Robustness and stability of nonlinear systems : a homogeneous point of view

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Contents

Notations & Usual definitions

Set theory and topology

- $\mathbb{N}$ is the set of natural integers $\{0, 1, \ldots\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.
- $\mathbb{R}$ is the field of real numbers and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.

In all this document, the integer $n \in \mathbb{N}^*$ represents the dimension of the state space $\mathbb{R}^n$.

- $|x|$ denotes the absolute value of $x \in \mathbb{R}$; $\|x\|$ denotes the Euclidian norm of $x \in \mathbb{R}^p$. If A is a linear mapping from $\mathbb{R}^p$ to $\mathbb{R}^q$, $\|A\|$ denotes the operator norm, that is $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$.
- $B(x, r)$ is the Euclidian open ball centered in x of radius r.
- $\bar{B}(x, r)$ is the Euclidian closed ball centered in x of radius r.
- If A is a subset of $\mathbb{R}^n$, we denote by $\mathring{A}$ the interior of A, that is the biggest open subset of A.
- Similarly, we denote by $\bar{A}$ the closure of A, that is the smallest closed set containing A.
- The boundary of A is defined by $\partial A = \bar{A} \setminus \mathring{A}$.
- The set $A \subset \mathbb{R}^n$ is convex iff for any $a, b \in A$ and for any $\lambda \in [0, 1]$, we have $\lambda a + (1 - \lambda) b \in A$.
- If $A \subset \mathbb{R}^n$, the convex hull of A, denoted $\mathrm{conv}(A)$, is defined as the smallest (in the sense of inclusion) convex set containing A. We denote also $\overline{\mathrm{conv}}(A)$ the closure of the convex hull of A. We recall that a compact set has a compact convex hull.
- If A and B are bounded subsets of $\mathbb{R}^n$, the distance between A and B is defined by $\mathrm{d}(A, B) = \inf\{|a - b|, a \in A, b \in B\}$.

Functions & Vector fields

- We call *function* any Lebesgue measurable mapping from $\mathbb{R}^n$ to $\mathbb{R}$.
- We call *vector field* any Lebesgue measurable mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$.
- For any $q \in \mathbb{N}^*$, if $\mathcal{U} \subset \mathbb{R}^n$ is an open set and $k \in \mathbb{N}$, $\mathcal{C}^k(\mathcal{U}, \mathbb{R}^q)$ denotes the set of mappings from $\mathcal{U}$ to $\mathbb{R}^q$ having continuous differential up to the order k. We also define $\mathcal{C}^\infty(\mathcal{U}, \mathbb{R}^q) = \cap_{k\in\mathbb{N}}\mathcal{C}^k(\mathcal{U}, \mathbb{R}^q)$. A mapping $f : \mathcal{U} \to \mathbb{R}^q$ is said to be of class k iff it belongs to $\mathcal{C}^k(\mathcal{U}, \mathbb{R}^q)$;
- If $f : \mathbb{R}^p \to \mathbb{R}^q$ is a differentiable mapping, we denote $d_x f$ its differential at the point $x \in \mathbb{R}^p$.
- If V is a differentiable function and f is a vector field, we denote $\mathcal{L}_f V = d_x V\, f(x)$ the Lie derivative of V along f.
- If f and g are differentiable vector fields, we denote $[f, g]$ the Lie bracket of f and g defined by $[f, g](x) = d_x g.f(x) - d_x f.g(x)$.
- For any $p, q \in \mathbb{N}^*$, if $\mathcal{U} \subset \mathbb{R}^p$ is an open set, $\mathcal{L}^\infty_{\mathsf{loc}}(\mathcal{U}, \mathbb{R}^q)$ denotes the set of locally essentially bounded measurable mapping from $\mathcal{U}$ to $\mathbb{R}^q$. For any $d \in \mathcal{L}^\infty_{\mathsf{loc}}(\mathcal{U}, \mathbb{R}^q)$ and any compact set $K \subset \mathcal{U}$, we denote $\|d\|_K = \operatorname{ess\,sup}_{z\in K} |d(z)|$.
- For any $p, q \in \mathbb{N}^*$, if $\mathcal{U} \subset \mathbb{R}^p$ is an open set, $\mathcal{L}^\infty(\mathcal{U}, \mathbb{R}^q)$ denotes the set of (globally) essentially bounded measurable mapping from $\mathcal{U}$ to $\mathbb{R}^q$. For any $d \in \mathcal{L}^\infty(\mathcal{U}, \mathbb{R}^q)$, we denote $\|d\|_\infty = \operatorname{ess\,sup}_{z\in\mathcal{U}} |d(z)|$.
- A function V is *positive definite*, denoted by $V \succeq 0$, if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. A function V is *negative definite* if $-V \succeq 0$.
- A mapping $\varphi : \mathbb{R}^p \to \mathbb{R}^q$ is proper if for any compact set $K \subset \mathbb{R}^q$, $\varphi^{-1}(K)$ is a compact set of $\mathbb{R}^p$.
- A multivalued map F from a set A to a set B, denoted $F : A \rightrightarrows B$, is a map $F : A \to \mathcal{P}(B)$, where $\mathcal{P}(B)$ denotes the set of subsets of B.

Miscellaneous

- For any $\alpha \geq 0$ and $x \in \mathbb{R}$, $\lfloor x\rceil^\alpha = \operatorname{sign}(x)|x|^\alpha$.

- The set $\mathcal{K}$ is defined as the set of strictly increasing continuous functions $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha(0) = 0$.
- We also define $\mathcal{K}_\infty = \{\alpha \in \mathcal{K} : \lim_{r \to +\infty} \alpha(r) = +\infty\}$.
- The set $\mathcal{KL}$ is defined as the set of continuous functions $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that:
$$\begin{cases} \forall t \in \mathbb{R}_+ & s \mapsto \beta(s,t) \in \mathcal{K}_\infty \\ \forall s \in \mathbb{R}_+ & t \mapsto \beta(s,t) \text{ is decreasing and } \lim_{t \to +\infty} \beta(s,t) = 0 \end{cases}.$$
- The identity mapping of a set E is denoted I_E. If the set E is clear from the context, we simply denote it I.
- The vector space $\mathbb{R}^n$ is endowed with the Borel σ-algebra and the Lebesgue measure. We denote $\mathcal{N}$ the set of all zero measure subsets of $\mathbb{R}^n$.
- If Φ is a diffeomorphism, Φ^* denotes the pullback by Φ. See Appendix B.

Abbreviations

Here is the list of the abbreviations used in this document. Those with a † are defined in details in Appendix A.

DI : Differential Inclusion.

FTS† : Finite-Time Stable.

GAS† : Globally Asymptotically Stable.

GFTS† : Globally Finite-Time Stable.

LAS† : Locally Asymptotically Stable.

LAT† : Locally ATtractive.

ODE : Ordinary Differential Equation.

SPI† : Strictly Positively Invariant.

General introduction

Homogeneous*, from Ancien Greek* $O\mu o\gamma \varepsilon \nu \eta \varsigma$*,*
"of the same race, family or kind",
from $O\mu o\varsigma$*, "same" and* $\gamma \varepsilon \nu o\varsigma$*, "kind".*

The *homogeneity* is an intrinsic property of an object on which the flow of a particular vector field operates as a scaling. This definition, rather simple, entails a lot of qualitative properties for a homogeneous object, and is of particular interest in view of stability purposes.

The study of the stability or the asymptotic stability of a dynamical system is a central problem in the control theory. Given that the equations of a system are very often impossible to integrate explicitely, indirect methods have to be used for getting qualitative properties. Even though the results of Kurzweil [Kurzweil 1963] and Clarke [Clarke 1998] prove the equivalence of the asymptotic stability and the existence of a smooth Lyapunov function, finding such a Lyapunov function may be a very difficult task. Qualitative results not involving the computation of a Lyapunov function are therefore of a great interest. This is why the homogeneity theory has been developed and used in control theory: the rigid properties of homogenous systems simplify the study of the stability and give sufficient conditions for deriving it.

The literature on the homogeneity theory is vast and detailed. A lot of theoretical and practical results have been proved in the last decades, and used in different context. The first Chapter of this work is devoted to a state of the art about homogeneity. The usual context of homogeneous systems as well as their main features are recalled. In that Chapter, we present the three steps of the definition of homogeneity. The classical definition, going back to Euler and his homogeneous function theorem, is very common in mathematics as well as in control theory. The weighted definition, firstly introduced in control theory by Zubov and Hermes, was extensively studied by Khomenuk, Kawski and Rosier afterwards. It is nowadays the most widely known and used notion of homogeneity.

The geometric definition is the last step of the homogeneity theory, giving it a unified and coordinate-free framework. The main results of the theory are stated in the first Chapter, among which various ways of checking homogeneity, the theorem of Rosier (homogeneous converse second Lyapunov's theorem), the equivalence between local attractiveness and global stability, the link between a negative degree of homogeneity and the finite-time stability, and the Hermes' theorem (homogeneous extension of the first Lyapunov's theorem). Finally this Chapter gives a quick introduction to the theory of local homogeneity.

The next Chapters are devoted to extensions, adaptations and applications of the usual theory. This work focuses on two main aspects of the homogeneity theory: first, deriving stability properties from the homogeneity or the local homogeneity; second, proving that the homogeneity of a system provides some useful robustness properties. The results are split into Chapters 2 to 5.

In Chapter 2, we extend the existing results on the homogenization of a nonlinear system. This technique has been already defined and studied in the framework of weighted homogeneity. We focus here on an extension to a more general setting of the geometric homogeneity. The advantages of the proposed extension include the coordinate-free definition, allowing us to define an approximation that is perserved under a change of coordinates, and its larger range of applicability due to a more general definition of geometric homogeneity. The main approximation theorems are extended and academic examples of use are given.

The third Chapter develops a theoretical framework for defining geometric homogeneity of discontinuous systems and/or systems described by a differential inclusion. Few results already exist in this direction; we propose a unified theoretical framework based on the geometric homogeneity. We show that the proposed definition is consistent with respect to the Filippov's regularization procedure. Then we give extensions of well-known qualitative properties of homogeneous systems, which have been presented herebefore. First, the converse homogeneous Lyapunov Theorem (Rosier's Theorem) is extended using the result of Clarke, Ledyaev and Stern about the existence of a Lyapunov pair for an asymptotically stable differential inclusion under standard assumptions. This allows us to link negative degree of homogeneity and finite-time stability. The equivalence between local attractiveness and global stability is also proved to hold.

Even though a nominal system is homogeneous, in applications the perturbations and unmodelled dynamics cannot be avoided. That is why Chapter 4 is devoted to a study of robustness properties of (weighted) homogeneous or homogenizable systems. We adress the question of the *input-to-state* and *integral input-to-state stability* property of homogeneous systems. In order to do that, we consider two different assumptions. The first one is algebraic and consists in the homogeneity of the system with respect to the perturbation. This assumption, although appearing at first sight to be very strong, is in fact tractable since a nonlinear change of coordinates can be performed on the perturbation. The second assumption is more analytic: we consider that the diffence between the perturbed and the nominal systems is bounded in an appropriate way. Both assumptions lead to a type of robustness linked to the degree of homogeneity. The results are compared to each other, and finally extended to the more general setting of homogenizable systems.

In the fifth Chapter, we study the example of the double integrator system. This system is very important in practice (in mechanics, electrical engineering...) since a lot of systems have a nominal form of the double integrator. However, in some applications, the usual exponential convergence is not sufficient. Our aim is hence to synthesize a family of (homogeneous) continuous finite-time stabilizing output feedbacks. The proposed algorithm is somehow between the linear control and the purely discontinuous control (twisting algorithm). The former does not achieve finite-time convergence, but the latter presents some chattering effects that are often an issue in practical situations. The proposed method is a mix between the preceding two, and displays some of their main features. Thereafter, we study the robustness of the closed loop system with respect to perturbations and the impact of the discretization by using techniques developed before. Simulations conclude the theoretical study of this system and illustrate its behavior.

Finally, a conclusion summarizes the results presented therein and proposes some ongoing or future works. Three appendices are given afterwards, the first two recalling classical definitions and results used in the document, and the third one presenting another result not linked with the topic of this document, but done in parallel and published in [Bernuau 2013d].

Chapter 1

Homogeneity

Contents

1.1 Introduction

Homogeneity has a long standing history. In the classical sense, a mapping is homogeneous if it maps an argument scaled by a given constant to the image of that argument, scaled by the same constant at a fixed power, called the degree. This property has been the subject of a huge amount of works, because it holds for a lot of very common mathematical objects, like linear mappings or norms. One of the most interesting property of homogeneous objects is that the scaling operation allows us to compare the behavior at any point with the behavior at a corresponding point on the sphere. This fact can be used for instance to reduce the dimension of a problem or to obtain symmetry properties, like the homogeneous function Theorem of Euler. The symmetry properties of the homogeneous polynomials were first studied by Euler and then more deeply during the nineteenth century, in view of projective geometry, algebraic geometry or in number theory. The classical homogeneity was also used to investigate stability properties

[Malkin 1952, Krasovskii 1963, Hahn 1967, Rothschild 1976, Goodman 1976] and a particular attention was paid for polynomial systems [Dayawansa 1989].

The first generalization of the classical homogeneity was introduced in Control Theory independently by V. I. Zubov [Zubov 1958] and H. Hermes [Hermes 1986, Hermes 1991b]. The main idea is to replace the classical scaling by a slightly more general transformation, namely a dilation. Indeed, each coordinate is scaled by the same constant but with different powers, called *weights*. This leads to a *weighted dilation* written under the form

$$\Lambda_{\mathbf{r}}(\lambda) : x \in \mathbb{R}^n \mapsto (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n) \in \mathbb{R}^n,$$

for $\lambda > 0$, where $r_i > 0$ are the weights and $\mathbf{r} = [r_1, \dots, r_n]$ is a *generalized weight*. Such a dilation leads to the extended notion called *weighted homogeneity*. The degrees of freedom given by the weights allow us to see much more objects as homogeneous.

Such homogeneity property was naturally considered looking at a local approximation of nonlinear systems: small time local controllability and local asymptotic stability is shown to be inherited by the original nonlinear system if this property holds for the homogeneous approximation [Hermes 1991b, Kawski 1988]. With this property, many results were obtained for stability, feedback stabilization [Kawski 1990, Kawski 1991b, Hermes 1995, Sepulchre 1996] or output feedback stabilization [Andrieu 2008]. Another very important result was obtained independently by Zubov [Zubov 1958] and Rosier [Rosier 1992a]: if a continuous homogeneous system is globally asymptotically stable, then there exists a homogeneous proper Lyapunov function.

This notion was also used in different contexts: switched systems [Orlov 2005b], self-triggered systems [Anta 2008, Anta 2010], time delay systems [Efimov 2011], control and analysis of oscillations [Efimov 2010]. Since investigation of the finite-time stability in [Haimo 1986] many papers were devoted to this concept (e.g. [Bhat 2000]) and its link with the homogeneity: the finite-time property is obtained if the system is locally asymptotically stable and homogeneous of a negative degree (see [Bhat 1998, Bacciotti 2005]). This result is exploited in [Bhat 1997, Orlov 2005a, Bhat 2005] and with application to controllers design in [Bhat 1998, Hong 2002b], observers design in [Perruquetti 2008, Shen 2008, Menard 2010], and output feedback in [Hong 2002a].

Extensions were given for vector fields of a degree of homogeneity that is a function of the state [Praly 1997] and to homogeneity in the bi-limit [Andrieu 2008], which provides homogeneous approximations at the origin and at infinity. The local homogeneity concept has been also introduced in [Efimov 2010]. These tools were useful for nonlinear observer and output feedback design. Let us finally mention the extensions provided by [Orlov 2005b] and [Levant 2005], defining weighted homogeneity for differential inclusions and making a short study of their properties.

In addition to all these works on weighted homogeneity, another approach has been considered. Since the weighted homogeneity is based on a dilation that is dependent on the coordinates, then there exist vector fields which are homogeneous for some coordinates, but not using other ones. In [Bacciotti 2005], it is even proved that the system

$$\begin{cases} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & -x_1 - x_1^2 x_2 \end{cases}$$

is globally asymptotically stable, but no change of coordinates could put it in a form for which the homogeneous approximation associated with any dilation is asymptotically stable. Moreover, seeing dilations as a one-parameter group, homogeneity can easily be stated without coordinates. This leads to a *geometric* definition of homogeneity. The very first geometric definitions appeared independently in [Khomenuk 1961, Kawski 1990] and [Rosier 1993]. The paper of S. Bhat and D. Bernstein [Bhat 2005] is written in this context, and proves a lot of theoretical results about homogeneous systems in the geometric sense. Let us also mention [Anta 2010], where the geometric homogeneity is used for self-triggered systems.

In this Chapter, we shall present the basics on the homogeneity theory. In the Section 1.2, we shall present the classical homogeneity and the related properties. The Section 1.3 will be devoted to the weighted homogeneity and the Section 1.4 to the geometric homogeneity. Each of these two sections will be an extension of the preceding. Finally, some extensions of the main theory shall be presented in Section 1.5. The contents of this Chapter have been submitted in a survey [Bernuau 2013b].

1.2 Standard homogeneity

Definition 1.1. *[Hahn 1967] Let n and m be two positive integers. A mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be homogeneous of degree $\kappa \in \mathbb{R}$ in the classical sense iff*

$$\forall \lambda > 0 : f(\lambda x) = \lambda^\kappa f(x).$$

Note that no regularity assumption is made on the mapping f. Let us see some examples.

- The function
$$x = (x_1, x_2) \mapsto \begin{cases} \frac{x_1^3+x_2^3}{x_1^2+x_2^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
is homogeneous of degree 1 and continuous, but it is not linear.

- The function defined by:
$$(x_1, x_2) \mapsto \begin{cases} \frac{\lfloor x_1 \rceil^{1/2} + \lfloor x_2 \rceil^{1/2}}{x_1+x_2} & \text{if } x_1 + x_2 \neq 0 \\ 0 & \text{else} \end{cases},$$
is homogeneous of degree $-\frac{1}{2}$ and not continuous.

There exists a necessary and sufficient condition for homogeneity.

Proposition 1.2 (Euler's Theorem for classical homogeneity)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable mapping. Then f is homogeneous of degree κ iff for all $i \in \{1, \ldots, m\}$*
$$\sum_{j=1}^{n} x_j \frac{\partial f_i}{\partial x_j}(x) = \kappa f_i(x), \qquad \forall x \in \mathbb{R}^n.$$

Let us mention that the regularity of a homogeneous mapping f is related to its degree:

- if $\kappa < 0$ then f is either discontinuous (at the origin) or the zero vector field;
- if $0 \leq \kappa < 1$ then either the Lipschitz condition is not satisfied by f at 0 or f is constant.

These conditions are necessary but not sufficient.

We will be particularly interested in homogeneous systems, e.g. systems like
$$\dot{x} = f(x) \tag{1.1}$$
where the vector field f is homogeneous. Let us consider some examples.

- Let $A \in \mathbb{R}^{n \times n}$ and $f(x) = Ax$. Then f is homogeneous of degree 1. Note that the flow of f, $x \mapsto \exp(At)x$ is homogeneous as well.

- The scalar vector field $f(x) = -\mathrm{sign}\,(x)$ is homogeneous of degree 0. For any $x_0 \in \mathbb{R}$, denote $x(t)$ (resp. $x_\lambda(t)$) a solution of $\dot{x} = -\mathrm{sign}\,(x)$ with initial condition $x(0) = x_0$ (resp. $x_\lambda(0) = \lambda x_0$). We have

$$x(t) = \begin{cases} \mathrm{sign}\,(x_0)\,(|x_0| - t) & t \in [0, |x_0|] \\ 0 & t > |x_0| \end{cases}$$

thus $x_\lambda(\lambda t) = \lambda x(t)$.

These examples lead us to the following proposition.

Proposition 1.3. *[Zubov 1964, Hahn 1967] Assume that the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree κ. For any solution $x(t)$ of (1.1) and for all $\lambda > 0$, the curve $t \mapsto \lambda x(\lambda^{\kappa-1} t)$ is a solution of $\dot{x} = f(x)$.*

If the system (1.1) admits a (semi-)flow $\Psi^t(x)$, we have

$$\lambda \Psi^{\lambda^{\kappa-1} t}(x) = \Psi^t(\lambda x). \tag{1.2}$$

Taking advantage of the Proposition 1.3, we can now state stability results.

Theorem 1.4. *[Krasovskii 1963] Consider the homogeneous system (1.1) with a continuous vector field f and with forward uniqueness of solutions. If the origin is a locally attractive equilibrium, then the origin is globally asymptotically stable.*

Theorem 1.5. *[Krasovskii 1963] Consider the homogeneous system (1.1) with a continuous vector field f. Then the origin is globally asymptotically stable iff there exists a homogeneous, proper and continuous function $V \in \mathcal{C}^1(\mathbb{R}^n \setminus \{0\})$, s.t. V is positive definite and $\dot{V}$ are negative definite.*

Corollary 1.6. *[Malkin 1952, Krasovskii 1963] Let $f_1, \ldots, f_p$ be continuous homogeneous vector fields with degree $\kappa_1 < \kappa_2 < \ldots < \kappa_p$ and denote $f = f_1 + \cdots + f_p$. Assume moreover that $f(0) = 0$. If the origin is globally asymptotically stable under f_1 then the origin is locally asymptotically stable under f.*

See the book [Hahn 1967] for more details. The preceeding results have been stated with the assumption of continuity of the vector field and the first one with the additional hypothesis of forward uniqueness of solutions. We will see in the Chapter 3 that the first hypothesis may be significantly weakened while the second may be dropped.

1.3 Weighted homogeneity

Let us first formally state the basic definitions of weighted homogeneity that have been evoked in the introduction.

Definition 1.7. *A* generalized weight *is a n-tuple* $\mathbf{r} = [r_1, \dots, r_n]$ *with* $r_i > 0$*. The* dilation *associated to the generalized weight* $\mathbf{r}$ *is the action of the group* $\mathbb{R}_+ \setminus \{0\}$ *on* $\mathbb{R}^n$ *given by:*

$$\begin{array}{rrcl} \Lambda_{\mathbf{r}}: & \mathbb{R}_+ \setminus \{0\} \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n, \\ & (\lambda, x) & \longmapsto & \Lambda_{\mathbf{r}}(\lambda)x = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n)^T. \end{array}$$

Remark 1.8. *Let us emphasize some facts.*

- *The definition of the dilation is coordinate-dependant. In all this work, homogeneity will be used for functions and vector fields defined on a vector space* $\mathbb{R}^n$*, with a positive integer* n*. For the sake of simplicity, we will always assume that the chosen basis is the canonical basis of* $\mathbb{R}^n$*, unless otherwise stated.*

- *In Chapter 4, we will sometimes allow the weights* r_i *to be non-negative.*

Definition 1.9. *[Zubov 1958, Hermes 1986] Let* $\mathbf{r}$ *be a generalized weight.*

- *A function* φ *is said to be* $\mathbf{r}$*-homogeneous of degree* κ *iff for all* $x \in \mathbb{R}^n$ *and all* $\lambda > 0$ *we have* $\lambda^{-\kappa}\varphi(\Lambda_{\mathbf{r}} x) = \varphi(x)$*;*

- *A vector field* f *is said to be* $\mathbf{r}$*-homogeneous with degree* κ *iff for all* $x \in \mathbb{R}^n$ *and all* $\lambda > 0$ *we have* $\lambda^{-\kappa}\Lambda_{\mathbf{r}}^{-1} f(\Lambda_{\mathbf{r}} x) = f(x)$*;*

- *The system (1.1) is* $\mathbf{r}$*-homogeneous iff* f *is so.*

Let us stress the links and the differences with the classical homogeneity. To begin with, taking $\mathbf{r} = [1, \dots, 1]$, we see that the dilation associated to $\mathbf{r}$ is $\Lambda_{\mathbf{r}}(\lambda) = \lambda I$. Hence a function φ is $\mathbf{r}$-homogeneous of degree κ iff $\varphi(\lambda x) = \lambda^\kappa \varphi(x)$, and we see that the Definitions 1.1 and 1.9 coincide. However, a vector field f is $\mathbf{r}$-homogeneous of degree κ iff $f(\lambda x) = \lambda^\kappa \lambda f(x)$. We see here a gap in the degrees of the two definitions: a vector field is homogeneous in the classical sense of degree κ iff it is $\mathbf{r}$-homogeneous of degree $\kappa - 1$. For instance, every linear vector field is $\mathbf{r}$-homogeneous of degree 0.

Remark 1.10. *A vector field* f *is* $\mathbf{r}$*-homogeneous of degree* κ *iff each coordinate function* f_i *is* $\mathbf{r}$*-homogeneous of degree* $\kappa + r_i$*.*

Obviously, some objects are weighted homogeneous without being homogeneous in the classical sense. Let us see some examples.

Example 1.11.
- *The function $\varphi : x \mapsto x_1 + x_2^2$ is $[2,1]$-homogeneous of degree 2;*
- *Let $\alpha_1, \ldots, \alpha_n$ be strictly positive. Consider the n-integrator system:*

$$\begin{cases} \dot{x}_1 & = & x_2, \\ \vdots & & \vdots \\ \dot{x}_{n-1} & = & x_n, \\ \dot{x}_n & = & \sum_i k_i \lfloor x_i \rceil^{\alpha_i}. \end{cases}$$

The system is $\mathbf{r}$-homogeneous of degree κ with $\mathbf{r} = [r_1, \ldots, r_n]$ iff the following relations hold:

$$\begin{cases} r_i & = & r_n + (i-n)\kappa, & \forall i \in \{1, \ldots, n\}, \\ r_i\alpha_i & = & r_n + \kappa, & \forall i \in \{1, \ldots, n\}. \end{cases}$$

Let us fix $r_n = 1$. We easily see that this assumption forces κ to be greater than -1. The equations become:

$$\begin{cases} r_i & = & 1 + (i-n)\kappa, & \forall i \in \{1, \ldots n\}, \\ \alpha_i & = & \frac{1+\kappa}{1+(i-n)\kappa}, & \forall i \in \{1, \ldots n\}. \end{cases}$$

If $\kappa = -1$, then the vector field defining the system is discontinuous on each coordinate axis. If $\kappa = 0$, then we recover a chain of integrators of n^{th}-order with a linear state feedback. This example will be treated with more details in Chapter 5 for the case $n = 2$.

Remark 1.12. *The generalized weight defining the homogeneity of a function or a vector field is not unique. Indeed, an object is $\mathbf{r}$-homogeneous of degree κ iff it is $\alpha\mathbf{r}$-homogeneous of degree $\alpha\kappa$ for all $\alpha > 0$. Let us also stress that some systems can be $\mathbf{r}$-homogeneous for different generalized weights $\mathbf{r}$ that are not colinear. For instance, the system $\dot{x} = x$ on $\mathbb{R}^n$ is $\mathbf{r}$-homogeneous (of degree 0) for any generalized weight $\mathbf{r}$.*

Let us check how the properties of classical homogeneity from Section 1.2 are extended into the framework of weighted homogeneity.

Proposition 1.13. *[Zubov 1958] Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathbf{r}$-homogeneous vector field of degree κ. For any solution $x(t)$ of (1.1) and for all $\lambda > 0$, the curve $t \mapsto \Lambda_{\mathbf{r}}(\lambda)x(\lambda^\kappa t)$ is a solution of (1.1).*

If the system (1.1) admits a (semi-)flow $\Psi^t(x)$, we have

$$\Lambda_{\mathbf{r}}\Psi^{\lambda^{\kappa}t}(x) = \Psi^t(\Lambda_{\mathbf{r}}x). \tag{1.3}$$

The Proposition 1.13 is the natural extention of the Proposition 1.3 in the framework of classical homogeneity.

Let us see now the Euler theorem.

Proposition 1.14 (Euler's Theorem for weighted homogeneity)**.** *[Zubov 1958] Let φ be a differentiable function. Then φ is $\mathbf{r}$-homogeneous of degree κ iff*

$$\sum_{j=1}^{n} r_j x_j \frac{\partial \varphi}{\partial x_j}(x) = \kappa\varphi(x), \qquad \forall x \in \mathbb{R}^n.$$

Let f be a differentiable vector field. Then f is $\mathbf{r}$-homogeneous of degree κ iff for all $i \in \{1, \ldots, n\}$

$$\sum_{j=1}^{n} r_j x_j \frac{\partial f_i}{\partial x_j}(x) = (k + r_i)f_i(x), \qquad \forall x \in \mathbb{R}^n.$$

Theorems 1.4 and 1.5 remain true in the weighted homogeneity framework [Zubov 1958] (in Russian), [Rosier 1992a]. The homogeneous converse Lyapunov theorem has been proved independently by Zubov and Rosier.

Theorem 1.15. *Let f be a continuous $\mathbf{r}$-homogeneous vector field. If the origin is a globally asymptotically equilibrium of f, then there exists a $\mathbf{r}$-homogeneous Lyapunov function for f of class C^1.*

Let us set other results, which are fundamental in the study of finite-time stability.

Theorem 1.16. *[Bhat 1997] Let f be a continuous $\mathbf{r}$-homogeneous vector field of degree $\kappa < 0$ with forward uniqueness of solutions. If the origin is a locally attractive equilibrium of (1.1), then the origin is globally finite-time stable (FTS).*

Corollary 1.17. *[Bhat 1997] Let $f_1, \ldots, f_p$ be continuous homogeneous vector fields of degrees $k_1 < k_2 < \ldots < k_p$ and denote $f = f_1 + \cdots + f_p$. Assume moreover that $f(0) = 0$. If the origin is globally asymptotically stable under f_1 then the origin is locally asymptotically stable under f. Moreover, if the origin is FTS under f_1 then the origin is FTS under f.*

Let us finally introduce a useful tool in the study of homogeneous systems.

Definition 1.18. *Let* $\mathbf{r}$ *be a generalized weight. A function* N *is said to be a* $\mathbf{r}$-homogeneous norm *if the three following assumptions hold:*

1. *N is positive definite;*

2. *N is $\mathbf{r}$-homogeneous of degree 1;*

3. *N is continuous.*

The following function gives us an example of a $\mathbf{r}$-homogeneous norm:

$$\|x\|_{\mathbf{r}} = (\sum_{i=1}^{n} |x_i|^{\frac{\rho}{r_i}})^{\frac{1}{\rho}}, \quad \rho = \prod_{i=1}^{n} r_i. \tag{1.4}$$

Lemma 1.19. *Let* $\mathbf{r} = [r_1, \dots r_n]$ *be a generalized weight and* N *be a* $\mathbf{r}$*-homogeneous norm. We denote* $r_{\min} = \min\{r_1, \dots, r_n\}$ *and* $r_{\max} = \max\{r_1, \dots, r_n\}$*. Then for all* $x \in \mathbb{R}^n$ *the following inequality holds*

$$\alpha_- v_-(N(x)) \leq \|x\| \leq \alpha_+ v_+(N(x)),$$

where α_- *and* α_+ *are positive constants and* v_- *and* v_+ *are class* $\mathcal{K}_\infty$ *functions defined by*

$$v_-(s) = \begin{cases} s^{r_{\max}} & \text{if } s \leq 1 \\ s^{r_{\min}} & \text{if } s \geq 1; \end{cases}$$

$$v_+(s) = \begin{cases} s^{r_{\min}} & \text{if } s \leq 1 \\ s^{r_{\max}} & \text{if } s \geq 1. \end{cases}$$

Proof. The inequality obviously holds for $x = 0$. Let us consider $x \neq 0$. There exists $y \in \{z \in \mathbb{R}^n : N(z) = 1\}$ such that $x = \Lambda_{\mathbf{r}}(\lambda) y$ with $\lambda = N(x)$. If $\lambda \geq 1$ we have

$$\|x\| = \|\Lambda_{\mathbf{r}}(\lambda) y\| = \left(\sum_i \lambda^{2r_i} y_i^2 \right)^{1/2},$$

which leads to

$$\begin{aligned} \lambda^{r_{\min}} \left(\sum_i y_i^2 \right)^{1/2} &\leq \|x\| \leq \lambda^{r_{\max}} \left(\sum_i y_i^2 \right)^{1/2} \\ \lambda^{r_{\min}} \|y\| &\leq \|x\| \leq \lambda^{r_{\max}} \|y\|. \end{aligned}$$

Setting $\alpha_- = \inf\{\|y\| : N(y) = 1\}$ and $\alpha_+ = \sup\{\|y\| : N(y) = 1\}$, we get

$$\alpha_- N(x)^{r_{\min}} \leq \|x\| \leq \alpha_+ N(x)^{r_{\max}}.$$

If $\lambda \leq 1$, we get similarly $\alpha_- N(x)^{r_{\max}} \leq \|x\| \leq \alpha_+ N(x)^{r_{\min}}$ which completes the proof. $\square$

1.4 Geometric homogeneity

As we have seen, the definition of the weighted homogeneity is based on a particular choice of a basis in the construction of the dilation and is therefore coordinate-dependent. For instance, consider the following vector field:

$$(x_1 + x_2 - x_1^2)\frac{\partial}{\partial x_1} + (x_2 + x_1^2 + 2x_1x_2 - 2x_1^3)\frac{\partial}{\partial x_2}.$$

According to the previous definitions, it is not weighted homogeneous. But setting $z = x_2 - x_1^2$, this vector field becomes:

$$(x_1 + z)\frac{\partial}{\partial x_1} + z\frac{\partial}{\partial z},$$

and in this form, the vector field is $(1, 1)$-homogeneous.

Since we are particularly looking at stability properties, a coordinate-free definition should be of a great interest. Let us take a look at the definition of weighted homogeneity. For a given $x \in \mathbb{R}^n$, the set $(\Lambda_{\mathbf{r}} x)_{\lambda>0}$ is a curve on $\mathbb{R}^n$. An object is homogeneous iff its variations along these curves reduces to the dilation and a scaling. With this point of view, the homogeneity property should be invariant under a change of coordinates if these curves are encoded in a geometric object.

This remark has been done very early in the development of the homogeneity theory [Khomenuk 1961, Kawski 1990, Rosier 1993]. The basic idea is to consider a vector field ν and to replace the curves $(\Lambda_{\mathbf{r}} x)_{\lambda>0}$ by the integral curves of ν. Even though this idea is widely shared in the literature, the authors do not always agree on the specific assumptions on this vector field. The definition we shall take here is not the most general, but allows us to translate the main properties of the two preceding sections into the geometric framework.

Definition 1.20. *[Kawski 1991a] A vector field $\nu \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ is said to be* Euler *if it is complete and if the origin is a GAS equilibrium of $-\nu$. We will always write Φ the flow* *of ν, that is $\Phi^s(x)$ is the current state at time s of the trajectory of ν starting from x at $s = 0$.*

Definition 1.21. *Let ν be an Euler vector field. A function φ or a vector field f is said to be ν-homogeneous of degree κ iff for all $s \in \mathbb{R}$ we have*[1]*:*

$$(\Phi^s)^*\varphi = e^{\kappa s}\varphi, \qquad (\Phi^s)^* f = e^{\kappa s} f. \tag{1.5}$$

Consider a generalized weight $\mathbf{r} = [r_1, \ldots, r_n]$ and the vector field $\nu = \sum_i r_i x_i \frac{\partial}{\partial x_i}$. It is straightforward to check that $\mathbf{r}$-homogeneity is equivalent to ν-homogeneity, and hence the weighted homogeneity is a particular case of the geometric homogeneity in a fixed basis.

We will be particularly interested in the homogeneity for a vector field. Since the definition of the geometric homogeneity needs to compute the flow Φ, it will be a difficult task in general. Nevertheless, there exist equivalent conditions for a vector field f to be homogeneous assuming regularity properties on f.

Proposition 1.22. *[Kawski 1995] Assume that the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is ν-homogeneous of degree κ. For any solution $x(t)$ of (1.1) and for all $s \in \mathbb{R}$, the curve $t \mapsto \Phi^s(x(e^{\kappa s}t))$ is a solution of (1.1).*

If the system (1.1) admits a (semi-)flow $\Psi^t(x)$, we have

$$\Phi^s \circ \Psi^{e^{\kappa s}t} = \Psi^t \circ \Phi^s. \tag{1.6}$$

The Proposition 1.22 is the natural extension of the Propositions 1.3 and 1.13 in the framework of classical homogeneity. The Euler's Theorem also admits an extension, that justifies the name "Euler vector field".

Proposition 1.23 (Euler's Theorem for geometric homogeneity)**.** *[Kawski 1995]*

Let φ be a differentiable function. Then φ is ν-homogeneous of degree κ iff

$$\mathcal{L}_\nu \varphi = \kappa\varphi.$$

Let f be a differentiable vector field. Then f is ν-homogeneous of degree κ iff

$$[\nu, f] = \kappa f.$$

[1] $(\Phi^s)^*$ denotes the pullback by the diffeomorphism Φ^s, see Appendix B.

Remark 1.24. *In [Rosier 1993], the property $[\nu, f] = \kappa f$ is the definition of a* symmetry*. This definition is based on the classical theory of partial differential equations as set in [Olver 1986]. The properties (1.5) and (1.6) are then seen as consequences of this definition in [Rosier 1993].*

The Theorems 1.4, 1.5, 1.16 and 1.17 remain true in the geometric homogeneity framework [Bhat 2005, Rosier 1992a]. The geometric homogeneous version of the Theorem 1.5 is often referred as the *Theorem of Rosier*. Some other results were proved in the geometric homogeneity framework and obviously hold for classical and weighted homogeneities.

Theorem 1.25. *[Bhat 2005] Consider a homogeneous continuous vector field f with forward uniqueness of solutions. If there exists a SPI compact set for f then f is GAS.*

The definition of a homogeneous norm remains unchanged for geometric homogeneity. However, the existence of such a homogeneous norm is not trivial.

Proposition 1.26. *Let ν be an Euler vector field. Then there exists a ν-homogeneous norm.*

Proof. Set $f = -\nu$. Since $[\nu, f] = 0$, f is ν-homogeneous of degree 0. By Theorem 3.22, there exists a continuous Lyapunov function of degree κ for any $\kappa > 0$. Take $\kappa = 1$. The obtained function N is definite positive, ν-homogeneous of degree 1 and continuous. □

Another natural question is to wonder whether a given Euler vector field ν corresponds to a weight $\mathbf{r}$ up to a change of coordinates.

Proposition 1.27. *Let ν be an Euler vector field. There exist coordinates $x_1, \ldots, x_n$ in which $\nu = \sum_i r_i x_i \frac{\partial}{\partial x_i}$ iff there exists ν-homogeneous functions $\varphi_1, \ldots, \varphi_n \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ of degree $r_1, \ldots, r_n$ such that rank$(d_0\varphi_1, \ldots, d_0\varphi_n) = n$.*

Proof. Assume that there exist coordinates $x_1, \ldots, x_n$ in which $\nu = \sum_i r_i x_i \frac{\partial}{\partial x_i}$. Then the functions $\varphi_i(x) = x_i$ are appropriate choices.

Conversely, assume that there exist ν-homogeneous functions $\varphi_1, \ldots, \varphi_n \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ such that $\operatorname{rank}(d_0\varphi_1, \ldots, d_0\varphi_n) = n$. Set $\Upsilon(x) = (\varphi_1(x), \ldots, \varphi_n(x))^T$. Let us prove first that Υ is a global diffeomorphism. The rank of the differential of Υ at 0 is

$$\operatorname{rank}(d_0\varphi_1, \ldots, d_0\varphi_n)^T = n.$$

The functions φ_i being of class $\mathcal{C}^1$, this rank is locally constant: there exists a neighborhood $\mathcal{U}$ of the origin such that for all $x \in \mathcal{U}$, $\operatorname{rank}(d_x\varphi_1, \ldots, d_x\varphi_n)^T = n$. By the inverse function Theorem, Υ is a local diffeomorphism. Now, let us show that Υ is injective. The condition $\Upsilon(x) = \Upsilon(y)$ is equivalent to $\varphi_i(x) = \varphi_i(y)$ for all i. Consider a ν-homogeneous norm N. Being continuous and homogeneous, it is proper (see [Bhat 2005]). Therefore the set $B = \{z \in \mathbb{R}^n : N(z) \leq 1\}$ is compact, and there exists $s \in \mathbb{R}$ such that $\Phi^s(B) = \{z \in \mathbb{R}^n : N(z) \leq e^s\} \subset \mathcal{U}$. There exists also $\sigma \in \mathbb{R}$ such that $\Phi^\sigma(x) \in \Phi^s(B)$ and $\Phi^\sigma(y) \in \Phi^s(B)$. Hence $\varphi_i(x) = \varphi_i(y)$ gives $e^{r_i\sigma}\varphi_i(x) = e^{r_i\sigma}\varphi_i(y)$ and $\varphi_i(\Phi^\sigma(x)) = \varphi_i(\Phi^\sigma(y)) \in \mathcal{U}$. Then $\Upsilon(\Phi^\sigma(x)) = \Upsilon(\Phi^\sigma(y))$ leads to $\Phi^\sigma(x) = \Phi^\sigma(y)$ because Υ is a diffeomorphism on $\mathcal{U}$, and finally Υ is injective on $\mathbb{R}^n$. We conclude by the Theorem of global inversion that Υ is a global diffeomorphism, and a straightforward verification shows that $\nu = \sum_i r_i\varphi_i\frac{\partial}{\partial\varphi_i}$. □

Remark 1.28. *Let us mention that such functions do not always exist. First, note that the conditions of the Proposition imply that the coefficients r_i are non zero, since a continuous homogeneous function of degree 0 is constant and therefore cannot verify the rank condition.*

Considering now a norm 1 vector x and a scalar $\alpha > 0$, we have:

$$\begin{aligned}
\mathcal{L}_\nu\varphi_i(\alpha x) &= r_i\varphi_i(\alpha x) \\
d_{\alpha x}\varphi_i\,\nu(\alpha x) &= r_i\varphi_i(\alpha x) \\
d_{\alpha x}\varphi_i\left(\nu(\alpha x) - \nu(0)\right) &= r_i\left(\varphi_i(\alpha x) - \varphi_i(0)\right) \\
d_{\alpha x}\varphi_i\left(d_0\nu\,(\alpha x) + o(\alpha)\right) &= r_i\left(d_0\varphi_i(\alpha x) + o(\alpha)\right) \\
d_{\alpha x}\varphi_i\left(d_0\nu\,x + o(1)\right) &= r_i\left(d_0\varphi_i x + o(1)\right).
\end{aligned}$$

Letting now $\alpha \to 0$, we get

$$d_0\varphi_i\,d_0\nu\,x = r_i d_0\varphi_i x.$$

Being true for any x in the unit sphere, the equality implies

$$d_0\varphi_i\,d_0\nu = r_i d_0\varphi_i,$$

that is $d_0\varphi_i^T$ is an eigenvector of $d_0\nu^T$ with associated eigenvalue r_i. Finally, the conditions of the Proposition imply that $d_0\nu^T$ admits n independant eigenvectors, that is, $d_0\nu^T$ is diagonizable. This is obviously not always true.

Let us finally mention that an example of an Euler vector field that does not reduce to a dilation is given in the Example 5.9 p. 194 of [Bacciotti 2005].

1.5 Extensions

Even though the successive improvements of the definitions of homogeneity allow us to see much more functions and vector fields as homogeneous, the interest of the theory is not only restricted to homogeneous objects. Sometimes, a homogeneous vector field can be seen as an approximation of non-homogeneous one, and some qualitative properties of the approximation hold for the approximated vector field. The first result in this direction is due to Hermes [Hermes 1991a], with an assumption of forward uniqueness of solutions and then generalized by Rosier in [Rosier 1992a] without this assumption. This theorem is a generalization of the Theorem of linearization of Lyapunov.

Theorem 1.29 (Hermes' Theorem)**.** *[Rosier 1992a] Let f be a continuous vector field and $\mathbf{r}$ be a generalized weight. Assume that there exists a continuous $\mathbf{r}$-homogeneous vector field h of degree κ such that*

$$\sup_{\|x\|=1} \|\lambda^{-\kappa}\Lambda_{\mathbf{r}}(\lambda)^{-1}f(\Lambda x) - h(x)\| \underset{\lambda\to 0}{\longrightarrow} 0.$$

If the origin is a GAS equilibrium for h, then it is a LAS equilibrium for f.

Remark 1.30. *The Hermes' theorem implies the first part of the Corollary 1.17.*

Following this idea, [Andrieu 2008] developed a theory on homogeneous approximations in the weighted homogeneous framework. To the approximation around the origin is added an approximation at the infinity; considering both lead to the notion of bilimit homogeneity. We refer to [Andrieu 2008] for more details. Let us also mention that similar ideas appear in [Zubov 1958]. In [Efimov 2010], these ideas are extended for getting approximation not only at the origin and at infinity.

Definition 1.31. *[Andrieu 2008, Efimov 2010] For a generalized weight $\mathbf{r}$, let us denote $\|\cdot\|_{\mathbf{r}}$ the $\mathbf{r}$-homogeneous norm defined by (1.4) and $\mathbb{S}_{\mathbf{r}} = \{x \in \mathbb{R}^n : \|x\|_{\mathbf{r}} = 1\}$.*

A function φ is $(\mathbf{r}, \lambda_0, \eta)$-homogeneous of degree *$\kappa \in \mathbb{R}$, with a generalized weight $\mathbf{r}$, $\lambda_0 \in [0, +\infty]$ and a $\mathbf{r}$-homogeneous function η of degree κ if we have:*

$$\sup_{x\in\mathbb{S}_{\mathbf{r}}} |\lambda^{-\kappa}\varphi(\Lambda_r(\lambda)x) - \eta(x)| \underset{\lambda\to\lambda_0}{\longrightarrow} 0.$$

A vector field f is $(\mathbf{r}, \lambda_0, h)$-homogeneous of degree *$\kappa \in \mathbb{R}$, with a generalized weight*

$\mathbf{r}$, $\lambda_0 \in [0, +\infty]$ *and a* $\mathbf{r}$*-homogeneous vector field* h *of degree* κ *if we have:*

$$\sup_{x \in \mathbb{S}_{\mathbf{r}}} \|\lambda^{-\kappa} \Lambda_{\mathbf{r}}(\lambda)^{-1} f(\Lambda_r(\lambda)x) - h(x)\| \underset{\lambda \to \lambda_0}{\longrightarrow} 0.$$

The Hermes' theorem can be easily reformulated in this setting. There exists also a similar result using the local homogeneity at infinity.

Theorem 1.32. *[Andrieu 2008] Let the vector field* f *be* ($\mathbf{r}$,$+\infty$,h)*–homogeneous with a continuous* h*. If the origin is (globally) asymptotically stable for the system* $\dot{x} = h(x)$*, then there exists an invariant compact set* $K \subset \mathbb{R}^n$ *containing the origin that is globally asymptotically stable for the system* $\dot{x} = f(x)$*.*

1.6 Conclusion

In this Chapter, we have seen the well-known definitions of the classical, weighted, geometric and local homogeneities. We have seen the main results of the theory, among which the Theorem of Rosier, that is a converse homogeneous Lyapunov theorem, the equivalence of local attractiveness and global stability for homogeneous systems and the Hermes' Theorem, that is a homogeneous extension of the first Theorem of Lyapunov.

This Chapter is the starting point of our work. In the Chapter 2, we shall extend the local homogeneity theory defined in Section 1.5 in the geometric setting. In the Chapter 3, we shall extend the geometric setting defined in Section 1.4 to discontinuous systems defined by a differential inclusion. In the Chapter 4, we shall see how the homogeneous systems behave under perturbations, and study their robustness. The Chapter 4 will be formulated in the weighted homogeneity framework defined in Section 1.3. Finally, the Chapter 5 will be an application of the preceding Chapters to the output stabilization of the double integrator.

Chapter 2

Coordinate-free transition from global to local homogeneity

Contents

2.1 Introduction

Do non-homogeneous objects exist? Even if this question may seem trivial, it is actually a bit delicate. Indeed, the geometric homogeneity allows us to see a lot of objects as homogeneous. Let us take a look at vector fields. It is obvious that a homogeneous vector field has no isolated equilibrium except the origin. However, in a lot of applications, global stabilization is achieved. Let us then restrict ourselves to vector fields with a unique possible equilibrium, namely the origin. Since a vector field always commutes with itself, we might think that any such vector field is eventually homogeneous. However, it is worth to stress that the Euler vector field defining homogeneity is specific: that is the main difference between homogeneity theory and symmetry theory. Restricting ourselves to complete $\mathcal{C}^1$ vector fields, we get:

Lemma 2.1. *The origin is GAS for a complete $\mathcal{C}^1$ vector field iff the origin is LAT for this vector field and there exists a Euler vector field ν for which this vector field is ν-homogeneous.*

When dealing with global stabilization of smooth systems, the geometric homogeneity approach may hence be applied. This lemma, despite its appearance of generality, gives us in return a way of designing a non-homogeneous vector field.

Example 2.2. *[Hahn 1967] Let us consider the following vector field on $\mathbb{R}^2$:*

$$\frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)}\frac{\partial}{\partial x_1} + \frac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)}\frac{\partial}{\partial x_2}.$$

It is shown in [Hahn 1967] that this vector field is globally attractive but unstable. If this vector field were ν-homogeneous for a given Euler vector field ν, attractiveness would imply stability. Thus for any Euler vector field ν, this vector field is not ν-homogeneous.

This example shows that the class of homogeneous vector fields, even though very large, will not be able to manage all the issues for the global stabilization of nonlinear systems. This setting is somehow similar to the linear setting. Indeed, many systems are linear, but the applicability of the linear systems approach becomes wider due to linearization. The same scheme can be used with homogeneity. If a vector field or a function fails to be homogeneous, sometimes we can compute a local homogeneous approximation of this object.

The study of a homogeneous approximation has a long history. Basically, to study a problem, the idea is to find another problem which is more easily solvable and which approximates in some sense the first problem. In the context of the stability theory, the method is to compute an approximation of a given vector field and to deduce the local stability of the initial system from the stability of the approximation. The second Theorem of Lyapunov may be consider to be the first attempt of formalizing this method: from the asymptotic stability of the linear approximation we deduce the local asymptotic stability of the nonlinear system.

Zubov [Zubov 1958] and Rosier [Rosier 1992a] proved independently an extension of this result for the weighted homogeneity. With this property, many results were obtained for stability/stabilization [Kawski 1990, Kawski 1991b, Hermes 1995, Sepulchre 1996] or output feedback [Andrieu 2008]. This notion was also used in different contexts: switched systems [Orlov 2005b], self-triggered systems [Anta 2010], control and analysis of oscillations [Efimov 2010], time delay systems [Efimov 2011]. Extensions were provided for the vector fields which are homogeneous of degrees of homogeneity that are functions of the state [Praly 1997] and to homogeneity in the bi-limit [Andrieu 2008], which makes the homogeneous approximation valid both at the origin and at infinity. The local homogeneity concept has been also introduced in [Efimov 2010]. These tools were useful for nonlinear observer and output feedback design [Menard 2013].

In this chapter, our aim is to extend the applicability of these homogeneous approximations by considering geometric homogeneity. First, we shall define the homogeneous approximation and its basic properties. Then we shall see conditions under which the approximation can be computed more easily. Finally, we shall recast the theorems of approximations in this setting and use them to treat examples. The contents of this Chapter have been submitted in [Bernuau 2013a].

2.2 Local homogeneous approximation

We recall that the functions and vector fields under consideration are assumed to be defined on $\mathbb{R}^n$. We assume moreover in all this Chapter that they are merely continuous, unless stronger regularity assumptions are explicitely stated.

Following [Andrieu 2008], we may now define the local approximation. The following definitions uses the uniform convergence on compact sets recalled in Appendix B.

Definition 2.3. *Let φ and η be functions and let f and h be vector fields.*

- *The function η is the ν-homogeneous approximation of degree κ at 0 of the function*

φ if:

$$e^{-\kappa s} \left(\Phi^s\right)^* \varphi \underset{s\to-\infty}{\overset{CUC}{\longrightarrow}} \eta. \tag{2.1}$$

- *The vector field h is the ν-homogeneous approximation of degree κ at 0 of the vector field f if:*

$$e^{-\kappa s} \left(\Phi^s\right)^* f \underset{s\to-\infty}{\overset{CUC}{\longrightarrow}} h. \tag{2.2}$$

If the uniform convergence is taken when $s \to +\infty$, we get the approximation at ∞.

Proposition 2.4.

- *Let φ be a function. Assume that the ν-homogeneous approximation of degree κ at 0 or at ∞ of φ exists and is denoted by η. Then η is a continuous ν-homogeneous function of degree κ.*

- *Let f be a vector field. Assume that the ν-homogeneous approximation of degree κ at 0 or at ∞ of f exists and is denoted by h. Then h is a continuous ν-homogeneous vector field of degree κ.*

Proof. We shall prove the proposition for a function and for homogeneous approximation at 0, the others cases being similar. Fix $x_0 \in \mathbb{R}^n$ and let us prove the continuity of η at x_0. Pick an $\varepsilon > 0$. The uniform convergence property gives:

$$\exists \sigma \in \mathbb{R} : \forall s \leq \sigma \, \forall x \in \bar{B}(x_0, 1) \quad |e^{-\kappa s} \left(\Phi^s\right)^* \varphi(x) - \eta(x)| \leq \varepsilon.$$

The function $e^{-\kappa\sigma} \left(\Phi^\sigma\right)^* \varphi$ is clearly continuous, hence there exists a neighborhood $\mathcal{U}$ of x_0, which we can choose contained in $\bar{B}(x_0, 1)$, such that:

$$\forall x \in \mathcal{U} \quad |e^{-\kappa\sigma} \left(\Phi^\sigma\right)^* \varphi(x) - e^{-\kappa\sigma} \left(\Phi^\sigma\right)^* \varphi(x_0)| \leq \varepsilon.$$

Finally for all $x \in \mathcal{U}$:

$$\begin{aligned} |\eta(x) - \eta(x_0)| \;\leq\; & |\eta(x) - e^{-\kappa\sigma} \left(\Phi^\sigma\right)^* \varphi(x)| \\ & + |e^{-\kappa\sigma} \left(\Phi^\sigma\right)^* \varphi(x) - e^{-\kappa\sigma} \left(\Phi^\sigma\right)^* \varphi(x_0)| \\ & + |e^{-\kappa\sigma} \left(\Phi^\sigma\right)^* \varphi(x_0) - \eta(x_0)| \\ \leq\; & 3\varepsilon. \end{aligned}$$

It remains to prove that η is ν-homogeneous of degree κ. For any $\sigma \in \mathbb{R}$ and $x \in \mathbb{R}^n$, we have:

$$\begin{aligned}
e^{-\kappa\sigma}\left(\Phi^{\sigma}\right)^{*}\eta(x) &= e^{-\kappa\sigma}\left(d_x\Phi^{\sigma}\right)^{-1}\eta(\Phi^{\sigma}(x)) \\
&= e^{-\kappa\sigma}\left(d_x\Phi^{\sigma}\right)^{-1}\lim_{s\to-\infty} e^{-\kappa s}\left(d_{\Phi^{\sigma}(x)}\Phi^{s}\right)^{-1}\varphi(\Phi^{\sigma+s}(x)) \\
&= \lim_{s\to-\infty} e^{-\kappa(\sigma+s)}\left(d_x\Phi^{\sigma}\right)^{-1}\left(d_{\Phi^{\sigma}(x)}\Phi^{s}\right)^{-1}\varphi(\Phi^{\sigma+s}(x)) \\
&= \lim_{s\to-\infty} e^{-\kappa(\sigma+s)}\left(d_x\Phi^{\sigma+s}\right)^{-1}\varphi(\Phi^{\sigma+s}(x)) \\
&= \lim_{s\to-\infty} e^{-\kappa(\sigma+s)}\left(\Phi^{\sigma+s}\right)^{*}\varphi(x) \\
&= \eta(x).
\end{aligned}$$

□

The following proposition shows that the uniform convergence on compact sets can be replaced with another property which is easier to check.

Proposition 2.5. *Let B be a compact subset of $\mathbb{R}^n$ such that the origin is in the interior of B.*

- *Let φ be a function. Assume that there exists a ν-homogeneous function η of degree κ such that $e^{-\kappa s}\left(\Phi^{s}\right)^{*}\varphi$ converges to η uniformly on B when $s\to-\infty$. Then η is the ν-homogeneous approximation of φ at 0 of degree κ.*

- *Let f be a vector field. Assume that there exists a ν-homogeneous vector field h of degree κ such that $e^{-\kappa s}\left(\Phi^{s}\right)^{*}f$ converges to h uniformly on B when $s\to-\infty$. Then h is the ν-homogeneous approximation of f at 0 of degree κ.*

Proof. We only give the proof for a vector field. Let K be a compact set. Since ν is Euler, there exists $\sigma\in\mathbb{R}$ such that $K\subset\Phi^{\sigma}(B)$. We have for all $y\in K$:

$$\begin{aligned}
\|e^{-\kappa s}\left(\Phi^{s}\right)^{*}(f-h)(y)\| &\leq \sup_{y\in\Phi^{\sigma}(B)}\|e^{-\kappa s}\left(\Phi^{s}\right)^{*}(f-h)(y)\| \\
&= \sup_{x\in B}\|e^{-\kappa s}\left(\Phi^{s}\right)^{*}(f-h)(\Phi^{\sigma}(x))\| \\
&= \sup_{x\in B}\|e^{\kappa\sigma}d_x\Phi^{\sigma}e^{-\kappa(s+\sigma)}\left(\Phi^{s+\sigma}\right)^{*}(f-h)(x)\| \\
&\leq C\sup_{x\in B}\|e^{-\kappa(s+\sigma)}\left(\Phi^{s+\sigma}\right)^{*}(f-h)(x)\|,
\end{aligned}$$

where $C=e^{\kappa\sigma}\sup_{x\in B}\|d_x\Phi^{\sigma}\|>0$. Finally, we have:

$$\sup_{y\in K}\|e^{-\kappa s}\left(\Phi^{s}\right)^{*}(f-h)(y)\| \underset{s\to-\infty}{\longrightarrow} 0$$

and h is the local approximation of f at 0. □

Example 2.6. *Consider the scalar vector field $\nu = \frac{x}{1+|x|}\frac{\partial}{\partial x}$. We want to compute the ν-approximation around 0 of the vector field $f = x\frac{\partial}{\partial x}$. The problem is that here we are not able to find a explicit expression of the flow of ν. However, we have:*

$$x\frac{\partial}{\partial x} = \left(\frac{x}{1+|x|} + |x|\frac{x}{1+|x|}\right)\frac{\partial}{\partial x}$$

and we find that:

$$(\Phi^s)^* f = \left(\frac{x}{1+|x|} + |\Phi^s(x)|\frac{x}{1+|x|}\right)\frac{\partial}{\partial x}.$$

Therefore for all $x \in B(0,1)$:

$$|\left(\Phi^s\right)^* f - \nu| \leq |\Phi^s(x)|,$$

and hence the uniform convergence of the flow of ν on the ball proves that the vector field ν is the ν-homogeneous approximation of f of degree 0 at 0.

When dealing with homogeneity, we often want to check a property on the unit sphere and extend it everywhere by homogeneity. The following proposition shows that, under mild conditions, we can check that a homogeneous function or vector field is the homogeneous approximation of a given function or vector field only on the sphere. In [Bacciotti 2005], the definition used for homogeneous approximation only takes the uniform convergence on the unit sphere, instead of taking it on all compact sets. This proposition shows also that these approaches match under some conditions.

Proposition 2.7. *Let us denote $\mathbb{S}$ the unit sphere of $\mathbb{R}^n$, λ_{min} the smallest real part of the eigenvalues of $d_0\nu$. Let φ be a function and η be a ν-homogeneous function of degree κ; let f be a vector field and h be a ν-homogeneous vector field of degree κ.*

- *If $\kappa > 0$ and if $\sup_{x\in\mathbb{S}} |e^{-\kappa s}(\varphi - \eta)(\Phi^s(x))| \to 0$ when $s \to -\infty$ then η is the ν-homogeneous approximation of degree κ at 0 of φ.*
- *If $\kappa > -\lambda_{min}$ and if $\sup_{x\in\mathbb{S}} \|e^{-\kappa s}\left(d_x\Phi^s\right)^{-1}(f-h)(\Phi^s(x))\| \to 0$ when $s \to -\infty$ then h is the ν-homogeneous approximation of degree κ at 0 of f.*

Proof. We will only prove the second point, the first being an easy adaptation. In this proof, we work in a fixed basis of $\mathbb{R}^n$ and we identify linear mappings with their corresponding matrices in this basis. Next we aim at comparing the speed of convergence of

$e^{-\kappa s}\left(d_x\Phi^s\right)^{-1}$ and $(f-h)(\Phi^s(x))$. To do that, we will first find a differential equation verified by $d_x\Phi^s$, and then use it to compare the speeds by an approximation of $d_{\Phi^s(x)}\nu$ by $d_0\nu$.

1. A look at the identity $\frac{d}{ds}\left(\Phi^s\right)^*\frac{\partial}{\partial x_i}=\left(\Phi^s\right)^*\left[\nu,\frac{\partial}{\partial x_i}\right]$ gives:

$$\frac{d}{ds}\left(d_x\Phi^s\right)^{-1}=-\left(d_x\Phi^s\right)^{-1}d_{\Phi^s(x)}\nu.$$

   Differentiating the identity $\left(d_x\Phi^s\right)\left(d_x\Phi^s\right)^{-1}=I$ leads to the desired differential equation:

$$\frac{d}{ds}\left(d_x\Phi^s\right)=d_{\Phi^s(x)}\nu\; d_x\Phi^s.$$

2. Let us now consider a given compact set $K\subset\mathbb{R}^n$ and prove that:

$$\sup_{x\in\mathbb{S},w\in K}\|e^{\kappa s}d_x\Phi^s w\|\underset{s\to-\infty}{\longrightarrow}0. \tag{2.3}$$

   Since $\kappa+\lambda_{min}>0$, the matrix $-d_0\nu-\kappa I$ is Hurwitz, and thus there exists a matrix $P=P^T>0$ and $\gamma>0$ such that:

$$P(d_0\nu+\kappa I)+(d_0\nu+\kappa I)^TP>\gamma P.$$

   Since the mapping

$$x\mapsto P(d_x\nu+\kappa I)+(d_x\nu+\kappa I)^TP-\gamma P$$

   is continuous, there exists a neighborhood $\mathcal{U}$ of the origin such that for all $x\in U$:

$$P(d_x\nu+\kappa I)+(d_x\nu+\kappa I)^TP>\gamma P.$$

   There exists $s_0\in\mathbb{R}$ such that for all $s<s_0$, $\Phi^s(\mathbb{S})\subset\mathcal{U}$. Therefore, for all $x\in\mathbb{S}$ and for all $s<s_0$:

$$P(d_{\Phi^s(x)}\nu+\kappa I)+(d_{\Phi^s(x)}\nu+\kappa I)^TP>\gamma P.$$

   Fix now a vector $w\in K$ and $x\in\mathbb{S}$. Set $y(s)=e^{\kappa s}d_x\Phi^s w$. We have:

$$\frac{d}{ds}y(s)=\left(d_{\Phi^s(x)}\nu+\kappa I\right)y(s).$$

If we denote $V(y) = y^T P y$, we have:

$$\frac{d}{ds}(V(y(s))) = y(s)^T \left(P(d_{\Phi^s(x)}\nu + \kappa I) + (d_{\Phi^s(x)}\nu + \kappa I)^T P\right) y(s).$$

Then for $s < s_0$

$$\frac{d}{ds}(V(y(s))) \geq \gamma V(y(s)),$$

and thus

$$V(y(s)) \leq V(y(s_0))e^{\gamma(s-s_0)}.$$

Given that

$$V(y(s_0)) \leq e^{2\kappa s_0} \sup_{x\in\mathbb{S}} \|d_x\Phi^{s_0}\|^2 \mu \|w\|^2,$$

with μ the biggest eigenvalue of P, there exists a constant C, independant of $x \in \mathbb{S}$, such that $\|y(s)\| \leq Ce^{\gamma s/2}\|w\|$. We conclude that $\|y(s)\| \to 0$ when $s \to -\infty$, uniformly on $x \in \mathbb{S}$, that is (2.3).

3. Let us prove that $f(0) = h(0)$. We still denote K a compact subset of $\mathbb{R}^n$. Assume that $0 \notin K$. For all $r > 0$, there exists $s_r \in \mathbb{R}$, such that for all $s < s_r$, for all $w \in K$:

$$\|e^{-\kappa s} \left(d_x\Phi^s\right)^{-1} w\| > r.$$

Set $x \in \mathbb{S}$ and denote $v = (f - h)(0)$. By continuity, $(f - h)(\Phi^s(x)) \to v$ when $s \to -\infty$. Hence, if $f(0) \neq h(0)$, there exists $s_1 \in \mathbb{R}$ such that for all $s \leq s_1$, $(f - h)(\Phi^s(x)) \in \bar{B}(v, |v|/2)$. Therefore:

$$e^{-\kappa s} \left(d_x\Phi^s\right)^{-1} (f - h)(\Phi^s(x)) \to \infty,$$

which is a contradiction. This proves that $f(0) = h(0)$ and hence:

$$0 = e^{-\kappa s} \left(d_0\Phi^s\right)^{-1} (f - h)(\Phi^s(0)) \underset{s\to-\infty}{\longrightarrow} 0.$$

4. Consider now a compact set L. We want to prove that:

$$\sup_{z\in L} \|e^{-\kappa s} \left(\Phi^s\right)^* (f - h)(z)\| \underset{s\to-\infty}{\longrightarrow} 0.$$

Since L is compact, there exists $\sigma_0 \in \mathbb{R}$ such that

$$L \subset \{0\} \cup \bigcup_{\sigma \leq \sigma_0} \Phi^\sigma(\mathbb{S}).$$

Let z be a non zero element of L. There exists $\sigma \leq \sigma_0$ and $x \in \mathbb{S}$ such that $z = \Phi^\sigma(x)$. An easy rewritting gives:

$$e^{-\kappa s} (\Phi^s)^* (f-h)(z) = e^{\kappa\sigma} d_x \Phi^\sigma \left[e^{-\kappa(s+\sigma)} \left(d_x \Phi^{s+\sigma}\right)^{-1} (f-h)(\Phi^{s+\sigma}(x)) \right].$$

The uniform convergence of $e^{-\kappa s} (\Phi^s)^* (f-h)$ on the sphere implies that for all $\varepsilon > 0$, there exists $s_1 \in \mathbb{R}$ such that for all $s \leq s_1$:

$$\sup_{x \in \mathbb{S}} \| e^{-\kappa s} (d_x \Phi^s)^{-1} (f-h)(\Phi^s(x)) \| \leq \varepsilon.$$

Thus for $s \leq s_0 = s_1 - \sigma_0$, we have $s + \sigma \leq s_1$ and therefore:

$$\sup_{x \in \mathbb{S}} \| e^{-\kappa(s+\sigma)} \left(d_x \Phi^{s+\sigma}\right)^{-1} (f-h)(\Phi^{s+\sigma}(x)) \| \leq \varepsilon.$$

But we have seen that

$$\sup_{x \in \mathbb{S}, |w| \leq \varepsilon} \| e^{\kappa\sigma} d_x \Phi^\sigma w \| \underset{\sigma \to -\infty}{\longrightarrow} 0.$$

Since $\sigma \leq \sigma_0$, we conclude that there exists $s_2 \in \mathbb{R}$ such that for all $s \leq s_2$

$$\| e^{-\kappa s} (\Phi^s)^* (f-h)(z) \| \leq \varepsilon.$$

Since $f(0) = h(0)$, we finally get the uniform convergence.

□

Remark 2.8. *In the Proposition 2.7, the unit sphere $\mathbb{S}$ can be replaced by any compact set S such that for all $x \in \mathbb{R}^n \setminus \{0\}$, there exists $\sigma \in \mathbb{R}$ such that $\Phi^\sigma(x) \in S$.*

Beforehand, we have given the definition of an homogeneous approximation for a given degree. We may naturally wonder about the possible choice for this degree. There is yet at most one degree of interest for a given function or vector field. Consider for instance the vector field case. Let m be such that the homogeneous approximation h of degree m at 0 of a vector field f exists. Then for all $k < m$, we have

$$e^{-\kappa s} (\Phi^s)^* f \underset{s \to -\infty}{\overset{CUC}{\longrightarrow}} 0.$$

Moreover, if $h \neq 0$ then for all $k > m$

$$e^{-\kappa s} (\Phi^s)^* f \underset{s \to -\infty}{\overset{CUC}{\longrightarrow}} \infty.$$

Finally, there exists at most one degree for which a non-vanishing approximation exists. This degree will be of a particular interest, because it may give us a non-vanishing approximation, and because this approximation will inherit qualitative properties of the initial object.

Definition 2.9. *Let ν be a Euler vector field and let $\varphi \neq 0$ be a function.*

1. *The* local degree of ν-homogeneity *of φ at 0 is defined as:*

$$\deg_0(\varphi) = \sup\{\kappa \in \mathbb{R} : e^{-\kappa s} (\Phi^s)^* \varphi \underset{s \to -\infty}{\overset{CUC}{\longrightarrow}} 0\}$$

   with convention $\sup \varnothing = -\infty$ *and* $\sup \mathbb{R} = +\infty$*.*

2. *The* local degree of ν-homogeneity *of φ at ∞ is defined as:*

$$\deg_\infty(\varphi) = \inf\{\kappa \in \mathbb{R} : e^{-\kappa s} (\Phi^s)^* \varphi \underset{s \to +\infty}{\overset{CUC}{\longrightarrow}}, 0\}$$

   with convention $\inf \varnothing = +\infty$ *and* $\inf \mathbb{R} = -\infty$*.*

The local degree of homogeneity of a vector field is defined similarly.

Example 2.10. *Set $\nu = x\frac{\partial}{\partial x}$ on $\mathbb{R}$. The scalar function $\varphi : x \mapsto e^x$ has a local degree of ν-homogeneity of 0 at the origin and the homogeneous approximation is the constant function* 1*. However, the limit of $e^{-\kappa s} (\Phi^s)^* \varphi$ is ∞ for all κ when $s \to +\infty$, and thus $deg_\infty(\varphi) = +\infty$.*

As we have seen, the degree may be infinite and therefore the local approximation may not exist. But even when the degree is finite, the local approximation may not be of any interest. Indeed, it is possible to go from a vanishing approximation to a diverging approximation when the degree is increasing, as in the following example.

Example 2.11. *Let ν be an Euler vector field. Let φ be a continuous ν-homogeneous function of degree $\kappa > 0$. We define:*

$$\psi(x) = \sum_{d \geq 1} \frac{|\varphi(x)|^{1/d}}{d!}.$$

This series converges, since for all $d \geq 1$, $|\varphi(x)|^{1/d} \leq 1 + |\varphi(x)|$, *and thus* $\psi(x) \leq e(1 + |\varphi(x)|)$. *The function* ψ *is continuous. Set* $m > 0$, *and let* d *be an integer such that* $d > \kappa/m$. *We have:*

$$\begin{aligned} e^{-ms}\psi(\Phi^s(x)) &\geq e^{-ms}\frac{|\varphi(\Phi^s(x))|^{1/d}}{d!} \\ &\geq e^{-ms}\frac{e^{\kappa s/d}|\varphi(x)|^{1/d}}{d!} \\ &\geq e^{(\kappa/d-m)s}\frac{|\varphi(x)|^{1/d}}{d!} \underset{s\to-\infty}{\longrightarrow} +\infty. \end{aligned}$$

Therefore, $\deg_0(\varphi) \leq 0$. *On the other hand, if* $m < 0$, *we have for all* $C > 0$:

$$\sup_{|x|\leq C} e^{-ms}\psi(\Phi^s(x)) \leq \sum_{d\geq 1}\frac{e^{\kappa s/d}}{d!}\sup_{|x|\leq C}|\varphi(x)|^{1/d} \underset{s\to-\infty}{\longrightarrow} 0,$$

thus $\deg_0(\varphi) = 0$, *but with a vanishing approximation. Finally, for all* $m \leq 0$, *the local* ν*-homogeneous approximation of* ψ *is vanishing, and for all* $m > 0$, *the local* ν*-homogeneous approximation is diverging.*

The previous example shows that sometimes, the only possible homogeneous approximation is vanishing. We have to be careful that even if the local degree of homogeneity is finite, say κ, it does not imply that there exists a local approximation of f of degree κ, even vanishing, as we can see in Example 2.12 (the same observation holds for [Andrieu 2008]).

Example 2.12. *Let* ν *be a Euler vector field and* φ *a* ν*-homogeneous function of degree* κ. *Consider* $\psi(x) = \varphi(x)\ln(|\varphi(x)|)$. *The function* ψ *is continuous. Let us compute the local degree of homogeneity of* ψ *at* 0. *We have*

$$e^{-ms}\varphi(\Phi^s(x))\ln(|\varphi(\Phi^s(x))|) = e^{(\kappa-m)s}\varphi(x)\kappa s + e^{(\kappa-m)s}\varphi(x)\ln(|\varphi(x)|),$$

thus the local degree of ψ *is* κ, *but*

$$e^{-\kappa s}\varphi(\Phi^s(x))\ln(|\varphi(\Phi^s(x))|) = \varphi(x)s + \varphi(x)\ln(|\varphi(x)|)$$

does not converge and the function ψ *has no* ν*-homogeneous approximation at* 0.

This example justifies the following definition.

Definition 2.13. *Let* φ *be a function. We say that* φ *is* ν-homogenizable *at* 0 *(resp. at* ∞*)*

if $\deg_0(\varphi)$ *(resp.* $\deg_\infty(\varphi)$*) is finite and* φ *admits a local* ν*-homogeneous approximation at* 0 *of degree* $\deg_0(\varphi)$ *(resp. a local* ν*-homogeneous approximation at* ∞ *of degree* $\deg_\infty(\varphi)$*).*

If φ *is* ν*-homogenizable of degree* d*, the* ν*-homogeneous approximation of* φ *of degree* d *is called the* ν-homogenization *of* φ*.*

The ν*-homogenizability and the* ν*-homogenization of a vector field are defined similarly.*

2.3 Stability results

The framework of homogenization allows us to give a precise meaning of the approximation evoked previously. In the preceeding section, we have given the basic properties of this homogenization, and we have seen how to compute it more easily. But up to now, we have not seen in which sense this homogenization approximates the initial object. In this section, we shall show that qualitative properties of a vector field are encoded in its homogenization.

Theorem 2.14. *Let* f *be a vector field* ν*-homogenizable at* 0 *with* $\kappa = \deg_0(F)$ *and denote by* h *its* ν*-homogenization at* 0*. If the origin is an asymptotically stable equilibrium for* h*, then it is a locally asymptotically stable equilibrium for* f*.*

Proof. From the theorem of Rosier [Rosier 1992a], there exists a homogeneous proper Lyapunov function $V \in \mathcal{C}^1(\mathbb{R}^n\backslash\{0\}, \mathbb{R})$ of degree $\mu > \max\{0, -\kappa\}$ for h. Set $S = \{V(x) = 1\}$ and $a = \max_{x\in S} \mathcal{L}_h V(x) < 0$. Since $V \in \mathcal{C}^1(\mathbb{R}^n\backslash\{0\}, \mathbb{R})$, d_xV is continuous on S. Therefore the number $b = \sup_{x\in S} \|d_xV\|$ is well defined and strictly positive. For all $x \in S$ and all $s \in \mathbb{R}$ we have $\mathcal{L}_f V(\Phi^s(x)) = \mathcal{L}_h V(\Phi^s(x)) + \mathcal{L}_{(f-h)} V(\Phi^s(x))$. Moreover, we find $\mathcal{L}_h V(\Phi^s(x)) = e^{(\mu+\kappa)s}\mathcal{L}_h V(x) \leq e^{(\mu+\kappa)s}a$ and

$$\begin{aligned}
\mathcal{L}_{(f-h)}V(\Phi^s(x)) &= e^{\kappa s}\mathcal{L}_{[e^{-\kappa s}(\Phi^s)^*(f-h)]}(V\circ\Phi^s)(x) \\
&= e^{(\mu+\kappa)s} d_xV\left[e^{-\kappa s}\left(\Phi^s\right)^*(f-h)(x)\right] \\
&\leq e^{(\mu+\kappa)s}\mu \sup_{x\in S}\|e^{-\kappa s}\left(\Phi^s\right)^*(f-h)(x)\|
\end{aligned}$$

Since h is the local homogeneous approximation of f, we have:

$$\sup_{x\in S}\|e^{-\kappa s}\left(\Phi^s\right)^*(f-h)(x)\| \underset{s\to-\infty}{\longrightarrow} 0$$

and thus there exists $T \in \mathbb{R}$ such that for all $s \leq T$:

$$\sup_{x \in S} \|e^{-\kappa s} (\Phi^s)^* (f - h)(x)\| < -\frac{a}{2b}.$$

Therefore for all $s \leq T$ we have

$$\mathcal{L}_f V(\Phi^s(x)) < -\frac{a}{2} e^{(\mu+\kappa)s}.$$

Given that $e^{(\mu+\kappa)s} = V(\Phi^s(x))^{\frac{\mu+\kappa}{\mu}}$, we get finally:

$$\mathcal{L}_f V(\Phi^s(x)) \leq -\frac{a}{2} V(\Phi^s(x))^{\frac{\mu+\kappa}{\mu}} \qquad \forall s \leq T \tag{2.4}$$

and the function $\mathcal{L}_f V$ is definite negative on $\{0\} \cup \bigcup_{s \leq T} \Phi^s(S) = \{V(y) \leq e^{\mu T}\}$, which is a neighborhood of 0. Finally, we conclude the proof by applying the Theorem of Lyapunov A.4. $\square$

Remark 2.15. *This theorem was proved in [Bacciotti 2005], Corollary 5.6, for vector fields of class $\mathcal{C}^1$.*

Moreover, if the degree of the homogenization in negative, finite-time stability is achieved locally.

Corollary 2.16. *Under the assumptions of Theorem 2.14, if $deg_0(f) < 0$, then the origin is a locally FTS equilibrium of f.*

Proof. We use the notations of the proof of Theorem 2.14. By (2.4), for all $y \in \{V(y) \leq e^{\mu T}\}$, we have:

$$\mathcal{L}_f V(y) \leq -\frac{a}{2} V(y)^{\frac{\mu+\kappa}{\mu}}. \tag{2.5}$$

Since $\kappa < 0$, $0 < \frac{\mu+\kappa}{\mu} < 1$ and then for any solution $x(t)$ of $\dot{x} = f(x)$ with initial condition $x_0 \in \{V(y) \leq e^{\mu T}\}$, $x(t)$ converges in a finite time to the origin. $\square$

A similar result exists for local approximation at ∞.

Theorem 2.17. *Let f be a vector field ν-homogenizable at ∞ with $\kappa = \deg_\infty(f)$ and denotes by h its ν-homogenization. If the origin is an asymptotically stable equilibrium of h, then there exists a SPI compact neighborhood of the origin K such that every solution of $\dot{x} = f(x)$ reaches K in a finite time.*

Proof. The vector field h is continuous, GAS and homogeneous of degree κ, thus, by the theorem of Rosier [Rosier 1992a], there exists a proper Lyapunov function V homogeneous of degree $\mu > \max\{0, -\kappa\}$ for h. Set $S = \{V(x) = 1\}$ and $a = \max_{x \in S} \mathcal{L}_h V(x) < 0$. For all $x \in S$ and all $s \in \mathbb{R}$ we have $\mathcal{L}_f V(\Phi^s(x)) = \mathcal{L}_h V(\Phi^s(x)) + \mathcal{L}_{(f-h)} V(\Phi^s(x))$. Moreover, we find $\mathcal{L}_h V(\Phi^s(x)) = e^{(\mu+\kappa)s} \mathcal{L}_h V(x) \leq e^{(\mu+\kappa)s} M$ and

$$\begin{aligned}
\mathcal{L}_{(f-h)} V(\Phi^s(x)) &= e^{\kappa s} \mathcal{L}_{[e^{-\kappa s}(\Phi^s)^*(f-h)]} (V \circ \Phi^s)(x) \\
&= e^{(\mu+\kappa)s} \mathcal{L}_{[e^{-\kappa s}(\Phi^s)^*(f-h)(x)]} V(x).
\end{aligned}$$

Since $V \in \mathcal{C}^1(\mathbb{R}^n \backslash \{0\}, \mathbb{R})$, $d_x V$ is continuous on S. Therefore the number $b = \sup_{x \in S} \|d_x V\|$ is well defined and strictly positive. Since h is the local homogeneous approximation of f, we have:

$$\sup_{x \in S} \|e^{-\kappa s} \left(\Phi^s\right)^* (f-h)(x)\| \underset{s \to +\infty}{\longrightarrow} 0$$

and thus there exists $T \in \mathbb{R}$ such that for all $s \geq T$:

$$\sup_{x \in S} \|e^{-\kappa s} \left(\Phi^s\right)^* (f-h)(x)\| < -\frac{a}{2b}.$$

Therefore for all $s \geq T$ we have

$$|\mathcal{L}_{[e^{-\kappa s}(\Phi^s)^*(f-h)]} V(x)| < -\frac{a}{2}.$$

Given that $e^{(\mu+\kappa)s} = V(\Phi^s(x))^{\frac{\mu+\kappa}{\mu}}$, we get finally:

$$\forall s \geq T \qquad \mathcal{L}_f V(\Phi^s(x)) \leq -\frac{a}{2} V(\Phi^s(x))^{\frac{\mu+\kappa}{\mu}} \tag{2.6}$$

which is equivalent to:

$$\forall y \in \{V(y) \geq e^{\mu T}\} \qquad \mathcal{L}_f V(y) \leq -\frac{a}{2} V(y)^{\frac{\mu+\kappa}{\mu}}. \tag{2.7}$$

Integrating the differential inequality (2.7) gives that for any solution $x(t)$ of $\dot{x} = f(x)$ with initial condition $x_0 \in \{V(y) \geq e^{\mu T}\}$ and for all $t \geq 0$ such that $V(x(t)) \geq e^{\mu T}$:

$$V(x(t)) \leq \begin{cases} \left(V(x_0)^{-\frac{\kappa}{\mu}} + \frac{a\kappa t}{2\mu}\right)^{-\frac{\mu}{\kappa}} & \text{if } \kappa \neq 0 \\ V(x_0) e^{-\frac{at}{2}} & \text{if } \kappa = 0 \end{cases}$$

which in turn implies that:

$$t \leq \begin{cases} \frac{2\mu}{a\kappa}\left(e^{-\kappa T} - V(x_0)^{-\frac{\kappa}{\mu}}\right) & \text{if } \kappa \neq 0 \\ \frac{2}{a}\left(\ln V(x_0) - \mu T\right) & \text{if } \kappa = 0 \end{cases}. \tag{2.8}$$

Therefore, for all t large enough, $V(x(t)) < e^{\mu T}$ and the finite-time attractivity is proved. $\square$

Moreover, if the degree of the homogenization is positive, the convergence to this compact set is achieved in a time which is uniform in the initial condition.

Corollary 2.18. *Under the assumptions of Theorem 2.17, if $deg_0(f) > 0$, then there exists a time $\bar{T} > 0$ such that all the solutions of $\dot{x} = f(x)$ reach the positively invariant compact K in time $\bar{T}$.*

Proof. We use the notations of the proof of Theorem 2.17. Since $\kappa > 0$, the function $z > 0 \mapsto \frac{2\mu}{a\kappa}\left(e^{-\kappa T} - z^{-\frac{\kappa}{\mu}}\right)$ is increasing and converges to $\frac{2\mu e^{-\kappa T}}{\kappa a}$ when $z \to +\infty$. Therefore $\frac{2\mu}{a\kappa}\left(e^{-\kappa T} - V(x_0)^{-\frac{\kappa}{\mu}}\right) \leq \frac{2\mu e^{-\kappa T}}{a\kappa} = \bar{T}$. $\square$

2.4 Examples

In this section, we want to illustrate the theory developed along this chapter and see how to use it on examples.

Example 1

The first example of this section shows how the geometric homogeneity framework generalizes the weighted one. Some objects can be proved to be homogeneous, even though they would not seem so *a priori*.

On $\mathbb{R}^2$, set $\nu = (x_1 + x_2)\frac{\partial}{\partial x_1} + (-x_1 + x_2)\frac{\partial}{\partial x_2}$. We easily prove that ν is Euler. Indeed, the flow of ν is given by $\Phi^s(x) = e^s R(s)x$ where $R(s) = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}$. Consider now

$$f = \|x\|\left(\cos(\ln\|x\|)\frac{\partial}{\partial x_1} + \sin(\ln\|x\|)\frac{\partial}{\partial x_2}\right).$$

The vector field f is continuous, and ν-homogeneous of degree 0. Indeed we have $f =$

$\|x\|R(\ln\|x\|)\frac{\partial}{\partial x_1}$ and:

$$\begin{aligned}(\Phi^s)^*(f) &= e^{-s}R(-s)\|e^s R(s)x\|R(\ln\|e^s R(s)x\|)\frac{\partial}{\partial x_1}\\ &= \|x\|R(-s)R(s+\ln\|x\|)\frac{\partial}{\partial x_1} = f.\end{aligned}$$

Example 2

In this example, we want to illustrate the techniques of homogenization developed in the Section 2.2 and the stability results of Section 2.3.

Consider the planar vector fields

$$\nu = x_1\frac{\partial}{\partial x_1} + (x_2 + x_1^2)\frac{\partial}{\partial x_2}$$

and

$$\mu = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}.$$

These vector fields are Euler as we can see by computing their flow. Now consider:

$$f_\alpha = -x_1(x_2 + x_1 - x_1^2)^2\frac{\partial}{\partial x_1} + [-\alpha(x_2 - x_1^2)^3 - 2x_2(x_2 + x_1 - x_1^2)^2)]\frac{\partial}{\partial x_2},$$

where $\alpha \geq 0$ is a real parameter. This vector field is $\mathcal{C}^1$. The direct computation of $[\nu\,,\,f_\alpha]$ shows f_α is ν-homogeneous of degree 2. Denote by h_α the local μ-homogeneous approximation of f_α around 0. We find:

$$h_\alpha = -x_1(x_1 + x_2)^2\frac{\partial}{\partial x_1} - x_2(2(x_1 + x_2)^2 + \alpha x_2^2)\frac{\partial}{\partial x_2}.$$

For $\alpha > 0$, we easily see that h_α is GAS, and then f_α is LAS. Since f_α is ν-homogeneous, it is GAS.

For $\alpha = 0$, we can directly write:

$$f_0 = (x_2 + x_1 - x_1^2)^2\left(-x_1\frac{\partial}{\partial x_1} - 2x_2\frac{\partial}{\partial x_2}\right).$$

Therefore f_0 is stable but not asymptotically stable, since every point in the set $x_2 + x_1 - x_1^2 = 0$ is an equilibrium.

Example 3

This example shows how the geometric homogeneity can manage issues on nonlinear problems. Consider the following system on $\mathbb{R}^{n+1}$, with $x \in \mathbb{R}$, $y \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$:

$$\left\{ \begin{array}{rcl} \dot{x} & = & \frac{-x^3 e^{2|x|}}{1+|x|} - \frac{x\|y\|}{1+|x|} \\ \dot{y} & = & Ay - \|y\|^2 y \end{array} \right. . \tag{2.9}$$

We denote $\nu = \frac{x}{1+|x|}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. This vector field is clearly Euler. We see that the system is defined by the sum of 3 ν-homogeneous vector fields: $Ay\frac{\partial}{\partial y}$, $-\frac{x\|y\|}{1+|x|}\frac{\partial}{\partial x}$ and $\frac{-x^3 e^{2|x|}}{1+|x|}\frac{\partial}{\partial x} - \|y\|^2 y \frac{\partial}{\partial y}$ which are respectively of degree 0, 1 and 2. The homogenization at infinity is therefore of degree 2 and given by:

$$\left\{ \begin{array}{rcl} \dot{x} & = & -\frac{x^3 e^{2|x|}}{1+|x|} \\ \dot{y} & = & -\|y\|^2 y \end{array} \right. . \tag{2.10}$$

The origin is clearly a globally asymptotically stable equilibrium for this approximation, and thus by Theorem 2.17, there exists a radius $r > 0$ such that all the trajectories of system (2.9) converge to the ball centered at the origin of radius r in finite time. Moreover, the degree of local homogeneity being strictly positive, the convergence to this ball is achieved in a time that is uniform in the initial conditions.

Example 4

Consider the following Euler vector field:

$$\nu = (x_1 + x_2)\frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}.$$

Being linear, one can easily compute its flow:

$$\Phi^s(x) = e^s \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} x.$$

We want to compute the ν-homogeneous approximation of the following vector field around the origin:

$$f = \frac{\lfloor x_2 \rceil^2}{\|x\|}\frac{\partial}{\partial x_2}.$$

We get, for a degree of -1:

$$e^s\,(d_x\Phi^s)^{-1}\,f(\Phi^s(x)) = \frac{1}{\sqrt{(x_1+sx_2)^2+x_2^2}}(-s\lfloor x_2\rceil^2\frac{\partial}{\partial x_1}+\lfloor x_2\rceil^2\frac{\partial}{\partial x_2}).$$

And this quantity converges pointwise to $h = x_2\frac{\partial}{\partial x_1}$. This means that the only possible ν-homogenization of the vector field f is the vector field h, that is, an orthogonal vector field!

Hopefully, the convergence is not uniform. Indeed:

$$\|e^s\,(d_x\Phi^s)^{-1}\,f(\Phi^s(x)) - h(x)\|^2 = \left(\frac{-s\lfloor x_2\rceil^2}{\sqrt{(x_1+sx_2)^2+x_2^2}} - x_2\right)^2 + \left(\frac{\lfloor x_2\rceil^2}{\sqrt{(x_1+sx_2)^2+x_2^2}}\right)^2$$

and for $x_1 = 1$ and $x_2 = -1/s$, we get:

$$\begin{aligned}
\|e^s\,(d_x\Phi^s)^{-1}\,f(\Phi^s(x)) - h(x)\|^2 &= \left(\frac{-s\lfloor x_2\rceil^2}{|x_2|} - x_2\right)^2 + \left(\frac{\lfloor x_2\rceil^2}{|x_2|}\right)^2 \\
&= (-sx_2 - x_2)^2 + x_2^2 \\
&= (1+\frac{1}{s})^2 + \frac{1}{s^2} \\
&= 2(\frac{1}{s}+\frac{1}{2})^2 + \frac{1}{2} \geq \frac{1}{2}.
\end{aligned}$$

The non-uniform convergence proves that f has no local ν-approximation around the origin. The conclusion on this example is twofold: first, it shows that it is necessary to pay attention at the *uniformity* of the convergence; secondly, it gives a hint to understand the necessity of this uniform convergence in the proof of the Theorems 2.14 and 2.17.

2.5 Conclusion

This chapter proposes an extension of the local homogeneity framework in the geometrical or coordinate-free setting. The stability theorems are recast for these generalized definitions for systems defined by continuous vector field, without any assumption on the uniqueness of the solutions. The theory is illustrated with examples of systems which can never be homogeneous (whatever the setting is), and others which are homogeneous only in the coordinate-free setting. Advances of this framework application for stability analysis are also demonstrated on these examples.

Chapter 3

Homogeneity: from ordinary differential equations to differential inclusions

Contents

3.1 Introduction

It is well known that the solution of a differential equation $\dot{x} = f(x)$, with f a continuous vector field on $\mathbb{R}^n$, is a differentiable curve $x(t)$ defined on a time interval. However, if we try to relax the assumption of continuity on f, and look for instance at a vector field f which admits a jump discontinuity at 0, a problem naturally arises. Indeed, consider $x(t)$ a solution with $x(0) = 0$, then $\dot{x}(t) = f(x(t))$ is discontinuous at $t = 0$, but the derivative of a differentiable function does not admit jump discontinuities, and we get a contradiction.

Indeed, consider the very simple and enlightening scalar example:

$$\dot{x} = -\operatorname{sign} x, \qquad x \in \mathbb{R}. \tag{3.1}$$

The solutions are defined by $x(t) = \operatorname{sign}(x_0)(|x_0| - t)$ for $t < |x_0|$, and therefore reach the origin in a finite time. The only possible solution starting at the origin is the constant solution. Finally, the general form of the solution should be:

$$x(t) = \begin{cases} \operatorname{sign} x_0(|x_0| - t) & t < |x_0| \\ 0 & t \geq |x_0| \end{cases},$$

but this curve is not differentiable at $t = -x_0$.

The study of discontinuous systems has a long history. Caratheodory developed in 1918 a well known generalization of the Cauchy–Peano–Arzéla theorem [Caratheodory 1918]. He used *absolutely continuous functions* to define solutions. This set of functions admits a derivative almost everywhere, and should hence verify the differential equation almost everywhere. This setting, although very general, does not handle all the situations. In particular, when a trajectory reaches a submanifold in finite-time and stays on this submanifold, this setting is not able to represent the behavior of the solutions.

Filippov's definitions have been introduced in this context. Instead of looking at the pointwise value of a vector field, Filippov's idea is to compute an "average" value of the vector field, using its values at the neighborhood of any point. This procedure, known as the *Filippov's regularization procedure*, transforms a vector field into a set-valued map: at each point, the value of this map consists in all the possible velocities. When the vector field is continuous, the set reduces to a singleton and this construction boils down to the usual setting. But when the vector field is discontinuous, the set-valued map encodes all the directions the system can go towards. Hence, the classical differential equation

$\dot{x} = f(x)$ is replaced by a *differential inclusion* $\dot{x} \in F(x)$.

Note that there exists also another similar framework, proposed by Krasovskii. The main difference between the Filippov and Krasovskii methods relies on the fact that the Filippov construction does not depend on the value of the vector field on any zero measure set, whereas the Krasovskii method does. This difference makes for instance the latter giving different differential inclusion for the system (3.1) for different values of the function sign at zero. Since we do not want to pay attention to the particular pointwise values of the vector field, we focus in this work on the Filippov framework only.

The theory of differential inclusions is well-established [Filippov 1988] [Aubin 1984]. Among others, it appears in optimal control theory or viability theory; when dealing with variable structure systems, systems with adaptive control, power electronic systems with switching devices or mechanical systems with friction, discontinuous right-hand sides appear naturally. Finally, the sliding mode control theory makes an important use of discontinuous controller to achieve finite-time stability as well as robustness.

As we have seen beforehand, the homogeneity theory provides tools to understand and describe nonlinear systems. If the theory of homogeneous system is well-known in the context of ODE, only few extensions to DI exist in the literature. First of all, Filippov [Filippov 1988] defined homogeneity for DI, but only in the context of classical homogeneity. A. Levant [Levant 2005] and Y. Orlov [Orlov 2005b] studied also the subject, but some properties of homogeneous ODE were not extended. Let us also mention that an extension of the Theorem of Rosier for locally essentially bounded vector fields is proved in [Rosier 1992b]. Finally, as far as we know, nothing has been done about geometric homogeneity for DI.

In this chapter we shall recall the basis of the differential inclusion theory, then define homogeneity for differential inclusions, connect this definition with the usual definition for vector fields, and extend all the useful properties of homogeneous ODE to homogeneous DI. We shall particularly see how the *flow commutation property* can be extended, demonstrate a Rosier's theorem on the existence of a Lyapunov function for DI and prove that the qualitative properties on homogeneous systems still apply with some slight changes. The contents of this Chapter have been partially published in [Bernuau 2013f].

3.2 Differential inclusions

3.2.1 Differential inclusions: basic concepts

Let us start with an ODE given by a vector field $f \in \mathcal{L}^{\infty}_{\mathsf{loc}}$:

$$\dot{x} = f(x). \tag{3.2}$$

The Filippov's regularization procedure consists in the construction of a set-valued map F defined by:

$$F(x) = \bigcap_{\varepsilon>0} \bigcap_{N \in \mathcal{N}} \overline{\mathrm{conv}}(f(B(x,\varepsilon) \setminus N)). \tag{3.3}$$

By construction, for all $x \in \mathbb{R}^n$, the set $F(x)$ is closed and convex. Moreover, if $f = g$ almost everywhere, then the Filippov's regularization of g is equal to the regularization of f. Hence, since the vector field f has been taken locally essentially bounded, we will assume for the sake of simplicity that f is locally bounded. For given $\varepsilon > 0$ and $N \in \mathcal{N}$, the set

$$\overline{\mathrm{conv}}(f(B(x,\varepsilon) \setminus N))$$

is a subset of the compact set

$$\overline{\mathrm{conv}}(f(B(x,\varepsilon)))$$

and therefore compact. Moreover the set

$$\bigcap_{N \in \mathcal{N}} \overline{\mathrm{conv}}(f(B(x,\varepsilon) \setminus N))$$

is nonempty if and only if for any finite family $\{N_1, \dots N_d\} \subset \mathcal{N}$ the set

$$\bigcap_{N \in \{N_1,\dots N_d\}} \overline{\mathrm{conv}}(f(B(x,\varepsilon) \setminus N))$$

is nonempty, which is true since

$$\overline{\mathrm{conv}}(f(B(x,\varepsilon) \setminus \bigcup_{i \in \{1,\dots,d\}} N_i))$$

is not empty. Finally, $F(x)$ is a nonempty compact and convex set.

Definition 3.1. *A set-valued map F is* upper semicontinuous*(USC) iff for any $x \in \mathbb{R}^n$ and any neighborhood $\mathcal{V}$ of $F(x)$, there exists a neighborhood $\mathcal{U}$ of x such that for any*

$y \in \mathcal{U}$, $F(y) \subset \mathcal{V}$.

Let us show that the Filippov's procedure leads to a USC set-valued map. Set $x \in \mathbb{R}^n$ and $\mathcal{V}$ a neighborhood of $F(x)$. Then there exists $\delta > 0$ such that for all $0 < \varepsilon \leq \delta$, $\bigcap_{N\in\mathcal{N}} \overline{\text{conv}}(f(B(x,\varepsilon) \setminus N)) \subset \mathcal{V}$. Set $\mathcal{U} = B(x,\delta)$. If $y \in \mathcal{U}$, for $\varepsilon = \delta - \|x - y\|$, $B(y,\varepsilon) \subset B(x,\delta)$. Therefore

$$\bigcap_{\varepsilon>0} \bigcap_{N\in\mathcal{N}} \overline{\text{conv}}(f(B(y,\varepsilon) \setminus N)) \subset \mathcal{V}$$

and F is USC.

In many applications, the differential inclusion is given by the set-valued map coming from the Filippov's procedure. We will therefore focus on set-valued map with the properties inherited by this procedure.

Definition 3.2. *Let F be a set-valued map. We say that F verifies the* standard assumptions *if F is USC and if for any $x \in \mathbb{R}^n$, $F(x)$ is a nonempty compact convex set.*

The general concept of solution of an ODE $\dot{x} = f(x)$ is given by a differentiable curve $x(t)$ defined on a nonempty time interval I such that $\frac{dx}{dt}(t) = f(x(t))$ for all $t \in I$. In the context of DI, this notion is replaced by the one of absolutely continuous function:

Definition 3.3. *Consider a curve $x : I \to \mathbb{R}^n$, where I denotes an interval. We say that x is* absolutely continuous *if for every positive number ε, there is a positive number δ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) of I satisfies:*

$$\sum_k \|y_k - x_k\| < \delta$$

then:

$$\sum_k \|f(y_k) - f(x_k)\| < \varepsilon.$$

This definition, although not really tractable in practice, leads to the following useful equivalence:

Proposition 3.4. *[Nielsen 1997] A curve $x : I \to \mathbb{R}^n$ is absolutely continuous iff it is differentiable almost everywhere, its derivative is Lebesgue integrable over I and for any $a < b \in I$:*

$$x(b) - x(a) = \int_a^b \dot{x}(t)dt.$$

This proposition allows us now to define the notion of solution of a differential inclusion.

Definition 3.5. *An absolutely continous curve x defined on I is a solution of the differential inclusion $\dot{x} = F(x)$ iff for almost all $t \in I$, $\dot{x}(t) \in F(x(t))$.*

This notion of solutions has numerous interests. Obviously, it boils down to the classical setting if the set-valued map comes from a continuous vector field. Moreover, such solutions do exist. For ODE, there exist classical sufficient conditions ensuring existence of solution for a Cauchy problem (e.g. continuity with the Cauchy–Peano–Arzéla theorem). These conditions depend on the smoothness of the function in the right-hand side of the ODE. For DI, the standard assumptions suffice to get the existence of a solution to any Cauchy problem [Aubin 1984]. This means that any differential equation with a locally essentially bounded right-hand side has a solution for any initial condition in the sense exposed above.

Additionally, the Lipschitz condition leads to the uniqueness of solutions for the classical setting. There exist definitions of a Lischitzean set-valued map leading to uniqueness of solutions as well. Note however that the Lipschitz setting for set-valued map will not be used in our work: the problems we are aiming to address are mostly never Lipschtizean.

Indeed, in a lot of problems involving discontinuous vector fields, uniqueness of solutions is not achieved. Actually, when the discontinuity comes from the control, like in sliding mode controls, the purpose of this choice is to get properties like finite-time stability or robustness with respect to some bounded perturbations. These properties are not compatible with uniqueness of solutions, at least backward. Handling the case of non-uniqueness of solutions needs a careful recast of the definitions of convergence, stability and even equilibrium. The revised definitions will have to be split in two parts: if *one* solution has a given property, we will say that this property is weak; if *all* the solutions have this property, we will say that this property is strong. Note that this split into weak and strong definitions is not due to the discontinuity itself but to the non-uniqueness of solutions.

Example 3.6. *Consider the continuous scalar system $\dot{x} = 2\lfloor x \rceil^{1/2}$. The vector field vanishes at the origin, and thus the constant zero curve is a solution. However, for all $t_0 > 0$, the curve*

$$t \mapsto \begin{cases} 0 & t \leq t_0 \\ (t - t_0)^2 & t \geq t_0 \end{cases}$$

is another solution to the Cauchy Problem with initial condition $x(0) = 0$. *Hence the origin, although a point of vanishing of the vector field, is only a* weak *equilibrium.*

3.2.2 Equilibria, attractiveness and stability

Definition 3.7. *A point* x_∞ *is a* weak equilibrium *of an ODE or a DI if the constant curve* $t \mapsto x_\infty$ *is a solution. It is a* strong equilibrium *if the constant curve is the only solution of an ODE or of a DI for any initial value* (t_0, x_∞), $t_0 \in \mathbb{R}$.

Similar definitions exist for attractiveness and stability.

Definition 3.8. *Consider an equilibrium* x_∞ *for an ODE or a DI. We say that* x_∞ *is:*

- weakly locally attractive *if there exists a neighborhood* $\mathcal{U}$ *of* x_∞ *such that for all* $x_0 \in U$, *there exists a solution* x *with* $x(0) = x_0$ *such that* $x(t) \to x_\infty$ *when* $t \to +\infty$;
- weakly globally attractive *if it is weakly locally attractive with* $\mathcal{U} = \mathbb{R}^n$;
- strongly locally attractive *if there exists a neighborhood* $\mathcal{U}$ *of* x_∞ *such that for all* $x_0 \in \mathcal{U}$ *and all solutions* x *with* $x(0) = x_0$, *we have* $x(t) \to x_\infty$ *when* $t \to +\infty$;
- strongly globally attractive *if it is strongly locally attractive with* $\mathcal{U} = \mathbb{R}^n$;
- weakly stable *if for any neighborhood* $\mathcal{U}$ *of* x_0 *there exists a neighborhood* $\mathcal{V}$ *of* x_0 *such that for all* $x_0 \in \mathcal{V}$, *there exists a solution* x *with* $x(0) = x_0$ *such that* $x(t) \in \mathcal{U}$ *for all* $t \geq 0$;
- strongly stable *if for any neighborhood* $\mathcal{U}$ *of* x_0 *there exists a neighborhood* $\mathcal{V}$ *of* x_0 *such that for all* $x_0 \in \mathcal{V}$ *and all solutions* x *with* $x(0) = x_0$, *we have* $x(t) \in \mathcal{U}$ *for all* $t \geq 0$;
- weakly locally asymptotically stable *if it is weakly locally attractive and weakly stable;*
- strongly locally asymptotically stable *if it is strongly locally attractive and strongly stable;*
- weakly globally asymptotically stable *if it is weakly globally attractive and weakly stable;*
- strongly globally asymptotically stable *if it is strongly globally attractive and strongly stable.*

In this work, our aim is often to get qualitative informations on the behavior of a system, that is to known how the system is going to behave in the future. In this setting, a property which is true only for one solution, but not necessarily for the others, will not be interesting. Hence, now and further on, when the distinction is not explicitly mentioned, it means that we are dealing with the strong properties.

3.2.3 Qualitative properties on the solutions of a Differential Inclusion

This last subsection exposes some properties of the set of solutions of a DI. These results, although well known, are presented here for the sake of completeness. In the classical setting of continuous vector fields with forward uniqueness of solutions, the flow or the semi-flow of the vector field provides a lot of qualitative informations about the system. When the forward uniqueness is lost, a flow does not exist anymore. In this subsection, we are going to define a *generalized flow* and study its properties. The results that we shall get will be helpful in the study of the next section. Remark also that similar results can be found in [Filippov 1988], [Aubin 1984], [Angeli 1999]...

Consider the autonomous differential inclusion defined by the set-valued map F verifying the standard assumptions:

$$\dot{x} \in F(x). \tag{3.4}$$

When dealing with ODE, a useful qualitative tool is the flow or at least the semiflow when we have only uniqueness of solutions in forward time. In the case of possible non-uniqueness, the situation is more complex. We say that a solution x of (3.4) *starts at* x_0 if x is defined on an interval containing 0 and $x(0) = x_0$. We will denote by $\mathcal{S}([0,T], A)$ the set of solutions of (3.4) defined on the interval $[0,T]$, $T > 0$, starting in $A \subset \mathbb{R}^n$. We also allow $T = +\infty$, and in this situation the interval $[0,T]$ has to be understood as $[0,+\infty)$. We will also denote $\mathcal{S}([0,T], x_0) = \mathcal{S}([0,T], \{x_0\})$.

Let $T \in]0,+\infty]$ be such that every solution of (3.4) starting in A is defined on $[0,T]$. We denote $\Psi^T(A) = \{x(T) : x \in \mathcal{S}([0,T], A)\}$. This set is the reachable set from A at time T, or the limit in case $T = +\infty$. Let us stress that with the assumption of uniqueness of solutions in forward time, Ψ^t corresponds to the semiflow of F; this remark justifies that we call Ψ the *generalized flow* of F.

Without the standard assumptions, the set $\mathcal{S}([0,T], \{x_0\})$ may not be well-defined. Indeed, consider the scalar differential inclusion defined by $\dot{x} \in F(x) = \mathbb{R}_+$. Then for any $T \in]0,+\infty]$, the curve $t \mapsto \tan\left(\frac{\pi t}{2T}\right)$ is a solution starting at 0, but is defined only on

$(-T,T)$ and cannot be extended to $[0,T]$. A phenomenon of finite-time blow up occurs for any positive time. However, the standard assumptions prevent such nasty behavior.

Lemma 3.9. *Let K be a compact subset of $\mathbb{R}^n$. There exists $T>0$ such that any solution of (3.4) starting in K can be extended to the whole interval $[0,T]$ and for all $t\in[0,T]$ the set $\Psi^t(K)$ is bounded.*

Proof. Consider a compact subset L of $\mathbb{R}^n$ such that $K\subset \mathring{L}$. Denote $\delta = \mathrm{d}(K,\partial L)>0$ and $M=\max\{\|v\| : v\in F(x), x\in L\}$. The positive number M is well-defined since the set $\{v\in F(x), x\in L\}$ is compact. If $M=0$, $\Psi^t(K)=K$ for any $t\geq 0$. We assume now that $M\neq 0$.

Let x be a solution of (3.4) starting in K and denote $\tau=\inf\{t\geq 0 : x(t)\in\partial L\}$. Hence, for all $0\leq t\leq\tau$, $x(t)\in L$ and thus:

$$\|x(\tau)-x(0)\|\leq\int_0^\tau \|\dot{x}(t)\|dt\leq M\tau.$$

Since $x(\tau)\in\partial L$ and $x(0)\in K$, we have $\|x(\tau)-x(0)\|\geq\delta$. Thus $\tau\geq\delta/M$. We can therefore extend any solution of (3.4) starting in K on the interval $[0,T]$, where $T=\delta/M$. Moreover, for any $t\in[0,T]$, we have $x(t)\in L$. □

Lemma 3.10. *Let I be a segment of $\mathbb{R}$, $y: I\to\mathbb{R}^n$ an integrable function. We assume moreover that there exists $\mathcal{V}$ and $\mathcal{W}$ two compacts subsets of $\mathbb{R}^n$ with $\mathcal{V}$ convex such that $y(I)\subset\mathcal{W}\subset\mathring{\mathcal{V}}$. Then for all $t\in I$ and $h\in\mathbb{R}$ such that $t+h\in I$:*

$$\frac{1}{h}\int_t^{t+h} y(\tau)d\tau\in\mathcal{V}.$$

Proof. Assume that y is a simple function. Then there exists a partition of $[t,t+h]$, $A_1,\ldots,A_d$ such that $y_{|A_i}=\alpha_i$. Then:

$$\frac{1}{h}\int_t^{t+h} y(\tau)d\tau=\sum_i\alpha_i\frac{\lambda(A_i)}{h}.$$

But $\sum_i\frac{\lambda(A_i)}{h}=\frac{\lambda([t,t+h])}{h}=1$, and by convexity of $\mathcal{V}$,

$$\frac{1}{h}\int_t^{t+h} y(\tau)d\tau\in\mathcal{V}.$$

Now in the general case, the function y being bounded, there exists a sequence of simple functions (y_k) converging to y uniformly on I. Thus for k large enough, $y_k(I)\subset\mathcal{V}$, and

$$\frac{1}{h}\int_t^{t+h} y(\tau)d\tau = \lim_k \frac{1}{h}\int_t^{t+h} y_k(\tau)d\tau \in \mathcal{V}$$

since $\mathcal{V}$ is closed. $\square$

Lemma 3.11. *Let $K \subset \mathbb{R}^n$ be a compact set. Let $(x_k) \in \mathcal{S}([0,T],K)$ be a sequence of solutions of (3.4) and let $L \subset \mathbb{R}^n$ be a compact set such that for all $t \in [0,T]$, $x_k(t) \in L$. Then there exists a subsequence of (x_k) converging to $x \in \mathcal{S}([0,T],K)$ uniformly on $[0,T]$.*

Proof. Let us denote $M = \sup_{x\in L}\{\|y\| : y \in F(x)\}$. Then for all $a < b \in [0,T]$, we have:

$$\|x_k(b) - x_k(a)\| \leq \int_a^b \|\dot{x}_k(t)\|dt \leq M(b-a).$$

Therefore the solutions x_k are Lipschitz with a constant M and the family (x_k) is equicontinuous. By the Arzela-Ascoli theorem, this sequence admits a subsequence (we do not relabel) uniformly converging to a continuous function x. Since all the functions x_k are Lipchitz with the same constant M, so is x; finally x is absolutly continuous.

Let $\mathcal{V}$ be a compact convex neighborhood of $F(x(t))$ and $\mathcal{W}$ a compact neighborhood of $F(x(t))$ such that $\mathcal{W} \subset \mathring{\mathcal{V}}$. By USC of F, there exists an open bounded neighborhood $\mathcal{U}$ of $x(t)$ such that for all $y \in \mathcal{U}$, $F(y) \subset \mathcal{W}$. Since x is continuous, there exists $\eta > 0$ such that for all $\tau \in [t-\eta, t+\eta]$, $x(\tau) \in \mathcal{U}$. Let us denote $I = \{x(\tau) : \tau \in [t-\eta, t+\eta]\}$. The set I is compact and is a subset of $\mathcal{U}$. Set $\alpha = \mathrm{d}(I, \partial U) > 0$. Since (x_k) converges uniformly to x, there exists $N > 0$ such that for all $k \geq N$ and for all $\tau \in [t-\eta, t+\eta]$, $\|x_k(\tau) - x(\tau)\| \leq \frac{\alpha}{2}$. Thus for all $k \geq N$ and for all $\tau \in [t-\eta, t+\eta]$, $x_k(\tau) \in \mathcal{U}$ and then $\dot{x}_k(t) \in \mathcal{W}$. Applying now the Lemma 3.10, we get that

$$\frac{x(t+h) - x(t)}{h} = \lim_{k\to\infty} \frac{1}{h}\int_t^{t+h} \dot{x}_k(\tau)d\tau \in \mathcal{V}.$$

Therefore, for any t where x is differentiable, $\dot{x}(t) \in \mathcal{V}$. Being compact and convex, $F(x(t))$ is equal to the intersection of all its compact and convex neighborhood. Therefore, $\dot{x}(t) \in F(x(t))$. Since $x(0) = \lim x_k(0)$ and $x_k(0) \in K$ compact, $x(0) \in K$. Finally, $x \in \mathcal{S}([0,T],K)$. $\square$

Proposition 3.12. *Let K be a compact subset of $\mathbb{R}^n$. Assume that there exists $T > 0$ such that every solution of (3.4) starting in K stay in the compact L for all $t \in [0,T]$. Then for all $t \in [0,T]$, the set $\Psi^t(K)$ is compact.*

Proof. For all $t \in [0,T]$ the set $\Psi^t(K)$ is bounded. Let us show that the set $\Psi^t(K)$ is then compact. Consider a sequence of points $(x_k(t)) \in \Psi^t(K)$ with corresponding solutions $(x_k) \in \mathcal{S}([0,t],K)$. By Lemma 3.11, there exists a subsequence of (x_k) converging to a solution x on $[0,t]$ and the sequence $(x_k(t))$ admits a converging subsequence, which proves the compactness of $\Psi^t(K)$. $\square$

Theorem 3.13. *Let $K \subset \mathbb{R}^n$ be compact and assume that all the solutions of (3.4) starting in K are defined on the whole interval $[0,T]$. Then there exists a compact set $L \subset \mathbb{R}^n$ such that for all $t \in [0,T]$, $\Psi^t(K) \subset L$ and the set $\Psi^t(K)$ is compact.*

Proof. We shall prove that there exists a compact set $L \subset \mathbb{R}^n$ such that for all $t \in [0,T]$, $\Psi^t(K) \subset L$. Then by the Proposition 3.12, the second assertion follows.

By contradiction, assume that for all $k \geq 1$, there exists $x_k \in \mathcal{S}([0,T],K)$ and $t_k \in [0,T]$ such that $d(x_k(t_k),K) > k$. Replacing x_k with a subsequence, we can assume that $x_k(0)$ converges. Without loss of generality, we assume that $x_k(0) \to 0$. Then $\|x_k(t_k)\| > k$.

For all $1 \leq m \leq k$, there exists $t_k^m \in (0,t_k)$ such that $\|x_k(t_k^m)\| = m$ and $x_k(t) \leq m$ for $t \in [0,t_k^m]$. Let us extract a subsequence of $(x_k(t_k^1))$ which is converging to q^1, with $\|q^1\| = 1$, and then extract from t_k a subsequence converging to t^1 (we do not relabel).

Let us denote $M = \sup_{x \in B(q^1,1)}\{\|y\| : y \in F(x)\}$. We assume that k is big enough so that $\|x_k(t_k^1) - q^1\| < 1/2$. For all $t \in [0,t_k^1]$, $\|x_k(t)\| \leq 1$. If $t^1 \leq t_k^1$, then x_k is bounded on $[0,t^1]$. Else, set:

$$t_*^1 = \sup\{t \leq t^1 : \||\|x_k(t) - x_k(t^1)\| \geq 1/2\}.$$

Then for all $t \in [t_*^1, t^1]$, $\|x_k(t) - x_k(t_1)\| < 1/2$ and then $\|x_k(t) - q^1\| < 1$. Thus $x_k(t) \in B(q^1,1)$ and then $\|\dot{x}_k(t)\| \leq M$. Hence:

$$\frac{1}{2} \leq \|x_k(t_*^1) - x_k(t^1)\| \leq \int_{t_k^1}^{t^1} \|\dot{x}_k(t)\| \leq M(t^1 - t_k^1).$$

We conclude that $t^1 - t_*^1 \geq \frac{1}{2M}$. Taking k big enough allows us to assume that $t^1 - t_k^1 \leq \frac{1}{2M}$ and then $t_k^1 \geq t_*^1$. Thus for all $t \in [t_k^1, t^1]$, $\|x_k(t) - q^1\| < 1$. Finally the sequence (x_k) is bounded on $[0,t^1]$ and then equicontinuous. By Lemma 3.11, there exists a subsequence of x_k converging to $\bar{x} \in \mathcal{S}([0,t^1],0)$ on $[0,t^1]$. Moreover, $\bar{x}(t^1) = q^1$. Indeed

$$\|\bar{x}(t^1) - q^1\| \leq \|\bar{x}(t^1) - x_k(t^1)\| + \|x_k(t^1) - x_k(t_k^1)\| + \|x_k(t_k^1) - q^1\|$$

and $\|x_k(t^1) - x_k(t_k^1)\| \leq |t^1 - t_k^1| M'$ where $M' = \sup_{x \in B(q^1,1) \cup B(0,1)} \{\|y\| : y \in F(x)\}$.

We reiterate this proof for $m = 2, 3, \ldots$ and construct this way a nested family of subsequences. A diagonal extraction leads to the extension of $\bar{x}$ on $[0, t^m]$ of all m, with $\|\bar{x}(t^m)\| = \|q^m\| = m$. Since the sequence (t^m) is bounded in $[0, T]$, we can extract a subsequence converging to $t^* \in [0, T]$ and $\bar{x}$ is defined on t^* by hypothesis. Then $\bar{x}(t^m) \to \bar{x}(t^*)$, but $\|\bar{x}(t^m)\| = m$ which leads to the desired contradiction. □

Corollary 3.14. *Let $T > 0$ be such that all solutions starting from the compact set $K \subset \mathbb{R}^n$ exist on $[0, T]$. Then $\mathcal{S}([0, T], K)$ is compact for the topology of uniform convergence on $[0, T]$.*

Proof. Let (x_k) be a sequence of $\mathcal{S}([0, T], K)$. By the Theorem 3.13, there exists L such that $x_k(t) \in L$ for all k and all $t \in [0, T]$. By the Lemma 3.11, there exists a subsequence of (x_k) which is converging to a solution $x \in \mathcal{S}([0, T], K)$ uniformly on $[0, T]$. □

Proposition 3.15. *Consider $T > 0$ such that the solutions of (3.4) starting from the compact set $K \subset \mathbb{R}^n$ are defined on $[0, T]$. For all neighborhood $\mathcal{V}$ of $\Psi^T(K)$, there exists a neighborhood $\mathcal{U}$ of K such that the solutions starting from $\mathcal{U}$ are defined on $[0, T]$ and $\Psi^t(\mathcal{U}) \subset V$.*

Proof. The fact that the solutions starting from a small enough neighborhood of K are defined on $[0, T]$ is an easy adaptation of the proof of Theorem 3.13.

Let us prove that $\cap_{k \geq 1} \Psi^T(K + \bar{B}(0, 1/k)) = \Psi^T(K)$. We have obviously $\Psi^T(K) \subset \Psi^T(K + \bar{B}(0, 1/k))$ for all $k \geq 1$ and then $\Psi^T(K) \subset \cap_{k \geq 1} \Psi^T(K + \bar{B}(0, 1/k))$. Let $y \in \cap_{k \geq 1} \Psi^T(K + \bar{B}(0, 1/k))$ be fixed. There exists a sequence of solutions $x_k \in \mathcal{S}([0, T], K + \bar{B}(0, 1/k))$ such that $x_k(T) = y$. By Theorem 3.13, there exists a compact set $L \subset \mathbb{R}^n$ such that $\Psi^t(K + \bar{B}(0, 1)) \subset L$ for all $t \in [0, T]$. By Lemma 3.11, we can therefore extract a subsequence converging to a solution x. Moreover, $x(0) \in K$ and $x(T) = y$, and thus $y \in \Psi^T(K)$, which proves that $\cap_{k \geq 1} \Psi^T(K + \bar{B}(0, 1/k)) = \Psi^T(K)$.

Let k be big enough such that $\Psi^T(K + \bar{B}(0, 1/k)) \subset \mathcal{V}$. Then $\mathcal{U} = K + B(0, 1/k)$ is a valid choice. □

3.3 Homogeneous differential inclusions

In this section, we continue to consider the DI (3.4) for which the standard assumptions hold. The aim of this section is to define a homogeneity notion consistent with the conventional definition and see how the nice properties of the homogeneous continuous systems can be generalized.

We adopt here a natural definition, which is also a straightforward extension of the definition of [Levant 2005] (given only for weighted homogeneity).

Definition 3.16. *Let ν be a Euler vector field. A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is ν-homogeneous of degree $\kappa \in \mathbb{R}$ if for all $x \in \mathbb{R}^n$ and for all $s \in \mathbb{R}$ we have:*

$$F(\Phi^s(x)) = e^{\kappa s} d_x \Phi^s F(x).$$

The system (3.4) is ν-homogeneous of degree κ if the set-valued map F is ν-homogeneous of degree κ.

Proposition 3.17. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued ν-homogeneous map of degree κ. Then for all $x_0 \in \mathbb{R}^n$ and any solution x of the system (3.4) starting at x_0 and all $s \in \mathbb{R}$, the absolute continuous curve $t \mapsto \Phi^s(x(e^{\kappa s}t))$ is a solution of the system (3.4) starting at $\Phi^s(x_0)$.*

Proof. Consider a solution x of (3.4) starting at x_0. The curve $t \mapsto \Phi^s(x(e^{\kappa s}t))$ is clearly an absolute continuous curve for all $s \in \mathbb{R}$. Moreover, for almost all $t \in \mathbb{R}$ we have:

$$\begin{aligned} \frac{d}{dt}\Phi^s(x(e^{\kappa s}t)) &= e^{\kappa s} d_{x(e^{\kappa s}t)} \Phi^s \dot{x}(e^{\kappa s}t) \\ &\in e^{\kappa s} d_{x(e^{\kappa s}t)} \Phi^s F(x(e^{\kappa s}t)). \end{aligned}$$

Since F is ν-homogeneous of degree κ, we find $\frac{d}{dt}\Phi^s(x(e^{\kappa s}t)) \in F(\Phi^s(x(e^{\kappa s}t)))$ and thus $t \mapsto \Phi^s(x(e^{\kappa s}t))$ is a solution of the system (3.4) for all $s \in \mathbb{R}$. □

Remark 3.18. *This proposition is the extension of Proposition 1.22. The proposition may also be recast using the generalized flow, stating that, for all $t \geq 0$, for all $s \in \mathbb{R}$ and all compact set $K \subset \mathbb{R}^n$:*

$$\Psi^t(\Phi^s(K)) = \Phi^s(\Psi^{e^{\kappa s}t}(K)). \tag{3.5}$$

Now, similarly to the usual setting, a lot of properties can be extended from a sphere to everywhere outside the origin by homogeneity.

Proposition 3.19. *Let F be a ν-homogeneous set-valued map of degree κ. Then $F(x)$ is compact for all $x \in \mathbb{R}^n \setminus \{0\}$ iff $F(x)$ is compact for all $x \in \mathbb{S}$, where $\mathbb{S} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. The same property holds for convexity or upper semi-continuity.*

Proof. The result about compactness or convexity is straightforward. Let us only prove that if $F(x)$ is USC on the sphere, so is F everywhere outside of the origin.

Set $y \neq 0$. There exists $s \in \mathbb{R}$ and $x \in \mathbb{S}$ such that $\Phi^s(x) = y$. Fix $\mathcal{V}$ a neighborhood of $F(y) = F(\Phi^s(x)) = e^{\kappa s} d_x \Phi^s F(x)$. Eventually replacing $\mathcal{V}$ by a bounded neighborhood of $F(y)$ included in $\mathcal{V}$, we assume that $\mathcal{V}$ is bounded. Consider a bounded neighborhood $\mathcal{V}_0 \subset \mathcal{V}$ of $F(y)$ such that there exists $\alpha > 0$ with $\mathrm{d}(\mathcal{V}_0, \partial\mathcal{V}) \geq \alpha$, and denote by $\tilde{\mathcal{V}}_0 = e^{-\kappa s}(d_x\Phi^s)^{-1}\mathcal{V}_0$. $\tilde{\mathcal{V}}_0$ is a neighborhood of $F(x)$. Let us denote $M = \sup_{v \in \mathcal{V}_0} \|v\| > 0$. Let us also denote by $\sigma_{max}(d_z\Phi^s(d_x\Phi^s)^{-1})$ the biggest singular value of the linear mapping $d_z\Phi^s(d_x\Phi^s)^{-1}$. The function $\varphi : z \mapsto |\sigma_{max}(d_z\Phi^s(d_x\Phi^s)^{-1}) - 1|$ is continuous and vanishes at $z = x$. Therefore, there exists a neighborhood $\tilde{\mathcal{U}}$ of x on which $\varphi(z) < \frac{\alpha}{M}$. By upper semi-continuity of F at x, there exists $\tilde{\mathcal{U}}_0$ a neighborhood of x such that for all $\tilde{z} \in \tilde{\mathcal{U}}_0$, $F(\tilde{z}) \subset \tilde{\mathcal{V}}_0$. Set $\mathcal{U} = \Phi^s(\tilde{\mathcal{U}} \cap \tilde{\mathcal{U}}_0)$, then $\mathcal{U}$ is a neighborhood of y. Let z be an element of $\mathcal{U}$. Then there exists $\tilde{z} \in \tilde{\mathcal{U}} \cap \tilde{\mathcal{U}}_0$ such that $z = \Phi^s(\tilde{z})$. Therefore $F(z) = F(\Phi^s(\tilde{z})) = e^{\kappa s} d_{\tilde{z}}\Phi^s F(\tilde{z}) \subset e^{\kappa s} d_{\tilde{z}}\Phi^s \tilde{\mathcal{V}}_0$ since $\tilde{z} \in \tilde{\mathcal{U}}_0$. But $\tilde{\mathcal{V}}_0 = e^{-\kappa s}(d_x\Phi^s)^{-1}\mathcal{V}_0$, thus $F(z) \subset d_{\tilde{z}}\Phi^s(d_x\Phi^s)^{-1}\mathcal{V}_0$. Let $v \in \mathcal{V}_0$ be fixed. We have:

$$\|d_{\tilde{z}}\Phi^s(d_x\Phi^s)^{-1}v - v\| \leq \varphi(\tilde{z})M.$$

Since $\tilde{z} \in \tilde{\mathcal{U}}$, we find $\|d_{\tilde{z}}\Phi^s(d_x\Phi^s)^{-1}v - v\| \leq \alpha$, and hence $d_{\tilde{z}}\Phi^s(d_x\Phi^s)^{-1}v \in \mathcal{V}$. Finally we conclude that $F(z) \subset \mathcal{V}$ and the proposition is proved. □

As we have seen, in many situations, the set-valued map F comes from the Filippov's regularization procedure of a discontinuous vector field f. Suppose that we have a vector field f, which is homogeneous in the sense of the usual definition 1.21. If we apply the regularization procedure, is the homogeneity property preserved? The answer is positive as shown in the following proposition.

Proposition 3.20. *Let $f \in \mathcal{L}^\infty_{loc}(\mathbb{R}^n, \mathbb{R}^n)$ be a vector field and F be the associated set-valued map obtained by the Filippov's regularization procedure (3.3). Suppose f is ν-homogeneous of degree κ. Then F is ν-homogeneous of degree κ.*

Proof. Since for all $\varepsilon > 0$ there exist $\varepsilon_- > 0$ and $\varepsilon_+ > 0$ such that $\Phi^s(B(x, \varepsilon_-)) \subset$

$B(\Phi^s(x),\varepsilon) \subset \Phi^s(B(x,\varepsilon_+))$ we have:

$$\begin{aligned}
F(\Phi^s(x)) &= \bigcap_{\varepsilon>0}\bigcap_{N\in\mathcal{N}} \overline{\mathrm{conv}}(f(y), y\in B(\Phi^s(x),\varepsilon)\setminus N)\\
&= \bigcap_{\varepsilon>0}\bigcap_{N\in\mathcal{N}} \overline{\mathrm{conv}}(f(y), y\in \Phi^s(B(x,\varepsilon))\setminus N)\\
&= \bigcap_{\varepsilon>0}\bigcap_{N\in\mathcal{N}} \overline{\mathrm{conv}}(f(\Phi^s(z)), z\in B(x,\varepsilon)\setminus N)\\
&= \bigcap_{\varepsilon>0}\bigcap_{N\in\mathcal{N}} \overline{\mathrm{conv}}(e^{\kappa s}d_z\Phi^s f(z), z\in B(x,\varepsilon)\setminus N)\\
&= e^{\kappa s}d_x\Phi^s \bigcap_{\varepsilon>0}\bigcap_{N\in\mathcal{N}} \overline{\mathrm{conv}}((d_x\Phi^s)^{-1}d_z\Phi^s f(z), z\in B(x,\varepsilon)\setminus N).
\end{aligned}$$

Let us denote by $\sigma_{max}((d_x\Phi^s)^{-1}d_z\Phi^s)$ the biggest singular value of the linear mapping $(d_x\Phi^s)^{-1}d_z\Phi^s$. The function $\varphi : z \mapsto |\lambda_{max}(d_z\Phi^s(d_x\Phi^s)^{-1})-1|$ is continuous and therefore bounded on $B(x,\varepsilon)$ and moreover vanishes at $z=x$. For all $z\in B(x,\varepsilon)$ we have:

$$\|d_z\Phi^s(d_x\Phi^s)^{-1}f(z) - f(z)\| \leq M(\varepsilon),$$

where $M(\varepsilon) = \sup_{B(x,\varepsilon)}\varphi \operatorname{ess\,sup}_{B(x,\varepsilon)}\|f\|$. The function $M(\varepsilon)$ is continuous at zero and $M(0)=0$. We have proved that $d_z\Phi^s(d_x\Phi^s)^{-1}f(z) \in B(f(z), M(\varepsilon))$.

Let K be a compact, convex neighborhood of $F(x)$. Following the previous consideration, there exists $\varepsilon_0 > 0$ such that for all $0<\varepsilon<\varepsilon_0$ and for all $z\in B(x,\varepsilon)$, we have $F(z) + B(0, M(\varepsilon)) \subset \mathring{K}$, where $\mathring{K}$ denotes the interior of the set K. Therefore $\overline{\mathrm{conv}}\{F(z) + B(0,M(\varepsilon))\} \subset K$ and finally $F(\Phi^s(x)) \subset e^{\kappa s}d_x\Phi^s K$. Being compact and convex, $F(x)$ is equal to the intersection of all its compact convex neighborhood and hence $F(\Phi^s(x)) \subset e^{\kappa s}d_x\Phi^s F(x)$. Applying the same proof to $y=\Phi^s(x)$, we find $F(x) = F(\Phi^{-s}(y)) \subset e^{-\kappa s}d_y\Phi^{-s}F(y) = e^{-\kappa s}(d_x\Phi^s)^{-1}F(\Phi^s(x))$ and thus $F(\Phi^s(x)) = e^{\kappa s}d_x\Phi^s F(x)$. □

Example 3.21. *Consider the n-integrator with an input $u(x) = -\sum_i k_i \mathrm{sign}\,(x_i)$, $k_i>0$:*

$$\left\{\begin{array}{lcl}
\dot{x}_1 &=& x_2\\
\vdots & & \vdots\\
\dot{x}_{n-1} &=& x_n\\
\dot{x}_n &=& -\sum_i k_i \mathrm{sign}\,(x_i)
\end{array}\right. .$$

It is easy to check that this vector field is homogeneous of degree -1 w.r.t. to the generalized weight $(n,\dots,2,1)$. The associated differential inclusion is therefore homogeneous w.r.t. $(n,\dots,2,1)$ of degree -1 as well.

3.4 Main results

In the previous section, we have seen how to define a homogeneous discontinuous system and the basic properties stemming from this definition. But the classical theory of homogeneity highlights a lot of very important and useful properties of homogeneous systems. Among those, the ones we are going to generalize in this section are:

- the Theorem of Rosier [Rosier 1992a], which is a homogeneous converse Lyapunov theorem;
- the link between negative degree of homogeneity and finite-time stability [Bhat 1998], and the properties of the settling-time function [Bhat 1997];
- the consequences of the existence of a strictly positively invariant compact set [Bhat 2005];
- the equivalence of the notions of local attractivity and global stability for homogeneous systems [Filippov 1988], [Bhat 2005].

3.4.1 Converse Lyapunov theorem for homogeneous differential inclusions

The following theorem asserts that a strongly globally asymptotically stable homogeneous differential inclusion admits a homogeneous Lyapunov function. This result is a generalization of the theorem proved by L. Rosier for ODE [Rosier 1992a].

Theorem 3.22. *Let F be a ν-homogeneous set-valued map of degree κ, satisfying the standard assumptions. Then the following statements are equivalent:*

- *The origin is (strongly) GAS for the system (3.4).*
- *For all $\mu > \max(-\kappa, 0)$, there exists a pair (V, W) of continuous functions, such that:*
  1. *$V \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, V is positive definite and homogeneous of degree μ;*
  2. *$W \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$, W is strictly positive outside the origin and homogeneous of degree $\mu + \kappa$;*
  3. *$\max_{v \in F(x)} d_x V v \leq -W(x)$ for all $x \neq 0$.*

Proof. By the result of [Clarke 1998], the two following statements are equivalent:

- The system (3.4) is strongly GAS.
- There exist a pair (V_0, W_0) of continuous functions, such that:
  1. $V_0 \in \mathcal{C}^\infty(\mathbb{R}^n)$, V_0 is positive definite;
  2. $W_0 \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$, W_0 is strictly positive outside the origin;
  3. $\max_{v \in F(x)} d_x V_0 v \leq -W_0(x)$ for all $x \neq 0$.

Hence, it suffices to prove that the homogeneity condition allows us to build a homogeneous Lyapunov pair. The sequel of the proof is widely inspired by the proof from [Rosier 1992a]. Let $a : [0, +\infty[\to [0, 1]$ be a class $\mathcal{C}^\infty$ function such that for all $t \leq 1$, $a(t) = 0$, for all $t \geq 2$, $a(t) = 1$ and for all $t \in]1, 2[$, $a'(t) > 0$. Set $\mu > \max(-\kappa, 0)$ and:

$$V(x) = \int_{\mathbb{R}} e^{-\mu s} a(V_0(\Phi^s(x)))ds,$$

then $V(0) = 0$. For all $x \neq 0$, there exists s_1 such that for all $s \leq s_1$, $V_0(\Phi^s(x)) \leq 1$. Similarly, there exists s_2 such that for all $s \geq s_2$, $V_0(\Phi^s(x)) \geq 2$. Hence:

$$V(x) = \int_{s_1}^{s_2} e^{-\mu s} a(V_0(\Phi^s(x)))ds + \frac{e^{-\mu s_2}}{\mu},$$

and V is well-defined.

The homogeneity of V is straightforward using a change of variable: $V(\Phi^\sigma(x)) = \int_{\mathbb{R}} e^{-\mu s} a(V_0(\Phi^s(\Phi^\sigma(x))))ds = e^{\mu\sigma} \int_{\mathbb{R}} e^{-\mu u} a(V_0(\Phi^u(x)))du = e^{\mu\sigma} V(x)$.

On the other hand, for all $s \in \mathbb{R}$, $e^{-\mu s} a(V_0(\Phi^s(x)))$ is $\mathcal{C}^\infty$ and $|e^{-\mu s} a(V_0(\Phi^s(x)))| \leq e^{-\mu s}$ which is integrable ($\mu > 0$). Thus V belongs to the class $\mathcal{C}^\infty$ on $\mathbb{R}^n$ and therefore proper [Bhat 2005]. Moreover, for all $v \in F(x)$:

$$d_x V v = \int_{\mathbb{R}} e^{-\mu s} a'(V_0(\Phi^s(x))) \left(d_{\Phi^s(x)} V_0\right) \left(d_x \Phi^s\right) v ds.$$

As F is homogeneous, there exists $\tilde{v} \in F(\Phi^s(x))$ such that $\tilde{v} = e^{\kappa s} d_x \Phi^s v$. Hence:

$$\begin{aligned} d_x V v &= \int_{\mathbb{R}} e^{-(\mu+\kappa)s} a'(V_0(\Phi^s(x))) \left(d_{\Phi^s(x)} V_0\right) \tilde{v} ds \\ &\leq -\int_{\mathbb{R}} e^{-(\mu+\kappa)s} a'(V_0(\Phi^s(x))) W_0(\Phi^s(x)) ds. \end{aligned}$$

Let us denote:

$$W(x) = \int_{\mathbb{R}} e^{-(\mu+\kappa)s} a'(V_0(\Phi^s(x))) W_0(\Phi^s(x)) ds,$$

thus $\max_{v\in F(x)} d_x V v \leq -W(x)$. It is clear that W is well-defined and strictly positive. The function W is clearly homogeneous of degree $\kappa+\mu$ (this fact can be also proven using a simple change of variable). Moreover, for all $s\in\mathbb{R}$, the function

$$x \mapsto e^{-(\mu+\kappa)s} a'(V_0(\Phi^s(x)))W_0(\Phi^s(x))$$

is in the class $\mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$. Let us show that $|e^{-(\mu+\kappa)s}a'(V_0(\Phi^s(x)))W_0(\Phi^s(x))|$ is locally upper-bounded by an integrable function. Set $U_x = \bar{B}(x, |x|/2)$. For $x\neq 0$, U_x is a neighborhood of x. Since ν is Euler, there exists s_1, s_2 such that for all $y\in U_x\subset \mathbb{R}^n\setminus\{0\}$, for all $s\leq s_1$, $V_0(\Phi^s(y))\leq 1$ and for all $s\geq s_2$, $V_0(\Phi^s(y))\geq 2$. Hence $a'(V_0(\Phi^s(y)))=0$ for all $s \notin]s_1,s_2[$ and for all $y\in U_x$. Denote $c_1 = \sup_{y\in U_x}\sup_{s\in[s_1,s_2]} W(\Phi^s(y))$ and $c_2=\sup_{t\in\mathbb{R}} a'(t)$. We get:

$$|e^{-(\mu+\kappa)s}a'(V_0(\Phi^s(x)))W_0(\Phi^s(x))| \leq e^{-(\mu+\kappa)s}\mathbf{1}_{[s_1,s_2]}c_1c_2$$

which is clearly integrable, where $\mathbf{1}_A$ is the characteristic function of the set A, that is $\mathbf{1}_A(x)=1$ if $x\in A$ and 0 else. Therefore, W is $\mathcal{C}^\infty$ on a neighborhood of x for all $x\in\mathbb{R}^n\setminus\{0\}$, i.e. W is $\mathcal{C}^\infty$ on $\mathbb{R}^n\setminus\{0\}$.

The only point remaining to prove is the continuity of W at the origin. Let $\varepsilon>0$ be fixed. There exists s_1 such that for all $s\leq s_1$, $V_0(\Phi^s(y))\leq 1$ for all $y\in B(0,\varepsilon)$. Thus, introducing the sets $A=\{V_0(\Phi^s(y))>2\}$ and $B=\{V_0(\Phi^s(y))\leq 2\}$, for all $y\in B(0,\varepsilon)$, we have

$$\begin{array}{rcl} W(y) & = & \int_{s_1}^{+\infty} e^{-(\mu+\kappa)s}a'(V_0(\Phi^s(y)))W_0(\Phi^s(y))ds \\ & = & \int_{s_1}^{+\infty} e^{-(\mu+\kappa)s}a'(V_0(\Phi^s(y)))W_0(\Phi^s(y))\mathbf{1}_A ds \\ & & + \int_{s_1}^{+\infty} e^{-(\mu+\kappa)s}a'(V_0(\Phi^s(y)))W_0(\Phi^s(y))\mathbf{1}_B ds. \end{array}$$

Since $a'(t)=0$ for $t>2$ the first part vanishes. But V_0 is proper, thus B is compact and W_0 is bounded by $c_3>0$ on this set. Therefore

$$\begin{array}{rcl} W(y) & = & \int_{s_1}^{+\infty} e^{-(\mu+\kappa)s}a'(V_0(\Phi^s(y)))W_0(\Phi^s(y))\mathbf{1}_B ds \\ & \leq & \int_{s_1}^{+\infty} e^{-(\mu+\kappa)s}c_2c_3 ds < +\infty, \end{array}$$

since $\mu+\kappa>0$. Finally, W is continuous at the origin and the proof is completed. □

3.4.2 Application to Finite-Time Stability

In this subsection, we aim at applying the Theorem 3.22 to Finite-Time Stability (FTS). Indeed, the existence of a smooth homogeneous Lyapunov function provides informations about the rate of convergence of such systems.

Definition 3.23. *Consider the system (3.4). The origin is said to be FTS if:*

1. *the origin is (strongly) stable;*
2. *there exists an open neighborhood $\mathcal{U}$ of the origin such that for all $x \in \mathcal{U}$, there exists $\tau \geq 0$ such that for all $t \geq \tau$, we have $\Psi^t(x) = \{0\}$ (strong finite-time convergence).*

The settling-time function is then defined for $x \in \mathcal{U}$ by $\mathsf{T}(x) = \inf\{\tau \geq 0 : \forall t \geq \tau, \Psi^t(x) = \{0\}\}$.

If the neighborhood $\mathcal{U}$ can be chosen to be $\mathbb{R}^n$, the system is said to be Globally FTS (GFTS).

Corollary 3.24. *Let F be a ν-homogeneous set-valued map of degree $\kappa < 0$, satisfying the standard assumptions. Assume also that the origin is GAS for F. Then the origin is GFTS for F and the settling-time function is continuous at zero and locally bounded.*

Proof. The origin is GAS for F and F is homogeneous; thus by Theorem 3.22, F admits a homogeneous Lyapunov pair (V, W). Let us apply Lemma 4.2 of [Bhat 2005] to the continuous functions V and W. We get that for all $x \in \mathbb{R}^n \setminus \{0\}$, and for all $v \in F(x)$:

$$d_x V v \leq -W(x) \leq -C\left(V(x)\right)^{\frac{\kappa+\mu}{k}}, \tag{3.6}$$

where $C = \min_{\{V=1\}} W$. Since $\frac{\kappa+\mu}{\mu} < 1$, V converges to zero in a finite time, giving us the finite-time convergence of the system, which is therefore GFTS. Moreover, a direct integration of the inequation (3.6) gives $\mathsf{T}(x) \leq \frac{\mu V(x)^{\frac{-\kappa}{\mu}}}{-\kappa C}$, where T denotes the settling-time function. Since V is continuous, T is locally bounded and continuous at zero. □

It has been shown in [Bhat 2005] that under the assumptions of homogeneity (of negative degree), continuity of the right-hand side and forward uniqueness of solutions, the settling-time function of a finite-time stable system is continuous. The two latter do obviously not hold in our context. We have seen that, however, under the standard assumptions, the settling-time function remains continuous at the origin and locally bounded. Let us emphasize that these conclusions are sharp and that the settling-time function is *not* continuous in general. See, for instance, [Polyakov 2012] or the following example.

Example 3.25. *(A counterexample to the 2^{nd} statement of Theorem 1 from [Levant 2005]) Consider the system defined on $\mathbb{R}^2$ by:*

$$\dot{x} = -(\mathrm{sign}\,(x_1) + 2)\frac{x}{\|x\|}.$$

This system is clearly strongly (uniformly [Levant 2005]) GFTS and ν-homogeneous of negative degree with $\nu = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$. A simple computation shows that the settling-time function is:

$$\mathsf{T}(x) = \begin{cases} \|x\| & x_1 \geq 0 \\ \frac{\|x\|}{3} & x_1 < 0 \end{cases},$$

which is discontinuous on $x_1 = 0$.

3.4.3 Sufficient conditions for Global Asymptotic Stability

In this subsection, we focus on the qualitative properties of homogeneous discountinuous systems that can lead to GAS. The first result is a generalization of Theorem 6.1 of [Bhat 2005].

Theorem 3.26. *Suppose that K is a strongly strictly positively invariant compact subset (SPI) of $\mathbb{R}^n$ for the homogeneous system (3.4). Then the origin is GAS for (3.4).*

Proof. Let us denote κ the degree of F. Since the solutions starting in K are bounded, they are defined for all $t \geq 0$, and thus $\Psi^t(K)$ is compact for all $t > 0$ by Proposition 3.12. From equation (3.5), we have:

$$\Psi^t(\Phi^s(K)) = \Phi^s(\Psi^{e^{\kappa s}t}(K)) \subset \Phi^s(\overset{\circ}{K}) = \overset{\circ}{\overbrace{\Phi^s(K)}}.$$

Therefore, the set $\Phi^s(K)$ is SPI for all $s \in \mathbb{R}$. We also note that $\Psi^{s+t}(K) = \Psi^t(\Psi^s(K)) \subset \Psi^t(K)$. Thus $(\Psi^t(K))_{t\geq 0}$ is a nested family of compact sets. Let us denote K_∞ their intersection; K_∞ is a non-empty compact, and is the biggest positively invariant compact subset of K. But for all $s \in \mathbb{R}$

$$\Phi^s(K_\infty) = \bigcap_{t\geq 0} \Phi^s(\Psi^t(K)) = \bigcap_{\tau\geq 0} \Psi^\tau(\Phi^s(K))$$

has the same property. Therefore $K_\infty = \Phi^s(K_\infty)$, that is K_∞ is an invariant subset for Φ. Since ν is Euler, we conclude that $K_\infty = \{0\}$ and every solution starting in K converges to the origin, thus $0 \in K$. The stability follows from the SPI of the sets $\Phi^s(K)$ for all $s \in \mathbb{R}$. □

Let us illustrate how this Theorem can be used to derive robustness properties for some homogeneous systems.

Example 3.27. *Assume we know $k_1, k_2 > 0$ such that the following system is GAS (the existence of such gains will be proved in Chapter 5):*

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -k_1 \mathrm{sign}\,(x_1) - k_2 \mathrm{sign}\,(x_2) \end{array} \right. .$$

As we have seen, this system is homogeneous w.r.t. the weight $(2,1)$ of degree -1. Therefore, there exists a homogeneous Lyapunov pair (V, W) of degrees $\kappa > 1$ and $\kappa - 1$. We denote F_0 the set valued map associated to this vector field and for $\alpha \in \mathbb{R}^2$ we denote F_α the set valued map associated to the vector field:

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -(k_1 + \alpha_1) \mathrm{sign}\,(x_1) - (k_2 + \alpha_2) \mathrm{sign}\,(x_2) \end{array} \right. .$$

We shall prove that the compact set $K = \{x \in \mathbb{R}^2 : V(x) \leq 1\}$ is SPI for F_α, for small values of α. Let $y \in K$ and $v \in F_\alpha(y)$. There exists $x \in S = \{x \in \mathbb{R}^n : V(x) = 1\}$ and $s \in \mathbb{R}$ such that $\Phi^s(x) = y$. By homogeneity, there also exists $w \in F_\alpha(x)$ such that $v = e^{-s} d_x \Phi^s w$. Therefore:

$$d_y V v = d_{\Phi^s(x)} V e^{-s} d_x \Phi^s w = e^{(\kappa-1)s} d_x V w.$$

Since $w \in F_\alpha(x)$, there exist $\sigma_1, \sigma_2 \in [-1, 1]$ such that $w = (x_2, -(k_1 + \alpha_1)\sigma_1 - (k_2 + \alpha_2)\sigma_2)^T$. Let us denote $\tilde{w} = (x_2, -k_1\sigma_1 - k_2\sigma_2)^T \in F_0(x)$. We have:

$$\begin{array}{rcl} d_y V v & = & e^{(\kappa-1)s} d_x V w \\ & = & e^{(\kappa-1)s} [d_x V \tilde{w} + d_x V(w - \tilde{w})] \\ & \leq & e^{(\kappa-1)s} \left[-W(x) + \sup\limits_{x \in S} \|d_x V\|.\|w - \tilde{w}\|| \right] \\ & \leq & e^{(\kappa-1)s} \left[-\inf\limits_{x \in S} W(x) + \sup\limits_{x \in S} \|d_x V\|.\| \sum\limits_{i=1}^{2} \alpha_i \sigma_i \| \right] \\ & \leq & e^{(\kappa-1)s} \left[-\inf\limits_{x \in S} W(x) + \sup\limits_{x \in S} \|d_x V\|. \sum\limits_{i=1}^{2} \|\alpha_i\| \right] . \end{array}$$

Therefore, if $|\alpha_1| + |\alpha_2| < \frac{\inf_{x \in S} W(x)}{\sup_{x \in S} \|d_x V\|}$, $d_y V v < 0$, which means that K is strictly positively invariant. The set valued map F_α being homogeneous, it is hence GAS for α small enough. Finally, any stabilizing control under the form $u(x) = -k_1 \mathrm{sign}\,(x_1) - k_2 \mathrm{sign}\,(x_2)$ is robust w.r.t. small errors on the gains k_i, like implementation errors.

As we have seen in the introduction, for continuous homogeneous systems (with forward uniqueness of solutions) the notions of local attractiveness and global asymptotic stability are merged. This fact admits a generalization in the discontinuous setting.

Theorem 3.28. *Let F be a ν-homogeneous set-valued map of degree κ with the standard assumptions. Assume moreover that all the solutions of the associated DI are defined for all $t \geq 0$ and tend to 0 when $t \to \infty$. Then the origin is strongly globally asymptotically stable.*

Proof. By contradiction, assume that the origin is unstable. Then there exists a neighborhood $\mathcal{U}$ of the origin such that for all neighborhoods of the origin $\mathcal{V} \subset \mathcal{U}$, there exists a solution starting in $\mathcal{V}$ which does not stay in $\mathcal{U}$. Taking $\mathcal{V} = B(0, 1/i)$, there exists a solution x_i such that $x_i(0) \in \mathcal{V}$, and there exists a real number t_i such that $x_i(t_i) \notin \mathcal{U}$. Therefore, when $i \to \infty$, $x_i(0) \to 0$ but $(x_i(t_i))$ does not converge to 0.

Let us denote by N a ν-homogeneous norm and denote $\delta_i = N(x_i(0))$. There exists $\varepsilon > 0$ such that $N(x_i(t_i)) \geq \varepsilon$. We can also assume that $\delta_i < \varepsilon$. Since $x_i(t_i) > 0$, we can finally assume that $\delta_i > 0$ by continuity.

Let us denote:

$$\begin{aligned} a_i &= \sup\{t \in [0, t_i] : N(x_i(t)) = \delta_i\}, \\ b_i &= \inf\{t \in [0, t_i] : N(x_i(t)) = \varepsilon\}. \end{aligned}$$

We define $y_i(t) = x_i(t + a_i)$. The curves y_i are solutions of (3.4) defined on $[0, b_i - a_i]$ and we have $N(y_i(0)) = \delta_i$, $N(y_i(t)) \in (\delta_i, \varepsilon)$ for all $t \in (0, b_i - a_i)$ and $N(y_i(b_i - a_i)) = \varepsilon$.

By Proposition 3.17, for all $s \in \mathbb{R}$, the curve $t \mapsto \Phi^s(y_i(e^{\kappa s} t))$ is a solution. Set $s_i = -\ln \delta_i$, $z_i(t) = \Phi^{s_i}(y_i(e^{\kappa s_i} t))$ and $t_i^* = \delta_i^\kappa (b_i - a_i)$. We find $N(z_i(t)) = e^{s_i} N(y_i(e^{\kappa s_i} t)) = N(y_i(e^{\kappa_i} t))/\delta_i$. Hence $N(z_i(0)) = 1$, for all $t \in (0, t_i^*)$ we have $N(z_i(t)) \geq 1$ and $N(z_i(t_i^*)) = \varepsilon/\delta_i$.

Assume that there exists a bounded subsequence of (t_i^*). By Theorem 3.13, the corresponding subsequence $(z_j(t_j^*))$ is bounded; however $N(z_j(t_j^*)) = \frac{\varepsilon}{\delta_j} \to \infty$. Then the sequence (t_i) tends to $+\infty$.

By Corollary 3.14, let us now extract a subsequence $(z_{\varphi_1(i)})$ converging to a solution $\bar{z}$ on $[0, 1]$. Then we extract a subsubsequence $(z_{\varphi_1 \circ \varphi_2(i)})$ converging to $\bar{z}$ on $[0, 2]$, etc. A diagonal extraction provides us the subsequence $(z_{\varphi_1 \circ \ldots \circ \varphi_i(i)})$ which is converging to $\bar{z}$ uniformly on $[0, j]$ for all $j \in \mathbb{N}$. For all $t > 0$, we have $N(\bar{z}(t)) = \lim_i N(z_{\varphi_1 \circ \ldots \circ \varphi_i(i)(t)})$. But for i large enough, $t \leq t^*_{\varphi_1 \circ \ldots \circ \varphi_i(i)}$ and thus $N(z_{\varphi_1 \circ \ldots \circ \varphi_i(i)(t)}) \geq 1$. Therefore $N(\bar{z}(t)) \geq 1$ and $\bar{z}(t)$ does not tend to zero, which is a contradiction. $\square$

Example 3.29. *Consider a double integrator endowed with an observer written with the error equation under the form:*

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= k_1 \operatorname{sign}(x_1) + k_2 \operatorname{sign}(x_2 - e_2) \\ \dot{e}_1 &= e_2 + l_1 \lfloor e_1 \rceil^{1/2} \\ \dot{e}_2 &= l_2 \operatorname{sign}(e_1) \end{cases} \tag{3.7}$$

with $k_1 < k_2 < 0$*,* $l_1, l_2 < 0$*. This system is weighted homogeneous of degree* -1 *with respect to the generalized weight* $\mathbf{r} = (2,1,2,1)$.

Consider first the error subsystem:

$$\begin{cases} \dot{e}_1 &= e_2 + l_1 \lfloor e_1 \rceil^{1/2} \\ \dot{e}_2 &= l_2 \operatorname{sign}(e_1) \end{cases} . \tag{3.8}$$

Using the function $V(e) = -l_2|e_1| + e_2^2/2$*, we see that the system (3.8) is FTS. Indeed, for* $e_1 \neq 0$*,* $\dot{V}(e) = -l_1 l_2 |e_1|^{1/2} < 0$*. Hence the compact set* $K = \{V \leq 1\}$ *is SPI, since the line* $e_1 = 0$ *is not invariant. Being of negative degree, we find that the system (3.8) is FTS.*

The system (3.7) being clearly forward complete, it becomes equivalent in a finite time to the system:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= k_1 \operatorname{sign}(x_1) + k_2 \operatorname{sign}(x_2) \end{cases} . \tag{3.9}$$

Using the Lyapunov function $V(x) = -k_1|x_1| + x_2^2$ *and* $k_1 < k_2$*, we find that the system (3.9) is FTS as well.*

Finally, all the solutions of the system (3.7) converge in finite-time to 0*, but stability is not straightforward. However, the Theorem 3.28 ensures us that the stability is a consequence of the attractiveness for homogeneous systems and we find that the system (3.7) is GFTS.*

3.5 Conclusion

In this Chapter, we have proposed a geometric definition of homogeneity for DI, and we have seen that it is consistent with respect to the Filippov's regularization procedure. With this framework, we have been able to state extensions to the DI setting of results holding for continuous homogeneous systems:

- The converse Lyapunov theorem of Rosier – if the origin is a globally asymptotically stable equilibrium, then there exists a Lyapunov function (indeed, a Lyapunov pair) for the DI.
- If the origin is a globally asymptotically stable equilibrium for a homogeneous DI of negative degree, then it is strongly FTS.
- The existence of a SPI compact set is equivalent to global asymptotic stability.
- The local attractiveness of the origin implies its global asymptotic stability.

Chapter 4

Robustness and stability

Contents

4.1 Introduction

The problem of robustness and stability analysis with respect to external inputs (like exogenous disturbances or measurement noises) for dynamical systems is in the center of attention of many research works [Doyle 1992, Hill 1980, van der Schaft 1996, Sontag 2001, Vidyasagar 1981, Willems 1972]. One of the most popular theories, which can be used for this robustness analysis for nonlinear systems, was originated more than twenty years ago [Sontag 1989] and it is based on the Input-to-State Stability (ISS) property and many related notions, see for instance the recent survey [Dashkovskiy 2011] and the references therein. The advantages of ISS theory include a list of necessary and sufficient conditions, existence of the Lyapunov method extension, a rich variety of stability concepts adopted for different control and estimation problems.

The main tool to check the ISS property for a nonlinear system consists in designing a Lyapunov function that satisfies some sufficient conditions. As usual, there is no generic approach to select a Lyapunov function for nonlinear systems. Therefore, computationally tractable approaches for ISS verification for particular classes of nonlinear systems are of great importance, and they are highly demanded in applications. In this chapter we are going to propose such a technique for checking ISS and integral ISS (iISS) for a class of homogeneous or locally homogeneous systems. We will restrict ourselves to the weighted homogeneity framework.

The ISS notion of homogeneous systems has been studied in [Hong 2001, Ryan 1995, Andrieu 2008]. In this chapter we are going to generalize the result of those works and extend it to the integral ISS (iISS) property. The underlying idea of the proposed results is that for a nonlinear system its asymptotic stability with zero disturbance implies a certain robustness (ISS or iISS) under homogeneity conditions.

The outline of this chapter is as follows. The section 4.2 is devoted to the necessary preliminaries, including the stability and robustness notions that we shall use afterwards. The ISS and iISS properties of homogeneous systems are studied in Section 4.3 and the same analysis for locally homogeneous systems is done in Section 4.4. The contents of this Chapter have been published in [Bernuau 2013e], submitted in [Bernuau 2013h] and used in [Bernuau 2013g].

4.2 Preliminaries

In the sequel the following nonlinear system is considered:

$$\dot{x} = f(x, d), \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the state, $d \in \mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}^m)$ is the external input and $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is a continuous vector field with $f(0,0) = 0$.

For a given input $d \in \mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}^m)$ we denote $\mathcal{S}_d(x_0)$ the set of solutions of (4.1) defined on a time interval containing $t = 0$ and with value x_0 at $t = 0$.

We will be interested in the following stability properties [Dashkovskiy 2011].

Definition 4.1. *The system (4.1) is called* input-to-state practically stable (ISpS*), if for any input $d \in \mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}^m)$ and any $x_0 \in \mathbb{R}^n$ there exist some functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and $c \geq 0$ such that for all $x \in \mathcal{S}_d(x_0)$:*

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(||d||_{[0,t]}) + c, \quad \forall t \geq 0.$$

The function γ is called the nonlinear asymptotic gain*. The system is called* ISS *if $c = 0$.*

Definition 4.2. *The system (4.1) is called* iISS, *if there exist some functions* $\alpha \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}$ *and* $\beta \in \mathcal{KL}$ *such that for any* $x_0 \in \mathbb{R}^n$, $d \in \mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}^m)$ *and* $x \in \mathcal{S}_d(x_0)$, *the following estimate holds:*

$$\alpha(\|x(t)\|) \leq \beta(\|x_0\|, t) + \int_0^t \gamma(\|d(s)\|)\, ds, \quad \forall t \geq 0.$$

The Definitions 4.1 and 4.2 have the following Lyapunov function characterizations.

Definition 4.3. *A smooth function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *is called an* ISpS Lyapunov function *for the system (4.1) if for all* $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ *and some* $r \geq 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ *and* $\theta \in \mathcal{K}$*:*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$
$$d_x V f(x, d) \leq r + \theta(\|d\|) - \alpha_3(\|x\|).$$

Such a function V *is called an* ISS Lyapunov function *if* $r = 0$. *It is an* iISS Lyapunov function *if* $r = 0$ *and if* $\alpha_3 : \mathbb{R}_+ \to \mathbb{R}_+$ *is only assumed to be positive definite instead of class* $\mathcal{K}_\infty$.

Note that an ISpS Lyapunov function can also satisfy the following equivalent condition for some $\gamma \in \mathcal{K}$ and $c \geq 0$:

$$\|x\| > c + \gamma(\|d\|) \Rightarrow d_x V f(x, d) \leq -\alpha_3(\|x\|).$$

Theorem 4.4. *[Sontag 1995] The system (4.1) is ISS (ISpS, iISS) iff it admits an ISS (ISpS, iISS) Lyapunov function.*

Note that if the system (4.1) is ISS, then it is also iISS.

4.3 Robustness of homogeneous systems

The ISS property of a $\mathbf{r}$–homogeneous system (4.1) of degree $\kappa > 1$ has been investigated in [Ryan 1995], while the ISS property of a $\mathbf{r}$–homogeneous system of the form

$$\dot{x} = f_0(x) + G_0(x)d \tag{4.2}$$

for any admissible degree $\kappa \geq -r_{\min}$ (with homogeneous f_0 and G_0) has been studied in [Hong 2001]. In this chapter we would like to propose conditions of ISS and iISS properties for a $\mathbf{r}$-homogeneous continuous system (4.1) of any degree κ.

Let us define

$$\tilde{f}(x,d) = [f(x,d)^T \ 0_m^T]^T \in \mathbb{R}^{n+m}.$$

The vector field $\tilde{f}$ is an extended auxiliary vector field for the system (4.1), where 0_m is the zero vector of dimension m.

Theorem 4.5. *Let the continuous vector field $\tilde{f}$ be homogeneous with the weights $\mathbf{r} = [r_1, \ldots, r_n] > 0$ and $\tilde{\mathbf{r}} = [\tilde{r}_1, \ldots, \tilde{r}_m] \geq 0$ of degree κ, i.e. for all $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ and all $\lambda > 0$ we have $f(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)d) = \lambda^\kappa \Lambda_r(\lambda) f(x,d)$. Assume that the system (4.1) is globally asymptotically stable for $d = 0$, then*

- *the system (4.1) is ISS if $\tilde{r}_{\min} > 0$,*
- *the system (4.1) is iISS if $\tilde{r}_{\min} = 0$ and $\kappa \leq 0$,*

where $\tilde{r}_{\min} = \min_{1 \leq j \leq m} \tilde{r}_j$.

Proof. Under the introduced conditions $f(\Lambda_r(\lambda)x, 0) = \lambda^\kappa \Lambda_r(\lambda) f(x,0)$ and the system $\dot{x} = f(x,0)$ is globally asymptotically stable, therefore by Theorem 1.5 there exists a $\mathbf{r}$-homogeneous, continuously differentiable, positive definite and proper Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ of degree μ such that $\mu + \kappa > 0$, $\mu > r_{max} := \max_{1 \leq j \leq m} r_j$ and

$$d_x V f(x,0) \leq -a, \quad \|d_x V\| \leq b, \quad \forall x \in \{V = 1\}, \tag{4.3}$$

where $a > 0$, $b > 0$.

There exists a function $\sigma \in \mathcal{K}$ such that:

$$\|f(y,d) - f(y,0)\| \leq \sigma(\|d\|) \quad \forall y \in \{V \leq 1\}. \tag{4.4}$$

Let us denote $f_d(x) = f(x,d)$. Consider now the time derivative of the Lyapunov function V computed for the system (4.1) for all $x \in \{z \in \mathbb{R}^n : V(z) = 1\}$, $\lambda > 0$ and $d \in \mathbb{R}^m$:

$$\begin{aligned}
\mathcal{L}_{f_d} V(\Lambda_r(\lambda)x) &= \mathcal{L}_{f_0} V(\Lambda_r(\lambda)x) + \mathcal{L}_{(f_d - f_0)} V(\Lambda_r(\lambda)x) \\
&= \lambda^{\kappa+\mu} \left[\mathcal{L}_{f_0} V(x) + d_x V \left(\lambda^{-\kappa} \Lambda_r(\lambda)^{-1} (f_d - f_0)(\Lambda_r(\lambda)x)\right)\right] \\
&= \lambda^{\kappa+\mu} \left[\mathcal{L}_{f_0} V(x) + d_x V \left((f_{\Lambda_{\tilde{r}}^{-1}(\lambda)d} - f_0)(x)\right)\right] \\
&\leq \lambda^{\kappa+\mu} \left[-a + b\sigma(\|\Lambda_{\tilde{r}}^{-1}(\lambda)d\|)\right]
\end{aligned}$$

We denote $\|\Lambda_{\tilde{r}}^{-1}(\lambda)\| = \delta(\lambda)$ with

$$\delta(s) = \begin{cases} s^{-\tilde{r}_{\max}} & \text{if } s \leq 1 \\ s^{-\tilde{r}_{\min}} & \text{if } s \geq 1 \end{cases}.$$

Now, if $\tilde{r}_{\min} > 0$, then δ is continuous on $\mathbb{R}_+ \setminus \{0\}$ and strictly decreasing. Consider the following function:

$$\theta(s) = \delta^{-1}\left(\frac{\sigma^{-1}(a/2b)}{s}\right).$$

The function θ is continuous on $\mathbb{R}_+$ with $\theta(0) = \lim_{s\to\infty} \delta^{-1}(s) = 0$ and strictly increasing, hence $\theta \in \mathcal{K}$. If $\lambda \geq \theta(D)$ then $\delta(\lambda) \leq \frac{\sigma^{-1}(a/2b)}{D}$ and thus for all $|d| \leq D$, we have $\|\Lambda_{\tilde{r}}^{-1}(\lambda)d\| \leq \gamma(\lambda)D \leq \sigma^{-1}(a/2b)$. Therefore:

$$\mathcal{L}_{f_d} V(\Lambda_r(\lambda)x) \leq -\frac{a}{2}\lambda^{\kappa+\mu},$$

that is $\mathcal{L}_{f_d} V(y) \leq -\frac{a}{2} V(y)^{\frac{\mu+\kappa}{\mu}}$ for all $V(y) \geq \theta(D)$ and for all $\|d\| \leq D$, and then the system (4.1) is ISS by Theorem 4.4.

Now assume that $\tilde{r}_{\min} = 0$ and $\kappa \leq 0$. First, note that we have $-r_{\min} < \kappa$. Indeed, if not, the coordinate of f_0 which corresponds to the weight $r_{\min}$ is **r**-homogeneous of degree $\kappa + r_{\min} \leq 0$ and continuous, thus constant. But this is impossible since the origin is a GAS equilibrium for f_0.

Consider now the function $W(x) = \ln(1 + V(x))$. This function is C^1, proper and positive definite. We get:

$$\begin{aligned} \mathcal{L}_{f_d} W(\Lambda_r(\lambda)x) &= \frac{\lambda^{\kappa+\mu}}{1+\lambda^\mu}\mathcal{L}_{f_0} V(x) + \frac{d_x V}{1+\lambda^\mu}\left(\lambda^\mu \Lambda_r^{-1}(\lambda)(f_d - f_0)(\Lambda_r(\lambda)x)\right) && (4.5) \\ &= \frac{\lambda^{\kappa+\mu}}{1+\lambda^\mu}\mathcal{L}_{f_0} V(x) + \frac{\lambda^{\kappa+\mu}}{1+\lambda^\mu} d_x V\left((f_{\Lambda_{\tilde{r}}^{-1}(\lambda)d} - f_0)(x)\right). && (4.6) \end{aligned}$$

For all $\lambda > 0$, we have $\frac{1}{1+\lambda^\mu} \leq 1$. For $\lambda \geq 1$, we have also $\frac{\lambda^{\kappa+\mu}}{1+\lambda^\mu} \leq 1$ and $\delta(\lambda) = 1$, and thus $\|\Lambda_{\tilde{r}}^{-1}(\lambda)d\| \leq \|d\|$. Therefore (4.6) gives:

$$\begin{aligned} \mathcal{L}_{f_d} W(\Lambda_r(\lambda)x) &\leq -a\frac{\lambda^{\kappa+\mu}}{1+\lambda^\mu} + b\sigma(\|\Lambda_{\tilde{r}}^{-1}(\lambda)d\|) \\ &\leq -a\frac{V(\Lambda_r(\lambda)x)^{\frac{\kappa+\mu}{\mu}}}{1+V(\Lambda_r(\lambda)x)} + b\sigma(\|d\|). \end{aligned}$$

For $\lambda \leq 1$, $V(\Lambda_r(\lambda)x) \leq 1$, thus $\|(f_d - f_0)(\Lambda_r(\lambda)x)\| \leq \sigma(\|d\|)$. Moreover, since $\mu > r_{\max}$,

$\|\lambda^{\mu}\Lambda_r^{-1}(\lambda)\| \leq 1$. Using (4.5), we find:

$$\mathcal{L}_{f_d} W(\Lambda_r(\lambda)x) \leq -a\frac{V(\Lambda_r(\lambda)x)^{\frac{\kappa+\mu}{\mu}}}{1+V(\Lambda_r(\lambda)x)} + b\sigma(\|d\|). \tag{4.7}$$

Finally, the relation (4.7) holds for any $\lambda > 0$, and therefore, for all $y \neq 0$:

$$\mathcal{L}_{f_d} W(y) \leq -a\frac{V(y)^{\frac{\kappa+\mu}{\mu}}}{1+V(y)} + b\sigma(\|d\|)$$

and the system is iISS by Theorem 4.4. □

As we can conclude from this result, for the homogeneous system (4.1) its robustness (ISS or iISS property) is a function of its degree of homogeneity.

Corollary 4.6. *Let a continuous vector field $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ be $\mathbf{r}$–homogeneous of degree κ and let the origin be a globally asymptotically stable equilibrium.*

- *If $f(x, d) = f_0(x) + d$, i.e. d is an additive disturbance, then the system (4.1) is ISS for $\kappa > -r_{\min}$, and iISS for $\kappa = -r_{\min}$.*
- *If $f(x, d) = f_0(x+d)$, i.e. d is a measurement noise, then the system (4.1) is always ISS.*

Proof. Take $\tilde{\mathbf{r}} = \mathbf{r} + \kappa$ and $\tilde{\mathbf{r}} = \mathbf{r}$ for the additive disturbance and measurement noise cases respectively. □

Thus to verify robustness of a homogeneous system with respect to an external input it is enough to establish its asymptotic stability for the case $d = 0$ and compute its degree of homogeneity performing some algebraic operations, which is a big advantage of the homogeneity approach, while in the conventional case an ISS/iISS Lyapunov function has to be found [Ning 2012]. However, the sole homogeneity of $\tilde{f}$ is not enough to claim iISS (ISS), and the case $\tilde{r}_{\min} = 0$ with $\kappa > 0$ is the only exclusion as in the following example for $\tilde{\mathbf{r}} = 0$ and $\mathbf{r} = 1$:

$$\dot{x} = (d-1)\lfloor x \rceil^{\alpha},\ \alpha > 1.$$

The asymptotically stable system (4.1) for $d = 0$ is finite-time stable if it is homogeneous with negative degree [Bhat 2000, Moulay 2005, Hong 2010]. Interestingly to note that the finite-time stability and iISS have a similar restriction on the degree of homogeneity (it has to be negative or non positive for iISS), thus the finite-time stability of a homogeneous system implies iISS, as stated in the following corollary.

Corollary 4.7. *Let the vector field $\tilde{f}$ be $(\mathbf{r}, \tilde{\mathbf{r}})$-homogeneous of degree $\kappa \leq 0$, with $\mathbf{r} > 0$ and $\tilde{\mathbf{r}} \geq 0$.*

- *If the origin is globally asymptotically stable for $d = 0$, then (4.1) is iISS.*
- *If $\tilde{r}_{\min} = 0$, then the origin is globally asymptotically stable for $d = 0$ if and only if (4.1) is iISS.*

This result can be applied, for example, to "bilinear" systems:

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(xd_i), \tag{4.8}$$

where all $i \in \{1, \ldots, m\}$, f_i are $\mathbf{r}$-homogeneous vector fields of the same degree and $f_i(0) = 0$ (the simplest example is $f_i(x) = A_i x$, where $A_i \in \mathbb{R}^{n \times n}$). According to Corollary 4.7, if the system $\dot{x} = f_0(x)$ is asymptotically stable and $\mathbf{r}$-homogeneous of a non-positive degree, then the system is iISS.

Theorem 4.5 also provides a quantitative estimate on the asymptotic gain of (4.1) in the ISS case.

Corollary 4.8. *We keep the assumptions and the notations of Theorem 4.5. Assume that $\tilde{r}_{\min} > 0$. Then there exists a constant $C > 0$ such that the following estimate of the asymptotic gain holds:*

$$\gamma(D) \leq C \begin{cases} D^{\frac{r_{\max}}{\tilde{r}_{\max}}} & \text{if } D \leq 1 \\ D^{\frac{r_{\min}}{\tilde{r}_{\min}}} & \text{if } D \geq 1 \end{cases}.$$

Proof. Let us recall first that by Lemma 1.19, there exist constants $\alpha_-, \alpha_+ > 0$ such that:

$$\alpha_- v_-(\|x\|_{\mathbf{r}}) \leq \|x\| \leq \alpha_+ v_+(\|x\|_{\mathbf{r}}),$$

where:

$$v_-(s) = \min(s^{r_{\min}}, s^{r_{\max}}) \quad v_+(s) = \max(s^{r_{\min}}, s^{r_{\max}}).$$

In the proof of Theorem 4.5, we have seen that $\mathcal{L}_{f_d} V(y) \leq -\frac{a}{2} V(y)$ as long as $\|y\|_r \geq \theta(D)$. That implies that there exists a function $\beta \in \mathcal{KL}$ such that for all $t \geq 0$ such that $\|x(t)\|_r \geq \theta(D)$, we have $\|x(t)\| \leq \beta(\|x(0)\|, t)$. But the condition $\|y\| \geq \alpha_+ v_+(\theta(D))$ implies $\|y\|_r \geq \theta(D)$. Finally, we get:

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \alpha_+ v_+(\theta(D)),$$

that is the system is ISS with asymptotic gain $\gamma(D) = \alpha_+ \upsilon_+(\theta(D))$. Straightforward computations give:

$$\gamma(D) \leq C \left\{ \begin{array}{ll} D^{\frac{r_{\min}}{\tilde{r}_{\max}}} & \text{if } D \leq 1 \\ D^{\frac{r_{\max}}{\tilde{r}_{\min}}} & \text{if } D \geq 1 \end{array} \right. ,$$

with $C > 0$. $\square$

The case $\tilde{r}_{\min} = 0$ is critical for Theorem 4.5, it is possible that the system (4.1) is ISS while $\tilde{r}_{\min} = 0$ as it is shown in the following example:

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -x_1^3 + x_2^2 d_1, \\ \dot{x}_2 & = & -\lfloor x_2 \rceil^{7/3} + |x_1|^{1/2} d_2, \end{array} \right. \tag{4.9}$$

where $\mathbf{r} = [1;\ 1.5]$, $\tilde{\mathbf{r}} = [0;\ 3]$, $\kappa = 2$ and its ISS Lyapunov function is $V(x) = x_1^2 + x_2^2$.

The conditions of Theorem 4.5 can be technically relaxed skipping homogeneity of $\tilde{f}$ (homogeneity with respect to d). It is worth stressing that homogeneity of $\tilde{f}$ is not a restrictive condition since d is an external input, and we can modify dimension or introduce nonlinear change of coordinates for d.

Theorem 4.9. *Assume that the system (4.1) is globally asymptotically stable for $d = 0$ and $\mathbf{r}$-homogeneous of degree κ, i.e. $f(\Lambda_r(\lambda)x, 0) = \lambda^\kappa \Lambda_r(\lambda) f(x, 0)$ for all $x \in \mathbb{R}^n$ and all $\lambda > 0$. Assume also that there exist functions $\psi, \varphi \in \mathcal{K}$ and $0 \leq \vartheta_{\min} \leq \vartheta_{\max}$ such that for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$:*

$$\|f(x, d) - f(x, 0)\| \leq \theta(\|x\|_r)\psi(\|d\|) + \varphi(\|d\|),$$

where

$$\theta(s) = \begin{cases} s^{\vartheta_{\min}} & \text{if } s \leq 1 \\ s^{\vartheta_{\max}} & \text{if } s > 1 \end{cases}.$$

Then the system (4.1) is

ISS if $\kappa > \vartheta_{\max} - r_{\min}$;

iISS if $\kappa \leq \vartheta_{\max} - r_{\min} \leq 0$.

Proof. Under introduced conditions, by Theorem 1.5 there exists a continuously differentiable, positive definite and proper Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that $V(\Lambda_r(\lambda)x) = \lambda^\mu V(x)$ for any $\lambda > 0$, with $\mu > r_{\max}$ and $\kappa + \mu > 0$, and the inequalities

(4.3) are satisfied for $a>0$, $b>0$. Consider the time derivative of the Lyapunov function V computed for the system (4.1) for all $x \in \{z \in \mathbb{R}^n : \|z\|_r = 1\}$ and $d \in \mathbb{R}^m$:

$$\begin{aligned}
\mathcal{L}_{f_d} V(\Lambda_r(\lambda)x) &= \mathcal{L}_{f_0} V(\Lambda_r(\lambda)x) + \mathcal{L}_{f_d - f_0} V(\Lambda_r(\lambda)x) \\
&= \lambda^{\kappa+\mu}\left[\mathcal{L}_{f_0} V(x) + d_x V \lambda^{-\kappa}\Lambda_r(\lambda)^{-1}(f_d - f_0)(\Lambda_r(\lambda)x)\right] \\
&\leq \lambda^{\kappa+\mu}\left[-a + b\varepsilon(\lambda)\left[\theta(\|\Lambda_r(\lambda)x\|_r)\psi(\|d\|) + \varphi(\|d\|)\right]\right] \\
&\leq \lambda^{\kappa+\mu}\left[-a + b\varepsilon(\lambda)\theta(\lambda)\psi(\|d\|) + b\varepsilon(\lambda)\varphi(\|d\|)\right],
\end{aligned}$$

where

$$\varepsilon(s) = \begin{cases} s^{-\kappa - r_{\max}} & \text{if } s \leq 1 \\ s^{-\kappa - r_{\min}} & \text{if } s \geq 1 \end{cases}.$$

Consider first the case $\kappa > \vartheta_{\max} - r_{\min}$. Let us denote

$$\eta(s) = \min\left\{\varphi^{-1}\left(\frac{-a}{4b\varepsilon(s)}\right); \psi^{-1}\left(\frac{-a}{4b\varepsilon(s)\theta(s)}\right)\right\}.$$

Clearly, if $\|d\| \leq \eta(\lambda)$, then $\varepsilon(\lambda)\theta(\lambda)\psi(\|d\|) \leq \frac{a}{4b}$ and $\varepsilon(\lambda)\varphi(\|d\|) \leq \frac{a}{4b}$. Therefore, $\mathcal{L}_{f_d} V(\Lambda_r(\lambda)x) \leq -\frac{a}{2}\lambda^{\kappa+\mu}$, i.e. for all $y \neq 0$, $\mathcal{L}_{f_d} V(y) \leq -\frac{a}{2}\|y\|_r^{\kappa+\mu}$ and the system is ISS, since $\eta \in \mathcal{K}$. Indeed, by the condition $\kappa > \vartheta_{\max} - r_{\min}$, the following functions belong to class $\mathcal{K}$:

$$s \mapsto \frac{1}{\varepsilon(s)} = \begin{cases} s^{\kappa + r_{\max}} & \text{if } s \leq 1 \\ s^{\kappa + r_{\min}} & \text{if } s \geq 1 \end{cases},$$

$$s \mapsto \frac{1}{\varepsilon(s)\theta(s)} = \begin{cases} s^{\kappa + r_{\max} - \vartheta_{\min}} & \text{if } s \leq 1 \\ s^{\kappa + r_{\min} - \vartheta_{\max}} & \text{if } s \geq 1 \end{cases}.$$

Now, if $\vartheta_{\max} - r_{\min} \leq 0$, we shall use the Lyapunov function $W = \ln(1+V)$. Denoting $\alpha = \max\{\sup\{V(x) : \|x\|_r = 1\}; 1\}$, we find:

$$\mathcal{L}_{f_d} W(\Lambda_r(\lambda)x) \leq -\frac{a\lambda^{\kappa+\mu}}{1+\alpha\lambda^{\mu}} + b\frac{\lambda^{\mu}}{1+\alpha\lambda^{\mu}}\left(\lambda^{\kappa}\varepsilon(\lambda)\theta(\lambda)\psi(\|d\|) + \lambda^{\kappa}\varepsilon(\lambda)\varphi(\|d\|)\right), \quad (4.10)$$

$$\mathcal{L}_{f_d} W(\Lambda_r(\lambda)x) \leq -\frac{a\lambda^{\kappa+\mu}}{1+\alpha\lambda^{\mu}} + b\frac{1}{1+\alpha\lambda^{\mu}}\lambda^{\kappa+\mu}\varepsilon(\lambda)(\theta(\lambda)\psi(\|d\|) + \varphi(\|d\|)). \quad (4.11)$$

If $\lambda \leq 1$ then $\frac{1}{1+\alpha\lambda^{\mu}} \leq 1$, $\theta(\lambda) \leq 1$ and $\lambda^{\kappa+\mu}\varepsilon(\lambda) = \lambda^{\mu - r_{\max}} \leq 1$ since $\mu - r_{\max} \geq 0$. Hence (4.11) gives:

$$\mathcal{L}_{f_d} W(\Lambda_r(\lambda)x) \leq -\frac{a\lambda^{\kappa+\mu}}{1+\alpha\lambda^{\mu}} + b\left(\psi(\|d\|) + \varphi(\|d\|)\right).$$

Otherwise, if $\lambda \geq 1$ then $\frac{\lambda^{\mu}}{1+\alpha\lambda^{\mu}} \leq 1/\alpha \leq 1$, $\lambda^{\kappa}\varepsilon(\lambda) = \lambda^{-r_{\min}} \leq 1$ and $\lambda^{\kappa}\varepsilon(\lambda)\theta(\lambda) =$

$\lambda^{\vartheta_{\max}-r_{\min}} \leq 1$ since $\vartheta_{\max} - r_{\min} \leq 0$. Hence (4.10) gives:

$$\mathcal{L}_{f_d} W(\Lambda_r(\lambda)x) \leq -\frac{a\lambda^{\kappa+\mu}}{1+\alpha\lambda^{\mu}} + b\left(\psi(\|d\|) + \varphi(\|d\|)\right). \tag{4.12}$$

Finally the relation (4.12) holds for any λ, which means that for any $y \neq 0$ we have:

$$\mathcal{L}_{f_d} W(y) \leq -\frac{a\|y\|_r^{\kappa+\mu}}{1+\alpha\|y\|_r^{\mu}} + b(\psi+\varphi)(\|d\|) \tag{4.13}$$

and therefore the system (4.1) is iISS. □

In the Theorem 4.9, two constants, namely $\vartheta_{\min}$ and $\vartheta_{\max}$, are involved in the definition of the function θ. Remark that $\vartheta_{\min} = 0$ is always a valid choice by the continuity of f. The only real condition is the existence of $\vartheta_{\max}$, which expresses the polynomial growth of the vector field $f_d - f_0$ when x becomes large.

The result of Theorem 4.9 can be applied for a larger class of systems, which are not necessarily homogeneous (the function $\tilde{f}$ may be non homogeneous). For example, to the system (4.2) with non homogeneous G_0 (the result of [Hong 2001] cannot be used in this case):

$$\begin{cases} \dot{x}_1 &= -x_1 + \frac{x_2 d_1}{1+|x_2|}, \\ \dot{x}_2 &= -x_2 + \lfloor x_1 \rceil^{1/3} d_2, \end{cases}$$

where $\mathbf{r} = [1;\ 1]$ and $\kappa = 0$ for $d = 0$, $\vartheta_{\min} = \vartheta_{\max} = 1/3$.

However, the conditions obtained in Theorem 4.9 also do not work for the critical case example (4.9), where $\vartheta_{\min} = 0.5$, $\vartheta_{\max} = 3$ and the equality $\kappa = \vartheta_{\max} - r_{\min}$ is satisfied. A reason of that is hidden in the conservatism of the function θ computation. Another explanation of this fact is that, in the case $\tilde{r}_{\min} = 0$ the system (4.1) *may not admit* a $\mathbf{r}$–homogeneous ISS Lyapunov function (both Theorems 4.5 and 4.9 are based on an ISS Lyapunov function of that type), see also the case of Proposition 4.10 below, where this fact is pointed out for the case $\tilde{\mathbf{r}} = 0_m$.

Proposition 4.10. *Considering d as a constant, let the vector field f be $\mathbf{r}$–homogeneous of degree κ independent of d, i.e. $f(\Lambda_r(\lambda)x, d) = \lambda^{\kappa}\Lambda_r(\lambda)f(x,d)$ for all $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ and all $\lambda > 0$.*

- *Assume that the system (4.1) admits a $\mathbf{r}$–homogeneous ISS Lyapunov function V. Then V is a Lyapunov function for f_d and the origin is a uniformly globally asymptotically stable equilibrium for (4.1).*

- *Assume that the system (4.1) admits a* $\mathbf{r}$*–homogeneous iISS Lyapunov function* V*. Then the origin is a uniformly stable equilibrium for (4.1).*

Proof. By definition, an ISS (resp. iISS) Lyapunov function for the system (4.1) verifies:

$$\mathcal{L}_{f_d}V(y) \le \theta(\|d\|) - \alpha_3(\|y\|), \quad \theta \in \mathcal{K}, \alpha_3 \in \mathcal{K}_\infty \text{ (resp. } \alpha_3 \succeq 0).$$

If V is $\mathbf{r}$-homogeneous, set $y = \Lambda_r(\lambda)x$, with $\|x\|_r = 1$. Then $\mathcal{L}_{f_d}V(y) = \lambda^{\kappa+\mu}\mathcal{L}_{f_d}V(x)$. Using Lemma 1.19, there exists a function $w \in \mathcal{K}_\infty$ (resp. $w \succeq 0$) such that:

$$\lambda^{\kappa+\mu}\mathcal{L}_{f_d}V(x) \le \theta(\|d\|) - w(\lambda\|x\|_r).$$

If $w \in \mathcal{K}_\infty$, then for any x such that $\|x\|_r = 1$ and any $d \in \mathbb{R}^m$, we find $\lambda^{\kappa+\mu}\mathcal{L}_{f_d}V(x) \to -\infty$ when $\lambda \to +\infty$. We obtain $\mathcal{L}_{f_d}V(x) < 0$ and $\lambda^{\kappa+\mu}\mathcal{L}_{f_d}V(x) = \mathcal{L}_{f_d}V(y) < 0$.

If $w \succeq 0$, then we just have $\mathcal{L}_{f_d}V(x) \le \lambda^{-(\kappa+\mu)}\theta(\|d\|)$, and when $\lambda \to +\infty$, we get $\mathcal{L}_{f_d}V(x) \le 0$, thus $\lambda^{\kappa+\mu}\mathcal{L}_{f_d}V(x) = \mathcal{L}_{f_d}V(y) \le 0$. □

To finish comparison of Theorems 4.5 and 4.9 note that the conditions of Theorem 4.9 may be more restrictive than in Theorem 4.5, as it can be seen in the following example:

$$\begin{cases} \dot{x}_1 &= -x_1^3 + \lfloor x_2 \rceil^{1/3} d_1 \\ \dot{x}_2 &= -\lfloor x_2 \rceil^{5/3} + x_1^3 d_2 \end{cases},$$

where $\mathbf{r} = [1;\ 3]$, $\tilde{\mathbf{r}} = [2;\ 2]$, $\kappa = 2$ and it is ISS by Theorem 4.5 (it also has a homogeneous ISS Lyapunov function $V(x) = x_1^6/6 + x_2^2/2$). However, Theorem 4.9 does not provide any conclusion since $\vartheta_{\min} = 1$, $\vartheta_{\max} = 3$ and $\kappa = \vartheta_{\max} - r_{\min} > 0$. In addition, the iISS condition in Theorem 4.9 implicitly needs $\vartheta_{\max} \le r_{\min}$. Another interpretation of the ISS condition of Theorem 4.9 is that the system (4.1) is $\mathbf{r}$-homogenizable at ∞ with homogenization $f(x, 0)$, and this approximation is uniform in d.

Finally consider an example, for which a strict Lyapunov function is not known, but using Theorems 4.5 or 4.9 it is possible to establish ISS property. Let us consider the following planar nonlinear system:

$$\begin{cases} \dot{e}_1 &= e_2 - l_1 \lfloor e_1 + d_1 \rceil^{\beta} \\ \dot{e}_2 &= -l_2 \lfloor e_1 + d_1 \rceil^{2\beta-1} + d_2 \end{cases}, \tag{4.14}$$

where $e_1 \in \mathbb{R}$, $e_2 \in \mathbb{R}$ are the states, $d_1 \in \mathbb{R}$, $d_2 \in \mathbb{R}$ are external inputs, $l_1 > 0$, $l_2 > 0$ and $\beta \in (\frac{1}{2}, 1)$ are the parameters. Such a system describes dynamics of estimation error when analyzing a finite-time observer/differentiator [Bernuau 2012], see Chapter 5. In this case d_1 represents the measurement noise and d_2 models an external disturbance or model mismatch. The system (4.14) is homogeneous for $\mathbf{r} = [1\ \beta]$ and $\tilde{\mathbf{r}} = [1\ 2\beta - 1]$ with degree $\kappa = \beta - 1$. To show that for $d_1 = d_2 = 0$ the system is asymptotically stable, we can consider a Lyapunov function

$$V(e_1, e_2) = \frac{l_2}{2\beta}|e_1|^{2\beta} + \frac{1}{2}e_2^2,$$

which derivative for (4.14) takes the form

$$\begin{aligned} \dot{V} &= l_2\lfloor e_1\rceil^{2\beta-1}(e_2 - l_1\lfloor e_1\rceil^{\beta}) - l_2 e_2\lfloor e_1\rceil^{2\beta-1} \\ &= -l_1 l_2 |e_1|^{3\beta-1}, \end{aligned}$$

i.e. V is not a strict Lyapunov function for the system (4.14). Therefore, the system is Lyapunov stable ($\dot{V} \leq 0$); moreover all its trajectories are attracted by the origin (the origin is the only invariant solution on the line $e_1 = 0$), thus the system (4.14) is globally asymptotically stable. In fact, since it is homogeneous with a negative degree it is finite-time stable. By Theorem 4.5, since $\tilde{r}_{\min} > 0$, the system is ISS with respect to d_1 and d_2.

In Chapter 5, we will see another example, for which a strict Lyapunov function is not known, but using homogeneity it is possible to prove ISS property for a nonlinear homogeneous controller from [Bhat 1998].

4.4 Robustness of locally homogeneous systems

The ISS property of locally homogeneous systems has been analyzed in [Andrieu 2008]. It was shown there that if the system (4.1) is locally homogeneous at 0 and $+\infty$, and all approximations and the system itself are globally asymptotically stable for $d = 0$, then (4.1) is ISS. First we are going to propose a variant of that proof for approximation at infinity and, next, we will extend it for systems that are not homogeneous with respect to d.

Theorem 4.11. *Let the vector field $\tilde{f}$ be $(\mathbf{r}, \tilde{\mathbf{r}})$–homogenizable at ∞ (with $\mathbf{r} > 0$ and $\tilde{\mathbf{r}} > 0$), i.e. for any compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$*

$$\sup_{(x,d)\in K} \|\lambda^{-\kappa}\Lambda_r^{-1}(\lambda) f(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)d) - h(x,d)\| \underset{\lambda\to+\infty}{\longrightarrow} 0,$$

where h is the homogenization of f. For $d \in R^m$, denote $h_d(x) = h(x, d)$. Assume that the system $\dot{x} = h_0(x)$ is globally asymptotically stable, then the system (4.1) is ISpS.

Proof. We shall here use the arguments of the proof of Theorems 4.5 and 2.17. We also denote $f_d(x) = f(x, d)$ and $h_d(x) = h(x, d)$.

There exists a function $\sigma \in \mathcal{K}$ such that:

$$\|h(y, d) - h(y, 0)\| \leq \sigma(\|d\|) \quad \forall y \in \{V \leq 1\}. \tag{4.15}$$

We denote $\kappa = \deg_\infty f$ and $K = \{z \in \mathbb{R}^n : V(z) = 1\} \times \{\delta \in \mathbb{R}^m : \|\delta\|_{\tilde{r}} \leq 1\}$. Let $V \in \mathcal{C}^1(\mathbb{R}^n, R)$ be a proper Lyapunov function for h_0, $\mathbf{r}$-homogeneous of degree $\mu > 0$. For all $x \in \{z \in \mathbb{R}^n : V(z) = 1\}$, $d \in \mathbb{R}^m$ and all $\lambda \geq 1$, we have:

$$\begin{aligned}
\mathcal{L}_{f_d} V(\Lambda_r(\lambda)x) &= \mathcal{L}_{h_0} V(\Lambda_r(\lambda)x) + \mathcal{L}_{h_d - h_0} V(\Lambda_r(\lambda)x) + \mathcal{L}_{f_d - h_d} V(\Lambda_r(\lambda)x) \\
&\leq \lambda^{\mu+\kappa} \big[a + \sigma(\|\Lambda_{\tilde{r}}(\lambda)^{-1} d\|) \\
&\quad + b\|\lambda^{-\kappa} \Lambda_r(\lambda)^{-1} (f - h)(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)\Lambda_{\tilde{r}}(\lambda)^{-1} d)\|\big],
\end{aligned}$$

where the constant $a > 0$ and $b > 0$ are defined as:

$$-a = \sup_{\{V(x)=1\}} \mathcal{L}_{h_0} V(x), \qquad b = \sup_{\{V(x)=1\}} \|d_x V\|. \tag{4.16}$$

By an argument similar to the proof of Theorem 4.5, there exists a function $\theta_1 \in \mathcal{K}$ such that if $\|\Lambda_r(\lambda)x\| \geq \theta_1(\|d\|)$, then $\sigma(\|\Lambda_{\tilde{r}}(\lambda)^{-1} d\|) \leq -a/4b$. There also exists a function $\theta_2 \in \mathcal{K}$ such that if $V(x) = 1$ and $\theta_2(\|d\|) \leq \|\Lambda_r(\lambda)x\|$, then $\|d\|_{\tilde{r}} \leq \lambda$. Let us denote $\theta(s) = \max\{\theta_1(s), \theta_2(s)\}$. Since

$$\sup_{(x,d)\in K} \|\lambda^{-\kappa} \Lambda_r^{-1}(\lambda) f(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)d) - h(x, d)\| \underset{\lambda \to +\infty}{\longrightarrow} 0,$$

there exists $c > 0$ such that if $\|\Lambda_r(\lambda)x\| \geq c$, then

$$\sup_{(x,d)\in K} \|\lambda^{-\kappa} \Lambda_r^{-1}(\lambda) f(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)d) - h(x, d)\| \leq \frac{a}{4b}.$$

Now, if $\|\Lambda_r(\lambda)x\| \geq c + \theta(\|d\|)$, then $\|\tilde{d}\|_{\tilde{r}} \leq 1$ where $\tilde{d} = \Lambda_{\tilde{r}}(\lambda)^{-1}d$ and

$$\begin{aligned}
& \|\lambda^{-\kappa}\Lambda_r(\lambda)^{-1}(f-h)(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)\Lambda_{\tilde{r}}(\lambda)^{-1}d)\| \\
= \ & \|\lambda^{-\kappa}\Lambda_r(\lambda)^{-1}(f-h)(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)\tilde{d})\| \\
\leq \ & \sup_{(x,\delta)\in K} \|\lambda^{-\kappa}\Lambda_r(\lambda)^{-1}(f-h)(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)\delta)\| \leq \tfrac{a}{4b}.
\end{aligned}$$

Finally, if $\|y\| \geq c + \theta(\|d\|)$, $\mathcal{L}_{f_d}V(y) \leq -\frac{a}{2}V(y)^{\frac{\mu+\kappa}{\mu}}$ and the system (4.1) is ISpS by Theorem 4.4. □

For example, consider the system:

$$\begin{cases} \dot{x}_1 = x_1 - x_1^3 + x_2|x_1|^{0.75}d \\ \dot{x}_2 = x_2 - \lfloor x_2 \rceil^2 + |x_1|^{3.5}|x_2|^{0.125}d \end{cases},$$

which is $(\mathbf{r}, \tilde{\mathbf{r}})$-homogenizable at ∞ with the weights $\mathbf{r} = [1, 2]$, $\tilde{\mathbf{r}} = 0.25$ and with

$$h(x,d) = \begin{pmatrix} -x_1^3 + x_2|x_1|^{0.75}d \\ -\lfloor x_2 \rceil^2 + |x_1|^{3.5}|x_2|^{0.125}d \end{pmatrix}$$

of degree 2. The linearization of the system is unstable and it is hard to simulate this system in order to check its stability since it is very stiff. However, since all conditions of Theorem 4.11 are satisfied, the system is ISpS.

Corollary 4.12. *Let all conditions of Theorem* 4.11 *be satisfied, and assume moreover that*

$$\sup_{(x,d)\in K} \|\lambda^{-\kappa}\Lambda_r^{-1}(\lambda)f(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)d) - h(x,d)\| < a,$$

where a is defined by (4.16). Then the system (4.1) is ISS.

Proof. If the additional condition holds, let us denote

$$\sup_{(x,d)\in K} \|\lambda^{-\kappa}\Lambda_r^{-1}(\lambda)f(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)d) - h(x,d)\| = a' < a.$$

There exists a function $\theta' \in \mathcal{K}$ such that if $\|y\| \geq \theta(\|d\|)$, then $\sigma(\|\Lambda_{\tilde{r}}(\lambda)^{-1}d\|) \leq \frac{a'-a}{2}$, with $V(y) = \lambda^{\mu}$, and we find that if $\|y\| \geq \theta'(\|d\|)$, $\mathcal{L}_{f_d}V(y) \leq -\frac{a'-a}{2}V(y)^{\frac{\mu+\kappa}{\mu}}$ and the system (4.1) is ISS by Theorem 4.4. □

Corollary 4.13. *Let a continuous vector field $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ be* $\mathbf{r}$*-homogenizable at* ∞ *and such that its homogenization is asymptotically stable. If $f(x,d) = f_0(x) + d$, i.e. d is* *an additive disturbance, or $f(x,d) = f_0(x+d)$, i.e. d is a measurement noise, then the system (4.1) is ISpS.*

Proof. Denote κ the local degree of homogeneity of f_0 at ∞. Being the degree of the approximation which is a GAS $\mathbf{r}$-homogenous vector field, $\kappa > r_{\min}$. Take $\tilde{\mathbf{r}} = \mathbf{r} + \kappa$ for the additive disturbance case, and $\tilde{\mathbf{r}} = \mathbf{r}$ for the measurement noise. The result now follows Theorem 4.11. □

There is a modification of Theorem 4.11, which skips homogeneity with respect to d. We keep denoting $f_d(x) = f(x,d)$.

Theorem 4.14. *Let the vector field f_0 be $\mathbf{r}$-homogenizable at ∞ and denote h_0 its homogenization. Assume also that there exist functions $\psi, \varphi \in \mathcal{K}$ and $0 \le \vartheta_{\min} \le \vartheta_{\max}$ such that for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$*

$$\|f(x,d) - f(x,0)\| \le \theta(\|x\|_r)\psi(\|d\|) + \varphi(\|d\|),$$

with

$$\theta(s) = \begin{cases} s^{\vartheta_{\min}} & \textit{if } s \le 1 \\ s^{\vartheta_{\max}} & \textit{if } s > 1 \end{cases}.$$

If $\kappa = \deg_\infty f_0 > \vartheta_{\max} - r_{\min}$ and if the system $\dot{x} = h_0(x)$ is globally asymptotically stable, then the system (4.1) is ISpS.

Proof. Under the introduced conditions the system $\dot{x} = h_0(x)$ is $\mathbf{r}$-homogeneous of degree κ and globally asymptotically stable. Then by the weighted homogeneous version of Theorem 1.5 there exists a continuously differentiable, positive definite and radially unbounded Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ $\mathbf{r}$-homogeneous of degree $\mu > 0$ and (4.16) holds for some $a > 0$, $b > 0$. Consider the time derivative of V computed for (4.1) for all $x \in \{z \in \mathbb{R}^n : V(z) = 1\}$ and $d \in \mathbb{R}^m$:

$$\begin{aligned} \mathcal{L}_{f_d} V(\Lambda_r(\lambda)x) &= \mathcal{L}_{h_0} V(\Lambda_r(\lambda)x) + \mathcal{L}_{f_0 - h_0} V(\Lambda_r(\lambda)x) + \mathcal{L}_{f_d - f_0} V(\Lambda_r(\lambda)x) \\ &\le \lambda^{\mu+\kappa}[-a + \sup_{V(x)=1} \|\lambda^{-\kappa}\Lambda_r^{-1}(\lambda) f_0(\Lambda_r(\lambda)x) - h_0(x)\| \\ &\quad + b\lambda^{-\kappa - r_{\min}}\theta(\lambda\|x\|_r)\psi(\|d\|) + b\lambda^{-\kappa - r_{\min}}\varphi(\|d\|)]. \end{aligned}$$

There exists $c_1 > 0$ such that if $\|\Lambda_r(\lambda)x\| \ge c_1$ then

$$\sup_{V(x)=1} \|\lambda^{-\kappa}\Lambda_r^{-1}(\lambda) f_0(\Lambda_r(\lambda)x) - h_0(x)\| \le \frac{a}{6b}.$$

There exists $c_2 > 0$ such that if $\|\Lambda_r(\lambda)x\| \geq c_2$ then $\lambda\|x\|_r \geq 1$. We set $c = \max\{c_1, c_2\}$. Then if $\|\Lambda_r(\lambda)x\| \geq c$ then

$$\mathcal{L}_{f_d} V(\Lambda_r(\lambda)x) \leq \lambda^{\mu+\kappa} \left[-a + \frac{a}{6} + bb'\lambda^{\vartheta_{\max}-\kappa-r_{\min}}\psi(\|d\|) + b\lambda^{-\kappa-r_{\min}}\varphi(\|d\|)\right],$$

with $b' = \sup_{V=1} \|x\|^{\vartheta_{\max}}$. Since $\kappa + r_{\min} - \vartheta_{\max} > 0$, there exist functions $\tilde{\psi}, \tilde{\varphi} \in \mathcal{K}$ such that if $\|\Lambda_r(\lambda)x\| \geq \tilde{\psi}(\|d\|)$ then $bb'\lambda^{\vartheta_{\max}-\kappa-r_{\min}}\psi(\|d\|) \leq a/6b$ and if $\|\Lambda_r(\lambda)x\| \geq \tilde{\varphi}(\|d\|)$ then $b\lambda^{-\kappa-r_{\min}}\varphi(\|d\|) \leq a/6b$. Hence, if $\|\Lambda_r(\lambda)x\| \geq c + (\tilde{\varphi} + \tilde{\psi})(\|d\|)$, we find

$$\mathcal{L}_{f_d} V(\Lambda_r(\lambda)x) \leq -\frac{a}{2}\lambda^{\mu+\kappa},$$

that is, for all $\|y\| \geq c + (\tilde{\varphi} + \tilde{\psi})(\|d\|)$,

$$\mathcal{L}_{f_d} V(y) \leq -\frac{a}{2}\lambda^{\mu+\kappa}$$

and by Theorem 4.4 the system (4.17) is ISpS. □

Theorems 4.11 and 4.14 extend the conditions of Theorems 4.5 and 4.9 on the case of local homogeneity at infinity. However, in the local case the difference between applicability conditions of Theorems 4.11 and 4.14 is minor. The main advantage is that the local approximation at infinity may fail to exist for both variables x and d (the case of Theorem 4.11), but it may exist for $d = 0$ and Theorem 4.14 can be applied in this case.

Consider the following example

$$\begin{cases} \dot{x}_1 = -x_1 + \frac{|x_1|^{1/6}d}{1+|x_2|^2} + x_2^2 \\ \dot{x}_2 = -x_2 + |x_1|^{1/6}d \end{cases} . \tag{4.17}$$

For $d \neq 0$ this system has no homogeneous approximation at infinity. Indeed, set weights $\mathbf{r} = [r_1;\ r_2] > 0$ and $\tilde{\mathbf{r}} = [\tilde{r}] \geq 0$ and degree $\kappa \in \mathbb{R}$. We find

$$\lim_{\lambda\to+\infty} \lambda^{-\kappa}\lambda^{-r_1}\frac{|\lambda^{r_1}x_1|^{1/6}\lambda^{\tilde{r}}d}{1+|\lambda^{r_2}x_2|^2} = \begin{cases} \infty & \text{if } -\kappa - \frac{5}{6}r_1 + \tilde{r} - 2r_2 > 0 \\ 0 & \text{if } -\kappa - \frac{5}{6}r_1 + \tilde{r} - 2r_2 < 0 \\ \frac{|x_1|^{1/6}d}{|x_2|^2} & \text{if } -\kappa - \frac{5}{6}r_1 + \tilde{r} - 2r_2 = 0 \end{cases} .$$

Therefore the only possible degree of local homogeneity of the system is $\kappa = -\frac{5}{6}r_1 + \tilde{r} - 2r_2$. However, if the system were homogenizable at ∞, the previous limit would have been continuous, which is not the case here. Hence for any $(\mathbf{r}, \tilde{\mathbf{r}})$, the system is not $(\mathbf{r}, \tilde{\mathbf{r}})$-

homogenizable and thus neither the Theorem 4.11 nor the result of [Andrieu 2008] can be applied here.

Now, if we choose $\mathbf{r} = [2;\ 0.5]$ the system (4.17) with $d = 0$ is $\mathbf{r}$-homogenizable at infinity with homogenization

$$h_0(x) = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$$

of degree $\kappa = 0$, which is clearly asymptotically stable. A direct calculation shows that $\vartheta_{\min} = \vartheta_{\max} = 1/3$ are valid values. Therefore, since $\kappa > \vartheta_{\max} - r_{\min}$, according to Theorem 4.14 the system (4.17) is ISpS.

4.5 Conclusion

In this Chapter, several conditions of the ISS and iISS properties have been developed based on the homogeneity theory. The advantage of these conditions is that the system robustness can be checked after its asymptotic stability in the unperturbed case provided that some algebraic homogeneity conditions are satisfied for the system equations (globally or locally). All results are obtained for generic nonlinear systems. Several examples are proposed showing efficiency of the proposed theory and its limitations. In the next Chapter, we will see how to use the ingredients developed in this Chapter for the concrete example of the double integrator.

Chapter 5

Application to the double integrator

Contents

5.1 Introduction

In many applications the nominal models have the double integrator form (mechanical systems, for instance). Despite its simplicity this model is rather important in the control theory since frequently a design method developed for the double integrator can be extended to a more general case (via backstepping, for example). Most of the current techniques for nonlinear feedback stabilization provide an asymptotic stability: the obtained closed-loop dynamics is locally Lipschitz and the system trajectories settle at the origin when the time is approaching infinity. However, there are situations where we need more precision on the speed of convergence. This is why the finite-time stability notion has been quickly developing during the last decades: solutions of a FTS system reach the equilibrium point in a finite time. Let us mention that finite-time convergence implies non-uniqueness of solutions (in backward time) which is not possible in the presence of Lipschitz-continuous dynamics, where different maximal trajectories never cross.

Engineers are interested in the FTS because one can manage the time for solutions to reach the equilibrium which is called the *settling time*. The *settling time function*, defined as the maximum time for all solutions to reach the desired equilibrium for a given initial condition, is hence an important tool in this context. However, like any quantitative information about the solutions of a differential equation, the settling time function is not always known. Some results exist, though, on qualitative properties of this settling time function. Its regularity, and particularly its continuity at the origin, has been studied in [Bhat 2000], under the assumption of forward uniqueness of solutions. That paper shows, among other results, that the continuity of the settling time function is equivalent to its continuity at the origin, which is not always achieved. Such properties allows an accurate numeric approximation of the settling time function, which is a good alternative to qualitative informations.

For continuous systems, necessary and sufficient conditions for FTS have been given (see [Haimo 1986, Moulay 2003]). In addition, necessary and sufficient conditions appear for finite-time stability for a class of discontinuous systems (see [Orlov 2005b]), and extended in Chapter 3. It was observed in many papers that FTS can be achieved if the system is locally asymptotically stable and homogeneous of negative degree [Bacciotti 2005, Bhat 1997]. We have also seen related results when the system is only homogenizable by a GAS homogenization of negative degree in the Chapter 2 (see Corollary 2.16). These results justify the central role played by the homogeneity in the FTS system design. The reader may found additional properties and results on homogeneity in [Bacciotti 2005, Bhat 1997, Hermes 1991a, Kawski 1995, Orlov 2003].

Our goal in this Chapter is to use the techniques developed along this work for the design of a FTS output feedback controller for the double integrator. Since the double integrator is controllable, open-loop control strategies can be used to drive the state to the origin in a finite time (see [Athans 1966, Ryan 1979, Wonham 1985] for a minimum time optimal control). Based on homogeneity, Bhat and Bernstein in their paper [Bhat 1998] provided a homogeneous FTS state controller for the double integrator under rather restrictive conditions on parameters of the controller. In [Orlov 2011] an output feedback control is proposed based on the homogeneous control from [Bhat 1998] and a sliding-mode observer. It is shown in [Orlov 2011] that this control has a certain robustness with respect to a particular form of disturbances (bounded by a function of the output). Our objective is to relax the applicability conditions for the control obtained in [Bhat 2005], and to improve robustness abilities of the FTS output control with respect to [Orlov 2011] proposing a purely continuous controller and observer.

The outline of this Chapter is as follows. We will begin by formulating the problem in Section 5.2. The output FTS controller will be designed in Section 5.3, while the robustness and the influence of the discretization will be studied in Section 5.4. Finally, we will present the results of computer simulations of the proposed control algorithm in Section 5.5. The contents of this Chapter have been submitted in [Bernuau 2012] and submitted in [Bernuau 2013c].

5.2 Problem formulation

Our contribution aims at designing a continuous FTS output feedback based on homogeneity for the following double-integrator system:

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & u(x_1, x_2), \\ y & = & x_1, \end{array}\right. \tag{5.1}$$

where x_1 and x_2 are the states of the system, u is the input and y is the output. We will proceed in four steps:

1. Design a continuous homogeneous state feedback control ensuring FTS for the double integrator:
$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & u. \end{array}\right. \tag{5.2}$$

2. Design a continuous homogeneous observer:
$$\left\{\begin{array}{rcl} \dot{\hat{x}}_1 & = & \hat{x}_2 - \chi_1(y - \hat{x}_1), \\ \dot{\hat{x}}_2 & = & u(x_1, x_2) - \chi_2(y - \hat{x}_1), \end{array}\right. \tag{5.3}$$

where χ_1 and χ_2 are functions to be designed such that the origin is FTS for the error $e = x - \hat{x}$ equation:

$$\begin{cases} \dot{e}_1 = e_2 + \chi_1(e_1), \\ \dot{e}_2 = \chi_2(e_1). \end{cases} \tag{5.4}$$

3. Show a separation principle such that the obtained observer-based closed loop system is FTS.

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u(y, \hat{x}_2), \\ y = x_1, \end{cases} \tag{5.5}$$

where $\hat{x}_2$ is obtained from (5.11).

4. Study the robustness of the closed loop system and the influence of the discretization of the control and of the observer.

5.3 Finite-time output feedback based on homogeneity

5.3.1 Finite-time control

Let us consider the double integrator (5.2). It is homogeneous of degree κ w.r.t. to the dilation $\Lambda_{\mathbf{r}}$ with weight $\mathbf{r} = [r_1, r_2]$ as soon as u is $\mathbf{r}$-homogeneous of degree r_u and

$$r_1 + \kappa = r_2, \quad r_2 + \kappa = r_u.$$

Thus fixing $r_2 = 1$ (without loss of generality) a necessary and sufficient condition for (5.2) to be homogeneous is

$$r_1 = 1 - \kappa, \quad r_u = 1 + \kappa. \tag{5.6}$$

To have FTS it is necessary and sufficient that (5.2) is LAS and that $\kappa < 0$. Let us find conditions for which the following feedback leads to LAS of the origin of the system (5.2):

$$u = k_1 \lfloor x_1 \rceil^{\alpha_1} + k_2 \lfloor x_2 \rceil^{\alpha_2}, \tag{5.7}$$

and $\kappa < 0$. The feedback (5.7) is homogeneous of degree r_u iff $r_u = 1 + \kappa = r_i \alpha_i$. From (5.6), setting $\alpha := \alpha_2$, we get:

$$\kappa = \alpha - 1, \quad r_1 = 2 - \alpha, \quad \alpha_1 = \frac{\alpha}{2 - \alpha}.$$

The condition $\kappa < 0$ is equivalent to $\alpha < 1$, which in turn implies that $\alpha_1 < 1$. For $\alpha < 0$, the feedback u is not well defined. For $\alpha = 0$, we recover the discontinuous system

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & k_1 \mathrm{sgn}(x_1) + k_2 \mathrm{sgn}(x_2) \end{array} \right. . \tag{5.8}$$

This system is very important in the sliding mode control theory and has already been extensively studied (see [Levant 2011] and references therein). However we focus here on continuous feedbacks and observers, and hence we assume $\alpha \in (0, 1)$.

Summarizing, the system (5.2) with the feedback (5.7) takes the form

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & k_1 \lfloor x_1 \rceil^{\frac{\alpha}{2-\alpha}} + k_2 \lfloor x_2 \rceil^{\alpha} \end{array} \right. , \tag{5.9}$$

and is $\mathbf{r}$-homogeneous of degree $\alpha - 1$, with $\mathbf{r} = [2 - \alpha;\ 1]$, and continuous.

We would like to find conditions on the coefficients k_i under which the origin is a LAS equilibrium for the system (5.9) (that due to homogeneity implies FTS). In the work [Bhat 2005] these conditions have been obtained for α sufficiently close to one. Here we consider let α being any number in $(0, 1)$.

Consider the following function:

$$V : x \mapsto \frac{-k_1}{1 + \alpha_1} |x_1|^{1+\alpha_1} + \frac{x_2^2}{2}. \tag{5.10}$$

The function V is continuously differentiable, proper, homogeneous of degree 2 with respect to $\Lambda_{\mathbf{r}}$, and $\dot{V} = k_2 |x_2|^{1+\alpha_2}$. If $k_1 < 0$ and $k_2 < 0$, the function V is definite positive, and $\dot{V}$ is negative semi-definite.

Theorem 5.1. *If $\alpha \in (0, 1)$, $k_1 < 0$ and $k_2 < 0$ then the system (5.9) is FTS.*

Proof. A direct application of the LaSalle invariance principle shows that the system (5.9) is GAS. Being homogenous of degree $\alpha - 1 < 0$, the system (5.9) is therefore FTS. □

Remark 5.2. *This result has been proved in [Orlov 2011] under the additional assumption $k_1 < k_2$.*

5.3.2 Finite-time observer design

A finite-time observer for a canonical observable form was constructed for the first time in [Perruquetti 2008]. The proof of finite-time stability is based on the homogeneity property. In the case of the double integrator, the observer is:

$$\begin{cases} \dot{\hat{x}}_1 &= \hat{x}_2 - l_1 \lfloor y - \hat{x}_1 \rceil^{\beta_1} \\ \dot{\hat{x}}_2 &= u - l_2 \lfloor y - \hat{x}_1 \rceil^{\beta_2} \end{cases} . \tag{5.11}$$

The powers β_i are defined such that the error dynamics can be written as follows:

$$\begin{cases} \dot{e}_1 = e_2 + l_1 \lfloor e_1 \rceil^{\beta_1} \\ \dot{e}_2 = l_2 \lfloor e_1 \rceil^{\beta_2} \end{cases} ,$$

where $e = x - \hat{x}$ and the right hand side is ρ-homogeneous with a negative degree where $\boldsymbol{\rho} = [\rho_1, \rho_2]$. The homogeneity holds as soon as the following relations hold

$$\rho_1 + \kappa = \rho_2 = \rho_1 \beta_2, \quad \rho_2 + \kappa = \rho_1 \beta_2.$$

Therefore, setting $\beta = \beta_1$, the homogeneity condition becomes

$$\rho_2 = \rho_1 \beta, \quad \beta_2 = 2\beta - 1, \quad \kappa = \rho_1(\beta - 1),$$

with $\beta \in (\frac{1}{2}, 1)$.

In [Perruquetti 2008], the FTS of the origin was only proved for any $\beta \in (1 - \varepsilon, 1)$ for a sufficiently small $\varepsilon > 0$. Here we show that the system is FTS for all $\beta \in (\frac{1}{2}, 1)$ and all $\rho_1 > 0$. The system can be recast as:

$$\begin{cases} \dot{e}_1 &= e_2 + l_1 \lfloor e_1 \rceil^{\beta} \\ \dot{e}_2 &= l_2 \lfloor e_1 \rceil^{2\beta - 1} \end{cases} , \tag{5.12}$$

and it is continuous and $\boldsymbol{\rho}$-homogeneous of degree $\rho_1(\beta - 1)$ with $\boldsymbol{\rho} = [\rho_1, \rho_1 \beta]$.

Theorem 5.3. *Set $\beta \in (\frac{1}{2}, 1)$ and consider the observer (5.11). Then the associated error system (5.12) is globally FTS for any $l_1 < 0$ and $l_2 < 0$.*

Proof. Consider the following function:

$$V(e) = -\frac{l_2}{2\beta} |e_1|^{2\beta} + \frac{e_2^2}{2}.$$

The function V is definite positive, proper, continuously differentiable and homogeneous of degree $2\rho_1 \beta$. Moreover, we compute $\dot{V}(e) = -l_1 l_2 |e_1|^{3\beta - 1} \leq 0$. Using the LaSalle invariance principle, it is then straightforward to prove that the system (5.12) is GAS. Being homogeneous, this system is therefore FTS. □

Thus the observer (5.11) ensures observation of the state of the system (5.1) in a finite time for any initial conditions.

5.3.3 Finite-time stable observer based control

Our aim is now to use the two preceeding subsections to build a finite-time observer based control. In view of Theorems 5.1 and 5.3, we assume here that $k_1 < 0$, $k_2 < 0$, $l_1 < 0$ and $l_2 < 0$. Let us rewrite the system (5.5) for the designed FTS control (5.7) and the FTS observer (5.12) (in the estimation error coordinates):

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & k_1\lfloor x_1\rceil^{\frac{\alpha}{2-\alpha}} + k_2\lfloor x_2 - e_2\rceil^{\alpha} \\ \dot{e}_1 & = & e_2 + l_1\lfloor e_1\rceil^{\beta} \\ \dot{e}_2 & = & l_2\lfloor e_1\rceil^{2\beta-1} \end{array}\right. . \tag{5.13}$$

Remark 5.4. *Note that $x_2 - e_2 = \hat{x}_2$, thus the control depends on the measured output x_1 only. Moreover, we could replace x_1 in this equation by $\hat{x}_1 = x_1 - e_1$ without changing the following results.*

To prove the FTS property of this system we need two auxiliary lemmas.

Lemma 5.5. *Set $\theta \in (0,1)$. The function $a \in \mathbb{R} \mapsto \lfloor a\rceil^{\theta} \in \mathbb{R}$ is θ-Hölder-continuous with constant $2^{1-\theta}$ on $\mathbb{R}$, that is*

$$|\lfloor a+b\rceil^{\theta} - \lfloor a\rceil^{\theta}| \leq 2^{1-\theta}|b|^{\theta} \quad \forall a, b \in \mathbb{R}.$$

Proof. Define for $a, b \in \mathbb{R}$ and $\theta \in (0,1)$:

$$g_{\theta}(a,b) = \lfloor a+b\rceil^{\theta} - \lfloor a\rceil^{\theta}.$$

Let us show that $|g_{\theta}(a,b)| \leq 2^{1-\theta}|b|^{\theta}$. It is clear that this inequality is true for $b = 0$. In the sequel, we assume $b \neq 0$. An easy verification shows that for all $\lambda > 0$:

$$\begin{array}{rcl} g_{\theta}(\lambda a, \lambda b) & = & \lambda^{\theta} g_{\theta}(a,b), \\ g_{\theta}(a,b) & = & \lfloor b\rceil^{\theta} g_{\theta}(\frac{a}{b}, 1). \end{array}$$

Let us denote $h_\theta : z \in \mathbb{R} \mapsto g_\theta(z, 1)$. The function h_θ is differentiable for all $z \notin \{-1, 0\}$ and $h'_\theta(z) = \theta(|1 + z|^{\theta-1} - |z|^{\theta-1})$. Hence h_θ is strictly increasing on $(-\infty, -1/2)$ and strictly decreasing on $(-1/2, +\infty)$. Thus, we find $0 \leq h_\theta(z) \leq h_\theta(-1/2) = 2^{1-\theta}$. Finally, we have $g_\theta(a, b) = \lfloor b \rceil^\theta g_\theta(\frac{a}{b}, 1) = \lfloor b \rceil^\theta h_\theta(\frac{a}{b})$, and therefore $|g_\theta(a, b)| \leq 2^{1-\theta}|b|^\theta$. □

Lemma 5.6. *The system*

$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & k_1 \lfloor x_1 \rceil^{\frac{\alpha}{2-\alpha}} + k_2 \lfloor x_2 - e_2 \rceil^\alpha \end{cases}$$

is ISS with respect to the input e_2.

Proof. We are in position to apply the Theorem 4.5. We see that the auxiliary vector field is $(\mathbf{r}, \tilde{\mathbf{r}})$-homogeneous, with $\tilde{\mathbf{r}} = r_2$, of degree $\alpha - 1$. Since $\tilde{r}_{\min} = r_2 > 0$, in view of Theorem 5.1, the system is ISS with respect to the input e_2. □

Now we are in position to formulate the main result.

Theorem 5.7. *The system (5.13) with $\hat{x}(t_0) = 0$ is globally FTS for any $\alpha \in (0, 1)$ and $\beta \in (1/2, 1)$ for any $k_1 < 0$, $k_2 < 0$, $l_1 < 0$ and $l_2 < 0$.*

Proof. By the stability of the observer and the ISS of the state equation, there exists $\gamma \in \mathcal{K}$ and $\alpha, \beta \in \mathcal{KL}$ such that for any $t_0 \geq 0$ and all $t \geq t_0$:

$$\begin{aligned} \|e(t)\| & \leq \alpha(\|e(t_0)\|, t - t_0), \\ \|x(t)\| & \leq \beta(\|x(t_0)\|, t - t_0) + \gamma(\sup_{\tau \in [t_0, t]} \|e(\tau)\|). \end{aligned}$$

We have obviously the estimate $\sup_{\tau \in [t_0, t]} \|e(\tau)\| \leq \alpha(\|e(t_0)\|, 0)$. Taking $\hat{x}(t_0) = 0$, we find $e(t_0) = x(t_0)$ and hence $\sup_{\tau \in [t_0, t]} \|e(\tau)\| \leq \alpha(\|x(t_0)\|, 0)$. Finally we have

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t) + \gamma \circ \alpha(\|x(t_0)\|, 0) \tag{5.14}$$

and therefore the system is stable. The finite-time convergence of the system is a direct consequence of the finite-time convergence of the error e and the finite-time convergence of the system (5.9). We conclude that the system (5.13) is globally FTS. □

Remark 5.8. *It is worth to stress that the system (5.13) is FTS in coordinates (e_1, e_2) (see Theorem 5.3) and it is FTS in coordinates (x_1, x_2, e_1, e_2) (Theorem 5.7). Moreover, equation (5.14) actually proves the stability of the isolated coordinates (x_1, x_2) provided that we choose $\hat{x}(t_0) = 0$.*

5.4 Robustness properties of the closed loop system with fixed parameters

If we choose $\beta = \frac{1}{2-\alpha}$ and $\rho_1 = 2-\alpha$ in (5.13), it is easy to see that the system becomes $\mathfrak{r}$-homogeneous of degree $\alpha - 1$ where $\mathfrak{r} = [r_1, r_2, \rho_1, \rho_2] = [2-\alpha, 1, 2-\alpha, 1]$. This choice provides another proof of the Theorem (5.7) without the help of ISS: being attractive and homogeneous, the system is stable.

In this section, we will study the robustness properties that we can get in this setting thanks to homogeneity. Indeed, we will be interested in the system:

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & k_1 \lfloor x_1 \rceil^{\frac{\alpha}{2-\alpha}} + k_2 \lfloor x_2 - e_2 \rceil^{\alpha} \\ \dot{e}_1 & = & e_2 + l_1 \lfloor e_1 \rceil^{\frac{1}{2-\alpha}} \\ \dot{e}_2 & = & l_2 \lfloor e_1 \rceil^{\frac{\alpha}{2-\alpha}} \end{array}\right. . \tag{5.15}$$

5.4.1 Robustness analysis

Assume that the system (5.15) is subject to disturbances:

1. a noise d_1 on the output x_1;
2. a perturbation d_2 which may appear in the transmission channel between the controller and the observer;
3. physical perturbations d_3 like frictions or unmodelled dynamics;
4. computationnal errors $\hat{d}_1$ and $\hat{d}_2$ on $\hat{x}_1$ and $\hat{x}_2$.

The disturbed system is now:

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & k_1 \lfloor x_1 + d_1 \rceil^{\frac{\alpha}{2-\alpha}} + k_2 \lfloor x_2 - e_2 + d_2 + \hat{d}_2 \rceil^{\alpha} + d_3 \\ \dot{e}_1 & = & e_2 - \hat{d}_2 + l_1 \lfloor e_1 - \hat{d}_1 + d_1 \rceil^{\frac{1}{2-\alpha}} \\ \dot{e}_2 & = & l_2 \lfloor e_1 - \hat{d}_1 + d_1 \rceil^{\frac{\alpha}{2-\alpha}} + d_3 \end{array}\right. . \tag{5.16}$$

Let us denote the disturbance $\mathfrak{d} = (d_1, d_2, d_3, \hat{d}_1, \hat{d}_2)$. We have the following robustness result:

Theorem 5.9. *The system (5.16) is ISS with respect to input $\mathfrak{d}$. Its asymptotic gain is upper bounded by the function γ_1 given by*

$$\gamma_1(s) = C_1 \begin{cases} s^{\frac{1}{2-\alpha}} & \text{if} \quad s \leq 1 \\ s^{\frac{2-\alpha}{\alpha}} & \text{if} \quad s \geq 1 \end{cases}, \tag{5.17}$$

and C_1 is a constant. We assume hereafter without loss of generality that $C_1 \geq 1$.

Proof. In view of Theorem 5.1, we are in position to apply the Theorem 4.5. We see that the auxiliary vector field is $(\mathfrak{r}, \tilde{\mathfrak{r}})$-homogeneous, with $\tilde{\mathfrak{r}} = [2-\alpha;\ 1;\ \alpha;\ 2-\alpha;\ 1]$, of degree $\alpha - 1$. Since $\tilde{r}_{\min} = \alpha > 0$, by Theorem 4.5, the system is ISS with respect to the input $\mathfrak{d}$. The Corollary 4.8 gives the estimation of the asymptotic gain. $\square$

5.4.2 Discretization effects

Similarly, we can study the influence of the discretization of the control and the observer in our observer-based feedback. We assume that there exists a sequence of sampling instants $(t_k)_{k\in\mathbb{N}}$ increasing to $+\infty$ at which the observer and the control are updated, such that $0 < t_{k+1} - t_k \leq h$. For $t \in (t_k, t_{k+1})$, the observer and the control remain constant. The system can be rewritten, for $t \in [t_k, t_{k+1})$:

$$\begin{cases} \dot{x}_1(t) &=& x_2(t) \\ \dot{x}_2(t) &=& k_1\lfloor x_1(t_k)\rceil^{\frac{\alpha}{2-\alpha}} + k_2\lfloor \hat{x}_2(t_k)\rceil^{\alpha} \\ \hat{x}_1(t_{k+1}) &=& \hat{x}_1(t_k) + (t_{k+1} - t_k)\times \\ && \left(\hat{x}_2(t_k) - l_1\lfloor x_1(t_k) - \hat{x}_1(t_k)\rceil^{\frac{1}{2-\alpha}}\right) \\ \hat{x}_2(t_{k+1}) &=& \hat{x}_2(t_k) + (t_{k+1} - t_k)\times \\ && \left(u(t_k) - l_2\lfloor x_1(t_k) - \hat{x}_1(t_k)\rceil^{\frac{\alpha}{2-\alpha}}\right) \end{cases}. \tag{5.18}$$

To compare this hybrid system with the continuous system (5.15), we need to define some other variables. We define, for $t \in [t_k, t_{k+1})$:

$$\begin{cases} \dot{\tilde{x}}_1(t) &=& \hat{x}_2(t_k) - l_1\lfloor x_1(t_k) - \hat{x}_1(t_k)\rceil^{\frac{1}{2-\alpha}} \\ \dot{\tilde{x}}_2(t) &=& u(t_k) - l_2\lfloor x_1(t_k) - \hat{x}_1(t_k)\rceil^{\frac{\alpha}{2-\alpha}} \end{cases}. \tag{5.19}$$

Setting $\tilde{x}_1(t_0) = \hat{x}_1(t_0)$ and $\tilde{x}_2(t_0) = \hat{x}_2(t_0)$ leads to $\tilde{x}_1(t_k) = \hat{x}_1(t_k)$ and $\tilde{x}_2(t_k) = \hat{x}_2(t_k)$ for any $k \in \mathbb{N}$. These variables are affine interpolations of the discrete system. We are naturally led to define new "observation errors" by $\varepsilon_1 = x_1 - \tilde{x}_1$ and $\varepsilon_2 = x_2 - \tilde{x}_2$. Finally, setting $\iota(t) = \max\{t_k, t_k \leq t\}$, and:

$$\begin{cases} d_1(t) &= x_1(\kappa(t)) - x_1(t) \\ \tilde{d}_2(t) &= \tilde{x}_2(\iota(t)) - \tilde{x}_2(t) \\ \tilde{d}_1(t) &= \tilde{x}_1(\iota(t)) - \tilde{x}_1(t) \end{cases}, \tag{5.20}$$

we get, for $t \in \mathbb{R}_+$:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= k_1 \lfloor x_1 + d_1 \rceil^{\frac{\alpha}{2-\alpha}} + k_2 \lfloor x_2 - \varepsilon_2 + \tilde{d}_2 \rceil^{\alpha} \\ \dot{\varepsilon}_1 &= \varepsilon_2 - \tilde{d}_2 + l_1 \lfloor \varepsilon_1 - \tilde{d}_1 + d_1 \rceil^{\frac{1}{2-\alpha}} \\ \dot{\varepsilon}_2 &= l_2 \lfloor \varepsilon_1 - \tilde{d}_1 + d_1 \rceil^{\frac{\alpha}{2-\alpha}} \end{cases}. \tag{5.21}$$

Therefore, setting $z = (x_1, x_2, \varepsilon_1, \varepsilon_2)$ and $\Delta = (d_1, \tilde{d}_1, \tilde{d}_2)$, we have the following corollary of Theorem 5.9:

Corollary 5.10. *The system (5.21) with $\alpha \in (0,1)$ is ISS w.r.t. the state z and the input Δ. Its asymptotic gain is upper bounded by γ_1.*

We shall now prove that the discretized system (5.18) is practically stable and converging to a ball which radius is a class $\mathcal{K}$ function of h.

Let us now study the variations of the input Δ through time.

$$\begin{aligned} |d_1(t)| &= \left| \int_{\iota(t)}^{t} \dot{x}_1(\tau) d\tau \right| \\ &\leq \int_{\iota(t)}^{t} |x_2(\tau) - x_2(\iota(t))| \, d\tau + h \, |x_2(\iota(t))| \\ &\leq \int_{\iota(t)}^{t} \int_{\iota(t)}^{\tau} |u(\iota(t))| \, ds d\tau + h \, |x_2(\iota(t))| \\ &\leq h^2 \, |u(\iota(t))| + h \, |x_2(\iota(t))| \, , \end{aligned}$$

where $u(\iota(t)) = k_1 \lfloor x_1(\iota(t)) \rceil^{\frac{\alpha}{2-\alpha}} + k_2 \lfloor \tilde{x}_2(\iota(t)) \rceil^{\alpha}$. Similarly, we get:

$$\begin{aligned} |\tilde{d}_2(t)| &\leq h \left| u(\iota(t)) - l_1 \lfloor \varepsilon_1(\iota(t)) \rceil^{\frac{1}{2-\alpha}} \right| \\ |\tilde{d}_1(t)| &\leq h \left| \tilde{x}_2(\iota(t)) - l_2 \lfloor \varepsilon_1(\iota(t)) \rceil^{\frac{\alpha}{2-\alpha}} \right| . \end{aligned}$$

We easily deduce that if $h \leq 1$, then $\|\Delta(t)\| \leq \gamma_2(\|z(\iota(t))\|)$, where:

$$\gamma_2(s) = C_2 h \begin{cases} s^{\frac{\alpha}{2-\alpha}} & \text{if} \quad s \leq 1 \\ s & \text{if} \quad s \geq 1 \end{cases} \tag{5.22}$$

and $C_2 > 0$ is a constant.

Now, we set $s_0 > 0$ such that:

$$s_0 < \min\left\{1, \left(C_1^{(2-\alpha)(4-\alpha^2)} C_2^{\alpha(2-\alpha)}\right)^{\frac{1}{(2-\alpha)(4-\alpha^2)-\alpha^2}}\right\}. \tag{5.23}$$

We claim that the following choice for h will be satisfactory:

$$h = \frac{1}{C_2}\left(\frac{s_0^{(2-\alpha)(4-\alpha^2)-\alpha^2}}{C_1^{(2-\alpha)(4-\alpha^2)}}\right)^{\frac{1}{\alpha(2-\alpha)}}. \tag{5.24}$$

Let us note that the condition (5.23) and (5.24) imply $0 < h < 1$. Denote $\theta(s) = s - \gamma_1 \circ \gamma_2(s)$, where γ_1 is given by (5.17) and γ_2 is given by (5.22).

Lemma 5.11. *For all $s > s_0$, we have $\theta(s) > 0$. Moreover the function θ is strictly increasing for $s > s_0$ and $\theta(s) \underset{s\to+\infty}{\longrightarrow} +\infty$.*

Proof. First of all, let us note that the definitions of h and $\alpha \in (0,1)$ imply the following inequality:

$$h < \frac{1}{C_2 C_1^{\frac{4-\alpha^2}{\alpha}}} < \frac{1}{C_1 C_2}. \tag{5.25}$$

In particular, equation (5.25) ensures that $\frac{1}{C_2 h} > C_1 \geq 1$. Let us hence distinguish 3 cases:

1. if $s \geq \frac{1}{C_2 h} > 1$, we have $\gamma_2(s) = C_2 hs \geq 1$ and $\gamma_1 \circ \gamma_2(s) = C_1 C_2 hs$. Hence $\theta(s) = (1 - C_1 C_2 h)s$ with $1 - C_1 C_2 h > 0$, thus θ is strictly increasing and positive.

2. if $s \in [1, \frac{1}{C_2 h}]$, we have similarly $\gamma_2(s) = C_2 hs \leq 1$. Therefore $\gamma_1 \circ \gamma_2(s) = C_1 (C_2 hs)^{\frac{\alpha}{4-\alpha^2}}$, that is $\theta(s) = s^{\frac{\alpha}{4-\alpha^2}}(s^{1-\frac{\alpha}{4-\alpha^2}} - C_1 (C_2 h)^{\frac{\alpha}{4-\alpha^2}})$. Using again (5.25) and $s \geq 1$, we get that θ is strictly increasing and positive.

3. finally if $s_0 < s < 1$, using the expression of h we have:

$$\gamma_1 \circ \gamma_2(s) = C_1 \left(C_2 h s^{\frac{\alpha}{2-\alpha}}\right)^{\frac{\alpha}{4-\alpha^2}} = s_0^{\frac{(2-\alpha)(4-\alpha^2)-\alpha^2}{(2-\alpha)(4-\alpha^2)}} s^{\frac{\alpha^2}{(2-\alpha)(4-\alpha^2)}}.$$

Therefore $\theta(s) = (s^{\frac{(2-\alpha)(4-\alpha^2)-\alpha^2}{(2-\alpha)(4-\alpha^2)}} - s_0^{\frac{(2-\alpha)(4-\alpha^2)-\alpha^2}{(2-\alpha)(4-\alpha^2)}}) \times s^{\frac{\alpha^2}{(2-\alpha)(4-\alpha^2)}}$ and θ is positive and strictly increasing for $s_0 < s < 1$.

Moreover, the first case gives $\theta(s) = (1 - C_1C_2h)s$ for $s \geq \frac{1}{C_2h}$ and equation (5.25) gives $1 - C_1C_2h > 0$, which ends the proof. □

Theorem 5.12. *If h is given by (5.24), then the ball centered at the origin of radius s_0 is globally asymptotically stable.*

Proof. Let us first show the stability. Since $\iota(t) \leq t$, we have $\sup_{\tau\in[t_0,t]} \|\Delta(\tau)\| \leq \sup_{\tau\in[t_0,t]} \gamma_2(\|z(\tau)\|)$. Therefore, by Corollary 5.10, there exists a class $\mathcal{KL}$ function β such that, for all $t \geq t_0$:

$$\|z(t)\| \leq \beta(\|z(t_0)\|, t - t_0) + \gamma_1 \circ \gamma_2(\sup_{\tau\in[t_0,t]} \|z(\tau)\|). \tag{5.26}$$

Let t_{max} belongs to the interval of definition of $z(t)$. For $t \in [t_0, t_{max}]$, we have:

$$\sup_{\tau\in[t_0,t_{max}]} \|z(\tau)\| \leq \beta(\|z(t_0)\|, 0) + \gamma_1 \circ \gamma_2(\sup_{\tau\in[t_0,t_{max}]} \|z(\tau)\|), \tag{5.27}$$

and thus $\theta\left(\sup_{\tau\in[t_0,t_{max}]} \|z(\tau)\|\right) \leq \beta(\|z(t_0)\|, 0)$. By Lemma 5.11, there exists s_1 such that for all $s > s_1$, $\theta(s) > \beta(\|z(t_0)\|, 0)$. Therefore, $\sup_{\tau\in[t_0,t_{max}]} \|z(\tau)\| \leq s_1$ is bounded. Since this inequality is true for all t_0 and t_{max}, $\|z(t)\|$ is uniformly bounded. Lemma 5.11 also implies that the function $\tilde{\theta} : \sigma \mapsto \theta(\sigma + s_0)$ is a class $\mathcal{K}_\infty$ function. If $\sup_{\tau\geq 0} \|z(\tau)\| > s_0$, then we have

$$\sup_{\tau\geq 0} \|z(\tau)\| \leq s_0 + \tilde{\theta}^{-1} \circ \beta(\|z(0)\|, 0). \tag{5.28}$$

Being true if $\sup_{\tau\geq 0} \|z(\tau)\| \leq s_0$ as well, this inequality always holds and proves the stability of the ball centered at the origin of radius s_0.

Let us now prove that $\limsup_{t\to\infty} \|z(t)\| \leq s_0$. By (5.26) and the first part, we have:

$$\|z(t)\| \leq \beta(D_0, t - t_0) + \gamma_1 \circ \gamma_2(\sup_{\tau\geq t_0} \|z(\tau)\|). \tag{5.29}$$

The function β being of class $\mathcal{KL}$, for all $\varepsilon > 0$ there exists $T_0 \geq 0$ such that for all

$t - t_0 \geq T_0$, we have $\beta(D_0, t - t_0) \leq \varepsilon$. Therefore, for all $t \geq t_0 + T_0$:

$$\begin{aligned}
\|z(t)\| &\leq \varepsilon + \gamma_1 \circ \gamma_2(\sup_{\tau \geq t_0} \|z(\tau)\|) \\
\sup_{\tau \geq t_0 + T_0} \|z(\tau)\| &\leq \varepsilon + \gamma_1 \circ \gamma_2(\sup_{\tau \geq t_0} \|z(\tau)\|) \\
\lim_{t_0 \to +\infty} \sup_{\tau \geq t_0 + T_0} \|z(\tau)\| &\leq \varepsilon + \gamma_1 \circ \gamma_2(\lim_{t_0 \to +\infty} \sup_{\tau \geq t_0} \|z(\tau)\|) \\
\limsup_{t \to \infty} \|z(t)\| &\leq \varepsilon + \gamma_1 \circ \gamma_2(\limsup_{t \to +\infty} \|z(t)\|) \\
\theta(\limsup_{t \to \infty} \|z(t)\|) &\leq \varepsilon.
\end{aligned}$$

This last inequality is true for any $\varepsilon > 0$, thus $\theta(\limsup_{t\to\infty} \|z(t)\|) \leq 0$ and therefore $\limsup_{t\to\infty} \|z(t)\| \leq s_0$ by Lemma 5.11. □

5.5 Simulations

Select $\alpha = 0.6$, $\beta = \frac{1}{2-\alpha}$ and $k_1 = -1$, $k_2 = -2$, $l_1 = -1$, $l_2 = -2$, then clearly all conditions of theorems 5.7, 5.9 and 5.12 are satisfied.

The results of the system simulation are presented in figures 5.1, 5.2. In figures 5.1.a, 5.2.a and 5.1.b, 5.2.b the examples of transients in time are given for the system state (x_1, x_2) and the estimation error (e_1, e_2) respectively. In the case of figure 5.1 all disturbances are selected to be zero, the step of simulation $h = 0.002$. In the case of figure 5.2 $d_1(t) = 0.1\sin(5t)$ and $d_3(t) = 0.1\cos(6t)$ with $h = 0.2$ (the disturbances $d_2(t)$, $\tilde{d}_1(t)$ and $\tilde{d}_2(t)$ are generated by the computational procedure used for simulation). As we can conclude from the results presented in figure 5.1, the system is converging to zero in a finite time for both pairs of variables, and the convergence is also monotone (that illustrates the theoretical results obtained above). From figure 5.2 we see that the trajectories stay bounded in the presence of disturbances and that they converge to some ball around the origin even for a rather large simulation step h. Non-smoothness of the trajectories in figure 5.2 is due to the structure of the developed algorith and to the large step of integration.

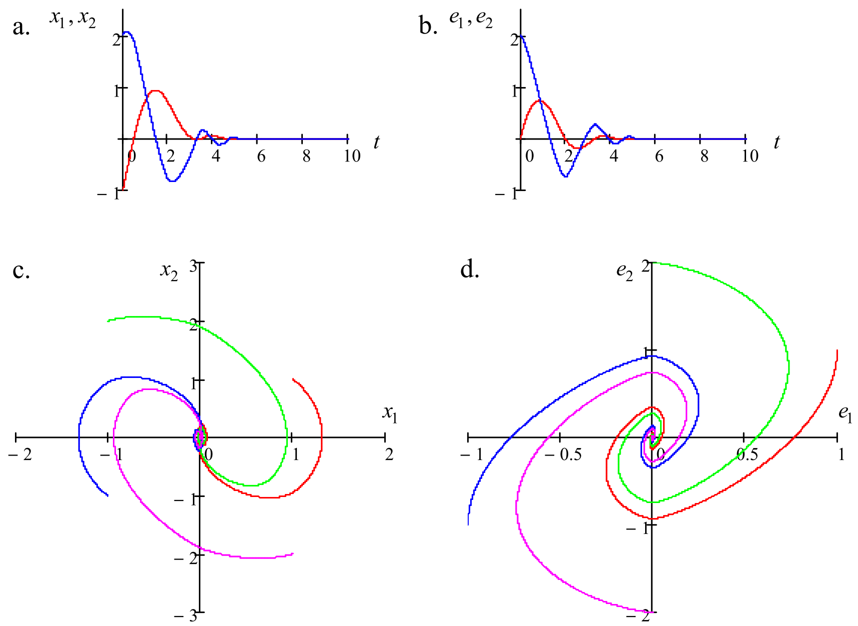

Figure 5.1: The results of simulation without disturbances, $h = 0.002$

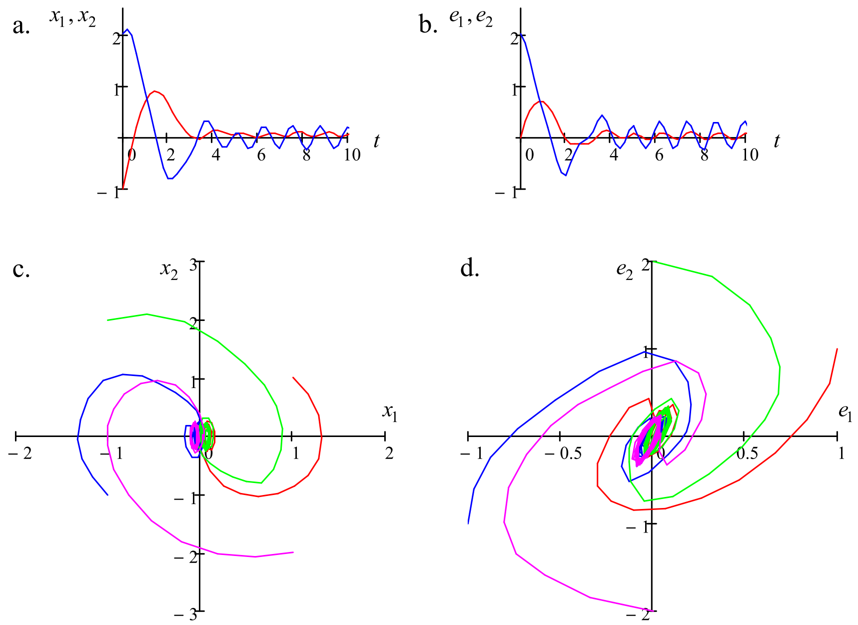

Figure 5.2: The results of simulation with disturbances, $h = 0.2$

5.6 Conclusion

In this Chapter, we have used some of the techniques developed herebefore, based on homogeneity and achieving finite-time stability and ISS. We have seen how the problems of finite-time control and estimation for the double integrator can be studied. We have designed a finite-time output control and proved the robustness of the proposed output control achieving an improvement of the result of [Orlov 2011]. We have shown that input and observer discretization does not destroy stability of the presented control algorithm. Finally, we have shown the efficiency of the obtained solution by computer simulations.

This work might be improved in several directions. For instance, the development of the approach to the case of n^{th}-dimensional integrator and the evaluation of the settling time function are possible future directions of the research.

General conclusion

In this work, we have studied stability and robustness properties of nonlinear systems using homogeneity-based methods.

The first Chapter has recalled the usual context of homogeneous systems as well as their main features. Standard, weighted and geometric definitions of homogeneity have been presented, and the main results of the theory stated. We have seen the link between negative degree of homogeneity and finite-time stability. Finally, this chapter gave an introduction to the theory of local homogeneity.

The second Chapter of this work extends the homogenization of nonlinear systems. This theory was already defined in the framework of weighted homogeneity, and an extension to the more general setting of the geometric homogeneity is presented here. The main approximation results are extended and academic examples of use are given.

The third Chapter develop a theoretical framework for defining geometric homogeneity of discontinuous systems and/or systems given by a differential inclusion. We have shown that the proposed definition is consistant with respect to the Filippov's procedure. Extensions of well-known qualitative properties of homogeneous systems have been presented in this context: the converse homogeneous Lyapunov Theorem, the equivalence between local attractiveness and global stability and the link between negative degree and finite-time stability.

The fourth Chapter consists in a study of the robustness properties of homogeneous or homogenizable systems. The *input-to-state* and *integral input-to-state stability* properties have been proved for homogeneous systems under two different principal assumptions: on the first hand, homogeneity with respect to the perturbation; on the other hand, a bound on the difference between the perturbed and the nominal systems. The type of robustness has been shown to be linked to the degree of homogeneity. These results have been compared to each other, and extended to the more general setting of homogenizable systems.

In the fifth Chapter, we studied the example of the double integrator system. We synthesize a homogeneous continuous finite-time stabilizing output feedback. We have studied its robustness with respect to perturbations and the impact of the discretization by using techniques developed before. Simulations conclude the theoretical study of this system and illustrate its behavior.

Homogeneity is a vast topic. Even though this work is extending existing results, a lot of work remains to do. Our future directions of research are the following.

- Develop homogenization for discontinuous systems, by using the techniques presented in Chapters 2 and 3.
- Extend the results of the Chapter 4 to geometric homogeneity.
- Extend the techniques developed in Chapter 5 for the n^{th} integrator and the corresponding observer.
- Extend the homogeneity definitions for more general systems, like hybrid systems and systems defined by distributions and currents.

Résumé étendu en français

Introduction générale

L'homogénéité est une propriété intrinsèque d'un objet sur lequel le flot d'un champ de vecteurs particulier agit comme une dilatation. Cette définition, très simple, implique un grand nombre de propriétés qualitatives importantes pour les objets qui la vérifient. Cette propriété, qui peut être vérifiée par des opérations algébriques, permet d'obtenir des conditions suffisantes de stabilité asymptotique qui simplifient l'étude de certains systèmes non-linéaires en permettant par exemple d'éviter le passage par une fonction de Lyapunov.

Dans le premier chapitre, nous présentons les définitions de l'homogénéité classique, à poids et géométrique, ainsi que l'homogénéité locale. Nous énonçons les propriétés usuelles des objets homogènes.

Dans le deuxième chapitre, nous développons la théorie de l'homogénéité locale au cadre de l'homogénéité géométrique. Les théorèmes d'approximation sont démontrés.

Dans le troisième chapitre, nous étendons la théorie de l'homogénéité aux inclusions différentielles. Nous y prouvons que cette extension est consistante avec la procédure de Filippov et que les propriétés usuelles des systèmes homogènes persistent dans ce cadre élargi. En particulier, un théorème de Lyapunov inverse homogène est démontré.

Dans le quatrième chapitre, nous étudions les propriétés de robustesse des systèmes homogènes. Nous prouvons, sous une hypothèse d'homogénéité par rapport à la perturbation ou de proximité entre le champ nominal et le champ perturbé, qu'un système homogène est ISS ou iISS. Ces résultats sont étendus aux systèmes localement homogènes.

Dans le cinquième chapitre, nous appliquons les techniques et les résultats obtenus précédemment à la stabilisation en temps fini par un retour de sortie d'un double intégrateur. Nous étudions les effets de la discrétisation sur un tel système.

Chapitre 1 – Homogénéité

Habituellement, on dit qu'une application est homogène si l'image d'un argument multiplié par un scalaire est égale à l'image de cet argument, multipliée par ce même scalaire à une puissance donnée, appelée le degré. On peut voir cette propriété comme une forme de symétrie de l'application le long des rayons issus de l'origine, celle-ci s'y comportant comme un monôme. Faisant suite à Euler et son théorème des fonctions homogènes, de nombreux auteurs se sont intéressés à cette propriété dans l'étude de la stabilité. Parmi les propriétés intéressantes, citons que l'attractivité locale de l'origine est équivalente à sa stabilité asymptotique globale.

La première généralisation de l'homogénéité classique a été introduite indépendamment par Zubov et Hermes. L'idée principale est de remplacer la multiplication par un scalaire par une opération plus générale appelée dilatation. Chaque coordonnée est multipliée par un même scalaire à une certaine puissance, appelée poids. On définit ainsi la dilatation associée aux poids, écrite sous la forme suivante:

$$\Lambda_{\mathbf{r}}(\lambda) : x \in \mathbb{R}^n \mapsto (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) \in \mathbb{R}^n,$$

pour $\lambda > 0$, où les $r_i > 0$ sont les poids et $\mathbf{r} = [r_1, \ldots, r_n]$ désigne le *poids généralisé*. Cette dilatation mène à la notion étendue d'*homogénéité à poids*.

Définition 1. *Soit* $\mathbf{r}$ *un poids généralisé.*

- *Une fonction* φ *est dite* $\mathbf{r}$*-homogène de degré* κ *ssi pour tout* $x \in \mathbb{R}^n$ *et tout* $\lambda > 0$ *nous avons* $\lambda^{-\kappa}\varphi(\Lambda_{\mathbf{r}} x) = \varphi(x)$*;*

- *Un champ de vecteurs* f *est dit* $\mathbf{r}$*-homogène de degré* κ *ssi pour tout* $x \in \mathbb{R}^n$ *et tout* $\lambda > 0$ *nous avons* $\lambda^{-\kappa}\Lambda_{\mathbf{r}}^{-1} f(\Lambda_{\mathbf{r}} x) = f(x)$*;*

- *Un système est* $\mathbf{r}$*-homogène s'il est défini par un champ de vecteurs* $\mathbf{r}$*-homogène.*

Cette propriété fut utilisée notamment pour approximer localement des systèmes non-linéaires et pour obtenir des résultats sur la stabilité ou la stabilisation. Un théorème très important fut démontré indépendamment par Zubov et Rosier : si un système homogène continu est globalement asymptotiquement stable, alors il existe une fonction de Lyapunov lisse homogène. Les applications de la théorie sont nombreuses. Citons notamment le lien avec la stabilité en temps fini : cette propriété est automatique pour les systèmes homogènes asymptotiquement stables de degré strictement négatif.

En plus de ces travaux sur l'homogénéité à poids, une approche géométrique, indépendante du choix des coordonnées, a été développée. Le poids généralisé est alors remplacé par un champ de vecteurs qui encode les directions de dilatation de l'espace.

Définition 2. *Un champ de vecteurs $\nu \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ est dit* Euler *s'il est complet et si l'origine est un équilibre GAS pour $-\nu$. Nous noterons toujours Φ le flot de ν, c'est-à-dire que $\Phi^s(x)$ est l'état au temps s de la trajectoire de ν issue de x à $s = 0$.*

Définition 3. *Soit ν un champ de vecteurs Euler. Une fonction φ ou un champ de vecteurs f est dit ν-homogène de degré κ ssi pour tout $s \in \mathbb{R}$ on a[2] :*

$$(\Phi^s)^*\varphi = e^{\kappa s}\varphi, \qquad (\Phi^s)^* f = e^{\kappa s} f. \tag{1}$$

Il existe des systèmes globalement asymptotiquement stables qu'aucun changement de coordonnées ne peut mettre dans une forme sous laquelle ils admettraient une approximation homogène asymptotiquement stable (au sens de l'homogénéité à poids).

Bien que les développements successifs de la théorie de l'homogénéité aient élargi son champ d'application, son intérêt ne réside pas uniquement dans les objets homogènes. Le théorème de Hermes qui suit permet d'approximer en un certain sens un champ de vecteurs non-homogène par un champ homogène, le premier héritant localement des propriétés de stabilité du second.

Théorème 4 (Théorème de Hermes)**.** *[Rosier 1992a] Soit f un champ de vecteurs continu et $\mathbf{r}$ un poids généralisé. Supposons qu'il existe un champ de vecteurs $\mathbf{r}$-homogène h de degré κ tel que*

$$\sup_{\|x\|=1} \|\lambda^{-\kappa}\Lambda_{\mathbf{r}}(\lambda)^{-1} f(\Lambda x) - h(x)\| \xrightarrow[\lambda \to 0]{} 0.$$

Si l'origine est un équilibre GAS pour h alors c'est un équilibre LAS pour f.

Cette idée a été largement développée dans [Andrieu 2008], menant à une théorie locale de l'homogénéité à poids.

Chapitre 2 – Transition géométrique de l'homogénéité globale à l'homogénéité locale

L'objet de ce chapitre est l'extension de l'homogénéité locale au cadre de l'homogénéité géométrique. On considère des fonctions et des champs de vecteurs continus dans ce

[2] $(\Phi^s)^*$ dénote le pullback par le difféomorphisme Φ^s, voir Appendice B.

chapitre.

Définition 5. *Soient φ et η des fonctions et soient f et h des champs de vecteurs.*

- *La fonction η est l'approximation ν-homogène de degré κ en 0 de la fonction φ si :*

$$e^{-\kappa s} (\Phi^s)^* \varphi \underset{s \to -\infty}{\overset{CUC}{\longrightarrow}} \eta. \tag{2}$$

- *Le champ de vecteurs h est l'approximation ν-homogenène de degré κ en 0 du champ de vecteurs f si :*

$$e^{-\kappa s} (\Phi^s)^* f \underset{s \to -\infty}{\overset{CUC}{\longrightarrow}} h. \tag{3}$$

Si la convergence uniforme est prise quand $s \to +\infty$, on obtient la définition de l'approximation en l'infini.

Toute approximation homogène d'une fonction ou d'un champ continu est continue. Dans la définition précédente, le degré est libre. En fait, un calcul simple montre qu'il existe au plus un degré pour lequel une approximation non-nulle peut exister.

Définition 6. *Soit ν un champ de vecteurs Euler et soit $\varphi \neq 0$ une fonction.*

1. *Le* degré local de ν-homogénéité *de φ en 0 est :*

$$\deg_0(\varphi) = \sup\{\kappa \in \mathbb{R} : e^{-\kappa s} (\Phi^s)^* \varphi \underset{s \to -\infty}{\overset{CUC}{\longrightarrow}} 0\}$$

   avec la convention $\sup \varnothing = -\infty$ *et* $\sup \mathbb{R} = +\infty$.

2. *Le* degré local de ν-homogénéité *de φ en ∞ est :*

$$\deg_\infty(\varphi) = \inf\{\kappa \in \mathbb{R} : e^{-\kappa s} (\Phi^s)^* \varphi \underset{s \to +\infty}{\overset{CUC}{\longrightarrow}}, 0\}$$

   avec la convention $\inf \varnothing = +\infty$ *et* $\inf \mathbb{R} = -\infty$.

Le degré local d'homogénéité est défini similairement pour les champs de vecteurs.

Il est tout à fait possible que pour tout degré, l'approximation homogène soit nulle ou divergente, donnant ainsi lieu à un degré local infini. Mais même quand un degré local fini existe, cela ne nous assure ni qu'il existe une approximation homogène de ce degré, ni qu'une telle approximation serait non-nulle. En revanche, quand une approximation existe pour le degré d'homogénéité locale, cette approximation est digne d'intérêt.

Définition 7. *Soit φ une fonction. Nous dirons que φ est ν-*homogénéisable *en* 0 *(resp. en l'infini) si* $\deg_0(\varphi)$ *(resp.* $\deg_\infty(\varphi)$*) est fini et si φ admet une approximation ν-homogène en* 0 *de degré* $\deg_0(\varphi)$ *(resp. une approximation ν-homogène en l'infini de degré* $\deg_\infty(\varphi)$*).*

*Si φ est ν-homogénéisable de degré d, l'approximation ν-homogène de φ de degré d est appelée la ν-*homogénéisation *de φ.*

La ν-homogénéisabilité et la ν-homogénéisation sont définis similairement pour un champ de vecteurs.

Ce cadre nous permet maintenant d'énoncer les théorèmes d'approximation au sens géométrique.

Théorème 8. *Soit f un champ de vecteurs ν-homogénéisable en* 0 *et soit h sa ν-homogénéisation en* 0*. Si l'origine est un équilibre asymptotiquement stable pour h, c'est un équilibre localement asymptotiquement stable pour f.*

Théorème 9. *Soit f un champ de vecteurs ν-homogénéisable en ∞ et soit h sa ν-homogénéisation en ∞. Si l'origine est un équilibre asymptotiquement stable pour h, alors il existe un compact strictement positivement invariant atteint par toutes les courbes intégrales de f.*

Chapitre 3 – Homogénéité: des équations différentielles ordinaires aux inclusions différentielles

Dans ce chapitre, nous étendons la définition de l'homogénéité géométrique aux inclusions différentielles, et nous présentons les extensions des résultats de stabilité des systèmes homogènes définis par des équations différentielles. Nous nous intéresserons uniquement à des inclusions différentielles définies par des applications multivaluées vérifiant les *hypothèses standard*, c'est-à-dire semi-continues supérieurement et telles que l'image de tout point soit compacte, convexe et non-vide.

Définition 10. *Soit ν un champ de vecteurs Euler. Une application multivaluée $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ est ν-homogène de degré $\kappa \in \mathbb{R}$ si pour tout $x \in \mathbb{R}^n$ et tout $s \in \mathbb{R}$ nous avons :*

$$F(\Phi^s(x)) = e^{\kappa s} d_x \Phi^s F(x).$$

Un système donné par une inclusion différentielle est dit ν-homogène de degré κ si l'application multivaluée qui le définit est ν-homogène de degré κ.

Cette définition étend naturellement la définition classique et elle est consistante avec la procédure de Filippov : si un champ de vecteurs est ν-homogène de degré κ, l'application multivaluée obtenue par la procédure de Filippov sera elle aussi ν-homogène de degré κ. Les résultats importants de la théorie classique s'étendent à ce nouveau cadre.

Théorème 11. *Soit F une application multivaluée ν-homogène de degré κ satisfaisant les hypothèses standard. Alors les assertions suivantes sont équivalentes :*

- *L'origine est (fortement) GAS pour le système $\dot{x} \in F(x)$.*
- *Pour tout $\mu > \max(-\kappa, 0)$, il existe une paire (V, W) de fonctions continues telles que :*
  1. *$V \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, V est définie positive et ν-homogène de degré μ;*
  2. *$W \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$, W est strictement positive en dehors de l'origine et ν-homogène de degré $\mu + \kappa$;*
  3. *$\max_{v \in F(x)} d_x V v \leq -W(x)$ pour tout $x \neq 0$.*

Ce théorème donne les mêmes caractérisations de la stabilité en temps fini que dans le cadre classique : un système homogène, GAS et de degré strictement négatif est stable en temps fini. On remarquera que la fonction de temps d'établissement est aussi homogène, continue en zéro et localement bornée, mais pas continue partout en général.

D'autres conditions suffisantes de stabilité asymptotique sont généralisées : l'existence d'un compact strictement positivement invariant ou l'attractivité locale impliquent la stabilité asymptotique.

Chapitre 4 – Robustesse et stabilité

Dans ce chapitre, nous nous intéressons à la robustesse des systèmes homogènes soumis à des entrées inconnues. On se restreint à des systèmes continus et au cadre de l'homogénéité à poids. Nous nous attachons à prouver que, sous certaines conditions, les systèmes homogènes sont ISS ou iISS.

Nous considérons le système suivant :

$$\dot{x} = f(x, d), \tag{4}$$

où $x \in \mathbb{R}^n$ désigne l'état du système tandis que $d \in \mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}^m)$ est l'entrée exogène et $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ est un champ de vecteurs continu vérifiant $f(0, 0) = 0$.

Théorème 12. *Supposons qu'il existe des poids* $\mathbf{r} = [r_1, \ldots, r_n] > 0$ *et* $\tilde{\mathbf{r}} = [\tilde{r}_1, \ldots, \tilde{r}_m] \geq 0$ *et un degré* κ *tels que* $f(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)d) = \lambda^\kappa \Lambda_r(\lambda) f(x,d)$ *pour tous* $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ *et* $\lambda > 0$. *On suppose enfin que le système (4) est globalement asymptotiquement stable pour* $d = 0$. *Alors le système (4) est*

- *ISS si* $\tilde{r}_{\min} > 0$,
- *iISS si* $\tilde{r}_{\min} = 0$ *et* $\kappa \leq 0$,

où $\tilde{r}_{\min} = \min_{1 \leq j \leq m} \tilde{r}_j$.

Les conditions du théorème sont en particulier vérifiées si on dispose d'un champ homogène continu f_0 et qu'on définit $f(x,d) = f(x) + d$ ou $f(x,d) = f(x+d)$. Dans le premier cas, le système est ISS si $\kappa > -r_{\min}$ et iISS si $\kappa = -r_{\min}$. Dans le second cas, le système est toujours ISS. On peut aussi obtenir une estimation du gain asymptotique.

Corollaire 13. *Nous conservons les hypothèses et les notations du théorème précédent. Supposons de plus que* $\tilde{r}_{\min} > 0$. *Alors il existe une constante* $C > 0$ *pour laquelle le gain asymptotique admet les estimations suivantes :*

$$\gamma(D) \leq C \begin{cases} D^{\frac{r_{\max}}{\tilde{r}_{\max}}} & \text{if } D \leq 1 \\ D^{\frac{r_{\min}}{\tilde{r}_{\min}}} & \text{if } D \geq 1 \end{cases}.$$

Le théorème précédent reposait sur une hypothèse d'homogénéité par rapport à la perturbation. On peut remplacer cette hypothèse par une condition de proximité entre le champ nominal non-perturbé et le champ perturbé.

Théorème 14. *Supposons que le système (4.1) est GAS pour* $d = 0$ *et* $\mathbf{r}$*-homogeneous de degré* κ, *i.e.* $f(\Lambda_r(\lambda)x, 0) = \lambda^\kappa \Lambda_r(\lambda) f(x,0)$ *pour tout* $x \in \mathbb{R}^n$ *et tout* $\lambda > 0$. *Supposons de plus qu'il existe des fonctions* $\psi, \varphi \in \mathcal{K}$ *et des réels positifs* $\vartheta_{\min} \leq \vartheta_{\max}$ *tels que pour tout* $x \in \mathbb{R}^n$ *et tout* $d \in \mathbb{R}^m$ *:*

$$\|f(x,d) - f(x,0)\| \leq \theta(\|x\|_r)\psi(\|d\|) + \varphi(\|d\|),$$

avec

$$\theta(s) = \begin{cases} s^{\vartheta_{\min}} & \text{if } s \leq 1 \\ s^{\vartheta_{\max}} & \text{if } s > 1 \end{cases}.$$

Alors le système (4.1) est

ISS si $\kappa > \vartheta_{\max} - r_{\min}$;

iISS si $\kappa \leq \vartheta_{\max} - r_{\min} \leq 0$.

Dans le manuscrit sont proposés des exemples pour lesquels le premier théorème s'applique mais pas le second, puis où le second s'applique mais pas le premier.

Les deux théorèmes précédents reposaient largement sur l'homogénéité du champ nominal. Lorsque le champ nominal n'est pas homogène, mais qu'il admet une homogénéisation en l'infini, on peut tout de même obtenir des résultats d'ISpS.

Théorème 15. *Supposons qu'il existe des poids* $\mathbf{r} > 0$ *et* $\tilde{\mathbf{r}} > 0$ *et une application continue* $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *tels que pour tout compact* $K \subset \mathbb{R}^n \times \mathbb{R}^m$

$$\sup_{(x,d)\in K} \|\lambda^{-\kappa}\Lambda_r^{-1}(\lambda) f(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)d) - h(x,d)\| \underset{\lambda\to+\infty}{\longrightarrow} 0.$$

Supposons que le système $\dot{x} = h(x,0)$ *est GAS, alors le système (4.1) est ISpS.*

Ce théorème correspond au théorème 12. Le théorème suivant correspond quant à lui au théorème 14.

Théorème 16. *Soit* $\mathbf{r}$ *un poids généralisé,* f_0 *un champ de vecteurs* $\mathbf{r}$*-homogénéisable en l'infini et notons* h_0 *sa* $\mathbf{r}$*-homogénéisation. Supposons aussi qu'il existe des fonctions* $\psi, \varphi \in \mathcal{K}$ *des réels positifs* $\vartheta_{\min} \leq \vartheta_{\max}$ *tels que pour tout* $x \in \mathbb{R}^n$ *et tout* $d \in \mathbb{R}^m$

$$\|f(x,d) - f(x,0)\| \leq \theta(\|x\|_r)\psi(\|d\|) + \varphi(\|d\|),$$

avec

$$\theta(s) = \begin{cases} s^{\vartheta_{\min}} & \text{if } s \leq 1 \\ s^{\vartheta_{\max}} & \text{if } s > 1 \end{cases}.$$

Si $\kappa = \deg_\infty f_0 > \vartheta_{\max} - r_{\min}$ *et si le système* $\dot{x} = h_0(x)$ *est globalement asymptotiquement stable, alors le système (4.1) est ISpS.*

Là encore, des exemples sont données pour comparer les champs d'application de ces deux théorèmes.

Chapitre 5 – Application au double intégrateur

Ce chapitre expose un schéma de stabilisation par retour de sortie d'un double intégrateur basé sur des techniques d'homogénéité. Les méthodes et les résultats obtenus précédemment y trouvent une application. Le système considéré est le suivant :

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= u(x_1, x_2), \\ y &= x_1, \end{cases} \tag{5}$$

où x_1 et x_2 désignent les états du système, u l'entrée et y la sortie.

Nous commençons par construire un retour d'état stabilisant. On prouve ainsi que, quelque soit la valeur du paramètre $\alpha \in]0, 1]$, le système suivant est $\mathbf{r}$-homogène de degré $\alpha - 1$ et stable en temps fini, avec $\mathbf{r} = [2 - \alpha;\ 1]$:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= k_1 \lfloor x_1 \rceil^{\frac{\alpha}{2-\alpha}} + k_2 \lfloor x_2 \rceil^{\alpha} \end{cases}, \tag{6}$$

pourvu que les gains k_1 et k_2 soient choisis strictement positifs.

On s'intéresse ensuite à l'observateur non-linéaire suivant :

$$\begin{cases} \dot{\hat{x}}_1 &= \hat{x}_2 - l_1 \lfloor y - \hat{x}_1 \rceil^{\beta} \\ \dot{\hat{x}}_2 &= u - l_2 \lfloor y - \hat{x}_1 \rceil^{2\beta-1} \end{cases}, \tag{7}$$

où l_1 et l_2 sont des réels et $\beta \in]\frac{1}{2}, 1]$. L'équation d'erreur associée est :

$$\begin{cases} \dot{e}_1 = e_2 + l_1 \lfloor e_1 \rceil^{\beta} \\ \dot{e}_2 = l_2 \lfloor e_1 \rceil^{2\beta-1} \end{cases},$$

Et on peut montrer que ce système est $\boldsymbol{\rho}$-homogène de degré $\rho_1(\beta - 1)$, avec $\boldsymbol{\rho} = [\rho_1, \beta\rho_1]$, et stable en temps fini.

On peut alors considérer le retour de sortie $u = k_1 \lfloor y \rceil^{\frac{\alpha}{2-\alpha}} + k_2 \lfloor \hat{x}_2 \rceil^{\alpha}$, ce qui donne, en boucle fermée, le système suivant :

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= k_1 \lfloor x_1 \rceil^{\frac{\alpha}{2-\alpha}} + k_2 \lfloor x_2 - e_2 \rceil^{\alpha} \\ \dot{e}_1 &= e_2 + l_1 \lfloor e_1 \rceil^{\beta} \\ \dot{e}_2 &= l_2 \lfloor e_1 \rceil^{2\beta-1} \end{cases}. \tag{8}$$

On montre alors que ce système est stable en temps fini en utilisant la stabilité en temps fini de chaque sous-système et l'ISS du système donné par les deux premières lignes par rapport à l'entrée e_2 grâce au théorème 12.

Dans la suite, on va s'intéresser à la robustesse de ce système en utilisant là encore les résultats du chapitre précédent. On va donc choisir les paramètres de manière à ce que le système soit homogène. En l'occurrence, il s'agit de sélectionner $\rho_1 = 2-\alpha$ et $\beta = \frac{1}{2-\alpha}$. Considérons alors le système perturbé suivant :

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= k_1\lfloor x_1 + d_1\rceil^{\frac{\alpha}{2-\alpha}} + k_2\lfloor x_2 - e_2 + d_2 + \hat{d}_2\rceil^{\alpha} + d_3 \\ \dot{e}_1 &= e_2 - \hat{d}_2 + l_1\lfloor e_1 - \hat{d}_1 + d_1\rceil^{\frac{1}{2-\alpha}} \\ \dot{e}_2 &= l_2\lfloor e_1 - \hat{d}_1 + d_1\rceil^{\frac{\alpha}{2-\alpha}} + d_3 \end{cases}, \tag{9}$$

où $d_1, d_2, d_3, \hat{d}_1$ et $\hat{d}_2$ sont des entrées inconnues. Une simple application du théorème 12 permet de prouver l'ISS de ce système perturbé. Enfin, nous allons appliquer ce résultat à l'étude des effets de la discrétisation. Considérons donc que l'entrée et l'observateur sont soumis à des dynamiques discrètes. On peut réécrire le système ainsi :

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= k_1\lfloor x_1(t_k)\rceil^{\frac{\alpha}{2-\alpha}} + k_2\lfloor \hat{x}_2(t_k)\rceil^{\alpha} \\ \hat{x}_1(t_{k+1}) &= \hat{x}_1(t_k) + (t_{k+1}-t_k)\times \\ & \quad \left(\hat{x}_2(t_k) - l_1\lfloor x_1(t_k) - \hat{x}_1(t_k)\rceil^{\frac{1}{2-\alpha}}\right) \\ \hat{x}_2(t_{k+1}) &= \hat{x}_2(t_k) + (t_{k+1}-t_k)\times \\ & \quad \left(u(t_k) - l_2\lfloor x_1(t_k) - \hat{x}_1(t_k)\rceil^{\frac{\alpha}{2-\alpha}}\right) \end{cases}. \tag{10}$$

En posant, pour $t \in [t_k, t_{k+1})$:

$$\begin{cases} \dot{\tilde{x}}_1(t) &= \hat{x}_2(t_k) - l_1\lfloor x_1(t_k) - \hat{x}_1(t_k)\rceil^{\frac{1}{2-\alpha}} \\ \dot{\tilde{x}}_2(t) &= u(t_k) - l_2\lfloor x_1(t_k) - \hat{x}_1(t_k)\rceil^{\frac{\alpha}{2-\alpha}} \end{cases}, \tag{11}$$

$\varepsilon_1 = x_1 - \tilde{x}_1$, $\varepsilon_2 = x_2 - \tilde{x}_2$, $\iota(t) = \max\{t_k, t_k \le t\}$, et :

$$\begin{cases} d_1(t) &= x_1(\iota(t)) - x_1(t) \\ \tilde{d}_2(t) &= \tilde{x}_2(\iota(t)) - \tilde{x}_2(t) \\ \tilde{d}_1(t) &= \tilde{x}_1(\iota(t)) - \tilde{x}_1(t) \end{cases}, \tag{12}$$

on peut réécrire le système sous la forme suivante :

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= k_1\lfloor x_1 + d_1\rceil^{\frac{\alpha}{2-\alpha}} + k_2\lfloor x_2 - \varepsilon_2 + \tilde{d}_2\rceil^{\alpha} \\ \dot{\varepsilon}_1 &= \varepsilon_2 - \tilde{d}_2 + l_1\lfloor \varepsilon_1 - \tilde{d}_1 + d_1\rceil^{\frac{1}{2-\alpha}} \\ \dot{\varepsilon}_2 &= l_2\lfloor \varepsilon_1 - \tilde{d}_1 + d_1\rceil^{\frac{\alpha}{2-\alpha}} \end{cases}. \tag{13}$$

On en déduit ainsi que le système est ISS par rapport à l'entrée $(d_1, \hat{d}_1, \hat{d}_2)$. Enfin, en étudiant les variations de l'entrée en fonction du temps et du pas d'échantillonnage h, on prouve le théorème suivant :

Théorème 17. *Si le pas d'échantillonnage est donné par :*

$$h = \frac{1}{C_2}\left(\frac{s_0^{(2-\alpha)(4-\alpha^2)-\alpha^2}}{C_1^{(2-\alpha)(4-\alpha^2)}}\right)^{\frac{1}{\alpha(2-\alpha)}},$$

alors la boule centrée à l'origine et de rayon s_0 est globalement asymptotiquement stable.

Enfin, des simulations viennent illustrer ces résultats.

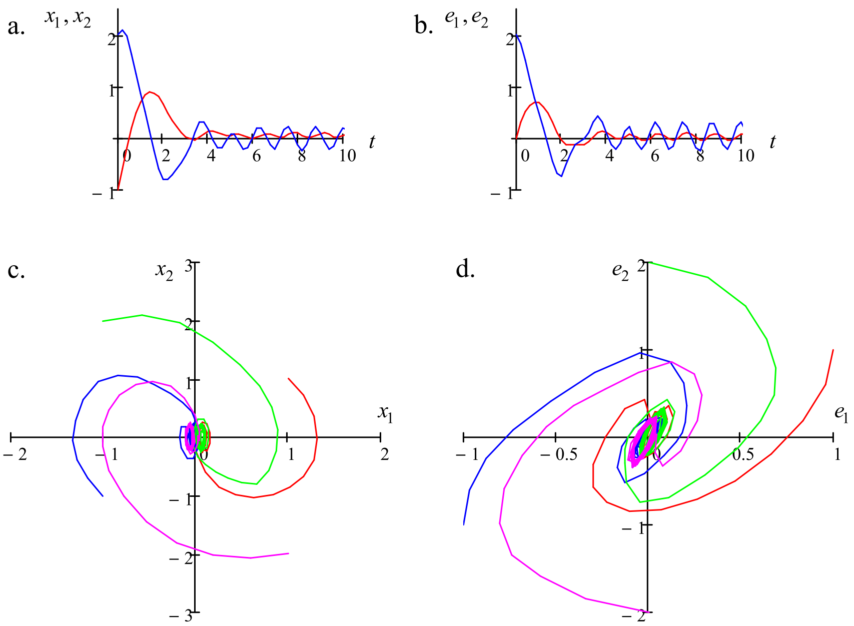

Figure 1: Résultats de simulation avec perturbations, $h = 0.2$

Conclusion

Dans ce travail nous avons étudié la stabilité et la robustesse de certains systèmes non-linéaires en utilisant des méthodes basées sur l'homogénéité. Même si ce travail étend des résultats existants, de nombreux points restent à approfondir, notamment l'homogénéisation et la robustesse des systèmes non-linéaires ou encore l'utilisation de ces techniques pour l'étude de l'intégrateur d'ordre n.

Appendix A

Stability and Lyapunov theory

Let us consider a continuous vector field f and the associated differential equation:

$$\dot{x} = f(x). \tag{A.1}$$

Definition A.1. *Let $x_\infty \in \mathbb{R}^n$ be fixed.*

- *The point x_∞ is an equilibrium of f if $f(x_0) = 0$.*
- *An equilibrum x_∞ is said to be* stable *if for any neighborhood $\mathcal{U}$ of x_∞, there exists a neighborhood $\mathcal{V}$ of x_∞ such that any solution x of (A.1) with $x(0) \in \mathcal{V}$ is defined for all $t \geq 0$ and $x(t) \in \mathcal{U}$ for all $t \geq 0$.*
- *An equilibrum x_∞ is said to be* locally attractive *(LAT) if there exists a neighborhood $\mathcal{V}$ of x_∞ such that any solution x of (A.1) with $x(0) \in \mathcal{V}$ is defined for all $t \geq 0$ and $x(t) \underset{t\to+\infty}{\longrightarrow} x_\infty$.*
- *An equilibrum x_∞ is said to be* globally attractive *(GAT) if in the previous definition we can choose $\mathcal{V} = \mathbb{R}^n$.*
- *An equilibrum x_∞ is said to be* locally finite-time attractive *if there exists a neighborhood $\mathcal{V}$ of x_∞ and a function $\mathsf{T} : \mathcal{V} \to \mathbb{R}_+$ such that any solution x of (A.1) with $x(0) \in \mathcal{V}$ is defined for all $t \geq 0$ and $x(t) = x_\infty$ for all $t \geq T(x_0)$.*
- *An equilibrum x_∞ is said to be* globally finite-time attractive *if in the previous definition we can choose $\mathcal{V} = \mathbb{R}^n$.*
- *An equilibrum x_∞ is said to be* locally asymptotically stable *(LAS) if it is stable and locally attractive.*

- *An equilibrum x_∞ is said to be* globally asymptotically stable *(GAS) if it is stable and globally attractive.*
- *An equilibrium x_∞ is said to be* finite-time stable *(FTS) if it is stable and locally finite-time attractive.*
- *An equilibrium x_∞ is said to be* globally finite-time stable *(GFTS) if it is stable and globally finite-time attractive.*
- *A compact set K is said to be* strictly positively invariant *(SPI) if any solution of (A.1) with initial condition in K belongs to $\mathring{K}$ for $t > 0$.*
- *The vector field f is said to be* complete *if the maximal solutions of (A.1) are defined for all $t \in \mathbb{R}$. It is* forward complete *if the maximal solutions of (A.1) are defined for all $t \in \mathbb{R}_+$.*

Given that an equilibrium can be translated to zero with a change of frame, the definitions and results given thereafter always assume that the equilibrium under consideration is the origin of $\mathbb{R}^n$. Let us also define the notions of stability and attractiveness for a set.

Definition A.2. *Let $K \subset \mathbb{R}^n$ be a compact set.*

- *The set K is said to be* stable *if for any neighborhood $\mathcal{U}$ of K, there exists a neighborhood $\mathcal{V}$ of K such that any solution x of (A.1) with $x(0) \in \mathcal{V}$ is defined for all $t \geq 0$ and $x(t) \in \mathcal{U}$ for all $t \geq 0$.*
- *The set K is said to be* locally attractive *if there exists a neighborhood $\mathcal{V}$ of K such that any solution x of (A.1) with $x(0) \in \mathcal{V}$ is defined for all $t \geq 0$ and $\mathrm{d}(\{x(t)\}, K) \underset{t \to +\infty}{\longrightarrow} 0$.*
- *The set K is said to be* globally attractive *if in the previous definition we can choose $\mathcal{V} = \mathbb{R}^n$.*
- *The set K is said to be* locally asymptotically stable *if it is stable and locally attractive.*
- *The set K is said to be* globally asymptotically stable *if it is stable and globally attractive.*

Definition A.3. *Let $\mathcal{U}$ be an open neighborhood of the origin and let $V \in \mathcal{C}^1(\mathcal{U}, \mathbb{R})$ be a positive definite function. The function V is said to be a* Lyapunov function *for the system (A.1) on $\mathcal{U}$ if $\mathcal{L}_f V(x) < 0$ for all $x \in \mathcal{U} \setminus \{0\}$.*

Theorem A.4 (Theorem of Lyapunov)**.** *Assume that there exists an open neighborhood $\mathcal{U}$ of the origin and a Lyapunov function for the system (A.1) on $\mathcal{U}$. Then the origin is LAS.*

If there exists a proper Lyapunov function for the system (A.1) on $\mathbb{R}^n$, then the origin is GAS.

Theorem A.5 (Kurzweil's converse Lyapunov theorem)**.** *[Kurzweil 1963] Assume that the system (A.1) is GAS. Then there exists a proper Lyapunov function for (A.1) on $\mathbb{R}^n$.*

Appendix B

Additional material

Definition B.1. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism of class $\mathcal{C}^1$, let φ be a function and f be a vector field.*

- *The pullback of φ by Φ is the function $\Phi^*\varphi : x \mapsto \varphi(\Phi(x))$.*
- *The pullback of f by Φ is the vector field $\Phi^* f : x \mapsto (d_x\Phi)^{-1} f(\Phi(x))$.*

Lemma B.2. *[Marsden 1998][Theorem 6.4.1 p365] Let ν be a $\mathcal{C}^1$ vector field, Φ its flow, φ a function and f a vector field, both of class $\mathcal{C}^1$. Then*

$$\begin{aligned} \frac{d}{ds}(\Phi^s)^*\varphi &= (\Phi^s)^*\mathcal{L}_\nu\varphi, \\ \frac{d}{ds}(\Phi^s)^*f &= (\Phi^s)^*[\nu, f]. \end{aligned}$$

Definition B.3. *Let $(\varphi_k)_{k\in\mathbb{N}}$ be a sequence of functions. We say that this sequence converges uniformly on compact sets to a function φ, denoted $\varphi_k \underset{k\to+\infty}{\overset{CUC}{\longrightarrow}} \varphi$, if for any compact set K and any $\varepsilon > 0$ there exists $k_0 > 0$ such that for all $k \geq k_0$, $\sup_{x\in K} |\varphi_k(x) - \varphi(x)| \leq \varepsilon$.*

Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of vector fields. We say that this sequence converges uniformly on compact sets to a vector field f, denoted $f_k \underset{k\to+\infty}{\overset{CUC}{\longrightarrow}} f$, if for any compact set K and any $\varepsilon > 0$ there exists $k_0 > 0$ such that for all $k \geq k_0$, $\|f_k - f\|_K \leq \varepsilon$, where $\|f\|_K = \sup_{x\in K} \|f(x)\|$.

Appendix C

Other works

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Technical communique

Retraction obstruction to time-varying stabilization ☆

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ABSTRACT

This paper addresses the problem of the global stabilization on a total space of a fiber bundle with a compact base space. We prove that, under mild assumptions (existence of a continuous section and forward unicity of solutions), no equilibrium of a continuous system defined on such a state space can be globally asymptotically uniformly stabilized using continuous time-varying feedback.

1. Introduction

Topological obstruction to stabilization is a long standing problem of control theory detailed in the introduction of Moulay and Hui (2011). There are two main topological obstructions to continuous stabilization. First, the *Brockett obstruction* is a local obstruction related to the structure of the controlled systems involving the nature of feedback controls (Brockett, 1983). Then, the *retraction obstruction* is a global obstruction related to the structure of the underlying state space: if the state space of the system has the structure of a vector bundle over a compact manifold, no continuous *static* feedback can globally stabilize an equilibrium. This result has been proved and its consequences are studied in detail in Bhat and Bernstein (2000).

The Brockett condition, which is necessary in the case of continuous time-invariant feedback controls, does not remain necessary for driftless controllable systems with time-varying feedback. The existence of such feedbacks has been proved in Coron (1992), while (Pomet, 1992) gave an explicit design under an additional condition on the Control Lie Algebra (see Pomet, 1992, Assumption 1). A natural question is, hence, to wonder whether continuous time-varying feedback controls could also avoid the retraction obstruction as suggested in the introduction of Nakamura, Yamashita, and Nishitani (2009).

In Bhat and Bernstein (2000, Remark 1), the authors mention that their result also handles the case of dynamic feedback. Indeed a dynamic feedback is usually seen as a dynamic extension where the augmented state is stable. Estimation of parameters or observer-based control are in this scope. In that case, the result of Bhat and Bernstein (2000) is applicable directly, with the method exposed in their remark. Nevertheless, a time-varying static feedback is also a dynamic extension using a timer $\dot{\tau} = 1$. However in this situation, there is no convergence to a single equilibrium point, but to a submanifold; the result of Bhat and Bernstein (2000) is therefore not applicable in this context, following (Bhat & Bernstein, 2000, Remark 1).

The aim of this paper is to prove that, in the second case, the obstruction still remains: a time-varying feedback control which is globally asymptotically uniformly stabilizing a system defined on a fiber bundle with a compact manifold as its base space does not exist.

2. Retraction obstruction

By a *manifold* we mean a smooth, positive dimensional, connected manifold without boundary. The definition of a *fiber bundle* is given for instance in Abraham and Marsden (2008).

Our purpose is to link the topological property of contractibility of a manifold to the existence of a globally asymptotically stable

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equilibrium. Let us introduce the definitions we will be using and some useful properties.

Definition 1. Let E be a topological space and $x_0 \in E$. A *retraction* of E on x_0 is a continuous mapping $h : [0, 1] \times E \to E$ such that for all $x \in E$, $h(0, x) = x$ and $h(1, x) = x_0$. A topological space E is said to be *contractible* if there exists a retraction of E.

Proposition 2 (*Guillemin & Pollack, 2010, Section 2.4*). *No compact manifold is contractible.*

Consider a controlled system defined by

$$\dot{y} = f_0(y, u) \quad y \in \mathcal{N},\ u \in \mathsf{U}, \tag{1}$$

with $\mathcal{N}$ a manifold and U a set of admissible controls, and where f_0 is a continuous vector field.

We wonder about the global stabilizability of the system via a time-varying feedback $u(y, t)$. To do that, we will set $\dot{\tau} = 1$ and look at the partial asymptotic stability of the closed-loop system:

$$\begin{aligned} \dot{y} &= f_0(y, u(y, \tau)) \\ \dot{\tau} &= 1. \end{aligned} \tag{2}$$

Hence, let us introduce the definitions of partial stability we will be using. Those definitions are adapted from Haddad and Chellaboina (2007).

Definition 3. Let $\mathcal{M} = \mathcal{N} \times \mathcal{T}$ be a product manifold. Consider $f = (f_1, f_2)$ a forward complete continuous vector field on $\mathcal{M}$ with the property of unicity of solutions in forward time. We denote by Φ the semiflow of f and p_1 the canonical projection on $\mathcal{N}$.

(1) We say that $y_\infty \in \mathcal{N}$ is a *partial equilibrium* if for all $\tau \in \mathcal{T}$, we have $f_1(y_\infty, \tau) = 0$.
(2) A partial equilibrium $y_\infty \in \mathcal{N}$ is said to be *partially stable uniformly in* τ if for all $U \subset \mathcal{N}$ neighborhood of y_∞ there exists $V \subset \mathcal{N}$ a neighborhood of y_∞ such that for all $y \in V$ and for all $\tau \in \mathcal{T}$, $p_1 \circ \Phi(t, (y, \tau)) \in U$ for all $t \geq 0$.
(3) A partial equilibrium $y_\infty \in \mathcal{N}$ is said to be *partially globally asymptotically stable uniformly in* τ if it is partially stable uniformly in τ and if for all $(y, \tau) \in \mathcal{M}$ we have $p_1 \circ \Phi(t, (y, \tau)) \to y_\infty$ when $t \to +\infty$.

Remark 4. The last item of Definition 3 is slightly different from the more standard ones. Indeed, to prove our result, we only need the stability to be uniform with respect to τ. The uniformity with respect to τ of the convergence, which is usually required, is not necessary here.

The following definition, inspired by Khalil (2002, Chapter 12) about time-varying stabilizability, is given here in the partial stability context.

Definition 5. The system (1) is said to be *globally asymptotically uniformly stabilizable by means of a continuous generalized time-varying feedback* if there exist a point $y_\infty \in \mathcal{N}$, a manifold $\mathcal{T}$, a continuous mapping $f_2 : \mathcal{N} \times \mathcal{T} \to \mathsf{T}\mathcal{T}$ with $f_2(y, \tau) \in \mathsf{T}_\tau\mathcal{T}$ for all $y \in \mathcal{N}$ and all $\tau \in \mathcal{T}$ and a continuous control law $u(y, \tau)$ such that y_∞ is a partial equilibrium of the closed loop system $(\dot{y}, \dot{\tau}) = (f_0(y, u(y, \tau)), f_2(y, \tau))$ and is partially globally asymptotically stable uniformly in τ.

Remark 6. Let us note that, taking $\mathcal{T} = \mathbb{R}$ and $f_2(y, \tau) = 1$, this definition boils down to the definition of global asymptotic stabilization by means of a continuous time-varying feedback. In the generalized time-varying setting, the variable τ can be stable or not, scalar or vector, bounded or not.

In Bhat and Bernstein (2000, Theorem 1), the authors use Proposition 2 to prove that if a manifold $\mathcal{N}$ admits a structure of fiber bundle over a compact manifold, then no continuous vector field over $\mathcal{N}$ can have a unique globally asymptotically stable equilibrium. Hence, they conclude that the system (1) cannot be globally asymptotically stabilized by means of a state feedback. Now let us prove that (Bhat & Bernstein, 2000, Theorem 1) can be extended in the following way to the time-varying setting.

Theorem 7. *Assume that $\mathcal{N}$ is a manifold with a structure of fiber bundle over a compact manifold $\mathcal{Q}$. If there exists a continuous section of the bundle, then the system (1) is not globally asymptotically uniformly stabilizable by means of a continuous generalized time-varying feedback in such a way that the augmented vector field has the forward unicity of solutions property.*

Proof. *Ad absurdum*, assume that there exists a continuous dynamic feedback which globally asymptotically uniformly stabilizes the system (1) in such a way that the augmented vector field has the forward unicity of solutions. Let us denote by τ the added variable, and $\tau \in \mathcal{T}$. We have $\dot{\tau} = f_2(y, \tau)$, and the system (1) can be rewritten in an extended form, with $f_1(y, \tau) = f_0(y, u(y, \tau))$:

$$\begin{pmatrix} \dot{y} \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} f_1(y, \tau) \\ f_2(y, \tau) \end{pmatrix}. \tag{3}$$

The Eq. (3) defines a continuous vector field f on the manifold $\mathcal{M} = \mathcal{N} \times \mathcal{T}$, with the forward unicity of solutions property. Moreover, there exists a partially globally asymptotically stable equilibrium uniformly in τ denoted by $y_\infty \in \mathcal{N}$.

Let us denote $\pi_0 : \mathcal{N} \to \mathcal{Q}$ the fiber bundle projection. Set $p_1 : \mathcal{M} \to \mathcal{N}$ the first canonical projection and set $\pi = \pi_0 \circ p_1$. We denote $q_\infty = \pi_0(y_\infty)$. Similarly, fix τ_0 in $\mathcal{T}$ and set $\sigma(q) = (\sigma_0(q), \tau_0)$ where $\sigma_0 : \mathcal{Q} \to \mathcal{N}$ is a continuous section of π_0. Clearly, σ is a continuous section of π. We also note that the manifold $\mathcal{M}$ trivially inherits a structure of fiber bundle over $\mathcal{Q}$ with projection π.

The vector field f is continuous and has the forward unicity of solutions property. Therefore, it admits a semiflow Φ (Bhatia & Hajek, 1969). Let us denote:

$$h : [0, 1] \times \mathcal{Q} \to \mathcal{Q}$$
$$(\lambda, q) \mapsto \begin{cases} \pi \circ \Phi\left(\ln\left(\dfrac{1}{1-\lambda}\right), \sigma(q)\right) & \text{if } \lambda \neq 1 \\ q_\infty & \text{if } \lambda = 1. \end{cases}$$

Since we clearly have $h(0, q) = q$ and $h(1, q) = q_\infty$, let us prove the continuity of h. This mapping is obviously continuous on $[0, 1) \times \mathcal{Q}$.

Let us show the continuity at $(1, q)$ for $q \in \mathcal{Q}$. Let $(\lambda_n, q_n) \in [0, 1) \times \mathcal{Q}$ be a sequence of points converging to $(1, q)$. We set

$$t_n = \ln\left(\frac{1}{1-\lambda_n}\right), \qquad x_n = \sigma(q_n), \quad x = \sigma(q).$$

We have $t_n \to +\infty$ and, by continuity of the section, $x_n \to x$. Let $U \subset \mathcal{Q}$ be a neighborhood of q_∞ and $U_0 = \pi_0^{-1}(U) \subset \mathcal{N}$ the corresponding neighborhood of y_∞. By partial stability uniformly in τ, there exists $V_0 \subset \mathcal{N}$ a neighborhood of y_∞ such that for all $y \in V_0$, all $\tau \in \mathcal{T}$ and all $t \geq 0$, we have $p_1 \circ \Phi(t, (y, \tau)) \subset U_0$.

On the other hand, the partial attractivity of y_∞ means that $p_1 \circ \Phi(t, x) \to y_\infty$ when $t \to \infty$. Thus, there exists $T > 0$ such that $p_1 \circ \Phi(T, x) \in V_0$. By continuity, there exists $N_1 > 0$ such that for all $n > N_1$ we have $p_1 \circ \Phi(T, x_n) \in V_0$. Therefore, for all $t \geq T$, we have $p_1 \circ \Phi(t, x_n) \in U_0$. But $t_n \to +\infty$, so there exists $N_2 > 0$ such that for all $n > N_2$, $t_n > T$. Thus, for all $n > N = \max(N_1, N_2)$, we have $p_1 \circ \Phi(t_n, x_n) \in U_0$. Hence, for all $n > N$, $h(\lambda_n, q_n) = \pi_0 \circ p_1 \circ \Phi(t_n, x_n) \in U$. Since U is an arbitrary neighborhood of q_∞, the mapping h is continuous.

However, the mapping h defines a retraction of the compact manifold $\mathcal{Q}$ on q_∞, which leads to the expected contradiction, thanks to Proposition 2. □

Example 8. Let us consider the following system, defined on the circle:

$$\ddot{\theta} = u, \quad \theta \in \mathbb{S}^1. \tag{4}$$

By using the angular velocity $\omega = \dot{\theta}$ we can rewrite the system as

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = u. \end{cases} \tag{5}$$

Here the state space is the tangent space of the circle, denoted by $\mathrm{T}\mathbb{S}^1$. The tangent space of a manifold has always a structure of vector bundle over that manifold, and therefore $\mathrm{T}\mathbb{S}^1$ has a structure of vector bundle over the compact manifold $\mathbb{S}^1$ with projection π_0 given by:

$$\pi_0(\theta, \omega) = \theta.$$

Moreover, being a vector bundle, the tangent bundle admits σ_0, the zero section, as a continuous section. One may wonder if it is possible to design a continuous feedback control $u(\theta, \omega, t)$ globally stabilizing a state $(\theta_0, 0)$ such that the closed-loop system has uniqueness of solution in forward time.

Since $\mathbb{S}^1$ is compact, from Theorem 7 the system (5) cannot be globally asymptotically uniformly stabilized.

Finally, taking into account Theorem 7 and (Bhat & Bernstein, 2000, Theorem 1), we have the following result: consider the system (1) defined on a manifold with a structure of fiber bundle over a compact manifold. If there exists a continuous section of the bundle, then no continuous dynamic feedback can globally asymptotically uniformly stabilize the system in such a way that the closed loop system has the forward unicity of solutions.

3. Conclusion

This paper extends (Bhat & Bernstein, 2000, Theorem 1) to the case of time-varying feedback control. We prove that under mild assumptions, no continuous time-varying feedback control can avoid the retraction obstruction; that is, no continuous time-varying feedback control can globally asymptotically uniformly stabilize an equilibrium on a state space which has a structure of fiber bundle over a compact manifold.

This topological obstruction on compact manifolds prevents us from having continuous globally asymptotically uniformly stabilizing feedback (neither static nor time-varying nor dynamic). Moreover it is proved in Mayhew and Teel (2011) that the obstruction still remains for discontinuous autonomous vector fields or differential inclusions. Finally the topological obstruction appears to be a strong constraint on stabilization, and few possibilities remain. First, since the uniformity property of the partial stability is indeed being used in our proof, the possibility of non-uniform global stabilizability still remains open. Second, hybrid feedbacks can be considered as suggested in Mayhew and Teel (2011). Finally, other notions of solutions for discontinuous systems exist; some of them might not inherit the same obstruction to global stabilization.

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Robustesse et stabilité des systèmes non-linéaires : un point de vue basé sur l'homogénéité

Résumé :

L'objet de ce travail est l'étude des propriétés de stabilité et de robustesse des systèmes non-linéaires *via* des méthodes basées sur l'homogénéité. Dans un premier temps, nous rappelons le contexte usuel des systèmes homogènes ainsi que leurs caractéristiques principales. La suite du travail porte sur l'extension de l'homogénéisation des systèmes non-linéaires, déjà proposée dans le cadre de l'homogénéité à poids, au cadre plus général de l'homogénéité géométrique. Les principaux résultats d'approximation sont étendus. Nous développons ensuite un cadre théorique pour définir l'homogénéité de systèmes discontinus et/ou donnés par des inclusions différentielles. Nous montrons que les propriétés bien connues des systèmes homogènes restent vérifiées dans ce contexte. Ce travail se poursuit par l'étude de la robustesse des systèmes homogènes ou homogénéisables. Nous montrons que sous des hypothèses peu restrictives, ces systèmes sont *input-to-state stable*. Enfin, la dernière partie de ce travail consiste en l'étude du cas particulier du double intégrateur. Nous développons pour ce système un retour de sortie qui le stabilise en temps fini, et pour lequel nous prouvons des propriétés de robustesse par rapport à des perturbations ou à la discrétisation en exploitant les résultats développés précédemment. Des simulations viennent compléter l'étude théorique de ce système et illustrer son comportement.

Mots clés : Stabilité, robustesse, systèmes homogènes, temps fini, inclusions différentielles.

Robustness and stability of nonlinear systems: a homogeneity point of view

Abstract: The purpose of this work is the study of stability and robustness properties of nonlinear systems using homogeneity-based methods. Firstly, we recall the usual context of homogeneous systems as well as their main features. The sequel of this work extends the homogenization of nonlinear systems, which was already defined in the framework of weighted homogeneity, to the more general setting of the geometric homogeneity. The main approximation results are extended. Then we develop a theoretical framework for defining homogeneity of discontinuous systems and/or systems given by a differential inclusion. We show that the well-known properties of homogeneous systems persist in this context. This work is continued by a study of the robustness properties of homogeneous or homogeneizable systems. We show that under mild assumptions, these systems are *input-to-state stable*. Finally, the last part of this work consists in the study of the example of the double integrator system. We synthesize a finite-time stabilizing output feedback, which is shown to be robust with respect to perturbations or discretization by using techniques developped before. Simulations conclude the theoretical study of this system and illustrate its behavior.

Keywords: Stability, robustness, homogeneous systems, finite-time, differential inclusions.