fading [36]* was first proposed by Nakagami because it matches empirical data for short wave ionospheric propagation. The distribution of the random fading is the Nakagami distribution, which is closely related to the Gamma distribution. The Nakagami model can model accurately the fading in many wireless communication scenarios where the signal is composed of several i.i.d. components, each following the Rayleigh fading model. Such scenarios can arise in either (a) the maximum ratio diversity combining where the fading in each diversity branch is Rayleigh or (b) in multipath scattering environments with relatively large delay-time spreads and with different clusters of reflected waves so that the aggregated signal from each cluster is Rayleigh.

Remark 1.1 *For notational convenience, we incorporate the transmission power into the fading. Thus, if the original fading power is an exponential r.v. of parameter μ (Rayleigh fading) and the transmission power is P, the new "fading" is an exponential r.v. of parameter μ/P. This "fading" is sometimes called the virtual power [6, p. 4].*

Remark 1.2 *For analysis simplicity, we always assume Rayleigh fading in all the models considered in this chapter.*

Usually, a network does not contain only one t.t.-r.t. pair but several such pairs co-located in some domain. By the pervasive nature of radio wave, the signal from one t.t. does not only propagate to its intended r.t. but also to other r.t.s. This signal then *interferes* with the intended signal at these r.t.s, making it harder for them to receive their intended message.

Let (T_i, R_i), $i = 1, \dots n$ be n t.t.-r.t. pairs in the same network. We assume that all the t.t.s transmit at the same time with the same unit power. Hence, the signal from T_i is received at R_i as the intended signal with power $PF_{ii}l(|R_i - T_i|)$ and is received at other R_j as interference with power $PF_{ij}l(|R_j - T_i|)$, where F_{ij} is the fading from T_i to R_j. With this notation, the *aggregated interference* at R_i is the sum of all the interference signal powers from all other t.t.s, i.e. $I_i = \sum_{j \neq i} PF_{ij}l(|R_i - T_j|)$.

For the r.t.s, a way to deal with interference is to treat it as ambient noise. The intended signal can be successfully received if the Signal to Interference and Noise (SINR) is larger than the decoding threshold T, i.e.

$$\mathrm{SINR}_i := \frac{F_{ii}l(|R_i - T_i|)}{N + I_i} > T$$

EXAMPLE. Consider a network with three t.t.-r.t. pairs (R_i, T_i) with $i = 1, 2, 3$. The geometrical pattern of these terminals is as in Figure 1.2. It is assumed further that there is no fading, so that $F_{ij} = 1$ for any $1 \leq i, j \leq 3$; the path loss exponent $\alpha = 3$; the decoding threshold T is 1; and there is no ambient noise.

If all three t.t.s transmit at the same time, the SINR at each r.t. is

$$\mathrm{SINR}_i = \frac{r^{-3}}{r^{-3} + (2r)^{-3}} = \frac{8}{9} < 1.$$

So, none of the r.t.s can successfully decode its intended message.

If only two of the t.t.s transmit, which we suppose without loss of generality (w.l.o.g.) to be T_1 and T_2, The SINR at R_1 is

$$\mathrm{SINR}_1 = \frac{r^{-3}}{r^{-3}} = 1,$$

and the SINR at R_2 is

$$\mathrm{SINR}_2 = \frac{r^{-3}}{(2r)^{-3}} = 8 > 1.$$

So, only R_2 can successfully receive its message.

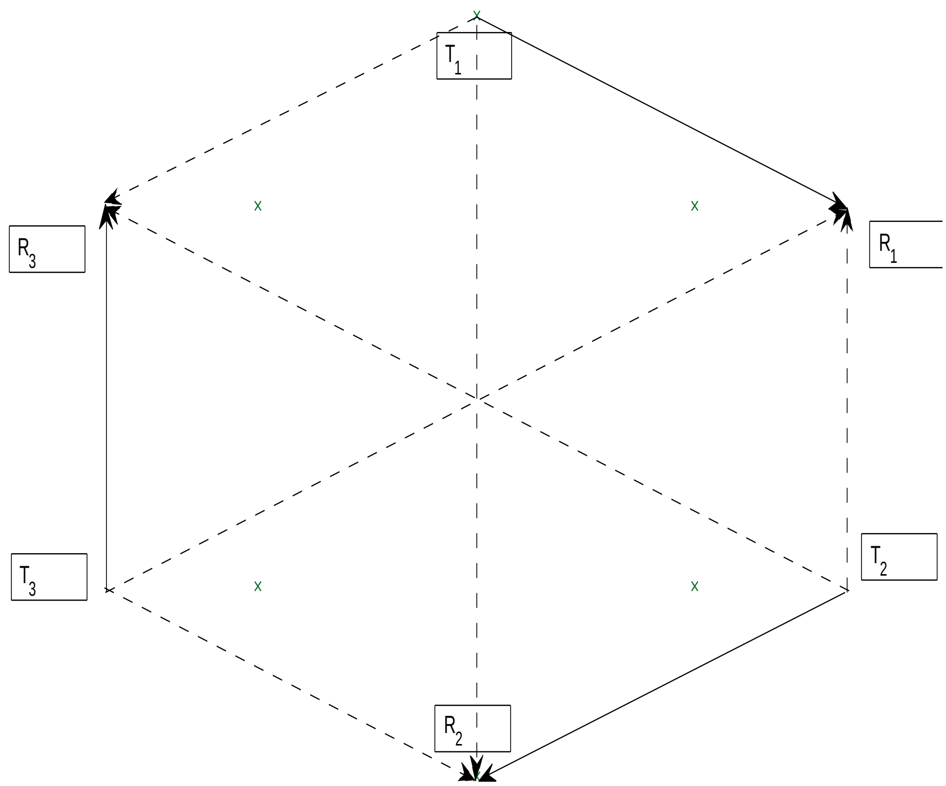

Figure 1.2: The positions of 3 t.t.-r.t.s, where the six points are positioned at the six vertices of a regular hexagon in this order $T_1, R_1, T_2, R_2, T_3, R_3$. The solid arrows are the intended links, the dashed arrows are the interference links.

1.1.2 MAC Protocols

Given that excessive interference can lead to low SINR and poor signal reception, it is then necessary to have a set of rules to allocate the common air medium to all of the t.t.s in the network, which is called a Multiple Access Control (MAC) protocol. We list below three groups of such protocols.

- *The TDMA protocol[13]* lets only one terminal transmit at a time so that the terminals do not interfere with each other. TDMA is mostly found in 2G cellular systems such as GMS, D-AMPS, PDC, iDEN. The advantage of TDMA is that it is very easy to analyse and implement. Its disadvantage is that it utilizes network resources, which is time in this case, poorly. A fixed time slot is preallocated to each terminal and the latter is allowed to transmit only in this slot. So, if a terminal does not have anything to transmit, its allocated slot is wasted. This waste is even larger for networks which have a lot of terminals, but the traffic load of each terminal is very small and irregular (bursty networks). A way to improve TDMA is to have token passing. Another way is to use reservation algorithms. TDMA and its variants are conflict-free protocols, i.e. the transmissions are organized in such a way that they do not interfere with each other (another example of conflict-free protocols is Frequency Division Multiple Access (FDMA), where transmissions are separated in the frequency domain). Taking into account the fact that traffic demand is usually irregular, another approach to the design of MAC protocols is to allow terminals to access the common medium randomly based on their loads and try to avoid interference by-the-fly. This leads to the random access protocols.

- *The ALOHA protocol* is the first random access MAC protocol for radio communication. It is first introduced in the ALOHAnet project of the University of Hawaii [4]. It has now become one of the most widely deployed and the most studied random access protocol due to its simple design. Some more sophisticated variants of ALOHA can be found in many wireless network deployments such as in the 2G systems for channel requesting or in the GPRS for user packet traffic handling. Even many 3G network designs reflect an increasing use of ALOHA random access for user packet data as well as for signalling and control purposes. The behaviour of a terminal in this protocol is as follows. It always transmits whenever it has a new packet. Then, it waits for some time for an acknowledge message (the ACK message). If the ACK message is not received, this terminal concludes that its message is not successfully received and hence retransmits its message at some later time (backlogging).

  There is also a slotted version of the ALOHA protocol, where time is synchronized into slots. The transmissions are forced to start at the beginning and to finish at the end of each slot. The purpose of this constraint is to minimize the time the transmissions of different terminals overlap (and hence interfere) with each other.

In conclusion, the virtue of ALOHA is simplicity. But this simplicity also leads to quite poor performance. Many studies have confirmed that the throughput of networks using ALOHA is quite low. One way to cope with this is to use hybrid schemes, where ALOHA is used in light load scenarios and a conflict-free protocol is used when the load is heavy. Another way is to use ALOHA in reservation algorithms where it is used to make reservations and whoever succeeds reserving the medium transmits freely without interference. The low throughput of ALOHA is mitigated by the very short reservation period compared to the transmission period.

- Another well-known class of random access protocols is *the Carrier Sensing Multiple Access (CSMA) protocols*. The poor performance of ALOHA is accounted for by the "impolite" behavior of terminals using that protocol, i.e. it transmits immediately whenever it has a message, taking no consideration of the others. On the other hand, a simple "listen before talk" rule, which is the design philosophy of CSMA, can benefit all. In CSMA, a terminal is required to listen for some time before transmitting its own message. If during this time the medium becomes busy, it has to refrain its own transmission.

  The first protocols of this kind are introduced and analysed in a series of papers [15, 29, 30] by L. Kleinrock and F.A. Tobagi. Different variants of CSMA differ from each other in many aspects. We give here a non exhaustive list of the most important ingredients that constitute these differences.

  – *Persistence:* when a terminal has a new message and it senses the medium busy, it can either (a) wait for some random "back-off" time then sense the medium again, which results in the non persistent CSMA; or (b) continuously monitor the medium and start transmitting once the medium becomes idle, which results in the persistent CSMA.

  – *Slot:* when a terminal transmits, it takes some short time (the propagation delay) before other terminals can actually "see" it. Thus, a collision can still happen if there is another nearby terminal who starts its transmission during this time window. This effect can be minimized by synchronizing the time into slots, where the slot duration is equal to the maximum propagation delay. Terminals are forced to start transmitting only at the beginning of each slot so that other terminals can see them at the end of this slot. Note that the slots here are much shorter than the time to transmit a message and are hence sometimes called the *mini slots*.

  – *Collision Detection:* once a terminal transmits, it holds the common medium for a duration which is long enough to transmit its packet. If a collision occurs, this whole transmission period is wasted. Thus, it will be better if terminals can detect collision during their transmissions so that they can stop transmitting immediately once collisions are detected and release the common medium for other terminals to use. This results in the CSMA/Collision detection (CD) protocol.

It is quite interesting to know that although CSMA is originally designed for radio communications, its popularity is attached to a class of wired networks, the Local Area Networks (LANs) [26, p. 102]. In LANs, terminals are all connected to a common wired infrastructure. This allows Carrier Sensing and Collision Detection to be implemented easily and efficiently. As a result, the performance of CSMA in these environments, when properly tuned, is quite high. The performance of CSMA for wireless networks is, however, subjected to the *hidden terminal* effect. This is the situation where there is a nearby terminal which will cause collision but the sensing terminal cannot detect it because of deep fading. Nevertheless, CSMA still provides wireless transmissions with better protection from interference than ALOHA.

1.2 Stochastic Geometry Framework—the Possion Case

The power of exploiting a Stochjastic Geometry framework lies in its ability to model the irregularity of terminals locations in large networks and to accurately model the effect of interference, taking into account all interferers. We illustrate these advantages by a classical problem that is set within this framework: the analysis of a slotted ALOHA network. All the tools and results from stochastic geometry employed in this section are provided in Appendix A for the sake of being self-contained. For a more complete treatment on the theory of stochastic geometry and point processes,we refer the reader to [27, 11].

1.2.1 The Spatial Model

In slotted ALOHA, time is synchronized into slots and messages are divided into packets in such a way that one packet can be completely transmitted within one slot. Any t.t. having a packet arriving during a slot schedules it for transmission in the next slot. If by the end of that slot, the t.t. does not receive the ACK for the transmitted packet, it assumes that there has been a collision and the packet will be scheduled to be transmitted in some later slot (backlogging). These mechanisms are captured in the following assumptions.

Assumptions

The terminal locations are assumed to form a realization of a Poisson Point Process (PPP) (see Appendix A). This means that the distributions of these locations in disjoint areas are mutually independent. This is a reasonable assumption for networks with high level of irregularity such as mobile ad hoc networks (MANETs) and unplanned dense wifi networks. Furthermore, we assume that each terminal has an intended r.t. associated with it. This is usually referred to as the bi-pole model in the literature [6].

The transmission dynamic is modelled as follows. Each terminal is assumed to always have some message to transmit. In each slot, it decides to transmit or not with probability p. The t.t.s that decide to transmit in a given slot are called the active t.t.s in that slot. This choice is independent of its choices at other slots and is independent of the choices of other t.t.s. This means that we make no distinction between newly generated messages and backlogged messages and we say that the system has an *offered load* of p message per terminal per slot, which accounts for both newly generated messages and backlogged messages. The offered load is normally larger than the *arrival load*, which accounts for only newly generated messages. In fact, the exact modelling of the transmitting process of ALOHA (and any random access based MAC protocol) is very complicated. Hence, one has to make necessary simplifications to make the model more tractable. The assumption that the choice of each terminal is independent of each other is linked to the "impolite" behaviour of terminals in ALOHA alluded to in Subsection 1.1.2.

The fading between each pair of terminals is assumed to be independent of other pairs. As noted before, we assume Rayleigh fading for analytical simplicity, so the fading value is an exponential r.v. of parameter $\mu > 0$ for each pair of terminals.

Problem Formulation

We now formulate the above assumptions in the language of stochastic geometry. The modelling consists of taking a snapshot of the system at a typical time slot. The system is represented by marked Poisson point process (MPPP, see Appendix A) with i.i.d. marks $\{(x, \mathbf{u}(x))\}$ for which the ground process $\Phi = \{x\}$ has intensity λ and represents the locations of the t.t.s. For each x, the mark $\mathbf{u}(x) = \big(r(x), e(x), \mathbf{f}(x)\big)$ where

- $r(x)$: the relative location w.r.t. x of its r.t., which is a uniformly distributed vector in the plane of fixed norm r. The absolute location of the r.t. is $x + r(x)$.

- $e(x)$: the indicator of the event that x chooses to transmit in the current slot, which is a $\{0, 1\}$ value r.v. with distribution $\mathbb{P}(e(x) = 1) = p$. The set of all active terminals is $\mathcal{D}(\Phi) = \{x \in \Phi \text{ s.t. } e(x) = 1\}$. $\mathcal{D}(\Phi)$ is an independent thinning of the homogeneous PPP Φ of intensity λ, hence it is also a homogeneous PPP but of intensity λp.

- $\mathbf{f}(x) = \{f_i(x),\ i = 1, 2, \ldots\}$ are i.i.d. exponential r.v.s of parameter μ which represent the fading from x to the r.t.s of other terminals. As Φ is a PPP, we can sort the points in it in the increasing order of their distances from x. In this order, we can write $\Phi = \{x_1, x_2, \ldots\}$ with $x_1 = x$. Then $f_i(x)$ is the fading from x to the r.t. of x_i (namely $x_i + r(x_i)$). For notational simplicity, we regard these marks as a function $f(x, y)$ taking pairs of points (x, y) in Φ as its argument.

Let l be the path-loss function. Consider a typical t.t. x in Φ, by the SINR condition discussed in Subsection 1.1.1, its r.t. can successfully receive its message iff it chooses to transmit, i.e. $e(x) = 1$, and

$$\mathrm{SINR}(x) := \frac{f(x, x)l(r)}{N + \mathcal{I}(x)} > T, \tag{1.2.1}$$

where N is the ambient noise power, T is the decoding threshold and $\mathcal{I}(x)$ is the Shot Noise (SN) interference process at the r.t. of x (namely $x + r(x)$), which is defined as

$$\mathcal{I}(x) = \sum_{y \in \mathcal{D}(\Phi) \setminus x} f(y, x) l(|y - x - r(x)|). \tag{1.2.2}$$

The performance metrics we want to compute here are the *coverage probability* and the *average throughput*. Note that $\mathcal{D}(\Phi)$ is stationary (Appendix A.1.4) , these metrics can be defined by considering a typical terminal x.

- The coverage probability (COP) is the probability that the SINR at the r.t. of a typical active terminal is higher than the decoding threshold T, i.e.

$$p_{\mathrm{COP}} := \mathbb{P}_{x, \mathcal{D}(\Phi)} \left(\mathrm{SINR}(x) > T \right) = \mathbb{P}_{o, \mathcal{D}(\Phi)} \left(\mathrm{SINR}(o) > T \right), \tag{1.2.3}$$

  where $\mathbb{P}_{x, \mathcal{D}(\Phi)}$ is the Palm distribution of $\mathcal{D}(\Phi)$ given a point at x and o is the centre of the plane (i.e. the origin of the coordinates we use).

- The average throughput (AT, denoted as $\mathcal{T}$ in the bellow formula) is the average number of successful transmissions taking place in the network per unit of area per time slot, i.e.

$$\mathcal{T} = \mathbb{E} \left[\sum_{x \in \mathcal{D}(\Phi) \cap B} \mathbf{1}_{\mathrm{SINR}(x) > T} \right], \tag{1.2.4}$$

  where B is any subset of $\mathbb{R}^2$ of unit area. Note that this choice of B is only possible because $\mathcal{D}(\Phi)$ is stationary.

One should always bear in mind that these two metrics described above are the averages taken over all possible geometric configurations of the terminal locations. They are not necessarily identical to the time averaged performances that a given node in the network experiences. More specifically, the COP is not necessarily equal to the ratio between the number of successful transmissions over a long period of time and number of transmissions over a long period of time; the average throughput is not necessarily equal to the average number of successful transmissions taking place in a fixed area of the network over a long period of time. This is only true when there is enough mobility in the network, so that the realizations of the terminal locations in different time slots are mutually independent or at least they form an ergodic system of PPs.

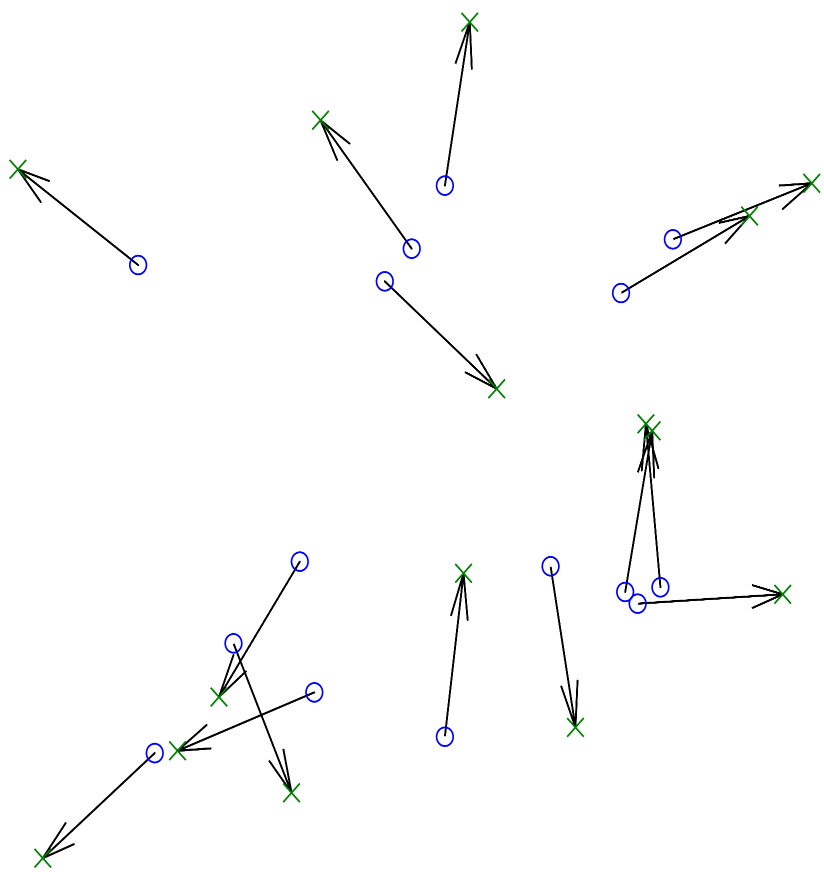

Figure 1.3: A snapshot of an ALOHA bi-pole network. The circles are the position of the t.t.s, which form a realization of a PPP of intensity .2 on a 10×10 square. Each t.t. has an arrow pointning to its intended r.t., which is represented by a cross.

1.2.2 Analysis

This Subsection contains the computations of the coverage probability and the average throughput.

Proposition 1.1 *Under the conditions given in Subsection 1.2.1, the coverage probability of a typical t.t., which is given the right to transmit in the current slot is*

$$p_{\mathrm{COP}} = \exp\left\{-\mu \frac{T}{l(r)}\right\} \exp\left\{-\lambda \int_{\mathbb{R}^2} \frac{yTl(|y|)}{Tl(|y|)+l(r)} dy\right\}. \tag{1.2.5}$$

Proof. As $f(o,o)$ is an exponential r.v. of parameter μ, we can write

$$\mathbb{P}_{o,\mathcal{D}(\Phi)}\big(\mathrm{SINR}(o) > T\big) = \mathbb{P}_{o,\mathcal{D}(\Phi)}\left(\frac{f(o,o)l(r)}{N + \sum_{y\in\mathcal{D}(\Phi)\setminus\{x\}} f(y,o)l(|y-r(o)|)} > T\right)$$

$$= \mathbb{P}_{o,\mathcal{D}(\Phi)}\left(f(o,o) > T\left(\frac{N}{l(r)} + \sum_{y\in\mathcal{D}(\Phi)\setminus\{x\}} f(y,o)\frac{l(|y-r(o)|)}{l(r)}\right)\right)$$

$$= \mathbb{E}_{o,\mathcal{D}(\Phi)}\left[\exp\left\{-\mu T\left(\frac{N}{l(r)} + \sum_{y\in\mathcal{D}(\Phi)\setminus\{x\}} f(y,o)\frac{l(|y-r(o)|)}{l(r)}\right)\right\}\right]$$

$$= \exp\left\{-\mu T\frac{N}{l(r)}\right\}\mathbb{E}_{o,\mathcal{D}(\Phi)}\left[\exp\left\{-\mu T\left(\sum_{y\in\mathcal{D}(\Phi)\setminus\{x\}} f(y,o)\frac{l(|y-r(o)|)}{l(r)}\right)\right\}\right]$$

$$= \exp\left\{-\mu T\frac{N}{l(r)}\right\}\mathbb{E}_{o,\mathcal{D}(\Phi)}\left[\prod_{y\in\mathcal{D}(\Phi)\setminus\{x\}} \exp\left\{-\mu T\left(f(y,o)\frac{l(|y-r(o)|)}{l(r)}\right)\right\}\right].$$

Under its Palm distribution given o, by Slivnyak's theorem $o \in \mathcal{D}(\Phi)$ a.s. and $\mathcal{D}(\Phi) \setminus x$ forms a realization of an homogeneous PPP of the same intensity. Furthermore, $\{f(x,o),\, x \in \mathcal{D}(\Phi)\}$ can be regarded as independent marks of the points in $\mathcal{D}(\Phi)$ where each mark is exponentially distributed with parameter μ. Hence, let $\mathcal{F}$ denotes the σ-algebra generated by the realization of the points in $\mathcal{D}(\Phi)$,

$$\mathbb{E}_{o,\mathcal{D}(\Phi)}\left[\prod_{y\in\mathcal{D}(\Phi)\setminus x} \exp\left\{-\mu T\left(f(y,o)\frac{l(|y-r(o)|)}{l(r)}\right)\right\}\middle|\mathcal{F}, r(o)\right]$$

$$= \prod_{y\in\mathcal{D}(\Phi)\setminus x} \mathbb{E}\left[\exp\left\{-\mu T\left(f(y,o)\frac{l(|y-r(o)|)}{l(r)}\right)\right\}\right]$$

$$= \prod_{y\in\mathcal{D}(\Phi)\setminus x} \frac{l(r)}{Tl(|y-r(o)|)+l(r)}.$$

Then by taking expectation w.r.t. $\mathcal{F}$ and using Theorem A.1, we get

$$\mathbb{E}_{o,\mathcal{D}(\Phi)}\left[\prod_{y\in\mathcal{D}(\Phi)\setminus x}\exp\left\{-\mu T\left(f(y,o)\frac{l(|y-r(o)|)}{l(r)}\right)\right\}\middle|r(o)\right]$$
$$=\mathbb{E}\left[\prod_{y\in\mathcal{D}(\Phi)\setminus x}\frac{l(r)}{Tl(|y-r(o)|)+l(r)}\middle|r(o)\right]=G_{\mathcal{D}(\Phi)}\left(\frac{l(r)}{Tl(|.-r(o)|)+l(r)}\right)$$
$$=\exp\left\{-\lambda p\int_{\mathbb{R}^2}\left(1-\frac{l(r)}{Tl(|y-r(o)|)+l(r)}\right)dy\right\}$$
$$=\exp\left\{-\lambda p\int_{\mathbb{R}^2}\frac{Tl(|y-r(o)|)}{Tl(|y-r(o)|)+l(r)}dy\right\},$$

where $G_{\mathcal{D}(\Phi)}$ is the p.g.fl of $\mathcal{D}(\Phi)$. By changing variable from y to $y-r(o)$, we have

$$\int_{\mathbb{R}^2}\frac{Tl(|y-r(o)|)}{Tl(|y-r(o)|)+l(r)}dy=\int_{\mathbb{R}^2}\frac{Tl(|y|)}{Tl(|y|)+l(r)}dy$$

for every possible value of $r(o)$ and the conclusion follows directly. □

Proposition 1.2 *Under the conditions given in Subsection 1.2.1, the average throughput of the network is*

$$\mathcal{T}=\lambda p p_{\text{COP}}. \tag{1.2.6}$$

Proof. By Campbell's formula,

$$\mathbb{E}\left[\sum_{x\in\mathcal{D}(\Phi)\cap B}\mathbf{1}_{\text{SINR}(x)>T}\right]=\int_B\mathbb{P}_{x,\mathcal{D}(\Phi)}\left(\text{SINR}(x)>T\right)\lambda p dx.$$

As $\mathbb{P}_{x,\mathcal{D}(\Phi)}\left(\text{SINR}(x)>T\right)=\mathbb{P}_{o,\mathcal{D}(\Phi)}\left(\text{SINR}(o)>T\right)=p_{\text{COP}}$ for every x in $\mathbb{R}^2$,

$$\begin{aligned}\mathcal{T}&=\mathbb{E}\left[\sum_{x\in\mathcal{D}(\Phi)\cap B}\mathbf{1}_{\text{SINR}(x)>T}\right]\\&=\lambda p p_{\text{COP}}.\end{aligned}$$

□

From the two proofs above, we see that there are four components that make up a successful analysis.

1 Given a realization of the t.t.s locations, there is a representation of the coverage probability as a product of a function taken over all points of the process of active t.t.s.

2 The expectation over all possible geometrical configurations of the process of active t.t.s under its Palm distribution of the aforementioned product is a probability generating functional (p.g.fl) of this PP.

3 This p.g.fl is computed using known results in stochastic geometry.

4 The computation of the average throughput is based on Campbell's formula.

Among these, the second and the fourth points are always true for models using any type of PPs to model the active t.t.s, the first point holds if that PP has independent fading marks, while the third point is true only for the Poisson case. As we will see in the next sections, this is the main reason that makes the analysis using different types of PPs to model the active t.t.s not as successful as in the Poisson case.

Bibliographical note

The analysis in Section 1.2 is based on [5]

Chapter 2

Wireless Network Modelling using Non-Poisson Point Processes

We gather in this chapter one of the two main contributions of this thesis-the modelling of wireless networks using non-Poisson PPs. In particular, we consider here three kinds of networks: the perfect CSMA Networks (Section 2.1), the Cognitive Radio Networks (Section 2.2) and the imperfect CSMA Networks (Section 2.3). In these networks, there is always some degree of *spatial separation*, which makes it impossible to use PPPs to model them. The analysis presented here is also the main motivation for our developments in Chapter 3.

2.1 Analysis of CSMA Networks

The analysis of the slotted ALOHA protocol benefits from the fact that an independent thinning of a PPP is again another PPP. In most of the other random access MAC protocols, terminals do no act independently, but adjust their behaviour to the situation in their neighbourhood. The CSMA protocol introduced in Subsection 1.1.2 is a very fine example of such protocols. Our object here is to extend the framework demonstrated in the last section to the case of CSMA, and by doing so we point out the main difficulties that one has to overcome when using stochastic geometry to analyse MAC protocols other than ALOHA.

2.1.1 Spatial Model

In CSMA, each t.t. having a packet to transmit does not transmit right away as in ALOHA, but monitors the network for sometime. If the medium is free until the end of this period, it can then transmit. Otherwise, it will reschedule its packet to a future time. This entails the following assumptions on the spatial model of the network.

Assumptions

All the basic assumptions are the same as in the model of the slotted ALOHA protocol, except the transmission process. As explained earlier, the exact transmission process of any random access MAC protocol is very complicated and the crucial point to a successful modelling is to identify the typical behaviour of the system. If the typical behaviour of ALOHA is the independence between terminals actions, the typical behaviour of CSMA is spatial separation. In other words, due to Carrier Sensing, a terminal can “see” other t.t.s if the power of the signals received at the tagged terminal is higher than some certain value called the *Carrier Sensing threshold*. For two terminals which can “see” each other, if one terminal transmits first, the other will see it when it attempts to make a transmission. These two terminals cannot transmit at the same time in a network using CSMA.

Thus, a key point in the spatial modelling of CSMA Networks is to find a proper PP to represent the set of active t.t.s. This PP does not only have to have the spatial separation property but also has to be able to reflect the irregular nature of the terminals locations. Ideally, it should be a subset (thinning) of a PPP, since the set of active t.t.s is a subset of the set of all terminals in the network, which is modelled by a PPP.

Within these criteria, there are two natural candidates that have been advocated for the last few years, the Matérn type II and the Matérn type III model, with a preference over the former due to the belief that it is more tractable. For the sake of easy reading, we only provide here brief and informal descriptions of these two models, leaving the formal constructions to Section 3.1.

Both models give each terminal an additional, artificial attribute which is a random real number in $[0, 1]$. For the reason explained in Subsection 3.1.1, we call this attribute the *timer* of the terminal, which is used in the retention procedure in the following way. We start with the Matérn type II model. In its retention procedure, each terminal compares its timer to those of the terminals it “sees” and it will be retained iff its timer has the smallest value. The retention procedure of the Matérn type III model is slightly more complicated. We first order the terminals in the increasing order of their timers (this can be done easily when there are finitely many terminals, the construction for infinitely many terminals is given in Subsection 3.1.3). Then we start afresh with an empty domain (i.e. the plane with no point in it), in which we sequentially add terminals in the aforementioned order. For each newly added terminal, we only keep it (or equivalently retain it) if it “sees” no terminal which has already been retained. Otherwise this terminal is rejected and erased.

In conclusion, the assumptions that we use for the analysis are:

- the time is slotted;
- the network consists of t.t.-r.t. pairs, where the locations of the t.t.s form a realization of a PPP;
- we assume independent Rayleigh fading, i.e. the fading between pairs of terminals are i.i.d. exponential r.v. of parameter μ; and
- the terminals that actually transmit in the current slot form a realization of either (a) a Matérn type II model or (b) a Matérn type III model.

The next step is to formulate the above assumptions in the stochastic geometry language.

Problem Formulation

We take a snapshot of the system at a typical time slot. The locations of all t.t.s are represented by a MPPP with i.i.d. marks $\{(x, \mathbf{u}(x))\}$ with its ground process $\Phi = \{x\}$ representing the locations of the t.t.s. For each x, the mark $\mathbf{u}(x)$ contains

- $r(x)$: the relative location w.r.t. x of its intended r.t. , which is a uniformly distributed random vector of fixed norm r in the plane;
- $t(x)$: the timer of x, which is uniformly distributed in $[0, 1]$;
- $\mathbf{f}(x) = \{f_i(x),\, i = 1, 2, \ldots\}$ are i.i.d. exponential r.v.s of parameter μ which represent the fading from x to the r.t.s of other t.t.s in the network, which can be regarded as a function $f(x, y)$ taking ordered pairs of points (x, y) in Φ as its parameter; and
- $\mathbf{g}(x) = \{g_i(x),\, i = 1, 2, \ldots\}$ are i.i.d. exponential r.v.s of parameter $\mu/2$. $\{\mathbf{g}(x), x \in \Phi\}$ and $\mathbf{f}$ are independent. The purpose of these r.v.s will be explained shortly below.

We want to construct a family of i.i.d. exponential r.v.s $\{g(x, y),\ (x, y)$ are unordered pairs in $\Phi\}$ of parameter μ which represents the fading from the t.t. at x to the t.t. at y. These fading values are used in the carrier sensing process modelling as follows. We associate $g_i(x)$ to y and $g_j(y)$ to x in the same way as $f_i(x)$ is associated to y and $f_j(y)$ is associated to x. Then we take $g(x, y) = \min(g_i(x), g_j(y))$. We can easily verify that $g(x, y)$ is independent for every unordered pair (x, y) in Φ and that

$$\mathbb{P}(g(x, y) > a) = \mathbb{P}(g_i(x) > a \text{ and } g_j(y) > a) = e^{-\mu a/2} e^{-\mu a/2} = e^{-\mu a}.$$

Moreover, this newly constructed family of r.v.s and $\mathbf{f}$ are independent due to the independence of $\mathbf{g}$ and $\mathbf{f}$. Two terminals at x and y are then said to “see” each other, which is written as $C_{cs}(x, y) = 1$ in the notation of Section 3.1.1, iff $g(x, y)l(|x - y|) > \rho$ with ρ be the sensing threshold and l is the path loss function. As $g(x, y)l(|x - y|)$ is the received signal power at one of the two terminals at x and y, given that the other terminal is transmitting, the random relation C_{cs} is a quite faithful model of the real carrier sensing process. Note that C_{cs} is what we call the random conflict relation in Section 3.1.1.

The collection of active t.t.s is represented by either (a) $\mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})$ or (b) $\mathcal{M}_{\mathrm{III}}(\Phi, C_{cs})$ (see definitions in 3.1.

Consider a typical terminal x in Φ. If it chooses to transmit in the current slot, the SINR at its r.t. will be

$$\mathrm{SINR}_{\mathrm{II}}(x) = \frac{f(x, x)l(r)}{N + \mathcal{I}_{\mathrm{II}}(x)} \tag{2.1.1}$$

in case (a) and be

$$\mathrm{SINR}_{\mathrm{III}}(x) = \frac{f(x, x)l(r}{N + \mathcal{I}_{\mathrm{III}}(x)} \tag{2.1.2}$$

in case (b). In the above formula, N is the ambient noise power and $\mathcal{I}_{\mathrm{II}}(x)$, $\mathcal{I}_{\mathrm{III}}(x)$ are the SN interference processes at $x + r(x)$ corresponding to $\mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})$ and $\mathcal{M}_{\mathrm{III}}(\Phi, C_{cs})$ respectively,

$$\mathcal{I}_{\mathrm{II}}(x) = \sum_{y \in \mathcal{M}_{\mathrm{II}}(\Phi, C_{cs}) \setminus x} f(y, x)l(|y - x - r(x)|); \tag{2.1.3}$$

$$\mathcal{I}_{\mathrm{III}}(x) = \sum_{y \in \mathcal{M}_{\mathrm{III}}(\Phi, C_{cs}) \setminus x} f(y, x)l(|y - x - r(x)|). \tag{2.1.4}$$

We consider three performance metrics, the *medium access probability* MAP, the *coverage probability* COP and the *average throughput* AT. By the discussion in Section 3.3, we know that under the assumptions given at the beginning of this subsection, both $\mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})$ and $\mathcal{M}_{\mathrm{III}}(\Phi, C_{cs})$ are stationary. Hence, the interested metrics can be defined by considering a typical terminal located at the centre o of the plane w.l.o.g..

- The MAP is the probability that a typical terminal transmits in a typical time slot, which is

  $$p_{\mathrm{MAP,II}} := \mathbb{P}_{x,\Phi}(x \in \mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})) = \mathbb{P}_{o,\Phi}(o \in \mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})) \tag{2.1.5}$$

  in case (a) and is

  $$p_{\mathrm{MAP,III}} := \mathbb{P}_{x,\Phi}(x \in \mathcal{M}_{\mathrm{III}}(\Phi, C_{cs})) = \mathbb{P}_{o,\Phi}(o \in \mathcal{M}_{\mathrm{III}}(\Phi, C_{cs})) \tag{2.1.6}$$

  in case (b) for any point x, where $\mathbb{P}_{x,\Phi}$ is the Palm distribution given a point at x of Φ. It is equivalent to the access probability p of ALOHA. Unlike p, the MAP of CSMA is not known a priory and hence needs to be computed explicitly.

- The COP is the probability that the SINR at the r.t. of a typical terminal is higher than the decoding threshold T, given that this terminal is active. In the other words, the COP for case (a) is

$$\begin{aligned} p_{\mathrm{COP,II}} &= \mathbb{P}_{x,\mathcal{M}_{\mathrm{II}}(\Phi,C_{cs})}(\mathrm{SINR}_{\mathrm{II}}(x) > T) \\ &= \mathbb{P}_{o,\mathcal{M}_{\mathrm{II}}(\Phi,C_{cs})}(\mathrm{SINR}_{\mathrm{II}}(o) > T) \end{aligned} \tag{2.1.7}$$

and for case (b) is

$$\begin{aligned} p_{\mathrm{COP,III}} &= \mathbb{P}_{x,\mathcal{M}_{\mathrm{III}}(\Phi,C_{cs})}(\mathrm{SINR}_{\mathrm{III}}(x) > T) \\ &= \mathbb{P}_{o,\mathcal{M}_{\mathrm{III}}(\Phi,C_{cs})}(\mathrm{SINR}_{\mathrm{III}}(o) > T), \end{aligned} \tag{2.1.8}$$

for any point x. $\mathbb{P}_{x,\mathcal{M}_{\mathrm{II}}}$ and $\mathbb{P}_{x,\mathcal{M}_{\mathrm{II}}}$ are the Palm distributions given a point at x of $\mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})$ and $\mathcal{M}_{\mathrm{III}}(\Phi, C_{cs})$ correspondingly.

- The AT is the average number of successful transmissions taking place in the network per slot and per unit of area, which can be defined as

$$\mathcal{T}_{\mathrm{II}} = \mathbb{E}\left[\sum_{x\in B\cap\mathcal{M}_{\mathrm{II}}(\Phi,C_{cs})} \mathbf{1}_{\mathrm{SINR}_{\mathrm{II}}(x)>T}\right] \tag{2.1.9}$$

in case (a) and as

$$\mathcal{T}_{\mathrm{III}} = \mathbb{E}\left[\sum_{x\in B\cap\mathcal{M}_{\mathrm{III}}(\Phi,C_{cs})} \mathbf{1}_{\mathrm{SINR}_{\mathrm{III}}(x)>T}\right], \tag{2.1.10}$$

in case (b) with B be any Borel set of unit area.

As with ALOHA, we would like to stress that the above metrics are the averages over all possible geometry configurations of the network and are not necessarily equal to the averages over time. This is only true if there is enough mobility in the network.

2.1.2 Probability Generating Functionals Representations of the Performance Metrics

Now we compute the metrics defined above in terms of the p.g.fls of the corresponding Matérn type II and type III models.

Proposition 2.1 *Under the conditions given in Subsection 2.1.1, the medium access probability of a typical t.t. is*

$$p_{\mathrm{MAP,II}} = (\pi r^2)^{-1} \left.\frac{d}{ds} G_{\mathcal{M}_{\mathrm{II}}(\Phi,C_{cs})}(e^{-s\mathbf{1}_{|\cdot|<r}})\right|_{s=0}$$

for case (a) and is

$$p_{\mathrm{MAP,III}} = (\pi r^2)^{-1} \left.\frac{d}{ds} G_{\mathcal{M}_{\mathrm{III}}(\Phi,C_{cs})}(e^{-s\mathbf{1}_{|\cdot|<r}})\right|_{s=0}$$

in case (b).

Proof. The proofs for case (a) and case (b) are similar. Hence, we only present the first one. By Campbell's formula and by stationarity,

$$\begin{aligned}\mathbb{E}\left[|\mathcal{M}_{\mathrm{II}}(\Phi, C_{cs}) \cap B(o,r)|\right] &= \mathbb{E}\left[\sum_{x \in B(o,r)} \mathbf{1}_{x \in \mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})}\right] \\ &= \int_{B(o,r)} \mathbb{P}_{x,\Phi}\left(x \in \mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})\right) dx \\ &= |B(o,r)| \mathbb{P}_{o,\Phi}\left(o \in \mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})\right) = 2\pi r^2 p_{\mathrm{MAP,II}},\end{aligned}$$

for every $r > 0$ with $B(o,r)$ the ball centred at o having radius r. By application of Proposition A.5, we have

$$\frac{d}{ds} G_{\mathcal{M}_{\mathrm{III}}(\Phi, C_{cs})}(e^{-s\mathbf{1}_{|.|<r}})\Big|_{s=0} = \mathbb{E}\left[|\mathcal{M}_{\mathrm{II}}(\Phi, C_{cs}) \cap B(o,r)|\right],$$

which directly implies (2.1.11). □

Proposition 2.2 *Under the conditions given in Subsection 2.1.1, the coverage probability of a typical active t.t. is*

$$p_{\mathrm{COP,II}} = \exp\left\{-\mu \frac{TN}{l(r)}\right\} G^{!}_{o,\mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})}\left(\frac{l(r)}{Tl(|. - r\boldsymbol{e}|) + l(r)}\right) \tag{2.1.11}$$

for case (a) and

$$p_{\mathrm{COP,III}} = \exp\left\{-\mu \frac{TN}{l(r)}\right\} G^{!}_{o,\mathcal{M}_{\mathrm{III}}(\Phi, C_{cs})}\left(\frac{l(r)}{Tl(|. - r\boldsymbol{e}|) + l(r)}\right) \tag{2.1.12}$$

for case (b), where o is the centre of the plane and $\boldsymbol{e}$ is a vector of unit length in the plane.

Proof. The proofs for both case are similar to the proof of Proposition 1.1. The only difference is that there is no closed form formula for the p.g.fls. □

Proposition 2.3 *Under the conditions given in Subsection 2.1.1, the AT of the network is*

$$\mathcal{T}_{\mathrm{II}} = \lambda p_{\mathrm{MAP,II}} p_{\mathrm{COP,II}} \tag{2.1.13}$$

in case (a) and is

$$\mathcal{T}_{\mathrm{III}} = \lambda p_{\mathrm{MAP,III}} p_{\mathrm{COP,III}} \tag{2.1.14}$$

in case (b).

Proof. This proof is similar to the proof of Proposition 1.2, which makes use of Campbell's formula. □

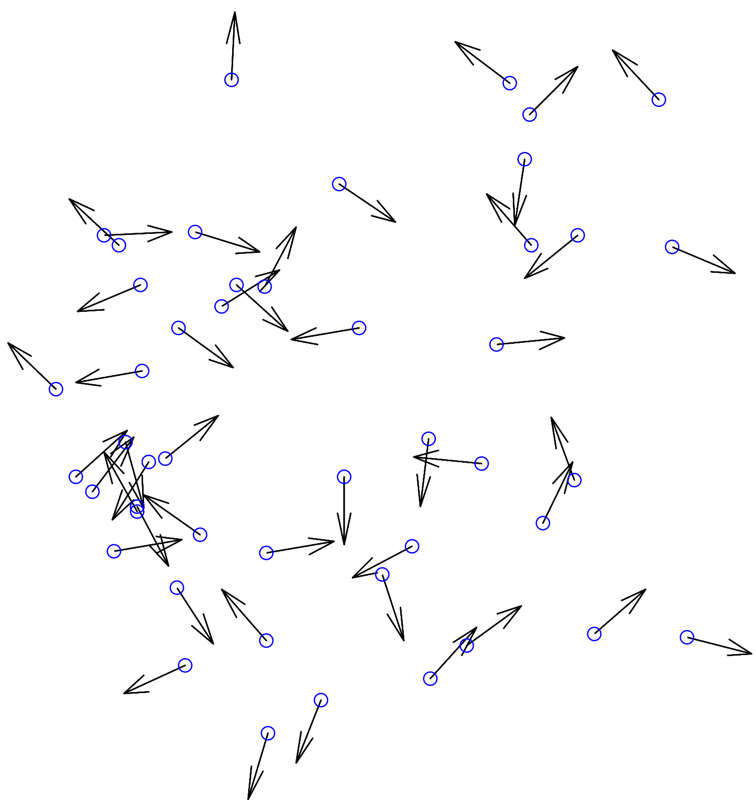

Figure 2.1: The set of all t.t.s in a dense network. The locations of the t.t.s are the circles. Each t.t. has an arrow pointing to its intended r.t. The distance from a t.t. to its r.t. is 1.

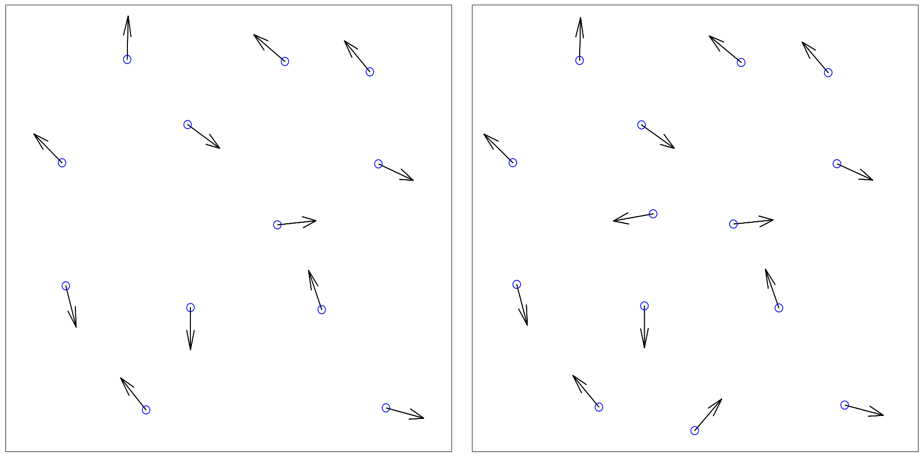

Figure 2.2: The set of active t.t.s in the previous dense CSMA Network. On the left is case (a) where the process of active t.t.s is modelled by the Matérn type II model. On the right is case (b) where this process is modelled by the Matérn type III model. There is no fading, $l(r) = r^{-4}$ and $\rho = 1/16$. We see that in (b) there are two more active t.t.s than (a).

2.1.3 Poisson Approximation of the Matérn type II Model

An heuristic method to compute the above p.g.fls is to approximate the reduced Palm distributions of the involved PPs by the distributions of the PPPs of the same intensity measures. However, such an approach is only applicable to the Matérn type II model since we can only compute the intensity under the reduced Palm distributions of this model explicitly. We start with the MAP.

Proposition 2.4 *Under the conditions given in Subsection 2.1.1, the MAP of a typical t.t. in (a) case is*

$$p_{\mathrm{MAP,II}} = \frac{1-\exp\{-\lambda\overline{\mathcal{N}}\}}{\lambda\overline{\mathcal{N}}}, \tag{2.1.15}$$

where

$$\overline{\mathcal{N}} = 2\pi\int_0^\infty \exp\{-\frac{\mu\rho}{l(r)}\}rdr. \tag{2.1.16}$$

Proof. Consider the distribution of Φ under its Palm distribution given o. Given that the timer $t(o)$ of this point is t, the probability that $o \in \mathcal{M}_{\mathrm{II}}(\Phi)$ is

$$\begin{aligned}\mathbb{P}_{o,\Phi}(o \in \mathcal{M}_{\mathrm{II}}(\Phi) \mid t(o) = t) &= \mathbb{P}_{o,\Phi}(C_{cs}(o,x) = 0 \text{ or } t(x) \geq t \text{ for all } x \in \Phi) \\ &= \mathbb{E}_{o,\Phi}\left[\prod_{x\in\Phi} \mathbf{1}_{C_{cs}(o,x)=0 \text{ or } t(x)\geq t}\right] \\ &= \mathbb{E}_{o,\Phi}\left[\prod_{x\in\Phi} \left(1 - \mathbf{1}_{C_{cs}(o,x)=1}\mathbf{1}_{t(x)<t}\right)\right].\end{aligned}$$

Denoting $\mathcal{F}$ the σ-algebra generated by the realization of the points in Φ,

$$\begin{aligned}\mathbb{E}_{o,\Phi}\left[\prod_{x\in\Phi} \left(1 - \mathbf{1}_{C_{cs}(o,x)=1}\mathbf{1}_{t(x)<t}\right) \middle| \mathcal{F}\right] &= \prod_{x\in\Phi} \mathbb{E}\left[1 - \mathbf{1}_{C_{cs}(o,x)=1}\mathbf{1}_{t(x)<t}\right] \\ &= \prod_{x\in\Phi}\left(1 - \exp\left\{-\frac{\mu\rho}{l(|x|)}\right\} t\right).\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{P}_{o,\Phi}(o \in \mathcal{M}_{\mathrm{II}}(\Phi) \mid t(o) = t) &= \mathbb{E}_{o,\Phi}\left[\prod_{x\in\Phi}\left(1 - \exp\left\{-\frac{\mu\rho}{l(|x|)}\right\} t\right)\right] \\ &= \exp\left\{-\lambda t \int_{\mathbb{R}^2} \exp\left\{-\frac{\mu\rho}{l(|x|)}\right\} dx\right\} \\ &= \exp\left\{-\lambda t \overline{\mathcal{N}}\right\},\end{aligned}$$

as

$$\int_{\mathbb{R}^2} \exp\left\{-\frac{\mu\rho}{l(|x|)}\right\} dx = \int_0^\infty \exp\left\{-\frac{\mu\rho}{l(r)}\right\} \pi 2r dr = \overline{\mathcal{N}}.$$

The proposition follows by taking integration w.r.t. t from 0 to 1. □

Remark 2.1 *Although the closed form expression derived in the above proposition is correct as confirmed by Proposition 3.8, its derivation is not rigorous. In fact, it is based on the implicit assumption that there is a realization of Φ and $\mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})$ in the same probability space in such a way that the Palm distribution given o of the latter can be expressed as a conditional Palm distribution of the former given that it has a point at o and that this point belongs to the latter. Such a claim is, in fact, quite hard to prove. By default, this assumption is used for all other results in this subsection.*

Proposition 2.5 *The intensity measure of $\mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})$ under its reduced Palm distribution given o is $b(\lambda, |x|)\lambda dx$ where*

$$b(\lambda, r) = \frac{2}{\lambda(\overline{\mathcal{N}} - \overline{\mathcal{N}}_2(r))}\left(\frac{1 - e^{-\lambda\overline{\mathcal{N}}}}{\overline{\mathcal{N}}} - \frac{1 - e^{-\lambda(2\overline{\mathcal{N}} - \overline{\mathcal{N}}_2(r))}}{2\overline{\mathcal{N}} - \overline{\mathcal{N}}_2(r)}\right)\frac{(1 - e^{-\frac{\mu\rho}{l(r)}})\overline{\mathcal{N}}}{1 - e^{-\lambda\overline{\mathcal{N}}}}, \tag{2.1.17}$$

and

$$\overline{\mathcal{N}}_2(r) = \int_0^{2\pi}\int_0^{\infty} \exp\left\{-\mu\rho\left(\frac{1}{l(z)} + \frac{1}{l(\sqrt{z^2+r^2-2rz\cos(\theta)})}\right)\right\} zdzd\theta. \tag{2.1.18}$$

Proof. There are two methods to obtain this result. We present here the first one which consists of using the implicit assumption alluded to above. The second one makes use of Corollary 3.3.

First, we rewrite $b(\lambda, |x|)$ as

$$\begin{aligned} b(\lambda,|x|) &:= \mathbb{P}_{o,x,\Phi}\left(x \in \mathcal{M}_{\mathrm{II}}(\Phi, C_{cs}) \text{ and } o \in \mathcal{M}_{\mathrm{II}}(\Phi, C_{cs}) \mid o \in \mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})\right) \\ &= \frac{\mathbb{P}_{o,x,\Phi}\left(x \in \mathcal{M}_{\mathrm{II}}(\Phi, C_{cs}) \text{ and } o \in \mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})\right)}{\mathbb{P}_{o,\Phi}\left(o \in \mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})\right)}. \end{aligned}$$

The numerator is computed as follow,

$$\begin{aligned} &\mathbb{P}_{o,x,\Phi}\left(o \in \mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})\right) \\ &= \mathbb{E}_{o,x,\Phi}\left[\mathbf{1}_{C_{cs}(o,x)=0} \prod_{y\in\Phi\setminus\{o,x\}} \mathbf{1}_{C_{cs}(o,y)=0 \text{ or } t(y)\geq t(o)} \mathbf{1}_{C_{cs}(x,y)=0 \text{ or } t(y)\geq t(x)}\right] \end{aligned}$$

By conditioning on $\mathcal{F}$, $t(o)$ and $t(x)$,

$$\begin{aligned} &\mathbb{E}_{o,x,\Phi}\left[\mathbf{1}_{C_{cs}(o,x)=0} \prod_{y\in\Phi\setminus\{o,x\}} \mathbf{1}_{C_{cs}(o,y)=0 \text{ or } t(y)\geq t(o)} \mathbf{1}_{C_{cs}(x,y)=0 \text{ or } t(y)\geq t(x)} \middle| \mathcal{F}, t(o), t(x)\right] \\ &= \mathbb{E}\left[\mathbf{1}_{C_{cs}(o,x)=0}\right] \prod_{y\in\Phi\setminus\{o,x\}} \mathbb{E}\left[\mathbf{1}_{C_{cs}(o,y)=0 \text{ or } t(y)\geq t(o)} \mathbf{1}_{C_{cs}(x,y)=0 \text{ or } t(y)\geq t(x)}\right] \\ &= \left(1 - e^{-\mu\rho/l(|x|)}\right) \prod_{y\in\Phi\setminus\{o,x\}} \Big(1 - \mathbb{E}\left[\mathbf{1}_{C_{cs}(o,y)=1}\mathbf{1}_{t(y)<t(o)}\right] \\ &\quad - \mathbb{E}\left[\mathbf{1}_{C_{cs}(o,y)=1}\mathbf{1}_{t(y)<t(x)}\right] + \mathbb{E}\left[\mathbf{1}_{C_{cs}(o,y)=1}\mathbf{1}_{t(y)<t(o)}\mathbf{1}_{C_{cs}(o,y)=1}\mathbf{1}_{t(y)<t(x)}\right]\Big) \\ &= \left(1 - \exp\left\{-\frac{\mu\rho}{l(|x|)}\right\}\right) \prod_{y\in\Phi\setminus\{o,x\}} \Bigg(1 - \exp\left\{-\frac{\mu\rho}{l(|y|)}\right\} t(o) - \exp\left\{-\frac{\mu\rho}{l(|y-x|)}\right\} t(x) \\ &\quad + \exp\left\{-\mu\rho\left(\frac{1}{l(|y|)} + \frac{1}{l(|y-x|)}\right)\right\} \min(t(o), t(x))\Bigg). \end{aligned}$$

By Slivnyak's theorem, $\Phi \setminus \{o,x\}$ is distributed as a PPP of intensity λ under $\mathbb{P}_{o,x,\Phi}$, so

$$\mathbb{E}_{o,x,\Phi}\Bigg[\prod_{y\in\Phi\setminus\{o,x\}}\Big(1-e^{-\mu\rho/l(|y|)}t(o)-e^{-\mu\rho/l(|y-x|)}t(x)+e^{-\mu\rho\left(\frac{1}{l(|y|)}+\frac{1}{l(|y-x|)}\right)}$$

$$\min(t(o),t(x))\Big)\Big|t(o),t(x)\Bigg]$$

$$=\exp\Bigg\{-\lambda\int_{\mathbb{R}^2}\Big(e^{-\mu\rho/l(|y|)}t(o)+e^{-\mu\rho/l(|y-x|)}t(x)-e^{-\mu\rho\left(\frac{1}{l(|y|)}+\frac{1}{l(|y-x|)}\right)}$$

$$\min(t(o),t(x))\Big)dy\Bigg\}.$$

Since

$$\int_{\mathbb{R}^2} e^{-\mu\rho/l(|y|)}dy=\int_{\mathbb{R}^2} e^{-\mu\rho/l(|y-x|)}dy=\overline{\mathcal{N}}$$

and

$$\int_{\mathbb{R}^2} e^{-\mu\rho\left(\frac{1}{l(|y|)}+\frac{1}{l(|y-x|)}\right)}dy=\overline{\mathcal{N}}_2(|x|)$$

for every x, we then have

$$\mathbb{P}_{o,x,\Phi}\left(o\in\mathcal{M}_{\mathrm{II}}(\Phi,C_{cs})\mid t(o),t(x)\right)$$
$$=\left(1-e^{-\mu\rho/l(|x|)}\right)\exp\{-\lambda t(o)\overline{\mathcal{N}}-\lambda t(x)\overline{\mathcal{N}}+\lambda\min(t(o),t(x))\overline{\mathcal{N}}_2(|x|)\}.$$

By taking integration w.r.t. $t(o)$ and $t(x)$ from 0 to 1, we get

$$\mathbb{P}_{o,x,\Phi}\left(o\in\mathcal{M}_{\mathrm{II}}(\Phi,C_{cs})\right)$$
$$=\frac{2}{\lambda(\overline{\mathcal{N}}-\overline{\mathcal{N}}_2(r))}\left(\frac{1-e^{-\lambda\overline{\mathcal{N}}}}{\lambda\overline{\mathcal{N}}}-\frac{1-e^{-\lambda(2\overline{\mathcal{N}}-\overline{\mathcal{N}}_2(r))}}{\lambda(2\overline{\mathcal{N}}-\overline{\mathcal{N}}_2(r))}\right)\left(1-e^{-\frac{\mu\rho}{l(r)}}\right).$$

As the denominator is $\frac{1-e^{-\lambda\overline{\mathcal{N}}}}{\lambda\overline{\mathcal{N}}}$, (2.1.17) follows directly. □

Then, the Poisson approximation gives us:

Proposition 2.6 *By approximating the reduced Palm distribution of $\mathcal{M}_{\mathrm{II}}(\Phi,C_{cs})$ given a point at o by that of a PPP of intensity measure $b(\lambda,|x|)\lambda dx$, the COP of a typical t.t. and the AT of the network in (a) case are approximated as*

$$p_{\mathrm{COP,II}}\approx\exp\left\{-\mu\frac{TN}{l(r)}+\lambda\int_{\mathbb{R}^2}\frac{Tl(|y-\mathbf{e}r|}{Tl(|y-\mathbf{e}r|)+l(r)}b(\lambda,|y|)dy\right\},\qquad(2.1.19)$$

and

$$\mathcal{T}_{\mathrm{II}}\approx\exp\left\{-\mu\frac{TN}{l(r)}+\lambda\int_{\mathbb{R}^2}\frac{Tl(|y-\mathbf{e}r|}{Tl(|y-\mathbf{e}r|)+l(r)}b(\lambda,|y|)dy\right\}\frac{(1-e^{-\frac{\mu\rho}{l(r)}})\overline{\mathcal{N}}}{1-e^{-\lambda\overline{\mathcal{N}}}}.\qquad(2.1.20)$$

Proof. The proof of (2.1.19) is a corollary of Proposition 2.2 and Theorem A.1. The proof of (2.1.20) is a corollary of (2.1.19), Proposition 2.4 and Proposition 2.3. □

As we can see, the basis of the above heuristic analysis is the assumption that the Palm distribution given o of $\mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})$ can be approximated by the distribution of a PPP of intensity measure $b(\lambda, |x|)\lambda dx$. It is then important to check the accuracy of this assumption. We do so by computing the p.g.fls of $\mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})$ under $\mathbb{P}_{o,\mathcal{M}_{\mathrm{II}}(\Phi,C_{cs})}$ by (i) Monte Carlo method and (ii) by the approximating formulas for different classes of functions v. We then check the consistency of the obtained results. Below are the plots of the results for (a) the functions $v_{a,r} = \mathbf{1}_{|x|>r}$, (b) the functions $v_{b,r} = \frac{|x|^4}{r+|x|^4}$ and (c) the functions $v_{c,r} = 1 - \exp\{-r|x|^4\}$. For these plots, we assume that the path-loss function is $l(|x|) = |x|^{-4}$ and the other parameters are $\mu = \rho = 1$. In all there cases, we observe good matches between the results obtained by two computation methods.

2.2 Analysis of Cognitive Radio Networks

The radio frequencies have always been a scarce resource in wireless communication. Traditionally, this resource is preallocated to each application (such as radio navigation, TV, aeronautical radio navigation, bluetooth, etc.) and to each network operator if the concerning application is for commercial purposes by regulatory bodies such as the FCC in the US, the OFCOM in the UK, the CEPT in Europe, the IDA in Singapore, etc. The report of the FCC special taskforce [1] in 2004, followed up by studies of other regulatory bodies all over the world shows that most of the allocated frequencies such as the military, the amateur radio and the paging frequencies, are inefficiently utilized, while cellular network bands are overloaded all over the world. An answer to this problem is the Dynamic Spectrum Access (DSA) [7], where bandwidth hungry applications are allowed to use the insufficiently utilized allocated frequencies, given that they do not cause to much degradation to the performance of the licensed applications.

Cognitive Radio (CR), a new wireless communication paradigm first introduced by J. Mitola in his thesis dissertation [18], is a major enabler of DSA. CR devices are wireless terminals that are able to monitor their network environment and configure their radio-system parameters accordingly. In the DSA context, a network, which is referred to as a CR Network from now on, consists of two classes of terminals: the primary terminals which are the devices of the licensed application and the secondary terminals which are the CR devices. Using their cognitive engine, the secondary terminals can detect the active primary terminals and try to be as far from the latter as possible by adjusting their radio transmission parameters. This leads to the spatial separation between the active primary terminals and the secondary terminals, a behaviour that the PPPs clearly fail to provide an accurate model. Moreover, each class (either primary

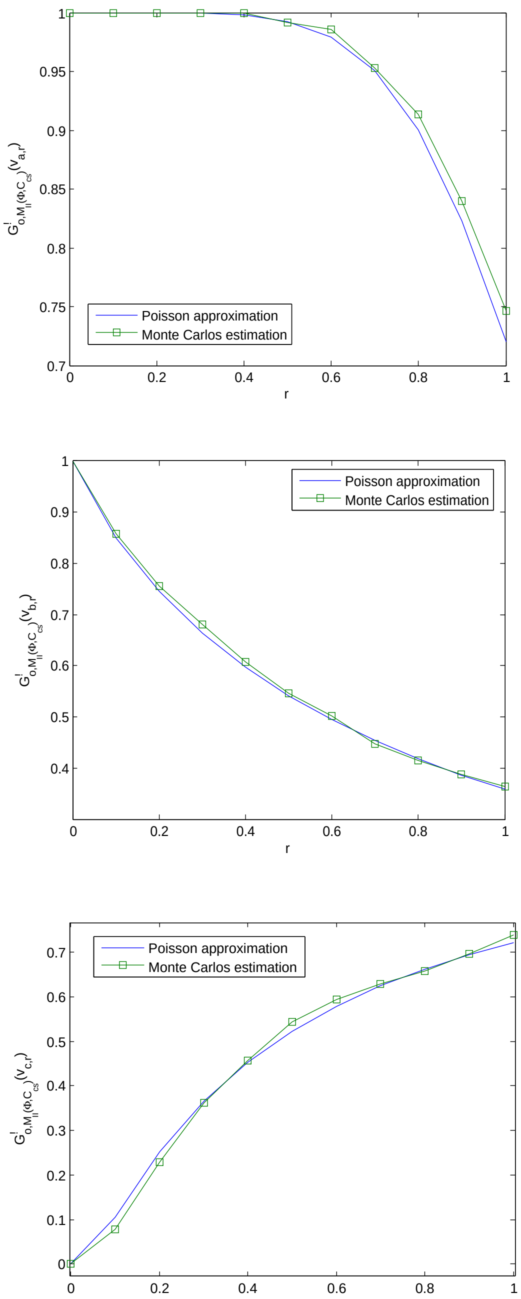

Figure 2.3: The plots of p.g.fls under the reduced Palm distribution of $\mathcal{M}_{\mathrm{II}}(\Phi, C_{cs})$ for different test function $v_{a,r}$. From top to bottom: plot of $G^!_{o,\mathcal{M}_{\mathrm{II}}(\Phi,C_{cs})}(v_{a,r})$ for r from $0, 1$, plot of $G^!_{o,\mathcal{M}_{\mathrm{II}}(\Phi,C_{cs})}(v_{b,r})$ for r from $0, 1$, plot of $G^!_{o,\mathcal{M}_{\mathrm{II}}(\Phi,C_{cs})}(v_{c,r})$ for r from $0, 1$; with $v_{a,r}(x) = \mathbf{1}_{|x|>r}$, $v_{b,r}(x) = \frac{|x|^4}{r+|x|^4}$, $v_{c,r}(x) = 1 - \exp\{-r|x|^4\}$.

or secondary) may employ different MAC schemes, entailing other behaviours that also need to be accounted for in the spatial modelling. For illustration, we consider here the analysis of three CR Networks of this type,

- *the single primary-ALOHA secondary (S/A) network* features a single primary t.t.-r.t. pair and a population of secondary t.t.-r.t. pairs using ALOHA to schedule their transmissions. In fact, this is a quite simple case which is presented here only to illustrate the modelling of the spatial separation between primary and secondary t.t.s and to prepare for the more involved models.
- *the ALOHA primary-ALOHA secondary (A/A) network* features a population of primary t.t.-r.t. pairs together with a population of secondary t.t.-r.t. pairs. Both of them schedule their transmissions using ALOHA.
- *the CSMA primary-CSMA secondary (C/C) network* features a population of primary t.t.-r.t. pairs together with a population of secondary t.t.-r.t. pairs. Both of them schedule their transmissions using CSMA.

2.2.1 Spatial Modelling

The spatial separation between the primary and the secondary terminals is modelled in the same way as we model the carrier sensing in CSMA, i.e. each active primary t.t. is surrounded by a protection zone, which is dictated by a protection parameter. This parameter is equivalent to the carrier sensing threshold in CSMA. Given this, the three CR Networks mentioned in the previous subsection are modelled by three different doubly stochastic PPPs [27, Section 5. 2]. In these PPs, the first level of randomness is the realization of the primary terminals locations; conditioned on this, the second level is the realization of secondary terminals locations.

Remark 2.2 *Although they are modelled similarly, the spatial separation in CR Networks and the carrier sensing in CSMA are very different in terms of physical requirements and implementations. In fact, the former is based on channel detection, whose physical design is a very active research domain, which is beyond the scope of this thesis.*

Assumptions

The assumptions are summarized below on a case by case basis.

- *S/A:* there is one primary t.t. at the centre of the plane. Around the primary t.t., we enforce a 'soft' protection zone which takes into account both path loss and fading in the same manner that we model the carrier sensing in CSMA.

  The process of secondary terminals is a PPP. Those that fall within the protection zone are automatically silenced. The others decide to transmit independently according to ALOHA.

- *A/A:* the locations of primary terminals are represented by a PPP and they decides to transmit independently according to ALOHA. Thus, the locations of the active primary terminals are represented by an independent thinning of a PPP, which is again another PPP. Around each active primary t.t., we enforce a 'soft' protection zone as in the previous case. The process of secondary terminals is also a PPP. Those that fall within any of the protection zones are automatically silenced. The others decide to transmit independently according to ALOHA. Thus, the process of active secondary terminals forms a realization of a Cox PP, [27], driven by the random measure that takes value 0 inside the protection zones.

- *C/C:* in the doubly stochastic PP representing this model, the process of primary terminals and the process of secondary terminals are two independent PPPs. The protection zones are modelled in the same manner as in the two above models. The process of active primary terminals can be modelled by the Matérn model of either type II or type III as in Subsection 2.1.1. The secondary terminals that fall within any of *the protection zones of the active primary terminals* are automatically silenced. As the others decide to transmit according to CSMA, they are represented by the Matérn model of either type II or type III corresponding to the Cox PP representing the secondary terminals who are not automatically silenced. We consider here only the case where the Matérn type II is used for both classes.

Problem Formulation

In the S/A network, there is only one primary t.t. whose location is represented by a point at the centre o of the plane, and a population of secondary t.t.s whose locations are represented by an homogeneous PPP Φ_s of intensity λ_s. The additional attributes of each t.t. are:

- with the primary t.t.,
  - an intended primary r.t. r, whose position is a uniformly distributed vector of norm r_p;
  - the fading f_{pp} to the intended primary r.t. which is an exponential r.v. of parameter μ_p;
  - the fading $\{g_{ps,i}\}$ to the secondary t.t.s, which is a family of i.i.d. exponential r.v.s of parameter μ_p. These r.v.s are associated to the secondary t.t.s by first sorting the points in Φ_s in the increasing order of the distance to the centre and then $g_{ps,i}$ is given to the i^{th} t.t. in this ordering. These fading values are used to model the cognitive channel detection;
  - the fading $\{f_{ps,i}\}$ to the secondary r.t.s, which is a family of i.i.d. exponential r.v.s of parameter μ_p. These r.v.s are associated to the

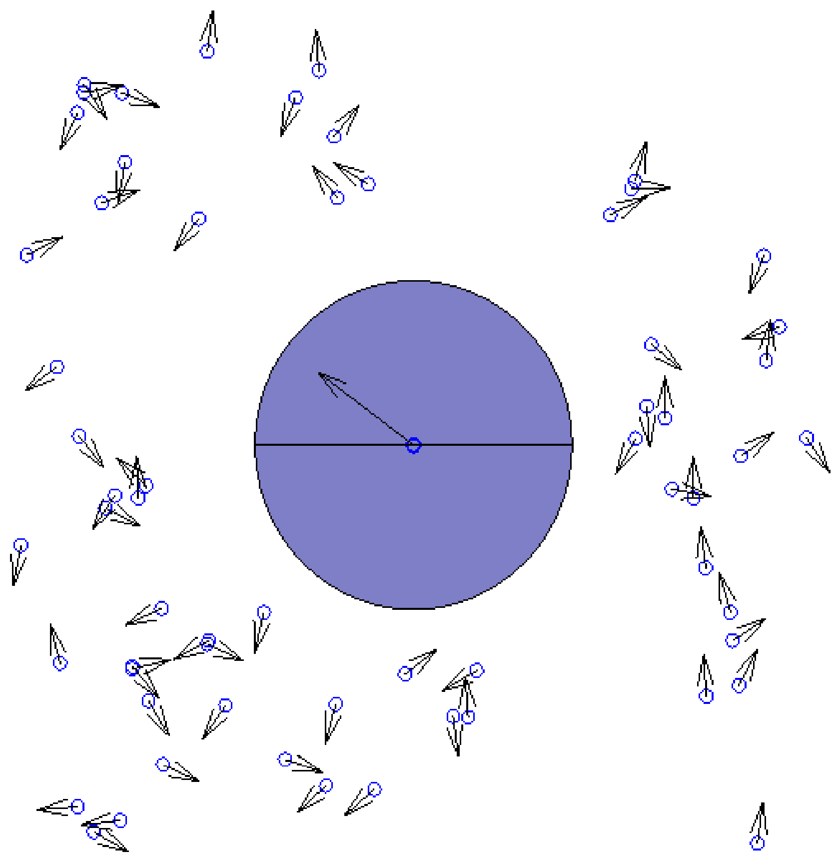

Figure 2.4: A snapshot of a CR S/A Network. It features a primary t.t. at the centre and a population of secondary t.t.s using ALOHA. Each t.t. has an arrow to its intended r.t. There is no fading. Other parameters are $r_p = 1.5$, $r_s = .5$, $l(r) = r^{-4}$, $\rho = 2^{-4}$. The shaded disk is the protection zone of the primary t.t.

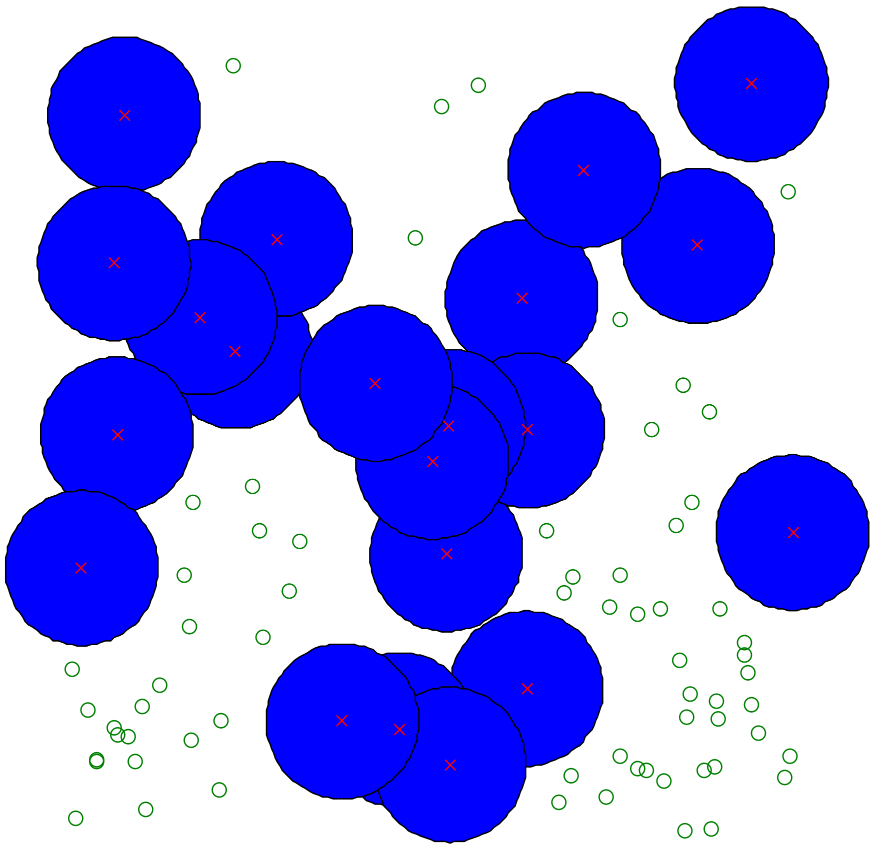

Figure 2.5: A snapshot of a CR A/A Network. It features a population of primary t.t.s and a population of secondary t.t.s, both using ALOHA. There is no fading. Other parameters are $l(r) = r^{-4}$, $\rho = 2^{-4}$. The positions of the active primary t.t.s are the red crosses. The blue disks are the protection zones of the primary t.t.s. The positions of the active secondary t.t.s are the small green circles. To keep the figure simple we do not plot the positions of the r.t.s.

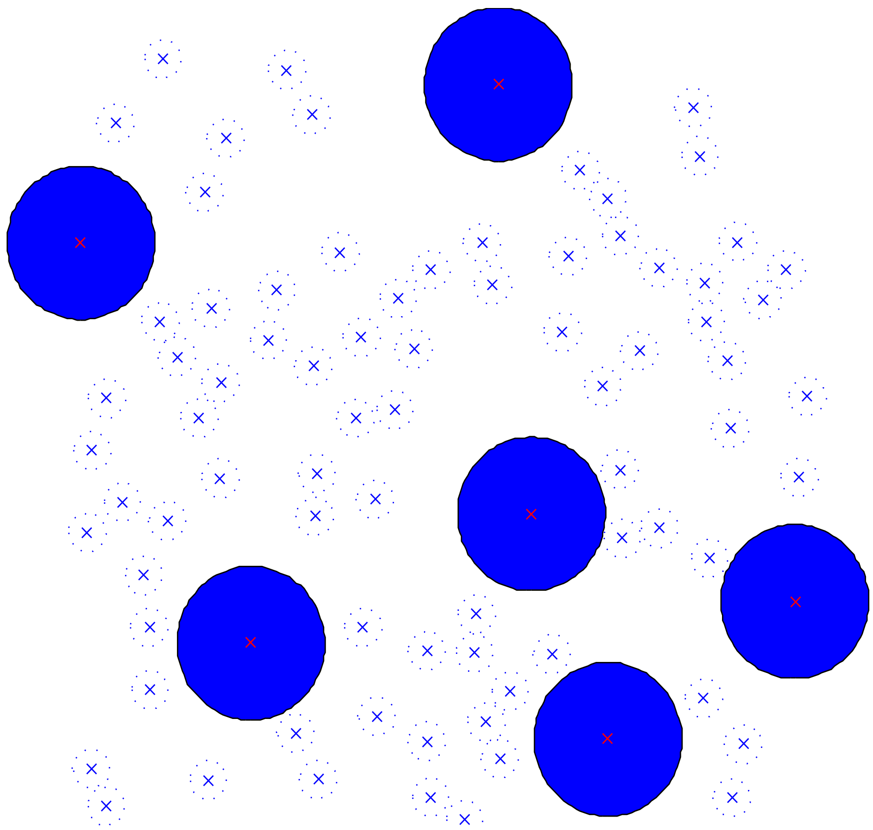

Figure 2.6: A snapshot of a CR C/C Network. It features a population of primary t.t.s and a population of secondary t.t.s, both using CSMA. There is no fading. Other parameters are $l(r) = r^{-4}$, $\rho = 2^{-4}$, $\rho_p = 5^{-4}$, $\rho_s = 1$. The positions of the active primary t.t.s are the red crosses. The blue disks are the protection zones of the primary t.t.s. The positions of the active secondary t.t.s are the blue crosses surrounded by the dotted circles which represents the protection zones of the secondary t.t.s enforced by CSMA. To keep the figure simple we do not plot the positions of the r.t.s.

secondary r.t.s by first sorting the points in Φ_s in the increasing order of the distance to the centre and then $f_{ps,i}$ is given to the r.t. of the i^{th} t.t. in this ordering (see below for the secondary r.t.s);

- with a secondary t.t. x,
  - the relative location $r(x)$ w.r.t. x of the intended r.t. which is a uniformly distributed vector in the plane of norm r;
  - a $\{0,1\}$-value r.v. $e(x)$ which takes value 1 with probability p_s. x transmits in the current slot iff $e(x)=1$;
  - the fading $f_{sp}(x)$ to the primary r.t. which is an exponential r.v. of parameter μ_s; and
  - the fading $\{f_{ss,i}(x)\}$ to the secondary r.t.s, which is a family of i.i.d. exponential r.v.s of parameter μ_s. These fading values are associated to the secondary r.t.s in the same manner as we do with $\{f_{ps,i}\}$.

For each t.t., its attributes are mutually independent; and the attributes of the t.t.s, considered as random vectors, are mutually independent too. For notational convenience, the $\{g_{ps,i}\}$ and $\{f_{ps,i}\}$ values are considered as two function g_{ps} and f_{ps} that takes points in Φ_s as parameter, and the values $\{\{f_{ss,i}(x)\}, x \in \Phi_s\}$ are considered as a function f_{ss} that takes elements of Φ_s^2 as parameter.

The primary t.t. is assumed to always transmit. This is a relevant assumption since the system behaviour when the primary t.t. does not transmit is merely that of an ALOHA network. A secondary t.t. at x is within the protection zone of the primary t.t. iff $E_{S/A}(x) := \mathbf{1}_{g_{ps}(x)l(|x|)<\rho} = 0$, where ρ is the detection threshold and l is the path loss. Thus, the process of secondary t.t.s which are not silenced by this criteria is $\Psi_{s,S/A} := \{x \in \Phi_s \text{ s.t. } E_{S/A}(x) = 1\}$. The process of active secondary t.t.s is hence

$$\Phi_{p,S/A} := \{x \in \Psi_{s,S/A} \text{ s.t. } e(x) = 1\} = \{x \in \Phi_s \text{ s.t. } E_{S/A}(x)e(x) = 1\}.$$

The SINR at the primary r.t. is

$$\mathrm{SINR}_p(o) = \frac{f_{pp}l(r_p)}{N + \mathcal{I}_{s,S/A}(o)},$$

and at the secondary r.t. corresponding to a secondary t.t. in $\Phi_{s,S/A}$ positioned at x is

$$\mathrm{SINR}_{s,S/A}(x) = \frac{f_{ss}l(r_s)}{N + f_{ps}(x)l(|x+r(x)|) + \mathcal{I}_{s,S/A}(x)}.$$

In the above formulas, $\mathcal{I}_{s,S/A}(.)$ is the SN interference associated to the active secondary t.t.s, which is

$$\sum_{y \in \Phi_{s,S/A} \setminus \{x\}} f_{ss}(y,x)l(|y-x-r(x)|)$$

for every x in $\Phi_{s,S/A}$ and is

$$\sum_{y \in \Phi_{s,S/A}} f_{sp}(y,x) l(|y - r(o)|)$$

for o.

Note that in this model, by the presence of the primary t.t. at the centre and by the spatial separation requirement, $\Phi_{s,S/A}$ is *not* a stationary PP. Thus, one cannot define the performance metrics by considering a typical t.t. Instead, we define these metrics as functions and measures in $\mathbb{R}^2$. In particular, the MAP function of the secondary t.t. is

$$p_{\mathrm{MAP,s,S/A}}(x) := \mathbb{P}_{x,\Phi_s}(E_{S/A}(x)e(x) = 1), \tag{2.2.1}$$

the COP of the primary t.t. is

$$p_{\mathrm{COP,p,S/A}} := \mathbb{P}\big(\mathrm{SINR}_p(o) > T\big), \tag{2.2.2}$$

the COP function of the secondary t.t.s is

$$p_{\mathrm{COP,s,S/A}}(x) = \mathbb{P}_{x,\Phi_{s,S/A}}\big(\mathrm{SINR}_s(x) > T\big), \tag{2.2.3}$$

and the AT measure of the secondary t.t.s is

$$\mathcal{T}_{s,S/A}(B) = \mathbb{E}\left[\sum_{x \in B \cap \Phi_{s,S/A}} \mathbf{1}_{\mathrm{SINR}_s(x) > T}\right]. \tag{2.2.4}$$

In the A/A network , the locations of the primary t.t.s and the locations of the secondary t.t.s are represented by two independent homogeneous PPPs Φ_p and Φ_s of intensities λ_p and λ_s, respectively. The attributes of these t.t.s are:

- for each t.t. (either primary or secondary) located at x,
  - the relative location $r(x)$ of its intended r.t, which is a uniformly distributed random vector of norm r_p if x is primary and of norm r_s if x is secondary;
  - the $\{0,1\}$-value r.v. $e(x)$ indicating whether x chooses to transmit in the current time slot or not. $\mathbb{P}\big(e(x) = 1\big)$ equals p_p if x is primary and equals p_s if x is secondary.
- the function f_{ps} (f_{pp}) taking elements in $\Phi_p \times \Phi_s$ (Φ_p^2) as parameter and representing the fading from primary t.t.s to secondary (primary) r.t.s. The realization of f_{ps} (f_{pp}) at each $(x,y) \in \Phi_p \times \Phi_s$ (Φ_p^2) is independent and is an exponential r.v. of parameter μ_p.
- the function f_{sp} (f_{ss}) taking elements in $\Phi_s \times \Phi_p$ (Φ_s^2) as parameter and representing the fading from primary t.t.s to primary r.t.s. The realization of f_{ps} (f_{ss}) at each $(x,y) \in \Phi_s \times \Phi_p$ (Φ_s^2) is independent and is an exponential r.v. of parameter μ_s.

- the function g_{ps} taking elements in $\Phi_p \times \Phi_s$ as parameter and representing the fading from primary t.t.s to secondary t.t.s. The realization of f_{ps} at each $(x, y) \in \Phi_p \times \Phi_s$ is independent and is an exponential r.v. of parameter μ_p. These fading is used in the modelling of the cognitive channel detection.
- the functions f_{pp}, f_{ps}, f_{sp}, f_{ss}, g_{ps} are constructed in the same way as we construct the function f_{ss} in the S/A model.

All the elements listed above are mutually independent.

The primary t.t.s schedule their transmissions using ALOHA. The process of active primary t.t.s is $\Phi_{p,A/A} := \{x \in \Phi_p \text{ s.t. } e(x) = 1\}$. A secondary t.t. at x is automatically silenced iff

$$E_{A/A}(x) := \prod_{y \in \Phi_{p,A/A}} \mathbf{1}_{g_{ps}(y,x)l(|y-x|)<\rho} = 0.$$

The remaining secondary t.t.s are $\Psi_{s,A/A} := \{x \in \Phi_s \text{ s.t. } E_{A/A}(x) = 1\}$ and the active secondary t.t.s are

$$\Phi_{s,A/A} := \{x \in \Psi_{s,A/A} \text{ s.t. } e(x) = 1\} = \{x \in \Phi_s \text{ s.t. } E_{A/A}(x)e(x) = 1\}.$$

The SINR at the intended r.t. associated to the t.t. positioned at x is

$$\mathrm{SINR}_{p,A/A}(x) := \frac{f_{pp}(x,x)l(r_p)}{N + \mathcal{I}_{pp,A/A}(x) + \mathcal{I}_{sp,A/A}(x)}$$

if x is primary and is

$$\mathrm{SINR}_{s,A/A}(x) := \frac{f_{ss}(x,x)l(r_s)}{N + \mathcal{I}_{ps,A/A}(x) + \mathcal{I}_{ss,A/A}(x)}$$

if x is secondary. The SN interference power are respectively

$$\begin{aligned}
\mathcal{I}_{pp,A/A}(x) &:= \sum_{y \in \Phi_{p,A/A} \setminus \{x\}} f_{pp}(y,x)l(|y - x - r(x)|);\\
\mathcal{I}_{sp,A/A}(x) &:= \sum_{y \in \Phi_{s,A/A}} f_{sp}(y,x)l(|y - x - r(x)|);\\
\mathcal{I}_{ps,A/A}(x) &:= \sum_{y \in \Phi_{p,A/A}} f_{ps}(y,x)l(|y - x - r(x)|);\\
\mathcal{I}_{ss,A/A}(x) &:= \sum_{y \in \Phi_{s,A/A} \setminus \{x\}} f_{ss}(y,x)l(|y - x - r(x)|).
\end{aligned}$$

Fortunately, both $\Phi_{p,A/A}$ and $\Phi_{s,A/A}$ are stationary. Thus, we can consider a typical primary (or secondary) t.t. to define the system performance metrics. In particular, we are interested in:

- for primary t.t.s,

- the COP: $p_{\mathrm{COP,p,A/A}} := \mathbb{P}_{o,\Phi_{p,A/A}}(\mathrm{SINR}_{p,A/A}(x) > T)$;
- the AT: $\mathcal{T}_{p,A/A} := \mathbb{E}[\sum_{x\in B\cap\Phi_{p,A/A}} \mathbf{1}_{\mathrm{SINR}_{p,A/A}(x)>T}]$ for any Borel set B of unit area;

- for secondary t.t.s,
  - the MAP: $p_{\mathrm{MAP,s,A/A}} := \mathbb{P}_{o,\Phi_s}(E_{A/A}(x)e(x) = 1)$;
  - the COP: $p_{\mathrm{COP,p,A/A}} := \mathbb{P}_{o,\Phi_{s,A/A}}(\mathrm{SINR}_{s,A/A}(x) > T)$;
  - the AT: $\mathcal{T}_{s,A/A} := \mathbb{E}[\sum_{x\in B\cap\Phi_{s,A/A}} \mathbf{1}_{\mathrm{SINR}_{s,A/A}(x)>T}]$ for any Borel set B of unit area;

In the C/C Network, the basic settings, i.e. the process of primary t.t.s, the process of secondary t.t.s and their attributes, are the same as in the A/A network with some modifications:

- the $\{0,1\}$-value r.v. $e(x)$ is replaced by the uniform r.v. $t(x)$ taking value in $[0,1]$. This r.v. is used in the determination of the Matérn type II model, which is used to model the set of active t.t.s determined by the CSMA protocol.

- apart from the functions f_{pp}, f_{ps}, f_{sp}, f_{ss}, we have the function g_{pp} (g_{ss}) taking unordered pair of t.t.s in Φ_p (Φ_s) as parameter. For each unordered pair (x,y) in Φ_p (Φ_s), $g_{pp}(x,y)$ ($g_{ss}(x,y)$) is an independent exponential r.v. of parameter μ_p (μ_s) that representing the fading from the t.t. at x to the t.t. at y and vice-versa. These two functions are constructed in the same way as in the construction of the function g in Subsection 2.1.1. The above fading values are used in the modelling of CSMA carrier sensing.

As in the A/A model, all the elements listed in the attributes are mutually independent. We define in Φ_p and Φ_s two random conflict relations $C_{cs,p}$ and $C_{cs,p}$ by

$$C_{cs,p}(x,y) = \mathbf{1}_{g_{pp}(x,y)l(|x-y|)>\rho_p}$$

for each unordered pair (x,y) in Φ_p and

$$C_{cs,s}(x,y) = \mathbf{1}_{g_{ss}(x,y)l(|x-y|)>\rho_s}$$

for each unordered pair (x,y) in Φ_s, where ρ_p and ρ_s are the CSMA carrier sensing thresholds of the primary and the secondary CSMA protocols respectively.

The process of active primary t.t.s is represented by the Matérn type II model of the PPPRCR $(\Phi_p, C_{cs,p})$ (see subsection 3.1.1 for its definition), namely $\Phi_{p,C/C} := \mathcal{M}_{\mathrm{II}}(\Phi_p, C_{cs,p})$. A secondary t.t. at x is silenced iff

$$E_{C/C}(x) := \prod_{y\in\Phi_{p,C/C}} \mathbf{1}_{g_{ps}(y,x)l(|y-x|)<\rho} = 0,$$

with ρ is the cognitive carrier sensing threshold. The remaining t.t.s are $\Psi_{s,C/C} := \{x \in \Phi_s \text{ s.t. } E_{C/C} = 1\}$ and the active secondary t.t.s are

$$\Phi_{s,C/C} := \mathcal{M}_{\mathrm{II}}(\Psi_{s,C/C}, C_{s,cs}|_{\Psi_{s,C/C}}),$$

where $C_{s,cs}|_{\Psi_{s,C/C}}$ is the restriction of $C_{s,cs}$ to $\Psi_{s,C/C}$.

The SINR at the intended r.t. of an active t.t. at x is

$$\mathrm{SINR}_{p,C/C} = \frac{f_{pp}(x,x)l(r_{p)}}{N + \mathcal{I}_{p,C/C}(x) + \mathcal{I}_{s,C/C}(x)}$$

if x is primary and is

$$\mathrm{SINR}_{s,C/C} = \frac{f_{ss}(x,x)l(r_{s)}}{N + \mathcal{I}_{p,C/C}(x) + \mathcal{I}_{s,C/C}(x)}$$

if x is secondary. The SN interference power are respectively,

$$\begin{aligned}
\mathcal{I}_{pp,C/C}(x) &:= \sum_{y \in \Phi_{p,C/C} \setminus \{x\}} f_{pp}(y,x)l(|y - x - r(x)|); \\
\mathcal{I}_{sp,C/C}(x) &:= \sum_{y \in \Phi_{s,C/C}} f_{sp}(y,x)l(|y - x - r(x)|); \\
\mathcal{I}_{ps,C/C}(x) &:= \sum_{y \in \Phi_{p,C/C}} f_{ps}(y,x)l(|y - x - r(x)|); \\
\mathcal{I}_{ss,C/C}(x) &:= \sum_{y \in \Phi_{s,C/C} \setminus \{x\}} f_{ss}(y,x)l(|y - x - r(x)|).
\end{aligned}$$

As the model defined here is also stationary, the performance metrics are then defined in the same manner as in the A/A model, namely

- for primary t.t.s,
  - the MAP: $p_{\mathrm{MAP,s,C/C}} := \mathbb{P}_{o,\Phi_s}(o \in \Phi_{p,C/C})$
  - the COP: $p_{\mathrm{COP,p,C/C}} := \mathbb{P}_{o,\Phi_{p,C/C}}(\mathrm{SINR}_{p,C/C}(x) > T)$;
  - the AT: $\mathcal{T}_{p,C/C} := \mathbb{E}[\sum_{x \in B \cap \Phi_{p,C/C}} \mathbf{1}_{\mathrm{SINR}_{p,C/C}(x) > T}]$ for any Borel set B of unit area;
- for secondary t.t.s,
  - the MAP: $p_{\mathrm{MAP,s,C/C}} := \mathbb{P}_{o,\Phi_s}(o \in \Phi_{s,C/C})$;
  - the COP: $p_{\mathrm{COP,p,C/C}} := \mathbb{P}_{o,\Phi_{s,C/C}}(\mathrm{SINR}_{s,C/C}(x) > T)$;
  - the AT: $\mathcal{T}_{s,C/C} := \mathbb{E}[\sum_{x \in B \cap \Phi_{s,C/C}} \mathbf{1}_{\mathrm{SINR}_{s,C/C}(x) > T}]$ for any Borel set B of unit area.

Remark 2.3 *In the above three models, we use different parameters, namely λ_p and λ_s, p_p and p_s, r_p and r_s, μ_p and μ_s, for each class. This reflects the fact that the primary and the secondary terminals may use different radio-system parameters. In particular, primary terminals in underutilized frequencies are usually sparsely deployed and they rarely use the air medium while secondary applications usually deploy a large population of terminals, who access the air medium quite frequently. More over, CR terminals deployed by secondary applications are usually small devices with weak signal power. Hence, they have quite small transmission range. To model correctly these facts, we usually take $\lambda_p < \lambda_s$, $p_p < p_s$, $r_p > r_s$ and $\mu_p < \mu_s$ (recall in Section 1.1 that the parameter μ of the fading r.v. takes into account the transmitting power and the stronger is the power, the smaller is μ).*

2.2.2 Analysis

The S/A network

Proposition 2.7 *The MAP function of the secondary t.t.s under the S/A model is*

$$p_{\mathrm{MAP,s,S/A}}(x) = (1 - \exp\{-\mu_p \rho l(|x|)\})\, p_s. \tag{2.2.5}$$

Proof. We have

$$\begin{aligned}
\mathbb{P}_{x,\Phi_s}(x \in \Phi_{s,S/A}) &= \mathbb{P}_{x,\Phi_s}\big(E_{S/A}(x)e(x) = 1\big) \\
&= \mathbb{P}_{x,\Phi_s}\big(E_{S/A}(x) = 1\big)\mathbb{P}_{x,\Phi_s}\big(e(x) = 1\big) \\
&= \mathbb{P}_{x,\Phi_s}\big(g_{sp}(x)l(|x|) < \rho\big)p_s \\
&= \big(1 - \exp\{-\mu_p \rho l(|x|)\}\big)p_s.
\end{aligned}$$

□

Since each secondary t.t. senses the network independently, the process of active secondary t.t.s forms an independent thinning of the process of all secondary t.t.s with the thinning probability only depends on the location of the terminal. As the latter forms an homogeneous PPP of intensity λ_s, the former is an inhomogeneous PPP of intensity measure $\lambda_s \left(1 - \exp\{-\mu_p \rho |x - R_0^I|^\alpha\}\right) dx$ and we can use Theorem A.1 to get,

Proposition 2.8 *The COP of the primary t.t. in S/A model is*

$$\begin{aligned}
p_{\mathrm{COP,p,A/S}} = \exp\Bigg\{ &-\frac{\mu_p T N}{l(r_p)} - \lambda_s \int_{\mathbb{R}^2} \frac{\mu_p T l(|x - r_p \mathbf{e}|)}{\mu_p T l(|x - r_p \mathbf{e}|) + \mu_s l(|r_p|)} \\
&\left(1 - \exp\left\{-\frac{\mu_p \rho}{l(|x|)}\right\}\right) p_s dx \Bigg\}.
\end{aligned} \tag{2.2.6}$$

Proof. First notice that the distribution of Φ_s is invariant under rotations, so we can assume w.l.o.g. that $r(o) = r_p\mathbf{e}$ We want to compute the following probability:

$$\mathbb{P}\left(\mathrm{SINR}_{p,S/A}(o) > T\right) = \mathbb{P}\left(\frac{f_{pp} l(r_p)}{N + \mathcal{I}_{s,S/A}(o)} > T\right).$$

Using the fact that f_{pp} is an exponential r.v. with parameter μ which is independent of all other random elements involved in the formula of $\mathrm{SINR}_{p,S/A}$,

$$\begin{aligned}\mathbb{P}\left(\mathrm{SINR}_{p,S/A}(o) > T\right) &= \mathbb{E}\left[\exp\left\{-\frac{\mu_p T(N + \mathcal{I}_{s,S/A}(o))}{l(r_p)}\right\}\right] \\ &= \exp\left\{-\frac{\mu_p T N}{l(r_p)}\right\} \mathbb{E}\left[\exp\left\{-\frac{\mu_p T(\mathcal{I}_{s,S/A}(o))}{l(r_p)}\right\}\right].\end{aligned}$$

For the second term in the last equality,

$$\begin{aligned}&\mathbb{E}\left[\exp\left\{-\frac{\mu_p T \mathcal{I}_{s,S/A}(o)}{l(r_p)}\right\}\right] \\ &= \mathbb{E}\left[\exp\left\{-\frac{\mu_p T\left(\sum_{x\in\Phi_{s,S/A}} f_{sp}(x) l|(x - r(o)|)\right)}{l(r_p)}\right\}\right] \\ &= \mathbb{E}\left[\exp\left\{-\mu_p T\left(\sum_{x\in\Phi_{s,S/A}} f_{sp}(x)\frac{l|(x - r(o)|)}{l(r_p)}\right)\right\}\right] \\ &= \mathbb{E}\left[\exp\left\{-\mu_p T\left(\sum_{x\in\Phi_s} E_{S/A}(x) e(x) f_{sp}(x)\frac{l|(x - r(o)|)}{l(r_p)}\right)\right\}\right] \\ &= \mathbb{E}\left[\prod_{x\in\Phi_s} \exp\left\{-\mu_p T\left(E_{S/A}(x) e(x) f_{sp}(x)\frac{l|(x - r(o)|)}{l(r_p)}\right)\right\}\right].\end{aligned}$$

As $E_{S/A}(x)$, $e(x)$, $f_{sp}(x)$ can be regarded as independent marks of Φ_s, we use Theorem A.1 to get

$$\begin{aligned}&\mathbb{E}\left[\prod_{x\in\Phi_s} \exp\left\{-\mu_p T\left(E_{S/A}(x) e(x) f_{sp}(x)\frac{l|(x - r(o)|)}{l(r_p)}\right)\right\}\right] \\ &= \exp\left\{-\lambda_s \int_{\mathbb{R}^2}\left(1 - \mathbb{E}\left[\exp\left\{-\mu_p T\left(E_{S/A}(x) e(x) f_{sp}(x)\frac{l|(x - r(o)|)}{l(r_p)}\right)\right\}\right]\right) dx\right\}.\end{aligned}$$

We can easily compute

$$\begin{aligned}&1 - \mathbb{E}\left[\exp\left\{-\mu_p T\left(E_{S/A}(x) e(x) f_{sp}(x)\frac{l|(x - r(o)|)}{l(r_p)}\right)\right\}\right] \\ &= \frac{\mu_p T l(|x - r(o)|)}{\mu_p T l(|x - r(o)|) + \mu_s l(|r_p|)}\left(1 - \exp\left\{-\frac{\mu_p \rho}{l(|x|)}\right\}\right) p_s.\end{aligned}$$

The conclusion follows directly by using $r(o) = r_p \mathbf{e}$. $\square$

Proposition 2.9 *The COP function of the secondary t.t.s in the S/A model is*

$$p_{\mathrm{COP,s,S/A}}(x) = \frac{\exp\left\{-\frac{\mu_s TN}{l(r_s)}\right\}}{2\pi}\int_0^{2\pi}\frac{\mu_p l(r_s)}{\mu_p l(r_s)+\mu_s Tl(|x+r_s\mathbb{S}_\theta(\boldsymbol{e})|))}$$
$$\exp\left\{-\lambda_s\int_{\mathbb{R}^2}\frac{Tl(|y-x-r_s\mathbb{S}_\theta(\boldsymbol{e})|)\left(1-\exp\left\{-\frac{\mu_p\rho}{l(|y|)}\right\}\right)p_s}{Tl(|y-x-r_s\mathbb{S}_\theta(\boldsymbol{e})|)+l(r_s)}dx\right\}d\theta, \tag{2.2.7}$$

where $\mathbb{S}_\theta$ denotes the rotation of angle θ and $\boldsymbol{e}$ is a fixed unit vector.

Proof. We need to compute

$$\begin{aligned}
&\mathbb{P}_{x,\Phi_{s,S/A}}(\mathrm{SINR}_{s,S/A}(x)>T)\\
&=\mathbb{P}_{x,\Phi_{s,S/A}}\left(\frac{f_{ss}(x,x)l(r_s)}{N+f_{ps}(x)l(|x+r(x)|)+\mathcal{I}_{s,S/A}(x)}>T\right)\\
&=\mathbb{P}_{x,\Phi_{s,S/A}}\left(f_{ss}(x,x)>\frac{T\left(N+f_{ps}(x)l(|x+r(x)|)+\mathcal{I}_{s,S/A}(x)\right)}{l(r_s)}\right)\\
&=\mathbb{E}_{x,\Phi_{s,S/A}}\left[\exp\left\{-\mu_s T\frac{\left(N+f_{ps}(x)l(|x+r(x)|)+\mathcal{I}_{s,S/A}(x)\right)}{l(r_s)}\right\}\right]\\
&=\exp\left\{-\frac{\mu_s TN}{l(r_s)}\right\}\mathbb{E}_{x,\Phi_{s,S/A}}\left[\exp\left\{-\mu_s T\frac{\left(f_{ps}(x)l(|x+r(x)|)+\mathcal{I}_{s,S/A}(x)\right)}{l(r_s)}\right\}\right].
\end{aligned}$$

Note that given $r(x)$, $f_{ps}(x)l(x+r(x))$ is independent of $\mathcal{I}_{s,S/A}(x)$ under $\mathbb{P}_{x,\Phi_{s,S/A}}$. Hence,

$$\begin{aligned}
&\mathbb{E}_{x,\Phi_{s,S/A}}\left[\exp\left\{-\mu_s T\frac{\left(f_{ps}(x)l(|x+r(x)|)+\mathcal{I}_{s,S/A}(x)\right)}{l(r_s)}\right\}\middle|r(x)\right]\\
&=\mathbb{E}_{x,\Phi_{s,S/A}}\left[\exp\left\{-\mu_s T\frac{f_{ps}(x)l(|x+r(x)|)}{l(r_s)}\right\}\middle|r(x)\right]\\
&\quad\mathbb{E}_{x,\Phi_{s,S/A}}\left[\exp\left\{-\mu_s T\frac{\mathcal{I}_{s,S/A}(x)}{l(r_s)}\right\}\middle|r(x)\right].
\end{aligned}$$

Putting $r(x)=r_s\mathbb{S}_\theta(\mathbf{e})$, the first term is

$$\mathbb{E}_{x,\Phi_{s,S/A}}\left[\exp\left\{-\mu_s T\frac{f_{ps}(x)l(|x+r(x)|)}{l(r_s)}\right\}\middle|r(x)\right]=\frac{\mu_p l(r_s)}{\mu_s l(r_s)+\mu_p Tl(|x+r_s\mathbb{S}_\theta(\mathbf{e})|)},$$

since $f_{ps}(x)$ is an exponential r.v. with parameter μ_p. The second term is

$$\begin{aligned}
&\mathbb{E}_{x,\Phi_{s,S/A}}\left[\exp\left\{-\mu_s T\frac{\mathcal{I}_{s,S/A}(x)}{l(r_s)}\right\}\Big| r(x)\right]\\
&= \mathbb{E}_{x,\Phi_{s,S/A}}\left[\exp\left\{-\mu_s T\frac{\sum_{y\in\Phi_{s,S/A}} f_{ss}(y,x)l(|y-x-r(x)|)}{l(r_s)}\right\}\Big| r(x)\right]\\
&= \mathbb{E}_{x,\Phi_{s,S/A}}\left[\prod_{y\in\Phi_{s,S/A}}\exp\left\{-\mu_s T\frac{f_{ss}(y,x)l(|y-x-r(x)|)}{l(r_s)}\right\}\Big| r(x)\right].
\end{aligned}$$

Since $\Phi_{s,S/A}$ is a PPP of intensity measure $\left(1-\exp\left\{-\frac{\mu_s\rho}{l(|y|)}\right\}\right)p_s dx$ and the thinning from Φ_s to $\Phi_{s,S/A}$ is independent of f_{ss}, we can consider $f_{ss}(y,x)$ as independent exponential marks with parameter μ_s of the points in $\Phi_{s,S/A}$. We then apply Slivnyak's theorem to get

$$\begin{aligned}
&\mathbb{E}_{x,\Phi_{s,S/A}}\left[\prod_{y\in\Phi_{s,S/A}}\exp\left\{-\mu_s T\frac{f_{ss}(y,x)l(|y-x-r(x)|)}{l(r_s)}\right\}\Big| r(x)\right]\\
&= \exp\left\{-\lambda_s\int_{\mathbb{R}_2}\left(1-\mathbb{E}\left[\exp\left\{-\mu_s T\frac{f_{ss}(y,x)l(|y-x-r(x)|)}{l(r_s)}\right\}\Big| r(x)\right]\right)\right.\\
&\quad\left.\left(1-\exp\left\{-\frac{\mu_s\rho}{l(|y|)}\right\}\right)p_s dx\right\}.
\end{aligned}$$

Now notice that $r(x)$ can be expressed as $r_s\mathbb{S}_\theta(\mathbf{e})$,

$$\begin{aligned}
&1-\mathbb{E}\left[\exp\left\{-\mu_s T\frac{f_{ss}(y,x)l(|y-x-r(x)|)}{l(r_s)}\right\}\Big| r(x)\right]\\
&= \frac{Tl(|y-x-r_s\mathbb{S}_\theta(\mathbf{e})|)}{Tl(|y-x-r_s\mathbb{S}_\theta(\mathbf{e})|)+l(r_s)},
\end{aligned}$$

and the conclusion follows directly. □

Proposition 2.10 *The average throughput measure of the secondary t.t.s in the S/A model is*

$$\mathcal{T}_{s,S/A}(B) = \lambda_s\int_B p_{\mathrm{MAP,s,S/A}}(x)p_{\mathrm{COP,s,S/A}}(x)dx.$$

Proof. This is just a corollary of Campbell's formula (see [27] or [6]) and Propositions 2.7 and 2.9. □

The A/A network

Proposition 2.11 *The MAP of a typical secondary t.t. in the A/A model is*

$$p_{\mathrm{MAP,s,A/A}} = \exp\left\{-\lambda_p p_p\int_{\mathbb{R}^2}\exp\left\{-\frac{\mu_p\rho}{l(|y|)}\right\}dy\right\}. \tag{2.2.8}$$

Proof. First we have,

$$\mathbb{P}_{o,\Phi_s}(E_{A/A}(x)=1 \text{ and } e(x)=1) = p_s \mathbb{E}_{o,\Phi_s}\left[\prod_{y\in\Phi_{p,A/A}} \mathbf{1}_{g_{ps}(y,o)l(|y|)<\rho}\right].$$

Let $\mathcal{F}$ be the σ-algebra generated by the realization of Φ_p, by the independence of Φ_p and Φ_s

$$\begin{aligned}\mathbb{E}_{o,\Phi_s}\left[\prod_{y\in\Phi_{p,A/A}} \mathbf{1}_{g_{ps}(y,o)l(|y|)<\rho}\middle|\mathcal{F}\right] &= \mathbb{E}_{o,\Phi_s}\left[\prod_{y\in\Phi_p} e(y)\mathbf{1}_{g_{ps}(y,o)l(|y|)<\rho}\middle|\mathcal{F}\right]\\ &= \prod_{y\in\Phi_p} p_p\left(1-\exp\left\{-\frac{\mu_p\rho}{l(|y|)}\right\}\right).\end{aligned}$$

Hence,

$$\begin{aligned}p_{\text{MAP,s,A/A}} &= p_s\mathbb{E}\left[\prod_{y\in\Phi_p} p_p\left(1-\exp\left\{-\frac{\mu_p\rho}{l(|y|)}\right\}\right)\right]\\ &= p_s\exp\left\{-\lambda_p p_p\int_{\mathbb{R}^2}\exp\left\{-\frac{\mu_p\rho}{l(|y|)}\right\}dy\right\}.\end{aligned}$$

□

Proposition 2.12 *The COP of a typical t.t. in the A/A model is upper bounded by*

$$\begin{aligned}&p_{\text{COP,p,A/A}} \le \exp\left\{-\mu_p\frac{TN}{l(r_p)}\right\}\\ &\left(e^{-\lambda_p p_p\mathfrak{c}(r_p,\mu_p,\mu_p)} - \frac{\left(1-e^{-\lambda_s p_s\mathfrak{c}(r_p,\mu_s,\mu_p)}\right)}{\mathfrak{c}(r_p,\mu_s,\mu_p)}\int_{\mathbb{R}^2}\frac{\mu_p Tl(|x-r(o)|)}{\mu_p Tl(|x-r(o)|)+\mu_s l(r_p)}\right.\\ &\left.\left(1-e^{\mu_p\rho/l(|x|)}\right)\exp\left\{-\lambda_p p_p\int_{\mathbb{R}^2}\frac{Tl(|y-r(o)|)+l(r_p)e^{\mu_p\rho/l(|y-x|)}}{l(r_p)+Tl(|y-r(o)|)}dy\right\}dx\right)\end{aligned} \tag{2.2.9}$$

with $r(o)=r_p\boldsymbol{e}$ *for primary t.t.s and*

$$\begin{aligned}&p_{\text{COP,s,A/A}}\\ &\le p_p\exp\left\{-\frac{\mu_s TN}{l(r_s)}\right\}\left(\exp\left\{-\lambda_p p_p\int_{\mathbb{R}^2}\frac{\mu_s Tl(|y-r(o)|)+\mu_p l(r_s)e^{-\mu_p\rho/l(|y|)}}{\mu_p l(r_s)+\mu_s Tl(|y-r(o)|)}dy\right\}\right.\\ &\quad-\frac{\left(1-e^{-\lambda_s p_s\mathfrak{c}(r_s,\mu_s,\mu_s)}\right)}{\mathfrak{c}(r_s,\mu_s,\mu_s)}\int_{\mathbb{R}^2}\frac{Tl(|x-r(o)|)}{Tl(|x-r(o)|)+l(r_s)}\exp\left\{-\lambda_p p_p\int_{\mathbb{R}^2}\right.\\ &\quad\left.\left.\left(1-\frac{\mu_p l(r_s)\left(1-e^{-\mu_p\rho/l(|y|)}\right)\left(1-e^{-\mu_p\rho/l(|y-x|)}\right)}{\mu_p l(r_s)+\mu_s Tl(|y-r(o)|)}\right)dy\right\}dx\right)p_{\text{MAP,s,A/A}}^{-1}\end{aligned} \tag{2.2.10}$$

with $r(o) = r_s\boldsymbol{e}$ for secondary t.t.s, where

$$\mathfrak{c}(r,\mu_1,\mu_2) = \int_{\mathbb{R}^2} \frac{\mu_2 T l(|y - r\boldsymbol{e}|)}{\mu_1 l(r) + \mu_2 T l(|y - r\boldsymbol{e}|)} dy. \tag{2.2.11}$$

Proof. We first note that the distribution of all PPs considered here are invariant under rotations (and hence so is their Palm distributions, see [27, p. 123]), so for a t.t. located at o, we can take $r(o) = r_p\mathbf{e}$ if o is primary and $r(o) = r_s\mathbf{e}$ if o is secondary. For o primary, let $\mathcal{F}_p$ and $\mathcal{F}_s$ be the σ-algebras generated by the realizations of $\Phi_{p,A/A}$ and Φ_s respectively,

$$\begin{aligned}
&\mathbb{P}_{o,\Phi_{p,A/A}}(\mathrm{SINR}_p(o) > T \mid \mathcal{F}_p, \mathcal{F}_s, \mathbf{g}_{ps}, r(o)) \\
&= \mathbb{P}_{o,\Phi_{p,A/A}}\left(\frac{f_{pp} l(r_p)}{N + \mathcal{I}_{pp}(o) + \mathcal{I}_{sp}(o)} > T \middle| \mathcal{F}_p, \mathcal{F}_s, \mathbf{g}_{ps}, r(o)\right) \\
&= \mathbb{E}_{o,\Phi_{p,A/A}}\left[\exp\left\{-\mu_p \frac{T\big(N + \mathcal{I}_{pp}(o) + \mathcal{I}_{sp}(o)\big)}{l(r_p)}\right\} \middle| \mathcal{F}_p, \mathcal{F}_s, \mathbf{g}_{ps}, r(o)\right] \\
&= \exp\left\{-\mu_p \frac{TN}{l(r_p)}\right\} \mathbb{E}_{o,\Phi_{p,A/A}}\left[\exp\left\{-\mu_p \frac{T\mathcal{I}_{pp}(o)}{l(r_p)}\right\} \middle| \mathcal{F}_p, \mathcal{F}_s, \mathbf{g}_{ps}, r(o)\right] \\
&\quad \mathbb{E}_{o,\Phi_{p,A/A}}\left[\exp\left\{-\mu_p \frac{T\mathcal{I}_{sp}(o)}{l(r_p)}\right\} \middle| \mathcal{F}_p, \mathcal{F}_s, \mathbf{g}_{ps}, r(o)\right]
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\mathbb{E}_{o,\Phi_{p,A/A}}\left[\exp\left\{-\mu_p \frac{T\mathcal{I}_{pp}(o)}{l(r_p)}\right\} \middle| \mathcal{F}_p, \mathcal{F}_s, \mathbf{g}_{ps}, r(o)\right] \\
&= \mathbb{E}_{o,\Phi_{p,A/A}}\left[\exp\left\{-\mu_p \frac{T\left(\sum_{x\in\Phi_{p,A/A}\setminus\{o\}} f_{pp}(x,o) l(|x - r(o)|)\right)}{l(r_p)}\right\} \middle| \mathcal{F}_p, \mathcal{F}_s, r(o)\right] \\
&= \prod_{x\in\Phi_{p,A/A}\setminus\{o\}} \mathbb{E}\left[\exp\left\{-\mu_p \frac{T f_{pp}(x,o) l(|x - r(o)|)}{l(r_p)}\right\}\right] \\
&= \prod_{x\in\Phi_{p,A/A}\setminus\{o\}} \frac{l(r_p)}{l(r_p) + Tl(|x - r(o)|)} = \prod_{y\in\Phi_{p,A/A}\setminus\{o\}} \frac{l(r_p)}{l(r_p) + Tl(|y - r(o)|)},
\end{aligned} \tag{2.2.12}$$

and

$$\begin{aligned}
&\mathbb{E}_{o,\Phi_{p,A/A}}\left[\exp\left\{-\mu_p\frac{T\mathcal{I}_{sp}(o)}{l(r_p)}\right\}\Big|\mathcal{F}_p,\mathcal{F}_s,\mathbf{g}_{ps},r(o)\right]\\
&=\mathbb{E}_{o,\Phi_{p,A/A}}\left[\exp\left\{-\mu_p\frac{T\left(\sum_{x\in\Phi_{s,A/A}}f_{sp}(x,o)l(|x-r(o)|)\right)}{l(r_p)}\right\}\Bigg|\mathcal{F}_p,\mathcal{F}_s,\mathbf{g}_{ps},r(o)\right]\\
&=\mathbb{E}_{o,\Phi_{p,A/A}}\left[\exp\left\{-\mu_p\frac{T\left(\sum_{x\in\Phi_s}E_{A/A}(x)f_{sp}(x,o)l(|x-r(o)|)\right)}{l(r_p)}\right\}\Bigg|\mathcal{F}_p,\mathcal{F}_s,\mathbf{g}_{ps},r(o)\right]\\
&=\mathbb{E}_{o,\Phi_{p,A/A}}\left[\prod_{x\in\Phi_s}\mathbb{E}\left[\exp\left\{-\mu_p\frac{TE_{A/A}(x)f_{sp}(x,o)l(|x-r(o)|)}{l(r_p)}\right\}\right]\Bigg|\mathcal{F}_p,\mathcal{F}_s,\mathbf{g}_{ps},r(o)\right].
\end{aligned}$$

Given $\mathcal{F}_p$ and $\mathbf{g}_{ps}$, $E_{A/A}(x)$ takes value in $\{0,1\}$ a.s. for every x, so

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{-\mu_p\frac{TE_{A/A}(x)f_{sp}(x,o)l(|x-r(o)|)}{l(r_p)}\right\}\right]\\
&=\mathbb{E}\left[\exp\left\{-\mu_p\frac{Tf_{sp}(x,o)l(|x-r(o)|)}{l(r_p)}\right\}E_{A/A}(x)+1-E_{A/A}(x)\right]\\
&=\frac{\mu_s l(r_p)}{\mu_p Tl(|x-r(o)|)+\mu_s l(r_p)}E_{A/A}(x)+1-E_{A/A}(x)\\
&=1-\frac{\mu_p Tl(|x-r(o)|)}{\mu_p Tl(|x-r(o)|)+\mu_s l(r_p)}E_{A/A}(x)dx.
\end{aligned}$$

Hence, by Theorem A.1,

$$\begin{aligned}
&\mathbb{E}_{o,\Phi_{p,A/A}}\left[\exp\left\{-\mu_p\frac{T\mathcal{I}_{sp}(o)}{l(r_p)}\right\}\Big|\mathcal{F}_p,\mathbf{g}_{ps},r(o)\right]\\
&=\exp\left\{-\lambda_s p_s\int_{\mathbb{R}^2}\left(1-\frac{\mu_s l(r_p)}{\mu_p Tl(|x-r(o)|)+\mu_s l(r_p)}\right)E_{A/A}(x)dx\right\}\\
&=\exp\left\{-\lambda_s p_s\int_{\mathbb{R}^2}\frac{\mu_p Tl(|x-r(o)|)}{\mu_p Tl(|x-r(o)|)+\mu_s l(r_p)}E_{A/A}(x)dx\right\}.
\end{aligned}$$

As $\exp\{-x\}\le 1-(1-\exp\{-y\})\frac{x}{y}$ for every $0\le x\le y$ and

$$\begin{aligned}
0&\le\int_{\mathbb{R}^2}\frac{\mu_p Tl(|x-r(o)|)}{\mu_p Tl(|x-r(o)|)+\mu_s l(r_p)}E_{A/A}(x)dx\\
&\le\int_{\mathbb{R}^2}\frac{\mu_p Tl(|x-r(o)|)}{\mu_p Tl(|x-r(o)|)+\mu_s l(r_p)}dx=\mathfrak{c}(r_p,\mu_s,\mu_p)\text{ a.s. },
\end{aligned}$$

we have,

$$\mathbb{E}_{o,\Phi_{p,A/A}}\left[\exp\left\{-\mu_p \frac{T\mathcal{I}_{sp}(o)}{l(r_p)}\right\}\Big|\mathcal{F}_p, \mathbf{g}_{ps}, r(o)\right]$$
$$\leq 1 - \left(1 - e^{-\lambda_s p_s \mathfrak{c}(r_p,\mu_s,\mu_p)}\right)\frac{\int_{\mathbb{R}^2}\frac{\mu_p Tl(|x-r(o)|)}{\mu_p Tl(|x-r(o)|)+\mu_s l(r_p)}E_{A/A}(x)dx}{\mathfrak{c}(r_p,\mu_s,\mu_p)}.$$

Combine this with (2.2.12),

$$\mathbb{P}_{o,\Phi_{p,A/A}}(\mathrm{SINR}_p(o) > T \mid \mathcal{F}_p, \mathbf{g}_{ps}, r(o))$$
$$\leq \exp\left\{-\mu_p \frac{TN}{l(r_p)}\right\}\prod_{y\in\Phi_{p,A/A}\setminus\{o\}}\frac{l(r_p)}{l(r_p)+Tl(|y-r(o)|)}\left(1 - \frac{\left(1-e^{-\lambda_s p_s \mathfrak{c}(r_p,\mu_s,\mu_p)}\right)}{\mathfrak{c}(r_p,\mu_s,\mu_p)}\right.$$
$$\left.\int_{\mathbb{R}^2}\frac{\mu_p Tl(|x-r(o)|)}{\mu_p Tl(|x-r(o)|)+\mu_s l(r_p)}E_{A/A}(x)dx\right).$$

As

$$E_{A/A}(x) = \mathbf{1}_{g_{ps}(o,x)l(|x|)<\rho}\prod_{y\in\Phi_{p,A/A}\setminus\{o\}}\mathbf{1}_{g_{ps}(y,x)l(|y-x|)<\rho}$$

by definition, by Slivnyak's theorem and Theorem A.1

$$\mathbb{E}_{o,\Phi_{p,A/A}}\left[\prod_{y\in\Phi_{p,A/A}\setminus\{o\}}\frac{l(r_p)}{l(r_p)+Tl(|y-r(o)|)}E_{A/A}(x)\right]$$
$$= \mathbb{E}_{o,\Phi_{p,A/A}}\left[\prod_{y\in\Phi_{p,A/A}\setminus\{o\}}\frac{l(r_p)}{l(r_p)+Tl(|y-r(o)|)}\prod_{y\in\Phi_{p,A/A}\setminus\{o\}}\mathbf{1}_{g_{ps}(y,x)l(|y-x|)<\rho}\right.$$
$$\left.\mathbf{1}_{g_{ps}(o,x)l(|x|)>\rho}\right]$$
$$= \mathbb{E}_{o,\Phi_{p,A/A}}\left[\left(\prod_{y\in\Phi_{p,A/A}\setminus\{o\}}\frac{l(r_p)\mathbf{1}_{g_{ps}(y,x)l(|y-x|)<\rho}}{l(r_p)+Tl(|y-r(o)|)}\right)\mathbf{1}_{g_{ps}(o,x)l(|x|)>\rho}\right]$$
$$= \mathbb{E}_{o,\Phi_{p,A/A}}\left[\left(\prod_{y\in\Phi_{p,A/A}\setminus\{o\}}\frac{l(r_p)\left(1-e^{\mu_p\rho/l(|y-x|)}\right)}{l(r_p)+Tl(|y-r(o)|)}\right)\left(1-e^{\mu_p\rho/l(|x|)}\right)\right]$$
$$= \left(1-e^{\mu_p\rho/l(|x|)}\right)\exp\left\{-\lambda_p p_p\int_{\mathbb{R}^2}\frac{Tl(|y-r(o)|)+l(r_p)e^{\mu_p\rho/l(|y-x|)}}{l(r_p)+Tl(|y-r(o)|)}dy\right\}$$

and

$$\mathbb{E}_{o,\Phi_{p,A/A}}\left[\prod_{y\in\Phi_{p,A/A}\setminus\{o\}}\frac{l(r_p)}{l(r_p)+Tl(|y-r(o)|)}\right] = e^{-\lambda_p p_p \mathfrak{c}(r_p,\mu_p,\mu_p)}.$$

So,

$$\mathbb{P}_{o,\Phi_{p,A/A}}(\mathrm{SINR}_p(o) > T) \leq \exp\left\{-\mu_p \frac{TN}{l(r_p)}\right\}$$
$$\left(e^{-\lambda_p p_p \mathfrak{c}(r_p,\mu_p,\mu_p)} - \frac{\left(1 - e^{-\lambda_s p_s \mathfrak{c}(r_p,\mu_s,\mu_p)}\right)}{\mathfrak{c}(r_p,\mu_s,\mu_p)} \int_{\mathbb{R}^2} \frac{\mu_p T l(|x - r(o)|)}{\mu_p T l(|x - r(o)|) + \mu_s l(r_p)}\right.$$
$$\left.\left(1 - e^{\mu_p \rho / l(|x|)}\right) \exp\left\{-\lambda_p p_p \int_{\mathbb{R}^2} \frac{T l(|y - r(o)|) + l(r_p) e^{\mu_p \rho / l(|y-x|)}}{l(r_p) + T l(|y - r(o)|)} dy\right\} dx\right).$$

For a typical active secondary t.t., note that

$$\begin{aligned}\mathbb{P}_{o,\Phi_{s,A/A}}(\mathrm{SINR}_o(x) > T) &= \mathbb{P}_{o,\Phi_s}(\mathrm{SINR}_o(x) > T \mid e(o) = 1, E_{A/A}(o) = 1) \\ &= \frac{\mathbb{P}_{o,\Phi_s}(\mathrm{SINR}_o(x) > T, e(o) = 1, E_{A/A}(o) = 1)}{p_{\mathrm{MAP,s,A/A}}}.\end{aligned}$$

By the same arguments as with the primary t.t.

$$\mathbb{P}_{o,\Phi_s}(\mathrm{SINR}_o(x) > T, e(o) = 1, E_{A/A}(o) = 1 \mid \mathcal{F}_p, \mathcal{F}_s, \mathbf{g}_{ps}, r(o))$$
$$= \exp\left\{-\frac{\mu_s TN}{l(r_s)}\right\} \mathbb{E}_{o,\Phi_s}\left[\exp\left\{-\mu_s T \frac{\mathcal{I}_{ps}(o)}{l(r_s)}\right\} E_{A/A}(o) \,\middle|\, \mathcal{F}_p, \mathcal{F}_s, \mathbf{g}_{ps}, r(o)\right]$$
$$\mathbb{E}^!_{o,\Phi_s}\left[\exp\left\{-\mu_s T \frac{\mathcal{I}_{ss}(o)}{l(r_s)}\right\} \,\middle|\, \mathcal{F}_p, \mathcal{F}_s, \mathbf{g}_{ps}, r(o)\right].$$

The second term above is computed as

$$\begin{aligned}&\mathbb{E}_{o,\Phi_s}\left[\exp\left\{-\mu_s T \frac{\mathcal{I}_{ps}(o)}{l(r_s)}\right\} E_{A/A}(o) \,\middle|\, \mathcal{F}_p, \mathcal{F}_s, \mathbf{g}_{ps}, r(o)\right] \\ &= \prod_{y \in \Phi_p} \mathbb{E}\left[\exp\left\{\frac{\mu_s T f_{ps}(y,o) l(|y - r(o)|)}{l(r_s)}\right\}\right] \prod_{y \in \Phi_p} \mathbf{1}_{g_{ps}(y,o) l(|y|) < \rho} \\ &= \prod_{y \in \Phi_p} \frac{\mu_p l(r_s)}{\mu_p l(r_s) + \mu_s T l(|y - r(o)|)} \prod_{y \in \Phi_p} \mathbf{1}_{g_{ps}(y,o) l(|y|) < \rho} \\ &= \prod_{y \in \Phi_p} \frac{\mu_p l(r_s) \mathbf{1}_{g_{ps}(y,o) l(|y|) < \rho}}{\mu_p l(r_s) + \mu_s T l(|y - r(o)|)}.\end{aligned}$$

For the third term,

$$\begin{aligned}
&\mathbb{E}_{o,\Phi_s}\left[\exp\left\{-\mu_s T\frac{\mathcal{I}_{ss}(o)}{l(r_s)}\right\}\middle|\mathcal{F}_p,\mathcal{F}_s,\mathbf{g}_{ps},r(o)\right]\\
&=\prod_{x\in\Phi_s\setminus\{o\}}\mathbb{E}\left[\exp\left\{-\frac{\mu_s T f_{ss}(x,o)l(|x-r(o)|)}{l(r_s)}E_{A/A}(x)e(x)\right\}\right]\\
&=\prod_{x\in\Phi_s\setminus\{o\}}\left(1-\left(1-\mathbb{E}\left[\exp\left\{-\frac{\mu_s T f_{ss}(x,o)l(|x-r(o)|)}{l(r_s)}\right\}\right]\right)E_{A/A}(x)e(x)\right)\\
&=\prod_{x\in\Phi_s\setminus\{o\}}\left(1-\frac{l(r_s)}{l(r_s)+Tl(|x-r(o)|)}E_{A/A}(x)e(x)\right).
\end{aligned}$$

Hence, by Theorem A.1,

$$\begin{aligned}
&\mathbb{E}_{o,\Phi_s}\left[\exp\left\{-\mu_s T\frac{\mathcal{I}_{ss}(o)}{l(r_s)}\right\}\middle|\mathcal{F}_p,\mathbf{g}_{ps},r(o)\right]\\
&=\exp\left\{-\lambda_s p_s\int_{\mathbb{R}^2}\frac{Tl(|x-r(o)|)}{Tl(|x-r(o)|)+l(r_s)}E_{A/A}(x)dx\right\}\\
&\le 1-\frac{\left(1-e^{-\lambda_s p_s \mathfrak{c}(r_s,\mu_s,\mu_s)}\right)}{\mathfrak{c}(r_s,\mu_s,\mu_s)}\int_{\mathbb{R}^2}\frac{Tl(|x-r(o)|)}{Tl(|x-r(o)|)+l(r_s)}E_{A/A}(x)dx.
\end{aligned}$$

Note that $\mathcal{F}_p$, $\mathcal{F}_s$, $\mathbf{g}_{ps}$ and $r(o)$ are mutually independent, by combining the three terms and taking expectation w.r.t. $\mathcal{F}_s$, we get

$$\begin{aligned}
&\mathbb{P}_{o,\Phi_s}\left(SINR_o(x)>T,e(o)E_{A/A}(o)=1\mid\mathcal{F}_p,\mathbf{g}_{ps},r(o)\right)\\
&\le p_p\exp\left\{-\frac{\mu_s TN}{l(r_s)}\right\}\prod_{y\in\Phi_p}\frac{\mu_p l(r_s)\mathbf{1}_{g_{ps}(y,o)l(|y|)<\rho}}{\mu_p l(r_s)+\mu_s Tl(|y-r(o)|)}\\
&\quad\left(1-\frac{\left(1-e^{-\lambda_s p_s \mathfrak{c}(r_s,\mu_s,\mu_s)}\right)}{\mathfrak{c}(r_s,\mu_s,\mu_s)}\int_{\mathbb{R}^2}\frac{Tl(|x-r(o)|)}{Tl(|x-r(o)|)+l(r_s)}E_{A/A}(x)dx\right).
\end{aligned}$$

By Theorem A.1, we take expectation w.r.t. $\mathcal{F}_p$ and $\mathbf{g}_{ps}$ to get,

$$\begin{aligned}
&\mathbb{E}_{o,\Phi_s}\left[\prod_{y\in\Phi_p}\frac{\mu_p l(r_s)\mathbf{1}_{g_{ps}(y,o)l(|y|)<\rho}}{\mu_p l(r_s)+\mu_s Tl(|y-r(o)|)}\middle|r(o)\right]\\
&=\mathbb{E}\left[\prod_{y\in\Phi_p}\frac{\mu_p l(r_s)\mathbf{1}_{g_{ps}(y,o)l(|y|)<\rho}}{\mu_p l(r_s)+\mu_s Tl(|y-r(o)|)}\middle|r(o)\right]\\
&=\exp\left\{-\lambda_p p_p\int_{\mathbb{R}^2}\frac{\mu_s Tl(|y-r(o)|)+l(r_s)e^{-\mu_p\rho/l(|y|)}}{\mu_p l(r_s)+\mu_s Tl(|y-r(o)|)}dy\right\},
\end{aligned}$$

and

$$\mathbb{E}\left[\prod_{y\in\Phi_p}\frac{\mu_p l(r_s)\mathbf{1}_{g_{ps}(y,o)l(|y|)<\rho}}{\mu_p l(r_s)+\mu_s Tl(|y-r(o)|)}E_{A/A}(x)\middle| r(o)\right]$$
$$=\mathbb{E}\left[\prod_{y\in\Phi_p}\frac{\mu_p l(r_s)\mathbf{1}_{g_{ps}(y,o)l(|y|)<\rho}\mathbf{1}_{g_{ps}(y,x)l(|y-x|)<\rho}}{\mu_p l(r_s)+\mu_s Tl(|y-r(o)|)}\middle| r(o)\right]$$
$$=\exp\left\{-\lambda_p p_p\int_{\mathbb{R}^2}\left(1-\frac{\mu_p l(r_s)\left(1-e^{-\mu_p\rho/l(|y|)}\right)\left(1-e^{-\mu_p\rho/l(|y-x|)}\right)}{\mu_p l(r_s)+\mu_s Tl(|y-r(o)|)}\right)dy\right\}.$$

Hence,

$$\mathbb{P}_{o,\Phi_s}(\mathrm{SINR}_s(o)>T,E(x)=1,t(x)=1)$$
$$\leq p_p\exp\left\{-\frac{\mu_s TN}{l(r_s)}\right\}\left(\exp\left\{-\lambda_p p_p\int_{\mathbb{R}^2}\frac{\mu_s Tl(|y-r(o)|)+l(r_s)e^{-\mu_p\rho/l(|y|)}}{\mu_p l(r_s)+\mu_s Tl(|y-r(o)|)}dy\right\}\right.$$
$$-\frac{\left(1-e^{-\lambda_s p_s\mathfrak{c}(r_s,\mu_s,\mu_s)}\right)}{\mathfrak{c}(r_s,\mu_s,\mu_s)}\int_{\mathbb{R}^2}\frac{Tl(|x-r(o)|)}{Tl(|x-r(o)|)+l(r_s)}\exp\left\{-\lambda_p p_p\int_{\mathbb{R}^2}\right.$$
$$\left.\left.\left(1-\frac{\mu_p l(r_s)\left(1-e^{-\mu_p\rho/l(|y|)}\right)\left(1-e^{-\mu_p\rho/l(|y-x|)}\right)}{\mu_p l(r_s)+\mu_s Tl(|y-r(o)|)}\right)dy\right\}dx\right).$$

This implies the conclusion directly. □

Proposition 2.13 *In the A/A CR Network, the AT of primary t.t.s is*

$$\mathcal{T}_{p,A/A}=p_p p_{\mathrm{COP,p,A/A}},\tag{2.2.13}$$

and the AT of secondary t.t.s is

$$\mathcal{T}_{p,A/A}=p_{\mathrm{MAP,s,A/A}}p_{\mathrm{COP,s,A/A}}.\tag{2.2.14}$$

Proof. This is a corollary of Campbell's formula, Propositions 2.11 and 2.12. □

The C/C network

The analysis of the C/C network in general is very complicated. Nevertheless, in the special case where $\mu_p=\mu_s$ and $\rho_p=\rho_s=\rho$, there is a lower bound of the union of $\Phi_{p,C/C}$ and $\Phi_{s,C/C}$, which can be represented as the Matérn type II model of some other PPP. Starting from that, an heuristic analysis can be carried out in the same spirit as the analysis in Subsection 2.1.3, which serves as an optimistic estimation of the C/C model. We present below this analysis.

Let $\Phi_{all}=\Phi_p\cup\Phi_s$ and

$$C_{cs,all}:=C_{cs,p}\cup C_{cs,s}\cup\left\{(x,y)\in\Phi_p\times\Phi_s\text{ s.t. }\mathbf{1}_{g_{ps}(x,y)l(|y_x|)>\rho}\right\}.$$

Every point x in Φ_{all} is associated with a *virtual timer* $t_v(x)$ which equals $t(x)\frac{\lambda_p}{\lambda_p+\lambda_s}$ if $x \in \Phi_p$ and equals $\frac{\lambda_p}{\lambda_p+\lambda_s} + t(x)\frac{\lambda_s}{\lambda_p+\lambda_s}$ if $x \in \Phi_s$. This virtual timer is used in the determination of the Matérn type II model of Φ_{all}. Then

Proposition 2.14 *When $\mu_p = \mu_s$ and $\rho_p = \rho_s = \rho$, we have*

$$\left(\Phi_{p,C/C} \cup \Phi_{s,C/C}\right) \supseteq \mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all}); \tag{2.2.15}$$

$$\Phi_{p,C/C} = T_{\lambda_p/(\lambda_p+\lambda_s)}\left(\mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all})\right); \tag{2.2.16}$$

$$\Phi_{p,C/C} \supseteq T_{\lambda_p/(\lambda_p+\lambda_s),1}\left(\mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all})\right). \tag{2.2.17}$$

Proof. As $\Phi_p = T_{\lambda_p/(\lambda_p+\lambda_s)}(\Phi_{all})$, (2.2.16) follows from the properties given in Subsection 3.1.4.

For (2.2.17), take any x in $T_{\lambda_p/(\lambda_p+\lambda_s),1}\left(\mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all})\right)$, we have by definition that $\lambda_p/(\lambda_p+\lambda_s) \le t_v(x) \le 1$ and $C_{cs,all}(x,y) = 0$ for all y in $T_{t_v(x)}(\Phi_{all})$. As $T_{t_v(x)}(\Phi_{all}) = \Phi_p \cup T_{t(x)}(\Phi_s)$, this is equivalent to $C_{cs,all}(x,y) = 0$ for all y in Φ_p and for all y in $T_{t(x)}(\Phi_s)$. Since $\Phi_p \supseteq \mathcal{M}_{\mathrm{II}}(\Phi_p, C_{cs,p})$, we deduce that $x \in \Psi_s$. Since $\Phi_s \supseteq \Psi_s$, we have that $T_{t(x)}(\Phi_s) \supseteq T_{t(x)}(\Psi_s)$ which implies that $x \in \mathcal{M}_{\mathrm{II}}(\Psi_s, C_{cs,s}|_{\Psi_s}) = \Phi_{s,C/C}$.

(2.2.15) is directly deduced from (2.2.16) and (2.2.17). □

From now on, we assume that $\Phi_{p,C/C} \approx T_{\lambda_p/(\lambda_p+\lambda_s),1}\left(\mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all})\right)$ and $\left(\Phi_{p,C/C} \cup \Phi_{s,C/C}\right) \approx \mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all})$. Under these assumptions,

Proposition 2.15 *The MAP for a typical t.t. is*

$$p_{p,\mathrm{MAP,C/C}} = \frac{1-\exp\{-\lambda_p\overline{\mathcal{N}}\}}{\lambda_p\overline{\mathcal{N}}}, \tag{2.2.18}$$

for primary t.t.s and

$$p_{p,\mathrm{MAP,C/C}} \approx \frac{1-\exp\{-\lambda_s\overline{\mathcal{N}}\}}{\lambda_s\overline{\mathcal{N}}} e^{-\lambda_p\overline{\mathcal{N}}} \tag{2.2.19}$$

for secondary t.t.s. Where

$$\overline{\mathcal{N}} = \int_{\mathbb{R}^2} \exp\left\{-\frac{\mu\rho}{l(|x|)}\right\} dx = 2\pi \int_0^\infty \exp\left\{-\frac{\mu\rho}{l(x)}\right\} x dx. \tag{2.2.20}$$

Proof. From Proposition 2.14, we have

$$\begin{aligned} p_{\mathrm{MAP,p,C/C}} &= \mathbb{P}_{o,\Phi_p}(o \in \Phi_{p,C/C}) \\ &= \mathbb{P}_{o,\Phi_{all}}\left(o \in \mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all}) \middle| t_v(x) \in \left[0, \frac{\lambda_p}{\lambda_p+\lambda_s}\right]\right); \\ p_{\mathrm{MAP,s,C/C}} &= \mathbb{P}_{o,\Phi_s}(o \in \Phi_{s,C/C}) \\ &\approx \mathbb{P}_{o,\Phi_{all}}\left(o \in \mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all}) \middle| t_v(x) \in \left[\frac{\lambda_p}{\lambda_p+\lambda_s}, 1\right]\right). \end{aligned}$$

By the same argument as in Proposition 2.4,

$$\mathbb{P}_{o,\Phi_{all}}\left(o \in \mathcal{M}_{\text{II}}(\Phi_{all}, C_{cs,all}) \middle| t_v(x) = t\right) = \exp\left\{-(\lambda_p + \lambda_s)t\overline{\mathcal{N}}\right\}.$$

We then integrate the above formula w.r.t. t from 0 to $\frac{\lambda_p}{\lambda_p+\lambda_s}$ to get $p_{\text{MAP,p,C/C}}$ and w.r.t. t from $\frac{\lambda_p}{\lambda_p+\lambda_s}$ to 1 to get $p_{\text{MAP,s,C/C}}$. □

To compute the COP of the primary and the secondary t.t.s, we need the following representations

$$\begin{aligned}
\text{SINR}_{p,C/C}(x) &= \frac{f_{pp}(x,x)l(r_p)}{N + \mathcal{I}_{pp,C/C}(x) + \mathcal{I}_{sp,C/C}(x)} \\
&= \frac{f_{all}(x,x)l(r_p)}{N + \mathcal{I}_{all}(x, r(x))} \text{ for } x \in \Phi_{p,C/C}; \\
\text{SINR}_{s,C/C}(x) &= \frac{f_{ss}(x,x)l(r_s)}{N + \mathcal{I}_{ps,C/C}(x) + \mathcal{I}_{ss,C/C}(x)} \\
&= \frac{f_{all}(x,x)l(r_s)}{N + \mathcal{I}_{all}(x, r(x))} \text{ for } x \in \Phi_{s,C/C},
\end{aligned}$$

where $\mathcal{I}_{all}(x, r(x)) = \sum_{y \in \mathcal{M}_{\text{II}}(\Phi_{all}, C_{cs,all})} f_{all}(y,x)l(|y - x - r(x)|)$ and f_{all} is a function in Φ_{all}^2 such that

$$\begin{aligned}
f_{all}(x,y) &= f_{pp}(x,y) \text{ for } (x,y) \in \Phi_p^2; \\
f_{all}(x,y) &= f_{ps}(x,y) \text{ for } (x,y) \in \Phi_p \times \Phi_s; \\
f_{all}(x,y) &= f_{sp}(x,y) \text{ for } (x,y) \in \Phi_s \times \Phi_p; \\
f_{all}(x,y) &= f_{ss}(x,y) \text{ for } (x,y) \in \Phi_s^2.
\end{aligned}$$

Since $\mu_p = \mu_s = \mu$, $\{f_{all}(x,y), (x,y) \in \Phi_{all}^2\}$ are i.i.d. exponential r.v.s of parameter μ. Then we have the Palm representation

$$\begin{aligned}
&\mathbb{P}_{o,\Phi_{p,C/C}}(\text{SINR}_{p,C/C}(o) > T) \\
&= \mathbb{P}_{o,\mathcal{M}_{\text{II}}(\Phi_{all},C_{cs,all})}\left(\frac{f_{all}(x,x)l(r_p)}{N + \mathcal{I}_{all}(x)} \middle| t_v(x) \in \left[0, \frac{\lambda_p}{\lambda_p + \lambda_s}\right]\right); \\
&\mathbb{P}_{o,\Phi_{s,C/C}}(\text{SINR}_{s,C/C}(o) > T) \\
&= \mathbb{P}_{o,\mathcal{M}_{\text{II}}(\Phi_{all},C_{cs,all})}\left(\frac{f_{all}(x,x)l(r_p)}{N + \mathcal{I}_{all}(x)} \middle| t_v(x) \in \left[\frac{\lambda_p}{\lambda_p + \lambda_s}, 1\right]\right).
\end{aligned}$$

To compute the above conditional probability, we need to know the Palm distribution of $\mathcal{M}_{\text{II}}(\Phi_{all}, C_{cs,all})$ given a point at o with virtual timer t. In the same spirit of the analysis in Subsection 2.1.3, we first compute the intensity of $\mathcal{M}_{\text{II}}(\Phi_{all}, C_{cs,all})$ under this distribution.

Proposition 2.16 *Let $\lambda_v = \lambda_p + \lambda_s$. The intensity measure of $\mathcal{M}_{\text{II}}(\Phi_{all}, C_{cs,all})$ under its Palm distribution given a point at o with virtual timer mark $t_v(o) = t$*

is

$$\mathfrak{b}(x,t,\lambda_v) = \left(\frac{1-e^{-t\lambda_v\overline{\mathcal{N}}_2(x)}}{\lambda_v\overline{\mathcal{N}}_2(x)} + e^{-t\lambda_v\overline{\mathcal{N}}_2(x)}\frac{1-e^{(1-t)\lambda_v\overline{\mathcal{N}}}}{\lambda_v\overline{\mathcal{N}}}\right)\left(1-e^{-\frac{\mu\rho}{l(|x|)}}\right), \tag{2.2.21}$$

where $\overline{\mathcal{N}}_2$ *is defined in Proposition 2.5.*

Proof. The argument is the same as in the proof of Proposition 2.5, except that we do not take expectation w.r.t. the virtual timer of o. In particular

$$\begin{aligned}&\mathbb{P}_{o,x,\Phi_{all}}\left(o \in \mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all}) \text{ and } x \in \mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all})\Big| t_v(o)=t, t_v(x)=t'\right)\\ &= \mathbf{1}_{t>t'}e^{-t\lambda_v\overline{\mathcal{N}}}e^{-t'\lambda_v(\overline{\mathcal{N}}-\overline{\mathcal{N}}_2(|x|))} + \mathbf{1}_{t\le t'}e^{-t'\lambda_v\overline{\mathcal{N}}}e^{-t\lambda_v(\overline{\mathcal{N}}-\overline{\mathcal{N}}_2(|x|))}.\end{aligned}$$

Taking expectation w.r.t. $t_v(x)$,

$$\begin{aligned}&\mathbb{P}_{o,x,\Phi_{all}}\left(o \in \mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all}) \text{ and } x \in \mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all})\Big| t_v(o)=t\right)\\ &= \left(\int_0^t e^{-t\lambda_v\overline{\mathcal{N}}}e^{-t'\lambda_v(\overline{\mathcal{N}}-\overline{\mathcal{N}}_2(|x|))}dt' + \int_t^1 e^{-t'\lambda_v\overline{\mathcal{N}}}e^{-t\lambda_v(\overline{\mathcal{N}}-\overline{\mathcal{N}}_2(|x|))}dt'\right)\\ &\quad\left(1-e^{-\frac{\mu\rho}{l(|x|)}}\right)\\ &= \left(e^{-t\lambda_v\overline{\mathcal{N}}}\frac{1-e^{-t\lambda_v\left(\overline{\mathcal{N}}-\overline{\mathcal{N}}_2(|x|)\right)}}{\lambda_v(\overline{\mathcal{N}}-\overline{\mathcal{N}}_2(|x|))} + e^{-t\lambda_v\overline{\mathcal{N}}}\frac{1-e^{-(1-t)\lambda_v\overline{\mathcal{N}}}}{\lambda_v\overline{\mathcal{N}}}e^{-t\lambda_v(\overline{\mathcal{N}}-\overline{\mathcal{N}}_2(|x|))}\right)\\ &\quad\left(1-e^{-\frac{\mu\rho}{l(|x|)}}\right).\end{aligned}$$

On the other hand,

$$\mathbb{P}_{o,\Phi_{all}}\left(o \in \mathcal{M}_{\mathrm{II}}\Big| t_v(o)=t\right) = e^{-t\lambda_v\overline{\mathcal{N}}}.$$

Hence,

$$\begin{aligned}&\mathbb{P}_{o,x,\Phi_{all}}\left(o \in \mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all})\Big| x \in \mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all}) \text{ and } t_v(o)=t\right)\\ &= \frac{\mathbb{P}_{o,x,\Phi_{all}}\left(o \in \mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all}) \text{ and } x \in \mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all})\Big| t_v(xo)=t\right)}{\mathbb{P}_{o,\Phi_{all}}\left(o \in \mathcal{M}_{\mathrm{II}}\Big| t_v(o)=t\right)}\\ &=\left(\frac{1-e^{-t\lambda_v(\overline{\mathcal{N}}-\overline{\mathcal{N}}_2(|x|))}}{\lambda_v\left(\overline{\mathcal{N}}-\overline{\mathcal{N}}_2(|x|)\right)} + \frac{1-e^{-(1-t)\lambda_v\overline{\mathcal{N}}}}{\lambda_v\overline{\mathcal{N}}}e^{-t\lambda_v(\overline{\mathcal{N}}-\overline{\mathcal{N}}_2(|x|))}\right)\left(1-e^{-\frac{\mu\rho}{l(|x|)}}\right).\end{aligned}$$

This implies the conclusion directly. □

This suggests us to approximate the Palm distribution of $\mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all})$ given a point at o with virtual timer $t_v(o) = t$ by an inhomogeneous PPP of intensity measure $\mathfrak{b}(|x|, t, \lambda_v)dx$.

Proposition 2.17 *By approximating the Palm distribution of $\mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all})$ given a point at o with virtual timer $t_v(o) = t$ by that of an inhomogeneous PPP of intensity measure $\mathfrak{b}(|x|, t, \lambda_v)dx$. The COP of primary and secondary t.t.s are respectively:*

1. *for primary t.t.s,*

$$p_{\mathrm{COP,p,C/C}} = \frac{\int_0^{\lambda_p/\lambda_v} \exp\left\{-\frac{\mu NT}{l(r_p)} - \int_{\mathbb{R}^2} \frac{Tl(|x-r_p\mathbf{e}|)}{Tl(|x-r_p\mathbf{e}|)+l(r_p)}\mathfrak{b}(x,t,\lambda_v)dx\right\}dt}{\frac{1-e^{-t\lambda_p\overline{\mathcal{N}}}}{1-e^{-\lambda_v\overline{\mathcal{N}}}}}; \tag{2.2.22}$$

2. *and for secondary t.t.s,*

$$p_{\mathrm{COP,s,C/C}} = \frac{\int_{\lambda_p/\lambda_v}^{1} \exp\left\{-\frac{\mu NT}{l(r_s)} - \int_{\mathbb{R}^2} \frac{Tl(|x-r_s\mathbf{e}|)}{Tl(|x-r_s\mathbf{e}|)+l(r_s)}\mathfrak{b}(x,t,\lambda_v)dx\right\}dt}{\frac{e^{-t\lambda_p\overline{\mathcal{N}}}\left(1-e^{-t\lambda_s\overline{\mathcal{N}}}\right)}{1-e^{-\lambda_v\overline{\mathcal{N}}}}}. \tag{2.2.23}$$

Proof.
Since $\mathcal{M}_{\mathrm{II}}(\Phi_{all}, C_{cs,all})$ invariant under rotations, so do its Palm distributions. We can assume w.l.o.g. that $r(o) = r\mathbf{e}$. We first compute the COP of a typical user conditioned on its timer t. Proceed as in Proposition 2.8 and using the fact that the receiver is uniformly distributed in the circle of radius r centred at o, we have:

$$\begin{aligned}
&\mathbb{P}_{o,\mathcal{M}_{\mathrm{II}}(\Phi_{all},C_{cs,all})}\left(\frac{f(o,o)l(r)}{N+\mathcal{I}_{all}\big(o,r(o)\big)} > T \middle| t_v(o) = t\right)\\
&\approx \exp\left\{-\frac{\mu NT}{l(r_p)} - \int_{\mathbb{R}^2} \frac{Tl(|x-r\mathbf{e}|)}{Tl(|x-r\mathbf{e}|)+l(r)}\mathfrak{b}(x,t,\lambda_v)dx\right\}
\end{aligned}$$

We write the COP of a typical primary t.t. as

$$\begin{aligned}
p_{\mathrm{COP,p,C/C}} &= \mathbb{P}_{o,\mathcal{M}_{\mathrm{II}}(\Phi_{all},C_{cs,all})}\left(\frac{f(o,o)l(r_p)}{N+\mathcal{I}_{all}\big(o,r(o)\big)} > T \middle| t_v(o) \in \left[0, \frac{\lambda_p}{\lambda_v}\right]\right)\\
&= \frac{\mathbb{P}_{o,\mathcal{M}_{\mathrm{II}}(\Phi_{all},C_{cs,all})}\left(\frac{f(o,o)l(r_p)}{N+\mathcal{I}_{all}(o,r(o))} > T \text{ and } t_v(o) \in \left[0, \frac{\lambda_p}{\lambda_v}\right]\right)}{\mathbb{P}_{o,\mathcal{M}_{\mathrm{II}}(\Phi_{all},C_{cs,all})}\left(t_v(o) \in \left[0, \frac{\lambda_p}{\lambda_v}\right]\right)}.
\end{aligned}$$

The numerator is computed as

$$\int_0^{\frac{\lambda_p}{\lambda_v}} \exp\left\{-\frac{\mu NT}{l(r_p)} - \int_{\mathbb{R}^2} \frac{Tl(|x-r_p\mathbf{e}|)}{Tl(|x-r_p\mathbf{e}|)+l(r_p)}\mathfrak{b}(x,t,\lambda_v)dx\right\}dt,$$

while the denominator is

$$
\begin{aligned}
&\mathbb{P}_{o,\mathcal{M}_{\text{II}}(\Phi_{all},C_{cs,all})}\left(t_v(o)\in\left[0,\frac{\lambda_p}{\lambda_v}\right]\right)\\
&=\mathbb{P}_{o,\Phi_{all}}\left(t_v(o)\in\left[0,\frac{\lambda_p}{\lambda_v}\right]\middle| o\in\mathcal{M}_{\text{II}}(\Phi_{all},C_{cs,all})\right)\\
&=\frac{\mathbb{P}_{o,\Phi_{all}}\left(t_v(o)\in\left[0,\frac{\lambda_p}{\lambda_v}\right] \text{ and } o\in\mathcal{M}_{\text{II}}(\Phi_{all},C_{cs,all})\right)}{\mathbb{P}_{o,\Phi_{all}}\left(o\in\mathcal{M}_{\text{II}}(\Phi_{all},C_{cs,all})\right)}\\
&=\frac{\int_0^{\frac{\lambda_p}{\lambda_v}} e^{-t\lambda_v\overline{\mathcal{N}}}dt}{\int_0^1 e^{-\lambda_v\overline{\mathcal{N}}}dt}=\frac{1-e^{-\lambda_p\overline{\mathcal{N}}}}{1-e^{-\lambda_v\overline{\mathcal{N}}}}.
\end{aligned}
$$

Hence,

$$
p_{\text{COP,p,C/C}}=\frac{\int_0^{\frac{\lambda_p}{\lambda_v}}\exp\left\{-\frac{\mu NT}{l(r_p)}-\int_{\mathbb{R}^2}\frac{Tl(|x-r_p\mathbf{e}|)}{Tl(|x-r_p\mathbf{e}|+l(r_p)}\mathfrak{b}(x,t,\lambda_v)dx\right\}dt}{\frac{1-e^{-\lambda_p\overline{\mathcal{N}}}}{1-e^{-\lambda_v\overline{\mathcal{N}}}}}.
$$

The COP of secondary t.t. is computed similarly. □

Corollary 2.1 *The AT of primary and secondary t.t.s in the C/C model are:*

1. *for primary t.t.s,*

$$
\mathcal{T}_{p,C/C}=\lambda_v\int_0^{\frac{\lambda_p}{\lambda_v}} exp\left\{-\frac{\mu NT}{l(r_p)}-\int_{\mathbb{R}^2}\frac{Tl(|x-r_p\boldsymbol{e}|)}{Tl(|x-r_p\boldsymbol{e}|)+l(r_p)}\mathfrak{b}(x,t,\lambda_v)dx\right\}dt;
$$

2. *for secondary t.t.s,*

$$
\mathcal{T}_{s,C/C}=\lambda_v\int_{\frac{\lambda_p}{\lambda_v}}^1 exp\left\{-\frac{\mu NT}{l(r_s)}-\int_{\mathbb{R}^2}\frac{Tl(|x-r_s\boldsymbol{e}|)}{Tl(|x-r_s\boldsymbol{e}|)+l(r_s)}\mathfrak{b}(x,t,\lambda_v)dx\right\}dt.
$$

Proof. This is a direct corollary of Proposition 2.15, 2.17 and Campbell formula. □

2.3 When does CSMA Become ALOHA

In the model of CSMA Networks in Section 2.1, we assume perfect separation between active t.t.s, i.e. there is always a minimum "distance" between any two active t.t.s. In fact, this is only an idealized assumption which may never be found in practice, as any two t.t.s transmitting within a time period smaller than the propagation delay can not "see" each other and there will be no spatial separation between them. Hence, there should always be some mechanism, which we refer to as congestion control mechanism, to avoid the situation where many nearby t.t.s transmit within one propagation delay interval (congestion). For instance, in the IEEE 802.11a CSMA based protocol, the congestion control mechanism is the adaptive Collision Window [16, 2]. Each t.t. wishing to transmit has to maintain a back-off counter which is originally uniformly generated from 0 to its Collision Window. After each back-off period that it sees the medium free, it decrements its back-off counter by 1. The duration of the sensing period is specified by the protocol. A t.t. can start transmitting its message once its counter expires. After that, it has to wait for an ACK message from its receiver. If the ACK is missing, it will assume that there is a congestion and have to double its Collision Window. Then it has to repeat the above procedure to retransmit its message until this message is successfully received.

Nevertheless, in some applications, it is impossible to implement such congestion control mechanisms. In particular, we consider here the IEEE 802.11p protocol, a half-clocked version of the IEEE 802.11a protocol used in Dedicated Short Range Communications (DSCR) in vehicular communications . Due to the broadcast nature of the messages in DSCR, there is no ACK message and hence the t.t.s cannot adapt the Collision Window accordingly. As a result, IEEE 802.11p uses a static Collision Window value which is equal to 15 [3].

Given its use for safety applications, the performance of such a broadcast scheme has received a lot of attention lately. In particular, many studies have observed that the IEEE 802.11p MAC has serious issues with congestion at high device densities. Authors in [8, 9, 35, 12] have observed undesirable message collisions and long delays between successful message receptions at high densities. Moreover, it is observed that message decoding fails frequently even for messages transmitted from a t.t. close-by. This indicates that when the device density is high, this CSMA based protocol somehow ceases to function appropriately. A recent work [28] shows by simulations that when device density grows high, the behaviour of IEEE 802.11p MAC appears more similar to ALOHA, a protocol without any sensing of other transmissions, than to CSMA.

Our object in this section is to take one step further in this direction. In particular, we quantify the above result by providing analytical bounds for the critical device intensity where the CSMA behaviour start breaking down to that of ALOHA. By doing that, we also show that the fundamental reason for CSMA-based IEEE 802.11p to behave like ALOHA is the *finite granularity* of the Collision Window size since there always exists a non-negligible probability that t.t.s may choose the same back-off counter and hence transmit at the same time.

2.3.1 Spatial Modelling

Basic assumptions

Although the IEEE 802.11p is an asynchronous protocol, to keep the analysis from being too complex we consider here a synchronous slotted time model where time is divided into slots of constant duration. Each slot has three components: it starts with an Inter Frame Spacing (IFS) period, followed by a number (the maximum number is determined by the Collision Window, W) of Back-off slots, and then the packet transmission period. Note that different slots may have slightly different durations depending on the actual back-off slots used. We ignore such differences for simplicity since a back-off slot ($9\mu s$) is negligible as compared to the packet transmission time. In other words, we assume the back-off slots to be of zero duration.

A major difference from the models considered earlier in this thesis is that we consider here the the time dependency between realizations of t.t.s locations in successive time slot (in previous models the realizations of t.t.s locations in different time slots are mutually independent). This dependency can be described as bellows.

In each slot, there is an influx of fresh terminals who arrive in the system with a packet to transmit. Their locations are represented by a PPP of intensity λ. The process of fresh terminals in different time slots are assume to be i.i.d. Each freshly arriving terminal participates in the channel contention protocol, which is abstracted from the CSMA mechanism in IEEE 802.11p MAC operation [15]. Each terminal arrives with a random back-off counter value chosen randomly from 0 to W, which is a fixed system parameter (15 in DSRC). If the back-off counter of a packet happens to be 0, the transmission happens immediately after IFS, without going through the back-off process. If the back-off counter is not zero, the terminal, assumed to be at position x, senses whether the medium is busy, i.e. whether there is another terminal transmitting within a neighbourhood $B(x, r)$ (a ball of radius r around x), which we refer to as the sensing area of this terminal. Suppose that the tagged terminal at x has back-off counter $w > 0$ at the beginning of the time slot t. There are two possibilities here: (a) if, after q ($q < w$) back-off slots, the counter of one of the other terminals within its sensing area expires and the other terminal starts to transmit; then the counter of the tagged terminal is $w - q + 1$ at this expiration time (since the tagged terminal decrements its counter by 1 at the beginning of each back-off slot) and is frozen until the end of the busy period; the tagged terminal will thus re-enter the competition in slot $t + 1$ with a new back-off counter $w - q - 1$. (b) If the tagged terminal detects no activity in the network during w backoff slots, it transmits in the current slot and leaves the system. An example is presented in Figure 2.7 to visualize this mechanism. Note that by this abstraction, we implicitly assume that there is no fading. This is a legitimate assumption since we are in the context of vehicular communications, there is usually a strong line of sight between vehicles and the transmission ranges are relatively small.

It is worth noting that this model is very similar to the A/A Cognitive model in the sense that the terminals are divided into different classes (in this model each class consists of all the terminals having the same back-off counter) and the spatial separation is enforced *inter-class* but not *intra-class*. The two models are, however, different in two aspects: (a) *the number of classes*, in this model we have $W + 1$ classes while the A/A model has only 2 classes; (b) *the distribution of the terminals in each class* in this model can not be modelled by PPPs as in the A/A model. Nevertheless, thanks to this similarity between

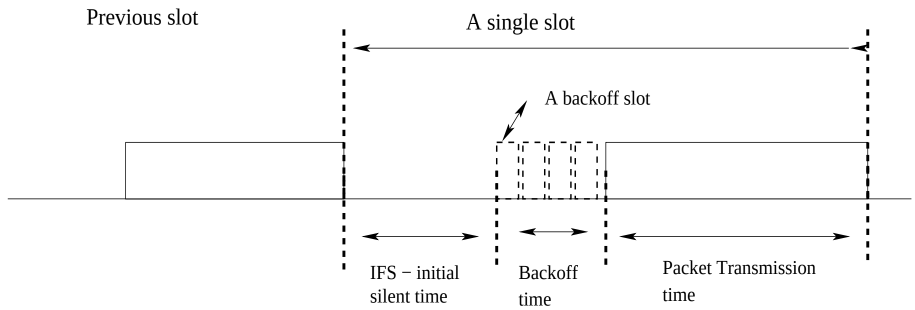

Figure 2.7: The back-off process of a typical terminal. It initially has back-off counter value 8. After an IFS, it senses the medium free for 4 back-off slots and decrements its counter value by 1 at the beginning of each slot. It senses the medium becoming busy during the fifth slot and freeze its counter at 3. Once the medium become free again, it resumes the counting down process. It continues until its counter value equals to 0 and then transmits. The duration between two long bars is a transmission slot, which consists of an IFS, then several back-off slots (the interval between two small bars), then a frozen period if the medium is busy or a transmission if the back-off counter expires.

them, in the analysis of this model we can use the technique which proves to be successful in the analysis of the A/A model.

Problem Formulation

At the beginning of each slot, we take a snapshot of the system. Let Φ^t (Ψ^t) be the set of t.t.s freshly arriving (having messages waiting to be scheduled) at slot t and Φ_i^t (Ψ_i^t) be the set of such t.t.s with counter equal to i ($i = 0, \dots, W$). $\{\Phi^t, t = 1, 2, \cdots\}$ is assumed to be a family of independent PPPs of intensity λ. As the back-off counters are independent marks of Φ^t, Φ_i^t, $i = 0, \dots W$, $t = 1, 2, \dots$, are independent PPPs of intensity $\lambda/(W+1)$. The point process of t.t.s with counter i that transmit in slot t is

$$\Pi_i^t = \Big\{x \in \Psi_i^t \text{ s.t. } \mathcal{C}(x, \Pi_k^t),\ k < i\Big\}.$$

We also identify the t.t.s which will decrement their counter to j ($j < i$) after the current slot. Let $\Xi_{i,j}^t$ be the process of such t.t.s,

$$\Xi_{i,j}^t = \Big\{x \in \Psi_i^t \text{ s.t. } \mathcal{C}(x, \Pi_k^t),\ k < i - j - 2 \text{ and } \overline{\mathcal{C}(x, \Pi_{i-j-1}^t)}\Big\},$$

where $\mathcal{C}(y, \Theta)$ is the event $B(y, r) \cap \Theta = \emptyset$ for every point y and every PP Θ and $\overline{\mathcal{E}}$ is the complement of the event $\mathcal{E}$. The parameter r is the carrier sensing range.

The following relations between the PPs defined above hold for every $t > 0$ and completely define the evolution of the system:

- For every $i \leq W$, $\Pi_i^t \cup \{\Xi_{i,j}^t\}_{j=0}^{i-1}$ form a partition of Ψ_i^t: each t.t. in Ψ_i^t either transmits in the current slot or decreases its counter from i to some $0 \leq j < i$.
- $\Phi_W^t = \Psi_W^t$: only freshly arriving t.t.s can take counter W.
- For every $i < W$, $\Phi_i^{t+1} \cup \{\Xi_{j,i}^t\}_{j=i+1}^{W}$ form a partition of Ψ_i^{t+1}: each t.t. in Ψ_i^t either freshly arrives into the system or has its counter decreased from some $j > i$ to i during the last time slot.
- $\Pi_0^t = \Psi_0^t$: t.t.s with counter 0 always transmit.

Note that when $W = 0$, the system corresponds to independent Poisson arrivals/transmissions in each slot without any contention or back-off.

Remark 2.4 *The system of PPs $\{(\Psi_0^t, \ldots, \Psi_W^t)\}_{t=1}^{\infty}$ is a Markov process in the sense that given $(\Psi_0^t, \ldots, \Psi_W^t)$, $\{(\Psi_0^k, \ldots, \Psi_W^k)\}_{k=t+1}^{\infty}$, and $\{(\Psi_0^k, \ldots, \Psi_W^k)\}_{k=0}^{t-1}$, are independent. We do not know whether this Markov process is ergodic and what is its invariant distribution. Instead, our purpose is only to extract some partial information about the limit of the distribution of this system when t goes to ∞, which is enough to establish bounds on the critical intensities.*

2.3.2 Critical intensity

Consider a scenario where the t.t.s of Ψ_0^t (i.e. the t.t.s that have a counter value of 0 at the beginning of slot t) cover most of the space within their Carrier Sensing zones. In such a scenario, most t.t.s would only be able to decrement their counters by 1 in each slot. Consequently, at slot $t' > t + W$, the $\Psi_0^{t'}$ t.t.s would be the union of t.t.s that arrived with a counter value k in slot $t' - k$ for some k from 0 to W. Notice that the active t.t.s then become the union of independent Poisson Point processes $\Psi_k^{t'-k}$ with the same intensity $\lambda/(W+1)$, which all transmit in the current slot. Hence, the process of transmitters forms a Poisson point process of with intensity λ and CSMA has a very negligible role in the process. Thus, an important quantity of interest is the fraction of space, denoted by $(1-\epsilon)$, that is covered by the carrier sensing regions of the t.t.s in Ψ_0^t. For a small ϵ, we refer to the system at this stage as $(1-\epsilon)$ ALOHA.

We define

$$\lambda^{\epsilon,W} := \inf\{\lambda \text{ s.t. } \limsup_{t\to\infty} \mathbb{P}\left(\mathcal{C}(o, \Psi_0^t)\right) \leq \epsilon\}, \tag{2.3.1}$$

with o the origin of the domain. As all the PPs considered here are stationary, we can replace o in the above definition by any other point x of the domain. This is the smallest t.t. intensity for which the system is $(1-\epsilon)$ ALOHA when the Collision Window value is W.

Note that for ALOHA, $\lambda^{\epsilon,0} = \frac{1}{\|B(0,r)\|} \ln(\frac{1}{\epsilon})$. This follows from the fact that the probability that there are no point of a PPP of intensity λ in the ball $B(0,r)$ is $e^{-\lambda|B(0,r)|}$. In the rest of this section, we provide bounds for $\lambda^{\epsilon,W}$, the intensity at which the system is $(1-\epsilon)$-ALOHA.

In order to obtain bounds for $\lambda^{\epsilon,W}$, we first describe two other PPs related to Ψ_0^t that have much simpler structures. For any $t > W$, for each k from 0 to W, let $\Omega_{k,0}^t$ be the PPs of the terminals which arrive in the system at time $t-k$, with counter value k and get to counter value 0 at time t by decrementing their counters by 1 in each of the k time slots from $t-k$ to $t-1$:

$$\Omega_k^t = \{x \in \Phi_k^{t-k} \text{ s.t. } \overline{\mathcal{C}(x, \Psi_0^{t-i})},\ i = 1 \ldots k-1\}.$$

For $k = 1$ to $W-1$, let

$$\Upsilon_k^t := \{x \in \bigcup_{j=k+1}^{W} \Phi_j^{t-k} \text{ s.t. } \exists 1 \le l \le k, \mathcal{C}(x, \Psi_0^{t-l})\}$$

be the set of t.t.s arriving at time $t-k$ with counter values larger than k and decreasing their counters by more than 1 in at least one of the next k time slots. The following lemma gives a lower bound and an upper bound on Ψ_0^t.

Lemma 2.1 *For every $t > W$, the following inclusions hold a.s.,*

$$\bigcup_{k=1}^{W+1} \Omega_k^t \subseteq \Psi_0^t \subseteq \left(\bigcup_{k=0}^{W} \Phi_k^{t-k}\right) \cup \left(\bigcup_{k=1}^{W-1} \Upsilon_k^t\right). \tag{2.3.2}$$

Proof. Note that $\Omega_k^t \subseteq \Psi_0^t$ for every k from 0 to W by definition, so the first inclusion follows directly. For the second inclusion, take any x in Ψ_0^t. It can either (a) arrive in the system at time $t-k$ with counter k for some k from 0 to W, which means that $x \in \left(\bigcup_{k=0}^{W} \Phi_k^{t-k}\right)$, or (b) arrive in the system at time k with some counter l ($0 < k < l \le W$). In case (b), x must decrements its counter by more than 1 in at least 1 of the k slots $t-k$, $t-k+1$, ..., $t-1$. This is equivalent to $x \in \Upsilon_k^t$. The second inclusion then follows directly. □

Using this lemma, we can bound $\lambda^{\epsilon,W}$ as follows,

Theorem 2.1 *Let $\epsilon < (W+1)/2$ be a small positive number. Then*

$$\lambda^{\epsilon,W} \le \lambda_u^{\epsilon,W} := \frac{W+1}{2\pi r^2} \ln\bigl(\rho(\epsilon)\bigr), \tag{2.3.3}$$

where $\rho(\epsilon)$ is the largest solution of the equation

$$x^{W+1} - \frac{W+1}{2} x^W + \frac{W+1}{2} = \epsilon^{-1}, \tag{2.3.4}$$

and

$$\lambda^{\epsilon,W} \ge \lambda_d^{\epsilon,W} := \frac{\iota(W,\epsilon)}{2\pi r^2}, \tag{2.3.5}$$

where $\iota(W,\epsilon)$ is the unique positive solution of the equation

$$x\left(1 + \frac{W(W-1)}{6\left(e^x - \frac{W+1}{2}\left(e^{\frac{Wx}{W+1}} - 1\right)\right)}\right) = -\ln(\epsilon). \tag{2.3.6}$$

Proof. The main idea of this proof is to bound $\limsup_{t\to\infty}\mathbb{P}(\mathcal{C}(o,\Psi_0^t))$. For this purpose, we first notice that $\mathcal{C}(o,\Theta)$ implies $\mathcal{C}(o,\Theta')$ if $\Theta\supseteq\Theta'$, and that $\mathcal{C}(o,\bigcup_{i=1}^n\Theta_i)$ is equivalent to the union of $\mathcal{C}(o,\Theta_1),\ldots,\mathcal{C}(o,\Theta_n)$ for any n PPs $\Theta_1,\ldots,\Theta_n$.

We start with the upper bound. By Lemma 2.1,

$$\mathbb{P}\left(\mathcal{C}(o,\Psi_0^t)\right)\le\mathbb{P}\Big(\mathcal{C}\left(o,\Omega_{0,0}^t\right),\ldots,\mathcal{C}\left(o,\Omega_{W,0}^t\right)\Big).$$

Given a realization of $\Psi_0^{t-W},\ldots,\Psi_0^{t-1}$, the PPs $\Omega_{0,0}^t,\ldots,\Omega_{W,0}^t$ are $W+1$ independent PPPs. The conditional intensity of $\Omega_{k,0}^t$ is $\frac{\lambda}{W+1}\prod_{i=1}^k\mathbf{1}_{\overline{\mathcal{C}(x,\Psi_0^{t-i})}}dx$, with the convention that the product over the empty set is 1. Hence,

$$\mathbb{P}\left(\mathcal{C}(o,\Psi_0^t)\right)\le\mathbb{E}\left[\exp\left\{-\frac{\lambda}{W+1}\int_{B(o,r)}\sum_{k=1}^{W+1}\prod_{i=1}^{k-1}\mathbf{1}_{\overline{\mathcal{C}(x,\Psi_0^{t-i})}}dx\right\}\right].$$

Moreover, as

$$\prod_{i=1}^{k-1}\mathbf{1}_{\overline{\mathcal{C}(x,\Psi_0^{t-i})}}\ge\max\left(1-\sum_{i=1}^{k-1}\mathbf{1}_{\mathcal{C}(x,\Psi_0^{t-i})},0\right)$$

for every x, we have

$$\begin{aligned}\sum_{k=1}^{W+1}\prod_{i=1}^{k-1}\mathbf{1}_{\overline{\mathcal{C}(x,\Psi_0^{t-i})}}&=1+\sum_{k=2}^{W+1}\prod_{i=1}^{k-1}\mathbf{1}_{\overline{\mathcal{C}(x,\Psi_0^{t-i})}}\\&\ge1+\max\left(W-\sum_{k=1}^{W}(W-k+1)\mathbf{1}_{\mathcal{C}(x,\Psi_0^{t-k})},0\right).\end{aligned}$$

So,

$$\begin{aligned}&\exp\left\{-\frac{\lambda}{W+1}\int_{B(o,r)}\sum_{k=1}^{W+1}\prod_{i=1}^{k-1}\mathbf{1}_{\overline{\mathcal{C}(x,\Psi^{t-i})}}dx\right\}\\&\le\exp\left\{-\frac{\lambda}{T(W+1)}\left(2\pi r^2+\int_{B(o,r)}\max\left(W-\sum_{k=1}^{W}(W-k+1)\mathbf{1}_{\mathcal{C}(x,\Psi_0^{t-k})},0\right)dx\right)\right\}\\&=\exp\left\{-\frac{\lambda}{T(W+1)}\left((W+1)2\pi r^2-\int_{B(o,r)}\min\left(\sum_{k=1}^{W}(W-k+1)\mathbf{1}_{\mathcal{C}(x,\Psi_0^{t-k})},W\right)dx\right)\right\}\end{aligned}$$

a.s.

Because e^t is a convex function, $e^a\le1-\frac{a}{b}+\frac{a}{b}e^b$ for any $0\le a\le b$. Applying this inequality to the last equation, we get

$$\begin{aligned}&\mathbb{E}\left[\exp\left\{\frac{\lambda}{W+1}\int_{B(o,r)}\min\left(\sum_{k=1}^{W}(W-k+1)\mathbf{1}_{\mathcal{C}(x,\Psi_0^{t-k})},W\right)dx\right\}\right]\\&\le1-\frac{\mathfrak{C}}{WN_\lambda}+\frac{\mathfrak{C}}{WN_\lambda}\exp\left\{\frac{WN_\lambda}{W+1}\right\}=1+\frac{\mathfrak{C}}{WN_\lambda}\left(\exp\left\{\frac{WN_\lambda}{W+1}\right\}-1\right),\end{aligned}$$

where $N_\lambda = \lambda 2\pi r^2$ and

$$\begin{aligned}
\mathfrak{C} &= \mathbb{E}\Big[\int_{B(o,r)} \min\Big(\sum_{k=1}^{W}(W-k+1)\mathbf{1}_{\mathcal{C}(x,\Psi_0^{t-k})}, W\Big)dx\Big] \\
&\leq \mathbb{E}\Big[\int_{B(o,r)} \Big(\sum_{k=1}^{W}(W-k+1)\mathbf{1}_{\mathcal{C}(x,\Psi_0^{t-k})}\Big)dx\Big] \\
&= \int_{B(o,r)} \Big(\sum_{k=1}^{W}(W-k+1)\mathbb{P}\left(\mathcal{C}(x,\Psi_0^{t-k})\right)\Big)dx \\
&= 2\pi r^2\Big(\sum_{k=1}^{W}(W-k+1)\mathbb{P}\left(\mathcal{C}(x,\Psi_0^{t-k})\right)\Big).
\end{aligned}$$

Hence,

$$\mathbb{P}\left(\mathcal{C}(o,\Phi_0^t)\right) \leq e^{-N_\lambda} + \left(e^{-\frac{N_\lambda}{W+1}} - e^{-N_\lambda}\right) W^{-1}\left(\sum_{k=1}^{W}(W-k+1)\mathbb{P}\left(\mathcal{C}(x,\Psi_0^{t-k})\right)\right).$$

Let τ_i, $i = 1, 2, \ldots$ be a sequence defined by

$$\begin{aligned}
&\tau_t = 1 \text{ for } t = 1, \ldots, W; \text{ and} \\
&\tau_t = e^{-N_\lambda} + \left(e^{-\frac{N_\lambda}{W+1}} - e^{-N_\lambda}\right)\left(\sum_{k=1}^{W}(W-k+1)\tau_{t-k}\right) W^{-1} \text{ otherwise .}
\end{aligned}$$

By an induction argument, we easily show that $\mathbb{P}\left(\mathcal{C}(o,\Psi_0^t)\right) \leq \tau_t$ for every $t > 0$. So, $\limsup_{t\to\infty} \mathbb{P}\big(\mathcal{C}(o,\Psi_0^t)\big) \leq \limsup_{t\to\infty} \tau_t$. We now consider the characteristic function of the sequence τ_t, which is

$$\mathfrak{T}(Z) := Z^{W+1} - \left(e^{-\frac{N_\lambda}{W+1}} - e^{-N_\lambda}\right)\left(\sum_{k=1}^{W} kZ^k\right) W^{-1}.$$

Note that when $\epsilon < (W+1)/2$ and $\lambda \geq \lambda^{\epsilon,W}$,

$$e^{N_\lambda} - \frac{W+1}{2}\left(e^{\frac{WN_\lambda}{W+1}} - 1\right) \geq \epsilon^{-1} > 0,$$

which is equivalent to

$$1 - \frac{W+1}{2}\left(e^{-\frac{N_\lambda}{W+1}} - e^{-N_\lambda}\right) > 0.$$

We have

$$\begin{aligned}\mathfrak{T}'(Z) &= (W+1)Z^W - \left(e^{-\frac{N_\lambda}{W+1}} - e^{-N_\lambda}\right)\left(\sum_{k=1}^{W} k^2 Z^{k-1}\right)W^{-1}\\ &\geq (W+1)Z^W - \left(e^{-\frac{N_\lambda}{W+1}} - e^{-N_\lambda}\right)\left(\sum_{k=1}^{W}(W+1)kZ^W\right)W^{-1}\\ &= (W+1)Z^W\left(1 - \frac{W+1}{2}\left(e^{-\frac{N_\lambda}{W+1}} - e^{-N_\lambda}\right)\right) > 0\end{aligned}$$

for all real $Z \geq 1$ and

$$\mathfrak{T}'(1) = 1 - \frac{W+1}{2}\left(e^{-\frac{N_\lambda}{W+1}} - e^{-N_\lambda}\right) > 0.$$

So, let $\zeta_1, \ldots, \zeta_{W+1}$ be the $W+1$ (complex) roots of $\mathfrak{T}$, then $|\zeta_i| \leq 1$ for $i = 1, \ldots, W+1$. Since τ_t can be expressed as

$$\left(e^{N_\lambda} - \frac{W+1}{2}\left(e^{\frac{N_\lambda W}{W+1}} - 1\right)\right)^{-1} + \eta_i \zeta_i^t.$$

for all t for some $W+1$ complex values $\eta_1, \ldots, \eta_{W+1}$. Then we have that

$$\limsup_{t\to\infty} \mathbb{P}\left(\mathcal{C}(o, \Psi_0^t)\right) \leq \left(e^{N_\lambda} - \frac{W+1}{2}\left(e^{\frac{N_\lambda W}{W+1}} - 1\right)\right)^{-1}, \tag{2.3.7}$$

when $\epsilon < (W+1)/2$ and $\lambda \geq \lambda^{\epsilon,W}$. The upper bound is concluded by noting that in this case the above quantity is smaller than ϵ.

For the lower bound, from the second inclusion in Lemma 2.1,

$$\mathbb{P}(\mathcal{C}(o, \Psi_0^t)) \geq \mathbb{P}(\mathcal{C}(o, \Phi_0^{t-W}), \ldots, \mathcal{C}(o, \Phi_0^t), \mathcal{C}(o, \Upsilon_{W-1,0}^t), \ldots, \mathcal{C}(o, \Upsilon_{1,0}^t)).$$

Given a realization of $\Psi_0^{t-W}, \ldots, \Psi_0^{t-1}$, $\{\Phi_W^{t-W}, \ldots, \Phi_1^{t-1}\}$ and $\{\Upsilon_{W-1,0}^t, \ldots, \Upsilon_{1,0}^t\}$ are $2W-1$ independent PPPs. For $k = 0, \ldots, W$ the (conditional) intensity of Φ_k^{t-k} is $\frac{\lambda}{W+1}$ and that of $\Upsilon_{k,0}^t$ is $\frac{(W-k)\lambda}{W+1}\left(1 - \prod_{l=1}^{k} \mathbf{1}_{\overline{\mathcal{C}(x,\Psi_0^{t-l})}}\right)dx$. As

$$\sum_{l=1}^{k} \mathbf{1}_{\mathcal{C}(x,\Psi_0^{t-l})} \leq \left(1 - \prod_{l=1}^{k} \mathbf{1}_{\overline{\mathcal{C}(x,\Psi_0^{t-l})}}\right) \text{ a.s.,}$$

we have

$$\mathbb{P}(\mathcal{C}(o, \Psi_0^t)) \geq \prod_{k=0}^{W} e^{-\frac{N_\lambda}{W+1}} \mathbb{E}\left[\prod_{k=1}^{W-1} e^{-\frac{(W-k)\lambda}{W+1}\int_{B(o,r)}\left(\sum_{l=1}^{k} \mathbf{1}_{\mathcal{C}(x,\Psi_0^{t-l})}\right)dx}\right].$$

By Jensen's inequality,

$$\mathbb{E}\left[\prod_{k=1}^{W-1} e^{-\frac{(W-k)\lambda}{W+1}\int_{B(o,r)}\left(\sum_{l=1}^{k} \mathbf{1}_{\mathcal{C}(x,\Psi_0^{t-l})}\right)dx}\right] \geq \prod_{k=1}^{W-1} e^{-\frac{N_\lambda}{W+1}\sum_{k=1}^{W-1}\sum_{l=1}^{k}\mathbb{P}(\mathcal{C}(o,\Psi_0^{t-l}))}.$$

So,

$$\mathbb{P}(\mathcal{C}(o,\Psi_0^t)) \geq e^{-\frac{N_\lambda}{W+1}} e^{-\frac{(W-k)N_\lambda}{W+1}\sum_{l=1}^{k}\mathbb{P}(\mathcal{C}(o,\Psi_0^{t-l}))}.$$

Note that

$$\limsup_{t\to\infty}\mathbb{P}(\mathcal{C}(o,\Psi_0^{t-l})) \leq \left(e^{N_\lambda} - \frac{W+1}{2}\left(e^{\frac{N_\lambda W}{W+1}}-1\right)\right)^{-1},$$

we deduce that

$$\limsup_{t\to\infty}\mathbb{P}(\mathcal{C}(o,\Psi_0^t)) \geq \liminf_{t\to\infty}\mathbb{P}(\mathcal{C}(o,\Psi_0^t)) \geq e^{-\frac{N_\lambda}{W+1}}$$
$$e^{-\frac{(W-k)N_\lambda}{W+1}\sum_{l=1}^{k}\left(e^{N_\lambda}-\frac{W+1}{2}\left(e^{\frac{N_\lambda W}{W+1}}-1\right)\right)^{-1}}.$$

Setting this to ϵ gives us the lower bound. □

Remark 2.5 *The lower bound and the upper bound are plotted against ϵ for different values of W in Figure 2.8. We can see that the bounds are loose for large W and large ϵ, but they are asymptotically tight as ϵ tends to 0 in the sense that $\lim_{\epsilon\to 0}\frac{\lambda_u^{\epsilon,W}2\pi r^2}{-\ln(\epsilon)} = \lim_{\epsilon\to 0}\frac{\lambda_d^{\epsilon,W}2\pi r^2}{-\ln(\epsilon)} = 1$. In other words, at small ϵ, a different value of W does not affect the critical intensity value, which is dictated by the ALOHA ($W=0$) critical intensity.*

Bibliographical note

The analysis in Section 2.1 is based on [6, Sect. IV.18]. The greater part of the results in Section 2.2 are published in [33, 19, 20]. The results in Section 2.3 are in [22], which is going to appear in INFOCOM 2013.

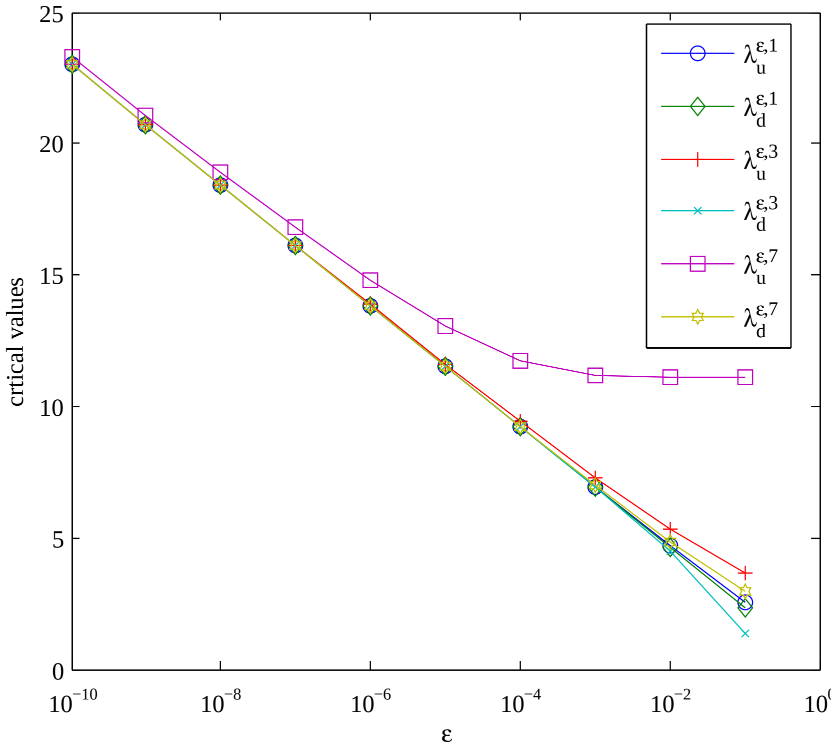

Figure 2.8: Plot of the critical value for different values of W

Chapter 3

The Probability Generating Functionals of Matérn Type II and Type III Models

The Matérn models of type II and type III are originally introduced in the seminal work of B.Matérn [17]. These models construct from a PP Φ, whose points may have conflict with each other, two conflict-free PPs $\mathcal{M}_{\mathrm{II}}(\Phi)$ and $\mathcal{M}_{\mathrm{III}}(\Phi)$. Apart from being used in the analysis of CSMA wireless networks as shown in Section 2.1, they also find applications in other disciplines, such as material science, geology, forestry, etc. While the type II model is first introduced by B. Matérn [17], the type III model is known long before and is widely used under many names. It first appears as the "Car Parking Problem" in a paper of Rényi [25]. In applications in network resources allocation, it is known as the "Random Interval Packing" [10] or the "On-line Packing Problem" , while in physics and material science it is known as the "Random Adsorption Model", see [23] and the citations herein.

As the original Matérn type III construction is only applicable when Φ is finite, a purpose of this chapter is to exhibit an extension of this model to the case where Φ is countably finite. Then, we study the p.g.fls (see Appendix A for their definitions) of $\mathcal{M}_{\mathrm{II}}(\Phi)$ and $\mathcal{M}_{\mathrm{III}}(\Phi)$ and we prove that under some mild conditions, these functionals are the unique solutions of some systems of differential equations. Similar results are also provided for the Palm versions of these p.g.fls (see Appendix A for the definition of Palm distributions) and for the special case when the considered PPs are stationary.

3.1 Constructions of the Matérn Models

Given a proposed PP Φ, each of the type II and type III models is made up of two components: a *conflict relation* between the points in Φ and a retention procedure (or equivalently, a construction procedure) to retain these points in such a way that conflicts do not arise among them. Here, we consider an instance of these two models where the conflict relation is random and its realizations at different pairs of points are mutually independent. We start by constructing such a random conflict relation in Subsection 3.1.1. The original constructions of Matérn are recalled in Subsection 3.1.2. The extension of the Matérn type III construction to the case where the number of points is countably infinite is given in Subsection 3.1.3. We then finish by introducing the notion of Poisson Rain with Random Conflict Relation in Subsection 3.1.4. This notion provides us a common structure to study the evolution of the distributions of the Matérn models when their intensities vary.

3.1.1 Point Process with Random Conflict Relation

In applications that require constructions of conflict-free point processes, there is always a conflict relation between points and it is required that any two points with a conflict between them do not appear at the same time in the constructed processes. Usually, the conflict is a deterministic relation based on the geometrical interaction between these points. For instance, it can be required that two points conflict with each other iff the distance between them is smaller than some threshold d. But for some applications, such as the CSMA wireless networks presented earlier, the conflict is a random relation.

In particular, a Point Process with Random Conflict Relation (PPRCR) is a pair (Φ, C) with Φ a PP and $C \in \{0,1\}^{\Phi^2}$ a random relation in Φ satisfying the following conditions.

- C is symmetric almost surely (a.s.).
- C is non-reflexive a.s., i.e. $C(x,x) = 0$ for all x in Φ a.s.
- Given a realization of Φ, $\{C(x,y),\ (x,y)$ unordered pairs of points in $\Phi\}$ is a family of independent $\{0,1\}-$ value r.v.s and

  $$\mathbb{P}\big(C(x,y) = 1 | \Phi \text{ and } x, y \in \Phi\big) = h(x,y),$$

  with h a symmetric function taking value in $[0,1]$ and satisfying $h(x,x) = 0$ for any x in the plane.

The existence of a such random structure is not obvious, and we dedicate this subsection to its construction. Let $\Psi = \{(x, \mathbf{u}(x))\}$ be an independently MPP such that for each point x, the mark $\mathbf{u}(x) = \{u_i(x),\ i = 1, 2, \dots, \}$ is a sequence of i.i.d. r.v.s uniformly distributed in $[0,1]$. These r.v.s are then associated to other points in Ψ in the following manner: we first enumerate the points in Ψ in the increasing order of their distances to x. If there are at least 2 points with the same distance to x, they are numbered in the counter-clockwise order. The i^{th} point in this ordering is associated to the r.v. $u_i(x)$. We now consider two points x and y in Ψ, let $u_i(x)$ be the r.v. associated to y in the mark of x and $u_j(y)$ be the r.v. associated to x in the mark of y. We let $C(x,y) = 1$ iff

$\min\big(u_i(x), u_j(y)\big) > 1 - \sqrt{h(x,y)}$. It is then easily verified that (Φ, C) defined in this manner is a PPRCR, where Φ is the ground PP of Ψ.

Remark 3.1 *This construction is adopted from [32]. The name of the model in [32] is the Random Connection Model (RCM). Although the model considered here is exactly the same, we do not use the same name since the physical meaning is different. In RCM, $C(x,y) = 1$ means that there is a connection between x and y (which is desirable), while here $C(x,y) = 1$ means that there is a conflict between x and y (which is undesirable).*

A PPRCR is called a Poisson Point Process with Random Conflict Relation (PPPRCR) iff its ground process Φ is a PPP. Let Λ be the intensity measure of Φ, it is easily seen that when (Φ, C) is a PPPRCR, the measure Λ and the function h are the two parameters that completely determine the distribution of (Φ, C). In this case, we call (Φ, C) a PPPRCR with intensity measure Λ and expected conflict function h. Without otherwise stated, we always assume that Λ is a locally finite measure.

3.1.2 The Original Construction of Matérn

Given a PPPRCR (Φ, C) with intensity measure Λ and expected conflict function h, the objectives of the Matérn models of type II and III are to construct from Φ the subsets $\mathcal{M}_{\mathrm{II}}(\Phi, C)$ and $\mathcal{M}_{\mathrm{III}}(\Phi, C)$ that are conflict-free, i.e. for every x, y in $\mathcal{M}_j(\Phi, C)$, $j = \mathrm{II}, \mathrm{III}$ we have $C(x,y) = 0$. In the rest of this chapter, as there is no confusion, the relation C in the above notations is omitted.

The Matérn Type II Model

The Matérn type II model gives each point x a mark $t(x)$ which takes value in $[0,1]$ as an additional attribute of x. This mark is interpreted as the time when a point 'arrives' in the system. For convenience, we refer to it as the timer. The conflict between any two points is resolved by a competition where the one arriving earlier wins. Only the winners belong to $\mathcal{M}_{\mathrm{II}}(\Phi)$, i.e.

$$\mathcal{M}_{\mathrm{II}}(\Phi) = \ \{x \in \Phi \text{ s.t. for all } y \in \Phi,\ C(x,y) = 1 \ \Rightarrow\ t(x) < t(y)\}. \qquad (3.1.1)$$

In the literature this construction is sometimes referred to as the Matérn hard-core model.

Remark 3.2 *Let us consider a simple example, where Φ has 3 points x, y, z, the conflict relation C is $C(x,y) = 1, C(y,z) = 1, C(x,z) = 0$. If the timers of the points in Φ are $t(x) = .1, t(y) = .2, t(z) = .3$, the Matérn type II model will be $\mathcal{M}_{\mathrm{II}}(\Phi) = \{x\}$ by definition. However, we can see that $\{x, z\}$ is another conflict-free subset of Φ, which has more points than $\mathcal{M}_{\mathrm{II}}(\Phi)$. In the other words, the subset retained by the retention procedure of the Matérn type II model is not a maximal conflict-free subset in the set theoretical sense.*

The Matérn Type III Model

The Matérn type III model is proposed with the purpose of resolving conflicts while retaining as many points as possible. In this sense, it can be viewed as an improvement of the Matérn type II model. The intuition behind this mechanism is as follows: when a point competes with others for space, it does not need to compete with those points that have already been defeated. When Φ contains only finitely many points, we can give an explicit construction of $\mathcal{M}_{\mathrm{III}}(\Phi)$. First, all the points in Φ are sorted in the increasing order of their timers. Let $\{x_i,\ i = 1, 2, \ldots\}$ be this ordering. We then construct an increasing sequence of sets $\{\Phi_{\mathrm{III}}^{(i)},\ i = 1, 2, \ldots\}$:

$$\Phi_{\mathrm{III}}^{(1)} = \{x_1\}; \tag{3.1.2}$$

$$\Phi_{\mathrm{III}}^{(i+1)} = \begin{cases} \Phi_{\mathrm{III}}^{(i)} \cup \{x_{i+1}\} & \text{if } C(x_{i+1}, x_j) = 0 \text{ for all } x_j \in \Phi_{\mathrm{III}}^{(i)}, \\ \Phi_{\mathrm{III}}^{(i)} & \text{otherwise.} \end{cases} \tag{3.1.3}$$

$\mathcal{M}_{\mathrm{III}}(\Phi)$ is defined as $\bigcup_{i=1}^{\infty} \Phi_{\mathrm{III}}^{(i)}$. It is easily seen that $\mathcal{M}_{\mathrm{III}}(\Phi)$ satisfies

$$\mathcal{M}_{\mathrm{III}}(\Phi) = \{x \in \Phi \text{ s.t. for all } y \in \mathcal{M}_{\mathrm{III}}(\Phi),\ C(x, y) = 1 \Rightarrow t(x) < t(y)\}.$$

Remark 3.3 *It is not difficult to see that the subset retained by the retention procedure of the Matérn type III model is a maximal conflict-free subset of* Φ*. Indeed, consider a conflict-free subset* Ξ *of* Φ *such that* $\mathcal{M}_{\mathrm{III}}(\Phi) \subseteq \Xi$*. Assume that* $\Xi \setminus \mathcal{M}_{\mathrm{III}}(\Phi) \neq \emptyset$*, we take a point* $x_j \in \Xi \setminus \mathcal{M}_{\mathrm{III}}(\Phi)$*. As* Ξ *is conflict-free,* $C(x_j, y) = 0$ *for every* y *in* Ξ*. Moreover, we have* $\Phi_{\mathrm{III}}^{(j-1)} \subseteq \mathcal{M}_{\mathrm{III}}(\Phi) \subseteq \Xi$*. So* $C(x_j, y) = 0$ *for every* y *in* $\Phi_{\mathrm{III}}^{(j-1)}$*, which implies that* x_j *is in* $\mathcal{M}_{\mathrm{III}}(\Phi)$ *by definition, which is a contradiction. So, we must have that* $\Xi = \mathcal{M}_{\mathrm{III}}(\Phi)$*, which means that* $\mathcal{M}_{\mathrm{III}}(\Phi)$ *is a maximal conflict-free subset of* Φ*.*

When Φ contains infinitely many points, there are configurations of Φ such that the type III construction is not applicable. A simple example is when $\Phi = \mathbb{Z}^+$ and $t(i) = i^{-1}$. In this case, there is no way to sort the points in Φ in the increasing order of the timers. Nevertheless, the construction of the Matérn type III model can still be extended to the case where $|\Phi| = \infty$ under some mild conditions. This construction is more involved and we leave its discussion to Subsection 3.1.3.

3.1.3 The Infinite Construction of the Matérn type III Model

Our aim here is to provide an extension of the Matérn type III construction to the case where Φ is countably infinite. We start by defining the *conflict graph* associated to Φ and C. Then we give our extension which is applicable only when the conflict graph has the *finite history* property (which will be defined shortly).

The Conflict graph

For any two points x and y in Φ, we put a directed edge from x to y if $C(x, y) = 1$ and $t(x) < t(y)$. Let $\mathcal{E}$ be the set of all such edges, i.e.

$$\mathcal{E} = \{(x, y) \text{ s.t. } C(x, y) = 1 \text{ and } t(x) < t(y)\}.$$

The *conflict graph* associated to Φ and C is the directed graph $\mathcal{G} = \{\Phi, \mathcal{E}\}$. It is not difficult to check that $\mathcal{G}$ is an acyclic graph.

We now define recursively the order (an asymmetrical, non-reflexive transitive binary relation) $\hookleftarrow$ in Φ as

$$x \hookleftarrow y \text{ if } \begin{cases} \text{either } (x, y) \in \mathcal{E}, \\ \text{or there exists } z \in \Phi \text{ s.t. } (z, y) \in \mathcal{E} \text{ and } x \hookleftarrow z. \end{cases} \tag{3.1.4}$$

We call this the ancestor order in Φ. For each x, let

$$\mathcal{A}(x) = \{y \in \Phi \text{ s.t. } y \hookleftarrow x\} \tag{3.1.5}$$

be the set of its *ancestors* . $\mathcal{G}$ is said to have the *finite history* property if $\mathcal{A}(x)$ is finite for all x in Φ.

The Matérn Type III Extension

Let e be a function from Φ to $\{0, 1\}$ satisfying

$$e(x) = \begin{cases} 1 & \text{if } \mathcal{A}(x) = \emptyset, \\ \displaystyle\prod_{y \in \Phi,\, (y,x) \in \mathcal{E}} (1 - e(y)) & \text{otherwise .} \end{cases} \tag{3.1.6}$$

We have,

Proposition 3.1 *When the conflict graph $\mathcal{G}$ has the finite history property, there exists a unique function e satisfying (3.1.6).*

Proof. It is sufficient to show that $e(x)$ is uniquely determined for every x in Φ. We do so by induction on $|\mathcal{A}(x)|$. The base case is when $|\mathcal{A}(x)| = 0$ so that $e(x) = 1$ by definition. Note that such x always exists by the finite history assumption (the argument is rather simple: if $|\mathcal{A}(x)| > 0$ for all x in Φ, we start with $x_0 = x$ and build an infinite chain $\{x_i,\, i = 0, 1, \ldots\}$ of mutually different points such that $x_0 \hookleftarrow x_1 \hookleftarrow x_2 \cdots$. By the transitivity of $\hookleftarrow$, we have that $\{x_i,\, i = 0, 1, \ldots\} \subseteq \mathcal{A}(x)$. Thus $|\mathcal{A}(x)| = \infty$, which is a contradiction).

Now suppose that $e(x)$ is uniquely determined for every $x \in \Phi$ such that $|\mathcal{A}(x)| < k$. Consider an $x \in \Phi$ such that $|\mathcal{A}(x)| = k$ (if such x exists), then

$$e(x) = \prod_{y \in \Phi, \text{ s.t. } (y,x) \in \mathcal{E}} (1 - e(y)) \tag{3.1.7}$$

by definition. As $|\mathcal{A}(x)| = k$ and $y \in \mathcal{A}(x)$ for all $y \in \Phi$ such that $(y, x) \in \mathcal{E}$, the left-hand side is a product of finitely many terms. Moreover, as $\hookleftarrow$ is an order, $\mathcal{A}(y) \subset \mathcal{A}(x)$ for all $y \in \mathcal{A}(x)$. Hence, $|\mathcal{A}(y)| < |\mathcal{A}(x)| = k$ for all $y \in \mathcal{A}(x)$, which implies, by the induction hypothesis, that every $e(y)$ term appearing in the left-hand side is uniquely determined, and so is $e(x)$. $\square$

The subset $\mathcal{M}_{\text{III}}(\Phi)$ produced by the Matérn type III model is defined as

$$\mathcal{M}_{\text{III}}(\Phi) = \{x \in \Phi \text{ s.t. } e(x) = 1\}. \tag{3.1.8}$$

It is easily checked that

$$\mathcal{M}_{\text{III}}(\Phi) = \{x \in \Phi \text{ s.t. for all } y \in \mathcal{M}_{\text{III}}(\Phi),\ C(x, y) = 1 \Rightarrow t(x) < t(y)\},$$

and that when Φ contains finitely many points, the PP constructed in this manner and the one constructed by the original Matérn type III model are identical.

Matérn type III Construction for PPPRCRs

Recall that the extended Matérn type III construction is applicable only when the conflict graph $\mathcal{G}$ associated to the realization of (Φ, C) has the finite history property. We prove here that the above is true a.s. if the measure Λ and the function h satisfy

$$\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} h(x, y)\, \Lambda(dy) = \overline{H} < \infty. \tag{3.1.9}$$

The main result of this section is

Proposition 3.2 *For every PPPRCR (Φ, C) with ground intensity measure Λ and expected conflict function h satisfying (3.1.9), its corresponding conflict graph has the finite history property a.s.*

Proof. Consider a typical point x in Φ. Recall that $\mathcal{A}(x)$ is the set of ancestors of x in the conflict graph. Let

$$\begin{aligned}\mathcal{A}^{(l)}(x) =& \{y \in \Phi \text{ s.t. exists } x_0, x_1, \ldots, x_l \in \Phi \\ & \text{s.t. } x_0 = x,\ x_l = y \text{ and } (x_1, x_0), \ldots, (x_l, x_{l-1}) \in \mathcal{E}\}.\end{aligned}$$

$\mathcal{A}(x)$ can be rewritten as

$$\mathcal{A}(x) = \bigcup_{l=1}^{\infty} \mathcal{A}^{(l)}(x).$$

We now bound $\mathbf{E}\left[|\mathcal{A}^{(l)}(x)|\right]$. First, notice that

$$\begin{aligned}|\mathcal{A}^{(l)}(x)| \leq & \sum_{x_0, \ldots, x_l \in \Phi, x_0 = x} \left(\prod_{k=1}^{l} \mathbf{1}_{(x_k, x_{k-1}) \in \mathcal{E}} \right) \\ = & \sum_{x_0, \ldots, x_l \in \Phi,\, x_0 = x} \left(\prod_{k=1}^{l} C(x_{k-1}, x_k) \mathbf{1}_{t(x_{k-1}) > t(x_k)} \right).\end{aligned}$$

Consider Φ under its Palm distribution given a point at x, we then apply the multivariate Campbell formula on it,

$$
\begin{aligned}
\mathbb{E}_x\left[|\mathcal{A}^{(l)}(x)|\right] &\leq \mathbb{E}_x\left[\sum_{x_1,\ldots,x_l \text{ mutually different in } \Phi,\, x_0=x}\ \prod_{k=0}^{l-1} C(x_k,x_{k+1})\mathbf{1}_{t(x_k)>t(x_{k+1})}\right] \\
&= \int_{(\mathbb{R}^2)^l}\int_{([0,1])^l} \mathbf{1}_{t(x)>t_1>t_2>\cdots>t_l}\ h(x,x_1)\ \prod_{k=1}^{l-1} h(x_k,x_{k+1})\ dt_1\ \ldots\ dt_l \\
&\quad \Lambda(dx_1)\ \ldots\ \Lambda(dx_l) \\
&= \int_{(\mathbb{R}^2)^l} \frac{t(x)^l}{l!}\ h(x_i,x_1)\ \prod_{k=1}^{l-1} h(x_k,x_{k+1})\ \Lambda(dx_1)\ \ldots\ \Lambda(dx_l).
\end{aligned}
$$

In the above formula, $\mathbb{E}_x$ is the expectation w.r.t. the corresponding Palm distribution. As h and Λ satisfy (3.1.9), we deduce that

$$
\begin{aligned}
\mathbb{E}_x\left[|\mathcal{A}^{(l)}(x)|\right] &\leq \int_{(\mathbb{R}^2)^l} \frac{t(x)^l}{l!}\ h(x,x_1)\ \prod_{k=1}^{l-1} h(x_k,x_{k+1})\ \Lambda(dx_1)\ \ldots\ \Lambda(dx_l) \\
&\leq \frac{t(x)^l}{l!}\overline{H}^l,
\end{aligned}
$$

where $\overline{H}$ is defined in (3.1.9). So

$$
\mathbb{E}_x\left[|\mathcal{A}(x)|\right] \leq \sum_{l=1}^{\infty} \mathbb{E}_x\left[|\mathcal{A}^{(l)}(x)|\right] \leq \sum_{l=1}^{\infty} \frac{t(x)^l}{l!}\overline{H}^l = \exp\{t(x)\overline{H}\} < \infty.
$$

This implies that $|\mathcal{A}(x)| < \infty$ a.s. for every x in Φ. Since Φ is at most countably infinite, it follows directly that the conflict graph has the finite history property a.s. $\square$

From now on, we assume that Λ and h satisfy (3.1.9), so that the extended Matérn type III construction is applicable a.s.

3.1.4 Poisson Rain with Random Conflict Relation and Its Matérn Models

In the next section, we are interested in the dynamic evolution of the Matérn models when the proposed PPP has more and more points. More concretely, we want to study the distribution of $\mathcal{M}_{\mathrm{II}}(\Phi_t)$, $\mathcal{M}_{\mathrm{III}}(\Phi_t)$ for different values of t, where Φ_t is a PPPRCR of intensity measure $t\Lambda$ and expected conflict function h. For this purpose, it is more convenient to consider Φ_t, $t \in \mathbb{R}^+$ as an increasing family of subsets of a *Poisson Rain with Random Conflict Relation* (PRRCR). First we define the notion of Poisson Rain.

Definition 3.1 *A Poisson Rain with ground intensity Λ is a PPP $\{(x,t(x))\}$ in $\mathbb{R}^2 \times \mathbb{R}^+$ of intensity measure $\Lambda \times \mathcal{L}$ with $\mathcal{L}$ the Lebesgue measure.*

Each 'point' in Φ is a pair $(x, t(x))$ with $x \in \mathbb{R}^2$ and $t(x) \in \mathbb{R}^+$. The x component is understood as the position of a point and $t(x)$ is understood as the timer of the point. The name Poisson Rain stems from the interpretation of Φ as a collection of raindrops falling from the sky, the timer of a point is the time when it hits the ground and its position is the place where it does so. By abuse of notation, for each $x \in \mathbb{R}^2$, we write $x \in \Phi$ for "there exists a t such that $(x,t) \in \Phi$". So, when we refer to a point of Φ as a pair of location-timer, we use the pair notation (x,t), and when we refer to it as a point in $\mathbb{R}^2$, we use the single element x notation.

Remark 3.4 *The Poisson Rain considered here is a special case of the extended Marked Point Process introduced in [11, Definition 9.1.VI, p. 7]. Note that the extended MPP in [11] is used to construct the counting measure of a purely atomic random measure, where the mark of a point is used to represent the mass of the random measure at that point. As the mark of a point is used here to represent its arrival time, we use here the name Poisson Rain instead of extended marked Poisson Point Process.*

The random conflict relation is introduced to the Poisson Rain as follow, where h is a function satisfying (3.1.9) .

Definition 3.2 *A Poisson Rain with Random Conflict Relation (PRRCR) with ground intensity Λ and expected conflict function h is a pair (Φ, C) where Φ is a Poisson Rain with ground intensity Λ and $C = \{C(x,y),\, x,y \in \Phi\}$ is a family of $\{0,1\}$ value r.v.s indexed by unordered pairs of locations in Φ satisfying*

1. *C is non-reflexive and symmetric a.s.;*
2. *given a realization of Φ, C is a family of independent r.v.s with the exceptions given by the condition 1; and*
3. *$\mathbb{P}\big(C(x,y) \mid \Phi$ and $x,y \in \Phi\big) = h(x,y)$.*

For completeness, we provide here a construction of a PRRCR, which is an extension of the construction in 3.1.1. Let $\{\Psi_i,\, i = 1,2,\ldots\}$ be a family of i.i.d. MPPPs with i.i.d. marks of ground intensity Λ. Each point x in Ψ_i is equipped with a mark $\mathbf{u}(x) = \{\tau(x), u_{j,k}(x),\, j,k = 1,2,\ldots\}$ which is a family of i.i.d. r.v.s uniformly distributed in $[0,1]$. Let

$$\Psi_i^{'} := \Big\{\big(x, \tau(x) + i, \{u_{j,k}(x)\}\big) \text{ for all } \big(x, \tau(x), \{u_{j,k}(x)\}\big) \in \Psi_i\Big\}.$$

The Poisson Rain Φ is defined as $\bigcup_{i=1}^{\infty} \Psi_i^{'}$. To determine the random conflict relation, for each x in Φ, we number the points in Ψ_i' in the same way as in the construction of Subsection 3.1.1 and associate to the j^{th} point in this numbering the r.v. $u_{i,j}(x)$ in the mark of x. Now, consider any two points x, y in Φ. Let $u_{i,j}(x)$ be the r.v. corresponding to y in the mark of x and $u_{k,l}(y)$ be the r.v. corresponding to x in the mark of y, we let $C(x,y) = 1$ iff $\min\big(u_{k,l}(y), u_{k,l}(y)\big) > 1 - \sqrt{h(x,y)}$. It is then easily verified that (Φ, C) defined in this manner is indeed a PRRCR with ground intensity Λ and expected conflict function h.

Given a PRRCR Φ, we define the restriction $T_{s,t}$ to the interval $[s,t)$ as

$$T_{s,t}(\Phi) = \{x \in \Phi \text{ s.t. } t(x) \in [s,t)\}. \tag{3.1.10}$$

This restricted version of Φ inherits the natural conflict relation from Φ. When $s = 0$, the above notation is reduced to $T_t(\Phi)$. Then the PPP Φ_t can be defined as $T_t(\Phi)$. Moreover, we can easily see that the restriction transformation can also be applied to any PPRCR and that a PRRCR can be seen as a PPPRCR with timers that do not take value in a bounded interval but 'uniformly distributed' in $\mathbb{R}^+$. With this view, we can verify that the Matérn type II model is applicable to PRRCR and the Matérn type III model is also applicable a.s. given that Λ and h satisfy (3.1.9). In particular, let the PPs constructed by the Matérn models inherit the natural conflict relation from their original PPRCR. The following facts can be easily proved:

- $T_t(\mathcal{M}_j(\Phi)) = \mathcal{M}_j(\Phi_t)$ for $j = \text{II}, \text{III}$;
- $\mathcal{M}_{\text{II}}(\Phi) = \{x \text{ s.t. } C(x,y) = 0 \; \forall \; y \in \Phi_{t(x)}\}$; and
- $\mathcal{M}_{\text{III}}(\Phi) = \{x \text{ s.t. } C(x,y) = 0 \; \forall \; y \in \mathcal{M}_{\text{III}}(\Phi_{t(x)})\}$.

In the other words, the first claim asserts that the restriction to $[0,t)$ of the Matérn models of Φ are the same as the the Matérn models of the restriction to $[0,t)$ of Φ while the two other claims are just reformulations of the Matérn models definitions. These three claims are based on the fact that the event $x \in \mathcal{M}_j(\Phi)$, $j = \text{II}, \text{III}$ depends only on the realization of the points in Φ whose timers are smaller than $t(x)$.

We finish with a result that will be used frequently in the next section. It allows us to approximate the PPs $T_{s,t}\big(\mathcal{M}_{\text{II}}(\Phi)\big)$ and $T_{s,t}\big(\mathcal{M}_{\text{III}}(\Phi)\big)$ by other PPs which have much simpler structures.

Proposition 3.3 *For every realization of the PRRCR (Φ, C) such that its associated conflict graph has the finite history property and for every $0 < s < t$,*

$$\begin{aligned} \Delta_{\text{II,d,s,t}} &\subseteq T_{s,t}\big(\mathcal{M}_{\text{II}}(\Phi)\big) \subseteq \Delta_{\text{II,u,s,t}}; \\ \Delta_{\text{III,d,s,t}} &\subseteq T_{s,t}\big(\mathcal{M}_{\text{III}}(\Phi)\big) \subseteq \Delta_{\text{III,u,s,t}}, \end{aligned} \tag{3.1.11}$$

where

$$\begin{aligned} \Delta_{\text{II,d,s,t}} &= \{x \in T_{s,t}(\Phi) \text{ s.t. } C(x,y) = 0 \; \forall \; y \in \Phi_t\}; \\ \Delta_{\text{II,u,s,t}} &= \{x \in T_{s,t}(\Phi) \text{ s.t. } C(x,y) = 0 \; \forall \; y \in \Phi_s\}; \\ \Delta_{\text{III,d,s,t}} &= \{x \in T_{s,t}(\Phi) \text{ s.t. } C(x,y) = 0 \; \forall \; y \in \mathcal{M}_{\text{III}}(\Phi_s) \cup T_{s,t}(\Phi)\}; \\ \Delta_{\text{III,u,s,t}} &= \{x \in T_{s,t}(\Phi) \text{ s.t. } C(x,y) = 0 \; \forall \; y \in \mathcal{M}_{\text{III}}(\Phi_s)\}. \end{aligned} \tag{3.1.12}$$

Proof. Since $\mathcal{M}_{\text{II}}(\Phi) = \{x \text{ s.t. } C(x,y) = 0 \; \forall \; y \in \Phi_{t(x)}\}$ and $\Phi_s \subseteq \Phi_{t(x)} \subseteq \Phi_t$ for every $x \in T_{s,t}(\Phi)$, the first double inclusion follows directly. The second one is proved similarly using the third claim above and taking into account that $\mathcal{M}_{\text{III}}(\Phi_s) \subseteq \mathcal{M}_{\text{III}}(\Phi_{t(x)}) \subseteq \mathcal{M}_{\text{III}}(\Phi_s) \cup T_{s,t}(\Phi)$ for every $x \in T_{s,t}(\Phi)$. $\square$

3.2 Principal Results

This section contains the main results of this chapter. In particular, we study here the p.g.fls of the Matérn type II and type III models and their reduced Palm versions. We first show that these p.g.fls are the solutions of some systems of differential equations and some integral equations in Subsections 3.2.1 and 3.2.2. Then we show that the systems p.g.fls of interest are indeed the only solutions of these equations.

3.2.1 The Probability Generating Functional

We now study the p.g.fls of $\mathcal{M}_{\mathrm{II}}(\Phi_t)$ and $\mathcal{M}_{\mathrm{III}}(\Phi_t)$ and compute their differentiations w.r.t. t. It turns out that although the Matérn type III model is more complicated to define, its p.g.fls are easier to study. For this reason, we first present the result for the Matérn type III model. The method is then extended to study the p.g.fls of the Matérn type II model.

The Matérn Type III Model

For every $t \geq 0$ and and every function v taking value in $[0, 1]$ such that

$$\int_{\mathbb{R}^2} \big(1 - v(x)\big)\, \Lambda(dx) < \infty, \tag{3.2.1}$$

we define

$$f_\Lambda(t, v) := G_{\mathcal{M}_{\mathrm{III}}(\Phi_t)}(v) = \mathbb{E}\left[\prod_{x \in T_t\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(x)\right]. \tag{3.2.2}$$

Remark 3.5 *Note that a sufficient condition for $f_\Lambda(t, v)$ to be non-trivial is that $\mathbb{E}\left[\sum_{x\in\mathcal{M}_{\mathrm{III}}(\Phi_t)} \big(1 - v(x)\big)\right] < \infty$. As*

$$\mathbb{E}\left[\sum_{x\in\mathcal{M}_{\mathrm{III}}(\Phi_t)} \big(1 - v(x)\big)\right] \leq \mathbb{E}\left[\sum_{x\in\Phi_t} \big(1 - v(x)\big)\right] = t\int_{\mathbb{R}^2} \big(1 - v(x)\big)\Lambda(dx),$$

condition (3.2.1) *is a little bit too strong. Nevertheless, we use it here since it is simpler to work with and all functions that we are interested in satisfy this condition. In particular, the distribution of $\mathcal{M}_{\mathrm{III}}(\Phi_t)$ is uniquely determined by the values of $f_\Lambda(t, v)$ at functions v such that $1-v$ have bounded support. Given that Λ is locally finite, such functions always satisfy* (3.2.1).

In order to compute the differentiation w.r.t. t of f_Λ, we first need it to be continuous in t.

Proposition 3.4 *For every function v satisfying condition (3.2.1), $f_\Lambda(t, v)$ is continuous in t.*

Proof. For every t and ϵ positive

$$\begin{aligned} f_\Lambda(t+\epsilon, v) &= \mathbb{E}\left[\prod_{x \in T_{t+\epsilon}\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(x)\right] \\ &= \mathbb{E}\left[\prod_{x \in T_t\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(x) \prod_{x \in T_{t,t+\epsilon}\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(x)\right]. \end{aligned}$$

As $T_{t,t+\epsilon}\big(\mathcal{M}_{\mathrm{III}}(\Phi)\big) \subseteq T_{t,t+\epsilon}(\Phi)$ a.s., we deduce that

$$1 \geq \prod_{x \in T_{[t,t+\epsilon)}\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(x) \geq \prod_{x \in T_{t,t+\epsilon}(\Phi)} v(x) \text{ a.s.}$$

Thus,

$$\begin{aligned} f_\Lambda(t, v) e^{-\epsilon \int_{\mathbb{R}^2}\left(1-v(x)\right)\Lambda(dx)} &= \mathbb{E}\left[\prod_{x \in T_t\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(x) \prod_{x \in T_{t,t+\epsilon}(\Phi)} v(x)\right] \\ &\leq \mathbb{E}\left[\prod_{x \in T_{t+\epsilon}\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(x)\right] \\ &= f_\Lambda(t+\epsilon, v) \leq f_\Lambda(t, v). \end{aligned}$$

Following the same method, we get

$$f_\Lambda(t-\epsilon, v) e^{-\epsilon \int_{\mathbb{R}^2} |1-v(x)|\, \Lambda(dx)} \leq f_\Lambda(t, v) \leq f_\Lambda(t-\epsilon, v).$$

Letting ϵ go to 0 completes this proof. □

Let H be the mapping that associates to a function $v : \mathbb{R}^2 \to [0,1]$ and a point $x \in \mathbb{R}^2$ the function

$$H(v, x) : y \longmapsto v(y)\big(1 - h(x, y)\big) \tag{3.2.3}$$

from $\mathbb{R}^2$ to $[0,1]$.

Theorem 3.1 *For any locally finite measure Λ, the functional f_Λ satisfies the following system of equations,*

$$\begin{aligned} f_\Lambda(0, v) &= 1; \\ \frac{df_\Lambda(t, v)}{dt} &= -\int_{\mathbb{R}^2} f_\Lambda\big(t, H(v, x)\big)\big(1 - v(x)\big)\Lambda(dx). \end{aligned} \tag{3.2.4}$$

The main idea behind the proof of this theorem is to divide $\mathcal{M}_{\mathrm{III}}(\Phi)$ into thin layers $T_{t,t+\epsilon}\big(\mathcal{M}_{\mathrm{III}}(\Phi)\big)$. The points in each layer are so sparse that there is almost no conflict between them. Then we can consider each layer as a PPP. In particular, we need to prove that

$$\lim_{\epsilon\to\infty} \frac{f_\Lambda(t+\epsilon,v)-f_\Lambda(t,v)}{\epsilon} = -\int_{\mathbb{R}^2} f_\Lambda\big(t,H(v,x)\big)\big(1-v(x)\big)\Lambda(dx); \quad (3.2.5)$$

$$\lim_{\epsilon\to\infty} \frac{f_\Lambda(t,v)-f_\Lambda(t-\epsilon,v)}{\epsilon} = -\int_{\mathbb{R}^2} f_\Lambda\big(t,H(v,x)\big)\big(1-v(x)\big)\Lambda(dx). \quad (3.2.6)$$

We will need the following lemmas

Lemma 3.1 *For every PP Ξ and every function v taking value in $[0,1]$ such that*

$$\mathbb{E}\left[\sum_{x\in\Xi}\big(1-v(x)\big)\right] < \infty,$$

we have

$$\prod_{x\in\Xi} v(x) = 1 + \sum_{i=1}^{\infty}(-1)^i \sum_{(x_1,\cdots,x_i)\in\Xi^{(i!)}} \prod_{j=1}^{i}\big(1-v(x_j)\big) \text{ a.s.}, \quad (3.2.7)$$

where $\Xi^{(i!)}$ is the set of unordered i-tuples of mutually different points in Ξ.

Proof. Since $\sum_{x\in\Xi}\log\big(v(x)\big) \le \sum_{x\in\Xi}|1-v(x)| < \infty$ a.s.,

$$\prod_{x\in\Xi} v(x) = \exp\left\{\sum_{x\in\Xi}\log(v(x))\right\}$$

is well-defined and is finite a.s.

Now we need to prove that the series in the right hand side of (3.2.7) converges. For this purpose, it is sufficient to show that

$$1 + \sum_{i=1}^{\infty} \sum_{(x_1,\cdots,x_i)\in\Xi^{i!}} \prod_{j=1}^{i}\big(1-v(x_j)\big) < \infty \text{ a.s.}$$

Note that

$$\begin{aligned}\sum_{(x_1,\cdots,x_i)\in\Xi^{i!}} \prod_{j=1}^{i}\big(1-v(x_j)\big) &= \frac{1}{i!}\sum_{x_1,\cdots,x_i \text{ mutually different}\in\Xi} \prod_{j=1}^{i}\big(1-v(x_j)\big)\\ &\le \frac{1}{i!}\left(\sum_{x\in\Xi}\big(1-v(x)\big)\right)^i.\end{aligned}$$

Hence,

$$1+\sum_{i=1}^{\infty}\sum_{(x_1,\cdots,x_i)\in\Xi^{i!}}\prod_{j=1}^{i}\big(1-v(x_j)\big)\leq 1+\sum_{i=1}^{\infty}\frac{1}{i!}\left(\sum_{x\in\Xi}\big(1-v(x)\big)\right)^i$$
$$=\exp\left\{\left(\sum_{x\in\Xi}\big(1-v(x)\big)\right)\right\}<\infty.$$

The equality can now be obtained by writing

$$\prod_{x\in\Xi}v(x)=\prod_{x\in\Xi}\Big(1-\big(1-v(x)\big)\Big).$$

□

Lemma 3.2 *Let* (Φ, C) *be a PRRCR with ground intensity* Λ *and expected conflict function* h*. For every* $t\geq 0$*, every* $\epsilon>0$ *small enough,*

$$\left|\mathbb{E}\left[\prod_{x\in T_{t,t+\epsilon}\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)}v(x)\,\middle|\,\Phi_t\right]-1+\epsilon\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda_{\mathrm{III,t}}(dx)\right|$$
$$\leq\epsilon^2\left(2\left(\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda(dx)\right)^2+\overline{H}\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda(dx)\right)\ \text{a.s.}\ , \tag{3.2.8}$$

where $\overline{H}$ *is defined in (3.1.9) and* $\Lambda_{\mathrm{III,t}}$ *is the random measure in* $\mathbb{R}^2$ *satisfying*

$$\Lambda_{\mathrm{III,t}}(dx)=\prod_{y\in T_t\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)}\big(1-h(x,y)\big)\,\Lambda(dx). \tag{3.2.9}$$

Proof. Every expectation in this proof should be understood as the conditional expectation given Φ_t. By Lemma 3.1,

$$\prod_{x\in T_{t,t+\epsilon}\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)}v(x)=1+\sum_{i=1}^{\infty}\sum_{(x_1,\ldots,x_i)\in\left(T_{t,t+\epsilon}\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)\right)^{(i!)}}\big(1-v(x)\big).$$

The first step is to show that for ϵ small enough,

$$\left|\mathbb{E}\left[\sum_{i=2}^{\infty}\sum_{(x_1,\ldots,x_i)\in\left(T_{t,t+\epsilon}\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)\right)^{(i!)}}\big(1-v(x)\big)\right]\right|\leq 2\epsilon^2\left(\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda(dx)\right)^2$$

a.s.

Since $T_{t,t+\epsilon}\big(\mathcal{M}_{\mathrm{III}}(\Phi)\big) \subseteq T_{t,t+\epsilon}(\Phi)$ a.s.,

$$\begin{aligned}
&\left|\mathbb{E}\left[\sum_{i=2}^{\infty} \sum_{(x_1,\ldots,x_i)\in(T_{t,t+\epsilon}(\mathcal{M}_{\mathrm{III}}(\Phi)))^{(i!)}} \prod_{j=1}^{i}\big(1-v(x_j)\big)\right]\right| \\
&\leq \mathbb{E}\left[\sum_{i=2}^{\infty} \sum_{(x_1,\ldots,x_i)\in(T_{t,t+\epsilon}(\Phi))^{(i!)}} \prod_{j=1}^{i}\big(1-v(x_j)\big)\right] \\
&= \sum_{i=2}^{\infty}\mathbb{E}\left[\sum_{(x_1,\ldots,x_i)\in(T_{t,t+\epsilon}(\Phi))^{(i!)}} \prod_{j=1}^{i}\big(1-v(x_j)\big)\right] \\
&= \sum_{i=2}^{\infty}\epsilon^i\left(\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda(dx)\right)^i \text{ a.s.,}
\end{aligned}$$

where in the last line we apply the multivariate Campbell formula to the PPP $T_{t,t+\epsilon}(\Phi)$ [27, p.112]. Take now any $\epsilon < \frac{1}{2}\left(\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda(dx)\right)^{-1}$,

$$\begin{aligned}
\sum_{i=2}^{\infty}\epsilon^i\left(\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda(dx)\right)^i &= \epsilon^2\frac{\left(\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda(dx)\right)^2}{1-\epsilon\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda(dx)} \\
&\leq 2\epsilon^2\left(\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda(dx)\right)^2.
\end{aligned}$$

The next step is to bound $\mathbb{E}\left[\sum_{x\in T_{t,t+\epsilon}(\mathcal{M}_{\mathrm{III}}(\Phi))}\big(1-v(x)\big)\right]$. Proposition 3.3 gives us

$$\Delta_{\mathrm{III,d,t,t+\epsilon}} \subseteq T_{t,t+\epsilon}\big(\mathcal{M}_{\mathrm{III}}(\Phi)\big) \subseteq \Delta_{\mathrm{III,u,t,t+\epsilon}} \text{ a.s.}$$

Then,

$$\sum_{x\in\Delta_{\mathrm{III,d,t,t+\epsilon}}}\big(1-v(x)\big) \leq \sum_{x\in T_{t,t+\epsilon}(\mathcal{M}_{\mathrm{III}}(\Phi))}\big(1-v(x)\big) \leq \sum_{x\in\Delta_{\mathrm{III,u,t,t+\epsilon}}}\big(1-v(x)\big) \text{ a.s.}$$

Given Φ_t, $\Delta_{\mathrm{III,u,t,t+\epsilon}}$ is a PPP of intensity $\epsilon\,\Lambda_{\mathrm{III,t}}$, so

$$\mathbb{E}\left[\sum_{x\in\Delta_{\mathrm{III,u,t,t+\epsilon}}}\big(1-v(x)\big)\right] = \epsilon\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda_{\mathrm{III,t}}(dx).$$

Moreover, we can compute the intensity of $\Delta_{\mathrm{III,d,t,t+\epsilon}}$ (conditioned on Φ_t) as follow. Take any bounded Borel set A in $\mathbb{R}^2$,

$$\begin{aligned}
&\mathbb{E}\left[|\Delta_{\mathrm{III,d,t,t+\epsilon}}\cap A|\right] \\
&= \mathbb{E}\left[\sum_{x\in T_{t,t+\epsilon}(\Phi)\cap A}\left(\left(\prod_{y\in T_{t,t+\epsilon}(\Phi)}\mathbf{1}_{C(x,y)=0}\right)\left(\prod_{y\in T_t(\mathcal{M}_{\mathrm{III}}(\Phi))}\mathbf{1}_{C(x,y)=0}\right)\right)\right].
\end{aligned}$$

Let $\mathbb{P}^!_x$ be the *reduced Palm* distribution of $T_{t,t+\epsilon}(\Phi)$ given a point at x. By Slivnyak's theorem, this reduced Palm distribution is the distribution of a PPP of intensity measure $\epsilon\Lambda$. Hence, by the refined Campbell formula,

$$\begin{aligned}
&\mathbb{E}\left[|\Delta_{\mathrm{III,d,t,t+\epsilon}} \cap A|\right] \\
&= \epsilon \int_A \mathbb{E}^!_x \left[\prod_{y \in T_{t,t+\epsilon}(\Phi)} \mathbf{1}_{C(x,y)=0} \right] \prod_{y \in T_t\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} \mathbf{1}_{C(x,y)=0}\, \Lambda(dx) \\
&= \epsilon \int_A \mathbb{E}^!_x \left[\prod_{y \in T_{t,t+\epsilon}(\Phi)} \big(1 - h(x,y)\big) \right] \prod_{y \in T_t\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} \big(1 - h(x,y)\big)\, \Lambda(dx) \\
&= \epsilon \int_A \exp\left\{ -\epsilon \int_{\mathbb{R}^2} h(x,y)\, \Lambda(dy) \right\} \Lambda_{\mathrm{III,t}}(dx).
\end{aligned}$$

Thus, the intensity measure of $\Delta_{\mathrm{III,d,t,t+\epsilon}}$ is

$$\epsilon \exp\left\{ -\epsilon \int_{\mathbb{R}^2} h(x,y)\, \Lambda(dy) \right\} \Lambda_{\mathrm{III,t}}(dx).$$

We now apply Campbell's formula to $\Delta_{\mathrm{III,d,t,t+\epsilon}}$,

$$\mathbb{E}\left[\sum_{x \in \Delta_{\mathrm{III,d,t,t+\epsilon}}} \big(1 - v(x)\big) \right] = \epsilon \int_{\mathbb{R}^2} \big(1 - v(x)\big) \exp\left\{ -\epsilon \int_{\mathbb{R}^2} h(x,y)\, \Lambda(dy) \right\} \Lambda_{\mathrm{III,t}}(dx).$$

As

$$\exp\left\{ -\epsilon \int_{\mathbb{R}^2} h(x,y)\, \Lambda(dy) \right\} \geq 1 - \epsilon \int_{\mathbb{R}^2} h(x,y)\, \Lambda(dy) \geq 1 - \epsilon\, \overline{H},$$

we get

$$\begin{aligned}
\epsilon \int_{\mathbb{R}^2} \big(1 - v(x)\big)\, \Lambda_{\mathrm{III,t}}(dx) &\geq \mathbb{E}\left[\sum_{x \in T_{t,t+\epsilon}\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} \big(1 - v(x)\big) \right] \\
&\geq \epsilon \int_{\mathbb{R}^2} \big(1 - v(x)\big)\, \Lambda_{\mathrm{III,t}}(dx) - \epsilon^2 \overline{H} \int_{\mathbb{R}^2} \big(1 - v(x)\big)\, \Lambda_{\mathrm{III,t}}(dx) \\
&\geq \epsilon \int_{\mathbb{R}^2} \big(1 - v(x)\big)\, \Lambda_{\mathrm{III,t}}(dx) - \epsilon^2 \overline{H} \left(\int_{\mathbb{R}^2} \big(1 - v(x)\big)\, \Lambda(dx) \right) \text{ a.s.}
\end{aligned}$$

The conclusion then follows directly. □

Note that given Φ_t, the p.g.fl of $\Delta_{\mathrm{III,up,t,t+\epsilon}}$, which is a PPP of intensity measure $\epsilon\Lambda_{\mathrm{III,t}}$, is

$$G_{\Delta_{\mathrm{III,up,t,t+\epsilon}}}(v) = \exp\left\{ -\epsilon \int_{\mathbb{R}^2} \big(1 - v(x)\big) \Lambda_{\mathrm{III,t}}(dx) \right\}.$$

Hence,

$$\lim_{\epsilon\to 0}\frac{G_{\Delta_{\mathrm{III,up,t,t+\epsilon}}}(v)-1}{\epsilon} = -\int_{\mathbb{R}^2}\big(1-v(x)\big)\Lambda_{\mathrm{III,t}}(dx)$$
$$= \lim_{\epsilon\to 0}\frac{G_{T_{t,t+\epsilon}\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)}(v)-1}{\epsilon}.$$

Thus, Lemma 3.2 justifies our intuition that when the time scale is small, the effect of conflict is negligible, and we can regard the thin layer $T_{t,t+\epsilon}\big(\mathcal{M}_{\mathrm{III}}(\Phi)\big)$ as a PPP. Such property, which we call the *quasi-Poisson* property, plays an important role in the subsequent studies.

Now we can proceed to the proof of Theorem 3.1, where the Λ subscript is dropped to avoid cumbersome notation. Note that for $r > s$

$$f(r,v)-f(s,v) = \mathbb{E}\left[\prod_{x\in T_s\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(x)\left(\prod_{y\in T_{s,r}\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(y)-1\right)\right].$$

In order to evaluate the last expression, we need the conditional probability

$$\mathbb{E}\left[\prod_{y\in T_{s,r}\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(y)-1\,\middle|\,\Phi_s\right]. \tag{3.2.10}$$

Put $s = t$ and $r = t+\epsilon$, by Lemma 3.2,

$$\left|\left(\mathbb{E}\left[\prod_{y\in T_{t,t+\epsilon}\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(y)\,\middle|\,\Phi_t\right]-1\right)\epsilon^{-1}+\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda_{\mathrm{III,t}}(dx)\right|$$
$$\leq \epsilon\left(2\left(\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda(dx)\right)^2+\overline{H}\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda(dx)\right)\ \text{a.s.}$$

Let $\mathfrak{C} = 2\left(\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda(dx)\right)^2+\overline{H}\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda(dx)$, we have

$$\left|\frac{f(t+\epsilon,v)-f(t,v)}{\epsilon}+\mathbb{E}\left[\prod_{y\in T_t\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(y)\left(\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda_{\mathrm{III,t}}(dx)\right)\right]\right|$$
$$\leq \epsilon\mathfrak{C}\mathbb{E}\left[\prod_{y\in T_t\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(y)\right] = \epsilon\mathfrak{C}f(t,v). \tag{3.2.11}$$

Moreover,

$$\mathbb{E}\left[\prod_{y\in T_t\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(y)\left(\int_{\mathbb{R}^2}\big(1-v(x)\big)\,\Lambda_{\mathrm{III,t}}(dx)\right)\right]$$

$$=\mathbb{E}\left[\prod_{y\in T_t\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(y)\left(\int_{\mathbb{R}^2}\big(1-v(x)\big)\left(\prod_{y\in T_t\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)}\big(1-h(x,y)\big)\right)\Lambda(dx)\right)\right]$$

$$=\mathbb{E}\left[\int_{\mathbb{R}^2}\big(1-v(x)\big)\left(\prod_{y\in T_t\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(y)\big(1-h(x,y)\big)\right)\Lambda(dx)\right]$$

$$=\mathbb{E}\left[\int_{\mathbb{R}^2}\big(1-v(x)\big)\left(\prod_{y\in T_t\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} H(v,x)(y)\right)\Lambda(dx)\right].$$

As the term inside the integration is a positive r.v., we can change the order of expectation and integration,

$$\mathbb{E}\left[\int_{\mathbb{R}^2}\big(1-v(x)\big)\prod_{y\in T_t\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} H(v,x)(y)\,\Lambda(dx)\right]$$

$$=\int_{\mathbb{R}^2}\big(1-v(x)\big)f\big(t,H(v,x)\big)\,\Lambda(dx).$$

Letting ϵ goes to 0 in (3.2.11) gives us (3.2.5). To prove (3.2.6), we put $s=t-\epsilon$ and $r=t$ in (3.2.10). Proceed as above, we obtain

$$\left|\frac{f(t,v)-f(t-\epsilon,v)}{\epsilon}+\mathbb{E}\left[\prod_{y\in T_{t-\epsilon}\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)} v(y)\left(\int_{\mathbb{R}^2}\big(1-v(x)\big)\Lambda_{\mathrm{III,t-\epsilon}}(dx)\right)\right]\right|$$

$$\le \epsilon\mathfrak{C}f(t-\epsilon,v),$$

We get (3.2.6) by letting ϵ goes to 0 and use the continuity of f in t (Proposition 3.4). □Another way to present the result just obtained above is

$$f_\Lambda(t,v)=1-\int_0^t\int_{\mathbb{R}^2} f_\Lambda\big(\tau,H(v,x)\big)\big(1-v(x)\big)\Lambda(dx)d\tau. \tag{3.2.12}$$

As the functional $f_\Lambda(t,v)$ is decreasing in t and is bounded below by 0, there must be a limit

$$0\le\lim_{t\to\infty} f_\Lambda(t,v)=1-\int_0^\infty\int_{\mathbb{R}^2} f_\Lambda\big(\tau,H(v,x)\big)\big(1-v(x)\big)\Lambda(dx)d\tau. \tag{3.2.13}$$

Hence, we have that $\lim_{t\to\infty} f_\Lambda\big(\tau, H(v,x)\big) = 0$ for all v satisfying (3.2.1). This is closely related to what is referred to as the saturated regime in the literature [34, 24]. In this regime, the space is saturated in the sense that every point must have a conflict with at least one point in the conflict-free PP a.s. So, the conflict-free PP cannot accept any more point and this is the reason why there exists the limit $\lim_{t\to\infty} f_\Lambda(t,v)$. Moreover, the probability that a point at x has no conflict with any point in $\mathcal{M}_{\mathrm{III}}(\Phi)$ is

$$\begin{aligned}\mathbb{E}\left[\prod_{x\in\mathcal{M}_{\mathrm{III}}(\Phi)}\big(1-h(x,y)\big)\right] &= \lim_{t\to\infty}\mathbb{E}\left[\prod_{x\in T_t(\mathcal{M}_{\mathrm{III}}(\Phi))}\big(1-h(x,y)\big)\right]\\ &= \lim_{t\to\infty} f_\Lambda\big(t, H(\mathbf{1},x)\big).\end{aligned}$$

In the saturated regime, we must have that $\lim_{t\to\infty} f_\Lambda\big(t, H(\mathbf{1},x)\big) = 0$, so $\lim_{t\to\infty} f_\Lambda\big(\tau, H(v,x)\big) = 0$ for all v satisfying (3.2.1) since $f_\Lambda\big(\tau, H(v,x)\big) \le f_\Lambda\big(\tau, H(\mathbf{1},x)\big)$ for all such v.

By the simple observation that the Matérn type III model of Φ_t is included in Φ_t a.s. for every t and using Theorem 3.1, we can obtain an upper bound and a lower bound of f_Λ.

Corollary 3.1 *For every $t > 0$ and every function v satisfying* (3.2.1)*,*

$$\begin{aligned}&\exp\left\{-t\int_{\mathbb{R}^2}\big(1-v(x)\big)\Lambda(dx)\right\} \le f_\Lambda(t,v)\\ &\le 1-\int_0^t\int_{\mathbb{R}^2}\frac{1-\exp\left\{-t\int_{\mathbb{R}^2}\Big(1-v(y)\big(1-h(x,y)\big)\Big)\Lambda(dy)\right\}}{\int_{\mathbb{R}^2}\Big(1-v(y)\big(1-h(x,y)\big)\Big)\Lambda(dy)}\big(1-v(x)\big)\Lambda(dx).\end{aligned} \tag{3.2.14}$$

Proof. The first inequality comes from the fact that $\mathcal{M}_{\mathrm{III}}(\Phi_t) \subseteq \Phi_t$ a.s. and from Theorem A.1. For the second inequality, we first use Theorem 3.1 to get

$$f_\Lambda(t,v) = 1-\int_0^t\int_{\mathbb{R}^2} f_\Lambda(\tau, H(v,x))\big(1-v(x)\big)\Lambda(dx)d\tau.$$

By the first inequality,

$$f_\Lambda(\tau, H(v,x)) \ge \exp\left\{-\tau\int_{\mathbb{R}^2}(1-v(y)\big(1-h(x,y)\big))\Lambda(dy)\right\}.$$

Hence,

$$\begin{aligned}&f_\Lambda(t,v)\\ &\le 1-\int_0^t\int_{\mathbb{R}^2}\exp\left\{-\tau\int_{\mathbb{R}^2}(1-v(y)\big(1-h(x,y)\big))\Lambda(dy)\right\}\big(1-v(x)\big)\Lambda(dx)d\tau.\end{aligned}$$

We then conclude by using Fubini's theorem. □

The Matérn Type II Model

To study the dynamic evolution of $T_t\big(\mathcal{M}_{\rm II}(\Phi)\big)$, we have to keep track not only on the points of itself but also on the other points in Φ_t. More concretely, we define for every $t \geq 0$ and every functions u, v taking value in $[0,1]$ satisfying (3.2.1),

$$g_\Lambda(t,u,v) = \mathbb{E}\left[\prod_{x\in\Phi_t} u(x) \prod_{x\in T_t(\mathcal{M}_{\rm II}(\Phi))} v(x)\right]. \tag{3.2.15}$$

As with f_Λ, we first show that g_Λ is continuous in t.

Proposition 3.5 *For every functions u, v satisfying (3.2.1), $g_\Lambda(t,u,v)$ is continuous in t.*

Proof. For every t and ϵ positive

$$\begin{aligned} g_\Lambda(t+\epsilon,u,v) &= \mathbb{E}\left[\prod_{x\in\Phi_{t+\epsilon}} u(x) \prod_{x\in T_{t+\epsilon}(\mathcal{M}_{\rm II}(\Phi))} v(x)\right] \\ &= \mathbb{E}\left[\prod_{x\in\Phi_t} u(x) \prod_{x\in T_t(\mathcal{M}_{\rm II}(\Phi_t))} v(x) \prod_{x\in T_{t,t+\epsilon}(\Phi)} u(x) \prod_{x\in T_{t,t+\epsilon}(\mathcal{M}_{\rm II}(\Phi))} v(x)\right]. \end{aligned}$$

By the same argument as in Proposition 3.4,

$$1 \geq \prod_{x\in T_{t,t+\epsilon}(\Phi)} u(x) \prod_{x\in T_{t,t+\epsilon}(\mathcal{M}_{\rm II}(\Phi))} v(x) \geq \prod_{x\in T_{t,t+\epsilon}(\Phi)} u(x)v(x) \text{ a.s.}$$

Thus,

$$g_\Lambda(t,v)\exp\left\{-\epsilon\int_{\mathbb{R}^2}\big(1-u(x)v(x)\big)\Lambda(dx)\right\} \leq\ g_\Lambda(t+\epsilon,u,v) \leq g_\Lambda(t,u,v).$$

Doing the same, replacing t by $t-\epsilon$ and $t+\epsilon$ by t,

$$g_\Lambda(t-\epsilon,v)\exp\left\{-\epsilon\int_{\mathbb{R}^2}\big(1-u(x)v(x)\big)\Lambda(dx)\right\} \leq\ g_\Lambda(t,u,v) \leq g_\Lambda(t-\epsilon,u,v).$$

Letting ϵ go to 0 completes this proof. $\square$

Now we have,

Theorem 3.2 *For any locally finite measure Λ, the functional g_Λ satisfies the following system of equations*

$$\begin{aligned} g_\Lambda(0,u,v) &= 1; \\ \frac{dg_\Lambda(t,u,v)}{dt} &= -\int_{\mathbb{R}^2} g_\Lambda\big(t,H(u,x),v\big)\big(u(x)-u(x)v(x)\big)\Lambda(dx) \\ &\quad -\int_{\mathbb{R}^2} g_\Lambda(t,u,v)\big(1-u(x)\big)\Lambda(dx). \end{aligned} \tag{3.2.16}$$

As with Theorem 3.1, the idea behind this theorem is to divide Φ into thin layers $T_{t,t+\epsilon}(\Phi)$ such that the effect of conflict is negligible within each layer. Then, there are two possibilities for each point x in $T_{t,t+\epsilon}(\Phi)$. The first one is that it belongs to $\mathcal{M}_{\rm II}(\Phi)$ and contributes $u(x)v(x)$ to the product inside the expectation in the definition of g_Λ. This corresponds to the first term in (3.2.16). The second possibility is that x belongs to $\Phi \setminus \mathcal{M}_{\rm II}(\Phi)$ and contributes $u(x)$ to the product. This corresponds to the second term of (3.2.16). To state this formally, we first need the quasi-Poisson property.

Lemma 3.3 *Let (Φ, C) be a PRRCR with ground intensity Λ and expected conflict function h. Then for every $t \geq 0$ and $\epsilon > 0$ small enough,*

$$\Bigg| \mathbb{E}\left[\prod_{x \in T_{t,t+\epsilon}(\Phi)} u(x) \prod_{x \in T_{t,t+\epsilon}(\mathcal{M}_{\rm II}(\Phi))} v(x) \,\Bigg|\, \Phi_t \right] - 1 + \epsilon \int_{\mathbb{R}^2} \big(1 - u(x)\big)\, \Lambda(dx)$$
$$+ \epsilon \int_{\mathbb{R}^2} \big(u(x) - u(x)v(x)\big)\, \Lambda_{\rm II,t}(dx) \Bigg|$$
$$\leq \epsilon^2 \left(2 \left(\int_{\mathbb{R}^2} \big(1 - u(x)v(x)\big)\, \Lambda(dx) \right)^2 + \overline{H} \int_{\mathbb{R}^2} \big(1 - u(x)v(x)\big)\, \Lambda(dx) \right) \; a.s. \;, \tag{3.2.17}$$

where $\overline{H}$ is defined in (3.1.9) and $\Lambda_{\rm II,t}$ is the random measure in $\mathbb{R}^2$ satisfying

$$\Lambda_{\rm II,t}(dx) = \prod_{y \in T_t\left(\mathcal{M}_{\rm II}(\Phi)\right)} \big(1 - h(x,y)\big)\, \Lambda(dx). \tag{3.2.18}$$

Proof. As in Lemma 3.2 , all expectations here are conditional expectations given Φ_t. By Lemma 3.1,

$$\prod_{x \in T_{t,t+\epsilon}(\Phi)} u(x) \prod_{x \in T_{t,t+\epsilon}(\mathcal{M}_{\rm II}(\Phi))} v(x)$$
$$= \prod_{x \in T_{t,t+\epsilon}(\Phi) \setminus T_{t,t+\epsilon}(\mathcal{M}_{\rm II}(\Phi))} u(x) \prod_{x \in T_{t,t+\epsilon}(\mathcal{M}_{\rm II}(\Phi))} u(x)v(x)$$
$$= 1 - \sum_{x \in T_{t,t+\epsilon}(\Phi) \setminus T_{t,t+\epsilon}(\mathcal{M}_{\rm II}(\Phi))} \big(1 - u(x)\big) - \sum_{x \in T_{t,t+\epsilon}(\mathcal{M}_{\rm II}(\Phi))} \big(1 - u(x)v(x)\big)$$
$$+ \sum_{k=2}^{\infty} (-1)^k \sum_{(x_1,\ldots,x_k) \in (T_{t,t+\epsilon}(\Phi))^{(k!)}} \prod_{i=1}^{k} \big(1 - \zeta(x_i)\big),$$

where $\zeta(x_i) = u(x_i)v(x_i)$ if $x_i \in \mathcal{M}_{\rm II}(\Phi)$ and $\zeta(x_i) = u(x_i)$ otherwise. Our first step is to bound

$$\mathbb{E}\left[\left| \sum_{k=2}^{\infty} (-1)^k \sum_{(x_1,\ldots,x_k) \in (T_{t,t+\epsilon}(\Phi))^{(k!)}} \prod_{i=1}^{k} \big(1 - \zeta(x_i)\big) \right| \right].$$

Note that since the value of u and v is not larger than 1, we have that $1 - u(x)v(x) \geq 1 - \zeta(x) \geq 0$ for every x in Φ. Hence,

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{k=2}^{\infty}(-1)^k \sum_{(x_1,\ldots,x_k)\in(T_{t,t+\epsilon}(\Phi))^{(k!)}} \prod_{i=1}^{k}\big(1-\zeta(x_i)\big)\right|\right] \\
&\leq \sum_{k=2}^{\infty}\mathbb{E}\left[\sum_{(x_1,\ldots,x_k)\in(T_{t,t+\epsilon}(\Phi))^{(k!)}} \prod_{i=1}^{k}\big(1-u(x_i)v(x_i)\big)\right] \\
&= \sum_{k=2}^{\infty}\left(\epsilon\int_{\mathbb{R}^2}\big(1-u(x)v(x)\big)\,\Lambda(dx)\right)^k = \epsilon^2\frac{\left(\int_{\mathbb{R}^2}\big(1-u(x)v(x)\big)\,\Lambda(dx)\right)^2}{1-\epsilon\int_{\mathbb{R}^2}\big(1-u(x)v(x)\big)\,\Lambda(dx)} \text{ a.s.,}
\end{aligned}$$

where we use multivariate Campbell formula in the third line. Taking any $\epsilon < \frac{1}{2}\left(\int_{\mathbb{R}^2}\big(1-u(x)v(x)\big)\,\Lambda(dx)\right)^{-1}$ gives us,

$$\begin{aligned}
&\mathbb{E}_t\left[\left|\sum_{k=2}^{\infty}(-1)^k \sum_{(x_1,\ldots,x_k)\in(T_{t,t+\epsilon}(\Phi))^{(k)!}} \prod_{i=1}^{k}\big(1-\zeta(x_i)\big)\right|\right] \\
&\leq 2\epsilon^2\left(\int_{\mathbb{R}^2}\big(1-u(x)v(x)\big)\,\Lambda(dx)\right)^2.
\end{aligned}$$

Next, we bound

$$\begin{aligned}
&\mathbb{E}_t\left[\sum_{x\in T_{t,t+\epsilon}(\mathcal{M}_{\mathrm{II}}(\Phi))}\big(1-u(x)v(x)\big) + \sum_{x\in T_{t,t+\epsilon}(\Phi)\setminus T_{t,t+\epsilon}(\mathcal{M}_{\mathrm{II}}(\Phi))}\big(1-u(x)\big)\right] \\
&\mathbb{E}_t\left[\sum_{x\in T_{t,t+\epsilon}(\mathcal{M}_{\mathrm{II}}(\Phi))}\big(u(x)-u(x)v(x)\big) + \sum_{x\in T_{t,t+\epsilon}(\Phi)}\big(1-u(x)\big)\right].
\end{aligned}$$

By Campbell's formula,

$$\mathbb{E}_t\left[\sum_{x\in T_{t,t+\epsilon}(\Phi)}\big(1-u(x)\big)\right] = \epsilon\int_{\mathbb{R}^2}\big(1-u(x)\big)\,\Lambda(dx).$$

By the same bounding technique as in Lemma 3.2,

$$\begin{aligned}
&\epsilon\int_{\mathbb{R}^2}\big(u(x)-u(x)v(x)\big)\,\Lambda_{\mathrm{II,t}}(dx) \geq \mathbb{E}_t\left[\sum_{x\in T_{t,t+\epsilon}(\mathcal{M}_{\mathrm{II}}(\Phi))}\big(u(x)-u(x)v(x)\big)\right] \\
&\geq \epsilon\int_{\mathbb{R}^2}\big(u(x)-u(x)v(x)\big)\,\Lambda_{\mathrm{II,t}}(dx) - \overline{H}\int_{\mathbb{R}^2}\big(1-u(x)v(x)\big)\,\Lambda(dx).
\end{aligned}$$

The conclusion follows directly. $\square$

Proof of Theorem 3.2. Given Lemma 3.3, the proof of Theorem 3.2 is just a verbatim expansion of the proof of Theorem 3.1 and is hence omitted. □

As with f_Λ, an upper and a lower bound on g_Λ are derived in the below corollary. Its proof is similar to the proof of Corollary 3.1 and is hence omitted.

Corollary 3.2 *For every $t > 0$ and every functions u, v satisfying* (3.2.1),

$$\exp\left\{-t\int_{\mathbb{R}^2}\big(1-u(x)v(x)\big)\Lambda(dx)\right\} \le g_\Lambda(t,u,v)$$

$$\le 1-\left(1-\exp\left\{-t\int_{\mathbb{R}^2}\big(1-u(x)v(x)\big)\Lambda(dx)\right\}\right)\frac{\int_{\mathbb{R}^2}\big(1-u(x)\big)\Lambda(dx)}{\int_{\mathbb{R}^2}\big(1-u(x)v(x)\big)\Lambda(dx)}-$$

$$\int_{\mathbb{R}^2}\frac{1-\exp\left\{-t\int_{\mathbb{R}^2}\Big(1-u(y)v(y)\big(1-h(x,y)\big)\Big)\Lambda(dy)\right\}}{\int_{\mathbb{R}^2}\Big(1-u(y)v(y)\big(1-h(x,y)\big)\Big)\Lambda(dy)}\big(1-v(x)\big)u(x)\Lambda(dx). \tag{3.2.19}$$

In particular,

$$\exp\left\{-t\int_{\mathbb{R}^2}\big(1-v(x)\big)\Lambda(dx)\right\} \le G_{\mathcal{M}_{\mathrm{II}}(\Phi_t)}(v)$$

$$\le 1-\int_0^t\int_{\mathbb{R}^2}\frac{1-\exp\left\{-t\int_{\mathbb{R}^2}\Big(1-v(y)\big(1-h(x,y)\big)\Big)\Lambda(dy)\right\}}{\int_{\mathbb{R}^2}\Big(1-v(y)\big(1-h(x,y)\big)\Big)\Lambda(dy)}\big(1-v(x)\big)\Lambda(dx). \tag{3.2.20}$$

3.2.2 The Reduced Palm Versions of The Probability Generating Functionals

As we have already seen in Section 2.1, in order to carry out the analysis of CSMA wireless networks, we need to compute not only the p.g.fls of the Matérn models, but also their reduced Palm versions. Thus, we derive in this subsection the integral equations that govern the evolution in t of the p.g.fls $\mathcal{M}_j(\Phi_t)$, $j = \mathrm{II},\mathrm{III}$ under their reduced Palm distributions. These p.g.fls are defined by

$$G^!_{x,\mathcal{M}_{\mathrm{II}}(\Phi_t)}(v) := \mathbb{E}^!_{x,\mathcal{M}_{\mathrm{II}}(\Phi_t)}\left[\prod_{y\in\mathcal{M}_{\mathrm{II}}(\Phi_t)}v(y)\right] = \mathbb{E}_{x,\mathcal{M}_{\mathrm{II}}(\Phi_t)}\left[\prod_{y\in\mathcal{M}_{\mathrm{II}}(\Phi_t)\setminus\{x\}}v(y)\right];$$

$$G^!_{x,\mathcal{M}_{\mathrm{III}}(\Phi_t)}(v) := \mathbb{E}^!_{x,\mathcal{M}_{\mathrm{III}}(\Phi_t)}\left[\prod_{y\in\mathcal{M}_{\mathrm{III}}(\Phi_t)}v(y)\right] = \mathbb{E}_{x,\mathcal{M}_{\mathrm{III}}(\Phi_t)}\left[\prod_{y\in\mathcal{M}_{\mathrm{III}}(\Phi_t)\setminus\{x\}}v(y)\right].$$

The Matérn Type III Model

Let

$$f_{x,\Lambda}(t,v) = G^{!}_{x,T_t\left(\mathcal{M}_{\mathrm{III}}(\Phi)\right)}(v). \tag{3.2.21}$$

The main result here is,

Proposition 3.6 *For Λ-almost every y, for every positive functions v satisfying (3.2.1) and every $t > 0$, the functional $f_{y,\Lambda}(t,v)$ satisfies the system of integral equations*

$$\begin{aligned}
&f_{y,\Lambda}(0,v) = 1;\\
&f_{y,\Lambda}(t,v) = \frac{\int_0^t f_\Lambda\big(\tau, H(v,y)\big)d\tau}{m_{\mathrm{III},\Lambda}(t,y)}\\
&\quad - \int_0^t \int_{\mathbb{R}^2} f_{y,\Lambda}\big(t, H(v,x)\big)\big(1-v(x)\big)\big(1-h(x,y)\big)\frac{m_{\mathrm{III},\Lambda}(\tau,y)}{m_{\mathrm{III},\Lambda}(t,y)}\Lambda(dx)d\tau,
\end{aligned}$$

where H is defined in (3.2.3) and

$$m_{\mathrm{III},\Lambda}(t,x) = \int_0^t f_\Lambda\big(\tau, H(1,x)\big)d\tau$$

is the Radon-Nikodym derivative w.r.t. Λ of the intensity measure of $\mathcal{M}_{\mathrm{III}}(\Phi_t)$.

Proof. As there is no ambiguity, we drop the Λ subscript in this proof. The first step is to compute the intensity measure of $\mathcal{M}_{\mathrm{III}}(\Phi_t)$. From Proposition A.5, for any bounded Borel set B,

$$\frac{d}{ds}f(t,e^{-s\mathbf{1}_B})\Big|_{s=0} = -\int_B f_x(t,1)m_{\mathrm{III}}(t,x)\,\Lambda(dx) = -\int_B m_{\mathrm{III}}(t,x)\,\Lambda(dx).$$

Moreover, by Theorem 3.1,

$$f(t,e^{-s\mathbf{1}_B}) = 1 - \int_0^t\int_{\mathbb{R}^2} f\big(\tau, H(e^{-s\mathbf{1}_B},x)\big)\left(1-e^{-s\mathbf{1}_B(x)}\right)\,\Lambda(dx)\,d\tau.$$

Hence,

$$\int_B m_{\mathrm{III}}(t,x)\Lambda(dx) = \frac{d}{ds}\int_0^t\int_{\mathbb{R}^2} f\big(\tau, H(e^{-s\mathbf{1}_B},x)\big)\left(1-e^{-s\mathbf{1}_B(x)}\right)\Lambda(dx)d\tau\Big|_{s=0}.$$

Now we want to move the derivative to the inside of the integrations. First, the conditions for this must be verified. Using Proposition A.5, we have for all s,

$$\left|\frac{d}{ds}f\big(\tau, H(e^{-s\mathbf{1}_B},x)\big)\right| = \left|\int_B f_y\big(\tau, H(e^{-s\mathbf{1}_B},x)\big)m_{\mathrm{III}}(\tau,y)\Lambda(dy)\right| \le \tau\Lambda(B),$$

since $m_{\mathrm{III}}(\tau, x) < \tau$ for all x (as the Matérn type III model is a thinning of Φ) and $0 \leq H(e^{-s\mathbf{1}_B}, x)(y) \leq 1$ for every x, y in $\mathbb{R}^2$. Hence,

$$\begin{aligned}
&\left| \frac{d}{ds} \left(f\big(\tau, H(e^{-s\mathbf{1}_B}, x)\big) \left(1 - e^{-s\mathbf{1}_B(x)}\right) \right) \right| \\
&\leq \mathbf{1}_B(x) e^{-s\mathbf{1}_B(x)} \left| f\big(\tau, H(e^{-s\mathbf{1}_B}, x)\big) \right| + \left(1 - e^{-s\mathbf{1}_B(x)}\right) \left| \frac{d}{ds} f\big(\tau, H(e^{-s\mathbf{1}_B}, x)\big) \right| \\
&\leq \mathbf{1}_B + s\mathbf{1}_B \tau \Lambda(B) \leq \mathbf{1}_B(x)\big(1 + s\tau\Lambda(B)\big).
\end{aligned}$$

In the third line, we use the inequality $1 - e^{-s\mathbf{1}_B(x)} \leq s\mathbf{1}_B(x)$. Since

$$\int_0^t \int_B \big(1 + s\tau\Lambda(B)\big)\, \Lambda(dx)\, d\tau = t\Lambda(B) + s\frac{t^2}{2}\Lambda(B)^2 < \infty,$$

we then have

$$\int_0^t \int_{\mathbb{R}^2} \left| \frac{d}{ds} \left(f\big(\tau, H(e^{-s\mathbf{1}_B}, x)\big) \left(1 - e^{-s\mathbf{1}_B(x)}\right) \right) \right| \Lambda(dx)\, d\tau < \infty.$$

So it is legitimate to change the order of the differentiation and the integrations,

$$\begin{aligned}
\int_B m(t, x)\Lambda(dx) &= \int_0^t \int_{\mathbb{R}^2} \frac{d}{ds} f\big(\tau, H(e^{-s\mathbf{1}_B}, x)\big) \left(1 - e^{-s\mathbf{1}_B(x)}\right) \Lambda(dx)\, d\tau \Big|_{s=0} \\
&= \int_0^t \int_B f\big(\tau, H(e^{-s\mathbf{1}_B}, x)\big) e^{-s\mathbf{1}_B(x)}\, \Lambda(dx)\, d\tau \Big|_{s=0} + \\
&\quad \int_0^t \int_B \frac{d}{ds} f\big(\tau, H(e^{-s\mathbf{1}_B}, x)\big) \left(1 - e^{-s\mathbf{1}_B(x)}\right) \Lambda(dx)\, d\tau \Big|_{s=0} \\
&= \int_0^t \int_B f\big(\tau, H(1, x)\big)\, \Lambda(dx)\, d\tau.
\end{aligned}$$

As this equality holds for all bounded Borel sets B, we must have

$$m_{\mathrm{III}}(t, x) = \int_0^t f\big(\tau, H(1, x)\big)\, d\tau$$

Λ-almost everywhere.

Now we compute $f_y(t, v)$. By Proposition A.5,

$$\frac{d}{ds} f(t, ve^{-s\mathbf{1}_B}) \Big|_{s=0} = -\int_B v(y) f_y(t, v) m_{\mathrm{III}}(t, y) \Lambda(dy).$$

By Theorem 3.1,

$$f(t, ve^{-s\mathbf{1}_B}) = 1 - \int_0^t \int_{\mathbb{R}^2} f\big(\tau, H(ve^{-s\mathbf{1}_B}, x)\big) \left(1 - v(x)e^{-s\mathbf{1}_B(x)}\right) \Lambda(dx)\, d\tau.$$

By the same argument as in the first step,

$$\begin{aligned}
&\int_B v(y) f_y(t,v) m_{\mathrm{III}}(t,y)\,\Lambda(dx)\\
&= \frac{d}{ds}\left(\int_0^t \int_{\mathbb{R}^2} f\big(\tau, H(ve^{-s\mathbf{1}_B},x)\big)\left(1 - v(x)e^{-s\mathbf{1}_B(x)}\right)\Lambda(dx)\,d\tau\right)\Bigg|_{s=0}\\
&= \int_0^t \int_{\mathbb{R}^2} \frac{d}{ds}\Big(f\big(\tau, H(ve^{-s\mathbf{1}_B},x)\big)\left(1 - v(x)e^{-s\mathbf{1}_B(x)}\right)\Big)\Bigg|_{s=0}\Lambda(dx)\,d\tau.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\frac{d}{ds}\Big(f\big(\tau, H(ve^{-s\mathbf{1}_B},x)\big)\left(1 - v(x)e^{-s\mathbf{1}_B(x)}\right)\Big)\Big|_{s=0}\\
&= \frac{d}{ds} f\big(\tau, H(ve^{-s\mathbf{1}_B},x)\big)\Big|_{s=0}\big(1 - v(x)\big) + \mathbf{1}_B(x)v(x)f(\tau, H(v,x)).
\end{aligned}$$

By Proposition A.5,

$$\begin{aligned}
\frac{d}{ds} f\big(\tau, H(ve^{-s\mathbf{1}_B},x)\big)\Big|_{s=0} &= \frac{d}{ds} f\big(\tau, H(v,x)e^{-s\mathbf{1}_B}\big)\Big|_{s=0}\\
&= -\int_B f_y\big(\tau, H(v,x)\big)v(y)\big(1 - h(x,y)\big)m_{\mathrm{III}}(\tau,y)\,\Lambda(dy)
\end{aligned}$$

for every x. Hence,

$$\begin{aligned}
&\int_B v(y) f_y(t,v) m_{\mathrm{III}}(t,y)\Lambda(dx)\\
&= -\int_0^t \int_{\mathbb{R}^2}\int_B f_y\big(\tau, H(v,x)\big)v(y)\big(1 - h(x,y)\big)m_{\mathrm{III}}(\tau,y)\,\Lambda(dy)\,\Lambda(dx)\,d\tau\\
&\quad + \int_0^t \int_B f\big(\tau, H(v,x)\big)v(x)\,\Lambda(dx)\,d\tau\\
&= -\int_0^t \int_{\mathbb{R}^2}\int_B f_y\big(\tau, H(v,x)\big)v(y)\big(1 - h(x,y)\big)m_{\mathrm{III}}(\tau,y)\,\Lambda(dy)\,\Lambda(dx)\,d\tau\\
&\quad + \int_0^t \int_B f\big(\tau, H(v,y)\big)v(y)\,\Lambda(dy)\,d\tau.
\end{aligned}$$

As this is true for any bounded Borel set B such that $v(x) > 0$ for all x in B, we must have

$$\begin{aligned}
f_y(t,v) &= \frac{\int_0^t f(\tau, H(v,y))d\tau}{m_{\mathrm{III}}(t,y)}\\
&\quad - \int_0^t \int_{\mathbb{R}^2} f_y\big(\tau, H(v,x)\big)\big(1 - v(x)\big)\big(1 - h(x,y)\big)\frac{m_{\mathrm{III}}(\tau,y)}{m_{\mathrm{III}}(t,y)}\Lambda(dx)d\tau
\end{aligned}$$

for Λ- almost every y such that $v(y) > 0$. For y such that $v(y) = 0$, we use Proposition A.6 to show that for any bounded Borel set B not in the support of v,

$$\frac{d}{ds} f(t, v + s\mathbf{1}_B)\Big|_{s=0} = \int_B f_y(t,v) m_{\mathrm{III}}(t,y)\Lambda(dy).$$

Again, by Theorem 3.1,

$$\begin{aligned} &f(t, ve^{-s\mathbf{1}_B}) \\ &= 1 - \int_0^t \int_{\mathbb{R}^2} f\big(\tau, H(v + s\mathbf{1}_B, x)\big)\big(1 - v(x) - s\mathbf{1}_B(x)\big)\, \Lambda(dx)\, d\tau. \end{aligned}$$

By using the same arguments as above and by noting that

$$\begin{aligned} \frac{d}{ds} f\big(\tau, H(v + s\mathbf{1}_B, x)\big)\Big|_{s=0} &= \frac{d}{ds} f\big(\tau, H(v,x) + s\mathbf{1}_B(1 - h(.,x))\big)\Big|_{s=0} \\ &= \int_B f_y\big(\tau, H(v,x)\big)\big(1 - h(x,y)\big) m_{\mathrm{III}}(\tau,y)\, \Lambda(dy), \end{aligned}$$

we have

$$\begin{aligned} f_y(t,v) = &\frac{\int_0^t f\big(\tau, H(v,y)\big) d\tau}{m_{\mathrm{III}}(t,y)} \\ &- \int_0^t \int_{\mathbb{R}^2} f_y\big(\tau, H(v,x)\big)\big(1 - v(x)\big)\big(1 - h(x,y)\big) \frac{m_{\mathrm{III}}(\tau,y)}{m_{\mathrm{III}}(t,y)} \Lambda(dx) d\tau \end{aligned}$$

for Λ- almost every x such that $v(x) = 0$. □

Matérn Type II Model

As $G_{\mathcal{M}_{\mathrm{II}}(\Phi_t)}$ is defined through the functional g_Λ, $G^!_{x,\mathcal{M}_{\mathrm{II}}(\Phi_t)}$ should be defined through the "reduced Palm version" of g_Λ, which is defined as follows. First of all, we can assume w.l.o.g. that $u(x) > 0$ for any x in $\mathbb{R}^2$ (as $G_{\mathcal{M}_{\mathrm{II}}(\Phi_t)}(v)$ is obtained by setting $u = \mathbf{1}$ in $g_\Lambda(t,u,v)$). By the same arguments as in Proposition A.5,

$$\begin{aligned} -\frac{d}{ds} g_\Lambda(t,u,ve^{-s\mathbf{1}_B})\Big|_{s=0} &= \mathbb{E}\left[\sum_{x \in \mathcal{M}_{\mathrm{II}}(\Phi_t) \cap B} \prod_{y \in \mathcal{M}_{\mathrm{II}}(\Phi_t)} v(y) \prod_{y \in \Phi_t} u(y) \right] \\ &\le \mathbb{E}\left[\sum_{x \in \mathcal{M}_{\mathrm{II}}(\Phi_t) \cap B} v(x) \right] = \int_B v(x) m_{\mathrm{II},\Lambda}(t,x)\Lambda(dx), \end{aligned}$$

where $m_{\mathrm{II},\Lambda}(t,x)$ is the Radon-Nikodym derivative of the intensity measure of $\mathcal{M}_{\mathrm{II}}(\Phi_t)$ w.r.t. Λ. So, for each u, v, we have that $-\frac{d}{ds} g_\Lambda(t,u,ve^{-s\mathbf{1}_\cdot})\big|_{s=0}$, considered as a measure in $\mathbb{R}^2$, is absolutely continuous w.r.t. $v(x)m_{\mathrm{II},\Lambda}(t,x)\Lambda(dx)$.

Hence, it admits a Radon-Nikodym derivative w.r.t. the latter, i.e.

$$-\frac{d}{ds} g_\Lambda(t,u,ve^{-s\mathbf{1}_B})\Big|_{s=0} = \int_B g_{x,\Lambda}(t,u,v)v(x)m_{\mathrm{II},\Lambda}(t,x)\Lambda(dx).$$

Now, by the same arguments as in Proposition A.6 , we can also prove that for every Borel set B not in the support of v,

$$\frac{d}{ds} g_\Lambda(t,u,v+s\mathbf{1}_B)\Big|_{s=0} = \int_B g_{x,\Lambda}(t,u,v)m_{\mathrm{II},\Lambda}(t,x)\Lambda(dx).$$

In particular, by taking $u=\mathbf{1}$, we get:

Proposition 3.7 *For every x every function v satisfying* (3.2.1)*,*

$$G^!_{x,\mathcal{M}_{\mathrm{II}}(\Phi_t)}(v) = g_{x,\Lambda}(t,\mathbf{1},v). \tag{3.2.22}$$

The next step is to derive a system of integral equation that governs the evolution in t of $g_{x,\Lambda}(t,u,v)$ in the same spirit as Proposition 3.6

Proposition 3.8 *For Λ-almost every y, every functions $u>0$, v satisfying* (3.2.1)*, the functional $g_{y,\Lambda}$ satisfies the integral equation*

$$g_{y,\Lambda}(0,u,v) = 1;$$

$$g_{y,\Lambda}(t,u,v) = \int_0^t \frac{g_\Lambda(\tau,H(u,y),v)}{m_{\mathrm{II},\Lambda}(t,y)}d\tau - \int_0^t \int_{\mathbb{R}^2} \Big(g_{y,\Lambda}(\tau,u,v)\big(1-u(x)\big) + g_{y,\Lambda}(\tau,H(u,x),v)u(x)\big(1-v(x)\big)\big(1-h(x,y)\big)\Big) \frac{m_{\mathrm{II},\Lambda}(\tau,y)}{m_{\mathrm{II},\Lambda}(t,y)}\Lambda(dx)d\tau,$$

where H is defined in (3.2.3) and

$$m_{\mathrm{II},\Lambda}(t,x) = \frac{1-e^{-t\int_{\mathbb{R}^2} h(x,y)\Lambda(dy)}}{\int_{\mathbb{R}^2} h(x,y)\Lambda(dy)}$$

is the Radon-Nikodym derivative w.r.t. Λ of the intensity measure of $\mathcal{M}_{\mathrm{II}}(\Phi_t)$.

Proof. We present here only a sketch of this proof. To complete this sketch, it is sufficient to prove the interchanging of derivatives and integrations in the same way as in the proof of Proposition 3.6. We start with computing $m_{\mathrm{II},\Lambda}$. By Proposition A.5, for every bounded Borel set B,

$$\begin{aligned}
\frac{d}{ds} g_\Lambda(t,1,e^{-s\mathbf{1}_B})\Big|_{s=0} &= \frac{d}{ds} G_{\mathcal{M}_{\mathrm{II}}(\Phi_t)}(e^{-s\mathbf{1}_B})\Big|_{s=0} \\
&= -\int_B G^!_{x,\mathcal{M}_{\mathrm{II}}(\Phi_t)}(\mathbf{1})m_{\mathrm{II},\Lambda}(t,x)\Lambda(dx) \\
&= -\int_B m_{\mathrm{II},\Lambda}(t,x)\Lambda(dx).
\end{aligned}$$

By Theorem 3.2,

$$g_\Lambda(t,1,e^{-s\mathbf{1}_B}) = 1 - \int_0^1 \int_{\mathbb{R}^2} g_\Lambda(\tau, H(\mathbf{1},x), e^{-s\mathbf{1}_B})(1-e^{-s\mathbf{1}_B(x)})\Lambda(dx)d\tau.$$

So,

$$\begin{aligned}
&\int_B m_{\mathrm{II},\Lambda}(t,x)\Lambda(dx) \\
&= \frac{d}{ds}\int_0^1 \int_{\mathbb{R}^2} g_\Lambda\left(\tau, H(\mathbf{1},x), e^{-s\mathbf{1}_B}\right)\left(1-e^{-s\mathbf{1}_B(x)}\right)\Lambda(dx)d\tau\Bigg|_{s=0} \\
&= \int_0^1 \int_{\mathbb{R}^2} \frac{d}{ds} g_\Lambda\left(\tau, H(\mathbf{1},x), e^{-s\mathbf{1}_B}\right)\left(1-e^{-s\mathbf{1}_B(x)}\right)\Bigg|_{s=0}\Lambda(dx)d\tau \\
&= \int_0^1 \int_{\mathbb{R}^2} \frac{d}{ds} g_\Lambda\left(\tau, H(\mathbf{1},x), e^{-s\mathbf{1}_B}\right)\Bigg|_{s=0}\left(1-e^{-s\mathbf{1}_B(x)}\right)\Bigg|_{s=0}\Lambda(dx)d\tau \\
&\quad + \int_0^1 \int_{\mathbb{R}^2} g_\Lambda\left(\tau, H(\mathbf{1},x), e^{-s\mathbf{1}_B}\right)\Bigg|_{s=0}\frac{d}{ds}\left(1-e^{-s\mathbf{1}_B(x)}\right)\Bigg|_{s=0}\Lambda(dx)d\tau.
\end{aligned}$$

As $\left(1-e^{-s\mathbf{1}_B(x)}\right)|_{s=0} = 0$, the first term is zero. The second term is

$$\int_0^1 \int_{\mathbb{R}^2} g_\Lambda(\tau, H(\mathbf{1},x), \mathbf{1})\mathbf{1}_B(x)\Lambda(dx)d\tau = \int_0^1 \int_B g_\Lambda(\tau, H(\mathbf{1},x), \mathbf{1})\Lambda(dx)d\tau.$$

Hence,

$$\int_B m_{\mathrm{II},\Lambda}(t,x)\Lambda(dx) = \int_0^1 \int_B g_\Lambda(\tau, H(\mathbf{1},x), \mathbf{1})\Lambda(dx)d\tau,$$

for every bounded Borel B, which implies that

$$m_{\mathrm{II},\Lambda}(t,x) = \int_0^1 g_\Lambda(\tau, H(\mathbf{1},x), \mathbf{1})d\tau$$

for Λ-almost every x. The closed form expression of $m_{\mathrm{II},\Lambda}$ is obtained by noting that

$$g_\Lambda(\tau, H(\mathbf{1},x), \mathbf{1}) = G_{\Phi_t}(H(\mathbf{1},x)) = \exp\left\{-t\int_{\mathbb{R}^2} h(x,y)\Lambda(dy)\right\}.$$

Now we compute the $g_{y,\Lambda}$ functionals. By definition, for any bounded Borel B in the support of v,

$$\frac{d}{ds} g_\Lambda(t,u,ve^{-s\mathbf{1}_B})\Bigg|_{s=0} = -\int_B g_{x,\Lambda}(t,u,v)u(x)v(x)m_{\mathrm{II},\Lambda}(t,x)\Lambda(dx).$$

By Theorem 3.2,

$$g_\Lambda(t,u,ve^{-s\mathbf{1}_B}) = 1 - \int_0^t \int_{\mathbb{R}^2} g_\Lambda\left(\tau,u,ve^{-s\mathbf{1}_B}\right)\big(1-u(x)\big)\Lambda(dx)d\tau - \int_0^t \int_{\mathbb{R}^2} g_\Lambda\left(\tau,H(u,x),ve^{-s\mathbf{1}_B}\right)u(x)\left(1-v(x)e^{-s\mathbf{1}_B(x)}\right)\Lambda(dx)d\tau.$$

So,

$$\begin{aligned}
&\int_B g_{x,\Lambda}(t,u,v)v(x)m_{\mathrm{II},\Lambda}(t,x)\Lambda(dx)\\
&= \frac{d}{ds}\int_0^t \int_{\mathbb{R}^2} g_\Lambda\left(\tau,u,ve^{-s\mathbf{1}_B}\right)\big(1-u(x)\big)\Lambda(dx)d\tau\Big|_{s=0}\\
&\quad + \frac{d}{ds}\int_0^t \int_{\mathbb{R}^2} g_\Lambda\left(\tau,H(u,x),ve^{-s\mathbf{1}_B}\right)u(x)\left(1-v(x)e^{-s\mathbf{1}_B(x)}\right)\Lambda(dx)d\tau\Big|_{s=0}.
\end{aligned}$$

The first term is computed as

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^2} \frac{d}{ds} g_\Lambda\left(\tau,u,ve^{-s\mathbf{1}_B}\right)\Big|_{s=0}\big(1-u(x)\big)\Lambda(dx)d\tau\\
&= -\int_0^t \int_{\mathbb{R}^2}\left(\int_B g_{y,\Lambda}(\tau,u,v)u(y)v(y)m_{\mathrm{II},\Lambda}(\tau,y)\Lambda(dy)\right)\big(1-u(x)\big)\Lambda(dx)d\tau\\
&= -\int_B\left(\int_0^t \int_{\mathbb{R}^2} g_{y,\Lambda}(\tau,u,v)\big(1-u(x)\big)\Lambda(dx)d\tau\right)u(y)v(y)m_{\mathrm{II},\Lambda}(\tau,y)\Lambda(dy).
\end{aligned} \tag{3.2.23}$$

For the second term,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^2} \frac{d}{ds} g_\Lambda\left(\tau,H(u,x),ve^{-s\mathbf{1}_B}\right)u(x)\left(1-v(x)e^{-s\mathbf{1}_B(x)}\right)\Big|_{s=0}\Lambda(dx)d\tau\\
&= \int_0^t \int_{\mathbb{R}^2} \frac{d}{ds} g_\Lambda\left(\tau,H(u,x),ve^{-s\mathbf{1}_B}\right)\Big|_{s=0}u(x)\left(1-v(x)e^{-s\mathbf{1}_B(x)}\right)\Big|_{s=0}\Lambda(dx)d\tau\\
&\quad + \int_0^t \int_{\mathbb{R}^2} g_\Lambda\left(\tau,H(u,x),ve^{-s\mathbf{1}_B}\right)\Big|_{s=0}u(x)\frac{d}{ds}\left(1-v(x)e^{-s\mathbf{1}_B(x)}\right)\Big|_{s=0}\Lambda(dx)d\tau.
\end{aligned}$$

The second term in the above equality is

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^2} g_\Lambda\left(\tau,H(u,x),ve^{-s\mathbf{1}_B}\right)\Big|_{s=0}u(x)\frac{d}{ds}\left(1-v(x)e^{-s\mathbf{1}_B(x)}\right)\Big|_{s=0}\Lambda(dx)d\tau\\
&= \int_0^t \int_{\mathbb{R}^2} g_\Lambda(\tau,H(u,x),v)u(x)v(x)\mathbf{1}_B(x)\Lambda(dx)d\tau\\
&= \int_B \int_0^t g_\Lambda(\tau,H(u,y),v)u(y)v(y)d\tau\Lambda(dy),
\end{aligned} \tag{3.2.24}$$

while the first term is

$$\int_0^t \int_{\mathbb{R}^2} \frac{d}{ds} g_\Lambda\left(t, H(u,x), ve^{-s\mathbf{1}_B}\right)\Big|_{s=0} u(x)\left(1 - v(x)e^{-s\mathbf{1}_B(x)}\right)\Big|_{s=0} \Lambda(dx)d\tau$$

$$= -\int_0^t \int_{\mathbb{R}^2} \int_B g_{y,\Lambda}(\tau, H(u,x), v) H(u,x)(y) v(y) m_{\mathrm{II},\Lambda}(\tau,y)\Lambda(dy) u(x)\big(1 - v(x)\big)$$
$$\Lambda(dx)d\tau$$

$$= -\int_0^t \int_{\mathbb{R}^2} \int_B g_{y,\Lambda}(\tau, H(u,x), v) u(y)\big(1 - h(x,y)\big) v(y) m_{\mathrm{II},\Lambda}(\tau,y)\Lambda(dy) u(x)$$
$$\big(1 - v(x)\big)\Lambda(dx)d\tau$$

$$= -\int_B \left(\int_0^t \int_{\mathbb{R}^2} g_{y,\Lambda}(\tau, H(u,x), v) u(x)\big(1 - v(x)\big)\big(1 - h(x,y)\big)\Lambda(dx)d\tau\right) u(y)$$
$$v(y) m_{\mathrm{II},\Lambda}(\tau,y)\Lambda(dy). \tag{3.2.25}$$

Putting together (3.2.23), (3.2.24) and (3.2.25), we get

$$\int_B g_{y,\Lambda}(t,u,v)u(y)v(y)m_{\mathrm{II},\Lambda}(t,y)\Lambda(dy) = \int_B \int_0^t g_\Lambda(\tau, H(u,y), v)u(y)v(y)d\tau\Lambda(dy)$$
$$-\int_B \left(\int_0^t \int_{\mathbb{R}^2} g_{y,\Lambda}(\tau,u,v)\big(1-u(x)\big)\Lambda(dx)d\tau\right) u(y)v(y)m_{\mathrm{II},\Lambda}(\tau,y)\Lambda(dy) - \int_B \Bigg(\int_0^t$$
$$\int_{\mathbb{R}^2} g_{y,\Lambda}(\tau, H(u,x), v)u(x)\big(1-v(x)\big)\big(1-h(x,y)\big)\Lambda(dx)d\tau\Bigg) u(y)v(y)m_{\mathrm{II},\Lambda}(\tau,y)\Lambda(dy)$$

for every bounded Borel B in the support of v, which proves the proposition for Λ-almost every x such that $v(x) > 0$. For x such that $v(x) = 0$, proceed as above and use the equality

$$\frac{d}{ds} g_\Lambda(t,u,v+s\mathbf{1}_B)\Big|_{s=0} = \int_B g_{x,\Lambda}(t,u,v) m_{\mathrm{II},\Lambda}(t,x)\Lambda(dx).$$

for every bounded Borel B not in the support of v. □

Remark 3.6 *The closed form expression of $m_{\mathrm{II},\Lambda}$ is not new, see for example [6, p. 71] and [27, p. 164]. Nevertheless, in previous computations, it is implicitly assumed that the reduced Palm distribution of $\mathcal{M}_{\mathrm{II}}(\Phi_t)$ is a conditional probability distribution of the reduced Palm distribution of Φ_t. An instance of such computation can be founded, for example, in Subsection 2.1.3. As there is no known construction of the reduced Palm distributions of $\mathcal{M}_{\mathrm{II}}(\Phi_t)$ and Φ_t in the same probability space in such a way that the former is a conditional probability of the latter, all the previous computations of $m_{\mathrm{II},\Lambda}$ are, strictly speaking, not rigorous. Our derivation of $m_{\mathrm{II},\Lambda}$ is based on the p.g.fls of $\mathcal{M}_{\mathrm{II}}(\Phi_t)$ and does not require the aforementioned assumption. Note that following this direction, we can also compute other interesting characteristics of $\mathcal{M}_{\mathrm{II}}(\Phi_t)$ such as the* *second moment measure, the reduced second moment measure, the K measure, etc. See [27, Section 4.3] for the definitions of these characteristics. We show below a result of this kind where the intensity measures under the reduced Palm distributions of $\mathcal{M}_{\mathrm{II}}(\Phi_t)$ is computed. This result is used in Subsection 2.1.3 to approximate the process of active terminals in a CSMA wireless network.*

Corollary 3.3 *The the Radon-Nikodym derivative w.r.t. Λ of the intensity measure of $\mathcal{M}_{\mathrm{II}}(\Phi_t)$ under its reduced Palm measure given a point at y is*

$$m_{y,\mathrm{II},\Lambda}(t,x) = \int_0^t g_{x,\Lambda}(\tau, H(\mathbf{1},y),\mathbf{1})\big(1-h(x,y)\big)\frac{m_{\mathrm{II},\Lambda}(\tau,dx)}{m_{\mathrm{II},\Lambda}(t,dy)}d\tau + \int_0^t g_{y,\Lambda}(\tau, H(\mathbf{1},x),\mathbf{1})\big(1-h(x,y)\big)\frac{m_{\mathrm{II},\Lambda}(\tau,dy)}{m_{\mathrm{II},\Lambda}(t,dy)}d\tau, \quad (3.2.26)$$

where

$$g_{y,\Lambda}(\tau, H(\mathbf{1},x),\mathbf{1}) = \frac{e^{-\tau\int_{\mathbb{R}^2} h(y,z)\Lambda(dz)} - e^{-\tau\int_{\mathbb{R}^2}\left(h(x,z)+h(y,z)-h(x,z)h(y,z)\right)\Lambda(dz)}}{m_{\mathrm{II},\Lambda}(\tau)\left(\int_{\mathbb{R}^2}\big(h(x,z)-h(x,z)h(y,z)\big)\Lambda(dz)\right)}.$$

Proof. We first derive (3.2.26). Consider any y in $\mathbb{R}^2$. By Proposition A.5, for every bounded Borel B,

$$\frac{d}{ds}g_{y,\Lambda}\left(t,\mathbf{1},e^{-s\mathbf{1}_B}\right)\Big|_{s=0} = \frac{d}{ds}G^!_{y,\mathcal{M}_{\mathrm{II}}(\Phi_t)}\left(e^{-s\mathbf{1}_B}\right)\Big|_{s=0} = -\int_B m_{y,\mathrm{II},\Lambda}(t,x)\Lambda(dx).$$

On the other hand, by Proposition 3.8,

$$g_{y,\Lambda}(t,\mathbf{1},e^{-s\mathbf{1}_B}) = \int_0^t \frac{g_\Lambda\left(\tau,H(\mathbf{1},y),e^{-s\mathbf{1}_B}\right)}{m_{\mathrm{II},\Lambda}(t,y)}d\tau$$
$$-\int_0^t\int_{\mathbb{R}^2} g_{y,\Lambda}\left(\tau,H(\mathbf{1},x),e^{-s\mathbf{1}_B}\right)\left(1-e^{-s\mathbf{1}_B}\right)\big(1-h(x,y)\big)\frac{m_{\mathrm{II},\Lambda}(\tau,y)}{m_{\mathrm{II},\Lambda}(t,y)}\Lambda(dx)d\tau.$$

Since

$$\frac{d}{ds}\int_0^t \frac{g_\Lambda\big(\tau,H(\mathbf{1},y),e^{-s\mathbf{1}_B}\big)}{m_{\mathrm{II},\Lambda}(t,y)}d\tau\Big|_{s=0} = \int_0^t \frac{\frac{d}{ds}g_\Lambda\big(\tau,H(\mathbf{1},y),e^{-s\mathbf{1}_B}\big)|_{s=0}}{m_{\mathrm{II},\Lambda}(t,y)}d\tau$$
$$= \int_0^t \frac{-\int_B g_{x,\Lambda}\big(\tau,H(\mathbf{1},y),\mathbf{1}\big)\big(1-h(x,y)\big)m_{\mathrm{II},\Lambda}(\tau,x)\Lambda(dx)}{m_{\mathrm{II},\Lambda}(t,y)}$$
$$= -\int_B\int_0^t g_{x,\Lambda}\big(\tau,H(\mathbf{1},y),\mathbf{1}\big)\big(1-h(x,y)\big)\frac{m_{\mathrm{II},\Lambda}(\tau,x)}{m_{\mathrm{II},\Lambda}(t,y)}\Lambda(dx),$$

and

$$\frac{d}{ds}\int_0^t\int_{\mathbb{R}^2} g_{y,\Lambda}\big(\tau,H(\mathbf{1},x),e^{-s\mathbf{1}_B}\big)\big(1-e^{-s\mathbf{1}_B}\big)\big(1-h(x,y)\big)\frac{m_{\mathrm{II},\Lambda}(\tau,y)}{m_{\mathrm{II},\Lambda}(t,y)}\Lambda(dx)d\tau\Big|_{s=0}$$
$$= \int_0^t\int_{\mathbb{R}^2} g_{y,\Lambda}\big(\tau,H(\mathbf{1},x),\mathbf{1}\big)\mathbf{1}_B(x)\big(1-h(x,y)\big)\frac{m_{\mathrm{II},\Lambda}(\tau,y)}{m_{\mathrm{II},\Lambda}(t,y)}\Lambda(dx)d\tau,$$

we get (3.2.26) directly.

To compute $g_{y,\Lambda}\big(t, H(\mathbf{1}, x), \mathbf{1}\big)$, we first use Proposition 3.8,

$$\begin{aligned} g_{y,\Lambda}\big(t, H(\mathbf{1}, x), \mathbf{1}\big) = &\int_0^t \frac{g_\Lambda\big(\tau, H(H(\mathbf{1}, x), y), \mathbf{1}\big)}{m_{\mathrm{II},\Lambda}(t, y)} d\tau \\ &- \int_0^t \int_{\mathbb{R}^2} g_{y,\Lambda}\big(t, H(\mathbf{1}, x), \mathbf{1}\big) h(x, y) \frac{m_{\mathrm{II},\Lambda}(\tau, y)}{m_{\mathrm{II},\Lambda}(t, y)} \Lambda(dx) d\tau. \end{aligned}$$

Putting $\mathfrak{g}(t) = g_{y,\Lambda}\Big(t, H\big(\mathbf{1}, x\big), \mathbf{1}\Big) m_{\mathrm{II},\Lambda}(t, y)$, the above equation is rewritten as

$$\begin{aligned} \mathfrak{g}(t) &= \int_0^t g_\Lambda\Big(\tau, H\big(H(\mathbf{1}, x), y\big), \mathbf{1}\Big) d\tau - \int_0^t \int_{\mathbb{R}^2} \mathfrak{g}(\tau) h(x, y) \Lambda(dx) d\tau \\ &= \int_0^t g_\Lambda\Big(\tau, H\big(H(\mathbf{1}, x), y\big), \mathbf{1}\Big) d\tau - \int_0^t \left(\int_{\mathbb{R}^2} h(x, y) \Lambda(dx)\right) \mathfrak{g}(\tau) d\tau, \end{aligned}$$

which is equivalent to the differential equation

$$\frac{d}{dt}\mathfrak{g}(t) = g_\Lambda g_\Lambda\Big(\tau, H\big(H(\mathbf{1}, x), y\big), \mathbf{1}\Big) - \left(\int_{\mathbb{R}^2} h(x, y) \Lambda(dx)\right) \mathfrak{g}(t).$$

As the only solution for the equation $df(t) = a(t) - bf(t)$, where a and f are functions in t and b is a scalar, is $f(t) = e^{-tb} \int_0^t a(\tau) e^{\tau b}$, we get

$$\mathfrak{g}(t) = e^{-t \int_{\mathbb{R}^2} h(x,y)\Lambda(dx)} \int_0^t g_\Lambda(\tau, H(H(\mathbf{1}, x), y), \mathbf{1}) e^{\tau \int_{\mathbb{R}^2} h(x,y)\Lambda(dx)}.$$

The conclusion follows by noting that

$$\begin{aligned} g_\Lambda(t, H(H(\mathbf{1}, x), y), \mathbf{1}) &= G_{\Phi_t}(H(H(\mathbf{1}, x), y)) \\ &= e^{-t \int_{\mathbb{R}^2} \big(h(x,z) + h(y,z) - h(x,z)h(y,z)\big) \Lambda(dz)}. \end{aligned}$$

□

3.2.3 Solutions of The Differential Equations

We now prove the converses of Theorems 3.1 and 3.2. More concretely, we show that $f_\Lambda(t, v)$ and $g_\Lambda(t, u, v)$ are the unique solutions of the systems of equations of the forms (3.2.4) and (3.2.16), respectively.

The Matérn Type III Model

We start with the converse of Theorem 3.1.

Proposition 3.9 *There is a unique functional $f(t, v)$ taking value on $[0, 1]$ satisfying system of equations* (3.2.4).

Proof. Consider a functional f taking value in $[0,1]$ and satisfies (3.2.4). We first show that f is infinitely differentiable in t and

$$\left|\frac{d^n}{d^n t} f(t,v)\right| \le \prod_{i=0}^{n-1} \left(\int_{\mathbb{R}^2} \big(1-v(x)\big)\,\Lambda(dx) + i\overline{H} \right) \tag{3.2.27}$$

for every v satisfying (3.2.1) where $\overline{H}$ is defined in (3.1.9).

For $n=1$, since f satisfies (3.2.4),

$$\frac{d}{dt} f(t,v) = -\int_{\mathbb{R}^2} f\big(t, H(v,x)\big)\big(1-v(x)\big)\,\Lambda(dx).$$

Hence,

$$\left|\frac{d}{dt} f(t,v)\right| \le \int_{\mathbb{R}^2} \big(1-v(x)\big)\,\Lambda(dx).$$

Now suppose that the claim holds for some $n \ge 1$. Consider

$$\begin{aligned}
\frac{d^{n+1}}{d^{n+1}t} f(t,v) &= \frac{d^n}{d^n t}\left(\frac{d}{dt} f(t,v)\right) \\
&= -\frac{d^n}{d^n t}\left(\int_{\mathbb{R}^2} f\big(t, H(v,x)\big)\big(1-v(x)\big)\,\Lambda(dx)\right).
\end{aligned}$$

By the induction hypothesis, $\frac{d^n}{d^n t} f\big(t, H(v,x)\big)$ exists and

$$\left|\frac{d^n}{d^n t} f\big(t, H(v,x)\big)\right| \le \prod_{i=0}^{n-1}\left(\int_{\mathbb{R}^2} \big(1-H(v,x)(y)\big)\,\Lambda(dy) + i\overline{H}\right).$$

Since

$$\begin{aligned}
\int_{\mathbb{R}^2} \big(1-H(v,x)(y)\big)\,\Lambda(dy) &= \int_{\mathbb{R}^2} \Big(1-\big(1-h(x,y)\big)v(y)\Big)\,\Lambda(dy) \\
&\le \int_{\mathbb{R}^2} \Big(\big(1-v(y)\big) + h(x,y)v(y)\Big)\,\Lambda(dy) \\
&\le \int_{\mathbb{R}^2} \big(1-v(y)\big)\,\Lambda(dy) + \overline{H},
\end{aligned}$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left|\frac{d^n}{d^n t} f\big(t, H(v,x)\big)\right| \big(1-v(x)\big)\,\Lambda(dx) \\
&\le \int_{\mathbb{R}^2} \prod_{i=1}^{n} \left(\int_{\mathbb{R}^2} \big(1-v(x)\big)\,\Lambda(dy) + i\overline{H}\right) \big(1-v(x)\big)\,\Lambda(dx) \\
&= \prod_{i=0}^{n}\left(\int_{\mathbb{R}^2} |1-v(x)|\,\Lambda(dx) + i\overline{H}\right) < \infty.
\end{aligned}$$

Hence, $\frac{d^{n+1}}{d^{n+1}t}f(t,v)$ exists and satisfies (3.2.27).

Now, let

$$\mathcal{T}_n(\tau,t,v) = f(t,v) + \sum_{i=1}^{n} \frac{(\tau-t)^i}{i!}\frac{d^i}{d^it}f(t,v).$$

By Taylor's theorem,

$$f_\Lambda(\tau,v) = \mathcal{T}_n(\tau,t,v) + \frac{\frac{d^{n+1}}{d^{n+1}t}f_\Lambda(t,v)|_{t=\xi}}{(n+1)!}(\tau-t)^{n+1}$$

for some ξ in $[t,\tau]$. For every $\tau \in [t, t+\left(\overline{H}\right)^{-1})$, we need to prove that $\lim_{n\to\infty}\mathcal{T}_n(\tau,t,v) = f(\tau,v)$. For this, it is sufficient to show that for all ξ in $[t,t+\left(\overline{H}\right)^{-1})$,

$$\lim_{n\to\infty}\left|\frac{\frac{d^{n+1}}{d^{n+1}t}f_\Lambda(t,v)|_{t=\xi}}{(n+1)!}(\tau-t)^{n+1}\right| = 0.$$

By (3.2.27),

$$\left|\frac{\frac{d^{n+1}}{d^{n+1}t}f_\Lambda(t,v)|_{t=\xi}}{(n+1)!}(\tau-t)^{n+1}\right| \le \prod_{i=0}^{n}\left(i\overline{H} + \int_{\mathbb{R}^2}|1-v(x)|\,\Lambda(dx)\right)\frac{(\tau-t)^{n+1}}{(n+1)!}.$$

Put $W = \left\lceil \frac{\int_{\mathbb{R}^2}|1-v(x)|\,\Lambda(dx)}{\overline{H}}\right\rceil$. As

$$\lim_{n\to\infty}\frac{\prod_{i=0}^{n}\left(i\overline{H} + \int_{\mathbb{R}^2}\left(1-v(x)\right)\Lambda(dx)\right)}{\overline{H}^{n+1}(n+1)!} \le (n+W)^W,$$

we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{\frac{d^{n+1}}{d^{n+1}t}f_\Lambda(t,v)|_{t=\xi}}{(n+1)!}(\tau-t)^{n+1}\right| &\le \lim_{n\to\infty}(n+W)^W\left((\tau-t)\overline{H}\right)^{n+1}\\ &= 0,\end{aligned}$$

since $(\tau-t)\overline{H} < 1$ and W does not depend on n.

Using the above result, we first show that for all $t \in [0, \frac{1}{2}\left(\overline{H}\right)^{-1}]$, f is uniquely determine as $\lim_{n\to\infty}\mathcal{T}_n(0,t,v)$. Then, for all $t \in [\frac{1}{2}\left(\overline{H}\right)^{-1}, \left(\overline{H}\right)^{-1}]$, f is uniquely determined as $\lim_{n\to\infty}\mathcal{T}_n\left(\frac{1}{2}\left(\overline{H}\right)^{-1}, t, v\right)$, and so on. $\square$

The Matérn Type II Model

The converse of Theorem 3.2 is stated in a similar manner.

Proposition 3.10 *There is a unique functional $g(t,u,v)$ taking value in $[0,1]$ satisfying system of equations* (3.2.16)*.*

Proof. The argument is similar to that in the proof of Proposition 3.9. We first show that g is infinitely differentiable and that

$$\left|\frac{d^n}{d^n t} g(t,u,v)\right| \leq \prod_{i=0}^{n}\left(\int_{\mathbb{R}^2}\big(1-u(x)v(x)\big)\Lambda(dx) + i\overline{H}\right),$$

with $\overline{H}$ is defined in (3.1.9). Then we show that the series

$$g(t,u,v) + \sum_{i=1}^{\infty}\frac{(\tau-t)^i}{i!}\frac{d^n}{d^n t} g(t,u,v)$$

converges to $g(\tau,u,v)$ for every $t>0$ and every τ in $[t, t+\big(\overline{H}\big)^{-1})$. This implies that $g(t,u,v)$ is uniquely determined for every $t>0$. $\square$

3.3 The Stationary Case

In the theory of PPs, the stationary PPs represent an important special case. Not only that they are simpler to study in many aspects, but also there are a wide range of applications for this kind of PPs as many phenomenons and objects in the nature are distributed in an even manner. In particular, in all the networks consider in Chapters 1 and 2 (except the Cognitive S/A model), the PPs representing their terminals locations are stationary. This is the reason why we systematically gather here the stationary versions of the results developed in the previous sections. We start by giving a sufficient condition for $\mathcal{M}_{\mathrm{II}}(\Phi,C)$ and $\mathcal{M}_{\mathrm{III}}(\Phi,C)$ to be stationary.

Proposition 3.11 *Let (Φ,C) be a PPPRCR of ground intensity Λ and expected conflict function h. $\mathcal{M}_{\mathrm{II}}(\Phi,C)$ and $\mathcal{M}_{\mathrm{III}}(\Phi,C)$ are stationary if (a) $\Lambda=\lambda\mathcal{L}$ for some positive λ where $\mathcal{L}$ is Lebesgue measure and (b) $h(x,y)=h(o,y-x)$ for all x in $\mathbb{R}^2$ where o is the center of $\mathbb{R}^2$.*

Proof. Using Lemma, it is sufficient to show that for every function v, $G_{\mathcal{M}_{\mathrm{II}}(\Phi,C)}(v) = G_{\mathcal{M}_{\mathrm{II}}(\Phi,C)}(v_x)$ and $G_{\mathcal{M}_{\mathrm{III}}(\Phi,C)}(v) = G_{\mathcal{M}_{\mathrm{III}}(\Phi,C)}(v_x)$ for every x in $\mathbb{R}^2$, where v_x is the function defined as $v_x(.) = v(.+x)$. By (a), we have $G_{\mathcal{M}_{\mathrm{II}}(\Phi,C)}(v) = g_{\mathcal{L}}(\lambda,\mathbf{1},v)$ and $G_{\mathcal{M}_{\mathrm{III}}(\Phi,C)}(v) = f_{\mathcal{L}}(\lambda,v)$. Hence, we need to prove that $g_{\mathcal{L}}(\lambda,\mathbf{1},v) = g_{\mathcal{L}}(\lambda,\mathbf{1},v_x)$ and $f_{\mathcal{L}}(\lambda,v) = f_{\mathcal{L}}(\lambda,v_x)$. By Theorems 3.1 and 3.2, the systems of differential equations corresponding to $f_{\mathcal{L}}(t,v)$ and $g_{\mathcal{L}}(t,u,v)$ are

$$\begin{aligned} f_{\mathcal{L}}(0,v) &= 1;\\ \frac{d}{dt} f_{\mathcal{L}}(t,v) &= -\int_{\mathbb{R}^2} f_{\mathcal{L}}(t,H(v,y))(1-v(y))dy, \end{aligned}$$

and

$$g_{\mathcal{L}}(0,u,v) = 1;$$
$$\frac{d}{dt}g_{\mathcal{L}}(t,u,v) = -\int_{\mathbb{R}^2}\Big(g_{\mathcal{L}}(t,u,v)\big(1-u(y)\big) + g_{\mathcal{L}}\big(t,H(u,y),v\big)u(y)\big(1-v(y)\big)\Big)dy,$$

respectively. Replacing u, v in the above systems of equations by u_x, v_x, we have

$$\begin{aligned} f_{\mathcal{L}}(0,v_x) &= 1; \\ \frac{d}{dt}f_{\mathcal{L}}(t,v_x) &= -\int_{\mathbb{R}^2} f_{\mathcal{L}}\big(t,H(v_x,y)\big)\big(1-v_x(y)\big)dy \\ &= -\int_{\mathbb{R}^2} f_{\mathcal{L}}\big(t,H(v_x,y)\big)\big(1-v(x+y)\big)dy, \end{aligned}$$

and

$$g_{\mathcal{L}}(0,u_x,v_x) = 1;$$
$$\begin{aligned} &\frac{d}{dt}g_{\mathcal{L}}(t,u_x,v_x) \\ &= -\int_{\mathbb{R}^2}\Big(g_{\mathcal{L}}\big(t,u_x,v_x\big)\big(1-u(y)\big) + g_{\mathcal{L}}\big(t,H(u_x,y),v_x\big)u_x(y)\big(1-v_x(y)\big)\Big)dy \\ &= -\int_{\mathbb{R}^2}\Big(g_{\mathcal{L}}\big(t,u_x,v_x\big)\big(1-u(x+y)\big) + g_{\mathcal{L}}\big(t,H(u_x,y),v_x\big)u(x+y)\big(1-v(x+y)\big)\Big)dy, \end{aligned}$$

respectively. By (b), $H(v_x,y-x)(z) = \big(1-h(y-x,z)\big)v_x(z) = \big(1-h(y,z+x)\big)v(x+z) = H(v,y)(x+z)$. Thus, by changing the variable from y to $y-x$, the above systems of equation is rewritten as

$$\begin{aligned} f_{\mathcal{L}}(0,v_x) &= 1; \\ \frac{d}{dt}f_{\mathcal{L}}(t,v_x) &= -\int_{\mathbb{R}^2} f_{\mathcal{L}}\big(t,H(v,y)_x\big)\big(1-v(y)\big)dy, \end{aligned}$$

and

$$g_{\mathcal{L}}(0,u_x,v_x) = 1;$$
$$\begin{aligned} &\frac{d}{dt}g_{\mathcal{L}}(t,u_x,v_x) \\ &= -\int_{\mathbb{R}^2}\Big(g_{\mathcal{L}}(t,u_x,v_x)\big(1-u(y)\big) + g_{\mathcal{L}}\big(t,H(u,y)_x,v_x\big)u(y)\big(1-v(y)\big)\Big)dy. \end{aligned}$$

So $f_{\mathcal{L}}(t,v)$ and $f_{\mathcal{L}}(t,v_x)$ as well as $g_{\Lambda}(t,u,v)$ and $g_{\Lambda}(t,u_x,v_x)$ satisfy the same systems of equations. Then, by applying Propositions 3.9 and 3.10, we have the conclusion. □

Remark 3.7 *Note that when (b) holds, $\overline{H} = \int_{\mathbb{R}^2} h(x,y)dy$ for every x in $\mathbb{R}^2$.*

From now on we consider (Φ, C) as a PRRCR of ground intensity $\mathcal{L}$ and expected conflict function h. For each $\lambda > 0$, (Φ_λ, C) is the restriction to $[0, \lambda)$ of (Φ, C), which is a PPPRCR. As $\mathcal{M}_{\mathrm{II}}(\Phi_\lambda, C)$ and $\mathcal{M}_{\mathrm{III}}(\Phi_\lambda, C)$ are stationary, their intensity measure equal to the Lebesgue measure multiplied by their *intensities*, which are computed as,

Corollary 3.4 *The intensity of $\mathcal{M}_{\mathrm{II}}(\Phi_\lambda, C)$ and $\mathcal{M}_{\mathrm{III}}(\Phi_\lambda, C)$ are*

$$m_{\mathrm{II},\mathcal{L}}(\lambda) = \int_0^\lambda e^{-t\overline{H}} = \frac{1e^{-\lambda\overline{H}}}{\overline{H}}, \tag{3.3.1}$$

and

$$m_{\mathrm{III},\mathcal{L}}(\lambda) = \int_0^\lambda f_{\mathcal{L}}(t, H(\mathbf{\mathit{1}}, o)). \tag{3.3.2}$$

The reduced Palm functionals $f_{y,\mathcal{L}}$ and $g_{y,\mathcal{L}}$ also take simpler forms

Corollary 3.5 *For every x in $\mathbb{R}^2$, we have*

$$f_{y,\mathcal{L}}(\lambda, v) = f_{o,\mathcal{L}}(\lambda, v_y); \qquad g_{y,\mathcal{L}}(\lambda, u, v) = g_{o,\mathcal{L}}(\lambda, u_y, v_y), \tag{3.3.3}$$

the functional $f_{o,\mathcal{L}}$ satisfies the system of integral equations

$$\begin{aligned} &f_{o,\mathcal{L}}(0, v) = 1; \\ &f_{o,\mathcal{L}}(\lambda, v) = \int_0^\lambda \frac{f_{\mathcal{L}}\big(t, H(v, o)\big)}{m_{\mathrm{III},\mathcal{L}}(\lambda)} dt \\ &- \int_0^\lambda \int_{\mathbb{R}^2} f_{o,\mathcal{L}}\big(t, H(v, x)\big)\big(1 - v(x)\big)\big(1 - h(x, o)\big) \frac{m_{\mathrm{III},\mathcal{L}}(t)}{m_{\mathrm{III},\mathcal{L}}(\lambda)} dx dt, \end{aligned} \tag{3.3.4}$$

and the functional $g_{y,\mathcal{L}}$ satisfies the system of integral equations

$$\begin{aligned} &g_{o,\mathcal{L}}(0, u, v) = 1; \\ &g_{o,\mathcal{L}}(\lambda, u, v) = \int_0^\lambda \frac{g_{\mathcal{L}}\big(t, H(u, o), v\big)}{m_{\mathrm{II},\mathcal{L}}(\lambda)} dt - \int_0^\lambda \int_{\mathbb{R}^2} \Big(g_{o,\mathcal{L}}(t, u, v)\big(1 - u(x)\big) \\ &+ g_{o,\mathcal{L}}\big(t, H(u, o), v\big) u(x) \big(1 - v(x)\big)\big(1 - h(x, o)\big)\Big) \frac{m_{\mathrm{III},\mathcal{L}}(t)}{m_{\mathrm{III},\mathcal{L}}(\lambda)} dx dt. \end{aligned} \tag{3.3.5}$$

Finally, we give closed form expression for the intensity measure of $\mathcal{M}_{\mathrm{II}}(\Phi_\lambda, C)$ under its reduced Palm distribution. This result is used in Subsection 2.1.3 to provide an approximation of the PPs representing the active t.t.s in a CSMA wireless network.

Corollary 3.6 *The intensity of $\mathcal{M}_{\mathrm{II}}(\Phi_\lambda, C)$ under its reduced Palm distribution is*

$$m_{y,\mathrm{II},\mathcal{L}}(\lambda) = \frac{2}{\overline{H} - \overline{H}_2(x-y)} \left(\frac{1-e^{-\lambda\overline{H}}}{\overline{H}} - \frac{1-e^{-\lambda(2\overline{H}-\overline{H}_2(x-y))}}{2\overline{H} - \overline{H}_2(x-y)} \right) \frac{(1-h(x,y))\overline{H}}{1-e^{-\lambda\overline{H}}}, \tag{3.3.6}$$

where

$$\overline{H}_2(x) = \int_{R^d} h(o,z)h(x,z)dz. \tag{3.3.7}$$

Bibliographical note

A part of the results in this chapter, in particular Theorem 3.1 appeared at IEEE INFOCOM 2012 [21].

Appendices

Appendix A

Preliminary on Point Processes

A.1 Basic Notions

A.1.1 Point Process

Let $\mathbb{N}$ be the set of all countable subsets n of $\mathbb{R}^d$ satisfying

- *Simple:* every point in $\mathbb{R}^d$ appears at most once in n; and
- *Non explosive:* for every bounded subset B of $\mathbb{R}^d$, $n(B) := |B \cap n| < \infty$.

Let $\mathcal{N}$ be the smallest σ-algebra that makes measurable the mappings $I_B : n \mapsto n(B)$ with B bounded Borel subsets of $\mathbb{R}^d$. A point process (PP) in $\mathbb{R}^d$ is a measurable mapping N from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathbb{N}, \mathcal{N})$.

Remark A.1 *Although the definitions and the results in this appendix are for the general case. In the thesis, we only need the special case where $d = 2$.*

By definition of $\mathcal{N}$, the distribution of a PP N is completely determined by its finite-dimensional (fi-di) distributions, which are the joint distributions of the k-tuples of r.v.s $\big(N(B_1), \dots, N(B_k)\big)$ for every integer k and every k mutually disjoint bounded Borel subsets $B_1, \dots, B_k$ of $\mathbb{R}^d$ [11, Proposition 9.2.II].

We can also verify easily that $B \mapsto m_N(B) := \mathbb{E}[N(B)]$ forms a measure in $\mathbb{R}^d$, which we call the intensity measure of N.

An important class of PPs in this theory is the Poisson PPs (PPPs).

Definition A.1 *Let Λ be a locally finite measure in $\mathbb{R}^d$. A PP Φ is a Poisson PP (PPP) of intensity measure Λ iff*

- *For every k mutually disjoint bounded Borel subsets B_1,…,B_k of $\mathbb{R}^d$, $N(B_1)$, …, $N(B_k)$ are k independent r.v.s; and*

- *For every bounded Borel subset B of $\mathbb{R}^d$, $N(B)$ is a Poisson r.v. of parameter $\Lambda(B)$.*

A special case is when $\Lambda = \lambda\mathcal{L}$ with $\mathcal{L}$ be the Lebesgue measure in $\mathbb{R}^d$. In this case we call Φ a homogeneous PPP of intensity λ.

A.1.2 Probability Generating Functional

For any PP N, we define for each function v taking value in $[0,1]$

$$G_N(v) = \mathbb{E}\left[\prod_{x\in N} v(x)\right]. \tag{A.1.1}$$

This is called the probability generating functional (p.g.fl) of N at v. For the p.g.fl to be well-defined and non-trivial, we need that $\left|\sum_{x\in N} log\big(v(x)\big)\right| < \infty$ a.s. As $\left|log\big(v(x)\big)\right| \le \big(1 - v(x)\big)$,this motivates us to consider only p.g.fls of N at functions v satisfying

$$\int_{\mathbb{R}^d} \big(1 - v(x)\big) m_N(dx) < \infty. \tag{A.1.2}$$

The fundamental role of p.g.fls in the study of PPs stems from the fact that all information about the distribution of a PP can be systematically extracted from its p.g.fls in the same way as the distribution of a r.v. can be extracted from its probability generating functions (p.g.f.s). In particular, the p.g.f.s of the fi-di distributions of N can be obtained from its p.g.fls by setting v to some specific functions,

$$G_{N(B_1),\ldots,N(B_k)}(z_1,\ldots,z_k) := \mathbb{E}\left[\prod_{i=1}^{k} z_i^{N(B_i)}\right] = G_N\left(\prod_{i=1}^{k} z_i^{\mathbf{1}_{B_i}}\right). \tag{A.1.3}$$

An important special case is the p.g.fls of PPPs, which can be computed in closed forms.

Theorem A.1 *Let Φ be a PPP of intensity measure Λ and v is a function taking value in $[0,1]$ such that $\int_{\mathbb{R}^d}\big(1 - v(x)\big)\Lambda(dx) < \infty$. Then,*

$$G_\Phi(v) = \exp\left\{-\int_{\mathbb{R}^d} \big(1 - v(x)\big)\Lambda(dx)\right\}. \tag{A.1.4}$$

A.1.3 Shot Noise Process

Let $l(\cdot,\cdot)$ be some function from $\left(\mathbb{R}^d\right)^2$ to $\mathbb{R}^+$ which is called the response function. $l(y,x)$ represents the individual effect of a "shot" from a source at y to a destination at x. The Shot Noise (SN) process associated with N and the response function l is defined by

$$I_{N,l}(x) = \sum_{y\in N} l(y,x), \tag{A.1.5}$$

for each x in $\mathbb{R}^d$.

In this thesis, we usually consider the realization of this SN process at a specific location x. In this point of view, $I_{N,l}(x)$ is a r.v. whose distribution is determined by its p.g.f.s, which can be represented in terms of the p.g.fls of N as

$$G_{I_{N,l}(x)}(z) := \mathbb{E}\left[z^{I_{N,l}(x)}\right] = G_N\left(z^{l(.,x)}\right). \tag{A.1.6}$$

In particular, for a PPP of intensity measure Λ,

$$G_{I_{\Phi,l}(x)}(z) = \exp\left\{-\int_{\mathbb{R}^d}\left(1 - z^{l(y,x)}\right)\Lambda(dy)\right\}. \tag{A.1.7}$$

A.1.4 Stationary Point Processes

In $\mathbb{R}^d$, we consider the translations,

$$T_u(A) = \{x + a \text{ for all } x \in A\} \qquad T_u(x) := T_u(\{x\}) = x + u. \tag{A.1.8}$$

An important result in the theory of point processes asserts that the translations are bijective and continuous in $(\mathbb{N}, \mathcal{N})$ [11, Lemma 12.1.I, p. 178]). So, if N is a PP, its translated version is another PP. N is stationary iff its distribution is invariant under translations. In the other words,

Definition A.2 *A PP N is stationary iff for every u in $\mathbb{R}^d$ and every $\mathfrak{B} \in \mathcal{N}$,*

$$\mathbb{P}(N \in \mathfrak{B}) = \mathbb{P}(T_v(N) \in \mathfrak{B}). \tag{A.1.9}$$

As the distribution of a PP is uniquely determined by its fi-di distributions, we have the following conditions for a PP to be stationary,

Proposition A.1 *A PP N is stationary iff for every u in $\mathbb{R}^d$, one of the following holds,*

(i) for every integer k and every k-tuple of mutually disjoint bounded Borel sets $(B_1, \ldots, B_k)$ and k-tuple of non-negative integers $(n_1, \ldots, n_k)$,

$$\begin{aligned}&\mathbb{P}\big(N(B_1) = n_1, \ldots, N(B_k) = n_k\big)\\ &= \mathbb{P}\Big(N\big(T_u(B_1)\big) = n_1, \ldots, N\big(T_u(B_k)\big) = n_k\Big);\end{aligned} \tag{A.1.10}$$

(ii) for every integer k and every k-tuple of bounded Borel sets $(B_1, \ldots, B_k)$ and every for every k-tuple of positive real numbers $(z_1, \ldots, z_k)$

$$G_N\left(\prod_{i=1}^{k} z_i^{\mathbf{1}_{B_i}(.)}\right) = G_N\left(\prod_{i=1}^{k} z_i^{\mathbf{1}_{T_v(B_i)}(.)}\right); \tag{A.1.11}$$

(iii) for every function v taking value in $[0,1]$ satisfying (A.1.2)*,*

$$G_N(v) = G_N(v_u), \tag{A.1.12}$$

with $v_u(x) = v(x+u)$.

Proof. Point (i) is a direct corollary of Definition 9.2.II in [11]. Point (ii) comes from the representation of the p.g.f.s of the fi-di distributions in terms of the p.g.fls of N. For point (iii), it is sufficient to show that (A.1.11) is equivalent to (A.1.12). The direction (A.1.12) implies (A.1.11) is easy. For the converse, we first note that for any two functions v, w taking value in $[0,1]$ and satisfying (A.1.2),

$$\begin{aligned}|G_N(v) - G_N(w)| &= \left|\mathbb{E}\left[\prod_{x\in N} v(x) - \prod_{x\in N} w(x)\right]\right| \\ &\leq \mathbb{E}\left[\sum_{x\in N} |v(x) - w(x)|\right] \\ &= \int_{\mathbb{R}^d} |v(x) - w(x)| m_N(dx).\end{aligned}$$

Now, there exists a sequence of function v_j, $j = 1, 2, \dots$ such that

- $v_j(x) = \prod_{i=1}^{k_j} z_i^{\mathbf{1}_{B_i}(x)}$ for some k_j positive number $z_1, \dots, z_{k_i}$ in $[0,1]$ and k_j Borel sets $B_1, \dots, B_{k_j}$.
- $\lim_{j\to\infty} \int_{\mathbb{R}^d} |v(x) - v_j(x)| m_N(x) = 0$.

Hence, $\lim_{j\to\infty} G_N(v_j) = G_N(v)$. As,

$$v_{j,u}(x) = \prod_{i=1}^{k_j} z_i^{\mathbf{1}_{T_u(B_i)}(x)}$$

we have by (A.1.11) that

$$G_N(v_{j,u}) = G_N(v_j).$$

Moreover, as T_u is a continuous mapping in $\mathbb{N}$, we have also that

$$\lim_{j\to\infty} \int_{\mathbb{R}^d} |v_{j,u}(x) - v_u(x)| m_N(dx) = 0.$$

This implies that

$$G_N(v) = \lim_{j\to\infty} G_N(v_j) = \lim_{j\to\infty} G_N(v_{j,u}) = G_N(v_u).$$

$\square$

A.1.5 Marked Point Process

Let $(\mathbb{T}, \mathcal{T})$ be a measurable space , which we refer to as the mark space. Let $\mathbb{N}_m$ be the set of all countable subset n_m of $\mathbb{R}^d \times \mathbb{T}$ satisfying

- *Simple:* every pair (x, t) in $\mathbb{R}^d \times \mathbb{T}$ appears at most once in n_m; and
- *Non explosive:* for any bounded Borel subset B of $\mathbb{R}^d$ and every measurable subset T of $\mathbb{T}$, $n_m(B \times T) := |B \times T \cap n_m| < \infty$.

Let $\mathcal{N}_m$ be the smallest σ-algebra that makes measurable the mappings $I_{B,T} : n_m \mapsto n_m(B \times T)$ for all bounded Borel subsets B of $\mathbb{R}^d$ and all measurable subset T of $\mathbb{T}$. A marked point process (MPP) is a measurable mapping N_m from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathbb{N}_m, \mathcal{N}_m)$. In this case, $N := \{x \in \mathbb{R}^d$ s.t. there exists $m \in \mathbb{T}$ s.t. $(x, m) \in N_m\}$ is called the ground PP of N_m and $\mathbb{T}$ is called the mark space of it. In this thesis, we are interested in the MPPs with i.i.d. marks, whose definition in terms of their fi-di distributions is as follow,

Definition A.3 *A MPP N_m with ground PP N and mark space $\mathbb{T}$ is a MPP with independent marks iff for every integer k, every k-tuple of mutually disjoint bounded Borel sets $B_1, \dots, B_k$ in $\mathbb{R}^d$ and every k-tuple of measurable sets $T_1, \dots, T_k$ in the mark space, $N_m(B_1 \times T_1), \dots, N_m(B_k \times T_k)$ are mutually independent given $N(B_1), \dots, N(B_k)$.*

Moreover, N_m is a MPP with i.i.d. marks iff it is a MPP with independent marks and there is a probability distribution $\mathfrak{d}$ in $\mathbb{T}$ such that for every bounded Borel set B in $\mathbb{R}^d$ and every measurable set T in the mark space,

$$\mathbb{P}\big(N_m(B \times T) = n_m \mid N(B) = n\big) = \binom{n}{n_m} \mathfrak{d}(T)^{n_m} \big(1 - \mathfrak{d}(T)\big)^{n - n_m}.$$

Intuitively, in a MPP, we associate to each point in the ground PP a mark in the mark space. In a MPP with i.i.d. marks, the marks at different ground points are mutually independent and have the same distribution. This distribution is $\mathfrak{d}$ in the above definition. This can be seen more clearly by considering the p.g.f.s of the fi-di distributions of the considered MPP.

Proposition A.2 *A MPP N_m with ground PP N and mark space $\mathbb{T}$ is a MPP with i.i.d. marks iff there is a probability distribution $\mathfrak{d}$ in $\mathbb{T}$ such that for every integer k, every k-tuple of mutually disjoint bounded Borel sets $B_1, \dots, B_k$ in $\mathbb{R}^d$ and every k-tuple of measurable sets $T_1, \dots, T_k$ in the mark space,*

$$\begin{aligned} & G_{N_m(B_1 \times T_1), \dots N_m(B_k \times T_k)}(z_1, \dots, z_k) \\ & = G_{N(B_1), \dots N(B_k)}\big(1 - (1 - z_1)\mathfrak{d}(T_1), \dots, 1 - (1 - z_k)\mathfrak{d}(T_k)\big) \end{aligned} \tag{A.1.13}$$

Proof. Suppose that N_m is a MPP with i.i.d. marks. We have,

$$G_{N_m(B_1 \times T_1), \dots N_m(B_k \times T_k)}(z_1, \dots, z_k) = \mathbb{E}\left[\prod_{i=1}^{k} z_i^{N_m(B_i \times T_i)}\right]$$

Given $N(B_1), \ldots, N(B_k)$, by the definition of MPPs with independent marks,

$$\mathbb{E}\left[\prod_{i=1}^{k} z_i^{N_m(B_i \times T_i)} \middle| N(B_1), \ldots, N(B_k)\right] = \prod_{i=1}^{k} \mathbb{E}\left[z_i^{N_m(B_i \times T_i)} \middle| N(B_i)\right]$$

By the definition of MPPs with i.i.d. marks,

$$\begin{aligned}
\mathbb{E}\left[z_i^{N_m(B_i \times T_i)} \middle| N(B_i)\right] &= \sum_{j=0}^{N(B_i)} \binom{N(B_i)}{j} \mathfrak{d}(T_i)^j \big(1 - \mathfrak{d}(T_i)\big)^{N(B_i)-j} z_i^j \\
&= \sum_{j=0}^{N(B_i)} \binom{N(B_i)}{j} \big(\mathfrak{d}(T_i) z_i\big)^j \big(1 - \mathfrak{d}(T_i)\big)^{N(B_i)-j} \\
&= \big(\mathfrak{d}(T_i) z_i + 1 - \mathfrak{d}(T_i)\big)^{N(B_i)} \\
&= \big(1 - \mathfrak{d}(T_i)(1 - z_i)\big)^{N(B_i)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&G_{N_m(B_1 \times T_1), \ldots N_m(B_k \times T_k)}(z_1, \ldots, z_k) \\
&= \mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^{k} z_i N_m(B_i \times T_i) \middle| N(B_1), \ldots, N(B_k)\right]\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{k} \big(1 - \mathfrak{d}(T_i)(1 - z_i)\big)^{N(B_i)}\right] \\
&= G_{N(B_1), \ldots; N(B_k)}\Big(\big(1 - \mathfrak{d}(T_1)(1 - z_1)\big), \ldots, \big(1 - \mathfrak{d}(T_k)(1 - z_k)\big)\Big).
\end{aligned}$$

The converse follows directly from the one–one corresponding between the joint distribution of r.v.s and their joint p.g.f.s. □

Corollary A.1 *If N_m is a MPP with i.i.d. marks of mark distribution $\mathfrak{d}$, its intensity measure can be decomposed as,*

$$m_{N_m}(dx, dt) = m_N(dx)\mathfrak{d}(dt), \tag{A.1.14}$$

where N is its ground PP.

Proof. It is sufficient to note that for every bounded Borel set B and for every measurable set T,

$$\begin{aligned}
m_{N_m}(B \times T) &= \frac{d}{dz} G_{N_m(B \times T)}(z)\bigg|_{z=0}, \\
m_N(B \times T) &= \frac{d}{dz} G_{N(B)}(z)\bigg|_{z=0},
\end{aligned}$$

and to use the above proposition. □

When the ground PP Φ is a PPP, we say that Φ_m is a Marked Poisson Point Process (MPPP). Similar definitions apply for MPPPs with independent marks and MPPPs with i.i.d. marks.

Probability Generating Functionals

Let $v : \mathbb{R}^d \times \mathbb{T} \longmapsto [0,1]$ be a function satisfying

$$\int_{\mathbb{R}^d \times \mathbb{T}} \big(1 - v(x,t)\big) m_{N_m}(dx, dt) < \infty. \tag{A.1.15}$$

Since a MPP can be considered as a PP in $\mathbb{R}^d \times \mathbb{T}$, we can define the p.g.fl of N_m at v as

$$G_{N_m}(v) := \mathbb{E}\left[\prod_{(x,t)\in N_m} v(x,t)\right], \tag{A.1.16}$$

which is well-defined and non-trivial when (A.1.15) is satisfied. For every distribution $\mathfrak{d}$ in $\mathbb{T}$, we define

$$\mathbb{E}_{\mathfrak{d}}[v](x) := \int_{\mathbb{T}} v(x,t)\mathfrak{d}(dt). \tag{A.1.17}$$

The definition of MPPs with i.i.d. marks can then be reformulated in terms of their p.g.fls as follows.

Proposition A.3 *A MPP N_m is a MPP with i.i.d. marks iff there exists a probability distribution $\mathfrak{d}$ in its mark space such that for every function v satisfying* (A.1.15)*,*

$$G_{N_m}(v) = G_N(E_{\mathfrak{d}}[v]). \tag{A.1.18}$$

Proof. It is sufficient to show that (A.1.18) is equivalent to (A.1.13). The direction (A.1.18) is implies (A.1.13) is easily verified by putting

$$v(x) = \prod_{i=1}^{k} z_i^{\mathbf{1}_{B_i \times T_i}(x)}. \tag{A.1.19}$$

For the other direction, (A.1.13) implies that (A.1.18) holds for every v of the form (A.1.19). Note that for every v satisfying (A.1.15), there exists a sequence of functions v_j of the form (A.1.19) such that,

$$\int_{\mathbb{R}^d \times \mathbb{T}} |v(x,t) - v_j(x,t)| m_{N_m}(dx, dt) = 0.$$

The conclusion then follows by noting that

$$|G_{N_m}(v) - G_{N_m}(v_j)| \leq \int_{\mathbb{R}^d \times \mathbb{T}} |v(x,t) - v_j(x,t)| m_{N_m}(dx, dt).$$

□

From this, we can deduce the following simple but very useful results.

Corollary A.2 *Φ_m is a MPPP with i.i.d. marks of intensity measure Λ and mark distribution $\mathfrak{d}$ iff for every function v satisfying* (A.1.15)*,*

$$G_{\Phi_m}(v) = \exp\left\{ - \int_{\mathbb{R}^d} \big(1 - \mathbb{E}_{\mathfrak{d}}\left[v\right](x)\big)\Lambda(dx) \right\} \tag{A.1.20}$$

Remark A.2 *The above results justify again the intuitive description of MPPs with i.i.d. marks as PPs with an independent mark attached to each point. The computation of the p.g.fls of a MPP with i.i.d. marks can be understood as follows. Given a realization of the ground PP, we first take the expectation over all the i.i.d. marks. This expectation at x is $\mathbb{E}_{\mathfrak{d}}[v](x)$. Then, we take the product of this function over all the points in the ground PP and take the expectation over all possible realizations of this PP. This intuitive argument, which can be made rigorous easily by using Proposition A.3 and Corollary A.2, is used extensively in Chapter 2.*

Shot Noise Process

The notion of Shot Noise process associated to a PP can also be extended to the case of MPP. This extension proves to be very useful in modelling the aggregated interference signal power in wireless networks with fading, where the fading values are represented by the i.i.d. marks.

First, we consider a response function $l : \mathbb{R}^d \times \mathbb{T} \times \mathbb{R}^d \longmapsto \mathbb{R}^+$ that takes into account both the position and the mark of the source. The SN process associated to the MPP N_m and the response function l is then,

$$I_{N_m,l}(x) = \sum_{(y,t)\in N_m} l(y,t,x). \tag{A.1.21}$$

The special case where N_m is a MPP with i.i.d. marks is of particular interest. In this case, the p.g.f. of the SN process value at x is

$$G_{I_{N_m,l}(x)}(z) = G_{N_m}\left(z^{l(\cdot,\cdot,x)}\right) = G_N\left(\mathbb{E}_{\mathfrak{d}}\left[z^{l(\cdot,\cdot,x)}\right]\right),$$

where $\mathfrak{d}$ is the mark distribution. Moreover, if Φ_m is a MPPP with i.i.d. marks of intensity measure Λ and mark distribution $\mathfrak{d}$,

$$G_{I_{N_m,l}(x)}(z) = \exp\left\{ - \int_{\mathbb{R}^d} \left(1 - \mathbb{E}_{\mathfrak{d}}\left[z^{l(y,\cdot,x)}\right]\right)\Lambda(dy) \right\}.$$

A.2 Palm Distribution

In the study of PPs, it is often necessary to "switch from the absolute frame of reference outside the process under study to a frame of reference inside the process" [11]. This is done by considering a *typical* point in this PP and study the distribution of the remaining points in it with reference to the typical point. Such a distribution is called the Palm distribution of the considered PP.

We first recall the formal definition of Palm distributions based on Radon-Nikodym's theorem. Then we study the p.g.fls of a PP under its Palm distribution and derive relations between these functionals. These results are extensively used in the developments in Subsection 3.2.2.

A.2.1 Radon-Nikodym Construction

Definition A.4 *For any PP N, its reduced Campbell measure is the measure*

$$\mathcal{C}^!(B,\Gamma) = \mathbb{E}\left[\sum_{x\in N\cap B} \mathbf{1}_{(N\setminus x)\in\Gamma}\right], \tag{A.2.1}$$

and its Campbell measure is the measure

$$\mathcal{C}(B,\Gamma) = \mathbb{E}\left[\sum_{x\in N\cap B} \mathbf{1}_{N\in\Gamma}\right], \tag{A.2.2}$$

in $\mathbb{R}^d \times \mathbb{N}$, where Γ is any set in $\mathcal{N}$.

It is clear from the definition that the reduced Campbell measure and the Campbell measure are refinements of the intensity measure of N. In fact, for any fixed Γ, the measures $\mathcal{C}^!(.,\Gamma)$ and $\mathcal{C}(.,\Gamma)$ are absolutely continuous w.r.t $m_N(.)$. Hence, they admit Radon-Nikodym derivatives $\mathbb{P}^!_{x,N}(\Gamma)$ and $\mathbb{P}_{x,N}(\Gamma)$ which satisfy

$$\mathcal{C}^!(B,\Gamma) = \int_{\mathbb{R}^d} \mathbb{P}^!_{x,N}(\Gamma) m_N(dx); \tag{A.2.3}$$

$$\mathcal{C}(B,\Gamma) = \int_{\mathbb{R}^d} \mathbb{P}_{x,N}(\Gamma) m_N(dx). \tag{A.2.4}$$

Moreover, if m_N is locally finite, so that $m_N(B) < \infty$ for every bounded Borel set B, it can be shown that $\mathbb{P}^!_{x,N}(.)$ and $\mathbb{P}_{x,N}(.)$ are indeed two probability distributions, which are called the *reduced Palm* distribution of N given a point at x and the *Palm* distribution of N given a point at x, respectively.

The importance of Palm theory stems from the Campbell formulas

Theorem A.2 *For every non-negative measurable function f defined on $\mathbb{R}^d \times \mathbb{N}$,*

$$\mathbb{E}\left[\sum_{x\in N} f(x, N\setminus x)\right] = \int_{\mathbb{R}^d} \mathbb{E}^!_x\left[f(x,N)\right] m_N(dx); \tag{A.2.5}$$

$$\mathbb{E}\left[\sum_{x\in N} f(x, N)\right] = \int_{\mathbb{R}^d} \mathbb{E}_x\left[f(x,N)\right] m_N(dx). \tag{A.2.6}$$

Proof. See [11, Proposition 13.1.IV]. □

Remark A.3 *We refer to (A.2.5) and (A.2.6) as the reduced Campbell formula and the Campbell formula respectively.*

When a PP is a PPP, Slivnyak–Meck's theorem below gives us its reduced Palm distributions explicitly.

Theorem A.3 *Let Φ be a PPP with a locally finite intensity measure Λ. Then Φ is a PPP if and only if for Λ-almost every x*

$$\mathbb{P}^!_x(.) = \mathbb{P}(\Phi \in .) \tag{A.2.7}$$

Proof. See the proof of Theorem 1.4.5 [11, Proposition 13.1.7]. □

This gives rise to the following approach for Palm distribution of PPP.

Corollary A.3 *For the PPP Φ one can take $\Phi^!_x = \Phi$ and $\Phi_x = \Phi \cup x$ for all $x \in R^d$.*

A.2.2 Probability Generating Functionals under Palm Distributions

As with the non-Palm version, the reduced Palm and the Palm distributions of a PP can also be characterized by its p.g.fls under the corresponding distributions. In particular, we define

$$G_{x,N}(v) := \mathbb{E}_{x,N}\left[\prod_{y\in N} v(y)\right]; \tag{A.2.8}$$

$$G^!_{x,N}(v) := \mathbb{E}^!_{x,N}\left[\prod_{y\in N} v(y)\right]. \tag{A.2.9}$$

The relation between the reduced Palm distribution and the Palm distribution is characterized as follows.

Proposition A.4 *For any PP N with intensity measure m_N,we have for m_N-almost every x,*

$$G_{x,N}(v) = v(x)G^!_{x,N}(v). \tag{A.2.10}$$

Proof Let $f(x,N) = \prod_{y\in N} v(y)$ and $g(x,N) = v(x)\prod_{y\in N} v(y)$. We have for any $x \in N$,

$$f(x,N) = v(x)f(x,N\setminus x) = g(x,N\setminus x).$$

So,

$$\begin{aligned}
\int_{\mathbb{R}^2} \mathbb{E}_x\left[f(x,N)\right] m_N(dx) &= \mathbb{E}\left[\sum_{x\in N} f(x,N)\right] \\
&= \mathbb{E}\left[\sum_{x\in N} g(x, N\setminus x)\right] \\
&= \int_{\mathbb{R}^d} \mathbb{E}^!_x\left[g(x,N)\right] m_N(dx) \\
&= \int_{\mathbb{R}^d} \mathbb{E}^!_x\left[v(x) f(x,N)\right] m_N(dx) \\
&= \int_{\mathbb{R}^d} v(x)\mathbb{E}^!_x\left[f(x,N)\right] m_N(dx).
\end{aligned}$$

The conclusion then follows directly. □

Hence, we can write

$$G^!_{x,N}(v) := \mathbb{E}^!_{x,N}\left[\prod_{y\in N} v(y)\right] = \mathbb{E}^!_{x,N}\left[\prod_{y\in N\setminus\{x\}} v(y)\right]. \tag{A.2.11}$$

And moreover, it is now sufficient to concentrate on the reduced Palm p.g.fls. The next two results give us the relation between the Palm and non Palm versions of the p.g.fl of a PP.

Proposition A.5 *Let N be a PP with locally finite intensity measure. Then for any function v and any bounded Borel set B,*

$$\begin{aligned}
\frac{d}{dt} G_N\left(ve^{-t\mathbf{1}_B}\right) &= -\int_B G_{x,N}\left(ve^{-t\mathbf{1}_B}\right) m_N(dx) \\
&= -\int_B v(x) e^{-t\mathbf{1}_B(x)} G^!_{x,N}\left(ve^{-t\mathbf{1}_B}\right) m_N(dx).
\end{aligned} \tag{A.2.12}$$

Proof. We have,

$$\begin{aligned}
\frac{d}{dt}\left(\prod_{x\in N} v(x) e^{-t\mathbf{1}_B(x)}\right) &= \frac{d}{dt}\left(\prod_{x\in N} v(x)\right) e^{-tN(B)} \\
&= -\left(\prod_{x\in N} v(x)\right) e^{-tN(B)} N(B) \\
&= -\left(\prod_{x\in N} v(x) e^{-t\mathbf{1}_B(x)}\right) N(B) \\
&= -\sum_{x\in B\cap N}\left(\prod_{y\in N} v(y) e^{-t\mathbf{1}_B(y)}\right).
\end{aligned}$$

Since the absolute value of the last line is bounded above by $N(B)$ and $\mathbb{E}[N(B)] = m_N(B) < \infty$ by definition, we have by bounded convergence theorem,

$$\begin{aligned}\frac{d}{dt}G_N(ve^{-t\mathbf{1}_B}) &= \frac{d}{dt}\mathbb{E}\left[\left(\prod_{x\in N} v(x)e^{-t\mathbf{1}_B(x)}\right)\right] \\ &= \mathbb{E}\left[\frac{d}{dt}\left(\prod_{x\in N} v(x)e^{-t\mathbf{1}_B(x)}\right)\right] \\ &= -\mathbb{E}\left[\sum_{x\in B\cap N}\left(\prod_{y\in N} v(y)e^{-t\mathbf{1}_B(y)}\right)\right].\end{aligned}$$

As the last line is equal to $-\int_B G_{x,N}(ve^{-t\mathbf{1}_B})m_N(dx)$ by the Campbell formula, the conclusion follows directly. $\square$

However, the previous proposition does not give us the relation between the p.g.fl and its reduced Palm versions at x such that $v(x) = 0$. For such x, we need the following result.

Proposition A.6 *Let N be a PP with locally finite intensity measure and t be positive number smaller than 1. For any function v and any Borel set B not in the support of v (i.e. $v(x) = 0$ for every x in B),*

$$\frac{d}{dt}G_N(v + t\mathbf{1}_B) = \int_B G^!_{x,N}(v + t\mathbf{1}_B)m_N(dx). \tag{A.2.13}$$

Proof. Since B is not in the support of v,

$$\begin{aligned}\frac{d}{dt}\left(\prod_{y\in N}(v(y) + \mathbf{1}_B(y))\right) &= \frac{d}{dt}\left(\prod_{y\in N\setminus B} v(y)\right)t^{N(B)} \\ &= \left(\prod_{y\in N\setminus B} v(y)\right)t^{N(B)-1}N(B) \\ &= \left(\prod_{y\in N\setminus B} v(y)\right)\left(\sum_{x\in N\cap B}\prod_{y\in(N\cap B)\setminus\{x\}} t\right) \\ &= \sum_{x\in N\cap B}\left(\prod_{y\in N\setminus B} v(y)\prod_{y\in N\cap B\setminus\{x\}} t\right) \\ &= \sum_{x\in N\cap B}\left(\prod_{y\in N\setminus\{x\}}(v(y) + \mathbf{1}_B(y))\right).\end{aligned}$$

Again, by bounded convergence theorem,

$$\begin{aligned}\frac{d}{dt}G_N(v+t\mathbf{1}_B) &= \mathbb{E}\left[\frac{d}{dt}\left(\prod_{y\in N}\big(v(y)+\mathbf{1}_B\big)\right)\right]\\ &= \mathbb{E}\left[\sum_{x\in N\cap B}\left(\prod_{y\in N\setminus x}\big(v(y)+\mathbf{1}_B\big)\right)\right].\end{aligned}$$

The conclusion follows directly from the fact that the last line equals

$$\int_B G^!_{x,N}(v+t\mathbf{1}_B)m_N(dx)$$

by reduced Campbell formula. □

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