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Weihua YANG

Supereulerian graphs, hamiltonicity of graphs and several extremal problems in graphs

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Composition du jury :

M.	Hao Li	(Directeur de thèse)
Mme.	Lianzhu Zhang	(Président du jury)
M.	Xiaofeng Guo	(Examinateurs)
M.	Yannis Manoussakis	(Examinateurs)
M.	Zdeněk Ryjáček	(Rapporteur)
Mme.	Guiying Yan	(Rapporteur)

Thèse préparée au

Département de Informatique d'Orsay

Laboratoire de Informatique (CNRS UMR 8623), Bât. 650

Université Paris-Sud

91,405 Orsay CEDEX

Dedication

This thesis is dedicated to my family, especially, my parents and my wife. All I have and will accomplish are only possible due to their love and sacrifices.

Contents

Résumé

Dans cette thèse, nous concentrons sur les sujets suivants: super-eulérien graphe, hamiltonien ligne graphes, le tolérants aux pannes hamiltonien laceabilité de Cayley graphe généré par des transposition arbres et plusieurs problèmes extrémaux concernant la (minimum et/ou maximum) taille des graphes qui ont la même propriété.

Cette thèse comprend six chapitres. Le premier chapitre introduit des définitions et indique la conclusion des résultats principaux de cette thèse, et dans le dernier chapitre, nous introduisons la recherche de furture de la thèse. Les travaux principaux sont montrés dans les chapitres 2-5 comme suit:

Dans le chapitre 2, nous explorons les conditions pour qu'un graphe soit super-eulérien.

Dans la section 1, nous prouvons que si pour tous les arcs $xy \in E(G)$, $d(x) + d(y) \geq n - 1 - p(n)$, alors G est collapsible sauf quelques bien définis graphes qui ont la propriété $p(n) = 0$ quand n est impair et $p(n) = 1$ quand n est pair. Comme corollaire, nous pourrions obtenir un caractérisation des graphes dont tous ces arcs satisfaisons $d(x) + d(y) \geq n - 1 - p(n)$.

Dans la section 2, nous caractérisons des graphes dont le dégrée minimum est au moins de 2 et le nombre de matching est au plus de 3. Utilisant cette caractérisation, nous renforçons les résultats de [93] et mentionnons une conjecture avec la papier d'avant.

Dans la section 3 de la Chapitre 2, nous concentrons sur la conjecture proposé par Chen and Lai [Conjecture 8.6 of [33]] que tous les graphes est pliable. Nous trouvons les conditions suffisantes pour que un graphe de 3-arcs connectés soit pliable.

Dans le chapitre 3, nous considérons surtout l'hamiltonien de 3-connecté ligne graphe.

Dans la première section de Chapitre 3, nous présentons plusieurs conditions pour que un ligne graphe soit hamiltonien. En particulier, nous montrons que chaque 3-connecté, essentiellement 11-connecté ligne graphe est hamiltonien-connecté. Cela renforce le résultat dans [91].

Dans la seconde section de Chapitre 3, nous montrons que chaque 3-connecté, essentiellement 10-connecté ligne graphe est hamiltonien-connecté.

Dans la troisième section de Chapitre 3, nous montrons que 3-connecté, essentiellement 4-connecté ligne graphe venant d'un graphe qui comprend au plus 9 sommets de degré 3 est hamiltonien. En plus, si un graphe G a 10 sommets de degré 3 et son ligne graphe n'est pas hamiltonien, G peut contractile à le graphe de Peterson.

Dans le chapitre 4, nous considérons arc tolérants aux pannes hamiltonicité de Cayley graphe généré par transposition arbre. Nous montrons d'abord que pour tous $F \subseteq E(Cay(B : S_n))$, si $|F| \leq n-3$ et $n \geq 4$, il existe un hamiltonien graphe dans $Cay(B : S_n) - F$ entre tous les paires de sommets qui sont dans les différents partite ensembles. De plus, nous renforçons le résultat figurant ci-dessus dans la seconde section montrant que $Cay(S_n, B) - F$ est bipancyclique si $Cay(S_n, B)$ n'est pas un star graphe, $n \geq 4$ et $|F| \leq n-3$.

Dans le chapitre 5, nous considérons plusieurs problèmes extrémaux concernant la taille des graphes.

Dans la section 1 de Chapitre 5, nous bornons la taille de sous-graphe provoqué par m sommets de hypercubes (n-cubes). Nous montrons que un sous-graphe provoqué par m (indiqué m par $\sum_{i=0}^{s} 2^{t_i}$, $t_0 = [\log_2 m]$ et $t_i = [\log_2(m - \sum_{r=0}^{i-1} 2^{t_r})]$ pour $i \geq 1$) sommets d'un hypercube a $\sum_{i=0}^{s} t_i 2^{t_i - 1} + \sum_{i=0}^{s} i \cdot 2^{t_i}$ arcs au plus. Pour ses applications, nous déterminons la m-extra arc-connectivité des hypercubes pour $m \leq 2^{[\frac{n}{2}]}$ et la g-extra-arc-connectivité ($\lambda_g(FQ_n)$) d'un hypercube FQ_n pour $g \leq n$.

Dans la section 2 de Chapitre 5, nous étudions partiellement la taille minimale d'un graphe savant son degré minimum et son degré d'arc. Pour une application, nous caractérisons quelques arc-connectés graphes limités savants leur arc nombre minimum.

Dans la section 3 de Chapitre 5, nous considérons la taille minimale des graphes satisfaisants la Ore-condition.

Mots Clés: Super-eulérien graphe, Hamiltonien cycle, Ligne graphe, Conjecture de Thomassen, Hypercube, Arc-connectivité, La théorie de graphe extrémal.

Abstract

In this thesis, we focus on the following topics: supereulerian graphs, hamiltonian line graphs, fault-tolerant hamiltonian laceability of Cayley graphs generated by transposition trees, and several extremal problems on the (minimum and/or maximum) size of graphs under a given graph property.

The thesis includes six chapters. The first one is to introduce definitions and summary the main results of the thesis, and in the last chapter we introduce the furture research of the thesis. The main studies in Chapters 2 - 5 are as follows.

In Chapter 2, we explore conditions for a graph to be supereulerian.

In Section 1, we characterize the graphs with minimum degree at least 2 and matching number at most 3. By using the characterization, we strengthen the result in [93] and we also address a conjecture in the paper.

In Section 2 of Chapter 2, we prove that if $d(x) + d(y) \geq n - 1 - p(n)$ for any edge $xy \in E(G)$, then G is collapsible except for several special graphs, where $p(n) = 0$ for n even and $p(n) = 1$ for n odd. As a corollary, a characterization for graphs satisfying $d(x) + d(y) \geq n - 1 - p(n)$ for any edge $xy \in E(G)$ to be supereulerian is obtained. This result extends the result in [21].

In Section 3 of Chapter 2, we focus on a conjecture posed by Chen and Lai [Conjecture 8.6 of [33]] that every 3-edge connected and essentially 6-edge connected graph is collapsible. We find a kind of sufficient conditions for a 3-edge connected graph to be collapsible.

In Chapter 3, we mainly consider the hamiltonicity of 3-connected line graphs.

In the first section of Chapter 3, we give several conditions for a line graph to be hamiltonian, especially we show that every 3-connected, essentially 11-connected line graph is hamilton-connected which strengthens the result in [91].

In the second section of Chapter 3, we show that every 3-connected, essentially 10-connected line graph is hamiltonian-connected.

In the third section of Chapter 3, we show that 3-connected, essentially 4-connected line graph of a graph with at most 9 vertices of degree 3 is hamiltonian.

Moreover, if G has 10 vertices of degree 3 and its line graph is not hamiltonian, then G can be contractible to the Petersen graph.

In Chapter 4, we consider edge fault-tolerant hamiltonicity of Cayley graphs generated by transposition trees. We first show that for any $F \subseteq E(Cay(B : S_n))$, if $|F| \leq n-3$ and $n \geq 4$, then there exists a hamiltonian path in $Cay(B : S_n) - F$ between every pair of vertices which are in different partite sets. Furthermore, we strengthen the above result in the second section by showing that $Cay(S_n, B) - F$ is bipancyclic if $Cay(S_n, B)$ is not a star graph, $n \geq 4$ and $|F| \leq n-3$.

In Chapter 5, we consider several extremal problems on the size of graphs.

In Section 1 of Chapter 5, we bounds the size of the subgraph induced by m vertices of hypercubes. We show that a subgraph induced by m (denote m by $\sum_{i=0}^{s} 2^{t_i}$, $t_0 = [\log_2 m]$ and $t_i = [\log_2(m - \sum_{r=0}^{i-1} 2^{t_r})]$ for $i \geq 1$) vertices of an n-cube (hypercube) has at most $\sum_{i=0}^{s} t_i 2^{t_i - 1} + \sum_{i=0}^{s} i \cdot 2^{t_i}$ edges. As its applications, we determine the m-extra edge-connectivity of hypercubes for $m \leq 2^{[\frac{n}{2}]}$ and g-extra-edge-connectivity ($\lambda_g(FQ_n)$) of the folded hypercube FQ_n for $g \leq n$.

In Section 2 of Chapter 5, we partially study the minimum size of graphs with a given minimum degree and a given edge degree. As an application, we characterize some kinds of restricted edge connected graphs with minimum edge number.

In Section 3 of Chapter 5, we consider the minimum size of graphs satisfying Ore-condition.

Keywords: Supereulerian graph, Hamiltonian cycle, Line graph, Thomassen's conjecture, Hypercube, Edge-connectivity, Extremal graph theory.

Chapter 1 Introduction

Graph theory is a very popular and interesting area of discrete mathematics. The earliest known paper on graph theory was given by E. Euler (1736), which told about the seven bridges of königsberg. In the recent decades, graph theory has developed very fast. There are many well-known problems on graph theory, e.g., Hamiltonian problem, four-color problem, Chinese postman problem, the optimal assignment problem etc.. Moreover, graph theory has wide applications to practical problems, such as chemistry, biology, computer science, communication networks.

This thesis focus on the following topics: supereulerian graphs, hamiltonian line graphs, fault-tolerant hamiltonian laceability of Cayley graphs generated by transposition trees, several extremal problems to determine the (minimum and/or maximum) size of graphs under a given graph property.

In this chapter, we give a short but relatively complete introduction. In the first section, some basic definitions and notation are given. In the second section we review the classic results on supereulerian graph and introduce our works on this topic. In the third section we review the classic results on hamiltonian line graphs and introduce our works on this topic. In the fourth section we introduce fault-tolerant hamiltonian laceability of Cayley graphs generated by transposition trees. Next, we introduce the main works on several extremal problems.

§1.1 Basic definitions and notation

A *graph* G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$, of edges, and an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = uv$, then e is said to *join* u and v; the vertices u and v are called the *ends* of e.

The ends of an edge are said to be *incident* with the edge, and vice versa. Two vertices which are incident with a common edge are *adjacent*, as are two edges which are incident with a common vertex. An edge with identical ends is called a *loop*, and an edge with distinct ends a *link*.

A graph is *simple* if it has no loops and no two of its links join the same pair of vertices. A non-simple graph is called multigraph.

A graph H is a *subgraph* of G (written $H \subseteq G$) if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$. When $H \subseteq G$ but $H \neq G$, we call H a *proper subgraph* of G. If H is a subgraph of G, G is a *supergraph* of H. A *spanning subgraph* of G is a subgraph H with $V(H) = V(G)$.

By deleting from G all loops and, for every pair of adjacent vertices, all but one link joining them, we obtain a simple spanning subgraph of G, called the *underlying simple graph* of G.

Suppose that V' is a nonempty subset of $V(G)$. The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both ends in V' is called the subgraph of G *induced* by V' and is denoted by $G[V']$; we say that $G[V']$ is an *induced subgraph* of G. The induced subgraph $G[V(G) \setminus V']$ is denoted by $G - V'$. If $V' = \{v\}$, we write $G - v$ for $G - \{v\}$.

Now suppose that E' is a nonempty subset of $E(G)$. The subgraph of G whose vertex set is the set of ends of edges in E' and whose edge set is E' is called the subgraph of G *induced* by E' and is denoted by $G[E']$; $G[E']$ is an *edge-induced subgraph* of G. The spanning subgraph of G with edge set $E(G) \setminus E'$ is written simply as $G - E'$. The graph obtained from G by adding a set of edges E' is denoted by $G + E'$. If $E' = \{e\}$, we write $G - e$ and $G + e$ instead of $G - \{e\}$ and $G + \{e\}$.

Let G_1 and G_2 be subgraph of G. We say that G_1 and G_2 are *disjoint* if they have no vertex in common, and *edge-disjoint* if they have no edge in common. The union $G_1 \cup G_2$ of G_1 and G_2 is the subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$; if G_1 and G_2 are disjoint, we sometimes denote their union by $G_1 + G_2$.

The *degree* $d_G(v)$ (If no confusion arises, simplified as $d(v)$. The same for other notation in the following.) of a vertex v in G is the number of edges of G incident with v, each loop counting as two edges. We denote $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees, respectively, of vertices of G.

A *walk* in G is a finite non-null sequence $W = v_0 e_1 v_1 e_2 v_2 \cdots e_k v_k$, whose terms are alternately vertices and edges, such that, for $1 \leq i \leq k$, the ends of e_i are v_{i-1}

and v_i. We say that W is a walk from v_0 to v_k, or a (v_0, v_k)-walk. The vertices v_0 and v_k are called the *origin* and *terminus* of W, respectively, and $v_1, \ldots, v_{k-1}$ its *internal vertices*. The integer k is the *length* of W. In a simple graph, a walk $v_0e_1v_1e_2v_2 \cdots e_kv_k$ can be specified simply by $v_0v_1 \cdots v_k$. If the edges $e_1, \ldots, e_k$ of a walk W are distinct, W is called a *trail*. If, in addition, the vertices $v_0, v_1, \ldots, v_k$ are distinct, W is called a *path*. Usually, denote the section $v_iv_{i+1} \cdots v_j$ of the path $P = v_0v_1 \cdots v_k$ by $P[v_i, v_j]$.

A walk is *closed* if it has positive length and its origin and terminus are the same. A closed trail whose origin and internal vertices are distinct is a *circuit*; and a closed path is a *cycle*. Similarly, we shall introduce eulerian trail (circuit), spanning (closed) trail, dominating trail (circuit), hamiltonian path (cycle), etc. in next section. We sometimes use the term 'path' or 'cycle' to denote a graph corresponding to a path or cycle.

Two vertices u and v of G are said to be *connected* if there is a (u, v)-path in G. Connection is an equivalence relation on the vertex set $V(G)$. Thus there is a partition of $V(G)$ into nonempty subsets $V_1, V_2, \ldots, V_w$ such that two vertices u and v are connected if and only if both u and v belong to the same set V_i. The subgraphs $G[V_1], G[V_2], \ldots, G[V_w]$ are called the *components* of G. If G has exactly one component, G is *connected*; otherwise, G is *disconnected*.

An *acyclic* graph is one that contains no cycle. A *tree* is a connected acyclic graph. A *spanning tree* of G is a spanning subgraph of G that is tree.

A *vertex cut* of G is a subset V' of $V(G)$ such that $G - V'$ is disconnected. If the vertex cut V' has only one vertex $\{v\}$, then call v as a *cut vertex*. A *k-vertex cut* is a vertex cut of k elements. If G has at least one pair of distinct nonadjacent vertices, the *connectivity* $\kappa(G)$ of G is the minimum k for which G has a k-vertex cut; otherwise, we define $\kappa(G)$ to be $|V(G)| - 1$. G is said to be *k-connected* if $\kappa(G) \geq k$.

Similarly, an *edge-cut* of G is a subset E' of $E(G)$ such that $G - E'$ is disconnected. If the edge-cut $E' = \{e\}$, then call e as a *cut-edge* or *bridge*. A *k-edge-cut* is an edge-cut of k elements. Define the *edge-connectivity* $\lambda(G)$ of G to be the minimum k for which G has a k-edge-cut. G is said to be *k-edge-connected* if $\lambda(G) \geq k$.

A vertex cut X (edge-cut) of G is *essential* if $G-X$ has at least two non-trivial components. For an integer $k > 0$, a graph G is *essentially k-connected* (*essentially k-edge-connected*) if G does not have an *essential vertex cut X* (*essential edge cut*) with $|X| < k$. In particular, the *essential edge-connectivity* of G, denote by $\lambda'(G)$, is the minimum cardinality over all essential edge-cut of G. Furthermore, an edge set F is said to be an *m-restricted edge cut* of a connected graph G if $G - F$ is disconnected and each component of $G - F$ contains at least m vertices. Let *m-restricted edge-connectivity* ($\lambda^m(G)$) be the minimum size of all m-restricted edge-cut.

In this thesis, we mainly consider simple graphs. Sometimes, we also use some properties of multigraphs. We conclude this section by introducing some special classes of graphs.

A simple graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. A *bipartite graph* is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition (X, Y) is called a *bipartition* of graph. A *complete bipartite graph* is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$. A *k-partite graph* is one whose vertex set can be partitioned into k subsets so that no edges has both ends in any one subset; a *complete k-partite graph* is one that is simple and in which each vertex is joined to every vertex that is not in the same subset.

The *line graph* of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common. From the definition of a line graph, if $L(G)$ is not a complete graph, then a subset $X \subseteq V(L(G))$ is a vertex cut of $L(G)$ if and only if X is an essential edge-cut of G.

The complete bipartite graph $K_{1,n}$ is called a *star*, and the $K_{1,3}$ is called a *claw*. A graph is *claw-free* if it contains no claw as its induced subgraph. Line graphs are claw-free.

The *n-dimensional hypercube* is the graph whose vertices are the ordered n-tuples of 0's and 1's, two vertices being joined if and only if they differ in exactly one coordinate.

Let $\mathcal{G}$ be a finite group and $X \subset \mathcal{G}$, the *Cayley graph* of $\mathcal{G}$ with respect to X is the graph $Cay(\mathcal{G}, X)$ with vertex set $\mathcal{G}$ in which g and gx are joined by an undirected edge for every $g \in \mathcal{G}$ and $x \in X$. If g and gx are joined by an undirected edge, then, we think of the edge (g, gx) as being labeled x. The more detailed properties for Cayley graphs will be given in Section 4.

§1.2 Supereulerian graphs

We organize this section as follows. We first introduce basic definitions and several related results on this topic. And then we introduce our main results on this topic.

§1.2.1 Basic definitions and backgrounds

An *eulerian trail* is a trail in a graph which visits every edge exactly once. Similarly, an *eulerian circuit* or *eulerian closed trail* is an eulerian trail which starts and ends on the same vertex. They were first discussed by Leonhard Euler while solving the famous Seven Bridges of Köigsberg problem in 1736. The problem was rather simple – the town of Königsberg consists of two islands and seven bridges. Is it possible, by beginning anywhere and ending anywhere, to walk through the town by crossing all seven bridges but not crossing any bridge twice?

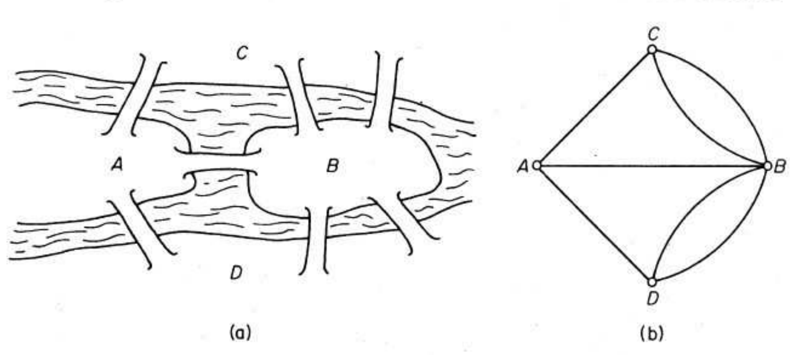


Fig. 1.1: The seven bridges and their graph.

A vertex is *odd* if its degree is odd and *even* if its degree is even. A graph is even if each of its vertices is even. A graph is *eulerian* if it is connected and even.

Theorem 1.2.1 ([52]) *An eulerian trail exists in a connected graph if and only if there are either no odd vertices or two odd vertices.*

Later, several equivalent theorems were obtained, as follows.

Theorem 1.2.2 *If G is not a trivial graph, then these are equivalent.*

(a) *G has a closed trail use each edge exactly once.*

(b) *G is eulerian.*

(c) *G is an edge disjoint union of cycles.*

(d) *The number of the sets of edges of G, each of which is contained in a spanning tree of G, is odd.*

(e) *Every edge of G lies on an odd number of cycles.*

Results (a) and (b) are equivalent due to C. Hierholzer [70] in 1873. Conditions $(c), (d)$, and (e), respectively, were due to Veblen [135], Shank [122], and Mckee's modification of Toida's theorem [128]. For the more detailed survey we refer to [25].

We call a graphs *supereulerian* if it has a spanning eulerian subgraph. Motivated by the Chinese Postman Problem, Boesch et al. [13] proposed the supereulerian graph problem: determine when a graph has a spanning eulerian subgraph. They indicated that this might be a difficult problem. Pulleyblank [119] showed that such a decision problem, even when restricted to planar graphs, is NP-complete. We refer the readers to [25, 33] for the supereulerian graph problem. So a natural problem is what conditions can guarantee a graph to be supereulerian? F. Jaeger in [80] reported the following well-known theorems.

Theorem 1.2.3 ([80]) *Let G be a graph and $F \subset E(G)$. There is an even subgraph H of G with edges F if and only if F contains no bonds (minimal edge-cuts) of G of odd cardinality.*

Theorem 1.2.4 ([80]) *If a graph contains two edge-disjoint spanning trees, then it is supereulerian.*

Tutte [134] and Nash-Williams [113] characterized the maximum number of edge-disjoint spanning trees in a given graph, which implies the following.

Theorem 1.2.5 *Every $2k$-edge-connected graph contains k-edge-disjoint spanning trees.*

The following corollary due to Jaeger [80] is clear.

Theorem 1.2.6 ([80]) *Every 4-edge-connected graph is supereulerian.*

We note that a graph with cut-edge is not supereulerian. The rest of the supereulerian problems focus on 3-edge-connected graphs and 2-edge-connected graphs. To explore the graphs with no two edge-disjoint trees, Catlin in [26] established a reduction method which was used to prove a lot of new results in this topic. We first introduce his method below.

For a graph G, let $O(G)$ denote the set of odd degree vertices of G. Given a subset R of $V(G)$, a subgraph Γ of G is called an R-subgraph if both $O(\Gamma) = R$ and $G - E(\Gamma)$ is connected. A graph G is collapsible if for any even subset R of $V(G)$, G has an R-subgraph. Note that when $R = \emptyset$, a spanning connected subgraph H with $O(H) = \emptyset$ is a spanning eulerian subgraph of G. Thus every collapsible graph is supereulerian. Catlin [26] showed that any graph G has a unique subgraph H such that every component of H is a maximally connected collapsible subgraph of G and every non-trivial connected collapsible subgraph of G is contained in a component of H. For a subgraph H of G, the graph G/H is obtained from G by identifying the two ends of each edge in H and then deleting the resulting loops. The contraction G/H is called the *reduction* of G if H is the maximal collapsible subgraph of G, i.e. there is no non-trivial collapsible subgraph in G/H. A graph G is *reduced* if it is the reduction of itself. Let $F(G)$ denote the minimum number of edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. The following summarizes some of the previous results concerning collapsible graphs.

Theorem 1.2.7 *Let G be a connected graph. Each of the following holds.*

(i)(Catlin [26]). If H is a collapsible subgraph of G, then G is collapsible if and only if G/H is collapsible; G is supereulerian if and only if G/H is supereulerian.

(ii)(Catlin, Theorem 5 of [26]). A graph G is reduced if and only if G contains no non-trivial collapsible subgraphs. As cycles of length less than 4 are collapsible, a reduced graph does not have a cycle of length less than 4.

(iii)(Catlin, Theorem 8 of [26]). If G is reduced and if $|E(G)| \geq 3$, then $\delta(G) \leq 3$, and $2|V(G)| - |E(G)| \geq 4$.

(iv)(Catlin [26]). If G is reduced and if $|E(G)| \geq 3$, then $\delta(G) \leq 3$ and $F(G) = 2|V(G)| - |E(G)| - 2$.

(v)(Catlin, Han and Lai [29]) Let G be a connected reduced graph. If $F(G) \leq 2$, then $G \in \{K_1, K_2, K_{2,t}\}(t \geq 1)$.

We have known that all complete graphs of order at least 3 are collapsible. All cycles of length at least 4 are not collapsible. Let G be a graph containing a 4-cycle $C = uvzwu$ with a partition $\pi =< \{u, z\}, \{v, w\} >$. Following Catlin [20], we define $G/\pi(C)$ to be the graph obtained from $G - E(C)$ by identifying u and z to form a vertex x, by identifying v and w to form a vertex y, and by adding an edge $e_\pi = xy$. Catlin in [20] introduced another useful technique for studying supereulerian graphs as follows.

Theorem 1.2.8 ([20]) . *Let G be a graph containing a 4-cycle C and let $G/\pi(C)$ be defined as above. Each of the following holds:*

(a) If $G/\pi(C)$ is collapsible, then G is collapsible;

(b) If $G/\pi(C)$ has a spanning Eulerian subgraph, then G has a spanning eulerian subgraph, i.e., if $G/\pi(C)$ is supereulerian, then G is supereulerian.

By using Catlin's method, several new results on 2- and/or 3-edge-connected graphs were reported around several parameters of graphs, e.g., the number of edge-cut of size at most 3, independent set, Chvatal-Erdös condition, diameter, and so on. In particular, Catlin and Lai in [30] showed the following under the restriction of the number of edge-cuts of size 3.

Theorem 1.2.9 ([30]) *A 3-edge-connected graph with at most 10 edge-cuts of size 3 is either supereulerian, or it is contractible to the Petersen graph.*

Later, Li and Lai in [97] improved the result above by showing the following.

Theorem 1.2.10 ([97]) *A 3-edge-connected graph with at most 11 edge-cuts of size 3 is either supereulerian, or it is contractible to the Petersen graph.*

Note that there are two reduced snarks of order 18. So 17 is best possible in the two theorems above. We conjecture A 3-edge-connected with at most 17 edge-cuts of size 3 is either supereulerian, or it is contractible to the Petersen graph. This problem is open. In fact, the theorems above are closely related to one kind of problems on supereulerian graphs. We discuss it below.

A minimum edge-cut of a graph G is called a bond of G. For two integers $l > 0$ and $k \geq 0$, let $C(l, k)$ denote the family of 2-edge-connected graphs such that a graph $G \in C(l, k)$ if and only if for every bond $S \subset E(G)$ with $S \leq 3$, each component of $G - S$ has order at least $(|V(G)| - k)/l$. Catlin and Li [32] first investigated the characterization of supereulerian graphs in the family of $C(5, 0)$, and they showed the following.

Theorem 1.2.11 ([32]) *Every graph in $C(5, 0)$ is either supereulerian or can be contracted to $K_{2,3}$.*

Later, Broersma and Xiong [15] improve the result above.

Theorem 1.2.12 ([15]) *If $G \in C(5, 2)$ and $|V(G)| \geq 13$, then G is either supereulerian, or G can be contracted to one of $K_{2,3}$ and $K_{2,5}$.*

Li et al. [96] continued the research and proved the following results concerning the characterization of supereulerian graphs in a $C(6, 0)$, as follows.

Theorem 1.2.13 ([96]) *If $G \in C(6, 0)$, then G is either supereulerian, or can be contracted to $K_{2,3}, K_{2,5}$ or $K'_{2,3}$, where $K'_{2,3}$ is the graph obtained from $K_{2,3}$ by replacing an edge $e \in E(K_{2,3})$ with a path of length 2.*

Similarly, Li et al. [98] reported a similar characterization on $C(6,5)$. One may note that the expectations in the results above are 2-edge-connected. So, we naturally ask what will happen if we restrict our research on the 3-edge-connected graphs. Chen [34] showed that a 3-edge-connected graph of order at most 13 is either supereulerian, or can be contracted to the Petersen graph. One may naturally conjecture a 3-edge-connected graph in $C(l,0), l \leq 13$ is either supereulerian, or can be contracted to the Petersen graph. In fact, we conjecture a 3-edge-connected graph in $C(l,0), l \leq 17$ is either supereulerian, or can be contracted to the Petersen graph. For the studies of the problems, we refer to [34, 99, 114]

Theorem 1.2.14 ([34]) *Let G be a 3-edge-connected graph. If $G \in C(10,0)$, then G is either supereulerian, or can be contracted to the Petersen graph.*

Niu et al. [114] improved the result above as follows.

Theorem 1.2.15 ([114]) *Let G be a 3-edge-connected graph with order at least $11k$. If $G \in C(10,k)$, then is either supereulerian, or can be contracted to the Petersen graph.*

Recently, Li et al. showed the following.

Theorem 1.2.16 ([34]) *Let G be a 3-edge-connected graph. If $G \in C(12,0)$, then G is either supereulerian, or can be contracted to the Petersen graph.*

Overall, the study of the supereulerian problem on $C(l,k)$ of 2- and/or 3-edge-connected graphs is still not complete. We shall list them as one of our further research topics.

We turn to another parameter–independent set of graphs. We first introduce a result due to Benhocine et al. [8].

Theorem 1.2.17 ([8]) *Let G be a 2-edge-connected graph on $n \geq 3$ vertices. If*

$$d(u) + d(v) \geq \frac{2n+3}{3}$$

whenever $uv \notin E(G)$, then G has a spanning closed trail.

Later, Catlin [27] and Chen [37] weakened the restriction as follows.

Theorem 1.2.18 ([27, 37]) *Let G be a 2-edge-connected graph with order n and girth $g \in \{3, 4\}$. If n is sufficiently large and if*

$$d(u) + d(v) \geq \frac{2}{g-2}(\frac{n}{5} - 4 + g)$$

whenever $uv \notin E(G)$, then G has a spanning closed trail.

Veldman weakened the condition above in [136] as follows.

Theorem 1.2.19 ([136]) *Let G be a connected graph with order $n \geq 5$. If*

$$d(u) + d(v) + d(w) \geq n - 1$$

for any 3-independent set $\{u, v, w\}$, then G has a spanning trail (possibly open).

The theorem above implies a spanning trail, may be not eulerian trail. So we would like to know what such kind conditions can guarantee an eulerian trail. Catlin [28] reported the following.

Theorem 1.2.20 ([28]) *Let G be a 2-edge-connected graph with order n. If*

$$d(u) + d(v) + d(w) \geq n + 1$$

for any 3-independent set $\{u, v, w\}$, then exactly one of the following holds:

(a) G is collapsible;

(b) $G \in \{C_4, C_5, K_{2,3}, G_a\}$, where G_a is a well defined graph of 6 vertices.

Besides, Chen and Xue [39] weakened the sum of the 3-independent set above to $n - 2$ for spanning trail and $n - 5$ for dominating trail.

In [36], Chen weakened the condition in Theorem 1.2.18 by using 3-independent set as follows.

Theorem 1.2.21 ([36]) *Let G be a 2-edge-connected graph with order n and girth $g \in \{3, 4\}$. Let G' be the reduction of G. If*

$$d(u) + d(v) + d(w) \geq \frac{2}{g-2}(\frac{n}{5} - 4 + g)$$

for every 3-independent set $\{u, v, w\}$ of $V(G)$, then exactly one of the following holds:

(a) *G is collapsible;*

(b) *$G' \in \{C_4, C_5\}$, and so G is supereulerian but not collapsible.*

(C) *$G' \in \{K_{2,3}, K'_{2,3}\}$, and so G is non-supereulerian.*

Naturally, when we restrict the result to the 3-edge-connected graphs, we have the following.

Theorem 1.2.22 ([36]) *Let G be a 3-edge-connected graph with order n and girth $g \in \{3, 4\}$. Let G' be the reduction of G. If n is sufficiently large and if*

$$d(u) + d(v) + d(w) \geq \frac{2}{g-2}(\frac{n}{11} - 4 + g)$$

for every 3-independent set $\{u, v, w\}$ of $V(G)$, then exactly one of the following holds:

(a) *G is supereulerian;*

(b) *G' is the Petersen graph.*

Similarly, Chen also stated the following.

Theorem 1.2.23 ([36]) *Let G be a 2-edge-connected graph with order n and girth $g \in \{3, 4\}$. Let G' be the reduction of G. If*

$$d(u) + d(v) + d(w) \geq \frac{n}{g-2} + 2(g-2)$$

for every 3-independent set $\{u, v, w\}$ of $V(G)$, then exactly one of the following holds:

(a) *G is supereulerian;*

(b) *$G' \in \{K_{2,3}, K'_{2,3}\}$.*

Theorem 1.2.24 ([36]) *Let G be a 3-edge-connected graph with order n and girth $g \in \{3, 4\}$. Let G' be the reduction of G. If n is sufficiently large and if*

$$d(u) + d(v) + d(w) \geq \frac{3}{g-2}(\frac{n}{14} - 4 + g)$$

for every 3-independent set $\{u, v, w\}$ of $V(G)$, then exactly one of the following holds:

(a) *G is supereulerian;*

(b) *G' is the Petersen graph.*

In [36], Chen also obtained a bound of matching number of the reduction of a graph. Similarly, Chen and Lai [35] bounded the matching number of the reduced graphs. The reduction of graphs of diameter 2 were characterized in [87]. Recently, Han et al. [65] studied supereulerian problem under the Chvatal-erdös condition. They showed that the graphs satisfying Chvatal-erdös condition are supereulerian but several well defined exceptions. Note that the supereulerian problem is NP-compiete. So, exploring conditions for a graph to be supereulerian is an interesting problem. In Chapter 2, we obtain several such conditions on matching number, edge-degree, and 3-restricted edge-connectivity. The main results will be introduced in the next subsection.

A *dominating closed trail* (DCT for short) is a closed trail T such that every edge has at least one end vertex on T. The following theorem turns the hamiltonian cycle (the definition will be introduced in the next section) of a line graph to the closed trail of its root graph.

Theorem 1.2.25 (Harary and Nash-Williams [66]) *Let G be a graph not a star. Then $L(G)$ is hamiltonian if and only if G has a dominating closed trail.*

Note that a spanning eulerian subgraph of a graph G is a dominating closed trail of G. Thus, the results on supereulerian graphs always imply the corresponding results on line graphs.

In this thesis, two conjectures on supereulerian graphs of Chen and Lai will be mentioned and several partial results on them will be obtained in Chapters 2 and 3 below.

Conjecture 1.2.26 (Chen and Lai Conjecture 8.6 of [33]) *Every 3-edge-connected and essentially 6-edge-connected graph is collapsible.*

Conjecture 1.2.27 (Chen and Lai Conjecture 8.7 [33]) *Every 3-edge-connected, essentially 5-edge-connected graph G is supereulerian.*

In the next section, we summarize our main results.

§1.2.2 Main results on supereulerian graphs

This subsection is corresponding to the main results in Chapter 2 of the thesis. This chapter includes three parts. The first section of the chapter was published in [P1] (we use [Pi] for ith paper in the author paper list). To understand the main result, we need the following backgrounds.

The *matching number* of a graph G is the size of the maximum matching in G, denoted by $\alpha'(G)$. We denote by $\delta(G)$ the minimum degree of G. Let m, n be two positive integers. Let $H_1 \cong K_{2,m}$ and $H_2 \cong K_{2,n}$ be two complete bipartite graphs. Let u_1, v_1 be two nonadjacent vertices of degree m in H_1, and u_2, v_2 be two nonadjacent vertices of degree n in H_2. Let $S_{n,m}$ denote the graph obtained from H_1 and H_2 by identifying v_1 and v_2, and by connecting u_1 and u_2 with a new edge u_1u_2. Note that $S_{1,1}$ is the same as C_5, the 5-cycle. Define $K_{1,3}(1,1,1)$ to be the graph obtained from a 6-cycle $C = u_1u_2u_3u_4u_5u_6u_1$ by adding one vertex u and three edges uu_1, uu_3 and uu_5.

Recently, Lai and Yan in [93] considered the supereulerian graph problem with the restriction of matching number of a graph. They characterized the non-supereulerian 2-edge-connected graphs with matching number 2, and posed a conjecture as follows.

Conjecture 1.2.28 ([93]) *If G is a 2-edge-connected simple graph with matching number at most 3, then G is supereulerian if and only if G is not one of $\{K_{2,t}, S_{n,m}, K_{1,3}(1,1,1)\}$ where n, m are natural numbers and t is an odd number.*

In the first section of Chapter 2, our main goal is to address the conjecture above. The main work of the section is as follows.

We characterize the graphs with a given small matching number. We characterize the graphs with minimum degree at least 2 and matching number at most 3. The characterization when the matching number is at most 2 strengthens the result of Lai and Yan's that characterized the non-supereulerian 2-edge-connected graphs with matching at most 2; The characterization of the graphs with matching number at most 3 addresses the conjecture of Lai and Yan in [93].

We next introduce the main result of the second section in Chapter 2, which was published in [P2]. The main work of this section is to find spanning trail in the graphs satisfying $d(x) + d(y) \geq n - 1 - p(n)$ for each edge $xy \in E(G)$, where $p(n) = 0$ for n even and $p(n) = 1$ for n odd, as follows.

Let G be a graph with $n \geq 4$ vertices. Catlin in [21] showed that if $d(x) + d(y) \geq n$ for each edge $xy \in E(G)$, then G has a spanning trail except for several defined graphs. In this work we obtain a similar result that if $d(x) + d(y) \geq n - 1 - p(n)$ for each edge $xy \in E(G)$, then G is collapsible except for several special graphs, which strengthens the result of Catlin's, where $p(n) = 0$ for n even and $p(n) = 1$ for n odd. As corollaries, a characterization for graphs satisfying $d(x)+d(y) \geq n-1-p(n)$ for each edge $xy \in E(G)$ to be supereulerian is obtained.

In the third section, we consider the Conjectures 1.2.26 and 1.2.27 under a restriction to 3-restricted edge connectivity of graphs. The results were published in [P3]. The main works are as follows.

For $e = uv \in E(G)$, define $d(e) = d(u) + d(v) - 2$ the *edge degree* of e, and $\xi(G) = \min\{d(e) : e \in E(G)\}$. Denote by $\lambda^m(G)$ the m-restricted edge-connectivity of G ($\lambda^3(G)$), and denote $D_i(G)$ the set of vertices of degree i. In this part, we prove that a 3-edge-connected graph with $\xi(G) \geq 7$, and $\lambda^3(G) \geq 7$ is collapsible; a 3-edge-connected simple graph with $\xi(G) \geq 7$, and $\lambda^3(G) \geq 6$ is collapsible; a 3-edge-connected graph with $\xi(G) \geq 6$, $\lambda^2(G) \geq 4$, and $\lambda^3(G) \geq 6$ with at most 24 vertices of degree 3 is collapsible; a 3-edge-connected simple graph with $\xi(G) \geq 6$, and $\lambda^3(G) \geq 5$ with at most 24 vertices of degree 3 is collapsible; a 3-edge-connected graph with $\xi(G) \geq 5$, and $\lambda^2(G) \geq 4$ with at most 9 vertices of degree 3 is collapsible. As a corollary, we show that a 4-connected line graph $L(G)$ with minimum degree at least 5 and $|D_3(G)| \leq 9$ is Hamiltonian.

§1.3 Hamiltonian line graphs

§1.3.1 Definitions and backgrounds

A *hamiltonian path* or *traceable path* is a path that visits each vertex exactly once. A graph that contains a hamiltonian path is called a traceable graph. A graph is $hamiltonian-connected$ if for every pair of vertices there is a hamiltonian path between the two vertices. A graph that contains a hamiltonian cycle is called a hamiltonian graph.

The hamiltonian problem, determining when a graph contains a spanning cycle, has long been fundamental in Graph Theory. Hamiltonian paths and cycles are named after William Rowan Hamilton who invented the Icosian Game. In 1856, Hamilton invented a mathematical game, the Icosian Game, consisting of a dodecahedron. Each of its twenty vertices was labeled with the name of a city and the problems was to find a hamiltonian cycle in this dodecahedron graph, to make a Voyage around the world.

The hamiltonian problem (graph theory) has been studied widely as one of the most important problems in graph theory. Determining whether hamiltonian cycles exist in graphs is NP-complete. Therefore it is natural and very interesting to study sufficient conditions for hamiltonicity.

On the hamiltonian problem, one may find many well known theorems in any graph theory textbook. So it is not necessary and impossible to give a detailed survey in this thesis. In particular, Li recently in [102] surveyed the results due to Dirac's theorem which started a new approach to develop sufficient conditions on degrees for a graph to be hamiltonian; For the survey on the hamiltonian problem on Cayley graphs, we refer to [140] by Witte and Gallian; Gould gave two nice surveys in [63, 64] in which contains many problems on generalizations of hamiltonian problem; Bauer et al. [7] gave a survey which focus on the toughness of graphs and the hamiltonian problem. We suggest the readers to refer to the surveys for different topics of the hamiltonian problem [9, 12, 112].

One of the important topics in hamiltonian graph theory is the hamiltonicity of claw-free graphs (i.e., a graph is called claw-free if it contain no induced claw,

$K_{1,3}$). Before we introduce our main results, we start by presenting a conjecture by Matthews and Sumner, as follows.

Conjecture 1.3.1 (Matthews and Sumner [109]) *Every 4-connected claw-free graph is hamiltonian.*

It is well known that every line graph is claw-free. Thomassen in 1986 posed the following conjecture:

Conjecture 1.3.2 (Thomassen [125]) *Every 4-connected line graph is hamiltonian.*

The line graph transformation is probably the most interesting of all graph transformations, and it is certainly the most widely studied. Much of this activity was stimulated by Ore's discussion of line graphs, and problems about them, in [116]. The line graph concept is quite natural, and has been introduced in several ways, see Whitney [139]. In the third chapter of this thesis we only concentrate on the hamiltonian line graphs.

Note that Conjecture 1.3.1 is stronger that Conjecture 1.3.2 since line graphs are claw-free. Herbert Fleischner asked whether they are equivalent? To answer the question, Ryjáček introduced a closure operation (It is based on adding edges without destroying the (non)hamiltonicity (similar to the Bondy-Chvátal closure for graphs with nonadjacent pairs with high degree sums). The edges are added by looking at a vertex v and the subgraph of G induced by $N(v)$ (neighborhood of v in G). If $G[N(v)]$ is connected and not a complete graph, all edges are added to turn $G[N(v)]$ into a complete graph. This procedure is repeated in the new graph, etc., until it is impossible to add any more edges. The resulting graph is called the closure of G, or simply $R-closure$ of G, denoted by $cl(G)$. By using the R-closure, Ryjáček [120] reported the following.

Theorem 1.3.3 ([120]) *Let G be a claw-free graph. Then*

(1) *the closure $cl(G)$ is uniquely determined;*

(2) *$cl(G)$ is hamiltonian if and only if G is hamiltonian;*

(3) *$cl(G)$ is the line graph of a triangle-free graph.*

In fact, Conjectures 1.3.1 and 1.3.2 are also equivalent to several other conjectures. Recall Theorem 1.2.25, one can see that Conjectures 1.3.1 and 1.3.2 are equivalent to the following.

Conjecture 1.3.4 *Every essentially 4-edge-connected graph has a DCT.*

If H is cubic, i.e., 3-regular, then a DCT becomes a dominating cycle (abbreviated DC). H is *cyclically* 4-edge-connected if H contains no edge-cut R such that $|R| < 4$ and at least two components of $H - R$ contain a cycle. It is not difficult to show that a cubic graph is essentially 4-edge-connected if and only if it is cyclically 4-edge-connected. Hence the following conjecture due to Ash and Jackson [5] is a specialization of Conjecture 1.3.4 to cubic graphs. By using inflation operation introduced by Fleischner and Jackson in [59], one can see that the following also implies Conjecture 1.3.4. Thus, they are equivalent.

Conjecture 1.3.5 ([59]) *Every cyclically 4-edge-connected cubic graph has a DC.*

Plummer [118] observed that Conjecture 1.3.5 is equivalent to the following two specializations of Conjecture 1.3.1.

Conjecture 1.3.6 ([118]) *Every 4-connected 4-regular claw-free graph is hamiltonian.*

Conjecture 1.3.7 ([118]) *Every 4-connected 4-regular claw-free graph in which each vertex lies on exactly two triangles is hamiltonian.*

A further restriction to cyclically 4-edge-connected cubic graphs that are not 3-edge-colorable, is due to Fleischner [58] who posed the following equivalent conjecture.

Conjecture 1.3.8 ([58]) *Every cyclically 4-edge-connected cubic graph that is not 3-edge-colorable has a DC.*

Furthermore, Broersma et al. proved that the following conjecture is also equivalent to others, as shown in [16].

Conjecture 1.3.9 ([16]) *Every snark has a dominating cycle.*

For Conjecture 1.3.2, the first nice result was due to Zhan [159], as follows.

Theorem 1.3.10 ([159]) *Every 7-connected line graph is hamiltonian-connected.*

By using R-closure, Ryjáček proved the following.

Theorem 1.3.11 ([120]) *Every 7-connected claw-free graph is hamiltonian.*

Very recently, an important progress towards the conjectures was obtained by Kaiser and Vrána [81] in which the following theorem is listed:

Theorem 1.3.12 ([81]) *Every 5-connected line graph with minimum degree at least 6 is hamiltonian.*

So we clearly have:

Corollary 1.3.13 *Every 6-connected line graph is hamiltonian.*

Using Ryjáček's line graph closure, the following corollary is obtained:

Corollary 1.3.14 ([81]) *Every 5-connected claw-free graph G with minimum degree at least 6 is hamiltonian.*

The above theorems are based on the restriction of high connectivity. Several authors considered Conjecture 1.3.2 under the restriction of the number of vertices of degree 3 in G. For example, Chen et al. in [38] reported that every 4-connected line graph $L(G)$ with $D_3(G) = \emptyset$ is hamiltonian. Li in [101] proved that every 6-connected claw-free graph with at most 33 vertices of degree 6 is hamiltonian. Let G be a 6-connected line graph. Hu, Tian and Wei in [79] showed that if $d_6(G) \leq 29$ or $G[D_6(G)]$ contains at most 5 vertex disjoint K_4's, then G is hamiltonian-connected. Let G be a 6-connected claw-free graph. Hu, Tian and Wei in [78] showed that if $d_6(G) \leq 44$ or $G[D_6(G)]$ contains at most 8 vertex disjoint K_4's, then G is hamiltonian. Let G be a 6-connected line graph. Zhan in [160] showed that if either $d_6(G) \leq 74$, or $d_6(G) \leq 54$ or $G[D_6(G)]$ contains at most 5 vertex disjoint K_4's, then G is hamiltonian.

We now turn to the 4-connected line (claw-free) graphs. Lai [88] considered the line graph of planar graphs as follows.

Theorem 1.3.15 ([88]) *Every 4-connected line graph of a planar graph is hamiltonian.*

Kriesell [85] considered the line graph of claw-free graphs as follows.

Theorem 1.3.16 ([85]) *All 4-connected line graphs of claw-free graphs are hamiltonian-connected.*

Lai et al. improved the result above to the following.

Theorem 1.3.17 ([89]) *Every 4-connected line graph of a quasi claw-free graph is hamilton- connected.*

Another topic on hamiltonicity of 4-connected line graphs is due to the restriction of forbidden subgraphs. A graph is called H-free if it contains no H as its induced subgraph. The forbidden subgraphs involving in hamiltionian line (claw-free) graph theme includes: Z_k (denote a graph obtained from the disjoint union of a P_{k+1} and a 3-cycle K_3 by identifying one endvertex of P_{k+1} with one vertex of K_3.), the generalized bull (denote by $B_{s,t}$, to be the graph obtained by attaching each of some two distinct vertices of a triangle to an end vertex of one of two vertex-disjoint paths of orders s and t.), net (denoted by N_{s_1,s_2,s_3}, to be the graph obtained by identifying each vertex of a K_3 with an end vertex of three disjoint paths $P_{s_1+1}, P_{s_2+1}, P_{s_3+1}$, respectively.) and so on. We refer to [63, 64] for the old results on this topic. We next summary several recently results on this topic.

In 1999, Brousek, Ryjáček and Favaron proved the following.

Theorem 1.3.18 ([18]) *Every 3-connected claw-free and Z_4-free graph is hamiltonian.*

Later, Lai et al. [92] improved the result above by the following.

Theorem 1.3.19 ([92]) *Every 3-connected claw-free and Z_8-free graph is hamiltonian.*

In [92], Lai et al. posed a conjecture which was addressed by Fujisawa [61].

Theorem 1.3.20 *Every 3-connected claw-free and Z_9-free graph G is hamiltonian unless G is the line graph of Q^*, where Q^* denote the graph obtained from the Petersen graph by adding one pendant edge to each vertex.*

Fujisawa with the other author Chiba [41] generalized the result above to 3-connected $B_{s,9-s}$-free and claw free graphs.

Theorem 1.3.21 ([41]) *Every 3-connected $B_{s,9-s}$-free line graph is hamiltonian for $1 \leq s \leq 4$.*

Very recently, Xiong et al. [142] considered claw-free graphs as follows (we state part of the original results, the complete results include the discussion for $B_{s,9-s}$-free claw-free graphs, and the discussion for N_{s_1,s_2,s_3}-free graph claw-free graphs when $s_i > 0, s_1 + s_2 + s_3 = 10$).

Theorem 1.3.22 ([142]) *Every 3-connected claw-free and N_{s_1,s_2,s_3}-free graph G is hamiltonian for $s_i > 0, s_1 + s_2 + s_3 \leq 9$.*

Luczak and F. Pfender [106] showed that every P_{11}-free claw free 3-connected graph is hamiltonian. The result was improved to P_{12}-free by Ma et al. [107].

So far, several different topics are mentioned on hamiltonian problem of line graph (i.e., claw-free graph). The following one lead to our research in this thesis. Lai et al in [91] considered the following problem:

Question 1 *For 3-connected line graphs, can high essential connectivity guarantee the existence of a hamiltonian cycle ?*

They proved the following theorem:

Theorem 1.3.23 (Lai et al. [91]) *Every 3-connected, essentially 11-connected line graph is hamiltonian.*

In Chapter 3, we consider what is the minimum integer k such that every 3-connected, essentially k-connected line graph is hamiltonian. In the next section, we summary our main results in Chapter 3.

§1.3.2 Main results

To discuss the main results of the third chapter, several new notations are needed. Esfahanian in [51] proved that if a connected graph G with $|V(G)| \geq 4$ is not a star $K_{1,n-1}$, then $\lambda'(G)$ exists and $\lambda'(G) \leq \xi(G)$. Thus, an essentially k-edge connected graph has edge-degree at least k. Denote $D_i(G)$ and $d_i(G)$ the set of vertices of degree i and $|D_i(G)|$, respectively. If no confusion arises, we directly use D_i and d_i for $D_i(G)$ and $d_i(G)$, respectively. For a subgraph $A \subseteq G$, $v \in V(G)$, $N_G(v)$ denotes the set of the neighbors of v in G and $N_G(A)$ denotes the set $(\bigcup_{v \in V(A)} N_G(v)) \setminus V(A)$. If no confusion arises, we use an edge uv for a subgraph with three elements of $\{u, v, uv\}$. Denote $G[X]$ the subgraph induced by the vertex set X of $V(G)$.

The main results of Chapter 3 includes three parts. In the first section of Chapter 3, we consider the hamiltonian-connectedness of 3-connected and essentially 11-connected line graphs. The result of this part was published in [P4]. The main results are as follows.

(1) Every 3-edge-connected, essentially 6-edge-connected graph with edge-degree at least 7 is collapsible. (2) Every 3-edge-connected, essentially 5-edge-connected graph with edge-degree at least 6 and at most 24 vertices of degree 3 is collapsible which implies that 5-connected line graph with minimum degree at least 6 of a graph with at most 24 vertices of degree 3 is hamiltonian. (3) Every 3-connected, essentially 11-connected line graph is hamiltonian-connected which strengthens the result in [Every 3-connected, essentially 11-connected line graph is Hamiltonian, Journal of Combinatorial Theory, Series B 96 (2006) 571-576] by Lai et al. (4) Every 7-connected line graph is Hamiltonian-connected which is proved

by a method different from Zhan's. By using the multigraph closure introduced by Ryjáček and Vrána which turns a claw-free graph into the line graph of a multigraph while preserving its hamiltonian-connectedness, the results (3) and (4) can be extended to claw-free graphs.

In the second section of Chapter 3, we show that every 3-connected, essentially 10-connected line graph is hamiltonian-connected. Using the line closure introduced by Ryjáček, we have that every 3-connected, essentially 10-connected claw-free graph is hamiltonian. The result of this part was published in [P5].

In the third section of Chapter 3, we consider the hamiltonicity of 3-connected and essentially 4-connected line graphs. The result of this part was published in [P6]. The main results are as follows.

We show the conjecture (every 3-connected, essentially 4-connected line graph is hamiltonian) posed by Lai et al is not true and there is an infinite family of counterexamples; we show that 3-connected, essentially 4-connected line graph of a graph with at most 9 vertices of degree 3 is hamiltonian; examples show that all conditions are sharp. Moreover, if G has 10 vertices of degree 3 and $L(G)$ is not hamiltonian, then G can be contracted to the Petersen graph.

§1.4 Fault-tolerant hamiltonian laceability of Cayley graphs generated by transposition trees

§1.4.1 Basic definitions and background

We would like to avoid the related topic of fault-tolerant hamiltonicity of general graphs. The main research in this part is due to the stimulation of several studies of computer scientists. As the Cayley graph is widely applied in the design of networks, several families of Cayley graphs have been received much attention. Cayley graphs generated by transposition trees is one of important families.

The interconnection network of a communication or distributed computer system is usually modeled by a (directed) graph in which the vertices represent the switching elements or processors and communication links are represented by (directed) edges. Clearly, properties of the (directed) graph determine how efficiently the system can run. Thus, the ones hope a graph has higher connectivity which increases the fault tolerance, smaller diameter which reduces the transmission delay, symmetry (vertex-and edge-transitivity) which reduce design and operation costs, recursive structure which simplifies design scheme. Another desired property is hamiltonicity of a graph which are widely applied to design computer algorithm, for example, broadcasting algorithms, gossiping algorithms and sorting algorithms.

Let S_n denote the symmetric group on $\{1, \cdots, n\}$, $(p_1p_2\cdots p_n)$ denote the permutation $\begin{pmatrix} 1\ 2\ \cdots n \\ p_1p_2\cdots p_n \end{pmatrix}$, and (ij) denote the permutation $\begin{pmatrix} 1\ 2\ \cdots i \cdots j \cdots n \\ 1\ 2 \cdots j \cdots i \cdots n \end{pmatrix}$ (it is obtained by exchanging the i^{th} and j^{th} objects of $\begin{pmatrix} 1\ 2\ \cdots n \\ 1\ 2 \cdots n \end{pmatrix}$) which is called a *transposition*. It is easy to see that $(p_1 \cdots p_i \cdots p_j \cdots p_n)(ij) = (p_1 \cdots p_j \cdots p_i \cdots p_n)$. Let B be a minimal generating set of S_n. If the minimal generating set B is a set of transpositions of S_n is called *minimal transposition generating set*. In this paper, we always assume the minimal generating set is minimal transposition generating set. For the convenience of discussion, we describe the the minimal transposition generating set B of S_n by a *transpositon generating graph*, written T_B, where the vertex set of T_B is $\{1, 2, \cdots, n\}$, the edge set is $\{(ij)|(ij) \in B\}$. Note that B is a minimal transposition generating set of S_n, it can be seen that T_B is a tree which is called *transposition generating tree*, see [62] for the details.

Cayley graph $Cay(S_n, B)$ is called *Cayley graph grenerated by transposition generating tree* if B is a minimal transposition generating set of S_n. It is well known that the Cayley graph $Cay(S_n, B)$ is called *star graph* and *bubble-sort graph* if $T_B \cong K_{1,n-1}$ and $T_B \cong P_n$ (P_n is a path with n vertices), respectively.

For a bipartite graph G, Wong introduced the following definition in [141]. A bipartite graph G is *hamiltonian laceable* if there is a hamiltonian path between every pair of vertices which are in different partite sets [141].

Since edge faults may happen when a network is put in use, it is significant to consider faulty networks. So, fault-tolerance ability is a very important factor of interconnection networks [76].

A bipartite graph G is *k-edge fault tolerant hamiltonian laceable* if $G - F$ is hamiltonian laceable for any $F \subseteq E(G)$ with $|F| \leq k$, where set F is called *$fault\ edge\ set$*, edges of F are called fault edges, and edges of $E(G) - F$ are called fault-free edges. A graph G is *pancyclic* if it contains cycles of all length l, $3 \leq l \leq |V(G)|$, see [82] for the details. We say that G is *vertex-pancyclic* if, for each vertex v of G and for every integer l with $3 \leq l \leq n$, there is an l-cycle that contains v. Furthermore, a graph is called *edge-pancyclic* if every edge lies on an l-cycle for every $3 \leq l \leq n$. Since bipartite graphs have no odd cycle, a bipartite graph G is called *bipancyclic* if it contains cycles of all even length from 4 to $|V(G)|$. For bipartite graphs, vertex-bipancyclicity and edge-bipancyclicity are defined similarly. A bipartite graph G is called *k-edge fault-tolerant bipancyclic* if $G - F$ remains bipancyclic for any set F of edges with $|F| \leq k$, where F is called a $fault\ edge\ set$ of the $fault\ edge$.

Hamiltonicity of graphs with fault edges have been studied by many authors, for examples, the works on hypercubes can be found in [4, 77, 129, 130], the works on star graphs and bubble-sort graphs can be found in [3, 60, 73, 100, 82, 131], the works on alternating group graphs and butterfly graphs can be found in [132, 141] etc..

In particular, Li et al. [100] and Araki et al. [3] showed that the n-dimensional star graph and n-dimensional bubble-sort graph are $(n-3)$-edge fault tolerant hamiltonian laceable, respectively. A common generalization of star graphs and bubble-sort graphs are studied in [2, 40, 62, 84, 127, 150], which is the class of Cayley graphs generated by transition trees. Motivated by the proofs of [3, 100], we consider edge fault-tolerant hamiltonicity of Cayley graphs generated by transposition trees.

In Chapter 4, we consider edge fault-tolerant hamiltonicity of Cayley graphs generated by transposition trees. We summary our main results in Chapter 4 as follows.

§1.4.2 Main results

The main results of Chapter 4 includes two parts. The result of the first section of Chapter 4 was published in [P7], in which we show that for any $F \subseteq$

$E(Cay(B : S_n))$, if $|F| \leq n-3$ and $n \geq 4$, then there exists a hamiltonian path in $Cay(B : S_n) - F$ between every pair of vertices which are in different partite sets. The result is optimal with respect to the number of edge faults.

The second section of Chapter 4 is included in a manuscript [P8]. In this section we strengthen the above result by showing that $Cay(S_n, B) - F$ is bipancyclic if $Cay(S_n, B)$ is not a star graph, $n \geq 4$ and $|F| \leq n-3$.

§1.5 Several extremal problems

Determining the minimum and/or maximum size of graphs with some given property is a classical problem in extremal graph theory. In Chapter 5, we consider several such problems of graphs. In particular, the hypercube (n-cube) is an interesting combinatorial structure, and there are many interesting conjectures on hypercubes. We would like to mention two conjectures due to Erdös. In [47], he conjectured the maximum size ($ex(Q_n, C_4)$) of subgraphs of an n-cube with no 4-cycle is $(\frac{1}{2} + o(1))e(Q_n)$ (note that $e(Q_n) = n2^{n-1} = |E(Q_n)|$). The conjecture stimulates many authors to study such problems. The best upper bound, $(0.6226 + o(1))n2^{n-1}$, is due to Thomason and Wagner [126], while Brass, Harborth and Nienborg [14] showed $\frac{1}{2}(n+\sqrt{n})2^{n-1} \leq ex(Q_n, C_4)$, when n is a positive integer power of 4, and $\frac{1}{2}(n + 0.9\sqrt{n})2^{n-1} \leq ex(Q_n, C_4)$ for all $n \geq 9$.

Furthermore, several authors generalized the problem to C_l for $l \geq 6$. Define $c_l := \lim_{n\to\infty} ex(Q_n, C_l)/e(Q_n)$. It is known that $1/3 \geq c_6 < 0.3941$ by Conder [44], $c_{4k} = 0$ for any integer $k \geq 2$ by Chung [42] and $c_{4k+2} \leq \frac{1}{\sqrt{2}}$ for $k \geq 1$ by Axenovich and Martin [6].

The other conjecture posed by Erdös and Guy [48] is the crossing number of the hypercube $cr(Q_n)$ satisfying $\lim \frac{cr(Q_n)}{4^n} = \frac{5}{32}$. The recently progress we refer to [108] by Madej and [53] by Faria.

Some other similar problems on hypercubes we refer to [6, 126] and we refer to [56, 83, 138, 155, 156, 157] for the studies related to the structure of hypercubes. We omit the detailed survey for these problems, as the topic is not the main focus of this thesis.

We shall consider the following extremal problem on n-cubes: How many edges can a subgraph of hypercube of m vertices contain in an n-cube have? In Section 1, we show that a subgraph induced by m (denote m by $\sum\limits_{i=0}^{s} 2^{t_i}$, $t_0 = [\log_2 m]$ and $t_i = [\log_2(m - \sum\limits_{r=0}^{i-1} 2^{t_r})]$ for $i \geq 1$) vertices of an n-dimensional hypercube has at most $\sum\limits_{i=0}^{s} t_i 2^{t_i - 1} + \sum\limits_{i=0}^{s} i \cdot 2^{t_i}$ edges.

As its applications, we determine the m-extra edge-connectivity (m-restricted edge-connectivity) of hypercubes for $m \leq 2^{[\frac{n}{2}]}$, and m-extra-edge-connectivity ($\lambda_m(FQ_n)$) of the folded hypercube FQ_n for $m \leq n$. The results on hypercubes was published in [P9]. In particular, the result on folded hypercubes will appear in [P10], in which we generalize the main result in [On reliability of the folded hypercubes. Information Sciences 177 (2007) 1782-1788.] on the 3-extra edge-connectivity of folded hypercubes. and it also implies the results in [On Extra Connectivity and Extra Edge Connectivity Measures of Folded Hypercubes, IEEE Transactions on Computers, Jan. 2013 DOI.10.1109/TC.2013.10.] on the 4-extra edge-connectivity of folded hypercubes.

In Section 2, we consider what is the minimum size of graphs with a given order n, a given minimum degree δ and a given minimum edge-degree $2\delta + k - 2$? In this thesis (results are included in [P11]), we partially answer the question for $k = 0, 1, 2$ and characterize the corresponding extremal graphs.

As an application, we characterize some kinds of restricted edge-connected graphs with minimum edge number. In fact, characterizing the extremal graphs with the given connectivity is a classical topic in graph theory. For example, In [45], Cozzens and Wu studied the minimum critically k-edge connected graph (a k-edge connected graph is critical if $\lambda(G - \{v\}) < k$ for any vertex $v \in V(G)$). In [110], Maurer and Slater defined the k-edge$^\sharp$-connectivity which is the other version of restricted edge connectivity and characterized the cases $k = 1, 2, 3$. Later, Peroche and Virlouvet gave some partial results on the cases $k = 4, 5$. Recently, Hong et al [72] showed that the minimally restricted k-edge connected graph is λ'-optimal (A restricted k-edge connected G is called *minimally restricted k-edge connected* if $\lambda'(G - e) < k$ for any edge e.). Here, we give some partial results on the minimum restricted edge connected graphs.

In Section 3, we consider the minimum size of graphs satisfying Ore condition, which is included in [P12].

Chapter 2 Supereulerian graphs

§2.1 A note on supereulerian graphs in terms of a given small matching number

The *matching number* of a graph G is the size of the maximum matching in G, denoted by $\alpha'(G)$. We denote by $\delta(G)$ the minimum degree of G. Let m, n be two positive integers. Let $H_1 \cong K_{2,m}$ and $H_2 \cong K_{2,n}$ be two complete bipartite graphs. Let u_1, v_1 be two nonadjacent vertices of degree m in H_1, and u_2, v_2 be two nonadjacent vertices of degree n in H_2. Let $S_{n,m}$ denote the graph obtained from H_1 and H_2 by identifying v_1 and v_2, and by connecting u_1 and u_2 with a new edge u_1u_2. Note that $S_{1,1}$ is the same as C_5, the 5-cycle. Define $K_{1,3}(1,1,1)$ to be the graph obtained from a 6-cycle $C = u_1u_2u_3u_4u_5u_6u_1$ by adding one vertex u and three edges uu_1, uu_3 and uu_5.

Recently, Lai and Yan in [93] considered supereulerian graph problem with the restriction of matching number of a graph and posed two conjectures as follows.

Conjecture 2.1.1 *If G is a 2-edge-connected simple graph with matching number at most 3, then G is supereulerian if and only if G is not one of $\{K_{2,t}, S_{n,m}, K_{1,3}(1,1,1)\}$ where n, m are natural numbers and t is an odd number*

Conjecture 2.1.2 *If G is a 3-edge-connected simple graph with matching number at most 5, then G is supereulerian if and only if G is not contractible to the Petersen graph.*

In [1], An and Xiong pointed that the second conjecture is a corollary of a result in Chen [35] and they also pointed that the first conjecture is not true by giving some counterexamples. They revised the first conjecture as a new conjecture in [1], but the revision is not complete yet. We do not list it here, see [1].

In this note we first obtain a characterization for graphs with minimum degree 2 and matching number at most 2 that strengthens the result of Lai and Yan in [93] (the main result in [93] is that every 2-edge-connected graph with $\alpha'(G) \leq 2$ is supereulerian if and only if G is not $K_{2,t}$ for some odd number t); Similarly, a characterization of the graphs with minimum degree at least 2 and matching number at most 3 is obtained which addresses the problem raised by Conjecture 1.

We first consider the characterization of graphs with minimum degree at least 2 and matching number at most 2.

In this section we characterize the graphs with minimum degree 2 and matching number at most 2. For a graph G, a cycle of G is called *dominating cycle* if the cycle contains at least one endvertex of any edge of G.

Theorem 2.1.3 *Let G be a graph with minimum degree at least 2 and maximum matching number at most 2. Then $G \in \mathcal{F}_1 = \{G : \delta(G) \geq 2$ and $|V(G)| \leq 5\} \cup \{K_{2,t}\} \cup \{K'_{2,t}\}$, where $K'_{2,t}$ is obtained from $K_{2,t}$ by adding an edge between the two vertices of degree t.*

Proof. Suppose C is the longest cycle of G with length l. If $l \leq 3$, then by $\delta(G) \geq 2$ and $\alpha'(G) \leq 2$ we clearly have either $G \cong K_3$ or G isomorphic to the hourglass, where the hourglass is a graph obtained from K_5 by removing the edges of a C_4. Thus, $G \in \mathcal{F}_1$. Note that $\alpha'(G) \leq 2$, then $l \leq 5$.

Case 1. $l = 5$.

We claim that C is a dominating cycle of G, since other cases will induce $\alpha'(G) \geq 3$. Note that G is connected. If $V(G - C) \neq \emptyset$, the case will induce a matching of size 3. Thus, $|V(G)| = 5$ and then $G \in \{G : \delta(G) \geq 2$ and $|V(G)| \leq 5\} \subseteq \mathcal{F}_1$.

Case 2. $l = 4$.

Suppose $C = x_1x_2x_3x_4x_1$. Clearly, C is a dominating cycle, since otherwise it will induce $\alpha'(G) \geq 3$. If $V(G) = V(C)$, then $G \in \mathcal{F}_1$. Otherwise, let $v \in V(G-C)$. Since $\delta(G) \geq 2$, v has exactly two neighbors in $V(C)$ (otherwise will induces a cycle of length 5). Since $\alpha'(G) \leq 2$, all the vertices in $V(G-C)$ have the same neighbors in $V(C)$. So we clearly have either $G \cong K_{2,t}$ or $G \cong K'_{2,t}$. □

Lemma 2.1.4 *A graph G with $\delta(G) \geq 2$ and $|V(G)| \leq 5$ is either supereulerian or $G = K_{2,3}$. Moreover, $K_{2,t}$ is supereulerian if t is an even number, and non-supereulerian otherwise.*

Proof. It is easy to obtain the first part by considering its longest cycle. The second part is obvious. □

Corollary 2.1.5 (Theorem 2 [93]) *If G is a 2-edge-connected simple graph with matching number at most 2, then G is supereulerian if and only if G is not $K_{2,t}$ for some odd number t.*

Proof. By the Observation 2.1.4, the corollary holds since $K'_{2,t}$ is supereulerian. □

In the following, we consider the graphs with matching number at most 3.

We first introduce some special graphs as follows.

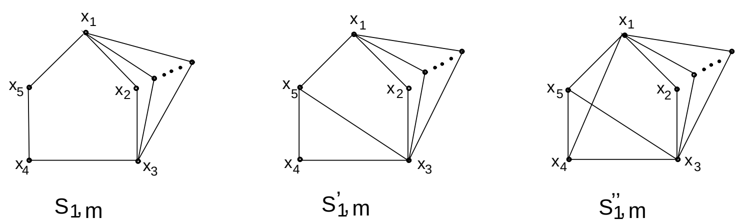


Fig. 2.1: The case $n = 1$

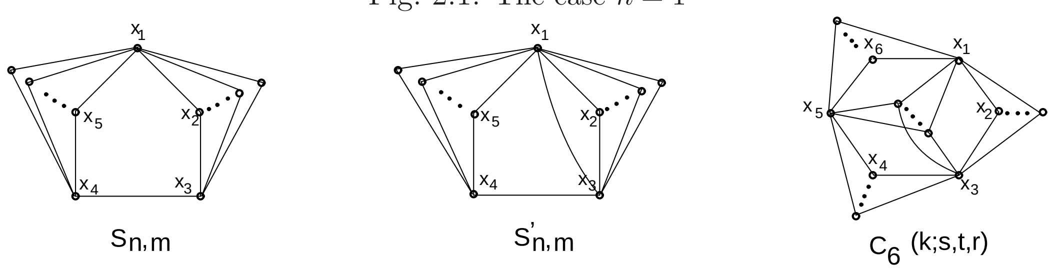


Fig. 2.2: The case $m \geq n \geq 2$

The following four graphs below are the simple variants of $S_{n.m}$: $S^*_{1,m} = S_{1,m} + x_1x_3, S'^*_{1,m} = S'_{1,m} + x_1x_3, S''^*_{1,m} = S'_{1,m} + x_1x_3, S''_{n,m} = S'_{n,m} + x_1x_4$. Similarly, $C_6(k; s, t)$ contains the following three variants that obtained by adding edges $\{x_1x_3\}, \{x_1x_3, x_3x_5\}$ and $\{x_1x_3, x_3x_5, x_5x_1\}$, respectively. We use $\mathcal{C}$ to denote the set of $C_6(k; s, t, r)$ and its three variants mentioned above. Let $\mathcal{S} = \{S_{1,m}, S'_{1,m}, S''_{1,m}, S^*_{1,m}, S'^*_{1,m}, S''^*_{1,m}, S_{n,m}, S'_{n,m}, S''_{n,m}, K_{2,t}, K'_{2,t}\}$

Theorem 2.1.6 *Let G be a graph with minimum degree 2 and matching number at most 3. Then G is in $\digamma_2 = \{G : |V(G)| \leq 7\} \cup \mathcal{S} \cup \mathcal{C}$.*

Proof. Suppose C is the longest cycle of G with length l. Similarly as in Theorem 4, we have $|V(G)| = 7$ if $l = 7$ and thus $G \in \digamma_2$.

If $l = 3$, then $G \in \{K_3, H, H', H''\}$, where H denotes the hourglass, H', H'' denote the two different graphs obtained from a hourglass and a triangle by identifying a vertex of a hourglass and a vertex of a triangle. Each of the cases implies $|V(G)| \leq 7$ and then $G \in \digamma_2$. Thus we may assume $l \geq 4$.

Case 1. $l = 6$

Let $C = x_1x_2x_3x_4x_5x_6x_1$ be a longest cycle of G. Clearly, C is a dominating cycle of G. Suppose $V(G - C) \neq \emptyset$. Since $\alpha'(G) \leq 3$ and $l = 6$, at most one of the two endvertices of an edge $ab \in E(C)$ has neighbors in $V(G - C)$. Thus at most three pairwise nonadjacent vertices of $V(C)$ have neighbors in $V(G-C)$ and assume that they are x_1, x_3, x_5. Suppose there are k vertices of $V(G-C)$ such that each of them is only adjacent to vertices of x_1, x_3, x_5. In this case, it is easy to see that $G \in \mathcal{C}$.

Case 2. $l = 5$

Let $C = x_1x_2x_3x_4x_5x_1$ be a longest cycle of G. Note that $l = 5$ and $\delta(G) \geq 2$. Then if C is not a dominating cycle of G, then G is the graph obtained from a cycle C_5 (a cycle of length 5) and a triangle by identifying a vertex of the C_5 and a vertex of triangle and adding some edges on the C_5. In this case, we have $|V(G)| \leq 7$. Suppose C is a dominating cycle of G. It is easy to see that G contains a $S_{n,m}$ as a subgraph for some m, n. If $m \geq n \geq 2$, then $G \in \{S_{n,m}, S'_{n,m}, S''_{n,m}\}$. If $n = 1$, then $G \in \{S_{1,m}, S'_{1,m}, S''_{1,m}\}$, see Fig. 2.3.

Case 3. $l = 4$

Let $C = x_1x_2x_3x_4x_1$ be a longest cycle of G. Similarly, if C is not a dominating cycle, then $|V(G)| \leq 7$. If C is a dominating cycle, we have $G \cong K_{2,t}$ or $G \cong K'_{2,t}$ for some integer t. □

Note that $S^*_{1,m}, S'^*_{1,m}, S''^*_{1,m}$, and $S''_{n,m}$ are supereulerian. If $G \cong C_6(k; s, t, r)$, $k = 1, s, t, r \geq 1$ and the parities of s, t, r are the same, then G is not supereulerian.

In fact, G has 4 vertices of odd degree and only one edge can be removed. So G is not supereulerian. If $k \geq 2$, the $k-1$ vertices of degree 3 can be used to adjust the parities of s, t, r such that one of them is different from others, then the resulting is supereulerian clearly. So we have the following theorem.

Theorem 2.1.7 *Let G be a graph with minimum degree 2 and $\alpha'(G) \leq 3$. Then G is not supereulerian if and only if one of the following holds:*

(1) *If $G \cong S_{n,m}, m \geq n \geq 1$, then one of n, m is an even number;*

(2) *If $G \in \{S'_{1,m}, S''_{1,m}\}$, then m is even number;*

(3) *If $G \cong S'_{n,m}, m \geq n \geq 2$, then x_4 is a vertex of odd degree.*

(4) *If $G \cong C_6(k; s, t, r)$, then $k = 0$ or 1. Moreover, if $k = 1$, then the parities of s, t, r are the same; If $k = 0$, then s, t, r are different.*

Proof. We only prove the necessity. We first claim that a graph $G \notin \mathcal{S} \cup \mathcal{C}$ with at most 7 vertices and $\delta(G) \geq 2$ is supereulerian. In fact, the claim is easily obtained by considering the longest cycle of G. Thus we assume $|V(G)| \geq 8$. By Theorem 2.1.6 and the discussion above, $(1), (2), (3), (4)$ are easy to obtain by consider the parities of n, m, s, t, r. □

Remark. Theorem 2.1.7 implies Conjecture 2.1.1 is not complete and one can revise it easily by using the theorem above.

§2.2 On spanning trail in terms of degree sum condition

Catlin in [21] showed the following:

Theorem 2.2.1 (Catlin [21]) *Let G be a connected graph on n vertices and let $u, v \in V(G)$. If*

$$d(x) + d(y) \geq n \tag{2.2.1}$$

for each edge $xy \in E(G)$, then exactly one of the following holds:

(i) G has a spanning (u, v)-trail.

(ii) $d(z) = 1$ *for some vertex* $z \notin \{u, v\}$.

(iii) $G = K_{2,n-2}, u = v$ *and* n *is odd.*

(iv) $G = K_{2,n-2}, u \neq v, uv \in E(G)$, n *is even, and* $d(u) = d(v) = n - 2$.

(v) $u = v$, *and* u *is the only vertex with degree* 1 *in* G.

The following stronger result modifies Catlin's theorem above:

Theorem 2.2.2 *Let* G *be a connected graph of order* $n \geq 4$ *and* G *satisfies* (2.2.1) *for every edge of* G. *Then exactly one of the following holds:*

(i) G *is collapsible.*

(ii) The reduction of G *is* $K_{1,t-1}$ *for* $t \geq 3$ *such that all of the vertices of degree 1 are trivial and they have the same neighbor in* G, $t \leq \frac{n}{2}$. *Moreover, if* $t = 2$, *then* $G - \{v\}$ *is collapsible for a vertex* v *in the* K_2.

(iii) G *is* $K_{2,n-2}$.

By the definition of the collapsible graph, Theorem 2.2.2 implies Theorem 2.2.1.

Theorem 2.2.2 implies a previous result by using Harary and Nash-Williams's theorem (we recall the theorem below).

Theorem 2.2.3 (Harary and Nash-Williams [66]) *Let* G *be a graph with at least 4 vertices. The line graph* $L(G)$ *is hamiltonian if and only if* G *has a closed trail that contains at least one vertex of each edge of* G.

Note that every graph in $\{K_{2,n-2}, K_{1,n-1}\}$ has a closed trail that contains at least one vertex of each edge of G. Then by Theorem 2.2.2 we have:

Theorem 2.2.4 (Brualdi and Shanny [19]) *Let* G *be a connected graph with* $n \geq 4$ *vertices. If* G *satisfies* (2.2.1) *for every edge of* G, *then* $L(G)$ *is hamiltonian.*

Theorem 2.2.4 was improved by Clark in [43] as follows:

Theorem 2.2.5 (Clark [43]) *Let G be a connected graph on $n \geq 6$ vertices, and let $p(n) = 0$ for n even and $p(n) = 1$ for n odd. If each edge of G satisfies*

$$d(x) + d(y) \geq n - 1 - p(n), \tag{2.2.2}$$

then $L(G)$ is hamiltonian.

Motivated by Theorem 2.2.5, Catlin in [21] considered replacing the condition (2.2.1) in Theorem 2.2.1 by (2.2.2), and classified the two exceptional cases that can arise. The following gives an analagous result for the collapsibility of graphs.

Theorem 2.2.6 *Let G be a connected graph of order $n \geq 4$ and let $p(n) = 0$ for n even and $p(n) = 1$ for n odd. If each edge of G satisfies* (2.2.2), *then exactly one of the following holds:*

(i) G is collapsible.

(ii) The reduction of G is $K_{1,t-1}$ for $t \geq 3$ such that all of the vertices of degree 1 are trivial and they have the same neighbor in G, $t \leq \frac{n}{2}$. Moreover, if $t = 2$, then $G - \{v\}$ is collapsible for some vertex v in the K_2.

(iii) G is in $\{C_5, G_7, G_7', K_{1,n-1}, K_{2,n-2}, K_{2,n-3}'\}$, where $K_{2,n-3}'$ is obtained from $K_{2,n-3}$ by adding a pendant edge on one of the vertices of degree $n-3$, and G_7, G_7' are the graphs shown in Fig.2.1.

The following example shows that the condition (2.2.2) in Theorem 2.2.6 is sharp: Let G be a graph of order $n \geq 6$, such that G has a bridge e and $G - e$ consisting of two complete graphs of order $\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil$ respectively. Clearly, if n is even, then $d(x) + d(y) \geq n - 2$ for each edge of G, and if n is odd, then $d(x) + d(y) \geq n - 3$. However, the reduction G' of G is K_2 and each vertices of G' is non-trivial. This violates Theorem 2.2.6.

Note that every graph of $\{G_7, G_7', K_{2,n-2}, k_{2,n-3}', K_{1,n-1}\}$ has a closed trail that contains at least one vertex of each edge of G. By the definition of a collapsible graph and Theorem 2.2.3, Theorem 2.2.6 implies Theorem 2.2.5. By Theorem 2.2.6 and the definition of a collapsible graph, we also have the following, which is similar to Theorem 2.2.1.

Corollary 2.2.7 *Let G be a connected graph on n vertices and let $u, v \in V(G)$. If*

$$d(x) + d(y) \geq n - 1 - p(n) \tag{2.2.3}$$

for each edge $xy \in E(G)$ and $p(n)$ defined as above, then exactly one of the following holds:

(i) G has a spanning (u, v)-trail.

(ii) $G \in \{G_7, G_7', K_{1,n-1}, K_{2,n-2}'\}$ or the reduction of G is $K_{1,t-1}$ for some integer $t \geq 2$.

(iii) $G = K_{2,n-2}, u = v$ and n is odd.

(iv) $G = K_{2,n-2}, u \neq v, uv \in E(G)$, n is even, and $d(u) = d(v) = n - 2$.

Fig. 2.3:

Assume H is a collapsible subgraph of G. It is easy to see that if a vertex $y \in V(G - H)$ and $|N(y) \cap V(H)| \geq 2$, then the subgraph induced by $\{y\} \cup V(H)$ is collapsible. We denote by $\xi(G)$ the value $\min\{d(x) + d(y) : xy \in E(G)\}$ and call $d(x) + d(y)$ the edge degree-sum of xy.

Lemma 2.2.8 *Let G be a graph on n vertices satisfying $\xi(G) \geq n - 1$. Suppose G' is the reduction of G. Then $\xi(G') \geq |V(G')| - 1$.*

Proof. Assume that u is a non-trivial vertex of G', i.e. it is the contraction of a maximal collapsible connected subgraph H. We call H the *preimage* of u and denote $PM(u) = H$. It is sufficient to consider the edges that are incident to u in $G/PM(u)$ and the others are clear. Assume that $yu \in E(G/PM(u))$. Then there is a vertex $x \in V(PM(u))$ such that $xy \in E(G)$. Note that the neighbors of x out of $PM(u)$ are also neighbors of u in G'. Then by $d(x) + d(y) \geq n - 1$ we have

$$\begin{aligned} d_{G/PM(u)}(u) + d_{G/PM(u)}(y) &\geq d(x) - (|PM(u)|) + 1 + d(y) \\ &\geq n - |PM(u)| + 1 - 1 \\ &= |G/PM(u)| - 1 \end{aligned}$$

The assertion follows from inequality (5.3.2). □

Lemma 2.2.9 *Let G be a graph on n vertices satisfying $\xi(G) \geq n - 1$. Suppose G' is of order t, G' is the reduction of G and $G \neq G'$. Then $G' \notin \{K_{2,t-2}, K'_{2,t-3}\}$ for $t \geq 4$.*

Proof. We only give the proof for the case $G \neq K_{2,t-2}$ as the other case is similar. By way of contradiction assume $K_{2,t-2}$ is the reduction of G. Let $v_1, \cdots, v_{t-2}$ be the vertices of degree 2 and u, v be the vertices of degree $t - 2$ in G'. If v_i is non-trivial for some i, we let G^* be the graph obtained by replacing v_i by $PM(v_i)$ and replacing the edges uv_i, vv_i by edges ux, vy, whenever x (y) in $PM(v_i)$ is adjacent to a vertex in $PM(u)$ in G. Assume $|V(G^*)| = n'$. Note that if $n' \geq t + 2$, then $d_{G^*}(v) + d_{G^*}(v_j) = t < n' - 1$ in G^*, a contradiction (by Lemma 2.2.8). Thus, every vertex of degree 2 is trivial. Similarly, u, v are also trivial. We complete the proof. □

Similar to Lemmas 2.2.8,2.2.9, we also have:

Lemma 2.2.10 *Let G be a graph on n vertices satisfying $\xi(G) \geq n - 2$. Suppose G' is the reduction of G. Then $\xi(G') \geq |V(G')| - 2$.*

Lemma 2.2.11 *Let G be a graph on n vertices satisfying $\xi(G) \geq n - 2$. Suppose G' (of order t) is the reduction of G and $G \neq G'$. Then $G' \notin \{K_{2,t-2}, K'_{2,t-3}\}$ for $t \geq 5$ if and only if $G \notin \{G_7, G'_7\}$, where $K'_{2,t-3}$ defined as in Theorem 2.2.6. Moreover, if $G \in \{G_7, G'_7\}$, then $t = 5$ and $G' = K_{2,3}$.*

Corollary 2.2.12 *Let G be a graph on n vertices satisfying $\xi(G) \geq n - 1 - p(n)$. Suppose G' of order t is the reduction of G and $G \neq G'$. Then $G' \notin \{K_{2,t-2}, K'_{2,t-3}\}$ for $t \geq 6$.*

Lemma 2.2.13 *Let G be a graph on n vertices satisfying $\xi(G) \geq n-1-p(n)$. Suppose G' is the reduction of G. Then G' contains at most one non-trivial vertex.*

Proof. Let G be a graph on n vertices satisfying $\xi(G) \geq n-1-p(n)$. Suppose G' is the reduction of G and $G \neq G'$. We shall show that G' contains at most one non-trivial vertex.

If there is a bridge $e = xy$ in G' such that both x and y are non-trivial, then e implies a bridge $e' = x'y' \in E(G)$ of G. We denote by G_1, G_2 the two components of $G - \{e'\}$. Since x, y are non-trivial, G_i contains an edge x_iy_i such that $x', y' \notin \{x_i, y_i\}$. It is easy to find a contradiction by computing the edge degree-sum of x_iy_i. So we assume G' contains no such bridge from now on. Thus $n \geq 7$ if G' contains at least two non-trivial vertices.

If G' contains at least three non-trivial vertices u_1, u_2, u_3, then there is an edge $e = xy \in E(G)$ such that $x \in V(PM(u_i))$ for some $i \in \{1, 2, 3\}$ and $y \notin V(PM(u_1) \cup PM(u_2) \cup PM(u_3))$. Without loss of generality we may assume $x \in V(PM(u_3))$. Let $n_i = |PM(u_i)|$ and $s = |V(G - \cup PM(u_i))|$. Note that x, y have at most two neighbors in $PM(u_1) \cup PM(u_2)$ and x, y have no common neighbor. We compute the edge degree-sum of $e = xy$: $d(e) \leq |V(G - \cup PM(u_i)) - \{y\}| + |V(PM(u_3)) - \{x\}| + |N_{PM(u_1) \cup PM(u_2)}(\{x, y\})| + |\{x, y\}| \leq s - 1 + n_3 - 1 + 2 + 2 = s + n_3 + 2 = n - n_1 - n_2 + 2 < n - 2$, a contradiction. Thus, G' has at most two non-trivial vertices.

Assume G' contains two non-trivial vertices u_1, u_2. If G' contains a trivial edge xy (i.e. x, y are both trivial), it is easy to find a contradiction by computing the edge degree-sum of xy. So G' contains no trivial edge. Thus, each of the vertices in $V(G) - V(PM(u_1) \cup PM(u_2))$ has degree at most 2 and then the degree of vertex in $N(V(G) - V(PM(u_1) \cup PM(u_2)))$ is at least $n - 1 - p(n) - 2$ (in fact, $n - 3$ for n even and $n - 4$ for n odd).

Case 1. There is a vertex of degree 1 in G'.

If u_i is a vertex of degree 1 in G' for some i, (without loss of generality) say u_1. Then there is a vertex of degree 2 in $V(G - PM(u_1) \cup PM(u_2))$, say x. Let $y_1 \in V(PM(u_1)), y_2 \in V(PM(u_2))$ be two vertices such that $xy_1, xy_2 \in E(G)$. So $|PM(u_1)| \geq |N_G(y_1) \cup \{y_1\} - \{x\}| \geq n - 1 - p(n) - d(x) - 1 = n - 4 -$

$p(n)$, $|V(G - PM(u_1) - \{x\})| \geq |N_G(y_2) \cup \{y_2\} - \{x\}| \geq n - 4 - p(n)$, and then $n \geq |N_G(y_1) \cup \{y_1\}| + |N_G(y_2) \cup \{y_2\}| + |\{x\}| \geq 2(n-4)+1$ for n even and $n \geq 2(n-5)+1$ for n odd. That is, $n \leq 6$ for n even and $n \leq 9$ for n odd. Recall that $n \geq 7$. Then n is odd and $n \leq 9$. If $n = 7$, then $PM(u_i) = K_3$. It is easy to find a contradiction by computing the edge degree-sum of an edge in $PM(u_i)$ which is not incident to x. Similarly, if $n = 9$, it is easy to find a contradiction.

Assume that there is a trivial vertex x of degree 1 and assume $xu_1 \in E(G')$ and $xy \in E(G)$. Note that u_2 is non-trivial. It is easy to see that there is an edge $x_1u_2 \in E(G')$ such that $x_1 \notin \{u_1, u_2\}$, $x_1u_1, x_1u_2 \in E(G')$ and then there is a vertex y_1 in $PM(u_2)$ such that $x_1y_1 \in E(G)$. Since $|PM(u_1)| \geq 3$ and $x_1u_1 \in E(G')$, $d(x_1)+d(y_1) \leq n-4+1 = n-3$ (note that $|(N(x_1) \cup N(y_1)) \cap (V(PM(u_1)) \cup \{x\})| \leq 1$ and x_1, y_1 have no common neighbor.), a contradiction.

Case 2. Each trivial vertex of G' has degree 2.

In this case, $G' = K_{2,t-2}$ for some t. A contradiction of this case follows from Lemma 2.2.11 since G_7, G'_7 contain exactly one non-trivial vertex.

Therefore, the claim holds. □

Lemma 2.2.14 (Catlin [24]) *The graphs $K_{3,3}$ and $K_{3,3} - e$ are collapsible, where $K_{3,3} - e$ is the graph obtained by removing an edge from $K_{3,3}$.*

Similar to Lemma 2.2.9, we have the following:

Lemma 2.2.15 *Let $G \notin \{G_7, G'_7\}$ be a graph of n vertices and $\xi(G) \geq n-1-p(n)$. Suppose G' is the reduction of G and $G' \neq G$. Then G' is $K_{1,t-1}$ for some integer t.*

Proof. Let u be the only non-trivial vertex of G'. It suffices to show that G' contains no trivial edge. By way of contradiction we assume that $xy \in E(G')$ is trivial and let $xu \in E(G')$.

If n is even, then $\xi(G) \geq n-1$ and thus $|V(G')| \geq n-1$. It is impossible since $|PM(u)| \geq 3$.

If n is odd, then $\xi(G) \geq n-2$ and thus $|V(G')| \geq n-2$. Therefore, $PM(u) = K_3$, $d(x)+d(y) = n-2$ and $V(G - PM(u) - \{x,y\}) \in N(\{x,y\})$. Let $S = V(G' -$

$\{u,x,y\}) = \{y_1,\cdots,y_{n-5}\}$ and let $N_{G'}(y) = \{y_1,\cdots,y_k\}$. Note that $|V(G')| = n-2$ and $d_{G'}(x) + d_{G'}(y) = n-2$. Then $N_{G'}(y_i) = N_{G'}(x) - \{y\}$ for $i = 1,\cdots,k$ ($yy_i, i = 1,\cdots,k$ are all trivial). So $G' = K_{d(x),d(y)}$. By Lemma 2.2.14, we have either $d(x) \le 2$ or $d(y) \le 2$. Note that $G' = K_{2,n-4}$, contradicts to Lemma 2.2.11. This implies that $G' = K_{1,t-1}$ for some t. We complete the proof. □

Lemma 2.2.16 *Let G be a reduced graph on $n \ge 5$ vertices satisfying $\xi(G) \ge n-1-p(n)$. Then G is in $\{C_5, K_{1,n-1}, K_{2,n-2}, K'_{2,n-3}\}$.*

Proof. Similarly, assume that G is reduced and $|V(G)| \ge 3$. By Theorem 1.2.7, we have $|E(G)| \le 2n-4$ and $\delta(G) \le 3$. So we may distinguish the following three cases for n even, or odd:

Case 1. n is even.

Note that $p(n) = 0$ and $\xi(G) \ge n-1$ in this case.

Subcase 1.1. $\delta(G) = 3$.

Let $d(u) = 3$ and $N(u) = \{u_1,u_2,u_3\}$. Since G contains no triangle, $\{u_1,u_2,u_3\}$ is independent. Note that n is even and $d(u) + d(u_i) \ge n-1$. Then, $|E(G)| \ge 3(n-4) > 2n-4$ for $n > 8$. Therefore, $n \le 8$. If $n = 6$, then $G = K_{3,3}$, but $K_{3,3}$ is collapsible, a contradiction. Thus $n = 8$ and then $12 = 3(n-4) \le |E(G)| \le 2n-4 = 12$. Thus $d(u_i) = 4$. It is easy to see that G is a bipartite graph and G contains a subgraph isomorphic to $K_{3,3} - e$, a contradiction.

Subcase 1.2. $\delta(G) = 2$.

Similarly, let $d(u) = 2$ and $N(u) = \{u_1,u_2\}$. Note that $d(u_i) \ge n-3$ and $\{u_1,u_2\}$ is independent. If one of u_1,u_2 has degree $n-2$, then $G = K_{2,n-2}$. If not, $d(u_1) = d(u_2) = n-3$, it can be seen that G contains a subgraph isomorphic to $K_{3,3} - e$ for $n \ge 6$.

Subcase 1.3. $\delta(G) = 1$.

Let $S = \{v_1,v_2,\cdots,v_s\} \ne \emptyset$ be the set of vertices of degree 1. We first claim that the neighbor of the s vertices $v_1,\cdots,v_s$ are the same since otherwise this implies $|E(G)| > 2n-4$. Denote the common neighbor by u. If $d(u) = n-1$, then G is $K_{1,n-1}$ since G contains no triangle. If not, there is a vertex $w \notin N(u)$, and let

$wv \in E(G)$ for some vertex v in $N(u) - S$. Clearly, $d(v) = 2$. Since $\xi(G) \geq n-1$ and G contains no triangle, we have $d(w) \geq n-3$, that is, $G = K'_{2,n-3}$.

Case 2. n is odd.

Subcase 2.1. $\delta(G) = 3$.

Assume that $d(u) = 3$ and $N(u) = \{u_1, u_2, u_3\}$. Note that $\xi(G) \geq n-2$ and $|E(G)| \leq 2n-4$. On the one hand, $3(n-5) \leq |E(G)| \leq 2n-4$ implies $n \leq 11$; On the other hand $\frac{3n+1}{2} \leq \frac{\sum_{x \in V(G)} d(x)}{2} = |E(G)| \leq 2n-4$ implies $n \geq 9$. If $n = 9$, then $\frac{3n+1}{2} = 2n-4 = |E(G)|$ and then there is an edge having edge degree-sum $6 < n-2$, a contradiction. Thus, $n = 11$. Note that $\frac{3n+1}{2} = 17 \leq |E(G)| \leq 2n-4 = 18$. It is easy to see that there is an edge that has edge degree-sum 6, a contradiction.

Subcase 2.2. $\delta(G) = 2$.

Assume that $d(u) = 2$ and $N(u) = \{u_1, u_2\}$. Note that $\xi(G) \geq n-2$. Then $d(u_i) \geq n-4$. A simple argument shows that $G \in \{C_5, K_{2,n-2}\}$, or G contains $K_{3,3} - e$ as a subgraph which implies a contradiction.

Subcase 2.3. $\delta(G) = 1$.

Assume that $d(u) = 1$ and $N(u_1) = \{u_1\}$. If $d(u_1) = n-1$, then $G = K_{1,n-1}$. A simple argument shows that if $d(u_1) = n-2$, then $G = K'_{2,n-3}$ $\square$

Proofs of Theorems 2.2.6 and 2.2.2. By Lemmas 2.2.11,2.2.15,2.2.16, Theorem 2.2.6 holds. Note that $G_7, G'_7, K'_{2,n-3}$ and C_5 violate condition (2.2.1). Then Theorem 2.2.6 implies Theorem 2.2.2. $\square$

§2.3 Collapsible graphs in terms of edge-degree conditions

Clearly, a minimal essential edge-cut is 2-restricted edge cut, and a 2-restricted edge cut is an essential edge-cut. So the essential edge-connectivity equals the 2-restricted edge-connectivity for a graph G. Esfahanian in [51] proved that if a connected graph G with $|V(G)| \geq 4$ is not a star $K_{1,n-1}$, then $\lambda^2(G)$ exists and $\lambda^2(G) \leq \xi(G)$. Thus, an essentially k-edge connected graph has edge-degree at least k.

Corresponding to the 3-restricted edge-cut, we define P_2-edge-cuts. An edge cut F of G is a P_2-edge-cut of G if at least two components of $G-F$ contain P_2, where P_2 denote a path with three vertices. Clearly, a minimal P_2-edge-cut of G is a 3-restricted edge-cut of G, and a 3-restricted edge-cut of G is a P_2-edge-cut of G. It is not difficult to see that a P_2-edge-cut of G implies a 3-restricted edge-cut. Thus, the size of a P_2-edge-cut of G is not less than the 3-restricted edge-connectivity of G.

Denote $D_i(G)$ the set of vertices of degree i and let $d_i(G) = |D_i(G)|$, respectively. If there is no confusion, we use D_i and d_i for $D_i(G)$ and $d_i(G)$, respectively. For a subgraph $A \subseteq G$, $v \in V(G)$, $N_G(v)$ denotes the set of the neighbors of v in G and $N_G(A)$ denotes the set $(\bigcup_{v\in V(A)} N_G(v)) \setminus V(A)$. If no confusion, we use an edge uv for a subgraph whose vertex set is $\{u, v\}$ and edge set $\{uv\}$. Denote $G[X]$ the subgraph induced by the vertex set X of $V(G)$.

This section considers the following conjecture.

Conjecture 2.3.1 (Chen and Lai Conjecture 8.6 of [33]) *Every 3-edge-connected and essentially 6-edge connected graph G is collapsible.*

Let G be a connected and essentially 3-edge-connected graph such that $L(G)$ is not a complete graph. The *core* of this graph G, denoted by G_0, is obtained by deleting all the vertices of degree 1 and contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$.

Lemma 2.3.2 (Shao [123]) *Let G be an essentially 3-edge-connected graph G.*

(*i*) *G_0 is uniquely defined, and $\lambda(G_0) \geq 3$.*

(*ii*) *If G_0 is supereulerian, then $L(G)$ is Hamiltonian.*

In the following lemma, the graph considered may have loops. Note that a loop is an edge with two same endpoints. For a graph G and $u \in V(G)$, denote $E_G(u)$ the set of edges incident with u in G. When the graph G is understood from the context, we write E_u for $E_G(u)$ simply. When a graph G is understood from the context, we use δ and n for $\delta(G)$ and $|V(G)|$, respectively.

Lemma 2.3.3 *Let G be a graph with minimum degree $\delta \geq 3, \xi(G) \geq 2\delta + k - 2$ and $k \geq 1$. Then $|E(G)| \geq 2n + \frac{\delta^2+(k-4)\delta-2k}{\delta+k} d_\delta$.*

Proof. Let $N(G) = N_G(D_\delta), T(G) = V(G) \setminus (N \cup D_\delta)$ (or simply, we use N and T for $N(G)$ and $T(G)$). Note that G is a graph with $\xi(G) \geq 2\delta - 1$, then D_δ is an independent set of G and the degree of the vertices in N is at least $\delta + k$, the vertices in T is at least $\delta + 1$. We prove this claim by induction on $|T|$.

We first let $|T| = \emptyset$. The degree of the vertex in N is at least $\delta + k$. If $|N| > \frac{\delta}{\delta+k} d_\delta$, we have

$$\begin{aligned}
|E(G)| = \frac{\sum i d_i}{2} &\geq \frac{\delta d_\delta}{2} + \frac{\delta + k}{2}|N| = \frac{\delta + k}{2} n - \frac{k}{2} d_\delta \\
&= 2n + \frac{\delta + k - 4}{2} n - \frac{k}{2} d_\delta \\
&= 2n + \frac{\delta + k - 4}{2}(d_\delta + |N|) - \frac{k}{2} d_\delta \\
&= 2n + \frac{\delta - 4}{2} d_\delta + \frac{\delta + k - 4}{2}|N| \\
&\geq 2n + \frac{\delta - 4}{2} d_\delta + \frac{\delta + k - 4}{2} \frac{\delta}{\delta + k} d_\delta \\
&= 2n + \frac{(\delta - 4)(\delta + k) + \delta(\delta + k - 4)}{2(\delta + k)} d_\delta \\
&= 2n + \frac{\delta^2 + (k - 4)\delta - 2k}{\delta + k} d_\delta.
\end{aligned}$$

If $|N| \leq \frac{\delta}{\delta+k} d_\delta$, we have

$$\begin{aligned}
|E(G)| &\geq \delta d_\delta = 2n + \delta d_\delta - 2n \\
&= 2n + \delta d_\delta - 2(\delta + |N|) = 2n + \delta d_\delta - 2|N| \\
&= 2n + (\delta + 2) d_\delta - \frac{2\delta}{\delta + k} d_\delta \\
&= 2n + \frac{\delta^2 + (k - 4)\delta - 2k}{\delta + k} d_\delta.
\end{aligned}$$

Now, we assume $|T| = 1$ and $T = \{u\}$. Clearly, $d(u) \geq \delta + 1 \geq 4$. We first suppose $d(u) = 2s$ for some $s \geq 2$. Assume that there is l loops on u and let $2s = 2l + 2t$. Now, we delete the l loops of u and label the $2t$ neighbors corresponding the $2t$ edges naturally. Denote the $2t$ neighbors by $N'(u) = \{u_1, u_2, \cdots, u_{2t}\}$ (it is not a set if $G[\{u\} \cup N(u)]$ contains some multi-edges), that is, $N'(u)$ contains v k times if there is k edges between u and v. We construct a graph G' by (i) : deleting vertex u and edges $uu_i, i = 1, 2, \cdots, 2t$; (ii) : adding new edges $u_1u_2, u_3u_4, \cdots, u_{2t-1}u_{2t}$. It can be seen that $D_\delta(G) = D_\delta(G'), V(G') = V(G) \setminus \{u\}, E(G') = (E(G) \setminus E_u) \cup \{u_1u_2, u_3u_4, \cdots, u_{2t-1}u_{2t}\}$. Hence, $|V(G')| = |V(G)| - 1, |E(G')| = |E(G)| - \frac{d(u)}{2}$, $\xi(G') \geq 2\delta + k - 2$. Note that the set $T(G')$ is $\emptyset$, then we have $|E(G')| \geq 2(n - 1) + \frac{\delta^2+(k-4)\delta-2k}{\delta+k}d_\delta$. Therefore,

$$\begin{aligned}
|E(G)| &= |E(G')| + \frac{d(u)}{2} \\
&\geq 2|V(G')| + \frac{\delta^2 + (k-4)\delta - 2k}{\delta + k}d_\delta + \frac{d(u)}{2} \\
&= 2(|V(G)| - 1) + \frac{\delta^2 + (k-4)\delta - 2k}{\delta + k}d_\delta + \frac{d(u)}{2} \\
&= 2|V(G)| + \frac{\delta^2 + (k-4)\delta - 2k}{\delta + k}d_\delta + (\frac{d(u)}{2} - 2) \\
&\geq 2|V(G)| + \frac{\delta^2 + (k-4)\delta - 2k}{\delta + k}d_\delta.
\end{aligned}$$

Next, we suppose $u \in T$ with l loops, $d(u) = 2s + 1$ and $2s + 1 = 2l + 2t + 1$ for some $s \geq 2$ and $N(u) = \{u_1, u_2, \cdots, u_{2t+1}\}$. Let $u' \in N$, we first construct G' by adding a new edge uu'. Now, u is in the $T(G')$ and $d_{G'}(u) \geq 6$ is even. Similarly, we construct a new graph G'' such that $T(G'')$ is empty. Note that $\frac{d_{G'}(u)}{2} \geq 3$, then

$$\begin{aligned}
|E(G')| &= |E(G'')| + \frac{d_{G'}(u)}{2} \\
&\geq 2|V(G'')| + \frac{\delta^2 + (k-4)\delta - 2k}{\delta + k}d_\delta + \frac{d_{G'}(u)}{2} \\
&= 2(|V(G')| - 1) + \frac{\delta^2 + (k-4)\delta - 2k}{\delta + k}d_\delta + \frac{d_{G'}(u)}{2} \\
&= 2|V(G)| + \frac{\delta^2 + (k-4)\delta - 2k}{\delta + k}d_\delta + (\frac{d_{G'}(u)}{2} - 2) \\
&\geq 2|V(G')| + \frac{\delta^2 + (k-4)\delta - 2k}{\delta + k}d_\delta + 1.
\end{aligned}$$

Thus, $|E(G)| = |E(G')| - 1 \geq 2|V(G)| + \frac{\delta^2+(k-4)\delta-2k}{\delta+k}d_\delta$.

Assume that the claim holds for $1 \leq |T| < m$ and $|T| = m \geq 2$ in the following. Take a vertex $u \in T$ such that $d(u) = \min\{d(v)|v \in T\}$. Clearly, by the argument above, if $d(u)$ is even, then, the claim holds by constructing a new graph

G' (similar to the case when $|T| = 1$, i.e. G' is constructed by deleting the vertex u, $l + t$ edges, and adding t new edges) with $|T| = m - 1$ and then by induction. Assume $d(u)$ is odd. Similar to the case when $|T| = 1$. We first construct a new graph G' by adding a new edge as the case $|T| = 1$. It can be seen that $d_{G'}(u)$ is even and $d_{G'}(u) \geq 6$. Then we construct a new graph G'' similar to that of $|T| = 1$, by induction and the argument similar to that of (4), the claim holds. We complete the proof of the claim. $\square$

In this paper, we only need the following three special cases of Lemma 3.1:

Corollary 2.3.4 *Let G is a graph with $\delta(G) \geq 3, \xi(G) \geq 7$. Then $|E(G)| \geq 2|V(G)|$.*

Corollary 2.3.5 *Let G be a graph with $\delta(G) \geq 3, \xi(G) \geq 6$. Then $|E(G)| \geq 2|V(G)| - \frac{d_3}{5}$.*

Corollary 2.3.6 *Let G be a graph with $\delta(G) \geq 3, \xi(G) \geq 5$. Then $|E(G)| \geq 2|V(G)| - \frac{d_3}{2}$.*

Let G' be the reduction of G. Note that contraction do not decrease the edge connectivity of G, then G' is either a k-edge connected graph or a trivial graph if G is k-edge connected. Assume that G' is the reduction of a 3-edge-connected graph and non-trivial. It follows from Theorem 1.2.7 and G' is 3-edge connected that $F(G') \geq 3$. Then by Theorem 1.2.7, we have $|E(G')| \leq 2|V(G')| - 5$.

A subgraph of G is called a 2-path or a P_2 subgraph of G if it is isomorphic to a $K_{1,2}$ or a 2-cycle. An edge cut X of G is a *2-path-edge-cut* of G if at least two components of $G - X$ contain 2-paths. Clearly, a P_2-edge-cut of a graph G is also a 2-path-edge-cut of G. By the definition of a line graph, for a graph G, if $L(G)$ is not a complete graph, then $L(G)$ is essentially k-connected if and only if G does not have a 2-path-edge-cut with size less than k. Since G_0 is a contraction of G, every P_2-edge-cut of G_0 is also a P_2-edge-cut of G.

Lemma 2.3.7 (Lai et al Lemma 2.3 of [91]) *Let $k > 2$ be an integer, and let G be a connected and essentially 3-edge-connected graph. If $L(G)$ is essentially k-connected, then every 2-path-edge-cut of G_0 has size at least k.*

We call a vertex of G' *non-trivial* if the vertex is obtained by contracting a collapsible subgraph of G_0, and *trivial*, otherwise. Assume that $k \geq 3$ is an integer, and G is a 3-edge-connected and essentially k-edge-connected graph. Thus G_0 has no non-trivial vertex of degree i such that $3 \leq i < k$.

Lemma 2.3.8 *Let G be a reduced 3-edge-connected non-trivial graph. Then $d_3 \geq 10$.*

Proof. Since $F(G') \geq 3$, we have

$$4|V(G)| - 10 \geq 2|E(G)| = \sum id_i \geq 3d_3 + 4(|V(G)| - d_3) = 4|V(G)| - d_3.$$

Thus, $d_3 \geq 10$. $\square$

If V_1 and V_2 are two disjoint subsets of $V(G)$, then $[V_1, V_2]_G$ denotes the set of edges in G with one end in V_1 and the other end in V_2. When the graph G is understood from the context, we also omit the subscript G and write $[V_1, V_2]$ for $[V_1, V_2]_G$. If H_1 and H_2 are two vertex disjoint subgraphs of G, then we write $[H_1, H_2]$ for $[V(H_1), V(H_2)]$. Assume that u is a non-trivial vertex of G', and it is obtained by contracting a maximal connected collapsible subgraph H of G. We call H the *preimage* of u and let $PM(u) = H$. If a subgraph X of G' is obtained by contracting some maximal connected collapsible subgraph U of G. We call U the *preimage* of X and let $PM(X) = U$. In particular, we call X non-trivial if $X \ncong U$.

Theorem 2.3.9 *A 3-edge-connected graph with $\xi(G) \geq 7$, and $\lambda^3(G) \geq 7$ is collapsible.*

Proof. Let G be a 3-edge-connected graph with $\xi(G) \geq 7$, and $\lambda^3(G) \geq 7$ and G' be the reduction of G. By way of contradiction, suppose that G' is non-trivial. Note that $F(G') \geq 3$ and thus $|E(G')| \leq 2|V(G')| - 5$, then we can obtain a contradiction by Corollary 2.3.4 if $\xi(G') \geq 7$. So we next show that the edge degree of G' is at least 7.

Suppose that there is an edge $e = uv$ with $d(e) < 7$ in G'. By Theorem 1.2.7 and Lemma 2.3.8, it is easy to see that $G' - \{u, v\}$ contains a component having at least three vertices. Note that the edge degree of uv is less than 7, then uv is clearly non-trivial. Thus, $[PM(uv), PM(G' - \{u, v\})]_G$ is a P_2-edge-cut of G, but its size is less than 7, a contradiction. We complete the proof. $\square$

Note that a simple graph contains no 2-cycle, then each non-trivial collapsible connected subgraph of a graph having at least three vertices. If we consider the simple graph, the condition of Theorem 2.3.9 can be weaken slightly.

Theorem 2.3.10 *A 3-edge-connected simple graph with* $\xi(G) \geq 7$, *and* $\lambda^3(G) \geq 6$ *is collapsible.*

Proof. Let G be a 3-edge-connected simple graph with $\xi(G) \geq 7$, and $\lambda^3(G) \geq 6$ and G' is the reduction of G. By way of contradiction, suppose that G' is non-trivial. Note that $F(G') \geq 3$ and thus $|E(G')| \leq 2|V(G')| - 5$, then we can obtain a contradiction by Corollary 2.3.4 if $\xi(G') \geq 7$. So we show that the edge degree of G' is at least 7. Not that G is 3-edge connected and so is G', then it is sufficient to show that the contraction does not product new vertices of degree less than 6.

By Theorem 1.2.7, $G' - \{u\}$ contains a component with at least three vertices, for any vertex $u \in V(G')$. Suppose that $u \in V(G')$ is a vertex obtained by contracting a maximal connected collapsible subgraph H of G. If u is non-trivial, then $|V(PM(u))| \geq 3$ sine G is simple graph. Then $[PM(u), PM(G' - \{u\})]$ is a P_2-edge-cut of G. If $d_{G'}(u) < 6$, then we get a P_2-edge-cut whose size is less than 6, a contradiction. That is, the edge degree of G' is at least 7. We complete the proof. $\square$

By Lemma 2.3.2 and Theorem 2.3.9, we have

Corollary 2.3.11 (Zhan [159]) *A 7-connected line graph is Hamiltonian.*

By a very similar proof to that of Theorems 2.3.9 and 2.3.10, we obtain the following theorem.

Theorem 2.3.12 *A 3-edge-connected graph with* $\xi(G) \geq 6$, $\lambda^2(G) \geq 4$, *and* $\lambda^3(G) \geq 6$ *with at most 24 vertices of degree 3 is collapsible.*

Proof. Let G be a 3-edge-connected graph with $\xi(G) \geq 6$, $\lambda^2(G) \geq 4$, and $\lambda^3(G) \geq 6$ and at most 24 vertices of degree 3, and G' be the reduction of G.

By an argument similar to that of Theorem 4.3, one can see that the edge degree of G' is at least 6. In fact, suppose that there is an edge $e = uv$ with $d(e) < 6$ in G'. By Theorem 1.2.7 and Lemma 2.3.8, it is easy to see that $G' - \{u, v\}$ contains a component having at least three vertices. Note that the edge degree of uv is less that 6, then uv is clearly non-trivial. Thus, $[PM(uv), PM(G' - \{u, v\})]_G$ is a P_2-edge-cut of G, but its size is less that 6, a contradiction.

Note that $\lambda^2(G) \geq 4$, then G' clearly contains no non-trivial vertex of degree 3, that is, $|D_3(G')| \leq |D_3(G)|$. By Corollary 2.3.5, we have $|E(G')| \geq 2|V(G')| - \frac{|D_3(G')|}{5}$. If $|D_3(G')| \leq |D_3(G)| \leq 24$, then $|E(G')| \geq 2|V(G')| - \frac{|D_3(G')|}{5} \geq 2|V(G)| - 4$ (note that the number of edges is an integer) which contradicts $|E(G')| \leq 2|V(G')| - 5$. Thus, the claim holds. □

Similar to Theorem 2.3.10, we have the following theorem:

Theorem 2.3.13 *A 3-edge-connected simple graph with $\xi(G) \geq 6$, and $\lambda^3(G) \geq 5$ with at most 24 vertices of degree 3 is collapsible.*

Proof. The proof is similar to that of Theorem 2.3.10. Let G be a 3-edge-connected simple graph with $\xi(G) \geq 6$, and $\lambda^3(G) \geq 5$, at most 24 vertices of degree 3, and G' is the reduction of G. By way of contradiction, suppose that G' is non-trivial. Note that $F(G') \geq 3$ and thus $|E(G')| \leq 2|V(G')| - 5$, then we obtain a contradiction by by an argument similar to that of Theorem 2.3.13 if $\xi(G') \geq 6$. So we show that the edge degree of G' is at least 6. Not that G is 3-edge-connected and so is G', then it is sufficient to show that the contraction does not product a new vertex of degree less than 5.

By Theorem 1.2.7, $G' - \{u\}$ contains a component with at least three vertices, for any vertex $u \in V(G')$. Suppose that $u \in V(G')$ is a vertex obtained by contracting a maximal connected collapsible subgraph H of G. If u is non-trivial, then $|V(PM(u))| \geq 3$ since G is simple graph. Then $[PM(u), PM(G' - \{u\})]$ is a P_2-edge-cut of G. If $d_{G'}(u) < 6$, then we get a is a P_2-edge-cut less than 6, a contradiction. Thus, the edge degree of G' is at least 6. We complete the proof. □

By Theorem 2.3.13, we have the following corollary:

Corollary 2.3.14 (Yang et al [149]) *For a 5-connected line graph $L(G)$ with minimum degree at least 6, if G is simple and $|D_3(G)| \leq 24$, them $L(G)$ is Hamiltonian.*

Similarly as above, we list the following results without proof.

Theorem 2.3.15 *A 3-edge-connected graph with $\xi(G) \geq 5$, and $\lambda^2(G) \geq 4$ with at most 9 vertices of degree 3 is collapsible.*

§2.4 Open problems

In Section 3 above, we consider the 3-edge connected graphs with a given edge degree and the restriction on λ_3. One may naturally ask the following.

Conjecture 2.4.1 *If G is a 3-edge-connected, essentially 4-edge-connected graph with $\xi(G) \geq 9$, then G has a spanning eulerian subgraph.*

Lai et al. in [90] studied the question, but the proof is not complete. So the question is still open.

According the current results on the supereulerian 3-edge connected graphs, one can see that the studies essentially depends on the assumption of few edge-cuts of size 3 (in other word, the graph considered almost contains two edge disjoint trees, i.e., F is small), for example, the results in [15, 30]. For more information on supereulerian graph, we refer to the survey [25] By Catlin. Catlin in [25] conjectured the following.

Conjecture 2.4.2 *A 3-edge-connected simple graph with at most 17 vertices is either supereulerian, or it is contractible to the Petersen graph.*

Moreover, Catlin and Lai in [30] showed that a 3-edge-connected with at most 10 edge cuts of size 3 is either supereulerian, or it is contractible to the Petersen graph. We naturally give the following conjecture.

Conjecture 2.4.3 *A 3-edge-connected with at most 17 edge cuts of size 3 is either supereulerian, or it is contractible to the Petersen graph.*

Note that there are two reduced snarks of order 18. So 17 is best possible.

So far, only a few results to consider 3-edge connected graphs whose reduction with many vertices of degree 3 (in other word, F is large.). Note that the description here is not very exact. So there is still a big gap to completely characterize the non-supereulerian 3-edge connected graphs.

Catlin in [23] showed that if a graph with $F(G) \leq 2k+1$, then either G contains a spanning subgraph with at most $2k$ vertices of odd degree, or G can be contracted to a tree of order $2k+2$ whose vertices all have odd degree. What will happen if a 3-edge connected graph with large F? Note a 3-regular graph is either supereulerian or it is a snark.

Conjecture 2.4.4 *Let G be a 3-edge-connected reduced graph with $F(G) = k$. Then G is either supereulerian, or G is contractible to a snark of order at most k.*

Fleischner and Jacson in [57] established an equivalent relation between cyclically 4-edge connected graphs and 3-regular graphs for considering the dominating trail. We ask the following.

Question 2 *Is there an equivalent relation between 3-edge connected graph and 3-regular graphs on the supereulerian problem?*

Chapter 3 Hamiltonian line graphs

§3.1 Collapsible graphs, and 3-connected and essentially 11-connected line graphs

§3.1.1 Collapsible graphs and hamiltonicity of line graphs

In the following lemma, the graph considered may have loops. Noticing that a loop is an edge with two same endpoints. For a graph G and $u \in V(G)$, denote $E_G(u)$ the set of edges incident with u in G. When the graph G is understood from the context, we write E_u for $E_G(u)$ simply. By using Lemma 2.3.3, we have the following.

Lemma 3.1.1 *Let G be a graph with $\delta(G) \geq 3, \xi(G) \geq 6$. Then $|E(G)| \geq 2|V(G)| - \frac{d_3}{5}$.*

Let G' be the reduction of G. Note that contraction can not decrease the edge connectivity of G, then G' is either a k-edge connected graph or a trivial graph if G is k-edge connected. Assume that G' is the reduction of a 3-edge connected graph and non-trivial. It follows from Theorem 1.2.7 and G' being 3-edge connected that $F(G') \geq 3$. Then by Theorem 1.2.7, we have $|E(G')| \leq 2|V(G')| - 5$.

We call a vertex of G' *non-trivial* if the vertex is obtained by contracting a collapsible subgraph of G, and *trivial*, otherwise. Assume that G is a 3-edge connected, essentially $k \geq 4$-edge connected graph. It is easy to see that G' contains no non-trivial vertex of degree i such that $3 \leq i < k$ (otherwise, an essentially edge cut of G with size less than k is found).

Theorem 3.1.2 *A 3-edge connected, essentially 5-edge connected graph with edge-degree at least 6 and at most 24 vertices of degree 3 is collapsible.*

Proof. Let G be a 3-edge connected, essentially 5-edge connected graph with edge-degree 6 and at most 24 vertices of degree 3, and G' be the reduction of G. Note that G is essentially 5-edge connected, then the contraction can not product new vertex of degree 3 or 4 by Theorem 1.2.7 (suppose u is vertex obtained by contracting a non-trivial maximal collapsible connected subgraph of G and $d_{G'}(u) < 5$. By Theorem 1.2.7, $G' - \{u\}$ contains at least one non-trivial component. It is not difficult to see that G contains an essential edge-cut with size less than than 5, a contradiction), that is, $|D_3(G')| \leq |D_3(G)|$ and G' is 3-edge connected graph with edge-degree 6. By Lemma 3.1, we have $|E(G')| \geq 2|V(G')| - \frac{|D_3(G')|}{5}$, that is, $|E(G')| \geq 2|V(G')| - \frac{|D_3(G')|}{5} \geq 2|V(G')| - 4$ which contradicts with $|E(G')| \leq 2|V(G')| - 5$. Thus, we complete the proof. □

Note that a 3-edge connected, essentially 6-edge connected graph has edge-degree at least 6, then we have the following two corollaries which are the partial results of Conjecture 1.2.26 and Conjecture 1.2.27, respectively:

Corollary 3.1.3 *A 3-edge connected, essentially 5-edge connected graph with edge-degree at least 6 and at most 24 vertices of degree 3 is supereulerian.*

Corollary 3.1.4 *A 3-edge connected, essentially 6-edge connected graph with at most 24 vertices of degree 3 is collapsible.*

For a graph G, if $L(G)$ is $k \geq 3$-connected, then G is essentially k-edge connected. Clearly, the core of G is 3-edge connected and essentially k-edge connected. Clearly, if the minimum degree of $L(G)$ is k, then the edge degree of G is at least k. Thus, by Lemma 2.3.2 and Theorem 3.1.2, we have the following corollaries:

Corollary 3.1.5 *A 5-connected line graph with minimum degree at least 6 of a graph with at most 24 vertices of degree 3 is hamiltonian.*

From the proof above, it can be seen that Lemma 2.3.3 plays a key role. Similarly, we pose the following lemma for considering the 3-edge connected graphs with $\xi(G) \geq 7$. The proof of the following is very similar to that of Lemma 2.3.3, we leave the complete proof to readers and only point the part which is different from the proof of Lemma 2.3.3.

Lemma 3.1.6 *Assume that G is a graph with $\delta(G) \geq 3, \xi(G) \geq 7$. Then $|E(G)| \geq 2|V(G)|$.*

Proof. By leting $k = 3$ in Lemma 2.3.3, one can see the claim. □

By Lemma 3.1.6, and the similar argument to that of the proof of Theorem 3.1.2, we have the following theorem which is another partial result of Conjecture 1.2.27. Note that Lemma 7 in [79] implies a stronger result: Every 3-edge connected essentially 6-edge-connected graph with edge-degree at least 7 has 2 edge disjoint spanning trees (also 44 edges edge-degree 6 are allowed), which also implies the following.

Theorem 3.1.7 *A 3-edge connected, essentially 6-edge connected graph with edge-degree at least 7 is collapsible.*

Similarly as Corollary 3.1.3, we have the following corollary which is posed by Chen and Lai [33].

Corollary 3.1.8 (Chen and Lai Theorem 7.3 of [33]) *A 3-edge connected, essentially 7-edge connected graph is collapsible.*

By Lemma 2.3.2 and Theorem 3.1.7, we give a weaker result (than the result of [79]):

Corollary 3.1.9 *A 6-connected line graph with minimum degree at least 7 is hamiltonian.*

By Corollary 3.1.9, the following corollary is clear.

Corollary 3.1.10 (Zhan [159]; Chen and Lai Theorem 7.2 of [33]) *A 7-connected line graph is hamiltonian.*

§3.1.2 3-connected and essentially 11-connected line graphs

Lemma 3.1.11 (Lai et al Theorem 2.3 (iii) [94]) *If G is collapsible, then for any pair of vertices $u, v \in V(G), G$ has a spanning (u, v)-trail.*

A dominating (e_1, e_2)-trail of G is an (e_1, e_2)-trail T of G such that every edge of G is incident with an internal vertex of T.

Lemma 3.1.12 (Lai et al Proposition 2.2. [94]) *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian connected if and only if for any pair of edges $e_1, e_2 \in E(G)$ has a dominating (e_1, e_2)-trail.*

For a graph G and any pair of edges $e_1, e_2 \in E(G)$, let $G(e_1, e_2)$ denote the graph obtained from G by subdividing both e_1 and e_2, and denote the new vertices by $v(e_1)$ and $v(e_2)$. Thus $V(G(e_1, e_2)) - V(G) = \{v(e_1), v(e_2)\}$. The following lemma is obtained by Lai et al by combining the definition of collapsible and Lemma 2.9 of [94]; Combining Lemmas 3.1.11 and 3.1.12, Zhan also stated the following lemma in (4.3) of [160].

Lemma 3.1.13 (Lai et al Lemma 2.9 [94], Zhan 4.3 of [160]) *Assume that G_0 is the core of G. If $G_0(e_1, e_2)$ is collapsible, then $L(G)$ is hamiltonian-connected.*

A subgraph of G isomorphic to a $K_{1,2}$ or a 2-cycle is called a 2-path or a P_2 subgraph of G. An edge cut X of G is a P_2-*edge cut* of G if at least two components of $G - X$ contain 2-paths. By the definition of a line graph, for a graph G, if $L(G)$ is not a complete graph, then $L(G)$ is essentially k-connected if and only if G does not have a P_2-edge cut with size less than k. Since the core G_0 is obtained from G by contractions (deleting a pendant edge is equivalent to contracting the same edge), every P_2-edge-cut of G_0 is also a P_2-edge-cut of G. Hence, the following lemma is easy:

Lemma 3.1.14 (Lai et al Lemma 2.3 of [91]) *Let $k > 2$ be an integer, and let G be a connected, essentially 3-edge connected graph. If $L(G)$ is essentially k-connected, then every P_2-edge cut of G_0 has size at least k.*

If V_1, V_2 are two disjoint subsets of $V(G)$, then $[V_1, V_2]_G$ denotes the set of edges in G with one end in V_1 and the other end in V_2. When the graph G is understood from the context, we also omit the subscript G and write $[V_1, V_2]$ for $[V_1, V_2]_G$. If H_1, H_2 are two vertex disjoint subgraphs of G, then we also write $[H_1, H_2]$ for $[V(H_1), V(H_2)]$.

Lemma 3.1.15 (Lai et al Lemma 3.1 of [91]) *Let G be graph such $L(G)$ is 3-connected and essentially 11-connected, and G' be the reduction of G_0. For each $u, v, w \in V(G')$ such that $P = uvw$ is a 2-path in G', the edge cut $X = [\{u, v, w\}, V(G') \setminus \{u, v, w\}]_{G'}$ is a P_2-edge cut of G' and $|X| \geq 11$.*

Recall that $f(x) = \frac{x-4}{x}$ and $l(u) = f(d(u))$ defined in [91]. The following lemma is an useful property of $f(x)$.

Lemma 3.1.16 (Lai et al Lemma 3.3 of [91]) *Each of the following holds.*

(i) $f(x)$ is an increasing function.

(ii) If $d(u) \geq k$, then $l(u) \geq f(k)$.

Lemma 3.1.17 *Let G be a graph such that $L(G)$ is 3-connected and essentially 11-connected. Then $G_0(e_1, e_2)$ is collapsible.*

Proof. Assume that G_0 is the core of G. Clearly, any P_2-edge cut of G_0 has size at least 11. Let G' be the reduction of $G_0(e_1, e_2)$. If G' is trivial, then, by Lemma 4.3, the assertion holds. By contradiction, suppose G' is non-trivial. Let $v \in D_3, N_{G'}(v) = \{v_1, v_2, v_3\}$, and let $s = |N_{G'}(v) \cap D_2(G')|$. We first show some claims as follows.

Claim 1. $F(G') \geq 3$.

Assume $F(G') \leq 2$. By Theorem 1.2.7, G' is a $K_{2,t}$. It is easy to find an edge cut of G_0 of size 2, which contradicts to that G_0 is 3-edge connected. So the claim holds.

By Theorem 1.2.7, we have $|E(G')| \leq 2|V(G)| - 5$.

Recall that we call a vertex of G' non-trivial if the vertex is obtained by contracting a non-trivial collapsible subgraph of $G_0(e_1, e_2)$, and trivial, otherwise. Assume that u is a non-trivial vertex of G', and it is the contraction of a maximal collapsible connected subgraph H. We call H the preimage of u and denote $PM(u) = H$.

Claim 2. Let xyz be a P_2 of G' and $\min\{d_{G'}(x), d_{G'}(y), d_{G'}(z)\} \geq 3$. Then $[G_0[PM(x) \cup PM(y) \cup PM(z)], G_0 - G_0[PM(x) \cup PM(y) \cup PM(z)]]_{G_0}$ is a P_2-edge cut of G_0.

Note that the degree (in G') of the vertex in $\{x, y, z\}$ is at least 3 and G' contains at most two vertices of degree 2, then it is easy to find a P_2 without vertex of degree 2 in $G' - \{x, y, z\}$ (Note that G' contains no cycles of lengths 3 and 2, then $N_{G'}(x) \cup N_{G'}(y) \cup N_{G'}(z) \setminus \{x, y, z\}$ contains at least 4 vertices of $G' - \{x, y, z\}$. Since there are at most two of $N_{G'}(x) \cup N_{G'}(y) \cup N_{G'}(z) \setminus \{x, y, z\}$ with degree 2, a simple argument shows that $G' - \{x, y, z\}$ contains a P_2 clearly.). Thus, $[G_0[PM(x) \cup PM(y) \cup PM(z)], G_0 - G_0[PM(x) \cup PM(y) \cup PM(z)]]_{G_0}$ is a P_2-edge cut of G_0.

Claim 3. Let $v \in D_3(G'), N_{G'}(v) = \{v_1, v_2, v_3\}$. For any two vertices $v_i, v_j \in N_{G'}(v) \setminus D_2(G')$, $d_{G'}(v_i) + d_{G'}(v_j) \geq 12$ for $i \neq j$ hold.

By Claim 2., this claim is clear.

Claim 4. Each component of $G'[D_3(G')]$ contains at most two vertices.

Suppose that there is a component in $G'[D_3(G')]$ contains a P_2, say xyz. By Claim 2, $[G_0[PM(x) \cup PM(y) \cup PM(z)], G_0 - G_0[PM(x) \cup PM(y) \cup PM(z)]]_{G_0}$ is a P_2-edge cut of G_0 with size less than 11, a contradiction.

Claim 5. Suppose that $v \in D_3(G')$ is an isolated vertex of $G'[D_3(G')]$ and $N_{G'}(v) \cap D_2(G') = \emptyset$. Then $l(v_1) + l(v_2) + l(v_3) \geq 1$.

By Lemma 3.1.16 and Claim 3, this claim is clear.

Claim 6. Suppose that $v, w \in D_3(G')$ and $vw \in E(G')$ and $N(vw) \cap D_2(G') = \emptyset$. If v_1, v_2 are the vertices adjacent to v in G' different from w and v_3, v_4 are the vertices adjacent to w in G' different from v, then (I) v_1, v_2, v_3, v_4 are mutually distinct vertices, and (II) both $l(v_1) + l(v_2) \geq 1$ and $l(v_3) + l(v_4) \geq 1$.

By Theorem 1.2.7, (I) is clearly true. By Lemma 3.1.16 and Claim 3, this claim is clear.

We now turn to prove Lemma 3.1.17. We first assume that $N_{G'}(v) \cap D_2(G') = \emptyset$ for all $v \in D_3(G')$. By Claims 5 and 6, we have the following inequality.

$$\begin{aligned} d_3 &= \sum_{v \in D_3} 1 \leq \sum_{v \in D_3} \sum_{uv \in E, u \notin D_3} l(u) = \sum_{i \geq 4} \sum_{u \in D_i} \sum_{uv \in E, v \in D_3} l(u) \\ &\leq \sum_{i \geq 4} \sum_{u \in D_i} i \cdot f(i) = \sum_{i \geq 4} \sum_{u \in D_i} (i-4) = \sum_{i \geq 4} (i-4) \cdot d_i. \end{aligned}$$

Now assume $N_{G'}(D_2(G')) \cap D_3(G') \neq \emptyset$. Notice that $|D_2(G')| \leq 2$. It is not difficult to see that at most 4 vertices of degree 3 do not satisfy Claims 5 and 6. Assume that S is the set of the vertices in $N_{G'}(D_2(G')) \cap D_3(G')$ and the vertices in $D_3(G')$ such that one of its neighbor is in $N_{G'}(D_2(G')) \cap D_3(G')$. Assume $|S| = t$. We claim that $N_{G'}(S) \cap D_3(G') = \emptyset$ and $|S| \leq 4$. In fact, suppose by the way of contradiction that $N_{G'}(S) \cap D_3(G') \neq \emptyset$ and let $x \in N_{G'}(S) \cap D_3(G')$. By the definition of S, there are two vertices $y, z \in S$ such that $xy \in E(G')$ and yz is a connected component of $G'[D_3(G')]$. By Claim 2 and Lemma 3.1.15, we induce a contradiction (note that the degree of each vertex of x, y, z is 3, this fact contradicts that $[G_0[PM(x) \cup PM(y) \cup PM(z)], G_0 - G_0[PM(x) \cup PM(y) \cup PM(z)]]_{G_0}$ should be a P_2-edge-cut of G_0 by Claim 2) and thus $N_{G'}(S) \cap D_3(G') = \emptyset$. By an argument similar to the proof of Claim 2, we have $|S| \leq 4$. If $|S| \geq 5, by |D_2(G')| \leq 2$, we can find x, y and z which satisfy the conditions of Claim 2.

Then, we have

$$\begin{aligned} d_3 &= t + \textstyle\sum_{v \in D_3 \setminus S} 1 \leq t + \sum_{v \in D_3 \setminus S} \sum_{uv \in E, u \notin D_3 \setminus S} l(u) = t + \sum_{i \geq 4} \sum_{u \in D_i} \sum_{uv \in E, v \in D_3 \setminus S} l(u) \\ &\leq t + \textstyle\sum_{i \geq 4} \sum_{u \in D_i} i \cdot f(i) = t + \sum_{i \geq 4} \sum_{u \in D_i} (i-4) = t + \sum_{i \geq 4} (i-4) \cdot d_i \\ &\leq 4 + \textstyle\sum_{i \geq 4} (i-4) \cdot d_i. \end{aligned}$$

On the other hand, by Claim 1, we have

$$\begin{aligned} 10 &\leq 2(2|V(G')| - |E(G')|) = 4|V(G')| - 2|E(G')| = 4|V(G')| - \sum_i i d_i \\ &= \sum_i (4-i) d_i = (4-2) d_2 + \sum_{i \geq 3} (4-i) d_i \leq 4 + d_3 - \sum_{i \geq 4} (i-4) d_i \\ &\leq 4 + 4 = 8, \end{aligned}$$

a contradiction. Thus, $G' = K_1$ and $G_0(e_1, e_2)$ is collapsible. □

By Lemmas 3.1.13 and 3.1.17, we have the following theorem.

Theorem 3.1.18 *Every 3-connected, essentially 11-connected line graph is hamiltonian-connected.*

Next, we consider the hamiltonicity of 7-connected line graph. If $L(G)$ is 7-connected, then G is essentially 7-edge connected and G_0 is 3-edge connected, essentially 7-edge connected. By an argument very similar to that of Theorems 3.1.17 and 3.1.18 (moreover, it is easier than the argument of 3.1.17 and 3.1.18), we have the following theorems:

Lemma 3.1.19 *Let G be 3-connected, essentially 7-edge connected graph. Then $G_0(e_1, e_2)$ is collapsible.*

Thus, by Lemma 3.1.13, we have

Theorem 3.1.20 *Every 7-connected line graph is hamiltonian-connected.*

In [121], Ryjáček and Vrána introduced a closure named multigraph closure which turns a claw-free graph into the line graph of a multigraph while preserving its hamiltonian-connectedness. Using Theorem 9 in [121], Theorem 3.1.18 can be extended to claw-free graphs.

Corollary 3.1.21 *Every 3-connected, essentially 11-connected claw-free graph is hamiltonian-connected.*

§3.2 3-connected and essentially 10-connected line graphs

We first recall several definitions. Let G be an essentially 3-edge-connected graph such that $L(G)$ is not a complete graph. In this section, the *core* of this graph G, denoted by G_0, is obtained by deleting all the vertices of degree at most 3 adjacent to at most one vertex (together with the incident edges), and contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$.

Lemma 3.2.1 ([159]) *Let G be an essentially 3-edge-connected graph G. Then the core G_0 of G satisfies the following.*

(i) G_0 is uniquely defined and $\delta(G) \geq 3$;

(ii) $V(G_0)$ is a dominating set of G;

(iii) If G_0 contains an eulerian spanning trail T, then T is a dominating trail of G;

(iv) If G_0 is spanning trailable, then G is dominating trailable;

(v) If G_0 is spanning trailable, then $L(G)$ is hamiltonian-connected.

Theorem 3.2.2 (Nash-Williams and Tutte [113, 134]) *A finite graph G can be decomposable into n connected factors if and only if*

$$|S| \geq n(\omega(G - S) - 1)$$

for each subset S of the edge set $E(G)$.

We shall prove the following theorem by using Theorem 3.2.2.

Theorem 3.2.3 *The core of an essentially 3-edge connected and conditional-P_2 10-edge connected graph contains two edge-disjoint spanning trees.*

Proof. Let S be a subset of $E(G_0)$ and let $G_1, \cdots, G_\omega$ be the components of $G_0 - S$. To prove the core G_0 of an essentially 3-edge connected and conditional-P_2 10-edge connected graph G contains two edge-disjoint spanning trees, it is sufficient to show that

$$|S| \geq 2(\omega(G_0 - S) - 1) = 2(\omega - 1)$$

for each subset S of the edge set $E(G_0)$ by Theorem 3.2.2.

Let $m(G_i)$ denote the number of edges exactly one of its endpoints in $V(G_i)$. If $\omega \leq 5$, it is easy to show that $|S| \geq 8$ and then (1) is true. We assume $\omega \geq 6$ in the following arguments. Assume G_a is a component with $m(G_a) = 3$ in $G_0 - S$ (in fact, such component is consisting of a single vertex of degree 3 since $\delta(G_0) \geq 3$ and P_2-$\lambda(G_0) \geq 10$) and G_b is a component in $G_0 - S$ which contains a vertex adjacent to a vertex of G_a. If $G_0 - V(G_a \cup G_b)$ contains a P_2, then G_b is either a component with $m(G_b) \geq 9$, or a component consisting of a single vertex since P_2-$\lambda(G_0) \geq 10$. If there is a component G_b in $G_0 - S$ such that $G_0 - V(G_a \cup G_b)$ contains no P_2, then (1) holds clearly (hints, this case implies that $G_0 - V(G_a \cup G_b)$ is consisting of some isolated vertices and isolated edges, and each of them satisfies $m(G_i) \geq 3$, thus $|S| \geq 3(\omega - 1)$). So we may assume $G_0 - V(G_a \cup G_b)$ contains a P_2 for any pair of components G_a, G_b.

By the argument above, we assume that $G_1, \cdots, G_{s_1}, \cdots, G_{s_2}, \cdots, G_{s_3}, \cdots, G_\omega$ denote all the components of $G_0 - S$, where $G_1, \cdots, G_{s_1}$ are the components consisting of a single vertex of degree 3 in G_0, $G_{s_1+1}, \cdots, G_{s_2}$ (if they exists) are the components consisting of a single vertex of degree either 4 or 5 and each of them is adjacent to a vertex of a component in $G_1, \cdots, G_{s_1}$; $G_{s_2+1}, \cdots, G_{s_3}$ (if they exists) are the components such that $m(G_i) \geq 6$ for $i = s_2 + 1, \cdots, s_3$ and each of them containing a vertex is adjacent to a vertex in some components of $G_1, \cdots, G_{s_1}$, $G_{s_3+1}, \cdots, G_\omega$ (if they exists) are the remaining components of $G_0 - S$ (note $m(G_i) \geq 4, i = s_3 + 1, \cdots, \omega$).

Let G be a graph with minimum degree $\delta \geq 3$, $T(G) = N_G(D_3(G))$ and $R(G) = V(G) - D_3(G) - T(G)$. To prove Theorem 11, a new claim is needed.

Claim 1. Let G be a graph with minimum degree $\delta \geq 3$. If $d_G(e) \geq 7$ for any edge $e \in E(G[D_3(G) \cup T(G)])$ and $d_G(v) \geq 4$ for any $v \in R(G)$, then $|E(G)| \geq 2|V(G)|$.

The detailed proof of Claim 1 can be found in Appendix.

Now we prove Theorem 11 by using Claim 1. We denote by G_0/G_i the graph obtained by contracting G_i, i.e. identify the two ends of each edge in G_i and then deleting the resulting loops. Let G' be the graph obtained from G_0 by contracting all the subgraphs $G_i, i \in \{1, \cdots, \omega\}$. Denote by v_{G_i} the vertex obtained by contracting G_i. Clearly, $|S| = |E(G')|$ and $\omega(G_0 - S) = |V(G')|$. Note that if $s_2 = s_1$, then $|E(G')| \geq 2|V(G')|$ by Claim 1, that is, $|S| \geq \omega(G_0 - S)$. We are done. So we assume $s_2 \neq s_1$ from now no.

Note that each component of $G_{s_1+1}, \cdots, G_{s_2}$ is a single vertex and is adjacent to at most one vertex in $G_1, \cdots, G_{s_1}$ since otherwise induces a P_2-edge cut with size no more than 8, a contraction. Let G_a, G_b be two components of $G_0 - S$ and $a \in \{1, \cdots, s_1\}, b \in \{s_1 + 1, \cdots, s_2\}$. Note that G_b is single vertex and there is a vertex of degree 3 is adjacent to G_b. Let $\{v\} = G_a, \{u\} = G_b$ and $uv \in S$, where $d_{G_0}(v) = 3$. It is not difficult to see that if $G_0 - \{v\} \cup \{u\} \cup G_t$, for any G_t that contains a vertex adjacent to at east one of u or v, contains a P_2, then $m(\{v\} \cup \{u\} \cup G_t) \geq 10$; if not, it is easy to see that (1) holds, we are done. Thus we always assume $m(\{v\} \cup \{u\} \cup G_t) \geq 10$, that is, $m(G_t) \geq 6$ and then $d_{G'}(v_{G_t}) \geq 6$. Assume $N_{G_0}(u) = \{v, u_1, u_2, u_3\}$ and $u_i \in V(G_{a_i}), i = 1, 2, 3$ (may $a_i = a_j, i \neq j$).

Now we define two operations for removing the vertices of degree 4 and 5 in $T(G')$.

(O_1): Assume that $v_{G_b} \in T(G')$ is a vertex of degree 4 in G' and $v_{G_b}v_{G_a}$, $v_{G_b}v_{G_{a_1}}$, $v_{G_b}v_{G_{a_2}}$, $v_{G_b}v_{G_{a_3}} \in E(G')$ (the multiedge is allowed in our argument). Remove vertex v_{G_b}, the edges $v_{G_b}v_{G_a}$, $v_{G_b}v_{G_{a_1}}$, $v_{G_b}v_{G_{a_2}}$, $v_{G_b}v_{G_{a_3}}$ and add new edges $v_{G_a}v_{G_{a_1}}$, $v_{G_{a_2}}v_{G_{a_3}}$ (The loop is allowed), see Fig.3.1.

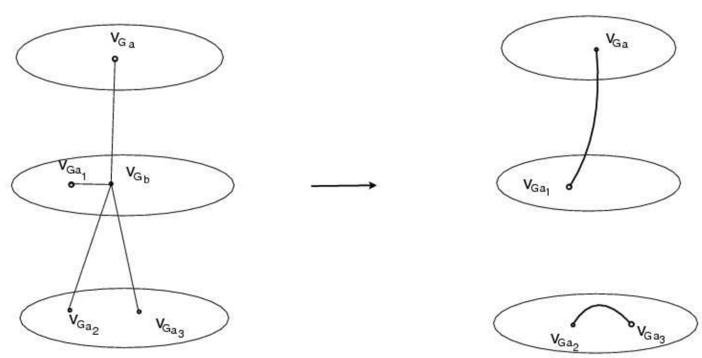


Fig. 3.1: O_1

(O_2): Assume that $v_{G_b} \in T(G')$ is a vertex of degree 5 in G' and $v_{G_b}v_{G_a}$, $v_{G_b}v_{G_{a_1}}$, $v_{G_b}v_{G_{a_2}}$, $v_{G_b}v_{G_{a_3}}$, $v_{G_b}v_{G_{a_4}} \in E(G')$. Remove vertex v_{G_b}, the edges $v_{G_b}v_{G_a}$, $v_{G_b}v_{G_{a_1}}$, $v_{G_b}v_{G_{a_2}}$, $v_{G_b}v_{G_{a_3}}$, $v_{G_b}v_{G_{a_4}}$ and add new edges $v_{G_a}v_{G_{a_1}}$, $v_{G_{a_2}}v_{G_{a_3}}$, $v_{G_{a_4}}v_{G_{a_4}}$ ($v_{G_{a_4}}v_{G_{a_4}}$ denotes a loop on vertex $v_{G_{a_4}}$), see Fig.3.2.

Apply operations O_1 (or O_2) on a graph G and denote the resulting graph by G^*. By the definitions of O_1 and O_2, we have that if $|E(G^*)| \geq 2|V(G^*)|$, then $|E(G)| \geq 2|V(G)|$.

Apply the operations O_1 and O_2 to the vertices of degree 4 and 5 in $T(G')$ and denote the resulting graph by G^*. Note that if $|E(G^*)| \geq 2|V(G^*)|$, then inequality (1) holds. Since $m(G_{a_i}) \geq 6$, we have $d_{G'}(v_{G_{a_i}}) \geq 6$. Thus, $|E(G^*)| \geq 2|V(G^*)|$ by Claim 1. We complete the proof. $\square$

By the definition of the core, we have the following corollary.

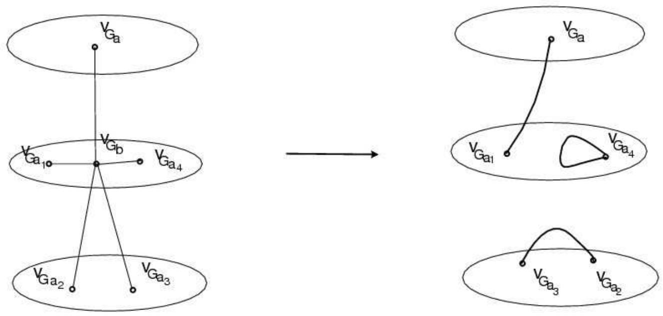


Fig. 3.2: O_2

Theorem 3.2.4 (Catlin and Lai [31]) *Let G be a graph and let $e_1, e_2 \in E(G)$. If G has two edge-disjoint spanning trees, then exactly one of the following holds:*

(*a*) *G has a spanning (e_1, e_2)-trail.*

(*b*) *$\{e_1, e_2\}$ is an edge-cut of G.*

Theorem 3.2.5 *The core G_0 of an essentially 3-edge connected and conditional-P_2 10-edge connected graph G is spanning trailable.*

Proof. By Theorem 5.2.1, G_0 contains two edge-disjoint trees. By Theorem 3.2.4, it is sufficient to show that G_0 is 3-edge connected (no edge-cut is consisting of two edges). Note that G is essentially 3-edge connected, then G_0 is 3-connected by the definition of the core. Then, we complete the proof. □

Combining Lemma 3.2.1 with Theorem 3.2.5, we have the following.

Corollary 3.2.6 *Every 3-connected, essentially 10-connected line graph is hamiltonian-connected.*

Using the multigraph closure introduced by Ryjáček [121], we have the following corollary.

Corollary 3.2.7 *Every 3-connected, essentially 10-connected claw-free graph is hamiltonian-connected.*

Appendix Proof of Claim 1.

Claim 1. Let G be a graph with minimum degree $\delta \geq 3$. If $d_G(e) \geq 7$ for any edge $e \in E(G[D_3(G) \cup T(G)])$ and $d_G(v) \geq 4$ for any $v \in R(G)$, then $|E(G)| \geq 2|V(G)|$.

Proof. Note that if a component of G has no vertex of degree 3, then the component satisfies the inequality. So we assume that each of the components of G contains some vertices of degree 3. Let $T = N_G(D_3), R = V \setminus (N \cup D_3)$. Note that $d_G(e) \geq 7$ for any edge $e \in E(G[D_3(G) \cup T(G)])$ and $d_G(v) \geq 4$ for any $v \in R(G)$, then D_3 is an independent set of G and the degree of the vertices in T is at least 6, the vertices in R is at least 4. We prove this claim by induction on $|R|$.

We first let $|R| = \emptyset$, then each of the vertices in T has degree at least 6. If $|T| > \frac{1}{2}d_3$, we have

$$\begin{aligned}
|E(G)| = \frac{\sum id_i}{2} &\geq \frac{3d_3}{2} + \frac{6(|V(G)| - d_3)}{2} = 2|V(G)| - \frac{3}{2}d_3 + |V(G)| \\
&= 2|V(G)| - \frac{3}{2}d_3 + d_3 + |T| \\
&> 2|V(G)| - \frac{3}{2}d_3 + d_3 + \frac{1}{2}d_3 \\
&= 2|V(G)|.
\end{aligned}$$

Thus, we may assume $|T| \leq \frac{1}{2}d_3$. It is easy to see that

$$\begin{aligned}
|E(G)| &\geq 3d_3 = 2d_3 + d_3 \\
&\geq 2(d_3 + |T|) \\
&= 2|V(G)|.
\end{aligned}$$

Now, we assume $|R| = 1$, that is, $R = \{u\}$ for some vertex v. Clearly, $d(u) \geq 4$. We first suppose $d(u) = 2k$ for some $k \geq 2$. Assume that there is l loops on u and let $2k = 2l + 2t$. Now, we delete the l loops of u and label the $2t$ neighbors corresponding the $2t$ edges naturally. Denote the $2t$ neighbors by $N'(u) = \{u_1, u_2, \cdots, u_{2t}\}$ (it is not a set if $G[\{u\} \cup N(u)]$ contains some multi-edges), that is, $N'(u)$ contains v k times if there is k edges between u and v. We construct a graph G' by (i) : deleting vertex u and edges $uu_i, i = 1, 2, \cdots, 2t$;

(ii) : adding new edges $u_1u_2, u_3u_4, \cdots, u_{2t-1}u_{2t}$. It can be seen that $D_3(G) = D_3(G'), V(G') = V(G) \setminus \{u\}, E(G') = (E(G) \setminus E_u) \cup \{u_1u_2, u_3u_4, \cdots, u_{2k-1}u_{2k}\}$. Hence, $|V(G')| = |V(G)| - 1, |E(G')| = |E(G)| - \frac{d(u)}{2}$. Note that the set $R(G')$ is $\emptyset$ and G' satisfies the conditions of Claim 1, then we have $|E(G')| \geq 2|V(G')|$. Therefore,

$$\begin{aligned}
|E(G)| &= |E(G')| + \frac{d(u)}{2} \\
&\geq 2|V(G')| + \frac{d(u)}{2} = 2(|V(G)| - 1) + \frac{d(u)}{2} \\
&= 2|V(G)| + (\frac{d(u)}{2} - 2) \\
&\geq 2|V(G)|.
\end{aligned}$$

Next, we suppose $u \in R$ with l loops, $d(u) = 2k + 1$ and $2k + 1 = 2l + 2t + 1$ for some $k \geq 2$ and similarly $N'(u) = \{u_1, u_2, \cdots, u_{2t+1}\}$. Let $u' \in T$, we first construct G' by adding an new edge uu'. Now, u is in the $R(G')$ and $d_{G'}(u) \geq 6$ is even. Similarly as above, we construct a new graph G'' such that the $R(G'')$ is empty. Note that $\frac{d_{G'}(u)}{2} \geq 3$, then

$$\begin{aligned}
|E(G')| &= |E(G'')| + \frac{d_{G'}(u)}{2} \\
&\geq 2|V(G'')| + \frac{d_{G'}(u)}{2} = 2(|V(G')| - 1) + \frac{d_{G'}(u)}{2} \\
&= 2|V(G)| + (\frac{d_{G'}(u)}{2} - 2) \\
&\geq 2|V(G')| + 1.
\end{aligned}$$

Thus, $|E(G)| = |E(G')| - 1 \geq 2|V(G)|$.

Assume that the claim holds for $1 \leq |R| < m$ and $|R| = m \geq 2$ in the following. Note that each of the components of G contains some vertex of degree 3, then there is a vertex u in R which is adjacent to some vertex of R. Clearly, by the argument above, if $d(u) = 2l + 2t$ is even, then, the claim holds by constructing a new graph G' (similarly as the case when $|R| = 1$, i.e. G' is constructed by deleting a vertex u, $l + t$ edges, and adding t new edges) with $|R| = m - 1$ and then by induction, we are done. Assume $d(u)$ is odd. Similarly as the case when $|R| = 1$. It can be seen that $d_{G'}(u)$ is even and $d_{G'}(u) \geq 6$. Then we construct a new graph

G'' similar to that of $|R| = 1$, by induction and the argument similar to that of (5), the claim holds. We complete the proof of the claim. $\square$

§3.3 3-connected and essentially 4-connected line graphs

Let G' be the reduction of G. Note that contraction do not decrease the edge connectivity of G, then G' is either a k-edge connected graph or a trivial graph if G is k-edge connected. Assume that G is the reduction of a 3-edge connected graph and non-trivial. It follows from Theorem 1.2.7 (iv) and G' is 3-edge connected that $F(G') \geq 3$. Then by Theorem 1.2.7 (iii), we have $|E(G')| \leq 2|V(G')| - 5$. Denote $D_i(G)$ and $d_i(G)$ the set of vertices of degree i and $|D_i(G)|$, respectively. For $X \subset V(G)$, denote $[X, V(G) \setminus X]$ the set of the edge one endpoint contained in X and the other one contained in $V(G)\setminus X$. Moreover, we also use $[G[X], G[V(G)\setminus X]]$ for the set $[X, V(G) \setminus X]$ if no confusion, where $G[X]$ denote the subgraph induced by vertex set X.

Lemma 3.3.1 *Let G be a reduced 3-edge connected non-trivial graph. Then $d_3 \geq$ 10.*

Proof. Since $F(G') \geq 3$, we have

$$4|V(G)| - 10 \geq 2|E(G)| = \sum id_i \geq 3d_3 + 4(|V(G)| - d_3) = 4|V(G)| - d_3.$$

Thus, $d_3 \geq 10$. $\square$

A subgraph of G isomorphic to a $K_{1,2}$ or a 2-cycle is called a 2-path or a P_2 subgraph of G. An edge cut X of G is a P_2-*edge-cut* of G if at least two components of $G - X$ contain 2-paths. By the definition of a line graph, for a graph G, if $L(G)$ is not a complete graph, then $L(G)$ is essentially k-connected if and only if G does not have a P_2-edge-cut with size less than k. Since the core G_0 is obtained from G by contractions (deleting a pendant edge is equivalent to contracting the same edge), every P_2-edge-cut of G_0 is also a P_2-edge-cut of G.

Lemma 3.3.2 *Let G be a 3-edge connected graph. If $L(G)$ is essentially 4-connected, then $L(G)$ is 4-connected.*

Proof. Since G is 3-edge connected, the minimum degree of G is at least 3. Thus, the minimum degree of $L(G)$ is at least 4. Noticing that $L(G)$ is essentially 4-connected. Thus, there is no vertex cut with less than 4 vertices, that is, $L(G)$ is 4-connected. □

Corollary 3.3.3 *Let G be a 3-edge connected graph. If $L(G)$ is essentially 4-connected, then G is essentially 4-edge connected.*

Lemma 3.3.4 *Let G be a 3-edge connected graph with at most 9 vertices of degree 3. If $L(G)$ is essentially 4-connected, then G is collapsible.*

Proof. Let G' be the reduction of G. If G' is trivial, we are done. Assume, G' is non-trivial. Note that G contains at most 9 vertices of degree 3, by Lemma 3.1, then there is a non-trivial vertex of degree 3 in G', say u. Note that $L(G)$ is essentially 4-connected, by Theorem 1.2.7 (ii) and Lemma 3.1, u can not be the reduction of a subgraph with more than two vertices (otherwise, the vertices of $L(G)$ induced by the edges in $[PM(u), PM(G - \{u\})]$ is an essential cut of $L(G)$ containing three vertices, which contradicts that $L(G)$ is essentially 4-connected). Thus, u is the reduction of a subgraph with order 2 and at least two multiple edges. By Theorem 1.2.7, $[PM(u), PM(G-\{u\})]$ is an essential edge-cut of G with three edges which contradict with Corollary 3.3. Thus, G is collapsible. □

Note that Petersen graph is not collapsible, then we clearly have that all conditions of Lemma 3.4 are sharp.

Theorem 3.3.5 *Let $L(G)$ be a 3-connected, essentially 4-connected line graph of a graph. If $d_3(G) \leq 9$, then $L(G)$ is hamiltonian. Moreover, if G has 10 vertices of degree 3 and $L(G)$ is not hamiltonian, then the reduction of G_0 is the Petersen graph.*

Proof. Let G be a graph with at most 9 vertices of degree 3 such that $L(G)$ is 3-connected, essentially 4-connected. Then by Lemma 2.2, the core of G is 3-edge connected with at most 9 vertices of degree 3. By Lemma 3.3.4, the core of G is collapsible. By Lemma 2.2, $L(G)$ is hamiltonian.

Suppose that G has 10 vertices of degree 3 and $L(G)$ is not hamiltonian. Note that G_0 is 3-edge connected and essentially 4-edge connected. By Lemma3.3.1, G_0 contains10 vertices of degree 3, then G_0 contains exactly 10 edge-cut of size 3. Then the reduction of G_0 is the Petersen graph (this due to a theorem in [30] by Catlin and Lai that any 3-edge-connected graph with at most 10 edge cuts of size 3 either has a spanning closed trail or it is contractible to the Petersen graph). $\square$

Concluding remarks. We shall show that all conditions of Theorem 3.5 are sharp.

We first show that the condition "3-connected" is sharp by the following example. Let u, v be the vertices of degree $2k+3$ in $K_{2,2k+3}$. Denote $K'_{2,2k+3}$ the graph obtained by subdividing all adjacent edges of u. Clearly, $L(K'_{2,2k+3})$ is 2-connected, essentially $2k+3$-connected, but it is not hamiltonian.

Second, let P' be the graph obtained by subdividing each edge of the Petersen graph exactly once. We add at least two pendant edges on each vertex of degree 3 in P', and denote the resulting graph by P''. Clearly, $L(P'')$ a is a 3-connected, essentially 3-connected graph without a hamiltonian cycle, then the condition "essentially 4-connected" is sharp.

Third, an example shows that the condition "$d_3(G) \leq 9$" in Theorem 5.1.13 is sharp: Petersen graph P has a perfect matching M with five edges. We construct a new graph P' by subdividing the five edges in M. Clearly, the resulting graph P' contains no dominating circuit (the dominating circuit of P' implies a hamiltonian cycle of P). Thus, $L(P')$ is not hamiltonian. It is not difficult to see that $L(P')$ is 3-connected, essentially 4-connected (this example is a special case of the following counterexamples, see the detailed proof below).

We will construct an infinite family of counterexamples for Conjecture 1.3. Two known results are needed.

Lemma 3.3.6 (Fleischner and Jackson Corollary 1 [54]) *A cubic graph is cyclically 4-edge connected if and only if it is essentially 4-edge connected.*

Theorem 3.3.7 (Petersen's Theorem, Corollary 5.4 [11]) *Any bridgeless cubic graph has a perfect matching.*

An infinite family of counterexamples: Let G be a snark. Noticing that G has a perfect matching M. We construct a new graph G' by subdividing the edges in M i.e. replace each edge of M by a path of length 2. Note that G is clearly non-hamiltonian (otherwise, it will be of class one), then G' has no dominating circuits. Therefore $L(G')$ is not hamiltonian. By Lemma 3.3.6, then $L(G')$ is 3-connected, essentially 4-connected (otherwise, $L(G')$ is 3-connected, essentially 3-connected, an essential-cut with three vertices of $L(G')$ induces an essential edge-cut of G by contracting one of the edge of each P_2 added by the subdivision, where a contraction of an edge is obtained by identifying the two ends of and deleting the resulting loops.).

§3.4 Open problems

Based on the results above, we naturally ask the following question.

Question 3 *What is the minimum integer k such that every 3-connected line graphs, essentially k-connected line graph has a hamiltonian cycle? The problem is still open. By the above remark, we have $5 \leq k \leq 11$. In particular, the next candidate will be $k = 5$.*

Question 4 *What is the minimum integer k such that every 3-connected line graphs, essentially k-connected line graph is hamiltonian connected if and only if it is hamiltonian?*

Question 5 *In Theorem 3.3.5, we show that if G has 10 vertices of degree 3 and $L(G)$ is not hamiltonian, then the reduction of G_0 is the Petersen graph. It is natural to ask what is the maximum t such that G has t vertices of degree 3 and $L(G)$ is not hamiltonian, then the reduction of G_0 is the Petersen graph.*

Chapter 4 Fault-tolerant hamiltonicity of Cayley graphs generated by transposition trees

We refer to the book [62] for notations and terminologies not described this chapter.

§4.1 Fault-tolerant hamiltonian laceability of Cayley graphs generated by transposition trees

Recall the definitions for Cayley graphs. $Cay(S_n, B)$ is said to be *Cayley graph grenerated by transposition tree* if B is a minimal transposition generating set of S_n. In particular, $Cay(S_n, B)$ is called *star graph*, written ST_n, and *bubble-sort graph*, written BS_n, if $T_B \cong K_{1,n-1}$ and $T_B \cong P_n$ (P_n is a path with n vertices), respectively.

It is convenient to denote $Cay(S_n, B)$ by G_n when no confusion arise. For any bipartite graph G, we use $V_0(G)$ and $V_1(G)$ to denote the two different partite sets of G, respectively.

Let x be a permutation. The i^{th} position of x is denoted by $x[i]$. For example, given $x = 4312 \in V(G_4)$, then, $x[1] = 4, x[2] = 3, x[3] = 1$ and $x[4] = 2$, respectively. Therefore, $\{x|\ x[n] = i\} = \{p_1p_2 \cdots p_{n-1}i|\ p_1p_2 \cdots p_{n-1}$ range over all permutations on $\{1, 2, \cdots, i-1, i+1, \cdots, n\}\}$. The following property present that the label of the vertex set of a transpositions tree T_B does not affect the structural property of the Cayley graph generated by T_B.

Lemma 4.1.1 ([62]) *Let B_1 and B_2 be two minimal transpositions generating sets of S_n. Then $Cay(S_n, B_1) \cong Cay(S_n, B_2)$ if and only if $T_{B_1} \cong T_{B_2}$.*

The following useful structural properties of G_n can be found in [40, 84, 150].

Lemma 4.1.2 *If n is a leaf of T_B, then G_n can be decomposed into n disjoint copies of G_{n-1}, say $G_n(1), G_n(2), \cdots, G_n(n)$, where $G_n(i)$ is a subgraph of G_n induced by vertex set $\{x|\ x[n] = i\}$.*

We say that the decomposition is induced by the leaf n. In the following, unless stated otherwise, we always assume that n is a leaf of T_B and $(n-1,n)$ is a pendant edge in T_B. Sometimes, we need a special decomposition by some other leaf in T_B, for example, if a decomposition is induced by leaf j, then $G_n(k)$ is the subgraph of G_n induced by vertex set $\{x|\ x[j]=k\}$.

Note that $(p_1\cdots p_i\cdots p_j\cdots p_n)(ij)=(p_1\cdots p_j\cdots p_i\cdots p_n)$. Let x be a vertex of $G_n(i)$, unless stated otherwise, we call the vertex $x'=x((n-1)n))$ the *outgoing neighbor* of x and (x,x') the *outgoing edge* of x.

Lemma 4.1.3 (1) *There are* $(n-2)!$ *edges between* $G_n(i)$ *and* $G_n(j)$ *for* $i,j=1,2,\cdots,n,\ i\neq j$. *We use* $E^{i,j}$ *to denote the set formed by such* $(n-2)!$ *edges between* $G_n(i)$ *and* $G_n(j)$.
(2) $E^{i,j}$ *is an independent edge set of* G_n *for any two distinct integers* i,j.
(3) *Half of the end vertices of the edges of* $E^{i,j}$ *contained in* $V(G_n(i))$ *are in* $V_0(G_n(i))$ *and the other* $\frac{(n-2)!}{2}$ *end vertices contained in* $V(G_n(i))$ *are in* $V_1(G_n(i))$.
(4) *The outgoing neighbors of* x *and* y *are contained in different subgraphs* $ST_n(i)$, *if* x *and* y *are adjacent in* $V(ST_n(j))$.

Note that $n-3<\frac{(n-2)!}{2}$ for $n\geq 5$, then we have the following observation by Lemma 4.1.3.

Lemma 4.1.4 *Let* F *be a fault edge set of* G_n. *If* $|F|\leq n-3$ *and* $n\geq 5$, *then there exists* $x\in V_0(G_n(i)), y\in V_1(G_n(i)), w\in V_1(G_n(j)), z\in V_0(G_n(j))$ *such that* $(x,w),(y,z)\in E^{i,j}-F$ *for* $i\neq j$.

For any $s\in B$, we think of the edge (g,gs) as being labeled s. An edge $e=(x,y)$ is a *pair-edge* if $x[n-1]=y[n-1]$ and $x[n]=y[n]$. So an edge with label (ij) is a pair-edge if and only if $\{i,j\}\cap\{n-1,n\}=\emptyset$. For a pair-edge e, $e[n-1]$ and $e[n]$ denote the $(n-1)^{th}$ and n^{th} coordinates of two ends of e, respectively. Clearly, if a pair-edge $e=(x,y)$ is in $G_n(i)$ with $e[n-1]=j$, then x' and y' are both in $G_n(j)$, and x' is adjacent to y', where x' and y' are the outgoing neighbors of x and y, respectively. Furthermore, the pair-edge $e'=(x',y')$ of $G_n(j)$ is called the *coupled pair-edge* of e, see [82] for the details.

Li et al. in [100] and Araki and Kikuchi in [3] reported the following results on ST_n and BS_n.

Lemma 4.1.5 ([100]) *Let $x \in V_0(BS_n), y \in V_1(BS_n)$ and $e \in E(BS_n)$ such that $e \neq (x, y)$[1]. If $n \geq 4$, then BS_n has a hamiltonian path from x to y containing e.*

Lemma 4.1.6 ([3]) *Let $e = (u, v) \in E(BS_n)$ and $x \in V(BS_n) - \{u, v\}$. If $n \geq 4$, then $BS_n - \{u, v\}$ has a hamiltonian path starting from x.*

Theorem 4.1.7 ([3],[100]) *Star graphs ST_n and bubble-sort graphs BS_n are $(n-3)$-edge fault tolerant hamiltonian laceable for $n \geq 4$.*

Theorem 4.1.8 ([127]) *Cayley graphs G_n are hamiltonian laceable for $n \geq 4$.*

§4.1.1 Johnson graphs and cartesian product

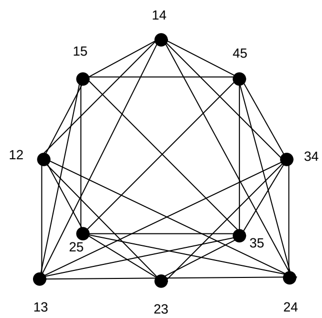
We shall introduce a well known graph family, namely Jonson graphs $J(v, k, k-1)$, and a useful relation between $J(v, k, k-1)$ and $Cay(S_n, B)$.

Let v, k, and i be fixed positive integers with $v \geq k \geq i$, and let Ω be a fixed set of size v. The vertices of $J(v, k, i)$ are the subsets of Ω with size k, and two vertices are adjacent if their intersection has size i.

By definition, $J(v, k, i)$ has $\binom{v}{k}$ vertices, and it is a $\binom{k}{i}\binom{v-k}{k-i}$-regular graph. For $v \geq 2k$, the graphs $J(v, k, k-1)$ are known as the *Johnson graph* and the graphs $J(v, k, 0)$ are known as the *Kneser graphs*. In particular, $J(5, 2, 0)$ is known as the *Petersen graph*. For the graphs $J(v, k, i)$, we suggest readers to refer to [62] for the details. The graphs $J(5, 2, 1)$ and $J(6, 3, 2)$ are given in Fig. 4.1 and Fig. 4.2.

A graph G is *vertex*(*resp. edge*) *transitive* if its automorphism group act transitively on $V(G)$(*resp.* $E(G)$. That is, for any two distinct vertices of G there exists an automorphism mapping one to the other. A connected graph G is *distance transitive* if given any two ordered pairs of vertices $\langle u, u' \rangle$ and $\langle v, v' \rangle$ such that $d(u, u') = d(v, v')$, there is an automorphism g of G such that $g(u) = u'$

[1] we mean that either x and y are nonadjacent, or edges e and (x, y) are different edges.



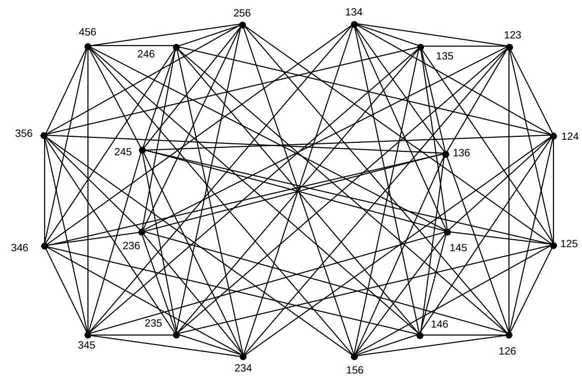


Fig.4.1. $J(5, 2, 1)$

Fig.4.2. $J(6, 3, 2)$

and $g(v) = v'$, where $d(u, u')$ denotes the distance between u and u'. It is easy to see that a distance-transitive graph is vertex-transitive, see [62] for the details.

Lemma 4.1.9 ([62]) *The graph $J(v, k, k-1)$ is distance transitive.*

It follows from Lemma 3.1 that the graph $J(v, k, k-1)$ is vertex transitive.

The *cartesian product* of G and H, written $G\Box H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $(v, v') \in E(H)$, or $v = v'$ and $(u, u') \in E(G)$.

For $X \subset V(G)$, $G[X]$ denotes the subgraph of G induced by X. When $D(T_B) = 3$, the following observation implies a relation between Johnson graphs and $Cay(S_n, B)$.

Lemma 4.1.10 *Given $B = \{((1m), (2m), \ldots, ((m-1)m), (m(m+1)), ((m+1)(m+2)), \ldots, ((m+1)n)\}$. Then there exists a mapping*

$$f : x_1x_2\cdots x_n \mapsto \{x_{m+1}, x_{m+2}, \ldots, x_n\}, \quad \forall x = x_1x_2\cdots x_n \in V(G_n) = S_n$$

from $V(Cay(S_n, B))$ to $V(J(n, n-m, n-m-1))$, such that the subgraph of $Cay(S_n, B)$ induced by the inverse image of $\{x_{m+1}, x_{m+2}, \ldots, x_n\}$ is a connected component of $Cay(B - \{((m-1)m)\} : S_n)$ isomorphic to $ST_m \Box ST_{n-m}$, which is denoted by $G_{x_{m+1}x_{m+2}\ldots x_n}$.

Lemma 4.1.11 *There exists a hamiltonian path between any two vertices in* $J(5,2,1)-F$ *for any* $F \subseteq E(J(5,2,1))$ *with* $|F| \leq 2$.

Proof. It is easy to see that $D(J(5,2,1)) = 2$. Let F be an edge set of $J(5,2,1)$ with $|F| \leq 2$. By Lemma 4.1.9, it is sufficient to show that there exists hamiltonian paths between 13 and 24, and between 35 and 45 in $J(5,2,1) - F$.

We first show that there is hamiltonian path between 13 and 24 in $J(5,2,1) - F$. Note that $P_1 = (13, 14, 34, 23, 12, 15, 45, 35, 25, 24)$ and $P_2 = (13, 34, 45, 14, 15, 35, 23, 25, 12, 24)$ are two edge-disjoint hamiltonian paths between 13 and 24, then there is nothing to do if one of them contains no fault edge. Supposes each of them contains an edge of F. Note that $P_3 = (13, 23, 34, 35, 45, 25, 15, 12, 14, 24)$ is another hamiltonian path between 13 and 24 in $J(5,2,1)$, and $P_3 \cap P_2 = \emptyset$, $P_3 \cap P_1 = \{(23, 34), (35, 45), (15, 12)\}$. If P_3 contains no fault edge, we are done. If not, P_3 contains a fault edge e, then $e \in \{(23, 34), (35, 45), (15, 12)\}$ since $|F| \leq 2$. Clearly, $P_4 = (13, 23, 12, 14, 34, 35, 25, 15, 45, 24)$ is a hamiltonian path between 13 and 24 in $J(5,2,1) - E(P_2) - \{(23, 34), (35, 45), (15, 12)\}$. That is, there is a fault-free hamiltonian path between 13 and 24 in $J(5,2,1) - F$.

Similarly, we can find a hamiltonian path between 35 and 45 in $J(5,2,1) - F$. We complete the proof. □

Lemma 4.1.12 *There exists a hamiltonian path between any two vertices in* $J(6,3,2)-F$ *for any* $F \subseteq E(J(6,3,2))$ *with* $|F| \leq 2$.

Proof. It is easy to see that $D(J(6,3,2)) = 3$. Let F be an edge set of $J(6,3,2)$ with $|F| \leq 2$. By Lemma 4.1.9, it is sufficient to show that there exists hamiltonian paths between 123 and 456, 123 and 256, and 123 and 234 in $J(6,3,2) - F$.

For the vertices 123 and 456, we can find three edge-disjoint hamiltonian paths as follows: $P_1 = (123, 134, 146, 126, 124, 245, 145, 125, 235, 135, 156, 136, 236, 246, 256, 356, 345, 234, 346, 456)$, $P_2 = (123, 124, 125, 126, 156, 145, 146, 136, 135, 134, 234, 246, 245, 256, 236, 235, 345, 346, 356, 456)$, $P_3 = (123, 125, 135, 145, 134, 136, 126, 256, 156, 146, 346, 246, 124, 234, 236, 356, 235, 245, 345, 456)$. Thus, there is a hamiltonian path between 123 and 456 in $J(6,3,2) - F$, since $|F| \leq 2$.

The other two cases are very similar to that of the case above. □

A hamiltonian cycle C is called *triangle-v hamiltonian cycle* if the two neighbors w, z of v on C are adjacent in G, denoted by C_v. Using the similar method of Lemma 4.1.11, we have the following two lemmas.

Lemma 4.1.13 *There exists a triangle-v hamiltonian cycle C_v in $J(5,2,1) - F$ for any $v \in V(J(5,2,1))$ and $F \subseteq E(J(5,2,1))$ with $|F| \leq 2$.*

Lemma 4.1.14 *There exists a triangle-v hamiltonian cycle C_v in $J(6,3,2) - F$ for any $v \in V(J(6,3,2))$ and $F \subseteq E(J(6,3,2))$ with $|F| \leq 3$.*

Let C_n denote the cycle of length n and K_n denote the complete graph with n vertices, respectively. It is not difficult to see that $C_6 \Box K_2$ and $C_6 \Box C_6$ are both vertex transitive and edge transitive bipartite graphs. The proof of Lemma 4.1.15 below is easy. So we omit the detailed proof of Lemma 4.1.15.

Lemma 4.1.15 *Suppose that $x \in V_0(C_6 \Box K_2), y \in V_1(C_6 \Box K_2)$, $e \in E(C_6 \Box K_2)$ and $e \neq (x, y)$. Then there is a hamiltonian path between x and y containing e in $C_6 \Box K_2$.*

Proof. Note that $C_6 \Box K_2$ is vertex transitive. We fix a vertex x and prove the claim by finding the hamiltonian path containing the given edge between the following four pairs of vertices: x and $y_i, i = 1, 2, 3, 4$ such that $d(x, y_i) = i$. It is not difficult to prove the result by constructing a hamiltonian path between x and y_i containing any given edge. □

Lemma 4.1.16 *Suppose $x \in V_0(C_6 \Box C_6), y \in V_1(C_6 \Box C_6), e \in E(C_6 \Box C_6)$ and $e \neq (x, y)$. Then there is a hamiltonian path between x and y containing e.*

Proof. Note that $C_6 \Box C_6$ contains three vertex disjoint copies of $C_6 \Box K_2$, say H_1, H_2 and H_3, such that there exists a matching $M_{i,j}$ between $V(H_i)$ and $V(H_j)$ which matches the vertices of a cycle C_6 in H_i to the vertices of a cycle C_6 in H_j for $i \neq j$. Let $x \in V_0(H_i)$ and $y \in V_1(H_j)$. Since $C_6 \Box C_6$ is edge transitive, without loss of generality we may assume $e \in E(H_1)$.

Case 1. $i = j = 1$.

By Lemma 3.6, there exists a hamiltonian path P_1 in H_1 between x and y containing e. Pick $f = (u, v) \in E(P_1)$ such that $(u, u'), (v, v') \in M_{1,2}$ (or $M_{1,3}$), where u' and v' are the matched vertices of u and v by $M_{1,2}$ (or $M_{1,3}$). Clearly, $f' = (u', v') \in E(C_6 \square C_6)$. Call $f' = (u', v')$ the matched edge of f. Without loss of generality, assume that $f' \in E(H_2)$. Similarly, take $g = (w, z) \in E(H_2)$ and $g' = (w', z') \in E(H_3)$ such that $g \neq f'$ and g' is the matched edge of g. By Lemma 4.1.15, there exists a hamiltonian path P_2 in H_2 between u' and v' containing f' and a hamiltonian path P_3 in H_3 between w' and z'. Thus, a hamiltonian path between x and y containing e is obtained from $P_1 - f, P_2 - f' - g, P_3 - g'$ and edges $(u, u'), (v, v'), (w, w'), (z, z')$.

Case 2. $i = 1$ or $j = 1$.

By the symmetry of H_1, H_2 and H_3, it is sufficient to consider the case $i = 1$ and $j = 3$. Take $u \in V_1(H_1), u' \in V_0(H_2), v \in V_1(H_2), v' \in V_0(H_3)$ such that $e \neq (x, u)$ and $(u, u'), (v, v') \in E(C_6 \square C_6)$. By Lemma 3.6, there exists a hamiltonian path P_1 in H_1 between x and u containing e, a hamiltonian path P_2 in H_2 between u' and v, a hamiltonian path P_3 in H_3 from v' to y. Thus, a hamiltonian path in $C_6 \square C_6$ between x and y containing e is formed by paths P_1, P_2, P_3 and edges $(u, u'), (v, v')$.

Case 3. $i = j \neq 1$.

Without loss of generality, we assume $i = 3$. It is not difficult to see that there exists $f = (u, v) \in E(H_1), f' = (u', v') \in E(H_2), g = (z, w) \in E(H_2)$ and $g' = (z', w') \in E(H_3)$ such that $e \neq f, g \neq f'$, and both f and g are matched by f' and g', respectively. By Lemma 4.1.15, there exists a hamiltonian path P_1 in H_1 between u and v containing e, a hamiltonian path P_2 in H_2 between z and w containing f', a hamiltonian path P_3 in H_3 between x and y containing g'. Thus, an expected hamiltonian path is obtained from path $P_1, P_2 - f', P_3 - g'$ and edges $(u, u'), (v, v'), (w, w'), (z, z')$.

Case 4. $i, j \neq 1$ and $i \neq j$.

Similarly as the argument of Case 2, we complete the proof. □

§4.1.2 G_n is $(n-3)$-edge fault tolerant hamiltonian laceable if $D(T_B) < 4$

Let H_1 and H_2 be two subgraphs of a bipartite graph H, and let $F \subseteq E(H)$ be a fault edge set. We say that H_1 and H_2 have *property* $\mathcal{P}$ with respect to F if there exists the vertices $x_1 \in V_0(H_1), y_1 \in V_1(H_1), x_2 \in V_1(H_2)$ and $y_2 \in V_0(H_2)$ such that $(x_1, x_2), (y_1, y_2) \in E(H) - F$.

Lemma 4.1.17 *Let $H = (V_0 \bigcup V_1, E)$ be a bipartite graph with fault edge set F, and $H_i, i = 1, 2, \cdots, m$, be the subgraphs of H such that $V_0(H) \bigcup V_1(H) = V(H_1) \bigcup V(H_2) \bigcup \cdots \bigcup V(H_m)$, $V(H_i) \bigcap V(H_j) = \emptyset$ for $i \neq j$. Suppose $x \in V_0(H_i), y \in V_1(H_j), i \neq j$, and $i_1, i_2, \cdots, i_m$ is a sequence of $\{1, 2, \cdots, m\}$ such that $i_1 = i, i_m = j$. If H_{i_r} and $H_{i_{r+1}}$ have property $\mathcal{P}$ with respect to F for each r, $1 \leq r \leq m-1$, and $H_{i_r} - F$ is hamiltonian laceable for $1 \leq r \leq m$, then there exists a hamiltonian path in H between x and y.*

Proof. By the definition of property $\mathcal{P}$, the lemma is clearly true. □

We have the following corollary by the above lemma.

Corollary 4.1.18 *Let $S \subseteq \{1, 2, \ldots, n\}$ and $F \subseteq E(G_n)$ with $|F| \leq n-3$. For any $x \in V_0(G_n(i)), y \in V_1(G_n(j)), i, j \in S$, $i \neq j$, if $n \geq 5$ and $G_n(k) - F$ is hamiltonian laceable for each $k \in S$, then there exists a hamiltonian path in $G_n[\bigcup_{i \in S} V(G_n(i))] - F$ between x and y.*

Note that $ST_2 \cong K_2$ and $ST_3 \cong C_6$, then it is easy to see that $Cay(\{(13), (23)\} : S_5)$ is the disjoint union of twenty cycles of length 6, and $Cay(\{(13), (23), (45)\} : S_5)$ is formed by ten subgraphs that each of them is isomorphic to $C_6 \square K_2$.

Lemma 4.1.19 *Suppose $B = \{(13), (23), (34), (45)\}$ and $F \subseteq E(Cay(B : S_5))$. If $|F| \leq 2$ and all edges of F have the same label (34), then $Cay(B, S_5) - F$ is hamiltonian laceable.*

Proof. Let $f(x_1 x_2 \cdots x_5) = \{x_4, x_5\}$ be the mapping from $V(Cay(B : S_5))$ to $V(J(5, 2, 1))$ in Lemma 4.1.10. Thus, the subgraph induced by the inverse image of $\{x_4, x_5\}$ is a connected component of $Cay(B - \{(34)\} : S_5)$, written $G_{x_4 x_5}$, which is isomorphic to $ST_3\Box ST_2$ ($C_6\Box K_2$). Furthermore, there are exactly two edges between $G_{x_4x_5}$ and $G_{y_4y_5}$ if $|\{x_4,x_5\}\bigcap\{y_4,y_5\}|=1$. In fact, we assume $x_5=y_5$, then $(x_1x_2y_4x_4x_5,\ x_1x_2x_4y_4y_5)$ and $(x_2x_1y_4x_4x_5, x_2x_1x_4y_4y_5)$ are such two edges between $G_{x_4x_5}$ and $G_{y_4y_5}$. Moreover, the two ends in $G_{x_4x_5}$ (or $G_{y_4y_5}$) of the above two edges are in different partite sets of $Cay(B,S_5)$. That is, $G_{x_4x_5}$ and $G_{y_4y_5}$ have the property $\mathcal{P}$ with respect to F.

For $J(5,2,1)$, we call edge $(\{x_4,x_5\},\{y_4,y_5\})$ fault edge of $J(5,2,1)$, if there exists some fault edges between $V(G_{x_4x_5})$ and $V(G_{y_4y_5})$ in G_5. Since $|F|\le 2$, $J(5,2,1)$ has at most two fault edges. Denote by F' the fault edge set of $J(5,2,1)$.

Now, we prove that $Cay(B:S_5)-F$ is hamiltonian laceable. Let $x\in V_0(G_{x_4x_5})$ and $y\in V_1(G_{y_4y_5})$. Consider the following two cases.

Case 1. $\{x_4,x_5\}=\{y_4,y_5\}$.

In this case, there exists a triangle-$\{x_4,x_5\}$ hamiltonian cycle $C_{\{x_4,x_5\}}$ in $J(5,2,1)-F'$ by Lemma 4.1.13. For convenience, we assume that $C_{\{x_4,x_5\}}=(U_1,U_2,\cdots,U_{10},U_1)$, where $U_1=\{x_4,x_5\}$. It is not difficult to see that there exists an edge $e=(u,v)$ in $G_{x_4x_5}$ such that $e\neq(x,y)$, and $(u,w),(v,z)\in E(Cay(B:S_5))$, where $w\in G_{U_2}$ and $z\in G_{U_{10}}$, (for example, if $U_2=\{x_4,t_5\}$ and $U_{10}=\{x_4,z_5\}$, where x_5,t_5 and z_5 are three distinct integers. Thus, at least one of $(t_5x_2z_5x_4x_5, z_5x_2t_5x_4x_5)$ and $(t_5x_2z_5x_5x_4, z_5x_2t_5x_5x_4)$ with label (13) is an such edge in $G_{x_4x_5}$. Similarly, the other cases is easy to obtained). By the definition of fault edge of $J(5,2,1)$, (u,w) and (v,z) are fault-free. By Lemma 4.1.15, there is a hamiltonian path P_1 in $G_{x_4x_5}$ between x and y containing edge (u,v). By the definition of fault edge of $J(5,2,1)$ and Lemma 4.1.17, there exists a path P_2 between z and w containing all vertices of $G_{U_i}, 2\le i\le 10$. Hence, a hamiltonian path in $Cay(B:S_5)-F$ is obtained from P_1-e, P_2, and edges (u,w) and (v,z).

Case 2. $\{x_4,x_5\}\neq\{y_4,y_5\}$.

By Lemma 4.1.11, there is a hamiltonian path in $Cay(B:S_5)-F$ between $\{x_4,x_5\}$ and $\{y_4,y_5\}$. Furthermore, by definition of fault edge of $J(5,2,1)$ and Lemma 4.1.17, there exists a hamiltonian path P in $Cay(B:S_5)-F$ between x and y. □

Lemma 4.1.20 *Suppose that* $B = \{((13), (23), (34), (45), (46)\}$ *and* $F \in E(Cay(B : S_6))$*. If* $|F| \leq 3$ *and all edges of* F *have label* (34)*, then* $Cay(B : S_6)$ *is hamiltonian laceable.*

Proof. Let $x \in V_0(Cay(B : S_6)), y \in V_1(Cay(B : S_6))$. We next show that there exists a hamiltonian path in $Cay(B : S_6) - F$ between x and y.

Similarly as the proof of Lemma 4.1.15, let f be the mapping from $V(Cay(B : S_6))$ to $V(J(6, 3, 2))$ in Lemma 4.1.10.

We definite the fault edge of $J(6, 3, 2)$ distinguishing the following two cases: when $x, y \in V(G_{x_4x_5x_6})$ for some set $\{x_4, x_5, x_6\}$, we call edge $(\{w_4, w_5, w_6\}, \{z_4, z_5, z_6\})$ fault edge of $J(6, 3, 2)$ if there exists fault edge between $V(G_{w_4w_5w_6})$ and $V(G_{z_4z_5z_6})$; when $x \in G_{x_4x_5x_6}, y \in G_{y_4y_5y_6}$ satisfying $\{x_4, x_5, x_6\} \neq \{y_4, y_5, y_6\}$, we call edge $(\{w_4, w_5, w_6\}, \{z_4, z_5, z_6\})$ fault edge of $J(6, 3, 2)$ if there exists at least two fault edges between $V(G_{w_4w_5w_6})$ and $V(G_{z_4z_5z_6})$.

It is easy to see there is exactly four edges between $G_{x_4x_5x_6}$ and $G_{y_4y_5y_6}$ if $|\{x_4, x_5, x_6\} \cap \{y_4, y_5, y_6\}| = 2$. By a similar argument as Lemma 4.1.19, this lemma follows from Lemma 4.1.12, 4.1.14, and 4.1.16. $\square$

Lemma 4.1.21 *Let* $F \subseteq E(ST_{n+1})$ *such that each edge of* F *has label* $(n(n + 1))$*. If* $|F| \leq 2n - 3$ *and* $n \geq 4$*, then* $ST_{n+1} - F$ *is a hamiltonian laceable.*

Proof. Assume $x \in V_0(ST_{n+1}(i_1)), y \in V_1(ST_{n+1}(i_2))$. Since each edge of F has label $(n(n+1))$, $E(ST_{n+1}(i)) \cap F = \emptyset$. Thus $ST_{n+1}(i) \cong ST_n, i = 1, 2, \cdots, n+1$, is hamiltonian laceable. We prove this lemma by distinguishing the following two cases.

Case 1. $i_1 \neq i_2$. Note that there are $\frac{(n+1-2)!}{2}$ edges between $V_0(ST_{n+1}(i))$ and $V_1(ST_{n+1}(j))$, and $\frac{(n+1-2)!}{2}$ edges between $V_1(ST_{n+1}(i))$ and $V_0(ST_{n+1}(j))$ if $i \neq j$ by Lemma 4.1.3. Since $2n - 3 < 2 \cdot \frac{(n+1-2)!}{2}$ for $n \geq 4$, at most one pair elements of $\{ST_{n+1}(i) | \, i = 1, 2, \cdots, n+1\}$ has no property $\mathcal{P}$ with respect to F. So there exists a sequence $ST_{n+1}(j_1), ST_{n+1}(j_2), \cdots, ST_{n+1}(j_{n+1})$ of $\{ST_{n+1}(i) \mid i = 1, 2, \cdots, n+1\}$ such that $i_1 = j_1, i_2 = j_{n+1}$ and $ST_{n+1}(k)$ and $ST_{n+1}(k + 1)$ have property $\mathcal{P}$ with respect to F, where, $1 \leq k \leq n$. By Lemma 4.1.17, there exists a fault-free hamiltonian path in $ST_{n+1} - F$ between x and y.

Case 2. $i_1 = i_2$.

Without loss of generality, assume $i_1 = i_2 = 1$, and let P_1 be a hamiltonian path of $ST_{n+1}(i_1)$ between x and y. Since P_1 has length $n! - 1$, there exists a matching in $ST_{n+1}(i_1)$ containing $\lfloor \frac{n!-1}{2} \rfloor$ edges of P_1. Since $\lfloor \frac{n!-1}{2} \rfloor > 2n - 3$ for $n \geq 4$ and all fault edges are outgoing edges, by the pigeonhole principle, we can pick an edge $(u, v) \in E(P_1)$ such that u, v are incident to no fault edge. Let u' and v' be the outgoing neighbors of u and v, respectively, and assume $v' \in V_1(ST_{n+1}(j_2))$ and $u' \in V_0(ST_{n+1}(j_{n+1}))$ ($j_2 \neq j_{n+1}$ by Lemma 4.1.3). Without loss of generality, assume $P_1 = (x, \cdots, u, v, \cdots, y)$.

If there exists a sequence $ST_{n+1}(j_2), ST_{n+1}(j_3), \cdots, ST_{n+1}(j_{n+1})$ of $\{ST_{n+1}(j) \mid j = 2, \cdots, n+1$ such that $ST_{n+1}(k)$ and $ST_{n+1}(k+1)$ have property $\mathcal{P}$ with respect to F, where $k = 2, \cdots, n$. By Lemma 4.1.17, there exists a hamiltonian path P_2 between u' and v' in $ST_{n+1}[\bigcup_{2 \leq i \leq n+1} V(ST_{n+1}(i))]$. Thus, $xP_1uu'P_2v'vP_1y$ is a hamiltonian path in $ST_{n+1} - F$ between x and y.

Otherwise, there exists no sequence $ST_{n+1}(j_2), ST_{n+1}(j_3), \cdots, ST_{n+1}(j_{n+1})$ of $\{ST_{n+1}(j), j = 2, \cdots, n+1\}$ such that $ST_{n+1}(k)$ and $ST_{n+1}(k+1)$ have property $\mathcal{P}$ with respect to F, where $k = 2, \cdots, n$. It only happens when $n = 4$ and at least 3 fault edges between $ST_5(j_3)$ and $ST_5(j_4)$ (otherwise, there exists such a sequence). But, $|E^{j_3 j_4}| = 6$ and $|F| \leq 5$, So, at least one edge $e = (w, w')$ in $E^{j_3 j_4}$ is fault-free. Without loss of generality, assume $w \in V_0(ST_5(j_3))$ and $w' \in V_1(ST_5(j_4))$. Since $|F - E_{j_3 j_4}| \leq 2$, each pair of $\{ST_5(j) \mid j = 2, 3, 4, 5\}$, other than $ST_5(j_3)$ and $ST_5(j_4)$, has property $\mathcal{P}$ with respect to F. Thus, by Lemma 4.1.17, there exists a hamiltonian path P_2 in $ST_5[V(ST_5(j_2)) \cup V(ST_5(j_3))] - F$ between v' and w, a hamiltonian path P_3 in $ST_5[V(ST_5(j_4)) \cup V(ST_5(j_5))] - F$ between u' and w'. So a desired hamiltonian path in $ST_5 - F$ is obtained from path $P_i, i = 1, 2, 3$ and edges $(u, u'), (v, v'), (w, w')$. $\square$

Lemma 4.1.22 *Let $F \subseteq E(G_n)$ such that each edge of F has label (lk). If $D(T_B) = 3$, $|F| \leq (n - 3)$, and neither l nor k is a leaf of T_B, then $G_n - F$ is hamiltonian laceable.*

Proof. Since $D(T_B) = 3$ and neither l nor k is a leaf of T_B, either $2d_{T_B}(l) \geq n$ or $2d_{T_B}(k) \geq n$. Without loss of generality, assume that $d_{T_B}(l) = m \geq n/2$.

Furthermore, set $l = m$, $k = m+1, N_{T_B}(m) = \{1, 2, \cdots, m-1, m+1\}$ and $N_{T_B}(m+1) = \{m, m+2, \cdots, n\}$. If $n = 4$, then $G_4 \cong BS_4$ follows from $D(T_B) = 3$ and the result holds by Theorem 2.1. Suppose $n \geq 5$, then $m \geq 3$ by $D(T_B) = 3$. If $m = 3$, then $n = 5, 6$ follows from $d_{T_B}(m) = m \geq n/2$. Therefore, the result holds for $m = 3$ by Lemmas 4.1.19 and 4.1.20. Suppose $m \geq 4$ in the following argument.

By induction on $n-m$. Clearly, $n-m \geq 2$. Let $x \in V_0(G_n(i_1)), y \in V_1(G_n(i_2))$. It is suffices to show that there exists a hamiltonian path in $G_n - F$ between x and y. Let $G_n(1), G_n(2), \ldots, G_n(n)$ be the decomposition induced by leaf n (*edge* $(m+1, n)$). If $n - m = 2$, then $G_n(i) \cong ST_{m+1}$. Since $n - 3 \leq 2m - 3$, $G_n(i) - F$ is hamiltonian laceable for each i by Lemma 4.1.21. It is easy to check that $G_n - F$ is hamiltonian laceable by considering either $i_1 \neq i_2$ or $i_1 = i_2$, and we omit it.

Suppose that this lemma holds for $n - m < k, k \geq 2$. It is not different to present that this lemma holds for $n - m = k$ by induction on $n - m$, and we omit it. □

Theorem 4.1.23 *Cayley graphs G_n are $(n-3)$-edge fault tolerant hamiltonian laceable for $n \geq 4$ if $D(T_B) \leq 3$.*

Proof. If $D(T_B) = 2$, then the result is true by Theorem 4.1.7. Thus, suppose $D(T_B) = 3$. By induction on n. If $n = 4$, then $G_4 \cong SB_4$ and the result holds by Theorem 4.1.7. We assume that the result holds for G_{n-1}, and prove that the result holds for G_n. If there exists no fault edge of F with label (l, k) such that $d_{T_B}(l) = 1$ or $d_{T_B}(k) = 1$, then the result holds by Lemma 4.1.22.

If not, that is, there exists some fault edges of F with label (lk) such that $d_{T_B}(l) = 1$ or $d_{T_B}(k) = 1$. Without loss of generality, we assume $l = n-1, k = n$ and $d_{T_B}(n) = 1$. Let $G_n(1), G_n(2), \ldots, G_n(n)$ be the decomposition induced by leaf n. Since at least one of F is outgoing edge, $|E(G_n(i)) \bigcap F| \leq n - 4$ for each i. Hence $G_n(i) - F$ is hamiltonian laceable by induction hypothesis. For $x \in V_0(G_n(i_1)), y \in V_1(G_n(i_2))$, it is suffices to show that there exists a hamiltonian path in $G_n - F$ between x and y.

If $i_1 \neq i_2$. There exists a hamiltonian path in $G_n - F$ between x and y by Lemma 4.1.18. Otherwise, that is, $i_1 = i_2$. Let P_1 be a hamiltonian path P_1

between x and y in $G_n(i_1) - F$. By Lemma 4.1.2, $G_n(i_1)$ contains $n-1$ copes of G_{n-2}. Thus P_1 contains at least $n-2$ edges with label $(r(n-1))$, where one of r and $(n-1)$ is a leaf of $T_B - n$. Since $|F| \leq n-3$ and edges with the same label are nonadjacent, by the pigeonhole principle, at least one edge, say $e = (u, v)$, of such $n-2$ edges is adjacent to no fault edge with label $((n-1)n)$. Let both u' and v' be the outgoing neighbors of u and v, respectively. Since (u, v) has label $(r(n-1))$, u', v' are in distinct subgraphs $G_n(j_k)$, say $G_n(j_2), G_n(j_n))$. By Lemma 4.1.17, $G_n[\bigcup_{2\leq k\leq n} V(G_n(j_k))] - F$ has a hamiltonian path P_2 between u' and v'. Therefore, a desired hamiltonian path is obtained from paths $P_1 - e, P_2$ and edges $(u, u'), (v, v')$. □

§4.1.3 G_n is $(n-3)$-edge fault tolerant hamiltonian laceable if $D(T_B) \geq 4$

In this section, we first show that if $n \geq 5$ and $D(T_B) \geq 3$, then there exists a particular decomposition by some leaf of T_B. This decomposition is the key of all the following results.

Lemma 4.1.24 *For arbitrary $e \in E(G_n)$, if $n \geq 5$ and $D(T_B) \geq 3$, then there exists a decomposition of G_n induced by some leaf m of T_B such that $D(T_B - m) \geq 3$ and e is no outgoing edge.*

Proof. First, if $D(T_B) \geq 4$, let P be a longest path in T_B with ends i, j, and let both i' and j' be neighbors of i and j on P, respectively. If e has no label (ii'), then the decomposition induced by leaf i is a desired decomposition.

Next, suppose $D(T_B) = 3$. Since $n \geq 5$, T_B has at least three leaves, and at least two leaves have a common neighbor in T_B. we may assume that i, j have common neighbor k. If e has no label (ik), then the decomposition induced by leaf i is a desired decomposition. □

Lemma 4.1.25 *Let $x \in V_0(G_n), y \in V_1(G_n)$ and $e \in E(G_n)$ such that $e \neq (x, y)$. If $D(T_B) \geq 3$ and $n \geq 4$, then there exists a hamiltonian path in G_n between x and y containing e .*

Proof. If $n = 4$, then $G_4 \cong BS_4$ and the result holds by Lemma 4.1.5. When $n \geq 5$, by Lemma 4.1.24, we can assume that $((n-1)n)$ is a pendent edge, and that the decomposition induced by leaf n satisfies $e \in E(G_n(k))$ and $D(T_B - n) \geq 3$. First, suppose that the result holds for G_{n-1} and let $x \in V_0(G_n(i_1))$ and $y \in V_1(G_n(i_2))$. It is easy to check that the result holds for G_n by considering the following four cases: (1) $i_1 = i_2 = k$; (2) $i_1 = k$ and $i_1 \neq i_2$; (3) $i_1 = i_2$ and $k \neq i_1$ and (4) i_1, i_2 and k are pairwise distinct. $\square$

Corollary 4.1.26 *Let $e = (u, v) \in E(G_n)$ and $x \in V(G_n)$. If $n \geq 4$ and $D(T_B) \geq 3$, then $G_n - \{u, v\}$ has a hamiltonian path starting from x.*

Proof. Without loss of generality, assume $x \in V_0(G_n)$ and $v \in V_1(G_n)$. Thus, there exists a hamiltonian path P in G_n between x and v containing e. So $P - \{u, v\}$ is a desired path. $\square$

Recall that for arbitrary $x \in V(G_n(i))$, unless stated otherwise, we call the vertex $x' = x((n-1)n))$ the outgoing neighbor of x and (x, x') the outgoing edge of x. Moreover, for arbitrary pair-edge e, we use e' to denote the coupled pair-edge.

Lemma 4.1.27 *Let $F \subseteq E(G_n), S \subseteq \{1, 2, \cdots, n\}$ with $n \geq 5$, and let $G_n(1)$, $G_n(2), \cdots, G_n(n)$ be the decomposition induced by a leaf n of T_B satisfying $D(T_B - n) \geq 3$. For arbitrary path P of $G_n - F$, if*
(1). $V(P) \bigcap V(G_n(i)) = \emptyset, i \in S$;
(2). there exists an edge $(u, v) \in E(P)$ such that $u[n-1], v[n-1] \in S$ and the outgoing edges of u and v are fault-free; and
(3). $F \bigcap E(G_n[\bigcup_{i \in S} V(G_n(i))]) = \emptyset$.
Then P can be extended to a path P^* in G_n such that P^* contains all edges of $P - e$ and all vertices of $G_n(i), i \in S$.

Proof. Since $D(T_B - n) \geq 3$, Lemma 4.1.25 holds for $G_n(i)$ satisfying $F \cap E(G_n(i)) = \emptyset$. Assume $u' \in G_n(i_1), v' \in G_n(i_2)$. If $|S| = 1$, then the result follows from Theorem 2.2. Suppose $|S| \geq 2$. If $i_1 \neq i_2$, then the result follows from Observation 4.1.

Otherwise $i_1 = i_2$. When $|S| = 2$, we set $S = \{i_1, j\}$. We can choose $e = (w, z) \in E(G_n(i_1))$ such that $e \neq (u', v')$, and w', z' are both in $G_n(j)$ since $D(T_B) \geq D(T_B - n) \geq 3$. By Lemma 4.1.25, there exists a hamiltonian path P_1 in $G_n(i_1)$

between x and y containing e, a hamiltonian path P_2 in $G_n(j)$ between w' and z'. Thus a desired path P^* is obtained from paths $P-(u,v), P_1-e, P_2$ and edges $(u,u'), (v,v'), (z,z'), (w,w')$.

When $|S|>2$, then we can choose $e=(w,z)\in E(G_n(i_1))$ such that $w'\in V(G_n(i_3))$, and $z'\in V(G_n(i_4))$, where $i_3,i_4\in S$ and $i_3\neq i_4$. (It is not difficult to see that there does exist a such edge since $D(T_B)\geq 3$, for example, if $(r(n-1))\in B$, then $(w_1\cdots w_{r-1}i_3w_{r+1}\cdots w_{n-3}i_4i_1, w_1\cdots w_{r-1}i_4w_{r+1}\cdots w_{n-3}i_3i_1)$ is a such edge with label $(r(n-1))$). Clearly $e\neq(u',v')$ since they have different outgoing edges. By Lemma 4.1.25, there exists a hamiltonian path P in $G_n(i_1)$ between u' and v' containing e. By Lemma 4.1.17, there exists a hamiltonian path P_2 in $G_n[\bigcup_{i\in S-\{i_1\}}V(G_n(i)]$ between w' and z'. So, an desired path P^* is obtained from paths $P-(u,v), P_1-e, P_2$ and edges $(u,u'), (v,v'), (z,z'), (w,w')$. □

Theorem 4.1.28 *Cayley graphs G_n are $(n-3)$-edge fault tolerant hamiltonian laceable for $n\geq 5$ if $D(T_B)\geq 4$.*

Proof. By induction on n. If $n=5$, then $G_5\cong BS_5$ and BS_5 is $(5-3)$-edges fault-tolerant hamiltonian laceable follows from $D(T_B)\geq 4$ and Theorem 4.1.7. We suppose that G_{n-1} is $(n-4)$-edges fault tolerant hamiltonian laceable for $n\geq 6$, and next present that this theorem holds for G_n. Let $F\in E(G_n)$ with $|F|\leq n-3$.

If there exists a edge $e\in F$ with label (lk) such that either $d_{T_B}(l)=1$ or $d_{T_B}(k)=1$. Without loss of generality, assume that $d_{T_B}(k)=1$, $k=n, l=n-1$. Hence, $|E(G_n(i))\bigcap F|\leq n-4$ for all $i\in\{1,2,\cdots,n\}$. If $D(T_B-n)\geq 4$, $G_n(i)-F$ is hamiltonian laceable by induction hypothesis. If not, that is, $D(T_B-n)=3$ since $D(T_B)\geq 4$. Thus $G_n(i)-F$ is hamiltonian laceable by Theorem 4.1.23. It is easy to check that the result holds in this case, and we omit it.

If there exists no edge e in F with label (lk) such that either $d_{T_B}(l)=1$ or $d_{T_B}(k)=1$. Suppose $|F|=n-3$ (add edges to F if necessary). If T_B is a path, then this case holds by Theorem 4.1.7. Thus we suppose that T_B is not a path. We have the following claim.

Claim. If T_B is not a path, then there exists a decomposition $G_n(1), G_n(2), \ldots, G_n(n)$ of G_n such that at least two fault edges with the same label are pair-edges.

Proof of Claim 1. Since T_B is not a path, it has at least three leaves. But no edge $e \in F$ has label (lk) such that either $d_{T_B}(l) = 1$ or $d_{T_B}(k) = 1$, fault edges occupy at most $n-4$ labels. Since $|F| \doteq n-3$, by the pigeonhole principle, at least two fault pair-edges have the common label, say (i, j). Clearly, there exists a leaf, without loss of generality, say n, such that n is not incident to the edge (i, j) in T_B. The decomposition $G_n(1), G_n(2), \ldots, G_n(n)$ of G_n induced by leaf n satisfies the requirement of this claim. This complete the proof of claim 1.

Assume that n has neighbor $n-1$ in T_B. Since $D(T_B) \geq 4$, $D(T_B - n) \geq 3$. Therefore, Lemmas 4.1.25 and 4.1.27 hold for $G_n(i)$ with $F \cap E(G_n(i)) = \emptyset$. We suppose that $|F \bigcap E(G_n(i))| = n-3$ for some i in the following. (Otherwise, that is, $|F \bigcap E(G_n(i))| \leq n-4$ for all $i \in \{1, 2, \cdots, n\}$, this case is easily checked and omitted.) Since $F \subseteq E(G_n(i))$, Lemmas 4.1.25 and 4.1.27 hold for $G_n(j), j \neq i$.

It is suffices to show that for any two vertices x and y with different partite sets, there exits a hamiltonian path in $G_n - F$ between x and y. Assume $x \in V_0(G_n(i_1)), y \in V_1(G_n(i_2))$, and consider the following four cases.

Case 1. $i_1 = i_2 = i$.

Suppose that $G_n(i) - F$ has a hamiltonian path P between x and y. Clearly, we can pick an edge $e = (u, v)$ of P such that $u[n-1], v[n-1] \neq i$. By Lemma 5.2, P can be extended to a hamiltonian path P^* in $G_n - F$ between x and y.

Next, suppose that $G_n(i) - F$ has no hamiltonian path from x to y. Pick $f = (w, z) \in F$ such that $f \neq (x, y)$, and let $F' = F - f$. Thus $G_n(i) - F'$ has a hamiltonian path P between x and y by induction hypothesis. Clearly, $f \in E(P)$. Thus P can be extended to a hamiltonian path P^* in $G_n - F$ between x and y by Lemma 4.1.27.

Case 2. $i_1 = i$ and $i_1 \neq i_2$.

Suppose that $G_n(i) - F$ has a hamiltonian path P_1 between x and some vertex v. If $v' \in V_0(G_n(i_2))$. Pick $e = (w, z) \in E(G_n(i_2))$ such that $w[n-1], z[n-1] \notin \{i, i_2\}$. By Lemma 4.1.25, there exists a hamiltonian path P_2 in $G_n(i_2)$ between v' and y containing e . By Lemma 4.1.27, (x, P_1, v, v', P_2, y) can be extended to a hamiltonian path in $G_n - F$ between x and y. If $v' \notin V_0(G_n(i_2))$, then, by

Lemma 4.1.18, there exists a hamiltonian path P_2 in $G_n - G_n(i) - F$ between v' and y. Hence a desired hamiltonian path is obtained from P_1, P_2 and edge (v, v').

We next suppose that $G_n(i) - F$ has no hamiltonian path starting from x.

If there exists a pair-edge $f = (u, v) \in F$ such that $f[n-1] \neq i_2$. Choose $z \in V_1(G_n(i))$ such that $f \neq (x, z)$ and $z' \in V_0(G_n(i_2))$. Let $F' = F - f$. By induction hypothesis, $G_n(i) - F'$ has a hamiltonian path P_1 between x and z. Clearly, $f \in E(P_1)$. By Theorem 4.1.8, $G_n(i_2)$ has a hamiltonian path P_2 between z' and y. By Lemma 4.1.27, (x, P_1, z, z', P_2, y) can be extended to a hamiltonian path in $G_n - F$ between x and y.

If not, that is, there exists no pair-edge $f \in F$ such that $f[n-1] \neq i_2$. By Claim 1, at least two fault pair-edges have the common label. So we can find a pair-edge $f = (u, v) \in F$ such that neither u nor v has y as the outgoing neighbor. Let $F' = F - \{f\}$ and the coupled pair-edge of f is $f' = (u', v') \in E(G_n(i_2))$. By Lemma 4.1.26, $G_n(i_2) - \{u', v'\}$ has a hamiltonian path P_2 starting from y to some vertex, say z.

If $z' \in V_1(G_n(i))$, then $G_n(i) - F'$ has a hamiltonian path P_1 between x and z' by induction hypothesis. Clearly, $f \in E(P_1)$. Without loss of generality, assume that $P_1 = (x, \cdots, v, u, \cdots, z')$. Pick an edge $e = (s, t) \in E(P_1)$ such that $s[n-1], t[n-1] \notin \{i, i_2\}$. (In fact, let s be a vertex in $G_n(i)$ different from x, u, v and z' such that $s[n-1] \neq i_2$. Clearly at least one of two incident edges of s on P_1 is a such edge since two incident edges of s on P_1 have different labels.) By Lemma 4.1.27, $(x, \cdots, v, v', u', u, \cdots, z', z, P_2, y)$ can be extended to a hamiltonian path in $G_n - F$ between x and y. If $z' \notin V_1(G_n(i))$, then we may assume that $z' \in V_1(G_n(i_3)), i_3 \notin \{i, i_2\}$. Choose $w \in V_1(G_n(i))$ such that $f \neq (x, w)$ and $w' \in V_0(G_n(i_4))$, where $i_4 \notin \{i, i_2, i_3\}$. Let P_1 be a hamiltonian path in $G_n(i) - F'$ between x and w. Clearly, $f \in E(P_1)$. By Lemma 4.1.18, $G_n - G_n(i) - G_n(i_2)$ has a hamiltonian path P_3 between w' and z'. Hence a desired hamiltonian path is obtained from $P_1 - f, P_2, P_3$ and edges $(u, u'), (v, v'), (u', v'), (w, w')$ and (z, z').

Case 3. $i_1 = i_2 \neq i$.

If there exists a pair-edge $f = (u, v) \in F$ such that its coupled pair-edge $f' = (u', v') \in E(G_n(i_1))$ and $f' \neq (x, y)$. By Lemma 4.1.25, $G_n(i_1)$ has a hamiltonian path P_1 between x and y containing f'. Without loss of generality, assume $P_1 = (x, \cdots, u', v', \cdots, y)$. By induction hypothesis, $G_n(i) - F$ has a hamiltonian path P_2 between u and v. Clearly, we can pick an edge $e = (w, z)$ of P_2 such that $w[n-1], z[n-1] \notin \{i, i_1\}$, then by Lemma 4.1.27, $(x, \cdots, u', u, P_2, v, v', \cdots, y)$ can be extended to a hamiltonian path in $G_n - F$ between x and y.

If not, that is, there exists no pair-edge $f = (u, v) \in F$ such that its coupled pair-edge $f' = (u', v') \in E(G_n(i_1))$, and $f' \neq (x, y)$. We have the following claim.

Claim There exists a pair-edge $f = (u, v) \in F$ such that its coupled pair-edge $f' = (u', v') \in E(G_n(i_3))$, $i_3 \notin \{i_1, i\}$.

Proof of Claim 2. By Claim 1, at least two fault pair-edges have the common label, say f_1 and f_2. First, clearly, $f_1[n-1], f_2[n-1] \neq i$ since $f_1, f_2 \in E(G_n(i))$. If $f_1[n-1] \neq i_1$, we are done. Thus, suppose $f_1[n-1] = i_1$ and $f_1 = (x, y)$(Otherwise, that is, $f_1[n-1] = i_1$ and $f_1 \neq (x, y)$, which contradicts the hypothesis that there exists a pair-edge $f = (u, v) \in F$ such that its coupled pair-edge $f' = (u', v') \in E(G_n(i_3))$, $i_3 \notin \{i_1, i\}$). We say $f_2[n-1] \neq i_1$. Otherwise, that is, $f_2[n-1] = i_1$. Since f_1 and f_2 have the common label and $f_1 = (x, y)$, we know $f_2 \neq (x, y)$, which contradicts the hypothesis that there exists a pair-edge $f = (u, v) \in F$ such that its coupled pair-edge $f' = (u', v') \in E(G_n(i_3))$, $i_3 \notin \{i_1, i\}$). This complete the proof of this claim.

By Claim 2, we can pick a pair-edge $f = (u, v) \in F$ such that its coupled pair-edge $f' = (u', v') \in E(G_n(i_3))$, $i_3 \notin \{i_1, i\}$. By induction hypothesis, $G_n(i) - F$ has a hamiltonian path P_1 from u to v. It is not different to construct a hamiltonian path P_2 in $G_n - G_n(i)$ between x and y containing f'. Thus, a desired hamiltonian path in $G_n - F$ is obtained from paths P_1, P_2 and edges $(u, u'), (v, v')$.

Case 4. i_1, i_2 and i are pairwise distinct.

If there exists a pair-edge $f = (u, v) \in F$ such that its coupled pair-edge $f' = (u', v') \in E(G_n(i_3)), i_3 \notin \{i_1, i_2\}$. Choose $z \in V_1(G_n(i_1))$, $w \in V_0(G_n(i_2))$ such that $z' \in V_0(G_n(i_3)), w' \in V_1(G_n(i_3))$. Furthermore, pick $e = (s, t) \in E(G_n(i_2))$ such that $e \neq (y, w), s[n-1], t[n-1] \notin \{i, i_1, i_2, i_3\}$. By Theorem 4.1.8, $G_n(i_1)$ has a hamiltonian path P_1 between x and z. By Lemma 4.1.20, there exists a hamiltonian path P_2 in $G_n(i_2)$ between y and w containing e, a hamiltonian path

P_3 in $G_n(i_3)$ between w' and z' containing f'. By induction hypothesis, there exists a fault-free hamiltonian path P_4 in $G_n(i) - F$ between u and v. Without loss of generality, assume $P_3 = (w', \cdots, u', v', \cdots, z')$. By Lemma 4.1.27, $(x, P_1, z, z', \cdots, v', v, P_4, u, u', \cdots, w', w, P_2, y)$ can be extended to a hamiltonian path in $G_n - F$ between x and y.

If not, that is, for any pair-edge $f \in F$, its coupled pair-edge f' is contained in either $G_n(i_1)$ or $G_n(i_2)$. By the symmetry of i_1 and i_2, we may assume that $f = (u, v)$ is a fault pair-edge and its coupled pair-edge $f' = (u', v') \in E(G_n(i_1))$. Choose a vertex $z \in V_1(G_n(i_1))$ such that $f' \neq (x, z)$ and $z' \in V_0(G_n(i_3))$, where $i_3 \notin \{i, i_1, i_2\}$. By Lemma 4.1.25, $G_n(i_1)$ has a hamiltonian path P_1 between x and z containing f'. Without loss of generality, assume $P_1 = (x, \cdots, u', v', \cdots, z)$. By induction hypothesis, $G_n(i) - F$ has a hamiltonian path P_2 between u and v. By Lemma 4.1.18, $G_n - G_n(i) - G_n(i_1)$ has a hamiltonian path P_3 between z' and y. Hence $(x, \cdots, u', u, P_2, v, v', \cdots, z, z', P_3, y)$ is a hamiltonian path in $G_n - F$ between x and y.

By this all possibilities have been exhausted and the proof is thus complete. $\square$

Combining Theorem 4.1.23 with Theorem 4.1.28, we have the following main result.

Theorem 4.1.29 *Cayley graphs G_n are $(n-3)$-edge fault tolerant hamiltonian laceable for $n \geq 4$.*

Note that G_n is $n-1$-regular. Thus the number of edge faults is sharp.

§4.2 Fault-tolerant bipancyclicity of Cayley graphs generated by transposition generating trees

Fault-tolerant pancyclicity of popular networks have been studied, such as star graphs [133], bubble-sort graphs [82], Möbius cubes [75, 147] and crossed cubes [76, 77]. In this section, we consider edge fault-tolerant bipancyclicity of Cayley graphs generated by transposition generating trees. For the other studies on Cayley graphs generated by transposition generating trees, we suggest readers to refer to [2, 3, 40, 60, 62, 73, 82, 84, 100, 127, 131, 151, 150].

We recall several notations related this section. Unless stated otherwise, we assume T_B is a tree and use G_n to denote the Cayley graph $Cay(S_n, B)$. Thus, each vertex of G_n is a permutation of S_n. We use $V_0(G_n)$ to denote the set of all odd permutations of S_n and $V_1(G_n)$ the set of all even permutations of S_n, respectively. Clearly, $V(G_n) = V_0(G_n)\bigcup V_1(G_n)$ and G_n is bipartite graph with partite sets $V_0(G_n)$ and $V_1(G_n)$. We use ST_n and BS_n to denote the n-dimensional star graph and n-dimensional bubble-sort graph, respectively.

By Lemma 4.1.1 in the last section, we have that the label of the vertex set of a transpositions generating tree T does not affect the structural property of the Cayley graph generated by T. We call the decomposition of Lemma **??** is induced by the leaf n. Unless stated otherwise, we always assume that $(n-1, n)$ is a pendant edge of T_B in the following. Let x be a vertex of $G_n(i)$. Sometimes, we may use other leaf to obtain some special decomposition, for example, if a decomposition is induced by j, then $G_n(k)$ is the subgraph induced by vertex set $\{x|\ x[j] = k\}$. We call the vertex $x' = x((n-1)n))$ the *outgoing neighbor* of x and (x, x') the *outgoing edge* of x. For convenience, we denote $V_0(G_n)\bigcap V(G_n(i))$ by $V_0(G_n(i))$, $V_1(G_n)\bigcap V(G_n(i))$ by $V_1(G_n(i))$, respectively.

A transposition generating tree T_B is called a *double-star* if it contains exactly one non-pendant edge. Assume that (ij) is the non-pendant edge of T_B. The following observation is easy to obtain.

Lemma 4.2.1 *Assume that $n \geq 5$ and $D(T_B) \geq 3$. Then there exists a decomposition of G_n induced by some leaf j such that $D(T_B - j) \geq 3$. Furthermore, if T_B is a double-star, we can get a decomposition of G_n induced by a pedant edge (ij) such that there exists another integer j' and (ij') is also a pedant edge of T_B.*

Lemma 4.2.2 ([104]) *Let F be a edge set of G_n and $n \geq 4, |F| \leq n-3$. Then $G_n - F$ is hamiltonian laceable.*

Combining Lemma 4.1.4 with the main results of the last section, we give the following corollary without proof.

Corollary 4.2.3 *Let $G_n(i_1), \cdots, G_n(i_s)$ be s subgraph of $G_n(i), i = 1, \cdots, n$, F be an edge set and $x \in V(G_n(i_1)), y \in V(G_n(i_s)$ be two vertex of different partite sets of G_n . If $F \leq n-4$, then there is a path of length $s(n-1)!$ in $G_n[V(G_n(i_1)) \cup \cdots \cup V(G_n(i_s))]$ between x and y.*

Lemma 4.2.4 ([124]) *The Cayley graph G_n is vertex-bipancyclic if and only if G_n is not a star graph.*

Lemma 4.2.5 ([124]) *For any two edges of Cayley graph G_n, if $n \geq 3$, then there is a hamiltonian cycle of G_n containing them.*

Lemma 4.2.6 ([124]) *For $n \geq 3$, every edge of G_n lies on an even cycle of length l for every $6 \leq l \leq n!$.*

Lemma 4.2.7 ([62]) *Let $\mathcal{T}$ be a set of transpositions from $Sym(n)$ and let g and h be elements of $\mathcal{T}$. If the transposition generating graph of $\mathcal{T}$ contains no triangles, then g and h have exactly one common neighbor in Cay($Sym(n)$,$\mathcal{T}$) if $gh \neq hg$, and exactly two common neighbors otherwise.*

Clearly, for any two distinct transpositions (ij) and (kl), $(ij)(kl) = (kl)(ij)$ if and only if $\{i, j\} \cap \{k, l\} = \emptyset$. It is not difficult to see that if $C_4 = v_1v_2v_3v_4$ is a 4-cycle of Γ_n, then there must exist two transpositions (ij) and (kl) such that $\{i, j\} \cap \{k, l\} = \emptyset$ and $v_2 = v_1(ij), v_3 = v_2(kl), v_4 = v_3(ij) = v_1(kl)$. In fact, there exists an automorphism of G_n, say $\psi \in Aut(G_n)$, such that $\psi(v_1) = id$ since G_n is vertex transitive, furthermore, $C'_4 = \psi(v_1v_2v_3v_4)$ is also a $4-$cycle of Γ_n. Let $g = \psi(v_2), h = \psi(v_4)$, clearly, g and h are two transpositions of $\mathcal{T}$. By lemma 2.1, we have that $gh = hg$ since they have two common neighbors. That is, $(ij) = \psi^{-1}(g), (kl) = \psi^{-1}(h)$.

By the above argument, we have the following corollary.

Corollary 4.2.8 *A cycle $uvxy$ is a 4-cycle of G_n such that edge (u, v) with label (ij) and edge (v, x) with label (st) if and only if (x, y) has label (ij), edges (u, y) has label (st), and $\{s, t\} \cap \{i, j\} = \emptyset$.*

Thus, the edge with label (ij) of the Cayley graph generated by a double-star with non-pendant edge (i, j) is not contained in a 4-cycle, that is, the Cayley graph generated by a double-star is not edge-bipancyclic. In [124], the following lemma is reported.

Lemma 4.2.9 ([124]) *If a tree T_B is neither a star nor a double-star, the Cayley graph $Cay(S_n, B)$ is edge-bipancyclic.*

Lemma 4.2.10 ([82]) *BS_n is $(n-3)$-edge fault-tolerant bipancyclic for $n \geq 4$.*

Note that the girth of star graph is 6. Xu et al. in [144] reported the following theorem.

Theorem 4.2.11 ([144]) *For any edge set F of ST_n with $|F| \leq n-3$ and any edge e of $ST_n - F$, there exists a cycle of an arbitrary even length from 6 to $n!$ in $ST_n - F$ containing e provide $n \geq 3$.*

Theorem 4.2.12 *G_n is $(n-3)$-edge fault-tolerant bipancyclic if G_n is not star graph.*

Proof By induction on n. If $n = 4$, then $G_n \cong BS_n$ and the result follows from Lemma 4.2.10 directly. Let F be a faulty edge set with $|F| \leq n-3$. Note that G_n is not star graph, then by Lemma 4.2.1 there exists a decomposition of G_n such that each $G_n(i), i = 1, 2, \cdots, n$ is not $(n-1)$-dimensional star graph if $n \geq 5$. Without loss of generality we assume the above decomposition is induced by leaf n and $(n-1, n)$ is a pedant edge of T_B. By Lemma 4.2.1, we can assume that $(j, n-1)$ is another pedant edge of T_B and $e = (x, y)$ is an edge of G_n with label $(j(n-1))$. Thus, we have that the outgong neighbors of x and y are in different subgraphs of $G_n(1), \cdots, G_n(n)$. For $n \geq 5$, we can assume that the result holds for $G_n(i)$. Let $F_i = F \cap E(G_n(i))$. Since $|F| \leq n-3$, there is a subgraph $G_n(k)$ such that $F_k = \emptyset$. Let l be any even integer such that $4 \leq l \leq n!$ and let $l = q(n-1)! + r$, where $0 \leq q \leq n-1$ and $0 < r \leq (n-1)!$. We next construct a cycle of length l in $G_n - F$.

Case 1. $|F_i| \leq n-4$ for all i.

In this case, we have that $G_n(i) - F_i$ is bipancyclic by induction.

Subcase 1.1 $q \geq 2$.

Since there are $\frac{(n-1)!}{2}$ edges with label $(j(n-1))$ in $G_n(k)$ and $\frac{(n-1)!}{4} > n-3$ for $n \geq 5$, there exists an edge $e_k = (u_k, v_k)$ is not adjacent to any fault edge. Assume that the outgoing neighbors u'_k and v'_k of u_k and v_k are in $G_n(i_1)$ and $G_n(i_{q-1})$, respectively. If $q = 2$ and since $\frac{(n-2)!}{2} > n-3$, then there exists an fault-free edge $f = (u_{i_1}, u'_{i_{q-1}})$ such that u'_k and u_{i_1} are in different partite sets. By Lemma 4.2.2, there exists two hamiltonian cycle C_{i_1} and $C_{i_{q-1}}$ in $G_n(i_1) - F_{i_1}$ and $G_n(i_{q-1}) - F_{i_{q-1}}$. By Lemma 2.5 and Lemma 4.2.1, we can take a cycle C' of length r containing e_k if $r \geq 4$. Thus there are a cycle of length l formed by C_{i_1}, $C_{i_{q-1}}$ and C'. If $q \geq 3$, then we choose other $q-3$ subgraphs of $G_n(1), \cdots, G_n(n)$, except for $G_n(i_1), G_n(i_2), G_n(k)$, say, $G_n(i_2), \cdots, G_n(i_{q-2})$. Similarly, there exists fault-free edges $(u_{i_2}, u'_{i_2}), \cdots, (u_{i_{q-2}}, u'_{i_{q-2}})$ such that u'_{i_j} and u_{i_j} are in different partite sets of $G_n(i_j)$. By an argument similar to above, we have a cycle of length l if $r \geq 4$. Clearly, if $r = 2$, then the cycle of length l is also obtained by above the edge e_k and other q cycles of length $(n-1)!$.

Subcase 1.2 $q \leq 1$.

If $q = 0$, then the result is clearly true by Lemma 4.2.4.

We next assume that $q = 1$. Note that $D(T_B) \geq 3$, then there exists a pedant edge (s,t) of T_B such that $\{s,t\} \cap \{n-1, n\} = \emptyset$. Since there are $\frac{(n-2)!}{2}$ edges with label (st) and $\frac{(n-2)!}{2} > n-3$, there exists an edge $e_k^* = (u_k^*, v_k^*)$ with label (st) such that e_k^* is not adjacent any fault edge. It is not difficult to see that the outgoing neighbors $u_k^{*\prime}$ and $v_k^{*\prime}$ of u_k^* and v_k^* are in a same subgraph of $G_n(1), \cdots, G_n(n)$, say, $G_n(i^*)$. By Lemma 4.2.2, there exists a hamiltonian cycle C_{i^*} of $G_n(i^*)$ between $u_k^{*\prime}$ and $v_k^{*\prime}$. By an argument similar to that of Subcase 1.1, a cycle of length l is obtained by C_{i^*} and a cycle of length r or an edge of $G_n(k)$.

Case 2. $|F_i| = n-3$ for some integer i.

Without loss of generality we assume that $i = 1$ and let $G_n^* = G_n[V(G_n(2) \cup \cdots \cup V(G_n(n))]$. Note that $D(T_B - (n-1, n)) \geq 3$, then we can see that there exists and pedant edge (s,t) of T_B such that $\{s,t\} \cap \{n-1,n\} = \emptyset$. Let (u_2, v_2) be an edge of $G_n(2)$ with label (st) and $u_2[n-1] = v_2[n-1] = 3$ (such edge exists, for example, let $u_2 = x_1 \cdots x_{n-3}32$ be a vertex of $G_n(2)$, then $(u_2, u_2(st))$ is a such edge of $G_n(2)$). We can see that the outgoing neighbors u_2', v_2' of u_2, v_2 are in $G_n(3)$ and u_2, v_2, v_2', u_2' induce a 4-cycle of G_n be Corollary 2.1. Similarly, we can take an edge (u_i, v_i) with label (st) in $G_n(i)$ such that the outgong neighbors u_i', v_i' of u_i, v_i are in $G_n(i+1)$ for $i = 2, \cdots, n-1$. Thus, we can see that there exists a hamiltonian cycle of $G_n(i)$ containing edges (u_i, v_i) and (u_i', v_i') by Lemma 2.10. By an argument similar to that of Subcase 1.1, we can see that there is a cycle of length l in G_n^* if $4 \le l \le (n-1)(n-1)!$.

Next, we show the existence of cycles of length l for $(n-1)(n-1)!+2 \le l \le n!$. Assume that $l = (n-1)(n-1)! + r$ If there exists a fault edge with label (st) such that $\{s,t\} \cap \{n-1,n\} = \emptyset$, say $e_1 = (u_1, v_1)$, then the outgoing neighbors u_1', v_1' of u_1, v_1 are in a same subgraph $G_n(i_2)$. We look the edge e_1 a fault free edge temporarily. By Lemma 4.2.2, we have that there is a hamiltonian path P_1 of $G_n(1)$ between u_1 and v_1. Thus, P_1, u_1' and u_2' form a cycle of length $(n-1)! + 2$. By an argument similar to that of above paragraph, we can construct an sequence of $\{u_1', v_1', u_{i_2}, v_{i_2}\}, \{u_{i_2}', v_{i_2}', u_{i_3}, v_{i_3}\}, \cdots, \{u_{i_{n-1}}', v_{i_{n-1}}', u_{i_n}, v_{i_n}\}$ and take a cycle (or an edge) of length r in $G_n(i_n)$. Thus, a cycle of length l is obtained clearly by above argument.

We next assume that each fault edge has label $(j(n-1))$ for some j. Note that $D(T_B) \ge 3$, then there exists a pedant edge (s,t) of T_B such that $n-1 \notin \{s,t\}$ and t is a leaf of T_B. Note that no fault edge has label (st). Thus, the leaf t induces a decomposition of G_n, say $G_n'(i), i = 1, 2, \cdots, n$. If $G_n'(i)$ is not isomorphic to star graph, then we can construct a cycle of length l by an argument similar to above paragraph (let $F_i' = G_n'(i) \cap F$. If $|F_i| \le n-4$, then by Case 1 the result is true; if $|F_k| = n-3$ for some k, then the result is obtained by an argument similar to that of above paragraph).

We next assume that each subgraph $G_n'(i)$ is isomorphic to $(n-1)$-dimensional star graph. Let $F_i' = G_n'(i) \cap F$. Then $|F_i'| \le n-3$. We omit the proof of the case of each $|F_i| \le n-4$ for all i (see Case 1). Assume that $F_k = n-3$ for some k. Look a fault edge $e_k = (u_k, v_k)$ the fault free edge temporarily. By Theorem 4.2.11, there exists a cycle of an arbitrary even length from 6 to $n!$ in $G'_n(k) - F_k$ containing e_k. Note that there is no fault edge out of $G'_n(k)$. It is not difficult to see that we can construct a cycle of length l for $(n-1)(n-1)! + 6 \leq l = q(n-1)! + r \leq n!$ by taking a r-cycle in $G'_n(k)$ containing e_k and a special $(n-1)!$-cycle in each other subgraph $G'_n(i), i \neq k$. If $l = (n-1)(n-1)! + 2$, then we can construct a cycle by taking an edge $e_k = (u_k, v_k)$ of $G'_n(k) - F$, the outgoing edges of u_k, v_k and a path of length $(n-1)(n-1)!$ of $G_n - G'_n(k)$ between u'_k and v'_k (if u'_k and v'_k are in a same subgraph, then by an the similar argument to above paragraph we have the $(n-1)(n-1)!$-path; if u'_k and v'_k are in the distinct subgraphs, then the path is also obtained by Corollary 4.2.3). Note that $(n-1)!/4 > n-3$ for $n \geq 5$, then there exists a path of length 4 in $G'_n(k) - F$, say P. Assume the ends of P are u_k, v_k. If $l = (n-1)(n-1)! + 4$, then we can construct a cycle by taking a path of length 4, the outgoing edges of u_k, v_k and another path of length $(n-1)(n-1)!$ between u_k and v_k in $G_n - G'_n(k)$. $\square$

Chapter 5 Several extremal graph problems

§5.1 Subgraph of hypercubes, extra-edge connectivity of hypercubes and folded hypercubes

§5.1.1 Subgraph of hypercubes

In this part we shall determine the maximum size of an induced subgraph of a vertex set with a given size. As an application, we determine a kind of edge-connectivity (extra edge-connectivity) of hypercubes in Section 3.

Let m be an integer and $m = \sum\limits_{i=0}^{s} 2^{t_i}$ be the decomposition of m such that $t_0 = \log_2 m$, and $t_i = [\log_2(m - \sum\limits_{r=0}^{i-1} 2^{t_r})]$ for $i \geq 1$. Suppose X is a vertex set of Q_n of size m. We denote by $\frac{ex_m}{2}$ the maximum size of the subgraph (of Q_n) induced by m vertices, i.e., $ex_m = \max\{2|E(Q_n[X])| : X \subset V(Q_n) \text{ and } |X| = m\}$, is the maximum possible sum of the degrees of the vertices in the subgraph (of Q_n) induced by m vertices. In particular, for an integer $i \in \{1, 2, \cdots, n\}$, we may express Q_n as $D_0 \oplus D_1$ such that D_0 and D_1 are the two $(n-1)$-subcubes of Q_n induced by the vertices with the ith coordinate 0 and 1 respectively. We will use decomposition $Q_n = D_0 \oplus D_1$ directly without explanation on its coordinate if no confusion arises. Sometimes we also use $X^{i-1}0X^{n-i}$ and $X^{i-1}1X^{n-i}$ to denote D_0 and D_1, where $X \in Z_2$.

Theorem 5.1.1 *Let X be a vertex set of Q_n with size m. Then* $ex_m = \sum\limits_{i=0}^{s} t_i 2^{t_i} + \sum\limits_{i=0}^{s} 2 \cdot i \cdot 2^{t_i}$.

Proof. We first claim that $ex_m \geq \sum\limits_{i=0}^{s} t_i 2^{t_i} + \sum\limits_{i=0}^{s} 2 \cdot i \cdot 2^{t_i}$. It is sufficient to show that there is a subgraph G_1 in Q_n such that $|V(G_1)| = m, |E(G_1)| = \frac{1}{2}(\sum\limits_{i=0}^{s} t_i 2^{t_i} + \sum\limits_{i=0}^{s} 2 \cdot i \cdot 2^{t_i})$. We pack G_1 as follows.

Take $s+1$ t_i-dimensional subcubes for $i = 0, \cdots, s$ as follows:

$$
\begin{array}{ll}
Q^0: & \underbrace{X_1 \cdots X_{t_0}}_{t_0} 0 \cdots 0 \\
Q^1: & \underbrace{\underbrace{X_1 \cdots X_{t_1}}_{t_1} 0 \cdots 0}_{t_0} 10 \cdots 0 \\
Q^2: & \underbrace{\underbrace{X_1 \cdots X_{t_2}}_{t_2} 0 \cdots 0}_{t_1} 10 \cdots 010 \cdots 0 \\
 & \cdots
\end{array}
$$

Note that Q^0 is given and Q^i is taken from a t_{i-1}-dimensional subcube which is obtained from Q^{i-1} by changing the 0 of $(t_{i-1}+1)th$-coordinate of Q^{i-1} to 1. Define $G_1 = Q_n[V(Q^0) \cup \cdots \cup V(Q^s)]$. For convenience sake we define $f(m) = \sum\limits_{i=0}^{s} t_i 2^{t_i} + \sum\limits_{i=0}^{s} 2 \cdot i \cdot 2^{t_i}$. It is not difficult to count the number of edges of G_1 by considering the edges within Q^i's $(\frac{1}{2}\sum\limits_{i=0}^{s} t_i 2^{t_i})$ and the edges between Q^i's $(\frac{1}{2}\sum\limits_{i=0}^{s} 2 \cdot i \cdot 2^{t_i})$. Thus, the claim holds, i.e., $ex_m \geq f(m) = \sum\limits_{i=0}^{s} t_i 2^{t_i} + \sum\limits_{i=0}^{s} 2 \cdot i \cdot 2^{t_i}$.

We next show that $ex_m \leq f(m)$ by induction on n, equivalently, $ex_m = f(m)$. Assume X is a vertex set such that $2|E(Q_n[X])| = ex_m$. By the definition and the property of hypercubes, we can check the equality directly for $n = 2, 3$. So we assume $n \geq 4$. Denote $X \cap D_0, X \cap D_1$ by X_0, X_1, respectively. Without loss of generality, we let $m_i = |X_i|, i = 0, 1$ and suppose $m_0 = \sum\limits_{i=0}^{s_0} 2^{t_{0i}} \leq m_1 = \sum\limits_{i=0}^{s_1} 2^{t_{1i}}$. By induction, if $m_0 = 0$, then $ex_m \leq f(m)$, then we may assume $m_0 > 0$ from now on.

Case 1. Assume $s = 0$, i.e., $m = 2^{t_0}$.

If $t_{00} = t_0 - 1$, then $m_0 = m_1 = 2^{t_0-1}$. In this case, $ex_m = 2|E(Q_n[X])| \leq 2|E(Q_n[X_0])| + 2|E(Q_n[X_1])| + 2m_0 \leq ex_{m_0} + ex_{m_1} + 2m_0 = f(m_0) + f(m_1) + 2m_0 = (t_0 - 1)2^{t_0-1} + (t_0 - 1)2^{t_0-1} + 2^{t_0} = f(m)$ (To understand the inequality clearly, we give two notes: 1. There are at most m_0 edges between $D_0[X_0]$ and $D_1[X_1]$; 2. The symbols ex_{m_0} and ex_{m_1} are restricted in D_0 and D_1, i.e., one can look them as $ex_{m_0}|_{D_0}$ and $ex_{m_1}|_{D_1}$. The similar context will be omitted from now on). So we may assume $t_{00} < t_0 - 1$ and $t_{10} = t_0 - 1$.

Claim: $f(m_0)+f(m_1)+2m_0 \le f(m)$ when $m=2^{t_0}=m_0+m_1$ and $m_1 > m_0 > 0$, and $f(m_0)+f(m_1)+2m_0 = f(m)$ when $m=2^{t_0}=m_0+m_1$ and $m_1=m_0$.

One can check the second part of the claim directly. So we only prove the first part.

Note that $m_0+m_1=2^{t_0}$. Suppose $m_0=\sum\limits_{i=0}^{s_0}2^{t_{0i}}$ and $m_1=\sum\limits_{i=0}^{s_1}2^{t_{1i}}$. It can be seen that $t_{0s_0}=t_{1s_1}$ and

$$\{t_{00},t_{01},\cdots,t_{0(s_0-1)},t_{10},\cdots,t_{1(s_1-1)}\}=\{t_0-1,t_0-2,\cdots,t_{0s_0}+1\} \qquad (2)$$

(In fact, one can see this recursively. Note that $m_1>m_0$, hence $t_{10}=t_0-1$. Clearly, $m_1-2^{t_0-1}+m_0=2^{t_0-1}$. Let $m_1'=m_1-2^{t_0-1}, m_0'=m_0$. If $m_1'>m_0'$, then $t_{11}=t_0-2$; If $m_0'>m_1'$, then $t_{00}=t_0-2$; If $m_0'=m_1'$, then $t_{11}=t_{00}=t_0-2$. Since 2^{t_0} is finite, one can find that $t_{0s_0}=t_{1s_1}$. Clearly, $|\{t_{00},t_{01},\cdots,t_{0(s_0-1)},t_{10},\cdots,t_{1(s_1-1)}\}|=|\{t_0-1,t_0-2,\cdots,t_{0s_0}+1\}|=s_0+s_1$). We put the two expressions ($m_0=\sum\limits_{i=0}^{s_0}2^{t_{0i}}$ and $m_1=\sum\limits_{i=0}^{s_1}2^{t_{1i}}$) together as follow:

$$m_0+m_1=2^{t_0}=\sum_{l=t_{0s_0}+1}^{t_0-1}2^l+2^{t_{0s_0}}+2^{t_{0s_1}}$$

$$=2^{t_0-1}+\cdots+2^{t_0-j}+\cdots+2^{t_{0s_0}+1}+2^{t_{0s_0}}+2^{t_{0s_0}} \qquad (3)$$

(Note that (3) contains exactly $(s_0+1)+(s_1+1)$ terms). Note that

$$f(2^{t_0})=t_02^{t_0}$$

$$=(t_0-1)2^{t_0-1}+[(t_0-2)2^{t_0-2}+2\cdot 2^{t_0-2}]+\cdots+[(t_0-i)2^{t_0-i}+2\cdot(i-1)\cdot 2^{t_0-i}] \qquad (4)$$

$$+\cdots+[(t_{0s_0})2^{t_{0s_0}}+2\cdot(t_0-t_{0s_0}-1)\cdot 2^{t_{0s_0}}]+[(t_{0s_0})2^{t_{0s_0}}+2\cdot(t_0-t_{0s_0})\cdot 2^{t_{0s_0}}]$$

(Recall the construction of the subgraph G_1 at the beginning of the proof, one can understand this equality by letting G_1 be the t_0-dimensional subcube. In fact, the equality (4) also follows from the definition of $f(m)$ and the expression of m). By (2), for any term $2^{t_{0i}}$ in $m_0=\sum\limits_{i=0}^{s_0}2^{t_{0i}}$, there is a term 2^{t_0-j} in (3) such that $t_0-j=t_{0i}$, and hence there is a term $[(t_0-j)2^{t_0-j}+2\cdot(j-1)\cdot 2^{t_0-j}]$ in (4) corresponding to the term. We write $(t_0-j)2^{t_0-j}+2\cdot(j-1)\cdot 2^{t_0-j}=t_{0i}2^{t_{0i}}+2\cdot i\cdot 2^{t_{0i}}+2\cdot(j-1-i)\cdot 2^{t_{0i}}$. Since $m_1>m_0$, we have $j-2\ge i$ (for example, $t_0 - 2 \geq t_{00}$ and hence $j = 2, i = 0, j-2 = 0 \geq i = 0$.). This implies $\sum_{i=0}^{s_0} 2 \cdot (j-1-i) \cdot 2^{t_{0i}} \geq \sum_{i=0}^{s_0} 2 \cdot \cdot 2^{t_{0i}} = 2m_0$. Thus $f(m_0) + f(m_1) + 2m_0 \leq f(m)$ when $m = 2^{t_0} = m_0 + m_1$ and $m_1 > m_0 > 0$. The claim holds.

By induction, we have $ex_{m_0} + ex_{m_1} = f(m_0) + f(m_1)$. Since $f(m_0) + f(m_1) + 2m_0 \leq f(m)$, we have $ex_m \leq ex_{m_0} + ex_{m_1} + 2m_0 = f(m_0) + f(m_1) + 2m_0 \leq f(m)$ for $m_0 \geq 0$.

Case 2. Assume $s > 0$.

We prove this case recursively. We define the following two operations:

(O_1). If $t_{10} > t_{00}$, let $T^1 \subset X_1$ be a subset of X_1 such that $|T^1| = 2^{t_{10}}$, and let $X^1 = X \setminus T^1, X_0^1 = X_0, X_1^1 = X_1 \setminus T^1$, and $m^1 = m - |T^1|, m_0^1 = m_0, m_1^1 = m_1 - |T^1|$.

We first claim $f(m) \geq f(m^1) + 2\min\{m^1, 2^{t_{10}}\} + t_{10}2^{t_{10}}$. In fact, we prove the claim by considering the following two subcases: $2^{t_{10}}$ is a term of the expression $m = \sum_{i=0}^{s} 2^{t_i}$, and the other.

If $2^{t_{10}}$ is a term of the expression $m = \sum_{i=0}^{s} 2^{t_i}$, then either $t_{10} = t_0$ or $t_{10} = t_1 = t_0 - 1$ (Note that $m_1 \geq \frac{m}{2}$ and $s > 0$. Then $m_1 > 2^{t_0 - 1}$ and hence $t_{10} \geq t_0 - 1$.). If $t_{10} = t_0$, then $f(m^1) = \sum_{i=1}^{s} t_i 2^{t_i} + \sum_{i=1}^{s} 2 \cdot (i-1) \cdot 2^{t_i}$ is obtained from $f(m) = \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i}$ by removing $t_0 2^{t_0} + 2(2^{t_1} + \cdots + 2^{t_s}) = t_0 2^{t_0} + 2m^1$, that is, $f(m) = f(m^1) + 2m^1 + t_0 2^{t_{10}}$; If $t_{10} = t_0 - 1$, then $f(m^1)$ is obtained from $f(m)$ by removing $t_1 2^{t_1} + 2(2^{t_1} + \cdots + 2^{t_s})$, that is, $f(m) \geq f(m^1) + 2 \cdot 2^{t_{10}} + t_{10} 2^{t_{10}}$.

If $2^{t_{10}}$ is not a term of the expression $m = \sum_{i=0}^{s} 2^{t_i}$, then $t_{10} = t_0 - 1 > t_1$. In this case, one can see $f(m) = (t_{10}+1)2^{t_{10}+1} + \sum_{i=1}^{s} t_i 2^{t_i} + \sum_{i=1}^{s} 2 \cdot i \cdot 2^{t_i}$ and $f(m^1) = t_{10}2^{t_{10}} + \sum_{i=1}^{s} t_i 2^{t_i} + \sum_{i=1}^{s} 2 \cdot i \cdot 2^{t_i}$. This $f(m) - f(m^1) = (t_{10}+1)2^{t_{10}+1} - t_{10}2^{t_{10}} = t_{10}2^{t_{10}} + 2^{t_{10}+1}$. Thus, the claim holds.

Note that $2|E(Q_n[X])| \leq 2|E(Q_n[X - T^1])| + 2\min\{m^1, 2^{t_{10}}\} + t_{10}2^{t_{10}}$. Combining this with $f(m) \geq f(m^1) + 2\min\{m^1, 2^{t_{10}}\} + t_{10}2^{t_{10}}$. It can be seen that $2|E(Q_n[X])| \leq f(m)$ follows from $2|E(Q_n[X - T^1])| = 2|E(Q_n[X^1])| \leq f(m^1)$.

(O_2). If $t_{10} = t_{00}$, let $T_i^1 \subset X_i$ be a subset of X_i such that $|T_i^1| = 2^{t_{10}} = 2^{t_{00}}$, and let $T^1 = T_0^1 \cup T_1^1, X^1 = X \setminus T^1, X_i^1 = X_i \setminus T_i$, and $m^1 = m - |T_0^1| - |T_1^1|, m_i^1 = m_i - |T_i^1|$.

Similarly, we note that $f(m) \geq f(m^1) + 2\min\{m^1, 2^{t_{10}+1}\} + (t_{10+1})2^{t_{10}+1}$ and $2|E(Q_n[X])| \leq 2|E(Q_n[X-(T_0 \cup T_1)])| + 2\min\{m^1, 2^{t_{10}+1}\} + (t_{10+1})2^{t_{10}+1}$. Thus, $2|E(Q_n[X])| \leq f(m)$ follows from $2|E(Q_n[X-T^1])| = 2|E(Q_n[X^1])| \leq f(m^1)$.

We repeat (O_1) and/or (O_2) i times ($i \leq \min\{s_0, s_1\}$) such that either $X^i \subset V(D_0)$ (or $V(D_1)$), or $m^i = 2^l$ for some integer l (i.e., Case 1 holds). Then we have $2|E(Q_n[X^i])| \leq f(m^i)$ by induction, or by Case 1. This completes the proof. $\square$

We point some properties of function ex_m as follows.

Lemma 5.1.2 *Let* $m = \sum_{i=0}^{s} 2^{t_i}$ *be an integer. If* $s \leq [\frac{n}{2}] - 1$, *then* $ex_{m+1} \leq ex_m + n$.

Proof. Note that $ex_m = \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i}$ and $s \leq [\frac{n}{2}] - 1$. Then the assertion follows from $ex_{m+1} - ex_m = f(m+1) - f(m) = 2s + 2 \leq n$. $\square$

Note that hypercube Q_n is n-regular and $s < [\frac{n}{2}]$ if $m < 2^{[\frac{n}{2}]} - 1$. Denote by E_X the set of edges in which each edge contains exactly one end vertex in X. By the observation above, we have the following.

Corollary 5.1.3 *Let* X *be a subset of* $V(Q_n)$ *with* $|X| = m$. *Then* E_X *contains at least* $n|X| - ex_m$ *edges. Moreover, the function* $e(m) = n|X| - ex_m$ *is strictly increasing (respect to* m*) if* $m < 2^{[\frac{n}{2}]}$.

Moreover, note that the hypercube is n-regular, and the size of G_1 is at most $\frac{ex_m}{2}$. If $m \leq 2^{n-2}$, then one can pick the subgraph G_1 in an $(n-2)$-dimensional subcube such that $2|E(G_1)| = ex_m$. Thus, we have the following.

Corollary 5.1.4 *If* $m \leq 2^{n-2}$, *then* $(n-2)m - ex_m \geq 0$.

In the next two subsections, we apply the result in this section. We consider the extra edge-connectivity of hypercubes and folded hypercubes.

§5.1.2 Extra-edge connectivity of hypercubes

In this section we give an application of Theorem 5.1.1 to determine the m-extra edge-connectivity of hypercubes which will be defined below.

It is well known that when the underlying topology of an interconnection network is modelled by a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network, the edge-connectivity $\lambda(G)$ is an important feature determining reliability and fault tolerance of the network. It can correctly reflect the fault tolerance of network systems with few processors. However, it always underestimates the resilience of large networks. There is a discrepancy because the occurrence of events which would disrupt a large network after a few processor or link failures is highly unlikely. Thus the disruption envisaged occurs in a worst case scenario. To overcome this shortcoming, Harary introduced the concept of conditional (edge-) connectivity [68]. One special case of the conditional connectivity is the extra (edge-) connectivity [50],[55], which receives a lot of attention in recent years, and is the focus of this paper.

For a connected graph G, a *m-extra edge-cut* of G is an edge-cut F of G such that every component of $G - F$ has at least m vertices. The cardinality of the minimum m-extra edge-cut of G is the *m-extra edge-connectivity* of G, denoted by $\lambda_m(G)$. A minimum m-extra edge-cut of G is abbreviated as a λ_m-cut of G. The extra edge-connectivity of hypercubes and cube-based graphs have attracted much attention these years as they are kinds of measures of fault-tolerance of networks, see for example [50]. For the extra edge-connectivity of cube-based graphs, there is no general results, several authors reported some results for the special cases $m = 2, 3, 4$ of hypercubes and several cube-based networks, see for examples [50, 105, 115, 161, 162]. We shall determine the g-extra edge-connectivity of hypercubes for $m \leq 2^{[\frac{n}{2}]}$ by using Theorem 5.1.1.

Lemma 5.1.5 *Let F be a λ_m-cut of Q_n such that the smallest component C of $Q_n - F$ contains $m \leq 2^{n-1}$ vertices. Then C contains $\frac{1}{2}ex_m$ edges.*

Proof. Note that F is a λ_m-cut. Then $Q_n - F$ contains exactly two components. Let C be the smaller one. It is sufficient to show that there exists a

m-extra edge-cut such that the smaller component having $m = \sum_{i=0}^{s} 2^{t_i}$ vertices and $\frac{1}{2}ex_m$ edges (since $|E(C)| \leq \frac{1}{2}ex_m$ edges, the claim holds). By (1) (in the proof of Theorem 5.1.1), we can pack a connected subgraph G_1 in Q_n such that $|V(G_1)| = m, |E(G_1)| = \frac{1}{2}ex_m$. We next show that $E_{V(G_1)}$ is a m-extra edge-cut by proving that $Q_n - V(G_1)$ is connected, where $E_{V(G_1)}$ is defined as E_X in Corollary 5.1.3. By the definition of Q_n and the subgraph G_1, it can be seen that there is a path from $11\cdots 1$ to any vertex of $Q_n - V(G_1)$ by changing 1 to 0 one by one in the suitable coordinates. We are done. $\square$

Since m-extra edge-connectivity of Q_n was determined for $m = 1, 2, 3, 4$ [50, 115, 105, 161], we only consider the case $5 \leq m \leq 2^{[\frac{n}{2}]}$ and then we assume $n \geq 6$ below.

Theorem 5.1.6 *The m-extra edge-connectivity of Q_n is $nm - ex_m$ for $n \geq 6, m \leq 2^{[\frac{n}{2}]}$.*

Proof. Assume $n \geq 6, m \leq 2^{[\frac{n}{2}]}$. By Lemma 5.1.5, we have $\lambda_m(Q_n) \leq nm - ex_m$ since there is a m-extra edge-cut of size $nm - ex_m$. We next show that $\lambda_m(Q_n) \geq nm - ex_m$.

Let F be a λ_m-cut of Q_n and C be the smaller component of $Q_n - F$ (since F is a minimal edge-cut, there are exactly two components in $Q_n - F$). If $m \leq |C| < 2^{[\frac{n}{2}]}$, then by Corollary 5.1.3 we have $|C| = m$ and $\lambda_m(Q_n) \geq nm - ex_m$, that is, $\lambda_m(Q_n) = nm - ex_m$. We may assume $|C| = m^* = \sum_{i=1}^{s} 2^{t_i} \geq 2^{[\frac{n}{2}]}$ from now on.

For convenience sake, we assume that n is even. Note that $2^{\frac{n}{2}} - 1 = 2^{\frac{n}{2}-1} + 2^{\frac{n}{2}-2} + \cdots + 2^1 + 2^0$ and $f(2^{\frac{n}{2}} - 1) = \frac{n}{2} \cdot 2^{\frac{n}{2}} - n$. Then, $n(2^{\frac{n}{2}} - 1) - f(2^{\frac{n}{2}} - 1) = n2^{\frac{n}{2}} - f(2^{\frac{n}{2}}) = \frac{n}{2} \cdot 2^{\frac{n}{2}}$. It is sufficient to show that $|E_{V(C)}| \geq nm^* - ex_{m^*} \geq \frac{n}{2} \cdot 2^{\frac{n}{2}}$ when $m^* \geq 2^{\frac{n}{2}}$. Denote by $m' = m^* - 2^{t_0}$. Clearly, $m' \leq 2^{n-2}$. By Corollary 5.1.4, it is easy to see the following:

$$\begin{aligned} nm^* - ex_{m^*} &= n2^{t_0} + n(2^{t_1} + \cdots + 2^{t_s}) - t_0 2^{t_0} - (\sum_{i=1}^{s} t_i 2^{t_i} + \sum_{i=1}^{s} 2 \cdot i \cdot 2^{t_i}) \\ &= (n - t_0)2^{t_0} + (n \sum_{i=1}^{s} 2^{t_i} - (\sum_{i=1}^{s} t_i 2^{t_i} + \sum_{i=1}^{s} 2 \cdot i \cdot 2^{t_i})) \end{aligned}$$

$$
\begin{aligned}
&= (n-t_0)2^{t_0} + nm' - ex_{m'} - 2m' \\
&\geq (n-t_0)2^{t_0} \\
&= (n-t_0)\cdot 2^{(t_0-\frac{n}{2})}\cdot 2^{\frac{n}{2}}
\end{aligned}
$$

Note that $\frac{n}{2} \leq t_0 < n-1$, then we have $(n-t_0)\cdot 2^{(t_0-\frac{n}{2})} \geq \frac{n}{2}$. So $\lambda_m(Q_n) = |F| = |E_{V(C)}| \geq nm - ex_m \geq \frac{n}{2}\cdot 2^{\frac{n}{2}}$ in this case.

If n is odd, the argument is similar. We omit the discussion for this case. We complete the proof. $\square$

Remark. We explain why we do not discuss the m-extra edge-connectivity of Q_n for $m > 2^{\frac{n}{2}}$. From Theorem 5.1.6, we have $\lambda_m(Q_n)$ is strict increasing respect to m when $m < 2^{[\frac{n}{2}]}$, but it is not true when $m \geq 2^{[\frac{n}{2}]}$. In fact, the function $\lambda_m(Q_n)$ is dependent on the decomposition of m, for example, $\lambda_{2^{\frac{n}{2}}}(Q_n) = \lambda_{2^{\frac{n}{2}}-1}(Q_n)$. To determine the exact value of m-extra edge-connectivity of Q_n, we have to determine the decomposition for each $2^l \leq m \leq 2^{l+1}, l = [\frac{n}{2}]+1,\cdots,n-1$. So we think it is imposable to give an exact value for every m, at least by the method posed here.

It is not difficult to see that if F is an edge-cut with size no more than $nm - ex_m$ for $m \leq 2^{[\frac{n}{2}]}$, then $B_n - F$ has two components such that the smaller one is of at most m vertices. Then we naturally consider the following problem.

Conjecture 5.1.7 *Let F be a λ_m-cut for $m \leq 2^{[\frac{n}{2}]}$. Then the smaller component of $Q_n - F$ is isomorphic to the subgraph G_1 of m vertices.*

To encourage readers to consider this conjecture, we give an evidence below. Latifi et al. in [95] proved that if H is a subgraph of Q_n such that $|V(H)| = 2^k, |E(H)| = k2^{k-1}$, then H is isomorphic to Q_k. This implies the following.

Corollary 5.1.8 *Let F be a λ_m-cut for $m = 2^k, k \leq [\frac{n}{2}]$. Then the smaller component of $Q_n - F$ is isomorphic to Q_k.*

§5.1.3 Extra-edge connectivity of folded hypercubes

In this part, we shall explore the g-extra edge-connectivity of folded hypercubes for more general $g \leq n$.

Folded hypercube FQ_n is superior to Q_n in some properties, see [49]. Thus, the folded hypercube FQ_n is an enhancement on the hypercube Q_n. FQ_n is obtained by adding a prefect matching M on the hypercube where $M = \{(u, \bar{u}) |\ u \in V(Q_n)\}$. In addition, denote by M_i the edge set $\{(x_1x_2\cdots x_{i-1}x_ix_{i+1}\cdots x_n, x_1x_2\cdots x_{i-1}\bar{x}_ix_{i+1}\cdots x_n) : x_i \in Z_2\}$. Clearly, $E(Q_n) = \cup_{i=1}^{n} M_i$ and $E(FQ_n) = E(Q_n) \cup M$, see [60].

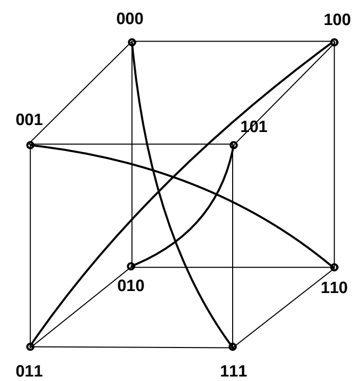


Fig. 5.1: The 3-dimensional Folded hypercube.

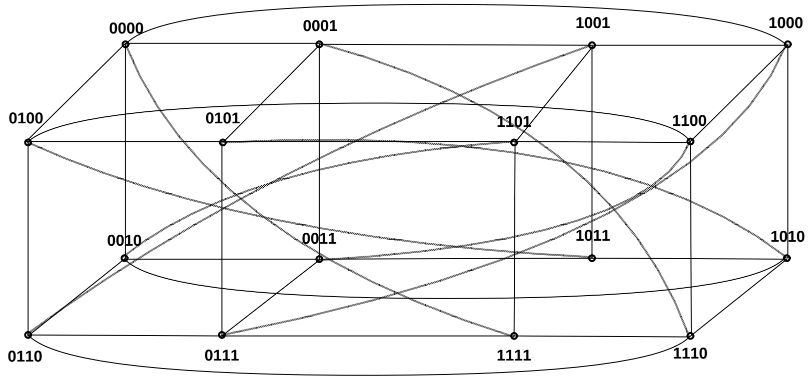


Fig. 5.2: The 4-dimensional Folded hypercube.

Several special cases of the g-extra edge-connectivity of folded hypercubes have been reported by following papers [49],[75],[145],[161],[162], in which the authors showed $\lambda_1(FQ_n) = n + 1$, $\lambda_2(FQ_n) = 2n$ for $n \geq 3$, $\lambda_3(FQ_n) = 3n - 1$ for $n \geq 4$, and $\lambda_4(FQ_n) = 4n - 4$ for $n \geq 4$. In this note, we show that $\lambda_g(FQ_n) = g(n+1) - (\sum\limits_{i=0}^{s} t_i 2^{t_i} + \sum\limits_{i=0}^{s} 2 \cdot i \cdot 2^{t_i})$, where $g = \sum\limits_{i=0}^{s} 2^{t_i} \leq n, t_0 = [\log_2 g], t_i = [\log_2(g - \sum\limits_{r=0}^{i-1} 2^{t_r})], n \geq 6$.

In the following, we shall determine $\lambda_g(FQ_n)$ for $g \leq n, n \geq 4$. Let g be an integer and $g = \sum_{i=0}^{s} 2^{t_i}$ be a decomposition of g such that $t_0 = [\log_2 m]$, and $t_i = [\log_2(m - \sum_{r=0}^{i-1} 2^{t_r})]$ for $i \geq 1$. For any integer $g = \sum_{i=0}^{s} 2^{t_i} \leq 2^{n-1}$, we first show that FQ_n contains an induced subgraph G_1 of order g such that $\sum_{u \in V(G_1)} d(u) = \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i}$. It is sufficient to show that there is a subgraph G_1 in Q_n such that $|V(G_1)| = g, |E(G_1)| = \frac{1}{2}(\sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i})$, and the subgraph contains no edges in M. We pack G_1 as defined in last section.

It is not difficult to count the number of edges of G_1 by considering the edges within Q^i's and the edges between Q^i's. In fact, each Q^i contains $\frac{1}{2}\sum_{i=0}^{s} t_i 2^{t_i}$ edges, and it is not difficult to see that each vertex in Q^i is adjacent to one vertex in each of $Q^{i-1}, Q^{i-2}, \cdots, Q^0$. Thus, $|V(G_1)| = g, |E(G_1)| = \frac{1}{2}(\sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i})$, and the subgraph contains no edges in M.

Let $E_{V(G_1)}(FQ_n)$ be the set of edges of FQ_n having exactly one endvertex in G_1 (Unless stated otherwise, we omit the subscript FQ_n). Then, $|E_{V(G_1)}| = g(n+1) - (\sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i})$.

Lemma 5.1.9 *There is a g-extra edge-cut of size $g(n+1) - (\sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i})$ for $g \leq 2^{n-1}$, i.e., $\lambda_g \leq g(n+1) - (\sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i})$.*

Proof. We shall that $E_{V(G_1)}$ is a g-extra edge-cut and then the lemma holds. It is sufficient to show that $FQ_n - E_{V(G_1)} - G_1$ is connected and contains more than g vertices. Note that $|V(G_1)| = g \leq 2^{n-1}$ and then $FQ_n - V(G_1)$ contains at least g vertices. So we only need to show that $Q_n - V(G_1)$ is connected. By the definition of Q_n, it can be seen that there is a path from $11 \cdots 1$ to any vertex of $Q_n - V(G_1)$ by changing 1 to 0 one by one in the suitable coordinates. We are done. $\square$

Lemma 5.1.10 ([103]) *Let X be a vertex set of Q_n with size m. Then $2|E(Q_n[X])| \leq \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i}$.*

Lemma 5.1.11 ([60]) *$FQ_n - M_i$ is isomorphic to Q_n for all i.*

Lemma 5.1.12 *Let H be connected subgraph induced by X in FQ_n.*

(i) If $|X| \leq g = \sum_{i=0}^{s} 2^{t_i} \leq n$ and $n \geq 5$, then $|E(H)| \leq \frac{1}{2}(\sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i})$.

(ii) If $|X| \geq n+1$, then $|E_X| \geq g(n+1) - (\sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i})$ for any $g \leq n, n \geq 6$.

Proof. We first prove (i). If H contains no edges of M, then by Lemma 5.1.10 the lemma holds. We claim that there exists an integer i such that H contains no edges of M_i. In fact, if H contains at least an edge of M_i for each i, then we pack one edge from $M_i \cap E(H)$ for each i and one edge from m. The $n+1$ edges induce a subgraph H^* of H. If H^* contains a cycle, then the cycle contains even number of edges of M_j if there is one edge of M_j on the cycle. Thus, H^* is a forest and then it contains at least $n+1$ vertices. This contradicts to $g \leq n$. So we may assume there exists an inter i such that $M_i \cap E(H) = \emptyset$. By Lemma 5.1.11 and Lemma 5.1.10, we have (i) holds.

We next show that (ii) holds. We suppose $n \geq 8$ in this claim and leave the proof for the cases $n = 6, 7$ in Appendix.

Let $f(g) = (\sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i})$. Note that $f(g+1) - f(g) = 2s + 2$ and $s < [\frac{n}{2}]$ since $g \leq n$ (one can obtain the inequalities by simply count, we omit the details). Then, $g(n+1) - f(g)$ is increasing for $g \leq n$ since $2s + 2 < n + 1$. Thus, it is sufficient to show that $|E_X| \geq n(n+1) - f(n)$.

Note that $|E_X| = |X|(n+1) - 2|E(FQ_n[X])|$. Clearly, $2|E(FQ_n[X]) \leq 2|E(FQ_n - M[X])| + |X|$ and then $2|E(FQ_n - M[X])| \leq f(|X|)$ by Lemma 5.1.10. Thus, $|E_X| = |X|(n+1) - 2|E(FQ_n[X])| \geq |X|(n+1) - 2|E(FQ_n - M[X])| - |X| = |X|n - 2|E(FQ_n - M[X])| \geq |X|n - f(|X|)$.

Consider the expression $h(|X|) = (|X|n - f(|X|)) - (n(n+1) - f(n)) = (|X| - n - 1)n - (f(|X|) - f(n))$ for $n+2 \leq |X| < 2^{[\frac{n}{2}]}$. Note that $|X|n - f(|X|)$ is increasing, then $h(|X|) \geq h(n+2)$. Let $n = \sum_{i=0}^{s} 2^{t_i}$. Then $h(n+2) = n - (f(n+2) - f(n)) = n - ((f(n+2) - f(n+1)) + (f(n+1) - f(n))) \leq n - (2(s+1) + 2 + 2s + 2) = n - (4s+6)$. It is easily checked that if $n \in [2^j, 2^{j+1}]$, then $s \leq j$. So $n - (4s+6) > 0$ for $n \geq 2^5$.

Moreover, $n-(4s+6)\geq 0$ for $16=2^4\leq n\leq 31=2^5-1$. Note that if we let $n+1=\sum\limits_{i=0}^{s'}2^{t_i}$, then

$$h(n+2)=n-((f(n+2)-f(n+1))+(f(n+1)-f(n)))=n-(2s'+2+2s+2). \qquad (1)$$

By (1), it can be seen that $h(n+2)\geq 0$ for $8\leq n\leq 15$. Thus, $h(n+2)\geq 0$ for $n+2\leq |X|<2^{[\frac{n}{2}]}$ and then $|E_X|\geq n(n+1)-f(n)$ for $n+2\leq |X|<2^{[\frac{n}{2}]}$.

Claim 1. Let X be a subset $V(Q_n)$ with $|X|\geq 2^{[\frac{n}{2}]}$. Then $|E_X(Q_n)|\geq [\frac{n}{2}]\cdot 2^{[\frac{n}{2}]}$.

By Lemma 5.1.10, it is sufficient to show that $|E_X(Q_n)|\geq n|X|-f(|X|)\geq [\frac{n}{2}]\cdot 2^{[\frac{n}{2}]}$ when $|X|\geq 2^{[\frac{n}{2}]}$. In fact, it is easy to see the following:

$$\begin{aligned}
n|X|-f(|X|) &= n2^{t_0}+n(2^{t_1}+\cdots+2^{t_s})-t_02^{t_0}-(\sum_{i=1}^{s}t_i2^{t_i}+\sum_{i=1}^{s}2\cdot i\cdot 2^{t_i})\\
&= (n-t_0)2^{t_0}+(n\sum_{i=1}^{s}2^{t_i}-(\sum_{i=1}^{s}t_i2^{t_i}+\sum_{i=1}^{s}2\cdot i\cdot 2^{t_i}))\\
&> (n-t_0)2^{t_0}\\
&= (n-t_0)\cdot 2^{(t_0-[\frac{n}{2}])}\cdot 2^{[\frac{n}{2}]}
\end{aligned}$$

Note that $[\frac{n}{2}]\leq t_0<n-1$, then we have $(n-t_0)\cdot 2^{(t_0-[\frac{n}{2}])}\geq [\frac{n}{2}]$.

Since FQ_n-M is isomorphic to Q_n and $|E_X|\geq [\frac{n}{2}]\cdot 2^{[\frac{n}{2}]}$, we have $|E_X(FQ_n)|\geq |E_X(Q_n)|\geq [\frac{n}{2}]\cdot 2^{[\frac{n}{2}]}$. It is not difficult to see that $[\frac{n}{2}]\cdot 2^{[\frac{n}{2}]}\geq n(n+1)-f(n)$ for $n\geq 8$.

The rest is to show that (ii) holds for $|X|=n+1$. Note that H is connected. If H contains either no edges of M_i for some i, or no edges of M, then by Lemma 5.1.11 that $|E_X|\geq (n+1)(n+1)-f(n+1)>n(n+1)-f(n)$ for $n\geq 5$ (Note that $f(n+1)-f(n)=2s+2$, the inequality follows from $n+1>2s+2$). So we may assume that H contains at least one edge of M_i for each i, and contains at least one edge of M.

Claim 2. $H-M$ is a path.

Take an edge e_i from $M_i\cap E(H)$ for each i. Then the subgraph H^* of H induced by $\{e_1,\cdots,e_n\}$ is a forest. Note that H is of order $n+1$. Then the forest is a tree. Since H contains an edge f of M, then H^*+f contains a cycle (of length $n+1$). Thus H^* is a path of length n. Note that the cycle contains no chords (by definition of FQ_n). Thus $H-M$ is a path.

So H is a cycle. Thus, $|E_X| \geq (n+1)(n+1)-2(n+1) > n(n+1)-f(n)$ for $n \geq 5$. $\square$

Theorem 5.1.13 *$\lambda_g(FQ_n) = g(n+1)-(\sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2\cdot i\cdot 2^{t_i})$, where $n \geq 6, g = \sum_{i=0}^{s} 2^{t_i} \leq n, t_0 = [\log_2 g]$, and $t_i = [\log_2(g-\sum_{r=0}^{i-1} 2^{t_r})]$ for $i \geq 1$.*

Proof. By Lemma 5.1.9, it is sufficient to show that $\lambda_g(FQ_n) \geq g(n+1)-(\sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2\cdot i\cdot 2^{t_i})$. Let F be a minimum g-extra edge-cut of FQ_n and C be the smaller one of the two components in FQ_n-F (note that FQ_n-F contains exactly two components since F is a minimum g-extra edge-cut). We next show that $|F| \geq g(n+1)-(\sum_{i=0}^{s} t_i 2^{t_i}+\sum_{i=0}^{s} 2\cdot i\cdot 2^{t_i})$. If $|V(C)| \leq n$, then by Lemma 5.1.12 (i) we have $|F| \geq |E_{V(C)}| = |V(C)|(n+1)-|E(C)| \geq g(n+1)-(\sum_{i=0}^{s} t_i 2^{t_i}+\sum_{i=0}^{s} 2\cdot i\cdot 2^{t_i})$ and then we are done. If $|V(C)| \geq n+1$, then $|F| \geq g(n+1)-(\sum_{i=0}^{s} t_i 2^{t_i}+\sum_{i=0}^{s} 2\cdot i\cdot 2^{t_i})$ by Lemma 5.1.12 (ii). We complete the proof. $\square$

Appendix. We shall show that $|E_X| \geq g(n+1)-(\sum_{i=0}^{s} t_i 2^{t_i}+\sum_{i=0}^{s} 2\cdot i\cdot 2^{t_i})$ for $n=6,7$. By Lemma 5.1.12 (i), it is sufficient to show that $|E_X| \geq n(n+1)-f(n)$ for $|X| \geq n+2, n=5,6,7$. By the proof of Lemma 5.1.12 (ii), we have $|E_X| \geq n(n+1)-f(n)$ for $|V(C)| = n+1, n \geq 5$. We may assume $|X| \geq n+2$. Let $z(n) = n(n+1)-f(n)$. Then $z(6) = 28, z(7) = 38$. To show that $|E_X| \geq n(n+1)-f(n)$, we first use an algorithm to exclude several cases. Note that $|E_X| \geq n|X|-f(|X|)$. We compute the function $R(x) = nx-f(x)-z(n)$ by using Matlab, where $x = |X| = \sum_{i=0}^{s} 2^{t_i}$ and then $f(x) = \sum_{i=0}^{s} t_i 2^{t_i}+\sum_{i=0}^{s} 2\cdot i\cdot 2^{t_i}$.

We list the algorithm as follows.

For $n=6$, we have $\{x=8, R(x)=-4\}, \{x=9, R(x)=0\}, \{x=10, R(x)=2\}, \{x=11, R(x)=4\}, \{x=12, R(x)=4\}, \{x=13, R(x)=6\}, \{x=14, R(x)=6\}, \{x=15, R(x)=6\}, \{x=16, R(x)=4\}, \{x=17, R(x)=8\}, \{x=18, R(x)=$

Algorithm 1 Input $\{n, z(n)\}$; Output $\{x, R(x)\}$

for $x = n + 2$ to 2^{n-1} **do**
    Initialize $j = 0, s = 0, f(x) = 0$;
    while $x > s$ **do**
        $t_j = [\log_2(x - s)]$;
        $s = s + 2^{t_j}$;
        $f(x) = f(x) + t_j 2^{t_j} + 2 * j * 2^{t_j}$;
        $j = j + 1$;
    end while
    $R(x) = nx - f(x) - z(n)$;
    Print $\{x, R(x)\}$;
    $x = x + 1$;
end for

$10\}, \{x = 19, R(x) = 12\}, \{x = 20, R(x) = 12\}, \{x = 21, R(x) = 14\}, \{x = 22, R(x) = 14\}, \{x = 23, R(x) = 14\}, \{x = 24, R(x) = 12\}, \{x = 25, R(x) = 14\}, \{x = 26, R(x) = 14\}, \{x = 27, R(x) = 14\}, \{x = 28, R(x) = 12\}, \{x = 29, R(x) = 12\}, \{x = 30, R(x) = 10\}, \{x = 31, R(x) = 8\}, \{x = 32, R(x) = 4\}$.

For $n = 7$, we have $\{x = 9, R(x) = -1\}, \{x = 10, R(x) = 2\}, \{x = 11, R(x) = 5\}, \{x = 12, R(x) = 6\}, \{x = 13, R(x) = 9\}, \{x = 14, R(x) = 10\}, \{x = 15, R(x) = 11\}, \{x = 16, R(x) = 10\}, \{x = 17, R(x) = 15\}, \{x = 18, R(x) = 18\}, \{x = 19, R(x) = 21\}, \{x = 20, R(x) = 22\}, \{x = 21, R(x) = 25\}, \{x = 22, R(x) = 26\}, \{x = 23, R(x) = 27\}, \{x = 24, R(x) = 26\}, \{x = 25, R(x) = 29\}, \{x = 26, R(x) = 30\}, \{x = 27, R(x) = 31\}, \{x = 28, R(x) = 30\}, \{x = 29, R(x) = 31\}, \{x = 30, R(x) = 30\}, \{x = 31, R(x) = 29\}, \{x = 32, R(x) = 26\}, \{x = 33, R(x) = 31\}, \{x = 34, R(x) = 34\}, \{x = 35, R(x) = 37\}, \{x = 36, R(x) = 38\}, \{x = 37, R(x) = 41\}, \{x = 38, R(x) = 42\}, \{x = 39, R(x) = 43\}, \{x = 40, R(x) = 42\}, \{x = 41, R(x) = 45\}, \{x = 42, R(x) = 46\}, \{x = 43, R(x) = 47\}, \{x = 44, R(x) = 46\}, \{x = 45, R(x) = 47\}, \{x = 46, R(x) = 46\}, \{x = 47, R(x) = 45\}, \{x = 48, R(x) = 42\}, \{x = 49, R(x) = 45\}, \{x = 50, R(x) = 46\}, \{x = 51, R(x) = 47\}, \{x = 52, R(x) = 46\}, \{x = 53, R(x) = 47\}, \{x =$

$54, R(x) = 46\}, \{x = 55, R(x) = 45\}, \{x = 56, R(x) = 42\}, \{x = 57, R(x) = 43\}, \{x = 58, R(x) = 42\}, \{x = 59, R(x) = 41\}, \{x = 60, R(x) = 38\}, \{x = 61, R(x) = 37\}, \{x = 62, R(x) = 34\}, \{x = 63, R(x) = 31\}, \{x = 64, R(x) = 26\}$.

Note that $R(x) < 0$ occurs only when $|X| = n+2, n = 6, 7$. We next prove the assertion for these cases. We suppose $|X| = n+2, n = 6, 7$. Note that H is connected. If H contains either no edges of M_i for some i, or no edges of M, then by Lemma 5.1.11 that $|E_X| \geq (n+1)(n+2) - f(n+2) \geq n(n+1) - f(n)$ for $n \geq 5$. So we may assume that H contains at least one edge of M_i for each i, and contains at least one edge of M. Take an edge e_i from $M_i \cap E(H)$ for each i. Then the subgraph of H induced by $\{e_1, \cdots, e_n\}$ is a forest. Note that H is of order $n+2$. Then the forest has exactly two components.

Suppose that there is an edge $f_i \in M_i$ such that the subgraph H^* induced by $\{e_1, \cdots, e_n, f_i\}$ is a tree. Take an edge f from $H \cap M$, then $H^* + f$ contains a cycle (of length $n+1$), say C. Clearly, C contains no chords. Note that $H^* - C$ is a isolated vertex, say u. Clearly, u is adjacent to at most one vertex on C by the definition of adjacent relation in FQ_n. So $|E(H)| = n+2$ and then $E_X = (n+1)|X| - 2|E(H)| = (n+1)(n+2) - 2(n+2) > n(n+1) - f(n)$ for $n = 6, 7$.

Suppose that there is no f_i in M_i between the two components of the forest. So $H - M$ has exactly two components, say H_1 and H_2. Let $X_1 = V(H_1), X_2 = V(H_2)$ and $x_1 = |X_1|, x_2 = |X_2|$. Without loss of generality we may assume $x_1 \geq x_2$. If $H[X_1]$ contains some edge in M, then $H[X_1]$ is a cycle and H_2 is a isolated vertex. Thus, H is the graph obtained from a cycle of length $n+1$ by adding a pendant edge. Therefore, $|E(H)| = n+2$ and $|E_X| > n(n+1) - f(n)$ for $n \geq 5$. So we may assume $x_2 \geq 2$. Note that H_i is a subgraph of $FQ_n - M$. Then $2|E(H_i)| \leq f(x_i)$. Since there is no edges of M in H_i, H contains at most $\min\{x_1, x_2\}$ edges in M. So, $2|E(H)| \leq f(x_1) + f(x_2) + 2\min\{x_1, x_2\} < f(x_1 + x_2)$ (Note that the last inequality is not difficult to obtain, see the detailed explanations in the proof of the Claim in Theorem 1 [104]). Thus $|E_H| \geq |X|(n+1) - f(|X|) = (n+1)(n+2) - f(n+2) > n(n+1) - f(n)$ for $n \geq 5$. We complete the proof.

Conclusions. We emphasize that the result in this part is a successor of the result in [158], in which we determined the g-extra connectivity of folded hypercubes FQ_n for $n \geq 8, g \leq n-4$.

Note that the assumption $g \leq n$ and the previous results cannot imply the case for $n = 5, g = 5$. Then Theorem 5.1.13 should contain the case $n = 5, g = 5$ but it miss the case. By the method posed in this note, we consider $R(x)$ for $n = 5$. We have $\{x = 7, R(x) = -3\}, \{x = 8, R(x) = -4\}, \{x = 9, R(x) = -1\}, \{x = 10, R(x) = 0\}, \{x = 11, R(x) = 1\}, \{x = 12, R(x) = 0\}, \{x = 13, R(x) = 1\}, \{x = 14, R(x) = 0\}, \{x = 15, R(x) = -1\}, \{x = 16, R(x) = -4\}$. So we have to discuss the case $|X| = 8, 9, 15, 16$ for $n = 5$. For these cases, we have no simple method to consider them. If we discuss them by the regular method, then it implies a long and tedious proof. So we omit the case in this note.

The key problem in this method is to determine the maximum number of edges of the subgraph induced by a vertex set with a given size. We conjecture the maximum number of edges of the subgraph induced by m vertices in folded hypercubes is $f(m)$ for $m \leq 2^{[\frac{n}{2}]}$.

§5.2 The minimum size of graphs under a given edge-degree condition

I first note that the main result in this subsection also appeared in my first Phd thesis in Xiamen University. In that thesis we consider the connectivity of graphs, in which I use the main results to characterized minimum restricted k-edge connected graphs for some integer k. Here, I would like to emphasize the extremal problem on the edge-condition.

Determining the minimum and/or maximum size of graphs with some given parameters is a classic extremal problem in graph theory, see [10]. In this paper, we consider the following problem: What is the minimum size of graphs with a given order n, a given minimum degree δ and a given minimum edge-degree $2\delta + k - 2$? We obtain a lower bound for the minimum size of graphs with a given order n, a given minimum degree δ and a given minimum edge-degree $2\delta+k-2$. Moreover, we characterize the extremal graphs for $k = 0, 1, 2$. As an application, we characterize some kinds of minimum restricted edge connected graphs.

In the following theorem, the graph considered may have loops (a loop is an edge with two same endpoints). For a graph G and $u \in V(G)$, let $E_G(u)$ be the set of edges incident with u in G. When the graph G is understood from the context, we write E_u for $E_G(u)$, δ and n for $\delta(G)$ and $|V(G)|$, respectively. Denote by $G_{n,\delta,k}$ a graph with order n, minimum degree δ, and minimum edge-degree $2\delta+k-2$, and let $\mathcal{G}_{n,\delta,k}$ be the set of all $G_{n,\delta,k}$'s.

Theorem 5.2.1 *Let G be a graph with minimum degree $\delta \geq 3, \xi(G) \geq 2\delta+k-2$. Then $|E(G)| \geq \frac{\delta}{2}n + \frac{\delta k}{2(\delta+k)}d_\delta$. Moreover, equality holds if and only if $d_\delta = \frac{\delta+k}{2\delta+k}n$ and $V(G) = D_\delta \cup N(D_\delta)$.*

Proof. Let $N(G) = N_G(D_\delta)$ and $T(G) = V \setminus (N \cup D_\delta)$ (or simply, we use N and T for $N(G)$ and $T(G)$). Note that $k \geq 0$ and the inequality holds if $k = 0$. Then we may assume that G is a graph with $\xi(G) \geq 2\delta - 1$, i.e. $k \geq 1$. Thus, D_δ is an independent set of G. The degrees of vertices in N are at least $\delta+k$ and the degrees of vertices in T are at least $\delta+1$. We consider the size of G by distinguishing the following two cases.

If $|N| > \frac{\delta}{\delta+k}d_\delta$, we have

$$\begin{aligned}
|E(G)| = \frac{\sum id_i}{2} \quad &\geq \quad \frac{\delta d_\delta}{2} + \frac{\delta+k}{2}|N| + \frac{\delta+1}{2}|T| \\
&= \quad \frac{\delta d_\delta}{2} + \frac{\delta+k}{2}|N| + \frac{\delta+1}{2}(n - d_\delta - |N|) \\
&= \quad \frac{\delta}{2}n + \frac{n}{2} - \frac{d_\delta}{2} - \frac{\delta+1}{2}|N| + \frac{d_\delta+k}{2}|N| \\
&= \quad \frac{\delta}{2}n + \frac{k-1}{2}|N| + \frac{\delta}{2}n - \frac{d_\delta}{2} \\
&> \quad \frac{\delta}{2}n + \frac{k-1}{2}(\frac{\delta}{\delta+k}d_\delta) + \frac{\delta}{2}n - \frac{d_\delta}{2} \\
&= \quad \frac{\delta}{2}n + \frac{\delta k}{2(\delta+k)}d_\delta - \frac{\delta}{2(\delta+k)}d_\delta + \frac{|N|+|T|}{2}
\end{aligned}$$

By the assumption, we have $|E(G)| \geq \frac{\delta}{2}n + \frac{\delta k}{2(\delta+k)}d_\delta$ since $\frac{|N|+|T|}{2} \geq \frac{\delta}{2(\delta+k)}d_\delta$.

If $|N| \leq \frac{\delta}{\delta+k}d_\delta$, we have

$$|E(G)| \quad \geq \quad \delta d_\delta + \frac{\delta+1}{2}|T|$$

$$
\begin{aligned}
&= \frac{\delta}{2}d_\delta + \frac{\delta}{2}\delta|N| + \frac{\delta}{2}d_\delta - \frac{\delta}{2}\delta|N| + \frac{\delta+1}{2}|T| \\
&= \frac{\delta}{2}n + \frac{\delta}{2}d_\delta - \frac{\delta}{2}|N| + \frac{1}{2}|T| \\
&\geq \frac{\delta}{2}n + \frac{\delta}{2}d_\delta - \frac{\delta}{2}\frac{\delta}{\delta+k}d_\delta + \frac{1}{2}|T| \\
&\geq \frac{\delta}{2}n + \frac{\delta k}{2(\delta+k)}d_\delta
\end{aligned}
$$

By the Inequality (2), one can see that if $|N| < \frac{\delta}{\delta+k}d_\delta$, then $|E(G)| > \frac{\delta}{2}n + \frac{\delta k}{2(\delta+k)}d_\delta$. Moreover, if the equality holds in Inequality (2), then $|T| = 0$. Note that $|N| = \frac{\delta}{\delta+k}d_\delta$ and $|T| = 0$ imply that $d_\delta = \frac{\delta+k}{2\delta+k}n$. Thus, the equality holds if and only if $d_\delta = \frac{\delta+k}{2\delta+k}n$ and $|N| = \frac{\delta}{2\delta+k}n$. We complete the proof. $\square$

We next consider the extremal graphs for $k = 0, 1, 2$. By Theorem 5.2.1, we have $|E(G_{n,\delta,k})| \geq \lceil \frac{\delta}{2}n \rceil$ if $k = 0$. The following graph $H_{s,m} \in \mathcal{G}_{n,\delta,k}$ such that $|E(H_{s,m})| = \lceil \frac{\delta}{2}n \rceil$.

For integers $s, n, n \geq s+1$, Harary [67] constructed classes of graphs $H_{s,n}$ that are minimum s-connected. The graph $H_{s,n}$ is as follows.

Case 1. $s = 2r, r > 0$. $H_{2r,n}$ is with vertex set $\{0, 1, 2, \cdots, n-1\}$ and two vertices i and j are adjacent if $i - r \leq j \leq i + r$, where addition is taken modulo n.

Case 2. $s = 2r + 1, r > 0$.

Case 2.1. n is even. Then $H_{2r+1,n}$ is obtained by adding edges joining vertex i to vertex $i + \frac{n}{2}$ for $1 \leq i < \frac{n}{2}$ on $H_{2r,n}$.

Case 2.2. n is odd. $H_{2r+1,n}$ is obtained by adding edges $[0, \frac{(n-1)}{2}]$ and $[0, \frac{(n+1)}{2}]$, and $[i, i + \frac{(n+1)}{2}]$ for $1 \leq i < \frac{(n-1)}{2}$ on $H_{2r,n}$.

Clearly, $H_{\delta,n}$ has minimum degree δ and the minimum edge-degree $2\delta + 0 - 2$. In particular, $H_{\delta,n}$ has $\lceil \frac{\delta}{2}n \rceil$ edges. A graph G is called *almost n-regular* if there is at most one vertex of degree $n+1$ and all other vertices have degree n. The following theorem follows immediately from the argument above.

Theorem 5.2.2 *$|E(G_{n,\delta,0})| \geq \lceil \frac{\delta}{2}n \rceil$ and $|E(G_{n,\delta,0})| = \lceil \frac{\delta}{2}n \rceil$ if and only if $G_{n,\delta,0}$ is an almost δ-regular graph.*

From now on, we always assume $k = 1, 2$ in this section. A bipartite graph is called $(\delta, \delta+k)$-graph if the degree the vertices in a part are all δ, and vertices in the other part have degree $\delta+k$. We denote a $(\delta, \delta+k)$-graph by $G(\delta, \delta+k)$. Note that if a graph $G_{n,\delta,k}$ is a $(\delta, \delta+k)$-graph, i.e. $d_\delta = \frac{\delta+k}{2\delta+k}n$, then the equality of Theorem 5.2.1 holds. By Theorem 5.2.1, we have $|E(G(\delta, \delta+k))| > \frac{\delta}{2}n + \frac{\delta k}{2(\delta+k)}d_\delta$ if $d_\delta \neq \frac{\delta+k}{2\delta+k}n, k = 1, 2$. That is, $|E(G_{n,\delta,k})| > \frac{\delta}{2}n + \frac{\delta k}{2(\delta+k)}d_\delta$ if $G_{n,\delta,k}$ is not a $(\delta, \delta+k)$-graph for $k = 1, 2$. Thus, the following holds.

Theorem 5.2.3 *If $k = 1, 2$ and $n = (2\delta+k)t$ for some integer t, then $|E(G_{n,\delta,k})| \geq \frac{\delta}{2}n + \frac{\delta k}{2(\delta+k)}d_\delta$. Moreover, $|E(G_{n,\delta,k})| = \frac{\delta}{2}n + \frac{\delta k}{2(\delta+k)}d_\delta$ if and only if $G_{n,\delta,k}$ is a $(\delta, \delta+k)$-graph.*

Proof. By Theorem 5.2.1, we have $|E(G(\delta, \delta+k))| \geq \frac{\delta}{2}n + \frac{\delta k}{2(\delta+k)}d_\delta$. Note that $k \geq 1$. Then $D_\delta(G)$ is an independent set. Since $d_\delta = \frac{\delta+k}{2\delta+k} \cdot n$ and $V(G) = D_\delta \cup N(D_\delta)$, we have that each vertex of $V(G) - D_\delta(G)$ has degree $\delta + k$ and $D_{\delta+k}(G)$ is an independent set. We complete the proof. $\square$

We define the function $f(n)$ for the minimum size of all $G_{n,\delta,k}$'s. In Theorem 5.2.3, we characterized the extremal graphs for $k = 1, 2$ and $n = (2\delta + k)t$ for some t. Next, we consider $f(n)$ for $k = 1, 2$ and $n \neq (2\delta + k)t$ for any t.

Lemma 5.2.4 *If $k = 1, 2$, then $f(n) + \lfloor \frac{\delta}{2} \rfloor \leq f(n+1)$.*

Proof. Let $G_{n+1,\delta,k}$ be a graph with $f(n+1)$ edges. Take a vertex u of degree δ in $G_{n+1,\delta,k}$. Note that there is no loops on u since $k \neq 0$. Assume that δ is even. Let $N'(u) = \{u_1, u_2, \cdots, u_\delta\}$. We construct a $G_{n,\delta,k}$ with $f(n+1) - \frac{\delta}{2}$ edges by deleting the vertex u and adding $\frac{\delta}{2}$ new edges $u_1u_2, u_3u_4, \cdots, u_{\delta-1}u_\delta$. If δ is odd, then we first add a new edge between u and a vertex in $D_{\geq\delta+k}$, and then we may construct a $G_{n,\delta,k}$ with $f(n+1) - \lfloor \frac{\delta}{2} \rfloor$ edges. That is, $f(n) + \lfloor \frac{\delta}{2} \rfloor \leq f(n+1)$. $\square$

Lemma 5.2.5 *If $k = 1$, then $f(n+1) \leq f(n) + \lfloor \frac{\delta}{2} \rfloor + 1$.*

Proof. We first assume that δ is even. Let $G_{n,\delta,k}$ be a graph with $f(n)$ edges. We shall construct a new graph $G_{n+1,\delta,k}$ with $f(n) + \lfloor \frac{\delta}{2} \rfloor + 1$ edges by adding a new vertex u, joining an edge between u and a vertex of $D_{\geq\delta+k}(G)$, and adding $\frac{\delta}{2}$ loops on u. Thus, $f(n+1) \leq f(n) + \lfloor \frac{\delta}{2} \rfloor + 1$ in this case.

Suppose δ is odd. Let $G_{n,\delta,k}$ be a graph with $f(n)$ edges. We shall construct a new graph $G_{n+1,\delta,k}$ with $f(n)+\lfloor\frac{\delta}{2}\rfloor+1$ edges by adding a new vertex u, joining an edge between u and vertex of $v\in D_{\geq\delta+k}(G)$, adding $\lfloor\frac{\delta}{2}\rfloor$ loops on u, deleting an edge $vv'\in E(G)$ and adding a new edge uv'. Thus, $f(n+1)\leq f(n)+\lfloor\frac{\delta}{2}\rfloor+1$. We complete the proof. □

Lemma 5.2.6 *If $k=2$, then $f(n+1)\leq f(n)+\lfloor\frac{\delta}{2}\rfloor+2$.*

Proof. Let $G_{n,\delta,k}$ be a graph with $f(n)$ edges. Similarly as the proof of Lemma 5.2.5, we construct a new graph $G_{n+1,\delta,k}$ with $f(n)+\lfloor\frac{\delta}{2}\rfloor+2$ edges by adding a new vertex u, joining an edge between u and vertex of $v\in D_{\geq\delta+k}(G)$, adding $\lfloor\frac{\delta}{2}\rfloor$ loops on u, deleting edges $vv',vv''\in E(G)$ and adding two new edges uv',uv''. Thus, $f(n+1)\leq f(n)+\lfloor\frac{\delta}{2}\rfloor+2$. □

Combining the Lemmas above, we have the following.

Theorem 5.2.7 *If $k=1$, then $f(n)+\lfloor\frac{\delta}{2}\rfloor\leq f(n+1)\leq f(n)+\lfloor\frac{\delta}{2}\rfloor+1$, and if $k=2$, then $f(n)+\lfloor\frac{\delta}{2}\rfloor\leq f(n+1)\leq f(n)+\lfloor\frac{\delta}{2}\rfloor+2$.*

Remark. Assume $k=1$. Note that $f((2\delta+k)(t-1))+\lfloor\frac{\delta}{2}\rfloor(2\delta+k)<f((2\delta+k)t)<f((2\delta+k)(t-1))+(\lfloor\frac{\delta}{2}\rfloor+1)(2\delta+k)$, then by Theorem 5.2.3 that the bounds of Theorem 5.2.7 for $k=1$ are sharp.

In the next section, we shall characterize some kinds of minimum restricted edge-connected graphs by applying the Theorems 5.2.2 and 5.2.3.

§5.2.1 Minimum restricted $(2\delta+k-2)$-edge connected graph for $k\in\{0,1,2\}$

We assume that the graph considered in the section has minimum degree δ. Esfahanian and Hakimi [51] introduced the concept of restricted edge-connectivity. The concept of restricted edge-connectivity is also a kind of conditional edge-connectivity proposed by Harary in [67]. An edge set F a *restricted edge-cut* of G if $G-F$ is disconnected and contains no isolated vertices. The minimum cardinality over all restricted edge-cuts is the *restricted edge-connectivity* of G, denoted by $\lambda'(G)$. If a graph has restricted edge-connectivity k, then the graph is called *restricted k-edge-connected*. A graph G is called *minimum restricted k-edge connected* if G is restricted k-edge connected and no restricted k-edge connected graphs with the same number of vertices has fewer edges than G. A graph G with $\lambda'(G) = \xi(G)$ is called *λ'-optimal*. Esfahanian in [51] proved that if a connected graph G with $|V(G)| \geq 4$ is not a star $K_{1,n-1}$, then $\lambda'(G)$ exists and

$$\lambda(G) \leq \lambda'(G) \leq \xi(G). \tag{5.2.1}$$

By inequality (5.2.12), the minimum edge degree of a restricted $2\delta + k - 2$-edge connected graph is at least $2\delta + k - 2$. For the studies of λ'-optimal graphs, we suggest the readers to refer to the survey [69].

In this section, we shall characterize minimum restricted $2\delta - 2$-edge connected graphs by using Theorem 5.2.2, and the minimum restricted $2\delta + k - 2$-edge connected graphs with order $(2\delta + k)t$ for $k = 1, 2$ by using Theorem 5.2.3. Note that if $\delta = 1$ and $k = 0, 1, 2$, then the minimum restricted $2\delta + k - 2$-edge connected graph is clear. Thus, we may assume $\delta \geq 2$ in the following.

In fact, characterizing the minimum graphs with some given property on the connectivity of graphs is a classic topic. For example, in [45], Cozzens and Wu studied the minimum critically k-edge connected graph (a k-edge connected graph is critical if $\lambda(G - \{v\}) < k$ for any vertex $v \in V(G)$). In [110], Maurer and Slater defined the k-edge$^{\sharp}$-connectivity which is the other version of restricted edge connectivity and characterized the cases $k = 1, 2, 3$. Later, Peroche and Virlouvet gave some partial results on the cases $k = 4, 5$. Recently, Hong et al [72] showed that the minimally restricted k-edge connected graph is λ'-optimal (A restricted k-edge connected graph G is called *minimally restricted k-edge connected* if $\lambda'(G - e) < k$ for any edge e.). Here, we give some partial results on the minimum restricted edge connected graphs.

To characterize the minimum restricted $2\delta - 2$-edge connectivity, the following lemma is needed. We shall give the proof of the following lemma in Section 3.

Lemma 5.2.8 *Harary graph $H_{\delta,n}$ is λ'-optimal.*

It can be seen that there exists a restricted $2\delta - 2$-edge connected graph with size $\lceil \frac{\delta n}{2} \rceil$ from Lemma 5.2.8. Thus, we have the following characterization for minimum restricted $2\delta-2$-edge connected graph.

Theorem 5.2.9 *The minimum restricted $2\delta-2$-edge connected graph with minimum degree δ is an almost δ-regular graph.*

Let $Aut(G)$ denote the automorphism group of G, i.e., the set of bijective mappings on its vertices that preserve incidence. For $x \in V(G)$, the set $\{g(x) : g \in Aut(G)\}$ is an *orbit of* $Aut(G)$; we abuse the terminology a little to call it *an orbit of* G. Let W be a subgroup of the symmetric group over a set S. We say that W *acts transitively* on a subset T of S if for any $h, l \in T$, there exists a permutation $\varphi \in W$ with $\varphi(h) = l$. Clearly, the automorphism group $Aut(G)$ acts transitively on each orbit of $Aut(G)$. We call a graph G a *double-orbit graph* if $Aut(G)$ has at most two orbits. A double-orbit bipartite graph is also called a *half vertex transitive graph*. When $Aut(G)$ has exactly two orbits, we denote them as V_1 and V_2. Next, we shall construct a double orbit bipartite graph $G(V_1, V_2)$ with $(2\delta+k)t$ vertices for each integer t.

We denote the vertex set of $G(V_1, V_2)$ by $V_1 \cup V_2$, and $V_1 = \{0, 1, \cdots, (\delta+k)t-1\} = Z_{(\delta+k)t}, V_2 = \{0, 1, \cdots, \delta t-1\} = Z_{\delta t}$. Add an edge between the vertex i of V_1 and the vertex $i+j$ of V_2 for $j = 0, 1, \cdots, \delta-1$ (the addition is taken modulo δ). Clearly, $G(V_1, V_2)$ is a double orbit bipartite graph such the vertex of V_1 has degree δ and the vertex in V_2 has degree $\delta+k$.

We shall give a characterization for the minimum restricted $2\delta+k-2$-edge connected graph for $k = 1, 2$. We need the following lemma and we leave its proof in Section 4.

Lemma 5.2.10 *The half vertex transitive graph $G(V_1, V_2)$ is λ'-optimal.*

By Lemma 5.2.10, there exists a restricted $2\delta+k-2$-edge connected $(\delta, \delta+k)$-graph. Then by Theorem 5.2.3, we have the following theorem.

Theorem 5.2.11 *If $k = 1, 2$ and $n = (2\delta+k)t$ for some positive integer t, then the minimum restricted $2\delta+k-2$-edge connected graph with minimum degree δ is a $(\delta, \delta+k)$-graph.*

Proof of Lemma 5.2.8

It is sufficient to show that $\lambda'(H_{\delta,n}) = \xi(H_{\delta,n}) = 2\delta - 2, n \geq 4$. We only prove the case $\lambda'(H_{\delta,n}) = \xi(H_{\delta,n}) = 2\delta - 2$ for $\delta = 2k+1$ and n is even, and other two cases can be shown by the similar arguments (we leave it to the readers). We write H for $H_{\delta,n}$ for simply.

We call edges $(i, i+\frac{n}{2})$ the *diagonal edges* and edges $(i, i+j)$ the j*-edges*. Clearly, all 1-edges form a cycle, denoted by $C = 12\dots n1$. We use $C_{i,j}, C_{\bar{i},j}$, $C_{i,\bar{j}}$, and $C_{\bar{i},\bar{j}}$ to denote paths $i(i+1), \dots, j$; $(i+1)(i+2), \dots, j$; $i(i+1), \dots, (j-1)$, and $(i+1), \dots, (j-1)$, respectively. For a graph G and two disjoint non-empty subsets A and B of $V(G)$, we denote by $[A, B]$ the set of edge with one end in A and the other one in B. We denote by $E_G(A)$ the set of edges (in $E(G)$) with exactly one end in A. For a subgraph $X \subset G$ and $e = uv \in E(G)$, we use $E_G(X)$ for $E_G(V(X))$ and $E_G(e)$ for $E_G(\{u, v\})$. Sometimes, we write x for the single vertex set $\{x\}$. For a subgraph $X \subset G$ and a vertex $u \notin X$, we use $[u, X]$ for $[\{u\}, V(X)]$. Moreover, if $\overline{A} = V \backslash A$, then we write $\omega(A)$ for $[A, \overline{A}]$.

Lemma 5.2.12 *$H_{\delta,n}$ is λ'-optimal for $4 \leq n \leq 7$.*

Lemma 5.2.13 *For any complete graph $K_m, m \geq 4$, $\lambda'(K_m) = 2m - 4$.*

The proof of Lemma 2.1. For an 1-edge e, it is easy to check that $E_H(e)$ is a restricted edge cut. Therefore, $\lambda'(H) \leq \xi(H) = 2\delta - 2$. We shall show that $\lambda'(H) \geq \xi(H) = 2\delta - 2$. Suppose that there exists a restricted edge cut $F, |F| < 2\delta - 2$, such that $H - F$ is disconnected and contains no trivial component. If there is a component of order $2 \leq t \leq 3$ in $H - F$, then we clearly have $|F| \geq 2\delta - 2$, a contradiction. Thus we may suppose that each component of $H - F$ has order at least 4. Let B be a component with minimum order in all the components of $H - F$.

Claim 1. $|F| > 2\delta - 3$ for $k = 1$ and $k = n/2 - 1$.

Proof of claim 1. Suppose $k = 1$. If the vertices of B are continuous, that is, $V(B) = \{x, x+1, \dots, y\}$, then edges $(x-1)x, y(y+1), x(x+n/2), \dots, y(y+n/2) \in F$. So $|F| \geq 4 > 2\delta - 3 = 3$, a contradiction. If not, there must be diagonal edge in B, without loss of generality, we assume that $0(n/2) \in E(B)$, then both $C_{0,n/2}$ and

$C_{n/2,0}$ contain vertices in $V(H)\backslash V(B)$ since B is the smallest component of $H-F$. So $|E(C_{0,n/2})\cap F|\geq 2$ and $|E(C_{n/2,0})\cap F|\geq 2$. Thus $|F|\geq 4>2\delta-3=3$.

If $k=n/2-1$, then H is a complete graph. It can be seen that $|F|>2\delta-3$. This completes the proof of the claim. $\square$

In the following arguments, we suppose that $n\geq 8, 2\leq k\leq n/2-2$ and each component of $H-F$ has order at least 4. We consider the following two cases.

Case 1. B contains two distinct vertices x and y such that $y-x\equiv l(modn), l>k$ and $V(C_{\overline{x},\overline{y}})\cap V(B)=\emptyset$, or $x-y\equiv l(modn), l>k$ and $V(C_{\overline{y},\overline{x}})\cap V(B)=\emptyset$.

Without loss of generality, we assume that $y-x\equiv l(modn), l>k$ and $V(C_{\overline{x},\overline{y}})\cap V(B)=\emptyset$, see Fig. 1. a.

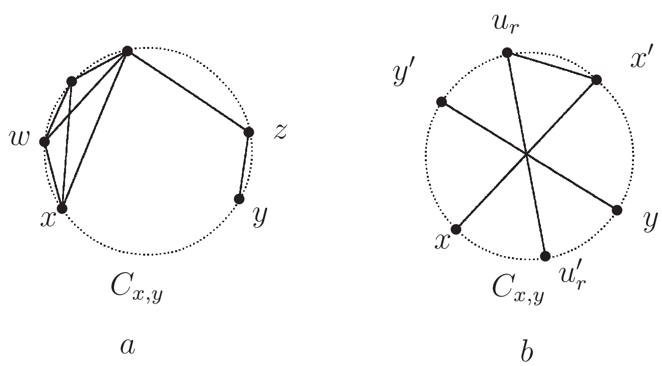


Fig. 1

Pick $w\in N_B(x)$ and $z\in N_B(y)$ such that $V(C_{w,x})\cap V(B)=\{x,w\}$ and $|V(C_{y,z})\cap V(B)=\{y,z\}$ (w and z may be the same). We use E'_x, E'_y, E'_w and E'_z to denote $[x,V(C_{\overline{x},y})]$, $[y,V(C_{x,\overline{y}})]$, $[w,V(C_{\overline{w},y})]\backslash\{xw\}$ and $[z,V(C_{x,\overline{z}})]\backslash\{yz\}$ respectively. By the hypothesis of Case 1 and the rules for picking w and z, we have $E'_x\subseteq F, E'_y\subseteq F, E'_w\subseteq F$ and $E'_z\subseteq F$. Note that $|E'_x|\geq k, |E'_y|\geq k, |E'_w|\geq k-1, |E'_z|\geq k-1$. Set $e=xx', f=yy'$ are diagonal edges incident to x,y respectively. We distinguish the following two subcases.

Case 1.1. $e,f\notin E(B)$.

Then, $|F|\geq|E'_x|+|E'_y|+|E'_w|+|E'_z|+2\geq 2\delta-2>2\delta-3$, a contradiction.

Case 1.2. At least one of $\{e,f\}$ is contained in B.

Without loss of generality, set that $e\in E(B)$ and $N_H(x')\cap V(C_{x',y'})=$

$\{u_1, u_2, \ldots, u_k\}$. We claim that for any $u_r \in N_H(x') \cap V(C_{x',y'})$, if $u_r x' \notin F$, that is, $u_r \in V(B)$, then $u_r u_r' \in F$. Otherwise, it contradicts the fact that $V(C_{\overline{x},\overline{y}}) \cap V(B) = \emptyset$, see Fig. 1. *b*. Similarly, the fact is true for y'. Thus there exist at least two edges in F except E_x', E_y', E_w' and E_z'. Therefore, $|F| \geq |E_x'| + |E_y'| + |E_w'| + |E_z'| + 2 \geq 2\delta - 2 > 2\delta - 3$, a contradiction.

Case 2. For any $x \in V(B)$, B contains a vertex in $\{x, x+1, \ldots, x+k\}$.

We first suppose $n \geq 4k$. Let H_i denote the subgraph of H induced by vertex set $\{ik, ik+1, \ldots, (i+1)k\}, 0 \leq i \leq 3$. By the definition of H, we have $H_i \cong K_{k+1}$. Furthermore, we denote $F \cap E(H_i)$ by F_i. Clearly, F_i is an edge cut of H_i. Since $\lambda(H_i) = k$, we have that $|F| \geq |F_0| + |F_1| + |F_2| + |F_3| \geq 4k = 2\delta - 2 > 2\delta - 3$, a contradiction.

We next suppose $n < 4k$. Note that B is the smallest component of $H - F$, then $|V(B)| \leq |V(H)|/2$. Thus, there exist two vertices $x, x+1 \in V(H)\backslash V(B)$. If $k = 2$, then B satisfies the condition of Case 1. We assume that $k \geq 3$ from now on.

Since $|V(B)| \geq 4$, $n < 4k$, and H is not a complete graph, there exist two disjoint cliques G_1 and G_2 (defined as H_i above, $G_1 \cong G_2 \cong K_{k+1}$ and $V(G_1) \cap V(G_2) = \emptyset$), such that $|V(B) \cap V(G_i)| \geq 2$ and $|(V(H)\backslash V(B)) \cap V(G_i)| \geq 2$ for $i = 1, 2$. Suppose $\{x, y\} \in V(B \cap G_1)$ and $F \cap E(G_i) = F_i$. Clearly, $|F_i| \geq \lambda'(G_i) = 2k - 2$. If $x(x+n/2), y(y+n/2) \in F$, then $|F| \geq |F_1| + |F_2| + 2 = 2\delta - 2 > 2\delta - 3$, a contradiction. If not, without loss of generality, we assume that $x(x+n/2) \notin F$, then at least one of the incident edges of vertex $(x+n/2)$ excepting the edges contained in $[x, V(B)\backslash x]$ is contained in F. Otherwise, $|V(B)| \geq 2k \geq |V(H)|/2$, this contradicts the hypothesis that B is the smallest component of $H - F$. The same fact is true for vertex $(y+n/2)$. Thus $|F| \geq |F_1| + |F_2| + 2 = 2\delta - 2 > 2\delta - 3$. This completes the proof. □

Proof of Lemma 5.2.10

A restricted edge cut F of G is called a λ'-cut if $|F| = \lambda'(G)$. It is easy to see that for any λ'-cut F, $G - F$ has exactly two connected non-trivial components. Let A be a proper subset of V. We denote $G[A]$ the subgraph of G induced by A. If $\omega(A)$ is a λ'-cut of G, then A is called a λ'-*fragment* of G. It is clear that if A is a λ'-fragment of G, then so is $\overline{A}$. Let $r(G)$=min$\{\ |A|$: A is a λ'-fragment of $G\}$. Obviously, $2 \le r(G) \le \frac{1}{2}|V|$. A λ'-fragment B is called a λ'-*atom* of G if $|B| = r(G)$.

In [143], Xu proved the following two theorems.

Theorem 5.2.14 ([143]) *Let $G = (V, E)$ be a connected graph with at least four vertices and $G \not\cong K_{1,m}$. Then G is λ'-optimal if and only if $r(G) = 2$.*

Theorem 5.2.15 ([143]) *Let $G = (V, E)$ be a connected graph with at least four vertices and $G \not\cong K_{1,m}$. If G is not λ'-optimal, then any two distinct λ'-atoms of G are disjoint.*

In what follows, we assume that $G = G(V_1, V_2)$ is a connected half vertex transitive graph with bipartitions V_1 and V_2. Assume that each vertex in V_1 has degree d_1 and each vertex in V_2 has degree d_2.

Lemma 5.2.16 *Let $G = G(V_1, V_2)$ be a connected half vertex transitive graph with $n = |V(G)| \ge 4$ and $G \not\cong K_{1,n-1}$. Assume that A is a λ'-atom, $Y = G[A]$ and $A_i = A \cap V_i$ $(i = 1, 2)$. If G is not λ'-optimal, then*

(i) $V(G)$ is a disjoint union of distinct λ'-atoms,

(ii) $Aut(Y)$ acts transitively both on A_1 and A_2.

Proof. (i) Clearly, $A_i = A \cap V_i \neq \emptyset$ (i=1,2). Since $Aut(G)$ acts transitively both on X_1 and X_2, each vertex of G lies in a λ'-atom. By Theorem 4.2, we have that $V(G)$ is a disjoint union of distinct λ'-atoms.

(ii) Since G is half vertex transitive, for any $x_{i1}, x_{i2} \in A_i$, there exists $\alpha \in Aut(G)$ such that $x_{i2} = \alpha(x_{i1})$. Note that $\alpha(A)$ is also a λ'-atom and $\alpha(A) \cap A \neq \emptyset$. Then $\alpha(A) = A$ by Theorem 4.2. Clearly, the restriction of α on A induces an automorphism of Y, which maps x_{i1} to x_{i2}. $\square$

Theorem 5.2.17 *Let $G = G(V_1, V_2)$ be a connected half vertex transitive graph with $n = |V(G)| \ge 4$ and $G \not\cong K_{1,n-1}$. Then G is λ'-optimal.*

Proof. Assume that G is not λ'-optimal. Let A be a λ'-atom of G. Obviously, $A_i = A \cap X_i \neq \emptyset$ for $i = 1, 2$. Suppose that each vertex in V_1 has degree d_1 and each vertex in V_2 has degree d_2. By Lemma 4.3, we have that $Y = G[A]$ is a connected half vertex transitive graph. Let $d_{G[A]}(x_{1i}) = k_1$ for any $x_{1i} \in A_1$ and $d_{G[A]}(x_{2i}) = k_2$ for any $x_{2i} \in A_2$. Thus we have $|A_1|(d_1 - k_1) + |A_2|(d_2 - k_2) < \xi(X) = d_1 + d_2 - 2$ which implies $(|A_1| - 1)(d_1 - k_1) + (|A_2| - 1)(d_2 - k_2) < k_1 + k_2 - 2 \leq |A_1| + |A_2| - 2$, and then $(|A_1| - 1)(d_1 - k_1 - 1) + (|A_2| - 1)(d_2 - k_2 - 1) < 0$. Thus, we may assume $(|A_1| - 1)(d_1 - k_1 - 1) < 0$ (or $(|A_2| - 1)(d_2 - k_2 - 1) < 0$). It follows $d_1 - k_1 - 1 < 0$, that is, $k_1 = d_1$. By Theorem 4.2, Lemma 4.3 and the half vertex transitivity of G, we may assume that $V(G) = \bigcup_{i=1}^{k} \alpha_i(A)$ (where $\alpha_i \in Aut(G)$ for $1 \leq i \leq k$) such that (1) $G[\alpha_i(A)] \cong X[\alpha_j(A)]$ and $\alpha_i(A) \cap \alpha_j(A) = \emptyset$ for any distinct $i, j \in \{1, 2, \cdots, k\}$, and (2) $X_j = \bigcup_{i=1}^{k} \alpha_i(A_j)$ for $j = 1, 2$. Thus we have that $[\alpha_i(A), \alpha_j(A)] = \emptyset$ for any distinct $i, j \in \{1, 2, \cdots, k\}$, that is, G is disconnected, a contradiction. □

The proof of Lemma 2.3. By Theorem 4.4, the result is clearly true. □

§5.3 The minimum size of graphs under Ore-conditon

Ore-conditon is a well known condition in several topics of graph theory. In this note we consider the minimum size of graphs of order n satisfying Ore-condition, i.e., the sum of the degrees of two non-adjacent vertices is at least n. Define $\mathcal{O}_n = \{G : G$ is of order n and satisfies Ore-condition $\}$, $ex_n = \min\{|E(G)| : G \in \mathcal{O}_n\}$. Clearly, any member of $\mathcal{O}_n$ is a connected graph with diameter 2. Let $G_{\frac{n-1}{2},\frac{n+1}{2}}$ denote the graph obtained by adding $\frac{n-1}{2}$ independent edges between a complete graph $K_{\frac{n-1}{2}}$ and a complete graph $K_{\frac{n+1}{2}}$, and adding an edge on the vertex of degree $\frac{n-1}{2}$ in the $K_{\frac{n+1}{2}}$ and a vertex in the $K_{\frac{n-1}{2}}$. We use $V_i(G)$ for the set of vertices of degrees i in G. Let $\mathcal{G}_{\frac{n-3}{2},\frac{n+3}{2}} = \{G : G \in \mathcal{O}_n, |V_{\frac{n-1}{2}}(G)| = \frac{n-3}{2}, |V_{\frac{n+1}{2}}(G)| = \frac{n+3}{2}$ and $G[V_{\frac{n-1}{2}}(G)]$ is complete graph $K_{\frac{n-3}{2}}$, $G[V_{\frac{n+1}{2}}(G)]$ is obtained from complete graph $K_{\frac{n+3}{2}}$ by removing $\frac{n-3}{2}$ edges$\}$.

Theorem 5.3.1 *For a given integer n, we have $ex_n = \frac{n^2}{4}$ for the even number n and $ex_n = \frac{n^2+3}{4}$ for the odd number n. Moreover, let G be the graph with minimum ex_n edges, then equalities hold if and only if G is a $\frac{n}{2}$-regular graph for even number*

n and either $G \cong G_{\frac{n-1}{2},\frac{n+1}{2}}$ or $G \in \mathcal{G}_{\frac{n-3}{2},\frac{n+3}{2}}$ for odd number n.

Proof. Note that the graph is of diameter 2 if it satisfies Ore-condition. The complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$ satisfies Ore-condition and has exactly $\frac{n^2}{4}$ edges for the even number n; and $G_{\frac{n-1}{2},\frac{n+1}{2}}$ has $\frac{n^2+3}{4}$ edges. Thus $ex_n \le \frac{n^2}{4}$ for the even number n and $ex_n \le \frac{n^2+3}{4}$ for the odd number n. We next show the two equalities hold.

By way of contradiction we assume that $ex_n < \frac{n^2}{4}$ for the even number n and G be a minimum counterexample. Thus the minimum degree δ of G is less than $\frac{n}{2}$. Let $d(u) = \delta$. Then the degrees of vertices in $N(u)$ are at least δ and the degrees of vertices in $V(G) - N(u) - \{u\}$ are at least $n - \delta$. Assume k is the number of edges between $N(u)$ and $V(G) - N(u) - \{u\}$. Thus

$$\begin{aligned}
|E(G)| &\ge \delta + (\frac{\delta^2 - \delta - k}{2}) + \frac{(n-\delta-1)(n-\delta) - k}{2} + k && (5.3.2)\\
&= \delta + \frac{\delta^2}{2} - \frac{\delta}{2} + \frac{n^2}{2} - n\delta + \frac{\delta^2}{2} - \frac{n}{2} + \frac{\delta}{2} && (5.3.3)\\
&= \delta^2 - (n-1)\delta + \frac{n^2}{2} - \frac{n}{2} && (5.3.4)
\end{aligned}$$

The function $f(\delta) = \delta^2 - (n-1)\delta + \frac{n^2}{2} - \frac{n}{2}$ is decreasing on the interval $[1, n-1]$. Since $\delta \le \frac{n-2}{2}$, $f(\delta) \ge f(\frac{n-2}{2}) = \frac{n^2}{4}$, a contradiction. Thus, $\delta \ge \frac{n}{2}$, that is, $ex_n \ge \frac{n^2}{4}$ and then $ex_n = \frac{n^2}{4}$. We complete the proof for the even number n.

We assume that n is odd, $ex_n < \frac{n^2+3}{4}$ and G be a minimum counterexample. We fist show that $\delta \ge \frac{n-1}{2}$. By contradiction, we assume $\delta < \frac{n-1}{2}$. By an argument similar to that the case of even number, we have $f(\delta) \ge f(\frac{n-3}{2}) = \frac{n^2+3}{4}$, a contradiction. So $\delta \ge \frac{n-1}{2}$. Let V' be the set of the vertices of degree $\frac{n-1}{2}$ and V'' be the set of the vertices of degree at least $\frac{n+1}{2}$. Thus

$$\begin{aligned}
|E(G)| &= \frac{1}{2}(\sum_{v\in V'} d(v) + \sum_{v\in V_2} d(v))\\
&\ge \frac{1}{2}(\frac{n-1}{2}|V'| + \frac{n+1}{2}|V''|)\\
&= \frac{1}{2}(\frac{n-1}{2}|V'| + \frac{n+1}{2}(n - |V'|))\\
&= \frac{1}{2}(\frac{n(n+1)}{2} - |V'|)
\end{aligned}$$

Note that $G[V']$ is complete and G is connected. Then $|V'| \leq \frac{n-1}{2}$. Thus, we have $|E(G)| \geq \frac{n^2+1}{4}$. Since n is odd, we have $|E(G)| \geq \lceil \frac{n^2+1}{4} \rceil = \frac{n^2+3}{4}$, a contradiction. That is, $ex_n = \frac{n^2+3}{4}$ for the even number n.

For the even number case, one can see that if $G \in \mathcal{O}_n$, then $|E(G)| = ex_n$ if and only if G is a $\frac{n}{2}$-regular graph. We now consider the odd number n. Combining the argument above, only two cases need to be discussed.

Case 1. There is a vertex $u \in V(G)$ such that $d(u) = \frac{n-3}{2}$.

Recall the function $f(\delta) \geq f(\frac{n-3}{2}) = \frac{n^2+3}{4}$. We have $d(v) = \frac{n-3}{2}$ each vertex $v \in N(u)$ since the equalities hold in (5.3.2). Note that the Ore-condition implies that $G[\{u\} \cup N(u)]$ is a $K_{\frac{n-1}{2}}$. Thus, there is no edges indecent to $\{u\} \cup N(u)$ excepting the edges in $K_{\frac{n-1}{2}}$, this contradict with G is connected.

Case 2. The minimum degree of G is $\frac{n-1}{2}$.

In this case, the equalities in (5.3.5) hold. Thus $|V'|$ is either equal to $\frac{n-1}{2}$ or $\frac{n-3}{2}$ by (5.3.5). Assume $|V'| = \frac{n-1}{2}$. It is easy to see that $G \cong G_{\frac{n-1}{2}, \frac{n+1}{2}}$ since the degrees of the vertices in $V(G) - \{u\} \cup N(u)$ are equal to $\frac{n+1}{2}$.

Similarly, if $|V'| = \frac{n-3}{2}$, then $G \in \mathcal{G}_{\frac{n-3}{2}, \frac{n+3}{2}}$. We complete the proof. $\square$

Generally, we want to explore the minimum size of graphs satisfying the sum of any two non-adjacent vertices at least $t < n$. Define $\mathcal{O}_n^t$ the set of graphs (order n) satisfying the sum of degrees of any two non-adjacent vertices at least $t < n$. Let $ex_n^t = \min\{|E(G)| : G \in \mathcal{O}_n^t\}$. We conjecture the following.

Conjecture 5.3.2 *For a given n, we have $ex_n^t = \frac{nt}{4}$ if t is even and $ex_n^t = \lceil \frac{n(t+1)-(t-1)}{4} \rceil$ if t is odd.*

§5.4 Open problems

In Section 2 and Section 3, we posed two problems under the degree sum condition, one is on the degree sum of two adjacent vertices and the other is on the degree sum of two non-adjacent vertices. Although we obtained some partial results, there is a gap to the solution of the problems. So the problems we posed, in some sense, are still open. In particular, characterizing the minimum/minimal restricted k-edge connected graphs is an interesting problem (itself).

Chapter 6 Future research

Note that we have summarized several open problems in the end of Chapter 2 and Chapter 3. Here, we would like to mention several our new studies related to this thesis which are not included in the thesis. as the extensions and open problems of the thesis.

§6.1 Supereulerian problem

For the supereulerian problem, we have mentioned several open problems in Chapter 2. We would like to mention another two kinds of sufficient conditions for a graph to supereulerian.

In 1972, Chvátal and Erdös showed that the graph G with independence number $\alpha(G)$ no more than its connectivity $\kappa(G)$ (i.e. $\kappa(G) \geq \alpha(G)$) is hamiltonian. This condition stimulates us to consider whether the condition edge-connectivity ($\lambda(G)$) of G no less than matching number (edge independence number) of G $\alpha'(G)$ can guarantee G containing a spanning closed trail (i.e. G is supereulerian).

We recently show that if $\lambda(G) \geq \alpha'(G)-1$, then G either is supereulerian or is in a well-defined family of graphs; we weaken the condition $\kappa(G) \geq \alpha(G)-1$ in [65] to $\lambda(G) \geq \alpha(G) - 1$ and obtain the similar characterization on non-supereulerian graphs. Moreover, we discuss the graph which contains a dominating trail in terms of $\lambda(G) \geq \alpha(G) - 2$ ($\lambda(G) \geq \alpha'(G) - 2$).

When we consider the topic above, we posed the following problems:

Question 6 *What is the minimum integer t such that a 3-edge connected graph G satisfying $\lambda(G) \geq \alpha'(G) - t$ is neither superian nor the Petersen graph ?*

Question 7 *Furthermore, what is the minimum integer t such that a 3-edge connected graph G satisfying $\lambda(G) \geq \alpha'(G) - t$ contains no dominating trail ?*

Question 8 *What is the minimum t such that a graph G satisfies $\lambda(G) \geq 2$ and $\alpha'(G) \leq \lambda(G) + t$, but G contains no dominating trail.*

Question 9 *Toughness and the hamiltonicity of graphs is a well known area in hamiltonian graph theory, see the survey [7]. What conditions can guarantee a 1-tough graph to be supereulerian ?*

§6.2 Hamiltonian line graphs

For the hamiltonicity problems of line graphs, Li posed two problems recently and were solved partially by our team.

It is well known that if a graph G contains a spanning closed trail, then its line graph $L(G)$ is Hamiltonian. Li asked the following.

Question 10 *Whether k spanning closed trails in a graph imply k edge-disjoint hamiltonian cycles in its line graph?*

We partially answer the problem as follows.

Theorem 6.2.1 *If a graph G with minimum degree $4k$ has k edge -disjoint spanning closed trails, then its line graph $L(G)$ contains k edge-disjoint Hamilton cycles.*

So the Li's problem is still open. Furthermore, one may weaken the condition spanning closed trail to the dominating trail. They are two different problems.

So far, a larger number of results on the hamiltonicity of line graphs with higher connectivity were reported. Li suggested a new idea (exploring the condition such that removing some edges of a hamiltonian line graph under the condition preserve the hamiltonicity) and found a condition as follows.

Theorem 6.2.2 *Let G be a closed (under R-closure) hamiltonian claw free-graph with $\delta(G) \geq 7$, and F be an edge set of G. For any maximal clique K_u, if $d_{K_u-F}(x) + d_{K_u-F}(y) \geq \frac{3}{4}(|K_u| + 1)$ for any pair of vertices $x, y \in V(K_u)$, then $G - F$ is hamiltonian.*

Question 11 *Exploring new conditions similar as the theorem above.*

§6.3 Fault-tolerant Hamiltonicity of Cayley graphs generated by transposition trees

We have improved the result in Section 1 of Chapter 4 by the following.

Theorem 6.3.1 *Let F be a fault edge set of G_n in which each vertex is incident with two or more fault-free edges. If $|F| \leq 2n-7$, then $G_n - F$ is hamiltonian for $n \geq 4$.*

Comparing the studies on star graphs, one may find that there are many generalizations of hamiltionian problems on this kind of Cayley graphs. For example, edge-hamiltonicity [154], and so on.

§6.4 Extremal problems

In Section 2 and Section 3 of Chapter 5, we posed two problems under the degree sum condition, one is on the degree sum of two adjacent vertices and the other is on the degree sum of two non-adjacent vertices. We have mentioned that the problems are still open. In this part, we would like to list a new kind of similar problems which were posed when we consider the connectivity of transitive graphs in [148]. As the backgrounds aroused the problems is not an area of this thesis, we do not mention it. We just pose them as extremal problems as follows.

Let $\Gamma_{n,g} = \{G : |V(G)| = n, g(G) \geq g\}$, $\Gamma^{*}_{n,g} = \{G : G \text{ is regular and } G \in \Gamma_{n,g}\}$, $\Gamma^{**}_{n,g} = \{G : G \text{ is bipartite and } G \in \Gamma^{*}_{n,g}\}$, $e(\Gamma_{n,g}) = max\{|E(G)| : G \in \Gamma_{n,g}\}$(resp. $e(\Gamma^{*}_{n,g}); e(\Gamma^{**}_{n,g})$), $k(\Gamma^{*}_{n,g}) = max\{k : k \text{ is the regular degree of } G, G \in \Gamma^{*}_{n,g}\}$(resp. $k(\Gamma^{**}_{n,g})$.

Question 12 *The relation between $e(\Gamma_{n,g})$ and $e(\Gamma'_{n,g})$, where $\Gamma_{n,g} = \{G : |V(G)| = n, g(G) \geq g\}; \Gamma'_{n,g} = \{G : |V(G)| = n, g(G) = g\}$. I conjecture that $e(\Gamma_{n,g})$=$e(\Gamma'_{n,g})$.*

Question 13 *The upper bound and lower bound of $e(\Gamma_{n,g})$. That is: what is the maximum size of a graph with order n and $g(G) \geq g$?*

Question 14 *Characterizing the extremal graph G such that $|E(G)| = e(\Gamma_{n,g})$. Turán's Theorem implies the best result of Problem 1 and 2 when $g = 4$. Can we give a Turán-type result? Even for some small g(=5,6...)?*

Question 15 *Consider the upper bound the minimum degree of the extremal graph. Similarly, a graph G is planar, then the minimum degree of G is not more than 5. Can we give an upper bound $f_\delta(n, g)$ such that the minimum degree of a graph with girth $g(G) \geq g$ and order n cannot more than $f_\delta(n, g)$?*

Question 16 *Consider the upper bound of $e(\Gamma^*_{n,g}), k(\Gamma^*_{n,g})$ (resp. $e(\Gamma^{**}_{n,g}), k(\Gamma^{**}_{n,g})$).*

The upper bound of $e(\Gamma^{**}_{n,g}), k(\Gamma^{**}_{n,g})$ may be is the easiest in above problems.

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1. Hao Li, Weihua He, Weihua Yang, Yandong Bai, A Note on Edge-Disjoint Hamilton Cycles in Line Graphs, prepare to submitted to Discrete Mathematics
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