Structural Ramsey theory with the Kechris-Pestov-Todorcevic correspondence in mind

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THÉORIE DE RAMSEY STRUCTURALE ET APPLICATIONS EN DYNAMIQUE VIA LA CORRESPONDANCE DE KECHRIS-PESTOV-TODORCEVIC

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- Distinguishing Number of Countable Homogeneous Relational Structures (en collaboration avec Claude Laflamme et Norbert Sauer), *Electron. J. Combin.*, **17 (1)**, 17 pages, 2010.

0. Introduction

Le but de ce mémoire est d'effectuer un survol des articles [NVTS09], [NVTS10], [NVTP10], [LNVTS10], [NVT10a], [NVT13a], [NVT13b], [LNVTPS11], [GNVT11], [JLNVTW12] et [NVTT12]. Ces derniers constituent l'essentiel de mes travaux depuis [NVT06] et sa version étendue [NVT10b]. Le sujet d'étude se situe à l'une des intersections entre la combinatoire, la dynamique topologique et la logique via le formalisme de ce qu'on appelle les structures *ultrahomogènes* et la théorie de Fraïssé[1]. Ce domaine a récemment connu un essor considérable grâce à deux contributions majeures : [KPT05] par Kechris, Pestov et Todorcevic, et [KR07] par Kechris et Rosendal. Mon travail part de [KPT05]. Plus précisément, il concerne l'interaction entre les propriétés de type Ramsey des structures combinatoires finies et dénombrables, et la dynamique des groupes topologiques. La théorie de Ramsey peut être vue comme l'étude de certains objets combinatoires au sein desquels un certain degré d'organisation apparaît lorsque les objets en question deviennent grands. Quant à la dynamique topologique, il s'agit surtout ici de l'étude des actions continues des groupes topologiques sur des espaces topologiques compacts. Ces sujets sont traditionnellement rattachés à des régions mathématiques différentes, mais on sait depuis un certain temps que théorie de Ramsey et dynamique peuvent être fortement liés. La démonstration de Furstenberg du théorème de Szemerédi et ses conséquences récentes en théorie additive des nombres constitue certainement l'un des exemples les plus emblématiques dans cette direction, mais il en existe beaucoup d'autres. Ce que l'article [KPT05] met en évidence est un lien nouveau entre théorie de Ramsey et dynamique. Après avoir permis de mieux appréhender un invariant topologique relatif à plusieurs groupes topologiques connus, cette connexion permet d'aborder avec un regard nouveau l'étude de plusieurs phénomènes remarquables. Par exemple, Gromov et Milman démontrèrent (cf [GM83]) que lorsqu'il agit continment sur un espace compact, le groupe unitaire de l'espace de Hilbert séparable ℓ_2 laisse toujours un point fixe. Grâce aux techniques développées dans [KPT05], il est concevable que ce phénomène soit la manifestation d'un résultat de type Ramsey purement combinatoire concernant les espaces métriques finis euclidiens. A titre de deuxième exemple, Odell et Schlumprecht publièrent dans [OS94] une solution au célèbre problème de la distorsion. Alors que la démonstration fait appel à des techniques d'analyse fonctionnelle extrêmement élaborées, il a récemment été démontré que dans certains contextes relativement proches, une approche purement combinatoire peut être adoptée. Cela laisse donc l'espoir de découvrir un argument direct et purement métrique. C'est entre autres pour ces raisons que mes travaux de recherche sont concentrés autour des deux thèmes suivants : Théorie de Ramsey structurale et dynamique topologique des groupes de transformation associés.

Remarque. Pour des raisons pratiques, seule l'introduction de ce mémoire a été rédigée en français. J'espère que cela ne rebutera pas le lecteur intéressé. Concernant les notations, j'ai, à quelques exceptions près, privilégié l'usage anglo-saxon. En particulier, les intervalles réels ouverts sont notés (a, b), mais le symbole d'inclusion est noté $\subset$. Lorsque k est un nombre naturel, $[k]$ désigne l'ensemble

[1]Pour la plupart des personnes qui ont travaillé sur les structures ultrahomogènes au cours des trente dernières années, l'ultrahomogénéité est simplement l'homogénéité. Je n'ai pas conservé cette terminologie ici car en topologie, et en particulier lorsqu'il s'agit de groupes de transformations, l'usage est de réserver le terme d'homogène pour les structures où le groupe d'automorphismes agit transitivement sur les points.

$\{0, 1, ..., k-1\}$. Concernant les références, tous les résultats énoncés le sont, dans la mesure du possible, avec des références exactes. Enfin, afin de différencier les résultats présentés dans ce mémoire et ceux qui y sont extérieurs, deux systèmes de numérotation ont été adoptés, à savoir chiffres arabes et chiffres romains.

0.1. **Théorie de Ramsey structurale finie.** Le résultat fondateur de théorie de Ramsey remonte à 1930. Il a été démontré par Ramsey et peut être énoncé comme suit. Pour un ensemble X et un entier l, on note $[X]^l$ l'ensemble des parties de X à l éléments:

Théorème I (Ramsey [Ram30]). *Pour tous $l, m \in \mathbb{N}$, il existe $p \in \mathbb{N}$ tel que pour tout ensemble X à p éléments, si $[X]^l$ est soumis à une partition en 2 classes $[X]^l = R \cup B$, alors il existe $Y \subset X$ à m éléments tel que $[Y]^l \subset R$ ou $[Y]^l \subset B$.*

Ce n'est malgré tout qu'au début des années soixante-dix que les idées essentielles de ce théorème furent reprises et développées pour donner naissance à la théorie de Ramsey structurale. Le but est alors d'établir des résultats semblables au théorème de Ramsey dans un contexte où plus de structure apparaît. Par exemple, si $\mathbf{H}$ est un graphe fini, alors il existe un graphe fini $\mathbf{K}$ possédant la propriété suivante : Pour tout coloriage des arêtes de $\mathbf{K}$ avec deux couleurs, il existe un sous-graphe induit de $\mathbf{K}$ et isomorphe à $\mathbf{H}$ où toutes les arêtes ont la même couleur. De nombreux autres résultats de ce type existent pour une grande variété de structures telles que les graphes, les hypergraphes et les systèmes d'ensembles (Abramson-Harrington [AH78], Nešetřil-Rödl [NR77, NR83]), les espaces vectoriels (Graham-Leeb-Rothschild [GLR72, GLR73]), les algèbres de Boole (Graham-Rothschild [GR71]), les arbres (Fouché [Fou99])...Toutes ces structures peuvent être appréhendées dans le langage général de la théorie des modèles, et plus précisément de la théorie de Fraïssé. Un premier pan de mes travaux de recherche est consacré l'enrichissement de cette liste. Certains de mes travaux de thèse portèrent en particulier sur les classes d'espaces métriques. L'article [NVT13b] traite des graphes dont le nombre chromatique est fixé, ou bien borné par une constante donnée à l'avance. L'article [LNVTS10] présente des résultats portant sur une classe de graphes orientés appelés *tournois circulaires*. Enfin, le travail en cours [JLNVTW12] porte sur les classes de Fraïssé de graphes dirigés. Tous ces résultats sont exposés en Section 3.

0.2. **Théorie de Ramsey structurale infinie.** L'une des particularités du théorème de Ramsey est d'admettre la version infinie suivante :

Théorème II (Ramsey [Ram30]). *Pour tout $l \in \mathbb{N}$, tout ensemble infini X et toute partition de $[X]^l$ en 2 classes $[X]^l = R \cup B$, il existe une partie infinie $Y \subset X$ telle que $[Y]^l \subset R$ ou $[Y]^l \subset B$.*

Dans certains contextes, ce dernier théorème se transpose au cadre structural mais peu de résultats de ce type sont connus (cf Devlin pour les propriétés de partitions des rationnels vus comme ordre total, cf Laflamme, Sauer et Vuksanovic pour les propriétés de partitions du graphe aléatoire dénombrable). L'article [LNVTS10] contient plusieurs résultats dans ce sens, notamment dans le cadre des graphes orientés circulaires.

Pour donner une idée des difficultés auxquelles on se retrouve confronté, prenons le cas où $l = 1$. Le théorème de Ramsey devient alors trivial : Pour toute partition finie de $\mathbb{N}$, on peut "plonger" $\mathbb{N}$ dans l'une des parties, et ainsi trouver dans l'une des parties une copie de $\mathbb{N}$ ($\mathbb{N}$ est ici supposé dénué de toute structure). Plaçons-nous désormais dans le cadre des espaces vectoriels. Fixons un corps F, et considérons l'espace vectoriel V_F de dimension dénombrable sur F. Supposons que l'on partitionne V_F en deux classes, $V_F = R \cup B$. L'une des parties contient-elle nécessairement un sous-ensemble isomorphe à V_F ? Lorsque c'est le cas quelle que soit la partition, on dit que V_F est *indivisible*. Dans le cas présent, la réponse dépend du corps, mais est négative en général. Néammoins, certains autres phénomènes peuvent se produire suivant la taille du corps, c'est ce dont traite l'article [LNVTPS11]. Les travaux [NVTS09] et [NVTS10] portent quant à eux sur des problèmes similaires dans le contexte métrique. En ce sens, ils étendent mes travaux de thèse où l'indivisibilité des espaces métriques est étudiée de manière approfondie. L'article [NVTS09] propose notamment, lorsqu'on lui adjoint les résultats de [LANVT08], une solution positive pour un espace métrique remarquable appelé *la sphère d'Urysohn* et noté ici $\mathbf{S}$. Les techniques utilisées pour démontrer ce résultat sont aussi appliquées dans [NVTS10], où elles permettent d'atteindre des résultats de partitions similaires dans le cas de certains espaces métriques non bornés.

Tous ces résultats sont présentés en Section 4. L'objectif initial de ces travaux était de développer de nouveaux outils qui seraient par la suite utilisables pour redémontrer de manière combinatoire le théorème d'Odell-Schlumprecht sur la distortion de ℓ_2, mais les résultats obtenus ne permettent pour le moment pas de réaliser ce projet.

0.3. **Flots minimaux universels et moyennabilité extrême.** L'une des avancées majeures de l'article [KPT05] est d'établir un pont entre la théorie de Ramsey structurale et la dynamique topologique. Pour un groupe topologique G, un *G-flot minimal compact* est un espace topologique compact X muni d'une action continue de G sur X pour laquelle l'orbite de tout point est dense. Un résultat général de dynamique affirme que tout groupe topologique G admet un G-flot minimal compact $M(G)$ *universel* (unique à isomorphisme près) dans le sens où il peut être envoyé via un homomorphisme surjectif sur n'importe quel G-flot compact minimal. Lorsque G est compact, $M(G)$ n'est autre que G lui-même muni de l'action à gauche mais lorsque G n'est pas compact, l'existence de $M(G)$ est garantie par des procédés hautement non constructifs. Néammoins, il est connu que l'emploi de méthodes de type Ramsey conduit parfois à une description très explicite de $M(G)$. Les techniques développées dans [KPT05] rendent cette connexion explicite (on parlera par la suite de *correspondance de Kechris-Pestov-Todorcevic*) et fournissent de nouveaux outils pour la détermination des flots minimaux compacts universels des groupes polonais non-archimédiens. Elles mettent en particulier l'accent sur le lien entre théorie de Ramsey et moyennabilité extrême (un groupe topologique G est *extrêmement moyennable* lorsque $M(G)$ est réduit à un point, c'est-à-dire que toute action continue de G sur tout espace compact admet un point fixe). Ces idées sont appliquées et généralisées dans [NVT13a] dans le contexte des *expansions précompactes*, dont il semble qu'elles forment le bon cadre de généralisation. Les expansions précompactes furent tout d'abord introduites pour calculer $M(G)$ lorsque G est le groupe des automorphismes d'un des graphes orientés dénombrables $S(2)$ ou $S(3)$, mais une analyse plus détaillée semble indiquer qu'elles devraient en fait permettre de capturer tous les graphes ultrahomogènes dénombrables, qu'ils soient dirigés ou non. Ce résultat apparaîtra dans [JLNVTW12] et on verra qu'il suggère un nouveau point de vue sur la théorie de Ramsey structurale finie. Les expansions précompactes sont également présentes dans [NVT13b], où l'étude de l'universalité de certains flots conduit à la notion de *moyennabilité extrême relative* (si H est un sous-groupe topologique de G, (G, H) est *relativement extrêmement moyennable* lorsque toute action continue de G sur tout espace compact admet un point fixé par H). L'article [GNVT11] est construit autour de cette dernière notion et porte sur l'existence d'interpolants pour les couples (G, H) qui sont relativement extrêmement moyennables (un *interpolant* pour un tel couple est un groupe topologique extrêmement moyennable K tel que $H \subset K \subset G$). Enfin, plus récemment, l'ensemble de ces résultats a été mis en commun pour établir un lien entre une propriété de type Ramsey d'un certain type, métrisabilité du flot minimal universel, et existence d'un “gros” sous-groupe extrêmement moyennable. Cela apparaîtra dans l'article [NVTT12]. Les résultats correspondants sont présentés en Section 5.

STRUCTURAL RAMSEY THEORY WITH THE KECHRIS-PESTOV-TODORCEVIC CORRESPONDENCE IN MIND

Contents

1. Introduction

The purpose of the present memoir is to present a survey of the papers [NVTS09], [NVTS10], [NVTP10], [LNVTS10], [NVT10a], [NVT13a], [NVT13b], [LNVTPS11], [GNVT11], [JLNVTW12] and [NVTT12]. Those constitute the essential part of my work since [NVT06] and its expanded version [NVT10b]. The corresponding material is located at one of the intersection points between combinatorics, topological dynamics and logic via the framework of so-called *ultrahomogeneous* structures and Fraïssé theory[2]. This area has recently known a considerable expansion thanks to two major contributions: [KPT05] by Kechris, Pestov and Todorcevic, and [KR07] by Kechris and Rosendal. My work elaborates on [KPT05]. More precisely, it deals with an interaction between Ramsey-type properties of finite and countable structures and dynamics of topological groups. Ramsey theory can be thought of as the study of certain combinatorial objects where a some degree of organization necessarily appears as the size of the objects increases. As for topological dynamics, it mostly deals here with the study of continuous actions of topological groups on compact spaces. Those fields traditionally belong to different parts of mathematics, but it has now been known for a number of years that Ramsey theory and dynamics can be very strongly tied. Furstenberg's proof of Szemerédi's theorem and its consequences in recent additive number theory certainly represent some of the most emblematic examples in that direction, but there are many more. What [KPT05] does is to establish a new link between Ramsey theory and dynamics. This connection provides a better grasp of an invariant related to several known topological groups, as well as a new light on several remarkable known facts. For example, Gromov and Milman showed (cf [GM83]) that when it acts continuously on a compact space, the unitary group of the separable Hilbert space ℓ_2 always leaves a point fixed. The techniques developed in [KPT05] suggest that this phenomenon could be the consequence of a purely combinatorial Ramsey-theoretic result concerning finite Euclidean metric spaces. As a second example, Odell and Schlumprecht published in [OS94] a solution to the so-called distortion problem. In essence, this problem asks whether every uniformly continuous map defined on the

[2]For most of the people who have worked on ultrahomogeneous structures in the past thirty years, ultrahomogeneity is simply called homogeneity. I have not kept this terminology here because in topology, and in particular in the study of transformation groups, the common habit is to call homogeneous a structure where the automorphism group acts transitively on points.

sphere of ℓ_2 with values in $\mathbb{R}$ can be made almost constant by passing to a closed infinite dimensional subspace. While the proof of the Odell-Schlumprecht theorem uses extremely sophisticated machinery from functional analysis, it has recently been proved that in some relatively close context, a purely combinatorial approach can be adopted. This leaves the hope for a direct and purely metric argument. It is for reasons of that kind that my research concentrates on two themes: structural Ramsey theory and topological dynamics of Polish transformation groups.

1.1. **Conventions.** All notations are standard, except maybe $\subset$ for the inclusion symbol and $[k]$ for the set $\{0, 1, ..., k-1\}$ when k is a natural number.

Concerning results, I tried to provide exact references whenever possible. In order to differenciate those results that are presented in this memoir from those that do not belong to it, I chose to use two indexing systems: Arabic and Roman numerals.

1.2. **Finite structural Ramsey theory.** The foundational result of Ramsey theory appeared in 1930. It was proved by Ramsey and can be stated as follows. For a set X and a positive integer l, $[X]^l$ denotes the set all of subsets of X with l elements:

Theorem I (Ramsey [Ram30]). *For every $l, m \in \mathbb{N}$, there exists $p \in \mathbb{N}$ such that for every set X with p elements, if $[X]^l$ is partitioned into two classes $[X]^l = R \cup B$, then there exists $Y \subset X$ with m elements such that $[Y]^l \subset R$ ou $[Y]^l \subset B$.*

However, it is only at the beginning of the seventies that the essential ideas behind this theorem crystalized and expanded to structural Ramsey theory. The goal was then to obtain results similar to Ramsey's theorem in a setting where more structure appears. For example, if $\mathbf{H}$ is a finite graph, there exists a finite graph $\mathbf{K}$ with the following property: for every coloring of the edges of $\mathbf{K}$ in two colors, there exists a finite induced subgraph of $\mathbf{K}$ isomorphic to $\mathbf{H}$ where all edges receive the same color. Many other results of the same kind exist for a wide variety of finite structures such as graphs, hypergraphs and set systems (Abramson-Harrington [AH78], Nešetřil-Rödl [NR77, NR83]), vector spaces (Graham-Leeb-Rothschild [GLR72, GLR73]), Boolean algebras (Graham-Rothschild [GR71]), trees (Fouché [Fou99])...The publication of [KPT05] has considerably revived the global interest towards results that enrich that list, and a part of my work follows that trend. During my PhD studies, I focused on classes of metric spaces. The article [NVT13b] deals with graphs whose chromatic number is fixed or bounded in advance. The paper [LNVTS10] presents results concerning a class of directed graphs called *circular tournaments*. Finally, the work in progress [JLNVTW12] deals with all the other Fraïssé classes of directed graphs. All those results are presented in Section 3.

1.3. **Infinite structural Ramsey theory.** One of the particularities of Ramsey's theorem is to admit the following infinite version:

Theorem II (Ramsey [Ram30]). *For every $l \in \mathbb{N}$, every infinite set X and every partition of $[X]^l$ into two classes $[X]^l = R \cup B$, there exists an infinite subset $Y \subset X$ so that $[Y]^l \subset R$ or $[Y]^l \subset B$.*

Analogs of this theorem when more structure is present are much more difficult to prove. In fact, only a very small number of results are known. The major ones are due to Devlin for partition properties of the rationals seen as a total order, and to Laflamme, Sauer and Vuksanovic for partition relations of the countable random graph. The paper [LNVTS10] contains several results in that direction for some enriched versions of the rationals and for circular directed graphs.

In order to give a flavor of the difficulties that have to be faced, consider the case $l = 1$. Ramsey's theorem then becomes trivial: for every partition of $\mathbb{N}$ into finitely many parts, one can of course "embed" $\mathbb{N}$ into one of the parts and so find a "copy" of $\mathbb{N}$ inside one of the parts ($\mathbb{N}$ is here considered without any structure). Let us now consider the case of vector spaces. Let F be a field and consider the vector space V_F of countable dimension over F. Assume that V_F is partitioned into two classes $V_F = R \cup B$. Does one of the parts necessarily contain a subset isomorphic to V_F? When the answer is positive for every partition, we say that V_F is *indivisible*. In the present case, the answer depends on the field, but is negative in general. Nevertheless, other partition phenomena appear depending on the size of the field. Those are studied in the paper [LNVTPS11]. As for the articles [NVTS09] and [NVTS10], they cover similar problems in a metric context. In that sense, they represent a continuation of my thesis work [NVT06] where indivisiblity was extensively studied for metric spaces. In particular, when combined with the results of [LANVT08] (which was included in [NVT06]), the paper [NVTS09] contains a positive solution for a remarkable metric space called the *Urysohn sphere* and denoted by $\mathbf{S}$. The techniques that are used in [NVTS09] are also applied in [NVTS10], where they allow to reach similar partition results for certain unbounded metric spaces.

All those results are presented in Section 4. I should say at that point that a common feature of all those projects was to develop new tools that could be used in order to obtain a combinatorial proof of the Odell-Schlumprecht theorem on the distortion of ℓ_2. Unfortunately, neither the techniques nor the results that were obtained so far allow to achieve that goal.

1.4. **Universal minimal flows and extreme amenability.** One of the major achievements of [KPT05] is to create a bridge between structural Ramsey theory and topological dynamics. For a topological group G, a *compact minimal G-flow* is a compact topological space X equipped with a continuous action of G on X for which every orbit is dense. A general result in topological dynamics states that every topological group G admits a (unique up to isomorphism) compact minimal G-flow $M(G)$ which is also *universal* in the sense that it can be mapped homomorphically onto any other compact minimal G-flow. When G is compact, $M(G)$ is simply G acting on itself by left translations but when G is not compact, the existence of $M(G)$ is only guaranteed by highly non-constructive methods. Nevertheless, it has been known for some time that Ramsey-theoretic methods sometimes lead to a very explicit description of $M(G)$. The techniques developed in [KPT05] make this connection explicit (this is the *Kechris-Pestov-Todorcevic correspondence* to which the title of this memoir refers), and provide new tools for the study of universal minimal flows of non-Archimedean Polish groups. In particular, they exhibit a strong link between Ramsey theory and *extreme amenability* (a topological group is *extremely amenable* when $M(G)$ is reduced to a single point, or equivalently when every continuous action of G on a compact space admits a fixed point). Those ideas are applied and generalized in [NVT13a] in the context of *precompact expansions*, which seem to be the right framework. Precompact expansions were first introduced in order to compute $M(G)$ when G is the automorphism group of the countable oriented graphs $S(2)$ and $S(3)$, but a further analysis has indicated that they should actually capture all countable undirected and directed graphs. This result will appear in [JLNVTW12] and we will see that it offers a new point of view on structural finite Ramsey theory. Precompact expansions are also present in [NVT13b] where the study of universality for certain flows leads to the notion of *relative extreme amenability* (if H is a topological subgroup of a topological group G, the pair (G, H) is *relatively extremely amenable* when every continuous action of G on a compact space admits a point that is fixed by H). The paper [GNVT11] is built around this latter notion and deals with the existence of interpolants for relatively extremely amenable pairs (G, H) (an *interpolant* for such a pair is an extremely amenable K such that $H \subset K \subset G$). Finally, more recently, all those results have been put together in order to establish an equivalence between a certain Ramsey-type property, metrizability of the universal minimal flow and existence of a large extremely amenable subgroup. This will appear in [NVTT12]. All the corresponding results are presented in Section 5.

2. Ultrahomogeneous structures and Fraïssé theory

The purpose of this section is to present ultrahomogeneous structures and Fraïssé theory. Section 2.1 will introduce ultrahomogeneous structures informally and provide several natural examples coming from various areas. In Section 2.2, we will go on with the formal definition and will present Fraïssé theory. Finally, Section 2.3 will introduce the notion of expansion as well as the more specific notion of precompact expansion.

2.1. **Ultrahomogeneous structures.** Ultrahomogeneous structures will be ubiquitous in this memoir but are not so commonly known. They will be defined precisely using the rigorous model-theoretic framework in Section 2.2, but for the moment, let us simply pretend that a structure $\mathbf{A}$ (which may be combinatorial, metric, algebraic,...) is *ultrahomogeneous* when every isomorphism between two of its finitely generated substructures can be extended to an automorphism of $\mathbf{A}$. In terms of groups, it implies that the automorphism group acts as transitively as possible on the set of all finite substructures. Therefore, when all points of $\mathbf{A}$ support substructures that are isomorphic, ultrahomogeneity can really be thought as a strengthening of the usual notion of point-homogeneity, where the automorphism group acts transitively on points. The set $\mathbb{N}$ without any structure is ultrahomogeneous: two finite isomorphic substructures are simply two finite sets of the same size, and every bijection between those extends to a bijection of $\mathbb{N}$. So is $\mathbb{Q}$ seen as a linear ordering. In a more algebraic setting, any vector space is ultrahomogeneous: finitely generated structures are finite dimensional spaces, and any isomorphism between two such objects can be extended to an isomorphism of the whole space.

There are two objects naturally attached to a given ultrahomogeneous structure: the class of its finitely generated structures and its automorphism group. While the former relates to combinatorics and the latter to transformation group theory, both can be thought as two sides of the same object, and passing from one to the other is sometimes extremely fruitful. This is typically what happens in [KPT05] but this aspect is also at the heart of several other recent developments including [KR07] and [AKL12].

In this memoir, we will mostly deal with two kinds of structures: graphs (directed or undirected) and metric spaces. Non-trivial ultrahomogeneous graphs are not that easy to visualize, but one of them is by now quite famous in combinatorics: it is the countable random graph, also called the Rado graph or the Erdős-Rényi graph. It is the object that appears almost surely when one equips $\mathbb{N}$ with an edge relation provided all edge appear independently with probability $0 < p < 1$ (more precisely, consider the Bernoulli measure on the space $[2]^{\mathbb{N}^2}$). It has been studied under various angles and many of its properties are now well understood (see for example [Cam97]). As for metric spaces, the most natural examples witnessing ultrahomogeneity are probably the real Hilbert vector spaces. Among those, we will see that the separable Hilbert space ℓ_2 and its unit sphere $\mathbb{S}^\infty$ are particularly interesting in view of the material we will cover.

A special attention will also be devoted to another remarkable space, namely the Urysohn space $\mathbf{U}$. This space, which appeared relatively early in the history of metric geometry (the definition of metric space is given in the thesis of Maurice Fréchet in 1906, [Fré06]), was constructed by Paul Urysohn in 1925. Up to isometry, it is the unique complete separable ultrahomogeneous metric space into which every finite metric space embeds. As a consequence, it can be proved that $\mathbf{U}$ is universal not only for the class of all finite metric spaces, but also for the class of all *separable* metric spaces. This latter universality property is essential and is precisely the reason for which Urysohn constructed $\mathbf{U}$: before it appeared, it was unknown whether a separable metric space could be universal for the class of all separable metric spaces. However, $\mathbf{U}$ virtually disappeared after Banach and Mazur showed that $\mathcal{C}([0,1])$ was also universal and it is only quite recently that it was brought back on the research scene, thanks in particular to the work of Katětov [Kat88] which was quickly followed by several results by Uspenskij [Usp90], [Usp04] and later supported by various contributions by Vershik [Ver04], [Ver08], Gromov [Gro07], Pestov [Pes02] and Bogatyi [Bog00], [Bog02]. Lately, the study of the space $\mathbf{U}$ has been a subject of active research and has being carried out by many different authors under many different lights, see [LPR$^+$08]. For example, Holmes proved in [Hol92] that $\mathbf{U}$ generates a unique Banach space in the following sense: there is a Banach space $\langle \mathbf{U} \rangle$ such that for every isometry $i : \mathbf{U} \longrightarrow \mathbf{Y}$ of the Urysohn space $\mathbf{U}$ into a Banach space $\mathbf{Y}$ such that $0_{\mathbf{Y}}$ is in the range of i, there is an isometric isomorphism between $\langle \mathbf{U} \rangle$ and the closed linear span of $i(\mathbf{U})$ in $\mathbf{Y}$. We will come back to this space in a moment.

Universality is an important aspect of infinite ultrahomogeneous structures, as it can often be shown that if a structure embeds all finite structures of a certain kind, then it also embeds all countable structures of the same kind. For example, every finite linear order embeds in $\mathbb{Q}$ and this can be used to show that every countable linear order also does. The same happens for graphs in the random graph: every finite graph embeds in it, and so does every countable graph. Among the aforementioned contributions concerning the Urysohn space, Uspenskij's one was essential to isolate another important aspect of ultrahomogeneous structures, namely that their automorphism groups also satisfy some kind of universality property. In the case of the Urysohn space, this is particularly spectacular: every Polish group embeds into the isometry group $\mathrm{iso}(\mathbf{U})$ of the Urysohn space equipped with the pointwise convergence topology. In [NVTP10], it was realized that the construction that was used to prove the previous result could also be used to prove:

Theorem 1 (NVT-Pestov [NVTP10])**.** *For a separable topological group G, the following conditions are equivalent:*

(1) Every continuous affine action of G on any Banach space by isometries has a fixed point.
(2) Every continuous affine action of G on the Holmes space $\langle \boldsymbol{U} \rangle$ by isometries has a fixed point.

Looking at this result, one could think that there is an interesting fixed-point property of groups to be studied here (that was actually the main motivation for the paper). Unfortunately, there is a third item to the theorem. It states that the above properties are equivalent to precompactness of the group...A not so interesting property.

Back to ultrahomogeneous structures, another aspect that makes them appear naturally is their connection to randomness and to genericity (to be understood as randomness in the Baire category sense). For example, every countable ultrahomogeneous structure is generic in a very precise sense (see [PR96]). On the other hand, modulo a simple combinatorial requirement, quite a number of them can be thought as random (see [AFP12] for a recent contribution). We will not touch this subject here, but the field is certainly full of beautiful problems in that direction.

2.2. **Fundamentals of Fraïssé theory.** In this section, we introduce formally the material needed in order to study ultrahomogeneous structures. A special attention will be devoted to the countable case (Fraïssé theory), where the link between a given ultrahomogeneous structure $\mathbf{F}$, the class of its finitely generated substructures $\mathrm{Age}(\mathbf{F})$ and its automorphism group $\mathrm{Aut}(\mathbf{F})$ is particularly elegant. We follow [KPT05] but a more detailed approach can be found in [Fra00] or [Hod93].

2.2.1. *Relational structures.* Let $L = \{R_i : \ i \in I\} \cup \{f_j : j \in J\}$ be a fixed language, that is to say a list of symbols to be interpreted later as relations and functions, each symbol having a corresponding integer called its *arity*. The arity of the relation symbol R_i is a positive integer $\alpha(i)$ and the arity of each function symbol f_j is a non-negative integer $\beta(j)$. Let $\mathbf{A}$ and $\mathbf{B}$ be two L-structures (that is, non empty sets A, B equipped with relations $R_i^{\mathbf{A}} \subset A^{\alpha(i)}$ and $R_i^{\mathbf{B}} \subset B^{\alpha(i)}$ for each $i \in I$ and functions $f_j^{\mathbf{A}} : A^{\beta(i)} \longrightarrow A$ and $f_j^{\mathbf{B}} : B^{\beta(i)} \longrightarrow B$ for each $j \in J$). An *embedding* from $\mathbf{A}$ to $\mathbf{B}$ is an injective map $\pi : A \longrightarrow B$ such that for every $i \in I$, $x_1, \ldots, x_{\alpha(i)} \in A$:

$$(x_1, \ldots, x_\alpha) \in R_i^{\mathbf{A}} \quad \text{iff} \quad (\pi(x_1), \ldots, \pi(x_{\alpha(i)})) \in R_i^{\mathbf{B}},$$

and every $j \in J$, $x_1, \ldots, x_{\beta(j)} \in A$:

$$\pi(f_j^{\mathbf{A}}(x_1, \ldots, x_{\beta(j)}) = f_j^{\mathbf{B}}(\pi(x_1), \ldots, \pi(x_{\beta(j)})).$$

An *isomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a surjective embedding while an *automorphism* of $\mathbf{A}$ is an isomorphism from $\mathbf{A}$ onto itself. Of course, $\mathbf{A}$ and $\mathbf{B}$ are *isomorphic* when there is an isomorphism from $\mathbf{A}$ to $\mathbf{B}$. This is written $\mathbf{A} \cong \mathbf{B}$. Finally, the set $\binom{\mathbf{B}}{\mathbf{A}}$ is defined as:

$$\binom{\mathbf{B}}{\mathbf{A}} = \{\widetilde{\mathbf{A}} \subset \mathbf{B} : \widetilde{\mathbf{A}} \cong \mathbf{A}\}.$$

Later on, we will refer to that set as the set of *copies* of $\mathbf{A}$ in $\mathbf{B}$. Above, we used the notation $\widetilde{\mathbf{A}} \subset \mathbf{B}$ to mean that $\widetilde{\mathbf{A}}$ is a substructure of $\mathbf{B}$, meaning that the underlying set of $\mathbf{A}$ was contained in the underlying set of $\mathbf{B}$, and that all relations and functions on $\widetilde{\mathbf{A}}$ were induced by those of $\mathbf{B}$. Note however that a subset of B may not support a substructure of $\mathbf{B}$, but that it always *generates* a substructure of $\mathbf{B}$ in an obvious way.

2.2.2. *Fraïssé theory.* A structure $\mathbf{F}$ is *ultrahomogeneous* when every isomorphism between finite substructures of $\mathbf{F}$ can be extended to an automorphism of $\mathbf{F}$. When in addition $\mathbf{F}$ is countable and every finite subset of F generates a finite substructure of $\mathbf{F}$ (we say in that case that $\mathbf{F}$ is *locally finite*), it is a *Fraïssé structure.*

Let $\mathbf{F}$ be an L-structure. The *age* of $\mathbf{F}$, denoted $\mathrm{Age}(\mathbf{F})$, is the collection of all finitely generated L-structures that can be embedded into $\mathbf{F}$. Observe also that if $\mathbf{F}$ is countable, then $\mathrm{Age}(\mathbf{F})$ contains only countably many isomorphism types. Abusing language, we will say that $\mathrm{Age}(\mathbf{F})$ is countable. Similarly, a class $\mathcal{K}$ of L-structures will be said to be countable if it contains only countably many isomorphism types.

If $\mathbf{F}$ is a Fraïssé L-structure, then observe that $\mathrm{Age}(\mathbf{F})$:

(1) is countable,
(2) is *hereditary*: for every L-structure $\mathbf{A}$ and every $\mathbf{B} \in \mathrm{Age}(\mathbf{F})$, if $\mathbf{A}$ embeds in $\mathbf{B}$, then $\mathbf{A} \in \mathrm{Age}(\mathbf{F})$.
(3) satisfies the *joint embedding property*: for every $\mathbf{A}, \mathbf{B} \in \mathrm{Age}(\mathbf{F})$, there is $\mathbf{C} \in \mathrm{Age}(\mathbf{F})$ such that $\mathbf{A}$ and $\mathbf{B}$ embed in $\mathbf{C}$.
(4) satisfies the *amalgamation property* (or is an *amalgamation class*): for every $\mathbf{A}$, $\mathbf{B}_0$, $\mathbf{B}_1 \in \mathrm{Age}(\mathbf{F})$ and embeddings $f_0 : \mathbf{A} \longrightarrow \mathbf{B}_0$ and $f_1 : \mathbf{A} \longrightarrow \mathbf{B}_1$, there is $\mathbf{C} \in \mathrm{Age}(\mathbf{F})$ and embeddings $g_0 : \mathbf{B}_0 \longrightarrow \mathbf{C}$, $g_1 : \mathbf{B}_1 \longrightarrow \mathbf{C}$ such that $g_0 \circ f_0 = g_1 \circ f_1$.
(5) contains structures of arbitrarily high finite size.

Any class of finitely generated structures satisfying those five items is called a *Fraïssé class*. The following theorem, due to Fraïssé, establishes that every Fraïssé class is actually the age of a Fraïssé structure.

Theorem III (Fraïssé [Fra54])**.** *Let L be a relational signature and let $\mathcal{K}$ be a Fraïssé class of L-structures. Then there is, up to isomorphism, a unique Fraïssé L-structure $\boldsymbol{F}$ such that* $\mathrm{Age}(\boldsymbol{F}) = \mathcal{K}$. *The structure $\boldsymbol{F}$ is called the* Fraïssé limit *of $\mathcal{K}$ and denoted* $\mathrm{Flim}(\mathcal{K})$.

2.2.3. *Examples of Fraïssé classes and Fraïssé limits.* Consider the class of all finite linear orders $\mathcal{LO}$. The language consists of one relational symbol $<$, which is binary (has arity 2). An element of $\mathcal{LO}$ is of the form $\mathbf{A} = (A, <^{\mathbf{A}})$, i.e. a set together with a linear order. The class $\mathcal{LO}$ is a Fraïssé class, and its Fraïssé limit is nothing else than the usual linear order $(\mathbb{Q}, <^{\mathbb{Q}})$.

As a second example, fix a finite field F and consider the class $\mathcal{V}_F$ of all finite vector spaces over F. The relevant language consists of one binary function symbol $+$ and finitely many unary function symbols M_λ ($\lambda \in F$). In a structure $\mathbf{A}$, $+$ is interpreted as a group operation on A, M_λ as the scalar multiplication by λ for each $\lambda \in F$, and all the usual axioms of vector spaces are satisfied. The class $\mathcal{V}_F$ is a Fraïssé class, and its limit is the vector space V_F of countable dimension over F.

As mentioned already, we will in this memoir mostly deal with graphs and metric spaces. All graphs will be simple and loopless. In the undirected case (which is the case we will refer to when we mention graphs without any further indication), the language is made of one binary relation symbol E. In a structure, E is interpreted as an irreflexive, symmetric relation. There are several Fraïssé classes of such objects, but all of them have been classified by Lachlan and Woodrow in [LW80]. An example of such a class is the class $\mathcal{G}$ of all finite graphs. The Fraïssé limit of $\mathcal{G}$ is the countable random graph alluded to in Section 2.1. For directed graphs, the language is made of one binary relation symbol $\leftarrow$ which is interpreted as an irreflexive, antisymmetric binary relation. Fraïssé classes of finite directed graphs have also been classified, but only much later than graphs. This classification is due to Cherlin in [Che98]. Both [LW80] and [Che98] are known to be remarkable achievements.

For metric spaces, the study of Fraïssé classes is still a very young subject. The language is made of binary symbols d_s, where s ranges over a set S of non-negative reals including zero. In a given structure $\mathbf{A}$, the relations $(d_s)_{s\in S}$ encode a distance, and $d_s^{\mathbf{A}}(x,y)$ means that the distance between x and y is less than s. There are at the moment very few classification results for Fraïssé classes of finite metric spaces, and they deal with extremely simple combinatorial situations. For example, the paper [DLPS07] isolates the four-value condition, which holds for a subset $S \subset \mathbb{R}_+$ exactly when the class $\mathcal{M}_S$ of all finite metric spaces is an amalgamation class. When S is countable, the metric space that arises as the Fraïssé limit is sometimes called the *Urysohn space attached to* $\mathcal{M}_S$, and is denoted $\mathbf{U}_S$. Various examples also appear in [NVT06] and [NVT10b]. In what follows, we will state several results involving $\mathbf{U}_\mathbb{Q}$ (the rational Urysohn space), $\mathbf{S}_\mathbb{Q} := \mathbf{U}_{\mathbb{Q}\cap[0,1]}$ (the rational Urysohn sphere), $\mathbf{U}_\mathbb{N}$ and $\mathbf{U}_m := \mathbf{U}_{\{0,1,\dots,m\}}$. All of them are closely related to the original Urysohn space $\mathbf{U}$ (which is not a Fraïssé limit, but can be thought as such in the more general setting of *metric Fraïssé theory*, see Section 6).

2.2.4. *Non-Archimedean Polish groups.* Another remarkable feature of Fraïssé structures is provided by their automorphism groups. Let $\mathbf{F}$ be a Fraïssé structure. Because its underlying set is countable, we may assume that this set is actually $\mathbb{N}$ and the group $\mathrm{Aut}(\mathbf{F})$ may be thought of as a subgroup of the permutation group of $\mathbb{N}$. Moreover, if g is a permutation of $\mathbb{N}$ failing to be an automorphism of $\mathbf{F}$, then there is a finite subset of $\mathbb{N}$ on which this failure is witnessed. Therefore, $\mathrm{Aut}(\mathbf{F})$ is a closed subgroup of S_∞, the permutation group of $\mathbb{N}$ equipped with the pointwise convergence topology. It turns out that *every* closed subgroup of S_∞ arises that way. The class of all closed subgroups of S_∞ can also be defined abstractly in several ways: it coincides with the class of all Polish groups that admit a basis at the identity consisting of open subgroups, but also with the class of all Polish groups that admit a compatible left-invariant ultrametric [BK96]. Recently, it has been referred to as the class of *non-Archimedean Polish groups* (see [Kec12]). It includes all countable discrete groups as well as all profinite groups, but in the sequel, we will mostly concentrate on non locally-compact groups.

2.3. **Precompact relational expansions.** Throughout this section, L is some at most countable language L and L^* is an at most countable language containing L such that $L^* \smallsetminus L = \{R_i : i \in I\}$ consists only of relation symbols.

Definition. *Let $\boldsymbol{A}$ be an L-structure. An L^*-structure $\boldsymbol{A}^*$ is an* expansion *of $\boldsymbol{A}$ in L^* when it is of the form $\boldsymbol{A}^* = (\boldsymbol{A}, (R_i^{\boldsymbol{A}})_{i\in I})$ (also written $(\boldsymbol{A}, \vec{R}^{\boldsymbol{A}})$). In that case, $\boldsymbol{A}$ is the* reduct *of $\boldsymbol{A}^*$ to L and is denoted by $\boldsymbol{A}^* \upharpoonright L$.*

Intuitively, $\mathbf{A}$ is simply obtained from $\mathbf{A}^*$ by forgetting all the relations coming from $L^* \smallsetminus L$. Note that it is also customary to use the restriction symbol $\upharpoonright$ to refer to substructures as opposed to reducts. Because the context almost always prevents the confusion between those two notations, we will use freely both of them, without any further indication.

Definition. *Let $\boldsymbol{A}$ be an L-structure. An L^*-structure $\boldsymbol{A}^*$ is a* precompact expansion *of $\boldsymbol{A}$ in L^* when every element of* $\mathrm{Age}(\boldsymbol{A})$ *has finitely many expansions in* $\mathrm{Age}(\boldsymbol{A}^*)$.

Definition. *Let $\mathcal{K}$ be a class of L-structures. An* expansion *of $\mathcal{K}$ in L^* is a class $\mathcal{K}^*$ of L^*-structures whose class of reducts to L is $\mathcal{K}$.*

Definition. *Let $\mathcal{K}$ be a class of L-structures, and let $\mathcal{K}$ be an expansion of $\mathcal{K}$ in L^*. The class $\mathcal{K}^*$ is a* precompact expansion *of $\mathcal{K}$ in L^* when every element of $\mathcal{K}$ has finitely many expansions in $\mathcal{K}^*$.*

Precompact expansions were introduced in [NVT13a]. The justification for the terminology and for their use will be given later on, in Section 5.2. Special cases of precompact expansions are provided by expansions for which the difference $L^* \smallsetminus L$ is finite. It is known after [KPT05] that of particular interest are what we will call here *pure order expansions* of Fraïssé classes. Those are obtained when $L^* = L \cup \{<\}$, where $<$ is a binary relation symbol that is not in L, and where all elements of $\mathcal{K}^*$ are of the form $\mathbf{A}^* = (\mathbf{A}, <^{\mathbf{A}})$, where $\mathbf{A} \in \mathcal{K}$ and $<^{\mathbf{A}}$ is a linear ordering on the universe A of $\mathbf{A}$. In the sequel, relational expansions will play an important rôle, especially when the following property is satisfied:

Definition. *Let $\mathcal{K}$ be a Fraïssé class in L and let $\mathcal{K}^*$ be a relational expansion of $\mathcal{K}$. The class $\mathcal{K}^*$ has the* expansion property *relative to $\mathcal{K}$ when for every $\boldsymbol{A} \in \mathcal{K}$, there exists $\boldsymbol{B} \in \mathcal{K}$ such that*

$$\forall \boldsymbol{A}^*, \boldsymbol{B}^* \in \mathcal{K}^* \quad (\boldsymbol{A}^* \upharpoonright L = \boldsymbol{A} \quad \wedge \quad \boldsymbol{B}^* \upharpoonright L = \boldsymbol{B}) \Rightarrow \boldsymbol{A}^* \leq \boldsymbol{B}^*.$$

In the case of pure order expansions, the expansion property is usually known as the *ordering property.*

3. Finite Ramsey theory

The purpose of this section is to present the part of my work that deals with finite Ramsey theory. In Section 3.1, an overview on finite structural Ramsey theory is given. It is followed in Section 3.2 by a presentation of the notion of Ramsey degree. Several examples involving graphs and directed graphs are then provided. Finally, Section 3.3 contains a discussion concerning the existence of Ramsey precompact expansions in a quite general context.

3.1. The Ramsey property.

Throughout this section, L is a fixed language. Let $k \in \mathbb{N}$, and $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be L-structures. Recall that the set of all *copies* of $\mathbf{A}$ in $\mathbf{B}$ is the set

$$\binom{\mathbf{B}}{\mathbf{A}} = \{\widetilde{\mathbf{A}} \subset \mathbf{B} : \widetilde{\mathbf{A}} \cong \mathbf{A}\}.$$

The standard arrow partition symbol

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_k$$

is used to mean that for every map $c : \binom{\mathbf{C}}{\mathbf{A}} \longrightarrow [k]$, thought as a k-coloring of the copies of $\mathbf{A}$ in $\mathbf{C}$, there is $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that c is constant on $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$.

Definition. *A class $\mathcal{K}$ of L-structures has the* Ramsey property*, or is a* Ramsey class*, when*

$$\forall k \in \mathbb{N} \quad \forall \boldsymbol{A}, \boldsymbol{B} \in \mathcal{K} \quad \exists \boldsymbol{C} \in \mathcal{K} \quad \boldsymbol{C} \longrightarrow (\boldsymbol{B})^{\boldsymbol{A}}_k.$$

When $\mathcal{K} = \mathrm{Age}(\mathbf{F})$, where $\mathbf{F}$ is a Fraïssé structure, this is equivalent, via a compactness argument, to:

$$\forall k \in \mathbb{N} \quad \forall \mathbf{A}, \mathbf{B} \in \mathcal{K} \quad \mathbf{F} \longrightarrow (\mathbf{B})^{\mathbf{A}}_k.$$

In other words, every finite coloring of the copies of $\mathbf{A}$ in $\mathbf{F}$ must be constant on arbitrarily large finite sets. The first example of a Ramsey class is provided by Ramsey's theorem, which states that the class of all finite sets (i.e. structures in the empty language) forms a Ramsey class. As indicated previously, the search for Ramsey classes generated a considerable activity in the seventies and in the early eighties. The most significant examples of Ramsey classes which appeared during that period are provided by finite Boolean algebras (Graham-Rothschild, [GR71]) and by finite vector spaces over a fixed finite field (Graham-Leeb-Rothschild, [GLR72, GLR73]). However, being Ramsey turns out to be very restrictive, and many natural classes of finite structures do not have the Ramsey property, for example finite equivalence relations, finite graphs, finite relational structures in a fixed language, finite K_n-free graphs, finite posets, etc...Nevertheless, it appears that those classes are in fact not so far from being Ramsey. In particular, they can be expanded into Ramsey classes simply by adding linear orderings. More precisely, the following classes are Ramsey: finite equivalence relations ordered by a linear ordering leaving the classes convex (Rado, [Rad54]), finite ordered graphs and more generally, finite ordered relational structures in a fixed language L (Abramson-Harrington [AH78] and Nešetřil-Rödl [NR77, NR83] independently), finite ordered K_n-free graphs (Nešetřil-Rödl [NR77, NR83]), finite posets ordered by a linear extension (announced by and attributed to Nešetřil-Rödl but the corresponding paper was never published, see Paoli-Trotter-Walker [PTW85] for the first proof in print).

The importance of linear orderings, and more generally of rigidity, in relation to the Ramsey property was realized pretty early. A structure is *rigid* when it admits no non-trivial automorphism. Essentially, all Ramsey classes must be made of rigid structures. Sets, Boolean algebras and vector spaces do not fall into that category, but those being Ramsey is equivalent to some closely related classes of rigid structures being Ramsey. To use the common jargon, rigidity prevents the appearance of Sierpiński type colorings, which do not stabilize on large sets.

Another restriction imposed by the Ramsey property appears in the following result.

Proposition IV (Nešetřil-Rödl [NR77], p.294, Lemma 1)**.** *Let $\mathcal{K}$ be a class of finite L-structures consisting of rigid elements. Assume that $\mathcal{K}$ has the hereditarity property, the joint embedding property, and the Ramsey property. Then $\mathcal{K}$ has the amalgamation property.*

This result explains why structural Ramsey theory and Fraïssé theory are so closely related: when a class of finite structures satisfies very common properties, it has to be Fraïssé whenever it is Ramsey. Amalgamation itself is a very restrictive feature, and was at the center of a very active area of research in the eighties. In particular, it led to spectacular classification results, the most significant ones being probably those we already mentioned concerning finite graphs (Lachlan-Woodrow, [LW80]), finite tournaments (Lachlan [Lac84], based on the work of

Woodrow [Woo76]) and more recently finite directed graphs (Cherlin, [Che98]). As indicated previously, any of those results represents a remarkable piece of work.

3.2. **Ramsey degrees.** It was already indicated that having the Ramsey property is extremely restrictive for a class of finite structures. For that reason, weaker partition properties were introduced. One of the most common ones is obtained by imposing that colorings should only take a small number of values on a large set, as opposed to being constant. This is captured by the following notion: for $k, l \in \mathbb{N} \smallsetminus \{0\}$ and L-structures $\mathbf{A}, \mathbf{B}, \mathbf{C}$, write

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,l}$$

when for any $c : \binom{\mathbf{C}}{\mathbf{A}} \longrightarrow [k]$ there is $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such c takes at most l-many values on $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$. Note that when $l = 1$, this is simply the partition property $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k}$ introduced previously.

Definition. *Let $\mathcal{K}$ be a class of L-structures. An element $\boldsymbol{A} \in \mathcal{K}$ has a* finite Ramsey degree in $\mathcal{K}$ *when there exists $l \in \mathbb{N}$ such that for any $\boldsymbol{B} \in \mathcal{K}$, and any $k \in \mathbb{N} \smallsetminus \{0\}$, there exists $\boldsymbol{C} \in \mathcal{K}$ such that:*

$$\boldsymbol{C} \longrightarrow (\boldsymbol{B})^{\boldsymbol{A}}_{k,l}.$$

The least such number l is denoted $\mathrm{t}_{\mathcal{K}}(\boldsymbol{A})$ *and is the* Ramsey degree of $\boldsymbol{A}$ in $\mathcal{K}$.

Equivalently, if $\mathcal{K}$ is Fraïssé and $\mathbf{F}$ denotes its limit, $\mathbf{A}$ has a finite Ramsey degree in $\mathcal{K}$ when there is $l \in \mathbb{N}$ such that for any $\mathbf{B} \in \mathcal{K}$, and any $k \in \mathbb{N} \smallsetminus \{0\}$,

$$\mathbf{F} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,l}.$$

The Ramsey degree is then equal to the least such number l. Note that it depends only on $\mathbf{A}$ and $\mathcal{K}$. Finite Ramsey degrees can be seen in two different ways. They reflect the failure of the Ramsey property within a given class $\mathcal{K}$, but also reflect that arbitrary finite colorings can always be reasonably controlled. A powerful method to compute them is to look for a precompact expansion $\mathcal{K}^*$ of $\mathcal{K}$ satisfying the Ramsey property and the expansion property relative to $\mathcal{K}$:

Proposition V. *Let $\mathcal{K}$ be a class of finite L-structures and let $\mathcal{K}^*$ be an expansion of $\mathcal{K}$ in L^*. Assume that both $\mathcal{K}$ and $\mathcal{K}^*$ have elements of arbitrarily high cardinality and satisfy the hereditary and the joint embedding property. Assume that $\mathcal{K}^*$ has the Ramsey property as well as the expansion property relative to $\mathcal{K}$. Then every element of $\mathcal{K}$ has a finite Ramsey degree in $\mathcal{K}$ equal to the number of non-isomorphic expansions it has in $\mathcal{K}^*$.*

This fact already appears in [KPT05] in the context of pure order expansions, but may have been known before. In fact, finding a precompact expansion $\mathcal{K}^*$ with both the Ramsey and the expansion property gives more than the Ramsey degrees, it also gives access to the most complicated colorings within $\mathcal{K}$. Let us detail in which sense. For simplicity, we will assume that $\mathcal{K}^*$ is of the form $\mathrm{Age}(\mathbf{F}^*)$ for some Fraïssé structure $\mathbf{F}^*$ which is of the form $\mathbf{F}^* = (\mathbf{F}, \vec{R}^*)$, where $\mathbf{F}$ is also a Fraïssé structure. Let $\mathbf{A} \in \mathcal{K}$. Every copy of $\mathbf{A}$ in $\mathbf{F}$ supports a substructure of $\mathbf{F}^*$, which turns out to be an expansion of $\mathbf{A}$ in $\mathcal{K}^*$. Therefore, there is a natural coloring of the copies of $\mathbf{A}$ in $\mathbf{F}$, which assigns to every copy of $\mathbf{A}$ the isomorphism type of the structure it supports in $\mathbf{F}^*$. The Ramsey property for $\mathcal{K}^*$ makes sure that for every $\mathbf{B} \in \mathcal{K}$ and every finite coloring c of $\binom{\mathbf{F}}{\mathbf{A}}$, there is a copy of $\mathbf{B}$ in $\mathbf{F}$ where two copies of $\mathbf{A}$ receive the same c-color whenever they are isomorphic in $\mathbf{F}^*$. On the other hand, the expansion property makes sure that there is some $\mathbf{B}$ for which all isomorphism types of $\mathbf{A}$ will always appear in every copy of $\mathbf{B}$ in $\mathbf{F}$. For this structure $\mathbf{B}$, there is consequently a copy where the color of any copy of $\mathbf{A}$ depends only on its isomorphism type as a substructure of $\mathbf{F}^*$. The coloring of copies of $\mathbf{A}$ using the isomorphism type in $\mathbf{F}^*$ is therefore the most complicated possible, and this is really what is behind the previous proposition.

To see on a concrete example how the previous proposition can be applied, consider the class $\mathcal{G}$ of finite graphs. It is not Ramsey, but the class $\mathcal{G}^<$ of finite ordered graphs does have the Ramsey property as well as the ordering property relative to $\mathcal{G}$. Therefore, every $\mathbf{A} \in \mathcal{G}$ has a finite Ramsey degree, which is equal to the number of non-isomorphic expansions of $\mathbf{A}$ in $\mathcal{G}^<$, i.e.

$$t_{\mathcal{G}}(\mathbf{A}) = |\mathbf{A}|!/|\mathrm{Aut}(\mathbf{A})|.$$

The expansion that satisfied the Ramsey and the expansion property in the previous example was a pure order expansion. There are many similar cases, but the following examples show that there are also some others, where general precompact expansions are necessary. This observation is not particularly deep, but it is apparently the first one referring to an interesting phenomemon, namely that many interesting classes of finite structures may actually have a precompact expansion with the Ramsey and the expansion property. See Section 3.3 for more details.

3.2.1. *n-chromatic graphs.* Our first example is taken from [NVT10a]. Consider the class χ_n of n-chromatic graphs, where n is a fixed integer (this class is *not* a Fraïssé class, as amalgamation does not hold). Write $\mathcal{M}_n^<$ for the class of all finite ordered monotone colored graphs with colors in $[n]$. Those are finite ordered graphs $(A, E^{\mathbf{A}}, <^{\mathbf{A}})$ together with a map $\lambda^{\mathbf{A}}$ that colors the vertices with colors in $[n] = \{0, \ldots, n-1\}$ in such a way that $\lambda^{\mathbf{A}}$ is increasing (when seen as a map from $(A, <^{\mathbf{A}})$ to $[n]$) and so that no two adjacent vertices receive the same color.

Proposition 1 (NVT [NVT10a])**.** *No pure order expansion of χ_n has the Ramsey property.*

Proposition 2 (NVT [NVT10a])**.** *The class $\mathcal{M}_n^<$ has the Ramsey property as well as the expansion property relative to χ_n.*

Corollary 1. *Every element $\boldsymbol{A}$ of χ_n has a finite Ramsey degree in χ_n equal to the number of non isomorphic expansions of $\boldsymbol{A}$ in $\mathcal{M}_n^<$.*

What is interesting about those results is that their proof uses one of the very standard theorems in structural Ramsey theory (namely, Nešetřil-Rödl theorem on forbidden configurations), but that nobody seems to have noticed previously that it could be used to derive the aforementioned partition result for such natural a class as the class χ_n.

3.2.2. *Circular directed graphs.* Another example, involving Fraïssé classes, was isolated in [LNVTS10]. The tournament $\mathbf{S}(2)$, called the dense local order, is defined as follows: let $\mathbb{T}$ denote the unit circle in the complex plane. Define an oriented graph structure on $\mathbb{T}$ by declaring that there is an arc from x to y (in symbols, $y \xleftarrow{\mathbb{T}} x$) iff $0 < \arg(y/x) < \pi$. Call $\overrightarrow{\mathbb{T}}$ the resulting oriented graph. The dense local order is then the substructure $\mathbf{S}(2)$ of $\overrightarrow{\mathbb{T}}$ whose vertices are those points of $\mathbb{T}$ with rational argument. It is represented in the picture below.

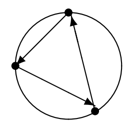

FIGURE 1. The tournament $\mathbf{S}(2)$

This structure is one of the only three countable ultrahomogeneous tournaments (a tournament is a directed graph where every pair of distinct points supports exactly one arc), the two other ones being the rationals $(\mathbb{Q}, <)$, seen as a directed graph where $x \overset{\mathbb{Q}}{\longleftarrow} y$ iff $x < y$, and the so-called random tournament. It is therefore a Fraïssé structure in the language $L = \{\leftarrow\}$ consisting of one binary relation. More information about this object can be found in [Woo76], [Lac84] or [Che98].

Proposition 3 (NVT [NVT13a])**.** *No pure order expansion of $\boldsymbol{S}(2)$ has an age with the Ramsey property.*

However, $\mathbf{S}(2)$ does allow a precompact relational expansion $\mathbf{S}(2)^*$ whose age has the Ramsey property and the expansion property. Such an expansion essentially appears in [LNVTS10], where the finite and the infinite Ramsey properties of $\mathbf{S}(2)$ were analyzed. The appropriate language is

$$L^* = L \cup \{P_j : j \in [2]\},$$

every symbol P_j being unary. We expand $\mathbf{S}(2)$ as $(\mathbf{S}(2), P_0^*, P_1^*)$ in L^*, where $P_0^*(x)$ holds iff x is in the right half plane, and $P_1^*(x)$ iff it is in the left half plane. Quite clearly, $\mathbf{S}(2)^*$ is a precompact relational expansion of $\mathbf{S}(2)$.

Proposition 4 (Laflamme-NVT-Sauer [LNVTS10])**.** *The class* $\mathrm{Age}(\boldsymbol{S}(2)^*)$ *has the Ramsey property and the expansion property relative to* $\mathrm{Age}(\boldsymbol{S}(2))$.

Corollary 2. *Every element $\boldsymbol{A}$ of* $\mathrm{Age}(\boldsymbol{S}(2))$ *has a finite Ramsey degree in* $\mathrm{Age}(\boldsymbol{S}(2))$ *equal to*

$$2|\boldsymbol{A}|/|\mathrm{Aut}(\boldsymbol{A})|.$$

The same technique also applies in the case of another directed graph, called $\mathbf{S}(3)$. The corresponding results appear in [NVT13a], but are really based on techniques from [LNVTS10]. The notation suggests that $\mathbf{S}(3)$ is a modified version of $\mathbf{S}(2)$, and it is indeed the case. Call $\overrightarrow{\mathbb{D}} = (\mathbb{T}, \overset{\mathbb{D}}{\longleftarrow})$ the directed graph defined on $\mathbb{T}$ by declaring that there is an arc from x to y iff $0 < \arg(y/x) < 2\pi/3$. The directed graph $\mathbf{S}(3)$ is then the substructure of $\overrightarrow{\mathbb{D}}$ whose vertices are those points of $\mathbb{T}$ with rational argument. It is represented in Figure 2.

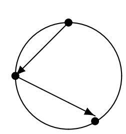

FIGURE 2. The directed graph $\mathbf{S}(3)$

Like $\mathbf{S}(2)$, $\mathbf{S}(3)$ is a Fraïssé structure in the language $L = \{\leftarrow\}$ consisting of one binary relation. The main obvious difference with $\mathbf{S}(2)$ is that it is not a tournament (some pairs of points do not support any arc). For more information about this object, we refer to [Che98]. For the same reason as in the case of $\mathbf{S}(2)$, no pure order expansion of $\mathbf{S}(3)$ has an age with the Ramsey and the expansion property, but there is a precompact expansion $\mathbf{S}(3)^*$ which does. The corresponding structure is described in Figure 3. The appropriate language is

$$L^* = L \cup \{P_j : j \in [3]\},$$

with every symbol P_i unary, and $\mathbf{S}(3)^*$ is defined by $\mathbf{S}(3)^* = (\mathbf{S}(3), P_0^*, P_1^*, P_2^*)$, where

$$P_j^*(x) \Leftrightarrow \left(\frac{2j\pi}{3} < \arg(x) + \frac{\pi}{6} < \frac{2(j+1)\pi}{3}\right)$$

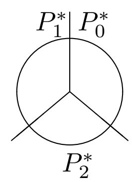

FIGURE 3. The expansion $\mathbf{S}(3)^*$

Proposition 5 (NVT [NVT13a])**.** *The class* Age($\boldsymbol{S}(3)^*$) *has the Ramsey property and the expansion property relative to* Age($\boldsymbol{S}(3)$)*.*

Corollary 3. *Every element* $\boldsymbol{A}$ *of* Age($\boldsymbol{S}(3)$) *has a finite Ramsey degree in* Age($\boldsymbol{S}(3)$) *equal to*

$$3|\boldsymbol{A}|/|\mathrm{Aut}(\boldsymbol{A})|.$$

For $\mathbf{S}(2)$ as well as for $\mathbf{S}(3)$, the proof of the previous results is based on a coding of the structures involving a particular kind of precompact expansions of the linear ordering $\mathbb{Q}$. Those are the structures that we will denote $\mathbb{Q}_n$. Given a natural n, the expansion $\mathbb{Q}_n$ is made by adding n many unary relation symbols $(P_i)_{i\in[n]}$ that are interpreted as a partition $(P_i^{\mathbb{Q}_n})_{i\in[n]}$ of $\mathbb{Q}$ with dense parts.

To see how $\mathbf{S}(2)$ relates to $\mathbb{Q}_2$, think of the ordering of $\mathbb{Q}_2$ as a directed graph relation where $x \longleftarrow y$ when $x < y$. Then, observe that the structure $\mathbb{Q}_2$ is simply obtained from $\mathbf{S}(2)^*$ by reversing all the arcs whose extremities do not belong to the same part of the partition. The simple reason behind that fact is that if $x, y \in \mathbf{S}(2)$ are such that $P_0^*(x)$ and $P_1^*(y)$, then $x \overset{\mathbb{T}}{\longleftarrow} y$ iff $(-y) \overset{\mathbb{T}}{\longleftarrow} x$, where $(-y)$ denotes the opposite of y. So one way to realize the transformation from $\mathbf{S}(2)^*$ to $\mathbb{Q}_2$ is to consider $\mathbf{S}(2)^*$, to keep the partition relation, but to replace the arc relation by the relation obtained by symmetrizing all the elements in the left half. Quite clearly, the new arc relation defines a total order, which is dense in itself and without extremity point, and where both parts of the partition are dense. Therefore, the resulting structure is $\mathbb{Q}_2$. Similarly, applying the same transformation to $\mathbb{Q}_2$ gives raise to $\mathbf{S}(2)^*$. Formally, $\mathbf{S}(2)^*$ and $\mathbb{Q}_2$ are said to be *first-order simply bi-definable.*

Very similarly, $\mathbf{S}(3)^*$ is first order simply bi-definable with $\mathbb{Q}_3$.

Once those connections are established, the Ramsey properties of the ages of $\mathbf{S}(2)^*$ and $\mathbf{S}(3)^*$ are a direct consequence of those of the ages of the structures $\mathbb{Q}_n$, which are in turn detailed in [KPT05], p.158.

3.3. **Are Ramsey classes so rare?** We saw above that for a class of finite structures, being Ramsey is extremely restrictive. Even among Fraïssé classes, a common point of view after the knowledge accumulated in the eighties is that Ramsey classes are quite exceptional objects. When analyzing how the most famous results of the field were obtained, it seems that two categories emerge. The first one corresponds to those "natural" classes where the Ramsey property holds: finite sets, finite Boolean algebras, finite vector spaces over a finite field. The second one corresponds to those classes where the Ramsey property fails but where this failure can be fixed by adjoining a linear ordering: finite graphs, finite K_n-free graphs, finite hypergraphs, finite partial orders, finite topological spaces, finite metric spaces... As for those classes where more than a linear ordering is necessary, besides the ones that appear in [KPT05] (finite equivalence relations with classes of size bounded by n, or equivalence relations with at most n classes) or those that were found more recently (namely, substructures of $\mathbf{S}(2)$ and $\mathbf{S}(3)$, posets that are unions of at most n many chains or that are obtained as a totally ordered set of antichains of size at most n [Sok12b], and boron tree structures [Jas13]), not so many cases are known, but it would be extremely surprising that nobody encountered such instances before. Quite likely, the corresponding results were not considered as true structural Ramsey results, and were therefore overlooked. However, precompact expansions seem to offer a reasonable general context. For example, it was realized recently that they allow to compute the Ramsey degrees in the case of all Fraïssé classes of graphs, posets, and tournaments. Those results naturally led to a more general study of Fraïssé classes of directed graphs. For those, a preliminary analysis has indicated that all of them seem to admit a precompact Ramsey expansion. The corresponding results will appear in [JLNVTW12]. In practice, it also appears that there is some sort of a standard scheme that can be applied in order to construct precompact Ramsey expansions whenever those exist. This motivates the following conjecture:

Conjecture 1. *Let $\mathcal{K}$ be a Fraïssé class where there are only finitely many non-isomorphic structures in every cardinality (equivalently, $\mathcal{K}$ is the age of a countable ultrahomogeneous ω-categorical structure). Then $\mathcal{K}$ admits a Ramsey precompact expansion.*

Contrary to the common opinion expressed at the top of the present paragraph, a positive answer would imply that after all, Ramsey classes are not so rare. We will come back to that conjecture later on, and see that there is indeed a characterization of those Fraïssé classes that admit a precompact Ramsey expansion. It is in terms of topological dynamics, and leaves open the possibility of a solution via techniques from dynamics and functional analysis. It also motivates the hypothesis made on $\mathcal{K}$, and shows that the conjecture is false when no restriction is placed on $\mathcal{K}$.

Note also that under some more restrictive conditions on $\mathcal{K}$, it is possible to prove that a given precompact expansion is Ramsey:

Proposition 6 (NVT [NVT13a], based on [KPT05], proof of Theorem 10.7)**.** *Let $\mathcal{K}$ be a Fraïssé class in L such that every $\boldsymbol{A} \in \mathcal{K}$ has a finite Ramsey degree in $\mathcal{K}$. Let $\mathcal{K}^*$ be a precompact expansion of $\mathcal{K}$ in L^* satisfying the hereditarity property, the joint embedding property, the expansion property relative to $\mathcal{K}$, and such that every $\boldsymbol{A} \in \mathcal{K}$ has a finite Ramsey degree in $\mathcal{K}$ whose value is at most the number of non-isomorphic expansions of $\boldsymbol{A}$ in $\mathcal{K}^*$. Then $\mathcal{K}^*$ has the Ramsey property.*

4. Infinite Ramsey theory

Recall that a Fraïssé class $\mathcal{K}$ of L-structures with limit $\mathbf{F}$ has the Ramsey property when for any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, and any $k \in \mathbb{N} \smallsetminus \{0\}$:

$$\mathbf{F} \longrightarrow (\mathbf{B})^{\mathbf{A}}_k.$$

Equivalently, for every $\mathbf{A} \in \mathcal{K}$, every finite coloring of the copies of $\mathbf{A}$ in $\mathbf{F}$ must be constant on arbitrarily large finite sets. When that happens, it is of course tempting to ask whether this conclusion can be replaced by the stronger following property: for every $\mathbf{A} \in \mathcal{K}$ and every $k \in \mathbb{N} \smallsetminus \{0\}$,

$$\mathbf{F} \longrightarrow (\mathbf{F})^{\mathbf{A}}_k.$$

In words, must any finite coloring of copies of $\mathbf{A}$ be constant on a copy of $\mathbf{F}$? This question leads to interesting problems already in the simple case $|\mathbf{A}| = 1$, which are called *indivisiblity* problems, and are detailed in Section 4.1. The case $|\mathbf{A}| \geq 2$ concentrates on the study of *big Ramsey degrees*, which are analogs of Ramsey degrees in the infinite setting. Those are presented in Section 4.2.

4.1. **Indivisibility and variations.**

Definition. *A structure $\boldsymbol{F}$ is* indivisible *when $\boldsymbol{F}$ embeds in R or B whenever $F = R \cup B$.*

Notice that for a structure to have a chance to be indivisible, all one-point substructures of $\mathbf{F}$ must be isomorphic. If $\mathbf{F}$ is ultrahomogeneous, this happens exactly when the language L does not contain any unary relation. There is no uniformity in the behavior of ultrahomogeneous structures relative to indivisibility, and there is no uniformity either in the range of difficulty that the corresponding problems cover. For example, proving indivisiblity for the Rado graph is very easy, but is much more difficult for Henson graphs or for countable Urysohn spaces. Furthermore, because quite a number of natural structures are divisible (i.e. not indivisible), a spectrum of related notions has been developed in order to capture weaker partition relations.

Definition. *A structure $\boldsymbol{F}$ is* weakly indivisible *when, for every finite substructure $\boldsymbol{A}$ of $\boldsymbol{F}$ and every partition $F = R \cup B$ (red and blue), either $\boldsymbol{A}$ embeds in R or $\boldsymbol{F}$ embeds in B.*

Weak indivisiblity is often compared to the following notion of *age-indivisiblity.*

Definition. *A structure $\boldsymbol{F}$ is* age-indivisible *when $\boldsymbol{A}$ embeds either in R or in B for every finite substructure $\boldsymbol{A}$ of $\boldsymbol{F}$ and every partition $F = R \cup B$.*

Age-indivisiblity is formally weaker than weak indivisibility but it is only recently (cf [LNVTPS11]) that the first examples of age-indivisible, not weakly indivisible, structures were found. In fact, when the language is finite, there is still no known example of a structure $\mathbf{F}$ which is age-indivisible without being weakly indivisible. This is quite surprising, as age-indivisiblity is a purely finite Ramsey theoretic notion (it is equivalent to a statement involving only finite structures via a standard compactness argument) while weak indivisiblity is definitely an infinite one. It is therefore expected that even in the case of finite languages, age-indivisiblity should be weaker than weak indivisiblity (see [Sau02] for a survey about this problem).

The aforementioned notions of indivisibility have been studied for many classical ultrahomogeneous relational structures including graphs, tournaments and most of the directed graphs. The results presented in Section 4.1.2 and 4.1.3 extend that list with a detailed analysis for vector spaces and metric spaces.

The reason for which I became interested in these structures is not a coincidence, and is strongly related to the so-called *distortion problem* from Banach space theory.

4.1.1. *The distortion problem.* Let $\mathbb{S}^\infty$ denote the unit sphere of the Hilbert space ℓ_2. Is it true that if $\varepsilon > 0$ and $f : \mathbb{S}^\infty \longrightarrow \mathbb{R}$ is uniformly continuous, then there is a closed infinite-dimensional subspace V of ℓ_2 such that

$$\sup\{|f(x) - f(y)| : x, y \in V \cap \mathbb{S}^\infty\} < \varepsilon?$$

Equivalently, for a metric space $\mathbf{X} = (X, d^{\mathbf{X}})$, a subset $Y \subset X$ and $\varepsilon > 0$, let

$$(Y)_\varepsilon = \{x \in X : \exists y \in Y \ \ d^{\mathbf{X}}(x, y) \leqslant \varepsilon\}.$$

The distortion problem for ℓ_2 asks: given a finite partition γ of $\mathbb{S}^\infty$, is there always $\Gamma \in \gamma$ such that $(\Gamma)_\varepsilon$ includes $V \cap \mathbb{S}^\infty$ for some closed infinite-dimensional subspace V of ℓ_2? That problem appeared in the early seventies when Milman's work led to the following property, which is at the heart of Dvoretzky's theorem:

Theorem VI (Milman [Mil71]). *Let γ be a finite partition of $\mathbb{S}^\infty$. Then for every $\varepsilon > 0$ and every $N \in \mathbb{N}$, there is $\Gamma \in \gamma$ and an N-dimensional subspace V of ℓ_2 such that $V \cap \mathbb{S}^\infty \subset (\Gamma)_\varepsilon$.*

In that context, the distortion problem for ℓ_2 really asks whether this result has an infinite dimensional analog. It is only a long time after Milman's theorem was established that the distortion problem for ℓ_2 was solved by Odell and Schlumprecht:

Theorem VII (Odell-Schlumprecht [OS94]). *There is a partition $\mathbb{S}^\infty = B \cup R$ and $\varepsilon > 0$ such that neither $\mathbb{S}^\infty \cap V \not\subset (B)_\varepsilon$ nor $\mathbb{S}^\infty \cap V \not\subset (R)_\varepsilon$ holds for any closed infinite dimensional vector subspace V of ℓ_2.*

The result and its proof bring forward two natural questions. The first one asks whether Odell-Schlumprecht's partition can be strengthened so that for some $N \in \mathbb{N}$, $\mathbb{S}^\infty \cap V \not\subset (B)_\varepsilon$ for any N-dimensional vector subspace V of ℓ_2. Using the terminology we introduced previously, the problem under consideration here is about weak indivisibility properties of the vector space ℓ_2. Even though we are far from a solution, this is one of the reasons for which I studied indivisibility properties in vector spaces. The corresponding results are presented in Section 4.1.2.

The second question originating from the distortion problem comes from the fact that even though its original formulation refers to the vector space structure of ℓ_2, it is also possible to state it in purely metric terms. For $\varepsilon \geqslant 0$, call a metric space $\mathbf{X}$ *ε-indivisible* when for every finite partition γ of $\mathbf{X}$, there is $\Gamma \in \gamma$ and $\widetilde{\mathbf{X}} \subset \mathbf{X}$ isometric to $\mathbf{X}$ such that

$$\widetilde{\mathbf{X}} \subset (\Gamma)_\varepsilon.$$

Then, say that $\mathbf{X}$ is *approximately indivisible* when $\mathbf{X}$ is ε-indivisible for every $\varepsilon > 0$. Note that $\mathbf{X}$ is indivisible exactly when $\mathbf{X}$ is 0-indivisible. Using this terminology, the theorem of Odell and Schlumprecht states that the sphere $\mathbb{S}^\infty$ is not approximately indivisible. However, because the proof is not based on the intrinsic geometry of ℓ_2, the impression somehow persists that something is still missing in our understanding of the metric structure of $\mathbb{S}^\infty$. That fact was one of the motivations for [LANVT08] as well as for [NVTS09]: our hope was that understanding indivisibility properties of another remarkable space, namely the Urysohn sphere $\mathbf{S}$, would help to reach a better grasp of $\mathbb{S}^\infty$. The final answer is that it does not, but that some non-trivial theorems are nevertheless on the way. Those are presented in Section 4.1.3.

4.1.2. *Vector spaces.* For a field F, recall that V_F refers to the countable infinite dimensional vector space over F. To my knowledge, indivisibility of such structures *per se* only appeared in [Bau75], but it is clear that questions of the same flavor were already considered before the publication of [LNVTPS11], especially after Gowers' work [Gow92] and its influence in Banach space theory. However, the first related result is much older. It is a reformulation of a theorem due to Hindman, originally stated in terms of partitions of the integers and finite sums:

Theorem VIII (Hindman [Hin74]). *The space $V_{\mathbb{F}_2}$ is indivisible.*

Of course, considering the hypothesis $F = \mathbb{F}_2$, it is natural to ask whether some similar phenomenon holds for other fields. The answer is negative:

Theorem IX (Folklore). *Let F be a finite field not equal to $\mathbb{F}_2$. Then V_F is divisible.*

The first proof seems to appear in print in [LNVTPS11], but the result was already known by some specialists in Ramsey theory. What was not known was whether some meaningful partition result could be found when $F \neq \mathbb{F}_2$. The answer crucially depends on the field:

Theorem X (Baumgartner [Bau75]). *Let F be an infinite field. Then V_F is not weakly indivisible. In fact, V_F can be divided into two parts so that neither part contains an affine line.*

Baumgartner's paper [Bau75] really only takes care of $F = \mathbb{Q}$, but already contains the important ideas of the slightly more general form above. On the other hand, when F is finite, there is a positive result:

Theorem 2 (Laflamme-NVT-Pouzet-Sauer [LNVTPS11]). *Let F be a finite field. Then V_F is weakly indivisible.*

4.1.3. *Metric spaces.* Most of the results that appear in the present section relate to the Urysohn sphere $\mathbf{S}$, i.e. the sphere of unit diameter in the Urysohn space $\mathbf{U}$. Apart from the fact that both $\mathbb{S}^\infty$ and $\mathbf{S}$ are complete, separable and ultrahomogeneous, it was for some time believed that the study of indivisiblity properties of $\mathbf{S}$ would be relevant for the distortion problem for ℓ_2 because, from a finite Ramsey-theoretic point of view, the spaces $\mathbb{S}^\infty$ and $\mathbf{S}$ behave in a very similar way. For example, the following analog of Milman's theorem holds for $\mathbf{S}$:

Theorem XI (Pestov [Pes02]). *Let γ be a finite partition of $\boldsymbol{S}$. Then for every $\varepsilon > 0$ and every compact $K \subset \boldsymbol{S}$, there is $\Gamma \in \gamma$ and an isometric copy $\widetilde{K}$ of K in $\boldsymbol{S}$ such that $\widetilde{K} \subset (\Gamma)_\varepsilon$.*

In fact, this analogy is only the most elementary form of a much more general Ramsey-theoretic theorem. More precisely, up to some small error, the classes of finite metric spaces of $\mathbb{S}^\infty$ and of $\mathbf{S}$ have the Ramsey property. It is also known that this latter result has a very elegant reformulation at the level of the surjective isometry groups $\mathrm{iso}(\mathbb{S}^\infty)$ and $\mathrm{iso}(\mathbf{S})$ (seen as topological groups when equipped with the pointwise convergence topology). We will study this connection in detail in Section 5, but let us mention here that on the group side, the common behavior of $\mathbb{S}^\infty$ and $\mathbf{S}$ we are refering to is contained in and reflected by the following two theorems. Recall that a topological group G is extremely amenable when every continuous action of G on a compact space admits a fixed point. Then on the one hand:

Theorem XII (Gromov-Milman [GM83])**.** *The group* $\mathrm{iso}(\mathbb{S}^\infty)$ *is extremely amenable.*

While on the other hand:

Theorem XIII (Pestov [Pes02])**.** *The group* $\mathrm{iso}(\boldsymbol{S})$ *is extremely amenable.*

Note that actually, even more is known as both $\mathrm{iso}(\mathbb{S}^\infty)$ and $\mathrm{iso}(\mathbf{S})$ are known to satisfy the so-called *Lévy property* (cf Gromov-Milman [GM83] for $\mathrm{iso}(\mathbb{S}^\infty)$ and Pestov [Pes07] for $\mathrm{iso}(\mathbf{S})$), a property known to be stronger than extreme amenability.

Back to indivisiblity, the problem for $\mathbf{S}$ was solved in two combinatorial steps. The first one is a discretization procedure carried out in [LANVT08] and largely inspired from the proof by Gowers of the stabilization theorem for the unit sphere $\mathbb{S}_{c_0}$ of c_0 and its positive part $\mathbb{S}^+_{c_0}$ (c_0 is the space of all real sequences converging to 0 equipped with the $\|\cdot\|_\infty$ norm, and $\mathbb{S}^+_{c_0}$ is the set of all those elements of $\mathbb{S}_{c_0}$ taking only positive values). Gowers' proof involves a family of approximation spaces called $(\mathrm{FIN}_k)_{k\geq 1}$. Ours uses the family $(\mathbf{U}_m)_{m\geqslant 1}$ of countable Urysohn metric spaces. For $m \geqslant 1$, the space $\mathbf{U}_m$ is defined as follows: up to isometry it is the unique countable ultrahomogeneous metric space with distances in $\{1, \ldots, m\}$ into which every countable metric space with distances in $\{1, \ldots, m\}$ embeds isometrically. In other words, it is the Fraïssé limit of the class of all finite metric spaces with distances in $\{1, \ldots, m\}$.

Theorem XIV (Lopez-Abad - NVT [LANVT08])**.** *The following are equivalent:*

(1) The space $\boldsymbol{S}$ *is approximately indivisible.*
(2) The space $\boldsymbol{S}_\mathbb{Q}$ *is approximately indivisible.*
(3) For every strictly positive $m \in \mathbb{N}$*,* $\boldsymbol{U}_m$ *is indivisible.*

Note that this result already appears in my doctoral dissertation. The second step is a proof of item (3). The starting point is a technique initiated by El-Zahar and Sauer in [EZS93] and refined later on in the series of papers [EZS94], [EZS05], [Sau02], [Sau03]. All those papers deal with graphs, hypergraphs and directed graphs. However, because metric spaces introduce different kinds of constraints, several new ideas were necessary in order to solve the problem.

Theorem 3 (NVT-Sauer [NVTS09])**.** *Let* $m \in \mathbb{N}$*,* $m \geqslant 1$*. Then* $\boldsymbol{U}_m$ *is indivisible.*

As a corollary:

Corollary 4. *The Urysohn sphere* $\boldsymbol{S}$ *is approximately indivisible.*

This result answers a question mentioned by Kechris, Pestov and Todorcevic in [KPT05], Hjorth in [Hjo08] and Pestov in [Pes98], and highlights a deep topological difference which, for the reasons mentioned previously, was not at all apparent until [NVTS09]. It also yields interesting consequences. First, observe that it is equivalent to approximate indivisiblity for every separable metric space $\mathbf{X}$ with finite diameter δ into which every separable metric space with diameter at most δ embeds isometrically. Next, recall that every separable metric space with diameter less or equal to 2 embeds isometrically into the unit sphere $\mathbb{S}_{\mathcal{C}([0,1])}$ of the Banach space $\mathcal{C}([0,1])$. It follows that:

Corollary 5. *The unit sphere of* $\mathcal{C}([0,1])$ *is approximately indivisible.*

Observe also that instead of $\mathcal{C}([0,1])$, we could have used the Holmes space $\langle \mathbf{U} \rangle$ introduced in Section 2.1. Very little is known about the space $\langle \mathbf{U} \rangle$, but it is easy to see that its unit sphere embeds isometrically every separable metric space with diameter less or equal to 2. Much less trivial is the fact that every separable Banach space embeds in $\langle \mathbf{U} \rangle$ via a *linear* isometry. The only known proof relies on the general result of Godefroy and Kalton (see [GK03]) according to which the following holds: if a separable Banach space embeds isometrically into another one, then it also embeds in it via a linear isometry. Observe also that the previous partition results for spheres do *not* say that for $\mathbf{X} = \mathcal{C}([0,1])$ or $\langle \mathbf{U} \rangle$, every finite partition γ of the unit sphere $\mathbb{S}_{\mathbf{X}}$ of $\mathbf{X}$ and every $\varepsilon > 0$, there is $\Gamma \in \gamma$ and a closed infinite dimensional subspace $\mathbf{Y}$ of $\mathbf{X}$ such that $\mathbb{S}_{\mathbf{X}} \cap \mathbf{Y} \subset (\Gamma)_\varepsilon$: according to the classical results about oscillation stability in Banach spaces, this latter fact is false for those Banach spaces into which every separable Banach space embeds linearly, and both $\mathcal{C}([0,1])$ and $\langle \mathbf{U} \rangle$ have that property.

The techniques used in [LANVT08] and [NVTS09] were subsequently used in two different projects. The first one is based on the fact that the Urysohn sphere is only a particular case of Urysohn-like space. Can anything be said in general about indivisibility and approximate indivisibility of Urysohn spaces? In particular, what about Urysohn spaces of the form $\mathbf{U}_S$, where S is a subset $(0, +\infty)$? Some partial answers can be found in [NVT10b] and can be derived directly via the techniques developed in [NVTS09], but the most satisfactory solutions were brought by Sauer in the sequence [Sau12b], [Sau13], [Sau12a]:

Theorem XV (Sauer [Sau12b])**.** *Let S be a finite subset of* $(0, +\infty)$*. Assume that there is a Urysohn space $\boldsymbol{U}_S$ with distance set S. Then $\boldsymbol{U}_S$ is indivisible.*

Theorem XVI (Sauer, [Sau12a])**.** *Every uncountable, complete, separable, bounded, ultrahomogeneous, universal metric space is approximately indivisible.*

The second project was completed when it was realized that the techniques from [NVTS09] could be applied to derive weak indivisibility type results for the spaces $\mathbf{U}_\mathbb{N}$, $\mathbf{U}_\mathbb{Q}$ and $\mathbf{U}$. All were known to be age-indivisible but being unbounded, they had no chance to be indivisible. Still:

Theorem 4 (NVT-Sauer [NVTS10])**.** *The space $\boldsymbol{U}_\mathbb{N}$ is weakly indivisible.*

Theorem 5 (NVT-Sauer [NVTS10])**.** *Let $U_\mathbb{Q} = B \cup R$ and $\varepsilon > 0$. Assume that there is a finite metric subspace $\boldsymbol{Y}$ of $\boldsymbol{U}_\mathbb{Q}$ that does not embed in B. Then $\boldsymbol{U}_\mathbb{Q}$ embeds in $(R)_\varepsilon$.*

Theorem 6 (NVT-Sauer [NVTS10])**.** *Let $U = B \cup R$ and $\varepsilon > 0$. Assume that there is a compact metric subspace $\boldsymbol{K}$ of $\boldsymbol{U}$ that does not embed in $(B)_\varepsilon$. Then $\boldsymbol{U}$ embeds in $(R)_\varepsilon$.*

Note that for $\mathbf{U}_\mathbb{Q}$, weak indivisibility is still open. It is not even clear whether $\mathbf{U}_\mathbb{Q}$ embeds in R when $\mathbf{U}_\mathbb{Q} = B \cup R$ and B does not contain two points that are distance one apart.

4.2. **Big Ramsey degrees.** When $|\mathbf{A}| \geq 2$, the property $\mathbf{F} \longrightarrow (\mathbf{F})^{\mathbf{A}}_{k}$ almost always fails (this was made formal by Hjorth in [Hjo08]), and we are naturally forced to consider weakenings. It would make sense to consider analogs of weak indivisibility, but for some reason, it is a direction which has remained almost completely unexplored so far. Much more common is the study of the analog of the Ramsey degrees. Recall that $\mathbf{A} \in \mathcal{K}$ has a finite Ramsey degree in $\mathcal{K}$ when there is $l \in \mathbb{N}$ such that for any $\mathbf{B} \in \mathcal{K}$, and any $k \in \mathbb{N} \smallsetminus \{0\}$,

$$\mathbf{F} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,l}.$$

If this latter result remains valid when $\mathbf{B}$ is replaced by $\mathbf{F}$, we say, following [KPT05], that $\mathbf{A}$ has a *big Ramsey degree in* $\mathcal{K}$. Its value $\mathrm{T}_{\mathcal{K}}(\mathbf{A})$ is the least $l \in \mathbb{N}$ such that for every $k \in \mathbb{N}$,

$$\mathbf{F} \longrightarrow (\mathbf{F})^{\mathbf{A}}_{k,l}.$$

There is only a small number of structures for which a full analysis of big Ramsey degrees can be provided. The most significant result of that kind is due to Devlin in [Dev79], and covers the case $\mathbf{F} = (\mathbb{Q}, <)$.

Theorem XVII (Devlin, [Dev79])**.** *Every* $\boldsymbol{A} \in \mathrm{Age}(\mathbb{Q}, <)$ *has a big Ramsey degree in* $\mathrm{Age}(\mathbb{Q}, <\ )$ *equal to* $\tan^{(2|\boldsymbol{A}|-1)}(0)$ *(the value at* 0 *of the* $(2|\boldsymbol{A}|-1)$*th derivative of the usual trigonometric tangent function).*

A result of the same kind is due to Laflamme, Sauer and Vuksanovic [LSV06], and shows that the value of the big Ramsey degree for a finite substructure $\mathbf{A}$ of the countable random graph and of the countable random tournament can be interpreted as the number of representations of $\mathbf{A}$ into a well identified finite structure. Therefore, there is an algorithm for the computation of the value of the Ramsey degree, but there is no direct expression for it. One of the goals when the project carried out in [LNVTS10] started was to produce more results of that kind. They are presented in the two following sections. It is worth noting that all those results make heavy use of a representation involving finitely branching trees, together with a Ramsey theorem for such trees called Milliken's theorem. In fact, to our knowledge, there is only one result about big Ramsey degrees that does not use that technique: edges have a big Ramsey degree equal to 2 in the class of finite triangle-free graphs (Sauer, [Sau98]). So far, the lack of tools to represent ultrahomogeneous structures is the major obstacle towards a better understanding of their infinite partition properties.

4.2.1. *Big Ramsey degrees in* $\mathrm{Age}(\mathbb{Q}_n)$*,* $\mathrm{Age}(\boldsymbol{S}(2))$ *and* $\mathrm{Age}(\boldsymbol{S}(3))$*.* The computation of big Ramsey degrees in $\mathrm{Age}(\mathbf{S}(2))$ is carried out in [LNVTS10]. The same technique also gives access to big Ramsey degrees in $\mathrm{Age}(\mathbf{S}(3))$. The first step consists of a computation of the big Ramsey degrees for the structures $\mathbb{Q}_n$, $n \geq 1$. Quite surprisingly, the values does not depend on n, and are not larger than Devlin's bounds, as one could have expected.

Theorem 7 (Laflamme-NVT-Sauer [LNVTS10])**.** *Let n be a positive natural. Then every element* $\boldsymbol{A}$ *of* $\mathrm{Age}(\mathbb{Q}_n)$ *has a big Ramsey degree in* $\mathrm{Age}(\mathbb{Q}_n)$ *equal to*

$$\tan^{(2|\boldsymbol{A}|-1)}(0)\ .$$

As in Section 3.2.2, the cases $n = 2, 3$ allow then to compute big Ramsey degrees in $\mathrm{Age}(\mathbf{S}(2))$ and in $\mathrm{Age}(\mathbf{S}(3))$.

Theorem 8 (Laflamme-NVT-Sauer [LNVTS10])**.** *Every element* $\boldsymbol{A}$ *of* $\mathrm{Age}(\boldsymbol{S}(2))$ *has a big Ramsey degree in* $\mathrm{Age}(\boldsymbol{S}(2))$ *equal to*

$$(2|\boldsymbol{A}|/|\mathrm{Aut}(\boldsymbol{A})|)\tan^{(2|\boldsymbol{A}|-1)}(0) \ .$$

Theorem 9 (essentially Laflamme-NVT-Sauer [LNVTS10])**.** *Every element* $\boldsymbol{A}$ *of* $\mathrm{Age}(\boldsymbol{S}(3))$ *has a big Ramsey degree in* $\mathrm{Age}(\boldsymbol{S}(3))$ *equal to*

$$(3|\boldsymbol{A}|/|\mathrm{Aut}(\boldsymbol{A})|)\tan^{(2|\boldsymbol{A}|-1)}(0) \ .$$

More concretely, for every natural $k > 0$ and every coloring $c : \mathbf{S}(2) \longrightarrow [k]$, there is an isomorphic copy of $\mathbf{S}(2)$ inside $\mathbf{S}(2)$ on which c takes only 2 colors, and 2 is the best possible bound. When coloring the arcs, this bound is 8. It is 32 for the circular triangles and 96 for the transitive ones.

4.2.2. *A colored version of Milliken's theorem.* The previous results are proved thanks to a variant of a theorem of Milliken. Because of its independent interest, we detail briefely its content in the present section. Consider a finitely branching tree (in the set-theoretic sense) T of infinite height, a number m, and a subset $S \subset T$. If S satisfies certain properties listed below, we say that S is a *strong subtree of* T *of height* m. Milliken's theorem states that if we partition the set of strong subtrees of height m into finitely many parts, then there exists a strong subtree of infinite height such that all strong subtrees of height m are contained in the same part. In the version we need in order to prove Theorem 7, each level of the tree is assigned a color (out of a finite set unrelated to the partition that is applied to the tree). We then consider only strong subtrees of height m with some given level-coloring structure and we look for a strong subtree of infinite height with a level-coloring structure similar to that of the original tree.

More precisely: a *tree* is a partially ordered set $(T, \leq)$ such that given any element $t \in T$, the set $\{s \in T : s \leq t\}$ is finite and linearly ordered by $\leq$. The number of predecessors of $t \in T$, $\mathrm{ht}(t) = |\{s \in T : s < t\}|$ is the *height of* $t \in T$. The m-th level of T is $T(m) = \{t \in T : \mathrm{ht}(t) = m\}$. The *height of* T is the least m such that $T(m) = \emptyset$ if such an m exists. When no such m exists, we say that T has infinite height. When $|T(0)| = 1$, we say that T is *rooted* and we denote the root of T by $root(T)$. T is *finitely branching* when every element of T has only finitely many immediate successors. When T is a tree, the tree structure on T induces a tree structure on every subset $S \subset T$. S is then called a *subtree* of T. Here, all the trees we will consider will be rooted subtrees of the tree $\mathbb{N}^{<\infty}$ of all finite sequences of naturals ordered by initial segment. Fix now a downwards closed finitely branching subtree T of $\mathbb{N}^{<\infty}$ with infinite height. Say that a subtree S of T is *strong* when

(1) S has a smallest element.
(2) Every level of S is included in a level of T.
(3) For every $s \in S$ not maximal in S and every immediate successor t of s in T there is exactly one immediate successor of s in S extending t.

An example of strong subtree in provided in Figure 4.2.2. For a natural $m > 0$, denote by $\mathcal{S}_m(T)$ the set of all strong subtrees of T of height m. Denote also by $\mathcal{S}_\infty(T)$ the set of all strong subtrees of T of infinite height.

Theorem XVIII (Milliken [Mil79])**.** *Let* T *be a nonempty downward closed finitely branching subtree of* $\mathbb{N}^{<\infty}$ *with infinite height. Let* $k, m > 0$ *be naturals. Then for every map* $c : \mathcal{S}_m(T) \longrightarrow [k]$*, there is* $S \in \mathcal{S}_\infty(T)$ *such that* c *is constant on* $\mathcal{S}_m(S)$.

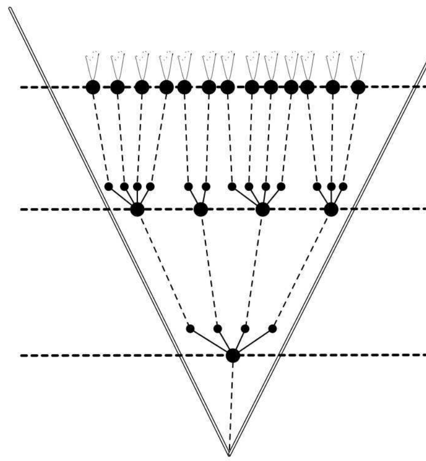

Figure 4. A strong subtree

In order to compute big Ramsey degrees in $\mathrm{Age}(\mathbb{Q}_n)$, what is needed is a strengthening of Milliken's theorem relative to *n-colored trees*. Let $\alpha \in \mathbb{N} \cup \{\infty\}$ and $n > 0$ be a natural. An n-colored tree of height α is a tree T of height α together with an *n-coloring sequence* τ assigning an element of $[n]$ (thought of as a color) to each of the levels of T ($\tau(i)$ then corresponds to the color of $T(i)$, the level i of T). If S is a strong subtree of T, τ induces an n-coloring sequence of S provided by a subsequence of τ. For $\beta \leq \alpha$ and σ a sequence of length β with values in $[n]$, let $\mathcal{S}_\sigma(T)$ denote the set of all strong subtrees of T such that the coloring sequence induced by τ is equal to σ.

Theorem 10 (Laflamme-NVT-Sauer [LNVTS10]). *Let T be a nonempty downward closed finitely branching subtree of $\mathbb{N}^{<\infty}$ with infinite height. Let $n > 0$ be a natural and Σ an n-coloring sequence of T taking each value $i \in [n]$ infinitely many times. Let $k > 0$ be a natural and σ an n-coloring sequence with finite length. Then for every map $c : \mathcal{S}_\sigma(T) \longrightarrow [k]$, there is $S \in \mathcal{S}_\Sigma(T)$ such that c is constant on $\mathcal{S}_\sigma(S)$.*

Trees are combinatorial objects that deserve attention in their own right, but the strength of Milliken's theorem also comes from their coding power. This latter aspect was indicated previously, but was also largely used in my thesis [NVT06] in order to deal with indivisiblity problems. It is also central for a very closely related result about partitions of trees, namely the Halpern-Laüchli theorem, which has been at the center of several remarkable recent advances (see the sequence [DKT12d], [DKT12a], [DKT12b], [DKT12e], [DKT12c]).

5. The Kechris-Pestov-Todorcevic correspondence

It was mentioned in the introduction that my work combined Ramsey theory and topological dynamics. The Kechris-Pestov-Todorcevic correspondence is what makes the connection possible. In particular, we will see how the results of Section 3 translate in dynamical terms. The present section is structured as follows: we start in 5.1 with an overview of the Kechris-Pestov-Todorcevic correspondence as it appears in [KPT05]. All basic notions from topological dynamics are introduced there. We then elaborate on this correspondence in the more general setting of precompact expansions. Those were already introduced, but we come back to them in 5.2. Subsection 5.3 deals with minimality and the expansion property. Subsection 5.4 deals with universality, relative extreme amenability and relative Ramsey property for embeddings. Subsection 5.5 covers universal minimal flows and the relative Ramsey property for structures. It also provides the example of an explicit computation based on results on circular directed graphs obtained in previous sections. Subsection 5.6 deals with relative extreme amenability and its relation to interpolation. Finally, the relation between Ramsey precompact expansions and metrizability of the universal minimal flow is presented in 5.7.

5.1. **The Kechris-Pestov-Todorcevic correspondence.** Let G be a topological group. A *G-flow* is a continuous action of G on a topological space X. We will often use the notation $G \curvearrowright X$. The flow $G \curvearrowright X$ is *compact* when the space X is. It is *minimal* when every $x \in X$ has dense orbit in X:

$$\forall x \in X \quad \overline{G \cdot x} = X$$

Finally, it is *universal* when every compact minimal $G \curvearrowright Y$ is a factor of $G \curvearrowright X$, which means that there exists $\pi : X \longrightarrow Y$ continuous and onto, so that

$$\forall g \in G \quad \forall x \in X \quad \pi(g \cdot x) = g \cdot \pi(x).$$

It turns out that when G is Hausdorff, there is, up to isomorphism of G-flows, a unique G-flow that is both minimal and universal. This flow is called *the universal minimal flow* of G, and is denoted $G \curvearrowright M(G)$. When the space $M(G)$ is reduced to a singleton, the group G is said to be *extremely amenable*. Equivalently, every compact G-flow $G \curvearrowright X$ admits a fixed point, ie an element $x \in X$ so that $g \cdot x = x$ for every $g \in G$. We refer to [KPT05] or [Pes06] for a detailed account on those topics. Let us simply mention that, concerning extreme amenability, it took a long time before even proving that such groups exist, but that several non-locally compact transformation groups are now known to be extremely amenable (the most remarkable ones being probably the isometry groups of the separable infinite dimensional Hilbert space (Gromov-Milman [GM83]), and of the Urysohn space (Pestov [Pes02])). As for universal minimal flows, prior to [KPT05], only a few cases were known to be both metrizable and non-trivial, the most important examples being provided by the orientation-preserving homeomorphisms of the torus (Pestov [Pes98]), S_∞ (Glasner-Weiss [GW02]), and the homeomorphism group of the Cantor space (Glasner-Weiss [GW03]). In that context, the paper [KPT05] established a link between Ramsey property and extreme amenability. For an L-structure $\mathbf{A}$, we denote by $\mathrm{Aut}(\mathbf{A})$ the corresponding automorphism group. When this group is trivial, we say that $\mathbf{A}$ is *rigid*.

Theorem XIX (Kechris-Pestov-Todorcevic [KPT05], essentially Theorem 4.8)**.** *Let $\boldsymbol{F}$ be a Fraïssé structure, and let $G = \mathrm{Aut}(\boldsymbol{F})$. The following are equivalent:*

i) The group G is extremely amenable.
ii) The class $\mathrm{Age}(\boldsymbol{F})$ has the Ramsey property and consists of rigid elements.

Because closed subgroups of S_∞ are all of the form $\mathrm{Aut}(\mathbf{F})$, where $\mathbf{F}$ is a Fraïssé structure, the previous theorem actually completely characterizes those closed subgroups of S_∞ that are extremely amenable. It also allows the description of many universal minimal flows via combinatorial methods. Indeed, when $\mathbf{F}^* = (\mathbf{F}, <^*)$ is an order expansion of $\mathbf{F}$, one can consider the space $\mathrm{LO}(\mathbf{F})$ of all linear orderings on $\mathbf{F}$, seen as a subspace of $[2]^{\mathbf{F}\times\mathbf{F}}$. In this notation, the factor $[2]^{\mathbf{F}\times\mathbf{F}}$ is thought as the set of all binary relations on $\mathbf{F}$. This latter space is compact, and G continuously acts on it: if $S \in [2]^{\mathbf{F}\times\mathbf{F}}$ and $g \in G$, then $g \cdot S$ is defined by

$$\forall x, y \in \mathbf{F} \quad g \cdot S(x, y) \Leftrightarrow S(g^{-1}(x), g^{-1}(y)).$$

It can easily be seen that $\mathrm{LO}(\mathbf{F})$ and $X^* := \overline{G\cdot <^*}$ are closed G-invariant subspaces.

Theorem XX (Kechris-Pestov-Todorcevic [KPT05], Theorem 7.4)**.** *Let $\boldsymbol{F}$ be a Fraïssé structure, and $\boldsymbol{F}^*$ a Fraïssé order expansion of $\boldsymbol{F}$. The following are equivalent:*

i) The flow $G \curvearrowright X^*$ is minimal.
ii) $\mathrm{Age}(\boldsymbol{F}^*)$ *has the ordering property relative to* $\mathrm{Age}(\boldsymbol{F})$*.*

The following result, which builds on the two preceding theorems, is then obtained:

Theorem XXI (Kechris-Pestov-Todorcevic [KPT05], Theorem 10.8)**.** *Let $\boldsymbol{F}$ be a Fraïssé structure, and $\boldsymbol{F}^*$ be a Fraïssé order expansion of $\boldsymbol{F}$. The following are equivalent:*

i) The flow $G \curvearrowright X^*$ is the universal minimal flow of G.
ii) The class $\mathrm{Age}(\boldsymbol{F}^*)$ *has the Ramsey property as well as the ordering property relative to* $\mathrm{Age}(\boldsymbol{F})$*.*

A direct application of those results allowed to find a wealth of extremely amenable groups and of universal minimal flows, see ([KPT05], Sections 6 and 8), but also [Neš07], [NVT06], [Sok12a], [Sok12b] and [Jas13]. However, some cases, which are very close to those described above, *cannot* be captured directly by those theorems. This is because, as already mentioned several times, some Fraïssé classes do not have an order expansion with the Ramsey and the ordering property, but do so when the language is enriched with additional symbols. Some examples already appear in [KPT05] (e.g. Theorem 8.4 dealing with equivalence relations with the number of classes bounded by a fixed number). It is also the case for equivalence relations whose classes have a size bounded by a fixed number, for the subtournaments of the dense local order (see [LNVTS10], or Section 3.2.2 of the present memoir), as well as for several classes of finite posets (see [Sok12b]). More recently, Jasiński showed that boron tree structures have the same property, see [Jas13]. For all those cases, a slight modification of the original framework, dealing with precompact relational expansions instead of order expansions, does allow to describe the universal minimal flow. The purpose of the following subsections is to explain where precompact expansions are coming from, and how they can be used in order to elaborate on the Kechris-Pestov-Todorcevic correspondence.

5.2. **What is precompact in precompact expansions.** The terminology "precompact expansions" is justified by the following construction. Consider Fraïssé structures $\mathbf{F}$ and $\mathbf{F}^*$ in L and L^* respectively. As before, L is some at most countable language, L^* is an at most countable language containing L such that $L^* \smallsetminus L = \{R_i : i \in I\}$ consists only of relation symbols, and $\mathbf{F}$ is an expansion of $\mathbf{F}$ in L^*. For $i \in I$, the arity of the symbol R_i is denoted $\alpha(i)$. We also assume that $\mathbf{F}$ and $\mathbf{F}^*$ are based on the set $\mathbb{N}$ of natural numbers.

The corresponding automorphism groups are denoted G and G^* respectively. The group G^* will be thought as a subgroup of G, and both are closed subgroups of S_∞, the permutation group of $\mathbb{N}$ equipped with the topology generated by sets of the form

$$U_{g,F} = \{h \in G : h \upharpoonright F = g \upharpoonright F\},$$

where g runs over G and F runs over all finite subsets of $\mathbb{N}$. This topology admits two natural uniform structures, a left-invariant one, $\mathcal{U}_L$, whose basic entourages are of the form

$$U_F^L = \{(g,h) : g^{-1}h \in U_{e,F}\}, \quad F \subset \mathbb{N} \text{ finite},$$

and a right-invariant one, $\mathcal{U}_R$, whose basic entourages are of the form

$$U_F^R = \{(g,h) : (g^{-1}, h^{-1}) \in U_F^L\}, \quad F \subset \mathbb{N} \text{ finite}.$$

In fact, those two uniform structures are respectively generated by the two following metrics: d_L, defined as

$$d_L(g,h) = \frac{1}{2^m}, \quad m = \min\{n \in \mathbb{N} : g(n) \neq h(n)\},$$

and d_R, given by

$$d_R(g,h) = d_L(g^{-1}, h^{-1}).$$

In what follows, we will be interested in the set of all expansions of $\mathbf{F}$ in L^*, which we think of as the product

$$P := \prod_{i \in I} [2]^{\mathbf{F}^{\alpha(i)}}.$$

In this notation, the factor $[2]^{\mathbf{F}^{\alpha(i)}} = \{0,1\}^{\mathbf{F}^{\alpha(i)}}$ is thought of as the set of all $\alpha(i)$-ary relations on $\mathbf{F}$. Each factor $[2]^{\mathbf{F}^{\alpha(i)}}$ is equipped with an ultrametric d_i, defined by

$$d_i(S,T) = \frac{1}{2^m}, \quad m = \min\{n \in \mathbb{N} : S \upharpoonright [n] \neq T \upharpoonright [n]\}$$

where $S_i \upharpoonright [m]$ (resp. $T_i \upharpoonright [m]$) stands for $S_i \cap [m]^{\alpha(i)}$ (resp. $T_i \cap [m]^{\alpha(i)}$). So $S_i \upharpoonright [m] = T_i \upharpoonright [m]$ means that

$$\forall y_1 \dots y_{\alpha(i)} \in [m] \quad S_i(y_1 \dots y_{\alpha(i)}) \Leftrightarrow T_i(y_1 \dots y_{\alpha(i)}).$$

The group G acts continuously on each factor as follows: if $i \in I$, $S_i \in [2]^{\mathbf{F}^{\alpha(i)}}$ and $g \in G$, then $g \cdot S_i$ is defined by

$$\forall y_1 \dots y_{\alpha(i)} \in \mathbf{F} \quad g \cdot S_i(y_1 \dots y_{\alpha(i)}) \Leftrightarrow S_i(g^{-1}(y_1) \dots g^{-1}(y_{\alpha(i)})).$$

This allows to define an action of G on the product P, where $g \cdot \vec{S}$ is simply defined as $(g \cdot S_i)_{i \in I}$ whenever $\vec{S} = (S_i)_{i \in I} \in P$ and $g \in G$. This action is continuous when P is equipped with the product topology (it is then usually referred to as the *logic action*), but also when it is endowed with the supremum distance d^P of all the distances d_i. The corresponding topology is finer than the product topology if I is infinite, but it is the one we will be interested in in the sequel because of its connection to the quotient G/G^*.

As a set, G/G^* can be thought as $G \cdot \vec{R}^*$, the orbit of $\vec{R}^*$ in P, by identifying $[g]$, the equivalence class of g, with $g \cdot \vec{R}^*$ (recall that $\vec{R}^*$ is defined as $\mathbf{F}^* = (\mathbf{F}, \vec{R}^*)$). Both uniform structures on G project onto uniform structures on G/G^*, but we will pay a particular attention to the projection of $\mathcal{U}_R$, whose basic entourages are of the form

$$V_F = \{([g],[h]) : g^{-1} \upharpoonright F = h^{-1} \upharpoonright F\}, \quad F \subset \mathbb{N} \text{ finite.}$$

Proposition 7 (NVT [NVT13a])**.** *The projection of $\mathcal{U}_R$ on $G/G^* \cong G\cdot\vec{R}^*$ coincides with the uniform structure induced by the restriction of d^P on $G \cdot \vec{R}^*$.*

Therefore, in the sequel, we can really think of the uniform space G/G^* as the metric subspace $G \cdot \vec{R}^*$ of P. Recall that a metric space X is precompact when its completion is compact. Equivalently, it can be covered by finitely many balls of arbitrary small diameter. When the space is only uniform as opposed to metric, this means that for every basic entourage V, there are finitely many $x_1, \ldots, x_n$ so that the family of sets $(\{x \in X : (x, x_i) \in V\})_{i\leq n}$ covers X. Here is what motivates the terminology "precompact expansions".

Proposition 8 (NVT [NVT13a])**.** *The space $G/G^* \cong G \cdot \vec{R}^*$ is precompact iff* Age($\boldsymbol{F}^*$) *is a precompact expansion of* Age($\boldsymbol{F}$)*. In that case, we denote by X^* the corresponding completion, ie*

$$X^* = \widehat{G/G^*} = \overline{G \cdot \vec{R}^*} \quad (\textit{where the closure in taken in } P).$$

We should emphasize at that point that the realization of spaces of relations as quotients of groups turned out to be an essential feature of all the constructions we are about to present.

5.3. **Minimality and the expansion property.** In [KPT05], minimality of the flow X^* associated to a pure order expansion appears to be related to the ordering property. The following results states that the link is preserved when passing to an arbitrary precompact expansion:

Theorem 11 (NVT [NVT13a])**.** *Let $\boldsymbol{F}$ be a Fraïssé structure, and $\boldsymbol{F}^*$ a Fraïssé precompact relational expansion of $\boldsymbol{F}$. The following are equivalent:*

i) The flow $G \curvearrowright X^*$ is minimal.
ii) The class Age($\boldsymbol{F}^*$) *has the expansion property relative to* Age($\boldsymbol{F}$)*.*

We will illustrate in Section 5.5 how this result can be applied in practice.

5.4. **Universality and the relative Ramsey property.** The study of universality of X^* by itself is not treated in [KPT05], but the question of whether it is equivalent to the Ramsey property is explicitly asked (see p.174). The following notions were introduced in [NVT13b] as an attempt for an answer:

Definition. *Let $H \leq G$ be topological groups. Say that the pair (G, H) is* relatively extremely amenable *when every continuous action of G on every compact space admits an H-fixed point.*

When $\mathbf{A}, \mathbf{B} \in$ Age($\mathbf{F}$), the set of all embedings from $\mathbf{A}$ into $\mathbf{B}$ is denoted

$$\binom{\mathbf{B}}{\mathbf{A}}_{Emb}$$

Let $\mathbf{B}^*$ an expansion of $\mathbf{B}$ in Age($\mathbf{F}^*$), and $a \in \binom{\mathbf{B}}{\mathbf{A}}_{Emb}$. The substructure of $\mathbf{B}^*$ supported by $a(\mathbf{A})$ is an expansion of $\mathbf{A}$ in Age($\mathbf{F}^*$). Using a, we can then define an expansion of $\mathbf{A}$ in $\mathrm{Age}(\mathbf{F}^*)$ as follows: for $i \in I$, call $\alpha(i)$ the arity of R_i. Then, set

$$\forall i \in I^* \quad R_i^a(x_1, \ldots, x_{\alpha(i)}) \Leftrightarrow R_i^{\mathbf{B}^*}(a(x_1), \ldots, a(x_{\alpha(i)})).$$

We will refer to $(a(\mathbf{A}), \vec{R}^a)$ as the *canonical expansion* induced by a on $\mathbf{A}$. If $a' \in \binom{\mathbf{B}}{\mathbf{A}}_{\mathrm{Emb}}$, write $a \cong_{\mathbf{B}^*} a'$ when the canonical expansions on $\mathbf{A}$ induced by a and a' are equal (not only isomorphic).

Definition. *Let $\mathcal{K}$ be a class of finite L-structures, and $\mathcal{K}^*$ an expansion of $\mathcal{K}$ in L^*. Say that the pair $(\mathcal{K}, \mathcal{K}^*)$ has the* relative Ramsey property for embeddings *when for every $k \in \mathbb{N}$, $\boldsymbol{A} \in \mathcal{K}$, $\boldsymbol{B}^* \in \mathcal{K}^*$, there exists $\boldsymbol{C} \in \mathcal{K}$ such that for every coloring $c : \binom{\boldsymbol{C}}{\boldsymbol{A}}_{\mathrm{Emb}} \longrightarrow [k]$, there exists $b \in \binom{\boldsymbol{C}}{\boldsymbol{B}}_{\mathrm{Emb}}$ such that:*

$$\forall a_0, a_1 \in \binom{\boldsymbol{B}}{\boldsymbol{A}}_{\mathrm{Emb}} \quad a_0 \cong_{\boldsymbol{B}^*} a_1 \;\Rightarrow\; c(ba_0) = c(ba_1).$$

Theorem 12 (NVT [NVT13b])**.** *Let $\boldsymbol{F}$ be a Fraïssé structure in L, and $\boldsymbol{F}^*$ a Fraïssé precompact relational expansion of $\boldsymbol{F}$ in L^*. Then the following are equivalent:*

i) The flow $G \curvearrowright X^*$ is universal.
ii) The pair (G, G^*) is relatively extremely amenable.
iii) The elements of $\mathrm{Age}(\boldsymbol{F}^*)$ *are rigid and the pair* $(\mathrm{Age}(\boldsymbol{F}), \mathrm{Age}(\boldsymbol{F}^*))$ *has the relative Ramsey property for embeddings.*

As a consequence, universality of $G \curvearrowright X^*$ is *not* equivalent to the Ramsey property for $\mathrm{Age}(\mathbf{F}^*)$ in full generality. For classes of rigid structures, the Ramsey property implies extreme amenability of G^* by the Kechris-Pestov-Todorcevic theorem, hence universality of $G \curvearrowright X^*$ because the pair (G, G^*) is then relatively extremely amenable. However, the converse may not hold. For example, consider the class $\mathcal{U}_S^{<}$ of finite ordered ultrametric spaces with distances in S, where S is a finite subset of $\mathbb{R}$. The corresponding Fraïssé limit is a countable ordered ultrametric space, denoted by $(\mathbf{U}_S^{ult}, <)$. As a linear ordering, it is isomorphic to $(\mathbb{Q}, <)$. Hence, $(\mathbf{U}_S^{ult}, <)$ can be thought as a precompact relational expansion of $(\mathbb{Q}, <)$, and the group $\mathrm{Aut}(\mathbf{U}_S^{ult}, <)$ can be thought of as a closed subgroup of $\mathrm{Aut}(\mathbb{Q}, <)$. Because this latter group is extremely amenable (see [Pes02]), the pair $(\mathrm{Aut}(\mathbb{Q}, <), \mathrm{Aut}(\mathbf{U}_S^{ult}, <))$ is relatively extremely amenable and the corresponding flow is universal. However, it is known that $\mathcal{U}_S^{<}$ does not have the Ramsey property, see [NVT06]. A similar situation occurs with finite posets, considering $(\mathbb{Q}, <)$ and the Fraïssé limit $(\mathbb{P}, <)$ of the class of all finite ordered posets. This class does not have the Ramsey property (cf [Sok10], [Sok12a]), but the corresponding flow is universal.

However, quite surprisingly, it is still unclear whether universality of $G \curvearrowright X^*$ implies Ramsey property of $\mathrm{Age}(\mathbf{F}^*)$ when $\mathbf{F}^*$ is a pure order Fraïssé expansion of $\mathbf{F}$. I believe that the answer should be negative, but was not able to construct any counterexample so far. In fact, results of Sokić (see [Sok10], [Sok11]) provide a positive answer in a number of cases.

5.5. **Universal minimal flows.** In this section, we put together results of the previous sections in order to derive the Kechris-Pestov-Todorcevic theorem on universal minimal flows for precompact expansions.

Definition. *Let $\mathcal{K}$ be a class of finite L-structures, and $\mathcal{K}^*$ an expansion of $\mathcal{K}$ in L^*. Say that the pair $(\mathcal{K}, \mathcal{K}^*)$ has the* relative Ramsey property for structures *when* *for every* $k \in \mathbb{N}$, $\boldsymbol{A} \in \mathcal{K}$ *and* $\boldsymbol{B}^* \in \mathcal{K}^*$, *there exists* $\boldsymbol{C} \in \mathcal{K}$ *such that for every coloring* $c : \binom{\boldsymbol{C}}{\boldsymbol{A}} \longrightarrow [k]$, *there exists* $b \in \binom{\boldsymbol{C}}{\boldsymbol{B}}_{\mathrm{Emb}}$ *such that:*

$$\forall \widetilde{\boldsymbol{A}}_0, \widetilde{\boldsymbol{A}}_1 \in \binom{\boldsymbol{B}}{\boldsymbol{A}} \ \left(\widetilde{\boldsymbol{A}}_0 \cong_{\boldsymbol{B}^*} \widetilde{\boldsymbol{A}}_1 \ \Rightarrow \ c(b(\widetilde{\boldsymbol{A}}_0)) = c(b(\widetilde{\boldsymbol{A}}_1))\right).$$

Above, $\widetilde{\boldsymbol{A}}_0 \cong_{\boldsymbol{B}^*} \widetilde{\boldsymbol{A}}_1$ *means that* $\boldsymbol{B}^* \upharpoonright \tilde{A}_0 \cong \boldsymbol{B}^* \upharpoonright \tilde{A}_1$.

Note that this property is formally weaker than the relative Ramsey property for embeddings, and is therefore, a priori, not sufficient in order to guarantee universality of the flow $G \curvearrowright X^*$. However, it *is* sufficient when combined with the expansion property:

Proposition 9 (NVT [NVT13b]). *Let* $\boldsymbol{F}$ *be a Fraïssé structure in* L, *and* $\boldsymbol{F}^*$ *a Fraïssé precompact expansion of* $\boldsymbol{F}$ *in* L^* *whose age consists of rigid structures. Assume that the pair* $(\mathrm{Age}(\boldsymbol{F}), \mathrm{Age}(\boldsymbol{F}^*))$ *has the relative Ramsey property for structures, and that* $\mathrm{Age}(\boldsymbol{F}^*)$ *has the expansion property relative to* $\mathrm{Age}(\boldsymbol{F})$. *Then* $\mathrm{Age}(\boldsymbol{F}^*)$ *has the Ramsey property.*

This result is useful in practice as from the existence of a precompact expansion with the relative Ramsey property for structures, it allows to construct a further precompact expansion with both the Ramsey and the expansion property. This is done by showing that going to a subclass, it is possible to satisfy the expansion property. Keeping in mind that the expansion property reflects minimality in some dynamical system, the result is maybe not so suprising, but more surprising is that there is currently no purely dynamical proof of that fact. Combining Theorem 11, Theorem 12 and Proposition 9, we obtain:

Theorem 13 (NVT [NVT13a]). *Let* $\boldsymbol{F}$ *be a Fraïssé structure, and* $\boldsymbol{F}^*$ *be a Fraïssé precompact relational expansion of* $\boldsymbol{F}$. *Assume that* $\mathrm{Age}(\boldsymbol{F}^*)$ *consists of rigid elements. The following are equivalent:*

i) The flow $G \curvearrowright X^*$ *is the universal minimal flow of* G.
ii) The class $\mathrm{Age}(\boldsymbol{F}^*)$ *has the Ramsey property as well as the expansion property relative to* $\mathrm{Age}(\boldsymbol{F})$.

Let us illustrate the previous result with the dense local order. Considering $G = \mathrm{Aut}(\mathbf{S}(2))$ and $G^* = \mathrm{Aut}(\mathbf{S}(2)^*)$, the universal minimal flow of G is, in virtue of Proposition 4 and Theorem 13, the action $G \curvearrowright X^*$, where $X^* := \overline{G \cdot (P_0^*, P_1^*)}$, the closure of $G \cdot (P_0^*, P_1^*)$ in $[2]^{\mathbf{S}(2)} \times [2]^{\mathbf{S}(2)}$. Moreover, it is possible to provide a concrete description of that action. In the unit circle $\mathbb{T}$, consider the set S supporting $\mathbf{S}(2)$, the set $(-S)$ of all its opposite points, and the set $C = \mathbb{T} \smallsetminus (S \cup (-S))$. Consider

$$\hat{\mathbb{T}} = C \cup ((S \cup (-S)) \times [2]).$$

Intuitively, it is obtained from the unit circle $\mathbb{T}$ by doubling the points in $S \cup (-S)$. Next, for $t \in \hat{\mathbb{T}}$, define $p(t)$ as the natural projection of t on $\mathbb{T}$, and for α, β in $S \cup (-S)$ so that $\alpha \xleftarrow{\mathbb{T}} \beta$, define $[\alpha, \beta]$ by:

$$[\alpha, \beta] := \{(\alpha, 0)\} \cup \{t \in \hat{\mathbb{T}} : \alpha \xleftarrow{\mathbb{T}} p(t) \xleftarrow{\mathbb{T}} \beta\} \cup \{(\beta, 1)\}.$$

This set is represented in Figure 5 (as the right part of the circle, together with the two black dots).

It turns out that sets of the form $[\alpha, \beta]$ form a basis of open sets for a topology on $\hat{\mathbb{T}}$, and the corresponding space is homeomorphic to X^*. This representation has

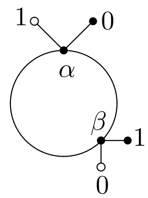

FIGURE 5. The set $[\alpha, \beta]$

at least two advantages. First, it is clear that X^* is homeomorphic to the Cantor space. Second, it allows to visualize pretty well the action of G on X^*, which is not so common when dealing with universal minimal flows. Using the same technique, a very similar result can be obtained for $\mathbf{S}(3)$.

Another remarkable instance where that happens is due to Pestov in [Pes98]. It deals with the orientation preserving homeomorphisms of $\mathbb{T}$, equipped with the pointwise convergence topology. That example provided the first known example of a metrizable, non-trivial, universal minimal flow, which is, in that case, the natural action on the circle by homeomorphisms.

It is worth noting that the cases of $\mathbf{S}(2)$ and $\mathbf{S}(3)$ are precisely the ones that motivated the introduction of precompact expansions. The purpose was to formulate a general framework for the Kechris-Pestov-Todorcevic correspondence to take place. As mentioned already when dealing with finite Ramsey-theoretic properties, it has then been gradually realized that after all, Fraïssé classes admitting a precompact Ramsey expansion may not be so rare. In view of Conjecture 1, recall that it is still unknown whether all Fraïssé classes in a finite language admit such an expansion. More generally, it is unknown whether every Fraïssé class where there are only finitely many nonisomorphic structures in every cardinality admits a precompact Ramsey expansion (Conjecture 1 reflects my view that it should be true).

5.6. **Interpolation of relatively extremely amenable pairs.** In this section, we come back to the notion of relative extreme amenability. In Section 5.4, we exhibited several examples of relatively extremely amenable pairs of groups (G, H) without H being extremely amenable. What made each pair relatively extremely amenable was the extreme amenability of G, but the same could have been achieved with any extreme amenable group sandwiched between H and G. This observation motivated the following definition:

Definition. *Let G be a topological group and $H \subset G$ a subgroup. An extremely amenable group E is an* extremely amenable interpolant *for the pair (G, H) if $H \subset E \subset G$.*

The question asking whether every relatively extremely amenable pair of groups admits such an interpolant is at the center of the paper [GNVT11]. Theorem 14 below will show that the answer is negative. This result is really due to Gutman. It is based on the following simple but crucial observation, which actually initiated the paper [GNVT11].

Proposition 10 (Gutman-NVT [GNVT11])**.** *Let G be a topological group and $H \subset G$, a subgroup. Then the following are equivalent:*

i) The pair (G, H) is relatively extremely amenable.
ii) $M(G)$ has an H-fixed point.

A simple argument then leads to:

Theorem 14 (Gutman-NVT [GNVT11])**.** *There exists a relatively extremely amenable pair (G, H) which which does not admit an extremely amenable interpolant, namely $(S_\infty, \mathrm{Aut}(\mathbb{Z}, <))$ (where we identify S_∞ with the permutation group of $\mathbb{Z}$).*

For pairs (G, G^*) of closed subgroups of S_∞ for which the quotient is precompact, the situation remains unclear, but the following result suggests that an interpolant may exist.

Theorem 15. *Let G, G^* be closed subgroups of S_∞, with G^* a closed subgroup of G. Assume that the quotient G/G^* is precompact, and that the pair (G, G^*) is relatively extremely amenable. Then G admits an extremely amenable subgroup G^{**} so that G/G^{**} is precompact.*

So far, the only proof of this result shares many common features with the proof of Theorem 10.7 from [KPT05]. The proofs of those results crucially make use of the same combinatorial argument, which heavily lies on the fact that G and G^* are closed subgroups of S_∞. However, it could still be that this hypothesis is unnecessary, and that the result actually holds for all Polish groups. There are two approaches to attack this problem. The first one would consist in removing the combinatorial content of the present proof via some use of dynamics. The other one would be to make use of continuous logic and metric Fraïssé structures, an approach that has already been adopted successfully by Melleray and Tsankov [MT11] in order to generalize the Kechris-Pestov-Todorcevic correspondence to all Polish groups (some recent work in collaboration with Melleray suggests that this latter approach should be successful).

5.7. **Metrizability of $M(G)$ and Ramsey precompact expansions.** In this section, I will present an unpublished result that indicates why precompact expansions should be the right framework to deal with the Kechris-Pestov-Todorcevic correspondence and with Conjecture 1. The question from where it originates is the following one. The Kechris-Pestov-Todorcevic theorem provides a characterization of Fraïssé classes $\mathcal{K}$ consisting of rigid elements and having the Ramsey property: they coincide with those whose group $G = \mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$ is extremely amenable, that is, for which the underlying space of the universal minimal flow $M(G)$ is as small as possible (a single point). Is there a similar characterization for those Fraïssé classes $\mathcal{K}$ almost having the Ramsey property, in the sense that they admit a Ramsey precompact expansion? Can such a property be translated in terms of smallness of the universal minimal flow? Theorem 13 already provides one part of the answer: if $\mathcal{K}$ admits a Ramsey precompact expansion consisting of rigid elements which also has the expansion property relative to $\mathcal{K}^*$, then $M(G)$ is metrizable (as the space X^* is metrizable). Being metrizable for a compact space is indeed a smallness condition (it is equivalent to being second-countable), and therefore appeared as a good candidate for the condition that we were looking for. In parallel, it also allowed to make the connection with another question, connected to metrizability of universal minimal flows. We mentioned already that the first example of a non-trivial, metrizable universal minimal flow was obtained in [Pes98]. Since then, several other examples were found, but all of them actually followed the same scheme: inside the group G under consideration, find a large extremely amenable closed subgroup H, where large means with precompact quotient G/H.

It was therefore quite reasonable to ask whether this is the only way to achieve metrizability of the universal minimal flow. The following result shows that metrizability does capture Ramsey precompact expansions, and that it indeed implies the existence of a large extremely amenable subgroup.

Theorem 16 (NVT-Tsankov [NVTT12])**.** *Let $\boldsymbol{F}$ be a Fraïssé structure, and let $G = \mathrm{Aut}(\boldsymbol{F})$. The following are equivalent:*

i) The structure $\boldsymbol{F}$ admits a Fraïssé precompact expansion $\boldsymbol{F}^*$ whose age has the Ramsey property and consists of rigid elements.
ii) The flow $G \curvearrowright M(G)$ is metrizable and has a generic orbit.
iii) The group G admits an extremely amenable closed subgroup G^* such that the quotient G/G^* is precompact.

Theorem 16 is the reason for which Conjecture 1 is only made for Fraïssé classes where there are only finitely many non isomorphic structures in every cardinality. Indeed, there are many known closed subgroups of S_∞ whose universal minimal flow is not metrizable (e.g. the countable discrete ones). Starting from those, the previous result produces some Fraïssé classes that do not have any precompact Ramsey expansion. For example, consider the structure $(\mathbb{Z}, d^{\mathbb{Z}}, <^{\mathbb{Z}})$ where $d^{\mathbb{Z}}$ and $<^{\mathbb{Z}}$ are the standard distance and ordering on $\mathbb{Z}$. Its automorphism group is $\mathbb{Z}$. Therefore, the corresponding age does not have any precompact Ramsey expansion (a fact which is actually easy to see directly).

Theorem 16 also allows to translate Conjecture 1 into purely dynamical terms. Call a closed subgroup of S_∞ *oligomorphic* when for every $n \in \mathbb{N}$, it induces only finitely many orbits on $\mathbb{N}^n$. Those groups are exactly the ones that appear as automorphism groups of Fraïssé structures whose age only has finitely many elements in every cardinality. Conjecture 1 then states that every closed oligomorphic subgroup of S_∞ should have a metrizable universal minimal flow with a generic orbit. At the moment, it is even possible that this should be true for a larger class of groups, called *Roelcke precompact*. A topological group is such when it is precompact with respect to the greatest lower bound of the left and right uniformities. For a closed subgroup of S_∞, being Roelcke precompact is equivalent to being an inverse limit of oligomorphic groups (see [Tsa12]), and it turns out that so far, all universal minimal flows coming from Roelcke precompact groups are metrizable with a generic orbit.

6. Open questions and perspectives

We close this memoir with a selection of open questions and perspectives related to the topics covered previously.

6.1. **Finite Ramsey theory.**

Concerning finite Ramsey theory, the first main open question is Conjecture 1 stating that every Fraïssé class where there are only finitely many nonisomorphic structures in every cardinality should have a precompact Ramsey expansion. It is an interesting one because of the change of point of view it would force on Ramsey classes (see discussion in Section 3.3). If one tries to tackle the problem in its full generality, the following, formally easier, question is also of interest:

Conjecture 2. *Let $\mathcal{K}$ be a Fraïssé class where there are only finitely many nonisomorphic structures in every cardinality. Then every element of $\mathcal{K}$ has a finite Ramsey degree.*

Conjecture 1 and Conjecture 2 are also attractive because it is possible to test them on a variety of classes of finite structures. Any new result, even if it only concerns a seemingly small class of objects, is potentially informative. In recent times, several contributions have been made in that direction, for example: [Jas13] by Jasiński, [Neš07] by Nešetřil, [Sok10], [Sok12a], [Sok12b] by Sokić and [DGMR11] by Dorais et al. Particular open problems about the Ramsey property include finite metric spaces with distances in some set S, Euclidean metric spaces (this problem appears in [KPT05]), projective Fraïssé classes (those are developed in [IS06] and are connected to Fraïssé classes of finite Boolean algebras) and equidistributed Boolean algebras (this problem appears in [KST12]). In view of those problems, some attention must be paid to the recent work [Sol10], [Sol12b] and [Sol12a] by Solecki. The papers [Sol10], [Sol12b] reprove and generalize some of the classical results from structural Ramsey theory, and [Sol12a] derives the main basic results from Ramsey theory (classical Ramsey, Graham-Rothschild, Hales-Jewett) thanks to a unified abstract framework.

Another question has to do with a conjecture of Thomas and was asked in the recent work of Bodirsky and Pinsker [BP11]. Following [BP11], let us say that a *reduct* of a relational structure $\mathbf{A}$ is a relational structure with the same domain as $\mathbf{A}$ all of whose relations can be defined by a first-order formula in $\mathbf{A}$. Thomas conjectured in [Tho91] that every Fraïssé relational structure $\mathbf{F}$ in a finite language only has finitely many reducts up to first-order interdefinability. Can anything be said if $\mathrm{Age}(\mathbf{F})$ consists of rigid elements and has the Ramsey property?

6.2. **Infinite Ramsey theory.** Concerning indivisibility, despite the various results (and in particular Sauer's results) that were mentioned in Section 4.1, there is still a lot to understand. In my opinion, metric spaces already provide a large class of structures on which many interesting problems can be studied. Of course, the Hilbert space still stands out in that program, but the search should by no means be restricted to it. Still, I would like to explicitly mention the following possible strengthening of Odell-Schlumprecht's theorem which appears in Section 4.1:

Question 1. *Let $N \in \mathbb{N}$. Is there a partition $\mathbb{S}^\infty = B \cup R$ and $\varepsilon > 0$ such that $\mathbb{S}^\infty \cap V \not\subset (B)_\varepsilon$ for every N-dimensional vector subspace V of ℓ_2 and $\mathbb{S}^\infty \cap W \not\subset (R)_\varepsilon$ for every infinite dimensional closed vector subspace W of ℓ_2? Equivalently, given a finite metric subspace $\boldsymbol{A}$ of $\mathbb{S}^\infty$, is there a partition $\mathbb{S}^\infty = B \cup R$ and $\varepsilon > 0$ such that $\boldsymbol{A}$ does not embed (isometrically) in $(B)_\varepsilon$, and $\mathbb{S}^\infty$ does not embed in $(R)_\varepsilon$?*

Concerning colorings of structures that are more complicated than points, even less is known and some very basic questions are still wide open.

Question 2. *Let $\boldsymbol{F}$ be a countable ultrahomogeneous (directed or undirected) graph. Does every element of $\mathrm{Age}(\boldsymbol{F})$ have a big Ramsey degree in $\mathrm{Age}(\boldsymbol{F})$? What about analogues of weak indivisiblity?*

In particular, what can be said when $\mathbf{F}$ is the K_n-free graph for some $n \geq 3$ or the countable generic poset?

6.3. **Dynamics.** The paper [KPT05] has recently been related to quite a number of promising developments. Several of them are directly coming from the Kechris-Pestov-Todorcevic correspondence. Those include the aforementioned works of Jasiński [Jas13], Nešetřil [Neš07], Sokić [Sok10], [Sok12a], [Sok12b] and Dorais et al. [DGMR11]. Some others take the correspondence to different contexts. It is the case for [Bar11] by Bartošová on structures that are not necessarily countable. It is also the case for the paper [MT11] by Melleray-Tsankov who show how it can be transferred to the so-called metric Fraïssé structures. What is interesting here is that the equivalence between Ramsey property and extreme amenability of non-Archimedean Polish groups becomes an equivalence between an approximate version of the Ramsey property and extreme amenability of *all* Polish groups. This equivalence actually captures some prior result obtained by Pestov in [Pes02], but because of the lack of technique to prove the approximate Ramsey property, it has not led to any practical result so far. Still, the parallel between classical and metric Fraïssé theory seems worth investigating.

More generally, the combinatorial translation of dynamical facts performed in [KPT05] opens a variety of perspectives connected to combinatorics and dynamics. For example, the usual notion of amenability can actually be studied via two different approaches. The first one has to do with universal minimal flows, since a topological group is amenable if and only if its universal minimal flow admits an invariant Borel probability measure. As a direct consequence, S_∞, the automorphism group of the countable random graph or the isometry group of the rational Urysohn space are amenable, but the automorphism groups of the countable atomless Boolean algebra or of the countable generic poset are not (see [KS12] by Kechris and Sokić). The second approach relative to amenability consists in expressing it directly in combinatorial terms using a "convex" version of the Ramsey property. This was done recently and independently by Moore in [Moo11] and by Tsankov (private communication). This approach did not lead to any concrete result so far, probably because nobody has really tried to develop techniques in direction of the convex Ramsey property.

Amenability is also connected to another combinatorial condition called the Hrushovski poperty. A Fraïssé class $\mathcal{K}$ of finite structures satisfies the Hrushovski property when for every $\mathbf{A} \in \mathcal{K}$, there exists $\mathbf{B} \in \mathcal{K}$ containing $\mathbf{A}$ so that every isomorphism between finite substructures of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$. It is proved by Kechris and Rosendal in [KR07] that the Hrushovski property translates nicely at the level of automorphism groups. Namely, it is equivalent to the fact that there is an increasing sequence of compact groups whose union is dense in $\mathrm{Aut}(\mathbf{F})$, where $\mathbf{F} = \mathrm{Flim}(\mathcal{K})$. Therefore, the Hrushovski property for $\mathcal{K}$ implies the amenability of $\mathrm{Aut}(\mathbf{F})$. It is also central in the study of other properties of Polish groups like the small index property, the automatic continuity property, and the existence of ample generics (see for example [KR07], [Sol05] or more recently [Kec12]). Nevertheless, there are still very natural classes of structures for which the Hrushovski property is not known to hold (e.g. the class of all finite tournaments) and those provide good, potentially difficult, combinatorial problems.

Still in connection with amenability, the paper [AKL12] has recently pointed out an intriguing fact: every known case of non-Archimedean Polish group G which is amenable and has a metrizable universal minimal flow turns out to be *uniquely ergodic*, in the sense that every compact minimal G-flow has a unique invariant Borel probablity measure (which is then necessarily ergodic). The question of knowing whether this always holds is open. Unique ergodicity also appears to be connected to new combinatorial phenomena, such as the uniqueness of so-called *consistent random orderings* or a quantitative version of the expansion property.

Last, it seems that a number of classical dynamical notions can be studied in the context of non-Archimedean Polish groups. For example, consider the class of proximal flows. For those flows, there is a natural notion of fixed point property, called *strong amenability*, which implies amenability (see for example Glasner [Gla76]). It turns out that some closed subgroups of S_∞ have that property. For example, the group $\mathrm{Aut}(\mathbb{Q}, B)$ is such, where B is the betweenness relation associated to $(\mathbb{Q}, <)$. As a result, the strongly proximal universal minimal flow of S_∞ is the natural action of S_∞ on the compact space of all betweenness relations on $\mathbb{N}$. Therefore, it makes sense to study whether an strategy similar to [KPT05] can be applied in the context of proximal flows. More generally, topological dynamics is full of many other natural classes of compact flows admitting universal minimal objects (e.g. equicontinuous flows, distal flows, almost periodic flows,...see [dV93]). Each of them is a potential candidate for an analog of [KPT05], with potential new combinatorial and dynamical phenomena.

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