Extreme Values and Recurrence for Deterministic and Stochastic Dynamics

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&

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Resumo

Neste trabalho, estudamos as propriedades estatísticas de sistemas dinâmicos determinísticos e estocásticos. Estamos particularmente interessados em valores extremos e recorrência. Provamos a existência de Leis de Valores Extremos (LVE) e Estatísticas do Tempo de Entrada (ETE) / Estatísticas de Tempo de Retorno (ETR) para sistemas com decaimento de correlações contra observáveis em L^1. Também realizamos o estudo da convergência dos Processos Pontuais de Acontecimentos Raros (PPAR).

Na primeira parte, investigamos o problema para dinâmica determinística e caracterizamos completamente o comportamento extremal de sistemas expansores. Mostramos que há uma dicotomia quanto 00E0 existência de um Índice de Extrema (IE). Nomeadamente, provamos que o IE é estritamente menor do que 1 em torno de pontos periódicos e é igual a 1 para pontos aperiódicos. Num contexto mais geral, mostramos que os PPAR convergem para um processo de Poisson simples ou um processo de Poisson composto, em que a distribuição de multiplicidade é geométrica, dependendo se o centro é um ponto aperiódico ou periódico, respectivamente. Além disso, realizamos uma análise da convergência dos PPAR em pontos de descontinuidade, o que conduziu à descoberta de convergência para um processo de Poisson composto com uma distribuição de multiplicidade diferente da usual distribuição geométrica.

Na segunda parte, consideramos dinâmica estocástica obtida por perturbação aleatória de um sistema determinístico por inclusão de um ruído aditivo. Apresentamos duas técnicas complementares que nos permitem obter LVE e as ETE na presença deste

tipo de ruído. A primeira abordagem é mais probabilística enquanto que a outra usa sobretudo teoria espectral. Conclui-se que, independentemente do centro escolhido, o IE é sempre igual a 1 e os PPAR convergem para o processo de Poisson simples.

Abstract

In this work, we study the statistical properties of deterministic and stochastic dynamical systems. We are particularly interested in extreme values and recurrence. We prove the existence of Extreme Value Laws (EVLs) and Hitting Time Statistics (HTS)/ Return Time Statistics (RTS) for systems with decay of correlations against L^1 observables. We also carry out the study of the convergence of Rare Event Point Processes (REPP).

In the first part, we investigate the problem for deterministic dynamics and completely characterise the extremal behaviour of expanding systems by giving a dichotomy relying on the existence of an Extremal Index (EI). Namely, we show that the EI is strictly less than 1 for periodic centres and is equal to 1 for non-periodic ones. In a more general setting, we prove that the REPP converges to a standard Poisson if the centre is non-periodic, and to a compound Poisson with a geometric multiplicity distribution for the periodic case. Moreover, we perform an analysis of the convergence of the REPP at discontinuity points which gives the convergence to a compound Poisson with a multiplicity distribution different than the usual geometric one.

In the second part, we consider stochastic dynamics by randomly perturbing a deterministic system with additive noise. We present two complementary methods which allow us to obtain EVLs and statistics of recurrence in the presence of noise. The first approach is more probabilistically oriented while the second one uses spectral theory. We conclude that, regardless of the centre chosen, the EI is always equal to 1 and the REPP converges to the standard Poisson.

Résumé

Dans ce travail, nous étudions les propriétés statistiques de certains systèmes dynamiques déterministes et stochastiques. Nous nous intéressons particulièrement aux valeurs extrêmes et à la récurrence. Nous montrons l'existence de Lois pour les Valeurs Extrêmes (LVE) et pour les Statistiques des Temps d'Entrée (STE) et des Temps de Retour (STR) pour des systèmes avec décroissance des correlations rapide. Nous étudions aussi la convergence du Processus Ponctuel d'Évènements Rares (PPER).

Dans la première partie, nous nous intéressons aux systèmes dynamiques déterministes, et nous caractérisons complètement les proprietés précedentes dans le cas des systèmes dilatants. Nous montrons l'existence d'un Indice Extrême (IE) strictement plus petit que 1 autour des points périodiques, et qui vaut 1 dans le cas non-périodique, mettant ainsi en évidence une dichotomie dans la dynamique caractérisée par l'indice extrême. Dans un contexte plus général, nous montrons que le PPER converge soit vers une distribution de Poisson pour des points non-périodiques, soit vers une distribution de Poisson mélangée avec une distribution multiple de type géométrique pour des points périodiques. De plus, nous déterminons explicitement la limite des PPER autour des points de discontinuité et nous obtenons des distributions de Poisson mélangées avec des distributions multiples différentes de la distribution géométrique habituelle.

Dans la deuxième partie, nous considérons des systèmes dynamiques stochastiques obtenus en perturbant de manière aléatoire un système déterministe donné. Nous élaborons deux méthodes nous permettant d'obtenir des lois pour les Valeurs Extrêmes et les

statistiques de la récurrence en présence de bruits aléatoires. La première approche est de nature probabiliste tandis que la seconde nécessite des outils d'analyse spectrale. Indépendamment du point choisi, nous montrons que le IE est constamment égal à 1 et que le PPER converge vers la distribution de Poisson standard.

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Chapter 1

Introduction

Deterministic discrete dynamical systems are often used to model physical phenomena. In many situations, inevitable observation errors make it more realistic to consider random dynamics, where the mathematical model is adjusted by adding random noise to the iterative process in order to account for these practical imprecisions. The behaviour of such random systems has been studied thoroughly in the last decades. We mention, for example, [K86a, KL06] for excellent expositions on the subject.

Laws of rare events for chaotic (deterministic) dynamical systems have also been exhaustively studied in the last years. When these results first appeared these notions were described as Hitting Times Statistics (HTS) or Return Times Statistics (RTS). In this setting, rare events correspond to entrances in small regions of the phase space and the goal is to prove distributional limiting laws for the normalised waiting times before hitting/returning to these asymptotically small sets. We refer to [S09] for an excellent review. More recently, rare events have also been studied through Extreme Value Laws (EVLs), *i.e.*, the distributional limit of the partial maxima of stochastic processes arising from such chaotic systems simply by evaluating an observable function along the orbits of the system. Very recently, in [FFT10, FFT11], the two perspectives have been proved to be linked so that, under general conditions on the observable functions, the existence of

HTS/RTS is equivalent to the existence of EVLs. These observable functions are chosen to achieve a maximum (possibly ∞) at some chosen point ζ in the phase space so that the rare event of occurring an exceedance of a high level corresponds to an entrance in a small ball around ζ. The study of rare events may be enhanced if we enrich the process by considering multiple exceedances (or hits/returns to target sets) that are recorded by Rare Events Point Processes (REPP), which count the number of exceedances (or hits/returns) in a certain time frame. Then one looks for limits in distribution for such REPP when time is adequately normalised.

Surprisingly, not much is known about rare events for stochastic dynamical systems. One of the main goals here is to establish what we believe to be the first result proving the existence of EVLs (or equivalently HTS/RTS) as well as the convergence of REPP, for randomly perturbed dynamical systems. Part of the difficulty in establishing this type of result derives from the fact that it is not immediately clear how to choose the best approach in order to prove the existence of HTS/RTS for stochastic dynamics. On the other hand, from the EVL perspective, it is quite straightforward how to address the existence of EVLs even when we have to deal with randomly perturbed systems. We exploit this fact, and the connection between EVLs and HTS/RTS, in order to settle the strategy.

We remark that in the recent paper [MR11] the authors defined the meaning of first hitting/return time in the random dynamical setting. To our knowledge this was the first paper to address this issue of recurrence for random dynamics. There, the authors define the concepts of quenched and annealed return times for systems generated by the composition of random maps. Moreover, they prove that for super-polynomially mixing systems, the random recurrence rate is equal to the local dimension of the stationary measure.

In the present work, we are interested in establishing the right setting in order to have the connection between EVL and HTS/RTS, for random dynamics, and, eventually, to prove the existence of EVLs and HTS/RTS for random orbits. Moreover, we also study the convergence of the REPP for randomly perturbed systems.

In general terms, we will consider uniformly expanding and piecewise expanding maps. Then we randomly perturb these discrete systems with additive, independent and identically distributed noise introduced at each iteration. The noise distribution is absolutely continuous with respect to Lebesgue measure.

The main ingredients will be decay of correlations against all L^1 observables (we mean decay of correlations of all observables in some Banach space against all observables in L^1, which will be made more precise in Definition 2.1.2 below) and the notion of first return time from a set to itself.

We realised that the techniques we were using to study the random scenario also allowed us to give an answer to one of the questions raised in [FFT12]. There the connection between periodicity, clustering of rare events and the Extremal Index (EI) was studied. In certain situations, like when rare events are defined as entrances in balls around (repelling) periodic points, the stochastic processes generated by the dynamics present clustering of rare events. The EI is a parameter $\vartheta \in (0,1]$ which quantifies the intensity of the clustering. In fact, in most situations the average cluster size is just $1/\vartheta$. No clustering means that $\vartheta = 1$ and strong clustering means that ϑ is close to 0. In [FFT12, Section 6], it is showed that, for uniformly expanding maps of the circle equipped with the Bernoulli measure, there is a dichotomy in terms of the possible EVL: either the rare events are centred at (repelling) periodic points and $\vartheta < 1$ or at non periodic points and the EI is 1. This was proved for cylinders, in the sense that rare events corresponded to entrances into dynamically defined cylinders (instead of balls) and one of the questions it raised was if this dichotomy could be proved more generally for balls and for more general systems. In [FP12], the authors build up on the work of [H93] and eventually obtain the dichotomy for balls and for conformal repellers.

One of our results here, Theorem A, allows to prove the dichotomy for balls and for systems with decay of correlations against L^1 which include, for example, piecewise expanding maps of the interval like Rychlik maps (Proposition 3.3.6) or piecewise expanding maps in higher dimensions, like the ones studied by Saussol in [S00], (Proposition 3.3.8). Moreover, as an end product of our approach, we can express the dichotomy for these systems in the following more general terms (see Propositions 3.3.6 and 3.3.8): either we have, at non periodic points, the convergence of the REPP to the standard Poisson process or we have, at repelling periodic points, the convergence of REPP to a compound Poisson process consisting of an underlying asymptotic Poisson process governing the positions of the clusters of exceedances and a multiplicity distribution associated to each such Poisson event, which is determined by the average cluster size. In fact, at repelling periodic points, we always get that the multiplicity distribution is the geometric distribution (see [HV09, FFT12a]).

We also consider discontinuity points of the map as centres of the rare events (see Proposition 3.4.2). A very interesting immediate consequence of this study is that, when we consider the REPP, we can obtain convergence to a compound Poisson process whose multiplicity distribution is not a geometric distribution. To our knowledge this is the first time these limits are obtained for the general piecewise expanding systems considered and in the balls' setting (rather than cylinders), in the sense that exceedances or rare events correspond to the entrance of the orbits in topological balls.

In the course of our investigation we came across a paper by Keller, [K12], where he proved the dichotomy of expanding maps with a spectral gap for the corresponding Perron-Frobenius operator (which also include Rychlik maps and the higher dimensional piecewise expanding maps studied by Saussol [S00], for example). He makes use of a powerful technique developed in [KL09], which is based on an eigenvalue perturbation formula. Our approach is different since we use an EVL kind of argument and our assumptions are based on decay of correlations against L^1 observables. Moreover, we also deal with the convergence of the REPP and obtain, in particular, the interesting fact that at discontinuity points we observe multiplicity distributions other than the geometric one.

We also mention the very recent paper [KR12], where the dichotomy for cylinders is established for mixing countable alphabet shifts, but also in the context of nonconventional ergodic sums. It also includes examples of non-convergence of the REPP, in the cylinder setting.

We remark that in most situations, decay of correlations against L^1 observables is a consequence of the existence of a gap in the spectrum of the map's corresponding Perron-Frobenius operator. However, in [D98], Dolgopyat proves exponential decay of correlations for certain Axiom A flows but along the way he proves it for semiflows against L^1 observables. This is done via estimates on families of twisted transfer operators for the Poincaré map, but without considering the Perron-Frobenius operator for the flow itself. This means that the discretisation of this flow by using a time 1 map, for example, provides an example of a system with decay of correlations against L^1 for which it is not known if there exists a spectral gap of the corresponding Perron-Frobenius operator. Apparently, the existence of a spectral gap for the map's Perron-Frobenius operator, defined in some nice function space, implies decay of correlations against L^1 observables. However, the latter is still a very strong property. In fact, from decay of correlations against L^1 observables, regardless of the rate, as long as it is summable, one can actually show that the system has exponential decay of correlations of Hölder observables against L^∞. See [AFL11, Theorem B]. So an interesting question is:

Question. *If a system presents summable decay of correlations against L^1 observables, is there a spectral gap for the system's Perron-Frobenius operator, defined in some appropriate function space?*

We note that, as we point out in Remark 3.2.1, we do not actually need decay of correlations against L^1 in its full strength.

Returning to the stochastic setting, our main result asserts that the dichotomy observed for deterministic systems vanishes and regardless of the centre being a periodic point or not, we always get standard exponential EVLs or, equivalently, standard exponential

HTS/RTS (which means that $\vartheta = 1$). Moreover, we also show that the REPP converges in distribution to a standard Poisson process. We will prove these results in Section 4.1 using an EVL approach, where the main assumption will be decay of correlations against L^1.

Still in the stochastic setting, motivated by the deep work of Keller, [K12], in Section 4.2, we prove results in the same directions as before but based on the spectral approach used by Keller and Liverani to study deterministic systems. As a byproduct we get an HTS/RTS formula with sharp error terms for randomly perturbed dynamical systems (see Proposition 4.2.1). In the beginning of Section 4.2, we will point out the differences between the two techniques (which we name here as *direct* and *spectral*, respectively). Let us simply stress that we implement the spectral technique in random situation only for one-dimensional systems and the existence of EI is proved for a substantially large class of noises. On the other hand, the direct technique works for systems in higher dimensions as well, but it requires additive noise with a continuous distribution. However, the latter is necessary to prove that EI is 1 in the spectral approach too.

Finally, let us mention that this work led to the preprint [AFV12].

Chapter 2

Preliminaries

2.1 General setting

Consider a discrete time dynamical system $(\mathcal{X}, \mathcal{B}, \mathbb{P}, T)$ which will denote two different but interrelated settings throughout this work. $\mathcal{X}$ is a topological space, $\mathcal{B}$ is the Borel σ-algebra, $T : \mathcal{X} \to \mathcal{X}$ is a measurable map and $\mathbb{P}$ is a T-invariant probability measure, *i.e.*,

$$\mathbb{P}(T^{-1}(B)) = \mathbb{P}(B) \text{ for all } B \in \mathcal{B}.$$

Also, given any $A \in \mathcal{B}$ with $\mathbb{P}(A) > 0$, let $\mathbb{P}_A$ denote the conditional measure on $A \in \mathcal{B}$, *i.e.*, $\mathbb{P}_A := \frac{\mathbb{P}|_A}{\mathbb{P}(A)}$.

First, it will denote a deterministic setting where $\mathcal{X} = \mathcal{M}$ is a compact Riemannian manifold, $\mathcal{B}$ is the Borel σ-algebra, $T = f : \mathcal{M} \to \mathcal{M}$ is a piecewise differentiable map and $\mathbb{P} = \mu$ is an f-invariant probability measure. Let $\mathrm{dist}(\cdot, \cdot)$ denote a Riemannian metric on $\mathcal{M}$ and Leb a normalised volume form on the Borel sets of $\mathcal{M}$ that we call Lebesgue measure.

Second, it will denote a stochastic setting which is constructed from the deterministic system via perturbing the original map with random additive noise. We assume that $\mathcal{M}$

is a quotient of a Banach vector space $\mathcal{V}$, like $\mathcal{M} = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, for some $d \in \mathbb{N}$. In the case $d = 1$, we will also denote the circle $\mathbb{T}^1$ by $\mathcal{S}^1$. Let $\text{dist}(\cdot, \cdot)$ denote the induced usual quotient metric on $\mathcal{M}$ and Leb a normalised volume form on the Borel sets of $\mathcal{M}$ that we call Lebesgue measure. Also denote the ball of radius $\varepsilon > 0$ around $x \in \mathcal{M}$ by

$$B_\varepsilon(x) := \{y \in \mathcal{M} : \text{dist}(x, y) < \varepsilon\}.$$

Consider the unperturbed deterministic system $f : \mathcal{M} \to \mathcal{M}$. For some $\varepsilon > 0$, let θ_ε be a probability measure defined on the Borel subsets of $B_\varepsilon(0)$, such that

$$\theta_\varepsilon = g_\varepsilon \text{Leb} \quad \text{and} \quad 0 < \underline{g_\varepsilon} \leq g_\varepsilon \leq \overline{g_\varepsilon} < \infty. \tag{2.1}$$

For each $\omega \in B_\varepsilon(0)$, we define the *random additive perturbation* of f that we denote by f_ω as the map $f_\omega : \mathcal{M} \to \mathcal{M}$, given by [1]

$$f_\omega(x) = f(x) + \omega. \tag{2.2}$$

Let $W_1, W_2, \dots$ be a sequence of independent and identically distributed random variables taking values on $B_\varepsilon(0)$ with common distribution given by θ_ε. Let $\Omega = B_\varepsilon(0)^{\mathbb{N}}$ denote the space of realisations of such a process and $\theta_\varepsilon^{\mathbb{N}}$ the product measure defined on its Borel subsets. Given a point $x \in \mathcal{M}$ and the realisation of the stochastic process $\underline{\omega} = (\omega_1, \omega_2, \dots) \in \Omega$, we define the *random orbit* of x as $x, f_{\underline{\omega}}(x), f^2_{\underline{\omega}}(x), \dots$ where, the evolution of x, up to time $n \in \mathbb{N}$, is obtained by the concatenation of the respective additive randomly perturbed maps in the following way:

$$f^n_{\underline{\omega}}(x) = f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1}(x), \tag{2.3}$$

[1] In the general theory of randomly perturbed dynamical systems one could consider perturbations other than the additive ones and distributions which are not necessarily absolutely continuous. Our choice is motivated by the fact that our main result for the extreme values in presence of noise can be showed relatively easily with those assumptions, but it is also clear from the proof where possible generalisations could occur. We are especially concerned in constructing the framework and in finding the good assumptions for the theory, which is surely satisfied for more general perturbations and probability distributions. Let us notice that other authors basically used additive noise when they studied statistical properties of random dynamical systems, [BBM02, BBM03, AA03], for instance.

with $f^0_{\underline{\omega}}$ being the identity map on $\mathcal{M}$. Next definition gives a notion that plays the role of invariance in the deterministic setting.

Definition 2.1.1. Given $\varepsilon > 0$, we say that the probability measure μ_ε on the Borel subsets of $\mathcal{M}$ is stationary if

$$\iint \phi(f_\omega(x))\,\mathrm{d}\mu_\varepsilon(x)\,\mathrm{d}\theta_\varepsilon(\omega) = \int \phi(x)\,\mathrm{d}\mu_\varepsilon(x),$$

for every $\phi : \mathcal{M} \to \mathbb{R}$ integrable with respect to μ_ε.

Introducing perturbation theory

We can interpret the above definition as

$$\int \mathcal{U}_\varepsilon \phi\,\mathrm{d}\mu_\varepsilon = \int \phi\,\mathrm{d}\mu_\varepsilon$$

where the operator $\mathcal{U}_\varepsilon : L^\infty(\mathrm{Leb}) \to L^\infty(\mathrm{Leb})$ is defined as

$$(\mathcal{U}_\varepsilon \phi)(x) = \int_{B_\varepsilon(0)} \phi(f_\omega(x))\,\mathrm{d}\theta_\varepsilon$$

and called the *random evolution operator*.

The adjoint of this operator is the so-called *random Perron-Frobenius operator*, $\mathcal{P}_\varepsilon : L^1(\mathrm{Leb}) \to L^1(\mathrm{Leb})$, and it acts by duality as

$$\int \mathcal{P}_\varepsilon \psi \cdot \phi\,\mathrm{dLeb} = \int \mathcal{U}_\varepsilon \phi \cdot \psi\,\mathrm{dLeb}$$

where $\psi \in L^1$ and $\phi \in L^\infty$.

It is immediate from the above definition to get another useful representation of this operator, namely for $\psi \in L^1$:

$$(\mathcal{P}_\varepsilon \psi)(x) = \int_{B_\varepsilon(0)} (\mathcal{P}_\omega \psi)(x)\,\mathrm{d}\theta_\varepsilon(\omega),$$

where $\mathcal{P}_\omega$ is the Perron-Frobenius operator associated to f_ω.

We recall that the stationary measure μ_ε is absolutely continuous with respect to the Lebesgue measure and with density h_ε if and only if such a density is a fixed point of the random Perron-Frobenius operator: $\mathcal{P}_\varepsilon h_\varepsilon = h_\varepsilon$. [2]

[2] The duality explains why we take $\mathcal{P}_\varepsilon$ acting on L^1 and $\mathcal{U}_\varepsilon$ on L^∞. Moreover our stationary measures will be absolutely continuous with density given by the fixed point of $\mathcal{P}_\varepsilon$.

Introducing the skew product system: stochastic $\xrightarrow{\text{skew product}}$ deterministic

We can give a deterministic representation of this stochastic setting using the following skew product transformation:

$$\begin{aligned} S: \ \mathcal{M} \times \Omega \ &\longrightarrow \quad \mathcal{M} \times \ \Omega \\ (x,\underline{\omega}) \ &\longmapsto \ (f_{\omega_1}, \sigma(\underline{\omega})), \end{aligned} \tag{2.4}$$

where $\sigma : \Omega \to \Omega$ is the one-sided shift $\sigma(\underline{\omega}) = \sigma(\omega_1, \omega_2, \ldots) = (\omega_2, \omega_3, \ldots)$. We remark that μ_ε is stationary if and only if the product measure $\mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}}$ is an S-invariant measure.

Hence, the random evolution can fit the original model $(\mathcal{X}, \mathcal{B}, \mathbb{P}, T)$ by taking the product space $\mathcal{X} = \mathcal{M} \times \Omega$, with the corresponding product Borel σ-algebra $\mathcal{B}$, where the product measure $\mathbb{P} = \mu_\varepsilon \times \theta_\epsilon^{\mathbb{N}}$ is defined. The system is then given by the skew product map $T = S$.

Main ingredient: decay against L^1

For systems we will consider, $\mathbb{P}$ has very good mixing properties, which in loose terms means that the system loses memory quite fast. In order to quantify the memory loss we look at the system's rates of decay of correlations with respect to $\mathbb{P}$.

Definition 2.1.2 (Decay of correlations)**.** Let $\mathcal{C}_1, \mathcal{C}_2$ denote Banach spaces of real-valued measurable functions defined on $\mathcal{X}$. We denote the *correlation* of non-zero functions $\phi \in \mathcal{C}_1$ and $\psi \in \mathcal{C}_2$ with respect to a measure $\mathbb{P}$ as

$$\mathrm{Cor}_{\mathbb{P}}(\phi, \psi, n) := \frac{1}{\|\phi\|_{\mathcal{C}_1} \|\psi\|_{\mathcal{C}_2}} \left| \int \phi \, (\psi \circ T^n) \, \mathrm{d}\mathbb{P} - \int \phi \, \mathrm{d}\mathbb{P} \int \psi \, \mathrm{d}\mathbb{P} \right|.$$

We say that we have *decay of correlations*, with respect to the measure $\mathbb{P}$, for observables in $\mathcal{C}_1$ *against* observables in $\mathcal{C}_2$ if for every $\phi \in \mathcal{C}_1$ and every $\psi \in \mathcal{C}_2$ we have

$$\mathrm{Cor}_{\mathbb{P}}(\phi, \psi, n) \to 0, \qquad \text{as } n \to \infty.$$

In the stochastic setting, we will only be interested in Banach spaces of functions that do not depend on $\underline{\omega} \in \Omega$, hence, we assume that ϕ,ψ are actually functions defined on

$\mathcal{M}$ and the correlation between these two observables can be written more simply as

$$
\begin{aligned}
\mathrm{Cor}_{\mathbb{P}}(\phi,\psi,n) :&= \frac{1}{\|\phi\|_{\mathcal{C}_1}\|\psi\|_{\mathcal{C}_2}}\left|\int\left(\int \psi\circ f^n_{\underline{\omega}}\,\mathrm{d}\theta_\varepsilon^{\mathbb{N}}\right)\phi\,\mathrm{d}\mu_\varepsilon - \int\phi\,\mathrm{d}\mu_\varepsilon\int\psi\,\mathrm{d}\mu_\varepsilon\right| \\
&= \frac{1}{\|\phi\|_{\mathcal{C}_1}\|\psi\|_{\mathcal{C}_2}}\left|\int \mathcal{U}^n_\varepsilon\psi\cdot\phi\,\mathrm{d}\mu_\varepsilon - \int\phi\,\mathrm{d}\mu_\varepsilon\int\psi\,\mathrm{d}\mu_\varepsilon\right| \qquad (2.5)
\end{aligned}
$$

where

$$
\begin{aligned}
(\mathcal{U}^n_\varepsilon\psi)(x) &= \int\cdots\int \psi(f_{\omega_n}\circ\cdots\circ f_{\omega_1}x)\,\mathrm{d}\theta_\varepsilon(\omega_n)\ldots\mathrm{d}\theta_\varepsilon(\omega_1) \\
&= \int \psi\circ f^n_{\underline{\omega}}(x)\,\mathrm{d}\theta_\varepsilon^{\mathbb{N}}.
\end{aligned}
$$

We say that we have *decay of correlations against L^1 observables*, *decay against L^1* in short, whenever we have decay of correlations, with respect to the measure $\mathbb{P}$, for observables in $\mathcal{C}_1$ against observables in $\mathcal{C}_2$ where $\mathcal{C}_2 = L^1(\mathrm{Leb})$ is the space of Leb-integrable functions on $\mathcal{M}$ and $\|\psi\|_{\mathcal{C}_2} = \|\psi\|_1 = \int|\psi|\,\mathrm{dLeb}$. Note that when μ, μ_ε are absolutely continuous with respect to Leb and the respective Radon-Nikodym derivatives are bounded above and below by positive constants, then $L^1(\mathrm{Leb}) = L^1(\mu) = L^1(\mu_\varepsilon)$.

Main purpose

Our goal is to study the statistical properties of such systems, with decay against L^1, regarding the occurrence of *rare events*. By rare events we mean that the event has a small probability. There are two approaches for this purpose that were recently proved to be equivalent: the existence of HTS/RTS and EVLs.

Throughout this work we consider the time series $X_0, X_1, X_2, \ldots$ arising from a system with decay against L^1 simply by evaluating a given random variable $\varphi : \mathcal{M}\to\mathbb{R}\cup\{+\infty\}$ along the orbits of the system:

$$
X_n = \varphi\circ f^n, \quad \text{for each } n\in\mathbb{N}. \qquad (2.6)
$$

Note that when we consider the random dynamics, the process will be

$$
X_n = \varphi\circ f^n_{\underline{\omega}}, \quad \text{for each } n\in\mathbb{N}, \qquad (2.7)
$$

which can also be written as $X_n = \bar{\varphi} \circ S^n$, where

$$\begin{aligned} \bar{\varphi}: \ \mathcal{M} \times \Omega &\longrightarrow \mathbb{R} \cup \{+\infty\} \\ (x, \underline{\omega}) &\longmapsto \varphi(x) \end{aligned} \tag{2.8}$$

Clearly, $X_0, X_1, \ldots$ defined in such a way is not an independent sequence. However, invariance of μ and stationarity of μ_ε guarantee that the stochastic process is stationary in both cases.

In what follows, an *exceedance* of the level $u \in \mathbb{R}$ at time $j \in \mathbb{N}$ means that the event $\{X_j > u\}$ occurs.

We denote by F the distribution function of X_0, *i.e.*,

$$F(x) = \mathbb{P}(X_0 \leq x).$$

Moreover, throughout this work, given any distribution function F, let $\bar{F} = 1 - F$, which is called the *tail* of the distribution function F, and u_F denote the *right endpoint* of the distribution function F, *i.e.*,

$$u_F = \sup\{x : F(x) < 1\}.$$

Regularity conditions on the observable function and the measure

We assume that the random variable $\varphi : \mathcal{M} \to \mathbb{R} \cup \{\pm\infty\}$ achieves a global maximum at $\zeta \in \mathcal{M}$ (we allow $\varphi(\zeta) = +\infty$). We also assume that φ and $\mathbb{P}$ are sufficiently regular so that:

(R1) for u sufficiently close to $u_F := \varphi(\zeta)$, the event

$$U(u) = \{X_0 > u\} = \{x \in \mathcal{M} : \ \varphi(x) > u\}$$

corresponds to a topological ball centred at ζ. Moreover, the quantity $\mathbb{P}(U(u))$, as a function of u, varies continuously on a neighbourhood of u_F.

2.2 Extreme Value Theory

The first result is the *classical* Extreme Value Theory, *i.e.*, the study of distributional properties of the maximum of n *independent and identically distributed* random variables as n becomes large. To be more precise, let $X_0, X_1, X_2, \dots$ be a sequence of such random variables and define the partial maximum to be

$$M_n = \max\{X_0, \dots, X_{n-1}\}. \tag{2.9}$$

Then, the aim is to find the appropriate normalising sequences in search of a non-degenerate asymptotic distribution law for M_n. More precisely, we want to know if there are normalising sequences $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $\{b_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\mathbb{P}\left(\{x : a_n(M_n - b_n) \leq y\}\right) = \mathbb{P}\left(\{x : M_n \leq u_n\}\right) \to H(y), \tag{2.10}$$

where

$$u_n := u_n(y) = \frac{y}{a_n} + b_n,$$

for some non-degenerate distribution function H, as $n \to \infty$. Here, non-degeneracy of H means that there is no $y_0 \in \mathbb{R}$ with $H(y_0) = 1$ and $H(y) = 0$ for all $y < y_0$.

In this sense, Extreme Value Theory is analogous to the Central Limit Theory where the study of partial maxima is replaced by that of partial sums.

One of the main results of the classical theory is the so-called *Extremal Types Theorem* which exhibits the possible limiting forms for the distribution of M_n under linear normalisations. To be more precise, the theorem asserts that, whenever the variable X_i's are independent and identically distributed, if for some constants $a_n > 0$, b_n, we have

$$\mathbb{P}(a_n(M_n - b_n) \leq y) \to H(y), \tag{2.11}$$

where the convergence occurs at continuity points of H, and H is non-degenerate, then $H(y) = \mathrm{e}^{-\tau(y)}$, where $\tau(y)$ is of one of the following three types (for some $\alpha > 0$):

$$\tau_1(y) = \mathrm{e}^{-y} \text{ for } y \in \mathbb{R}, \quad \tau_2(y) = y^{-\alpha} \text{ for } y > 0 \quad \text{and} \quad \tau_3(y) = (-y)^{\alpha} \text{ for } y \leq 0. \tag{2.12}$$

They are first given by Fisher and Tippett [FT28], but then completely determined by Gnedenko [G43].

EVL type, tail behaviour, and corresponding normalising constants

Asymptotic properties of the maximum can be deduced with a little information about the distribution function F. We emphasise that, as observed in [G43], for independent and identically distributed sequences of random variables, the limiting distribution type of the partial maxima is completely determined by the tail of the distribution function F, *i.e.*, $\bar{F} = 1 - F$. Namely, as it can also be found in [LLR83, Theorem 1.6.2], in order to obtain the respective domain of attraction for maxima we have the following sufficient and necessary conditions on the tail of the distribution function F:

Type 1: (Gumbel) $H(y) = \mathrm{e}^{-\tau_1(y)}$ iff there exists some strictly positive function $h : \mathbb{R} \to \mathbb{R}$ such that for all $y \in \mathbb{R}$

$$\lim_{s \to u_F} \frac{\bar{F}(s + yh(s))}{\bar{F}(s)} = \mathrm{e}^{-y}; \tag{2.13}$$

Type 2: (Fréchet) $H(y) = \mathrm{e}^{-\tau_2(y)}$ iff $u_F = +\infty$ and there exists $\beta > 0$ such that for all $y > 0$

$$\lim_{s \to u_F} \frac{\bar{F}(sy)}{\bar{F}(s)} = y^{-\beta}; \tag{2.14}$$

Type 3: (Weibull) $H(y) = \mathrm{e}^{-\tau_3(y)}$ iff $u_F < +\infty$ and there exists $\beta > 0$ such that for all $y > 0$

$$\lim_{s \to 0} \frac{\bar{F}(u_F - sy)}{\bar{F}(u_F - s)} = y^{\beta}. \tag{2.15}$$

Moreover, as it is given in [LLR83, Corollary 1.6.3] the normalising constants a_n and b_n to get the corresponding extreme value laws are as follows:

Type 1: $a_n = [g(\gamma_n)]^{-1}$, $b_n = \gamma_n$;

Type 2: $a_n = \gamma_n^{-1}$, $b_n = 0$;

Type 3: $a_n = (u_F - \gamma_n)^{-1}$, $b_n = u_F$,

where $\gamma_n = F^{-1}(1 - 1/n) = \inf\{x; F(x) \geq 1 - 1/n\}$.

Max-stable distributions

As a further remark, we may add that EVLs are identified with the so-called *max-stable distributions*, a class of distributions with a certain stability property. A non-degenerate distribution function H is *max-stable* if for each $n = 2, 3, \ldots$, there are constants $a_n > 0$ and b_n such that

$$H^n(a_n x + b_n) = H(x).$$

We may say that two distribution functions H_1, H_2 are of the same type if

$$H_2(x) = H_1(ax + b)$$

for some constants $a > 0, b$. Then a non-degenerate distribution function H is max-stable if for each $n = 2, 3, \ldots$, the distribution function H^n is of the same type as H. The distribution functions may be divided into equivalence classes (types) by saying that H_1 and H_2 are equivalent if $H_2(x) = H_1(ax + b)$ for some $a > 0, b$. From Khintchine theorem, see [LLR83, Theorem 1.2.3], we get the following:

- if H_1, H_2 are of the same type, then $Domain(H_1) = Domain(H_2)$;
- if F belongs to both $Domain(H_1)$ and $Domain(H_2)$, then H_1 and H_2 are of the same type. Hence $Domain(H_1)$ and $Domain(H_2)$ are identical if H_1 and H_2 are of the same type, and disjoint otherwise.

As a result, we can say that the domain of attraction of a distribution function depends only on its type.

Existence of EVLs

Definition 2.2.1. We say that we have an *EVL* for M_n if there is a non-degenerate distribution function $H : \mathbb{R} \to [0, 1]$ with $H(0) = 0$ and, for every $\tau > 0$, there exists a sequence of levels $u_n = u_n(\tau)$, $n = 1, 2, \ldots$, such that

$$n\,\mathbb{P}(X_0 > u_n) \to \tau, \quad \text{as } n \to \infty, \tag{2.16}$$

and for which the following holds:

$$\mathbb{P}(M_n \leq u_n) \to \bar{H}(\tau), \quad \text{as } n \to \infty. \tag{2.17}$$

The motivation for using a normalising sequence u_n satisfying (2.16) comes from the classical theory. When $X_0, X_1, X_2, \dots$ are independent and identically distributed, it is clear that $\mathbb{P}(M_n \leq u) = (F(u))^n$. Hence, condition (2.16) implies that

$$\mathbb{P}(M_n \leq u_n) = (1 - \mathbb{P}(X_0 > u_n))^n \sim \left(1 - \frac{\tau}{n}\right)^n \to \mathrm{e}^{-\tau},$$

as $n \to \infty$. Moreover, the converse is also true. Note that in this case $H(\tau) = 1 - \mathrm{e}^{-\tau}$ is the standard exponential distribution function.

For every sequence $(u_n)_{n \in \mathbb{N}}$ satisfying (2.16) we define:

$$U_n := \{X_0 > u_n\} \tag{2.18}$$

Stationary sequences, dependence conditions and extremal index

The second step in the study of extremes for stochastic processes is as follows. When $X_0, X_1, X_2, \dots$ are not independent but satisfy some mixing condition $D(u_n)$ introduced by Leadbetter in [L73], we can still say something about H. Let $F_{i_1,\dots,i_n}$denote the joint distribution function of $X_{i_1}, \dots, X_{i_n}$, and set $F_{i_1,\dots,i_n}(u) = F_{i_1,\dots,i_n}(u, \dots, u)$.

Condition $(D(u_n))$**.** We say that $D(u_n)$ holds for the sequence $X_0, X_1, \dots$ if for any integers $i_1 < \dots < i_p$ and $j_1 < \dots < j_k$ for which $j_1 - i_p > m$, and any large $n \in \mathbb{N}$,

$$\left|F_{i_1,\dots,i_p,j_1,\dots,j_k}(u_n) - F_{i_1,\dots,i_p}(u_n) F_{j_1,\dots,j_k}(u_n)\right| \leq \gamma(n,t),$$

where $\gamma(n, t_n) \xrightarrow[n \to \infty]{} 0$ for some sequence $t_n = o(n)$.

If $D(u_n)$ holds for $X_0, X_1, \dots$ and the limit (2.17) exists for some $\tau > 0$ then there exists $0 < \vartheta \leq 1$ such that $\bar{H}(\tau) = \mathrm{e}^{-\vartheta\tau}$ for all $\tau > 0$ (see [L83, Theorem 2.2] or [LLR83, Theorem 3.7.1]).

Definition 2.2.2. We say that $X_0, X_1, \ldots$ has an *Extremal Index* (EI) $0 < \vartheta \leq 1$ if we have an EVL for M_n with $\bar{H}(\tau) = \mathrm{e}^{-\vartheta\tau}$ for all $\tau > 0$.

The notion of the EI was latent in the work of Loynes [L65] but was established formally by Leadbetter in [L83]. It gives a measure of the strength of the dependence of $X_0, X_1, \ldots$ in such a way that $\vartheta = 1$ indicates that the process has practically no memory while $\vartheta = 0$, conversely, reveals extremely long memory. Another way of looking at the EI is that it gives some indication of how much exceedances of high levels have a tendency to "cluster". Namely, for $\vartheta > 0$ this interpretation of the EI is that ϑ^{-1} is the mean number of exceedances of a high level in a cluster of large observations, *i.e.*, is the "mean size of the clusters".

In fact, as it is given in [L73], if an anti-clustering condition, namely $D'(u_n)$, as defined below, holds in addition to $D(u_n)$, one can show that the EI is 1. Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of integers such that

$$k_n \to \infty \quad \text{and} \quad k_n t_n = o(n), \tag{2.19}$$

for some sequence $t_n = o(n)$.

Condition $(D'(u_n))$**.** We say that $D'(u_n)$ holds for the sequence $X_0, X_1, \ldots$ if there exists a sequence $\{k_n\}_{n\in\mathbb{N}}$ satisfying (2.19) and such that

$$\lim_{n\to\infty} n \sum_{j=1}^{\lfloor n/k_n \rfloor} \mathbb{P}(X_0 > u_n, X_j > u_n) = 0. \tag{2.20}$$

We remark that under conditions $D(u_n)$ and $D'(u_n)$ we have an EVL for M_n such that $\bar{H}(\tau) = \mathrm{e}^{-\tau}$, as in the independent case.

Condition $D(u_n)$ can be seen as a long range mixing condition. It requires that when considering two blocks of random variables, with a time gap between them, the dependence of the events corresponding to no exceedances among these blocks fades away as the size of the gap increases. On the other hand, condition $D'(u_n)$ is a short range dependence condition. If we break the first n random variables into blocks of size $\lfloor n/k_n \rfloor$, then $D'(u_n)$ restricts the existence of more than one exceedance in each block, that is to say that the exceedances of high thresholds should be scattered in the time line.

The study of EVLs in the context of dynamical systems started with the pioneer work of Collet, [C01]. Although it is not explicitly written in his paper, it is clear that he realises the connection between EVLs and HTS in the sense that studying maxima corresponds to hitting some set near ζ. Another important aspect of Collet's paper lies in the proof of [C01, Lemma 3.3]. There, he uses a very simple event to use decay of correlations against L^1. Motivated by this fact, in [FF08a], the authors introduced a weaker version of the condition $D(u_n)$, which they named as $D_2(u_n)$. The importance and ease of condition $D_2(u_n)$ comes from the fact that the first block of random variables in $D(u_n)$ consists only of the first one in the stochastic process. It is proved in [FF08a, Section 2] that it replaces the condition $D(u_n)$ with an additional use of $D'(u_n)$. The main advantage of $D_2(u_n)$ is that it follows immediately for stochastic processes derived from dynamical systems with sufficiently fast decay of correlations, which is also the case for us. Hence, we use this condition in our investigation.

Condition $(D_2(u_n))$**.** We say that $D_2(u_n)$ holds for the sequence $X_0, X_1, \dots$ if for all ℓ, t and n

$$|\mathbb{P}\left(X_0 > u_n \cap \max\{X_t, \dots, X_{t+\ell-1} \leq u_n\}\right) - \mathbb{P}(X_0 > u_n)\mathbb{P}(M_\ell \leq u_n)| \leq \gamma(n,t),$$

where $\gamma(n,t)$ is decreasing in t for each n and $n\gamma(n,t_n) \to 0$ when $n \to \infty$ for some sequence $t_n = o(n)$.

By [FF08a, Theorem 1], if conditions $D_2(u_n)$ and $D'(u_n)$ hold for $X_0, X_1, \dots$ then there exists an EVL for M_n and $H(\tau) = 1 - \mathrm{e}^{-\tau}$.

Proposition 2.2.3. *$D_2(u_n)$ and $D'(u_n)$ are invariant under conjugacy.*

Proof. Let $T_i : \mathcal{X}_i \to \mathcal{X}_i$, $i = 1, 2$, be two systems with stationary measures $\mathbb{P}_i$, $i = 1, 2$.

Assume that they are conjugate, *i.e.*, there is a homeomorphism h such that $h \circ T_1 = T_2 \circ h$, and $h_* \mathbb{P}_1 = \mathbb{P}_2$. Observe that:

1. If we have a stochastic process $X_0, X_1, \dots$ derived from the first system in a way that $X_i = \varphi \circ T_1^i$ then, using h, we can write $Y_i = (\varphi \circ h^{-1}) \circ T_2^i$ for the stochastic process derived from the second system.

2. Using conjugacy we can write $Y_j = \varphi \circ T_1^j \circ h^{-1}$. In particular $Y_0 = \varphi \circ h^{-1}$.

Assume that $D_2(u_n)$ holds for T_1:

$$\left|\mathbb{P}_1\big(\{\varphi > u_n\} \cap \{M_{t,\ell} \le u_n\}\big) - \mathbb{P}_1\big(\{\varphi > u_n\}\big)\mathbb{P}_1\big(\{M_\ell \le u_n\}\big)\right| \le \gamma(n,t)$$

for some integers ℓ, t and n. We have to show that

$$\left|\mathbb{P}_2\big(\{\varphi \circ h^{-1} > u_n\} \cap \{M'_{t,\ell} \le u_n\}\big) - \mathbb{P}_2\big(\{\varphi \circ h^{-1} > u_n\}\big)\mathbb{P}_2\big(\{M'_\ell \le u_n\}\big)\right| \le \gamma'(n,t).$$

But this is a simple consequence of the previous observations together with the following relation:

$$\{M'_{t,\ell} \le u_n\} = h^{-1}\{M_{t,\ell} \le u_n\},$$

where $M_{t,\ell} := \max\{X_t, X_{t+1}, \dots, X_{t+\ell-1}\}$ and $M'_{t,\ell} := \max\{Y_t, Y_{t+1}, \dots, Y_{t+\ell-1}\}$.

Now, assume that $D'(u_n)$ holds for T_1:

$$\lim_{n\to\infty} n \sum_{j=1}^{\lfloor n/k_n \rfloor} \mathbb{P}_1\big(\{\varphi > u_n\} \cap \{\varphi \circ T_1^j > u_n\}\big) = 0.$$

We want to show that

$$\lim_{n\to\infty} n \sum_{j=1}^{\lfloor n/k_n \rfloor} \mathbb{P}_2\big(\{Y_0 > u_n\} \cap \{Y_j > u_n\}\big) = 0. \tag{2.21}$$

We have:

$$\begin{aligned}\sum_{j=1}^{\lfloor n/k_n\rfloor} \mathbb{P}_2\big(\{Y_0>u_n\}\cap\{Y_j>u_n\}\big) &= \sum_{j=1}^{\lfloor n/k_n\rfloor} \mathbb{P}_2\big(\{\varphi\circ h^{-1}>u_n\}\cap\{\varphi\circ T_1^j\circ h^{-1}>u_n\}\big)\\ &= \sum_{j=1}^{\lfloor n/k_n\rfloor} h_*\mathbb{P}_1\big(\{\varphi\circ h^{-1}>u_n\}\cap\{\varphi\circ T_1^j\circ h^{-1}>u_n\}\big)\\ &= \sum_{j=1}^{\lfloor n/k_n\rfloor} \mathbb{P}_1\big(h^{-1}\big(\{\varphi\circ h^{-1}>u_n\}\cap\{\varphi\circ T_1^j\circ h^{-1}>u_n\}\big)\big),\end{aligned}$$

and since we clearly have

$$h^{-1}\big(\{\varphi\circ h^{-1}>u_n\}\cap\{\varphi\circ T_1^j\circ h^{-1}>u_n\}\big)=\{\varphi>u_n\}\cap\{\varphi\circ T_1^j>u_n\},$$

one easily deduces (2.21). The proposition follows. □

Clustering and periodicity

As we already mentioned above, condition $D'(u_n)$ prevents the existence of clusters of exceedances, which implies that the EVL is standard exponential, $\bar{H}(\tau)=\mathrm{e}^{-\tau}$. However, when $D'(u_n)$ does not hold, clustering of exceedances is responsible for the appearance of a parameter $0<\vartheta<1$ in the EVL, called the EI, which implies that, in this case, $\bar{H}(\tau)=\mathrm{e}^{-\vartheta\tau}$. In [FFT12], the authors established a connection between the existence of an EI less than 1 and periodic behaviour. This was later generalised for REPP in [FFT12a]. Namely, this phenomenon of clustering appeared when ζ is a repelling periodic point. We assume that the invariant measure $\mathbb{P}$ and the observable φ are sufficiently regular so that besides (R1), we also have the following condition:

(R2) If $\zeta\in\mathcal{X}$ is a repelling periodic point, of prime period[3] $p\in\mathbb{N}$, then we have that the periodicity of ζ implies that for all large u, $\{X_0>u\}\cap f^{-p}(\{X_0>u\})\neq\emptyset$ and the fact that the prime period is p implies that $\{X_0>u\}\cap f^{-j}(\{X_0>u\})=\emptyset$ for all $j=1,\dots,p-1$. Moreover, the fact that ζ is repelling means that we

[3]i.e., the *smallest* $n\in\mathbb{N}$ such that $f^n(\zeta)=\zeta$. Clearly $f^{ip}(\zeta)=\zeta$ for any $i\in\mathbb{N}$.

have backward contraction which means that there exists $0 < \vartheta < 1$ such that $\bigcap_{j=0}^{i} f^{-jp}(X_0 > u)$ corresponds to another ball of smaller radius around ζ with $\mathbb{P}\left(\bigcap_{j=0}^{i} f^{-jp}(X_0 > u)\right) \sim (1-\vartheta)^i \mathbb{P}(X_0 > u)$,[4] for all u sufficiently close to u_F.

The main obstacle when dealing with periodic points is that they create plenty of dependence in the short range. In particular, using (R2) we have that for all u sufficiently large,

$$\mathbb{P}(\{X_0 > u\} \cap \{X_p > u\}) \sim (1-\vartheta)\mathbb{P}(X_0 > u)$$

which implies that $D'(u_n)$ is not satisfied, since for the levels u_n as in (2.16) it follows that

$$n \sum_{j=1}^{[n/k_n]} \mathbb{P}(X_0 > u_n, X_j > u_n) \geq n\mathbb{P}(X_0 > u_n, X_p > u_n) \xrightarrow[n\to\infty]{} (1-\vartheta)\tau.$$

To overcome this difficulty around periodic points the key observation is that around periodic points one just needs to replace the topological ball $\{X_0 > u_n\}$ by the topological annulus

$$Q_p(u) := \{X_0 > u,\ X_p \leq u\}. \tag{2.22}$$

Then much of the analysis works out as in the absence of clustering. Note that $Q_p(u)$ is obtained by removing from $U(u)$ the points that were doomed to return after p steps, which form the smaller ball $U(u) \cap f^{-p}(U(u))$. Then, the crucial observation is that the limit law corresponding to no entrances up to time n into the ball $U(u_n)$ is equal to the limit law corresponding to no entrances into the annulus $Q_p(u_n)$ up to time n.

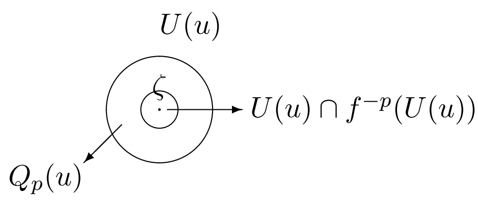

Fig.1 - Annulus $Q_p(u)$

[4] $A_u \sim B_u \Leftrightarrow \lim_{u\to u_F} \frac{A_u}{B_u} = 1$

To give an intuitive idea of the EI, let us state the following. As it is easily seen from the Fig.1, the set $U(u)$ can be decomposed into non-intersecting components $Q_p(u)$ and $U(u) \cap f^{-p}(U(u))$. Using this, (R2) and the definition of $Q_p(u)$ given in (2.22), we can write the following:

$$\mathbb{P}(\{X_0 > u\} \cap \{X_p \leq u\}) \sim \vartheta \mathbb{P}(X_0 > u), \tag{2.23}$$

which means that the EI, ϑ, measures the proportion of the points, or the amount of the mass, that escapes the periodic phenomena.

For future purposes, let us note that $Q_{p,0}(u_n) := Q_p(u_n)$.

In what follows for every $A \in \mathcal{B}$, we denote the complement of A as $A^c := \mathcal{X} \setminus A$. For $s \leq \ell \in \mathbb{N}_0$, we define

$$\mathscr{Q}_{p,s,\ell}(u) = \bigcap_{i=s}^{s+\ell-1} f^{-i}(Q_p(u))^c, \tag{2.24}$$

which corresponds to no entrances in the annulus from time s to $s+\ell-1$. Sometimes to abbreviate we also write: $\mathscr{Q}_\ell(u) := \mathscr{Q}_{p,0,\ell}(u)$.

Theorem 2.2.4 ([FFT12, Proposition 1])**.** *Let $X_0, X_1, , \ldots$ be a stochastic process defined by* (2.6) *where φ achieves a global maximum at a repelling periodic point $\zeta \in \mathcal{X}$, of prime period $p \in \mathbb{N}$, so that conditions (R1) and (R2) above hold. Let $(u_n)_n$ be a sequence of levels such that* (2.16) *holds. Then,* $\lim_{n\to\infty} \mathbb{P}(M_n \leq u_n) = \lim_{n\to\infty} \mathbb{P}(\mathscr{Q}_n(u_n))$.

Hence, the idea to cope with clustering caused by periodic points is to adapt conditions $D_2(u_n)$ and $D'(u_n)$, letting annuli replace balls. In order to make the theory as general as possible, motivated by the above considerations for stochastic processes generated by dynamical systems around periodic points, some abstract conditions were given in [FFT12] to prove the existence of an EI less than 1 for general stationary stochastic processes.

The first one establishes exactly the type of periodic behaviour assumed, namely:

Condition ($\mathrm{SP}_{p,\vartheta}(u_n)$)**.** We say that $X_0, X_1, X_2, \ldots$ satisfies condition $\mathrm{SP}_{p,\vartheta}(u_n)$ for $p \in \mathbb{N}$ and $\vartheta \in [0,1]$ if

$$\lim_{n\to\infty} \sup_{1\le j<p} \mathbb{P}(X_j > u_n | X_0 > u_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathbb{P}(X_p > u_n | X_0 > u_n) \to (1-\vartheta) \tag{2.25}$$

and moreover

$$\lim_{n\to\infty} \sum_{i=0}^{[\frac{n-1}{p}]} \mathbb{P}(X_0 > u_n, X_p > u_n, X_{2p} > u_n, \ldots, X_{ip} > u_n) = 0. \tag{2.26}$$

Condition (2.25), when $\vartheta < 1$, imposes some sort of periodicity of period p among the exceedances of high levels u_n, since if at some point the process exceeds the high level u_n, then, regardless of how high u_n is, there is always a strictly positive probability of another exceedance occurring at the (finite) time p. In fact, if the process is generated by a deterministic dynamical system $f : \mathcal{X} \to \mathcal{X}$ and f is continuous then (2.25) implies that ζ is a periodic point of period p.

The next two conditions concern the dependence structure of $X_0, X_1, \ldots$ and can be described as being obtained from $D_2(u_n)$ and $D'(u_n)$ by replacing balls by annuli.

Condition ($D^p(u_n)$)**.** We say that $D^p(u_n)$ holds for the sequence $X_0, X_1, X_2, \ldots$ if for any integers ℓ, t and n $|\mathbb{P}(Q_{p,0}(u_n) \cap \mathscr{Q}_{p,t,\ell}(u_n)) - \mathbb{P}(Q_{p,0}(u_n))\mathbb{P}(\mathscr{Q}_{p,0,\ell}(u_n))| \le \gamma(n,t)$, where $\gamma(n,t)$ is non increasing in t for each n and $n\gamma(n,t_n) \to 0$ as $n \to \infty$ for some sequence $t_n = o(n)$.

As with $D_2(u_n)$, the main advantage of this condition when compared to Leadbetter's $D(u_n)$ (or others of the same sort) is that it follows directly from sufficiently fast decay of correlations as observed in [F12, Section 5.1], on the contrary to $D(u_n)$.

Assuming $D^p(u_n)$ holds let $(k_n)_{n\in\mathbb{N}}$ be a sequence of integers such that (2.19) holds.

Condition ($D'_p(u_n)$)**.** We say that $D'_p(u_n)$ holds for $X_0, X_1, X_2, \ldots$ if there exists a sequence $(k_n)_{n\in\mathbb{N}}$ satisfying (2.19) and such that

$$\lim_{n\to\infty} n \sum_{j=1}^{[n/k_n]} \mathbb{P}(Q_{p,0}(u_n) \cap Q_{p,j}(u_n)) = 0. \tag{2.27}$$

One of the main results in [FFT12] is:

Theorem 2.2.5 ([FFT12, Theorem 1])**.** *Let $(u_n)_{n\in\mathbb{N}}$ be such that* (2.16) *holds. Consider a stationary stochastic process $X_0, X_1, \ldots$ be a stochastic process defined by* (2.6) *where φ achieves a global maximum at a repelling periodic point $\zeta \in \mathcal{X}$, of prime period $p \in \mathbb{N}$, so that conditions (R1) and (R2) above hold. Assume further that conditions $D^p(u_n)$ and $D'_p(u_n)$ hold. Then* $\lim_{n\to\infty} \mathbb{P}(M_n \le u_n) = \lim_{n\to\infty} \mathbb{P}(\mathscr{Q}_{p,0,n}(u_n)) = \mathrm{e}^{-\vartheta\tau}$.

Computing the EI

In order to prove the existence of an EI around a repelling periodic point, we may use Theorem 2.2.5 and, basically, observe that, once conditions $D^p(u_n)$ and $D'_p(u_n)$ are verified, by (R2) the EI may be computed from the formula:

$$\vartheta = \lim_{n\to\infty} \frac{\mathbb{P}(Q_{p,0}(u_n))}{\mathbb{P}(U_n)}. \tag{2.28}$$

2.3 Rare Event Point Processes

Closely related to the properties of extremes are those of exceedances and up crossings of high levels by sequences and continuous parameter processes. By considering such exceedances and up crossings one may obtain some quite general results proving convergence to Poisson and related point processes. If we consider multiple exceedances we are led to point processes of rare events counting the number of exceedances in a certain time frame. For every $A \subset \mathbb{R}$ we define

$$\mathscr{N}_u(A) := \sum_{i\in A\cap\mathbb{N}_0} \mathbf{1}_{X_i > u}.$$

In the particular case where $A = I = [a, b)$ we simply write $\mathscr{N}_{u,a}^b := \mathscr{N}_u([a, b))$. Observe that $\mathscr{N}_{u,0}^n$ counts the number of exceedances amongst the first n observations of the process $X_0, X_1, \ldots, X_n$ or, in other words, the number of entrances in $U(u)$ up to time n. Also, note that

$$\{\mathscr{N}_{u,0}^n = 0\} = \{M_n \le u\}. \tag{2.29}$$

In order to define a point process that captures the essence of an EVL and HTS through (2.29), we need to re-scale time using the factor $v := 1/\mathbb{P}(X > u)$ given by Kac's Theorem. However, before giving the definition, we need some formalism. Let $\mathcal{S}$ denote the semi-ring of subsets of $\mathbb{R}_0^+$ whose elements are intervals of the type $[a,b)$, for $a,b \in \mathbb{R}_0^+$. Let $\mathcal{R}$ denote the ring generated by $\mathcal{S}$. Recall that for every $J \in \mathcal{R}$ there are $k \in \mathbb{N}$ and k intervals $I_1, \dots, I_k \in \mathcal{S}$ such that $J = \cup_{i=1}^k I_j$. In order to fix notation, let $a_j, b_j \in \mathbb{R}_0^+$ be such that $I_j = [a_j, b_j) \in \mathcal{S}$. For $I = [a,b) \in \mathcal{S}$ and $\alpha \in \mathbb{R}$, we denote $\alpha I := [\alpha a, \alpha b)$ and $I + \alpha := [a+\alpha, b+\alpha)$. Similarly, for $J \in \mathcal{R}$ define $\alpha J := \alpha I_1 \cup \cdots \cup \alpha I_k$ and $J + \alpha := (I_1 + \alpha) \cup \cdots \cup (I_k + \alpha)$.

Definition 2.3.1. We define the *rare event point process* (REPP) by counting the number of exceedances (or hits to $U(u_n)$) during the (re-scaled) time period $v_n J \in \mathcal{R}$, where $J \in \mathcal{R}$. To be more precise, for every $J \in \mathcal{R}$, set

$$N_n(J) := \mathscr{N}_{u_n}(v_n J) = \sum_{j \in v_n J \cap \mathbb{N}_0} \mathbf{1}_{X_j > u_n}. \tag{2.30}$$

Under similar dependence conditions to the ones just seen above, the REPP just defined converges in distribution to a standard Poisson process, when no clustering is involved and to a compound Poisson process with intensity ϑ and a geometric multiplicity distribution function, otherwise. For completeness, we define here what we mean by a Poisson and a compound Poisson process. We refer to [K86] for more details on this subject.

Definition 2.3.2. Let $T_1, T_2, \dots$ be an independent and identically distributed sequence of random variables with common exponential distribution of mean $1/\vartheta$. Let $D_1, D_2, \dots$ be another independent and identically distributed sequence of random variables, independent of the previous one, and with distribution function π. Given these sequences, for $J \in \mathcal{R}$, set

$$N(J) = \int \mathbf{1}_J \, \mathrm{d}\left(\sum_{i=1}^{\infty} D_i \delta_{T_1 + \ldots + T_i}\right),$$

where δ_t denotes the Dirac measure at $t > 0$. Whenever we are in this setting, we say that N is a compound Poisson process of intensity ϑ and multiplicity distribution function π.

Remark 2.3.3. In this work, the multiplicity will always be integer valued which means that π is completely defined by the values $\pi_k = \mathbb{P}(D_1 = k)$, for every $k \in \mathbb{N}_0$. Note that, if $\pi_1 = 1$ and $\vartheta = 1$, then N is the standard Poisson process and, for every $t > 0$, the random variable $N([0,t))$ has a Poisson distribution of mean t.

Remark 2.3.4. When clustering is involved, we will see that π is actually a geometric distribution of parameter $\vartheta \in (0,1]$, *i.e.*, $\pi_k = \vartheta(1-\vartheta)^k$, for every $k \in \mathbb{N}_0$. This means that, as in [HV09], here, the random variable $N([0,t))$ follows a Pólya-Aeppli distribution, *i.e.*:

$$\mathbb{P}(N([0,t)) = k) = \mathrm{e}^{-\vartheta t} \sum_{j=1}^{k} \vartheta^j (1-\vartheta)^{k-j} \frac{(\vartheta t)^j}{j!} \binom{k-1}{j-1},$$

for all $k \in \mathbb{N}$ and $\mathbb{P}(N([0,t)) = 0) = \mathrm{e}^{-\vartheta t}$.

When $D'(u_n)$ holds, since there is no clustering, then, due to a criterion proposed by Kallenberg [K86, Theorem 4.7], which applies only to simple point processes, without multiple events, ***i.e.,*** $\pi_1 = 1$, we can simply adjust condition $D_2(u_n)$ to this scenario of multiple exceedances in order to prove that the REPP converges in distribution to a standard Poisson process. We denote this adapted condition by:

Condition $(D_3(u_n))$**.** Let $A \in \mathcal{R}$ and $t \in \mathbb{N}$. We say that $D_3(u_n)$ holds for the sequence $X_0, X_1, \ldots$ if

$$|\mathbb{P}\left(\{X_0 > u_n\} \cap \{\mathscr{N}(A+t) = 0\}\right) - \mathbb{P}(\{X_0 > u_n\})\mathbb{P}(\mathscr{N}(A) = 0)| \leq \gamma(n,t),$$

where $\gamma(n,t)$ is non-increasing in t for each n and $n\gamma(n,t_n) \to 0$ as $n \to \infty$ for some sequence $t_n = o(n)$, which means that $t_n/n \to 0$ as $n \to \infty$.

Condition $D_3(u_n)$ follows, as easily as $D_2(u_n)$, from sufficiently fast decay of correlations.

In [FFT10, Theorem 5] a strengthening of [FF08a, Theorem 1] is proved, which essentially says that, under $D_3(u_n)$ and $D'(u_n)$, the REPP N_n defined in (2.30) converges in distribution to a standard Poisson process.

Regarding the convergence of the REPP, when there is clustering, one cannot use the aforementioned criterion of Kallenberg because the point processes are not simple anymore and possess multiple events. This means that a much deeper analysis must be done in order to obtain convergence of the REPP. The detailed investigation is carried out in [FFT12a], below we describe the main results and conditions we need. First, we define the sequence $\left(U^{(\kappa)}(u)\right)_{\kappa\geq 0}$ of nested balls centred at ζ given by:

$$U^{(0)}(u) = U(u) \quad \text{and} \quad U^{(\kappa)}(u) = f^{-p}(U^{(\kappa-1)}(u)) \cap U(u), \quad \text{for all } \kappa \in \mathbb{N}. \tag{2.31}$$

For $i, \kappa, \ell, s \in \mathbb{N} \cup \{0\}$, we define the following events:

$$Q^{\kappa}_{p,i}(u) := f^{-i}\left(U^{(\kappa)}(u) - U^{(\kappa+1)}(u)\right). \tag{2.32}$$

Observe that for each κ, the set $Q^{\kappa}_{p,0}(u)$ corresponds to an annulus centred at ζ. Besides, $U(u) = \bigcup_{\kappa=0}^{\infty} Q^{\kappa}_{p,0}(u)$, which means that the ball centred at ζ which corresponds to $U(u)$ can be decomposed into a sequence of disjoint annuli where $Q^{0}_{p,0}(u)$ is the most outward ring and the inner ring $Q^{\kappa+1}_{p,0}(u)$ is sent outward by f^p to the ring $Q^{\kappa}_{p,0}(u)$, i.e., $f^p(Q^{\kappa+1}_{p,0}(u)) = Q^{\kappa}_{p,0}(u)$. See the pictures below.

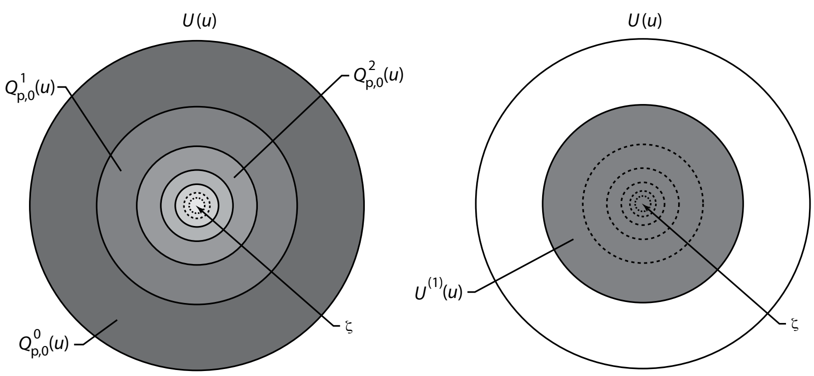

Fig.2 - Annuli $Q^{\kappa}_{p,0}(u)$ and sets $U^{(\kappa)}(u)$

We are now ready to state:

Condition $(D_p(u_n)^*)$**.** We say that $D_p(u_n)^*$ holds for the sequence $X_0, X_1, X_2, \ldots$ if for any integers $t, \kappa_1, \ldots, \kappa_\varsigma$, n and any $J = \cup_{j=2}^{\varsigma} I_j \in \mathcal{R}$ with $\inf\{x : x \in J\} \geq t$,

$$\left|\mathbb{P}\left(Q_{p,0}^{\kappa_1}(u_n) \cap \left(\cap_{j=2}^{\varsigma} \mathscr{N}_{u_n}(I_j) = \kappa_j\right)\right) - \mathbb{P}\left(Q_{p,0}^{\kappa_1}(u_n)\right) \mathbb{P}\left(\cap_{j=2}^{\varsigma} \mathscr{N}_{u_n}(I_j) = \kappa_j\right)\right| \leq \gamma(n,t),$$

where for each n we have that $\gamma(n,t)$ is non-increasing in t and $n\gamma(n,t_n) \to 0$ as $n \to \infty$, for some sequence $t_n = o(n)$.

This mixing condition is stronger than $D^p(u_n)$ because it requires a uniform bound for all possible integer values of κ_1, nonetheless, it still is much weaker than the original $D(u_n)$ from Leadbetter [L73] or any of the kind. As all the other preceding conditions (D_2, D_3, D^p) it can be easily verified for systems with sufficiently fast decay of correlations (see [F12, Section 5.1]).

In [FFT12a], for technical reasons only, the authors also introduced a slight modification of $D'_p(u_n)$. The new condition was denoted by $D'_p(u_n)^*$ and was given as follows.

Condition $(D'_p(u_n)^*)$**.** We say that $D'_p(u_n)^*$ holds for the sequence $X_0, X_1, X_2, \ldots$ if there exists a sequence $\{k_n\}_{n\in\mathbb{N}}$ satisfying (2.19) and such that

$$\lim_{n\to\infty} n \sum_{j=1}^{[n/k_n]} \mathbb{P}(Q_{p,0}(u_n) \cap \{X_j > u_n\}) = 0. \tag{2.33}$$

We can now state the main theorem in [FFT12a].

Theorem 2.3.5 ([FFT12a, Theorem 1])**.** *Let $X_0, X_1, \ldots$ be given by* (2.6)*, where φ achieves a global maximum at the repelling periodic point ζ, of prime period p, and conditions (R1) and (R2) hold. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence satisfying* (2.16)*. Assume that conditions $D_p(u_n)^*$, $D'_p(u_n)^*$ hold. Then the REPP N_n converges in distribution to a compound Poisson process N with intensity ϑ and multiplicity distribution function π given by $\pi(\kappa) = \vartheta(1-\vartheta)^\kappa$, for every $\kappa \in \mathbb{N}_0$, where the extremal index ϑ is given by the expansion rate at ζ stated in (R2).*

Computing the multiplicity distribution

In order to compute the multiplicity distribution of the limiting compound Poisson process for the REPP, when ζ is a repelling periodic point, we can use the following estimate :

Lemma 2.3.6 ([FFT12a, Corollary 2.4]). *Assuming that φ achieves a global maximum at the repelling periodic point ζ, of prime period p, and conditions (R1) and (R2) hold, there exists $C > 0$ depending only on ϑ given by property (R2) such that for any s,$\kappa \in \mathbb{N}$ and u sufficiently close to $u_F = \varphi(\zeta)$ we have for $\kappa > 0$*

$$\left|\mathbb{P}\big(\mathscr{N}_{u,0}^{s+1} = \kappa\big) - s\left(\mathbb{P}(Q_{p,0}^{\kappa-1}(u)) - \mathbb{P}(Q_{p,0}^{\kappa}(u))\right)\right| \\ \leq 4s\sum_{j=p+1}^{s}\mathbb{P}(Q_{p,0}^{0}(u)\cap\{X_j > u\}) + 2C\,\mathbb{P}(X_0 > u_n),$$

and in the case $\kappa = 0$

$$\left|\mathbb{P}\big(\mathscr{N}_{u,0}^{s+1} = 0\big) - \big(1 - s\mathbb{P}(Q_{p,0}^{0}(u))\big)\right| \leq 2s\sum_{j=p+1}^{s}\mathbb{P}(Q_{p,0}^{0}(u)\cap\{X_j > u\}) + C\,\mathbb{P}(X_0 > u).$$

The idea then is to realise that in the proof of Theorem 2.3.5 one splits the first n random variables $X_0, \ldots, X_{n-1}$ into blocks of size $\lfloor n/k_n\rfloor$ with a time gap of size t_n between them. Then using the asymptotic "independence" obtained from $D^p(u_n)^*$ and $D'_p(u_n)^*$ we get the compound Poisson limit with multiplicity distribution determined by the distributional limit of the number of exceedances in each block of size $\lfloor n/k_n\rfloor$, given that at least one exceedance occurs. Hence, we need to compute, for all $\kappa \in \mathbb{N}$:

$$\lim_{n\to\infty}\mathbb{P}\left(\mathscr{N}_{u_n,0}^{\lfloor n/k_n\rfloor+1} = \kappa | \mathscr{N}_{u_n,0}^{\lfloor n/k_n\rfloor+1} > 0\right).$$

Since, by $D'_p(u_n)^*$, we have that

$$\lfloor n/k_n\rfloor\sum_{j=p+1}^{\lfloor n/k_n\rfloor}\mathbb{P}(Q_{p,0}^{0}(u_n)\cap\{X_j > u_n\}) = o(1/k_n),$$

then it follows from Lemma 2.3.6 that we have

$$\pi(\kappa) = \lim_{n\to\infty}\mathbb{P}\left(\mathscr{N}_{u_n,0}^{\lfloor n/k_n\rfloor+1} = \kappa | \mathscr{N}_{u_n,0}^{\lfloor n/k_n\rfloor+1} > 0\right) = \lim_{n\to\infty}\frac{\left(\mathbb{P}(Q_{p,0}^{\kappa-1}(u_n)) - \mathbb{P}(Q_{p,0}^{\kappa}(u_n))\right)}{\mathbb{P}(Q_{p,0}^{0}(u_n))} \tag{2.34}$$

for every $\kappa \in \mathbb{N}$.

2.4 Hitting/Return Time Statistics

Now, we turn to the other approach in the study of statistical properties of rare events which regards the existence of HTS and RTS. One of the main ingredients in our study is the first return time from a set to itself. Let us introduce some definitions. We first consider the deterministic case, take a set $A \in \mathcal{B}$ in that setting.

Definition 2.4.1. We define a function that we refer to as *first hitting time function* to A and denote by $r_A : \mathcal{X} \to \mathbb{N} \cup \{+\infty\}$ where

$$r_A(x) = \min \left\{ j \in \mathbb{N} \cup \{+\infty\} : \ f^j(x) \in A \right\}.$$

The restriction of r_A to A is called the *first return time function* to A.

Definition 2.4.2. We define the *first return time* to A, which we denote by $R(A)$, as the minimum of the return time function to A, *i.e.*,

$$R(A) = \min_{x \in A} r_A(x).$$

Normalising sequences

The normalising sequences to obtain HTS/RTS are motivated by Kac's Lemma, [K47].

Theorem 2.4.3 (Kac's Lemma)**.** *Given $A \in \mathcal{B}$ with $\mu(A) > 0$, we have*

$$\int_A r_A \, \mathrm{d}\mu = \mu(\{r_A < \infty\}).$$

In particular, if μ is ergodic, then $\int_A r_A \, \mathrm{d}\mu_A = 1/\mu(A)$, which is the expected value of r_A with respect to μ_A.

So in studying the fluctuations of r_A on A, the relevant normalising factor should be $1/\mu(A)$.

Relation between HTS and RTS

The existence of exponential HTS is equivalent to the existence of exponential RTS. In fact, according to the Main Theorem in [HLV05], a system has HTS G if and only if it has RTS $\tilde{G}$ and

$$G(t) = \int_0^t (1 - \tilde{G}(s))\,\mathrm{d}s. \tag{2.35}$$

In the random case, we have to a make choice regarding the type of definition we want to play the roles of the first hitting/return times (functions). Essentially, there are two possibilities. The *quenched* perspective which consists of fixing a realisation $\underline{\omega} \in \Omega$ and define the objects in the same way as in the deterministic case. The *annealed* perspective consists of defining the same objects by averaging over all possible realisations $\underline{\omega}$. Here, we will use the quenched perspective to define hitting/return times because it will facilitate the connection between EVLs and HTS/RTS in the random setting. (We refer to [MR11] for more details on both perspectives.)

Definition 2.4.4. For some $\underline{\omega} \in \Omega$ fixed, some $x \in \mathcal{M}$ and $A \subset \mathcal{M}$ measurable, we may define the *first random hitting time*

$$r_A^{\underline{\omega}}(x) := \min\{j \in \mathbb{N} : \ f_{\underline{\omega}}^j(x) \in A\} \tag{2.36}$$

and the *first random return* from A to A as

$$R^{\underline{\omega}}(A) = \min\{r_A^{\underline{\omega}}(x) : \ x \in A\}. \tag{2.37}$$

Existence of HTS/RTS

Definition 2.4.5. Given a sequence of measurable subsets of $\mathcal{X}$, $(V_n)_{n\in\mathbb{N}}$, so that $\mathbb{P}(V_n) \to 0$, the system has (random) *HTS* G for $(V_n)_{n\in\mathbb{N}}$ if for all $t \geq 0$

$$\mathbb{P}\left(r_{V_n} \leq \frac{t}{\mathbb{P}(V_n)} \right) \to G(t) \ \text{ as } n \to \infty, \tag{2.38}$$

and the system has (random) *RTS* $\tilde{G}$ for $(V_n)_{n\in\mathbb{N}}$ if for all $t \geq 0$

$$\mathbb{P}_{V_n}\left(r_{V_n} \leq \frac{t}{\mathbb{P}(V_n)} \right) \to \tilde{G}(t) \ \text{ as } n \to \infty. \tag{2.39}$$

In deterministic case,

$$\mathcal{X} = \mathcal{M},\ \mathbb{P} = \mu \text{ and } T = f.$$

In stochastic case,

$$\mathcal{X} = \mathcal{M} \times \Omega,\ \mathbb{P} = \mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}},\ T = S \text{ defined in (2.4)},$$
$$V_n = V_n^* \times \Omega, \text{ where } V_n^* \subset \mathcal{M} \text{ and } \mu_\varepsilon(V_n^*) \to 0, \text{ as } n \to \infty.$$

Note that

$$\mathbb{P}\left(r_{V_n} \leq \frac{t}{\mathbb{P}(V_n)}\right) = \mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}}\left(r^\omega_{V_n^*} \leq \frac{t}{\mu_\varepsilon(V_n^*)}\right).$$

In [FFT10], the authors established a link between HTS/RTS (for balls) and EVLs of stochastic processes given by (2.6). This was done for invariant measures which are absolutely continuous with respect to Leb. Essentially, it was proved that if such time series have an EVL H then the system has HTS H for balls "centred" at ζ and vice versa. (Recall that having HTS H is equivalent to saying that the system has RTS $\tilde{H}$, where H and $\tilde{H}$ are related by (2.35)). This was based on the elementary observation that for stochastic processes given by (2.6) we have:

$$\{M_n \leq u\} = \{r_{\{X_0 > u\}} > n\}. \tag{2.40}$$

This connection was exploited to prove EVLs using tools from HTS/RTS and the other way around. In [FFT11], the authors took this connection further to include more general measures, which, in particular, allows us to obtain the connection in the random setting. To check that we just need to use the skew product map to look at the random setting as a deterministic system and take the observable $\bar{\varphi} : \mathcal{M} \times \Omega \to \mathbb{R} \cup \{+\infty\}$ defined as in (2.8) with $\varphi : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ as in [FFT11, equation (4.1)]. Then Theorems 1 and 2 from [FFT11] guarantee that if we have an EVL, in the sense that (2.17) holds for some distribution function H, then we have HTS for sequences $\{V_n\}_{n \in \mathbb{N}}$, where $V_n = B_{\delta_n} \times \Omega$ and $\delta_n \to 0$ as $n \to \infty$, with $G = H$ and vice-versa.

Remark 2.4.6. We remark that one of the advantages of the EVL approach for the study of rare events for stochastic dynamics is that its definition follows straightforwardly from the deterministic case. In fact, the only difference is that for randomly perturbed dynamical systems, the random variable M_n's are defined on $\mathcal{M} \times \Omega$ where we use the measure $\mathbb{P} = \mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}}$ as opposed to the deterministic case where the ambient space is $\mathcal{M}$ and $\mathbb{P} = \mu$.

Chapter 3

Deterministic Dynamics

In this chapter, we study aforementioned statistical properties of deterministic dynamical systems.

3.1 Statement of the main results

First, we give an abstract result that allows us to check conditions $D_2(u_n)$ and $D'(u_n)$ for any stochastic process $X_0, X_1, \ldots$ arising from a system which has sufficient decay of correlations against L^1 observables. As a consequence of this result in Sections 3.3 and 3.4, more precisely in Propositions 3.3.6, 3.3.8 and 3.4.2, we will obtain the announced dichotomy for the EI and the convergence of REPP based on the periodicity of the point ζ; clearly, the latter is a more general setting. By doing so, we completely characterise the extremal behaviour of expanding maps in our applications.

Theorem A. *Consider a dynamical system $(\mathcal{M}, \mathcal{B}, \mu, f)$ for which there exists a Banach space $\mathcal{C}$ of real-valued functions such that for all $\phi \in \mathcal{C}$ and $\psi \in L^1(\mu)$,*

$$Cor_\mu(\phi, \psi, n) \leq Cn^{-2}, \tag{3.1}$$

where $C > 0$ is a constant independent of both ϕ,ψ. Let $X_0, X_1, \ldots$ be given by (2.6), *where φ achieves a global maximum at some point ζ for which condition (R1) holds. Let u_n be such that* (2.16) *holds, U_n be defined as in* (2.18) *and set $R_n := R(U_n)$.*

If there exists $C' > 0$ such that for all n we have $\mathbf{1}_{U_n} \in \mathcal{C}$, $\|\mathbf{1}_{U_n}\|_{\mathcal{C}} \leq C'$ and $R_n \to \infty$, as $n \to \infty$, then conditions $D_2(u_n)$ and $D'(u_n)$ hold for $X_0, X_1, \ldots$. This implies that there is an EVL for M_n defined in (2.9) *and $H(\tau) = 1 - \mathrm{e}^{-\tau}$.*

In light of the connection between EVLs and HTS/RTS it follows immediately:

Corollary B. *Under the same hypothesis of Theorem A we have HTS/RTS for balls around ζ with $G(t) = \tilde{G}(t) = 1 - \mathrm{e}^{t}$.*

Since, under the same assumptions of Theorem A, condition $D_3(u_n)$ holds trivially then applying [FFT10, Theorem 5] we obtain:

Corollary C. *Under the same hypothesis of Theorem A, the REPP N_n defined in* (2.30) *is such that $N_n \xrightarrow{d} N$, as $n \to \infty$, where N denotes a Poisson Process with intensity* 1.

Remark 3.1.1. Note that condition $R_n \to \infty$, as $n \to \infty$, is easily verified if the map is continuous at every point of the orbit of ζ. We will state this formally in Lemma 3.2.2.

Remark 3.1.2. Observe that decay of correlations against $L^1(\mu)$ observables as in (3.1) is a very strong property. In fact, regardless of the rate (in this case n^{-2}), as long as it is summable, one can actually show that the system has exponential decay of correlations of Hölder observables, *i.e.*, $\mathcal{C}$ is the space of Hölder continuous functions with the positive Hölder constant, against $L^\infty(\mu)$. (See [AFL11, Theorem B].)

3.2 Twofold dichotomy for deterministic systems

In this section we will give the proof of Theorem A together with that of Corollary C and a simple lemma asserting that continuity is enough to guarantee that $R_n \to \infty$, as $n \to \infty$.

Proofs of Theorem A and Corollary C. As explained in [F12, Section 5.1], conditions $D_2(u_n)$ and $D_3(u_n)$ are designed to follow easily from decay of correlations. In fact, if we choose $\phi = \mathbf{1}_{U_n}$ and $\psi = \mathbf{1}_{\{M_\ell \le u_n\}}$, in the case of $D_2(u_n)$, and $\psi = \mathbf{1}_{\mathscr{N}(A)=0}$, for some $A \in \mathcal{R}$, in the case of $D_3(u_n)$, we have that we can take $\gamma(n,t) = C^* t^{-2}$, where $C^* = CC'$. Hence, conditions $D_2(u_n)$ and $D_3(u_n)$ are trivially satisfied for the sequence $(t_n)_n$ given by $t_n = n^{2/3}$, for example.

Now, we turn to condition $D'(u_n)$. Taking $\psi = \phi = \mathbf{1}_{U_n}$ in (3.1) and since $\|\mathbf{1}_{U_n}\|_{\mathcal{C}} \le C'$ we easily get

$$\mu\left(U_n \cap f^{-j}(U_n)\right) \le (\mu(U_n))^2 + C\,\|\mathbf{1}_{U_n}\|_{\mathcal{C}}\,\|\mathbf{1}_{U_n}\|_{L^1(\mu)}\,j^{-2} \le (\mu(U_n))^2 + C^*\mu(U_n)j^{-2}, \tag{3.2}$$

where $C^* = CC' > 0$. By definition of R_n, estimate (3.2) and since $n\mu(U_n) \to \tau$, as $n \to \infty$, it follows that there exists some constant $D > 0$ such that

$$\begin{aligned} n\sum_{j=1}^{\lfloor n/k_n \rfloor} \mu(U_n \cap f^{-j}(U_n)) &= n\sum_{j=R_n}^{\lfloor n/k_n \rfloor} \mu(U_n \cap f^{-j}(U_n)) \\ &\le n\lfloor \tfrac{n}{k_n} \rfloor \mu(U_n)^2 + n\,C^*\mu(U_n)\sum_{j=R_n}^{\lfloor n/k_n \rfloor} j^{-2} \\ &\le \frac{(n\mu(U_n))^2}{k_n} + n\,C^*\mu(U_n)\sum_{j=R_n}^{\infty} j^{-2} \\ &\le D\left(\frac{\tau^2}{k_n} + \tau\sum_{j=R_n}^{\infty} j^{-2}\right) \xrightarrow[n\to\infty]{} 0. \end{aligned}$$

□

Remark 3.2.1. In the above demonstration it is important to use L^1-norm to obtain the factor $\mu(U_n)$ in the second summand of the last term in (3.2), which is crucial to kill off the n factor coming from the definition of $D'(u_n)$. However, note that we actually do not need decay of correlations against L^1 in its full strength, that is to say that it holds for all L^1 functions. In fact, in order to prove $D'(u_n)$ we only need it to hold for the function $\mathbf{1}_{U_n}$.

Also, note that we do not need such a strong statement regarding the decay of correlations of the system in order to prove $D_2(u_n)$ or $D_3(u_n)$. In particular, even if $\mathbf{1}_{U_n} \notin \mathcal{C}$ (as when $\mathcal{C}$ is the space of Hölder continuous functions), we can still verify these conditions by using a suitable Hölder approximation. (See [F12, Proposition 5.2].)

According to Theorem A, in general terms, if the system has decay of correlations against L^1 observables, then to prove $D'(u_n)$ one basically has to show that $R_n \to \infty$, as $n \to \infty$. Next lemma gives us a sufficient condition for that to happen.

Lemma 3.2.2. *Assume that ζ is not a periodic point and f is continuous at every point of the orbit of ζ, namely $\zeta, f(\zeta), f^2(\zeta), \ldots$, then $\lim_{n\to\infty} R_n = \infty$, where R_n is as in Theorem A.*

Proof. Let $j \in \mathbb{N}$. We will show that if $n \in \mathbb{N}$ is sufficiently large, then $R_n > j$. Let $\epsilon = \min_{i=1,\ldots,j} \operatorname{dist}(f^i(\zeta), \zeta)$. Our assumptions assure that each f^i, for $i = 1, \ldots, j$, is continuous at ζ. Hence, for every $i = 1, \ldots, j$, there exists $\delta_i > 0$ such that $f^i(B_{\delta_i}(\zeta)) \subset B_{\epsilon/2}(f^i(\zeta))$. Let $U := \bigcap_{i=1}^{j} B_{\delta_i}(\zeta)$. If we choose N sufficiently large that $U_n \subset U$ for all $n \geq N$, then using the definition of ϵ it is clear that $f^i(U_n) \cap U_n = \emptyset$, for all $i = 1, \ldots, j$, which implies that $R_n > j$. $\square$

3.3 The dichotomy for specific systems

One of the results in [FFT12] is that for uniformly expanding systems like the doubling map there is a dichotomy in terms of the type of laws of rare events that one gets at every possible centre ζ. Namely, it was showed that either ζ is non-periodic in which case one always gets a standard exponential EVL/HTS or ζ is a periodic (repelling) point and one obtains an exponential law with an EI $0 < \vartheta < 1$ given by the expansion rate at ζ (see [FFT12, Section 6]). This was proved for cylinders rather than balls, meaning that the set U_n's are dynamically defined cylinders (see [FFT12, Section 5] or [FFT11, Section 5], for details). Results for cylinders are weaker than the ones for balls, since, in rough terms, it means that the limit is only obtained for certain subsequences of $n \in \mathbb{N}$ rather than the whole sequence.

In [FFT12], it was conjectured that this dichotomy should hold in greater generality, namely for balls rather than cylinders and for more general systems. As a consequence of Theorem A we will be able to show both. We remark that from the results in [FP12], one can also derive the dichotomy for conformal repellers and, in [K12], the dichotomy is also obtained for maps with a spectral gap for their Perron-Frobenius operator. In both these papers, the results were obtained by studying the spectral properties of the Perron-Frobenius operator.

Now, we will give some examples of systems to which we can apply Theorem A in order to prove a dichotomy regarding the existence of an EI equal to 1 or less than 1. Moreover, we will see that Corollary C yields the second, and more general, aspect of this dichotomy: convergence type of REPP, namely, standard or compound Poisson. In its both aspects, the dichotomy depends on the centre ζ being a non-periodic or periodic point, respectively. We will study the dichotomy for uniformly expanding and piecewise expanding maps, when all points in the orbit of ζ are continuity points of the map.

3.3.1 Rychlik maps

We will introduce a class of dynamical systems considered by Rychlik in [R83]. This class includes, for example, piecewise C^2 uniformly expanding maps of the unit interval with the relevant physical measures. Recall that a *physical measure* is a Borel probability measure μ on $\mathcal{M}$ for which there exists a positive Lebesgue measure set of points $x \in \mathcal{M}$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \phi(f^j(x)) = \int \phi \, \mathrm{d}\mu,$$

for any continuous function $\phi : \mathcal{M} \to \mathbb{R}$. First, we need some definitions.

Definition 3.3.1. Given a potential $\psi : Y \to \mathbb{R}$ on an interval Y, the *total variation* of

ψ is defined as

$$\mathrm{Var}(\psi) := \sup \left\{ \sum_{i=0}^{n-1} |\psi(x_{i+1}) - \psi(x_i)| \right\},$$

where the supremum is taken over all finite ordered sequences $(x_i)_{i=0}^n \subset Y$.

We use the norm $\|\psi\|_{BV} = \sup |\psi| + \mathrm{Var}(\psi)$, which makes

$$BV := \{\psi : Y \to \mathbb{R} : \|\psi\|_{BV} < \infty\}$$

into a Banach space. We also define

$$S_n\psi(x) := \psi(x) + \cdots + \psi \circ f^{n-1}(x).$$

Definition 3.3.2. For a measurable potential $\psi : \mathcal{X} \to \mathbb{R}$, we define the *pressure* of $(\mathcal{X}, f, \phi)$ to be

$$P(\phi) := \sup_{\mathbb{P} \in \mathcal{M}_f} \left\{ h(\mathbb{P}) + \int \phi \, \mathrm{d}\mathbb{P} : -\int \phi \, \mathrm{d}\mathbb{P} < \infty \right\},$$

where $\mathcal{M}_f$ is the set of f-invariant probability measures and $h(\mathbb{P})$ denotes the metric entropy of the measure $\mathbb{P}$, see [W82] for details. If $\mathbb{P}$ is an invariant probability measure such that $h(\mathbb{P}_\phi) + \int \phi \, \mathrm{d}\mathbb{P} = P(\phi)$, then we say that $\mathbb{P}$ is an *equilibrium state.*

Definition 3.3.3. A measure m is called a *ϕ-conformal* measure if $m(\mathcal{M}) = 1$ and if whenever $f : A \to f(A)$ is a bijection, for a Borel set A, then $m(f(A)) = \int_A e^{-\phi} \, \mathrm{d}m$. Therefore, if $f^n : A \to f^n(A)$ is a bijection then $m(f^n(A)) = \int_A e^{-S_n\phi} \, \mathrm{d}m$.

Definition 3.3.4 (Rychlik system)**.** (Y, f, ψ) is a *Rychlik system* if Y is an interval, $\{Y_i\}_i$ is an at most countable collection of open intervals such that $\bigcup_i \overline{Y}_i \supset Y$ (where $\overline{Y}_i$ is the closure of Y_i), $f : \bigcup_i Y_i \to Y$ is a function continuous on each Y_i, which admits a continuous extension to the closure of Y_i that we denote by $f_i : \overline{Y}_i \to Y$ and $\psi : Y \to [-\infty, \infty)$ is a potential such that

1. $f_i : \overline{Y}_i \to f(\overline{Y}_i)$ is a diffeomorphism;

2. Var $\mathrm{e}^{\psi} < +\infty$, $\psi = -\infty$ on $Y \setminus \bigcup_i Y_i$ and $P(\psi) = 0$;

3. there is a ψ-conformal measure m_ψ on Y;

4. (f,ψ) is expanding: $\sup_{x\in Y} \psi(x) < 0$.

Rychlik [R83] proved that these maps have exponential decay of correlations against L^1 observables. To be more precise, if (Y, f, ψ) is a topologically mixing Rychlik system, then there exists an equilibrium state $\mu_\psi = hm_\psi$ where $h \in BV$and m_ψ and μ_ψ are non-atomic and (Y, f, μ_ψ) has exponential decay of correlations, i.e., there exists $C > 0$ and $\gamma \in (0, 1)$ such that

$$\left| \int \varsigma \circ f^n \cdot \phi \, \mathrm{d}\mu_\psi - \int \varsigma \, \mathrm{d}\mu_\psi \int \phi \, \mathrm{d}\mu_\psi \right| \leq C \|\varsigma\|_{L^1(\mu_\psi)} \|\phi\|_{BV} \eta^n, \tag{3.3}$$

for any $\varsigma \in L^1(\mu_\psi)$ and $\phi \in BV$.

Note that, in the original statement, instead of the $L^1(\mu_\psi)$-norm, the $L^1(m_\psi)$-norm appeared. However, we will assume that $h > c$, for some $c > 0$, which means that we can write (3.3) as it is. We remark that h being bounded below by a positive constant is not very restrictive. That is the case if, for example, h is lower semi-continuous (see [BG97, Theorem 8.2.3]) or if the system has summable variations as uniformly expanding systems with Hölder continuous potentials do.

Moreover, the fact that these maps have decay of correlations of observables in a strong norm like BV against L^1 observables allows us to prove the following lemma which is very similar to the first computations in the proof of [BSTV03, Theorem 3.2].

Lemma 3.3.5. *There exists $C' > 0$ such that for all $j \in \mathbb{N}$*

$$\mu_\psi \left(Q_p(u_n) \cap f^{-j}(Q_p(u_n)) \right) \leq \mu_\psi(Q_p(u_n)) \left(C' e^{-\beta j} + \mu_\psi(Q_p(u_n)) \right).$$

Proof. Taking $\varsigma = \phi = \mathbf{1}_{Q_p(u_n)}$ in (3.3) we easily get

$$\mu_\psi \left(Q_p(u_n) \cap f^{-j}(Q_p(u_n)) \right) \leq \mu_\psi(Q_p(u_n))^2 + C \left\| \mathbf{1}_{Q_p(u_n)} \right\|_{BV} m_\psi(Q_p(u_n)) e^{-\beta j}.$$

Since we have assumed that $\frac{d\mu_\psi}{dm_\psi} \in BV$ and is strictly positive, and since $\left\|1_{Q_p(u_n)}\right\|_{BV} \leq 5$ there is $C' > 0$ as required. $\square$

Let $\mathbb{S} = Y \setminus \bigcup_i Y_i$ and define $\Lambda := \{x \in Y : f^n(x) \notin \mathbb{S}, \text{ for all } n \in \mathbb{N}_0\}$. As a consequence of Theorem A and Lemma 3.2.2 it follows immediately:

Proposition 3.3.6. *Suppose that (Y, f, ψ) is a topologically mixing Rychlik system, ψ is Hölder continuous on each $\overline{Y}_i$, and $\mu = \mu_\psi$ is the corresponding equilibrium state such that $\frac{d\mu_\psi}{dm_\psi} > c$, for some $c > 0$. Let $X_0, X_1, \ldots$ be given by (2.6), where φ achieves a global maximum at some point ζ. Then we have an EVL for M_n and*

1. *if $\zeta \in \Lambda$ is not a periodic point then the EVL is such that $\bar{H}(\tau) = \mathrm{e}^{-\tau}$ and the REPP N_n converges in distribution to a standard Poisson process N of intensity 1.*
2. *if $\zeta \in \Lambda$ is a (repelling) periodic point of prime period p then the EVL is such that $\bar{H}(\tau) = \mathrm{e}^{-\vartheta\tau}$ where the EI is given by $\vartheta = 1 - \mathrm{e}^{S_p\psi(\zeta)}$ and the REPP N_n converges in distribution to a compound Poisson process N with intensity ϑ and multiplicity distribution function π given by $\pi(\kappa) = \vartheta(1-\vartheta)^\kappa$ for every $\kappa \in \mathbb{N}_0$.*

Proof. For proving statement (1), first note that for Rychlik maps, (3.3) clearly implies that condition (3.1) is satisfied. Besides since U_n must be an interval then $\mathbf{1}_{U_n} \in BV$ and $\|\mathbf{1}_{U_n}\|_{BV} \leq 2$. Moreover, by definition of Λ, we can apply Lemma 3.2.2 and consequently obtain that $\lim_{n\to\infty} R_n = \infty$. Hence, we are now in condition to apply Theorem A and Corollary C in order to obtain the results.

Regarding statement (2), let us note that we follow the proof of [FFT12, Proposition 2] for the first argument. First, take $\phi = \mathbf{1}_{Q_p(u_n)}$, $\varsigma = \mathbf{1}_{\mathscr{Q}_{p,t,\ell}(u_n)}$ for proving Condition $D^p(u_n)$. Let $C' > 0$ be such that $\mathrm{Var}(\mathbf{1}_{Q_p(u_n)}) \leq C'$, for all $n \in \mathbb{N}$. Then (3.3) implies that Condition $D^p(u_n)$ holds with $\gamma(n,t) = \gamma(t) := c\eta^t$ and for the sequence t_n such that $n\eta(t_n) \to 0$, as $n \to \infty$, c collects all the constants. Observe that the existence of such $C' > 0$ derives from the fact that $Q_p(u_n)$ depends only on X_0 and X_p.

The idea to prove $D'_p(u_n)$ is that it takes at least something of order $\log n$ iterations for a point to return to $Q_p(u_n)$. Then after $\log n$ iterates, the decay of correlation estimates take over to give $D'_p(u_n)$. First let $\overline{V} \ni \zeta$ denote a domain such that $x \in V$ implies $\mathrm{dist}(f^p(x), \zeta) > \mathrm{dist}(x, \zeta)$. In order for a point in $Q_p(u_n)$ to return to $Q_p(u_n)$ at time $k \in \mathbb{N}$, there must be some time $\ell \leq k/p$ such that image $f^{\ell p}(Q_p(u_n))$ must have only just escaped from the domain V. Therefore we must have $\mu(f^{(\ell-1)p}(Q_p^*(u_n))) \geq C\mu(V)$ for some $C > 0$ which depends only on V and ζ. Recall that

$$m_\psi(f^{(\ell-1)p}(Q_p^*(u_n))) = \int_{Q_p^*(u_n)} e^{-S_{(\ell-1)p}\psi} \, \mathrm{d}m_\psi.$$

Hence

$$\begin{aligned}
\mu_\psi(f^{(\ell-1)p}(Q_p^*(u_n))) &= \int_{f^{(\ell-1)p}Q_p^*(u_n)} h \, \mathrm{d}m_\psi > C\, m_\psi(f^{(\ell-1)p}(Q_p^*(u_n))) \\
&= C \int_{Q_p^*(u_n)} e^{-S_{(\ell-1)p}\psi} \, \mathrm{d}m_\psi \approx C\, e^{-(\ell-1)S_p\psi(\zeta)} m_\psi(Q_p^*(u_n)).
\end{aligned}$$

Since $m_\psi(Q_p^*(u_n)) \sim (1-\vartheta)\tau/n$ and $\vartheta = 1 - \mathrm{e}^{S_p\psi(\zeta)}$, we can write

$$\mu_\psi(f^{(\ell-1)p}(Q_p^*(u_n))) > C\, \mathrm{e}^{-\ell S_p\psi(\zeta)} \frac{\tau}{n}.$$

Now, as $e^{S_p\psi(\zeta)} \in (0,1)$ we must have ℓ, and therefore k, greater than $B\log n$ for some $B > 0$, depending on C, V and $\frac{d\mu}{dm}$. Using this and Lemma 3.3.5,

$$\begin{aligned}
&n \sum_{j=1}^{[n/k_n]} \mathbb{P}(\{X_0 \in Q_p(u_n)\} \cap \{X_j \in Q_p(u_n)\}) \\
&\quad\approx n \sum_{j=B\log n}^{[n/k_n]} \mathbb{P}(\{X_0 \in Q_p(u_n)\} \cap \{X_j \in Q_p(u_n)\}) \\
&\quad\leq n\,([n/k_n] - B\log n)\, \mu(Q_p(u_n))^2 + n\mu(Q_p(u_n)) e^{-B\beta\log n} \sum_{j=1}^{[n/k_n]-B\log n} e^{-\beta j} \\
&\quad\leq \frac{(n\mu(Q_p(u_n)))^2}{k_n} + C_\beta n\mu(Q_p(u_n)) n^{-B\beta}
\end{aligned}$$

where $C_\beta := \sum_{j=0}^{\infty} e^{-j\beta}$. Since $n\mu(Q_p(u_n)) \to \tau\theta$ as $n \to \infty$ we have for some $D > 0$

$$\lim_{n\to\infty} n \sum_{j=1}^{[n/k_n]} \mathbb{P}(\{X_0 \in Q_p(u_n)\} \cap \{X_j \in Q_p(u_n)\}) \leq \lim_{n\to\infty} D\frac{(\tau\theta)^2}{k_n} = 0.$$

For the convergence of REPP, we refer the reader to [FFT12a, Corollary 3]. □

3.3.2 Piecewise expanding maps in higher dimensions

Our second example is multidimensional piecewise uniformly expanding maps. We follow the definition given by Saussol in [S00]. As it is pointed out in [AFL11], these maps generalise Markov maps which also contain one-dimensional piecewise uniformly expanding maps.

We need some notation: $\mathrm{dist}(\cdot, \cdot)$ being the usual metric in $\mathbb{R}^N$, we introduce

$$B_\varepsilon(x) = \{y \in \mathbb{R}^N : \mathrm{dist}(x, y) < \varepsilon\}$$

given $\varepsilon > 0$. Moreover, Z being a compact subset of $\mathbb{R}^N$, for any $A \subset Z$ and given a real number $c > 0$, we write

$$B_c(A) = \{x \in \mathbb{R}^N : \mathrm{dist}(x, A) \leq c\}.$$

Z° stands for the interior of Z, and $\overline{Z}$ is the closure.

Definition 3.3.7 (Multidimensional piecewise expanding system)**.** (Z, f, μ) is a *multidimensional piecewise expanding system* if Z is a compact subset of $\mathbb{R}^N$ with $\overline{Z^\circ} = Z$, $f : Z \to Z$ and $\{Z_i\}$ is a family of at most countably many disjoint open sets such that $\mathrm{Leb}(Z \setminus \bigcup_i Z_i) = 0$ and there exist open sets $\widetilde{Z}_i \supset \overline{Z_i}$ and $C^{1+\alpha}$ maps $f_i : \widetilde{Z}_i \to \mathbb{R}^N$, for some real number $0 < \alpha \leq 1$ and some sufficiently small real number $\varepsilon_1 > 0$ such that for all i,

1. $f_i(\widetilde{Z}_i) \supset B_{\varepsilon_1}(f(Z_i))$;

2. for $x, y \in f(Z_i)$ with $\mathrm{dist}(x, y) \leq \varepsilon_1$,

$$|\det Df_i^{-1}(x) - \det Df_i^{-1}(y)| \leq c|\det Df_i^{-1}(x)|\mathrm{dist}(x, y)^\alpha;$$

3. there exists $s = s(f) < 1$ such that $\forall x, y \in f(\widetilde{Z}_i)$ with $\mathrm{dist}(x, y) \leq \varepsilon_1$, we have

$$\mathrm{dist}(f_i^{-1}x, f_i^{-1}y) \leq s\,\mathrm{dist}(x, y);$$

4. set $G(\varepsilon,\varepsilon_1) := \sup_x G(x,\varepsilon,\varepsilon_1)$ where

$$G(x,\varepsilon,\varepsilon_1) := \sum_i \frac{\mathrm{Leb}(f_i^{-1}B_\varepsilon(\partial f Z_i) \cup B_{(1-s)\varepsilon_1}(x))}{\mathrm{Leb}(B_{(1-s)\varepsilon_1}(x))} \tag{3.4}$$

and assume that $\sup\limits_{\delta\leq\varepsilon_1}\left(s^\alpha + 2\sup\limits_{\varepsilon\leq\delta}\frac{G(\varepsilon)}{\varepsilon^\alpha}\delta^\alpha\right) < 1$.

Now, let us introduce the space of quasi-Hölder functions, in which we want to investigate the spectrum of the corresponding Perron-Frobenius operator. Given a Borel set $\Gamma \subset Z$, we define the oscillation of $\varphi \in L^1(\mathrm{Leb})$ over Γ as

$$\mathrm{osc}(\varphi,\Gamma) := \operatorname*{ess\,sup}_\Gamma \varphi - \operatorname*{ess\,inf}_\Gamma \varphi.$$

It is easy to verify that $x \mapsto \mathrm{osc}(\varphi, B_\varepsilon(x))$ defines a measurable function (see [S00, Proposition 3.1]). Given real numbers $0 < \alpha \leq 1$ and $\varepsilon_0 > 0$, we define α-seminorm of φ as

$$|\varphi|_\alpha = \sup_{0<\varepsilon\leq\varepsilon_0} \varepsilon^{-\alpha}\int_{\mathbb{R}^{\mathbb{N}}} \mathrm{osc}(\varphi, B_\varepsilon(x))\,\mathrm{dLeb}(x).$$

Let us consider the space of functions with bounded α-seminorm

$$V_\alpha = \{\varphi \in L^1(\mathrm{Leb}) : |\varphi|_\alpha < \infty\},$$

and endow V_α with the norm

$$\|\cdot\|_\alpha = \|\cdot\|_{L^1(\mathrm{Leb})} + |\cdot|_\alpha,$$

which makes it into a Banach space. We note that V_α is independent of the choice of ε_0. According to [S00, Theorem 5.1], there exists an absolutely continuous invariant probability measure μ. Also in [S00, Theorem 6.1], it is showed that on the mixing components μ enjoys exponential decay of correlations against L^1 observables on V_α, more precisely, if the map f is as defined above and if μ is the mixing absolutely continuous invariant probability measure, then there exist constants $C < \infty$ and $\gamma < 1$ such that

$$\left|\int_Z \psi\circ f^n\, h\,\mathrm{d}\mu\right| \leq C\|\psi\|_{L^1}\|h\|_\alpha\gamma^n,\ \forall\psi\in L^1,\ \text{where} \int\psi\,\mathrm{d}\mu = 0 \text{ and } \forall h\in V_\alpha. \tag{3.5}$$

One can find the exact values of the above constants in [S00].

Let $\mathbb{S} = Z \setminus \bigcup_i Z_i$ and define $\Lambda := \{x \in Z : f^n(x) \notin \mathbb{S}, \text{ for all } n \in \mathbb{N}_0\}$. As a consequence of Theorem A, Corollary C and Lemma 3.2.2 it follows immediately:

Proposition 3.3.8. *Suppose that (Z, f, μ) is a topologically mixing multidimensional piecewise expanding system and μ is its absolutely continuous invariant probability measure with a Radon-Nikodym density bounded away from zero. Let $X_0, X_1, \ldots$ be given by* (2.6)*, where φ achieves a global maximum at some point ζ. Then we have an EVL for M_n and*

1. *if $\zeta \in \Lambda$ is not a periodic point then the EVL is such that $\bar{H}(\tau) = \mathrm{e}^{-\tau}$ and the REPP N_n converges in distribution to a standard Poisson process N of intensity 1;*

2. *if $\zeta \in \Lambda$ is a (repelling) periodic point of prime period p then the EVL is such that $\bar{H}(\tau) = \mathrm{e}^{-\vartheta\tau}$ where the EI is given by $\vartheta = 1 - |\det D(f^{-p})(\zeta)|$ and the REPP N_n converges in distribution to a compound Poisson process N with intensity ϑ and multiplicity distribution function π given by $\pi(\kappa) = \vartheta(1-\vartheta)^\kappa$ for every $\kappa \in \mathbb{N}_0$.*

Proof. For proving (1), we can start by remarking that the condition (3.1) is satisfied since we have (3.5). Since U_n corresponds to a ball, by definition of $|\cdot|_\alpha$, it follows easily that $\mathbf{1}_{U_n} \in V_\alpha$ and $\|\mathbf{1}_{U_n}\|_\alpha$ is uniformly bounded from above. Now, considering the definition of Λ, we can apply Lemma 3.2.2 and consequently obtain that $\lim_{n\to\infty} R_n = \infty$. The result then follows by applying Theorem A and Corollary C.

Statement (2) has already been proved in [FFT12a, Corollary 4]. $\square$

3.4 The extremal behaviour at discontinuity points

In this section, we go back to Rychlik maps introduced in Section 3.3.1, however this time we consider only the ones with finitely many branches, and study the extremal behaviour of these systems when the orbit of ζ hits a discontinuity point of the map.

Consider a point $\zeta \in Y$. Note that here we have at most finitely many collection of open intervals such that $\bigcup_i \overline{Y}_i \supset Y$. If $\zeta \in \Lambda$ then we say that ζ is a *simple point*. If ζ is a *non-simple point*, which means that $r_{\mathbb{S}}(\zeta)$ is finite, then let $\ell = r_{\mathbb{S}}(\zeta)$ and $z = f^\ell(\zeta)$. We will always assume that $z \in \mathbb{S}$ is such that: there exist $i^+, i^- \in \mathbb{N}$ so that z is the right end point of Y_{i^-} and the left end point of Y_{i^+}. We consider that the point z is doubled and has two versions: $z^+ \in Y_{i^+}$ and $z^- \in Y_{i^-}$, so that $f(z^+) := f_{i^+}(z) = \lim_{x\to z,\, x\in Y_{i^+}} f(x)$ and $f(z^-) := f_{i^-}(z) = \lim_{x\to z,\, x\in Y_{i^-}} f(x)$. When ζ is a non-simple point we consider that its orbit bifurcates when it hits $\mathbb{S}$ and consider its two possible evolutions. We express this fact by saying that when ζ is non-simple we consider the "orbits" of ζ^+ and ζ^- which are defined in the following way:

- for $j = 1, \dots, \ell$ we let $f^j(\zeta^\pm) := f^j(\zeta)$;
- for $j = \ell + 1$, we define $f^j(\zeta^\pm) := f_{i^\pm}(f^{j-1}(\zeta^\pm))$
- for $j > \ell + 1$ we consider two possibilities:
  - if $j - 1$ is such that $f^{j-1}(\zeta^\pm) \notin \mathbb{S}$, then we set $f^j(\zeta^\pm) := f(f^{j-1}(\zeta^\pm))$
  - otherwise we set $f^j(\zeta^\pm) := f_i(f^{j-1}(\zeta^\pm))$, where i is such that $f^{j-\ell}(z^\pm) \in Y_i$

Remark 3.4.1. Note that for the "orbits" of $\zeta^\pm$ just defined above, there is a sequence $(i_j^\pm)_{j\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, we have $f^n(\zeta^\pm) \in \overline{Y}_{i_n^\pm}$ and $f^n(\zeta^\pm) = f_{i_n^\pm} \circ \cdots \circ f_{i_1^\pm}(\zeta)$. Also observe that, in the notation above, $i_\ell^\pm = i^\pm$.

A non-simple point ζ is *aperiodic* if for all $j \in \mathbb{N}$ we have $f^j(\zeta^+) \neq \zeta \neq f^j(\zeta^-)$.

If there exists $p^\pm$ such that $f^{p^\pm}(\zeta^\pm) = \zeta$ and for $j = 1, \dots, p^\pm - 1$ we have $f^j(\zeta^\pm) \neq \zeta$, but, for all $j \in \mathbb{N}$, we have $f^j(\zeta^\mp) \neq \zeta$, then we say that ζ is *singly returning*. If ζ is singly returning and $f^{\pm}(\zeta^{\pm}) = \zeta^{\pm}$, which means that $f^{p^{\pm}}(z^{\pm}) \in Y_{i^{\pm}}$, then we say that ζ is a *singly periodic* point of period $p^{\pm}$.

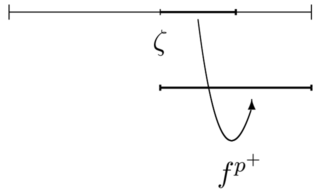

Fig.3 - Singly returning, singly periodic ζ

If ζ is singly returning and $f^{p^{\pm}}(\zeta^{\pm}) = \zeta^{\mp}$, which means that $f^{p^{\pm}}(z^{\pm}) \in Y_{i^{\mp}}$, then we say that ζ is an *eventually aperiodic* point.

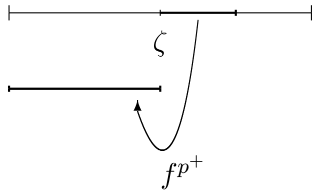

Fig.4 - Singly returning, eventually aperiodic ζ

If there exist p^+ and p^- such that $f^{p^+}(\zeta^+) = \zeta = f^{p^-}(\zeta^-)$ and for $j = 1, \ldots, p^+ - 1$ and $k = 1, \ldots, p^- - 1$ we have $f^j(\zeta^+) \neq \zeta \neq f^k(\zeta^-)$, then we say that ζ is *doubly returning*. In the case, ζ is a doubly returning point and both $f^{p^+}(\zeta^+) = \zeta^+$ and $f^{p^-}(\zeta^-) = \zeta^-$, then we say that ζ is *doubly periodic* with periods p^+ and p^-, respectively.

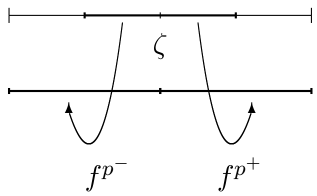

Fig.5 - Doubly returning, doubly periodic ζ (no switches)

If ζ is doubly returning, $f^{p^{\pm}}(\zeta^{\pm}) = \zeta^{\pm}$ and $f^{p^{\mp}}(\zeta^{\mp}) = \zeta^{\pm}$ then we say that ζ is *doubly returning with one switch.*

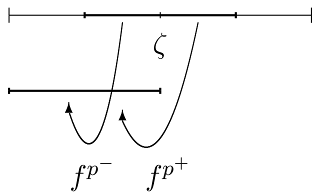

Fig.6 - Doubly returning ζ with one switch

If ζ is doubly returning, $f^{p^{\pm}}(\zeta^{\pm}) = \zeta^{\mp}$ and $f^{p^{\mp}}(\zeta^{\mp}) = \zeta^{\pm}$ then we say that ζ is *doubly returning with two switches.*

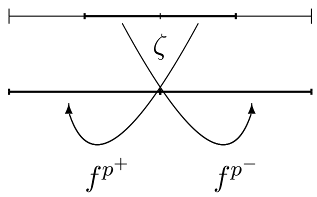

Fig.7 - Doubly returning ζ with two switches

In what follows consider that

$$U_n^{\pm} = U_n \cap f^{-\ell}(Y_{i^{\pm}}).$$

The main goals of this section are to compute the EI and also the limit for the REPP at non-simple points as defined above. In the case of aperiodic non-simple points, the analysis is very similar to the one held for non-periodic points, in the previous sections, and we get an EI equal to 1 and the convergence of the REPP to the standard Poisson process. In the case of singly returning and doubly returning points, we have periodicity and consequently clustering. This means that the analysis should follow the footsteps of [FFT12, FFT12a] with the necessary adjustments.

Proposition 3.4.2. *Suppose that (Y, f, ψ) is a topologically mixing Rychlik system with finitely many branches, ψ is Hölder continuous on each $\overline{Y}_i$, and $\mu = \mu_\psi$ is the corresponding equilibrium state. Let $X_0, X_1, \dots$ be given by (2.6), where φ achieves a global maximum at some point $\zeta \in Y \setminus \Lambda$. Let u_n be such that (2.16) holds and U_n be defined as in (2.18). We assume that $\mu(U_n^\pm) \sim \alpha^\pm \mu(U_n)$, where $0 < \alpha^-, \alpha^+ < 1$ and $\alpha^- + \alpha^+ = 1$. Then we have an EVL for M_n and*

1. *if ζ is an aperiodic non-simple point then the EVL is such that $\bar{H}(\tau) = \mathrm{e}^{-\tau}$;*
2. *if ζ is a non-simple, repelling singly returning point then the EVL is such that $\bar{H}(\tau) = \mathrm{e}^{-\vartheta\tau}$ where the EI is given by $\vartheta = 1 - \alpha^\pm \mathrm{e}^{S_{p^\pm}\psi(\zeta^\pm)}$;*
3. *if ζ is a non-simple, repelling doubly returning point, then the EVL is such that $\bar{H}(\tau) = \mathrm{e}^{-\vartheta\tau}$ where the EI is given by $\vartheta = 1 - \alpha^+ \mathrm{e}^{S_{p^+}\psi(\zeta^+)} - \alpha^- \mathrm{e}^{S_{p^-}\psi(\zeta^-)}$, when ζ has no switches; $\vartheta = 1 - \alpha^\pm(\mathrm{e}^{S_{p^+}\psi(\zeta^+)} + \mathrm{e}^{S_{p^-}\psi(\zeta^-)})$, when ζ has one switch; $\vartheta = 1 - \alpha^- \mathrm{e}^{S_{p^+}\psi(\zeta^+)} - \alpha^+ \mathrm{e}^{S_{p^-}\psi(\zeta^-)}$, when ζ has two switches.*

Remark 3.4.3. We remark that, in the particular case when μ_ψ is absolutely continuous with respect to the Lebesgue measure and the invariant density is continuous at the points ζ considered in the proposition above, the formulas for the EI can be seen as special cases of the formula in [K12, Remark 8].

Proof. If ζ *is an aperiodic non-simple point* then we just have to mimic the argument for non-periodic points in the previous sections. The proof of $D_2(u_n)$ is done exactly as before. Using decay of correlations against L^1, stated in 3.3, the proof that $D'(u_n)$ holds for these points follows the same footsteps except for the adjustments in order to consider the two possible evolutions corresponding to the "orbits" of ζ^+ and ζ^-. For example, to prove that $R(U_n) \to \infty$, as $n \to \infty$, in the argument of Lemma 3.2.2 we would define

$$\epsilon = \min\left\{\min_{k=1,\dots j} \operatorname{dist}(f^k(\zeta^+), \zeta), \min_{k=1,\dots j} \operatorname{dist}(f^k(\zeta^-), \zeta)\right\},$$

and proceed as before.

When ζ *is a non-simple (singly or doubly) returning point*, we just need to adjust the definition (2.22) of $Q_p(u_n)$ to cope with the two possibly different evolutions of ζ^+ and ζ^-. Everything else, namely the proofs of conditions $D^p(u_n)$ and $D'_p(u_n)$ follow from decay of correlations against L^1, stated in 3.3, exactly in the same lines as in the proof of [FFT12a, Theorem 2]. Hence, essentially, for each different case we have to define coherently $Q_p(u_n)$ and compute the EI using formula (2.28).

Assume first that ζ *is a singly returning (eventually aperiodic or not) non-simple point.* Without loss of generality, we also assume that there exists p such that $f^p(\zeta^+) = \zeta^+$. In this case, we should define $Q_p(u_n) = U_n^- \cup (U_n^+ \setminus f^{-p}(U_n))$, as seen in the figure below.

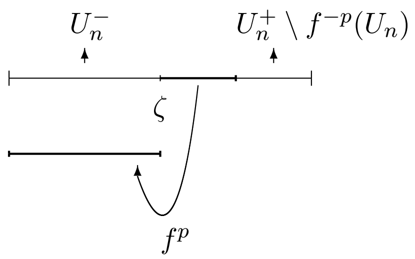

Fig.8 - $Q_p = U_n^- \cup (U_n^+ \setminus f^{-p}(U_n))$

We can now compute the EI:

$$\begin{aligned}\vartheta &= \lim_{n\to\infty} \frac{\mu(Q_p(u_n))}{\mu(U_n)} = \lim_{n\to\infty} \frac{\mu(U_n^-) + (1 - \mathrm{e}^{S_p\psi(\zeta^+)})\mu(U_n^+)}{\mu(U_n)} \\ &= \lim_{n\to\infty} \frac{\alpha^-\mu(U_n) + \alpha^+(1 - \mathrm{e}^{S_p\psi(\zeta^+)})\mu(U_n)}{\mu(U_n)} = 1 - \alpha^+\mathrm{e}^{S_p\psi(\zeta^+)}.\end{aligned}$$

Let ζ *be a non-simple, repelling doubly returning point* and p^-, p^+ be such that $f^{p^-}(\zeta^-) = \zeta$ and $f^{p^+}(\zeta^+) = \zeta$. For definiteness, we assume without loss of generality that $p^- < p^+$.

First we consider the case where *no switching* occurs. In this case, we have two different "periods", hence we should define $Q_{p^-,p^+}(u_n) = \left(U_n^- \setminus f^{-p^-}(U_n^-)\right) \cup \left(U_n^+ \setminus f^{-p^+}(U_n^+)\right)$, as seen in the figure below.

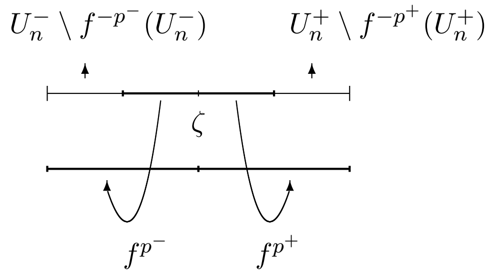

Fig.9 - $Q_{p^-,p^+}(u_n) = \left(U_n^- \setminus f^{-p^-}(U_n^-)\right) \cup \left(U_n^+ \setminus f^{-p^+}(U_n^+)\right)$

It follows that

$$\begin{aligned}\vartheta &= \lim_{n\to\infty} \frac{\alpha^-(1-\mathrm{e}^{S_p\psi(\zeta^-)})\mu(U_n) + \alpha^+(1-\mathrm{e}^{S_p\psi(\zeta^+)})\mu(U_n)}{\mu(U_n)} \\ &= 1-\alpha^-\mathrm{e}^{S_p\psi(\zeta^-)} - \alpha^+\mathrm{e}^{S_p\psi(\zeta^+)}.\end{aligned}$$

Next, we consider the case with *one switch*. In this case, we also have two different "periods" and for definiteness we assume without loss of generality that $f^{p^-}(\zeta^-) = \zeta^-$ and $f^{p^+}(\zeta^+) = \zeta^-$. Then we define $Q_{p^-,p^+}(u_n) = \left(U_n^- \setminus f^{-p^-}(U_n^-)\right) \cup \left(U_n^+ \setminus f^{-p^+}(U_n^-)\right)$, as seen in the figure below.

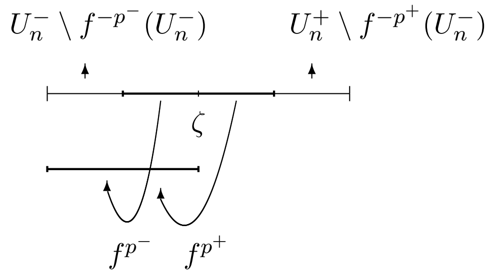

Fig.10 - $Q_{p^-,p^+}(u_n) = \left(U_n^- \setminus f^{-p^-}(U_n^-)\right) \cup \left(U_n^+ \setminus f^{-p^+}(U_n^-)\right)$

It follows that

$$\begin{aligned}\vartheta &= \lim_{n\to\infty} \frac{\alpha^-(1-\mathrm{e}^{S_p\psi(\zeta^-)})\mu(U_n) + \alpha^-(1-\mathrm{e}^{S_p\psi(\zeta^+)})\mu(U_n)}{\mu(U_n)} \\ &= 1-\alpha^-\mathrm{e}^{S_p\psi(\zeta^-)} - \alpha^-\mathrm{e}^{S_p\psi(\zeta^+)}.\end{aligned}$$

Finally, we consider the case with *two switches*. In this case, we also have two different "periods" and we should define $Q_{p^-,p^+}(u_n) = \left(U_n^- \setminus f^{-p^-}(U_n^+)\right) \cup \left(U_n^+ \setminus f^{-p^+}(U_n^-)\right)$, as seen the figure below.

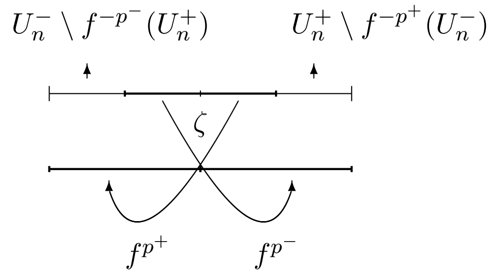

Fig.11 - $Q_{p^-,p^+}(u_n) = \left(U_n^- \setminus f^{-p^-}(U_n^+)\right) \cup \left(U_n^+ \setminus f^{-p^+}(U_n^-)\right)$

It follows that

$$\begin{aligned}\vartheta &= \lim_{n\to\infty} \frac{\alpha^+(1-\mathrm{e}^{S_p\psi(\zeta^-)})\mu(U_n) + \alpha^-(1-\mathrm{e}^{S_p\psi(\zeta^+)})\mu(U_n)}{\mu(U_n)} \\ &= 1 - \alpha^-\mathrm{e}^{S_p\psi(\zeta^+)} - \alpha^+\mathrm{e}^{S_p\psi(\zeta^-)}.\end{aligned}$$

Hence, the proposition follows. □

Next result gives the convergence of the REPP at non-simple points. Note that, contrarily to the usual geometric distribution obtained, for example, in [HV09, CCC09, FFT12a], in here, the multiplicity distribution is quite different. In fact, for eventually aperiodic returning points, for example, we have that $\pi(\kappa) = 0$ for all $\kappa \geq 3$.

Proposition 3.4.4. *Let $a^\pm := \mathrm{e}^{S_{p^\pm}\psi(\zeta^\pm)}$. Under the same assumptions of Proposition 3.4.2, we have:*

1. *if ζ is an aperiodic non-simple point then the REPP converges to a standard Poisson process of intensity 1;*

2. *if ζ is a non-simple, singly returning point*

(a) *not eventually aperiodic then the REPP converges to a compound Poisson process of intensity ϑ, given in Proposition 3.4.2, and multiplicity distribution defined by:*

$$\pi(1) = \tfrac{\vartheta-(1-\vartheta)(1-a^{\pm})}{\vartheta}, \quad \pi(\kappa) = \tfrac{\alpha^{\pm}(1-a^{\pm})^2(a^{\pm})^{-(\kappa-1)}}{\vartheta}, \quad \kappa \geq 2.$$

(b) *eventually aperiodic then the REPP converges to a compound Poisson process of intensity ϑ, given in Proposition 3.4.2, and multiplicity distribution defined by:*

$$\pi(1) = \tfrac{2\vartheta-1}{\vartheta}, \quad \pi(2) = \tfrac{1-\vartheta}{\vartheta}, \quad \pi(\kappa) = 0, \quad \kappa \geq 3.$$

3. *if ζ is a non-simple, repelling doubly returning point*

(a) *with no switches then the REPP converges to a compound Poisson process of intensity ϑ, given in Proposition 3.4.2, and multiplicity distribution defined by:*

$$\begin{aligned}\pi(1) &= \tfrac{2\vartheta-1+\alpha^-(a^-)^2+\alpha^+(a^+)^2}{\vartheta}\\ \pi(\kappa) &= \tfrac{\alpha^-(1-a^-)^2(a^-)^{-(\kappa-1)}+\alpha^-(1-a^+)^2(a^+)^{-(\kappa-1)}}{\vartheta}, \ \kappa \geq 2.\end{aligned}$$

(b) *with one switch then the REPP converges to a compound Poisson process of intensity ϑ, given in Proposition 3.4.2, and multiplicity distribution defined by:*

$$\pi(1) = \tfrac{2\vartheta-1+a^{\pm}(1-\vartheta)}{\vartheta}, \ \pi(\kappa) = \tfrac{(1-\vartheta)(a^{\pm})^{\kappa-2}(1-a^{\pm})^2}{\vartheta}, \ \kappa \geq 2.$$

(c) *with two switches then the REPP converges to a compound Poisson process of intensity ϑ, given in Proposition 3.4.2, and multiplicity distribution defined by:*

$$\begin{aligned}\pi(1) &= \tfrac{1-2(1-\vartheta)+a^-a^+}{\vartheta}, \quad \pi(2j) = \tfrac{(a^-a^+)^{j-1}\left((1-\vartheta)(1+a^-a^+)-2a^-a^+\right)}{\vartheta},\\ \pi(2j+1) &= \tfrac{(a^-a^+)^j\left(1-2(1-\vartheta)+a^-a^+\right)}{\vartheta}, \ j \geq 1.\end{aligned}$$

Proof. When ζ *is an aperiodic non-simple point* then as we have seen in Proposition 3.4.2, condition $D'(u_n)$ holds. Clearly, $D_3(u_n)$ follows from decay of correlations and, by [FFT10, Theorem 5], we easily conclude that the REPP converges to the standard Poisson process of intensity 1.

When ζ *is a non-simple (singly or doubly) returning point*, we just need to adjust the definition of the sets $U^{(\kappa)}$ given in (2.31), which ultimately affects the sets $Q_p^\kappa(u)$, given in (2.32), in order to cope with the two possibly different evolutions of ζ^+ and ζ^-. Everything else, namely the proofs of conditions $D^p(u_n)^*$ and $D'_p(u_n)^*$ follow from decay of correlations against L^1, stated in (3.3), exactly in the same lines as in the proof of [FFT12a, Theorem 2]. Hence, essentially, for each different case we have to define coherently the sets $U^{(\kappa)}$ and compute the multiplicity distribution using formula (2.34).

In all cases, $U^{(0)} = U_n = U_n^- \cup U_n^+$.

Let ζ *be a singly returning non-simple point which is not eventually aperiodic*. Without loss of generality, we assume that there exist p such that for all $j \in \mathbb{N}$,

$$f^p(\zeta^+) = \zeta^+ \text{ and } f^j(\zeta^-) \neq \zeta.$$

For every $\kappa \in \mathbb{N}$, we define:

$$U^{(\kappa)} := \left(\bigcap_{i=0}^{\kappa} f^{-ip}(U_n^+) \right)$$

Using (2.32), we can now easily define:

$$Q^\kappa := U^{(\kappa)} \setminus U^{(\kappa+1)}, \text{ for all } \kappa \geq 0$$

We have:

$$\mathbb{P}(Q^0) \sim \mathbb{P}(U_n) - a^+\mathbb{P}(U_n^+) \sim \mathbb{P}(U_n)(1 - \alpha^+ a^+)$$

The same computations would lead us to:

$$\mathbb{P}(Q^\kappa) \sim \mathbb{P}(U_n)(\alpha^+(1 - a^+)(a^+)^\kappa)$$

Using formula (2.34), it follows:

$$\pi(1) = \lim_{n\to\infty} \frac{\mathbb{P}(Q^0)-\mathbb{P}(Q^1)}{\mathbb{P}(Q^0)} = \frac{(1-\alpha^+a^+)-(\alpha^+(1-a^+)a^+)}{(1-\alpha^+a^+)} = \frac{\vartheta-(1-\vartheta)(1-a^\pm)}{\vartheta}$$
$$\pi(\kappa) = \lim_{n\to\infty} \frac{\mathbb{P}(Q^{\kappa-1})-\mathbb{P}(Q^\kappa)}{\mathbb{P}(Q^0)} = \frac{(\alpha^+(1-a^+)(a^+)^{\kappa-1})-(\alpha^+(1-a^+)(a^+)^\kappa)}{(1-\alpha^+a^+)} = \frac{\alpha^+(1-a^+)^2(a^+)^{\kappa-1}}{\vartheta}.$$

Let ζ *be a singly returning non-simple point which is eventually aperiodic.* Without loss of generality, we assume that there exist p such that for all $j \in \mathbb{N}$,

$$f^p(\zeta^+) = \zeta^- \text{ and } f^j(\zeta^-) \neq \zeta.$$

For every $\kappa \in \mathbb{N}$, we define:

$$U^{(1)} := \left(U_n^+ \cap f^{-ip}(U_n^-)\right), \quad U^{(\kappa)} := \emptyset, \quad \kappa \geq 2$$

Note that $Q^0 = U_n \setminus U^{(1)}$, $Q^1 = U^{(1)}$ and $Q^\kappa = \emptyset$, for all $\kappa \geq 2$. We have:

$$\mathbb{P}(Q^0) \sim \mathbb{P}(U_n) - a^+\mathbb{P}(U_n^-) \sim \mathbb{P}(U_n)(1 - \alpha^- a^+)$$

and

$$\mathbb{P}(Q^1) \sim \mathbb{P}(U_n)\alpha^- a^+, \mathbb{P}(Q^\kappa) = 0, \text{ for all } \kappa \geq 2$$

Using formula (2.34), it follows:

$$\begin{aligned}
\pi(1) &= \lim_{n\to\infty} \tfrac{\mathbb{P}(Q^0)-\mathbb{P}(Q^1)}{\mathbb{P}(Q^0)} = \tfrac{(1-\alpha^-a^+)-(\alpha^-a^+)}{(1-\alpha^-a^+)} = \tfrac{2\vartheta-1}{\vartheta} \\
\pi(2) &= \lim_{n\to\infty} \tfrac{\mathbb{P}(Q^1)-\mathbb{P}(Q^2)}{\mathbb{P}(Q^0)} = \tfrac{\alpha^-a^+}{(1-\alpha^-a^+)} = \tfrac{1-\vartheta}{\vartheta}, \quad \pi(\kappa) = 0, \quad k \geq 3.
\end{aligned}$$

Let ζ *be a doubly returning non-simple point with no switches.* Let p^-, p^+ be such that

$$f^{p^-}(\zeta^-) = \zeta^- \text{ and } f^{p^+}(\zeta^+) = \zeta^+.$$

For every $\kappa \in \mathbb{N}$, we define:

$$U^{(\kappa)} := \left(\bigcap_{i=0}^{\kappa} f^{-ip^-}(U_n^-)\right) \bigcup \left(\bigcap_{i=0}^{\kappa} f^{-ip^+}(U_n^+)\right)$$

Note that using (2.32), we can now easily define $Q^\kappa := U^{(\kappa)} \setminus U^{(\kappa+1)}$, for all $\kappa \geq 0$. We have:

$$\mathbb{P}(Q^0) \sim \mathbb{P}(U_n) - a^-\mathbb{P}(U_n^-) - a^+\mathbb{P}(U_n^+) \sim \mathbb{P}(U_n)(1 - \alpha^- a^- - \alpha^+ a^+)$$

The same computations would lead us to:

$$\mathbb{P}(Q^\kappa) \sim \mathbb{P}(U_n)(\alpha^-(1 - a^-)(a^-)^\kappa + \alpha^+(1 - a^+)(a^+)^\kappa)$$

Using formula (2.34), it follows:

$$\begin{aligned}
\pi(1) &= \lim_{n\to\infty} \frac{\mathbb{P}(Q^0)-\mathbb{P}(Q^1)}{\mathbb{P}(Q^0)} = \frac{(1-\alpha^-a^- -\alpha^+a^+)-(\alpha^-(1-a^-)a^- +\alpha^+(1-a^+)a^+)}{(1-\alpha^-a^- -\alpha^+a^+)} \\
&= \frac{2\vartheta-1+\alpha^-(a^-)^2+\alpha^+(a^+)^2}{\vartheta} \\
\pi(\kappa) &= \lim_{n\to\infty} \frac{\mathbb{P}(Q^{\kappa-1})-\mathbb{P}(Q^\kappa)}{\mathbb{P}(Q^0)} = \frac{\alpha^-(1-a^-)^2(a^-)^{\kappa-1}+\alpha^+(1-a^+)^2(a^+)^{\kappa-1}}{\vartheta}.
\end{aligned}$$

Let ζ *be a doubly returning non-simple point with one switch.* Without loss of generality, we assume that there exist p^-, p^+ such that

$$f^{p^-}(\zeta^-) = \zeta^- \text{ and } f^{p^+}(\zeta^+) = \zeta^-.$$

For every $\kappa \in \mathbb{N}$, we define:

$$U^{(\kappa)} := \left(\bigcap_{i=0}^{\kappa} f^{-ip^-}(U_n^-)\right) \bigcup \left(U_n^+ \cap f^{-p^+}(U_n^-) \cap \bigcap_{i=0}^{\kappa} f^{-p^+-ip^-}(U_n^-)\right)$$

Using (2.32), we can now easily define $Q^\kappa := U^{(\kappa)} \setminus U^{(\kappa+1)}$, for all $\kappa \geq 0$. We have:

$$\mathbb{P}(Q^0) \sim \mathbb{P}(U_n) - a^-\mathbb{P}(U_n^-) - a^+\mathbb{P}(U_n^-) \sim \mathbb{P}(U_n)(1-\alpha^-a^- - \alpha^-a^+)$$

The same computations would lead us to:

$$\mathbb{P}(Q^\kappa) \sim \mathbb{P}(U_n)(\alpha^-(1-a^-)(a^-)^\kappa + \alpha^-(1-a^-)a^+(a^-)^{\kappa-1})$$

Using formula (2.34), it follows:

$$\begin{aligned}
\pi(1) &= \lim_{n\to\infty} \frac{\mathbb{P}(Q^0)-\mathbb{P}(Q^1)}{\mathbb{P}(Q^0)} = \frac{(1-\alpha^-(a^-+a^+))-(\alpha^-(1-a^-)a^- +\alpha^-(1-a^-)a^+)}{(1-\alpha^-a^- -\alpha^-a^+)} = \frac{2\vartheta-1+a^-(1-\vartheta)}{\vartheta} \\
\pi(\kappa) &= \lim_{n\to\infty} \frac{\mathbb{P}(Q^{\kappa-1})-\mathbb{P}(Q^\kappa)}{\mathbb{P}(Q^0)} = \frac{\alpha^-(1-a^-)^2(a^-)^{\kappa-1}+\alpha^-(1-a^-)^2a^+(a^-)^{\kappa-2}}{\vartheta} = \frac{(1-\vartheta)(a^-)^{\kappa-2}(1-a^-)^2}{\vartheta}.
\end{aligned}$$

Let ζ *be a doubly returning non-simple point with two switches.* Without loss of generality, we assume that there exist p^-, p^+ such that

$$f^{p^-}(\zeta^-) = \zeta^+ \text{ and } f^{p^+}(\zeta^+) = \zeta^-.$$

For every $j \in \mathbb{N}_0$, we define:

$$U^{(2j+1)} := \left(U_n^- \cap \bigcap_{i=1}^{j+1} f^{-ip^- -(i-1)p^+}(U_n^+) \cap \bigcap_{i=1}^{j} f^{-ip^- -ip^+}(U_n^-) \right)$$
$$\bigcup \left(U_n^+ \cap \bigcap_{i=1}^{j+1} f^{-ip^+ -(i-1)p^-}(U_n^-) \cap \bigcap_{i=1}^{j} f^{-ip^+ -ip^-}(U_n^+) \right)$$
$$U^{(2j)} := \left(U_n^- \cap \bigcap_{i=1}^{j} f^{-ip^- -(i-1)p^+}(U_n^+) \cap \bigcap_{i=1}^{j} f^{-ip^- -ip^+}(U_n^-) \right)$$
$$\bigcup \left(U_n^+ \cap \bigcap_{i=1}^{j} f^{-ip^+ -(i-1)p^-}(U_n^-) \cap \bigcap_{i=1}^{j} f^{-ip^+ -ip^-}(U_n^+) \right)$$

Using (2.32), we can now easily define $Q^\kappa := U^{(\kappa)} \setminus U^{(\kappa+1)}$, for all $\kappa \geq 0$. We have:

$$\mathbb{P}(Q^0) \sim \mathbb{P}(U_n) - a^-\mathbb{P}(U_n^+) - a^+\mathbb{P}(U_n^-) \sim \mathbb{P}(U_n)(1 - \alpha^+a^- - \alpha^-a^+)$$

The same computations would lead us to:

$$\mathbb{P}(Q^{2j}) \sim \mathbb{P}(U_n)(1 - \alpha^+a^- - \alpha^-a^+)(a^-a^+)^j$$

and

$$\mathbb{P}(Q^{2j}) \sim \mathbb{P}(U_n)(\alpha^+a^- + \alpha^-a^+ - a^-a^+)(a^-a^+)^j.$$

Using formula (2.34), it follows that, for every $j \in \mathbb{N}$:

$$\pi(1) = \lim_{n\to\infty} \frac{\mathbb{P}\big(Q^0\big) - \mathbb{P}\big(Q^1\big)}{\mathbb{P}(Q^0)} = \frac{(1-\alpha^+a^- - \alpha^-a^+) - (\alpha^+a^- + \alpha^-a^+ - a^-a^+)}{(1-\alpha^+a^- - \alpha^-a^+)} = \frac{1-2(1-\vartheta)+a^-a^+}{\vartheta}$$
$$\pi(2j) = \lim_{n\to\infty} \frac{\mathbb{P}\big(Q^{2j-1}\big) - \mathbb{P}\big(Q^{2j}\big)}{\mathbb{P}(Q^0)} = \frac{(a^-a^+)^{j-1}(\alpha^+a^-(1+a^-a^+) + \alpha^-a^+(1+a^-a^+) - 2a^-a^+)}{\vartheta}$$
$$\pi(2j+1) = \lim_{n\to\infty} \frac{\mathbb{P}\big(Q^{2j}\big) - \mathbb{P}\big(Q^{2j+1}\big)}{\mathbb{P}(Q^0)} = \frac{(a^-a^+)^j(1-2\alpha^+a^- - 2\alpha^-a^+ + a^-a^+)}{\vartheta}.$$

The proposition follows. $\square$

Chapter 4

Stochastic Dynamics

In this chapter, we study some statistical properties of stochastic dynamical systems. In particular, we deal with randomly perturbed dynamical systems. To our knowledge, this is a first attempt for proving the existence of EVLs and HTS/RTS as well as the convergence of REPP in the random setting.

4.1 Extremes for stochastic dynamics: direct approach

4.1.1 Statement of the main results

We give an abstract result which concludes by stating that by adding random noise to the original system considered, we always get an EI equal to 1 regardless of the chosen point ζ.

Theorem D. *Consider a dynamical system* $(\mathcal{M} \times \Omega, \mathcal{B}, \mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}}, S)$*, where* $\mathcal{M} = \mathbb{T}^d$*, for some* $d \in \mathbb{N}$*,* $f : \mathcal{M} \to \mathcal{M}$ *is a deterministic system which is randomly perturbed as in* (2.2) *with noise distribution given by* (2.1) *and* S *is the skew product map defined in* (2.4)*. Assume that there exists* $\eta > 0$ *such that* $dist(f(x), f(y)) \leq \eta dist(x, y)$*, for all* $x, y \in \mathcal{M}$*. Assume also that the stationary measure* μ_ε *is such that* $\mu_\varepsilon = h_\varepsilon \mathrm{Leb}$, *with* $0 < \underline{h}_\varepsilon \le h_\varepsilon \le \overline{h}_\varepsilon < \infty$*. Suppose that there exists a Banach space* $\mathcal{C}$ *of real-valued functions defined on* $\mathcal{M}$ *such that for all* $\phi \in \mathcal{C}$ *and* $\psi \in L^1(\mu_\varepsilon)$*,*

$$Cor_{\mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}}}(\phi,\, \psi, n) \le Cn^{-2}, \tag{4.1}$$

where $Cor_{\mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}}}(\cdot)$ *is defined as in* (2.5) *and* $C > 0$ *is a constant independent of both* ϕ,ψ *.*

For any point $\zeta \in \mathcal{M}$*, consider that* $X_0, X_1, \ldots$ *is defined as in* (2.7)*, let* u_n *be such that* (2.16) *holds and assume that* U_n *is defined as in* (2.18)*.If there exists* $C' > 0$ *such that for all* n *we have* $\mathbf{1}_{U_n} \in \mathcal{C}$ *and* $\|\mathbf{1}_{U_n}\|_{\mathcal{C}} \le C'$*, then the stochastic process* $X_0, X_1, \ldots$ *satisfies* $D_2(u_n)$ *and* $D'(u_n)$*, which implies that we have an EVL for* M_n *such that* $\bar{H}(\tau) = \mathrm{e}^{-\tau}$*.*

Again, using the connection between EVLs and HTS/RTS we get

Corollary E. *Under the same hypothesis of Theorem D we have exponential HTS/RTS for balls around* ζ*, in the sense that* (2.38) *and* (2.39) *hold with* $G(t) = \tilde{G}(t) = 1 - \mathrm{e}^t$ *and* $V_n = B_{\delta_n}(\zeta) \times \Omega$*, where* $\delta_n \to 0$*, as* $n \to \infty$*.*

Moreover, appealing to [FFT10, Theorem 5] once again, we have

Corollary F. *Under the same hypothesis of Theorem D, the stochastic process* $X_0, X_1, \ldots$ *satisfies* $D_3(u_n)$ *and* $D'(u_n)$*, which implies that the REPP* N_n *defined in* (2.30) *is such that* $N_n \xrightarrow{d} N$*, as* $n \to \infty$*, where* N *denotes a Poisson Process with intensity* 1*.*

Remark 4.1.1. We remark that we do not need to consider that $\mathcal{M}$ is a d-dimensional torus in order to apply our theory. Basically, we only need that $f_\omega(\mathcal{M}) \subset \mathcal{M}$, for all $\omega \in B_\varepsilon(0)$.

4.1.2 Laws of rare events for stochastic dynamics

In this section we will start with the proof of Theorem D which states that the dichotomy observed in Section 3.2 vanishes when we add absolutely continuous (with respect to

Lebesgue) noise to the original system and for every chosen point in the phase space we have a standard exponential distribution for the EVL and HTS/RTS weak limits. We will also certify that the REPP converges to a Poisson Process with intensity 1. Next, we will give some examples of random dynamical systems for which we can prove the existence of EVLs and HTS/RTS as well as the convergence of REPP.

In what follows, we denote the *diameter* of set a $A \subset \mathcal{M}$ by

$$|A| := \sup\{\mathrm{dist}(x, y) : \ x, y \in A\},$$

and for any $x \in \mathcal{M}$ we define the *translation* of A by x as the set

$$A + x := \{a + x : \ a \in A\}.$$

Proof of Theorem D. We want to start by showing that the condition $D_2(u_n)$ can be deduced from the decay of correlations as in the deterministic case.

From our assumption, the random dynamical system has (annealed) decay of correlations, *i.e.*, there exists a Banach space $\mathcal{C}$ of real-valued functions such that for all $\phi \in \mathcal{C}$ and $\psi \in L^1(\mu_\varepsilon)$,

$$\Big| \int \phi(\mathcal{U}_\varepsilon^t \psi)(x)\, \mathrm{d}\mu_\varepsilon - \int \phi\, \mathrm{d}\mu_\varepsilon \int \psi\, \mathrm{d}\mu_\varepsilon \Big| \leq C\|\phi\|_{\mathcal{C}}\|\psi\|_{L^1(\mu_\varepsilon)} t^{-2} \tag{4.2}$$

where $C > 0$ is a constant independent of both φ and ψ.

In proving $D_2(U_n)$, the main point is to choose the right observable. We take

$$\phi(x) = \mathbf{1}_{\{X_0 > u_n\}} = \mathbf{1}_{\{\varphi(x) > u_n\}}, \ \psi(x) = \int \mathbf{1}_{\{\varphi(x), \varphi\circ f_{\tilde{\omega}_1}(x), \dots, \varphi\circ f_{\underline{\tilde{\omega}}}^{\ell-1}(x) \leq u_n\}}\, \mathrm{d}\theta_\varepsilon^{\ell-1}(\underline{\tilde{\omega}}).$$

Substituting ψ in the random evolution operator, we get

$$(\mathcal{U}_\varepsilon^t \psi)(x) = \iint \mathbf{1}_{\{\varphi\circ f_{\underline{\omega}}^t(x), \dots, \varphi\circ f_{\underline{\tilde{\omega}}}^{\ell-1}\circ f_{\underline{\omega}}^t(x) \leq u_n\}}\, \mathrm{d}\theta_\varepsilon^{\ell-1}(\underline{\tilde{\omega}}) \mathrm{d}\theta_\varepsilon^t(\underline{\omega}).$$

Since all ω_i's and $\tilde{\omega}_j$'s are chosen in an independent and identically distributed structure, we can rename the random iterates, *i.e.*, we lose no information in writing

$$(\mathcal{U}_\varepsilon^t \psi)(x) = \int \mathbf{1}_{\{\varphi\circ f_{\underline{\omega}}^t(x), \dots, \varphi\circ f_{\underline{\omega}}^{t+\ell-1}(x) \leq u_n\}} \mathrm{d}\theta_\varepsilon^{\mathbb{N}}(\underline{\omega}).$$

Therefore, we get

$$\int \phi(x)\,(\mathcal{U}_\varepsilon^t\psi)(x)\,\mathrm{d}\mu_\varepsilon$$
$$= \int \mu_\varepsilon\Big(\varphi(x) > u_n, \varphi\circ f_{\underline{\omega}}^t(x) \le u_n,\,\ldots,\,\varphi\circ\, f_{\underline{\omega}}^{t+\ell-1}(x) \le u_n\Big)\mathrm{d}\theta_\varepsilon^{\mathbb{N}}(\underline{\omega}).$$

On the other hand,

$$\begin{aligned}
\int \phi(x)\mathrm{d}\mu_\varepsilon &= \mu_\varepsilon(X_0(x) > u_n) = \int \mu_\varepsilon(X_0(x) > u_n)\,\mathrm{d}\theta_\varepsilon^{\mathbb{N}}(\underline{\omega})\\
\int \psi(x)\mathrm{d}\mu_\varepsilon &= \int\Big(\int \mathbf{1}_{\{\varphi(x),\varphi\circ f_{\omega_1}(x),\,\ldots\,,\varphi\circ f_{\underline{\omega}}^{\ell-1}(x)\le u_n\}}\mathrm{d}\mu_\varepsilon\Big)\mathrm{d}\theta_\varepsilon^{\mathbb{N}}(\underline{\omega})\\
&= \int \mu_\varepsilon\Big(\varphi(x) \le u_n, \varphi\circ f_{\omega_1}(x) \le u_n,\,\ldots\,,\,\varphi\circ\, f_{\underline{\omega}}^{\ell-1}(x) \le u_n\Big)\mathrm{d}\theta_\varepsilon^{\mathbb{N}}(\underline{\omega}).
\end{aligned}$$

Now, the decay of correlations can be written as

$$\Big|\int \mu_\varepsilon\big(X_0(x) > u_n, \varphi\circ f_{\underline{\omega}}^t(x) \le u_n,\,\ldots\,,\,\varphi\circ\, f_{\underline{\omega}}^{t+\ell-1}(x) \le u_n\big)\,\mathrm{d}\theta_\varepsilon^{\mathbb{N}}(\underline{\omega}) -$$
$$\int \mu_\varepsilon(\varphi(x) > u_n)\,\mathrm{d}\theta_\varepsilon^{\mathbb{N}}(\underline{\omega}) \int \mu_\varepsilon\big(\varphi(x) \le u_n, \varphi\circ f_{\omega_1}(x) \le u_n,\,\ldots\,,\,\varphi\circ f_{\underline{\omega}}^{\ell-1}(x) \le u_n\big)\,\mathrm{d}\theta_\varepsilon^{\mathbb{N}}(\underline{\omega})\Big|$$
$$\le C\|\phi\|_{\mathcal{C}}\|\psi\|_{L^1(\mu_\varepsilon)}t^{-2}$$

which leads us to the conclusion that the condition $D_2(u_n)$ holds with

$$\gamma(n,t) = \gamma(t) = C^* t^{-2} \tag{4.3}$$

for some $C^* > 0$ and $t_n = n^\beta$, with $1/2 < \beta < 1$.

For proving $D'(u_n)$, the basic idea is to use the fact that we have decay of correlations against L^1 as in Theorem A and then to show that except for a small set of $\underline{\omega}$'s, $R^{\underline{\omega}}(U_n)$ grows at a sufficiently fast rate. Hence, we split Ω into two parts: the $\underline{\omega}$'s for which $R^{\underline{\omega}}(U_n) > \alpha_n$, where $(\alpha_n)_n$ is some sequence such that

$$\alpha_n \to \infty \qquad \text{and} \qquad \alpha_n = o(\log k_n), \tag{4.4}$$

which is designed, on one hand, to guarantee that for the $\underline{\omega}$'s for which $R^{\underline{\omega}}(U_n) > \alpha_n$, the argument using decay of correlations against L^1 is still applicable and, on the other hand, the set of the $\underline{\omega}$'s for which $R^{\underline{\omega}}(U_n) \leq \alpha_n$ has $\theta_\varepsilon^{\mathbb{N}}$ small measure. To show the latter we make an estimate on the $\underline{\omega}$'s that take the orbit of ζ too close to itself.

First, note that since f is continuous (which implies that $f_{\underline{\omega}}^j$ is also continuous for all $j \in \mathbb{N}$) and η is the highest rate at which points can separate, the diameter of $f_{\underline{\omega}}^j(U_n)$ grows at most at a rate given by η^j, so, for any $\underline{\omega} \in \Omega$ we have $|f_{\underline{\omega}}^j(U_n)| \leq \eta^j |U_n|$. This implies that

$$\text{if } \operatorname{dist}(f_{\underline{\omega}}^j(\zeta), \zeta) > 2\eta^j|U_n| > |U_n| + \eta^j|U_n| \text{ then } f_{\underline{\omega}}^j(U_n) \cap U_n = \emptyset. \tag{4.5}$$

Note that, by equation (4.5), if for all $j = 1, \ldots, \alpha_n$ we have $\operatorname{dist}(f_{\underline{\omega}}^j(\zeta), \zeta) > 2\eta^j|U_n|$ then clearly $R^{\underline{\omega}}(U_n) > \alpha_n$. Hence, we may write that

$$\left\{\underline{\omega} :\ R^{\underline{\omega}}(U_n) \leq \alpha_n\right\} \subset \bigcup_{j=1}^{\alpha_n} \left\{\underline{\omega} :\ f_{\underline{\omega}}^j(\zeta) \in B_{2\eta^j|U_n|}(\zeta)\right\}.$$

It follows that there exists some $C > 0$ such that

$$\begin{aligned}
\theta_\varepsilon^{\mathbb{N}}\left(\left\{\underline{\omega} :\ R^{\underline{\omega}}(U_n) \leq \alpha_n\right\}\right) &\leq \sum_{j=1}^{\alpha_n} \int \theta_\varepsilon\Big(\Big\{\omega_j : f\left(f_{\underline{\omega}}^{j-1}(\zeta)\right) + \omega_j \in B_{2\eta^j|U_n|}(\zeta)\Big\}\Big) \mathrm{d}\theta_\varepsilon^{\mathbb{N}} \\
&= \sum_{j=1}^{\alpha_n} \int \theta_\varepsilon\Big(\Big\{\omega_j : \omega_j \in B_{2\eta^j|U_n|}(\zeta) - f\left(f_{\underline{\omega}}^{j-1}(\zeta)\right)\Big\}\Big) \mathrm{d}\theta_\varepsilon^{\mathbb{N}} \\
&= \sum_{j=1}^{\alpha_n} \iint_{B_{2\eta^j|U_n|}(\zeta) - f\left(f_{\underline{\omega}}^{j-1}(\zeta)\right)} g_\varepsilon(x)\, \mathrm{dLeb}\, \mathrm{d}\theta_\varepsilon^{\mathbb{N}} \\
&\leq \sum_{j=1}^{\alpha_n} \overline{g_\varepsilon}\, \mathrm{Leb}\left(B_{2\eta^j|U_n|}(\zeta)\right) \\
&= \sum_{j=1}^{\alpha_n} \overline{g_\varepsilon}\, C\eta^j \mathrm{Leb}(U_n) \\
&\leq C\, \overline{g_\varepsilon}\, \mathrm{Leb}(U_n) \frac{\eta}{\eta - 1} \eta^{\alpha_n}.
\end{aligned}$$

Now, observe that

$$\begin{aligned}
n \sum_{j=1}^{\lfloor n/k_n \rfloor} \mathbb{P}(U_n \cap f_{\underline{\omega}}^{-j}(U_n)) &\leq n \sum_{j=\alpha_n}^{\lfloor n/k_n \rfloor} \mathbb{P}\Big(\left\{(x, \underline{\omega}) : x \in U_n,\ f_{\underline{\omega}}^j(x) \in U_n\right\}\Big) \\
&\quad + n \sum_{j=1}^{\lfloor n/k_n \rfloor} \mathbb{P}\Big(\left\{(x, \underline{\omega}) : x \in U_n, R^{\underline{\omega}}(U_n) \leq \alpha_n\right\}\Big) := I + II.
\end{aligned}$$

Let us start by estimating I, which will be dealt as in Section 3.2. Taking $\psi = \phi = \mathbf{1}_{U_n}$ in (4.2) and since $\|\mathbf{1}_{U_n}\|_{\mathcal{C}} \leq C'$ we get

$$\mathbb{P}\big(\{(x,\underline{\omega}) : x \in U_n,\ f^j_{\underline{\omega}}(x) \in\ U_n\}\big) \leq (\mu_\varepsilon(U_n))^2 + C\,\|\mathbf{1}_{U_n}\|_{\mathcal{C}}\,\|\mathbf{1}_{U_n}\|_{L^1(\mu_\varepsilon)}\,j^{-2}$$
$$\leq (\mu_\varepsilon(U_n))^2 + C^*\mu_\varepsilon(U_n)j^{-2}, \tag{4.6}$$

where $C^* = CC' > 0$. Now observe that by definition of U_n and (2.16), we have that $\mu_\epsilon(U_n) \sim \tau/n$. Using this observation together with the definition of $R^{\underline{\omega}}_n$ and the estimate (4.6), it follows that there exists some constant $D > 0$ such that

$$n\sum_{j=\alpha_n}^{\lfloor n/k_n\rfloor}\mathbb{P}\big(\{(x,\underline{\omega}) : x \in U_n,\ f^j_{\underline{\omega}}(x) \in\ U_n\}\big) \leq n\big\lfloor \tfrac{n}{k_n}\big\rfloor\mu_\varepsilon(U_n)^2 + n\,C^*\mu_\varepsilon(U_n)\sum_{j=\alpha_n}^{\lfloor n/k_n\rfloor} j^{-2}$$
$$\leq \frac{(n\mu_\varepsilon(U_n))^2}{k_n} + n\,C^*\mu_\varepsilon(U_n)\sum_{j=\alpha_n}^{\infty} j^{-2} \leq D\left(\frac{\tau^2}{k_n} + \tau\sum_{j=\alpha_n}^{\infty} j^{-2}\right) \xrightarrow[n\to\infty]{} 0.$$

For the term II, as $\mu_\varepsilon(U_n) \sim \tau/n$ and since $d\mu_\varepsilon/d\mathrm{Leb}$ is bounded from below and above by positive constants, there exists some positive constant $C^* > 0$ so that

$$n\sum_{j=1}^{\lfloor n/k_n\rfloor}\mathbb{P}\big(\{(x,\underline{\omega}) : x \in U_n, R^{\underline{\omega}}(U_n) \leq \alpha_n\}\big) \leq \frac{n^2}{k_n}\mu_\varepsilon(U_n)C\overline{g_\varepsilon}\mathrm{Leb}(U_n)\frac{\eta}{\eta-1}\eta^{\alpha_n}$$
$$\leq C^*\frac{\eta^{\alpha_n}}{k_n} \xrightarrow[n\to\infty]{} 0 \quad \text{by (4.4).} \tag{4.7}$$

□

Proof of Corollary F. The only extra step we need to do is to check that $D_3(u_n)$ also holds. To do that we just have to change slightly the definition of ψ that we used to prove $D_2(u_n)$ by using (4.2). Let $A \in \mathcal{R}$. We set:

$$\psi(x) = \int \mathbf{1}_{\bigcap_{i\in A\cap\mathbb{N}}\{f^i_{\underline{\tilde{\omega}}}(x)\leq u_n\}}\,\mathrm{d}\theta^{\mathbb{N}}_\varepsilon(\underline{\tilde{\omega}}).$$

The rest of the proof follows exactly as in the proof of $D_2(u_n)$ in the proof of Theorem D.

□

4.1.3 Expanding and piecewise expanding maps on the circle with a finite number of discontinuities

We give a general definition from [V97] of piecewise expanding maps on the circle which also includes the particular case of the continuous expanding maps:

(1) there exist $\ell \in \mathbb{N}_0$ and $0 = a_0 < a_1 < \cdots < a_\ell = 1 = 0 = a_0$ for which the restriction of f to each$\Xi_i = (a_{i-1}, a_i)$ is of class C^1, with $|Df(x)| > 0$ for all $x \in \Xi_i$ and $i = 1, \ldots, \ell$. In addition, for all $i = 1, \ldots, \ell$, $g_{\Xi_i} = 1/|Df|_{\Xi_i}|$ has bounded variation for $i = 1, \ldots, \ell$.

We assume that $(f|_{\Xi_i})$ and g_{Ξ_i} admit continuous extensions to$\Xi_i = [a_{i-1}, a_i]$, for each $i = 1, \ldots, \ell$. Since modifying the values of a map over a finite set of points does not change its statistical properties, we may assume that f is either left-continuous or right-continuous (or both) at a_i, for each $i = 1, \ldots, \ell$ (possibly for all i's at the same time). Then let $\mathcal{P}^{(1)}$ be some partition of $\mathcal{S}^1$ into intervals Ξ such that$\Xi \subset \Xi_i$ for some i and $(f|\Xi)$ is continuous. Furthermore, for $n \geq 1$, let $\mathcal{P}^{(n)}$ be the partition of $\mathcal{S}^1$ such that $\mathcal{P}^{(n)}(x) = \mathcal{P}^{(n)}(y)$ if and only if $\mathcal{P}^{(1)}(f^j(x)) = \mathcal{P}^{(1)}(f^j(y))$ for all $0 \leq j < n$. Given $\Xi \in \mathcal{P}^{(n)}$, denote $g_\Xi^{(n)} = 1/|Df^n|_\Xi|$;

(2) there exist constants $C_1 > 0, \lambda_1 < 1$ such that $\sup g_\Xi^{(n)} \leq C_1 \lambda_1^n$ for all$\Xi \in \mathcal{P}^{(n)}$ and all $n \geq 1$;

(3) for every subinterval J of $\mathcal{S}^1$, there exists some $n \geq 1$ such that $f^n(J) = \mathcal{S}^1$.

According to [V97, Proposition 3.15], one has exponential decay of correlations for randomly perturbed systems derived from maps satisfying conditions (1) – (3) above, taking $\mathcal{C}$ as the space of functions with bounded variation (BV), *i.e.*, given φ in BV and $\psi \in L^1(\text{Leb})$,

$$\left| \int (\mathcal{U}_\varepsilon \psi) \varphi \, \text{dLeb} - \int \psi \, d\mu_\varepsilon \int \varphi \, \text{dLeb} \right| \leq C \lambda^n \|\varphi\|_{BV} \|\psi\|_{L^1(\text{Leb})}, \tag{4.8}$$

where $0 < \lambda < 1$ and $C > 0$ is a constant independent of both φ,ψ .

Hence, in the particular case of f being a continuous expanding map of the circle, (4.8), Theorem D, Corollaries E and F allow us to obtain

Corollary 4.1.2. *Let $f : \mathcal{S}^1 \to \mathcal{S}^1$ be a continuous expanding map satisfying* (1) – (3) *above, which is randomly perturbed as in* (2.2) *with noise distribution given by* (2.1). *For any point $\zeta \in \mathcal{M}$, consider that $X_0, X_1, \dots$ is defined as in* (2.7) *and let u_n be such that* (2.16) *holds. Then the stochastic process $X_0, X_1, \dots$ satisfies $D_2(u_n)$, $D_3(u_n)$ and $D'(u_n)$, which implies that we have an EVL for M_n such that $\bar{H}(\tau) = \mathrm{e}^{-\tau}$ and we have exponential HTS/RTS for balls around ζ. Moreover, the REPP N_n defined in* (2.30) *is such that $N_n \xrightarrow{d} N$, as $n \to \infty$, where N denotes a Poisson Process with intensity* 1.

In the proof of Theorem D, we used the continuity of the map, in particular, in (4.5). However, we can adapt the argument in order to allow a finite number of discontinuities for expanding maps of the circle.

Proposition 4.1.3. *Let $f : \mathcal{S}^1 \to \mathcal{S}^1$ be a map satisfying conditions* (1) – (3) *above, which is randomly perturbed as in* (2.2) *with noise distribution given by* (2.1). *For any point $\zeta \in \mathcal{M}$, consider that $X_0, X_1, \dots$ is defined as in* (2.7) *and let u_n be such that* (2.16) *holds. Then the stochastic process $X_0, X_1, \dots$ satisfies $D_2(u_n)$, $D_3(u_n)$ and $D'(u_n)$, which implies that we have an EVL for M_n such that $\bar{H}(\tau) = \mathrm{e}^{-\tau}$ and we have exponential HTS/RTS for balls around ζ. Moreover, the REPP N_n defined in* (2.30) *is such that $N_n \xrightarrow{d} N$, as $n \to \infty$, where N denotes a Poisson Process with intensity* 1.

Proof. The proof of $D_2(u_n)$ follows from (4.8) in much the same way as in the continuous case. Regarding the proof of $D'(u_n)$, in order to use the same arguments as in the continuous case, we want to avoid coming close to the discontinuity points along the random orbit of ζ (up to time α_n). Since there are finitely many discontinuity points, say ξ_i's for $i = 1, \dots, \ell$, we can control this by asking for some "safety regions" around each of them. By doing so, we ensure that the random orbit of ζ is sufficiently far away from ξ_i's so that the iterates of U_n consist of only one connected component. We can formulate these "safety regions" as

$$\operatorname{dist}(f^j_{\underline{\omega}}(\zeta), \xi_i) > 2\eta^j |U_n| \text{ for all } i = 1, \ldots, \ell\,. \tag{4.9}$$

Now, we make an estimate on the $\underline{\omega}$'s that take the orbit of ζ too close to the discontinuity points as well as close to ζ itself and our aim is to show that the $\theta_\varepsilon^{\mathbb{N}}$ measure of this set is small. Let us set $\xi_0 = \zeta$ to simplify the notation. Then,

$$\left\{\underline{\omega} :\ R^{\underline{\omega}}(U_n) \le \alpha_n\right\} \subset \bigcup_{j=1}^{\alpha_n} \bigcup_{i=0}^{\ell} \left\{\underline{\omega} :\ f^j_{\underline{\omega}}(\zeta) \in B_{2\eta^j|U_n|}(\xi_i)\right\}.$$

Thus, we have

$$\begin{aligned}
\theta_\varepsilon^{\mathbb{N}}\left(\left\{\underline{\omega} : R^{\underline{\omega}}(U_n) \le \alpha_n\right\}\right) &\le \sum_{i=0}^{\ell}\sum_{j=1}^{\alpha_n} \int \theta_\varepsilon\Big(\Big\{\omega_j : f\left(f^{j-1}_{\underline{\omega}}(\zeta)\right) + \omega_j \in B_{2\eta^j|U_n|}(\xi_i)\Big\}\Big) \mathrm{d}\theta_\varepsilon^{\mathbb{N}} \\
&\le \sum_{i=0}^{\ell}\sum_{j=1}^{\alpha_n} \overline{g_\varepsilon}\left|B_{2\eta^j|U_n|}(\xi_i)\right| = \sum_{i=0}^{\ell}\sum_{j=1}^{\alpha_n} \overline{g_\varepsilon} 4\eta^j |U_n| \le 4(\ell+1)\overline{g_\varepsilon}|U_n| \frac{\eta}{\eta-1}\eta^{\alpha_n}.
\end{aligned}$$

The proof now follows the same lines as the proof of Theorem D and Corollary F. $\square$

4.1.4 Expanding and piecewise expanding maps in higher dimensions

Let us now consider the multidimensional piecewise expanding systems defined in Section 3.3.2 but only with a finite number, K, of domains of local injectivity. Moreover, let us restrict ourselves to a mixing component which, for simplicity, we take as the whole space Z. We take μ as the unique absolutely continuous invariant measure with density h. In addition, we ask each ∂Z_i to be included in piecewise C^1 codimension-1 embedded compact submanifolds and for

$$Z(f) = \sup_x \sum_{i=1}^{K} \#\{\text{ smooth pieces intersecting } \partial Z_i \text{ containing } x\}$$

we require

$$s^\alpha + \frac{4s}{1-s} Z(f) \frac{\gamma_{N-1}}{\gamma_N} < 1, \tag{4.10}$$

where γ_N is the N-volume of the N-dimensional unit ball of $\mathbb{R}^N$. Then, we know that by Lemma 2.1. in [S00], item (4) in Definition 3.3.7 is satisfied [1]. Notice that formula (4.10) gives exponential decay of correlations for the adapted pair: L^1 functions against functions in the quasi-Hölder space V_α.

We will perturb this kind of maps with additive noise asking that the image of Z is strictly included in Z. We will also require that h is bounded from below by the positive constant h_m. We will now prove the exponential decay of correlations for the random evolution operator $\mathcal{U}_\varepsilon$ by using the perturbation theory in [KL09], which we will also quote and use later on in Section 4.2.2. This theory ensures that the perturbed Perron-Frobenius operator $\mathcal{P}_\epsilon$ is mixing on the adapted pair (L^1, V_α) whenever we have:

(i) a uniform Lasota-Yorke inequality for $\mathcal{P}_\epsilon$, *i.e.*, all the constants in that inequality are independent of the noise ε,

(ii) the closeness property (see also hypothesis **H4** in Section 4.2.2): there exists a monotone upper semi-continuous function $p : \Omega \to [0, \infty)$ such that $\lim_{\varepsilon \to 0} p_\varepsilon = 0$ and $\forall \varphi \in V_\alpha$, $\forall \varepsilon \in \Omega : ||\mathcal{P}\varphi - \mathcal{P}_\varepsilon \varphi||_1 \leq p_\varepsilon ||\varphi||_\alpha$.

Condition (i) follows easily from observing that the derivatives of the original and the perturbed maps are the same, and this does not change the contraction factor s, and the multiplicity of the boundaries' intersection, $Z(f)$, is invariant too. Finally we evoke the observation written in the preceding footnote. Therefore the Perron-Frobenius operators $\mathcal{P}_\omega$ associated to the perturbed maps f_ω verify the same Lasota-Yorke inequality and therefore the same is true for $\mathcal{P}_\varepsilon$.

Our next step is to prove condition (ii), in particular we have:

[1] The inequality (4.10) ensures that for the unperturbed map the quantity $\eta(\varepsilon_1) < 1$; see the definition of this quantity after the formula (3.4). The value of $\eta(\varepsilon_1)$ is one of the constants in the Lasota-Yorke inequality , see item (i) below, and we will require that it will be independent of the noise. This will be the case for the additive noise since the determinant of the perturbed maps will not change and this is what is used in (3.4) to control the Lebesgue measure of $f_i^{-1} B_\varepsilon(\partial f Z_i)$. The other factor in the Lasota-Yorke inequality is also given in terms of the quantity (3.4).

Proposition 4.1.4. *There exists a constant C such that for any $\varphi \in V_\alpha$ we have*

$$\|\mathcal{P}\varphi - \mathcal{P}_\varepsilon\varphi\|_1 \le C\varepsilon^\alpha ||\varphi||_\alpha.$$

Proof. We have

$$\|\mathcal{P}\varphi - \mathcal{P}_\varepsilon\varphi\|_1 \le \int_Z \int_\Omega |\mathcal{P}_\omega\varphi(x) - \mathcal{P}\varphi(x)| \mathrm{d}\theta_\varepsilon(\omega)\mathrm{d}x.$$

Putting $G(x) = \frac{1}{|\det Df(x)|}$, we can write

$$\begin{aligned} |\mathcal{P}_\omega\varphi(x) - \mathcal{P}\varphi(x)| \le \sum_{Z_i, i=1,\dots,K} &|\varphi(f_i^{-1}x)G(f_i^{-1}x)\mathbf{1}_{fZ_i}(x) - \varphi(f_{\omega,i}^{-1}x)G(f_i^{-1}x)\mathbf{1}_{f_\omega Z_i}(x)| \\ &+ \sum_{Z_i, i=1,\dots,K} |\varphi(f_{\omega,i}^{-1}x)||G(f_i^{-1}x) - G(f_{\omega,i}^{-1}x)|\mathbf{1}_{f_\omega Z_i}(x) \\ &:= I + II \end{aligned} \tag{4.11}$$

where $f_\omega(x) = f(x) + \omega$ and ω is a vector in $\mathbb{R}^N$ with each component being less than ε in modulus. Moreover $f_{\omega,i}^{-1}$ denotes the inverse of the restriction of f_ω to Z_i, which is denoted by $f_{\omega,i}$ itself. We now bound the first summand in (4.11), I, by considering two cases:

(i) Let us first suppose that $x \in fZ_i \cap f_{\omega,i}Z_i$. Then since both f and $f_{\omega,i}$ are injective, there will be two points, y_i and $y_{\omega,i}$ in Z_i such that $x = f(y_i) = f_{\omega,i}(y_{\omega,i}) = f(y_{\omega,i}) + \omega$. This immediately implies that $\mathrm{dist}(y_i, y_{\omega,i}) \le s\sqrt{N}\varepsilon$, if dist is the Euclidean distance. For such an x we continue to bound I as:

$$I \le \sum_{Z_i, i=1,\dots,K} G(f_i^{-1}x)\mathrm{osc}(\varphi, B_{s\sqrt{N}\varepsilon}(f_i^{-1}(x)))\mathbf{1}_{fZ_i}(x)$$

By integrating over Z we get

$$\begin{aligned} \int_Z \Big(\sum_{Z_i, i=1,\dots,K} G(f_i^{-1}x)\mathrm{osc}(\varphi, B_{s\sqrt{N}\varepsilon}(f_i^{-1}(x)))\mathbf{1}_{fZ_i}(x) \Big)\mathrm{d}x &= \int_Z \mathcal{P}(\mathrm{osc}(\varphi, B_{s\sqrt{N}\varepsilon}(x)))\mathrm{d}x \\ &= \int_Z \mathrm{osc}(\varphi, B_{s\sqrt{N}\varepsilon}(x))\mathrm{d}x \le (s\sqrt{N}\varepsilon)^\alpha |\varphi|_\alpha. \end{aligned}$$

(ii) We now consider the case when $x \in fZ_i \Delta f_{\omega,i} Z_i$; the Lebesgue measure of this last set is bounded by ε times the codimension-1 volume of $\partial f Z_i$: let r denote the maximum of those volumes for $i = 1, \cdots, k$. Thus we get

$$\int_Z |\mathcal{P}_\omega \varphi(x) - \mathcal{P}\varphi(x)| \mathrm{d}x \leq r\varepsilon \|\varphi\|_\infty \|\mathcal{P}1\|_\infty. \tag{4.12}$$

We notice that the inclusion $V_\alpha \hookrightarrow L^\infty_m$ is bounded, that is to say that there exists c_v such that $\|\varphi\|_\infty \leq c_v \|\varphi\|_\alpha$. We therefore continue (4.12) as

$$(4.12) \leq r\varepsilon c_v \|\varphi\|_\alpha \|\mathcal{P}\frac{h}{h}\|_\infty \leq r\varepsilon c_v \|\varphi\|_\alpha \frac{\|h\|_\infty}{h_m}.$$

We now come to the second summand in (4.11), II; we begin by observing that

$$\begin{aligned} |G(f_i^{-1}x) - G(f_{\omega,i}^{-1}x)| &= \left| \frac{1}{|\det Df(f_i^{-1}x)|} - \frac{1}{|\det Df(f_{\omega,i}^{-1}x)|} \right| \\ &= \Big| |\det Df_i^{-1}(x)| - |\det Df_i^{-1}(z)| \Big| \\ &\leq |\det Df_i^{-1}(x) - \det Df_i^{-1}(z)| \end{aligned}$$

where $z = f(y_{\omega,i})$ and $\mathrm{dist}(x,z) \leq \sqrt{N}\varepsilon$. By using the Hölder assumption (2) in Definition 3.3.7, we have

$$\begin{aligned} II &\leq c(\sqrt{N}\varepsilon)^\alpha \sum_{Z_i, i=1,\ldots,K} |\varphi(f_{\omega,i}^{-1}x)| |\det Df_i^{-1}(z)| \mathbf{1}_{f_\omega Z_i}(x) \\ &\leq c(\sqrt{N}\varepsilon)^\alpha \sum_{Z_i, i=1,\ldots,K} |\varphi(f_{\omega,i}^{-1}x)| \frac{1}{|\det Df(f_{\omega,i}^{-1}(x))|} \mathbf{1}_{f_\omega Z_i}(x). \end{aligned}$$

By integrating over Z we get the contribution

$$c(\sqrt{N}\varepsilon)^\alpha \int_Z \mathcal{P}_\omega(|\varphi|) \mathrm{d}x \leq c(\sqrt{N}\varepsilon)^\alpha \int_Z |\varphi| \mathrm{d}x \leq c(\sqrt{N}\varepsilon)^\alpha ||\varphi||_{L^1(\mathrm{Leb})}.$$

In conclusion, we get $\|\mathcal{P}\varphi - \mathcal{P}_\varepsilon \varphi\|_1 \leq C\varepsilon^\alpha \|\varphi\|_\alpha$, where the constant C collects the various constants introduced above. □

As a consequence of Proposition 4.1.4, we obtain exponential decay of correlations of quasi-Hölder functions (in V_α), against L^1 functions, in particular, for uniformly expanding maps on the torus $\mathbb{T}^d$. Since, $\mathbf{1}_{U_n} \in V_\alpha$, $\|\mathbf{1}_{U_n}\|_\alpha$ is uniformly bounded from above, then it follows from Theorem D and Corollary E that:

Corollary 4.1.5. *Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a C^2 uniformly expanding map on $\mathbb{T}^d$, which is randomly perturbed as in* (2.2) *with noise distribution given by* (2.1)*. For any point $\zeta \in \mathcal{M}$, consider that $X_0, X_1, \dots$ is defined as in* (2.7) *and let u_n be such that* (2.16) *holds. Then the stochastic process $X_0, X_1, \dots$ satisfies $D_2(u_n)$, $D_3(u_n)$ and $D'(u_n)$, which implies that we have an EVL for M_n such that $\bar{H}(\tau) = \mathrm{e}^{-\tau}$ and we have exponential HTS/RTS for balls around ζ. Moreover, the REPP N_n defined in* (2.30) *is such that $N_n \xrightarrow{d} N$, as $n \to \infty$, where N denotes a Poisson Process with intensity* 1.

As in the previous case of maps on the circle, we may adapt the argument used in the continuous case to consider more general piecewise expanding maps of Definition 3.3.7, as long as there is a finite number of domains of local injectivity.

Proposition 4.1.6. *Suppose that (Z, f, μ) is a topologically mixing multidimensional piecewise expanding system as in Definition 3.3.7, μ is the absolutely continuous invariant probability measure with a Radon-Nikodym density bounded away from 0. We assume that there are $K \in \mathbb{N}$ domains of injectivity of the map and there exists $\eta > 1$ such that for all $i = 1, \dots, K$ and all $x, y \in Z_i$ we have $dist(f(x), f(y)) \le \eta\, dist(x, y)$. Consider that such a map is randomly perturbed with additive noise as in* (2.2) *with noise distribution given by* (2.1) *and such that the image of Z is strictly included in Z. For any point $\zeta \in \mathcal{M}$, consider that $X_0, X_1, \dots$ is defined as in* (2.7) *and let u_n be such that* (2.16) *holds. Then the stochastic process $X_0, X_1, \dots$ satisfies $D_2(u_n)$, $D_3(u_n)$ and $D'(u_n)$, which implies that we have an EVL for M_n such that $\bar{H}(\tau) = \mathrm{e}^{-\tau}$ and we have exponential HTS/RTS for balls around ζ. Moreover, the REPP N_n defined in* (2.30) *is such that $N_n \xrightarrow{d} N$, as $n \to \infty$, where N denotes a Poisson Process with intensity* 1.

Proof. Previously, for maps on the circle, by putting some "safety regions" around the discontinuity points we guaranteed that the iterates of $f^j_{\underline{\omega}}(U_n)$, $j = 0, 1, \dots, \alpha_n$ had one connected component. Since in this case the border of the domains of injectivity are codimension-1 submanifolds instead of single points (as in the 1-dimensional case), we must proceed to a more thorough analysis. To that end, for each $\underline{\omega}$ and for $j = 1$ let $1 \le$

$l_1 \leq K$ be the number of intersections with non-empty interior between $f_{\underline{\omega}}(U_n)$ and Z_i, with $i = 1, \ldots, K$. For each $\ell = 1, \ldots, l_1$, let i_ℓ denote the index of the partition element Z_{i_ℓ} for which such intersection has non-empty interior, define $U_n^{(1,\ell)} := f_{\underline{\omega}}(U_n) \cap Z_{i_\ell}$ and let $\zeta_{1,\ell}$ be a point in the interior of $U_n^{(1,\ell)}$. For any $j = 2, \ldots, \alpha_n$, given the sets $U_n^{(j-1,k)}$, with $k = 1, \ldots, l_{j-1}$, let l_j be the total number of intersections of non-empty interior between $f_{\sigma^{j-1}(\underline{\omega})}\left(U_n^{(j-1,k)}\right)$ and Z_i, with $i = 1, \ldots, K$. For each $\ell = 1, \ldots, l_j$, let i_ℓ denote the index of the partition element Z_{i_ℓ} and k_ℓ the super index of the sets $U_n^{(j-1,k)}$ for which the intersection between $f_{\sigma^{j-1}(\underline{\omega})}\left(U_n^{(j-1,k_\ell)}\right)$ and Z_{i_ℓ} has non-empty interior, define $U_n^{(j,\ell)} = f_{\sigma^{j-1}(\underline{\omega})}\left(U_n^{(j-1,k_\ell)}\right) \cap Z_{i_\ell}$ and let $\zeta_{j,\ell}$ be a point in the interior of $U_n^{(j,\ell)}$.

In order to avoid the first return time to U_n occurring before α_n in a similar way as in the previous proofs, we require that:

$$\operatorname{dist}(f_{\sigma^{j-1}(\underline{\omega})}(\zeta_{j-1,\ell}), \zeta) > 2\eta^j |U_n| \text{ for all } j = 2, \ldots, \alpha_n,\ \ell = 1, \ldots, l_{j-1}. \tag{4.13}$$

Note that, similarly to the proof of Theorem D, for any $\underline{\omega} \in \Omega$, we have $|U_n^{(j,\ell)}| \leq \eta^j |U_n|$. This implies that

$$\text{if } \operatorname{dist}(f_{\sigma^{j-1}(\underline{\omega})}(\zeta_{j-1,\ell}), \zeta) > 2\eta^j |U_n| > |U_n| + \eta^j |U_n| \text{ then } U_n^{(j,\ell)} \cap U_n = \emptyset. \tag{4.14}$$

Note that, by equation (4.14), if (4.13) holds then clearly $R^{\underline{\omega}}(U_n) > \alpha_n$. Hence, letting $l_0 = 1$ and $\zeta_{0,1} = \zeta$, we may write that

$$\left\{\underline{\omega} :\ R^{\underline{\omega}}(U_n) \leq \alpha_n\right\} \subset \bigcup_{j=1}^{\alpha_n} \bigcup_{\ell=1}^{l_{j-1}} \left\{\underline{\omega} :\ f_{\sigma^{j-1}(\underline{\omega})}(\zeta_{j-1,\ell}) \in B_{2\eta^j |U_n|}(\zeta)\right\}.$$

Recalling that $l_j \leq K^j$, for all $j = 1, \ldots, \alpha_n$, it follows that, there exists some $C > 0$ such that

$$\begin{aligned}
\theta_\varepsilon^{\mathbb{N}}\left(\left\{\underline{\omega} :\ R^{\underline{\omega}}(U_n) \leq \alpha_n\right\}\right) &\leq \sum_{j=1}^{\alpha_n} \sum_{\ell=1}^{l_{j-1}} \int \theta_\varepsilon\Big(\Big\{\omega_j : f\left(\zeta_{j-1,\ell}\right) + \omega_j \in B_{2\eta^j |U_n|}(\zeta)\Big\}\Big) \mathrm{d}\theta_\varepsilon^{\mathbb{N}} \\
&\leq \sum_{j=1}^{\alpha_n} \sum_{\ell=1}^{l_{j-1}} \overline{g_\varepsilon} \mathrm{Leb}\left(B_{2\eta^j |U_n|}(\zeta)\right) \leq \sum_{j=1}^{\alpha_n} K^j \overline{g_\varepsilon} C \eta^j \mathrm{Leb}(U_n) \leq C \overline{g_\varepsilon} \mathrm{Leb}(U_n) \frac{\eta K}{\eta K - 1} (\eta K)^{\alpha_n}.
\end{aligned}$$

Now, the proof follows exactly in the same way as the proofs of Theorem D and Corollary F, except that in the final estimate (4.7), η should be replaced by ηK, which will not make any difference by the choice of α_n defined in (4.4). $\square$

4.2 Extremes for stochastic dynamics: spectral approach

In this section, we want to prove our results for the stochastic case using another approach introduced by Keller in [K12]. His technique is based on an eigenvalue perturbation formula which was given in [KL09] under a certain number of assumptions. We recall them in the first subsection and adapt to our situation. We check those assumptions in Section 4.2.3 for a large class of maps of the interval whose properties are listed in the conditions (**H1-H5**). Possible generalisations deserve to be investigated and we point out here a major difficulty in higher dimensions. In this case one should control (any kind of) variation/oscillation on the boundaries of the preimages of the complements of balls (the set U_m^c in the proof of Proposition 4.2.2 below; it is important that such variation/oscillation grows at most sub-exponentially). To sum up, the direct technique introduced in Section 4.1 and the spectral one in this section are complementary. The direct technique is easily adapted to higher dimensions but it requires assumptions on the noise in order to control the short returns (see the quantity $R^{\underline{\omega}}(U_n)$ in Proposition 4.1.6), which follows easily for additive noise. The spectral technique is an alternative method and for the moment particularly adapted to the 1-D case and, as we will see in a moment, the noise could be chosen in a quite general way to prove the existence of the EI, formula (4.21). Instead, if we want to characterise such an EI and show that it is always equal to 1, we need to consider special classes of uniformly expanding maps and particularly the noise should be chosen as additive and with a continuous distribution (Proposition 4.2.3). The fact that the existence of EI follows for general classes of noises is clear by looking at the proof of Proposition 4.2.2. Indeed, what is really necessary is that the derivatives of the randomly chosen maps are close enough to each other in order to guarantee the uniformity of the Lasota-Yorke inequality for the perturbed Perron-Frobenius operator. This could be achieved quite widely and with discrete distributions as well. Nevertheless, in order to make the exposition simpler and coherent with the previous sections, we will consider additive noise, together with *any kind of distribution* to prove Proposition 4.2.2 and with *absolutely continuous distributions* to prove Proposition 4.2.3.

4.2.1 The setting

Given a Banach space $(V, \|\cdot\|)$, and a set of parameters E which is equipped with some topology, let us suppose there are $\lambda_\varepsilon \in \mathbb{C}$, $\varphi_\varepsilon \in V$, $\nu_\varepsilon \in V'$ (V' denotes the dual of V) and linear operators $P_\varepsilon, Q_\varepsilon : V \to V$ such that

$$\lambda_\varepsilon^{-1} P_\varepsilon = \varphi_\varepsilon \otimes \nu_\varepsilon + Q_\varepsilon \text{ (assume } \lambda_0 = 1) \ , \tag{4.15}$$

$$P_\varepsilon(\varphi_\varepsilon) = \lambda_\varepsilon \varphi_\varepsilon, \nu_\varepsilon P_\varepsilon = \lambda_\varepsilon \nu_\varepsilon,\ Q_\varepsilon(\varphi_\varepsilon) = 0,\ \nu_\varepsilon Q_\varepsilon = 0, \tag{4.16}$$

$$\sum_{n=0}^{\infty} \sup_{\varepsilon \in E} \|Q_\varepsilon^n\| =: C_1 < \infty, \tag{4.17}$$

$$\exists C_2 > 0, \forall \varepsilon \in E : \nu_0(\varphi_\varepsilon) = 1 \text{ and } \|\varphi_\varepsilon\| \le C_2 < \infty, \tag{4.18}$$

$$\lim_{\varepsilon \to 0} \|\nu_0(P_0 - P_\varepsilon)\| = 0, \tag{4.19}$$

$$\|\nu_0(P_0 - P_\varepsilon)\| \cdot \|(P_0 - P_\varepsilon)\varphi_0\| \le \text{const} \cdot |\Delta_\varepsilon| \tag{4.20}$$

where

$$\Delta_\varepsilon := \nu_0((P_0 - P_\varepsilon)(\varphi_0)).$$

Under these assumptions, Keller and Liverani got the following formula as the main result in [KL09]:

$$1 - \lambda_\varepsilon = \Delta_\varepsilon \vartheta(1 + o(1)) \text{ in the limit as } \varepsilon \to 0 \tag{4.21}$$

where ϑ is said to be a constant to take care of short time correlations, which is later identified as the extremal index in extreme value theory context as mentioned in [K12, Section 1.2]. Actually ϑ is given by an explicit and, in some cases, computable formula, and, in fact, we will be able to compute it for our random systems. This formula is the content of Theorem 2.1 in [KL09] and states that under the above assumptions, in particular when$\Delta_\varepsilon \neq 0$, for ε small enough, and whenever the following limit exists

$$q_k := \lim_{\varepsilon \to 0} q_{k,\varepsilon} := \lim_{\varepsilon \to \infty} \frac{\nu_0((P_0 - P_\varepsilon)P_\varepsilon^k(P_0 - P_\varepsilon)(\varphi_0))}{\Delta_\varepsilon}, \tag{4.22}$$

we have

$$\lim_{\varepsilon\to 0}\frac{1-\lambda_\varepsilon}{\Delta_\varepsilon}=\vartheta := 1-\sum_{k=0}^{\infty} q_k. \tag{4.23}$$

We now state equivalent ways to verify assumptions (4.15)-(4.20), we refer the reader to [K12] for the details.

(A1) There are constants $A>0, B>0, D>0$ and a second norm $|\cdot|_\omega \le \|\cdot\|$ on V (it is enough to be a seminorm) such that:

$$\forall \varepsilon\in E, \forall\psi\in V, \forall n\in\mathbb{N} : |P_\varepsilon^n\psi|_\omega \le D|\psi|_\omega \tag{4.24}$$

$$\exists\alpha\in(0,1), \forall \varepsilon\in E, \forall\psi\in V, \forall n\in\mathbb{N} : \|P_\varepsilon^n\psi\| \le A\alpha^n\|\psi\| + B|\psi|_\omega \tag{4.25}$$

Moreover the closed unit ball of $(V, \|\cdot\|)$, is $|\cdot|_\omega$-compact.

(A2) The unperturbed operator verifies the mixing condition

$$P=\varphi\otimes\nu+Q_0 \text{ (assume } \lambda_0=1) \tag{4.26}$$

(A3) $\exists C>0$ such that

$$\eta_\varepsilon := \sup_{\|\psi\|\le 1}\left|\int (P_0-P_\varepsilon)\psi \, d\nu_0\right| \to 0, \text{ as } \varepsilon\to 0 \tag{4.27}$$

(A4) and

$$\eta_\varepsilon \ \|(P_0-P_\varepsilon)\varphi_0\| \le C\ \Delta_\varepsilon \tag{4.28}$$

Keller called this framework *Rare events Perron-Frobenius operators*, REPFO. We will construct a *perturbed* Perron-Frobenius operator which satisfies the previous assumptions and which will give us information on extreme value distributions and statistics of first returns to small sets.

Before continuing, we should come back to our extreme distributions, namely to the quantity $\{M_m\le u_m\}=\{r_{\{\phi>u_m\}}>m\}$ where $\{\phi>u_m\}=: U_m$ is a topological ball shrinking to the point ζ (see (2.18): we changed U_n into U_m here). Now we consider the first time the point x enters U_m under the realisation $\underline{\omega}$, $r^{\omega}_{U_m}(x)$. For simplicity we indicate it by $r^{\omega}_{m}(x)$ and consider its annealed distribution:

$$(\mu_\varepsilon \text{ x } \theta_\varepsilon^{\mathbb{N}})((x,\underline{\omega}) : r^{\omega}_{m}(x) > m) = (\mu_\varepsilon \text{ x } \theta_\varepsilon^{\mathbb{N}})(M_m \leq u_m) \tag{4.29}$$

Let us write the measure on the left-hand side of (4.29) in terms of integrals:

$$\iint_{\{r^{\omega}_{m} > m\}} \mathrm{d}(\mu_\varepsilon \text{ x } \theta_\varepsilon^{\mathbb{N}}) = \iint h_\varepsilon \mathbf{1}_{U_m^c}(x)\mathbf{1}_{U_m^c}(f_{\omega_1}x)\cdots \mathbf{1}_{U_m^c}(f_{\omega_{m-1}}\circ\cdots\circ f_{\omega_1}x)\,\mathrm{dLeb}\,\mathrm{d}\theta_\varepsilon^{\mathbb{N}} \tag{4.30}$$

which is in turn equal to

$$\int_M \widetilde{\mathcal{P}}^m_{\varepsilon,m} h_\varepsilon(x)\,\mathrm{dLeb} \tag{4.31}$$

where we have now defined

$$\widetilde{\mathcal{P}}_{\varepsilon,m}\psi(x) := \mathcal{P}_\varepsilon(\mathbf{1}_{U_m^c}\psi)(x). \tag{4.32}$$

Let us note that the operator $\widetilde{\mathcal{P}}_{\varepsilon,m}$ depends on m via the set U_m, and not on ε which is kept fixed and that $\widetilde{\mathcal{P}}_{\varepsilon,m}$ "reduces" to $\mathcal{P}_\varepsilon$ as $m \to \infty$. It is therefore tempting to consider $\widetilde{\mathcal{P}}_{\varepsilon,m}$ as a small perturbation of $\mathcal{P}_\varepsilon$ when m is large and to check if it shares the spectral properties of a REPFO. We will show in a moment that it will be the case; let us now see what that implies for our theory.

4.2.2 Limiting distributions

We now indicate the correspondences between the general notations of Keller's results and our own quantities:

$$P_0 \Rightarrow \mathcal{P}_\varepsilon$$

$$P_\varepsilon \Rightarrow \widetilde{\mathcal{P}}_{\varepsilon,m};\ Q_\varepsilon \Rightarrow Q_{\varepsilon,m}$$

$$\varphi_\varepsilon \Rightarrow \varphi_{\varepsilon,m};\ \varphi_0 \Rightarrow h_\varepsilon$$

$$\lambda_\varepsilon \Rightarrow \lambda_{\varepsilon,m}$$

$$\nu_\varepsilon \Rightarrow \nu_{\varepsilon,m};\ \nu_0 \Rightarrow \mathrm{Leb}$$

$$\Delta_\varepsilon \Rightarrow \Delta_{\varepsilon,m} = \mu_\varepsilon(U_m) = \mathrm{Leb}((\mathcal{P}_\varepsilon - \widetilde{\mathcal{P}}_{\varepsilon,m})h_\varepsilon)$$

The framework for which we will prove the assumptions **(A1)-(A4)** for our REPFO $\widetilde{\mathcal{P}}_{\varepsilon,m}$ are those behind the system and its perturbations which we introduced in the previous sections and summarise here:

Hypotheses on the system and its perturbations

We consider piecewise expanding maps f of the circle or of the interval I which verify:

H1 The map f admits a (unique) absolutely continuous invariant probability measure which is mixing.

H2 We will require that

$$\inf_{x\in I} |Df(x)| \geq \beta > 1. \tag{4.33}$$

and

$$\sup_{x\in I00B2} \left| \frac{D^2 f(x)}{Df(x)} \right| \leq C_1 < \infty, \tag{4.34}$$

whenever the first and the second derivatives are defined.

H3 The couple of adapted spaces upon which the REPFO will act are: the space of functions of bounded variation (as in Definition 3.3.1, we will indicate with Var the total variation), and $L^1(\text{Leb})$, with norm $\|\cdot\|_1$; this time, we will write $\|\cdot\|_{BV} = \text{Var}(\cdot) + \|\cdot\|_1$ for the associated Banach norm.

H4 There exists a monotone upper semi-continuous function $p : \Omega \to [0, \infty)$ such that $\lim_{\varepsilon\to 0} p_\varepsilon = 0$ and $\forall f \in \text{BV}, \forall \varepsilon \in \Omega : ||\mathcal{P}f - \mathcal{P}_\varepsilon f||_1 \leq p_\varepsilon ||f||_{BV}$[2].

H5 The density h_ε of the stationary measure is bounded from below Leb-a.e. and we call this bound $\underline{h_\varepsilon}$.

[2]This condition can be checked in several cases. We did it, for instance, in Section 4.1.4. A general theorem is presented in Lemma 16 in [K82] for piecewise expanding maps of the interval endowed with our pair of adapted spaces and with the noise given by a convolution kernel. This means that θ_ε is absolutely continuous with respect to Lebesgue on the space Ω with density s_ε, and our two operators are related by the convolution formula $\mathcal{P}_\varepsilon g(x) = \int_\Omega (\mathcal{P}g)(x-\omega)s_\varepsilon(\omega)\mathrm{d}\omega$, where $g \in BV$. In the case of additive noise, it is straightforward to check that the previous formula is equivalent to $\mathcal{P}_\varepsilon g(x) = \int_\Omega (\mathcal{P}_\omega g)(x)s_\varepsilon(\omega)\mathrm{d}\omega$, where $\mathcal{P}_\omega$ is the Perron-Frobenius operator of the transformation f_ω.

Extreme values

Let us therefore write

$$\widetilde{\mathcal{P}}_{\varepsilon,m}\varphi_{\varepsilon,m} = \lambda_{\varepsilon,m}\varphi_{\varepsilon,m}, \nu_{\varepsilon,m}\widetilde{\mathcal{P}}_{\varepsilon,m} = \lambda_{\varepsilon,m}\nu_{\varepsilon,m}, \text{ and } \lambda_{\varepsilon,m}^{-1}\widetilde{\mathcal{P}}_{\varepsilon,m} = \varphi_{\varepsilon,m} \otimes \nu_{\varepsilon,m} + Q_{\varepsilon,m}.$$

Then formula (4.21) implies that $1 - \lambda_{\varepsilon,m} = \Delta_{\varepsilon,m}\vartheta_\varepsilon(1 + o(1))$. We can therefore write:

$$\begin{aligned}(\mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}})(M_m \leq u_m) &= \int_M \widetilde{\mathcal{P}}_{\varepsilon,m}^m h_\varepsilon(x)\,\mathrm{dLeb} = \lambda_{\varepsilon,m}^m \int h_\varepsilon\,\mathrm{d}\nu_{\varepsilon,m} + \lambda_{\varepsilon,m}^m \int Q_{\varepsilon,m}h_\varepsilon\,\mathrm{dLeb} \\ &= e^{-(\vartheta_\varepsilon m\,\mu_\varepsilon(U_m)+m\,o(\mu_\varepsilon(U_m)))} \int h_\varepsilon\,\mathrm{d}\nu_{\varepsilon,m} + \mathcal{O}(\lambda_{\varepsilon,m}^m \|Q_{\varepsilon,m}\|_{BV})\end{aligned}$$

Remember that we are under the assumption that

$$m\,(\mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}})(\phi > u_m) = m\,\mu_\varepsilon(\phi > u_m) = m\,\mu_\varepsilon(U_m) \to \tau,$$

when $m \to \infty$; moreover it follows from the theory of [KL09] that

$$\int h_\varepsilon\,\mathrm{d}\nu_{\varepsilon,m} \to \int h_\varepsilon\,\mathrm{dLeb} = 1,$$

as m goes to infinity. In conclusion we get

$$(\mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}})(M_m \leq u_m) = e^{-\tau\vartheta_\varepsilon}(1 + o(1)),$$

in the limit $m \to \infty$ and where ϑ_ε will be the extremal index and this will be explicitly computed later on for some particular maps thanks to formula (4.23) and showed to be equal to 1 for any point ζ, see Proposition 4.2.3 below.

Random hitting times

Let us denote again the first entrance into the ball U_m by $r_{U_m}^\omega(x)$. A direct application of [K12, Proposition 2] and which is true for REPFO, allows us to get the following result, which we adapted to our situation and which provides an explicit formula for the statistics of the first hitting times in the annealed case. Notice that this result strengthens our Corollary E since it provides the error in the convergence to the exponential law.

Proposition 4.2.1. *For the REPFO $\widetilde{\mathcal{P}}_{\varepsilon,m}$ which verifies the hypotheses* **H1**-**H5**, *and using the notations introduced above, there exists a constant $C > 0$ such that for all m big enough there exists $\xi_m > 0$ s.t. for all $t > 0$*

$$\left|(\mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}})\left\{r_{U_m}^{\omega} > \frac{t}{\xi_m\,\mu_\varepsilon(U_m)}\right\} - e^{-t}\right| \le C\delta_m(t \vee 1)e^{-t}$$

where $\delta_m = O(\eta_m \log \eta_m)$,

$$\eta_m = \sup\left\{\int_{U_m} \psi \,\mathrm{dLeb}; \|\psi\|_{BV} \le 1\right\}$$

and ξ_m goes to ϑ_ε as $m \to \infty$.

4.2.3 Checking assumptions (A1)-(A4)

Proposition 4.2.2. *For the REPFO $\widetilde{\mathcal{P}}_{\varepsilon,m}$ which verifies the hypotheses* **H1**-**H5**, *the assumptions* **(A1)**-**(A4)** *hold.*

Proof. Condition (**A1**) means to prove the Lasota-Yorke inequality for the operator $\widetilde{\mathcal{P}}_{\varepsilon,m}$. We recall that the constants A and B there must be independent of the perturbation parameter, which is m in our case. We start with the total variation.

The structure of $\widetilde{\mathcal{P}}_{\varepsilon,m}$'s iterates is

$$(\widetilde{\mathcal{P}}^n_{\varepsilon,m}\psi) = \int \cdots \int \mathcal{P}_{\omega_n}(\mathbf{1}_{U_m^c}\mathcal{P}_{\omega_{n-1}}(\mathbf{1}_{U_m^c} \cdots \mathcal{P}_{\omega_1}(\psi\,\mathbf{1}_{U_m^c})))\,\mathrm{d}\theta_\varepsilon(\omega_1)\cdots\mathrm{d}\theta_\varepsilon(\omega_n). \tag{4.35}$$

Let us call A_{l,ω_j} the l-domain of injectivity of the map f_{ω_j} and denote by f^{-1}_{l,ω_j} the inverse of f_{ω_j} restricted to A_{l,ω_j}. We have

$$\begin{aligned}\Upsilon_{\omega_1,\ldots,\omega_n} &:= \mathcal{P}_{\omega_n}(\mathbf{1}_{U_m^c}\mathcal{P}_{\omega_{n-1}}(\mathbf{1}_{U_m^c} \cdots \mathcal{P}_{\omega_1}(\psi\,\mathbf{1}_{U_m^c})))(x)\\ &= \sum_{k_n,\ldots,k_1} \frac{(\psi \cdot \mathbf{1}_{U_m^c} \cdot \mathbf{1}_{U_m^c} \circ f_{\omega_1} \cdots \mathbf{1}_{U_m^c} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1})((f^{-1}_{k_1,\omega_1} \circ \cdots \circ f^{-1}_{k_n,\omega_n})(x))}{|D(f_{\omega_n} \circ \cdots \circ f_{\omega_1})((f^{-1}_{k_1,\omega_1} \circ \cdots \circ f^{-1}_{k_n,\omega_n})(x))|}\\ &\qquad\qquad\qquad \times \mathbf{1}_{f_{\omega_n}\circ\cdots\circ f_{\omega_1}\Omega^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n}}(x).\end{aligned}$$

The sets

$$\Omega^{k_1,\dots,k_n}_{\omega_1,\dots,\omega_n} := f^{-1}_{k_1,\omega_1}\circ\dots\circ f^{-1}_{k_{n-1},\omega_{n-1}}A_{k_n,\omega_n}\cap f^{-1}_{k_1,\omega_1}\circ\dots\circ f^{-1}_{k_{n-2},\omega_{n-2}}A_{k_{n-1},\omega_{n-1}}\cap\cdots$$
$$\cdots\cap f^{-1}_{k_1,\omega_1}A_{k_2,\omega_2}\cap A_{k_1,\omega_1}$$

are intervals and they give a mod-0 partition of $I=[0,1]$. Moreover, the image

$$H^{k_1,\dots,k_n}_{\omega_1,\dots,\omega_n} := f_{\omega_n}\circ\dots\circ f_{\omega_1}\Omega^{k_1,\dots,k_n}_{\omega_1,\dots,\omega_n}$$

for a given n-tuple $\{k_n,\dots,k_1\}$ is a connected interval. We also note for future purposes that we can equivalently write:

$$g_n := \mathbf{1}_{U^c_m}\cdot\mathbf{1}_{U^c_m}\circ f_{\omega_1}\cdot\ldots\cdot\mathbf{1}_{U^c_m}f_{\omega_{n-1}}\circ\dots\circ f_{\omega_1} = \mathbf{1}_{U^c_m\cap f^{-1}_{\omega_1}U^c_m\cap\dots\cap f^{-1}_{\omega_1}\circ\dots\circ f^{-1}_{\omega_{n-1}}U^c_m}.$$

Observe that the set

$$U^c_m(n) := U^c_m\cap f^{-1}_{\omega_1}U^c_m\cap f^{-1}_{\omega_1}\circ f^{-1}_{\omega_2}U^c_m\cap\dots\cap f^{-1}_{\omega_1}\circ\dots\circ f^{-1}_{\omega_{n-1}}U^c_m\cap\Omega^{k_1,\dots,k_n}_{\omega_1,\dots,\omega_n}$$

is actually given by:

$$U^c_m(n) := U^c_m\cap f^{-1}_{k_1,\omega_1}U^c_m\cap f^{-1}_{k_1,\omega_1}\circ f^{-1}_{k_2,\omega_2}U^c_m\cap\dots\cap f^{-1}_{k_1,\omega_1}\circ\dots\circ f^{-1}_{k_{n-1},\omega_{n-1}}U^c_m\cap\Omega^{k_1,\dots,k_n}_{\omega_1,\dots,\omega_n}.$$

Since U^c_m is the disjoint union of two connected intervals, the number of connected intervals in $U^c_m(n)$ is bounded from above by $n+1$ and it is important that it grows linearly with n. We now take the total variation $\mathrm{Var}(\Upsilon_{\omega_1,\dots,\omega_n})$. We begin to remark that, by standard techniques:

$$\mathrm{Var}\left(\frac{(\psi g_n)((f^{-1}_{k_1,\omega_1}\circ\dots\circ f^{-1}_{k_n,\omega_n})(x))}{|D(f_{\omega_n}\circ\dots\circ f_{\omega_1})((f^{-1}_{k_1,\omega_1}\circ\dots\circ f^{-1}_{k_n,\omega_n})(x))|}\mathbf{1}_{f_{\omega_n}\circ\dots\circ f_{\omega_1}\Omega^{k_1,\dots,k_n}_{\omega_1,\dots,\omega_n}}(x)\right)$$
$$\le 2\mathrm{Var}_{H^{k_1,\dots,k_n}_{\omega_1,\dots,\omega_n}}\left(\frac{(\psi g_n)((f^{-1}_{k_1,\omega_1}\circ\dots\circ f^{-1}_{k_n,\omega_n})(x))}{|D(f_{\omega_n}\circ\dots\circ f_{\omega_1})((f^{-1}_{k_1,\omega_1}\circ\dots\circ f^{-1}_{k_n,\omega_n})(x))|}\right)$$
$$+\frac{2}{\beta^n}\frac{1}{\mathrm{Leb}(\Omega^{k_1,\dots,k_n}_{\omega_1,\dots,\omega_n})}\int_{\Omega^{k_1,\dots,k_n}_{\omega_1,\dots,\omega_n}}|\psi g_n|\,\mathrm{dLeb},$$

where β is given by (4.33) in **H2**.

The variation above can be further estimated by standard techniques:

$$\leq \frac{2}{\beta^n}\mathrm{Var}_{\Omega^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n}}(\psi g_n) + \frac{2}{\beta^n}\frac{1}{\mathrm{Leb}(\Omega^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n})}\int_{\Omega^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n}} |\psi g_n| \,\mathrm{dLeb} \\ + 2\sup_{\zeta,\omega_1,\ldots,\omega_n}\frac{|D^2(f_{\omega_n}\circ\cdots\circ f_{\omega_1})(\zeta)|}{[D(f_{\omega_n}\circ\cdots\circ f_{\omega_1})(\zeta)]^2}\int_{\Omega^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n}} |\psi g_n| \,\mathrm{dLeb} \quad (4.36)$$

We now have

$$\frac{|D^2(f_{\omega_n}\circ\cdots\circ f_{\omega_1})(\zeta)|}{[D(f_{\omega_n}\circ\cdots\circ f_{\omega_1})(\zeta)]^2} = \sum_{k=0}^{n-1}\frac{D^2 f_{\omega_{n-k}}\left(\prod_{l=1}^{n-1-k} T_{\omega_{n-l}}(\zeta)\right)}{\left[Df_{\omega_{n-k}}\left(\prod_{l=1}^{n-1-k} f_{\omega_{n-l}}(\zeta)\right)\right]^2 \prod_{j=0}^{k} Df_{\omega_{n-j+1}}\left(\prod_{l=1}^{n-j} f_{\omega_{n-l}}(\zeta)\right)}.$$

By (4.34) in **H2** and using again (4.33), the previous sum will be bounded by C_1 times the sum of a geometric series of reason β^{-1}: we call C the upper bound thus found. Our variation above is therefore bounded by:

$$(4.36) \leq \frac{2}{\beta^n}\mathrm{Var}_{\Omega^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n}}(\psi g_n) + \frac{2}{\beta^n}\frac{1}{\mathrm{Leb}(\Omega^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n})}\int_{\Omega^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n}} |\psi g_n|\mathrm{dLeb} \\ + 2C\int_{\Omega^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n}} |\psi g_n|\mathrm{dLeb} \quad (4.37)$$

Now

$$\mathrm{Var}_{\Omega^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n}}(\psi g_n) \leq \mathrm{Var}_{\Omega^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n}}(\psi) + 2(n+1)\sup_{\Omega^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n}}|\psi| \\ \leq [2(n+1)+1]\mathrm{Var}_{\Omega^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n}}(\psi) + \frac{1}{\mathrm{Leb}(\Omega^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n})}\int_{\Omega^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n}} |\psi| \,\mathrm{dLeb}, \quad (4.38)$$

where $2(n+1)$ is an estimate from above of the number of jumps of g_n. We now observe that for a finite realisation of length n, $\omega_1,\ldots,\omega_n$, the quantity

$$\Psi_{n,\omega_1,\ldots,\omega_n} = \inf_{k_1,\ldots,k_n} \mathrm{Leb}(\Omega^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n}),$$

where each k_j runs over the finite branches of f_{ω_j}, is surely strictly positive and this will also implies that$\Psi\ _n^{-1} := \int \Psi^{-1}_{n,\omega_1,\ldots,\omega_n} d\theta^{\mathbb{N}}_{\varepsilon} > 0$. We now replace (4.38) into (4.37), we sum over the $k_1, \ldots, k_n$ and integrate with respect to $\theta_\varepsilon^{\mathbb{N}}$; finally we get

$$\mathrm{Var}(\widetilde{\mathcal{P}}^n_{\varepsilon,m}\psi) \leq \frac{2}{\beta^n}(2n+3)\mathrm{Var}(\psi) + \left[\frac{4}{\beta^n}\frac{1}{\Psi_n} + 2C\right] \int_I |\psi| \, \mathrm{dLeb}.$$

In order to get the Lasota-Yorke inequality one should get a certain n_0 and a number $\beta > \kappa > 1$ and such that

$$\frac{2}{\beta^{n_0}}(2n_0+3) < \kappa^{-n_0}; \tag{4.39}$$

then the Lasota-Yorke inequality, (4.25), follows from standard arguments.[3]

We now compute the L^1-norm of our operator. In fact, we need to compute $\|\widetilde{\mathcal{P}}^n_{\varepsilon,m}\psi\|_1$. By splitting ψ into the sum of its positive and negative parts and by using the linearity of the transfer operator, we may suppose that ψ is non-negative. This allows us to interchange the integrals with respect to the Lebesgue measure and $\theta_\varepsilon^{\mathbb{N}}$ and to use duality for each of the $\mathcal{P}_\omega$'s. Hence we get

$$\|\widetilde{\mathcal{P}}^n_{\varepsilon,m}\psi\|_1 \leq \int |\psi| h_\varepsilon \mathbf{1}_{U_m^c}(x)\mathbf{1}_{U_m^c}(f_{\omega_1}x)\cdots\mathbf{1}_{U_m^c}(f_{\omega_{n-1}}\circ\cdots\circ f_{\omega_1}x)\,\mathrm{dLeb} \leq \|\psi\|_1.$$

This concludes the proof of the Lasota-Yorke inequality, (**A1**). Now, we have to show that the operator $\mathcal{P}_\varepsilon$, which is the unperturbed operator with respect to $\widetilde{\mathcal{P}}_{\varepsilon,m}$, verifies the mixing condition (**A2**). Notice that the Perron-Frobenius operator, $\mathcal{P}$, for the original map, f, which is in turn the unperturbed operator with respect to $\mathcal{P}_\varepsilon$, is mixing (1 is the only eigenvalue of finite multiplicity on the unit circle) since f was chosen to be mixing (hypothesis **H1**), and therefore, by the perturbation theory in [KL09] and the closeness of the two operators expressed by assumption **H4**, $\mathcal{P}_\varepsilon$ is also a mixing operator. For assumption (**A3**), let us bound the following quantity, for any ψ of bounded variation

[3] By defining $A = 2(2n_0+3)$ and $B = \left[\frac{4}{\Psi_{n_0}} + 2C\right]\frac{2}{1-\kappa^{-n_0}}$, we have

$$\mathrm{Var}(\widetilde{\mathcal{P}}^n_{\varepsilon,m}\psi) \leq A\kappa^{-n}\mathrm{Var}(\psi) + B\int_I |\psi| \, \mathrm{dLeb}.$$

and of total variation less than or equal to 1:

$$\begin{aligned}\left|\int_I (\widetilde{\mathcal{P}}_{\varepsilon,m}\psi(x) - \mathcal{P}_\varepsilon\psi(x))\,\mathrm{dLeb}(x)\right| &= \left|\int_I \mathcal{P}_\varepsilon(\mathbf{1}_{U_m}\psi)(x)\,\mathrm{dLeb}(x)\right| \\ &\leq \left|\int \left(\int_I \mathcal{P}_\omega(\mathbf{1}_{U_m}\psi)\,\mathrm{dLeb}\right)\mathrm{d}\theta_\varepsilon(\omega)\right| \\ &\leq \|\psi\|_\infty \ \mathrm{Leb}(U_m),\end{aligned}$$

where $\|\psi\|_\infty \leq \|\psi\|_{BV}$ and $\mathrm{Leb}(U_m) \to 0$ as $m \to \infty$. Finally we check assumption (**A4**) under the hypothesis **H5**. We have

$$\|(\widetilde{\mathcal{P}}_{\varepsilon,m} - \mathcal{P}_\varepsilon)h_\varepsilon\|_{BV} = \|\mathcal{P}_\varepsilon(\mathbf{1}_{U_m}h_\varepsilon)\|_{BV} \leq A\kappa^{-1}\|\mathbf{1}_{U_m}h_\varepsilon\|_{BV} + B\|\mathbf{1}_{U_m}h_\varepsilon\|_1.$$

The right-hand side is bounded by a constant C^* which is independent of m. We recall that in our case$\Delta_{\varepsilon,m} = \mu_\varepsilon(U_m)$, and

$$\eta_{\varepsilon,m} := \sup_{\|\psi\|_{BV}\leq 1}\left|\int_I (\widetilde{\mathcal{P}}_{\varepsilon,m}\psi(x) - \mathcal{P}_\varepsilon\psi(x))\,\mathrm{dLeb}(x)\right| \leq \mathrm{Leb}(U_m)$$

(see computation above). Then

$$\|(\widetilde{\mathcal{P}}_{\varepsilon,m} - \mathcal{P}_\varepsilon)h_\varepsilon\|_{BV} \leq C^*\frac{\mu_\varepsilon(U_m)}{\underline{h}_\varepsilon\mathrm{Leb}(U_m)} \leq C^*\frac{\Delta_{\varepsilon,m}}{\eta_{\varepsilon,m}}.$$

□

4.2.4 Extremal index

In this part, we investigate the following quantity, see (4.22) and (4.23):

$$q_{k,m} = \frac{\mathrm{Leb}((\mathcal{P}_\varepsilon - \widetilde{\mathcal{P}}_{\varepsilon,m})\widetilde{\mathcal{P}}^k_{\varepsilon,m}(\mathcal{P}_\varepsilon - \widetilde{\mathcal{P}}_{\varepsilon,m})(h_\varepsilon))}{\mu_\varepsilon(U_m)}.$$

We recall that $U_m := U_m(\zeta)$ represents a ball around the point ζ.

Proposition 4.2.3. *Let us suppose that f is either a C^2 expanding map of the circle or a piecewise expanding map of the circle with finite branches and verifying hypotheses* **H1**-**H4**. *Then for each k,*

$$\lim_{m\to\infty} q_{k,m} \equiv 0,$$

i.e., *the limit in the definition of* q_k *in* (4.22) *exists and equals zero. Also the extremal index verifies* $\vartheta = 1 - \sum_{k=0}^{\infty} q_k = 1$ *and this is independent of the point* ζ*, the centre of the ball* U_m*.*

Proof. Let us define $G_{k,m} \equiv \int (\mathcal{P}_\varepsilon - \widetilde{\mathcal{P}}_{\varepsilon,m})\widetilde{\mathcal{P}}_{\varepsilon,m}^k(\mathcal{P}_\varepsilon - \widetilde{\mathcal{P}}_{\varepsilon,m})h_\varepsilon \,\mathrm{dLeb}$.

As $(\mathcal{P}_\varepsilon - \widetilde{\mathcal{P}}_{\varepsilon,m})\psi = \mathcal{P}_\varepsilon(\mathbf{1}_{U_m}\psi)$, we may write $G_{k,m} = \int \mathbf{1}_{U_m}(x)\widetilde{\mathcal{P}}_{\varepsilon,m}^k(\mathcal{P}_\varepsilon - \widetilde{\mathcal{P}}_{\varepsilon,m})h_\varepsilon \,\mathrm{dLeb}$. By using (4.35) we get

$$G_{k,m} = \iint \mathbf{1}_{U_m}(f_{\omega_{k+1}}\circ\cdots\circ f_{\omega_1}x)\mathbf{1}_{U_m^c}(f_{\omega_k}\circ\cdots\circ f_{\omega_1}x)\ldots\mathbf{1}_{U_m^c}(f_{\omega_1}x)\mathbf{1}_{U_m}(x)h_\varepsilon(x)\,\mathrm{dLeb}\,\mathrm{d}\theta_\varepsilon^{\mathbb{N}}.$$

In order to simplify the notation let us put

$$\psi_{k,U_m,\underline{\omega}}(x) = \mathbf{1}_{U_m}(f_{\omega_{k+1}}\circ f_{\omega_k}\circ\cdots\circ f_{\omega_1}x)\mathbf{1}_{U_m^c}(f_{\omega_k}\circ\cdots\circ f_{\omega_1}x)\ldots\mathbf{1}_{U_m^c}(f_{\omega_1}x)\mathbf{1}_{U_m}(x).$$

Now let us prove that $q_{k,m}$ converges to 0. Our approach is very similar to what we did to prove $D'(u_m)$. We split the proof into two according to the regularity of the map.

(i) Suppose that $f : \mathcal{S}^1 \to \mathcal{S}^1$ is a C^2, expanding map, *i.e.*, there exists $|Df(x)| > \lambda > 1$, for all $x \in \mathcal{S}^1$. First, note that since S^1 is compact and f is C^2, there exists $\sigma > 1$ such that $|Df(x)| \le \sigma$. Hence the set U_m grows at most at a rate given by σ, so, for any $\underline{\omega} \in \Omega^{\mathbb{N}}$ we have $|f_{\underline{\omega}}^j(U_m)| \le \sigma^j|U_m|$. This implies that

$$\text{if } \operatorname{dist}(f_{\underline{\omega}}^j(\zeta),\zeta) > 2\sigma^j|U_m| > |U_m| + \sigma^j|U_m| \text{ then } f_{\underline{\omega}}^j(U_m)\cap U_m = \emptyset. \tag{4.40}$$

Note that, by inequality (4.40), if for all $j = 1,\ldots,k+1$ we have $\operatorname{dist}(f_{\underline{\omega}}^j(\zeta),\zeta) > 2\sigma^j|U_m|$, then clearly $\psi_{k,B_m,\underline{\omega}}(x) = 0$, for all x. We define

$$W_{k,m} = \bigcap_{j=1}^{k+1}\left\{\underline{\omega} \in (-\varepsilon,\varepsilon)^{\mathbb{N}} : \operatorname{dist}(f_{\underline{\omega}}^j(\zeta),\zeta) > 2\sigma^j|U_m|\right\}. \tag{4.41}$$

Note that on $W_{k,m}$ we have $\psi_{k,U_m,\underline{\omega}} = 0$. We want to compute the measure of $W_{k,m}^c$.

Observe that $W_{k,m}^c \subset \bigcup_{j=1}^{k+1} \left\{\underline{\omega} : \; f_{\underline{\omega}}^j(\zeta) \in B_{2\sigma^j |U_m|}(\zeta)\right\}$. Hence, we have

$$\begin{aligned}
\theta_\varepsilon^{\mathbb{N}}(W_{k,m}^c) &\leq \sum_{j=1}^{k+1} \int \theta_\varepsilon\Big(\Big\{\omega_j : f\left(f_{\underline{\omega}}^{j-1}(\zeta)\right) + \omega_j \in B_{2\sigma^j|U_m|}(\zeta)\Big\}\Big) \mathrm{d}\theta_\varepsilon^{\mathbb{N}} \\
&\leq \sum_{j=1}^{k+1} \overline{g_\varepsilon} \left|B_{2\sigma^j|U_m|}(\zeta)\right| = \sum_{j=1}^{k+1} \overline{g_\varepsilon} 4\sigma^j |U_m| \leq 4\overline{g_\varepsilon}|U_m| \frac{\sigma}{\sigma-1}\sigma^{k+1}.
\end{aligned}$$

Using this estimate we obtain:

$$\begin{aligned}
G_{k,m} &= \iint\limits_{W_{k,m}} \psi_{k,U_m,\underline{\omega}}(x) h_\varepsilon(x) \,\mathrm{dLeb}\,\mathrm{d}\theta_\varepsilon^{\mathbb{N}} + \iint\limits_{W_{k,m}^c} \psi_{k,U_m,\underline{\omega}}(x) h_\varepsilon(x) \,\mathrm{dLeb}\,\mathrm{d}\theta_\varepsilon^{\mathbb{N}} \\
&= 0 + \iint\limits_{W_{k,m}^c} \psi_{k,U_m,\underline{\omega}}(x) h_\varepsilon(x) \,\mathrm{dLeb}\,\mathrm{d}\theta_\varepsilon^{\mathbb{N}} \quad \text{and because } \psi_{k,U_m,\underline{\omega}}(x) \leq \mathbf{1}_{U_m}(x), \text{ we have:} \\
&\leq \iint\limits_{W_{k,m}^c} \mathbf{1}_{U_m}(x) h_\varepsilon(x) \,\mathrm{dLeb}\,\mathrm{d}\theta_\varepsilon^{\mathbb{N}} \leq \mu_\varepsilon(U_m)\, \theta_\varepsilon^{\mathbb{N}}(W_{k,m}^c) \\
&\leq \mu_\varepsilon(U_m) 4\overline{g_\varepsilon}|U_m| \frac{\sigma}{\sigma-1}\sigma^{k+1}.
\end{aligned}$$

Now recall that $q_{k,m} = \frac{G_{k,m}}{\mu_\varepsilon(U_m)}$. It follows that

$$q_{k,m} \leq \frac{\mu_\varepsilon(U_m)4\overline{g_\varepsilon}|U_m|\frac{\sigma}{\sigma-1}\sigma^{k+1}}{\mu_\varepsilon(U_m)} \leq 4\overline{g_\varepsilon}|U_m|\frac{\sigma}{\sigma-1}\sigma^{k+1} \xrightarrow[m\to\infty]{} 0.$$

(ii) Using the same ideas as in the previous section, we can extend this result to the piecewise expanding maps with finite branches. Recall that we need to define some "safety regions" in order to use the same arguments as in the continuous case. So, if for all $j = 1, \dots, k+1$ and $i = 1, \dots, \ell$, where ℓ stands for the number of discontinuity points, we have

$$\mathrm{dist}(f_{\underline{\omega}}^j(\zeta), \xi_i) > 2\sigma^j |U_m|, \tag{4.42}$$

then the set U_m consists of one connected component at each iteration. Also we have $f_{\underline{\omega}}^j(U_m) \cap U_m = \emptyset$ which means $\psi_{k,U_m,\underline{\omega}}(x) = 0$, for all x. Now let us define

$$W_{k,m} = \bigcap_{j=1}^{k+1} \bigcap_{i=0}^{\ell} \left\{\underline{\omega} \in (-\varepsilon,\varepsilon)^{\mathbb{N}} : \mathrm{dist}(f_{\underline{\omega}}^j(\zeta), \xi_i) > 2\sigma^j|U_m|\right\}. \tag{4.43}$$

Observe that in this case

$$W_{k,m}^c \subset \bigcup_{j=1}^{k+1} \bigcup_{i=0}^{\ell} \{\underline{\omega} : \ f_{\underline{\omega}}^j(\zeta) \in B_{2\sigma^j |U_m|}(\xi_i)\}.$$

Hence, we have

$$\begin{aligned}
\theta_\varepsilon^{\mathbb{N}}(W_{k,m}^c) &\le \sum_{i=0}^{\ell} \sum_{j=1}^{k+1} \int \theta_\varepsilon\Big(\Big\{\omega_j : f\left(f_{\underline{\omega}}^{j-1}(\zeta)\right) + \omega_j \in B_{2\sigma^j|U_m|}(\xi_i)\Big\}\Big) \mathrm{d}\theta_\varepsilon^{\mathbb{N}} \\
&\le \sum_{i=0}^{\ell} \sum_{j=1}^{k+1} \overline{g_\varepsilon} \left|B_{2\sigma^j|U_m|}(\xi_i)\right| = \sum_{i=0}^{\ell} \sum_{j=1}^{k+1} \overline{g_\varepsilon} 4\sigma^j |U_m| \le 4(\ell+1)\overline{g_\varepsilon}|U_m| \frac{\sigma}{\sigma-1}\sigma^{k+1}.
\end{aligned}$$

Using this estimate we obtain

$$\begin{aligned}
G_{k,m} &= \iint\limits_{W_{k,m}} \psi_{k,U_m,\underline{\omega}}(x) h_\varepsilon(x) \,\mathrm{dLeb}\,\mathrm{d}\theta_\varepsilon^{\mathbb{N}} + \iint\limits_{W_{k,m}^c} \psi_{k,U_m,\underline{\omega}}(x) h_\varepsilon(x) \,\mathrm{dLeb}\,\mathrm{d}\theta_\varepsilon^{\mathbb{N}} \\
&= 0 + \iint\limits_{W_{k,m}^c} \psi_{k,U_m,\underline{\omega}}(x) h_\varepsilon(x) \,\mathrm{dLeb}\,\mathrm{d}\theta_\varepsilon^{\mathbb{N}}.
\end{aligned}$$

Since $\psi_{k,U_m,\underline{\omega}}(x) \le \mathbf{1}_{U_m}(x)$, we have

$$\begin{aligned}
G_{k,m} &\le \iint\limits_{W_{k,m}^c} \mathbf{1}_{U_m}(x) h_\varepsilon(x) \,\mathrm{dLeb}\,\mathrm{d}\theta_\varepsilon^{\mathbb{N}} \le \mu_\varepsilon(U_m)\theta_\varepsilon^{\mathbb{N}}(W_{k,m}^c) \\
&\le \mu_\varepsilon(U_m) 4(\ell+1)\overline{g_\varepsilon}|U_m| \frac{\sigma}{\sigma-1}\sigma^{k+1}.
\end{aligned}$$

Finally, as $q_{k,m} = \frac{G_{k,m}}{\mu_\varepsilon(U_m)}$, we get

$$q_{k,m} \le \frac{\mu_\varepsilon(U_m) 4(\ell+1)\overline{g_\varepsilon}|U_m| \frac{\sigma}{\sigma-1}\sigma^{k+1}}{\mu_\varepsilon(U_m)} \le 4(\ell+1)\overline{g_\varepsilon}|U_m| \frac{\sigma}{\sigma-1}\sigma^{k+1} \xrightarrow[m\to\infty]{} 0.$$

□

Remark 4.2.4. Let us note that $D'(u_m)$ implies that all q_k's are well defined and equal to 0. Assume that there exist $k \in \mathbb{N}$ and a subsequence $(m_i)_{i\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} \frac{G_{k,m_j}}{\mu_\varepsilon(U_{m_j})} = \alpha > 0.$$

Let us prove that $D'(u_m)$ does not hold in this situation. Recall that if $D'(u_m)$ holds then

$$\lim_{m\to\infty} m \sum_{j=1}^{\lfloor m/k_m \rfloor} \mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}}(X_0 > u_m, X_j > u_m) = 0,$$

where k_m (which should not be confused with k, here) is a sequence diverging to ∞ but slower than m, which implies that $\lfloor m/k_m \rfloor \to \infty$, as $m \to \infty$. Hence, let M_0 be sufficiently large so that for all $m > M_0$ we have $\lfloor m/k_m \rfloor > k$. Hence, for i sufficiently large so that $m_i > M_0$, we may write

$$\begin{aligned} m_i \sum_{j=1}^{\lfloor m_i/k_{m_i} \rfloor} \mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}}(X_0 > u_{m_i}, X_j > u_{m_i}) &\geq m_i\, \mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}}(X_0 > u_{m_i}, X_{k+1} > u_{m_i}) \\ &\geq m_i\, G_{k,m_i} \sim \frac{\tau\, G_{k,m_i}}{\mu_\varepsilon(U_{m_i})} \to \tau\alpha > 0, \text{ as } i \to \infty, \end{aligned}$$

since B_m is such that $m\mu_\varepsilon(U_m) \to \tau$, as $m \to \infty$. This implies that $D'(u_m)$ does not hold.

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Glossary

EI : Extremal Index, 17

EVL : Extreme Value Law, 15

HTS : Hitting Time Statistics, 31

REPFO : Rare Events Perron-Frobenius Operators, 74

REPP : Rare Events Point Processes, 25

RTS : Return Time Statistics, 31

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