Couches initiales et limites de relaxation aux systèmes d'Euler-Poisson et d'Euler-Maxwell

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RÉSUMÉ

Mes travaux concernent deux systèmes d'équations utilisés dans la modélisation mathématique de semi-conducteurs et de plasmas : le système d'Euler-Poisson et le système d'Euler-Maxwell. Le premier système est constitué des équations d'Euler pour la conservation de la masse et de la quantité de mouvement couplées à l'équation de Poisson pour le potentiel électrostatique. Le second système décrit le phénomène d'électro-magnétisme. C'est un système couplé, qui est constitué des équations d'Euler pour la conservation de la masse et de la quantité de mouvement et les équations de Maxwell, aussi appelées équations de Maxwell-Lorentz. Les équations de Maxwell sont dues aux lois fondamentales de la physique. Elles constituent les postulats de base de l'électromagnétisme, avec l'expression de la force électromagnétique de Lorentz.

En utilisant une technique de développement asymptotique, nous étudions les limites en zéro du système d'Euler-Poisson dans les modèles unipolaire et bipolaire. Il est bien connu que la limite formelle du système d'Euler-Poisson est gouvernée par les équations de dérive-diffusion lorsque le temps de relaxation tend vers zéro. Par des estimations d'énergie aux systèmes hyperboliques symétriques, nous justifions rigoureusement cette limite lorsque les conditions initiales sont bien préparées. Le phénomène des conditions initiales mal préparées est interprété par l'apparition de couches initiales. Dans ce cas, nous faisons une analyse mathématique de ces couches initiales en ajoutant des termes de correction dans le développement asymptotique.

En utilisant les techniques itératives des systèmes hyperboliques symétrisables et la technique de développement asymptotique, nous étudions la limite de relaxation en zéro du système d'Euler-Maxwell, avec des conditions initiales bien préparées ainsi que l'étude des couches initiales, dans le modèle évolutif bipolaire et unipolaire.

Mots clés : équations d'Euler-Poisson, équations d'Euler-Maxwell, équations de dérive-diffusion, la limite de relaxation en zéro, couches initiales,...

ABSTRACT

My work is concerned with two different systems of equations used in the mathematical modeling of semiconductors and plasmas : the Euler-Poisson system and the Euler-Maxwell system. The first is given by the Euler equations for the conservation of the mass and momentum, with a Poisson equation for the electrostatic potential. The second system describes the phenomenon of electromagnetism. It is given by the Euler equations for the conservation of the mass and momentum, with a Maxwell equations for the electric field and magnetic field which are coupled to the electron density through the Maxwell equations and act on electrons via the Lorentz force.

Using an asymptotic expansion method, we study the zero relaxation limit of unipolar Euler-Poisson system and of two-fluid multidimensional Euler-Poisson equations, we prove the existence and uniqueness of profiles to the asymptotic expansion and some error estimate. By employing the classical energy estimate for symmetrizable hyperbolic equations, we justify rigorously the convergence of Euler-Poisson system with well-prepared initial data. For ill-prepared initial data, the phenomenon of initial layers occurs. In this case, we also add the correction terms in the asymptotic expansion.

Using an iterative method of symmetrizable hyperbolic systems and asymptotic expansion method, we study the zero-relaxation limit of unipolar and bipolar Euler-Maxwell system. For well-prepared initial data, we construct an approximate solution by an asymptotic expansion up to any order. For ill-prepared initial data, we also construct initial layer corrections in the asymptotic expansion.

Keywords : Euler-Poisson equations, Euler-Maxwell equations, drift-diffusion equations, zero-relaxation limit, initial layer correction,...

Sommaire

Part I

Introduction

Chapter 1

Préliminaires

L'objet de cette étude et de mettre en oeuvre des outils mathématiques pour l'analyse de modèles des plasmas. Selon la structure des dispositifs, le transport de particules peut être très différent, en raison de plusieurs phénomènes physiques comme la diffusion, la dispersion ou les effets quantiques. Par conséquent, il existe plusieurs modèles mathématiques qui peuvent être utilisés dans la modélisation de ces dispositifs.

Le mot Plasma vient du grec et veut dire ”ouvrage façonné”. Le terme "plasma" a été introduit en 1928 par le physicien américain I.Langmuir. La physique des plasmas étudie les propriétés des gaz ionisés. Cependant, il est nécessaire de préciser la notion de gaz ionisé lorsque l'on parle de plasma. On dit souvent que le plasma est le quatrième état de la matière. Cette appellation vient du fait qu'au fur et à mesure la température d'un corps est augmentée. Il change d'état en passant successivement de l'état solide à l'état liquide puis à l'état gazeux. Si la température atteint environ 10.000 Kelvin à 100.000 Kelvin, la plupart de la matière est ionisée : on a alors l'état de plasma. A une température de l'ordre de 105 Kelvin correspond une énergie d'environ 10 électron-volt, ce qui est approximativement les énergies d'ionisation. Bien entendu, les plasmas existent à des températures bien inférieures pour autant que l'on fournisse un processus d'ionisation dont le taux soit supérieur à celui des pertes.

En préambule à ce travail, nous présentons au **chapitre 2** le système d'Euler-Poisson qui modélise un plasma et donnons les différents théorèmes obtenus dans les limites de relaxation en zéro du système. Le **troisième chapitre** est consacré à la représentation du système d'Euler-Maxwell. Nous donnons des résultats sur la limite de relaxation en zéro vers le système de dérive diffusion où les conditions initiales sont bien préparées et mal préparées. La seconde partie est dédiée à l'étude de la limite de relaxation en zéro du système d'Euler-Poisson et est divisée en deux chapitres. Nous étudierons dans les **chapitres 4 et 5** la limite de relaxation en zéro du système d'Euler-Poisson dans les cas unipolaire et bipolaire, respectivement. Nous faisons une analyse des couches initiales quand les conditions initiales sont mal-préparées en ajoutons des termes de correction dans le développement asymptotique. La troisième partie traitera la limite de relaxation en zéro du système d'Euler-Maxwell, et est divisée en 3 chapitres. Nous aborderons dans les **chapitres 6 et 7** la limite de relaxation en zéro du système d'Euler-Maxwell dans les cas unipolaire et bipolaire, respectivement. Nous faisons une étude des couches initiales en ajoutant des termes de correction de sorte qu'ils décroissent rapidement. Nous nous attacherons particulièrement dans le **chapitre 8** à justifier la convergence du système d'Euler-Maxwell avec des conditions initiales mal-préparées où les termes de correction dans le développement asymptotique ne sont pas donnés explicitement et ne décroissent pas rapidement.

Dans le chapitre 4, on considère un modèle de fluide multi-dimensionnel de plasma constitué d'électrons et d'ions, considéré comme un fluide unipolaire (ions fixes) de particules chargés soumis à un champ électrique donné par l'équation de Poisson. Les électrons sont décrits par leur densité $n(x,t)$ et leur vitesse $u(x,t)$. Ces variables macroscopiques vérifient le système des équations d'Euler (conservation de la masse et de la quantité de mouvement) en prenant en compte des conditions initiales périodiques

$$\partial_t n + \operatorname{div}(nu) = 0, \quad t > 0, \quad x \in \mathbb{T}, \tag{1.0.1}$$

$$m_e \partial_t(nu) + m_e \operatorname{div}(nu \otimes u) + \nabla p(n) = n\nabla\phi - \frac{nu}{\tau}, \tag{1.0.2}$$

où la fonction $p = p(s)$ représente la pression, ϕ le champ électrostatique et $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^d$ le tore. Dans le modèle unipolaire étudié, les équations décrivant les ions sont adimensionnées de la façon suivante : la densité des ions est une fonction donnée $n_i = b(x)$, la vitesse des ions est nulle. Les paramètres physiques m_e et τ représentent respectivement la masse et le temps de relaxation d'électrons. Le potentiel électrostatique est alors donné par l'équation de Poisson qui s'écrit sous forme adimensionnée :

$$-\left(\frac{\lambda'}{L}\right)^2 \Delta\phi = b(x) - n, \tag{1.0.3}$$

où $\lambda' = \sqrt{\dfrac{\varepsilon_0 K_B T_e}{n_0 q^2}}$ est la longueur de Debye, avec ε_0 la permittivité du vide, $K_B > 0$ est la constante de Boltzmann, $T_e > 0$ est la température d'électron, $q > 0$ la charge d'ion et $n_0 > 0$ est la densité moyenne du plasma. Pour simplifier les notations, on a pris $L = \lambda'$ dans ce chapitre, sans restreindre la généralité du problème à longueur de Debye fixée. Cependant, dans de nombreuse situations physiques le paramètre $\lambda = \dfrac{\lambda'}{L}$ est très petit et il est alors indispensable de faire une analyse asymptotique de modèle (ce qui correspond physiquement à la limite de quasi-neutre du plasma, voir [17] pour une présentation physique de ce passage à la limite). Cette analyse a motivé de nombreux travaux mathématiques pour des modèles simplifiés (par exemple, ions fixes [23], apparition de mesures de défaut [24], modèles de semiconducteurs [53], modèles stationnaires [5], [55] et [49]). D'autre part, des résultats ont été obtenus pour des modèles de fluides à la fois pour les électrons et les ions (solutions ondes progressives [13], théorème d'existence [12]). L'adimensionnement décrit ci-dessus (en choisissant $m_e = 1$, $\lambda' = L$) conduit à la forme suivante du système d'Euler-Poisson unipolaire :

$$\partial_t n + \operatorname{div}(nu) = 0, \tag{1.0.4}$$

$$\partial_t(nu) + \operatorname{div}(nu \otimes u) + \nabla p(n) = n\nabla\phi - \frac{nu}{\tau}, \tag{1.0.5}$$

$$-\Delta\phi = b(x) - n, \tag{1.0.6}$$

auquel on ajoute les conditions initiales suivantes :

$$n(0,x) = n_0(x), \quad u(0,x) = u_0(x), \quad x \in \mathbb{T}. \tag{1.0.7}$$

Lorsque l'on fait tendre τ tend vers 0 dans (1.0.4)-(1.0.6), en remplaçant u par τu, on obtient formellement :

$$\partial_t n + \operatorname{div}(nu) = 0, \tag{1.0.8}$$

$$\nabla h(n) = \nabla\phi - u, \tag{1.0.9}$$

$$-\Delta\phi = b(x) - n. \tag{1.0.10}$$

Ce dernier système est un système de dérive-diffusion classique qui admet une solution unique dans la classe $m(\phi) = \displaystyle\int_{\mathbb{T}} \phi(t,.)dx = 0$.

On utilise la méthode de développement asymptotique on étudie la limite de relaxation en zéro du système (1.0.4)-(1.0.6). Le but de cette méthode est de trouver une solution approchée du problème (1.0.4)-(1.0.6) dépendant d'un petit paramètre τ. Cette solution est obtenue sous forme d'un développement (appelé "asymptotique") en fonction du petit paramètre τ. Par des estimations d'énergie aux systèmes hyperboliques symétriques, nous justifions rigoureusement cette limite lorsque les conditions initiales sont bien préparées. Le phénomène des conditions initiales mal préparées est interprété par l'apparition de couches initiales. Dans ce cas, nous faisons une analyse mathématique de ces couches initiales en ajoutant des termes de correction dans le développement asymptotique.

Dans le chapitre 5, on considère un modèle de fluide multi-dimensionnel de plasma constitué d'électrons et d'ions, considéré comme un fluide bipolaire de particules chargées soumis à un champ électrique donné par l'équation de Poisson. Ce modèle s'ecrit sous la forme suivante :

$$\partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \tag{1.0.11}$$

$$m_\nu \partial_t(n_\nu u_\nu) + m_\nu \operatorname{div}(n_\nu u_\nu \otimes u_\nu) + \nabla p_\nu(n_\nu) = -q_\nu n_\nu \nabla\phi - \frac{m_\nu n_\nu u_\nu}{\tau_\nu}, \tag{1.0.12}$$

$$-\lambda\Delta\phi = n_i - n_e, \tag{1.0.13}$$

où $\nu = e, i$, $n_e = n_e(t,x)$, $u_e = u_e(t,x)$ (respectivement, $n_i = n_i(t,x), u_i = u_i(t,x)$) sont la densité et la vitesse des électrons (respectivement, ions) et ϕ est le potentiel électrostatique. Pour simplifier les notations, on suppose dans ce chapitre que $m_\nu = 1$ et $\lambda = 1$, sans restreindre la généralité du problème à la longueur de Debye et à la masse des électrons. La limite formelle du système bipolaire (1.0.11)-(1.0.13) quand τ tend vers zéro donne le système de dérive-diffusion suivant :

$$\partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \quad \nu = e, i, \tag{1.0.14}$$

$$\nabla h_\nu(n_\nu) = -q_\nu \nabla\phi - u_\nu, \quad \nu = e, i, \tag{1.0.15}$$

$$-\Delta\phi = n_i - n_e, \quad m(\phi) = 0. \tag{1.0.16}$$

Dans ce chapitre, on étudie la limite de relaxation en zéro et les couches initiales du modèle bipolaire (1.0.11)-(1.0.13) en utilisant la méthode de développement asymptotique en τ et des estimations d'energie.

Dans le chapitre 6, nous étudions la limite de relaxation et les couches initiales du système d'Euler-Maxwell dans le modèle unipolaire. Le modèle unipolaire du système d'Euler-Maxwell prend la forme suivante :

$$\partial_t n + \operatorname{div}(nu) = 0, \tag{1.0.17}$$

$$m_e \partial_t u + m_e (u \cdot \nabla) u + \nabla h(n) = -(E + \gamma u \times B) - \frac{u}{\tau}, \tag{1.0.18}$$

$$\gamma \lambda^2 \partial_t E - \nabla \times B = -\gamma(b(x) - nu), \quad -\lambda^2 \operatorname{div} E = n - b(x), \tag{1.0.19}$$

$$\gamma \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0. \tag{1.0.20}$$

Les variables n, u, E et B représentent respectivement la densité, la vitesse des électrons, le champ électrique et le champ magnétique. La fonction $p = p(s)$ représente la pression. Les paramètres physiques m_e, λ et τ représentent respectivement la masse des électrons, la longueur de Debye et le temps de relaxation. Le paramètre physique γ est définit comme suit :

$$\gamma = \frac{1}{\varepsilon_0^{1/2} c},$$

où ε_0 et c représentent respectivement la permittivité du vide et la vitesse de la lumière.

L'étude de la limite de relaxation en zero ($\tau \to 0$) et les couches initiales du système (1.0.17)-(1.0.20) nous permet d'appliquer la théorie de Madja du système hyperbolique symétrique et des estimations d'énergie. Bien que le résultat principal de ce paragraphe ne concerne que le cas où les conditions initiales satisfont des conditions de compatibilité, on obtient aussi des résultat de convergence même avec des conditions initiales générales. Ainsi, nous sommes en mesure d'étudier les couches initiales en ajoutant des termes de correction dans le développement asymptotique.

Dans le chapitre 7, on étudie le modèle fluide bipolaire du système d'Euler-Maxwell. Le système s'ecrit ainsi :

$$\partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \tag{1.0.21}$$

$$m_\nu \partial_t u_\nu + m_\nu (u_\nu \cdot \nabla) u_\nu + \nabla h_\nu(n_\nu) = q_\nu (E + \gamma u_\nu \times B) - \frac{u_\nu}{\tau_\nu}, \tag{1.0.22}$$

$$\gamma \lambda^2 \partial_t E - \nabla \times B = -\gamma(q_e n_e u_e + q_i n_i u_i), \quad \lambda^2 \operatorname{div} E = q_e n_e + q_i n_i, \tag{1.0.23}$$

$$\gamma \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0, \tag{1.0.24}$$

où $\nu = e, i$, $n_e = n_e(t,x)$, $u_e = u_e(t,x)$ (respectivement, $n_i = n_i(t,x), u_i = u_i(t,x)$) sont la densité et la vitesse des électrons (respectivement, ions). Pour simplifier les notations, on suppose que $m_\nu = 1$, $\lambda = 1$ et $\gamma = 1$. La limite formelle du système bipolaire (1.0.21)-(1.0.24) quand τ tend vers zéro donne le système de dérive-diffusion suivant :

$$\partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \quad \nu = e, i, \tag{1.0.25}$$

$$\nabla h_\nu(n_\nu) = q_\nu E - u_\nu, \quad \nu = e, i, \tag{1.0.26}$$

$$\nabla \times E = 0, \ \ \operatorname{div} E = n_i^0 - n_e^0, \tag{1.0.27}$$

$$\nabla \times B = \partial_t E + n_i^0 u_i^0 - n_e^0 u_e^0, \ \ \operatorname{div} B = 0 \tag{1.0.28}$$

On étudie la limite de relaxation en zéro et les couches initiales du modèle bipolaire (1.0.21)-(1.0.24) en utilisant la méthode de développement asymptotique en τ et des estimations d'energie.

Finalement, le dernier chapitre est consacré à l'étude des couches initiales du système (1.0.17)-(1.0.20) à l'aide d'un développement asymptotique indiqué dans [64]. Dans [64], les auteurs ont étudié la limite de relaxation du système d'Euler-Maxwell unipolaire en utilisant un développement asymptotique de la forme suivante :

$$(n_\tau^m, u_\tau^m, E_\tau^m, B_\tau^m) = \sum_{j=0}^{m} \tau^j (n^j, \tau E^j, B^j). \tag{1.0.29}$$

Ce développement asymptotique présente deux inconvénients majeurs. Premièrement, la convergence n'est pas valide pour $m = 0$. Deuxièmement, les termes des couches initiales n'ont pas de décroissance exponentielle. Pour surmonter ces difficultés, nous sommes amenés à ajouter des termes de correction dans (1.0.29). Bien qu'en ajoutant des termes de correction dans (1.0.29), les couches initiales ne décroissent pas rapidement, nous avons malgré tout réussi à montrer la convergence du système d'Euler-Maxwell.

Chapter 2

Le système d'Euler-Poisson

Dans ce chapitre, nous présentons le système d'Euler-Poisson et donnons différents résultats de la limite de relaxation en zéro du système dans les cas unipolaires et bipolaires.

2.1 Présentation des travaux existants

Le système d'Euler-Poisson présente un outil important dans la modélisation mathématique et la simulation numérique de plasmas [9, 54]. On considère un plasma magnétisé constitué d'électrons de charge $q_e = -1$ et d'ions de charge $q_i = 1$. Le plasma ou le semi-conducteur est décrit par le système d'Euler-Poisson. Il contient des équations d'Euler couplées à une équation de Poisson. Dans le modèle bipolaire, les équations d'Euler-Poisson sont composées de :

- l'équation de **conservation de la masse**, ou encore équation de continuité, elle s'écrit :

$$\partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \tag{2.1.1}$$

- les équations de **conservation de la quantité de mouvement**, que l'on appelle aussi équation de moments :

$$m_\nu \partial_t(n_\nu u_\nu) + m_\nu \operatorname{div}(n_\nu u_\nu \otimes u_\nu) + \nabla p_\nu(n_\nu) = -q_\nu n_\nu \nabla\phi - \frac{m_\nu n_\nu u_\nu}{\tau_\nu}, \tag{2.1.2}$$

et l'équation linéaire de Poisson :

$$-\lambda\Delta\phi = n_i - n_e, \tag{2.1.3}$$

où $\nu = e, i$, $n_e = n_e(t,x)$, $u_e = u_e(t,x)$ (respectivement, $n_i = n_i(t,x), u_i = u_i(t,x)$) sont la densité et la vitesse des électrons (respectivement, ions) et ϕ est le potentiel électrostatique. Ces derniers sont des inconnues de x et t, où $x \in \mathbb{T}$ et $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^d$ est le tore et le paramètre $t > 0$ désigne le temps. La quantité

$$J_\nu = n_\nu u_\nu$$

est la densité de courant des électrons et des ions. La fonction $p_\nu = p_\nu(s)$ représente la pression. Puisque nous ne considérons que le fluide isentropique, l'équation d'énergie du modèle hydrodynamique est remplacée par une relation pression-densité $p_\nu = p_\nu(n_\nu)$. Dans ce qui suit, la fonction de pression $p_\nu = p_\nu(n_\nu)$ est supposée suffisamment régulière et strictement croissante pour $n_\nu > 0$. En pratique, la fonction de pression est généralement gouvernée par une loi de type γ-loi donnée par $p(s) = cs^\gamma$, où $c > 0$ et $\gamma \geq 1$ sont deux constantes. La température est constante dans le cas $\gamma = 1$ qui correspond au flux isotherme. Les paramètres physiques m_e, τ_e (respectivement, m_i, τ_i) représentent la masse, le temps de relaxation d'électron (respectivement, d'ion) et λ est la longueur de Debye. Ces derniers paramètres sont petits par rapport à d'autres paramètres physiques du système. Il est donc important d'étudier les limites asymptotiques du système d'Euler-Poisson lorsque m_ν ou τ_ν ou λ tend vers zéro indépendamment.

Quand la solution (n_ν, u_ν, ϕ) est assez régulière, pour $n_\nu > 0$, l'équation (2.1.2) est équivalente à :

$$m_\nu \partial_t u_\nu + m_\nu (u_\nu \cdot \nabla) u_\nu + \nabla h_\nu(n_\nu) = -q_\nu \nabla\phi - \frac{u_\nu}{\tau_\nu}, \quad \nu = e, i, \tag{2.1.4}$$

où h_ν est la fonction d'enthalpie qui satisfait :

$$h_\nu(n_\nu) = \int_1^{n_\nu} \frac{p'_\nu(s)}{s} ds. \tag{2.1.5}$$

En effet, pour $n_\nu > 0$, en multipliant l'équation (2.1.2) par $\dfrac{1}{n_\nu}$, on obtient :

$$\frac{m_\nu}{n_\nu} \partial_t(n_\nu u_\nu) + \frac{m_\nu}{n_\nu} \operatorname{div}(n_\nu u_\nu \otimes u_\nu) + \frac{1}{n_\nu} \nabla p_\nu(n_\nu) = -q_\nu \nabla\phi - \frac{m_\nu u_\nu}{\tau_\nu}. \tag{2.1.6}$$

Alors, compte tenu de l'équation (2.1.1), on trouve évidemment :

$$\frac{m_\nu}{n_\nu} \partial_t(n_\nu u_\nu) = -\frac{m_\nu}{n_\nu} u_\nu \operatorname{div}(n_\nu u_\nu) + m_\nu \partial_t u_\nu. \tag{2.1.7}$$

Par ailleurs, un calcul simple donne :

$$\frac{m_\nu}{n_\nu} \operatorname{div}(n_\nu u_\nu \otimes u_\nu) = \frac{m_\nu}{n_\nu} u_\nu \operatorname{div}(n_\nu u_\nu) + m_\nu (u_\nu \cdot \nabla) u_\nu. \tag{2.1.8}$$

En combinant (2.1.6)-(2.1.8), on obtient (2.1.4).

Dans la littérature (voir [59]), il existe beaucoup d'études du système d'Euler-Poisson stationnaire. Dans [59], les auteurs ont étudié le système d'Euler-Poisson stationnaire dans le modèle unipolaire en ajoutant une condition supplémentaire du flux potentiel. Concernant l'existence de solutions régulières stationnaires d'une dynamique de gaz, avec un flux potentiel subsonique, nous renvoyons le lecteur à [22]. Dans [15], différentes approches analytiques du système d'Euler-Poisson stationnaire, avec des conditions aux limites, ont été étudiées par une utilisation explicite de l'équation de Poisson. L'existence et l'unicité de la solution du système unipolaire dans un espace de dimension 1 avec des conditions aux limites de Dirichlet ont été démontrées par Peng et Violet dans [59]. Dans [19, 20], les solutions stationnaires transsonique ont été étudiées en utilisant une méthode de viscosité artificielle. L'existence de telles solutions a été prouvée par le passage à la limite de la solution approchée du système d'Euler-Poisson quand le coefficient de la viscosité tend vers zéro. Les solutions non isentropiques dans un espace de dimension 1 ont été étudiées dans [4]. Dans le cas stationnaire, isentropique, avec des conditions aux limites de Dirichlet, l'existence et l'unicité de solutions régulières du système d'Euler-Poisson ont été obtenues dans [15].

Le système d'Euler-Poisson évolutif a été largement étudié par de nombreux auteurs. L'existence globale et la stabilité en temps long de solutions régulières ont été obtenues dans [3, 29] quand les solutions régulières sont proches d'un état d'équilibre constant. L'existence de solutions globales régulières proches d'un état stationnaire est prouvée dans [25].

Il est bien connu que la limite formelle du système d'Euler-Poisson est gouvernée par les équations de dérive-diffusion lorsque le temps de relaxation tend vers zéro. Dans l'espace de dimension 1, cette limite a été prouvée dans [51, 36, 37, 34] pour des solutions faibles et globales par la méthode de compacité par compensation. Dans le cas des solutions régulières multidimensionnelles et avec des conditions initiales bien préparées, la limite de relaxation du système d'Euler-Poisson unipolaire en zéro a été justifiée dans [74]. Cette convergence a été justifiée dans un intervalle de temps indépendant de τ, en utilisant la méthode de développement asymptotique d'ordre zéro et des estimations d'énergie, voir [43, 42]. En prenant $n_i = b$ et $u_i = 0$ et en éliminant les équations d'Euler pour les ions avec $m_e = \lambda = 1$, on obtient le système d'Euler-Poisson unipolaire. En notant (n, u) au lieu de (n_e, u_e) et τ au lieu de τ_e. Le système d'Euler-Poisson unipolaire prend la forme suivante :

$$\partial_t n_\tau + \operatorname{div}(n_\tau u_\tau) = 0, \tag{2.1.9}$$

$$m_e \partial_t u_\tau + m_e (u_\tau \cdot \nabla) u_\tau + \nabla h(n_\tau) = \nabla \phi_\tau - \frac{u_\tau}{\tau}, \tag{2.1.10}$$

$$-\Delta \phi_\tau = b - n_\tau. \tag{2.1.11}$$

L'échelle de temps habituelle pour étudier le système (2.1.9)-(2.1.11) est $t = \tau\xi$. En utilisant t au lieu de ξ et en prenant $m_e = 1$, le système d'Euler-Poisson unipolaire (2.1.9)-(2.1.11) devient :

$$\partial_t n_\tau + \frac{1}{\tau} \operatorname{div}(n_\tau u_\tau) = 0, \tag{2.1.12}$$

$$\partial_t u_\tau + \frac{1}{\tau}(u_\tau \cdot \nabla) u_\tau + \frac{1}{\tau} \nabla h(n_\tau) = \frac{\nabla \phi_\tau}{\tau} - \frac{u_\tau}{\tau^2}, \tag{2.1.13}$$

$$-\Delta \phi_\tau = b - n_\tau, \tag{2.1.14}$$

qui sera complété par les conditions initiales périodiques :

$$t = 0 \;:\quad (n, u) = (n_0^\tau, u_0^\tau). \tag{2.1.15}$$

La limite formelle du système (2.1.12)-(2.1.14) quand le temps de relaxation τ tend vers zéro donne :

$$\partial_t n + \operatorname{div}(nu) = 0, \tag{2.1.16}$$

$$\nabla h(n) = \nabla \phi - u, \tag{2.1.17}$$

$$-\Delta \phi = b - n. \tag{2.1.18}$$

Il est bien connu que (2.1.16)-(2.1.18) forme un système de dérive-diffusion classique. Il implique la condition de compatibilité suivante :

$$u_0 = -\nabla\big(h(n_0) - \phi_0\big), \tag{2.1.19}$$

où ϕ_0 est donnée par :

$$-\Delta\phi_0 = b(x) - n_0, \quad m(\phi_0) = \int_{\mathbb{T}} \phi_0(x)dx = 0. \tag{2.1.20}$$

Ces conditions de compatibilité (2.1.19)-(2.1.20) ont permis à W.A.Yong de montrer, dans [74], la convergence dans $H^s(\mathbb{T})$ de $(n_\tau, u_\tau, \phi_\tau)_{\tau>0}$ vers (n, u, ϕ), solution du système (2.1.16)-(2.1.18) quand le temps de relaxation τ tend vers zéro, pour tout $s > \frac{d}{2} + 1$. Plus précisément, il a prouvé que :

$$\|n_\tau - n, u_\tau - \tau u\|_{H^s(\mathbb{T})} \leq A_3\tau^2,$$

où $A_3 > 0$ est une constante indépendante de τ.

2.2 Quelques résultats sur le système d'Euler-Poisson

Dans le **chapitre 3**, nous utilisons la méthode de développement asymptotique et des estimations d'énergie du système hyperbolique symétrique pour étudier la limite de relaxation en zéro et les couches initiales du système d'Euler-Poisson unipolaire. En remplaçant u par τu dans le système (2.1.12)-(2.1.14), ce dernier prend la forme suivante :

$$\partial_t n + \operatorname{div}(nu) = 0. \tag{2.2.1}$$

$$\tau^2\big(\partial_t u + (u\cdot\nabla)u\big) + \nabla h(n) = \nabla\phi - u, \tag{2.2.2}$$

$$-\Delta\phi = b(x) - n. \tag{2.2.3}$$

Nous remarquons l'apparition d'un seul paramètre assez petit "τ^2" dans le système (2.2.1)-(2.2.3), qui justifie notre choix de développement asymptotique (2.2.4).

En utilisant la méthode de développement asymptotique et en prouvant l'existence et l'unicité de chaque terme, nous obtenons le théorème suivant.

Théorème 2.2.1 *Soient $s > \frac{d}{2} + 1$, $m \geq 0$ deux entiers fixés et $(n_j, u_j) \in H^{s+1}(\mathbb{T})$ pour $j = 0, 1, \cdots, m$, avec $n_0 \geq$ constante > 0 dans $\mathbb{T}$ et satisfaisant les conditions de compatibilité (2.1.19)-(2.1.20). On pose*

$$n_\tau^m = \sum_{j=0}^{m} \tau^{2j} n^j, \quad u_\tau^m = \tau\sum_{j=0}^{m} \tau^{2j} u^j, \quad \phi_\tau^m = \sum_{j=0}^{m} \tau^{2j}\phi^j. \tag{2.2.4}$$

On suppose qu'il existe une constante $C_1 > 0$ telle que :

$$\|(n_0^\tau, u_0^\tau) - \sum_{j=0}^{m} \tau^{2j}(n_j, \tau u_j)\|_s \leq C_1\tau^{2(m+1)}. \tag{2.2.5}$$

Alors il existe un temps $T > 0$, indépendant de τ, tel que le système (2.1.12)-(2.1.14) admette une solution unique

$$(n^\tau, u^\tau, \phi^\tau) \in C([0,T], H^s) \cap C^1([0,T], H^{s-1}).$$

De plus, on a les estimations suivantes :

$$\|(n^\tau - n_\tau^m)(t)\|_{H^s(\mathbb{T})} \leq C_2 \tau^{2(m+1)}, \quad \|(u^\tau - u_\tau^m)(t)\|_{H^s(\mathbb{T})} \leq C_2 \tau^{2(m+1)},$$

$$\|(\phi^\tau - \phi_\tau^m)(t)\|_{H^{s+1}(\mathbb{T})} \leq C_2 \tau^{2(m+1)}$$

et

$$\|(u^\tau - \tau u^0)\|_{L^2(0,T_1;H^s(\mathbb{T}))} \leq C_2 \tau^{2m+3},$$

pour tout $t \in [0,T]$, où C_2 est une constante indépendante de τ.

Nous abordons aussi dans le chapitre 3 l'étude des couches initiales du système d'Euler-Poisson dans le modèle unipolaire lorsque la condition de compatibilité (2.1.19)-(2.1.20) n'est pas satisfaite. Dans ce cas, nous proposons un développement asymptotique d'ordre 1 en ajoutant des termes de correction :

$$n_{\tau,I} = \sum_{j=0}^{1} \tau^{2j}(n^j + n_I^j), \quad u_{\tau,I} = \tau \sum_{j=0}^{1} \tau^{2j}(u^j + u_I^j), \quad \phi_{\tau,I} = \sum_{j=0}^{1} \tau^{2j}(\phi^j + \phi_I^j), \tag{2.2.6}$$

où, (n_I^j, u_I^j, ϕ_I^j) sont les termes de couches initiales qui seront déterminés explicitement. Le théorème de convergence du développement asymptotique est énoncé comme suit :

Théorème 2.2.2 *Soit $s > \frac{5}{2} + 1$. On suppose qu'il existe une constante $C_1 > 0$ telle que*

$$\|(n_0^\tau, u_0^\tau) - (n_0, \tau u_0)\|_s \leq C_1 \tau^2. \tag{2.2.7}$$

Alors il existe un temps $T > 0$ et une constante $C_2 > 0$, indépendants de τ, tels que le système (2.1.16)-(2.1.18) admette une solution unique

$$(n^\tau, u^\tau, \phi^\tau) \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$$

De plus, on a les estimations suivantes :

$$\|(n^\tau - n_{\tau,I})(t)\|_{H^s(\mathbb{T})} \leq C_3 \tau^2, \quad \|(u^\tau - u_{\tau,I})(t)\|_{H^s(\mathbb{T})} \leq C_3 \tau^2,$$

$$\|(\phi^\tau - \phi_{\tau,I})(t)\|_{H^{s+1}(\mathbb{T})} \leq C_2 \tau^2$$

et

$$\|(u^\tau - u_{\tau,I})\|_{L^2(0,T_1;H^s(\mathbb{T}))} \leq C_2 \tau^3,$$

pour tout $t \in [0,T]$.

Nous avons réussi à donner une seule preuve répondant aux deux théorèmes 2.2.2 et 2.2.2.

Dans le **chapitre 5**, nous étudions la limite de relaxation en zéro et les couches initiales du système d'Euler-Poisson bipolaire en utilisant un symétriseur différent de celui utilisé dans le chapitre 4.

Chapter 3

Le système d'Euler-Maxwell

Dans ce chapitre, nous présentons le système d'Euler-Maxwell et donnons différents résultats obtenus de la limite de relaxation en zéro dans les modèles unipolaire et bipolaire.

3.1 Présentation générale des travaux existants

Le système d'Euler-Maxwell décrit le phénomène d'électromagnétisme. Il contient des équations d'Euler et des équations de Maxwell. Les équations d'Euler sont composées de l'équation de conservation de la masse, ou encore équation de continuité, elle s'écrit :

$$\partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0$$

et des équations de conservation de la quantité de mouvement, que l'on appelle aussi équation des moments :

$$m_\nu \partial_t(n_\nu u_\nu) + m_\nu \operatorname{div}(n_\nu u_\nu \otimes u_\nu) + \nabla p_\nu(n_\nu) = q_\nu n_\nu(E + u_\nu \times B') - \frac{m_\nu n_\nu u_\nu}{\tau_\nu}. \tag{3.1.1}$$

Entre 1850 et 1861, James Clerk Maxwell a démontré l'unicité des champs électrique et magnétique en régime variable et a émis l'hypothèse, inconcevable à l'époque, que la lumière pourrait n'être qu'une forme d'onde électromagnétique. J.C. Maxwell a proposé de généraliser la loi d'Ampère en remplaçant le courant j par le courant total $j + j_D$, ce qui a donné la loi de Maxwell-Ampère. Ajoutant à cela les lois de Gauss, Faraday et de conservation du champ magnétique, on obtient les équations de Maxwell :

$$\begin{cases} \varepsilon_0 \partial_t E - \mu_0^{-1} \nabla \times B' = -(q_e n_e u_e + q_i n_i u_i), \quad \varepsilon_0 \operatorname{div} E = q_e n_e + q_i n_i, \\ \partial_t B' + \nabla \times E = 0, \quad \operatorname{div} B' = 0, \end{cases} \tag{3.1.2}$$

où $\nu = e, i$, $q_i = 1 > 0$ est la charge d'ion, $q_e = -1$ est la charge d'électron et les variables n_i et u_i (respectivement n_e et u_e) représentent respectivement la densité et la vitesse des ions (respectivement des électrons). De manière générale, p_ν est un terme de pression pouvant dépendre de plusieurs quantités selon les contextes. Dans notre cas, $p_\nu = p_\nu(n_\nu)$ est une fonction régulière et strictement croissante pour $n_\nu > 0$. Les variables $E = E(t,x)$ et $B' = B'(t,x)$ représentent respectivement le champ électrique et le champ magnétique. Les constantes ε_0, μ_0, m_i (respectivement m_e) et τ_ν représentent respectivement la permittivité du vide, la perméabilité du vide, la masse d'ions (respectivement électrons) et le temps de relaxation. Nous rappelons que la vitesse de la lumière c et la longueur de Debye λ' sont définis par :

$$c = (\varepsilon_0\mu_0)^{-\frac{1}{2}}, \quad \lambda' = \Big(\frac{\varepsilon_0 K_B T_e}{n_0 q^2}\Big)^{1/2},$$

où $K_B > 0$ est la constante de Boltzmann, $T_e > 0$ est la température d'électron et $n_0 > 0$ est la densité moyenne du plasma ([9], p. 350). Nous définissons également :

$$\lambda = \varepsilon_0^{1/2}, \quad \gamma = \frac{1}{\varepsilon_0^{1/2} c}.$$

Alors la longueur de Debye $\lambda > 0$ à une échelle près est proportionnelle à λ'. Nous remarquons aussi que $\gamma \to 0$ quand $c \to \infty$.

Nous introduisons $B' = \gamma B$, alors les équations de Maxwell bipolaires sont de la forme suivante :

$$\begin{cases} \gamma\lambda^2 \partial_t E - \nabla \times B = -\gamma(q_e n_e u_e + q_i n_i u_i), \quad \lambda^2 \operatorname{div} E = q_e n_e + q_i n_i, \\ \gamma \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0, \quad \nu = e, i. \end{cases}$$

Quand la solution (n_ν, u_ν, E, B) est assez régulière, pour $n_\nu > 0$, l'équation (3.1.1) est équivalente à :

$$m_\nu \partial_t u_\nu + m_\nu (u_\nu \cdot \nabla) u_\nu + \nabla h_\nu(n_\nu) = q_\nu (E + \gamma u_\nu \times B) - \frac{u_\nu}{\tau_\nu}, \tag{3.1.3}$$

où h_ν est la fonction d'enthalpie, qui satisfait :

$$h_\nu(n_\nu) = \int_1^{n_\nu} \frac{p'_\nu(s)}{s} ds. \tag{3.1.4}$$

Ainsi, le système d'Euler-Maxwell peut se réécrire comme suit :

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \\ m_\nu \partial_t u_\nu + m_\nu (u_\nu \cdot \nabla) u_\nu + \nabla h_\nu(n_\nu) = q_\nu (E + \gamma u_\nu \times B) - \dfrac{u_\nu}{\tau_\nu}, \\ \gamma\lambda^2 \partial_t E - \nabla \times B = -\gamma(q_e n_e u_e + q_i n_i u_i), \quad \lambda^2 \operatorname{div} E = q_e n_e + q_i n_i, \\ \gamma \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0, \end{cases} \tag{3.1.5}$$

pour tout $(t,x) \in]0,\infty[\times \mathbb{T}$, où $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^3$ est le tore.
Le système (3.1.5) est complété par des conditions initiales périodiques :

$$t = 0 \quad : (n_\nu, u_\nu, E, B) = (n^\tau_{\nu,0}, u^\tau_{\nu,0}, E^\tau_0, B^\nu_0), \quad \nu = e, i. \tag{3.1.6}$$

La première étude du système d'Euler-Maxwell avec le terme de relaxation est donnée par Chen, dans [9], où l'existence globale de solutions faibles est prouvée dans un espace de dimension 1. Récemment, Peng et Wang ont établi une série de résultats sur des limites du système lorsque des petits paramètres tendent vers zéro (voir [61]-[60]). La limite non-relativiste $c \to +\infty$ du système et sa convergence vers le système d'Euler-Poisson compressible ont été étudiées dans [61]. La convergence du système d'Euler-Maxwell vers le système d'e-MHD a été prouvée dans [63]. La limite combinée $c \to 0$ et $\lambda \to 0$ a été étudiée dans [62]. Dans [60], les auteurs présentent une analyse asymptotique formelle du système d'Euler-Maxwell (3.1.5).

Dans la suite, nous rappelons quelques travaux de Peng et Wang concernant le système d'Euler-Maxwell (3.1.5). Dans [61], Peng et Wang ont étudié la limite non-relativiste ($c \to +\infty$ i.e $\gamma \to 0$) du système en prenant $\lambda = \tau = 1$. Le système d'Euler-Maxwell unipolaire est de la forme suivante :

$$\begin{cases} \partial_t n_c + \operatorname{div}(n_c u_c) = 0, \\ \partial_t u_c + (u_c \cdot \nabla) u_c + \nabla h(n_c) = -(E_c + \dfrac{1}{c} u_c \times B_c) - u_c, \\ \dfrac{1}{c}\partial_t E_c - \nabla \times B_c = \dfrac{1}{c} n_c u_c, \quad \operatorname{div} E_c = b(x) - n_c, \\ \dfrac{1}{c}\partial_t B_c + \nabla \times E_c = 0, \quad \operatorname{div} B_c = 0. \end{cases} \tag{3.1.7}$$

En effectuant la limite formelle $c \to +\infty$ dans le système (3.1.7), nous obtenons le système d'Euler-Poisson suivant :

$$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ \partial_t u + (u \cdot \nabla) u + \nabla h(n) = -E, \\ \operatorname{div} E = b(x) - n, \quad \nabla \times E = 0, \\ \operatorname{div} B = 0, \quad \nabla \times B = 0, \end{cases} \tag{3.1.8}$$

avec les conditions initiales suivantes :

$$(n, u)(0, x) = (n_0, u_0), \quad x \in \mathbb{T}.$$

La limite non-relativiste $c \to +\infty$ du système (3.1.7) donne des conditions de compatibilité sur les conditions initiales :

$$E_0 = -\nabla \phi_0, \quad B_0 = 0, \tag{3.1.9}$$

où ϕ_0 est déterminé par :

$$-\Delta \phi_0 = b(x) - n_0, \quad m(\phi_0) = \int_{\mathbb{T}} \phi_0(x) dx = 0. \tag{3.1.10}$$

Avec les conditions de compatibilité (3.1.9)-(3.1.10), Peng et Wang ont justifié dans [61] la convergence dans $H^s(\mathbb{T})$ de $(n_c, u_c, E_c, B_c)_{c>0}$ vers (n, u, E, B), solution du système (3.1.8) lorsque c tend vers $+\infty$, pour tout $s > \frac{5}{2}$. Plus précisément, sous les hypothèses (3.1.9)-(3.1.10), on a les estimations suivantes :

$$\|n_c - n\|_{H^s(\mathbb{T})} \le A_1 \frac{1}{c}, \quad \|u_c - u\|_{H^s(\mathbb{T})} \le A_1 \frac{1}{c}, \quad \|B_c, E_c - B\|_{H^s(\mathbb{T})} \le A_1 \frac{1}{c},$$

où, $A_1 > 0$ est une constante indépendante de c.

La limite combinée non-relativiste et quasi-neutralité ($\gamma = \lambda^2$) du système d'Euler-Maxwell bipolaire a été établie dans [62] avec $\tau = O(1)$. Alors la solution $(n_\nu^\gamma, u_\nu^\gamma, E^\gamma, B^\gamma)$ satisfait le système suivant :

$$\begin{cases} \partial_t n_\nu^\gamma + \operatorname{div}(n_\nu^\gamma u_\nu^\gamma) = 0, \\ \partial_t u_\nu^\gamma + (u_\nu^\gamma \cdot \nabla) u_\nu^\gamma + \nabla h_\nu(n_\nu^\gamma) = q_\nu(E^\gamma + \gamma u_\nu^\gamma \times B^\gamma) - u_\nu^\gamma, \\ \gamma^2 \partial_t E^\gamma - \nabla \times B^\gamma = \gamma n_\nu^\gamma u_\nu^\gamma, \quad \gamma \operatorname{div} E^\gamma = n_i^\gamma - n_e^\gamma, \\ \gamma \partial_t B^\gamma + \nabla \times E^\gamma = 0, \quad \operatorname{div} B^\gamma = 0. \end{cases} \tag{3.1.11}$$

La limite formelle du système (3.1.11) quand le petit paramètre γ tend vers zéro donne le système d'Euler compressible suivant :

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \\ \partial_t u_\nu + (u_\nu \cdot \nabla) u_\nu + \nabla h_\nu(n_\nu) = q_\nu E - u_\nu, \\ \nabla \times E = 0, \\ \operatorname{div} B = 0, \quad \nabla \times B = 0, \\ n_i = n_e. \end{cases} \tag{3.1.12}$$

Dans [62], avec des conditions de compatibilité sur les conditions initiales, les auteurs ont démontré la convergence dans $H^s(\mathbb{T})$ de $(n_\nu^\gamma, u_\nu^\gamma, E^\gamma, B^\gamma)_{\gamma>0}$ vers (n_ν, u_ν, E, B) solution du système (3.1.12) lorsque le petit paramètre γ tend vers zéro. Plus précisément, ils ont prouvé les estimations suivantes :

$$\|n_\nu^\gamma - n_\nu\|_{H^s(\mathbb{T})} \leq A_2 \gamma, \quad \|u_\nu^\gamma - u_\nu\|_{H^s(\mathbb{T})} \leq A_2 \gamma, \quad \|B^\gamma, E^\gamma - E\|_{H^s(\mathbb{T})} \leq A_2 \gamma,$$

où $A_2 > 0$ est une constante indépendante de γ.

3.2 Quelques résultats sur le système d'Euler-Maxwell

Dans le **chapitre 5**, nous nous intéressons l'étude de la limite de relaxation et des couches initiales du système d'Euler-Maxwell unipolaire en utilisant la méthode de développement asymptotique. L'échelle de temps habituelle pour étudier le système d'Euler-Maxwell (3.1.5) est $t = \tau\xi$. En prenant $m = \lambda = \gamma = 1$ et en utilisant t au lieu de ξ, le système d'Euler-Maxwell unipolaire prend la forme :

$$\begin{cases} \partial_t n + \dfrac{1}{\tau} \operatorname{div}(nu) = 0, \\ \partial_t u + \dfrac{1}{\tau}(u \cdot \nabla) u + \dfrac{1}{\tau} \nabla h(n) = -\dfrac{E}{\tau} - \dfrac{u \times B}{\tau} - \dfrac{u}{\tau^2}, \\ \partial_t E - \dfrac{1}{\tau} \nabla \times B = \dfrac{nu}{\tau}, \quad \operatorname{div} E = b - n, \\ \partial_t B + \dfrac{1}{\tau} \nabla \times E = 0, \quad \operatorname{div} B = 0. \end{cases} \tag{3.2.1}$$

En remplaçant u par τu et B par τB dans (3.2.1), nous obtenons le système suivant :

$$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ \tau^2\big(\partial_t u + (u \cdot \nabla)u\big) + \nabla h(n) = -E - \tau^2 u \times B - u, \\ \partial_t E - \nabla \times B = nu, \quad \operatorname{div} E = b - n, \\ \tau^2 \partial_t B - \nabla \times E = 0, \quad \operatorname{div} B = 0, \end{cases}$$

Nous remarquons l'apparition d'un seul paramètre assez petit "τ^2" dans le système précédent, qui justifie notre choix de développement asymptotique (3.2.2)-(3.2.3).

Nous utilisons la méthode de développement asymptotique et après avoir prouvé l'existence et l'unicité de chaque terme du développement asymptotique, nous obtenons le théorème suivant.

Théorème 3.2.1 *Soient $s > \frac{5}{2}$, $m \geq 0$ deux entiers fixés et $(n_j, u_j, E_j, B_j) \in H^{s+1}(\mathbb{T})$ pour $j = 0, 1, \cdots, m$, avec $n_0 \geq$ constante > 0 dans $\mathbb{T}$ et satisfaisant les conditions de compatibilité. On pose*

$$n_\tau^m = \sum_{j=0}^{m} \tau^{2j} n^j, \quad u_\tau^m = \tau \sum_{j=0}^{m} \tau^{2j} u^j, \tag{3.2.2}$$

$$E_\tau^m = \sum_{j=0}^{m} \tau^{2j} E^j + \tau^{2(m+1)} E_c^{m+1}, \quad B_\tau^m = \tau \sum_{j=0}^{m} \tau^{2j} B^j. \tag{3.2.3}$$

On suppose qu'il existe une constante $C_1 > 0$ telle que :

$$\|(n_0^\tau, u_0^\tau, E_0^\tau, B_0^\tau) - \sum_{j=0}^{m} \tau^{2j}(n_j, \tau u_j, E_j, \tau B_j)\|_s \leq C_1 \tau^{2(m+1)}. \tag{3.2.4}$$

Alors il existe un temps $T > 0$ et une constante $C_2 > 0$, indépendants de τ, tel que le système (3.2.1) admette une solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ unique et régulière. De plus on a les estimations suivantes :

$$\|(n^\tau - n_\tau^m)(t)\|_{H^s(\mathbb{T})} \leq C_2 \tau^{2(m+1)}, \quad \|(u^\tau - u_\tau^m)(t)\|_{H^s(\mathbb{T})} \leq C_2 \tau^{2(m+1)},$$

$$\|(E^\tau - E_\tau^m)(t)\|_{H^{s+1}(\mathbb{T})} \leq C_2 \tau^{2(m+1)}, \quad \|(B^\tau - B_\tau^m)(t)\|_{H^{s+1}(\mathbb{T})} \leq C_2 \tau^{2(m+1)}$$

et

$$\|(u^\tau - \tau u^0)\|_{L^2(0,T_1;H^s(\mathbb{T}))} \leq C_2 \tau^{2m+3},$$

pour tout $t \in [0, T]$.

Nous abordons aussi dans le chapitre 6 l'étude des couches initiales du système d'Euler-Maxwell unipolaire quand les conditions de compatibilité ne sont pas satisfaites. Dans ce cas, nous proposons un développement asymptotique d'ordre 1 en ajoutant des termes de correction :

$$n_{\tau,I} = \sum_{j=0}^{1} \tau^{2j}(n^j + n_I^j), \quad u_{\tau,I} = \tau \sum_{j=0}^{1} \tau^{2j}(u^j + u_I^j), \tag{3.2.5}$$

$$E_{\tau,I} = \sum_{j=0}^{1} \tau^{2j}(E^j + E_I^j) + \tau^4 E_c^2, \quad B_{\tau,I} = \tau \sum_{j=0}^{1} \tau^{2j}(B^j + B_I^j) \tag{3.2.6}$$

où, $(n_I^j, u_I^j, E_I^j, B_I^j)_{j\in\{0,1\}}$ sont les termes de couches initiales qui seront déterminés explicitement. Alors, le théorème du développement asymptotique est donné comme suit :

Théorème 3.2.2 *Soit $s > \frac{5}{2}$. On suppose qu'il existe une constante $C_3 > 0$ telle que*

$$\|(n_0^\tau, u_0^\tau, E_0^\tau, B_0^\tau) - (n_0, \tau u_0, E_0, \tau B_0)\|_s \leq C_3\tau^2. \tag{3.2.7}$$

Alors il existe un temps $T > 0$ et une constante $C_4 > 0$, indépendants de τ, tel que le système (3.2.1) admette une solution unique et régulière qui satisfait les estimations suivantes :

$$\|(n^\tau - n_{\tau,I})(t)\|_{H^s(\mathbb{T})} \leq C_4\tau^2, \quad \|(u^\tau - u_{\tau,I})(t)\|_{H^s(\mathbb{T})} \leq C_4\tau^2,$$

$$\|(E^\tau - \phi_{\tau,I})(t)\|_{H^{s+1}(\mathbb{T})} \leq C_4\tau^2, \quad \|(B^\tau - B_{\tau,I})(t)\|_{H^s(\mathbb{T})} \leq C_4\tau^2$$

et

$$\|(u^\tau - u_{\tau,I})\|_{L^2(0,T_1;H^s(\mathbb{T}))} \leq C_4\tau^3,$$

pour tout $t \in [0, T]$.

La justification rigoureuse de la limite de relaxation en zéro du système d'Euler-Maxwell unipolaire est effectuée dans une seule preuve. Nous utilisant les techniques du système hyperbolique symétrique et des estimations d'énergie pour justifier rigoureusement cette limite.

Dans le **chapitre 7**, nous étudions la limite de relaxation en zéro et les couches initiales du système d'Euler-Maxwell bipolaire en utilisant un symétriseur différent à de celui utilisé dans le chapitre 6.

Dans [64], les auteurs ont étudié la limite de relaxation du système d'Euler-Maxwell unipolaire en utilisant un développement asymptotique de la forme :

$$(n_\tau^m, u_\tau^m, E_\tau^m, B_\tau^m) = \sum_{j=0}^{m} \tau^j(n^j, \tau u^j, E^j, B^j). \tag{3.2.8}$$

Ce développement asymptotique présente deux inconvénients majeurs. Premièrement, la convergence n'est pas valide pour $m = 0$. Deuxièmement, les termes des couches initiales n'ont pas de décroissance exponentielle. Pour surmonter ces difficultés, nous sommes amenés à ajouter des termes de correction dans (3.2.8). Bien qu'en ajoutant des termes de correction dans (3.2.8), les couches initiales ne décroissent pas rapidement, nous avons malgré tout réussi dans le **chapitre** 8 à montrer la convergence du système d'Euler-Maxwell.

Part II

The zero-relaxation limits of multidimentional Euler-Poisson systems

Chapter 4

Initial layers and zero-relaxation limits of one-fluid Euler-Poisson equations

Abstract. In this chapter we consider zero-relaxation limits for periodic smooth solutions of the time-dependent Euler-Poisson system. For well-prepared initial data, we construct an approximate solution by an asymptotic expansion up to any order. For ill-prepared initial data, we also construct initial layer corrections in an explicit way. In both cases, the asymptotic expansions are valid in a time interval independent of the relaxation time and their convergence is justified by establishing uniform energy estimates.

Keywords : Euler-Poisson equations, drift-diffusion equations, zero-relaxation limit, initial layer correction

4.1 Introduction

The Euler-Poisson system plays an important role in mathematical modeling and numerical simulation for plasmas and semiconductors [9, 54]. This work is concerned with initial layer analysis and relaxation limits of smooth solutions of the compressible Euler-Poisson system. We consider a plasma or semiconductor consisting of electrons of charge $q_e = -1$ and a single species of ions of charge $q_i = 1$. Denoting by n_e, u_e (respectively, n_i, u_i) the scaled density and velocity vector of the electrons (respectively, ions) and by ϕ the electric potential. They are functions of the time $t > 0$ and the position $x \in \mathbb{R}^d$. Throughout this chapter, we restrict to the case of periodic functions. Then $x \in \mathbb{T}$, with $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^d$ being the d-dimensional torus.

The two-fluid isentropic Euler-Poisson system consists of a set of nonlinear conservation laws for density and momentum coupled to a Poisson equation. It reads :

$$\begin{cases} \partial_t n_\alpha + \operatorname{div}(n_\alpha u_\alpha) = 0, \\ \partial_t(n_\alpha u_\alpha) + \operatorname{div}(n_\alpha u_\alpha \otimes u_\alpha) + \nabla p_\alpha(n_\alpha) = -q_\alpha n_\alpha \nabla\phi - \dfrac{n_\alpha u_\alpha}{\tau_\alpha}, \\ -\Delta\phi = n_i - n_e, \end{cases} \tag{4.1.1}$$

for $\alpha = e, i$ and $(t,x) \in (0,\infty) \times \mathbb{T}$, where $\otimes$ stands for the tensor product, $p_\alpha = p_\alpha(n_\alpha)$ is the pressure function which is supposed to be smooth and strictly increasing for $n_\alpha > 0$, and $\tau_\alpha > 0$ is a small parameter for the momentum relaxation time.

For smooth solutions with $n_\alpha > 0$, the second equation of (4.1.1) is equivalent to

$$\partial_t u_\alpha + (u_\alpha \cdot \nabla) u_\alpha + \nabla h_\alpha(n_\alpha) = -q_\alpha \nabla\phi - \frac{u_\alpha}{\tau_\alpha}, \tag{4.1.2}$$

where $''\cdot''$ denotes the inner product of $\mathbb{R}^d$ and the enthalpy function h_α is defined by

$$h_\alpha(n_\alpha) = \int_1^{n_\alpha} \frac{p'_\alpha(s)}{s} ds. \tag{4.1.3}$$

In the plasma when the ions are non-moving and become a uniform background, by letting $n_i = b$, $u_i = 0$ and deleting the Euler equations for ions, a one-fluid Euler-Poisson model is formally derived. Replacing (n_e, u_e) by (n, u) and τ_e by τ, the one-fluid isentropic Euler-Poisson system becomes

$$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ \partial_t u + (u \cdot \nabla) u + \nabla h(n) = \nabla\phi - \dfrac{u}{\tau}, \\ -\Delta\phi = b - n, \end{cases} \tag{4.1.4}$$

where b also stands for the doping profile in the semiconductor and in general it depends only on x. Since we consider periodic smooth solutions, b is supposed to be periodic and smooth. It is well-known that in the zero-relaxation limit the two-fluid Euler-Poisson system doesn't present additional difficulties compared to the one-fluid Euler-Poisson system. That is why we only deal with the one-fluid model in what follows.

The usual time scaling for studying system (4.1.4) is $t = \tau\xi$. Rewriting still ξ by t, then system (4.1.4) becomes (see [51, 36, 34] etc.)

$$\begin{cases} \partial_t n + \dfrac{1}{\tau} \operatorname{div}(nu) = 0, \\ \partial_t u + \dfrac{1}{\tau}(u \cdot \nabla) u + \dfrac{1}{\tau} \nabla h(n) = \dfrac{\nabla\phi}{\tau} - \dfrac{u}{\tau^2}, \\ -\Delta\phi = b(x) - n, \end{cases} \tag{4.1.5}$$

for $t > 0$, $x \in \mathbb{T}$. It is complemented by periodic initial conditions :

$$t = 0 \ : \quad (n, u) = (n_0^\tau, u_0^\tau). \tag{4.1.6}$$

In problem (4.1.5)-(4.1.6), ϕ is not determined in a unique way. To avoid this, we add a contraint condition

$$m(\phi) \stackrel{def}{=} \int_{\mathbb{T}} \phi(\cdot, x)dx = 0. \tag{4.1.7}$$

System (4.1.5) has a nice structure in the study of the zero-relaxation limit $\tau \to 0$, as mentioned in [74]. Its local existence of smooth solutions is a well-known result due to Kato [38] for symmetric hyperbolic systems. Indeed, the Poisson equation in (4.1.5) gives estimate

$$\|\nabla\phi\|_{H^s(\mathbb{T})} \leq C\|n-b\|_{H^s(\mathbb{T})}. \tag{4.1.8}$$

Then, regarding $\nabla\phi$ as a function of n, (n, u) satisfies a symmetric hyperbolic system in which $\nabla\phi$ appeared on the right hand side of (4.1.5) is a low order term. Moreover, estimate (4.1.8) implies that $\phi \in C\big([0, T), H^{s+1}(\mathbb{T})\big)$ as soon as $n \in C\big([0, T), H^s(\mathbb{T})\big)$ for some $T > 0$ and $s > 0$.

The global existence and the long time stability of smooth solutions have been obtained in [1, 3, 29] when the solutions are close to a constant equilibrium. The long time stability of smooth solutions near a stationary potential flow was given in [25].

The zero-relaxation limit $\tau \to 0$ of the Euler-Poisson system (4.1.5) has been extensively studied by many authors. It is known that its limit is the classical drift-diffusion system. For one dimensional global weak solutions, this limit was first analyzed in [51, 36, 37, 34] by the compensated compactness method. For the result in two-fluid Euler-Poisson system we refer to [56] or [45, 44] by assuming the L^∞ bounds of weak solutions with respect to τ. In multidimensional case with well-prepared initial data which mean that compatibility conditions are fulfilled, the zero-relaxation limit was justified in [74] in a time interval independent of τ, by establishing uniform energy estimates like those of [43, 42] together with an argument on the time extension of smooth solutions. Recently, this last result has been extended to the non-isentropic Euler-Poisson system (see [72]).

The goal of this chapter is to justify the zero-relaxation limit $\tau \to 0$ to problem (4.1.5)-(4.1.7) by the method of asymptotic expansions. Assuming that the initial data admit an asymptotic expansion, we construct an asymptotic expansion for smooth solutions and prove its convergence up to any order for well-prepared initial data. In particular, we obtain the result in [74] in which the expansion corresponds to zero order. For ill-prepared initial data, the above convergence result is not valid because of the formation of initial layers. In this case, we construct initial layer corrections and prove the convergence of the asymptotic expansion of zero order. In both cases, the convergence rates are given.

This chapter is organized as follows. In the next section, we derive asymptotic expansions of solutions and state the convergence result to problem (4.1.5)-(4.1.7) in the case of well-prepared initial data. Section 3 is devoted to the analysis of initial layers in the case of ill-prepared initial data. We construct the initial layer corrections which exponentially decay to zero and state the convergence result. The proof of both two asymptotic expansions is given in the last section. For this purpose, we prove a more general convergence theorem which implies those in both cases of well-prepared initial data and ill-prepared initial data.

We introduce some notations which will be used in the sequel of this thesis. Let $\Omega \subset \mathbb{R}^d$ be an open domain, where $d = 3$ in chapter 6, 7 and 8. When Ω is bounded, we denote by $|\Omega|$ the Lebesgue measure of Ω. Note that when $\Omega = \mathbb{T}$, we have $|\mathbb{T}| = 1$. For a multi-index

$\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$, we denote :

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \quad \text{with} \quad |\alpha| = \alpha_1 + \cdots + \alpha_d.$$

For an integer $s > 0$ and a real number $p \geq 1$, $\| \cdot \|_{s,p}$ stands for the norm of the Sobolev space $W^{s,p}(\Omega)$ defined by :

$$W^{s,p}(\Omega) = \big\{ f;\ \partial_x^\alpha f \in L^p(\Omega),\ \forall\, |\alpha| \leq s \big\}.$$

We denote also $H^s(\Omega) = W^{s,2}(\Omega)$, and by $\| \cdot \|$ and $\| \cdot \|_\infty$ the norms of $L^2(\Omega)$ and $L^\infty(\Omega)$, respectively.

The following lemmas are needed in the proofs of theorems of this thesis.

Lemma 4.1.1 *(Moser-type calculus inequalities, see [43, 50]) Let $s \geq 1$ be an integer and $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}$. Suppose $u \in H^s(\Omega)$, $\nabla u \in L^\infty(\Omega)$ and $v \in H^{s-1}(\Omega) \cap L^\infty(\Omega)$. Then for all multi-index $\alpha \in \mathbb{N}^d$ with $1 \leq |\alpha| \leq s$, we have $\partial_x^\alpha(uv) - u\partial_x^\alpha v \in L^2(\Omega)$ and*

$$\|\partial_x^\alpha(uv) - u\partial_x^\alpha v\| \leq C_s\big(\|\nabla u\|_\infty \|D^{|\alpha|-1} v\| + \|D^{|\alpha|} u\| \|v\|_\infty\big),$$

where

$$\|D^s u\| = \sum_{|\alpha|=s} \|\partial_x^\alpha u\|, \quad \forall\, s \in \mathbb{N}.$$

Moreover, if $s > 1 + \frac{d}{2}$, $A \in C_b^s(G)$ and $V \in H^s(\Omega, G)$ where $G \subset \mathbb{R}^n$, then the embedding $H^{s-1}(\Omega) \hookrightarrow L^\infty(\Omega)$ is continuous and we have :

$$\|uv\|_{s-1} \leq C_s \|u\|_{s-1} \|v\|_{s-1},$$

$$\|\partial_x^\alpha(uv) - u\partial_x^\alpha v\| \leq C_s \|u\|_s \|v\|_{s-1}, \quad \forall\, |\alpha| \leq s.$$

and

$$\big\|A\big(V(.)\big)\big\|_s \leq C_s \|A\|_s \big(1 + \|V\|_s^s\big).$$

Lemma 4.1.2 *(Poincaré inequality, see [11]) Let $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^d$ be a bounded connected open domain with a Lipschitz boundary. Then there exists a constant $C > 0$ depending only on p and Ω such that*

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}, \quad \forall\, u \in W^{1,p}(\Omega), \tag{4.1.9}$$

where

$$u_\Omega = \frac{1}{|\Omega|} \int_\Omega u(x)\, dx$$

is the average value of u over Ω.

Proposition 4.1.1 *(Local existence of smooth solutions) Let $s > 1 + \frac{d}{2}$ and $(n_0^\tau, u_0^\tau) \in H^s(\mathbb{T})$ with $n_0^\tau \geq \kappa$, for some given constant $\kappa > 0$, independent of τ. Then there exist $T_1^\tau > 0$ and a unique smooth solution $(n^\tau, u^\tau, \phi^\tau)$ to Cauchy problem (4.1.5)-(4.1.7) defined in the time interval $[0, T_1^\tau)$, with*

$$(n^\tau, u^\tau) \in C\big([0, T_1^\tau), H^s(\mathbb{T})\big) \cap C^1\big([0, T_1^\tau), H^{s-1}(\mathbb{T})\big),$$

$$\phi^\tau \in C\big([0,T_1^\tau), H^{s+1}(\mathbb{T})\big) \cap C^1\big([0,T_1^\tau), H^s(\mathbb{T})\big).$$

4.2 Case of well-prepared initial data

4.2.1 Formal asymptotic expansions

We seek an approximate solution $(n_\tau, u_\tau, \phi_\tau)$ to system (4.1.5) under the form of an asymptotic expansion of a power series in τ. From the second equation of (4.1.5), we have formally $u_\tau \to 0$ as $\tau \to 0$. Then the first term of the asymptotic expansion of u_τ should be equal to zero. Moreover, if we replace u by τu, then system (4.1.5) becomes

$$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ \tau^2\big(\partial_t u + (u\cdot\nabla)u\big) + \nabla h(n) = \nabla\phi - u, \\ -\Delta\phi = b(x) - n, \end{cases}$$

in which the only small parameter is τ^2. This suggests the following asymptotic expansion for both the initial data and the solution :

$$(n_\tau, u_\tau)(0,x) = \sum_{j\ge 0} \tau^{2j}(n_j, \tau u_j)(x), \quad x \in \mathbb{T}, \tag{4.2.1}$$

$$(n_\tau, u_\tau, \phi_\tau)(t,x) = \sum_{j\ge 0} \tau^{2j}(n^j, \tau u^j, \phi^j)(t,x), \quad t>0, \ \ x\in\mathbb{T}, \tag{4.2.2}$$

where $(n_j, u_j)_{j\ge 0}$ is sufficiently smooth given data with $n_0 \ge \text{constant} > 0$ in $\mathbb{T}$.

Now it needs to determine the profiles (n^j, u^j, ϕ^j) for all $j \ge 0$. Substituting expansion (4.2.2) into system (4.1.5), (4.1.7) and identifying the coefficients in power of τ, we obtain

$$\begin{cases} \partial_t n^0 + \operatorname{div}(n^0 u^0) = 0, \\ \nabla h(n^0) = \nabla\phi^0 - u^0, \\ -\Delta\phi^0 = b - n^0, \ \ m(\phi^0) = 0, \end{cases} \tag{4.2.3}$$

and for $j \ge 1$,

$$\begin{cases} \partial_t n^j + \displaystyle\sum_{k=0}^{j} \operatorname{div}\left(n^k u^{j-k}\right) = 0, \\ \partial_t u^{j-1} + \displaystyle\sum_{k=0}^{j-1} (u^k\cdot\nabla)u^{j-1-k} + \nabla\Big(h'(n^0)n^j + h^{j-1}\big((n^k)_{k\le j-1}\big)\Big) = \nabla\phi^j - u^j, \\ \Delta\phi^j = n^j, \ \ m(\phi^j) = 0, \end{cases} \tag{4.2.4}$$

where $h^0 = 0$ and $h^{j-1}\big((n^k)_{k\le j-1}\big)$ is defined for $j \ge 2$ by

$$h\Big(\sum_{j\ge 0}\tau^j n^j\Big) = h(n^0) + h'(n^0)\sum_{j\ge 1}\tau^j n^j + \sum_{j\ge 2}\tau^j h^{j-1}\big((n^k)_{k\le j-1}\big).$$

From (4.2.3) and (4.1.6) we deduce that (n^0, ϕ^0) solves a classical drift-diffusion system :

$$\begin{cases} \partial_t n^0 - \mathrm{div}\,\big(n^0\nabla\big(h(n^0) - \phi^0\big)\big) = 0, \\ -\Delta\phi^0 = b - n^0, \quad m(\phi^0) = 0, \end{cases} \qquad t > 0, \quad x \in \mathbb{T} \tag{4.2.5}$$

with an initial condition :

$$n^0(0, x) = n_0, \quad x \in \mathbb{T}. \tag{4.2.6}$$

The existence of smooth solutions to problem (4.2.5)-(4.2.6) can be easily established, at least locally in time. See for instance [54]. Then u^0 is determined by

$$u^0 = -\nabla\big(h(n^0) - \phi^0\big). \tag{4.2.7}$$

from which we get the zero-order compatibility condition :

$$u_0 = -\nabla\big(h(n_0) - \phi_0\big), \tag{4.2.8}$$

where ϕ_0 is determined by

$$-\Delta\phi_0 = b - n_0 \quad \text{in } \mathbb{T} \quad \text{and} \quad m(\phi_0) = 0. \tag{4.2.9}$$

The profile (n^j, u^j, ϕ^j) for $j \ge 1$ is determined by induction in j. Assume that $\big(n^k, u^k, \phi^k\big)_{0\le k\le j-1}$ is smooth and determined in previous steps. Then, from (4.2.4) (n^j, ϕ^j) solves a linearized system of drift-diffusion equations :

$$\begin{cases} \partial_t n^j - \mathrm{div}\,\big(n^0\nabla(h'(n^0)n^j - \phi^j)\big) + \mathrm{div}\,\big(n^j u^0\big) \\ = f^j\big((V^k, \partial_t V^k, \partial_x V^k, \partial_t\partial_x V^k, \partial_x^2 V^k)_{0\le k\le j-1}\big), \qquad t > 0, \quad x \in \mathbb{T} \\ \Delta\phi^j = n^j, \quad m(\phi^j) = 0, \end{cases} \tag{4.2.10}$$

with an initial condition :

$$n^j(0, x) = n_j(x), \quad x \in \mathbb{T} \tag{4.2.11}$$

and u^j is given by

$$u^j = \nabla\big(\phi^j - h'(n^0)n^j - h^{j-1}((n^k)_{k\le j-1})\big) - \Big(\partial_t u^{j-1} + \sum_{k=0}^{j-1}(u^k\cdot\nabla)u^{j-1-k}\Big), \tag{4.2.12}$$

where f^j is a given smooth function and $V^k = (n^k, u^k, \phi^k)$. Thus, from (4.2.12) we get the jth-order compatibility conditions for $j \geq 1$:

$$u_j = \nabla\big(\phi_j - h'(n_0)n_j - h^{j-1}((n_k)_{k\leq j-1})\big) - \Big(\partial_t u^{j-1}\big|_{t=0} + \sum_{k=0}^{j-1}(u_k\cdot\nabla)u_{j-1-k}\Big), \tag{4.2.13}$$

where ϕ_j is determined by

$$\Delta\phi_j = n_j \quad \text{in } \mathbb{T} \quad \text{and} \quad m(\phi_j) = 0. \tag{4.2.14}$$

We conclude the above discussion in the following result.

Proposition 4.2.1 *Assume that for each $j \geq 1$ the initial datum (n_j, u_j) is sufficiently smooth, with $n_0 \geq$ constant > 0 in $\mathbb{T}$, and well-prepared, i.e. the compatibility conditions (4.2.8) and (4.2.13) hold. Then there exists a unique asymptotic expansion up to any order of the form (4.2.2), i.e. there exist $T_1 > 0$ and a unique smooth solution (n^j, u^j, ϕ^j) in the time interval $[0, T_1]$ to problems (4.2.5)-(4.2.7) and (4.2.10)-(4.2.12) for $j \geq 1$. Moreover, $n^0 \geq$ constant > 0 in $[0, T_1] \times \mathbb{T}$. In particular, the formal zero-relaxation limit $\tau \to 0$ of the Euler-Poisson system (4.1.5) is the classical drift-diffusion system (4.2.5) and (4.2.7).*

4.2.2 Convergence results

Let $m \geq 0$ be a fixed integer and $(n^\tau, u^\tau, \phi^\tau)$ be the exact solution to problem (4.1.5)-(4.1.7) defined in the time interval $[0, T_1^\tau)$. We denote by

$$(n_\tau^m, u_\tau^m, \phi_\tau^m) = \sum_{j=0}^{m} \tau^{2j}(n^j, \tau u^j, \phi^j) \tag{4.2.15}$$

an approximate solution of order m, where $(n^j, u^j, \phi^j)_{0\leq j\leq m}$ is constructed in the previous subsection. The convergence of the asymptotic expansion (4.2.2) is to establish the limit $(n^\tau, u^\tau, \phi^\tau) \to (n_\tau^m, u_\tau^m, \phi_\tau^m)$ and its convergence rate as $\tau \to 0$ in a time interval independent of τ, when $(n^\tau, u^\tau, \phi^\tau) \to (n_\tau^m, u_\tau^m, \phi_\tau^m)$ at $t = 0$. For $m = 0$, this result was proved in [74]. Now we consider a more general case $m \geq 0$.

From the construction of the approximate solution, we have

$$\begin{cases} \partial_t n_\tau^m + \dfrac{1}{\tau}\operatorname{div}(n_\tau^m u_\tau^m) = R_n^{\tau,m}, \\ \partial_t u_\tau^m + \dfrac{1}{\tau}(u_\tau^m\cdot\nabla)u_\tau^m + \dfrac{1}{\tau}\nabla h(n_\tau^m) = \dfrac{\nabla\phi_\tau^m}{\tau} - \dfrac{u_\tau^m}{\tau^2} + R_u^{\tau,m}, \\ -\Delta\phi_\tau^m = b - n_\tau^m, \ \ m(\phi_\tau^m) = 0, \end{cases} \tag{4.2.16}$$

where $R_n^{\tau,m}$ and $R_u^{\tau,m}$ are remainders. It is clear that the convergence rate depends strongly on the order of the remainders with respect to τ. Since the profile $(n^j, u^j, \phi^j)_{j\geq 0}$ is sufficiently smooth, a straightforward computation gives the following result.

Proposition 4.2.2 *For all integer $m \geq 0$, the remainders $R_n^{\tau,m}$ and $R_u^{\tau,m}$ satisfy*

$$\sup_{0\leq t\leq T_1}\|R_n^{\tau,m}(t,\cdot)\|_s \leq C_m\tau^{2(m+1)}, \quad \sup_{0\leq t\leq T_1}\|R_u^{\tau,m}(t,\cdot)\|_s \leq C_m\tau^{2m+1}, \tag{4.2.17}$$

where $C_m > 0$ is a constant independent of τ.

The convergence result of this section is stated as follows of which the proof is given in section 4.

Theorem 4.2.1 *Let $m \geq 0$ and $s > 1 + \frac{d}{2}$ be any fixed integers. Let $(n_j, u_j) \in H^{s+1}(\mathbb{T})$ for $j = 0, 1, \cdots, m$, with $n_0 \geq$ constant > 0 in $\mathbb{T}$ satisfying the compatibility conditions (4.2.8) and (4.2.13) for $j \geq 1$, respectively. Suppose*

$$\Big\|(n_0^\tau, u_0^\tau) - \sum_{j=0}^{m} \tau^{2j}(n_j, \tau u_j)\Big\|_s \leq C_1 \tau^{2(m+1)}, \tag{4.2.18}$$

where $C_1 > 0$ is a constant independent of τ. Then there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \geq T_1$ and the solution $(n^\tau, u^\tau, \phi^\tau)$ to the periodic problem (4.1.5)-(4.1.7) satisfies

$$\big\|(n^\tau - n_\tau^m, u^\tau - u_\tau^m)(t)\big\|_s \leq C_2 \tau^{2(m+1)}, \quad \forall\, t \in [0, T_1]$$

and

$$\big\|\phi^\tau(t) - \phi_\tau^m(t)\big\|_{s+1} \leq C_2 \tau^{2(m+1)}, \quad \forall\, t \in [0, T_1].$$

Moreover,

$$\big\|u^\tau - u_\tau^m\big\|_{L^2(0,T_1;H^s(\mathbb{T}))} \leq C_2 \tau^{2m+3}.$$

4.3 Case of ill-prepared initial data

4.3.1 Initial layer corrections

In Theorem 4.2.1, compatibility conditions are made on the initial data. These conditions are restrictions on the initial data since u_j should be determined in terms of n_j for all $j \geq 0$. If these conditions are not satisfied, the asymptotic expansion (4.2.2) cannot generally converge for $t > 0$ because the approximate solution cannot satisfy the prescribed initial conditions. In this section, we consider the case of the so called ill prepared initial data by adding an initial layer correction in the asymptotic expansion.

To avoid tedious computation, we only consider a zero-order asymptotic expansion in the case that condition (4.2.8) is violated, i.e.

$$u_0 \neq \nabla\big(\phi_0 - h(n_0)\big).$$

We seek a simplest possible form of an asymptotic expansion such that its remainders are at least of order $O(\tau)$. Let the initial data of an approximate solution $(n_\tau, u_\tau, \phi_\tau)$ have an asymptotic expansion of the form

$$(n_\tau, u_\tau)\big|_{t=0} = (n_0, \tau u_0) + O(\tau^2). \tag{4.3.1}$$

In general, an asymptotic expansion including initial layer corrections is of the form :

$$\begin{aligned}(n_\tau,u_\tau,\phi_\tau)(t,x) &= (n^0,\tau u^0,\phi^0)(t,x)+(n_I^0,\tau u_I^0,\phi_I^0)(z,x)\\ &\quad +\tau^2\big((n^1,\tau u^1,\phi^1)(t,x)+(n_I^1,\tau u_I^1,\phi_I^1)(z,x)\big)+O(\tau^4),\end{aligned} \tag{4.3.2}$$

where $z=t/\tau^2\in\mathbb{R}$ is the fast variable and the subscript I stands for the initial layer variables. A direct computation shows that we may take

$$(n^1,u^1,\phi^1,u_I^1)=0,$$

since this expansion gives the remainders of order $O(\tau)$, which is the case of well-prepared initial data for $m=0$. Then we propose the following ansatz :

$$(n_\tau,u_\tau,\phi_\tau)(t,x)=(n^0,\tau u^0,\phi^0)(t,x)+\big((n_I^0,\tau u_I^0,\phi_I^0)+\tau^2(n_I^1,0,\phi_I^1)\big)(z,x)+O(\tau^4). \tag{4.3.3}$$

Obviously, (n^0,u^0,ϕ^0) still satisfies the drift-diffusion system (4.2.5) with (4.2.7). It is easy to see that the asymptotic expansions (4.3.1) and (4.3.3) imply that

$$n^0(0,x)+n_I^0(0,x)=n_0(x),\quad u^0(0,x)+u_I^0(0,x)=u_0(x), \tag{4.3.4}$$

which give the initial values of n_I^0 and u_I^0. It remains to determine the initial-layer profiles (n_I^0,u_I^0,ϕ_I^0), n_I^1 and ϕ_I^1.

Putting expression (4.3.3) into system (4.1.5) and using (4.2.5) and (4.2.7), we have

$$n_I^0=0,\quad \phi_I^0=0\quad\text{and}\quad \partial_z u_I^0+u_I^0=0. \tag{4.3.5}$$

Equation $n_I^0=0$ means that there is no zero-order initial layer on variable n. Therefore, (4.3.4) gives

$$n^0(0,x)=n_0(x). \tag{4.3.6}$$

From the third equation of (4.3.5) together with (4.3.4), we obtain

$$u_I^0(z,x)=u_I^0(0,x)e^{-z}=\big(u_0(x)-u^0(0,x)\big)e^{-z}. \tag{4.3.7}$$

Similarly, the second order initial layers n_I^1 and ϕ_I^1 satisfy

$$\partial_z n_I^1(z,x)+\operatorname{div}\big(n^0(0,x)u_I^0(z,x)\big)=0 \tag{4.3.8}$$

and

$$\Delta\phi_I^1(z,x)=n_I^1(z,x),\quad m(\phi_I^1)=0. \tag{4.3.9}$$

Let n_1 be an arbitrary smooth function and let $n_I^1(0,x)=n_1(x)$. Together with (4.3.7), we have

$$n_I^1(z,x)=n_1(x)-\operatorname{div}\big(n^0(0,x)(u_0(x)-u^0(0,x))\big)\big(1-e^{-z}\big). \tag{4.3.10}$$

Thus, the initial layer profiles (n_I^0,u_I^0,ϕ_I^0), n_I^1 and ϕ_I^1 are completely determined by (4.3.5), (4.3.7), (4.3.9) and (4.3.10). They are periodic and smooth functions of (z,x).

4.3.2 Convergence results

Let

$$(n_{\tau,I}, u_{\tau,I}, \phi_{\tau,I})(t,x) = (n^0, \tau u^0, \phi^0)(t,x) + \big((0, \tau u_I^0, 0) + \tau^2(n_I^1, 0, \phi_I^1)\big)(t/\tau^2, x). \tag{4.3.11}$$

By the construction above, we have

$$\begin{cases} \partial_t n_{\tau,I} + \dfrac{1}{\tau}\operatorname{div}(n_{\tau,I}u_{\tau,I}) = R_n^{\tau,I}, \\ \partial_t u_{\tau,I} + \dfrac{1}{\tau}(u_{\tau,I}\cdot\nabla)u_{\tau,I} + \dfrac{1}{\tau}\nabla h(n_{\tau,I}) = \dfrac{\nabla\phi_{\tau,I}}{\tau} - \dfrac{u_{\tau,I}}{\tau^2} + R_u^{\tau,I}, \\ -\Delta\phi_{\tau,I} = b - n_{\tau,I}, \quad m(\phi_{\tau,I}) = 0, \\ t = 0 \ : \quad \big(n_{\tau,I}, u_{\tau,I}\big) = \big(n_0 + \tau^2 n_1, \tau u_0\big), \end{cases} \tag{4.3.12}$$

where the expressions of the remainders $R_{n,I}^\tau$ and $R_{u,I}^\tau$ are given by

$$\begin{aligned} R_n^{\tau,I} &= \partial_t\big(n^0 + \tau^2 n_I^1\big) + \operatorname{div}\big((n^0 + \tau^2 n_I^1)(u^0 + u_I^0)\big) \\ &= \partial_z n_I^1 + \operatorname{div}(n^0 u_I^0) + \tau^2 \operatorname{div}\big(n_I^1(u^0 + u_I^0)\big) \\ &= \operatorname{div}\big((n^0(t,x) - n^0(0,x))u_I^0(z,x)\big) + \tau^2 \operatorname{div}\big(n_I^1(u^0 + u_I^0)\big) \end{aligned}$$

and

$$\begin{aligned} R_u^{\tau,I} &= \tau\big(\partial_t(u^0 + u_I^0) + (u^0 + u_I^0)\cdot\nabla(u^0 + u_I^0)\big) \\ &\quad + \frac{1}{\tau}\big(\nabla(h(n^0 + \tau^2 n_I^1) - (\phi^0 + \tau^2\phi_I^1)) + (u^0 + u_I^0)\big) \\ &= \frac{1}{\tau}\big(\nabla(h(n^0) - \phi^0) + u^0\big) + \frac{1}{\tau}(\partial_z u_I^0 + u_I^0) + \frac{1}{\tau}\nabla\big(h(n^0 + \tau^2 n_I^1) - h(n^0)\big) \\ &\quad + \tau\big(\partial_t u^0 + (u^0 + u_I^0)\cdot\nabla(u^0 + u_I^0)\big) + \tau\nabla\phi_I^1 \\ &= \tau\big(\partial_t u^0 + (u^0 + u_I^0)\cdot\nabla(u^0 + u_I^0)\big) + \tau\nabla\phi_I^1 + \frac{1}{\tau}\nabla\big(h(n^0 + \tau^2 n_I^1) - h(n^0)\big). \end{aligned}$$

For $R_n^{\tau,I}$, there is $\eta \in [0,t] \subset [0,T_1]$ such that

$$n^0(t,x) - n^0(0,x) = t\partial_t n^0(\eta,x) = \tau^2 z\partial_t n^0(\eta,x).$$

Noting that function $z \longmapsto ze^{-z}$ is bounded for $z \geq 0$, from

$$\partial_t n^0 = -\operatorname{div}(n^0 u^0)$$

and (4.3.7) we deduce from that

$$\operatorname{div}\big((n^0(t,x) - n^0(0,x))u_I^0(z,x)\big) = O(\tau^2).$$

Thus

$$R_n^{\tau,I} = O(\tau^2).$$

For $R_u^{\tau,I}$, it is clear that

$$h(n^0 + \tau^2 n_I^1) - h(n^0) = O(\tau^2),$$

which implies that

$$R_u^{\tau,I} = O(\tau).$$

Thus, we obtain the following estimates on the remainders.

Proposition 4.3.1 *For given smooth data, the remainders $R_{n,I}^\tau$ and $R_{u,I}^\tau$ satisfy*

$$\sup_{0\le t\le T_1} \|R_n^{\tau,I}(t,\cdot)\|_s \le C\tau^2, \qquad \sup_{0\le t\le T_1} \|R_u^{\tau,I}(t,\cdot)\|_s \le C\tau, \tag{4.3.13}$$

where $C>0$ is a constant independent of τ.

The convergence result with initial layers can be stated as follows.

Theorem 4.3.1 *Let $s > 1 + \frac{d}{2}$ be a fixed integer. Let $(n_0, u_0) \in H^{s+1}(\mathbb{T})$ with $n_0 \ge constant > 0$ in $\mathbb{T}$. Suppose*

$$\left\| \left(n_0^\tau - n_0, u_0^\tau - \tau u_0 \right) \right\|_s \le C_1 \tau^2, \tag{4.3.14}$$

where $C_1 > 0$ is a constant independent of τ. Then there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \ge T_1$ and the solution $(n^\tau, u^\tau, \phi^\tau)$ to the periodic problem (4.1.5)-(4.1.7) satisfies

$$\left\| (n^\tau - n_{\tau,I}, u^\tau - u_{\tau,I})(t) \right\|_s \le C_2\tau^2, \quad \forall\, t \in [0, T_1]$$

and

$$\left\| \phi^\tau(t) - \phi_{\tau,I}(t) \right\|_{s+1} \le C_2\tau^2, \quad \forall\, t \in [0, T_1]$$

Moreover,

$$\left\| u^\tau - u_{\tau,I} \right\|_{L^2(0,T_1;H^s(\mathbb{T}))} \le C_2\tau^3.$$

4.4 Justification of asymptotic expansions

4.4.1 Statement of the main result

In this section, we justify rigorously the asymptotic expansions of solutions to the periodic problem (4.1.5)-(4.1.7) constructed in section 4.2 and 4.3. To this end, we prove a more general convergence result which implies both Theorems 4.2.1-4.3.1.

Let $(n^\tau, u^\tau, \phi^\tau)$ be the exact solution to (4.1.5)-(4.1.7) and $(n_\tau, u_\tau, \phi_\tau)$ be an approximate periodic solution defined on $[0, T_1]$, with

$$(n_\tau, u_\tau) \in C([0, T_1], H^{s+1}(\mathbb{T})) \cap C^1([0, T_1], H^s(\mathbb{T})),$$

$$\phi_\tau \in C([0, T_1], H^{s+2}(\mathbb{T})) \cap C^1([0, T_1], H^{s+1}(\mathbb{T})).$$

We define the remainders of the approximate solution by

$$\begin{cases} R_n^\tau = \partial_t n_\tau + \dfrac{1}{\tau}\operatorname{div}(n_\tau u_\tau), \\ R_u^\tau = \partial_t u_\tau + \dfrac{1}{\tau}(u_\tau \cdot \nabla) u_\tau + \dfrac{1}{\tau}\nabla(h(n_\tau) - \phi_\tau) + \dfrac{u_\tau}{\tau^2}. \end{cases} \tag{4.4.1}$$

Suppose

$$-\Delta \phi_\tau = b - n_\tau, \tag{4.4.2}$$

$$\sup_{0\le t\le T_1} \|n_\tau(t,\cdot)\|_s \le C_1, \quad \sup_{0\le t\le T_1} \|u_\tau(t,\cdot)\|_s \le C_1\tau, \tag{4.4.3}$$

$$\left\|\left(n_0^\tau - n_\tau(0,\cdot), u_0^\tau - \tau u_\tau(0,\cdot)\right)\right\|_s \le C_1\tau^{\lambda+1}, \tag{4.4.4}$$

$$\sup_{0\le t\le T_1} \|R_n^\tau(t,\cdot)\|_s \le C_1\tau^{\lambda+1}, \qquad \sup_{0\le t\le T_1} \|R_u^\tau(t,\cdot)\|_s \le C_1\tau^{\lambda}, \tag{4.4.5}$$

where $\lambda \ge 0$ and $C_1 > 0$ are constants independent of τ.

Theorem 4.4.1 *Under the assumptions above, there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \ge T_1$ and the solution $(n^\tau, u^\tau, \phi^\tau)$ to the periodic problem (4.1.5)-(4.1.7) satisfies*

$$\left\|(n^\tau - n_\tau, u^\tau - u_\tau)(t)\right\|_s \le C_2\tau^{\lambda+1}, \quad \forall\, t \in [0, T_1]$$

and

$$\left\|\phi^\tau(t) - \phi_\tau(t)\right\|_{s+1} \le C_2\tau^{\lambda+1}, \quad \forall\, t \in [0, T_1].$$

Moreover,

$$\left\|u^\tau - u_\tau\right\|_{L^2(0,T_1;H^s(\mathbb{T}))} \le C_2\tau^{\lambda+2}.$$

It is clear that Theorem 4.4.1 implies Theorems 4.2.1-4.3.1. In particular, $\lambda = 2m+1$ with $m \ge 0$ in section 2 and $\lambda = 1$ in section 3. The next subsection is devoted to the proof of Theorem 4.4.1.

4.4.2 Proof of the main result

By proposition 4.1.1, the exact solution $(n^\tau, u^\tau, \phi^\tau)$ is defined on a time interval $[0, T_1^\tau)$ with $T_1^\tau > 0$. Since $n^\tau \in C([0, T_1^\tau), H^s(\mathbb{T}))$ and the embedding from $H^s(\mathbb{T})$ to $C(\mathbb{T})$ is continuous, we have $n^\tau \in C([0, T_1^\tau) \times \mathbb{T})$. From assumption $n_0^\tau \ge \kappa > 0$, we deduce that there exist a constant $C_0 > 0$ and a maximal existence time $T_2^\tau \in (0, T_1^\tau]$, independent of τ, such that $\frac{\kappa}{2} \le n^\tau(t,x) \le C_0$ for all $(t,x) \in [0, T_2^\tau) \times \mathbb{T}$. Then we define $T^\tau = \min(T_1, T_2^\tau) > 0$ so that the exact solution and the approximate solution are both defined in the time interval $[0, T^\tau)$. In this time interval, we denote by

$$(N^\tau, U^\tau, \Phi^\tau) = (n^\tau - n_\tau, u^\tau - u_\tau, \phi^\tau - \phi_\tau). \tag{4.4.6}$$

Obviously, $(N^\tau, U^\tau, \Phi^\tau)$ satisfies the following problem :

$$\begin{cases} \partial_t N^\tau + \dfrac{1}{\tau}\big((U^\tau + u_\tau)\cdot\nabla\big)N^\tau + \dfrac{1}{\tau}(N^\tau + n_\tau)\operatorname{div} U^\tau \\ \quad = -\dfrac{1}{\tau}\big(N^\tau \operatorname{div} u_\tau + (U^\tau\cdot\nabla)n_\tau\big) - R_n^\tau, \\ \partial_t U^\tau + \dfrac{1}{\tau}\big((U^\tau + u_\tau)\cdot\nabla\big)U^\tau + \dfrac{1}{\tau}h'(N^\tau + n_\tau)\nabla N^\tau \\ \quad = -\dfrac{1}{\tau}\big(\nabla\Phi^\tau + (U^\tau\cdot\nabla)u_\tau + \big(h'(N^\tau + n_\tau) - h'(n_\tau)\big)\nabla n_\tau\big) - \dfrac{U^\tau}{\tau^2} - R_u^\tau, \\ \Delta\Phi^\tau = N^\tau, \;\; m(\Phi^\tau) = 0, \\ t = 0 \; : \quad (N^\tau, U^\tau) = \big(n_0^\tau - n_\tau(0,\cdot), u_0^\tau - u_\tau(0,\cdot)\big). \end{cases} \tag{4.4.7}$$

Set

$$W^\tau = \begin{pmatrix} N^\tau \\ U^\tau \end{pmatrix}, \quad W_0^\tau = \begin{pmatrix} n_0^\tau - n_\tau(0,\cdot) \\ u_0^\tau - u_\tau(0,\cdot) \end{pmatrix},$$

$$A_i(n^\tau, u^\tau) = \begin{pmatrix} u_i^\tau & n^\tau e_i^t \\ h'(n^\tau)e_i & u_i^\tau I_d \end{pmatrix}, \quad i = 1, \cdots, d,$$

$$H_1(W^\tau) = \begin{pmatrix} -(U^\tau\cdot\nabla)n_\tau - N^\tau \operatorname{div} u_\tau \\ -(U^\tau\cdot\nabla)u_\tau - \big(h'(N^\tau + n_\tau) - h'(n_\tau)\big)\nabla n_\tau \end{pmatrix},$$

$$H_2(\Phi^\tau) = \begin{pmatrix} 0 \\ -\nabla\Phi^\tau \end{pmatrix}, \quad H_3(W^\tau) = \begin{pmatrix} 0 \\ -U^\tau \end{pmatrix}, \quad R^\tau = \begin{pmatrix} R_n^\tau \\ R_u^\tau \end{pmatrix},$$

where $(e_1, \cdots, e_d)$ is the canonical basis of $\mathbb{R}^d$, y_i denotes the ith-component of $y \in \mathbb{R}^d$ and I_d is the $d \times d$ matrix. Thus problem (4.4.7) for unknown W^τ can be rewritten as

$$\partial_t W^\tau + \frac{1}{\tau}\sum_{i=1}^d A_i(n^\tau, u^\tau)\partial_{x_i} W^\tau = \frac{1}{\tau}\big(H_1(W^\tau) + H_2(\Phi^\tau)\big) + \frac{1}{\tau^2}H_3(W^\tau) - R^\tau, \tag{4.4.8}$$

in which Φ^τ and W^τ are linked by the Poisson equation

$$\Delta\Phi^\tau = N^\tau. \tag{4.4.9}$$

The initial condition of (4.4.8) is

$$t = 0 \; : \; W^\tau = W_0^\tau. \tag{4.4.10}$$

It is not difficult to see that equation (4.4.8) is symmetrizable hyperbolic with symmetrizer

$$A_0(n^\tau) = \begin{pmatrix} (n^\tau)^{-1} & 0 \\ 0 & \big(h'(n^\tau)\big)^{-1} I_d \end{pmatrix},$$

which is a positively definite matrix when $0 < \frac{\kappa}{2} \le n^\tau = N^\tau + n_\tau \le C_0$. Then

$$\tilde{A}_i(n^\tau, u^\tau) = A_0(n^\tau) A_i(n^\tau, u^\tau) = u_i^\tau A_0(n^\tau) + D_i, \tag{4.4.11}$$

which is symmetric for all $1 \le i \le d$, where each D_i is a constant matrix

$$D_i = \begin{pmatrix} 0 & e_i^t \\ e_i & 0 \end{pmatrix}.$$

The existence and uniqueness of smooth solutions to (4.1.5)-(4.1.7) is equivalent to that of (4.4.8)-(4.4.10). Thus, using standard arguments, in order to show Theorem 4.4.1, it suffices to establish uniform estimates of W^τ with respect to τ.

In what follows, we denote by $C > 0$ various constants independent of τ and for $\alpha \in \mathbb{N}^d$, $W_\alpha^\tau = \partial_x^\alpha W^\tau$ etc. The main estimate of solutions is contained in the following lemma.

Lemma 4.4.1 *Under the assumptions of Theorem 4.4.1, for all $t \in (0, T^\tau)$, as $\tau \to 0$ we have*

$$\|W^\tau(t)\|_s^2 + \frac{1}{\tau^2}\int_0^t \|U^\tau(\xi)\|_s^2 d\xi \le C\int_0^t \Big(\|W^\tau(\xi)\|_s^2 + \|W^\tau(\xi)\|_s^4\Big) d\xi + \ C\tau^{2(\lambda+1)}. \tag{4.4.12}$$

Proof. For $\alpha \in \mathbb{N}^d$ with $|\alpha| \le s$, differentiating equations (4.4.8) with respect to x yields

$$\begin{aligned} & \partial_t W_\alpha^\tau + \frac{1}{\tau}\sum_{i=1}^d A_i(n^\tau, u^\tau)\partial_{x_i} W_\alpha^\tau \\ = \ & \frac{1}{\tau}\partial_x^\alpha H_1(W^\tau) + \frac{1}{\tau}\partial_x^\alpha H_2(\Phi^\tau) + \frac{1}{\tau^2}\partial_x^\alpha H_3(W^\tau) - \partial_x^\alpha R^\tau \\ & + \frac{1}{\tau}\sum_{i=1}^d \Big(A_i(n^\tau, u^\tau)\partial_{x_i} W_\alpha^\tau - \partial_x^\alpha\big(A_i(n^\tau, u^\tau)\partial_{x_i} W^\tau\big)\Big). \end{aligned} \tag{4.4.13}$$

Multiplying (4.4.13) by $A_0(n^\tau)$ and taking the inner product of the resulting equations with W_α^τ, by employing the classical energy estimate for symmetrizable hyperbolic equations, we obtain

$$\begin{aligned} & \frac{d}{dt}\big(A_0(n^\tau) W_\alpha^\tau, W_\alpha^\tau\big) - \ \frac{2}{\tau^2}\big(A_0(n^\tau)\partial_x^\alpha H_3(W^\tau), W_\alpha^\tau\big) \\ = \ & \frac{2}{\tau}\big(A_0(n^\tau)\partial_x^\alpha H_1(W^\tau), W_\alpha^\tau\big) + \frac{2}{\tau}\big(A_0(n^\tau)\partial_x^\alpha H_2(\Phi^\tau), W_\alpha^\tau\big) + \frac{2}{\tau}\big(J_\alpha^\tau, W_\alpha^\tau\big) \\ & + \big(\operatorname{div} A_\tau(n^\tau, u^\tau) W_\alpha^\tau, W_\alpha^\tau\big) - 2\big(A_0(n^\tau)\partial_x^\alpha R^\tau, W_\alpha^\tau\big), \end{aligned} \tag{4.4.14}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\mathbb{T})$,

$$J_\alpha^\tau = -\sum_{i=1}^d A_0(n^\tau)\big(\partial_x^\alpha(A_i(n^\tau,u^\tau)\partial_{x_i}W^\tau) - A_i(n^\tau,u^\tau)\partial_x^\alpha(\partial_{x_i}W^\tau)\big)$$

and

$$\operatorname{div} A_\tau(n^\tau,u^\tau) = \partial_t A_0(n^\tau) + \frac{1}{\tau}\sum_{i=1}^d \partial_{x_i}\widetilde{A}_i(n^\tau,u^\tau). \tag{4.4.15}$$

Let us estimate each term of equations (4.4.14). First, $A_0(n^\tau)$ being positively definite, we have

$$\big(A_0(n^\tau)W_\alpha^\tau, W_\alpha^\tau\big) \ge C^{-1}\|W_\alpha^\tau\|^2. \tag{4.4.16}$$

Moreover, a direct computation gives

$$-\big(A_0(n^\tau)\partial_x^\alpha H_3(W^\tau), W_\alpha^\tau\big) = \big(n^\tau U_\alpha^\tau, U_\alpha^\tau\big) \ge \frac{\kappa}{2}\|U_\alpha^\tau\|^2. \tag{4.4.17}$$

Since

$$\begin{aligned} W_\alpha^\tau \cdot A_0(n^\tau)\partial_x^\alpha H_1(W^\tau) &= -h'(n^\tau)N_\alpha^\tau\partial_x^\alpha\big((U^\tau\cdot\nabla)n_\tau + N^\tau \mathrm{div} u_\tau\big) \\ &\quad - n^\tau U_\alpha^\tau \cdot \partial_x^\alpha\big((U^\tau\cdot\nabla)u_\tau + (h'(N^\tau+n_\tau) - h'(n_\tau))\nabla n_\tau\big), \end{aligned}$$

by Lemma 4.1.1 and (4.4.3) for u_τ, we get

$$\begin{aligned} \frac{2}{\tau}\big(A_0(n^\tau)\partial_x^\alpha H_1(W^\tau), W_\alpha^\tau\big) &\le \frac{C}{\tau}\big(\|N^\tau\|_s\|U^\tau\|_s + \tau\|N^\tau\|_s^2 + \tau\|U^\tau\|_s^2\big) \\ &\le \frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2 + C_\varepsilon\|W^\tau\|_s^2. \end{aligned} \tag{4.4.18}$$

Here and hereafter, ε denotes a small constant independent of τ and $C_\varepsilon > 0$ denotes a constant depending only on ε.

For the term containing $H_2(\Phi^\tau)$, we have

$$\frac{2}{\tau}\big(A_0(n^\tau)\partial_x^\alpha H_2(\Phi^\tau), W_\alpha^\tau\big) = -\frac{2}{\tau}\int_{\mathbb{T}}(p'(n^\tau))^{-1}\nabla\Phi_\alpha^\tau U_\alpha^\tau dx.$$

From the Poisson equation (4.4.9), we have

$$\Delta\Phi_\alpha^\tau = N_\alpha^\tau,$$

which implies that

$$\|\nabla\Phi_\alpha^\tau\| \le C\|N_\alpha^\tau\| \le C\|N^\tau\|_s.$$

It follows that

$$\frac{2}{\tau}\big(A_0(n^\tau)\partial_x^\alpha H_2(\Phi^\tau), W_\alpha^\tau\big) \le \frac{C}{\tau}\|N^\tau\|_s\|U^\tau\|_s \le \frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2 + C_\varepsilon\|W^\tau\|_s^2. \tag{4.4.19}$$

Now we consider the estimate for the term containing J^τ_α. Let us first point out that a direct application of Lemma 4.1.1 to J^τ_α does not yield the desired result. We have to develop the terms in the summation of J^τ_α to see the appearance of terms U^τ or $U^\tau+u_\tau$. By the definition of $A_i(n^\tau,u^\tau)$, we have

$$\begin{aligned}
&\partial_x^\alpha\big(A_i(n^\tau,u^\tau)\partial_{x_i}W^\tau\big)-A_i(n^\tau,u^\tau)\partial_x^\alpha(\partial_{x_i}W^\tau)\\
=&\begin{pmatrix}\partial_x^\alpha\big((U^\tau+u_\tau)_i\partial_{x_i}N^\tau\big)-(U^\tau+u_\tau)_i\partial_x^\alpha\partial_{x_i}N^\tau\\ \partial_x^\alpha\big(h'(N^\tau+n_\tau)\partial_{x_i}N^\tau e_i\big)-h'(N^\tau+n_\tau)\partial_x^\alpha\partial_{x_i}N^\tau e_i\end{pmatrix}\\
&+\begin{pmatrix}\partial_x^\alpha\big((N^\tau+n_\tau)_i\partial_{x_i}U^\tau\cdot e_i^t\big)-(N^\tau+n_\tau)\partial_x^\alpha\partial_{x_i}U^\tau\cdot e_i^t\\ \partial_x^\alpha\big((U^\tau+u_\tau)_i\partial_{x_i}U^\tau\big)-(U^\tau+u_\tau)_i\partial_x^\alpha\partial_{x_i}U^\tau\end{pmatrix}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\big(A_0(n^\tau)(\partial_x^\alpha(A_i(n^\tau,u^\tau)\partial_{x_i}W^\tau)-A_i(n^\tau,u^\tau)\partial_x^\alpha(\partial_{x_i}W^\tau)),W_\alpha^I\big)\\
=&\ (n^\tau)^{-1}\big(\partial_x^\alpha\big((U^\tau+u_\tau)_i\partial_{x_i}N^\tau\big)-(U^\tau+u_\tau)_i\partial_x^\alpha\partial_{x_i}N^\tau\big)N_\alpha^\tau\\
&+\big(h'(n^\tau)\big)^{-1}\big(\partial_x^\alpha\big((U^\tau+u_\tau)_i\partial_{x_i}U^\tau\big)-(U^\tau+u_\tau)_i\partial_x^\alpha\partial_{x_i}U^\tau\big)U_\alpha^\tau\\
&+(n^\tau)^{-1}\big(\partial_x^\alpha\big((N^\tau+n_\tau)_i\partial_{x_i}U^\tau\cdot e_i^t\big)-(N^\tau+n_\tau)\partial_x^\alpha\partial_{x_i}U^\tau\cdot e_i^t\big)N_\alpha^\tau\\
&+\big(h'(n^\tau)\big)^{-1}\big(\partial_x^\alpha\big(h'(N^\tau+n_\tau)\partial_{x_i}N^\tau e_i\big)-h'(N^\tau+n_\tau)\partial_x^\alpha\partial_{x_i}N^\tau e_i\big)U_\alpha^\tau\\
=&\ J_{i1}+J_{i2}+J_{i3}+J_{i4}.
\end{aligned}$$

Noting (4.4.3) for u_τ and applying Lemma 4.1.1 to each term on the right hand side of the above equation gives

$$\big|J_{i1}+J_{i2}\big|\le C\big(\tau+\|U^\tau\|_s\big)\|W^\tau\|_s^2\le\frac{\varepsilon}{\tau}\|U^\tau\|_s^2+C_\varepsilon\tau\big(\|W^\tau\|_s^2+\|W^\tau\|_s^4\big),$$

$$\big|J_{i3}+J_{i4}\big|\le C\big(1+\|N^\tau\|_s\big)\|N^\tau\|_s\|U^\tau\|_s\le\frac{\varepsilon}{\tau}\|U^\tau\|_s^2+C_\varepsilon\tau\big(\|W^\tau\|_s^2+\|W^\tau\|_s^4\big).$$

This implies that

$$\frac{2}{\tau}\big(J_\alpha^\tau,W_\alpha^\tau\big)\le\frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2+C_\varepsilon\big(\|W^\tau\|_s^2+\|W^\tau\|_s^4\big). \tag{4.4.20}$$

Using the expression of $A_0(n^\tau)$, we have obviously,

$$-2\big(A_0(n^\tau)\partial_x^\alpha R^\tau,W_\alpha^\tau\big)=-2\int_{\mathbb{T}}\big((n^\tau)^{-1}N_\alpha^\tau\partial_x^\alpha R_n^\tau+(h'(n^\tau))^{-1}U_\alpha^\tau\partial_x^\alpha R_u^\tau\big)dx.$$

Then (4.4.5) implies that

$$-2\big(A_0(n^\tau)\partial_x^\alpha R^\tau,W_\alpha^\tau\big)\le C\|W^\tau\|_s^2+\frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2+C_\varepsilon\tau^{2(\lambda+1)}. \tag{4.4.21}$$

Finally, for $i = 1, \cdots, d$, it follows from (4.4.11) and (4.4.15) that

$$\begin{aligned} \operatorname{div} A_\tau(n^\tau, u^\tau) &= \big(A_0\big)'(n^\tau)\partial_t n^\tau + \frac{1}{\tau}\sum_{i=1}^d \partial_{x_i}\big(u_i^\tau A_0(n^\tau)\big) \\ &= \big(A_0\big)'(n^\tau)\big(\partial_t n^\tau + \frac{1}{\tau}\nabla n^\tau \cdot u^\tau\big) + \frac{1}{\tau}\operatorname{div} u^\tau A_0(n^\tau) \end{aligned}$$

Using the first equation of (4.1.5), we deduce that

$$\operatorname{div} A_\tau(n^\tau, u^\tau) = \frac{\operatorname{div} u^\tau}{\tau}\big(A_0(n^\tau) - n^\tau\big(A_0\big)'(n^\tau)\big).$$

Noting

$$\frac{\kappa}{2} \le n^\tau = N^\tau + n_\tau \le C_0, \quad u^\tau = U^\tau + u_\tau, \quad u_\tau = O(\tau),$$

we obtain from the continuous embedding $H^{s-1}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ that

$$\|\operatorname{div} u^\tau\|_\infty \le C\|\operatorname{div}(U^\tau + u_\tau)\|_{s-1} \le C(\|U^\tau\|_s + \tau).$$

Therefore,

$$\|\operatorname{div} A_\tau(n^\tau, u^\tau)\|_\infty \le C\big(1 + \frac{1}{\tau}\|U^\tau\|_s\big).$$

Thus,

$$\big(\operatorname{div} A_\tau(n^\tau, u^\tau)W_\alpha^\tau, W_\alpha^\tau\big) \le \frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2 + C_\varepsilon\big(\|W^\tau\|_s^2 + \|W^\tau\|_s^4\big). \tag{4.4.22}$$

Together with (4.4.14) and (4.4.17)-(4.4.22), we obtain, for all $|\alpha| \le s$,

$$\frac{d}{dt}\big(A_0(n^\tau)W_\alpha^\tau, W_\alpha^\tau\big) + \frac{\kappa}{\tau^2}\|U_\alpha^\tau\|^2 \le \frac{C\varepsilon}{\tau^2}\|U^\tau\|_s^2 + C_\varepsilon\big(\|W^\tau\|_s^2 + \|W^\tau\|_s^4\big) + C_\varepsilon\tau^{2(\lambda+1)}.$$

Integrating this equation over $(0,t)$ with $t \in (0, T^\tau) \subset (0, T_1)$ and summing up over all $|\alpha| \le s$, taking $\varepsilon > 0$ sufficiently small such that the term including $C\varepsilon\|U^\tau\|_s^2/\tau^2$ can be controlled by the left hand side, noting (4.4.16) and condition (4.4.4) for the initial data, we get (4.4.12). □

Proof of Theorem 4.4.1. For $t \in (0, T^\tau)$, let

$$y(t) = C\int_0^t \Big(\|W^\tau(\xi)\|_s^2 + \|W^\tau(\xi))\|_s^4\Big)\, d\xi + C\tau^{2(\lambda+1)}.$$

Then it follows from Lemma 4.4.1 that

$$\|W^\tau(t)\|_s^2 \le y(t), \quad \frac{1}{\tau^2}\int_0^t \|U^\tau(\xi)\|_s^2 d\xi \le y(t), \quad \forall\, t \in (0, T^\tau) \tag{4.4.23}$$

and

$$y'(t) = C\Big(\|W^\tau(t)\|_s^2 + \|W^\tau(t))\|_s^4\Big) \le C\big(y(t) + y^2(t)\big),$$

with

$$y(0) = C\tau^{2(\lambda+1)}.$$

A straightforward computation yields

$$y(t) \le C\tau^{2(\lambda+1)}e^{Ct} \le C\tau^{2(\lambda+1)}e^{CT_1}, \quad \forall\, t \in [0, T^\tau].$$

Therefore, from (4.4.23) we obtain

$$\|W^\tau(t)\|_s \le \sqrt{y(t)} \le C\tau^{\lambda+1}, \quad \int_0^t \|U^\tau(\xi)\|_s^2 \le \tau^2 y(t) \le C\tau^{2(\lambda+2)}, \quad \forall\, t \in [0, T^\tau].$$

By a standard argument on the time extension of smooth solutions, we obtain $T_2^\tau \ge T_1$, i.e. $T^\tau = T_1$. Then $T_1^\tau \ge T_1$. This gives the uniform estimate for (n^τ, u^τ). The uniform estimate for ϕ^τ follows from the Poisson equation $\Delta\Phi^\tau = N^\tau$. This finishes the proof of Theorem 4.4.1. □

Chapter 5

Initial layers and zero-relaxation limits of two-fluid Euler-Poisson equations

Abstract. The Euler-Poisson system consists of the balance laws for electron density and current density coupled to the Poisson equation for the electrostatic potential. The limit of vanishing relaxation time of this multidimensional system with both well- and ill-prepared initial data on the two spaces case is discussed in this chapter. We study, by means of asymptotic expansions, the zero-relaxation limit. For this limit with well-prepared initial data, we show the existence and uniqueness of an asymptotic expansion up to any order. For general data, an asymptotic expansion up to order 1 of the relaxation limit is constructed by taking into account the initial layers. Finally, we justify the convergence by establishing uniform energy estimates.

5.1 Introduction

The Euler-Poisson system arises in semiconductors or plasma physics to study the time evolution of charged fluids. It can be obtained from the Boltzmann equation for charged particles, i.e. electrons and ions (or holes in semiconductors), see [35, 54]. It consists of the balance laws for the electron (ion) density and for the current density for electron (ion), coupled to the Poisson equation for the electrostatic potential. More precisely, we consider the (scaled) hydrodynamic equations for the electron density n_e with charge $q_e = 1$, the density n_i of the positively charged ions with charge $q_i = +1$, the respective velocities u_e, u_i and the electrostatic potential ϕ;

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \\ m_\nu \partial_t(n_\nu u_\nu) + m_\nu \operatorname{div}(n_\nu u_\nu \otimes u_\nu) + \nabla p_\nu(n_\nu) = -q_\nu n_\nu \nabla\phi - \dfrac{m_\nu n_\nu u_\nu}{\tau_\nu}, \\ -\lambda^2 \Delta\phi = n_i - n_e, \end{cases} \tag{5.1.1}$$

where $\nu = e, i$, and we use the notation $\partial_t = \partial/\partial t$. They are functions of d-dimensional position vector $x \in \mathbb{T}$ and time $t > 0$, where $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^d$ is the torus. Throughout this chapter, we restrict ourselves to the case of periodic functions. This system is complemented by initial conditions for the variables n_ν and u_ν, which are periodic in x.

In the above equations, p_ν are the pressure functions, usually given by $p_\nu(n_\nu) = a_\nu^2 n_\nu^{\gamma_\nu}$, where $a_\nu > 0$ and $\gamma_\nu > 0$ are constants. In this work, we only assume that p_ν is smooth and strictly increasing. The (scaled) physical parameters are the particle mass m_ν, the relaxation time τ_ν and the Debye length λ.

In the literature, there are many mathematical works devoted to the Euler-Poisson system, both on well-posedness and on different kinds of limit problems. Here we mention only a few of them. In the stationary case, Degond and Markowich [16] discussed the existence and uniqueness of solutions in the subsonic case, while Gamba [19] studied the same problem in the transonic case. The existence of global smooth solutions in the multidimensional case is proved in [3]. For asymptotic limits in multi-dimensional problems for a potential flow, we refer to [58, 65, 70]. In the time evolution case, Zhang [76], Marcati and Natalini [51] and Poupaud *et al* [67] got the global existence of entropy solutions of the Cauchy problem using the compensated compactness argument. The relaxation limit of the Euler-Poisson system to the drift-diffusion equations, which has been studied in [51], has been solved in [34, 36, 37] for weak entropy solutions. The same limit has been studied in [57] in a one-dimensional domain, proving the decay of the initial layer which develops for initial data not in equilibrium. For smooth solutions, Luo *et al* [48], Hsiao and Yang [30] and Li et al [47] investigated the asymptotic behaviour of solutions to the Cauchy and initial boundary value problem, respectively.

In addition to the relaxation limit ($\tau_{i,e} \to 0$) mentioned above, there are also two other kinds of relevant singular limit problems, that is, the quasi-neutral limit ($\lambda \to 0$) and the zero mass limit ($m_e/m_i \to 0$). The quasi-neutral limit in the Euler-Poisson system has been analysed for transient smooth solutions by Cordier and Grenier [14] in the one-dimensional case (see also [76] for sign-changing doping profile) and independently in [66, 71] in the multi-dimensional case.

In this chapter we consider smooth periodic solutions to the Euler-Poisson system (5.1.1). Then for smooth solutions with $n_\nu > 0$, the second equation of (5.1.1) is equivalent to :

$$m_\nu \partial_t u_\nu + m_\nu (u_\nu \cdot \nabla) u_\nu + \nabla h_\nu(n_\nu) = -q_\nu \nabla\phi - \frac{m_\nu u_\nu}{\tau_\nu}, \quad \nu = e, i, \tag{5.1.2}$$

where $''\cdot''$ denotes the inner product of $\mathbb{R}^d$ and the enthalpy function $h_\nu(n_\nu)$ is defined by :

$$h_\nu(n_\nu) = \int_1^{n_\nu} \frac{p'_\nu(s)}{s} ds. \tag{5.1.3}$$

Since $p_\nu(n_\nu)$ is strictly increasing, so is $h_\nu(n_\nu)$. Thus, $h_\nu(n_\nu)$ has an inverse $n_\nu = n_\nu(h_\nu)$.

We study the zero-relaxation limit $\tau_\nu \to 0$ for $\nu = e, i$ with $m_e = m_i = \lambda = 1$. For simplifying the notations, we assume that $\tau_i = \tau_e = \tau$. To perform the limit $\tau \to 0$ we introduce a time scaling :

$$t = \tau\xi. \tag{5.1.4}$$

Using still t by ξ, the Euler-Poisson system (5.1.1) becomes (see [51, 36, 34] etc.) :

$$\begin{cases} \partial_t n_\nu + \dfrac{1}{\tau}\operatorname{div}(n_\nu u_\nu) = 0, \\ \partial_t u_\nu + \dfrac{1}{\tau}(u_\nu \cdot \nabla)u_\nu + \dfrac{1}{\tau}\nabla h_\nu(n_\nu) = -q_\nu \dfrac{\nabla\phi}{\tau} - \dfrac{u_\nu}{\tau^2}, \qquad \nu = e, i, \\ -\Delta\phi = n_i - n_e, \end{cases} \tag{5.1.5}$$

for $t > 0$, $x \in \mathbb{T}$. It is complemented by periodic initial conditions :

$$t = 0 \;\;:\quad (n_\nu, u_\nu) = (n^\tau_{\nu,0}, u^\tau_{\nu,0}). \tag{5.1.6}$$

In problem (5.1.5)-(5.1.6), ϕ is not determined in a unique way. To avoid this, we add a restriction condition :

$$m(\phi) \overset{def}{=} \int_{\mathbb{T}} \phi(\cdot, x)dx = 0. \tag{5.1.7}$$

By the Poincaré inequality (see Lemma 4.1.2), for all integer $s \geq 0$ the Poisson equation in system (5.1.5) with (5.1.7) gives the estimate :

$$\|\nabla\phi\|_{H^s(\mathbb{T})} \leq C\|n_i - n_e\|_{H^s(\mathbb{T})}. \tag{5.1.8}$$

Then, regarding $\nabla\phi$ as a function of n_e and n_i, (n_ν, u_ν) for $\nu = e, i$ still satisfy a symmetric hyperbolic system in which $\nabla\phi$ on the right hand side of (5.1.5) is a low order term. Moreover, estimate (5.1.8) implies that $\phi \in C\big([0,T), H^{s+1}(\mathbb{T})\big)$ as soon as $n_\nu \in C\big([0,T), H^s(\mathbb{T})\big)$ for some $T > 0$ and all integer $s \geq 0$.

In this chapter, we will show the zero-relaxation limit $\tau \to 0$ to problem (5.1.5)-(5.1.6) for both well- and ill-prepared initial data on $\mathbb{T}$. As a first step, we will construct an asymptotic expansion for smooth solutions and prove its convergence up to any order for well-prepared initial data. For ill-prepared initial data, the above convergence result is not valid because the approximate solution cannot satisfy the prescribed initial conditions. In this case, we construct initial layer corrections and prove the convergence of the asymptotic expansion of zero order. In both cases, the convergence rates are given.

This chapter is organized as follows. In section 2, we derive asymptotic expansions of solutions and state the convergence result to problem (5.1.5)-(5.1.6) in the case of well-prepared initial data. Section 3 is devoted to the initial layer analysis in the relaxation limit. We construct the initial layer corrections which exponentially decay to zero and state the convergence result. Finally, in the last section, we also show the justification of both two asymptotic expansions. The justification is given by using another symmetrizer different from that used in chapter 4. For this purpose, we prove a more general convergence theorem which implies those in both cases of well-prepared initial data and ill-prepared initial data.

Proposition 5.1.1 *(Local existence of smooth solutions) Let* $s > 1 + \frac{d}{2}$ *and* $(n^\tau_{\nu,0}, u^\tau_{\nu,0}) \in H^s(\mathbb{T})$ *with* $n^\tau_{\nu,0} \geq \kappa$, *for some given constant* $\kappa > 0$, *independent of* τ. *Then there exist* $T^\tau_1 > 0$ *and a unique smooth solution* $(n^\tau_\nu, u^\tau_\nu, \phi^\tau)$ *to Cauchy problem (5.1.5)-(5.1.6) defined in the time interval* $[0, T^\tau_1)$, *with*

$$(n^\tau_\nu, u^\tau_\nu) \in C\big([0, T^\tau_1), H^s(\mathbb{T})\big) \cap C^1\big([0, T^\tau_1), H^{s-1}(\mathbb{T})\big),$$

$$\phi^\tau \in C\big([0,T_1^\tau), H^{s+1}(\mathbb{T})\big) \cap C^1\big([0,T_1^\tau), H^s(\mathbb{T})\big).$$

5.2 Preliminaries

In this section, we write the hydrodynamical models as symmetrizable hyperbolic systems and review the convergence-stability lemma from [45].

Now we write (5.1.5) as a symmetric hyperbolic system. To do this, we set :

$$\rho_\nu(h_\nu) = p_\nu'\big(n_\nu(h_\nu)\big).$$

Then, for smooth solutions, (5.1.5) is equivalent to :

$$\begin{cases} \big(\rho_\nu(h_\nu)\big)^{-1}\Big(\partial_t h_\nu + \dfrac{1}{\tau}(u_\nu\cdot\nabla)h_\nu\Big) + \dfrac{1}{\tau}\operatorname{div}(u_\nu) = 0, \\ \partial_t u_\nu + \dfrac{1}{\tau}(u_\nu\cdot\nabla)u_\nu + \dfrac{1}{\tau}\nabla h_\nu = -q_\nu\dfrac{\nabla\phi}{\tau} - \dfrac{u_\nu}{\tau^2}, \qquad\qquad t>0,\ x\in\mathbb{T} \\ -\Delta\phi = n_i(h_i) - n_e(h_e), \quad \nu = e,i. \\ t=0\ :\ \ (h_\nu,u_\nu) = \big(h_\nu(n_{\nu,0}^\tau), u_{\nu,0}^\tau\big), \end{cases} \tag{5.2.1}$$

with the following initial datas :

$$t=0\ :\ (h_\nu,u_\nu) = (h_\nu(n_{\nu,0}^\tau), u_{\nu,0}^\tau), \tag{5.2.2}$$

or

$$\partial_t\begin{pmatrix} h_\nu \\ u_\nu\end{pmatrix} + \frac{1}{\tau}\sum_{i=1}^d A_i(h_\nu,u_\nu)\partial_{x_i}\begin{pmatrix} h_\nu \\ u_\nu\end{pmatrix} = \frac{1}{\tau^2}\begin{pmatrix} 0 \\ -\tau q_\nu\nabla\phi\end{pmatrix}. \tag{5.2.3}$$

Here the coefficients have the following structure :

$$\begin{aligned} &A_i(h_\nu,u_\nu) = A_0^{-1}(h_\nu)C_i + (u_\nu\cdot e_i^t)I_{d+1}, \\ &A_0(h_\nu) = \operatorname{diag}\Big(\frac{1}{\rho_\nu(h_\nu)}, I_d\Big), \end{aligned} \tag{5.2.4}$$

$$\text{each } C_i \text{ is a constant symmetric matrix,}$$

$$\text{and the first element } C_i^{11} \text{ in the first row of } C_i \text{ is zero,}$$

where I_k denotes the unit matrix of order k and $(e_1,\dots e_d)$ is the canonical basis of R^d. Thus, (5.2.3) is a symmetrizable hyperbolic system with A_0 the symmetrizer.

Remark 5.2.1 *Although (5.2.3) is of the form of the systems studied in [46], it is essentially different from them. In fact, a crucial assumption in [46] is that the coefficients A_i and the symmetrizer A_0 depend on the unknown $W \equiv (h_\nu,u_\nu)$ only through τW, that is, $A_i = A_i(\tau W)$ and $A_0 = A_0(\tau W)$. This assumption is obviously not satisfied by our present system* (5.2.3).

The local-in-time existence theory for periodic IVPs (initial-value problems) of first-order symmetrizable hyperbolic systems can be well applied to (5.2.3). Moreover, we recall the *convergence-stability lemma* in [10] for general singular limit problems of IVPs for quasi-linear first-order symmetrizable hyperbolic systems depending (singularly) on parameters in several space variables :

$$\begin{aligned} & U_t + \sum_{i=1}^{d} A_i(U,\tau)U_{x_i} = Q(U,\tau) \\ & U(x,0) = \bar{U}(x,\tau). \end{aligned} \tag{5.2.5}$$

Here τ represents a parameter in a topological space, $A_i(U,\tau)\,(i = 1, 2, \cdots, d)$ and $Q(U,\tau)$ are sufficiently smooth functions of $U \in G \subset \mathbb{R}^n$, and $\bar{U}(x,\tau)$ is a given initial-value function. For simplicity, we assume that $\bar{U}(x,\tau)$ is periodic in x.

Assume $\bar{U}(x,\tau) \in G_0 \subset\subset G$ for all (x,τ) and $\bar{U}(.,\tau) \in H^s$ with $s > d/2+1$ an integer. Fix τ. According to the local existence theory for IVPs of symmetrizable hyperbolic systems (see Theorem 2.1 in [50]), there is a time interval $[0, T_1^\tau]$ so that (5.2.5) has a unique H^s-solution :

$$U^\tau \in (C[0, T_1^\tau], H^s).$$

Define

$$T_2^\tau = \sup\{T_1^\tau > 0 : \; U^\tau \in C[0, T_1^\tau], H^s\}. \tag{5.2.6}$$

Namely, $[0, T_2^\tau)$ is the maximal time interval of H^s existence. Note that T_2^τ depends on G and may tend to zero as τ goes to a certain singular point, say 0.

In order to show that $\underline{\lim}_{\tau\to 0} T_2^\tau > 0$, which means the stability (see [43, 50]), we make the following assumption.

Convergence assumption. There exists $T_1 > 0$ and $U_\tau \in L^\infty\big([0, T_1], H^s\big)$ for each τ, satisfying :

$$\bigcup_{x,t,\tau} \{U_\tau(t,x)\} \subset\subset G,$$

such that for $t \in [0, T^\tau = \min\{T_1, T_2^\tau\})$,

$$\sup_{t,x} |U^\tau(t,x) - U_\tau(t,x)| = o(1),$$

$$\sup_{t} ||U^\tau(t,.) - U_\tau(t,.)||_s = O(1),$$

as τ tends to the singular point.

With such a convergence assumption, we are in a position to state the following fact established in [10].

Lemma 5.2.1 *Suppose $\bar{U}(x,\tau) \in G_0 \subset\subset G$ for all (x,τ), $\bar{U}(.,\tau) \in H^s$ with an integer $s > d/2+1$, and that the convergence assumption holds. Let $[0, T_2^\tau)$ be the maximal time interval such that (5.2.5) has a unique H^s-solution : $U^\tau \in C([0, T_2^\tau), H^s)$.*
Then

$$T_2^\tau > T_1$$

for all τ in a neighborhood of the singular point.

Thanks to Lemma 5.2.1, our task is reduced to finding a $U_\tau(t,x)$ such that the convergence assumption holds. Below, we will use this lemma with G replaced by its compact subsets.

5.3 Case of well-prepared initial data

5.3.1 Formal asymptotic expansions

We look for an approximate solution $(n_\nu^\tau, u_\nu^\tau, \phi^\tau)$ to system (5.1.5) under the form of a power series in τ. From the momentum equations for u_ν^τ it is easy to see that the leading terms in u_ν^τ should be equal to zero. For convenience we replace u_ν^τ by τu_ν^τ and still denote the latter by u_ν^τ. In this case, the Euler-Poisson system (5.1.5) is rewritten as :

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \\ \tau^2\big(\partial_t u_\nu + (u_\nu \cdot \nabla)u_\nu\big) + \nabla h(n_\nu) = -q_\nu \nabla\phi - u_\nu, \\ -\Delta\phi = n_i - n_e, \quad x \in \mathbb{T}, \quad \nu = e, i, \end{cases}$$

in which the only small parameter is τ^2. This suggests the following asymptotic expansion for both the initial data and the solution :

$$(n_{\nu,\tau}, u_{\nu,\tau})(0,x) = \sum_{j\ge 0} \tau^{2j}(n_{\nu,j}, \tau u_{\nu,j})(x), \quad x \in \mathbb{T}, \quad \nu = e, i, \tag{5.3.1}$$

where $(n_{\nu,j}, u_{\nu,j})_{j\ge 0}$ is sufficiently smooth given data with $n_{\nu,0} \ge$ constant > 0 in $\mathbb{T}$. Then we make the following ansatz :

$$(n_{\nu,\tau}, u_{\nu,\tau}, \phi_\tau)(t,x) = \sum_{j\ge 0} \tau^{2j}(n_\nu^j, \tau u_\nu^j, \phi^j)(t,x), \quad t > 0, \;\; x \in \mathbb{T}, \quad \nu = e, i. \tag{5.3.2}$$

Now it needs to determine the profiles $(n_\nu^j, u_\nu^j, \phi^j)$ for all $j \ge 0$. Substituting expansion (5.3.2) into system (5.1.5), we obtain a series of equations verified by the profiles $(n_\nu^j, u_\nu^j, \phi^j)_j \ge 0$.

1. The leading profiles $(n_\nu^0, u_\nu^0, \phi^0)$ satisfy the following system :

$$\begin{cases} \partial_t n_\nu^0 + \operatorname{div}(n_\nu^0 u_\nu^0) = 0, \\ \nabla h_\nu(n_\nu^0) = -q_\nu \nabla\phi^0 - u_\nu^0, \qquad\qquad \nu = e, i, \\ -\Delta\phi^0 = n_i^0 - n_e^0, \;\; m(\phi^0) = 0. \end{cases} \tag{5.3.3}$$

   Then (n_e^0, n_i^0, ϕ^0) satisfies the classical drift-diffusion equations :

$$\begin{cases} \partial_t n_\nu^0 - \operatorname{div}\big(n_\nu^0 \nabla\big(h_\nu(n_\nu^0) + q_\nu\phi^0\big)\big) = 0, \quad \nu = e, i, \\ -\Delta\phi^0 = n_i^0 - n_e^0, \;\; m(\phi^0) = 0, \end{cases} \qquad t > 0, \quad x \in \mathbb{T} \tag{5.3.4}$$

with the initial conditions :

$$n_\nu^0(0,x) = n_{\nu,0}(x), \quad x \in \mathbb{T}, \quad \nu = e, i. \tag{5.3.5}$$

For smooth initial data $n_{\nu,0}$ satisfying $n_{\nu,0} > 0$ in $\mathbb{T}$, the periodic problem (5.3.4)-(5.3.5) has a unique smooth solution (n_e^0, n_i^0, ϕ^0) in the class $m(\phi^0) = 0$, defined in a time interval $[0, T_1]$ with $T_1 > 0$. The solution satisfies $n_e > 0$ and $n_i > 0$ in $[0, T_1] \times \mathbb{T}$. Finally, we obtain from the second equation of (5.3.3) that :

$$u_\nu^0 = -\nabla\big(h_\nu(n_\nu^0) + q_\nu \phi^0\big), \quad \nu = e, i. \tag{5.3.6}$$

This implies that the initial data $u_{\nu,0}$ is not arbitrary. They should be given in terms of $n_{\nu,0}$. Precisely :

$$u_{\nu,0} = -\nabla\big(h_\nu(n_{\nu,0}) + q_\nu \phi_0\big), \quad \nu = e, i, \tag{5.3.7}$$

where ϕ_0 is determined by :

$$-\Delta\phi_0 = n_{i,0} - n_{e,0} \quad \text{in } \mathbb{T} \quad \text{and} \quad m(\phi_0) = 0. \tag{5.3.8}$$

Thus we need compatibility type conditions (5.3.7)-(5.3.8) for the initial data of the leading terms $(n_\nu^0, u_\nu^0, \phi^0)$.

2. For $j \geq 1$, the profiles $(n_\nu^j, u_\nu^j, \phi^j)$ are obtained by induction. Assume that $(n_\nu^k, u_\nu^k, \phi^k)_{0 \leq k \leq j-1}$ are smooth and already determined in previous steps.
Then $(n_\nu^j, u_\nu^j, \phi^j)$ satisfy the linear system :

$$\begin{cases} \partial_t n_\nu^j + \operatorname{div}\left(n_\nu^0 u_\nu^j + n_\nu^j u_\nu^0\right) = -\displaystyle\sum_{k=1}^{j-1} \operatorname{div}\left(n_\nu^k u_\nu^{j-k}\right), \quad \nu = e, i, \\ \partial_t u_\nu^{j-1} + \displaystyle\sum_{k=0}^{j-1} u_\nu^k \cdot \nabla u_\nu^{j-1-k} + \nabla\big(h'_\nu(n_\nu^0) n_\nu^j + h_\nu^{j-1}\big((n_\nu^k)_{k \leq j-1}\big)\big) = -q_\nu \nabla\phi^j - u_\nu^j, \\ -\Delta\phi^j = n_i^j - n_e^j, \quad m(\phi^j) = 0, \end{cases} \tag{5.3.9}$$

where $h_\nu^0 = 0$ and $h_\nu^{j-1}\big((n_\nu^k)_{k \leq j-1}\big)$ is defined for $j \geq 2$ by :

$$h_\nu\Big(\sum_{j \geq 0} \tau^j n_\nu^j\Big) = h_\nu(n_\nu^0) + h'_\nu(n_\nu^0) \sum_{j \geq 1} \tau^j n_\nu^j + \sum_{j \geq 2} \tau^j h_\nu^{j-1}\big((n_\nu^k)_{k \leq j-1}\big), \ \ \nu = e, i.$$

Therefore, in the class $m(\phi^j) = 0$, $(n_\nu^j, u_\nu^j, \phi^j)$ solve a linearized drift-diffusion system :

$$\begin{cases} \partial_t n_\nu^j - \operatorname{div}\left[n_\nu^0 \nabla(h'_\nu(n_\nu^0) n_\nu^j + q_\nu \phi^j) - n_\nu^j u_\nu^0\right] \\ = f^j\big((V_\nu^k, \partial_t V_\nu^k, \partial_x V_\nu^k, \partial_t \partial_x V_\nu^k, \partial_x^2 V_\nu^k)_{0 \leq k \leq j-1}\big), \qquad t > 0, \quad x \in \mathbb{T}, \quad \nu = e, i, \\ -\Delta\phi^j = n_i^j - n_e^j, \end{cases} \tag{5.3.10}$$

together with the initial conditions :

$$n_\nu^j(0,x) = n_{\nu,j}(x), \quad x \in \mathbb{T}, \quad \nu = e, i. \tag{5.3.11}$$

Finally, u_ν^j are given by :

$$u_\nu^j = \nabla\big(- q_\nu \phi^j - h_\nu'(n_\nu^0) n_\nu^j - h_\nu^{j-1}((n_\nu^k)_{k\le j-1})\big) - \Big(\partial_t u_\nu^{j-1} + \sum_{k=0}^{j-1} (u_\nu^k \cdot \nabla) u_\nu^{j-1-k}\Big), \quad (5.3.12)$$

where f^j is a given smooth function and $V_\nu^k = (n_\nu^k, u_\nu^k)$. Thus, the following compatibility conditions should be imposed :

$$\begin{aligned} u_{\nu,j} \;&=\; \nabla\big(- q_\nu \phi_j - h_\nu'(n_{\nu,0}) n_{\nu,j} - h_\nu^{j-1}((n_{\nu,k})_{k\le j-1})\big) \\ &\quad -\Big(\partial_t u_\nu^{j-1}\big|_{t=0} + \sum_{k=0}^{j-1} (u_{\nu,k}\cdot\nabla) u_{\nu,j-1-k}\Big), \quad \nu = e, i, \end{aligned} \quad (5.3.13)$$

where ϕ_j is determined by :

$$-\Delta \phi_j = n_{i,j} - n_{e,j} \quad \text{in } \mathbb{T} \quad \text{and} \quad m(\phi_j) = 0. \quad (5.3.14)$$

Proposition 5.3.1 *Assume that the initial data $(n_{\nu,j}, u_{\nu,j})_{j\ge 0}$ are sufficiently smooth, with $n_{\nu,0} \ge$ constant > 0 in $\mathbb{T}$ and satisfy the compatibility conditions (5.3.7)-(5.3.8) and (5.3.13)-(5.3.14). Then there exists a unique asymptotic expansion up to any order of the form (5.3.2), i.e. there exist $T_1 > 0$ and a unique smooth profiles $(n_\nu^j, u_\nu^j, \phi^j)_{j\ge 0}$ in the time interval $[0, T_1]$ to problems (5.3.4)-(5.3.6) and (5.3.10)-(5.3.12) for $j \ge 1$. In particular, the formal zero-relaxation limit $\tau \to 0$ of the two-fluid Euler-Poisson system (5.1.5) is the bipolar drift-diffusion system(5.3.4) and (5.3.6).*

5.3.2 Convergence results

Having constructed the formal approximation $(n_{\nu,\tau}^m, u_{\nu,\tau}^m, \phi_\tau^m)$ of the hydrodynamical model (5.2.3), let $m \ge 0$ be a fixed integer and $(n_\nu^\tau, u_\nu^\tau, \phi^\tau)$ be the exact solution to problem (5.1.5)-(5.1.6) defined in the time interval $[0, T_1^\tau)$. We denote by :

$$(n_{\nu,\tau}^m, u_{\nu,\tau}^m, \phi_\tau^m) = \sum_{j=0}^{m} \tau^{2j} (n_\nu^j, \tau u_\nu^j, \phi^j), \quad \nu = e, i \quad (5.3.15)$$

an approximate solution of order m, where $(n_\nu^j, u_\nu^j, \phi^j)_{0\le j\le m}$ are constructed in the previous subsection. The convergence of the asymptotic expansion (5.3.2) is to establish the limit $(n_\nu^\tau, u_\nu^\tau, \phi^\tau) \to (n_{\nu,\tau}^m, u_{\nu,\tau}^m, \phi_\tau^m)$ and its convergence rate as $\tau \to 0$ in a time interval independent of τ, when $(n_\nu^\tau, u_\nu^\tau, \phi^\tau) \to (n_{\nu,\tau}^m, u_{\nu,\tau}^m, \phi_\tau^m)$ at $t = 0$.

From the construction of the approximate solution, for $(t,x) \in [0,T_1] \times \mathbb{T}$ we have :

$$\begin{cases} \partial_t n^m_{\nu,\tau} + \dfrac{1}{\tau}\,\mathrm{div}(n^m_{\nu,\tau} u^m_{\nu,\tau}) = R^{\tau,m}_{n_\nu}, \\ \partial_t u^m_{\nu,\tau} + \dfrac{1}{\tau}(u^m_{\nu,\tau}\cdot\nabla)u^m_{\nu,\tau} + \dfrac{1}{\tau}\nabla h_\nu(n^m_{\nu,\tau}) = -q_\nu \dfrac{\nabla\phi^m_\tau}{\tau} - \dfrac{u^m_{\nu,\tau}}{\tau^2} + R^{\tau,m}_{u_\nu}, \\ -\Delta\phi^m_\tau = n^m_{i,\tau} - n^m_{e,\tau}, \quad m(\phi^m_\tau) = 0, \quad \nu = e,i, \end{cases} \tag{5.3.16}$$

or

$$\begin{cases} \rho_\nu(h^m_{\nu,\tau})^{-1}\big(\partial_t h^m_{\nu,\tau} + \dfrac{1}{\tau}(u^m_{\nu,\tau}\cdot\nabla)h^m_{\nu,\tau}\big) + \dfrac{1}{\tau}\,\mathrm{div}(u^m_{\nu,\tau}) = \dfrac{R^{\tau,m}_{n_\nu}}{n^m_{\nu,\tau}}, \\ \partial_t u^m_{\nu,\tau} + \dfrac{1}{\tau}(u^m_{\nu,\tau}\cdot\nabla)u^m_{\nu,\tau} + \dfrac{1}{\tau}\nabla h^m_{\nu,\tau} = -\dfrac{q_\nu}{\tau}\nabla\phi^m_\tau - \dfrac{u^m_{\nu,\tau}}{\tau^2} + R^{\tau,m}_{u_\nu}, \end{cases} \tag{5.3.17}$$

where $h^m_{\nu,\tau} = h_\nu(n^m_{\nu,\tau})$ and, $R^{\tau,m}_{n_\nu}, R^{\tau,m}_{u_\nu}$ are remainders. It is clear that the convergence rate depends strongly on the order of the remainders with respect to τ. Since the profile $(n^j_\nu, u^j_\nu, \phi^j)_{j\geq 0}$ is sufficiently smooth, a straightforward computation gives the following result.

Proposition 5.3.2 *Let the assumptions of Proposition 5.3.1 hold. Then for all integers $m \geq 0$, the remainders $R^{\tau,m}_{n_\nu}$ and $R^{\tau,m}_{u_\nu}$ satisfy :*

$$\sup_{0\leq t\leq T_1} \|R^{\tau,m}_{n_\nu}(t,\cdot)\|_s \leq C_m \tau^{2(m+1)}, \quad \sup_{0\leq t\leq T_1} \|R^{\tau,m}_{u_\nu}(t,\cdot)\|_s \leq C_m \tau^{2m+1}, \quad \nu = e,i, \tag{5.3.18}$$

where $C_m > 0$ is a constant independent of τ.

We prove in the section 5 the validity of the approximation $(n^m_{\nu,\tau}, u^m_{\nu,\tau}, \phi^m_\tau)$ under some regularity assumptions on the given data and an existence result. The main result of this section is stated as follows :

Theorem 5.3.1 *Let $m \geq 0$ and $s > 1 + \frac{d}{2}$ be any fixed integers. Let $(n_{\nu,j}, u_{\nu,j})$ be sufficiently smooth functions for $j = 0,1,\cdots,m$, with $n_0 \geq$ constant > 0 in $\mathbb{T}$ and satisfying the compatibility conditions (5.3.7) and (5.3.13) for $j \geq 1$, respectively. Suppose*

$$\Big\|(n^\tau_{\nu,0}, u^\tau_{\nu,0}) - \sum_{j=0}^m \tau^{2j}(n_{\nu,j}, \tau u_{\nu,j})\Big\|_s \leq C_1 \tau^{2(m+1)}, \quad \nu = e,i, \tag{5.3.19}$$

where $C_1 > 0$ is a constant independent of τ. Then there exists a constant $C_2 > 0$, independent of τ but dependent on $T_1 < \infty$ and the solution $(n^\tau_\nu, u^\tau_\nu, \phi^\tau)$ to the periodic problem (5.1.5)-(5.1.6) satisfies :

$$\sup_{t\in[0,T_1]} \big\|(n^\tau_\nu - n^m_{\nu,\tau}, u^\tau_\nu - u^m_{\nu,\tau})(t,.)\big\|_s \leq C_2 \tau^{2(m+1)}$$

and

$$\sup_{t\in[0,T_1]} \big\|\big(\phi^\tau - \phi^m_\tau\big)(t,\cdot)\big\|_{s+1} \leq C_2 \tau^{2(m+1)}.$$

Moreover,

$$\big\|u^\tau_\nu - u^m_{\nu,\tau}\big\|_{L^2(0,T_1;H^s(\mathbb{T}))} \leq C_2 \tau^{2m+3}.$$

5.4 Case of ill-prepared initial data

5.4.1 Initial layer corrections

In the discussion of the limit in section 3, the compatibility conditions are made on the initial data. These conditions are restrictions on the initial data, since $u_{\nu,j}$ should be determined in terms of $n_{\nu,j}$ for all $j \geq 0$. If these compatibility conditions do not hold, the phenomenon of the initial layers occurs. In this section, we consider the case of the so called ill prepared initial data by adding an initial layer corrections in the asymptotic expansion.

To avoid tedious computation, we only consider a zero-order asymptotic expansion in the case that condition (5.3.7) is violated, i.e.

$$u_{\nu,0} \neq -\nabla\big(h_\nu(n_{\nu,0}) + q_\nu \phi_0\big), \quad \nu = e, i.$$

We seek a simplest possible form of an asymptotic expansion such that its remainders are at least of order $O(\tau)$. Let the initial data of an approximate solution $(n_{\nu,\tau}, u_{\nu,\tau}, \phi_\tau)$ have an asymptotic expansion of the form :

$$(n_{\nu,\tau}, u_{\nu,\tau})\big|(0,x) = (n_{\nu,0}, \tau u_{\nu,0}) + O(\tau^2), \quad \nu = e, i, \tag{5.4.1}$$

where $(n_{\nu,0}, u_{\nu,0})$ are given smooth functions. We want to construct an asymptotic expansion of the approximate solution $(n_{\nu,\tau}, u_{\nu,\tau}, \phi_\tau)$ up to order 1. Then we may take the following ansatz of the form :

$$\begin{aligned}(n_{\nu,\tau}, u_{\nu,\tau}, \phi_\tau)(t,x) &= (n_\nu^0, \tau u_\nu^0, \phi^0)(t,x) + (n_{\nu,I}^0, \tau u_{\nu,I}^0, \phi_I^0)(z,x) \\ &\quad + \tau^2\big((n_\nu^1, \tau u_\nu^1, \phi^1)(t,x) + (n_{\nu,I}^1, \tau u_{\nu,I}^1, \phi_I^1)(z,x)\big) + O(\tau^4), \end{aligned} \tag{5.4.2}$$

where $z = t/\tau^2 \in \mathbb{R}$ is the fast variable and the subscript $''I''$ stands for the initial layer variables. A direct computation shows that we may take :

$$(n_\nu^1, u_\nu^1, \phi^1, u_{\nu,I}^1) = 0, \quad \nu = e, i,$$

since this expansion gives the remainders of order $O(\tau)$, which is the case of well-prepared initial data for $m = 0$. Then we propose the following ansatz :

$$\begin{aligned}(n_{\nu,\tau}, u_{\nu,\tau}, \phi_\tau)(t,x) &= (n_\nu^0, \tau u_\nu^0, \phi^0)(t,x) + (n_{\nu,I}^0, \tau u_{\nu,I}^0, \phi_I^0)(z,x) \\ &\quad + \tau^2(n_{\nu,I}^1, 0, \phi_I^1)(z,x) + O(\tau^4), \quad \nu = e, i. \end{aligned} \tag{5.4.3}$$

Substituting the expression (5.4.3) into the problem (5.1.5), we have :

1. The leading profiles $(n_\nu^0, u_\nu^0, \phi^0)$ are determined by two-fluid compressible Euler-Poisson system (5.3.4)-(5.3.6). The smooth solutions $(n_\nu^0, u_\nu^0, \phi^\tau)$ is defined in the time interval $[0, T_1]$ in the class $m(\phi^0) = 0$, without any compatibility conditions. From (5.3.6) we have :

$$u_\nu^0(0,.) = -\nabla(h_\nu(n_{\nu,0}) + q_\nu \phi_0), \quad \nu = e, i, \tag{5.4.4}$$

where ϕ_0 is given by (5.3.8). From the asymptotic expansion (5.4.1) and (5.4.3), it is easy to see that :

$$n_\nu^0(0,x) + n_{I,\nu}^0(0,x) = n_{\nu,0}(x), \quad u_\nu^0(0,x) + u_{\nu,I}^0(0,x) = u_{\nu,0}(x), \quad \nu = e,i, \tag{5.4.5}$$

which give the initial values of $n_{\nu,I}^0$ and $u_{\nu,I}^0$.

2. The leading correction terms $(n_{\nu,I}^0, u_{\nu,I}^0, \phi_I^0)$ satisfy the following equations :

$$n_{\nu,I}^0 = 0, \quad \phi_I^0 = 0 \quad \text{and} \quad \partial_z u_{\nu,I}^0 + u_{\nu,I}^0 = 0, \quad \nu = e,i. \tag{5.4.6}$$

Equation $n_{\nu,I}^0 = 0$ means that there is no zero-order initial layer on variable n_ν. Therefore, (5.4.5) gives :

$$n_\nu^0(0,x) = n_{\nu,0}(x), \quad \nu = e,i. \tag{5.4.7}$$

From the third equation of (5.4.6) together with (5.4.5), we obtain :

$$u_{\nu,I}^0(z,x) = u_{\nu,I}^0(0,x)e^{-z} = \big(u_{\nu,0}(x) - u_\nu^0(0,x)\big)e^{-z}, \quad \nu = e,i. \tag{5.4.8}$$

The second order correction terms $(n_{\nu,I}^1, \phi_I^1)$ satisfy :

$$\partial_z n_{\nu,I}^1(z,x) + \mathrm{div}\,\big(n_\nu^0(0,x)u_{\nu,I}^0(z,x)\big) = 0, \quad \nu = e,i \tag{5.4.9}$$

and

$$-\Delta\phi_I^1(z,x) = n_{i,I}^1(z,x) - n_{e,I}^1(z,x), \quad m(\phi_I^1) = 0. \tag{5.4.10}$$

Let $n_{\nu,1}$ be an arbitrary smooth function and let $n_{\nu,I}^1(0,x) = n_{\nu,1}(x)$. Together with (5.4.8), we have :

$$n_{\nu,I}^1(z,x) = n_{\nu,1}(x) - \mathrm{div}\,\big(n_\nu^0(0,x)(u_{\nu,0}(x) - u_\nu^0(0,x))\big)\big(1 - e^{-z}\big), \quad \nu = e,i. \tag{5.4.11}$$

Thus, the asymptotic expansion is constructed up to order 1 for general initial data.

Proposition 5.4.1 *Assume that the initial data $(n_{\nu,0}, u_{\nu,0})$ are sufficiently smooth with $n_{\nu,0} > 0$ in $\mathbb{T}$. Then there exists a unique asymptotic expansion up to order 1 of the form (5.4.3), in which the correction terms are determined by (5.4.6), (5.4.8), (5.4.10) and (5.4.11).*

5.4.2 Convergence results

Let

$$(n_{\nu,\tau}^I, u_{\nu,\tau}^I, \phi_\tau^I)(t,x) = (n_\nu^0, \tau u_\nu^0, \phi^0)(t,x) + \big((0, \tau u_{\nu,I}^0, 0) + \tau^2(n_{\nu,I}^1, 0, \phi_I^1)\big)(t/\tau^2, x). \tag{5.4.12}$$

By the construction above, we have :

$$\begin{cases} \partial_t n^I_{\nu,\tau} + \dfrac{1}{\tau}\operatorname{div}(n^I_{\nu,\tau}u^I_{\nu,\tau}) = R^{\tau,I}_{n_\nu}, \\ \partial_t u^I_{\nu,\tau} + \dfrac{1}{\tau}(u^I_{\nu,\tau}\cdot\nabla)u^I_{\nu,\tau} + \dfrac{1}{\tau}\nabla h_\nu(n^I_{\nu,\tau}) = -q_\nu\dfrac{\nabla\phi^I_{\nu,\tau}}{\tau} - \dfrac{u^I_{\nu,\tau}}{\tau^2} + R^{\tau,I}_{u_\nu}, \\ -\Delta\phi^I_\tau = n^I_{i,\tau} - n^I_{e,\tau}, \quad m(\phi^I_\tau) = 0, \\ t=0\ :\quad \big(n^I_{\nu,\tau}, u^I_{\nu,\tau}\big) = \big(n_{\nu,0} + \tau^2 n_{\nu,1}, \tau u_{\nu,0}\big), \end{cases} \tag{5.4.13}$$

or

$$\begin{cases} \rho_\nu(h^I_{\nu,\tau})^{-1}\big(\partial_t h^I_{\nu,\tau} + \dfrac{1}{\tau}(u^I_{\nu,\tau}\cdot\nabla)h^I_{\nu,\tau}\big) + \dfrac{1}{\tau}\operatorname{div}(u^I_{\nu,\tau}) = \dfrac{R^{\tau,I}_{n_\nu}}{n^I_{\nu,\tau}}, \\ \partial_t u^I_{\nu,\tau} + \dfrac{1}{\tau}(u^I_{\nu,\tau}\cdot\nabla)u^I_{\nu,\tau} + \dfrac{1}{\tau}\nabla h^I_{\nu,\tau} = -\dfrac{q_\nu}{\tau}\nabla\phi^I_{\nu,\tau} - \dfrac{u^I_{\nu,\tau}}{\tau^2} + R^{\tau,I}_{u_\nu}, \end{cases} \tag{5.4.14}$$

where $h^I_{\nu,\tau} = h_\nu(n^I_{\nu,\tau})$ and the expressions of the remainders $R^{\tau,I}_{n_\nu}$ and $R^{\tau,I}_{u_\nu}$ are given by :

$$\begin{aligned} R^{\tau,I}_{n_\nu} &= \partial_t\big(n^0_\nu + \tau^2 n^1_{\nu,I}\big) + \operatorname{div}\big((n^0_\nu + \tau^2 n^1_{\nu,I})(u^0_\nu + u^0_{\nu,I})\big) \\ &= \partial_z n^1_{\nu,I} + \operatorname{div}(n^0_\nu u^0_{\nu,I}) + \tau^2\operatorname{div}\big(n^1_{\nu,I}(u^0_\nu + u^0_{\nu,I})\big) \\ &= \operatorname{div}\Big(\big(n^0_\nu(t,x) - n^0_\nu(0,x)\big)u^0_{\nu,I}(z,x)\Big) + \tau^2\operatorname{div}\big(n^1_{\nu,I}(u^0_\nu + u^0_{\nu,I})\big) \end{aligned}$$

and

$$\begin{aligned} R^{\tau,u}_{\nu,I} &= \tau\big(\partial_t(u^0_\nu + u^0_{\nu,I}) + (u^0_\nu + u^0_{\nu,I})\cdot\nabla(u^0_\nu + u^0_{\nu,I})\big) \\ &\quad + \frac{1}{\tau}\big(\nabla\big(h_\nu(n^0_\nu + \tau^2 n^1_{\nu,I}) + q_\nu(\phi^0 + \tau^2\phi^1_I)\big) + (u^0_\nu + u^0_{\nu,I})\big) \\ &= \frac{1}{\tau}\big(\nabla(h_\nu(n^0_\nu) + q_\nu\phi^0) + u^0_\nu\big) + \frac{1}{\tau}(\partial_z u^0_{\nu,I} + u^0_{\nu,I}) + \frac{1}{\tau}\nabla\big(h_\nu(n^0_\nu + \tau^2 n^1_{\nu,I}) - h_\nu(n^0_\nu)\big) \\ &= \tau\big(\partial_t u^0_\nu + (u^0_\nu + u^0_{\nu,I})\cdot\nabla(u^0_\nu + u^0_{\nu,I})\big) + \tau\nabla\phi^1_I + \frac{1}{\tau}\nabla\big(h_\nu(n^0_\nu + \tau^2 n^1_{\nu,I}) - h_\nu(n^0_\nu)\big). \end{aligned}$$

Now we establish error estimates for $(R^{\tau,I}_{n_\nu}, R^{\tau,I}_{u_\nu})$. For $R^{\tau,I}_{n_\nu}$, there is $\eta\in[0,t]\subset[0,T_1]$ such that :

$$n^0_\nu(t,x) - n^0_\nu(0,x) = t\partial_t n^0_\nu(\eta,x) = \tau^2 z\partial_t n^0_\nu(\eta,x).$$

Since function $z\longmapsto ze^{-z}$ is bounded for $z\geq 0$, from

$$\partial_t n^0_\nu = -\operatorname{div}(n^0_\nu u^0_\nu),$$

it follows from (5.4.8) that :

$$\big(n^0_\nu(t,x) - n^0_\nu(0,x)\big)u^0_{\nu,I}(z,x) = O(\tau^2).$$

Thus

$$R_{n_\nu}^{\tau,I} = O(\tau^2).$$

Finally, for $R_{u_\nu}^{\tau,I}$, we have :

$$h_\nu(n_\nu^0 + \tau^2 n_{\nu,I}^1) - h_\nu(n_\nu^0) = O(\tau^2).$$

Then

$$R_{u_\nu}^{\tau,I} = O(\tau).$$

From the previous discussions on the remainders, we obtain the following error estimates.

Proposition 5.4.2 *For given smooth data, the remainders $R_{n_\nu}^{\tau,I}$ and $R_{u,I}^{\tau,I}$ satisfy*

$$\sup_{0\le t\le T_1} \|R_{n_\nu}^{\tau,I}(t,\cdot)\|_s \le C\tau^2, \qquad \sup_{0\le t\le T_1} \|R_{u_\nu}^{\tau,I}(t,\cdot)\|_s \le C\tau, \tag{5.4.15}$$

where $C > 0$ is a constant independent of τ.

The convergence result with initial layers can be stated as follows.

Theorem 5.4.1 *Let $s > 1+\frac{d}{2}$ be a fixed integer and $(n_{\nu,0}, u_{\nu,0}) \in H^{s+1}(\mathbb{T})$ with $n_{\nu,0} \ge constant > 0$ in $\mathbb{T}$. Suppose*

$$\left\|\left(n_{\nu,0}^\tau - n_{\nu,0}, u_{\nu,0}^\tau - u_{\nu,0}\right)\right\|_s \le C_1\tau^2, \tag{5.4.16}$$

where $C_1 > 0$ is a constant independent of τ. Then there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \ge T_1$ and the solution $(n_\nu^\tau, u_\nu^\tau, \phi^\tau)$ to the periodic problem (5.1.5)-(5.1.6) satisfies

$$\sup_{0\le t\le T_1} \left\|(n_\nu^\tau - n_{\nu,\tau}^I, u_\nu^\tau - u_{\nu,\tau}^I)(t,.)\right\|_s \le C_2\tau^2,$$

and

$$\sup_{0\le t\le T_1} \left\|(\phi^\tau - \phi_\tau^I)(t,.)\right\|_{s+1} \le C_2\tau^2, \quad \forall\, t \in [0, T_1]$$

Moreover,

$$\left\|u_\nu^\tau - u_{\nu,\tau}^I\right\|_{L^2(0,T_1;H^s(\mathbb{T}))} \le C_2\tau^3.$$

5.5 Justification of asymptotic expansions

5.5.1 Statement of the main result

In this section, we justify rigorously the asymptotic expansions of solutions $(n_\nu^\tau, u_\nu^\tau, \phi^\tau)$ to the periodic problem (5.1.5)-(5.1.6) constructed in section 3-4. We prove a more general convergence result which implies both Theorems 5.3.1-5.4.1. As a consequence, we obtain the existence of exact solutions $(n_\nu^\tau, u_\nu^\tau, \phi^\tau)$ in a time interval independent of τ. To justify rigorously the asymptotic expansions (5.3.15) and (5.4.12), it suffices to obtain the uniform estimates of the smooth solutions to (5.1.5) with respect to the parameter τ.

Let $(n_\nu^\tau, u_\nu^\tau, \phi^\tau)$, $(h_\nu^\tau, u_\nu^\tau, \phi^\tau)$ be the exact solution to (5.1.5) and (5.2.1) (respectively) with initial data $(n_{\nu,0}^\tau, u_{\nu,0}^\tau)$, $(h_\nu(n_{\nu,0}^\tau), u_{\nu,0}^\tau)$ (respectively) and $(n_{\nu,\tau}, u_{\nu,\tau}, \phi_\tau)$ be an approximate periodic solution defined on $[0, T_1]$, with

$$(n_{\nu,\tau}, u_{\nu,\tau}) \in C\big([0,T_1], H^{s+1}(\mathbb{T})\big) \cap C^1\big([0,T_1], H^s(\mathbb{T})\big),$$

$$\phi_\tau \in C\big([0,T_1], H^{s+2}(\mathbb{T})\big) \cap C^1\big([0,T_1], H^{s+1}(\mathbb{T})\big).$$

We define the remainders of the approximate solution by :

$$\begin{cases} R_{n_\nu}^\tau = \partial_t n_{\nu,\tau} + \dfrac{1}{\tau}\operatorname{div}(n_{\nu,\tau} u_{\nu,\tau}), \\ R_{u_\nu}^\tau = \partial_t u_{\nu,\tau} + \dfrac{1}{\tau}(u_{\nu,\tau}\cdot\nabla)u_{\nu,\tau} + \dfrac{1}{\tau}\nabla\big(h_\nu(n_{\nu,\tau}) + q_\nu\phi_\tau\big) + \dfrac{u_{\nu,\tau}}{\tau^2}. \end{cases} \tag{5.5.1}$$

Suppose

$$-\Delta\phi_\tau = n_{i,\tau} - n_{e,\tau}, \tag{5.5.2}$$

$$\sup_{0\le t\le T_1} \|n_{\nu,\tau}(t,\cdot)\|_s \le C_1, \quad \sup_{0\le t\le T_1} \|u_{\nu,\tau}(t,\cdot)\|_s \le C_1\tau, \tag{5.5.3}$$

$$\big\|\big(n_{\nu,0}^\tau - n_{\nu,\tau}(0,\cdot), u_{\nu,0}^\tau - u_{\nu,\tau}(0,\cdot)\big)\big\|_s \le C_1\tau^{\lambda+1}, \tag{5.5.4}$$

$$\sup_{0\le t\le T_1} \|R_{\nu,n}^\tau(t,\cdot)\|_s \le C_1\tau^{\lambda+1}, \qquad \sup_{0\le t\le T_1} \|R_{\nu,u}^\tau(t,\cdot)\|_s \le C_1\tau^{\lambda}, \tag{5.5.5}$$

where $\lambda \ge 0$ and $C_1 > 0$ are constants independent of τ.

Theorem 5.5.1 *Let $\lambda \ge 0$. Under the above assumptions , there exists a constant $C_2 > 0$, independent of τ, and the solution $(n_\nu^\tau, u_\nu^\tau, \phi^\tau)$ to the periodic problem (5.1.5)-(5.1.6) satisfies :*

$$\sup_{0\le t\le T_1} \big\|(n_\nu^\tau - n_{\nu,\tau}, u_\nu^\tau - u_{\nu,\tau})(t,.)\big\|_s \le C_2\tau^{\lambda+1}, \tag{5.5.6}$$

and

$$\sup_{0\le t\le T_1} \big\|(\phi^\tau - \phi_\tau)(t,.)\big\|_{s+1} \le C_2\tau^{\lambda+1}. \tag{5.5.7}$$

Moreover,

$$\big\|u_\nu^\tau - u_{\nu,\tau}\big\|_{L^2(0,T_1;H^s(\mathbb{T}))} \le C_2\tau^{\lambda+2}.$$

It is clear that Theorem 5.5.1 implies Theorems 5.3.1-5.4.1. In particular, $\lambda = 2m+1$ with $m \ge 0$ in section 3 and $\lambda = 1$ in section 4. The next subsection is devoted to the proof of Theorem 5.5.1.

5.5.2 Proof of the main result

By proposition 5.1.1, the exact solution $(n_\nu^\tau, u_\nu^\tau, \phi^\tau)$ of system (5.1.5) is defined on a time interval $[0, T_1^\tau)$ with $T_1^\tau > 0$. Since $n_\nu^\tau \in C\big([0,T_1^\tau), H^s(\mathbb{T})\big)$ and the embedding from $H^s(\mathbb{T})$ to $C(\mathbb{T})$

is continuous, we have $n_\nu^\tau \in C\big([0,T_1^\tau) \times \mathbb{T}\big)$. From assumption $n_{\nu,0}^\tau \geq \kappa > 0$, we deduce that there exists a maximal existence time $T_2^\tau \in (0,T_1^\tau]$, independent of τ, where the symmetrizable hyperbolic system (5.1.5) with initial data (5.1.6) has a unique H^s-solution $(n_\nu^\tau, u_\nu^\tau, \phi^\tau)$ with values $(a,2b)\times\mathbb{R}^n\times\mathbb{R}$. Since $n_\nu^\tau \in \big([0,T_2^\tau), H^s\big)$ has a positive lower bound, there are two positive numbers κ and C_0 such that :

$$k \leq h_\nu(n_\nu^\tau) \leq 2C_0 \quad \forall\,(t,x)\in[0,T_2^\tau]\times\mathbb{T}, \quad \nu=e,i.$$

Similarly, the function $t \longmapsto \big\|\big(h_\nu(n_\nu^\tau)(t,\cdot),u^\tau(t,\cdot)\big)\big\|_s$ is continuous in $C\big([0,T_2^\tau)\big)$. From (5.5.3), the sequence $\big(\|(n_\nu^\tau(0,\cdot),u_\nu^\tau(0,\cdot))\|_s\big)_{\tau>0}$ is bounded. Then there exists $T_3^\tau \in (0,T_2^\tau)$ and a constant, still denoted by C_0, such that :

$$\big\|\big(h_\nu(n_\nu^\tau)(t,\cdot),u_\nu^\tau(t,\cdot)\big)\big\|_s \leq C_0, \quad \forall\, t\in(0,T_3^\tau], \quad \nu=e,i. \tag{5.5.8}$$

Thus the symmetrizable hyperbolic system (5.2.3) has a unique H^s-solution (h_ν^τ,u_ν^τ) with values in $(\kappa,2C_0)\times\mathbb{R}^n\equiv G$. Thanks to Lemma 5.2.1, it suffices to prove the error estimate in Theorem 5.5.1 for $t\in[0,T^\tau=\min\{T_1,T_3^\tau\})$. In this time interval, we denote by :

$$\begin{aligned}(H_\nu^\tau,U_\nu^\tau,\Phi^\tau) &= (h_{\nu,\tau}-h_\nu^\tau, u_{\nu,\tau}-u_\nu^\tau, \phi_\tau-\phi^\tau)\\ &= (h_\nu(n_{\nu,\tau})-h_\nu(n_\nu^\tau), u_{\nu,\tau}-u_\nu^\tau, \phi_\tau-\phi^\tau).\end{aligned} \tag{5.5.9}$$

To this end, we set :

$$W_\nu^\tau = \begin{pmatrix} H_\nu^\tau \\ U_\nu^\tau \end{pmatrix}, \quad W_{\nu,0}^\tau = \begin{pmatrix} h_\nu\big(n_{\nu,\tau}(0,.)\big)-h_\nu(n_{\nu,0}^\tau) \\ u_{\nu,\tau}(0,.)-u_{\nu,0}^\tau \end{pmatrix}, \quad F_{1,\nu}(W_\nu^\tau) = \begin{pmatrix} 0 \\ -U_\nu^\tau \end{pmatrix},$$

$$F_{2,\nu}(\Phi^\tau) = \begin{pmatrix} 0 \\ -q_\nu\nabla\Phi^\tau \end{pmatrix}, \quad R_\nu^\tau = \begin{pmatrix} h_\nu'(n_{\nu,\tau})R_{n_\nu}^\tau \\ R_{u_\nu}^\tau \end{pmatrix}.$$

Obviously, the error W_ν^τ satisfies :

$$\partial_t W_\nu^\tau + \frac{1}{\tau}\sum_{i=1}^d A_i(h_\nu^\tau,u_\nu^\tau)\partial_{x_i}W_\nu^\tau = \frac{1}{\tau^2}F_{1,\nu}(W_\nu^\tau) + \frac{1}{\tau}\big(F_{2,\nu}(\Phi^\tau)+F_{3,\nu}(W_\nu^\tau)\big) + R_\nu^\tau, \tag{5.5.10}$$

where A_i is given by (5.2.4), Φ^τ is linked by the Poisson equation :

$$\Delta\Phi^\tau = N_i^\tau - N_e^\tau, \tag{5.5.11}$$

and

$$F_{3,\nu}(W_\nu^\tau) = \sum_{i=1}^d [A_i(h_\nu^\tau,u_\nu^\tau)-A_i(h_{\nu,\tau},u_{\nu,\tau})]\begin{pmatrix}\partial_{x_i}h_{\nu,\tau}\\ \partial_{x_i}u_{\nu,\tau}\end{pmatrix}.$$

The initial condition of (5.5.10) is :

$$t=0 \ : \ W_\nu^\tau = W_{\nu,0}^\tau. \tag{5.5.12}$$

We remember that the system (5.5.10) is symmetrizable hyperbolic with symmetrizer :

$$A_0(h_\nu^\tau) = \operatorname{diag}\left(\frac{1}{\rho_\nu(h_\nu)(h_\nu^\tau)}, I_d\right).$$

The existence and uniqueness of smooth solutions to (5.1.5)-(5.1.6) is equivalent to that of (5.5.10)-(5.5.12). Thus, using standard arguments, in order to show Theorem 5.5.1, it suffices to establish uniform estimates of W_ν^τ with respect to τ.

In what follows, we denote by $C > 0$ various constants independent of τ, for $\alpha \in \mathbb{N}^d$, $W_{\nu,\alpha}^\tau = \partial_x^\alpha W_\nu^\tau$, $F_i^\alpha = \partial_x^\alpha F_i$ for $i = 1, 2, 3$ and $R_{\nu,\alpha}^\tau = \partial_x^\alpha R_\nu^\tau$. etc. We differentiate the equation (5.5.10) with respect to x for a multi-index α satisfying $|\alpha| \le s$ to get :

$$\begin{aligned} &\partial_t W_{\nu,\alpha}^\tau + \frac{1}{\tau}\sum_{i=1}^d A_i(h_\nu^\tau, u_\nu^\tau)\partial_{x_i} W_{\nu,\alpha}^\tau \\ = \;& \frac{1}{\tau^2} F_{1,\nu}^\alpha(W_\nu^\tau) + \frac{1}{\tau}\big(F_{2,\nu}^\alpha(\Phi^\tau) + F_{3,\nu}^\alpha(W_\nu^\tau) + F_{4,\nu}^\alpha(W_\nu^\tau)\big) + R_{\nu,\alpha}^\tau, \end{aligned} \tag{5.5.13}$$

where,

$$F_{4,\nu}^\alpha(W_\nu^\tau) = \sum_{i=1}^d \big(A_i(h_\nu^\tau, u_\nu^\tau)\partial_{x_i} W_{\nu,\alpha}^\tau - \partial_x^\alpha(A_i(h_\nu^\tau, u_\nu^\tau)\partial_{x_i} W_\nu^\tau)\big),$$

For the sake of clarity, we divide the following arguments into lemmas :

Lemma 5.5.1 *Under the conditions of Theorem 5.5.1, we have*

$$\begin{aligned} \frac{d}{dt}\big(W_{\nu,\alpha}^\tau, A_0(h_\nu^\tau) W_{\nu,\alpha}^\tau\big) + \frac{2}{\tau^2}\|U_{\nu,\alpha}^\tau\|^2 \;\le\; & \frac{2}{\tau}\|U_{\nu,\alpha}^\tau\|\|F_{2,\nu}^\alpha\| + \frac{C}{\tau}\int_{\mathbb{T}} |\operatorname{div} u_\nu^\tau||W_{\nu,\alpha}^\tau|^2 dx \\ & + \frac{C}{\tau}\|W_{\nu,\alpha}^\tau\|\big(\|F_{3,\nu}^\alpha\| + \|F_{4,\nu}^\alpha\|\big) + 2\big(A_0^\tau R_{\nu,\alpha}^\tau, W_{\nu,\alpha}^\tau\big), \end{aligned} \tag{5.5.14}$$

where $(\cdot, \cdot)$ is the inner product of $L^2(\mathbb{T})$, and C is a given generic constant depending only on the range $[\kappa, 2C_0]$ of h_ν^τ.

Proof. Since matrices $A_0^\tau \equiv A_0(h_\nu^\tau)$ and $A_0^\tau A_j(h_\nu^\tau, u_\nu^\tau)$ are symmetric, we multiply (5.5.13) by A_0^τ and taking the inner product of the resulting equations with $W_{\nu,\alpha}^\tau$, by employing the classical energy estimate for symmetrizable hyperbolic equations, we obtain :

$$\begin{aligned} &\frac{d}{dt}\big(W_{\nu,\alpha}^\tau, A_0^\tau W_{\nu,\alpha}^\tau\big) + \frac{1}{\tau}\sum_{i=1}^d \big(W_{\nu,\alpha}^\tau, A_0^\tau A_i^\tau W_{\nu,\alpha}^\tau\big)_{x_i} \\ = \;& \frac{2}{\tau^2}\big(A_0^\tau F_{1,\nu}^\alpha(W_\nu^\tau), W_{\nu,\alpha}^\tau\big) + \frac{2}{\tau}\big(A_0^\tau[F_{2,\nu}^\alpha(\Phi^\tau) + F_{3,\nu}^\alpha(W_\nu^\tau) + F_{4,\nu}^\alpha(W_\nu^\tau)], W_{\nu,\alpha}^\tau\big) \\ & + \big(\operatorname{div} A_\tau(h_\nu^\tau, u_\nu^\tau) W_{\nu,\alpha}^\tau, W_{\nu,\alpha}^\tau\big) + 2\big(A_0^\tau R_{\nu,\alpha}^\tau, W_{\nu,\alpha}^\tau\big), \end{aligned} \tag{5.5.15}$$

where $\operatorname{div} A_\tau(h_\nu^\tau, u_\nu^\tau)$ is given by :

$$\operatorname{div} A_\tau(h_\nu^\tau, u_\nu^\tau) = \partial_t A_0^\tau + \frac{1}{\tau}\sum_{i=1}^d \partial_{x_i}(A_0^\tau A_i^\tau). \tag{5.5.16}$$

Let us estimate each term of equations (5.5.15).
Recall from (5.2.4) that

$$A_0^\tau = \operatorname{diag}\Big(\frac{1}{\rho_\nu(h_\nu^\tau)}, I_d\Big).$$

It is obvious that :

$$\frac{2}{\tau^2}\big(A_0^\tau F_{1,\nu}^\alpha(W_\nu^\tau), W_{\nu,\alpha}^\tau\big) = \frac{-2}{\tau^2}\|U_{\nu,\alpha}^\tau\|^2 \tag{5.5.17}$$

and

$$\frac{2}{\tau}\big(A_0^\tau F_{2,\nu}^\alpha(\Phi^\tau), W_{\nu,\alpha}^\tau\big) \le \frac{2}{\tau}\|U_{\nu,\alpha}^\tau\|\|F_{2,\nu}^\alpha\|. \tag{5.5.18}$$

On the other hand, since h_ν^τ takes values in the compact $[\kappa, 2C_0]$, we get :

$$\frac{2}{\tau}\big(A_0^\tau F_{3,\nu}^\alpha(W_\nu^\tau), W_{\nu,\alpha}^\tau\big) \le \frac{C}{\tau}\|W_{\nu,\alpha}^\tau\|\|F_{3,\nu}^\alpha\|. \tag{5.5.19}$$

Moreover, we use the relation in (5.2.4) and the h_ν-equation in (5.2.3) to compute :

$$\begin{aligned}
\operatorname{div} A_\tau(h_\nu^\tau, u_\nu^\tau) &= \partial_t A_0(h_\nu^\tau) + \frac{1}{\tau}\sum_{i=1}^d \partial_{x_i}\big(A_0(h_\nu^\tau)A_i(h_\nu^\tau, u_\nu^\tau)\big)\\
&= \big(A_0\big)'(h_\nu^\tau)\big(\partial_t h_\nu^\tau + \frac{1}{\tau}\sum_{i=1}^d (u_\nu^\tau\cdot e_i^t)\partial_{x_i}h_\nu^\tau\big) + \frac{1}{\tau}\sum_{i=1}^d \partial_{x_i}(u_\nu^\tau\cdot e_i^t)A_0(h_\nu^\tau)\\
&= \frac{\operatorname{div} u_\nu^\tau}{\tau}\big(A_0(h_\nu^\tau) - \rho_\nu(h_\nu^\tau)A_0'(h_\nu^\tau)\big).
\end{aligned}$$

Thus, we have :

$$\big(\operatorname{div} A_\tau(h_\nu^\tau, u_\nu^\tau)W_{\nu,\alpha}^\tau, W_{\nu,\alpha}^\tau\big) \le \frac{C}{\tau}\int_{\mathbb{T}} |\operatorname{div} u_\nu^\tau||W_{\nu,\alpha}^\tau|^2 dx. \tag{5.5.20}$$

Finally, using the periodicity of the initial data, the second term of left side of the equation (5.5.15) vanish, which finish the lemma. □

For the right-hand side of the inequality in Lemma 5.5.1 we have the following claim.

Lemma 5.5.2 *Under the conditions of Theorem 5.5.1. Then, for $\tau < 1$, we have :*

$$\frac{1}{\tau}\int_{\mathbb{T}} |\operatorname{div} u_\nu^\tau||W_{\nu,\alpha}^\tau|^2 dx \le \frac{\varepsilon}{5\tau^2}\|U_\nu^\tau\|_s^2 + C_\varepsilon\big(\|W_\nu^\tau\|_s^2 + \|W_\nu^\tau\|_s^4\big),$$

$$2\big(A_0^\tau R_{\nu,\alpha}^\tau, W_{\nu,\alpha}^\tau\big) \le \frac{\varepsilon}{5\tau^2}\|U_\nu^\tau\|_s^2 + C_\varepsilon\|W_\nu^\tau\|_s^2 + C_\varepsilon\tau^{2(\lambda+1)},$$

$$\frac{1}{\tau}\|U_\nu^\tau\|\|F_{2,\nu}^\alpha\| \le \frac{\varepsilon}{5\tau^2}\|U_\nu^\tau\|_s^2 + C_\varepsilon\big(\|H_\nu^\tau\|_s^2 + \|H_\nu^\tau\|_s^4\big),$$

$$\frac{1}{\tau}\|W^\tau_{\nu,\alpha}\|\|F^\alpha_{3,\nu}\| \le \frac{\varepsilon}{5\tau^2}\|U^\tau_\nu\|_s^2 + C_\varepsilon\big(\|W^\tau_\nu\|_s^2 + \|W^\tau_\nu\|_s^4\big),$$
$$\frac{1}{\tau}\|W^\tau_{\nu,\alpha}\|\|F^\alpha_{4,\nu}\| \le \frac{\varepsilon}{5\tau^2}\|U^\tau_\nu\|_s^2 + C_\varepsilon\big(\|W^\tau_\nu\|_s^2 + \|W^\tau_\nu\|_s^4\big).$$

Here and hereafter, ε denotes a small constant independent of τ and $C_\varepsilon > 0$ denotes a constant depending only on ε.

Proof. Recall that :

$$u^\tau_\nu = u_{\nu,\tau} - U^\tau_\nu, \quad u_{\nu,\tau} = O(\tau). \tag{5.5.21}$$

Thus, for $s > \frac{d}{2} + 1$, we use the continuous embedding $H^{s-1}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ to obtain :

$$|\operatorname{div} u^\tau_\nu| \le C\|\operatorname{div}(u_{\nu,\tau} - U^\tau_\nu)\|_{s-1} \le C(\|U^\tau_\nu\|_s + \tau).$$

Therefore,

$$\begin{aligned}\frac{1}{\tau}\int_{\mathbb{T}}|\operatorname{div} u^\tau_\nu||W^\tau_{\nu,\alpha}|^2dx &\le C\big(1 + \frac{1}{\tau}\|U^\tau_\nu\|_s\big)\|W^\tau_\nu\|_s^2 \\ &\le \frac{\varepsilon}{5\tau^2}\|U^\tau_\nu\|_s^2 + C_\varepsilon\big(\|W^\tau_\nu\|_s^2 + \|W^\tau_\nu\|_s^4\big).\end{aligned}$$

Thus,

$$\frac{1}{\tau}\int_{\mathbb{T}}|\operatorname{div} u^\tau_{\nu,\alpha}||W^\tau_\nu|^2dx \le \frac{\varepsilon}{5\tau^2}\|U^\tau_\nu\|_s^2 + C_\varepsilon\big(\|W^\tau_\nu\|_s^2 + \|W^\tau_\nu\|_s^4\big). \tag{5.5.22}$$

Using the expression of A^τ_0, we have obviously :

$$2\big(A^\tau_0 R^\tau_{\nu,\alpha}, W^\tau_{\nu,\alpha}\big) = 2\big((\rho_\nu(h^\tau_\nu))^{-1}H^\tau_{\nu,\alpha}, h'_\nu(n_{\nu,\tau})\partial^\alpha_x R^\tau_{n_\nu}\big) + 2\big(U^\tau_{\nu,\alpha}, \partial^\alpha_x R^\tau_{u_\nu}\big).$$

Together with (5.5.5) and the fact that the function h^τ_ν take values in the compact set $[\kappa, 2C_0]$ we have the following estimate :

$$2\big(A^\tau_0 R^\tau_{\nu,\alpha}, W^\tau_{\nu,\alpha}\big) \le C_\varepsilon\|W^\tau_\nu\|_s^2 + \frac{\varepsilon}{4\tau^2}\|U^\tau_\nu\|_s^2 + C_\varepsilon\tau^{2(\lambda+1)}. \tag{5.5.23}$$

Next we estimate $\frac{2}{\tau}\|U^\tau_{\nu,\alpha}\|\|F^\alpha_{2,\nu}\|$. Since

$$n_\nu(h^\tau_\nu) - n_\nu(h_{\nu,\tau}) = H^\tau_\nu\int_0^1 n'_\nu(h_{\nu,\tau} - \sigma H^\tau_\nu)d\sigma$$

and the convexity of $[\kappa, 2C_0]$ gives :

$$h_{\nu,\tau}(t,x) - \sigma H^\tau_\nu(t,x) \in [\kappa, 2C_0],$$

for all $(t,x,\sigma) \in [0, T^\tau = \min\{T^\tau_2, T_1\}) \times \mathbb{T} \times [0,1]$ and $\tau > 0$, it follows from Lemma 4.1.1 and

(5.5.8) that :

$$
\begin{aligned}
\|\big(n_\nu(h^\tau_\nu) - n_\nu(h_{\nu,\tau})\big)_\alpha\| &\le C\|H^\tau_\nu\|_{|\alpha|}\Big\|\int_0^1 n'_\nu(h_{\nu,\tau} - \sigma H^\tau_\nu)d\sigma\Big\|_s \\
&\le C\|H^\tau_\nu\|_{|\alpha|}\int_0^1 \|n'_\nu(h_{\nu,\tau} - \sigma H^\tau_\nu)\|_s d\sigma \\
&\le C\|H^\tau_\nu\|_{|\alpha|}\int_0^1 \big(1 + \|h_{\nu,\tau} - \sigma H^\tau_\nu\|^s_s\big)d\sigma \qquad (5.5.24)\\
&\le C\|H^\tau_\nu\|_{|\alpha|}\big(1 + \|H^\tau_\nu\|^s_s\big) \\
&\le C\|H^\tau_\nu\|_{|\alpha|}\big(1 + \|H^\tau_\nu\|^2_s\big).
\end{aligned}
$$

In the last inequality we had used also the boundedness of $\|h_{\nu,\tau}\|_s$.
From the Poisson equation (5.5.11), we have :

$$-\Delta\Phi^\tau_\alpha = \partial^\alpha_x N^\tau_i - \partial^\alpha_x N^\tau_e,$$

which implies that :

$$\|\nabla\Phi^\tau_\alpha\| \le \big(\|N^\tau_i\|_s + \|N^\tau_e\|_s\big).$$

Thus, we deduce that :

$$
\begin{aligned}
\frac{1}{\tau}\|U^\tau_{\nu,\alpha}\|\|F^\alpha_{2,\nu}\| &\le \frac{C}{\tau}\|U^\tau_{\nu,\alpha}\|\|\big(\|N^\tau_i\|_s + \|N^\tau_e\|_s\big) \\
&\le \frac{C}{\tau}\|U^\tau_{\nu,\alpha}\|\sum_{\nu=e,i}\|\big(n_\nu(h_{\nu,\tau}) - n_\nu(h^\tau_\nu)\big)_\alpha\| \\
&\le \frac{C}{\tau}\|U^\tau_{\nu,\alpha}\|\|H^\tau_\nu\|_{|\alpha|}\big(1 + \|H^\tau_\nu\|^2_s\big) \\
&\le \frac{\varepsilon}{5\tau^2}\|U^\tau_\nu\|^2_s + C_\varepsilon\big(\|H^\tau_\nu\|^2_s + \|H^\tau_\nu\|^4_s\big).
\end{aligned}
$$

Now we turn to estimate $\dfrac{1}{\tau}\|W^\tau_{\nu,\alpha}\|\|F^\alpha_{3,\nu}\|$ with the help of the Lemma 4.1.1. From (5.2.4) we have :

$$A^\tau_i - A_i(h_{\nu,\tau}, u_{\nu,\tau}) = \big((u^\tau_\nu \cdot e^t_i) - (u_{\nu,\tau} \cdot e^t_i)\big)I_{d+1} + \big((A^\tau_0)^{-1} - A^{-1}_0(h_{\nu,\tau})\big)C_i.$$

Since $C^{11}_i = 0$ as in (5.2.4) and

$$(A^\tau_0)^{-1} - A^{-1}_0(h_{\nu,\tau}) = \text{diag}\,\big(\rho_\nu(h^\tau_\nu) - \rho_\nu(h_{\nu,\tau}), 0\big),$$

it is clear that :

$$\big((A^\tau_0)^{-1} - A^{-1}_0(h_{\nu,\tau})\big)C_i\begin{pmatrix} h_{\nu,\tau} \\ u_{\nu,\tau}\end{pmatrix} = \big(\rho_\nu(h^\tau_\nu) - \rho_\nu(h_{\nu,\tau})\big)O(|u_{\nu,\tau}|).$$

We use the same fashion as that used for (5.5.24), we have :

$$\|\rho_\nu(h_{\nu,\tau}) - \rho_\nu(h^\tau_\nu)\|_{|\alpha|} \le \|H^\tau_\nu\|_{|\alpha|}\big(1 + \|H^\tau_\nu\|^2_s\big). \qquad (5.5.25)$$

Thus, we use the Lemma 4.1.1, the boundedness of $\|(h_{\nu,\tau}, u_{\nu,\tau})\|_{s+1}$ and (5.5.21), (5.5.25) to conclude that :

$$\begin{aligned}\|F^\alpha_{3,\nu}\| &\leq C\sum_{i=1}^d \|\big(h_{\nu,\tau}, u_{\nu,\tau}\big)_{x_i}\|_s \|(u^\tau_\nu - u_{\nu,\tau})\cdot e^t_i\|_{|\alpha|} \\ &\quad + C\sum_{i=1}^d \|\rho_\nu(h^\tau_\nu) - \rho_\nu(h_{\nu,\tau})\|_{|\alpha|}\|\partial_{x_i}(u_{\nu,\tau})\|_s \\ &\leq C\|U^\tau_\nu\|_{|\alpha|} + C\tau(1+\|H^\tau_\nu\|^2_s)\|H^\tau_\nu\|_s.\end{aligned}$$

Thus, obviously we have :

$$\frac{1}{\tau}\|W^\tau_{\nu,\tau}\|\|F^\alpha_{3,\nu}\| \leq \frac{\varepsilon}{5\tau^2}\|U^\tau_\nu\|^2_s + C_\varepsilon\big(\|W^\tau_\nu\|^2_s + \|W^\tau_\nu\|^4_s\big). \tag{5.5.26}$$

Finally, we estimate $\frac{1}{\tau}\|W^\tau_{\nu,\tau}\|\|F^\alpha_{4,\nu}\|$. Since we have :

$$\begin{aligned}F^\alpha_{4,\nu} &= \sum_{i=1}^d \Big((u^\tau_\nu\cdot e^t_i)(\partial_{x_i}W^\tau_\nu)_\alpha - \big((u^\tau_\nu\cdot e^t_i)\partial_{x_i}W^\tau_\nu\big)_\alpha\Big) \\ &\quad + \sum_{i=1}^d \Big((A^\tau_0)^{-1}C_i(\partial_{x_i}W^\tau_\nu)_\alpha - \big((A^\tau_0)^{-1}C_i\partial_{x_i}W^\tau_\nu\big)_\alpha\Big),\end{aligned}$$

due to the Lemma 4.1.1 and (5.5.8), $F^\alpha_{4,\nu}$ can be bounded as :

$$\begin{aligned}\|F^\alpha_{4,\nu}\| &\leq C_s\sum_{i=1}^d \Big(\|\nabla(u^\tau_\nu\cdot e^t_i)\|_\infty\|D^{s-1}\partial_{x_i}W^\tau_\nu\| + \|D^s(u^\tau_\nu\cdot e^t_i)\|\|\partial_{x_i}W^\tau_\nu\|_\infty\Big) \\ &\quad + C_s\sum_{i=1}^d \Big(\|\nabla\big((A^\tau_0)^{-1}C_i\big)\|_\infty\|D^{s-1}\partial_{x_i}U^\tau_\nu\| + \|D^s(u^\tau_\nu\cdot e^t_i)\|\|\partial_{x_i}U^\tau_\nu\|_\infty\Big) \\ &\leq C_s\|u^\tau_\nu\|_s\|W^\tau_\nu\|_s + C_s\|\rho_\nu(h^\tau_\nu)\|_s\|U^\tau_\nu\|_s \\ &\leq C_s\big(\tau + \|U^\tau_\nu\|_s)\|W^\tau_\nu\|_s + C_s(1+\|h^\tau_\nu\|^s_s)\|U^\tau_\nu\|_s \\ &\leq C_s\big(\tau + \|U^\tau_\nu\|_s)\|W^\tau_\nu\|_s + C_s(1+\|H^\tau_\nu\|_s)\|U^\tau_\nu\|_s.\end{aligned}$$

Thus, we have :

$$\frac{1}{\tau}\|W^\tau_{\nu,\alpha}\|\|F^\alpha_{4,\nu}\| \leq \frac{\varepsilon}{5\tau^2}\|U^\tau_\nu\|^2_s + C_\varepsilon\big(\|W^\tau_\nu\|^2_s + \|W^\tau_\nu\|^4_s\big). \tag{5.5.27}$$

This completes the proof. □

Proof of Theorem 5.5.1.

Substituting the estimates in Lemma 5.5.2 into inequality in Lemma 5.5.1 yields :

$$\frac{d}{dt}\int_{\mathbb{T}}\big(W^\tau_{\nu,\alpha}, A_0(h^\tau_\nu)W^\tau_{\nu,\alpha}\big) + \frac{1}{\tau^2}\|U^\tau_{\nu,\alpha}\|^2 \leq C\tau^{2(\lambda+1)} + C\big(\|W^\tau_\nu\|^2_s + \|W^\tau_\nu\|^4_s\big).$$

Integrating this equation over $(0,t)\in(0,T^\tau)\subset(0,T_1)$, noting,

$$C^{-1}\|W^\tau_{\nu,\alpha}\|^2 \le \big(W^\tau_{\nu,\alpha}, A_0(h^\tau_\nu)W^\tau_{\nu,\alpha}\big) \le C\|W^\tau_{\nu,\alpha}\|^2$$

and condition (5.5.4) for the initial data we obtain :

$$\|W^\tau_{\nu,\alpha}\|^2 + \frac{1}{\tau^2}\int_0^t \|U^\tau_{\nu,\alpha}(\xi)\|^2 d\xi \le CT\tau^{2(\lambda+1)} + C\int_0^t \big(\|W^\tau_\nu(\xi)\|_s^2 + \|W^\tau_\nu(\xi)\|_s^4\big)d\xi.$$

Summing up the last inequality over all α satisfying $|\alpha|\le s$, we get :

$$\|W^\tau_\nu\|_s^2 + \frac{1}{\tau^2}\int_0^t \|U^\tau_\nu(\xi)\|_s^2 d\xi \le CT_1\tau^{2(\lambda+1)} + C\int_0^t \big(\|W^\tau_\nu(\xi)\|_s^2 + \|W^\tau_\nu(\xi)\|_s^4\big)d\xi. \tag{5.5.28}$$

For $t\in[0,T^\tau]$, let

$$y(t) = C\int_0^t \big(\|W^\tau_\nu(\xi)\|_s^2 + \|W^\tau_\nu(\xi)\|_s^4\big)d\xi + CT_1\tau^{2(\lambda+1)}.$$

Then it follows from (5.5.28) that :

$$\|W^\tau_\nu(t)\|_s^2 \le y(t), \quad \frac{1}{\tau^2}\int_0^t \|U^\tau_\nu(\xi)\|_s^2 d\xi \le y(t), \quad \forall\, t\in[0,T^\tau] \tag{5.5.29}$$

and

$$y'(t) = C\big(\|W^\tau_\nu(t)\|_s^2 + \|W^\tau_\nu(t)\|_s^4\big) \le C\big(y(t)+y^2(t)\big),$$

with

$$y(0) = CT_1\tau^{2(\lambda+1)}.$$

Applying the nonlinear Gronwall-type inequality in Lemma 4.1.2 to the last inequality yields :

$$y(t) \le CT_1\tau^{2(\lambda+1)}e^{CT_1}, \quad \forall\, t\in[0,T^\tau].$$

Therefore, from (5.5.28) we obtain :

$$\|W^\tau_\nu(t)\|_s \le \sqrt{y(t)} \le C\tau^{\lambda+1}, \qquad \int_0^t \|U^\tau_\nu(\xi)\|_s^2 d\xi \le \tau^2 y(t) \le C\tau^{2(\lambda+2)},$$

for any $t\in[0,T^\tau]$. Thus gives the uniform estimate for (n^τ_ν, u^τ_ν). The uniform estimate for ϕ^τ follows from the Poisson equation $-\Delta\Phi^\tau = N^\tau_i - N^\tau_e$. By Lemma 5.2.1, we obtain $T^\tau_2 \ge T_1$ i.e $T^\tau = T_1$. Then $T^\tau_1 \ge T_1$. This finishes the proof of Theorem 5.5.1. □

Part III

The zero-relaxation limits of Euler-Maxwell systems

Chapter 6

Initial layers and zero-relaxation limits of one-fluid Euler-Maxwell equations

Abstract. In this chapter we consider zero-relaxation limits for periodic smooth solutions of Euler-Maxwell systems. For well-prepared initial data, we propose an approximate solution based on a new asymptotic expansion up to any order. For ill-prepared initial data, we construct initial layer corrections in an explicit way. In both cases, the asymptotic expansions are valid in time intervals independent of the relaxation time and their convergence is justified by establishing uniform energy estimates.

6.1 Introduction

Euler-Maxwell equations appear in the modeling of plasmas under conditions on the frequency collision of particles. One example is the modeling of ionospheric plasmas. For a magnetized plasma composed of electrons and ions, let n_e and u_e (respectively, n_i and u_i) be the density and velocity vector of the electrons (respectively, ions), E and B' be respectively the electric field and magnetic field. Theses variables are functions of a three-dimensional position vector $x \in \mathbb{R}^3$ and of the time $t > 0$. The fields E and B' are coupled to the electron density through the Maxwell equations and act on electrons via the Lorentz force. In this chapter, we consider the periodic case in a torus $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^3$.

In vacuum, variables (n_ν, u_ν, E, B') satisfy a two-fluid Euler-Maxwell equations (see [6, 9, 68]) :

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \\ m_\nu \partial_t(n_\nu u_\nu) + m_\nu \operatorname{div}(n_\nu u_\nu \otimes u_\nu) + \nabla p_\nu(n_\nu) = q_\nu n_\nu(E + u_\nu \times B') - \dfrac{m_\nu n_\nu u_\nu}{\tau_\nu}, \\ \varepsilon_0 \partial_t E - \mu_0^{-1} \nabla \times B' = -(q_e n_e u_e + q_i n_i u_i), \quad \varepsilon_0 \operatorname{div} E = q_e n_e + q_i n_i, \\ \partial_t B' + \nabla \times E = 0, \quad \operatorname{div} B' = 0, \end{cases} \tag{6.1.1}$$

for $\nu = e, i$ and $(t, x) \in (0, \infty) \times \mathbb{T}$, where $\otimes$ stands for the tensor product and $p_\nu = p_\nu(n_\nu)$ is the pressure function which is sufficiently smooth and strictly increasing for $n_\nu > 0$. In (6.1.1) the physical parameters are the charges of the electron $q_e = -q$ and of the ion $q_i = q > 0$, the electron mass $m_e > 0$ and the ion mass $m_i > 0$, the momentum relaxation times $\tau_e > 0$ and $\tau_i > 0$, and the vacuum permittivity $\varepsilon_0 > 0$ and the vacuum permeability $\mu_0 > 0$. Recall that the speed of light c and the Debye length λ' are defined by

$$c = (\varepsilon_0 \mu_0)^{-\frac{1}{2}}, \quad \lambda' = \Big(\frac{\varepsilon_0 K_B T_e}{n_0 q^2}\Big)^{1/2},$$

where $K_B > 0$ is the Boltzmann constant, $T_e > 0$ is the temperature of the electron, and $n_0 > 0$ is the mean density of the plasma ([9], p. 350). Let us define

$$\lambda = \varepsilon_0^{1/2}, \quad \gamma = \frac{1}{\varepsilon_0^{1/2} c}.$$

Then $\lambda > 0$ is a scaled Debye length since it is proportional to λ'. Remark that $\gamma \to 0$ as $c \to \infty$.

Let us introduce a scaling for the magnetic field $B' = \gamma B$. Then the scaled two-fluid Euler-Maxwell equations are written as :

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \\ m_\nu \partial_t(n_\nu u_\nu) + m_\nu \operatorname{div}(n_\nu u_\nu \otimes u_\nu) + \nabla p_\nu(n_\nu) = q_\nu n_\nu(E + \gamma u_\nu \times B) - \dfrac{m_\nu n_\nu u_\nu}{\tau_\nu}, \\ \gamma \lambda^2 \partial_t E - \nabla \times B = -\gamma(q_e n_e u_e + q_i n_i u_i), \quad \lambda^2 \operatorname{div} E = q_e n_e + q_i n_i, \\ \gamma \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0, \quad \nu = e, i. \end{cases} \tag{6.1.2}$$

For smooth solutions with $n_\nu > 0$, the second equation of (6.1.2) is equivalent to

$$m_\nu \partial_t u_\nu + m_\nu (u_\nu \cdot \nabla) u_\nu + \nabla h_\nu(n_\nu) = q_\nu(E + \gamma u_\nu \times B) - \frac{m_\nu u_\nu}{\tau_\nu}, \tag{6.1.3}$$

where $\cdot$ denotes the inner product of $\mathbb{R}^3$ and the enthalpy function h_ν is defined by

$$h_\nu(n_\nu) = \int_1^{n_\nu} \frac{p'_\nu(s)}{s} ds, \quad \nu = e, i. \tag{6.1.4}$$

Since p_ν is sufficiently smooth and strictly increasing on $(0, +\infty)$, so is h_ν.

In the plasma when the ions are non-moving and become a uniform background with a given stationary density, by letting $n_i = b$, $u_i = 0$ and deleting the Euler equations for ions, a one-fluid

Euler-Maxwell model is formally derived. For simplifying the discussion, in the sequel we take $q = 1$. Replacing (n_e, u_e) by (n, u), m_e by m and τ_e by τ, the one-fluid Euler-Maxwell equations read :

$$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ m\partial_t u + m(u \cdot \nabla)u + \nabla h(n) = -E - \gamma u \times B - \dfrac{mu}{\tau}, \\ \gamma\lambda^2 \partial_t E - \nabla \times B = \gamma nu, \quad \lambda^2 \operatorname{div} E = b - n, \\ \gamma \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0, \end{cases} \tag{6.1.5}$$

for $(t, x) \in (0, \infty) \times \mathbb{T}$. It is complemented by periodic initial conditions :

$$t = 0 : \quad (n, u, E, B) = (n_0^\tau, u_0^\tau, E_0^\tau, B_0^\tau). \tag{6.1.6}$$

The given function b depends only on x. This is compatible with system (6.1.5). Indeed, we have

$$\partial_t n = -\operatorname{div}(nu) = -\partial_t(\lambda^2 \operatorname{div} E) = \partial_t n - \partial_t b,$$

which implies that $\partial_t b = 0$. Moreover, since we consider periodic smooth solutions, b is supposed to be sufficiently smooth and periodic.

In (6.1.2) the physical parameters m_ν, τ_ν, γ and λ^2 can be chosen independently of each other according to physical situations. They are very small compared to the physical size of the other quantities. Therefore, it is important to study the limits of system (6.1.2) or (6.1.5) as these parameters go to zero. The formal asymptotic limits of the two-fluid Euler-Maxwell equations (6.1.2) have been investigated in [60]. In the one-fluid Euler-Maxwell equations (6.1.5), the non-relativistic limit $\gamma \to 0$, the quasi-neutral limit $\lambda \to 0$ and the limit of their combination $\gamma = \lambda \to 0$ have been rigorously justified in [61], [63] and [62], respectively. The results show that these limits of (6.1.5) are respectively a compressible Euler-Poisson system, an electron magnetohydrodynamics system and incompressible Euler equations. The justifications are valid for smooth periodic solutions in time intervals independent of the parameters γ and λ. We mention also that in the two-fluid Euler-Maxwell equations the non-relativistic limit can be justified in a similar way [73], however, the justifications of the quasi-neutral limit and the combined limit $\gamma = \lambda \to 0$ are still open problems. For the kinetic version of the above limits in Vlasov-Maxwell equations, we refer to [8] for the combined limit and to [7] for the non-relativistic limit.

Another interesting problems rely on the limit of the mass ratio between electrons and ions in system (6.1.2). Since the electron mass is much smaller than the ion mass, for fixing the idea we let $m_i = 1$. Then we may consider the zero-electron mass limit $m_e \to 0$ and the combined limit $m_e \to 0$ with $\tau_e,\ \tau_i \to 0$ in (6.1.2). The formal equations of limits can be easily derived (see Appendix), however, the mathematical justification of these limits is a quite open problem. We leave these problems for a future investigation. On this topic, we refer to [2] for a rigorous justification of the electron mass limit in Euler-Poisson equations.

In what follows, we only consider the one-fluid Euler-Maxwell system (6.1.5), which is symmetrizable hyperbolic in the sense of Friedrichs [18]. Its local existence of smooth solutions is a well-known result due to Kato [38]. The global existence and the long-time stability of smooth solutions have been recently obtained in [64] when the solutions are close to a constant equilibrium. In a simplified one dimensional Euler-Maxwell system, the global existence of entropy solutions has been studied in [11] by the compensated compactness method.

In this chapter, we are interested in the zero-relaxation limit $\tau \to 0$ of system (6.1.5) under the conditions $m = O(1)$, $\gamma = O(1)$ and $\lambda = O(1)$. We assume, throughout this chapter, that $m = \gamma = \lambda = 1$. The usual time scaling for studying the limit $\tau \to 0$ is $t' = \tau t$. Since $t = 0$ if and only if $t' = 0$, this change of scaling does not affect the initial condition (6.1.6). Rewriting still t' by t, system (6.1.5) becomes (see [60, 64])

$$\begin{cases} \partial_t n + \dfrac{1}{\tau}\operatorname{div}(nu) = 0, \\ \partial_t u + \dfrac{1}{\tau}(u\cdot\nabla)u + \dfrac{1}{\tau}\nabla h(n) = -\dfrac{E}{\tau} - \dfrac{u\times B}{\tau} - \dfrac{u}{\tau^2}, \\ \partial_t E - \dfrac{1}{\tau}\nabla\times B = \dfrac{nu}{\tau}, \quad \operatorname{div} E = b - n, \\ \partial_t B + \dfrac{1}{\tau}\nabla\times E = 0, \quad \operatorname{div} B = 0. \end{cases} \tag{6.1.7}$$

Remark that the time scaling $t' = \tau t$ may reveal the long-time asymptotic behavior of solutions. Indeed, $t = t'\tau^{-1} = O(\tau^{-1})$ for fixed $t' > 0$. Then for a fixed time $T_1 > 0$, a local-in-time convergence for system (6.1.7) on the time interval $[0, T_1]$ means the convergence for system (6.1.5) on a long-time interval $[0, T_1\tau^{-1}]$. On the other hand, as $\tau \to 0$, a convergence error $O(\tau^r)$ with $r > 0$ implies a rate $O(t^{-r})$ of the long-time asymptotics (see (6.4.8) in Remark 6.4.2).

For $m \geq 1$, the authors of [64] proposed an asymptotic expansion to (6.1.7) of the form :

$$(n_\tau^m, u_\tau^m, E_\tau^m, B_\tau^m) = \sum_{j=0}^{m} \tau^j (n^j, \tau u^j, E^j, B^j).$$

They established the convergence in Sobolev spaces of the solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ of (6.1.7) to $(n_\tau^m, u_\tau^m, E_\tau^m, B_\tau^m)$ with order $O(\tau^m)$ when the initial data are well-prepared and the initial error has the same order. Here the well-prepared initial data mean that compatibility conditions hold. Unfortunately, this result cannot deal with the case of ill-prepared data and the case $m = 0$ of the well-prepared initial data in which the error disappears. The goal of this chapter is to improve the above result in two directions.

First, we propose a different asymptotic expansion to (6.1.7) of the form :

$$(n_\tau^m, u_\tau^m, E_\tau^m, B_\tau^m) = \sum_{j=0}^{m} \tau^{2j} (n^j, \tau u^j, E^j, \tau B^j), \tag{6.1.8}$$

where the first order profile (n^0, u^0, E^0) satisfies a drift-diffusion system, as shown in [64]. The motivation of this expansion is the following consideration. If we replace u by τu and B by τB, then system (6.1.7) becomes

$$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ \tau^2\big(\partial_t u + (u\cdot\nabla)u\big) + \nabla h(n) = -E - \tau^2 u\times B - u, \\ \partial_t E - \nabla\times B = nu, \quad \operatorname{div} E = b - n, \\ \tau^2\partial_t B - \nabla\times E = 0, \quad \operatorname{div} B = 0, \end{cases}$$

in which the only small parameter is τ^2. With expansion (6.1.8), for $m \geq 0$ we prove the convergence of the solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ of (6.1.7) to $(n_\tau^m, u_\tau^m, E_\tau^m, B_\tau^m)$ with a higher order $O(\tau^{2(m+1)})$ when the initial data are well-prepared and the initial error has the same order. This includes the case $m = 0$. In the proof of the result, we have to treat the order of the remainder $R_B^{\tau;m}$ for variable B. Indeed, there is a loss of one order for $R_B^{\tau;m}$ in comparison with those for variables n, u and E. This is overcome by introducing a correction term into E_τ^m so that the new remainder for B becomes zero without changing the order of the other remainders.

Second, for ill-prepared initial data, the above convergence result is not valid because the approximate solution cannot satisfy the prescribed initial conditions. In this case, we construct initial layer corrections with exponential decay to zero and prove the convergence of the first order asymptotic expansion. The analysis shows that there are no first order initial layers on variables n, E and B. However, we have to consider the second order initial layer corrections to obtain the desired order of remainders.

The zero-relaxation limit $\tau \to 0$ in the Euler-Poisson system was extensively studied by many authors. See [51, 36, 37, 34, 45, 44, 3, 74, 27] and the references therein. Remark that the Euler-Maxwell system and the Euler-Poisson system are essentially different due to the coupling terms and to the difference between Poisson equation and Maxwell equations. Finally, assuming $\tau_e = \tau_i = \tau$, so that the change of scaling of time is possible, the zero-relaxation limit $\tau \to 0$ in the two-fluid Euler-Maxwell system can be carried out in a similar way. Indeed, here the essential point in the proof is that the equations for u_ν are dissipative. Then we may treat the energy estimates of the Euler equations for both $\nu = e, i$ in a similar way to the one-fluid case. In Appendix, we give a formal description of the zero-relaxation limit in the two-fluid Euler-Maxwell equations and write down the limit equations. For avoiding the tedious calculations, the rigorous justification of the limit is omitted.

Lemma 6.1.1 *(See [61]) Let $s \geq 0$ be an integer and $f \in H^s(\mathbb{T})$ and $g \in H^s(\mathbb{T})$. Then problem*

$$\nabla \times B = f, \quad \operatorname{div} B = g, \quad \operatorname{div} f = 0, \quad m(g) = 0 \tag{6.1.9}$$

has a unique solution $B \in H^{s+1}(\mathbb{T})$ in the class $m(B) = 0$, where

$$m(B) = \int_{\mathbb{T}} B dx.$$

Proposition 6.1.1 *(Local existence of smooth solutions, see [38, 50]) Let $s \geq 3$ be an integer and $(n_0^\tau, u_0^\tau, E_0^\tau, B_0^\tau) \in H^s(\mathbb{T})$ with $n_0^\tau \geq \kappa$ for some given constant $\kappa > 0$, independent of τ. Then there exist $T_1^\tau > 0$ and a unique smooth solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ to the periodic problem (6.1.5)-(6.1.6) defined in the time interval $[0, T_1^\tau)$, with $(n^\tau, u^\tau, E^\tau, B^\tau) \in C^1\big([0, T_1^\tau]; H^{s-1}(\mathbb{T})\big) \cap C\big([0, T_1^\tau]; H^s(\mathbb{T})\big)$.*

This chapter is organized as follows. In the next section, we derive asymptotic expansions of solutions and state the convergence result to problem (6.1.6)-(6.1.7) in the case of well-prepared initial data. In particular, we add a correction term to derive desired error estimates. In section 3 we consider the asymptotic expansions in the case of ill-prepared initial data by constructing initial layer corrections which exponentially decay to zero. The justification of both the two asymptotic expansions is given in the last section. For this purpose, we prove a more general convergence theorem which implies the convergence of both expansions. Finally, in Appendix, we consider the formal derivation of the combined zero-electron mass and zero-relaxation limits.

6.2 Case of well-prepared initial data

6.2.1 Formal asymptotic expansions

In this section we consider the zero-relaxation limit $\tau \to 0$ in problem (6.1.6)-(6.1.7) with well-prepared initial data. Based on the discussion on the asymptotic expansion, we make the following ansatz for both the approximate solution and its initial data :

$$(n_\tau, u_\tau, E_\tau, B_\tau)(0,x) = \sum_{j\geq 0} \tau^{2j}(n_j, \tau u_j, E_j, \tau B_j)(x), \quad x \in \mathbb{T}, \tag{6.2.1}$$

$$(n_\tau, u_\tau, E_\tau, B_\tau)(t,x) = \sum_{j\geq 0} \tau^{2j}(n^j, \tau u^j, E^j, \tau B^j)(t,x), \quad t>0, \ \ x \in \mathbb{T}, \tag{6.2.2}$$

where $(n_j, u_j, E_j, B_j)_{j\geq 0}$ are given sufficiently smooth data with $n_0 \geq$ constant > 0 in $\mathbb{T}$. The validity of expansions (6.2.1)-(6.2.2) is discussed in section 4 (see Theorem 6.4.1).

Now let us determine the profiles (n^j, u^j, E^j, B^j) for all $j \geq 0$. Putting expression (6.2.2) into system (6.1.7) and identifying the coefficients in powers of τ, we see that $(n^j, u^j, E^j, B^j)_{j\geq 0}$ are solutions of the following systems :

$$\begin{cases} \partial_t n^0 + \operatorname{div}(n^0 u^0) = 0, \\ \nabla h(n^0) = -(E^0 + u^0), \\ \nabla \times E^0 = 0, \ \ \operatorname{div} E^0 = b - n^0, \\ \nabla \times B^0 = \partial_t E^0 - n^0 u^0, \ \ \operatorname{div} B^0 = 0, \end{cases} \tag{6.2.3}$$

and for $j \geq 1$,

$$\begin{cases} \partial_t n^j + \displaystyle\sum_{k=0}^{j} \operatorname{div}\left(n^k u^{j-k}\right) = 0, \\ \partial_t u^{j-1} + \displaystyle\sum_{k=0}^{j-1} (u^k \cdot \nabla) u^{j-1-k} + \nabla\big(h'(n^0) n^j + h^{j-1}\big((n^k)_{k\leq j-1}\big)\big) \\ \quad = -E^j - \displaystyle\sum_{k=0}^{j-1} u^k \times B^{j-1-k} - u^j, \\ \nabla \times E^j = -\partial_t B^{j-1}, \ \ \operatorname{div} E^j = -n^j, \\ \nabla \times B^j = \partial_t E^j - \displaystyle\sum_{k=0}^{j} n^k u^{j-k}, \ \ \operatorname{div} B^j = 0, \end{cases} \tag{6.2.4}$$

where $h^0 = 0$ and h^{j-1} is a function depending only on $(n^k)_{0\leq k\leq j-1}$ and is defined for $j \geq 2$ by

$$h\Big(\sum_{j\geq 0} \tau^{2j} n^j\Big) = \sum_{j\geq 0} c_j \tau^{2j},$$

with

$$c_0 = h(n^0), \quad c_1 = h'(n^0)n^1, \quad c_j = h'(n^0)n^j + h^{j-1}\big((n^k)_{k\leq j-1}\big), \quad \forall\, j \geq 2.$$

In (6.2.3), equation $\nabla \times E^0 = 0$ implies the existence of a potential ϕ^0 such that $E^0 = -\nabla\phi^0$. Then (n^0, ϕ^0) solves a classical system of drift-diffusion equations :

$$\begin{cases} \partial_t n^0 - \operatorname{div}\,\big(n^0\nabla\big(h(n^0) - \phi^0\big)\big) = 0, \\ -\Delta\phi^0 = b - n^0, \end{cases} \quad t > 0, \quad x \in \mathbb{T} \tag{6.2.5}$$

with the initial condition :

$$n^0(0, x) = n_0, \quad x \in \mathbb{T}. \tag{6.2.6}$$

The existence of smooth solutions to problem (6.2.5)-(6.2.6) can be easily established, at least locally in time. The solution ϕ^0 is unique in the class $m(\phi^0) = 0$. See for instance [54]. Then (u^0, E^0) are given by

$$u^0 = -\nabla\big(h(n^0) - \phi^0\big), \;\; E^0 = -\nabla\phi^0. \tag{6.2.7}$$

Since (n^0, u^0, E^0) are known, B^0 solves the linear system of curl-div equations of type (6.1.9) in the class $m(B^0) = 0$. More precisely, using $\nabla \times E^0 = 0$ and formula

$$\nabla \times \nabla \times B^0 = \nabla \operatorname{div} B^0 - \Delta B^0,$$

we obtain

$$\Delta B^0 = \nabla \times (n^0 u^0) \quad \text{in } \mathbb{T} \quad \text{and} \quad m(B_0) = 0.$$

From (6.2.7) and the fourth equation of system (6.2.3) we get the first order compatibility conditions :

$$u_0 = -\nabla\big(h(n_0) - \phi_0\big), \;\; E_0 = -\nabla\phi_0, \;\; B_0 = B^0(0, .), \tag{6.2.8}$$

where ϕ_0 is determined by

$$-\Delta\phi_0 = b - n_0 \quad \text{in } \mathbb{T} \quad \text{and} \quad m(\phi_0) = 0. \tag{6.2.9}$$

For $j \geq 1$, the profiles (n^j, u^j, E^j, B^j) are obtained by induction in j. Assume that $\big(n^k, u^k, E^k, B^k\big)_{0\leq k\leq j-1}$ are smooth and have already been determined in previous steps. Equations for B^j are of curl-div type and determine a unique smooth B^j in the class $m(B^j) = 0$. Moreover, from $\operatorname{div} B^j = 0$, we deduce the existence of a given vector ψ^j such that $B^j = -\nabla\times\psi^j$. Then, equation $\nabla\times E^j = -\partial_t B^{j-1}$ in (6.2.4) becomes $\nabla\times(E^j - \partial_t\psi^{j-1}) = 0$. It follows that there is a potential function ϕ^j such that

$$E^j = \partial_t\psi^{j-1} - \nabla\phi^j. \tag{6.2.10}$$

From (6.2.4) we also get

$$\begin{aligned} u^j &= \nabla\big(\phi^j - h'(n^0)n^j - h^{j-1}((n^k)_{k\le j-1})\big) \\ &\quad -\Big(\partial_t u^{j-1} + \partial_t \psi^{j-1} + \sum_{k=0}^{j-1}\big((u^k\cdot\nabla)u^{j-1-k} + u^k\times B^{j-1-k}\big)\Big). \end{aligned} \tag{6.2.11}$$

Therefore, in the class $m(\phi^j)=0$, (n^j,ϕ^j) solves a linearized system of drift-diffusion equations :

$$\begin{cases} \partial_t n^j - \mathrm{div}\,\big(n^0\nabla(h'(n^0)n^j - \phi^j)\big) + \mathrm{div}\,\big(n^j u^0\big) \\ \;= f^j\big((V^k,\partial_t V^k,\partial_x V^k,\partial_t\partial_x V^k,\partial_x^2 V^k)_{0\le k\le j-1}\big) + \mathrm{div}(n^0\partial_t\psi^{j-1}), \quad t>0,\ x\in\mathbb{T} \\ \Delta\phi^j = n^j + \partial_t(\mathrm{div}\,\psi^{j-1}), \end{cases} \tag{6.2.12}$$

with the initial condition :

$$n^j(0,x) = n_j(x), \quad x\in\mathbb{T}, \tag{6.2.13}$$

where f^j is a given smooth function and $V^k=(n^k,u^k,\psi^k)$. Problem (6.2.12)-(6.2.13) is linear. It admits a unique global smooth solution. Then (u^j,E^j) are given by (6.2.10)-(6.2.11). Thus, we get the high-order compatibility conditions for $j\ge 1$:

$$\begin{aligned} u_j &= \nabla\big(\phi_j - h'(n_0)n_j - h^{j-1}((n_k)_{k\le j-1})\big) \\ &\quad -\Big(\partial_t u^{j-1}\big|_{t=0} + \psi^{j-1}\big|_{t=0} + \sum_{k=0}^{j-1}\big(\,(u_k\cdot\nabla)u_{j-1-k} + u_k\times B^{j-1-k}(0,.)\,\big)\Big), \end{aligned} \tag{6.2.14}$$

$$E_j = \partial_t\psi^{j-1}(0,.) - \nabla\phi_j, \quad B_j = B^j(0,.), \tag{6.2.15}$$

where ϕ_j is determined by

$$\Delta\phi_j = n_j + \partial_t(\mathrm{div}\,\psi^{j-1})|_{t=0} \quad \text{in } \mathbb{T} \quad \text{and} \quad m(\phi_j)=0. \tag{6.2.16}$$

We conclude the above discussion with the following result.

Proposition 6.2.1 *Let $s\ge 3$ be an integer. Assume $(n_j,u_j,E_j,B_j)\in H^{s+1}(\mathbb{T})$ for $j\ge 0$, with $n_0\ge$ constant >0 in $\mathbb{T}$, and satisfy the compatibility conditions (6.2.8)-(6.2.9), and (6.2.14)-(6.2.16) for $j\ge 1$. Then there exists a unique asymptotic expansion up to any order of the form (6.2.2), i.e. there exist $T_1>0$ and a unique smooth solution $(n^j,u^j,E^j,B^j)_{j\ge 0}$ in the time interval $[0,T_1]$ of problems (6.2.5)-(6.2.7) and (6.2.10)-(6.2.13) for $j\ge 1$. Moreover, $n^0\ge$ constant >0 in $[0,T_1]\times\mathbb{T}$ and*

$$(n^j,u^j,E^j,B^j)\in C^1(0,T_1;H^s(\mathbb{T}))\cap C(0,T_1;H^{s+1}(\mathbb{T})), \quad \forall\, j\ge 0.$$

In particular, the formal zero-relaxation limit $\tau\to 0$ of the Euler-Maxwell system (6.1.7) is the classical drift-diffusion system (6.2.5) and (6.2.7).

6.2.2 Convergence results

Let $m \geq 0$ be a fixed integer. We denote by

$$(n_\tau^m, u_\tau^m, \tilde{E}_\tau^m, B_\tau^m) = \sum_{j=0}^{m} \tau^{2j}(n^j, \tau u^j, E^j, \tau B^j), \tag{6.2.17}$$

an approximate solution of order m, where $(n^j, u^j, E^j, B^j)_{0\leq j\leq m}$ are constructed in the previous subsection. From the construction of the approximate solution, for $(t,x) \in [0, T_1] \times \mathbb{T}$ we have

$$\operatorname{div} \tilde{E}_\tau^m = b - n_\tau^m, \ \ \operatorname{div} B_\tau^m = 0. \tag{6.2.18}$$

We define the remainders $R_n^{\tau,m}$, $R_u^{\tau,m}$ and $R_E^{\tau,m}$ and $\tilde{R}_B^{\tau,m}$ by

$$\begin{cases} \partial_t n_\tau^m + \dfrac{1}{\tau}\operatorname{div}(n_\tau^m u_\tau^m) = R_n^{\tau,m}, \\ \partial_t u_\tau^m + \dfrac{1}{\tau}(u_\tau^m \cdot \nabla)u_\tau^m + \dfrac{1}{\tau}\nabla h(n_\tau^m) = -\dfrac{\tilde{E}_\tau^m}{\tau} - \dfrac{u_\tau^m}{\tau^2} - \dfrac{u_\tau^m \times B_\tau^m}{\tau} + R_u^{\tau,m}, \\ \partial_t \tilde{E}_\tau^m - \dfrac{1}{\tau}\nabla \times B_\tau^m = \dfrac{n_\tau^m u_\tau^m}{\tau} + R_E^{\tau,m}, \\ \partial_t B_\tau^m + \dfrac{1}{\tau}\nabla \times \tilde{E}_\tau^m = \tilde{R}_B^{\tau,m}. \end{cases} \tag{6.2.19}$$

It is clear that the convergence rate depends strongly on the order of the remainders with respect to τ. Since the last equation in (6.2.19) is linear, for sufficiently smooth profiles $(n^j, u^j, E^j, B^j)_{j\geq 0}$, it is easy to see that

$$\tilde{R}_B^{\tau,m} = \tau^{2m+1}\partial_t B^m. \tag{6.2.20}$$

Moreover, a further computation gives

$$R_n^{\tau,m} = O\big(\tau^{2(m+1)}\big), \quad R_E^{\tau,m} = O\big(\tau^{2(m+1)}\big), \quad R_u^{\tau,m} = O\big(\tau^{2m+1}\big). \tag{6.2.21}$$

In (6.2.20)-(6.2.21), there is a loss of one order for the remainders $R_u^{\tau,m}$ and $\tilde{R}_B^{\tau,m}$. For $R_u^{\tau,m}$ this loss will be recovered in the error estimate of convergence due to the dissipation term for u. However, the situation is different for $\tilde{R}_B^{\tau,m}$ since the equation for B is not dissipative. A simple way to remedy this is to introduce a correction term into $\tilde{E}_\tau^m$ so that

$$E_\tau^m = \tilde{E}_\tau^m + \tau^{2(m+1)}E_c^{m+1} = \sum_{j=0}^{m} \tau^{2j}E^j + \tau^{2(m+1)}E_c^{m+1}. \tag{6.2.22}$$

In view of (6.2.18)-(6.2.20), E_c^{m+1} should be defined by

$$\nabla \times E_c^{m+1} = -\partial_t B^m, \quad \operatorname{div} E_c^{m+1} = 0, \quad m(E_c^{m+1}) = 0 \tag{6.2.23}$$

so that the new remainder $R_B^{\tau,m}$ of B satisfies

$$R_B^{\tau,m} \overset{def}{=} \partial_t B_\tau^m + \frac{1}{\tau}\nabla \times E_\tau^m = 0 \tag{6.2.24}$$

and we still have
$$\operatorname{div} E_\tau^m = b - n_\tau^m, \quad \operatorname{div} B_\tau^m = 0. \tag{6.2.25}$$
Since the correction term is of order $O\big(\tau^{2(m+1)}\big)$, the orders of the remainders $R_n^{\tau,m}$, $R_u^{\tau,m}$ and $R_E^{\tau,m}$ are not changed. Moreover, the correction term does not affect assumption (6.2.28) below.

We conclude the above discussion with the following result.

Proposition 6.2.2 *Let the assumptions of Proposition 6.2.1 hold. For all integers $m \geq 0$ and $s \geq 3$ the remainders $R_n^{\tau,m}$, $R_u^{\tau,m}$, $R_E^{\tau,m}$ and $R_B^{\tau,m}$ satisfy (6.2.24) and*
$$\sup_{0\le t\le T_1} \|(R_n^{\tau,m}, R_E^{\tau,m})(t,\cdot)\|_s \le C_m \tau^{2(m+1)}, \quad \sup_{0\le t\le T_1} \|R_u^{\tau,m}(t,\cdot)\|_s \le C_m \tau^{2m+1}, \tag{6.2.26}$$
where $C_m > 0$ is a constant independent of τ.

The main result of this section is stated as follows :

Theorem 6.2.1 *Let $m \geq 0$ and $s \geq 3$ be any fixed integers. Let the assumption of Proposition 6.2.1 hold. Suppose*
$$\operatorname{div} E_0^\tau = b - n_0^\tau, \quad \operatorname{div} B_0^\tau = 0 \quad in \quad \mathbb{T} \tag{6.2.27}$$
and
$$\Big\|(n_0^\tau, u_0^\tau, E_0^\tau, B_0^\tau) - \sum_{j=0}^m \tau^{2j}(n_j, \tau u_j, E_j, \tau B_j)\Big\|_s \le C_1 \tau^{2(m+1)}, \tag{6.2.28}$$
where $C_1 > 0$ is a constant independent of τ. Then there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \geq T_1$ and the solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ to the periodic problem (6.1.6)-(6.1.7) satisfies
$$\big\|(n^\tau, u^\tau, E^\tau, B^\tau)(t) - (n_\tau^m, u_\tau^m, E_\tau^m, B_\tau^m)(t)\big\|_s \le C_2 \tau^{2(m+1)}, \quad \forall\, t \in [0, T_1].$$
Moreover,
$$\big\|u^\tau - u_\tau^m\big\|_{L^2(0,T_1;H^s(\mathbb{T}))} \le C_2 \tau^{2m+3}.$$

6.3 Case of ill-prepared initial data

6.3.1 Initial layer corrections

In Theorem 6.2.1, compatibility conditions are made on the initial data. These conditions means that the initial profiles $(u^j, E^j, B^j)(0,\cdot)$ are determined through the resolution of the problems (6.2.3)-(6.2.4) for (n^j, u^j, E^j, B^j). Then $(u_0^\tau, E_0^\tau, B_0^\tau)$ cannot be given explicitly. If these conditions are not satisfied, the phenomenon of initial layers occurs. In this section, we consider this situation for so called ill-prepared initial data. We seek a simplest possible form of an asymptotic expansion with initial layer corrections such that its remainders are at least of order $O(\tau)$ for variable u.

Let the initial data of an approximate solution $(n_\tau, u_\tau, E_\tau, B_\tau)$ have an asymptotic expansion of the form :

$$(n_\tau, u_\tau, E_\tau, B_\tau)\big|_{t=0} = \big(n_0, \tau u_0, E_0, \tau B_0\big) + O(\tau^2), \tag{6.3.1}$$

where (n_0, u_0, E_0, B_0) are given smooth functions. Taking into account the expansion in the case of well-prepared initial data, the simplest form of an asymptotic expansion including initial layer corrections is

$$\begin{aligned}(n_\tau, u_\tau, E_\tau, B_\tau)(t,x) \;&=\; (n^0, \tau u^0, E^0 + \tau^2 E_c^1, \tau B^0)(t,x) \\ &\quad + \big((n_I^0, \tau u_I^0, E_I^0, \tau B_I^0) + \tau^2(n_I^1, \tau u_I^1, E_I^1, \tau B_I^1)\big)(z,x) + O(\tau^2),\end{aligned} \tag{6.3.2}$$

where $z = t/\tau^2 \in \mathbb{R}$ is the fast variable, the subscript I stands for the initial layer variables and E_c^1 is the correction term defined by (6.2.23) with $m = 0$. As we will see below, this expansion is enough to give the remainders at least of order $O(\tau)$ for variable u, which is the case of well-prepared initial data for $m = 0$.

Obviously, (n^0, u^0, E^0, B^0) still satisfies the drift-diffusion system (6.2.3). It remains to determine the initial-layer profiles $(n_I^0, u_I^0, E_I^0, B_I^0)$ and $(n_I^1, u_I^1, E_I^1, B_I^1)$. Putting expression (6.3.2) into system (6.1.7) and using (6.2.3), we obtain

$$\partial_z n_I^0 = 0, \quad \partial_z E_I^0 = 0, \quad \partial_z B_I^0 + \nabla \times E_I^0 = 0 \tag{6.3.3}$$

and

$$\partial_z u_I^0 + u_I^0 = 0. \tag{6.3.4}$$

Equations (6.3.3) imply that there are no first order initial layers for variables n, E and B. Therefore, up to a constant for variable B, we may take

$$n^0(0,x) = n_0(x), \;\; E^0(0,x) = E_0(x) \;\text{ and } B^0(0,x) = B_0(x). \tag{6.3.5}$$

Moreover, expressions (6.3.1) and (6.3.2) for u imply that

$$u^0(0,x) + u_I^0(0,x) = u_0(x), \tag{6.3.6}$$

which determines the initial value of u_I^0, where $u^0(0,\cdot)$ is given by (6.2.8)-(6.2.9). Together with (6.3.4), we obtain

$$u_I^0(z,x) = u_I^0(0,x)e^{-z} = \big(u_0(x) - u^0(0,x)\big)e^{-z}. \tag{6.3.7}$$

Similarly, the second order initial layers n_I^1 and E_I^1 satisfy

$$u_I^1 = 0, \tag{6.3.8}$$

$$\partial_z n_I^1(z,x) + \mathrm{div}\,\big(n^0(0,x)u_I^0(z,x)\big) = 0, \tag{6.3.9}$$

$$\partial_z E_I^1(z,x) = n^0(0,x)u_I^0(z,x) \tag{6.3.10}$$

and

$$\partial_z B_I^1(z,x) + \nabla \times E_I^1(z,x) = 0. \tag{6.3.11}$$

Let (n_1, E_1, B_1) be smooth functions such that

$$E_1(x) = -n^0(0,x)(u_0(x) - u^0(0,x)) \tag{6.3.12}$$

and

$$n_1 = \operatorname{div} E_1, \quad \operatorname{div} B_1 = 0. \tag{6.3.13}$$

Set

$$(n_I^1, E_I^1, B_I^1)(0,x) = (n_1, E_1, B_1)(x).$$

Together with (6.3.7) and (6.3.9)-(6.3.12), it is easy to obtain

$$n_I^1(z,x) = n_1(x) - \operatorname{div}\big(n^0(0,x)(u_0(x) - u^0(0,x))\big)\big(1 - e^{-z}\big), \tag{6.3.14}$$

$$E_I^1(z,x) = -n^0(0,x)\big(u_0(x) - u^0(0,x)\big)e^{-z} \tag{6.3.15}$$

and

$$B_I^1(z,x) = B_1(x) + \nabla \times \big[n^0(0,x)(u_0(x) - u^0(0,x))\big](1 - e^{-z}). \tag{6.3.16}$$

Finally, from (6.3.13) we have

$$\operatorname{div} E_I^1 + n_I^1 = 0, \quad \operatorname{div} B_I^1 = 0. \tag{6.3.17}$$

Thus, the initial layer profiles $(n_I^0, u_I^0, E_I^0, B_I^0)$ and $(n_I^1, u_I^1, E_I^1, B_I^1)$ are completely determined by (6.3.3), (6.3.7)-(6.3.8) and (6.3.14)-(6.3.16). They are smooth functions of (z,x) and bounded with respect to z.

6.3.2 Convergence results

According to the asymptotic expansions above, set

$$\begin{cases} n_{\tau,I}(t,x) = n^0(t,x) + \tau^2 n_I^1(t/\tau^2, x), \\ u_{\tau,I}(t,x) = \tau\big(u^0(t,x) + u_I^0(t/\tau^2, x)\big), \\ E_{\tau,I}(t,x) = E^0(t,x) + \tau^2\big(E_c^1(t,x) + E_I^1(t/\tau^2, x)\big), \\ B_{\tau,I}(t,x) = \tau\big(B^0(t,x) + \tau^2 B_I^1(t/\tau^2, x)\big). \end{cases} \tag{6.3.18}$$

Then we have

$$t = 0 \ : \ \big(n_{\tau,I}, u_{\tau,I}, E_{\tau,I}, B_{\tau,I}\big) = \big(n_0, \tau u_0, E_0, \tau B_0\big) + \tau^2\big(n_1, 0, E_1 + E_c^1(0,\cdot), \tau B_1\big). \tag{6.3.19}$$

Moreover, equations (6.2.3), (6.2.25) and (6.3.17) imply that

$$\operatorname{div} E_{\tau,I} = b - n_{\tau,I}, \quad \operatorname{div} B_{\tau,I} = 0. \tag{6.3.20}$$

Define the remainders $R_n^{\tau,I}$, $R_u^{\tau,I}$, $R_E^{\tau,I}$ and $R_B^{\tau,I}$ by

$$\begin{cases} \partial_t n_{\tau,I} + \dfrac{1}{\tau}\operatorname{div}(n_{\tau,I}u_{\tau,I}) = R_n^{\tau,I}, \\ \partial_t u_{\tau,I} + \dfrac{1}{\tau}(u_{\tau,I}\cdot\nabla)u_{\tau,I} + \dfrac{1}{\tau}\nabla h(n_{\tau,I}) = -\dfrac{E_{\tau,I}}{\tau} - \dfrac{u_{\tau,I}}{\tau^2} - \dfrac{u_{\tau,I}\times B_{\tau,I}}{\tau} + R_u^{\tau,I}, \\ \partial_t E_{\tau,I} - \dfrac{1}{\tau}\nabla\times B_{\tau,I} = \dfrac{n_{\tau,I}u_{\tau,I}}{\tau} + R_E^{\tau,I}, \\ \partial_t B_{\tau,I} + \dfrac{1}{\tau}\nabla\times E_{\tau,I} = R_B^{\tau,I}. \end{cases} \tag{6.3.21}$$

Using equations (6.2.3), (6.2.23) for $(n^0, u^0, E^0, B^0, E_c^1)$ and (6.3.4), (6.3.9)-(6.3.11) for $(u_I^0, n_I^1, E_I^1, B_I^1)$, we obtain

$$\begin{aligned} R_n^{\tau,I} &= \partial_t\big(n^0+\tau^2 n_I^1\big) + \operatorname{div}\big((n^0+\tau^2 n_I^1)(u^0+u_I^0)\big) \\ &= \partial_z n_I^1 + \operatorname{div}(n^0 u_I^0) + \tau^2\operatorname{div}\big(n_I^1(u^0+u_I^0)\big) \\ &= \operatorname{div}\big((n^0(t,x)-n^0(0,x))u_I^0(z,x)\big) + \tau^2\operatorname{div}\big(n_I^1(u^0+u_I^0)\big), \end{aligned}$$

$$\begin{aligned} R_u^{\tau,I} &= \tau\big(\partial_t(u^0+u_I^0) + (u^0+u_I^0)\cdot\nabla(u^0+u_I^0) + (u^0+u_I^0)\times(B^0+\tau^2 B_I^1)\big) \\ &\quad + \frac{1}{\tau}\big(\nabla h(n^0+\tau^2 n_I^1) + (E^0+\tau^2 E_c^1+\tau^2 E_I^1) + (u^0+u_I^0)\big) \\ &= \frac{1}{\tau}\big(\nabla h(n^0)+E^0+u^0\big) + \frac{1}{\tau}(\partial_z u_I^0 + u_I^0) + \frac{1}{\tau}\nabla\big(h(n^0+\tau^2 n_I^1) - h(n^0)\big) \\ &\quad + \tau\big(\partial_t u^0 + (u^0+u_I^0)\cdot\nabla(u^0+u_I^0) + (u^0+u_I^0)\times B^0\big) \\ &\quad + \tau(E_I^1+E_c^1) + \tau^3(n^0+n_I^1)\times B_I^1 \\ &= \tau\big(\partial_t u^0 + (u^0+u_I^0)\cdot\nabla(u^0+u_I^0) + (u^0+u_I^0)\times B^0\big) \\ &\quad + \tau(E_I^1+E_c^1) + \frac{1}{\tau}\nabla\big(h(n^0+\tau^2 n_I^1) - h(n^0)\big) + \tau^3(n^0+n_I^1)\times B_I^1, \end{aligned}$$

$$\begin{aligned} R_E^{\tau,I} &= \partial_t(E^0+\tau^2 E_c^1+\tau^2 E_I^1) - \nabla\times(B^0+\tau^2 B_I^1) - (n^0+\tau^2 n_I^1)(u^0+u_I^0) \\ &= \big(\partial_t E^0 - \nabla\times B^0 - n^0 u^0\big) + \partial_z E_I^1 - n^0 u_I^0 \\ &\quad + \tau^2\big(n_I^1(u^0+u_I^0) + \partial_t E_c^1 - \nabla\times B_I^1\big) \\ &= \big(n^0(0,x) - n^0(t,x)\big)u_I^0(z,x) + \tau^2\big(n_I^1(u^0+u_I^0) + \partial_t E_c^1 - \nabla\times B_I^1\big) \end{aligned}$$

and

$$\begin{aligned}
R_B^{\tau,I} &= \tau\partial_t(B^0+\tau^2 B_I^1)+\frac{1}{\tau}\nabla\times\big(E^0+\tau^2E_c^1+\tau^2E_I^1\big)\\
&= \frac{1}{\tau}\nabla\times E^0+\tau\big(\partial_t B^0+\nabla\times E_c^1\big)+\tau\big(\partial_z B_I^1+\nabla\times E_I^1\big)\\
&= 0.
\end{aligned}$$

Now we establish error estimates for $(R_n^{\tau,I}, R_u^{\tau,I}, R_E^{\tau,I}, R_B^{\tau,I})$. For $R_n^{\tau,I}$ and $R_E^{\tau,I}$, there is $\eta\in[0,t]\subset[0,T_1]$ such that

$$n^0(t,x)-n^0(0,x)=t\partial_t n^0(\eta,x)=\tau^2 z\partial_t n^0(\eta,x).$$

Since function $z\longmapsto ze^{-z}$ is bounded for $z\ge 0$ and

$$\partial_t n^0=-\operatorname{div}(n^0u^0)$$

it follows from (6.3.7) that

$$\big(n^0(t,x)-n^0(0,x)\big)u_I^0(z,x)=O(\tau^2).$$

Then

$$R_n^{\tau,I}=O(\tau^2)\quad\text{and}\quad R_E^{\tau,I}=O(\tau^2).$$

Finally, for $R_u^{\tau,I}$, we have

$$h(n^0+\tau^2n_I^1)-h(n^0)=O(\tau^2).$$

Thus

$$R_u^{\tau,I}=O(\tau).$$

From the previous discussions on the remainders, we obtain the following error estimates.

Proposition 6.3.1 *Let $s\ge 3$ be an integer. For given smooth data, the remainders $R_n^{\tau,I}$, $R_u^{\tau,I}$, $R_E^{\tau,I}$ and $R_B^{\tau,I}$ satisfy*

$$\sup_{0\le t\le T_1}\|\big(R_n^{\tau,I},R_E^{\tau,I}\big)(t,\cdot)\|_s\le C\tau^2,\qquad \sup_{0\le t\le T_1}\|R_u^{\tau,I}(t,\cdot)\|_s\le C\tau,\quad R_B^{\tau,I}=0, \tag{6.3.22}$$

where $C>0$ is a constant independent of τ.

The convergence result with initial layers can be stated as follows.

Theorem 6.3.1 *Let $s\ge 3$ be a fixed integer and $(n_0,u_0,E_0,B_0)\in H^{s+1}(\mathbb{T})$ with $n_0\ge$ constant >0 in $\mathbb{T}$. Suppose*

$$\operatorname{div}E_0^\tau=b-n_0^\tau,\quad \operatorname{div}B_0^\tau=0\quad in\quad \mathbb{T} \tag{6.3.23}$$

and

$$\Big\|\big(n_0^\tau,u_0^\tau,E_0^\tau,B_0^\tau\big)-(n_0,\tau u_0,E_0,\tau B_0)\Big\|_s\le C_1\tau^2, \tag{6.3.24}$$

where $C_1 > 0$ is a constant independent of τ. Then there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \geq T_1$ and the solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ to the periodic problem (6.1.6)-(6.1.7) satisfies

$$\left\|\left(n^\tau, u^\tau, E^\tau, B^\tau\right) - \left(n^0, u_{\tau,I}, E^0, B^0\right)(t)\right\|_s \leq C_2\tau^2, \quad \forall\, t \in [0, T_1].$$

Moreover,

$$\left\|u^\tau - u_{\tau,I}\right\|_{L^2(0,T_1;H^s(\mathbb{T}))} \leq C_2\tau^3.$$

6.4 Justification of asymptotic expansions

6.4.1 Statement of the main result

In this section, we justify rigorously the asymptotic expansions of solutions $(n^\tau, u^\tau, E^\tau, B^\tau)$ to the periodic problem (6.1.6)-(6.1.7) developed in section 2-3. We prove a more general convergence result which implies both Theorems 6.2.1-6.3.1. As a consequence, we obtain the existence of exact solutions $(n^\tau, u^\tau, E^\tau, B^\tau)$ in a time interval independent of τ. To justify rigorously the asymptotic expansions (6.2.2) and (6.3.18), it suffices to obtain the uniform estimates of the smooth solutions to (6.1.7) with respect to the parameter τ.

Let $(n^\tau, u^\tau, E^\tau, B^\tau)$ be the exact solution to (6.1.7) with initial data $(n_0^\tau, u_0^\tau, E_0^\tau, B_0^\tau)$ and $(n_\tau, u_\tau, E_\tau, B_\tau)$ be an approximate periodic solution defined on $[0, T_1]$, with

$$(n_\tau, u_\tau, E_\tau, B_\tau) \in C([0, T_1], H^{s+1}(\mathbb{T})) \cap C^1([0, T_1], H^s(\mathbb{T})).$$

We define the remainders of the approximate solution by

$$\begin{cases} R_n^\tau = \partial_t n_\tau + \dfrac{1}{\tau}\operatorname{div}(n_\tau u_\tau), \\ R_u^\tau = \partial_t u_\tau + \dfrac{1}{\tau}(u_\tau \cdot \nabla)u_\tau + \dfrac{1}{\tau}\nabla h(n_\tau) + \dfrac{E_\tau}{\tau} + \dfrac{u_\tau \times B_\tau}{\tau} + \dfrac{u_\tau}{\tau^2}, \\ R_E^\tau = \partial_t E_\tau - \dfrac{1}{\tau}\nabla \times B_\tau - \dfrac{n_\tau u_\tau}{\tau}, \\ R_B^\tau = \partial_t B_\tau + \dfrac{1}{\tau}\nabla \times E_\tau. \end{cases} \tag{6.4.1}$$

Suppose

$$\operatorname{div} E_\tau = b - n_\tau, \quad \operatorname{div} B_\tau = 0, \tag{6.4.2}$$

$$\sup_{0\leq t\leq T_1} \|(n_\tau, E_\tau, B_\tau)(t,\cdot)\|_s \leq C_1, \quad \sup_{0\leq t\leq T_1} \|u_\tau(t,\cdot)\|_s \leq C_1\tau, \tag{6.4.3}$$

$$\left\|\left(n_0^\tau - n_\tau(0,\cdot), u_0^\tau - u_\tau(0,\cdot), E_0^\tau - E_\tau(0,\cdot), B_0^\tau - B_\tau(0,\cdot)\right)\right\|_s \leq C_1\tau^{\lambda+a}, \tag{6.4.4}$$

$$\sup_{0\leq t\leq T_1} \|(R_n^\tau, R_E^\tau)(t,\cdot)\|_s \leq C_1\tau^{\lambda+a}, \quad \sup_{0\leq t\leq T_1} \|R_u^\tau(t,\cdot)\|_s \leq C_1\tau^\lambda, \quad R_B^\tau = 0, \tag{6.4.5}$$

where $\lambda \geq 0$, $C_1 > 0$ and $0 < a \leq 1$ are constants independent of τ.

Theorem 6.4.1 *Let $s \geq 3$ be an integer, $\lambda \geq 0$ and $0 < a \leq 1$. Under the above assumptions, there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \geq T_1$ and the solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ of the periodic problem (6.1.6)-(6.1.7) satisfies*

$$\left\|(n^\tau, u^\tau, E^\tau, B^\tau)(t) - (n_\tau, u_\tau, E_\tau, B_\tau)(t)\right\|_s \leq C_2 \tau^{\lambda+a}, \quad \forall\, t \in [0, T_1]. \tag{6.4.6}$$

Moreover,

$$\left\|u^\tau - u_\tau\right\|_{L^2(0,T_1;H^s(\mathbb{T}))} \leq C_2 \tau^{\lambda+a+1}. \tag{6.4.7}$$

Remark 6.4.1 *It is clear that Theorem 6.4.1 implies Theorems 6.2.1-6.3.1. In particular, $\lambda = 2m+1$, $a = 1$ with $m \geq 0$ in section 2 and $\lambda = a = 1$ in section 3, since*

$$\|(n_{\tau,I}, E_{\tau,I}, B_{\tau,I})(t) - (n^0, E^0, B^0)(t)\|_s = O(\tau^2),$$

uniformly with respect to t.

Remark 6.4.2 *Theorem 6.4.1 holds in the scaled time variable $t' = \tau t$. It can be written down in the normal time variable $t = t'/\tau$. Precisely, since Theorem 6.4.1 is valid for $t' \in [0, T_1]$, it is valid for $t \in [0, T_1\tau^{-1}]$, so that the periodic problem (6.1.5)-(6.1.6) admits a unique solution $(\tilde{n}^\tau, \tilde{u}^\tau, \tilde{E}^\tau, \tilde{B}^\tau)$ on a long-time interval $[0, T_1\tau^{-1}]$ as $\tau \to 0$. This solution satisfies*

$$(\tilde{n}^\tau, \tilde{u}^\tau, \tilde{E}^\tau, \tilde{B}^\tau) \in C([0, T_1\tau^{-1}], H^s(\mathbb{T})) \cap C^1([0, T_1\tau^{-1}], H^{s-1}(\mathbb{T})).$$

With the notations of Theorem 6.4.1, we have

$$(\tilde{n}^\tau, \tilde{u}^\tau, \tilde{E}^\tau, \tilde{B}^\tau)(t, x) = (n^\tau, u^\tau, E^\tau, B^\tau)(\tau t, x).$$

Then estimate (6.4.6) becomes

$$\left\|(\tilde{n}^\tau, \tilde{u}^\tau, \tilde{E}^\tau, \tilde{B}^\tau)(t) - (n_\tau, u_\tau, E_\tau, B_\tau)(\tau t)\right\|_s \leq C_2 \tau^{\lambda+a}, \quad \forall\, t \in [0, T_1\tau^{-1}]. \tag{6.4.8}$$

On the other hand, the change of variable $t = t'/\tau$ gives

$$\int_0^{T_1} \left\|u^\tau(t') - u_\tau(t')\right\|_s^2 dt' = \tau \int_0^{T_1\tau^{-1}} \left\|\tilde{u}^\tau(t) - u_\tau(\tau t)\right\|_s^2 dt.$$

Hence, (6.4.7) implies that

$$\int_0^{T_1\tau^{-1}} \left\|\tilde{u}^\tau(t) - u_\tau(\tau t)\right\|_s^2 dt \leq C_2 \tau^{2(\lambda+a)+1}. \tag{6.4.9}$$

6.4.2 Proof of the main result

By proposition 6.1.1, the exact solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ is defined in a time interval $[0, T_1^\tau)$ with $T_1^\tau > 0$. Since $n^\tau \in C\big([0, T_1^\tau], H^s(\mathbb{T})\big)$ and the embedding from $H^s(\mathbb{T})$ to $C(\mathbb{T})$ is continuous, we have $n^\tau \in C([0, T_1^\tau] \times \mathbb{T})$. From (6.4.3)-(6.4.4) and assumption $n_0^\tau \geq \kappa > 0$, we deduce that there exist $T_2^\tau \in (0, T_1^\tau]$ and a constant $C_0 > 0$, independent of τ, such that

$$\frac{\kappa}{2} \le n^\tau(t,x) \le C_0 \quad \forall\,(t,x) \in [0, T_2^\tau] \times \mathbb{T}.$$

Similarly, the function $t \longmapsto \big\|\big(n^\tau(t,\cdot), u^\tau(t,\cdot), E^\tau(t,\cdot), B^\tau(t,\cdot)\big)\big\|_s$ is continuous in $C\big([0,T_2^\tau]\big)$. From (6.4.3), the sequence $\big(\|(n^\tau(0,\cdot), u^\tau(0,\cdot), E^\tau(0,\cdot), B^\tau(0,\cdot))\|_s\big)_{\tau>0}$ is bounded. Then there exists $T_3^\tau \in (0, T_2^\tau]$ and a constant, still denoted by C_0, such that

$$\big\|\big(n^\tau(t,\cdot), u^\tau(t,\cdot), E^\tau(t,\cdot), B^\tau(t,\cdot)\big)\big\|_s \le C_0, \quad \forall\, t \in (0, T_3^\tau].$$

Then we define $T^\tau = \min(T_1, T_3^\tau) > 0$ so that the exact solution and the approximate solution are both defined in the time interval $[0, T^\tau]$. In this time interval, we denote by

$$(N^\tau, U^\tau, F^\tau, G^\tau) = (n^\tau - n_\tau, u^\tau - u_\tau, E^\tau - E_\tau, B^\tau - B_\tau). \tag{6.4.10}$$

Obviously, $(N^\tau, U^\tau, F^\tau, G^\tau)$ satisfies the following problem :

$$\begin{cases} \partial_t N^\tau + \dfrac{1}{\tau}\big((U^\tau + u_\tau)\cdot\nabla\big)N^\tau + \dfrac{1}{\tau}(N^\tau + n_\tau)\operatorname{div} U^\tau \\ \quad = -\dfrac{1}{\tau}\big(N^\tau \operatorname{div} u_\tau + (U^\tau\cdot\nabla)n_\tau\big) - R_n^\tau, \\ \partial_t U^\tau + \dfrac{1}{\tau}\big((U^\tau + u_\tau)\cdot\nabla\big)U^\tau + \dfrac{1}{\tau}h'(N^\tau + n_\tau)\nabla N^\tau \\ \quad = -\dfrac{1}{\tau}\big[(U^\tau\cdot\nabla)u_\tau + \big(h'(N^\tau + n_\tau) - h'(n_\tau)\big)\nabla n_\tau\big] - \dfrac{U^\tau}{\tau^2} \\ \quad -\dfrac{1}{\tau}\big[F^\tau + (U^\tau + u_\tau)\times G^\tau + U^\tau \times B_\tau\big] - R_u^\tau, \\ \partial_t F^\tau - \dfrac{1}{\tau}\nabla\times G^\tau = \dfrac{1}{\tau}\big(N^\tau U^\tau + N^\tau u_\tau + n_\tau U^\tau\big) - R_E^\tau, \quad \operatorname{div} F^\tau = -N^\tau, \\ \partial_t G^\tau + \dfrac{1}{\tau}\nabla\times F^\tau = 0, \quad \operatorname{div} G^\tau = 0, \\ t = 0\ :\quad (N^\tau, U^\tau, F^\tau, G^\tau) = \big(n_0^\tau - n_\tau(0,\cdot), u_0^\tau - u_\tau(0,\cdot), \\ \qquad\qquad\qquad\qquad E_0^\tau - E_\tau(0,\cdot), B_0^\tau - B_\tau(0,\cdot)\big). \end{cases} \tag{6.4.11}$$

Set

$$W_I^\tau = \begin{pmatrix} N^\tau \\ U^\tau \end{pmatrix}, \quad W_{II}^\tau = \begin{pmatrix} F^\tau \\ G^\tau \end{pmatrix}, \quad W^\tau = \begin{pmatrix} W_I^\tau \\ W_{II}^\tau \end{pmatrix},$$

$$A_i^I(n^\tau, u^\tau) = \begin{pmatrix} u_i^\tau & n^\tau e_i^t \\ h'(n^\tau)e_i & u_i^\tau I_3 \end{pmatrix}, \quad i = 1,2,3,$$

$$H_1(W_I^\tau) = \begin{pmatrix} -(U^\tau\cdot\nabla)n_\tau - N^\tau \operatorname{div} u_\tau \\ -(U^\tau\cdot\nabla)u_\tau - \big(h'(N^\tau + n_\tau) - h'(n_\tau)\big)\nabla n_\tau \end{pmatrix}, \quad H_2(W_I^\tau) = \begin{pmatrix} 0 \\ -U^\tau \end{pmatrix},$$

$$H_3(W^\tau) = \begin{pmatrix} 0 \\ -F^\tau - (U^\tau + u_\tau) \times G^\tau - U^\tau \times B_\tau \end{pmatrix}, \quad R^\tau = \begin{pmatrix} R_n^\tau \\ R_u^\tau \end{pmatrix},$$

where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$, y_i denotes the i^{th} component of $y \in \mathbb{R}^3$ and I_3 is the 3×3 unit matrix. Then system (6.4.11) for unknown W_I^τ can be rewritten as

$$\partial_t W_I^\tau + \frac{1}{\tau}\sum_{i=1}^{3} A_i^I(n^\tau, u^\tau)\partial_{x_i} W_I^\tau = \frac{1}{\tau}\big(H_1(W_I^\tau) + H_3(W^\tau)\big) + \frac{1}{\tau^2}H_2(W_I^\tau) - R^\tau. \tag{6.4.12}$$

It is symmetrizable hyperbolic with symmetrizer

$$A_0^I(n^\tau) = \begin{pmatrix} (n^\tau)^{-1} & 0 \\ 0 & \big(h'(n^\tau)\big)^{-1} I_3 \end{pmatrix},$$

which is a positive definite matrix when $0 < \frac{\kappa}{2} \le n^\tau = N^\tau + n_\tau \le C_0$. Moreover,

$$\tilde{A}_i^I(n^\tau, u^\tau) = A_0^I(n^\tau)A_i^I(n^\tau, u^\tau) = u_i^\tau A_0^I(n^\tau) + D_i, \tag{6.4.13}$$

is symmetric for all $1 \le i \le 3$, where each D_i is a constant matrix

$$D_i = \begin{pmatrix} 0 & e_i^t \\ e_i & 0 \end{pmatrix}.$$

The existence and uniqueness of smooth solutions to (6.1.6)-(6.1.7) is equivalent to that of (6.4.11). Thus, in order to prove Theorem 6.4.1, it suffices to establish uniform estimates of W^τ with respect to τ. In what follows, we denote by $C > 0$ various constants independent of τ and for $\alpha \in \mathbb{N}^3$, $(W_{I\alpha}^\tau, W_{II\alpha}^\tau) = \partial_x^\alpha (W_I^\tau, W_{II}^\tau)$ etc. The main estimates are contained in the following two lemmas for W_I^τ and W_{II}^τ, respectively. We first consider the estimate for W_I^τ.

Lemma 6.4.1 *Under the assumptions of Theorem 6.4.1, for all $t \in (0, T^\tau]$, as $\tau \to 0$ we have*

$$\|W_I^\tau(t)\|_s^2 + \frac{1}{\tau^2}\int_0^t \|U^\tau(\xi)\|_s^2\, d\xi \le C\int_0^t \Big(\|W^\tau(\xi)\|_s^2 + \|W^\tau(\xi)\|_s^4\Big)\, d\xi + \ C\tau^{2(\lambda+1)}. \tag{6.4.14}$$

Proof. For $\alpha \in \mathbb{N}^3$ with $|\alpha| \le s$, differentiating equations (6.4.12) with respect to x yields

$$\begin{aligned} &\partial_t W_{I\alpha}^\tau + \frac{1}{\tau}\sum_{i=1}^{3} A_i^I(n^\tau, u^\tau)\partial_{x_i} W_{I\alpha}^\tau \\ = \ &\frac{1}{\tau}\big[\partial_x^\alpha H_1(W_I^\tau) + \partial_x^\alpha H_3(W^\tau)\big] + \frac{1}{\tau^2}\partial_x^\alpha H_2(W_I^\tau) - \partial_x^\alpha R^\tau \\ &+ \frac{1}{\tau}\sum_{i=1}^{3}\big[A_i^I(n^\tau, u^\tau)\partial_{x_i} W_{I\alpha}^\tau - \partial_x^\alpha\big(A_i^I(n^\tau, u^\tau)\partial_{x_i} W_I^\tau\big)\big]. \end{aligned} \tag{6.4.15}$$

Multiplying (6.4.15) by $A_0^I(n^\tau)$ and taking the inner product of the resulting equations with $W_{I\alpha}^\tau$, by employing the classical energy estimate for symmetrizable hyperbolic equations, we obtain

$$\begin{aligned} & \frac{d}{dt}\big(A_0^I(n^\tau)W_{I\alpha}^\tau, W_{I\alpha}^\tau\big) - \frac{2}{\tau^2}\big(A_0^I(n^\tau)\partial_x^\alpha H_2(W_I^\tau), W_{I\alpha}^\tau\big) \\ = \; & \frac{2}{\tau}\big(A_0^I(n^\tau)\big[\partial_x^\alpha H_1(W_I^\tau) + \partial_x^\alpha H_3(W^\tau)\big], W_{I\alpha}^\tau\big) + \frac{2}{\tau}\big(J_\alpha^\tau, W_{I\alpha}^\tau\big) \\ & + \big(\operatorname{div} A_\tau^I(n^\tau, u^\tau)W_{I\alpha}^\tau, W_{I\alpha}^\tau\big) - 2\big(A_0^I(n^\tau)\partial_x^\alpha R^\tau, W_{I\alpha}^\tau\big), \end{aligned} \tag{6.4.16}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\mathbb{T})$,

$$J_\alpha^\tau = -\sum_{i=1}^3 A_0^I(n^\tau)\big[\partial_x^\alpha\big(A_i^I(n^\tau,u^\tau)\partial_{x_i}W_I^\tau\big) - A_i^I(n^\tau,u^\tau)\partial_x^\alpha(\partial_{x_i}W_I^\tau)\big]$$

and

$$\operatorname{div} A_\tau^I(n^\tau,u^\tau) = \partial_t A_0^I(n^\tau) + \frac{1}{\tau}\sum_{i=1}^3 \partial_{x_i}\widetilde{A}_i^I(n^\tau,u^\tau). \tag{6.4.17}$$

Let us estimate each term of equations (6.4.16). First, a direct computation gives

$$-\big(A_0^I(n^\tau)\partial_x^\alpha H_2(W_I^\tau), W_{I\alpha}^\tau\big) = \big((h'(n^\tau))^{-1}U_\alpha^\tau, U_\alpha^\tau\big) \geq C^{-1}\|U_\alpha^\tau\|^2. \tag{6.4.18}$$

On the other hand, since $A_0^I(n^\tau)$ is a positive definite matrix, we get :

$$\big(A_0^I(n^\tau)W_{I\alpha}^\tau, W_{I\alpha}^\tau\big) \geq C^{-1}\|W_{I\alpha}^\tau\|^2. \tag{6.4.19}$$

Moreover, we use the expression of $H_1(W_I^\tau)$ to compute :

$$\begin{aligned} \big(A_0^I(n^\tau)\partial_x^\alpha H_1(W_I^\tau), W_{I\alpha}^\tau\big) = \; & -(n^\tau)^{-1}N_\alpha^\tau\partial_x^\alpha\big[(U^\tau\cdot\nabla)n_\tau + N^\tau \operatorname{div} u_\tau\big] \\ & - (h'(n^\tau))^{-1}U_\alpha^\tau\cdot\partial_x^\alpha\big[(U^\tau\cdot\nabla)u_\tau + \big(h'(N^\tau+n_\tau) - h'(n_\tau)\big)\nabla n_\tau\big]. \end{aligned}$$

By Lemma 4.1.1 and $u_\tau = O(\tau)$, we get

$$\begin{aligned} \frac{2}{\tau}\big(A_0^I(n^\tau)\partial_x^\alpha H_1(W_I^\tau), W_{I\alpha}^\tau\big) & \leq \frac{C}{\tau}\big(\|N^\tau\|_s\|U^\tau\|_s + \tau\|N^\tau\|_s^2 + \tau\|U^\tau\|_s^2\big) \\ & \leq \frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2 + C_\varepsilon\|W^\tau\|_s^2. \end{aligned} \tag{6.4.20}$$

Here and hereafter, ε denotes a small constant independent of τ and $C_\varepsilon > 0$ denotes a constant depending only on ε.

For the term containing H_3, we have

$$\begin{aligned}
&\frac{2}{\tau}\big(A_0^I(n^\tau)\partial_x^\alpha H_3(W^\tau), W_{I\alpha}^\tau\big)\\
= \;& -\frac{2}{\tau}\big((h'(n^\tau))^{-1}U_\alpha^\tau, \partial_x^\alpha[F^\tau + (U^\tau+u_\tau)\times G^\tau + U^\tau \times B_\tau]\big)\\
&\le \frac{\varepsilon}{\tau^2}\|U_\alpha^\tau\|^2 + C_\varepsilon\int_{\mathbb{T}}\big|\partial_x^\alpha[F^\tau + (U^\tau+u_\tau)\times G^\tau + U^\tau\times B_\tau]\big|^2 dx\\
&\le \frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2 + C_\varepsilon\big(\|W^\tau\|_s^2 + \|U^\tau\|_s^2\|G^\tau\|_s^2\big).
\end{aligned}$$

Therefore,

$$\frac{2}{\tau}\big(A_0^I(n^\tau)\partial_x^\alpha H_3(W^\tau), W_{I\alpha}^\tau\big) \le \frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2 + C_\varepsilon\big(\|W^\tau\|_s^2 + \|W^\tau\|_s^4\big). \tag{6.4.21}$$

Now we consider the estimate for the term containing J_α^τ. Let us first point out that a direct application of Lemma 4.1.1 to J_α^τ does not yield the desired result. We have to develop the terms in the summation of J_α^τ to see the appearance of terms U^τ or $U^\tau + u_\tau$. By the definition of $A_i^I(n^\tau, u^\tau)$, we have

$$\begin{aligned}
&\partial_x^\alpha\big(A_i^I(n^\tau,u^\tau)\partial_{x_i}W_I^\tau\big) - A_i^I(n^\tau,u^\tau)\partial_x^\alpha(\partial_{x_i}W_I^\tau)\\
= \;& \begin{pmatrix} \partial_x^\alpha\big((U^\tau+u_\tau)_i\partial_{x_i}N^\tau\big) - (U^\tau+u_\tau)_i\partial_x^\alpha\partial_{x_i}N^\tau \\ \partial_x^\alpha\big(h'(N^\tau+n_\tau)\partial_{x_i}N^\tau e_i\big) - h'(N^\tau+n_\tau)\partial_x^\alpha\partial_{x_i}N^\tau e_i \end{pmatrix}\\
&+ \begin{pmatrix} \partial_x^\alpha\big((N^\tau+n_\tau)_i\partial_{x_i}U^\tau\cdot e_i^t\big) - (N^\tau+n_\tau)\partial_x^\alpha\partial_{x_i}U^\tau\cdot e_i^t \\ \partial_x^\alpha\big((U^\tau+u_\tau)_i\partial_{x_i}U^\tau\big) - (U^\tau+u_\tau)_i\partial_x^\alpha\partial_{x_i}U^\tau \end{pmatrix}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\big(A_0^I(n^\tau)(\partial_x^\alpha(A_i^I(n^\tau,u^\tau)\partial_{x_i}W_I^\tau) - A_i^I(n^\tau,u^\tau)\partial_x^\alpha(\partial_{x_i}W_I^\tau)), W_\alpha^\tau\big)\\
= \;& (n^\tau)^{-1}\big[\partial_x^\alpha\big((U^\tau+u_\tau)_i\partial_{x_i}N^\tau\big) - (U^\tau+u_\tau)_i\partial_x^\alpha\partial_{x_i}N^\tau\big]N_\alpha^\tau\\
&+ \big(h'(n^\tau)\big)^{-1}\big[\partial_x^\alpha\big((U^\tau+u_\tau)_i\partial_{x_i}U^\tau\big) - (U^\tau+u_\tau)_i\partial_x^\alpha\partial_{x_i}U^\tau\big]U_\alpha^\tau\\
&+ (n^\tau)^{-1}\big(\partial_x^\alpha\big((N^\tau+n_\tau)_i\partial_{x_i}U^\tau\cdot e_i^t\big) - (N^\tau+n_\tau)\partial_x^\alpha\partial_{x_i}U^\tau\cdot e_i^t\big)N_\alpha^\tau\\
&+ \big(h'(n^\tau)\big)^{-1}\big[\partial_x^\alpha\big(h'(N^\tau+n_\tau)\partial_{x_i}N^\tau e_i\big) - h'(N^\tau+n_\tau)\partial_x^\alpha\partial_{x_i}N^\tau e_i\big]U_\alpha^\tau\\
= \;& J_{i1} + J_{i2} + J_{i3} + J_{i4}.
\end{aligned}$$

Noting (6.4.3) for u_τ and applying Lemma 4.1.1 to each term on the right-hand side of the above equation gives

$$\big|J_{i1} + J_{i2}\big| \le C\big(\tau + \|U^\tau\|_s\big)\|W_I^\tau\|_s^2 \le \frac{\varepsilon}{\tau}\|U^\tau\|_s^2 + C_\varepsilon\tau\big(\|W^\tau\|_s^2 + \|W^\tau\|_s^4\big),$$

$$\big|J_{i3}+J_{i4}\big| \le C\big(1+\|N^\tau\|_s\big)\|N^\tau\|_s\|U^\tau\|_s \le \frac{\varepsilon}{\tau}\|U^\tau\|_s^2 + C_\varepsilon\tau\big(\|W^\tau\|_s^2+\|W^\tau\|_s^4\big),$$

which imply that

$$\frac{2}{\tau}\big(J_\alpha^\tau, W_\alpha^\tau\big) \le \frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2 + C_\varepsilon\big(\|W^\tau\|_s^2+\|W^\tau\|_s^4\big). \tag{6.4.22}$$

Using the expression of $A_0^I(n^\tau)$, we have obviously,

$$-2\big(A_0^I(n^\tau)\partial_x^\alpha R^\tau, W_{I\alpha}^\tau\big) = -2\big((n^\tau)^{-1}N_\alpha^\tau, \partial_x^\alpha R_n^\tau\big) - 2\big((h'(n^\tau))^{-1}U_\alpha^\tau, \partial_x^\alpha R_u^\tau\big).$$

Together with (6.4.5) and with $0 < a \le 1$ yields

$$\begin{aligned} -\,2\big(A_0^I(n^\tau)\partial_x^\alpha R^\tau, W_{I\alpha}^\tau\big) &\le C\|W^\tau\|_s^2 + \frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2 + C_\varepsilon\tau^{2(\lambda+1)} + C_\varepsilon\tau^{2(\lambda+a)} \\ &\le C\|W^\tau\|_s^2 + \frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2 + C_\varepsilon\tau^{2(\lambda+a)} \end{aligned} \tag{6.4.23}$$

Finally, for $i = 1,2,3$, it follows from (6.4.13) and (6.4.17) that

$$\begin{aligned} \operatorname{div} A_\tau^I(n^\tau,u^\tau) &= \big(A_0^I\big)'(n^\tau)\partial_t n^\tau + \frac{1}{\tau}\sum_{i=1}^{3}\partial_{x_i}\big[u_i^\tau A_0^I(n^\tau)\big] \\ &= \big(A_0^I\big)'(n^\tau)\big(\partial_t n^\tau + \frac{1}{\tau}\nabla n^\tau\cdot u^\tau\big) + \frac{1}{\tau}\operatorname{div} u^\tau A_0^I(n^\tau). \end{aligned}$$

Using the first equation of (6.1.7), we deduce that

$$\operatorname{div} A_\tau^I(n^\tau,u^\tau) = \frac{\operatorname{div} u^\tau}{\tau}\big[A_0^I(n^\tau) - n^\tau\big(A_0^I\big)'(n^\tau)\big].$$

Noting

$$\frac{\kappa}{2} \le n^\tau = N^\tau + n_\tau \le C_0, \quad u^\tau = U^\tau + u_\tau, \quad u_\tau = O(\tau),$$

we obtain

$$\|\operatorname{div} u^\tau\|_\infty \le C\|\operatorname{div}(U^\tau+u_\tau)\|_{s-1} \le C(\|U^\tau\|_s + \tau).$$

Here in the last inequality we have used the continuous embedding $H^{s-1}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$. Therefore,

$$\|\operatorname{div} A_\tau^I(n^\tau,u^\tau)\|_\infty \le C\big(1+\frac{1}{\tau}\|U^\tau\|_s\big).$$

We conclude that

$$\big(\operatorname{div} A_\tau^I(n^\tau,u^\tau)W_{I\alpha}^\tau, W_{I\alpha}^\tau\big) \le \frac{\varepsilon}{\tau^2}\|U^\tau\|_s^2 + C_\varepsilon\big(\|W^\tau\|_s^2+\|W^\tau\|_s^4\big). \tag{6.4.24}$$

Thus, together with (6.4.16) and (6.4.19)-(6.4.24), we obtain, for all $|\alpha| \le s$,

$$\frac{d}{dt}\big(A_0^I(n^\tau)W_{I\alpha}^\tau, W_{I\alpha}^\tau\big) + \frac{\kappa}{\tau^2}\|U_\alpha^\tau\|^2 \le \frac{C\varepsilon}{\tau^2}\|U^\tau\|_s^2 + C_\varepsilon\big(\|W^\tau\|_s^2+\|W^\tau\|_s^4\big) + C_\varepsilon\tau^{2(\lambda+a)}.$$

Integrating this equation over $(0,t)$ with $t \in (0,T^\tau) \subset (0,T_1)$ and summing up over all $|\alpha| \le s$, taking $\varepsilon > 0$ sufficiently small such that the term including $\frac{C\varepsilon}{\tau^2}\|U^\tau\|_s^2$ can be controlled by the left-hand side, together with condition (6.4.4) for the initial data and noting (6.4.19), we get (6.4.14). $\square$

Now, let us establish the estimate for W^τ_{II}.

Lemma 6.4.2 *Under the assumptions of Theorem 6.4.1, for all* $t \in (0, T^\tau]$, *as* $\tau \to 0$ *we have*

$$\|W^\tau_{II}(t)\|_s^2 \le \int_0^t \Big(\frac{1}{2\tau^2}\|U^\tau(\xi)\|_s^2 + C\|W^\tau(\xi)\|_s^2 + C\|W^\tau(\xi)\|_s^4\Big)\,d\xi + C\tau^{2(\lambda+1)}. \tag{6.4.25}$$

Proof. For a multi-index $\alpha \in \mathbb{N}^3$ with $|\alpha| \le s$, differentiating the third and fourth equations of (6.4.11) with respect to x, we have

$$\begin{cases} \partial_t F^\tau_\alpha - \dfrac{1}{\tau}\nabla \times G^\tau_\alpha = \dfrac{1}{\tau}\partial_x^\alpha\big(N^\tau U^\tau + N^\tau u_\tau + n_\tau U^\tau\big) - \partial_x^\alpha R^\tau_E, \\ \partial_t G^\tau_\alpha + \dfrac{1}{\tau}\nabla \times F^\tau_\alpha = 0, \end{cases} \tag{6.4.26}$$

where $W^\tau_{II\alpha} = (F^\tau_\alpha, G^\tau_\alpha) = \partial_x^\alpha(F^\tau, G^\tau)$.

By the vector analysis formula :

$$\mathrm{div}(f \times g) = (\nabla \times f)\cdot g - (\nabla \times g)\cdot f,$$

the singular term appearing in Sobolev energy estimates vanishes, i.e,

$$\int_{\mathbb{T}} \Big(-\frac{1}{\tau}\nabla \times G^\tau_\alpha \cdot F^\tau_\alpha + \frac{1}{\tau}\nabla \times F^\tau_\alpha \cdot G^\tau_\alpha\Big)dx = \frac{1}{\tau}\int_{\mathbb{T}} \mathrm{div}(F^\tau_\alpha \times G^\tau_\alpha)dx = 0.$$

Hence, using $u_\tau = O(\tau)$ and (6.4.5), a standard energy estimate for (6.4.26) yields

$$\begin{aligned} \frac{d}{dt}\|W^\tau_{II\alpha}\|^2 &\le \frac{2}{\tau}\int_{\mathbb{T}} \big(|\partial_x^\alpha(N^\tau U^\tau)| + |\partial_x^\alpha(N^\tau u_\tau)| + |\partial_x^\alpha(n_\tau U^\tau)|\big)|F^\tau_\alpha|dx + \int_{\mathbb{T}} |\partial_x^\alpha R^\tau_E|\,|F^\tau_\alpha|dx \\ &\le \frac{1}{2\tau^2}\|U^\tau\|_s^2 + C\big(\|W^\tau\|_s^2 + \|W^\tau\|_s^4\big) + C\tau^{2(\lambda+a)}. \end{aligned} \tag{6.4.27}$$

Integrating (6.4.27) over $(0, t)$, with $t \in (0, T^\tau)$, summing up over α satisfying $|\alpha| \le s$ and using (6.4.4) we obtain the Lemma. $\square$

Proof of Theorem 6.4.1. Let $\tau \to 0$ and $\varepsilon > 0$ be sufficiently small. By Lemma 6.4.1 and Lemma 6.4.2, for $t \in (0, T^\tau]$ we have

$$\|W^\tau(t)\|_s^2 + \frac{1}{\tau^2}\int_0^t \|U^\tau(\xi)\|_s^2\,d\xi \le C\int_0^t \Big(\|W^\tau(\xi)\|_s^2 + \|W^\tau(\xi)\|_s^4\Big)\,d\xi + C\tau^{2(\lambda+a)}. \tag{6.4.28}$$

Let

$$y(t) = C\int_0^t \Big(\|W^\tau(\xi)\|_s^2 + \|W^\tau(\xi)\|_s^4\Big)\,d\xi + C\tau^{2(\lambda+a)}.$$

Then it follows from (6.4.28) that

$$\|W^\tau(t)\|_s^2 \le y(t), \quad \frac{1}{\tau^2}\int_0^t \|U^\tau(\xi)\|_s^2\, d\xi \le y(t), \quad \forall\, t \in (0, T^\tau] \tag{6.4.29}$$

and

$$y'(t) = C\Big(\|W^\tau(t)\|_s^2 + \|W^\tau(t)\|_s^4\Big) \le C\big(y(t) + y^2(t)\big),$$

with

$$y(0) = C\tau^{2(\lambda+a)}.$$

A straightforward computation yields

$$y(t) \le C\tau^{2(\lambda+a)}e^{Ct} \le C\tau^{2(\lambda+a)}e^{CT_1}, \quad \forall\, t \in [0, T^\tau].$$

Therefore, from (6.4.29) we obtain

$$\|W^\tau(t)\|_s \le \sqrt{y(t)} \le C\tau^{\lambda+a}, \quad \int_0^t \|U^\tau(\xi)\|_s^2 \le \tau^2 y(t) \le C\tau^{2(\lambda+a+1)}, \quad \forall\, t \in [0, T^\tau].$$

In particular, this implies that W^τ is bounded in $L^\infty\big(0, T^\tau; H^s(\mathbb{T})\big)$, so is $(n^\tau, u^\tau, E^\tau, B^\tau)$. By a standard argument on the time extension of smooth solutions, we obtain $T_3^\tau \ge T_1$, i.e. $T^\tau = T_1$. This finishes the proof of Theorem 6.4.1. □

Appendix. Formal derivation of combined limits

We give a formal derivation of the combined zero-relaxation and zero-electron mass limits in system (6.1.2). For simplicity, let $\tau_i = \tau_e = \tau$, $q_i = -q_e = 1$, $m_i = 1$ and $\lambda = \gamma = 1$. Then there are only two parameters τ and m_e in (6.1.2). Similarly to the one-fluid equations, we make the time scaling by replacing t by t/τ. We replace also u by τu. With this simplification, system (6.1.2) is written as :

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \quad \nu = e, i, \\ m_\nu \tau^2\big(\partial_t u_\nu + (u_\nu\cdot\nabla)u_\nu\big) + \nabla h_\nu(n_\nu) = q_\nu(E + \tau u_\nu \times B) - m_\nu u_\nu, \\ \partial_t E - \dfrac{1}{\tau}\nabla\times B = n_e u_e - n_i u_i, \quad \operatorname{div} E = n_i - n_e, \\ \partial_t B + \dfrac{1}{\tau}\nabla\times E = 0, \quad \operatorname{div} B = 0. \end{cases} \tag{6.4.30}$$

There are several situations of limits in view of the two parameters. First, the formal limit equations in the zero-electron mass limit $m_e \to 0$ of (6.4.30) are

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \quad \nu = e, i, \\ \nabla h_e(n_e) = -(E + \tau u_e \times B), \\ \tau^2\big(\partial_t u_i + (u_i \cdot \nabla)u_i\big) + \nabla h_i(n_i) = E + \tau u_i \times B - u_i, \\ \partial_t E - \dfrac{1}{\tau}\nabla \times B = n_e u_e - n_i u_i, \quad \operatorname{div} E = n_i - n_e, \\ \partial_t B + \dfrac{1}{\tau}\nabla \times E = 0, \quad \operatorname{div} B = 0, \end{cases} \tag{6.4.31}$$

in which only the momentum equations for electrons are changed. Up to our knowledge this limit system has not been analyzed mathematically. Furthermore, replacing B by τB and letting $\tau \to 0$ in (6.4.31), we obtain the limit equations

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \quad \nu = e, i, \\ \nabla h_e(n_e) = \nabla\phi = -E, \\ \nabla h_i(n_i) = -u_i - \nabla\phi, \\ -\Delta\phi = n_i - n_e, \\ \nabla \times B = n_i u_i - n_e u_e - \partial_t \nabla\phi, \quad \operatorname{div} B = 0, \end{cases} \tag{6.4.32}$$

from which we deduce relations

$$n_e = h_e^{-1}(\phi), \quad u_i = -\nabla(h_i(n_i) + \phi), \quad E = -\nabla\phi \tag{6.4.33}$$

and a drift-diffusion type system

$$\begin{cases} \partial_t n_i - \operatorname{div}\big(n_i \nabla(h_i(n_i) + \phi)\big) = 0, \\ -\Delta\phi = n_i - h_e^{-1}(\phi), \end{cases} \tag{6.4.34}$$

where h_e^{-1} is the inverse function of h_e. Hence, we may determine (n_i, ϕ) by (6.4.34) and (n_e, u_i, E) by (6.4.33). Note that when n_e is given, the first equation in (6.4.32) for u_e is an incompressibility-type condition, which is not sufficient to determine u_e. The determination of u_e requires also another equations which can be derived from a high order asymptotic expansion.

Second, in the zero-relaxation limit $\tau \to 0$ of (6.4.30), still replacing B by τB, we obtain the classical drift-diffusion equations :

$$\begin{cases} m_\nu \partial_t n_\nu - \operatorname{div}\big(n_\nu \nabla(h_\nu(n_\nu) + q_\nu\phi)\big) = 0, \quad \nu = e, i, \\ -\Delta\phi = n_i - n_e, \end{cases} \tag{6.4.35}$$

together with

$$\begin{cases} m_\nu u_\nu = -\big(\nabla h_\nu(n_\nu) + q_\nu \nabla\phi\big), \quad \nu = e, i, \quad E = -\nabla\phi, \\ \nabla \times B = n_i u_i - n_e u_e - \partial_t \nabla\phi, \quad \operatorname{div} B = 0. \end{cases} \tag{6.4.36}$$

Taking the zero-electron mass limit $m_e \to 0$ in (6.4.35)-(6.4.36) we still obtain (6.4.32), which is also the combined limit system (6.4.30) as $(m_e, \tau) \to 0$. Therefore, the three limits $m_e \to 0$ then

$\tau \to 0$, $\tau \to 0$ then $m_e \to 0$ and $(m_e, \tau) \to 0$ are formally commutative.

Another interesting limit is $m = m_i = m_e \to 0$ and $m\tau^2 = 1$. Hence, $\tau \to +\infty$. Replace B by B/τ, the limit $m \to 0$ and $m\tau^2 = 1$ in (6.4.30) gives equations

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \quad \nu = e, i, \\ \partial_t u_\nu + (u_\nu \cdot \nabla) u_\nu + \nabla h_\nu(n_\nu) = q_\nu(E + \tau u_\nu \times B), \\ \partial_t E = n_e u_e - n_i u_i, \quad \operatorname{div} E = n_i - n_e, \\ \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0. \end{cases} \tag{6.4.37}$$

This is still a symmetrizable hyperbolic system for (n_ν, u_ν), $\nu = e, i$, coupled to a degenerate Maxwell system. Therefore, it is hopeful to prove a result on the local-in-time smooth solutions to (6.4.37). However, the justification of the limit is also an unsolved problem.

Chapter 7

Initial layers and zero-relaxation limits of two-fluid Euler-Maxwell equations

Abstract : In this chapter we consider zero-relaxation limits for periodic problems of the two-fluid compressible Euler-Maxwell equations arising in the modeling of magnetized plasmas. These equations are symmetrizable hyperbolic in the sense of Friedrichs. For well-prepared initial data, we construct an approximate solution of the two-fluid compressible Euler-Maxwell equations by an asymptotic expansion up to any order. For ill-prepared initial data, an asymptotic expansion up to order 1 of the relaxation limit is constructed by taking into account the initial layers. In both cases, the asymptotic expansions are valid in a time interval independent of the relaxation time and their convergence is justified by establishing uniform energy estimates.

7.1 Introduction

We consider a plasma consisting of electrons of charge $q_e = -1$ and a single species of ions of charge $q_i = 1$. Denoting by n_e, u_e (respectively, ions) and by E, B the scaled electric field and magnetic field. They are functions of a three-dimensional position vector $x \in \mathbb{T}$ and of the time $t > 0$, where $\mathbb{T} \stackrel{def}{=} \left(\mathbb{R}/\mathbb{Z}\right)^3$ is the torus. The fields E and B are coupled to the particles through the Maxwell equations and act on the particles via the Lorentz force. The variables satisfy the scaled Euler-Maxwell system (see chapter 6 and [6, 9, 68]) :

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \\ m_\nu \partial_t(n_\nu u_\nu) + m_\nu \operatorname{div}(n_\nu u_\nu \otimes u_\nu) + \nabla p_\nu(n_\nu) = q_\nu n_\nu(E + \gamma u_\nu \times B) - \dfrac{m_\nu n_\nu u_\nu}{\tau_\nu}, \\ \gamma\lambda^2 \partial_t E - \nabla \times B = -\gamma(q_e n_e u_e + q_i n_i u_i), \quad \lambda^2 \operatorname{div} E = n_i - n_e, \\ \gamma \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0, \quad \nu = e, i, \end{cases} \tag{7.1.1}$$

for $\nu = e, i$ and $(t, x) \in (0, \infty) \times \mathbb{T}$, where $\otimes$ stands for the tensor product and $p_\nu = p_\nu(n_\nu)$ is the pressure function. this system is complemented by initial conditions for the variables n_ν, u_ν, E and B, which are periodic in x.

In the system (7.1.1), $n_i - n_e = q_i n_i + q_e n_e$ and $q_i n_i u_i + q_e n_e u_e$ stand for the free charge and current densities and $n_\nu(E + \gamma u_\nu \times B)$ is the Lorentz force for the particle. The two first equations of (7.1.1) are the mass and momentum balance laws, while the two last equations of (7.1.1) are the Maxwell equations. Here the energy equations are replaced by the state equations of pressures $p_\nu = p_\nu(n_\nu)$ $(\nu = e, i)$ which are supposed to be smooth and strictly increasing for $n_\nu > 0$. Usually, they are of the form :

$$p_\nu(n_\nu) = a_\nu^2 n_\nu^{\gamma'}, \quad \nu = e, i.$$

where $\gamma' \geq 1$ and $a_\nu > 0$ are constants. In this chapter we consider smooth periodic solutions to the Euler-Maxwell system (7.1.1). Then for smooth solutions with $n_\nu > 0$, the second equation of (7.1.1) is equivalent to

$$m_\nu \partial_t u_\nu + m_\nu (u_\nu \cdot \nabla) u_\nu + \nabla h_\nu(n_\nu) = q_\nu (E + \gamma u_\nu \times B) - \frac{m_\nu u_\nu}{\tau_\nu}, \quad \nu = e, i, \tag{7.1.2}$$

where $\cdot$ denotes the inner product of $\mathbb{R}^3$ and h_ν is the enthalpy function defined by :

$$h_\nu(n_\nu) = \int_1^{n_\nu} \frac{p'_\nu(s)}{s} ds, \quad \nu = e, i. \tag{7.1.3}$$

Since p_ν is strictly increasing on $(0, +\infty)$, so is h_ν.

The dimensionless parameters τ_ν, $\lambda > 0$ and $\gamma > 0$ can be chosen independently on each other, according to the desired scaling. Physically, τ_ν stands for the momentum relaxation time, λ stands for the scaled Debye length and γ can be chosen to be proportional $\frac{1}{c}$, where $c = (\varepsilon_0 \mu_0)^{-\frac{1}{2}}$ is the speed of light, with ε_0, μ_0 being the vacuum permittivity and permeability, $m_e > 0$ being the electron mass and $m_i > 0$ being the ion mass. These parameters are small compared to the physical size of the known variables. Thus, regarding τ_ν, γ, λ and m_ν as singular perturbation parameters, we can study the limits in the system (7.1.1) as these parameters tend to zero. The limit $\lambda \to 0$ leads to $n_e = n_i$, which is the quasi-neutrality of the plasma. Hence, $\lambda \to 0$ is called the quasi-neutral limit. Also, $\tau_\nu \to 0$ and $\gamma \to 0$ are physically called the zero-relaxation limit and the non-relativistic limit, respectively. For the other physical meaning of the dimensionless parameters λ and γ, we refer to [8] for a similar choice of the scaling in the Vlasov-Maxwell equations.

The first mathematical study of the one-fluid Euler-Maxwell equations is due to Chen et al [11], where the global existence of weak solutions in one-dimensional case is established by the fractional step Godunov scheme together with a compensated compactness argument. Paper [11] exhibits also some applications of the model (7.1.1) in the semiconductor theory. Since then a few progress have been made. In [61, 63], we justify the non-relativistic limit $\gamma \to 0$ and the quasi-neutral limit $\lambda \to 0$ in multi-dimensional case. The results show that the limit $\gamma \to 0$ is the (one-fluid) compressible Euler-Poisson system and the limit $\lambda \to 0$ is the electron magnetohydrodynamics equations. Furthermore, we prove also that the combined non-relativistic and quasi-neutral limit $\gamma = \lambda^2$ is the (one-fluid) incompressible Euler equations [62]. The justification of these limits is rigorous for smooth periodic solutions in time intervals independent of the parameters γ and λ.

In this chapter, we are interested in the zero-relaxation limit $\tau_\nu \to 0$ of system (7.1.1) under the conditions $\gamma = O(1)$, $m_\nu = O(1)$ and $\lambda = O(1)$. We assume, throughout this chapter, that $\gamma = \lambda = m_\nu = 1$. For simplifying the notations, we assume that $\tau_i = \tau_e = \tau$. The usual time scaling for studying the limit $\tau \to 0$ is $t = \tau\xi$. Rewriting still ξ by t, system (7.1.1) becomes (see [60, 64])

$$\begin{cases} \partial_t n_\nu + \dfrac{1}{\tau}\operatorname{div}(n_\nu u_\nu) = 0, \\ \partial_t u_\nu + \dfrac{1}{\tau}(u_\nu \cdot \nabla) u_\nu + \dfrac{1}{\tau}\nabla h_\nu(n_\nu) = \dfrac{q_\nu}{\tau}\big(E + u_\nu \times B\big) - \dfrac{u_\nu}{\tau^2}, \\ \partial_t E - \dfrac{1}{\tau}\nabla \times B = \dfrac{1}{\tau}\big(n_e u_e - n_i u_i\big), \quad \operatorname{div} E = n_i - n_e, \\ \partial_t B + \dfrac{1}{\tau}\nabla \times E = 0, \quad \operatorname{div} B = 0. \end{cases} \qquad \nu = e, i \tag{7.1.4}$$

Since $t = 0$ if and only if $\xi = 0$, system (7.1.4) is complemented by periodic initial conditions :

$$t = 0: \quad (n_\nu, u_\nu, E, B) = (n^\tau_{\nu,0}, u^\tau_{\nu,0}, E^\tau_0, B^\tau_0), \quad \nu = e, i. \tag{7.1.5}$$

The Euler-Maxwell system (7.1.4) is nonlinear and symmetrizable hyperbolic for $n_\nu > 0$ in the sense of Friedrichs. Then, according to the result of Kato [38], the Cauchy problem or the periodic problem (7.1.4)-(7.1.5) has a unique local smooth solution when the initial data are smooth.

In this chapter, we propose an asymptotic expansion to system (7.1.4) under the form of a power series in τ. From momentum equations for u^τ_ν it is easy to see that the leading terms in u^τ_ν should be to zero. For convenience if we replace u_ν by τu_ν and B by τB, then system (7.1.4) becomes :

$$\begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \\ \tau^2\big(\partial_t u_\nu + (u_\nu \cdot \nabla) u_\nu\big) + \nabla h_\nu(n_\nu) = q_\nu\big(E + u_\nu \times B\big) - u_\nu, \\ \partial_t E - \nabla \times B = n_e u_e - n_i u_i, \quad \operatorname{div} E = n_i - n_e, \\ \tau^2 \partial_t B - \nabla \times E = 0, \quad \operatorname{div} B = 0, \end{cases} \qquad \nu = e, i$$

in which the only small parameter is τ^2. This suggests that the asymptotic expansion as the form :

$$(n^m_{\nu,\tau}, u^m_{\nu,\tau}, E^m_\tau, B^m_\tau) = \sum_{j=0}^{m} \tau^{2j}(n^j_\nu, \tau u^j_\nu, E^j, \tau B^j), \quad \nu = e, i.$$

With this expansion, for $m \geq 0$ we prove the convergence of the solution $(n^\tau_\nu, u^\tau_\nu, E^\tau, B^\tau)$ of (7.1.4) to $(n^m_{\nu,\tau}, u^m_{\nu,\tau}, E^m_\tau, B^m_\tau)$ with order $O(\tau^{m+1})$ when the initial data are well-prepared and the initial error has the same order. This includes the case $m = 0$. In the proof of the result, we have to treat the order of the remainder $R^{\tau,m}_B$ for variable B. Indeed, there is a loss of one order for $R^{\tau,m}_B$ comparing to variables n_ν, u_ν and E. This is overcome by introducing a correction term into E^m_τ so that the new remainder for B becomes zero without changing the order of the other remainders. For ill-prepared initial data, the above convergence result is not valid because the approximate solution cannot satisfy the prescribed initial conditions. In this case, we construct initial layer corrections with exponential decay to zero and prove the convergence of the first order asymptotic expansion. The analysis shows that there are no first order initial layers on variables n_ν, E and B. However, it is necessary to consider the second order initial layer corrections to obtain the desired order of remainders.

This chapter is organized as follows. In the next section, we summarize some basic properties of our system. In particular, we point out that our system admits a strict convex energy and is written in the form of a symmetric hyperbolic system. In section 3, we derive asymptotic expansions of solutions and state the convergence result to problem (7.1.4)-(7.1.5) in the case of well-prepared initial data. In section 4 we study the initial layers in the case of ill-prepared initial data. We construct the initial layer corrections which exponentially decay to zero and state the convergence result. The justification of both two asymptotic expansions is given in the last section. The justification is given by using another symmetrizer different from that used in chapter 6. For this purpose, we prove a more general convergence theorem which implies those in both cases of well-prepared initial data and ill-prepared initial data.

Proposition 7.1.1 *(Local existence of smooth solutions, see [38, 50]) Let $s > \frac{5}{2}$ be an integer and $(n^\tau_{\nu,0}, u^\tau_{\nu,0}, E^\tau_0, B^\tau_0) \in H^s(\mathbb{T})$ with $n^\tau_{\nu,0} \geq \kappa$ for some given constant $\kappa > 0$, independent of τ. Then there exist $T^\tau_1 > 0$ and a unique smooth solution $(n^\tau_\nu, u^\tau_\nu, E^\tau, B^\tau)$ to the periodic problem (7.1.4)-(7.1.5) defined in the time interval $[0, T^\tau_1)$, with $(n^\tau_\nu, u^\tau_\nu, E^\tau, B^\tau) \in C^1\big([0, T^\tau_1); H^{s-1}(\mathbb{T})\big) \cap C\big([0, T^\tau_1); H^s(\mathbb{T})\big)$.*

7.2 Basic properties of the system

We review some basic properties of the system (7.1.4), which will be used in the proof of our main theorem. First, it is well known that the constraint equations :

$$\operatorname{div} E = n_i - n_e, \quad \operatorname{div} B = 0 \tag{7.2.1}$$

are compatible with the another equations in (7.1.1). They hold for $t > 0$ if and only if the prescribed initial data satisfy (7.2.1) i.e :

$$\operatorname{div} E_0 = n_{i,0} - n_{e,0}, \quad \operatorname{div} B_0 = 0. \tag{7.2.2}$$

Second, we remark that our system (7.1.4) admits the following energy balance law :

$$\begin{aligned} \Big\{ \sum_{\nu=e,i} n_\nu \mathcal{E}_\nu + \frac{1}{2}(|E|^2 + |B|^2) \Big\}_t + \frac{1}{\tau} \sum_{\nu=e,i} \operatorname{div}\big(n_\nu u_\nu \mathcal{E}_\nu + p_\nu(n_\nu) u_\nu\big) \\ + \frac{1}{\tau}\operatorname{div}(E \times B) + \frac{1}{\tau^2} \sum_{\nu=e,i} n_\nu |u_\nu|^2 = 0, \end{aligned} \tag{7.2.3}$$

where the function $\mathcal{E}_\nu = \mathcal{E}_\nu(n_\nu, u_\nu)$ is defined as :

$$\mathcal{E}_\nu(n_\nu, u_\nu) := \frac{1}{2}|u_\nu|^2 + \Phi_\nu(n_\nu), \qquad \Phi_\nu(n_\nu) := \int^{n_\nu} \frac{p(\eta)}{\eta^2} d\eta, \quad \nu = e, i.$$

Note that we do not use (7.2.1) in deriving (7.2.3). It is easy to see that the total energy $\mathcal{H} := \sum_{\nu=e,i} n_\nu \mathcal{E}_\nu + (|E|^2 + |B|^2)/2$ is a strictly convex function of the conserved quantities $\tilde{w_\nu} := (n_\nu, n_\nu u_\nu, E, B)$; for a similar convexity property of the total energy, we refer to [40, 39]. This total energy can be regarded as a mathematical entropy defined in [41] for symmetric hyperbolic systems with dissipation (cf. [28, 75]). Also, the potential energy $\Phi_\nu(n_\nu)$ can be regarded as a strictly convex function of $v_\nu := 1/n_\nu$, so that $\mathcal{E}_\nu$ is also a strictly convex function of (v_ν, u_ν).

Finally, we can rewrite (7.1.4) in the form :

$$A^0_\nu(w_\nu)\partial_t w_\nu + \frac{1}{\tau}\sum_{j=1}^{3} A^j(w_\nu)\partial_{x_j} w_\nu + \frac{1}{\tau^2} L^\tau(w_\nu) w_\nu = 0, \quad \nu = e, i, \tag{7.2.4}$$

where $w_\nu = (n_\nu, u_\nu, E, B)^t$ and the coefficient matrices are given explicitly as :

$$A^0_\nu(w_\nu) = \begin{pmatrix} h'_\nu(n_\nu) & & & \\ & I & & \\ & & I & \\ & & & I \end{pmatrix}, \qquad L^\tau(w_\nu) = \begin{pmatrix} 0 & & & \\ & (I + q_\nu \tau \Omega_B) & \tau q_\nu I & \\ & n_i u_i - n_e u_e & O & \\ & & & O \end{pmatrix},$$

$$\sum_{j=1}^{3} A^j(w_\nu)\xi_j = \begin{pmatrix} h'_\nu(n_\nu)(u_\nu \cdot \xi) & h'_\nu(n_\nu)\xi & & \\ n_\nu h'_\nu(n_\nu)\xi^t & (u_\nu \cdot \xi) I & & \\ & & O & -\Omega_\xi \\ & & \Omega_\xi & O \end{pmatrix}.$$

Here I and O denotes the 3×3 identity matrix and zero matrix, respectively, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, and Ω_ξ is the skew-symmetric matrix defined by :

$$\Omega_\xi = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & O \end{pmatrix},$$

for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, so that we have $\Omega_\xi E^t = (\xi \times E)^t$ (as a column vector in $\mathbb{R}^3$) for $E = (E_1, E_2, E_3) \in \mathbb{R}^3$. It should be noted that (7.2.4) is a symmetric hyperbolic system because $A^0_\nu(w_\nu)$ is real symmetric and positive definite and $A^j(w_\nu)$, $j = 1, 2, 3$, are real symmetric. Also, the matrix $L^\tau(w_\nu)$ is nonnegative definite, so that it is regarded as a dissipation matrix.

7.3 Case of well-prepared initial data

7.3.1 Formal asymptotic expansions

In this section we consider the limit $\tau \to 0$ in problem (7.1.4)-(7.1.5) with well-prepared initial data. Based on the discussion on the asymptotic expansion, we make the following ansatz for both the approximate solution and its initial data :

$$(n_{\nu,\tau}, u_{\nu,\tau}, E_\tau, B_\tau)(0,x) = \sum_{j\geq 0} \tau^{2j}(n_{\nu,j}, \tau u_{\nu,j}, E_j, \tau B_j)(x), \quad x \in \mathbb{T}, \quad \nu = e, i, \tag{7.3.1}$$

$$(n_{\nu,\tau}, u_{\nu,\tau}, E_\tau, B_\tau)(t,x) = \sum_{j\geq 0} \tau^{2j}(n_\nu^j, \tau u_\nu^j, E^j, \tau B^j)(t,x), \quad t > 0, \ \ x \in \mathbb{T}, \ \ \nu = e, i, \tag{7.3.2}$$

where $(n_{\nu,j}, u_{\nu,j}, E_j, B_j)_{j\geq 0}$ are given sufficiently smooth data with $n_{\nu,0} \geq$ constant > 0 in $\mathbb{T}$.

Now it needs to determine the profiles $(n_\nu^j, u_\nu^j, E^j, B^j)$ for all $j \geq 0$. Substituting expansion (7.3.2) into system (7.1.4), we obtain a series of equations verified by the profiles $(n_\nu^j, u_\nu^j, E^j, B^j)_{j\geq 0}$.

1. The leading profiles $(n_\nu^0, u_\nu^0, E^0, B^0)$ satisfy the following system :

$$\begin{cases} \partial_t n_\nu^0 + \operatorname{div}(n_\nu^0 u_\nu^0) = 0, \\ \nabla h_\nu(n_\nu^0) = q_\nu E^0 - u_\nu^0, \\ \nabla \times E^0 = 0, \ \ \operatorname{div} E^0 = n_i^0 - n_e^0, \\ \nabla \times B^0 = \partial_t E^0 + n_i^0 u_i^0 - n_e^0 u_e^0, \ \ \operatorname{div} B^0 = 0, \end{cases} \qquad \nu = e, i, \tag{7.3.3}$$

Equation $\nabla \times E^0 = 0$ implies the existence of a potential ϕ^0 such that $E^0 = -\nabla\phi^0$. Then (n_e^0, n_i^0, ϕ^0) satisfies the classical drift-diffusion equations :

$$\begin{cases} \partial_t n_\nu^0 - \operatorname{div}\big(n_\nu^0 \nabla\big(h_\nu(n_\nu^0) + q_\nu \phi^0\big)\big) = 0, \\ -\Delta\phi^0 = n_i^0 - n_e^0, \quad \nu = e, i, \end{cases} \qquad t > 0, \quad x \in \mathbb{T} \tag{7.3.4}$$

with the initial conditions :

$$n_\nu^0(0,x) = n_{\nu,0}, \quad x \in \mathbb{T}, \quad \nu = e, i. \tag{7.3.5}$$

For smooth initial data $n_{\nu,0}$ satisfying $n_{\nu,0} > 0$ in $\mathbb{T}$, the periodic problem (7.3.4)-(7.3.5) has a unique smooth solution (n_e^0, n_i^0, ϕ^0) in the class $m(\phi^0) = 0$, defined in a time interval $[0, T_1]$ with $T_1 > 0$. The solution satisfies $n_e > 0$ and $n_i > 0$ in $[0, T_1] \times \mathbb{T}$. Then (u_ν^0, E^0) are given by :

$$u_\nu^0 = -\nabla\big(h_\nu(n_\nu^0) + q_\nu\phi^0\big), \quad E^0 = -\nabla\phi^0, \quad \nu = e, i. \tag{7.3.6}$$

Since (n_ν^0, u_ν^0, E^0) are known, B^0 solves the linear system of curl-div equations of type (6.1.9)in the class $m(B^0) = 0$. More precisely, using $\nabla \times E^0 = 0$ and formula :

$$\nabla \times (\nabla \times B^0) = \nabla \operatorname{div} B^0 - \Delta B^0,$$

we obtain :

$$\Delta B^0 = \nabla \times (n_e^0 u_e^0 - n_i^0 u_i^0) \quad \text{in } \mathbb{T} \quad \text{and} \quad m(B^0) = 0.$$

This implies that the initial data $(u_{\nu,0}, E_0, B_0)$ are not arbitrary. They should be given in terms of $n_{\nu,0}$. Precisely :

$$u_{\nu,0} = -\nabla\big(h_\nu(n_{\nu,0}) + q_\nu\phi_0\big), \quad E_0 = -\nabla\phi_0, \quad B_0 = B^0(0,.), \quad \nu = e, i, \tag{7.3.7}$$

where ϕ_0 is determined by :

$$-\Delta\phi_0 = n_{i,0} - n_{e,0} \quad \text{in } \mathbb{T} \quad \text{and} \quad m(\phi_0) = 0. \tag{7.3.8}$$

Thus we need compatibility type conditions (7.3.7)-(7.3.8) for the initial data of the leading terms $(n_\nu^0, u_\nu^0, E^0, B^0)$.

2. For any $j \geq 1$, the profiles $(n_\nu^j, u_\nu^j, E^j, B^j)$ are obtained by induction. Assume that $(n_\nu^k, u_\nu^k, E^k, B^k)_{0\leq k\leq}$ are smooth and already determined in previous steps.
Then $(n_\nu^j, u_\nu^j, E^j, B^j)$ satisfy the linear system :

$$\begin{cases} \partial_t n_\nu^j + \operatorname{div}\big(n_\nu^0 u_\nu^j + n_\nu^j u_\nu^0\big) = -\displaystyle\sum_{k=1}^{j-1} \operatorname{div}\big(n_\nu^k u_\nu^{j-k}\big), \\ \partial_t u_\nu^{j-1} + \displaystyle\sum_{k=0}^{j-1} (u_\nu^k \cdot \nabla) u_\nu^{j-1-k} + \nabla\Big(h_\nu'(n_\nu^0) n_\nu^j + h_\nu^{j-1}\big((n_\nu^k)_{k\leq j-1}\big)\Big) \\ \quad = q_\nu\Big(E^j + \displaystyle\sum_{k=0}^{j-1} u_\nu^k \times B^{j-1-k}\Big) - u_\nu^j, \\ \nabla \times E^j = -\partial_t B^{j-1}, \ \ \operatorname{div} E^j = n_i^j - n_e^j, \\ \nabla \times B^j = \partial_t E^j + \displaystyle\sum_{k=0}^{j} \big(n_i^k u_i^{j-k} - n_e^k u_e^{j-k}\big), \ \ \operatorname{div} B^j = 0, \end{cases} \qquad \nu = e, i, \tag{7.3.9}$$

where $h^0 = 0$ and h^{j-1} is a function depending only on $(n^k)_{0\leq k\leq j-1}$ and is defined for $j \geq 2$ by :

$$h_\nu\Big(\sum_{j\geq 0} \tau^j n_\nu^j\Big) = h_\nu(n_\nu^0) + h_\nu'(n_\nu^0)\sum_{j\geq 1} \tau^j n_\nu^j + \sum_{j\geq 2} \tau^j h_\nu^{j-1}\big((n_\nu^k)_{k\leq j-1}\big).$$

Equations for B^j in (7.3.9) are of curl-div type and determine a unique smooth B^j in the class $m(B^j) = 0$. Moreover, from $\operatorname{div} B^j = 0$, we deduce the existence of a given vector ψ^j such that $B^j = -\nabla \times \psi^j$. Then, the equation $\nabla \times E^j = -\partial_t B^{j-1}$ in (7.3.9) becomes $\nabla \times (E^j - \partial_t \psi^{j-1}) = 0$. It follows that there is a potential function ϕ^j such that :

$$E^j = \partial_t \psi^{j-1} - \nabla\phi^j. \tag{7.3.10}$$

Therefore, in the class $m(\phi^j) = 0$, (n_ν^j, ϕ^j) solve a linearized drift-diffusion system :

$$\begin{cases} \partial_t n_\nu^j - \operatorname{div}\big[n_\nu^0 \nabla(h_\nu'(n_\nu^0) n_\nu^j + q_\nu \phi^j) - n_\nu^j u_\nu^0\big] \\ = f^j\big((V_\nu^k, \partial_t V_\nu^k, \partial_x V_\nu^k, \partial_t\partial_x V_\nu^k, \partial_x^2 V_\nu^k)_{0\leq k\leq j-1}\big), \qquad t > 0, \quad x \in \mathbb{T}, \quad \nu = e, i, \\ -\Delta\phi^j = n_i^j - n_e^j - \partial_t(\operatorname{div}\psi^{j-1}), \end{cases} \tag{7.3.11}$$

together with the initial conditions :

$$n_\nu^j(0, x) = n_{\nu,j}(x), \quad x \in \mathbb{T}, \quad \nu = e, i. \tag{7.3.12}$$

Finally, E^j is given by (7.3.10) and

$$\begin{aligned} u_\nu^j &= q_\nu\big(\partial_t\psi^{j-1} + u_\nu^k \times B^{j-1-k}\big) - \nabla\big(q_\nu\phi^j + h'_\nu(n_\nu^0)n_\nu^j + h_\nu^{j-1}((n_\nu^k)_{k\le j-1})\big) \\ &\quad -\Big(\partial_t u_\nu^{j-1} + \sum_{k=0}^{j-1}(u_\nu^k\cdot\nabla)u_\nu^{j-1-k}\Big) \end{aligned} \tag{7.3.13}$$

where f^j is a given smooth function and $V_\nu^k = (n_\nu^k, u_\nu^k, \psi^k)$. Problem (7.3.11)-(7.3.12) is linear. It admits a unique global smooth solution. Thus, we get the high-order compatibility conditions for $j \ge 1$:

$$\begin{aligned} u_{\nu,j} &= -\nabla\Big(q_\nu\phi_j + h'_\nu(n_{\nu,0})n_{\nu,j} + h_\nu^{j-1}((n_{\nu,k})_{k\le j-1})\Big) \\ &\quad +q_\nu\Big(\partial_t\psi^{j-1}\big|_{t=0} + u_{\nu,k}\times B^{j-1-k}(0,.)\Big) \\ &\quad -\Big(\partial_t u_\nu^{j-1}\big|_{t=0} + \sum_{k=0}^{j-1}(u_{\nu,k}\cdot\nabla)u_{\nu,j-1-k}\Big), \quad \nu = e, i, \end{aligned} \tag{7.3.14}$$

$$E_j = \partial_t\psi^{j-1}(0,.) - \nabla\phi_j, \quad B_j = B^j(0,.), \tag{7.3.15}$$

where ϕ_j is determined by :

$$-\Delta\phi_j = n_{i,j} - n_{e,j} \quad \text{in } \mathbb{T} \quad \text{and} \quad m(\phi_j) = 0. \tag{7.3.16}$$

Proposition 7.3.1 *Assume that the initial data $(n_{\nu,j}, u_{\nu,j}, E_j, B_j)_{j\ge0}$ are sufficiently smooth for $j \in \mathbb{N}$, with $n_{\nu,0} \ge constant > 0$ in $\mathbb{T}$, and satisfy the compatibility conditions (7.3.7)-(7.3.8) and (7.3.14)-(7.3.16) for $j \ge 1$. Then there exists a unique asymptotic expansion up to any order of the form (7.3.2), i.e. there exist $T_1 > 0$ and a unique smooth solution $(n_\nu^j, u_\nu^j, E^j, B^j)_{j\ge0}$ in the time interval $[0, T_1]$ of problems (7.3.4)-(7.3.6) and (7.3.10)-(7.3.13) for $j \ge 1$. Moreover, $n_\nu^0 \ge constant > 0$ in $[0, T_1]\times\mathbb{T}$. In particular, the formal zero-relaxation limit $\tau \to 0$ of the two-fluid Euler-Maxwell system (7.1.4) is the classical drift-diffusion system (7.3.4) and (7.3.6).*

7.3.2 Convergence results

Let $m \ge 0$ be a fixed integer. We denote by :

$$(n_{\nu,\tau}^m, u_{\nu,\tau}^m, \tilde{E}_\tau^m, B_\tau^m) = \sum_{j=0}^m \tau^{2j}(n_\nu^j, \tau u_\nu^j, E^j, \tau B^j), \quad \nu = e, i, \tag{7.3.17}$$

an approximate solution of order m, where $(n_\nu^j, u_\nu^j, E^j, B^j)_{0\le j\le m}$ are constructed in the previous subsection. From the construction of the approximate solution, for $(t,x) \in [0,T_1]\times\mathbb{T}$ we have :

$$\operatorname{div}\tilde{E}_\tau^m = n_{i,\tau}^m - n_{e,\tau}^m, \quad \operatorname{div} B_\tau^m = 0. \tag{7.3.18}$$

We define the remainders $R^{\tau,m}_{n_\nu}$, $R^{\tau,m}_{u_\nu}$ and $R^{\tau,m}_{E}$ and $\tilde{R}^{\tau,m}_{B}$ by :

$$\begin{cases} \partial_t n^m_{\nu,\tau} + \dfrac{1}{\tau}\operatorname{div}(n^m_{\nu,\tau}u^m_{\nu,\tau}) = R^{\tau,m}_{n_\nu}, \quad \nu = e,i, \\ \partial_t u^m_{\nu,\tau} + \dfrac{1}{\tau}(u^m_{\nu,\tau}\cdot\nabla)u^m_{\nu,\tau} + \dfrac{1}{\tau}\nabla h_\nu(n^m_{\nu,\tau}) = q_\nu\dfrac{\tilde{E}^m_\tau}{\tau} - \dfrac{u^m_{\nu,\tau}}{\tau^2} + q_\nu\dfrac{u^m_{\nu,\tau}\times B^m_\tau}{\tau} + R^{\tau,m}_{u_\nu}, \\ \partial_t \tilde{E}^m_\tau - \dfrac{1}{\tau}\nabla\times B^m_\tau = \dfrac{1}{\tau}\big(n^m_{e,\tau}u^m_{e,\tau} - n^m_{i,\tau}u^m_{i,\tau}\big) + R^{\tau,m}_E, \\ \partial_t B^m_\tau + \dfrac{1}{\tau}\nabla\times\tilde{E}^m_\tau = \tilde{R}^{\tau,m}_B. \end{cases} \tag{7.3.19}$$

It is clear that the convergence rate depends strongly on the order of the remainders with respect to τ. Since the last equation in (7.3.19) is linear, for sufficiently smooth profiles $(n^j_\nu, u^j_\nu, E^j, B^j)_{j\geq 0}$, it is easy to see that :

$$\tilde{R}^{\tau,m}_B = \tau^{2m+1}\partial_t B^m. \tag{7.3.20}$$

Moreover, a further computation gives :

$$R^{\tau,m}_{n_\nu} = O\big(\tau^{2(m+1)}\big), \quad R^{\tau,m}_{E} = O\big(\tau^{2(m+1)}\big), \quad R^{\tau,m}_{u_\nu} = O\big(\tau^{2m+1}\big). \tag{7.3.21}$$

In (7.3.20)-(7.3.21), there is a loss of one order for the remainders $R^{\tau,m}_{u_\nu}$ and $\tilde{R}^{\tau,m}_B$. For $R^{\tau,m}_{u_\nu}$ this loss will be recovered in the error estimate of convergence due to the dissipation term for u_ν. However, the situation is different for $\tilde{R}^{\tau,m}_B$ since the equation for B is linear. A simple way to remedy this is to introduce a correction term into $\tilde{E}^m_\tau$ so that :

$$E^m_\tau = \tilde{E}^m_\tau + \tau^{2(m+1)}E^{m+1}_c = \sum_{j=0}^m \tau^{2j}E^j + \tau^{2(m+1)}E^{m+1}_c. \tag{7.3.22}$$

In view of (7.3.18)-(7.3.20), E^{m+1}_c should be defined by :

$$\nabla\times E^{m+1}_c = -\partial_t B^m, \quad \operatorname{div} E^{m+1}_c = 0, \quad m(E^{m+1}_c) = 0, \tag{7.3.23}$$

so that the new remainder $R^{\tau,m}_B$ of B satisfies :

$$R^{\tau,m}_B \stackrel{def}{=} \partial_t B^m_\tau + \frac{1}{\tau}\nabla\times E^m_\tau = 0 \tag{7.3.24}$$

and we still have

$$\operatorname{div} E^m_\tau = n^m_{i,\tau} - n^m_{e,\tau}, \quad \operatorname{div} B^m_\tau = 0. \tag{7.3.25}$$

Since the correction term is of order $O\big(\tau^{2(m+1)}\big)$, the orders of the remainders $R^{\tau,m}_{n_\nu}$, $R^{\tau,m}_{u_\nu}$ and $R^{\tau,m}_E$ are not changed. Moreover, the correction term does not affect on assumption (7.3.28) below.

We conclude the above discussion as follows.

Proposition 7.3.2 *Under the assumption of Proposition 7.3.1, for all integer $m \geq 0$, the remainder $R^{\tau,m}_B$ satisfies (7.3.24) and the remainders $R^{\tau,m}_{n_\nu}$, $R^{\tau,m}_{u_\nu}$, $R^{\tau,m}_E$ satisfy :*

$$\sup_{0\leq t\leq T_1}\|(R^{\tau,m}_{n_\nu}, R^{\tau,m}_E)(t,\cdot)\|_s \leq C_m\tau^{2(m+1)}, \quad \sup_{0\leq t\leq T_1}\|R^{\tau,m}_{u_\nu}(t,\cdot)\|_s \leq C_m\tau^{2m+1}, \tag{7.3.26}$$

where $C_m > 0$ is a constant independent of τ.

The convergence result of this section is stated as follows of which the proof is given in section 5.

Theorem 7.3.1 *Let $m \geq 0$ and $s > \frac{5}{2}$ be any fixed integers. Let the assumption of Proposition 7.3.1 hold. Suppose*

$$\operatorname{div} E_0^\tau = n_{i,0}^\tau - n_{e,0}^\tau, \quad \operatorname{div} B_0^\tau = 0 \quad in \quad \mathbb{T}, \tag{7.3.27}$$

and

$$\left\| (n_{\nu,0}^\tau, u_{\nu,0}^\tau, E_0^\tau, B_0^\tau) - \sum_{j=0}^{m} \tau^{2j}(n_{\nu,j}, \tau u_{\nu,j}, E_j, \tau B_j) \right\|_s \leq C_1 \tau^{2(m+1)}, \quad \nu = e, i, \tag{7.3.28}$$

where $C_1 > 0$ is a constant independent of τ. Then there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \geq T_1$ and the solution $(n_\nu^\tau, u_\nu^\tau, E^\tau, B^\tau)$ to the periodic problem (7.1.4)-(7.1.5) satisfies :

$$\left\| (n_\nu^\tau, u_\nu^\tau, E^\tau, B^\tau)(t) - (n_{\nu,\tau}^m, u_{\nu,\tau}^m, E_\tau^m, B_\tau^m)(t) \right\|_s \leq C_2 \tau^{2(m+1)}, \quad \forall\, t \in [0, T_1], \quad \nu = e, i.$$

Moreover,

$$\left\| u_\nu^\tau - u_{\nu,\tau}^m \right\|_{L^2(0,T_1;H^s(\mathbb{T}))} \leq C_2 \tau^{2m+3}, \quad \nu = e, i.$$

7.4 Case of ill-prepared initial data

7.4.1 Initial layer corrections

In the discussion of the limit in section 3, the compatibility conditions are made on the initial data. These conditions means that the initial profiles $(u_\nu^j, E^j, B^j)(0, \cdot)$ are determined through the resolution of the problems (7.3.3), (7.3.9) for $(n_\nu^j, u_\nu^j, E^j, B^j)$. Then $(u_{\nu,0}^\tau, E_0^\tau, B_0^\tau)$ cannot be given explicitly. In this section, we consider the case of the so called ill prepared initial data by adding an initial layer correction in the asymptotic expansion. We seek the simplest possible form of an asymptotic expansion with initial layer corrections such that its remainders are at least of order $O(\tau)$ for variable u_ν.

Let the initial data of an approximate solution $(n_{\nu,\tau}, u_{\nu,\tau}, E_\tau, B_\tau)$ have an asymptotic expansion of the form :

$$(n_{\nu,\tau}, u_{\nu,\tau}, E_\tau, B_\tau)\big|_{t=0} = \big(n_{\nu,0}, \tau u_{\nu,0}, E_0, \tau B_0\big) + O(\tau^2), \quad \nu = e, i, \tag{7.4.1}$$

where $(n_{\nu,0}, u_{\nu,0}, E_0, B_0)$ are given smooth functions. We wand to construct the simplest form an asymptotic expansion of the approximate solution $(n_{\nu,\tau}, u_{\nu,\tau}, E_\tau, B_\tau)$ up to order 1. Then we may take the following ansatz :

$$\begin{aligned}(n_{\nu,\tau}, u_{\nu,\tau}, E_\tau, B_\tau)(t,x) &= (n_\nu^0, \tau u_\nu^0, E^0 + \tau^2 E_c^1, \tau B^0)(t,x) \\ &\quad + \big((n_{\nu,I}^0, \tau u_{\nu,I}^0, E_I^0, \tau B_I^0) + \tau^2(n_{\nu,I}^1, \tau u_{\nu,I}^1, E_I^1, \tau B_I^1)\big)(z,x) + O(\tau^2),\end{aligned} \tag{7.4.2}$$

where $z = t/\tau^2 \in \mathbb{R}$ is the fast variable, the subscript I stands for the initial layer variables and E_c^1 is the correction term defined by (7.3.23) with $m = 0$. As we will see below, this expansion is enough to give the remainders at least of order $O(\tau)$ for variable u_ν, which is the case of well-prepared initial data for $m = 0$.
Substituting the expression (7.4.2) into the problem (7.1.4), we have :

1. The leading profiles $(n_\nu^0, u_\nu^0, E^0, B^0)$ are determined by the drift-diffusion system (7.3.4)-(7.3.6). The smooth solution $(n_\nu^0, u_\nu^0, E^0, B^0)$ is defined in the time interval $[0, T_1]$ in the class $m(B^0) = 0$, without any compatibility conditions. From (7.3.6) we have :

$$u_\nu^0(0,.) = -\nabla(h_\nu(n_{\nu,0}) + q_\nu \phi_0), \quad \nu = e, i, \tag{7.4.3}$$

   where ϕ_0 is given by (7.3.8).

   Now we determine the initial-layer profiles $(n_{\nu,I}^0, u_{\nu,I}^0, E_I^0, B_I^0)$ and $(n_{\nu,I}^1, u_{\nu,I}^1, E_I^1, B_I^1)$.

2. The leading correction terms $(n_{\nu,I}^0, u_{\nu,I}^0, E_I^0, B_I^0)$ satisfy the following equations :

$$\partial_z n_{\nu,I}^0 = 0, \quad \partial_z E_I^0 = 0, \quad \partial_z B_I^0 + \nabla \times E_I^0 = 0, \quad \nu = e, i \tag{7.4.4}$$

   and

$$\partial_z u_{\nu,I}^0 + u_I^0 = 0, \quad \nu = e, i. \tag{7.4.5}$$

   Equations (7.4.4) imply that there are no first order initial layers on variables n_ν, E and B. Therefore, up to a constant, we may take :

$$n_\nu^0(0,x) = n_{\nu,0}(x), \quad E^0(0,x) = E_0(x) \text{ and } B^0(0,x) = B_0(x), \quad \nu = e, i. \tag{7.4.6}$$

   Moreover, expressions (7.4.1) and (7.4.2) for u_ν imply that :

$$u_\nu^0(0,x) + u_{\nu,I}^0(0,x) = u_{\nu,0}(x), \quad \nu = e, i, \tag{7.4.7}$$

   which determines the initial value of $u_{\nu,I}^0$, where $u_\nu^0(0,\cdot)$ is given by (7.4.3). Together with (7.4.5), we obtain :

$$u_{\nu,I}^0(z,x) = u_{\nu,I}^0(0,x)e^{-z} = \big(u_{\nu,0}(x) - u_\nu^0(0,x)\big)e^{-z}, \quad \nu = e, i. \tag{7.4.8}$$

   The second order correction terms $(n_{\nu,I}^1, u_{\nu,I}^1, E_I^1, B_I^1)$ satisfy :

$$u_{\nu,I}^1 = 0, \quad \nu = e, i, \tag{7.4.9}$$

$$\partial_z n_{\nu,I}^1(z,x) + \operatorname{div}\big(n_\nu^0(0,x)u_{\nu,I}^0(z,x)\big) = 0, \quad \nu = e, i, \tag{7.4.10}$$

$$\partial_z E_I^1(z,x) = n_e^0(0,x)u_{e,I}^0(z,x) - n_i^0(0,x)u_{i,I}^0(z,x), \tag{7.4.11}$$

   and

$$\partial_z B_I^1(z,x) + \nabla \times E_I^1(z,x) = 0. \tag{7.4.12}$$

Let $(n_{\nu,1}, E_1, B_1)$ be smooth functions such that :

$$E_1(x) = -n_e^0(0,x)\big(u_{e,0}(x) - u_e^0(0,x)\big) + n_i^0(0,x)\big(u_{i,0}(x) - u_i^0(0,x)\big), \qquad (7.4.13)$$

and

$$n_{i,1} - n_{e,1} = \operatorname{div} E_1, \quad \operatorname{div} B_1 = 0. \qquad (7.4.14)$$

Set

$$(n_{\nu,I}^1, E_I^1, B_I^1)(0,x) = (n_{\nu,1}, E_1, B_1)(x), \quad \nu = e, i.$$

Together with (7.4.8) and (7.4.10)-(7.4.13), it is easy to obtain :

$$n_{\nu,I}^1(z,x) = n_{\nu,1}(x) - \operatorname{div}\big(n_\nu^0(0,x)(u_{\nu,0}(x) - u_\nu^0(0,x))\big)\big(1 - e^{-z}\big), \quad \nu = e, i, \qquad (7.4.15)$$

$$E_I^1(z,x) = -\Big[n_e^0(0,x)\big(u_{e,0}(x) - u_e^0(0,x)\big) - n_i^0(0,x)\big(u_{i,0}(x) - u_i^0(0,x)\big)\Big]e^{-z}, \qquad (7.4.16)$$

and

$$\begin{aligned} B_I^1(z,x) \;\; = \;\; & \nabla \times \Big[n_e^0(0,x)\big(u_{e,0}(x) - u_e^0(0,x)\big) - n_i^0(0,x)\big(u_{i,0}(x) - u_i^0(0,x)\big)\Big](1 - e^{-z}) \\ & + B_1(x). \end{aligned} \qquad (7.4.17)$$

Finally, from (7.4.14) we have :

$$\operatorname{div} E_I^1 + n_{e,I}^1 - n_{i,I}^1 = 0, \quad \operatorname{div} B_I^1 = 0. \qquad (7.4.18)$$

Thus, the asymptotic expansion is constructed up to order 1 for general initial data.

Proposition 7.4.1 *Assume that the initial data $(n_{\nu,0}, u_{\nu,0}, E_0, B_0)$ are sufficiently smooth with $n_{\nu,0} > 0$ in $\mathbb{T}$. Then there exists a unique asymptotic expansion up to order 1 of the form (7.4.2), in which the correction terms are determined by (7.4.4), (7.4.8), (7.4.9) and (7.4.15)-(7.4.17).*

7.4.2 Convergence results

According to the asymptotic expansions above, set :

$$\begin{cases} n_{\nu,\tau}^I(t,x) = n_\nu^0(t,x) + \tau^2 n_{\nu,I}^1(t/\tau^2, x), \\ u_{\nu,\tau}^I(t,x) = \tau\big(u_\nu^0(t,x) + u_{\nu,I}^0(t/\tau^2, x)\big), \\ E_\tau^I(t,x) = E^0(t,x) + \tau^2\big(E_c^1(t,x) + E_I^1(t/\tau^2, x)\big), \\ B_\tau^I(t,x) = \tau\big(B^0(t,x) + \tau^2 B_I^1(t/\tau^2, x)\big). \end{cases} \quad \nu = e, i \qquad (7.4.19)$$

Then we have :

$$t = 0 \;:\; \big(n_{\nu,\tau}^I, u_{\nu,\tau}^I, E_\tau^I, B_\tau^I\big) = \big(n_{\nu,0}, \tau u_{\nu,0}, E_0, \tau B_0\big) + \tau^2\big(n_{\nu,1}, 0, E_1 + E_c^1(0,\cdot), \tau B_1\big). \qquad (7.4.20)$$

Moreover, equations (7.3.3), (7.3.25) and (7.4.18) imply that :

$$\operatorname{div} E^I_\tau = n^I_{i,\tau} - n^I_{e,\tau}, \quad \operatorname{div} B^I_\tau = 0. \tag{7.4.21}$$

By the construction above, we have :

$$\begin{cases} \partial_t n^I_{\nu,\tau} + \dfrac{1}{\tau}\operatorname{div}(n^I_{\nu,\tau} u^I_{\nu,\tau}) = R^{\tau,I}_{n_\nu}, \quad \nu = e, i, \\ \partial_t u^I_{\nu,\tau} + \dfrac{1}{\tau}(u^I_{\nu,\tau}\cdot\nabla)u^I_{\nu,\tau} + \dfrac{1}{\tau}\nabla h_\nu(n^I_{\nu,\tau}) = q_\nu \dfrac{E^I_\tau}{\tau} - \dfrac{u^I_{\nu,\tau}}{\tau^2} + q_\nu \dfrac{u^I_{\nu,\tau}\times B^I_\tau}{\tau} + R^{\tau,I}_{u_\nu}, \\ \partial_t E^I_\tau - \dfrac{1}{\tau}\nabla\times B^I_\tau = \dfrac{1}{\tau}\big(n^I_{e,\tau}u^I_{e,\tau} - n^I_{i,\tau}u^I_{i,\tau}\big) + R^{\tau,I}_E, \\ \partial_t B^I_\tau + \dfrac{1}{\tau}\nabla\times E^I_\tau = R^{\tau,I}_B. \end{cases} \tag{7.4.22}$$

Using equations (7.3.3), (7.3.23) for $(n^0_\nu, u^0_\nu, E^0, B^0, E^1_c)$ and (7.4.5),(7.4.10)-(7.4.11) for $(u^0_{\nu,I}, n^1_{\nu,I}, E^1_I, B^1_I)$, we obtain :

$$\begin{aligned} R^{\tau,I}_{n_\nu} &= \partial_t\big(n^0_\nu + \tau^2 n^1_{\nu,I}\big) + \operatorname{div}\big((n^0_\nu + \tau^2 n^1_{\nu,I})(u^0_\nu + u^0_{\nu,I})\big) \\ &= \operatorname{div}\big((n^0_\nu(t,x) - n^0_\nu(0,x))u^0_I(z,x)\big) + \tau^2\operatorname{div}\big(n^1_{\nu,I}(u^0_\nu + u^0_{\nu,I})\big), \end{aligned}$$

$$\begin{aligned} R^{\tau,I}_{u_\nu} &= \tau\big(\partial_t(u^0_\nu + u^0_{\nu,I}) + (u^0_\nu + u^0_{\nu,I})\cdot\nabla(u^0_\nu + u^0_{\nu,I}) - q_\nu(u^0_\nu + u^0_{\nu,I})\times(B^0 + \tau^2 B^1_I)\big) \\ &\quad + \frac{1}{\tau}\big(\nabla h_\nu(n^0_\nu + \tau^2 n^1_{\nu,I}) - q_\nu(E^0 + \tau^2 E^1_c + \tau^2 E^1_I) + (u^0_\nu + u^0_{\nu,I})\big) \\ &= \tau\big(\partial_t u^0_\nu + (u^0_\nu + u^0_{\nu,I})\cdot\nabla(u^0_\nu + u^0_{\nu,I}) - q_\nu(u^0_\nu + u^0_{\nu,I})\times B^0\big) \\ &\quad - q_\nu\,\tau(E^1_I + E^1_c) + \frac{1}{\tau}\nabla\big(h_\nu(n^0_\nu + \tau^2 n^1_{\nu,I}) - h_\nu(n^0_\nu)\big) + \tau^3(u^0_\nu + u^1_{\nu,I})\times B^1_I, \end{aligned}$$

$$\begin{aligned} R^{\tau,I}_E &= \partial_t(E^0 + \tau^2 E^1_c + \tau^2 E^1_I) - \nabla\times(B^0 + \tau^2 B^1_I) \\ &\quad + (n^0_i + \tau^2 n^1_{i,I})(u^0_i + u^0_{i,I}) - (n^0_e + \tau^2 n^1_{e,I})(u^0_e + u^0_{e,I}) \\ &= \big(n^0_e(0,x) - n^0_e(t,x)\big)u^0_{e,I}(z,x) + \big(n^0_i(t,x) - n^0_i(0,x)\big)u^0_{i,I}(z,x) \\ &\quad + \tau^2\big(n^1_{i,I}(u^0_i + u^0_{i,I}) - n^1_{e,I}(u^0_e + u^0_{e,I}) + \partial_t E^1_c - \nabla\times B^1_I\big) \end{aligned}$$

and

$$\begin{aligned} R^{\tau,I}_B &= \tau\partial_t(B^0 + \tau^2 B^1_I) + \frac{1}{\tau}\nabla\times\big(E^0 + \tau^2 E^1_c + \tau^2 E^1_I\big) \\ &= \frac{1}{\tau}\nabla\times E^0 + \tau\big(\partial_t B^0 + \nabla\times E^1_c\big) + \tau\big(\partial_z B^1_I + \nabla\times E^1_I\big) \\ &= 0. \end{aligned}$$

Now we establish error estimates for $(R^{\tau,I}_{n_\nu}, R^{\tau,I}_{u_\nu}, R^{\tau,I}_E, R^{\tau,I}_B)$. For $R^{\tau,I}_{n_\nu}$ and $R^{\tau,I}_E$, there is $\eta \in$

$[0,t] \subset [0,T_1]$ such that :

$$n_\nu^0(t,x) - n_\nu^0(0,x) = t\partial_t n_\nu^0(\eta,x) = \tau^2 z \partial_t n_\nu^0(\eta,x), \quad \nu = e,i.$$

Since function $z \longmapsto ze^{-z}$ is bounded for $z \geq 0$ and

$$\partial_t n_\nu^0 = -\operatorname{div}(n_\nu^0 u_\nu^0), \quad \nu = e,i,$$

it follows from (7.4.8) that :

$$\big(n_\nu^0(t,x) - n_\nu^0(0,x)\big) u_{\nu,I}^0(z,x) = O(\tau^2), \quad \nu = e,i.$$

Thus

$$R_{n_\nu}^{\tau,I} = O(\tau^2) \quad \text{and} \quad R_E^{\tau,I} = O(\tau^2), \quad \nu = e,i.$$

Finally, for $R_{u_\nu}^{\tau,I}$, we have :

$$h_\nu(n_\nu^0 + \tau^2 n_{\nu,I}^1) - h_\nu(n_\nu^0) = O(\tau^2), \quad \nu = e,i.$$

Then

$$R_{u_\nu}^{\tau,I} = O(\tau), \quad \nu = e,i.$$

From the previous discussions on the remainders, we obtain the following error estimates.

Proposition 7.4.2 *For given smooth data, the remainders $R_{n_\nu}^{\tau,I}$, $R_{u_\nu}^{\tau,I}$, $R_E^{\tau,I}$ and $R_B^{\tau,I}$ satisfy :*

$$\sup_{0\leq t\leq T_1} \|\big(R_{n_\nu}^{\tau,I}, R_E^{\tau,I}\big)(t,\cdot)\|_s \leq C\tau^2, \qquad \sup_{0\leq t\leq T_1} \|R_{u_\nu}^{\tau,I}(t,\cdot)\|_s \leq C\tau, \quad R_B^{\tau,I} = 0, \tag{7.4.23}$$

where $C > 0$ is a constant independent of τ.

The convergence result with initial layers can be stated as follows.

Theorem 7.4.1 *Let $s > 5/2$ be a fixed integer and $(n_{\nu,0}, u_{\nu,0}, E_0, B_0) \in H^{s+1}(\mathbb{T})$ with $n_{\nu,0} \geq$ constant > 0 in $\mathbb{T}$. Suppose (7.3.27) holds and*

$$\Big\|\big(n_{\nu,0}^\tau, u_{\nu,0}^\tau, E_0^\tau, B_0^\tau\big) - (n_{\nu,0}, \tau u_{\nu,0}, E_0, \tau B_0)\Big\|_s \leq C_1\tau^2, \quad \nu = e,i, \tag{7.4.24}$$

where $C_1 > 0$ is a constant independent of τ. Then there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \geq T_1$ and the solution $(n_\nu^\tau, u_\nu^\tau, E^\tau, B^\tau)$ to the periodic problem (7.1.4)-(7.1.5) satisfies :

$$\big\|\big(n_\nu^\tau, u_\nu^\tau, E^\tau, B^\tau\big) - \big(n_\nu^0, u_{\nu,\tau}^I, E^0, B^0\big)(t)\big\|_s \leq C_2\tau^2, \quad \forall\, t \in [0,T_1], \quad \nu = e,i.$$

Moreover,

$$\big\|u_\nu^\tau - u_{\nu,\tau}^I\big\|_{L^2(0,T_1;H^s(\mathbb{T}))} \leq C_2\tau^3, \quad \nu = e,i.$$

7.5 Justification of asymptotic expansions

7.5.1 Statement of the main result

In this section, we justify rigorously the asymptotic expansions of solutions $(n_\nu^\tau, u_\nu^\tau, E^\tau, B^\tau)$ to the periodic problem (7.1.4)-(7.1.5) developed in section 3-4. We prove a more general convergence result which implies both Theorems 7.3.1-7.4.1. As a consequence, we obtain the existence of exact solutions $(n_\nu^\tau, u_\nu^\tau, E^\tau, B^\tau)$ in a time interval independent of τ. To justify rigorously the asymptotic expansions (7.3.2) and (7.4.19), it suffices to obtain the uniform estimates of the smooth solutions to (7.1.4) with respect to the parameter τ.

Let $(n_\nu^\tau, u_\nu^\tau, E^\tau, B^\tau)$ be the exact solution to (7.1.4) with initial data $(n_{\nu,0}^\tau, u_{\nu,0}^\tau, E_0^\tau, B_0^\tau)$ and $(n_{\nu,\tau}, u_{\nu,\tau}, E_\tau, B_\tau)$ be an approximate periodic solution defined on $[0, T_1]$, with

$$(n_{\nu,\tau}, u_{\nu,\tau}, E_\tau, B_\tau) \in C([0,T_1], H^{s+1}(\mathbb{T})) \cap C^1([0,T_1], H^s(\mathbb{T})), \quad \nu = e, i.$$

We define the remainders of the approximate solution by :

$$\begin{cases} R_{n_\nu}^\tau = \partial_t n_{\nu,\tau} + \dfrac{1}{\tau}\,\mathrm{div}(n_{\nu,\tau} u_{\nu,\tau}), \\ R_{u_\nu}^\tau = \partial_t u_{\nu,\tau} + \dfrac{1}{\tau}(u_{\nu,\tau}\cdot\nabla) u_{\nu,\tau} + \dfrac{1}{\tau}\nabla h_\nu(n_{\nu,\tau}) - q_\nu \dfrac{E_\tau}{\tau} - q_\nu \dfrac{u_{\nu,\tau}\times B_\tau}{\tau} + \dfrac{u_{\nu,\tau}}{\tau^2}, \\ R_E^\tau = \partial_t E_\tau - \dfrac{1}{\tau}\nabla\times B_\tau + \dfrac{1}{\tau}\big(n_{i,\tau}u_{i,\tau} - n_{e,\tau}u_{e,\tau}\big), \\ R_B^\tau = \partial_t B_\tau + \dfrac{1}{\tau}\nabla\times E_\tau, \quad \nu = e, i. \end{cases} \tag{7.5.1}$$

Suppose

$$\mathrm{div}\, E_\tau = n_{i,\tau} - n_{e,\tau}, \quad \mathrm{div}\, B_\tau = 0, \quad \nu = e, i, \tag{7.5.2}$$

$$\sup_{0\le t\le T_1} \|(n_{\nu,\tau}, E_\tau, B_\tau)(t,\cdot)\|_s \le C_1, \quad \sup_{0\le t\le T_1} \|u_{\nu,\tau}(t,\cdot)\|_s \le C_1\tau, \tag{7.5.3}$$

$$\left\|\big(n_{\nu,0}^\tau - n_{\nu,\tau}(0,\cdot), u_{\nu,0}^\tau - u_{\nu,\tau}(0,\cdot), E_0^\tau - E_\tau(0,\cdot), B_0^\tau - B_\tau(0,\cdot)\big)\right\|_s \le C_1\tau^{\lambda+1}, \tag{7.5.4}$$

$$\sup_{0\le t\le T_1} \|(R_{n_\nu}^\tau, R_E^\tau)(t,\cdot)\|_s \le C_1\tau^{\lambda+1}, \quad \sup_{0\le t\le T_1} \|R_{u_\nu}^\tau(t,\cdot)\|_s \le C_1\tau^{\lambda}, \quad R_B^\tau = 0, \tag{7.5.5}$$

where $\lambda \ge 0$ and $C_1 > 0$ are constants independent of τ.

Theorem 7.5.1 *Let $\lambda \ge 0$. Under the above assumptions, there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \ge T_1$ and the solution $(n_\nu^\tau, u_\nu^\tau, E^\tau, B^\tau)$ of the periodic problem (7.1.4)-(7.1.5) satisfies :*

$$\left\|(n_\nu^\tau, u_\nu^\tau, E^\tau, B^\tau)(t) - (n_{\nu,\tau}, u_{\nu,\tau}, E_\tau, B_\tau)(t)\right\|_s \le C_2\tau^{\lambda+1}, \quad \forall\, t\in[0,T_1], \quad \nu = e, i. \tag{7.5.6}$$

Moreover,

$$\left\|u_\nu^\tau - u_{\nu,\tau}\right\|_{L^2(0,T_1;H^s(\mathbb{T}))} \le C_2\tau^{\lambda+2}, \quad \nu = e, i. \tag{7.5.7}$$

Remark 7.5.1 *It is clear that Theorem 7.5.1 implies Theorems 7.3.1-7.4.1. In particular,* $\lambda = 2m+1$ *with* $m \geq 0$ *in section* 3 *and* $\lambda = 1$ *in section* 5*, since*

$$\|(n_{\tau,I}, E_{\tau,I}, B_{\tau,I})(t) - (n^0, E^0, B^0)(t)\|_s = O(\tau^2),$$

uniformly with respect to t*.*

7.5.2 Proof of the main result

By proposition 7.1.1, the exact solution $(n_\nu^\tau, u_\nu^\tau, E^\tau, B^\tau)$ is defined in a time interval $[0, T_1^\tau)$ with $T_1^\tau > 0$. Since $n_\nu^\tau \in C\big([0,T_1^\tau), H^s(\mathbb{T})\big)$ and the embedding from $H^s(\mathbb{T})$ to $C(\mathbb{T})$ is continuous, we have $n_\nu^\tau \in C([0,T_1^\tau) \times \mathbb{T})$. From assumption $n_{\nu,0}^\tau \geq \kappa > 0$, we deduce that there exist a maximal existence time $T_2^\tau \in (0, T_1^\tau]$ and a constant $C_0 > 0$, independent of τ, such that $\frac{\kappa}{2} \leq n_\nu^\tau(t,x) \leq C_0$ for all $(t,x) \in [0,T_2^\tau) \times \mathbb{T}$. Then we define $T^\tau = \min(T_1, T_2^\tau) > 0$ so that the exact solution and the approximate solution are both defined in the time interval $[0, T^\tau)$. In this time interval, we denote by :

$$(N_\nu^\tau, U_\nu^\tau, F^\tau, G^\tau) = (n_\nu^\tau - n_{\nu,\tau}, u_\nu^\tau - u_{\nu,\tau}, E^\tau - E_\tau, B^\tau - B_\tau), \quad \nu = e, i. \tag{7.5.8}$$

Obviously, $(N_\nu^\tau, U_\nu^\tau, F^\tau, G^\tau)$ satisfies the following problem :

$$\begin{cases} \partial_t N_\nu^\tau + \dfrac{1}{\tau}\big((U_\nu^\tau + u_{\nu,\tau})\cdot\nabla\big)N_\nu^\tau + \dfrac{1}{\tau}(N_\nu^\tau + n_{\nu,\tau})\operatorname{div} U_\nu^\tau \\ \quad = -\dfrac{1}{\tau}\big(N_\nu^\tau \operatorname{div} u_{\nu,\tau} + (U_\nu^\tau\cdot\nabla)n_{\nu,\tau}\big) - R_n^\tau, \\ \partial_t U_\nu^\tau + \dfrac{1}{\tau}\big((U_\nu^\tau + u_{\nu,\tau})\cdot\nabla\big)U_\nu^\tau + \dfrac{1}{\tau}h_\nu'(N_\nu^\tau + n_{\nu,\tau})\nabla N_\nu^\tau \\ \quad = -\dfrac{1}{\tau}\big[(U_\nu^\tau\cdot\nabla)u_{\nu,\tau} + \big(h_\nu'(N_\nu^\tau + n_{\nu,\tau}) - h'(n_{\nu,\tau})\big)\nabla n_{\nu,\tau}\big] - \dfrac{U_\nu^\tau}{\tau^2} \\ \quad +q_\nu\dfrac{1}{\tau}\big[F^\tau + (U_\nu^\tau + u_{\nu,\tau})\times G^\tau + U_\nu^\tau \times B_\tau\big] - R_{u_\nu}^\tau, \\ \partial_t F^\tau - \dfrac{1}{\tau}\nabla\times G^\tau = \dfrac{1}{\tau}\big(N_e^\tau U_e^\tau + N_e^\tau u_{e,\tau} + n_{e,\tau}U_e^\tau\big) \\ \qquad\qquad\qquad -\dfrac{1}{\tau}\big(N_i^\tau U_i^\tau + N_i^\tau u_{i,\tau} + n_{i,\tau}U_i^\tau\big) - R_E^\tau, \\ \operatorname{div} F^\tau = N_i^\tau - N_e^\tau, \\ \partial_t G^\tau + \dfrac{1}{\tau}\nabla\times F^\tau = 0, \quad \operatorname{div} G^\tau = 0, \\ t = 0 \;:\; \quad (N_\nu^\tau, U_\nu^\tau, F^\tau, G^\tau) = \big(n_{\nu,0}^\tau - n_{\nu,\tau}(0,\cdot), u_{\nu,0}^\tau - u_{\nu,\tau}(0,\cdot), \\ \qquad\qquad\qquad E_0^\tau - E_\tau(0,\cdot), B_0^\tau - B_\tau(0,\cdot)\big). \end{cases} \quad \nu = e, i, \tag{7.5.9}$$

Set

$$W_{\nu,I}^\tau = \begin{pmatrix} N_\nu^\tau \\ U_\nu^\tau \end{pmatrix}, \quad W_{II}^\tau = \begin{pmatrix} F^\tau \\ G^\tau \end{pmatrix}, \quad W_\nu^\tau = \begin{pmatrix} W_{\nu,I}^\tau \\ W_{II}^\tau \end{pmatrix},$$

The existence and uniqueness of smooth solutions to (7.1.4)-(7.1.5) is equivalent to that of

(7.5.9). Thus, in order to prove Theorem 7.5.1, it suffices to establish uniform estimates of W_ν^τ with respect to τ. In what follows, we denote by $C > 0$ various constants independent of τ and for $\alpha \in \mathbb{N}^3$, $(W_{\nu,I\alpha}^\tau, W_{II\alpha}^\tau) = \partial_x^\alpha(W_{\nu,I}^\tau, W_{II}^\tau)$ etc. The main estimates are contained in the following two lemmas for $W_{\nu,I}^\tau$ and W_{II}^τ, respectively. We first consider the estimate for $W_{\nu,I}^\tau$.

Lemma 7.5.1 *Under the assumptions of Theorem 7.5.1, for all $t \in (0, T^\tau)$, as $\tau \to 0$ we have :*

$$\|W_{\nu,I}^\tau(t)\|_s^2 + \frac{1}{\tau^2}\int_0^t \|U_\nu^\tau(\xi)\|_s^2\, d\xi \le C\int_0^t \Big(\|W_\nu^\tau(\xi)\|_s^2 + \|W_\nu^\tau(\xi)\|_s^4\Big)\, d\xi + \ C\tau^{2(\lambda+1)}. \tag{7.5.10}$$

Proof. For $\alpha \in \mathbb{N}^3$ with $|\alpha| \le s$, differentiating the two first equations of (7.5.9) with respect to x and multiplying the first resulting equation by $\dfrac{h'_\nu(n_\nu^\tau)}{n_\nu^\tau}$, we have :

$$\begin{cases} \dfrac{h'_\nu(n_\nu^\tau)}{n_\nu^\tau}\Big(\partial_t\partial_x^\alpha N_\nu^\tau + \dfrac{1}{\tau}\big((U_\nu^\tau + u_{\nu,\tau})\cdot\nabla\big)\partial_x^\alpha N_\nu^\tau\Big) + \dfrac{1}{\tau}h'_\nu(n_\nu^\tau)\operatorname{div}\partial_x^\alpha U_\nu^\tau \\ \quad = \dfrac{h'_\nu(n_\nu^\tau)}{\tau n_\nu^\tau}\big(f_1^\alpha(W_\nu^\tau) + f_2^\alpha(W_\nu^\tau)\big) - \dfrac{h'_\nu(n_\nu^\tau)}{n_\nu^\tau}\partial_x^\alpha R_{n_\nu}^\tau, \\ \partial_t\partial_x^\alpha U_\nu^\tau + \dfrac{1}{\tau}\big((U_\nu^\tau + u_{\nu,\tau})\cdot\nabla\big)\partial_x^\alpha U_\nu^\tau + \dfrac{1}{\tau}h'_\nu(n_\nu^\tau)\nabla\partial_x^\alpha N_\nu^\tau + \dfrac{1}{\tau^2}\partial_x^\alpha U_\nu^\tau \\ \quad = \dfrac{1}{\tau}\displaystyle\sum_{i=3}^5 f_i^\alpha(W_\nu^\tau) - \partial_x^\alpha R_{u_\nu}^\tau, \end{cases} \qquad \nu = e, i \tag{7.5.11}$$

where we put

$$f_1^\alpha(W_\nu^\tau) := -\big[\partial_x^\alpha, (U_\nu^\tau + u_{\nu,\tau})\cdot\nabla\big]N_\nu^\tau - [\partial_x^\alpha, (N_\nu^\tau + n_{\nu,\tau})\operatorname{div}]U_\nu^\tau,$$

$$f_2^\alpha(W_\nu^\tau) := -\partial_x^\alpha\big(N_\nu^\tau \operatorname{div} u_{\nu,\tau} + (U_\nu^\tau\cdot\nabla)n_{\nu,\tau}\big),$$

$$f_3^\alpha(W_\nu^\tau) := -\big[\partial_x^\alpha, (U_\nu^\tau + u_{\nu,\tau})\cdot\nabla\big]U_\nu^\tau - \big[\partial_x^\alpha, h'_\nu(n_\nu^\tau)\nabla\big]N_\nu^\tau + q_\nu\big[\partial_x^\alpha, U_\nu^\tau \times\big]B_\tau,$$

$$f_4^\alpha(W_\nu^\tau) := -\partial_x^\alpha\big((U_\nu^\tau\cdot\nabla)u_{\nu,\tau} + \big(h'_\nu(N_\nu^\tau + n_{\nu,\tau}) - h'_\nu(n_{\nu,\tau})\big)\nabla n_{\nu,\tau}\big)$$

$$f_5^\alpha(W_\nu^\tau) := q_\nu\big(\partial_x^\alpha F^\tau + \partial_x^\alpha\big((U_\nu^\tau + u_{\nu,\tau})\times G^\tau\big) + \partial_x^\alpha U_\nu^\tau \times B_\tau\big)$$

and $[\,,\,]$ denotes the commutator defined by $[\partial_x^\alpha, A]B := \partial_x^\alpha(AB) - A\partial_x^\alpha B$. Note that system (7.5.11) is a symmetric form which is different from (7.2.4). We multiply the first equation of (7.5.11) by $\partial_x^\alpha N_\nu^\tau$ and we take the inner product of the second equation with $\partial_x^\alpha U_\nu^\tau$. Then, adding the resulting equations, we obtain :

$$\partial_t\mathcal{H}_\nu^\alpha + \frac{1}{\tau^2}|\partial_x^\alpha U_\nu^\tau|^2 + \frac{1}{\tau}\operatorname{div}\mathcal{Q}_\nu^\alpha = \frac{1}{\tau}\mathcal{R}_\nu^{\alpha,\tau} + \frac{1}{\tau}\mathcal{S}_\nu^\alpha + \frac{1}{\tau}\mathcal{T}_\nu^\alpha + \mathcal{V}_\nu^\alpha, \quad \nu = e, i, \tag{7.5.12}$$

where $\mathcal{H}_\nu^\alpha$, $\mathcal{Q}_\nu^\alpha$, $\mathcal{R}_\nu^{\alpha,\tau}$, $\mathcal{S}_\nu^\alpha$ and $\mathcal{T}_\nu^\alpha$ are defined as :

$$\mathcal{H}_\nu^\alpha := \frac{1}{2}\Big\{\frac{h'_\nu(n_\nu^\tau)}{n_\nu^\tau}(\partial_x^\alpha N_\nu^\tau)^2 + |\partial_x^\alpha U_\nu^\tau|^2\Big\}, \quad \nu = e, i,$$

$$\mathcal{Q}_\nu^\alpha := \frac{1}{2}(U_\nu^\tau + u_{\nu,\tau})\Big\{\frac{h'_\nu(n_\nu^\tau)}{n_\nu^\tau}(\partial_x^\alpha N_\nu^\tau)^2 + |\partial_x^\alpha U_\nu^\tau|^2\Big\} + h'_\nu(n_\nu^\tau)\partial_x^\alpha(N_\nu^\tau + n_{\nu,\tau})\partial_x^\alpha U_\nu^\tau, \quad \nu = e, i,$$

$$\begin{aligned}\mathcal{R}_\nu^{\alpha,\tau} \quad := \quad & -\frac{1}{2}\Big\{\tau\partial_t\Big(\frac{h_\nu'(n_\nu^\tau)}{n_\nu^\tau}\Big) + \operatorname{div}\Big(\frac{h_\nu'(n_\nu^\tau)}{n_\nu^\tau}(U_\nu^\tau + u_{\nu,\tau})\Big)\Big\}(\partial_x^\alpha N_\nu^\tau)^2 \\ & -\frac{1}{2}\operatorname{div}(U_\nu^\tau + u_{\nu,\tau})|\partial_x^\alpha U_\nu^\tau|^2 - \nabla(h_\nu'(n_\nu^\tau))\cdot\partial_x^\alpha N_\nu^\tau \partial_x^\alpha U_\nu^\tau,\end{aligned}$$

$$\mathcal{S}_\nu^\alpha := \partial_x^\alpha N_\nu^\tau f_2^\alpha + \partial_x^\alpha U_\nu^\tau \cdot (f_4^\alpha + f_5^\alpha), \quad \nu = e, i,$$

$$\mathcal{T}_\nu^\alpha := \partial_x^\alpha N_\nu^\tau f_1^\alpha(W_\nu^\tau) + \partial_x^\alpha U_\nu^\tau \cdot f_3^\alpha(W_\nu^\tau), \quad \nu = e, i,$$

and

$$\mathcal{V}_\nu^\alpha := -\Big(\frac{h_\nu'(n_\nu^\tau)}{n_\nu^\tau}\partial_x^\alpha N_\nu^\tau \partial_x^\alpha R_{n_\nu}^\tau + \partial_x^\alpha U_\nu^\alpha \cdot \partial_x^\alpha R_{u_\nu}^\tau\Big).$$

Integrating (7.5.12) over $\mathbb{T}$, we have :

$$\frac{d}{dt}\int_{\mathbb{T}} \mathcal{H}_\nu^\alpha dx + \frac{1}{\tau^2}\|\partial_x^\alpha U_\nu^\tau\|^2 = \frac{1}{\tau}R_\nu^{\alpha,\tau} + \frac{1}{\tau}S_\nu^\alpha + \frac{1}{\tau}T_\nu^\alpha + V_\nu^\alpha, \quad \nu = e, i, \tag{7.5.13}$$

where we put :

$$R_\nu^{\alpha,\tau} := \int_{\mathbb{T}} \mathcal{R}_\nu^{\alpha,\tau} dx, \quad S_\nu^\alpha := \int_{\mathbb{T}} \mathcal{S}_\nu^\alpha dx, \quad T^\alpha := \int_{\mathbb{T}} \mathcal{T}_\nu^\alpha dx, \quad V^\alpha := \int_{\mathbb{T}} \mathcal{V}_\nu^\alpha dx.$$

Here we see that $\mathcal{H}^\alpha$ is equivalent to the quadratic function $|\partial_x^\alpha W_I^\tau|^2$.

In the next we estimate each term of the right hand side of equation (7.5.13). First, we use the expressions of $f_1^\alpha(W_\nu^\tau)$ and $(f_3^\alpha(W_\nu^\tau))$ to compute :

$$\begin{aligned}T_\nu^\tau = \quad - \quad & \Big(\partial_x^\alpha\big(\ ((U_\nu^\tau + u_{\nu,\tau})\cdot\nabla)N_\nu^\tau\ \big) - \big((U_\nu^\tau + u_{\nu,\tau})\cdot\nabla\big)\partial_x^\alpha N_\nu^\tau, N_{\nu,\alpha}^\tau\Big) \\ - \quad & \Big(\partial_x^\alpha\big(\ ((U_\nu^\tau + u_{\nu,\tau})\cdot\nabla)U_\nu^\tau\ \big) - \big((U_\nu^\tau + u_{\nu,\tau})\cdot\nabla\big)\partial_x^\alpha U_\nu^\tau, U_{\nu,\alpha}^\tau\Big) \\ - \quad & \Big(\partial_x^\alpha\big((N_\nu^\tau + n_{\nu,\tau})\operatorname{div} U_\nu^\tau\big) - (N_\nu^\tau + n_{\nu,\tau})\operatorname{div}\partial_x^\alpha U_\nu^\tau, N_{\nu,\alpha}^\tau\Big) \\ - \quad & \Big(\partial_x^\alpha\big(h_\nu'(N_\nu^\tau + n_{\nu,\tau})\nabla N_\nu^\tau - h_\nu'(N_\nu^\tau + n_{\nu,\tau})\nabla\partial_x^\alpha N_\nu^\tau, U_{\nu,\alpha}^\tau\Big) \\ - \quad & \big(\partial_x^\alpha(U_\nu^\alpha \times B_\tau) - U_\nu^\tau \times \partial_x^\alpha B_\tau, U_{\nu,\alpha}^\tau\big) \\ = \quad & J_1 + J_2 + J_3 + J_4 + J_5,\end{aligned}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\mathbb{T})$.
Noting (7.5.3) for $u_{\nu,\tau}$ and applying Lemma 4.1.1 to each term on the right hand side of the above equation gives :

$$\big|J_1 + J_2\big| \le C\big(\tau + \|U_\nu^\tau\|_s\big)\|W_{\nu,I}^\tau\|_s^2 \le \frac{\varepsilon}{\tau}\|U_\nu^\tau\|_s^2 + C_\varepsilon\tau\big(\|W_\nu^\tau\|_s^2 + \|W_\nu^\tau\|_s^4\big),$$

$$\big|J_3 + J_4\big| \le C\big(1 + \|N_\nu^\tau\|_s\big)\|N_\nu^\tau\|_s\|U_\nu^\tau\|_s \le \frac{\varepsilon}{\tau}\|U_\nu^\tau\|_s^2 + C_\varepsilon\tau\big(\|W_\nu^\tau\|_s^2 + \|W_\nu^\tau\|_s^4\big),$$

$$\big|J_5\big| \le C\|U_\nu^\tau\|_s^3 \le \frac{\varepsilon}{\tau}\|U_\nu^\tau\|_s^2 + C_\varepsilon\tau\|W_\nu^\tau\|_s^4.$$

Here and hereafter, ε denotes a small constant independent of τ and $C_\varepsilon > 0$ denotes a constant depending only on ε. Which imply that :

$$\frac{1}{\tau}T_\nu^\tau \leq \frac{\varepsilon}{\tau^2}\|U_\nu^\tau\|_s^2 + C_\varepsilon\big(\|W_\nu^\tau\|_s^2 + \|W_\nu^\tau\|_s^4\big), \quad \nu = e, i. \tag{7.5.14}$$

Now we consider the estimate for the term $R_\nu^{\alpha,\tau}$. A direct computation gives :

$$\begin{aligned} R_\nu^{\alpha,\tau} = & -\ \frac{1}{2}\big(\operatorname{div} A_\nu^\tau(n_\nu^\tau, u_\nu^\tau)N_{\nu,\alpha}^\tau, N_{\nu,\alpha}^\tau\big) \\ & -\ \frac{1}{2}\big(\operatorname{div}(U_\nu^\tau + u_{\nu,\tau})U_{\nu,\alpha}^\tau, U_{\nu,\alpha}^\tau\big) \\ & -\ \big(N_{\nu,\alpha}^\tau \nabla\big(h_\nu'(n_\nu^\tau)\big), U_{\nu,\alpha}^\tau\big) \\ = &\ g_1 + g_2 + g_3, \end{aligned}$$

where,

$$\operatorname{div} A_\nu^\tau(n_\nu^\tau, u_\nu^\tau) := \tau\partial_t\Big(\frac{h_\nu'(n_\nu^\tau)}{n_\nu^\tau}\Big) + \operatorname{div}\Big(\frac{h_\nu'(n_\nu^\tau)}{n_\nu^\tau}u_\nu^\tau\Big), \quad \nu = e, i.$$

Let us first point out that a direct application of Cauchy-Schwartz inequality to g_1 does not yield the desired result. Then we more developed the term of g_1, we have :

$$\operatorname{div} A_\nu^\tau(n_\nu^\tau, u_\nu^\tau) \ = \ \Big(\frac{h_\nu'(n_\nu^\tau)}{n_\nu^\tau}\Big)'\Big(\tau\partial_t n_\nu^\tau + \nabla n_\nu^\tau \cdot u_\nu^\tau\Big) + \frac{h_\nu'(n_\nu^\tau)}{n_\nu^\tau}\operatorname{div} u_\nu^\tau, \quad \nu = e, i.$$

Using the first equation of (7.1.4), we deduce that :

$$\operatorname{div} A_\nu^\tau(n_\nu^\tau, u_\nu^\tau) = \operatorname{div} u_\nu^\tau\Big[\frac{h_\nu'(n_\nu^\tau)}{n_\nu^\tau} - \Big(\frac{h_\nu'(n_\nu^\tau)}{n_\nu^\tau}\Big)'n_\nu^\tau\Big], \quad \nu = e, i.$$

Noting

$$\frac{\kappa}{2} \leq n_\nu^\tau = N_\nu^\tau + n_{\nu,\tau} \leq C_0, \quad u_\nu^\tau = U_\nu^\tau + u_{\nu,\tau}, \quad u_{\nu,\tau} = O(\tau), \quad \nu = e, i,$$

we obtain :

$$\|\operatorname{div} u_\nu^\tau\|_\infty \leq C\|\operatorname{div}(U_\nu^\tau + u_{\nu,\tau})\|_{s-1} \leq C(\|U_\nu^\tau\|_s + \tau), \quad \nu = e, i.$$

Here in the last inequality we have used the continuous embedding $H^{s-1}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$. Therefore,

$$\big|g_1\big| \leq C\big(\tau + \|U_\nu^\tau\|_s\big)\|N_\nu^\tau\|_s^2 \leq \frac{\varepsilon}{\tau}\|U_\nu^\tau\|_s^2 + C_\varepsilon\tau\big(\|W_\nu^\tau\|_s^2 + \|W_\nu^\tau\|_s^4\big),$$

and

$$\big|g_2 + g_3\big| \leq C\big(\tau + \|U_\nu^\tau\|_s\big)\|U_\nu^\tau\|_s^2 + C\|N_\nu^\tau\|\|U_\nu^\tau\| \leq \frac{\varepsilon}{\tau}\|U_\nu^\tau\|_s^2 + C_\varepsilon\tau\big(\|W_\nu^\tau\|_s^2 + \|W_\nu^\tau\|_s^4\big),$$

Thus,

$$\frac{1}{\tau}R_\nu^\tau \leq \frac{\varepsilon}{\tau^2}\|U_\nu^\tau\|_s^2 + C_\varepsilon\big(\|W_\nu^\tau\|_s^2 + \|W_\nu^\tau\|_s^4\big), \quad \nu = e, i. \tag{7.5.15}$$

For the term S_ν^α, we use the expressions of $f_2^\alpha(W_\nu^\tau)$ and $f_4^\alpha(W_\nu^\tau)$ to compute :

$$\begin{aligned}\big(N_{\nu,\alpha}^\tau, f_2^\alpha(W_\nu^\tau)\big) + \big(U_{\nu,\alpha}^\tau, f_4^\alpha(W_\nu^\tau)\big) &= -\big(N_{\nu,\alpha}^\tau, \partial_x^\alpha\big[(U_\nu^\tau\cdot\nabla)n_{\nu,\tau} + N_\nu^\tau \operatorname{div} u_{\nu,\tau}\big]\big)\\ &\quad - \big(U_{\nu,\alpha}^\tau, \partial_x^\alpha\big[(U_\nu^\tau\cdot\nabla)u_{\nu,\tau} + \big(h_\nu'(N_\nu^\tau + n_{\nu,\tau}) - h_\nu'(n_{\nu,\tau})\big)\nabla n_{\nu,\tau}\big]\big).\end{aligned}$$

By Lemma 4.1.1and $u_{\nu,\tau} = O(\tau)$, we get

$$\begin{aligned}\big(N_{\nu,\alpha}^\tau, f_2^\alpha(W_\nu^\tau)\big) + \big(U_{\nu,\alpha}^\tau, f_4^\alpha(W_\nu^\tau)\big) &\le \big(\|N_\nu^\tau\|_s\|U_\nu^\tau\|_s + \tau\|N_\nu^\tau\|_s^2 + \tau\|U_\nu^\tau\|_s^2\big)\\ &\le \frac{\varepsilon}{\tau}\|U_\nu^\tau\|_s^2 + C_\varepsilon\tau\|W_\nu^\tau\|_s^2.\end{aligned}$$

Since

$$U_{\nu,\alpha}^\tau\cdot U_{\nu,\alpha}^\tau\times B_\nu = 0, \quad \nu = e,i.$$

Then, we have the following estimate for the term containing $f_\nu^5(W_\nu^\tau)$ in (7.5.12) :

$$\begin{aligned}\frac{2}{\tau}\big(U_{\nu,\alpha}^\tau, f_5^\alpha(W_\nu^\tau)\big) &= -\big(U_{\nu,\alpha}^\tau, \partial_x^\alpha[F^\tau + (U_\nu^\tau + u_{\nu,\tau})\times G^\tau]\big)\\ &\le \frac{\varepsilon}{\tau^2}\|U_\nu^\tau\|^2 + C_\varepsilon\int_{\mathbb{T}}\big|\partial_x^\alpha[F^\tau + (U_\nu^\tau + u_{\nu,\tau})\times G^\tau]\big|^2 dx\\ &\le \frac{\varepsilon}{\tau^2}\|U_\nu^\tau\|_s^2 + C_\varepsilon\big(\|W_\nu^\tau\|_s^2 + \|U_\nu^\tau\|_s^2\|G^\tau\|_s^2\big).\end{aligned}$$

Thus,

$$\frac{1}{\tau}S_\nu^\tau \le \frac{\varepsilon}{\tau^2}\|U_\nu^\tau\|_s^2 + C_\varepsilon\big(\|W_\nu^\tau\|_s^2 + \|W_\nu^\tau\|_s^4\big), \quad \nu = e,i. \tag{7.5.16}$$

Finally, a direct computation gives :

$$V_\nu^\tau = -\big(\frac{h_\nu'(n_\nu^\tau)}{n_\nu^\tau}N_{\nu,\alpha}^\tau, \partial_x^\alpha R_{n_\nu}^\tau\big) - \big(U_{\nu,\alpha}^\tau, \partial_x^\alpha R_{u_\nu}^\tau\big), \quad \nu = e,i.$$

Together with (7.5.5) yields :

$$V_\nu^\tau \le C\|W_\nu^\tau\|_s^2 + \frac{\varepsilon}{\tau^2}\|U_\nu^\tau\|_s^2 + C_\varepsilon\tau^{2(\lambda+1)}, \quad \nu = e,i. \tag{7.5.17}$$

Together with (7.5.13) and (7.5.14)-(7.5.17), we obtain, for all $|\alpha| \le s$,

$$\frac{d}{dt}\|W_{\nu,I\alpha}^\tau\|^2 + \frac{1}{\tau^2}\|U_{\nu,\alpha}^\tau\|^2 \le \frac{C\varepsilon}{\tau^2}\|U_\nu^\tau\|_s^2 + C_\varepsilon\big(\|W_\nu^\tau\|_s^2 + \|W_\nu^\tau\|_s^4\big) + C_\varepsilon\tau^{2(\lambda+1)}, \quad \nu = e,i.$$

Integrating this equation over $(0,t)$ with $t\in(0,T^\tau)\subset(0,T_1)$ and summing up over all $|\alpha|\le s$, taking $\varepsilon > 0$ sufficiently small such that the term including $\frac{C\varepsilon}{\tau^2}\|U_\nu^\tau\|_s^2$ can be controlled by the left hand side, together with condition (7.5.4) for the initial data, we get (7.5.10). □

Now, let us establish the estimate for $W_{\nu,II}^\tau$.

Lemma 7.5.2 *Under the assumptions of Theorem 7.5.1, for all $t \in (0, T^\tau)$, as $\tau \to 0$ we have :*

$$\|W^\tau_{\nu,II}(t)\|_s^2 \le \int_0^t \Big(\frac{1}{2\tau^2}\|U^\tau_\nu(\xi)\|_s^2 + C\|W^\tau_\nu(\xi)\|_s^2 + C\|W^\tau_\nu(\xi)\|_s^4\Big)\, d\xi + C\tau^{2(\lambda+1)}. \tag{7.5.18}$$

Proof. For a multi-index $\alpha \in \mathbb{N}^3$ with $|\alpha| \le s$, differentiating the third and fourth equations of (7.5.9) with respect to x, we have :

$$\begin{cases} \partial_t F^\tau_\alpha - \dfrac{1}{\tau}\nabla \times G^\tau_\alpha = \dfrac{1}{\tau}\partial_x^\alpha\big(N^\tau_\nu U^\tau_\nu + N^\tau_\nu u_{\nu,\tau} + n_{\nu,\tau}U^\tau\big) - \partial_x^\alpha R^\tau_E, \\ \partial_t G^\tau_\alpha + \dfrac{1}{\tau}\nabla \times F^\tau_\alpha = 0, \end{cases} \qquad \nu = e, i \tag{7.5.19}$$

where $W^\tau_{II\alpha} = (F^\tau_\alpha, G^\tau_\alpha) = \partial_x^\alpha(F^\tau, G^\tau)$.

By the vector analysis formula :

$$\operatorname{div}(f \times g) = (\nabla \times f)\cdot g - (\nabla \times g)\cdot f,$$

the singular term appearing in Sobolev energy estimates vanishes, i.e,

$$\int_{\mathbb{T}} \big(-\frac{1}{\tau}\nabla \times G^\tau_\alpha \cdot F^\tau_\alpha + \frac{1}{\tau}\nabla \times F^\tau_\alpha \cdot G^\tau_\alpha\big)dx = \frac{1}{\tau}\int_{\mathbb{T}} \operatorname{div}(F^\tau_\alpha \times G^\tau_\alpha)dx = 0.$$

Hence, using $u_{\nu,\tau} = O(\tau)$ and (7.5.5), a standard energy estimate for (7.5.19) yields :

$$\begin{aligned} \frac{d}{dt}\|W^\tau_{\nu,II\alpha}\|^2 &\le \frac{2}{\tau}\int_{\mathbb{T}} \big(|\partial_x^\alpha(N^\tau_\nu U^\tau_\nu)| + |\partial_x^\alpha(N^\tau_\nu u_{\nu,\tau})| + |\partial_x^\alpha(n_{\nu,\tau}U^\tau_\nu)|\big)|F^\tau_\alpha|dx + \int_{\mathbb{T}} |\partial_x^\alpha R^\tau_E|\,|F^\tau_\alpha|dx \\ &\le \frac{1}{2\tau^2}\|U^\tau_\nu\|_s^2 + C\big(\|W^\tau_\nu\|_s^2 + \|W^\tau_\nu\|_s^4\big) + C\tau^{2(\lambda+1)}. \end{aligned} \tag{7.5.20}$$

Integrating (7.5.20) over $(0, t)$, with $t \in (0, T^\tau)$, summing up over α satisfying $|\alpha| \le s$ and using (7.5.4) we obtain the Lemma. □

Proof of Theorem 7.5.1. Let $\tau \to 0$ and $\varepsilon > 0$ be sufficiently small. By Lemma 7.5.1 and Lemma 7.5.2, for $t \in (0, T^\tau)$ we have :

$$\|W^\tau_\nu(t)\|_s^2 + \frac{1}{\tau^2}\int_0^t \|U^\tau_\nu(\xi)\|_s^2\, d\xi \le C\int_0^t \Big(\|W^\tau_\nu(\xi)\|_s^2 + \|W^\tau_\nu(\xi)\|_s^4\Big)\, d\xi + C\tau^{2(\lambda+1)}. \tag{7.5.21}$$

Let

$$y_\nu(t) = C\int_0^t\Big(\|W^\tau_\nu(\xi)\|_s^2 + \|W^\tau_\nu(\xi))\|_s^4\Big)\, d\xi + C\tau^{2(\lambda+1)}, \quad \nu = e, i.$$

Then it follows from (7.5.21) that :

$$\|W^\tau_\nu(t)\|_s^2 \le y_\nu(t), \quad \frac{1}{\tau^2}\int_0^t \|U^\tau_\nu(\xi)\|_s^2\, d\xi \le y_\nu(t), \quad \forall\, t \in (0, T^\tau), \quad \nu = e, i \tag{7.5.22}$$

and

$$y_\nu'(t) = C\Big(\|W_\nu^\tau(t)\|_s^2 + \|W_\nu^\tau(t)\|_s^4\Big) \le C\big(y_\nu(t) + y_\nu^2(t)\big),$$

with

$$y(0) = C\tau^{2(\lambda+1)}, \quad \nu = e, i.$$

A straightforward computation yields

$$y_\nu(t) \le C\tau^{2(\lambda+1)}e^{Ct} \le C\tau^{2(\lambda+1)}e^{CT_1}, \quad \forall\, t \in [0, T^\tau], \quad \nu = e, i.$$

Therefore, from (7.5.22) we obtain :

$$\|W_\nu^\tau(t)\|_s \le \sqrt{y_\nu(t)} \le C\tau^{\lambda+1}, \quad \int_0^t \|U_\nu^\tau(\xi)\|_s^2 \le \tau^2 y_\nu(t) \le C\tau^{2(\lambda+2)}, \quad \forall\, t \in [0, T^\tau].$$

Thus, by a standard argument on the time extension of smooth solutions, we obtain $T_2^\tau \ge T_1$, i.e. $T^\tau = T_1$. This finishes the proof of Theorem 7.5.1. $\square$

Chapter 8

A study of initial layers without exponential decay to zero in a one-fluid Euler-Maxwell system

Abstract : In this chapter we study the zero relaxation limit and the initial layers of a one-fluid Euler-Maxwell system. For well prepared initial data the convergence of solutions is rigorously justified by an analysis of different asymptotic expansions, which considers all the powers of the small parameter, removing the smallness assumption on the magnetic field. The authors perform also an initial layer analysis for general initial data which gives an initial layers without exponential decay to zero and prove the convergence of asymptotic expansions up to first order.

8.1 Introduction

The Euler-Maxwell system plays an important role in the mathematical modeling and numerical simulation for plasmas and semiconductors. It consists of two nonlinear equations given by the conservation of density and momentum, called Euler equations, plus a Maxwell equations for the electric and magnetic fields. This chapter deals with the relaxation limit of compressible Euler-Maxwell equations, which are problems mentioned in the chapters 6 and 7. Let n and u be the density and velocity vector of the electric particles in a plasma, E and B be respectively the electric field and magnetic field. They are functions of a three-dimensional position vector $x \in \mathbb{T}$ and of the time $t > 0$, where $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^3$ is the torus. The fields E and B are coupled to the electron density through the Maxwell equations and act on electrons via the Lorentz force. The dynamics of the compressible electrons for plasma physics in a uniform background of non-moving ions with fixed density $b(x)$ obey the (scaled) one-fluid Euler-Maxwell system :

$$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ m\partial_t u + m(u\cdot\nabla)u + \nabla h(n) = -E - \gamma u \times B - \dfrac{mu}{\tau}, \\ \gamma\lambda^2\partial_t E - \nabla\times B = \gamma nu, \quad \lambda^2 \operatorname{div} E = b - n, \\ \gamma\partial_t B + \nabla\times E = 0, \quad \operatorname{div} B = 0, \end{cases} \tag{8.1.1}$$

for $x \in \mathbb{T}$. In the above equations, $h = h(n)$ is the enthalpy function, assumed to be smooth and strictly increasing for $n > 0$, $j = nu$ is the current density and $E + \gamma u \times B$ represents the Lorentz force. We assume that b is smooth function and there is a constant $\underline{b}$ such that : $b(x) \geq \underline{b}$ for all $x \in \mathbb{T}$. The physical parameters $c = \frac{1}{\gamma} = (\varepsilon_0\nu_0)^{-\frac{1}{2}}$, where ε_0 and ν_0 are the vacuum permittivity and permeability, λ, τ and m stand for, the scaled Debye length, the momentum relaxation times of the system and the electron mass, respectively. They are small compared with the characteristic length of physical interest.

There have been a lot of studies on the Euler-Poisson equations and their asymptotic analysis contrarily to the study on the Euler-Maxwell equations. In particular, the convergence of compressible Euler-Poisson equations to incompressible Euler equations is shown independently in [66, 71]. The first mathematical study of the Euler-Maxwell equations with extra relaxation terms is due to Chen et al [11], where a global existence result to weak solutions in one-dimensional case is established by the fractional step Godunov scheme together with a compensated compactness argument. The paper [11] exhibits also some applications of the model (8.1.1) in semiconductor theory. Since then few progress have been made on the Euler-Maxwell equations.

In this chapter, we study the zero-relaxation-time limit $\tau \to 0$ when $\lambda > 0$, $\gamma > 0$ and $m > 0$ are fixed and the initial layers with not exponential decay to zero of a unipolar model for electrons. For this purpose, we use the method of asymptotic expansions, which considers all the powers of the small parameter, removing the smallness assumption on the magnetic field. The convergence of the expansions is achieved through the energy estimates for error equations derived from the asymptotic expansions and the Euler-Maxwell equations.

In the sequel, we assume that $\lambda = \gamma = m = 1$. The usual time scaling for studying the limit $\tau \to 0$ is $t' = \tau t$. Since $t = 0$ if and only if $t' = 0$. Rewriting still ξ by t', system (8.1.1) becomes (see [60, 64])

$$\begin{cases} \partial_t n + \dfrac{1}{\tau}\operatorname{div}(nu) = 0, \\ \partial_t u + \dfrac{1}{\tau}(u\cdot\nabla)u + \dfrac{1}{\tau}\nabla h(n) = -\dfrac{E}{\tau} - \dfrac{u\times B}{\tau} - \dfrac{u}{\tau^2}, \\ \partial_t E - \dfrac{1}{\tau}\nabla\times B = \dfrac{nu}{\tau}, \quad \operatorname{div} E = b(x) - n, \\ \partial_t B + \dfrac{1}{\tau}\nabla\times E = 0, \quad \operatorname{div} B = 0. \end{cases} \tag{8.1.2}$$

for $t > 0$, $x \in \mathbb{T}$. It is complemented by periodic initial conditions :

$$t = 0 \ : \quad (n, u, E, B) = (n_0^\tau, u_0^\tau, E_0^\tau, B_0^\tau). \tag{8.1.3}$$

For $m \geq 1$, the authors of [64] proposed an asymptotic expansion to (8.1.2) of the form :

$$(n_\tau^m, u_\tau^m, E_\tau^m, B_\tau^m) = \sum_{j=0}^{m} \tau^j (n^j, \tau u^j, E^j, B^j). \tag{8.1.4}$$

They established the convergence in Sobolev spaces of the solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ of (8.1.2) to $(n_\tau^m, u_\tau^m, E_\tau^m, B_\tau^m)$ with order $O(\tau^m)$ when the initial data are well-prepared and the initial error has the same order. Here the well-prepared initial data mean that compatibility conditions holds. Unfortunately, this result does not apply to the case of ill-prepared initial data. The goal of this chapter is to improve the convergence of system (8.1.2) with the case of ill-prepared initial data by using the asymptotic expansion as the form (8.1.4).

The remainder of the chapter is arranged as follows. In section 2 we remember the asymptotic expansions of solutions as the form (8.1.4) and the convergence result to problem (8.1.2)-(8.1.3) in the case of well-prepared initial data indicated in paper [64]. In section 3 we study the initial layers in the case of ill-prepared initial data. We add the initial layer corrections in asymptotic expansion and state the convergence result. For this propose, we give a more general convergence theorem which implies those in both cases of well-prepared initial data and ill-prepared initial data.

The following lemmas are needed in the proofs of Proposition 8.3.1.

Lemma 8.1.1 *(Generalized Holder inequality, see [68]*
Let $1 \leq p_i \leq \infty$, $i = 1, \cdots, k$, such that

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} = \frac{1}{p}; \quad 1 \leq p \leq \infty.$$

If $f_i \in L^{p_i}(\mathbb{R}^n)$, for all $i = 1, \cdots, k$ then

$$\Big(\int |f_1 f_2 \cdots f_k|^p dx\Big)^{\frac{1}{p}} \leq \prod_{i=1}^{k} \Big(\int |f_i|^{p_i} dx\Big)^{\frac{1}{p_i}}$$

Lemma 8.1.2 *(See [33]) Let $\alpha \in]0,1[$, $C > 0$. Let $\varphi(t)$ and $m(t)$ be two nonnegative continuous functions defined on $[0,T]$ which satisfy the integral inequality :*

$$\forall t \in [0,T] \quad \varphi(t) \leq C + \int_0^t m(s)\varphi(s)^\alpha ds,$$

then

$$\forall t \in [0,T] \quad \varphi(t) \leq \Big\{C^{1-\alpha} + (1-\alpha)\int_0^t m(s) ds\Big\}^{\frac{1}{1-\alpha}}$$

8.2 Case of well-prepared initial data

In this section we recall the asymptotic expansions of solutions as the form (8.1.4) and the convergence result to problem (8.1.2)-(8.1.3) in the case of well-prepared initial data indicated in paper [64].

8.2.1 Formal asymptotic expansions

We consider the limit $\tau \to 0$ in problem (8.1.2)-(8.1.3) with well-prepared initial data. From the momentum equation for u^τ in (8.1.2), we have formally $u^\tau \to 0$ as $\tau \to 0$. We make the following ansatz for both the approximate solution and its initial data :

$$(n_\tau, u_\tau, E_\tau, B_\tau)(0,x) = \sum_{j\geq 0} \tau^j (n_j, \tau u_j, E_j, B_j)(x), \quad x \in \mathbb{T}, \tag{8.2.1}$$

$$(n_\tau, u_\tau, E_\tau, B_\tau)(t,x) = \sum_{j\geq 0} \tau^j (n^j, \tau u^j, E^j, B^j)(t,x), \quad t>0, \;\; x \in \mathbb{T}, \tag{8.2.2}$$

where $(n_j, u_j, E_j, B_j)_{j\geq 0}$ are given sufficiently smooth data with $n_0 \geq$ constant > 0 in $\mathbb{T}$.

Now we need to determine the profiles (n^j, u^j, E^j, B^j) for all $j \geq 0$. Putting expression (8.2.2) into system (8.1.2) and identifying the coefficients in powers of τ, we see that $(n^j, u^j, E^j, B^j)_{j\geq 0}$ are solutions of the following systems :

$$\begin{cases} \partial_t n^0 + \mathrm{div}(n^0 u^0) = 0, \\ \nabla h(n^0) = -(E^0 + u^0), \\ \nabla \times E^0 = 0, \;\; \mathrm{div}\, E^0 = b - n^0, \\ \nabla \times B^0 = 0, \;\; \mathrm{div}\, B^0 = 0, \end{cases} \tag{8.2.3}$$

$$\begin{cases} \partial_t n^1 + \mathrm{div}(n^1 u^0 + n^0 u^1) = 0, \\ \nabla\big(h'(n^0) n^1\big) = -(E^1 + u^0 \times B^0 + u^1), \\ \nabla \times E^1 = -\partial_t B^0, \;\; \mathrm{div}\, E^1 = -n^1, \\ \nabla \times B^1 = \partial_t E^0 - n^0 u^0, \;\; \mathrm{div}\, B^1 = 0, \end{cases} \tag{8.2.4}$$

and for $j \geq 2$,

$$\begin{cases} \partial_t n^j + \displaystyle\sum_{k=0}^{j} \mathrm{div}\left(n^k u^{j-k}\right) = 0, \\ \partial_t u^{j-2} + \displaystyle\sum_{k=0}^{j-2} (u^k \cdot \nabla) u^{j-2-k} + \nabla\big(h'(n^0) n^j + h^{j-1}\big((n^k)_{k\leq j-1}\big)\big) \\ \quad = -E^j - \displaystyle\sum_{k=0}^{j-1} u^k \times B^{j-1-k} - u^j, \\ \nabla \times E^j = -\partial_t B^{j-1}, \;\; \mathrm{div}\, E^j = -n^j, \\ \nabla \times B^j = \partial_t E^{j-1} - \displaystyle\sum_{k=0}^{j-1} n^k u^{j-1-k}, \;\; \mathrm{div}\, B^j = 0, \end{cases} \tag{8.2.5}$$

where $h^0 = 0$ and h^{j-1} is a function depending only on $(n^k)_{0\leq k\leq j-1}$ and is defined for $j \geq 2$ by :

$$h\Big(\sum_{j\geq 0} \tau^j n^j\Big) = h(n^0) + h'(n^0) \sum_{j\geq 1} \tau^j n^j + \sum_{j\geq 2} \tau^j h^{j-1}\big((n^k)_{k\leq j-1}\big).$$

From equations in (8.2.3), we make take $B^0 = 0$ and equation $\nabla \times E^0 = 0$ implies the existence of a potential ϕ^0 such that $E^0 = -\nabla\phi^0$. Then (n^0, ϕ^0) solves a classical system drift-diffusion equations :

$$\begin{cases} \partial_t n^0 - \operatorname{div}\big(n^0 \nabla(h(n^0) - \phi^0)\big) = 0, \\ -\Delta\phi^0 = b - n^0, \end{cases} \quad t > 0, \quad x \in \mathbb{T} \tag{8.2.6}$$

with the initial condition :

$$n^0(0, x) = n_0, \quad x \in \mathbb{T}. \tag{8.2.7}$$

The existence of smooth solutions to problem (8.2.6)-(8.2.7) can be easily established, at least locally in time. The solution ϕ^0 is unique in the class $m(\phi^0) = 0$. See for instance [54]. Then (u^0, E^0, B^0) are given by :

$$u^0 = -\nabla\big(h(n^0) - \phi^0\big), \;\; E^0 = -\nabla\phi^0, \;\; B^0 = 0. \tag{8.2.8}$$

From (8.2.8) we get the zero-order compatibility conditions :

$$u_0 = -\nabla\big(h(n_0) - \phi_0\big), \;\; E_0 = -\nabla\phi_0, \;\; B_0 = 0, \tag{8.2.9}$$

where ϕ_0 is determined by :

$$-\Delta\phi_0 = b - n_0 \quad \text{in } \mathbb{T} \quad \text{and} \quad m(\phi_0) = 0. \tag{8.2.10}$$

For system (8.2.4), since (n^0, u^0, E^0) are known, B^1 solves the linear system of curl-div equations of type $C1.7$ in the class $m(B^1) = 0$. Moreover, $\nabla \times E^1 = 0$ determines a potential function ϕ^1 such that $E^1 = -\nabla\phi^1$. Then (n^1, ϕ^1) solves a linearized system of drift-diffusion equations :

$$\begin{cases} \partial_t n^1 - \operatorname{div}\big(n^0 \nabla(h'(n^0)n^1 - \phi^1)\big) + \operatorname{div}(n^1 u^0) = 0, \\ \Delta\phi^1 = n^1, \end{cases} \quad t > 0, \quad x \in \mathbb{T} \tag{8.2.11}$$

with the initial condition :

$$n^1(0, x) = n_1, \quad x \in \mathbb{T}. \tag{8.2.12}$$

This linear problem has a unique smooth solution. Then (u^1, E^1) are given by :

$$u^1 = -\nabla\big(h'(n^0)n^1 - \phi^1\big), \;\; E^1 = -\nabla\phi^1. \tag{8.2.13}$$

Since B^1 is solved by the fourth equations in (8.2.4). Then, we get the first-order compatibility conditions :

$$u_1 = -\nabla\big(h'(n_0)n_1 - \phi_1\big), \;\; E_1 = -\nabla\phi_1, \;\; B_1 = B^1(0, .), \tag{8.2.14}$$

where ϕ_1 is determined by :

$$\Delta\phi_1 = n_1 \quad \text{in } \mathbb{T} \quad \text{and} \quad m(\phi_1) = 0. \tag{8.2.15}$$

For $j \geq 2$, the profiles (n^j, u^j, E^j, B^j) are obtained by induction in j. Assume that $\big(n^k, u^k, E^k, B^k\big)_{0 \leq k \leq j-1}$ are smooth and already determined in previous steps. Equations for B^j are of curl-div type and determine a unique smooth B^j in the class $m(B^j) = 0$. Moreover, from

$\operatorname{div} B^j = 0$, we deduce the existence of a given vector ψ^j such that $B^j = -\nabla \times \psi^j$. Then, the equation $\nabla \times E^j = -\partial_t B^{j-1}$ in (8.2.5) becomes $\nabla \times (E^j - \partial_t \psi^{j-1}) = 0$. It follows that there is a potential function ϕ^j such that :

$$E^j = \partial_t \psi^{j-1} - \nabla \phi^j. \tag{8.2.16}$$

From (8.2.5) we also get :

$$\begin{aligned} u^j &= \nabla\big(\phi^j - h'(n^0)n^j - h^{j-1}((n^k)_{k\le j-1})\big) \\ &\quad -\Big(\partial_t u^{j-2} + \partial_t \psi^{j-1} + \sum_{k=0}^{j-2}(u^k\cdot\nabla)u^{j-2-k} + \sum_{k=0}^{j-1} u^k \times B^{j-1-k}\Big). \end{aligned} \tag{8.2.17}$$

Therefore, in the class $m(\phi^j) = 0$, (n^j, ϕ^j) solves a linearized system of drift-diffusion equations :

$$\begin{cases} \partial_t n^j - \operatorname{div}\big(n^0 \nabla (h'(n^0)n^j - \phi^j)\big) + \operatorname{div}\big(n^j u^0\big) \\ \ = f^j\big((V^k, \partial_t V^k, \partial_x V^k, \partial_t\partial_x V^k, \partial_x^2 V^k)_{0\le k\le j-1}\big) + \operatorname{div}(n^0 \partial_t \psi^{j-1}), \quad t>0,\ x\in\mathbb{T} \\ \Delta \phi^j = n^j + \partial_t(\operatorname{div}\psi^{j-1}), \end{cases} \tag{8.2.18}$$

with the initial condition :

$$n^j(0,x) = n_j(x), \quad x\in\mathbb{T}, \tag{8.2.19}$$

where f^j is a given smooth function and $V^k = (n^k, u^k, \psi^k)$.
Problem (8.2.18)-(8.2.19) is linear. It admits a unique global smooth solution. Then (u^j, E^j) are given by (8.2.16)-(8.2.17). Thus, we get the high-order compatibility conditions for $j \ge 2$:

$$\begin{aligned} u_j &= \nabla\big(\phi_j - h'(n_0)n_j - h^{j-1}((n_k)_{k\le j-1})\big) \\ &\quad -\Big(\partial_t u^{j-2}\big|_{t=0} + \psi^{j-1}\big|_{t=0} + \sum_{k=0}^{j-2}(u_k\cdot\nabla)u_{j-2-k} + \sum_{k=0}^{j-1} u_k \times B^{j-1-k}(0,.)\Big), \end{aligned} \tag{8.2.20}$$

$$E_j = \partial_t \psi^{j-1}(0,.) - \nabla\phi_j, \quad B_j = B^j(0,.), \tag{8.2.21}$$

where ϕ_j is determined by :

$$\Delta \phi_j = n_j + \partial_t(\operatorname{div}\psi^{j-1})|_{t=0} \quad \text{in } \mathbb{T} \quad \text{and} \quad m(\phi_j) = 0. \tag{8.2.22}$$

We conclude the above discussion in the following result.

Proposition 8.2.1 *Assume that the initial data (n_j, u_j, E_j, B_j) are sufficiently smooth for $j\in\mathbb{N}$, with $n_0 \ge$ constant > 0 in $\mathbb{T}$, and satisfy the compatibility conditions (8.2.9)-(8.2.10), (8.2.14)-(8.2.15) and (8.2.20)-(8.2.22) for $j\ge 2$. Then there exists a unique asymptotic expansion (8.2.2), i.e. there exist $T_1 > 0$ and a unique smooth solution $(n^j, u^j, E^j, B^j)_{j\ge 0}$ in the time interval $[0, T_1]$ of problems (8.2.6)-(8.2.8), (8.2.11)-(8.2.13) and (8.2.16)-(8.2.19) for $j \ge 2$. Moreover, $n^0 \ge$ constant > 0 in $[0,T_1]\times\mathbb{T}$. In particular, the formal zero-relaxation limit $\tau \to 0$ of the Euler-Maxwell system (8.1.2) is the classical drift-diffusion system (8.2.6) and (8.2.8).*

8.2.2 Convergence results

Let $m \geq 0$ be a fixed integer. We denote by :

$$(n_\tau^m, u_\tau^m, \tilde{E}_\tau^m, \tilde{B}_\tau^m) = \sum_{j=0}^{m} \tau^j (n^j, \tau u^j, E^j, B^j), \tag{8.2.23}$$

an approximate solution of order $m \geq 0$, where $(n^j, u^j, E^j, B^j)_{0 \leq j \leq m}$ are constructed in the previous subsection. From the construction of the approximate solution, for $(t, x) \in [0, T_1] \times \mathbb{T}$ we have :

$$\operatorname{div} \tilde{E}_\tau^m = b - n_\tau^m, \quad \operatorname{div} \tilde{B}_\tau^m = 0. \tag{8.2.24}$$

We define the remainders $R_n^{\tau,m}$, $R_u^{\tau,m}$, $\tilde{R}_E^{\tau,m}$ and $\tilde{R}_B^{\tau,m}$ by :

$$\begin{cases} \partial_t n_\tau^m + \dfrac{1}{\tau} \operatorname{div}(n_\tau^m u_\tau^m) = R_n^{\tau,m}, \\ \partial_t u_\tau^m + \dfrac{1}{\tau}(u_\tau^m \cdot \nabla) u_\tau^m + \dfrac{1}{\tau} \nabla h(n_\tau^m) = -\dfrac{\tilde{E}_\tau^m}{\tau} - \dfrac{u_\tau^m}{\tau^2} - \dfrac{u_\tau^m \times \tilde{B}_\tau^m}{\tau} + R_u^{\tau,m}, \\ \partial_t \tilde{E}_\tau^m - \dfrac{1}{\tau} \nabla \times \tilde{B}_\tau^m = \dfrac{n_\tau^m u_\tau^m}{\tau} + \tilde{R}_E^{\tau,m}, \\ \partial_t \tilde{B}_\tau^m + \dfrac{1}{\tau} \nabla \times \tilde{E}_\tau^m = \tilde{R}_B^{\tau,m}. \end{cases} \tag{8.2.25}$$

It is clear that the convergence rate depends strongly on the order of the remainders with respect to τ. For $m = 0$, the expressions of $R_n^{\tau,0}$, $R_u^{\tau,0}$, $\tilde{R}_E^{\tau,0}$ and $\tilde{R}_B^{\tau,0}$ are given by :

$$R_n^{\tau,0} = 0, \quad R_u^{\tau,0} = u^0 \times B^0 + \tau\big(\partial_t u^0 + (u^0 \cdot \nabla) u^0\big),$$

$$\tilde{R}_E^{\tau,0} = \partial_t E^0 - n^0 u^0, \quad \tilde{R}_B^{\tau,0} = \partial_t B^0.$$

Thus

$$R_u^{\tau,0} = O(1), \quad \tilde{R}_E^{\tau,0} = O(1), \quad \tilde{R}_B^{\tau,0} = O(1). \tag{8.2.26}$$

It is clear that we have the convergence of system (8.1.2) if the remainders at least order $O(\tau^\alpha)$ or equal to zero, with $\alpha > 0$. But this condition is not verified in (8.2.26). For $R_u^{\tau,0}$, the loss of this order will be recovered in the error estimate of convergence due to the dissipation term u. However, the situation is different for $\tilde{R}_E^{\tau,0}$ and $\tilde{R}_B^{\tau,0}$, since the equations for E and B are not dissipatives. A simple way to remedy this is to introduce a correction terms into $\tilde{E}_\tau^0$ and $\tilde{B}_\tau^0$ so that :

$$E_\tau^0 = \tilde{E}_\tau^0 + \tau E_c^1 = E^0 + \tau E_c^1, \tag{8.2.27}$$

$$B_\tau^0 = \tilde{B}_\tau^0 + \tau B_c^1 = B^0 + \tau B_c^1. \tag{8.2.28}$$

In view of (8.2.24)-(8.2.28), E_c^1 and B_c^1 should be defined by :

$$\nabla \times E_c^1 = -\partial_t B^0, \quad \operatorname{div} E_c^1 = 0, \quad m(E_c^1) = 0, \tag{8.2.29}$$

$$\nabla \times B_c^1 = \partial_t E^0 - n^0 u^0, \quad \operatorname{div} B_c^1 = 0, \quad m(B_c^1) = 0, \tag{8.2.30}$$

so that the new remainder $R_E^{\tau,0}$ and $R_B^{\tau,0}$ of E and B, respectively satisfies :

$$R_E^{\tau,0} \stackrel{def}{=} \partial_t E_\tau^0 - \frac{1}{\tau}\nabla \times B_\tau^0 - \frac{1}{\tau} n_\tau^0 u_\tau^0 = \tau \partial_t E_c^1 = O(\tau), \tag{8.2.31}$$

$$R_B^{\tau,0} \stackrel{def}{=} \partial_t B_\tau^0 + \frac{1}{\tau}\nabla \times E_\tau^0 = \tau \partial_t B_c^1 = O(\tau) \tag{8.2.32}$$

and we still have

$$\operatorname{div} E_\tau^0 = b - n_\tau^m, \quad \operatorname{div} B_\tau^0 = 0. \tag{8.2.33}$$

For $m \geq 1$, a further computation gives :

$$R_n^{\tau,m} = O(\tau^{m+1}), \quad R_u^{\tau,m} = O(\tau^m), \quad \tilde{R}_E^{\tau,m} = O(\tau^m), \quad \tilde{R}_B^{\tau,m} = O(\tau^m). \tag{8.2.34}$$

In (8.2.34), there is loss of one order for the remainder $R_u^{\tau,m}$, $\tilde{R}_E^{\tau,m}$ and $\tilde{R}_B^{\tau,m}$. For $R_u^{\tau,m}$ this loss will be recovered in the error estimate of convergence due to the dissipation term for u. However, the situation is different for $\tilde{R}_E^{\tau,m}$ and $\tilde{R}_B^{\tau,m}$ since the equations for E and B are not dissipatives. A simply way to remedy this is to introduce a correction terms into $\tilde{R}_E^{\tau,m}$ and $\tilde{R}_B^{\tau,m}$ so that :

$$\begin{aligned} E_\tau^m &= \tilde{E}_\tau^m + \tau^{m+1} E_c^{m+1} + \tau^{m+2} E_{c'}^{m+2} \\ &= \sum_{j=0}^m \tau^j E^j + \tau^{m+1} E_c^{m+1} + \tau^{m+2} E_{c'}^{m+2}, \end{aligned} \tag{8.2.35}$$

$$B_\tau^m = \tilde{B}_\tau^m + \tau^{m+1} B_c^{m+1} = \sum_{j=0}^m \tau^j B^j + \tau^{m+1} B_c^{m+1}. \tag{8.2.36}$$

In view of (8.2.24)-(8.2.25), $\left(E_c^{m+1}, E_{c'}^{m+2}, B_c^{m+1}\right)$ should be defined by :

$$\nabla \times B_c^{m+1} = \partial_t E^m - \sum_{k=0}^m n^k u^{m-k}, \quad \operatorname{div} B_c^{m+1} = 0, \quad m(B_c^{m+1}) = 0, \tag{8.2.37}$$

$$\nabla \times E_c^{m+1} = -\partial_t B^m, \quad \operatorname{div} E_c^{m+1} = 0, \quad m(E_c^{m+1}) = 0, \tag{8.2.38}$$

$$\nabla \times E_{c'}^{m+2} = -\partial_t B_c^{m+1}, \quad \operatorname{div} E_{c'}^{m+2} = 0, \quad m(E_{c'}^{m+2}) = 0. \tag{8.2.39}$$

so that the new remainder $R_E^{\tau,m}$ and $R_B^{\tau,m}$ of E and B, respectively satisfies :

$$R_E^{\tau,m} \stackrel{def}{=} \partial_t E_\tau^m - \frac{1}{\tau}\nabla \times B_\tau^m - \frac{1}{\tau} n_\tau^m u_\tau^m = O(\tau^{m+1}) \tag{8.2.40}$$

and

$$
\begin{aligned}
R_B^{\tau,m} &\overset{def}{=} \partial_t B_\tau^m + \frac{1}{\tau}\nabla \times E_\tau^m \\
&= \frac{1}{\tau}\nabla \times E^0 + \sum_{j=0}^{m-1} \tau^j \big(\partial_t B^j + \nabla \times E^{j+1}\big) \\
&\quad + \tau^m \big(\partial_t B^m + \nabla \times E_c^{m+1}\big) + \tau^{m+1}\big(\partial_t B_c^{m+1} + \nabla \times E_{c'}^{m+2}\big) \\
&= 0
\end{aligned}
\tag{8.2.41}
$$

and we still have

$$
\operatorname{div} E_\tau^m = b - n_\tau^m, \quad \operatorname{div} B_\tau^m = 0. \tag{8.2.42}
$$

Since the correction term is of order $O\big(\tau^{m+1}\big)$, the orders of the remainders $R_n^{\tau,m}$ and $R_u^{\tau,m}$ are not changed. Moreover, the correction term does not affect on assumption (8.2.46) below.

We conclude the above discussion as follows.

Proposition 8.2.2 *Under the assumption of Proposition 8.2.1, for all integer $m \geq 1$, the remainder $R_B^{\tau,m}$ satisfies (8.2.41) and the remainders $R_n^{\tau,m}$, $R_u^{\tau,m}$ and $R_E^{\tau,m}$ satisfy :*

$$
\sup_{0\leq t\leq T_1} \|(R_n^{\tau,m}, R_E^{\tau,m})(t,\cdot)\|_s \leq C_m \tau^{m+1}, \quad \sup_{0\leq t\leq T_1} \|R_u^{\tau,m}(t,\cdot)\|_s \leq C_m \tau^m, \tag{8.2.43}
$$

where $C_m > 0$ is a constant independent of τ.

The convergence results of this section is stated as follows.

Theorem 8.2.1 *Let $s > \frac{5}{2}$ be any fixed integer. Let the assumption of Proposition 8.2.1 hold. Suppose*

$$
\operatorname{div} E_0^\tau = b - n_0^\tau, \quad \operatorname{div} B_0^\tau = 0 \quad in \quad \mathbb{T} \tag{8.2.44}
$$

and

$$
\Big\|(n_0^\tau, u_0^\tau, E_0^\tau, B_0^\tau) - (n_0, \tau u_0, E_0, B_0)\Big\|_s \leq C_1 \tau, \tag{8.2.45}
$$

where $C_1 > 0$ is a constant independent of τ. Then there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \geq T_1$ and the solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ to the periodic problem (8.1.2)-(8.1.3) satisfies :

$$
\Big\|(n^\tau, u^\tau, E^\tau, B^\tau)(t) - \big(n^0, \tau u^0, E^0 + \tau E_c^1, B^0 + \tau B_c^1\big)(t)\Big\|_s \leq C_2 \tau, \quad \forall\, t \in [0, T_1]
$$

Moreover,

$$
\big\|u^\tau - \tau u^0\big\|_{L^2(0,T_1;H^s(\mathbb{T}))} \leq C_2 \tau^2.
$$

Theorem 8.2.2 *Let $m \geq 1$ and $s > \frac{5}{2}$ be any fixed integers. Let the assumption of Proposition 8.2.1 and (8.2.44) holds. Suppose*

$$
\Big\|(n_0^\tau, u_0^\tau, E_0^\tau, B_0^\tau) - \sum_{j=0}^{m} \tau^j (n_j, \tau u_j, E_j, B_j)\Big\|_s \leq C_1 \tau^{m+1}, \tag{8.2.46}
$$

where $C_1 > 0$ is a constant independent of τ. Then there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \geq T_1$ and the solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ to the periodic problem (8.1.2)-(8.1.3) satisfies :

$$\left\|(n^\tau, u^\tau, E^\tau, B^\tau)(t) - (n_\tau^m, u_\tau^m, E_\tau^m, B_\tau^m)(t)\right\|_s \leq C_2\tau^{m+1}, \quad \forall\, t \in [0, T_1]$$

Moreover,

$$\left\|u^\tau - u_\tau^m\right\|_{L^2(0,T_1;H^s(\mathbb{T}))} \leq C_2\tau^{m+2}.$$

8.3 Case of ill-prepared initial data

8.3.1 Initial layer corrections

In Theorem 8.2.1, compatibility conditions are made on the initial data. These conditions means that the initial profiles $(u^j, E^j, B^j)(0,\cdot)$ are determined through the resolution of the problems (8.2.3)-(8.2.5) for (n^j, u^j, E^j, B^j). Then $(u_0^\tau, E_0^\tau, B_0^\tau)$ cannot be given explicitly. If these conditions are not satisfied, the phenomenon of initial layers occurs. In this section, we consider this situation of so called ill-prepared initial data. We seek a simplest possible form of an asymptotic expansion with initial layer corrections such that its remainders are at least of order $O(\tau^\alpha)$ or equal to zero, with $\alpha > 0$.

We assume that the initial data of an approximate solution $(n_\tau, u_\tau, E_\tau, B_\tau)$ have the asymptotic expansion with respect to τ of the form :

$$(n_\tau, u_\tau, E_\tau, B_\tau)\big|_{t=0} = \big(n_0 + \tau n_1, \tau u_0, E_0 + \tau E_1, B_0 + \tau B_1\big) + O(\tau^2), \tag{8.3.1}$$

where (n_0, u_0, E_0, B_0) are given smooth functions. Taking into account the expansion in the case of well-prepared initial data, the simplest form of an asymptotic expansion including initial layer corrections is :

$$\begin{aligned}(n_\tau, u_\tau)(t,x) &= (n^0, \tau u^0)(t,x) + (n_I^0, \tau u_I^0)(z,x) \\ &\quad + \tau\big((n^1, \tau u^1)(t,x) + (n_I^1, \tau u_I^1)(z,x)\big) + O(\tau^2),\end{aligned} \tag{8.3.2}$$

$$\begin{aligned}(E_\tau, B_\tau)(t,x) &= (E^0, B^0)(t,x) + (E_I^0, B_I^0)(z,x) \\ &\quad + \tau\big((E^1 + \tau E_c^2, B^1)(t,x) + (E_I^1, B_I^1)(z,x)\big) + O(\tau^2),\end{aligned} \tag{8.3.3}$$

where $z = t/\tau \in \mathbb{R}$ is the fast variable, the subscript I stands for the initial layer variables and E_c^2 is the correction term defined by (8.2.29) with $m = 1$.

Now it needs to determine the profiles (n^j, u^j, E^j, B^j) and $(n_I^j, u_I^j, E_I^j, B_I^j)$ for all $j \in \{0,1\}$. Putting expressions (8.3.2) and (8.3.3) into system (8.1.2) and identifying the coefficients in power of τ, we have :

1. The leading profiles (n^0, u^0, E^0, B^0) satisfy the drift-diffusion system (8.2.3) in which $B^0 = 0$, i.e.,
$$\begin{cases} \partial_t n^0 + \operatorname{div}(n^0 u^0) = 0, \\ \nabla h(n^0) = -\big(E^0 + u^0\big), \\ \nabla \times E^0 = 0, \quad \operatorname{div} E^0 = b - n^0, \\ B^0 = 0, \\ t = 0 \ : \ \ (n^0, u^0) = (n_0, u_0). \end{cases} \tag{8.3.4}$$
The second order profiles (n^1, u^1, E^1, B^1)satisfy the equations in (8.2.4). Since $B^0 = 0$, we may take :
$$n^1 = 0, \quad u^1 = 0, \quad E^1 = 0. \tag{8.3.5}$$
Then B^1 is determined by :
$$\operatorname{div} B^1 = 0, \quad \nabla \times B^1 = \partial_t E^0 - n^0 u^0. \tag{8.3.6}$$
Now we determine the initial-layer profiles $(n_I^0, u_I^0, E_I^0, B_I^0)$ and $(n_I^1, u_I^1, E_I^1, B_I^1)$

2. The leading correction terms $(n_I^0, u_I^0, E_I^0, B_I^0)$ satisfy :
$$\begin{cases} \partial_z E_I^0 - \nabla \times B_I^0 = 0, \\ \partial_z B_I^0 + \nabla \times E_I^0 = 0, \\ \operatorname{div} E_I^0 = 0, \quad \operatorname{div} B_I^0 = 0, \end{cases} \tag{8.3.7}$$
$$n_I^0 = 0 \quad \text{and} \quad u_I^0 = -E_I^0. \tag{8.3.8}$$
From the two last equations of (8.3.7) we need :
$$\operatorname{div} E_I^0(0, x) = \operatorname{div} B_I^0(0, x) = 0.$$
Therefore,
$$\operatorname{div} E_0 = b - n_0, \quad \operatorname{div} B_0 = 0. \tag{8.3.9}$$
Equation $n_I^0 = 0$ imply that there is not first order initial layer on variable n. Therefore, up to a constant, we may take
$$n^0(0, x) = n_0(x). \tag{8.3.10}$$
Moreover, Equation $B^0 = 0$ and expressions (8.3.1)-(8.3.3) for u and E imply that :
$$u^0(0, x) + u_I^0(0, x) = u_0(x), \quad E^0(0, x) + E_I^0(0, x) = E_0(x), \quad B_I^0(0, x) = B_0(x), \tag{8.3.11}$$
which determines the initial value of u_I^0, E_I^0 and B_I^0, where $u^0(0, \cdot)$ and $E^0(0, \cdot)$ are given by (8.2.8).

The second order correction terms $(n_I^1, u_I^1, E_I^1, B_I^1)$ satisfy :

$$\begin{cases} \partial_z E_I^1 - \nabla \times B_I^1 = n^0(0,x) u_I^0(z,x), \\ \partial_z B_I^1 + \nabla \times E_I^1 = 0, \\ \operatorname{div} E_I^1 = -n_I^1, \quad \operatorname{div} B_I^1 = 0, \end{cases} \tag{8.3.12}$$

$$u_I^1 = -E_I^1, \tag{8.3.13}$$

$$\partial_z n_I^1(z,x) + \operatorname{div}\big(n^0(0,x) u_I^0(z,x)\big) = 0. \tag{8.3.14}$$

From the two last equations of (8.3.12) we need :

$$\operatorname{div} E_I^1(0,x) = -n_I^1(0,x), \quad \operatorname{div} B_I^1(0,x) = 0.$$

Therefore,

$$\operatorname{div} E_1 = -n_1, \quad \operatorname{div} B_1 = 0. \tag{8.3.15}$$

Moreover, equations (8.3.5) and expressions (8.3.1)-(8.3.3) for B imply that :

$$n_I^1(0,x) = n_1(x), \quad u_I^1(0,x) = u_1(x), \quad E_I^1(0,x) = E_1(x), \tag{8.3.16}$$

$$B^1(0,x) + B_I^1(0,x) = B_1(x), \tag{8.3.17}$$

which determines the initial value of the second order initial layers.

Since, $u_I^0 = -E_I^0$ and with (8.3.10), (8.3.13)-(8.3.14) and (8.3.16), it is easy to obtain that the equivalent of the Maxwell system (8.3.12) read :

$$\begin{cases} \partial_z E_I^1 - \nabla \times B_I^1 = -n_0(x) E_I^0(z,x), \\ \partial_z B_I^1 + \nabla \times E_I^1 = 0, \\ \operatorname{div} E_I^1 = -n_I^1, \quad \operatorname{div} B_I^1 = 0 \end{cases} \tag{8.3.18}$$

and the equivalent of the identity (8.3.14) read :

$$n_I^1(0,x) = n_1(x) + \Big(\int_0^z E_I^0(\xi,x) d\xi \cdot \nabla\Big) n_0(x). \tag{8.3.19}$$

Thus, the initial layer profiles $(n_I^0, u_I^0, E_I^0, B_I^0)$ and $(n_I^1, u_I^1, E_I^1, B_I^1)$ are completely determined by, (8.3.7)-(8.3.8), (8.3.12)-(8.3.13) and (8.3.18)-(8.3.19) which not determined with exponential decay to zero. They are smooth functions of (z,x) and different from the initial layers indicated in chapters 6 and 7. The proof of the convergence of system (8.1.3) to an asymptotic expansion containing these initial layers is more complicated compared to an asymptotic expansion containing initial layers with exponential decay.

8.3.2 Convergence results

According to the asymptotic expansions above, set

$$\begin{cases} n_{\tau,I}(t,x) = n^0(t,x) + \tau n_I^1(t/\tau,x), \\ u_{\tau,I}(t,x) = \tau\big(\ u^0(t,x) + u_I^0(t/\tau,x) + \tau u_I^1(t/\tau,x)\ \big), \\ E_{\tau,I}(t,x) = E^0(t,x) + E_I^0(t/\tau,x) + \tau E_I^1(t/\tau,x) + \tau^2 E_c^2(t,x), \\ B_{\tau,I}(t,x) = B_I^0(t/\tau,x) + \tau\big(\ B^1(t,x) + B_I^1(t/\tau,x)\ \big). \end{cases} \tag{8.3.20}$$

Then we have :

$$t = 0\ :\ \big(n_{\tau,I}, u_{\tau,I}, E_{\tau,I}, B_{\tau,I}\big) = \big(n_0, \tau u_0, E_0, B_0\big) + \tau\big(n_1, u_1, E_1, B_1\big). \tag{8.3.21}$$

Moreover, equations (8.3.4), (8.3.6)-(8.3.7) and (8.3.18) imply that :

$$\operatorname{div} E_{\tau,I} = b - n_{\tau,I}, \quad \operatorname{div} B_{\tau,I} = 0. \tag{8.3.22}$$

Define the remainders $R_n^{\tau,I}$, $R_u^{\tau,I}$, $R_E^{\tau,I}$ and $R_B^{\tau,I}$ by :

$$\begin{cases} \partial_t n_{\tau,I} + \dfrac{1}{\tau}\operatorname{div}(n_{\tau,I}u_{\tau,I}) = R_n^{\tau,I}, \\ \partial_t u_{\tau,I} + \dfrac{1}{\tau}(u_{\tau,I}\cdot\nabla)u_{\tau,I} + \dfrac{1}{\tau}\nabla h(n_{\tau,I}) = -\dfrac{E_{\tau,I}}{\tau} - \dfrac{u_{\tau,I}}{\tau^2} - \dfrac{u_{\tau,I}\times B_{\tau,I}}{\tau} + R_u^{\tau,I}, \\ \partial_t E_{\tau,I} - \dfrac{1}{\tau}\nabla\times B_{\tau,I} = \dfrac{n_{\tau,I}u_{\tau,I}}{\tau} + R_E^{\tau,I}, \\ \partial_t B_{\tau,I} + \dfrac{1}{\tau}\nabla\times E_{\tau,I} = R_B^{\tau,I}. \end{cases} \tag{8.3.23}$$

Using equations (8.3.4), (8.3.6) for $(n^0, u^0, E^0, B^0, B^1)$ and (8.3.7)-(8.3.8), (8.3.12)-(8.3.14) for $(u_I^0, E_I^0, B_I^0, n_I^1, u_I^1, E_I^1, B_I^1)$, we obtain :

$$\begin{aligned} R_n^{\tau,I} &= \partial_t\big(n^0 + \tau n_I^1\big) + \operatorname{div}\big((n^0+\tau n_I^1)(u^0 + u_I^0 + \tau u_I^1)\big) \\ &= \partial_z n_I^1 + \operatorname{div}(n^0 u_I^0) + \tau\operatorname{div}\big(n^0 u_I^1 + n_I^1(u^0 + u_I^0)\big) + \tau^2\operatorname{div}(n_I^1 u_I^1) \\ &= \operatorname{div}\big((n^0(t,x) - n^0(0,x))u_I^0(z,x)\big) + \tau\operatorname{div}\big(n^0 u_I^1 + n_I^1(u^0+u_I^0)\big) \\ &\quad + \tau^2\operatorname{div}(n_I^1 u_I^1), \end{aligned}$$

$$
\begin{aligned}
R_u^{\tau,I} &= \tau\big(\partial_t(u^0+u_I^0+\tau u_I^1)+(u^0+u_I^0+\tau u_I^1)\cdot\nabla(u^0+u_I^0+\tau u_I^1)\big)\\
&\quad+\big(u^0+u_I^0+\tau u_I^1\big)\times\big(B_I^0+\tau(B^1+B_I^1)\big)\\
&\quad+\frac{1}{\tau}\big(\nabla h(n^0+\tau n_I^1)+(E^0+E_I^0+\tau E_I^1+\tau^2E_c^2)+(u^0+u_I^0+\tau u_I^1)\big)\\
&= \frac{1}{\tau}\big(\nabla h(n^0)+E^0+u^0\big)+\frac{1}{\tau}(E_I^0+u_I^0)+\frac{1}{\tau}\nabla\big(h(n^0+\tau n_I^1)-h(n^0)\big)\\
&\quad+\partial_z u_I^0+(u^0+u_I^0)\times B_I^0+(E_I^1+u_I^1)\\
&\quad+\tau\big(\partial_t u^0+\partial_z u_I^1+(u^0+u_I^0)\cdot\nabla(u^0+u_I^0)+u_I^1\times B_I^0+(u^0+u_I^0)\times(B^1+B_I^1)+E_c^2\big)\\
&\quad+\tau^2\big((u^0+u_I^0)\cdot\nabla u_I^1+u_I^1\cdot\nabla(u^0+u_I^0)+u_I^1\times(B^1+B_I^1)\big)+\tau^3(u_I^1\cdot\nabla)u_I^1\\
&= -\nabla\times B_I^0+(u^0+u_I^0)\times B_I^0+\frac{1}{\tau}\nabla\big(h(n^0+\tau n_I^1)-h(n^0)\big)\\
&\quad+\tau\big(\partial_t u^0-\nabla\times B_I^1+n_0E_I^0+(u^0+u_I^0)\cdot\nabla(u^0+u_I^0)+u_I^1\times B_I^0+(u^0+u_I^0)\times(B^1+B_I^1)+E_c^2\big)\\
&\quad+\tau^2\big((u^0+u_I^0)\cdot\nabla u_I^1+u_I^1\cdot\nabla(u^0+u_I^0)+u_I^1\times(B^1+B_I^1)\big)+\tau^3(u_I^1\cdot\nabla)u_I^1,
\end{aligned}
$$

$$
\begin{aligned}
R_E^{\tau,I} &= \partial_t\big(E^0+E_I^0+\tau E_I^1+\tau^2E_c^2\big)-\frac{1}{\tau}\nabla\times\big(B_I^0+\tau(B^1+B_I^1)\big)-(n^0+\tau n_I^1)(u^0+u_I^0+\tau u_I^1)\\
&= \frac{1}{\tau}\big(\partial_z E_I^0-\nabla\times B_I^0\big)+\big(\partial_t E^0-\nabla\times B^1-n^0u^0\big)\\
&\quad+\partial_z E_I^1-n^0u_I^0-\nabla\times B_I^1-\tau\big(n^0u_I^1+n_I^1(u^0+u_I^0)\big)+\tau^2\big(E_c^2-n_I^1u_I^1\big)\\
&= \big(n^0(0,x)-n^0(t,x)\big)u_I^0(z,x)-\tau\big(n^0u_I^1+n_I^1(u^0+u_I^0)\big)+\tau^2\big(E_c^2-n_I^1u_I^1\big)
\end{aligned}
$$

and

$$
\begin{aligned}
R_B^{\tau,I} &= \partial_t(B_I^0+\tau(B^1+B_I^1))+\frac{1}{\tau}\nabla\times\big(E^0+E_0^I+\tau E_I^1+\tau^2E_c^2\big)\\
&= \frac{1}{\tau}\big(\nabla\times E^0+(\partial_z B_I^0+\nabla\times E_I^0)\big)+\partial_z B_I^1+\nabla\times E_I^1+\tau\big(\partial_t B^1+\nabla\times E_c^2\big)\\
&= 0.
\end{aligned}
$$

Now we establish error estimates for $(R_n^{\tau,I},R_u^{\tau,I},R_E^{\tau,I},R_B^{\tau,I})$. Thus, from the previous discussions on the remainders, we obtain the following error estimates.

Proposition 8.3.1 *For given smooth data, the remainders $R_n^{\tau,I}$, $R_u^{\tau,I}$, $R_E^{\tau,I}$ and $R_B^{\tau,I}$ satisfy :*

$$\sup_{0\le t\le T_1}\|\big(R_n^{\tau,I},R_E^{\tau,I}\big)(t,\cdot)\|_s\le C\tau^{1/2},\quad R_B^{\tau,I}=0, \tag{8.3.24}$$

$$\sup_{0\le t\le T_1}\|R_u^{\tau,I}(t,\cdot)\|_s\le C \tag{8.3.25}$$

where $C>0$ is a constant independent of τ.

It is clear that the convergence rate depends strongly on the order of the remainders with respect to τ. In (8.3.25), there is a loss of half order for the remainders $R_u^{\tau,m}$, this loss will be recovered in the error estimate of convergence due to the dissipation term for u.

The convergence result with initial layers can be stated as follows.

Theorem 8.3.1 *Let $s > 5/2$ be a fixed integer and $(n_j, u_j, E_j, B_j)_{j\in\{0,1\}} \in H^{s+1}(\mathbb{T})$ with $n_0 \geq$ constant > 0 in $\mathbb{T}$. Suppose (8.2.45) holds and*

$$\Big\|\big(n_0^\tau, u_0^\tau, E_0^\tau, B_0^\tau\big) - \sum_{j=0,1} \tau^j (n_j, \tau u_j, E_j, B_j)\Big\|_s \leq C_1 \tau^{1/2}, \tag{8.3.26}$$

where $C_1 > 0$ is a constant independent of τ. Then there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \geq T_1$ and the solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ to the periodic problem (8.1.2)-(8.1.3) satisfies :

$$\big\|\big(n^\tau, u^\tau, E^\tau, B^\tau\big) - \big(n^0, u_{\tau,I}, E^0, B^0\big)(t)\big\|_s \leq C_2 \tau^{1/2}, \quad \forall\, t \in [0, T_1].$$

Moreover,

$$\big\|u^\tau - u_{\tau,I}\big\|_{L^2(0,T_1;H^s(\mathbb{T}))} \leq C_2 \tau^{3/2}.$$

Before proving the Proposition 8.3.1 we need to introduce the following lemmas, of which Lemma 8.3.1 can be found in [68]. For completness, a proof of Lemma 8.3.1 is given in Appendix.

Lemma 8.3.1 *Let the following problem :*

$$\begin{cases} \partial_t^2 y - \Delta y = 0, \\ t = 0, \quad y = 0, \quad \partial_t y = g, \end{cases} \tag{8.3.27}$$

where $y = y(t,x)$ is a real-valued function, $t \geq 0$, $x \in \mathbb{R}^3$ and g is assumed to be smooth for the moment. Then there exist an integer $c > 0$, independent of g such that the following holds :

$$\|y(t)\|_\infty \leq (1+t)^{-1} \|g\|_{2,1}, \tag{8.3.28}$$

and

$$\|Dy(t)\|_\infty \leq (1+t)^{-1} \|g\|_{3,1}, \tag{8.3.29}$$

where $D(y) = \begin{pmatrix} \partial_t \\ \nabla \end{pmatrix} y(t,x)$.

Corollary 8.3.1 *There exist an integer $c > 0$ such that for all $g \in W^{3,1}(\mathbb{R}^3)$, for all $t \geq 0$,*

$$\|y(t)\|_\infty \leq c(1+t)^{-1} \|g\|_{3,1}, \tag{8.3.30}$$

and

$$\|Dy(t)\|_\infty \leq (1+t)^{-1} \|g\|_{3,1}. \tag{8.3.31}$$

Proof. Since, we have $C_0^\infty(\mathbb{R}^3)$ is a dense space in $W^{3,1}(\mathbb{R}^3)$. Then we obtain (8.3.30) and (8.3.31). □

Lemma 8.3.2 *Let the Maxwell equations in* $\mathbb{R}^3$ *:*

$$\begin{cases} \partial_t E - \nabla \times B = 0, \\ \partial_t B + \nabla \times E = 0. \end{cases} \tag{8.3.32}$$

Additionally one has the initial conditions :

$$E(t=0) = E(0,x), \quad B(t=0) = B(0,x) \tag{8.3.33}$$

and the contraint

$$\operatorname{div} E = 0, \quad \operatorname{div} B = 0. \tag{8.3.34}$$

Then the solution (E,B) *of the systems (8.3.32)-(8.3.34) satisfies :*

$$\|(E,B)(t)\| = \|(E,B)(0,.)\| \quad , t \geq 0, \tag{8.3.35}$$

$$\|(E,B)(t)\|_\infty \leq c(1+t)^{-1}\|(E,B)(0,.)\|_{3,1}, \quad t \geq 0, \tag{8.3.36}$$

where C *is independent of* t.

Proof. It is easy to get an L^2-L^2-estimate for E and B. Multiplying both sides of the first equation of (8.3.32) with E in L^2 and both sides of the second equation of (8.3.32) with B in L^2 we end up with :

$$\frac{1}{2}\frac{d}{dt}\|E(t)\|^2 - \big(\nabla \times B, E\big) = 0,$$
$$\frac{1}{2}\frac{d}{dt}\|B(t)\|^2 + \big(\nabla \times E, B\big) = 0.$$

Adding the last two equations we get :

$$\|(E,B)(t)\| = \|(E,B)(0,.)\|, \quad t \geq 0.$$

From the linearized initial value problem (8.3.32), we obtain by differentiation :

$$\partial_t^2 E + \nabla \times (\nabla \times E) = 0,$$

and

$$\partial_t^2 B + \nabla \times (\nabla \times B) = 0.$$

Using the formula :

$$\Delta = \nabla \operatorname{div} - \nabla \times (\nabla \times)$$

and (8.3.34) we obtain the equations :

$$\partial_t^2 E - \Delta E = 0$$

and

$$\partial_t^2 B - \Delta B = 0.$$

Then $u = (E, B)$ is the solution of the linear initial value problem :

$$\partial_t^2 u - \Delta u = 0,$$

with the initial conditions :

$$u(t=0) = \big(E(0,x), B(0,x)\big) = u_0, \quad \partial_t u(t=0) = \big(\nabla \times E(0,x), \nabla \times B(0,x)\big) = u_1.$$

Let the operator $w(t)$ be defined through :

$$\big(w(t)g\big)(x) := y(t,x),$$

where y is the solution of the linear initial value problem :

$$\partial_t^2 y - \Delta y = 0,$$

$$y(t=0) = 0, \quad \partial_t y(t=0) = g.$$

Then $u = (E, B)$ is given by :

$$u(t) = \big(w(t)u_1\big) + \partial_t\big(w(t)u_0\big),$$

because the function $v(t)$ defined by :

$$v(t,x) = \partial_t\big(w(t)g\big)(x)$$

solves the initial value problem :

$$\partial_t^2 v - \Delta v = 0,$$

$$v(t=0) = g, \quad \partial_t v(t=0) = \partial_t^2\big(w(t)g\big)(t=0) = \Delta\big(w(t)g\big)(t=0) = 0.$$

Then from the Corollary 8.3.1 we obtain (8.3.36). $\square$

8.3.3 Proof of Proposition 8.3.1

We know the terms (E_I^0, B_I^0) in system (8.3.7) satisfy the Maxwell system (8.3.32), then from the Lemma 8.3.2 we have :

$$\|E_I^0(z,.)\|_s + \|B_I^0(z,.)\|_s = \|E_I^0(0,.)\|_s + \|B_I^0(0,.)\|_s, \quad \forall s > 0.$$

Then we deduce that :

$$(E_I^0, B_I^0) \in C\big([0, +\infty[, \mathbb{T}\big), \tag{8.3.37}$$

with the uniform estimate :

$$\|E_I^0(z,.)\|_s + \|B_I^0(z,.)\|_s \le K_1, \quad \forall s > 0, \tag{8.3.38}$$

where $K_1 > 0$ is a constant independent of τ.

From (8.3.8) and the uniform estimate (8.3.38) we deduce the following estimate of u_I^0 :

$$\|u_I^0(z,.)\|_s \leq K_2, \tag{8.3.39}$$

where $K_2 > 0$ is a constant independent of τ.

On the other hand we have that the initial layer n_I^1 satisfies :

$$n_I^1(z,x) = n_1(x) + \Big(\int_0^z E_I^0(\xi,x)d\xi \cdot \nabla\Big)n_0(x),$$

which is non-local with respect to the fast variable z. In order to establish a uniform estimate of n_I^1 with $z > 0$, we get :

$$\xi(z,x) = \int_0^z E_I^0(t,x)dt, \quad \eta(z,x) = \int_0^z B_I^0(t,x)dt. \tag{8.3.40}$$

From (8.3.7), it is easy to see that ξ and η solve the following problem on $[0,+\infty) \times \mathbb{T}$:

$$\begin{cases} \partial_z\xi(z,x) - \nabla \times \eta(z,x) = E_I^0(0,.), \\ \partial_z\eta(z,x) + \nabla \times \xi(z,x) = B_I^0(0,.), \\ \operatorname{div}\xi(z,x) = 0, \quad \operatorname{div}\eta(z,x) = 0, \\ z = 0, \quad \xi = \eta = 0. \end{cases}$$

Since

$$\operatorname{div} E_I^0(0,x) = \operatorname{div} B_I^0(0,x) = 0,$$

there exist functions Φ, $\Psi \in C\big([0,\infty), H^{s+1}(\mathbb{T})\big)$ such that :

$$E_I^0(0,x) = \nabla \times \Phi(x), \quad B_I^0(0,x) = -\nabla \times \Psi(x).$$

Therefore, ξ and η satisfy in $[0,+\infty[\times\mathbb{T}$:

$$\begin{cases} \partial_z\big(\xi + \Psi(x)\big)(z,x) - \nabla \times \big(\eta + \Phi(x)\big)(z,x) = 0, \\ \partial_z\big(\eta + \Phi(x)\big)(z,x) + \nabla \times \big(\xi + \Psi(x)\big)(z,x) = 0, \\ \operatorname{div}\xi(z,x) = 0, \quad \operatorname{div}\eta(z,x) = 0, \\ s = 0, \quad \xi = \eta = 0. \end{cases} \tag{8.3.41}$$

This yields the energy estimate by differentiation and by multiplying both sides of the first equation of (8.3.41) with $\partial_x^\alpha\big(\xi + \Psi(x)\big)$ in L^2 and both sides of the second equation of (8.3.41) with $\partial_x^\alpha\big(\eta + \Phi(x)\big)$ in L^2 :

$$\int_{\mathbb{T}} \Big(\big|\partial_x^\alpha\big(\xi(z,x) + \Psi(x)\big)\big|^2 + \big|\partial_x^\alpha\big(\eta(z,x) + \Phi(x)\big)\big|^2\Big)dx = \int_{\mathbb{T}} \Big(\big|\partial_x^\alpha\Phi(x)\big|^2 + \big|\partial_x^\alpha\Psi(x)\big|^2\Big)dx,$$

for all $z > 0$ and $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq s+1$, from which we deduce a uniform estimates of ξ and Ψ

with respect to s. Thus from (8.3.19) and with the last equality we have :

$$\|n_I^1(z,.)\|_s \leq K_3, \quad \forall s > 0, \tag{8.3.42}$$

where $K_3 > 0$ is a constant independent of τ.

Now, we get an estimate for E_I^1 and B_I^1. Let

$$\mathcal{E}(z) = \frac{1}{2}\int_{\mathbb{T}} \big(|\partial_x^\alpha E_I^1(z,x)|^2 + |\partial_x^\alpha B_I^1(z,x)|^2\big)dx \quad \text{and} \quad j_\alpha(z,x) = -\partial_x^\alpha\big(n_0(x)E_I^0(z,x)\big).$$

We differentiate the system (8.3.18) with ∂_x^α (in x) for a multi-index α with $|\alpha| \leq s$ and we multiply both sides of the first equation of (8.3.18) with $\partial_x^\alpha E_I^1$ in L^2 and both sides of second equation of (8.3.18) with $\partial_x^\alpha B_I^1$ in L^2 we obtain :

$$\frac{d}{dz}\mathcal{E}(z) - \big(j_\alpha(z,x), \partial_x^\alpha E_I^1(z,x)\big) = 0. \tag{8.3.43}$$

It is obvious that :

$$\begin{aligned} \big(j_\alpha(z,x), \partial_x^\alpha E_I^1(z,x)\big) &\leq \|j_\alpha(z,.)\|\|\partial_x^\alpha E_I^1(z,.)\| \\ &\leq \|j_\alpha(z,.)\|\mathcal{E}(z)^{\frac{1}{2}}, \end{aligned}$$

which imply :

$$\frac{d}{dz}\mathcal{E}(z) \leq \|j_\alpha(z,.)\|\mathcal{E}(z)^{\frac{1}{2}}. \tag{8.3.44}$$

Applying Lemma 8.1.2 in (8.3.44) we obtain :

$$\begin{aligned} \mathcal{E}(z)^{\frac{1}{2}} &\leq \mathcal{E}(0)^{\frac{1}{2}} + \frac{1}{2}\int_0^z \|j_\alpha(\xi,.)\|d\xi \\ &\leq \mathcal{E}(0)^{\frac{1}{2}} + \frac{1}{2}\int_0^z \|\partial_x^\alpha\big(n_0E_I^0\big)(\xi,.)\|d\xi \end{aligned}$$

Thus, Generalized Holder inequality and Lemma (8.3.2), implies :

$$\begin{aligned} \mathcal{E}(z)^{\frac{1}{2}} &\leq C\Big(\mathcal{E}(0)^{\frac{1}{2}} + \|n_0\|_{|\alpha|}\int_0^z \|\partial_x^\alpha(E_I^0, B_I^0)(\xi,.)\|_\infty d\xi\Big) \\ &\leq C\Big(\mathcal{E}(0)^{\frac{1}{2}} + \|n_0\|_{|\alpha|}\|\partial_x^\alpha(E_I^0, B_I^0)(0,.)\|_{3,1}\int_0^z (1+t)^{-1}dt\Big) \\ &\leq C\Big(\mathcal{E}(0)^{\frac{1}{2}} + \|n_0\|_{|\alpha|}\|\partial_x^\alpha(E_I^0, B_I^0)(0,.)\|_{3,1}\ln(1+z)\Big). \end{aligned}$$

Then we have :

$$\begin{aligned} \|E_I^1(z,.)\|_s + \|B_I^1(z,.)\|_s \leq\ & C\Big(\|E_I^1(0,.)\|_s + \|B_I^1(0,.)\|_s \\ & + \|n_0\|_s\|(E_I^0, B_I^0)(0,.)\|_{s+3,1}\ln(1+z)\Big), \end{aligned}$$

which implies :

$$\|E_I^1(z,.)\|_s + \|B_I^1(z,.)\|_s \;\leq\; C\big(C + C\ln(1+z)\big) \leq C\big(C + (1+z)^{\frac{1}{2}}\big).$$

Then we obtain :

$$\|E_I^1(z,.)\|_s + \|B_I^1(z,.)\|_s \leq K_4\tau^{-\frac{1}{2}}, \tag{8.3.45}$$

where K_4 is a constant independent of τ.

From (8.3.13) and using the estimates (8.3.45) we obtain :

$$\|u_I^1(z,.)\|_s \leq K_5\tau^{-\frac{1}{2}}, \tag{8.3.46}$$

where K_5 is a constant independent of τ.

On the other hand, there is $\eta \in [0,t] \subset [0,T_1]$ such that :

$$n^0(t,x) - n^0(0,x) = t\partial_t n^0(\eta,x) = \tau z\partial_t n^0(\eta,x).$$

From (8.3.8), it is easy to obtain :

$$\big(n^0(t,x) - n^0(0,x)\big)u_I^0 = -\tau z\partial_t n^0(\eta,x)E_I^0. \tag{8.3.47}$$

To complete the proof, we get an estimate for (zE_I^0, zB_I^0). From (8.3.7), it is easy to obtain that (zE_I^0, zB_I^0) satisfies the following system :

$$\begin{cases} \partial_z(zE_I^0) - \nabla\times(zB_I^0) = E_I^0, \\ \partial_z(zB_I^0) + \nabla\times(zE_I^0) = B_I^0, \\ \operatorname{div} E_I^0 = 0, \quad \operatorname{div} B_I^0 = 0. \end{cases} \tag{8.3.48}$$

For $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq s$, differentiating equations (8.3.48) with respect to x and multiplying both sides of the first equation of (8.3.48) with $\partial_x^\alpha(zE_I^0)$ in L^2 and both sides of the second equation of (8.3.48) with $\partial_x^\alpha(zB_I^0)$ in L^2 yields :

$$\frac{d}{dz}\mathcal{E}'(z) - \big(\partial_x^\alpha\big(zE_I^0(z,x)\big), \partial_x^\alpha E_I^0(z,x)\big) - \big(\partial_x^\alpha(zB_I^0), \partial_x^\alpha B_I^0(z,x)\big) = 0, \tag{8.3.49}$$

where,

$$\mathcal{E}'(z) = \frac{1}{2}\int_{\mathbb{T}}\Big(|\partial_x^\alpha\big(zE_I^0(z,x)\big)|^2 + |\partial_x^\alpha\big(zB_I^0(z,x)\big)|^2\Big)dx.$$

It is obvious that :

$$\frac{d}{dz}\mathcal{E}'(z) \leq \|\partial_x^\alpha\big(B_I^0, E_I^0\big)(z,.)\|\mathcal{E}'(z)^{\frac{1}{2}}. \tag{8.3.50}$$

Applying Lemma 8.1.2 in (8.3.50) we obtain :

$$\mathcal{E}'(z)^{\frac{1}{2}} \;\leq\; C\Big(\mathcal{E}'(0)^{\frac{1}{2}} + \int_0^z \|\partial_x^\alpha\big(B_I^0, E_I^0\big)(\xi,.)\|d\xi\Big)$$

Thus, Generalized Holder inequality and Lemma (8.3.2), imply :

$$\begin{aligned}
\mathcal{E}'(z)^{\frac{1}{2}} &\leq C|\mathbb{T}|\int_0^z \|\partial_x^\alpha\big(B_I^0, E_I^0\big)(\xi,.)\|_\infty d\xi \\
&\leq C\|\partial_x^\alpha(E_I^0, B_I^0)(0,.)\|_{3,1}|\mathbb{T}|\int_0^z (1+t)^{-1}dt \\
&\leq C\|\partial_x^\alpha(E_I^0, B_I^0)(0,.)\|_{3,1}|\mathbb{T}|\ln(1+z).
\end{aligned}$$

Then we have :

$$\|zE_I^0(z,.)\|_s + \|zB_I^0(z,.)\|_s \leq C\|(E_I^0, B_I^0)(0,.)\|_{s+3,1}|\mathbb{T}|\ln(1+z),$$

which implies :

$$\begin{aligned}
\|zE_I^0(z,.)\|_s + \|zB_I^0(z,.)\|_s &\leq C\ln(1+z) \\
&\leq C(1+z)^{\frac{1}{2}}.
\end{aligned}$$

Then we obtain :

$$\|zE_I^0(z,.)\|_s + \|zB_I^0(z,.)\|_s \leq K_6\tau^{-\frac{1}{2}}, \tag{8.3.51}$$

where K_6 is a constant independent of τ.
It follows from (8.3.47) and (8.3.51) that :

$$\big(n^0(t,x) - n^0(0,x)\big)u_I^0(z,x) = O(\tau^{1/2}). \tag{8.3.52}$$

Thus, from (8.3.39), (8.3.42), (8.3.46) and (8.3.52) we have :

$$R_n^{\tau,I} = O(\tau^{1/2}) \quad \text{and} \quad R_E^{\tau,I} = O(\tau^{1/2}).$$

Finally, for $R_u^{\tau,I}$, we have

$$h(n^0 + \tau n_I^1) - h(n^0) = O(\tau).$$

Then, from (8.3.38)-(8.3.39), (8.3.42), (8.3.45)-(8.3.46) we obtain :

$$R_u^{\tau,I} = O(1).$$

This completes the proof. □

8.3.4 General convergence result

Let $(n^\tau, u^\tau, E^\tau, B^\tau)$ be the exact solution to (8.1.2) with initial data $(n_0^\tau, u_0^\tau, E_0^\tau, B_0^\tau)$ and $(n_\tau, u_\tau, E_\tau, B_\tau)$ be an approximate periodic solution defined on $[0, T_1]$, with

$$(n_\tau, u_\tau, E_\tau, B_\tau) \in C([0,T_1], H^{s+1}(\mathbb{T})) \cap C^1([0,T_1], H^s(\mathbb{T})).$$

We define the remainders of the approximate solution by :

$$\begin{cases} R_n^\tau = \partial_t n_\tau + \dfrac{1}{\tau}\operatorname{div}(n_\tau u_\tau), \\ R_u^\tau = \partial_t u_\tau + \dfrac{1}{\tau}(u_\tau\cdot\nabla)u_\tau + \dfrac{1}{\tau}\nabla h(n_\tau) + \dfrac{E_\tau}{\tau} + \dfrac{u_\tau\times B_\tau}{\tau} + \dfrac{u_\tau}{\tau^2}, \\ R_E^\tau = \partial_t E_\tau - \dfrac{1}{\tau}\nabla\times B_\tau - \dfrac{n_\tau u_\tau}{\tau}, \\ R_B^\tau = \partial_t B_\tau + \dfrac{1}{\tau}\nabla\times E_\tau. \end{cases} \tag{8.3.53}$$

Suppose

$$\operatorname{div} E_\tau = b - n_\tau, \quad \operatorname{div} B_\tau = 0, \tag{8.3.54}$$

$$\sup_{0\le t\le T_1} \|(n_\tau, E_\tau, B_\tau)(t,\cdot)\|_s \le C_1, \quad \sup_{0\le t\le T_1} \|u_\tau(t,\cdot)\|_s \le C_1\tau, \tag{8.3.55}$$

$$\left\|\left(n_0^\tau - n_\tau(0,\cdot), u_0^\tau - u_\tau(0,\cdot), E_0^\tau - E_\tau(0,\cdot), B_0^\tau - B_\tau(0,\cdot)\right)\right\|_s \le C_1\tau^{\lambda+a}, \tag{8.3.56}$$

$$\sup_{0\le t\le T_1} \|(R_n^\tau, R_E^\tau)(t,\cdot)\|_s \le C_1\tau^{\lambda+a}, \quad \sup_{0\le t\le T_1} \|R_u^\tau(t,\cdot)\|_s \le C_1\tau^{\lambda}, \quad R_B^\tau = 0, \tag{8.3.57}$$

where $\lambda \ge 0$, $C_1 > 0$ and $0 < a \le 1$ are constants independent of τ.

Theorem 8.3.2 *Let $\lambda \ge 0$ and $a > 0$. Under the above assumptions, there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \ge T_1$ and the solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ of the periodic problem (8.1.2)-(8.3.3) satisfies :*

$$\left\|(n^\tau, u^\tau, E^\tau, B^\tau)(t) - (n_\tau, u_\tau, E_\tau, B_\tau)(t)\right\|_s \le C_2\tau^{\lambda+a}, \quad \forall\, t\in[0,T_1]. \tag{8.3.58}$$

Moreover,

$$\left\|u^\tau - u_\tau\right\|_{L^2(0,T_1;H^s(\mathbb{T}))} \le C_2\tau^{\lambda+a+1}. \tag{8.3.59}$$

Remark 8.3.1 *It is clear that Theorem 8.3.2 implies Theorems 8.3.1-8.2.2. In particular, $\lambda = m$ and $a = 1$ with $m \ge 1$ in Theorem 8.2.2, and $a = 0$ and $\lambda = 1/2$ in Theorem 8.3.1.*

The proof of Theorem 8.3.2 is given in section 6.4.2 and the proof of Theorem 8.2.1 is similar to that of Theorem 8.3.2.

Appendix 1 : Proof of Lemma 8.1.2

We define the following function :

$$G(t) = C + \int_0^t m(s)\varphi(s)^\alpha ds.$$

Since G is differentiable, we have :

$$G'(t) = m(t)G(t)^\alpha$$

Using the assumption $\varphi(t) \leq G(t)$ and the function $x \to x^\alpha$ is increasing with $\alpha > 0$, we have :

$$G(t)^{-\alpha} G'(t) \leq m(t).$$

We integrate the last inequality from 0 to t gives :

$$\frac{1}{1-\alpha}\{G(t)^{1-\alpha} - C^{1-\alpha}\} \leq \int_0^t m(s)ds.$$

Since $\alpha < 1$, $1 - \alpha > 0$ and the function $x \to x^{\frac{1}{1-\alpha}}$ is increasing, we obtain :

$$G(t) \leq \{C^{1-\alpha} + (1-\alpha)\int_0^t m(s)ds\}^{\frac{1}{1-\alpha}}.$$

Which complete the Lemma. □

Appendix 2 : Proof of Lemma 8.3.1

From Kirchhoff's formula says that y defined by :

$$y(t,x) := \frac{t}{4\pi}\int_{S^2} g(x+tz)dz, \tag{8.3.60}$$

is the solution of (8.3.27), where $S^2 = \partial B(0,1)$ denotes the unit sphere in $\mathbb{R}^3$. This is easily checked. From (8.3.60) we obtain :

$$\begin{aligned} y(t=0) &= 0, \\ 4\pi\partial_t^2 y(t,x) &= \int_{S^2} g(x+tz)dz + t\int_{S^2}(\nabla g)(x+tz)zdz, \\ \partial_t^2 y(t=0) &= g. \end{aligned}$$

Moreover

$$4\pi\nabla y(t,x) = \int_{S^2}(\nabla g)(x+tz)dz,$$

hence

$$\begin{aligned} 4\pi\partial_t^2 y(t,x) &= 2\int_{S^2}(\nabla g)(x+tz)zdz + t\int_{S^2}\nabla\{(\nabla g)(x+tz)z\}zdz \\ &= 3t\int_{B(0,1)}(\Delta g)(x+tz)dz + t^2\int_{B(0,1)}(\nabla\Delta g)(x+tz)zdz, \\ 4\pi\Delta y(t,x) &= t\int_{S^2}(\Delta g)(x+tz)dz = t\int_{S^2}\{(\Delta g)(x+tz)z\}zdz \\ &= t^2\int_{B(0,1)}(\nabla\Delta g)(x+tz)zdz + 3t\int_{B(0,1)}(\Delta g)(x+tz)dz. \end{aligned}$$

This implies

$$\partial_t^2 y - \Delta y = 0.$$

Now we shall prove (8.3.28) and (8.3.29)
First let $t \geq 1$:
1.

$$\begin{aligned} - \quad \int_{S^2} g(x+tz)dz &= \int_{S^2}\int_t^\infty \frac{d}{ds} g(x+sz)dsdz \\ = \quad \int_{S^2}\int_t^\infty (\nabla g)(x+sz)zdsdz &= \int_{S^2}\int_t^\infty \frac{s^2}{s^3}(\nabla g)(x+sz)szdsdz \\ = \quad \int_{|z|>t} |z|^{-3}(\nabla g)(x+z)zdz. & \end{aligned}$$

This implies

$$|\int_{S^2} g(x+tz)dz| \leq t^{-2}\int_{|z|>t} |(\nabla g)(x+z)|dz \leq t^{-2}\|g\|_{1,1}.$$

2. Analogously one obtains

$$|t\int_{S^2} (\nabla g)(x+tz)zdz| \leq t^{-1}\|g\|_{2,1}$$

and

$$|t\int_{S^2} (\nabla g)(x+tz)dz| \leq t^{-1}\|g\|_{2,1}.$$

Hence we get for $t \geq 1$:

$$\|y(t)\|_\infty + \|Dy(t)\|_\infty \leq t^{-1}\|g\|_{2,1}.$$

3. Now let $0 \leq t \leq 1$:

$$\begin{aligned} \int_{S^2} g(x+tz)dz &= -\int_{S^2}\int_t^\infty \frac{d}{ds} g(x+sz)dsdz \\ &= \int_{S^2}\int_t^\infty (s-t)\frac{d^2}{ds^2} g(x+sz)dsdz \\ &= \int_{|z|>t} \frac{(|z|-t)}{|z|^4}\sum_{i,j=1}^{3} z_i z_j (\partial_i\partial_j g)(x+z)dz. \end{aligned}$$

This implies :

$$\begin{aligned} |t\int_{S^2} g(x+tz)dz| &\leq t\sum_{i,j=1}^{3}\int_{|z|>t}\frac{1}{z}|\partial_i\partial_j g(x+z)|dz \\ &\leq t\frac{1}{t}\sum_{i,j=1}^{3}\int_{|z|>1}|\partial_i\partial_j g(x+z)|dz \\ &\leq \|g\|_{2,1}. \end{aligned}$$

Similarly we obtain :

$$\begin{aligned} \int_{S^2} g(x+tz)dz &= \int_{S^2}\int_t^{\infty}\frac{(s-t)^2}{2}\frac{d^3}{ds^3}g(x+tz)dsdz \\ &= -\int_{|z|>t}\frac{(|z|-t)^2}{2|z|^5}\sum_{i,j,k=1}^{3} z_i z_j z_k(\partial_i\partial_j\partial_k g)(x+z)dz. \end{aligned}$$

This implies

$$|\int_{S^2} g(x+tz)dz| \leq c\sum_{i,j,k=1}^{3}\int_{|z|>t}|\partial_i\partial_j\partial_k g(x+z)|dz \leq c\|g\_3,1.$$

In a similar way we obtain :

$$|t\int_{S^2}(\nabla g)(x+tz)zdz| \leq c\|g\|_{3,1},$$

and

$$|t\int_{S^2}(\nabla g)(x+tz)dz| \leq c\|g\|_{3,1},$$

which complete the proof. □

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