Métriques naturelles associées aux familles de variétés Kahlériennes compactes

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UNIVERSITÉ DE GRENOBLE

THÈSE

Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ DE GRENOBLE

Spécialité : **Mathématiques**

Arrêté ministériel :

Présentée par

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préparée au sein **de l'Institut Fourier**
et de **l'École Doctorale Mathématiques,**
Sciences et Technologies de l'Information, Informatique

Natural metrics associated to families of compact Kähler manifolds

Thèse soutenue publiquement le **28 Novembre 2012**,
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Thomas Jefferson kept ferrets, that gnawed on his body at night, and eventually poisoned his blood.

– Hunter S. Thompson.

Remerciements

Je remercie Jean-Pierre Demailly du fond de mon coeur pour la direction de cette thèse, pour ses conseils, sa disponibilité, sa patience et sa générosité ces dernières années. Je ne peux pas imaginer un meilleur directeur de thèse. Takk. Takk fyrir allt.

Je suis très honoré d'avoir un tel expert de la géométrie kählerinne, Olivier Biquard, comme rapporteur de cette thèse. Une grande partie de mon temps à été consacré à la lecture des travaux de Georg Schumacher sur la métrique de Weil–Petersson, et ça me fait plaisir qu'il soit un rapporteur de ma thèse. Je suis aussi fier que Frédéric Campana, Philippe Eyssidieux et Christophe Mourougane ont accepté d'être examinateurs et je les remercie tous.

Je tiens à remercier tous les collègues et chercheurs qui m'ont invité et accueilli lors de diverses conférences. Ces rencontres m'ont fait decouvrir à la fois les mathématiques et le monde. Pour ces deux choses, je suis très reconnaissant.

J'adresse mes meilleures voeux à mes grands frères mathématiques Christophe, Mihai et Simone d'avoir pris le temps de discuter de mathématiques et d'autres choses avec moi. Et également, bien sûr, à mes frères Mickael et Junyan, qui ont été soumis à un nombre inquiétant de questions bêtes de ma part pendant des années.

Je voudrais remercier le personnel de l'Institut Fourier pour leurs efforts. Ma vie aurait été beaucoup plus difficile sans eux. Merci à Françoise pour les impressions positives, Marie-Hélène pour les cours de peinture, Yan pour les voyages, Fanny pour sa signature, Francesca pour les bonnes notes, Hervé pour la fin du règne de la terreur de KDE, et Géraldine pour la Chamechaude.

Les amateurs de la littérature savent bien qu'il faut avoir des expériences variés pour mieux comprendre les divers romans. Graçe au CIES j'ai une nouvelle appréciation des œvres de Franz Kafka.

Enfin, je dédie cette partie des remerciements à la lutte menée depuis plusieurs années par Gérard contre le chocolat.

L'écrivain japonais Haruki Murakami s'est mis à courir régulièrement après avoir consacré une partie de sa vie à l'écriture. Il avait besoin à la fois de faire du sport et de vider sa tête après le travail. Il raconte qu'il s'était choisi une mantra à répéter pour le moment où les kilomètres commencent à s'accumuler et où l'envie de tout arrêter s'installe. « La douleur est inévitable, mais la souffrance est un choix. » Je n'avais malheureseument aucun mantra pendant ces années de thèse. Comme plusiers d'autres, je suis arrivé ici avec un peu d'aide de mes amis.

D'abord, j'aimerais vous remercier tous, les thésards de l'Institut Fourier. Vous avez réussi à convertir un bloc de béton soviétique en travaux permanents en une vraie maison. Un prix Nobel de la paix à tous ! Vous allez me manquer, tous, même si je vais vous faire le coup de salaud de ne pas vous citer dans ce qui suit.

À mes cobureaux[1], un grand merci. Mickael, encore, de m'avoir subi au quotidien pendant trois ans ; et G3 pour toujours avoir eu quelque chose à dire.

Il y a beaucoup de gens que je ne sais vraiment pas comment remercier. Eliza, Bashar, Mélaine, Thomas, Soizic, Adrien, Simon, Camille, Jean, Ariadna, Fred, Aline, Maxime, Fred, Mr. and Ms. Determinant, Elise, Carlos, Claudia, Alix, Alvaro, Ximena, Max, Vincent . . . . Vous étiez ma famille pendant ces années de thèse. Je vous aime.

Aux cothésards[2] oubliés, un cri ! Jeff, Marco, Binbin, Filippo, Rolandbot 2000, Delphine, Koju, Ronan, Marianne, Camille, Kévin, Axelle, Héloise, . . . merci à tous.

Jessica, merci d'avoir partagé des bons moments avec moi en Argentine, à Banff et à la Fontaine.

Je remercie aussi Silvio, Sheila, Alex, Emmy et tous les autres chez MSP, pour les jeux de société incroyablements durs, les fêtes des cerf-volants et les mauvais accents autour des repas.

Merci à mes parents, ma soeur, mes grand-parents, Söru, Þór, et les autres pour avoir été là pour moi depuis toujours.

Enfin, merci à Aby, pour ses tentatives de m'apprendre à danser. Te amo.

[1] Les cobureaux sont les mêmes que les bureaux, mais toutes les flèches sont inversées.

[2] Les cothésards sont les mêmes que les thésards . . . ah, non, on a déjà fait celle-là.[a]

[a] Il n'y a que deux types de comédie[i] : la comédie de la répétition et la comédie de la répétition.

[i] La comédie est la même que la médie, mais toutes les flèches sont inversées.

Introduction

Consider a complex manifold of dimension n. One of the objectives of complex geometry is to classify such manifolds in a suitable sense. Since the pioneering work of Bernhardt Riemann in the 19th century is has been clear that the basic invariants of complex manifolds come in two forms; discrete and continuous.

The discrete invariants have much in common with those of smooth or topological manifolds. They include the dimension of the manifold, its fundamental group and its cohomology ring and may or may not enjoy relations with the holomorphic structure. However, in contrast with the case of smooth manifolds, complex manifolds usually come in entire families of non-isomorphic varieties. These represent continuous invariants of the manifold, or moduli. The set of the isomorphism classes of complex manifolds of a given type is called a moduli space, and may often itself be given the structure of a complex space. By doing so, one may apply powerful methods of complex differential geometry to classification problems.

This theory of deformations of complex manifolds may be defined in all generality, but concrete results are often unavailable without some simplifying assumptions. First, one must often assume the manifolds under consideration are compact to obtain reasonably interesting and finite-dimensional objects. Second, to ensure a strong link with algebraic geometry and to avoid wild phenomena, we suppose our manifolds are Kähler.

A compact Kähler manifold X depends on a certain number of moduli. Familiar to many are the parameters of the underlying complex structure. By the fundamental work of Kodaira and Spencer [**KS58, KS60**] and Kuranishi [**Kur62**] these infinitesimal parameters lie in a subvariety of the finite-dimensional complex vector space $H^1(X, T_X)$. But we also have the moduli of the underlying Kähler structure.

In Riemannian geometry one considers each Riemannian metric on a given manifold as defining a unique and separate structure. Despite its close ties to the Riemannian world, the same is not quite true in Kähler geometry. There the cohomology class of the symplectic form of a Kähler metric seems to play a more important role than the metric itself, as exemplified by the hard Lefschetz theorem. One may thus define the Kähler moduli space of a compact manifold as the collection of the cohomology classes of Kähler metrics on it. This set is the Kähler cone of the manifold, and as the name suggests it is an open cone in the finite-dimensional real vector space $H^{1,1}(X, \mathbb{R})$.

This thesis is on the interplay between moduli of complex and Kähler structures.

The geometry of complex moduli

The study of complex moduli spaces started with Riemann. He showed that a compact complex curve, or a compact Riemann surface, depends on a certain number of continuous parameters, or moduli. Later, Teichmüller showed that these moduli fit into a complex manifold, and the question of how best to investigate the geometry of this manifold arose. Weil and Petersson answered it independently.

Consider a family $\pi : \mathcal{X} \to S$ of compact Riemann surfaces of genus g over a smooth base S. Since the complex structure on the projective line is unique, and the case of elliptic curves may be handled directly, we suppose that the genus $g \geq 2$. In this case, each curve X_s is equipped with a "best" Kähler metric ω_s, or the unique Kähler metric of constant curvature -1 on X_s. Combining the Hodge L^2 metric induced by this Kähler metric on cohomology, and the Kodaira–Spencer morphism

$$\rho_s : T_{S,s} \longrightarrow H^1(X_s, T_{X_s})$$

one obtains a hermitian form h_{WP} on the base of deformations S. Under some additional hypotheses on the family, this is an honest hermitian metric, the Weil–Petersson metric.

The Weil–Petersson metric associated to a family of complex curves is a Kähler metric. It is furthermore negatively curved, and may be used to prove that the moduli space of smooth complex curves of a given genus g may be embedded in projective space. All this gives important information about the moduli space and the original curves.

The generalization of this construction to families of higher-dimensional manifolds was delayed by the lack of a suitable "best" Kähler metric in higher dimensions. The existence of the first suitable candidates for these metrics, the Kähler–Einstein metrics, was proven by Aubin and Yau [**Aub78, Yau78**] in the 70's following ideas of Calabi. Soon after, Siu [**Siu86**] gave a construction of a Weil–Petersson metric on the base of deformations of compact Kähler manifolds with ample canonical bundle, which may be regarded as a generalization of curves of genus $g \geq 2$ to higher dimensions.

However, with more freedom of movement came new problems. While the Aubin–Calabi–Yau theorem predicted the existence of canonical metrics on compact manifolds with zero first Chern class—the higher dimensional cousins of elliptic curves—unicity of such metrics was far from guaranteed. In fact, each manifold with zero first Chern class supports an infinite number of such metrics, one in each Kähler class. This poses a problem, since having a unique "best" metric on a manifold is important for ensuring smoothness and establishing functorial properties of the induced Weil–Petersson metric.

Nannicini and Schumacher both attacked this problem soon after the original work of Siu [**Nan86, Sch85**], and found that if one supposes that if all the manifolds in our family share a Kähler metric, that is a metric whose cohomology class stays constant as the complex parameters vary, then one obtains a good notion of a Weil–Petersson metric. The choice of such a Kähler metric is called a polarization of the family. There are two problems with this approach.

The first is perhaps not so bad. To get a Weil–Petersson metric one much choose a polarization, so one would expect that choosing a different polarization results in a different Weil–Petersson metric. This would not matter if the geometries properties of the resulting metrics were all more or less the same, which turns out to be the case. In fact, the different polarizations all result in the same metric, though this is far from obvious from the classical constructions.

The second and more serious problem is that not every family of compact Kähler manifolds with zero first Chern class can be polarized. As a simple example, consider the family of complex tori of dimension $n \geq 2$ parametrized by $n \times n$ complex matrices s with $\det \operatorname{Im} s \neq 0$. An exercise in variation of Hodge structures shows that given any non-zero cohomology class of degree $(1, 1)$, there always exists an infinitesimal direction on the moduli space that one may not deform the class into, if one wants its cohomology class to remain constant and of type $(1, 1)$. In particular, one cannot find a Kähler class that stays Kähler on the whole family.

It seems this problem troubled the sleep of absolutely no one. One reason seems to be that unpolarized families may admit far too many automorphism to form a good moduli space, so one may not even end up with a holomorphic stack after quotienting them out [**KS60**], while polarized families admit good, honest orbifolds as coarse moduli spaces [**Sch84**]. Another may be that the discovery of mirror symmetry brought the rise of Calabi–Yau manifolds. These are projective manifolds with zero first Chern class, whose families always admit local polarizations, which is enough to obtain a Weil–Petersson metric.

The geometry of Kähler moduli

The credo of mirror symmetry says that families of complex manifolds do not appear one by one, but in pairs [**SYZ96, Kon95**]. Near degenerating points of the respective moduli spaces, one expects to see an exchange of complex and Kähler moduli, so the complex moduli of one family correspond to Kähler moduli of the other and vice versa. Since we possess Weil–Petersson metrics on complex moduli it is natural to look for "mirror" metrics on Kähler moduli.

It seems Wilson [**Wil04**] was the first person to take this idea seriously. The Kähler classes on a compact manifold X are parametrized by an open subset of the vector space $H^{1,1}(X, \mathbb{R})$. Given a Kähler class ω, we obtain a bilinear form on this space by taking the cup product of two elements of it against a suitable power of the Kähler class. This form has mixed signature, but we can obtain an honest inner product by adding a correction term to it. Alternatively, one can observe that $-\log \operatorname{Vol}$ defines a smooth function on the Kähler cone, and that its Hessian is positive-definite by the hard Lefschetz theorem.

Thus we obtain a Riemannian metric on the Kähler cone of X. If one is so inclined, one may complexify the Kähler cone and obtain a Kähler metric on its complexification. This metric may be calculated in some cases, where it displays curvature properties quite similar to that of the Weil–Petersson metric on complex moduli. Wilson succeeded in expliciting several formulas for the curvature tensor of this metric in the general case, expressed in terms of the intersection product on the cohomology ring of X. However, these formulas contain terms whose positivity is hard to control, leaving us with less complete results than in the case of complex moduli.

Organization of the text

Chapter 1. The starting point of this thesis is an attempt to do away with the hypothesis of polarization of a family for families of compact Kähler manifolds with zero first Chern class. To describe how this works it is necessary to introduce some terminology.

If $\pi : \mathcal{X} \to S$ is a family of compact Kähler manifolds over a smooth base S, then there exists a complex manifold $\mathcal{K}$ equipped with a submersion $p : \mathcal{K} \to S$, whose fiber over a point s is the complexified Kähler cone of the manifold X_s. We call $\mathcal{K}$ the relative Kähler cone of the family $\pi : \mathcal{X} \to S$. It is in fact an open subset of the total space of the holomorphic vector bundle $\mathcal{R}^1\pi_*\Omega^1_{\mathcal{X}/S}$ over S. As such it comes equipped with a smooth connection induced by the Gauss–Manin connection on $\mathcal{R}^2\pi_*\mathbb{C}$ and the Hodge decomposition of cohomology groups. A polarization of the family, if one exists, corresponds to a parallel holomorphic section of the relative Kähler cone.

Our replacement for a polarization is the Aubin–Calabi–Yau theorem, which guarantees the existence of a unique Kähler metric in each Kähler class whose Ricci-form is equal to 0. Using these metrics, we obtain a smooth hermitian metric on the total space of the manifold $\mathcal{K}$, which may be suggestively described as being pieced together from the metrics on Kähler moduli and Weil–Petersson metrics. Pushing the same idea slightly further, we obtain a smooth hermitian metric on the fiber product $\mathcal{X} \times_S \mathcal{K}$, which may be thought of as a universal family over the "twisted" moduli space $\mathcal{K}$. We can resume all this as follows:

Theorem. *Let $\pi : \mathcal{X} \to S$ be a family of compact Kähler manifolds with zero first Chern class over a smooth base S. There exists a complex manifold equipped with a submersion $p : \mathcal{K} \to S$ such that the fiber $\mathcal{K}_s$ is the complexified Kähler cone of the manifold X_s. The spaces $\mathcal{K}$ and $\mathcal{X} \times_S \mathcal{K}$ are equipped with hermitian metrics that glue together over base change.*

This construction encompasses several more well known situations. For example both the twistor space of a hyperkähler manifold and, of course, a polarized family of manifolds with zero first Chern class, both equipped with the natural hermitian metrics their situations provide, may be realized by embeddings into a suitable fiber product $\mathcal{X} \times_S \mathcal{K}$.

Chapter 2. The above constructions are somewhat complicated, and while traversing them it is easy to get lost in a notational jungle. We thus restrict ourselves to a simple situation and examine the Riemannian metric on the Kähler cone of a single fixed manifold, which is the case previously studied by Wilson. Since the complex moduli are now fixed we are free to consider an arbitrary compact Kähler manifold X.

We are able to express the curvature tensor of the metric on the Kähler cone through L^2 scalar products and harmonic forms on the given manifold. Unfortunately we were not able to improve on Wilson's results on its positivity, but we have also not had time to exhaust all approaches to the problem. In summary:

Theorem. *Let K be the Kähler cone of a compact Kähler manifold X and let g be the Riemannian metric on K. Let $\mathcal{K}$ be the space of all hermitian metrics on X and let G be the L^2 Hodge metric on $\mathcal{K}$. Then (K, g) embeds into $(\mathcal{K}, G)$ as a Riemannian manifold. The curvature tensor of the embedding is a perturbation of the curvature tensor of a negatively curved symmetric space.*

The key to our effort is again the Aubin–Calabi–Yau theorem, which permits us to embed the Kähler cone into the infinite-dimensional space of all hermitian metrics on the given manifold. This space is equipped with a tautological Riemannian metric, whose sectional curvature is seminegative. We are able to explicit the second fundamental form of the embedding of the Kähler cone into this space. To finish the chapter we briefly discuss the failure of the metric to be complete.

Chapter 3. We return to our original interests to study these constructions in the case of compact Kähler manifolds with trivial canonical bundle. In this case the Weil–Petersson metric on the base of deformations is the curvature form of a holomorphic line bundle [**Tia87**]. Previous proofs of this fact made use of a polarization, which we do not require.

Next we observe that, even in the general case, the hermitian metric on the complexified relative Kähler cone is induced by the curvature form of a holomorphic line bundle over the space $\mathcal{K}$. For manifolds with trivial canonical bundle, we elaborate on the link between the hermitian metric on $\mathcal{K}$ and the curvature form of the tensor product of these two holomorphic line bundles, but are unable to show that the two coincide. This raises:

Question. Let $\pi : \mathcal{X} \to S$ be a family of compact Kähler manifolds with trivial canonical bundle over a smooth base S.

(1) Is the natural hermitian metric on the relative Kähler cone Kähler?
(2) Is the hermitian metric on the line bundle over the Kähler cone positive?

We hope to address these questions soon.

Chapter 4. To finish on a light note, we study some simple examples and applications of our work.

Accenting that with great power comes great responsibility, we remark that the existence of Ricci-flat metrics can be used for evil. Indeed, one can find examples of compact holomorphic symplectic manifolds that admit automorhpisms of infinite order. By basic results of Riemannian geometry, such an automorphism must move every Kähler class. An elementary construction then produces a compact non-Kähler manifold, obtained as the total space of a fibration of holomorphic symplectic manifolds over a complex torus. Surprisingly enough, some of these manifolds violate the abundance conjecture for non-Kähler manifolds.

We do note that, unbeknownst to us at the time of study, variants of this construction were considered by the Japanese school in birational geometry in the 70's [**Uen75**].

Next we explicit our constructions in the case of the standard family of elliptic curves over the Poincaré half-plane. Due to the very simple nature of the objects involved, things work as in a dream. For example, the hermitian metric on the universal curve is Kähler.

Somewhat interestingly, we find in this case an explicit non-holomorphic self-diffeomorphism of the universal curve over the relative Kähler cone. Morally speaking, this diffeomorphism exchanges the complex and Kähler moduli of the family, and acts like the naïve interpretation of mirror symmetry for elliptic curves. It is an isometry with respect to our metrics.

Finally we point out how adding the relative Kähler cone into the variation of complex structures can rigidify moduli problems without unnecessarily throwing away automorphisms: While there may not be a good moduli space of complex structures of manifolds with zero first Chern class, the total space of the relative Kähler cone will be an orbifold is the category of complex spaces, and the metrics we have constructed are orbifold metrics.

Introduction

Généralités

Nous rappelons ici quelques généralités sur la géométrie complexe qui peuvent être utiles pour fixer les idées. Nous supposons que le lecteur est familier avec les notions de la géométrie différentielle, et renvoyons à [**Dem12, Voi02, Huy05**] pour les détails.

Connexion de Chern. Soit X une variété complexe lisse de dimension n, et soit $E \to X$ un fibré vectoriel holomorphe de rang r au-dessus de X. Nous supposons E muni d'une métrique hermitienne h, c'est-à-dire un produit scalaire hermitien h_x sur chaque fibré $E_x \cong \mathbb{C}^r$ qui varie de manière lisse avec le paramètre $x \in X$.

Soit D une connexion sur E. C'est un outil qui nous permet de dériver les sections du fibré E et obtenir une forme différentielle à valeurs dans les sections de E. On demande que D satisfasse la règle de Leibnitz :

$$D(f\sigma) = df \otimes \sigma + fD\sigma,$$

où f est une fonction lisse sur X et σ une section locale de E.

Soit σ une (p,q)-forme à valeurs dans E, ou une section lisse du fibré $\bigwedge^{p,q} T_X^* \otimes E$. Nous pouvons alors appliquer la connexion D à σ et obtenir $D\sigma$, une $(p+q+1)$-forme à valeurs dans E. En composant cette forme avec les projections sur les espaces de $(p+1,q)$- et $(p,q+1)$-formes nous obtenons alors une décomposition de la connexion D :

$$D\sigma = D'\sigma + D''\sigma,$$

où D' envoie les (p,q)-formes dans les $(p+1,q)$-formes, et D'' envoie les (p,q)-formes dans les $(p,q+1)$-formes, toutes à valeurs dans E.

Nous pouvons imposer une condition de compatibilité entre la métrique h et la connexion D. Moralement, nous demandons que les sections parallèles de E, c'est-à-dire les sections lisses locales σ telles que $D\sigma = 0$, soient de norme constante par rapport à h. La condition

$$dh(\sigma, \overline{\tau}) = h(D\sigma, \overline{\tau}) + h(\sigma, \overline{D\tau}),$$

où σ et τ sont des sections locales quelconques de E, exprime cette compatibilité. Une connexion qui vérifie cette condition est dite *hermitienne*.

Nous notons que nous possédons d'un opérateur privilégié en géométrie complexe : l'opérateur $\bar{\partial}$ agit naturellement sur les sections de n'importe quel fibré vectoriel holomorphe. Un calcul montre maintenant qu'étant donné un fibré hermitien (E,h), il existe une unique connexion D compatible avec h telle que $D'' = \bar{\partial}$. Cette connexion est la *connexion de Chern* du fibré (E,h). Moralement nous pouvons dire que la connexion de Chern est l'unique connexion compatible avec la métrique h qui “agit le plus comme l'opérateur $\bar{\partial}$”. Ceci distingue la géométrie complexe de la géométrie riemannienne, où il n'y a pas une connexion privilégiée attachée à un fibré vectoriel général (E, g).

Métriques kähleriennes. Regardons maintenant le fibré tangent holomorphe T_X de X. À l'aide des coordonnées locales et d'une partition d'unité, nous pouvons munir ce fibré d'une métrique hermitienne h. C'est l'analogue complexe du fait que chaque variété lisse peut être munie d'une métrique riemannienne.

Fixons une telle métrique h et notons par D sa connexion de Chern. C'est une connexion affine, donc elle admet un tenseur de torsion

$$\tau(\xi, \eta) = D_\xi \eta - D_\eta \xi - [\xi, \eta].$$

Notons maintenant par M la variété lisse obtenue en oubliant la structure complexe sur X, et par g la métrique riemannienne induite par la partie réelle de h sur M. La connexion de Chern induit alors une connexion sur T_M compatible avec g, et son tenseur de torsion s'identifie à τ.

Chaque métrique riemannienne g admet une connexion canonique, la connexion de Levi-Civita. C'est l'unique connexion qui est compatible avec la métrique g et qui est de torsion nulle. Quand les deux coïncident, c'est-à-dire quand la connexion de Chern de h est de torsion nulle et s'identifie alors à la connexion de Levi-Civita de g, nous disons que la métrique h est *kählerienne.*

Pour obtenir un lien plus fort avec la géométrie complexe ou algébrique, nous posons $\omega := -\operatorname{Im} h$. C'est une forme différentielle réelle lisse de type $(1, 1)$ sur X, et elle est positive définie. Un calcul dans des coordonnées locales montre alors que l'identité

$$\partial\omega(\xi, \eta, \overline{\nu}) = \omega\big(\tau(\xi, \eta), \overline{\nu}\big),$$

qui relie la forme extérieure $\partial\omega$ et le tenseur de torsion τ de h, est vérifiée pour tout champ de vecteurs holomorphes locales ξ, η et ν sur X. Vu que la forme ω est réelle, nous avons que $\partial\omega = \bar{\partial}\omega = \frac{1}{2} d\omega$.

On en conclut qu'une métrique hermitienne h est kählerienne si et seulement si la forme ω est fermée, $d\omega = 0$. Cette identité implique qu'il y a un lien fort entre les géométries kähleriennes et symplectiques.

Nous n'entrons pas plus dans les détails ici, mais nous avons en fait une équivalence entre les quatres affirmations suivantes :

(1) La métrique h est kählerienne.
(2) La forme ω est fermée.
(3) La métrique h peut être approximée localement à l'ordre 2 par la métrique Euclidienne.
(4) Localement, il existe un potentiel φ tel que $\omega = i\partial\bar{\partial}\varphi$.

L'équivalence entre ces affirmations est détaillée dans [**Dem12, Voi02, Zhe00**].

Cohomologie des variétés kähleriennes. Supposons que notre variété X est compacte. Un théorème de Cartan et Serre dit alors que les groupes de cohomologie $H^q(X, \mathcal{F})$ sont de dimension finie pour n'importe quel faisceau cohérent $\mathcal{F}$. Ce théorème s'applique en particulier aux fibrés vectoriels holomorphes $\Omega_X^p := \bigwedge^p T_X^*$, et un théorème de Dolbeault donne un isomorphisme canonique

$$H^q(X, \Omega_X^p) = H^{p,q}(X, \mathbb{C}),$$

où les groupes $H^{p,q}(X, \mathbb{C})$ sont définis comme le quotient des (p, q)-formes fermées sur X par les images des $(p + q - 1)$-formes.

Un miracle est que si X est une variété kählerienne, c'est-à-dire si elle admet une métrique kählerienne, alors les groupes $H^{p,q}(X, \mathbb{C})$ sont des sous-groupes de $H^{p+q}(X, \mathbb{C})$, ce qui donne un lien entre la cohomologie "analytique" $H^q(X, \Omega_X^p)$ et la cohomologie de De Rham $H^k(X, \mathbb{C})$. Ce lien est en fait encore plus fort, car le théorème de décomposition de Hodge affirme qu'il y un isomorphisme

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}),$$

et que la conjugaison complexe induit des isomorphismes réels $H^{p,q}(X, \mathbb{C}) \to H^{q,p}(X, \mathbb{C})$.

Ce miracle est dû aux identités kähleriennes. *A priori* nous avons trois Laplaciens associées à la métrique h sur X :

$$\Delta = d^*d + dd^*, \quad \Delta' = \partial^*\partial + \partial\partial^*, \quad \Delta'' = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*,$$

où $*$ dénote l'adjoint L^2 de chaque opérateur. Chaqu'un de ces opérateurs permet de définir des formes harmoniques sur X, qui fournissent des isomorphismes entre des groupes de cohomologie et les espaces des formes harmoniques en question, à là géométrie riemannienne. Les identités kähleriennes montrent que $\Delta = \Delta' + \Delta''$ et $\Delta' = \Delta''$, donc chaque opérateur définit les mêmes formes harmoniques, ce qui fournit la décomposition de Hodge, voir [**Dem12**].

Prenons maintenant une métrique kählerienne ω sur X. Il est clair que si u est une (p, q)-forme sur X, alors $\omega \wedge u$ est une $(p + 1, q + 1)$-forme. La formule

$$d(\omega \wedge u) = d\omega \wedge u + \omega \wedge du = \omega \wedge du$$

montre alors que si u est fermée, alors $\omega \wedge u$ l'est aussi. La forme ω induit alors un morphisme

$$L : H^k(X, \mathbb{C}) \longrightarrow H^{k+2}(X, \mathbb{C}), \quad [u] \longmapsto [\omega \wedge u]$$

au niveau de groupes de cohomologie. Par dualité de Poincaré, les espaces vectoriels $H^{n-k}(X, \mathbb{C})$ et $H^{n+k}(X, \mathbb{C})$ sont isomorphes, et donc de la même dimension, pour tout $k \leq n$. En itérant le morphisme L nous avons donc un morphisme linéaire

$$L^k : H^{n-k}(X, \mathbb{C}) \longrightarrow H^{n+k}(X, \mathbb{C})$$

entre deux espaces vectoriels de même dimension pour chaque k.

Or, l'algèbre linéaire montre que le morphisme $\omega^k : \bigwedge T_{X,x}^{n-k} \to \bigwedge T_{X,x}^{n+k}$ est un isomorphisme *en chaque point x*. Un deuxième miracle est que le même est vrai du morphisme L^k au niveau de cohomologie si ω est kählerienne ; c'est une partie du théorème de Lefschetz difficile.

Ce miracle est encore dû aux identités kähleriennes, qui montrent que L^k envoie les formes harmoniques sur les formes harmoniques, puis il n'est pas difficile de vérifier que l'opérateur L^k est injectif au niveau des formes différentielles. Ces identités fournissent aussi d'une action de l'algèbre $\mathfrak{sl}_2$ sur le groupe $H^k(X,\mathbb{C})$, via les formes harmoniques. Cet action donne ensuite une décomposition

$$H^k(X,\mathbb{C}) = \bigoplus_{p\geq 0} L^p H^{k-2p}(X,\mathbb{C})_{\mathfrak{p}},$$

où $H^{k-2p}(X,\mathbb{C})_{\mathfrak{p}} = \{u \in H^{k-2p} \mid u \wedge \omega^{n+1-k+2p} = 0\}$ sont les groupes des classes ω-primitives. Avec les relations de Hodge–Riemann, qui disent que si u est une (p,q)-classe primitive alors l'accouplement

$$i^{p-q}(-1)^{\binom{p+q}{2}} \int_X u \wedge \overline{u} \wedge \omega^{n-p-q}$$

est positif, ce résultat forme la deuxième partie du théorème Lefschetz difficile.

Classification des variétés

Considérons une variété complexe de dimension n. L'un des objectifs de la géométrie complexe est de classifier de telles variétés dans un sens convenable. Depuis les travaux de Riemann pendant le 19ème siècle il est clair que les invariants fondamentaux des variétés complexes apparaissent sous deux formes : discrets et continus.

Les invariants discrets ont beaucoup en commun avec ceux des variétés lisses ou topologiques. Des exemples de ces invariants sont la dimension de la variété, son groupe fondamental et son anneau de cohomologie, et ils peuvent (ou non) être en lien avec la structure holomorphe de la variété. Contrairement au cas des variétés lisses, les variétés complexes forment souvent des familles de variétés non-isomorphes. Ces familles représentent les invariants continus, ou modules, de la variété en question. L'ensemble des classes d'isomorphismes de variétés complexes d'un type donné est appelé un espace de modules et peut souvent être muni d'une structure d'espace complexe. En procédant ainsi, on peut appliquer les méthodes puissantes de la géométrie complexe aux problèmes de classification.

Cette théorie des déformations de variétés complexes peut être définie en toute généralité, mais les résultats concrets sont souvent hors de portée sans hypothèses simplificatrices. D'abord, on doit souvent supposer que les variétés en question sont compactes pour obtenir des objets intéressants de dimension finie. D'autre part, pour assurer un lien fort avec la géométrie algébrique et d'éviter les phénomènes trop "sauvages", nous supposons que nos variétés sont kähleriennes.

Une variété kählerienne compacte X dépend d'un certain nombre de modules. Comme il est bien connu, on a d'abord les paramètres de la structure complexe sous-jacente à la variété. D'après les travaux fondamentaux de Kodaira et Spencer [**KS58, KS60**] et Kuranishi [**Kur62**] ces paramètres infinitésimaux habitent dans une sous-variété de l'espace vectoriel $H^1(X, T_X)$, qui est de dimension finie. Mais on dispose aussi des modules de la structure kählerienne.

En géométrie riemannienne l'on considère que chaque métrique riemannienne sur une variété donnée définit une unique structure riemannienne.

Malgré le lien avec le monde riemannien, le point de vue est en général différent en géométrie kählerienne. Dans ce cas, la classe de cohomologie de la forme symplectique définie par une métrique kählerienne semble jouer un rôle plus important que la métrique elle-même ; l'exemple prototypique de ce phénomène est le théorème de Lefschetz difficile. Nous pouvons donc définir l'espace de modules des structures kähleriennes sur une variété comme la collection de classes de cohomologie des métriques kähleriennes portées par celle-ci. Cet ensemble est par définition le cône de Kähler de la variété. Comme son nom suggère, c'est un cône ouvert dans l'espace vectoriel réel $H^{1,1}(X, \mathbb{R})$, qui est de dimension finie.

Cette thèse porte sur l'interaction entre les modules des structures complexes et les modules des structures kähleriennes.

La géométrie des modules complexes

L'étude des modules des structures complexes a commencé avec Riemann. Cer dernier a montré qu'une courbe complexe, autrement dit une "surface de Riemann" compacte, dépend d'un certain nombre de paramètres, ou modules. Plus tard, Teichmüller a montré que ces modules forment eux-mêmes une variété complexe, et la question de savoir comment étudier la géométrie de cette variété est devenue l'objet de recherches de plus en plus poussées. Weil et Petersson lui ont donné une réponse précise.

Considérons une famille $\pi : \mathcal{X} \to S$ de surfaces de Riemann compactes de genre g au-dessus d'une base lisse S. Étant donné que la structure complexe sur la droite projective est unique et que le cas de courbes elliptiques est très explicite, nous supposerons ici que le genre g est 2. Dans ce cas, chaque courbe X_s peut être munie d'une "meilleure" métrique kählerienne de courbure constante -1 sur X_s. En combinant la métrique L^2 de Hodge induite par cette métrique kählerienne sur X_s avec le morphisme de Kodaira–Spencer

$$\rho_s : T_{S_s} \longrightarrow H^1(X_s, T_{X_s})$$

nous obtenons une forme hermitienne h_{WP} sur la base de déformations S. Sous quelques hypothèses supplémentaires, cette forme est une vraie métrique hermitienne, appelée métrique de Weil–Petersson.

La métrique de Weil–Petersson associée à une famille de courbes complexes est une métrique kählerienne. De plus, elle est courbée négativement, et peut servir à montrer que l'espace de modules des courbes lisses de genre g peut être plongé dans un espace projectif. Tout ceci donne des informations importantes sur l'espace de modules et sur les courbes associées.

La généralisation de cette construction aux familles de variétés de dimension quelconque a été retardée par la difficulté de construction d'une "meilleure" métrique kählerienne en dimension supérieure. L'existence des premières candidates convenables à ce nom, à savoir les métriques de Kähler–Einstein, a été démontrée par Aubin et Yau [**Aub78, Yau78**] dans les années 70 en suivant des idées de Calabi. Peu temps après, Siu [**Siu86**] a donné une construction d'une métrique de Weil–Petersson sur la base de déformations d'une variété de fibre canonique ample. Ces dernières sont l'une des généralisations des courbes de genre $g \geq 2$ en dimension quelconque.

Cependant, plus on dispose de liberté pour effectuer des manoeuvres, plus les soucis potentiels peuvent survenir. Alors que le théorème de Aubin–Calabi–Yau prédit l'existence de métriques canoniques sur une variété de première classe de Chern nulle—variétés qui fournissent une généralisation des courbes elliptiques en dimension supérieure—l'unicité de telles métriques est loin d'être garanti. En fait, chaque variété de première classe de Chern nulle possède un nombre infini de "meilleures" métriques, une dans chaque classe de Kähler. Ce fait pose un problème, car avoir une unique "meilleure" métrique sur une variété est important pour assurer la lissité de la métrique induite et pour démontrer les bonnes propriétés fonctorielles de cette métrique.

Nannicini et Schumacher ont attaqué tous les deux ce problème peu après le travail de Siu [**Nan86, Sch85**] et ont trouvé que si l'on suppose que toutes les variétés dans la famille donnée partagent une même métrique kählerienne, c'est-à-dire une métrique dont la classe de cohomologie reste constante lorsque les paramètres complexes varie, alors on obtient une bonne métrique de Weil–Petersson. Le choix d'une telle métrique est appelé une polarisation de la famille. Cette approche présente deux problèmes.

Le premier n'est pas si grave. Pour obtenir une métrique de Weil–Petersson nous sommes conduits à choisir une polarisation, donc nous nous attendons à ce qu'un choix différent produise une métrique différente. Cela aura peu d'importance si les propriétés géométriques des métriques obtenues sont toutes plus ou moins les mêmes, ce qui s'avère être le cas. En fait, les différentes polarisations donnent toutes la même métrique, mais ceci est loin d'être évident du point de vue de la construction classique.

Le deuxième problème, plus sérieux, est qu'il existe des familles de variétés kähleriennes compactes de première classe de Chern nulle qui ne peuvent pas être polarisées. Désignons une famille de telles variétés par $\pi : \mathcal{X} \to S$ et supposons que ω est une polarisation de la famille, alors ω_s est une métrique kählerienne sur X_s pour tout $s \in S$. Si $\rho : T_{S,s} \to H^1(X_s, T_{X_s})$ est le morphisme de Kodaira–Spencer associé à la famille en un point s, nous avons nécessairement que $\rho(\xi) \cup \omega_s = 0$ pour tout champ de vecteurs $\xi \in T_{S,s}$, où $\cup$ est le cup-produit. Les déformations infinitésimales de la famille s'envoient donc dans un sous-espace propre de $H^1(X_s, T_{X_s})$. Mais les déformations des variétés kähleriennes à première classe de Chern nulle sont non-obstruées [**Tia87**]. Nous sommes donc bien capables de construire une famille $\pi : \mathcal{X} \to S$ dont le morphisme de Kodaira–Spencer est un bijection, et cette famille ne peut pas être polarisée.

Nous laissons au lecteur le soin de construire un exemple très explicite de ce phénomène : prenons pour $\mathcal{X}$ la famille tautologique de tores de dimension $n \geq 2$ paramétrés par les matrices complexes s carrées de taille n dont la partie imaginaire est inversible (ce condition sert à garantir que l'ensemble composé de la base canonique $(e_1, \ldots, e_n)$ de $\mathbb{C}^n$ et les images $(se_1, \ldots, se_n)$ soit un réseau dans $\mathbb{C}^n$). On démontre alors qu'étant donné une $(1,1)$-forme constante $\alpha \neq 0$ sur un tore X_s, il existe toujours une direction de déformation infinitésimale ξ dans laquelle α ne peut pas être déformée.

Ce problème semble n'avoir dérangé personne. Nous pouvons émettre une hypothèse expliquant pourquoi c'est le cas : les familles non-polarisées admettent trop d'automorphismes pour définir de bons espaces de modules, donc on n'est même pas sûr d'avoir un champ holomorphe après avoir quotienté par l'action de ces automorphismes [**KS58**]. En revanche, les familles polarisées admettent comme espaces de modules des espaces mieux compris, à savoir des orbifolds [**Sch84**].

La géométrie des modules kähleriens

La symétrie miroir prédit que les familles de variétés kähleriennes apparaissent sous forme de paires. Près des points dégénérés de leurs espaces de modules respectifs, nous nous attendons ansi à un échange de modules de structures complexes et kähleriennes. Puisque nous possédons une métrique de Weil–Petersson sur les modules de structures complexes il est naturel de chercher une métrique "miroir" sur les modules de structures kähleriennes.

Il semble que Wilson [**Wil04**] ait été l'un des premiers mathématiciens à prendre cette idée sérieusement en considération. Les classes de Kähler d'une variété compacte sont paramétrées par un ouvert de l'espace vectoriel $H^{1,1}(X,\mathbb{R})$. Étant donné une classe de Kähler ω, nous obtenons une forme bilinéaire sur cet espace en prenant le cup-produit de ses éléments là-dedans par une puissance convenable de la classe de Kähler. Cette forme est de signature mixte, mais nous pouvons obtenir un vrai produit scalaire à partir d'elle en lui rajoutant un facteur correctif. De manière équivalent, nous pouvons observer que la fonction $-\log \mathrm{Vol}$ est lisse sur le cône de Kähler, et que son Hessien est positif-défini par le théorème de Lefschetz difficile.

Ainsi nous obtenons une métrique riemannienne sur le cône de Kähler de X. Si nous le voulons, nous pouvons complexifier ce cône de Kähler et obtenir une métrique kählerienne sur sa complexification de la même manière. Cette métrique peut-être explicitée dans certains cas, et elle se révèle avoir des propriétés similaires à celles de la métrique de Weil–Petersson sur les modules de structures complexes. Wilson a pu donner plusieurs formules pour le tenseur de courbure de cette métrique dans le cas général, exprimées en termes du produit d'intersection sur l'anneau de cohomologie de X. Malheureusement ces formules contiennent des termes dont nous ne savons pas contrôler la positivité, ce qui nous conduit à des résultats moins forts que dans le cas des modules de structures complexes.

Résumé du texte

Chapitre 1. Le point de départ de cette thèse est une tentative d'éliminer l'hypothèse de polarisation pour une famille de variétés kähleriennes compactes de première classe de Chern nulle. Nous présentons ici une description détaillée du procédé que nous avons envisagé dans cet objectif.

Généralités sur des familles de variétés kähleriennes. Soit $\pi : \mathcal{X} \to S$ une famille de variétés kähleriennes compactes au-dessus d'une base lisse S. La suite exacte

$$0 \longrightarrow T_{\mathcal{X}/S} \longrightarrow T_{\mathcal{X}} \longrightarrow \pi^* T_S \longrightarrow 0$$

de fibrés vectoriels au-dessus de $\mathcal{X}$ donne lieu aux faisceaux analytiques cohérents $\mathcal{R}^q\pi_*\Omega^p_{\mathcal{X}/S}$ au-dessus de S. Sous nos hypothèses, ces faisceaux sont des vrais fibrés vectoriels holomorphes, notés $E^{p,q}$ dont les fibrés s'identifient à

$$E^{p,q}_s = \mathcal{R}^q\pi_*\Omega^p_{\mathcal{X}/S,s} = H^q(X_s, \Omega^p_{X_s}) = H^{p,q}(X_s, \mathbb{C})$$

par le théorème de Dolbeault.

Nous pouvons aussi considérer le système local $\mathcal{R}^k\pi_*\mathbb{Z}$ de faisceaux au-dessus de S, dont les fibrés sont les groupes $\mathcal{R}^k\pi_*\mathbb{Z}_s = H^k(X_s, \mathbb{Z})$, tous isomorphes entre eux pour un k fixé. En tensorisant par $\mathcal{O}_S$ nous obtenons un fibré vectoriel holomorphe E. Ce fibré est muni d'une connexion plate ∇_{GM}, appelée la connexion de Gauss–Manin.

Nos variétés étant kähleriennes, chaque fibré $E^{p,q}$ peut-être interprété comme étant un sous-fibré vectoriel de E^{p+q}. Sur chaque fibré, ceci provient simplement de l'injection canonique

$$H^{p,q}(X_s, \mathbb{C}) \hookrightarrow H^{p+q}(X_s, \mathbb{C}).$$

Malheureusement, en tant que sous-fibré, $E^{p,q}$ est seulement réel analytique en général. Cépendant, nous pouvons définir une filtration décroissante de E^k par

$$F^pE^k = \bigoplus_{q\geq p} E^{q,k-q}$$

et le sous-fibré vectoriel $F^pE^k \subset E^k$ est alors holomorphe d'après un théorème de Griffiths. Notons que le fibré quotient $F^pE^{p+q}/F^{p+1}E^{p+q}$ s'identifie canoniquement au fibré holomorphe $E^{p,q}$.

Ceci nous donne une connexion ∇ sur $E^{p,q}$. Elle se définit en suivant simplement les flèches

$$E^{p,q} \hookrightarrow E^{p+q} \xrightarrow{\nabla_{GM}} E^{p+q} \otimes \Omega^1_S \longrightarrow E^{p,q} \otimes \Omega^1_S,$$

où la derniere flèche est obtenue à partir de la projection réelle analytique $E^{p+q} \to E^{p,q}$ définie par la décomposition de Hodge. Cette connexion a une courbure non-triviale en général. Nous rappellons au lecteur qu'une connexion sur un fibré vectoriel $p : E \to S$ définit un relèvement lisse des fibrés tangents $p^*T_S \to T_E$; voir [**KN96**].

Nous allons maintenent définir le cône de Kähler relatif de la famille $\pi : \mathcal{X} \to S$. Soit u une classe de type $(1,1)$ sur une variété X_s. Nous appellons u une *classe de Kähler complexifiée* si sa partie imaginaire $\operatorname{Im} u$ est une classe de Kähler, c'est-à-dire si elle contient une métrique kählerienne. L'ensemble des classes de Kähler complexes sur X_s sera noté $K(X_s)$. Le *cône de Kähler complexifié relatif* de la famille $\pi : \mathcal{X} \to S$ est alors le sous-ensemble $\mathcal{K} \subset E^{1,1}$ composé des cônes de Kähler complexifiés de chaque variété X_s.

Rappelons que Kodaira et Spencer ont démontré que les petites déformations d'une variété kählerienne sont encore kähleriennes [**KS60**]. Nous pouvons modifier aisément la preuve de leur théorème pour voir que le cône de Kähler complexifié relatif est ouvert dans l'espace total du fibré holomorphe $E^{1,1}$. Il s'agit donc d'une variété complexe munie d'un submersion holomorphe $p : \mathcal{K} \to S$.

Variétés dont la première classe de Chern est nulle. Nous supposons désormais que nos variétés sont de première classe de Chern nulle. Dans ce cadre nous profitons d'une version du théorème de Aubin–Calabi–Yau [**Yau78**], qui nous dit que chaque classe de Kähler $[\omega]$ contient une unique métrique kählerienne ω dont la courbure de Ricci est nulle. Ceci nous permet de définir une métrique hermitienne sur $\mathcal{K}$. Notons que la submersion $p : \mathcal{K} \to S$ nous donne une suite exacte courte

$$0 \longrightarrow T_{\mathcal{K}/S} \longrightarrow T_{\mathcal{K}} \xrightarrow{p_*} p^*T_S \longrightarrow 0$$

de fibrés vectoriels au-dessus de $\mathcal{K}$.

Nous définissons d'abord une métrique sur le sous-fibré $T_{\mathcal{K}/S} \subset T_{\mathcal{K}}$. Soit (s, a) un point de $\mathcal{K}$. Nous notons Ω_a la métrique Ricci-plate dans $\omega = \operatorname{Im} a$. Étant donné deux sections locales u et v de $T_{\mathcal{K}/S}$ nous posons

$$g_{\mathcal{K}/S}(u, \overline{v}) = \frac{1}{\operatorname{Vol}(X_s, \Omega_a)} \int\limits_{X_s} \langle U, \overline{V}; \Omega_a \rangle \, dV_{\Omega_a},$$

où U et V sont les représentants Ω_a-harmoniques de u et v. La forme hermitienne définie par cette expression, qui n'est rien d'autre que le produit scalaire L^2 de Hodge sur $H^{1,1}(X_s, \mathbb{C})$, est positive-définie, comme on le voit aisément par inspection directe. Elle est aussi lisse car les métriques Ω_a sont des solutions d'une équation de Monge–Ampère, qui est un équation aux dérivées partielles non-linéaire elliptique, et qui varient donc de manière lisse avec a.

Nous passons ensuite au fibré p^*T_S au-dessus de $\mathcal{K}$. Le morphisme de Kodaira–Spencer de la famille est un morphisme

$$\rho : T_S \to \mathcal{R}^1\pi_* T_{\mathcal{X}/S}$$

de fibrés vectoriels. Le fibré $p^*R^1\pi_*T_{\mathcal{X}/S}$ est muni d'une métrique hermitienne L^2 ; si u et v sont des sections locales de ce fibré sur un voisinage de (s, a) nous avons

$$\langle u, \overline{v} \rangle = \frac{1}{\operatorname{Vol}(X_s, \Omega_a)} \int\limits_{X_s} \langle U, \overline{V}; \Omega_a \rangle \, dV_{\Omega_a}$$

avec des notations similaires à celles qu'on a déjà utilisées. Nous obtenons maintenant une forme hermitienne g_S sur p^*T_S en tirant cette métrique L^2 en arrière par le morphisme de Kodaira–Spencer. La forme g_S peut être dégénérée, mais elle est une vraie métrique si le morphisme de Kodaira–Spencer est injectif, ce qui revient à demander que la famille $\pi : \mathcal{X} \to S$ soit effective.

La connexion ∇ sur $E^{1,1}$ nous fournit maintenant un scindage lisse

$$T_{\mathcal{K}} \cong T_{\mathcal{K}/S} \oplus p^*T_S$$

qui nous permet de recoller les métriques $g_{\mathcal{X}/S}$ et g_s et d'obtenir une métrique hermitienne h sur $\mathcal{K}$. (En réalité h est une vraie métrique seulement lorsque la famille est effective, mais ce fait ne sera pas important.)

Nous mentionnons brièvement comment obtenir une métrique hermitienne sur le produit fibré $\mathcal{X} \times_S \mathcal{K}$. Dans ce cas nous avons une suite exacte

$$0 \longrightarrow p_{\mathcal{X}}^*T_{\mathcal{X}/S} \oplus p_{\mathcal{K}}^*T_{\mathcal{K}/S} \longrightarrow T_{\mathcal{X}\times_S\mathcal{K}} \longrightarrow \nu^*T_S \longrightarrow 0$$

de fibrés vectoriels au-dessus du produit fibré. Nous laissons au lecteur le soin de produire des métriques hermitiennes sur les facteurs de cette suite situés aux extrémités. Celà achèvé, il ne reste plus qu'à trouver un scindage de la suite.

Nous renvoyons au chapitre 1 pour tous les détails, mais disons pour simplifier qu'il suffit de faire scinder la suite

$$0 \longrightarrow T_{\mathcal{X}/S} \longrightarrow T_{\mathcal{X}} \longrightarrow \pi^* T_S \longrightarrow 0,$$

une fois qu'elle a été tirée en arrière sur le produit fibré. Ceci peut être fait à l'aide des relèvements harmoniques définis par Siu [**Siu86**] : Soit Ω_a la métrique Ricci-plate associée à un point (s, a) comme ci-dessus, et soit ξ une section locale de $\pi^* T_S$. Il existe alors un unique[3] champ de vecteurs lisse Ξ sur $\mathcal{X}$ tel que $\bar\partial \Xi_{|X_s}$ soit le représentant Ω_a-harmonique de $\rho(\xi)_s$. L'association $\xi \to \Xi$ définit un relèvement lisse du fibré tangent de S dans $T_{\mathcal{X}}$, ce qui nous donne le scindage cherché.

Nous récupérons ainsi une forme hermitienne sur le produit fibré $\mathcal{X} \times_S \mathcal{K}$, qui produit une vraie métrique lorsque la famille en question est effective.

Les deux métriques, sur $\mathcal{K}$ et $\mathcal{X} \times_S \mathcal{K}$, passent aux changements de base $f : B \to S$ comme décrit par le diagramme suitant :

$$\begin{tikzcd}
f^*\mathcal{X} \times_S \mathcal{K}_S \arrow[rr] \arrow[dr] & & \mathcal{X} \times_S \mathcal{K}_S \arrow[dr] & \\
& f^*\mathcal{X} \arrow[rr] \arrow[d] & & \mathcal{X} \arrow[d] \\
& B \arrow[rr] & & S
\end{tikzcd}$$

Elles définissent alors des métriques hermitiennes sur les objets correspondants au-dessus de n'importe quel espace de modules de variétés kähleriennes compactes à première classe de Chern nulle.

Ces métriques généralisent bien les objets déjà connus : on vérifie sans difficultés qu'une polarisation $s \mapsto \omega_s$ de la famille $\pi : \mathcal{X} \to S$ donne lieu à une section holomorphe $\sigma : S \to \mathcal{K}$, $s \mapsto [\omega_s]$, qui est de plus parallèle par rapport à la connexion sur $\mathcal{K}$. Le tiré en arrière $\sigma^* h$ de la métrique sur $\mathcal{K}$ est alors la métrique de Weil–Petersson classique définie par la polarisation ω.

Chapitre 2. Comme le lecteur a pu le constater, les métriques que nous considérons dans cette thèse sont des objets compliqués. Il peut donc s'avérer être une bonne idée de considérer une situation simplifiée pour se familiariser avec elles. Dans le chapitre 2 nous procédons exactement ainsi et nous nous plaçons dans le cas d'une famille au-dessus d'un point, ou, autrement dit, nous regardons notre métrique sur le cône de Kähler d'une variété X fixée.

Vu que la variété X est fixée, et que nous n'allons pas avoir besoin de son fibré canonique, nous désignons par K le cône de Kähler de X. Il est

[3]Grosso modo.

muni d'une fonction lisse

$$\mathrm{Vol} : K \longrightarrow \mathbb{R}_+, \quad \omega \longmapsto \int_X \frac{\omega^n}{n!}$$

qui associe à une classe de Kähler ω son volume. Cette fonction a le bon goût de définir le potentiel d'une métrique riemannienne sur K.

Proposition. *La fonction $\varphi = -\log \mathit{Vol}$ est strictement convexe et définit une métrique riemannienne g sur K. Si Ω est une métrique kählerienne dans la classe ω et U et V sont les représentants harmoniques de deux classes u et v, alors*

$$g(u,v)(\omega) = \frac{1}{\mathit{Vol}(X,\Omega)} \int_X \langle U, V\rangle \, dV_\Omega,$$

où le produit scalaire est celui induit par Ω sur les $(1,1)$-formes lisses sur X.

Esquisse de preuve: En dérivant la fonction φ dans la direction des vecteurs u et v nous voyons que

$$g(u,v) = \frac{1}{\mathrm{Vol}(X,\omega)} \int_X u \wedge \frac{\omega^{n-1}}{(n-1)!} \frac{1}{\mathrm{Vol}(X,\omega)} \int_X v \wedge \frac{\omega^{n-1}}{(n-1)!} - \frac{1}{\mathrm{Vol}(X,\omega)} \int_X u \wedge v \wedge \frac{\omega^{n-2}}{(n-2)!}.$$

Nous écrivons maintenant $u = u_0\omega + u_1$ et $v = v_0\omega + v_1$, où u_0, v_0 sont des scalaires et u_1, v_1 sont des classes primitives par rapport à ω. En substituant ces classes dans l'expression ci-dessus et appliquant le théorème de Lefschetz difficile nous voyons que la forme g est positive-définie. Elle définit donc bien une métrique riemannienne sur K.

Pour la deuxième partie, nous écrivons encore $U = U_0\Omega + U_1$ et $V = V_0\Omega + V_1$. C'est à nouveau le théorème de Lefschetz difficile qui nous dit alors que $g(u,v)_\omega$ coïncide avec le produit scalaire L^2 de Hodge $\langle\langle U, V\rangle\rangle$ défini par Ω. □

Dans le chapitre 1 nous avons considéré une métrique hermitienne définie sur le cône de Kähler complexifié de X. Ici nous pouvons bien rejouer le même jeu et regarder la métrique kählerienne définie par le potentiel φ sur ce cône. Le même argument que ci-dessus montre alors qu'il s'agit bien de la métrique L^2 considérée dans le chapitre 1. Cet observation montre à la fois le fait non-trivial que cette métrique L^2 est kählerienne et qu'elle peut être définie sur le cône de Kähler de n'importe quelle variété kählerienne compacte, et non seulement les variétés à première classe de Chern nulle.

Notons aussi que l'expression de $g(u,v)$ à partir du produit d'intersection sur X suggère que la métrique pourrait provenir d'une décomposition de K comme le produit de deux variétés. C'est bien le cas ; si $\lambda > 0$ est un réel nous pouvons considérer le difféomorphisme

$$\mathbb{R}_+ \times K_\lambda \longrightarrow K \quad (t,\omega) \mapsto t\omega.$$

Le tiré en arrière de g sur le produit $\mathbb{R}_+ \times K_\lambda$ par ce difféomorphisme, à multiplication par une constante près, est alors le produit de la métrique standard complète sur $\mathbb{R}_+$ par la métrique définie par l'intégration de deux classes primitives contre une puissance d'une classe de Kähler.

Cette remarque montre que la géométrie intéressante du cône de Kähler est concentrée dans les ensembles K_λ. Nous imaginons que c'est pour cette raison que les autres chercheurs qui ont considéré cette métrique ont choisi de travailler avec les classes primitives. Nous allons travailler sur le cône entier, mais ce n'est qu'une question de goût.

Remarque — Observons qu'il est, bien sûr, entièrement possible d'exprimer le tenseur de courbure de la métrique g à l'aide du produit d'intersection sur l'anneau de cohomologie de la variété X. C'est exactement ce qu'a fait Wilson [**Wil04**]. Cependant, cette approche pose des problèmes difficiles car Wilson fait intervenir des termes dont nous ne comprenons pas la positivité.

Nous notons aussi que sur les courbes la question est triviale, car on obtient à chaque fois la métrique standard complète sur $\mathbb{R}_+^*$. Le cas des surfaces compactes peut aussi être totalement résolu ; un calcul brutal montre que la métrique g est à courbure sectionnelle constante, son tenseur de courbure est

$$R(x,y,z,w) = -(g(x,z)g(y,w) - g(x,w)g(y,z)),$$

ce qui rend sa courbure sectionnelle négative.

Cela se montre en choisissant un point ω de $K(X)$, puis on considère le voisinage affine centré en ω défini par une base $(u_1, \ldots, u_{N-1}, \omega)$, où les classes u_j sont primitives par rapport à ω. Un point de ce voisinage est alors donné par les coordonnées $(t_1, \ldots, t_N) \mapsto \sum t_j u_j + t_N \omega$.

Nous définissons ensuite les champs de vecteurs

$$v_j(t) = u_j + \lambda_j(t)\omega, \quad j = 1, \ldots, N-1, \quad v_N(t) = (1 + \lambda_N(t))\omega,$$

où les fonctions lisses λ_j (qui se calculent explicitement) sont choisis pour rendre la classe $v_j(t)$ primitive par rapport à classe de Kähler $\omega_t := (1 + t_N)\omega + \sum t_j u_j$. Sur un petit voisinage centré en ω, ces champs de vecteurs engendrent l'espace tangent à $K(X)$.

L'étape suivante de notre approche n'est pas très élégante, mais elle a le mérite d'aboutir. En inversant les séries convergentes en t et approximant le tout par des polynômes d'ordre trois—ceci dans le but de calculer ce qu'il faut pour appliquer la formule de Koszul et approximer la connexion de Levi-Civita à l'ordre nécessaire—on parvient à calculer le tenseur de courbure de g en ω.

Cette méthode n'est pas très utile en dimension supérieure, car elle repose sur le fait que le cup produit des surfaces ne fait intervenir ici que l'intersection de deux classes, ce qui rend le calcul faisable. Il est probable que des efforts supplémentaires permettraient sans doute de retrouver les formules de Wilson concernées pour le tenseur de courbure en dimension supérieure.

Je vais maintenant essayer de résumer le reste du chapitre 2. Pour cela il sera nécessaire d'introduire un certain nombre de notions techniques et de notations. L'outil principal dont nous nous servons est le théorème de Aubin–Calabi–Yau.

Théorème (Aubin–Calabi–Yau)**.** *Soit X une variété kählerienne compacte et soit ρ une forme représentant la classe $c_1(X)$. Dans chaque classe de Kähler $[\omega]$ sur X il existe une unique métrique kählerienne ω tel que $Ric\,\omega = \rho$.*

Pour nous, ce théorème peut être décrit comme suit : notons par $\mathcal{K}(X)$ l'ensemble de toutes les métriques kähleriennes sur X. C'est un cône dans un espace vectoriel de dimension infinie. Le passage d'une forme fermée à sa classe de cohomologie induit un morphisme $p : \mathcal{K}(X) \to K(X)$. Le théorème de Aubin–Calabi–Yau affirme alors l'existence d'un morphisme injectif $j : K(X) \to \mathcal{K}(X)$ tel que $p \circ j = \mathrm{id}_{K(X)}$, au moins une fois qu'on a fixé une forme ρ dans $c_1(X)$.

Autrement dit, le cône de Kähler peut-être plongé dans l'espace de toutes métriques kähleriennes (ou même hermitiennes) sur X. Ce qui rend cette interprétation intéressante est que ce plongement est riemannien. En effet, l'espace des métriques kähleriennes (ou hermitiennes) est naturellement muni de la métrique L^2 de Hodge, ce qui lui donne la structure d'une variété riemannienne de dimension infinie. La proposition ci-dessus nous dit maintenant que le plongement considéré est riemannien.

La voie est alors claire. Au lieu d'avoir à nous occuper du produit d'intersection sur l'anneau de cohomologie de X, nous allons calculer le tenseur de courbure de la métrique induite par le plongement, en espérant que le résultat de ces calculs sera plus facilement manipulable que par une approche purement cohomologique.

C'est du moins ce que nous pensions avant de nous lancer dans les calculs. Mais il est apparu que la situation n'était pas si simple, et il y a de bonnes et de mauvaises nouvelles. Les bonnes d'abord : le tenseur de courbure de la métrique L^2 sur l'espace de métriques hermitiennes sur X est totalement explicitable.

Théorème. *Soit $\mathcal{M}$ l'espace de métriques hermitiennes sur X. Le tenseur de courbure de la métrique L^2, notée G, sur $\mathcal{M}$ est*

$$R(U, V, Z, W) = \tfrac{1}{4} G(\{Z, W\}, \{U, V\})$$

et sa courbure sectionnelle est négative ou nulle.

Ici $\{U, V\}$ désigne le commutateur sur l'espace d'endomorphismes du fibré tangent de X, transporté sur l'espace des $(1,1)$-formes à l'aide de l'isomorphisme $T_X \to \overline{T}_X^*$ fourni par la métrique Ω en chaque point de $\mathcal{M}$. Ce qu'il y a à retenir ici est que ce tenseur de courbure a la même forme qu'un tenseur de courbure d'un espace symétrique.

Arrivé à ce point, il ne reste plus qu'à calculer la contribution de la seconde forme fondamentale du plongement $K(X) \to \mathcal{M}$ pour conclure l'affaire. C'est ici que la nuit tombe !

D'abord, il aurait été vraiment plus facile de conclure si l'espace de métriques kähleriennes avait été totalement géodésique dans l'espace de métriques hermitiennes, car alors on démontre facilement que la courbure sectionnelle du tenseur cherché est semi-négative. Malheureusement ce n'est pas le cas. On est donc obligé de travailler dans l'espace de toutes métriques hermitiennes.

Ceci ne sera cependant pas si grave si on parvient à expliciter la seconde forme fondamentale de $\mathcal{K}$ dans cet espace. En effet, on peut y parvenir. Cette forme est donnée par

$$\mathrm{II}(U, V) = \Delta Gr \nabla_V U,$$

et les seules choses qu'il faut savoir sont que $\nabla_V U$ est une $(1,1)$-forme lisse à peu près arbitraire sur X, que Δ est le Laplacien associé à une métrique kählerienne et que Gr est son opérateur de Green.

Pour dire quoi que ce soit sur la positivité de la courbure sectionnelle de notre métrique, il faudrait contrôler la contribution de cette seconde forme fondamentale à la courbure. Pour l'instant nous n'y sommes pas arrivé. Nous avons essayé de faire intervenir une formule de Bochner–Weitzenböck, mais celle-ci ne donne pas d'information utile. Pour aller plus loin il faudra probablement mettre en oeuvre des techniques d'analyse. Ce qui complique leur application est le terme $\nabla_V U$, qui mélange les formes U et V d'une façon qui rend difficile toute tentative d'en extraire d'information à partir de ce que nous savons de U et V. Il n'est pas impossible qu'on arrive à comprendre davantage la positivité de cette courbure, mais il faudra des efforts beaucoup plus sophistiqués que ceux que nous avons été capable de mener pour l'instant.

Dans la toute dernière partie de ce chapitre nous discutons les propriétés faibles de fonctorialité de cette métrique et de sa complétude.

D'abord, il apparaît que cette métrique n'est pas très stable via le morphisme induit par pullback selon un morphisme $f : X \to Y$, en premier lieu parce que le cône de Kähler ne l'est pas. Actuellement, nous ne possédons pas de description générale des morphismes qui préservent les cônes de Kähler des variétés, pour autant qu'il en existe une. Nous nous contentons alors de montrer que notre métrique est stable via le pullback par un morphisme fini entre variétés kähleriennes compactes.

Ensuite, il s'avère que notre métrique n'est quasiment jamais complète. Pour que ce soit le cas, il faut et il suffit que le cône de Kähler coïncide avec une composante connexe du cône défini par les classes de type $(1,1)$ à volume positif. En général ce deuxième cône est beaucoup plus grand que le cône de Kähler. En effet, dans le cas où il est possible de se déplacer dans le cône de Kähler et d'atteindre son bord en “écrasant” le volume d'une sous-variété propre, sans pour autant annuler le volume de la variété ambiante, alors on aboutit à une métrique qui n'est pas complète. Cette situation se présente très souvent : pour la produire il suffit de prendre n'importe quelle variété kählerienne compacte et d'éclater un point dans celle-ci.

Chapitre 3. Dans ce chapitre nous reprenons à peu de choses près le cadre général de notre problème. Rappelons que si X est une variété kählerienne compacte dont la première classe de Chern est nulle, alors il existe une autre telle variété X', dont le fibré canonique est trivial, et un revêtement fini $X' \to X$. Les variétés compactes kähleriennes à fibré canonique triviale forment trois classes :

(1) Les tores complexes, ou les groupes de Lie analytiques et compactes.
(2) Les variétés hyperkähleriennes. Ces sont les variétés compactes kähleriennes X simplement connexes telles que $H^0(X, \Omega^{2p}) = \mathbb{C}\sigma$ et $H^0(X, \Omega^{2p+1}) = 0$, où σ est une $(2,0)$-forme holomorphe non-nulle.
(3) Les variétés de Calabi–Yau. Ces sont les variétés compactes kähleriennes X simplement connexes telles que $H^0(X, \Omega^p) = 0$ pour $0 < p < n = \dim_{\mathbb{C}} X$.

L'intersection entre les classes (2) et (3) est composé des surfaces K3. Parmi ces variétés, les tores sont bien sûr les mieux comprises. Une belle théorie de variétés hyperkähleriennes commence à voir le jour, grâce au fait que ces

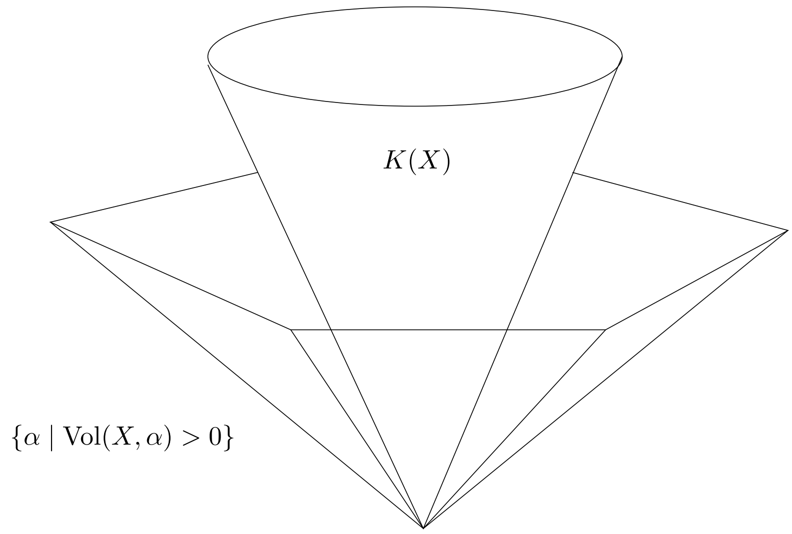

FIGURE 1. Le cône de Kähler et le cône de classes à volume positif.

variétés portent une structure rigide, mais nous ne possédons toujours pas de nombreux exemples de ces variétés. Enfin nous avons des milliers d'exemples de variétés Calabi–Yau, mais elles ne semblent pas d'avoir beaucoup en commun entre elles.

Nous travaillerons donc ici sous l'hypothèse que nos variétés sont du fibré canonique trivial, ce qui ne différe de la situation générale que par passage à un quotient. L'intérêt de cette hypothèse est que si $\pi : \mathcal{X} \to S$ est une famille de telles variétés, alors l'image directe $\mathcal{R}^0\pi_* K_{\mathcal{X}/S}$ du faisceau canonique relatif est non-trivial. En général, ce n'est pas le cas ; par exemple il est facile de produire un exemple d'un quotient libre d'un tore de dimension ≥ 2 dont le fibré canonique n'admet pas de section.

Le but de la première partie de ce chapitre est de jouer le même jeu que dans le chapitre précédent, c'est-à-dire de trouver une expression cohomologique de la métrique L^2 sur le fibré $\pi^* T_S$.

Pour cela, nous avons d'abord besoin d'effectuer un certain nombre de calculs d'algèbre linéaire pour relier les propriétés des métriques Ricci-plates à celles des formes de volumes holomorphes. La situation ponctuelle (qui revient à l'algèbre linéaire) est simple : Si ω est une $(1,1)$-forme non-dégénérée sur un espace vectoriel T de dimension n, et si Ω est une $(n,0)$-forme non-nulle, alors les deux sont réliées par une formule

$$\frac{\omega^n}{n!} = c\, i^{n^2} \Omega \wedge \overline{\Omega},$$

où c est une constante réelle. Dans le cas d'une variété kählerienne compacte X à fibré canonique triviale, une telle identité est vérifiée globalement sur X, avec une constante réelle c, si nous prenons pour ω une métrique Ricci-plate et pour Ω une $(n,0)$-forme holomorphe qui trivialise le fibré canonique.

Une fois que c'est fait, nous observons que l'image directe $L := \mathcal{R}^0\pi_* K_{\mathcal{X}/S}$ du fibré canonique relatif est naturellement muni d'une forme hermitienne, qui est une vraie métrique dès que la famille $\pi : \mathcal{X} \to S$ est effective. La forme de courbure de cette métrique est donnée par

$$\frac{i}{2\pi}\Theta_{L,h} = -i\partial\bar{\partial}\log\int_{X_s} i^{n^2}\sigma\wedge\overline{\sigma},$$

où σ est une section locale non-nulle de L. Au moyen d'un calcul un peu plus poussé on montre que cette forme s'écrit comme

$$\frac{i}{2\pi}\Theta_{L,h}(\xi,\overline{\eta}) = \frac{-i^{n^2}}{\mathrm{Vol}(X,\sigma)}\int_{X_s}\rho(\xi)\cup\sigma\wedge\overline{\rho(\eta)\cup\sigma},$$

où ρ est le morphisme de Kodaira–Spencer de la famille en question, et où ξ et η sont des champs de vecteurs locaux sur S.

Cette égalité nous sert à faire le lien avec la métrique L^2 qui a été construite dans le chapitre 1. On montre alors :

Théorème. *La métrique h_{WP} de Weil–Petersson sur p^*T_S et le pullback de (L,h) à p^*T_S satisfont l'égalité*

$$2^n h_{WP} = p^*\frac{i}{2\pi}\Theta_{L,h}.$$

Ce théorème montre deux choses. D'abord, la métrique h_{WP} que nous avons construite à partir des métriques Ricci-plates ne dépend pas du tout des métriques choisies. Ceci n'est absolument pas clair à partir de sa construction. Comme c'est le cas, nous disposons donc d'une métrique de Weil–Petersson sur la base de déformations de n'importe quelle variété kählerienne compacte, dont la première classe de Chern est nulle, sans avoir fait appel à une polarisation.

Ensuite, le théorème nous dit que la métrique h sur l'image directe L est positive si la famille $\pi : \mathcal{X} \to S$ est effective, car nous savons déjà que la métrique de Weil–Petersson est positive dans ce cas. De ce fait, le fibré $L \to S$ est positif.

Nous nous appuyons maintenant sur des résultats de Wang [**Wan03**]. Ce dernier a explicité le tenseur de courbure de la métrique de Weil–Petersson associée aux variétés à fibré canonique trivial dans le cas polarisé. En lisant sa preuve on se rend compte qu'il n'a pas besoin de la polarisation pour effectuer ses calculs, donc Wang calcule en réalité le tenseur de courbure de notre métrique dans le cas général. Ceci donne le :

Théorème. *Le tenseur de courbure de la métrique kählerienne $h = \frac{i}{2\pi}\Theta_{L,h}$ est*

$$\begin{aligned} R(\xi,\overline{\eta},\nu,\overline{\zeta}) = -\big(h(\xi,\overline{\eta})\,h(\nu,\overline{\zeta}) + h(\xi,\overline{\zeta})\,h(\eta,\overline{\nu})\big) \\ + \frac{i^{n^2}}{Vol(X_s,\Omega)}\int_{X_s}\rho(\xi)\cup\rho(\nu)\cup\Omega\wedge\overline{\rho(\eta)\cup\rho(\zeta)\cup\Omega}, \end{aligned}$$

où ρ est le morphisme de Kodaira–Spencer de la famille $\pi : \mathcal{X} \to S$.

À partir de cette information on conclut comme à l'habitude que la métrique de Weil–Petersson satisfait des propriétés de négativité.

Pour terminer le chapitre nous montrons comment obtenir la métrique sur le fibré tangent relatif $T_{\mathcal{K}/S}$ du cône de Kähler relatif $p : \mathcal{K} \to S$ à partir d'un fibré en droites. Cette observation est valable pour n'importe quelle famille de variétés lisses $\pi : \mathcal{X} \to S$.

En effet, une telle famille est munie du fibré en droites holomorphe $E := \mathcal{R}^{2n}\pi_*\mathbb{C} \otimes \mathcal{O}_S = \mathcal{R}^n\pi_*\Omega^n_{\mathcal{X}/S}$. Le pullback p^*E de ce fibré sur l'espace total de $\mathcal{K}$ est muni d'une métrique hermitienne ; si α et β sont des sections holomorphes locales de p^*E, on écrit $\alpha = a(\omega^n/n!)/\operatorname{Vol}(X,\omega)$ et $\beta = b(\omega^n/n!)/\operatorname{Vol}(X,\omega)$, où a et b sont des fonctions holomorphes. Alors

$$g(\alpha, \overline{\beta})_{(\omega,s)} := a(s)\overline{b}(s)\operatorname{Vol}(X,\omega).$$

On voit maintenant facilement que la forme de courbure de cette métrique est

$$\frac{i}{2\pi}\Theta_{E,g} = i\partial\bar\partial \log \operatorname{Vol}(X,\omega),$$

et d'après les résultats dans le chapitre 2 on sait que l'expression à droite coïncide, à une signe près, avec la métrique L^2 sur le cône de Kähler relatif. Si on savait que cette forme s'annulait sur les vecteurs tangets horisontaux, on pourrait conclure qu'elle était la même que la métrique naturelle L^2, ce qui entrenerait que la métrique L^2 sur $\mathcal{K}$ serait kählerienne. Pour l'instant ne nous savons pas si c'est le cas. Nous posons donc deux questions :

(1) Est-ce que la métrique naturelle L^2 sur $\mathcal{K}$ est kählerienne ?
(2) Est-ce que le fibré en droites $E \to \mathcal{K}$ est semipositif ?

Chapitre 4. Le dernier chapitre de cette thèse est consacré à l'exposé de quelques applications et exemples.

Abondance. Si $L \to X$ est un fibré en droites sur une variété complexe compacte, nous pouvons définir deux notions de "grosseur" L. D'abord, nous avons la dimension de Kodaira,

$$\kappa(L) = \limsup_{m\to+\infty} \frac{\log \dim_{\mathbb{C}} H^0(X, mL)}{\log m},$$

qui mesure moralement les dimensions des images des morphismes $X \to \mathbb{P}^N$ définis par les sections globales des puissances du fibré L. Nous avons aussi la notion de sa dimension numérique,

$$nd(L) = \sup\{k \geq 0 \mid c_1(L)^k \neq 0\}.$$

La conjecture d'abondance affirme que si X est projective (ou même kählerienne), alors $\kappa(K_X) = nd(K_X)$. Cet énoncé a bien du sens sur les variétés complexes quelconques, mais il n'est malheureusement pas vrai dans ce cadre plus général.

Ce fait est connu depuis les années 70 [**Uen75**]. Nous présentons ici quelques exemples qui montrent à nouveau l'échec de cette conjecture dans ce cadre général. Ces exemples ressemblent à ceux qu'on trouve dans [**Uen75**], mais nous les avons retrouvés indépendamment, faute d'avoir repéré à temps le travail de Ueno quand nous les avons construits.

Le point de départ de notre construction est l'observation classique que si X est une variété kählerienne compacte dont la première classe de Chern est nulle, et si f est un automorphisme de X qui fixe une classe de Kähler $[\omega]$, alors l'ordre de f est fini. La raison est que f fixe alors forcément la métrique Ricci-plate dans $[\omega]$, et que le groupe d'isométries de telles métriques est fini d'après les résultats généraux de la géométrie riemannienne.

Nous choisissons alors une surface K3 M qui admet un automorphisme f d'ordre infini. Soit V un espace vectoriel complexe de dimension finie, et soit Γ un réseau dans V. Le réseau Γ agit alors sur le produit $M \times V$; un générateur γ de Γ agit comme

$$\gamma \cdot (x, v) = (f(x), v + \gamma),$$

et l'action d'un élément général de Γ se déduit de celles des générateurs. Cet action est libre et sans point fixe, donc le quotient $X = (M \times V)/\Gamma$ est une variété complexe compacte.

Proposition. *La variété X ainsi obtenue est compacte et non-Kählerienne. Sa première classe de Chern réelle est nulle, alors que son dimension de Kodaira et soit nulle soit négative.*

Esquisse de preuve: La variété X ne peut pas être kählerienne, car une métrique kählerienne sur X nous donnerait une classe de Kähler sur M invariante sous l'action de f, ce qui est impossible.

On observe maintenant que $c_1(K_X) = 0$. En fait, on peut montrer beaucoup plus, car il est possible de produire une métrique hermitienne plate sur le fibré canonique K_X. Cette métrique se construit à partir de la forme de volume d'une métrique Ricci-plate sur M (qui est nécessairement fixée par l'automorphisme f, même si la métrique Ricci-plate ne l'est pas) et d'une forme de volume constante sur le tore V/Γ.

Mais par ailleurs, la dimension de Kodaira se relève être une notion plus délicate. Soit σ une forme volume holomorphe sur la surface K3 M. Nous avons nécessairement $f^*\sigma = \lambda\sigma$, où λ est un nombre complexe tel que $|\lambda| = 1$. Ce nombre λ peut être une racine d'unité. S'il l'est, la dimension de Kodaira de X est zéro, s'il ne l'est pas la dimension de Kodaira de X est négative. $\square$

Celà implique alors l'échec de la conjecture d'abondance dans le cas non-kählerien.

Courbes elliptiques. Soit $\mathbb{H}$ le demi-plan de Poincaré. Chaque élément τ de $\mathbb{H}$ définit un tore complexe $X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, ce qui nous donne une famille $\pi : \mathcal{X} \to \mathbb{H}$ de courbes elliptiques. Le cône de Kähler complexifié relatif $p : \mathcal{K} \to \mathbb{H}$ s'identifie à une fibration en demi-plans de Poincaré, et on peut décrire le produit fibré $\mathcal{X} \times_{\mathbb{H}} \mathcal{K}$ comme le quotient de l'espace $\mathbb{C} \times \mathbb{H} \times \mathbb{H}$ par l'action du groupe

$$\tilde{G} = \{g_{n,m} \mid g_{n,m}(z, a, s) = (z + n + ms, a, s)\} \cong \mathbb{Z}^2.$$

D'après les résultats du chapitre 3 nous savons que l'espace total $\mathcal{K}$ est une variété kählerienne. En général nous ne nous attendons pas à ce que le même résultat soit vrai pour le produit fibré $\mathcal{X} \times_{\mathbb{H}} \mathcal{K}$, mais ici c'est le cas. La raison est que toute famille de courbes elliptiques est automatiquement polarisée ; l'unique classe de Kähler de volume 1 fournit une section holomorphe et plate du fibré $\mathcal{R}^2\pi_*\mathbb{C} \otimes \mathcal{O}_S$. On en déduit que la métrique naturelle sur le produit fibré $\mathcal{X} \times_{\mathbb{H}} \mathcal{K}$ est induite par une métrique hermitienne sur un fibré en droites.

Nous sortons maintenant de notre chapeau un difféomorphisme

$$\varphi : \mathcal{X} \times_{\mathbb{H}} \mathcal{K} \to \mathcal{X} \times_{\mathbb{H}} \mathcal{K}, \quad \varphi(z, a, s) = (L(a, s)(z), s, a),$$

où

$$L(a, s)(z) = \frac{a - \bar{s}}{s - \bar{s}} z + \frac{s - a}{s - \bar{s}} \, \overline{z}.$$

C'est une involution du produit fibré, $\varphi \circ \varphi = \mathrm{id}$, qui échange les modules kähleriens et complexes. C'est aussi une isométrie par rapport à la métrique naturelle sur le produit fibré.

L'une des raisons originales pour laquelle nous nous sommes lancé dans ce travail était d'essayer de décrire la symétrie miroir à l'aide des outils de la géométrie différentielle. On peut voir l'existence de cette isométrie comme le signe que ce n'est pas entièrement désespéré, mais nous n'avons pas encore une bonne idée de la forme précise sous laquelle cet espoir pourrait se réaliser dans le cas général.

Espace de modules. Soit X une variété kählerienne compacte à première classe de Chern nulle. Nous pouvons voir X comme étant une variété lisse M munie d'une structure complexe J, comme ils ont fait Kodaira et Spencer. Si $\mathcal{J}$ est l'ensemble de toutes les structures complexes sur M, nous obtenons une famille tautologique $M \times \mathcal{J} \to \mathcal{J}$ de variétés complexes.

Cette famille contient bien sûr un nombre impressionnant de variétés isomorphes entre-elles. Pour réduire ce nombre nous pouvons quotienter ce produit par l'action du groupe $\mathcal{D}^+$ de difféomorphismes de M qui préservent l'orientation définie par J. Grosso modo, nous aimerions travailler avec l'espace de modules $\mathcal{J}/\mathcal{D}^+$. Le problème est que ce quotient peut être très sauvage, il n'admet même pas forcement une structure d'espace complexe.

La solution classique de ce problème est de polariser les variétés en question. L'effet du choix d'une polarisation est de réduire la quantité d'automorphismes par lesquels il faut quotienter, ce qui rend le quotient final plus facile à comprendre.

Un effet corrélatif de notre construction de variétés non-kähleriennes est que l'on peut quotienter par tous les automorphismes que l'on veut et obtenir un espace raisonnable, quitte à agrandir l'espace avec lequel on travaille. Au lieu de $\mathcal{J}$ nous considérons alors l'espace total du cône de Kähler relatif $\mathcal{K} \to \mathcal{J}$. Le groupe $\mathcal{D}^+$ agit naturellement sur cet espace, et le quotient $\mathcal{K}/\mathcal{D}^+$ est une orbifold, c'est-à-dire une variété singulière dont un certain nombre de point admet des groupes d'automorphismes finis.

Directions futures de recherche. Nous avons quelques pistes de recherche et questions que nous aimerions explorer dans l'avenir.

1. Le tenseur de courbure de la métrique L^2 sur le cône de Kähler d'une variété compacte X a été explicité, d'après nos résultats dans le chapitre 2 il s'écrit

$$\begin{aligned} R^{\mathcal{K}}(U,V,Z,W) &= R^{\mathcal{M}}(U,V,Z,W) \\ &\qquad + \langle(\mathrm{II}(U,W),\mathrm{II}(V,Z)\rangle - \langle \mathrm{II}(U,Z),\mathrm{II}(V,W)\rangle, \end{aligned}$$

où $R^{\mathcal{M}}$ est le tenseur de courbure d'un espace symétrique, et où II est la deuxième forme fondamentale du plongement du cône de Kähler dans l'espace des métriques hermitiennes sur X.

Nous nous attendons à un lien entre cette métrique et la métrique Weil–Petersson sur les espaces de modules de structures complexes. Cette deuxième métrique est de courbure sectionnelle holomorphe négative, et on peut espérer que la même propriété soit vraie pour la métrique sur le cône.

Dans cette thèse nous avons essayé de démontrer un résultat plus fort, c'est-à-dire que la courbure sectionnelle de la métrique sur le cône de Kähler est négative.

Nous avons donc deux pistes à suivre : Nous ne savons toujours pas si la courbure sectionnelle de la métrique sur le cône est négative ou pas. Cette question peut être attaquée avec les méthodes d'analyse poussées, dans le but de montrer qu'elle a une réponse positive. Il est aussi envisageable de chercher des exemples concrets de variétés dont l'anneau de cohomologie est relativement explicite, cette fois dans le but de chercher un contre-exemple à la négativité de la courbure sectionnelle. Une remarque due à Jason Starr affirme que l'on peut décrire l'anneau de cohomologie et le cône de Kähler d'une hypersurface de dimension $n \geq 3$ dans $\mathbb{P}^a \times \mathbb{P}^b$ (ou dans n'importe quelle variété torique). Il serait intéressant d'effectuer le calcul pour cette famille d'exemples.

La deuxième piste est plus modeste : Nous essayons simplement de montrer que la courbure holomorphe sectionnelle de la métrique sur le cône de Kähler complexifié est négative. Comme nous avons dit, c'est le cas pour la métrique de Weil–Petersson, donc il est possible que ce soit aussi vrai pour notre métrique.

2. L'un des objectifs de départ de cette thèse était de décrire la symétrie miroir à l'aide des outils de la géométrie différentielle. À cause des difficultés techniques, ceci n'a pas pu être fait pendant les trois ans consacrés au travail de la thèse.

Le premier pas dans cette direction est d'expliciter complètement le tenseur de courbure de la métrique sur le cône de Kähler relatif complexifié associé à une famille $\pi : \mathcal{X} \to S$ de variétés compactes kähleriennes à fibré canonique trivial. Nous ne sommes pas très loin d'attendre ce but ; nous connaissons les tenseurs de courbure de la métrique Weil–Petersson et de la métrique sur le cône de Kähler, ce qui manque est de décrire de manière satisfaisante la contribution du scindage du fibré tangent $T_{\mathcal{K}}$ au tenseur de courbure de $\mathcal{K}$. Ceci ne pose aucune difficulté.

Le deuxième pas devrait être d'étudier le cas de tores de dimension $n \geq 2$ en détail. Nous connaissons la symétrie miroir sur les tores, et nous devrions être capables d'expliciter complètement nos métriques et tenseurs dans ce cas. Le cas de courbes elliptiques était en définitive trop simple pour donner des idées utiles, mais il y a des chances que nous puissions voir comment la symétrie miroir s'exprime avec nos métriques sur les tores de dimension supérieure. Les calculs de base nécessaires pour ce travail pourraient être faits assez rapidement, mais il est plus difficile de prédire le temps que la compréhension de la situation demanderait.

3. Enfin il peut être utile de travailler sur quelques petits projets qui ne sont par directement liés au travail de la thèse. Ceci permettrait à la fois l'apprentissage de nouvelles techniques mathématiques, et pourrait laisser le subconscient travailler le "vrai" projet lorsqu'il ne progresse pas aussi vite que nous aimerions.

Soient X et Y des variétés projectives. Comme Grothendieck l'a montré [**Gro95**], nous pouvons mettre une structure de schéma sur l'espace $\mathrm{Hom}(X, Y)$ de morphismes $f : X \to Y$. Ce schéma est un sous-schéma d'un schéma de Hilbert, donc il peut être aussi singulier et non-réduit qu'on veut [**Vak06**]. Cependant, une grande partie de ce schéma peut être lisse, et nous pouvons donc essayer de construire une métrique hermitienne sur cette partie et espérer qu'elle se prolonge en tant que courant sur tout l'espace.

Nous rappelons que l'espace tangent de $\mathrm{Hom}(X,Y)$ en un point lisse $[f]$ est

$$T_{\mathrm{Hom}(X,Y),[f]} = H^0(X, f^*T_Y).$$

Si ω_X et ω_Y sont des métriques hermitiennes sur X et Y, nous pouvons définit un produit scalaire ω sur l'espace tangent de $\mathrm{Hom}(X,Y)$ par

$$\omega_{[f]}(\xi,\overline{\eta}) = \int_X \omega_Y(\xi(x),\overline{\eta}(x))_{f(x)}dV_X(x),$$

où $dV_X := \omega_X^n/n!$ est la forme volume définie par ω_X. La forme ω ainsi définit est alors une métrique hermitienne lisse, au moins où l'espace $\mathrm{Hom}(X,Y)$ est lisse. Cette métrique est *a posteriori* intéressante grâce au résultat suivant, qu'on montre sans difficultés :

Proposition. 1. *La métrique ω est kählerienne si la métrique ω_Y l'est. Elle est invariante sous l'action des groupes $\mathrm{Aut}\,X$ et $\mathrm{Aut}\,Y$ sur l'espace $\mathrm{Hom}(X,Y)$ si les formes dV_X et ω_Y le sont.*

2. *Soit R_Y le tenseur de courbure de la métrique ω_Y. Le tenseur de courbure de ω est*

$$R(\xi,\overline{\eta},\nu,\overline{\chi})_{[f]} = \int_X R_Y(\xi(x),\overline{\eta}(x),\nu(x),\overline{\chi}(x))_{f(x)}dV_X(x).$$

Un corollaire est que si ω_Y est Kähler–Einstein avec $\mathrm{Ric}\,\omega_Y = \lambda\omega_Y$, alors la métrique ω est également Kähler–Einstein et $\mathrm{Ric}\,\omega = \lambda\omega$. La géométrie de l'espace d'arrivée Y semble alors permettre de contrôler la géométrie de l'espace $\mathrm{Hom}(X,Y)$.

Pour rendre ce résultat plus utile, il faudrait démontrer que la métrique ω s'étend en un courant positif fermé sur tout l'espace $\mathrm{Hom}(X,Y)$. Un résultat de ce genre a déjà été montré par Axelsson et Schumacher [**AS06**] dans le cadre d'espaces de Douady. Leur travail donne effectivement une métrique sur l'espace $\mathrm{Hom}(X,Y)$, qui peut être écrite comme suit, avec de nouveau beaucoup d'abus de notations que nous n'expliquerons pas :

$$\omega_{AS} = \pi_*(\omega_X + ev^*\omega_Y)^{n+1}.$$

Axelsson et Schumacher ont montré que cette forme s'étend en un courant de Kähler sur l'espace $\mathrm{Hom}(X,Y)$. Notre métrique correspond grosso modo au facteur $\pi_*(\omega_X^n \wedge ev^*\omega_Y)$ dans l'expression ci-dessus, ce qui donne l'espoir qu'on peut la prolonger sur tout l'espace de morphismes.

Stack.

CHAPTER 1

Natural metrics associated to families of compact Kähler manifolds

1. Variation of Hodge structures

1.1. Let us briefly review the Hodge bundles and Gauss-Manin connection associated to a family. This material is standard and is treated in detail in [**Voi02**] and [**BDIP96**].

Let $\pi : \mathcal{X} \to S$ be a family of compact Kähler manifolds over a smooth base S. We work locally over the base S, so we may suppose it to be connected and contractible. Under these hypotheses the family $\mathcal{X}$ is diffeomorphic to a trivial family $X_0 \times S$. Thus the cohomology groups $H^k(X_s, \mathbb{Z})$ of the manifolds X_s of the family $\mathcal{X}$ are isomorphic one to another.

These groups form a local system over S. That is, there is a locally trivial sheaf of abelian groups over S whose stalk over a point s is the group $H^k(X_s, \mathbb{Z})$. We denote by $p : E^k \to S$ the holomorphic vector bundle we obtain by tensoring this sheaf by $\mathcal{O}_S$. It is equipped with a flat connection ∇_{GM}, called the Gauss–Manin connection.

By applying the Hodge decomposition theorem on each manifold X_s we obtain smooth subbundles of E^k, whose stalk over a point s is given by the Dolbeault group $H^{p,q}(X_s, \mathbb{C})$. As subbundles of E^k, these bundles are almost never holomorphic. We can however consider the Hodge filtration of each group $H^k(X_s, \mathbb{C})$. This defines a subbundle F^pE^k of E^k whose stalk over a point s is

$$F^pE^k_s = \bigoplus_{q \geq p} H^{q,k-q}(X_s, \mathbb{C}).$$

By a theorem of Griffiths, these subbundles F^pE^k are holomorphic. We now define holomorphic vector bundles $E^{p,q}$ over the base S by setting

$$E^{p,q} = F^pE^{p+q}/F^{p+1}E^{p+q}.$$

The fiber of $E^{p,q}$ over a point s is then equal to the Dolbeault group $H^{p,q}(X_s, \mathbb{C})$. We remark that in our case one may also define these bundles as the underlying vector bundles of the direct image sheaves $\mathcal{R}^q\pi_*\Omega^p_{\mathcal{X}/S}$.

1.2. The Hodge bundles E^k are equipped with a flat connection ∇, called the Gauss–Manin connection. This connection does not preserve the subbundles defined by the Hodge filtration. However, it does satisfy Griffiths transversality, which means that

$$\nabla F^pE^k \subset F^{p-1}E^k.$$

Now let γ be a k-cycle on a manifold X_s. As all the nearby manifolds in the family $\pi : \mathcal{X} \to S$ are diffeomorphic we may consider γ as a k-cycle on nearby manifolds as well. Let $\sigma : S \to E^k$ be a section of a Hodge bundle. We then obtain a complex function on S by considering

$$s \mapsto \int_\gamma \sigma(s).$$

The Gauss–Manin connection lets us calculate the derivative of this function. Explicitly, let ξ be a holomorphic vector field on S. Then

$$\xi \cdot \int_\gamma \sigma(s) = \int_\gamma \nabla_\xi \sigma(s).$$

1.3. Recall that the Hodge decomposition theorem permits us to write each bundle E^k as a direct sum of the Hodge bundles $E^{p,q}$ with $p+q=k$. This decomposition is only real-analytic and not holomorphic. However, we can still use it to define a connection ∇ on each bundle $E^{p,q}$ by following the arrows

$$E^{p,q} \hookrightarrow E^{p+q} \xrightarrow{\nabla_{GM}} E^{p+q} \otimes \Omega^1_S \longrightarrow E^{p,q} \otimes \Omega^1_S,$$

where the last arrow is induced by the projection defined by the Hodge decomposition of E^k.

If $H^k(X_s, \mathbb{C}) = H^{p,q}(X_s, \mathbb{C})$, which happens very rarely, then the connection ∇ coincides with the Gauss–Manin connection. Otherwise it has non-trivial curvature.

1.4. Let $p : E \to S$ be a vector bundle over a smooth base S. Associated to this bundle is a short exact sequence

$$0 \longrightarrow T_{E/S} \longrightarrow T_E \longrightarrow p^*T_S \longrightarrow 0$$

of vector bundles over the total space E. We note that the bundle $T_{E/S}$ is nothing but the pullback p^*E of the bundle E to itself.

This short exact sequence does not split naturally. Suppose however that the vector bundle E is equipped with a connection. A basic fact of differential geometry is that a connection on the vector bundle $E \to S$ defines a smooth lift of the vector bundle p^*T_S into T_E, and thus a smooth splitting of the exact sequence.

Proposition 1.1. *Let $p : (E, \nabla) \to S$ be a holomorphic vector bundle with a connection ∇ over a smooth base S. Let (s, e) be a point of E, and let σ be a holomorphic local section of E such that $\sigma(s) = e$. The lift $p^*T_S \to T_E$ defined by ∇ is given by the section*

$$\beta = d\sigma - \nabla\sigma$$

of $(p^*T_S)^* \otimes T_E = Hom(p^*T_S, T_E)$ at the point (s, e).

Proof: Let σ be a local section of $\mathcal{K}$ such that $\sigma(s) = e$. Let U be a neighborhood of s and let $\theta : E_{|U} \to U \times \mathbb{C}^r$ be a trivialization of E over U. We set $\tau = \theta \circ \sigma$. Then there is a matrix A of 1-forms on S such that $\nabla\sigma \simeq_\theta \mathrm{d}\tau + A \wedge \tau$, and we get

$$\mathrm{d}\sigma - \nabla\sigma \simeq_\theta -A \wedge \tau.$$

At the point s we then find that $\mathrm{d}\sigma(s) - \nabla\sigma(s) \simeq_\theta -A(s) \wedge \tau(s) = -A(s) \wedge e$. Thus the section β does not depend on the choice of σ.

Note that $\nabla\sigma$ is a 1-form with values in $T_{E/S} = \operatorname{Ker} p_*$. Also, if σ is a section of E then $p \circ \sigma = \mathrm{id}_S$. This entails that $p_* \circ \beta = \mathrm{id}_{p^* T_S}$, so β is indeed a lift.

Finally we remark that if σ is parallel along a path that defines a tangent vector in $T_{S,s}$, then this lift coincides with the horizontal lift described in Kobayashi–Nomizu [**KN96**]. □

2. The relative Kähler cone

Recall that a real $(1,1)$ cohomology class a on a compact Kähler manifold X is called a Kähler class if it contains a Kähler metric. A $(1,1)$ class a will be called a *complexified Kähler class* if its imaginary part $\operatorname{Im} a$ is a Kähler class. We denote the set of complexified Kähler classes on X by $K(X)$. It is a convex open cone in the finite-dimensional vector space $H^{1,1}(X, \mathbb{C})$.

Definition. The *relative Kähler cone* of a family $\pi : \mathcal{X} \to S$ is the subset $\mathcal{K}$ of $p : E^{1,1} \to S$ which consists of the complexified Kähler cones of each manifold X_s.

Proposition 2.1. *The relative Kähler cone $\mathcal{K}$ is open in the total space of the vector bundle $E^{1,1}$.*

Proof: Let (a_0, s_0) be a point in $\mathcal{K}$. Then there is a Kähler metric Ω_0 in the class $\operatorname{Im} a_0$. By Kodaira–Spencer [**KS60**] there exists a relative Kähler metric $\Omega_{\mathcal{X}/S}$ on the family $\mathcal{X} \to S$, such that $\Omega_{\mathcal{X}/S}|_{X_{s_0}} = \Omega_0$. These metrics define a L^2-inner product on the space of relative $(1,1)$-forms of the family. The metrics also define a smooth bundle isomorphism between the bundle $E^{1,1}$ and the bundle $\mathcal{H}^{1,1}$, whose stalk over s consists of the Ω_s-harmonic forms on X_s.

Let us now restrict our attention to a relatively compact neighborhood U of s_0. As the morphism $\pi : \mathcal{X} \to S$ is proper, then the preimage of $\overline{U}$ in $\mathcal{X}$ is compact It follows that the unit sphere fibration in $T_{\mathcal{X}/S}$ defined by the relative metric $\Omega_{\mathcal{X}/S}$ is compact in the total space $T_{\mathcal{X}/S}$, at least once restricted to $\overline{U}$.

The positivity of forms can be measured on the unit sphere. The compactness of the unit sphere fibration thus lets us find an open ball in $E^{1,1}$ around the section $\Omega_{\mathcal{X}/S}$ which is contained in the relative Kähler cone $\mathcal{K}$. This proves the proposition. □

As the set $\mathcal{K}$ is open in $E^{1,1}$, then it has the structure of a complex manifold. It also inherits the connection ∇ on the ambient vector bundle. We continue to denote the restriction of the projection $p : E^{1,1} \to S$ to the set $\mathcal{K}$ by p in a slight abuse of notation. The projection p is a holomorphic submersion, and its fiber over a point s is the set $K(X_s)$ of complexified Kähler classes on the manifold X_s.

Remark — Note that we didn't need to complexify the Kähler cone. We could have considered the usual real Kähler cone and gotten a smooth space $p : \mathcal{K}_{\mathbb{R}} \to S$ whose fiber over a point s is the real Kähler cone of X_s.

Later we will construct a connection and hermitian metrics on the space $\mathcal{K}$. These constructions only depend on having a Kähler class, so the only gain from having a complexified Kähler class is that the constructions give hermitian metrics instead of Riemannian ones. I believe the complexified

Kähler classes should play a non-trivial role, but I haven't found what it is yet.

I point this out because in the second chapter we will consider only the real Kähler cone. Since our constructions do not yet use the complexified Kähler classes, this will not complicate our lives.

3. A first hermitian metric

From now on all manifolds will have zero first Chern class. Consider the short exact sequence

$$0 \longrightarrow T_{\mathcal{K}/S} \longrightarrow T_{\mathcal{K}} \xrightarrow{p_*} p^*T_S \longrightarrow 0$$

of holomorphic vector bundles over the total space $\mathcal{K}$. We recall that the relative tangent bundle of a vector bundle E is $T_{E/S} = p^*E$, so we may identify the relative tangent bundle of $\mathcal{K}$ with $p^*E^{1,1}$.

If Ω is a Kähler metric on X, then the Ricci curvature of Ω may be defined as the curvature form of the hermitian metric that Ω induces on the canonical bundle K_X. In local holomorphic coordinates we have $2\pi \operatorname{Ric}\Omega = -i\partial\bar\partial \log\det \Omega_{j\bar k}$, where $\Omega_{j\bar k}$ are the composants of the metric Ω in the chosen local coordinates. The Ricci-form of Ω represents the first Chern class of X.

Theorem (Aubin–Calabi–Yau, [**Aub78, Yau78**])**.** *Let X be a compact Kähler manifold and let α be a smooth form in $-c_1(X)$. Let ω be a Kähler class on X. Then there exists a unique Kähler metric Ω in the class ω such that $Ric\,\Omega = \alpha$.*

Our manifolds have zero first Chern class, to the Aubin–Calabi–Yau theorem entails that there exists a unique Ricci-flat Kähler metric in each Kähler class.

Definition-Proposition 3.1. *The Ricci-flat Kähler metrics on the manifolds in the family $\pi : \mathcal{X} \to S$ induce a smooth hermitian metric $g_{\mathcal{K}/S}$ on the vector bundle $T_{\mathcal{K}/S}$ over $\mathcal{K}$.*

Proof: Let $x = (s, a)$ be a point of $\mathcal{K}$, and let Ω_a be the unique Ricci-flat Kähler metric in $\operatorname{Im} a$. Let u and v be classes in $T_{\mathcal{K}/S,x} = H^{1,1}(X_s, \mathbb{C})$. The Kähler metric Ω_a defines an isomorphism $H^{1,1}(X_s, \mathbb{C}) \to \mathcal{H}^{1,1}_{\Omega_a}$, where $H^{1,1}_{\Omega_\alpha}$ is the space of Ω_a-harmonic $(1,1)$-forms on X_s. We denote by U and V the images of the classes u and v under this isomorphism, and define

$$g_{\mathcal{K}/S}(u, \overline{v}) = \frac{1}{\operatorname{Vol}(X_s, \Omega_a)} \int\limits_{X_s} \langle U, \overline{V}; \Omega_a\rangle \, dV_{\Omega_a}. \tag{1}$$

Here $\langle \cdot, \bar{\cdot}\,; \Omega_a\rangle$ is the inner product on $(1,1)$-forms defined by Ω_a. The form $g_{\mathcal{K}/S}$ is then a hermitian scalar product on $T_{\mathcal{K}/S,x}$ by inspection.

Now, when Ω_a varies smoothly, the isomorphism $H^{1,1}(X_s, \mathbb{C}) \cong \mathcal{H}^{1,1}_{\Omega_a}$ varies smoothly as well, as the Laplacian is an elliptic linear differential operator whose coefficients depend smoothly on Ω_a. The inner product $\langle \cdot, \cdot\rangle_{\Omega_a}$ also depends smoothly on Ω_a, so the expression (1) depends smoothly on Ω_a.

We recall that the metric Ω_a is the solution of a Monge-Ampère equation $MA(x)$. This is a non-linear elliptic partial differential equation, so its solutions depend smoothly on x. Thus $g_{\mathcal{K}/S}$ is smooth. $\square$

Remark — The normalization by the volume of each manifold is not standard and does not matter for the existence theorems we want to establish in this chapter. However, this normalization is essential for later results. Until that point, which justifies this modification, I ask that the reader accept the normalization as an eccentricity.

Proposition 3.2. *The Ricci-flat Kähler metrics on the manifolds in the family $\pi : \mathcal{X} \to S$ induce a smooth hermitian form g_S on the vector bundle p^*T_S over $\mathcal{K}$. This hermitian form is a metric if the family $\pi : \mathcal{X} \to S$ is effective (that is, the Kodaira–Spencer morphism of the family is injective).*

Proof: Denote by $E^1(T_{\mathcal{X}/S}) \to S$ the holomorphic vector bundle whose fiber over a point s is $H^1(X_s, T_{X_s})$, and pull it back to the total space of $\mathcal{K}$ by the projection p. Let $x = (s, a)$ be a point of $\mathcal{K}$, and let Ω_a be the unique Ricci-flat Kähler metric in $\operatorname{Im} a$.

The Kähler metric Ω_a defines an isomorphism $H^1(X_s, T_{X_s}) \to \mathcal{H}^1_{\Omega_a}(T_{X_s})$, where $H^1_{\Omega_a}(T_{X_s})$ is the space of Ω_a-harmonic 1-forms on X_s with values in T_{X_s}. Let u and v be two classes in $H^1(X_s, T_{X_s})$. We denote by U and V the images of the classes u and v under this isomorphism, and define

$$\langle u, \overline{v} \rangle = \frac{1}{\operatorname{Vol}(X_s, \Omega_a)} \int_{X_s} \langle U, \overline{V}; \Omega_a \rangle \, dV_{\Omega_a}. \tag{2}$$

Here $\langle \cdot, \cdot; \Omega_a \rangle$ is the inner product on 1-forms with values in T_{X_s} defined by Ω_a. The form defined in this way is then a hermitian scalar product on $p^*E^1_x$ by inspection. As in the proof of the last proposition, it varies smoothly with x.

The hermitian form g_S is the pullback of this hermitian metric to p^*T_S via the Kodaira–Spencer morphism ρ. By elementary linear algebra, the hermitian form g_S is non-degenerate if the morphism ρ is injective, which happens by definition if and only if the family $\pi : \mathcal{X} \to S$ is effective. $\square$

Theorem 3.3. *Let $\pi : \mathcal{X} \to S$ be a family of compact Kähler manifolds over a smooth base S. Let $p : (\mathcal{K}, \nabla) \to S$ be the associated relative Kähler cone over S. Then the Ricci-flat Kähler metrics on the manifolds in the family $\mathcal{X}$ and the connection ∇ induce a smooth hermitian form h on $\mathcal{K}$. This form is a hermitian metric if the family is effective.*

Proof: We consider the short exact sequence

$$0 \longrightarrow T_{\mathcal{K}/S} \longrightarrow T_{\mathcal{K}} \xrightarrow{p_*} p^*T_S \longrightarrow 0$$

of holomorphic vector bundles over the total space $\mathcal{K}$. There is a smooth hermitian metric $g_{\mathcal{K}/S}$ on the bundle $T_{\mathcal{K}/S}$ and a smooth hermitian form g_S on the bundle p^*T_S. The connection ∇ now gives a smooth splitting of the tangent bundle of $\mathcal{K}$ into a direct sum of $T_{\mathcal{K}/S}$ and p^*T_S. The hermitian metric we want is induced by this splitting and the metrics $g_{\mathcal{K}/S}$ and g_S. $\square$

4. A second hermitian metric

Consider the family $\pi : \mathcal{X} \to S$. As π is a submersion, then we have a short exact sequence

$$0 \longrightarrow T_{\mathcal{X}/S} \longrightarrow T_{\mathcal{X}} \xrightarrow{\pi_*} \pi^*T_S \longrightarrow 0$$

of holomorphic vector bundles over the total space $\mathcal{X}$. We would like to play the same game as in the last section and get hermitian metrics on the bundles in the sequence. To do this we must add the information about the complexified Kähler cones on the manifolds in $\mathcal{X}$ into the mix.

We thus consider the fiber product $\mathcal{X} \times_S \mathcal{K}$. As the spaces $\mathcal{X}$, $\mathcal{K}$ and S are smooth, then the fiber product is a smooth complex manifold. The fiber product is equipped with projection maps $p_{\mathcal{X}} : \mathcal{X} \times_S \mathcal{K} \to \mathcal{X}$ and $p_{\mathcal{K}} : \mathcal{X} \times_S \mathcal{K} \to \mathcal{K}$. We set $\nu = \pi \circ p_{\mathcal{X}} = p \circ p_{\mathcal{K}}$. The holomorphic map $\nu : \mathcal{X} \times_S \mathcal{K} \to S$ is a submersion and its fiber over a point s is $X_s \times K(X_s)$ by definition of the fiber product.

The space $p_{\mathcal{K}} : \mathcal{X} \times_S \mathcal{K} \to \mathcal{K}$ may be regarded as a universal manifold over the relative Kähler cone. Its fiber over a point is a manifold X_s along with the choice of a complexified Kähler class a on X_s.

We now pull back the above short exact sequence to the total space of the fiber product $\mathcal{X} \times_S \mathcal{K}$. This gives the short exact sequence

$$0 \longrightarrow p_{\mathcal{X}}^* T_{\mathcal{X}/S} \longrightarrow p_{\mathcal{X}}^* T_{\mathcal{X}} \xrightarrow{\nu_*} \nu^* T_S \longrightarrow 0$$

of holomorphic vector bundles over $\mathcal{X} \times_S \mathcal{K}$.

Proposition 4.1. *The Ricci-flat Kähler metrics on the manifolds in the family $\pi : \mathcal{X} \to S$ induce a smooth hermitian metric $g_{\mathcal{X}/S}$ on the vector bundle $p_{\mathcal{X}}^* T_{\mathcal{X}/S}$ over $\mathcal{X} \times_S \mathcal{K}$.*

Proof: Let $x = (z, a, s)$ be a point of the fiber product, and let Ω_a be the Ricci-flat Kähler metric in the class $\operatorname{Im} a$. We define $g_{\mathcal{X}/S}(x)$ to be the normalized metric $\frac{1}{\operatorname{Vol}(X, \Omega_a)} \Omega_a$. This metric clearly varies smoothly with x. □

We remark that there is also a smooth hermitian metric on $\nu^* T_S$. It is just the pullback of g_S from the last section to $\nu^* T_S$ via the projection $p_{\mathcal{K}} : \mathcal{X} \times_S \mathcal{K} \to \mathcal{K}$. What is missing to get a hermitian metric on the bundle $p_{\mathcal{X}}^* T_{\mathcal{X}}$ is a smooth splitting of our short exact sequence. Such a splitting is provided by harmonic lifts in the sense of Siu.[1]

Let ρ be the Kodaira–Spencer morphism of the family $\pi : \mathcal{X} \to S$. If $x = (z, a, s)$ is a point of the fiber product $\mathcal{X} \times_S \mathcal{K}$, then we denote by Ω_a the Ricci-flat metric in the class $\operatorname{Im} a$.

Definition. A smooth morphism of vector bundles $\beta : \nu^* T_S \to pr_{\mathcal{X}}^* T_{\mathcal{X}}$ is a *harmonic lift* if $\nu_* \beta = \mathrm{id}_{\nu^* T_S}$ and if $\bar{\partial} \beta(\xi)|_{X_s}$ is the Ω_a-harmonic representant of $\rho_s(\xi)$ for any point x and any tangent field ξ defined on a small neighborhood of x.

Proposition 4.2. *The Ricci-flat Kähler metrics on the manifolds in the family $\pi : \mathcal{X} \to S$ define a harmonic lift of the bundle $\nu^* T_S$ into $p_{\mathcal{X}}^* T_{\mathcal{X}}$.*

Proof: Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}(p_{\mathcal{X}}^* T_{\mathcal{X}/S}) \longrightarrow \mathcal{O}(p_{\mathcal{X}}^* T_{\mathcal{X}}) \xrightarrow{\nu_*} \mathcal{O}(\nu^* T_S) \longrightarrow 0$$

[1]These are often called *canonical lifts* in the literature. The lifts depend on the choice of hermitian metrics on the fibers of a family, which can rarely be made in any canonical fashion. I feel the word *harmonic* better describes what is going on and reminds us of the metrics lurking in the background.

of sheaves of holomorphic sections over the space $\mathcal{X} \times_S \mathcal{K}$. As all the manifolds X_s are Kähler, and the base S is smooth, then the exact sequence consists of sheaves of sections of holomorphic vector bundles, and not just coherent sheaves.

Let $x = (z, a, s)$ be a point in $\mathcal{X} \times_S \mathcal{K}$, and denote by Ω_a the unique Ricci-flat Kähler metric on X_s in the Kähler class $\operatorname{Im} a$. As before, the metric Ω_a depends smoothly on the point x.

Denote by $\bar{\partial}^*_{\mathcal{X}/S}$ the formal Ω_a-adjoint of the $\bar{\partial}$ operator on X_s. Also denote by $\Delta_{\mathcal{X}/S}$ the Laplace operator corresponding to Ω_a, by $h_{\mathcal{X}/S}$ the Ω_a-harmonic projection map $\mathcal{C}^\infty(X_s, T_{X_s}) \to \mathcal{H}(X_s, T_{X_s})$, and by $G_{\mathcal{X}/S}$ the Green operator associated to Ω_a. These operators depend smoothly on the choice of Ω_a, and thus depend smoothly on x. Recall that these operators are related by the identity

$$G_{\mathcal{X}/S}\Delta_{\mathcal{X}/S} + h_{\mathcal{X}/S} = id_{\mathcal{X}/S},$$

where $id_{\mathcal{X}/S}$ is the identity map on the sheaf $\mathcal{O}(p^*_{\mathcal{X}} T_{\mathcal{X}/S})$.

Now, pick any smooth lift $\tilde{\beta} : \mathcal{O}(\nu^* T_S) \to \mathcal{O}(p^*_{\mathcal{X}} T_{\mathcal{X}})$. Such a lift may for example be constructed by choosing a hermitian metric on $T_{\mathcal{X}}$ and identifying $\nu^* T_S$ with the orthogonal complement of $p^*_{\mathcal{X}} T_{\mathcal{X}/S}$. By looking at the definition of Dolbeault cohomology, one sees that the cohomology class of $\bar{\partial}\tilde{\beta}(\xi)|_{X_s}$ is equal to $\rho_s(\xi)$, where ξ is a section of $\nu^* T_S$ on a neighborhood of x, and ρ is the Kodaira–Spencer morphism of the family $\pi : \mathcal{X} \to S$.

The lift $\tilde{\beta}$ may not be harmonic. However, define a new smooth vector bundle morphism $\beta : \nu^* T_S \to p^*_{\mathcal{X}} T_{\mathcal{X}}$ by

$$\beta(x) := \tilde{\beta}(x) - G(x)\,\bar{\partial}^*(x)\,\bar{\partial}\,\tilde{\beta}(x).$$

If ξ is a section of $\nu^* T_S$, then this new morphism satisfies

$$\begin{aligned}
\bar{\partial}\beta(\xi)|_{X_s} &= \bar{\partial}\tilde{\beta}(\xi)|_{X_s} - \bar{\partial}\,G(x)\,\bar{\partial}^*(x)\,\bar{\partial}\tilde{\beta}(\xi)|_{X_s} \\
&= \bar{\partial}\tilde{\beta}(\xi)|_{X_s} - (G(x)\,\bar{\partial}\bar{\partial}^*(x))\,\bar{\partial}\tilde{\beta}(\xi)|_{X_s} \\
&= \bar{\partial}\tilde{\beta}(\xi)|_{X_s} - (id_{X_s} - h(x))(\bar{\partial}\tilde{\beta}(\xi))|_{X_s} \\
&= h(x)(\bar{\partial}\tilde{\beta}(\xi))|_{X_s},
\end{aligned}$$

so it is harmonic.

To see that β is a lift, i.e. that $\nu_*\beta = id_{\nu^* T_S}$, we note that the relative harmonic and Green operators give a smooth orthogonal splitting $\mathcal{O}(p^*_{\mathcal{X}} T_{\mathcal{X}/S}) = \mathcal{H} \oplus \mathcal{G}$, where $\mathcal{H}$ and $\mathcal{G}$ are the sheaves of harmonic sections and the image of sections under the Green operator, respectively. These fit into the short exact sequence

$$0 \longrightarrow \mathcal{H} \oplus \mathcal{G} \longrightarrow \mathcal{O}(p^*_{\mathcal{X}} T_{\mathcal{X}}) \xrightarrow{\nu_*} \mathcal{O}(\nu^* T_S) \longrightarrow 0.$$

As $\mathcal{H} \oplus \mathcal{G} \subset \ker \nu_*$ and $\mathcal{G}$ injects into the direct image, we see that $\nu_* G\xi = 0$ for all sections ξ of $p^*_{\mathcal{X}} T_{\mathcal{X}/S}$. It follows that $\nu_*\beta = \nu_*\tilde{\beta} = id_{\nu^* T_S}$. $\square$

Corollary 4.3. *The Ricci-flat Kähler metrics on the manifolds in the family $\pi : \mathcal{X} \to S$ induce a smooth hermitian metric $g_{\mathcal{X}}$ on the vector bundle $p^*_{\mathcal{X}} T_{\mathcal{X}}$ over $\mathcal{X} \times_S \mathcal{K}$.*

Proof: Siu's harmonic lifts give a smooth splitting $p_{\mathcal{X}}^* T_{\mathcal{X}} = p_{\mathcal{X}}^* T_{\mathcal{X}/S} \oplus \nu^* T_S$. The metric in question is obtained via this splitting and the metrics $g_{\mathcal{X}/S}$ and g_S. □

Theorem 4.4. *Let $\pi : \mathcal{X} \to S$ be a family of compact Kähler manifolds with zero first Chern class over a smooth base S. Let $p : (\mathcal{K}, \nabla) \to S$ be the associated relative Kähler cone over S, and let $\mathcal{X} \times_S \mathcal{K}$ be the fiber product. Then the Ricci-flat Kähler metrics on the manifolds in the family $\mathcal{X}$, the connection ∇, and Siu's harmonic lifts induce a smooth hermitian form h on $\mathcal{X} \times_S \mathcal{K}$. This form is a hermitian metric if the family is effective.*

Proof: Denote by $p_{\mathcal{X}} : \mathcal{X} \times_S \mathcal{K} \to \mathcal{X}$ and $p_{\mathcal{K}} : \mathcal{X} \times_S \mathcal{K} \to \mathcal{K}$ the projections associated to the fiber product. The relative tangent bundle of a fiber product splits holomorphically as

$$T_{\mathcal{X} \times_S \mathcal{K}/S} = p_{\mathcal{X}}^* T_{\mathcal{X}/S} \oplus p_{\mathcal{K}}^* T_{\mathcal{K}/S}.$$

The tangent bundle of the fiber product thus fits into the short exact sequence

$$0 \longrightarrow p_{\mathcal{X}}^* T_{\mathcal{X}/S} \oplus p_{\mathcal{K}}^* T_{\mathcal{K}/S} \longrightarrow T_{\mathcal{X} \times_S \mathcal{K}} \longrightarrow \nu^* T_S \longrightarrow 0$$

of holomorphic vector bundles. We can also identify the tangent bundle $T_{\mathcal{X} \times_S \mathcal{K}}$ with the subbundle of $p_{\mathcal{X}}^* T_{\mathcal{X}} \oplus p_{\mathcal{K}}^* T_{\mathcal{K}}$ which consists of the elements (ξ, η) such that $\pi_* \xi = p_* \eta$. The connection ∇ and the harmonic lifts now split the above short exact sequence. The metrics $g_{\mathcal{X}/S}$, $g_{\mathcal{K}/S}$ and g_S then induce a hermitian metric h on the fiber product. □

Remark — It should be possible to generalize this construction somewhat. The key is to have a unique choice of a Kähler metric in each Kähler class. It is tempting to choose a smooth form C representing the Chern class $c_1(X_s)$ for some s and using the Aubin–Calabi–Yau theorem to pick metrics on each manifold and in each Kähler class whose Ricci-form is C. However this doesn't quite work, as the form C has no reason to be of pure type $(1, 1)$ on all manifolds X_s, even though it represents the first Chern class of the underlying smooth manifold as a 2-form.

A happy middle ground might consist of Kähler metrics of constant scalar curvature, or cscK metrics. These may not be unique in each Kähler class, but the cscK metrics in a given Kähler class are parametrized by a group of holomorphic isomorphisms of the underlying complex manifold, so by adding these parameters into the mix one should be able to repeat the above construction pretty much verbatim.

5. Base change

Let $\pi : \mathcal{X} \to S$ be a family of compact Kähler manifolds over a smooth base. If B is a complex manifold and $f : B \to S$ a holomorphic morphism then we obtain a family $f^*\mathcal{X} \to B$ over the base B by pullback. In order to show that our constructions can claim to be natural, we must show that they enjoy functorial properties with respect to base change. Amongst other things, this will ensure that the objects we construct pass to any reasonable moduli space.

In this section we prove that our objects are functorial with respect to base change. This section is entirely formal. The work done only consists of chasing Cartesian diagrams and checking definitions.

Let $\pi : \mathcal{X} \to S$ be a family of compact Kähler manifolds over a smooth base S. Let B be another smooth complex manifold, and let $f : B \to S$ be a holomorphic map. We then get a family $f^*\mathcal{X} \to T$ via pullback, and a Cartesian diagram

$$\begin{tikzcd} f^*\mathcal{X} \arrow[r, "pr_{\mathcal{X}}"] \arrow[d, "pr_B"] & \mathcal{X} \arrow[d, "\pi"] \\ B \arrow[r, "f"] & S \end{tikzcd}$$

where $pr_{\mathcal{X}}$ and pr_B are the projection morphisms of $\mathcal{X} \times B$ onto each factor, restricted to the fiber product $f^*\mathcal{X} \subset \mathcal{X} \times B$.

We will denote a vector or fiber bundle or a fibration E over the base S by E_S.

The first step towards showing that our metrics pull back nicely under base change is to show that our spaces behave as expected. We thus need to show that the Kähler cone fibration of $f^*\mathcal{X} \to B$ is the pullback of the Kähler cone fibration on $\mathcal{X} \to S$. Fiber by fiber there is clearly no problem, but we do need to show that the holomorphic structure of the fibrations coincide.

Proposition 5.1. *Let $E_S^{p,q}$ be the Hodge bundle over S. Then $E_B^{p,q} = f^*E_S^{p,q}$.*

Proof: First note that the relative cotangent sheaf $\Omega^1_{f^*\mathcal{X}/B}$ is equal to the pullback $pr_{\mathcal{X}}^*\Omega^1_{\mathcal{X}/S}$. It follows that the sheaf of relative p-forms on the pullback is a pullback as well, or $\Omega^p_{f^*\mathcal{X}/B} = pr_{\mathcal{X}}^*\Omega^p_{\mathcal{X}/S}$.

The fibers of $\mathcal{X}$ are Kähler manifolds, and the base S is connected, so the function $s \mapsto \dim H^q(X_s, \Omega^p_{X_s})$ is constant for any $q \geq 0$. In addition, the sheaf $\pi_*\Omega^p_{\mathcal{X}/S}$ is locally free by Grauert's theorem. Under these conditions, cohomology and base change commute for the locally free sheaf $\Omega^p_{\mathcal{X}/S}$.

The sheaf of sections of the Hodge bundle $E_S^{p,q}$ over S is $\mathcal{R}^q\pi_*\Omega^p_{\mathcal{X}/S}$. We now find that

$$f^*\mathcal{R}^q\pi_*\Omega^p_{\mathcal{X}/S} = \mathcal{R}^q pr_{B_*}(pr_{\mathcal{X}}^*\Omega^p_{\mathcal{X}/S}) = \mathcal{R}^q pr_{B_*}\Omega^p_{f^*\mathcal{X}/B},$$

which implies that $E_B^{p,q} = f^*E_S^{p,q}$. □

Corollary 5.2. *The relative Kähler cone of the pullback $f^*\mathcal{X} \to B$ is the pullback of the relative Kähler cone $\mathcal{K}_S$ by the morphism f.*

Proof: Fiber by fiber we have $f^*\mathcal{K}_{S,b} = K(X_{f(b)}) = \mathcal{K}_{B,b}$, so the fibers of each space coincide set-theoretically. The equality $f^*E_S^{1,1} = E_B^{1,1}$ then entails that they have the same holomorphic structure. □

Corollary 5.3. *The fiber product $f^*\mathcal{X} \times_B \mathcal{K}_B$ associated to the pullback $f^*\mathcal{X} \to B$ is the pullback of the fiber product $\mathcal{X} \times_S \mathcal{K}_S$ to the manifold B via the morphism f.*

We now embark on the thankless task of showing that our metrics and lifts pull back as expected, and thus again assume that the manifolds in the family have zero first Chern class. The two relative of Kähler cones fit into the Cartesian diagram

$$\begin{tikzcd} f^*\mathcal{K}_S \arrow[r, "F"] \arrow[d, "p_B"] & \mathcal{K}_S \arrow[d, "p_S"] \\ B \arrow[r, "f"] & S \end{tikzcd}$$

Proposition 5.4. *The metric $g_{\mathcal{K}_B/B}$ on $T_{\mathcal{K}_B/B}$ is the pullback of the metric $g_{\mathcal{K}_S/S}$ on $T_{\mathcal{K}_S/S}$ by the morphism F.*

Proof: We first note that the relative tangent bundle of $\mathcal{K} \to S$ is $T_{\mathcal{K}_S/S} = p_S^* E_S^{1,1}$. Then

$$T_{\mathcal{K}_B/B} = p_B^* E_B^{1,1} = p_B^* f^* E_S^{1,1} = F^* p_S^* E_S^{1,1} = F^* T_{\mathcal{K}_S/S}$$

because the above diagram is Cartesian. That $g_{\mathcal{K}_B/B} = F^* g_{\mathcal{K}_S/S}$ is now an immediate consequence of the definition of the metrics. $\square$

Proposition 5.5. *The metric g_B on $T_{\mathcal{K}_B/B}$ is the pullback of the metric g_S on $T_{\mathcal{K}_S/S}$ by the morphism F.*

Proof: Let $E^1(T_{\mathcal{X}/S})$ be the holomorphic vector bundle over S whose fiber over a point is $E^1(T_{\mathcal{X}/S})_s = H^1(X_s, T_{X_s})$. The proof of Proposition 5.1 can be modified without difficulty to show that $f^*E^1(T_{\mathcal{X}/S}) = E^1(T_{f^*\mathcal{X}/B})$. That the hermitian metric on the bundle $p_B^* E^1(T_{f^*\mathcal{X}/B})$ over $\mathcal{K}_B$ is the pullback of the equivalent metric over $\mathcal{K}_S$ by F then follows from their definition.

The metric g_S is the pullback of the metric on $p_S^* E^1(T_{\mathcal{X}/S})$ by the Kodaira–Spencer morphism ρ_S of the family $\pi : \mathcal{X} \to S$. The Kodaira–Spencer morphism of the pullback family is $\rho_B = f^*\rho_S$. Looking again at the definition of our metric, we find that $F^* g_S = g_B$. $\square$

Proposition 5.6. *Let $g_{f^*\mathcal{X}/B}$ be the smooth hermitian metric on the bundle $p_{f^*\mathcal{X}}^* T_{f^*\mathcal{X}/B}$. Then $g_{f^*\mathcal{X}/B} = f^* g_{\mathcal{X}/S}$.*

Proof: The first order of business is to show that the relative tangent bundle $T_{\mathcal{X}/S}$ and its various pullbacks all function as we would like. We start by looking at the Cartesian diagram

$$\begin{tikzcd} T_{f^*\mathcal{X}/B} \arrow[dr] & & T_{\mathcal{X}/S} \arrow[dr] & \\ & f^*\mathcal{X} \arrow[rr, "F"] & & \mathcal{X} \\ f^*\mathcal{X} \times_S \mathcal{K}_S \arrow[ur, "pr_{f^*\mathcal{X}}"] \arrow[rr, "\tilde{F}"] & & \mathcal{X} \times_S \mathcal{K}_S \arrow[ur, "pr_{\mathcal{X}}"] & \end{tikzcd}$$

Here the objects in the bottom two rows are complex manifolds, while the two objects in the top row are holomorphic vector bundles over the manifolds that their arrows point to.

The identity we want to show is $\tilde{F}^* pr_{\mathcal{X}}^* T_{\mathcal{X}/S} = pr_{f^*\mathcal{X}}^* T_{f^*\mathcal{X}/B}$. By commutativity of the diagram this identity follows from $F^* T_{\mathcal{X}/S} = T_{f^*\mathcal{X}/B}$.

The pullback $f^*\mathcal{X}$ is given by the subset $f^*\mathcal{X} = \{(x,b) \,|\, \pi(x) = f(b)\}$ of $\mathcal{X} \times B$. Its tangent space can thus be described as the set of tangent vectors (ξ, η) of $\mathcal{X} \times B$ that satisfy $\pi_*(\xi) = f_*(\eta)$. The relative tangent space is then the set of tangent vectors (ξ, η) such that $\pi_*(\xi) = f_*(\eta) = 0$, which implies that the space $T_{f^*\mathcal{X}/B}$ is the pullback $F^* T_{\mathcal{X}/S}$, like we wanted.

Showing that $\tilde{F}^* g_{\mathcal{X}/S} = g_{f^*\mathcal{X}/B}$ is now just a matter of fixing a point in the space and writing down the metrics involved. □

Proposition 5.7. *The connection ∇_B on $p_B : \mathcal{K}_B \to B$ is the pullback of the connection ∇_S on $p_S : \mathcal{K}_S \to S$ by the morphism f.*

Proof: This is clearly true for the Gauss–Manin connection ∇^2 on the Hodge bundle E^2. We now see that the Hodge decomposition of E_B^2 is the pullback of the Hodge decomposition of E_S^2 by f. Thus we get $f^*\nabla_S = \nabla_B$ on the bundle $E^{1,1}$, and thus also on the relative Kähler cone. □

Corollary 5.8. *Denote by h_S the smooth hermitian metric on $\mathcal{K}_S$ constructed in Theorem 3.3. Then h_B on $\mathcal{K}_B$ is the pullback of h_S by the morphism F.*

Proposition 5.9. *The harmonic lift β_B of $f^*\nu^* T_B$ into $pr_{f^*\mathcal{X}}^* T_{f^*\mathcal{X}}$ is the pullback of the canonical lift β_S by the morphism $\tilde{F}$.*

Proof: As β_S is a lift, then the pullback $\tilde{F}^*\beta_S$ is again a lift. The morphism $\tilde{F}$ is holomorphic, so

$$\bar{\partial}(\tilde{F}^*\beta_S(\xi))|_{X_{f(b)}} = \tilde{F}^*(\bar{\partial}\beta_S(\xi)|_{X_{f(b)}}).$$

The form on the right is harmonic, and it represents $\rho_B(\xi)$ by the functorality of the Kodaira–Spencer morphism. Thus $\tilde{F}^*\beta_S = \beta_B$. □

Corollary 5.10. *Denote by h_S the smooth hermitian metric on $\mathcal{X} \times_S \mathcal{K}_S$ constructed in Theorem 4.4. Then h_B on $\mathcal{X} \times_S \mathcal{K}_B$ is the pullback of h_S by the morphism $\tilde{F}$.*

Remark — The corollaries justify our use of the word "natural" for the hermitian metrics constructed in this chapter, as they permit us to construct global metrics on various moduli spaces by gluing together local bases of deformations. Thus we obtain smooth hermitian metrics on the relative Kähler cone and fiber product associated to *any* smooth base of deformations, and not just connected and contractible ones.

6. Examples

It might be useful to look at some special cases of this construction.

Example 6.1. Following Schumacher [**Sch84**] we say that a family $\pi : \mathcal{X} \to S$ of compact Kähler manifolds over a smooth base S is *polarized* if there is a family of Kähler classes ω_s on each manifold X_s such that the map that sends s to the degree two cohomology class ω_s on the underlying smooth manifold is constant. Another way to express this condition is to say that the section $s \mapsto \omega_s$ is parallel with respect to the Gauss–Manin connection on $\mathcal{R}^2\pi_*\mathbb{C}$. Examples of such families include families of manifolds with positive or negative canonical bundle, and families equipped with a relatively ample line bundle $\mathcal{L} \to \mathcal{X}$.

Since the morphism $s \mapsto \omega_s$ is parallel with respect to the Gauss–Manin connection, and since ω_s is a $(1,1)$-class on each manifold X_s, then the polarization gives rise to a holomorphic section ω of the relative Kähler cone $\mathcal{K}$. Let us denote by $\mathcal{X}_\omega$ the restriction of the fiber product $\mathcal{X} \times_S \mathcal{K}$ to the image of the section ω in $\mathcal{K}$. Then the restriction of the natural hermitian metric on $\mathcal{X} \times_S \mathcal{K}$ to $\mathcal{X}_\omega$ is closely related to the constructions of a Weil–Petersson metric on S in [**Nan86, Sch85, Siu86**].

Indeed, the canonical lifts considered in those papers are exactly the harmonic lifts we have constructed, once restricted to $\mathcal{X}_\omega$, which permits to construct the Weil–Petersson metric as we have done. Generalizing this construction of Nannicini–Schumacher–Siu was one of the earliest motivations of our work.

Example 6.2. A somehow orthogonal approach to the earlier example is to fix a compact Kähler manifold X and only regard its complexified Kähler cone $K(X)$. We can then pick a complex manifold B and a morphism $f : B \to K(X)$, and obtain a "family" $X \times B \to B$ where the complexified Kähler class $f(b)$ on X varies holomorphically with b. By picking a representative of the first Chern class of X and using the Kähler metrics in each class thus obtained, we find ourselves in the situation of variation of Kähler structures on the complex manifold X.

These two examples can of course be combined by picking an arbitrary holomorphic morphism $B \to \mathcal{K}$, which is not necessarily a section of $\mathcal{K} \to S$.

Example 6.3. Let (M, g) be a compact simply connected hyperkähler manifold, where g is a Ricci-flat Riemannnian metric with holonomy group equal to $Sp(k)$ for $k = \frac{1}{4}\dim_{\mathbb{R}} M$. There exist three complex structures I, J and K on M for which g is a Kähler metric. If $t = (a, b, c) \in S^2$ then in fact any tensor of the form $I_t = aI + bJ + cK$ is an integrable almost complex structure on M. One can show that in this way we obtain a holomorphic family $\pi : \mathcal{X} \to \mathbb{P}^1$, where $\mathbb{P}^1 \simeq S^2$ and $\mathcal{X}_t = (M, I_t)$, see [**GHJ03**]. The space $\mathcal{X}$ is called the *twistor space* of the hyperkähler manifold (M, g).

There is a natural section $\mathbb{P}^1 \to \mathcal{K}$ given by sending t to the Kähler form of g with respect to the complex structure I_t, which gives an embedding $\mathcal{X} \hookrightarrow \mathcal{X} \times_{\mathbb{P}^1} \mathcal{K}$. In the literature one finds a construction of a lift of the tangent bundle of $\mathbb{P}^1$ into the twistor space $\mathcal{X}$, and consequently of a hermitian metric on the twistor space $\mathcal{X}$. These are the restrictions of the harmonic lift and metric on the fiber product $\mathcal{X} \times_{\mathbb{P}^1} \mathcal{K}$.

CHAPTER 2

The geometry of Kähler cones

Let X be a compact Kähler manifold and let ω be a Kähler class on X, that is, is a real $(1,1)$-class that contains a Kähler metric. The hard Lefschetz theorem says that the symmetric bilinear form

$$(u,v) \longmapsto -u \wedge v \wedge \omega^{n-2}/(n-2)!$$

on the vector space $H^{1,1}(X,\mathbb{R})$ has signature $(h^{1,1}-1,1)$. If we add a correction factor this becomes an honest inner product, and by varying the Kähler class ω we obtain a Riemannian metric g on the Kähler cone of X. One can then reverse this construction and find that the metric g is given by the Hessian of the logarithm of the volume function on the Kähler cone.

Wilson studied this metric in [**Wil04**]. He obtained explicit formulas for its curvature tensor, expressed in terms of the intersection product on the cohomology ring of X, and asked if the sectional curvature of the metric is bounded between $-1/2n(n-1)$ and 0.

In the same paper, and later with Trenner in [**TW11**], Wilson also proposed that this metric on the Kähler cone should correspond to the Weil–Petersson metric on the base of deformations of a family of Kähler manifolds under mirror symmetry.

We investigate the curvature tensor of this metric in this paper and obtain an explicit expression of its curvature tensor in Theorem 6.2.

The main idea is to use the Aubin–Calabi–Yau theorem to embed the Kähler cone into the infinite dimensional manifold $\mathcal{M}$ of hermitian metrics on X. The Hodge L^2 metric defines a Riemannian metric on this space, and the embedding of the Kähler cone therein is Riemannian. The sectional curvature of $\mathcal{M}$ is seminegative and we are able to give an explicit formula for the contribution of the second fundamental form of the embedding to the curvature of the Kähler cone. Unfortunately, we are not able to estimate its positivity at this point.

This chapter is organized thus: We define the Riemannian metric on the Kähler cone and give some examples in Sections 1 and 2. The buildup and proof of our main theorem are carried out in Sections 3–6. We then finish by discussing functorial properties of the metric and precising when it is complete in Sections 7 and 8.

1. The Kähler cone

Let X be a compact Kähler manifold of dimension $\dim_{\mathbb{C}} X = n$. Let

$$K := \{\omega \in H^{1,1}(X,\mathbb{R}) \mid \omega \text{ contains a Kähler metric}\}$$

be the Kähler cone of X. The set K is an open cone in the finite-dimensional vector space $H^{1,1}(X,\mathbb{R})$.

We will regard the cone K as a smooth manifold. There is a tautological smooth function on the space K, defined by sending each Kähler class to its volume. Explicitly,

$$\mathrm{Vol} : K \longrightarrow \mathbb{R}_+, \quad \omega \longmapsto \int_X \frac{\omega^n}{n!}.$$

This function is a submersion and its fiber over a point λ is the set

$$K_\lambda = \{\omega \in K \mid \mathrm{Vol}(X, \omega) = \lambda\}.$$

All the fibers of Vol are clearly diffeomorphic one to another.

Proposition 1.1. *The smooth function $\varphi = -\log$ Vol is strictly convex and defines a Riemannian metric g on K. If Ω is a Kähler metric in a class ω and U and V are the harmonic representatives of u and v, then*

$$g(u,v)(\omega) = \frac{1}{Vol(X,\Omega)} \int_X \langle U, V \rangle \, dV_\Omega,$$

where the inner product is the one induced by Ω on smooth $(1,1)$-forms.

Proof: Fix a point ω of K and tangent vectors u and v at ω. Differentiating once we find

$$D_v \varphi = \frac{-1}{\mathrm{Vol}(X,\omega)} \int_X v \wedge \frac{\omega^{n-1}}{(n-1)!},$$

and differentiating again gives

$$\begin{aligned} D_u D_v \varphi = & \frac{1}{\mathrm{Vol}(X,\omega)} \int_X u \wedge \frac{\omega^{n-1}}{(n-1)!} \frac{1}{\mathrm{Vol}(X,\omega)} \int_X v \wedge \frac{\omega^{n-1}}{(n-1)!} \\ & - \frac{1}{\mathrm{Vol}(X,\omega)} \int_X u \wedge v \wedge \frac{\omega^{n-2}}{(n-2)!}. \end{aligned}$$

Now write $u = u_0 \, \omega + u_1$ and $v = v_0 \, \omega + v_1$ for the primitive decomposition of the classes u and v. Substituting these into the integrals above we get

$$D_u D_v \varphi = n \, u_0 v_0 - \frac{1}{\mathrm{Vol}(X,\omega)} \int_X u_1 \wedge v_1 \wedge \frac{\omega^{n-2}}{(n-2)!}. \tag{3}$$

Here the classes u_0 and v_0 are simply real numbers, and the classes u_1 and v_1 are primitive. The hard Lefschetz theorem [**Huy05**, Chapter 3.3] then entails that the above expression is positive semidefinite in u and v. We also have that $D_u D_u \varphi = 0$ if and only if $u_0 = u_1 = 0$, which happens only if $u = 0$ by the unicity of the primitive decomposition. The Hessian of φ is thus positive definite, so φ is convex.

For the second part of the proposition, simply decompose the forms U and V into their primitive components. Using again the hard Lefschetz theorem we find that the L^2 inner product of U and V, once normalized by the volume $\mathrm{Vol}(X, \Omega)$, is exactly the expression (3), with U and V in the place of u and v. But this expression only depends on the cohomology classes of the forms involved. □

Remark — We can complexify the Kähler cone and consider the cone $K_\mathbb{C} = H^{1,1}(X, \mathbb{R}) + iK$ in $H^{1,1}(X, \mathbb{C})$. The same function φ then defines a Kähler metric on $K_\mathbb{C}$. It is expressed in exactly the same way as the metric g, modulo a factor of $1/4$ coming from the complexification and a substitution of $\overline{v}$ for v.

Equation (3) strongly suggests that the metric g arises as a product metric. This is indeed the case; to see this one can fix a real number λ and consider the diffeomorphism

$$\mathbb{R}_+ \times K_\lambda \longrightarrow K \quad (t, \omega) \mapsto t\omega.$$

The pullback of g to the product space $\mathbb{R}_+ \times K_\lambda$ by this diffeomorphism is then, up to multiples of some constants, the product of the standard complete metric on $\mathbb{R}_+$ by the metric on K_λ given by integrating the product of two primitive classes against ω^{n-2}.

This illustrates a point. The interesting geometry of the Kähler cone takes place in the level sets of K_λ, or the set of classes of volume λ, since the behavior of classes under scaling is completely understood. I imagine this is why other authors have chosen to pick and focus on a set K_λ instead of the entire Kähler cone K. We choose to work with the entire Kähler cone, for no good reason.

It is entirely possible to calculate the curvature tensor of g in terms of the intersection product on $H^*(X, \mathbb{R})$, see [**Wil04**]. However, this approach produces terms whose positivity is hard to control, in part because we do not have a satisfactory description of the the Levi-Civita connection of the metric on the Kähler cone in terms of the intersection product on degree $(1,1)$-cohomology.

2. Examples

Example 2.1. Let X be a compact Kähler manifold with Hodge number $h^{1,1}(X) = 1$. Then the Kähler cone of X is isomorphic to the positive real line, for if we let ω_1 be the unique Kähler class of volume 1 in $K(X)$ then any other Kähler class is a positive multiple of ω_1. As $\mathrm{Vol}(X, t\omega_1) = t^n$ then the metric g is given by the Hessian of $-n \log t$. The reader may be more familiar with the complexification of this metric, which is just the Poincaré metric on the upper half plane.

Example 2.2. Let X be the blowup of the projective plane $\mathbb{P}^2$ in a point. Let H be the pullback of a hyperplane divisor on $\mathbb{P}^2$ to X and let E be the exceptional divisor of the blowup. We may pick H so that it does not contain the point we blew up, so the intersection numbers of these classes are $H^2 = 1$, $E^2 = -1$ and $EH = 0$.

The divisors E and H span the group $H^{1,1}(X, \mathbb{R})$, and the Nakano–Moishezon criterion shows that the Kähler cone of X is

$$K(X) = \{-aE + bH \mid a > 0,\, b > 0,\, b > a\}.$$

Indeed, the first two conditions on a and b ensure that the intersection of any class in $K(X)$ with either E or H is positive, and the last condition is needed to ensure that $\mathrm{Vol}(X, -aE + bH) = (b^2 - a^2)/2$ is positive. One can calculate that

$$g(a, b) = \frac{1}{2\,\mathrm{Vol}^2} \begin{pmatrix} a^2 + b^2 & -2ab \\ -2ab & a^2 + b^2 \end{pmatrix}.$$

The thing to note here is that this metric is not complete. In fact the length of any path that approach a boundary point of the type $(0, b)$ with $b > 0$ will be finite. Geometrically this corresponds to collapsing the volume of the exceptional divisor E to zero.

Here the metric is not complete, morally speaking, because it is given by the Hessian of a function that does not blow up along the entire boundary. In more geometric terms, there is a big class H on X that is not Kähler. Proposition 8.2 shows that this occurs often and that the metric g is almost never complete.

Example 2.3. Let V be a complex vector space of dimension n and let Γ be a lattice in V. Then $X = V/\Gamma$ is a complex torus. Its degree $(1,1)$ cohomology group is canonically isomorphic to $\bigwedge^{1,1} V^*$. If we pick a basis of V, then an element ω in $\bigwedge^{1,1} V^*$ is given by a $n \times n$ matrix Ω of complex numbers. The element ω is real if Ω is hermitian, and a Kähler class if Ω is positive-definite. One may calculate that the metric on $K(X)$ is given by

$$g(U, V)(\Omega) = \operatorname{tr}(\Omega^{-1} U \Omega^{-1} V).$$

A slightly painful way to see this is to calculate the Hessian of the volume function $-\log \operatorname{Vol}(X, \Omega) = -\log \det \Omega$. A better way is through deep inner reflection and linear algebra.

The general linear group $GL(V)$ acts on $\bigwedge^{1,1} V^*$. This action preserves the Kähler cone of X and is transitive on it. If Ω is a positive-definite hermitian matrix and G is in $GL(V)$, then

$$\operatorname{Vol}(X, G \cdot \Omega) = \det(G \Omega^t \overline{G}) = |\deg G|^2 \det \Omega = |\det G| \operatorname{Vol}(X, \Omega).$$

Once we take the logarithm and Hessian of both sides we find that G is an isometry on the Kähler cone of X. Thus $GL(V)$ is a transitive group of isometries on $K(X)$, so the metric g is complete.

In this case the Kähler cone is a homogeneous Riemannian manifold with nonpositive sectional curvature. One can calculate that it is actually zero in certain directions. Theorem 6.2 shows that this situation generalizes somewhat; the curvature tensor of g is always a perturbation of the curvature tensor of a symmetric space.

3. The Aubin–Calabi–Yau theorem

Let us recall a version of the Aubin–Calabi–Yau theorem [**Bes08**, Chapter 11] which appears in [**Huy01**] and is well suited to our interests:

Theorem 3.1. *Let X be a compact Kähler manifold. If dV is a smooth volume form*[1] *on X, then every Kähler class ω contains a unique Kähler metric Ω whose volume form is*

$$dV_\Omega = \frac{\Omega^n}{n!} = c\, dV,$$

where the constant c is $Vol(X, \Omega)/\, Vol(X, dV)$.

[1] To be completely precise we need dV to be compatible with the orientation defined by the complex structure on X, that is, we want $\operatorname{Vol}(X, dV) > 0$.

The reader may recall that the Aubin–Calabi–Yau theorem is usually stated as saying that if a smooth form ρ represents the class $2\pi c_1(X)$, then every Kähler class contains a unique metric Ω whose Ricci-form is ρ. However, choosing a form ρ results in the same metrics in each class as choosing a volume form dV. We'll sketch the equivalence between the two because it's fun:

First, if we fix a Kähler metric Ω_0 whose Ricci-form is ρ, we can take the volume form $dV = \Omega_0^n/n!$. The volume form of any other Kähler metric Ω is then $\Omega^n/n! = f\,dV$, where f is a smooth function. If $\operatorname{Ric}\Omega = \rho$ as well, then a quick calculation shows that the function f is pluriharmonic and thus constant.

Conversely, suppose we fix a volume form dV. This form gives a hermitian metric h on the canonical bundle K_X, for if α and β are local sections of K_X, then we can define h by the equality

$$i^{n^2}\alpha \wedge \overline{\beta} = h(\alpha, \overline{\beta})\,dV.$$

Now take ρ to be the curvature form of the metric h. If Ω is a Kähler metric whose Ricci-form is ρ, then we see as before that its volume form must be a constant multiple of dV.

Let $\mathcal{M}$ be the space of all hermitian metrics Ω on X. It is an infinite dimensional manifold that has the structure of an open set in the vector space of smooth $(1,1)$-forms on X. The space $\mathcal{M}$ is equipped with a Riemannian metric

$$G(U,V)(\Omega) = \frac{1}{\operatorname{Vol}(X,\Omega)}\int_X \langle U, V\rangle\, dV_\Omega,$$

where the inner product under the integral sign is the one induced by Ω on the space of smooth $(1,1)$-forms on X. The non-normalized version of this metric is known as the Ebin metric [**Ebi70**] and has received much attention in the Riemannian world, see for example [**CR11**].

Now, and for the rest of the paper, we fix a volume form dV that is compatible with the orientation defined by the complex structure of X. Let $\mathcal{M}_K \subset \mathcal{M}$ be the closed subspace of Kähler metrics on X. It is a smooth submanifold of $\mathcal{M}$. Following Huybrechts [**Huy01**] we define the *non-linear Kähler cone* of X by

$$\mathcal{K} = \{\Omega \in \mathcal{M}_K \mid dV_\Omega = c\,dV, \quad c > 0\} = \{\Omega \in \mathcal{M}_K \mid \operatorname{Ric}\Omega = \rho\}$$

where ρ is the curvature form of the hermitian metric defined by dV on the canonical bundle of X. Note that there is a smooth map $p : \mathcal{M}_K \to K$, given by sending a Kähler metric to its cohomology class. The Aubin–Calabi–Yau theorem now says that the restriction of p to $\mathcal{K}$ is a bijection. We refer to [**Huy01**, Section 1] for the proof of:

Proposition 3.2. *The set $\mathcal{K}$ is a smooth submanifold of $\mathcal{M}_K$, whose tangent space at Ω is the space of Ω-harmonic $(1,1)$-forms on X. The smooth map $p : \mathcal{K} \to K$ is a diffeomorphism.*

Denote by $f : K \to \mathcal{K} \hookrightarrow \mathcal{M}$ the composition of inverse of the diffeomorphism p and the injection of $\mathcal{K}$ into $\mathcal{M}$. By the above, it is an embedding of the Kähler cone K into the space $\mathcal{M}$ of hermitian metrics on X. Perhaps unsurprisingly, we have:

Proposition 3.3. *The morphism $f : K \to \mathcal{M}$ is an isometric embedding of Riemannian manifolds.*

Proof: Let ω be a point of K and denote by Ω its image under f. If u is a tangent vector of K at ω, then its pushforward f_*u is a Ω-harmonic form on X. Since the composition $p \circ f$ is the identity map on K, and p_* sends a form to the cohomology class it represents, the form f_*u must be the Ω-harmonic form that represents the class u.

The pullback of G to K is now given by

$$f^*G(u, v) = G(f_*u, f_*v),$$

but the right hand side here is equal to $g(u, v)$ by Proposition 1.1. □

Our attack vector should now be clear. Instead of grappling with the intersection product in the cohomology ring of X, we will try our luck with the restriction of the metric G to the non-linear Kähler cone $\mathcal{K}$.

4. The Levi-Civita connection on $\mathcal{M}$

Our objective is to calculate the curvature of the submanifold $\mathcal{K}$ of $\mathcal{M}$. The first step in this direction is to describe the geometry of the embedding $\mathcal{K} \longrightarrow \mathcal{M}$ and the Levi-Civita connection of the metric on $\mathcal{K}$. This gives the curvature of $\mathcal{M}$. If we can then describe the second fundamental form of $\mathcal{K}$ then we have all we need.

The curvature tensor of $\mathcal{M}$ seems to be known, it is basically the curvature tensor of a locally symmetric space of noncompact type. However I had a devil of a time finding a suitable reference for this fact, so we will calculate this tensor here. To do this we need to perform differential calculus on the infinite dimensional manifold $\mathcal{M}$. As explained in Chapter 2 of [**Lan99**] this need not strike fear into our hearts; the usual Lie and exterior derivatives exist and interact as in the finite-dimensional case. We'll denote the exterior derivative on M by D to avoid confusion with the exterior derivative d on X.

Let's fix some notation. The space of smooth (p, q)-forms on X will be denoted by $\mathcal{A}^{p,q}$. We note that the tangent bundle $T_{\mathcal{M}}$ is the trivial bundle with fiber $\mathcal{A}^{1,1}$, so the exterior derivative on $\mathcal{M}$ defines a flat connection on $\mathcal{M}$. Remark that we possess a smooth vector bundle over the manifold $\mathcal{M}$. If we denote it by $\mathcal{H}$, then its fiber of a point Ω is

$$\mathcal{H}_\Omega = \mathcal{H}^{1,1}(\Omega),$$

the space of Ω-harmonic $(1, 1)$-forms on X. The tangent bundle of $\mathcal{K}$ is just the restriction of $\mathcal{H}$ to the space $\mathcal{K}$. Hodge theory shows that the quotient bundle of $\mathcal{H}$ in $T_{\mathcal{M}}$ identifies with the bundle whose fibers consists of the forms that are either d or d^*-exact.

An interlude on linear algebra. Let T be a complex vector space of dimension n. One should think of $T = T_{X,x}$ for some point x in X.

There is a canonical isomorphism $\bigwedge^{1,1} T^* = \mathrm{Hom}_{\mathbb{C}}(T, \overline{T}^*)$, so we may view a $(1, 1)$-form u on T as a linear morphism $T \to \overline{T}^*$. The latter space is conjugate dual to itself, and the form u is real if and only if ${}^t\overline{u} = u$ as a linear morphism.

In particular, a hermitian metric ω on T corresponds to an auto-adjoint isomorphism $\omega : T \to \overline{T}^*$. We note that if ω is a hermitian metric and u a $(1,1)$-form, then the composition of linear morphisms $\omega^{-1}u$ is an endomorphism of T. If v is another $(1,1)$-form, then the composition $v\,\omega^{-1}u$ will again be a $(1,1)$-form.

Given a hermitian metric ω on T, we obtain a hermitian metric on the space $\mathrm{Hom}_{\mathbb{C}}(T, \overline{T}^*)$. If u and v are elements of this space, then we have

$$\langle u, \overline{v}\rangle = \mathrm{tr}(\omega^{-1}u\,\omega^{-1t}\overline{v}).$$

One may now verify that if we equip $\bigwedge^{1,1} T^*$ with the metric induced by ω on $(1,1)$-forms, then the canonical isomorphism $\bigwedge^{1,1} T^* = \mathrm{Hom}_{\mathbb{C}}(T, \overline{T}^*)$ is an isometry. We will use this isometry to express the metric on $\bigwedge^{1,1} T^*$ without remark in what follows.

The reader may enjoy comparing the following expression of the Levi-Civita connection with the one given in Section 3 of [**CR11**]. Recall that D denotes the exterior derivative on the infinite-dimensional space $\mathcal{M}$.

Proposition 4.1. *Let U and Z be tangent fields on a neighborhood of a point Ω_0. The Levi-Civita connection is given by*

$$\begin{aligned}\nabla_Z U &= \tfrac{1}{2}\big(\langle Z,\Omega\rangle - G(Z,\Omega)\big)\,U - \tfrac{1}{2}\big(Z\Omega^{-1}U + U\Omega^{-1}Z\big) + D_Z U\\ &=: T(Z)\,U + S(Z,U) + D_Z U.\end{aligned}$$

In particular, if Ω is Kähler and Z is Ω-harmonic, then

$$\nabla_Z U = -\tfrac{1}{2}\big(Z\Omega^{-1}U + U\Omega^{-1}Z\big) + D_Z U.$$

Proof: Let V be another vector field. The metric G is given by

$$G(U,V) = \frac{1}{\mathrm{Vol}(X,\Omega)}\int_X \mathrm{tr}(\Omega^{-1}U\Omega^{-1}V)\,dV_\Omega$$

and the Levi-Civita connection is characterized by the equality

$$Z\cdot G(U,V) = G(\nabla_Z U, V) + G(U, \nabla_Z V)$$

and the symmetry condition $\nabla_Z U - \nabla_U Z = [Z,U]$. To differentiate the function $G(U,V)$ in the direction of a vector field Z we must differentiate three terms: the volume form dV_Ω, the volume $\mathrm{Vol}(X,\Omega)$ and the inner product $\langle U,V\rangle$ inside the integral.

First consider the inner product. Regard the metric Ω as a linear morphism $T_X \to \overline{T}^*_X$. Since $D_Z\Omega = Z$ we get $D_Z\Omega^{-1} = -\Omega^{-1}Z\,\Omega^{-1}$ by using standard formulas for the derivative of the inverse of a linear morphism. Next we find that

$$\begin{aligned}Z\cdot\langle U,V\rangle &= Z\cdot\mathrm{tr}(\Omega^{-1}U\Omega^{-1}V)\\ &= -\,\mathrm{tr}(\Omega^{-1}Z\Omega^{-1}U\Omega^{-1}V) - \mathrm{tr}(\Omega^{-1}U\Omega^{-1}Z\Omega^{-1}V)\\ &\quad + \langle D_Z U, V\rangle + \langle U, D_Z V\rangle\\ &= -\,\tfrac{1}{2}\big(\langle Z\Omega^{-1}U, V\rangle + \langle U, Z\Omega^{-1}V\rangle\big)\\ &\quad - \tfrac{1}{2}\big(\langle U\Omega^{-1}Z, V\rangle + \langle U, V\Omega^{-1}Z\rangle\big)\\ &\quad + \langle D_Z U, V\rangle + \langle U, D_Z V\rangle\end{aligned}$$

on a neighborhood of Ω_0. Here the entries in the first pair of parentheses come from the first trace, and similarly for the second pair. We have split them in this way so the symmetry condition of the Levi-Civita connection will be satisfied. These terms give the tensor $-\frac{1}{2}(Z\Omega^{-1}U+U\Omega^{-1}Z)+D_ZU = S(Z,U)+D_ZU$.

Next recall that the volume form of a hermitian metric is $dV_\Omega = \Omega^n/n!$. Differentiating this in the direction of Z we get

$$Z\cdot dV_\Omega = Z\wedge\frac{\Omega^{n-1}}{(n-1)!} = \mathrm{tr}_\Omega(Z)\,dV_\Omega = \langle Z,\Omega\rangle\,dV_\Omega.$$

The derivative of the volume is then

$$Z\cdot\mathrm{Vol}(X,\Omega) = \int_X \mathrm{tr}_\Omega(Z)\,dV_\Omega = G(Z,\Omega)\,\mathrm{Vol}(X,\Omega).$$

Thus he contributions of the volume and the volume form to $Z\cdot G(U,V)$ are

$$\frac{1}{\mathrm{Vol}(X,\Omega)}\int_X\langle U,V\rangle\langle Z,\Omega\rangle dV_\Omega - G(Z,\Omega)G(U,V).$$

We split each factor in two, and incorporate one into U and the other into V as before. This gives the tensor $T(Z)\,U$ announced in the proposition.

Now, if Ω is Kähler and Z is harmonic, then the function $\langle Z,\Omega\rangle = \mathrm{tr}_\Omega(Z) = \Lambda Z$ is harmonic because the operators Δ and Λ commute. It is thus constant on X, so $G(Z,\Omega)=\langle Z,\Omega\rangle$, and the above term vanishes. □

Note that even if we take the forms U and Z to be harmonic, it is absolutely not clear that the form $\nabla_U V$ is closed and thus represents a vector tangent to the space of Kähler metrics. In fact, this almost never happens and will represent a major headache when we try to estimate the curvature of our metric.

Since $\mathcal{M}$ is an open set in a vector space, there is a canonical smooth vector field on $\mathcal{M}$, given by $\Omega\mapsto\Omega$. We can calculate the covariant derivative of this vector field.

Corollary 4.2. *If Z is any tangent field on $\mathcal{M}$, then*

$$\nabla_Z\Omega = T(Z)\,\Omega.$$

In particular, if Ω is Kähler and Z is harmonic then $\nabla_Z\Omega = 0$.

Proof: The tangent field Ω is just the identity on $\mathcal{M}$, so $D_Z\Omega = Z$. We see that $Z\Omega^{-1}\Omega+\Omega\Omega^{-1}Z = 2Z$, so the statement follows. As before, if Z is Ω-harmonic then $\langle Z,\Omega\rangle = G(Z,\Omega)$ so $T(Z)=0$. □

5. The curvature tensor on $\mathcal{M}$

The curvature tensor of G is given by

$$R(Z,W)U = \nabla_Z\nabla_W U - \nabla_W\nabla_Z U - \nabla_{[Z,W]}U.$$

We will calculate this tensor. The road is long, with many a winding turn.

Lemma 5.1. *Define a smooth 1-form T on the tangent bundle of $\mathcal{M}$ that takes values in the smooth functions on X by $T(Z)=\frac{1}{2}(\langle Z,\Omega\rangle - G(Z,\Omega))$. If Z and W are tangent fields on $\mathcal{M}$, then*

$$\nabla_Z T(W) - \nabla_W T(Z) = T([Z,W])$$

Proof: The identity to be proved is $\mathbb{R}$-linear in T, so we may multiply everything by 2 to get rid of the factor $\frac{1}{2}$. It is also enough to consider each of the terms $\langle Z, \Omega\rangle$ and $G(Z, \Omega)$ separately.

For the first term we get

$$\nabla_Z \langle W, \Omega\rangle = -\langle W, Z\rangle + \langle D_Z W, \Omega\rangle.$$

We note that

$$D_Z W - D_W Z = D_Z D_W \Omega - D_W D_Z \Omega = D_{[Z,W]}\Omega = [Z, W],$$

and thus

$$\nabla_W \langle Z, \Omega\rangle - \nabla_Z \langle W, \Omega\rangle = \langle [Z, W], \Omega\rangle.$$

For the second term we have

$$\nabla_Z G(W, \Omega) = G(\nabla_Z W, \Omega) + G(W, \nabla_Z \Omega) = G(\nabla_Z W, \Omega) + G(W, T(Z)\Omega),$$

Going back to the definitions of the metric G and the tensor T, we note that

$$\begin{aligned} G(W, T(Z)\Omega) &= \frac{1}{\mathrm{Vol}(X, \Omega)} \int_X \mathrm{tr}(\Omega^{-1} W(\langle Z, \Omega\rangle + G(Z, \Omega))\Omega\Omega^{-1}) dV_\Omega \\ &= \frac{1}{\mathrm{Vol}(X, \Omega)} \int_X \langle W, \Omega\rangle\langle Z, \Omega\rangle \, dV_\Omega + G(W, \Omega)\, G(Z, \Omega) \end{aligned}$$

is symmetric in Z and W. Thus

$$\nabla_Z G(W, \Omega) - \nabla_W G(Z, \Omega) = G(\nabla_Z W - \nabla_W Z, \Omega) = G([Z, W], \Omega).$$

Putting the two together we obtain the statement of the lemma. □

Let us define an affine connection ∇' on $\mathcal{M}$ by setting

$$\nabla'_Z U = S(Z, U) + D_Z U.$$

It differs from the Levi-Civita connection ∇ only by the tensor T. We'll also write R' for the curvature tensor of the connection ∇'.

Lemma 5.2. *The curvature tensors R and R' are equal.*

Proof: Let U, Z and W be tangent fields on $\mathcal{M}$. We have

$$\begin{aligned} \nabla_Z \nabla_W U &= \nabla_Z (T(W)U + \nabla'_W U) \\ &= (\nabla_Z T(W))U + T(W)\nabla_Z U + \nabla_Z \nabla'_W U \\ &= (\nabla_Z T(W))U + T(W)T(Z)U \\ &\quad + T(W)\nabla'_Z U + T(Z)\nabla'_W U + \nabla'_Z \nabla'_W U, \end{aligned}$$

and similarly

$$\begin{aligned} \nabla_W \nabla_Z U &= (\nabla_W T(Z))U + T(Z)T(W)U \\ &\quad + T(Z)\nabla'_W U + T(W)\nabla'_Z U + \nabla'_W \nabla'_Z U. \end{aligned}$$

Note that each of the terms $T(Z)T(W)U$, $T(Z)\nabla'_W U$ and $T(W)\nabla'_Z U$ appears once in each expression. Thus

$$\begin{aligned} \nabla_Z \nabla_W U - \nabla_W \nabla_Z U &= (\nabla_Z T(W) - \nabla_W T(Z))U + \nabla'_Z \nabla'_W U - \nabla'_W \nabla'_Z U \\ &= T([Z, W])U + \nabla'_Z \nabla'_W U - \nabla'_W \nabla'_Z U. \end{aligned}$$

Finally $\nabla_{[Z,W]} U = T([Z, W])\, U + \nabla'_{[Z,W]} U$, so $R(Z, W)U = R'(Z, W)U$ as promised. □

Some notation will be useful before going further. If Z and W are $(1,1)$-forms, we set

$$\{Z, W\} := Z\Omega^{-1}W - W\Omega^{-1}Z.$$

This is again a $(1,1)$-form, and real if Z and W are real. This bracket is antisymmetric and satisfies the Jacobi identity, as the reader may find pleasure in verifying.[2]

Theorem 5.3. *The curvature tensor of $\mathcal{M}$ is*

$$R(U, V, Z, W) = G(R(Z, W)U, V) = \tfrac{1}{4}G(\{Z, W\}, \{U, V\})$$

and the sectional curvature of $\mathcal{M}$ is non-positive.

Proof: It is enough to show that the identity holds for the curvature tensor R'. We start by noting that

$$\begin{aligned}\nabla'_Z\nabla'_W U &= -\tfrac{1}{2}\nabla'_Z(W\Omega^{-1}U + U\Omega^{-1}W) + \nabla'_Z D_W U\\ &= \tfrac{1}{4}(Z\Omega^{-1}W\Omega^{-1}U + Z\Omega^{-1}U\Omega^{-1}W + W\Omega^{-1}U\Omega^{-1}Z + U\Omega^{-1}W\Omega^{-1}Z)\\ &\quad - \tfrac{1}{2}D_Z(W\Omega^{-1}U + U\Omega^{-1}W) - \tfrac{1}{2}(Z\Omega^{-1}D_W U + D_W U\Omega^{-1}Z) + D_Z D_W U.\end{aligned}$$

Next we see that

$$\begin{aligned}D_Z(W\Omega^{-1}U + U\Omega^{-1}W) &= D_Z W\Omega^{-1}U - W\Omega^{-1}Z\Omega^{-1}U + W\Omega^{-1}D_Z U\\ &\quad + D_Z U\Omega^{-1}W - U\Omega^{-1}Z\Omega^{-1}W + U\Omega^{-1}D_Z W,\end{aligned}$$

so in total

$$\begin{aligned}\nabla'_Z\nabla'_W U &= \tfrac{1}{4}(Z\Omega^{-1}W\Omega^{-1}U + U\Omega^{-1}W\Omega^{-1}Z)\\ &\quad + \tfrac{1}{4}(Z\Omega^{-1}U\Omega^{-1}W + W\Omega^{-1}U\Omega^{-1}Z)\\ &\quad + \tfrac{1}{2}(W\Omega^{-1}Z\Omega^{-1}U + U\Omega^{-1}Z\Omega^{-1}W)\\ &\quad - \tfrac{1}{2}(D_Z W\Omega^{-1}U + U\Omega^{-1}D_Z W)\\ &\quad - \tfrac{1}{2}(D_Z U\Omega^{-1}W + D_W U\Omega^{-1}Z)\\ &\quad - \tfrac{1}{2}(Z\Omega^{-1}D_W U + W\Omega^{-1}D_Z U)\\ &\quad + D_Z D_W U.\end{aligned}$$

We encourage the reader to stare at this expression for a little while, and to appreciate that we have moved some terms between parentheses.

Remark that the term $\nabla'_W\nabla'_Z U$ can be obtained by formally exchanging the fields Z and W. Do so, and write the resulting mess next to the above expression so we can compare them line for line. The first line of the difference between the two is

$$\begin{aligned}&\tfrac{1}{4}(Z\Omega^{-1}W\Omega^{-1}U + U\Omega^{-1}W\Omega^{-1}Z - W\Omega^{-1}Z\Omega^{-1}U - U\Omega^{-1}Z\Omega^{-1}W)\\ &\quad = \tfrac{1}{4}(\{Z, W\}\Omega^{-1}U + U\Omega^{-1}\{W, Z\}) = \tfrac{1}{4}\{\{Z, W\}, U\}.\end{aligned}$$

We note that the second line of the expression is symmetric in Z and W, so it contributes nothing to the curvature tensor. Now, the third line of the

[2] Just note that this is the commutator on the space of global sections of $\operatorname{End} T_X$ under the isometry $\Omega : \operatorname{End} T_X \to \bigwedge^{1,1} T_X^*$.

difference is

$$\tfrac{1}{2}(W\Omega^{-1}Z\Omega^{-1}U + U\Omega^{-1}Z\Omega^{-1}W - Z\Omega^{-1}W\Omega^{-1}U - U\Omega^{-1}W\Omega^{-1}Z)$$
$$= \tfrac{1}{2}(\{W,Z\}\Omega^{-1}U + U\Omega^{-1}\{Z,W\}) = -\tfrac{1}{2}\{\{Z,W\},U\}.$$

The fourth and seventh lines together give

$$-\tfrac{1}{2}(D_Z W\Omega^{-1}U + U\Omega^{-1}D_Z W - D_W Z\Omega^{-1}U - U\Omega^{-1}D_W Z) + D_Z D_W U - D_W D_Z U$$
$$= -\tfrac{1}{2}([Z,W]\Omega^{-1}U + U\Omega^{-1}[Z,W]) + D_{[Z,W]}U = \nabla'_{[Z,W]}U,$$

which looks very promising. This leaves the fifth and sixth lines. But both of them are symmetric in Z and W and thus contribute nothing to the curvature tensor. Taken together, we have

$$\nabla'_Z\nabla'_W U - \nabla'_W\nabla'_Z U = -\tfrac{1}{4}\{\{Z,W\},U\} + \nabla'_{[Z,W]}U,$$

which gives $R(Z,W)\,U = -\frac{1}{4}\{\{Z,W\},U\}$.

We now claim that the identity

$$\langle\{\{Z,W\},U\},V\rangle = -\langle\{Z,W\},\{U,V\}\rangle$$

holds pointwise on X. In order to prove this, we pick a hermitian metric Ω and an orthonormal frame at some point x and represent the forms U, V, Z and W by hermitian matrices u, v, z and w. We note that the bracket $\{U,V\}$ is just the commutator $[u,v]$, and that the identity $\operatorname{tr}(u[z,w]) = \operatorname{tr}([u,z]w)$ holds by elementary calculations. This gives

$$\begin{aligned}\langle\{\{Z,W\},U\},V\rangle &= \operatorname{tr}([[z,w],u]\,{}^t\overline{v}) = \operatorname{tr}([z,w],[u,v])\\ &= -\operatorname{tr}([z,w],{}^t\overline{[u,v]}) = -\langle\{\{Z,W\},\{U,V\}\rangle\end{aligned}$$

and implies that the curvature tensor has the stated form. If the tangent fields U and V have unit norm, the sectional curvature of the metric is

$$K(U,V) = R(U,V,V,U) = \tfrac{1}{4}G(\{U,V\},\{V,U\}) = -\tfrac{1}{4}G(\{U,V\},\{U,V\}),$$

which is non-positive. $\square$

6. The curvature of $\mathcal{K}$

Recall that the nonlinear Kähler cone $\mathcal{K}$ is the subspace of $\mathcal{M}$ defined by

$$\mathcal{K} = \{\Omega \in \mathcal{M} \mid d\Omega = 0 \quad \text{and} \quad \operatorname{Ric}\Omega = \rho\}$$

where ρ is a fixed smooth $(1,1)$-form that represents the Chern class $-c_1(X)$. Huybrechts [**Huy01**, Section 1] showed that the tangent space of $\mathcal{K}$ at a point Ω is the space of real harmonic $(1,1)$-forms. We thus get a short exact sequence

$$0 \longrightarrow T_{\mathcal{K}} \longrightarrow T_{\mathcal{M}|\mathcal{K}} \longrightarrow N_{\mathcal{K}/\mathcal{M}} \longrightarrow 0$$

of vector bundles over $\mathcal{K}$.

Proposition 6.1. *The second fundamental form of $\mathcal{K}$ in $\mathcal{M}$ at a point Ω is*

$$\mathrm{I\!I}(U,V) = \Delta Gr\,\nabla_V U,$$

where Δ and Gr is the Laplacian and the Green operator associated to the metric Ω.

Proof: We can decompose the identity morphism on the space of smooth $(1,1)$-forms as

$$\mathrm{id} = h_\Omega + \Delta Gr,$$

where h_Ω is the projection onto the space of harmonic forms and Gr is the Green operator. This decomposition is orthogonal by Hodge theory and the operator h_Ω identifies with the projection onto $T_\mathcal{K}$. The operator ΔGr thus identifies with the orthogonal projection pr onto the normal bundle $N_{\mathcal{K}/\mathcal{M}}$. One expression for the second fundamental form is $\mathrm{II} = pr(\nabla_U V)$. □

Classical formulas now express the curvature tensor of the subspace $\mathcal{K} \subset \mathcal{M}$ in terms of the second fundamental form and the curvature of $\mathcal{M}$.

Theorem 6.2. *The curvature tensor of the space $\mathcal{K}$ at a point Ω is*

$$\begin{aligned} R^\mathcal{K}(U,V,Z,W) &= R^\mathcal{M}(U,V,Z,W) \\ &\quad + G(\mathrm{II}(U,W), \mathrm{II}(V,Z)) - G(\mathrm{II}(U,Z), \mathrm{II}(V,W)). \end{aligned}$$

Remark — Wilson [**Wil04**] asked for which manifolds X is the sectional curvature of the Kähler cone seminegative? I would have liked to make progress on this question, but so far my earth is barren. Here are some approaches to this question that do not work:

1. We know that the sectional curvature of $\mathcal{M}$ is seminegative. Taking U and V orthonormal to simplify matters, we also see that

$$K^\mathcal{K}(U,V) = K^\mathcal{M}(U,V) + G(\mathrm{II}(U,U), \mathrm{II}(V,V)) - G(\mathrm{II}(U,V), \mathrm{II}(U,V)),$$

so it is most tempting to try and prove that $\mathrm{II}(U,U) = 0$ for certain manifolds X and thus that the sectional curvature is seminegative. This doesn't work because the second fundamental form is a bilinear form, so the polarization identity

$$\mathrm{II}(U,V) = \tfrac{1}{4}\big(\mathrm{II}(U+V, U+V) - \mathrm{II}(U-V, U-V)\big)$$

would imply that II is identically zero if $\mathrm{II}(U,U) = 0$ for all U. But then $\mathcal{K}$ would be totally geodesic in $\mathcal{M}$, so $\nabla_V U$ should be tangent to $\mathcal{K}$. In particular, $\nabla_V U$ should be closed, which it most certainly is not.

2. Nevertheless, we can try to simplify the expressions involved. Note that if U and V are harmonic forms, then

$$\nabla_V U = S(U,V) + D_V U.$$

By differentiating the identity $U = h_\Omega U + \Delta Gr U$ in the direction of V, one sees that the smooth $(1,1)$-form $D_V U$ is d-exact when U and V are harmonic. This information is of surprisingly little use.

3. We can also note that $\langle S(U,V), \Omega\rangle = -\langle U, V\rangle$. Taking the Laplacian of the right hand side—because why not?—and applying a Bochner–Weitzenböck formula yields

$$\Delta\langle U,V\rangle = \langle (\mathrm{Ric}\,\Omega)\,\Omega^{-1}U, V\rangle - 2\mathrm{tr}_\Omega\langle \nabla U, \nabla V\rangle + \langle U, (\mathrm{Ric}\,\Omega)\,\Omega^{-1}V\rangle.$$

The middle term may need some explanation. First we apply the connection ∇ induced by the Levi-Civita connection on X to the $(1,1)$-form U, thus obtaining a 1-form ∇U with values in the $(1,1)$-forms on X. Then we do the same to V and take the scalar product of the $(1,1)$-parts, thus obtaining a 2-tensor $\langle \nabla_U, \nabla_V \rangle$. Finally we take the trace of this tensor with respect to the metric.

Unfortunately, this is *not* the scalar product on the space of 1-forms with values in $(1,1)$-forms, so we can't say anything about it. If this were so, then we could for example place ourselves in the case of Ricci-flat manifolds and obtain

$$\Delta |U|^2 = -2|\nabla U|^2 \leq 0.$$

This implies that the function $|U|^2$ is superharmonic on the compact space X, and thus constant. But then the scalar product $\langle U, V \rangle$ of any two harmonic forms would be constant on X by a polarization identity.

This cannot be true: Consider a K3 surface X. Fix a Ricci-flat Kähler metric Ω on X and take U_1, U_2 and U_3 to be primitive, real and harmonic $(1,1)$-forms on X such that $\langle U_j, U_k \rangle = \delta_{jk}$. This can be done since the scalar product $\langle U, V \rangle$ of any harmonic forms is constant. The space of real 2-forms at a given point of X has real dimension 6, and splits orthogonally into self- and anti-self-dual forms. The identity $U_j \wedge *U_k = -\langle U_j, U_k \rangle \Omega^2/2$ implies that each of the forms U_j is anti-self-dual. Thus they form a basis of the space of anti-self-dual forms on X at any point, so any primitive harmonic form V with $\langle U_j, V \rangle = 0$ for $j = 1, 2, 3$ is in fact zero. This implies that the dimension of the space of real primitive forms on X is 3, while we know that it is 19.

There is no known example that shows that the sectional curvature of g is not seminegative for an arbitrary manifold X. Perhaps we can use the above approach of harmonic forms to obtain bounds on the sectional curvature of g. The moral of the above remarks, and the failure of my work over the past few months, seems to be that such a bound will not be obtained by simple means, but requires a detailed analytical approach.

If we continue to take the analogy between the metric g and the Weil–Petersson metric seriously, then I think that the techniques of heat kernels are promising candidates for a more technical approach. These methods have been applied by Siu and Schumacher, among others, to estimate the positivity of the curvature tensor of Weil–Petersson metrics. A serious study of their techniques, with and eye towards our situation, seems worth pursuing.

7. Finite morphisms

Let Y be another compact Kähler manifold and let $f : X \to Y$ be a holomorphic morphism. Then f induces a pullback morphism $f^* : H^*(Y) \to H^*(X)$ that respects the grading of each cohomology algebra. If ω is a Kähler class on Y then its pullback $f^*\omega$ is not a Kähler class on X in general.

However we can impose some conditions on f which ensure that this is the case and thus get a well defined holomorphic morphism of complexified Kähler cones $f^* : K(Y) \to K(X)$. For example, this is the case if f is either a finite morphism or the inclusion of a submanifold into X. We can say something about at least one of those cases:

Proposition 7.1. *Let $f : X \to Y$ be a finite surjective morphism. Let g_X and g_Y be the Riemannian metrics on the Kähler cones of X and*

Y, respectively. Then the pullback morphism $f^* : K(Y) \to K(X)$ is a Riemannian embedding.

Proof: Let ω be a point in $K(Y)$. The volume of X with respect to $f^*\omega$ is

$$\mathrm{Vol}(X, f^*\omega) = p\ \mathrm{Vol}(Y, \omega)$$

as f is finite of degree p. Let us denote the exterior derivatives on $K(X)$ and $K(Y)$ by D. The metric g_X is given by the Hessian of $-\log \mathrm{Vol}_X$, so the pullback f^*g_X is

$$(f^*)^*g_X = -f^* \operatorname{Hess} \log \mathrm{Vol}_X = -\operatorname{Hess} \log(p \mathrm{Vol}_Y) = g_Y,$$

so f^* is an embedding. □

Examples showing that the morphism f^* need not be surjective are plentiful. For example, one can consider a projective manifold X of dimension $\dim_{\mathbb{C}} = n$ with Hodge number $h^{1,1} > 1$. Any projective manifold of dimension n admits a finite surjective morphism $f : X \to \mathbb{P}^n$, but the projective space has Hodge number $h^{1,1} = 1$. The Kähler cone of $\mathbb{P}^n$ is thus a ray, which will be embedded into the higher dimensional cone of X, but will not coincide with the entire cone.

Corollary 7.2. *The group Aut(X) of holomorphic automorphisms of X acts by isometries on the Kähler cone $K(X)$.*

A closer look reveals that this last statement contains somewhat less information than first meets the eye. The automorphism group $\mathrm{Aut}(X)$ of a compact complex manifold is a Lie group and it splits roughly into two parts; a positive-dimensional group given by the flows of holomorphic vector fields, or elements of $H^0(X, T_X)$, and a discrete part consisting of "other" automorphisms. Now, the flow of any vector field is, almost by definition, homotopic to the identity morphism, and its pullback thus acts trivially on the cohomology ring of X. The only part of $\mathrm{Aut}(X)$ that possibly acts by non-trivial isometries on $K(X)$ is thus discrete.

Remark — If we agree with Grothendieck's yoga of using functoriality to take compass bearings in mathematics, then the bad functorial properties of the Kähler cone and the metric are cause for worry. There are at least two ways to better the situation.

First, we could simply restrict our morphisms to those that preserve the Kähler cone. This leaves finite surjective morphisms, embeddings of submanifolds, and certainly other things we may have to describe on a case-by-case basis. Somehow this seems unsatisfactory.

Second, we could enlarge the cone. The problem here is that some of the natural candidates for this enlargement, like the movable Kähler cone or the big cone, have singular boundaries. Also, even if we can extend the bilinear form g to these cones, it is quite unclear what its signature is on these larger cones.

8. On completeness

The Kähler cone of a compact complex manifold X is described by the following result:

Theorem (Demailly-Paun, [**DP04**]). *Let X be a compact Kähler manifold. Then the Kähler cone of X is one of the connected components of the set of real $(1,1)$ cohomology classes a which are numerically positive on analytic cycles, i.e. such that $\int_Z a^p > 0$ for every irreducible analytic set Z in X of dimension p.*

The boundary of the Kähler cone of a compact complex manifold then consists of three parts. The first two have simple descriptions in terms of the volume of the manifold X, but the third is slightly more subtle.

The first contains of the limits of Kähler classes a_t whose volume $\mathrm{Vol}(X, a_t) = \frac{1}{n!}\int_X a_t^n$ tends to zero, or, in other words, of classes that are nef but neither Kähler nor big. The second part then consists of the limits of classes whose volume $\mathrm{Vol}(X, a_t) = \frac{1}{n!}\int_X a_t^n$ tends to infinity.

Finally, the third part consists of limits of classes whose volume $\mathrm{Vol}(X, a_t) = \frac{1}{n!}\int_X a_t^n$ tends to some positive real number, or classes that are both nef and big, but not Kähler. Here there exists a proper irreducible complex subspace Z of X of dimension $p \geq 1$ such that $\mathrm{Vol}(Z, a_t) = \frac{1}{p!}\int_Z a_{t|Z}{}^p$ tends to zero.

These last classes can also be described as those that lie on the intersection of the boundary of the Kähler cone with the open cone of big classes. We note that the first two parts of the boundary are always present, as one can always follow the one-dimensional ray defined by any Kähler class to zero or infinity. However, the third part of the boundary may be empty; this happens exactly when the Kähler cone is strictly contained in the big cone of the manifold, or equivalently, when the nef cone is strictly contained in the pseudoeffective cone.

Lemma 8.1. *Let $I = [a, b]$ be a compact interval in the real numbers $\mathbb{R}$, and let $\gamma : I \to K(X)$ be a smooth path in $K(X)$. The length of the path γ satisfies*

$$L(\gamma) \geq \frac{1}{\sqrt{n}} \left|\log \mathit{Vol}(X, \gamma(b)) - \log \mathit{Vol}(X, \gamma(a))\right|.$$

Proof: The length of γ with respect to our metric is

$$L(\gamma) = \int_I \sqrt{h(\gamma'(t), \gamma'(t))}\,\mathrm{d}t.$$

Let u be any $(1,1)$-class on X and let ω be a Kähler class on X. Applying the Cauchy-Scwarz inequality to the classes u and ω we find

$$|g(u, \omega)|^2 \leq g(u, u) \cdot g(\omega, \omega) = g(u, u)\, n$$

on one hand. On the other, we calculate that

$$\begin{aligned} g(u, \omega) &= \frac{n}{\mathrm{Vol}(X, \omega)} \int_X u \wedge \frac{\omega^{n-1}}{(n-1)!} - \frac{n-1}{\mathrm{Vol}(X, \omega)} \int_X u \wedge \frac{\omega^{n-1}}{(n-1)!} \\ &= \frac{1}{\mathrm{Vol}(X, \omega)} \int_X u \wedge \frac{\omega^{n-1}}{(n-1)!} = -u \cdot \log \mathrm{Vol}(X, \omega). \end{aligned}$$

Combining the two and applying the triangle inequality for integrals we then get

$$L(\gamma) \geq \frac{1}{\sqrt{n}} \left| \int_I D_{\gamma'(t)} \log \mathrm{Vol}(X, \gamma(t)) \right| = \frac{1}{\sqrt{n}} \left| \int_I \frac{\mathrm{d}}{\mathrm{d}t} \log \mathrm{Vol}(X, \gamma(t)) \right|.$$

The fundamental theorem of calculus now supplies the bound in the statement of the lemma. □

Let us conspire to call $\mathcal{V} = \{\alpha \in H^{1,1}(X,\mathbb{R}) \mid \alpha^n > 0\}$ the cone of volume classes on X. It contains the Kähler cone, but is in almost all cases bigger than it.

Proposition 8.2. *The metric on the Kähler cone of X is complete if and only if the Kähler cone is a connected component of the volume cone.*

Proof: We first show that the classes on the first two parts of the boundary pose no problems. Let I be an interval in the real numbers and let $\gamma : I \to K(X)$ be a smooth path in $K(X)$ that approaches the boundary of $K(X)$. Let $I_m = [a, b_m]$ be an increasing exhaustion of I by compact intervals and let γ_m be the restriction of γ to I_m.

Suppose that the volume $\mathrm{Vol}(X, \gamma_m)$ tends to either zero or infinity as m tends to infinity. Applying Lemma 8.1 on each interval I_m then gives that

$$L(\gamma) = \lim_{m\to+\infty} L(\gamma_m) = +\infty.$$

Thus the limit class $\lim \gamma(t)$ on the boundary cannot be approached by paths in $K(X)$ of finite length.

If the Kähler and volume cones of X coincide, then these are the the only classes on the boundary and we are done. If not, then there exists a class α on the boundary of $K(X)$ that is both nef and big. This means that $\mathrm{Vol}(X, \alpha) > 0$, but that there is a proper complex subspace $Z \subset X$ such that $\mathrm{Vol}(Z, \alpha) = 0$.

As α is on the boundary of the Kähler cone, then there exists a Kähler class ω such that $\gamma(t) := \alpha + t\omega$ is in the Kähler cone for all $t > 0$. We will show that the path $t \mapsto \gamma(t)$ has finite length with respect to our metric, thus completing the proof.

The tangent vectors of the path γ are $\gamma'(t) = \omega$, and the norm of $\gamma'(t)$ at the point $\gamma(t)$ is

$$\begin{aligned} g(\omega,\omega)(\gamma(t)) = &\left(\frac{1}{\mathrm{Vol}(X,\gamma(t))}\int_X \omega \wedge \frac{(\alpha+t\omega)^{n-1}}{(n-1)!}\right)^2 \\ &- \frac{1}{\mathrm{Vol}(X,\gamma(t))}\int_X \omega^2 \wedge \frac{(\alpha+t\omega)^{n-2}}{(n-2)!}. \end{aligned}$$

We may consider each of the integral in this expression as a function of t on some small interval $[0, t_0]$. To show that the length $L(\gamma)$ is finite it is enough to show that each of these functions is continuous as $t \to 0$, since then we find ourselves integrating bounded continuous real-valued functions over a compact interval.

Despite their intimidating appearance, these integrals are actually quite gentle. Indeed, by applying Newton's binomial formula we find that the first integral is a polynomial in t:

$$\int_X \omega \wedge \frac{(\alpha+t\omega)^{n-1}}{(n-1)!} = \sum_{k=0}^{n-1} c_k t^k, \quad \text{where} \quad c_k = \int_X \binom{n-1}{k} \alpha^{n-1-k} \wedge \omega^{k+1}.$$

By hypothesis the limit $\lim_{t\to 0} \mathrm{Vol}(X, \gamma(t)) = \mathrm{Vol}(X, \alpha)$ is a positive real number, so it follows that the function

$$t \longmapsto \frac{1}{\mathrm{Vol}(X, \gamma(t))} \int_X \omega \wedge \frac{(\alpha + t\omega)^{n-1}}{(n-1)!}$$

is continuous on the interval $[0, t_0]$.

A completely analogous argument, which is left to the reader, shows that the second integral also defines a continuous function on the interval $[0, t_0]$.

The length of the path γ is thus equal to the integral of a continuous function over a compact interval, and is thus finite. $\square$

CHAPTER 3

Manifolds with trivial canonical bundle

We now turn our attention to compact Kähler manifolds with trivial canonical bundle. These are complex tori, Calabi–Yau manifolds and hyperkähler manifolds, and products of such manifolds [**Bea83**]. The Weil–Petersson metric associated to polarized families of these manifolds is given by the curvature form of a hermitian line bundle [**Tia87, Wan03**]. We first provide a detailed account of this fact for the benefit of the reader and show that in fact no polarization is required for the construction of the Weil–Petersson metric.

Next we observe that there is a natural hermitian line bundle over the relative Kähler cone associated to a family of such manifolds. There is a link between the curvature form of this metric and the natural hermitian metric constructed in Chapter 1, but more work is needed to see if it is positive or the hermitian metric on $\mathcal{K}$ in fact Kähler.

1. Holomorphic volume forms

Let $\pi : \mathcal{X} \to S$ be a family of compact Kähler manifolds of dimension $\dim_{\mathbb{C}} X_s = n$ over a smooth base S. We will suppose that the manifolds X_s have trivial canonical bundle $K_{X_s} \cong \mathcal{O}_{X_s}$. In this case each manifold admits a holomorphic volume form σ, or a global holomorphic section of K_{X_s}. This form is unique up to multiplication by a non-zero scalar.

There is an advantage to working with manifolds that admit a holomorphic volume form. Indeed, let X be such a manifold and let σ be a holomorphic volume form. Then cup product against σ defines a linear isomorphism

$$H^1(X, T_X) \longrightarrow H^{n-1,1}(X, \mathbb{C})$$

of cohomology groups. This section is devoted to showing that if ω is a Ricci-flat Kähler metric on X then we can choose σ in such a way that this isomorphism becomes an isometry with respect to the inner products that ω induces on the cohomology groups.

The following two propositions are the key steps towards proving this fact. The first one is the local, or linear algebraic, version of this isometry. The second shows how to reduce to this local version in the case of a Ricci-flat metric.

Proposition 1.1. *Let T be a complex vector space of dimension n. Let ω be the $(1,1)$-form of a hermitian inner product on V and let σ be a $(n,0)$-form on T. Suppose that*

$$\frac{\omega^n}{n!} = \frac{i^{n^2}}{2^n}\sigma \wedge \overline{\sigma}.$$

Then the linear morphism

$$\begin{cases} \iota : T & \longrightarrow & \bigwedge^{n-1} T^* \\ u & \longmapsto & \iota_u \sigma \end{cases}$$

is an isometry with respect to ω *and the inner product induced by* ω *on* $\bigwedge^{n-1} T^*$.

Proof: Let $(v_1, \ldots, v_n)$ be an orthonormal basis of T, and let $(v_1^*, \ldots, v_n^*)$ be the dual basis of T^*. The inner product induced by ω on $\bigwedge^{n-1} T^*$ may be defined by declaring the basis $(v_1^* \wedge \ldots \wedge \widehat{v_k^*} \wedge \ldots \wedge v_n^*)_{k=1,\ldots,n}$ to be orthonormal.

In this basis the volume form of ω is

$$\frac{\omega^n}{n!} = \left(\frac{i}{2}\right)^n v_1^* \wedge \overline{v}_1^* \wedge \ldots \wedge v_n^* \wedge \overline{v}_n^*.$$

We write $\sigma = \lambda\, v_1^* \wedge \ldots \wedge v_n^*$ for some scalar λ. I claim that $|\lambda| = 1$.

To verify the claim, we first calculate that

$$\frac{i^{n^2}}{2^n} \sigma \wedge \overline{\sigma} = |\lambda|^2 \frac{i^n}{2^n} v_1^* \wedge \overline{v}_1^* \wedge \ldots \wedge v_n^* \wedge \overline{v}_n^* = |\lambda|^2 \frac{\omega^n}{n!}.$$

The second equality follows trivially from the first. To verity the first equality, set $\sigma_k = v_1^* \wedge v_2^* \wedge \ldots \wedge v_k^*$ and $\alpha_k = v_k^* \wedge \overline{v}_k^*$ for $1 \leq k \leq n$. The reader may quickly verify that

$$\sigma_{k+1} \wedge \overline{\sigma}_{k+1} = (-1)^k \, \sigma_k \wedge \overline{\sigma}_k \wedge \alpha_{k+1}.$$

We note that $\sigma_1 \wedge \overline{\sigma}_1 = \alpha_1$. As $\sigma_n \wedge \overline{\sigma}_n = \alpha_1 \wedge \ldots \wedge \alpha_n$ the announced equality follows by induction and by noting that $(-1)^{n(n-1)/2} = i^{n(n-1)}$.

Our hypothesis on the relationship between ω and σ now gives

$$\frac{\omega^n}{n!} = \frac{i^{n^2}}{2^n} \sigma \wedge \overline{\sigma} = |\lambda|^2 \frac{\omega^n}{n!},$$

from which we conclude that $|\lambda|^2 = 1$.

Now let v_k be one of the basis vectors of T. Then we calculate that

$$\iota(v_k) := \iota_{v_k} \sigma = \lambda\, v_1^* \wedge \ldots \wedge \widehat{v_k^*} \wedge \ldots \wedge v_n^*.$$

As $|\lambda| = 1$ then the images of the basis vectors v_k are orthonormal with respect to the metric induced by ω on $\bigwedge^{n-1} T^*$. This concludes the proof. $\square$

Corollary 1.2. *Let* T *be a complex vector space of dimension* n. *Let* ω *be the* $(1,1)$*-form of a hermitian inner product on* V *and let* σ *be a* $(n,0)$*-form on* T. *Suppose that*

$$\frac{\omega^n}{n!} = \frac{i^{n^2}}{2^n} \sigma \wedge \overline{\sigma}.$$

Then the linear morphism

$$\begin{cases} \iota : T \otimes \overline{T}^* & \longrightarrow & \bigwedge^{n-1} T^* \otimes \overline{T}^* \\ u & \longmapsto & \iota_u \sigma \end{cases}$$

is an isometry with respect the inner products induced by ω *on each vector space.*

Proof: We have already seen that cup product defines an isometry $T \to \bigwedge^{n-1} T^*$, and the identity morphism $\overline{T}^* \to \overline{T}^*$ is an isometry with respect to any inner product. The above morphism $T \otimes \overline{T}^* \to \bigwedge^{n-1} T^* \otimes \overline{T}^*$ is simply the tensor product of these two morphisms.

Now, the inner product induced by ω on $T \otimes \overline{T}^*$ the tensor product of the inner products induced by ω on T and $\overline{T}^*$. The same is true for $\bigwedge^{n-1} T^* \otimes \overline{T}^*$.

The morphism under consideration is then the tensor product of two isometries, and is thus an isometry with respect to the induced inner product. $\square$

Proposition 1.3. *Let ω be a Kähler metric on X and let σ be a holomorphic volume form on X. Let g be a positive smooth function on X such that*

$$\frac{\omega^n}{n!} = g\,\frac{i^{n^2}}{2^n}\,\sigma \wedge \overline{\sigma}.$$

Then $i\partial\bar{\partial} \log g = -\,Ric(\omega)$.

Proof: To begin with, we note that the proof of Proposition 1.1 shows that the (n,n)-form $i^{n^2}\sigma \wedge \overline{\sigma}$ is positive on X, so the function g is indeed positive.

Let $(z_1, \ldots, z_n)$ be holomorphic local coordinates centered on a point x_0. Let

$$\omega = \frac{i}{2} \sum_{j,\overline{k}} \omega_{j\overline{k}}\, \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k, \qquad \text{and} \qquad \sigma = \lambda\, \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n$$

be the local expressions of the forms ω and σ. Note that λ is a holomorphic function on the coordinate neighborhood around x_0. Referring again to the proof of Proposition 1.1 we see that

$$\frac{i^{n^2}}{2^n}\sigma \wedge \overline{\sigma} = |\lambda|^2 \left(\frac{i}{2}\right)^n \mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \wedge \ldots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n.$$

We also recall that

$$\frac{\omega^n}{n!} = \det \omega_{j\overline{k}} \left(\frac{i}{2}\right)^n \mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \wedge \ldots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n.$$

from which we gather that $g = \det \omega_{j\overline{k}} / |\lambda|^2$, and thus

$$\log g = \log \det \omega_{j\overline{k}} - \log |\lambda|^2.$$

The result now follows from applying the $i\partial\bar{\partial}$ operator to the last equality, since the function $\log|\lambda|^2$ is pluriharmonic on the coordinate neighborhood in question. $\square$

Proposition 1.4. *Let ω be a Ricci-flat Kähler metric on X. If σ is a holomorphic volume form on X such that*

$$\int_X \frac{i^{n^2}}{2^n}\sigma \wedge \overline{\sigma} = \mathit{Vol}(X, \omega),$$

then cup product against the form σ defines an isometry of cohomology groups

$$H^1(X, T_X) \longrightarrow H^{n-1,1}(X, \mathbb{C}),$$

where the groups are equipped with the L^2 inner products defined by ω.

Proof: First suppose that σ is any holomorphic volume form on X. Then there exists a smooth function g on X such that

$$\frac{\omega^n}{n!} = g\, \frac{i^{n^2}}{2^n}\, \sigma \wedge \overline{\sigma}.$$

By Proposition 1.3 the function g satisfies $i\partial\bar{\partial} \log g = 0$. Then its logarithm is pluriharmonic on X which entails that g is constant. By integrating over X we see that the constant g satisfies

$$\mathrm{Vol}(X, \omega) = g \cdot \int_X \frac{i^{n^2}}{2^n} \sigma \wedge \overline{\sigma}.$$

If we choose σ as in the statement of this Proposition, which we can do simply by multiplying it by a suitable constant, then $g = 1$.

We now use the Hodge isomorphism theorem to represent both groups $H^1(X, T_X)$ and $H^{n-1,1}(X, \mathbb{C})$ by ω-harmonic differential forms. The hypotheses of Corollary 1.2 are then satisfied at all points of the manifold X, so the cup product defines a pointwise isometry on X. The inner products on the cohomology groups are defined by integrating these pointwise inner products over the manifold, so the cup product is an isometry of cohomology groups. $\square$

The choice of σ in the above proposition is clearly not unique, as any multiple $\lambda\,\sigma$ with $|\lambda| = 1$ will also satisfy the conditions of the proposition. Aside from these multiples, there are no other choices of σ that work given a fixed metric ω.

On the other hand, given a fixed holomorphic volume form σ, there are lots of Ricci-flat metrics for which the cup product against σ defines an isometry of cohomology groups. Indeed, this will be the case for any two metrics of the same volume.

2. The relative canonical bundle

Let $\pi : \mathcal{X} \to S$ be a family of compact Kähler manifolds with trivial canonical bundle over a smooth base as before. As the manifolds in the family are compact and Kähler the direct image sheaf $L := \mathcal{R}^0\pi_* K_{\mathcal{X}/S}$ is a holomorphic vector bundle over the base S. Its fiber over a point s is $L_s = H^0(X_s, K_{X_s})$, so in our case L is a line bundle.

The intersection product on the cohomology rings of the manifolds in our family defines a hermitian metric on L. Indeed, if α and β are local sections of L, then

$$h(\alpha, \overline{\beta}) = \int_{X_s} i^{n^2} \alpha \wedge \overline{\beta}$$

is positive-definite, since the integrand is a positive (n, n)-form on each manifold X_s. If we pick a local trivializing section σ of L, that is, a nowhere zero holomorphic volume form $\sigma(s)$ on each manifold X_s that varies holomorphically with s, then the curvature form of (L, h) is given by

$$\frac{i}{2\pi}\Theta_{L,h} = -i\partial\bar{\partial}\log\int_{X_s} i^{n^2}\sigma\wedge\overline{\sigma}.$$

This follows from standard formulas for the curvature of a hermitian line bundle, see [**Dem09**].

Standard techniques from variation of Hodge structures permit us to calculate the curvature form explicitly. We introduce some space saving notation before embarking on these calculations. If σ is a holomorphic volume form on X, then we call the positive real number

$$\mathrm{Vol}(X, \sigma) := \int_X i^{n^2}\sigma\wedge\overline{\sigma}$$

the volume of X with respect to the form σ.

Proposition 2.1. *Let $\pi : \mathcal{X} \to S$ be a family of compact manifolds with trivial canonical bundle over a smooth base S. Let ξ and η be holomorphic vector fields on S. Let σ be a local holomorphic nowhere zero section of L over S, and let ρ be the Kodaira-Spencer morphism of the family. Then*

$$\frac{i}{2\pi}\Theta_{L,h}(\xi,\overline{\eta}) = \frac{-i^{n^2}}{Vol(X,\sigma)}\int_{X_s}\rho(\xi)\cup\sigma\wedge\overline{\rho(\eta)\cup\sigma}.$$

Proof: Griffiths transversality entails that $\nabla_\xi\sigma = \nabla_\xi\sigma^{(n,0)} + \nabla_\xi\sigma^{(n-1,1)}$. We claim that $\nabla_\xi\sigma^{(n-1,1)} = \rho(\xi)\cup\sigma$: If $\mathcal{P}^{n,n}$ denotes the period map associated to the variation of Hodge structures $F^nE^n \to E^n$,[1] then

$$d\mathcal{P}^{n,n}_s(\xi)(\alpha) = \rho(\xi)\cup\alpha$$

for any $\xi \in T_{S,s}$ and $\alpha \in H^0(X_s, \Omega^n_{X_s})$ by [**Voi02**, Théorème 10.21]. The morphism $dP^{n,n}_s$ identifies by adjuction with the morphism

$$\overline{\nabla}^n_s : F^nH^n(X_s,\mathbb{C}) \to H^n(X_s,\mathbb{C})/F^nH^n(X_s,\mathbb{C})\otimes\Omega_{S,s}$$

defined by the composition

$$F^nE^n \xrightarrow{\nabla} E^n\otimes\Omega_S \longrightarrow (E^n/F^nE^n)\otimes\Omega_S,$$

where ∇ is the Gauss–Manin connection, by [**Voi02**, Lemme 10.19]. We conclude that $\overline{\nabla}^n_{\xi,s}\sigma = (\nabla_{\xi,s}\sigma)^{(n-1,1)}$ by Griffiths transversality and by using the Hodge decomposition theorem, so $(\nabla_{\xi,s}\sigma)^{(n-1,1)} = \rho(\xi)\cup\sigma$.

As canonical bundle of X_s is trivialized by the form σ, there is a complex function f_ξ on S such that $\nabla_\xi\sigma^{(n,0)} = f_\xi\sigma$, so

$$\nabla_\xi\sigma = f_\xi\,\sigma + \rho(\xi)\cup\sigma.$$

A similar decomposition holds for $\nabla_\eta\sigma$. This entails that

$$\nabla_\xi\sigma\wedge\overline{\sigma} = f_\xi\,\sigma\wedge\overline{\sigma},$$

and

$$\nabla_\xi\sigma\wedge\overline{\nabla_\eta\sigma} = f_\xi\,\overline{f}_\eta\,\sigma\wedge\overline{\sigma} + \rho(\xi)\cup\sigma\wedge\overline{\rho(\eta)\cup\sigma}$$

by type considerations.

[1] We refer to Chapter 1 for the notations used here.

We now calculate $\frac{i}{2\pi}\Theta_{L,h}$ by differentiating the function $\log \mathrm{Vol}(X, \sigma(s))$ twice, first in the direction of $\overline{\eta}$ and then in the direction of ξ. What we find is

$$\begin{aligned}
\frac{i}{2\pi}\Theta_{L,h}(\xi,\overline{\eta}) &= \frac{-1}{\mathrm{Vol}(X,\sigma)}\int_{X_s} i^{n^2}\nabla_\xi\sigma\wedge\overline{\nabla_\eta\sigma} \\
&\quad + \frac{1}{\mathrm{Vol}(X,\sigma)}\int_{X_s} i^{n^2}\nabla_\xi\sigma\wedge\overline{\sigma}\cdot\frac{1}{\mathrm{Vol}(X,\sigma)}\int_{X_s} i^{n^2}\sigma\wedge\overline{\nabla_\eta\sigma} \\
&= \frac{-1}{\mathrm{Vol}(X,\sigma)}\int_{X_s} i^{n^2}\left(f_\xi\,\overline{f}_\eta\,\sigma\wedge\overline{\sigma}+\rho(\xi)\cup\sigma\wedge\overline{\rho(\eta)\cup\sigma}\right) \\
&\quad + \frac{f_\xi}{\mathrm{Vol}(X,\sigma)}\int_{X_s} i^{n^2}\sigma\wedge\overline{\sigma}\cdot\frac{\overline{f}_\eta}{\mathrm{Vol}(X,\sigma)}\int_{X_s} i^{n^2}\sigma\wedge\overline{\sigma} \\
&= \frac{-1}{\mathrm{Vol}(X,\sigma)}\int_{X_s} i^{n^2}\rho(\xi)\cup\sigma\wedge\overline{\rho(\eta)\cup\sigma}
\end{aligned}$$

because the functions f_ξ and f_η are constant on X_s, so we can pull them out of the relevant integrals. □

The reader should take the time to compare the above formula for the curvature form with Proposition 1.4, which says that the cup product against a normalized holomorphic volume form is an isometry. Here we integrate cup products against a holomorphic volume form. We smell blood in the water.

3. The Weil–Petersson metric

We must compare the intersection product of $(n-1,1)$-forms with the L^2 inner product defined by a Kähler metric on our manifold. This can be achieved by applying the hard Lefschetz theorem.

Proposition 3.1. *Let X be a compact Kähler manifold of dimension $\dim_{\mathbb{C}} X = n$ and let ω be a Kähler metric on X. Let α and β be harmonic $(n-1,1)$-forms on X. Then*

$$\langle\!\langle \alpha, \overline{\beta}\rangle\!\rangle = -i^{n^2}\int_X \alpha\wedge\overline{\beta},$$

where the expression on the left hand side is the L^2 inner product defined by the metric ω.

Proof: Let us start by decomposing the given forms as

$$\begin{aligned}
\alpha &= \alpha_0\wedge\omega+\alpha_1, \\
\beta &= \beta_0\wedge\omega+\beta_1,
\end{aligned}$$

where the forms α_0 and β_0 are primitive $(n-2,0)$-forms on X, and the forms α_1 and β_1 are primitive $(n-1,1)$-forms. As the forms α and β are harmonic, then so are the forms in their primitive decompositions.

We now observe that

$$\langle\!\langle \alpha, \overline{\beta}\rangle\!\rangle = \langle\!\langle \alpha_0\wedge\omega, \overline{\beta}_0\wedge\omega\rangle\!\rangle + \langle\!\langle \alpha_1, \overline{\beta}_1\rangle\!\rangle,$$

because any mixed terms of the type $\langle\!\langle \alpha_0 \wedge \omega, \overline{\beta}_1 \rangle\!\rangle$ are zero since the forms α_1 and β_1 are primitive. We also see that

$$\int_X \alpha \wedge \overline{\beta} = \int_X \alpha_0 \wedge \overline{\beta}_0 \wedge \omega^2 + \int_X \alpha_1 \wedge \overline{\beta}_1 \tag{4}$$

because as α_1 and β_1 are primitive $(n-1,1)$-forms then $\alpha_1 \wedge \omega = \beta_1 \wedge \omega = 0$. Thus any mixed terms of the type $\alpha_0 \wedge \omega \wedge \overline{\beta}_1$ are zero. As the forms α_1 and β_1 are primitive, then the hard Lefschetz theorem gives that

$$\langle\!\langle \alpha_1, \overline{\beta}_1 \rangle\!\rangle = -i^{n^2} \int_X \alpha_1 \wedge \overline{\beta}_1, \tag{5}$$

see, for example, [**Huy05**, Chapter 1.2].

Lemma 3.2. *Let α_0 and β_0 be primitive $(n-2,0)$-forms on X. Then*

$$\langle\!\langle \alpha_0 \wedge \omega, \overline{\beta}_0 \wedge \omega \rangle\!\rangle = -i^{n^2} \int_X \alpha_0 \wedge \overline{\beta}_0 \wedge \omega^2.$$

Proof: Let γ be a primitive (p,q)-form and set $k = p + q$. The main tool for the proof of the lemma is the following formula, which calculates the Hodge star and Lefschetz operators applied to γ:

$$*L^j\gamma = (-1)^{k(k+1)/2} \frac{j!}{(n-k-j)!} L^{n-k-j} i^{p-q} \gamma. \tag{6}$$

For a proof, see [**Huy05**, Chapter 1.2].

Let $\Lambda = *^{-1}L*$ be the formal adjoint of the Lefschetz operator. Then we have

$$\langle\!\langle \alpha_0 \wedge \omega, \overline{\beta}_0 \wedge \omega \rangle\!\rangle = \langle\!\langle \alpha_0, \Lambda(\omega \wedge \overline{\beta}_0) \rangle\!\rangle = \langle\!\langle \alpha_0, *^{-1}L * L\overline{\beta}_0 \rangle\!\rangle$$

by the adjoint property, the fact that $\overline{\beta}_0 \wedge \omega = \omega \wedge \overline{\beta}_0$, and the definitions of the operators L and Λ.

We apply equation (6) to the $(0, n-2)$-form $\overline{\beta}_0$ and find that

$$*L\overline{\beta}_0 = (-1)^{(n-2)(n-1)/2} i^{2-n} L\overline{\beta}_0.$$

Next we note that $L^2\overline{\beta}_0$ is a $(2,n)$-form. The Hodge star operator on $(2,n)$-forms satisfies $*^2 = (-1)^{(n+2)(n-2)}\,\mathrm{id} = (-1)^n\,\mathrm{id}$, so its inverse is $*^{-1} = (-1)^n *$. We apply 6 again to the $(0, n-2)$-form $\overline{\beta}_0$ and get that

$$\begin{aligned} *^{-1}L * L\overline{\beta}_0 &= (-1)^n \, (-1)^{(n-2)(n-1)/2} i^{2-n} * L^2\overline{\beta}_0 \\ &= (-1)^n \left((-1)^{(n-2)(n-1)/2} i^{2-n} \right)^2 2\,\overline{\beta}_0 = 2\,\overline{\beta}_0. \end{aligned}$$

Finally we apply the hard Lefschetz theorem and obtain

$$\begin{aligned} \langle\!\langle \alpha_0 \wedge \omega, \overline{\beta}_0 \wedge \omega \rangle\!\rangle &= 2\,\langle\!\langle \alpha_0, \overline{\beta}_0 \rangle\!\rangle \\ &= 2\frac{1}{2!} i^{n-2} (-1)^{(n-2)(n-1)/2} \int_X \alpha_0 \wedge \overline{\beta}_0 \wedge \omega^2 \\ &= -i^{n^2} \int_X \alpha_0 \wedge \overline{\beta}_0 \wedge \omega^2, \end{aligned}$$

which completes the proof of the lemma. $\square$

The proposition now follows. Indeed, we use equation (5) and the lemma and find that

$$\begin{aligned}\langle\!\langle \alpha, \overline{\beta} \rangle\!\rangle &= -i^{n^2} \int_X \alpha_0 \wedge \overline{\beta}_0 \wedge \omega^2 - i^{n^2} \int_X \alpha_1 \wedge \overline{\beta}_1 \\ &= -i^{n^2} \int_X \alpha \wedge \overline{\beta},\end{aligned}$$

where the second equality is given by equation (4). □

Let us stress that we need the forms α and β to be harmonic for the proposition to hold. If we plug $\alpha + \mathrm{d}\gamma$ and $\beta + \mathrm{d}\delta$ into the formula given in the proposition, then the right hand side does not change, as it only depends on the cohomology classes of the forms in question, while the left hand side will change by $\langle\!\langle \mathrm{d}\gamma, \mathrm{d}\overline{\delta} \rangle\!\rangle$, which is non-zero if the forms γ and δ are not closed.

We are now in a position to compare the curvature form of L with the hermitian form h_{WP}. To do this we must first pull the form $\frac{i}{2\pi}\Theta_{L,h}$ back to the vector bundle p^*T_S over the relative Kähler cone $\mathcal{K}$.

Theorem 3.3. *The Weil–Petersson form and the pullback of the curvature of (L, h) to p^*T_S satisfy*

$$2^n\, h_{WP} = p^* \frac{i}{2\pi} \Theta_{L,h}.$$

In particular, the form h_{WP} descends to the tangent space of the base S, and the curvature form $\frac{i}{2\pi}\Theta_{L,h}$ is positive, and thus defines a Kähler metric on S, if the family $\pi : \mathcal{X} \to S$ is effective.

Proof: Let (a, s) be a point of $\mathcal{K}$, and let Ω be the Ricci-flat Kähler metric in the class $\omega := \operatorname{Im} a$. Let σ be a holomorphic volume form on X_s that satisfies the conditions of Proposition 1.4. Let ξ and η be two vectors in $p^*T_{S,(a,s)} = T_{S,s}$.

Let ρ be the Kodaira-Spencer morphism of the family $\pi : \mathcal{X} \to S$. Proposition 3.1 shows that

$$p^* \frac{i}{2\pi} \Theta_{L,h}(\xi, \overline{\eta})(a, s) = \frac{1}{\operatorname{Vol}(X_s, \sigma)} \langle\!\langle \rho(\xi) \cup \sigma, \overline{\rho(\eta) \cup \sigma} \rangle\!\rangle,$$

where the inner product is the one induced by Ω on $(n-1, 1)$-classes. We note that the conditions of Proposition 1.4 may be written as

$$\frac{1}{2^n} \operatorname{Vol}(X_s, \sigma) = \operatorname{Vol}(X_s, \Omega).$$

We then get that

$$p^* \frac{i}{2\pi} \Theta_{L,h}(\xi, \overline{\eta})(a, s) = 2^n \frac{1}{\operatorname{Vol}(X_s, \omega)} \langle\!\langle \rho(\xi), \overline{\rho(\eta)} \rangle\!\rangle = 2^n h_{WP}(\xi, \overline{\eta})(a, s),$$

first by applying Proposition 1.4 and then by appealing to the definition of the form h_{WP}. □

Remark — This result shows that we have a Kähler Weil-Petersson metric on the base of deformations of an arbitrary effective family of compact Kähler manifolds with trivial canonical bundle, without any appeal to a polarization. In particular, the theorem applies to complex tori and holomorphic symplectic manifolds.

4. The curvature of the Weil–Petersson metric

We tread lightly in this section and only point out similarities between our work and what has already been done. The reference of choice for this section is [**Wan03**].

In this article, Wang considers an effectively parametrized and polarized family $\pi : \mathcal{X} \to S$ of compact Kähler manifolds with trivial canonical bundle. He then defines a Weil-Petersson metric for such a family from a Hodge theoretic point of view and calculates its curvature tensor. In our notation, the metric Wang considers is the curvature form $\frac{i}{2\pi}\Theta_{L,h}$.

We now invite the reader to verify that none of the results Wang proves in Section 2.1 of his paper actually rely on having a polarized family. In short, this is because he only uses the intersection product on the level of n-forms to perform his calculations, but there the intersection product is a topological invariant that does not depend on the choice of a polarization. Thus he effectively proves:

Theorem 4.1. *The curvature tensor of the Kähler metric h defined by $\frac{i}{2\pi}\Theta_{L,h}$ is*

$$R(\xi,\overline{\eta},\nu,\overline{\zeta}) = -\left(h(\xi,\overline{\eta})\,h(\nu,\overline{\zeta}) + h(\xi,\overline{\zeta})\,h(\eta,\overline{\nu})\right) + \frac{i^{n^2}}{Vol(X_s,\Omega)}\int_{X_s}\rho(\xi)\cup\rho(\nu)\cup\Omega\wedge\overline{\rho(\eta)\cup\rho(\zeta)\cup\Omega}$$

where ρ is the Kodaira-Spencer morphism of $\pi : \mathcal{X} \to S$.

This result then of course gives the curvature tensor of the form h_{WP} when combined with Theorem 3.3.

Remark — It is also possible to calculate the curvature tensor of h_{WP} by the methods of Nannicini and Schumacher. The key is again Theorem 3.3, which tells us that the possibly messy reliance of h_{WP} on the Kähler moduli is not an issue. Thus the calculations needed actually become similar enough to the case of a polarized family of manifolds as to make no difference.

The usual results on the negativity of the curvature of the Weil–Petersson metric also carry over without modification. For example, we have:

Theorem 4.2. *The holomorphic sectional and Ricci curvatures of the Kähler metric $\frac{i}{2\pi}\Theta_{L,h}$ are negative. They satisfy the bounds*

$$H \geq -2 \quad \text{and} \quad r \geq -(n+1),$$

where n is the dimension of the manifolds X_s.

We direct the reader to [**Wan03**, Theorem 3.1] for details and further information.

5. The relative Kähler cone

We momentarily abandon the world of manifolds with trivial canonical bundle, and consider a family $\pi : \mathcal{X} \to S$ of compact Kähler manifolds over a smooth base S. Let $p : \mathcal{K} \to S$ be the relative Kähler cone associated to the family. We fix a smooth form ρ in $c_1(X_s)$ as in Chapter 1.

Consider the sheaf $E := \mathcal{R}^{2n}\pi_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_S$ over the space S. Since the manifolds of our family are compact and Kähler and the base S is smooth, E

is a holomorphic vector bundle. Its fiber over a point s is $E_s = H^{n,n}(X_s, \mathbb{C})$, so E is a holomorphic line bundle.

We now pull this line bundle back to the total space of the relative Kähler cone $\mathcal{K}$. Then we can define a smooth hermitian metric g on the pullback p^*E: if α and β are local holomorphic sections of p^*E, we write

$$\alpha = \frac{a}{\operatorname{Vol}(X,\omega)} \frac{\omega^n}{n!} \quad \text{and} \quad \beta = \frac{b}{\operatorname{Vol}(X,\omega)} \frac{\omega^n}{n!},$$

where a and b are holomorphic functions (see the proof of Propostion 5.1). We then set

$$g(\alpha, \overline{\beta})_{(\omega,s)} = a(s)\overline{b}(s) \operatorname{Vol}(X,\omega).$$

Proposition 5.1. *The curvature form of g is given by*

$$\frac{i}{2\pi}\Theta_{E,g} = i\partial\bar{\partial} \log \operatorname{Vol}(X,\omega).$$

In particular, the restriction of the curvature form to a fiber of $\mathcal{K}$ is the negative of the metric on the complexified Kähler cone of X_s considered in Chapter 2.

Proof: I claim that the section $\tau(a,s) = (\omega^n/n!)/\operatorname{Vol}(X,\omega)$ of p^*E, where $\omega = \operatorname{Im} a$, is holomorphic. To verify the claim, first note that the section τ is constant on the fibers of $\mathcal{K}$. Indeed, if we fix the parameter s and take a $(1,1)$-class u on X_s, then $d_u\omega = \frac{1}{2i}u$. Standard calculations then show that

$$d_u\tau(a,s) = \frac{1}{2i}\frac{1}{\operatorname{Vol}(X,\omega)}\left(u \wedge \frac{\omega^{n-1}}{(n-1)!} - \frac{\omega^n/n!}{\operatorname{Vol}(X,\omega)}\int_{X_s} u \wedge \frac{\omega^{n-1}}{(n-1)!}\right).$$

If the class u is ω-primitive, then $u \wedge \omega^{n-1} = 0$ so $d_u\tau = 0$. If $u = \lambda\omega$ for some complex number λ, then $u \wedge \omega^{n-1}/(n-1)! = n\lambda\omega^n/n!$. Substituting this into the above formula, we find $d_u\tau = 0$.

Next note that the section τ satisfies

$$\int_{X_s} \tau(a,s) = \frac{1}{\operatorname{Vol}(X_s,\Omega)}\int_{X_s} dV_\Omega = 1$$

at all points (a,s) of the space $\mathcal{K}$. It is thus dual to the fundamental class of each manifold X_s, so it is parallel with respect to the pullback of the Gauss–Manin connection on E to $\mathcal{K}$. The holomorphic structure on E may be defined by declaring the parallel sections to be holomorphic, so τ is holomorphic in s and independent of a, and thus holomorphic on $\mathcal{K}$.

Since τ is a nowhere zero holomorphic section of the line bundle E, the curvature form of g is given by $-i\partial\bar{\partial}\log|\tau|_g^2$. If we pick a Kähler metric Ω in the class ω, then $|dV_\Omega|_\Omega = 1$. Thus $|\tau|_g^2 = 1/\operatorname{Vol}(X,\omega)$, which implies the result.

Once we restrict to a fiber $K(X_s)$ we only differentiate the function $\log \operatorname{Vol}$ with respect to $(1,1)$-classes on X_s. We saw in Chapter 2 that this gives the negative of the natural metric on the complexified Kähler cone of X_s. ☐

Proposition 5.2. *Let $\pi : \mathcal{X} \to S$ be a family of compact Kähler manifolds with trivial canonical bundle over a smooth base S, and let $p : \mathcal{K} \to S$ be the relative Kähler cone of the family. The curvature form of the line* *bundle $L \otimes E^*$ over $\mathcal{K}$ is positive-definite on vertical tangent vectors and defines the same smooth splitting of $T_{\mathcal{K}}$ as the natural hermitian metric on $\mathcal{K}$.*

Proof: Let's start by recalling the definition of the hermitian metric h on $\mathcal{K}$. Since the differential of $p : \mathcal{K} \to S$ is surjective there is a short exact sequence

$$0 \longrightarrow T_{\mathcal{K}/S} \longrightarrow T_{\mathcal{K}} \longrightarrow p^*T_S \longrightarrow 0$$

of vector bundles over $\mathcal{K}$. The bundle $T_{\mathcal{K}/S}$ is equipped with the hermitian metric g given by the natural metric on each complexified Kähler cone $K(X_s)$, and the bundle p^*T_S is equipped with the Weil–Petersson form defined by the Ricci-flat metrics on the manifolds X_s. The metric on $\mathcal{K}$ is then constructed from these two metrics by splitting the short exact sequence via the connection ∇ induced by the Gauss–Manin connection on $\mathcal{R}^2\pi_*\mathbb{C}$ and the Hodge decomposition.

By Theorem 3.3 the Weil–Petersson form on p^*T_S is the curvature form of the line bundle L, and by Proposition 5.1 the restriction of the curvature form Θ of $L \otimes E^*$ to $T_{\mathcal{K}/S}$ agrees with the hermitian metric on that bundle. In particular, the restriction of Θ to $T_{\mathcal{K}/S}$ is positive-definite, and thus defines a smooth splitting of $T_{\mathcal{K}}$. This lift may be described as sending a tangent vector Z on $\mathcal{K}$ to the unique section ξ of $T_{\mathcal{K}/S}$ that satisfies

$$\Theta(Z, \overline{\nu}) = \Theta(\xi, \overline{\nu}) = h(\xi, \overline{\nu})$$

for all sections ν of $T_{\mathcal{K}/S}$. It remains to show that this splitting is the same as the one defined by the connection ∇.

To this end, take a tangent field Z on $\mathcal{K}$ near a point (a, s) and let $\alpha : S \to \mathcal{K}$ be a parallel section of $\mathcal{K}$ such that $\alpha(s) = a$. The splitting defined by ∇ is then given by

$$Z = (Z - d_{p_*Z}\alpha) + p_*Z,$$

and we see that

$$\Theta(Z - d_{p_*Z}\alpha, \overline{\nu}) = \Theta(Z, \overline{\nu}) - \Theta(d_{p_*Z}\alpha, \overline{\nu})$$

for all sections ν of $T_{\mathcal{K}/S}$. The smooth inclusion $\mathcal{K} \hookrightarrow \mathcal{R}^2\pi_*\mathbb{C}$ lets us consider α as a smooth section of $\mathcal{R}^2\pi_*\mathbb{C}$, and since this bundle is flat the Gauss–Manin connection on this bundle identifies with the exterior derivative d. Now, α is parallel with respect to ∇ if and only if the class $d\alpha$ decomposes into $(2,0)$ and $(0,2)$-classes only. Then

$$\Theta(d_{p_*Z}\alpha, \overline{\nu}) = h(d_{p_*Z}\alpha, \overline{\nu}) = 0$$

because the $(1,1)$-class ν is orthogonal to $(2,0)$ and $(0,2)$-classes with respect to the intersection product on the degree 2 cohomology on X_s. The two smooth splittings of the tangent bundle of $\mathcal{K}$ thus agree. □

Remark — If the curvature form were degenerate on horizontal tangent vectors, or equivalently, if $\log \mathrm{Vol} : \mathcal{K} \to \mathbb{R}$ were pluriharmonic in horizontal directions, then the above would entail that the natural metric on $\mathcal{K}$ coincides with the curvature form, and thus that it is Kähler. It is not clear to me that this is the case, but one can perhaps still show either that the line bundle in question is positive (by expressing it in terms of the natural metric plus a perturbation term), or that the natural metric is Kähler by showing that the difference between the curvature form and the Kähler form of the metric is closed. Unfortunately there was not time to do this before the thesis defence. Thus we simply state two questions:

(1) Is the natural hermitian metric on $\mathcal{K}$ Kähler?
(2) Is the hermitian metric on $L \otimes E^*$ positive?

We hope to address these questions soon.

CHAPTER 4

Applications and examples

1. Abundance fails on compact complex manifolds

Let X be a compact complex manifold of dimension n. The generalized version of the abundance conjecture says that if X is Kähler then the numerical dimension of the canonical bundle K_X should be equal to its Kodaira dimension [**Dem09**, Chapter 18]. A consequence of this conjecture is the Iitaka $C_{n,m}$ conjecture, which says that if $f : X \to Y$ is a holomorphic morphism of compact Kähler manifolds, then $\kappa(X) \geq \kappa(Y) + \kappa(f_y)$, where f_y is a general fiber of f and κ denotes the Kodaira dimension.

These conjectures were originally stated for projective varieties, but their statements make sense for Kähler manifolds and indeed any compact complex manifold. In this section we produce a examples of compact non Kähler manifolds that violate both the abundance and the Iitaka conjectures.

The construction of these manifolds is simple. A folklore result says that if M is a simply connected Kähler manifold with trivial canonical bundle that admits an automorphism f of infinite order, then f must move every Kähler class on M. Given such a manifold, we let a lattice in a complex vector space V act on $M \times V$ by translation on V and by mapping each generator of the lattice to f. The quotient manifold is then a compact non Kähler manifold, with flat canonical bundle, but whose Kodaira dimension is negative in some cases.

I must point out that similar examples were already known, though I didn't know of them when I was writing this thesis. In the book [**Uen75**] one finds the example of the total space of a fibration of a particular torus over an elliptic curve, and the properties we remark are already present in that example.

Automorphisms and Kähler classes. Let M be a compact simply connected Kähler manifold of complex dimension $\dim_{\mathbb{C}} M = n$ with trivial canonical bundle.

Proposition 1.1. *An automorphism f of M fixes a Kähler class $[\omega]$ on M if and only if the order of f is finite.*

Proof: The condition is clearly sufficient, since if the degree of f is d then the Kähler class $[\omega] + f^*[\omega] + \cdots + (f^*)^{d-1}[\omega]$ is invariant under f.

Suppose now that f fixes a Kähler class $[\omega]$ and let ω be the unique Ricci flat metric in this class. Then $f^*\omega$ is again Ricci flat, and thus equal to ω by unicity. Thus f is an element of the isometry group of (M, ω). A general result of Riemannian geometry [**Bal06**, Corollary 6.2] now says that the isometry group of a simply connected manifold with non positive Ricci curvature is finite. □

The condition that M be simply connected serves to exclude complex tori, for tori admit non zero holomorphic vector fields. These fields generate automorphisms homotopic to the identity, which thus act trivially on the cohomology of the torus, despite usually being of infinite order.

This result points the way to a construction of non Kähler manifolds: Let M be a compact simply connected Kähler manifold with trivial canonical bundle. Suppose M admits an automorphism f of infinite order. Let V be a complex vector space of dimension p and let Γ be a lattice in V, we denote by $B = V/\Gamma$ the complex torus defined by Γ. We define a representation $\Gamma \longrightarrow \operatorname{Aut} M$ by mapping every generator of Γ to the automorphism f. The lattice Γ then acts on the product $M \times V$ by

$$\gamma \cdot (z, t) = (\gamma(z), t + \gamma).$$

We set $X := X(M, B) = (M \times V)/\Gamma$.

Theorem 1.2. *The complex space X is a smooth compact non Kähler manifold. It is the total space of a holomorphic fibration $\pi : X \to B$, whose fibers are all isomorphic to M.*

Proof: The lattice Γ clearly acts without fixed points on $M \times V$. Its action is also properly discontinuous, since any compact set in $M \times V$ may be translated as far to infinity in V as desired. The quotient X is thus a smooth complex manifold, and compact for the same reason that the torus V/Γ is compact.

The projection map $pr : M \times V \longrightarrow V$ is invariant by the action of Γ and thus defines a holomorphic morphism $\pi : X \to B$. It is proper as the manifold X is compact, and a submersion because the projection morphism is a submersion. Let t be a point of B. The preimage $\pi^{-1}(t)$ may be identified with the product $M \times \Gamma + t$. If we pick an element γ in the lattice Γ, then the restriction of the quotient map $q : M \times V \to X$ identifies with the automorphism $\gamma \cdot f : M \to M$ and defines an isomorphism $M \to X_t$.

Finally, suppose that X were Kähler. If ω were a Kähler metric on X, then by restriction we would obtain a Kähler metric ω_0 on the fiber M_0 that would be invariant under the action of the group generated by f. This is impossible since f is of infinite order. $\square$

Remark — It seems hard to extract precise topological information about X, aside from that which follows trivially from general facts about fibrations. For example, the naive road to the Betti numbers of X passes through the space of closed forms on $M \times V$ that are invariant under the automorphism f. Since f is quite wild I have no idea how one could calculate this in practice.

The canonical bundle of M is trivial, so there is a nowhere zero holomorphic $(n, 0)$-form σ on M. As $f^*\sigma$ is again a $(n, 0)$-form on M, we must have $f^*\sigma = \lambda\sigma$ for some complex number λ. Note that the (n, n)-form $i^{n^2}\sigma \wedge \overline{\sigma}$ is real and positive on M, and that $f^*(\sigma \wedge \overline{\sigma}) = |\lambda|^2 \sigma \wedge \overline{\sigma}$. Integrating over M, we find $|\lambda| = 1$.

Proposition 1.3. *The Kodaira dimension of X is zero if λ is a root of unity and negative otherwise.*

Proof: Suppose α is a global section of mK_X for some $m \geq 1$. If $q : M \times V \longrightarrow X$ is the quotient map, then $q^*\alpha$ is a global section of $mK_{M\times V}$. We may thus write

$$q^*\alpha = \theta(z,v)\left(\sigma_M \otimes \sigma_V\right)^{\otimes m}$$

where σ_M is a trivializing section of K_M, $\sigma_V = dv_1 \wedge \ldots \wedge dv_n$ is the standard holomorphic volume form on V, and θ is a holomorphic function on $M \times V$. We note that since M is compact, θ is actually just a holomorphic function on V.

Since α is a section of mK_X, the pullback $q^*\alpha$ must be invariant under the action of Γ on $M \times V$. The holomorphic volume form σ_V is invariant under the action of Γ, so if γ_i is one of the generators of Γ we find

$$\theta(v)\left(\sigma_M \otimes \sigma_V\right)^{\otimes m} = q^*\alpha = \gamma_i \cdot q^*\alpha = \lambda^m \theta(v+\gamma_i)\left(\sigma_M \otimes \sigma_V\right)^{\otimes m}.$$

If $\gamma = \sum_i a_i \gamma_i$ is an element of Γ, we set $\deg \gamma := \sum_i a_i$. Using the above we then get $\theta(v) = \lambda^{m \deg \gamma}\theta(v+\gamma)$ for any γ and v. This entails that $|\theta(v)| = |\theta(v+\gamma)|$ for all v and γ, but then $|\theta|$ takes its maximum on V in the fundamental parallelogram of Γ, so θ is constant. The complex number λ must then satisfy $\lambda^m = 1$.

We thus see that if λ is an m^{th} root of unity, then every m^{th} power of K_F admits a unique non-zero holomorphic section. In this case, the Kodaira dimension of X is zero. Likewise, if λ is not a root of unity, then no power of K_M admits a global section, so the Kodaira dimension of X is negative. □

Proposition 1.4. *The numerical dimension of K_X is zero.*

Proof: We will show that the canonical bundle K_X admits a flat hermitian metric. Its first Chern class is thus zero, which implies the proposition.

Since $M \to X \to B$ is a fibration there is a short exact sequence

$$0 \longrightarrow T_{X/M} \longrightarrow T_X \longrightarrow \pi^* T_B \longrightarrow 0$$

of tangent bundles over X. Note that since B is a torus the bundle π^*T_B is trivial. The adjunction formula now says that the canonical bundle of X is $K_X = K_{X/M}$. Let $q : M \times V \to X$ be the quotient morphism and consider the pullback bundle $q^*K_{X/M} = p_M^* K_M$, where $p_M : M \times V \to M$ is the projection.

Now pick a Ricci-flat Kähler metric ω on M, and let $dV = \omega^n/n!$ be its volume form. Recall that the volume form of any other Ricci-flat Kähler metric is a constant multiple of dV. The form dV defines a smooth hermitian metric on $p_M^*K_M$ by the formula $h(\alpha, \overline{\beta})\, dV = i^{n^2}\alpha \wedge \overline{\beta}$, where α and β are sections of $p_M^*K_M$. The curvature form of this metric is the Ricci-form of ω, so it is flat.

If σ_M is a trivializing holomorphic volume form on M, then $f^*\sigma_M = \lambda\sigma_M$, where λ is a complex number with absolute value 1. Also note that $f^*\omega$ is again a Ricci-flat Kähler metric on M, and that

$$\mathrm{Vol}(M, f^*\omega) = \int_M \frac{f^*\omega^n}{n!} = \int_M \frac{\omega^n}{n!} = \mathrm{Vol}(M, \omega)$$

because $f : M \to M$ is a finite morphism of degree one. Thus $f^*dV = dV$. From these two facts it follows that

$$f^*\big(h(\alpha, \overline{\beta})\big)dV = f^*\big(h(\alpha, \overline{\beta})dV\big) = i^{n^2} f^*\alpha \wedge \overline{f^*\beta} = h(f^*\alpha, \overline{f^*\beta})dV,$$

so the metric h is invariant under the action of Γ and thus defines a flat hermitian metric on $K_{X/M} = K_X$. $\square$

Automorphisms of hyperkähler manifolds. As before we let M be a compact simply connected Kähler manifold with trivial canonical bundle. The automorphism group of M admits a natural representation

$$\operatorname{Aut} M \longrightarrow \operatorname{Aut} H^2(M, \mathbb{C}),$$

obtained by sending each automorphism to the pullback morphism on cohomology. If M is a K3 surface, then the global Torelli theorem entails that this group morphism is actually injective. The order of an automorphism f is thus equal to the order of its pullback f^* on degree two cohomology.

One may obtain examples of higher dimensional holomorphic symplectic manifolds from a K3 surface, see [**Bea83**]. The idea is to consider the symmetric product $M^n/\mathfrak{S}_n$. This space is singular, but the Douady space $M^{[n]}$ of 0-dimensional subspaces of M of length n is a desingularization of the symmetric product. The Douady space is then a holomorphic symplectic manifold of dimension $2n$.

The second cohomology of the Douady space is isomorphic to

$$H^2(M^{[n]}, \mathbb{C}) = H^2(M, \mathbb{C}) \oplus \mathbb{C} \cdot E,$$

where E is an exceptional divisor of the desingularization $M^{[n]} \to M^n/\mathfrak{S}_n$. Any automorphism f of the K3 surface M induces an automorphism of the Douady space $M^{[n]}$. This new automorphism acts like f on the part of the second cohomology coming from M, and trivially on the exceptional divisor. In particular, if f is of infinite order on M, then the induced automorphism on $M^{[n]}$ is of infinite order.

Recall that the holomorphic symplectic form σ on M is unique up to scalars. It follows that σ is an eigenvector of any automorphism f of M, and as before one sees that the eigenvalue of σ must have absolute value 1. Oguiso gives much more precise results in [**Ogu08**]; for the moment we will contend ourselves with the following special case of his Theorem 2.4:

Proposition 1.5. *Let M is a projective K3 surface and f an automorphism of M. Let λ be the eigenvalue of f^* on the space $H^0(M, K_M)$. Then λ is a root of unity.*

By our discussion of Douady spaces, the same is true of the holomorphic symplectic space constructed from a projective K3 surface.

Example 1.6. Let $\mathbb{P} := \mathbb{P}^1_1 \times \mathbb{P}^1_2 \times \mathbb{P}^1_3$. This space comes equipped with three projections $p_i : \mathbb{P} \to \mathbb{P}_i$. Let

$$L := p_1^* \mathcal{O}(1)_{\mathbb{P}^1_1} \otimes p_2^* \mathcal{O}(1)_{\mathbb{P}^1_2} \otimes p_3^* \mathcal{O}(1)_{\mathbb{P}^1_3}$$

be an ample line bundle on $\mathbb{P}$, so that $K_{\mathbb{P}} = -2L$. The adjunction formula shows that if τ is a general section of $2L$, then the zero variety $X = \tau^{-1}(0)$ is a smooth K3 surface.

We can now consider the projections $p_{jk} : \mathbb{P} \to \mathbb{P}_j \times \mathbb{P}_k$. Restricted to the K3 surface M, these define ramified coverings $M \to \mathbb{P}_j \times \mathbb{P}_k$ of degree 2. The Galois groups of these coverings give three holomorphic involution ι_i of M, and we have

$$\operatorname{Aut} X = \langle \iota_1, \iota_2, \iota_3 \rangle \simeq \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2,$$

where $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. Both identities in the above formula are non trivial, but they are proved in [**Ogu11a**]. The automorphism group of M thus contains several elements of infinite order.

Example 1.7. We again refer to Oguiso's paper [**Ogu08**, Examples 2.5 and 2.6], from which one may extract that there exists a K3 surface M which admits an automorphism f such that the eigenvalue of f^* on $H^0(M, K_M)$ has infinite order. As before, it follows that there exist higher dimensional hyperkähler manifolds with the same property.

We now consider the non Kähler manifold $X = X(M, B)$. This manifold has negative Kodaira dimension by our earlier results. By construction there is a holomorphic map $\pi : X \longrightarrow B$ whose fiber at every point is M. Both M and B have Kodaira dimension zero, so

$$\kappa(X) < \kappa(M) + \kappa(B).$$

The manifold X is thus shows that the Iitaka $C_{n,m}$ conjecture is false for general compact complex manifolds. Furthermore, since $\kappa(X)$ is negative, but the canonical bundle K_X has numerical dimension zero, the manifold also shows that the generalized abundance conjecture is false for general complex manifolds.

Example 1.8. Oguiso and Schröer show in [**Ogu11b**] that the universal cover $\widetilde{M}^{[n]}$ of the Douady space $M^{[n]}$ of an Enriques surface M is a Calabi–Yau manifold. They also show that there exists an Enriques surface M with $\operatorname{Aut} M = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, similarly to the hyperkähler manifolds considered above. The fibration $X(\widetilde{M}^{[n]}, B)$ then provides an example of a non Kähler manifold $X \to B$ with a Calabi–Yau fiber.

2. Elliptic curves

An elliptic curve X is a compact complex curve of genus 1. Its canonical bundle is trivial, so it is a compact Kähler manifold with zero first Chern class. Choosing a point of X makes the curve into a commutative Lie group. A *marking* of an elliptic curve is the choice of a basis of $H_1(X, \mathbb{Z})$, or equivalently, the choice of a lattice Λ such that $X = \mathbb{C}/\Lambda$. One can always choose a marking such that $\Lambda = \mathbb{Z} \oplus s\mathbb{Z}$ for some s in the upper half-plane $\mathbb{H}$.

Elliptic curves with the choice of a point are parametrized by the orbifold $\mathcal{M}_{1,1}$, and marked elliptic curves are parametrized by the half-plane $\mathbb{H}$. There is a morphism $\mathbb{H} \to \mathcal{M}_{1,1}$ given by passage to the quotient by the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$. The space $\mathbb{H}$ is a fine moduli space for marked elliptic curves. We refer to [**Hai09**] for an excellent introduction to the moduli of elliptic curves.

The metrics. There is a universal family $\pi : \mathcal{X} \to \mathbb{H}$. It can be constructed as the quotient of the space $\mathbb{C} \times \mathbb{H}$ by the action of the group

$$G = \{g_{n,m} \mid g_{n,m}(z, s) = (z + n + ms, s)\}.$$

A universal family over $\mathcal{M}_{1,1}$ is then obtained by quotienting again by $SL_2(\mathbb{Z})$, whose action lifts to the family over $\mathbb{H}$.

The $(1,1)$-form $\frac{i}{2} dz \wedge d\overline{z}$ on $\mathbb{C}$ descends to each elliptic curve X_s, so it defines a smooth section of the Hodge bundle E^2 over S. We note that

$$\int_{X_s} \frac{i}{2} dz \wedge d\overline{z} = \operatorname{Im} s,$$

so the form $\frac{1}{\operatorname{Im} s} \frac{i}{2} dz \wedge d\overline{z}$ is dual to the fundamental class of each curve X_s. It thus defines a trivializing holomorphic section of E^2 which is flat with respect to the Gauss–Manin connection. The Kähler cone fibration $\mathcal{K} \to \mathbb{H}$ is thus isomorphic to the trivial cone fibration with fiber $\mathbb{H}$; an element (a, s) of $\mathbb{H} \times \mathbb{H}$ corresponds to the complexified Kähler class $\frac{a}{\operatorname{Im} s} \frac{i}{2} dz \wedge d\overline{z}$.

Note that $\operatorname{Vol}(X_s, (a, s)) = \operatorname{Im} a$ is independent of the parameter s. The curvature form of the hermitian metric on the line bundle $\mathcal{R}^2 \pi_* \mathbb{C}$ is then

$$i\Theta = i\partial\bar{\partial} \log \operatorname{Im} a = -\frac{i}{(\operatorname{Im} a)^2} da \wedge \mathrm{d}\overline{a}.$$

This form is seminegative, but the dual metric on the dual bundle has semipositive curvature.

Similarly the $(1,0)$-form dz defines a smooth section of the Hodge bundle $E^{1,0} \to S$. The homology group $H^1(X_s, \mathbb{Z})$ is spanned by 1 and s, so the frame $(1^*, s^*)$ of E^1 is flat with respect to the Gauss–Manin connection. We find that $dz = 1 \cdot 1^* + s \cdot s^*$, so dz is a holomorphic section of $E^{1,0}$. Note that

$$\int_{X_s} i dz \wedge d\overline{z} = 2 \operatorname{Im} s,$$

so $-\log 2 \operatorname{Im} s$ is a potential for the Weil–Petersson metric on $\mathbb{H}$. Replacing the sections dz by a constant multiple, we may take $-\log \operatorname{Im} s$ as our potential. This is of course just the usual potential of the hyperbolic Poincaré metric on the upper half-plane.

Put together, the Kähler metric on $\mathcal{K}$ is then given by

$$\alpha = -i\partial\bar{\partial} \log \operatorname{Im} a \operatorname{Im} s = \frac{i}{(\operatorname{Im} a)^2} da \wedge \mathrm{d}\overline{a} + \frac{i}{(\operatorname{Im} s)^2} ds \wedge \mathrm{d}\overline{s}$$

in the coordinates we have chosen. We note that there is an evident isometry between $\mathcal{K}$ and $(\mathbb{H}, h_P)^2$, where h_P is the hyperbolic Poincaré metric on the upper half plane.

The universal family. The fiber product $\mathcal{X} \times_{\mathbb{H}} \mathcal{K}$ may be described as the quotient of the space $\mathbb{C} \times \mathbb{H} \times \mathbb{H}$ by the action of the group

$$\tilde{G} = \{g_{n,m} \mid g_{n,m}(z, a, s) = (z + n + ms, a, s)\},$$

because the action induced by the group G on the line bundle E^2 is trivial. We will identify objects on the space $\mathcal{X} \times_{\mathbb{H}} \mathcal{K}$ with those objects on $\mathbb{C} \times \mathbb{H} \times \mathbb{H}$ that are invariant under the action of $\tilde{G}$. We denote by $p_{\mathcal{K}} : \mathcal{X} \times_{\mathbb{H}} \mathcal{K} \to \mathcal{K}$ the morphism induced by the projection.

Consider the smooth lift of vector bundles $\beta : p_{\mathcal{K}}^* T_{\mathcal{K}} \to T_{\mathcal{X}\times_{\mathbb{H}}\mathcal{K}}$ given by

$$v(z,a,s)\frac{\partial}{\partial a} + w(z,a,s)\frac{\partial}{\partial s} \longmapsto -\frac{z-\overline{z}}{s-\overline{s}}\frac{\partial}{\partial z} + v(z,a,s)\frac{\partial}{\partial a} + w(z,a,s)\frac{\partial}{\partial s}.$$

Note that if the functions v and w are invariant under the action of $\tilde{G}$, then the this morphism is invariant under the action of $\tilde{G}$ and thus really gives a morphism of the above vector bundles.

Let $\xi = w\frac{\partial}{\partial s}$ be a holomorphic vector field on S and pull it back to $\mathcal{K}$. As the reader may verify, the form

$$\bar{\partial}\beta(\xi)_{|X_s} = \frac{w}{s-\overline{s}}\frac{\partial}{\partial z} \otimes d\overline{z}$$

is the harmonic representative of the Kodaira–Spencer class $\rho(\xi)$ with respect to any flat Kähler metric on X_s. The morphism β thus coincides with the harmonic lift induced by the flat metrics on the manifolds of the family $\mathcal{X} \to S$.

Using the induced splitting of the tangent bundle of the fiber product $\mathcal{X} \times_{\mathbb{H}} \mathcal{K}$, we see that the hermitian metric on this family is given by

$$H = \begin{pmatrix} \frac{1}{\operatorname{Im} s} & 0 & -\frac{z-\overline{z}}{s-\overline{s}}\frac{1}{\operatorname{Im} s} \\ 0 & \frac{1}{(\operatorname{Im} a)^2} & 0 \\ -\frac{z-\overline{z}}{s-\overline{s}}\frac{1}{\operatorname{Im} s} & 0 & \frac{1}{(\operatorname{Im} s)^2} + \frac{1}{\operatorname{Im} s}\left(\frac{z-\overline{z}}{s-\overline{s}}\right) \end{pmatrix}.$$

Now, a hermitian metric $H = (h_{j\overline{k}})$ is Kähler if and only if its coefficients satisfy the partial differential equations

$$\frac{\partial h_{j\overline{k}}}{\partial z_l} = \frac{\partial h_{l\overline{k}}}{\partial z_j}$$

for all j, k and l. We cordially invite the reader to verify that this is indeed the case here, so H is a Kähler metric.[1]

An isometry. Since the relative Kähler cone is isometric to two copies of the Poincaré half-plane, it comes as little surprise that the exchange of the two copies of the half-plane induces a holomorphic isometry of the relative Kähler cone. Somewhat more interestingly, this isometry extends to the universal family over the relative Kähler cone.

Consider the self-diffeomorphism φ of $\mathbb{C} \times \mathbb{H} \times \mathbb{H}$ given by

$$\varphi(z,a,s) = (L(a,s)(z), s, a),$$

where

$$L(a,s)(z) = \frac{a-\bar{s}}{s-\bar{s}}z + \frac{s-a}{s-\bar{s}}\,\overline{z}$$

is the unique $\mathbb{R}$-linear automorphism of $\mathbb{C}$ that sends the basis $(1,s)$ to $(1,a)$. It is invariant under the action of $\tilde{G}$ and thus defines a self-diffeomorphism of the universal family $\mathcal{X} \times_{\mathbb{H}} \times \mathcal{K}$ over $\mathcal{K}$.

[1] There is a more conceptual way to see this: the family of elliptic curves is polarized by the line bundle $\mathcal{L} \to \mathcal{X}$ whose restriction to each X_s is the ample generator of $\operatorname{Pic} X_s$. Along with the Weil–Petersson metric on S, this induces a Kähler metric on $\mathcal{X}$. Our claim is now just that the fiber product of Kähler manifolds is Kähler and that the induced metric on the fiber product is indeed the above hermitian metric.

To see that this is an isometry, we first note that we can write the Kähler form of the metric H as

$$\omega(z,a,s) = \frac{-1}{s-\overline{s}}\beta \wedge \overline{\beta} - \frac{1}{(a-\overline{a})^2}\frac{i}{2}da \wedge d\overline{a} - \frac{1}{(s-\overline{s})^2}\frac{i}{2}ds \wedge d\overline{s}$$

where β is the 1-form

$$\beta(z,a,s) = dz - \frac{z-\overline{z}}{s-\overline{s}}ds.$$

Standard calculations now show that $\varphi^*\beta(z,a,s) = \beta(z,s,a)$, which entails that $\varphi^*\omega(z,a,s) = \omega(z,s,a)$, so φ is an isometry of the total space of the universal family.

Remark — One of the original motivations for the work done in this thesis was to try to express mirror symmetry of manifolds with zero first Chern class in metric terms. Since the diffeomorphism in this example is directly inspired by the statement of mirror symmetry for elliptic curves, the example at least shows that this effort is not hopeless.

In the general case, or even in the case of complex tori, this endeavor remains hindered by my lack of a conceptual understanding of the phenomenon of mirror symmetry. Work continues.

3. A remark on a moduli space for Ricci-flat manifolds

Let X be a compact Kähler manifold with zero first Chern class. We can view X as a smooth manifold M equipped with a complex structure J. Let $\mathcal{J}$ be the set of all complex structures on M. It is an infinite dimensional complex manifold, and by considering the product $M \times \mathcal{J} \to \mathcal{J}$ we obtain a quite large family of complex manifolds, in some sense universal.

This family contains a large number of isomorphic manifolds. To cut this number down we must identify those complex structures that differ by a diffeomorphism of the underlying smooth manifold M. In detail, the group $\mathcal{D}^+$ of diffeomorphisms of M that preserve the orientation defined by the complex structures acts on $\mathcal{J}$, and two complex manifolds (M,J) and (M,J') are isomorphic if and only if the structures J and J' belong to the same coset of $\mathcal{D}^+$. Thus we would very much like to define the moduli space of complex structures on M by the quotient $\mathcal{J}/\mathcal{D}^+$.

This is indeed what Kodaira and Spencer tried to do in their fundamental papers on deformation theory [**KS58, KS60**]. However they found that the quotient $\mathcal{J}/\mathcal{D}^+$ can be quite wild, since some manifolds may admit discrete automorphism groups of infinite order the quotient will not have the structure of a complex manifold or orbifold. The classical example of this phenomena is the space of complex structures on a torus [**KS58**, p. 413].

The subgroup $\mathcal{D}^0$ of $\mathcal{D}^+$ that consists of diffeomorphisms homotopic to the identity on M also acts on $\mathcal{J}$. The Bogomolov–Tian–Todorov theorem [**Tia87**] entails that the Teichmüller space $\mathcal{T} = \mathcal{J}/\mathcal{D}^0$ is at least a smooth complex manifold. Thus problems only appear when trying to pass to the quotient by the action of the mapping class group $\mathcal{D}^+/\mathcal{D}^0$.

People have handled this problem by polarizing the families under consideration [**Sch84**]. This has the effect of throwing away a number of automorphisms and the resulting quotient then often has the structure of an honest manifold. We would like to point out that one can avoid polarizing and quotient by all automorphisms, at least in the case of manifolds with zero first Chern class. The main step in this direction is provided by the following result, which we state in our language:

Theorem 3.1 ([**Bes08**, Theorem 12.103])**.** *Let $\mathcal{K}_{\mathbb{R}}$ be the real relative Kähler cone over the Teichmüller space $\mathcal{T}$. The group $\mathcal{D}^+$ of diffeomorphisms of M acts naturally on $\mathcal{K}_{\mathbb{R}}$ and the quotient $\mathcal{K}_{\mathbb{R}}/\mathcal{D}^+$ has the structure of an orbifold.*

From this one can easily deduce that the total space of the complexified Kähler cone $\mathcal{K}$ has a quotient that admits the structure of a holomorphic orbifold. Indeed, there is a submersion $\mathcal{K} \to \mathcal{K}_{\mathbb{R}}$ given by sending a complexified Kähler class $a = b + i\omega$ to ω, and the action of $\mathcal{D}^+$ commutes with this submersion. Now consider both relative cones over the Teichmuller space $\mathcal{T}$. Since the quotient $\mathcal{K}_{\mathbb{R}}/\mathcal{D}^+$ is an orbifold, then so is the quotient $\mathcal{K}/\mathcal{D}^+$, and it is holomorphic since $\mathcal{D}^+$ acts holomorphically on $\mathcal{K}$.

One can also consider the space $M \times \mathcal{K}$ over $\mathcal{K}$ and get a universal holomorphic manifold over the quotient $\mathcal{K}/\mathcal{D}^+$. We remark that the metrics we constructed in this thesis are invariant under the action of automorphisms on each manifold in a family, so they pass to the quotient and define orbifold metrics on each space.

Bibliography

[AS06] Reynir Axelsson and Georg Schumacher. Kähler geometry of Douady spaces. *Manuscripta Math.*, 121(3):277–291, 2006.

[Aub78] Thierry Aubin. Équations du type Monge-Ampère sur les variétés kählériennes compactes. *Bull. Sci. Math. (2)*, 102(1):63–95, 1978.

[Bal06] Werner Ballmann. *Lectures on Kähler manifolds*. ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2006.

[BDIP96] José Bertin, Jean-Pierre Demailly, Luc Illusie, and Chris Peters. *Introduction à la théorie de Hodge*, volume 3 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 1996.

[Bea83] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.*, 18(4):755–782 (1984), 1983.

[Bes08] Arthur L. Besse. *Einstein manifolds*. Classics in Mathematics. Springer-Verlag, Berlin, 2008. Reprint of the 1987 edition.

[CR11] Brian Clarke and Yanir Rubinstein. Conformal deformations of the Ebin metric and a generalized Calabi metric on the space of Riemannian metrics. *arXiv*, arXiv:1104.4314v1, 2011.

[Dem09] Jean-Pierre Demailly. Analytic Methods in Algebraic Geometry. `http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/analmeth.pdf`, 2009.

[Dem12] Jean-Pierre Demailly. Complex analytic and differential geometry. `http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`, 2012.

[DP04] Jean-Pierre Demailly and Mihai Paun. Numerical characterization of the Kähler cone of a compact Kähler manifold. *Ann. of Math. (2)*, 159(3):1247–1274, 2004.

[Ebi70] David G. Ebin. The manifold of Riemannian metrics. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968)*, pages 11–40. Amer. Math. Soc., Providence, R.I., 1970.

[GHJ03] M. Gross, D. Huybrechts, and D. Joyce. *Calabi-Yau manifolds and related geometries*. Universitext. Springer-Verlag, Berlin, 2003. Lectures from the Summer School held in Nordfjordeid, June 2001.

[Gro95] Alexander Grothendieck. Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert. In *Séminaire Bourbaki, Vol. 6*, pages Exp. No. 221, 249–276. Soc. Math. France, Paris, 1995.

[Hai09] Hain, R. Lectures on Moduli Spaces of Elliptic Curves. *arXiv*, 0812.1803v2, 2009.

[Huy01] Daniel Huybrechts. Products of harmonic forms and rational curves. *Doc. Math.*, 6:227–239 (electronic), 2001.

[Huy05] Daniel Huybrechts. *Complex geometry*. Universitext. Springer-Verlag, Berlin, 2005. An introduction.

[KN96] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. I*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1996. Reprint of the 1963 original, A Wiley-Interscience Publication.

[Kon95] Maxim Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 120–139, Basel, 1995. Birkhäuser.

[KS58] K. Kodaira and D. C. Spencer. On deformations of complex analytic structures. I, II. *Ann. of Math. (2)*, 67:328–466, 1958.

[KS60] K. Kodaira and D. C. Spencer. On deformations of complex analytic structures. III. Stability theorems for complex structures. *Ann. of Math. (2)*, 71:43–76, 1960.

[Kur62] M. Kuranishi. On the locally complete families of complex analytic structures. *Ann. of Math. (2)*, 75:536–577, 1962.

[Lan99] Serge Lang. *Fundamentals of differential geometry*, volume 191 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[Nan86] Antonella Nannicini. Weil-Petersson metric in the moduli space of compact polarized Kähler-Einstein manifolds of zero first Chern class. *Manuscripta Math.*, 54(4):405–438, 1986.

[Ogu08] Keiji Oguiso. Bimeromorphic automorphism groups of non-projective hyperkähler manifolds—a note inspired by C. T. McMullen. *J. Differential Geom.*, 78(1):163–191, 2008.

[Ogu11a] Oguiso, Keiji. Birational automorphism groups and the movable cone theorem for Calabi-Yau manifolds of Wehler type via universal Coxeter groups. `http://arxiv.org/abs/1107.5862`, 2011.

[Ogu11b] Oguiso, Keiji and Schröer, Stefan. Enriques manifolds. `http://arxiv.org/abs/1001.4912`, 2011.

[Sch84] Georg Schumacher. Moduli of polarized Kähler manifolds. *Math. Ann.*, 269(1):137–144, 1984.

[Sch85] Georg Schumacher. On the geometry of moduli spaces. *Manuscripta Math.*, 50:229–267, 1985.

[Siu86] Yum Tong Siu. Curvature of the Weil-Petersson metric in the moduli space of compact Kähler-Einstein manifolds of negative first Chern class. In *Contributions to several complex variables*, Aspects Math., E9, pages 261–298. Vieweg, Braunschweig, 1986.

[SYZ96] Andrew Strominger, Shing-Tung Yau, and Eric Zaslow. Mirror symmetry is T-duality. *Nuclear Phys. B*, 479(1-2):243–259, 1996.

[Tia87] Gang Tian. Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric. In *Mathematical aspects of string theory (San Diego, Calif., 1986)*, volume 1 of *Adv. Ser. Math. Phys.*, pages 629–646. World Sci. Publishing, Singapore, 1987.

[TW11] Thomas Trenner and P. M. H. Wilson. Asymptotic curvature of moduli spaces for Calabi-Yau threefolds. *J. Geom. Anal.*, 21(2):409–428, 2011.

[Uen75] Kenji Ueno. *Classification theory of algebraic varieties and compact complex spaces*. Lecture Notes in Mathematics, Vol. 439. Springer-Verlag, Berlin, 1975. Notes written in collaboration with P. Cherenack.

[Vak06] Ravi Vakil. Murphy's law in algebraic geometry: badly-behaved deformation spaces. *Invent. Math.*, 164(3):569–590, 2006.

[Voi02] Claire Voisin. *Théorie de Hodge et géométrie algébrique complexe*, volume 10 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2002.

[Wan03] C.L. Wang. Curvature properties of the Calabi-Yau moduli. *Documenta Mathematica*, 8:577–590, 2003.

[Wil04] P. M. H. Wilson. Sectional curvatures of Kähler moduli. *Math. Ann.*, 330(4):631–664, 2004.

[Yau78] Shing Tung Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Comm. Pure Appl. Math.*, 31(3):339–411, 1978.

[Zhe00] Fangyang Zheng. *Complex differential geometry*, volume 18 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2000.

Abstract. In this thesis we consider families $\pi : \mathcal{X} \to S$ of compact Kähler manifolds with zero first Chern class over a smooth base S. We construct a relative complexified Kähler cone $p : \mathcal{K} \to S$ over the base of deformations. Then we prove the existence of natural hermitian metrics on the total spaces $\mathcal{K}$ and $\mathcal{X} \times_S \mathcal{K}$ that generalize the classical Weil–Petersson metrics associated to polarized families of such manifolds.

As a byproduct we obtain a Riemannian metric on the Kähler cone of any compact Kähler manifold. We obtain an expression of its curvature tensor via an embedding of the Kähler cone into the space of hermitian metrics on the manifold. We also prove that if the manifolds in our family have trivial canonical bundle, then our generalized Weil–Petersson metric is the curvature form of a positive holomorphic line bundle. We then give some examples and applications.

Résumé. Dans cette thèse nous considérons des familles $\pi : \mathcal{X} \to S$ de variétés compactes kähleriennes au-dessus d'une base lisse S. Nous construisons un cône de Kähler relatif $p : \mathcal{K} \to S$ au-dessus de la base de déformations. Nous démontrons ensuite l'existence de métriques hermitiennes naturelles sur les espaces totaux $\mathcal{K}$ et $\mathcal{X} \times_S \mathcal{K}$ qui généralisent la métrique de Weil–Petersson classique associée aux familles polarisées de telles variétés.

Nous obtenons aussi une métrique riemannienne sur le cône de Kähler d'une variété compacte kählerienne quelconque. Nous exprimons son tenseur de courbure à l'aide d'un plongement du cône de Kähler dans l'espace des métriques hermitiennes sur la variété. Nous démontrons aussi que si les variétés en question sont de fibré canonique trivial, alors la métrique construite précédemment est la forme de courbure d'un fibré en droites holomorphe. Nous donnons ensuite quelques exemples et applications.