Independence-Friendly Modal Logic. Studies in its Expressive Power and Theoretical Relevance.

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Introduction

1. The aims of the dissertation. The present dissertation aims at studying the notion of independence in connection with modal logic. It attempts to provide basically two sorts of contributions: *systematical* and *logical*.

The *systematical* contribution of the present dissertation is the definition of an *independence-friendly* version of any *basic modal logic*, analogously to the way in which the independence-friendly (IF) first-order logic of Jaakko Hintikka and Gabriel Sandu is a version of traditional first-order logic.[1]

In the framework offered by the thesis, it is natural to draw a distinction between two types of independence manifest in modal logic – *logical* and *relational*. With the IF modal-logical apparatus at our disposal, we are then able to address the systematical linguistic question of the existence of tenses as operators in natural languages. I will argue that there are irreducible instances of tenses as operators in English, whence the idea of the operator-like nature of tenses – severely criticized in contemporary linguistic literature – cannot be totally dismissed. Furthermore, and on a more particular level, it will be shown how a certain interpretation (the so-called 'backwards-looking

[1] For IF first-order logic, see e.g. Hintikka & Sandu (1989), Sandu (1989), Hintikka (1996), Hintikka & Sandu (1996). For a short presentation, see *Sect.* 1.1 of the present thesis.

operators' interpretation) of the language of IF modal logic in fact overcomes many, but not all, of the problems traditionally attributed to the very idea of tenses as operators.

The *logical* contribution of the present dissertation is a number of theorems comparing the relative expressive powers of IF modal logic and basic modal logic relative to different classes of modal structures. IF temporal logics (or, as we will call them, IF *tense* logics) are studied as special cases of modal logics. Further, the translatability of IF modal logic into usual *first-order logic* is proven. It is also shown how making a small change in the syntax of IF modal logic results in an extended IF modal logic that can no longer be translated into usual first-order logic, but can express some genuine second-order properties of modal structures.

2. Different notions of independence in modal logic. For a given class **prop** of propositional atoms, the well-formed formulae of *basic modal logic* of k modality types (or $\mathbf{ML}[k]$) are given by the rule:

$$\varphi := p \mid \neg p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \Diamond_i(\varphi) \mid \Box_i(\varphi)$$

where $p \in \mathbf{prop}$, and $i < k$. "Modality type" is simply a term for a collection of modal operators that make use of the same accessibility relation in their semantics. Interest in mixing distinct modality types is abundant in philosophically motivated logics: consider, for instance, *temporality* and *knowledge*, *temporality* and *necessity*, *belief* and *normative obligation*, or *multiagent epistemic logic*. In the present thesis I discern the basic two modalities for each modality type $i < k$: $\Diamond_i$ and $\Box_i$.

A *k-ary modal structure* $\mathcal{M}$ is a tuple $(D, R_0, \ldots, R_{k-1}, \mathfrak{h})$, where the R_i are binary relations (*accessibility relations*) on the non-empty domain D, and $\mathfrak{h}$ assigns a subset of D for each propositional atom. It is possible to give semantics to $\mathbf{ML}[k]$

by associating a *semantical game* between two players (*Abélard*, *Héloïse*) with each triple $(\varphi, \mathcal{M}, d)$, consisting of a formula $\varphi \in \mathbf{ML}[k]$, a modal structure $\mathcal{M}$ and a point d from the domain of $\mathcal{M}$. The class of all *plays* (or *histories*) of $\mathrm{G}(\varphi, \mathcal{M}, d)$ is defined by the following game rules:

- If $\varphi \in \{p, \neg p\}$, no move is made. If $\varphi = p$ and $d \in \mathfrak{h}(p)$, or if $\varphi = \neg p$ and $d \notin \mathfrak{h}(p)$, then *Héloïse* wins the play and *Abélard* loses. Otherwise *Abélard* wins and *Héloïse* loses.

- If $\varphi = (\theta \wedge \psi)$ [*resp.* $\varphi = (\theta \vee \psi)$], then *Abélard* picks out a conjunct $\chi \in \{\theta, \psi\}$ [*resp. Héloïse* picks out a disjunct $\chi \in \{\theta, \psi\}$], and the play goes on as $\mathrm{G}(\chi, \mathcal{M}, d)$.

- Let $i < k$. If $\varphi = \Box_i(\psi)$ [*resp.* $\varphi = \Diamond_i(\psi)$], then *Abélard* [*resp. Héloïse*] picks out, if possible, a state d' with $R_i(d, d')$; and the play continues as $\mathrm{G}(\psi, \mathcal{M}, d')$. If, however, such a choice is not possible (i.e. if d is R_i-maximal), then *Héloïse* [*resp. Abélard*] wins and *Abélard* [*resp. Héloïse*] loses.

The plays can be viewed as finite *sequences*

$$((\varphi_0, a_0), (\varphi_1, a_1), \ldots, (\varphi_{n-1}, a_{n-1}))$$

of positions, starting with the initial position $(\varphi, d) = (\varphi_0, a_0)$, and moving on by transitions $(\varphi_i, a_i) \longmapsto (\varphi_{i+1}, a_{i+1})$ made in accordance with the above game rules. A *strategy* of a player in game $\mathrm{G}(\varphi, \mathcal{M}, d)$ is a function specifying a move for this player in any situation in which, according to the rules of the game, it is his or her turn to move. A strategy of a player is a *winning strategy* (*w.s.*) in game $\mathrm{G}(\varphi, \mathcal{M}, d)$, if the player in question wins any play of the game following this strategy.

Truth and falsity for $\mathbf{ML}[k]$ formulae are then defined game-theoretically as follows:

- $\mathcal{M} \models^{+}_{\mathrm{GTS}} \varphi[d] \iff$ there exists a w.s. for *Héloïse* in $\mathrm{G}(\varphi, \mathcal{M}, d)$.

- $\mathcal{M} \models^{-}_{\mathrm{GTS}} \varphi[d] \iff$ there exists a w.s. for *Abélard* in $\mathrm{G}(\varphi, \mathcal{M}, d)$.

In the present dissertation I define a logic I have termed *IF modal logic of k modality types*, or **IFML**$[k]$. Its novelty as compared with **ML**$[k]$ is that formulae of the form

$$O_{i_1,1} \ldots O_{i_{n-1},n-1}(O_{i_n,n}/W)\varphi$$

are also allowed, where:

- $\varphi \in$ **ML**$[k]$.

- Every $O_{i_j,j}$ $(j := 1, \ldots, n)$ is either $\Diamond_{i_j,j}$ or $\Box_{i_j,j}$ with $i_j < k$.

- $W \subseteq [1, n-1]$.

Being familiar with basic modal logic might in fact lead one to wonder whether it is even possible to define IF modal logic. The evaluation of modal operators in basic modal logic is *local*: such evaluation involves a transition from a state s to a state s' along a given accessibility relation; and the alternative accessible states s' are *determined* by previous transitions (if any) leading to s. In other words, the class of states s' accessible from a fixed state s is always the same, no matter which transitions are made to reach the state s. Therefore it might seem difficult to see where to pose the requirement of independence, as the earlier transitions are in any case irrelevant as regards the possibility of any particular transition $s \longmapsto s'$ proceeding from s.

The first person to have formulated an IF modal logic was Julian Bradfield in his paper “Independence: Logics and Concurrency” (2000). In his joint work with Sibylle Fröschle, “Independence-Friendly Modal Logic and True Concurrency” (2002), the authors provide another formulation of IF modal logic. Bradfield explicitly recognized the apparent problem just described (cf. 2002, p. 104). His tactics in implementing the idea of logical independence in a modal-logical setting is to require that logically independent transitions be *concurrent* (processed in parallel).

We will consider three alternative semantics to the language of **IFML** $[k]$, which differ in the way they interpret the slash sign “/”: *uniformity interpretation* (**UNI**), *‘backwards-looking operators’ interpretation* (**BLO**) and *algebraic interpretation* (**ALG**).

The uniformity interpretation aims at being a very straightforward modal-logical analogue of the IF first-order logic of Hintikka and Sandu: it makes use of semantical games, and implements the notion of independence by imposing appropriate conditions of uniformity on winning strategies. Nothing corresponding to the relation of concurrency between transitions − used by Bradfield for defining his IF modal logic − is employed in its formulation. In *Section* 1.3 Bradfield’s formulations of IF modal logic are discussed, and some basic differences and similarities between his approach and that of this thesis are pointed out.

Under **UNI** interpretation, the intuitive reading of the expression $(\Diamond_{i,n}/W)$ is: “there is a state s_n such that $R_i(s_{n-1}, s_n)$, *independently* of the states s_j with $j \in W$.”

The intuitive reading of $(\Box_{i,n}/W)$ is analogous. A semantical game corresponding to any of the new formulae $O_1 \ldots O_{n-1}(O_n/W)\varphi$ is provided by stipulating that the *game rule* for $(\Diamond_{i,n}/W)$ and $(\Box_{i,n}/W)$ is *precisely the same* as the game rule of $\mathbf{ML}[k]$ games for $\Diamond_i$ *resp.* $\Box_i$. The novelty in the semantics will be a condition imposed on strategies that can count as winning. This condition is that of *uniformity.* The following example illustrates what is at stake.

Example 1 (*Uniformity Interpretation*) Consider the formula $\psi = \Box_1(\Diamond_2/\{1\})\top$ and the unary pointed modal structures $(\mathcal{M}, a)$ *resp.* $(\mathcal{N}, a)$ depicted in Figure 1. ($\top$ stands for *verum*, i.e. an atom true everywhere.)

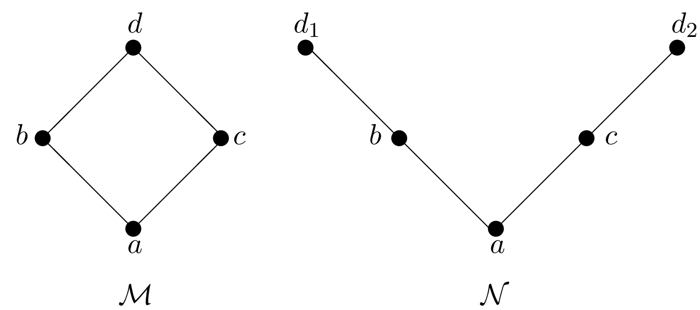

FIGURE 1

In both games $\mathrm{G}(\psi, \mathcal{M}, a)$ and $\mathrm{G}(\psi, \mathcal{N}, a)$, *Abélard* begins by choosing a successor $x \in \{b, c\}$ to a, whereafter it is *Héloïse*'s turn to pick out a successor y to x. *Héloïse*'s strategy f for choosing $y = f(x)$ is said to be $\{1\}$-uniform, if

$$f(b) = f(c),$$

i.e. if the value of f is the same regardless of the choice for x by *Abélard*. Strategy f is said to be a w.s. for *Héloïse* in a game corresponding to ψ, if f is $\{1\}$-uniform, and no matter what the move $x \in \{b, c\}$ made by *Abélard* to interpret $\Box_1$ is, f gives a successor to x (at which $\top$ is true).

Hence there is a w.s. for *Héloïse* in $\mathrm{G}(\psi, \mathcal{M}, a)$, because there is a *common* successor to both b and c, namely d. By contrast, there is no w.s. for her in $G(\psi, \mathcal{N}, a)$, because in the structure $\mathcal{N}$ there is no common successor to b and c. In this game *Abélard* also lacks a w.s., since there is no move x available to him after which *Héloïse* could not move to *some* successor of x. The fact that neither of the players has a w.s. is expressed by saying that the game is *non-determined.* ■

Under **BLO** interpretation, the independence indication "$/W$", appearing as in (O_n/W), is understood as referring back to an earlier round in a play of the relevant semantical game. Then the element a_i that was introduced in that round by one of the players will be the element relative to which the choice a_n interpreting (O_n/W) is made. Hence this element a_n must satisfy $R(a_i, a_n)$, where R is the accessibility relation associated with the operator O_n. This is in contrast to both the case of basic modal logic and **UNI** interpretation of IF modal logic, where the element a_n interpreting the operator O_n indexed with n must always satisfy $R(a_{n-1}, a_n)$, i.e. be made relative to the element introduced in the immediately preceding round. The following example serves to illustrate **BLO** semantics.

Example 2 (*Backwards-Looking Operators*) Consider the formula $\psi = \Box_1(\Diamond_2/\{1\})p$ and the unary pointed modal structures $(\mathcal{M}, a)$ *resp.* $(\mathcal{N}, a)$ depicted in Figure 2.

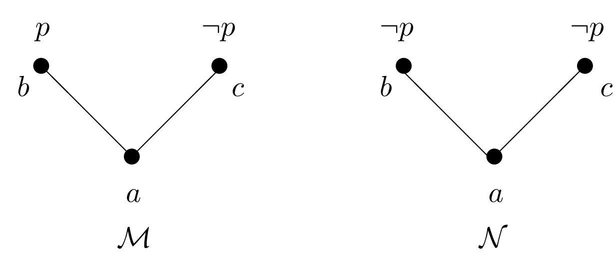

FIGURE 2

Under **BLO** interpretation the independence indication in an expression "(O_n/W)" refers to a unique earlier position in a play, namely the position identified by the number

$$i := max(\{0, 1, \ldots, n-1\}\backslash W).$$

Now in both games $G(\psi, \mathcal{M}, a)$ and $G(\psi, \mathcal{N}, a)$, *Abélard* first chooses a successor $x_1 \in \{b, c\}$ to a. Thereafter it is *Héloïse*'s turn to pick out an element x_2 satisfying the following:

$$R(x_0, x_2),$$

where $x_0 := a$ and the index 0 is obtained as $max(\{0, 1\}\backslash\{1\})$. A winning strategy for *Héloïse* in a game corresponding to ψ is simply any strategy that gives an element x_2 at which p is true and that can be obtained from a along R, no matter what move *Abélard* makes to interpret $\Box_1$.

Clearly in both $G(\psi, \mathcal{M}, a)$ and $G(\psi, \mathcal{N}, a)$, the existence of *Héloïse*'s w.s. does not depend on *Abélard*'s moves, since *Héloïse*'s choice must be made along R starting from a and not from a point chosen by her opponent. There is now a w.s. for *Héloïse* in $G(\psi, \mathcal{M}, a)$, because $R(a, b)$ and p is true at b. And there is no w.s. for her in the game $G(\psi, \mathcal{N}, a)$ because neither of the choices (b and c) available to her satisfies p in the structure $\mathcal{N}$. In this game *Abélard* has trivially a w.s., which consists of his simply making some move, i.e. choosing for instance b to interpret $\Box_1$. ■

In **BLO** interpretation no uniformity conditions are imposed on strategies of the players of this game. The sense of "independence" induced by this interpretation is obtained by *constraining the moves* available to a player when he or she is making a move, and not by *constraining the strategies*, as is the case with **UNI** interpretation.[2] **BLO** interpretation of IF

[2] The distinction between these two possible ways of interpreting the independence indication – constraints on moves and constraints on strategies – is due to Prof. Gabriel Sandu (personal communication).

modal logic will turn out to be useful when discussing natural language tenses later in *Chapter* 5. In fact, **BLO** interpretation implements the notion of *relational independence*, not *functional independence*, the properly logical notion. However, as will be argued in *Chapter* 5, the sense of independence relevant in considering the operator status of natural language tenses − the sense in which an interpretation of one tense may be independent of that of another − is precisely the relational one.

Under **ALG** interpretation, the role of the independence indication "$/W$" is to indicate 'subtraction'. The algebraic interpretation is defined *only* for IF *tense* logic, evaluated over what we will call *algebraic* models

$$\mathcal{M} = (T, <, \mathfrak{h}, o),$$

where $<$ is a linear order on T and (T, o) is an Abelian group. By its syntax IF tense logic is the IF version of Priorean tense logic with universal and existential operators for both future (F *resp.* G) and past (P *resp.* H). Technically, IF tense logic is just **IFML**[2] in whose semantics the operators P and H employ the converse of the accessibility relation by means of which the semantics of F and G is given.

Truth under **ALG** interpretation is defined by recursion on the complexity of IF tense-logical formulae. The clause for formulae of the form $O_1 \ldots O_{n-1}(O_n/W)\varphi$, in the special case of the additive group $(\mathbb{R}, +)$ of the reals, is as follows:

$$\mathcal{M} \models O_1 \ldots O_{n-1}(O_n/W)\varphi[t] \iff$$

$$Q_1 \ldots Q_n : \mathcal{M} \models \varphi[t + \textstyle\sum_{i \in \{1,\ldots,n\}}(x_i) - \sum_{i \in W}(x_i)],$$

where each Q_i is one of the relativized quantifiers $(\exists x_i > 0)$, $(\forall x_i > 0)$, $(\exists x_i < 0)$ or $(\forall x_i < 0)$ − depending on whether the operator O_i is F_i, G_i, P_i or H_i, respectively. The semantics for an arbitrary Abelian group (T, o) is slightly more complicated and will be given in *Section* 4.2. The example below serves to illustrate what the algebraic interpretation amounts to.

Example 3 (*Algebraic Interpretation*) Consider the formula $\psi = G_1(P_2/\{1\})p$ and the pointed algebraic models $(\mathcal{M}, 0)$ *resp.* $(\mathcal{N}, 0)$ depicted in Figure 3, in which the circles indicate the unique point (-1 *resp.* 0) at which the respective model makes the propositional atom p true.

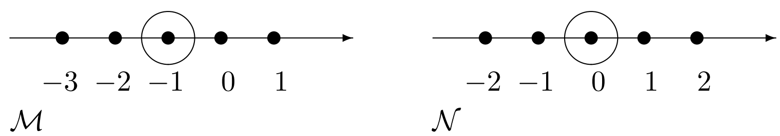

FIGURE 3

The formula ψ is true in $\mathcal{M}$ at 0, since indeed for every $r_1 > 0$ there is $r_2 < 0$ such that p is true at $(0+r_1+r_2-r_1) = r_2$. For all r_1, -1 can be chosen as r_2. By contrast, ψ is false in $\mathcal{N}$ at 0, because there is $r_1 > 0$ such that for all $r_2 < 0$, p is false at $(0 + r_1 + r_2 - r_1) = r_2$. In fact, any positive real number is such an r_1; p is false at all $r_2 < 0$. ■

The algebraic interpretation can be seen to incorporate the effects of the other two interpretations in one maneuver. In it, the relational sense of independence as implemented by **BLO** interpretation, and the logical sense as implemented by **UNI** interpretation, are tied together: the semantic effects of these two interpretations always appear simultaneously.

3. The logical results. If $\mathcal{L}$ and $\mathcal{L}'$ are modal logics (IF or not) and $\mathcal{K}$ is a class of modal structures on which the semantics of these logics is defined, it is said that $\mathcal{L}$ is *embeddable* in $\mathcal{L}'$ over $\mathcal{K}$ (symbolically $\mathcal{L} \leq_{\mathcal{K}} \mathcal{L}'$), if for each formula $\varphi \in \mathcal{L}$ there exists a formula $\psi_\varphi \in \mathcal{L}'$ such that for all $\mathcal{M} \in \mathcal{K}$ and all $d \in dom(\mathcal{M})$,

$$\varphi \text{ is true in } \mathcal{M} \text{ at } d \quad \textit{iff} \quad \psi_\varphi \text{ is true in } \mathcal{M} \text{ at } d.$$

If $\mathcal{L}$ is embeddable in $\mathcal{L}'$ over $\mathcal{K}$ but not *vice versa*, it is said that $\mathcal{L}'$ has *greater expressive power* than $\mathcal{L}$ over $\mathcal{K}$ (in symbols $\mathcal{L} <_{\mathcal{K}} \mathcal{L}'$). If $\mathcal{L}$ is embeddable in $\mathcal{L}'$ and *vice versa*, then it is said that they have *the same expressive power* over $\mathcal{K}$ (symbolically $\mathcal{L} =_{\mathcal{K}} \mathcal{L}'$).

The main logical results of the present thesis can be summarized as follows. Concerning the interpretations **BLO** and **ALG**, we have:

- **[1]** **BLO** does *not* yield **IFML**$[k]$ extra expressive power over **ML**$[k]$. By contrast, under **BLO** the two logics have the same expressive power relative to the class of all k-ary modal structures.

- **[2]** Relative to **ALG**, basic tense logic **TL**$[1]$ and IF tense logic **IFTL**$[1]$ have the same expressive power over the class of all algebraic models.

For the rest of the results, **UNI** interpretation of **IFML**$[k]$ is employed.

- **[3]** The expressive power of **IFML**$[k]$ is greater than that of **ML**$[k]$ over the class $\mathcal{C}_k$ of arbitrary k-ary modal structures: for all $k \geq 1$, **ML**$[k] <_{\mathcal{C}_k}$ **IFML**$[k]$.

- **[4]** Over the class **LO** of unary *linear* temporal structures, the tense logics **IFTL**$[1]$ and **TL**$[1]$ have the same expressive power: **TL**$[1] =_{\mathbf{LO}}$ **IFTL**$[1]$. Likewise the modal logics **IFML**$[1]$ and **ML**$[1]$ coincide relative to the class of all unary linear modal structures.

- **[5]** By contrast, relative to the class $\mathbf{LO}[k,n]$ of all k-ary modal structures of whose k accessibility relations n are linear orders ($n \geq 2$), $\mathbf{IFML}[k]$ has a greater expressive power than $\mathbf{ML}[k]$: for all $n \geq 2$ and for all $k \geq n$, we have $\mathbf{ML}[k] <_{\mathbf{LO}[k,n]} \mathbf{IFML}[k]$. An analogous result holds true for $\mathbf{IFTL}[k]$, $\mathbf{TL}[k]$ and the class of all k-ary temporal structures with at least 2 linear orders corresponding to distinct temporal modality types.

The first result mentioned under [5] says that result [4] cannot be generalized from the class of unary linear *temporal* structures − which are all binary *modal* structures whose linear accessibility relations are each other's converses − to the class of all binary linear modal structures. The second result under [5] says that the coincidence result of [4] between $\mathbf{IFTL}[k]$ and $\mathbf{TL}[k]$ is proper to $k = 1$ and does not generalize to any $k \geq 2$.

- **[6]** Further, I show that $\mathbf{IFML}[k]$ can be translated into traditional first-order logic. Hence the well-known fact that basic modal logic $\mathbf{ML}[k]$ has a *standard translation* into first-order logic generalizes to the class $\mathbf{IFML}[k]$.

- **[7]** I define *Extended IF modal logic* of k *modality types*, or $\mathbf{EIFML}[k]$, which allows modal operators to be independent of propositional connectives (conjunctions, disjunctions), and show that this logic is not translatable into first-order logic but can express properties of structures that are *not first-order definable* (e.g. *having even cardinality*).

The results from [1] to [7] are to be found in the thesis under the following names:

1. **Theorem** 4.1.5
2. **Theorem** 4.2.7
3. **Theorem** 3.4.4
4. **Theorem** 3.4.11 and **Theorem** 3.4.12
5. **Theorem** 3.4.13
6. **Theorem** 3.2.4
7. **Theorem** 3.3.9

4. Wider theoretical relevance. In the fifth and last chapter of the present thesis I point out that the tools employed to discuss independence in modal logic − offered by the different interpretations of IF modal logic − prove to be useful for the systematical study of the notion of operator in connection with the semantics of tenses in natural languages. It will be shown that there are in natural languages instances of tenses as operators. More specifically, it will be argued that the critics of natural language tense operators have not sufficiently appreciated the conceptual difference between the two types of (in)dependence manifest in tense logic − *logical* and *relational*. **BLO** interpretation of IF tense logic is an illustration of how to speak of these two types of (in)dependence within one and the same formalism.

There is a tradition stemming from Prior (1967) of construing natural language tenses as sentential operators. Most contemporary linguists (e.g. Enç, Hornstein, Kamp) oppose, in one way or another, construing tenses as operators. On the other hand, not much would be required to show that tenses really are indispensable in linguistics. In fact, it would suffice that the interpretation of one tense could be logically (functionally) dependent on the interpretation of another tense in an English sentence – analogously to the way in which the interpretation of the quantifier $\exists y$ is dependent on the interpretation of the quantifier $\forall x$ in a formula of the form $\forall x \exists y \varphi$. If no such functional dependencies ever occur with tenses, then their sole role is to introduce times, and they may as well be taken for pronominal expressions, or for adverbs (which act only relatively locally). But if such dependency can occur, then there are cases where tenses in fact behave as operators.

The aim of this thesis is of course *not* to argue that *all* linguistically relevant features of tenses can be accounted for by taking them to be operators, only that some such features must be explained in this way. In *Chapter* 5 it will be argued that such functional dependencies are expressible by using plain English sentences. A case in point is a sentence like

(1) *Harry has not realized he has not called Debbie.*

The times that interpret the present perfect tenses of "has realized" and "has called" are, in evaluation, both chosen to be earlier than the time of speech (indexicality). This fact would be expressed in the traditional scope theory of tense by saying that both instances of past tense have in (1) 'wide scope' (or, 'matrix scope'). But in fact, the time of Harry's calling Debbie is, according to (1), functionally dependent on the past time chosen to interpret "has realized". More technically, according to (1) for every past time $t < NOW$ (from some contextually understood interval) there is a scenario compatible with all that Harry realizes, and a time $t' < NOW$ such that Harry calls Debbie at t' in that scenario. Hence the dependence of t' on t according to (1) is functional.

The proper logical notion of scope ('priority scope') is *not* concerned with relational conditions that objects introduced in the evaluation of these operators must satisfy, but rather with expressing *logical priorities*. Precisely when one operator lies in the priority scope of another, the latter is logically prior to the former; in other words, the evaluation of the former is functionally dependent on the evaluation of the latter. The logical notion of scope *is* essentially operative in (1), while the sentence does not say anything about the temporal relation (in terms of the relation *earlier than*) between the times t and t'. The sentence (1) is an example of where a tense (that of "has called") lies in the priority scope of another tense (that of "has realized", which is to say, notably, that the tense of "has realized" behaves as an operator.

It will be argued in *Chapter* 5 that an important part of the criticism targeted against tenses-as-operators in the literature is in fact a critique of the particular way in which tense operators appear in basic tense logic − which is *not* representative of what tense operators in general are, as witnessed by **BLO** interpretation of IF tense logic. The basic tense-logical setting makes it possible to confuse the question of *temporal relations* among the times which serve to interpret tenses, with the proper *logical relations* (functional dependencies) between such times interpreting tenses. These two contextually manifesting aspects of tenses always go together in basic tense logic, whereby contingent features of tense operators are easily perceived wrongly as being essential if basic tense logic is relied upon as a standard. By contrast, looking at the critique of tense operators from the vantage point of IF tense logic makes a clearer assessment of the conceptual situation possible.

Chapter 1

Preliminaries

We begin by briefly presenting the IF first-order logic of Hintikka and Sandu, discussing how to treat logical independence in a modal-logical setting, and surveying the work of Julian Bradfield, who is the first to have defined an IF modal logic. The present chapter is devoted to these preliminary considerations.

1.1 Logical Priority and First-Order Logic

Let us consider a formulation of first-order logic (**FO**) employing the existential quantifier ($\exists x$) and the universal quantifier ($\forall x$), the binary Boolean connectives disjunction ($\vee$) and conjunction ($\wedge$), and atomic negation ($\neg$) which is a unary Boolean connective applicable only to atomic formulae.[1] For a given vo-

[1] Since the formulae of **FO** given under any formulation have a negation normal form (i.e. for any such formula there exists a logically equivalent **FO** formula in which the negation sign only occurs in front of atomic subformulae), the proposed way of considering **FO** formulae is not restrictive from the point of view of expressive power. For the corresponding fact as

cabulary τ consisting of relation symbols with specified arities, the well-formed formulae of first-order logic of this vocabulary τ (or, **FO** $[\tau]$) are given by the rule:

$$\varphi := R_i(x_1, \ldots, x_n) \mid \neg R_i(x_1, \ldots, x_n) \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid$$

$$\mid (\exists x_k)\varphi \mid (\forall x_k)\varphi,$$

where $R_i \in \tau$ is of arity $n < \omega$. The semantics to $\mathbf{FO}[\tau]$ can be given by defining for each formula $\varphi \in \mathbf{FO}[\tau]$ a semantical game G(φ,$\mathcal{M}$,γ) between two players (*Héloïse* and *Abélard*) relative to a first-order τ-structure $\mathcal{M}$ and a variable assignment $\gamma : \{x_1, x_2, \ldots\} \to dom(\mathcal{M})$. The class of all plays of G(φ,$\mathcal{M}$,γ) is defined by the following game rules:

- If $\varphi \in \{R_i(\overline{x}), \neg R_i(\overline{x})\}$, then no move is made. *Héloïse* wins the play if $\varphi = R_i(\overline{x})$ and $\mathcal{M}, \gamma \models \varphi$, or $\varphi = \neg R_i(\overline{x})$ and $\mathcal{M}, \gamma \not\models \varphi$. Otherwise *Abélard* wins the play.

- If $\varphi = \psi \vee \chi$, *Héloïse* picks out a disjunct $\theta \in \{\psi, \chi\}$, and the play goes on as with G(θ,$\mathcal{M}$,γ).

- If $\varphi = \psi \wedge \chi$, *Abélard* picks out a conjunct $\theta \in \{\psi, \chi\}$, and the play goes on as with G(θ ,$\mathcal{M}$,γ).

- If $\varphi = (\exists x_k)\psi$, *Héloïse* picks out an individual $a \in dom(\mathcal{M})$, and the play goes on as with G(ψ,$\mathcal{M}$,$\gamma[x_k/a]$).[2]

- If $\varphi = (\forall x_k)\psi$, *Abélard* picks out an individual $a \in dom(\mathcal{M})$), and the play goes on as with G(ψ,$\mathcal{M}$,$\gamma[x_k/a]$).

A *strategy* for a player is a function that provides a reply to that player in all plays for which it is his or her turn to

regards basic modal logic, see Fact 2.2.1 below.

[2] $\gamma[x_k/a]$ is an assignment obtained from γ by replacing the pair $\langle x_k, \gamma(x_k)\rangle$ of γ by the pair $\langle x_k, a\rangle$.

move. A strategy provides a move at a stage in a play of a game G(φ,$\mathcal{M}$,γ) as a function of all moves made earlier in that play. A strategy of a player is a *winning strategy* (*w.s.*) in game G(φ,$\mathcal{M}$,γ), if the player in question wins any play of the game following this strategy. In particular, the truth and falsity of an **FO** *sentence* φ is defined as follows:

- The sentence φ is *true* in the model $\mathcal{M}$, if for every variable assignment γ there exists a w.s. for *Héloïse* in G(φ,$\mathcal{M}$,γ).
- The sentence φ is *false* in the model $\mathcal{M}$, if for every variable assignment γ there exists a w.s. for *Abélard* in G(φ,$\mathcal{M}$,γ).

It is not difficult to show that this game-theoretical truth-definition agrees with the standard definition of the truth of an **FO** sentence (which is by recursion on the complexity of the formula).[3] The satisfiability and non-satisfiability of an **FO** *formula* φ in a model $\mathcal{M}$ by the assignment γ are definable as the existence of a w.s. for *Héloïse* resp. *Abélard* in the game G(φ,$\mathcal{M}$,γ).

Now the rules of formation of traditional first-order logic − historically due to Frege, Russell and Whitehead − determine a particular, syntactically given relation of *logical priority* among logical operators, i.e. quantifiers and Boolean connectives. A quantifier $Qx \in \{\exists x, \forall x\}$ bears logical priority precisely to all logical operators occurring in the subformula φ of any formula of the form $(Qx)\varphi$. Likewise, a binary Boolean connective o $\in \{\vee, \wedge\}$ is logically prior to precisely all logical operators occurring in either of the subformulae φ and ψ of any formula of the

[3] To establish the direction from the standard to the game-theoretical semantics, *Axiom of Choice* is required, if the domain of the model is infinite.

form φ o ψ. The atomic negation $\neg$ bears logical priority to precisely all logical operators occurring in the subformula φ to which it is applied – but there are no such operators, as it is only applied to *atomic* formulae. The relation of logical priority, thus defined by syntactic means, has the properties of *irreflexivity*, *asymmetry* and *transitivity*. The relation is certainly *not* connected: for instance, the occurrences of the quantifiers $\exists x$ and $\forall y$ in the formula

$$(\exists x)\varphi \vee (\forall y)\psi$$

are not related by logical priority in the sense defined above, but are incomparable with respect to this relation. Observe, however, the *left-linearity*: if operators (quantifiers or propositional connectives) O_1 and O_2 are logically prior to an operator O, then O_1 and O_2 are necessarily comparable with respect to logical priority.

The relation of logical priority can indeed easily be read from the syntactic tree of any **FO** formula φ. First, identify the nodes of the tree with the 'outmost forms' of the subformulae of φ, such a form being by stipulation either some of the logical operators or an atomic formula. Then, simply enough, if and only if a node O is not an atomic formula and is a successor of a node O' in the syntactic tree, we see that O' is logically prior to O. Particularly, in the case of **FO** there is no way in which a non-atomic node could escape from being logically subordinate to any of its predecessor nodes, or, equivalently, no way in which an operator could escape from being logically prior to each of the non-atomic nodes succeeding it.

There is clearly something contingent about the notion of logical priority characterized above. The built-in relation of logical priority of **FO** is *left-linear*: in particular the set of operators logically prior to a given operator is ordered *linearly* by the relation of logical priority. The intuitive semantic import of this fact is that whatever has been achieved in the evaluation of a formula up to a given stage can in no way be annulled any longer, but the evaluation inevitably proceeds progressively on the basis of the history of evaluation produced by then.

The prejudice of the left-linearity of the relation of logical priority − inherent in the first-order logic of Frege and Russell − was challenged for the first time in the theory of branching quantifiers (Henkin quantifiers).[4] The IF first-order logic of Hintikka and Sandu admits full freedom in explicitly indicating in linear notation the logical priorities among logically active expressions in first-order formulae. Because Hintikka holds that requiring any specific properties of the relation of logical priority is theoretically unfounded (see e.g. Hintikka 1996, *Ch.* 3), it is his view that IF first-order logic is *the* first-order logic − *the* logic of quantifiers − and the traditional, received, first-order logic is what really should be identified by a qualification, be that qualification "dependence-handicapped", "independence-challenged", "ordinary" or just "traditional".[5] From the combinatorial viewpoint, it is hard to disagree with him.

Let the logical priorities among the operators $O_1, \ldots, O_{n-1}$ be fixed, and let O be a new operator.[6] Think of these n operators as being distributed over nodes of a tree in some fixed way. Now, why should it not be possible to consider all of the combinatorially possible 2^{n-1} ways in which O may relate, in terms of logical priority, to the operators from the set $\{O_1, \ldots, O_{n-1}\}$, instead of prejudging that precisely one such way is the only one possible? In **FO**, the relative position of O with respect to

[4] Henkin quantifiers were introduced in Henkin (1961). Classical papers on Henkin quantifiers include Walkoe (1970), Enderton (1970) and Barwise (1979).

[5] For the terminology, cf. e.g. Hintikka (2002), p. 408.

[6] These need not be $n-1$ distinct operator types; it suffices that they are *distinct occurrences* of operators.

the operators $O_1, \ldots, O_{n-1}$ in the syntactic tree of a formula containing all these operators uniquely determines which of the operators $O_1, \ldots, O_{n-1}$ bear logical priority to O and which do not. There is no room for variation, given the position of O in the tree. Conceptually, however, it would be perfectly possible to *dissociate* the relation of logical priority and that of being a subformula under the formation rules of **FO**: one might indicate for each node the other nodes to which this is logically subordinate − completely independently of the subformula relation. (This is a reasonable way of expressing things for the purpose of illustration. Once the idea behind IF logic is sufficiently appreciated, it might be more natural to forget about the formation rules of **FO** altogether, and define formulae of IF first-order logic as relational structures $\varphi = (N, R)$, where N would be a set of occurrences of logical operators and atomic formulae of φ, and R a relation of logical priority among them, whence the relation of logical priority and the subformula relation would in effect be reconnected.[7])

Generally, then, for logical operators $O_0, \ldots, O_{n-1}$ there would exist $(2^{n-1})^n$ possible mutual relations of logical priority. In the case where the operators $O_0, \ldots, O_{n-1}$ are by the standards of **FO** linearly ordered with respect to logical priority as $O_0 < \ldots < O_{n-1}$, allowing only a relaxing of the priority requirements (but not the bringing in of more priori-

[7] Logical formulae are typically understood as strings of symbols, and e.g. a Henkin quantifier formulae $H\varphi$ can be thought of as consisting of a matrix (the Henkin prefix H) concatenated with a string (the first-order formula φ). On the other hand, Henkin prefixes can naturally be defined as structures where a dependency relation is specified between existential and universal variables; see Walkoe (1970), Krynicki & Mostowski (1995). The non-wellfounded modal logic studied by Prof. Lauri Hella (personal communication) is an example of a logic which could not even be defined without thinking of its formulae as structures. This logic allows loops in the relation of logical priority.

ties), there would exist $2^0 \cdot 2^1 \cdot \ldots \cdot 2^{n-1}$ combinatorially possible patterns among the $O_0, \ldots, O_{n-1}$ operators, instead of one, viz. that given by $O_0 < \ldots < O_{n-1}$.

Above, we have treated logical priority as a syntactically given relation, which in the case of **FO** goes together with the subformula relation, but which can generally be separated from it. The semantic content of logical priority is as follows. Precisely when an operator O is logically prior to another, O', the *evaluation* of O' is *dependent* on the evaluation of the operator O. Seen from the viewpoint of the game-theoretical semantics, this means that the strategy function of the player making a move corresponding to O' employs as one of its arguments the move that has previously been made for the operator O. By contrast, the absence of logical priority of O over O' means that the evaluation of O' is made independently – 'in ignorance' – of the result of the evaluation of O. The move for O' must not depend on the move corresponding to O.

IF first-order logic breaks the connection between the subformula relation and the relation of logical priority, and allows an operator to be logically prior to an *arbitrary subset* of operators appearing in the corresponding subformulae. IF first-order logic of vocabulary τ (or, $\mathbf{IF}[\tau]$) can be defined as the closure of atomic and negated atomic $\mathbf{FO}\ [\tau]$ formulae under the operations

(*i*) $\wedge /W$

(*ii*) $\vee /W$

(*iii*) $\forall x_n/W$

(*iv*) $\exists x_n/W$,

where for some subset $\{i_1, \ldots, i_k\}$ of natural numbers, W is the set $\{x_{i_1}, \ldots, x_{i_k}\}$ of individual variables; and in (iii) and (iv) it is further required that $n \notin \{i_1, \ldots, i_k\}$.

The set $Free(\varphi)$ of free variables of an $\mathbf{IF}[\tau]$ formula φ is defined recursively by:

- $Free(R(x_{i_1}, \ldots, x_{i_k})) = \{x_{i_1}, \ldots, x_{i_k}\} = Free(\neg R(x_{i_1}, \ldots, x_{i_k}))$
- $Free(\varphi \wedge /W\ \psi) = Free(\varphi) \cup Free(\psi) \cup W = Free(\varphi \vee /W\ \psi)$
- $Free(\forall x_n/W\ \varphi) = (Free(\varphi)\backslash\{x_n\}) \cup W = Free(\exists x_n/W\ \varphi)$.

An $\mathbf{IF}[\tau]$ formula φ is a *sentence* of $\mathbf{IF}[\tau]$, if $Free(\varphi)$ is empty. The class of IF first-order logical sentences is the fragment of $\mathbf{IF}[\tau]$ that Hintikka and Sandu have considered in their work on IF logic. Their semantics for this fragment is not compositional. Wilfrid Hodges (1997 [a], [b]) and Jouko Väänänen (2002) have presented a compositional semantics for the whole of $\mathbf{IF}[\tau]$.[8] This general semantics will not be discussed here. The basic idea in the semantics for $\mathbf{IF}[\tau]$ sentences is that in a relevant semantical game *Héloïse* must specify her strategy (in order for it to be a w.s.) so that when choosing the move, say, for $\exists x_n/W$, she "does not know" the choices that have been made corresponding to the variables in W earlier in the game. The same holds for *Abélard*'s strategy. The appropriate sense of "does not know" is captured by the way winning strategies for the respective players are defined. The following example illustrates the semantics.

Example 1.1.1 *Fix the vocabulary as $\tau = \{R\}$, where R is a quaternary relation symbol, and let $\mathcal{M} = (D, R^{\mathcal{M}})$ be a first-order structure. Then consider the* $\mathbf{IF}[\tau]$ *sentence*

$$\varphi : \qquad \forall x_0 \exists x_1 \forall x_2 \exists x_3/\{x_0, x_1\} R(x_0, x_1, x_2, x_3).$$

[8] For a discussion on compositionality and IF logic, see Hintikka and Sandu (2001).

For each expression for which according to the game rules it is Héloïse's turn to move, a strategy function for Héloïse in the game G(φ,$\mathcal{M}$,$\emptyset$) is given. The function $f_{\exists x_1} : D \to D$ corresponding to $\exists x_1$ gives an element of the domain depending on the preceding game history, which consists simply of one choice by Abélard from the domain. The function $f_{\exists x_3/\{x_0,x_1\}} : D^3 \to D$ which corresponds to $\exists x_3/\{x_0, x_1\}$, likewise gives a value from D depending on the preceding game history, which now is a 3-tuple of elements instead of a single element from D. A strategy for Héloïse is a set of strategy functions, one for each expression for which it is her turn to move. Hence Héloïse's strategy here is the set $\{f_{\exists x_1}, f_{\exists x_3/\{x_0,x_1\}}\}$.

A strategy function $f_{\exists x_3/\{x_0,x_1\}}$ is {0,1}-uniform if its value only depends on arguments that do not correspond to interpreting the variables x_0, x_1; more exactly, if for all 3-tuples (a_0, a_1, a_2), $(b_0, b_1, b_2) \in D^3$ for which $a_2 = b_2$,

$$f_{\exists x_3/\{x_0,x_1\}}(a_0, a_1, a_2) = f_{\exists x_3/\{x_0,x_1\}}(b_0, b_1, b_2).$$

Hence whatever values the arguments corresponding to the indices 0 and 1 get, whenever the third argument is the same in two argument sequences, the value of the strategy function must be the same as well.

A winning strategy for Héloïse in the game G(φ,$\mathcal{M}$,$\emptyset$) is a set of strategy functions $\{f_{\exists x_1}, f_{\exists x_3/\{x_0,x_1\}}\}$, where the function $f_{\exists x_3/\{x_0,x_1\}}$ is {0,1}-uniform, and for all Abélard's choices a_0 and a_2 interpreting the variables x_0 resp. x_2, we have

$$\langle \mathcal{M}, a_0, a_1, a_2, a_3 \rangle \models R(x_0, x_1, x_2, x_3),$$

where $a_1 = f_{\exists x_1}(a_0)$ and $a_3 = f_{\exists x_3/\{x_0,x_1\}}(a_0, a_1, a_2)$. Consider then the following concrete cases, (α) and (β).

(α) If $R^{\mathcal{M}} := \{(a, b, b, a), (b, a, b, b)\}$, there is no w.s. for Héloïse in G(φ,$\mathcal{M}$,$\emptyset$). For if she had a w.s., it would contain

a {0,1}-uniform strategy function f *corresponding to the expression* $\exists x_3/\{x_0, x_1\}$*, satisfying* $f(a,b,b) = f(b,a,b)$*. If this value was not* b*, Abélard could interpret* x_0 *as* b *and make sure Héloïse loses. But if it was* b*, Abélard could interpret* x_0 *as* a *and again make sure that Héloïse does not win.*

(β) *If* $R^{\mathcal{M}} := \{(a,b,b,a),(b,a,b,a)\}$*, let* $f(a) = b$ *and* $f(b) = a$*. And further, let* $g(a,b,b) = g(b,a,b) = a$*. Then the set* $\{f,g\}$ *is clearly a w.s. for Héloïse.* ■

1.2 Logical Priority and IF Modal Logic

An operator's being independent of other operators corresponds game-theoretically to an appropriate player's *imperfect information* concerning moves made earlier in the course of a play of a semantical game. A move for a quantifier $\exists x/W_i$ or a disjunction $\vee/W_i$ appearing in a formula of $\mathbf{IF}[\tau]$ is made 'in ignorance' of moves corresponding to the quantifiers identified by the indices in the set W_i. This can be modeled by means of the W_i-uniformity of a strategy function. Unlike in the case of IF first-order logic, the idea of a *move made in ignorance* becomes potentially ambiguous in the modal-logical setting, there being various types of ignorance.

Expressed game-theoretically, it is possible, for example, to know that the choice of an individual has been made from a particular (proper) subclass of the universe of discourse without knowing which individual has been chosen (*weak ignorance*). On the other hand, one might fail to know *both* the specific class from which the choice is made *and* the individual chosen (*strong ignorance*). In the latter case it is only known that the choice is made from the universe of discourse − from the domain of the model considered. In IF first-order logic the distinction between these two types of ignorance would become crucial if *bounded quantification* was taken seriously, that is, if quantifiers always went together with a subclass of the domain from which the values of the quantified variable would have to be chosen.

An essential feature that serves to distinguish modal logic from first-order logic is that modal logic is 'guarded' so that all choices mandated by modal operators in an evaluation game are made along a relation specified for that operator; more exactly, a chosen state s' must satisfy $R(s, s')$, where s is the most recently attained state and R is given by the semantics of the modal operator being considered.

Under the uniformity interpretation of IF modal logic, as briefly sketched above in the *Introduction* (Paragraph 2), resorting to weak ignorance is what comes out naturally: it is natural to think that the constitution of a considered modal-logical formula is anyway known to the players of a semantical game – and hence the relations along which the choices are made in the evaluation of that formula up to any given stage are known also. Weak ignorance is only ignorance about the particular choices made along such relations. From the point of view of modal logic, strong ignorance would practically mean losing sight of the accessibility relations altogether, or, to put it another way, acting as if all accessibility relations in fact equalled the universal relation, $dom(\mathcal{M}) \times dom(\mathcal{M})$. To illustrate, assume I had to choose a time point *earlier* than a time point chosen by you, but without knowing which particular time point you chose. We could think of my task as that of winning a play of a two-player game. I would win a play of this game if I managed to choose a time t' which indeed was earlier than the time t chosen by you, otherwise you would win. Now what would count as a winning strategy for me in this game – what would be the recipe for me to win irrespective of what you did? Well, as the game is described above, I can have no winning strategy at all, for such a strategy would have to propose a particular, constant time t' as my reply for each and every time chosen by you; but it is obvious that you would beat this strategy of mine by letting your choice be any time t satisfying $t \leq t'$, a choice you could perfectly well make. In other words, *if your choice was not bounded* – if I was 'strongly ignorant' of your choice – I could not conceivably have a winning strategy in the game.

However, things would be different if your choice was bounded. Let us assume that you would have to choose a *future* time, a time t later than the specified present time t_0 ($t > t_0$). This would change my odds of winning dramatically: indeed, I would end up having a winning strategy in this new game. My strategy would consist of choosing the present time or any earlier one: a time t' satisfying $t' \leq t_0$. For, no matter which future time you would care to pick out, my choice t' would certainly be earlier than that. In the latter game I would only be 'weakly ignorant' of your choice – knowing a proper subset of the domain from which you make your choice – which is indeed what allows me to have a winning strategy in the game.

Unary modal operators are by their semantics bounded quantifiers. It is well known that formulae of basic modal logic can be translated into **FO**:[9] if R_i is the accessibility relation in terms of which the semantics of the operator $\Diamond_i$ *resp.* $\Box_i$ is given, and if $ST_x(\varphi)$ is a first-order translation (so-called *standard translation*) of the modal formula φ, then the modal formulae $\Diamond_i(\varphi)$ and $\Box_i(\varphi)$ are respectively translated by the first-order formulae

$$\exists y(\mathrm{R}_i(x,y) \wedge ST_{x/y}(\varphi))$$

and

$$\forall y(\mathrm{R}_i(x,y) \to ST_{x/y}(\varphi)),$$

[9] For the so-called standard translation, see *Sect.* 2.1 below.

where "R_i" is a binary relation symbol interpreted as the relation R_i, and "$ST_{x/y}(\varphi)$" stands for the result of having first changed, if necessary, variables in $ST_x(\varphi)$ so that y will be free for x in the resulting formula,[10] and having then substituted y for x in that resulting formula. So the quantifiers in terms of which the semantics of modal operators is given, are actually bounded quantifiers: the set in which the quantified variable ranges is the set of states accessible from a given state according to a particular accessibility relation. If the state relative to which, say, a subformula of the form $\Diamond_i(\varphi)$ being evaluated is s, and the accessibility relation associated with the operator $\Diamond_i$ is R_i, then the range of the (existential) quantifier in terms of which the semantics of $\Diamond_i$ is given, is bounded to the set $\{s' : R_i(s, s')\}$.

It is the two-fold nature of modal operators − semantically bounded quantifiers without syntactically given variables − that will be responsible for the particular way in which we define IF modal logic under its *uniformity* interpretation. Their nature as bounded quantifiers will introduce the possibility of weak ignorance in the sense explained above. However, their lacking of syntactically manifest variables will, on the other hand, make it uninteresting to try utilizing the idea of strong ignorance in connection with them. To appreciate the latter point, consider evaluating the formula

$$\varphi := \Box_1 \Diamond_2/\{1\}\top$$

relative to a structure $\mathcal{M}$ and a state s − interpreting the independence indication "$/\{1\}$" as declaring that $\Diamond_2$ is 'strongly independent' of $\Box_1$. Hence the formula φ would be true at s iff *either* no move along R is possible from s, *or else* a state s'' can be chosen so that for any s' from the domain of $\mathcal{M}$, $R(s', s'')$

[10] A variable x is *free for* a variable y in a formula φ, if y does not occur free in the binding scope of any quantifier Qx which appears in φ.

holds. In other words, φ evaluated locally would serve to state quite a global property of $\mathcal{M}$, namely that

- s is R-maximal *or* the relation R has a maximum.[11]

In the case of IF first-order logic, strong ignorance is the natural sense of independence and weak ignorance would only make sense if quantifiers were explicitly bounded. But since modal operators are semantically just bounded quantifiers, the situation is reversed in modal logic: weak ignorance is natural and strong ignorance is its extremely robust special case, where independence becomes so strong that not only the moves, but also the relations along which these moves are made, become completely invisible to the player moving for an independent operator.

Offering a particular way in which the slash symbol "/" is to be interpreted in semantics is one of the tasks that any formulation of any IF logic faces. Its intended interpretation is that it should express *logical independence*, or that it should serve to remove certain logical priorities to which an operator otherwise would be subjected. But there is no reason why other readings of this symbol could not be investigated. In particular, taking it anyway to stand for some sort of 'independence', different interpretations of the slash sign "/" might manage to capture some interesting notions of independence *other* than that of logical independence. While in *Section* 2.3 below I define in detail the uniformity interpretation of **IFML**$[k]$, being intended to capture the sense of logical independence expressed by the slash sign "/", in *Chapter* 4 I will in fact provide two additional ways to interpret it, which I call the *'backwards-looking*

[11] We say here that y is R-maximal, if there is no x such that $R(x, y)$, and that x is a maximum of R, if for all x, $R(x, y)$. (This definition requires that an R-maximal element x does not satisfy $R(x, x)$, and that a maximum x of R indeed satisfies $R(x, x)$.)

operators' interpretation and the *algebraic* interpretation of IF modal logic. The former of these alternative semantics proves useful in *Chapter* 5, where we investigate the need for tense operators in linguistics.

For logical subordination (viz. the converse of the relation of logical priority), we may use the term "functional dependence"; synonymously we might speak of "informational dependence". And we could employ the term "informational independence" to denote the absence of logical subordination.

The term "scope" is customarily used where I have chosen to speak of logical priority. Instead of saying that an operator O bears logical priority over an operator O', we could say that the operator O' lies in the *priority scope* of O. In the special case of quantifiers, there is a conceptually distinct notion to be considered, namely the notion Hintikka has distinguished from that of priority scope by speaking of *binding scope* (see Hintikka, 1997, esp. pp. 515-8). 'Being in priority scope of' is a relation between an operator (such as a quantifier) and other *operators*; while 'being in binding scope of' is a relation between a quantifier and a set of occurrences of a *variable*. Hintikka has stressed, in agreement with his basic insight underlying the combinatorial freedom that must go together with IF logic, that these two types of scope − priority scope and binding scope − can also in fact be exemplified separately: there is no reason in principle why a quantifier could not for instance bind a variable which does not occur in the priority scope of this quantifier. Priority scopes of operators (e.g. of quantifiers) determine the relations of functional dependency between values chosen when interpreting these operators. By contrast, an occurrence of a variable, x, bound by a quantifier (Qx) serves to 'refer back' to a previously determined value of the variable. This value − being available once it has been specified − does *not* conceptually presuppose that the variable will appear within the priority scope of the quantifier (Qx) whose evaluation has introduced the value in question. Examples of formulae in which the two notions appear distinct would be:[12]

(1) $\exists x[\{A(x)] \rightarrow B(x)\}$

(2) $\Diamond[\forall x\{A(x)] \rightarrow B(x)\}$

(3) $\forall x(A(x) \vee \exists y/\forall x B(x,y))$.

In formulae (1) and (2), the curly brackets $\{,\}$ indicate binding scope, and the square brackets [,] mark priority scope. Formulae (1) and (2) are ill-formed in the usual formulations of first-order logic (IF or not), but they still are understandable and serve to make the point. Observe that in formula (3) $-$ which is well-formed in IF first-order logic $-$ the slash notation appearing in $\exists y/\forall x$ serves to remove the logical priority that the universal quantifier $\forall x$ would otherwise have over the existential quantifier $\exists y$. As a result, the occurrence of the variable x in the subformula $B(x,y)$ is in the binding scope of $\forall x$, while the priority scope of the quantifier $\forall x$ does not extend to the subformula $\exists y/\forall x B(x,y)$.

The present thesis is interested in propositional modal logic in which syntactically manifested binding of a variable cannot occur; we will therefore not discuss binding scope at more length.

Be it noted that while logical priorities are of course syntactically manifested in any formalism worth the name of logic, they indicate important *semantic* properties. It is indeed possible to speak of changing the relation of logical priority already before deciding how to present this syntactically, as is clear

[12] Formula (2) is mentioned by Hintikka and Sandu (1989); it goes back to Kaplan (1973, p. 504). Together with this formula, formula (1) appears in Hintikka (1997).

when one considers for instance the usual formalism of **FO**. In it, the syntactically given subformula-relation and the relation of logical priority go together, but thinking of changing the latter does not presuppose a pre-existing idea of how to indicate this syntactically. Hintikka (2002 [b], p. 405) in effect points out that the semantics of independent quantifiers is actually easier to master than their syntax.

1.3 The Independence-Friendly Modal Logic of Julian Bradfield

The first person to have defined and studied an IF modal logic was the British computer scientist Julian Bradfield, in his paper "Independence: Logics and Concurrency" from the year 2000. In his joint paper with Sibylle Fröschle, "Independence-Friendly Modal Logic and True Concurrency" (2002), a semantics differing from the one given in Bradfield (2000) is provided to IF modal logic (IFML), and IFML equivalence between models is investigated by comparison to known equivalences from concurrency theory.

The purpose of Bradfield's studies was to investigate how the logical notion of independence – as this appears in IF logics – relates to 'independence' in the sense prominent in computer science, namely the relation of concurrency given as a component in a model (transition system), specifying certain transitions as 'concurrent' (being processed in parallel). Bradfield's IF modal logic turns out to be much more expressive than the **IFML**[k] we are going to study. His logic is actually designed for defining second-order properties (and notably it expresses the Henkin quantifier), whereas I will prove further below that **IFML** [k] can be translated into a fragment of traditional first-order logic.

In Bradfield (2000) a logic termed *Henkin modal logic* is defined. This logic extends basic modal logic (having certain given basic actions) with *concurrent modalities*

$$\otimes_{i:=1,\ldots,n} Q_i^1(A_i^1)\ldots Q_i^m(A_i^m),$$

each $\mathrm{Q}_i^j(\mathrm{A}_i^j)$ $(j := 1,\ldots,m)$ being one of the modal operators $[A_i^j]$ or $\langle A_i^j \rangle$, where A_i^j is one of the previously specified basic actions, or the 'idle action' $\perp$. (The basic actions simply correspond to distinct accessibility relations of a traditional modal structure, and the idle action corresponds to the identity relation.) The semantics of these concurrent modalities uses 'distributed system models' $||_{i:=1,\ldots,n}^{\mathcal{S}} P_i$, which are parallel compositions of n components P_i. Here $\mathcal{S}$ is a synchronization relation that relates a basic action a with an n-tuple $(\alpha_1,\ldots,\alpha_n)$ of basic actions (some or all of which may be idle) in such a way that

$$||_{i:=1,\ldots,n}^{\mathcal{S}} P_i \xrightarrow{a} ||_{i:=1,\ldots,n}^{\mathcal{S}} P_i' \quad \textit{if and only if}$$

$$P_i \xrightarrow{\alpha_i} P_i' \ (i := 1,\ldots,n) \ \textit{and} \ \mathcal{S}(\alpha_1,\ldots,\alpha_n,a).$$

Hence the synchronization relation serves to 'collect' *local transitions* $P_i \xrightarrow{\alpha_i} P_i'$ into *global* ones, namely into transitions $||_{i:=1,\ldots,n}^{\mathcal{S}} P_i \xrightarrow{a} ||_{i:=1,\ldots,n}^{\mathcal{S}} P_i'$.

Now, for example, the semantics of the following concurrent modality (a so-called *Henkin modality*) of length $m := 2$, relative to a 2-component system model $P_1 \ ||^{\mathcal{S}} \ P_2$ is:

$$P_1 \ ||^{\mathcal{S}} \ P_2 \models \begin{array}{c}[A_1]\langle B_1\rangle \\ [A_2]\langle B_2\rangle\end{array} \ \varphi \quad \textit{if and only if}$$

$$\exists f_1 \exists f_2 \forall \alpha_1 \in A_1, P_1' \ \forall \alpha_2 \in A_2, P_2':$$

$$f_1(\alpha_1) \in B_1, P_1'' \text{ and } f_2(\alpha_2) \in B_2, P_2'' \text{ and}$$

$$(P_1||^{\mathcal{S}}P_2) \xrightarrow{\alpha_1 \otimes \alpha_2} (P_1'||^{\mathcal{S}}P_2') \xrightarrow{f_1(\alpha_1)\otimes f_2(\alpha_2)} (P_1''||^{\mathcal{S}}P_2'')\varphi,$$

where $a := \alpha_1 \otimes \alpha_2$ and $b := f_1(\alpha_1) \otimes f_2(\alpha_2)$ satisfy: $\mathcal{S}(\alpha_1, \alpha_2, a)$ and $\mathcal{S}(f_1(\alpha_1), f_2(\alpha_2), b)$. (The modal Henkin prefix is an optional notation for the 2-composition

$$\otimes_{i:=1,2} Q_i^1(A_i^1) Q_i^2(A_i^2),$$

where $Q_1^1(A_1^1) = [A_1]$, $Q_1^2(A_1^2) = \langle B_1 \rangle$, $Q_2^1(A_2^1) = [A_2]$, and $Q_2^2(A_2^2) = \langle B_2 \rangle$.)

The paper of Bradfield and Fröschle (2002) provides a semantics for IF modal logic by explicitly giving the relation of concurrency as a primitive in the model; this is in contrast to concurrency being indirectly defined via local transitions, as done in Bradfield (2000). A *transition system with concurrency* (or, TSC) is defined by specifying a set S of states, a set L of labels, a ternary transition relation $\rightarrow \subseteq S \times L \times S$, a binary concurrency relation $\mathcal{C} \subseteq \rightarrow \times \rightarrow$ between $\rightarrow$-transitions, and an initial state s_0. In addition, a relation $\prec$ between transitions having the same label is defined, which is further used for defining a relation $\sim$ which groups transitions into 'events'. (For the exact definition of a TSC, see Bradfield and Fröschle, 2002, pp. 105-6.) Now the semantics for a formula of IF modal logic is defined in terms of a model-checking game of imperfect information between *Héloïse* and *Abélard* on a TSC. A *position* in such a game is a pair consisting of a *tagged run* ρ and a subformula Ψ, denoted as $\rho \vdash \Psi$. Tagged runs are sequences

$$s_0 \underset{\alpha_0}{\overset{a_0}{\rightarrow}} \dots \underset{\alpha_{n-1}}{\overset{a_{n-1}}{\rightarrow}} s_n,$$

where the α_i are distinct tags. (These tags correspond to distinct indices identifying occurrences of modal operators in the syntax of **IFML**$[k]$.) In particular, the game rule for formulae of the form

$$\langle b \rangle_{\beta/\alpha_1,\dots,\alpha_m} \Psi,$$

to be read "there is a b-transition, independently of the choices made in the modalities tagged by the α_i, such that Ψ", is this:

At

$$s_0 \underset{\alpha_0}{\overset{a_0}{\to}} \ldots \underset{\alpha_{n-1}}{\overset{a_{n-1}}{\to}} s_n \vdash \langle b \rangle_{\beta/\alpha_{i_1},\ldots,\alpha_{i_m}} \Psi,$$

Héloïse chooses a transition $s_n \underset{\beta}{\overset{b}{\to}} t$ that is concurrent with all transitions

$$s_{i_j} \underset{\alpha_{i_j}}{\overset{a_{i_j}}{\to}} s_{i_{j+1}} (j := 1, \ldots, m) \ ,$$

and the new position is

$$s_0 \underset{\alpha_0}{\overset{a_0}{\to}} \ldots \underset{\alpha_{n-1}}{\overset{a_{n-1}}{\to}} s_n \underset{\beta}{\overset{b}{\to}} t \vdash \Psi.$$

Hence this game rule makes use of 'independence as given by the model', i.e. the relation of concurrency. By contrast, the logical notion of independence is introduced by means of a condition that will be imposed on a strategy of *Héloïse*, in order for it to count as winning. This condition is that of *uniformity.* *Héloïse*'s strategy f is now said to be *uniform* if the values of f at $\langle\rangle$-positions are uniform in the sense that in a position

$$s_0 \overset{a_0}{\to} \ldots \overset{a_{n-1}}{\to} s_n \models \langle b \rangle_{\beta/\alpha_{i_1},\ldots,\alpha_{i_m}} \Psi,$$

the transition $s_n \overset{b}{\to} t$ provided by f satisfies that if

$$s'_0 \overset{a_0}{\to} \ldots \overset{a_{n-1}}{\to} s'_n \models \langle b \rangle_{\beta/\alpha_{i_1},\ldots,\alpha_{i_m}} \Psi$$

is any other position such that $j \notin \{i_1, \ldots, i_m\} \Longrightarrow (s_j \overset{a_j}{\to} s_{j+1}) \sim (s'_j \overset{a_j}{\to} s'_{j+1})$, then f provides a transition $(s'_n \overset{b}{\to} t') \sim (s_n \overset{b}{\to} t)$.

In other words, f must map tagged runs whose constituent transitions $-$ with the possible exception of those referred to by $i_1, \ldots, i_m$ $-$ are pairwise in the same events, to the same event.

Hence what makes the model-checking game for the IF modal logic of Bradfield and Fröschle (2002) a game of imperfect information is the additional condition of uniformity that *Héloïse*'s strategy must satisfy in order to be winning. On the other hand, the relation of concurrency – 'the computer scientist's notion of independence' – is made use of when defining the game rule for slashed modal formulae, i.e. formulae to which one wishes to give a semantics involving a choice *logically* independent of certain earlier choices, this *desideratum* then being met via the condition of uniformity on *Héloïse*'s winning strategies.

Indeed, a basic observation by Bradfield in defining his IF modal logic was that on the face of it, it does not make much sense to speak of independent choices in a modal-logical setting, because "in a standard transition system semantics for modal logic, the choices available at a modality are determined by the choices made in earlier modalities" (2002, p. 104). His solution to this difficulty is what simultaneously makes IF modal logic interesting from a computer scientist's point of view, namely, taking the modalities that are declared logically independent of each other as satisfying some condition that makes these modalities in an appropriate sense 'independent' of each other – and Bradfield takes precisely *concurrency* between transitions (made along relevant modalities) to be such a condition.

IFML$[k]$ – evaluated under its uniformity interpretation – has a much more modest expressive power than the Henkin modal logic of Bradfield (2000) or the IF modal logic of Bradfield and Fröschle (2002). Indeed, it turns out that **IFML**$[k]$ is translatable into traditional first-order logic, while the latter logics most certainly are not. (This follows from the fact that the Henkin quantifier is not first-order definable.) On the other hand, it will be shown that **IFML**$[k]$ already has a greater expressive power than **ML**$[k]$. Furthermore, the existence of $\mathbf{IFML}[k]$ shows that it is not, after all, impossible to turn basic modal logic into an IF modal logic without imposing some additional conditions, such as concurrency, on the relevant modalities. On the contrary: there is nothing in the modal structures that will be used for evaluating $\mathbf{IFML}[k]$ that does not already exist in the case of basic modal logic. Only one notion of independence is operative in $\mathbf{IFML}[k]$, namely the logical one, implemented by means of imposing a uniformity constraint on *Héloïse*'s winning strategies in the semantical games used for evaluating $\mathbf{IFML}[k]$ formulae.

In fact, once the game-theoretical approach to semantics is adopted, the logical notion of independence *does* make perfect sense in a modal-logical setting: yes, the choices available at a modality are determined by the choices made for earlier modalities, but in the presence of ignorance as to which particular choices some of those earlier ones were, the choice at hand is to be made logically independently of those earlier choices, in such a way that the choice will be possible no matter what those earlier choices in fact were. Nothing more is required for independence to make sense. And such independent choices can be modeled in a game-theoretical setting, as game histories can always be made use of in the semantics. Bradfield and Fröschle (2002, p. 116) claim to have shown that "a modal version of the Hintikka-Sandu independence-friendly logic...naturally requires true concurrent models." However, the sole definition of the $\mathbf{IFML}[k]$ that we will study in the present thesis shows that if their claim is taken at the face value, it is false. Concurrency is not required, since IF modal logic can be defined without resorting to anything but logical independence interpreted via the uniformity condition on winning strategies. This is, of course, not to say that studying the interplay of concurrency and logical independency would lack interest. It just means that introducing 'independence in the model' in the form of a relation such as concurrency is not conceptually necessary for defining a variant of IF modal logic.

Chapter 2

IF Modal Logic and its Uniformity Interpretation

In the present chapter we define and study the **IFML**$[k]$ logics. For $k < \omega$, **IFML**$[k]$ is called *IF modal logic of* k *modality types*. One of the actual challenges related to the work at hand is providing the means for studying *arbitrary relations of logical priority* among modal operators. For this purpose we will make use of the game-theoretical notion of *informational independence*. First, the language of IF modal logic (or **IFML**$[k]$) is specified as an extension of the language of basic modal logic **ML**$[k]$. Then a semantics for **IFML**$[k]$ is presented in detail in *Section* 2.3. The semantics given will be referred to as the *uniformity interpretation* of **IFML**$[k]$, as distinguished from other interpretations of the language of **IFML**$[k]$ that will be introduced and studied in *Chapter* 4.

2.1 Basic Modal Logic

We begin by giving a detailed description of the syntax and semantics of *basic modal logic* $\mathbf{ML}[k]$ of k modality types, already outlined in the *Introduction* above. Throughout the present thesis we deal exclusively with *propositional* modal logic; there is no reason for explicitly qualifying the considered modal logics as propositional.

Let a countable class **prop** of propositional atoms be given, and let $k < \omega$. The syntax for basic modal logic $\mathbf{ML}[k]$ of k modality types is given by the following rule:

$$\varphi := p \mid \neg p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \Diamond_i(\varphi) \mid \Box_i(\varphi),$$

where $p \in \mathbf{prop}$, and $i < k$. By the syntax, the negation sign $\neg$ may appear in front of propositional atoms only.[1]

For all $i < k$, the operators $\Diamond_i$, $\Box_i$ are called *modal operators*. The operators $\Diamond_i$ and $\Box_i$ are said to be the *duals* of each other.[2] The operators having a common index $i < k$ are said to be of the same *modality type*, and for any $i \neq j$, operators from $\{\Diamond_i, \Box_i\}$ and operators from $\{\Diamond_j, \Box_j\}$ are of distinct modality types. Hence the notion of modality type is given *syntactically*. Modality type is to be contrasted with *modality* simpliciter. All distinct members of $\{\Diamond_i, \Box_i\}_{i<k}$ are operators for mutually distinct modalities. Different occurrences (tokens) of the operators are distinguished by making use of double indexing. For instance, we write $\Diamond_{i,n}$ to mark a particular occurrence, identified by the index $n < \omega$, of the modal operator $\Diamond_i$ − an operator which is by definition of the modality type i.

We will mainly be interested in the *number* of the modality types − *not* in the *Eigenart* of the modalities (i.e. knowledge,

[1] See Fact 2.1.1 below for closing the class $\mathbf{ML}[k]$ under negation.

[2] In general, we will denote the dual of an operator O_i by $(O_i)^d$. Hence $(\Diamond)^d = \Box$; $(\Box)^d = \Diamond$. Further, $((O)^d)^d = O$ always.

temporality, logical possibility, provability, and so on). So we simply use the notation $\mathbf{ML}[k]$ − where k is the number of the modality types that the logic employs − to refer to these logics.

An exception to the exclusive interest in the number of modality types involved is the case of *tense logic*. We define *basic tense logic* of k *temporal modality types*, or $\mathbf{TL}[k]$, to be syntactically $\mathbf{ML}[2k]$. By convention, the notation for the modal operators with an index $i < k$ is chosen to be kept as $\Diamond_i$, $\Box_i$; the rest − those with an index $i + k$ (with $i < k$) − being then symbolized as $\Diamond_i^{-1}$ and $\Box_i^{-1}$ (instead of being written as $\Diamond_{i+k}$ and $\Box_{i+k}$).

The semantics for basic modal logic $\mathbf{ML}[k]$ of k modality types is given in terms of *k-ary modal structures*, i.e. structures

$$\mathcal{M} = (D, R_0, \ldots, R_{k-1}, \mathfrak{h}) \ ,$$

where:

- D is a non-empty set;
- for each $i < k$, R_i is a binary relation on D;
- $\mathfrak{h}$ is a function from **prop** to the power set of D.

The set D is said to be the *domain* of $\mathcal{M}$; in symbols $D = dom(\mathcal{M})$. The relations R_i $(i < k)$ are the *accessibility relations* of $\mathcal{M}$. And the function $\mathfrak{h} : \mathbf{prop} \to Pow(D)$ is the *assignment function* of $\mathcal{M}$. The elements of D are the indices relative to which the truth-conditions of $\mathbf{ML}[k]$ formulae will be defined.[3]

If $\mathcal{M} = (D, R_0, \ldots, R_{k-1}, \mathfrak{h})$ is a k-ary modal structure, we say that the relational structure $(D, R_0, \ldots, R_{k-1})$ is its frame.

[3] Here we always have only two modalities ($\Diamond$, $\Box$) in each modality type. Observe, however, that the notion of modality type would naturally admit of generalization, taking any number of operators whose semantics is defined in terms of a fixed accessibility relation to be of the same modality type.

If $d \in dom(\mathcal{M})$ and $\mathcal{M}$ is a k-ary modal structure, the structure $(\mathcal{M}, d)$ is termed a *pointed* k-ary modal structure. A pointed modal structure is hence simply a modal structure with a designated element. The evaluation of modal-logical formulae is in effect formulated in terms of such pointed modal structures.

We write $\mathcal{M} \models^+ \varphi[d]$ to make the judgment that the formula $\varphi \in \mathbf{ML}[k]$ is true in $\mathcal{M}$ at d, and we write $\mathcal{M} \models^- \varphi[d]$ to say that $\varphi \in \mathbf{ML}[k]$ is *not* true in $\mathcal{M}$ at d. The semantic clauses for formulae of $\mathbf{ML}[k]$ are:

- $\mathcal{M} \models^+ p[d] \iff d \in \mathfrak{h}(p)$
- $\mathcal{M} \models^+ \neg p[d] \iff d \notin \mathfrak{h}(p)$
- $\mathcal{M} \models^+ \varphi \vee \psi[d] \iff$ for some $\theta \in \{\varphi, \psi\}$: $\mathcal{M} \models^+ \theta[d]$
- $\mathcal{M} \models^+ \varphi \wedge \psi[d] \iff$ for every $\theta \in \{\varphi, \psi\}$: $\mathcal{M} \models^+ \theta[d]$

For each modality type $i < k$:

- $\mathcal{M} \models^+ \Diamond_i(\varphi)[d] \iff$ for some $c \in D$ with $R_i(d, c)$: $\mathcal{M} \models^+ \varphi[c]$
- $\mathcal{M} \models^+ \Box_i(\varphi)[d] \iff$ for every $c \in D$ with $R_i(d, c)$: $\mathcal{M} \models^+ \varphi[c]$.

$\mathbf{ML}[0]$ evaluated over degenerate modal structures $(\{d_0\}, \mathfrak{h})$ is simply Propositional Logic.

It is easy to check that the above recursively defined semantics to $\mathbf{ML}[k]$ and the game-theoretical semantics sketched for it above in the *Introduction* (Paragraph 2), coincide:[4]

- $\mathcal{M} \models^+ \varphi[d] \iff \mathcal{M} \models^+_{\mathrm{GTS}} \varphi[d]$

[4] In general, the directions from left (the recursive definition) to right (the GTS definition) require *Axiom of Choice*.

- $\mathcal{M} \models^{-} \varphi[d] \Longleftrightarrow \mathcal{M} \models^{-}_{\mathrm{GTS}} \varphi[d]$ ■

From the viewpoint of expressive power it is not a restriction that the negation sign ($\neg$) can only appear as prefixed to a propositional atom in $\mathbf{ML}[k]$ formulae:

Fact 2.1.1 *Let $\mathbf{ML}[k, \neg]$ be the class of formulae obtained from $\mathbf{ML}[k]$ by closing it under negation. The semantics to this logic is obtained from that of $\mathbf{ML}[k]$ by having available the recursive clause*

$$\mathcal{M} \models^{+} \neg\varphi[d] \Longleftrightarrow \mathcal{M} \not\models^{+} \varphi[d].$$

We say that a formula φ of $\mathbf{ML}[k, \neg]$ is in negation normal form, if φ is in particular (by syntactical criteria) a formula of $\mathbf{ML}[k]$. Hence such a φ contains appearances of the negation sign ($\neg$), at most in front of propositional atoms. In fact, it is easy to define a syntactic transformation

$$nnf : \mathbf{ML}[k, \neg] \to \mathbf{ML}[k]$$

mapping each formula φ of $\mathbf{ML}[k, \neg]$ to a formula $nnf(\varphi)$ in negation normal form, in such a way that the transformation preserves truth: for every formula $\varphi \in \mathbf{ML}[k, \neg]$ and every pointed modal structure ($\mathcal{M}$,d):

$$\mathcal{M} \models^{+} \varphi[d] \Longleftrightarrow \mathcal{M} \models^{+} nnf(\varphi)[d]. \blacksquare$$

We may observe that the semantics for $\mathbf{ML}[k]$ could well be defined more generally than above, by allowing evaluation relative to *tuples* of elements from a specified Cartesian product $D_0 \times \ldots \times D_{N_k - 1}$ with $1 \leq N_k \leq k$. Then for each (binary) accessibility relation R_i a component $D_{\pi(i)}$ of the Cartesian product would have to be associated by some surjection $\pi : \{0, \ldots, k-1\} \to \{0, \ldots, N_k - 1\}$. A more flexible semantics such as this would be useful for many applications. For instance, if we wished to consider two modality types such as physical necessity and temporality, we would probably find it desirable to have one co-ordinate for possible worlds and another for time in our semantics. On the other hand, in multiagent epistemic logic,[5] for example, k would equal the number of the agents while the value of the parameter N_k would be 1, since the formulae would be evaluated relative to single possible worlds. In the present thesis we do not consider many-dimensional modal logics, but in effect stay with the dimension $N_k = 1$.

The semantics of *basic tense logic* $\mathbf{TL}[k] = \mathbf{ML}\ [2k]$ employs $2k$-ary modal structures

$$\mathcal{M} = (D, R_0, \ldots, R_{2k-1}, \mathfrak{h})$$

with the characteristic features that: **(i)** for each $i < k$, the relation R_{i+k} is the *converse* of the relation R_i; **(ii)** the accessibility relations are *irreflexive partial orders*, in other words irreflexive and transitive binary relations. The motivation for the restriction (i) is that the inverse of an operator O (which by definition always exists in a tense logic) must make use of the converse of precisely the accessibility relation associated with O. The feature (ii) is required, because at least irreflexivity and transitivity are thought of as essential to the temporal '*earlier than*' relation.

$2k$-ary modal structures $\mathcal{M}$ satisfying the above conditions (i) and (ii) are termed k-ary *temporal* structures. By contrast, we will call $2k$-ary modal structures meeting requirement (i) but not requirement (ii) *quasi-temporal* structures.

For any k-ary modal structure $\mathcal{M} = (D, R_0, \ldots, R_{k-1}, \mathfrak{h})$, there is a corresponding first-order structure, obtained as follows. If $\langle p_i : i < \kappa \rangle$ is an enumeration of the class **prop** of

[5] Such as the one studied in Fagin, Halpern, Moses and Vardi (1996).

propositional atoms ($\kappa \leq \omega$), let $\tau := \{\mathrm{R}_0, \ldots, \mathrm{R}_{k-1}\} \cup \{\mathrm{P}_i\}_{i<\kappa}$ be a vocabulary, where the R_i are binary and the P_i unary. Then for each $i < k$, interpret the relation symbol R_i by the accessibility relation R_i. For each unary relation symbol P_i, its interpretation is taken to be $\mathfrak{h}(p_i)$. Then define:

$$\mathcal{M}^{\mathbf{FO}} := (D, R_0, \ldots, R_{k-1}, \{\mathfrak{h}(p_i)\}_{i<\kappa}).$$

We associate with each $\mathbf{ML}[k]$ formula φ a formula $ST_x(\varphi)$ of $\mathbf{FO}[\tau]$ with precisely one free variable, x. If x and y are individual variables, and $ST_x(\varphi)$ is a first-order formula, let us agree on writing $ST_{x/y}(\varphi)$ for the result of having first changed, if necessary, variables in $ST_x(\varphi)$ so that y will be free for x in the resulting formula, and having then substituted y for x in that resulting formula. Then define:[6]

- If $p_j \in \mathbf{prop}$, then $ST_x(p_j) = \mathrm{P}_j(x)$; and

  $ST_x(\neg p_j) = \neg \mathrm{P}_j(x)$.
- $ST_x(\varphi \vee \psi) = ST_x(\varphi) \vee ST_x(\psi)$.
- $ST_x(\varphi \wedge \psi) = ST_x(\varphi) \wedge ST_x(\psi)$.
- $ST_x(\Diamond_i(\varphi)) = \exists y(R_i(x, y) \wedge ST_{x/y}(\varphi))$.
- $ST_x(\Box_i(\varphi)) = \forall y(R_i(x, y) \to ST_{x/y}(\varphi))$.

We say that the first-order formula $ST_x(\varphi)$ is the *standard translation* of the $\mathbf{ML}[k]$ formula φ. That the operation ST_x really induces a *translation* of $\mathbf{ML}[k]$ to $\mathbf{FO}[\tau]$ is expressed in the following result:

[6] Implication ($\to$) is used via its definition in terms of negation ($\neg$) and disjunction:

$$(\varphi \to \psi) := \neg\varphi \vee \psi.$$

For the definition of standard translation we may in fact assume that the antecedent φ of an implication is always an *atomic* first-order formula.

Proposition 2.1.2 *For all $\varphi \in \mathbf{ML}[k]$, all $\mathcal{M} = (D, R_0, \ldots, R_{k-1}, \mathfrak{h})$ and all $d \in dom(\mathcal{M})$, we have:*

$$\mathcal{M} \models \varphi[d] \Longleftrightarrow \textit{for all } \gamma \textit{ with } \gamma(x) = d : \langle \mathcal{M}^{\mathbf{FO}}, \gamma \rangle \models ST_x(\varphi).$$

Proof. Easy induction on the complexity of the $\mathbf{ML}[k]$ formula φ. ■

Remark 2.1.3 *The above clauses for $ST_x(\Diamond_i(\varphi))$ and $ST_x(\Box_i(\varphi))$ could be replaced by the following clauses without changing the truth of Proposition* 2.1.2*:*

- $ST_x(\Diamond_i(\varphi)) = \exists y(R_i(x,y) \wedge \exists x(x = y \wedge ST_x(\varphi)))$.
- $ST_x(\Box_i(\varphi)) = \forall y(R_i(x,y) \rightarrow \exists x(x = y \wedge ST_x(\varphi)))$.

Making use of the variant of the definition for standard translation hence obtained, changing variables (required by the definition of the operation $ST_{x/y}$) would be avoided. Furthermore, this alternative definition for standard translation establishes that the formulae in the class

$$ST_x(\mathbf{ML}[k]) := \{ST_x(\varphi) : \varphi \in \mathbf{ML}[k]\}$$

can all be written with two variables: the total number of *free and bound* variables that is sufficient for writing down, up to logical equivalence, the formulae of the class $ST_x(\mathbf{ML}[k])$ equals two.

It is easily seen that the requisite number is precisely two; one variable (obviously) does not suffice. To see how this is strictly proven, define first-order structures $\mathcal{M}$ and $\mathcal{N}$ as follows. The domain of $\mathcal{M}$ is $\{a, b, c\}$, and that of $\mathcal{N}$ is $\{a', b', c'\}$. The unary predicate P is interpreted as $\{c\}$ in $\mathcal{M}$, and as $\{a'\}$ in $\mathcal{N}$. The binary predicate R is interpreted as $\{(b, c)\}$ in $\mathcal{M}$, and as $\{(b', c')\}$ in $\mathcal{N}$. (See Figure 4 below.)

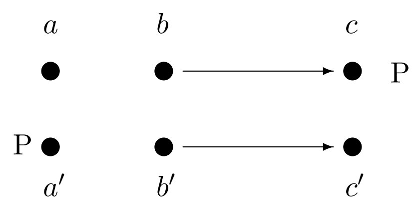

FIGURE 4

Now, it is possible to show that the first-order {P,R}-structures $\langle \mathcal{M}, b\rangle$ and $\langle \mathcal{N}, b'\rangle$ satisfy the same first-order formulae written with precisely *one* variable (be its occurrences free or bound). This is established by showing that for all $m < \omega$, *Duplicator* has a winning strategy in the *pebble game* $\mathrm{G}^1_m(\mathcal{M}, b, \mathcal{N}, b')$, played with precisely one pebble for m rounds.[7] In fact, such a winning strategy is constituted by the following rules for *Duplicator*:

- reply to b by b' and *vice versa*;
- reply to c by a' and *vice versa*;
- reply to a by c' and *vice versa.*

On the other hand, however, the first-order formula

$$ST_x(\Box(p)) = \forall y(\mathrm{R}(x, y) \to \mathrm{P}(y)),$$

which is written with *two* variables, does distinguish the structures $\langle \mathcal{M}, b\rangle$ and $\langle \mathcal{N}, b'\rangle$, being true in the former but false in the latter. Because this formula is in the class $ST_x(\mathbf{ML}[k])$, we may conclude that *precisely* two variables are needed for writing down all standard translations of $\mathbf{ML}[k]$ formulae.

[7] For a definition of a pebble game, and the requisite characterization of $\mathbf{FO}^s$-equivalence up to quantifier rank at most m of structures $(\mathcal{M}, \overline{a})$ and $(\mathcal{N}, \overline{b})$ in terms of a pebble game $\mathrm{G}^s_m(\mathcal{M}, \overline{a}, \mathcal{N}, \overline{b})$, see e.g. Ebbinghaus & Flum (1999, pp. 49-50).

2.2 The Language of IF Modal Logic

A modification of the syntax of $\mathbf{ML}[k]$ will now be given, yielding a syntax for a logic that presents a wider range of combinatorially possible patterns of logical priority between modal operators than $\mathbf{ML}[k]$. The resulting formalism will be referred to as $\mathbf{IFML}[k]$, or *independence-friendly (IF) modal logic of k modality types*. I have chosen to use the above terminology despite the fact that only a much more general logic would fully merit the name − this $\mathbf{IFML}[k]$ being only a fragment of such a general IF modal logic. The restriction to the particular fragment $\mathbf{IFML}[k]$ given here is, however, motivated by our interest in expressive power: already this fragment has a greater expressive power than $\mathbf{ML}[k]$ over arbitrary modal structures, as we will see in *Section* 3.4. The $\mathbf{EIFML}[k]$ logic, briefly discussed in *Section* 3.3, is one possible formulation of a more general IF modal logic, and is still 'ideologically' on a par with $\mathbf{IFML}[k]$ since it implements independence simply by means of the uniformity condition imposed on winning strategies.

The convention of using subscripts $j < \omega$ for indexing the modal operators adopted above in connection with $\mathbf{ML}[k]$ (these operators being already indexed once by their modality type $i < k$) will be extended to the syntax of $\mathbf{IFML}[k]$. IF modal logic of k modality types, or $\mathbf{IFML}[k]$, is simply defined by:

$$\mathbf{IFML}[k] :=$$

$$\mathbf{ML}[k] \cup \{O_1 \ldots O_{n-1}(O_n/W)\varphi : \varphi \in \mathbf{ML}[k], n \geq 1\},$$

where:

- for all $j \in \{1, \ldots, n\}$, O_j is one of the modal operators $\Diamond_{i,j}$, $\Box_{i,j}$ for some $i < k$.
- W is a (possibly empty) subset of the interval $[1, n-1]$ of natural numbers.

Hence in addition to all $\mathbf{ML}[k]$ formulae, strings obtained by prefixing a finite block $O_1 \ldots O_{n-1}(O_n/W)$ to a formula of $\mathbf{ML}[k]$ are also formulae of $\mathbf{IFML}[k]$. Notice that for $n := 1$, the sequence $O_1 \ldots O_{n-1}$ is empty, and because the interval $[1, n-1] = [1, 0]$ is then empty, $(O_1/W)\varphi$ can only be a formula if $W = \varnothing$. For any $n > 1$, $[1, n-1]$ is non-empty and so may be W.

Remark 2.2.1 *The class of well-formed formulae of IF modal logic of k modality types could be defined more generally as follows. We could first generate a class $\mathcal{L}^*$ of incomplete IF modal-logical formulae from propositional atoms and their negations by the rules of closure under conjunction ($\wedge$), disjunction ($\vee$), and application of any of the modal operators $\Diamond_{i,j}/W$ and $\Box_{i,j}/W$ ($i < k$ arbitrary; j otherwise arbitrary but the same j would be allowed to appear as an index only once in any one formula), W being empty or any finite set of natural numbers. The set $Free(\varphi)$ of free indices of an $\mathcal{L}^*$ formula φ would then be defined recursively by stipulating that if φ is a (negated) propositional atom, $Free(\varphi)$ is empty, $Free(\varphi \wedge \psi) = Free(\varphi) \cup Free(\psi) = Free(\varphi \vee \psi)$, and finally, $Free(O_{i,j}/W\varphi) = (Free(\varphi)\backslash\{j\}) \cup W$. Then by stipulation the class $\mathcal{L}$ of (complete) formulae of the "general IF modal logic of k modality types" would be the class of those $\mathcal{L}^*$ formulae φ for which $Free(\varphi)$ is empty.*

If the set W of indices referred to by operators $O_{i,j}/W$ in a formula $\varphi \in \mathcal{L}$ is taken as identifying precisely those occurrences of operators in φ of which $O_{i,j}/W$ is independent – $O_{i,j}/W$ being then dependent on all other operators in the formula (no matter where these other operators are syntactically situated in the formula) – then the class $\mathcal{L}$ would be able to present all patterns of logical priority among occurrences of modal operators $\Diamond_i$, $\Box_i$ that can be obtained by relaxing the pri-

ority scope relations of $\mathbf{ML}[k]$*. If we wished to totally relax the relations of logical priority involved, we should allow even conjunctions and disjunctions to be independent of other operators, and allow modal operators to be independent of these Boolean connectives. The present thesis mostly adheres to the definition of* $\mathbf{IFML}[k]$ *given above. An exception is the Extended IF Modal Logic* (*or* $\mathbf{EIFML}[k]$) *that we briefly examine in Section* 3.3. ■

Observe that not only is $\mathbf{IFML}[k]$ incapable of presenting many possible patterns of *independence* among modal operators, when it comes to possible *dependencies* that can occur among modal operators in our syntax for $\mathbf{IFML}[k]$, it is observed that *mutual* dependencies are *not* among them. In connection with IF logic (modal or not) it is to be borne in mind that, not only arbitrary independencies, but also arbitrary dependencies, must be expressible in the most general formulation of the IF logic in question. (Cf. here esp. Hintikka 2002 [a], Hintikka 2002 [b].)

IF *tense* logic of k temporal modality types, or $\mathbf{IFTL}[k]$, is defined to be $\mathbf{IFML}[2k]$ with the same stipulations about notation as were introduced above in connection with basic tense logic $\mathbf{TL}[k] = \mathbf{ML}[2k]$.

We define the class of *proper substrings* of an $\mathbf{IFML}[k]$ formula by defining recursively the set $Sub(\varphi)$ for strings φ *including* all $\mathbf{IFML}[k]$ formulae, but *not all* of which are formulae of this logic.[8] The clauses for (negated) propositional atoms and Boolean connectives are standard: for an atom $p \in \mathbf{prop}$, $Sub(p) = Sub(\neg p)$ is empty, and $Sub(\varphi \wedge \psi) = Sub(\varphi \vee \psi) =$

[8] Here, we cannot simply define the class of *subformulae* of $\mathbf{IFML}[k]$, since substrings $O_i \dots O_{n-1}(O_n/W)\varphi$ – which we wish to always have in the class $Sub(\varphi)$ – are by the syntax *not* formulae of $\mathbf{IFML}$ $[k]$, if W is not contained in $\{i, \dots, n-1\}$.

$\{\varphi, \psi\} \cup Sub(\varphi) \cup Sub(\psi)$. If $W \subseteq \{1, \ldots, n-1\}$ and $1 \leq i \leq n$, we define:

$$Sub(O_i \ldots O_{n-1}(O_n/W)\varphi) =$$

$$\{O_{i+1} \ldots O_{n-1}(O_n/W)\varphi\} \cup Sub(O_{i+1} \ldots O_{n-1}(O_n/W)\varphi).$$

Finally, by stipulation the string `FAIL` is in $Sub(\varphi)$ for all $\varphi \in$ **IFML**$[k]$. The strings in $Sub(\varphi) \cap$ **IFML**$[k]$ are *subformulae* of φ.

2.3 Uniformity Interpretation of IF Modal Logic

We shall define a semantics for **IFML**$[k]$, to be referred to as its *uniformity interpretation*. As the name already indicates, the basic idea is to interpret independence − indicated syntactically by means of the slash-notation − semantically in terms of the constraint of *uniformity* imposed on winning strategies of the players in the semantical games associated with formulae of **IFML**$[k]$. Hence the slash sign "/" functions here as a *(winning) strategy constraining device.*

We associate a semantical game

$$G_A(\varphi, \mathcal{M}, d) = \langle \{\forall, \exists\}, H, Z, P, \{u_\forall, u_\exists\}, \{I_\forall, I_\exists\} \rangle$$

in extensive normal form with each triple $(\varphi, \mathcal{M}, d)$ consisting of a formula φ of **IFML**$[k]$, a k-ary modal structure $\mathcal{M} = (D, R_0, \ldots, R_{k-1}, \mathfrak{h})$, and an element $d \in D$. *Positions* in this game are pairs (ψ, a) from the set

$$A = (Sub(\varphi) \cup \{\varphi\}) \times (dom(\mathcal{M}) \cup \{\star\}).$$

The object $\star$ goes together with the substring `FAIL`; the pair (`FAIL`,$\star$) is a position that can be chosen in situations where there was otherwise no choice complying with the game rules available. The game $\mathrm{G}_A(\varphi, \mathcal{M}, d)$ is a game between two players, *Héloïse* (or $\exists$) and *Abélard* (or $\forall$). The players are said to be *opponents* of each other. The set H of *plays* (or, *histories*) of $\mathrm{G}_A(\varphi, \mathcal{M}, d)$ is a set of *finite sequences of positions* (ψ, a), defined recursively (on the subformula structure of φ) as follows:

- $(\varphi, d) \in H$.
- If the last position in $h \in H$ is $(\theta \vee \psi, a)$, then $h^\frown(\theta, a) \in H$ and $h^\frown(\psi, a) \in H$.[9] It is *Héloïse* who makes a choice from $\{(\theta, a), (\psi, a)\}$ to extend h. We write $P(h) = \exists$ to indicate that the move corresponding to the history h is made by $\exists$.
- The case where the last position in $h \in H$ is $(\theta \wedge \psi, a)$ is defined analogously, the only difference being that here it is *Abélard* who makes the choice: $P(h) = \forall$.

For $i < k$:

- If the last position in $h \in H$ is from

$$\{((\Diamond_{i,n}/W)(\psi), a), (\Diamond_{i,n}(\psi), a)\},$$

then for all $a' \in dom(\mathcal{M})$ with $R_i(a, a')$, $h^\frown(\psi, a') \in H$; if no such $a' \in dom(\mathcal{M})$ exists, then $h^\frown(\texttt{FAIL}, \star) \in H$. Further, $P(h) = \exists$.
- The definitions for $\Box_{i,n}/W$ and $\Box_{i,n}$ are like those for $\Diamond_{i,n}/W$ and $\Diamond_{i,n}$, except that here it is $\forall$'s turn to move.

[9] If $h = (a_1, \ldots, a_m)$ and $a \in A$, we write "$h^\frown a$" (read: h *extended* by a) to denote the sequence $(a_1, \ldots, a_m, a)$.

All sequences in the set H are finite. The set Z of *terminal histories* is defined simply as the set of maximally long histories, i.e. sequences from H which cannot be extended by any position so as to yield a sequence in H. By definition, then, the last position in a terminal history has as its substring-component a propositional atom, the negation of a propositional atom, or the label FAIL. The function $P : H \backslash Z \to \{\forall, \exists\}$, constructed above simultaneously with the definition of H, will be called the *player function*.

The *utility functions* $u_\forall$ and $u_\exists$ for the two players of the game are maps from the class Z of terminal histories to the values 1 (a win) and -1 (a loss) as follows. For any $h \in Z$, $u_\exists(h) = 1$ and $u_\forall(h) = -1$, if the last position of h is of the form (p, a) and $a \in \mathfrak{h}(p)$, or is of the form $(\neg p, a)$ and $a \notin \mathfrak{h}(p)$, or is the position $(\mathtt{FAIL}, \star)$ chosen by $\forall$. Otherwise $u_\forall(h) = 1$ and $u_\exists(h) = -1$.

The *information partitions* $I_\forall$ and $I_\exists$ for the two players still need to be defined. For an arbitrary history $h = ((\varphi_0, a_0), \ldots, (\varphi_m, a_m))$, we call the sequence $pr_1(h) = (\varphi_0, \ldots, \varphi_m)$ its *left projection*, and the sequence $pr_2(h) = (a_0, \ldots, a_m)$ its *right projection*. If $(s_0, \ldots, s_n)$ is any sequence and $i \leq n$, we write $(s_0, \ldots, s_n)[i]$ for its member s_i. Now the sets $P^{-1}(\{\exists\})$ and $P^{-1}(\{\forall\})$ are partitioned into equivalence classes under the following equivalence relations $\sim_\exists$ and $\sim_\forall$, respectively.

- $h_1 \sim_\exists h_2 \iff$

  (for some $a_1, a_2 \in dom(\mathcal{M})$, the last position in h_1 is $((\Diamond_{i,n}/W)(\theta), a_1)$ and the last position in h_2 is $((\Diamond_{i,n}/W)(\theta), a_2)$; and $pr_1(h_1) = pr_1(h_2)$; and for all $j \notin W$, $pr_2(h_1)[j] = pr_2(h_2)[j]$) **or**
  ($h_1 = h_2$ and its last position is of the form (ψ, a) for some $\psi \in \{\Diamond_{i,n}(\theta), (\theta \vee \chi)\}$ and $a \in dom(\mathcal{M})$).

The condition under which $h_1 \sim_\forall h_2$ holds is similar to the condition for $\sim_\exists$ and is obtained from it by replacing the occurrences of the symbols "$\Diamond$" and "$\vee$" by the symbols "$\Box$" and "$\wedge$", respectively. The promised partitions of $P^{-1}(\{\exists\})$ and $P^{-1}(\{\forall\})$ then are:

$$I_\exists = \{[h]_{\sim_\exists} : h \in H \backslash Z\} \text{ and } I_\forall = \{[h]_{\sim_\forall} : h \in H \backslash Z\}.$$

The sets $I_\exists$ and $I_\forall$ themselves are called *information partitions*, and their members (i.e. the cells of these partitions) are called *information sets*.

The definition of the semantical game

$$G_A(\varphi, \mathcal{M}, d) = \langle \{\forall, \exists\}, H, Z, P, \{u_\forall, u_\exists\}, \{I_\forall, I_\exists\} \rangle$$

in its extensive form has now been completed. By definition it is a *zero-sum game*: for all $h \in Z$, $u_\exists(h) + u_\forall(h) = 0$. In particular, it is a *win-loss game* in the sense that it is a zero-sum game satisfying $u_j(h) \in \{-1, 1\}$ for all $j \in \{\exists, \forall\}$, $h \in Z$. Moreover, $\mathrm{G}_A(\varphi, \mathcal{M}, d)$ is a game of *imperfect* information. (For a discussion on the properties of the information partitions of these semantical games, as compared with information partitions of games of imperfect information typically encountered in game theory, see *Subsect.* 2.3.1 below.)

With information partitions the idea intuitively is that histories in the same cell of the information partition of the player $j \in \{\exists, \forall\}$ are indistinguishable for this player. Unless there is some way of ruling out a subset of such an information set as histories that definitely cannot have been reached in the course of a play of the game (cf. *Subsect* . 2.3.1), any move j makes after one history belonging to this set, j must be able to make after *any* history from this set, in order for j's strategy for making these moves to be winning: j's winning strategy has to agree on all these histories. To get a better grasp of these partitions, let us take an example.

Example 2.3.1 *Let* $\mathcal{M} = (\{0,1,2,3,4\}, <, \mathfrak{h})$ *be a modal structure, where* $<$ *is the ordering of the natural numbers* $0,1,2,3,4$ *by magnitude, the assignment* $\mathfrak{h}$ *being arbitrary. Consider the formulae:*

(**a**) $\varphi := \Box_1 \Diamond_2 \Diamond_3 / \{1\} p.$

(**b**) $\psi := \Box_1 \Diamond_2 \Diamond_3 / \{2\} p.$

Up to indicating the relevant information partitions, the extensive form of both semantical games, $G_A(\varphi, \mathcal{M}, 0)$ *and* $G_A(\psi, \mathcal{M}, 0)$*, is depicted as in Figure 5.*

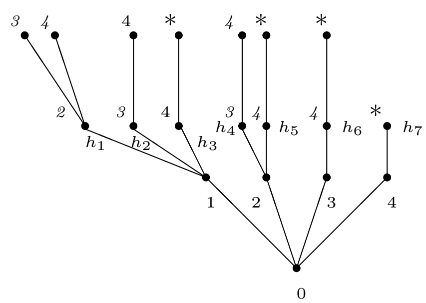

FIGURE 5

In the figure, the histories of length 2 *are listed as* $h_1, \ldots, h_7$*. Now in the case of formula (a), the information partition for Héloïse contains, in addition to four singletons each consisting of a history of length* 1*, the following cells whose members are histories of length* 2*:*

$$\{h_1\}, \{h_2, h_4\}, \{h_3, h_5, h_6\} \text{ and } \{h_7\}.$$

In the same cell are those histories of length 2 *that have their last member in common. By contrast, in the case of formula (b), the cells of Héloïse's information partition consisting of histories of length* 2 *are these:*

$$\{h_1, h_2, h_3\}, \{h_4, h_5\}, \{h_6\} \text{ and } \{h_7\}.$$

Here the histories in the same cell share their second member (i.e. the first member chosen after the initial position $(\psi, 0)$*).* ■

To make use of the games $\mathrm{G}_A(\varphi, \mathcal{M}, d)$ in defining a semantics for **IFML**$[k]$, we need the notion of strategy of a player. A *strategy* of the player $j \in \{\exists, \forall\}$ is simply any function

$$f_j : P^{-1}(\{j\}) \to A.$$

Observe that strategies are not components in the extensive form of a semantical game. If $h \in H$ is a history on which a strategy f_j is defined, we say that the move $f_j(h)$ given by the strategy is *legal*, if $h^\frown f_j(h) \in H$, i.e. when the extension to h determined by f_j remains inside the set H of histories of the relevant game. Otherwise the move is said to be *illegal*.

The following terminology is employed. If s, s_1 and s_2 are sequences such that $s = s_1{}^\frown s_2$, then s_1 is said to be an *initial segment* of s. If s_2 is non-empty, s_1 is a *proper* initial segment of s. When S is a set of sequences, we write $Cl(S)$ for the closure of S under forming initial segments, i.e. for the set of initial segments of members of S.

A strategy f_j of the player j is a *winning strategy* (*w.s.*), if there exists a non-empty subset $W \subseteq Z$ of terminal histories satisfying the following four conditions:

(a) If $h \in Cl(W)$ and $P(h)$ is j, then $h^\frown f_j(h) \in Cl(W)$.

(b) If $h \in Cl(W)$ and $P(h)$ is the opponent of j, then

for every $u \in A$ such that $h^\frown u \in H$, we have

$h^\frown u \in Cl(W)$.

(c) For every $h, h' \in Cl(W)$: if $h, h' \in \mathrm{I} \in I_j$, then

$f_j(h) = f_j(h')$.

(d) Every $h \in W$ is a win for j.

Condition (a) simply says that $Cl(W)$ is closed under applications of the strategy f_j, and (c) expresses the 'uniformity condition' for this strategy. Condition (b) in meant to ensure that j wins against any move of his opponent. Together these conditions are meant to ensure that only the plays that are "reached" given that j uses the strategy f_j are relevant for its being winning: the only plays in which the strategy f_j ever needs to be used, are those from $Cl(W)$.

A set $W \subseteq Z$ which thus establishes that a strategy is winning, is called a *plan of action*. We say that the set W *establishes* that f_j is a winning strategy, or that f_j is a winning strategy *based on* W. We observe that W guarantees the existence of a minimal (though not necessarily unique) partial function from $P^{-1}(\{j\})$ to A which specifies a move for all combinatorially possible moves of the opponent of j and always yields a win for j.

If there exists a set $W \subseteq Z$ satisfying conditions (a), (b) and (c) but not necessarily (d), we say that a corresponding *strategy* f_j is *based on the plan of action* W. But unless condition (d) is satisfied, such a strategy is *not* winning.

Fact 2.3.2 *No more than one of the players has a w.s. in a game* $\Gamma = G_A(\varphi, \mathcal{M}, d)$.

Proof. Assume for contradiction that strategies $f_\exists$ and $f_\forall$ are both winning in Γ. Fix W *resp.* W' as corresponding plans of action. We show that there is a history h such that $h \in W$ and $h \in W'$, which yields a contradiction by condition (d).

We construct h in steps. For the initial position $\alpha_1 = (\varphi, d)$, we trivially have that $(\alpha_1) \in Cl(W)$ and $(\alpha_1) \in Cl(W')$. Assume then that we have reached $(\alpha_1, \ldots, \alpha_m)$ such that $(\alpha_1, \ldots, \alpha_m) \in Cl(W)$ and $(\alpha_1, \ldots, \alpha_m) \in Cl(W')$. If $P((\alpha_1, \ldots, \alpha_m)) = \forall$, then by (a), $(\alpha_1, \ldots, \alpha_m, f_{\forall}(\alpha_1, \ldots, \alpha_m)) \in Cl(W')$. But then by clause (b), $(\alpha_1, \ldots, \alpha_m, f_{\forall}(\alpha_1, \ldots, \alpha_m)) \in Cl(W)$. The same reasoning applies if $P((\alpha_1, \ldots, \alpha_m) = \exists$. Hence in a finite number of steps a terminal history $h = (\alpha_1, \ldots, \alpha_N)$ is reached that is both in W and in W', and so, by (d), h is a win for both players, which is impossible. ■

Now that the plethora of requisite definitions are all given, we can state the semantics for **IFML**$[k]$. Let $\mathcal{M} = (D, R_0, \ldots, R_{k-1}, \mathfrak{h})$ be any k-ary modal structure, and let $d \in D$. Then:

- $\varphi \in$ **IFML**$[k]$ is true in $\mathcal{M}$ at d, in symbols $\mathcal{M} \models^+ \varphi[d]$, iff there exists a w.s. for *Héloïse* in $\mathrm{G}_A(\varphi, \mathcal{M}, d)$.
- $\varphi \in$ **IFML**$[k]$ is false in $\mathcal{M}$ at d, in symbols $\mathcal{M} \models^- \varphi[d]$, iff there exists a w.s. for *Abélard* in $\mathrm{G}_A(\varphi, \mathcal{M}, d)$.
- $\varphi \in$ **IFML**$[k]$ is non-determined in $\mathcal{M}$ at d, in symbols $\mathcal{M} \models^0 \varphi[d]$, iff $\mathcal{M} \not\models^+ \varphi[d]$ and $\mathcal{M} \not\models^- \varphi[d]$.

When a formula is determined (non-determined), the corresponding game is likewise said to be *determined* (*non-determined*). Further, we say that a class $\mathcal{L}$ of formulae is *determined* (*non-determined*), when every (some) $\varphi \in \mathcal{L}$ is determined (non-determined).

The semantics just given to **IFML**$[k]$ will be referred to as its *uniformity interpretation* (or, **UNI**).

For the subclass **ML**$[k]$ of **IFML**$[k]$ it is easily established that the recursively defined semantics of **ML**$[k]$ and the game-theoretical semantics given to $\mathbf{IFML}[k]$ yield the same truth-values for all formulae of $\mathbf{ML}[k]$.[10]

For $k = 0$, $\mathbf{IFML}[k]$ and $\mathbf{ML}[k]$ are the same languages. Hence, $\mathbf{IFML}[0] = \mathbf{ML}[0]$ evaluated over degenerate modal structures $(\{d_0\}, \mathfrak{h})$ is indeed Propositional Logic. For $k \geq 1$, $\mathbf{ML}[k]$ is a proper subclass of $\mathbf{IFML}[k]$.

The formulae of IF *tense logic* $\mathbf{IFTL}[k] = \mathbf{IFML}[2k]$ are evaluated relative to k-ary temporal structures. (Recall that k-ary temporal structures are by definition a subclass of all $2k$-ary modal structures.)

In the present thesis we will encounter a number of examples of non-determined $\mathbf{IFML}[k]$ formulae, i.e. formulae $\varphi \in \mathbf{IFML}[k]$ for which there exists a pointed modal structure $(\mathcal{M}, d)$ such that for neither *Abélard* nor *Héloïse* there exists a winning strategy in $\mathrm{G}_A(\varphi, \mathcal{M}, d)$. Several examples will be provided in *Subsect.* 2.3.2 below. Hence, in a qualified sense, the *law of excluded middle* fails in $\mathbf{IFML}[k]$, just as it does in IF first-order logic. What this failure in $\mathbf{IF}[\tau]$ means is that *not* all instances of the schema

$$\varphi \ \vee \ \neg\varphi$$

are satisfied in all models, given that "$\neg$" stands for *dual negation*, signifying the switch of the roles of the players in a semantical game (from verifier to falsifier and *vice versa*).

It goes together with the game-theoretical viewpoint that the falsity of a formula φ corresponds to the existence of *Abélard*'s (initial falsifier's) winning strategy in the game correlated with φ – whence the notion of falsity is not given in terms of *contradictory negation*. Contradictory negation ($\sim$) is a connective that has no corresponding game rule and which can only appear sentence-initially in $\mathbf{IF}[\tau]$ when game-theoretical

[10] Proving this requires *Axiom of Choice* in the direction from the recursive semantics to the game-theoretical semantics.

semantics is used. Its truth-condition is given by stipulating that a sentence $\sim S$ is true iff S is *not* true, where "*not*" is a contradictory negation in the metalanguage. Hence $\sim S$ is true iff S is false or S is non-determined. The version of *tertium non datur* restricted to sentences only and formulated using contradictory negation – i.e. the claim of the necessary truth of instances of the schema

$$S \vee \sim S$$

– is *not* refuted by game-theoretical semantics. From the point of view of game-theoretical semantics and IF logic, it is one of the fortunate accidents of traditional first-order logic that in it, dual negation and contradictory negation coincide.[11]

The possibility of an IF logical formula being neither true nor false – the failure of the necessity of the schema $S \vee \neg S$ – has nothing to do with eventual epistemic restrictions of the human mind, or with anything tantamount to considering truth and falsehood as somehow unevenly polarized alethic modes of propositions. What is at stake in the failure of *tertium non datur* in IF logic is a brute mathematical (esp. combinatorial) fact about the existence of a structure not allowing a winning strategy for either of the two players of the semantical game associated with a formula of the form $S \vee \neg S$.

2.3.1 Two species of informational independence

Above, we have defined semantics for **IFML**$[k]$ by first associating a semantical game of imperfect information with each

[11] This coincidence depends on the fact that atomic formulae are always either true or false in the models of **FO**, and also on the fact that the evaluation games for **FO** are games of perfect information. For a proof of this coincidence, see *Theorem* 2 in Sandu (1993).

formula of $\mathbf{IFML}[k]$, and then defining the truth and falsehood of such a formula as the existence of a winning strategy for *Héloïse* resp. *Abélard* in such a game. The notion of winning strategy was defined in terms of the existence of a set W of terminal histories called a plan of action. Hence the definition of our semantics, and thereby eventually our interpretation of logical independence as it appears in modal logic, appeals both to the *extensive form* of a game − involving information sets − and to a *strategy* (*plan of action*) made use of in playing such a game.

Consider now the two cases:

(a) Modeling informational independence exclusively in terms of extensive forms of games.

(b) Modeling informational independence by reference to *strategies* made use of when playing games (that are given in their extensive form).

Each of these cases leads to its own species of informational independence:

(a') A player being subject to *imperfect information*, as standardly defined in game theory.

(b') A player being subject to a *norm demanding uniform strategic action.*

Let us look at these options in more detail.

In standard game theory the player j's *imperfect information of past histories* of a game is presented by having available in the extensive form of the game an information partition I_j of the class of those histories at which it is player j's turn to move, and the effect of imperfect information is brought in by requiring that all histories in any $I \in I_j$ have the same possible extensions,[12] schematically

$$(\star)\ h, h' \in I \in I_j \implies$$
$$A(h) = \{a : h^\frown a \in H\} = \{a : h'\,^\frown a \in H\} = A(h').$$

Hence if the partitions $I_\forall$ and $I_\exists$ in a structure

$$\Gamma = \langle \{\forall, \exists\}, H, Z, P, \{u_\forall, u_\exists\}, \{I_\forall, I_\exists\} \rangle$$

do not satisfy the above condition $(\star)$, the structure Γ is thereby not a game of imperfect information.

Now the histories in a given cell $I \in I_j$ of the information partition I_j are customarily interpreted as being *indistinguishable* to the player j. Thereby the requirement $(\star)$ receives, as a part of modeling imperfect information, the following rationale: if, as Osborne and Rubinstein (1994, p. 201) put it, we had $A(h) \neq A(h')$, the player j could deduce, when faced with $A(h)$, that the history was not h', which would contradict the *desideratum* that the histories in the information set I should be indistinguishable to j. (For all histories h, the player is required to be aware of the set $A(h)$ of possible choices from which he can choose his move.)

Hence a consequence of the decision to *interpret* members of an information set as indistinguishable is readily observed: imperfect information means a *strict lack of knowledge* about the route by means of which playing a game has proceeded to a situation where it is a given player's turn to move. He knows what his alternatives are for the next move, and he knows a set containing the sequence of moves actually made, but does not know which of the histories in the set is the actual one.

Now what would be the result of *interpreting the role of information sets* in some other way? What if we thought of these

[12] See e.g. Osborne and Rubisntein (1994, pp. 200-1).

histories as indeed being totally visible to the player whose turn it is to move, and interpreted their being gathered into the same cell of the information partition as inducing a 'behavioral restriction' for the player, instead of an epistemic blackout? A different intuitive view on information sets might well render the condition $(\star)$ dispensable.

There is, in fact, such a different view, and it has been concretely implemented above in defining evaluation games for IF modal logic. These games do *not* in general meet the condition $(\star)$, as is easily seen.

Example 2.3.3 *Consider the semantical game corresponding to the formula*

$$\chi := \Box_1 \Diamond_2 / \{1\} \top$$

and the pointed modal structure $(\mathcal{M}, a) = (M, R, \mathfrak{h}, a)$ *with*

$$M = \{a, b, c, d, e, f\},$$

$$R = \{(a, b), (a, c), (b, d), (b, e), (c, e), (c, f)\}.$$

This structure is illustrated in Figure 6:

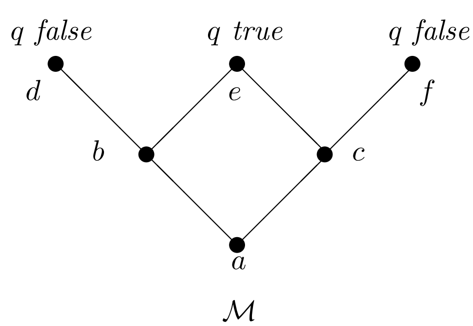

FIGURE 6

The information partition corresponding to the operator $\Diamond_2/\{1\}$ *is now* $\{\{(a,b),(a,c)\}\}$,[13] *while the sets of the extensions of* (a,b) *and* (a,c)*, respectively, are not the same:*

$$A((a,b)) = \{d,e\} \neq \{e,f\} = A((a,c)).$$

Hence the semantic game associated with χ *does not meet the condition* ($\star$)*. What is more,* χ *is true in* $(\mathcal{M},a)$ *by our truth-definition, as* e *is a uniform, truth-establishing choice for Héloïse corresponding to* $\Diamond_2/\{1\}$. ■

So the relevant notion of independence in connection with the evaluation games for **IFML**$[k]$ can be said *not* to be modeled by imperfect information, if the above game-theoretic implementation (a) is taken as a standard of what indeed counts as imperfect information. Instead, independence is, in connection with these games, modeled by *imposing a norm on actions governed by winning strategies:* the norm of *uniformity.* Put differently, the requirement is that *winning strategies yield actions that are universalizable.*

Roughly, what the information sets do under interpretation (b) is specify which histories must be treated as equals – what a player does with one of them, he must be prepared to do with the rest.[14] For a concrete example of the same phenomenon, think of actions or series of actions that can be morally or legally punishable. They can be grouped into classes by taking into account different parameters (e.g. under what circumstances, by which means and with what intention a crime, say, was committed), but the punishment – to be legal or moral

[13] Explicitly writing down the subformula components of actions is refrained from, whenever this may not cause confusion.

[14] As will be spelled out below, this description is subject to the specification that the only histories which matter are those that can be reached when the specified recipe for behavior (plan of action) is followed.

− cannot differ according to the identity of the person responsible for the action in question. When judging a person, one must act in a manner that can be made universal − so that the result would be the same if another person had committed the same crime. Being aware of the identity of the person who has committed the crime does not change this fact in the least.

Here, technically, informational independence is brought into the picture *not* on the level of extensive forms of games but on the level of winning strategies used for playing these games. (Strategies that are not winning need not respect informational independence in any way.) That is, when the player j uses a strategy f to make a move at a history $h \in I \in I_j$ − and while we may think that he *can* distinguish h from every other history in I − he nevertheless must, in order to win the play of the game, move in such a way that the same move could be made to also extend all other histories from I that could be reached given that j follows the strategy f. This is what the norm concerning winning strategies dictates.

Consider the examples of evaluation games designed according to the two ways of modeling informational independence in connection with modal logic, (a) and (b). In both cases consider evaluating the formulae

(i) $\Diamond_1 \Diamond_2/\{1\}\top$ in the structure $(\mathcal{N}, 0) = (\mathbb{N}, <, \mathfrak{h}, 0)$. [15]

(ii) $\Box_1 \Diamond_2/\{1\}\top$ in the structure $(\mathcal{M}, a) = (M, R, \mathfrak{h}, a)$

specified above.

Example 2.3.4 *("Imperfect information")* ***(i)*** *Write* Γ *for the imperfect information game associated with the formula*

[15] The relation $<$ is the ordering of the set of natural numbers by magnitude.

$\Diamond_1\Diamond_2/\{1\}\top$. *Then Héloïse's information partition in* Γ *contains a single information set*

$$I = \{(0,n) : n > 0\},$$

corresponding to the operator $\Diamond_2/\{1\}$*, and any extension* m *of a history* $(0,n)$ *satisfies* $m > n$*. But then, by the condition* $(\star)$*, in fact* $A(h) = \varnothing$ *for all* $h \in I$*. Hence there is no w.s. for Héloïse in the game, and the formula is not true under this interpretation. In fact it is false, Abélard's w.s. consisting of doing precisely nothing.*

(ii) *Here the relevant information set is*

$$I = \{(a,b),(a,c)\}\,,$$

and, due to the condition $(\star)$*,* $A((a,b)) = A((a,c)) = \{e\}$*. Hence there is a w.s.* $(f_\exists)$ *for Héloïse in the correlated game, defined by putting* $f_\exists((a,b)) = e = f_\exists((a,c))$*.* ■

Remark 2.3.5 *Notice that due to the condition* $(\star)$*,* $(0,1,2)$*, for example, is not a history in the game corresponding to case (i), and* (a,b,d) *is not a history in the game associated with case (ii). It is not only for example that Héloïse's w.s. cannot choose* 2 *to extend* $(0,1)$*, but that* 2 *is not among the possible choices.*

Example 2.3.6 *("Uniform strategic actions")* ***(i)*** *Put* $W := \{(0,1,2)\}$*, and define a strategy* $f_\exists$ *by setting* $f_\exists((0)) = 1$*,* $f_\exists((0,1)) = 2$*.*

We observe that the terminal history $(0,1,2)$ *is a win for Héloïse. Hence* $f_\exists$ *is a winning strategy based on the plan of action* W*.*

(ii) *See Example* 2.3.3 *above.* ■

Remark 2.3.7 *Under uniformity interpretation, informational independence rests upon the requirement that of those moves that can be made using a given strategy f, any move made to extend a history from a given information set I must also be a legal extension for all the other histories from I that could have been realized using f. Hence for case (i) the uniformity requirement is in fact vacuous, and in case (ii) a w.s. for Héloïse exists, as there is a common extension, e, to (a, b) and (a, c). This is true even if (a, b) and (a, c) do not have all their respective possible extensions in common.*

The distinction between the two ways of modeling informational independence has repercussions in the discussion about the sorts of strategies that players are assumed to be using when playing games. In game-theoretical literature, a *pure strategy* of the player j in a game of imperfect information is any function

$$f_j : h \mapsto A(h),$$

associating each history from $P^{-1}(\{j\})$ with a possible extension, and respecting the information partition I_j in the sense that if $h, h' \in I \in I_j$, then $f_j(h) = f_j(h')$.

By contrast, a *strategy based on a plan of action* W, as we have defined this notion above, is any function

$$f_j : P^{-1}(\{j\}) \rightarrow A$$

for which there exists a set W of terminal histories in the correlated semantical game satisfying criteria (a), (b) and (c) laid down therein. A plan of action specifies only a minimal subset of all game histories, namely *Cl(W)*, which suffices to cover all combinatorially possible histories that can be reached depending on the moves of the opponent. Whether a plan of action is winning or not depends only on this set *Cl(W)*.

Ariel Rubinstein (1998, p. 66) comments on the difference between pure strategies and plans of actions, saying that

> "[In the game-theoretic tradition, the definition of a pure strategy] requires that the decision maker specify his actions after histories he will *not* reach if he follows the strategy."[16]

He goes on to state that:

> "[t]he more natural definition of a strategy is as a *plan of action:* a function that assigns an action only to histories reached with a positive probability."

Plans of action, as I have defined above, precisely specify the actions of the player only for such histories that he or she may actually confront when applying this plan.[17] Also, the way in which we have chosen to model informational independence, option (b) above, precisely restricts the attention to what can happen in a play when it is fixed how the player in whose winning strategy we are interested, is going to move. Uniformity is only required relative to such histories that can be realized.

By contrast, the requirement behind the definition of pure strategies, on the one hand, and the condition ($\star$) imposable on information sets, on the other, are similar in spirit. Both stem from a globalist viewpoint on games in which the particular strategy-governed way of playing the game is not given any special attention: a pure strategy must specify what to do even in situations in which one cannot end up using that strategy, and the condition ($\star$) requires uniform behavior in the extreme sense that an admissible move must be possible at any history in an information set, irrespective of any considerations of how

[16] Italics mine.

[17] Given a strategy f of the player j, it is precisely the histories whose realizability is not ruled out by the sole fact that j employs this strategy, which have positive probability (for j).

such a history might have been reached in the course of a play of a game.

When the idea of employing a specified strategy for playing a game is taken seriously, and if combined with the interpretation of information sets as sets of indistinguishable histories (i.e. according to the interpretation preferred in standard game theory), it would become reasonable to interpret as indistinguishable only the histories in such a *subset* I_f of a given information set I that can be attained given that the strategy f is employed. When this approach is not taken − as when proceeding from interpretation (a) − interpreting histories in I as simply indistinguishable means that this interpretation deliberately ignores the strategic aspect of playing games.

In what follows, we carry on making use of interpretation (b) of informational independence. Games will continue to be presented as defined in their extensive forms, but in the semantics we will essentially resort to the strategies (plans of action) used for playing these games.

2.3.2 Examples of evaluations by uniformity interpretation

For the sake of illustration, it will be shown how to evaluate, relative to the structures specified below, the following **IFML**$[k]$ formulae:

(1) $\Diamond_{0,1}\Diamond_{0,2}/\{1\}q$

(2) $\Diamond^{-1}_{0,1}\Diamond_{0,2}/\{1\}q$

(3) $\Box_{0,1}\Diamond_{0,2}/\{1\}q$

(4) $\Box^{-1}_{0,1}\Diamond_{0,2}/\{1\}q$

(5) $\Box_{0,1}\Box^{-1}_{0,2}\Diamond^{-1}_{0,3}\ /\{1,2\}q$

(6) $\Box_{0,1}\Diamond_{1,2}/\{1\}q$

For formulae (1)-(5) we have $k \geq 1$, and for formula (6), $k \geq 2$. For the former group of cases, k can be assumed to equal 1, and we can drop the first member of the double indexing of the modal operators, as this first index will be 0 for each operator. Accordingly we write formula (1), for example, as $\Diamond_1\Diamond_2/\{1\}q$.

When describing strategies below, explicit mention of subformula components of histories is omitted, whenever it is possible to do so without causing confusion.

Example 2.3.8 *For any modal structure $\mathcal{M} = (D, R, \mathfrak{h})$ and any point $d \in D$, we have:*

$$\Diamond_1\Diamond_2/\{1\}q \text{ and } \Diamond\Diamond q$$

are both true or both false in $\mathcal{M}$ at d.

This is seen as follows. If the set $\{z$: there is y such that $R(d, y)$ and $R(y, z)\}$ is empty, both formulae are trivially false. So assume the set is not empty, and define a strategy f for Héloïse as follows:

$$f(d) = d', f(d, d') = d'',$$

where d' and d'' are fixed so that $R(d, d')$, $R(d', d'')$, and if possible also $d'' \in \mathfrak{h}(q)$. Then clearly, if indeed $d'' \in \mathfrak{h}(q)$, then the set $W = \{(d, d', d'')\}$ establishes that f is a w.s. for Héloïse in the game $G_A(\Diamond_1\Diamond_2/\{1\}q, \mathcal{M}, d)$, and thereby the same strategy f is also a w.s. for her in $G_A(\Diamond\Diamond q, \mathcal{M}, d)$. If, again, d'' cannot be chosen so as to satisfy q, both formulae $\Diamond_1\Diamond_2/\{1\}q$ and $\Diamond\Diamond q$ are false.

Example 2.3.9 *For any model $\mathcal{M}$ and any point d:*

$$\Diamond_1^{-1}\Diamond_2/\{1\}q \text{ and } \Diamond^{-1}\Diamond q$$

are both true or both false in $\mathcal{M}$ at d.

As in Example 2.3.8*, again the independence indication has no effect, since all moves are made by the same player (Héloïse).*

Example 2.3.10 *Consider evaluating the* **IFML**[1] *formula $\Box_1\Diamond_2/\{1\}q$ in a model $\mathcal{M} = (D, R, \mathfrak{h})$ at a point d, assuming that R is an irreflexive partial order, i.e. an irreflexive (antisymmetric) and transitive binary relation.*

In order for there to exist a w.s. for Héloïse in $G_A(\Box_1\Diamond_2/\{1\}q, \mathcal{M}, d)$, there must be an element c such that for all d', if $R(d, d')$, then $R(d', c)$.

(a) *If d has no R-successors (i.e. if d is an R-maximal element), then Héloïse has trivially a winning strategy. This strategy then consists of doing precisely nothing, and is based on the plan of action $\{\langle(\Box_1\Diamond_2/\{1\}q, d), (\mathtt{FAIL}, \star)\rangle\}$.*

(b) *If there are R-successors to d, then no winning strategy for Héloïse can exist. For contradiction, assume f is such a strategy, and $f(d, d') = c$ for all d' with $R(d, d')$. But then there is a play in which Abélard chooses c to interpret $\Box_1$. (By assumption there is d' such that $R(d, d')$ and $R(d', c)$. Hence, by transitivity of R, we have that $R(d, c)$.) But then, by irreflexivity of R, in this play c is not a legal reply for Héloïse. Hence f cannot be a w.s. for her.*

On the other hand, under assumption (b) there is a w.s. for Abélard in $G_A(\Box_1\Diamond_2/\{1\}q, \mathcal{M}, d)$ if and only if there is an R-successor d' to d such that all of its R-successors satisfy $\neg q$. Otherwise the game is non-determined.

To sum up:

$$\mathcal{M} \models^+ \Box_1\Diamond_2/\{1\}q[d] \iff \mathcal{M} \models^+ \Box\bot[d].$$

$$\mathcal{M} \models^- \Box_1\Diamond_2/\{1\}q[d] \iff \mathcal{M} \models^+ \Diamond\Box\neg q[d].$$

$$\mathcal{M} \models^0 \Box_1\Diamond_2/\{1\}q[d] \iff \mathcal{M} \models^+ (\Diamond\top \wedge \Box\Diamond q)[d].$$

Example 2.3.11 *Consider evaluating the formula*

$$\Box_1^{-1}\Diamond_2/\{1\}q$$

in a modal structure $\mathcal{M} = (\mathbb{Q}, <, \mathfrak{h})$ *whose frame* $(\mathbb{Q}, <)$ *is the set of the rationals ordered by their magnitude. Let us take* $0 \in \mathbb{Q}$ *as the point of evaluation. We claim:*

$$\mathcal{M} \models^+ \Box_1^{-1}\Diamond_2/\{1\}q[0] \Longleftrightarrow \mathcal{M} \models^+ q \vee \Diamond q[0].$$

If the right side of the equivalence holds, define a strategy f *for Héloïse by:*

$$\textit{for all } d < 0 : f(0, d) = c,$$

where $c \geq 0$ *is a fixed rational satisfying* q*. The existence of at least one such rational is guaranteed precisely by the condition* $\mathcal{M} \models^+ q \vee \Diamond q[0]$*. But then* f *is a w.s. for Héloïse based on the plan of action* $W = \{(0, d, c) : d < 0\}$*. And if the left-hand side holds, then the existence of a w.s. for Héloïse in the game associated with the formula* $\Box_1^{-1}\Diamond_2/\{1\}q$ *guarantees that there exists a point* $c \geq 0$ *at which* q *holds* $-$ *whence* $q \vee \Diamond q$ *is true in* $\mathcal{M}$ *at* 0.

On the other hand, the formula $\Box_1^{-1}\Diamond_2/\{1\}q$ *is false, iff there is some negative rational* x *such that always after* x, q *is false:*

$$\mathcal{M} \models^- \Box_1^{-1}\Diamond_2/\{1\}q[0] \Longleftrightarrow \mathcal{M} \models^+ \Diamond^{-1}\Box\neg q[0].$$

And finally, the formula is non-determined, iff arbitrarily near 0 *on the left there are rationals at which* q *holds, but at* 0 *and always after it* q *is false:*

$$\mathcal{M} \models^0 \Box_1^{-1}\Diamond_2/\{1\}q[0] \Longleftrightarrow \mathcal{M} \models^+ (\neg q \wedge \Box\neg q) \wedge \Box^{-1}\Diamond q[0].$$

Hence in this case $\mathfrak{h}(q)$ *and the set of non-negative rationals are disjoint, and the supremum of the set* $\{x \in \mathfrak{h}(q) : x < 0\}$ *is* 0.

Example 2.3.12 *Consider evaluating the formula*

$$\Box_1 \Box_2^{-1} \Diamond_3^{-1}/\{1,2\}q$$

in a structure $\mathcal{M} = (\mathbb{Q}, <, \mathfrak{h})$ *at the point* $0 \in \mathbb{Q}$. *This formula is not true in* $\mathcal{M}$ *at* 0. *To see this, assume for contradiction that there is a w.s.* f *for Héloïse in the correlated game. Then there is some rational* c *such that for all sequences* $(0, x, y)$ *and* $(0, x', y')$ *from her information set*

$$I = \{(0, x, y) : 0 < x, y < x\}$$

corresponding to the operator $\Diamond_3^{-1}/\{1,2\}$, *the following holds:*

$$f(0, x, y) = f(0, x', y') = c.$$

But then there is in I *the sequence* $(0, |c| \cdot max\{c, 2\}, c)$ *and choosing* c *to extend this sequence by* f *cannot yield a win for Héloïse (since* $c \not< c$*).*

On the other hand, the formula is false, iff there is some rational below which q *is always false:*

$$\mathcal{M} \models^- \Box_1 \Box_2^{-1} \Diamond_3^{-1}/\{1,2\}q[0] \iff \mathcal{M} \models^+ \Diamond^{-1}\Box^{-1}\neg q[0].$$

(Note that if there is some such rational, then there is such a rational below 0*.) Finally, the formula is non-determined, iff below any rational there is a rational at which* q *is true:*

$$\mathcal{M} \models^0 \Box_1 \Box_2^{-1} \Diamond_3^{-1}/\{1,2\}q[0] \iff \mathcal{M} \models^+ \Box^{-1}\Diamond^{-1}q[0].$$

Example 2.3.13 *Consider the formula* $\Box_{0,1}\Diamond_{1,2}/\{1\}q$, *in which there appear modal operators of two distinct modality types. Let us think of evaluating the formula relative to the binary modal structure* $M = (\mathbb{Q}, <, \prec, \mathfrak{h})$ *at the point* 0, *given that*

- $<$ *is the order of rationals by their magnitude*

- $\prec$ *is the linear order of the relational structure*

$$(\mathbb{Q}, \prec) := \mathbb{Q}^{\geq 0} \oplus \mathbb{Q}^{<0},$$

i.e. the ordered sum of the ordered set $\mathbb{Q}^{\geq 0}$ *of non-negative rationals and the ordered set* $\mathbb{Q}^{<0}$ *of negative rationals (both sets ordered by magnitude). For an illustration of the order* $\prec$*, see that:*

$$0 \prec \tfrac{7}{1000} \prec \tfrac{1}{2} \prec \tfrac{2}{3} \prec \tfrac{1000}{7} \prec$$
$$\prec -\tfrac{1000}{7} \prec -\tfrac{2}{3} \prec -\tfrac{1}{2} \prec -\tfrac{7}{1000}.$$

- $\mathfrak{h}(q) = \{x : x \geq 0\}$.

The orders $<$ *(above),* $\prec$ *(below) can be depicted as in Figure 7 (the box indicating zero).*

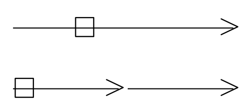

FIGURE 7

We claim that $\Box_{0,1}\Diamond_{1,2}/\{1\}q$ *is not true in* $\mathcal{M}$ *at* 0*. Now in order for there to be a w.s. for Héloïse, there must exist a rational* c *such that for all* d*:*

$$\textit{if } 0 < d, \textit{ then } d \prec c.$$

However, no positive rational can be a $\prec$*-successor to all positive rationals, while only (and all) negative rationals would count as such a number* c*. But* q *is not true at any negative rational. Hence there cannot exist a w.s. for Héloïse.*

Further, $\Box_{0,1}\Diamond_{1,2}/\{1\}q$ *is also not false in* $\mathcal{M}$ *at* 0*. Namely, for every* $d > 0$*, Héloïse can for instance choose the number* $c := d + 1$*; this choice satisfies* $d \prec c$*, and because it always is a positive rational, it also makes* q *true.*

From what we have noted we may conclude that the formula is non-determined in $\mathcal{M}$ *at* 0*:*

$$\mathcal{M} \models^0 \Box_{0,1}\Diamond_{1,2}/\{1\}q[0].$$

Chapter 3

The Expressive Power of IF Modal Logic

We now move on to study the relative expressive power of IF modal logic. In *Section* 2 of this chapter I will show that **IFML**$[k]$ can be translated into traditional first-order logic (**FO**). On the other hand, in *Section* 3 it will be shown that a variant of IF modal logic that allows modal operators independence even from conjunctions and disjunctions (**EIFML**$[k]$, $k \geq 3$) has genuine second-order expressive power, i.e. cannot be translated into **FO**. In *Section* 4 the expressive powers of **IFML**$[k]$ and **ML**$[k]$ are compared relative to various classes of modal structures. *Section* 1 provides some requisite basic definitions and facts. Throughout the present chapter the *uniformity interpretation* of **IFML**$[k]$ will be employed as semantics.

3.1 Facts about Expressive Power

If $\mathcal{L}$ and $\mathcal{L}'$ are modal logics (such as **ML**$[k]$ and **IFML**$[k]$) and $\mathcal{K}$ is a class of modal structures on which the semantics of both logics is defined, the relations

- $\mathcal{L} \leq_{\mathcal{K}} \mathcal{L}'$ ($\mathcal{L}$ is *embeddable* in $\mathcal{L}'$ over $\mathcal{K}$)
- $\mathcal{L} <_{\mathcal{K}} \mathcal{L}'$ ($\mathcal{L}'$ has *greater expressive power* than $\mathcal{L}$ over $\mathcal{K}$)
- $\mathcal{L} =_{\mathcal{K}} \mathcal{L}'$ ($\mathcal{L}$ and $\mathcal{L}'$ have *the same expressive power* over $\mathcal{K}$)

are defined as was explained in the *Introduction* (Paragraph 3).[1] By convention, we write $\mathcal{C}_k$ for the class of all k-ary modal structures. For an arbitrary $k < \omega$, we trivially have the following embeddability result:

Fact 3.1.1 *Over* $\mathcal{C}_k$, $\mathbf{ML}[k]$ *is embeddable in* $\mathbf{IFML}[k]$.

This is simply because $\mathbf{ML}[k]$ is literally a subclass of $\mathbf{IFML}[k]$: any formula $\varphi \in \mathbf{ML}[k]$ is its own translation in $\mathbf{IFML}[k]$. ■

3.1.1 The determined fragment of IFML

We say that the class $\mathbf{IFML}_{det}[k] :=$

$$\{\varphi \in \mathbf{IFML}[k] : \text{for all } \mathcal{M} \in \mathcal{C}_k \text{ and all } d \in dom(\mathcal{M}),$$
$$\mathcal{M} \models^+ \varphi[d] \text{ or } \mathcal{M} \models^- \varphi[d]\}$$

is the *determined fragment* of $\mathbf{IFML}[k]$. The following theorem shows that relative to the class $\mathcal{C}_k$ of all k-ary modal structures, the expressive powers of $\mathbf{IFML}_{det}[k]$ and $\mathbf{ML}[k]$ are the same.

[1] *Ch.* 4 looks at two sorts of embeddability: weak and strong. In the present chapter we will only deal with weak embeddability, and call it simply "embeddability". This is in keeping with the fact that **UNI** interpretation, on which we concentrate here, allows non-determined $\mathbf{IFML}[k]$ formulae, and so strong embeddability (the translation and the translated having everywhere coincident truth-values) is, in general, expressly *not* attainable.

Theorem 3.1.2 $\mathbf{IFML}_{det}[k] =_{\mathcal{C}_k} \mathbf{ML}[k]$.

Proof. All formulae of $\mathbf{ML}[k]$ are determined (everywhere either true or false), and they are also (directly by syntactical criteria) formulae of $\mathbf{IFML}[k]$, so $\mathbf{ML}[k] \leq_{\mathcal{C}_k} \mathbf{IFML}_{det}[k]$. For the other direction, let $\varphi \in \mathbf{IFML}_{det}[k]$, $\mathcal{M} \in \mathcal{C}_k$ and $d \in dom(\mathcal{M})$ all be arbitrary. We must find a formula $\psi_\varphi \in \mathbf{ML}[k]$, satisfying that

there is a $w.s.$ for $\exists$ in $\mathrm{G}(\varphi, \mathcal{M}, d) \Longleftrightarrow$

there is a $w.s.$ for $\exists$ in $\mathrm{G}(\psi_\varphi, \mathcal{M}, d)$.

Now if in particular $\varphi \in \mathbf{ML}[k]$, then we may clearly take ψ_φ to be φ itself. If, again, φ is of the form $O_1 \ldots O_{n-1}(O_n/W)\chi$ with $\chi \in \mathbf{ML}[k]$, then put $\psi_\varphi := O_1 \ldots O_{n-1}O_n\chi$. What can be done in greater ignorance can be done in lesser ignorance: clearly if some $f_\exists$ is a w.s. for $\exists$ in $\mathrm{G}(\varphi, \mathcal{M}, d)$, the same strategy is a w.s. for her in $\mathrm{G}(\psi_\varphi, \mathcal{M}, d)$. Assume then that there is *no* w.s. for $\exists$ in $\mathrm{G}(\varphi, \mathcal{M}, d)$. But then, by the assumption of determinacy, there is a w.s. for $\forall$ in $\mathrm{G}(\varphi, \mathcal{M}, d)$, which is clearly also a w.s. for him in $\mathrm{G}(\psi_\varphi, \mathcal{M}, d)$. This means, in turn, that there is no w.s. for $\exists$ in $\mathrm{G}(\psi_\varphi, \mathcal{M}, d)$. ∎

3.1.2 Bisimulations

A specific logical tool we will make use of when proving the main results concerning the relative expressive powers of $\mathbf{IFML}[k]$ and $\mathbf{ML}[k]$ is *bisimulation*. A bisimulation is a binary relation between elements from the domains of two modal structures. When two k-ary modal structures are bisimilar, they make true precisely the same $\mathbf{ML}[k]$ formulae. It turns out that while bisimilarity thus preserves the truth of $\mathbf{ML}[k]$ formulae, in general the same does *not* hold true for all $\mathbf{IFML}[k]$ formulae as some are capable of distinguishing bisimilar modal structures (by being true in one but not in the another), hence the usefulness of bisimilarity for our purposes. For, if there is a formula of a logic ($\mathcal{L}$) that can distinguish models from a class $\mathcal{K}$, but no formula of another logic ($\mathcal{L}'$) can do this, then it cannot be the case that $\mathcal{L}$ were embeddable in $\mathcal{L}'$ over $\mathcal{K}$. If we happen to be able to establish, furthermore, that $\mathcal{L}'$ is in turn embeddable in $\mathcal{L}$ over $\mathcal{K}$, we have in effect shown that over $\mathcal{K}$, $\mathcal{L}$ has a strictly greater expressive power than $\mathcal{L}'$.

In the present subsection we discuss some basic properties of bisimulations connected with basic modal logic. How to give a natural definition to the relation of **IFML** bisimulation, i.e. a relation that holds between pointed modal structures if and only if they satisfy exactly the same **IFML** formulae, is a separate question for further research.

3.1.2.1. Bisimilarity proper

The relation of bisimulation for k-ary modal structures is defined as follows.

Definition 3.1.3 *Let* $\mathcal{M} = (D, R_0, \ldots, R_{k-1}, \mathfrak{h})$ *and* $\mathcal{N} = (D', R'_0, \ldots, R'_{k-1}, \mathfrak{h}')$ *be k-ary modal structures. A bisimulation between the modal structures* $\mathcal{M}$ *and* $\mathcal{N}$ *is a binary relation* $\equiv_{D,D'} \subseteq D \times D'$*, satisfying the following conditions* (1)*,* (2) *and* (3)*.*

(1) *Atomic Harmony:*
$d \equiv_{D,D'} d' \Longrightarrow$ *for all* $p \in \mathbf{prop}$*:* $d \in \mathfrak{h}(p) \Longleftrightarrow d' \in \mathfrak{h}'(p)$.

(2) *Zigzag Forwards: for all* $i < k$*,*
$d \equiv_{D,D'} d'$ *and* $R_i(d, c) \Longrightarrow \exists c'(R'_i(d', c')$ *and* $c \equiv_{D,D'} c')$.

(3) *Zigzag Backwards: for all* $i < k$*,*
$d \equiv_{D,D'} d'$ *and* $R'_i(d', c') \Longrightarrow \exists c(R_i(d, c)$ *and* $c \equiv_{D,D'} c')$.

If $\equiv_{D,D'} \subseteq D \times D'$ is a bisimulation between $\mathcal{M}$ and $\mathcal{N}$, we may indicate this by writing

$$\equiv_{D,D'} : \mathcal{M} \rightleftarrows \mathcal{N}.$$

A *bisimilarity* $B_{D,D'}$ (as opposed to a bisimulation) between the modal structures $\mathcal{M} = (D, R_0, \ldots, R_{k-1}, \mathfrak{h})$ and $\mathcal{N} = (D', R'_0, \ldots, R'_{k-1}, \mathfrak{h}')$ is the *union* of all bisimulations between $\mathcal{M}$ and $\mathcal{N}$:

$$B_{D,D'} := \bigcup_{\equiv_{D,D'} : \mathcal{M} \rightleftarrows \mathcal{N}} \equiv_{D,D'} .$$

The pointed modal structures $(\mathcal{M}, d)$ and $(\mathcal{N}, d')$ are said to be *bisimilar*, if the pair (d, d') is a member of the bisimilarity $B_{D,D'}$:

$$(d, d') \in B_{D,D'}.$$

We say that the pointed modal structures $(\mathcal{M}, d)$ and $(\mathcal{N}, d')$ are $\mathbf{ML}[k]$ *equivalent*, if for all $\varphi \in \mathbf{ML}[k]$,

$$\mathcal{M} \models^+ \varphi[d] \Longleftrightarrow \mathcal{N} \models^+ \varphi[d'].$$

The notion of $\mathbf{IFML}[k]$ equivalence of $(\mathcal{M}, d)$ and $(\mathcal{N}, d')$ is defined similarly.

Now for formulae of $\mathbf{ML}[k]$ the following *Invariance Lemma* holds. The lemma says that bisimilarity preserves truth in $\mathbf{ML}[k]$ in the sense that if $(d, d') \in B_{dom(\mathcal{M}),dom(\mathcal{N})}$, then $(\mathcal{M}, d)$ and $(\mathcal{N}, d')$ are $\mathbf{ML}[k]$ equivalent.

Lemma 3.1.4 *(Invariance Lemma) Let $\mathcal{M} = (D, R_0, \ldots, R_{k-1}, \mathfrak{h})$ and $\mathcal{N} = (D', R'_0, \ldots, R'_{k-1}, \mathfrak{h}')$ be k-ary modal structures. Then for all $\phi \in \mathbf{ML}[k]$ and all $(d, d') \in D \times D'$:*

$$(d, d') \in B_{D,D'} \Longrightarrow (\mathcal{M} \models^+ \phi[d] \Longleftrightarrow \mathcal{N} \models^+ \phi[d']).$$

Proof. The proof is by induction on the complexity of the formula $\phi \in \mathbf{ML}[k]$. The case for (negated) propositional atoms, and the inductive cases for disjunction and conjunction are completely trivial. Consider the case $\phi = \Diamond_i \varphi$ $(i < k)$, assuming inductively that φ satisfies the claim:

$$\forall x \in D \forall x' \in D' : (x, x') \in B_{D,D'} \Longrightarrow$$

$$(\mathcal{M} \models^+ \varphi\ [x] \Longleftrightarrow \mathcal{N} \models^+ \varphi[x']).$$

Let then $x \in D$, $x' \in D'$ be arbitrary elements satisfying $(x, x') \in B_{D,D'}$. If $\mathcal{M} \models^+ \Diamond_i \varphi[x]$, then there is y such that $R_i(x, y)$ and $\mathcal{M} \models^+ \varphi[y]$. But then, because $(x, x') \in B_{D,D'}$, we obtain by condition (2) of the definition of bisimulation that there exists y' such that $R'_i(x', y')$ and $(y, y') \in B_{D,D'}$. (More specifically: since $(x, x') \in B_{D,D'}$, there is a bisimulation $\equiv_{D,D'}$ such that $(x, x') \in \equiv_{D,D'}$. By the definition of bisimulation, then, the above pair (y, y') satisfies $(y, y') \in \equiv_{D,D'}$, and so belongs to $B_{D,D'}$.) Hence, by the inductive hypothesis we further get that $\mathcal{N} \models^+ \varphi[y']$, whence we may conclude that $\mathcal{N} \models^+ \Diamond_i \varphi[x']$. The converse implication,

$$\mathcal{N} \models^+ \Diamond_i \varphi[x'] \Longrightarrow \mathcal{M} \models^+ \Diamond_i \varphi[x],$$

is shown to hold exactly similarly, using condition (3) of the definition of bisimulation.

The inductive case for formulae of the form $\phi = \Box_i \varphi$ $(i < k)$ is proven analogously to the above case for $\phi = \Diamond_i \varphi$. ∎

Observe that the implication in the statement of *Invariance Lemma* cannot be converted: it is *not* the case that if pointed modal structures $(\mathcal{M}, d)$ and $(\mathcal{N}, d')$ are indistinguishable in terms of $\mathbf{ML}[k]$ formulae then $(\mathcal{M}, d)$ and $(\mathcal{N}, d')$ are bisimilar. If instead of $\mathbf{ML}[k]$, the infinitary modal logic $\mathbf{ML}[k, \infty]$

(otherwise like **ML**$[k]$ but allowing the forming of *infinite* disjunctions and conjunctions) is considered, then we indeed have an equivalence:

> $(*)$ Pointed k-ary modal structures $(\mathcal{M}, d)$ and $(\mathcal{N}, d')$
>
> are bisimilar $\Longleftrightarrow$ $(\mathcal{M}, d)$ and $(\mathcal{N}, d')$ cannot be
>
> distinguished by an **ML**$[k, \infty]$ formula.

Bisimilarity therefore does *not* characterize **ML**$[k]$ equivalence (but characterizes **ML**$[k, \infty]$ equivalence instead). Informally expressed, **ML**$[k]$ equivalence of modal structures $\mathcal{M}$ and $\mathcal{N}$ would be characterized by the following condition: For all natural numbers $n < \omega$, it is possible to choose n pairs from $dom(\mathcal{N}) \times dom(\mathcal{N})$ in such a way that the chosen pairs comply with the zigzag conditions (1) and (2) of Definition 3.1.3, and satisfy precisely the same propositional atoms from all finite subsets of the class **prop**. This requires that it be possible to choose *any finite number* of pairs meeting these conditions, but leaves open the alternative that it would not be possible to choose an *infinite number* of such pairs, which is essentially what bisimilarity proper requires. It also leaves open the option that (i) there is $a \in dom(\mathcal{M})$ which satisfies atoms from an infinite set $S \subseteq$ **prop**, such that for all *finite subsets* S_F of S there is $b \in dom(\mathcal{N})$ correlated with a, making all atoms of S_F true, but (ii) no element $b \in dom(\mathcal{N})$ is correlated with a so that b would satisfy *all* the atoms from the infinite set S.

The situation in first-order logic (**FO**) is analogous. First-order structures $\mathfrak{A}$ and $\mathfrak{B}$ (written in a finite vocabulary) are said to be *elementarily equivalent* when they satisfy exactly the same first-order sentences, viz. when for all $n < \omega$, $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same first-order sentences of quantifier rank at most n. The elementary equivalence of $\mathfrak{A}$ and $\mathfrak{B}$ is characterized by the condition that for all $n < \omega$, *Duplicator* has a winning strategy in the n-round Ehrenfeucht-Fraïssé game $\mathrm{G}_n(\mathfrak{A}, \mathfrak{B})$. By contrast, the condition stating that *Duplicator* has a winning strategy in the Ehrenfeucht-Fraïssé game $\mathrm{G}_\infty(\mathfrak{A}, \mathfrak{B})$ with *infinitely many* rounds, serves to characterize the $\mathcal{L}_{\infty\omega}$ equivalence[2] of the first-order structures $\mathfrak{A}$ and $\mathfrak{B}$ (written in an arbitrary vocabulary), where $\mathcal{L}_{\infty\omega}$ is an infinitary logic, otherwise like **FO**, but allowing disjunctions and conjunctions to be formed out of sets of formulae with *arbitrary* cardinality.[3]

3.1.2.2. Bisimilarity up to a modal depth

With each formula φ of $\mathbf{ML}[k]$, we associate by recursion a natural number, $md(\varphi)$, to be called the *modal depth* of φ:

- $md(p) = 0 = md(\neg p)$.
- $md(\varphi \vee \psi) = max\{md(\varphi), md(\psi)\} = md(\varphi \wedge \psi)$.
- $md(\Diamond_i(\varphi)) = md(\varphi) + 1 = md(\Box_i(\varphi))$.

The notion of modal depth of an $\mathbf{ML}[k]$ formula is an exact analogue of the usual notion of quantifier rank in first-order logic.[4]

To help formulate a criterion that characterizes $\mathbf{ML}[k]$ equivalence, we introduce a doubly relativized notion of bisimilarity: that of (n, l)-bisimilarity, where n indicates intuitively that n pairs of elements can be chosen from the domains of (n, l)-bisimilar modal structures so that these pairs satisfy the zigzag conditions (2) and (3) in the definition of bisimulation proper,

[2] I.e. equivalence relative to $\mathcal{L}_{\infty\omega}$ *sentences*.

[3] This result is known as *Karp's Theorem*. For game-theoretical characterizations of **FO** and $\mathcal{L}_{\infty\omega}$, see e.g. Ebbinghaus & Flum (1999), or Hodges (1997 [c]).

[4] For quantifier rank, see Definition 3.3.6 below.

and l is the size of a fixed set of propositional atoms relative to which *Atomic Harmony* prevails.

We take the class **prop** (of a cardinality $\kappa \le \omega$) itself to always have a canonical enumeration of the form

$$\mathbf{prop} = \langle p_i : i < \kappa \rangle,$$

and given a countable ordinal $l \le \kappa$, the set $\mathbf{prop}(l)$ is then determined by the initial segment of the enumeration $\langle p_i : i < \kappa \rangle$ corresponding to l:

$$\mathbf{prop}(l) = \langle p_i : i < l \rangle.$$

In particular if $\kappa = Card(\mathbf{prop})$, then $\mathbf{prop}(\kappa) = \mathbf{prop}$. The relation of (n, l)-bisimilarity is defined as follows:

Definition 3.1.5 *Let $\mathcal{M} = (D, R_0, \ldots, R_{k-1}, \mathfrak{h})$ and $\mathcal{N} = (D', R'_0, \ldots, R'_{k-1}, \mathfrak{h}')$ be k-ary modal structures, and let $a \in D$, $b \in D'$. If $l \le Card(\mathbf{prop})$ and n is a positive integer, an (n, l)-bisimulation between the pointed modal structures $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ is a decreasing sequence of binary relations in $D \times D'$,*

$$\equiv_0 \supseteq \ldots \supseteq \equiv_n,$$

satisfying the following conditions (1)*,* (2)*,* (3) *and* (4)*.*

(1) *Initial co-ordination: $a \equiv_n b$.*

(2) *Atomic harmony on $\mathbf{prop}(l)$: $d \equiv_0 d' \Longrightarrow$*

for all $p \in \mathbf{prop}(l)$: $d \in \mathfrak{h}(p) \Longleftrightarrow d' \in \mathfrak{h}'(p)$.

(3) *Zigzag Forwards: for all $i < k$ and all $j + 1 \le n$:*

$$d \equiv_{j+1} d' \text{ and } R_i(d, c) \Longrightarrow \exists c' (R'_i(d', c') \text{ and } c \equiv_j c').$$

(4) *Zigzag Backwards: for all $i < k$ and all $j + 1 \leq n$:*

$$d \equiv_{j+1} d' \textit{ and } R_i'(d', c') \Longrightarrow \exists c(R_i(d, c) \textit{ and } c \equiv_j c').$$

Observe that since $\equiv_0$ is required to contain all $\equiv_i$ with $i \leq n$, (2) suffices for establishing *Atomic Harmony* at all 'levels':

$0 \leq i \leq n$ **and** $d \equiv_i d' \Longrightarrow$

d and d' agree on all propositional atoms from $\mathbf{prop}(l)$.

If a sequence $(\equiv_j)_{j \leq n}$ is an (n, l)-bisimulation between $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$, we may write

$$(\equiv_j)_{j \leq n} : (\mathcal{M}, a) \rightleftarrows_{n,l} (\mathcal{N}, b),$$

and say that "$(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ are (n, l)-*bisimilar via* $(\equiv_j)_{j \leq n}$", or "$(\equiv_j)_{j \leq n}$ *establishes* the (n, l)-*bisimilarity* between $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$".

By stipulation we say that structures $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ are (n, l)-*equivalent*, if they satisfy exactly the same $\mathbf{ML}[k]$ formulae of modal depth at most n involving only propositional atoms from $\mathbf{prop}(l)$.

Now we have, first of all, that (n, l)-bisimilarity characterizes (n, l)-equivalence:

Proposition 3.1.6 *Let $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ be arbitrary pointed k-ary modal structures. For all $n < \omega$ and for all $l < Card(\mathbf{prop})$ we have: $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ are (n, l)-equivalent if and only if $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ are (n, l)-bisimilar.*

Proof. Cf. *Proposition* 2.31 of Blackburn, de Rijke & Venema (2002), p. 75. ■

All $\mathbf{ML}[k]$ formulae have a finite modal depth, and involve only a finite number of propositional atoms. So structures

$(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ are $\mathbf{ML}[k]$ equivalent iff for all natural numbers n and for all natural numbers l, $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ are (n, l)-equivalent. But then it follows by Proposition 3.1.6 that in fact the $\mathbf{ML}[k]$ equivalence itself between $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ is characterized by the condition that for all $n, l < \omega$, $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ are (n, l)-bisimilar. This relation between pointed modal structures might be termed *bisimilarity in the finite*, whence the characterization result becomes expressible as:

Corollary 3.1.7 *Let $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ be arbitrary pointed k-ary modal structures. We have: $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ are $\mathbf{ML}[k]$ equivalent if and only if $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ are bisimilar in the finite.* ■

It was already suggested at the end of the previous subsection that bisimilarity proper is a needlessly coarse-grained tool for simply establishing $\mathbf{ML}[k]$ equivalence of given pointed modal structures $(\mathcal{M}, d)$ and $(\mathcal{N}, d')$: the existence of a bisimulation between these structures is a sufficient but not necessary condition for $\mathbf{ML}[k]$ equivalence of $(\mathcal{M}, d)$ and $(\mathcal{N}, d')$. Now we are in possession of the requisite tools for in effect showing that this is so. Consider the structures depicted in Figure 8 below.[5]

[5] Blackburn, de Rijke & Venema (2002, pp. 68-9) employ these structures to show that modal equivalence does not imply bisimilarity.

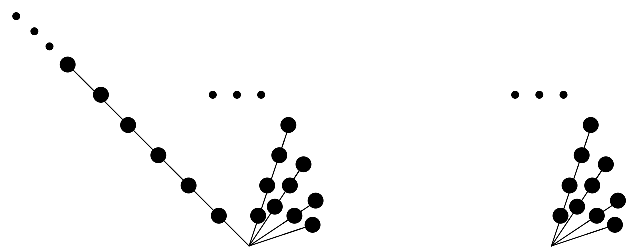

FIGURE 8

Call the structure on the left $\mathcal{M}$, and the one on the right $\mathcal{N}$. Their domains are subsets of $(\omega + 1) \times \omega$:

- $dom(\mathcal{N}) = \{(n_1, n_2) : n_1 = n_2 = 0 \vee 1 \leq n_2 \leq n_1 < \omega\}$
- $dom(\mathcal{M}) = dom(\mathcal{N}) \cup \{(\omega, n) : n \geq 1\}$.

The accessibility relations R of $\mathcal{M}$ and R' of $\mathcal{N}$ are:

- $R' = \{(\langle 0, 0\rangle, \langle b, 1\rangle) : b \geq 1\} \cup$
$$\{(\langle b, n\rangle, \langle b, n+1\rangle) : b \geq 1, 1 \leq n \leq b-1\}$$
- $R = R' \cup \{(\langle 0, 0\rangle, \langle \omega, 1\rangle)\} \cup \{(\langle \omega, n\rangle, \langle \omega, n+1\rangle) : n \geq 1\}$.

All propositional atoms from a given set **prop** are assumed false at all points in both structures. The first co-ordinate (b) in a point $\langle b, n\rangle$ of a domain indicates a *branch*, and the second (n) indicates the distance of this point $\langle b, n\rangle$ from the root $\langle 0, 0\rangle$ measured in steps from the root to the point $\langle b, n\rangle$ along the accessibility relation. In $\mathcal{M}$ the branch ω is infinitely high; otherwise $\mathcal{M}$ is like $\mathcal{N}$ and has one branch for each finite height.

Now first of all, **(i)** the pointed modal structures $(\mathcal{M}, \langle 0, 0\rangle)$ and $(\mathcal{N}, \langle 0, 0\rangle)$ are *not* bisimilar. Assume for contradiction that $\equiv$ is a bisimulation between them. Hence there is a point $\langle b, 1\rangle$

$\in dom(\mathcal{N})$ such that $\langle\omega, 1\rangle \equiv \langle b, 1\rangle$. But b is of some finite height $h \geq 1$. Then there is only a finite path $\langle b, 1\rangle R' \dots R'\langle b, h\rangle$ forwards along R', but an infinite path $\langle\omega, 1\rangle R \dots R\langle\omega, h\rangle R \dots$ forwards along R – a contradiction. On the other hand, **(ii)** the structures are bisimilar in the finite. This is established by constructing an (n, l)-bisimulation between these structures for all $n, l < \omega$. Since the structures are homogeneous relative to the atoms, the parameter l has no effect. It is otherwise completely trivial how to build up a sequence witnessing the (n, l)-bisimilarity for a given n except for the case where points from the infinite branch in $\mathcal{M}$ must be correlated to some elements in $\mathcal{N}$. But with the point $\langle\omega, 1\rangle$ correlate $\langle n, 1\rangle$ (where n is the required length of the bisimulation sequence); hence for the remaining at most $n - 1$ zigzag steps, the required extensions of the path from $\langle n, 1\rangle$ *can* be constructed if successors to $\langle\omega, 1\rangle$ are chosen in the opposite structure.

From (ii) it follows by Corollary 3.1.7 that $(\mathcal{M}, \langle 0, 0\rangle)$ and $(\mathcal{N}, \langle 0, 0\rangle)$ are **ML** [1] equivalent. Since by (i) these structures are not bisimilar, we have justified the claim that bisimilarity is not a necessary condition for **ML**[1] equivalence. ■

Let us then move on to consider another example, which will serve to establish a number of facts about modal equivalences and the variants of the notion of bisimulation considered here.

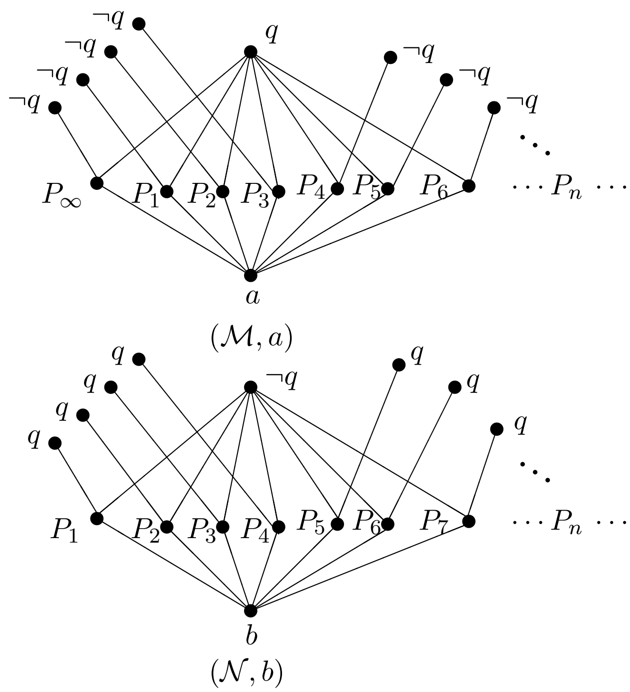

FIGURE 9

In Figure 9 above, call the upper structure $\mathcal{M}$ and the lower $\mathcal{N}$. They have a common frame (D, R), with the domain

$$D = \{(0,0)\} \cup \{(c,2)\} \cup \{(b,h) : b \geq 1 \text{ and } h := 1,2\};$$

and the accessibility relation

$$R = \{(\langle 0,0\rangle, \langle b,1\rangle) : b \geq 1\} \cup \{(\langle b,1\rangle, \langle b,2\rangle) : b \geq 1\} \cup$$

$$\{(\langle b,1\rangle, \langle c,2\rangle) : b \geq 1\}.$$

So $(c,2)$ in particular is the *common* second-generation successor of all first-generation successors of $(0,0)$. Let $\langle p_i : i < \omega\rangle$ be an enumeration of the class **prop**, which we assume here to be countably infinite.[6] (Observe that the atom q referred to in Figure 9 is by definition among these p_i.) Then define assignments $\mathfrak{h}$ and $\mathfrak{h}'$ for $\mathcal{M}$ *resp.* $\mathcal{N}$ as follows.

The assignment $\mathfrak{h}$ for $\mathcal{M}$:

- (0,0) and all points from $\{(b, 2) : b \geq 1\}$ make all atoms false according to $\mathfrak{h}$.
- (c,2) makes q true according to $\mathfrak{h}$, but all other atoms false.
- For all $n \geq 1$ and $p \in$ **prop**: $(n + 1, 1) \in \mathfrak{h}(p) \iff p \in \{p_0, p_2, \ldots, p_{2n}\}$.
- For all $p \in$ **prop**: $(1, 1) \in \mathfrak{h}(p) \iff p \in \{p_0, p_2, \ldots, p_{2n}, \ldots\} = \{p_{2n} : n < \omega\}$.

The assignment $\mathfrak{h}'$ for $\mathcal{N}$:

- $(0, 0)$ and $(c, 2)$ make all atoms false according to $\mathfrak{h}'$.
- All points from $\{(b, 2) : b \geq 1\}$ make q true, but all other atoms false according to $\mathfrak{h}'$.
- For all $n \geq 1$ and $p \in$ **prop**: $(n, 1) \in \mathfrak{h}'(p) \iff p \in \{p_0, p_2, \ldots, p_{2n}\}$.

[6] There is nothing peculiar in taking **prop** to be infinite. E.g. Blackburn, de Rijke & Venema (2002, p. 10) regard, as a standard assumption, that **prop** is countably infinite.

Hence for any $n \geq 1$, the point $(n+1, 1)$ in $\mathcal{M}$ *resp.* the point $(n, 1)$ in $\mathcal{N}$ satisfies precisely the atoms whose index in the enumeration of **prop** is zero or even, and not greater than $2n$. Notice that these points $(n+1, 1)$ *resp.* $(n, 1)$ are indicated in Figure 9 by the label P_n. And the point $(1, 1)$ in $\mathcal{M}$ makes true exactly the atoms whose index is not odd, hence an infinite number of atoms. This point is labeled with the symbol P_∞.

For simplicity write $w = (0, 0) = v$. We can learn a good number of facts by considering the two structures $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$.

Observation 3.1.8 *The structures $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ are not (n, ω)-bisimilar for any $n \geq 1$.*

REASON: *Notice first that $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ are not $(1, \omega)$-bisimilar. For consider the successor $x := (1, 1)$ of w in $\mathcal{M}$, labeled in the picture with the symbol P_∞. Infinitely many propositional atoms are true at x. On the other hand, each successor of v from $\mathcal{N}$ only makes true a finite number of such atoms. So it is not possible to correlate x with any successor y of v in such a way that x and y would make true precisely the same atoms. A fortiori, then, $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ are not (n, ω)-bisimilar for any $n \geq 1$.* ■

Observation 3.1.9 *For all $n, l < \omega$: the structures $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ are (n, l)-bisimilar.*

REASON: *Let $n, l < \omega$ be arbitrary. We describe an (n, l)-bisimulation between $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ as follows. Let $T_n = \{(w, v)\}$. If $n \geq 1$, to form T_{n-1}, associate with the successor of w [resp. v] labeled with P_m ($m < \omega$) the successor of v [resp. w] with this same label. With the successor of w whose label is P_∞, correlate any successor of v whose label P_m satisfies $m \geq l$, e.g. the label P_l. (Because only the atoms $\{p_0, \ldots, p_{l-1}\}$ are considered, the chosen successor will make precisely the same of these* *true as does the successor of w with the label P_∞.) If $n \geq 2$, to define T_{n-2}, correlate all elements of $\mathcal{N}$ that make q true and are found among the second-generation successors of v, with the unique second-generation successor of w that makes q true, i.e. $(c, 2)$. Similarly, correlate the unique second-generation successor $(c, 2)$ of v making q false with all second-generation successors of w that make q false in $\mathcal{M}$. (Effect such a correlation also if l is too small to make q included in the set of atoms considered.) No further T_i can be defined. As the structures $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ make by their construction precisely the same atoms true at their roots, it is by its construction clear that the sequence (T_n, T_{n-1}, T_{n-2}) is a (n, l)-bisimulation between $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$.* ■

Directly by the above two observations we have:

Fact 3.1.10 *The condition "(n, l)-bisimilar for all $n, l < \omega$" does not imply the condition "(n, ω)-bisimilar for some $n \geq 1$".*

Notice the contrariness: even if for *all* $n, l < \omega$ two structures are (n, l)-bisimilar, it does not follow that they were (n, ω)-bisimilar for *any* $n \geq 1$.

Observation 3.1.11 *For all $n < \omega$, the structures $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ make true precisely the same* **ML**[1] *formulae of modal depth at most n involving arbitrary propositional atoms.*

REASON: *By Proposition 3.1.6, the statement follows, if for all $n, l < \omega$, $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ are (n, l)-bisimilar. But by Observation 3.1.9 this is the case.* ■

Fact 3.1.12 *Let us say that pointed k-ary modal structures $(\mathcal{M}, d)$ and $(\mathcal{M}', d')$ are n-equivalent, if they satisfy exactly the same* **ML**[k] *formulae of modal depth at most n involving arbitrary propositional atoms. Then we have:*

- *The condition "*$(\mathcal{M}, d)$ *and* $(\mathcal{M}', d')$ *are n-equivalent" does not imply that* $(\mathcal{M}, d)$ *and* $(\mathcal{M}', d')$ *are* (n, ω)*-bisimilar.*
- $(\mathcal{M}, d)$ *and* $(\mathcal{M}', d')$ *are n-equivalent if and only if for all* $l < \omega$, $(\mathcal{M}, d)$ *and* $(\mathcal{M}', d')$ *are* (n, l)*-bisimilar.*

The latter equivalence is by Proposition 3.1.6 and the fact that any **ML**$[k]$ formula involves only a finite number of propositional atoms. This is in contrast to the failure of the implication from n-equivalence to (n, ω)-bisimilarity, which is by Observation 3.1.11 and Observation 3.1.8 − both concerning unary modal structures. These structures can, however, easily be turned into arbitrary k-ary structures which satisfy analogues of Observations 3.1.8 and 3.1.11 by adding to both structures $k - 1$ instances of, say, the relation $\{(\langle 0, 0\rangle, \langle 0, 0\rangle)\}$.

Fact 3.1.13 *The structures* $(\mathcal{M}, w)$ *and* $(\mathcal{N}, v)$ *are not* **IFML**[1] *equivalent.*

Reason: The **IFML**[1] formula $\chi = \Box_1 \Diamond_2/\{1\} q$ is true in $\mathcal{M}$ at w, but is non-determined in $\mathcal{N}$ at v. In the game $\mathrm{G}(\chi, \mathcal{M}, w)$ *Héloïse*'s w.s. consists of always choosing for $\Diamond_2/\{1\}$ the unique second-generation successor $(c, 2)$ of w, which satisfies the atom q. By contrast, in the game $\mathrm{G}(\chi, \mathcal{N}, v)$ she cannot have a w.s., as there is no second-generation successor of v that would make q true *and* would be a *common* successor to all first-generation successors of v. However, since for all first-generation successors $(b, 1)$ of v there is some second-generation successor of v making q true, namely $(b, 2)$, neither does *Abélard* have a w.s. in $\mathrm{G}(\chi, \mathcal{N}, v)$. ∎

The fact just noted already tells us at this point that **IFML**[1] is capable of distinguishing pointed modal structures that are not distinguishable in terms of **ML**[1]. (The

ML[1] equivalence of the structures $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ follows from Observation 3.1.11.) This means that the expressive power of **IFML**[1] is strictly greater than that of **ML**[1], since **ML**[1] is trivially embeddable in **IFML**[1].

The issue of expressive power of IF modal logic will occupy us throughout *Chapter* 3. However, the observation just made in Fact 3.1.12 is of special interest, in that the structures $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ are *not* bisimilar. They are not even (n, ω)-bisimilar for any $n \geq 1$. So from the above example we know specifically that

- **IFML**[1] can distinguish **ML**[1] equivalent, non-bisimilar modal structures.

In fact, we even know that

- **IFML**[1] can distinguish **ML**[1] equivalent modal structures that are not (n, ω)-bisimilar for any $n \geq 1$.

3.2 IF Modal Logic and First-Order Logic

We now move on to show that **IFML**$[k]$ can be translated into usual first-order logic (**FO**). By contrast, in *Section* 3.3 an **EIFML**$[k]$ modal logic will be defined in which modal operators can be independent of conjunctions and disjunctions; this logic can then be shown not to be translatable into **FO**.

3.2.1 Translatability of IFML into FO

It is said that an **IFML**$[k]$ formula φ can be *translated* into **FO** if there exists a first-order formula ψ_φ of one free variable, written in appropriate vocabulary, such that over all pointed k-ary modal structures $(\mathcal{M}, d)$, the relation

$$\mathcal{M} \models \varphi[d] \iff \langle \mathcal{M}^{\mathbf{FO}}, d\rangle \models \psi_\varphi(x_0)$$

holds, where $\mathcal{M}^{\mathbf{FO}}$ is the first-order structure corresponding to $\mathcal{M}$ (as defined in *Sect.* 2.1). It will be shown that $\mathbf{IFML}[k]$ − interpreted by its *uniformity* interpretation − is indeed translatable into **FO**. Hence in particular the proper resources of IF first-order logic are *not* needed in the translation.

Our strategy in showing this translatability is as follows. We first prove a lemma which will considerably simplify the treatment of $\mathbf{IFML}[k]$ formulae of the form

$$O_1 \ldots O_{n-1}(O_n/W)\varphi,$$

where W is an arbitrary subset of the interval $[1, n-1]$. The lemma says that the set W can in fact *salva veritate* − or, *salva existentia strategemam vincentem Helenae* − be assumed to be the continuous segment $[K+1, n-1]$ of the interval $[1, n-1]$, where $K = max([0, n-1] \backslash W)$.

We then prove two lemmata, providing second-order and first-order translations respectively to $\mathbf{IFML}[k]$ formulae of the form

$$O_1 \ldots O_{n-1}(O_n/\{1, \ldots, n-1\})\varphi,$$

whereafter the theorem we are looking for follows easily.

Lemma 3.2.1 *Let* $W \subseteq [1, n-1]$*, and let* $U = [K+1, n-1] \subseteq W$*, where* $K \notin W$ $(0 \leq K \leq n-1)$*. For all pointed* k*-ary modal structures* ($\mathcal{M}$,d) *and all* $\varphi \in \mathbf{IFML}[k]$ *we have:*

$$\textit{Héloïse has a w.s. in } G(O_1 \ldots O_{n-1}(O_n/W)\varphi, \mathcal{M}, d) \Longleftrightarrow$$

$$\textit{Héloïse has a w.s. in } G(O_1 \ldots O_{n-1}(O_n/U)\varphi, \mathcal{M}, d).$$

Proof. The set $W \subseteq [1, n-1]$ is uniquely presented as a disjoint union $W = V' \cup U$, where $K \notin W$ and $U = [K+1, n-1]$. The statement of the lemma holds trivially if $O_n = \Box_{i,n}$. Namely, we have indeed for *any* $V \subseteq [1, n-1]$

that $\exists$ has a w.s. in $\mathrm{G}(O_1 \ldots O_{n-1}(O_n/V)\varphi, \mathcal{M}, d) \Longleftrightarrow \exists$ has a w.s. in $\mathrm{G}(O_1 \ldots O_{n-1}(O_n/\varnothing)\varphi, \mathcal{M}, d)$. For in either game $\exists$'s eventual w.s. leads to a win irrespective of the move $\forall$ makes for O_n, and in both games $\forall$ has the same moves he can make for O_n.

Consider then the case $O_n = \Diamond_{i,n}$. The implication from left to right is trivial: whatever can be done in greater ignorance can be done in lesser ignorance. So consider the converse implication. Write

$$\Gamma := G(O_1 \ldots O_{n-1}(O_n/U)\varphi, \mathcal{M}, d);$$

$$\Gamma^* := G(O_1 \ldots O_{n-1}(O_n/W)\varphi, \mathcal{M}, d).$$

Assume that there exists a w.s. (f) for $\exists$ in Γ, and define a strategy f^* for her in Γ^* as follows. Let I_Γ be the set of the information sets corresponding to (O_n/U) in Γ. Then a particular information set in I_Γ is determined by a particular distribution of moves for the operators $O_1 \ldots O_K$: if I_1 and I_2 are information sets in I_Γ, then

$$I_1 = I_2 \Longleftrightarrow$$

for all $(d, s_1, \ldots, s_{n-1}) \in I_1$ and all $(d, s'_1, \ldots, s'_{n-1}) \in I_2$:

$$(s_1, \ldots, s_K) = (s'_1, \ldots, s'_K).$$

Further, let I_{Γ^*} be the set of the information sets corresponding to (O_n/W) in Γ^*, and let

$$g_f : I \in I_{\Gamma^*} \longmapsto g_f(I) = (d, s_1, \ldots, s_{n-1}) \in I$$

be a choice function, associating each information set I from I_{Γ^*} with such a history $g_f(I)$ from this set I that is the result of using f against some sequence of moves by $\forall$. (If at least one set $I \in I_{\Gamma^*}$ contains an infinite number of histories constructible using f against some sequence of moves by $\forall$, this maneuver requires, in general, *Axiom of Choice*.) Then to define f^*, proceed as follows. If $h = (d, s_1^*, \ldots, s_m^*)$ is a history of Γ^* with $P(h) = \exists$, put:

$$f^*(h) := \begin{cases} f(h) \ \text{ if the length of } h < K \\ f(d, g_f(I)[1], \ldots, g_f(I)[K], s_{K+1}^*, \ldots, s_m^*) \\ \qquad \text{if } K \leq m < n-1 \text{ and } I \in I_{\Gamma^*} \text{ is} \\ \qquad\quad \text{the information set determined} \\ \qquad\quad \text{by the sequence } (d, s_1^*, \ldots, s_K^*) \\ f(g_f(I)) \ \text{ if } h \in I \\ f(h) \ \text{ if the length of } h \geq n \end{cases}$$

We must show that f^*, thus defined, in fact legally extends all such histories from any given $I \in I_{\Gamma^*}$ in whose construction $\exists$ has applied f^*.

Fix a set $I \in I_{\Gamma^*}$, and let I' be the information set from I_Γ determined by the prefix of the length K of the history $g_f(I) = (d, s_1, \ldots, s_{n-1})$. Then let $h^* = (d, s_1^*, \ldots, s_{n-1}^*) \in I$ be an arbitrary history (of the length $n-1$, of the game Γ^*) in whose construction $\exists$ has applied f^*. (In particular, h^* need not be in I'.) Assume for contradiction that $f(g_f(I))$ does *not* legally extend h^*. Now $(s_K^*, \ldots, s_{n-1}^*)$ in particular constitutes a transition along the accessibility relations associated respectively with the operators $O_K, \ldots, O_{n-1}$, whereas by assumption $(s_{n-1}^*, f(g_f(I))) \notin R$, where R is the relation associated with O_n.

But as $I' \subseteq I$, we have $s_K^* = s_K$ $(= g_f(I)[K])$. So it is even possible to extend the history $(d, s_1, \ldots, s_K)$ with $(s_{K+1}^*, \ldots, s_{n-1}^*)$ by moving similarly along the same relations, hence producing a history belonging to the set I', in whose construction $\exists$ has employed f. But as $f(g_f(I))$ does not legally extend this history, this move after all does *not* legally extend all histories from I' that are constructed so that $\exists$ follows the strategy f. But this is a contradiction, as $I' \in I_\Gamma$, and f is a w.s. for $\exists$ in Γ.

As h^* was assumed to be arbitrary, we may conclude that in fact $f(g_f(I))$ legally extends all histories from I. ∎

The following lemma on a *second-order* translation of an **IFML**$[k]$ formula of the form $O_1 \dots O_{n-1}(\Diamond_n/[1, n-1])\psi$ is a straightforward consequence of the semantics of **IFML**$[k]$.

Lemma 3.2.2 *Any* **IFML**$[k]$ *formula* φ *of the form*

$$O_1 \dots O_{n-1}(\Diamond_n/[1, n-1])\psi$$

is translated into second-order logic by the formula $\Theta_\varphi(x_0) :=$

$$\exists f_1 \dots \exists f_m \exists f_n^{unif} \forall x_{i_1} \dots \forall x_{i_k} (R_{l_1}(x_0, x_1)\ o_1 \dots$$

$$(R_{l_{n-1}}(x_{n-2}, x_{n-1})\ o_{n-1} R_{l_n}(x_{n-1}, f_n(x_1, \dots, x_{n-1})) \wedge$$

$$ST_{x/f_n^{unif}(x_1,\dots,x_{n-1})}(\psi)),$$

where:

(i) $m + k = n - 1$.

(ii) $i_1 < \dots < i_k$ *are the indices of the* $\Box$*-operators in* $O_1 \dots O_{n-1}$.

(iii) *if* $j_1 < \dots < j_m$ *are the indices of the* $\Diamond$*-operators in* $O_1 \dots O_{n-1}$*, then*

$$x_{j_1} = f_1(x_1, \dots, x_{j_1 - 1}), \dots, x_{j_m} = f_m(x_1, \dots, x_{j_m - 1}).$$

(iv) *the quantifier* $\exists f_n^{unif}$ *asserts the existence of a* $\{x_1, \dots, x_{n-1}\}$*-uniform function, i.e. a constant function.*

(v) *for* $i \in \{1, \dots, n-1\}$, $o_i := \begin{cases} \wedge, \text{ if } O_i = \Diamond_i \\ \rightarrow, \text{ if } O_i = \Box_i \end{cases}$

(vi) *for each* $i \in \{1, \dots, n-1\}$, $l_i < k$ *identifies the accessibility relation associated with the operator* O_i.

Proof. Let φ be an arbitrary formula of the form $O_1 \dots O_{n-1}(\Diamond_n/[1, n-1])\psi$, and let $\mathcal{M}$ and d be arbitrary. We show that

$$\mathcal{M} \models \varphi\,[d] \quad \Leftrightarrow \langle \mathcal{M}^{\mathbf{FO}}, d\rangle \models \Theta_\varphi(x_0).$$

($\Longrightarrow$) Supposing a w.s. f for $\exists$ in $G(\varphi, \mathcal{M}, d)$ is given, define witness functions $f_1, \dots, f_m, f_n^{unif}$ for the quantifiers in the block $\exists f_1 \dots \exists f_m \exists f_n^{unif}$ by

- $f_k(x_1, \dots, x_{j_k-1}) := f(x_1, \dots, x_{j_k-1)} \quad (k := \{1, \dots, m\})$
- $f_n^{unif}(x_1, \dots, x_{n-1}) := f(x_1, \dots, x_{n-1})$

Hence in particular f_n^{unif} becomes a constant function, as f is $[1, n-1]$-uniform. Because f is a w.s. for $\exists$ in $G(\varphi, \mathcal{M}, d)$, we further have that $\langle \mathcal{M}^{\mathbf{FO}}, c\rangle \models ST_{x/x_n}(\psi)$, where c is the constant value of f_n^{unif}. Hence clearly $\langle \mathcal{M}^{\mathbf{FO}}, d\rangle \models \Theta_\varphi(x_0)$.

($\Longleftarrow$) Conversely, if we have $\langle \mathcal{M}^{\mathbf{FO}}, d\rangle \models \Theta_\varphi(x_0)$, let $f_1, \dots, f_m, f_n^{unif}$ be witness functions for the quantifiers in $\exists f_1 \dots \exists f_m \exists f_n^{unif}$, and define a strategy f of $\exists$ in $G(\varphi, \mathcal{M}, d)$ by putting:

- $f(x_1, \dots, x_{j_k-1}) := f_j(x_1, \dots, x_{j_k-1)} \quad (k := \{1, \dots, m\})$
- $f(x_1, \dots, x_{n-1}) := f_n^{unif}(x_1, \dots, x_{n-1})$

Hence f in particular will give a constant value c on histories of length $n-1$, and for this value c we have that $\mathcal{M} \models \psi[c]$. So we have that f is a w.s. for $\exists$ in $G(\varphi, \mathcal{M}, d)$. ■

Making use of the above lemma we immediately find a *first-order* translation to IF modal-logical formulae of the form $O_1 \dots O_{n-1}(\Diamond_n/[1, n-1])\psi$.

Lemma 3.2.3 *Any* $\mathbf{IFML}[k]$ *formula* φ *of the form*

$$O_1 \dots O_{n-1}(\Diamond_n/[1, n-1])\psi$$

is translated into **FO** *by the formula* $\theta_\varphi(x_0) :=$

$$\exists x_n Q x_1 \dots Q x_{n-1}(R_{l_1}(x_0, x_1)\, o_1 \dots$$

$$(R_{l_{n-1}}(x_{n-2}, x_{n-1})\, o_{n-1} R_{l_n}(x_{n-1}, x_n) \wedge ST_{x/x_n}(\psi)),$$

where:

$$(Qx_i, o_i) := \begin{cases} (\forall x_i, \rightarrow_i) & if\ O_i = \Box_i, \\ (\exists x_i, \wedge_i) & if\ O_i = \Diamond_i \end{cases}$$

Proof. Let φ be an arbitrary formula of the form

$$O_1 \dots O_{n-1}(\Diamond_n/[1, n-1])\psi;$$

and let $\Theta_\varphi(x_0) :=$

$$\exists f_1 \dots \exists f_m \exists f_n^{unif} \forall x_{i_1} \dots \forall x_{i_k}(R_{l_1}(x_0, x_1)\ o_1 \dots$$

$$(R_{l_{n-1}}(x_{n-2}, x_{n-1}) o_{n-1} R_{l_n}(x_{n-1}, f_n(x_1, \dots, x_{n-1})) \wedge$$

$$ST_{x/f_n^{unif}(x_1, \dots, x_{n-1})}(\psi))$$

be its second-order translation provided by Lemma 3.2.2 above. Let $\mathcal{M}$ and d be arbitrary.

Now for every $k \in \{1, \dots, m\}$, a witness function f_k of the quantifier $\exists f_k$ in the prefix of $\Theta_\varphi(x_0)$ takes as its argument the sequence of values of the variables $x_1, \dots, x_{j_k-1}$, where j_k is the index of the k-th $\Diamond$-operator of the block $O_1 \dots O_{n-1}$. Hence each argument of f_k contains a component corresponding to each operator in the block $O_1 \dots O_{j_k-1}$, whence each existential second-order quantifier $\exists f_k$ has the semantic force of an existential first-order quantifier being in the syntactical scope of precisely the first-order quantifiers corresponding to the modal operators $O_1 \dots O_{j_k-1}$. Further, as f_n^{unif} is a constant function, this means that $\Theta_\varphi(x_0)$ is equivalent to the Skolem form of the first-order formula $\theta_\varphi(x_0)$. Hence these two formulae in particular are logically equivalent, and therefore $\theta_\varphi(x_0)$ is a first-order translation of φ. ∎

Using Lemma 3.2.3 proven above, we obtain the theorem about the translatability of **IFML**$[k]$ into **FO**.

Theorem 3.2.4 *Given a number $k < \omega$ and a class* **prop**, *let a vocabulary τ be defined as explained in Section* 2.1. *Then for all $\chi \in$* **IFML**$[k]$ *there is $\theta_\chi(x_0) \in$* **FO**$[\tau]$ *with precisely one free variable, x_0, such that for all k-ary modal structures $\mathcal{M} = (D, R_0, \dots, R_{k-1}, \mathfrak{h})$ and for all $d \in dom(\mathcal{M})$:*

$$\mathcal{M} \models^+ \chi[d] \Longleftrightarrow \langle \mathcal{M}^{\mathbf{FO}}, d\rangle \models \theta_\chi(x_0).$$

Proof. Let $\chi \in$ **IFML**$[k]$ be arbitrary. If in particular $\chi \in$ **ML**$[k]$, for $\theta_\chi(x_0)$ we may take the standard translation $ST_{x_0}(\chi)$ of χ. (Cf. Proposition 2.1.2.) Further, as trivially for all $\mathcal{M}$ and d,

$$\mathcal{M} \models^+ O_1 \dots O_{n-1}(\Box_n/W)[d] \Longleftrightarrow \mathcal{M} \models^+ O_1 \dots O_{n-1}\Box_n\varphi[d],$$

we have that if χ is of the form $O_1 \dots O_{n-1}(\Box_n/W)$, it has a first-order translation, namely $ST_{x_0}(O_1 \dots O_{n-1}\Box_n\varphi)$. So assume that χ is of the form

$$O_1 \dots O_{n-1}(\Diamond_n/W)\varphi.$$

By Lemma 3.2.1, we may indeed assume that $W = \{1, \dots, n-1\}$. For W can, in any case, be written as a disjoint union $W = V' \cup [K+1, n-1]$ for $K := max([0, n-1] \backslash W)$. And assuming that we already have a first-order translation $\theta_\psi(x_0)$ for the **IFML**$[k]$ formula

$$\psi := O_{K+1} \dots O_{n-1}(\Diamond_n/[K+1, n-1])\varphi,$$

we may by Lemma 3.2.1 take $\theta_\chi(x_0)$ to be the first-order translation of the **IFML**$[k]$ formula $O_1 \dots O_K \psi$. And clearly the first-order formula

$$(Qx_1) \dots (Qx_K)(R_{i_1}(x_0, x_1)\ o_1 \dots (R_{i_K}(x_{K-1}, x_K)\ o_K$$
$$\theta_\psi[x_0/x_K]))$$

is such a translation. [7]

It remains to be shown that if χ is of the form

$$O_1 \dots O_{n-1}(\Diamond_n/[1, n-1])\varphi,$$

it can be translated into **FO**. But that this is so, is precisely stated by Lemma 3.2.3. This observation completes the proof. ∎

3.2.2 First-order translations of ML and IFML

Let us consider the difference between basic modal logic and IF modal logic by viewing the difference in their respective translations into **FO**.

[7] Here for each $j \in \{1, \dots, K\}$, $i_j < k$ identifies the accessibility relation associated with the operator O_j, and if ϕ is a first-order formula in which x_0 occurs free, and x ia a variable, then the notation "$\phi[x_0/x]$" stands for the result of having first changed, if necessary, variables in $\phi(x_0)$ so that x will be free for x_0 in the resulting formula, and having then substituted x for x_0 in that resulting formula. Further, $Qx_i := \exists x_i$ and $o_i := \wedge$, if $O_i = \Diamond_i$; and $Qx_i := \forall x_i$ and $o_i := \rightarrow$, if $O_i = \Box_i$.

Definition 3.2.5 *An* **FO** *formula* $\beta(x, x_{i_1}, \ldots, x_{i_n}; \varphi[x_{i_n}]) :=$

$$(R_{j_1}(x, x_{i_1}) o_{j_1} (R_{j_2}(x_{i_1}, x_{i_2}) \ldots o_{j_{n-2}}$$
$$(R_{j_{n-1}}(x_{i_{n-2}}, x_{i_{n-1}})\, o_{j_{n-1}}\, (R_{j_n}(x_{i_{n-1}}, x_{i_n})\, o_{j_n} \varphi[x_{i_n}])) \ldots))$$

is said to be a transition if the R_{j_i} *are binary relations, the* $\boldsymbol{o}_{j_i} \in \{\wedge, \rightarrow\}$ *and* $\varphi[x_{i_n}]$ *is a standard translation of an* **ML** *formula.*

If $\beta(x, x_1, \ldots, x_n)$ is a transition and π is a permutation of the set $\{x_1, \ldots, x_n\}$, write θ_π for the first-order formula

$$Q_1 x_1 \ldots Q_n x_n \beta(x, \pi(x_1), \ldots, \pi(x_n)).$$

We write '$\Leftrightarrow$' for logical equivalence between **FO** formulae, in the sense of 'satisfied in precisely the same models under the same variable assignments'. An **FO** formula θ_φ is a translation of an **ML** formula φ, if for all $\mathcal{M}, d$ we have $\mathcal{M} \models \varphi[d] \iff \langle \mathcal{M}^{\mathbf{FO}}, d\rangle \models \theta_\varphi(x)$.

The following fact states when a formula of the form θ_π is a translation of an $\mathbf{ML}[k]$ formula:

Fact 3.2.6 *Let* $\beta(x, x_1, \ldots, x_n)$ *be any transition, and let* π *be an arbitrary permutation of the set* $\{x_1, \ldots, x_n\}$. *Then the following conditions (i), (ii) and (iii) are equivalent:*

- **(i)** θ_π *is a translation of an* $\mathbf{ML}[k]$ *formula*
- **(ii)** *there is* $\theta_{\pi'} \in \mathbf{FO}[\tau]$ *such that* $\theta_\pi \Leftrightarrow \theta_{\pi'}$, *and*

  $\pi' = \mathtt{id}_{\{x_1, \ldots, x_n\}}$
- **(iii)** θ_π *is logically equivalent to the* $\sum_1^1[\tau]$ *formula*

  $$\exists f_1 \ldots \exists f_m \forall x_{l_1} \ldots \forall x_{l_k} (x_{j_1} = f_1(x_1, \ldots, x_{j_1 - 1}) \wedge \ldots$$

$$\wedge\, x_{j_m} = f_m(x_1, \ldots, x_{j_m-1}) \to \beta\,(x, x_1, \ldots, x_n)),$$

where $\{x_1, \ldots, x_n\}$ *can be written as the disjoint union*

$$\{x_{l_1}, \ldots, x_{l_k}\} \cup \{x_{j_1}, \ldots, x_{j_m}\}\,,$$

and $l_1 < \ldots < l_k$ *and* $j_1 < \ldots < j_m$*.* ■

That is, in order to be a translation of an $\mathbf{ML}[k]$ formula, the formula

$$Q_1x_1 \ldots Q_nx_n\beta\,(x, \pi(x_1), \ldots, \pi(x_n))$$

must be logically equivalent to the formula

$$Q_1x_1 \ldots Q_nx_n\beta\,(x, x_1, \ldots, x_n)\,.$$

What is distinctive about this **FO** formula is that in it, the order of the variables $x_1, \ldots, x_n$ on the path fixed by the transition $\beta\,(x, x_1, \ldots, x_n)$ is *the same* as the order of quantifiers in the relevant quantifier prefix.

To prove that θ_π's being a translation of an $\mathbf{ML}[k]$ formula implies its being logically equivalent to a formula of the form $\overline{Qx}\beta\,(x, \overline{x})$, it would suffice to show by double induction on the complexity of the formula $\varphi \in \mathbf{ML}[k]$ and on the length of the block $\overline{Qx}$ of quantifiers, that *either*

$$ST_x\,(\varphi) \Leftrightarrow \overline{Qx}\beta\,(x, \overline{x})$$

or there is no transition $\beta\,(x, \pi(\overline{x}))$ such that

$$ST_x\,(\varphi) \Leftrightarrow \overline{Qx}\beta\,(x, \pi(\overline{x}))\,.$$

Other statements involved in the fact are implied by well-known facts about the relation of the $\sum_1^1[\tau]$ fragment of second-order logic and **FO** (Skolem forms of **FO** formulae), the existence of a standard translation to basic modal logic, and the following validities of **FO**:

- $\models (\alpha \wedge \exists x\beta) \leftrightarrow \exists x(\alpha \wedge \beta)$,
  where x does not occur free in α
- $\models (\alpha \rightarrow \forall x\beta) \leftrightarrow \forall x(\alpha \rightarrow \beta)$,
  where x does not occur free in α.

By contrast, the next fact indicates when a formula of the form θ_π translates an **IFML**$[k]$ formula:

Fact 3.2.7 *Let $\beta\,(x, x_1, \ldots, x_n)$ be any transition, and let π be an arbitrary permutation of the set $\{x_1, \ldots, x_n\}$. Then the following conditions (i), (ii) and (iii) are equivalent:*

(i) *θ_π is a translation of an* **IFML**$[k]$ *formula.*

(ii) *there is $\theta_{\pi\prime} \in \mathbf{FO}[\tau]$ such that $\theta_\pi \Leftrightarrow \theta_{\pi'}$, where for some (possibly empty) $[i, j] \subseteq \{1, \ldots, n\}$ with $Q_i = \exists_i$,*

$$\pi'(x_k) = \begin{cases} x_j & \textit{if } k = i \\ x_{k-1} & \textit{if } i < k \leq j \\ x_k & \textit{otherwise} \end{cases}$$

(iii) *θ_π is logically equivalent to the $\sum_1^1[\tau]$ formula*

$$\exists g_1 \ldots \exists g_m \forall x_{l_1} \ldots \forall x_{l_k}(x_{j_1} = g_1(x_1, \ldots, x_{j_1 - 1}) \wedge \ldots$$

$$\wedge\ x_{j_m} = g_m(x_1, \ldots, x_{j_m - 1}) \rightarrow \beta\,(x, x_{i_1}, \ldots, x_{i_n})),$$

where $x_{l_1}, \ldots, x_{l_k}, x_{j_1}, \ldots, x_{j_m}$ are exactly like in Fact 3.2.6*,*

and furthermore the second-order quantifier $\exists g_k$ with

$j_k = j \geq i$, if there is such a quantifier, asserts the

existence of a $[i, j-1]$-uniform function $g_k = f_k^{unif}$. ■

In other words, a first-order translation of an **IFML**$[k]$ formula can differ from a translation of an **ML**$[k]$ formula precisely by having the corresponding permutation $\pi : \{x_1, \dots, x_n\} \to \{x_1, \dots, x_n\}$ involve exactly one 'circle': for some subset $[i, j] \subseteq \{1, \dots, n\}$,

$$\pi(x_i) = x_j, \pi(x_{i+1}) = x_i, \dots, \pi(x_j) = x_{j-1}.$$

For proving the fact, it would again suffice to consider the direction from left to right of the first equivalence, other statements being obvious. First one observes that if

$$\theta_\pi = Q_1 x_1 \dots Q_n x_n \beta \left(x, \pi(x_1), \dots, \pi(x_n); \psi\left[\pi(x_n)\right]\right)$$

is indeed a translation of an **IFML**$[k]$ formula, then such a formula looks like this:

$$O_1 \dots O_n \varphi,$$

where for at most one $i \in \{1, \dots, n\}$, $O_i := (O_i/W)$, otherwise the operators do not involve the slash, and $\varphi \in$ **ML**$[k]$. (The transition β determines the modality types of the relations involved in the block $O_1 \dots O_n$.) But then by Theorem 3.2.4 $O_1 \dots O_n \varphi$ has a translation

$$Q_1 x_1 \dots Q_n x_n \beta \left(x, \pi'(x_1), \dots, \pi'(x_n); ST_{\pi'(x_n)}(\varphi)\right),$$

where π' satisfies the condition mentioned in the consequent of the implication being proven. But then $\theta_{\pi'}$ and θ_π are in fact logically equivalent.

The following examples illustrate the effect of being able to impose the condition of independence (the effect of 'IF-ing') on first-order logic and basic modal logic, respectively. As is well known, the impact of IF-ing in the former case results in second-order expressive power (IF first-order logic has the expressive power of the $\sum_1^1[\tau]$ fragment of second-order logic), while we just proved in Theorem 3.2.4 that IF-ing basic modal logic does not give the resulting logic an expressive power greater than that of **FO**. Still, IF modal logic has extra expressive power as compared with basic modal logic, as will be proven in Theorem 3.4.4 below. (Cf. also Fact 3.1.13.)

Example 3.2.8 (**IF**$[\tau] = \sum_1^1[\tau]$) *Let t_0 be a constant, and let $\tau = \{t_0\}$. Consider evaluating the IF sentence $\chi_{t_0} :=$*

$$\forall x \forall y \exists z / \{y\} \exists v / \{x, z\} \varphi [t_0, x, y, z, v]$$

in an arbitrary first-order structure $\langle \mathcal{M}, (t_0, t_0^{\mathcal{M}}) \rangle$, given that $\varphi [t_0, x, y, z, v]$ is the formula

$$[(x = y \leftrightarrow z = v) \wedge z \neq t_0] .$$

Since the IF sentence $\exists x(x = t_0 \wedge \chi_{t_0})$ (which states the domain of its model to be infinite) is obviously equivalent to the so-called Ehrenfeucht sentence,[8] *which is well known not to be* **FO***-translatable, neither is χ_{t_0} translatable into* **FO**$[\tau]$*.*

Let us then mimic the formula χ_{t_0} of the above example in IF modal logic.[9]

Example 3.2.9 (**IFML** *on arbitrary modal structures*) *Consider the* **IFML** *formula $\chi :=$*

$$\Box_1 \Box_2 \Diamond_3 / \{2\} \Diamond_4 / \{1, 3\} q.$$

By exactly the same argument as used in the proof of Lemma 3.2.1*, we see that*

[8] For the Ehrenfeucht sentence, see e.g. Krynicki & Mostowski (1995).

[9] In fact, we employ an extended version of **IFML** to enable several independence indications in one formula. On how to give precise semantics to this logic, see *Sect.* 3.3 below.

$$\Box_1\Box_2\Diamond_3/\{2\}\Diamond_4/\{1,3\}\,q \Leftrightarrow \Box_1\Box_2\Diamond_3/\{2\}\Diamond_4/\{3\}\,q \Leftrightarrow$$

$$\Box_1\Box_2\Diamond_3/\{2\}\Diamond_4 q.$$

Hence χ has the **FO** *equivalent*

$$\forall x\exists z\forall y\exists v(R(t_0,x) \rightarrow (R(x,y) \rightarrow (R(y,z) \wedge$$

$$(R(z,v) \wedge Q(v))))).$$

It is easily seen (cf. the proof of Theorem 3.4.4*) that the formula χ has no* **ML** *equivalent when evaluated over arbitrary unary modal structures.*

The evaluation of modal logic is transitional, each transition being 'guarded' by an accessibility relation, and depends only on where the previous transition led. Hence the semantics of modal logic in particular blocks the possibility of having the effect of the Henkin quantifier

$$\begin{matrix} \forall x\exists z \\ \forall y\exists v \end{matrix}$$

that is responsible for the import of the Ehrenfeucht sentence made use of in the first example. In particular, we cannot have the two operators $\Diamond_3/\{2\}$ and $\Diamond_4/\{1,3\}$ both genuinely independent of the indicated previous choices, precisely due to locality.

3.3 Extended IF Modal Logic and Second-Order Expressive Power

IFML$[k]$, then, can be translated into **FO**. On the other hand, we noticed above (*Sect.* 1.3) that the IF modal logic of Julian Bradfield can express some genuine second-order properties of its models. The framework within which Bradfield works is more complex than ours: Bradfield (2000) uses the synchronization relation between local transitions to give content to logical independence, and Bradfield and Fröschle (2002) use the relation of concurrency, explicitly given in a model, for this purpose. By contrast, **IFML**$[k]$ does not conceptually presuppose anything of its models that would not already be present in the case of basic modal logic. Essentially only the extra requirement of uniformity is imposed upon players' winning strategies. The systematical question arises: is there anything one can do to modify **IFML**$[k]$, so as to produce a logic that still employs plain k-ary modal structures, but which *is* capable of expressing second-order properties?

The answer is affirmative and is due to Dr. Tapani Hyttinen:[10] a very small change in the syntax of **IFML**$[k]$ suffices, essentially just allowing modal operators to be independent of Boolean connectives. We shall refer to this new logic as *Extended IF modal logic*, or **EIFML**$[k]$. To be more exact, the formulae of **EIFML**$[k]$ are modal analogues of Vaught formulae:[11] an evaluation game corresponding to **EIFML**$[k]$ formulae consists of two players choosing in a given order a finite number of indices and elements corresponding to conjunctions/disjunctions *resp.* universal/existential modalities. The moves made for the Boolean connectives by the end of a play of the game determine a particular (negated) propositional atom, whose truth *resp.* falsity then determines the winner of the play. The logic is made IF by the fact that specified choices for modalities must be made *uniformly* in order for a player to have a winning strategy.

[10] Personal communication.

[11] For Vaught formulae, see e.g. Hodges (1997 [c], pp. 583-5); Makkai (1977, pp. 254-61).

3.3.1 The Language of EIFML

The syntax is given as follows. Let a countable class **prop** of propositional atoms be given, and write

$$\pm\mathbf{prop} = \mathbf{prop} \cup \{\neg p : p \in \mathbf{prop}\}.$$

Formulae of **EIFML**$[k]$ are of the form

$$(X_1, \ldots, X_n)\varphi_{\bar{i}_{n+1}},$$

where the components X_j $(1 \leq j \leq n)$ of the *prefix* $(X_1, \ldots, X_n)$ and the *matrix formulae* $\varphi_{\bar{i}_{n+1}}$ are as follows. Each of the X_j is one of the following:

(i) $\vee_{i_j \in I_j}$, where I_j is a finite (non-empty) set.

(ii) $\wedge_{i_j \in I_j}$, where I_j is a finite (non-empty) set.

(iii) $\Diamond(R^j_{\bar{i}_j})/W_j$, where $\bar{i}_j = \langle i_k : X_k = \vee, \wedge \text{ and } k < j\rangle$, $W_j \subseteq [1, j-1]$, and $R_{\bar{i}_j} \in \{R_0, \ldots, R_{k-1}\}$.

(iv) $\Box(R^j_{\bar{i}_j})/W_j$, where $\bar{i}_j = \langle i_k : X_k = \vee, \wedge \text{ and } k < j\rangle$, $W_j \subseteq [1, j-1]$, and $R_{\bar{i}_j} \in \{R_0, \ldots, R_{k-1}\}$.

The matrix formulae $\varphi_{\bar{i}_{n+1}}$ are (negated) propositional atoms, i.e. members of $\pm\mathbf{prop}$, where

$$\bar{i}_{n+1} = \langle i_k : X_k = \vee, \wedge \text{ and } k \leq n\rangle.$$

The idea is (as will become clear when the semantics is given) that for an expression X_j of the form (iii) and (iv), the sequence $\bar{i}_j$ of all earlier choices corresponding to disjunctions and conjunctions determines an accessibility relation $R_{\bar{i}_j}$, and once choices have been made for all expressions in the prefix, a unique matrix formula is determined again by the sequence of all earlier choices for disjunctions and conjunctions. Therefore we require that for each expression X_j of the forms (iii) and (iv), there is a function

$$R^j : \times\langle I_k : X_k = \vee, \wedge \text{ and } k < j\rangle \to \{R_0, \ldots, R_{k-1}\}$$

giving the relation corresponding to the index vectors $\bar{i}_j$, and that likewise there is a function

$$\varphi : \times\langle I_k : X_k = \vee, \wedge \text{ and } k \leq n\rangle \to \pm\mathbf{prop}$$

specifying the matrix formulae. A degenerate case of a Cartesian product is the Cartesian product of the empty sequence of sets, which contains precisely one element, namely the empty sequence:

$$\text{If } n = 0, \text{ then } \times_{i<n} I_i = \{\emptyset\}.$$

Thereby the functions R^j and φ are never empty (and so formulae involving no conjunctions or disjunctions are also well defined).

It is possible to obtain *propositional logic*, *basic modal logic* ($\mathbf{ML}[k]$) and *IF modal logic* ($\mathbf{IFML}[k]$) all as versions of $\mathbf{EIFML}[k]$. This involves taking all index sets as two-element sets {L,R}, defining matrix formulae $\varphi_{\bar{i}_{n+1}}$ generally for subsequences of vectors $\bar{i}_{n+1} = \langle i_k : X_k = \vee, \wedge \text{ and } k \leq n\rangle$, and likewise defining the functions R^j corresponding to the expressions $\Diamond(R^j_{\bar{i}_j})/W_j$ and $\Box(R^j_{\bar{i}_j})/W_j$ as partial functions.

Syntax for Extended IF *tense* logic $\mathbf{EIFTL}[k]$ is obtained as $\mathbf{EIFML}[2k]$, where in particular the relations corresponding to the expressions $\Diamond(R^j_{\bar{i}_j})/W_j$ and $\Box(R^j_{\bar{i}_j})/W_j$ are from the class

$$\{R_0, \ldots, R_{k-1}, R_0^{-1}, \ldots, R_{k-1}^{-1}\},$$

instead of being from an arbitrary class $\{R_0, \ldots, R_{2k-1}\}$ of $2k$ accessibility relations.

We move on to give semantics to $\mathbf{EIFML}[k]$. When doing so, the following notation is used: if $(a_0, \ldots, a_n)$ is a sequence and $i \leq n$, we write $(a_0, \ldots, a_n)[i]$ for its member a_i.

Semantics

First of all, for (negated) propositional atoms we agree on the following notation: if $\mathfrak{h}$ is the assignment function of a modal structure $\mathcal{M}$, we write

- "$\mathcal{M} \models^+ p[d]$" for "$d \in \mathfrak{h}(p)$";
- "$\mathcal{M} \models^+ \neg p[d]$" for "$d \notin \mathfrak{h}(p)$".

Now with each formula $\psi = (X_1, \ldots, X_n)\varphi_{\bar{i}_{n+1}} \in \mathbf{EIFML}[k]$ and each pointed k-ary modal structure $(\mathcal{M}, d)$, we associate a semantical game

$$\mathrm{G}_A(\psi, \mathcal{M}, a_0) = \langle \{\forall, \exists\}, H, Z, P, \{u_\forall, u_\exists\}, \{I_\forall, I_\exists\} \rangle$$

in extensive normal form. It is a game between two players, $\forall$belard and $\exists$loise. The game is played on the set

$$A = \bigcup \{I_k : X_k = \vee, \wedge \text{ and } k \leq n\} \cup dom(\mathcal{M}) \cup \{\star\}^+$$

of *actions*.[12] *Histories* (or *plays*) are sequences of length at most n consisting of the initial position and at most n consecutive moves. The set H of all histories is defined recursively as follows:

- $(a_0) \in H$

[12] When defining the game, we take $\star$ to be an object that is neither in the domain nor in any of the index sets involved. If A is a set, A^+ is the set of non-empty strings over elements of A. The set of non-empty strings over $\{\star\}$ with at most n members would in fact suffice here, where n is the number of expressions X_j in the prefix of ψ.

- If $(a_0, \ldots, a_{j-1}) \in H$ $(j \leq n)$ and $X_j = \wedge$ (*resp.* $X_j = \vee$), then ∀belard (∃loise) picks out an index $i_j \in I_j$ and $(a_0, \ldots, a_{j-1}, i_j) \in H$.
- If $(a_0, \ldots, a_{j-1}) \in H$ $(j \leq n)$ and $X_j = \Box(R^j_{\bar{i}_j})/W_j$ (*resp.* $X_j = \Diamond(R^j_{\bar{i}_j})/W_j$), then ∀belard (∃loise) picks out, if possible, an element d_j such that

$$R^j_{\bar{i}_j}(a, d_j),$$

where

$$a := (a_0, a_1, \ldots, a_{j-1})[max\{k : k < j \text{ and}$$

$$(X_k = \Diamond, \Box \text{ or } k = 0)\}],$$

and

$$(a_0, \ldots, a_{j-1}, d_j) \in H.$$

Otherwise, he (*resp.* she) picks out a sequence $(\star, \ldots, \star)$ of $(n-j)+1$ occurrences of the object $\star$, and $(a_0, \ldots, a_{j-1}, \star, \ldots, \star) \in H$.

Whenever $(a_0, \ldots, a_{j-1})$ is a history $(j \leq n)$ for which it is ∀belard who makes the choice, we put $P(a_0, \ldots, a_{j-1}) = \forall$, and when it is ∃loise who makes the choice, $P(a_0, \ldots, a_{j-1}) = \exists$. Observe that by the above rules, if $(a_0, \ldots, a_{j-1}, \star, \ldots, \star)$ is a history, it is necessarily of length n, and its longest proper initial segment that moreover is a history, is $(a_0, \ldots, a_{j-1})$.

The set Z of terminal histories is the subset of H consisting of histories of length n. Hence Z contains precisely those histories from H which cannot be extended by any move so as to yield a sequence in H. The utility functions $u_\exists : Z \to \{1, -1\}$, $u_\forall : Z \to \{1, -1\}$ for the two players are defined as follows:

- $u_\exists(a_0, \dots, a_n) = 1$ and $u_\forall(a_0, \dots, a_n) = -1$, if *either* $a_n = \star$ and $\forall$belard has chosen a_n, or else

$$\mathcal{M} \models^+ \varphi_{\bar{i}_{n+1}}[a_m],$$

  where

$$m := max\{k : k \leq n \text{ and } (X_k = \Diamond, \Box \text{ or } k = 0)\},$$

$$\bar{i}_{n+1} := \langle i_k : X_k = \vee, \wedge \text{ and } k \leq n\rangle.$$

  It is then said that $\exists$loise *wins* (and $\forall$belard *loses*) the play $(a_0, \dots, a_n)$, or that this play is a *win* for her (and a *loss* for him).

- Otherwise $u_\forall(a_0, \dots, a_n) = 1$ and $u_\exists(a_0, \dots, a_n) = -1$.

The information partitions $I_\exists$ and $I_\forall$ for the two players still need to be defined. The sets $P^{-1}(\{\exists\})$ and $P^{-1}(\{\forall\})$ are partitioned into equivalence classes under the following equivalence relations $\sim_\exists$ and $\sim_\forall$, respectively. If $h = (a_0, \dots, a_{j-1})$ and $h' = (a'_0, \dots, a'_{j-1})$ are histories ($j - 1 \leq n$), put:

$$h \sim_\exists h' \Longleftrightarrow$$

$$[X_j = \Diamond(R^j_{\bar{i}_j})/W_j \textbf{ and } \text{for all } k \in [1, j-1]\backslash W :$$

$$a_k = a'_k] \textbf{ or } [X_j = \vee \text{ and } h = h'].$$

$\exists$loise's *information partition* $I_\exists$ then is the set

$$I_\exists = \{[h]_{\sim_\exists} : h \in H\backslash Z\}.$$

$\forall$belard's *information partition* $I_\forall$ is defined analogously. The cells of information partitions are called *information sets*.

The definition of the semantical game $\mathrm{G}_A(\psi, \mathcal{M}, a_0)$ in its extensive form has been completed. By definition it is a zero-sum game, and it is a game of *imperfect* information.

A *strategy* of the player $j \in \{\exists, \forall\}$ in the game $\mathrm{G}_A(\psi, \mathcal{M}, d)$ is any function

$$f_j : P^{-1}(\{j\}) \to A.$$

The strategy f_j is a *winning strategy* (*w.s.*) if there exists a subset $W \subseteq Z$ of terminal histories satisfying the following four conditions:

(a) If $h \in Cl(W)$ and $P(h)$ is j, then $h^\frown f_j(h) \in Cl(W)$.

(b) If $h \in Cl(W)$ and $P(h)$ is the opponent of j, then for every $a \in A$ such that $h^\frown a \in H$, $h^\frown a \in Cl(W)$.

(c) For every $h, h' \in Cl(W)$: if $h, h' \in \mathrm{I} \in I_j$, then $f_j(h) = f_j(h')$.

(d) Every $h \in W$ is a win for j.

Such a set $W \subseteq Z$ is called a *plan of action*. We say that W *establishes* that f_j is a winning strategy, or that f_j is a winning strategy *based on* W. It is easy to verify that there cannot exist a w.s. for both players:

Fact 3.3.1 *At most one of the players has a w.s. in a game* $G_A(\varphi, \mathcal{M}, d)$.

Proof. See the proof of Fact 2.3.2. ∎

The semantics of **EIFML**$[k]$, then, is simply this. If $(\mathcal{M}, d)$ is a pointed k-ary modal structure and $\varphi \in$ **EIFML**$[k]$, then:

- $\mathcal{M} \models^+ \varphi[d] \iff$

  there exists a w.s. for ∃loise in $\mathrm{G}_A(\varphi, \mathcal{M}, d)$.

- $\mathcal{M} \models^- \varphi[d] \iff$

  there exists a w.s. for ∀belard in $\mathrm{G}_A(\varphi, \mathcal{M}, d)$.

- $\mathcal{M} \models^0 \varphi[d] \iff \mathcal{M} \not\models^+ \varphi[d]$ and $\mathcal{M} \not\models^- \varphi[d]$.

Extended IF tense logic $\mathbf{EIFTL}[k] = \mathbf{EIFML}[2k]$ is evaluated relative to k-ary *temporal* structures instead of arbitrary $2k$-ary modal structures.

For an illustration of the semantics, consider the two examples below.

Example 3.3.2 *Define relations R_λ and R_ρ as follows:*

$$R_\lambda = \{(0, n) : n \textit{ is even}\}; \quad R_\rho = \{(0, n) : n \textit{ is prime}\}.$$

Then consider evaluating the $\mathbf{EIFML}[2]$ *formula*

$$\chi := \wedge_{i_1 \in \{\lambda,\rho\}} \Diamond(R^2_{i_1})/\{1\}\varphi_{i_1}$$

in the modal structure $\mathcal{M} = (\mathbb{N}, R_\lambda, R_\rho, \mathfrak{h})$ at 0, given that $\mathfrak{h}(q) = \{3\}$, and $\varphi_\lambda = q = \varphi_\rho$. We claim that there is no w.s. for $\exists$loise in $G_A(\chi, \mathcal{M}, 0)$. For, in order for $\exists$loise to have a w.s. $f_\exists$, there must be a number $c \in \mathbb{N}$ such that

$$R_\lambda(0, c) \textit{ and } R_\rho(0, c).$$

Let then $f_\exists(\lambda) = f_\exists(\rho) = c$. But by the definition of the relations R_λ and R_ρ, c is then both even and prime. Hence, in fact, $c = 2$. But then $f_\exists$ is not winning, since $2 \notin \{3\} = \mathfrak{h}(q)$.

On the other hand, there is a w.s. for $\forall$belard in the game $G_A(\chi, \mathcal{M}, 0)$. For let him choose $i_1 = \lambda$. All points x that are R_λ-accessible from 0 are now even. But $\mathfrak{h}(q) = \{3\}$ and 3 is not even. So $\varphi_\lambda = q$ is false at any such x. Hence χ is false in $\mathcal{M}$ at 0.

Example 3.3.3 *Write $\mathbb{N}^+$, $\mathbb{N}^{even}$, and $\mathbb{N}^{odd}$ for the set of positive, even and odd natural numbers, respectively, and define a relation $\prec$ as follows:*

$$\prec \; := (\{0\} \times \mathbb{N}^+) \cup (\mathbb{N}^{even} \times \{\omega\}) \cup (\mathbb{N}^{odd} \times \{\omega + 1\}).$$

Then consider evaluating the **EIFML**[1] *formula*

$$\chi := \Box(R^1_\emptyset) \vee_{i_2 \in \{\lambda,\rho\}} \Diamond(R^3_{i_2})/\{1\}\varphi_{i_2}$$

in the modal structure $\mathcal{M} = (\omega+2, \prec, \mathfrak{h})$ *at* 0*, given that* $\mathfrak{h}(q) = \{\omega\}$*, and* $\varphi_\lambda = q$*,* $\varphi_\rho = \neg q$*,* $R_\emptyset = R_\lambda = R_\rho = \prec$*. Define a function* $f_\exists$ *as follows:*

$$f_\exists(0, n) = \begin{cases} \lambda & \text{if } n \text{ is even} \\ \rho & \text{otherwise} \end{cases}$$

$$f_\exists(0, n, i_2) = \begin{cases} \omega & \text{if } i_2 = \lambda \\ \omega + 1 & \text{if } i_2 = \rho \end{cases}$$

Let $W = \{(0, n, f_\exists(n), f_\exists(n, i_2)) : n \in \mathbb{N}^+, i_2 \in \{\lambda, \rho\}\}$*. We claim that in the game* $G_A(\chi, \mathcal{M}, 0)$*,* $f_\exists$ *is a w.s. for* $\exists$*loise, based on the plan of action* W*. To see this, notice first that the information sets corresponding to the expression* $\Diamond(R^3_{i_2})/\{1\}$ *are*

$$\{(0, n, \lambda) : n \in \mathbb{N}^+\} \text{ and } \{(0, n, \rho) : n \in \mathbb{N}^+\}.$$

Now $f_\exists$ *agrees on these sets: for any* $n, n' \in \mathbb{N}^+$*:*

$$f_\exists(0, n, \lambda) = \omega = f_\exists(0, n', \lambda);$$

and

$$f_\exists(0, n, \rho) = \omega + 1 = f_\exists(0, n', \rho).$$

To see that $f_\exists$ *always yields a win for* $\exists$*loise, observe that if* n *is even, then*

$$n \prec \omega = f_\exists(0, n, \lambda) = f_\exists(0, n, f_\exists(0, n)).$$

And $\mathcal{M} \models^+ \varphi_\lambda[\omega]$*, since* $\varphi_\lambda = q$ *and* $\mathfrak{h}(q) = \{\omega\}$*. And if* n *is odd, then*

$$n \prec \omega + 1 = f_\exists(0, n, \rho) = f_\exists(0, n, f_\exists(0, n)).$$

And $\mathcal{M} \models^+ \varphi_\rho[\omega + 1]$*, as* $\varphi_\rho = \neg q$ *and* $\omega + 1 \notin \{\omega\} = \mathfrak{h}(q)$*. Hence* $f_\exists$ *is in fact a w.s. for* $\exists$*loise in the game* $G_A(\chi, \mathcal{M}, 0)$*.*

3.3.2 EIFML and second-order expressive power

It will now be proven that Extended IF modal logic is *not* translatable into **FO**, but can instead express second-order properties of modal structures.

Of course the claim will *not* thereby be made that **EIFML** would 'have a greater expressive power' than **FO**. For this would mean not only showing there are properties that are expressible in **EIFML** but not in **FO** (this much we *are* going to do), but crucially also that **EIFML** is *expressively complete* relative to **FO** (over all k-ary modal structures) in the sense that for every first-order formula φ of one free variable, written in an appropriate vocabulary, there is a formula χ_φ of $\mathbf{EIFML}[k]$ such that for all pointed k-ary modal structures $(\mathcal{M}, d)$:

$$\mathcal{M} \models \chi_\varphi[d] \Longleftrightarrow \langle \mathcal{M}^{\mathbf{FO}}, d \rangle \models \varphi(x).$$

Hans Kamp (1968) proved that his propositional modal logic of two binary connectives *Until* and *Since* is indeed in this sense expressively complete relative to **FO**, over the particular class of unary modal structures whose accessibility relation is any Dedekind-complete linear order (i.e. a linear order satisfying the completeness axiom of the reals). In the present thesis the question is left open of the exact relation between the expressive powers of **EIFML** and the relevant fragment of **FO**, consisting of formulae with exactly one free variable written in a vocabulary with k binary relation symbols and countably many unary ones. Instead, then, we content ourselves with showing negatively that **EIFML** is *not* translatable into **FO**.

For the purpose of the proof, we state *Ehrenfeucht's theorem*, which says that the equivalence of two first-order structures up to a quantifier rank n is characterized by the existence of a winning strategy for *Duplicator* in the *Ehrenfeucht-Fraïssé* game of n rounds. In order to state the theorem, the three following definitions are needed.

Definition 3.3.4 (*Partial isomorphism*) *Let τ be a vocabulary containing no constant symbols or function symbols. Let $\mathcal{M}$ and $\mathcal{N}$ be first-order τ-structures, and let f be a function whose domain is included in dom($\mathcal{M}$) and whose range is included in dom($\mathcal{N}$). The function f is said to be a partial isomorphism from $\mathcal{M}$ to $\mathcal{N}$ if:* ***(i)*** *f is injective, and* ***(ii)*** *for every n-ary R $\in \tau$ and all $a_1, \ldots, a_n \in dom(f)$:*

$$\langle \mathcal{M}, a_1, \ldots, a_n \rangle \models R(x_1, \ldots, x_n) \Longleftrightarrow$$

$$\langle \mathcal{N}, f(a_1), \ldots, f(a_n) \rangle \models R(x_1, \ldots, x_n).$$

The set of partial isomorphisms from $\mathcal{M}$ to $\mathcal{N}$ is denoted by "Part($\mathcal{M}$, $\mathcal{N}$)".

Definition 3.3.5 (*Ehrenfeucht-Fraïssé game*) *Let $\mathcal{M}$ and $\mathcal{N}$ be first-order τ-structures, let $\overline{a} = (a_1, \ldots, a_s) \in dom(\mathcal{M})^s$, $\overline{b} = (b_1, \ldots, b_s) \in dom(\mathcal{N})^s$, and let $n < \omega$. The Ehrenfeucht-Fraïssé game (EF game)*

$$EF_n(\mathcal{M}, \overline{a}, \mathcal{N}, \overline{b})$$

between two players, Spoiler and Duplicator, has the following rules. In the course of a play, Spoiler and Duplicator must both make n moves. The players take turns. In his i-th move, Spoiler selects a structure ($\mathcal{M}$ or $\mathcal{N}$) and an element of the domain of this structure. If Spoiler chooses an element e_i in $\mathcal{M}$, then Duplicator in her i-th move must choose an element f_i in $\mathcal{N}$. (Similarly, if Spoiler chooses an element f_i in $\mathcal{N}$, then Duplicator in her i-th move has to answer by choosing an element e_i in $\mathcal{M}$.)

After n moves, n choices have been made from $dom(\mathcal{M})$, and equally many choices have been made from $dom(\mathcal{N})$. We stipulate that the notation

$$\overline{a}e_1 \dots e_n \mapsto \overline{b}f_1 \dots f_n$$

stands for the set

$$\{(a_i, b_i) : 1 \leq i \leq s\} \cup \{(e_i, f_i) : 1 \leq i \leq n\}.$$

We say that Duplicator wins a play, if $\overline{a}e_1 \dots e_n \mapsto \overline{b}f_1 \dots f_n \in Part(\mathcal{M}, \mathcal{N})$. Otherwise, Spoiler wins the play. We say that there is a winning strategy (w.s.) for Duplicator in $EF_n(\mathcal{M}, \overline{a}, \mathcal{N}, \overline{b})$ if she can, against any moves made by Spoiler, make her moves in such a way that she wins the corresponding play.

Definition 3.3.6 *The quantifier rank $qr(\varphi)$ of a first-order formula φ of a vocabulary $\tau = \{\mathrm{R}_i\}_{i<\kappa}$ is defined as follows:*

- $qr(\mathrm{R}_i(x_1, \dots, x_n)) = 0$
- $qr(\neg\varphi) = qr(\varphi)$
- $qr(\varphi \vee \psi) = max\{qr(\varphi), qr(\psi)\} = qr(\varphi \wedge \psi)$
- $qr(\exists x_i \varphi) = qr(\varphi) + 1 = qr(\forall x_i \varphi)$

Proposition 3.3.7 (*Ehrenfeucht's Theorem*) *Let $\mathcal{M}$ and $\mathcal{N}$ be first-order τ-structures, $\overline{a} = (a_1, \dots, a_s) \in dom(\mathcal{M})^s$ and $\overline{b} = (b_1, \dots, b_s) \in dom(\mathcal{N})^s$. Further, let $n < \omega$ be arbitrary. The following are equivalent:*

(i) *There is a w.s. for Duplicator in $EF_n(\mathcal{M}, \overline{a}, \mathcal{N}, \overline{b})$.*

(ii) *For every first-order formula $\varphi(\overline{x})$ of quantifier rank at most n, with free variables among $\{x_1, \dots, x_s\}$:*

$$\langle \mathcal{M}, \overline{a} \rangle \models \varphi(\overline{x}) \Longleftrightarrow \langle \mathcal{N}, \overline{b} \rangle \models \varphi(\overline{x}).$$

Proof. See, for example, the proof of *Theorem* 2.2.8. in Ebbinghaus & Flum (1999, pp. 18-9). ■

Structures $\langle \mathcal{M}, \overline{a} \rangle$ and $\langle \mathcal{N}, \overline{b} \rangle$ that satisfy precisely the same first-order formulae $\varphi(\overline{x})$ of quantifier rank at most n can be said to be "n-equivalent."

It will now be shown that **EIFML** is *not* translatable into **FO**. We prove this specifically for the number $k = 3$ of accessibility relations, from which the result trivially follows for all **EIFML**$[k]$ with $k \geq 3$. We do not prove what happens in the case $k = 2$, but conjecture that **EIFML**$[2]$ *is not* translatable into **FO**, and we informally describe a proof to the effect that, by contrast, **EIFML**$[1]$ in fact *is* translatable into **FO**.

3.3.2.1 Idea of the proof of the non-translatability of EIFML into FO

Let $n < \omega$, and consider a modal structure $\mathcal{M}_n$ involving three accessibility relations, R_n, P_n and Q_n. The domain of the structure consists of:

- A circle $C_n = \{c_1, \ldots, c_n\}$ formed by the relation P_n:

  $P_n(c_1, c_2)$ and $\ldots$ and $P_n(c_{n-1}, c_n)$ and $P_n(c_n, c_1)$.

- Three points $0, a$ and b outside of C_n.

From the point 0 one can move along R_n to any point inside the circle. Further, from any point x within the circle one can get to a and b along R_n, to x itself along Q_n, and to the immediate P_n-successor of x.

Now consider the following game, played on the structure $(\mathcal{M}_n, 0)$:

(a) $\forall$ chooses any point x from the circle C_n.

(b) $\exists$ labels the point hence chosen as black or white.

(c) $\forall$ chooses a relation, P_n or Q_n, and then makes a move from x along the relation he chose, ending up at a point y.

(d) Finally, $\exists$ chooses either a or b along R_n.

But, crucially, when making her choice, $\exists$ is informed only of the following:

(d.i) the *color* she herself gave to x (but she is not informed of the point x itself);

(d.ii) the point y.

Finally it is stipulated that $\exists$ wins iff:

- she chose a, and $\forall$ employed Q_n, or
- she chose b, and $\forall$ employed P_n.

Now because of the restriction regarding the information that $\exists$ has at her disposal when choosing a or b, $\exists$ can only win if she can paint the circle C_n in black and white in such a way that the colors of points x and y reveal whether $\forall$ has moved forward by P_n, or stood still employing Q_n. But this is obviously possible precisely when the circle C_n is of even size.

It is not difficult to show, by an argument employing Ehrenfeucht-Fraïssé games, that there is no first-order formula which could, for all $n < \omega$, distinguish the models corresponding to the modal structures $(\mathcal{M}_n, 0)$ and $(\mathcal{M}_{n+1}, 0)$. However, there is an **EIFML** formula that does the job: the instructions for defining the above game can be incorporated into such a formula,

$$\Box(R^1_\emptyset) \vee_{i_2\in\{\lambda,\rho\}} \wedge_{i_3\in\{\lambda,\rho\}} \Box(R^4_{i_2 i_3})\Diamond(R^5_{i_2 i_3})/\{1,3\}\varphi_{i_2 i_3}$$

(see the proof below for details). The crucial thing here is that by requiring the existential modality's *independence of a conjunction*, we in effect are requiring that $\exists$ *loses sight of the accessibility relation* along which her opponent makes his move corresponding to $\Box(R^4_{i_2 i_3})$. In the game at hand, she can only have a winning strategy if she can nevertheless always infer which relation $\forall$ in fact employed. And such an inference is not possible for odd circles C_n, as in them some P_n-adjacent points necessarily receive the same color $-$ whence the fact that points x and y have the same color does not guarantee that y has been obtained from x along Q_n, unlike in the even case. It should be noted that in plain IF modal logic (**IFML** $[k]$) the players are always fully informed about the accessibility relations involved in the moves.

3.3.2.2 The proof of the non-translatability

Lemma 3.3.8 **EIFML**[3] *is not translatable into* **FO**.

Proof. Let a,b be distinct, fixed negative integers, and let **prop** $= \{s\}$. For each natural number n, define a modal structure

$$\mathcal{M}_n = (D_n, R_n, P_n, Q_n, \mathfrak{h}_n)$$

by setting:

- $D_n = \{0, 1, \ldots, n\} \cup \{a, b\}$.
- $R_n = \{(0, x) : 1 \leq x \leq n\} \cup (\{1, \ldots, n\} \times \{a, b\})$.
- $P_n = \{(x, x + 1) : 1 \leq x < n\} \cup \{(n, 1)\}$.

- $Q_n = \{(x, x) : 1 \leq x \leq n\}$.
- $\mathfrak{h}_n(s) = a$.

See Figure 10 for an illustration of a structure $\mathcal{M}_n$ $(n := 6)$.

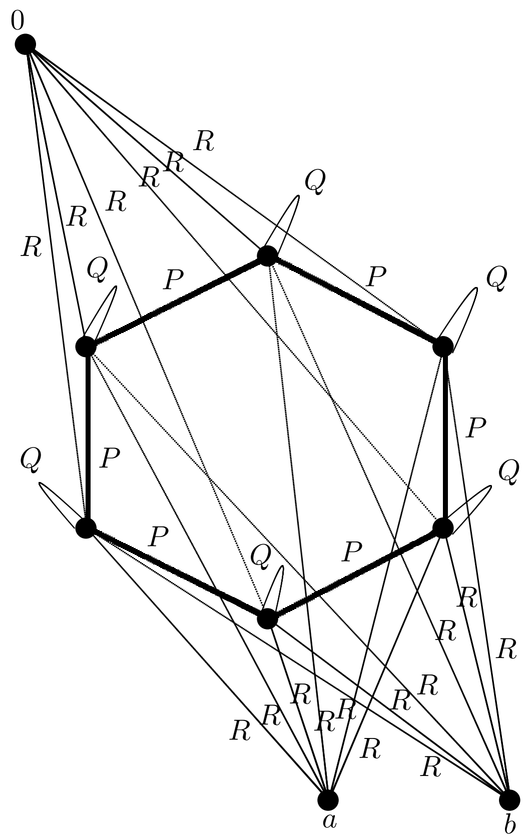

FIGURE 10

Given the modal structure $\mathcal{M}_n = (D_n, R_n, P_n, Q_n, \mathfrak{h}_n)$, let τ := {R,P,Q,S} be a vocabulary, where R, P, Q are binary and S is unary. Then there corresponds to $\mathcal{M}_n$ a first-order

τ-structure,

$$\mathcal{M}_n^{\mathbf{FO}} := (D_n, R_n, P_n, Q_n, \mathfrak{h}_n(s)),$$

where the unary relation $\mathfrak{h}_n(s)$ interprets the predicate S, and the binary relations R_n, P_n and Q_n interpret respectively the predicates R, P and Q.

Then consider evaluating the **EIFML**[3] formula

$$\chi := \Box(R^1_\emptyset) \vee_{i_2\in\{\lambda,\rho\}} \wedge_{i_3\in\{\lambda,\rho\}} \Box(R^4_{i_2 i_3})\Diamond(R^5_{i_2 i_3})/\{1,3\}\varphi_{i_2 i_3},$$

relative to modal structures $\mathcal{M}_n$ ($n < \omega$), given that:

- $R^1_\emptyset = R_n$
- $R^4_{i_2 i_3} = \begin{cases} Q_n & \text{if } i_3 = \lambda \\ P_n & \text{if } i_3 = \rho \end{cases}$
- For all $i_2, i_3 \in \{\lambda, \rho\}$: $R^5_{i_2 i_3} = R_n$
- $\varphi_{i_2 i_3} = \begin{cases} s & \text{if } \; i_2 = i_3 \\ \neg s & \text{if } \; i_2 \neq i_3 \end{cases}$

Observe that the formula χ does *not* depend on the number n: for every $n < \omega$ the expressions $\Box(R^1_\emptyset)$ and $\Diamond(R^5_{i_2 i_3})$ speak of the accessibility relation of the modality type 0 of the structure $\mathcal{M}_n$ (i.e. R_n), and in each case the function associated with the expression $\Box(R^4_{i_2 i_3})$ yields the relation of modality type 1 or 2 (P_n *resp.* Q_n) for the same vectors $i_2 i_3$ of indices in both cases. It is therefore always one and the same formula, and only the interpretations of its constituent expressions vary with the structure $\mathcal{M}_n$ of the evaluation.

We move on to prove a series of three claims:

(1) For every *even* $N < \omega$: $\mathcal{M}_N \models^+ \chi[0]$.

(2) For every *odd* $N < \omega$: $\mathcal{M}_N \models^0 \chi[0]$.

(3) For all $n < \omega$, and for all first-order formulae $\varphi(x)$ of one free variable, with quantifier rank at most $n+1$:

$$\langle \mathcal{M}^{\mathbf{FO}}_{2^n}, 0\rangle \models \varphi(x) \Longleftrightarrow \langle \mathcal{M}^{\mathbf{FO}}_{2^n+1}, 0\rangle \models \varphi(x).$$

CLAIM 1. If $N < \omega$ is even, then χ is true in $\mathcal{M}_N$ at 0.

Proof of Claim 1: Let N be an arbitrary even number. Define a function $f_\exists$ as follows:

- $f_\exists(0,x) = \begin{cases} \lambda & \text{if } x \text{ is even} \\ \rho & \text{otherwise} \end{cases}$
- $f_\exists(0,x,i_2,i_3,y) = \begin{cases} a & \text{if } y \text{ is even} \\ b & \text{otherwise} \end{cases}$

We show that $f_\exists$ is a w.s. for $\exists$loise in $\mathrm{G}_A(\chi, \mathcal{M}_N, 0)$, being based on the plan of action $W =$

$\{\langle 0, x, f_\exists(0,x), i_3, y, f_\exists(0,x,f_\exists(0,x),i_3,y)\rangle$:

$R(0,x)$ and $i_3 \in \{\lambda, \rho\}$ and $R^4_{f_\exists(0,x)i_3}(x,y)\}$.

The union of the information sets $\mathrm{I}(i_2, y)$ corresponding to the expression $\Diamond(R^5_{i_2 i_3})/\{1,3\}$ is this:

$\bigcup_{i_2,y} I(i_2,y) := \bigcup_{i_2,y}\{(0,x,i_2,i_3,y) : R(0,x)$ and

$i_3 \in \{\lambda,\rho\}$ and $R^4_{i_2 i_3}(x,y)\}$.

Sequences of the following forms *cannot* in fact appear in the *intersection* of any set $\mathrm{I}(i_2, y)$ with $Cl(W)$:

- $(0, x, \lambda, \rho, y)$ and $(0, x, \rho, \lambda, y)$, where y is even.

- $(0, x, \lambda, \lambda, y)$ and $(0, x, \rho, \rho, y)$,[13] where y is odd.

This is seen as follows.

(i) Assume $(0, x, \lambda, \rho, y) \in Cl(W)$, where y is even. Because λ is chosen as $f_\exists(0, x)$, x must be even. But y satisfies $R^4_{\lambda\rho}(x, y)$, and $R^4_{\lambda\rho} = P_N$, whence $y = x+1$ (if $x < N$) or $y = 1$ (if $x = N$). So y is not even.

(ii) Assume $(0, x, \rho, \lambda, y) \in Cl(W)$, where y is even. Then x must be odd. On the other hand, y satisfies $R^4_{\rho\lambda}(x, y)$, and $R^4_{\rho\lambda} = Q_N$, so $x = y$. Thus y is not even.

(iii) Assume $(0, x, \lambda, \lambda, y) \in Cl(W)$, where y is odd. Hence x is even, and y satisfies $R^4_{\lambda\lambda}(x, y)$ for $R^4_{\lambda\lambda} = Q_N$. So $x = y$, whereby y is not odd.

(iv) Assume $(0, x, \rho, \rho, y) \in Cl(W)$, where y is odd. Hence x is odd. On the other hand, y satisfies $R^4_{\rho\rho}(x, y)$, and $R^4_{\rho\rho} = P_N$. Hence $y = x + 1$ (since $x < N$), and so y is not odd.

Note: Case (iv) is special as compared with the rest of the cases in that it would *not* be generally valid if the circle formed by the relation P_N $(N < \omega)$ consisted of an *odd number* of elements, and not of an even number as here. (Here there are N elements in the circle.) We could have x and y both odd, still satisfying $P_N(x, y)$, without contradiction: namely, y could be 1 and x could be N, where by assumption N would be odd! This is not possible if N is even, for if x is odd it cannot be the maximum of the set $\{1, \ldots, N\}$, and so x is related by P_N to $x + 1$.

Now, due to the fact that in the present case we have a P-circle of an even size (N), all of the above four sequences are known not to appear in the intersections $\mathrm{I}(i_2, y) \cap Cl(W)$, and

[13] As will be subsequently observed, the sequence $(0, x, \rho, \rho, y)$ *is* indeed possible when the set $dom(\mathcal{M})\backslash\{0, a, b\}$ is of odd size. This fact is crucial for the whole of the present proof.

hence we know that sequences of precisely the following four forms appear in (mutually disjoint) sets $\mathrm{I}(i_2, y) \cap Cl(W)$:

Case (α): $(0, x, \lambda, \lambda, y)$ and $(0, x, \rho, \rho, y)$, where y is even.

Case (β): $(0, x, \lambda, \rho, y)$ and $(0, x, \rho, \lambda, y)$, where y is odd.

To show that the function $f_\exists$ is winning for $\exists$loise, we show that it agrees on the sets $\mathrm{I}(i_2, y) \cap Cl(W)$, and that it always yields a win for her.

Case (α): Assume y is even, and let $(0, x_1, i_2, i_3, y)$, $(0, x_2, i_2, i_3', y) \in \mathrm{I}(i_2, y) \cap Cl(W)$ be arbitrary. From what was just observed, here $i_2 = i_3 = i_3'$. But as y is even, directly by the definition of the function $f_\exists$ we have

$$f_\exists(0, x_1, i_2, i_2, y) = a = f_\exists(0, x_2, i_2, i_2, y).$$

Hence $f_\exists$ agrees on the set $\mathrm{I}(i_2, y) \cap Cl(W)$. But the matrix formula corresponding to both terminal histories $(0, x_1, i_2, i_2, y, a)$ and $(0, x_2, i_2, i_2, y, a)$ is $\varphi_{i_2 i_2} = s$, and $a \in \mathfrak{h}_N(s)$. Hence $\mathcal{M}_N \models^+ \varphi_{i_2 i_2}[a]$. That is, in the case ($\alpha$) every history from W that is obtained by $f_\exists$ from a history in $\mathrm{I}(i_2, y) \cap Cl(W)$ is a win for $\exists$loise.

Case (β): Assume y is odd, and let $(0, x_1, i_2, i_3, y)$, $(0, x_2, i_2, i_3', y) \in \mathrm{I}(i_2, y) \cap Cl(W)$ be arbitrary. By the earlier observation the pairs (i_2, i_3), (i_2, i_3') are (not necessarily distinct) elements of the set $\{(\lambda, \rho), (\rho, \lambda)\}$. As y is odd, directly by the definition of the function $f_\exists$ we have

$$f_\exists(0, x_1, i_2, i_3, y) = b = f_\exists(0, x_2, i_2, i_3', y).$$

Hence $f_\exists$ agrees on the set $\mathrm{I}(i_2, y) \cap Cl(W)$. But the matrix formula corresponding to both terminal histories $(0, x_1, i_2, i_3, y, b)$

and $(0, x_2, i_2, i_3', y, b)$ is $\varphi_{i_2 i_3} = \neg s = \varphi_{i_2 i_3'}$, and $b \notin \{a\} = \mathfrak{h}_N(s)$. Hence

$$\mathcal{M}_N \models^+ \varphi_{i_2 i_3}[b] \text{ and } \mathcal{M}_N \models^+ \varphi_{i_2 i_3'}[b].$$

But this means that in the case (β) every history from W that is obtainable by $f_\exists$ from a history in $\mathrm{I}(i_2, y) \cap Cl(W)$ is a win for ∃loise.

As the cases (α) and (β) exhaust the intersection

$$\bigcup_{i_2, y} I(i_2, y) \cap Cl(W),$$

we may conclude that $f_\exists$ really is a w.s. for ∃loise based on the plan of action W. We have thus shown that the formula χ is true in $\mathcal{M}_N$ at 0. (■)

CLAIM 2. If $N < \omega$ is odd, then χ is non-determined in $\mathcal{M}_N$ at 0.

Proof of Claim 2. Let N be an arbitrary odd number. We show first that *if* there is a w.s. $f_\exists$ for ∃loise in the game $\mathrm{G}_A(\chi, \mathcal{M}_N, 0)$, *then* this strategy must map P_N-adjacent elements to distinct 'colors' from $\{\lambda, \rho\}$:

$$P_N(x, y) \Longrightarrow f_\exists(0, x) \neq f_\exists(0, y).$$

Assume for contradiction that $f_\exists$ is a w.s. for ∃loise such that for some $d_1, d_2 \in D_N$ with $P_N(d_1, d_2)$: $f_\exists(0, d_1) = f_\exists(0, d_2)$. We may assume that $f_\exists(0, d_1) = f_\exists(0, d_2) = \lambda$. (The option that $f_\exists(0, d_1) = f_\exists(0, d_2) = \rho$ can be dealt with similarly.) Consider the following two alternative histories h and h' of the game $\mathrm{G}_A(\chi, \mathcal{M}_N, 0)$:

- $h = (0, d_1, \lambda, \rho, d_2)$

  (here d_1 and d_2 satisfy $R^4_{\lambda\rho}(d_1, d_2)$, and $R^4_{\lambda\rho} = P_N$)

- $h' = (0, d_2, \lambda, \lambda, d_2)$

  (here d_2 satisfies $R^4_{\lambda\lambda}(d_2, d_2)$, and $R^4_{\lambda\lambda} = Q_N$)

These histories belong to the same information set $\mathrm{I}(\lambda, d_2)$ corresponding to the expression $\Diamond(R^5_{i_2 i_3})/\{1,3\}$. Because $f_\exists$ is a w.s., it must in particular agree on $\mathrm{I}(\lambda, d_2)$, and so

$$f_\exists(0, d_1, \lambda, \rho, d_2) = f_\exists(0, d_2, \lambda, \lambda, d_2).$$

Let us write $c := f_\exists(h) = f_\exists(h')$. In order for $f_\exists$ to be a w.s., both $h^\frown c$ and $h'^\frown c$ must be wins for $\exists$loise. Now the matrix formula corresponding to h is $\varphi_{\lambda\rho} = \neg s$, and the matrix formula corresponding to h' is $\varphi_{\lambda\lambda} = s$. On the other hand, $\mathfrak{h}_N(s) = \{a\}$. Hence it is *not* the case that

$$\mathcal{M}_N \models^+ \varphi_{\lambda\rho}[c] \text{ and } \mathcal{M}_N \models^+ \varphi_{\lambda\lambda}[c].$$

So $f_\exists$ cannot be a w.s. for $\exists$loise. We may conclude that indeed any w.s. $f_\exists$ for $\exists$loise satisfies the following: for all $x, y \in \{1, \ldots, N\}$,

$$P_N(x, y) \Longrightarrow f_\exists(0, x) \neq f_\exists(0, y).$$

Assume then that there is in fact a w.s. $g_\exists$ for $\exists$loise in $\mathrm{G}_A(\chi, \mathcal{M}_N, 0)$. Hence if $N = 1$, then $g_\exists(0,1) \neq g_\exists(0,1)$. And if $N > 1$, then

$$g_\exists(0,1) \neq g_\exists(0, N-1) \neq g_\exists(0, N) \neq g_\exists(0,1).$$

Here, because $g_\exists$ is a w.s., necessarily $g_\exists(0, x) \in \{\lambda, \rho\}$ for all $x \in \{1, \ldots, N\}$. But then $g_\exists(0,1) = g_\exists(0,N)$, and so $g_\exists(0,1) \neq g_\exists(0,1)$. As this is impossible, we may conclude there is *no* w.s. for $\exists$loise in $\mathrm{G}_A(\chi, \mathcal{M}_N, 0)$, and so χ is *not* true in $\mathcal{M}_N$ at 0.

Neither is there a w.s. for ∀belard in the game $\mathrm{G}_A(\chi, \mathcal{M}_N, 0)$. For, define a strategy $f_\exists$ for ∃loise as follows: for all x, $f_\exists(0, x) = \lambda$, and

$$f_\exists(0, x, \lambda, i_3, y) = \begin{cases} a & \text{if } i_3 = \lambda \\ b & \text{otherwise} \end{cases}$$

But because

$$\mathcal{M}_N \models^+ \varphi_{\lambda\lambda}[a] \text{ and } \mathcal{M}_N \models^+ \varphi_{\lambda\rho}[b],$$

∃loise wins every individual play of $\mathrm{G}_A(\chi, \mathcal{M}_N, 0)$ following the strategy $f_\exists$. ($f_\exists$ is *not* a *winning* strategy, as it does not agree on the relevant information sets.) *A fortiori*, then, there is no w.s. for ∀belard in the game $\mathrm{G}_A(\chi, \mathcal{M}_N, 0)$, and χ is *not* false in $\mathcal{M}_N$ at 0.

Hence the formula χ is in fact non-determined in $\mathcal{M}_N$ at 0. (■)

We proceed to show that for every natural number n, the first-order counterparts of the pointed modal structures $\mathcal{M}_{2^n}$ and $\mathcal{M}_{2^n+1}$ are '$(n+1)$-equivalent'. More exactly, the following is shown.

CLAIM 3. The first-order structures $\langle \mathcal{M}^{\mathbf{FO}}_{2^n}, 0\rangle$ and $\langle \mathcal{M}^{\mathbf{FO}}_{2^n+1}, 0\rangle$ satisfy precisely the same first-order formulae of one free variable with quantifier rank at most $n+1$, i.e. for all $\varphi(x)$ with $qr(\varphi) \leq n+1$:

$$\langle \mathcal{M}^{\mathbf{FO}}_{2^n}, 0\rangle \models \varphi(x) \Longleftrightarrow \langle \mathcal{M}^{\mathbf{FO}}_{2^n+1}, 0\rangle \models \varphi(x).$$

Proof of Claim 3. We show that *Duplicator* has a w.s. in the Ehrefeucht-Fraïssé game

$$\mathrm{EF}_{n+1}(\mathcal{M}^{\mathbf{FO}}_{2^n}, 0, \mathcal{M}^{\mathbf{FO}}_{2^n+1}, 0).$$

By *Ehrenfeucht's Theorem* this will, then, establish the statement of the claim.

To make the describing of *Duplicator*'s strategy easier, let us define "directed distance" in the set $\{1, \ldots, k\}$ $(k < \omega)$ as the following function $\delta : \{1, \ldots, k\}^2 \to \{0, \ldots, k-1\}$: [14]

$$\delta(x, y) = \begin{cases} y - x & \text{if } x \leq y \\ (k - x) + y & \text{if } y < x \end{cases}$$

Hence directed distance δ in the subset $\{1, \ldots, n\}$ of the domain of a modal structure $\mathcal{M}_n$ $(n < \omega)$ measures the minimum amount of steps from x to y along the 'circular' relation P_n.

We describe a strategy for *Duplicator* by the following set of recipes:

- Reply to any $x \in \{0, a, b\}$ by x itself, irrespective of whether it comes from D_{2^n} or D_{2^n+1}.
- If none of the players has yet chosen from $\{1, \ldots, 2^n\} \subseteq D_{2^n}$, and k is *Spoiler*'s **first** choice from there, reply by k itself.
- If none of the players has yet chosen from $\{1, \ldots, 2^n+1\} \subseteq D_{2^n+1}$, and k is *Spoiler*'s **first** choice from there, reply by k itself, if $k \leq 2^n$; if however $k = 2^n + 1$, then reply by 2^n.

For the rest of *Spoiler*'s choices, do as follows:

- If the previous pair of choices from $\{1, \ldots, 2^n\} \times \{1, \ldots, 2^n + 1\}$ was (m, m'), and *Spoiler* now chooses $k \in$

[14] By its definition δ satisfies that ($x = y$ iff $\delta(x, y) = 0$) and it satisfies the *triangle inequality*; but it does *not* satisfy the symmetry $\delta(x, y) = \delta(y, x)$. (The only instance of symmetry that holds is $\delta(\frac{k}{2}, k) = \delta(k, \frac{k}{2})$ when k is even.) This is, of course, in keeping with its being a "directed" distance function.

$\{1, \ldots, 2^n\} \subseteq D_{2^n}$, reply by the unique element $k' \in \{1, \ldots, 2^n + 1\} \subseteq D_{2^n+1}$ satisfying the two conditions

$$\begin{cases} \min\{\delta(m,k), \delta(k,m)\} = \min\{\delta(m',k'), \delta(k',m')\} \\ \delta(m',k') = \delta(m,k) \text{ or } \delta(k',m') = \delta(k,m) \end{cases}$$

The upper condition requires that k' be chosen so that its directed distance from m' (in some order) equals the minimum directed distance between m and k. This does not yet determine k', as the required $\min\{\delta(m,k), \delta(k,m)\}$ steps could be taken either along P_{2^n+1} or its converse. To make k' unique, we further require what is stated in the lower condition, namely that k' be reached from m' in these minimal amount of steps by moving in the direction of P_{2^n+1} iff k is similarly reached from m by going in the direction of P_{2^n}. Notice that because the size of D_{2^n+1} is one greater than that of D_{2^n}, we have that if $\delta(m',k') = \delta(m,k)$, then $\delta(k',m') = \delta(k,m) + 1 \neq \delta(k,m)$.

We claim that the strategy thus described is a w.s. for *Duplicator* in $\mathrm{EF}_{n+1}(\mathcal{M}^{\mathbf{FO}}_{2^n}, 0, \mathcal{M}^{\mathbf{FO}}_{2^n+1}, 0)$. To see this, let

$$(0, a_1, \ldots, a_{n+1}) \in D^{n+2}_{2^n} \text{ and } (0, b_1, \ldots, b_{n+1}) \in D^{n+2}_{2^n+1}$$

be arbitrary $(n+2)$-sequences constructed by playing the EF game $n+1$ rounds, assuming that *Duplicator* has followed the strategy described above. We must show that the relation

$$p : (0, a_1, \ldots, a_{n+1}) \mapsto (0, b_1, \ldots, b_{n+1})$$

is a partial isomorphism from $\mathcal{M}^{\mathbf{FO}}_{2^n}$ to $\mathcal{M}^{\mathbf{FO}}_{2^n+1}$. Write $dom(p) := \{0, a_1, \ldots, a_{n+1}\}$; $rng(p) := \{0, b_1, \ldots, b_{n+1}\}$.

(i) p is a function: Directly by the definition of p, if $c \in dom(p)$, there is $d \in rng(p)$ such that $p(c,d)$ holds: (c,d) is either (0,0) or else (a_i, b_i) for some i. We still must check that if $p(c, b_i)$ and $p(c, b_j)$, then $b_i = b_j$. But given *Duplicator*'s above strategy, a maximally quick way for *Spoiler* to force *Duplicator* to choose distinct elements $b_i \neq b_j$ for one and the same $c \in D_{2^n}$ in a play of the relevant EF game is clearly to pick out successively from D_{2^n+1} the elements

$$2^0, 2^1, \ldots, 2^{n-1}, 2^n, 2^n + 1.$$

The respective sequence of replays by *Duplicator* from D_{2^n} then is

$$2^0, 2^1, \ldots, 2^{n-1}, 2^n, 2^0.$$

Hence the first $n+1$ choices by *Duplicator* respect functionality, whereas if the $(n+2)$-th round was still played, 2^0 would end up being a reply to both 2^0 and 2^n+1, and the respective sequences would no longer serve to define a function. But as the plays of the EF game are of length $n+1$, *Duplicator* can always survive $n+1$ rounds correlating any given move of *Spoiler* with only one reply of her own. But then p in particular is a function.

(ii) p is an injection: Because the above-described strategy for *Duplicator* is used in constructing p, we have that if $c \in \{0, a, b\}$,

$$p(c, x) \Longrightarrow \ x = c.$$

Elements from $\{1, \ldots, 2^n\}$ are not related by p to any of the elements from $\{0, a, b\}$, so in order to see that p is an injection, it suffices to establish that any distinct $a_i, a_j \in \{1, \ldots, 2^n\} \cap$ *dom*(p) satisfy $p(a_i) \neq p(a_j)$. But due to *Duplicator*'s strategy, indeed for all arguments a_i, a_j we have:

- $\min\{\delta(a_i, a_j), \delta(a_j, a_i)\} =$
  $\min\{\delta(p(a_i), p(a_j)), \delta(p(a_j), p(a_i))\}$

- $\delta(p(a_i), p(a_j)) = \delta(a_i, a_j)$ or $\delta(p(a_j), p(a_i)) = \delta(a_j, a_i)$.

But this implies that p is an injection: the directed distance between the arguments is reproduced on the side of values of p.

(iii) It is trivial that the satisfaction of the unary predicate S (which is satisfied precisely by a in both structures) is preserved under p. The rule for moves within $\{1, \ldots, 2^n\}$ *resp.* $\{1, \ldots, 2^n + 1\}$ is so formulated that preservation of the satisfaction of the binary predicate P under the function p is trivial as well. Further (because p is an injective function),

$$\langle \mathcal{M}^{\mathbf{FO}}_{2^n}, a_i, a_j \rangle \models Q(x, y) \iff a_i = a_j \in \{1, \ldots, 2^n\} \iff$$
$$p(a_i) = p(a_j) \in \{1, \ldots, 2^n + 1\} \iff$$
$$\langle \mathcal{M}^{\mathbf{FO}}_{2^n+1}, p(a_i), p(a_j) \rangle \models Q(x, y).$$

Finally, consider the binary predicate R. Let $a_i \in dom(p)$ be arbitrary. Then:

$$\langle \mathcal{M}^{\mathbf{FO}}_{2^n}, 0, a_i \rangle \models R(x, y) \iff a_i \in \{1, \ldots, 2^n\} \iff$$
$$p(a_i) \in \{1, \ldots, 2^n + 1\} \iff$$
$$\langle \mathcal{M}^{\mathbf{FO}}_{2^n+1}, p(0), p(a_i) \rangle \models R(x, y).$$

Further, recall that if $c \in \{a, b\}$, then $p(c) = c$. Hence:

$$\langle \mathcal{M}^{\mathbf{FO}}_{2^n}, a_i, c \rangle \models R(x, y) \iff c \in \{a, b\} \iff$$
$$\langle \mathcal{M}^{\mathbf{FO}}_{2^n+1}, p(a_i), p(c) \rangle \models R(x, y).$$

We may conclude that a w.s. exists for *Duplicator* in the game

$$\mathrm{EF}_{n+1}(\mathcal{M}^{\mathbf{FO}}_{2^n}, 0, \mathcal{M}^{\mathbf{FO}}_{2^n+1}, 0).$$

By *Ehrenfeucht's Theorem* we conclude that $\langle \mathcal{M}^{\mathbf{FO}}_{2^n}, 0 \rangle$ and $\langle \mathcal{M}^{\mathbf{FO}}_{2^n+1}, 0 \rangle$ satisfy precisely the same first-order formulae of one free variable with quantifier rank at most $n + 1$. ($\blacksquare$)

We have thus proven:

(1) For all even $N < \omega : \mathcal{M}_N \models^+ \chi[0]$.

(2) For all odd $N < \omega : \mathcal{M}_N \models^0 \chi[0]$.

(3) For all $n < \omega$: $\langle \mathcal{M}^{\mathbf{FO}}_{2^n}, 0\rangle$ and $\langle \mathcal{M}^{\mathbf{FO}}_{2^n+1}, 0\rangle$ are '$(n+1)$-equivalent'.

We go on to derive the statement of Lemma 3.3.8. Still writing χ for the **EIFML**[3] formula

$$\Box(R^1_\emptyset) \vee_{i_2 \in \{\lambda,\rho\}} \wedge_{i_3 \in \{\lambda,\rho\}} \Box(R^4_{i_2 i_3}) \Diamond(R^5_{i_2 i_3})/\{1,3\}\varphi_{i_2 i_3},$$

assume for contradiction that there indeed exists a first-order formula $\varphi_\chi(x)$ such that for all modal structures $\mathcal{M}$ and for all $d \in dom(\mathcal{M})$:

$$\mathcal{M} \models^+ \chi[d] \Longleftrightarrow \langle \mathcal{M}^{\mathbf{FO}}, d\rangle \models \varphi_\chi(x).$$

By Claim 3 we know that for all $n < \omega$, the structures $\langle \mathcal{M}^{\mathbf{FO}}_{2^n}, 0\rangle$ and $\langle \mathcal{M}^{\mathbf{FO}}_{2^n+1}, 0\rangle$ are '$(n+1)$-equivalent.' Write then r for the quantifier rank of φ_χ. Hence, by Claim 3, in particular the structures

$$\langle \mathcal{M}^{\mathbf{FO}}_{2^r}, 0\rangle \text{ and } \langle \mathcal{M}^{\mathbf{FO}}_{2^r+1}, 0\rangle$$

satisfy precisely the same first-order formulae of quantifier rank at most $r+1$. But since $qr(\varphi_\chi) = r < r+1$, we have

$$(*) \qquad \langle \mathcal{M}^{\mathbf{FO}}_{2^r}, 0\rangle \models \varphi_\chi(x) \Longleftrightarrow \langle \mathcal{M}^{\mathbf{FO}}_{2^r+1}, 0\rangle \models \varphi_\chi(x).$$

Now on the one hand, since by assumption $\varphi_\chi(x)$ is a translation of χ, we have in particular

$$\begin{cases} \mathcal{M}_{2^r} \models^+ \chi[0] \Longleftrightarrow \langle \mathcal{M}^{\mathbf{FO}}_{2^r}, 0\rangle \models \varphi_\chi(x) \ ; \\ \mathcal{M}_{2^r+1} \models^+ \chi[0] \Longleftrightarrow \langle \mathcal{M}^{\mathbf{FO}}_{2^r+1}, 0\rangle \models \varphi_\chi(x). \end{cases}$$

Hence by $(*)$:

$$\mathcal{M}_{2^r} \models^+ \chi[0] \iff \mathcal{M}_{2^r+1} \models^+ \chi[0].$$

But on the other hand, by Claim 1 and Claim 2 we know that if $r > 0$, then

$$\mathcal{M}_{2^r} \models^+ \chi[0] \text{ but } \mathcal{M}_{2^r+1} \not\models^+ \chi[0];$$

and if $r = 0$, then

$$\mathcal{M}_{2^r} \not\models^+ \chi[0] \text{ but } \mathcal{M}_{2^r+1} \models^+ \chi[0].$$

Hence, whichever natural number r is, we reach a contradiction. So we may conclude that the **EIFML**[3] formula χ has no translation into **FO**. This finishes the proof of Lemma 3.3.8. ■

We thus trivially obtain the following general result:

Theorem 3.3.9 *For every $k \geq 3$, there exists no translation of* **EIFML**$[k]$ *into* **FO**.

Proof. The result follows immediately from Lemma 3.3.8. ■

For the value $k := 1$, there is only one accessibility relation in the models relative to which **EIFML**$[k]$ is evaluated. Therefore only one relation in particular can be associated with an expression of the form $\Diamond(R^j_{\bar{i}_j})/W_j$, and whether or not a set W_j contains indices of some conjunctions $(\wedge_{i_k \in I_k})$ makes no semantic difference. For, if corresponding to one index $i_k \in I_k$ it is possible to choose b so that $R(a, b)$, this is possible for all indices $i_k \in I_k$. It would be different if there were several relations, for then the relevant relation could be different for different indices. Making this observation explicit, it would be possible to show that in fact **EIFML**[1] is translatable into **FO**.

By contrast, we conjecture that for the value $k := 2$, **EIFML**[k] cannot be given a first-order translation, but in fact **EIFML**[2] already has some second-order expressive power. Providing a proof to this claim will be left for another occasion.

3.4 IF Modal Logic and Basic Modal Logic

In the present section the expressive powers of IF modal logic (**IFML**[k]) and its traditional sibling (**ML**[k]) are compared with respect to certain classes of modal structures. The relative expressive powers of IF tense logic (**IFTL**[k]) and basic tense logic (**TL**[k]) are also considered.[15]

More specifically, it is shown that for arbitrary k-ary modal structures and all $k \geq 1$, **IFML**[k] is strictly more expressive than **ML**[k] (Theorem 3.4.4). An analogous result is shown to hold for **TL**[k] and **IFTL**[k] relative to genuine k-ary temporal structures (Lemma 3.4.8).

Further, it is proven that for $k = 1$, over the class of k-ary linear temporal structures, **IFTL**[k] and **TL** [k] have the same expressive power (Lemma 3.4.11). From the proof of this result it immediately follows that the expressive powers of **IFML**[1] and **ML**[1] indeed coincide relative to unary linear modal structures (Lemma 3.4.12).

In Lemma 3.4.13.(i) it is shown that for $k \geq n \geq 2$ and for k-ary modal structures of whose accessibility relations n are linear, **IFML**[k] is strictly more expressive than **ML**[k]. This shows that Lemma 3.4.12 cannot be generalized beyond $k = 1$, and that in Lemma 3.4.11 the class of unary linear temporal

[15] The results of *Subsect.* 3.4 have appeared in Tulenheimo (2003).

structures cannot be replaced by the class of all binary modal structures linear in both dimensions. Lemma 3.4.13.(ii) shows that Lemma 3.4.11 is not generalizable to any $k > 1$.

3.4.1 Arbitrary k-ary structures

It will be proven here that over the class of all unary modal structures, IF modal logic with one modality type (i.e. **IFML**[1]) is indeed strictly more expressive than basic modal logic with one modality type (i.e. **ML**[1]). In particular, the proof for this claim will establish that **IFML**[1] is capable of distinguishing such unary modal structures $\mathcal{M}$ and $\mathcal{N}$ which have *mutually isomorphic frames* and which cannot be distinguished by any **ML**[1] formula, and so is capable of distinguishing genuine properties of *models* as opposed to mere properties of frames.

An analogous result regarding the relationship of Priorean tense logic (or **TL**[1]) and IF tense logic (or **IFTL**[1]) will be established as well.

Write $\mathcal{C}_1$ for the class of all unary modal structures. Observe that as noticed in Fact 3.1.1, **ML**[1] is anyway embeddable in **IFML**[1] over $\mathcal{C}_1$. Hence, to show that in fact

$$\mathbf{ML}[1] <_{\mathcal{C}_1} \mathbf{IFML}[1],$$

we need only to find modal structures $\mathcal{M}$,$\mathcal{N} \in \mathcal{C}_1$ and points $a \in dom(\mathcal{M})$, $b \in dom(\mathcal{N})$ such that:

- for each **ML**[1] formula φ, $\mathcal{M} \models^+ \varphi[a] \Longleftrightarrow \mathcal{N} \models^+ \varphi[b]$,

and

- $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ can be distinguished by an **IFML**[1] formula.

In particular, for the purpose of simply proving that $\mathbf{ML}[1] <_{\mathcal{C}_1} \mathbf{IFML}[1]$, we need *not* require that the frames of $\mathcal{M}$ and $\mathcal{N}$ be isomorphic. Actually, the structures $\mathcal{M}^\star$ and $\mathcal{N}^\star$, described next, serve to establish the claim.

Example 3.4.1 *We say that the relation R in a modal structure $(D, R, \mathfrak{h})$ is weakly confluent at the point $d \in D$, if there exists $c \in D$ such that for all $d' \in D$: $R(d, d') \Longrightarrow R(d', c)$.*[16] *We observe that for any modal structure $\mathcal{M} = (D, R, \mathfrak{h})$ and any $d \in D$,*

$$\mathcal{M} \models^+ \Box_1 \Diamond_2/\{1\}(q \vee \neg q)[d] \Longleftrightarrow R \textit{ is weakly confluent at } d.$$

Write $\chi := \Box_1 \Diamond_2/\{1\}(q \vee \neg q)$. The implication holds from right to left: define a strategy f for Héloïse by putting $f(d, d') := c$ for all R-successors d' of d, where c is a point (given by the weak confluence of R) satisfying $R(d', c)$ for all d' with $R(d, d')$. Obviously f, then, is winning in the game associated with χ. The implication also holds from left to right: letting f be a w.s. for Héloïse in the game associated with χ, there is a point c such that $c = f(d, d')$ for all R-successors d' of d. But this means that R is weakly confluent.

Then put $\mathcal{M}^\star := (M, R, \mathfrak{h})$, and $\mathcal{N}^\star := (N, R', \mathfrak{h}')$, where:

[16] Weak confluence is in contrast to confluence *simpliciter*. Let R be a binary relation on D, $d \in D$. Then *confluence* of R at d and *weak confluence* of R at d are defined by the following first-order conditions:

Confluence:

$\langle (D, R), d\rangle \models \forall z[\exists y(R(x, y) \wedge R(y, z)) \rightarrow \forall y(R(x, y) \rightarrow R(y, z))]$

Weak confluence:

$\langle (D, R), d\rangle \models \exists z \forall y(R(x, y) \rightarrow R(y, z))$.

Confluence *simpliciter* at a point is characterized in *basic modal logic* by the *validity* of the formula $\Diamond\Box p \rightarrow \Box\Diamond p$, i.e. the truth of this formula at d in *all* modal structures based on the frame (D, R).

$$M = \{1, \ldots, 5\}; \qquad N = \{1, \ldots, 4\}$$

$$R = \{(1,2), (1,3), (2,4), (3,5), (1,4), (1,5)\} \cup$$
$$\{(x, x) : x \in M\}$$

$$R' = \{(1,2), (1,3), (2,4), (3,4), (1,4)\} \cup \{(x, x) : x \in N\}$$

$$\mathfrak{h}(q) = \ \{2, 3, 4, 5\}; \qquad \mathfrak{h}'(q) = \{2, 3, 4\}$$

Since the domains M and N are not equipotent, the frames (M, R) and (N, R') of the structures $\mathcal{M}^\star$ and $\mathcal{N}^\star$ are of course not isomorphic. The structures can be depicted as in Figure 11 below. While it is not indicated in the figure, all elements in the domains of these models are related to themselves (reflexivity).

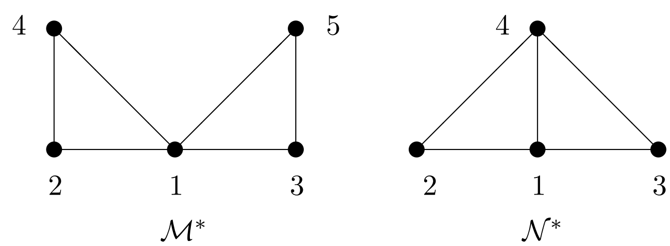

FIGURE 11

It is easy to see that the relation

$$\{(1,1), (2,2), (3,3), (4,4), (5,4)\}$$

is a bisimulation between the structures $\mathcal{M}^\star$ and $\mathcal{N}^\star$. So in particular the pointed modal structures $(\mathcal{M}^\star, 1)$ and $(\mathcal{N}^\star, 1)$ are not distinguished by an $\mathbf{ML}[1]$ *formula (by Invariance Lemma). On the other hand, the* $\mathbf{IFML}[1]$ *formula*

$$\Box_1 \Diamond_2 / \{1\} (q \vee \neg q)$$

is true in $\mathcal{N}^\star$ *at* 1 (R' *is weakly confluent at* 1) *but non-determined in* $\mathcal{M}^\star$ *at* 1 (R *is not weakly confluent at* 1*, however for all* R*-successors* x *of* 1 *there is an* R*-successor* y *of* x).

Hence weak confluence is a property of points in frames that is characterizable in **IFML**[1]*, but formulae of* **ML**[1] *cannot distinguish points that are weakly confluent from points that are not. The latter fact can be expressed in another way by saying that weak confluence is a property that is not preserved under bisimulations, as just seen.* ■

Example 3.4.2 *Define temporal structures* $\mathcal{M}^\star := (M, R, R^{-1}, \mathfrak{h})$*, and* $\mathcal{N}^\star := (N, R', R'^{-1}, \mathfrak{h}')$ *as follows:*

$$M = \{a, b_1, b_2, c_1, c_2\} \qquad (5 \textit{ distinct elements})$$

$$N = \{a', b', c'_1, c'_2\} \qquad (4 \textit{ distinct elements})$$

$$R = \{(a, x) : x \neq a\} \cup \{(b_1, c_1), (b_2, c_2)\}$$

$$R' = \{(a', x) : x \neq a'\} \cup \{(b', c'_1), (b', c'_2)\}$$

$$\mathfrak{h}(\top) = M, \qquad \mathfrak{h}'(\top) = N.$$

Hence R *and* R' *in particular are irreflexive and transitive relations.* (*See Figure 12 below for a picture of* $\mathcal{M}^\star$ *and* $\mathcal{N}^\star$.)

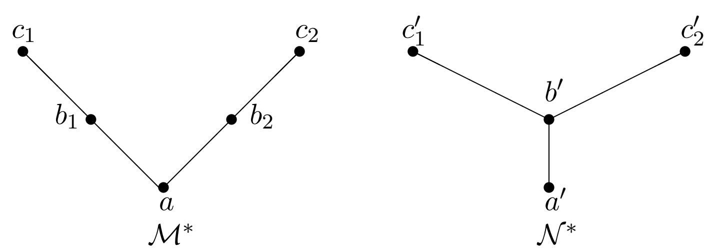

FIGURE 12

The temporal structures $\mathcal{M}^\star$ *and* $\mathcal{N}^\star$ *are bisimilar, as is witnessed by the relation*

$$\{(a, a')\} \cup \{(b_i, b') : i \in \{1,2\}\} \cup \{(c_i, c'_j) : i, j \in \{1,2\}\}.$$

The pointed temporal structures $(\mathcal{M}^\star, a)$ *and* $(\mathcal{N}^\star, a')$ *are therefore not distinguished by any formula of* $\mathbf{TL}[1] = \mathbf{ML}[2]$. *On the other hand, they are distinguished by the formula* φ *of* $\mathbf{IFTL}[1] = \mathbf{IFML}[2]$,

$$\varphi := \Box_1 \Box_2 \Diamond_3^{-1} / \{2\} \top.$$

This formula is non-determined in $\mathcal{M}^\star$ *at* a. [*Héloïse does not have a w.s., since there is no legal uniform reply available: no choice would legally extend both histories* (a, b_1, c_1), (a, b_2, c_2). *Abélard has none either, since Héloïse has some legal reply in both cases.*] *On the other hand, the formula is true in* $\mathcal{N}^\star$ *at* a'. [*Héloïse can choose the element* b' *as a legal extension for both histories* (a', b', c'_1), (a', b', c'_2).] ■

By *Fact* 3.1.1 we know that $\mathbf{ML}[k] \leq_{\mathcal{C}_k} \mathbf{IFML}[k]$, for all $k < \omega$. Hence Example 3.4.1 above already suffices to establish the claim that

$$\mathbf{ML}[1] <_{\mathcal{C}_1} \mathbf{IFML}[1],$$

and Example 3.4.2 is enough to guarantee that

$$\mathbf{TL}[1] <_{\mathcal{T}_1} \mathbf{IFTL}[1],$$

where $\mathcal{T}_1$ is the class of all unary temporal structures. However, we can do better than resort to the observation above: we can prove the claim of the strictly greater expressive power of $\mathbf{IFML}[1]$ as compared with that of $\mathbf{ML}[1]$ by using unary modal structures with *isomorphic frames*, as suggested above. The same can be done in the case of tense logics of one temporal modality type.

Lemma 3.4.3 *Over the class $\mathcal{C}_1$ of all unary modal structures,* **IFML**[1] *is strictly more expressive than* **ML**[1]. *In particular, there are pointed modal structures $(\mathcal{M}^\star, m)$ and $(\mathcal{N}^\star, n)$ having isomorphic frames and being* **ML**[1] *equivalent but not* **IFML**[1] *equivalent.*

Proof. Construct unary modal structures $\mathcal{M}$ and $\mathcal{M}'$ as follows. Put $\mathcal{M} = (M, R, \mathfrak{h})$ and $\mathcal{M}' = (M', R', \mathfrak{h}')$, where

$$M = M' = \{a, b, c, d, e, f\} \qquad \text{(6 distinct elements)}$$

$$R = R' = \{(a,b), (a,c), (b,d), (b,e), (c,e), (c,f)\}$$

$$\mathfrak{h}(q) = \{e\}, \quad \mathfrak{h}'(q) = \{d, f\}.$$

These modal structures are illustrated in Figure 13 below.

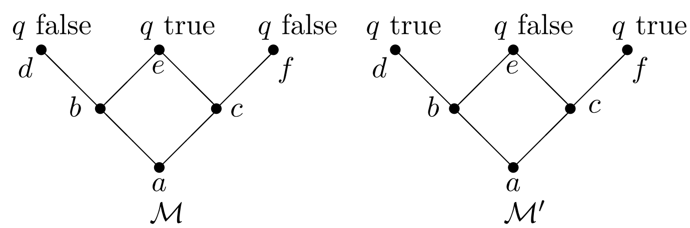

FIGURE 13

We observe:

(1) The frame of $\mathcal{M}$ is the same as the frame of $\mathcal{M}'$, so the frames are isomorphic via the identity map $\mathtt{id}_M : M \to M'$.

(2) Write $\chi := \Box_{0,1}\Diamond_{0,2}/\{1\}q$. We have that

$$\mathcal{M} \models^+ \chi[a] \text{ but } \mathcal{M}' \models^0 \chi[a].$$

The formula χ is non-determined in $\mathcal{M}'$ at a: *Héloïse* does not have a w.s., since there is no reply available to her that would legally extend both of the histories (a, b) and (a, c) *and* at which q would be true. Neither does *Abélard* have a w.s., since *Héloïse* has *some* reply making q true in both cases, a reply which is not the same in the two cases. On the other hand, χ is true in $\mathcal{M}$ at a: the point e is a legal extension to both (a, b) and (a, c), and q is true in $\mathcal{M}$ at e.

Hence the pointed modal structures $(\mathcal{M}, a)$ and $(\mathcal{M}', a)$ are *not* **IFML**[1] equivalent.

(3) It is easy to check that the relation $\equiv :=$

$$\{(a, a), (b, b), (c, c), (b, c), (c, b), (d, e), (f, e), (e, d), (e, f)\}$$

is a bisimulation between the modal structures $\mathcal{M}$ and $\mathcal{M}'$. In fact, it is even seen that the relation $\equiv$ is a bisimulation between the *unary temporal* structures

$$\mathcal{M}^{+} = (M, R, R^{-1}, \mathfrak{h}) \text{ and } \mathcal{M}'^{+} = (M', R', R'^{-1}, \mathfrak{h}') ,$$

obtained from the unary modal structures $\mathcal{M} = (M, R, \mathfrak{h})$ and $\mathcal{M}' = (M', R', \mathfrak{h}')$ simply by adding to the respective unary modal structure the *converse* of its accessibility relation as another accessibility relation. (This fact will be used when deducing Corollary 3.4.6 below.)

Since in particular a $\equiv$ a, we have by *Invariance Lemma* that the modal structures $(\mathcal{M}, a)$ and $(\mathcal{M}', a)$ are **ML**[1] equivalent, and that the temporal structures $(\mathcal{M}^{+}, a)$ and $(\mathcal{M}'^{+}, a)$ are **TL**[1] equivalent.

Now, by (2), (3) and Fact 3.3.1, **IFML**[1] is strictly more expressive than **ML**[1] over the class $\mathcal{C}_1$ of all unary modal structures. And (1), (2) and (3) together establish that there are unary modal structures $\mathcal{M}^{\star}$ and $\mathcal{N}^{\star}$ with isomorphic frames, and points m and n in the respective domains, such that $(\mathcal{M}^\star, m)$ and $(\mathcal{N}^\star, n)$ are **ML**[1] equivalent but not **IFML**[1] equivalent. This completes the proof of Lemma 3.4.3. ■

Lemma 3.4.3 has the straightforward consequence that for any positive $k < \omega$, **IFML**$[k]$ is strictly more expressive than **ML**$[k]$ over the class $\mathcal{C}_k$ of arbitrary k-ary modal structures. Namely, since the formula $\Box_1 \Diamond_2/\{1\}q$ employed in the proof of this lemma can be written in **IFML**[1], it can *a fortiori* be written in any **IFML**$[k]$, $k \geq 1$. Then simply replace the unary modal structures $\mathcal{M} = (M, R, \mathfrak{h})$ and $\mathcal{M}' = (M', R', \mathfrak{h}')$, utilized in the above proof, by the k-ary modal structures

$$\mathcal{M}^\star = (M, R, S_2, \ldots, S_k, \mathfrak{h}),$$

$$\mathcal{M}^{\star\star} = (M', R', S'_2, \ldots, S'_k, \mathfrak{h}'),$$

where the new relations S_i, S'_j are otherwise arbitrary binary relations on $M = M'$, but they satisfy $S_i = S'_i$ $(i := 2, \ldots, k)$. (Another possibility in view of our purpose would be to put $S_2 = \ldots = S_k = R$ and $S'_2 = \ldots = S'_k = R'$.) Hence $\mathcal{M}^\star$ and $\mathcal{M}^{\star\star}$ will automatically be bisimilar (because $\mathcal{M}$ and $\mathcal{M}'$ are), and it follows that $(\mathcal{M}^\star, a)$ and $(\mathcal{M}^{\star\star}, a)$ are **ML**$[k]$ equivalent, while these pointed modal structures are *not* **IFML**$[k]$ equivalent – as they are distinguished by the formula $\Box_1 \Diamond_2/\{1\}q$. Hence we have:

Theorem 3.4.4 *For any positive $k < \omega$,*

$$\mathbf{ML}[k] <_{\mathcal{C}_k} \mathbf{IFML}[k]. \ ■$$

We may notice that unlike the respective accessibility relations R and R' of the modal structures $\mathcal{M}$ and $\mathcal{M}'$ made use of in the proof of Lemma 3.4.3, the relations R of $\mathcal{M}^\star$ and R' of $\mathcal{N}^\star$ employed in Example 3.4.1 above are transitive. In fact, the latter two relations are *reflexive*, *antisymmetric* and *transitive*, viz. *reflexive partial orders*. Hence, writing **PO** for the subclass of $\mathcal{C}_1$ consisting of precisely those unary modal structures whose frames are *reflexive partial orders*, we have by Example 3.4.1:

Corollary 3.4.5 $\mathbf{ML}[1] <_{\mathbf{PO}} \mathbf{IFML}[1]$. ■

Let us say that a k-ary modal structure $\mathcal{M}$ is a *partial order in n dimensions*, if $\mathcal{M}$ has $n \leq k$ accessibility relations which are reflexive partial orders, and let us adopt the convention of writing $\mathbf{PO}[k,n]$ for the class of k-ary modal structures which are partial orders in n dimensions. Then Corollary 3.4.5 trivially implies that for all $k \geq 1$ and all $n \in \{1,\ldots,k\}$,

$$\mathbf{ML}[k] <_{\mathbf{PO}[k,n]} \mathbf{IFML}[k].$$

Let then $\mathcal{T}_k$ be the class of all k-ary *temporal* structures, i.e. $2k$-ary modal structures

$$\mathcal{M} = (D, R_0, \ldots, R_{2k-1}, \mathfrak{h})$$

with the characteristic features that: **(i)** for each $i < k$, the relation R_{i+k} is the *converse* of the relation R_i, and **(ii)** each R_i is irreflexive and transitive. Recall that those modal structures that meet condition (i) but not (ii) we have agreed (in *Sect.* 2.1) to call *quasi-temporal structures*.

Now *if* we do not require the properties of irreflexivity and transitivity of temporal accessibility relations (which, however, is customary), then we get a *tense-logical analogue* of Theorem 3.4.4 for free, due to our proof of Lemma 3.4.3 above. Directly by Theorem 3.4.4 we have, of course, that for all positive $k < \omega$, $\mathbf{IFTL}[k] = \mathbf{IFML}[2k]$ is strictly more expressive than $\mathbf{TL}[k] = \mathbf{ML}[2k]$ over the class $\mathcal{C}_{2k}$ of all $2k$-ary modal structures. But we can do better. By the above proof of Lemma 3.4.3, we can indeed show the same even for the proper subclass QT_k of $\mathcal{C}_{2k}$ which consists of all quasi-temporal structures.

Corollary 3.4.6 *For any positive $k < \omega$, let QT_k be the class of quasi-temporal structures from $\mathcal{T}_k$. Then we have:*

$$\mathbf{TL}[k] <_{QT_k} \mathbf{IFTL}[k].$$

Proof. The statement of the theorem will trivially follow, if we manage to establish this statement for the value $k = 1$. To do so, it suffices to show that there are $\mathbf{ML}[2]$ equivalent, but $\mathbf{IFML}[2]$ distinguishable, pointed modal structures $(\mathcal{N}, c)$, $(\mathcal{N}', d)$ with $\mathcal{N}, \mathcal{N}' \in QT_k$. (Hence $\mathcal{N}$ and $\mathcal{N}'$ are required to be quasi-temporal structures and not just any binary modal structures.)

Let $\mathcal{M} = (M, R, \mathfrak{h})$ and $\mathcal{M}' = (M', R', \mathfrak{h}')$ be the unary modal structures defined in the proof of Lemma 3.4.3. Then the structures $\mathcal{M}^+ = (M, R, R^{-1}, \mathfrak{h})$ and $\mathcal{M}'^+ = (M', R', R'^{-1}, \mathfrak{h}')$, obtained from $\mathcal{M}$ and $\mathcal{M}'$ by adding to these the converses of their accessibility relations, are in the class QT_1. But the relation $\equiv$ defined in the proof of Lemma 3.4.3 is in fact a bisimulation between the binary modal structures $\mathcal{M}^+$ and $\mathcal{M}'^+$, as noticed when proving this lemma. As well, the formula $\Box_1 \Diamond_2/\{1\} q$ is true in $\mathcal{M}^+$ at a, but non-determined in $\mathcal{M}'^+$ at a. Hence the statement follows. ■

The above observation is tense-logical in a rather abstract sense, in which a modal logic is taken for a tense logic on the sole basis that this logic is never able to speak of an accessibility relation without also being able to speak of its converse. However, it is possible to construct even genuine temporal structures (structures whose accessibility relations are irreflexive partial orders) – with isomorphic frames, moreover – in such a way that they establish that $\mathbf{IFTL}[k]$ has a greater expressive power than $\mathbf{TL}[k]$.

Lemma 3.4.7 *There are pointed temporal structures* $(\mathcal{M}, s)$ *and* $(\mathcal{N}, s')$ *with isomorphic frames which are* **TL**[1] *equivalent but not* **IFTL**[1] *equivalent.*

Proof. Let $(C_1, <_1)$ be the set of positive rationals ordered by magnitude. Write $C_2 := C_1 \times \{0\}$, and let $(C_2, <_2)$ be an isomorphic copy of $(C_1, <_1)$. Finally, let t, t_1, t_2 be three *irrational* numbers. Define a frame (D, R) as follows:

- $D := \mathbb{Q} \cup C_2 \cup \{t, t_1, t_2\}$
- $R := < \cup [\mathbb{Q} \times \{t_1\}] \ \cup \ [(\mathbb{Q} \setminus C_1) \times (C_2 \cup \{t_2\})] \ \cup \ <_2 \ \cup$

  $[C_2 \times \{t_2\}] \cup \{(t, t_1), (t, t_2)\}$

where $\mathbb{Q}$ stands for the set of all rational numbers, and $<$ for their order by magnitude. Finally, let **prop** $= \{q\}$, and define assignments $\mathfrak{h}$ and $\mathfrak{h}'$ by setting:

- $\mathfrak{h}(q) = \{1, 1'\}, \qquad \mathfrak{h}'(q) = \{0\}$

where $1'$ is the image of 1 under a fixed isomorphism that establishes the isomorphism between $(C_1, <_1)$ and $(C_2, <_2)$. Figure 14 illustrates the temporal structures $\mathcal{M} = (D, R, R^{-1}, \mathfrak{h})$ and $\mathcal{N} = (D, R, R^{-1}, \mathfrak{h}')$.

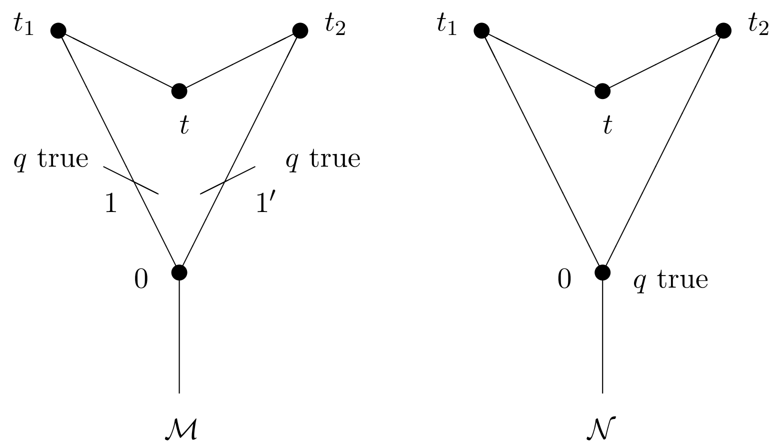

FIGURE 14

The relation R is clearly irreflexive and transitive. Observe that in the structures the 'times' t_1 and t_2 are both in the future of t (and are mutually R-incomparable). Further, each of the times t_1 and t_2 has a past that branches into two: one branch contains only the time t, while the other branch is infinite. Furthermore, the infinite pasts of t_1 and t_2 meet at the rational number 0, which itself has a unique past.

The structures $\mathcal{M}$ and $\mathcal{N}$ are genuine temporal structures. They also have a *common* frame (D, R, R^{-1}), and so *a fortiori* their frames are isomorphic. Furthermore, the temporal structures in question are bisimilar, as is witnessed by the relation

$$\{(t,t)\} \cup [\{t_1,t_2\} \times \{t_1,t_2\}] \ \cup \ \{(1,0),(1',0)\} \cup$$

$$[\{x : 1 <_1 x <_1 t_1 \text{ or } 1' <_2 x <_2 t_2\}\times$$

$$\{x : 0 <_1 x <_1 t_1 \text{ or } 0 <_2 x <_2 t_2\}] \cup$$

$$[\{x : x <_1 1 \text{ or } x <_2 1'\} \times \{x : x <_1 0\}].$$

Hence the pointed temporal structures $(\mathcal{M}, t)$ and $(\mathcal{N}, t)$ cannot be distinguished by any **TL**[1] formula. On the other hand, the **IFTL**[1] formula

$$\varphi := \Box_1 \Diamond_2^{-1}/\{1\} q$$

serves to distinguish the structures. Namely, φ is non-determined in $\mathcal{M}$ at t, because *Héloïse* can choose a past time at which q holds, but only depending on *Abélard*'s future choice of t_1 *resp.* t_2: for t_1 she must pick out 1 and for t_2 the element $1' \neq 1$. But φ is true in $\mathcal{N}$ at t: independently of whether *Abélard* chooses t_1 or t_2, *Héloïse* can pick out 0, which makes q true. Hence we may conclude that **IFTL**[1] is more expressive than **TL**[1] over the class of all unary temporal structures. ■

From Lemma 3.4.7 we directly obtain the general result:

Theorem 3.4.8 *For any positive $k < \omega$,*

$$\mathbf{TL}[k] <_{\mathcal{T}_k} \mathbf{IFTL}[k]. \;\blacksquare$$

3.4.2 Tense logic and unary linear structures

We move on to consider unary modal structures with linear frames, i.e. frames whose accessibility relation is *antisymmetric*, *transitive* and *connected*: in short, a *linear order*. Furthermore, we will assume throughout that the relation is *irreflexive*.

Let us first establish some terminology. We say that a k-ary modal structure $\mathcal{M}$ is *linear in n dimensions*, if $\mathcal{M}$ has $n \leq k$ accessibility relations which are linear orders. A k-ary *temporal* structure is said to be linear in n dimensions, if it has $2n \leq 2k$ accessibility relations which are linear orders. We adopt the convention of writing $\mathbf{LO}[k, n]$ for the class of k-ary modal structures linear in n dimensions. By stipulation we write $\mathbf{LO}[k] := \mathbf{LO}[k, k]$. The class of k-ary temporal structures linear in n dimensions are denoted by writing $\mathbf{LO}_{temp}[k,n]$, and we stipulate that $\mathbf{LO}_{temp}[k] := \mathbf{LO}_{temp}[k,k]$.

We write $\mathbf{TL} := \mathbf{TL}[1]$, $\mathbf{IFTL} := \mathbf{IFTL}\ [1]$, and $\mathbf{LO} := \mathbf{LO}_{temp}[1]$. By a typical tense-logical convention, we write F, P, G and H for the operators $\Diamond$, $\Diamond^{-1}$, $\Box$ and $\Box^{-1}$, respectively. We call the logic **IFTL** simply *IF tense logic*, and **TL** *basic tense logic*.[17]

Given a linear frame $(T,<)$, we say that a point $t \in T$ is *dense to the right*, if t is *not* $<$-maximal, and for all $t' > t$ there is s such that $t < s < t'$. The meaning of "dense to the left" is analogously defined. Further, if there exists an element $t' > t$ such that

$$(\forall s)(t < s \Longrightarrow\ t' \leq s),$$

we say t' is the *immediate successor* of t. (Because $<$ is antisymmetric, such a t' is unique when it exists.) What it means for a point to have an *immediate predecessor* is analogously defined.

Recall the definition of *transition* from Definition 3.2.5: an **FO** formula $\beta(x, x_{i_1}, \ldots, x_{i_n}; \varphi[x_{i_n}]) :=$

$$(R_{j_1}(x, x_{i_1})\, o_{j_1} (R_{j_2}(x_{i_1}, x_{i_2}) \ldots o_{j_{n-2}}$$
$$(R_{j_{n-1}}(x_{i_{n-2}}, x_{i_{n-1}})\, o_{j_{n-1}}\ (R_{j_n}(x_{i_{n-1}}, x_{i_n})\, o_{j_n} \varphi[x_{i_n}])) \ldots))$$

is a transition when the R_{j_i} are binary relations ($j_i < k$), the $\mathrm{o}_{j_i} \in \{\wedge, \rightarrow\}$, and $\varphi\,[x_{i_n}]$ is a standard translation of an $\mathbf{ML}[k]$ formula.

For the purposes of the proof below, recall that by the syntax of **TL**, the negation sign ($\neg$) can only appear as prefixed to a propositional atom in formulae of this logic. However,

[17] In fact, **TL** coincides with *Priorean tense logic*, to be introduced in *Sect.* 4.2.

by Fact 2.1.1 this is not a restriction from the viewpoint of expressive power, as every formula of **TL**[$\neg$] has an equivalent (its *negation normal form*) in the class **TL**. Below, we allow writing $\neg\varphi$ when φ is an arbitrary **TL** formula. By stipulation we then take the string "$\neg\varphi$" to stand for the **TL** formula $nnf(\neg\varphi)$.

We now proceed to prove that relative to unary linear temporal structures, the expressive powers of **TL** and **IFTL** coincide. To do this, we first prove a lemma concerning the first-order translations of **IFTL** formulae $-$ as these were provided by Theorem 3.2.4 above. This lemma in fact takes care of everything that is not completely trivial in the proof of the coincidence theorem. It will further follow from the proof that the modal logics **ML**[1] and **IFML**[1] of one modality type have the same expressive power over the class **LO**[1] of unary linear modal structures (Theorem 3.4.12 below).

Lemma 3.4.9 *For any $n < \omega$, any $Q_1, \ldots, Q_n \in \{\exists, \forall\}$, and any transition β involving no other binary relation symbols than $<$ and $>$, the formula*

$$\exists y Q_1 x_1 \ldots Q_n x_n \beta\left(x_0, \ldots, x_n, y; \varphi\left[y\right]\right)$$

has the property $(*)$ of being equivalent over linear orders[18] *to a standard translation of a* **TL** *formula.*

Proof. We prove the claim by induction on the length n of the block $Q_1 x_1 \ldots Q_n x_n$. If $\varphi = ST_x(\psi)$, we write by convention $\psi = \varphi^*$. And if φ and θ are **FO**[τ] formulae, we write $\varphi \Longleftrightarrow \theta$ to say that these formulae are satisfied in precisely the same τ-structures.

For the base case, $n := 0$, we have:

[18] That is, equivalent over the models $\mathcal{M}^{\mathbf{FO}}$, where $\mathcal{M}$ is a unary, linearly ordered temporal structure.

$$\exists y Q_1 x_1 \ldots Q_0 x_0 \beta\,(x_0, \ldots, x_0, y; \varphi\,[y]) =$$
$$\exists y \beta\,(x_0, y; \varphi\,[y]) \in$$
$$\{\exists y(y < x_0 \wedge \varphi\,[y]), \exists y(y > x_0 \wedge \varphi\,[y])\} =$$
$$\{ST_{x_0}(P\,(\varphi^*)), ST_{x_0}(F\,(\varphi^*))\}.$$

Assume inductively that for a fixed $n < \omega$, any formula of the form

$$\exists y Q_1 x_1 \ldots Q_n x_n \beta\,(x_0, \ldots, x_n, y; \varphi\,[y])$$

has the property $(*)$, and let $\chi :=$

$$\exists y Q_1 x_1 \ldots Q_{n+1} x_{n+1} \beta\,(x_0, \ldots, x_{n+1}, y; \varphi\,[y])$$

be arbitrary. Then χ is logically equivalent to the formula

$$\exists y Q_1 x_1 \ldots Q_n x_n (R_1(x_0, x_1)\, o_1\, \ldots$$
$$R_n(x_{n-1}, x_n)\, o_n\, Q_{n+1} x_{n+1}(R_{n+1}(x_n, x_{n+1})\, o_{n+1}$$
$$R_{n+2}(x_{n+1}, y) \wedge \varphi[y])).$$

Now consider the subformula $\phi :=$

$$Q_{n+1} x_{n+1}(R_{n+1}(x_n, x_{n+1})\ o_{n+1}\ R_{n+2}(x_{n+1}, y) \wedge \varphi\,[y])).$$

We notice that ϕ can in different cases (which are jointly exhaustive of the combinatorial possibilities) be written as follows:

(1.) If $\langle R_{n+1}, R_{n+2}\rangle = \langle <, < \rangle$, then

$$\phi := \begin{cases} (1.a)\ \exists x(x_n < x \wedge (x < y \wedge \varphi[y])),\ \text{if } Q_{n+1} = \exists \\ (1.b)\ \top(x_n), \\ \qquad \text{if}\, x_n = max(dom(\mathcal{M})) \text{ and } Q_{n+1} = \forall \\ (1.c)\ \bot(x_n) \text{ otherwise} \end{cases}$$

(2.) If $\langle R_{n+1}, R_{n+2}\rangle = \langle <, > \rangle$, then

$$\phi := \begin{cases} (2.a)\ (x_n \neq max(dom(\mathcal{M})) \neq y) \ \wedge \ \phi[y], \\ \qquad \text{if } Q_{n+1} = \exists \\ (2.b)\ \top(x_n), \\ \qquad \text{if} x_n = max(dom(\mathcal{M})) \text{ and } Q_{n+1} = \forall \\ (2.c)\ y \leq x_n \ \wedge \ \phi[y] \quad \text{otherwise} \end{cases}$$

(3.) If $\langle R_{n+1}, R_{n+2}\rangle = \langle >, < \rangle$, then

$$\phi := \begin{cases} (3.a)\ (x_n \neq min(dom(\mathcal{M})) \neq y) \ \wedge \ \phi[y], \\ \qquad \text{if } Q_{n+1} = \exists \\ (3.b)\ \top(x_n), \\ \qquad \text{if } x_n = min(dom(\mathcal{M})) \text{ and } Q_{n+1} = \forall \\ (3.c)\ y \geq x_n \ \wedge \ \phi[y] \quad \text{otherwise} \end{cases}$$

(4.) If $\langle R_{n+1}, R_{n+2}\rangle = \langle >, > \rangle$, then

$$\phi := \begin{cases} (4.a)\ \exists x(x_n > x \wedge (x > y) \wedge \phi[y])),\ \text{if } Q_{n+1} = \exists \\ (4.b)\ \top(x_n), \\ \qquad \text{if } x_n = min(dom(\mathcal{M})) \text{ and } Q_{n+1} = \forall \\ (4.c)\ \bot(x_n) \quad \text{otherwise} \end{cases}$$

Hence in the respective cases the formula χ can be written in a way which, directly by inductive hypothesis, implies that it has the property $(*)$. Let us write $\overline{Qx} := Q_1x_1 \ldots Q_nx_n$.

(1.a) Consider first the case that the formula χ has the form

$$\exists y \overline{Qx} R_1(x_0, x_1)\ o_1 \ \ldots$$

$$(R_n(x_{n-1}, x_n)\ o_n\ \exists x(x_n < x \wedge x < y \wedge \varphi[y]))).$$

Consider further the following formula θ_1:

$$\exists y\overline{Qx}(R_1(x_0, x_1)\ o_1 \ldots (R_n(x_{n-1}, x_n)\ o_n$$
$$(x_n < y \wedge \varphi[y]) \vee (x_n < y \wedge ST_y(F(\varphi^* \wedge G\neg\varphi^*) \wedge$$
$$\neg FF\varphi^*))).$$

χ is seen to be equivalent to θ_1 as follows. Let $y_0 = f_{\exists y}(\langle\rangle)$ be provided by *Héloïse*'s winning strategy f in $G_A(\chi, \mathcal{M}, t)$. Then if y_0 is either dense to the left, or is not the greatest y with $\varphi\,[y]$, the strategy f induces a w.s. g_l for *Héloïse* in the game $G_A(\theta_1, \mathcal{M}, t)$: g_l chooses the leftmost disjunct in the subformula $\delta :=$

$$(x_n < y \wedge \varphi[y]) \vee$$
$$(x_n < y\ \wedge ST_y(F(\varphi^* \wedge G\neg\varphi^*) \wedge \neg FF\varphi^*)),$$

and is otherwise like f. On the other hand, if there is an immediate predecessor $(y_0 - 1)$ to y_0 and y_0 is the greatest y with $\varphi[y]$, then f induces a w.s. g_r in $G_A(\theta_1, \mathcal{M}, t)$ as follows:

- $g_{r,\exists y}(\langle\rangle) = y_0 - 1$
- If $Q_i = \exists_i (i := 1, \ldots, n)$, then $g_{r,Q_i x_i}(\overline{x}) = f_{Q_i x_i}(\overline{x})$
- g_r chooses the right disjunct of δ
- For the quantifier corresponding to F, g_r gives y_0

Conversely, from a w.s. g for *Héloïse* in the game $G_A(\theta_1, \mathcal{M}, t)$, a w.s. f for her in $\mathrm{G}(\chi)$ is obtained in an analogous manner.

Consider further the following formula θ_2:

$$\exists y\overline{Qx}(R_1(x_0,x_1)\, o_1 \ldots (R_n(x_{n-1},x_n)\, o_n$$
$$(x_n < y \wedge \varphi[y]))) \vee$$
$$\exists y\overline{Qx}(R_1(x_0,x_1)\, o_1 \ldots (R_n(x_{n-1},x_n)\, o_n$$
$$(x_n < y \wedge ST_y(F(\varphi^* \wedge G\neg\varphi^*) \wedge \neg FF\varphi^*)))).$$

Then it is seen that there is a w.s. for *Héloïse* in $G_A(\theta_1, \mathcal{M}, t)$ iff there is a w.s. for her in $G_A(\theta_2, \mathcal{M}, t)$. For we may assume that to interpret $\exists y$, *Héloïse*'s w.s. in the former game always picks out, if possible, the greatest y with $\varphi\,[y]$, provided this y has an immediate predecessor: if this is possible, she thereby already arranges that she will choose the right disjunct in δ; if this is not possible, she will choose the left disjunct. Hence the choice of disjunct can already be made in the beginning of the game, yielding a w.s. for her in $G_A(\theta_2, \mathcal{M}, t)$. The other direction is immediate. Hence, by induction hypothesis, χ is seen to have the property $(*)$.

(1.b) In this case,

$$\chi \Leftrightarrow \overline{Qx}(R_1(x_0,x_1)\, o_1 \ldots R_n(x_{n-1},x_n)\, o_n \top(x_n)).$$

Hence χ is trivially in the set $\{ST_{x_0}(\psi) : \psi \in \mathbf{TL}\}$.

(1.c) In this case,

$$\chi \Leftrightarrow \overline{Qx}(R_1(x_0,x_1)\, o_1 \ldots R_n(x_{n-1},x_n)\, o_n \bot(x_n)).$$

Again, χ has, trivially, a translation in **TL**.

(2.a) In this case we clearly have:

$$\chi \Leftrightarrow \exists y(ST_y(F(\top) \wedge \varphi^*) \wedge \overline{Qx}R_1(x_0,x_1)\, o_1 \ldots$$
$$(R_n(x_{n-1},x_n)\, o_n\, (ST_{x_n}(F(\top)))).$$

As, moreover

$$\exists y(ST_y(F(\top) \wedge \varphi^*) \Leftrightarrow$$
$$ST_{x_0}(P(F(\top) \wedge \varphi^*) \vee (F(\top) \wedge \varphi^*) \vee F(F(\top) \wedge \varphi^*)),$$

the formula χ has the property $(*)$.

(2.b) Exactly as case (1.b).

(2.c) Clearly the following equivalences now hold:

$$\chi \Leftrightarrow$$
$$\exists y\overline{Qx}(R_1(x_0, x_1)\ o_1 \ldots (R_n(x_{n-1}, x_n)\ o_n$$
$$((y < x_n \wedge \varphi[y]) \vee \varphi[x_n]))) \Leftrightarrow$$
$$\exists y\overline{Qx}(R_1(x_0, x_1)\ o_1 \ldots$$
$$(R_n(x_{n-1}, x_n)\ o_n(y < x_n \wedge \varphi[y]))) \vee$$
$$\overline{Qx}(R_1(x_0, x_1)\ o_1 \ldots (R_n(x_{n-1}, x_n)\ o_n\varphi[x_n])).$$

In the last formula the right disjunct has the property $(*)$ trivially, while the left disjunct has the property $(*)$ by inductive hypothesis.

The subcases (3.a-c) and (4.a-c) are analogous to the cases (2.a-c) and (1.a-c), respectively. ■

Now notice that as **IFTL** formulae are really formulae of **IFML** [2], we already know by Theorem 3.2.4 that they are translatable into **FO**. Such a first-order translation is not, however, generally equivalent to the standard translation of any basic modal-logical formula, as shown by Theorem 3.4.4. By the above lemma, however, such an equivalence always precisely pertains in the special case of **IFTL** evaluated over linear frames. The following example contrasts with Example 3.2.9 above.

Example 3.4.10 *Consider evaluating the* (**E**)**IFTL** *formula* $\chi := G_1 G_2 P_3 / \{2\}\, F_4 / \{1,3\}\, q$ *relative to the class* **LO** *of all unary linear temporal structures.* χ *is true in a pointed modal structure iff the formula* $G_1 G_2 P_3 / \{2\}\, F_4\, q$ *is true therein. And it is easily seen that relative to* **LO***, the* **FO** *equivalent*

$$\forall x \exists z \forall y \exists v (t_0 < x \rightarrow (x < y \rightarrow (y > z \wedge (z < v \wedge Q(v)))))$$

of χ *can be written as*

$$\forall x \exists z \exists v (t_0 < x \rightarrow [x \neq \max(dom(\mathcal{M})) \rightarrow$$

$$(x \geq z \wedge (z < v \wedge Q(v)))]),$$

and χ *is equivalent to the* **TL** *formula*

$$G(G\bot \vee (PFq \vee Fq)). \blacksquare$$

Now, using Lemma 3.4.9 the coincidence theorem readily follows:

Theorem 3.4.11 **IFTL** $=_{\mathbf{LO}}$ **TL**.

Proof. **TL** is trivially embeddable in **IFTL** over any class of structures, therefore over **LO** in particular. To show embeddability in the other direction, let $\chi \in$ **IFTL** be arbitrary. The result follows trivially, if $\chi \in$ **TL** $\cap$ **IFTL**. Suppose then, that χ is of the form

$$O_1 \ldots O_{n-1}(O_n/W)\, \varphi,$$

where $\varphi \in$ **TL** and $W \subseteq [1, n-1]$. If in particular $O_n \in \{G_n, H_n\}$, we trivially have that for all $\mathcal{M}$,t:

$$\mathcal{M} \models^+ \chi[t] \Longleftrightarrow \mathcal{M} \models^+ O_1 \ldots O_{n-1} O_n \varphi[t]$$

and hence χ is translated in **TL**. If, again, $O_n \in \{F_n, P_n\}$, by Lemma 3.2.1 we may, without loss of generality, assume that in fact $W = \{1, \ldots, n-1\}$ (cf. the proof of Theorem 3.2.4). Then by Lemma 3.2.3, χ is translated into **FO** by the formula $\theta_\chi(x_0) :=$

$$\exists y Q x_1 \ldots Q x_{n-1}(R_{l_1}(x_0, x_1)\ o_1 \ldots$$

$$(R_{l_{n-1}}(x_{n-2}, x_{n-1})\ o_{n-1}\ R_{l_n}(x_{n-1}, y)\ \wedge\ ST_{x/y}(\varphi)),$$

where:

$$(Qx_i, o_i) := \begin{cases} (\forall x_i, \rightarrow) \text{ if } O_i = \Box_i \\ (\exists x_i, \wedge) \text{ if } O_i = \Diamond_i \end{cases}$$

and $l_i < k$ identifies the accessibility relation $(<, >)$ associated with the operator O_i. Thus, by Lemma 3.4.9 above, χ has a translation into **TL**. ∎

As a further coincidence result, we have that the expressive powers of the *modal* (as opposed to temporal) logics of one modality type, **ML**[1] and **IFML**[1], coincide over unary linear modal structures.

Theorem 3.4.12 **ML**[1] $=_{\mathbf{LO}[1]}$ **IFML**[1].

Proof. Trivially **ML**[1] $\leq_{\mathbf{LO}[1]}$ **IFML**[1]. For the other direction, it suffices to consider formulae of the form $O_1 \ldots O_{n-1}(\Diamond_n/[1, n-1])\varphi$, the other cases being trivial. But cases (1) and (4) of Lemma 3.4.9 guarantee that there is an **ML**[1] translation for any **IFML**[1] formula of this form. ∎

3.4.3 Modal structures linear in two dimensions

We now proceed to show that over the class **LO**[2] of binary modal structures linear in both dimensions (i.e. whose two accessibility relations are linear orders), IF modal logic with two modality types is strictly more expressive than the corresponding basic modal logic with two modality types. A similar result is shown to hold for tense logic as well.

We may observe that on the one hand, Theorem 3.4.11 proven above shows that $\mathbf{IFML}[2]$ does *not* have more expressive power than $\mathbf{ML}[2]$ over the *subclass* $\mathbf{LO}_{temp}[1] = \mathbf{LO}$ of $\mathbf{LO}[2]$ consisting of *such* binary modal structures linear in both dimensions whose two accessibility relations are *converses of each other* (i.e. over unary linear *temporal* structures). On the other hand, by Theorem 3.4.4 we know that $\mathbf{IFML}[2]$ *has* a greater expressive power than $\mathbf{ML}[2]$ over *arbitrary* binary modal structures. The case we now move on to below is hence an intermediate one.

$$\mathbf{LO} \subset \mathbf{LO}[2] \subset \mathcal{C}_2$$

is of course a chain of proper inclusions, and thereby Theorems 3.4.4 and 3.4.11 do not suffice for deciding the question whether the relation

$$\mathbf{ML}[2] <_{\mathbf{LO}[2]} \mathbf{IFML}[2]$$

holds or not. The following theorem implies that in fact $\mathbf{IFML}[2]$ is more expressive than $\mathbf{ML}[2]$ already relative to the class $\mathbf{LO}[2]$. More generally, the theorem states that the equalities

$$\mathbf{ML}[k] =_{\mathbf{LO}[k,n]} \mathbf{IFML}[k]$$

$$\mathbf{TL}[k] =_{\mathbf{LO}_{temp}[k,n]} \mathbf{IFTL}[k]$$

which by Theorem 3.4.11 *resp.* Theorem 3.4.12 are established for the value $k = n = 1$, cannot be generalized to any values $k \geq n \geq 2$. We proceed to prove:

Theorem 3.4.13 *For all $n \geq 2$ and for all $k \geq n$, we have:*

(*i*) $\mathbf{ML}[k] <_{\mathbf{LO}[k,n]} \mathbf{IFML}[k]$.

(ii) $\mathbf{TL}[k] <_{\mathbf{LO}_{temp}[k,n]} \mathbf{IFTL}[k]$.

Proof. We shall show that each of the claims (i) and (ii) holds for the values $k := 2$ and $n := 2$. The results then trivially generalize to all $k \geq n \geq 2$. Now let $\mathcal{M} = (D, <_0, <_1, \mathfrak{h})$ and $\mathcal{M}' = (D', <'_0, <'_1, \mathfrak{h}')$ be binary modal structures whose components are given as follows.

- D is the set of rational numbers, and $<_0 = <_1$ is the order of rationals by magnitude.

Let then $C = D \times \{0\}$, and let $(C, <^+_0, <^+_1)$ be an isomorphic copy of $(D, <_0, <_1)$. By construction $C \cap D = \varnothing$. Then put:

- $D' := D \cup C$
- $<'_0 := <_0 \cup <^+_0 \cup (D \times C)$
- $<'_1 := <_1 \cup <^+_1 \cup (C \times D)$

Finally, assume for simplicity that $\mathbf{prop} = \{\top\}$ and put $\mathfrak{h}(\top) = D$ and $\mathfrak{h}'(\top) = D'$. The structures $\mathcal{M}$ and $\mathcal{M}'$ can be depicted as in Figure 15 below.

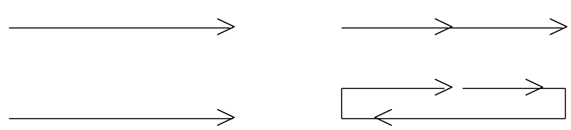

FIGURE 15

It is clear that the relation $D \times D'$ is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$. Then fix points $\zeta \in D$ and $\xi \in D' \cap C$. It follows by *Invariance Lemma* that the pointed modal structures $(\mathcal{M},\zeta)$ and $(\mathcal{M}',\xi)$ cannot be distinguished by any formula of $\mathbf{ML}[2]$. But we observe that the $\mathbf{IFML}[2]$ formula $\Box_{0,1}\Diamond_{1,2}/\{1\}\top$ is true in $\mathcal{M}'$ at ξ (*Héloïse* can choose for $\Diamond_{1,2}$ any $\xi'' \in D' \cap D$, as any such ξ'' will be a $<'_1$-successor of any possible ξ' with $\xi <'_0 \xi'$ chosen by *Abélard*). On the other hand this formula is non-determined in $\mathcal{M}$ at ζ (not true, since no $\zeta'' \in D$ is a $<_1$-successor to all $\zeta' \in D$ with $\zeta <_0 \zeta'$; not false, as for all $\zeta' \in D$ there is some $\zeta'' \in D$ such that $\zeta <_0 \zeta' <_1 \zeta''$). It follows that **IFML**[2] is not embeddable in **ML**[2]. Claim (i) of the theorem is proved for the value $k := 2$ by appealing to Fact 3.1.1. Hence, trivially, the generalized claim (i) follows.

Then observe that as both $<_0$ and $<_1$ are irreflexive and transitive relations, the structures

$$\mathcal{M}^+ = (D, <_0, <_1, <_0^{-1}, <_1^{-1}, \mathfrak{h})$$

and

$$\mathcal{M}'^+ = (D', <'_0, <'_1, <'^{-1}_0, <'^{-1}_1, \mathfrak{h}')$$

are temporal structures. But the relation $D \times D'$ is certainly a bisimulation even between the binary temporal structures $\mathcal{M}^+$ and $\mathcal{M}'^+$. Hence ($\mathcal{M}^+$,ζ) and ($\mathcal{M}'^+$,ξ) are **TL**[2] equivalent but can, of course, be distinguished by the formula $\Box_{0,1}\Diamond_{1,2}/\{1\}\top$. Hence claim (ii) follows. ■

Chapter 4

Alternative Interpretations of IF Modal Logic

In *Chapter* 3 we studied the language of **IFML**$[k]$, using as its semantics its *uniformity* interpretation, introduced in *Chapter* 2. In the present chapter we will consider two alternative interpretations of this language, to be called the 'backwards-looking operators' interpretation and the algebraic interpretation of IF modal logic, respectively. These semantics essentially provide alternative interpretations of the slash sign "/" of **IFML**$[k]$.

The 'backwards-looking operators' interpretation (**BLO**) is designed to provide a semantics for **IFML**$[k]$ on arbitrary k-ary modal structures. It means interpreting the slash sign "/" as a *device for constraining moves* of the players of a semantical game, instead of imposing restrictions on their winning strategies, as the uniformity interpretation does. The algebraic interpretation (**ALG**) is of a more special character: it is only meant for evaluating IF tense logic relative to unary linear temporal structures equipped with a commutative group operation. This interpretation treats the slash as indicating 'subtraction'.

In the concluding chapter of the present thesis (*Ch.* 5), the semantic mechanism behind **BLO** interpretation is shown to be useful in connection with natural language analysis. Algebraic interpretation, in turn, gives a very concrete and robust sense to 'independence'. In fact, this interpretation essentially comes down to erasing the operators referred to in the independence indication "$/W$". It may be noted that it would be perfectly possible to investigate languages where there are several different types of slash signs available – e.g. $/_{\mathbf{UNI}}$, $/_{\mathbf{BLO}}$, $/_{\mathbf{ALG}}$ – each type of slash having a semantics of its own. We do not, however, examine such 'multi-slash' languages here.

4.1 Backwards-Looking Operators

Let us proceed to define the 'backwards-looking operators' interpretation of **IFML**[k], and to study its expressive power.

We associate a semantical game

$$\mathrm{G}_A(\varphi, \mathcal{M}, d) = \langle\{\forall, \exists\}, H, Z, P, \{u_\forall, u_\exists\}\rangle$$

in extensive normal form with each triple $(\varphi, \mathcal{M}, d)$ consisting of a formula φ of **IFML**[k], a k-ary modal structure $\mathcal{M} = (D, R_0, \ldots, R_{k-1}, \mathfrak{h})$ and a point $d \in D$. This game is one of *perfect* information and hence contains no information partitions. (Or, equivalently, the respective information sets are singletons, each history belonging to an information set of its own.) The game is otherwise defined exactly as the respective game made use of in defining the **UNI** interpretation of **IFML**[k] above, except for the following difference in the definition of the class H of histories.

The clauses for $(\Diamond_{i,n}/W)$ and $(\Box_{i,n}/W)$ in the recursive definition of the set H of *plays* (or, *histories*) of $\mathrm{G}_A(\varphi, \mathcal{M}, d)$

are these:

- If the last position in $h \in H$ is of the form

  $$((\Diamond_{i,n}/W)\ \psi, a_{n-1}),$$

  find the element[1]

  $$b := pr_2(h)[max(\{0, 1, \ldots, n-1\} \backslash W)].$$

  Then for all $a' \in dom(\mathcal{M})$ with $R_i(b, a')$ we have $h^\frown(\psi, a') \in H$; if no such a' exists in $dom(\mathcal{M})$, then $h^\frown(\texttt{FAIL}, \star) \in H$. Further, $P(h) = \exists$.

- The definition for $\Box_{i,n}/W$ is the same as that for $\Diamond_{i,n}/W$ except that here it is $\forall$'s turn to move.

Remark 4.1.1 *Hence the evaluation of $\Diamond_{i,n}/W$ (resp. $\Box_{i,n}/W$) is relative to the element that the evaluation of the operator with the index $max(\{0, 1, \ldots, n-1\} \backslash W) + 1$ has introduced, or else is relative to the initial point of evaluation ($a_0 := d$). This is in fact the key point of* **BLO** *semantics, allowing more choices for "relational dependence" than does the semantics of basic modal logic: in the latter, the evaluation of an operator O is always relative to the state introduced by the evaluation of the (unique) modal operator in whose scope O immediately lies. Here the evaluation may happen relative to the state introduced by any logically superordinate modal operator.*

A *strategy* of the player $j \in \{\exists, \forall\}$ in $\mathrm{G}_A(\varphi, \mathcal{M}, d)$ is any function from finite sequences of positions to positions. A strategy f_j of the player j is a *winning strategy* (*w.s.*), if there exists a subset $W \subseteq Z$ of terminal histories satisfying the following three conditions:

[1] Observe that as W is a subset of $[1, n-1]$ and hence always $0 \notin W$, the difference set $\{0, 1, \ldots, n-1\} \backslash W$ is never empty.

(a) If $h \in Cl(W)$ and $P(h)$ is j, then $h^\frown f_j(h) \in Cl(W)$.

(b) If $h \in Cl(W)$ and $P(h)$ is the opponent of j, then for every $u \in A$ such that $h^\frown u \in H$, $h^\frown u \in Cl(W)$.

(c) Every $h \in W$ is a win for j.

A set $W \subseteq Z$ which thus establishes that a strategy f_j is winning, is called a *plan of action*. We say that f_j is a winning strategy *based on* W. Observe that unlike the case of **UNI** interpretation, there is of course no uniformity constraint imposed here upon $f_\exists$. This is naturally in keeping with the fact that **BLO** games are games of *perfect* information, while the games by means of which the **UNI** interpretation was given, were games of imperfect information.

In terms of the games $\mathrm{G}_A(\varphi, \mathcal{M}, d)$ we then define truth *resp.* falsity:

- $\mathcal{M} \models^{+}_{\mathrm{BLO}} \varphi[d] \iff$

  there exists a w.s. for *Héloïse* in $\mathrm{G}_A(\varphi, \mathcal{M}, d)$.

- $\mathcal{M} \models^{-}_{\mathrm{BLO}} \varphi[d] \iff$

  there exists a w.s. for *Abélard* in $\mathrm{G}_A(\varphi, \mathcal{M}, d)$.

In Corollary 4.1.8 below, it is proven that the logic **IFML**$[k]$ is *determined*, i.e. in all k-ary modal structures $\mathcal{M}$ and at all points $d \in dom(\mathcal{M})$ any formula φ of **IFML**$[k]$ is in the above sense either true or false. This result is only to be expected, because by a well-known theorem of von Neumann & Morgenstern (1944) every two-player zero-sum game of perfect information is determined.[2]

[2] Cf. von Neumann & Morgenstern (1944), *Ch.* 15, esp. *Sect.* 15.6, where it is proven that every zero-sum two-player game of perfect informa-

It is obvious that the semantics for a formula $\varphi \in \mathbf{ML}[k]$, given in terms of the semantical game $\mathrm{G}_A(\varphi, \mathcal{M}, d)$ as above, coincides with the recursive semantics for $\mathbf{ML}[k]$ given in *Section* 2.1: if $\varphi \in \mathbf{ML}[k]$, then

$$\mathcal{M} \models^{+} \varphi[d] \Longleftrightarrow \mathcal{M} \models^{+}_{\mathrm{BLO}} \varphi[d].$$

We now move on to prove that under **BLO** interpretation, $\mathbf{IFML}[k]$ is *not* more expressive than $\mathbf{ML}[k]$. This will establish that **BLO** interpretation does not yield to the slash sign "/" a meaning which would serve to distinguish the expressive powers of $\mathbf{IFML}[k]$ and $\mathbf{ML}[k]$.

Remark 4.1.2 *In accordance with our earlier decision* (*Sect.* 2.1)*, we restrict our attention in the present thesis exclusively to one-dimensional semantics. However, it should be observed here that the semantic mechanism of backwards-looking could well be made use of in connection with many-dimensional modal logics. Providing and studying the* **BLO** *semantics of many-dimensional IF modal logic might be especially natural in view of analyzing natural language discourse involving both tense and modalities. And in fact there is a natural definition of many-dimensional IF modal logic under which this logic can be shown to have a greater expressive power than the corresponding many-dimensional basic modal logic.*[3] *Proving this is not difficult, but remains outside the scope of the present thesis. (Basically it suffices to define an appropriate notion of bisimulation, and construct bisimilar models of which one does*

tion is "strictly determined". The property "strictly determined" implies determinacy in the sense relevant here, i.e. it implies that in any such game one of the players has a winning strategy. See *Appendix* B of the present thesis for details.

[3] This semantics requires that the transitions be made *componentwise*. Hence when a move in time-dimension is made, for example, the co-ordinate for possible worlds remains the same.

and the other does not make the English sentence "John believed that Mary will turn up" true, given that "will" is read indexically.)

4.1.1 The expressive power of IFML under BLO

It will first be shown that the languages **ML**$[k]$ and **IFML**$[k]$ *have the same expressive power* over the class $\mathcal{C}_k$ of all k-ary modal structures in the sense that these languages are *weakly embeddable* to each other relative to $\mathcal{C}_k$: if $\mathcal{L}, \mathcal{L}' \in$ {**ML**$[k]$,**IFML**$[k]$}, $\mathcal{L} \neq \mathcal{L}'$, for each $\varphi \in \mathcal{L}$ there is $\psi \in \mathcal{L}'$ such that for all $\mathcal{M} \in \mathcal{C}_k$ and all $d \in dom(\mathcal{M})$:

$$\mathcal{M} \models^+_{\text{BLO}} \varphi[d] \Longleftrightarrow \mathcal{M} \models^+_{\text{BLO}} \psi[d].$$

Only then do we proceed to prove that **IFML**$[k]$ is *determined* relative to $\mathcal{C}_k$ under **BLO** interpretation, which finally helps us to infer that indeed **ML**$[k]$ and **IFML**$[k]$ are *strongly embeddable* to each other: for every formula $\varphi \in$ **IFML**$[k]$ there is a formula $\psi \in$ **ML**$[k]$ such that for all $\mathcal{M} \in \mathcal{C}_k$ and all $d \in dom(\mathcal{M})$:

$$(\mathcal{M} \models^+_{\text{BLO}} \varphi[d] \textbf{ and } \mathcal{M} \models^+_{\text{BLO}} \psi[d]) \textbf{ or}$$
$$(\mathcal{M} \models^-_{\text{BLO}} \varphi[d] \textbf{ and } \mathcal{M} \models^-_{\text{BLO}} \psi[d]);$$

the direction from **ML**$[k]$ to **IFML**$[k]$ being trivial.

We write $\bot$ for *falsum* and $\top$ for *verum*. Letting $(\mathcal{M}, d)$ be an arbitrary pointed modal structure, we first observe the following:

- $\mathcal{M} \models^+ O_1 \ldots O_{n-1}\bot[d] \Longleftrightarrow$
  for some $i \in \{1, \ldots, n-1\}$ up to which *Héloïse* is able to make her moves (if any) without choosing $(\texttt{FAIL}, \star)$, it is *Abélard*'s turn to move, and he is forced to reply by $(\texttt{FAIL}, \star)$.

- $\mathcal{M} \models^+ O_1 \dots O_{n-1}\top[d] \Longleftrightarrow$
  for all $i \in \{1, \dots, n-1\}$, the player whose turn it is to move is able to reply by a move other than $(\texttt{FAIL}, \star)$, or else $\mathcal{M} \models^+ O_1 \dots O_{n-1}\bot[d]$.

Lemma 4.1.3 *Assume $\varphi \in \mathbf{ML}[k]$, and let $n \geq 1$. For all $j \in \{1, \dots, n\}$, let $O_{i,j} \in \{\Diamond_{i,j}, \Box_{i,j}\}$ for some $i < k$. Then for all pointed k-ary modal structures $(\mathcal{M}, d)$ we have:*

$$\mathcal{M} \models^+_{BLO} O_1 \dots O_{n-1}(O_n/\{1, \dots, n-1\})\varphi[d] \Longleftrightarrow$$

$$\mathcal{M} \models^+ (O_1 \dots O_{n-1}\bot \vee (O_n\varphi \wedge O_1 \dots O_{n-1}\top))[d].$$

Proof. The result is immediate from the semantics of the operator $(O_n/\{1, \dots, n-1\})$. ■

Lemma 4.1.4 *Assume $\varphi \in \mathbf{ML}[k]$, and let $n \geq 1$. Let $W \subseteq [1, n-1]$ be given. Now write W as the disjoint union $W = V' \cup V$, where*

$$V = [K+1, n-1] \textit{ and } V' \subseteq [1, K-1]$$

for K such that $0 \leq K \leq n-1$ and $K \notin W$. (Such a K is uniquely determined.) Then for all pointed k-ary modal structures $(\mathcal{M}, d)$ we have

$$\mathcal{M} \models^+_{BLO} O_1 \dots O_{n-1}(O_n/W)\varphi[d] \Longleftrightarrow$$

$$\mathcal{M} \models^+_{BLO} O_1 \dots O_{n-1}(O_n/V)\varphi[d].$$

Proof. Notice that by the definition of K,

$$max(\{0, 1, \dots, n-1\} \backslash W) = K = max(\{0, 1, \dots, n-1\} \backslash V).$$

The result then obviously follows. ■

Theorem 4.1.5 *Let the language* **IFML**$[k]$ *be interpreted by* **BLO** *interpretation. Then the expressive powers of* **ML**$[k]$ *and* **IFML**$[k]$ *are the same over the class* $\mathcal{C}_k$ *of arbitrary* k*-ary modal structures.*

Proof. Trivially **ML**$[k]$ is weakly embeddable to **IFML**$[k]$ over $\mathcal{C}_k$. For the other direction, let $\varphi \in$ **IFML** $[k]$, $\mathcal{M} \in \mathcal{C}_k$ and $d \in dom(\mathcal{M})$ all be arbitrary. We must find a formula $\psi_\varphi \in$ **ML**$[k]$, satisfying that

$$\mathcal{M} \models^+_{\mathrm{BLO}} \varphi[d] \Longleftrightarrow \mathcal{M} \models^+_{\mathrm{BLO}} \psi_\varphi[d].$$

Now if in particular $\varphi \in$ **ML**$[k]$, then we may clearly take ψ_φ to be φ itself. Assume, then, that φ is of the form $O_1 \dots O_{n-1}(O_n/W)\chi$ with $\chi \in$ **ML**$[k]$. The set $W \subseteq [1, n-1]$ can be written as $W = V' \cup V$, where $V = [K+1, n-1]$ and $V' \subseteq [1, K-1]$ for a unique K with $0 \leq K \leq n-1$, $K \notin W$. Hence, by Lemma 4.1.4, we have that

$$\mathcal{M} \models^+_{\mathrm{BLO}} O_1 \dots O_{n-1}(O_n/W)\varphi[d] \Longleftrightarrow$$

$$\mathcal{M} \models^+_{\mathrm{BLO}} O_1 \dots O_K O_{K+1} \dots O_{n-1}(O_n/[K+1, n-1])\varphi)[d].$$

But now, by Lemma 4.1.3, we may conclude that

$$\mathcal{M} \models^+_{\mathrm{BLO}} O_1 \dots O_K O_{K+1} \dots$$

$$O_{n-1}(O_n/[K+1, n-1])\varphi)[d] \Longleftrightarrow$$

$$\mathcal{M} \models^+_{\mathrm{BLO}} O_1 \dots O_K (O_{K+1} \dots$$

$$O_{n-1}\bot \vee (O_n\varphi \wedge O_{K+1} \dots O_{n-1}\top))[d].$$

Hence we can choose $\psi_\varphi :=$

$$O_1 \ldots O_K(O_{K+1} \ldots O_{n-1}\bot \vee (O_n\varphi \wedge O_{K+1} \ldots O_{n-1}\top)).$$

Hereby the proof is completed. ■

Definition 4.1.6 *We define recursively for every* **IFML**$[k]$ *formula* φ *its dual* $\Delta(\varphi)$ *as follows:*

- $\Delta(p) = \neg p$ *and* $\Delta(\neg p) = p$
- $\Delta(\varphi \wedge \psi) = \Delta(\varphi) \vee \Delta(\psi)$
- $\Delta(\varphi \vee \psi) = \Delta(\varphi) \wedge \Delta(\psi)$
- $\Delta(\Diamond_i\varphi) = \Box_i\Delta(\varphi)$
- $\Delta((\Diamond_i/W)\varphi) = (\Box_i/W)\Delta(\varphi)$
- $\Delta(\Box_i\varphi) = \Diamond_i\Delta(\varphi)$
- $\Delta((\Box_i/W)\varphi) = (\Diamond_i/W)\Delta(\varphi)$

Derivatively the *operators* $\Diamond_i$ and $\Box_i$ are said to be duals of each other, symbolically

- $(\Diamond_i)^d = \Box_i$
- $(\Box_i)^d = \Diamond_i$.

Given a block $O_1 \ldots O_m$ of operators, we write $(O_1 \ldots O_m)^d$ for the block of their duals:

$$(O_1 \ldots O_m)^d = (O_1)^d \ldots (O_m)^d.$$

Corollary 4.1.7 **IFML**$[k]$ *is determined relative to* **BLO** *interpretation.*

Proof. Let $\varphi \in \mathbf{IFML}[k]$, $\mathcal{M} \in \mathcal{C}_k$ and $d \in dom(\mathcal{M})$ all be arbitrary. If $\varphi \in \mathbf{ML}[k]$, there is nothing to prove, as it is well known that $\mathbf{ML}[k]$ is determined. In particular, it is known that $\varphi \in \mathbf{ML}[k]$ is not true in $\mathcal{M}$ at d iff $\Delta(\varphi)$ is true in $\mathcal{M}$ at d.

Consider then the case that $\varphi = O_1 \dots O_{n-1}(O_n/W)\chi$ with $\chi \in \mathbf{ML}[k]$. Assume $\mathcal{M} \not\models^+_{\mathrm{BLO}} \varphi[d]$. We wish to show that then, positively, $\mathcal{M} \models^-_{\mathrm{BLO}} \varphi[d]$. Now by the proof of Theorem 4.1.5, we have for the $\mathbf{ML}[k]$ formula ψ_φ :=

$$O_1 \dots O_K(O_{K+1} \dots O_{n-1}\bot \vee (O_n\chi \wedge O_{K+1} \dots O_{n-1}\top))$$

that $\mathcal{M} \not\models^+ \psi_\varphi[d]$. Hence, by the aforementioned basic fact about $\mathbf{ML}[k]$, we have that $\mathcal{M} \models^+ \Delta(\psi_\varphi)[d]$.

Then consider the formula

$$\Delta(\varphi) = (O_1 \dots O_{n-1})^d((O_n)^d/W)\Delta(\chi) \in \mathbf{IFML}[k].$$

Assume for contradiction that $\Delta(\varphi)$ is not true in $\mathcal{M}$ at d. Then we have for its $\mathbf{ML}[k]$ translation

$$\psi_{\Delta(\varphi)} = (O_1 \dots O_K)^d((O_{K+1} \dots O_{n-1})^d\bot \vee$$
$$((O_n)^d\Delta(\chi) \wedge (O_{K+1} \dots O_{n-1})^d\top))$$

(which is provided by the proof of Theorem 4.1.5) that $\mathcal{M} \not\models^+ \psi_{\Delta(\varphi)}[d]$, and so $\mathcal{M} \models^+ \Delta(\psi_{\Delta(\varphi)})[d]$. Therefore both of the two formulae $\Delta(\psi_\varphi)$ and $\Delta(\psi_{\Delta(\varphi)})$ are true in $\mathcal{M}$ at d:

- $\Delta(\psi_\varphi) = (O_1 \dots O_K)^d((O_{K+1} \dots O_{n-1})^d\top \wedge$
  $((O_n)^d\Delta(\chi) \vee (O_{K+1} \dots O_{n-1})^d\bot));$

- $\Delta(\psi_{\Delta(\varphi)}) = O_1 \ldots O_K(O_{K+1} \ldots O_{n-1}\top \wedge$
$$(O_n\chi \vee O_{K+1} \ldots O_{n-1}\bot)).$$

Now since $\Delta(\psi_{\Delta(\varphi)})$ is true in $\mathcal{M}$ at d,

$$O_1 \ldots O_K((O_{K+1} \ldots O_{n-1}\top \wedge O_n\chi) \vee$$
$$(O_{K+1} \ldots O_{n-1}\top \wedge O_{K+1} \ldots O_{n-1}\bot))$$

is also true in $\mathcal{M}$ at d, and hence so is

$$\Delta(\Delta(\psi_\varphi)) =$$
$$O_1 \ldots O_K((O_{K+1} \ldots O_{n-1}\top \wedge O_n\chi) \vee O_{K+1} \ldots O_{n-1}\bot).$$

Thus both $\Delta(\Delta(\psi_\varphi)) = \psi_\varphi$ and $\Delta(\psi_\varphi)$ are true in $\mathcal{M}$ at d, which is impossible (by the aforementioned basic fact about $\mathbf{ML}[k]$). We may conclude that $\Delta(\varphi)$ is true in $\mathcal{M}$ at d.

Any w.s. of *Héloïse* in $\mathrm{G}_A(\Delta(\varphi), \mathcal{M}, d)$ is a w.s. of *Abélard* in $\mathrm{G}_A(\varphi, \mathcal{M}, d)$. So $\mathcal{M} \models^-_{\mathrm{BLO}} \varphi[d]$. ■

Corollary 4.1.8 $\mathbf{IFML}[k]$ *and* $\mathbf{ML}[k]$ *are strongly embeddable to each other over the class* $\mathcal{C}_k$ *of all* k*-ary modal structures relative to* **BLO** *interpretation of* $\mathbf{IFML}[k]$*.*

Proof. If $\varphi \in \mathbf{ML}[k]$, it is its own translation in $\mathbf{IFML}[k]$. So suppose $\varphi \in \mathbf{IFML}[k]$, and let $\mathcal{M} \in \mathcal{C}_k$ and $d \in dom(\mathcal{M})$ both be arbitrary. Theorem 4.1.5 guarantees that there is a formula $\psi_\varphi \in \mathbf{ML}[k]$ such that

$$\mathcal{M} \models^+_{\mathrm{BLO}} \varphi[d] \Longleftrightarrow \mathcal{M} \models^+_{\mathrm{BLO}} \psi_\varphi[d].$$

But this means, in view of the determinacy of $\mathbf{ML}[k]$ and Corollary 4.1.7, that the following holds:

$$\mathcal{M} \models^-_{\mathrm{BLO}} \varphi[d] \Longleftrightarrow \mathcal{M} \not\models^+_{\mathrm{BLO}} \varphi[d] \Longleftrightarrow$$
$$\mathcal{M} \not\models^+_{\mathrm{BLO}} \psi_\varphi[d] \Longleftrightarrow \mathcal{M} \models^-_{\mathrm{BLO}} \psi_\varphi[d].$$

This completes the proof. ■

4.2 Algebraic Interpretation of IF Tense Logic

Here we will give an alternative semantics for **IFTL**[1], to be called its *algebraic* interpretation (**ALG**). Theorem 4.2.7 below will show that under this interpretation, **IFTL**[1] and its traditional counterpart **TL**[1] (i.e. *Priorean tense logic* of past and future) have the same expressive power over the class $\mathcal{A}$ of all linear temporal structures equipped with a commutative group operation.

4.2.1 Priorean tense logic

By the definition given in *Section* 2.1, the basic tense logic **TL**[1] is syntactically the basic modal logic **ML**[2] of two modality types, with operators written as $\Diamond$, $\Box$, $\Diamond^{-1}$, $\Box^{-1}$. Its semantics being relative to unary temporal structures, this logic is in fact what can be termed *Priorean tense logic* (or **PTL**).[4] In **PTL**, the symbols F, G, P, H are customarily used for the operators $\Diamond$, $\Box$, $\Diamond^{-1}$, $\Box^{-1}$, respectively, and the same convention is adopted when discussing **TL**[1], as well as when considering the corresponding IF tense logic, **IFTL**[1]. Furthermore, the number (1) of temporal modality types is not mentioned in this connection; only the name "**IFTL**" for the IF counterpart of **PTL** is used.

So the formulae of **PTL** are generated from a class **prop** of propositional atoms and negations of propositional atoms by the rules of closure under conjunction ($\wedge$), disjunction ($\vee$), and application of any of the (unary) tense operators F, G, P, H. The semantics for **PTL** is defined in terms of *temporal struc-*

[4] This is the basic tense logic of past and future considered by Prior, see e.g. Prior (1967, pp. 34-8). In present-day literature, this logic is often referred to simply as Propositional Temporal Logic.

tures $\mathcal{M} = (T, R, \mathfrak{h})$, where T is a non-empty domain, R is an irreflexive and transitive binary relation on T, and $\mathfrak{h}$ is a function assigning a subset of T to each propositional atom from **prop**. (Explicitly indicating the converse of R in the structure is refrained from when discussing **PTL**.) The formulae of **PTL** are evaluated relative to points $t \in T$. The semantics for **PTL** is obtained from the definition of $\mathbf{ML}[k]$ semantics given in *Section* 2.1. We may simply refer to the semantics thus obtained as the *standard semantics* of **PTL**.

We call $\mathbf{IFTL}[1]$ simply *IF tense logic.* Hence we notice that IF tense logic is the class

$$\mathbf{PTL} \cup \{O_1 \ldots O_{n-1}(O_n/W)\varphi : \varphi \in \mathbf{PTL}, n \geq 1\}$$

where:

- for all $j \in \{1, \ldots, n\}$, O_j is one of the tense operators F_j, G_j, P_j, H_j. [5]
- $W \subseteq [1, n-1]$.

4.2.2 The intuitive import of algebraic interpretation: a special case

Any set T equipped with an operation $o : T \times T \rightarrow\ T$ is a *group*, if

(1) o is *associative*, i.e. for all $x, y, z \in T$,
$x\ o\ (y\ o\ z) = (x\ o\ y)\ o\ z$;

(2) there is $e \in T$ such that for all $x \in T$,
$(e\ o\ x) = (x\ o\ e) = e$;

[5] j is here an index identifying an occurrence of an operator from $\{F,G,P,H\}$.

(3) for all $x \in T$, there exists $x^{-1} \in T$ such that $(x\ o\ x^{-1}) = (x^{-1}\ o\ x) = e$.

An element $e \in T$ given by (2) is necessarily unique, and is called the *identity element* of the group (T, o). Also, for any $x \in T$ in a group there necessarily exists precisely one element $x^{-1} \in T$ as provided by (3). For a given x, this x^{-1} is called the *inverse* of x. Further, if the operation o is *commutative*, i.e. if

(4) for all $x, y \in T$, $(x\ o\ y) = (y\ o\ x)$,

then the group (T, o) is said to be *Abelian.*

Now consider the structure

$$\mathcal{M} = (\mathbb{R}, <, \mathfrak{h}, +),$$

where $(\mathbb{R}, +)$ is the additive Abelian group of the reals and $(\mathbb{R}, <, \mathfrak{h})$ is a unary temporal structure, $(R, <)$ being the order of the reals by magnitude. We call the structure $\mathcal{M}$ an *arithmetic model* of IF tense logic. (With the terminology of *Subsect.* 4.2.3 below it will be an instance of a general algebraic model.)

We fix for each occurrence of a tense operator O_i (this occurrence being identified by the subscript i) a unique relativized (metalanguage) quantifier $Q_i := \mathbb{Q}(O_i)$ as follows:

- $\mathbb{Q}(F_i) := (\exists x_i > 0)$
- $\mathbb{Q}(G_i) := (\forall x_i > 0)$
- $\mathbb{Q}(P_i) := (\exists x_i < 0)$
- $\mathbb{Q}(H_i) := (\forall x_i < 0)$

Let $\mathcal{M} = (\mathbb{R}, <, \mathfrak{h}, +)$ be as above. Relative to such a particular model, the algebraic interpretation of **IFTL** is defined as follows.

- $\mathcal{M} \models_A^+ p[t] \Longleftrightarrow t \in \mathfrak{h}(p)$
- $\mathcal{M} \models_A^+ \neg p[t] \Longleftrightarrow t \notin \mathfrak{h}(p)$
- $\mathcal{M} \models_A^+ (\varphi \wedge \psi)[t] \Longleftrightarrow \mathcal{M} \models_A^+ \varphi[t]$ and $\mathcal{M} \models_A^+ \psi[t]$
- The clause for disjunction ($\vee$) is analogous to that for conjunction.
- $\mathcal{M} \models_A^+ F(\varphi)[t] \Longleftrightarrow$ there is $t' > 0$ such that

  $\mathcal{M} \models_A^+ \varphi[t + t']$.
- $\mathcal{M} \models_A^+ \mathrm{G}(\varphi)[t] \Longleftrightarrow$ for all $t' > 0$: $\mathcal{M} \models_A^+ \varphi[t + t']$.
- $\mathcal{M} \models_A^+ \mathrm{P}(\varphi)[t] \Longleftrightarrow$ there is $t' < 0$ such that

  $\mathcal{M} \models_A^+ \varphi[t + t']$.
- $\mathcal{M} \models_A^+ \mathrm{H}(\varphi)[t] \Longleftrightarrow$ for all $t' < 0$: $\mathcal{M} \models_A^+ \varphi[t + t']$.

Finally, for the proper **IFTL** formulae, we put:

- $\mathcal{M} \models_A^+ O_1 \ldots O_{n-1}(O_n/W)\varphi[t] \Longleftrightarrow$

  $Q_1 \ldots Q_n$: $\mathcal{M} \models_A^+ \varphi[t + \sum_{i \in \{1,\ldots,n\}}(x_i) - \sum_{i \in W}(x_i)]$.

The letter "A" in the subscript of the symbol "$\models$" simply stands for *algebraic*.

Example 4.2.1 *Let us consider the arithmetical models* $\mathcal{M} = (\mathbb{R}, <, \mathfrak{h}, +)$ *and* $\mathcal{N} = (\mathbb{R}, <, \mathfrak{h}', +)$*, where in particular* $\mathfrak{h}(p) = \{-1\}$ *and* $\mathfrak{h}'(p) = \{0\}$ *(see Figure 16).*

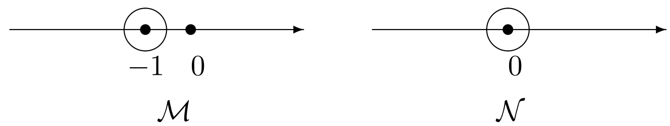

FIGURE 16

Then consider evaluating the formula $\psi = G_1(P_2/\{1\})p$ *in* $(\mathcal{M}, 0)$ *resp.* $(\mathcal{N}, 0)$*. Now* ψ *is true in* $\mathcal{M}$ *at* 0*, since indeed for every positive real number* r *there is a negative real number* s *such that* p *is true at* $(0 + r + s - r) = s$*. For all* r*,* s *can be chosen to be* -1*. By contrast,* ψ *is false in* $\mathcal{N}$ *at* 0*, because there is a real number* $r > 0$ *such that for all reals* $s < 0$*,* p *is false at* $(0+r+s-r) = s$*. In fact, any positive real number is such an* r*;* p *is false at all* $s < 0$*.*

The following results, concerning the expressive power of **IFTL**, hold.

Fact 4.2.2 *Let* $\mathcal{M} = (\mathbb{R}, <, \mathfrak{h}, +)$ *be an arithmetic model of* **IFTL***, and put* $\mathcal{N} = (\mathbb{R}, <, \mathfrak{h})$*. The standard and algebraic semantics coincide for* **PTL** *on arithmetical models, i.e. for all* $\varphi \in$ **PTL** *and all* $t \in \mathbb{R}$*,*

$$\mathcal{N} \models^+ \varphi[t] \Longleftrightarrow \mathcal{M} \models^+_A \varphi[t].$$

Proof. See below the proof for the general case of $\mathcal{M} = (T, <, \mathfrak{h}, o)$, where (T, o) is any Abelian group (Lemma 4.2.6). ■

In the following theorem it is shown that **IFTL** and **PTL** are weakly embeddable in each other over arithmetical models.

Theorem 4.2.3 *Let* $\mathcal{M} = (\mathbb{R}, <, \mathfrak{h}, +)$ *be an arithmetic model of* **IFTL***.* **PTL** *and* **IFTL** *have the same expressive power relative to* $\mathcal{M}$ *in the sense that for each formula* $\varphi \in$ **PTL** (*resp.* $\varphi \in$ **IFTL**) *there is a formula* $\psi \in$ **IFTL** (*resp.* $\psi \in$ **PTL**) *such that for all* $t \in \mathbb{R}$*,* $\mathcal{M} \models^+_A \varphi[t]$ *iff* $\mathcal{M} \models^+_A \psi[t]$*.*

Proof. Because **PTL** is a subclass of **IFTL**, it is trivial that **PTL** is embeddable in **IFTL**. (By Fact 4.2.2 the algebraic semantics can be used for both logics.)

Let then $\varphi \in$ **IFTL** and $t \in \mathbb{R}$ be arbitrary. If in particular $\varphi \in$ **PTL**, there is nothing to prove, as φ counts as its own translation. So suppose that $\varphi = O_1 \dots O_{n-1}(O_n/W)\chi$ for some $\chi \in$ **PTL**. Let $i_1, \dots, i_c$ be the list of the elements of $\{1, \dots, n-1\} \backslash W$ in their order by magnitude. (Here $c = 0$ iff $Card(W) = 0$.) Then, by definition,

$$O_{i_1} \dots O_{i_c}$$

is the sequence of operators obtained from the sequence $O_1 \dots O_{n-1}$ by erasing from it all operators O_i with $i \in W$ and keeping intact the relative order of the remaining operators (namely those O_i with $i \in \{1, \dots, n-1\} \backslash W$). We claim that the **PTL** formula

$$\psi := O_{i_1} \dots O_{i_c} O_n \chi$$

is a required translation for φ. Indeed, trivially

$$\mathcal{M} \models^+_A \varphi[t] \iff$$

$$Q_1 \dots Q_n : \mathcal{M} \models^+_A \chi[t + \textstyle\sum_{i \in \{1,\dots,n\}}(x_i) - \sum_{i \in W}(x_i)] \iff {}^6$$

$$Q_{i_1} \dots Q_{i_c} Q_n : \mathcal{M} \models^+_A \chi[t + \textstyle\sum_{i \in \{1,\dots,n\} \backslash W}(x_i)] \iff$$

$$\mathcal{M} \models^+_A O_{i_1} \dots O_{i_c} O_n \chi[t].$$

Hence the proof is completed. ■

IFTL and **PTL** are even strongly embeddable in each other over arithmetic models, as will be witnessed by Corollary 4.2.9 below.

[6] This equivalence is *not* generally valid but holds here because $(\mathbb{R}, <)$ has no extrema. See the proof of Theorem 4.2.7 below for details concerning the general case.

Remark 4.2.4 *We observe that* **ALG** *interpretation gives the same semantic effect as simultaneously requiring independence in the sense of* **BLO** *interpretation* (*"independence concerning accessibility relations"*) *and independence in the sense of* **UNI** *interpretation* (*logical independence, or freedom of specified priority scopes*). *In effect, this interpretation, on arithmetic models, means simply removing altogether those operators that are referred to by the independence indication.*

Let us move on to consider the evaluation on general algebraic models.

4.2.3 General algebraic interpretation

We say that a structure $\mathcal{M}$ is a *general algebraic model* of IF tense logic, if

$$\mathcal{M} = (T, <, \mathfrak{h}, o),$$

where:

- $(T, <)$ is a non-empty linear order;
- $(T, <, \mathfrak{h})$ is a unary temporal structure;
- (T, o) is an Abelian group.

We write $\mathcal{A}$ for the class of all general algebraic models of **IFTL**. For all $t \in T$, define

- $FUT_t := \{t' : (t \ o \ t') > t\}$;
- $PAST_t := \{t' : (t \ o \ t') < t\}$.

We observe the following.

Fact 4.2.5 *For arbitrary $t, t' \in T$ we have:*

(a) $t < t' \Longleftrightarrow$ *there is an* $s \in FUT_t$ *such that* $t' = (t \; o \; s)$.

(b) $t' < t \Longleftrightarrow$ *there is an* $s \in PAST_t$ *such that* $t' = (t \; o \; s)$.

Proof. (a) ($\Longrightarrow$) Assume $t < t'$. Now $t' = ((t \; o \; t^{-1}) \; o \; t') = (t \; o \; (t^{-1} \; o \; t'))$. Hence $(t^{-1} \; o \; t') \in FUT_t$. ($\Longleftarrow$) Assume there is an $s \in FUT_t$ such that $t' = (t \; o \; s)$. But $s \in FUT_t$ means that $t < (t \; o \; s)$. Hence $t < t'$. Case (b) is proven similarly. ■

The semantics fixes for each occurrence of a tense operator O_i a unique relativized (metalanguage) quantifier $Q_i = \mathbb{Q}(O_i)$. The definition essentially depends on the index $i < \omega$ by which the operator token O_i is identified.

- $\mathbb{Q}(F_i) := (\exists x_i \in FUT_{x_{i-1}})$
- $\mathbb{Q}(G_i) := (\forall x_i \in FUT_{x_{i-1}})$
- $\mathbb{Q}(P_i) := (\exists x_i \in PAST_{x_{i-1}})$
- $\mathbb{Q}(H_i) := (\forall x_i \in PAST_{x_{i-1}})$

The general algebraic interpretation of **IFTL** is then given relative to general algebraic models as follows.

- The clauses for (negated) propositional atoms as well as those for conjunction and disjunction are defined just as above in the special case of the models $(\mathbb{R}, <, \mathfrak{h}, +)$.
- $\mathcal{M} \models^+_A F(\varphi)[t] \Longleftrightarrow$ there is $t' \in FUT_t$ such that

  $\mathcal{M} \models^+_A \varphi[t \; o \; t']$.

- $\mathcal{M} \models^+_A G(\varphi)[t] \iff$ for every $t' \in FUT_t$, $\mathcal{M} \models^+_A \varphi[t \; o \; t']$.
- $\mathcal{M} \models^+_A P(\varphi)[t] \iff$ there is $t' \in PAST_t$ such that

  $\mathcal{M} \models^+_A \varphi[t \; o \; t']$.
- $\mathcal{M} \models^+_A H(\varphi)[t] \iff$ for every $t' \in PAST_t$,

  $\mathcal{M} \models^+_A \varphi[t \; o \; t']$.
- $\mathcal{M} \models^+_A O_1 \ldots O_{n-1}(O_n/W)\varphi[t] \iff$

  $Q_1 \ldots Q_n$: $t = x_0$ and

  $\mathcal{M} \models^+_A \varphi[t \; o \; \mathrm{O}_{i\in\{1,\ldots,n\}}(x_i) \; o \; \mathrm{O}_{i\in W} \; (x_i^{-1})]$. [7]

We now prove some basic properties of the expressive power of **IFTL** under the general **ALG** interpretation.

Lemma 4.2.6 *Let* $\mathcal{M} = (T, <, \mathfrak{h}, o)$ *be a general algebraic model of* **IFTL**, *and put* $\mathcal{N} = (T, <, \mathfrak{h})$. *The standard and the general algebraic semantics coincide for* **PTL**, *i.e. for all* $\varphi \in$ **PTL** *and all* $t \in T$*:* $\mathcal{N} \models^+_A \varphi[t] \iff \mathcal{M} \models^+_A \varphi[t]$.

Proof. The lemma is proven by induction on the complexity of the **PTL** formula φ. It suffices to consider the cases for formulae of the forms F, P, G, H, as the other cases are immediate. So assume inductively that for all $t \in T$: $\mathcal{N} \models^+_A \varphi[t] \iff \mathcal{M} \models^+_A \varphi[t]$.

Let $t \in T$ be arbitrary. Now if $\mathcal{N} \models^+_A F(\varphi)[t]$, then there is $t' > t$ such that $\mathcal{N} \models^+_A \varphi[t']$, and so, by the inductive hypothesis, $\mathcal{M} \models^+_A \varphi[t']$. But then $\mathcal{M} \models^+_A F(\varphi)[t]$, because there is $s \in FUT_t$ such that $t' = (t \; o \; s)$. Namely,

[7] The generalized notation $\mathrm{O}_{i\in I} \; (x_i)$ has the following obvious meaning: for a finite set I of indices, let $i_1, \ldots, i_k$ be the list of its elements in a fixed order. Assuming that an element x_i for each $i \in I$ is defined, $\mathrm{O}_{i\in I} \; (x_i)$ is identified by definition with the following element: $(((x_{i_1} \; o \; x_{i_2}) \ldots) \; o \; x_{i_k})$.

$$(t\ o\ (t'\ o\ t^{-1})) = t' > t,$$

and so $(t'\ o\ t^{-1}) \in FUT_t$. Hence we may take $s = (t'\ o\ t^{-1})$. In the other direction, if $\mathcal{M} \models^+_A F(\varphi)[t]$, then there is $s \in FUT_t$ such that $\mathcal{M} \models^+_A \varphi[t\ o\ s]$. So, by the inductive hypothesis, $\mathcal{N} \models^+_A \varphi[t\ o\ s]$. But then $\mathcal{N} \models^+_A F(\varphi)[t]$, since $(t\ o\ s) > t$. (This, again, is because $s \in FUT_t$.) The cases for *G, P, H* are dealt with similarly. ∎

Letting $\mathcal{M} \in \mathcal{A}$ and $t \in dom(\mathcal{M})$ be arbitrary, we recall from *Subsection* 4.1.1 the following facts that are most conveniently expressed in game-theoretical terms:

- $\mathcal{M} \models O_1 \ldots O_{n-1}\bot[t] \iff$ for some $i \in \{1, \ldots, n-1\}$ up to which *Héloïse* is able to make her moves (if any) without failing to choose, it is *Abélard* 's turn to move, and he's forced to fail in choosing.

- $\mathcal{M} \models O_1 \ldots O_{n-1}\top[t] \iff$ for all $i \in \{1, \ldots, n-1\}$, the player whose turn it is to move is able to reply without failing to choose, or else $\mathcal{M} \models O_1 \ldots O_{n-1}\bot[t]$.

We proceed to prove the following theorem of bidirectional weak embeddability:

Theorem 4.2.7 PTL *and* **IFTL** *have the same expressive power relative to the class $\mathcal{A}$ of all general algebraic models of* **IFTL**.

Proof. There is nothing to prove concerning the direction from **PTL** to **IFTL**. It will now be shown that **IFTL** is embeddable in **PTL** over the class $\mathcal{A}$. Let $\varphi \in$ **IFTL**, $\mathcal{M} \in \mathcal{A}$ and $t \in T$ all be arbitrary. The only case that really needs to be checked is the case for $\varphi = O_1 \ldots O_{n-1}(O_n/W)\chi$ where $\chi \in$ **PTL**, because the result is trivial for the subclass

PTL of **IFTL**. Let then $i_1, \ldots, i_c$ be the list of the elements of $\{1, \ldots, n-1\} \backslash W$ in their order by magnitude. We claim that the **PTL** formula

$$\psi := (O_1 \ldots O_{n-1}\bot \vee (O_{i_1} \ldots O_{i_c} O_n \chi \wedge O_1 \ldots O_{n-1}\top))$$

is a required translation for φ.

Observe that this general translation ψ is slightly more complex than the formula $O_{i_1} \ldots O_{i_c} O_n \chi$, shown above in Theorem 4.2.3 to be a translation of $O_1 \ldots O_{n-1}(O_n/W)\chi$ relative to the models $\mathcal{M} = (\mathbb{R}, <, \mathfrak{h}, +)$. The reason is that the frame $(\mathbb{R}, <)$ of such models $\mathcal{M}$ has by definition no extrema (no minimum, no maximum). On the other hand, for linear orders, it is precisely the extrema that potentially cause a situation where one of the players of a relevant semantical game cannot make a choice. Hence in general − when nothing special is assumed of the frame $(T, <)$ of the algebraic model except for its being linear − the translation must, when it comes to the size of the domain of the model, be precisely as demanding as the formula being translated. This would not generally be the case if the plain $O_{i_1} \ldots O_{i_c} O_n \chi$ was offered as a translation in the general case as well. For linear orders *with at least one end-point* (out of the two possible), such a 'translation' would then not always be correct.

We have:

$\mathcal{M} \models^+_A \varphi[t] \Longleftrightarrow$

$Q_1 \ldots Q_n$: $t = x_0$ **and**

$\mathcal{M} \models^+_A \chi[t \; o \; \mathrm{O}_{i \in \{1,\ldots,n\}}(x_i) \; o \; \mathrm{O}_{i \in W}(x_i^{-1})] \Longleftrightarrow$ [8]

[8] By associativity and commutativity of o.

$Q_1 \dots Q_n$: $t = x_0$ **and**

$\mathcal{M} \models^+_A \chi[t\ o\ \mathrm{O}_{i \in \{1,\dots,n\} \setminus W}(x_i)] \Longleftrightarrow^9$

$(Q_1 \dots Q_n$: $t = x_0$ **and** $\mathcal{M} \models^+_A \bot[t])$ **or**

$((Q_1 \dots Q_n$: $t = x_0$ **and** $\mathcal{M} \models^+_A \top[t])$ **and**

$Q_{i_1} \dots Q_{i_c} Q_n : t = x_{i_1 - 1}$ **and**

$\mathcal{M} \models^+_A \chi[t\ o\ \mathrm{O}_{i \in \{1,\dots,n\} \setminus W}(x_i)]) \Longleftrightarrow$

$\mathcal{M} \models^+_A (O_1 \dots O_{n-1} \bot \vee$

$(O_{i_1} \dots O_{i_c} O_n \chi \wedge O_1 \dots O_{n-1} \top))[t].$

This completes the proof. ■

From the last lines of the proof, notice that it is not possible to erase the metalanguage quantifiers Q_i with $i \in W$ from the block $Q_1 \dots Q_n$ and obtain

$$Q_{i_1} \dots Q_{i_c} Q_n\colon\ \mathcal{M} \models^+_A \chi[t\ o\ \mathrm{O}_{i \in \{1,\dots,n\} \setminus W}(x_i)]$$

as a statement equivalent to $\mathcal{M} \models^+_A \varphi[t]$. Even though these quantifiers bind no metalanguage variable in the metalanguage expression

$$(t\ o\ \mathrm{O}_{i \in \{1,\dots,n\} \setminus W}(x_i)),$$

they serve to assert something concerning the model, because they are relativized quantifiers. They make assertions about how times can be chosen in relation to other times (in terms of the relation appearing in the relativizing clause of these quantifiers). A very simple example is provided by the formula

[9] Here the metalanguage quantifiers with indices in W *bind vacuously*. But as they are *relativized* quantifiers, they cannot simply be erased.

$P_1F_2/\{1\}p$ evaluated at the minimum of the domain of the algebraic model $\mathcal{M} = (\omega, <, \mathfrak{h}, +)$. The formula $P_1F_2/\{1\}p$ is true in $\mathcal{M}$ at 0 iff

$$(\exists x_1 < 0)(\exists x_2 > 0)\colon\ \mathcal{M} \models^+_A p[0 + x_1 + x_2 - x_1].$$

Accordingly, the formula $P_1F_2/\{1\}p$ is *not* true in $\mathcal{M}$ at 0, because there is no natural number smaller than 0, unlike what the truth-condition of the formula $P_1F_2/\{1\}p$ however requires. On the other hand, the judgment

$$(\exists x_2 > 0)\colon\ \mathcal{M} \models^+_A p[0 + x_2]$$

is perfectly true, provided that $\mathfrak{h}(p)\backslash\{0\} \neq \varnothing$.

We say that a formula $\varphi \in \mathbf{IFTL}$ is *false* in $\mathcal{M}$ at t under the algebraic interpretation, if the dual $\Delta(\varphi)$ of φ satisfies the following:

$$\mathcal{M} \models^+_A \Delta(\varphi)[t].$$

(Dual is understood in the sense of Definition 4.1.7 above.) We write

$$\mathcal{M} \models^-_A \varphi[t]$$

to indicate the falsehood of φ in this sense. Now, using Theorem 4.2.7, it is a straightforward task to prove the following analogues of Corollaries 4.1.8 and 4.1.9.

Corollary 4.2.8 IFTL *is determined relative to the algebraic interpretation, i.e. for all* $\varphi \in \mathbf{IFTL}$*, all* $\mathcal{M} \in \mathcal{A}$ *and all* $t \in$ *dom*$(\mathcal{M})$*, either* $\mathcal{M} \models^+_A \varphi[t]$ *or else* $\mathcal{M} \models^-_A \varphi[t]$. ■

Corollary 4.2.9 **IFTL** *and* **PTL** *are strongly embeddable to each other over the class* $\mathcal{A}$ *of all algebraic models relative to* **ALG** *interpretation of* **IFTL**, *i.e. for every formula* $\varphi \in$ **IFTL** (*resp.* $\varphi \in$ **PTL**) *there is a formula* $\psi \in$ **PTL** (*resp.* $\psi \in$ **IFTL**) *such that for all* $\mathcal{M} \in \mathcal{A}$ *and all* $t \in dom(\mathcal{M})$*:*

$$(\mathcal{M} \models^{+}_{A} \varphi[t] \textbf{ and } \mathcal{M} \models^{+}_{A} \psi[t]) \textbf{ or}$$

$$(\mathcal{M} \models^{-}_{A} \varphi[t] \textbf{ and } \mathcal{M} \models^{-}_{A} \psi[t]). \ \blacksquare$$

Chapter 5

Tense Operators and Linguistic Theorizing about Tense

In *Section* 2.1 above we have adopted the convention of viewing tense logic as a special sort of modal logic. More specifically, a basic modal logic $\mathbf{ML}[k]$ (*resp.* an IF modal logic $\mathbf{IFML}[k]$) is considered to be a temporal logic when (i) each modal operator O has an inverse O^{-1} interpreted by means of the converse R^{-1} of the accessibility relation R in terms of which the semantics of O is given, and (ii) all accessibility relations employed by the semantics of the logic have the properties *irreflexivity* and *transitivity*, which are held as minimal assumptions about the nature of the relation *Earlier than* among points in time.

Various logics have been formulated in the literature under the heading "temporal logics", not all of which are tense logics in the sense given above: by no means have all of these logics been special cases of basic modal logics $\mathbf{ML}[k]$. The logics studied by Kamp (1968), Gabbay (1981), Halpern and Shoham (1986), Venema (1990), and Moszkowski (1986, 1994) are all cases in point. The logic of the binary connectives *Until* and *Since* of Kamp (1968) is not an instance of any modal logic $\mathbf{ML}[k]$, as all modal connectives in $\mathbf{ML}[k]$ are unary. In general, the tense logics with d-dimensional n-ary connectives of Gabbay (1981) are potentially much richer semantically than $\mathbf{ML}[k]$ (and of course their formulae cannot be written syntactically in any $\mathbf{ML}[k]$ for $n \geq 2$). Halpern and Shoham (1986) define an interval tense logic, whose expressive power is studied by Venema (1990). The logics $\mathbf{ML}[k]$ with $k \geq 2$ evaluated over modal structures of dimension 2 may indeed be seen as interval logics. However, the natural way of presenting the semantics for the interval logic of Halpern and Shoham requires defining the accessibility relations as 4-ary and not binary, as is done with $\mathbf{ML}[2]$. Changing $\mathbf{ML}[k]$ so that it can instantiate Halpern and Shoham's logic would not be difficult: since the modal connectives of this logic are in any case unary, changing the definition of the accessibility relations would suffice. Moszkowski's *Interval Tense Logic* also uses intervals in its semantics. His logic is, however, definitely far from $\mathbf{ML}[k]$: it has quantifiers, a binary modal operator *chop* (;) and the unary operator *chop-star* (*). In particular, a formula φ^* says of an interval that it has a finite partition into such subintervals that φ holds at each of them. This operator * is thus not even first-order definable.[1]

In order to call a logic *temporal*, it is indeed perfectly sufficient that the structures employed as its models have properties that are structurally similar to possible properties of time. A structural property customarily associated with time is *linearity*. On the other hand, it would not be particularly far removed to model (experienced) time by means of a branching time structure (a *tree*), where the branches in the direction of the future represent alternative future continuations of a fixed

[1] For a logic reminiscent of Moszkowski's *Interval Tense Logic*, cf. Hella & Tulenheimo (2003, in progress).

past. A variety of other properties may be considered, such as *discreteness*, *density*, *Dedekind-completeness*, the existence or non-existence of *extrema*, *well-orderedness*, and *cyclicity*.[2] From the viewpoint of tense logic it is *not* necessary to fix once and for all a particular class of temporal structures that would then be studied.

In the present thesis we have understood the syntax of temporal logics in the narrow sense defined above, and will continue to do so. This type of temporal logics has a very close connection to Priorean tense logic (cf. *Subsect.* 4.2.1), so close indeed that **ML**[2] with the operators $\Diamond$, $\Box$, $\Diamond^{-1}$ and $\Box^{-1}$ (or, more colloquially, F, G, P and H), evaluated relative to binary modal structures $\mathcal{M} = (D, R, R^{-1}, \mathfrak{h})$, actually *coincides* with **PTL**.

5.1 The Character of Prior's Tense-Logic

Arthur Prior (1914-1969) is generally viewed as the founding father of modern temporal logic. In *Past, Present and Future* (pp. 1, 8-10) he mentions John N. Findlay as a person who could be regarded as the initiator of what was to become tense logic.[3] Prior refers specifically to Findlay's article "Time: A

[2] For the definitions of these properties of relations, see *Appendix* A. For literature related to studying different temporal structures, see Gabbay, Hodkinson & Reynolds (1994); for cyclical time, see esp. Reynolds (1994). For a short discussion on different "conceptualizations" of time in connection with tense logic, see Gamut (1991, pp. 35-7). Gamut points out that for tense logic, it is natural to proceed by choosing semantics − fixing a temporal structure − and only then go on to study the syntactic principles (such as the validity of formulae) the semantics gives rise to. This is in contradistinction to the case of modal logic of alethic modalities, where our understanding primarily pertains to valid schemata, and not to the relations between possible worlds.

[3] According to Øhrstrøm & Hasle (1995, p. 167), in 1934 Findlay was Prior's teacher in logic and ethics, at Otago University in Dunedin, New

Treatment of Some Puzzles", published in 1941 in the *Australasian Journal of Psychology and Philosophy*, where Findlay points out (p. 233) that our linguistic conventions regarding tenses are so well worked out that ingredients exist for a calculus of tenses. He suggests (footnote 17, p. 233) that such a calculus should indeed have been included in the development of the modal logics of his time.[4] Prior mentions (1967, p. 10) that to the best of his knowledge, he himself was, in the early 1950s, the first to actually attempt to produce a calculus of tenses of the type Findlay had meant. Historically, temporal notions have been a source for logical studies at least since Aristotle's famous 'Sea Battle' passage in *De interpretatione* 9. Characters appearing in this tradition are Diodorus Cronus (a Megarian logician active around 300 B.C.), Thomas Aquinas (1225-1274), Paul of Venice (c. 1368-1429), Jean Buridan (c. 1300-1358) and William of Ockham (c. 1285-1347).[5] Prior himself was well informed about the history of logic; he was particularly interested in Aristotle, Diodorus Cronus and the scholastic philosophers.

Prior used the term "tense-logic" (spelled thus) as a general term designating all the various systems of temporal logic that might be designed for purposes of representing temporal discourse logically. All of the tense-logical systems he formulated

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Zealand.

[4] The development of modern modal logic can be seen as having started with the work of Clarence Irving Lewis on strict implication, published as Chapter V of his *A Survey of Symbolic Logic* (1918). Apparently Lewis felt he was rather alone in his interest of modal notions in logic. He ends the chapter on strict implication with (p. 339): "We shall not further prolong a tedious discussion by any special plea for the 'propriety' of strict implication as against material implication and formal implication. Anyone who has read through so much technical and uninteresting matter has demonstrated his right and his ability to draw his own conclusions."

[5] For a simple overview of the logic of temporal notions during antiquity and the Middle Ages, see Part 1 in Øhrstrøm & Hasle (1995).

shared one basic insight, namely the particular way in which he chose to deal with time in logic: treating temporality as it appears in natural language by means of *operators* in logic.

The different systems of tense logic that Prior formulated were typically motivated by philosophical or other 'real-life' concerns. They were not systems of logic for their own sake, but on the contrary Prior's intention was to employ tense logics as tools for analyzing the use of language involving temporality, for characterizing phenomena and defining notions involving time. In *Past, Present and Future* Prior discusses, for example, the so-called *Master Argument* of Diodorus Cronus,[6] and the seeming incompatibility of foreknowledge and indeterminism, and he considers how to characterize structural properties of time such as circularity, discreteness, density and Dedekind-completeness by tense-logical means.

Prior's approach to tense logic is axiomatic (proof-theoretic). A particular tense-logical system is defined by laying down its axioms, and its study thereafter consists of deducing theorems from these axioms. The semantics employed is an intuitively understood semantics, not a formal one along the lines of contemporary model-theoretic semantics. Of course, semantics still has a crucial role for Prior, since he is primarily interested in what tense-logical formulae serve to say or repre-

[6] The Master Argument states that the following three propositions cannot all be true: **(i)** *Every proposition true about the past is necessary*; **(ii)** *An impossible proposition cannot follow from (or after) a possible one*; **(iii)** *There is a proposition which is possible, but which neither is nor will be true*. Diodorus himself thought that (iii) was false, and that (i) and (ii) were true. To properly reconstruct the argument, one would have to decide how "proposition" in (ii) is to be understood, and whether in (iii) "follows" is to be read logically or temporally. For an introductory discussion of the Master Argument, see Øhrstrøm & Hasle (1995, pp. 15-32). For further discussion, see Mates (1953), Geach (1955) and Prior (1967, esp. pp. 17, 32-5).

sent, not in the formulae themselves as syntactic objects.

Prior was very explicit about the relative nature of systems of tense logic; he was not constructing the 'ultimate logic of tenses', but various alternative tense-logical formalisms. He writes (1967, p. 59):

> "The logician must be rather like a lawyer − not in ... [the sense of] reasoning less rigorously than a mathematician − but in the sense that he is there to give the metaphysician, perhaps even the physicist, the tense-logic that he wants, provided that it be consistent. He must tell his client what the consequences of a given choice will be ... and what alternatives are open to him; but I doubt whether he can, *qua* logician, do more. We must develop, in fact, alternative tense-logics, rather like alternative geometries."

Hence his views agree with the way axiom systems are naturally viewed in modern mathematics: any consistent set of axioms is a proper subject of study, whose interest is proportionate to the properties of the models it has.[7]

The theoretical background on the basis of which Prior made his choice of introducing temporality to logic via operators, instead of any other mechanism, as well as the theoretical meaning and import of this choice for Prior himself, are by no means self-evident from the logical systems that he formulated, or, for that matter, from the present-day expositions of propositional temporal logic that can be read in textbooks on modal logic. But Prior had his reasons for his preference for operators.

[7] Hence there is no prejudice to choose our axioms so that a previously known piece of reality will become a model − perhaps up to isomorphism a unique model − of these axioms.

Actually, his motivation was a rather sophisticated, philosophical one. The way he formulated his tense-logic was thus by no means arbitrarily chosen. It is of some interest – systematical as well as historical – to briefly sketch how Prior's thoughts on this subject arose. It turns out that it was the particular way in which Prior came to understand the character of temporal sentences that is ultimately responsible for the form Prior's tense-logic was to take: having initially thought that all token-reflexivity must be removed from natural language temporal sentences to make them possible objects of logical study, Prior came to think of sentences whose distinct occurrences may have distinct truth-values as capable of complete meaning.

5.1.1 Two types of temporal sentences

In what follows, we shall consider *utterances* (*tokens*) of sentences as primary *truth-bearers*. Sentences are seen as *types*, admitting of indefinitely many 'realizations', namely tokens or utterances of these sentences. For instance, the sentence "It rains" uttered on December 1, 2002, in Helsinki, and the sentence "It rains" uttered on June 1, 1900, in London, constitute *two* utterances of *the same* sentence.

Indicative sentences may be divided into two disjoint classes according to whether the statement that can be made by uttering such a sentence is or is not fully determined without having specified its context of utterance. Those whose assertive force is determined may be termed *eternal sentences*, and those whose utterances vary in the statement they make according to the context, we may here simply term *non-eternal*. Hence, for example, the sentence "*It rains on June 1, 1900, in the immediate vicinity of Sacre Cœur in Paris*" is eternal (given that "rains" is construed as atemporal), while the sentences "*It rains*" and "*I am here now*" are not. All utterances of an eternal sentence have the same truth-value, while distinct occurrences of a non-eternal sentence may vary in truth-value. (The latter is not necessarily the case with non-eternal sentences, however, as witnessed by sentences like "*I am here now*" or "*I exist*".) W. V. O. Quine (1908-2000) points out on various occasions that any indicative sentence can be *eternalized*, i.e. made eternal by objectively indicating persons, times and places spoken of in the sentence, and cancelling the tenses of the verbs occurring therein (see e.g. (1960), esp. pp. 193-5, 208; (1970), pp. 13-4). In particular, any actual utterance of a non-eternal sentence of course serves to determine specific values for its context-dependent parameters.

Let us move on to consider instances of eternal and non-eternal sentences, and whether such sentences specify the relevant time-parameters absolutely, or leave them for the context to specify. Consider sentences of the following two types:

(1) p (now),

(2) p at t_0,

where 'p' represents an indicative sentence of some natural language. Further, (1) involves the word 'now' tacitly or explicitly; in (2), t_0 is a time-point specified independently of the moment the sentence is uttered.[8] For simplicity we may assume here that there appears no indexical references to places or persons in either type of sentence. In both (1) and (2) the sentence represented by 'p' may contain any tenses.

Sentences of type (1) are token-reflexive. By the terminology introduced above, they are non-eternal sentences: what they state is sensitive to the time of their utterance. By contrast, sentences of type (2) are tied to a chronology, i.e. to

[8] Hintikka discusses these two types of sentences in his book *Time and Necessity* (1973, *Ch.* 8), in connection with his investigation into Aristotle's 'Sea Battle' passage in *De Int.* 9.

a time scale whose ordering is available absolutely, instead of being given by reference to a particular perceptual situation or in some other perspectival way. Hence in the above terminology these sentences are eternal sentences (up to their possibly containing tenses of verbs).

As instances of temporal sentences of these two types, we may consider sentences

(1′) *Socrates is speaking (now)*

(2′) *Socrates is speaking at noon, December 1, 400* B.C.

The utterance of (1′) may well be *false* at dawn, December 1, 400 B.C., while its utterance might be *true* at noon the same day. It is *not* necessary that the distinct utterances of a sentence of type (1) have the same truth-value. On the other hand, if some utterance of (2′) is true, all its utterances are true. This is precisely because this sentence speaks of an individual, absolutely (non-perspectively) specified time. What sentence (2′) states is completely indifferent to the context of its utterance; the fact that something happened or did not happen at a given time, or that something is or is not true at a given time, is itself omnitemporally true. In (2′), the objective time-reference temporally anchors the evaluation of the constituent sentence '*Socrates is speaking*' in such a way that the time of utterance of the resulting complex, (2′), becomes completely immaterial to the truth or falsehood of that utterance.

Observe, however, that if a sentence of type (2) involves tenses, generally there are past or future times at which its utterance results in an *ungrammatical* sentence.[9] For example, unless it is accepted that 'is' in (2′) be construed as atemporal, (2′) is grammatical only when uttered on December 1,

[9] Hintikka (1973) takes up this objection on p. 152 when discussing the distinction between sentences of type (1) and type (2).

400 B.C. *Before* that date we should really utter "*Socrates will be speaking at noon, December 1, 400* B.C.", and after that date the grammatical formulation would be "*Socrates was speaking at noon, December 1, 400* B.C." − instead of (2′).

The distinction between sentences of types (1) and (2) is most easily appreciated by construing the verb forms in type (2) sentences as atemporal. Otherwise, the existence of a difference between these types of sentences can in any case be seen from the fact that the grammaticality vs. ungrammaticality of sentences of type (2) varies with time, whereas it is truth vs. falsity of sentences of type (1) that varies with time.

The distinction between sentences of the above types (1) and (2) proves to be important for Prior's formulation of tense logic. Prior (1967, pp. 15-6) writes that for him, before 1949,

> "it was not only correct but also 'traditional' to think of propositions as incomplete, and not ready for accurate logical treatment, until all time-references had been so filled in that we had something that was either unalterably true or unalterably false."

We are not going to discuss here the potential metaphysical connotations related to Prior's use of the word "proposition", but simply take him to be concerned about finding the proper object of study for a logic of temporal discourse. Such objects might reasonably be *eternal sentences*, or sentences of type (2).[10] In the above quote Prior expresses that he had thought that precisely eternal sentences − which as such do

[10] Strictly speaking, Prior's quote only requires that the sentences he used to think of as proper objects of a logic of temporality be devoid of *temporally* indexical expressions (and of tenses of the verbs); he does not discusse the possibility that they involve indexical references to places and persons.

not contain contextual references to times – would count as proper objects of a logical study of temporal language.

In 1949 Prior read P. T. Geach's critical notice in *Mind* concerning Julius Rudolph Weinberg's book on the thought of Nicolaus de Ultricuria (c. 1300-1350). There, Geach (p. 244) points out (as the only major flaw in Weinberg's understanding of his subject due specifically to the six hundred years of history separating them) that

> "[s]uch expressions as 'at time *t*' ... are out of place in expounding scholastic views of time and motion. For a scholastic, 'Socrates is sitting' is a complete proposition, *enuntiabile*, which is sometimes true, sometimes false; *not* an incomplete expression requiring a further phrase like 'at time *t*' to make it into an assertion."

As observed by Prior (1967, p. 17), Geach (1955, p. 144) repeats this comment relating to the classical view on propositions when reviewing Benson Mates's book, *Stoic Logic* (1953), saying:

> "... it is quite unhistorical of [Mates] to write: "Diodorus usually predicates necessity of what are in effect propositional functions. ... Consider the function 'Socrates dies at *t*' " (p. 39). The Stoics neither had a pair of terms answering to the Peano-Russell distinction between a proposition and a propositional function, nor gave any example that could suitably be translated by an expression like "Socrates dies at *t*." ... Moreover, to introduce this distinction would spoil the Stoics' examples in propositional logic. For they held, e.g., that

> > "If Dion is alive, then Dion is breathing; but Dion is alive; therefore Dion is breathing"
>
> was of the form "if p, then q; but p; therefore q." But this form is not to be found in:
>
> > "For any t, if Dion is alive at t, then Dion is breathing at t; but Dion is alive *now*; therefore Dion is breathing *now*."
>
> May not the Stoics well have thought that, though the truth-value of "Dion is alive" changes at Dion's death, the sentence *still expresses the same complete meaning* (*lecton*)? Arguments that might be produced against this view of propositions are not sufficient evidence that the Stoics did not hold it."

Here Geach wishes to point out that historically it has not been thought that sentences with time-references explicitly filled out – eternal sentences, sentences of type (2) – would serve to express complete sentential meanings. By contrast, it can be thought that tensed sentences like "Dion is alive" do in effect express such complete meanings (*lecta*), as Geach suggests the Stoics in fact had thought. Sentences in the same spirit as "Dion is alive" would even be "Dion was alive" and "Dion is going to be alive tomorrow", in general non-eternal sentences containing no explicit references to absolutely specified times.

Geach resorts to the notion of proposition in his comments, but as to the purely linguistic object of the study of temporal logic, what could we learn of options, from what he says? If not eternal sentences, then on the basis of Geach's view what could be the proper linguistic items for our logical study? The nearest linguistic analogue to a proposition corresponding to a sentence S would actually be a function specifying for any context c of utterance of S the truth condition of S relative to c. Namely, whatever a *lecton* correlated with a non-eternal sentence S precisely is, it must, in any event, serve to determine for all occurrences of S the statement that the occurrence serves to make.[11]

This would mean, then, that sentences of type (2), or eternal sentences, do not have the privilege of possessing 'complete meaning' in a reasonable sense. By contrast, sentences of type (1) possess such a meaning, despite the fact that distinct occurrences of these latter sentences may serve to make distinct statements. So these sentences as well may be taken as full-fledged objects of logical study. As Geach points out, the Stoics and scholastics went even further, and preferred studying sentences without explicit references to times.

Prior (1967, p. 16) describes the effect Geach's remark (1949) had on him:

> "Geach's remark sent me to the sources. The 'Socrates is sitting' example was not only in the scholastics but in Aristotle, who says that 'statements and opinions' vary in their truth and falsehood with the times at which they are made or held, just as concrete things have different qualities at different times."

The moral Prior drew from Geach's remarks was quite radical. He decided to start working exclusively on sentences of type (1), i.e. the token-reflexive temporal sentences Geach had

[11] Speaking of complete sentential meanings as functions from contexts to truth-conditions is certainly anachronistic from the Stoics' viewpoint, but does not prevent the description from being accurate.

explained as possessing a complete meaning even though what they assert varies contextually.

It is illuminating to realize that the shift in Prior's understanding of what sort of sentences admit of complete meaning (or, the shift in his understanding of the notion of proposition) – influenced by Geach – made him formulate tense logic in the way he did. The same approach is still repeated time and again in at least most contemporary studies on temporal logic conducted under the general heading of modal logic. Due to Prior's newly acquired understanding of sentences having a complete meaning, he thought tense logic should *not* be formulated for sentences with explicit references to times on an objective chronology. By contrast, he decided instead to formulate his calculus in terms of tenses – formalized as *operators*.

5.1.2 The idea of tenses as operators

It is characteristic of Prior's tense-logical formulae that *syntactically* they are *devoid of any mechanism of referring to times*. By contrast, one could easily imagine using first-order logic to speak about time, making use of individual variables having time-points as their values. Prior's way of dealing with temporal discourse is decidedly not the latter.

The insistence of any Priorean system to refrain from explicitly speaking of times leads to a variety of problems when one seriously attempts to make use of such a system in the analysis of natural language.[12] From a more principal viewpoint, a relevant difference between Prior's tense logic and first-order logic is as follows. If $(Qx)\varphi$ is a first-order formula, the quantifier Qx *binds* all appearances of the quantified variable, x, that

[12] Such problems – related, among other things, to natural language expressions explicitly referring to, or quantifying over, times – are discussed e.g. in Kamp & Reyle (1993, pp. 491-8, 611-89).

occur free in the formula φ. Semantically this means that when evaluating the formula $(Qx)\varphi$, once the variable x gets a value in the interpretation of the quantifier Qx, all occurrences of x that are free in φ thereby get the same value in that evaluation. Hence the mechanism of binding, as this appears in first-order logic, allows making use of a value – once fixed to a quantified variable x in an evaluation – arbitrarily many times in the binding scope of the corresponding quantifier Qx. This is not the case in **PTL**. Consider, for instance, the existential future tense formula $F\varphi$ in **PTL**, with the first-order translation

$$\exists x_1(x_0 < x_1 \wedge ST_{x_0/x_1}(\varphi)),$$

where $ST_{x_0}(\varphi)$ is the standard translation of the **PTL** formula φ. One way of grasping the semantical effect of the operator F is by observing that the 'bounded quantifier'

$$\exists x_1(x_0 < x_1 \wedge \ldots)$$

binds the free occurrences of x_1 in $ST_{x_0/x_1}(\varphi)$, but crucially this formula $ST_{x_0/x_1}(\varphi)$ *cannot* have such free occurrences of x_1 in arbitrary locations – by no means can it be an arbitrary first-order formula with one free variable, x_1. The bounded quantifiers that can appear in **FO** translations of **PTL** formulae are of the forms

$$\exists x_i(R(x_{i-1}, x_i) \wedge \ldots) \text{ and } \forall x_i(R(x_{i-1}, x_i) \rightarrow \ldots),$$

where $R \in \{<, >\}$. And in fact all free occurrences of x_1 in $ST_{x_0/x_1}(\varphi)$ are necessarily *outside* the binding scopes of *all* bounded quantifiers appearing in $ST_{x_0/x_1}(\varphi)$; further, x_1 can only appear in the bounding clause(s) "$R(x_{i-1}, x_i)$" of the quantifier(s) immediately logically subordinate to the quantifier

$$\exists x_1(x_0 < x_1 \wedge \ldots).$$

Concretely, these severe restrictions on the binding effects of tense operators mean that at most two states need to be considered at any step in the evaluation of a tense-logical formula, and that such states are precisely those where one is reached along an accessibility relation from the other. Formally this is seen directly from the fact (see Remark 2.1.3) that the first-order translations of **PTL** formulae can be written with a total of two variables (free and bound). At a stage in an evaluation game for first-order logic, by contrast, a player must in general take into account assignments

$$x_0 \longmapsto a_0, \ldots, x_k \longmapsto a_k$$

for all variables $x_0, \ldots, x_k$ interpreted at previous stages in the relevant play of the game. There are no restrictions to the positions in which variables can appear in formulae of first-order logic, and in particular any subformula not yet evaluated may contain all of these variables. By contrast, the semantics of **PTL**, and of basic modal logics in general, is characteristically in terms of *local* transitions − the moves in evaluation games are only concerned with two states at a time.

Simply put, tense operators are quantifiers semantically, but not syntactically, as they lack syntactically manifest variables.[13] This fundamental difference between modal logics and abstract logics (such as **FO**) makes the question of trying to express fragments of abstract logics by means of a modal logic − i.e. the question of *expressive completeness* of a modal logic

[13] This can be seen precisely as a major motivation for the Hybrid Logic of Patrick Blackburn and his associates: in this logic, specific symbols called 'nominals' in a sense do the job of individual variables. (Nominals are really propositional atoms that are by definition true at precisely one state, i.e. they are a kind of global unique description.) For a short introduction to Hybrid Logic, see e.g. Blackburn & Tzakova (2000); Blackburn, de Rijke & Venema (2002, pp. 434-45).

with respect to an abstract logic – a challenging one.[14]

From what we noted in the previous subsection directly above, we are now in a position to understand the reasons for Prior's insistence on not wishing to mention times in his formalism. It is not at all accidental that he formulated his logic in a manner which from the beginning ruled out the possibility of referring explicitly to times from an absolute time scale. Prior himself describes, in *Past, Present and Future* (1967, p. 17) how he came to think about his particular choice for formalism. In the following excerpt he refers to Benson Mates's *Stoic Logic* (1953), in which Mates discusses, among other things, Diodorus Cronus's views on alethic modalities.

> "Mates, in attempting to formalize the thought of Diodorus, made free use of expressions like 'p at time t'. (Geach [1955], reviewing *Stoic Logic...* naturally did not miss this, and amplified his remarks on Weinberg); I wondered if it could be done some other way, and tried writing Fp for 'It will be that p', by analogy with the usual modal Mp for 'It could be that p'."

Prior continues by explaining how his motivation at that point – at a stage where no tense logic as yet existed – was not only to fill in the gaps in the Diodorean Master Argument (to which Mates's 1949 article "Diodorean Implication" had already drawn his attention) but also something much more general. He wanted to know the particular system of modal logic that the Diodorean definition of *possibility*, *necessity* and *impossibility* would yield (i.e. the definition of *possible* as that which is or will be true, *necessary* as that which both is and

[14] For studies in expressive completeness of modal logics, cf. e.g. Kamp (1968), Gabbay (1981), Venema (1990), Hodkinson (1994), Hella & Tulenheimo (2003).

always will be true, and *impossible* as that which both is and always will be false). Prior (1967, p. 17) closes his description of the opening act in the development of tense logic:

> "Definitions [such as the Diodorean ones for modalities] alone, however, yield nothing at all; to get a logic of the possible from its definition in terms of future, one must also have a *logic* of futurity. The construction, or at least the adumbration, of a calculus of tenses could not wait much longer."

In view of the quotes just made, it hardly remains doubtful that Geach's point in his notes (1949, 1955) about the historically accurate understanding of sentences expressing 'complete meanings' had initiated an important change in Prior's thinking: Prior became interested in sentences whose distinct occurrences may serve to make distinct statements, sentences containing no explicit reference to times on an absolute time scale. He decided to base the logical treatment of temporality on *tenses* − formalized as operators − instead of thinking in terms of sentences of the type 'p at time t'.

Prior thereby adopted an *internal* view on temporality, a viewpoint on time from within. The British philosopher John McTaggart Ellis McTaggart (1866-1925) is well known for having pointed out that there are two ways of viewing temporal order: time as forming **(i)** an *A-series*, and time as forming **(ii)** a *B-series*. McTaggart explains that the A-series is the order of positions in time as *past*, *present* and *future*, whereas the B-series is the order of positions in time as *earlier* or *later* (McTaggart, 1968 [a], pp. 9-10; McTaggart, 1968 [b], pp. 110-1). As Geach (1979, p. 90) and C. D. Broad (1976 [b], pp. 289-91) both separately point out, it is also useful to make a distinction between *A-characteristics* and *B-characteristics*.[15] *Being past,*

[15] Not least because the terminology thus yielded is more rigorous than

being present, being future, being yesterday and *being ten years ago* are A-characteristics: they are all characteristics (qualities or relations) that can only be ascribed to entities from a fixed point of view which is taken to be the present. By contrast, *being earlier than some event* or *being later than some event, lasting an hour* and *being ten years apart in birthdays* are B-characteristics: possessing such characteristics is not relative to any fixed now-point. The idea behind the distinctions is obvious: the A-series and A-characteristics are indexical and perspectival by nature, while the B-series and B-characteristics are absolute.

Hence we see that the way in which Prior decided to consider time for the purposes of his formalism was indeed from the vantage point of McTaggart's A-series. The alternative he chose to reject would have been an external, *sub specie aeternitatis* conception of time, i.e. the one according to McTaggart's B-series. Descriptions in terms of the B-series would have been naturally carried out in first-order logic. The option Prior chose was the modal-logical one – the view describing first-order structures from within.

From the logical point of view, it is indeed illustrative to think of the choice between the A-series conception of time

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the talk about a series, notably in the case of the A-series. As Geach (1976, p. 90) puts it: "[A]s McTaggart presents this idea [of an A-series] it is not at all clear why he speaks of a *series*, or what is supposed to be the ordering relation of the series." Obviously pastness, presentness and futurity only partition the class of temporal positions in three slices $(PAST, \{now\}, FUT)$ but *do not order* it to *any* extent. McTaggart's own wording, however, recognizes quantitative variations in pastness and futurity (see e.g. 1968 [a], p. 10), and the order McTaggart means is obtained by first separately ordering the past slice and future slice according to their *decreasing degree* of pastness *resp. increasing degree* of futurity, and then defining the desired order as the 'ordered sum' of $(PAST, <)$, $(\{now\}, \varnothing)$ and $(FUT, <)$. This maneuver of course requires that the A-characteristics of *being past* (resp. *future*) *to a greater degree* be available.

vs. the B-series conception as leading to a logical treatment in terms of tense logic *resp.* first-order logic, in such a way that in both cases we are talking about the same sort of structures (namely, first-order structures with an arbitrary number of unary predicates and one binary predicate). In the first-order logical setting we have a God's eye view on the structure: with one glimpse we are able to detect it from without and express connections between the remotest parts of this structure by first-order formulae. By contrast, the tense-logical viewpoint puts us in a particular location in this structure, and we are never allowed to ascend out of the structure to see a picture of the whole: instead, we must move within it and only make use of local resources, so to speak. The B-series view allows for the full machinery of describing time as a given totality, while the A-series view restricts our possibilities to perspectively given resources.[16] Be it noted that if one would wish to carry out time-references – perspectival time-references – under the A-series view, there would be no conceptual problems involved. At stake, would be precisely the perspectival mode of identification, the one employed in indexical (deictic) reference. This option is not available in the system of **PTL** we are discussing, but could be incorporated in it.[17]

Prior meant to consider type (1) sentences, i.e. sentences of the type "p now", and leave it to the context to take care of temporal specification. Truth was thereby temporally relativized: two non-simultaneous utterances of one and the same sentence could differ in truth-value. The idea is that an occur-

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[16] For a characterization of modal logic as providing an *internal* viewpoint on first-order structures, contrasted with the *external* point of view offered by first-order logic, see Blackburn, de Rijke & Venema (2002, esp. pp. xii-xiii).

[17] In fact, introducing nominals as in Hybrid Logic can be seen as precisely such an incorporation.

rence of a *tense operator* shifts the temporal context attained in an evaluation process so that relative to the resulting new context, a token-reflexive "φ now" is evaluated, where, by definition, "now" refers to the most recent context formed. More formally, under this construal, e.g. for a future tense sentence $F\varphi$, we have:

$$(F\varphi \text{ now}) \text{ is true}[t] \Longleftrightarrow \text{there is a time } t' \text{ later than } t$$

$$\text{such that } (\varphi \text{ now}) \text{ is true}[t'].$$

To sum up, Prior − influenced by Geach's comments on temporal sentences capable of complete meaning − formulated his tense-logic so that it does not make use of explicit time-reference relative to an absolute time scale. Instead, temporality is represented in his logic by operators, which mimic tenses of natural language sentences; more generally, operators can be seen as a logical means of dealing with temporal token-reflexivity. The semantics of tense operators is essentially connected to the fact that the sentences under consideration yield differing temporal statements (and, in general, differing truth-values) depending on when they are uttered. Prior's work can naturally be seen as modal-logical by nature, but it was by no means self-evident − as his own, above-cited considerations concerning the nature of 'temporal propositions' make clear − that temporality found its way into Prior's writings as a phenomenon studied by modal-logical means.

5.2 Operators *vs.* Quantifiers

We observed above in *Subsection* 5.1.2 that the basic dissimilarity between first-order quantifiers and modal operators is precisely in their respective ability or disability to serve as bases for repeated use of an individual once introduced in the evaluation process. An individual interpreting a quantified variable can be referred to indefinitely in the binding scope of a quantifier; but when one state is reached from another in accordance with the semantics of a modal operator, there no longer remains a mechanism for referring back to the former.

There is also an important similarity between quantifiers and modal operators. In fact, the semantics of operators shares the two-fold semantic character of quantifiers – the two features of quantifiers that Hintikka has stressed in so many words in his recent writings. One of these features is that quantifiers *range over a class of values*, and the other stems from what different combinations of logical priorities among quantifiers serve to express, namely *functional dependencies* between the values of the variables associated with these quantifiers. (See e.g. Hintikka, 1996, *Ch.* 3; Hintikka, 2002 [b]; Hintikka & Sandu, 1996.)

In the usual formulation of first-order logic, the class over which quantifiers range – i.e. the class of individuals employed for interpreting quantified variables – is the whole domain of the model employed. In a perfect analogy, we can say that a modal operator O ranges over a class of values, namely the class of those objects (states, worlds, times) that are possible choices, in a semantical game, for the player whose move is mandated by the operator O in question. The range of a modal operator in a modal-logical formula is more strictly connected to the location of that operator in the formula than is the range of a quantifier in a first-order formula. For instance, if the formula

$$\Box_1\Diamond_2\Box_3\Diamond_4\, q$$

is evaluated in a modal structure $\mathcal{M} = (D, R, \mathfrak{h})$ at d, then

$$\Box_1 \text{ ranges in the set } \{x_1 : R(d, x_1)\}$$

and

$$\Diamond_2 \text{ ranges in the set } \bigcup_{R(d,x_1)} \{x_2 : R(x_1, x_2)\}.$$

By the sorts of ranges that modal operators then have, modal operators are reminiscent of *bounded quantifiers* in a suitable formulation of first-order logic. In first-order logic, employing such bounded quantifiers would mean restricting the class of values from which the value of the quantified variable, associated with a given quantifier, can be picked out.[18]

The other feature of quantifiers is perhaps less obvious, or at least has not historically been recognized with the same vigor with which the 'ranging over' feature has been observed. As Hintikka has time and again pointed out, the real gist of quantifiers cannot be fully appreciated when quantifiers are considered in isolation from each other. The interplay of quantifiers allows expressing functional dependencies between the quantified variables. For instance, a formula

(1) $\forall x \exists y \; \mathrm{S}[x, y]$

serves to make the claim that a witness individual b for y can be chosen as a function of a value a of x in such a way that the choice satisfies $\mathrm{S}[a, b]$. This is precisely what the equivalent second-order Skolem-form of (1) says:

(2) $\exists f \forall x \; \mathrm{S}[x, f(x)]$.

[18] In bounded quantification, quantifiers are assumed to be of the form $(Qx \in S)$ and the value of x must be chosen from the subset S of the domain. A language-relative formulation of such boundedness is obtained by considering quantifiers of the form $(Qx : \varphi)$, where φ is a formula of the abstract logic considered (e.g. of **FO**) in which x occurs free. The choice a for x in the evaluation then has to satisfy $\langle \mathcal{M}, a \rangle \models \varphi(x)$. (Whether additional free variables are allowed in φ has to be settled separately.)

The functional dependencies among variables are in effect expressed in first-order logic by the relative ordering of the priority scopes of quantifiers binding these variables. In traditional first-order logic (**FO**) the priority scope relation becomes necessarily asymmetric and transitive (cf. *Sect.* 2.1 above). But obviously there are conceivable functional dependencies which cannot be expressed by such scope patterns, and in any event it would seem to be systematically advantageous to allow arbitrary combinatorially possible patterns of logical priority among the quantifiers in first-order formulae. When this idea of arbitrary patterns of priority scopes is incorporated into first-order logic, the result is the IF first-order logic of Hintikka and Sandu. For instance a Henkin quantifier formula

$$(3)\quad \begin{matrix}\forall x \exists y \\ \forall z \exists v\end{matrix} \; R[x,y,z,v]$$

which is not expressible in **FO**, can be expressed in IF first-order logic as

$$(4)\ \forall x \forall z \exists y/\{z\} \exists v/\{x,y\}\ \mathrm{R}[x,y,z,v].$$

Observe that as the only reasonable criterion for a logic being first-order is that its quantifiers range over individuals only, (3) = (4) in particular is a perfectly good instance of a first-order formula − despite the fact that it has no equivalent in **FO**. Its second-order equivalent Skolem-form is:

$$\begin{aligned}(5)\ & \exists f \exists g \forall x \forall z (\mathrm{R}[x, f(x,z), z, g(x,z,f(x,z))] \wedge \\ & \forall x \forall z \forall x' \forall z' ([x = x' \rightarrow f(x,z) = f(x',z')] \wedge \\ & [(z = z' \rightarrow g(x,z,f(x,z)) = g(x',z',f(x',z'))]).\end{aligned}$$

Now it is as true of modal operators as it is of quantifiers that an essential part of their semantics is constituted by their interplay with other logical operators. This interplay is mediated by the relative priority scopes among operators, and the way in which priority scopes appear in traditional modal logic (in $\mathbf{ML}[k]$) is equally restrictive from a general point of view as is the way in which relative patterns of logical priorities are manifested in **FO**. To enable expressing a wider range of patterns of relative priority scopes among modal operators than can be expressed by $\mathbf{ML}[k]$, we have defined the modal logics $\mathbf{IFML}[k]$ and their *uniformity interpretation.* So the lesson we have learned from the formulation of IF first-order logic − that the way in which the priority scopes of quantifiers relate to each other is a key constituent of their semantics − is generalizable as such to the case of modal logic, and the IF modal logics ($\mathbf{IFML}[k]$) we have formulated offer a technical tool for investigating the import of this semantic feature in the case of modal logic.

5.3 Critique of Tense Operators in Linguistics

We now pose the following two questions, and will endeavor to answer them in the light of the modal-logical tools now at our disposal:

> Question 1: *Are there instances of natural language tenses acting logically as operators?*
>
> Question 2: *Considering the problems of* **PTL** *in analyzing natural language temporal discourse, is there a natural way of overcoming these problems by making only small 'ideological' changes to Prior's formalism?*

Our aim in the present section is to show that the answer to Question 1 is affirmative, notwithstanding comments to the contrary by many contemporary linguists, and that the 'backwards-looking operators' interpretation of IF tense logic provides a viable tool for showing how to effect a small change in **PTL**, so that the resulting logic is able to cope with many problems that **PTL** itself cannot deal with. One such change is the possibility of formally distinguishing between *functional* and *relational* dependence in tense logic: an operator is functionally dependent on another when lying in the latter's priority scope, while the interpretation t' of an operator O' is relationally dependent on the interpretation t of a syntactically superordinate operator O, when t' is according to the semantics of the logic obtained from t along the accessibility relation associated with O.

Now whatever the properties are of tense logics in describing temporal structures – however good, however bad – this does not by itself in any way serve to make a linguistically relevant connection between tense logics and natural language temporal discourse. Both formal temporal logics and fragments of a natural language (say, English) involving temporal expressions in fact make statements concerning time. It is even probable that at least some of the things that can be said by such a fragment of English can be captured in a given temporal logic,[19] and that at least a part of the formulae of the temporal logic can be given a natural reading in English. Insofar as only the things expressed interest us – not the means of expressing them – we may of course treat in particular a given fragment of a natural language by means of a formal logic of our choice. Traditional first-order logic surely expresses most of the temporal properties that the man (or woman) in the street manages to formulate in the course of an everyday discourse – and IF first-

[19] " Captured" in the sense that for each sentence of that fragment of English there is a formula of the temporal logic in question that expresses the truth-condition of that English sentence.

order logic can do much more. If for some reason and for some purpose of expression not even the latter would suffice, the class of abstract logics will not be exhausted: the choice of formalism remains free.

Linguists, however, by definition *are* interested in the means of expressing things. Accordingly, in order for a temporal logic to be interesting to a linguist, this logic would have to have a closer – a syntactic – connection to natural language, in addition to being able to capture the truth-conditions of sentences in a fragment of natural language temporal discourse.

For the purposes of our discussion, a crucial factor about **PTL** is that its syntax contains *operators*, by means of which the temporal content of the formulae of this logic is expressed. Insofar as **PTL** is applied to natural languages, it treats *natural language tenses* as *operators*. Admittedly, **PTL** treats any other natural language temporal constructions, that it is able to treat at all, likewise by means of operators (e.g. temporal adverbs such as "always"), but theoretically the most controversial – and genetically the most original – aspect of **PTL** is that it deals with natural language tensed sentences by means of operators. So a crucial criterion for **PTL**'s usefulness in linguistics – or indeed for the usefulness of any logic utilizing the idea of tenses as operators – would be that natural language tenses in fact act as operators. For if they do not, the 'isomorphism' between natural language itself and its putative presentation in terms of **PTL** (or the like) – which is needed for a linguistic analysis – is hopelessly missing.

In fact, **PTL** and the idea of tenses as operators is not currently popular in linguistics. A good number of renowned linguists very clearly oppose construing tenses as operators. Among the fiercest critics are:[20] Mürvet Enç (1986, 1987), Norbert Hornstein (1981, 1990), Hans Kamp & Uwe Reyle (1993),

[20] The order of the names is alphabetical.

as well as Barbara Partee (1973, 1984). There, unanimity is found as regards the insufficiency of **PTL**. Other critics who view the Priorean approach to natural language tense as unmotivated or otherwise problematic include[21] Johan van Benthem (1977), David Dowty (1982), Alessandra Giorgi & Fabio Pianesi (1997), as well as Hans Kamp (1971).

I will now move on to discuss the critique of tenses as operators put forward by Norbert Hornstein. A near-complete survey and discussion of the state of the art in the linguistics of tense is impossible within the confines of this thesis, therefore our attention will focus on one particular linguist. I have chosen to discuss Hornstein, as his arguments bring out many issues that make it easier to understand why linguists oppose tense operators in the first place. We will eventually see more clearly what linguists really wish to reject, when rejecting tenses as operators.[22]

5.3.1 Hornstein's background

Hornstein's critique of tenses as operators is presented in his book *As Time Goes By* (1990). However, this is already anticipated in his survey article "The Study of Meaning in Natural Language: Three Approaches to Tense" from 1981.

The framework within which Hornstein writes is influenced in particular by Hans Reichenbach's (1891-1953) views on natural language tense and in general by Noam Chomsky's theory of language.

In his *Elements of Symbolic Logic* (1947) Reichenbach presented a short, original account of tense phenomena in natural language, and it is this account that Hornstein (1990) elab-

[21] Again in alphabetical order.

[22] I am indebted to Mr. Atle Grønn for his very useful remarks and criticisms on an earlier draft of *Sect.* 5.3.

orates upon in a very detailed fashion. Reichenbach's basic insight was to characterize tenses, as they in fact appear in natural languages, in terms of three temporal parameters S, E and R – or *point of speech, point of event* and *point of reference*. Given a tensed sentence, the parameter S represents the time of utterance of the sentence. E corresponds to the temporal position, relative to S, of the event spoken of by the sentence. (Hence E does not represent a particular, identified point of time in the same way S does.) The role of the point R is perhaps *prima facie* the least easy to grasp, but by the same token its introduction to the theory is usually held to be quite an original idea from Reichenbach. R is customarily said to mediate the relationship between S and E; it represents a further relative temporal position, one whose temporal relation is determined with respect to both S and E. Its intuitive import is to be a 'point of view' or 'vantage point' on the relation between S and E; semantically its importance and reality equals theirs.[23] A Reichenbachian tense consists of a specification, in terms of the relations *earlier than* ($<$) and *contemporaneousness* ($=$), of the relative positions of the points S and E on the one hand, and the points R and E on the other.[24] Formally, then, a Reichenbachian tense becomes a relational structure

$$(\{\mathrm{S},\mathrm{E},\mathrm{R}\}, <, =),$$

which determines for both of the pairs (S,R) and (R,E) precisely

[23] Hintikka (1982, p. 10) suggests that Reichenbach's "mythical reference time" is uncalled for if a game-theoretical approach to natural language semantics is assumed.

[24] The Reichenbachian idea can be formulated in a variety of slightly differing ways. The most obvious would be to take a Reichenbachian tense as any quintuple (A, X, B, Y, C), where (A, B, C) is any permutation of the three-element set {S,E,R}, and X and Y are both binary relations from the set $\{<, =\}$. Hence a maximum of $24 = 3{\cdot}2{\cdot}2{\cdot}2{\cdot}1$ such tenses is yielded. The formulation presented in the text above is the one Hornstein prefers. For a discussion, see Hornstein (1990), esp. pp. 87-90, 108-11.

one of the relations $<$, $>$ or $=$ as the relation whose member the pair is. (The relations $<$, $>$ and $=$ will hence be disjoint and jointly exhaustive of the set $\{(\text{S,R}), (\text{R,E})\}$.) To reiterate, it must be realized that while in a given speech situation S is fixed as the moment of speech, E and R actually indicate generalizations, much like an existential quantification of a variable does in a formula of first-order logic: their theoretical role is to mark a temporal position relative to S, not a fixed location on a time scale.

For the purposes of our discussion here, we need not dwell on the particulars of Reichenbach's views on tense, or on the details of the neo-Reichenbachian theory of natural language tenses put forward by Hornstein (1990). The reasons Hornstein gives for his dismissal of tenses as operators (*ibidem*, esp. pp. 142-6, pp. 166-8) are largely independent of his particular theory of tense. It is only when he presents his positive view of the nature of tenses – a view according to which tenses are adverbs (*ibidem*, esp. pp. 168-79) – that he actually resorts to the general neo-Reichenbachian framework within which he works.

We now move on to consider whether the grounds on which Hornstein wishes to discard tenses as operators in linguistic theorizing are as conclusive as he takes them to be. We have just seen above, in *Section* 5.1, how Prior was led to formalize tenses as operators in his logic of tenses. Genetically, it is really the Priorean view that Hornstein opposes.[25]

It is a separate question how Prior himself would have reacted to Hornstein's positive neo-Reichenbachian account of the nature of tenses – for at least considering them as adverbs is not to make temporal sentences into eternal ones in the sense of tying them once and for all to an absolute time

[25] Hornstein (1990, footnote 30, p. 223) explicitly mentions Prior's central role in theories categorizing tenses as operators.

scale, which was ultimately what Prior wanted to avoid in his formalism. On the other hand, Prior (1967, pp. 12-5) indeed criticizes Reichenbach's ideas on natural language tense by indicating that the Reichenbachian trichotomy is unfortunate, as Reichenbach's theory should, says Prior, for systematical reasons really allow any number of reference points, whence ultimately the sharp distinction between the point of speech and the point(s) of reference would become unnecessary and misleading. It is Prior's opinion that due to the impossibility of generalizing Reichenbach's account, Reichenbach's theory had in fact been more of a hindrance than a help in formulating a logic of tenses (*ibidem*, p. 13).

Whatever one's taste for the Reichenbachian theory of tense is or may be, there still exists the systematical issue of tenses as operators, and I will argue that the basic idea of tenses as operators cannot be refuted as easily as Hornstein makes us believe. Of course, at this point we have to be clear in our use of terminology, to avoid tilting at windmills. The crucial word here is "operator". And more generally, it is of crucial importance to be clear about what one's logic of time or theory of tenses is designed for. There are at least three viewpoints on logical approaches to temporal discourse, which might be dubbed (in want of better terminology) as **(i)** *logic-internal*, **(ii)** *descriptive*, and **(iii)** *linguistic*.

Understood logic-internally, a logic such as **PTL** is a potential object for mathematical study and presupposes nothing whatsoever about anything empirical. In particular, its being called a logic of tenses can be understood purely heuristically, and no discoveries about, say, features of natural language verbs bear any conceivable relevance to its status as a logical formalism.

As a matter of fact, it is obvious that Arthur Prior did not intend his tense-logic exclusively for mathematical use, but also wished to employ its expressive resources for studying real-life claims and inferences involving temporality. His objects of study included the Diodorean Master Argument and Aristotle's 'Sea Battle' passage in *De Interpretatione* 9. Such intended cases of application naturally move the focus from the logic-internal view of **PTL** to its descriptive function. In the descriptive approach *the empirical* attracts some interest. Characteristically, what is important under this approach are the things expressed and expressible – not the particular way of doing this. That is, from the solely descriptive vantage point we would be happy to be able to present the future perfect sentence

(1) *John will have met Mary*

for example, by means of Priorean tense operators as

(2) $FP(\texttt{John meets Mary})$,

while at the same time being absolutely convinced that natural language tenses – such as those represented by the auxiliary "will" and the present perfect tense of "have met" – can in no conceivable way be understood as operators. For, if capturing the truth-condition of sentence (1) – its content, what it states – is what interests us, it is *of no relevance* that the syntactic components F and P (which are operators) in the logical formula correspond to nothing with the same character in the natural language sentence (as by assumption the tenses in the sentence would not be correctly analyzed as operators). All the same, a logical formula may have the same descriptive capacity as a natural language sentence.

It is only in what I proposed to call the linguistic view of the logic of temporality where the overall correspondence of the logic employed and natural language is required. Under the linguistic viewpoint, what is at stake is finding something like the *logic of natural language*. Characteristically, then, it is not only the things expressed that interest us, but that the syntax of the logic should allow presenting the actual 'logical form' of the natural language sentences. The idea would be that if the tenses of verbs are for instance presented by means of entities of category C in the logical analysis, these very tenses of verbs really *are* themselves of category C. It would not be accepted under this viewpoint that tenses in fact were, for example, adverbs but still were treated as operators, and not as adverbs, in the formal analysis.

Hornstein is interested precisely in a linguistic theory of tenses. He is not building up a *logic* of temporal discourse at all, but the logical counterpart of his theoretical account in terms of Reichenbachian tenses would certainly be an account where there would have to prevail an 'isomorphism' between the logic used for analyzing and the natural language analyzed, i.e. his viewpoint on logic would be 'linguistic' in the sense specified above. Now even if Hornstein's own positive account of tenses was right – in fact, as an empirical linguistic theory, his theory of tenses very elegantly treats the data he has chosen to consider – the systematical reasons he offers for rejecting 'operator' as the correct categorization of natural language tenses are not, so I will argue, satisfactory. Interestingly from the viewpoint of the present thesis, it is precisely the deepened understanding of the nature of tense operators – made easier to attain by the availability of the different possible interpretations of IF modal and IF tense logics – that provides us with the requisite tools for pinpointing certain inaccuracies in Hornstein's argument against tenses as operators.

5.3.2 Hornstein's argument and a counter-argument in brief

Hornstein wants to show that natural language tenses are not operators. His tactics are as follows. He bases his argument on assuming that tense operators act like quantifiers, more specifically he takes quantifiers as 'canonical instances' of operators (see e.g. (1990) pp. 144-6, 166; (1981) pp. 124, 127, and esp. pp. 139-40, 145). Secondly, he detects a property of configurations of quantifiers that natural language tenses do not exemplify. He then concludes that tenses cannot be quantifiers, from which it follows by his basic premise that tenses cannot be operators either.

He gives essentially the following argument:

[1] Operators and quantifiers have all relevant properties in common.

[2] There is a structural property P that tenses do not possess but quantifiers do.

$\therefore$ Tenses are not operators.

Premise [1] is merely definitional to Hornstein: he assimilates tense operators to temporal quantifiers, thereby looking upon them explicitly from the perspective of first-order logic. Strictly speaking, this view of tense operators is different from that found in **PTL**, since in the latter, tense operators, semantically, are without a doubt quantifiers, but expressly lack syntactically manifest variables. Hornstein wishes to make tense operators 'semantically transparent', and when discussing Hornstein's argument below, I shall also treat them as quantifiers.

The structural property P that Hornstein thinks of in premise [2] is the great freedom in which quantifiers can affect the interpretation of other quantifiers. This can happen, on the one hand, via the relations of logical priority among quantifiers. For instance, in order for the sentence

(3) $\forall x \exists y \, \mathrm{S}[x, y]$

to be true, it must be possible to choose the value of y as a function of the value of x; in this sense the universal quantifier $\forall x$ affects the interpretation of the existential quantifier $\exists y$. On the other hand, a quantifier can have a more modest effect on the interpretation of another quantifier. For example, the formula

(4) $\exists x(x < t \wedge \exists y(y < x \wedge Q(y)))$

is satisfied in a model only if the value of the quantified variable y can be chosen to be earlier than the value of the quantified variable x. The fact that the bounding clause of the 'bounded quantifier'

$$\exists y(y < x \wedge \ldots)$$

involves the free variable x, results in the interpretation of this quantifier being affected by the superordinate quantifier $\exists x$. No functional dependence of y on x is at stake, but in order for the formula to be satisfied, the value of y must satisfy a condition involving a value of x, and in this, *completely different* sense, the value of y is dependent on that of x.

Hornstein takes operators to be bounded quantifiers. For instance, the past tense operator $PAST_x$ is in effect the relativized quantifier

$$\exists x(x < x_0 \wedge \ldots).$$

Generally, tense operators always involve a relational condition in terms of a temporal relation (such as *earlier than*). Logically, the fundamental factor about any operator is its priority scope, and the real import of such priority scopes comes to the fore as a structural property of the configurations of operators. Precisely when an operator is logically prior to another, the latter receives its semantical interpretation as a function of the interpretation of the former. This corresponds to the kind of dependence manifested among quantifiers in sentence (3) above. The dependence of the type exemplified by the above formula (4) is not logical at all. In this formula, all that is at stake is a rule regulating a 'context-shift': the value of the variable y must be attained along the relation $>$ from the value of x. This is *not* a logical requirement but a *relational* one, expressed in terms of a relational atomic formula "$x > y$".

I will argue that Hornstein fails to make a clear distinction between relevant *temporal* and *logical* relations involved in configurations of tense operators. When evaluating tense-logical formulae – or natural language sentences involving tenses – we choose times (or temporal intervals) relative to other times (or intervals). But for this we do not really need a logic at all. The real power of logic comes into play when such choices – be they relative to times as in tense logic, relative to possible worlds as in modal logic, or not relative to any context as in first-order logic – are functionally dependent on other choices, made earlier in the evaluation. Now Hornstein correctly criticizes **PTL** for not being able to deal with a sufficiently large variety of patterns of context-shifts in terms of temporal relations, but accuses logic – the practice of presenting tenses logically as operators – for what in fact is the fault of the conceptually distinct matter of how a particular logic, i.e. **PTL** or a version of it formulated in terms of temporal quantifiers, presents 'context-shift' in its formalism. I will show that **BLO** interpretation of IF tense logic (**IFTL**) serves to illustrate that these two features can well be kept apart.

These considerations attempt, then, to show that the particular reasons Hornstein offers for dismissing the idea of tenses as operators are not conclusive. I wish to go even further, however. In *Subsection* 5.3.4 below I will go on to argue that there are examples of natural language sentences that serve to express functional dependencies between interpretations of natural language tenses. Because operators are entities among which, by definition, the relation of logical priority only can hold, these natural language examples require operators of some shape or form for their analysis. Of course, this is not to say that operators are a universal solution to all linguistic problems of tense and temporality, but it does aim at showing that operators cannot be totally dispensed with in linguistics. And already this is a strong rebuttal of Hornstein's position, as well as the position of any other linguist claiming to be able to do without tense operators altogether.

I will now proceed to discuss Hornstein's line of thought in more detail, beginning with premise [2] sketched above.

5.3.3 Quantifiers *vs.* tenses

Before we commence, a word of warning is in order. Throughout the following discussion, the tense-logical formalism employed is linguistically simplistic in treating temporal events as temporally indivisible wholes and in its evaluation of tensed clauses as always being relative to individual time-points. A linguistically more realistic approach would recognize intervals and their subintervals in addition to time-points, events spoken of could have a duration, and clauses could be evaluated relative to intervals. In refraining from the use of intervals in our formalism, we make it impossible in particular to deal with aspectual features of verbs. However, the critique Hornstein aims at operators is explicitly intended to apply already at the level of temporal events that are not further temporally analyzed; in

As Time Goes By Hornstein wishes to leave out any aspectual considerations.[26]

Premise [2] actually involves *three* distinct conceptual components:

- an empirical observation: the interpretive interactions between natural language tenses are severely restricted,
- the assumption that in the tense-logical approach these interpretive dependencies are presented by the priority scopes of quantifiers,
- the observation that there is great freedom in the forming of possible patterns of priority scopes of quantifiers.

As to the first component, the following rules (I) and (II) regulate interpretation of syntactically embedded English tenses.

(I) Embedded tenses can always be interpreted indexically, viz. as anchored to the moment of speech, no matter how deep in a complex clause they are to be found.

As examples of this phenomenon, Hornstein (1990, p. 166) gives the sentences (5) and (6).

(5) *John heard that Mary said that Bill denied that Fred* is *in New York.*

[26] Hornstein (1990, p. 9) writes: "Tense and aspect are no doubt intimately related, and interact quite extensively. However, I will assume that they form separate modules rather than a single inclusive system. Tenses ... locate the events that sentences represent in time. This is to be contrasted with the internal 'temporal contour' of the event, which is specified within the aspectual system. Assuming that this widely respected distinction between tense and aspect is tenable, I will concentrate on the former."

(6) *John said that Bill saw the man that* is *at the next table.*

According to the most natural reading of (5), Fred is in New York at the moment of utterance of (5); the two intervening clauses after the past tense matrix clause, both involving a past tense verb, make no difference to the interpretation of "is" as anchored to the moment of speech. Similarly, in (6) a past tense matrix clause is followed by a past tense clause and a noun phrase; nonetheless according to (6) the man spoken of is *currently*, when (6) is uttered, at a nearby table.

(II) The possibilities for interpreting a tense are *polarized*: if a tense is not interpreted indexically, it must be interpreted relative to the interpretation of the tense that immediately precedes it syntactically. More exactly, this interpretive restriction manifests itself as follows:[27]

(i) A tense in an embedded clause can only be temporally dependent on the tense of the clause under which it is immediately embedded.

(ii) A tense within a relative clause, regardless of how deeply embedded the relative clause is, is never temporally dependent on any other tense. In other words, it is always temporally interpreted relative to the moment of speech.

For an example of (i), consider the interpretation of "was" in the sentence

(7) *Fred knew that John said that Mary was pregnant.*

[27] Cases (i) and (ii) are stated here as given by Hornstein (1990, p. 166). For more details about case (i), see *ibidem* pp. 142-6; for case (ii), see pp. 138-42.

As an instance of a sequence-of-tense structure, the sentence suggests Mary being pregnant at the moment of John's saying so: "was" is read as a morphological variant of "is" and interpreted as anchored to the event time with which "said" is associated. In the terminology of Enç (1987) such a reading of "was" is its *simultaneous reading*, since the temporal interpretation of "was" is identical to the time at which John's saying takes place. In particular, it is not possible to interpret "is" as being anchored to the time of Fred's knowing.[28] Hence (7) would have the paraphrase

(8) *Fred knew that John said "Mary is pregnant".*

Sentence (7) also has what Enç (1987) calls its *shifted interpretation* – where "was" is evaluated as a past tense with respect to John's statement – which is paraphrasable as

(9) *Fred knew that John said "Mary was pregnant".*

All the same, "was" is interpreted as being interpretively (temporally) dependent on the time that the evaluation process gives when interpreting the tense of the verb form "said" in the clause under which the clause "Mary was pregnant" is immediately embedded.[29]

Examples in which rule (ii) is operative, would be (10) and (11), given by Hornstein (1990, p. 138).

[28] Of course, if "said" itself is given a simultaneous reading (which is a very natural possibility), then the time to which "is" gets anchored *is* the time of Fred's past knowing, but from the point of view of the theory of tense interactions this is accidental.

[29] The usage by Enç (1987, see esp. p. 635) of the term "shifted reading" must not be confused with Hornstein's way of employing the word "shifted": Hornstein occasionally uses the term "shifted interpretation" of the sequence-of-tense reading – that is, of the phenomenon for which Enç uses the term "simultaneous reading" (see 1990, p. 121). When I have to make a terminological choice in the text, I stay with Enç's terminology.

(10) *We spoke to the man who was crying.*

(11) *John insulted the man who is walking toward us.*

In both cases the tense of the relative clause must be interpreted with respect to the moment of speech. No sequence-of-tense phenomenon is at stake, and indeed the very words indicating past tense *resp.* present tense (aspectual) progressive – "was crying" and "is walking" (instead of any morphological variants of these) – are to be evaluated with respect to the utterance time. As a consequence, (10) indicates that in the past we spoke and that a certain man was crying, but as to the temporal relations of these two past events (10) does not assert anything specific and leaves the relation undetermined (given our convention of treating events here as temporally indivisible).[30] In turn, uttering (11) reports John having insulted a man before the moment of utterance, and says that the walking of the man in question is contemporaneous with the moment of utterance. Hence, even though a sequence-of-tense structure is not at stake here either, it so happens that the relative temporal order of the events spoken of can be deduced: "insulted" evaluated now makes insulting past, while "is walking" interpreted now has walking taking place presently.

The first conceptual component of premise [2] in Hornstein's argument consists, then, of an empirical observation about semantical tense interactions in natural language (or, at least, in a natural language such as English). It sets limits to what an accurate description of, or an account for, natural language tense phenomena should accomplish. In particular, it says what

[30] Here the inability to recognize temporal relations between the two past events would be removed if the (unrealistic) assumption that events are point-like was given up. Namely, the progressive aspect of "crying" in (10) shows that the interval of our speaking to the man was included in the larger interval of the man's crying.

tenses are *not* capable of. Arbitrary interpretive dependencies between tenses in complex sentences cannot occur, but all interpretation of tenses is either indexical or else complies with the aforementioned rule (II.i) regulating the temporal interpretation of tensed embedded clauses.

We may notice that the third conceptual component of premise [2] is relatively obvious: a great variety of logical priorities among quantifiers can be represented by well-formed formulae of **FO**. (An even greater, in fact, full, variety of such priority scope patterns could of course be presented by means of IF first-order logic, but for Hornstein's purposes **FO** is already quite sufficient.)

Let us move on to consider the second conceptual component of premise [2], which I stated above to be a theoretical assumption concerning how to represent tense interaction logically: by means of priority scopes of quantifiers.

5.3.3.1. Formal counterpart to interpretive dependency

Basically, construing tenses as operators is an attempted move from *pretheoretical* to *theoretical*. Among the pretheoretical ideas to which one thereby would like to find a logical counterpart is the idea of interpretive dependencies between tenses. In tense logics such as **PTL** these interpretive dependencies are simply presented by priority scope relations between temporal quantifiers (tense operators) – to the extent they can be presented at all. Hornstein (1990) shows that in approaches resorting to temporal quantifiers, many *ungrammatical* English sentences receive a logical rendering. And since there is no systematical way of ruling out such logical forms of ungrammatical sentences from the overall stock of logical forms of tensed English sentences allowed by patterns of priority scopes among temporal quantifiers, the very construal of tenses as quantifiers becomes questionable. Or so Hornstein wishes to argue.

Before commenting further on Hornstein's argument, I will show that admitting the fact that interpretive dependencies between tenses are not a matter of relative priority scopes of temporal quantifiers, does not necessitate giving up the idea of tenses as operators. Interpretive dependencies are a matter of *temporal relations* between times interpreting tenses, instead of being a matter of logical priority between operators.

In *As Time Goes By*, Hornstein asks us (p. 144) to conceive of three temporal quantifiers $\mathtt{P}_x$, $\mathtt{Pres}_x$ and $\mathtt{F}_x$. These quantifiers are bounded quantifiers; if for example $\mathtt{P}_x$ is written so that it becomes semantically transparent, we see that it is the bounded quantifier

$$(\exists x : x < x_i).$$

Hornstein points out that some natural language tensed sentences are easily presented by employing such quantifiers. Cases in point are sentences such as (12) and (13).

(12) *John said that Mary was pregnant.*

(13) *John said that Mary is pregnant.*

The verb form "was" in (12) is given a sequence-of-tense reading, and in (13) "is" appears indexically. The truth-conditions of these sentences can be captured by (14) *resp.* (15):

(14) $\mathtt{P}_x[\mathtt{John\ says\ at}\ x\ [\ \mathtt{Pres}_y[\mathtt{Mary\ pregnant\ at}\ y]]]$

(15) $\mathtt{Pres}_y[\ \mathtt{P}_x[\mathtt{John\ says\ at}\ x\ [\mathtt{Mary\ pregnant\ at}\ y]]].$

Here, then, the interpretive (temporal) dependencies between tenses are indeed reflected in the respective scope orderings of the quantifiers $\mathtt{P}_x$ and $\mathtt{Pres}_y$. But it should be noted that the interpretive dependencies at stake here are essentially connected to *temporal relations* between the times involved in the evaluation, while it is more of a contingency from the viewpoint of the phenomenon in question that here the way of presenting the interpretive dependencies matches the way of expressing *logical* dependencies between quantifiers. To appreciate this fact, consider the following formula expressed in a version of IF tense logic interpreted under its **BLO** semantics, henceforth referred to as **IFTL (BLO)**:

(16) $\mathrm{P}^1_x[\texttt{John says at } x\ [\mathtt{Pres}^2_y/_{\mathbf{BLO}}\{1\}\ \texttt{Mary}$

$\texttt{pregnant at } y]]]$.

The semantics of the operator $\mathtt{Pres}^2_y/_{\mathbf{BLO}}\{1\}$ in (16) is the following: if in a play of the correlated semantical game the history (a_0, a_1) is already formed ($a_0 = t_0$ being the point of evaluation, and a_1 the interpretation of x), the variable y is given the value

$$a_2 := (a_0, a_1)[max(\{0, 1\}\backslash\{1\})] = (a_0, a_1)[0] = a_0.$$

Hence we see that (16) formalizes precisely the same statement as (15), but unlike the latter, in (16) the relative order of the priority scopes of the quantifiers $\mathtt{P}_x$ and $\mathtt{Pres}_y$ is exactly the same as in formula (14). So it is not, after all, the relative priority scope order that is essential to presenting interpretive dependencies. What is important is how times in evaluation are chosen in relation to other times. Typical formalisms of temporal logic are unable to separate the proper effects of bounded quantifiers (functional dependencies expressed via priority scopes) from mechanisms for context-shift. But if they *can* be distinguished within a formalism construing tenses as bounded quantifiers, the possible problems that some particular operator-based system may have in presenting temporal context-shifts are not sufficient grounds for claiming that tenses are not operators.

On the subject of interpretative relations among operators, Hornstein (1990, p. 143) writes:

> "[I]nterpretive dependency is reflected in scope relations [of operators, e.g. of quantifiers]. Thus, if the interpretation of one operator depends on the interpretation of a second operator, the first must be in the scope of the second. Similarly, if the interpretation of a given operator is independent of the interpretation of another operator, the first must be outside the scope of this second. In sum: Interpretive facts have syntactic consequences of a rather particular sort within scope theory."

This quote is an expression of Hornstein's observation that on operator-based approaches to tense (or to anything else, for that matter), interpretive dependencies between operators are presented by the relative scope order between these operators. As witnessed above by formalization (16), it is, however, perfectly possible to present the interpretive independency of the present tense of "is" from the past tense in "said" in sentence (13) so that while the formalization renders tenses as quantifiers, the analysis of the interpretive independence is definitely *not* a matter of scope order. The relevant notion of *interpretive dependency*, then, is not essentially a matter of scope relations, in other words, not a matter of *functional* dependencies.

A temporal quantifier is by its semantics a relativized quantifier; for instance the past tense quantifier $\mathtt{P}_x$ is essentially of the form $(\exists x : x < y)$. Put very simply, the issue of interpretive dependence concerns the value of the variable y appearing in this quantifier. The question is: how is the time (given as a value of y) determined, relative to which the value of x is to be chosen. Whatever the particular answer is, or can be, depends on the particular logic employed, but in every case the tense formalized by $\mathsf{P}_x = (\exists x : x < y)$ is 'interpretively dependent' on that very tense which provides a value for the free variable y in $(\exists x : x < y)$. This is what interpretive dependency *means.* As we saw in the above example (16), it is possible to formulate a logic, namely **IFTL (BLO)**, where a quantified variable can be related to a time which is *not* introduced by the immediately superordinate quantifier, the latter being, by contrast, the general mechanism of **PTL** and the temporal logic considered by Hornstein.

What is essential to interpretive dependency is not relative priority scopes, but what might be termed context-shift. From the viewpoint of these interpretive dependencies, all that is at stake in the evaluation of complex sentences involving embedded tenses (enumerated by the numbers $1, \ldots, n$), is the construction of a sequence

$$(t_0, t_1, \ldots, t_n),$$

where for each $t_i \neq t_0$ there is a relation $\rho_i \in \{<, >, =\}$ and a number $j_i < i$ such that the condition

$$(t_{j_i}, t_i) \in \rho_i$$

is satisfied. As an empirical fact $-$ by the rules (I) and (II) mentioned in the beginning of *Subsection* 5.3.3 $-$ the number j_i must always be either 0 or else $i-1$. Of course the semantics of tenses is not exhausted in a construction of such a sequence, but the description of their interpretive dependencies in fact is.

It is of some interest to notice that in the notation Hornstein uses for discussing the operator-based approach in his survey article from the year 1981, he makes free use of first-order quantifiers ($\exists x, \forall x$), and introduces expressions $P^{(x)}$ and $F^{(x)}$ with a free variable, x. He calls these expressions "tense operators". Then, for instance, a formula $\exists x P^{(x)} q$ says the same as the usual **PTL** formula Pq or the first-order formula $\exists x(x < t_0 \wedge Q(x))$. As we have seen above, in *As Time Goes By* Hornstein assimilates tense operators into quantifiers, i.e. there he treats operators for what they are worth from the viewpoint of first-order logic. By contrast, the 'operators' $P^{(x)}$ and $F^{(x)}$ of the 1981 paper are really binary atomic formulae

$$x < x_0 \text{ and } x > x_0,$$

i.e. they are the respective bounding clauses characteristic of the tenses in question. Calling binary predicates "operators" is certainly not in the spirit of what operators are in logic. In fact, it could not be more clearly illustrated that the tense operators of Hornstein (1981) are really vehicles of context-shift − designed for expressing interpretive dependencies − and have absolutely nothing to do with the logical notion of operator.

5.3.3.2. The overrepresentativeness of tense logic

Hornstein's attack on tenses as quantifier-like expressions is based on the fact that by means of quantifiers it is possible to represent indefinitely many patterns of interpretive dependencies between tenses that correspond to ungrammatical natural language sentences. He wishes to ask that *since* real-life tenses in English, for example, *cannot* do much of what quantifiers and operators typically *can* do, why should they ever be regarded as operators? That is, he points out that quantifiers give an overrepresentative tool for representing natural language tenses.

Let us consider two examples.

Example 1 Consider the pair of schemata:

(17) $\texttt{Pres}_y[\ \texttt{P}_x[\ldots\texttt{at}\ x\ldots[\ldots\texttt{at}\ y\ldots[\texttt{F}_z[\ldots\texttt{at}\ z\ldots]]]]]$

(18) $\texttt{P}_x[\ldots\texttt{at}\ x\ldots[\ \texttt{Pres}_y[\ldots\texttt{at}\ y\ldots[\texttt{F}_z[\ldots\texttt{at}\ z\ldots]]]]]$.

In a tense-logical approach based on **PTL**, these would be taken as providing logical forms to the following sentences, respectively:

(19) **John said that Harry believes that Fred would be here.*

(20) *John said that Harry believed that Fred would be here.*

Schema (17) is a perfectly well-formed formula in the language of temporal quantifiers, but the English sentence (19) whose logical form it gives is *ungrammatical*.[31] The explication of the ungrammaticality is that the morphological variant "would" of "will" indicates the presence of a sequence-of-tense construction. Such a morphological variation can only occur under a past tense clause or in a string of similarly varied morphological forms with a simultaneous reading. Still, "would" is embedded under an indexically interpreted present-tense form of "believe". (For more details about sentence (19)'s unacceptability, see Hornstein, 1990, pp. 137-8.)

Sentence (20), by contrast, is grammatical. The only difference between (19) and (20) is that in the latter the present tense of "believe"[32] is interpretively dependent on the past tense of

[31] Hornstein (1990, p. 144) uses the schema when arguing against the 'operatorhood' of tenses.

[32] I.e. the simultaneous reading of the past tense of "believe".

"say", whereas in the former this interpretive dependency does *not* hold, and the present tense of "believe" is interpreted as anchored to the moment of speech (indexically).

If scope order was the correct logical counterpart of the pretheoretically understood interpretive dependency, then, as regards the tense-logical approach, we could say that (19) and (20) differ only in the scope order of their temporal quantifiers, the relevant scope orders being respectively

$$\mathtt{Pres}_y < \mathtt{P}_x < \mathtt{F}_z \text{ and } \mathtt{P}_x < \mathtt{Pres}_y < \mathtt{F}_z,$$

as indicated by schemata (17) and (18). This is the way in which Hornstein interprets the tense-logical approach, and accordingly he suggests that the tenses-as-operators view is not credible, because supposing that tenses really were operators, it would be odd if simply putting such operators in a different order produced an ungrammatical sentence out of a grammatical one. And this is precisely what happens with sentences (19) and (20).

I argued in the previous subsection that instead of the priority scope order among temporal quantifiers, the real issue in connection with interpretive dependence is that times in the evaluation of tensed sentences are chosen *in relation* to times already chosen in that evaluation. And for a given temporal quantifier, the *scope pattern* of which it partakes does *not* have to determine the 'context' relative to which the quantifier is interpreted. In tense logics like **PTL** the scope pattern in fact precisely determines the context, but in my **IFTL (BLO)** this is not the case. The difference between sentences (19) and (20) can, from the more general perspective provided by **IFTL (BLO)**, be described by writing down, to begin with, the relevant *relational* dependencies in an arbitrary evaluation (play of a semantical game):

- *now* $> t_1$, *now* $= t_2$, $t_2 < t_3$ [evaluating (19)]
- *now* $> t_1$, $t_1 = t_2$, $t_2 < t_3$ [evaluating (20)]

Accordingly, the logical form of (19) becomes expressible using a variant of **IFTL (BLO)** as

(21) $\mathtt{P}^1_x[\ldots \mathtt{at}\ x \ldots [\ \mathtt{Pres}^2_y/_{\mathbf{BLO}}\{1\}[\ldots \mathtt{at}\ y \ldots$
$[\ \mathtt{F}^3_z[\ldots \mathtt{at}\ z \ldots]]]]]$.

The fact that interpretive dependence corresponds to a relational context-shift rather than scope order of temporal quantifiers, does not, however, change the basic source of Hornstein's complaint, namely, the fact that the formal machinery employed can produce logical forms of ungrammatical sentences. Here the reason for such overrepresentativeness of the formal machinery is as follows. Given that Qx is a temporal quantifier and that the quantifiers logically prior to Qx are $Q_1, \ldots, Q_{n-1}$, there are n different ways to determine the time relative to which the value of x is chosen: it can be either the 'time of speech' t_0, or the time given as an interpretation of any of the quantifiers Q_i $(1 \leq i \leq n-1)$. All other choices except t_0 and the interpretation of Q_{n-1} always contradict the empirical rules (i) and (ii) about tense interactions (presented above in the beginning of *Subsect.* 5.3.3), and as the ungrammatical sentence (19) directly above demonstrates, it is even possible to adhere to these rules $-$ always choosing relatively either the moment of speech or the immediately previous choice $-$ and still end up formalizing an ungrammatical sentence.

Example 2 Consider the following pair of sentences:

(22) *John said that Frank would believe that Sam* will *be in London.*

(23) *John said that Frank would believe that Sam* would *be in London.*

Their respective logical forms are provided by the schemata

(24) $\mathtt{P}^1_x[\ldots\mathtt{at}\ x\ldots[\ \mathtt{F}^2_y\ \ldots\mathtt{at}\ y\ldots[\ \mathrm{F}^3_z\ /_{\mathbf{BLO}}\{1{,}2\}[\ldots \mathtt{at}\ z\ldots]]]]$,

(25) $\mathtt{P}_x[\ldots\mathtt{at}\ x\ldots[\ \mathtt{F}_y\ldots\mathtt{at}\ y\ldots[\ \mathtt{F}_z[\ldots\mathtt{at}\ z\ldots]]]]$.

The difference in meaning between (22) and (23) is due to the temporal interpretation of the respective words "will" and "would" indicating the future tense within an embedded clause. In (22) the indexical future tense of "will be" is anchored to the moment of speech.

By contrast, in (23) the future tense of "would be" has simultaneous reading (proper to sequence-of-tense structures). In an evaluation of sentence (23), to what time (t_3) will the interpretation of this second occurrence of "would" in (23) be anchored? One might perhaps conceive of two temporal positions as such anchors:

[i] the time t_2 by which the (simultaneous) reading of "would believe" is interpreted, which must satisfy:

$$\text{now} > t_1 < t_2$$

(t_1 being the interpretation of "said");

[ii] the time (t_1) introduced by the interpretation of "said".

Option [i] gives the natural – and in particular, a grammatical – reading of sentence (23). Option [ii], though by itself intelligible from a systematical viewpoint, produces an *ungrammatical* reading of (23). Namely, this reading would not comply with the rule stating that a tense in an embedded clause can only be interpretively (temporally) dependent on the tense of the clause under which it is *immediately* embedded: the tense of "would be" is immediately embedded under the tense of "would believe" and is only mediately subordinate to the tense of "said" of the matrix clause.

Once again, there are no problems in providing a formalization to the unwelcome, ungrammatical reading of (23); the **IFTL (BLO)** formula

(26) $\mathtt{P}_x^1[\ldots \mathtt{at}\ x \ldots [\ \mathtt{F}_y^2 \ldots \mathtt{at}\ y \ldots [\ \mathrm{F}_z^3\ /_{\mathbf{BLO}}\{2\}[\ldots$
$\mathtt{at}\ z \ldots]]]]$

does the job.

Now, both examples 1 and 2 above attest to the fact that our logical machinery – be it a traditional tense logic like **PTL** with its presentation of interpretive dependencies in terms of scope relations, or **IFTL (BLO)** that deals with interpretive dependency on the model 'times chosen in relation to other times' – is overrepresentative of linguistic reality. Either approach is able to produce well-formed formulae that are logical forms of ungrammatical sentences. Then, Hornstein insists, if tenses were operators, and tense interactions accordingly were essentially interactions between operators, the property of grammaticality of sentences would be left unaccounted for: why would some patterns of operators yield logical forms of grammatical sentences, and others not?

Should, then, the mechanism itself that provides the logical forms of relevant English tensed sentences really also offer a criterion for the grammaticality of such sentences? Could not the tense operators of, say, **IFTL** provide a correct means of producing the logical forms – it being a condition of some other level that bans certain 'forms' as ungrammatical? A serious answer to this question may vary depending on the vantage point adopted. For instance, such theoretical overrepresentation of which Hornstein accuses the operator-based approaches to tense is certainly less of a problem if only a description of the logical forms of natural language tensed sentences is called for, but it would be more problematic if the apparatus used for the description should possess some 'psychological reality' in the sense that it should make it comprehensible how a child learns to use tensed language. (In fact the latter is explicitly among Hornstein's *desiderata*.[33] However, this is an area we cannot touch upon in the present thesis.)

The simplest conceivable defence for tenses as operators is to show that we clearly cannot dispense with operators when analyzing tenses, because *there are tensed sentences whose logical forms cannot be presented without operators*. This is in fact exactly what I am going to do. I will show in *Subsection* 5.3.4 that certain English sentences express functional dependencies between interpretations of tenses, which means precisely that the relation of *logical priority* is at stake. And the terms of that relation are by definition *operators*.

[33] See esp. pp. 1-7, 82-7, 188-96 in Hornstein (1990). Hornstein points out (p. 2) that a linguistic theory is to be evaluated not only in terms of how many sentences it handles correctly, but also in terms of how well it addresses the "logical problem of language acquisition": how is it even possible that children come to master their native languages, given the many-faceted poverty of the linguistic stimulus? (Linguistic input is finite and is in terms of sentential utterances instead of well-formed sentences, and it seems impossible that the acquisition of grammatical principles producing complex sentences with properties diverging from those of simple ones would be data-driven.)

5.3.4 A criterion for the operatorhood of tenses

We have seen above how Hornstein attempts to show in *As Time Goes By* that tenses are not operators, and that in essence the correct way of accounting for interpretive dependencies between tenses is not by resorting to their relative scope order.

Hornstein (1990, footnote 26, p. 222) recognizes the crucial role of Prior (1967) in all scope theories of tense. And in fact, to the extent that it can present interpretive dependencies between tenses at all, **PTL** does so precisely in terms of the scope order between tense operators. A divergent conception on the nature of tense is the view treating tenses as *pronominal*, or as *referential expressions*. Such a view is advocated notably by Partee (1973) and Enç (1986, 1987). In Hornstein's opinion the pronominal theory is less obviously inadequate than the scope theory. Still, he points out problems that this view on tense encounters given that certain of his basic observations are correct. (Cf. Hornstein, 1990, pp. 186-7, footnote 32, p. 223.) Hornstein's own understanding of tenses is that tenses are *adverbs*. He characterizes adverbs by saying (*ibidem*, pp. 188-9): "Adverbs 'modify' and 'specify'. They do not bind. They do not have scopes. Their domains of interpretive efficacy are not scope domains, nor are they binding domains. Their domains are much more restricted. The interpretive reach of a tense element is the domain it governs." Under both the pronominal view and the adverbial view, tenses are expressly much more local and isolated as to their semantic effects than tenses under a 'scope theoretic' view.

The crucial test for deciding whether or not tenses can, after all, behave like operators, is to check if functional dependencies may exist between interpretations of tenses. If they do, then their effects are *not* ultimately only local, and can only be accounted for by means of the notion of priority scope. If not, then in the final analysis there is no reason to assume that the tenses were operators.

Hornstein (1990) lists the following as potential tenses:[34]

> *present* (sing), *past* (sang), *future* (will sing), *present perfect* (has sung), *past perfect* (had sung), *future perfect* (will have sung), *distant future* (will be going to sing), *future in past* (was [going] to sing), *proximate future* (is about to sing).

For simplicity, verbs whose semantics is in terms of *momentary events* will henceforth be referred to as verbs of category C_1, in short, C_1 verbs, while verbs expressing *ongoing activities* or *states* will be referred to as verbs of category C_2, or C_2 verbs.[35] Hence for instance the verbs *to finish, to open* and *to pay* are C_1 verbs: accordingly

> *finishing a job, opening a window, paying the rent*

all are momentary events. By contrast, *to stay, to walk* and *to build* are C_2 verbs:

> *staying in Paris, walking home* and *building a house*

[34] In his neo-Reichenbachian theory, Hornstein is able to present systematical reasons why the list is precisely this, but we cannot discuss the reasons here.

[35] A more fine-grained analysis of the verbs falling under the categories C_1 and C_2 could of course be accomplished. Such an analysis could e.g. be based on the tripartite division among what Comrie (1978, p. 13) calls "situations": states, events and processes. States are static, i.e. continue as before unless changed; events and processes are dynamic, i.e. require a continuous input of energy in order to go on. Events are dynamic situations viewed as complete wholes (perfectively), whereas processes are viewed in progress, from within (imperfectively).

are all enduring events. A more exact criterion for a verb's being of category C_2 is the *subinterval property*: if I is an interval relative to which an event expressed by a C_2 verb takes place, this event takes place also relative to all non-empty subintervals of I. A concrete example goes as follows: assume that the sentence "Marie stays in Paris" is true relative to an interval I. If I is the interval from the 1st of June 2002 to the 15th of August 2004, then "Marie stays in Paris" must be true in particular on the 10th of July 2003.

Now tenses of C_1 verbs are, logically speaking, interpreted by means of *existential quantifiers*, while the interpretation of tenses of C_2 verbs is in general *ambiguous between the universal and the existential reading* (cf. e.g. Hintikka, 1982, pp. 9-10). Relevant tenses are any of the nine listed above. Consider the following examples:

(27) *Mary opened the window.*

(28) *Mary will open the window.*

(29) *Marie stayed in Paris (last summer).*

(30) *Marie will stay in Paris (next summer).*

According to (27), there is a past time such that at that time, Mary opens the window; similarly, (28) says that at some future time Mary opens the window. The verb *to open* is of category C_1, and the respective tenses are interpreted existentially. Also, they could *not* be read universally. For example, (27) cannot possibly be interpreted as saying that relative to all moments in an extended past interval, Mary opens the window.

By contrast, (29) can be read as saying that during the whole of last summer Marie stayed in Paris (universal reading), but also as saying that there was a time last summer when Marie stayed in Paris (existential reading). The case of (30) is analogous. Game-theoretically the ambiguity between these interpretations is characterized as a difference in the player who makes a choice to interpret the tense of a C_2 verb. For instance if (30) is interpreted universally, it is *Abélard* (*Falsifier*) who chooses a future time (among the times that make up the next year) to interpret the past tense of "stayed"; in the case of the existential interpretation the relevant choice is made by *Héloïse* (*Verifier*).

All nine tenses listed above admit of an 'existential reading': as tenses of C_1 verbs, this is, moreover, the only option, but even as tenses of C_2 verbs such an existential reading is possible. On the face of it, it might be tempting to think that there *are no* 'universal tenses' in natural language, i.e. that there are no tense expressions – grammaticalized expressions imprinted in the verb forms – that would semantically correspond to universal quantifications in the way that all of the aforementioned types of natural language tense can correspond to existential quantifications. This appearance, however, is mistaken, as shown by the existence of the universal reading of tenses of C_2 verbs.

Admittedly the universal reading of a C_2 verb tense tends to call for a specification of the period to which the universality pertains – an interval or frame within which the event expressed takes place. If we overhear a conversation where a speaker utters: "Marie stayed in Paris", we may reasonably assume that the overall context specifies such a period, and that the specification is known to both the speaker and the addressee. Frame specification arguably is not, however, essential to the possibility of universal readings of C_2 verb tenses. For instance, in *Genesis* 1:2[36] we read: "And the earth was without form, and void." The verb form "was" is an instance of a state verb, whose simple past tense is to be construed universally.

[36] *The Holy Bible, King James Version.*

And here a further specification for a time period is obviously not presupposed.

I will now move on to present my argument in favor of the need for operators in the analysis of natural language tensed sentences. The argument can be presented briefly as follows. There are in fact at least two sources (indicated in (i) and (ii) below) that can be used for generating sentences that express functional dependencies between interpretations of tenses.

(i) Exclusively employing verbs with existentially read tenses and negation. (We would get the same effect by exclusively employing verbs with universally construed tenses.) If T_1 is a tense with an existential interpretation, we may form the complex expression "*not*: T_1: *not* ..." whose interpretation is universal. Letting, then, another existentially read tense T_2 be in a syntactically embedded position relative to the clause in which the expression "*not*: T_1: *not* ..." appears, the structure

$$\textit{not}\text{: } T_1\text{: } \textit{not} \ldots (T_2 \ldots)$$

is yielded. There the interpretation of T_2 will be functionally dependent on the interpretation of T_1.

(ii) Having C_1 verbs combined with C_2 verbs in one and the same sentence. Forming a complex sentence exemplifying the structure

$$T_1 \ldots (T_2 \ldots),$$

where T_1 is a *universally interpreted* tense of a C_2 verb, and T_2 is a tense of a C_1 verb (hence existentially interpreted), we obtain a sentence according to which the interpretation of T_2 is functionally dependent on the interpretation of T_1.

The relevance of constructions (i) and (ii) to our primary concern is as follows: That a tense is functionally dependent on another *means* that the latter is *logically* prior to the former. But the terms of the relation of logical priority are none other than *operators*. Hence there are natural language tensed sentences in which a tense appears as an operator.

Let us consider these two options in more detail.

5.3.4.1. Functional dependencies by negation

We need to have *dual negation* at our disposal in the fragment of English we are going to consider. If S is an English sentence and "$\neg$" stands for dual negation, we have:

$\neg S$ is true $\Longleftrightarrow$ S is false $\Longleftrightarrow$ there exists a winning

strategy for *Falsifier* in the semantical game $\mathrm{G}(S)$.

In general, natural language negation behaves rather like *contradictory negation*. Given that "$\sim$" stands for contradictory negation, we have for a sentence S that

$\sim S$ is true $\Longleftrightarrow$ S is not true $\Longleftrightarrow$ $\neg S$ is true, or there

exists for neither *Verifier* nor *Falsifier* a w.s. in $\mathrm{G}(S)$.

Even though the contradictory negation – and not the dual negation – is the favored negation of our *Sprachlogik*, it behaves precisely like dual negation in any fragment of English that is formalizable in **FO**. Accordingly, for any natural language sentence that can be formalized in **FO** by a formula whose atomic constituents are either true or false in the models employed, we may construe the appearances of negation as instances of dual negation. (Observe that the formalization need not be a *sentence* of **FO**; e.g. the first-order counterparts of modal-logical formalizations of natural language sentences are actually formulae of one free variable.)

Formally, there exists for each English tense T its *dual* T^d, satisfying:

- $T^d = \neg T \neg$
- $T = \neg T^d \neg$

For instance, for the simple future tense in English there corresponds as its dual the construction

not [`will`] *not*. . .

which has for C_1 verbs the semantic force of "always in the future". It must be observed that neither the above construction, nor the locution "always in the future", is itself a *tense*. The latter is an English phrase that happens to have the semantics of the dual of the simple future tense in connection with C_1 verbs.

Despite the abstract and to some extent artificial flavor of duals of tenses, it is an incontestable fact that the existence of dual negation induces such duals for English tenses – insofar as dual negation is applicable in appropriate positions in English tensed sentences.

Consider the following sentences (31) and (32):

(31) *Harry knew that Debbie would not admit that she would not become a novelist.*

(32) *John knew that Sally would not admit that she had not turned off the stove.*

To capture what these sentences serve to assert, let us employ epistemic tense logic. (Obviously we are *not* claiming to thereby give the *logical forms* of the sentences. Claiming so would beg the question of finding out whether or not tenses as operators exist.)

The notation is: G stands for the dual of F; $[K]_c$ is read "c knows that"; $[A]_c$ is read "c admits that"; $\langle A\rangle_c$ is the dual of $[A]_c$. We may safely take all occurrences of negation in (31) and (32) to be instances of dual negation since (i), we can assume that Debbie's becoming a novelist/Sally's turning off the stove are determined propositions (i.e. they are true or false in every world at each time) and (ii), the sentences (31) and (32) clearly admit of formalization in **FO**. We take all three verbs *to admit, to become* and *to turn off* to be C_1 verbs. (These verbs do not possess the subinterval property.) Clearly, negation has priority over the past perfect in "had not".

The schemata (33) and (34) both capture the truth-condition of (31):

(33) $P[K]_{Harry}\neg F[A]_{Debbie}\neg F(\texttt{she becomes a novelist})$

(34) $P[K]_{Harry}G\langle A\rangle_{Debbie}F(\texttt{she becomes a novelist})$.

Similarly, sentence (32) has the equivalent formal representations (35) and (36):

(35) $P[K]_{John}\neg F[A]_{Sally}\neg P(\texttt{she turns off the stove})$

(36) $P[K]_{John}G\langle A\rangle_{Sally}P(\texttt{she turns off the stove})$.

Schema (34) shows that sentence (31) states the following:

($*$) for some $t^* < now$ there exists a function

$$f : W_{Harry} \times \{s : s > t^*\} \rightarrow W_{Debbie} \times T$$

such that for all $(u,t) \in dom(f)$, the value $f(u,t) = (u',t')$ satisfies: $t' > t$ (and u' is R^{admit}_{Debbie}-accessible from u) and Debbie becomes a novelist in u' at t'.

Hence we see that in (31), the interpretation of the future tense of “would become” is functionally dependent on the interpretation of the future tense of “would admit”. Game-theoretically expressed, the consequence of the appearances of the negation in the sentence is that the interpretation of “would admit” is chosen by *Abélard*, while “would become”, appearing in the deepest embedded clause, is interpreted by *Héloïse*. This explains the functional dependence.

Completely analogously, schema (36) reveals that sentence (32) involves the functional dependence of the interpretation of “had turned off” on the interpretation of “would admit”.

In order to discuss the next example, we must observe in passing the following fact about the interaction of negation with the *present perfect* of a C_1 verb in English. When evaluating for instance the sentence

(37) *I have not turned off the stove*,

the negation applies first, and only then is a past time chosen [$\neg P$]. Equivalently, then, (37) says that at all past times − typically from some contextually provided interval, stretching to the moment of speech − the sentence

(38) *I am not turning off the stove*

is true. Hintikka (1982, pp. 8-9) points out that while the truth-conditions for the simple past sentence (39) and the present perfect sentence (40)

(39) *I turned off the stove*

(40) *I have turned off the stove*

are exactly the same, the truth-conditions of their negations are not the same. He calls attention to the fact that the difference is (partly) a matter of scope order. Namely, in the evaluation of

(41) *I did not turn off the stove*,

the negation applies only after a past time has been chosen $[P\neg]$, while in the evaluation of (37) the order of negation and the choice of a past time was precisely the reverse. There is of course another relevant difference, discussed by Partee (1973) from whom the very example (41) stems: namely, sentence (41) does not in the first instance communicate that *there is* some past time at which "I am not turning off the stove" is true, but implies that some such past time is *contextually given*. Accordingly, (41) is a statement about a particular past time – at *that* time "I am not turning off the stove" is true. Partee employs sentence (41) in her argument attempting to show that tenses are pronominal.

Now to obtain still a third illustration of how English sentences can get involved in expressing functional dependencies between interpretations of tenses, consider sentence (42):

(42) *Harry has not realized he has not called Debbie.*

Here, notably, both occurrences of "has" – namely those in "has not realized" and "has not called" – are indexical in the sense that their interpretation introduces a past time relative to the utterance time of the sentence. Both verbs *to realize* and *to call* are to be read as instances of C_1 verbs, i.e. they both refer to momentary events (the momentary act of something becoming clear to a person *resp.* the act of making a phone call). From the above observations we recall that in connection with a C_1 verb, negation has priority over the present perfect.

The schemata (43) and (44) written in epistemic IF tense logic (interpreted by **BLO** semantics), then, both capture the truth-condition of (42):

(43) $\neg P_1[\texttt{realizes}]_{Harry}\neg P_2/_{\mathbf{BLO}}\{1\}(\texttt{he calls Debbie})$

(44) $H_1\langle\texttt{realizes}\rangle_{Harry}P_2/_{\mathbf{BLO}}\{1\}(\texttt{he calls Debbie})$.

We cannot state the truth-condition in traditional epistemic tense logic,[37] because the embedded past tense (esp. present perfect) clause "he has not called Debbie" must be evaluated relative to the moment of speech – it has in (42) *neither* a simultaneous reading (*Harry has not realized: " I am not dialing Debbie's number"*) *nor* a shifted reading (*Harry has not realized: "I did not dial Debbie's number"*). Once again it must be borne in mind that the schemata (43) and (44) are here just to explicate the truth-condition of (42), and I am not claiming that either of them would represent its logical form in a linguistically relevant sense. Whether or not one of them does so is another matter, but if we were to claim that (43) or (44) provides such a logical form – while arguing for the necessity of construing at least some appearances of tenses as operators – we would be assuming what we are trying to prove; for these schemata definitely employ operators to represent tenses.

We observe, then, that sentence (42) states the following:

(∗∗) there exists a function

$$f : \{t : t < now\} \rightarrow W_{Harry} \times \{t : t < now\}$$

such that for all $t < now$, the value $f(t) = (u, s)$ satisfies: $s < now$ (and u is compatible with all that Harry actually realizes) and Harry calls Debbie in u at s.

[37] Unless we strengthen it with the (two-dimensional) *NOW* operator of Kamp (1971), or something analogous.

That is, in (42) the indexical interpretation of the past tense of "has called" is functionally dependent on the indexical interpretation of the past tense of "has realized".

It is to be noted that both past times in the interpretation are – being indexical – actually interpreted relative to the moment of speech, whereby the tense of "has called" in particular is *not interpretively dependent* (in the sense in which this locution has been used in the present subsection) on the tense of "has realized". As sentence (42) demonstrates, this does not stop the interpretation of the former tense from being *functionally dependent* on the interpretation of the latter.

By the same token, sentence (42) is one more illustration of the fact that interpretive dependence cannot be a matter of scope relations. For if that was the case, a tense could not fail to be interpretively dependent on another while still being functionally so.

The above schema

(44) $H_1 \, \langle \texttt{realizes} \rangle_{Harry} \, P_2 \, /_{\mathbf{BLO}}\{1\}(\texttt{he calls Debbie})$

has, then, the same truth-condition as schema (43), and both serve to capture what the English sentence

(42) *Harry has not realized he has not called Debbie*

asserts. If one would wish to translate (44) back to English respecting the syntactic structure of this **IFTL** formula, one would obtain something like

(45) *It has always been compatible with all that Harry*

realizes that he has called Debbie.

While (45), then, says the same as (42), there is from the viewpoint of the theory of tense an important difference between these sentences: (42) is entirely in terms of verb tenses, while (45) makes essential use of the *temporal adverb* "always". Because our aim using the examples in this subsection is to show that interpretations of *tenses* can give rise to functional dependencies, sentences such as (45) are not of interest to us. Sentence (45) does, of course, express a functional dependency, too. For it says that the interpretation of the tense of "has called" is functionally dependent on the interpretation of the adverb "always". But this would not suffice to establish that there are functional dependencies between *tenses*.

5.3.4.2. Combining C_1 verbs and C_2 verbs

I suggested above that alongside the use of negation in connection with tenses of C_1 verbs, there is another way to force functional dependencies to the fore between interpretations of natural language tenses, and this would be by combining in an appropriate way tenses of C_1 verbs with tenses of C_2 verbs. Concretely, consider the sentence

(46) *John believed yesterday that Laura would turn up.*

Here *to believe* is a C_2 verb (it has subinterval property), while *to turn up* expresses a momentary event and is a C_1 verb.

In (46) we have the temporal frame adverbial "yesterday", which indicates that the interval relative to which John's believing is considered consists of yesterday. In fact, then, if we employ temporal quantifiers to capture the truth-condition of (46), we may present the past tense of "believed" in the matrix clause by the bounded quantifier

$$(Qy : y < x \wedge \mathtt{YESTERDAY}(y)),$$

where `YESTERDAY` is simply a unary predicate interpreted as the set of times that make up the calendar day preceding the day to which the time of utterance of (46) belongs.

As we observed further above, tenses of C_2 verbs can be interpreted universally. That the simple past of "believed" is read universally in (46), means that John is taken to believe *during the whole of yesterday* that Laura would turn up. This corresponds to specifying the above relativized quantifier by setting $Q := \forall$. If the tense was read existentially, the statement would just be that at some point yesterday, John believed that Laura would turn up. This, then, would mean setting $Q := \exists$.

Let us consider the *universal* reading of the tense of "believed" in sentence (46). The truth-condition of the reading thus obtained is captured by the following formula of epistemic temporal predicate logic employing bounded quantifiers:

(47) $(\forall t_1 : t_1 < t_0 \wedge \texttt{YESTERDAY}[t_1])$ $(\forall w : w$ compatible with all that John actually knows$)(\exists t_2 : t_2 > t_1)$ $(\texttt{Laura turns up}[t_2, w])$.

The same would be captured in propositional epistemic tense logic by the formula:

(48) $H(\texttt{yesterday} \rightarrow [B]_{John} F(\texttt{Laura turns up}))$,

where the propositional atom `yesterday` has the same interpretation as the predicate `YESTERDAY` above in schema (47). Hence it is seen that the reading of (46) under consideration asserts the existence of such a function f that if $t < \textit{now}$ is any time from yesterday and w is compatible with all that John believes, then

$$f(t, w) > t, \text{ and Laura turns up at } f(t, w) \text{ in } w.$$

That is, according to sentence (46), the interpretation of the simple future in "would turn up" is functionally dependent on the interpretation of the simple past in "believed".

The conclusions reached in *Subsection* 5.3.4 can be summed up as follows:

(A) *There are instances of natural language tenses acting as logical operators.*

(B) **BLO** *interpretation of IF tense logic is clearly more adequate than* **PTL** *as a logic of temporal discourse, while differing from it only in incorporating a slight conceptual generalization of a mechanism already present in* **PTL**.

We are now in a position to say that *tense operators are unavoidable in linguistics.* For, we have just seen that there are sentences in natural languages (such as English) that express functional dependencies between the interpretations of the tenses they contain. That functional dependencies exist between interpretations of tenses *means* that one tense is logically prior to another. And the terms of the relation of logical priority are *operators.*

It should be noted that the truth of (A) in no way presupposes the existence of any particular logical formalism, particularly not the one referred to in (B), or the 'backwards-looking operators' interpretation of IF tense logic. The truth of (A) is observed as soon as the notion of priority scope and its conceptual unrelatedness to interpretive dependency (or what we called relational dependency) are understood.

On the other hand, once the difference between the two types of dependency − functional and relational − is grasped, it becomes natural to incorporate them in one and the same logic so that they need not go hand in hand as they do in **PTL** − whereby the **BLO** interpretation of IF tense logic is obtained. This logic is otherwise like **PTL**, but has the built-in capacity to avoid the problems to which **PTL** unavoidably is driven by the sole fact of not being able to separate the logical phenomenon of dependence (functional dependence) from the interpretive (or relational) one.

It must be stressed that the result formulated in (A) does *not* involve any claim to the effect that the whole sphere of tenses and temporal expressions in linguistics can be dealt with exclusively in terms of operators. I claim only to have provided an argument to the effect that *there are natural language sentences in which tenses appear as operators*. In particular, only complex clauses were considered in which the relevant tenses were syntactically embedded one under the other. Temporal adverbs, and expressions directly referring to or quantifying over times were not considered at all. As well, we did not study tenses other than the present, simple past, simple future, present perfect and past perfect. And we worked under the grossly simplifying idealization that the semantics of tenses is, throughout, in terms of time-points, instead of often having recourse to temporal intervals. But irrespective of whatever overall theory of tense one might formulate, it will *not* be a matter of free decision whether some tenses must be presented as operators (i.e. whether there are instances of tenses that act as operators in logical forms of English sentences containing these tenses). Some must – this is what the examples here have shown. *There are instances, then, of tenses as operators*.

A general conclusion arising from our considerations related to natural language tense is that semantic interactions of logically active expressions must never be underestimated. There is no general reason why the semantic properties of an expression should be restricted to those that it has in isolation. Notably, configurations of logically active expressions can have properties that could not be manifested if those expressions were considered simply on their own. The sentences (31), (32), (42) and (46) are illustrative examples of this phenomenon. If all appearances of negation were removed in (31), (32) and (42), no functional dependencies would be expressed. Introducing negation – a species of logically active expression – into these sentences changes the scene: the configuration of appearances of negation that serves to express duality makes one of the tenses in each of these sentences have universal force, thereby making the operator-nature of a tense visible. What is at stake in (46) is the logical interaction of tenses of two verbs having respectively different readings – universal and existential. Structural properties decidedly *can* indicate semantic effects of expressions that would not be visible if these expressions were viewed in isolation.

Appendix A

Some Properties of Binary Relations

Any subset $R \subseteq D \times D$ is a binary relation on the set D. Whenever $(x, y) \in R$, we say that y is *an R-successor of x*, and indicate this by writing $R(x, y)$. We write $dom(R)$ for the set $\{x : \exists y R(x, y)\}$ and $rng(R)$ for the set $\{y : \exists x R(x, y)\}$. The properties of binary relations mentioned in this thesis are defined below, where throughout, "R" stands for a binary relation.

- *Reflexivity*

$(D, R) \models \forall x \mathrm{R}(x, x)$

- *Irreflexivity*

$(D, R) \models \forall x \neg \mathrm{R}(x, x)$

- *Antisymmetry*

$(D, R) \models \forall x \forall y ((x \neq y \wedge \mathrm{R}(x, y)) \rightarrow \neg \mathrm{R}(y, x))$

- *Transitivity*

$(D, R) \models \forall x \forall y \forall z ((\mathrm{R}(x, y) \wedge \mathrm{R}(y, z)) \rightarrow \mathrm{R}(x, z))$

- *Partial order*

R is a *partial order* of D if R is *antisymmetric* and *transitive* on D.

- *Temporal frame*

A pair (D, R) is a *temporal frame* if R is an *irreflexive* and *transitive* relation on D. Observe that R is thus any *irreflexive partial order* on D.

- *Connectedness*

$(D, R) \models \forall x \forall y(\mathrm{R}(x, y) \vee \mathrm{R}(y, x) \vee x = y)$

- *Comparability*

Elements $x, y \in D$ are said to be R-*comparable* if $R(x, y)$ or $R(y, x)$ or $x = y$. If $x, y \in D$ are not R-comparable, they are said to be R-*incomparable*. Observe that R is therefore connected iff all elements $x, y \in D$ are R-comparable.

- *Linearity*

R is a *linear order* of D if it is a *connected partial order* of D.

- *Local linearity*

R is *locally linear* on D if for all $x \in D$, the sets $\{y : R(x, y)\}$ and $\{y : R(y, x)\}$ are linearly ordered by R.

- *Discreteness*

If S is a binary predicate on D, write $\Phi(\mathrm{S})$ for the formula

$$\forall x \forall y[(\mathrm{S}(x, y) \wedge x \neq y) \rightarrow$$

$$\exists y_0(x \neq y_0 \wedge \mathrm{S}(x, y_0) \wedge (y_0 = y \vee \mathrm{S}(y_0, y)) \wedge$$

$$\forall z[(\mathrm{S}(x, z) \wedge \mathrm{S}(z, y_0)) \rightarrow (z = x \vee z = y_0)])].$$

Then R is *discrete* on D if the formula

$$\Phi(\mathrm{R}) \wedge \Phi(\mathrm{R}^{-1})$$

is true in (D, R).

- *Density*

R is *dense* on D, if the formula

$$\forall x \forall y[(x \neq y \wedge \mathrm{R}(x, y)) \rightarrow$$

$$\exists z(z \notin \{x, y\} \wedge \mathrm{R}(x, z) \wedge \mathrm{R}(z, y))]$$

is true in (D, R).

- *Upper bound, lower bound, supremum, infimum*

Let $C \subseteq D$ and $d \in D$.

(a) If for all $x \in C$: $R(x, d)$, then d is said to be an *R-upper bound* of C.

(b) If for all $x \in C$: $R(d, x)$, then d is said to be an *R-lower bound* of C.

(c) If $d = \inf\{d' : d'$ is an R-upper bound of C$\}$, then d is an *R-supremum* of C.

(d) Similarly, if $d = \sup\{d' : d'$ is an R-lower bound of $C\}$, then d is said to be an *R-infimum* of C.

- *Dedekind-completeness*

A pair (D, R) is *Dedekind-complete* if every non-empty subset of D which has an R-upper bound has an R-*supremum*.

- *Maximal* and *minimal* element*; maximum* and *minimum*

Let $d \in dom(R)$.

(a) If there is no $x \neq d$ such that $R(d, x)$, d is an *R-maximal element* of D.

(b) If there is no $x \neq d$ such that $R(x, d)$, d is an *R-minimal element* of D.

(c) If for all $x \in D$*:* $R(x, d)$ or $x = d$, then d is an *R-maximum* of D.

In other words, d is an *R-maximum*, whenever d is *R-maximal* and all *R-maximal* elements of D are R-comparable.

(d) If for all $x \in D$: $R(d, x)$ or $x = d$, then $d \in D$ is an *R-minimum* of D.

An element is an *R-minimum* precisely when it is *R-minimal* and all *R-minimal* elements of D are R-comparable.

- *Well-orderedness*

R is a *well-order* if R is *linear* and every non-empty subset of $dom(R)$ has an *R-minimum.*

For instance, the order $<$ of *real numbers* by their magnitude is *not* a well-order: this order is linear, but e.g. the *infimum* $\sqrt{2}$ of the set $C = \{x : x^2 > 2\}$ is not the minimum of C. The set C has *no* minimum. Also the order of the set $\mathbb{Z}$ of integers by magnitude is *not* a well-order, because the set $\mathbb{Z}$ is its own non-empty subset and has no minimum. On the other hand, any *finite* subset of reals ordered by magnitude is a well-order, and so is the order by magnitude of *natural numbers.*

- *Cyclicity*[1]

R is *cyclical* if R is anti-symmetric, connected, locally linear and, furthermore, *non-transitive* in the sense that the formula

$$\exists x \exists y \exists z (\mathrm{R}(x, y) \wedge \mathrm{R}(y, z) \wedge \mathrm{R}(z, x))$$

is true in (D, R).

- *Tree*

A structure (D,R) is a *tree* if R is a partial order, for all $x \in D$, the elements of the set $\{y : R(y, x)\}$ are R-comparable, and there exists an R-minimum of D (the root of the tree).

[1] This definition is given in Reynolds (1994).

Appendix B

Determinacy and Zero-sum Two-player Games of Perfect Information

In John von Neumann and Oskar Morgenstern's *Theory of Games and Economic Behavior* (*Ch.* 15 esp. *Sect.* 15.6) it is proven:

Theorem B.1 *Every zero-sum two-player game of perfect information is strictly determined.*

The relevant definitions behind the theorem can be given as follows. First, given any set X of finite sequences, let us agree on writing $\mathtt{is}(X) :=$

$$\{h : \text{for some } z \in X \text{ and some non-empty } h', h^\frown h' = z\}.$$

Hence $\mathtt{is}(X)$ is the set of proper initial segments of members of X.

Definition B.2 *A two-player zero-sum game* Γ *in extensive form is a tuple*

$$(Z, P, \upsilon, \{I_1, I_2\}),$$

where

- Z *is a set of finite sequences*
- $P : \boldsymbol{is}(Z) \to \{1, 2\}$ *is a function, associating each proper initial segment of* Z *with one of the players*
- $\upsilon : Z \to \mathbb{R}$ *is a function specifying the outcomes for Player* 1
- I_i $(i := 1, 2)$ *is a partition of the set* $P^{-1}(\{i\})$.

The game is a zero-sum game, i.e. the outcome of every sequence $z \in Z$ *for Player* 2 *is the number* $-\upsilon(z)$. *The game is one of perfect information if the members of both partitions* I_i *are all singletons. This means that when a player makes a choice at a sequence* $h \in \boldsymbol{is}(Z)$, *the player is fully informed about the earlier moves in the play of the game, i.e. of the members of the sequence* h.

Definition B.3 *A two-player zero-sum game* Γ *in normalized form is a tuple*

$$(\beta_1, \beta_2, \kappa),$$

where the β_i *are any ordinals, and* κ *is two-place function associating a real number with each pair* $(\tau_1, \tau_2) \in \beta_1 \times \beta_2$. *The choices available to Player* 1 *are the ordinals smaller than* β_1, *and the choices available to Player* 2 *are the ordinals smaller than* β_2. *A play of the game consists of Player* 1 *choosing an ordinal* $\tau_1 < \beta_1$ *and Player* 2 *choosing an ordinal* $\tau_2 < \beta_2$; *each player makes his choice in complete ignorance of the other player's choice. The outcome of the play* (τ_1, τ_2) *for Player*

1 is the real number $\kappa(\tau_1, \tau_2)$*, and for Player 2 the number* $-\kappa(\tau_1, \tau_2)$*.*

Von Neumann & Morgenstern (1944) only consider cases where β_1, β_2 are finite.

Observation B.4 *If* $\Gamma^* = (Z, P, \upsilon, \{I_1, I_2\})$ *is a game in extensive form, the set of strategies for Player* $i := 1, 2$ *is* $F(i) :=$

$$\{f : dom(f) = P^{-1}(\{i\}) \text{ and always } h^\frown f(h) \in \boldsymbol{is}(Z) \cup Z\}.$$

If the cardinalities of the sets $F(1)$ *and* $F(2)$ *are respectively* α *and* γ*, then by letting* $\beta_1 := \alpha$ *and* $\beta_2 := \gamma$*, we can (by Axiom of Choice) enumerate the* $F(i)$ *as*

$$\{f_j : j < \beta_i\}.$$

There is then a one-one correspondence between pairs of strategies $(f, g) \in F(1) \times F(2)$ *and elements* $z \in Z$*. Defining then for all* $(\tau_1, \tau_2) \in \beta_1 \times \beta_2$*:*

$$\kappa(\tau_1, \tau_2) :=$$

$$\upsilon(\text{“the member of } Z \text{ determined by } f_{\tau_1} \text{ and } f_{\tau_2}\text{”}),$$

we have obtained a game $\Gamma = (\beta_1, \beta_2, \kappa)$ *in normalized form, satisfying:*

the outcome of Player i *in* Γ^* *for the* $z \in Z$ *determined*

by f_{τ_1} *and* f_{τ_2} *is the same as (=)*

the outcome of Player i *in* Γ *for* (τ_1, τ_2)*.*

Definition B.5 *The game in normalized form is strictly determined if the following condition holds:*

$$Max_{\tau_1} Min_{\tau_2}\ \kappa(\tau_1, \tau_2) = Min_{\tau_2} Max_{\tau_1}\ \kappa(\tau_1, \tau_2).$$

Intuitively this condition means that the outcome of the play would be the same if Player 2 found out the choice τ_1 of Player 1 and if the choice τ_2 of Player 2 was found out by Player 1.

Definition B.6 *The game in extensive form with the two possible outcomes* 1 *and* -1 *is said to be determined if one of the players has a strategy that yields him the outcome* 1 *against any sequence of moves by the other player.*

Proposition B.7 *Let* $\kappa : \beta_1 \times \beta_2 \to \{-1, 1\}$ *be any function.* (*Hence the* β_i *may be of any cardinality.*) *Then the following holds:*

$$Max_{\tau_1} Min_{\tau_2}\ \kappa(\tau_1, \tau_2) = Min_{\tau_2} Max_{\tau_1}\ \kappa(\tau_1, \tau_2).$$

Proof. [1] Write $v_1 := Max_{\tau_1} Min_{\tau_2}\ \kappa(\tau_1, \tau_2)$, and $v_2 := Min_{\tau_2} Max_{\tau_1}\ \kappa(\tau_1, \tau_2)$. Assume for contradiction that $v_1 \neq v_2$. Hence either $v_1 = 1$ and $v_2 = -1$, or *vice versa*. Consider the former case first.

(1.1) Because $v_1 = 1$, there is τ_1 such that for all τ_2:

$\kappa(\tau_1, \tau_2) = 1$. Let σ_1 be some such τ_1.

(1.2) Since $v_2 = -1$, there is τ_2 such that for all τ_1:

$\kappa(\tau_1, \tau_2) = -1$. Choose σ_2 to be such τ_2.

But now, by (1.2) we have $\kappa(\sigma_1, \sigma_2) = -1$. And by (1.1) we have that $\kappa(\sigma_1, \sigma_2) = 1$. This is impossible.

Consider then the latter case: $v_1 = -1$ and $v_2 = 1$.

(2.1) We have $v_1 = -1$ iff for all τ_1 there is τ_2 such that $\kappa(\tau_1, \tau_2) = -1$. But the latter condition means that there exists a function f such that for all τ:

[1] For the proof, cf. von Neumann & Morgenstern (1944), pp. 97-8.

$$Max_\tau Min_{f(\tau)}\ \kappa(\tau, f(\tau)) = -1.$$

Hence:

$$-1 = Max_\tau Min_\sigma\ \kappa(\tau, \sigma) =$$

$$= Max_\tau Min_{f(\tau)}\ \kappa(\tau, f(\tau)) = Max_\tau\ \kappa(\tau, f(\tau)).$$

A fortiori, then, by the last equality:

$$Min_g Max_\tau\ \kappa(\tau, g(\tau)) \leq Max_\tau\ Min_{f(\tau)}\ \kappa(\tau, f(\tau)).$$

Now

$$Min_g \kappa(\tau, g(\tau)) = Min_\sigma \kappa(\tau, \sigma).$$

Consequently

$$Max_\tau Min_g\ \kappa(\tau, g(\tau)) = Max_\tau Min_\sigma\ \kappa(\tau, \sigma).$$

Hence:

$$Min_g Max_\tau\ \kappa(\tau, g(\tau)) \leq Max_\tau Min_g\ \kappa(\tau, g(\tau)) = -1.$$

(2.2) On the other hand, because $v_2 = 1$, we have:

$$1 = Min_{\tau_2} Max_{\tau_1}\ \kappa(\tau_1, \tau_2) =$$

$$= Min_g Max_\tau\ \kappa(\tau, g(\tau)) \leq Max_\tau Min_g\ \kappa(\tau, g(\tau)) =$$

$$= -1.$$

A contradiction.

Hence the proof is completed. ■

Theorem B.8 *Every 2-player zero-sum game of perfect information with outcomes in the set* $\{1, -1\}$ *is determined.*

Proof. Let Γ^* be any 2-player zero-sum game of perfect information with outcomes in $\{1, -1\}$, and let $\Gamma = (\beta_1, \beta_2, \kappa)$ be the corresponding game in normalized form. If the value

$$Max_{\tau_1} Min_{\tau_2}\ \kappa(\tau_1, \tau_2) = Min_{\tau_2} Max_{\tau_1}\ \kappa(\tau_1, \tau_2) = 1,$$

then there exists a w.s. for Player 1, and any τ_1 such that

$$Min_{\tau_2}\ \kappa(\tau_1, \tau_2) = 1$$

counts as such a strategy. Otherwise

$$Max_{\tau_1} Min_{\tau_2}\ \kappa(\tau_1, \tau_2) = Min_{\tau_2} Max_{\tau_1}\ \kappa(\tau_1, \tau_2) = -1,$$

and there exists a w.s. for Player 2; any τ_2 such that

$$Max_{\tau_1}\ \kappa(\tau_1, \tau_2) = -1$$

being such a strategy. ■

The result of von Neumann and Morgenstern (the above Theorem B.1) allows any payoffs, i.e. allows the range of κ to be any subset of reals. Its proof would be more difficult than that of Theorem B.8. On the other hand, the present result is more general in that it does not restrict the size of the domain of κ in any way, as β_1 and β_2 may be any ordinals.

Summary

In the present thesis I have defined a modification of the language **ML** of basic modal logic: *IF modal logic*. Three semantics have been offered for **IFML** − interpretations in terms of *uniformity of winning strategies*, *backwards-looking operators* and *linear temporal structures equipped with a commutative group operation*. The issue of the expressive power of IF modal logic compared with basic modal logic has been explored in connection with each interpretation, and it has been shown that while interpreting independence as *logical* independence − in uniformity semantics − *does* in general give **IFML** extra expressive power over **ML**, the expressive resources of **IFML** under the other two interpretations in fact *coincide* with the resources of **ML**. The 'backwards-looking operators' interpretation turned out to be useful, however, when discussing the notion of *tense operator* from the linguistic point of view.[2]

The uniformity interpretation (**UNI**) of IF modal logic is a straightforward analogue of the IF first-order logic of Jaakko Hintikka and Gabriel Sandu (1989): logical independence is implemented by the condition of uniformity on players' winning strategies in semantical games. The models employed by

[2] And, as was noted above in *Ch.* 4 (Remark 4.1.2), a many-dimensional **BLO** interpretation of IF modal logic would indeed make it more expressive than the corresponding many-dimensional version of basic modal logic.

IFML are precisely the same as the models of basic modal logic. Julian Bradfield, who was the first to define a version of IF modal logic (Bradfield 2000; Bradfield & Fröschle 2002), has proceeded from the conviction that an extra relation between transitions − whose intuitive interpretation is that these transitions are 'independent' of each other − is needed before one can make sense of *logical* independence in connection with modal logic. A more straightforward approach along the lines of IF first-order logic has not seemed possible, because the transitional semantics of modal logic makes the evaluation essentially local: whenever a state s is reached in the evaluation game, the available moves in the game do not depend on the history that led to s − the available moves depend only on s itself. Therefore it may seem that there is no room for independence here. Bradfield (2002) has taken the relation of *concurrency* between transitions as a primitive in his models. The condition requiring that a move x in an evaluation game be logically independent of specified earlier moves is implemented in his work by requiring that (i) the move x be concurrent with the specified earlier moves, *and* (ii) a strategy of the relevant player can only be winning if it gives the move x uniformly with respect to those same earlier moves. However, the **UNI** interpretation of IF modal logic presented in this thesis establishes that no additional ingredients in the models of the logic are needed for defining IF modal logic: uniformity constraint on winning strategies suffices.

The main logical results of the thesis are as follows:

- **IFML** has a strictly greater expressive power than **ML** over the class of all k-ary modal structures.
- *IF tense logic* and *basic tense logic* have the same expressive power relative to unary temporal structures with a linear accessibility relation. An analogous result holds

true for IF modal logic, basic modal logic and unary modal structures.

- When evaluated relative to modal (temporal) structures with at least two linear accessibility relations, IF modal (tense) logic has a strictly greater expressive power than basic modal (tense) logic.

- **IFML** has a translation into **FO**.

- A modification of IF modal logic (dubbed **EIFML**) allowing modal operators to be independent even from conjunctions and disjunctions cannot be translated into **FO**.

The concluding chapter (*Ch.* 5) of the present thesis provided an example of the theoretical relevance of the IF modal-logical framework, in its capacity to allow a broadened vista over modal logic. In the case at hand, a relevant modification of basic modal logic was shown to result from the 'backwards-looking operators' interpretation of IF modal logic. I discussed the linguistic critique of the operator-based view on natural language tenses, and, by making use of the 'backwards-looking operators' interpretation of **IFTL**, it was shown how one can systematically distinguish two aspects of 'tense operators' – one concerning *logical relations* (patterns of operators expressing functional dependencies), the other pertaining to *temporal relations* (how times are chosen in relation to other times when interpreting a sentence). Basic tense logic cannot make a distinction between these two features. I argued that while the critique against tenses as operators correctly points out that the operator-based approach leads to an overflow in possible logical forms of temporal sentences (many ungrammatical sentences receive a putative logical form), it fails by insisting that this is a reason for rejecting tenses as operators. The over-representativeness of tense logic as a source of logical forms of tensed sentences arises from possible patterns of temporal relations (*earlier than*, *contemporaneousness*) between times (i.e. between values associated with tense operators in evaluation). This, however, justifies giving up tense operators *only if* there appear *no* functional dependencies between tenses, that is to say, if the logical aspect just mentioned is never realized in natural language. Further, I showed that there are English sentences which serve to express functional dependencies between interpretations of tenses. This means, then, that there are instances of tenses as operators. While this does *not* claim to be a universal solution to the body of linguistic problems relating to tense, this *does* aim to be a rebuttal of the position in linguistics which attempts to dispense with tense operators altogether.

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