# Some applications of BSDE theory: fractional BDSDEs and regularity properties of Integro-PDEs

Shuai Jing

université de bretagne occidentale

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**Shuai Jing**

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**Rainer Buckdahn**

## Quelques Applications de la Théorie d'EDSR : EDDSR Fractionnaire et Propriétés de Régularité des EDP-Intégrales

## Some Applications of BSDE Theory : Fractional BDSDEs and Regularity Properties of Integro-PDEs

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après avis des rapporteurs :

**Philippe BRIAND**

Professeur Université de Savoie, France

**Yaozhong HU**

Professeur Université de Kansas, États Unis

devant le jury composé de :

**Vlad BALLY**

Professeur, Université de Marne-la-Vallée, France ;

**Philippe BRIAND**

Professeur, Université de Savoie, France ;

**Rainer BUCKDAHN, directeur de thèse**

Professeur, UBO, France ;

**Yaozhong HU**

Professeur, Université de Kansas, États Unis ;

**Catherine RAINER**

MCF/HDR, UBO, France.

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# Contents

# Résumé (en français)

L'objectif principal de cette thèse est d'étudier d'une part les équations différentielles doublement stochastiques rétrogrades régies par un mouvement brownien standard et un mouvement brownien fractionnaire indépendent, ainsi que les équations différentielles partielles stochastiques associées régies par le mouvement brownien fractionnaire. D'autre part nous avons aussi étudié la régularité, à savoir la continuité de Lipschitz conjointe et la semiconcavité conjointe, de la solution de viscosité pour une classe générale d'équations aux dérivées partielles non locales de type Hamilton-Jacobi-Bellman.

Bien que ces deux sujets de recherche mentionnés ci-dessus semblent être très différents à première vue, les deux ont en commun que les équations différentielles stochastiques rétrogrades et les méthodes connexes sont parmi leurs principaux outils.

Dans la première partie (les chapitres 1 et 2) nous nous sommes intéressés aux équations différentielles doublement stochastiques rétrogrades régies par un mouvement brownien standard et d'un mouvement brownien fractionnaire indépendent. Ce type d'équation combine la théorie de l'équation différentielle stochastique rétrograde avec celle du mouvement brownien fractionnaire. La présentation de cette partie est principalement basée sur les deux travaux suivants:

1. S. J. et J. A. León, *Semilinear Backward Doubly Stochastic Differential Equations and SPDEs Driven by Fractional Brownian Motion with Hurst Parameter in (0,1/2),* accepté pour publication dans *Bulletin des Sciences Mathématiques*, doi:10.1016/j.bulsci.2011.06.003.
2. S. J., *Fractional Backward Doubly Stochastic Differential Equations with Hurst Parameter in (1/2,1),* soumis pour publication.

La deuxième partie du présent projet de doctorat est consacrée à l'étude des propriétés de régularité de la solution de viscosité pour une certaine classe des équations différentielles partielles-intégrales. Pour cela est utilisé l'interprétation stochastique de ces équations comme un problème de contrôle stochastique dont le fontionnel de coût est défini par une équation différentielle stochastique rétrograde avec sauts. Cette partie est principalement basée sur le travail:

3. S. J., *Regularity Properties of Viscosity Solutions of Integro-Partial Differential Equations of Hamilton-Jacobi-Bellman Type,* soumis pour publication.

L'histoire des équations différentielles stochastiques rétrogrades remonte à 1973, quand Bismut [13] a été le premier à utiliser les équations différentielles stochastiques rétrogrades linéaires dans son étude sur les systèmes de contrôle optimal stochastique. Les équations différentielles stochastiques rétrogrades non linéaires généraux sous les conditions de Lipschitz ont d'abord été étudiées en 1990 par Pardoux et Peng dans leur article pionnier [83].

En 1992, Duffie et Epstein [37] ont introduit indépendamment des équations différentielles stochastiques rétrogrades dans leur étude des problèmes dans les modèles d'utilité récursive en finance. Mais leurs équations représentent un cas particulier de celles étudiées dans [83]. Soit $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ la filtration naturelle générée par un mouvement brownien $W$. Alors l'équation différentielle stochastique rétrograde non linéaire générale s'écrit comme suit:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \quad t \in [0, T], \tag{0.1}$$

où $\xi$ est une variable aléatoire de carré intégrable, mesurable par rapport à $\mathcal{F}_T$. Une solution de cette équation différentielle stochastique rétrograde est un couple de processus $\{Y_t, Z_t\}_{t\in[0,T]}$ adaptés par rapport à $\mathbb{F}$, vérifiant certaines hypothèses d'intégrabilité. Pardoux et Peng [83] ont prouvé que sous la condition de Lipschitz sur $f$, l'équation différentielle stochastique rétrograde (0.1) admet une unique solution adaptée de carré intégrable.

Depuis les travaux pionniers de Pardoux et Peng, la théorie d'équations différentielles stochastiques rétrogrades a été étudiée largement et profondement. En bref, il existe plusieurs options dans cette direction. Nous les expliquerons séparément dans les paragraphes suivants.

L'un des objectifs majeurs de la recherche a concerné l'affaiblissement des conditions dans lesquelles une équation différentielle stochastique rétrograde a une solution: Lepeltier et San Martín [71] ont étudié les équations différentielles stochastiques rétrogrades aux coefficients continus avec une croissance linéaire en 1997; en 2001 Bahlali [8] a travaillé sur les équations différentielles stochastiques rétrogrades multidimensionnelles aux coefficients localements lipschitziens; les équations différentielles stochastiques rétrogrades à croissance quadratique sont d'abord étudiées par Kobylanski [61] et [62] dans sa thèse de doctorat. Stimulé par ces travaux pionniers, de nombreux auteurs ont travaillé sur ce type d'équations différentielles stochastiques rétrogrades, et nous mentionnons ici, en particulier, sans être exhaustif, les travaux de Lepeltier et San Martín [72] pour des coefficients superlinéaires quadratiques, de Briand et Hu [14], [15] pour des conditions terminales non bornées, de Ankirchner, Imkeller et Dos Reis [5] pour la différentiabilité de solutions, de Morlais [78] pour des martingales continues directrices, de Imkeller et Dos Reis [54] pour des équations à croissance quadratique tronquée, de Delbaen, Hu et Bao [35] de croissance super-quadratique.

Peng [92] a été le premier à montrer l'interprétation probabiliste des équations différentielles partielles paraboliques via les équations différentielles stochastiques rétrogrades en 1991, connue comme la formule de Feynman-Kac non-linéaire; Pardoux et Peng [85] ont encore étudié les solutions de viscosité des équations différentielles partielles paraboliques par l'interprétation comme équation différentielle stochastique rétrograde en 1992.

Antonelli [6] a été le premier à étudier les équations différentielles stochastiques progressives - rétrogrades pour un petit horizon de temps en 1993; Ma, Protter et Yong [74] en 1994 ont développé un schéma en quatre-étapes pour résoudre les équations différentielles stochastiques progressives-rétrogrades en utilisant une méthode d'équation aux dérivées partielles; Hu et Peng [51] ont étudié l'existence et l'unicité sous des hypothèses de monotonie en 1995.

Citons également qu'El Karoui et al. [40] ont travaillé sur les équations différentielles stochastiques rétrogrades réfléchies avec une barrière en 1997, qui sont étroitement liées aux problèmes d'arrêt optimal. Par ailleurs, ils ont fourni des interprétations probabilistes aux équations aux dérivées partielles paraboliques avec obstacle. Cvitanić et Karatzas [31] ont généralisé leurs travaux aux équations différentielles stochastiques rétrogrades réfléchies à deux barrières et les ont appliqués aux jeux de Dynkin.

Un autre type d'équations rétrogrades, que sont les équations différentielles doublement stochastiques rétrogrades, joue un rôle central dans les deux premiers chapitres de ce projet de thèse, tandis que les équations différentielles stochastiques rétrogrades avec sauts sont utilisées dans le troisième manuscrit (voir le chapitre 3). Nous allons les présenter plus en détail plus tard.

Il est également intéressant de noter qu', en utilisant les équations différentielles stochastiques rétrogrades, Peng [88] a défini une espérance non-linéaire, en 1997, appelée la $g$-espérance, qui est une généralisation de la notion d'espérance classique. Inspiré par les travaux sur la $g$-espérance et les mesures de risque, Peng [90] [91] a défini récemment un nouveau type d'espérance non linéaire plus générale, appelé la $G$-espérance, et a développé un calcul d'Itô sous cette espérance.

Citons aussi que le développement de la théorie des équations différentielles stochastiques rétrogrades a été stimulé depuis ses débuts par ses diverses applications, parmi lesquelles en particulier ceux en finance (voir par exemple, El Karoui, Peng et Quenez [42] en 1997, Chen et Epstein [25] en 2002, Hamadène et Jeanblanc [46] en 2007), mais aussi ceux en contrôle stochastique et théorie des jeux (par exemple, Peng [87], [89], Hamadène et Lepeltier [47], Hamadène et Zhang [48], El Karoui et Hamadène [39], Buckdahn et Li [20], [21], Hu et Tang [52], Peng et Wu [93], Peng et Xu [94], Yong et Zhou [108]).

Maintenant nous allons introduire les équations différentielles doublement stochastiques rétrogrades. Ce type d'équation, d'abord présenté et étudié par Pardoux et Peng [84], en 1994, a la forme suivante:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s + \int_t^T g(s, Y_s, Z_s) \downarrow \mathrm{d}B_s - \int_t^T Z_s \mathrm{d}W_s, \quad t \in [0, T],$$

où $B$ et $W$ sont deux mouvements browniens indépendants; l'intégrale stochastique par rapport à $B$ est une d'Itô rétrograde, qui par rapport à $W$ est une d'Itô progessive classique. Dans [84], les auteurs ont prouvé l'existence et l'unicité des solutions des équations différentielles doublement stochastiques rétrogrades sous des conditions de Lipschitz sur $f$ et $g$, et il convient de noter que la constante de Lipschitz de $g$ par rapport à $z$ est requis pour être inférieur à un. En particulier, par analogie avec la formule de Feynman-Kac non-linéaire, ils ont donné une interprétation stochastique d'une certaine classe d'équations difféntielles partielles stochastiques paraboliques en termes de solutions des équations différentielles doublement stochastiques rétrogrades correspondantes. La solution de l'équation difféntielle partielle stochastique considérée dans [84] a été dans le sens classique. Bally et Matoussi [9] ont appliqué la méthode d'équations différentielles doublement stochastiques rétrogrades à l'étude de solutions faibles pour les équations difféntielles partielles stochastiques en 1999.

En 2001, en appliquant la transformation de Doss-Sussmann, Buckdahn et Ma [22], [23] ont donné une définition de la solution de viscosité stochastique pour un type particulier d'équations difféntielles partielles stochastiques régies par un mouvement brownien. La transformation de Doss-Sussmann, qui a été étudiée par Doss [36] en 1977 et, indépendamment, par Sussmann [103] en 1978, peut être utilisée pour transformer des équations différentielles stochastiques en équations différentielles ordinaires trajectorielles. Pour cela, le coefficient de diffusion doit être une fonction du processus de la solution seulement et doit satisfaire à certaines conditions de régularit. Par ailleurs, l'intégrale stochastique de l'équation différentielle stochastique doit être considérée dans le sens de Stratonovich. En utilisant cette méthode, les auteurs ont réussi à transformer une équation différentielle doublement stochastique rétrograde non linéaire en une équation différentielle stochastique rétrograde à croissance quadratique. La relation ainsi obtenue entre les équations différentielles doublement stochastiques rétrogrades et les équations différentielles stochastiques rétrogrades à croissance quadratique se reflète aussi dans une relation correspondante entre les équations différéntielles partielles stochastiques associées aux équations différentielles doublement stochastiques rétrogrades, et les équations avec dérivées partielles aux coefficients aléatoires associées aux équations différentielles stochastiques rétrogrades à croissance quadratique. En appliquant ces relations, ils ont défini la notion de solution de viscosité stochastique, qui était une extension notable de la définition classique de la solution de viscosité déterministe (voir, Crandall, Ishii et Lions [28]).

L'histoire du mouvement brownien fractionnaire est beaucoup plus ancienne que celle de la théorie des équations différentielles stochastiques rétrogrades. Kolmogorov a été le premier d'avoir a travaillé sur ce sujet; notons que le mouvement brownien fractionnaire a été appelé par lui la "spirale de Wiener" [63]. Après les travaux statistiques sur l'inondation du Nil par Hurst [53], après lequel le paramètre $H$ a été nommé (cependant, la lettre $H$ a été donnée par Mandelbrot en l'honneur des deux Hurst et Hölder), et les travaux pionniers de Mandelbrot et van Ness [75] sur le calcul stochastique fractionnaire, le mouvement brownien fractionnaire a attiré de plus en plus de mathématiciens à ce sujet. Depuis les années 1990, les modèles portant sur le mouvement brownien fractionnaire ont trouvé leur place dans les modèles issus du marché financier. L'intérêt pour le mouvement brownien fractionnaire en la finance s'explique par le fait que, contrairement au mouvement brownien, le mouvement brownien fractionnaire permet de décrire les effets de mémoire à long terme lorsque le paramètre de Hurst $H \in (1/2, 1)$, et à courte portée de dépendance (ou mémoire) quand $H \in (0, 1/2)$. Ce comportement est assez différent de celui du mouvement brownien classique dont les accroissements sont indépendants.

Il est bien connu que le mouvement brownien fractionnaire n'est pas une semimartingale, sauf pour le cas $H = 1/2$. Ainsi, la théorie bien développée du calcul d'Itô ne peut pas être appliquée directement ici. En raison de la différence significative entre $H \in (1/2, 1)$ et $H \in (0, 1/2)$, les définitions de l'intégrale stochastique par rapport au mouvement brownien fractionnaire sont divisées en deux groupes.

Pour $H \in (1/2, 1)$, dans le cadre du calcul de Malliavin, l'opérateur divergence peut être utilisé comme la définition de l'intégrale stochastique, voir Decreusefont et Üstünel [33], Ducan, Hu et Pasik-Duncan [38], Alòs, Mazet et Nualart [3], ainsi que Alòs et Nualart [4]. La continuité höldérienne (avec ordre strictement inférieur à $H$) permet également de définir une intégrale stochastique trajectorielle. Nous nous référons, par exemple, à l'intégrale généralisée de Riemann-Stieltjes développée par Zähle [109], [110], à l'intégration dans le cadre de la théorie des trajectoires rugueuses étudiée dans Coutin et Qian [30], et aussi à l'intégrale de Russo-Vallois [99], [100], [101].

Pour le paramètre de Hurst $H \in (0, 1/2)$ les trajectoires du mouvement brownien fractionnaire sont plus irrégulières. Elles n'ont que la propriété de continuité höldérienne avec l'ordre strictement inférieur à $H$, c'est à dire, l'ordre est strictement inférieur à $1/2$. Ceci a comme conséquence que la situation est plus compliquée ici. Alòs, León, Mazet et Nualart ont publié plusieurs articles discutant de ce problème: [1], [2], [3]. L'opérateur divergence peut être utilisé pour définir une intégrale stochastique, mais comme il est montré dans le travail de Cheridito et Nualart [27], le domaine de l'opérateur divergence ne contient pas le mouvement brownien fractionnaire lui-même quand $H \leq 1/4$. Toutefois, Cheridito et Nualart [27], et León et Nualart [67] ont généralisé la définition de l'opérateur divergence et l'ont appelé l'opérateur divergence généralisé. Par ailleurs, ils ont prouvé que la formule d'Itô, la formule de Tanaka ainsi que le théorème de Fubini sont vrais pour les mouvements browniens fractionnaires avec n'importe quel $H \in (0, 1/2)$ dans un cadre général.

Les équations différentielles stochastiques régies par un mouvement brownien fractionnaire ont été étudiées par plusieurs auteurs utilisant des définitions différentes de l'intégrale stochastique. Sans être exhaustif, citons ici les travaux suivants: León et San Martín [68], León et Tindel [69] pour $H \in (0, 1/2)$, Maslowski et Nualart [76], Nualart et Răşcanu [81] pour $H \in (1/2, 1)$, Tindel, Tudor et Viens [105] pour $H \in (0, 1)$.

Bender [11] était le premier en 2001 à réunir deux théories de développements dynamiques, celle des équations différentielles stochastiques rétrogrades et celle du mouvement brownien fractionnaire, dans le but d'étudier des équations différentielles stochastiques rétrogrades fractionnaires linéaires, pour lesquelles il a donné des formules de solutions explicites. Hu et Peng [50] ont été les premiers en 2009 d'étudier les équations différentiels stochastiques rétrogrades fractionnaires non linéaires. Pour cela ils ont utilisé la notion d'espérance quasi-conditionnelle introduite par Hu et Øksendal [49]. Nous remarquons que l'utilisation de l'espérance quasi-conditionnelle conduit à des conditions assez restrictives sur les coefficients.

En 2009, Jien et Ma [57] ont étendu la méthode de la transformation de Girsanov anticipative utilisée dans Buckdahn [16] au cas du mouvement brownien fractionnaire. Leurs résultats peuvent être utilisés pour résoudre les équations différentielles stochastiques fractionnaires éventuellement anticipatives. Motivé par le travail de Jien et Ma [57], notre premier chapitre de cette thèse vise à combiner la théorie des équations différentielles stochastiques rétrogrades et celle du mouvement brownien fractionnaire d'une manière différente que celle de Bender [11], Hu et Peng [51]. En effet, nous étudions les équations différentielles doublement stochastiques rétrogrades semilinéaires régies par un mouvement brownien classique $W$ et par un mouvement brownien fractionnaire $B$:

$$Y_t = \xi + \int_0^t f(s, Y_s, Z_s) \mathrm{d}s - \int_0^t Z_s \downarrow \mathrm{d}W_s + \int_0^t \gamma_s Y_s \mathrm{d}B_s, \quad t \in [0, T]. \tag{0.2}$$

Ici, les intégrales stochastiques par rapport au mouvement brownien $W$ et au mouvement brownien fractionnaire $B$ sont l'intégrale d'Itô rétrograde et l'opérateur divergence généralisé, respectivement. Nous soulignons qu'une légère différence existe dans la forme entre notre équation différentielle doublement stochastique rétrograde fractionnaire et la plus classique. Nous utilisons une forme de temps inversé. Ainsi, ici la condition terminale habituelle $\xi$ se révèle être une condition initiale, mesurable par rapport à la $\sigma$-algèbre générée par le mouvement brownien.

Comme dans [57], où la transformation de Girsanov anticipative a été utilisée pour transformer les équations différentielles stochastiques fractionnaires éventuellement anticipatives régies par un opérateur divergence (l'intégrale de Skorohod), en équations différentielles ordinaires trajectorielles, nous appliquons la transformation de Girsanov à l'équation (0.2) afin d'éliminer l'intégrale par rapport à $B$. Plus précisément, nous pouvons montrer que l'équation (0.2) et l'équation différentielle stochastique rétrograde trajectorielle suivante

$$\widehat{Y}_t = \xi + \int_0^t f\left(s, \widehat{Y}_s \varepsilon_s(T_s), \widehat{Z}_s \varepsilon_s(T_s)\right) \varepsilon_s^{-1}(T_s) \mathrm{d}s - \int_0^t \widehat{Z}_s \downarrow \mathrm{d}W_s, \quad t \in [0,T]. \tag{0.3}$$

sont équivalentes (Pour plus de détails, voir le théorème 3.9 du chapitre 1). Dans le même esprit suivant Buckdahn et Ma [22], nous associons à (0.2) l'équation différentielle partielle stochastique semilinéaire régie par le mouvement brownien fractionnaire $B$:

$$\begin{cases} \mathrm{d}u(t,x) = \left[\mathcal{L}u(t,x) + f\left(t,x,u(t,x),\nabla_x u(t,x)\sigma(x)\right)\right]\mathrm{d}t + \gamma_t u(t,x)\mathrm{d}B_t, \quad t \in [0,T], \\ u(0,x) = \Phi(x). \end{cases} \tag{0.4}$$

par étudier l'équation aux dérivées partielles trajectorielle associée avec (0.3):

$$\begin{cases} \frac{\partial}{\partial t}\widehat{u}(t,x) = \left[\mathcal{L}\widehat{u}(t,x) + f\left(t,x,\widehat{u}(t,x)\varepsilon_t(T_t), \nabla_x \widehat{u}(t,x)\sigma(x)\varepsilon_t(T_t)\right)\varepsilon_t^{-1}(T_t)\right], \quad t \in [0,T]; \\ \widehat{u}(0,x) = \Phi(x), \end{cases}$$

La principale difficulté ici est de prouver la propriété de continuité du champ aléatoire $u$ définie par l'équation (0.2). En effet, $u$ est liée à la solution continue $\widehat{u}$ de l'équation (0.3) par la transformation de Girsanov. Mais, dans le cas général, un champ aléatoire continue $\widehat{u}$ peut perdre sa propriété de continuité après la transformation de Girsanov (voir la remarque 4.13 dans le chapitre 1). Toutefois, en appliquant un calcul fractionnaire assez technique et des estimations sur les équations différentielles stochastiques rétrogrades, nous réussissons à prouver la continuité du champ aléatoire $u$ et nous montrons que c'est la solution de viscosité de l'équation différentielle partielle stochastique (0.4).

La méthode de la transformation de Girsanov anticipative nous impose de ne traiter que le cas semilinéaire. Bien que cette approche employée est spécifique pour $H \in (0, 1/2)$, on peut, néanmoins, utiliser l'opérateur divergence (au lieu de l'opérateur divergence généralisé) également pour le cas $H \in (1/2, 1)$, et les résultats correspondants sont toujours vrais, avec seulement quelques changements mineurs. Mais, en réalité, nous pouvons faire encore mieux dans le cas de $H \in (1/2, 1)$: en effet, voici l'intégrale de Russo-Vallois permet d'envisager un coefficient non linéaire comme l'intégrand de l'intégrale par rapport au mouvement brownien fractionnaire $B$.

Ainsi, au chapitre 2, nous nous occupons des équations différentielles doublement stochastiques rétrogrades fractionnaires non linéaires avec paramètre de Hurst $H \in (1/2, 1)$:

$$Y_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\mathrm{d}r + \int_0^s g(Y_r^{t,x})\mathrm{d}B_r - \int_0^s Z_r^{t,x} \downarrow \mathrm{d}W_r, \ s \in [0,t]. \tag{0.5}$$

Dans le travail [22], Buckdahn et Ma ont appliqué la transformation de Doss-Sussmann pour étudier les équations différentielles doublement stochastiques rétrogrades régies par deux mouvements browniens indépendants. Cela correspond au cas $H = 1/2$, i.e., le processus de $B$ dans l'équation différentielle doublement stochastique rétrograde est un mouvement brownien, mais l'intégrale stochastique par rapport à ce mouvement brownien $B$ a été considérée par les auteurs dans le sens de Stratonovich. En conséquence, nous étendons leur approche aux équations différentielles doublement stochastiques rétrogrades fractionnaires non linéaires avec le paramètre de Hurst $\mathrm{H} \in (1/2, 1)$. Alors l'intégrale stochastique par rapport au mouvement brownien fractionnaire doit être remplacée par une intégrale trajectorielle, et nous prenons celle de Russo-Vallois. En fait, dans leur papier original [100], sur la généralisation des intégrales stochastiques classiques, Russo et Vallois ont défini trois intégrales, appelées les intégrales progressives, rétrogrades et symétriques, qui sont les extensions correspondantes des intégales d'Itô classiques, d'Itô rétrogrades et de Stratonovich. Il doit être remarqué que dans le cas d'un mouvement brownien fractionnaire avec le paramètre de Hurst $H \in (1/2, 1)$ en tant qu'intégrateur, ces trois intégrales coïncident. Un nouveau type de la formule d'Itô pour les processus qui contiennent une intégrale d'Itô par rapport à un mouvement brownien classique $W$ et l'intégrale Russo-Vallois par rapport au mouvement brownien fractionnaire $B$, est crucial pour l'application de la transformation de Doss-Sussmann. Cependant, la difficulté ici est que, contrairement au cas semilinéaire, nous ne pouvons pas trouver un espace de processus invariants sous la transformation de Doss-Sussmann, auquel les deux processus de solutions $Y$ et $Z$ appartiennent. C'est aussi la principale différence entre notre travail et celui de Buckdahn et Ma [22]: dans le cas classique, on peut résoudre directement l'équation différentielle doublement stochastique rétrograde et obtenir l'intégrabilité carrée des processus de solutions $Y$ et $Z$. Mais ici, dans le cas fractionnaire, nous ne pouvons pas résoudre directement l'équation (0.5), nous ne pouvons résoudre que l'équation différentielle stochastique rétrograde trajectorielle associée

$$U_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s \tilde{f}(r, X_r^{t,x}, U_r^{t,x}, V_r^{t,x})\mathrm{d}r - \int_0^s V_r^{t,x} \downarrow \mathrm{d}W_r. \tag{0.6}$$

Mais pour cette équation différentielle stochastiques rétrogrades trajectorielle, nous avons seulement obtenu le résultat que les processus de solution de $U$ et $V$ sont de carré intégrable conditionnellement connaissant le mouvement brownien fractionnaire $B$, i.e., l'espérance conditionnelle de l'intégrale du carré de le processus de $V$ est finie presque sûrement. Ceci est lié au fait que l'équation différentielle stochastique rétrograde (0.6) a une croissance quadratique en $V$. Par ailleurs, dans notre étude de ce type d'équation différentielle doublement stochastique rétrograde, on considère également l'équation différentielle partielle stochastique fractionnaire associée

$$\begin{cases} \mathrm{d}u(t,x) = \left[\mathcal{L}u(t,x) + f\left(t,x,u(t,x),\nabla_x u(t,x)\sigma(x)\right)\right]\mathrm{d}t + g(u(t,x))\mathrm{d}B_t, \quad t \in [0,T], \\ u(0,x) = \Phi(x), \end{cases}$$

et sa solution de viscosité stochastique. Nous soulignons aussi que notre équation différentielle partielle stochastique est un peu plus générale que celle qui a été prise en compte dans Maslowski et Nualart [76], où la fonction $f$ ne dépend que de $u$, et aussi différente de celui dans Issoglio [56], où le mouvement brownien fractionnaire est en fait défini par rapport au paramètre d'espace au lieu du paramètre de temps.

Enfin, nous devons aussi souligner que les maniéres dont les solutions de viscosité stochastiques sont définies dans les Chapitres 1 et 2 sont différentes. Dans les deux cas, leurs définitions sont étroitement liées aux principales méthodes utilisées pour résoudre les équations différentielles doublement stochastiques rétrogrades associées. Dans le cas du paramètre de Hurst $H \in (0,1/2)$, c'est la transformation de Girsanov, qui change les trajectoires de l'échantillon; dans le cas $H \in (1/2,1)$, c'est la transformation de Doss-Sussmann, qui ne les change pas.

La deuxième partie de cette thèse, i.e., le chapitre 3, concerne l'étude des propriétés de régularité, à savoir la continuité de Lipschitz conjointe et la semiconcavité conjointe dans l'espace et dans le temps, de la solution de viscosité pour les équations différentielles partielles-intégrales de type Hamilton-Jacobi-Bellman.

Pour cela, l'équation différentielle partielle-intégrale est étudiée à travers son interprétation stochastique comme un problème de contrôle stochastique impliquant des équations différentielles stochastiques rétrogrades avec sauts. Ces équations rétrogrades avec sauts ont d'abord été étudiées par Li et Tang [104] en 1994. En utilisant leurs résultats, mais dans une forme plus générale, Barles, Buckdahn et Pardoux [10] en 1997 ont étudié la solution de viscosité de l'équation différentielle partielle-intégrale associée. Dans leur papier, la solution de viscosité $\{v(t,x) : t,x \in [0,T] \times \mathbb{R}^d\}$ de l'équation différentielle partielle-intégrale se révèle être lipschitzienne en $x$, mais seulement 1/2-höldérienne en $t$.

D'autre part, Buckdahn, Cannarsa et Quincampoix [18] ainsi que Buckdahn, Huang et Li [19] ont montré que les solutions de viscosité des équations aux dérivées partielles paraboliques pas nécessairement non dégénérées (sans et avec obstacles) sont lipschitziennes conjointements en $(t,x) \in [0,T-\delta] \times \mathbb{R}^d$, pour tout $\delta > 0$, et, en outre, sous des hypothèses convenables, ils sont aussi semiconcaves conjointements en $(t,x) \in [0,T-\delta] \times \mathbb{R}^d$, pour tout $\delta > 0$. Soulignons que l'importance de la semiconcavité conjointe provient du théorème d'Alexandrov, en raison de laquelle nous pouvons conclure que la solution de viscosité possède $\mathrm{d}t\mathrm{d}x$ presque partout un développement limité d'ordre deux. Par ailleurs, Cannarsa et Sinestrari [24] ont montré que ces propriétés de régularité sont les optimales qu'on peut s'attendre. Par ailleurs, ils ont montré (voir aussi [18]) qu', en général, la propriété de continuité de Lipschitz conjointe et la semiconcavité conjointe ne peuvent pas tenir sur tout le domaine $[0,T] \times \mathbb{R}^d$.

Soulignons aussi que les travaux [18], [19] et [24] ont amélioré non seulement les résultats déjà connus sur la continuité de Lipschitz pour les solutions de viscosité des équations aux dérivées partielles, mais aussi les résultats anciens sur la semiconcavité. En effet, la semiconcavité par rapport à $x$, pour $t \in [0,T]$ fixé, a été déjà prouvée par Ishii et Lions [55] (par l'intermédiaire d'une approche purement analytique) et par Yong et Zhou [108] (par les estimations stochastiques dans le problème de contrôle stochastique associé).

Dans notre travail, nous étendons les résultats de [18] et [19] aux équations différentielles partielles-intégrales de la forme

$$\begin{cases} \frac{\partial}{\partial t} v(t,x) + \inf_{u \in U}\{(\mathcal{L}^u + B^u)v(t,x) + f(t,x,v(t,x),(D_x v\sigma)(t,x), \\ \qquad\qquad v(t,x+\beta(t,x,u,\cdot)) - v(t,x))\} = 0, & (t,x) \in (0,T) \times \mathbb{R}^d; \\ v(T,x) = \Phi(x), & x \in \mathbb{R}^d, \end{cases}$$

où $U$ est un espace métrique compact, $\mathcal{L}^u$ est l'opérateur différentielle linéaire de second ordre

$$\mathcal{L}^u \varphi(x) = \mathrm{tr}\left(\frac{1}{2}\sigma\sigma^T(t,x,u) D^2_{xx}\varphi(x)\right) + b(t,x,u) \cdot D_x\varphi(x), \quad \varphi \in C^2(\mathbb{R}^d),$$

et $B^u$ est l'opérateur différentiel-intégral:

$$B^u\varphi(x) = \int_E [\varphi(x+\beta(t,x,u,e)) - \varphi(t,x) - \beta(t,x,u,e)\cdot D_x\varphi(x)]\Pi(\mathrm{d}e), \quad \varphi \in C_b^1(\mathbb{R}^d).$$

Nous étendons les approches de [18] et [19] au cadre de l'équation différentielle partielle-intégrale ci-dessus. Pour cela, nous considérons le problème de contrôle stochastique associé composé d'une équation progressive contrôlée et une équation différentielle stochastique rétrograde contrôlée, toutes les deux régies par un mouvement brownien $B$ et une mesure aléatoire de Poisson $\mu$. Alors que nous adaptons la méthode de changement de temps pour le mouvement brownien à partir de [18] et [19], nous introduisons la transformation de Kulik [65] et [66] pour la mesure aléatoire de Poisson $\mu$. La combinaison de ces outils à la fois permet d'obtenir la continuité de Lipschitz conjointe et la semiconcavité conjointe pour la solution de viscosité de l'équation différentielle intégrale-partielle ci-dessus.

Dans la suite, nous résumons dans une manière plus précise les principaux résultats obtenus dans le cadre de ce projet de thèse:

## Principaux Résultats de cette Thèse

**Principaux résultats au chapitre 1.** *Équations doublement stochastiques rétrogrades semilinéaires et EDP stochastiques régies par un mouvement brownien fractionnaire avec paramètre de Hurst dans (0,1/2).*

Cette introduction ainsi que le chapitre 1 sont basés sur le travail [58] écrit en collaboration avec Jorge A. León (CINVESTAV-Instituto Politecnico nationale à Ciudad de México, Mexique), accepté pour publication par le *Bulletin des Sciences Mathématiques.*

Nous considérons un espace de probabilité complet $(\Omega, \mathcal{F}, P)$ sur lequel un mouvement brownien $W$ et un mouvement brownien fractionnaire indépendant $B$ avec le paramètre de Hurst $H \in (0, 1/2)$ sont définis. Soit $\mathcal{F} = \mathcal{F}_T$ la $\sigma$-algèbre générée par $W$ et $B$, augmentée par tous les ensembles $P$-nuls. Par $\mathbb{G} = \{\mathcal{G}_t\}_{t\in[0,T]}$ on note la famille de $\sigma$-algèbres $\mathcal{G}_t$ générés par $\{W_s - W_t, s \in [t, T]\}$ et $\{B_s, s \in [0, t]\}$.

Notre objectif est d'étudier l'équation différentielle partielle stochastique semi-linéaire suivante régie par le mouvement brownien fractionnaire $B$:

$$\begin{cases} \mathrm{d}u(t,x) = [\mathcal{L}u(t,x) + f(t,x,u(t,x),\nabla_x u(t,x)\sigma(x))]\,\mathrm{d}t + \gamma_t u(t,x)\mathrm{d}B_t, & t\in[0,T], \\ u(0,x) = \Phi(x), \end{cases} \tag{0.7}$$

où $\mathcal{L}$ est l'opérateur différentiel du second ordre

$$\mathcal{L} := \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(x)D^2_{xx}) + b(x)\nabla_x.$$

Les coefficients sont supposés satisfaire les conditions de Lipschitz et $\gamma \in L^2(0, T)$. Notre approche est basée sur une interprétation (doublement) stochastique de l'équation différentielle partielle stochastique (0.7) par une équation différentielle doublement stochastique rétrograde. Nous avons donc d'abord étudié l'équation différentielle doublement stochastique rétrograde fractionnaire régie par le mouvement brownien fractionnaire $B$ et un mouvement brownien indépendant $W$,

$$Y_t = \xi + \int_0^t f(s, Y_s, Z_s)\mathrm{d}s - \int_0^t Z_s \downarrow \mathrm{d}W_s + \int_0^t \gamma_s Y_s \mathrm{d}B_s, \quad t \in [0,T]. \tag{0.8}$$

Voici l'intégrale stochastique par rapport au mouvement brownien $W$ qui est l'intégrale d'Itô rétrograde, tandis que l'intégrale stochastique par rapport au mouvement brownien fractionnaire est l'opérateur divergence généralisé.

L'idée principale de notre approche pour résoudre l'équation différentielle doublement stochastique rétrograde (0.8) consiste dans les applications d'une transformation de Girsanov

$$T_t(\omega) = \omega + \int_0^{\cdot\wedge t} (\mathcal{K}\gamma 1_{[0,t]})(r)\mathrm{d}r,$$

(La définition de l'opérateur $\mathcal{K}$ est donné ci-dessous), avec l'aide de laquelle nous transformons l'équation différentielle doublement stochastique rétrograde fractionnaire en une équation différentielle stochastique rétrograde trajectorielle dont les coefficients dépendent des trajectoires de $B$ et qui est régie uniquement par le mouvement brownien $W$:

$$\widehat{Y}_t = \xi + \int_0^t f\left(s, \widehat{Y}_s\varepsilon_s(T_s), \widehat{Z}_s\varepsilon_s(T_s)\right)\varepsilon_s^{-1}(T_s)\mathrm{d}s - \int_0^t \widehat{Z}_s \downarrow \mathrm{d}W_s, \quad t \in [0,T]. \tag{0.9}$$

En étudiant l'équation différentielle stochastique rétrograde (0.9) ainsi que son équation différentielle partielle associée

$$\begin{cases} \frac{\partial}{\partial t}\widehat{u}(t,x) = \left[\mathcal{L}\widehat{u}(t,x) + f\left(t, x, \widehat{u}(t,x)\varepsilon_t(T_t), \nabla_x\widehat{u}(t,x)\sigma(x)\varepsilon_t(T_t)\right)\varepsilon_t^{-1}(T_t)\right], \quad t \in [0,T]; \\ \widehat{u}(0,x) = \Phi(x), \end{cases} \tag{0.10}$$

nous obtenons la solution de l'équation différentielle partielle stochastique (0.7).

Après avoir expliqué ci-dessus le régime général, nous devenons un peu plus précis. Pour cela nous introduisons quelques éléments du calcul fractionnaire qui seront utilisés dans ce qui suit, comme les définitions des dérivées fractionnaires à droite (gauche) et des intégrales fractionnaires à droite (gauche): Soit $T > 0$ un horizon de temps positif, et soit $f : [0,T] \to \mathbb{R}$ une fonction intégrable, et $\alpha \in (0,1)$. La *dérivée fractionnaire à droite* de $f$ d'ordre $\alpha$ est donnée par

$$I_{T-}^{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^T \frac{f(u)}{(u-x)^{1-\alpha}}\mathrm{d}u, \quad \text{for a.a. } x \in [0,T],$$

et la dérivée fractionnaires à droite

$$(D_{T-}^{\alpha}f)(s) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(s)}{(T-s)^{\alpha}} + \alpha\int_s^T \frac{f(s)-f(u)}{(u-s)^{1+\alpha}}\mathrm{d}u\right).$$

Continuons par définir l'espace $\Lambda_T^{1/2-H}$ qui est important pour la définition de l'opérateur divergence généralisé:

$$\Lambda_T^{1/2-H} := \left\{ f : \exists\, \varphi_f \in L^2(0,T) \text{ s.t. } f(u) = u^{1/2-H} I_{T-}^{1/2-H}\left(s^{H-1/2}\varphi_f(s)\right)(u) \right\}.$$

Nous observons que toutes les fonctions indicatrices $I_{[0,t]}, t \in [0,T]$, appartiennent à $\Lambda_T^{1/2-H}$. L'application de $\mathcal{K} : \Lambda_T^{1/2-H} \to L^2([0,T])$, définie par

$$(\mathcal{K}\varphi)(s) = C_H\Gamma(H+1/2)s^{1/2-H}\left(D_{T-}^{1/2-H}u^{H-1/2}\varphi(u)\right)(s), \quad s \in [0,T],$$

est une isométrie telle que

$$B_t = \int_0^t (\mathcal{K}I_{[0,t]})(s)\mathrm{d}W_s,\ t \in [0,T].$$

Maintenant, nous pouvons préciser la définition de l'opérateur divergence généralisé pour notre mouvement brownien fractionnaire.

**Définition 0.1.** *Soit $u \in L^2(\Omega, \mathcal{F}, P; L^2([0,T]))$ Nous disons que $u$ appartient à Dom $\delta^B$ s'il existe $\delta^B(u) \in L^2(\Omega, \mathcal{F}, P)$ tel que*

$$E\left[\langle \mathcal{K}^*\mathcal{K}DF, u\rangle_{L^2([0,T])}\right] = E\left[F\delta^B(u)\right], \quad \text{pour chaque} \quad F \in \mathcal{S}_{\mathcal{K}},$$

*où $\mathcal{S}_{\mathcal{K}}$ est l'espace des fonctionnels lisses du mouvement brownien fractionnaire $B$ et $DF$ est la dérivée de Malliavin de $F \in \mathcal{S}_{\mathcal{K}}$ (pour les définitions précises de $\mathcal{S}_{\mathcal{K}}$ et $DF$, le lecteur est renvoyé au chapitre 1). Dans ce cas, la variable aléatoire $\delta^B(u)$ est appelée l'opérateur divergence généralisé de $u$.*

Afin d'utiliser l'opérateur divergence généralisé, nous devons imposer l'hypothèse suivante sur $\gamma$ dans ce chapitre:

**(H1)** $\gamma 1_{[0,t]}$ appartient à $\Lambda_T^{1/2-H}$, pour tout $t \in [0,T]$.

Cette hypothèse a déjà été utilisée, par exemple, dans le travail [57] par León et San Martín pour les équations différentielles stochastiques linéaires régies par un mouvement brownien fractionnaire.

L'outil principal de notre approche est la transformation de Girsanov. Plus précisément, nous considérons la transformation

$$T_t : \Omega \to \Omega,\ T_t(\omega) = \omega + \int_0^{\cdot\wedge t} (\mathcal{K}\gamma 1_{[0,t]})(r)\mathrm{d}r, \quad \omega \in \Omega$$

et son inverse

$$A_t : \Omega \to \Omega, \quad A_t(\omega) = \omega - \int_0^{\cdot\wedge t} (\mathcal{K}\gamma 1_{[0,t]})(r)\mathrm{d}r, \quad \omega \in \Omega.$$

Nous introduisons également le terme

$$\varepsilon_t = \exp\left(\int_0^t \gamma_r \mathrm{d}B_r - \frac{1}{2}\int_0^t ((\mathcal{K}\gamma 1_{[0,t]})(r))^2 \mathrm{d}r\right).$$

Un rôle important dans notre approche est joué par la quantité

$$I_T^* := \sup_{t\in[0,T]} \left|\int_0^t \gamma_s \mathrm{d}B_s\right|,$$

pour laquelle nous prouvons que $E[\exp\{pI_T^*\}] < \infty$, pour tout $p \geq 1$. Il permet d'introduire l'espace $L_{\mathbb{G}}^{2,*}(0,T;\mathbb{R}\times\mathbb{R}^d)$ comme l'espace de tous les processus $(Y,Z)$ $\mathbb{G}$-adaptés qui sont tels que

$$E\left[\exp\{pI_T^*\}\int_0^T \left(|Y_t|^2 + |Z_t|^2\right)\mathrm{d}t\right] < \infty, \text{ for all } p \geq 1.$$

L'importance de l'espace $L_{\mathbb{G}}^2(0,T;\mathbb{R}\times\mathbb{R}^d)$ provient de son invariance par les transformations de Girsanov $T_t, A_t, t \in [0,T]$. Ainsi, il permet de démontrer le résultat clé suivant:

**Proposition 0.2.** *Pour tous les processus* $(Y,Z) \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$ *nous avons:*
*i)* $(\widetilde{Y}_t,\widetilde{Z}_t) := (Y_t(T_t)\varepsilon_t^{-1}(T_t), Z_t(T_t)\varepsilon_t^{-1}(T_t)) \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$ *et*
*ii)* $(\overline{Y}_t,\overline{Z}_t) := (Y_t(A_t)\varepsilon_t, Z_t(A_t)\varepsilon_t) \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$.

Après la préparation ci-dessus, nous pouvons étudier l'équation différentielle doublement stochastique rétrograde fractionnaire (0.8) et l'équation différentielle stochastique rétrograde trajectorielle associée (0.9). Notre premier résultat principal concerne la propriété d'intégrabilité de la solution de l'équation différentielle stochastique rétrograde trajectorielle.

**Théorème 0.3.** *Sous nos hypothèses standards (voir **(H3)** dans le chapitre 1), l'équation différentielle stochastique rétrograde*

$$\widehat{Y}_t = \xi + \int_0^t f(s,\widehat{Y}_s\varepsilon_s(T_s),\widehat{Z}_s\varepsilon_s(T_s))\varepsilon_s^{-1}(T_s)\mathrm{d}s - \int_0^t \widehat{Z}_s \downarrow \mathrm{d}W_s,\ \ t\in[0,T], \tag{0.11}$$

*possède une unique solution* $(\widehat{Y},\widehat{Z}) \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$.

*Par ailleurs, il existe une constante positive $C$ telle que,*

$$E\left[\sup_{t\in[0,T]}\left|\widehat{Y}_t\right|^2 + \int_0^T \left|\widehat{Z}_t\right|^2 \mathrm{d}t|\mathcal{F}_T^B\right] \le C\exp\{2I_T^*\}.$$

En utilisant les résultats ci-dessus et en appliquant la transformation de Girsanov, nous obtenons notre deuxième résultat principal. Il combine l'équation différentielle doublement stochastique rétrograde fractionnaire et l'équation différentielle stochastique rétrograde trajectorielle en montrant que nous obtenons la solution d'une équation en résolvant l'autre et vice versa.

**Théorème 0.4.** 1) *Soit* $(\widehat{Y},\widehat{Z}) \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$ *une solution de l'équation différentielle stochastique rétrograde* (0.11). *Alors*

$$\{(Y_t,Z_t), t\in[0,T]\} = \{(\widehat{Y}_t(A_t)\varepsilon_t, \widehat{Z}_t(A_t)\varepsilon_t), t\in[0,T]\} \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$$

*est une solution de l'équation* (0.8) *avec* $\gamma Y 1_{[0,t]} \in Dom\ \delta^B$, *pour tout* $t\in[0,T]$.

2) *Réciproquement, étant donné une solution arbitraire* $(Y,Z) \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$ *de l'équation* (0.8) *avec* $\gamma Y 1_{[0,t]} \in Dom\ \delta^B$, *pour tout* $t\in[0,T]$, *le processus*

$$\{(\widehat{Y}_t,\widehat{Z}_t)_{t\in[0,T]}\} = \{(Y_t(T_t)\varepsilon_t^{-1}(T_t), Z_t(T_t)\varepsilon_t^{-1}(T_t))_{t\in[0,T]}\} \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$$

*est un solution de l'équation différentielle stochastique rétrograde* (0.11).

La partie restante du chapitre 1 est consacrée à l'équation différentielle partielle stochastique associée. Pour cette fin, nous associons notre équation différentielle doublement stochastique rétrograde fractionnaire à une équation différentielle stochastique progressive.

Nous désignons par $(X_s^{t,x})_{0\le s\le t}$ l'unique solution de l'équation différentielle stochastique suivante:

$$\begin{cases} \mathrm{d}X_s^{t,x} = -b(X_s^{t,x})\mathrm{d}s - \sigma(X_s^{t,x}) \downarrow \mathrm{d}W_s, \quad s\in[0,t], \\ X_t^{t,x} = x \in \mathbb{R}^d, \end{cases} \tag{0.12}$$

que nous associons à l'équation différentielle doublement stochastique rétrograde

$$Y_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \mathrm{d}r - \int_0^s Z_r^{t,x} \downarrow \mathrm{d}W_r + \int_0^s \gamma_r Y_r^{t,x} \mathrm{d}B_r, \quad (0.13)$$

$s \in [0, t]$. A partir des résultats de la première partie, avec $f(s, y, z) := f(s, X_s^{t,x}, y, z)$, nous obtenons que cette équation a une unique solution $\left(Y^{t,x}, Z^{t,x}\right)$ donnée par

$$(Y_s^{t,x}, Z_s^{t,x}) = (\widehat{Y}_s^{t,x}(A_s)\varepsilon_s, \widehat{Z}_s^{t,x}(A_s)\varepsilon_s), s \in [0, t],$$

où $(Y, Z) \in L_{\mathbb{G}}^{2,*}(0, T; \mathbb{R} \times \mathbb{R}^d)$ est l'unique solution de l'équation

$$\widehat{Y}_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f(r, X_r^{t,x}, \widehat{Y}_r^{t,x}\varepsilon_r(T_r), \widehat{Z}_r^{t,x}\varepsilon_r(T_r))\varepsilon_r^{-1}(T_r) \mathrm{d}r - \int_0^s \widehat{Z}_r^{t,x} \downarrow \mathrm{d}W_r, \quad (0.14)$$

$s \in [0, t]$.

Introduisons maintenant un champ aléatoire: $\widehat{u}(t, x) = \widehat{Y}_t^{t,x}, (t, x) \in [0, T] \times \mathbb{R}^d$. Nous allons montrer que $\widehat{u}$ est une solution de viscosité de l'équation différentielle partielle trajectorielle (0.10). Pour cette cause, nous allons d'abord prouver qu'il a une version continue.

**Lemme 0.5.** *Le processus de* $\{\widehat{Y}_s^{t,x}; (s, t) \in [0, T]^2, x \in \mathbb{R}^d\}$ *possède une version continue. Par ailleurs,* $|\widehat{u}(t, x)| \leq C \exp\{I_T^*\}(1 + |x|)$, *P-p.s.*

Puis nous prouvons que le champ aléatoire $\widehat{u}$ est l'unique solution de viscosité trajectorielle de l'équation (0.10).

**Théorème 0.6.** *Le champ aléatoire* $\widehat{u}$ *défini par* $\widehat{u}(t, x) = \widehat{Y}_t^{t,x}$ *est une solution de viscosité de l'équation trajectorielle* (0.10)*, où* $\widehat{Y}^{t,x}$ *est la solution de l'équation* (0.14)*. Par ailleurs, cette solution* $\widehat{u}(t, x)$ *est unique dans la classe des champs stochastiques continus* $\tilde{u} : \Omega \times [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}$ *tels que, pour une certaine variable aléatoire* $\eta \in L^0(\mathcal{F}_T^B)$*,*

$$|\tilde{u}(t, x)| \leq \eta(1 + |x|), (t, x) \in [0, T] \times \mathbb{R}^d, P - p.s.$$

Afin d'identifier le champ $u(t, x) := \widehat{u}(A_t, t, x)\varepsilon_t, (t, x) \in [0, T] \times \mathbb{R}^d$ comme une solution de viscosité stochastique de l'équation (0.7), nous devons garantir que $u$ a une version continue. En fait, nous avons un contre-exemple qui montre que, en général, un champ aléatoire continu, après notre transformation de Girsanov, peut être discontinu. Toutefois, une preuve assez technique en utilisant le forme spéciale de $u$ permet de montrer:

**Lemme 0.7.** *Le champ aléatoire* $u$ *a une version continue.*

Motivé par la relation classique entre des solutions de l'équation différentielle partielle stochastique (0.7) et l'équation aux dérivées partielles (0.10) (voir Proposition 4.11 dans le chapitre 1), nous donnons la définition suivante:

**Définition 0.8.** *Un champ aléatoire continu* $u : [0, T] \times \mathbb{R}^d \times \Omega' \mapsto \mathbb{R}$ *est une solution de viscosité* (*stochastique*) *de léquation* (0.7) *si* $\widehat{u}(t, x) = u(T_t, t, x)\varepsilon_t^{-1}(T_t), (t, x) \in [0, T] \times \mathbb{R}^d$ *est une solution de viscosité de l'équation trajectorielle* (0.10).

Pour résumer nos résultats, nous pouvons énoncer le théorème suivant:

**Théorème 0.9.** *Le champ aléatoire continu* $u(t,x) := \widehat{u}(A_t,t,x)\varepsilon_t = \widehat{Y}_t^{t,x}(A_t)\varepsilon_t = Y_t^{t,x}$ *est une solution de viscosité de l'équation différentielle partielle stochastique semilinéaire* (0.7)*. Cette solution est unique à l'intérieur de la classe des champs stochastiques continus* $\tilde{u} : \Omega' \times [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}$ *telle que,*

$$|\tilde{u}(t,x)| \leq C \exp\{I_T^*\}(1+|x|), (t,x) \in [0,T] \times \mathbb{R}^d, P-p.s.,$$

*pour une certaine constante* $C$ *pouvant dependre de* $\tilde{u}$*.*

**Principaux résultats au chapitre 2.** *Équations différentielles doublement stochastiques rétrogrades avec paramètre de Hurst dans (1/2,1)*

Ce chapitre est basé sur le manuscrit [59] soumis pour publication.

Après avoir examiné le cas semi-linéaire de l'équation différentielle partielle stochastique parabolique (0.7) régie par un mouvement brownien fractionnaire, nous nous intéressons au cas non linéaire, ce qui signifie que l'intégrand de l'intégrale stochastique est non linéaire, i.e., l'équation de la forme

$$\begin{cases} \mathrm{d}v(t,x) = \left[\mathcal{L}v(t,x) - f(t,x,v(t,x),\sigma(x)^T \frac{\partial}{\partial x} v(t,x))\right] \mathrm{d}t + g(v(t,x))\mathrm{d}B_t, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in (0,T) \times \mathbb{R}^n; \\ v(0,x) = \Phi(x), \qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^n. \end{cases} \tag{0.15}$$

Cette équation différentielle partielle stochastique est étudiée par son interprétation stochastique d'une équation différentielle doublement stochastique rétrograde non linéaire régie par le mouvement brownien fractionnaire $B$ avec le paramètre de Hurst $H \in (0,1/2)$ et un mouvement brownien indépendant $W$:

$$Y_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\mathrm{d}r + \int_0^s g(Y_r^{t,x})\mathrm{d}B_r - \int_0^s Z_r^{t,x} \downarrow \mathrm{d}W_r,\ s \in [0,t]. \tag{0.16}$$

La non-linéarité impose que l'utilisation de l'outil du chapitre 1, la transformation de Girsanov, n'est plus applicable dans ce cas. Par conséquent, nous appliquons la transformation de Doss-Sussmann, qui a été utilisée par Buckdahn et Ma [22] et [23] dans leur étude des solutions de viscosité stochastiques des équations différentielles partielles stochastiques régies par un mouvement brownien.

Nous utilisons les mêmes notations pour l'espace probabilisé $(\Omega, \mathcal{F}, P)$ et en plus nous notons par $\mathbb{F}_{[t,T]}^W = \{\mathcal{F}_{[t,T]}^W\}_{t\in[0,T]}$ la filtration générée par $\{W_s - W_t, s \in [t,T]\}$ et par $\mathbb{H}$ dont la filtration $\mathbb{F}_{[t,,T]}^W$ augmenté par la $\sigma$-algèbre générée par $\{B_s\}_{s\in[0,T]}$.

D'abord nous donnons quelques notations qui seront utilisées dans ce qui suit:

Soit $C(\mathbb{H},[0,T];\mathbb{R}^m)$ l'espace des processus continus $\{\varphi_t, t \in [0,T]\}$ à valeurs dans $\mathbb{R}^m$ tels que $\varphi_t$ est $\mathcal{H}_t$-mesurable, $t \in [0,T]$;

Soit $\mathcal{M}^2(\mathbb{F}^W,[0,T];\mathbb{R}^m)$ l'espace des processus de carrés intégrables $\{\psi_t, t \in [0,T]\}$ à valeurs dans $\mathbb{R}^m$ tels que $\psi_t$ est $\mathcal{F}_{[t,T]}^W$-mesurable, $t \in [0,T]$;

Soit $\mathcal{H}_T^\infty(\mathbb{R})$ l'ensemble des processus $\mathbb{H}$-progressivement mesurables, qui sont presque sûrement bornés par certaine variable aléatoire $\mathcal{F}_T^B$-mesurable à valeurs réelles;

Soit $\mathcal{H}_T^2(\mathbb{R}^d)$ l'ensemble des processus $\mathbb{H}$-progressivement mesurables $\gamma = \{\gamma_t : t \in [0,T]\}$ à valeurs dans $\mathbb{R}^d$ tels que $\mathbb{E}\left[\int_0^T |\gamma_t|^2 \mathrm{d}t | \mathcal{F}_T^B\right] < +\infty$, $\mathbb{P}$-p.s.

Contrairement à l'utilisation de l'opérateur divergence gńéralisé comme l'intégrale stochastique par rapport au mouvement brownien fractionnaire $B$ dans [58], nous choisissons ici l'intégrale de Russo-Vallois. L'intégrale de Russo-Vallois est définie dans [100] comme suit.

**Définition 0.10.** *Soit $X$ et $Y$ deux processus continus. Pour $\varepsilon > 0$, nous posons*

$$I(\varepsilon, t, X, \mathrm{d}Y) \triangleq \frac{1}{\varepsilon}\int_0^t X(s)(Y(s) - Y(s-\varepsilon))\mathrm{d}s, \ t \in [0,T].$$

*Alors l'intégrale de Russo-Vallois est définie comme la limite uniforme en probabilité lorsque $\varepsilon \to 0^+$, si la limite existe.*

Comme notre premier résultat concerne une formule d'Itô, qui implique à la fois l'intégrale d'Itô rétrograde par rapport au mouvement brownien et l'intégrale de Russo-Vallois par rapport au mouvement brownien fractionnaire.

**Théorème 0.11.** *Soient $\alpha \in C(\mathbb{H}, [0,T]; \mathbb{R})$ un processus de la forme*

$$\alpha_t = \alpha_0 + \int_0^t \beta_s \mathrm{d}s + \int_0^t \gamma_s \downarrow \mathrm{d}W_s, \quad t \in [0,T],$$

*où $\beta$ et $\gamma$ sont des processus $\mathbb{H}$-adaptés tels que $\mathbb{P}\{\int_0^T |\beta_s|\mathrm{d}s < +\infty\} = 1$ et $\mathbb{P}\{\int_0^T |\gamma_s|^2 \mathrm{d}s < +\infty\} = 1$, respectivement. Supposons que $F \in C^2(\mathbb{R}\times\mathbb{R})$. Alors l'intégrale de Russo-Vallois $\int_0^t \frac{\partial F}{\partial y}(\alpha_s, B_s)\mathrm{d}B_s$ existe pour tout $0 \le t \le T$, et elle vérifie que, $\mathbb{P}$-p.s., pour tout $0 \le t \le T$,*

$$\begin{aligned} F(\alpha_t, B_t) =& F(\alpha_0, 0) + \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s)\beta_s \mathrm{d}s + \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s)\gamma_s \downarrow \mathrm{d}W_s \\ &+ \int_0^t \frac{\partial F}{\partial y}(\alpha_s, B_s)\mathrm{d}B_s - \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(\alpha_s, B_s)|\gamma_s|^2 \mathrm{d}s. \end{aligned}$$

Nous introduisons maintenant la transformation de Doss-Sussmann. Nous désignons par $\eta$ le flux stochastique défini comme l'unique solution de l'équation différentielle stochastique suivante:

$$\eta(t,y) = y + \int_0^t g(\eta(s,y))\mathrm{d}B_s, \ t \in [0,T]. \tag{0.17}$$

La solution de cette équation différentielle stochastique peut être écrite comme $\eta(t,y) = \alpha(y, B_t)$, où $\alpha(y,z)$ est la solution de l'équation différentielle ordinaire

$$\begin{cases} \frac{\partial \alpha}{\partial z}(y,z) = g(\alpha(y,z)), \ z \in \mathbb{R}, \\ \alpha(y,0) = y. \end{cases} \tag{0.18}$$

Sous des hypothèses appropriées, nous avons que $\eta(t,\cdot) = \alpha(\cdot, B_t) : \mathbb{R} \to \mathbb{R}$ est un difféomorphisme et nous définissons $\mathcal{E}(t,y) := \eta(t,\cdot)^{-1}(y) = h(y, B_t), (t,y) \in [0,T]\times\mathbb{R}$. Nous avons les estimations suivantes pour $\eta$ et $\mathcal{E}$.

**Lemme 0.12.** *Il existe une constante $C > 0$ dépendant seulement de la borne de $g$ et de ses dérivées partielles, telle que pour $\xi = \eta, \mathcal{E}$, $P$-p.s., pour tout $(t, y)$,*

$$
\begin{aligned}
&|\xi(t,y)| \leq |y| + C|B_t|, \quad \exp\{-C|B_t|\} \leq \left|\frac{\partial}{\partial y}\xi\right| \leq \exp\{C|B_t|\},\\
&\left|\frac{\partial^2}{\partial y^2}\xi\right| \leq \exp\{C|B_t|\}, \qquad\qquad \left|\frac{\partial^3}{\partial y^3}\xi\right| \leq \exp\{C|B_t|\}.
\end{aligned}
$$

Soit $(X_s^{t,x})_{0\leq s\leq t}$ l'unique solution de l'équation différentielle stochastique suivante:

$$
\begin{cases}
\mathrm{d}X_s^{t,x} = -b(X_s^{t,x})\mathrm{d}s - \sigma(X_s^{t,x}) \downarrow \mathrm{d}W_s, \quad s \in [0,t]\,,\\
X_t^{t,x} = x.
\end{cases}
\tag{0.19}
$$

Voici l'intégrale stochastique $\int_0^t \cdot \downarrow \mathrm{d}W_s$ qui est de nouveau considérée comme celle d'Itô rétrograde. La condition de Lipschitz sur $b$ et $\sigma$ garantit l'existence et l'unicité de la solution $(X_s^{t,x})_{0\leq s\leq t}$ dans $\mathcal{M}^2(\mathbb{F}^W, [0,T]; \mathbb{R}^n)$. Notre objectif est d'étudier l'équation différentielle doublement stochastique rétrograde suivante:

$$
Y_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\mathrm{d}r + \int_0^s g(Y_r^{t,x})\mathrm{d}B_r - \int_0^s Z_r^{t,x} \downarrow \mathrm{d}W_r, \ s \in [0,t]. \tag{0.20}
$$

Soit $(U^{t,x}, V^{t,x})$ l'unique solution de l'équation différentielle stochastique rétrograde suivante:

$$
U_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s \tilde{f}(r, X_r^{t,x}, U_r^{t,x}, V_r^{t,x})\mathrm{d}r - \int_0^s V_r^{t,x} \downarrow \mathrm{d}W_r, \tag{0.21}
$$

où

$$
\tilde{f}(t,x,y,z) = \frac{1}{\frac{\partial}{\partial y}\eta(t,y)} \left\{ f\left(t, x, \eta(t,y), \frac{\partial}{\partial y}\eta(t,y)z\right) + \frac{1}{2}\mathrm{tr}\left[z^T \frac{\partial^2}{\partial y^2}\eta(t,y)z\right]\right\}.
$$

Nous remarquons qu'en raison de la forme spéciale de $\tilde{f}$, notre équation différentielle stochastique rétrograde trajectorielle est une équation différentielle stochastique rétrograde à croissance quadratique; ces équations ont d'abord été étudiées par Kobylanski [61] et [62].

Cependant, contrairement à l'équation différentielle stochastique rétrograde à croissance quadratique classique, nous ne pouvons pas obtenir l' intégrabilité carré des processus de la solution, au lieu que cela nous prouvons le résultat suivant.

**Théorème 0.13.** *Sous nos hypothèses standards sur les coefficients $\sigma, b, f$ et $\Phi$, l'équation (0.21) admet une unique solution $(U^{t,x}, V^{t,x})$ dans $\mathcal{H}_t^\infty(\mathbb{R}) \times \mathcal{H}_t^2(\mathbb{R}^d)$. Par ailleurs, il existe un processus positif croissant $\theta \in L^0(\mathbb{H}, \mathbb{R})$ tel que*

$$
|U_s^{t,x}| \leq \theta_s, \quad \mathbb{E}\left[\int_0^\tau |V_s^{t,x}|^2 \mathrm{d}s | \mathcal{H}_\tau\right] \leq \exp\{\exp\{C \sup_{s\in[0,t]} |B_s|\}\}, \ \mathbb{P}-\text{p.s.},
$$

*pour tout $\mathbb{H}$-temps d'arrêt $\tau$ $(0 \leq \tau \leq t)$, où $C$ est une constante choisie de manière adéquate. Par ailleurs, le processus $(U^{t,x}, V^{t,x})$ est $\mathbb{G}$-adapté.*

Maintenant, nous pouvons effectuer la transformation de Doss-Sussmann et obtenir la solution de l'équation différentielle doublement stochastique rétrograde fractionnaire (0.20).

**Théorème 0.14.** *Nous définissons une nouvelle paire de processus* $(U^{t,x}, V^{t,x})$ *par*

$$Y_s^{t,x} = \eta(s, U_s^{t,x}), \quad Z_s^{t,x} = \frac{\partial}{\partial y}\eta(s, U_s^{t,x})V_s^{t,x},$$

*où* $(U^{t,x}, V^{t,x})$ *est la solution de l'équation différentielle stochastique rétrograde* (0.21). *Alors* $(Y^{t,x}, Z^{t,x}) \in \mathcal{H}_t^\infty(\mathbb{R}) \times \mathcal{H}_t^2(\mathbb{R}^d)$ *est la solution de l'équation différentielle doublement stochastique rétrograde* (0.20).

En appliquant la transformation de Doss-Sussmann, nous pouvons également établir le lien entre les équations différentielles partielles stochastiques suivantes: l'équation différentielle partielle stochastique semilinéaire (0.15) régie par le mouvement brownien fractionnaire $B$, et l'équation aux dérivées partielles semilinéaire avec coefficients aléatoires,

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \mathcal{L}u(t,x) - \tilde{f}(t,x,u(t,x),\sigma(x)^T\frac{\partial}{\partial x}u(t,x)), & (t,x) \in (0,T) \times \mathbb{R}^n; \\ u(0,x) = \Phi(x), & x \in \mathbb{R}^n. \end{cases} \tag{0.22}$$

Motivé par cette relation, nous définissons la solution de viscosité stochastique de (0.15).

**Définition 0.15.** *Un champ aléatoire continu* $\hat{u} : \Omega' \times [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *est appelé une solution de viscosité* (*stochastique*) *de l'équation* (0.15) *si* $u(t,x) = \mathcal{E}(t, \hat{u}(t,x))$, $(t,x) \in [0,T] \times \mathbb{R}^n$ *est la solution de viscosité de l'équation* (0.22).

Bien que la définition ci-dessus peut ressembler à celle de la solution de viscosité stochastique définie au chapitre 1, elles ne sont pas du même type. Leurs différences s'expliquent par les différents outils utilisés dans les chapitres précédents. En effet, alors que dans le chapitre 1 la transformation de Girsanov change les trajectoires du mouvement brownien fractionnaire, dans le chapitre 2 la transformation de Doss-Sussmann ne les change pas.

Enfin, nous donnons une interprétation stochastique pour la solution de viscosité stochastique de (0.15) en utilisant la solution de l'équation différentielle doublement stochastique rétrograde fractionnaire (0.20).

**Théorème 0.16.** *Le champ stochastique* $\hat{u} : \Omega' \times [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *défini par* $\hat{u}(t,x) \triangleq \alpha(U_t^{t,x}, B_t)$ *est une solution de viscosité* (*stochastique*) *de l'équation différentielle partielle stochastique* (0.15).

Enfin, présentons les

## Principaux résultats au chapitre 3. *Propriétés de régularité de la solution de viscosité pour des EDP-intégrales de type Hamilton-Jacobi-Bellman*

Ce chapitre est basé sur le manuscrit [60] soumis pour publication.

Nous considérons l'équation différentielle partielle-intégrale suivante de type Hamilton-Jacobi-Bellman:

$$\begin{cases} \frac{\partial}{\partial t}V(t,x) + \inf_{u \in U}\{(\mathcal{L}^u + B^u)V(t,x) + f(t,x,V(t,x),(D_xV\sigma)(t,x), \\ \qquad\qquad\qquad V(t,x+\beta(t,x,u,\cdot)) - V(t,x),u)\} = 0; \\ V(T,x) = \Phi(x), \end{cases} \tag{0.23}$$

où $U$ est un espace métrique compact, $\mathcal{L}^u$ est l'opérateur différentiel linéaire de second ordre

$$\mathcal{L}^u\varphi(x) = \mathrm{tr}\left(\frac{1}{2}\sigma\sigma^T(t,x,u)D^2_{xx}\varphi(x)\right) + b(t,x,u)\cdot D_x\varphi(x), \quad \varphi \in C^2(\mathbb{R}^d),$$

et $B^u$ est l'opérateur différentiel-intégral:

$$B^u\varphi(x) = \int_E [\varphi(x+\beta(t,x,u,e)) - \varphi(t,x) - \beta(t,x,u,e)\cdot D_x\varphi(x)]\Pi(\mathrm{d}e), \quad \varphi \in C^2_b(\mathbb{R}^d).$$

Ici $\Pi$ désigne une mesure finie de Lévy sur l'espace $(E,\mathcal{B}(E))$, où $E = \mathbb{R}^n\backslash\{0\}$.

Dans ce chapitre, nous étudions pour l'équation ci-dessus la régularité conjointe (la continuité de Lipschitz et la semiconcavité) de la solution de viscosité de $V(t,x)$ en $(t,x)$ par son interprétation stochastique comme un probléme de contrôle stochastique constitué d'une équation progressive et d'une équation différentielle stochastique rétrograde avec sauts. Plus précisément, soit $(t,x)\in[0,T]\times\mathbb{R}^d$, soit $B=(B_s)_{s\in[t,T]}$ un mouvement brownien $d$-dimensionnel avec la valeur initiale zéro au temps $t$, et soit $\mu$ une mesure aléatoire de Poisson avec compensateur $\mathrm{d}s\Pi(\mathrm{d}e)$ sur $[t,T]\times E$. Nous désignons par $\mathbb{F}^{B,\mu}$ la filtration générée par $B$ et $\mu$, et par $\mathcal{U}^{B,\mu}(t,T)$ l'ensemble des processus de contrôle $\mathbb{F}^{B,\mu}$-prévisibles à valeurs dans $U$. Il est maintenant évident que l'équation différentielle stochastique régie par le mouvement brownien $B$ et la mesure aléatoire de Poisson compensée $\tilde{\mu}$:

$$X^{t,x,u}_s = x + \int_t^s b(r,X^{t,x,u}_r,u_r)\mathrm{d}r + \int_t^s \sigma(r,X^{t,x,u}_r,u_r)\mathrm{d}B_r + \int_t^s\int_E \beta(r,X^{t,x,u}_{r-},u_r,e)\tilde{\mu}(\mathrm{d}r,\mathrm{d}e),$$

$s\in[t,T]$, a une unique solution sous des hypothèses appropriées pour les coefficients. Avec cette équation différentielle stochastique, nous associons l'équation différentielle stochastique rétrograde avec sauts

$$\begin{aligned} Y^{t,x,u}_s &= \Phi(X^{t,x,u}_T) + \int_s^T f(r,X^{t,x,u}_r,Y^{t,x,u}_r,Z^{t,x,u}_r,U^{t,x,u}_r,u_r)\mathrm{d}r - \int_s^T Z^{t,x,u}_r\mathrm{d}B_r \\ &\quad - \int_s^T\int_E U^{t,x,u}_r(e)\tilde{\mu}(\mathrm{d}r,\mathrm{d}e), \quad s\in[t,T]. \end{aligned} \tag{0.24}$$

En ce que concerne les hypothèses sur les coefficients, nous nous référons aux hypothèses **(H1) - (H5)** dans la section 2 et la section 3 du chapitre 3. De Barles, Buckdahn et Pardoux [10] nous savons que l'équation différentielle stochastique rétrograde avec sauts (0.24) ci-dessus a une unique solution de carré intégrable $(Y^{t,x,u},Z^{t,x,u},U^{t,x,u})$. En outre, parce que $Y^{t,x,u}$ est $\mathbb{F}^{B,\mu}$-adapté, $Y^{t,x,u}_t$ est déterministe. Il résulte de Barles, Buckdahn et Pardoux [10] ou Pham [95] que la fonction valeur

$$V(t,x) = \inf_{u\in\mathcal{U}^{B,\mu}(t,T)} Y^{t,x,u}_t, \quad (t,x)\in[0,T]\times\mathbb{R}^d$$

est la solution de viscosité de notre équation différentielle partielle-intégrale.

Parce que notre système implique non seulement le mouvement brownien $B$, mais aussi la mesure aléatoire de Poisson $\mu$, la méthode de changement de temps pour le mouvement brownien seule, ce qui a été utilisée dans [18] et [19] pour travailler sur les propriétés de régularité des équations aux dérivées partielles paraboliques du second ordre, n'est pas suffisantes pour notre approche ici. Nous combinons la méthode de changement de temps pour le mouvement brownien avec la transformation de Kulik de la mesure aléatoire de Poisson (voir [65], [66]). A notre connaissance, l'utilisation de la transformation de Kulik pour l'étude des problèmes de contrôle stochastique est nouvelle. En raison des difficultés à obtenir l'estimation de la norme $L^p$ des intégrales régies par la mesure aléatoire de Poisson compensée (voir, par exemple, Pham [95]), nous nous restreindrons au cas d'une mesure de Lévy avec $\Pi(E) < +\infty$. Le cas le plus général où $\int_E (1 \wedge |e|^2)\Pi(\mathrm{d}e) < +\infty$ est resté encore ouverte.

En appliquant la méthode de changement de temps pour le mouvement brownien $B$ combinée avec la transformation de Kulik pour la mesure aléatoire de Poisson $\mu$, on obtient les résultats suivants:

**Théorème 0.17.** *Soit $\delta \in (0, T)$ arbitrairement fixé. Sous nos hypothèses (**H1**) et (**H2**), la fonction valeur $V(\cdot, \cdot)$ est lipschitzienne conjointement sur $[0, T - \delta] \times \mathbb{R}^d$, c'est à dire, pour une certaine constante $C_\delta$, nous avons que, pour tout $(t_0, x_0), (t_1, x_1) \in [0, T - \delta] \times \mathbb{R}^d$:*

$$|V(t_0, x_0) - V(t_1, x_1)| \leq C_\delta(|t_0 - t_1| + |x_0 - x_1|).$$

**Théorème 0.18.** *Sous les hypothèses (**H1**) − (**H5**), pour chaque $\delta \in (0, T)$, il existe une certaine constante $C_\delta > 0$ telle que, pour tout $(t_0, x_0), (t_1, x_1) \in [0, T - \delta] \times \mathbb{R}^d$, et pour tout $\lambda \in [0, 1]$:*

$$\lambda V(t_0, x_0) + (1 - \lambda)V(t_1, x_1) - V(t_\lambda, x_\lambda) \leq C_\delta \lambda(1 - \lambda)(|t_0 - t_1|^2 + |x_0 - x_1|^2),$$

*où $t_\lambda = \lambda t_0 + (1 - \lambda)t_1$ et $x_\lambda = \lambda x_0 + (1 - \lambda)x_1$.*

A la fin nous montrons à travers un contre-exemple (voir aussi Buckdahn, Cannarsa et Quincampoix [18]) qu'en général, les propriétés ci-dessus de la continuité de Lipschitz et la semiconcavité ne sont pas valables pour tout l'espace $[0, T] \times \mathbb{R}^d$.

**Exemple 0.19.** *Nous étudions le problème*

$$X_s^{t,x} = x + B_s, \quad s \in [t, T], \quad x \in \mathbb{R};$$

$$Y_s^{t,x} = -\mathbb{E}\left[\left|X_T^{t,x}\right| |\mathcal{F}_s\right] = -\mathbb{E}\left[|x + B_T||\mathcal{F}_s\right], \quad s \in [t, T],$$

*sans contrôle ni sauts. Puis*

$$V(t, x) = Y_t^{t,x} = -\mathbb{E}\left[|x + B_T|\right],$$

*et, pour $x = 0$, en rappelant que $B$ est un mouvement brownien avec $B_t = 0$, nous avons*

$$V(t, 0) = -\mathbb{E}[|B_T|] = -\sqrt{\frac{2}{\pi}}\sqrt{T - t}, \quad t \in [0, T].$$

*Évidemment, $V(\cdot, x)$ n'est ni lipschitzienne, ni semiconcave en $t$ pour $t = T$. Cependant, $V$ est lipschitzienne et semiconcave conjointement sur $[0, T -\ delta] \times \mathbb{R}$, pour $\delta \in (0, T)$.*

# Introduction

The main objective of this dissertation is to study on one hand backward doubly stochastic differential equations driven by both a standard Brownian motion and an independent fractional Brownian motion, as well as the associated stochastic partial differential equations governed by the fractional Brownian motion. On the other hand we also investigate the regularity properties, namely the joint Lipschitz continuity and the joint semiconcavity, of the viscosity solution for a general class of nonlocal Hamilton-Jacobi-Bellman equations.

Although these both research topics mentioned above seem to be very different at first glance, both have in common that backward stochastic differential equations and related methods are among the major tools.

In the first part (Chapter 1 and Chapter 2) we are concerned with backward doubly stochastic differential equations driven by both a standard Brownian motion and a fractional Brownian motion. This kind of equation combines the theory of backward stochastic differential equations with that of the fractional Brownian motion. The presentation of this part is mainly based on the following two papers:

1. S. J. and J. A. León, *Semilinear Backward Doubly Stochastic Differential Equations and SPDEs Driven by Fractional Brownian Motion with Hurst Parameter in (0,1/2).* Accepted for publication by *Bulletin des Sciences Mathématiques.*

2. S. J., *Fractional Backward Doubly Stochastic Differential Equations with Hurst Parameter in (1/2,1).* Submitted for publication.

The second part of the present PhD project is devoted to the study of regularity properties of the viscosity solution for a certain class of integro-partial differential equations. For this the stochastic interpretation of such equations as a stochastic control problem which cost functional is defined through a backward stochastic differential equation with jumps is used. This part is mainly based on the work:

3. S. J., *Regularity Properties of Viscosity Solutions of Integro-Partial Differential Equations of Hamilton-Jacobi-Bellman Type.* Submitted for publication.

The history of backward stochastic differential equations can be traced back to 1973, when Bismut [13] was the first to use linear backward stochastic differential equations in his study of stochastic optimal control systems. General nonlinear backward stochastic differential equations under Lipschitz conditions were first investigated in 1990 by Pardoux and Peng in their pioneering paper [83]. In 1992, Duffie and Epstein [37] independently introduced backward stochastic differential equations in their study of recursive utility problems in financial market models. However, their equations are a special case of those studied in [83]. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ be the natural filtration generated by a Brownian motion $W$. Then the general nonlinear backward stochastic differential equation writes as follows:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \quad t \in [0,T], \tag{0.25}$$

where $\xi$ is a square integrable, $\mathcal{F}_T$-measurable random variable. A solution of this backward stochastic differential equation is a pair of $\mathbb{F}$-adapted processes $\{Y_t, Z_t\}_{t\in[0,T]}$ satisfying some integrability assumptions. Pardoux and Peng [83] proved that under the Lipschitz condition on $f$, the backward stochastic differential equation (0.25) has a unique adapted, square integrable solution.

Since the pioneering work by Pardoux and Peng, the theory of backward stochastic differential equations has been widely and thoroughly investigated. Briefly speaking, there are several directions in this topic. We explain them separately in the following.

One of the major objectives of the research has concerned the weakening of conditions under which a backward stochastic differential equation has a solution: Lepeltier and San Martín [71] studied backward stochastic differential equations with continuous coefficients with linear growth in 1997; in 2001 Bahlali [8] worked on multidimensional backward stochastic differential equations with locally Lipschitz coefficients; Backward stochastic differential equations with quadratic growth were first studied by Kobylanski [61] and [62] in her PhD thesis. Stimulated by her pioneering work, many authors have worked on this type of backward stochastic differential equation, and without being exhaustive, we mention here in particular the works by Lepeltier and San Martín [72] for superlinear-quadratic coefficients, by Briand and Hu [14], [15] for unbounded terminal conditions, by Ankirchner, Imkeller and Dos Reis [5] for differentiability of the solutions, that by Morlais [78] for continuous martingale drivers, that by Imkeller and Dos Reis [54] for truncated quadratic growth, and finally, that by Delbaen, Hu and Bao [35] for superquadratic growth.

Peng [92] was the first to state in 1991 the probabilistic interpretation for parabolic partial differential equations via backward stochastic differential equations, known as the nonlinear Feynman-Kac formula. Pardoux and Peng [85] further investigated viscosity solutions for parabolic partial differential equations through the backward stochastic differential equations interpretation in 1992.

Antonelli [6] was the first to study forward-backward stochastic differential equations for a small time horizon in 1993; Ma, Protter and Yong [74] in 1994 stated a four-step scheme for solving forward-backward stochastic differential equations by using a partial differential equation method, while Hu and Peng [51] studied existence and uniqueness under certain monotonicity assumptions in 1995.

Let us also mention that El Karoui et al. [40] investigated reflected backward stochastic differential equations with one barrier in 1997, which are tightly related to optimal stopping problems. Moreover, they provided a probabilistic interpretation for obstacle problems associated with parabolic partial differential equations. Cvitanić and Karatzas [31] generalized their work to reflected backward stochastic differential equations with two barriers and applied them to Dynkin games.

Another type of backward equations, backward doubly stochastic differential equations, plays a central role in the first two chapters of this dissertation project, while backward stochastic differential equations with jumps are used in the third manuscript (see Chapter 3). We will introduce them in more details later.

It is also worth noting that, by using backward stochastic differential equations, Peng [88] defined a nonlinear expectation in 1997, called the $g$-expectation, which is a generalization of the concept of classical expectation. Inspired by the work on $g$-expectation and risk measures, Peng [90] [91] recently defined a new type of more general nonlinear expectation, called the $G$-expectation, and developed the Itô calculus based on it.

Let us also mention that the development of the theory of backward stochastic differential equations has been stimulated from its early beginning by its various applications, among them in particular those in finance (see e.g., El Karoui, Peng and Quenez [42] in 1997, Chen and Epstein [25] in 2002, Hamadène and Jeanblanc [46] in 2007), but also those in stochastic control and game theory (for example, Peng [87], [89], Hamadène and Lepeltier [47], Hamadène and Zhang [48], El Karoui and Hamadène [39], Buckdahn and Li [20], [21], Hu and Tang [52], Peng and Wu [93], Peng and Xu [94], Yong and Zhou [108]).

Now let us introduce backward doubly stochastic differential equations. This type of equations, first introduced and studied by Pardoux and Peng [84] in 1994, has the following form:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s + \int_t^T g(s, Y_s, Z_s) \downarrow \mathrm{d}B_s - \int_t^T Z_s \mathrm{d}W_s, \quad t \in [0, T],$$

where $B$ and $W$ are two independent Brownian motions; the stochastic integral with respect to $B$ is the Itô backward one, that with respect to $W$ the usual Itô forward one. In [84], the authors proved the existence and uniqueness of the solutions for backward doubly stochastic differential equations under Lipschitz conditions on $f$ and $g$, and it should be noted that the Lipschitz constant of $g$ with respect to $z$ is required to be less than one. In particular, in analogy to the nonlinear Feynman-Kac formula, they gave a stochastic interpretation of a certain class of parabolic stochastic partial differential equations in terms of the solution of the corresponding backward doubly stochastic differential equations. The solution of the stochastic partial differential equation considered in [84] was in the classical sense. Bally and Matoussi [9] applied the method of backward doubly stochastic differential equations in the study of weak solutions for stochastic partial differential equations in 1999.

In 2001, by applying the Doss-Sussmann transformation, Buckdahn and Ma [22], [23] gave a definition of the stochastic viscosity solution for a special kind of stochastic partial differential equations driven by a Brownian motion. The Doss-Sussmann transformation, which was studied by Doss [36] in 1977 and, independently, by Sussmann [103] in 1978, can be used to transform special types of stochastic differential equations into pathwise ordinary differential equations. For this the diffusion coefficient has to be only a function of the solution process and has to satisfy some regular conditions. Moreover, the stochastic integral in the stochastic differential equation has to be considered in the Stratonovich sense. By using this method, the authors successfully transformed a nonlinear backward doubly stochastic differential equation into a backward stochastic differential equation with quadratic growth. The thus got relationship between backward doubly stochastic differential equations and backward stochastic differential equations with quadratic growth, reflects also in a corresponding relationship between the stochastic partial differential equations associated with the backward doubly stochastic differential equations, and partial differential equations with random coefficients associated with the backward stochastic differential equation with quadratic growth. By applying these relations, they defined the notion of the stochastic viscosity solution, which was a notable extension of the classical definition of the deterministic viscosity solution (see, Crandall, Ishii and Lions [28]).

The history of the fractional Brownian motion is much longer than that of the theory of backward stochastic differential equations. Kolmogorov was the first who actually worked on it, but at that time it was called "Wiener spiral" [63]. After the statistical work on the flood of Nile river by Hurst [53], after whom the parameter $H$ was named (however, the letter $H$ was given by Mandelbrot in honor of both Hurst and Hölder), and the pioneering work of Mandelbrot and van Ness [75] on the stochastic calculus, the fractional Brownian motion has been attracting more and more mathematicians to work on. From the 1990s, models involving the fractional Brownian motion have found their place in the financial market models. The interest for the fractional Brownian motion in finance stems from the fact that, unlike the Brownian motion, the fractional Brownian motion allows to describe long memory effects when the Hurst parameter $H \in (1/2, 1)$, and short range dependence (or memory) when $H \in (0, 1/2)$. This behavior is quite different from that of the classical Brownian motion which increments are independent.

It is well known that the fractional Brownian motion is not a semimartingale except for the case $H = 1/2$. Thus the well-developed theory of Itô calculus cannot be applied directly here. Because of the significant difference between $H \in (0, 1/2)$ and $H \in (1/2, 1)$, the definitions of the stochastic integral with respect to the fractional Brownian motion are divided into two groups.

For $H \in (1/2, 1)$, within the framework of the Malliavin calculus, the divergence operator can be used as the definition of the stochastic integral, see Decreusefont and Üstünel [33], Ducan, Hu and Pasik-Duncan [38], Alòs, Mazet and Nualart [3], as well as Alòs and Nualart [4]. The Hölder continuity (with order strictly less than $H$) also allows to define a pathwise stochastic integral. We refer, for instance, to the generalized Riemann-Stieltjes integral developed by Zähle [109], [110], to the integration in the frame of the rough path theory studied in Coutin and Qian [30], and also to the Russo-Vallois integral [99], [100], [101].

For $H \in (0, 1/2)$ the paths of the fractional Brownian motion are more irregular. They only have Hölder continuity with order less than $H$, i.e., the order is strictly less than 1/2. This has as consequence that the situation is more complicated here. Alòs, León, Mazet and Nualart have published several papers discussing this problem: [1], [2], [3]. The divergence operator can be used to define a stochastic integral, but as it is shown in the paper of Cheridito and Nualart [27], the domain of the divergence operator does not contain the fractional Brownian motion itself when $H \leq 1/4$. However, Cheridito and Nualart [27], and León and Nualart [67] generalized the definition of the divergence operator and called it the extended divergence operator. Furthermore, they proved that the Itô formula, the Tanaka formula as well as the Fubini theorem hold true for fractional Brownian motions with any $H \in (0, 1/2)$ under the extended framework.

Stochastic differential equations driven by a fractional Brownian motion have been studied by several authors using different definitions of the stochastic integral. Without being exhaustive, let us mention here the following works: León and San Martín [68], León and Tindel [69] for $H \in (0, 1/2)$, Maslowski and Nualart [76], Nualart and Răşcanu [81] for $H \in (1/2, 1)$, Tindel, Tudor and Viens [105] for $H \in (0, 1)$.

Bender [11] was the first in 2001 to bring together two dynamically developing theories, that of backward stochastic differential equations and that of the fractional Brownian motion, by studying linear fractional backward stochastic differential equations, for which he gave explicit solution formulas. Hu and Peng [50] were the first in 2009 to investigate nonlinear fractional backward stochastic differential equations. For this they used the notion of the quasi-conditional expectation introduced by Hu and Øksendal [49]. We notice that the use of the quasi-conditional expectation leads to rather restrictive conditions on the coefficients.

In 2009, Jien and Ma [57] extended the method of the anticipative Girsanov transformation in Buckdahn [16] to the fractional Brownian motion case. Their results can be used to solve possibly anticipative fractional stochastic differential equations. Motivated by Jien and Ma's work [57], our first chapter of this thesis aims to combine the theory of backward stochastic differential equations and that of the fractional Brownian motion in a different way than that in Bender [11], Hu and Peng [51]. Indeed, we study the semilinear backward doubly stochastic differential equations driven by a classical Brownian motion $W$ and a fractional Brownian motion $B$:

$$Y_t = \xi + \int_0^t f(s, Y_s, Z_s)\mathrm{d}s - \int_0^t Z_s \downarrow \mathrm{d}W_s + \int_0^t \gamma_s Y_s \mathrm{d}B_s, \quad t \in [0, T]. \tag{0.26}$$

Here the stochastic integrals with respect to the Brownian motion $W$ and the fractional Brownian motion $B$ are the backward Itô integral and the extended divergence operator, respectively. We emphasize that a slight difference in the form between our fractional backward doubly stochastic differential equation and the classical one is that we use a time-reversed form. Hence, here the usual terminal condition $\xi$ turns out to be an initial condition, measurable with respect to the $\sigma$-field generated by the Brownian motion.

Like in [57], where the anticipative Girsanov transformation was used to transform anticipative semilinear stochastic differential equations driven by a divergence operator (Skorohod integral) into pathwise ordinary differential equations, we apply the Girsanov transformation to (0.26) in order to eliminate the integral with respect to $B$. More precisely, we can show that equation (0.26) and the following pathwise backward stochastic differential equation

$$\widehat{Y}_t = \xi + \int_0^t f\left(s, \widehat{Y}_s \varepsilon_s(T_s), \widehat{Z}_s \varepsilon_s(T_s)\right) \varepsilon_s^{-1}(T_s)\mathrm{d}s - \int_0^t \widehat{Z}_s \downarrow \mathrm{d}W_s, \quad t \in [0, T]. \tag{0.27}$$

are equivalent (For more details, see Theorem 3.9 in Chapter 1.) In the same spirit following Buckdahn and Ma [22], we associate with (0.26) the semilinear stochastic partial differential equation driven by the fractional Brownian motion $B$:

$$\begin{cases} \mathrm{d}u(t,x) = \left[\mathcal{L}u(t,x) + f\left(t, x, u(t,x), \nabla_x u(t,x)\sigma(x)\right)\right]\mathrm{d}t + \gamma_t u(t,x)\mathrm{d}B_t, & t \in [0,T], \\ u(0,x) = \Phi(x). \end{cases} \tag{0.28}$$

by studying the pathwise partial differential equation associated with (0.27):

$$\begin{cases} \frac{\partial}{\partial t}\widehat{u}(t,x) = \left[\mathcal{L}\widehat{u}(t,x) + f\left(t,x,\widehat{u}(t,x)\varepsilon_t(T_t), \nabla_x\widehat{u}(t,x)\sigma(x)\varepsilon_t(T_t)\right)\varepsilon_t^{-1}(T_t)\right], \quad t \in [0,T]; \\ \widehat{u}(0,x) = \Phi(x), \end{cases}$$

The main difficulty here is to prove the continuity property of the random field $u$ defined through equation (0.26). Indeed, $u$ is related with the continuous solution $\widehat{u}$ of equation (0.27) by Girsanov transformation. But, in the general case, a continuous random field $\widehat{u}$ may lose its continuity property after the Girsanov transformation (see Remark 4.13 in Chapter 1). However, by applying some rather technical fractional calculus and estimates on backward stochastic differential equations, we succeed in proving the continuity of the random field $u$ and we show that it is the stochastic viscosity solution of the stochastic partial differential (0.28).

The anticipative Girsanov transformation method imposes us to deal only with the semilinear case. While this employed approach is specified for $H \in (0, 1/2)$, we could, nevertheless, use the divergence operator (instead of the extended divergence operator) also for the case $H \in (1/2, 1)$, and the corresponding results still hold true, only with some slight changes. But, actually, we can do even better in the case of $H \in (1/2, 1)$: Indeed, here the Russo-Vallois integral allows to consider a nonlinear coefficient as the integrand of the integral with respect to the fractional Brownian motion $B$.

So, in Chapter 2, we deal with nonlinear fractional backward doubly stochastic differential equations with Hurst parameter $H \in (1/2, 1)$:

$$Y_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\mathrm{d}r + \int_0^s g(Y_r^{t,x})\mathrm{d}B_r - \int_0^s Z_r^{t,x} \downarrow \mathrm{d}W_r, \ s \in [0,t]. \tag{0.29}$$

In the paper [22], Buckdahn and Ma applied the Doss-Sussmann transformation to study backward doubly stochastic differential equations driven by two independent Brownian motions. This corresponds to the case $H = 1/2$, i.e., also the process $B$ in the backward doubly stochastic differential equation is a Brownian motion, but the stochastic integral with respect this Brownian motion $B$ was considered by the authors in the Stratonovich sense. Accordingly, we extend their approach to nonlinear fractional backward doubly stochastic differential equations with Hurst parameter $H \in (1/2, 1)$. For this the stochastic integral with respect to the fractional Brownian motion has to be replaced by a pathwise integral, and we take that of Russo-Vallois. In fact, in their original paper [100], in generalization of classical stochastic integrals, Russo and Vallois defined three integrals, called forward, backward and symmetric integrals, which are the corresponding extension of the classical Itô integral, the Itô backward integral and the Stratonovich integral. It should be noticed that in the case of a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ as integrator, these three integrals coincide. A new type of Itô formula for processes which contain both a backward Itô integral with respect to a classical Brownian motion $W$ and the Russo-Vallois integral with respect to the fractional Brownian motion $B$, is crucial for the application of the Doss-Sussmann transformation. However, the difficulty here is that, unlike the semilinear case, we cannot find a space of processes invariant under Doss-Sussmann transformation to which both solution processes $Y$ and $Z$ belong. This is also the main difference between our work and that of Buckdahn and Ma [22]: In the classical case, one can solve directly the backward doubly stochastic differential equation and get the square integrability of the solution processes $Y$ and $Z$. But here in the fractional case, we cannot solve directly equation (0.29), we only can solve through the associated pathwise backward stochastic differential equation

$$U_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s \tilde{f}(r, X_r^{t,x}, U_r^{t,x}, V_r^{t,x})\mathrm{d}r - \int_0^s V_r^{t,x} \downarrow \mathrm{d}W_r. \tag{0.30}$$

But for this pathwise backward stochastic differential equation, we only obtain the result that the solution process $U$ and $V$ are conditionally square integrable knowing the fractional Brownian motion $B$, i.e., the conditional expectation of the integral of the square of the process $V$ is finite almost surely. This is related with the fact that the backward stochastic differential equation (0.30) has quadratic growth in $V$. Moreover, in our study of the above type of backward doubly stochastic differential equations we also consider the associated fractional stochastic partial differential equation

$$\begin{cases} \mathrm{d}u(t,x) = [\mathcal{L}u(t,x) + f\,(t,x,u(t,x),\nabla_x u(t,x)\sigma(x))]\,\mathrm{d}t + g(u(t,x))\mathrm{d}B_t, & t \in [0,T], \\ u(0,x) = \Phi(x), \end{cases}$$

and its stochastic viscosity solution. We also emphasize that our stochastic partial differential equation is a bit more general than that was considered in Maslowski and Nualart [76], where the function $f$ depends only on $u$, and also different from that in Issoglio [56], where the the fractional Brownian motion is in fact defined with respect to the space parameter instead of the time parameter.

Finally, let us also point out that the ways in which the stochastic viscosity solutions are defined in Chapter 1 and Chapter 2 are different. In both cases their definitions are tightly related to the main methods used to solve the associated backward doubly stochastic differential equations. In the case of the Hurst parameter $H \in (0, 1/2)$, it is the Girsanov transformation, which changes the sample paths; in the case $H \in (1/2, 1)$, it is the Doss-Sussmann transformation, which does not change the sample paths.

The second main part of this thesis, i.e., Chapter 3, concerns the study of regularity properties, namely, the joint Lipschitz continuity and the joint semiconcavity in space and in time, of the viscosity solution for integro-partial differential equations of Hamilton-Jacobi-Bellman type.

For this, the integro-partial differential equation is investigated through its stochastic interpretation as a stochastic control problem involving backward stochastic differential equations with jumps. Such backward equations with jumps were first studied by Li and Tang [104] in 1994. By using Li and Tang's results but in a more general form, Barles, Buckdahn and Pardoux [10] in 1997 investigated the viscosity solution of the associated integro-partial differential equation. In their paper, the viscosity solution $\{v(t,x) : t, x \in [0,T] \times \mathbb{R}^d\}$ of the integro-partial differential equation is shown to be Lipschitz in $x$ but only 1/2-Hölder in $t$.

On the other hand, Buckdahn, Cannarsa and Quincampoix [18] as well as Buckdahn, Huang and Li [19] showed that the viscosity solutions of not necessarily non-degenerate parabolic partial differential equations (without and with obstacle) are jointly Lipschitz in $(t,x) \in [0, T-\delta] \times \mathbb{R}^d$, for all $\delta > 0$, and, moreover, under suitable assumptions they are also jointly semiconcave in $(t,x) \in [0, T-\delta] \times \mathbb{R}^d$, for all $\delta > 0$. Let us emphasize that the importance of the joint semiconcavity stems from Alexandrov's theorem, due to which we can conclude that the viscosity solution possesses $\mathrm{d}t\mathrm{d}x$ almost everywhere a second order limit expansion. Furthermore, Cannarsa and Sinestrari [24] showed that these regularity properties are the optimal ones one can expect. Moreover, they showed (see also [18]) that, in general, the joint Lipschitz continuity property and the joint semiconcavity cannot hold over the whole domain $[0,T]\times\mathbb{R}^d$.

Let us also emphasize that the works [18], [19] and [24] improved not only already known results on Lipschitz continuity for the viscosity solutions of partial differential equations but also former results on semiconcavity. Indeed, the semiconcavity with respect to $x$, for fixed $t\in[0,T]$, was already proved by Ishii and Lions [55] (via a purely analytical approach) and by Yong and Zhou [108] (by stochastic estimates in the associated stochastic control problem).

In our work, we extend the results of [18] and [19] to the integro-partial differential equations of the form

$$\begin{cases} \frac{\partial}{\partial t}v(t,x)+\inf_{u\in U}\{(\mathcal{L}^u+B^u)v(t,x)+f(t,x,v(t,x),(D_xv\sigma)(t,x), \\ \qquad\qquad v(t,x+\beta(t,x,u,\cdot))-v(t,x))\}=0, & (t,x)\in(0,T)\times\mathbb{R}^d; \\ v(T,x)=\Phi(x), & x\in\mathbb{R}^d, \end{cases}$$

where $U$ is a compact metric space, $\mathcal{L}^u$ is the linear second order differential operator

$$\mathcal{L}^u\varphi(x)=\mathrm{tr}\left(\frac{1}{2}\sigma\sigma^T(t,x,u)D^2_{xx}\varphi(x)\right)+b(t,x,u)\cdot D_x\varphi(x),\quad \varphi\in C^2(\mathbb{R}^d),$$

and $B^u$ is the integro-differential operator:

$$B^u\varphi(x)=\int_E[\varphi(x+\beta(t,x,u,e))-\varphi(t,x)-\beta(t,x,u,e)\cdot D_x\varphi(x)]\Pi(\mathrm{d}e),\quad \varphi\in C_b^2(\mathbb{R}^d).$$

We extend the approaches of [18] and [19] to the framework of the above integro-partial differential equation. For this we consider the associated stochastic control problem composed of a controlled forward and a controlled backward stochastic differential equation, both governed by a Brownian motion $B$ and a Poisson random measure $\mu$. While we adapt the method of time change for the underlying Brownian motion from [18] and [19], we introduce Kulik's transformation [65] and [66] for the Poisson random measure $\mu$. The combination of these both main tools allows to obtain the joint Lipschitz continuity and the joint semiconcavity for the above integro-partial differential equation.

In the following we summarize in a more precise manner the main results obtained in the frame of this PhD project:

## Main Results in this Thesis

**Main results in Chapter 1.** *Semilinear Backward Doubly Stochastic Differential Equations and stochastic partial differential equations Driven by Fractional Brownian Motion with Hurst Parameter in (0,1/2).*

This introduction as well as Chapter 1 is based on the paper [58] written in collaboration with Jorge A. León (Cinvestav-Instituto Politecnico National à Ciudad, Mexico), accepted by *Bulletin des Sciences Mathématiques.*

We consider a complete probability space $(\Omega, \mathcal{F}, P)$ on which a Brownian motion $W$ and an independent fractional Brownian motion $B$ with Hurst parameter $H \in (0, 1/2)$ are defined. We let $\mathcal{F} = \mathcal{F}_T$ be the $\sigma$-field generated by $W$ and $B$, augmented by all $P$-null sets. By $\mathbb{G} = \{\mathcal{G}_t\}_{t\in[0,T]}$ we denote the family of $\sigma$-fields $\mathcal{G}_t$ generated by $\{W_s - W_t, s \in [t, T]\}$ and $\{B_s, s \in [0, t]\}$.

Our aim is to study the following semilinear stochastic partial differential equation driven by the fractional Brownian motion $B$:

$$\begin{cases} \mathrm{d}u(t,x) = \left[\mathcal{L}u(t,x) + f\left(t, x, u(t,x), \nabla_x u(t,x)\sigma(x)\right)\right]\mathrm{d}t + \gamma_t u(t,x)\mathrm{d}B_t, \quad t \in [0,T], \\ u(0,x) = \Phi(x), \end{cases} \tag{0.31}$$

where $\mathcal{L}$ is the second order differential operator

$$\mathcal{L} := \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(x)D^2_{xx}) + b(x)\nabla_x.$$

The coefficients are assumed to satisfy Lipschitz conditions and $\gamma \in L^2(0,T)$. Our approach is based on a (doubly) stochastic interpretation of the stochastic partial differential equation (0.31) by a backward doubly stochastic differential equation. So we first investigate the fractional backward doubly stochastic differential equation driven by both the fractional Brownian motion $B$ and an independent Brownian motion $W$,

$$Y_t = \xi + \int_0^t f(s, Y_s, Z_s)\mathrm{d}s - \int_0^t Z_s \downarrow \mathrm{d}W_s + \int_0^t \gamma_s Y_s \mathrm{d}B_s, \quad t \in [0,T]. \tag{0.32}$$

Here the stochastic integral with respect to the Brownian motion $W$ is the Itô backward integral, while the stochastic integral with respect to the fractional Brownian motion is the extended divergence operator.

The main idea in our approach for solving backward doubly stochastic differential equation (0.32) consists in the applications of a Girsanov transformation

$$T_t(\omega) = \omega + \int_0^{\cdot\wedge t} (\mathcal{K}\gamma 1_{[0,t]})(r)\mathrm{d}r,$$

(the definition of the operator $\mathcal{K}$ is given below), with the help of which we transform the fractional backward doubly stochastic differential equation into a pathwise backward stochastic differential equation which coefficients depend on the paths of $B$ and which is driven only by the Brownian motion $W$:

$$\widehat{Y}_t = \xi + \int_0^t f\left(s, \widehat{Y}_s\varepsilon_s(T_s), \widehat{Z}_s\varepsilon_s(T_s)\right)\varepsilon_s^{-1}(T_s)\mathrm{d}s - \int_0^t \widehat{Z}_s \downarrow \mathrm{d}W_s, \quad t \in [0,T]. \tag{0.33}$$

By studying the backward stochastic differential equation (0.33) as well as its associated partial differential equation

$$\begin{cases} \frac{\partial}{\partial t}\widehat{u}(t,x) = \left[\mathcal{L}\widehat{u}(t,x) + f\left(t, x, \widehat{u}(t,x)\varepsilon_t(T_t), \nabla_x\widehat{u}(t,x)\sigma(x)\varepsilon_t(T_t)\right)\varepsilon_t^{-1}(T_t)\right], \quad t \in [0,T]; \\ \widehat{u}(0,x) = \Phi(x), \end{cases} \tag{0.34}$$

we obtain the solution of stochastic partial differential equation (0.31).

After having explained above the general scheme, let us become a bit more precise. For this we introduce some elements of the fractional calculus which will be used in what follows, such as the definitions of right- (left-) sided fractional derivatives and right- (left-) sided fractional integrals: Let $T > 0$ denote a positive time horizon, and let $f : [0, T] \to \mathbb{R}$ be an integrable function, and $\alpha \in (0, 1)$. The *right–sided fractional integral* of $f$ of order $\alpha$ is given by

$$I^{\alpha}_{T-}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^T \frac{f(u)}{(u-x)^{1-\alpha}} \mathrm{d}u, \quad \text{for a.a. } x \in [0, T],$$

and the right–sided fractional derivative

$$(D^{\alpha}_{T-}f)(s) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(s)}{(T-s)^{\alpha}} + \alpha \int_s^T \frac{f(s) - f(u)}{(u-s)^{1+\alpha}} \mathrm{d}u \right).$$

Let us continue with defining the space $\Lambda_T^{1/2-H}$ which is important for the definition of the extended divergence operator:

$$\Lambda_T^{1/2-H} := \left\{ f : \exists\, \varphi_f \in L^2(0, T) \text{ s.t. } f(u) = u^{1/2-H} I_{T-}^{1/2-H} \left( s^{H-1/2} \varphi_f(s) \right)(u) \right\}.$$

We observe that all indicator functions $I_{[0,t]}, t \in [0, T]$, belong to $\Lambda_T^{1/2-H}$. The map $\mathcal{K} : \Lambda_T^{1/2-H} \to L^2([0, T])$, defined by

$$(\mathcal{K}\varphi)(s) = C_H \Gamma(H + 1/2) s^{1/2-H} \left( D_{T-}^{1/2-H} u^{H-1/2} \varphi(u) \right)(s), \quad s \in [0, T],$$

is an isometry such that

$$B_t = \int_0^t (\mathcal{K} I_{[0,t]})(s) \mathrm{d}W_s, \ t \in [0, T].$$

Now we can state the definition of the extended divergence operator for our fractional Brownian motion.

**Definition 0.20.** *Let $u \in L^2(\Omega, \mathcal{F}, P; L^2([0, T]))$. We say that $u$ belongs to Dom $\delta^B$ if there exists $\delta^B(u) \in L^2(\Omega, \mathcal{F}, P)$ such that*

$$E\left[\langle \mathcal{K}^*\mathcal{K}DF, u\rangle_{L^2([0,T])}\right] = E\left[F\delta^B(u)\right], \quad \text{for every} \quad F \in \mathcal{S}_{\mathcal{K}},$$

*where $\mathcal{S}_{\mathcal{K}}$ is the space of smooth functions with respect to the fractional Brownian motion $B$ and $DF$ is the Malliavin derivative of $F$ (for the precise definitions of $\mathcal{S}_{\mathcal{K}}$ and $DF$, the reader is referred to Chapter 1). In this case, the random variable $\delta^B(u)$ is called the extended divergence of $u$.*

In order to use the extended divergence operator, we have to impose the following hypothesis on $\gamma$ throughout this chapter:

**(H1)** $\gamma 1_{[0,t]}$ belongs to $\Lambda_T^{1/2-H}$, for every $t \in [0, T]$.

Such an assumption has already been used, for instance, in the paper [57] by León and San Martín on linear stochastic differential equations driven by fractional Brownian motions.

The main tool in our approach is the Girsanov transformation. More precisely, we consider the transformations

$$T_t : \Omega \to \Omega, \ T_t(\omega) = \omega + \int_0^{\cdot \wedge t} (\mathcal{K}\gamma 1_{[0,t]})(r) \mathrm{d}r, \quad \omega \in \Omega$$

and its inverse

$$A_t : \Omega \to \Omega, \quad A_t(\omega) = \omega - \int_0^{\cdot\wedge t} (\mathcal{K}\gamma 1_{[0,t]})(r)\mathrm{d}r, \quad \omega \in \Omega.$$

We also introduce the term

$$\varepsilon_t = \exp\left(\int_0^t \gamma_r \mathrm{d}B_r - \frac{1}{2}\int_0^t ((\mathcal{K}\gamma 1_{[0,t]})(r))^2 \mathrm{d}r\right).$$

An important role in our approach is played by the quantity $I_T^* := \sup_{t\in[0,T]} \left|\int_0^t \gamma_s \mathrm{d}B_s\right|$, for which we prove that $E[\exp\{pI_T^*\}] < \infty$, for all $p \geq 1$. It allows to introduce the space $L_{\mathbb{G}}^{2,*}(0,T;\mathbb{R}\times\mathbb{R}^d)$ as the space of all $\mathbb{G}$-adapted processes $(Y,Z)$ which are such that

$$E\left[\exp\{pI_T^*\}\int_0^T \left(|Y_t|^2 + |Z_t|^2\right)\mathrm{d}t\right] < \infty, \text{ for all } p \geq 1.$$

The importance of the space $L_{\mathbb{G}}^2(0,T;\mathbb{R}\times\mathbb{R}^d)$ stems from its invariance under the Girsanov transformations $T_t, A_t, t \in [0,T]$. So it allows to prove the following key result:

**Proposition 0.21.** *For all processes $(Y,Z) \in L_{\mathbb{G}}^{2,*}(0,T;\mathbb{R}\times\mathbb{R}^d)$ we have:*
*i) $(\widetilde{Y}_t, \widetilde{Z}_t) := (Y_t(T_t)\varepsilon_t^{-1}(T_t), Z_t(T_t)\varepsilon_t^{-1}(T_t)) \in L_{\mathbb{G}}^{2,*}(0,T;\mathbb{R}\times\mathbb{R}^d)$ and*
*ii) $(\overline{Y}_t, \overline{Z}_t) := (Y_t(A_t)\varepsilon_t, Z_t(A_t)\varepsilon_t) \in L_{\mathbb{G}}^{2,*}(0,T;\mathbb{R}\times\mathbb{R}^d)$.*

After the above preparation we can study the fractional backward doubly stochastic differential equation (0.32) and the associated pathwise backward stochastic differential equation (0.33). Our first main result concerns the integrability properties of the solutions of the pathwise backward stochastic differential equation.

**Theorem 0.22.** *Under our standard assumptions (see (**H3**) in Chapter 1), the backward stochastic differential equation*

$$\widehat{Y}_t = \xi + \int_0^t f(s, \widehat{Y}_s\varepsilon_s(T_s), \widehat{Z}_s\varepsilon_s(T_s))\varepsilon_s^{-1}(T_s)\mathrm{d}s - \int_0^t \widehat{Z}_s \downarrow \mathrm{d}W_s,\ t \in [0,T], \tag{0.35}$$

*has a unique solution $(\widehat{Y}, \widehat{Z}) \in L_{\mathbb{G}}^{2,*}(0,T;\mathbb{R}\times\mathbb{R}^d)$.*

*Furthermore, there exists a positive constant $C$ such that,*

$$E\left[\sup_{t\in[0,T]}\left|\widehat{Y}_t\right|^2 + \int_0^T \left|\widehat{Z}_t\right|^2 \mathrm{d}t|\mathcal{F}_T^B\right] \leq C\exp\{2I_T^*\}.$$

By using the above results and applying the Girsanov transformation, we obtain our second main result. It combines the fractional backward doubly stochastic differential equation and the pathwise backward stochastic differential equation by showing that we obtain the solution of one equation by solving the other one and vice versa.

**Theorem 0.23.** 1) *Let $(\widehat{Y}, \widehat{Z}) \in L_{\mathbb{G}}^{2,*}(0,T;\mathbb{R}\times\mathbb{R}^d)$ be a solution of backward stochastic differential equation* (0.35)*. Then*

$$\{(Y_t, Z_t), t \in [0,T]\} = \{(\widehat{Y}_t(A_t)\varepsilon_t, \widehat{Z}_t(A_t)\varepsilon_t), t \in [0,T]\} \in L_{\mathbb{G}}^{2,*}(0,T;\mathbb{R}\times\mathbb{R}^d)$$

*is a solution of equation* (0.32) *with $\gamma Y 1_{[0,t]} \in Dom\ \delta^B$, for all $t \in [0,T]$.*

2) *Conversely, given an arbitrary solution* $(Y,Z) \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$ *of the equation* (0.32) *with* $\gamma Y 1_{[0,t]} \in Dom\ \delta^B$, *for all* $t\in[0,T]$, *the process*

$$\{(\widehat{Y}_t,\widehat{Z}_t)_{t\in[0,T]}\} = \{(Y_t(T_t)\varepsilon_t^{-1}(T_t), Z_t(T_t)\varepsilon_t^{-1}(T_t))_{t\in[0,T]}\} \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$$

*is a solution of backward stochastic differential equation* (0.35).

The remaining part of Chapter 1 is devoted to the associated stochastic partial differential equation. To this end we shall associate our backward doubly stochastic differential equation with a forward stochastic differential equation.

We denote by $(X_s^{t,x})_{0\le s\le t}$ the unique solution of the following stochastic differential equation:

$$\begin{cases} \mathrm{d}X_s^{t,x} = -b(X_s^{t,x})\mathrm{d}s - \sigma(X_s^{t,x}) \downarrow \mathrm{d}W_s, \quad s\in[0,t], \\ X_t^{t,x} = x\in\mathbb{R}^d, \end{cases} \tag{0.36}$$

which we associate with the backward doubly stochastic differential equation

$$Y_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\mathrm{d}r - \int_0^s Z_r^{t,x}\downarrow \mathrm{d}W_r + \int_0^s \gamma_r Y_r^{t,x}\mathrm{d}B_r, \tag{0.37}$$

$s\in[0,t]$. From the results of the first part, with $f(s,y,z) := f(s,X_s^{t,x},y,z)$, we get that this equation has a unique solution $\left(Y^{t,x},Z^{t,x}\right)$ given by

$$(Y_s^{t,x},Z_s^{t,x}) = (\widehat{Y}_s^{t,x}(A_s)\varepsilon_s, \widehat{Z}_s^{t,x}(A_s)\varepsilon_s), s\in[0,t],$$

where $(Y,Z)\in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$ is the unique solution of the equation

$$\widehat{Y}_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f(r,X_r^{t,x},\widehat{Y}_r^{t,x}\varepsilon_r(T_r),\widehat{Z}_r^{t,x}\varepsilon_r(T_r))\varepsilon_r^{-1}(T_r)\mathrm{d}r - \int_0^s \widehat{Z}_r^{t,x}\downarrow\mathrm{d}W_r, \tag{0.38}$$

$s\in[0,t]$.

Let us now introduce the random field: $\widehat{u}(t,x) = \widehat{Y}_t^{t,x}, (t,x)\in[0,T]\times\mathbb{R}^d$. We will show that $\widehat{u}$ is a viscosity solution of the pathwise partial differential equation (0.34). For this sake, we first prove that it has a continuous version.

**Lemma 0.24.** *The process* $\{\widehat{Y}_s^{t,x};(s,t)\in[0,T]^2, x\in\mathbb{R}^d\}$ *possesses a continuous version. Moreover,* $|\widehat{u}(t,x)|\le C\exp\{I_T^*\}(1+|x|)$, *P-a.s.*

Then we prove the random field $\widehat{u}$ is the unique pathwise viscosity solution of (0.34).

**Theorem 0.25.** *The random field* $\widehat{u}$ *defined by* $\widehat{u}(t,x) = \widehat{Y}_t^{t,x}$ *is a pathwise viscosity solution of equation* (0.34), *where* $\widehat{Y}^{t,x}$ *is the solution of equation* (0.38). *Furthermore, this solution* $\widehat{u}(t,x)$ *is unique in the class of continuous stochastic fields* $\tilde{u}:\Omega\times[0,T]\times\mathbb{R}^d\mapsto\mathbb{R}$ *such that, for some random variable* $\eta\in L^0(\mathcal{F}_T^B)$,

$$|\tilde{u}(t,x)|\le\eta(1+|x|),(t,x)\in[0,T]\times\mathbb{R}^d, P\text{-}a.s.$$

In order to identify the field $u(t,x) := \widehat{u}(A_t,t,x)\varepsilon_t, (t,x)\in[0,T]\times\mathbb{R}^d$ as a stochastic viscosity solution of (0.31), we have to guarantee that $u$ has a continuous version. In fact we have a counterexample which shows that, in general, a continuous random field after our Girsanov transformation may not be continuous anymore. However, a rather technical proof using the special form of $u$ allows to show:

**Lemma 0.26.** *The random field $u$ has a continuous version.*

Motivated by the relation for classical solutions between stochastic partial differential equation (0.31) and partial differential equation (0.34) (see Proposition 4.11 in Chapter 1), we give the following definition:

**Definition 0.27.** *A continuous random field $u : [0,T] \times \mathbb{R}^d \times \Omega' \mapsto \mathbb{R}$ is a (stochastic) viscosity solution of equation (0.31) if $\widehat{u}(t,x) = u(T_t, t, x)\varepsilon_t^{-1}(T_t), (t,x) \in [0,T] \times \mathbb{R}^d$ is a pathwise viscosity solution of equation (0.34).*

Summarizing our results, we can state the following theorem:

**Theorem 0.28.** *The continuous stochastic field $u(t,x) := \widehat{u}(A_t, t, x)\varepsilon_t = \widehat{Y}_t^{t,x}(A_t)\varepsilon_t = Y_t^{t,x}$ is a stochastic viscosity solution of the semilinear stochastic partial differential equation (0.31). This solution is unique inside the class of continuous stochastic field $\tilde{u} : \Omega' \times [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}$ such that,*

$$|\tilde{u}(t,x)| \leq C \exp\{I_T^*\}(1 + |x|), (t,x) \in [0,T] \times \mathbb{R}^d, P - a.s.,$$

*for some constant $C$ only depending on $\tilde{u}$.*

**Main results in Chapter 2.** *Fractional Backward Doubly Stochastic Differential Equations with Hurst Parameter in (1/2,1)*

This chapter is based on the manuscript [59] submitted for publication.

After having considered the semi-linear case of the parabolic stochastic partial differential equation (0.31) driven by a fractional Brownian motion, we are now interested in the nonlinear case, which means that the integrand in the stochastic integral is nonlinear, i.e., the equation has the form

$$\begin{cases} \mathrm{d}v(t,x) = \left[\mathcal{L}v(t,x) - f(t,x,v(t,x),\sigma(x)^T \frac{\partial}{\partial x} v(t,x))\right] \mathrm{d}t + g(v(t,x))\mathrm{d}B_t, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in (0,T) \times \mathbb{R}^n; \\ v(0,x) = \Phi(x), \qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^n. \end{cases} \tag{0.39}$$

This stochastic partial differential equation is studied by its stochastic interpretation of a nonlinear backward doubly stochastic differential equation driven by the fractional Brownian motion $B$ with Hurst parameter $H \in (0, 1/2)$ and an independent Brownian motion $W$:

$$Y_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\mathrm{d}r + \int_0^s g(Y_r^{t,x})\mathrm{d}B_r - \int_0^s Z_r^{t,x} \downarrow \mathrm{d}W_r, \ s \in [0,t]. \tag{0.40}$$

The nonlinearity makes the main tool used in Chapter 1, the Girsanov transformation, nonapplicable in this case. Hence, we apply the Doss-Sussmann transformation, which was used by Buckdahn and Ma [22] [23] in their study of stochastic viscosity solutions of stochastic partial differential equations driven by a Brownian motion.

We use the same notations for the probability space $(\Omega, \mathcal{F}, P)$ and in addition we denote by $\mathbb{F}_{[t,,T]}^W = \{\mathcal{F}_{[t,T]}^W\}_{t\in[0,T]}$ the backward filtration generated by $\{W_s - W_t, s \in [t,T]\}$ and by $\mathbb{H}$ the filtration $\mathbb{F}_{[t,,T]}^W$ enlarged by the $\sigma$-field generated by $\{B_s\}_{s\in[0,T]}$.

First we give some notations which will be used in what follows:

$C(\mathbb{H},[0,T];\mathbb{R}^m)$ : the space of the $\mathbb{R}^m$-valued continuous processes $\{\varphi_t, t\in[0,T]\}$ such that $\varphi_t$ is $\mathcal{H}_t$-measurable, $t\in[0,T]$;

$\mathcal{M}^2(\mathbb{F}^W,[0,T];\,\mathbb{R}^m)$ : the space of the $\mathbb{R}^m$-valued square-integrable processes $\{\psi_t, t\in[0,T]\}$ such that $\psi_t$ is $\mathcal{F}^W_{[t,T]}$-measurable, $t\in[0,T]$;

$\mathcal{H}^\infty_T(\mathbb{R})$ : the set of $\mathbb{H}$-progressively measurable processes which are almost surely bounded by some real-valued $\mathcal{F}^B_T$-measurable random variable;

$\mathcal{H}^2_T(\mathbb{R}^d)$ : the set of all $\mathbb{R}^d$-valued $\mathbb{H}$-progressively measurable processes $\gamma=\{\gamma_t : t\in[0,T]\}$ such that $\mathbb{E}\left[\int_0^T|\gamma_t|^2\mathrm{d}t|\mathcal{F}^B_T\right]<+\infty$, $\mathbb{P}$-a.s.

Unlike our use of the extended divergence operator as the stochastic integral with respect to the fractional Brownian motion $B$ in [58], we choose here the Russo-Vallois integral. The Russo-Vallois integral is defined in [100] as follows.

**Definition 0.29.** *Let $X$ and $Y$ be two continuous processes. For $\varepsilon>0$, we set*

$$I(\varepsilon,t,X,\mathrm{d}Y)\triangleq\frac{1}{\varepsilon}\int_0^t X(s)(Y(s)-Y(s-\varepsilon))\mathrm{d}s,$$

*Then the Russo-Vallois integral is defined as the uniform limit in probability as $\varepsilon\to 0^+$, if the limit exists.*

As our first result we present the following special form of Itô's formula, which involves both the backward Itô integral with respect to the Brownian motion and the Russo-Vallois integral with respect to the fractional Brownian motion.

**Theorem 0.30.** *Let $\alpha\in C(\mathbb{H},[0,T];\mathbb{R})$ be a process of the form*

$$\alpha_t=\alpha_0+\int_0^t\beta_s\mathrm{d}s+\int_0^t\gamma_s\downarrow\mathrm{d}W_s,\quad t\in[0,T],$$

*where $\beta$ and $\gamma$ are $\mathbb{H}$-adapted processes such that $\mathbb{P}\{\int_0^T|\beta_s|\mathrm{d}s<+\infty\}=1$ and $\mathbb{P}\{\int_0^T|\gamma_s|^2\mathrm{d}s<+\infty\}=1$, respectively. Suppose that $F\in C^2(\mathbb{R}\times\mathbb{R})$. Then the Russo-Vallois integral $\int_0^t\frac{\partial F}{\partial y}(\alpha_s,B_s)\mathrm{d}B_s$ exists for all $0\le t\le T$, and it holds that, $\mathbb{P}$-almost surely, for all $0\le t\le T$,*

$$\begin{aligned}F(\alpha_t,B_t)=&F(\alpha_0,0)+\int_0^t\frac{\partial F}{\partial x}(\alpha_s,B_s)\beta_s\mathrm{d}s+\int_0^t\frac{\partial F}{\partial x}(\alpha_s,B_s)\gamma_s\downarrow\mathrm{d}W_s\\&+\int_0^t\frac{\partial F}{\partial y}(\alpha_s,B_s)\mathrm{d}B_s-\frac{1}{2}\int_0^t\frac{\partial^2F}{\partial x^2}(\alpha_s,B_s)|\gamma_s|^2\mathrm{d}s.\end{aligned}$$

We introduce now the Doss-Sussmann transformation. We denote by $\eta$ the stochastic flow defined as the unique solution of the following stochastic differential equation:

$$\eta(t,y)=y+\int_0^t g(\eta(s,y))\mathrm{d}B_s,\ t\in[0,T].$$

The solution of such a stochastic differential equation can be written as $\eta(t,y)=\alpha(y,B_t)$, where $\alpha(y,z)$ is the solution of the ordinary differential equation

$$\begin{cases}\frac{\partial\alpha}{\partial z}(y,z)=g(\alpha(y,z)),\ z\in\mathbb{R},\\ \alpha(y,0)=y.\end{cases}$$

Under appropriate assumptions we have that $\eta(t,\cdot)=\alpha(\cdot,B_t):\mathbb{R}\to\mathbb{R}$ is a diffeomorphism and we define $\mathcal{E}(t,y):=\eta(t,\cdot)^{-1}(y)=h(y,B_t),(t,y)\in[0,T]\times\mathbb{R}$. We have the following estimates for $\eta$ and $\mathcal{E}$.

**Lemma 0.31.** *There exists a constant $C>0$ depending only on the bound of $g$ and its partial derivatives such that for $\xi=\eta,\mathcal{E}$, it holds that, $P$-a.s., for all $(t,y)$,*

$$\begin{array}{ll} |\xi(t,y)|\le|y|+C|B_t|, & \exp\{-C|B_t|\}\le\left|\frac{\partial}{\partial y}\xi\right|\le\exp\{C|B_t|\},\\ \left|\frac{\partial^2}{\partial y^2}\xi\right|\le\exp\{C|B_t|\}, & \left|\frac{\partial^3}{\partial y^3}\xi\right|\le\exp\{C|B_t|\}. \end{array}$$

Let $(X_s^{t,x})_{0\le s\le t}$ be the unique solution of the following stochastic differential equation:

$$\left\{\begin{array}{l} \mathrm{d}X_s^{t,x}=-b(X_s^{t,x})\mathrm{d}s-\sigma(X_s^{t,x})\downarrow\mathrm{d}W_s,\quad s\in[0,t],\\ X_t^{t,x}=x. \end{array}\right.$$

Here the stochastic integral $\int_0^t\cdot\downarrow\mathrm{d}W_s$ is again understood as the backward Itô one. The Lipschitz condition on $b$ and $\sigma$ guarantees the existence and the uniqueness of the solution $(X_s^{t,x})_{0\le s\le t}$ in $\mathcal{M}^2(\mathbb{F}^W,[0,T];\mathbb{R}^n)$. Our aim is to study the following backward doubly stochastic differential equation:

$$Y_s^{t,x}=\Phi(X_0^{t,x})+\int_0^s f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\mathrm{d}r+\int_0^s g(Y_r^{t,x})\mathrm{d}B_r-\int_0^s Z_r^{t,x}\downarrow\mathrm{d}W_r,\ s\in[0,t].\quad(0.41)$$

We let $(U^{t,x},V^{t,x})$ be the unique solution of the following backward stochastic differential equation:

$$U_s^{t,x}=\Phi(X_0^{t,x})+\int_0^s\tilde{f}(r,X_r^{t,x},U_r^{t,x},V_r^{t,x})\mathrm{d}r-\int_0^s V_r^{t,x}\downarrow\mathrm{d}W_r,\qquad(0.42)$$

where

$$\tilde{f}(t,x,y,z)=\frac{1}{\frac{\partial}{\partial y}\eta(t,y)}\left\{f\left(t,x,\eta(t,y),\frac{\partial}{\partial y}\eta(t,y)z\right)+\frac{1}{2}\mathrm{tr}\left[z^T\frac{\partial^2}{\partial y^2}\eta(t,y)z\right]\right\}.$$

We notice that due to the special form of $\tilde{f}$, our pathwise backward stochastic differential equation becomes a backward stochastic differential equation with quadratic growth; such equations were first studied by Kobylanski [61] [62].

However, unlike the classical backward stochastic differential equation with quadratic growth, we cannot obtain the square integrability of the solution processes, instead we prove the following result.

**Theorem 0.32.** *Under our standard assumptions on the coefficients $\sigma,b,f$ and $\Phi$, equation* (0.42) *admits a unique solution $(U^{t,x},V^{t,x})$ in $\mathcal{H}_t^\infty(\mathbb{R})\times\mathcal{H}_t^2(\mathbb{R}^d)$. Moreover, there exists a positive increasing process $\theta\in L^0(\mathbb{H},\mathbb{R})$ such that*

$$|U_s^{t,x}|\le\theta_s,\quad\mathbb{E}\left[\int_0^\tau|V_s^{t,x}|^2\mathrm{d}s|\mathcal{H}_\tau\right]\le\exp\{\exp\{C\sup_{s\in[0,t]}|B_s|\}\},\ \mathbb{P}-\text{a.s.},$$

*for all $\mathbb{H}$-stopping times $\tau$ $(0\le\tau\le t)$, where $C$ is a constant chosen in an adequate way. Furthermore, the process $(U^{t,x},V^{t,x})$ is $\mathbb{G}$-adapted.*

Now we can perform the Doss-Sussmann transformation and obtain the solution of the fractional backward doubly stochastic differential equation (0.41).

**Theorem 0.33.** *Let us define a new pair of processes* $(U^{t,x}, V^{t,x})$ *by*

$$Y_s^{t,x} = \eta(s, U_s^{t,x}), \quad Z_s^{t,x} = \frac{\partial}{\partial y}\eta(s, U_s^{t,x})V_s^{t,x},$$

*where* $(U^{t,x}, V^{t,x})$ *is the solution of the backward stochastic differential equation* (0.42). *Then* $(Y^{t,x}, Z^{t,x}) \in \mathcal{H}_t^\infty(\mathbb{R}) \times \mathcal{H}_t^2(\mathbb{R}^d)$ *is the solution of backward doubly stochastic differential equation* (0.41).

By applying the Doss-Sussmann transformation, we can also establish the link between the following stochastic partial differential equations: the semilinear stochastic partial differential equation (0.39) driven by the fractional Brownian motion $B$, and the semilinear partial differential equation with random coefficients,

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \mathcal{L}u(t,x) - \tilde{f}(t,x,u(t,x),\sigma(x)^T \frac{\partial}{\partial x}u(t,x)), & (t,x) \in (0,T)\times\mathbb{R}^n; \\ u(0,x) = \Phi(x), & x \in \mathbb{R}^n. \end{cases} \tag{0.43}$$

Motivated by this relation, we define the stochastic viscosity solution of (0.39).

**Definition 0.34.** *A continuous random field* $\hat{u} : \Omega' \times [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is called a (stochastic) viscosity solution of equation* (0.39) *if* $u(t,x) = \mathcal{E}(t, \hat{u}(t,x))$, $(t,x) \in [0,T] \times \mathbb{R}^n$ *is the viscosity solution of equation* (0.43).

Although the above definition may look similar to that of the stochastic viscosity solution defined in Chapter 1, they are not of the same type. Their difference explains by the different tools used in the both chapters. Indeed, while in Chapter 1 the Girsanov transformation changes the sample paths of the fractional Brownian motion, in Chapter 2 the Doss-Sussmann transformation does not change them.

Finally we give a stochastic interpretation for the stochastic viscosity solution of (0.39) by using the solution of the fractional backward doubly stochastic differential equation (0.41).

**Theorem 0.35.** *The stochastic field* $\hat{u} : \Omega' \times [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *defined by* $\hat{u}(t,x) \triangleq \alpha(U_t^{t,x}, B_t)$ *is a (stochastic) viscosity solution of stochastic partial differential equation* (0.39).

Finally, let us present the

**Main results in Chapter 3.** *Regularity of Viscosity Solutions of Integro-Partial Differential Equations of Hamilton-Jacobi-Bellman Type*

This chapter is based on the manuscript [60] submitted for publication.

We consider the following integro-partial differential equation of Hamilton-Jacobi-Bellman type:

$$\begin{cases} \dfrac{\partial}{\partial t}V(t,x) + \inf\limits_{u\in U}\{(\mathcal{L}^u + B^u)V(t,x) + f(t,x,V(t,x),(D_xV\sigma)(t,x), \\ \qquad\qquad\qquad\qquad V(t,x+\beta(t,x,u,\cdot)) - V(t,x),u)\} = 0; \\ V(T,x) = \Phi(x), \end{cases} \tag{0.44}$$

where $U$ is a compact metric space, $\mathcal{L}^u$ is the linear second order differential operator

$$\mathcal{L}^u\varphi(x) = \mathrm{tr}\left(\frac{1}{2}\sigma\sigma^T(t,x,u)D^2_{xx}\varphi(x)\right) + b(t,x,u)\cdot D_x\varphi(x), \quad \varphi \in C^2(\mathbb{R}^d),$$

and $B^u$ is the integro-differential operator:

$$B^u\varphi(x) = \int_E [\varphi(x+\beta(t,x,u,e)) - \varphi(t,x) - \beta(t,x,u,e)\cdot D_x\varphi(x)]\Pi(\mathrm{d}e), \quad \varphi \in C^1_b(\mathbb{R}^d).$$

Here $\Pi$ denotes a finite Lévy measure on a measurable space $(E, \mathcal{B}(E))$, where $E = \mathbb{R}^n \backslash \{0\}$.

In this chapter we study for the above equation the joint regularity (Lipschitz continuity and semiconcavity) of the viscosity solution $V(t,x)$ in $(t,x)$ through its stochastic interpretation as a stochastic control problem composed of a forward and a backward stochastic differential equation. More precisely, let $(t,x) \in [0,T]\times\mathbb{R}^d$, $B = (B_s)_{s\in[t,T]}$ be a $d$-dimensional Brownian motion with initial value zero at time $t$, and let $\mu$ be a Poisson random measure with compensator $\mathrm{d}s\Pi(\mathrm{d}e)$ on $[t,T]\times E$. We denote by $\mathbb{F}^{B,\mu}$ the filtration generated by $B$ and $\mu$, and by $\mathcal{U}^{B,\mu}(t,T)$ the set of all $\mathbb{F}^{B,\mu}$-predictable control processes with values in $U$. It is by now standard that the stochastic differential equation driven by the Brownian motion $B$ and the compensated Poisson random measure $\tilde{\mu}$:

$$X^{t,x,u}_s = x + \int_t^s b(r, X^{t,x,u}_r, u_r)\mathrm{d}r + \int_t^s \sigma(r, X^{t,x,u}_r, u_r)\mathrm{d}B_r + \int_t^s\int_E \beta(r, X^{t,x,u}_{r-}, u_r, e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e),$$

$s \in [t,T]$, has a unique solution under appropriate assumptions for the coefficients. With this stochastic differential equation we associate the backward stochastic differential equation with jumps

$$\begin{aligned} Y^{t,x,u}_s &= \Phi(X^{t,x,u}_T) + \int_s^T f(r, X^{t,x,u}_r, Y^{t,x,u}_r, Z^{t,x,u}_r, U^{t,x,u}_r, u_r)\mathrm{d}r - \int_s^T Z^{t,x,u}_r \mathrm{d}B_r \\ &\quad - \int_s^T\int_E U^{t,x,u}_r(e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e), \quad s\in[t,T]. \end{aligned} \tag{0.45}$$

As concerns the assumptions on the coefficients, we refer to the hypotheses **(H1)-(H5)** in Section 2 and Section 3 of Chapter 3. From Barles, Buckdahn and Pardoux [10] we know that the above backward stochastic differential equation with jumps (0.45) has a unique square integrable solution $(Y^{t,x,u}, Z^{t,x,u}, U^{t,x,u})$. Moreover, since $Y^{t,x,u}$ is $\mathbb{F}^{B,\mu}$-adapted, $Y^{t,x,u}_t$ is deterministic. It follows from Barles, Buckdahn and Pardoux [10] or Pham [95] that the value function

$$V(t,x) = \inf_{u\in\mathcal{U}^{B,\mu}(t,T)} Y^{t,x,u}_t, \quad (t,x)\in[0,T]\times\mathbb{R}^d$$

is the viscosity solution of our integro-partial differential equation.

Since our system involves not only the Brownian motion $B$ but also the Poisson random measure $\mu$, the method of time change for the Brownian motion alone, which was used in [18] and [19] to investigate regularity properties of parabolic second order partial differential equations, is not sufficient for our approach here. We combine the method of time change for the Brownian motion with Kulik's transformation of Poisson random measures (see [65], [66]). To our best knowledge, the use of Kulik's transformation for the study of stochastic control problems is new. Because of the difficulty to obtain suitable $L^p$-estimates of the stochastic integrals with respect the compensated Poisson random measure (see, for example, Pham [95]), we have to restrict ourselves to the case of a finite Lévy measure $\Pi(E) < +\infty$. The more general case where $\int_E (1 \wedge |e|^2)\Pi(\mathrm{d}e) < +\infty$ remains still open.

By applying the method of time change for the Brownian motion $B$ combined with Kulik's transformation for the Poisson random measure $\mu$, we obtain the following main results:

**Theorem 0.36.** *Let $\delta \in (0, T)$ be arbitrarily fixed. Under our assumptions ($\boldsymbol{H1}$) and ($\boldsymbol{H2}$), the value function $V(\cdot,\cdot)$ is jointly Lipschitz continuous on $[0, T-\delta] \times \mathbb{R}^d$, i.e., for some constant $C_\delta$ we have that, for all $(t_0, x_0), (t_1, x_1) \in [0, T-\delta] \times \mathbb{R}^d$:*

$$|V(t_0, x_0) - V(t_1, x_1)| \leq C_\delta(|t_0 - t_1| + |x_0 - x_1|).$$

**Theorem 0.37.** *Under the assumptions ($\boldsymbol{H1}$) $-$ ($\boldsymbol{H5}$), for every $\delta \in (0, T)$, there exists some constant $C_\delta > 0$ such that, for all $(t_0, x_0), (t_1, x_1) \in [0, T-\delta] \times \mathbb{R}^d$, and for all $\lambda \in [0, 1]$:*

$$\lambda V(t_0, x_0) + (1-\lambda)V(t_1, x_1) - V(t_\lambda, x_\lambda) \leq C_\delta \lambda(1-\lambda)(|t_0 - t_1|^2 + |x_0 - x_1|^2),$$

*where $t_\lambda = \lambda t_0 + (1-\lambda)t_1$ and $x_\lambda = \lambda x_0 + (1-\lambda)x_1$.*

At the end let us show through a counterexample (see also Buckdahn, Cannarsa and Quincampoix [18]) that, in general, the above properties of Lipschitz continuity and semiconcavity do not hold true for the whole space $[0, T] \times \mathbb{R}^d$.

**Example 0.38.** *We study the problem*

$$X_s^{t,x} = x + B_s, \quad s \in [t, T], \quad x \in \mathbb{R};$$

$$Y_s^{t,x} = -\mathbb{E}\left[\left|X_T^{t,x}\right| |\mathcal{F}_s\right] = -\mathbb{E}\left[|x + B_T||\mathcal{F}_s\right], \quad s \in [t, T],$$

*without control neither jumps. Then*

$$V(t, x) = Y_t^{t,x} = -\mathbb{E}\left[|x + B_T|\right],$$

*and, for $x = 0$, recalling that $B$ is a Brownian motion with $B_t = 0$, we have*

$$V(t, 0) = -\mathbb{E}[|B_T|] = -\sqrt{\frac{2}{\pi}}\sqrt{T-t}, \quad t \in [0, T].$$

*Obviously, $V(\cdot, x)$ is neither Lipschitz nor semiconcave in $t$ for $t = T$. However, $V$ is jointly Lipschitz and semiconcave on $[0, T-\delta] \times \mathbb{R}$, for $\delta \in (0, T)$.*

# Chapter 1

# Semilinear Backward Doubly Stochastic Differential Equations and SPDEs Driven by Fractional Brownian Motion with Hurst Parameter in (0,1/2)

**Abstract**: We study the existence of a unique solution to semilinear fractional backward doubly stochastic differential equation driven by a Brownian motion and a fractional Brownian motion with Hurst parameter less than 1/2. Here the stochastic integral with respect to the fractional Brownian motion is the extended divergence operator and the one with respect to Brownian motion is Itô's backward integral. For this we use the technique developed by R. Buckdahn [16] to analyze stochastic differential equations on the Wiener space, which is based on the Girsanov theorem and the Malliavin calculus, and we reduce the backward doubly stochastic differential equation to a backward stochastic differential equation driven by the Brownian motion. We also prove that the solution of semilinear fractional backward doubly stochastic differential equation defines the unique stochastic viscosity solution of a semilinear stochastic partial differential equation driven by a fractional Brownian motion.

**Key words:** fractional Brownian motion; semilinear fractional backward doubly stochastic differential equation; semilinear stochastic partial differential equation; extended divergence operator; Girsanov transformation; stochastic viscosity solution.

**MSC 2000**: 60G22; 60H15; 35R60.

# 1 Introduction

This chapter investigates semilinear fractional backward doubly stochastic differential equations (BDSDEs) and semilinear stochastic partial differential equations (SPDEs) driven by fractional Brownian motion. Fractional Brownian motions (fBMs) and backward stochastic differential equations (BSDEs) have been extensively studied in recent twenty years. However, up to now there are only few works that combine both topics. Bender [11] considered a class of linear fractional BSDEs and gave their explicit solutions. There are two major obstacles depending on the properties of fBM: Firstly, the fBM is not a semimartingale except for the case of Brownian motion (Hurst parameter $H = 1/2$), hence the classical Itô calculus which is based on semimartingales cannot be transposed directly to the fractional case. Secondly, there is no martingale representation theorem with respect to the fBM. However, such a martingale representation property with respect to the Brownian motion is the main tool in BSDE theory. Hu and Peng's paper [50] overcame the second obstacle for the case of $H > 1/2$ by using the quasi-conditional expectation and by studying nonlinear fractional BSDEs in a special case only.

Nevertheless, there are many papers considering stochastic differential equations driven by fractional Brownian motion with Hurst parameter $H > 1/2$ ([12], [77] and references therein) or $H < 1/2$ ([68]), or covering both cases ([57]). For the case $H < 1/2$, one of the main difficulties is how to properly define the stochastic integral with respect to the fBM. In the paper of Cheridito and Nualart [27], and then generalized by León and Nualart [67], the authors have defined the extended divergence operator which can be applied to the fBM for $H < 1/2$ as a special case. In this chapter we will use such definition for the stochastic integration with respect to the fBM, and then apply the non-anticipating Girsanov transformation developed by Buckdahn [16] to transform the semilinear fractional doubly backward stochastic differential equation driven by the Brownian motion $W$ and the fractional Brownian motion $B$:

$$Y_t = \xi + \int_0^t f(s, Y_s, Z_s)\mathrm{d}s - \int_0^t Z_s \downarrow \mathrm{d}W_s + \int_0^t \gamma_s Y_s \mathrm{d}B_s, \quad t \in [0, T], \tag{1.1}$$

into a pathwise (in the sense of fBM) BSDE

$$\widehat{Y}_t = \xi + \int_0^t f\left(s, \widehat{Y}_s \varepsilon_s(T_s), \widehat{Z}_s \varepsilon_s(T_s)\right) \varepsilon_s^{-1}(T_s)\mathrm{d}s - \int_0^t \widehat{Z}_s \downarrow \mathrm{d}W_s, \quad t \in [0, T]. \tag{1.2}$$

More precisely, the solutions $(Y, Z)$ and $\left(\widehat{Y}, \widehat{Z}\right)$ are linked together by the following relations:

$$\{(Y_t, Z_t), t \in [0, T]\} = \left\{\left(\widehat{Y}_t(A_t)\varepsilon_t, \widehat{Z}_t(A_t)\varepsilon_t\right), t \in [0, T]\right\}$$

and

$$\left\{\left(\widehat{Y}_t, \widehat{Z}_t\right), t \in [0, T]\right\} = \left\{\left(Y_t(T_t)\varepsilon_t^{-1}(T_t), Z_t(T_t)\varepsilon_t^{-1}(T_t)\right), t \in [0, T]\right\},$$

where $A_t$ and $T_t$ are Girsanov transformations. It is worth noting that such kind of method was also used by Jien and Ma [57] to deal with fractional stochastic differential equations.

It is well known that the solution of a BSDE can be regarded as a viscosity solution of an associated parabolic partial differential equation (PDE) (cf. [42] [82] and [92]), and the solution of BDSDE driven by two independent Brownian motions can be regarded as a stochastic viscosity solution of an SPDE (cf. [22] and [84]). So it is natural to consider the relationship between the solutions of our fractional BDSDE and the associated SPDE. We show that the solution of the above fractional BDSDE, which is a random field, is a stochastic viscosity solution of an SPDE driven by our fractional Brownian motion. To be more precise, the random field $u(t,x)$ defined by the solution of a fractional BSDE over the time interval $[0,t]$ instead of $[0,T]$, see (4.4), will be shown to possess a continuous version and to be the stochastic viscosity solution of the following semilinear SPDE

$$\begin{cases} \mathrm{d}u(t,x) = [\mathcal{L}u(t,x) + f(t,x,u(t,x),\nabla_x u(t,x)\sigma(x))]\,\mathrm{d}t + \gamma_t u(t,x)\mathrm{d}B_t, \quad t\in[0,T], \\ u(0,x) = \Phi(x). \end{cases} \tag{1.3}$$

Taking a Brownian motion instead of the fBM, equation (1.1) becomes a classical BDSDE, which was first studied by Pardoux and Peng [84]. The associated stochastic viscosity solution of SPDE (1.3) (with $H = 1/2$) was studied by Buckdahn and Ma [22]. Let us point out that, unlike [22] considering the stochastic integral with respect to $B$ ($H = 1/2$) as the Stratonovich one and using a Doss-Sussman transformation as main tool, we have to do here with an extended divergence operator ($H < 1/2$), which compels us to use the Girsanov transformation as main argument. However, this restricts us to semilinear equations. We will investigate the general case in Chapter 3, but with a different approach.

The chapter is organized as follows: In Section 2 we recall some preliminaries which will be used in what follows: Malliavin calculus for fractional Brownian motion, the definition of extended divergence operator and the Girsanov transformation. In Section 3 we prove existence and uniqueness results for stochastic differential equations driven by a fractional Brownian motion and backward doubly stochastic differential equations driven by a Brownian motion as well as a fractional Brownian motion. The relationship between the stochastic viscosity solution of the stochastic partial differential equation (1.3) driven by fractional Brownian motion and that of an associated pathwise partial differential equation is given in Section 4.

# 2 Preliminaries

The purpose of this section is to describe the framework that will be used in this chapter. Namely, we introduce briefly the transformations on the Wiener space, appearing in the construction of the solution to our equations, some preliminaries of the Malliavin calculus for the fBM, and the left and right-sided fractional derivatives, which are needed to understand the definition of the extension of the divergence operator with respect to the fBM.

## 2.1 Fractional Calculus

For a detailed account on the fractional calculus theory, we refer, for instance, to Samko et al. [102].

Let $T > 0$ denote a positive time horizon, fixed throughout our chapter, and let $f : [0,T] \to \mathbb{R}$ be an integrable function, and $\alpha \in (0,1)$. The *right–sided fractional integral* of $f$ of order $\alpha$ is given by

$$I^{\alpha}_{T-}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^T \frac{f(u)}{(u-x)^{1-\alpha}} \mathrm{d}u, \quad \text{for a.a. } x \in [0,T].$$

Note that $I^{\alpha}_{T-}(f)$ is well-defined because the Fubini theorem implies that it is a function in $L^p([0,T])$, $p \geq 1$, whenever $f \in L^p([0,T])$.

We denote by $I^{\alpha}_{T-}(L^p)$, $p \geq 1$, the family of all functions $f \in L^p([0,T])$ such that

$$f = I^{\alpha}_{T-}(\varphi), \tag{2.1}$$

for some $\varphi \in L^p([0,T])$. Samko et al. [102] (Theorem 13.2) provide a characterization of the space $I^{\alpha}_{T-}(L^p)$, $p > 1$. Namely, a measurable function $f$ belongs to $I^{\alpha}_{T-}(L^p)$ (i.e., it satisfies (2.1)) if and only if $f \in L^p([0,T])$ and the integral

$$\int_{s+\varepsilon}^T \frac{f(s) - f(u)}{(u-s)^{1+\alpha}} \mathrm{d}u \tag{2.2}$$

converges in $L^p([0,T])$ as $\varepsilon \downarrow 0$. In this case a function $\varphi$ satisfying (2.1) coincides with the right–sided fractional derivative

$$(D^{\alpha}_{T-}f)(s) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(s)}{(T-s)^{\alpha}} + \alpha \int_s^T \frac{f(s) - f(u)}{(u-s)^{1+\alpha}} \mathrm{d}u \right), \tag{2.3}$$

where the integral is the $L^p([0,T])$–limit of (2.2). Moreover, it has also been shown in [102] (Lemma 2.5) that there is at most one solution $\varphi$ to the equation (2.1). Consequently, the inversion formulae

$$I^{\alpha}_{T-}\left(D^{\alpha}_{T-}f\right) = f, \quad \text{for all } f \in I^{\alpha}_{T-}(L^p),$$

and

$$D^{\alpha}_{T-}\left(I^{\alpha}_{T-}(f)\right) = f, \quad \text{for all } f \in L^1([0,T])$$

hold.

Similarly, the *left–sided fractional integral and the derivative* of $f$ of order $\alpha$, which are given, respectively, by

$$I^{\alpha}_{0+}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(u)}{(x-u)^{1-\alpha}} \mathrm{d}u, \quad \text{for a.a. } x \in [0,T],$$

and

$$(D^{\alpha}_{0+}f)(s) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(s)}{s^{\alpha}} + \alpha\int_0^s \frac{f(s)-f(u)}{(s-u)^{1+\alpha}}\mathrm{d}u\right), \tag{2.4}$$

satisfy the inversion formulae

$$I^{\alpha}_{0+}\left(D^{\alpha}_{0+}f\right) = f, \quad \text{for all } f \in I^{\alpha}_{0+}(L^p),$$

and

$$D^{\alpha}_{0+}\left(I^{\alpha}_{0+}(f)\right) = f, \quad \text{for all } f \in L^1([0,T]).$$

## 2.2 Fractional Brownian Motion

In this subsection we will recall some basic facts of the fBM. The reader can consult Mishura [77] and Nualart [79] and the references therein for a more complete presentation of this subject.

Henceforth $(\Omega, \mathcal{F}, P)$ and $W^0 = \{W_t^0 : t \in [0,T]\}$ are the canonical Wiener space on the interval $[0,T]$ and the canonical Wiener process, respectively. This means, in particular, that $\Omega = C_0([0,T])$ is the space of all continuous functions $h : [0,T] \mapsto \mathbb{R}$ with $h(0) = 0$, $\{W_t^0(\omega) = \omega(t), t \in [0,T], \omega \in \Omega\}$ is the coordinate process on $\Omega$, $P$ is the Wiener measure on $(\Omega, \mathcal{B}(\Omega))$ and $\mathcal{F}$ is the completion of $\mathcal{B}(\Omega) = \sigma\{W_s^0, s \in [0,T]\}$ with respect to $P$. The noise under consideration is the process

$$B_t = \int_0^t K_H(t,s)\mathrm{d}W_s^0, \quad t \in [0,T],$$

where $K_H$ is the kernel of the fBM with parameter $H \in (0,1/2)$. That is,

$$K_H(t,s) = C_H \left[ \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-1/2\frac{1}{2}} - (H-1/2)s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}\mathrm{d}u \right],$$

where $C_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H,H+1/2)}}$. The process $B = \{B_t : t \in [0,T]\}$ is an fBM with Hurst parameter $H$, defined on $(\Omega, \mathcal{F}, P)$, i.e., $B$ is a Gaussian process with zero mean and covariance function

$$\begin{aligned} R_H(t,s) :=& E\left[B_t B_s\right] = \int_0^{s\wedge t} K_H(t,r)K_H(s,r)\mathrm{d}r \\ =& \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \in [0,T]. \end{aligned}$$

Let $\mathcal{H}_H$ be the Hilbert space defined as the completion of the space $L^e(0,T)$ of step functions on $[0,T]$ with respect to the norm generated by the inner product

$$\left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{\mathcal{H}_H} = R_H(t,s) = E\left[B_t B_s\right], \quad t,s \in [0,T].$$

From Pipiras and Taqqu [96] (see also [79]), it follows that $\mathcal{H}_H$ coincides with the Hilbert space

$$\begin{aligned} \Lambda_T^{1/2-H} :=& \Big\{ f : [0,T] \to \mathbb{R} : \exists\, \varphi_f \in L^2(0,T) \text{ such that} \\ & \qquad f(u) = u^{1/2-H} I_{T-}^{1/2-H}\left(s^{H-1/2}\varphi_f(s)\right)(u) \Big\} \end{aligned}$$

equipped with scalar product

$$\langle f, g \rangle_{\Lambda_T^{1/2-H}} = C_H^2 \Gamma(H+1/2)^2 \langle \varphi_f, \varphi_g \rangle_{L^2(0,T)}.$$

So $1_{[0,t]} \mapsto B_t$ can be extended to an isometry of $\Lambda_T^{1/2-H}$ onto a Gaussian closed subspace of $L^2(\Omega, \mathcal{F}, P)$. This isometry is denoted by $\varphi \mapsto B(\varphi)$. Moreover, by the transfer principle (see Nualart [79]), the map $\mathcal{K} : \Lambda_T^{1/2-H} \to L^2([0,T])$, defined by (see (2.3))

$$(\mathcal{K}\varphi)(s) = C_H \Gamma(H+1/2) s^{1/2-H} \left(D_{T-}^{1/2-H} u^{H-1/2}\varphi(u)\right)(s), \quad s \in [0,T],$$

is an isometry such that

$$B(\varphi) = \int_0^T (\mathcal{K}\varphi)(s)\mathrm{d}W_s^0 \quad \text{and} \quad \mathcal{K}1_{[0,t]} = K_H(t,\cdot)1_{[0,t]},\ t \in [0,T].$$

Using the properties of $\mathcal{K}$, Cheridito and Nualart [27] have extended the domain of the divergence operator with respect to the fBM $B$. This extension of the divergence operator in the sense of Malliavin calculus holds also true for some suitable Gaussian processes (see León and Nualart [67]). For $B$, this extension is introduced as follows.

The following result identifies the adjoint of the operator $\mathcal{K}$ (see [67]). It uses the left–sided fractional derivative $D^{\alpha}_{0+}$ defined in (2.4).

**Proposition 2.1.** *Let $g : [0,t] \to \mathbb{R}$ be a function such that $u \mapsto u^{1/2-H}g(u)$ belongs to $I_{0+}^{1/2-H}(L^q([0,a]))$ for some $q > (1/2-H)^{-1} \vee H^{-1}$. Then, $g \in$ Dom $\mathcal{K}^*$, and for all $u \in [0,T]$,*

$$(\mathcal{K}^*g)(u) = C_H\Gamma(H+1/2)u^{H-1/2}D_{0+}^{1/2-H}\left(s^{1/2-H}g(s)\right)(u).$$

Let $\mathcal{S}$ (resp. $\mathcal{S}_{\mathcal{K}}$) denote the class of smooth random variables of the form

$$F = f(B(\varphi_1), \ldots, B(\varphi_n)), \tag{2.5}$$

where $\varphi_1, \ldots, \varphi_n$ are in $\Lambda_T^{1/2-H}$ (resp. in the domain of the operator $\mathcal{K}^*\mathcal{K}$) and $f \in C_p^\infty(\mathbb{R}^n)$. Here, $C_p^\infty(\mathbb{R}^n)$ is the set of $C^\infty$ functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $f$ and all its partial derivatives have polynomial growth.

The derivative of the smooth random variable $F$ given by (2.5) is the $\Lambda_T^{1/2-H}$–valued random variable $DF$ defined by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_i.$$

Now we can introduce the stochastic integral that we use in this chapter. It is an extended divergence operator with respect to $B$.

**Definition 2.2.** *Let $u \in L^2(\Omega, \mathcal{F}, P; L^2([0,T]))$. We say that $u$ belongs to Dom $\delta$ if there exists $\delta(u) \in L^2(\Omega)$ such that*

$$E\left[\langle \mathcal{K}^*\mathcal{K}DF, u\rangle_{L^2([0,T])}\right] = E\left[F\delta(u)\right], \quad \text{for every} \quad F \in \mathcal{S}_{\mathcal{K}}. \tag{2.6}$$

*In this case, the random variable $\delta(u)$ is called the* extended divergence *of $u$.*

**Remark 2.3.** *i) In [67] it is shown that the domain of $\mathcal{K}^*\mathcal{K}$ is a dense subset of $\Lambda_T^{1/2-H}$. Therefore, there is at most one square integrable random variable $\delta(u)$ such that* (2.6) *holds.*

(*ii*) *In [27] and [67], it is proven that the domain of $\delta$ is bigger than that of the classical divergence operator, which is defined by the chaos decomposition approach* (*see Nualart [80]*).

(*iii*) *In Section* 3 *and Section* 4*, we use the convention*

$$\int_0^t u_s \mathrm{d}B_s = \delta(u1_{[0,t]}),$$

*whenever $u1_{[0,t]} \in$ Dom $\delta$.*

## 2.3 Girsanov Transformations

In this section we introduce the Girsanov transformations on $\Omega$, which we consider in this chapter.

In what follows, we assume that $\gamma$ is a square-integrable function satisfying the following hypothesis:

**(H1)** $\gamma 1_{[0,t]}$ belongs to $\Lambda_T^{1/2-H}$, for every $t \in [0, T]$.

We emphasize that León and San Martín [68] (Lemma 2.3) have shown the existence of square-integrable functions satisfying the above hypothesis.

Now, for $t \in [0, T]$, we consider the transformations on $\Omega$ of the form

$$T_t(\omega) = \omega + \int_0^{\cdot \wedge t} (\mathcal{K}\gamma 1_{[0,t]})(r)\mathrm{d}r$$

and

$$A_t(\omega) = \omega - \int_0^{\cdot \wedge t} (\mathcal{K}\gamma 1_{[0,t]})(r)\mathrm{d}r.$$

Notice that $A_t T_t$ and $T_t A_t$ are the identity operator of $\Omega$, and that the Girsanov theorem leads to write

$$B(\varphi)(T_t) = B(\varphi) + \int_0^T (\mathcal{K}\gamma 1_{[0,t]})(r)(\mathcal{K}\varphi)(r)\mathrm{d}r = B(\varphi) + \int_0^t \gamma_r (\mathcal{K}^*\mathcal{K}\varphi)(r)\mathrm{d}r,$$

for all $\varphi \in \mathrm{Dom}\ (\mathcal{K}^*\mathcal{K})$, and

$$E[F] = E[F(A_t)\varepsilon_t],$$

with

$$\varepsilon_t = \exp\left(\int_0^t \gamma_r \mathrm{d}B_r - \frac{1}{2}\int_0^t ((\mathcal{K}\gamma 1_{[0,t]})(r))^2 \mathrm{d}r\right).$$

We will need the following estimate of the above exponential of the integral with respect to the fractional Brownian motion:

**Lemma 2.4.** *Let $\gamma : [0, T] \mapsto \mathbb{R}$, $\gamma \in L_p[0, T] \cap D_p^B[0, T]$, for some $p > 1/H$, where*

$$D_p^B[0, T] = \{\gamma : [0, T] \mapsto \mathbb{R} | \int_0^T (\int_x^T \varphi(x, t)\mathrm{d}t)^p \mathrm{d}x < \infty\}$$

*and we set $\varphi(x, t) = \frac{|\gamma(t)-\gamma(x)|}{(t-x)^{3/2-H}} 1_{\{0<x<t\leq T\}}$. Then there exists a constant $C(H, p)$ only depending on $H$ and $p$, such that*

$$E\left[\exp\left\{\sup_{0\leq 0\leq T}\left|\int_0^t \gamma_s \mathrm{d}B_s\right|\right\}\right] \leq 2\exp\left\{1/2\left(C(H, p)G_p(0, T, \gamma) + 4\sqrt{2}\right)^2\right\}, \qquad (2.7)$$

*where $G_p(0, T, \gamma) := \|\gamma\|_{L^p[0,T]} \cdot T^{H-1/p} + T^{1/2-1/p}\left(\int_0^T\left(\int_x^T \varphi(x, t)\mathrm{d}t\right)^p\right)^{1/p}$.*

*Proof:* Let $I_T^* = \sup_{0\leq t\leq T} |\int_0^t \gamma_s \mathrm{d}B_s|$. We denote by $\rho$ the semi-metric on $[0, T]$ generated by the process $\int_0^\cdot \gamma_s \mathrm{d}B_s$: $\rho^2(s, t) = E\left|\int_s^t \gamma_r \mathrm{d}B_r\right|^2$. For any $\delta > 0$, we denote by $\mathcal{N}([0, T], \varepsilon)$ the minimum number of centers of closed $\delta$-balls covering $[0, T]$, then the Dubley integral

$D(T,\delta)$ is defined by $\int_0^\delta |\log \mathcal{N}([0,T],r)|^{1/2}\mathrm{d}r$. According to Lifshitz [73] Theorem 1, P.141, and its corollary, for all $r > 4\sqrt{2}D(T,\lambda/2)$, we have the inequality

$$P\{I_T^* > r\} \leq 2\left(1-\Phi\left(\frac{r-4\sqrt{2}D(T,\lambda/2)}{\lambda}\right)\right), \tag{2.8}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x \exp\{-y^2/2\}\mathrm{d}y$ and $\lambda^2 = \sup_{t\in[0,T]} E\left[\left(\int_0^t \gamma_s \mathrm{d}B_s\right)^2\right]$.

Since $E\left[\exp\{I_T^*\}\right] = 1 + \int_0^{+\infty}\exp\{x\}P(I_T^* > x)\mathrm{d}x$, by using the estimate of (2.8) we have

$$\begin{aligned}
&E\left[\exp\{I_T^*\}\right]\\
=&1+\int_0^{4\sqrt{2}D(T,\lambda/2)}\exp\{x\}P(I_T^*>x)\mathrm{d}x+\int_{4\sqrt{2}D(T,\lambda/2)}^{+\infty}\exp\{x\}P(I_T^*>x)\mathrm{d}x\\
\leq&\exp\left\{4\sqrt{2}D(T,\lambda/2)\right\}+2\int_{4\sqrt{2}D(T,\lambda/2)}^{+\infty}\exp\{x\}\left(1-\Phi\left(\frac{x-4\sqrt{2}D(T,\lambda/2)}{\lambda}\right)\right)\mathrm{d}x\\
=&\exp\left\{4\sqrt{2}D(T,\lambda/2)\right\}+2\int_0^{+\infty}\exp\left\{4\sqrt{2}D(T,\lambda/2)+x\right\}\\
&\qquad\left(1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x/\lambda}\exp\{-y^2/2\}\mathrm{d}y\right)\mathrm{d}x\\
\leq&2\exp\left\{\lambda^2/2+4\sqrt{2}D(T,\lambda/2)\right\}.
\end{aligned}$$

Moreover, from Theorem 1.10.6 of Mishura[77] and its proof we know that

$$\lambda \leq C_1(H,p)G_p(0,T,\gamma)$$

and

$$D(T,\lambda/2) \leq C_2(p)G_p(0,T,\gamma).$$

By substituting them to the former inequality, we easily get the wished result. ∎

# 3 Semilinear Fractional SDEs and Fractional Backward Doubly SDEs

## 3.1 Fractional Anticipating Semilinear Equations

In this subsection we discuss the existence and uniqueness of solutions to anticipating semilinear equations driven by a fractional Brownian motion $B$ with Hurst parameter $H \in (0,1/2)$. This type of equation was studied by Jien and Ma [57], and since it motivates the approach in our work, we give it in details.

We consider the fractional anticipating equation

$$X_t = \xi + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \gamma_s X_s \mathrm{d}B_s, \quad t \in [0,T]. \tag{3.1}$$

Here $\xi \in L^p(\Omega)$, $p > 2$, and $b : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that:

**(H2)** There exist $\nu \in L^1([0,T])$, $\nu \geq 0$, and a positive constant $L$ such that

$$\begin{aligned} |b(\omega,t,x) - b(\omega,t,y)| &\leq \nu_t|x-y|, \quad \int_0^T \nu_t \mathrm{d}t \leq L, \\ |b(\omega,t,0)| &\leq L, \end{aligned}$$

for all $x, y \in \mathbb{R}$ and almost all $\omega \in \Omega$.

We observe that the above assumption guarantees that the pathwise equation

$$\zeta_t(x) = x + \int_0^t \varepsilon_s^{-1}(T_s) b(T_s, s, \varepsilon_s(T_s)\zeta_s(x))\mathrm{d}s, \quad t \in [0,T],$$

has a unique solution. Henceforth, we denote it by $\zeta$.

Now we can state the existence of a unique solution equation (3.1):

**Theorem 3.1.** *Under Hypotheses (**H1**) and (**H2**), the process*

$$X_t = \varepsilon_t \zeta_t(A_t, \xi(A_t)) \tag{3.2}$$

*is the unique solution in $L^2(\Omega \times [0,T])$ of the equation (3.1), such that $\gamma X 1_{[0,t]} \in$ Dom $\delta$, for all $t \in [0,T]$.*

*Proof:* We first show that the process $X$ given in (3.2) is a solution of equation (3.1). For this we first observe that $X$ belongs to $L^2(\Omega \times [0,T])$ and we let $F \in \mathcal{S}_{\mathcal{K}}$. Then, by the integration by parts formula and the Girsanov theorem, together with the fact that $\frac{\mathrm{d}F(T_t)}{\mathrm{d}t} = \gamma_t(\mathcal{K}^*\mathcal{K}DF)(T_t,t)$, we have

$$\begin{aligned} E\left[FX_t - F\xi\right] &= E\left[F(T_t)\zeta_t(\xi) - F\zeta_0(\xi)\right] = E\left[\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(F(T_s)\zeta_s(\xi))\mathrm{d}s\right] \\ &= \int_0^t \gamma_s E\left[(\mathcal{K}^*\mathcal{K}DF)(T_s,s)\zeta_s(\xi)\right]\mathrm{d}s + \int_0^t E\left[F(T_s)\varepsilon_s^{-1}(T_s)b(T_s,s,\varepsilon_s(T_s)\zeta_s(\xi))\right]\mathrm{d}s \\ &= \int_0^t \left(\gamma_s E\left[(\mathcal{K}^*\mathcal{K}DF)(s)X_s\right] + E\left[Fb(s,X_s)\right]\right)\mathrm{d}s. \end{aligned}$$

Hence, since $\gamma X 1_{[0,t]}$ is square integrable, Definition 2.2 implies that $\gamma X 1_{[0,t]}$ belongs to Dom $\delta$ and the equality in (3.1) holds, for all $t \in [0,T]$.

Now we deal with the uniqueness of equation (3.1). For this end, let $Y$ be another solution of equation (3.1), $F \in \mathcal{S}_{\mathcal{K}}$ and $t \in [0,T]$. Then,

$$E\left[Y_t F(A_t)\right] = E\left[\xi F(A_t)\right] + E\left[\int_0^t F(A_t) b(s, Y_s) \mathrm{d}s\right] + E\left[\int_0^t \gamma_s Y_s \left(\mathcal{K}^* \mathcal{K} D F(A_t)\right)(s) \mathrm{d}s\right].$$

Therefore, the integration by parts formula, Fubini's theorem as well as the fact that $\frac{\mathrm{d}F(A_s)}{\mathrm{d}s} = -\gamma_s (\mathcal{K}^* \mathcal{K} D F(A_s))(s)$ yield

$$\begin{aligned}
&E\left[Y_t F\right] \\
=&E\left[\xi F(A_t)\right] - E\left[\xi \int_0^t \gamma_s (\mathcal{K}^* \mathcal{K} D F(A_s))(s) \mathrm{d}s\right] \\
&+ E\left[\int_0^t F(A_s) b(s, Y_s) \mathrm{d}s\right] - E\left[\int_0^t \gamma_r (\mathcal{K}^* \mathcal{K} D F(A_r))(r) \int_0^r b(s, Y_s) \mathrm{d}s \mathrm{d}r\right] \\
+&E\left[\int_0^t \gamma_s Y_s (\mathcal{K}^* \mathcal{K} D F(A_s))(s) \mathrm{d}s - \int_0^t \int_0^r \gamma_r (\mathcal{K}^* \mathcal{K} D((\mathcal{K}^* \mathcal{K} D F(A_r))(s)))(r) \gamma_s Y_s \mathrm{d}s \mathrm{d}r\right].
\end{aligned}$$

Hence, by using that $Y$ is a solution of (3.1), Definition 2.2 and the relation

$$(\mathcal{K}^* \mathcal{K} D((\mathcal{K}^* \mathcal{K} D F(A_r))(s)))(r) = (\mathcal{K}^* \mathcal{K} D((\mathcal{K}^* \mathcal{K} D F(A_r))(r)))(s),$$

we obtain

$$E\left[Y_t F(A_t)\right] = E\left[\xi F\right] + E\left[\int_0^t F(A_s) b(s, Y_s) \mathrm{d}s\right].$$

Consequently, by using the Girsanov theorem again, we get

$$Y_t(T_t) \varepsilon_t^{-1}(T_t) = \xi + \int_0^t b(T_s, s, Y_s(T_s)) \varepsilon_s^{-1}(T_s) \mathrm{d}s,$$

which implies that $Y_t(T_t)\varepsilon_t^{-1}(T_t) = \zeta(\xi)$. That is, $Y_t = \varepsilon_t \zeta_t(A_t, \xi(A_t))$, and therefore the proof is complete. ■

## 3.2 Fractional Backward Doubly Stochastic Differential Equations

In this section we state some of the main results of this chapter. Namely, the existence and uniqueness of backward doubly stochastic differential equations driven by both a fractional Brownian motion $B$ and a standard Brownian motion $W$.

Let $\{B_t : 0 \le t \le T\}$ be a one-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$, defined on the classical Wiener space $(\Omega', \mathcal{F}^B, P^B)$ with $\Omega' = C_0([0,T];\mathbb{R})$, and $\{W_t = (W_t^1, W_t^2, \cdots, W_t^d) : 0 \le t \le T\}$ a d-dimensional canonical Brownian motion defined on the classical Wiener space $(\Omega'', \mathcal{F}^W, P^W)$ with $\Omega'' = C_0([0,T];\mathbb{R}^d)$. We put $(\Omega, \mathcal{F}^0, P) = (\Omega', \mathcal{F}^B, P^B) \otimes (\Omega'', \mathcal{F}^W, P^W)$ and let $\mathcal{F} = \mathcal{F}^0 \vee \mathcal{N}$, where $\mathcal{N}$ is the class of the $P$-null sets. We denote again by $B$ and $W$ their canonical extension from $\Omega'$ and $\Omega''$, respectively, to $\Omega$.

We let $\mathcal{F}_{t,T}^W = \sigma\{W_T - W_s, t \le s \le T\} \vee \mathcal{N}$, $\mathcal{F}_t^B = \sigma\{B_s, 0 \le s \le t\} \vee \mathcal{N}$, and $\mathcal{G}_t = \mathcal{F}_{t,T}^W \vee \mathcal{F}_t^B, t \in [0,T]$. Let us point out that $\mathcal{F}_{t,T}^W$ is decreasing and $\mathcal{F}_t^B$ is increasing in $t$, but $\mathcal{G}_t$ is neither decreasing nor increasing. We denote the family of $\sigma$-fields $\{\mathcal{G}_t\}_{0\le t\le T}$ by $\mathbb{G}$. Moreover, we shall also introduce the backward filtrations $\mathbb{H} = \{\mathcal{H}_t = \mathcal{F}_{t,T}^W \vee \mathcal{F}_T^B\}_{t\in[0,T]}$ and $\mathbb{F}^W = \{\mathcal{F}_{t,T}^W\}_{t\in[0,T]}$.

Let $\mathcal{S}'_{\mathcal{K}}$ denote the class of smooth random variables of the form

$$F = f(B(\varphi_1), \dots, B(\varphi_n), W(\psi_1), \dots, W(\psi_m)), \tag{3.3}$$

where $\varphi_1, \dots, \varphi_n$ are elements of the domain of the operator $\mathcal{K}^*\mathcal{K}$, $\psi_1, \dots, \psi_m \in C([0,T], \mathbb{R}^d)$, $f \in C_p^\infty(\mathbb{R}^{n+m})$ and $n, m \ge 1$. Here, $C_p^\infty(\mathbb{R}^{n+m})$ is the set of all $C^\infty$ functions $f : \mathbb{R}^{n+m} \to \mathbb{R}$ such that $f$ and all its partial derivatives have polynomial growth.

The Malliavin derivative of the smooth random variable $F$ w.r.t. $B$ is the $\Lambda_T^{1/2-H}$–valued random variable $D^B F$ defined by

$$D^B F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(\varphi_1), \dots, B(\varphi_n), W(\psi_1), \dots, W(\psi_m))\varphi_i,$$

and the Malliavin derivative $D^W F$ of the smooth random variable $F$ w.r.t. $W$ is given by

$$D^W F = \sum_{i=1}^m \frac{\partial f}{\partial x_{n+i}}(B(\varphi_1), \dots, B(\varphi_n), W(\psi_1), \dots, W(\psi_m))\psi_i.$$

**Definition 3.2.** (*Skorohod integral w.r.t. B. Extension of Definition 2.2.*) *We say that $u \in L^2(\Omega \times [0,T])$ belongs to Dom $\delta^B$ if there exists a random variable $\delta^B(u) \in L^2(\Omega)$ such that*

$$E\left[\langle \mathcal{K}^*\mathcal{K} D^B F, u\rangle_{L^2([0,T])}\right] = E\left[F\delta^B(u)\right], \quad \text{for all} \quad F \in \mathcal{S}'_{\mathcal{K}}.$$

*We call $\delta^B(u)$ the Skorohod integral with respect to $B$.*

**Definition 3.3.** (*Skorohod integral w.r.t. W.*) *We say that $u \in L^2(\Omega \times [0,T])$ belongs to Dom $\delta^W$ if there exists a random variable $\delta^W(u) \in L^2(\Omega)$ such that*

$$E\left[\int_0^T (D_s^W F)u_s \mathrm{d}s\right] = E\left[F\delta^W(u)\right], \quad \text{for all} \quad F \in \mathcal{S}'_{\mathcal{K}}.$$

*We call $\delta^W(u)$ the Skorohod integral with respect to $W$.*

For the the Skorohod integral with respect to $W$ we have, in particular, the following well known result:

**Proposition 3.4.** *Let $u \in L^2(\Omega \times [0,T])$ be $\mathbb{H}$-adapted. Then the Itô backward integral $\int_0^T u_s \downarrow \mathrm{d}W_s$ coincides with the Skorohod integral with respect to $W$:*

$$\int_0^T u_s \downarrow \mathrm{d}W_s = \delta^W(u).$$

The extension to $\Omega$ of the operators $T_t$ and $A_t$ introduced in subsection 2.3 as acting over $\Omega'$, is done in a canonical way:

$$T_t(\omega',\omega'') := (T_t\omega',\omega''), \quad A_t(\omega',\omega'') := (A_t\omega',\omega''), \quad \text{for } (\omega',\omega'') \in \Omega = \Omega' \times \Omega''.$$

We denote by $L^2_{\mathbb{G}}(0,T;\mathbb{R}^n)$ (resp., $L^2_{\mathbb{H}}(0,T;\mathbb{R}^n)$) the set of $n$-dimensional measurable random processes $\{\varphi_t, t \in [0,T]\}$ which satisfy:
i) $E\left[\int_0^T |\varphi_t|^2 \mathrm{d}t\right] < +\infty$,
ii) $\varphi_t$ is $\mathcal{G}_t$- (resp., $\mathcal{H}_t$-) measurable, for a.e. $t \in [0,T]$.

We also shall introduce a subspace of $L^2_{\mathbb{G}}(0,T;\mathbb{R}^n)$, which stems its importance from its invariance with respect to a class of Girsanov transformations. Recalling the notation

$$I_T^* := \sup_{t\in[0,T]} \left| \int_0^t \gamma_s dB_s \right|$$

from subsection 2.3, we define $L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$ to be the space of all $\mathbb{G}$-adapted processes $(Y,Z)$ which are such that

$$E\left[\exp\{pI_T^*\} \int_0^T \left(|Y_t|^2 + |Z_t|^2\right) \mathrm{d}t\right] < \infty, \ for\ all\ p \geq 1.$$

For the space $L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$ we have the following invariance property:

**Proposition 3.5.** *For all processes $(Y,Z) \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$ we have:*
*i)* $\left(\widetilde{Y}_t, \widetilde{Z}_t\right) := \left(Y_t(T_t)\varepsilon_t^{-1}(T_t), Z_t(T_t)\varepsilon_t^{-1}(T_t)\right) \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$ *and*
*ii)* $\left(\overline{Y}_t, \overline{Z}_t\right) := (Y_t(A_t)\varepsilon_t, Z_t(A_t)\varepsilon_t) \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$.

*Proof:* Since the proofs of i) and ii) are similar, we only prove i):

For the case of $(\widetilde{Y},\widetilde{Z})$, from the Girsanov transformation and Lemma 2.4 we have

$$\begin{aligned}
& E\left[\exp\{pI_T^*\} \int_0^T \left(\left|\widetilde{Y}_t\right|^2 + \left|\widetilde{Z}_t\right|^2\right) \mathrm{d}t\right] \\
= & \int_0^T E\left[\exp\{pI_T^*\}(|Y_t(T_t)|^2 + |Z_t(T_t)|^2)\varepsilon_t^{-2}(T_t)\right] \mathrm{d}t \\
= & \int_0^T E\left[\exp\{pI_T^*(A_t)\}(|Y_t|^2 + |Z_t|^2)\varepsilon_t^{-1}\right] \mathrm{d}t \\
\leq & \ E\Bigg[\exp\Bigg\{p \sup_{0\leq t\leq T}\left|\int_0^t \gamma_s \mathrm{d}B_s\right| + \sup_{\substack{0\leq t\leq T \\ 0\leq r\leq T}} \int_0^T (\mathcal{K}\gamma 1_{[0,t]})(s)(\mathcal{K}\gamma 1_{[0,r]})(s)\mathrm{d}s\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^T (|Y_t|^2 + |Z_t|^2)\varepsilon_t^{-1} \mathrm{d}t \Big] \\
\leq\ & CE\left[\exp\{(p+1)I_T^*\} \int_0^T (|Y_t|^2 + |Z_t|^2)\mathrm{d}t\right] \\
<\ & +\infty, \quad \textit{for all } p \geq 1.
\end{aligned}
$$

Hence, the proof is complete. ■

We now consider the following type of backward doubly stochastic differential equation driven by the Brownian motion $W$ and the fractional Brownian motion $B$:

$$
Y_t = \xi + \int_0^t f(s, Y_s, Z_s)\mathrm{d}s - \int_0^t Z_s \downarrow \mathrm{d}W_s + \int_0^t \gamma_s Y_s \mathrm{d}B_s, \quad t \in [0,T]. \tag{3.4}
$$

Here $\xi \in L^2(\Omega, \mathcal{F}_{0,T}^W, P)$ and $f : \Omega'' \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ is a measurable function such that

**(H3) i)**. $f(\cdot, t, y, z)$ is $\mathcal{F}_{t,T}^W$-measurable, for all $t \in [0,T]$, and for all $(y,z) \in \mathbb{R} \times \mathbb{R}^d$;
**(H3) ii)**. $f(\cdot, 0, 0) \in L^2(\Omega \times [0,T])$;
**(H3) iii)**. There exists a constant $C \in \mathbb{R}^+$ such that for all $(y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$,

$$
|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|), \quad a.e., a.s.
$$

**Remark 3.6.** *Let us refer to some special cases of the above BDSDE:*
*i) If $\gamma = 0$, equation (3.4) becomes a classical BSDE (Pardoux and Peng [83]) with a unique solution $(Y, Z) \in L^2_{\mathbb{F}^W}(0,T; \mathbb{R} \times \mathbb{R}^d)$;*
*ii) If $\xi \in \mathbb{R}$ and $f : [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are deterministic, we can choose $Z = 0$ and $Y \in L^2(\Omega' \times [0,T])$ with $\gamma Y 1_{[0,t]} \in Dom(\delta^B)$, $t \in [0,T]$, as the unique solution of the fractional SDE*

$$
Y_t = \xi + \int_0^t f(s, Y_s, 0)\mathrm{d}s + \int_0^t \gamma_s Y_s \mathrm{d}B_s, \ t \in [0,T],
$$

*which can be solved due to subsection 3.1.*
*iii) Pardoux and Peng [84] considered backward doubly SDEs in the case that $B$ and $W$ are two independent Brownian motions and in a nonlinear framework.*

We let $\widetilde{\Omega}' := \{\omega' \in \Omega' | I_T^*(\omega') < \infty\}$, which satisfies $P^B(\widetilde{\Omega}') = 1$ from Lemma 2.4. We first establish the following theorem:

**Theorem 3.7.** *For all $\omega' \in \widetilde{\Omega}'$, the backward stochastic differential equation*

$$
\begin{aligned}
\widehat{Y}_t(\omega', \cdot) =& \xi + \int_0^t f\left(s, \widehat{Y}_s(\omega', \cdot)\varepsilon_s(T_s, \omega'), \widehat{Z}_s(\omega', \cdot)\varepsilon_s(T_s, \omega')\right) \varepsilon_s^{-1}(T_s, \omega')\mathrm{d}s \\
& - \int_0^t \widehat{Z}_s(\omega', \cdot) \downarrow \mathrm{d}W_s,
\end{aligned} \tag{3.5}
$$

*$t \in [0,T]$, has a unique solution $\left(\widehat{Y}(\omega', \cdot), \widehat{Z}(\omega', \cdot)\right) \in L^2_{\mathbb{F}^W}(0,T; \mathbb{R} \times \mathbb{R}^d)$.*

*Moreover, putting $\left(\widehat{Y}_t(\omega', \cdot), \widehat{Z}_t(\omega', \cdot)\right) := (0,0)$, for $\omega' \in \widetilde{\Omega}'^c$, the random variable $\left(\widehat{Y}_t(\omega', \omega''), \widehat{Z}_t(\omega', \omega'')\right)$ is jointly measurable in $(\omega', \omega'')$, and $\left(\widehat{Y}, \widehat{Z}\right) \in L^{2,*}_{\mathbb{G}}(0,T; \mathbb{R} \times \mathbb{R}^d)$.*

*Furthermore, there exists a positive constant $C$ (only depending on the $L^2$-norm of $\xi$ and $\mathcal{K}\gamma 1_{[0,t]}$, $L^2$-bound of $f(\cdot,0,0)$ and the Lipschitz constant of $f$) such that, for all $\omega' \in \widetilde{\Omega}'$:*

$$E^W\left[\sup_{t\in[0,T]}\left|\widehat{Y}_t(\omega',\cdot)\right|^2+\int_0^T\left|\widehat{Z}_t(\omega',\cdot)\right|^2\mathrm{d}t\right]\le C\exp\{2I_T^*(\omega')\}. \tag{3.6}$$

*Proof:* We put $F_s(\omega',y,z)=f\left(s,y\varepsilon_s(T_s,\omega'),z\varepsilon_s(T_s,\omega')\right)\varepsilon_s^{-1}(T_s,\omega')$, $s\in[0,T]$, $(y,z)\in\mathbb{R}\times\mathbb{R}^d$, $\omega'\in\widetilde{\Omega}'$. Obviously,
i) $F_s(\cdot,y,z)$ is $\mathcal{G}_s$-measurable and $F_s(\omega',\cdot,y,z)$ is $\mathcal{F}^W_{s,T}$-measurable, $\omega'\in\widetilde{\Omega}'$;
ii) $F_s(\omega',\omega'',y,z)$ is Lipschitz in $(y,z)$, uniformly with respect to $(s,\omega',\omega'')$;
iii) $|F_s(\omega',\cdot,0,0)|\le C|f(\cdot,s,0,0)|\exp\{I_T^*(\omega')\}$.

Using $F_s$, equation (3.5) can be rewritten as follows:

$$\widehat{Y}_t(\omega')=\xi+\int_0^t F_s\left(\omega',\widehat{Y}_s(\omega',\cdot),\widehat{Z}_s(\omega',\cdot)\right)\mathrm{d}s-\int_0^t\widehat{Z}_s(\omega',\cdot)\downarrow\mathrm{d}W_s,\ t\in[0,T],\ \omega'\in\widetilde{\Omega}'. \tag{3.7}$$

**Step 1**: We begin by proving the *existence*: From the conditions i)-iii) and standard BSDE arguments (see: Pardoux and Peng [83]) we know that, for all $\omega'\in\widetilde{\Omega}'$, there is a unique solution $\left(\widehat{Y}(\omega',\cdot),\widehat{Z}(\omega',\cdot)\right)\in L^2_{\mathbb{F}^W}(0,T;\mathbb{R}\times\mathbb{R}^d)$. On the other hand, the joint measurability of $F_s$ with respect to $(\omega',\omega'')$ allows to show that, extended to $\Omega'\times\Omega''$ by putting $\left(\widehat{Y}_t(\omega',\cdot),\widehat{Z}_t(\omega',\cdot)\right):=(0,0)$, $\omega'\in\widetilde{\Omega}'^c$, the process $\left(\widehat{Y},\widehat{Z}\right)$ is $\mathbb{H}$-adapted. Let us show that $\left(\widehat{Y},\widehat{Z}\right)\in L^2_{\mathbb{H}}(0,T;\mathbb{R}\times\mathbb{R}^d)$. For this end, it suffices to prove (3.6).

Let $\omega'\in\widetilde{\Omega}'$ be arbitrarily fixed. By applying Itô's formula to $\left|\widehat{Y}_t\right|^2$ we have at $\omega'$, $P^W$-a.s.,

$$\mathrm{d}\left|\widehat{Y}_t\right|^2=2\widehat{Y}_t\left(F_t\left(\widehat{Y}_t,\widehat{Z}_t\right)\mathrm{d}t-\widehat{Z}_t\downarrow\mathrm{d}W_t\right)-\left|\widehat{Z}_t\right|^2\mathrm{d}t.$$

It follows that at $\omega'$, $P^W$-a.s.,

$$\begin{aligned}\left|\widehat{Y}_t\right|^2+\int_0^t\left|\widehat{Z}_s\right|^2\mathrm{d}s&=\xi^2+2\int_0^t\widehat{Y}_sF_s\left(\widehat{Y}_s,\widehat{Z}_s\right)\mathrm{d}s-2\int_0^t\widehat{Y}_s\widehat{Z}_s\downarrow\mathrm{d}W_s\\&\le\xi^2+\int_0^t\left(C_1\left|\widehat{Y}_s\right|^2+\frac{1}{2}\left|\widehat{Z}_s\right|^2+C_2|F_s(0,0)|^2\right)\mathrm{d}s-2\int_0^t\widehat{Y}_s\widehat{Z}_s\downarrow\mathrm{d}W_s.\end{aligned}$$

Hence, at $\omega'$, $P^W$-a.s.,

$$\left|\widehat{Y}_t\right|^2+\frac{1}{2}\int_0^t\left|\widehat{Z}_s\right|^2\mathrm{d}s\le\xi^2+\int_0^t\left(C_1\left|\widehat{Y}_s\right|^2+C_3\varepsilon_s^{-2}\right)\mathrm{d}s-2\int_0^t\widehat{Y}_s\widehat{Z}_s\downarrow\mathrm{d}W_s.$$

Taking the expectation with respect to $P^W$, we notice that

$$\begin{aligned}&E^W\left[\left(\int_0^T\left|\widehat{Y}_t(\omega',\cdot)\widehat{Z}_t(\omega',\cdot)\right|^2\mathrm{d}t\right)^{1/2}\right]\\&\le\left(E^W\left[\sup_{t\in[0,T]}\left|\widehat{Y}_t(\omega',\cdot)\right|^2\right]\right)^{1/2}\left(E^W\left[\int_0^T\left|\widehat{Z}_t(\omega',\cdot)\right|^2\mathrm{d}t\right]\right)^{1/2}<+\infty.\end{aligned}$$

Consequently, $E^W\left[\int_0^t \widehat{Y}_s(\omega',\cdot)\widehat{Z}_s(\omega',\cdot)\downarrow \mathrm{d}W_s\right]=0$, and by taking the conditional expectation with respect to $\mathcal{F}_T^B$, we obtain

$$\begin{aligned}
&E^W\left[\left|\widehat{Y}_t(\omega',\cdot)\right|^2+\frac{1}{2}\int_0^t\left|\widehat{Z}_s(\omega',\cdot)\right|^2\mathrm{d}s\right]\\
\le&E^W\left[\xi^2\right]+\int_0^t C_1E^W\left[\left|\widehat{Y}_s(\omega',\cdot)\right|^2\right]\mathrm{d}s+C_4\exp\{2I_T^*(\omega')\}.
\end{aligned}$$

Thus, from Gronwall's inequality, we have

$$E^W\left[\left|\widehat{Y}_t(\omega',\cdot)\right|^2\right]\le\left(E\left[\xi^2\right]+C_4\exp\left\{2I_T^*\left(\omega'\right)\right\}\right)\exp\{C_1t\}, t\in[0,T],$$

which, combined with the previous estimate, yields

$$E^W\left[\int_0^T\left(\left|\widehat{Y}_t(\omega',\cdot)\right|^2+\left|\widehat{Z}_t(\omega',\cdot)\right|^2\right)\mathrm{d}t\right]\le C\exp\{2I_T^*(\omega')\}.$$

In order to get the estimate (3.6), it suffices now to estimate $E^W\left[\sup_{t\in[0,T]}\left|\widehat{Y}_t(\omega',\cdot)\right|^2\right]$ by using equation (3.7), Burkhölder-Davis-Gundy's inequality and the above estimate.

To prove that $\left(\widehat{Y},\widehat{Z}\right)$ belongs even to $L^2_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$, we have to prove the *uniqueness* in $L^2_{\mathbb{H}}(0,T;\mathbb{R}\times\mathbb{R}^d)$.

**Step 2:** We suppose that $\left(\widehat{Y}^1,\widehat{Z}^1\right)$ and $\left(\widehat{Y}^2,\widehat{Z}^2\right)$ are two solutions of equation (3.7) belonging to $L^2_{\mathbb{H}}(0,T;\mathbb{R}\times\mathbb{R}^d)$ (Notice that $L^2_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)\subset L^2_{\mathbb{H}}(0,T;\mathbb{R}\times\mathbb{R}^d)$). Putting $\Delta\widehat{Y}_t=\widehat{Y}_t^1-\widehat{Y}_t^2$ and $\Delta\widehat{Z}_t=\widehat{Z}_t^1-\widehat{Z}_t^2$, we have

$$\Delta\widehat{Y}_t=\int_0^t\left[F_s\left(\widehat{Y}_s^1,\widehat{Z}_s^1\right)-F_s\left(\widehat{Y}_s^2,\widehat{Z}_s^2\right)\right]\mathrm{d}s-\int_0^t\Delta\widehat{Z}_s\downarrow\mathrm{d}W_s,\quad t\in[0,T].$$

By applying Itô's formula to $\left|\Delta\widehat{Y}_t\right|^2$, we get that

$$\begin{aligned}
E\left|\Delta\widehat{Y}_t\right|^2+E\left[\int_0^t\left|\Delta\widehat{Z}_s\right|^2\mathrm{d}s\right]&=2E\left[\int_0^t\Delta\widehat{Y}_s\left[F_s\left(\widehat{Y}_s^1,\widehat{Z}_s^1\right)-F_s\left(\widehat{Y}_s^2,\widehat{Z}_s^2\right)\right]\mathrm{d}s\right]\\
&\le E\left[\int_0^t\left[2(C_0+C_0^2)\left|\Delta\widehat{Y}_s\right|^2+\frac{1}{2}\left|\Delta\widehat{Z}_s\right|^2\right]\mathrm{d}s\right],
\end{aligned}$$

and finally from Gronwall's lemma, we conclude that $\Delta\widehat{Y}_t=0$, $\Delta\widehat{Z}_t=0$, a.s., a.e.

**Step 3:** Let us now show that $\left(\widehat{Y},\widehat{Z}\right)$ is not only in $L^2_{\mathbb{H}}(0,T;\mathbb{R}\times\mathbb{R}^d)$ but even in $L^2_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$. For this we consider for an arbitrarily given $\tau\in[0,T]$ equation (3.7) over the time interval $[0,\tau]$:

$$\widehat{Y}_t^\tau=\xi+\int_0^t F_s\left(\widehat{Y}_s^\tau,\widehat{Z}_s^\tau\right)\mathrm{d}s-\int_0^t\widehat{Z}_s^\tau\downarrow\mathrm{d}W_s,\quad t\in[0,\tau]. \tag{3.8}$$

Let $\mathcal{H}_t^\tau:=\mathcal{F}_{t,T}^W\vee\mathcal{F}_\tau^B$, $t\in[0,\tau]$. Then $\mathbb{H}^\tau=\{\mathcal{H}_t^\tau\}_{t\in[0,\tau]}$ is a filtration with respect to which $W$ has the martingale representation property. Since $F_t(y,z)$ is $\mathcal{G}_t$- and hence also

$\mathcal{H}_t^\tau$-measurable, d$t$ a.e. on $[0,\tau]$, it follows from the classical BSDE theory that BSDE (3.8) admits a solution $\left(\widehat{Y},\widehat{Z}\right) \in L^2_{\mathbb{H}^\tau}(0,\tau;\mathbb{R}\times\mathbb{R}^d)$. Due to the first step this solution is unique in $L^2_{\mathbb{H}^\tau}(0,T;\mathbb{R}\times\mathbb{R}^d)$. Hence, $\left(\widehat{Y}_t,\widehat{Z}_t\right) = \left(\widehat{Y}_t^\tau,\widehat{Z}_t^\tau\right)$, d$t$ a.e., for $t<\tau$. Consequently, $\left(\widehat{Y}_t,\widehat{Z}_t\right)$ is $\mathcal{H}_t^\tau$- measurable, d$t$ a.e., for $t<\tau$. Therefore, letting $\tau\downarrow t$ we can deduce from the right continuity of the filtration $\mathbb{F}^B$ that $\left(\widehat{Y},\widehat{Z}\right)$ is $\mathbb{G}$-adapted.

It still remains to prove that $\left(\widehat{Y},\widehat{Z}\right) \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$.

**Step 4:** For the proof that $\left(\widehat{Y},\widehat{Z}\right)$ belongs to $L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$ we notice that, by the above estimates and Lemma 2.4,

$$
\begin{aligned}
&E\left[\exp\{pI_T^*\}\int_0^T\left(\left|\widehat{Y}_t\right|^2+\left|\widehat{Z}_t\right|^2\right)\mathrm{d}t\right] = E\left[\exp\{pI_T^*\}E\left[\int_0^T\left(\left|\widehat{Y}_t\right|^2+\left|\widehat{Z}_t\right|^2\right)\mathrm{d}t|\mathcal{F}_T^B\right]\right]\\
&\leq E\left[C\exp\{(2+p)I_T^*\}\right]<\infty,\quad for\ all\ \ p\geq 1.
\end{aligned}
$$

Hence, the proof is complete. ∎

**Corollary 3.8.** *The process $\left(\widehat{Y},\widehat{Z}\right)$ given by Theorem 3.7 is the unique solution of equation* (3.5) *in $L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$.*

Now we state the main result of this subsection:

**Theorem 3.9.** 1) *Let $\left(\widehat{Y},\widehat{Z}\right) \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$ be a solution of BSDE* (3.5). *Then*

$$
\{(Y_t,Z_t),t\in[0,T]\} = \left\{\left(\widehat{Y}_t(A_t)\varepsilon_t,\widehat{Z}_t(A_t)\varepsilon_t\right),t\in[0,T]\right\} \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)
$$

*is a solution of equation* (3.4) *with $\gamma Y 1_{[0,t]}\in Dom\ \delta^B$, for all $t\in[0,T]$.*

2) *Conversely, given an arbitrary solution $(Y,Z)\in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$ of the equation* (3.4) *with $\gamma Y 1_{[0,t]}\in Dom\ \delta^B$, for all $t\in[0,T]$, the process*

$$
\left\{\left(\widehat{Y}_t,\widehat{Z}_t\right)_{t\in[0,T]}\right\} = \left\{\left(Y_t(T_t)\varepsilon_t^{-1}(T_t),Z_t(T_t)\varepsilon_t^{-1}(T_t)\right)_{t\in[0,T]}\right\} \in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)
$$

*is a solution of BSDE* (3.5).

From the Theorems 3.7 and 3.9 we can immediately conclude the following

**Corollary 3.10.** *The solution of equation* (3.4) *in $L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$ exists and is unique.*

*Proof* (*of Theorem 3.9*)*:* We first prove that, given the solution $\left(\widehat{Y},\widehat{Z}\right)$ of equation (3.5), $(Y,Z)$ defined in the theorem solves (3.4). For this we notice that for $F$ being an arbitrary but fixed element of $\mathcal{S}'_{\mathcal{K}}$, from Girsanov transformation and from the equation (3.5), it follows

$$
\begin{aligned}
&E\left[FY_t-F\xi\right] = E\left[F(T_t)\widehat{Y}_t-F\widehat{Y}_0\right]\\
=&E\left[F(T_t)\widehat{Y}_0+F(T_t)\int_0^t F_s\left(\widehat{Y}_s,\widehat{Z}_s\right)\mathrm{d}s-F(T_t)\int_0^t\widehat{Z}_s\downarrow\mathrm{d}W_s-F\widehat{Y}_0\right].
\end{aligned}
$$

We recall that $E\left[F(T_t)\int_0^t \widehat{Z}_s \downarrow \mathrm{d}W_s\right] = E\left[\int_0^t D_s^W F(T_t)\widehat{Z}_s \mathrm{d}s\right]$. Thus, from the fact that $\frac{\mathrm{d}}{\mathrm{d}t}F(T_t) = \gamma_t(\mathcal{K}^*\mathcal{K}D^B F)\ (t, T_t)$, we have

$$
\begin{aligned}
&E\left[FY_t - F\xi\right]\\
=&E\left[\widehat{Y}_0 \int_0^t \gamma_s(\mathcal{K}^*\mathcal{K}D^B F)(s, T_s)\mathrm{d}s\right]\\
&+E\left[\int_0^t F(T_s)F_s\left(\widehat{Y}_s, \widehat{Z}_s\right)\mathrm{d}s + \int_0^t \int_s^t \gamma_r(\mathcal{K}^*\mathcal{K}D^B F)(r, T_r)\mathrm{d}r F_s\left(\widehat{Y}_s, \widehat{Z}_s\right)\mathrm{d}s\right]\\
&-E\left[\int_0^t D_s^W F(T_s)\widehat{Z}_s\mathrm{d}s + \int_0^t \int_s^t D_s^W \gamma_r(\mathcal{K}^*\mathcal{K}D^B F)(r, T_r)\mathrm{d}r\widehat{Z}_s\mathrm{d}s\right].
\end{aligned}
$$

Moreover, from Fubini's theorem, the definition of the Skorohod integral with respect to $W$, and from Proposition 3.4, we obtain that

$$
\begin{aligned}
&E\left[\int_0^t \int_s^t D_s^W \gamma_r(\mathcal{K}^*\mathcal{K}D^B F)(r, T_r)\mathrm{d}r\widehat{Z}_s\mathrm{d}s\right]\\
=&\int_0^t \gamma_r E\left[\int_0^r D_s^W(\mathcal{K}^*\mathcal{K}D^B F)(r, T_r)\widehat{Z}_s\mathrm{d}s\right]\mathrm{d}r\\
=&\int_0^t \gamma_r E\left[(\mathcal{K}^*\mathcal{K}D^B F)(r, T_r)\int_0^r \widehat{Z}_s \downarrow \mathrm{d}W_s\right]\mathrm{d}r.
\end{aligned}
$$

Thus, by applying the inverse Girsanov transformation as well as Fubini's theorem, we obtain

$$
\begin{aligned}
&E\left[FY_t - F\xi\right]\\
=&E\left[\int_0^t F(T_s)F_s\left(\widehat{Y}_s, \widehat{Z}_s\right)\mathrm{d}s - \int_0^t D_s^W F(T_s)\widehat{Z}_s\mathrm{d}s\right]\\
&+E\left[\int_0^t \gamma_s(\mathcal{K}^*\mathcal{K}D^B F)(s, T_s)\left(\widehat{Y}_0 + \int_0^s F_r\left(\widehat{Y}_r, \widehat{Z}_r\right)\mathrm{d}r - \int_0^s \widehat{Z}_r \downarrow \mathrm{d}W_r\right)\mathrm{d}s\right]\\
=&E\left[\int_0^t Ff(s, Y_s, Z_s)\mathrm{d}s\right] - E\left[\int_0^t D_s^W F Z_s\mathrm{d}s\right] + E\left[\int_0^t \gamma_s(\mathcal{K}^*\mathcal{K}D^B F)(s)Y_s\mathrm{d}s\right],
\end{aligned}
$$

where, for the latter expression, we have used that $\widehat{Y}_s$ is a solution of (3.5). Since $Z \in L_{\mathbb{G}}^{2,*}(0, T; \mathbb{R}\times\mathbb{R}^d)\ \left(\subset L_{\mathbb{G}}^2(0, T; \mathbb{R}\times\mathbb{R}^d)\right)$, it holds

$$
E\left[\int_0^t D_s^W F Z_s\mathrm{d}s\right] = E\left[F\int_0^t Z_s \downarrow \mathrm{d}W_s\right].
$$

Consequently,

$$
E\left[\int_0^t (\mathcal{K}^*\mathcal{K}D^B F)(s)\gamma_s Y_s\mathrm{d}s\right] = E\left[F\left\{Y_t - \xi - \int_0^t f(s, Y_s, Z_s)\mathrm{d}s + \int_0^t Z_s \downarrow \mathrm{d}W_s\right\}\right]
$$

holds for all $F \in \mathcal{S}'_{\mathcal{K}}$. From Proposition 3.5 we know that both, $\left(\widehat{Y}, \widehat{Z}\right)$ and $(Y, Z)$, belong to $L_{\mathbb{G}}^{2,*}(0, T; \mathbb{R}\times\mathbb{R}^d)$. Consequently,

$$
Y_t - \xi - \int_0^t f(s, Y_s, Z_s)\mathrm{d}s + \int_0^t Z_s \downarrow \mathrm{d}W_s \in L^2(\Omega, \mathcal{F}, P),
$$

for all $t \in [0,T]$. Moreover, using (3.6),

$$\begin{aligned}
&E\left[\int_0^T \left|\gamma_r 1_{[0,t]}(r) Y_r\right|^2 \mathrm{d}r\right] \\
=&E\left[\int_0^T \left|\gamma_r 1_{[0,t]}(r) \widehat{Y}_r(A_r)\right|^2 \varepsilon_r^2 \mathrm{d}r\right] \\
=&E\left[\int_0^T \left|\gamma_r 1_{[0,t]}(r) \widehat{Y}_r\right|^2 \varepsilon_r(T_r) \mathrm{d}r\right] \\
\leq&CE\left[\int_0^T \left|\gamma_r 1_{[0,t]}(r) \widehat{Y}_r\right|^2 \exp\{I_T^*\} \mathrm{d}r\right] \\
\leq&CE\left[\int_0^T \left|\gamma_r 1_{[0,t]}(r)\right|^2 \sup_{r\in[0,T]} \left|\widehat{Y}_r\right|^2 \exp\{I_T^*\} \mathrm{d}r\right] \\
\leq&C\int_0^T \left|\gamma_r 1_{[0,t]}(r)\right|^2 E\left[\exp\{CI_T^*\}\right] \mathrm{d}r \\
\leq&C\int_0^T \left|\gamma_r 1_{[0,t]}(r)\right|^2 \mathrm{d}r < \infty.
\end{aligned}$$

Thus, according to the definition of the Skorohod integral with respect to $B$ we then conclude $\gamma Y 1_{[0,t]} \in Dom\ \delta^B$ and

$$\int_0^t \gamma_s Y_s \mathrm{d}B_s = Y_t - \left(\xi + \int_0^t f(s,Y_s,Z_s)\mathrm{d}s - \int_0^t Z_s \downarrow \mathrm{d}W_s\right), t \in [0,T].$$

Hence $Y_t = \xi + \int_0^t f(s,Y_s,Z_s)\mathrm{d}s - \int_0^t Z_s \downarrow \mathrm{d}W_s + \int_0^t \gamma_s Y_s \mathrm{d}B_s, t \in [0,T]$.

Let us prove now the second assertion of the Theorem. For this end we let $(Y,Z) \in L_{\mathbb{G}}^{2,*}(0,T;\mathbb{R}\times\mathbb{R}^d)$ be an arbitrary solution of equation (3.4) and $F$ be an arbitrary but fixed element of $\mathcal{S}_{\mathcal{K}}'$. Then we have

$$\begin{aligned}
E\left[Y_t F(A_t)\right] =&E\left[\xi F(A_t)\right] + E\left[\int_0^t F(A_t) f(s,Y_s,Z_s)\mathrm{d}s\right] - E\left[F(A_t)\int_0^t Z_s \downarrow \mathrm{d}W_s\right] \\
&+ E\left[F(A_t)\int_0^t \gamma_s Y_s \mathrm{d}B_s\right] \\
=&I_1 + I_2 - I_3 + I_4,
\end{aligned}$$

where, using the fact that $\frac{\mathrm{d}}{\mathrm{d}t}F(A_t) = -\gamma_t(\mathcal{K}^*\mathcal{K}D^B F(A_t))(t)$ and Fubini's theorem,

$$I_1 =E\left[\xi F\right] - E\left[\xi \int_0^t \gamma_s(\mathcal{K}^*\mathcal{K}D^B F(A_s))(s)\mathrm{d}s\right],$$

$$I_2 =E\left[\int_0^t F(A_s) f(s,Y_s,Z_s)\mathrm{d}s\right] - E\left[\int_0^t \gamma_r(\mathcal{K}^*\mathcal{K}D^B F(A_r))(r)\int_0^r f(s,Y_s,Z_s)\mathrm{d}s\mathrm{d}r\right].$$

From the duality between the Itô backward integral and the Malliavin derivative $D^W$

(recall that $Z$ is square integrable), we get

$$
\begin{aligned}
I_3 =& E\left[\int_0^t Z_s D_s^W F(A_s)\mathrm{d}s\right] - E\left[\int_0^t Z_s D_s^W \left[\int_s^t \gamma_r (\mathcal{K}^*\mathcal{K}D^B F(A_r))(r)\mathrm{d}r\right]\mathrm{d}s\right]\\
=& E\left[\int_0^t Z_s D_s^W F(A_s)\mathrm{d}s\right] - E\left[\int_0^t \int_0^r Z_s \gamma_r D_s^W (\mathcal{K}^*\mathcal{K}D^B F(A_r))(r)\mathrm{d}s\mathrm{d}r\right]\\
=& E\left[\int_0^t Z_s D_s^W F(A_s)\mathrm{d}s\right] - E\left[\int_0^t \gamma_s (\mathcal{K}^*\mathcal{K}D^B F(A_s))(s)\int_0^s Z_r \downarrow \mathrm{d}W_r \mathrm{d}s\right].
\end{aligned}
$$

Moreover, from the duality between the integral w.r.t. $B$ and $D^B$ (observe that $\gamma Y 1_{[0,t]} \in$ *Dom* $\delta^B$) we obtain

$$
\begin{aligned}
I_4 =& E\left[\int_0^t \gamma_s Y_s (\mathcal{K}^*\mathcal{K}D^B F(A_s))(s)\mathrm{d}s\right]\\
&- E\left[\int_0^t \int_0^r \gamma_r (\mathcal{K}^*\mathcal{K}D^B (\mathcal{K}^*\mathcal{K}D^B F(A_r)(r))(s))\gamma_s Y_s \mathrm{d}s\mathrm{d}r\right]\\
=& E\left[\int_0^t \gamma_s Y_s (\mathcal{K}^*\mathcal{K}D^B F(A_s))(s)\mathrm{d}s\right] - E\left[\int_0^t \gamma_s (\mathcal{K}^*\mathcal{K}D^B F(A_s))(s)\int_0^s \gamma_r Y_r \mathrm{d}B_r \mathrm{d}s\right].
\end{aligned}
$$

Consequently, using the fact that $(Y,Z)$ is a solution of equation (3.4), i.e.,

$$
Y_t = \xi + \int_0^t f(s,Y_s,Z_s)\mathrm{d}s - \int_0^t Z_s \downarrow \mathrm{d}W_s + \int_0^t \gamma_s Y_s \mathrm{d}B_s, \quad t \in [0,T],
$$

we obtain that

$$
E\left[Y_t F(A_t)\right] = E\left[\xi F\right] + E\left[\int_0^t F(A_s) f(s,Y_s,Z_s)\mathrm{d}s\right] - E\left[\int_0^t Z_s D_s^W F(A_s)\mathrm{d}s\right].
$$

Therefore, by applying Girsanov transformation again and taking into account the arbitrariness of $F \in \mathcal{S}'_{\mathcal{K}}$, it follows that for all $F \in \mathcal{S}'_{\mathcal{K}}$,

$$
\begin{aligned}
& E\left[\int_0^t Z_s(T_s)\varepsilon_s^{-1}(T_s) D_s^W F \mathrm{d}s\right]\\
&= \left[F\left\{\xi - Y_t(T_t)\varepsilon_t^{-1}(T_t) + \int_0^t f(s,Y_s(T_s),Z_s(T_s))\varepsilon_s^{-1}(T_s)\mathrm{d}s\right\}\right].
\end{aligned}
$$

Now, since according to Proposition 3.5, $(Y_t(T_t)\varepsilon_t^{-1}(T_t), Z_t(T_t)\varepsilon_t^{-1}(T_t))$
$\in L^{2,*}_{\mathbb{G}}(0,T;\mathbb{R}\times\mathbb{R}^d)$, we have

$$
Y_t(T_t)\varepsilon_t^{-1}(T_t) - \xi - \int_0^t f(s,Y_s(T_s),Z_s(T_s))\varepsilon_s^{-1}(T_s)\mathrm{d}s \in L^2(\Omega,\mathcal{G}_t,P).
$$

Therefore,

$$
Y_t(T_t)\varepsilon_t^{-1}(T_t) = \xi + \int_0^t f\left(s,Y_s\left(T_s\right),Z_s\left(T_s\right)\right)\varepsilon_s^{-1}(T_s)\mathrm{d}s - \int_0^t Z_s(T_s)\varepsilon_s^{-1}(T_s)\downarrow \mathrm{d}W_s, \tag{3.9}
$$

a.s., for all $t \in [0,T]$, which means that

$$
(\widehat{Y},\widehat{Z}) := \left\{Y_t(T_t)\varepsilon_t^{-1}(T_t), Z_t(T_t)\varepsilon_t^{-1}(T_t), t \in [0,T]\right\}
$$

is a solution of equation (3.5). Hence, the proof is complete. ∎

# 4 The Associated Stochastic Partial Differential Equations

In this section we will use the following standard assumptions:

**(H4) i)**. The functions $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\Phi : \mathbb{R}^d \to \mathbb{R}$ are Lipschitz.

**(H4) ii)**. The function $f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ is continuous, $f(t,\cdot,\cdot,\cdot)$ is Lipschitz, uniformly with respect to $t$ and $f(\cdot,0,0,0) \in L^2(\Omega \times [0,T])$.

We denote by $(X_s^{t,x})_{0\le s\le t}$ the unique solution of the following stochastic differential equation:

$$\begin{cases} \mathrm{d}X_s^{t,x} = -b\left(X_s^{t,x}\right)\mathrm{d}s - \sigma\left(X_s^{t,x}\right)\downarrow \mathrm{d}W_s, \quad s \in [0,t], \\ X_t^{t,x} = x \in \mathbb{R}^d. \end{cases} \tag{4.1}$$

We note that this equation looks like a backward stochastic differential equation, but due to the backward Itô integral, the SDE (4.1) is indeed a classical forward SDE. Under our standard assumptions on $\sigma$ and $b$, it has a unique strong solution which is $\{\mathcal{F}_{s,t}^W\}_{s\in[0,t]}$-adapted. The following result provides some standard estimates for the solution of equation (4.1) (cf. [85] Lemma 2.7).

**Lemma 4.1.** *Let $X^{t,x} = \{X_s^{t,x}, s \in [0,t]\}$ be the solution of the SDE* (4.1)*. Then*

*(i) There exists a continuous version of $X^{t,x}$ such that $(s,x) \mapsto X_s^{t,x}$ is locally Hölder $\left(C^{\alpha,2\alpha}\right.$, for all $\left.\alpha \in (0,1/2)\right)$;*

*(ii) For all $q \ge 1$, there exists $C_q > 0$ such that, for $t,t' \in [0,T]$ and $x,x' \in \mathbb{R}^d$,*

$$E\left[\sup_{0\le s\le t}\left|X_s^{t,x}\right|^q\right] \le C_q(1+|x|^q), \tag{4.2}$$

$$E\left[\sup_{0\le s\le T}\left|X_{s\wedge t}^{t,x} - X_{s\wedge t'}^{t',x'}\right|^q\right] \le C_q\left[(1+|x|^q+|x'|^q)|t-t'|^{q/2} + |x-x'|^q\right]. \tag{4.3}$$

With the forward SDE we associate a BDSDE with driving coefficient $f$: for $s \in [0,t]$,

$$Y_s^{t,x} = \Phi\left(X_0^{t,x}\right) + \int_0^s f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right)\mathrm{d}r - \int_0^s Z_r^{t,x}\downarrow \mathrm{d}W_r + \int_0^s \gamma_r Y_r^{t,x}\mathrm{d}B_r. \tag{4.4}$$

By Theorem 3.7 and Theorem 3.9, the above BDSDE has a unique solution $\left(Y^{t,x}, Z^{t,x}\right)$ given by

$$\left(Y_s^{t,x}, Z_s^{t,x}\right) = \left(\widehat{Y}_s^{t,x}(A_s)\varepsilon_s, \widehat{Z}_s^{t,x}(A_s)\varepsilon_s\right), s \in [0,t],$$

where, for all $\omega' \in \widetilde{\Omega}'$, $P^W$-a.s.,

$$\begin{aligned} \widehat{Y}_s^{t,x}\left(\omega',\cdot\right) = &\int_0^s f\left(r, X_r^{t,x}, \widehat{Y}_r^{t,x}\left(\omega',\cdot\right)\varepsilon_r(T_r,\omega'), \widehat{Z}_r^{t,x}\left(\omega',\cdot\right)\varepsilon_r(T_r,\omega')\right)\varepsilon_r^{-1}(T_r,\omega')\mathrm{d}r \\ &+ \Phi\left(X_0^{t,x}\right) - \int_0^s \widehat{Z}_r^{t,x}\left(\omega',\cdot\right)\downarrow \mathrm{d}W_r, \quad s \in [0,t]. \end{aligned} \tag{4.5}$$

Pardoux and Peng [83] [84] have studied BSDEs in a Markovian context in which the driver $F_r(x,y,z)$ is deterministic; here, in our framework the driver, for $(r,x,y,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$,

$$F_r(\omega', x, y, z) := f\left(r, x, y\varepsilon_r(T_r,\omega'), z\varepsilon_r(T_r,\omega')\right)\varepsilon_r^{-1}(T_r,\omega'),$$

is random but it depends only on $B$ and is independent of the driving Brownian motion $W$. In the following we shall define $X_s^{t,x}$, $\widehat{Y}_s^{t,x}$ and $\widehat{Z}_s^{t,x}$ for all $(s,t) \in [0,T]^2$ by setting $X_s^{t,x} = x$, $\widehat{Y}_s^{t,x} = \widehat{Y}_t^{t,x}$ and $\widehat{Z}_s^{t,x} = 0$, for $t < s \leq T$.

We have the following standard estimates for the solution:

**Lemma 4.2.** *For all $p \geq 1$, there exists a constant $C_p \in \mathbb{R}_+$ such that for all $(t,x)$, $(t',x') \in [0,T] \times \mathbb{R}^d$, $\omega' \in \widetilde{\Omega}'$, $P^W$-a.s.,*

$$
\begin{aligned}
&E^W\left[\sup_{0\leq s\leq t}\left|\widehat{Y}_s^{t,x}(\omega',\cdot)\right|^p + \left(\int_0^t \left|\widehat{Z}_s^{t,x}(\omega',\cdot)\right|^2 \mathrm{d}s\right)^{p/2}\right] \\
\leq& C_p\left(1+|x|^p\right)\exp\left\{pI_T^*(\omega')\right\};
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
&E^W\left[\sup_{0\leq s\leq t\wedge t'}\left|\widehat{Y}_s^{t,x}(\omega',\cdot) - \widehat{Y}_s^{t',x'}(\omega',\cdot)\right|^p + \left(\int_0^{t\wedge t'}\left|\widehat{Z}_s^{t,x}(\omega',\cdot) - \widehat{Z}_s^{t',x'}(\omega',\cdot)\right|^2 \mathrm{d}s\right)^{p/2}\right] \\
\leq& C_p\exp\left\{pI_T^*(\omega')\right\}\left(|x-x'|^p + (1+|x|^p+|x'|^p)|t-t'|^{p/2}\right).
\end{aligned}
\tag{4.7}
$$

*Proof*: Let us fix any $\omega' \in \widetilde{\Omega}'$. For $p \geq 2$, by applying Itô's formula to $|\widehat{Y}_s^{t,x}|^p$, it follows that, $P^W$-a.s., for $s \in [0,t]$,

$$
\begin{aligned}
&\left|\widehat{Y}_s^{t,x}(\omega',\cdot)\right|^p + \frac{p(p-1)}{2}\int_0^s \left|\widehat{Y}_r^{t,x}(\omega',\cdot)\right|^{p-2}\left|\widehat{Z}_r^{t,x}(\omega',\cdot)\right|^2 \mathrm{d}r \\
=&\left|\Phi\left(X_0^{t,x}\right)\right|^p + p\int_0^s \left|\widehat{Y}_r^{t,x}(\omega',\cdot)\right|^{p-2}\widehat{Y}_r^{t,x}(\omega',\cdot) \\
\times&\quad \left(F_r\left(\omega', X_r^{t,x}, \widehat{Y}_r^{t,x}(\omega',\cdot), \widehat{Z}_r^{t,x}(\omega',\cdot)\right)\mathrm{d}r + \widehat{Z}_r^{t,x}(\omega',\cdot)\downarrow \mathrm{d}W_r\right).
\end{aligned}
$$

Let $0 \leq s \leq t' \leq t \leq T$. We take the conditional expectation with respect to $\mathcal{F}_{t',t}^W$ on both sides of the above equality, and we obtain

$$
\begin{aligned}
&E^W\left[\left|\widehat{Y}_s^{t,x}(\omega',\cdot)\right|^p|\mathcal{F}_{t',t}^W\right] + E^W\left[\frac{p(p-1)}{2}\int_0^s\left|\widehat{Y}_r^{t,x}(\omega',\cdot)\right|^{p-2}\left|\widehat{Z}_r^{t,x}(\omega',\cdot)\right|^2\mathrm{d}r|\mathcal{F}_{t',t}^W\right] \\
=&E^W\left[\left|\Phi\left(X_0^{t,x}\right)\right|^p|\mathcal{F}_{t',t}^W\right] + pE^W\left[\int_0^s\left|\widehat{Y}_r^{t,x}(\omega',\cdot)\right|^{p-2}\widehat{Y}_r^{t,x}(\omega',\cdot)\right. \\
&\qquad\left.\times F_r\left(\omega', X_r^{t,x}, \widehat{Y}_r^{t,x}(\omega',\cdot), \widehat{Z}_r^{t,x}(\omega',\cdot)\right)\mathrm{d}r|\mathcal{F}_{t',t}^W\right] \\
\leq&E^W\left[\left|\Phi\left(X_0^{t,x}\right)\right|^p|\mathcal{F}_{t',t}^W\right] + pE^W\left[\int_0^s\left(C_p\left|\widehat{Y}_r^{t,x}(\omega',\cdot)\right|^p + C_p\left|X_r^{t,x}\right|^p\right.\right. \\
&\qquad\left.\left.+\frac{p(p-1)}{4}\left|\widehat{Y}_r^{t,x}(\omega',\cdot)\right|^{p-2}\left|\widehat{Z}_r^{t,x}(\omega',\cdot)\right|^2 + C_p\varepsilon_r^p(\omega')\right)\mathrm{d}r|\mathcal{F}_{t',t}^W\right].
\end{aligned}
$$

Thus, from Gronwall's inequality and **(H4)** we have

$$
\begin{aligned}
\left|\widehat{Y}_s^{t,x}(\omega',\cdot)\right|^p \leq& C_p\left(E^W\left[\sup_{0\leq r\leq s}\left|X_r^{t,x}\right|^p|\mathcal{F}_{s,t}^W\right] + \exp\left\{pI_T^*(\omega')\right\}\right) \\
\leq& C_p\left(1+\left|X_s^{t,x}\right|^p\right)\exp\left\{pI_T^*(\omega')\right\}.
\end{aligned}
$$

Consequently, by using Doob's inequality, we get from the arbitrariness of $p \geq 1$:

$$E^W\left[\sup_{0\leq s\leq t}\left|\widehat{Y}_s^{t,x}\left(\omega',\cdot\right)\right|^p\right] \leq C_p\left(1+|x|^p\right)\exp\left\{pI_T^*(\omega')\right\}.$$

The first result follows from Burkhölder-Davis-Gundy inequality applied to $\left|\int_0^{t'}\widehat{Z}_s^{t,x}\left(\omega',\cdot\right)\downarrow \mathrm{d}W_s\right|^p$ (see, e.g. [85]).

Concerning the second assertion, without loss of generality, we can suppose $t \geq t'$. Let $0 \leq s \leq t'' \leq t'$.

Using an argument similar to that developed by Pardoux and Peng [85], we see that, for some constants $\theta > 0$ and $C > 0$,

$$\begin{aligned}
&E^W\left[\left|\widehat{Y}_s^{t,x}\left(\omega',\cdot\right)-\widehat{Y}_s^{t',x'}\left(\omega',\cdot\right)\right|^p|\mathcal{F}_{t'',t}^W\right]\\
&+\theta E^W\left[\int_0^s\left|\widehat{Y}_r^{t,x}\left(\omega',\cdot\right)-\widehat{Y}_r^{t',x'}\left(\omega',\cdot\right)\right|^{p-2}\left|\widehat{Z}_r^{t,x}\left(\omega',\cdot\right)-\widehat{Z}_r^{t',x'}\left(\omega',\cdot\right)\right|^2\mathrm{d}r|\mathcal{F}_{t'',t}^W\right]\\
\leq& CE^W\left[\left|\Phi\left(X_0^{t,x}\right)-\Phi\left(X_0^{t',x'}\right)\right|^p|\mathcal{F}_{t'',t}^W\right]\\
&+CE^W\left[\int_0^s\left[\varepsilon_r^{-p}\left(T_r,\omega'\right)\left|X_r^{t,x}-X_r^{t',x'}\right|^p+\left|\widehat{Y}_r^{t,x}\left(\omega',\cdot\right)-\widehat{Y}_r^{t',x'}\left(\omega',\cdot\right)\right|^p\right]\mathrm{d}r|\mathcal{F}_{t'',t}^W\right]\\
\leq& CE^W\left[\left|\Phi\left(X_0^{t,x}\right)-\Phi\left(X_0^{t',x'}\right)\right|^p|\mathcal{F}_{t'',t}^W\right]\\
&+C\exp\left\{pI_T^*\left(\omega'\right)\right\}\left(E\left[\int_0^s\left|X_r^{t,x}-X_r^{t',x'}\right|^{2p}\mathrm{d}r|\mathcal{F}_{t'',t}^W\right]\right)^{1/2}\\
&+CE^W\left[\int_0^s\left|\widehat{Y}_r^{t,x}\left(\omega',\cdot\right)-\widehat{Y}_r^{t',x'}\left(\omega',\cdot\right)\right|^p\mathrm{d}r|\mathcal{F}_{t'',t}^W\right].
\end{aligned}$$

Consequently, from Gronwall's lemma and according to Lemma 4.1,

$$\left|\widehat{Y}_s^{t,x}\left(\omega',\cdot\right)-\widehat{Y}_s^{t,x}\left(\omega',\cdot\right)\right|^p \leq C_p\left|X_s^{t,x}-X_s^{t',x'}\right|^p\exp\left\{pI_T^*\left(\omega'\right)\right\}, 0\leq s\leq t'\leq t,$$

and

$$\begin{aligned}
&E^W\left[\sup_{0\leq s\leq t'}\left|\widehat{Y}_s^{t,x}\left(\omega',\cdot\right)-\widehat{Y}_s^{t,x}\left(\omega',\cdot\right)\right|^p\right]\\
\leq& C_p\exp\left\{pI_T^*\left(\omega'\right)\right\}\left(\left(1+|x|^p+|x'|^p\right)|t-t'|^{p/2}+|x-x'|^p\right).
\end{aligned}$$

Finally, with the help of the Burkhölder-Davis-Gundy inequality together with Lemma 2.4, we deduce that for all $p \geq 2$, there exists $C_p$ such that

$$\begin{aligned}
&E^W\left[\sup_{0\leq s\leq t'}\left|\widehat{Y}_s^{t,x}\left(\omega',\cdot\right)-\widehat{Y}_s^{t',x'}\left(\omega',\cdot\right)\right|^p+\left(\int_0^{t'}\left|\widehat{Z}_s^{t,x}\left(\omega',\cdot\right)-\widehat{Z}_s^{t',x'}\left(\omega',\cdot\right)\right|^2\mathrm{d}s\right)^{p/2}\right]\\
\leq& C_p\exp\left\{pI_T^*\left(\omega'\right)\right\}\left(|x-x'|^p+(1+|x|^p+|x'|^p)|t-t'|^{p/2}\right).
\end{aligned}$$

The case $p \geq 1$ follows easily from the case $p \geq 2$. This completes the proof of the lemma. ∎

We now introduce the random field: $\widehat{u}(t,x) = \widehat{Y}_t^{t,x}, (t,x) \in [0,T]\times\mathbb{R}^d$, which has the following regularity properties:

**Proposition 4.3.** *The random field $\widehat{u}(t,x)$ is $\mathcal{F}_t^B$-measurable and we have $\widehat{Y}_s^{t,x}(\omega',\cdot) = \widehat{Y}_s^{s,X_s^{t,x}}(\omega',\cdot) = \widehat{u}(\omega', s, X_s^{t,x})$, $P^W$-a.s., $0 \le s \le t \le T$, $\omega' \in \widetilde{\Omega}'$.*

*Proof:* From Theorem 3.7 with terminal time $t$, we know that $\widehat{Y}_s^{t,x}$ is $\mathcal{F}_{s,t}^W \vee \mathcal{F}_s^B$-measurable. Hence $\widehat{u}(t,x) = \widehat{Y}_t^{t,x}$ is $\mathcal{F}_{t,t}^W \vee \mathcal{F}_t^B$-measurable. By applying Blumenthal zero-one law we deduce that $\widehat{u}$ is $\mathcal{F}_t^B$-measurable and independent of $W$. The second assertion is a direct result from the uniqueness property of the solutions of (4.1) and (4.5)(cf. [42]). ∎

**Lemma 4.4.** *The process $\{\widehat{Y}_s^{t,x}; (s,t) \in [0,T]^2, x \in \mathbb{R}^d\}$ possesses a continuous version. Moreover, $|\widehat{u}(t,x)| \le C \exp\{I_T^*\}(1+|x|)$, $P$-a.s.*

*Proof:* Recall that, for $s \in [0,t]$,

$$\widehat{Y}_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f\left(r, X_r^{t,x}, \widehat{Y}_r^{t,x}\varepsilon_r(T_r), \widehat{Z}_r^{t,x}\varepsilon_r(T_r)\right)\varepsilon_r^{-1}(T_r)\mathrm{d}r - \int_0^s \widehat{Z}_r^{t,x} \downarrow \mathrm{d}W_r.$$

Let $0 \le t \le t' \le T$, $x, x' \in \mathbb{R}^d$. Then by Proposition 4.3, we have,

$$\widehat{u}(t,x) - \widehat{u}(t',x') = E\left[\widehat{Y}_t^{t,x} - \widehat{Y}_{t'}^{t',x'} | \mathcal{F}_T^B\right]$$

and, thus,

$$\begin{aligned}
\left|\widehat{u}(t,x) - \widehat{u}(t',x')\right|^p &= \left|E\left[\widehat{Y}_t^{t,x} - \widehat{Y}_{t'}^{t',x'} | \mathcal{F}_T^B\right]\right|^p \\
&\le CE\left[\left|\widehat{Y}_t^{t,x} - \widehat{Y}_t^{t',x'}\right|^p | \mathcal{F}_T^B\right] + C\left|E\left[\widehat{Y}_t^{t',x'} - \widehat{Y}_{t'}^{t',x'} | \mathcal{F}_T^B\right]\right|^p,
\end{aligned}$$

where, $P$-a.s., by Lemma 4.2,

$$E\left[\left|\widehat{Y}_t^{t,x} - \widehat{Y}_t^{t',x'}\right|^p | \mathcal{F}_T^B\right] \le C\exp\{pI_T^*\}\left((1+|x|^p+|x'|^p)|t-t'|^{p/2} + |x-x'|^p\right),$$

and

$$\begin{aligned}
&\left|E\left[\widehat{Y}_t^{t',x'} - \widehat{Y}_{t'}^{t',x'} | \mathcal{F}_T^B\right]\right|^p \\
\le&\left|E\left[\int_t^{t'}\left|f\left(s, X_s^{t',x'}, \widehat{Y}_s^{t',x'}\varepsilon_s(T_s), \widehat{Z}_s^{t',x'}\varepsilon_s(T_s)\right)\varepsilon_s^{-1}(T_s)\right|\mathrm{d}s|\mathcal{F}_T^B\right]\right|^p \\
\le&\left|E\left[\int_t^{t'} C\left(\varepsilon_s^{-1}(T_s)f(s,0,0,0)\left(1+\left|X_s^{t',x'}\right|\right) + \left|\widehat{Y}_s^{t',x'}\right| + \left|\widehat{Z}_s^{t',x'}\right|\right)\mathrm{d}s|\mathcal{F}_T^B\right]\right|^p \\
\le& C(1+|x'|^p)|t-t'|^p\exp\{pI_T^*\} \\
&+ C|t-t'|^{p/2}\left(E\left[\int_t^{t'}\left(\left|\widehat{Y}_s^{t',x'}\right|^2 + \left|\widehat{Z}_s^{t',x'}\right|^2\right)\mathrm{d}s|\mathcal{F}_T^B\right]\right)^{p/2} \\
\le& C(1+|x'|^p)\exp\{pI_T^*\}|t-t'|^{p/2}.
\end{aligned}$$

Consequently, for all $(t,x), (t',x') \in [0,T]\times\mathbb{R}^d$, $p \ge 1$, $P$-a.s.,

$$|\widehat{u}(t,x) - \widehat{u}(t',x')| \le C\exp\{I_T^*\}\left((1+|x|+|x'|)|t-t'|^{1/2} + |x-x'|\right).$$

Hence,

$$E\left[|\widehat{u}(t,x) - \widehat{u}(t',x')|^p\right] \le C_p\left((1+|x|+|x'|)^p|t-t'|^{p/2} + |x-x'|^p\right),$$

and Kolmogorov's continuity criterion gives the existence of a continuous version of $\widehat{u}$. ■

Henceforth we denote by $\mathcal{L}$ the second-order differential operator:

$$\mathcal{L} := \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(x)D^2_{xx}) + b(x)\nabla_x,$$

and we consider the following stochastic partial differential equations:

$$\begin{cases} \mathrm{d}\widehat{u}(t,x) = \left[\mathcal{L}\widehat{u}(t,x) + f\left(t,x,\widehat{u}(t,x)\varepsilon_t(T_t), \nabla_x\widehat{u}(t,x)\sigma(x)\varepsilon_t(T_t)\right)\varepsilon_t^{-1}(T_t)\right]\mathrm{d}t, \\ \hfill t\in[0,T]; \\ \widehat{u}(0,x) = \Phi(x). \end{cases} \tag{4.8}$$

and

$$\begin{cases} \mathrm{d}u(t,x) = \left[\mathcal{L}u(t,x) + f\left(t,x,u(t,x),\nabla_x u(t,x)\sigma(x)\right)\right]\mathrm{d}t + \gamma_t u(t,x)\mathrm{d}B_t, \quad t\in[0,T]; \\ u(0,x) = \Phi(x). \end{cases} \tag{4.9}$$

Our objective is to characterize $\widehat{u}(t,x) = \widehat{Y}_t^{t,x}$ and $u(t,x) = \widehat{u}(A_t,t,x)\varepsilon_t = Y_t^{t,x}$ as the viscosity solutions of the above stochastic partial differential equations (4.8) and (4.9), respectively.

**Remark 4.5.** *In fact, equation* (4.8) *is a partial differential equation with random coefficients which can be solved pathwisely, and the equation* (4.9) *is a stochastic partial differential equation driven by the fractional Brownian motion $B$.*

First we give the definition of a pathwise viscosity solution of SPDE (4.8).

**Definition 4.6.** *A real valued continuous random field $\widehat{u} : \Omega' \times [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}$ is called a pathwise viscosity solution of equation* (4.8) *if there exists a subset $\overline{\Omega}'$ of $\Omega'$ with $P'(\overline{\Omega}') = 1$, such that for all $\omega' \in \overline{\Omega}'$, $\widehat{u}(\omega',\cdot,\cdot)$ is a viscosity solution for the PDE* (4.8) *at $\omega'$.*

For the definition of the viscosity solution, which is a well-known concept by now, we refer to the *User's Guide* by Crandall et al. [28].

For the proof that $\widehat{u}(t,x)$ is the pathwise viscosity solution of equation (4.8), we need the following two auxiliary results.

**Lemma 4.7.** (*Comparison result.*) *Let* (**H3**) *hold. Let $\left(\widehat{Y}^1(\omega',\cdot), \widehat{Z}^1(\omega',\cdot)\right)$ and $(\widehat{Y}^2(\omega',\cdot), \widehat{Z}^2(\omega',\cdot))$ be the solutions of BSDE* (3.5) *with coefficients $(\xi_1,f_1)$ and $(\xi_2,f_2)$, respectively. Then, if $\xi_1 \le \xi_2$ and $f_1 \le f_2$, it holds that $\widehat{Y}_t^1(\omega',\cdot) \le \widehat{Y}_t^2(\omega',\cdot)$, $t\in[0,T]$, $P^W$-a.s., for all $\omega' \in \widetilde{\Omega}'$.*

*Proof:* For the proof the reader is referred to the comparison for BSDEs by Peng [89], or also to Buckdahn and Ma [16]. ■

**Lemma 4.8.** (*A priori estimate.*) *Let $\omega' \in \widetilde{\Omega}'$ and let $\left(\widehat{Y}^1, \widehat{Z}^1\right)$ and $\left(\widehat{Y}^2, \widehat{Z}^2\right)$ be the solutions of BSDE* (3.5) *with coefficients $(\xi_1,f_1)$ and $(\xi_2,f_2)$, respectively, and put $\delta\widehat{Y}(\omega',\cdot) = \widehat{Y}^1(\omega',\cdot) - \widehat{Y}^2(\omega',\cdot)$, $\delta\xi = \xi^1 - \xi^2$ and $\delta f_s(\omega',\cdot) = (f_1 - f_2)\big(s, \widehat{Y}_s^2(\omega',\cdot)\varepsilon_s(T_s,\omega'), \widehat{Z}_s^2(\omega',\cdot)$ $\varepsilon_s(T_s,\omega')\big)\ \varepsilon_s^{-1}(T_s,\omega')$. Moreover, let $\widetilde{C}$ be the Lipschitz constant of $f_1$. Then there exist $\beta \ge \widetilde{C}(2+\nu^2)+\mu^2, \nu>0, \mu>0$, such that $P^W$-a.s., for $0 \le s \le T$, and for all $\omega' \in \widetilde{\Omega}'$,*

$$\begin{aligned} &E^W\left[\exp\{\beta(T-s)\}\left|\delta\widehat{Y}_s(\omega',\cdot)\right|^2\right] \\ \le& E^W\left[\exp\{\beta T\}\,|\delta\xi|^2 + \frac{1}{\mu^2}\int_0^s \exp\{\beta(T-r)\}|\delta f_r(\omega',\cdot)|^2\mathrm{d}r\right], \end{aligned} \tag{4.10}$$

$$E^W\left[\sup_{0\le r\le T}\left|\delta\widehat{Y}_r(\omega',\cdot)\right|^2+\int_0^T\left|\delta\widehat{Z}_r(\omega',\cdot)\right|^2\mathrm{d}r\right]\le CE^W\left[|\delta\xi|^2+\int_0^T|\delta f_r(\omega',\cdot)|^2\mathrm{d}r\right]. \tag{4.11}$$

*Proof:* Let $\omega'\in\widetilde{\Omega}'$. By applying Itô's formula to $\left|\delta\widehat{Y}_s(\omega',\cdot)\right|^2\exp\{\beta(T-s)\}$, we obtain

$$\begin{aligned}
&\left|\delta\widehat{Y}_s(\omega',\cdot)\right|^2\exp\{\beta(T-s)\}+\int_0^s\exp\{\beta(T-r)\}\left(\beta\left|\delta\widehat{Y}_r(\omega',\cdot)\right|^2+\left|\delta\widehat{Z}_r(\omega',\cdot)\right|^2\right)\mathrm{d}r\\
&\quad+2\int_0^s\exp\{\beta(T-r)\}\delta\widehat{Y}_r(\omega',\cdot)\delta\widehat{Z}_r(\omega',\cdot)\downarrow\mathrm{d}W_r\\
=&|\delta\xi|^2\exp\{\beta T\}+2\int_0^s\delta\widehat{Y}_r(\omega',\cdot)\exp\{\beta(T-r)\}\varepsilon_r^{-1}(T_r,\omega')\\
&\qquad\times\Big[f_1\left(r,\widehat{Y}_r^1(\omega',\cdot)\varepsilon_r(T_r,\omega'),\widehat{Z}_r^1(\omega',\cdot)\varepsilon_r(T_r,\omega')\right)\\
&\qquad\quad-f_2\left(r,\widehat{Y}_r^2(\omega',\cdot)\varepsilon_r(T_r,\omega'),\widehat{Z}_r^2(\omega',\cdot)\varepsilon_r(T_r,\omega')\right)\Big]\mathrm{d}r,
\end{aligned}$$

and by taking the expectation with respect to $P^W$ on both sides we get, for $\nu>0,\mu>0$, $P$-a.s.,

$$\begin{aligned}
&E^W\Big[\exp\{\beta(T-s)\}\left|\delta\widehat{Y}_s(\omega',\cdot)\right|^2\\
&\qquad+\int_0^s\exp\{\beta(T-r)\}\left(\beta\left|\delta\widehat{Y}_r(\omega',\cdot)\right|^2+\left|\delta\widehat{Z}_r(\omega',\cdot)\right|^2\right)\mathrm{d}r\Big]\\
\le\ &E^W\left[|\delta\xi|^2\exp\{\beta T\}\right]+E^W\left[\int_0^s\exp\{\beta(T-r)\}\big((2+\nu^2)\widetilde{C}+\mu^2\big)\left|\delta\widehat{Y}_r(\omega',\cdot)\right|^2\right]\mathrm{d}r\\
&+E^W\left[\int_0^s\exp\{\beta(T-r)\}\left[\widetilde{C}\left|\delta\widehat{Z}_r(\omega',\cdot)\right|^2/\nu^2+|\delta f_r(\omega',\cdot)|^2/\mu^2\right]\mathrm{d}r\right].
\end{aligned}$$

Finally, by choosing $\beta\ge\widetilde{C}(2+\nu^2)+\mu^2$, with $\nu^2>\widetilde{C}$, we obtain that $P$-a.s.,

$$\begin{aligned}
&E^W\left[\exp\{\beta(T-s)\}\left|\delta\widehat{Y}_s(\omega',\cdot)\right|^2\right]\\
\le&E^W\left[\exp\{\beta T\}\,|\delta\xi|^2+\int_0^s\exp\{\beta(T-r)\}\frac{1}{\mu^2}|\delta f_r(\omega',\cdot)|^2\mathrm{d}r\right],
\end{aligned}$$

which is exactly (4.10).

Estimate (4.11) can be proven by the arguments developed for (4.7). ■

Let us now turn to the solutions of our SPDEs. The next theorem is one of the main results of this section. For this let $\widehat{\Omega}'=\left\{\omega'\in\widetilde{\Omega}'|\widehat{u}(\omega',\cdot,\cdot)\text{ is continuous}\right\}$ and notice that in light of Lemma 4.4, $P^B(\widehat{\Omega}')=1$.

**Theorem 4.9.** *The random field $\widehat{u}$ defined by $\widehat{u}(\omega',t,x)=\widehat{Y}_t^{t,x}(\omega')$ for all $\omega'\in\widehat{\Omega}'$ is a pathwise viscosity solution of equation (4.8), where $\widehat{Y}^{t,x}(\omega',\cdot)$ is the solution of equation (4.5). Furthermore, this solution $\widehat{u}(\omega',t,x)$ is unique in the class of continuous stochastic fields $\tilde{u}:\Omega'\times[0,T]\times\mathbb{R}^d\mapsto\mathbb{R}$ such that, for some random variable $\eta\in L^0(\mathcal{F}_T^B)$,*

$$|\tilde{u}(\omega',t,x)|\le\eta(\omega')(1+|x|),(t,x)\in[0,T]\times\mathbb{R}^d,P^B(\mathrm{d}\omega')\text{-}a.s.$$

**Remark 4.10.** *The uniqueness of the solution is to be understood as a $P$-almost sure one: let $\widehat{u}_i (i=1,2)$ be such that $\widehat{u}_i(\omega')$ is a viscosity solution of PDE (4.8) at $\omega'$, for all $\omega' \in \widehat{\Omega}'_i$. Then, by the uniqueness result of viscosity solution of deterministic PDEs (see: Pardoux [82], Theorem 6.14) we know that $\widehat{u}_1(\omega',\cdot) = \widehat{u}_2(\omega',\cdot)$, for all $\omega' \in \widehat{\Omega}'_1 \cap \widehat{\Omega}'_2$. In particular, $\widehat{u}_1 = \widehat{u}_2$, $P$-a.s.*

*Proof of Theorem 4.9:* We adapt the method used in the paper of El Karoui et al.[42] to show that $\widehat{u}$ is a pathwise viscosity subsolution of equation (4.8). Recall that the set $\widetilde{\Omega}' := \{\omega' \in \Omega' | I^*_T(\omega') < +\infty\}$ satisfies $P^B(\widetilde{\Omega}') = 1$. We work here on the set $\widehat{\Omega}' := \Big\{\omega' \in \widetilde{\Omega}' | \widehat{u}(\omega',\cdot,\cdot) \text{ is continuous}\Big\}$ which satisfies $P^B(\widehat{\Omega}') = 1$ in light of Lemma 4.4.

Now, according to the definition of the viscosity solution, for an arbitrarily chosen $\omega' \in \widehat{\Omega}'$, we fix arbitrarily a point $(t,x) \in [0,T] \times \mathbb{R}^d$ and a test function $\varphi \in C_b^\infty$ such that $\varphi(t,x) = \widehat{u}(\omega',t,x)$ and $\varphi \geq \widehat{u}(\omega',\cdot,\cdot)$.

For $t \in [0,T]$ and $h \geq 0$, we have, thanks to the Proposition 4.3 and equation (4.5),

$$\begin{aligned}\widehat{u}\left(\omega',t,x\right) =& \widehat{u}\left(\omega',t-h,X^{t,x}_{t-h}\right) + \int_{t-h}^t F_r\left(\omega', X^{t,x}_r, \widehat{Y}^{t,x}_r(\omega',\cdot), \widehat{Z}^{t,x}_r(\omega',\cdot)\right)\mathrm{d}r \\ & - \int_{t-h}^t \widehat{Z}^{t,x}_r(\omega',\cdot) \downarrow \mathrm{d}W_r.\end{aligned}$$

We emphasize that for fixed $\omega'$, this BSDE can be viewed as a classical BSDE with respect to $W$ and we recall that $\widehat{Y}^{t,x}_{t-h}(\omega',\cdot) = \widehat{u}\left(\omega',t-h,X^{t,x}_{t-h}\right)$. Now for the fixed $\omega' \in \widehat{\Omega}'$, it holds that

$$\begin{aligned}\varphi(t,x) \leq& \varphi\left(t-h,X^{t,x}_{t-h}\right) + \int_{t-h}^t F_r\left(\omega', X^{t,x}_r, \widehat{Y}^{t,x}_r(\omega',\cdot), \widehat{Z}^{t,x}_r(\omega',\cdot)\right)\mathrm{d}r \\ & - \int_{t-h}^t \widehat{Z}^{t,x}_r(\omega',\cdot) \downarrow \mathrm{d}W_r.\end{aligned}$$

Let $\left(\overline{Y}^{t,x,h}(\omega',\cdot), \overline{Z}^{t,x,h}(\omega',\cdot)\right) \in L^2_{\mathbb{G}}\left(t-h,t;\mathbb{R}\times\mathbb{R}^d\right)$ be the solution of the following equation evaluated at $\omega'$: for $s \in [t-h,t]$,

$$\begin{aligned}\overline{Y}^{t,x,h}_s(\omega',\cdot) =& \varphi\left(t-h,X^{t,x}_{t-h}\right) + \int_{t-h}^s F_r\left(\omega', X^{t,x}_r, \overline{Y}^{t,x,h}_r(\omega',\cdot), \overline{Z}^{t,x,h}_r(\omega',\cdot)\right)\mathrm{d}r \\ & - \int_{t-h}^s \overline{Z}^{t,x,h}_r(\omega',\cdot) \downarrow \mathrm{d}W_r.\end{aligned} \tag{4.12}$$

From Lemma 4.7, it follows that $\overline{Y}^{t,x,h}_t(\omega',\cdot) \geq \varphi(t,x) = \widehat{u}(\omega',t,x)$. Now we put

$$\widetilde{Y}^{t,x}_s(\omega',\cdot) = \overline{Y}^{t,x,h}_s(\omega',\cdot) - \varphi\left(s,X^{t,x}_s\right) - \int_{t-h}^s G(\omega',r,x)\mathrm{d}r \tag{4.13}$$

and

$$\widetilde{Z}^{t,x}_s(\omega',\cdot) = \overline{Z}^{t,x,h}_s(\omega',\cdot) - (\nabla_x\varphi\sigma)(s,X^{t,x}_s),$$

where $G(\omega',s,x) = \partial_s\varphi(s,x) - \mathcal{L}\varphi(s,x) - F_s\big(\omega',x,\varphi(s,x),\nabla_x\varphi(s,x)\sigma(x)\big)$. From the equations (4.12), (4.13) and Itô's formula we have

$$\widetilde{Y}^{t,x}_s(\omega',\cdot) \quad = \quad \int_{t-h}^s \Big[F_r\Big(\omega', X^{t,x}_r, \varphi(r,X^{t,x}_r) + \widetilde{Y}^{t,x}_r(\omega',\cdot) + \int_{t-h}^r G(\omega',s,x)\mathrm{d}s,$$

$$(\nabla_x\varphi\sigma)(r,X_r^{t,x})+\widetilde{Z}_r^{t,x}(\omega',\cdot)\Big)-(\partial_r\varphi-\mathcal{L}\varphi)(r,X_r^{t,x})+G(\omega',r,x)\Big]\mathrm{d}r$$
$$-\int_{t-h}^{s}\widetilde{Z}_r^{t,x}(\omega',\cdot)\downarrow\mathrm{d}W_r.$$

Putting

$$\begin{aligned}\delta(\omega',r,h)=&F_r\left(\omega',X_r^{t,x},\varphi(r,X_r^{t,x})+\int_{t-h}^{r}G(\omega',s,x)\mathrm{d}s,(\nabla_x\varphi\sigma)(r,X_r^{t,x})\right)\\&-(\partial_r\varphi-\mathcal{L}\varphi)(r,X_r^{t,x})+G(\omega',r,x),\end{aligned}$$

we have $|\delta(\omega',r,h)|\le\kappa(\omega')\left|X_r^{t,x}-x\right|, r\in[0,t]$, for some $\mathcal{F}_T^B$-measurable and $P^B$-integrable $\kappa:\Omega'\mapsto\mathbb{R}^+$. From the a priori estimate (4.10), it follows that

$$\begin{aligned}&E^W\left[\sup_{t-h\le s\le t}\left|\widetilde{Y}_s^{t,x}(\omega',\cdot)\right|^2+\int_{t-h}^{t}\left|\widetilde{Z}_r^{t,x}(\omega',\cdot)\right|^2\mathrm{d}r\right]\\&\le CE^W\left[\int_{t-h}^{t}\left|\delta(\omega',r,h)\right|^2\mathrm{d}r\right]=h\rho(\omega',h).\end{aligned}\tag{4.14}$$

where $\rho(\omega',h)$ tends to 0 as $h\to 0$. Consequently, it yields

$$E^W\left[\int_{t-h}^{t}\left(\left|\widetilde{Y}_r^{t,x}(\omega',\cdot)\right|+\left|\widetilde{Z}_r^{t,x}(\omega',\cdot)\right|\right)\mathrm{d}r\right]=h\sqrt{\rho}(\omega',h).\tag{4.15}$$

Furthermore, we have $\widetilde{Y}_t^{t,x}(\omega')=E^W\left[\widetilde{Y}_t^{t,x}(\omega',\cdot)\right]=E^W\left[\int_{t-h}^{t}\delta'(\omega',r,h)\mathrm{d}r\right]$, where

$$\begin{aligned}\delta'(\omega',&r,h)=-(\partial_r\varphi-\mathcal{L}\varphi)(r,X_r^{t,x})+G(\omega',r,x)+F_r\big(\omega',X_r^{t,x},\\&\varphi(r,X_r^{t,x})+\widetilde{Y}_r^{t,x}(\omega',\cdot)+\int_{t-h}^{r}G(\omega',s,x)\mathrm{d}s,(\nabla_x\varphi\sigma)(r,X_r^{t,x})+\widetilde{Z}_r^{t,x}(\omega',\cdot)\big).\end{aligned}$$

From the fact that $f$ is Lipschitz and the estimates (4.14) and (4.15), we get

$$\left|\widetilde{Y}_t^{t,x}(\omega')\right|\le E^W\left[\int_{t-h}^{t}\left(|\delta(\omega',r,h)|+\left|\widetilde{Y}_r^{t,x}(\omega',\cdot)\right|+\left|\widetilde{Z}_r^{t,x}(\omega',\cdot)\right|\right)\mathrm{d}r\right]=h\rho(\omega',h).$$

Thus, from (4.13) (for $s=t$) and since $\overline{Y}_t^{t,x}(\omega',\cdot)\ge\varphi(t,x)$, we obtain $\int_{t-h}^{t}G(\omega',r,x)\mathrm{d}r\ge -h\rho(\omega',h)$. Consequently, $\frac{1}{h}\int_{t-h}^{t}G(\omega',r,x)\mathrm{d}r\ge-\rho(\omega',h)$. By letting $h$ tend to 0, we finally get for $\omega'\in\widehat{\Omega}'$,

$$G(\omega',t,x)=\partial_t\varphi(t,x)-\mathcal{L}\varphi(t,x)-F_t\big(\omega',x,\varphi(t,x),\nabla_x\varphi(t,x)\sigma(x)\big)\ge 0.$$

Hence $\widehat{u}(\omega',t,x)$ is a pathwise viscosity subsolution of (4.8). The proof of $\widehat{u}$ being a pathwise viscosity supersolution is similar.

The proof of uniqueness becomes clear from Remark 4.10. ■

In analogy to the relation between the solutions $(Y,Z)$ and $(\widehat{Y},\widehat{Z})$ of the associated BDSDE and the BSDE, respectively, we shall expect that $u(t,x)=Y_t^{t,x}=\widehat{Y}_t^{t,x}(A_t)\varepsilon_t$, $(t,x)\in[0,T]\times\mathbb{R}^d$, is a solution of SPDE (4.9). This claim is confirmed by

**Proposition 4.11.** *Suppose that $u, \widehat{u}$ are $C^{0,2}$-stochastic fields over $\Omega' \times [0,T] \times \mathbb{R}^d$ such that there exist $\delta > 0$ and a constant $C_{\delta,x} > 0$ (only depending on $\delta$ and $x$) with: for $t \in [0,T]$, $w = u, \widehat{u}$,*

$$E\left[|w(t,x)|^{2+\delta} + \int_0^t \left(|\nabla_x w(s,x)|^{2+\delta} + |D_{xx}^2 w(s,x)|^{2+\delta}\right) \mathrm{d}s\right] \leq C_{\delta,x}. \tag{4.16}$$

*Then $\widehat{u}(t,x)$ is a classical pathwise solution of equation (4.8) if and only if $u(t,x)$ is a classical solution of SPDE (4.9).*

*Proof:* We restrict ourselves to show $u(t,x)$ solves equation (4.9) whenever $\widehat{u}$ solves (4.8). For this, we proceed in analogy to the proof of Theorem 3.9. Let $F$ be an arbitrary but fixed element of $\mathcal{S}'_{\mathcal{K}}$. By using the Girsanov transformation we have

$$\begin{aligned}
&E\left[u(t,x)F - u(0,x)F\right] \\
=&E\left[\widehat{u}(A_t,t,x)\varepsilon_t F - \widehat{u}(0,x)F\right] \\
=&E\left[F(T_t)\widehat{u}(t,x) - F\widehat{u}(0,x)\right] \\
=&E\left[F(T_t)\widehat{u}(0,x) - F\widehat{u}(0,x)\right] \\
&+ E\left[F(T_t)\int_0^t \left[\mathcal{L}\widehat{u}(s,x) + f\big(s,x,\widehat{u}(s,x)\varepsilon_s(T_s), \nabla_x \widehat{u}(s,x)\sigma(x)\varepsilon_s(T_s)\big)\varepsilon_s^{-1}(T_s)\right] \mathrm{d}s\right].
\end{aligned}$$

As in the proof of Theorem 3.9 we use the fact that $\frac{\mathrm{d}}{\mathrm{d}t}F(T_t) = \gamma_t(\mathcal{K}^*\mathcal{K}D^B F)(t,T_t)$ to deduce the following

$$\begin{aligned}
&E\left[u(t,x)F - u(0,x)F\right] \\
=&E\left[\widehat{u}(0,x)\int_0^t \gamma_s(\mathcal{K}^*\mathcal{K}D^B F(T_s))(s)\mathrm{d}s\right] + E\left[\int_0^t \int_s^t \gamma_r(\mathcal{K}^*\mathcal{K}D^B F(T_r))(r)\mathrm{d}r\mathcal{L}\widehat{u}(s,x)\mathrm{d}s\right] \\
&+ E\left[\int_0^t F(T_s)\left[\mathcal{L}\widehat{u}(s,x) + f\big(s,x,\widehat{u}(s,x)\varepsilon_s(T_s), \nabla_x \widehat{u}(s,x)\sigma(x)\varepsilon_s(T_s)\big)\varepsilon_s^{-1}(T_s)\right] \mathrm{d}s\right] \\
+&E\left[\int_0^t \int_s^t \gamma_r(\mathcal{K}^*\mathcal{K}D^B F(T_r))(r)\mathrm{d}r f\big(s,x,\widehat{u}(s,x)\varepsilon_s(T_s), \nabla_x \widehat{u}(s,x)\sigma(x)\varepsilon_s(T_s)\big)\varepsilon_s^{-1}(T_s)\mathrm{d}s\right].
\end{aligned}$$

Thanks to the assumption that $\widehat{u}(t,x)$ is a pathwise classical solution of equation (4.8), we obtain

$$\begin{aligned}
&E\left[u(t,x)F - u(0,x)F\right] \\
=&E\left[\int_0^t F(T_s)\left[\mathcal{L}\widehat{u}(s,x) + f\big(s,x,\widehat{u}(s,x)\varepsilon_s(T_s), \nabla_x \widehat{u}(s,x)\sigma(x)\varepsilon_s(T_s)\big)\varepsilon_s^{-1}(T_s)\right] \mathrm{d}s\right] \\
&+ E\left[\int_0^t \gamma_s(\mathcal{K}^*\mathcal{K}D^B F(T_s))(s)\widehat{u}(s,x)\mathrm{d}s\right] \\
=&E\left[\int_0^t F\left[\mathcal{L}u(s,x) + f\big(s,x,u(s,x), \nabla_x u(s,x)\sigma(x)\big)\right] \mathrm{d}s\right] \\
&+ E\left[\int_0^t \gamma_s(\mathcal{K}^*\mathcal{K}D^B F)(s)u(s,x)\mathrm{d}s\right].
\end{aligned}$$

Consequently,

$$\begin{aligned}
&E\left[\int_0^t \gamma_s(\mathcal{K}^*\mathcal{K}D^B F)(s)u(s,x)\mathrm{d}s\right] \\
=&E\left[F\Big(u(t,x) - u(0,x) - \int_0^t \left[\mathcal{L}u(s,x) + f\big(s,x,u(s,x), \nabla_x u(s,x)\sigma(x)\big)\right] \mathrm{d}s\Big)\right].
\end{aligned}$$

From the integrability condition (4.16) we know that

$$u(t,x) - u(0,x) - \int_0^t \left[\mathcal{L}u(s,x) + f\big(s,x,u(s,x),\nabla_x u(s,x)\sigma(x)\big)\right] \mathrm{d}s \in L^2(\Omega,\mathcal{F},P).$$

Moreover, $\gamma 1_{[0,t]} u \in L^2([0,T]\times\Omega)$. Indeed,

$$\begin{aligned} &E\left[\int_0^T |\gamma_r 1_{[0,t]}(r) u(r,x)|^2 \mathrm{d}r\right] = \int_0^T |\gamma_r 1_{[0,t]}(r)|^2 E[|u(r,x)|^2]\mathrm{d}r \\ \le& C_{\delta,x}\int_0^T |\gamma_r 1_{[0,t]}(r)|^2 \mathrm{d}r < \infty. \end{aligned}$$

By Definition 3.2 we get

$$\begin{aligned} &E\left[F\int_0^t \gamma_s u(s,x)\mathrm{d}B_s\right] \\ =&E\left[F\Big(u(t,x) - u(0,x) - \int_0^t \left[\mathcal{L}u(s,x) + f\big(s,x,u(s,x),\nabla_x u(s,x)\sigma(x)\big)\right] \mathrm{d}s\Big)\right]. \end{aligned}$$

It then follows from the arbitrariness of $F \in \mathcal{S}'_{\mathcal{K}}$ that

$$u(t,x) = \Phi(x) + \int_0^t \left[\mathcal{L}u(s,x) + f\big(s,x,u(s,x),\nabla_x u(s,x)\sigma(x)\big)\right] \mathrm{d}s + \int_0^t \gamma_s u(s,x)\mathrm{d}B_s.$$

The proof is complete now. ∎

**Remark 4.12.** *The regularity of $\widehat{u}$ in the above proposition is difficult to get under not too restrictive assumptions (like coefficients of class $C^3_{l,b}$, linearity of $f$ in $z$).*

**Remark 4.13.** *Notice that generally speaking, a continuous random field after Girsanov transformation $A_t$ is not necessarily continuous in $t$ any more. We give a simple counterexample:*
*Let $0 < s < T$ be fixed and*

$$\widehat{u}(t,x) = \begin{cases} (t-s)B_t, & 0 \le t \le s; \\ (t-s)\mathrm{sgn}(B_s), & s < t \le T. \end{cases}$$

*It is obvious that $\widehat{u}(t,x)$ is $\mathcal{F}^B_t$-measurable and continuous in $t$. But after Girsanov transformation $A_t$, it becomes*

$$u(t,x) = \widehat{u}(A_t,t,x)\varepsilon_t = \begin{cases} (t-s)\left(B_t - \int_0^t (\mathcal{K}\gamma \boldsymbol{1}_{[0,t]})(r)\mathrm{d}r\right)\varepsilon_t, & 0 \le t \le s; \\ (t-s)\mathrm{sgn}\left(B_s - \int_0^s (\mathcal{K}\gamma \boldsymbol{1}_{[0,t]})(r)\mathrm{d}r\right)\varepsilon_t, & s < t \le T, \end{cases}$$

*which is not continuous in $t$ on*

$$\left\{\omega' : \inf_{t\in[s,T]}\left(B_s - \int_0^s \left(\mathcal{K}\gamma 1_{[0,t]}\right)(r)\mathrm{d}r\right) < 0 < \sup_{t\in[s,T]}\left(B_s - \int_0^s \left(\mathcal{K}\gamma 1_{[0,t]}\right)(r)\mathrm{d}r\right)\right\}.$$

However, as we state below, the random field $u$ has a continuous version in our case. To this end we need the following technical result:

**Lemma 4.14.** *Let $\gamma$ be such that* (**H1**) *holds. Then there exist positive constants $C$ and $q$ such that for all $r, v, v' \in [0,T]$, $v \leq v'$,*

$$\left| \int_0^T (\mathcal{K}\gamma 1_{[v',v]})(s)(\mathcal{K}\gamma 1_{[0,r]})(s)\mathrm{d}s \right| \leq C|v-v'|^q.$$

*Proof:* We have

$$\begin{aligned}
&\left| \int_0^T (\mathcal{K}\gamma 1_{[v',v]})(s)(\mathcal{K}\gamma 1_{[0,r]})(s)\mathrm{d}s \right| \\
\leq &\left( \int_0^T (\mathcal{K}\gamma 1_{[v',v]})^2(s)\mathrm{d}s \right)^{1/2} \left( \int_0^T (\mathcal{K}\gamma 1_{[0,r]})^2(s)\mathrm{d}s \right)^{1/2} \\
\leq &C \left( \int_0^T (\mathcal{K}\gamma 1_{[v',v]})^2(s)\mathrm{d}s \right)^{1/2},
\end{aligned}$$

where the last inequality follows from [68] (Lemma 2.3). Also, by the proof of Lemma 2.3 in [68] and using the notation $\alpha = 1/2 - H$, we have

$$\begin{aligned}
&(\mathcal{K}\gamma 1_{[v',v]})(s) \\
=&1_{[v',v]}(s) \left( \phi_\gamma(s) + \frac{\alpha s^\alpha}{\Gamma(1-\alpha)} \int_v^T \frac{r^{-\alpha}\gamma_r}{(r-s)^{1+\alpha}}\mathrm{d}r \right) \\
&\qquad - 1_{[0,v']}(s) \frac{\alpha s^\alpha}{\Gamma(1-\alpha)} \int_{v'}^v \frac{r^{-\alpha}\gamma_r}{(r-s)^{1+\alpha}}\mathrm{d}r \\
=&I_1(s) + I_2(s).
\end{aligned} \tag{4.17}$$

Now, applying [68] (Lemma 2.3) again, we obtain

$$\begin{aligned}
&\int_0^T I_1(s)^2 \mathrm{d}s = \int_0^T 1_{[v',v]}(s) \left( 1_{[0,v]}(s) \left[ \phi_\gamma(s) + \frac{\alpha s^\alpha}{\Gamma(1-\alpha)} \int_v^T \frac{r^{-\alpha}\gamma_r}{(r-s)^{1+\alpha}}\mathrm{d}r \right] \right)^2 \mathrm{d}s \\
\leq &|v-v'|^{(p'-2)/p'} \|\phi_{\gamma 1_{[0,v]}}\|^2_{L^{p'}} \leq C|v-v'|^{(p'-2)/p'}.
\end{aligned} \tag{4.18}$$

On the other hand, for $m = 1+\eta$ and $q_m = m/\eta$, with $\eta$ small enough, we can write

$$\begin{aligned}
&\int_{v'}^v \frac{r^{-\alpha}\gamma_r}{(r-s)^{1+\alpha}}\mathrm{d}r \\
\leq \quad & |v-v'|^{1/q_m} \left[ \int_{v'}^v \frac{r^{-m\alpha}|\gamma_r|^m}{(r-s)^{m(1+\alpha)}}\mathrm{d}r \right]^{1/m} \\
= \quad & \frac{1}{\Gamma(\alpha)} |v-v'|^{1/q_m} \left[ \int_{v'}^v \frac{\left( \int_r^T \phi_\gamma(\theta)\theta^{-\alpha}(\theta-r)^{\alpha-1}\mathrm{d}\theta \right)^m}{(r-s)^{m(1+\alpha)}}\mathrm{d}r \right]^{1/m} \\
\leq \quad & C|v-v'|^{\frac{1}{q_m}} \left[ \int_{v'}^T \frac{\int_r^T |\phi_\gamma(\theta)|^m \theta^{-m\alpha}(\theta-r)^{m(\alpha-1)}\mathrm{d}\theta}{(r-s)^{m(1+\alpha)}}\mathrm{d}r \right]^{1/m} \\
\leq \quad & C|v-v'|^{1/q_m} \left[ \int_{v'}^T |\phi_\gamma(\theta)|^m \theta^{-\eta} \int_{v'}^\theta (r-s)^{-m(1+\alpha)} r^{-m\alpha+\eta}(\theta-r)^{m(\alpha-1)}\mathrm{d}r\mathrm{d}\theta \right]^{1/m} \\
\leq \quad & C|v-v'|^{1/q_m}(v'-s)^{-2\eta/m}
\end{aligned}$$

$$\times \left[\int_{v'}^{T} |\phi_\gamma(\theta)|^m \theta^{-\eta} \int_{v'}^{\theta} (r-s)^{-m\alpha+\eta-1} r^{-m\alpha+\eta} (\theta-r)^{m\alpha-\eta-1} \mathrm{d}r \mathrm{d}\theta\right]^{1/m}.$$

Hence, [68] (Lemma 2.2) gives

$$\begin{aligned}
&\int_{v'}^{v} \frac{r^{-\alpha}\gamma_r}{(r-s)^{1+\alpha}} \mathrm{d}r \\
\leq\ & C|v-v'|^{1/q_m}(v'-s)^{-2\eta/m} \\
&\times \left[\int_{v'}^{T} |\phi_\gamma(\theta)|^m \theta^{-\eta} v'^{-m\alpha+\eta} (v'-s)^{-m\alpha+\eta} (\theta-s)^{-1} (\theta-v')^{m\alpha-\eta} \mathrm{d}\theta\right]^{1/m} \\
\leq\ & C|v-v'|^{1/q_m}(v'-s)^{-\alpha-\eta/m} \left[\int_{v'}^{T} |\phi_\gamma(\theta)|^m \theta^{-\eta} v'^{-m\alpha+\eta} (\theta-s)^{-1+m\alpha-\eta} \mathrm{d}\theta\right]^{1/m},
\end{aligned}$$

which, together with (4.17), implies, for $p < \frac{1}{\alpha}$, $p' = \frac{pm}{2(1-(m\alpha-\eta)p)}$ and $\frac{1}{p'} + \frac{1}{q'} = 1$,

$$\begin{aligned}
\int_0^T I_2(s)^2 \mathrm{d}s \leq & C|v-v'|^{\frac{2}{q_m}} \left(\int_0^{v'} (v'-s)^{2q'(-\alpha-\eta/m)} \mathrm{d}s\right)^{\frac{1}{q'}} \\
& \times \left(\int_0^{v'} \left[\int_{v'}^{T} |\phi_\gamma(\theta)|^m \theta^{-\eta} (\theta-s)^{-1+m\alpha-\eta} \mathrm{d}\theta\right]^{\frac{2p'}{m}} \mathrm{d}s\right)^{\frac{1}{p'}} \\
\leq & C|v-v'|^{2/q_m}.
\end{aligned}$$

Thus, we get the wished result. ■

**Lemma 4.15.** *The random field $u(t,x) := \widehat{u}(A_t,t,x)\varepsilon_t$, $(t,x) \in [0,T]\times\mathbb{R}^d$ has a continuous version.*

*Proof:* In the following, for simplicity of notations, we put

$$\Theta_r^{t,x,v,v'} = \left(r, X_r^{t,x}, \widehat{Y}_r^{t,x}(A_v)\varepsilon_r(T_r A_{v'}), \widehat{Z}_r^{t,x}(A_v)\varepsilon_r(T_r A_{v'})\right).$$

For $0 \leq v' \leq v \leq T$, we notice that $\left(\widehat{Y}^{t,x}(A_v)\right)$ is the solution of the BSDE

$$\widehat{Y}_s^{t,x}(A_v) = \Phi\left(X_0^{t,x}\right) + \int_0^s f\left(\Theta_r^{t,x,v,v}\right)\varepsilon_r^{-1}(T_r A_v)\mathrm{d}r - \int_0^s \widehat{Z}_r^{t,x}(A_v) \downarrow \mathrm{d}W_r, \tag{4.19}$$

while $(\widehat{Y}^{t,x}(A_{v'}))$ is the solution of the BSDE

$$\widehat{Y}_s^{t,x}(A_{v'}) = \Phi\left(X_0^{t,x}\right) + \int_0^s f\left(\Theta_r^{t,x,v',v'}\right)\varepsilon_r^{-1}(T_r A_{v'})\mathrm{d}r - \int_0^s \widehat{Z}_r^{t,x}(A_{v'}) \downarrow \mathrm{d}W_r.$$

We set $J_r^v = \exp\left\{\int_0^T (\mathcal{K}\gamma 1_{[0,v]})(s)(\mathcal{K}\gamma 1_{[0,r]})(s)\mathrm{d}s\right\}$, and we observe that $\varepsilon_r^{-1}(T_r A_v) = \varepsilon_r^{-1}(T_r)J_r^v$. Moreover, equation (4.19) can be written as follows:

$$\begin{aligned}
\widehat{Y}_s^{t,x}(A_v) = & \Phi\left(X_0^{t,x}\right) + \int_0^s f\left(r, X_r^{t,x}, \left(\widehat{Y}_r^{t,x}(A_v), \widehat{Z}_r^{t,x}(A_v)\right)\varepsilon_r(T_r)(J_r^v)^{-1}\right)\varepsilon_r^{-1}(T_r)J_r^v \mathrm{d}r \\
& - \int_0^s \widehat{Z}_r^{t,x}(A_v) \downarrow \mathrm{d}W_r, \quad s \in [0,t],
\end{aligned}$$

and a comparison with (4.5) suggests the similarity of arguments which can be applied. So, by a standard BSDE estimate, see, for instance, the proof of Lemma 4.2, we have

$$
\begin{aligned}
&E\left[\sup_{0\le s\le T}\left|\widehat{Y}^{t,x}_s(A_v)-\widehat{Y}^{t,x}_s(A_{v'})\right|^p\right]\\
\le& CE\left[\int_0^T\left|f(\Theta^{t,x,v,v}_r)\varepsilon_r^{-1}(T_rA_v)-f(\Theta^{t,x,v,v'}_r)\varepsilon_r^{-1}(T_rA_{v'})\right|^p\mathrm{d}r\right]\\
\le& CE\Bigg[\int_0^T\Bigg[\left|f(\Theta^{t,x,v,v}_r)\left(\varepsilon_r^{-1}(T_rA_v)-\varepsilon_r^{-1}(T_rA_{v'})\right)\right|^p\\
&\qquad+\left|\left(f(\Theta^{t,x,v,v}_r)-f(\Theta^{t,x,v,v'}_r)\right)\varepsilon_r^{-1}(T_rA_{v'})\right|^p\Bigg]\mathrm{d}r\Bigg].
\end{aligned}
\tag{4.20}
$$

Recalling that $\varepsilon_r^{-1}(T_rA_v)=\varepsilon_r^{-1}(T_r)J^v_r$ and applying Lemma 4.14, we get

$$
\begin{aligned}
&\left|\varepsilon_r^{-1}(T_rA_v)-\varepsilon_r^{-1}(T_rA_{v'})\right|^p=\varepsilon_r^{-p}(T_r)\left|J^v_r-J^{v'}_r\right|^p\\
\le& C\varepsilon_r^{-p}(T_r)\left|\int_0^T(\mathcal{K}\gamma 1_{[0,v]})(s)(\mathcal{K}\gamma 1_{[0,r]})(s)\mathrm{d}s-\int_0^T(\mathcal{K}\gamma 1_{[0,v']})(s)(\mathcal{K}\gamma 1_{[0,r]})(s)\mathrm{d}s\right|^p\\
=& C\varepsilon_r^{-p}(T_r)\left|\int_0^T(\mathcal{K}\gamma 1_{[v',v]})(s)(\mathcal{K}\gamma 1_{[0,r]})(s)\mathrm{d}s\right|^p\le C\varepsilon_r^{-p}(T_r)\left|v-v'\right|^{pq}.
\end{aligned}
\tag{4.21}
$$

On the other hand,

$$
\begin{aligned}
&\left|f(\Theta^{t,x,v,v}_r)-f(\Theta^{t,x,v,v'}_r)\right|^p\varepsilon_r^{-p}(T_rA_{v'})\\
\le& C\left(\left|\widehat{Y}^{t,x}_r(A_v)\right|^p+\left|\widehat{Z}^{t,x}_r(A_v)\right|^p\right)\left|\varepsilon_r(T_rA_v)-\varepsilon_r(T_rA_{v'})\right|^p\varepsilon_r^{-p}(T_rA_{v'})\\
=& C\left(\left|\widehat{Y}^{t,x}_r(A_v)\right|^p+\left|\widehat{Z}^{t,x}_r(A_v)\right|^p\right)\left|\exp\left\{-\int_0^T(\mathcal{K}\gamma 1_{[v',v]})(s)(\mathcal{K}\gamma 1_{[0,r]})(s)\mathrm{d}s\right\}-1\right|^p\\
\le& C\left(\left|\widehat{Y}^{t,x}_r(A_v)\right|^p+\left|\widehat{Z}^{t,x}_r(A_v)\right|^p\right)\left|v-v'\right|^{pq}.
\end{aligned}
\tag{4.22}
$$

Plugging estimates (4.21) and (4.22) into the equation (4.20), Lemma 4.2 yields that

$$
\begin{aligned}
&E\left[\sup_{0\le s\le T}\left|\widehat{Y}^{t,x}_s(A_v)-\widehat{Y}^{t,x}_s(A_{v'})\right|^p\right]\\
\le& CE\left[\int_0^T\left(\left|f(\Theta^{t,x,v,v}_r)\left(\varepsilon_r^{-1}(T_r)\right)\right|^p+\left|\widehat{Y}^{t,x}_r(A_v)\right|^p+\left|\widehat{Z}^{t,x}_r(A_v)\right|^p\right)\mathrm{d}r\right]\left|v-v'\right|^{pq}\\
\le& C(1+|x|^p)\left|v-v'\right|^{pq}.
\end{aligned}
$$

For the latter inequality we have used the following estimate of $\widehat{Z}$, which proof will be postponed until the end of the current proof.

**Lemma 4.16.** *There exists a constant $C_p$ such that* $E\left[\sup_{0\le s\le T}\left|\widehat{Z}^{t,x}_s\right|^p\right]\le C_p(1+|x|^p)$.

We continue our proof of Lemma 4.15: According to the proof of Lemma 4.4 we have

$$\begin{aligned}
&E\left[\sup_{0\le s\le T}\left|\widehat{Y}_s^{t',x'}(A_v)-\widehat{Y}_s^{t,x}(A_v)\right|^p\right]=E\left[E\left[\sup_{s\in[0,T]}\left|\widehat{Y}_s^{t',x'}-\widehat{Y}_s^{t,x}\right|^p\Big|\mathcal{F}_T^B\right]\circ A_v\right]\\
\le&\left(E\left[\sup_{s\in[0,T]}\left|\widehat{Y}_s^{t',x'}-\widehat{Y}_s^{t,x}\right|^{2p}\right]\right)^{1/2}\left(E\left[\varepsilon_v^{-2}(T_v)\right]\right)^{1/2}\\
\le&C\left((1+|x|^p+|x'|^p)|t-t'|^{p/2}+|x-x'|^p\right).
\end{aligned}$$

Hence, by combining the above estimates we obtain

$$\begin{aligned}
&E\left[\sup_{0\le s\le T}\left|\widehat{Y}_s^{t',x'}(A_{v'})-\widehat{Y}_s^{t,x}(A_v)\right|^p\right]\\
\le&C\left(E\left[\sup_{0\le s\le T}\left|\widehat{Y}_s^{t',x'}(A_{v'})-\widehat{Y}_s^{t',x'}(A_v)\right|^p\right]+E\left[\sup_{0\le s\le T}\left|\widehat{Y}_s^{t',x'}(A_v)-\widehat{Y}_s^{t,x}(A_v)\right|^p\right]\right)\\
\le&C\left((1+|x|^p+|x'|^p)(|t-t'|^{p/2}+|v-v'|^{pq})+|x-x'|^p\right).
\end{aligned}$$

Consequently, from the Kolmogorov continuity criterion we know the process $\{\widehat{Y}_s^{t,x}(A_v); s,t,v\in[0,T], x\in\mathbb{R}^d\}$ has an a.s. continuous version. From Lemma 4.14 we have that $\varepsilon_t$ is continuous in $t$. It then follows that $u(t,x)=\widehat{Y}_t^{t,x}(A_t)\varepsilon_t$ has a version which is jointly continuous in $t$ and $x$. ∎

*Proof of Lemma 4.16:* For simplicity we suppose all the functions $\Phi, f, \sigma, b$ are smooth. For the proof of the general case, the Lipschitz functions have to be approximated by smooth functions with the same Lipschitz constants. We define $(\underline{Y}^{t,x}, \underline{Z}^{t,x})$ to be the solution of the equation

$$\begin{aligned}
\underline{Y}_s^{t,x}=&\Phi'(X_0^{t,x})\nabla_x X_0^{t,x}+\int_0^s\left[f_x'(\Xi_r')\varepsilon_r^{-1}(T_r)\nabla_x X_r^{t,x}+f_y'(\Xi_r')\underline{Y}_r^{t,x}+f_z'(\Xi_r')\underline{Z}_r^{t,x}\right]\mathrm{d}r\\
&-\int_0^s\underline{Z}_r^{t,x}\downarrow\mathrm{d}W_r,
\end{aligned}\tag{4.23}$$

where $\Xi_r'=\left(r, X_r^{t,x}, \widehat{Y}_r^{t,x}\varepsilon_r(T_r), \widehat{Z}_r^{t,x}\varepsilon_r(T_r)\right)$ and

$$\nabla_x X_r^{t,x}=I-\int_r^t\nabla_x X_s^{t,x}b'\left(X_s^{t,x}\right)\mathrm{d}s-\int_r^t\nabla_x X_s^{t,x}\sigma'\left(X_s^{t,x}\right)\downarrow\mathrm{d}W_s,\ r\in[0,t].\tag{4.24}$$

With the arguments used in the proof of Lemma 4.2, we get

$$E\left[\sup_{0\le s\le T}|\underline{Y}_s^{t,x}|^p+\left(\int_0^T|\underline{Z}_s^{t,x}|^2\mathrm{d}s\right)^{p/2}\right]\le C_p(1+|x|^p).\tag{4.25}$$

By arguments which by now are standard, it can be seen that the processes $X^{t,x}, \widehat{Y}^{t,x}$ and $\widehat{Z}^{t,x}$ are Malliavin differentiable with respect to $W$, and thus, for $s\le u\le t\le T$,

$$\begin{aligned}
D_u^W\widehat{Y}_s^{t,x}=&\Phi'(X_0^{t,x})D_u^W X_0^{t,x}-\int_0^s D_u^W\widehat{Z}_r^{t,x}\downarrow\mathrm{d}W_r\\
&+\int_0^s\left[f_x'(\Xi_r')\varepsilon_r^{-1}(T_r)D_u^W X_r^{t,x}+f_y'(\Xi_r')D_u^W\widehat{Y}_r^{t,x}+f_z'(\Xi_r')D_u^W\widehat{Z}_r^{t,x}\right]\mathrm{d}r.
\end{aligned}\tag{4.26}$$

On the other hand, from the equation, for $\theta < s \le u \le T$,

$$\widehat{Y}_s^{t,x} = \widehat{Y}_\theta^{t,x} + \int_\theta^s f\left(r, X_r^{t,x}, \widehat{Y}_r^{t,x}\varepsilon_r(T_r), \widehat{Z}_r^{t,x}\varepsilon_r(T_r)\right)\varepsilon_r^{-1}(T_r)\mathrm{d}r - \int_\theta^s \widehat{Z}_r^{t,x} \downarrow \mathrm{d}W_r,$$

we get

$$\begin{aligned} D_u^W \widehat{Y}_s^{t,x} =& - \widehat{Z}_u^{t,x} + \int_s^u D_u^W \widehat{Z}_r^{t,x} \downarrow \mathrm{d}W_r \\ &+ \int_s^u \left[ f_x'(\Xi_r')\varepsilon_r^{-1}(T_r) D_u^W X_r^{t,x} + f_y'(\Xi_r') D_u^W \widehat{Y}_r^{t,x} + f_z'(\Xi_r') D_u^W \widehat{Z}_r^{t,x} \right] \mathrm{d}r, \end{aligned} \tag{4.27}$$

From the above three equations (4.23), (4.26) and (4.27) and the relation

$$D_r^W(X_s^{t,x}) = \nabla_x X_s^{t,x} (\nabla_x X_r^{t,x})^{-1} \sigma(X_r^{t,x}) \tag{4.28}$$

(see: Pardoux and Peng [84]), we obtain that, for $s \le u \le t \le T$,

$$D_u^W \widehat{Y}_s^{t,x} = \underline{Y}_s^{t,x} (\nabla_x X_u^{t,x})^{-1} \sigma(X_u^{t,x}),$$

$$\widehat{Z}_s^{t,x} = -\underline{Y}_s^{t,x} (\nabla_x X_s^{t,x})^{-1} \sigma(X_s^{t,x}).$$

Finally, from standard estimates for (4.24) and from the estimate (4.25), we get

$$\begin{aligned} & E\left[\sup_{0\le s\le T} \left|\widehat{Z}_s^{t,x}\right|^p\right] \\ \le & E\left[\sup_{0\le s\le T} \left|\underline{Y}_s^{t,x}\right|^p \sup_{0\le s\le T} \left|\nabla_x X_s^{t,x}\right|^{-p} \sup_{0\le s\le T} \left|\sigma(X_s^{t,x})\right|^p\right] \\ \le & E\left[\sup_{0\le s\le T} \left|\underline{Y}_s^{t,x}\right|^{pq_1}\right]^{1/q_1} E\left[\sup_{0\le s\le T} \left|\nabla_x X_s^{t,x}\right|^{-pq_2}\right]^{1/q_2} E\left[\sup_{0\le s\le T} \left|\sigma(X_s^{t,x})\right|^{pq_3}\right]^{1/q_3} \\ \le & C_p(1+|x|^p), \end{aligned}$$

where $1/q_1 + 1/q_2 + 1/q_3 = 1$, with $q_1, q_2, q_3 > 1$. The proof is complete. ∎

The above proposition motivates the following definition:

**Definition 4.17.** *A continuous random field $u : [0,T] \times \mathbb{R}^d \times \Omega' \mapsto \mathbb{R}$ is a (stochastic) viscosity solution of equation (4.9) if and only if $\widehat{u}(t,x) = u(T_t, t, x)\varepsilon_t^{-1}(T_t), (t,x) \in [0,T] \times \mathbb{R}^d$ is a pathwise viscosity solution of equation (4.8).*

As a consequence of our preceding discussion, we can formulate the following statement:

**Theorem 4.18.** *The continuous stochastic field $u(t,x) := \widehat{u}(A_t, t, x)\varepsilon_t = \widehat{Y}_t^{t,x}(A_t)\varepsilon_t = Y_t^{t,x}$ is a stochastic viscosity solution of the semilinear SPDE (4.9). This solution is unique inside the class $C_p^B$ of continuous stochastic field $\tilde{u} : \Omega' \times [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}$ such that,*

$$|\tilde{u}(t,x)| \le C \exp\{I_T^*\}(1+|x|), (t,x) \in [0,T] \times \mathbb{R}^d, P-a.s.,$$

*for some constant $C$ only depending on $\tilde{u}$.*

**Remark 4.19.** 1) *It can be easily checked that* $\tilde{u} \in C_p^B$ *if and only if* $(\tilde{u}(A_t, t, x)\varepsilon_t) \in C_p^B$ *if and only if* $(\tilde{u}(T_t, t, x)\varepsilon_t^{-1}(T_t)) \in C_p^B$.
2) *As a consequence of the preceding theorem we have that* $u(t, x) = Y_t^{t,x}$ *is the unique (in* $C_p^B$*) stochastic viscosity solution of SPDE* (4.9)*. This extends the Feynman-Kac formula to SPDEs driven by a fractional Brownian motion.*

We conclude the main theorems of Section 4 with the following relation diagram, which shows the mutual relationship between fractional backward SDEs and SPDEs:

$$\boxed{\begin{array}{ccc} \widehat{u}(t,x):\ viscosity\ solution\ of\ (4.8) & \overset{GT}{\longleftrightarrow} & u(t,x):\ viscosity\ solution\ of\ (4.9) \\ \updownarrow & & \updownarrow \\ (\widehat{Y}, \widehat{Z}):\ solution\ of\ BSDE\ (4.5) & \overset{GT}{\longleftrightarrow} & (Y,Z):\ solution\ of\ BDSDE\ (4.4) \end{array}}$$

where $'GT'$ stands for 'Girsanov transformation'.

Finally, in order to illustrate how our method works, we give the example of a linear fractional backward doubly stochastic differential equation.

**Example 4.20.** *We let* $d = 1$ *and* $f(s, x, y, z) = f_s^1 x + f_s^2 y + f_s^3 z$*, where the coefficients* $f_s^1, f_s^2$ *and* $f_s^3$ *are bounded and deterministic functions. The associated fractional backward doubly stochastic differential equation is linear and writes:*

$$Y_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s \left(f_r^1 X_r^{t,x} + f_r^2 Y_r^{t,x} + f_r^3 Z_r^{t,x}\right) \mathrm{d}r - \int_0^s Z_r^{t,x} \downarrow \mathrm{d}W_r + \int_0^s \gamma_r Y_r^{t,x} \mathrm{d}B_r. \tag{4.29}$$

*After Girsanov transformation, it becomes*

$$\widehat{Y}_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s \left(f_r^1 X_r^{t,x} \varepsilon_r^{-1}(T_r) + f_r^2 \widehat{Y}_r^{t,x} + f_r^3 \widehat{Z}_r^{t,x}\right) \mathrm{d}r - \int_0^s \widehat{Z}_r^{t,x} \downarrow \mathrm{d}W_r.$$

*and has the following solution:*

$$\begin{aligned} \widehat{Y}_s^{t,x} =& E_Q\Bigg[\int_0^s \left(f_r^1 X_r^{t,x} \varepsilon_r^{-1}(T_r) \exp\left\{\int_r^s f_u^2 \mathrm{d}u\right\}\right) \mathrm{d}r \\ &+ \exp\left\{\int_0^s f_r^2 \mathrm{d}r\right\} \Phi(X_0^{t,x}) \Big| \mathcal{F}_{s,t}^W \vee \mathcal{F}_t^B\Bigg], \end{aligned}$$

*where* $E_Q$ *is the expectation with respect to* $Q = \exp\left\{\int_0^t f_r^3 \mathrm{d}W_r - \frac{1}{2}\int_0^t (f_r^3)^2 \mathrm{d}r\right\} P$*. According to Theorem 3.9, the solution of* (4.29) *is then*

$$\begin{aligned} Y_s^{t,x} = E_Q\Bigg[&\varepsilon_s \int_0^s \left(f_r^1 X_r^{t,x} \varepsilon_r^{-1}(T_r A_s) \exp\left\{\int_r^s f_u^2 \mathrm{d}u\right\}\right) \mathrm{d}r \\ &+ \varepsilon_s \exp\left\{\int_0^s f_r^2 \mathrm{d}r\right\} \Phi(X_0^{t,x}) \Big| \mathcal{F}_{s,t}^W \vee \mathcal{F}_t^B\Bigg]. \end{aligned}$$

# Chapter 2

# Fractional Backward Doubly Stochastic Differential Equations with Hurst Parameter in (1/2,1)

**Abstract**: We first state a special type of Itô formula involving stochastic integrals of both standard and fractional Brownian motions. Then we use Doss-Sussman transformation to establish the link between backward doubly stochastic differential equations, driven by both standard and fractional Brownian motions, and backward stochastic differential equations, driven only by standard Brownian motions. Following the same technique, we further study associated nonlinear stochastic partial differential equations driven by fractional Brownian motions and partial differential equations with stochastic coefficients.

**Key words**: fractional Brownian motion, backward doubly stochastic differential equation, stochastic partial differential equation, Russo-Vallois integral, Doss-Sussman transformation, stochastic viscosity solution.

**MSC 2000**: 60G22; 60H15; 35R60.

# 1 Introduction

In this chapter, we deal with BDSDEs for which the integrand of the integral with respect to the fBM is not necessarily linear with the solution process, and the Hurst coefficient $H$ is supposed to belong t o the interval$(1/2, 1)$. Unlike the more irregular case $H < 1/2$, the stochastic integrals with respect to an fBM with $H > 1/2$ can be defined in different ways. So they can be defined with the help of the divergence operator in the frame of the Malliavin calculus, see Decreusefont and Üstünel [33] and Alòs *et al.* [3] (Notice that the Wick-Itô integral defined in Duncan *et al.* [38] coincides with the first one). They can also be defined pathwise as generalized Riemann-Stieltjes integral (see Zähle [109] and [110]) or with the help of the rough path theory (see Coutin and Qian [30]). For a complete list of references we refer to the two books by Biagini *et al.* [12] and Mishura [77].

Our approach to BDSDEs with an fBM is inspired by the work of Buckdahn and Ma [22]. In their study of stochastic PDEs driven by a Brownian motion $B$ the authors of [22] used BDSDEs driven by $B$ as well as an independent Brownian motion; the integral with respect to $B$ is interpreted in Stratonovich sense. This allowed the application of the Doss-Sussmann transformation in order to transform the BDSDE into a BSDE without integral with respect to $B$. On the other hand, the pathwise integral with respect to the fBM plays a role which is comparable with that of the Stratonovich integral in the classical theory. Nualart and Răşcanu [81] used the pathwise integral to solve (forward) stochastic differential equations driven by an fBM. For some technical reasons (such as the lack of Hölder continuity, see Remark 4.12), we shall make use of the Russo-Vallois integral developed by Russo and Vallois in a series of papers ([100], [100], [100], *etc.*). Under standard assumptions which allow to apply the Doss-Sussmann transformation, we associate the BDSDE driven by both a standard Brownian motion $W$ and an fBM $B$, for $s \in [0, t]$,

$$U_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f(r, X_r^{t,x}, U_r^{t,x}, V_r^{t,x})\mathrm{d}r + \int_0^s g(U_r^{t,x})\mathrm{d}B_r - \int_0^s V_r^{t,x} \downarrow \mathrm{d}W_r, \text{ [1]} \quad (1.1)$$

with the BSDE driven only by the Brownian motion $W$,

$$Y_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s \tilde{f}(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\mathrm{d}r - \int_0^s Z_r^{t,x} \downarrow \mathrm{d}W_r, \; s \in [0, t]. \quad (1.2)$$

Here $\tilde{f}$ will be specified in Section 4; it is a driver with quadratic growth in $z$. We point out that the classical BDSDEs were first studied by Pardoux and Peng [84] and our BSDE (1.2) is a quadratic growth BSDE, which was studied first by Kobylanski [61]. In the works of Pardoux and Peng [84] and Buckdahn and Ma [22], i.e., when the Hurst parameter $H = 1/2$, one can solve the BDSDE directly and get the square integrability of the solution process. However, in the fractional case ($H \neq 1/2$), to our best knowledge, there does not exist a direct way to solve the BDSDE (1.1), and as it turns out in Theorem 4.7, we can only get that the conditional expectation of $\int_0^t \left| Z_r^{t,x} \right|^2 \mathrm{d}r$ is bounded by an a.s. finite process. This is also the reason that instead of using the space of square integrable processes, we use the space of a.s. conditionally square integrable processes (see the definition of the space $\mathcal{H}_T^2(\mathbb{R}^d)$ in Section 2).

1. $\int_0^t \cdot \downarrow \mathrm{d}W_s$ indicates that the integral is considered as the Itô backward one.

A celebrated contribution of the BSDE theory consists in giving a form of probabilistic interpretation, nonlinear Feynman-Kac formula, to the solutions of PDEs (see, for instance, Peng [92], Pardoux and Peng [85]). As indicated in Kobylanski [61], the quadratic growth BSDE (1.2) is connected to the semilinear parabolic PDE

$$\left\{\begin{array}{lcll} \frac{\partial u}{\partial t}(t,x) & = & \mathcal{L}u(t,x) - \tilde{f}(t,x,u(t,x),\sigma(x)^T\frac{\partial}{\partial x}u(t,x)), & (t,x)\in(0,T)\times\mathbb{R}^n; \\ u(0,x) & = & \Phi(x), & x\in\mathbb{R}^n, \end{array}\right. \tag{1.3}$$

where $\mathcal{L}$ is the infinitesimal operator of a Markov process. Hence, it is natural for us to consider the form of equation (1.3) after Doss-Sussmann transformation and we prove that it becomes the following semilinear SPDE

$$\left\{\begin{array}{ll} \mathrm{d}u(t,x) = \left[\mathcal{L}u(t,x) - f(t,x,u(t,x),\sigma(x)^T\frac{\partial}{\partial x}u(t,x))\right]\mathrm{d}t + g(u(t,x))\mathrm{d}B_t, & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in(0,T)\times\mathbb{R}^n; & \\ u(0,x) = \Phi(x), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x\in\mathbb{R}^n. & \end{array}\right. \tag{1.4}$$

We emphasize that this chapter can not be considered as a generalization of Chapter 1. The reason is that, firstly, the Hurst parameters are distinct; secondly, the stochastic integrals with respect to the fBM are of different types.

We organize the chapter as follows. In section 2, we recall some basic facts about the fBM with Hurst parameter $H\in(0,1/2)$, we give the general framework of our work, and we recall the definition of the backward Russo-Vallois integral as well as some of its properties. In section 3 we prove a type of Itô formula involving integrals with respect to both standard and fractional Brownian motions, which will play an important role in the following sections. We perform a Doss-Sussmann transformation in Section 4 to transform a nonlinear BDSDE (1.1) into a BSDE (1.2) and show the relationship between their solutions. In particular, we show that BSDE (1.2) has a unique solution $(Y^{t,x},Z^{t,x})$, and the couple of processes $(U^{t,x},V^{t,x})$ associated with $(Y^{t,x},Z^{t,x})$ by the inverse Doss-Sussmann transformation is the unique solution of BDSDE (1.1). Finally, the stochastic PDE associated with BDSDE (1.1) is briefly discussed in Section 5.

# 2 Preliminaries

## 2.1 Fractional Brownian Motion and General Setting

In this subsection we recall some basic results on the fBM and the related setting. For a more complete overview of the theory of fBM, we refer the reader to Biagini *et al.* [12] and Mishura [77].

Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a classical Wiener space with time horizon $T > 0$, i.e., $\Omega' = C_0([0,T];\mathbb{R})$ denotes the set of real-valued continuous functions starting from zero at time zero, endowed with the topology of the uniform convergence, $\mathcal{B}(\Omega')$ is the Borel $\sigma$-algebra on $\Omega'$ and $\mathbb{P}'$ is the unique probability measure on $(\Omega, \mathcal{B}(\Omega'))$ with respect to which the coordinate process $W_t^0(\omega') = \omega'(t)$, $t \in [0,T]$, $\omega' \in \Omega'$ is a standard Brownian motion. By $\mathcal{F}'$ we denote the completion of $\mathcal{B}(\Omega')$ by all $\mathbb{P}'$-null sets in $\Omega'$. Given $H \in (1/2,1)$, we define

$$B_t = \int_0^t K_H(t,s)\mathrm{d}W_s^0, \quad t \in [0,T],$$

where $K_H$ is the kernel of the fBM with parameter $H \in (1/2,1)$:

$$K_H(t,s) = C_H s^{1/2-H} \int_s^t u^{H-1/2}(u-s)^{H-3/2}\mathrm{d}u,$$

with $C_H = \sqrt{\frac{H}{(2H-1)\beta(2-2H,H-1/2)}}$. It is well known that such defined process $B$ is a one-dimensional fBM, i.e., it is a Gaussian process with zero mean and covariance function

$$R_H(t,s) := \mathbb{E}\left[B_t B_s\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \in [0,T].$$

We let $\{W_t : 0 \le t \le T\}$ be the coordinate process on the classical Wiener space $(\Omega'', \mathcal{F}'', \mathbb{P}'')$ with $\Omega'' = C_0([0,T];\mathbb{R}^d)$, which is a $d$-dimensional Brownian motion with respect to the Wiener measure $\mathbb{P}''$. We put $(\Omega, \mathcal{F}^0, \mathbb{P}) = (\Omega', \mathcal{F}', \mathbb{P}') \otimes (\Omega'', \mathcal{F}'', \mathbb{P}'')$ and let $\mathcal{F} = \mathcal{F}^0 \vee \mathcal{N}$, where $\mathcal{N}$ is the class of the $\mathbb{P}$-null sets. We denote again by $B$ and $W$ the canonical extensions of the fBM $B$ and of the Brownian motion $W$ from $\Omega'$ and $\Omega''$, respectively, to $\Omega$.

We let $\mathcal{F}_{[t,T]}^W = \sigma\{W_T - W_s, t \le s \le T\} \vee \mathcal{N}$, $\mathcal{F}_t^B = \sigma\{B_s, 0 \le s \le t\} \vee \mathcal{N}$, and $\mathcal{G}_t = \mathcal{F}_{[t,T]}^W \vee \mathcal{F}_t^B, t \in [0,T]$. Let us point out that $\mathcal{F}_{[t,T]}^W$ is decreasing and $\mathcal{F}_t^B$ is increasing in $t$, but $\mathcal{G}_t$ is neither decreasing nor increasing. We denote the family of $\sigma$-fields $\{\mathcal{G}_t\}_{0 \le t \le T}$ by $\mathbb{G}$. Moreover, we also introduce the backward filtrations $\mathbb{H} = \{\mathcal{H}_t = \mathcal{F}_{[t,T]}^W \vee \mathcal{F}_T^B\}_{t \in [0,T]}$ and $\mathbb{F}^W = \{\mathcal{F}_{[t,T]}^W\}_{t \in [0,T]}$.

Finally, we denote by $C(\mathbb{H}, [0,T]; \mathbb{R}^m)$ the space of the $\mathbb{R}^m$-valued continuous processes $\{\varphi_t, t \in [0,T]\}$ such that $\varphi_t$ is $\mathcal{H}_t$-measurable, $t \in [0,T]$, and $\mathcal{M}^2(\mathbb{F}^W, [0,T];\ \mathbb{R}^m)$ the space of the $\mathbb{R}^m$-valued square-integrable processes $\{\psi_t, t \in [0,T]\}$ such that $\psi_t$ is $\mathcal{F}_{[t,T]}^W$-measurable, $t \in [0,T]$. Let $\mathcal{H}_T^\infty(\mathbb{R})$ be the set of $\mathbb{H}$-progressively measurable processes which are almost surely bounded by some real-valued $\mathcal{F}_T^B$-measurable random variable, and let $\mathcal{H}_T^2(\mathbb{R}^d)$ denote the set of all $\mathbb{R}^d$-valued $\mathbb{H}$-progressively measurable processes $\gamma = \{\gamma_t : t \in [0,T]\}$ such that $\mathbb{E}\left[\int_0^T |\gamma_t|^2 \mathrm{d}t | \mathcal{F}_T^B\right] < +\infty$, $\mathbb{P}$-a.s.

## 2.2 Russo-Vallois Integral

In a series of papers ([99], [100], [101], *etc.*), Russo and Vallois defined new types of stochastic integrals, namely forward, backward and symmetric integrals, which are extensions of the classical Riemann-Stieltjes integral, and in fact these three integrals coincide, when the integrator is a fBM with Hurst parameter $H \in (1/2, 1)$. Here we will mainly use the backward Russo-Vallois integral in this chapter. It turns out to be a convenient definition for stochastic integral with respect to our fBM $B$.

Let us recall some results by Russo and Vallois which we will use later. In what follows, we make the convention that all continuous processes $\{X_t, t \in [0,T]\}$ are extended to the whole line by putting $X_t = X_0$, for $t < 0$, and $X_t = X_T$, for $t > T$.

**Definition 2.1.** *Let $X$ and $Y$ be two continuous processes. For $\varepsilon > 0$, we set*

$$I(\varepsilon, t, X, \mathrm{d}Y) \triangleq \frac{1}{\varepsilon}\int_0^t X(s)(Y(s) - Y(s-\varepsilon))\mathrm{d}s,$$

$$C_\varepsilon(X,Y)(t) \triangleq \frac{1}{\varepsilon}\int_0^t (X(s) - X(s-\varepsilon))(Y(s) - Y(s-\varepsilon))\mathrm{d}s,\ t \in [0,T].$$

*Then the backward Russo-Vallois integral is defined as the uniform limit in probability as $\varepsilon \to 0^+$, if the limit exists. The generalized bracket $[X, Y]$ is the uniform limit in probability of $C_\varepsilon(X,Y)$ as $\varepsilon \to 0^+$ (of course, again under the condition of existence).*

We recall that (cf. Protter [97]) a sequence of processes $(H_n; n \geq 0)$ converges to a process $H$ uniformly in probability if

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} |H_n(t) - H(t)| > \alpha\right) = 0 \quad \text{for every} \quad \alpha > 0.$$

In [101] (Theorem 2.1) Russo and Vallois derived the Itô formula for the backward Russo-Vallois integral.

**Theorem 2.2.** *Let $f \in C^2(\mathbb{R})$ and $X$ be a continuous process admitting the generalized bracket, i.e., $[X,X]$ exists in the sense of Definition 2.1. Then for every $t \in [0,T]$, the backward Russo-Vallois integral $\int_0^t f'(X(s))\mathrm{d}X(s)$ exists and*

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))\mathrm{d}X(s) - \frac{1}{2}\int_0^t F''(X(s))\mathrm{d}[X,X](s),$$

*for every $t \geq 0$.*

We list some properties of Russo-Vallois integral, which will be used later in this chapter.

**Proposition 2.3.** (1). *If $X$ is a finite quadratic variation process (i.e., $[X,X]$ exists and $[X,X]_T < +\infty$, $\mathbb{P}$-a.s.) and $Y$ is a zero quadratic variation process (i.e., $[Y,Y]$ exists and equals to zero), then the mutual generalized bracket $[X,Y]$ exists and vanishes, $\mathbb{P}$-a.s.*

(2). *If $X$ and $Y$ have $\mathbb{P}$-a.s. Hölder continuous paths with order $\alpha$ and $\beta$, respectively, such that $\alpha > 0$, $\beta > 0$ and $\alpha + \beta > 1$, then $[X,Y] = 0$.*

(3). *We assume that $X$ and $Y$ are continuous and admit a mutual bracket. Then, for every continuous process $\{H(s) : s \in [0,T]\}$,*

$$\int_0^\cdot H(s)\mathrm{d}C_\varepsilon(X,Y)(s) \quad \textit{converges to} \quad \int_0^\cdot H(s)\mathrm{d}[X,Y](s).$$

The following proposition, which can be found in Russo and Vallois [99], states the relationship between the Young integral (see Young [107]) and the backward Russo-Vallois integral.

**Proposition 2.4.** *Let $X, Y$ be two real processes with paths being $\mathbb{P}$-a.s. in $C^\alpha$ and $C^\beta$, respectively, with $\alpha > 0$, $\beta > 0$ and $\alpha + \beta > 1$. Then the backward Russo-Vallois integral $\int_0^{\cdot} Y \mathrm{d} X$ coincides with the Young integral $\int_0^{\cdot} Y \mathrm{d}^{(y)} X$.*

## 3 A Generalized Itô Formula

In this section we state a generalized Itô formula involving an Itô backward integral with respect to the Brownian motion $W$ and the Russo-Vallois integral with respect to the fBM $B$. It will play an important role in this chapter. It is noteworthy that this Itô formula corresponds to Lemma 1.3 in the paper of Pardoux and Peng [84] for the case of an fBM with Hurst parameter $H = 1/2$, i.e., when $B$ is a Brownian motion.

**Theorem 3.1.** *Let $\alpha \in C(\mathbb{H}, [0,T]; \mathbb{R})$ be a process of the form*

$$\alpha_t = \alpha_0 + \int_0^t \beta_s \mathrm{d}s + \int_0^t \gamma_s \downarrow \mathrm{d}W_s, \quad t \in [0,T],$$

*where $\beta$ and $\gamma$ are $\mathbb{H}$-adapted processes and $\mathbb{P}\{\int_0^T |\beta_s|\mathrm{d}s < \infty\} = 1$ and $\mathbb{P}\{\int_0^T |\gamma_s|^2\mathrm{d}s < +\infty\} = 1$, respectively. Suppose that $F \in C^2(\mathbb{R}\times\mathbb{R})$. Then the Russo-Vallois integral $\int_0^t \frac{\partial F}{\partial y}(\alpha_s, B_s)\mathrm{d}B_s$ (defined as the uniform limit in probability of $\frac{1}{\varepsilon}\int_0^t (B_s - B_{s-\varepsilon})\frac{\partial F}{\partial y}(\alpha_s, B_s)\mathrm{d}s$) exists for $0 \le t \le T$, and it holds that, $\mathbb{P}$-almost surely, for all $0 \le t \le T$,*

$$\begin{aligned} F(\alpha_t, B_t) =& F(\alpha_0, 0) + \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s)\beta_s \mathrm{d}s + \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s)\gamma_s \downarrow \mathrm{d}W_s \\ &+ \int_0^t \frac{\partial F}{\partial y}(\alpha_s, B_s)\mathrm{d}B_s - \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(\alpha_s, B_s)|\gamma_s|^2\mathrm{d}s. \end{aligned} \tag{3.1}$$

*Proof:* **Step 1**. First we suppose $F \in C_b^2(\mathbb{R}\times\mathbb{R})$ (i.e., the function $F$ is twice continuously differentiable and has bounded derivatives of order less than or equal to two) and there is a positive constant $C$ such that $\int_0^T |\beta_s|\mathrm{d}s \le C$ and $\int_0^T |\gamma_s|^2\mathrm{d}s \le C$. It is direct to check that

$$F(\alpha_t, B_t) - F(\alpha_0, 0) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t (F(\alpha_s, B_s) - F(\alpha_{s-\varepsilon}, B_{s-\varepsilon}))\mathrm{d}s.$$

For simplicity we put $\alpha_{a,\varepsilon,s} \triangleq \alpha_s - a(\alpha_s - \alpha_{s-\varepsilon})$ and $B_{a,\varepsilon,s} \triangleq B_s - a(B_s - B_{s-\varepsilon})$, for any $a \in [0,1]$, $s \in [0,T]$, $\varepsilon > 0$. We have

$$\begin{aligned} &F(\alpha_s, B_s) - F(\alpha_{s-\varepsilon}, B_{s-\varepsilon}) \\ =&(\alpha_s - \alpha_{s-\varepsilon})\frac{\partial F}{\partial x}(\alpha_s, B_s) + (B_s - B_{s-\varepsilon})\frac{\partial F}{\partial y}(\alpha_s, B_s) \\ &- (\alpha_s - \alpha_{s-\varepsilon})^2\int_0^1 \frac{\partial^2 F}{\partial x^2}(\alpha_{a,\varepsilon,s}, B_{a,\varepsilon,s})(1-a)\mathrm{d}a \\ &- 2(\alpha_s - \alpha_{s-\varepsilon})(B_s - B_{s-\varepsilon})\int_0^1 \frac{\partial^2 F}{\partial x\partial y}(\alpha_{a,\varepsilon,s}, B_{a,\varepsilon,s})(1-a)\mathrm{d}a \\ &- (B_s - B_{s-\varepsilon})^2\int_0^1 \frac{\partial^2 F}{\partial y^2}(\alpha_{a,\varepsilon,s}, B_{a,\varepsilon,s})(1-a)\mathrm{d}a. \end{aligned} \tag{3.2}$$

By applying the stochastic Fubini theorem, we get that

$$\begin{aligned} &\frac{1}{\varepsilon}\int_0^t (\alpha_s - \alpha_{s-\varepsilon})\frac{\partial F}{\partial x}(\alpha_s, B_s)\mathrm{d}s \\ =&\frac{1}{\varepsilon}\int_0^t \left(\int_{s-\varepsilon}^s \beta_r \mathrm{d}r + \int_{s-\varepsilon}^s \gamma_r \downarrow \mathrm{d}W_r\right)\frac{\partial F}{\partial x}(\alpha_s, B_s)\mathrm{d}s \\ =&\frac{1}{\varepsilon}\int_0^t \int_r^{(r+\varepsilon)\wedge t} \beta_r \frac{\partial F}{\partial x}(\alpha_s, B_s)\mathrm{d}s\mathrm{d}r + \frac{1}{\varepsilon}\int_0^t \int_r^{(r+\varepsilon)\wedge t} \gamma_r \frac{\partial F}{\partial x}(\alpha_s, B_s)\mathrm{d}s \downarrow \mathrm{d}W_r. \end{aligned}$$

Since $\frac{1}{\varepsilon}\int_r^{(r+\varepsilon)\wedge t}\frac{\partial F}{\partial x}(\alpha_s, B_s)\mathrm{d}s$ is $\mathcal{H}_t$-measurable and converges to $\frac{\partial F}{\partial x}(\alpha_r, B_r)$ when $\varepsilon \to 0$, it follows that, thanks to the continuity of $\frac{\partial F}{\partial x}(\alpha_s, B_s)$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\left|\int_0^t \beta_r\left(\frac{1}{\varepsilon}\int_r^{(r+\varepsilon)\wedge t}\frac{\partial F}{\partial x}(\alpha_s, B_s)\mathrm{d}s - \frac{\partial F}{\partial x}(\alpha_r, B_r)\right)\mathrm{d}r\right| \\
\le &\lim_{\varepsilon\to 0}\sup_{r\in[0,T]}\left|\frac{1}{\varepsilon}\int_r^{(r+\varepsilon)\wedge t}\frac{\partial F}{\partial x}(\alpha_s, B_s)\mathrm{d}s - \frac{\partial F}{\partial x}(\alpha_r, B_r)\right|\int_0^T|\beta_r|\mathrm{d}r = 0, \quad \text{in probability.}
\end{aligned}$$

Thus, in virtue of the boundedness of $\frac{\partial F}{\partial x}$, by the Dominated Convergence Theorem,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \gamma_r\left(\frac{1}{\varepsilon}\int_r^{(r+\varepsilon)\wedge t}\frac{\partial F}{\partial x}(\alpha_s, B_s)\mathrm{d}s - \frac{\partial F}{\partial x}(\alpha_r, B_r)\right)\downarrow \mathrm{d}W_r\right|^2\middle|\mathcal{F}_T^B\right] \\
\le &\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{r\in[0,T]}\left|\frac{1}{\varepsilon}\int_r^{(r+\varepsilon)\wedge t}\frac{\partial F}{\partial x}(\alpha_s, B_s)\mathrm{d}s - \frac{\partial F}{\partial x}(\alpha_r, B_r)\right|^2\int_0^t|\gamma_r|^2\mathrm{d}r\middle|\mathcal{F}_T^B\right] \\
=&0, \ \mathbb{P}-\text{a.s.}
\end{aligned}$$

(Recall that $\left(\frac{1}{\varepsilon}\int_r^{(r+\varepsilon)\wedge t}\frac{\partial F}{\partial x}(\alpha_s, B_s)\mathrm{d}s - \frac{\partial F}{\partial x}(\alpha_r, B_r)\right)_{r\in[0,T]}$ is $\mathbb{H}$-adapted and $W_\cdot - W_T$ is an $\mathbb{H}$-(backward) Brownian motion.) Thus, we get

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t(\alpha_s-\alpha_{s-\varepsilon})\frac{\partial F}{\partial x}(\alpha_s, B_s)\mathrm{d}s = \int_0^t\beta_r\frac{\partial F}{\partial x}(\alpha_r, B_r)\mathrm{d}r + \int_0^t\gamma_r\frac{\partial F}{\partial x}(\alpha_r, B_r)\downarrow \mathrm{d}W_r, \tag{3.3}$$

uniformly in probability. We notice that the generalized bracket of $\alpha$ is the same as the classical one, i.e., $[\alpha,\alpha]_s = \int_0^s|\gamma_r|^2\mathrm{d}r$, $s\in[0,T]$. We also have

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t(\alpha_s-\alpha_{s-\varepsilon})^2\int_0^1\frac{\partial^2 F}{\partial x^2}(\alpha_{a,\varepsilon,s}, B_{a,\varepsilon,s})(1-a)\mathrm{d}a\mathrm{d}s \\
=&\frac{1}{2\varepsilon}\int_0^t(\alpha_s-\alpha_{s-\varepsilon})^2\frac{\partial^2 F}{\partial x^2}(\alpha_s, B_s)\mathrm{d}s + A_{\varepsilon,t}, \quad t\in[0,T],
\end{aligned}\tag{3.4}$$

where

$$A_{\varepsilon,t} = \frac{1}{\varepsilon}\int_0^t(\alpha_s-\alpha_{s-\varepsilon})^2\int_0^1\left(\frac{\partial^2 F}{\partial x^2}(\alpha_{a,\varepsilon,s}, B_{a,\varepsilon,s}) - \frac{\partial^2 F}{\partial x^2}(\alpha_s, B_s)\right)(1-a)\mathrm{d}a\mathrm{d}s.$$

Proposition 2.3 yields that

$$\frac{1}{2\varepsilon}\int_0^\cdot(\alpha_s-\alpha_{s-\varepsilon})^2\frac{\partial^2 F}{\partial x^2}(\alpha_s, B_s)\mathrm{d}s \ \text{ converges to } \int_0^\cdot\frac{\partial^2 \mathrm{F}}{\partial \mathrm{x}^2}(\alpha_\mathrm{s}, \mathrm{B}_\mathrm{s})\mathrm{d}[\alpha,\alpha]_\mathrm{s}$$

and the continuity of $\frac{\partial^2 F}{\partial x^2}$, $\alpha$ and $B$ implies that $A_{\varepsilon,t}$ converges to zero. A similar argument shows that

$$\frac{1}{\varepsilon}\int_0^\cdot 2(\alpha_s-\alpha_{s-\varepsilon})(B_s-B_{s-\varepsilon})\int_0^1\frac{\partial^2 F}{\partial x\partial y}(\alpha_{a,\varepsilon,s}, B_{a,\varepsilon,s})(1-a)\mathrm{d}a\mathrm{d}s$$

and the term

$$\frac{1}{\varepsilon}\int_0^\cdot(B_s-B_{s-\varepsilon})^2\int_0^1\frac{\partial^2 F}{\partial y^2}(\alpha_{a,\varepsilon,s}, B_{a,\varepsilon,s})(1-a)\mathrm{d}a\mathrm{d}s$$

converge in probability, respectively, to

$$2\int_0^{\cdot} \frac{\partial^2 F}{\partial x \partial y}(\alpha_s, B_s)\mathrm{d}[\alpha, B]_s \quad \text{and} \quad \int_0^{\cdot} \frac{\partial^2 F}{\partial x \partial y}(\alpha_s, B_s)\mathrm{d}[B, B]_s.$$

However, these both latter expressions are zero due to the fact that $H \in (1/2, 1)$ and Proposition 2.3 (Observe that the fBM has Hölder continuous paths of any positive order less than $H$ almost surely).

Combining the above results with (3.2), (3.3) and (3.4), we see that

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t (B_s - B_{s-\varepsilon})\frac{\partial F}{\partial y}(\alpha_s, B_s)\mathrm{d}s = F(\alpha_t, B_t) - F(\alpha_0, 0) - \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s)\beta_s \mathrm{d}s \\ - \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s)\gamma_s \downarrow \mathrm{d}W_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(\alpha_s, B_s)|\gamma_s|^2 \mathrm{d}s,\end{aligned}$$

uniformly in $t \in [0, T]$, in probability. Consequently, the integral $\int_0^t \frac{\partial F}{\partial y}(\alpha_s, B_s)\mathrm{d}B_s$ exists in Russo-Vallois' sense (Recall Definition 2.1) and we get the Itô formula (3.1) for $F \in C_b^2(\mathbb{R}\times\mathbb{R})$.

**Step 2**. Now we deal with the general case that

$$\mathbb{P}\{\int_0^T |\beta_s|\mathrm{d}s < \infty\} = \mathbb{P}\{\int_0^T |\gamma_s|^2\mathrm{d}s < \infty\} = 1.$$

For each $n \in \mathbb{N}$, we define a sequence of $\mathbb{H}$-stopping times by $\tau_n = \sup\{t \le T : \int_t^T |\beta_s|\mathrm{d}s + \int_t^T |\gamma_s|^2\mathrm{d}s > n\}\vee 0$, so we know that the processes $\{\beta_t^n := \beta_t \mathbf{1}_{[\tau_n, T]}(t), t \in [0, T]\}$ and $\{\gamma_t^n := \gamma_t \mathbf{1}_{[\tau_n, T]}(t), t \in [0, T]\}$ satisfy $\int_0^T |\beta_s^n|\mathrm{d}s \le n$ and $\int_0^T |\gamma_s^n|^2\mathrm{d}s \le n$, $\mathbb{P}$-a.s. Furthermore, as $n \to \infty$, $\tau_n \to 0$, $\mathbb{P}$-a.s. We consider the Itô formula for the process $\alpha_t^n = \alpha_0 + \int_0^t \beta_s^n \mathrm{d}s + \int_0^t \gamma_s^n \downarrow \mathrm{d}W_s, t \in [0, T]$. Thanks to the result of **Step 1**, we have that, for every $n \in \mathbb{N}$,

$$\begin{aligned}F(\alpha_t^n, B_t) =& F(\alpha_0, 0) + \int_0^t \frac{\partial F}{\partial x}(\alpha_s^n, B_s)\beta_s^n \mathrm{d}s + \int_0^t \frac{\partial F}{\partial y}(\alpha_s^n, B_s)\mathrm{d}B_s \\ &+ \int_0^t \frac{\partial F}{\partial x}(\alpha_s^n, B_s)\gamma_s \downarrow \mathrm{d}W_s - \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(\alpha_s^n, B_s)|\gamma_s|^2\mathrm{d}s.\end{aligned}$$

Since $\alpha^n$ converges to $\alpha$ uniformly in probability on $[0, T]$, by letting $n \to \infty$ in the above equation, we deduce that the limit

$$\begin{aligned}\lim_{n\to\infty}\int_0^t \frac{\partial F}{\partial y}(\alpha_s^n, B_s)\mathrm{d}B_s &= \lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_0^t \frac{1}{\varepsilon}\frac{\partial F}{\partial y}(\alpha_s^n, B_s)(B_s - B_{s-\varepsilon})\mathrm{d}s \\ =& \lim_{\varepsilon\to 0}\int_0^t \frac{1}{\varepsilon}\frac{\partial F}{\partial y}(\alpha_s, B_s)(B_s - B_{s-\varepsilon})\mathrm{d}s\end{aligned}$$

exists, it is the Russo-Vallois integral $\int_0^t \frac{\partial F}{\partial y}(\alpha_s, B_s)\mathrm{d}B_s$ and it equals to

$$\begin{aligned}&F(\alpha_t, B_t) - F(\alpha_0, 0) - \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s)\beta_s \mathrm{d}s - \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s)\gamma_s \downarrow \mathrm{d}W_s \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(\alpha_s, B_s)|\gamma_s|^2\mathrm{d}s.\end{aligned}$$

**Step 3.** Finally we consider the case $F \in C^2(\mathbb{R}\times\mathbb{R})$. We let $\{\varphi_N\}_{N\in\mathbb{N}}$ be a sequence of infinitely differentiable functions with compact support such that $\varphi_N(x) = x$ for $\{(x_1, x_2) : \max(|x_1|, |x_2|) \leq N\}, N \in \mathbb{N}$. We set $F_N(x) = F(\varphi_N(x))$, so that $F_N(x) \in C_b^2(\mathbb{R} \times \mathbb{R})$ for every $N > 0$. We notice that $F_N(\alpha., B.)$ and $F(\alpha., B.)$ coincide on the set $\Omega_N = \{\omega \in \Omega : \sup_{0\leq s\leq t} |\alpha_s| \leq N, \sup_{0\leq s\leq t} |B_s| \leq N\}$ and that $\Omega = \bigcup_{N\geq 1} \Omega_N$. Due to **Step 1** and **Step 2**, for every $N$, we have

$$\begin{aligned}F_N(\alpha_t, B_t) =& F_N(\alpha_0, 0) + \int_0^t \frac{\partial F_N}{\partial x}(\alpha_s, B_s)\beta_s \mathrm{d}s + \int_0^t \frac{\partial F_N}{\partial y}(\alpha_s, B_s)\mathrm{d}B_s \\ &+ \int_0^t \frac{\partial F_N}{\partial x}(\alpha_s, B_s)\gamma_s \downarrow \mathrm{d}W_s - \frac{1}{2}\int_0^t \frac{\partial^2 F_N}{\partial x^2}(\alpha_s, B_s)|\gamma_s|^2 \mathrm{d}s, \quad t \in [0, T].\end{aligned}$$

Therefore, for every $N$, it holds on $\Omega_N$ that

$$\begin{aligned}F(\alpha_t, B_t) =& F(\alpha_0, 0) + \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s)\beta_s \mathrm{d}s + \int_0^t \frac{\partial F}{\partial y}(\alpha_s, B_s)\mathrm{d}B_s \\ &+ \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s)\gamma_s \downarrow \mathrm{d}W_s - \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(\alpha_s, B_s)|\gamma_s|^2 \mathrm{d}s, \quad t \in [0, T].\end{aligned}$$

Finally, by letting $N$ tend to $+\infty$, we get the wished result. The proof is complete. ∎

# 4 Doss-Sussmann Transformation of Fractional BDSDE

In what follows, we use the following hypotheses:

(H1) The functions $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ and $b : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz continuous.

(H2) The function $f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is Lipschitz in $(x, y, z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$ with $|f(t, 0, 0, 0)| \le C$ uniformly in $t \in [0, T]$, the function $g : \mathbb{R}^n \to \mathbb{R}$ belongs to $C_b^3(\mathbb{R})$ and the function $\Phi$ is bounded.

We fix an arbitrary $t \in [0, T] \subset \mathbb{R}^+$. Let $(X_s^{t,x})_{0\le s\le t}$ be the unique solution of the following stochastic differential equation:

$$\begin{cases} \mathrm{d}X_s^{t,x} = -b(X_s^{t,x})\mathrm{d}s - \sigma(X_s^{t,x}) \downarrow \mathrm{d}W_s, & s \in [0, t]\,, \\ X_t^{t,x} = x. \end{cases} \tag{4.1}$$

Here the stochastic integral $\int_0^t \cdot \downarrow \mathrm{d}W_s$ is again understood as the backward Itô one. The condition (H1) guarantees the existence and uniqueness of the solution $(X_s^{t,x})_{0\le s\le t}$ in $\mathcal{M}^2(\mathbb{F}^W, [0, T]; \mathbb{R}^n)$. Our aim is to study the following backward doubly stochastic differential equation:

$$U_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f(r, X_r^{t,x}, U_r^{t,x}, V_r^{t,x})\mathrm{d}r + \int_0^s g(U_r^{t,x})\mathrm{d}B_r - \int_0^s V_r^{t,x} \downarrow \mathrm{d}W_r,\ s \in [0, t]. \tag{4.2}$$

We emphasize that the integral with respect to the fBM $B$ is interpreted in the Russo-Vallois sense, while the integral with respect to the Brownian motion $W$ is the Itô backward one. If $B$ is a standard Brownian motion, equation (4.2) coincides with the BDSDE which was first studied by Pardoux and Peng [84] in 1994 (apart of a time inversion).

Before we investigate the BDSDE (4.2), we first give the definition of its solution.

**Definition 4.1.** *A solution of equation* (4.2) *is a couple of processes* $(U_s^{t,x}, V_s^{t,x})_{s\in[0,t]}$ *such that:*
*1).* $(U_s^{t,x}, V_s^{t,x})_{s\in[0,t]} \in \mathcal{H}_t^\infty(\mathbb{R}) \times \mathcal{H}_t^2(\mathbb{R}^d)$;
*2). The Russo-Vallois integral* $\int_0^\cdot g(U_r^{t,x})\mathrm{d}B_r$ *is well defined on* $[0, t]$;
*3). Equation* (4.2) *holds* $\mathbb{P}$*-a.s.*

Unlike the classical case, the lack of the semimartingale property of the fBM $B$ gives an extra difficulty in solving BDSDE (4.2) directly. However, the work of Buckdahn and Ma [22] indicates another possibility to investigate this equation: by using the Doss-Sussmann transformation. Let us develop the idea. We denote by $\eta$ the stochastic flow which is the unique solution of the following stochastic differential equation:

$$\eta(t, y) = y + \int_0^t g(\eta(s, y))\mathrm{d}B_s,\ t \in [0, T], \tag{4.3}$$

where the integral is interpreted in the sense of Russo-Vallois. The solution of such a stochastic differential equation can be written as $\eta(t, y) = \alpha(y, B_t)$ via Doss transformation (see, for example, Zähle [110]), where $\alpha(y, z)$ is the solution of the ordinary differential equation

$$\begin{cases} \frac{\partial\alpha}{\partial z}(y, z) = g(\alpha(y, z)),\ z \in \mathbb{R}, \\ \alpha(y, 0) = y. \end{cases} \tag{4.4}$$

By the classical PDE theory we know that, for every $z \in \mathbb{R}$, the mapping $y \mapsto \alpha(y,z)$ is a diffeomorphism over $\mathbb{R}$ and $(y,z) \mapsto \alpha(y,z)$ is $C^2$. In particular, we can define the $y$-inverse of $\alpha(y,z)$ and we denote it by $h(y,z)$, such that we have $\alpha(h(y,z)) = y$, $(y,z) \in \mathbb{R} \times \mathbb{R}$. Hence, it follows that

$$\frac{\partial \alpha}{\partial y}(h(y,z),z)\frac{\partial h}{\partial y}(y,z) = 1 \text{ and } \frac{\partial \alpha}{\partial z}(h(y,z),z) + \frac{\partial \alpha}{\partial y}(h(y,z),z)\frac{\partial h}{\partial z}(y,z) = 0.$$

Therefore,

$$\frac{\partial h}{\partial z}(y,z) = -\left(\frac{\partial \alpha}{\partial y}(h(y,z),z)\right)^{-1}\frac{\partial \alpha}{\partial z}(h(y,z),z) = -\frac{\partial h}{\partial y}(y,z)g(y).$$

As a direct consequence we have that also $\eta(t,\cdot) = \alpha(\cdot, B_t) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism and, thus, we can define $\mathcal{E}(t,y) := \eta(t,\cdot)^{-1}(y) = h(y,B_t)$, $(t,y) \in [0,T] \times \mathbb{R}$. Moreover, by the Itô formula (Theorem 3.1), we have

$$\mathrm{d}\mathcal{E}(t,y) = \mathrm{d}h(y,B_t) = \frac{\partial}{\partial z}h(y,B_t)\mathrm{d}B_t = -\frac{\partial}{\partial y}\mathcal{E}(t,y)g(y)\mathrm{d}B_t,\ t \in [0,T],$$

i.e., the process $\mathcal{E}$ satisfies the following equation:

$$\mathcal{E}(t,y) = y - \int_0^t \frac{\partial}{\partial y}\mathcal{E}(s,y)g(y)\mathrm{d}B_s,\ t \in [0,T]. \tag{4.5}$$

Furthermore, we have the following estimates for $\eta$ and $\mathcal{E}$.

**Lemma 4.2.** *There exists a constant $C > 0$ depending only on the bound of $g$ and its partial derivatives such that for $\xi = \eta, \mathcal{E}$, it holds that, $P$-a.s., for all $(t,y)$,*

$$\begin{array}{ll} |\xi(t,y)| \le |y| + C|B_t|, & \exp\{-C|B_t|\} \le \left|\frac{\partial}{\partial y}\xi\right| \le \exp\{C|B_t|\}, \\ \left|\frac{\partial^2}{\partial y^2}\xi\right| \le \exp\{C|B_t|\}, & \left|\frac{\partial^3}{\partial y^3}\xi\right| \le \exp\{C|B_t|\}. \end{array}$$

*Proof:* The first three estimates are similar to those in Buckdahn and Ma [22]. So we only prove the last one. For this end we define $\gamma(\theta,y,z) = \alpha(y,\theta z)$, for $(\theta,y,z) \in [0,1] \times \mathbb{R} \times \mathbb{R}^n$. It follows from (4.4) that

$$\gamma(\theta,y,z) = y + \int_0^\theta \langle g(\gamma(r,y,z)), z\rangle \mathrm{d}r.$$

By differentiating this latter equation, we obtain,

$$\begin{cases} \frac{\partial^4 \gamma}{\partial\theta\partial y^3}(\theta,y,z) = & \frac{\partial^3 g}{\partial y^3}(\gamma(\theta,y,z))z\left(\frac{\partial \gamma}{\partial y}(\theta,y,z)\right)^3 + \frac{\partial g}{\partial y}(\gamma(\theta,y,z))z\frac{\partial^3\gamma}{\partial y^3}(\theta,y,z) \\ & +3\frac{\partial^2 g}{\partial y^2}(\gamma(\theta,y,z))z\frac{\partial\gamma}{\partial y}(\theta,y,z)\frac{\partial^2\gamma}{\partial y^2}(\theta,y,z); \\ \frac{\partial^3\gamma}{\partial y^3}(0,y,z) = 0, & \end{cases} \tag{4.6}$$

and from the variation of parameter formula it follows that

$$\begin{aligned} \frac{\partial^3}{\partial y^3}\gamma(1,y,z) = \int_0^1 \exp\left\{\int_u^1 \frac{\partial g}{\partial y}(\gamma(v,y,z))z\mathrm{d}v\right\}&\left(\frac{\partial^3 g}{\partial y^3}(\gamma(u,y,z))z\left(\frac{\partial\gamma}{\partial y}(u,y,z)\right)^3\right. \\ &\left.+ 3\frac{\partial^2 g}{\partial y^2}(\gamma(u,y,z))z\frac{\partial\gamma}{\partial y}(u,y,z)\frac{\partial^2\gamma}{\partial y^2}(u,y,z)\right)\mathrm{d}u. \end{aligned}$$

Thus, by using the first three estimates of this Lemma, we get

$$\left|\frac{\partial^3}{\partial y^3}\eta(t,y)\right| = \left|\frac{\partial^3}{\partial y^3}\alpha(y,B_t)\right| \le \exp\{C|B_t|\}.$$

Hence we have completed the proof. ■

Lemma 4.2 plays an important role in the rest of the chapter thanks to the following lemma.

**Lemma 4.3.** *For any $C \in \mathbb{R}$, we have*

$$\mathbb{E}\left[\exp\left\{C \sup_{s\in[0,T]} |B_s|\right\}\right] < \infty.$$

*Proof:* The proof is similar as Lemma 2.4 in [58] (and even easier), so we omit it. ■

We denote by $\widetilde{\Omega}'$ the subspace of $\Omega'$ such that

$$\widetilde{\Omega}' = \left\{\omega' \in \Omega' : \sup_{s\in[0,T]} |B_s| < \infty\right\}.$$

It is clear that $\mathbb{P}'(\widetilde{\Omega}') = 1$.

We let $(Y^{t,x}, Z^{t,x})$ be the unique solution of the following BSDE:

$$Y_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s \tilde{f}(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\mathrm{d}r - \int_0^s Z_r^{t,x} \downarrow \mathrm{d}W_r, \tag{4.7}$$

where

$$\tilde{f}(t,x,y,z) = \frac{1}{\frac{\partial}{\partial y}\eta(t,y)}\left\{f\left(t,x,\eta(t,y),\frac{\partial}{\partial y}\eta(t,y)z\right) + \frac{1}{2}\mathrm{tr}\left[z^T\frac{\partial^2}{\partial y^2}\eta(t,y)z\right]\right\}.$$

This BSDE, studied over $\Omega = \Omega' \times \Omega''$ and driven by the Brownian motion $W_r(\omega) = W_r(\omega'') = \omega''(r)$, $r \in [0,T]$, can be interpreted as an $\omega'$-pathwise BSDE, i.e., as a BSDE over $\Omega''$, considered for every fixed $\omega' \in \Omega'$. However, subtleties of measurability make us preferring to consider the BSDE over $\Omega$, with respect to the filtration $\mathbb{H}$. We point out that the coefficient $\tilde{f}$ has a quadratic growth in $z$, while the terminal value is bounded. BSDEs of this type have been studied by Kobylanski [61]. We state an existence and uniqueness result for this kind of BSDE, but with a slight adaptation to our framework. For this we consider a driving coefficient $G$ satisfying the following assumptions:

(H3) The coefficient $G : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d$ is measurable, for every fixed $(y,z)$, progressively measurable, with respect to the backward filtration $\mathbb{H}$ and $G$ is continuous in $(t,y,z)$;

(H4) There exists some real-valued $\mathcal{F}_T^B$-measurable random variable $K : \Omega' \to \mathbb{R}$ such that $|G(t,y,z)| \le K(1+|z|^2)$.

(H5) There exist real-valued $\mathcal{F}_T^B$-measurable random variables $C > 0$, $\varepsilon > 0$, and $\mathcal{F}_T^B \otimes \mathcal{B}([0,T])$-measurable functions $k$, $l_\varepsilon : \Omega' \times [0,T] \to \mathbb{R}$ such that

$$\left|\frac{\partial G}{\partial z}(t,y,z)\right| \le k(t) + C|z|, \text{ for all } (t,y,z), \ \mathbb{P}-\text{a.s.},$$

$$\frac{\partial G}{\partial y}(t,y,z) \le l_\varepsilon(t) + \varepsilon|z|^2, \text{ for all } (t,y,z), \ \mathbb{P}-\text{a.s.}$$

**Remark 4.4.** *Due to the Lemmata 4.2 and 4.3, the function $\tilde{f}$ in the equation (4.7) satisfies $(H3)-(H5)$. In particular, $K = \exp\{C \sup_{t\in[0,T]} |B_t|\}$ for $C \in \mathbb{R}^+$ appropriately chosen.*

Adapting the results by Kobylanski [61] (Theorem 2.3 and Theorem 2.6), we can state the following:

**Theorem 4.5.** *Let $G$ be a driver such that hypotheses $(H3)$-$(H5)$ hold and let $\xi$ be a real-valued $\mathcal{H}_0$-measurable random variable, which is bounded by a real-valued $\mathcal{F}_T^B$-measurable random variable. Then there exists a unique solution $(Y, Z) \in \mathcal{H}_T^\infty(\mathbb{R}) \times \mathcal{H}_T^2(\mathbb{R}^d)$ of BSDE*

$$Y_t = \xi + \int_0^t G(s, Y_s, Z_s)\mathrm{d}s - \int_0^t Z_s \downarrow \mathrm{d}W_s,\ t \in [0, T]. \tag{4.8}$$

*Moreover, there exists a real-valued $\mathcal{F}_T^B$-measurable random variable $C$ depending only on $\operatorname{esssup}_{[0,T]\times\Omega''} |Y_t(\omega', \omega'')|$ and $K$, such that*

$$\mathbb{E}\left[\int_0^T |Z_s|^2 \mathrm{d}s |\mathcal{F}_T^B\right] \leq C,\ \mathbb{P}-\text{a.s.}$$

**Remark 4.6.** *The conditional expectation $\mathbb{E}\left[\cdot|\mathcal{F}_T^B\right]$ is here understood in the generalized sense: if $\xi$ is a nonnegative $\mathcal{H}_0$-measurable random variable,*

$$\mathbb{E}[\xi|\mathcal{F}_T^B] := \lim_{n\to\infty} \uparrow \mathbb{E}[\xi \wedge n|\mathcal{F}_T^B] (\leq \infty)$$

*is a well defined $\mathcal{F}_T^B$-measurable random variable. If $\xi$ is not nonnegative we decompose $\xi = \xi^+ - \xi^-, \xi^+ = \max\{\xi, 0\}, \xi^- = -\min\{\xi, 0\}$ and we put $\mathbb{E}[\xi|\mathcal{F}_T^B] = \mathbb{E}[\xi^+|\mathcal{F}_T^B] - \mathbb{E}[\xi^-|\mathcal{F}_T^B]$ on $\{\min\{\mathbb{E}[\xi^+|\mathcal{F}_T^B], \mathbb{E}[\xi^-|\mathcal{F}_T^B]\} < \infty\}$.*

*Proof of Theorem 4.5:* We observe that the Brownian motion $W$ possesses the (backward) martingale representation with respect to the backward filtration $\mathbb{H}$, i.e., given an $\mathcal{H}_0$-measurable random variable $\xi$ such that $\mathbb{E}\left[\xi^2|\mathcal{F}_T^B\right] < \infty$, $\mathbb{P}$-a.s., there exists a unique process $\gamma \in \mathcal{H}_T^2(\mathbb{R}^d)$ such that

$$\xi = \mathbb{E}[\xi|\mathcal{F}_T^B] - \int_0^T \gamma_r \downarrow \mathrm{d}W_r,\ \mathbb{P}-\text{a.s.}$$

This martingale representation property allows to show the existence and uniqueness of a solution of (4.8) when $G$ is of linear growth and Lipschitz in $(y, z)$. Combining this with the approach by Kobylanski [61] allows to obtain the result stated in Theorem 4.5. ∎

By using Theorem 4.5, we are now able to characterize more precisely the solution of BSDE (4.7).

**Theorem 4.7.** *Under our standard assumptions on the coefficients $\sigma, b, f$ and $\Phi$, BSDE (4.7) admits a unique solution $(Y^{t,x}, Z^{t,x})$ in $\mathcal{H}_t^\infty(\mathbb{R}) \times \mathcal{H}_t^2(\mathbb{R}^d)$. Moreover, there exists a positive increasing process $\theta \in L^0(\mathbb{H}, \mathbb{R})$ such that*

$$|Y_s^{t,x}| \leq \theta_s,\quad \mathbb{E}\left[\int_0^\tau |Z_s^{t,x}|^2 \mathrm{d}s |\mathcal{H}_\tau\right] \leq \exp\{\exp\{C \sup_{s\in[0,t]} |B_s|\}\},\ \mathbb{P}-\text{a.s.},$$

*for all $\mathbb{H}$-stopping times $\tau$ $(0 \leq \tau \leq t)$, where $C$ is a constant chosen in an adequate way. Furthermore, the process $(Y^{t,x}, Z^{t,x})$ is $\mathbb{G}$-adapted.*

*Proof:* Due to Theorem 4.5, equation (4.7) has a unique solution $(Y^{t,x}, Z^{t,x})$ in $\mathcal{H}_t^\infty(\mathbb{R}) \times \mathcal{H}_t^2(\mathbb{R}^d)$.

**Step 1**. In order to give the estimates of $(Y^{t,x}, Z^{t,x})$, we proceed as in Lemma 5.3 in [22]. In particular, we can show that there exists an increasing positive process $\theta \in L^0(\mathbb{F}^B, [0,T])$ such that $\mathbb{P}$-a.s., $|Y_s^{t,x}| \leq \theta_s, 0 \leq s \leq t \leq T$. This process $(\theta_s)$ can be chosen as the solution of the following ordinary differential equation

$$\frac{\mathrm{d}\theta_s}{\mathrm{d}s} = \exp\{C \sup_{0\leq r\leq t} |B_r|\}(1+\theta_s);\ \theta(0) = |\Phi(X_0^{t,x})|,$$

for some suitably chosen real constant $C$, i.e.,

$$\theta_s = (|\Phi(X_0^{t,x})| + 1)\exp\left\{\exp\left\{C \sup_{0\leq r\leq t} |B_r|\right\} s\right\} - 1.$$

Indeed, for $M > 0$, let $\varphi_M(y)$ be a $C^\infty$ function such that $0 \leq \varphi_M \leq 1$, $\varphi_M(y) = 1$ for $|y| \leq M$ and $\varphi_M(y) = 0$ for $|y| \geq M+1$. Defining a new function $\tilde{f}^M$ by $\tilde{f}^M(t,x,y,z) \triangleq \tilde{f}(t,x,y,z)\varphi_M(y)$ we see that the function $\tilde{f}^M$ also satisfies conditions (H3)-(H5). According to Theorem 4.5, there exists a unique solution $(Y^{M,t,x}, Z^{M,t,x})$ of equation (4.7) with $\tilde{f}$ being replaced with $\tilde{f}^M$. The stability result in Kobylanski [61] shows that, when $M \to +\infty$, there exists a subsequence of $Y^{M,t,x}$ converging to $Y^{t,x}$ uniformly in probability. Therefore, following Buckdahn and Ma's approach and slightly adapted, we only need to prove that $Y_s^{M,t,x}$ is uniformly bounded by $\theta_s$. We apply the Tanaka formula to $|Y^{M,t,x}|$ to get that

$$\begin{aligned}\left|Y_s^{M,t,x}\right| =& |\Phi(X_0^{t,x})| + \int_0^s \mathrm{sgn}\left(Y_r^{M,t,x}\right) \tilde{f}^M(r, X_r^{t,x}, Y_r^{M,t,x}, Z_r^{M,t,x})\mathrm{d}r \\ &- \int_0^s \mathrm{sgn}\left(Y_r^{M,t,x}\right) Z_r^{M,t,x} \downarrow \mathrm{d}W_r + L_s - L_0,\end{aligned}$$

for a local-time-like process $L$ such that $L_t = 0$ and $L_s = \int_s^t 1_{\{Y_r^{M,t,x}=0\}}\mathrm{d}L_r$. Now we define a new function $\psi(y)$ by letting $\psi(y) = e^{2Ky} - 1 - 2Ky - 2K^2y^2$, for $y > 0$, and $\psi(y) = 0$ for $y \leq 0$, where $K$ is the bound in Remark 4.4. Then we apply the Itô formula to $\psi(\left|Y_s^{M,t,x}\right| - \theta_s)$ and get

$$\begin{aligned}&\psi\left(\left|Y_s^{M,t,x}\right| - \theta_s\right) \\ =& \int_0^s \psi'\left(\left|Y_r^{M,t,x}\right| - \theta_r\right)\mathrm{sgn} \\ &\times \left(Y_r^{M,t,x}\right)\left(\tilde{f}^M\left(r, X_r^{t,x}, Y_r^{M,t,x}, Z_r^{M,t,x}\right) - \exp\{C \sup_{0\leq u\leq t}|B_u|\}(1+\theta_r)\right)\mathrm{d}r \\ &- \int_0^s \psi'\left(\left|Y_r^{M,t,x}\right| - \theta_r\right)\mathrm{sgn}\left(Y_r^{M,t,x}\right) Z_r^{M,t,x} \downarrow \mathrm{d}W_r + \int_0^s \psi'\left(\left|Y_r^{M,t,x}\right| - \theta_r\right)\mathrm{d}L_r \\ &- \frac{1}{2}\int_0^s \psi''\left(\left|Y_r^{M,t,x}\right| - \theta_r\right)\left|Z_r^{M,t,x}\right|^2 \mathrm{d}r\end{aligned} \tag{4.9}$$

The property of $\psi$ shows that $\int_0^s \psi'\left(\left|Y_r^{M,t,x}\right| - \theta_r\right)\mathrm{d}L_r = \int_0^s \psi'\left(-\theta_r\right)\mathrm{d}L_r = 0$. We also have

$$\begin{aligned}
&\int_0^s \psi'\left(\left|Y_r^{M,t,x}\right| - \theta_r\right) \operatorname{sgn} \\
&\quad \times \left(Y_r^{M,t,x}\right)\left(\tilde{f}^M\left(r, X_r^{t,x}, Y_r^{M,t,x}, Z_r^{M,t,x}\right) - \exp\{C \sup_{0\le u\le t} |B_u|\}(1+\theta_r)\right) \mathrm{d}r \\
\le &\int_0^s \psi'\left(\left|Y_r^{M,t,x}\right| - \theta_r\right)\left(K\left(\varphi(Y_r^{M,t,x})\left|Y_r^{M,t,x}\right| - \theta_r\right) + K|Z_r^{M,t,x}|^2\right) \mathrm{d}r.
\end{aligned}$$

Since $\psi'' - 2K\psi' \ge 0$, we get from equation (4.9) that

$$\begin{aligned}
\psi\left(\left|Y_s^{M,t,x}\right| - \theta_s\right) \le & K\int_0^s \psi'\left(\left|Y_r^{M,t,x}\right| - \theta_r\right)\left(\varphi(Y_r^{M,t,x})\left|Y_r^{M,t,x}\right| - \theta_r\right) \mathrm{d}r \\
&- \int_0^s \psi'\left(\left|Y_r^{M,t,x}\right| - \theta_r\right) \operatorname{sgn}\left(Y_r^{M,t,x}\right) Z_r^{M,t,x} \downarrow \mathrm{d}W_r.
\end{aligned}$$

Thus, we deduce that

$$\begin{aligned}
&\mathbb{E}\left[\psi\left(\left|Y_s^{M,t,x}\right| - \theta_s\right)\middle| \mathcal{H}_s\right] \\
\le &\mathbb{E}\left[K\int_0^s \psi'\left(\left|Y_r^{M,t,x}\right| - \theta_r\right)\left(\varphi(Y_r^{M,t,x})\left|Y_r^{M,t,x}\right| - \theta_r\right) \mathrm{d}r \middle| \mathcal{H}_s\right].
\end{aligned}$$

From the definition of $\psi$ we get that $\psi'(y) = 2K(\psi(y) + 2K^2y^2)$. Hence, we have

$$\begin{aligned}
&\mathbb{E}\left[\psi\left(\left|Y_s^{M,t,x}\right| - \theta_s\right)\middle| \mathcal{H}_s\right] \\
\le &\mathbb{E}\Big[\int_0^s 2K^2\psi\left(\left|Y_r^{M,t,x}\right| - \theta_r\right)\left(\varphi(Y_r^{M,t,x})\left|Y_r^{M,t,x}\right| - \theta_r\right) \\
&\quad + 4K^4\left(\left|Y_r^{M,t,x}\right| - \theta_r\right)^3 \mathrm{d}r\Big|\mathcal{H}_s\Big].
\end{aligned}$$

There also exists a $\tilde{K}$ such that $y^3 \le \tilde{K}\psi(y)$, for all $y \in \mathbb{R}$. Consequently, we get that

$$\begin{aligned}
&\mathbb{E}\left[\psi\left(\left|Y_s^{M,t,x}\right| - \theta_s\right)\middle| \mathcal{H}_s\right] \\
\le &2K^2(M + \|\theta\|_{\infty,[0,t]} + 2K^2\tilde{K})\int_0^s \mathbb{E}\left[\psi\left(\left|Y_r^{M,t,x}\right| - \theta_r\right)\middle| \mathcal{H}_s\right] \mathrm{d}r.
\end{aligned}$$

Finally, the Gronwall inequality shows that $\psi\left(\left|Y_s^{M,t,x}\right| - \theta_s\right) = 0$, for any $s \in [0,t]$, $\mathbb{P}$-a.s. Therefore, $\left|Y_s^{M,t,x}\right| \le \theta_s$, for any $s \in [0,t]$, $\mathbb{P}$-a.s.

**Step 2**. We apply the Itô formula to $e^{aY_s^{t,x}}$, with $a$ being a real-valued $\mathcal{F}_T^B$-measurable random variable to be determined later, and we obtain

$$\begin{aligned}
e^{aY_s^{t,x}} &= e^{a\Phi(X_0^{t,x})} + \int_0^s ae^{aY_s}\tilde{f}(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\mathrm{d}r - \int_0^s \frac{1}{2}a^2e^{aY_r^{t,x}}|Z_r^{t,x}|^2\mathrm{d}r \\
&\quad - \int_0^s ae^{aY_r^{t,x}} Z_r^{t,x} \downarrow \mathrm{d}W_r \\
&\le e^{a\Phi(X_0^{t,x})} + \int_0^s \left(-\frac{1}{2}a^2 + |a|K\right) e^{aY_r^{t,x}}|Y_r^{t,x}|^2\mathrm{d}r + \int_0^s |a|Ke^{aY_r^{t,x}}\mathrm{d}r \\
&\quad - \int_0^s ae^{aY_r^{t,x}} Z_r^{t,x} \downarrow \mathrm{d}W_r.
\end{aligned}$$

Let $\zeta(\omega') := \operatorname{esssup}_{[0,t]\times\Omega''} |Y_s^{t,x}(\omega',\omega'')|(< +\infty, \omega' \in \widetilde{\Omega}')$. Hence, by taking the conditional expectations $\mathbb{E}[\cdot|\mathcal{H}_t]$ on both sides, we deduce that for any $\mathbb{H}$-stopping times $\tau \in [0,T]$,

$$
\begin{aligned}
&\left(\frac{1}{2}a^2 - |a|K\right) e^{-|a|\zeta}\mathbb{E}\left[\int_0^\tau |Z_r^{t,x}|^2 \mathrm{d}r \middle| \mathcal{H}_\tau\right] \\
\leq &\left(\frac{1}{2}a^2 - |a|K\right) \mathbb{E}\left[\int_0^\tau e^{aY_r^{t,x}} |Z_r|^2 \mathrm{d}r \middle| \mathcal{H}_\tau\right] \\
\leq &\mathbb{E}\left[e^{a\Phi(X_0^{t,x})} - e^{aY_\tau^{t,x}} + \int_0^\tau |a|K e^{aY_r^{t,x}} \mathrm{d}s \middle| \mathcal{H}_\tau\right].
\end{aligned}
$$

We can choose $a = 4K$ such that $\frac{1}{2}a^2 - |a|K = 4K^2$ and we get, keeping in mind that here $K$ is a random variable bounded by $\exp\{C \sup_{s\in[0,t]} |B_s|\}$ ($C \in \mathbb{R}^+$ is a real constant, see Remark 4.4),

$$
\mathbb{E}\left[\int_0^\tau |Z_r^{t,x}|^2 \mathrm{d}r \middle| \mathcal{H}_\tau\right] \leq \frac{(2+4K^2 t)}{4K^2} e^{8K} \leq \exp\left\{\exp\left\{C \sup_{s\in[0,t]} |B_s|\right\}\right\}. \tag{4.10}
$$

**Step 3**. Let us show that the process $(Y^{t,x}, Z^{t,x})$ is not only $\mathbb{H}$- but also $\mathbb{G}$-adapted. For this we consider for an arbitrarily given $\tau \in [0,t]$ equation (4.7) over the time interval $[0,\tau]$:

$$
Y_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s \tilde{f}\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) \mathrm{d}r - \int_0^s Z_r^{t,x} \downarrow \mathrm{d}W_r, \quad s \in [0,\tau]. \tag{4.11}
$$

Let $\mathcal{H}_t^\tau := \mathcal{F}_{t,T}^W \vee \mathcal{F}_\tau^B$, $t \in [0,\tau]$. Then $\mathbb{H}^\tau = \{\mathcal{H}_t^\tau\}_{t\in[0,\tau]}$ is a backward Brownian filtration enlarged by a $\sigma$-algebra generated by the fBM $B$, which is independent of the Brownian filtration. Thus, with respect to $\mathbb{H}^\tau$, the Brownian motion $W$ has the martingale representation property. Since $\tilde{f}(r,x,y,z)$ is $\mathcal{G}_t$- and, hence, also $\mathcal{H}_t^\tau$-measurable, $\mathrm{d}r$ a.e. on $[0,\tau]$, it follows from the classical BSDE theory (or Theorem 4.5) that BSDE (4.11) admits a unique solution $\left(Y^{t,x,\tau}, Z^{t,x,\tau}\right) \in \mathcal{H}_\tau^\infty(\mathbb{R}) \times \mathcal{H}_\tau^2(\mathbb{R}^d)$. On the other hand, also $\left(Y_r^{t,x,\tau}, Z_r^{t,x,\tau}\right)_{r\in[0,\tau]}$ is a solution of (4.7). Hence, $\left(Y_r^{t,x}, Z_r^{t,x}\right) = \left(Y_r^{t,x,\tau}, Z_r^{t,x,\tau}\right)$, $\mathrm{d}r$ a.e., for $t < \tau$. Consequently, $\left(Y_r^{t,x}, Z_r^{t,x}\right)$ is $\mathcal{H}_r^\tau$- measurable, $\mathrm{d}r$ a.e., for $r < \tau$. Therefore, letting $\tau \downarrow t$ we can deduce from the right continuity of the filtration $\mathbb{F}^B$ that $\left(Y^{t,x}, Z^{t,x}\right)$ is $\mathbb{G}$-adapted. ∎

**Remark 4.8.** *We remind the reader that the bound we get in* (4.10) *is only $\mathbb{P}$-a.s. finite, but not square-integrable. As a matter of fact, it is hard to prove directly that $Z^{t,x}$ is a square-integrable process, which constitutes the main reason that we use instead the space $\mathcal{H}_t^2(\mathbb{R}^d)$. Hence, the major difference between our work and Buckdahn and Ma [22] is: In the classical case, a priori we can solve the BDSDE in the first step to get the square integrability of $Z$, but in the fractional case, there is not a direct way to solve the BDSDE.*

Now we are ready to give the main result of this section by linking the BSDEs (4.2) and (4.7) with the help of the Doss-Sussmann transformation.

**Theorem 4.9.** *Let us define a new pair of processes $(U^{t,x}, V^{t,x})$ by*

$$
U_s^{t,x} = \eta(s, Y_s^{t,x}), \quad V_s^{t,x} = \frac{\partial}{\partial y}\eta(s, Y_s^{t,x}) Z_s^{t,x},
$$

*where $(Y^{t,x}, Z^{t,x})$ is the solution of BSDE* (4.7)*. Then $(U^{t,x}, V^{t,x}) \in \mathcal{H}_t^\infty(\mathbb{R}) \times \mathcal{H}_t^2(\mathbb{R}^d)$ is the solution of BDSDE* (4.2)*.*

**Remark 4.10.** *The above theorem can be considered as a counterpart of Theorem 3.9 in Jing and León [58] for the semi-linear case when $H < 1/2$. However, since we use here a different Hurst parameter $H$ and a different type of stochastic integral with respect to fBM $B$, the above theorem obviously does not cover the result in [58].*

*Proof of Theorem 4.9:* The fact that $(U^{t,x}, V^{t,x}) \in \mathcal{H}_t^\infty(\mathbb{R}) \times \mathcal{H}_t^2(\mathbb{R}^d)$ follows directly from Theorem 4.7 and Lemma 4.2. In order to prove the remaining part of the theorem, we just have to apply the Itô formula (Theorem 3.1) to $\alpha(Y_s^{t,x}, B_s)$, noticing $\alpha(Y_s^{t,x}, B_s) = \eta(s, Y_s^{t,x})$, to obtain that, for $(s,x) \in [0,t] \times \mathbb{R}^d$, the Russso-Vallois integral $\int_0^s g(U_r^{t,x})\mathrm{d}B_r$ exists and

$$\begin{aligned} U_s^{t,x} =& \Phi_0(X_0^{t,x}) + \int_0^s \frac{\partial}{\partial y}\alpha(Y_r^{t,x}, B_r)\Big[ \Big( \frac{\partial}{\partial y}\eta(r, Y_r^{t,x}) \Big)^{-1} \Big\{ f\Big(r, X_r^{t,x}, \eta(s, Y_r^{t,x}), \\ & \frac{\partial}{\partial y}\eta(r, Y_r^{t,x}) Z_r^{t,x}\Big) + \frac{1}{2}\mathrm{tr}\left[ (Z_r^{t,x})^T \frac{\partial^2}{\partial y^2}\eta(r, Y_r^{t,x}) Z_r^{t,x} \right] \Big\} \mathrm{d}r - Z_r^{t,x} \downarrow \mathrm{d}W_r \Big] \\ & - \frac{1}{2}\int_0^s \mathrm{tr}\left[ (Z_r^{t,x})^T \frac{\partial^2}{\partial y^2}\eta(r, Y_r^{t,x}) Z_r^{t,x} \right] \mathrm{d}r + \int_0^s \frac{\partial}{\partial z}\alpha(Y_r^{t,x}, B_r)\mathrm{d}B_r. \end{aligned} \tag{4.12}$$

Consequently,

$$U_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f(r, X_r^{t,x}, U_r^{t,x}, V_r^{t,x})\mathrm{d}r + \int_0^s g(U_r^{t,x})\mathrm{d}B_r - \int_0^s V_r^{t,x} \downarrow \mathrm{d}W_r. \tag{4.13}$$

The proof is complete. ■

Now we close this section by a simple example to illustrate the idea of the above procedure.

**Example 4.11.** *Let us consider the linear case ( for simplicity of notations we omit the superscript $(t, x)$):*

$$\begin{cases} \mathrm{d}U_s = U_s \mathrm{d}B_s + f(U_s, V_s)\mathrm{d}s + V_s \downarrow \mathrm{d}W_s, \ s \in [0,t]; \\ U_0 = \Phi(X_0^{t,x}). \end{cases} \tag{4.14}$$

*It is elementary to show that $\alpha(y,z) = ye^z$. Hence, the solution $(U, V)$ of equation (4.14) is given by $U_s = Y_s e^{B_s}$ and $V_s = Z_s e^{B_s}$, where $(Y, Z)$ is the solution of*

$$\begin{cases} \mathrm{d}Y_s = \hat{f}(Y_s, Z_s)\mathrm{d}s + Z_s \downarrow \mathrm{d}W_s, \ s \in [0,t]; \\ Y_0 = \Phi(X_0^{t,x}). \end{cases} \tag{4.15}$$

*and the function $\hat{f}$ is defined by $\hat{f}(y,x) = e^{-B_t} f(ye^{B_t}, ze^{B_t})$. Obviously $\hat{f}$ is Lipschitz in $(y,z)$ as far as $f$ is Lipschitz. By following a classical Malliavin calculus method (see, e.g., Pardoux and Peng [84]), we can get that $Z$ is $\mathbb{P}$-a.s. uniformly bounded and, thus, the process $\{Y_s, s \in [0,t]\}$ is $(1/2-\varepsilon)$-Hölder continuous in $s$, for all $\varepsilon \in (0, 1/2)$. Therefore, we have, for every $r, s \in [0,t]$, almost all $\omega' \in \Omega'$,*

$$\begin{aligned} & |U_r - U_s| = |Y_r e^{B_r} - Y_s e^{B_s}| \\ \le & \|Y\|_\infty |e^{B_r} - e^{B_s}| + e^{|B_s|}|Y_r - Y_s| \\ \le & \exp\Big\{ \sup_{0\le u\le t} |B_u| \Big\} \|Y\|_\infty C(\omega')|r-s|^\alpha + \exp\Big\{ \sup_{0\le u\le t} |B_u| \Big\} C(\omega')|r-s|^{1/2-\varepsilon}, \end{aligned}$$

*where we can choose* $\alpha$ *to be in* $(1/2, H)$. *That is to say, we can choose* $0 < \varepsilon < \alpha - 1/2$ *and get that the process* $\{U_s, s \in [0,t]\}$ *has* $\alpha$*-Hölder continuous paths. Hence, instead of using the Russo-Vallois integral with respect to the fractional Brownian motion in equation* (4.14) *we can use the classical Young integral. Furthermore, thanks to Proposition 2.3, these two integrals coincide.*

**Remark 4.12.** In the latter example, the Young integral and the Russo-Vallois integral coincide. Unfortunately, the Hölder continuity seems to be very hard to deduce in the general, nonlinear case. As a consequence, we have to work with the more general Russo-Vallois integral.

# 5 Associated Stochastic Partial Differential Equations

In this section we discuss briefly the relationship between an associated SPDE and a PDE with stochastic coefficients. For simplicity, we only show the relationship for the case of classical solutions. For a complete discussion of the case of (stochastic) viscosity solutions, one can proceed by adapting the approaches in Buckdahn and Ma [22] and Chapter 1.

Let $\mathcal{L}$ be the second order elliptic differential operator:

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i},$$

which means that it is the infinitesimal generator of the Markovian process $\{X_s^{t,x}, s \in [0,t]\}$ defined by equation (4.1).

Our aim is to study the following semilinear SPDE driven by the fBM $B$:

$$\begin{cases} \mathrm{d}v(t,x) = \left[\mathcal{L}v(t,x) - f(t,x,v(t,x),\sigma(x)^T\frac{\partial}{\partial x}v(t,x))\right]\mathrm{d}t + g(v(t,x))\mathrm{d}B_t, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in (0,T)\times\mathbb{R}^n; \\ v(0,x) = \Phi(x), \qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^n. \end{cases} \tag{5.1}$$

In the case that $B$ is a Brownian motion (an fBM with Hurst parameter $H = 1/2$), it is well known (see, Pardoux and Peng [84], Buckdahn and Ma [22]) that the random field $v(t,x) := U_t^{t,x}$ solves (5.1) (in the viscosity sense if the coefficients are Lipschitz, and in the classical sense if the coefficients are $C_b^3$), where $U^{t,x}$ is the solution of BDSDE (4.2). Thus, it is natural to raise the following question: Can we also solve the SPDE (5.1) driven by the fBM $B$, by studying the properties of the solution of its associated BDSDE (4.2)? The answer is positive. Indeed, by applying the Doss-Sussmann transformation, we will show that the PDE:

$$\begin{cases} \mathrm{d}u(t,x) = \left(\mathcal{L}u(t,x) - \tilde{f}(t,x,u(t,x),\sigma(x)^T\frac{\partial}{\partial x}u(t,x))\right)\mathrm{d}t, \quad (t,x) \in (0,T)\times\mathbb{R}^n; \\ u(0,x) = \Phi(x), \qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^n, \end{cases} \tag{5.2}$$

is transformed into SPDE (5.1), where we recall that

$$\tilde{f}(t,x,y,z) = \left(\frac{\partial}{\partial y}\eta(t,y)\right)^{-1}\left\{f(t,x,\eta(t,y),\frac{\partial}{\partial y}\eta(t,y)z) + \frac{1}{2}\mathrm{Tr}\left[z\frac{\partial^2}{\partial y^2}\eta(t,y)z^T\right]\right\}.$$

We also observe that following a similar argument as in Kobylanski [61], under some smoothness assumptions, $u(t,x) := Y_t^{t,x}, (t,x) \in [0,T]\times\mathbb{R}^n$ is the solution of equation (5.2), where $Y^{t,x}$ is the solution of BSDE (4.7).

First we give the definition for the classical solutions of equations (5.1) and (5.2).

**Definition 5.1.** *We say a stochastic field $w : \Omega' \times [0,T]\times\mathbb{R}^n \to \mathbb{R}$ is a classical solution of equation* (5.1) *(resp.,* (5.2)*), if $w \in C_{\mathbb{F}^B}^{0,2}$* [2] *and satisfies equation* (5.1) *(resp.,* (5.2)*).*

We have the following proposition.

**Proposition 5.2.** *Suppose that $u$ is a classical solution of equation* (5.2)*, then $\hat{u}(t,x) \triangleq \eta(t,u(t,x)) = \alpha(u(t,x),B_t)$ is a classical solution of SPDE* (5.1)*. The converse holds also true: Every classical solution $\hat{u}$ of equation* (5.1) *defines a classical solution $u(t,x) = \mathcal{E}(t,\hat{u}(t,x))$ of equation* (5.2)*.*

2. $C_{\mathbb{F}^B}^{0,2}$ is the space of $\mathbb{F}^B$-adapted processes $u(t,x)$ which are continuous in $t$ and $C^2$ in $x$.

*Proof:* The claim that $\hat{u}(t,x) \in C^{0,2}_{\mathbb{F}^B}$ follows from the regularity property of the functions $\alpha$ and $u$ and the fact that $u$ is $\mathbb{F}^B$-adapted. Moreover, we first observe that $\hat{u}(0,x) = \eta(0, u(0,x)) = u(0,x) = \Phi(x)$, and we apply the Itô formula to $\hat{u}(t,x)$ to obtain:

$$\begin{aligned}
\mathrm{d}\hat{u}(t,x) =& \frac{\partial}{\partial y}\alpha(u(t,x),B_t)\left(\mathcal{L}u(t,x) - \tilde{f}\left(t,x,u(t,x),\sigma(x)^T\frac{\partial}{\partial x}u(t,x)\right)\right)\mathrm{d}t \\
&+ \frac{\partial}{\partial z}\alpha(u(t,x),B_t)\mathrm{d}B_t \\
=& \Bigg[\frac{\partial}{\partial y}\alpha(u(t,x),B_t)\mathcal{L}u(t,x) - f\Bigg(t,x,\eta(t,u(t,x)),\frac{\partial}{\partial y}\eta(t,u(t,x))\sigma(x)^T \\
&\frac{\partial}{\partial x}u(t,x)\Bigg) - \frac{1}{2}\mathrm{Tr}\left[\sigma(x)^T\frac{\partial}{\partial x}u(t,x)\frac{\partial^2}{\partial y^2}\eta(t,u(t,x))\frac{\partial}{\partial x}u(t,x)^T\sigma(x)\right]\Bigg]\mathrm{d}t \\
&+ \frac{\partial}{\partial z}\alpha(u(t,x),B_t)\mathrm{d}B_t.
\end{aligned} \tag{5.3}$$

We notice that

$$\begin{aligned}
\frac{\partial}{\partial x}\hat{u}(t,x) &= \frac{\partial}{\partial x}[\alpha(u(t,x),B_t)] = \left(\frac{\partial}{\partial y}\alpha\right)(u(t,x),B_t)\frac{\partial}{\partial x}u(t,x), \\
\frac{\partial^2}{\partial x^2}\hat{u}(t,x) =& \frac{\partial}{\partial x}u(t,x)\left(\frac{\partial^2}{\partial y^2}\alpha\right)(u(t,x),B_t)\left(\frac{\partial}{\partial x}u(t,x)\right)^T \\
&+ \left(\frac{\partial}{\partial y}\alpha\right)(u(t,x),B_t)\frac{\partial^2}{\partial x^2}u(t,x).
\end{aligned}$$

Thus, by substituting the above relations into (5.3), we can simplify the equation (5.3) and we get

$$\mathrm{d}\hat{u}(t,x) = \left[\mathcal{L}\hat{u}(t,x) - f\left(t,x,\hat{u}(t,x),\sigma(x)^T\frac{\partial}{\partial x}\hat{u}(t,x)\right)\right]\mathrm{d}t + g(\hat{u}(x,t))\mathrm{d}B_t. \tag{5.4}$$

Consequently, $\hat{u}$ is a solution of SPDE (5.1). The proof of the converse direction is analogous. The proof is complete. ■

In order to summarize, we have constructed a solution of SPDE (5.1) with the help of the fractional BDSDE (4.2), by passing through the quadratic BSDE (4.7) and the associated PDE (5.2) with random coefficients:

$$\begin{array}{ccc}
\textit{fract. BDSDE}\ (4.2) & \overset{Thm.\ 4.9}{\Longrightarrow} & \textit{quadratic BSDE}\ (4.7) \\
 & & \Downarrow \\
\textit{fract. SPDE}\ (5.1) & \overset{Prop.\ 5.2}{\Longleftrightarrow} & \textit{PDE}\ (5.2)\ (\textit{with random coefficients}).
\end{array}$$

Proposition 5.2 shows that, in the case of a classical solution, the Doss-Sussmann transformation establishes a link between PDE (5.2) and SPDE (5.1). This motivates us to give the following definition of the stochastic viscosity solution. For the cases $H \in (0,1/2]$, the reader is referred to Buckdahn and Ma [22], Jing and León [58]. For the classical definition of the viscosity solution, we refer to Crandall et al. [28].

**Definition 5.3.** *A continuous random field $\hat{u} : \Omega' \times [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is called a (stochastic) viscosity solution of equation (5.1) if and only if $u(t,x) = \mathcal{E}(t,\hat{u}(t,x))$, $(t,x) \in [0,T] \times \mathbb{R}^n$ is the viscosity solution of equation (5.2).*

By following a similar argument as that developed in the proof of Theorem 4.9 in Jing and León [58], our preceding discussion leads to the following theorem:

**Theorem 5.4.** *The stochastic field* $\hat{u} : \Omega' \times [0, T] \times \mathbb{R}^n \to \mathbb{R}$ *defined by* $\hat{u}(t, x) \triangleq \alpha(Y_t^{t,x}, B_t)$ *is a* (*stochastic*) *viscosity solution of SPDE* (5.1).

# Chapter 3

# Regularity Properties of Viscosity Solutions of Integro-Partial Differential Equations of Hamilton-Jacobi-Bellman Type

**Abstract**: We study the regularity properties of a certain class of integro-partial differential equations of Hamilton-Jocobi-Bellman type with terminal condition, which can be interpreted through a stochastic control system, composed of a forward and a backward stochastic differential equation, both driven by a Brownian motion and a compensated Poisson random measure. More precisely, we prove that, under appropriate assumptions, the viscosity solution of such equations is jointly Lipschitz and jointly semiconcave in $(t, x) \in \Delta \times \mathbb{R}^d$, for all compact time intervals $\Delta$ excluding the terminal time. Our approach is based on the method of time change for the Brownian motion and on Kulik's transformation for the Poisson random measure. It extends earlier works by Buckdahn, Cannarsa and Quincampoix (2010) and by Buckdahn, Huang and Li (2011) on regularity properties for viscosity solutions of Hamilton-Jacobi-Bellman partial differential equations without and with obstacle.

**Keywords**: backward stochastic differential equations; Brownian motion; Poisson random measure; time change; Kulik transformation; Lipschitz continuity; semiconcavity; viscosity solution; value function.

**MSC 2000**: 35D10, 60H30, 93E20.

# 1 Introduction

We are interested in the regularity properties of the viscosity solution for a certain class of integro-partial differential equations (IPDEs) of Hamilton-Jacobi-Bellman (HJB) type. In order to be more precise, let us consider the following possibly degenerate equation:

$$\begin{cases} \frac{\partial}{\partial t}V(t,x) + \inf_{u\in U}\{(\mathcal{L}^u + B^u)V(t,x) + f\big(t,x,V(t,x),(D_xV\sigma)(t,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad V(t,x+\beta(t,x,u,\cdot)) - V(t,x),u\big)\} = 0; \\ V(T,x) = \Phi(x), \end{cases} \tag{1.1}$$

where $U$ is a compact metric space, $\mathcal{L}^u$ is the linear second order differential operator

$$\mathcal{L}^u\varphi(x) = \mathrm{tr}\left(\frac{1}{2}\sigma\sigma^T(t,x,u)D^2_{xx}\varphi(x)\right) + b(t,x,u)\cdot D_x\varphi(x), \quad \varphi\in C^2(\mathbb{R}^d),$$

and $B^u$ is the integro-differential operator:

$$B^u\varphi(x) = \int_E [\varphi(x+\beta(t,x,u,e)) - \varphi(x) - \beta(t,x,u,e)\cdot D_x\varphi(x)]\Pi(\mathrm{d}e), \quad \varphi\in C^1_b(\mathbb{R}^d).$$

Here, $\Pi$ denotes a finite Lévy measure on $E=\mathbb{R}^n\backslash\{0\}$. Our main results say that, under appropriate assumptions, for all $\delta>0$, the viscosity solution $V$ is jointly Lipschitz and jointly semiconcave on $[0,T-\delta]\times\mathbb{R}^d$, i.e., there is some constant $C_\delta$ such that

$$|V(t_0,x_0)-V(t_1,x_1)| \le C_\delta(|t_0-t_1|+|x_0-x_1|),$$

$$\lambda V(t_0,x_0)+(1-\lambda)V(t_1,x_1) \le V\big(\lambda(t_0,x_0)+(1-\lambda)(t_1,x_1)\big)+C_\delta\lambda(1-\lambda)(|t_0-t_1|^2+|x_0-x_1|^2),$$

for all $(t_0,x_0),(t_1,x_1)\in[0,T-\delta]\times\mathbb{R}^d$. The joint semiconcavity of $V$ stems its importance from the fact that, due to Alexandrov's theorem, it implies that $V$ has a second order expansion in $(t,x)$, $\mathrm{d}t\mathrm{d}x$-a.e. Such expansions are important, for instance, for the study of the propagation of singularities.

Although, at least for PDEs of HJB type, the regularity of the solution of strictly elliptic equations (with $\sigma\sigma^T\ge\alpha I$, for $\alpha>0$) has been well understood for a long time, the joint regularity (Lipschitz continuity and semiconcavity) in $(t,x)$ for the viscosity solution of such equations, for which $\sigma\sigma^T$ is not necessarily strictly elliptic, have been studied only recently. However, under suitable hypotheses, the Lipschitz continuity and the semiconcavity of $V(t,x)$ in $x$ as well as the Hölder continuity of $V(t,x)$ in $t$ has already been known for a longer time. As concerns the semiconcavity of $V(t,x)$ in $x$, a purely analytical proof was given by Ishii and Lions [55]; for a stochastic proof the reader is referred, for example, to Yong and Zhou [108]. Concerning the Lipschitz continuity of $V$ in $x$ and the Hölder continuity in $t$ (with Hölder coefficient 1/2), we refer, for example, to Pham [95]. Krylov [64] suggested the joint Lipschitz continuity of $V(t,x)$ on $(t,x)$. However, counterexamples show that, in general, one cannot get the Lipschitz continuity or semiconcavity in $(t,x)$ for the whole domain $[0,T]\times\mathbb{R}^d$. In Buckdahn, Cannarsa and Quincampoix [18], it was shown that the viscosity solution of PDEs of HJB type (with $\beta=0$) is Lipschitz and semiconcave over $[0,T-\delta]\times\mathbb{R}^d$ for $\delta>0$. In Buckdahn, Huang and Li [19], these results were extended to PDEs with obstacle. The approach in [18] and [19] consists in the study of the viscosity solution $V$ with the help of its stochastic interpretation as a value function of an associated stochastic control problem; it uses, in particular, the method of time change, which was translated to backward stochastic differential equations (BSDEs) in [19].

In this chapter we study the joint regularity of $V(t,x)$ in $(t,x)$ through the stochastic interpretation of the above HJB equation as a stochastic control problem composed of a forward and a backward stochastic differential equation (SDE). More precisely, let $(t,x) \in [0,T]\times\mathbb{R}^d$, $B=(B_s)_{s\in[t,T]}$ be a $d$-dimensional Brownian motion with initial value zero at time $t$, and let $\mu$ be a Poisson random measure on $[t,T]\times E$. We denote by $\mathbb{F}$ the filtration generated by $B$ and $\mu$, and by $\mathcal{U}^{B,\mu}(t,T)$ the set of all $\mathbb{F}$-predictable control processes with values in $U$. It is by now standard that the SDE driven by the Brownian motion $B$ and the compensated Poisson random measure $\tilde{\mu}$:

$$X_s^{t,x,u} = x+\int_t^s b(r,X_r^{t,x,u},u_r)\mathrm{d}r+\int_t^s \sigma(r,X_r^{t,x,u},u_r)\mathrm{d}B_r+\int_t^s\int_E \beta(r,X_{r-}^{t,x,u},u_r,e)\tilde{\mu}(\mathrm{d}r,\mathrm{d}e), \tag{1.2}$$

$s\in[t,T]$, has a unique solution under appropriate assumptions for the coefficients. With this SDE we associate the BSDE with jumps

$$\begin{aligned} Y_s^{t,x,u} &= \Phi(X_T^{t,x,u})+\int_s^T f(r,X_r^{t,x,u},Y_r^{t,x,u},Z_r^{t,x,u},U_r^{t,x,u},u_r)\mathrm{d}r-\int_s^T Z_r^{t,x,u}\mathrm{d}B_r \\ &\quad -\int_s^T\int_E U_r^{t,x,u}(e)\tilde{\mu}(\mathrm{d}r,\mathrm{d}e), \quad s\in[t,T]. \end{aligned} \tag{1.3}$$

(As concerns the assumptions on the coefficients, we refer to the hypotheses **(H1)-(H5)** in Section 2 and Section 3.) From Barles, Buckdahn and Pardoux [10] we know that the above BSDE with jumps (1.3) has a unique square integrable solution $(Y^{t,x,u},Z^{t,x,u},U^{t,x,u})$. Moreover, since $Y^{t,x,u}$ is $\mathbb{F}$-adapted, $Y_t^{t,x,u}$ is deterministic. It follows from Barles, Buckdahn and Pardoux [10] or Pham [95] that the value function

$$V(t,x)=\inf_{u\in\mathcal{U}^{B,\mu}(t,T)} Y_t^{t,x,u}, \quad (t,x)\in[0,T]\times\mathbb{R}^d \tag{1.4}$$

is the viscosity solution of our IPDE.

Since unlike [18] and [19], our system involves not only the Brownian motion $B$ but also the Poisson random measure $\mu$, the method of time change for the Brownian motion alone is not sufficient for our approach here. So we combine the method of time change for the Brownian motion by Kulik's transformation for Poisson random measures (see, [65] [66]). To our best knowledge, the use of Kulik's transformation for the study of stochastic control problems is new. Because of the difficulty to obtain suitable $L^p$-estimates of the stochastic integrals with respect the compensated Poisson random measure (see, for example, Pham [95]) we have to restrict ourselves to the case of a finite Lévy measure $\Pi(E)<+\infty$. The more general case where $\int_E(1\wedge|e|^2)\Pi(\mathrm{d}e)<+\infty$ remains still open.

This chapter is organized as follows. In Section 2 we introduce our main tools, i.e., the method of time change for the Brownian motion and Kulik's transformation for the Poisson random measure, with the help of which we study the joint Lipschitz continuity for the viscosity solution of the IPDEs of HJB type. This method of time change for the Brownian motion combined with Kulik's transformation is extended in Section 3 to the study of the semiconcavity property for the viscosity solution of IPDE (1.1). The proof of more technical statements and estimates used in Section 3 is shifted in the Appendix.

## 2 Lipschitz Continuity

In this section, we prove the joint Lipschitz continuity of the viscosity solution of a certain class of integro-differential Hamilton-Jacobi-Bellman (HJB) equations.

Let $T$ be an arbitrarily fixed time horizon, $U$ a compact metric space, $E = \mathbb{R}^d\backslash\{0\}$ and $\mathcal{B}(E)$ be the Borel $\sigma$-algebra over $E$. We are concerned with the integro-partial differential equation of HJB type (1.1). The coefficients

$$b : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^d,\ \sigma : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times d},\ \beta : [0,T] \times \mathbb{R}^d \times U \times E \to \mathbb{R}^d,$$

$$f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times L^2(E,\mathcal{B}(E),\Pi;\mathbb{R}) \times U \to \mathbb{R} \ \text{ and } \ \Phi : \mathrm{R^d} \to \mathbb{R}$$

are bounded continuous functions which satisfy the following conditions:
(**H1**) There exists a constant $K > 0$ such that, for any $\xi_i = (s_i, y_i) \in [0,T] \times \mathbb{R}^d, u \in U, i = 1,2,$

$$\begin{aligned}&|b(\xi_1,u) - b(\xi_2,u)| + |\sigma(\xi_1,u) - \sigma(\xi_2,u)| + \left(\int_E |\beta(\xi_1,u,e) - \beta(\xi_2,u,e)|^4 \Pi(\mathrm{d}e)\right)^{1/4}\\ \leq &K(|s_1 - s_2| + |y_1 - y_2|).\end{aligned}$$

(**H2**) The function $f$ is Lipschitz in $(t,x,y,z,p)$, uniformly with respect to $u \in U$, and the function $\Phi$ is a Lipschitz function.

The integro-PDE (1.1), as is well-known by now (see, for instance, [10]), has a unique continuous viscosity solution $V(t,x)$ in the class of the continuous functions with at most polynomial growth.

Let $\{B_s^0\}_{s\geq 0}$ be a $d$-dimensional Brownian motion defined on a complete space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$, and $\eta$ be a Poisson random measure defined on a complete probability space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$. We introduce $(\Omega, \mathcal{F}, \mathbb{P})$ as the product space $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1, \mathcal{F}_1, \mathbb{P}_1) \otimes (\Omega_2, \mathcal{F}_2, \mathbb{P}_2) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$. The processes $B^0$ and $\eta$ are canonically extended from $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, respectively, to the product space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote the compensated Poisson random measure associated with $\eta$ by $\tilde{\eta}$, i.e., $\tilde{\eta}(\mathrm{d}t, \mathrm{d}e) = \eta(\mathrm{d}t, \mathrm{d}e) - \mathrm{d}t\Pi(\mathrm{d}e)$. We assume throughout this paper that the Lévy measure $\Pi$ is a finite measure on $(E, \mathcal{B}(E))$.

We define the process $\{B_s\}_{s\geq t}$ by putting

$$B_s = B_s^0 - B_t^0, \quad s \in [t,T], \tag{2.1}$$

so that $\{B_s\}_{s\geq t}$ is a Brownian motion beginning at time $t$ with $B_t = 0$. Furthermore, we denote by $\mu$ be the restriction of the Poisson random measure $\eta$ from $[0,T] \times E$ to $[t,T] \times E$, and by $\tilde{\mu}$ its compensated measure.

We put

$$\mathcal{F}_s^B = \sigma\{B_r, r \in [t,s]\} \vee \mathcal{N}_{\mathbb{P}_1}, \quad \mathcal{F}_s^\mu = \sigma\{\mu((t,r] \times \Delta) : \Delta \in \mathcal{B}(E), r \in [t,s]\} \vee \mathcal{N}_{\mathbb{P}_2},$$

and

$$\mathcal{F}_s = \left(F_s^B \otimes \mathcal{F}_s^\mu\right) \vee \mathcal{N}_{\mathbb{P}}, \quad s \in [t,T],$$

where $\mathcal{N}_{\mathbb{P}_1}$, $\mathcal{N}_{\mathbb{P}_2}$ and $\mathcal{N}_{\mathbb{P}}$ are the collections of the null sets under the corresponding probability measure.

Let us also introduce the following spaces of stochastic processes over $(\Omega, \mathcal{F}, \mathbb{P})$ which will be needed in what follows. By $\mathcal{S}^2(t,T;\mathbb{R}^d)$ we denote the set of all $\mathbb{F}$-adapted càdlàg processes $\{Y_s; t \le s \le T\}$ such that

$$\|Y\|_{\mathcal{S}^2(t,T;\mathbb{R}^d)} = \mathbb{E}\left[\sup_{t\le s\le T} |Y_s|^2\right] < \infty.$$

Let $\mathbb{L}^2(t,T;\mathbb{R}^d)$ denote the set of all $\mathbb{F}$-predictable $d$-dimensional processes $\{Z_s : t \le s \le T\}$ such that

$$\|Z\|_{\mathbb{L}^2(t,T;\mathbb{R}^d)} = \left(\mathbb{E}\left[\int_t^T |Z_s|^2 \mathrm{d}s\right]\right)^{1/2} < \infty.$$

Finally, we also introduce the space $\mathbb{L}^2(t,T;\tilde{\mu},\mathbb{R})$ of mappings $U : \Omega \times [0,T] \times E \to \mathbb{R}$ which are $\mathbb{F}$-predictable and measurable such that

$$\|U\|_{\mathbb{L}^2(t,T;\tilde{\mu},\mathbb{R})} = \left(\mathbb{E}\left[\int_t^T \int_E |U_s(e)|^2 \Pi(\mathrm{d}e)\mathrm{d}s\right]\right)^{1/2} < \infty.$$

Let us now consider the following stochastic differential equation driven by the Brownian motion $B$ and the compensated Poisson random measure $\tilde{\mu}$:

$$X_s^{t,x,u} = x + \int_t^s b(r, X_r^{t,x,u}, u_r)\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x,u}, u_r)\mathrm{d}B_r + \int_t^s\int_E \beta(r, X_{r-}^{t,x,u}, u_r, e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e), \tag{2.2}$$

$s \in [t,T]$, where the process $u : [t,T]\times\Omega \to U$ is an admissible control, i.e., an $\mathbb{F}$-predictable process with values in $U$; the space of admissible controls over the time interval $[t,T]$ is denoted by $\mathcal{U}^{B,\mu}(t,T)$. The following theorem is by now classical:

**Theorem 2.1.** *Assume the Lipschitz condition (**H1**). For any fixed admissible control $u(\cdot) \in \mathcal{U}(t,T)$, there exists a unique adapted càdlàg solution $(X_s^{t,x,u})_{s\in[t,T]} \in \mathcal{S}^2(t,T;\mathbb{R}^d)$ of the stochastic differential equation* (2.2)*.*

We associate SDE (2.2) with the backward stochastic differential equation

$$\begin{aligned} Y_s^{t,x,u} &= \Phi(X_T^{t,x,u}) + \int_s^T f(r, X_r^{t,x,u}, Y_r^{t,x,u}, Z_r^{t,x,u}, U_r^{t,x,u}, u_r)\mathrm{d}r - \int_s^T Z_r^{t,x,u}\mathrm{d}B_r \\ &\quad - \int_s^T \int_E U_r^{t,x,u}(e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e), \quad s \in [t,T]. \end{aligned}$$

Then from Barles, Buckdahn and Pardoux [10], Tang and Li [104], we know that this BSDE has a unique solution

$$(Y^{t,x,u}, Z^{t,x,u}, U^{t,x,u}) \in \mathcal{S}^2(t,T;\mathbb{R}) \times \mathbb{L}^2(t,T;\mathbb{R}^d) \times \mathbb{L}^2(t,T;\tilde{\mu},\mathbb{R}).$$

Notice that $Y_t^{t,x,u}$ is $\mathcal{F}_t$-measurable, hence it is deterministic in the sense that it coincides $\mathbb{P}$-a.s. with a real constant, with which it is identified. Thus, we have

$$Y_t^{t,x,u} = \mathbb{E}\left[Y_t^{t,x,u}\right] = \mathbb{E}\left[\int_t^T f(r, X_r^{t,x,u}, Y_r^{t,x,u}, Z_r^{t,x,u}, U_r^{t,x,u}, u_r)\mathrm{d}r + \Phi(X_T^{t,x,u})\right].$$

As usual in stochastic control problems, we define the cost functional $J(t,x;u)$ associated with $u \in \mathcal{U}^{B,\mu}(0,T)$ by setting $J(t,x;u) := Y_t^{t,x,u}$, and the value function is defined as follows:

$$V(t,x) = \inf_{u(\cdot)\in\mathcal{U}^{B,\mu}(t,T)} J(t,x;u), \quad (t,x) \in [0,T]\times\mathbb{R}^d.$$

It is well known by now that $V = \{V(t,x) : (t,x) \in [0,T]\times\mathbb{R}^d\}$ is a continuous viscosity solution of the HJB equation (1.1). Moreover, $V$ is the unique viscosity solution in the class of continuous functions with at most polynomial growth (see: [95], [106]).

Our main result in this section is the following theorem.

**Theorem 2.2.** *Let $\delta \in (0,T)$ be arbitrary but fixed. Under our assumptions (**H1**) and (**H2**), the value function $V(\cdot,\cdot)$ is jointly Lipschitz continuous on $[0,T-\delta]\times\mathbb{R}^d$, i.e., for some constant $C_\delta$ we have, for all $(t_0,x_0),(t_1,x_1) \in [0,T-\delta]\times\mathbb{R}^d$:*

$$|V(t_0,x_0) - V(t_1,x_1)| \le C_\delta(|t_0-t_1| + |x_0-x_1|).$$

**Remark 2.3.** *In general we cannot expect to get the joint Lipschitz continuity over the whole domain $[0,T]\times\mathbb{R}^d$. In [18] is given an easy counterexample: We study the problem*

$$X_s^{t,x} = x + B_s, \quad s \in [t,T], \quad x \in \mathbb{R};$$

$$Y_s^{t,x} = -\mathbb{E}\left[\left|X_T^{t,x}\right| |\mathcal{F}_s\right] = -\mathbb{E}\left[|x+B_T||\mathcal{F}_s\right], \quad s \in [t,T],$$

*without control neither jumps. Then*

$$V(t,x) = Y_t^{t,x} = -\mathbb{E}\left[|x+B_T|\right],$$

*and, for $x=0$, recalling that $B$ is a Brownian motion with $B_t = 0$, we have*

$$V(t,0) = -\mathbb{E}[|B_T|] = -\sqrt{\frac{2}{\pi}}\sqrt{T-t}, \quad t \in [0,T].$$

*Obviously, $V(\cdot,x)$ is not Lipschitz in $t$ for $t=T$. However, $V$ is jointly Lipschitz on $[0,T-\delta]\times\mathbb{R}$, for $\delta \in (0,T)$.*

Let us introduce now Kulik's transformation in our framework. The reader interested in more details on this transformation is referred to the papers [65] and [66].

Let $t_0, t_1 \in [0,T]$ and let, for $t = t_0$, $\mu$ be the Poisson random measure which we have introduced as restriction of $\eta$ from $[0,T]\times E$ to $[t_0,T]\times E$. With the help of $\mu$ we define now a random measure $\tau(\mu)$ on $[t_0,T]\times E$. Denoting by

$$\tau : [t_1,T] \to [t_0,T]$$

the linear time change

$$\tau(s) = t_0 + \frac{T-t_0}{T-t_1}(s-t_1), \; s \in [t_1,T],$$

we put

$$\tau(\mu)([t_1,s]\times\Delta) := \mu([\tau(t_1),\tau(s)]\times\Delta), \; t_1 \le s \le T, \; \Delta \in \mathcal{B}(E).$$

Observing that $\dot\tau = \dot\tau(s) = \frac{T-t_0}{T-t_1}$, we put

$$\gamma = \ln(\dot\tau) = \ln\left(\frac{T-t_0}{T-t_1}\right).$$

From Lemma 1.1 in Kulik [65] we know that, for all $\{s_1,\cdots,s_n\} \subset [t_1,T]$, $\Delta_1,\cdots,\Delta_n \in \mathcal{B}(E)$ and all Borel function $\varphi : \mathbb{R}^n \to \mathbb{R}^+$, we have

**Lemma 2.4.**
$$\begin{aligned}&\mathbb{E}\left[\varphi(\tau(\mu)([t_1,s_1]\times\Delta_1),\cdots,\tau(\mu)([t_1,s_n]\times\Delta_n))\right]\\ =&\mathbb{E}\left[\rho_\tau\varphi(\eta([t_1,s_1]\times\Delta_1),\cdots,\eta([t_1,s_n]\times\Delta_n))\right],\end{aligned}$$

*where*
$$\rho_\tau=\exp\{\gamma\eta([t_1,T]\times E)-(t_1-t_0)\Pi(E)\}.$$

For the convenience of the reader we sketch the proof. However, we restrict to a special case ($n=1$), the proof of the general case $n\geq 1$ can be carried out with a similar argument and can be consulted for the more general case $\Pi(E)=+\infty$ in [65].

*Proof*: (for $n=1$). Observing that for $\Delta_2=E\backslash\Delta_1$,
$$\eta([t_1,s_1]\times\Delta_1),\qquad \eta([s_1,T]\times\Delta_1)\quad\text{ and }\quad \eta([t_1,T]\times\Delta_2)$$
are independent Poisson distributed random variables with the intensities $(s_1-t_1)\Pi(\Delta_1)$, $(T-s_1)\Pi(\Delta_1)$ and $(T-t_1)\Pi(\Delta_2)$, respectively, we have
$$\begin{aligned}&\mathbb{E}[\rho_\tau\varphi(\eta([t_1,s_1]\times\Delta_1))]\\ =&\Bigg\{\sum_{k,l\geq 0}\varphi(k)\exp\{\gamma k+\gamma l-(t_1-t_0)\Pi(\Delta_1)\}\times\\ &\qquad\times\exp\{-(T-t_1)\Pi(\Delta_1)\}\frac{((s_1-t_1)\Pi(\Delta_1))^k}{k!}\frac{((T-s_1)\Pi(\Delta_1))^l}{l!}\Bigg\}\\ &\times\Bigg\{\sum_{m\geq 0}\exp\{\gamma m-(t_1-t_0)\Pi(\Delta_2)\}\exp\{-(T-t_1)\Pi(\Delta_2)\}\frac{((T-t_1)\Pi(\Delta_2))^m}{m!}\Bigg\}\\ =&I_1\times I_2.\end{aligned}$$

But, taking into account the definition of $\gamma$ and that of $\tau(s_1)$ we have
$$\begin{aligned}I_1=&\Bigg\{\sum_{k\geq 0}\varphi(k)\exp\{-(T-t_0)\Pi(\Delta_1)\}\frac{1}{k!}\left(\left(\frac{T-t_0}{T-t_1}\right)(s_1-t_1)\Pi(\Delta_1)\right)^k\Bigg\}\times\\ &\times\Bigg\{\sum_{l\geq 0}\frac{1}{l!}\left(\left(\frac{T-t_0}{T-t_1}\right)(T-s_1)\Pi(\Delta_1)\right)^l\Bigg\}\\ =&\Bigg\{\sum_{k\geq 0}\varphi(k)\exp\{-(T-t_0)\Pi(\Delta_1)\}\frac{1}{k!}\left((\tau(s_1)-t_0)\Pi(\Delta_1)\right)^k\Bigg\}\exp\{(T-\tau(s_1))\Pi(\Delta_1)\}\\ =&\sum_{k\geq 0}\varphi(k)\exp\{-(\tau(s_1)-t_0)\Pi(\Delta_1)\}\frac{1}{k!}\left((\tau(s_1)-t_0)\Pi(\Delta_1)\right)^k\\ =&\mathbb{E}[\varphi(\tau(\mu)([t_1,s_1]\times\Delta_1))].\end{aligned}$$

In analogy to the computation for $I_1$, but now with $\varphi\equiv 1$, we get that $I_2=1$. Consequently,
$$\mathbb{E}[\rho_\tau\varphi(\eta([t_1,s_1]\times\Delta_1))]=\mathbb{E}[\varphi(\tau(\mu)([t_1,s_1]\times\Delta_1))].$$

Hence the proof is complete. ■

From the above lemma we have, for all $n \geq 1$, $\{s_1, \cdots, s_n\} \subset [t_1, T], \Delta_1, \cdots, \Delta_n \in \mathcal{B}(E)$ and $\varphi : \mathbb{R}^n \to \mathbb{R}^+$ Borel function,

$$\begin{aligned}
&\mathbb{E}[\varphi(\eta([t_1, s_1] \times \Delta_1), \cdots, \eta([t_1, s_n] \times \Delta_n))] \\
=&\mathbb{E}\big[\rho_\tau \varphi(\eta([t_1, s_1] \times \Delta_1), \cdots, \eta([t_1, s_n] \times \Delta_n)) \\
&\qquad \times \exp\{-\gamma\eta([t_1, T] \times E) + (t_1 - t_0)\Pi(E)\}\big] \\
=&\mathbb{E}\big[\varphi(\tau(\mu)([t_1, s_1] \times \Delta_1), \cdots, \eta([t_1, s_n] \times \Delta_n)) \\
&\qquad \times \exp\{-\gamma\tau(\mu)([t_1, T] \times E) + (t_1 - t_0)\Pi(E)\}\big] \\
=&\mathbb{E}\left[g_\tau \varphi(\tau(\mu)([t_1, s_1] \times \Delta_1), \cdots, \eta([t_1, s_n] \times \Delta_n))\right],
\end{aligned}$$

for $g_\tau = \exp\{-\gamma\tau(\mu)([t_1, T] \times E) + (t_1 - t_0)\Pi(E)\}$.

This allows to show that under the probability measure $\mathbb{Q}_\tau = g_\tau \mathbb{P}$, the point process $\tau(\mu)$ defined over $[t_1, T] \times E$, has the same law as the Poisson random measure $\eta$ restricted to $[t_1, T] \times E$, under $\mathbb{P}$. Consequently, under $\mathbb{Q}_\tau = g_\tau \mathbb{P}$, $\tau(\eta)$ is a Poisson random measure with compensator $\mathrm{d}s\Pi(\mathrm{d}e)$.

We use the same time change $\tau : [t_1, T] \to [t_0, T]$ in order to introduce the process

$$W_t = \frac{1}{\sqrt{\dot{\tau}}} B_{\tau(t)}, \ t \in [t_1, T].$$

We observe that $W = (W_t)_{t \in [t_1, T]}$ is a Brownian motion under the probability $\mathbb{P}$ but also under $\mathbb{Q}_\tau = g_\tau \mathbb{P}$ (Indeed, $B$ and $g_\tau$ are independent under $\mathbb{P}$), and $W$ and $\tau(\mu)$ are independent under both $\mathbb{P}$ and $\mathbb{Q}_\tau$.

Let $\varepsilon > 0$. From the definition of the value function $V$,

$$V(t_0, x_0) = \inf_{u \in \mathcal{U}^{B,\mu}(t_0, T)} J(t_0, x_0, u),$$

we get the existence of an admissible control $u^0 \in \mathcal{U}^{B,\mu}(t_0, T)$ such that

$$J(t_0, x_0, u^0) \leq V(t_0, x_0) + \varepsilon.$$

We define

$$u^1(t) = u^0(\tau(t)), \ t \in [t_1, T].$$

Then, obviously, $u^1 \in \mathcal{U}^{W,\tau(\mu)}(t_1, T)$, i.e., $u^1$ is a $U$-valued process predictable with respect the filtration

$$\mathcal{F}_t^{W,\tau(\mu)} = \sigma\{W_s, \tau(\mu)([t_1, s] \times \Delta), s \in [t_1, t], \Delta \in \mathcal{B}(E)\} \vee \mathcal{N}_\mathbb{P}, \quad t \in [t_1, T],$$

generated by $W$ and $\tau(\mu)$.

Let now $X^0 = \left\{X_s^0\right\}_{s \in [t_0, T]}$ be the solution of the forward equation

$$X_s^0 = x_0 + \int_{t_0}^s b(r, X_r^0, u_r^0)\mathrm{d}r + \int_{t_0}^s \sigma(r, X_r^0, u_r^0)\mathrm{d}B_r + \int_{t_0}^s \int_E \beta(r, X_{r-}^0, u_r^0, e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e), \quad (2.3)$$

$s \in [t_0, T]$, under the probability $\mathbb{P}$, and let $X^1 = \left\{X_s^1\right\}_{s \in [t_1, T]}$ be the solution of the equation

$$X_s^1 = x_1 + \int_{t_1}^s b(r, X_r^1, u_r^1)\mathrm{d}r + \int_{t_1}^s \sigma(r, X_r^1, u_r^1)\mathrm{d}W_r + \int_{t_1}^s \int_E \beta(r, X_{r-}^1, u_r^1, e)\widetilde{\tau(\mu)}^{\mathbb{Q}_\tau}(\mathrm{d}r, \mathrm{d}e), \quad (2.4)$$

$s \in [t_1, T]$, under probability measure $\mathbb{Q}_\tau$. Notice that the compensated Poisson random measure $\tilde{\mu}$ under $\mathbb{P}$ is of the form

$$\tilde{\mu}(\mathrm{d}s, \mathrm{d}e) = \mu(\mathrm{d}s, \mathrm{d}e) - \mathrm{d}s\Pi(\mathrm{d}e),\ (s,e) \in [t_0, T] \times E,$$

while the compensated Poisson random measure for $\tau(\mu)$ under $\mathbb{Q}_\tau$ has the form

$$\widetilde{\tau(\mu)}^{\mathbb{Q}_\tau}(\mathrm{d}s, \mathrm{d}e) = \tau(\mu)(\mathrm{d}s, \mathrm{d}e) - \mathrm{d}s\Pi(\mathrm{d}e),\ (s,e) \in [t_1, T] \times E.$$

We employ the BSDE method to prove the Lipschitz continuity of the value function $V$. For this we associate the above SDEs with the following BSDEs with jumps:

$$Y_s^0 = \Phi(X_T^0) + \int_s^T f(r, X_r^0, Y_r^0, Z_r^0, U_r^0, u_r^0)\mathrm{d}r - \int_s^T Z_r^0 \mathrm{d}B_r - \int_s^T \int_E U_r^0(e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e), \quad (2.5)$$

$s \in [t_0, T]$, under probability $\mathbb{P}$, and

$$Y_s^1 = \Phi(X_T^1) + \int_s^T f(r, X_r^1, Y_r^1, Z_r^1, U_r^1, u_r^1)\mathrm{d}r - \int_s^T Z_r^1 \mathrm{d}W_r - \int_s^T \int_E U_r^1(e)\widetilde{\tau(\mu)}^{\mathbb{Q}_\tau}(\mathrm{d}r, \mathrm{d}e), \quad (2.6)$$

$s \in [t_1, T]$, under probability $\mathbb{Q}_\tau$. From [10], we know the above two BSDEs have unique solutions $(Y^0,\ Z^0,\ U^0)= (Y_s^0, Z_s^0, U_s^0)_{s\in[t_0,T]}$ and $(Y^1, Z^1, U^1) = (Y_s^1, Z_s^1, U_s^1)_{s\in[t_1,T]}$, respectively. While $Y^0$ is adapted and $Z^0$ and $U^0$ are predictable with respect to the filtration generated by $B$ and $\mu$, $Y^1$ is adapted and $Z^1$ and $U^1$ are predictable with respect the filtration generated by $W$ and $\tau(\mu)$. Thus, $Y_{t_0}^0$ and $Y_{t_1}^1$ are deterministic, and from the definition of the cost functionals we have

$$Y_{t_0}^0 - \varepsilon = J(t_0, x_0; u^0) - \varepsilon \le V(t_0, x_0) \left(= \inf_{u\in\mathcal{U}^{B,\mu}(t_0,T)} J(t_0, x_0; u)\right), \quad (2.7)$$

and

$$Y_{t_1}^1 = J(t_1, x_1; u^1) \ge V(t_1, x_1) \left(= \inf_{u\in\mathcal{U}^{W,\tau(\mu)}(t_1,T)} J(t_1, x_1; u)\right). \quad (2.8)$$

Here we have used that the stochastic interpretation of $V$ does not depend on the special choice of the underlying driving Brownian motion and the underlying Poisson random measure with compensator $\mathrm{d}s\Pi(\mathrm{d}e)$. In order to show the Lipschitz property of $V$ in $(t, x)$, we have to estimate

$$\begin{aligned} V(t_0, x_0) - V(t_1, x_1) \ge& J(t_0, x_0; u^0) - J(t_1, x_1; u^1) - \varepsilon \\ =& Y_{t_0}^0 - Y_{t_1}^1 - \varepsilon. \end{aligned}$$

However, in order to estimate the difference between the processes $Y^0$ and $Y^1$, we have to make their both BSDEs comparable, i.e., we need them over the same time interval, driven by the same Brownian motion and by the same compensated Poisson random measure. For this reason we apply to SDE (2.4) and BSDE (2.6) the inverse time change $\tau^{-1} : [t_0, T] \to [t_1, T]$. So we introduce the process $\widetilde{X}^1 = \{\widetilde{X}_s^1\}_{s\in[t_0,T]}$ by setting $\widetilde{X}_s^1 = X_{\tau^{-1}(s)}^1$. We also observe that $W_{\tau^{-1}(r)} = \frac{1}{\sqrt{\bar{\tau}}} B_r$ and $u_{\tau^{-1}(r)}^1 = u_r^0$, $r \in [t_0, T]$. Obviously, $\widetilde{X}^1 \in \mathcal{S}^2(t_0, T; \mathbb{R})$ is the unique solution of the SDE

$$\widetilde{X}_s^1 \ = \ x_1 + \int_{t_0}^s b(\tau^{-1}(r), \widetilde{X}_r^1, u_{\tau^{-1}(r)}^1)\mathrm{d}\tau^{-1}(r) + \int_{t_0}^s \sigma(\tau^{-1}(r), \widetilde{X}_r^1, u_{\tau^{-1}(r)}^1)\mathrm{d}W_{\tau^{-1}(r)}$$

$$
\begin{aligned}
&+\int_{t_0}^{s}\int_E \beta(\tau^{-1}(r),\widetilde{X}^1_{r-},u^1_{\tau^{-1}(r)},e)\widetilde{\tau(\mu)}^{\mathbb{Q}_\tau}(\mathrm{d}\tau^{-1}(r),\mathrm{d}e)\\
=\;& x_1+\int_{t_0}^{s}\frac{1}{\dot\tau}b(\tau^{-1}(r),\widetilde{X}^1_r,u^0_r)\mathrm{d}r+\int_{t_0}^{s}\frac{1}{\sqrt{\dot\tau}}\sigma(\tau^{-1}(r),\widetilde{X}^1_r,u^0_r)\mathrm{d}B_r\\
&+\int_{t_0}^{s}\int_E \beta(\tau^{-1}(r),\widetilde{X}^1_{r-},u^0_r,e)\left(\tilde\mu(\mathrm{d}r,\mathrm{d}e)+\left(1-\frac{1}{\dot\tau}\right)\Pi(\mathrm{d}e)\mathrm{d}r\right). \qquad (2.9)
\end{aligned}
$$

This time change in equation (2.4) makes the processes $X^0$ and $\widetilde{X}^1$ comparable. More precisely, we have

**Lemma 2.5.** *There exists some constant $C_\delta$, only depending on the bounds of $\sigma, b, \beta$, their Lipschitz constants, as well as on $\Pi(E)$ and $\delta$, such that, for all $t\in[t_0,T]$,*

$$
\mathbb{E}\left[\sup_{t\le s\le T}\left|X^0_s-\widetilde{X}^1_s\right|^2\Big|\mathcal{F}_t\right]\le C_\delta\left(|t_0-t_1|^2+|X^0_t-\widetilde{X}^1_t|^2\right).
$$

For the proof of this lemma we need the following estimates gotten by an elementary straightforward computation (see [18], [19]).

**Lemma 2.6.** *There is a constant $C_\delta$ only depending on $\delta>0$, such that for all $r\in[t_0,T]$, we have*

$$
\left|1-\frac{1}{\dot\tau}\right|+\left|\tau^{-1}(r)-r\right|+\left|1-\sqrt{\dot\tau}\right|\le C_\delta|t_0-t_1|.
$$

*Proof (of Lemma 2.5):* By taking the difference between the SDEs (2.3) and (2.9) and after the conditional expectation of the supremum of its square, we get from Lemma 2.6 and the assumptions on the coefficients, for $s\in[t,T]$,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{r\in[t,s]}\left|X^0_r-\widetilde{X}^1_r\right|^2\Big|\mathcal{F}_t\right]\\
=\;&\mathbb{E}\Bigg[\Bigg(\left|X^0_t-\widetilde{X}^1_t\right|+\int_t^s\left|b(v,X^0_v,u^0_v)-\frac{1}{\dot\tau}b(\tau^{-1}(v),\widetilde{X}^1_v,u^0_v)\right|\mathrm{d}v\\
&+\int_t^s\int_E\left|\beta(\tau^{-1}(v),\widetilde{X}^1_{v-},u^0_v,e)\right|\left|1-\frac{1}{\dot\tau}\right|\Pi(\mathrm{d}e)\mathrm{d}v\\
&+\sup_{r\in[t,s]}\left|\int_t^r\left(\sigma(v,X^0_v,u^0_v)-\frac{1}{\sqrt{\dot\tau}}\sigma(\tau^{-1}(v),\widetilde{X}^1_v,u^0_v)\right)\mathrm{d}B_v\right|\\
&+\sup_{r\in[t,s]}\left|\int_t^r\int_E\left(\beta(v,X^0_{v-},u^0_v,e)-\beta(\tau^{-1}(v),\widetilde{X}^1_{v-},u^0_v,e)\right)\tilde\mu(\mathrm{d}v,\mathrm{d}e)\right|\Bigg)^2\Bigg|\mathcal{F}_t\Bigg]\\
\le\;& C\left|X^0_t-\widetilde{X}^1_t\right|^2+C\mathbb{E}\left[\left(\int_t^s\left(\left|1-\frac{1}{\dot\tau}\right|+\left|\tau^{-1}(v)-v\right|+\left|X^0_v-\widetilde{X}^1_v\right|\right)\mathrm{d}v\right)^2\Bigg|\mathcal{F}_t\right]\\
&+C\mathbb{E}\left[\int_t^s\left(\left|1-\frac{1}{\sqrt{\dot\tau}}\right|+\left|\tau^{-1}(v)-v\right|+\left|X^0_v-\widetilde{X}^1_v\right|\right)^2\mathrm{d}v\Bigg|\mathcal{F}_t\right]\\
\le\;& C_\delta\left(\left|X^0_t-\widetilde{X}^1_t\right|^2+|t_0-t_1|^2\right)+C_\delta\int_t^s\mathbb{E}\left[\left|X^0_v-\widetilde{X}^1_v\right|^2|\mathcal{F}_t\right]\mathrm{d}v.
\end{aligned}
$$

Finally, from Gronwall's inequality, we have

$$
\mathbb{E}\left[\sup_{s\in[t,T]}\left|X^0_s-\widetilde{X}^1_s\right|^2|\mathcal{F}_t\right]\le C_\delta\left(|t_0-t_1|^2+|X^0_t-\widetilde{X}^1_t|^2\right).
$$

Hence the proof of Lemma 2.5 is complete now. ∎

After having made comparable $X^0$ and $X^1$ by the time change of $X^1$, we make now $Y^0$ and $Y^1$ comparable. For this we put $\widetilde{Y}_s^1 = Y^1_{\tau^{-1}(s)}$, $\widetilde{Z}_s^1 = \frac{1}{\sqrt{\dot\tau}} Z^1_{\tau^{-1}(s)}$ and $\widetilde{U}_s^1 = U^1_{\tau^{-1}(s)}$, $s \in [t_0, T]$. Then $(\widetilde{Y}^1, \widetilde{Z}^1, \widetilde{U}^1) = (\widetilde{Y}_s^1, \widetilde{Z}_s^1, \widetilde{U}_s^1)_{s\in[t_0,T]} \in \mathcal{S}^2(t_0,T;\mathbb{R})\times\mathbb{L}^2(t_0,T;\mathbb{R}^d)\times \mathbb{L}^2(t_0,T;\tilde\mu,\mathbb{R})$ is the solution of the BSDE

$$\begin{aligned}\widetilde{Y}_s^1 =&\Phi(\widetilde{X}_T^1) + \int_s^T \frac{1}{\dot\tau} f(\tau^{-1}(r), \widetilde{X}_r^1, \widetilde{Y}_r^1, \sqrt{\dot\tau}\widetilde{Z}_r^1, \widetilde{U}_r^1, u_r^0)\mathrm{d}r - \int_s^T \widetilde{Z}_r^1 \mathrm{d}B_r \\ &- \int_s^T\int_E \widetilde{U}_r^1 \left(\tilde\mu(\mathrm{d}r,\mathrm{d}e) + \left(1-\frac{1}{\dot\tau}\right)\Pi(\mathrm{d}e)\mathrm{d}r\right), \ s\in[t_0,T],\end{aligned} \tag{2.10}$$

with respect to the same filtration $\mathbb{F}$ as $(Y^0, Z^0, U^0)$.

For the above BSDE, we have the following a priori estimates which can be proven by a straight-forward standard argument:

**Lemma 2.7.** *Under hypothesis* (**H2**)*, there exists some constant $C_\delta$, only depending on the bounds of $\sigma, b, \beta$, their Lipschitz constants, as well as on $\Pi(E)$ and $\delta$, such that,*

$$\mathbb{E}\left[\sup_{s\in[t,T]} |\widetilde{Y}_s^1|^2 + \int_t^T \left|\widetilde{Z}_r^1\right|^2 \mathrm{d}r + \int_t^T\int_E \left|\widetilde{U}_r^1(e)\right|^2 \Pi(\mathrm{d}e)\mathrm{d}r \middle| \mathcal{F}_t\right] \le C_\delta < +\infty, \quad t\in[t_0,T]. \tag{2.11}$$

Now we can state the key lemma for proving the joint Lipschitz continuity of $V$.

**Lemma 2.8.** *Under our standard assumptions* (**H1**) *and* (**H2**)*, we have*

$$\begin{aligned}&\mathbb{E}\left[\sup_{s\in[t,T]} |Y_s^0 - \widetilde{Y}_s^1|^2 + \int_t^T \left|Z_r^0 - \widetilde{Z}_r^1\right|^2 \mathrm{d}r + \int_t^T\int_E \left|U_r^0(e) - \widetilde{U}_r^1(e)\right|^2 \Pi(\mathrm{d}e)\mathrm{d}r \middle| \mathcal{F}_t\right] \\ \le& C_\delta\left(|t_0-t_1|^2 + |X_t^0 - \widetilde{X}_t^1|^2\right), \quad t\in[t_0,T].\end{aligned} \tag{2.12}$$

*Proof*: First we notice that, for $s \ge t$,

$$\begin{aligned}Y_s^0 - \widetilde{Y}_s^1 =&\Phi(X_T^0) - \Phi(\widetilde{X}_T^1) \\ &+ \int_s^T \left(f(r, X_r^0, Y_r^0, Z_r^0, U_r^0, u_r^\lambda) - \frac{1}{\dot\tau} f(\tau_1^{-1}(r), \widetilde{X}_r^1, \widetilde{Y}_r^1, \sqrt{\dot\tau}\widetilde{Z}_r^1, \widetilde{U}_r^1, u_r^\lambda)\right)\mathrm{d}r \\ &- \int_s^T \left(Z_r^0 - \widetilde{Z}_r^1\right)\mathrm{d}B_r - \int_s^T\int_E \left(U_r^0(e) - \widetilde{U}_r^1(e)\right)\tilde\mu(\mathrm{d}r,\mathrm{d}e) \\ &+ \int_s^T\int_E \left(1-\frac{1}{\dot\tau}\right)\widetilde{U}_r^1(e)\mathrm{d}r\Pi(\mathrm{d}e).\end{aligned}$$

We apply Itô's formula to $|Y_s^0 - \widetilde{Y}_s^1|^2$ and, using the boundedness and the Lipschitz continuity of $\Phi$ and $f$, as well as Lemma 2.6, we deduce that

$$\begin{aligned}&|Y_s^0 - \widetilde{Y}_s^1|^2 + \int_s^T \left|Z_r^0 - \widetilde{Z}_r^1\right|^2 \mathrm{d}r + \int_s^T\int_E \left|U_r^0(e) - \widetilde{U}_r^1(e)\right|^2 \Pi(\mathrm{d}e)\mathrm{d}r \\ \le& \left|\Phi(X_T^0) - \Phi(\widetilde{X}_T^1)\right|^2 + C\int_s^T \left|X_r^0 - \widetilde{X}_r^1\right|^2 \mathrm{d}r + C\int_s^T |Y_r^0 - \widetilde{Y}_r^1|^2\mathrm{d}r\end{aligned}$$

$$
\begin{aligned}
&-2\int_s^T \left(Y_r^0-\widetilde{Y}_r^1\right)\left(Z_r^0-\widetilde{Z}_r^1\right)\mathrm{d}B_r + C|t_0-t_1|^2\\
&-\int_s^T\int_E \left(2\left(Y_r^0-\widetilde{Y}_r^1\right)\left(U_r^0(e)-\widetilde{U}_r^1(e)\right)+\left|U_r^0(e)-\widetilde{U}_r^1(e)\right|^2\right)\tilde{\mu}(\mathrm{d}r,\mathrm{d}e)\\
&+C|t_0-t_1|^2\int_s^T\left(\left|\widetilde{Z}_r^1\right|^2+\int_E\left|\widetilde{U}_r^1(e)\right|^2\Pi(\mathrm{d}e)\right)\mathrm{d}r.
\end{aligned}
$$

By taking the conditional expectation on both sides, using Lemma 2.5, the a priori estimate (2.11) and Gronwall's lemma, we obtain, for $t_0 \le t \le s \le T$,

$$
\begin{aligned}
&\mathbb{E}\left[|Y_s^0-\widetilde{Y}_s^1|^2+\int_s^T\left|Z_r^0-\widetilde{Z}_r^1\right|^2\mathrm{d}r+\int_s^T\int_E\left|U_r^0(e)-\widetilde{U}_r^1(e)\right|^2\Pi(\mathrm{d}e)\mathrm{d}r\middle|\mathcal{F}_t\right]\\
\le& C_\delta\left(|t_0-t_1|^2+|X_t^0-\widetilde{X}_t^1|^2\right).
\end{aligned}
$$

Then the Burkholder-Davis-Gundy inequality allows to show that

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[t,T]}|Y_s^0-\widetilde{Y}_s^1|^2+\int_t^T\left|Z_r^0-\widetilde{Z}_r^1\right|^2\mathrm{d}r+\int_t^T\int_E\left|U_r^0(e)-\widetilde{U}_r^1(e)\right|^2\Pi(\mathrm{d}e)\mathrm{d}r\middle|\mathcal{F}_t\right]\\
\le& C_\delta\left(|t_0-t_1|^2+|X_t^0-\widetilde{X}_t^1|^2\right),\quad t\in[t_0,T].
\end{aligned}
$$

The proof is now complete. ■

Now we are ready to give the proof of Theorem 2.2.

*Proof of Theorem 2.2*: By taking $t=t_0$ in (2.12), we have

$$
\begin{aligned}
&|Y_{t_0}^0-Y_{t_1}^1|^2=|Y_{t_0}^0-\widetilde{Y}_{t_0}^1|^2\\
\le& C_\delta\left(|t_0-t_1|^2+|X_{t_0}^0-\widetilde{X}_{t_0}^1|^2\right)=C_\delta\left(|t_0-t_1|^2+|X_{t_0}^0-X_{t_1}^1|^2\right)\\
=& C_\delta\left(|t_0-t_1|^2+|x_0-x_1|^2\right).
\end{aligned}
$$

Therefore, from (2.7) and (2.8), we get that

$$
\begin{aligned}
&V(t_0,x_0)-V(t_1,x_1)\\
\ge& J(t_0,x_0;u^0)-J(t_1,x_1;u^1)-\varepsilon\\
=& Y_{t_0}^0-\widetilde{Y}_{t_0}^1-\varepsilon\\
\ge& -C_\delta(|t_0-t_1|+|x_0-x_1|)-\varepsilon,
\end{aligned}
$$

for some $C_\delta$ only depending on $\delta$ but not on $(t_0,x_0)$, $(t_1,x_1)\in[0,T-\delta]\times\mathbb{R}^d$. Thus, from the arbitrariness of $\varepsilon$, we deduce that

$$
V(t_0,x_0)-V(t_1,x_1)\ge -C_\delta(|t_0-t_1|+|x_0-x_1|).
$$

Symmetrical argument yields the converse relation. Consequently, the joint Lipschitz continuity of $V$ over $[0,T-\delta]\times\mathbb{R}^d$. ■

# 3 Semiconcavity

We study in this section the semiconcavity property of the viscosity solution $V$ and to extend for this the method of time change and Kulik's transformation used in the preceding section.

For the semiconcavity property, we need more assumptions on the coefficients:
**(H3)** The function $\Phi(x)$ is semiconcave, and $f(\cdot,\cdot,\cdot,\cdot,\cdot,u)$ is semiconcave in $(t,x,y,z,p) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times L^2(E,\mathcal{B}(E),\Pi;\mathbb{R})$, uniformly with respect to $u\in U$, i.e., there exists a constant $C>0$, such that, for any $\xi_1 \triangleq (t_1,x_1,y_1,z_1,p_1)$, $\xi_2 \triangleq (t_2,x_2,y_2,z_2,p_2)$ in $[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times L^2(E,\mathcal{B}(E),\Pi;\mathbb{R})$, and $\lambda\in[0,1]$, $u\in U$,

$$\begin{aligned}&\lambda f(\xi_1,u)+(1-\lambda)f(\xi_2,u)-f(\lambda\xi_1+(1-\lambda)\xi_2,u)\\ \le& C\lambda(1-\lambda)\left(|t_1-t_2|^2+|x_1-x_2|^2+|y_1-y_2|^2+|z_1-z_2|^2+\int_E|p_1(e)-p_2(e)|^2\Pi(\mathrm{d}e)\right).\end{aligned}$$

**(H4)** The first-order derivatives $\nabla_{t,x}b$, $\nabla_{t,x}\sigma$ and $\nabla_{t,x}\beta$ of $b$, $\sigma$ and $\beta$ with respect to $(t,x)$ exist and are continuous in $(t,x,u)$ and Lipschitz continuous in $(t,x)$, uniformly with respect to $u$.
**(H5)** There exist two constants $-1<C_1<0$ and $C_2>0$ such that, for all $(t,\xi) := (t,x,y,z)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$, $u\in U$, and $p,p'\in L^2(E,\mathcal{B}(E),\Pi;\mathbb{R})$,

$$f(t,\xi,p,u)-f(t,\xi,p',u)\le\int_E(p(e)-p'(e))\gamma_t^{\xi,u;p,p'}(e)\Pi(\mathrm{d}e),$$

where $\{\gamma_t^{\xi,u;p,p'}(e)\}_{t\in[0,T]}$ is a measurable function such that, for every $t\in[0,T]$,

$$C_1(1\wedge|e|)\le\gamma_t^{\xi,u;p,p'}(e)\le C_2(1\wedge|e|).$$

Our main result in this section is the following:

**Theorem 3.1.** *Under the assumptions (**H1**) − (**H5**), for every $\delta\in(0,T)$, there exists some constant $C_\delta>0$ such that, for all $(t_0,x_0),(t_1,x_1)\in[0,T-\delta]\times\mathbb{R}^d$, and for all $\lambda\in[0,1]$:*

$$\lambda V(t_0,x_0)+(1-\lambda)V(t_1,x_1)-V(t_\lambda,x_\lambda)\le C_\delta\lambda(1-\lambda)(|t_0-t_1|^2+|x_0-x_1|^2),$$

*where $t_\lambda=\lambda t_0+(1-\lambda)t_1, x_\lambda=\lambda x_0+(1-\lambda)x_1$.*

**Remark 3.2.** *Again as in the case of the Lipschitz continuity, we cannot hope, in general, that the property of semiconcavity holds over the whole domain $[0,T]\times\mathbb{R}^d$. Indeed, let us consider the example given in the preceding section (Remark 2.3). In particular, we have gotten there that*

$$V(s,0)=\mathbb{E}[\Phi(X_T^{s,0})]=\mathbb{E}[-|B_T-B_s|]=-\sqrt{\frac{2}{\pi}}\sqrt{T-s},\quad s\in[0,T].$$

*However, it is easy to check that this function $V$ is not semiconcave in $[0,T]\times\mathbb{R}^d$, but it has this semiconcavity property on $[0,T-\delta]\times\mathbb{R}^d$, for all $\delta>0$.*

The proof of Theorem 3.1 will be based again on the method of time change. But unlike the proof of the Lipschitz property, we have to work here with two time changes. In order to be more precise, for given $\delta > 0$, $(t_0, x_0), (t_1, x_1) \in [0, T-\delta] \times \mathbb{R}^d$, let us consider the both following linear time changes:

$$\tau_i : [t_i, T] \to [t_\lambda, T],\ \tau_i(t) = t_\lambda + \frac{T - t_\lambda}{T - t_i}(t - t_i),$$

with the derivatives $\dot{\tau}_i = \frac{T-t_\lambda}{T-t_i},\ t \in [t_i, T],\ i = 0, 1$.

For $t = t_\lambda$, we let $B = \{B_s\}_{s\in[t_\lambda,T]}$ be a Brownian motion starting from zero at $t_\lambda$: $B_{t_\lambda} = 0$. Then $\{W_s^i = \frac{1}{\sqrt{\dot{\tau}_i}} B_{\tau_i(s)}\}_{s\in[t_i,T]}$ is a Brownian motion on $[t_i, T]$, starting from zero at time $t_i$, $i = 0, 1$. For $t = t_\lambda$, let $\mu(\mathrm{d}r, \mathrm{d}e)$ be our Poisson random measure on $[t_\lambda, T] \times E$ under probability $\mathbb{P}$. Then $\tau_i(\mu), i = 0, 1$, defined as the Kulik transformation of $\mu$,

$$\tau_i(\mu)([t_i, t_i + s] \times \Delta) \triangleq \mu([t_\lambda, \tau_i(t_i + s)] \times \Delta), 0 \le s \le T - t_i, \Delta \in \mathcal{B}E$$

is a new Poisson random measure but under probability $\mathbb{Q}_i$, where

$$\frac{\mathrm{d}\mathbb{Q}_i}{\mathrm{d}\mathbb{P}} = \exp\left\{ -\ln\left(\frac{T - t_\lambda}{T - t_i}\right) \mu([t_i, T] \times E) + (t_i - t_\lambda)\Pi(E) \right\}.$$

We denote the corresponding compensated Poisson random measures under $\mathbb{P}$ and $\mathbb{Q}_i$ by $\tilde{\mu}$ and $\widetilde{\tau_i(\mu)},\ i = 0, 1$, respectively:

$$\tilde{\mu}(\mathrm{d}s, \mathrm{d}e) = \mu(\mathrm{d}s, \mathrm{d}e) - \mathrm{d}s\Pi(\mathrm{d}e),\ (s, e) \in [t_\lambda, T] \times E,$$

and

$$\widetilde{\tau_i(\mu)}(\mathrm{d}s, \mathrm{d}e) = \tau_i(\mu)(\mathrm{d}s, \mathrm{d}e) - \mathrm{d}s\Pi(\mathrm{d}e),\ (s, e) \in [t_i, T] \times E.$$

Let us now fix an arbitrary $u^\lambda \in \mathcal{U}^{B,\mu}(t_\lambda, T)$ (Recall the definition of $\mathcal{U}^{B,\mu}(t_\lambda, T)$). Then, obviously, $u_s^i \triangleq u^\lambda_{\tau_i(s)}, s \in [t_i, T]$, is an admissible control in $\mathcal{U}^{W^i,\tau_i(\mu)}(t_i, T)$ with respect to $W^i$ and $\tau_i(\mu)$.

We let $\{X_s^\lambda\}_{s\in[t_\lambda,T]}$ be the unique solution of the SDE,

$$X_s^\lambda = x_\lambda + \int_{t_\lambda}^s b(r, X_r^\lambda, u_r^\lambda)\mathrm{d}r + \int_{t_\lambda}^s \sigma(r, X_r^\lambda, u_r^\lambda)\mathrm{d}B_r + \int_{t_\lambda}^s \int_E \beta(r, X_{r-}^\lambda, u_r^\lambda, e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e), \quad (3.1)$$

$s \in [t_\lambda, T]$. We also make use of the unique solution $\{X_s^i\}_{s\in[t_i,T]}$ of the following SDE,

$$X_s^i = x_i + \int_{t_i}^s b(r, X_r^i, u_r^i)\mathrm{d}r + \int_{t_i}^s \sigma(r, X_r^i, u_r^i)\mathrm{d}W_r^i + \int_{t_i}^s \int_E \beta(r, X_{r-}^i, u_r^i, e)\widetilde{\tau_i(\mu)}(\mathrm{d}r, \mathrm{d}e), \quad (3.2)$$

$s \in [t_i, T],\ i = 0, 1$.

As in the preceding section, we associate the forward equations (3.1) and (3.2) with BSDEs. Let $(Y_s^\lambda, Z_s^\lambda, U_s^\lambda)_{s\in[t_\lambda,T]}$ and $(Y_s^i, Z_s^i, U_s^i)_{s\in[t_i,T]}$, $i = 0, 1$, be the unique solutions of the BSDEs

$$Y_s^\lambda = \Phi(X_T^\lambda) + \int_s^T f(r, X_r^\lambda, Y_r^\lambda, Z_r^\lambda, U_r^\lambda, u_r^\lambda)\mathrm{d}r - \int_s^T Z_r^\lambda \mathrm{d}B_r - \int_s^T \int_E U_r^\lambda(e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e),$$

$s \in [t_\lambda, T]$, and

$$Y_s^i = \Phi(X_T^i) + \int_s^T f(r, X_r^i, Y_r^i, Z_r^i, U_r^i, u_r^i)\mathrm{d}r - \int_s^T Z_r^i \mathrm{d}W_r^i - \int_s^T \int_E U_r^i(e)\widetilde{\tau_i(\mu)}(\mathrm{d}r, \mathrm{d}e),$$

$s \in [t_i, T]$, respectively. Then from the adaptedness of the solutions $Y^\lambda$ and $Y^i$ with respect to the filtrations generated by $(B, \mu)$ and $(W^i, \tau_i(\mu))$, respectively, we know that $Y^\lambda_{t_\lambda}$ and $Y^i_{t_i}$ are deterministic and equal to the cost functionals $J(t_\lambda, x_\lambda; u^\lambda)$ and $J(t_i, x_i; u^i)$, respectively.

For the proof of the Theorem 3.1 it is crucial to estimate $\lambda Y^0_{t_0} + (1-\lambda) Y^1_{t_1} - Y^\lambda_{t_\lambda}$, for $\lambda \in (0,1)$. Since the processes $Y^0, Y^1$ and $Y^\lambda$ are solutions of BSDEs over different time intervals, driven by different Brownian motions and different Poisson random measures, we have to make them comparable with the help of the inverse time change.

In a first step we carry out this inverse time change for the forward equations. For this end we introduce the time-changed processes: $\widetilde{X}^i_s = X^i_{\tau_i^{-1}(s)}$, $s \in [t_\lambda, T]$, $i = 0, 1$. Then we have, for $i = 0, 1$,

$$\begin{aligned} \widetilde{X}^i_s =& x_i + \int_{t_\lambda}^s \frac{1}{\dot{\tau}_i} b(\tau_i^{-1}(r), \widetilde{X}^i_r, u^\lambda_r) \mathrm{d}r + \int_{t_\lambda}^s \frac{1}{\sqrt{\dot{\tau}_i}}\, \sigma(\tau_i^{-1}(r), \widetilde{X}^i_r, u^\lambda_r) \mathrm{d}B_r \\ &+ \int_{t_\lambda}^s \int_E \beta(\tau_i^{-1}(r), \widetilde{X}^i_{r-}, u^\lambda_r, e) \left( \tilde{\mu}(\mathrm{d}r, \mathrm{d}e) + \left(1 - \frac{1}{\dot{\tau}_i}\right) \Pi(\mathrm{d}e)\mathrm{d}r \right), \ s \in [t_\lambda, T]. \end{aligned} \tag{3.3}$$

Comparable with Lemma 2.5 but now with arbitrary power $p \geq 2$, we can show

**Lemma 3.3.** *Let $p \geq 2$. Then there exists a constant $C_{\delta,p}$ depending only on the bounds of $\sigma, b, \beta$, their Lipschitz constants, $\Pi(E)$, $\delta$ as well as $p$, such that, for all $t \in [t_\lambda, T]$,*

$$\mathbb{E}\left[ \sup_{t \leq s \leq T} \left| \widetilde{X}^0_s - \widetilde{X}^1_s \right|^p \middle| \mathcal{F}_t \right] \leq C_{\delta,p} \left( \left| \widetilde{X}^0_t - \widetilde{X}^1_t \right|^p + |t_0 - t_1|^p \right). \tag{3.4}$$

Moreover, in addition to Lemma 3.3, which gives a kind of "first order estimate", we also have the following kind of "second order estimate". For this we introduce the process $\widetilde{X}^\lambda = \lambda \widetilde{X}^0 + (1-\lambda)\widetilde{X}^1$.

**Lemma 3.4.** *Let $p \geq 2$. There exists a constant $C_{p,\delta}$ depending only on the bounds of $\sigma, b, \beta$, their Lipschitz constants, $\Pi(E)$, $\delta$ and $p$, such that, for all $t \in [t_\lambda, T]$,*

$$\begin{aligned} &\mathbb{E}\left[ \sup_{t \leq s \leq T} \left| \widetilde{X}^\lambda_s - X^\lambda_s \right|^p \middle| \mathcal{F}_t \right] \\ \leq& C_{p,\delta} \left| \widetilde{X}^\lambda_t - X^\lambda_t \right|^p + C_{p,\delta} (\lambda(1-\lambda))^p \left( |t_0 - t_1|^{2p} + \left| \widetilde{X}^0_t - \widetilde{X}^1_t \right|^{2p} \right). \end{aligned} \tag{3.5}$$

For the proof of the both above lemmata the reader is referred to the Appendix.

After having applied the inverse time changes to the forward equations, let us do it now for the BSDEs. Thus, for $i = 0, 1$, we introduce the processes $\widetilde{Y}^i_s \triangleq Y^i_{\tau_i^{-1}(s)}$, $\widetilde{Z}^i_s \triangleq \frac{1}{\sqrt{\dot{\tau}_i}} Z^i_{\tau_i^{-1}(s)}$, and $\widetilde{U}^i_s \triangleq U^i_{\tau_i^{-1}(s)}$, $s \in [t_\lambda, T]$. Obviously, $\left(Y^\lambda, Z^\lambda, U^\lambda\right)$ and $(\widetilde{Y}^i, \widetilde{Z}^i, \widetilde{U}^i)$ belong to $\mathcal{S}^2(t_\lambda, T; \mathbb{R}) \times \mathbb{L}^2(t_\lambda, T; \mathbb{R}^d) \times \mathbb{L}^2(t_\lambda, T; \tilde{\mu}, \mathbb{R})$, and

$$\begin{aligned} \widetilde{Y}^i_s =& \Phi(\widetilde{X}^i_T) + \int_s^T \frac{1}{\dot{\tau}_i} f(\tau_i^{-1}(r), \widetilde{X}^i_r, \widetilde{Y}^i_r, \sqrt{\dot{\tau}_i}\widetilde{Z}^i_r, \widetilde{U}^i_r, u^\lambda_r) \mathrm{d}r - \int_s^T \widetilde{Z}^i_r \mathrm{d}B_r \\ &- \int_s^T \int_E \widetilde{U}^i_r(e) \left( \tilde{\mu}(\mathrm{d}r, \mathrm{d}e) + \left(1 - \frac{1}{\dot{\tau}_i}\right) \Pi(\mathrm{d}e)\mathrm{d}r \right), \ s \in [t_\lambda, T]. \end{aligned}$$

With the help of standard BSDE estimates we can show

**Lemma 3.5.** *For $p \geq 2$, there exists some constant $C_p$ only depending on $p$ and the bounds of the coefficients $f, \Phi$, such that, for all $s \in [t_\lambda, T]$, $i = 0, 1$,*

$$\mathbb{E}\left[\sup_{s\leq r\leq T} |\widetilde{Y}^i|^p + \left(\int_s^T |\widetilde{Z}_r^i|^2 \mathrm{d}r\right)^{p/2} + \left(\int_s^T \int_E |\widetilde{U}_r^i(e)|^2 \Pi(\mathrm{d}e)\mathrm{d}r\right)^{p/2} \Bigg| \mathcal{F}_s\right] \leq C_p,$$

*and*

$$\mathbb{E}\left[\sup_{s\leq r\leq T} |Y^\lambda|^p + \left(\int_s^T |Z_r^\lambda|^2 \mathrm{d}r\right)^{p/2} + \left(\int_s^T \int_E |U_r^\lambda(e)|^2 \Pi(\mathrm{d}e)\mathrm{d}r\right)^{p/2} \Bigg| \mathcal{F}_s\right] \leq C_p.$$

As the proof uses simple BSDE estimates which by now are standard (see, for instance, [26]), the proof is omitted.

Recall that we have defined $\widetilde{X}^\lambda = \lambda \widetilde{X}^0 + (1-\lambda)\widetilde{X}^1$. In the same manner, we introduce the processes $\widetilde{Y}^\lambda = \lambda \widetilde{Y}^0 + (1-\lambda)\widetilde{Y}^1$, $\widetilde{Z}^\lambda = \lambda \widetilde{Z}^0 + (1-\lambda)\widetilde{Z}^1$ and $\widetilde{U}^\lambda = \lambda \widetilde{U}^0 + (1-\lambda)\widetilde{U}^1$. Then we get that $(\widetilde{Y}^\lambda, \widetilde{Z}^\lambda, \widetilde{U}^\lambda) \in \mathcal{S}^2(t_\lambda, T; \mathbb{R}) \times \mathbb{L}^2(t_\lambda, T; \mathbb{R}^d) \times \mathbb{L}^2(t_\lambda, T; \tilde{\mu}, \mathbb{R})$ is the unique solution of the following BSDE

$$\begin{aligned}
\widetilde{Y}_s^\lambda &= \lambda\Phi\left(\widetilde{X}_T^0\right) + (1-\lambda)\Phi\left(\widetilde{X}_T^1\right) - \int_s^T \widetilde{Z}_r^\lambda \mathrm{d}B_r - \int_s^T \int_E \widetilde{U}_r^\lambda(e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e) \\
&+ \int_s^T \left[\frac{\lambda}{\dot{\tau}_0} f(\tau_0^{-1}(r), \widetilde{X}_r^0, \widetilde{Y}_r^0, \sqrt{\dot{\tau}_0}\widetilde{Z}_r^0, \widetilde{U}_r^0, u_r^\lambda) + \frac{1-\lambda}{\dot{\tau}_1} f(\tau_1^{-1}(r), \widetilde{X}_r^1, \widetilde{Y}_r^1, \sqrt{\dot{\tau}_1}\widetilde{Z}_r^1, \widetilde{U}_r^1, u_r^\lambda)\right] \mathrm{d}r \\
&- \int_s^T \int_E \left[\lambda\left(1 - \frac{1}{\dot{\tau}_0}\right)\widetilde{U}_r^0(e) + (1-\lambda)\left(1 - \frac{1}{\dot{\tau}_1}\right)\widetilde{U}_r^1(e)\right] \Pi(\mathrm{d}e)\mathrm{d}r, \ s \in [t_\lambda, T].
\end{aligned}$$

In analogy to Lemma 3.3 we have for the associated BSDE the following statement, which proof is postponed in the Appendix:

**Lemma 3.6.** *For all $p \geq 2$, there exists a constant $C_\delta$ depending only on the bounds of $\sigma, b, \beta$, their Lipschitz constants, $\Pi(E)$, $\delta$ and $p$, such that, for any $t \in [t_\lambda, T]$,*

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\leq s\leq T} |\widetilde{Y}_s^0 - \widetilde{Y}_s^1|^p + \left(\int_t^T \left|\widetilde{Z}_s^0 - \widetilde{Z}_s^1\right|^2 \mathrm{d}s\right)^{\frac{p}{2}} + \left(\int_t^T \int_E \left|\widetilde{U}_s^0(e) - \widetilde{U}_s^1(e)\right|^2 \Pi(\mathrm{d}e)\mathrm{d}s\right)^{\frac{p}{2}} \Bigg| \mathcal{F}_t\right] \\
&\leq C_\delta\left(|\widetilde{X}_t^0 - \widetilde{X}_t^1|^p + |t_0 - t_1|^p\right).
\end{aligned}$$

Our objective is to estimate

$$\lambda Y_{t_0}^0 + (1-\lambda)Y_{t_1}^1 - Y_{t_\lambda}^\lambda = \widetilde{Y}_{t_\lambda}^\lambda - Y_{t_\lambda}^\lambda.$$

For this end some auxiliary processes shall be introduced. So let us introduce the increasing càdlàg processes

$$A_t := |t_0 - t_1| + \sup_{s\in[t_\lambda, t]} |\widetilde{X}_s^0 - \widetilde{X}_s^1|,$$

and

$$B_t := \sup_{s\in[t_\lambda, t]} |\widetilde{X}_s^\lambda - X_s^\lambda|, \quad t \in [t_\lambda, T].$$

For some suitable $C$ and $C_\delta$ which will be specified later, we also introduce the increasing càdlàg process

$$D_t = CB_t + C_\delta\lambda(1-\lambda)A_t^2, \quad t \in [t_\lambda, T].$$

We can obtain easily from the Lemmata 3.3 and 3.4 the following estimate for $D_t$.

**Corollary 3.7.** *For any $p \geq 2$, there exists a constant $C_p$ such that*

$$\mathbb{E}\left[|D_s|^p|\mathcal{F}_t\right] \leq C_p|D_t|^p, \quad \textit{for all} \;\; s,t \in [t_\lambda, T], \quad \textit{with} \;\; t \leq s.$$

We observe that, in particular,

$$\mathbb{E}\left[|D_T|^p\right] \leq C_p(|t_0 - t_1|^p + |x_0 - x_1|^p) < +\infty, \quad p \geq 2.$$

We let $(\widehat{Y}^\lambda, \widehat{Z}^\lambda, \widehat{U}^\lambda) \in \mathcal{S}^2(t_\lambda, T; \mathbb{R}) \times \mathbb{L}^2(t_\lambda, T; \mathbb{R}^d) \times \mathbb{L}^2(t_\lambda, T; \tilde{\mu}, \mathbb{R})$ be the unique solution of the following BSDE,

$$\begin{aligned}
\widehat{Y}_s^\lambda = \Phi\left(X_T^\lambda\right) + D_T - \int_s^T \widehat{Z}_r^\lambda \mathrm{d}B_r - \int_s^T \int_E \widehat{U}_r^\lambda(e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e) \\
+ \int_s^T \Bigg[ f(r, X_r^\lambda, \widehat{Y}_r^\lambda - D_r, \widehat{Z}_r^\lambda, \widehat{U}_r^\lambda, u_r^\lambda) + CD_r \\
+ C_\delta^0 \lambda(1-\lambda)\left(|t_0 - t_1|^2\left(1 + |\widetilde{Z}_r^1|^2\right) + |\widetilde{Z}_r^0 - \widetilde{Z}_r^1|^2\right) + \int_E \left|\widetilde{U}_r^0(e) - \widetilde{U}_r^1(e)\right|^2 \Pi(\mathrm{d}e)\Bigg]\mathrm{d}r,
\end{aligned}$$

$s \in [t_\lambda, T]$. The process $\widehat{Y}^\lambda$ stems its importance from the fact that it majorizes $\widetilde{Y}^\lambda$ in a suitable manner. More precisely, we have

**Lemma 3.8.** $\widetilde{Y}_s^\lambda \leq \widehat{Y}_s^\lambda$, *$\mathbb{P}$-a.s., for any $s \in [t_\lambda, T]$.*

For a better readability of the paper, also this proof is postponed to the Appendix.

In addition to Lemma 3.8, we also have to estimate the difference between $\widetilde{Y}^\lambda$ and $Y^\lambda$. For this we introduce the process $\overline{Y}_t^\lambda = \widehat{Y}_t^\lambda - D_t$, $t \in [t_\lambda, T]$, and we identify $(\overline{Y}^\lambda, \widehat{Z}^\lambda, \widehat{U}^\lambda)$ as the unique solution of the BSDE

$$\begin{aligned}
\overline{Y}_s^\lambda =& \Phi\left(X_T^\lambda\right) + \int_s^T \Bigg[ f(r, X_r^\lambda, \overline{Y}_t^\lambda, \widehat{Z}_r^\lambda, \widehat{U}_r^\lambda, u_r^\lambda) + CD_r \\
&+ C_\delta^0 \lambda(1-\lambda)\left(|t_0 - t_1|^2\left(1 + |\widetilde{Z}_r^1|^2\right) + |\widetilde{Z}_r^0 - \widetilde{Z}_r^1|^2\right) \\
&+ \int_E \left|\widetilde{U}_r^0(e) - \widetilde{U}_r^1(e)\right|^2 \Pi(\mathrm{d}e)\Bigg]\mathrm{d}r - \int_s^T \widehat{Z}_r^\lambda \mathrm{d}B_r - \int_s^T\int_E \widehat{U}_r^\lambda(e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e) + \int_s^T \mathrm{d}D_r,
\end{aligned}$$

$s \in [t_\lambda, T]$. We observe that we have the following statement, which proof is given in the Appendix.

**Lemma 3.9.** *For $t \in [t_\lambda, T]$, we have*

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[t,T]}\left|\overline{Y}_s^\lambda - Y_s^\lambda\right|^2 + \int_t^T \left|\widehat{Z}_s^\lambda - Z_s^\lambda\right|^2 \mathrm{d}s + \int_t^T \int_E \left|\widehat{U}_s^\lambda(e) - U_s^\lambda(e)\right|^2 \mu(\mathrm{d}s, \mathrm{d}e)\middle|\mathcal{F}_t\right] \\
&\leq C_\delta D_t^2.
\end{aligned} \tag{3.6}$$

Now we are ready to give the proof of Theorem 3.1.

*Proof of Theorem 3.1*: We know from the stochastic interpretation of the viscosity solution $V$ as value function (see (1.4)) that, for any $\lambda \in (0,1)$ and $\varepsilon > 0$, there exists an admissible control process $u^\lambda \in \mathcal{U}^{B,\mu}(t_\lambda, T)$ such that $Y_{t_\lambda}^\lambda \leq V(t_\lambda, x_\lambda) + \varepsilon$. On the other hand, using again (1.4), but now for $\mathcal{U}^{W^i,\tau_i(\mu)}$, we obtain: $V(t_i,x_i)\le Y^i_{t_i}$, $i=0,1$. From the Lemmata 3.8 and 3.9 we deduce that

$$\begin{aligned}
& \lambda V(t_0,x_0)+(1-\lambda)V(t_1,x_1)\\
\le\ & \lambda Y^0_{t_0}+(1-\lambda)Y^1_{t_1}\\
=\ & \lambda \widetilde{Y}^0_{t_\lambda}+(1-\lambda)\widetilde{Y}^1_{t_\lambda}=\widetilde{Y}^\lambda_{t_\lambda}\\
\le\ & \widehat{Y}^\lambda_{t_\lambda}=\overline{Y}^\lambda_{t_\lambda}+D_{t_\lambda}\\
\le\ & Y^\lambda_{t_\lambda}+C_\delta D_{t_\lambda}\\
\le\ & V(t_\lambda,x_\lambda)+C_\delta B_{t_\lambda}+C_\delta\lambda(1-\lambda)A^2_{t_\lambda}+\varepsilon\\
\le\ & V(t_\lambda,x_\lambda)+C_\delta\lambda(1-\lambda)(|t_0-t_1|^2+|x_0-x_1|^2)+\varepsilon.
\end{aligned}$$

Here we have used that $D_{t_\lambda}=CB_{t_\lambda}+C_\delta\lambda(1-\lambda)A^2_{t_\lambda}$ and $B_{t_\lambda}=0$. From the arbitrariness of $\varepsilon$, it follows that

$$\lambda V(t_0,x_0)+(1-\lambda)V(t_1,x_1)\le V(t_\lambda,x_\lambda)+C_\delta\lambda(1-\lambda)(|t_0-t_1|^2+|x_0-x_1|^2).$$

Hence, the semiconcavity of $V$ is proved. ∎

# 4 Appendix

The appendix is devoted to the proof of the Lemmata 3.3–3.9.

First we give the following lemma, which will be used in what follows. It can be checked by a straightforward computation and, hence, its proof is omitted.

**Lemma 4.1.** *There exists some positive constant $C_\delta$ only depending on $T$ and $\delta$ such that, for $s\in[t_\lambda,T]$,*

$$|\tau_0^{-1}(s)-\tau_1^{-1}(s)|+|\frac{1}{\dot\tau_0}-\frac{1}{\dot\tau_1}|+|\frac{1}{\sqrt{\dot\tau_0}}-\frac{1}{\sqrt{\dot\tau_1}}|\le C_\delta|t_0-t_1|,$$

$$\lambda\left|1-\frac{1}{\sqrt{\dot\tau_0}}\right|+(1-\lambda)\left|1-\frac{1}{\sqrt{\dot\tau_1}}\right|\le\frac{1}{2\delta}\lambda(1-\lambda)|t_0-t_1|,$$

$$\left|\lambda(1-\frac{1}{\sqrt{\dot\tau_0}})+(1-\lambda)(1-\frac{1}{\sqrt{\dot\tau_1}})\right|\le\frac{1}{\delta^2}\lambda(1-\lambda)|t_0-t_1|^2.$$

*Moreover, for all $s\in[t_\lambda,T]$,*

$$\lambda(1-\frac{1}{\dot\tau_0})=-(1-\lambda)(1-\frac{1}{\dot\tau_1})=\frac{\lambda(1-\lambda)}{T-t_\lambda}(t_0-t_1),\ \ \lambda\tau_0^{-1}(s)+(1-\lambda)\tau_1^{-1}(s)=s.$$

We begin with the proof of Lemma 3.3.

*Proof of Lemma 3.3:* Let us put

$$\Delta\beta(r,e)=\beta(\tau_0^{-1}(r),\widetilde{X}^0_{r-},u^\lambda_r,e)-\beta(\tau_1^{-1}(r),\widetilde{X}^1_{r-},u^\lambda_r,e),$$

for $(r,e)\in[t_\lambda,T]\times E$, and consider, for $t\in[t_\lambda,T]$,

$$M_s=\int_t^s\int_E\Delta\beta(r,e)\tilde\mu(\mathrm{d}r,\mathrm{d}e),\ \ s\in[t,T].$$

Since $\Delta\beta$ is bounded and predictable, $M$ is a $p$-integrable martingale, for all $p \geq 2$. We deduce from Itô's formula that,

$$N_s := |M_s|^p - \int_t^s \int_E \left(|M_r + \Delta\beta(r,e)|^p - |M_r|^p - p|M_r|^{p-2} M_r \Delta\beta(r,e)\right) \Pi(\mathrm{d}e)\mathrm{d}r,\ s \in [t,T],$$

is a martingale with $N_t = 0$ (see, Fujiwara and Kunita [44]). Moreover, since $\beta$ is Lipschitz in $(t,x)$, uniformly with respect to $(u,e)$,

$$\begin{aligned}
&|M_r + \Delta\beta(r,e)|^p - |M_r|^p - p|M_r|^{p-2} M_r \Delta\beta(r,e) \\
\leq & C_p \left(|\Delta\beta(r,e)|^2 |M_r|^{p-2} + |\Delta\beta(r,e)|^p\right) \\
\leq & C_p \left( \left( |\widetilde{X}_r^0 - \widetilde{X}_r^1|^2 + |\tau_0^{-1}(r) - \tau_1^{-1}(r)|^2 \right) |M_r|^{p-2} + |\widetilde{X}_r^0 - \widetilde{X}_r^1|^p + |\tau_0^{-1}(r) - \tau_1^{-1}(r)|^p \right) \\
\leq & C_p \left( \left( |\widetilde{X}_r^0 - \widetilde{X}_r^1|^2 + |t_0 - t_1|^2 \right) |M_r|^{p-2} + |\widetilde{X}_r^0 - \widetilde{X}_r^1|^p + |t_0 - t_1|^p \right).
\end{aligned}$$

It follows that, for $s \in [t,T]$,

$$\begin{aligned}
& \mathbb{E}\left[|M_s|^p | \mathcal{F}_t\right] \\
= \ & \mathbb{E}\left[ \int_t^s \int_E \left(|M_r + \Delta\beta(r,e)|^p - |M_r|^p - p|M_r|^{p-2} M_r \Delta\beta(r,e)\right) \Pi(\mathrm{d}e)\mathrm{d}r \Big| \mathcal{F}_t \right] \\
\leq \ & C_p \mathbb{E}\left[ \int_t^s \left( \left( |\widetilde{X}_r^0 - \widetilde{X}_r^1|^2 + |t_0 - t_1|^2 \right) |M_r|^{p-2} + |\widetilde{X}_r^0 - \widetilde{X}_r^1|^p + |t_0 - t_1|^p \right) \mathrm{d}r \Big| \mathcal{F}_t \right] \\
\leq \ & C_p \mathbb{E}\left[ \int_t^s \left( |\widetilde{X}_r^0 - \widetilde{X}_r^1|^p + |t_0 - t_1|^p \right) \mathrm{d}r \Big| \mathcal{F}_t \right] + C_p \mathbb{E}\left[ \int_t^s |M_r|^p \mathrm{d}r \Big| \mathcal{F}_t \right],
\end{aligned}$$

and from Gronwall's inequality we obtain

$$\mathbb{E}\left[|M_s|^p | \mathcal{F}_t\right] \leq C_p \mathbb{E}\left[ \int_t^s \left( |\widetilde{X}_r^0 - \widetilde{X}_r^1|^p + |t_0 - t_1|^p \right) \mathrm{d}r \Big| \mathcal{F}_t \right],\ s \in [t,T]. \tag{4.1}$$

Noticing that, for any $t_\lambda \leq t \leq v \leq T$,

$$\begin{aligned}
& \widetilde{X}_v^0 - \widetilde{X}_v^1 \\
= \ & \widetilde{X}_t^0 - \widetilde{X}_t^1 + \int_t^v \left( \frac{1}{\dot{\tau}_0} b(\tau_0^{-1}(r), \widetilde{X}_r^0, u_r^\lambda) - \frac{1}{\dot{\tau}_1} b(\tau_1^{-1}(r), \widetilde{X}_r^1, u_r^\lambda) \right) \mathrm{d}r \\
& + \int_t^v \left( \frac{1}{\sqrt{\dot{\tau}_0}} \sigma(\tau_0^{-1}(r), \widetilde{X}_r^0, u_r^\lambda) - \frac{1}{\sqrt{\dot{\tau}_1}} \sigma(\tau_1^{-1}(r), \widetilde{X}_r^1, u_r^\lambda) \right) \mathrm{d}B_r + M_v \qquad (4.2) \\
& + \int_t^v \int_E \left( \left(1 - \frac{1}{\dot{\tau}_0}\right) \beta(\tau_0^{-1}(r), \widetilde{X}_r^0, u_r^\lambda, e) - \left(1 - \frac{1}{\dot{\tau}_1}\right) \beta(\tau_1^{-1}(r), \widetilde{X}_r^1, u_r^\lambda, e) \right) \Pi(\mathrm{d}e)\mathrm{d}r,
\end{aligned}$$

we get, by a standard argument (Recall that $\Pi(E) < \infty$) and Lemma 4.1, the existence of a constant $C_{\delta,p}$ such that

$$\begin{aligned}
& \mathbb{E}\left[ \sup_{t \leq v \leq s} \left| \widetilde{X}_v^0 - \widetilde{X}_v^1 \right|^p | \mathcal{F}_t \right] \\
\leq \ & C_{\delta,p} \left| \widetilde{X}_t^0 - \widetilde{X}_t^1 \right|^p + C_p \mathbb{E}\left[ \left| \int_t^s \left( \left| \frac{1}{\dot{\tau}_0} - \frac{1}{\dot{\tau}_1} \right| + \left| \frac{1}{\dot{\tau}_1} \right| \left| \widetilde{X}_r^0 - \widetilde{X}_r^1 \right| \right) \mathrm{d}r \right|^p \Big| \mathcal{F}_t \right] \\
& + C_{\delta,p} \mathbb{E}\left[ \left( \int_t^s \left( \left| \frac{1}{\sqrt{\dot{\tau}_0}} - \frac{1}{\sqrt{\dot{\tau}_1}} \right| + \left| \frac{1}{\sqrt{\dot{\tau}_1}} \right| \left| \widetilde{X}_r^0 - \widetilde{X}_r^1 \right| \right)^2 \mathrm{d}r \right)^{p/2} \Big| \mathcal{F}_t \right]
\end{aligned}$$

$$+C_{\delta,p}\mathbb{E}\left[|M_s|^p\,|\mathcal{F}_t\right]+C_{\delta,p}|t_0-t_1|^p.$$

Thus, taking into account (4.1), we obtain

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\le v\le s}\left|\widetilde{X}^0_v-\widetilde{X}^1_v\right|^p\Big|\mathcal{F}_t\right]\\ \le\;& C_{\delta,p}\left(\left|\widetilde{X}^0_t-\widetilde{X}^1_t\right|^p+|t_0-t_1|^p\right)+C_{\delta,p}\int_t^s\mathbb{E}\left[\sup_{t\le v\le r}\left|\widetilde{X}^0_v-\widetilde{X}^1_v\right|^p\Big|\mathcal{F}_t\right]\mathrm{d}r,\;\; s\in[t_\lambda,T],\end{aligned}$$

and, finally, Gronwall's lemma yields that

$$\mathbb{E}\left[\sup_{t\le v\le T}\left|\widetilde{X}^0_v-\widetilde{X}^1_v\right|^p\Big|\mathcal{F}_t\right]\le C_{\delta,p}\left(|\widetilde{X}^0_t-\widetilde{X}^1_t|^p+|t_0-t_1|^p\right).$$

The proof is complete now. ■

Let us now prove Lemma 3.4:
*Proof of Lemma 3.4:* We observe that, for $s\in[t_\lambda,T]$,

$$\begin{aligned}&\widetilde{X}^\lambda_s-X^\lambda_s\\ =&\int_{t_\lambda}^s\left(\frac{\lambda}{\dot\tau_0}b(\tau_0^{-1}(r),\widetilde{X}^0_r,u^\lambda_r)+\frac{1-\lambda}{\dot\tau_1}b(\tau_1^{-1}(r),\widetilde{X}^1_r,u^\lambda_r)-b(r,X^\lambda_r,u^\lambda_r)\right)\mathrm{d}r\\ &+\int_{t_\lambda}^s\left(\frac{\lambda}{\sqrt{\dot\tau_0}}\sigma(\tau_0^{-1}(r),\widetilde{X}^0_r,u^\lambda_r)+\frac{1-\lambda}{\sqrt{\dot\tau_1}}\sigma(\tau_1^{-1}(r),\widetilde{X}^1_r,u^\lambda_r)-\sigma(r,X^\lambda_r,u^\lambda_r)\right)\mathrm{d}B_r\\ &+\int_{t_\lambda}^s\int_E\left(\lambda\beta(\tau_0^{-1}(r),\widetilde{X}^0_{r-},u^\lambda_r,e)+(1-\lambda)\beta(\tau_1^{-1}(r),\widetilde{X}^1_{r-},u^\lambda_r,e)-\beta(r,X^\lambda_{r-},u^\lambda_r,e)\right)\tilde\mu(\mathrm{d}r,\mathrm{d}e)\\ &+\int_{t_\lambda}^s\int_E\left(\lambda\left(1-\frac{1}{\dot\tau_0}\right)\beta(\tau_0^{-1}(r),\widetilde{X}^0_r,u^\lambda_r,e)+(1-\lambda)\left(1-\frac{1}{\dot\tau_1}\right)\beta(\tau_1^{-1}(r),\widetilde{X}^1_r,u^\lambda_r,e)\right)\Pi(\mathrm{d}e)\mathrm{d}r.\end{aligned}\tag{4.3}$$

Taking into account that, as a consequence of the assumptions on the coefficients, we have on one hand that the functions $b,\sigma,\beta$ but also $-b,-\sigma,-\beta$ are semiconcave in $(t,x)$, uniformly with respect to $u$ and $(u,e)$, respectively, and that, on the other hand, $\lambda(1-\frac{1}{\dot\tau_0})=-(1-\lambda)(1-\frac{1}{\dot\tau_1})=\frac{\lambda(1-\lambda)}{T-t_\lambda}(t_0-t_1)$ (see Lemma 4.1), we deduce

$$\begin{aligned}&\left|\frac{\lambda}{\dot\tau_0}b(\tau_0^{-1}(r),\widetilde{X}^0_r,u^\lambda_r)+\frac{1-\lambda}{\dot\tau_1}b(\tau_1^{-1}(r),\widetilde{X}^1_r,u^\lambda_r)-b(r,X^\lambda_r,u^\lambda_r)\right|\\ \le\;&\left|\lambda b(\tau_0^{-1}(r),\widetilde{X}^0_r,u^\lambda_r)+(1-\lambda)b(\tau_1^{-1}(r),\widetilde{X}^1_r,u^\lambda_r)-b(r,X^\lambda_r,u^\lambda_r)\right|\\ &+\left|\lambda\left(\frac{1}{\dot\tau_0}-1\right)b(\tau_0^{-1}(r),\widetilde{X}^0_r,u^\lambda_r)+(1-\lambda)\left(\frac{1}{\dot\tau_1}-1\right)b(\tau_1^{-1}(r),\widetilde{X}^1_r,u^\lambda_r)\right|\\ \le\;& C_\delta\lambda(1-\lambda)\left(|t_0-t_1|^2+\left|\widetilde{X}^0_r-\widetilde{X}^1_r\right|^2\right)+C_\delta\left|\widetilde{X}^\lambda_r-X^\lambda_r\right|,\;\; r\in[t_\lambda,T].\end{aligned}$$

Similarly, we get

$$\begin{aligned}&\left|\frac{\lambda}{\sqrt{\dot\tau_0}}\sigma(\tau_0^{-1}(r),\widetilde{X}^0_r,u^\lambda_r)+\frac{1-\lambda}{\sqrt{\dot\tau_1}}\sigma(\tau_1^{-1}(r),\widetilde{X}^1_r,u^\lambda_r)-\sigma(r,X^\lambda_r,u^\lambda_r)\right|\\ \le\;& C_\delta\lambda(1-\lambda)\left(|t_0-t_1|^2+\left|\widetilde{X}^0_r-\widetilde{X}^1_r\right|^2\right)+C_\delta\left|\widetilde{X}^\lambda_r-X^\lambda_r\right|,\;\; r\in[t_\lambda,T],\end{aligned}$$

and

$$\begin{aligned}
&\left|\lambda\beta(\tau_0^{-1}(r),\widetilde{X}_r^0,u_r^\lambda,e)+(1-\lambda)\beta(\tau_1^{-1}(r),\widetilde{X}_r^1,u_r^\lambda,e)-\beta(r,X_r^\lambda,u_r^\lambda,e)\right|\\
\leq\ & C_\delta\lambda(1-\lambda)\left(|t_0-t_1|^2+\left|\widetilde{X}_r^0-\widetilde{X}_r^1\right|^2\right)+C_\delta\left|\widetilde{X}_r^\lambda-X_r^\lambda\right|,\ r\in[t_\lambda,T].
\end{aligned}$$

Moreover, by using again that $\lambda(1-\frac{1}{\dot\tau_0})=-(1-\lambda)(1-\frac{1}{\dot\tau_1})=\frac{\lambda(1-\lambda)}{T-t_\lambda}(t_0-t_1)$, we obtain

$$\begin{aligned}
&\left|\lambda\left(1-\frac{1}{\dot\tau_0}\right)\beta(\tau_0^{-1}(r),\widetilde{X}_r^0,u_r^\lambda,e)+(1-\lambda)\left(1-\frac{1}{\dot\tau_1}\right)\beta(\tau_1^{-1}(r),\widetilde{X}_r^1,u_r^\lambda,e)\right|\\
\leq& C_\delta\lambda(1-\lambda)\left(|t_0-t_1|^2+\left|\widetilde{X}_r^0-\widetilde{X}_r^1\right|^2\right),\ r\in[t_\lambda,T].
\end{aligned}$$

Consequently, by combining the above estimates with the argument which has lead to (4.1) in the proof of Lemma 3.3, we get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\leq v\leq s}\left|\widetilde{X}_v^\lambda-X_v^\lambda\right|^p\middle|\mathcal{F}_t\right]\\
\leq\ & C_{\delta,p}\left|\widetilde{X}_t^\lambda-X_t^\lambda\right|^p+C_\delta(\lambda(1-\lambda))^p\left(|t_0-t_1|^{2p}+\mathbb{E}\left[\int_t^s\left|\widetilde{X}_r^0-\widetilde{X}_r^1\right|^{2p}\mathrm{d}r\middle|\mathcal{F}_t\right]\right)\\
&+C_{\delta,p}\int_t^s\mathbb{E}\left[\sup_{t\leq v\leq r}\left|\widetilde{X}_v^\lambda-X_v^\lambda\right|^p\middle|\mathcal{F}_t\right]\mathrm{d}r,\quad t_\lambda\leq t\leq s\leq T,
\end{aligned}$$

and, thus, Lemma 3.3 yields

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\leq v\leq s}\left|\widetilde{X}_v^\lambda-X_v^\lambda\right|^p\middle|\mathcal{F}_t\right]\\
\leq\ & C_{\delta,p}\left|\widetilde{X}_t^\lambda-X_t^\lambda\right|^p+C_{\delta,p}(\lambda(1-\lambda))^p\left(|t_0-t_1|^{2p}+\left|\widetilde{X}_t^0-\widetilde{X}_t^1\right|^{2p}\right)\\
&+C_{\delta,p}\int_t^s\mathbb{E}\left[\sup_{t\leq v\leq r}\left|\widetilde{X}_v^\lambda-X_v^\lambda\right|^p\middle|\mathcal{F}_t\right]\mathrm{d}r,\quad t_\lambda\leq t\leq s\leq T.
\end{aligned}$$

Hence, Gronwall's inequality gives

$$\mathbb{E}\left[\sup_{t\leq s\leq T}\left|\widetilde{X}_s^\lambda-X_s^\lambda\right|^p\middle|\mathcal{F}_t\right]\leq C_{\delta,p}\left|\widetilde{X}_t^\lambda-X_t^\lambda\right|^p+C_{\delta,p}(\lambda(1-\lambda))^p\left(|t_0-t_1|^{2p}+\left|\widetilde{X}_t^0-\widetilde{X}_t^1\right|^{2p}\right).$$

The proof of Lemma 3.4 is complete. ■

We prove now the analogue estimates for our BSDEs stated in Lemma 3.6.
*Proof of Lemma 3.6:* We notice that, for $t_\lambda\leq s\leq t\leq T$,

$$\begin{aligned}
&\widetilde{Y}_s^0-\widetilde{Y}_s^1\\
=&\Phi(\widetilde{X}_T^0)-\Phi(\widetilde{X}_T^1)\\
&+\int_s^T\left(\frac{1}{\dot\tau_0}f(\tau_0^{-1}(r),\widetilde{X}_r^0,\widetilde{Y}_r^0,\sqrt{\dot\tau_0}\widetilde{Z}_r^0,\widetilde{U}_r^0,u_r^\lambda)-\frac{1}{\dot\tau_1}f(\tau_1^{-1}(r),\widetilde{X}_r^1,\widetilde{Y}_r^1,\sqrt{\dot\tau_1}\widetilde{Z}_r^1,\widetilde{U}_r^1,u_r^\lambda)\right)\mathrm{d}r\\
&-\int_s^T\left(\widetilde{Z}_r^0-\widetilde{Z}_r^1\right)\mathrm{d}B_r-\int_s^T\int_E\left(\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)\right)\widetilde{\mu}(\mathrm{d}r,\mathrm{d}e)\\
&-\int_s^T\int_E\left(\left(1-\frac{1}{\dot\tau_0}\right)\widetilde{U}_r^0(e)-\left(1-\frac{1}{\dot\tau_1}\right)\widetilde{U}_r^1(e)\right)\Pi(\mathrm{d}e)\mathrm{d}r.
\end{aligned}\tag{4.4}$$

By applying Ito's formula to $|\widetilde{Y}_s^0-\widetilde{Y}_s^1|^2$, we get from the boundedness and the Lipschitz continuity of $f$ that, for $t_\lambda \le s \le t \le T$,

$$\begin{aligned}
&|\widetilde{Y}_s^0-\widetilde{Y}_s^1|^2+\int_s^T \left|\widetilde{Z}_r^0-\widetilde{Z}_r^1\right|^2 \mathrm{d}r+\int_s^T\int_E \left|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)\right|^2 \Pi(\mathrm{d}e)\mathrm{d}r\\
\le\ & \left|\Phi(\widetilde{X}_T^0)-\Phi(\widetilde{X}_T^1)\right|^2+C|t_0-t_1|^2+C\int_s^T \left|\widetilde{X}_r^0-\widetilde{X}_r^1\right|^2\mathrm{d}r+C\int_s^T|\widetilde{Y}_r^0-\widetilde{Y}_r^1|^2\mathrm{d}r\\
&-2\int_s^T\left(\widetilde{Y}_r^0-\widetilde{Y}_r^1\right)\left(\widetilde{Z}_r^0-\widetilde{Z}_r^1\right)\mathrm{d}B_r\\
&+|t_0-t_1|^2\int_s^T\left(\left|\widetilde{Z}_r^1\right|^2+\int_E\left(\left|\widetilde{U}_r^0(e)\right|^2+\left|\widetilde{U}_r^1(e)\right|^2\right)\Pi(\mathrm{d}e)\right)\mathrm{d}r\\
&-\int_s^T\int_E\left(2\left(\widetilde{Y}_r^0-\widetilde{Y}_r^1\right)\left(\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)\right)+\left|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)\right|^2\right)\tilde{\mu}(\mathrm{d}r,\mathrm{d}e). \qquad (4.5)
\end{aligned}$$

Then taking the conditional expectation and applying Gronwall's inequality and the Burkholder-Davis-Gundy inequality we obtain from the Lemmata 3.5 and 3.3 (Recall also the arguments given in the proof of Lemma 3.4) that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\le r\le T}|\widetilde{Y}_r^0-\widetilde{Y}_r^1|^2+\int_t^T\left|\widetilde{Z}_r^0-\widetilde{Z}_r^1\right|^2\mathrm{d}r+\int_t^T\int_E\left|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)\right|^2\Pi(\mathrm{d}e)\mathrm{d}r\middle|\mathcal{F}_t\right]\\
\le\ & C_\delta\left(\left|\widetilde{X}_t^0-\widetilde{X}_t^1\right|^2+|t_0-t_1|^2\right),\ t\in[t_\lambda,T].
\end{aligned}$$

In particular, we have

$$|\widetilde{Y}_t^0-\widetilde{Y}_t^1|^2\le C_\delta\left(\left|\widetilde{X}_t^0-\widetilde{X}_t^1\right|^2+|t_0-t_1|^2\right),\quad t\in[t_\lambda,T].$$

Consequently, from Lemma 3.4, we have, for $p\ge 2$,

$$\mathbb{E}\left[\sup_{t\le r\le T}|\widetilde{Y}_r^0-\widetilde{Y}_r^1|^p\middle|\mathcal{F}_t\right]\le C_\delta\left(\left|\widetilde{X}_t^0-\widetilde{X}_t^1\right|^p+|t_0-t_1|^p\right),\ t\in[t_\lambda,T]. \qquad (4.6)$$

From (4.5), by using that $\tilde{\mu}(\mathrm{d}r\mathrm{d}e)=\mu(\mathrm{d}r\mathrm{d}e)-\Pi(\mathrm{d}e)\mathrm{d}r$, we get also that

$$\begin{aligned}
&\int_s^T\left|\widetilde{Z}_r^0-\widetilde{Z}_r^1\right|^2\mathrm{d}r+\int_s^T\int_E\left|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)\right|^2\mu(\mathrm{d}r,\mathrm{d}e)\\
\le\ & \left|\Phi(\widetilde{X}_T^0)-\Phi(\widetilde{X}_T^1)\right|^2+C|t_0-t_1|^2+C\int_s^T \left|\widetilde{X}_r^0-\widetilde{X}_r^1\right|^2\mathrm{d}r+C\int_s^T|\widetilde{Y}_r^0-\widetilde{Y}_r^1|^2\mathrm{d}r\\
&-2\int_s^T\left(\widetilde{Y}_r^0-\widetilde{Y}_r^1\right)\left(\widetilde{Z}_r^0-\widetilde{Z}_r^1\right)\mathrm{d}B_r\\
&+|t_0-t_1|^2\int_s^T\left(\left|\widetilde{Z}_r^1\right|^2+\int_E\left(\left|\widetilde{U}_r^0(e)\right|^2+\left|\widetilde{U}_r^1(e)\right|^2\right)\Pi(\mathrm{d}e)\right)\mathrm{d}r\\
&-\int_s^T\int_E\left(2\left(\widetilde{Y}_r^0-\widetilde{Y}_r^1\right)\left(\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)\right)\right)\tilde{\mu}(\mathrm{d}r,\mathrm{d}e).
\end{aligned}$$

Hence, by applying Lemma 3.3 and Burkholder-Davis-Gundy's inequality, we obtain from the above inequality that

$$\mathbb{E}\left[\left(\int_s^T\left|\widetilde{Z}_r^0-\widetilde{Z}_r^1\right|^2\mathrm{d}r\right)^{\frac{p}{2}}+\left(\int_s^T\int_E\left|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)\right|^2\mu(\mathrm{d}r,\mathrm{d}e)\right)^{\frac{p}{2}}\middle|\mathcal{F}_t\right]$$

$$
\begin{aligned}
&\le \; C\left(\left|\widetilde{X}^0_t-\widetilde{X}^1_t\right|^p+|t_0-t_1|^p\right)+C\mathbb{E}\Bigg[\left(\int_s^T\left|\widetilde{Y}^0_r-\widetilde{Y}^1_r\right|^2\left|\widetilde{Z}^0_r-\widetilde{Z}^1_r\right|^2\mathrm{d}r\right)^{\frac{p}{4}}\\
&\qquad +2\left(\int_s^T\int_E\left|\widetilde{Y}^0_r-\widetilde{Y}^1_r\right|^2\left|\widetilde{U}^0_r(e)-\widetilde{U}^1_r(e)\right|^2\mu(\mathrm{d}r,\mathrm{d}e)\right)^{\frac{p}{4}}\Bigg|\mathcal{F}_t\Bigg]\\
&\le \; C\left(\left|\widetilde{X}^0_t-\widetilde{X}^1_t\right|^p+|t_0-t_1|^p\right)+\mathbb{E}\Bigg[C\sup_{s\le r\le T}\left|\widetilde{Y}^0_r-\widetilde{Y}^1_r\right|^p+\frac{1}{2}\left(\int_s^T\left|\widetilde{Z}^0_r-\widetilde{Z}^1_r\right|^2\mathrm{d}r\right)^{\frac{p}{2}}\\
&\qquad +\frac{1}{2}\left(\int_s^T\int_E\left|\widetilde{U}^0_r(e)-\widetilde{U}^1_r(e)\right|^2\mu(\mathrm{d}r,\mathrm{d}e)\right)^{\frac{p}{2}}\Bigg|\mathcal{F}_t\Bigg]. \qquad (4.7)
\end{aligned}
$$

Combining (4.6) and (4.7), we get

$$
\begin{aligned}
&\mathbb{E}\Bigg[\sup_{s\le r\le T}|\widetilde{Y}^0_r-\widetilde{Y}^1_r|^p+\left(\int_s^T\left|\widetilde{Z}^0_r-\widetilde{Z}^1_r\right|^2\mathrm{d}r\right)^{p/2}\\
&\qquad +\left(\int_s^T\int_E\left|\widetilde{U}^0_r(e)-\widetilde{U}^1_r(e)\right|^2\mu(\mathrm{d}r,\mathrm{d}e)\right)^{p/2}\Bigg|\mathcal{F}_t\Bigg]\\
&\le \; C\left(|\widetilde{X}^0_t-\widetilde{X}^1_t|^p+|t_0-t_1|^p\right). \qquad (4.8)
\end{aligned}
$$

In order to replace in the left-hand side of (4.8) the term

$$
\mathbb{E}\Bigg[\left(\int_s^T\int_E\left|\widetilde{U}^0_r(e)-\widetilde{U}^1_r(e)\right|^2\mu(\mathrm{d}r,\mathrm{d}e)\right)^{p/2}\Bigg|\mathcal{F}_t\Bigg]
$$

by

$$
\mathbb{E}\Bigg[\left(\int_s^T\int_E\left|\widetilde{U}^0_r(e)-\widetilde{U}^1_r(e)\right|^2\Pi(\mathrm{d}e)\mathrm{d}r\right)^{p/2}\Bigg|\mathcal{F}_t\Bigg],
$$

we make the following estimate

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_s^T\int_E\left|\widetilde{U}^0_r(e)-\widetilde{U}^1_r(e)\right|^2\Pi(\mathrm{d}e)\mathrm{d}r\right)^{p/2}\Bigg|\mathcal{F}_t\right]\\
&\le \; (\Pi(E)T)^{\frac{p-2}{2}}\mathbb{E}\left[\int_s^T\int_E\left|\widetilde{U}^0_r(e)-\widetilde{U}^1_r(e)\right|^p\Pi(\mathrm{d}e)\mathrm{d}r\Bigg|\mathcal{F}_t\right]\\
&= \; (\Pi(E)T)^{\frac{p-2}{2}}\mathbb{E}\left[\int_s^T\int_E\left|\widetilde{U}^0_r(e)-\widetilde{U}^1_r(e)\right|^p\mu(\mathrm{d}r,\mathrm{d}e)\Bigg|\mathcal{F}_t\right].
\end{aligned}
$$

Let us denote by $N$ the Poisson process $N_s=\mu([t_\lambda,s]\times E), s\in[t_\lambda,T]$, with intensity $\Pi(E)$, the associated sequence of jump times by $\tau_i=\inf\{s\ge t_\lambda : N_s=i\}, i\ge 1$, and by $p_i:\{\tau_i\le T\}\to E$ s.t. $\mu(\{(\tau_i,p_i)\})=1$ on $\{\tau_i\le T\}$, the associated sequence of marks. We observe that

$$
\begin{aligned}
&\int_s^T\int_E\left|\widetilde{U}^0_r(e)-\widetilde{U}^1_r(e)\right|^p\mu(\mathrm{d}r,\mathrm{d}e)=\sum_{i\ge 1}\left|\widetilde{U}^0_{\tau_i}(p_i)-\widetilde{U}^1_{\tau_i}(p_i)\right|^p\\
&\le \; \left(\sum_{i\ge 1}\left|\widetilde{U}^0_{\tau_i}(p_i)-\widetilde{U}^1_{\tau_i}(p_i)\right|^2\right)^{p/2}=\left(\int_s^T\int_E\left|\widetilde{U}^0_r(e)-\widetilde{U}^1_r(e)\right|^2\mu(\mathrm{d}r,\mathrm{d}e)\right)^{p/2}.
\end{aligned}
$$

Consequently, from the previous estimate we get

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_s^T\int_E\left|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)\right|^2\Pi(\mathrm{d}e)\mathrm{d}r\right)^{p/2}\middle|\mathcal{F}_t\right]\\
\leq\ & (\Pi(E)T)^{\frac{p-2}{2}}\mathbb{E}\left[\int_s^T\int_E\left|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)\right|^p\mu(\mathrm{d}r,\mathrm{d}e)\middle|\mathcal{F}_t\right]\\
\leq\ & (\Pi(E)T)^{\frac{p-2}{2}}\mathbb{E}\left[\left(\int_s^T\int_E\left|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)\right|^2\mu(\mathrm{d}r,\mathrm{d}e)\right)^{p/2}\middle|\mathcal{F}_t\right].
\end{aligned}$$

This latter estimate allows to deduce from (4.8) the wished result. ■

Let us now give the proof of Lemma 3.8.

*Proof of Lemma 3.8*: First we notice that, for $r\in[t_\lambda,T]$,

$$\begin{aligned}
I :=\ & \frac{\lambda}{\dot\tau_0}f(\tau_0^{-1}(r),\widetilde{X}_r^0,\widetilde{Y}_r^0,\sqrt{\dot\tau_0}\widetilde{Z}_r^0,\widetilde{U}_r^0,u_r^\lambda)+\frac{1-\lambda}{\dot\tau_1}f(\tau_1^{-1}(r),\widetilde{X}_r^1,\widetilde{Y}_r^1,\sqrt{\dot\tau_1}\widetilde{Z}_r^1,\widetilde{U}_r^1,u_r^\lambda)\\
=\ & \lambda f(\tau_0^{-1}(r),\widetilde{X}_r^0,\widetilde{Y}_r^0,\sqrt{\dot\tau_0}\widetilde{Z}_r^0,\widetilde{U}_r^0,u_r^\lambda)+(1-\lambda)f(\tau_1^{-1}(r),\widetilde{X}_r^1,\widetilde{Y}_r^1,\sqrt{\dot\tau_1}\widetilde{Z}_r^1,\widetilde{U}_r^1,u_r^\lambda)\\
& -\lambda(1-\lambda)\frac{t_1-t_0}{T-t_\lambda}\Big(f(\tau_0^{-1}(r),\widetilde{X}_r^0,\widetilde{Y}_r^0,\sqrt{\dot\tau_0}\widetilde{Z}_r^0,\widetilde{U}_r^0,u_r^\lambda)-f(\tau_1^{-1}(r),\widetilde{X}_r^1,\widetilde{Y}_r^1,\sqrt{\dot\tau_1}\widetilde{Z}_r^1,\widetilde{U}_r^1,u_r^\lambda)\Big).
\end{aligned}$$

We also observe that, by applying Lemma 4.1,

$$\begin{aligned}
&\left|\lambda\sqrt{\dot\tau_0}\widetilde{Z}_r^0+(1-\lambda)\sqrt{\dot\tau_1}\widetilde{Z}_r^1-\widetilde{Z}_r^\lambda\right|\\
=&\left|\lambda(\sqrt{\dot\tau_0}-1)\widetilde{Z}_r^0+(1-\lambda)(\sqrt{\dot\tau_1}-1)\widetilde{Z}_r^1\right|\\
\leq&\lambda|1-\sqrt{\dot\tau_0}||\widetilde{Z}_r^0-\widetilde{Z}_r^1|+|\lambda(1-\sqrt{\dot\tau_0})+(1-\lambda)(1-\sqrt{\dot\tau_1})||\widetilde{Z}_r^1|\\
\leq&C_\delta\lambda(1-\lambda)(|t_0-t_1||\widetilde{Z}_r^0-\widetilde{Z}_r^1|+|t_0-t_1|^2|\widetilde{Z}_r^1|).
\end{aligned}$$

Thus, by using the estimates from Lemma 4.1, the semiconcavity as well as the Lipschitz continuity of $f$, we have

$$\begin{aligned}
I\ \leq\ & f(r,\widetilde{X}_r^\lambda,\widetilde{Y}_r^\lambda,\lambda\sqrt{\dot\tau_0}\widetilde{Z}_r^0+(1-\lambda)\sqrt{\dot\tau_1}\widetilde{Z}_r^1,\widetilde{U}_r^\lambda,u_r^\lambda)+C_\delta\lambda(1-\lambda)\Bigg(|t_0-t_1|^2\left(1+\left|\widetilde{Z}_r^1\right|^2\right)\\
&+\left|\widetilde{X}_r^0-\widetilde{X}_r^1\right|^2+\left|\widetilde{Y}_r^0-\widetilde{Y}_r^1\right|^2+\left|\widetilde{Z}_r^0-\widetilde{Z}_r^1\right|^2+\int_E\left|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)\right|^2\Pi(\mathrm{d}e)\Bigg)\\
\leq\ & f(r,\widetilde{X}_r^\lambda,\widetilde{Y}_r^\lambda,\widetilde{Z}_r^\lambda,\widetilde{U}_r^\lambda,u_r^\lambda)+C_\delta\lambda(1-\lambda)\Bigg(|t_0-t_1|^2\left(1+\left|\widetilde{Z}_r^1\right|^2\right)+\left|\widetilde{X}_r^0-\widetilde{X}_r^1\right|^2\\
&+\left|\widetilde{Y}_r^0-\widetilde{Y}_r^1\right|^2+\left|\widetilde{Z}_r^0-\widetilde{Z}_r^1\right|^2+\int_E\left|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)\right|^2\Pi(\mathrm{d}e)\Bigg).
\end{aligned}$$

Finally, by taking into account the Lipschitz continuity of $f$ as well as the definition of $D_r$, we get

$$\begin{aligned}
I\ \leq\ & f(r,\widetilde{X}_r^\lambda,\widetilde{Y}_r^\lambda-D_r,\widetilde{Z}_r^\lambda,\widetilde{U}_r^\lambda,u_r^\lambda)+C_\delta\lambda(1-\lambda)\Bigg(|t_0-t_1|^2\left(1+\left|\widetilde{Z}_r^1\right|^2\right)+\left|\widetilde{X}_r^0-\widetilde{X}_r^1\right|^2\\
&+\left|\widetilde{Y}_r^0-\widetilde{Y}_r^1\right|^2+\left|\widetilde{Z}_r^0-\widetilde{Z}_r^1\right|^2+\int_E\left|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)\right|^2\Pi(\mathrm{d}e)\Bigg)+CD_r.
\end{aligned}$$

Moreover, thanks to the semiconcavity of $\Phi$, we have

$$\lambda\Phi\left(\widetilde{X}_T^0\right)+(1-\lambda)\Phi\left(\widetilde{X}_T^1\right)\leq\Phi\left(\widetilde{X}_T^\lambda\right)+CB_T+C_\delta\lambda(1-\lambda)A_T^2.$$

By virtue of assumption **(H5)**, we can use the comparison theorem in [98] (Theorem 2.5) in order to conclude that

$$\widetilde{Y}_s^\lambda\leq\widehat{Y}_s^\lambda,\ \text{for } s\in[t_\lambda,T].$$

The proof of Lemma 3.8 is complete. ■

*Proof of Lemma 3.9:* For some constant $\gamma>0$ which will be specified later, we apply Ito's formula to $e^{\gamma s}(\overline{Y}_s^\lambda-Y_s^\lambda)^2$, and we get

$$\begin{aligned}
&\mathrm{d}e^{\gamma s}(\overline{Y}_s^\lambda-Y_s^\lambda)^2\\
=\ &-2e^{\gamma s}(\overline{Y}_s^\lambda-Y_s^\lambda)\Big[f(s,X_s^\lambda,\overline{Y}_s^\lambda,\widehat{Z}_s^\lambda,\widehat{U}_s^\lambda,u_s^\lambda)-f(s,X_s^\lambda,Y_s^\lambda,Z_s^\lambda,U_s^\lambda,u_s^\lambda)+CD_s\\
&+C_\delta^0\lambda(1-\lambda)\left(|t_0-t_1|^2\left(1+|\widetilde{Z}_s^1|^2\right)+|\widetilde{Z}_s^0-\widetilde{Z}_s^1|^2\right)+\int_E\left|\widetilde{U}_s^0(e)-\widetilde{U}_s^1(e)\right|^2\Pi(\mathrm{d}e)\Big]\mathrm{d}s\\
&+2e^{\gamma s}(\overline{Y}_s^\lambda-Y_s^\lambda)\left(\widehat{Z}_s^\lambda-Z_s^\lambda\right)\mathrm{d}B_s+2e^{\gamma s}(\overline{Y}_{s-}^\lambda-Y_{s-}^\lambda)\int_E\left(\widehat{U}_s^\lambda(e)-U_s^\lambda(e)\right)\tilde{\mu}(\mathrm{d}s,\mathrm{d}e)\\
&-2e^{\gamma s}(\overline{Y}_{s-}^\lambda-Y_{s-}^\lambda)\mathrm{d}D_s+e^{\gamma s}\left(\gamma\left|\overline{Y}_s^\lambda-Y_s^\lambda\right|^2+\left|\widehat{Z}_s^\lambda-Z_s^\lambda\right|^2\right)\mathrm{d}s\\
&+e^{\gamma s}\mathrm{d}\left[\overline{Y}^\lambda-Y^\lambda\right]_s^d,\ s\in[t_\lambda,T],
\end{aligned}$$

where $\left[\overline{Y}^\lambda-Y^\lambda\right]_s^d$ denotes the purely discontinuous part of the quadratic variation of $\overline{Y}^\lambda-Y^\lambda$:

$$\left[\overline{Y}^\lambda-Y^\lambda\right]_s^d=\sum_{t_\lambda<r\leq s}\left(\Delta\overline{Y}_r^\lambda-\Delta Y_r^\lambda\right)^2,\ s\in[t_\lambda,T].$$

We notice that, for $s\in[t_\lambda,T]$,

$$\begin{aligned}
&\int_s^T e^{\gamma r}\mathrm{d}\left[\overline{Y}^\lambda-Y^\lambda\right]_r^d\\
=\ &\sum_{s\leq r\leq T}e^{\gamma r}\left(\Delta\overline{Y}_r^\lambda-\Delta Y_r^\lambda\right)^2\\
=\ &\sum_{s\leq r\leq T}e^{\gamma r}\left(\int_E\left(\widehat{U}_r^\lambda(e)-U_r^\lambda(e)\right)\mu(\{r\},\mathrm{d}e)-\Delta D_r\right)^2\\
\geq\ &\frac{1}{2}\int_s^T\int_E e^{\gamma r}\left(\widehat{U}_r^\lambda(e)-U_r^\lambda(e)\right)^2\mu(\mathrm{d}r,\mathrm{d}e)-C_\gamma\int_s^T|\Delta D_r|\mathrm{d}D_r\\
\geq\ &\frac{1}{2}\int_s^T\int_E e^{\gamma r}\left(\widehat{U}_r^\lambda(e)-U_r^\lambda(e)\right)^2\mu(\mathrm{d}r,\mathrm{d}e)-C_\gamma(D_T-D_s)^2,
\end{aligned}$$

where $C_\gamma=e^{\gamma T}$. Hence, by integrating from $s\in[t_\lambda,T]$ to $T$ and taking the conditional expectation on both sides, we deduce that

$$e^{\gamma s}\left|\overline{Y}_s^\lambda-Y_s^\lambda\right|^2+\mathbb{E}\Big[\int_s^T e^{\gamma r}\left(\gamma\left|\overline{Y}_r^\lambda-Y_r^\lambda\right|^2+\left|\widehat{Z}_r^\lambda-Z_r^\lambda\right|^2\right)\mathrm{d}r$$

$$
\begin{aligned}
&+\frac{1}{2}\int_s^T\int_E e^{\gamma r}\left|\widehat{U}_r^\lambda(e)-U_r^\lambda(e)\right|^2\mu(\mathrm{d}r,\mathrm{d}e)\Big|\mathcal{F}_s\Big]\\
\le\ & \mathbb{E}\Big[\int_s^T 2e^{\gamma r}(\overline{Y}_r^\lambda-Y_r^\lambda)\Big\{f(r,X_r^\lambda,\overline{Y}_r^\lambda,\widehat{Z}_r^\lambda,\widehat{U}_r^\lambda,u_r^\lambda)-f(r,X_r^\lambda,Y_r^\lambda,Z_r^\lambda,U_r^\lambda,u_r^\lambda)+CD_r\\
&+C_\delta^0\lambda(1-\lambda)\left(|t_0-t_1|^2\left(1+|\widetilde{Z}_r^1|^2\right)+|\widetilde{Z}_r^0-\widetilde{Z}_r^1|^2\right)\\
&+\int_E\left|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)\right|^2\Pi(\mathrm{d}e)\Big\}\mathrm{d}r+\int_s^T 2e^{\gamma r}(\overline{Y}_{r-}^\lambda-Y_{r-}^\lambda)\mathrm{d}D_r+C_\gamma(D_T-D_s)^2\Big|\mathcal{F}_s\Big].
\end{aligned}
$$

Then, by a standard argument and from the Lipschitz continuity of $f$, we get

$$
\begin{aligned}
&e^{\gamma s}\left|\overline{Y}_s^\lambda-Y_s^\lambda\right|^2+\mathbb{E}\Big[\int_s^T e^{\gamma r}\left(\gamma\left|\overline{Y}_r^\lambda-Y_r^\lambda\right|^2+\left|\widehat{Z}_r^\lambda-Z_r^\lambda\right|^2\right)\mathrm{d}r\\
&+\frac{1}{2}\int_s^T\int_E e^{\gamma r}\left|\widehat{U}_r^\lambda(e)-U_r^\lambda(e)\right|^2\mu(\mathrm{d}r,\mathrm{d}e)\Big|\mathcal{F}_s\Big]\\
\le\ & \mathbb{E}\Big[C_K\int_s^T e^{\gamma r}\left|\overline{Y}_r^\lambda-Y_r^\lambda\right|^2\mathrm{d}r+\frac{1}{2}\int_s^T e^{\gamma r}\left|\widehat{Z}_r^\lambda-Z_r^\lambda\right|^2\mathrm{d}r\\
&+\frac{1}{4}\int_s^T\int_E e^{\gamma r}\left|\widehat{U}_r^\lambda(e)-U_r^\lambda(e)\right|^2\Pi(\mathrm{d}e)\mathrm{d}r\Big|\mathcal{F}_s\Big]\\
&+C_{\delta,\gamma}\mathbb{E}\Big[\sup_{s\le r\le T}\left|\overline{Y}_r^\lambda-Y_r^\lambda\right|D_{s,T}+C_\gamma(D_T-D_s)^2\Big|\mathcal{F}_s\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
D_{s,T}=&D_T+\lambda(1-\lambda)\left(|t_0-t_1|^2\left(1+\int_s^T|\widetilde{Z}_r^1|^2\mathrm{d}r\right)+\int_s^T|\widetilde{Z}_r^0-\widetilde{Z}_r^1|^2\mathrm{d}r\right)\\
&+\int_s^T\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^2\Pi(\mathrm{d}e)\mathrm{d}r.
\end{aligned}
$$

Therefore, by choosing $\gamma$ large enough and applying Lemma 3.3 and Corollary 3.7, we have the following estimate:

$$
\begin{aligned}
&\left|\overline{Y}_s^\lambda-Y_s^\lambda\right|^2+\mathbb{E}\left[\int_s^T\left|\widehat{Z}_r^\lambda-Z_r^\lambda\right|^2\mathrm{d}r+\int_s^T\int_E\left|\widehat{U}_r^\lambda(e)-U_r^\lambda(e)\right|^2\mu(\mathrm{d}r,\mathrm{d}e)\Big|\mathcal{F}_s\right]\\
\le\ & C_{\delta,\gamma}\mathbb{E}\Big[\sup_{s\le r\le T}\left|\overline{Y}_r^\lambda-Y_r^\lambda\right|D_{s,T}\Big|\mathcal{F}_s\Big]+C_\gamma D_s^2,\ s\in[t_\lambda,T].
\end{aligned}\tag{4.9}
$$

From Lemma 3.6 and Corollary 3.7, we get that for $p\ge 2$,

$$
\mathbb{E}\left[|D_{t,T}|^p|\mathcal{F}_t\right]\le C(|\widetilde{X}_t^0-\widetilde{X}_t^1|^{2p}+|t_0-t_1|^{2p})\le CD_t^p.\tag{4.10}
$$

Let $1<p<2$ and $q>2$ be two constants such that $\frac{1}{p}+\frac{1}{q}=1$. Then we have, for $\varepsilon>0$,

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{s\le r\le T}\left|\overline{Y}_r^\lambda-Y_r^\lambda\right|D_{s,T}\Big|\mathcal{F}_s\Big]\\
\le&\mathbb{E}\Big[\sup_{s\le r\le T}\left|\overline{Y}_r^\lambda-Y_r^\lambda\right|^p\Big|\mathcal{F}_s\Big]^{\frac{1}{p}}\mathbb{E}\Big[|D_{s,T}|^q\Big|\mathcal{F}_s\Big]^{\frac{1}{q}}\\
\le&\varepsilon M_{s,t}^{\frac{2}{p}}+\frac{1}{\varepsilon}\mathbb{E}\Big[|D_{s,T}|^q\Big|\mathcal{F}_s\Big]^{\frac{2}{q}}\\
\le&\varepsilon M_{s,t}^{\frac{2}{p}}+\frac{1}{\varepsilon}C_{\delta,q}D_s^2,\ t_\lambda\le t\le s\le T
\end{aligned}
$$

where

$$M_{s,t} = \mathbb{E}\left[\sup_{t\leq r\leq T}\left|\overline{Y}_r^\lambda - Y_r^\lambda\right|^p \middle| \mathcal{F}_s\right],\ t_\lambda \leq t \leq s \leq T.$$

Thus, Doob's inequality allows to show that, since $1 < p < 2$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[t,T]} M_{s,t}^{\frac{2}{p}}|\mathcal{F}_t\right] \\
\leq &\left(\frac{2}{2-p}\right)^{\frac{2}{p}}\mathbb{E}\left[M_{T,t}^{\frac{2}{p}}|\mathcal{F}_t\right] \\
\leq &\left(\frac{2}{2-p}\right)^{\frac{2}{p}}\mathbb{E}\left[\sup_{s\in[t,T]}\left|\overline{Y}_s^\lambda - Y_s^\lambda\right|^2|\mathcal{F}_t\right],\ t\in[t_\lambda, T].
\end{aligned}$$

Therefore, we can deduce from (4.9) that

$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|\overline{Y}_s^\lambda - Y_s^\lambda\right|^2|\mathcal{F}_t\right] \leq C_{\delta,\varepsilon}D_t^2 + C_{\delta,\gamma}\varepsilon\left(\frac{2}{2-p}\right)^{\frac{2}{p}}\mathbb{E}\left[\sup_{s\in[t,T]}\left|\overline{Y}_s^\lambda - Y_s^\lambda\right|^2|\mathcal{F}_t\right].$$

By choosing $\varepsilon$ small enough such that $C_{\delta,\gamma}\varepsilon\left(\frac{2}{2-p}\right)^{\frac{2}{p}} < 1$, we get

$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|\overline{Y}_s^\lambda - Y_s^\lambda\right|^2|\mathcal{F}_t\right] \leq C_\delta D_t^2.$$

Hence, it follows easily from (4.9) and (4.10) that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[t,T]}\left|\overline{Y}_s^\lambda - Y_s^\lambda\right|^2 + \int_t^T\left|\widehat{Z}_s^\lambda - Z_s^\lambda\right|^2\mathrm{d}s + \int_t^T\int_E\left|\widehat{U}_s^\lambda(e) - U_s^\lambda(e)\right|^2\mu(\mathrm{d}s,\mathrm{d}e)\middle|\mathcal{F}_t\right] \\
\leq &C_\delta D_t^2.
\end{aligned}$$

The proof of Lemma 3.9 is complete now. ■

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