Contribution à l'étude des lacets markoviens

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UNIVERSITÉ PARIS-SUD

FACULTÉ DES SCIENCES D'ORSAY

THÈSE

présentée pour obtenir

LE GRADE DE DOCTEUR EN SCIENCES

DE L'UNIVERSITÉ PARIS XI

Spécialité : Mathématiques

par

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Titre :

Contribution à l'étude des lacets markoviens.

Soutenue le 3 Juin 2013 devant la Commission d'examen :

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Résumé

Nous nous intéressons aux lacets markoviens définis dans le cadre de la théorie des chaînes de Markov à temps continu sur un espace d'états discret. Ce sujet a notamment été étudié par Le Jan [LJ11] et Sznitman [Szn12]. En contraste avec ces références, nous ne supposerons pas la symétrie de la chaîne et nous intéresserons plutôt au cas infini. Tous les résultats sont présentés en termes de générateur de semi-groupe. En comparaison avec [LJ11], certaines preuves ont été détaillées ou améliorées.

Nous fournissons par ailleurs quelques résultats sur les amas de boucles (voir [LJL12] dans le cas symétrique). Nous traitons notamment l'exemple du cercle discret. Nous étudions aussi les arbres couvrants définis par l'algorithme de Wilson dans le cas asymétrique.

Dans la dernière partie, nous considérons la proportion des lacets couvrants l'espace. En utilisant la limite du spectre, nous donnons une expression générale de la limite de cette proportion pour une suite de graphes. Comme applications, nous donnons deux exemples concrets dans lesquels une transition de phase apparaît.

Mots clés : lacets markoviens, amas des lacets, l'arbre couvrant, lacets couvrant.

Abstract

We are interested in Markov laces defined in the framework of the theory of Markov chains in continuous time on a discrete state space. This particular subject has been studied by Le Jan [LJ11] and Sznitman [Szn12]. In contrast to these references, we do not assume the reversibility of the chain and we are mostly interested in the case of countable state space. All the results are presented in terms of the generator of semigroup. In comparison with [LJ11], some demonstration has been detailed or improved.

We also provide some results on the loop clusters (see [LJL12] in the reversible case). In particular, we study the example of discrete circle. We also study the spanning tree algorithm defined by Wilson in the non-symmetric case.

In the last part, we consider the proportion of loops covering the whole space. Using the limit of the spectrums, we give a general expression for the limit of this ratio for a sequence of graphs. As an application, we give two examples in which a phase transition occurs.

Keywords: Markovian loops, loop cluster, spanning tree, covering loop.

Remerciements

Je tiens tout d'abord à exprimer ma gratitude la plus profonde envers mon directeur de thèse, Yves Le Jan. Avec sa gentillesse et sa grande expérience, il m'a bien guidé et apporté son soutien tout au long de la thèse. Je vous remercie vivement pour tout le temps que vous m'avez consacré, pour vos encouragements et vos conseils avisés et aussi pour m'avoir appris le métier de chercheur.

Je remercie vivement Christophe Sabot et Zhan Shi pour avoir accepté de rapporter ma thèse et faire partie du jury. Merci pour tous vos conseils qui m'ont aidé à améliorer ce travail. Je remercie également Nathalie Eisenbaum et Jean-François Le Gall pour avoir accepté de faire partie de mon jury.

Je souhaite exprimer ma reconnaissance à Sophie Lemaire pour sa relecture très attentive. Merci beaucoup d'avoir pris le temps de me guider à rédiger mes textes.

Mes vifs remerciements vont également à Zhiying Wen qui m'a aidé dans mes premiers pas en analyse. Merci pour le soutien et les encouragements constants pendant ces 9 années.

Je voudrais remercier Max Von Renesse pour m'avoir invité à Leipzig et pour ses discussions avec moi. J'adresse également mes remerciements à Artem Sapozhnikov pour son intérêt à mon travail et pour des discussions intéressantes.

Je voudrais adresser mes chaleureux remerciements à tous les membres du laboratoire de Mathématiques de l'Université de Paris-Sud dont Raphaël Cerf, Wendelin Werner, Grégory Miermont, Mylène Maïda, Pierre Pansu, Valérie Lavigne, David Harari, Nessim Sibony, ...

Merci à Édouard Maurel-Segala pour m'avoir recueilli dans son TD et m'avoir aidé à améliorer mes enseignements.

Merci à Xinxin Chen, Xiong Jin, Shen Lin, Shu Shen, Minmin Wang, Hao Wu pour le partage des connaissance et pour l'ambiance dans le séminaire étudiant.

Merci également à tous mes amis : Li Chen, Miaofen Chen, Wen Deng, Yiwen Ding, Yong Hu, Yongquan Hu, Yi Huang, Le Li, Yongqi Liang, Xiangyu Liang, Yue Ma, Xun Jiang, Zhi Jiang, Yanqi Qiu, Xu Shen, Shenghao Sun, Zhe Sun, Shun Tang, Chunhui Wang, Haoran

Wang, Jian Wang, Shanwen Wang, Qiaoling Wei, Haiyan Xu, Wenyuan Yang, Zhihui Yuan, Guodong Zhou, Jialin Zhu et plus particulièrement à Jiatu Cai, Hatem Hajri et Xin Nie.

Mes pensées vont maintenant à Jingzhi. Merci pour ton accompagnement durant ces années et pour tout le bonheur partagé. Enfin, Je remercie du fond du cœur mes parents pour tout ce qu'ils ont fait pour moi. Cette thèse leur est dédiée.

Contents

Chapter 1

Introduction

Dans cette thèse nous nous intéressons aux lacets markoviens dans un espace S d'états dénombrables. La notion de la soupe de lacets fut introduite par Lawler et Werner. Dans la référence [LW04], la soupe de lacets associés à un mouvement brownien dans $\mathbb{R}^2$ est définie et son analogue pour une marche aléatoire simple est étudié dans [LTF07]. Dans le cadre des processus réversibles, ce sujet a été étudié en détail par Le Jan et Sznitman (voir [Szn12] et [LJ11]).

Étant donné un générateur infinitésimal L, nous pouvons y associer une chaîne de Markov minimale (voir [Nor98]). Désignons par $\mathbb{P}^x$ la loi de la chaîne issue de x et par $\mathbb{P}_t^{x,y}$ la mesure de pont de durée t :

$$\mathbb{P}_t^{x,y}[f(X_s, s \in [0,t])] = \mathbb{P}^x[f(X_s, s \in [0,t]), X_t = y].$$

où X désigne la chaîne de Markov. Un lacet basé l de durée $|l| = t$ est simplement une trajectoire càdlàg de durée t dont les positions au temps initial et final sont égaux : $l : [0,t] \to S$ telle que $l(0) = l(t-) = l(t)$. Nous définissons une mesure μ^b $\sigma-$finie par

$$\mu^b(dl) = \int_0^\infty \sum_{x \in S} \mathbb{P}_t^{x,x}[dl]\, dt.$$

Deux lacets basés sont équivalents ssi ils sont égaux à une rotation près. Un lacet est une classe d'équivalence des lacets basé (voir Définition 3.1.5). La mesure des lacets markovien est la mesure d'image μ de la mesure μ^b sur l'ensemble des lacets.

Le temps d'occupation d'un lacet (basé) est défini par $l^x = \int\limits_{0<s<|l|} 1_{\{l(s)=x\}}\, ds$ pour tout $x \in S$.

Le champ de multi-occupation s'écrit comme

$$\langle l,f\rangle = \sum_{j=0}^{n-1} \int_{0<s^1<\cdots<s^n<t} f(l(s^{1+j}),\ldots,l(s^{n+j}))\,ds^1\cdots ds^n,$$

où f désigne une fonction sur S. L'espace vectoriel engendré par le champs de multi-occupation est une algèbre qui engendre la tribu borélienne. Par conséquence, la loi du champ de multi-occupation détermine la mesure des lacets markoviens[1]. Sous l'hypothèse de transience, nous pouvons calculer une variante de la transformée de Laplace du champ d'occupation (voir 3.5.1) : Soit χ une fonction à support compact. Notons $\rho(M_{\sqrt{\chi}}VM_{\sqrt{\chi}})$ le rayon spectral de $M_{\sqrt{\chi}}VM_{\sqrt{\chi}}$. Pour $z\in D=\{z\in\mathbb{C}:\mathrm{Re}(z)<\frac{1}{\rho(M_{\sqrt{\chi}}VM_{\sqrt{\chi}})}\}$, nous avons

$$\mu(e^{z\langle l,\chi\rangle}-1) = -\ln\det(I-zM_{\sqrt{\chi}}VM_{\sqrt{\chi}}).$$

Par contre, dans le complémentaire de D, nous avons $\mu(|e^{z\langle l,\chi\rangle}-1|)=\infty$. En utilisant ce résultat, nous pouvons déterminer la mesure des lacets non-triviaux qui rencontrent un sous-ensemble fini $F\subset S$:

$$\mu(l \text{ est non-trivial et } l\cap F\neq\phi) = \ln(\prod_{x\in F}(-L^x_x)\det(V_F)).$$

Plus généralement, pour n ensembles finis $F_1,\ldots,F_n$, nous avons

$$\mu(l\cap F_i\neq\phi \text{ pour tout } i=1,\ldots,n) = -\sum_{A\subset\{1,\ldots,n\},A\neq\phi}(-1)^{|A|}\ln\det(V_{F_A}) + \sum_{x\in\bigcap_{i=1}^n F_i}\ln(-L^x_x)$$

où $F_A \overset{\text{déf}}{=} \bigcup_{i\in A} F_i$ pour $A\subset\{1,\ldots,n\}$ (voir Proposition 3.5.8). La mesure des lacets markoviens est compatible avec le changement du temps et la perturbation de Feynman-Kac. En particulier, elle est compatible avec les notions de "trace" et "restriction" (voir Proposition 3.2.1 et Proposition 3.4.1.)

Soit $(\mathcal{L}_\alpha,\alpha\geq 0)$ l'ensemble poissonnien des lacets d'intensité $\alpha\mu$. Le champ d'occupation de cet ensemble est défini par $\sum_{l\in\mathcal{L}_\alpha} l^x$. Les moments sont sous la forme de $\alpha-$permenant :

$$\mathbb{E}[\hat{\mathcal{L}}_\alpha^{x_1}\cdots\hat{\mathcal{L}}_\alpha^{x_n}] = \mathrm{Per}_\alpha((V^{x_l}_{x_m})_{1\leq m,l\leq n}).$$

En utilisant la formule de Campbell, nous fournissons la transformée de Laplace de $\langle\mathcal{L}_\alpha,\chi\rangle$ (voir Proposition 4.1.5). La formule de Campbell nous permet aussi de calculer la probabilité

[1]En effet, la mesure μ est déterminée par l'espérance du champ (voir 3.4.9).

pour qu'un sous-ensemble F ne soit pas couvert par des lacets non-triviaux :

$$\mathbb{P}[\nexists l \in \mathcal{L}_\alpha; l \text{ est non-trivial et } l \cap F \neq \phi] = \inf_{A \subset F, |A| < \infty} (\prod_{x \in A} (-L_x^x) \det(V_A))^{-\alpha}.$$

Par ailleurs, d'après la généralisation du théorème de l'isomorphisme de Dynkin (voir [LJ08]), nous pouvons identifier le champ d'occupation avec le carré d'un champ "gaussien" de densité complexe. En conséquence, on obtient une expression de la densité de $(\mathcal{L}_\alpha^{x_1}, \ldots, \mathcal{L}_\alpha^{x_n})$ pour $\alpha = 1$. Pour α quelconque, nous trouvons la densité de $(\mathcal{L}_\alpha^{x_1}, \ldots, \mathcal{L}_\alpha^{x_n})$ à partir de sa transformée de Laplace (voir Proposition 4.5.3). Nous sommes également intéressés par le comportement conditionnel du champ d'occupation. Plus précisément, nous nous intéressons à la distribution du champ d'occupation connaissant la trace de l'ensemble de lacets dans un sous-ensemble F. Étant donné un lacet l qui rencontre F, en supprimant les excursions en dehors de F et en collant le reste dans l'ordre de lacet original, nous obtenons la trace l_F du lacet l dans F qui est encore un lacet. La proposition 4.6.2 montre que l'ensemble des excursions à l'extérieur de F est une mesure aléatoire de Poisson conditionnée au l_F. En effet, cette mesure conditionnée dépend seulement de :

- Le champs d'occupation sur la frontière ∂F de F.
- Le nombre de saut de $x \in \partial F$ vers $y \in \partial F$ pour la trace l_F.
- La mesure d'excursion en dehors de F.

En conséquence, nous donnons une expression explicite de $\mathbb{E}[e^{-\langle \hat{\mathcal{L}}_\alpha, \chi \rangle} | \sigma(\mathcal{L}_\alpha|_F)]$ dans la Proposition 4.6.3 où $\mathcal{L}_\alpha|_F$ sont les trajectoires observées sous la fenêtre F. Dans la section 4.3, nous nous intéressons au comportement asymptotique de $\frac{1}{\alpha}(\mathcal{L}_\alpha^{x_1}, \ldots, \mathcal{L}_\alpha^{x_n})$. D'après un résultat général de A. de Acosta [dA94], ils vérifient un principe de grandes déviations dont les fonctions de taux ont été précisées pour $n \leq 2$ (voir Proposition 4.3.1 et la remarque suivante). Dans la dernière section de ce chapitre, nous considérons l'amas de lacets [2]. Ce sujet a été introduit et bien étudié par Yves Le Jan et Sophie Lemaire dans le cas d'une chaîne réversible (voir [LJL12]). Comme dans [LJL12], nous pouvons calculer la probabilité pour que certaines arêtes soient fermées et la probabilité pour que la partition de l'espace d'états définie par l'amas de lacets soit plus fine qu'une partition fixée. Parmis des nombreux exemples étudiés dans [LJL12], on trouve l'amas de lacets sur $\mathbb{N}$. Il est prouvé dans [LJL12] que ses arêtes fermées forment un subordinateur à la limite dont le potentiel est donné explicitement. A la

[2] Une arête est dite "ouverte" ssi. elle est couverte par au moins un lacet dans l'ensemble Poissonnien des lacets.

fin de ce chapitre, nous considérons la version conditionnelle de ce modèle : l'amas de lacet sur le cercle discret avec une arête fermée. Nous identifions sa limite avec un subordinateur conditionné (voir Proposition 4.7.7).

La partie 5 étudie les liens entre l'algorithme de Wilson qui génère un arbre couvrant aléatoire d'une chaîne de Markov et l'ensemble Poissonnien des lacets associé à cette même chaîne. Les propositions 5.1.3 et 5.1.4 permettent de relier l'ensemble des lacets effacés dans l'algorithme de Wilson à l'ensemble Poissonnien des lacets. Dans la dernière section, nous étudions l'arbre couvrant orienté qui peut être construit par l'algorithme de Wilson. Nous donnons une preuve élémentaire du théorème de transfert du courant (voir Théorème 5.3.3 et Corollaire 5.3.4). Ces résultats sont dûs à Burton and Permantle pour les chaîne réversibles (voir [BP93]) dont les approches requièrent l'hypothèse de symétrie. Notre méthode s'applique à des graphes orientés arbitraires.

Dans le sixième et dernier chapitre nous étudions des lacets couvrants munis de la mesure de lacets markoviens. Le problème du temps de couverture est un problème classique dans l'étude des chaînes de Markov. Nous considérons une suite croissante de graphes G_n avec une suite de paramètres de meurtre c_n. La mesure des lacets non triviaux s'écrit comme

$$\mu_n^{p*}(\text{k sauts}, \xi_1 = x_1, \ldots, \xi_k = x_k) = \frac{1}{k}\left(\frac{1}{1+c_n}\right)^k (Q_n)^{x_1}_{x_2}(Q_n)^{x_2}_{x_3}\cdots(Q_n)^{x_k}_{x_1}$$

où Q_n est la matrice de transition associée à G_n. Soit $\mathfrak{P}_n = \frac{\mu_n^{p*}}{\mu_n^{p*}(1)}$. Le théorème 6.1.1 donne la limite $\mathfrak{P}_n[l \text{ couvre tous les sommets de } G_n]$ en fonction de la limite $a \overset{def}{=} \lim\limits_{n\to\infty} -\frac{\ln c_n}{m_n}$ avec la limite spectrale de G_n sous les hypothèses suivantes :

- (H1) Les degrés sont uniformément bornés.
- (H2) Les conductances des arêtes sont uniformément bornés.
- (H3) Il existe une distribution ν sur $[-1, 1]$ telle que pour tout $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{Tr} Q_n^k = \int x^k \nu(dx).$$

(Autrement dit, soient $1 \geq \lambda_1 \geq \cdots \geq \lambda_n \geq -1$ des valeurs propres de Q_n, alors $\sum\limits_{i=1}^{n} \frac{1}{n}\delta_{\lambda_i} \xrightarrow{\text{loi}} \nu$.)

Intuitivement, en diminuant les taux de meurtre, les longueurs de lacets augmentent sous $\mathfrak{P}_n$ ce qui suggère l'augmentation de $\mathfrak{P}_n$. Pour établir le Théorème 6.1.1, nous avons estimé d'une part la longueur des lacets et d'autre part la probabilité conditionnelle

$$\{l \text{ couvre tous les sommets de } G_n\}$$

sachant que $\{\text{longueur} = k\}$ pour k assez grand. Pour la première partie, nous avons utilisé une borne inférieure du trou spectral dûe à Persi Diaconis et Daniel Stroock (voir [DS91]) et une borne supérieure de $\operatorname{Tr} Q_n^k$ dûe à E. A. Carlen, S. Kusuoka et D. W. Stroock (voir [CKS87]) qui est ensuite rétablie par un argument simple par Aldous pour les graphes réguliers (voir [AF]). Avec peu de modifications, nous adaptons la méthode d'Aldous dans le cas des graphes vérifiant (H1) et (H2). Pour la deuxième partie, nous utilisons un résultat classique concernant la borne supérieure de l'espérance de temps de recouvrement. Nous fournissons deux exemples dans lesquels nous explicitons les distributions ν à limite. Dans la dernière section, nous considérons des graphes complets qui ne vérifient pas (H1) pour lesquels nous ne pouvons plus appliquer Théorème 6.1.1. Néanmoins, nous établissons le Théorème 6.6.3 en comparant une variable géométrique modifiée au temps de recouvrement de la chaîne de Markov dans les graphes complets.

Introduction (English version)

We fix a countable state space S. Given an infinitesimal generator L, one can construct a minimal continuous Markov chain X, see [Nor98]. Denote by $\mathbb{P}^x$ its law starting from x. Let $\mathbb{P}_t^{x,y}$ be the non-normalized bridge measure of duration t defined by

$$\mathbb{P}_t^{x,y}[f(X_s, s \in [0,t])] = \mathbb{P}^x[f(X_s, s \in [0,t]), X_t = y].$$

By a based loop with time duration t, we mean a càdlàg path $l : [0,t] \to S$ that $l(0) = l(t-) = l(t)$. Define the based loop measure μ^b by

$$\mu^b(dl) = \int_0^\infty \sum_{x \in S} \mathbb{P}_t^{x,x}[dl]\, dt.$$

By forgetting the initial point, we get a $\sigma-$finite measure on the space of loops[3], namely the Markovian loop measure μ. Given a non-negative function χ on S, let M_χ be the operator defined by $M_\chi f(x) = \chi(x) f(x)$.

For a (based) loop, its occupation time at $x \in S$ is defined by $l^x = \int\limits_{0<s<t} 1_{\{l(s)=x\}}\, ds$. The multi-occupation field is defined by

$$\langle l, f \rangle = \sum_{j=0}^{n-1} \int\limits_{0<s^1<\cdots<s^n<t} f(l(s^{1+j}), \ldots, l(s^{n+j}))\, ds^1 \cdots ds^n$$

where f is a function on S. The linear space generated by these multi-occupation fields form an algebra and they generate the Borel$-\sigma-$field of the loops. In particular, their distributions determine the loop measure.[4] Under the assumption of transience, we are able to calculate, for example, the "Laplace transform" of the occupation field:

$$\mu(e^{z\langle l,\chi\rangle} - 1) = -\ln\det(I - zM_{\sqrt{\chi}} V M_{\sqrt{\chi}}).$$

We also show the compatibility of the loop measure with time change and killing. In particular, we can take the trace and the restriction of a Markovian loop. The push forward measure is a Markovian loop measure associated with the trace and the restriction of the Markov process.

By using the loop measure as intensity measure, we construct a Poisson point process of loops $(\mathcal{L}_\alpha, \alpha \geq 0)$. We define the occupation field of this loop ensemble by $\sum\limits_{l \in \mathcal{L}_\alpha} l^x$. We can

[3]See Definition 3.1.5.

[4]In fact, the expectation is enough under the assumption of transience and Markov, see 3.4.9.

calculate the moments of the occupation field $(\mathcal{L}_\alpha^x, x \in S)$:

$$\mathbb{E}[\hat{\mathcal{L}}_\alpha^{x_1} \cdots \hat{\mathcal{L}}_\alpha^{x_n}] = \mathrm{Per}_\alpha((V_{x_m}^{x_l})_{1\leq m,l\leq n}).$$

Moreover, we give an explicit expression for the Laplace transform of $\langle \mathcal{L}_\alpha, \chi\rangle$, see Proposition 4.1.5. By a non-symmetric generalization of the Dynkin's isomorphism [LJ08], we give an expression for the density of $\mathcal{L}_1^{x_1}, \ldots, \mathcal{L}_1^{x_n}$ using complex integration. We are also interested in the conditioned behavior of the occupation field. More precisely, what is the occupation field like if we know the trace of the loop ensemble on a subset F? For a loop intersecting F, by removing the excursions out of F and collecting the remainder in the original order, we get the trace l_F of a loop l on F which is again a loop. Proposition 4.6.2 shows that conditionally on the collection of loop traces, the collection of the excursions is a conditioned Poisson random measure. That conditioned Poisson random measure depends on three types of quantities:

- The occupation time at the boundary of F.
- The jumping times at the boundary of F for the trace l_F.
- The excursion measure out of F.

Further calculations give the conditioned occupation field in Proposition 4.6.3. In the end of this chapter, we consider loop clusters[5]. The problem first appears in [LJL12]. An example considered in this paper is the loop cluster on $\mathbb{N}$. It is shown that the closed edges form a subordinator with explicit renewal measure in the proper scaling limit. We consider conditioned version of this model: the loop clusters on the discrete circle conditioned to be closed at a specific edge. For the scaling limit, we obtain a conditioned subordinator.

In Chapter 5, we explain the way to recover the Poisson ensemble of loops for $\alpha = 1$ (resp. $\alpha \in]0,1[$) from the loops removed in Wilson's algorithm by cutting them according to some additional Poisson-Dirichlet distributions (resp. gamma subordinators), see Proposition 5.1.4 and Remark 16. In the last section, we give an elementary proof of the transfer current theorem for the non-symmetric directed random spanning tree measure with a given root weight p, see Theorem 5.3.3 and Corollary 5.3.4.

[5] An edge is open iff. it is covered by some loop.

In the last chapter, we consider the loops on a sequence of undirected finite graph G_n. By adding a killing rate c_n, we get a non-trivial loop measure μ_n on G_n and the total mass of μ_n is finite. Denote by $\mathcal{C}$ the set of covering loops, i.e. $\mathcal{C} = \{l \text{ covers the whole space.}\}$. We are interested in the limit proportion of the "covering loops", i.e. $\lim\limits_{n\to\infty} \frac{\mu_n(\mathcal{C})}{\mu_n(\text{non-trivial loops})}$. We calculate it under the following crucial assumption:

i) The empirical distribution of the eigenvalues of the transition matrix Q_n[6] converges to a probability measure on ν.

ii) The sequence of the graphs has uniformly bounded degrees and weights.

As an application, we give two concrete examples: the sequence of discrete tori and the sequence of increasing balls in a regular tree. The limit distribution for the discrete model converges to the convolution of the semi-circle law and the second converges to a purely atomic distribution given by the roots of a family of polynomials. We also calculate the case of the complete graphs as a counter-example.

[6]The transition matrix Q_n is associated with G_n.

Chapter 2

Preliminaries

In this section, we present some basic results about continuous time Markov chains, including a discrete version of Feynman-Kac and the transformation by time change.

2.1 Notations

1. Suppose E_1, E_2 are two countable sets, $(A^i_j, i \in E_1, j \in E_2)$ is a matrix. For $F_1 \subset E_1$ and $F_2 \subset E_2$, let $(A|_{F_1 \times F_2}, i \in F_1, j \in F_2)$ be the sub-matrix defined by $(A|_{F_1 \times F_2})^i_j = A^i_j$. By convention, the absolute value $|A|$ will denote the matrix: $(|A|)^i_j = |A^i_j|$.

2. $\mathcal{E}(\lambda), \lambda \in [0, \infty]$ denotes a random variable, exponentially distributed with parameter λ with the convention that $\mathcal{E}(0) = \infty$ and $\mathcal{E}(\infty) = 0$.

3. If k is a non-negative finite function on the state space S, M_k will denote the matrix, $(M_k)^x_y = k(x)\delta^x_y$ where $\delta^x_y = 1$ iff. $x = y$.

4. $x \in \mathbb{R}^n$ can be extended to an periodic series, $x^{nm+k} = x^k, m \in \mathbb{Z}, k = 0, \ldots, n-1$. Given $x \in \mathbb{R}^n$, each time we write x_{n+j}, we extend x to the n-periodical series.

5. For any countable set A, $\#A$ and $|A|$ will denote the number of elements in A.

6. Let $\mathfrak{S}_k$ be the collection of permutations on $\{1, \ldots, k\}$ and S some state space. For a permutation $\sigma \in \mathfrak{S}_k$ and $x = (x_1, \ldots, x_k) \in S^k$, define $\sigma(x) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$. Accordingly, a permutation σ can be viewed as a function from S^k to S^k. Define the circular permutation r_j as follows: $r_j(1, \ldots, k) = (j+1, \ldots, k, 1, \ldots, j)$. Define $\mathfrak{R}_k$ to be the subset of $\mathfrak{S}_k$ consisting of circular permutations on $\{1, \ldots, k\}$. Note that σ plays two roles, a function on $\{1, \ldots, k\}$ mapping an integer to another integer and a

function on some S^k mapping a k-uple to another k-uple (for example, $r_1(2,1,3,4) = (4,2,1,3) \neq (r_1(2), r_1(3), r_1(1), r_1(4)) = (3,4,2,1)$). We have $\sigma_1(\sigma_2(x_1,\ldots,x_n)) = (x_{(\sigma_1\circ\sigma_2)^{-1}(1)},\ldots,x_{(\sigma_1\circ\sigma_2)^{-1}(n)})$.

2.2 Minimal continuous-time sub-Markov chain in a countable space

Let S be a countable set equipped with the discrete topology. Add an additional cemetery point ∂ to S and set $\overline{S} = S\bigcup\{\partial\}$ (compactification).

Definition 2.2.1 (Generator)**.** A matrix $L = (L^x_y, x, y \in S)$ is called a sub-Markovian (Markovian resp.) generator iff.

$$\begin{aligned} &0 \le -L^x_x < \infty && \text{for all } x \in S,\\ &L^x_y \ge 0 && \text{for all } x \neq y,\\ &\sum_j L^x_y \le 0 \quad (\sum_j L^x_y = 0 \text{ resp.}) && \text{for all } x \in S. \end{aligned}$$

In case $L^x_x < 0$, set $Q^x_y = \frac{L^x_y}{-L^x_x}$ for $x \neq y$ and $Q^x_x = 0$. In case $L^x_x = 0$, set $Q^x_y = \delta^x_y$.

Convention 2.2.2. A sub-Markovian generator L on S can be extended to a Markovian generator $\overline{L}$ on $\overline{S}$ as follows: $\overline{L}^x_y = L^x_y$ for $x, y \in S$, $\overline{L}^x_\partial = -\sum\limits_{y\in S} L^x_y$ for $x \in S$, $\overline{L}^\partial_x = 0$ for $x \in \overline{S}$.

Construction of the probability on the space of right-continuous[1] paths

Let μ, a probability measure on S, be the initial distribution. Let ξ_0 be a random variable with distribution μ and $(\tau_{ix}, i \in \mathbb{N}, x \in S)$ be independent random variables, exponentially distributed with parameter $-L^x_x$. Let $(J_{ix}, i \in \mathbb{N}, x \in S)$ be independent random variables such that for $y \in S$

$$\mathbb{P}(J_{ix} = y) = Q^x_y.$$

[1] In a discrete space, any right-continuous Markov chain has left limit in its lifetime $[0, \zeta[$ if the path stays at the cemetery ∂ after there has been infinitely many jumps. Besides, on $\zeta < \infty$, the left limit at time ζ is the cemetery point for the process.

Moreover, assume that ξ_0, $\tau = (\tau_{ix}, i \in \mathbb{N}, x \in S)$ and $J = (J_{ix}, i \in \mathbb{N}, x \in S)$ are independent. For any configuration of (μ, τ, J), recursively define:

$$\begin{aligned}
&\xi_n = J_{n\xi_{n-1}} \text{ for } n \geq 1 && \text{(discrete Markov chain)}\\
&T_0 = 0, T_{n+1} = T_n + \tau_{n\xi_n} && \text{(jumping time)}\\
&\zeta = \lim_{n\to\infty} T_n && \text{(explosion time).}
\end{aligned}$$

Then define the path as follows:

$$\begin{aligned}
X_t &= \xi_i \quad \text{for } T_i \leq t < T_{i+1},\\
X_t &= \partial \quad \text{for } t \geq \zeta.
\end{aligned}$$

Remark 1. From this construction, it is clear that $\{x \in S : L_x^x = 0\}$ is the set of absorbing states.

Theorem 2.2.1 (Markov Property). *Set $(P_t)_y^x = \mathbb{P}[X_t = y | X_0 = x]$. Use $\mathbb{P}^\mu$ to stand for the law of the process $(X_t, t \geq 0)$. $(X_t, t \geq 0)$ defined above is a Markov process with initial distribution μ. Its semi-group will be denoted P_t and $(P_t)_y^x$ is right-continuous in t.*

Proof. Let $T_0 < T_1 < \cdots$ be the sequence of jumping times. For all initial distribution μ, a fixed time t cannot be the jumping time, i.e. $\mathbb{P}^\mu[\exists n > 0, T_n = t] = 0$. Let $t + T_{t,0} < t + T_{t,1} < \cdots$ be the sequence of jumping times after time t. It is enough to show the conditional law of $T_{t,0}$ is $\mathcal{E}(-L_{X_t}^{X_t})$ under the sigma field $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$. Let N_t be the number of jumps in $[0,t]$. For $t_1, \ldots, t_k, s > 0$ with $\sum_1^k t^i < t$,

$$\begin{aligned}
&\mathbb{P}[N_t = k, X_{T_0} = x_0, \ldots, X_{T_k} = x_k, T_1 - T_0 \in dt^1, \ldots, T_k - T_{k-1} \in dt^k, T_{t,0} \in ds]\\
&\qquad = L_{x_1}^{x_0} \cdots L_{x_k}^{x_{k-1}} (-L_{x_k}^{x_k}) e^{L_{x_0}^{x_0} t^1}\, dt^1 \cdots e^{L_{x_{k-1}}^{x_{k-1}} t^k}\, dt^k e^{L_{x_k}^{x_k}(t - \sum_1^k t^i + s)}\, ds\\
&\qquad = L_{x_1}^{x_0} \cdots L_{x_k}^{x_{k-1}} e^{L_{x_0}^{x_0} t^1} \cdots e^{L_{x_{k-1}}^{x_{k-1}} t^k} e^{L_{x_k}^{x_k}(t - \sum_1^k t^i)}\, dt^1 \cdots dt^k \times (-L_{x_k}^{x_k}) e^{L_{x_k}^{x_k} s}\, ds
\end{aligned}$$

From this expression, we find the conditional law of $T_{t,0}$. Then, we get the desired result. □

The following theorem is taken from the book [Nor98].

Theorem 2.2.2.

a) ***Backward Equation.***

P_t is the minimal non-negative solution of the backward equation:

$$\frac{dP_t}{dt} = LP_t,$$
$$P_0 = I \text{ (identity)}.$$

*b) **Forward Equation.***

P_t is the minimal non-negative solution of the forward equation:

$$\frac{dP_t}{dt} = P_t L,$$
$$P_0 = I \ \textit{(identity)}.$$

(These equations are viewed as an infinite system of differential equations.)

Proof.

a) **Backward Equation.**

a.1) We will show that P_t satisfy the backward equation.

Since $\tau_{0,x}$ is exponentially distributed with parameter L^x_x,

$$\mathbb{P}^x[X_t = y, t < T_1] = e^{tL^x_x}\delta^x_y.$$

By the strong Markov property,

$$\mathbb{P}^x[T_1 < t, X_{T_1} = z, X_t = y] = \int_0^t -L^x_x e^{L^x_x s} Q^x_z (P_{t-s})^z_y \, ds.$$

Therefore,

$$\begin{aligned}
(P_t)^x_y =& \mathbb{P}^x[X_t = y] \\
=& \mathbb{P}^x[X_t = y, t < T_1] + \sum_z \mathbb{P}^x[T_1 < t, X_{T_1} = z, X_t = y] \\
=& e^{tL^x_x}\delta^x_y + \sum_z \int_0^t -L^x_x e^{L^x_x s} Q^x_z (P_{t-s})^z_y \, ds \\
=& e^{tL^x_x}(\delta^x_y + \int_0^t \sum_z -L^x_x e^{-L^x_x s} Q^x_z (P_u)^z_y \, du).^2
\end{aligned}$$

Multiply $e^{-tL^x_x}$ on both sides,

$$e^{-tL^x_x}(P_t)^x_y = \delta^x_y + \int_0^t \sum_z -L^x_x e^{-L^x_x s} Q^x_z (P_t)^z_y(s) \, ds.$$

We see that $e^{-tL^x_x}(P_t)^x_y$ is continuous and the convergence in the integral is uniform on $[0, t]$ by the following estimation:

$$\sum_z |-L^x_x e^{-L^x_x s} Q^x_z (P_{t-s})^z_y| \le \sum_z |-L^x_x e^{-L^x_x s} Q^x_z| \le -L^x_x e^{L^x_x s}.^3$$

[3] This estimation ensures that $\sum_z |L^x_z|(P_t)^z_y < \infty$.

Then $e^{-tL_x^x}(P_t)_y^x$ is differentiable and

$$\frac{d}{dt}\left(e^{-tL_x^x}(P_t)_y^x\right) = \sum_z -L_x^x e^{-tL_x^x} Q_z^x (P_t)_y^z.$$

Finally, $\dfrac{d(P_t)_y^x}{dt} = \sum_z L_z^x (P_t)_y^z$.

a.2) We will show that P_t is the smallest non-negative solution.

Suppose $\tilde{P}$ is another non-negative solution of the backward equation. Then by reversing the steps in part a.1), we find that $\tilde{P}$ also satisfies the backward equation in the following integral form:

$$(\tilde{P}_t)_y^x = e^{tL_x^x}\delta_y^x + \sum_z \int_0^t -L_x^x e^{L_x^x s} Q_z^x (\tilde{P}_{t-s})_y^z \, ds. \tag{*}$$

By applying the strong Markov property at time T_1,

$$\mathbb{P}^x[X_t = y, t < T_{n+1}] = e^{L_x^x t} + \sum_{z \in S\setminus\{x\}} \int_0^t -L_x^x e^{L_x^x s} Q_z^x \mathbb{P}^z[X_{t-s} = y, t < T_n] \, ds.$$

For $t \geq 0$, we will prove that $(\tilde{P}_t)_y^x \geq \mathbb{P}^x[X_t = y, t < T_n]$ for $n \in \mathbb{N}$ by recurrence. Obviously, for $y \neq x$, $(\tilde{P}_t)_y^x \geq 0 = \mathbb{P}^x[X_t = y, t < T_0]$. From the equation (*), we see that $(\tilde{P}_t)_x^x \geq e^{L_x^x t}$. Next, suppose $(\tilde{P}_t)_y^x \geq \mathbb{P}^x[X_t = y, t < T_n]$ is true for $n \leq m$ and $t \geq 0$, then

$$\begin{aligned}
\mathbb{P}^x[X_t = y, t < T_{m+1}] =& e^{L_x^x t} + \sum_{z \in S\setminus\{x\}} \int_0^t -L_x^x e^{L_x^x s} Q_z^x \mathbb{P}^z[X_{t-s} = y, t < T_m] \, ds \\
\leq& e^{L_x^x t} + \sum_{z \in S\setminus\{x\}} \int_0^t -L_x^x e^{L_x^x s} Q_z^x (\tilde{P}_{t-s})_y^z \, ds \\
=& (\tilde{P}_t)_y^x
\end{aligned}$$

By induction, we have $(\tilde{P}_t)_y^x \geq \mathbb{P}^x[X_t = y, t < T_n]$ for all $n \in \mathbb{N}$ and $t > 0$. Letting n tend to infinity, the right-hand side tends to $(P_t)_y^x$. Finally, P is the smallest non-negative solution.

b) **Forward Equation.**

b.1) We will show that P_t satisfies the forward equation. First, consider the case when $-L_x^x > 0$ for all $x \in S$[4]:

[4] In other words, there is no absorbing state.

$$
\begin{aligned}
&- L^{x_n}_{x_n}\mathbb{P}^{x_0}[T_n \le t \le T_{n+1}, X_{T_0} = x_0, \dots, X_{T_n} = x_n] \\
&\qquad = -L^{x_n}_{x_n} \prod_{k=0}^{n-1} Q^{x_k}_{x_{k+1}} \int\limits_{\substack{t_0+\cdots+t_{n-1}\le t \\ t\le t_0+\cdots+t_n \\ t_i\ge 0 \text{ for } i=0,\dots,n}} \prod_{k=0}^{n} -L^{x_k}_{x_k} e^{L^{x_k}_{x_k} t_k}\, dt_k \\
&\qquad = -L^{x_n}_{x_n} \prod_{k=0}^{n-1} Q^{x_k}_{x_{k+1}} \int\limits_{\substack{t_0+\cdots+t_{n-1}\le t \\ t_i\ge 0 \text{ for } i=0,\dots,n-1}} e^{L^{x_n}_{x_n}(t-t_0-\cdots-t_{n-1})} \times \\
&\qquad\qquad\qquad\qquad \prod_{k=0}^{n-1} -L^{x_k}_{x_k} e^{L^{x_k}_{x_k} t_k}\, dt_k
\end{aligned}
$$

We change the variables: $u_0 = t - t_0 - \cdots - t_{n-1}, u_1 = t_{n-1}, \dots, u_{n-1} = t_1$. The above quantity equals

$$
\begin{aligned}
&\prod_{k=0}^{n-1} Q^{x_k}_{x_{k+1}} \int\limits_{\substack{u_0+\cdots+u_{n-1}\le t \\ u_i\ge 0 \text{ for } i=0,\dots,n-1}} -L^{x_n}_{x_n} e^{L^{x_n}_{x_n} u_0} - L^{x_0}_{x_0} e^{L^{x_0}_{x_0}(t-u_0-\cdots-u_{n-1})} \\
&\qquad\qquad du_0 \prod_{k=1}^{n-1} -L^{x_k}_{x_k} e^{L^{x_k}_{x_k} u_{n-k}}\, du_{n-k} \\
&= \prod_{k=0}^{n-1} Q^{x_k}_{x_{k+1}} \int\limits_{\substack{u_0+\cdots+u_{n-1}\le t \\ u_i\ge 0 \text{ for } i=0,\dots,n-1}} -L^{x_0}_{x_0} e^{L^{x_0}_{x_0}(t-u_0-\cdots-u_{n-1})} \prod_{k=0}^{n-1} -L^{x_{n-k}}_{x_{n-k}} e^{L^{x_{n-k}}_{x_{n-k}} u_k}\, du_k
\end{aligned}
$$

Then,

$$
\begin{aligned}
&- L^{x_{n-1}}_{x_{n-1}}\mathbb{P}^{x_0}[T_{n-1} \le t \le T_n, X_{T_0} = x_0, \dots, X_{T_{n-1}} = x_{n-1}] \\
&= \prod_{k=0}^{n-2} Q^{x_k}_{x_{k+1}} \int\limits_{\substack{u_0+\cdots+u_{n-2}\le t \\ u_i\ge 0 \text{ for } i=0,\dots,n-2}} -L^{x_0}_{x_0} e^{L^{x_0}_{x_0}(t-u_0-\cdots-u_{n-2})} \prod_{k=0}^{n-2} -L^{x_{n-1-k}}_{x_{n-1-k}} e^{L^{x_{n-1-k}}_{x_{n-1-k}} u_k}\, du_k \\
&= \prod_{k=0}^{n-2} Q^{x_k}_{x_{k+1}} \int\limits_{\substack{u_1+\cdots+u_{n-1}\le t \\ u_i\ge 0 \text{ for } i=1,\dots,n-1}} -L^{x_0}_{x_0} e^{L^{x_0}_{x_0}(t-u_1-\cdots-u_{n-1})} \prod_{k=1}^{n-1} -L^{x_{n-k}}_{x_{n-k}} e^{L^{x_{n-k}}_{x_{n-k}} u_k}\, du_k
\end{aligned}
$$

By comparing the expression for n and $n-1$, we see that

$$
\begin{aligned}
&- L^{x_n}_{x_n}\mathbb{P}^{x_0}[T_n \le t \le T_{n+1}, X_{T_0} = x_0, \dots, X_{T_n} = x_n] \\
&\qquad = Q^{x_{n-1}}_{x_n} \int_0^t \mathbb{P}^{x_0}[T_{n-1} \le t-u_0 \le T_n, X_{T_0} = x_0, \dots, X_{T_{n-1}} = x_{n-1}] \\
&\qquad\qquad\qquad\qquad \times (-L^{x_n}_{x_n})(-L^{x_{n-1}}_{x_{n-1}}) e^{L^{x_n}_{x_n}}\, du_0
\end{aligned}
$$

Dividing by $-L_{x_n}^{x_n}$ on both sides,

$$\begin{aligned}
&\mathbb{P}^{x_0}[T_n \le t \le T_{n+1}, X_{T_0} = x_0, \dots, X_{T_n} = x_n] \\
&\qquad = L_{x_n}^{x_{n-1}} \int_0^t \mathbb{P}^{x_0}[T_{n-1} \le t - u_0 \le T_n, X_{T_0} = x_0, \dots, X_{T_{n-1}} = x_{n-1}] e^{L_{x_n}^{x_n}} \, du_0
\end{aligned}$$

For a general L, set $L^{(\epsilon)} = L - \epsilon$. By taking $\epsilon \to 0$, the above result is true in general. For the rest of the proof, suppose we have a general L.

$$\begin{aligned}
(P_t)_y^x =& \mathbb{P}^x[X_t = y] \\
=& \delta_y^x \mathbb{P}^x[t \le T_1] + \sum_{n \ge 1} \sum_{\{x_0,\dots,x_n\} \in S^{n+1}} 1_{\{x_0 = x, x_n = y\}} \\
& \mathbb{P}^x[T_n \le t \le T_{n+1}, X_{T_0} = x_0, \dots, X_{T_n} = x_n] \\
=& \delta_y^x e^{L_x^x t} + \int_0^t L_y^{x_{n-1}} e^{L_y^y s} \, ds \sum_{n \ge 1} \sum_{x_{n-1} \in S} \sum_{\{x_0,\dots,x_{n-2}\} \in \{x\} \times S^{n-2}} \\
& \mathbb{P}^x[T_{n-1} \le t - s \le T_n, X_{T_0} = x_0, \dots, X_{T_{n-1}} = x_{n-1}] \\
=& \delta_y^x e^{L_x^x t} + \int_0^t \sum_{z \in S \setminus \{y\}} (P_{t-s})_z^x L_y^z e^{L_y^y s} \, ds
\end{aligned}$$

Consequently,

$$(P_t)_y^x e^{-L_y^y t} = \delta_y^x + \int_0^t \sum_{z \in S \setminus \{y\}} (P_u)_z^x L_y^z e^{-L_y^y u} \, du.$$

It follows that $((P_t)_y^x e^{-L_y^y t}, t \ge 0)$ is finite non-decreasing. Then, $\sum_{z \in S \setminus \{y\}} (P_u)_z^x L_y^z e^{-L_y^y u}$ converges uniformly on $[0, t]$. We find the forward equation by calculating the derivatives on both sides.

b.2) In the part b.1), we see that

$$\begin{aligned}
&\mathbb{P}^{x_0}[T_n \le t \le T_{n+1}, X_{T_0} = x_0, \dots, X_{T_n} = x_n] \\
&\qquad = L_{x_n}^{x_{n-1}} \int_0^t \mathbb{P}^{x_0}[T_{n-1} \le t - u_0 \le T_n, X_{T_0} = x_0, \dots, X_{T_{n-1}} = x_{n-1}] e^{L_{x_n}^{x_n}} \, du_0
\end{aligned}$$

Therefore,

$$\mathbb{P}^x[X_t = y, t < T_{n+1}] = \delta_y^x e^{L_y^y t} + \int_0^t \sum_{z \in S \setminus \{y\}} \mathbb{P}^x[X_{t-s} = z, t < T_n] L_y^z e^{L_y^y s} \, ds.$$

Then by an argument similar to the part a.2) for the backward equation, one can prove that P_t is the smallest non-negative solution for the forward equation.

□

Remark 2. The process we constructed is minimal in the sense of its semi-group as the solution of the forward backward equations. In a more probabilistic language, it is the least conservative process. To be more precise, for any sub-Markovian process with generator L, if we kill the process as long as it jumps infinitely many times, we get the minimal sub-Markov process with generator L.

Definition 2.2.3. The potential V is defined as follows:

$$V_y^x = \mathbb{E}^x[\int_0^\infty 1_{\{X_t=y\}}\,dt] = \int_0^\infty (P_t)_y^x\,dt.$$

Let ν be the counting measure on S. V is viewed as a kernel as follows: $V(x,dy) = V_y^x \nu(dy)$. Define the Green function to be the density of $V(x,dy)$ with respect to the counting measure ν. In this case, we find that $G_y^x = V_y^x$.

Theorem 2.2.3 (Feynman-Kac). *For a non-negative function k on S, define*

$$(P_{t,k})_y^x = \mathbb{E}^x\left(e^{-\int_0^t k(X_s)\,ds} 1_{\{X_t=y\}}\right).$$

Then, it is the minimal positive solution of the following equation:

$$\frac{\partial u}{\partial t}(t,x) = (L - M_k)u(t,x) \quad ^{5}$$

with initial condition $u(0,x) = \delta_y^x$. We denote by V_k the associated potential. Denote by $\mathbb{P}_k$ the law of the canonical minimal Markov process with generator $L - M_k$. Then,

$$\left.\frac{d\mathbb{P}_k}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{-\int_0^t k(X_s)\,ds}$$

where $\mathcal{F}_t = \sigma(X_s, s \in [0,t])$.

Proof. Suppose there is no absorbing point.[6] Given $L - M_k$, we can construct a minimal right-continuous Markov process Y. Let $((P_{(k)})_t, t \geq 0)$ be its semi-group. It is enough to prove that

$$u(L,k,t,x,y) \overset{def}{=} \mathbb{E}^x\left(\exp\left(-\int_0^t k(X_s)\,ds\right) 1_{\{X_s=y\}}\right) = (P_{t,k})_y^x.$$

[5] Recall that $(M_k f)(x) = k(x)f(x)$.

[6] Otherwise, split every absorbing point into two points. Let the process jump between the two state as soon as the process hits the absorbing point.

For $(X_t, t \geq 0)$, let $T_i, i \geq 1$ be the sequence of jumping times before explosion. Set $T_0 = 0$. Then, $\xi_i = X_{T_i}, i \in \mathbb{N}$ is the discrete chain and $\tau_{i-1} = T_i - T_{i-1}, i \in \mathbb{N}_+$ is the sequence of corresponding holding times. Set $Q^x_y = \dfrac{L^x_y}{-L^x_x}$. Let $J[0,t]$ denote the number of jumps in the time interval $[0,t]$. For $J[0,t] = 0$, $\mathbb{P}^{x_0}[J[0,t] = 0] = e^{L^{x_0}_{x_0} t}$; for $J[0,t] \in \mathbb{N}_+$,

$$
\begin{aligned}
&\mathbb{P}^{x_0}(J[0,t] = n, \xi_1 = x_1, \ldots, \xi_n = x_n, \tau_0 \in dt^0, \ldots, \tau_{n-1} \in dt^{n-1}) \\
&\qquad = Q^{x_0}_{x_1} \cdots Q^{x_{n-1}}_{x_n} 1_{\left\{\sum\limits_{i=0}^{n-1} t^i < t\right\}} e^{L^{x_n}_{x_n}(t - t^0 - \cdots - t^{n-1})} \left(\prod_{i=0}^{n-1} (-L^{x_i}_{x_i}) e^{t^i L^{x_i}_{x_i}} \, dt^i \right)
\end{aligned}
$$

Then,

$$
\begin{aligned}
u(L,k,t,x,y) = &\sum_{n=0}^{\infty} \mathbb{E}^x \left(e^{-\int\limits_0^t k(X_s)\, ds} 1_{\{X_s = y\}}, J[0,t] = n \right) \\
= &\delta^x_y \exp((L^x_x - k(x))t) \\
&+ \sum_{n=1}^{\infty} \sum_{\substack{x_i \in S \text{ for } i=0,\ldots,n; \\ x_0 = x, x_n = y; \\ x_i \neq x_{i+1} \text{ for } i=0,\ldots,n-1}} L^{x_0}_{x_1} \cdots L^{x_{n-1}}_{x_n} 1_{\{t^0 + \cdots + t^{n-1} < t\}} \\
&\left(\prod_{i=0}^{n-1} \exp\{t^i (L^{x_i}_{x_i} - k(x_i))\} \right) \\
&\exp\{(L^{x_n}_{x_n} - k(x_n))(t - \sum_{i=0}^{n-1} t^i)\} \, dt^0 \cdots dt^n
\end{aligned}
$$

For $(Y_t, t \geq 0)$, do the same calculation with $k = 0$,

$$
\begin{aligned}
u(L - M_k, 0, t, x, y) = &\delta^x_y \exp((L_Y)^x_x t) \\
&+ \sum_{n=1}^{\infty} \sum_{\substack{x_i \in S \text{ for } i=0,\ldots,n; \\ x_0 = x, x_n = y; \\ x_i \neq x_{i+1} \text{ for } i=0,\ldots,n-1}} (L_Y)^{x_0}_{x_1} \cdots (L_Y)^{x_{n-1}}_{x_n} 1_{\{t^0 + \cdots + t^{n-1} < t\}} \\
&\left(\prod_{i=0}^{n-1} \exp\{t^i (L_Y)^{x_i}_{x_i}\} \right) \exp\{(L_Y)^{x_n}_{x_n}(t - \sum_{i=0}^{n-1} t^i)\} \, dt^0 \cdots dt^n
\end{aligned}
$$

where $L_Y = L - M_k$. We see that $u(L,k,t,x,y) = u(L - M_k, 0, t, x, y) = (P_{t,k})^x_y$. Consequently, we can conclude the desired result. □

Proposition 2.2.4. *Suppose V is transient, i.e. $V^x_y < \infty$ for all x and y, then $LV = VL = -Id$.*[7]

[7] To be more precise, as long as $V|f| < \infty$ at point x, we have $-LVf(x) = f(x)$ point wise.

Proof.

a) $-LV = Id$:

$$(LV)_y^x = \sum_z L_z^x V_y^z = \int_0^\infty \sum_z L_z^x (P_t)_y^z \, dt = \int_0^\infty \frac{d(P_t)_y^x}{dt} \, dt = -\delta_y^x.$$

In order to use Fubini's theorem, one needs $(|L|V)_y^x < \infty$ for $x, y \in S$ under the assumption of transience. Recall that we use the following equation in the proof of the backward equation:

$$(P_t)_y^x = e^{L_x^x t} \delta_y^x + \int_0^t \sum_{z \in S \setminus \{x\}} e^{L_x^x (t-s)} L_z^x (P_s)_y^z \, ds.$$

By integrating both sides from 0 to $+\infty$ with respect to t, we see that

$$\begin{aligned} V_y^x =& \frac{1}{-L_x^x} \delta_y^x + \int_{0<s<t<\infty} \sum_{z \in S \setminus \{x\}} L_z^x (P_s)_y^z e^{L_x^x (t-s)} \, ds \, dt \\ =& \frac{1}{-L_x^x} \left(\delta_y^x + \int_0^\infty \sum_{z \in S \setminus \{x\}} L_z^x (P_s)_y^z \, ds \right) \\ =& \frac{1}{-L_x^x} (\delta_y^x + \sum_{z \in S \setminus \{x\}} L_z^x V_y^z). \end{aligned}$$

Then, $(|L|V)_y^x = 2(-L_x^x) V_y^x - \delta_y^x$. Under the assumption of transience, $(|L|V)_y^x < \infty$ for $x, y \in S$ and the proof is complete.

b) $-VL = Id$:

The proof is similar to part a). The following equation appears in the proof of the forward equation:

$$(P_t)_y^x = e^{L_x^x t} \delta_y^x + \int_0^t \sum_{z \in S \setminus \{y\}} (P_s)_z^x L_y^z e^{L_y^y (t-s)} \, ds.$$

Then,

$$\begin{aligned} V_y^x =& \frac{1}{-L_x^x} \delta_y^x + \int_{0<s<t<\infty} \sum_{z \in S \setminus \{y\}} (P_s)_z^x L_y^z e^{L_y^y (t-s)} \, ds \, dt \\ =& \frac{1}{-L_y^y} \left(\delta_y^x + \int_0^\infty \sum_{z \in S \setminus \{y\}} (P_s)_z^x L_y^z \, ds \right) \end{aligned}$$

$$=\frac{1}{-L^y_y}(\delta^x_y+\sum_{z\in S\setminus\{y\}}V^x_zL^z_y).$$

Consequently, $(V|L|)^x_y=2V^x_y(-L^y_y)-\delta^x_y<\infty$ for $x,y\in S$. Then we can use Fubini's theorem:

$$(VL)^x_y=\sum_z V^x_zL^z_y=\int_0^\infty\sum_z(P_t)^x_zL^z_y\,dt=\int_0^\infty\frac{d(P_t)^x_y}{dt}\,dt=-\delta^x_y.$$

□

Lemma 2.2.5. *Suppose $k:S\to[0,\infty]$, then V_kM_k is sub-Markovian, i.e. $V_kk(x)\le 1$ for all x.*

Proof.

$$(V_kk)(x)=\mathbb{E}^x[\int_0^\infty e^{-\int_0^{t\wedge\zeta}k(X_s)\,ds}k(X_t)\,dt]$$

If k is bounded, $(V_kk)(x)=\mathbb{E}^x[1-e^{-\int_0^{\zeta}k(X_s)\,ds}]\le 1$. For unbounded case, it is enough to use Fatou's lemma. □

Theorem 2.2.6 (Resolvent equation). *The following identities hold:*

a) $V_k+VM_kV_k=V$.

b) $V_k+V_kM_kV=V$.

c) $V_kM_kV=VM_kV_k$.

Proof.

a) We will prove this in the sense of matrices.[8] Set $\tilde{k}(x)=k(x)-L^x_x$. By monotone convergence, we can replace V,V_k by $V_{\frac{1}{n}+\frac{\tilde{k}}{n}},\left(V_{\frac{1}{n}+\frac{\tilde{k}}{n}}\right)_k$. Since $V_{\frac{1+\tilde{k}}{n}}(1+\tilde{k})\le n$, one could suppose $V(1+\tilde{k})$ is bounded. Intuitively, by multiplying V_k on both sides of the equation $(M_k-L)(V-V_k)=M_kV$, we get $V-V_k=V_k(M_k-L)(V-V_k)=V_kM_kV$. But associativity $(AB)C=A(BC)$ is not true for general infinite matrices. In order to use Fubini, one needs to check the integrability, which is correct under the assumption that "$V(1+\tilde{k})$ is bounded".

b) The argument is similar.

[8] Since they are all non-negative, it will be correct in the sense of operator.

c) First, take k bounded positive. By a) and b), $VM_kV_k = V_kM_kV$. Then by monotone convergence, c) is proved.

$\square$

2.3 The time change induced by a non-negative function

Let $(X_t, t \geq 0)$ be a minimal Markovian process on S, with generator L and lifetime ζ. Given $\lambda : S \to [0, \infty]$, define

$$A_t = \int_0^{t\wedge\zeta} \lambda(X_s)\, ds, \quad \sigma_t = \inf\{s \geq 0, A_s > t\}, \quad \hat{\zeta} = \inf\{s \geq 0, \sigma_s = \sigma_\infty\}$$

with the convention that $\inf \phi = \infty$. Then, σ_t are stopping times for all t and they are right-continuous with respect to $t \geq 0$. Set $Y_t = X_{\sigma_t}$ for $0 \leq t \leq \hat{\zeta}$ and let Y be killed at time $\hat{\zeta}$. By the strong Markov property, Y_t is also a càdlàg sub-Markov process with lifetime $\hat{\zeta}$. It could be constructed directly from its generator $\hat{L}$ as before.

Proposition 2.3.1.

a) If $0 < \lambda < \infty$, then $\hat{L}^x_y = \dfrac{L^x_y}{\lambda_x}$ (change of jumping rates).

b) If $\lambda = 1_A + 1_{A^c} \cdot \infty$, then

$$\hat{L}^x_y = \begin{cases} L^x_y & \text{for } x, y \in A^c \\ 0 & \text{elsewhere.} \end{cases}$$

(Y is the restriction of X to A.)

c) If $\lambda = 1_A$, Y is called the trace of X on A. The generator $\hat{L}$ of Y will be denoted by L_A. In this case, $(Y_t, t \geq 0)$ has the same potential as $(X_t, t \geq 0)$. On $A \times A$:

$$V^x_y = \mathbb{E}^x[\int_0^\zeta 1_{\{X_s = y\}}\, ds] = \mathbb{E}^x[\int_0^{\hat{\zeta}} 1_{\{Y_s = y\}}\, ds].$$

Let T_1 be the first jumping time and $T_{1,A} = \inf\{s \geq T_1, X_s \in A\}$.
Define $(R^A)^x_y = \mathbb{E}^x[X_{T_{1,A}} = y, T_{1,A} < \infty]$ for $y \in S$ and $(R^A)^x_\partial = 1 - \sum\limits_y (R^A)^x_y$. Then, the generator L_A of Y satisfies:

$$(L_A)^x_x = L^x_x(1 - (R^A)^x_x) \text{ and } (L_A)^x_y = -L^x_x(R^A)^x_y \text{ for } x \neq y.$$

Proof. Define $T_A = \inf\{t \geq 0, X_t \in A\}$ and $(H_A)^x_y = \mathbb{E}^x[X_{T_A} = y, T_A < \infty]$. As usual, set $Q^x_y = -\frac{L^x_y}{L^x_x}$ for $y \neq x$, $Q^x_x = 0$ and $Q^x_y = \delta^x_y$ if $L^x_x = 0$. For any subset B of S, define $(J_B)^x_y = 1_{\{x \in B\}} \delta^x_y$, $G_B = I + QJ_B + QJ_BQJ_B + \cdots$. Then

$$H_A = J_A + J_{A^c}QH_A = J_A + J_{A^c}QJ_A + J_{A^c}QJ_{A^c}QJ_A + \cdots .$$

Next, we see that $(R^A)^x_y = \mathbb{E}^x[X_{T_{1,A}} = y] = Q^x_y + \sum_{z \in A^c} Q^x_z (H_A)^z_y = (G_{A^c}QJ_A)^x_y$ for $x, y \in A$. Then, Y can be described as follows: from x, it waits for an $\mathcal{E}(-L^x_x)$-time, then jumps to $y \in A \cup \{\partial\}$ according to $(R^A)^x_y$ (it does not actually jump if $y = x$). Finally, it is not hard to see that $(L_A)^x_x = L^x_x(1 - (R^A)^x_x)$ and $(L_A)^x_y = -(R^A)^x_y L^x_x$ for $y \neq x$. □

Definition 2.3.1. For $A \subset S$, define $V_A = V|_{A \times A}$. V_A is the potential of the trace of the Markov process on A and $L_A = -(V_A)^{-1}$ is its generator. Let $L^A = L|_{A \times A}$ denote the generator of the Markov process restricted in A (i.e. killed at entering A^c) and let $V^A = (-L^A)^{-1}$ be its potential.

Proposition 2.3.2. *Assume that V is transient, χ is a non-negative function on S and that $F \subset S$ contains the support of χ. Then, $(V_\chi)_F = (V_F)_\chi$.*

Proof. For $x, y \in F$, let $(X_t, t \geq 0)$ be the minimal Markov process with initial point x and potential V. Let $A_t = \int_0^{t \wedge \zeta} 1_{\{X_s \in F\}} \, ds, \sigma_t = \inf\{s \geq 0, A_s > t\}$ with the convention that $\inf \phi = \infty$. Then, $(Y_t = X_{\sigma_t}, t \geq 0)$ is the trace of X_t on F. Moreover, its potential is V_F. Using the assumption $\mathrm{supp}(\chi) \subset F$ and Lebesgue's change of time formula,

$$\begin{aligned}
((V_F)_\chi)^x_y &= \mathbb{E}^x[\int_0^\infty 1_{\{Y_t = y\}} e^{-\int_0^t \chi(Y_s)\, ds} dt] \\
&= \mathbb{E}^x[\int_0^\infty 1_{\{X_{\sigma_t} = y\}} e^{-\int_0^{\sigma_t} \chi(X_{\sigma_s})\, ds} dt] \\
&= \mathbb{E}^x[\int_0^\infty 1_{\{X_{\sigma_t} = y\}} e^{-\int_0^{\sigma_t} \chi(X_s)\, ds} dt] \\
&= \mathbb{E}^x[\int_0^\infty 1_{\{X_t = y\}} e^{-\int_0^t \chi(X_s)\, ds} dt] \\
&= (V_\chi)^x_y \text{ for } x, y \in F.
\end{aligned}$$

□

Chapter 3

Loops and Markovian loop measure

In this section, we introduce the loop measure associated with a continuous time Markov chain. Its properties under various transformations (time change, trace, restriction, Feynman-Kac) are studied as well as the associated occupation and multi-occupation field.

3.1 Definitions and basic properties

Definition 3.1.1 (Based loops). A based loop l is an element $(\xi_1,\tau_1,\ldots,\xi_p,\tau_p,\xi_{p+1},\tau_{p+1})$ in $\bigcup\limits_{p\in\mathbb{N}}(S\times]0,+\infty[)^{p+1}$ such that $\xi_{p+1}=\xi_1$ and $\xi_{i+1}\neq\xi_i$ for $i=1,\ldots,p$. We call p the number of jumps in l and denote it by $p(l)$. Define $T=\tau_1+\cdots+\tau_{p+1}, T_0=0, T_i=\tau_1+\cdots+\tau_i$. Then, a based loop can be viewed as a càdlàg piecewise constant path l on $[0,T]$ such that $l(t)=\xi_{i+1}$ for $t\in[T_i,T_{i+1}[, i=1,\ldots,p$ and $l(T)=\xi_{p+1}=\xi_1$. Clearly, we have $l(T)=l(T-)$.

Let $\mathbb{P}^x$ be the law of the minimal sub-Markovian process started from x with semi-group $(P_t,t\geq 0)$ (or with generator L equivalently). It induces a probability measure on the space of paths l indexed by $[0,t]$, namely $\mathbb{P}^x_t$. $\mathbb{P}^x_t$ is carried by the space of paths with finite many jumps such that $l(0)=l(0+)=x$. Define the non-normalized bridge measure $\mathbb{P}^x_{t,y}$ from x to y with duration time t as follows: $\mathbb{P}^x_{t,y}(\cdot)=\mathbb{P}^x_t(\cdot\cap 1_{\{l(t)=y\}})$.

Definition 3.1.2. The measure on the based loops is defined as $\mu^b=\sum\limits_{x\in S}\int\limits_0^\infty\frac{1}{t}\mathbb{P}^x_{t,x}\,dt$.

Proposition 3.1.1 (Expression of the based loop measure)**.** *For $k\geq 2$,*

$$\begin{aligned}&\mu^b(p(l)=k,\xi_1=x_1,\ldots,\xi_k=x_k,\xi_{k+1}=x_{k+1},\tau_1\in dt^1,\ldots,\tau_{k+1}\in dt^{k+1})\\&=1_{\{x_1=x_{k+1}\}}Q^{x_1}_{x_2}\cdots Q^{x_k}_{x_1}\frac{1}{t^1+\cdots+t^{k+1}}(-L^{x_1}_{x_1})e^{L^{x_1}_{x_1}t^1}\cdots(-L^{x_k}_{x_k})e^{L^{x_k}_{x_k}t^k}e^{L^{x_{k+1}}_{x_{k+1}}t^{k+1}}\,dt^1\cdots dt^{k+1}\end{aligned}$$

For $k=1$,

$$\mu^b(p(l)=1,\xi=x,\tau\in dt)=\frac{1}{t}e^{L_x^x t}\,dt$$

Proof. For $k\geq 2$ and all sequence of positive measurable functions $(f_i, i\geq 1)$, denote by $(*)$ the value of $\mu^b(p(l)=k,\xi_1=x_1,\dots,\xi_k=x_k,\xi_{k+1}=x_{k+1},f_1(\tau_1)\cdots f_{k+1}(\tau_{k+1}))$.

$$\begin{aligned}(*)&=\int_0^\infty\frac{dt}{t}\sum_{x\in S}\mathbb{P}^x_{t,x}[p(l)=k,\xi_1=x_1,\dots,\xi_{k+1}=x_{k+1},f_1(\tau_1)\cdots f_k(\tau_k)f_{k+1}(t-\sum_{j=1}^k\tau_j)]\\&=\int_0^\infty\frac{dt}{t}\sum_{x\in S}\mathbb{P}^x_t[p(l)=k,\xi_1=x_1,\dots,\xi_{k+1}=x_{k+1},\\&\qquad f_1(\tau_1)\cdots f_k(\tau_k)f_{k+1}(t-\sum_{j=1}^k\tau_j),l(t)=x]\\&=\int_0^\infty\frac{dt}{t}\mathbb{P}^{x_1}_t[p(l)=k,\xi_1=x_1,\dots,\xi_{k+1}=x_{k+1},\\&\qquad f_1(\tau_1)\cdots f_k(\tau_k)f_{k+1}(t-\sum_{j=1}^k\tau_j),l(t)=x_1].\end{aligned}$$

By definition of $\mathbb{P}^x_t$,

$$\begin{aligned}(*)&=1_{\{x_1=x_{k+1}\}}\int_0^\infty\frac{1}{t}\,dtQ^{x_1}_{x_2}\cdots Q^{x_{k-1}}_{x_k}Q^{x_k}_{x_1}\int\limits_{\substack{s^1,\dots,s^{k+1}>0\\ s^1+\dots+s^k<t\\ s^1+\dots+s^{k+1}>t}}f_1(s^1)\cdots f_k(s^k)f_{k+1}(t-\sum_{j=1}^k s_j)\\&\qquad(\prod_{i=1}^{k+1}(-L^{x_i}_{x_i})e^{L^{x_i}_{x_i}s^i}\,ds^i)\\&=1_{\{x_1=x_{k+1}\}}\int_0^\infty\frac{dt}{t}Q^{x_1}_{x_2}\cdots Q^{x_{k-1}}_{x_k}Q^{x_k}_{x_1}\int\limits_{\substack{s^1,\dots,s^k>0\\ s^1+\dots+s^k<t}}f_1(s^1)\cdots f_k(s^k)f_{k+1}(t-s^1-\dots-s^k)\\&\qquad e^{L^{x_1}_{x_1}(t-s^1-\dots-s^k)}(\prod_{i=1}^k(-L^{x_i}_{x_i})e^{L^{x_i}_{x_i}s^i}\,ds^i).\end{aligned}$$

Now, change the variables as follows: $t^1=s^1,\dots,t^k=s^k,t^{k+1}=t-s^1-\dots-s^k$.

$$\begin{aligned}&\mu^b(p(l)=k,\xi_1=x_1,\dots,\xi_k=x_k,\xi_{k+1}=x_1,f_1(\tau_1)\cdots f_k(\tau_k)f_{k+1}(\tau_{k+1}))\\&=1_{\{x_1=x_{k+1}\}}\int\limits_{t^1,\dots,t^{k+1}>0}\frac{1}{t^1+\dots+t^{k+1}}Q^{x_1}_{x_2}\cdots Q^{x_{k-1}}_{x_k}Q^{x_k}_{x_1}f_1(t^1)\cdots f_k(t^k)f_{k+1}(t^{k+1})e^{L^{x_1}_{x_1}t^{k+1}}\\&\qquad\prod_{i=1}^k(-L^{x_i}_{x_i})e^{L^{x_i}_{x_i}t^i}\prod_{i=1}^{k+1}dt^i.\end{aligned}$$

Consequently, for $k \geq 2$,

$$\mu^b(p(l) = k, \xi_1 = x_1, \ldots \xi_k = x_k, \xi_{k+1} = x_{k+1}, \tau_1 \in dt^1, \ldots, \tau_{k+1} \in dt^{k+1})$$
$$= 1_{\{x_1 = x_{k+1}\}} Q^{x_1}_{x_2} \cdots Q^{x_k}_{x_1} \frac{1}{t^1 + \cdots + t^{k+1}} (-L^{x_1}_{x_1}) e^{L^{x_1}_{x_1} t^1} \cdots (-L^{x_k}_{x_k}) e^{L^{x_k}_{x_k} t^k} e^{L^{x_{k+1}}_{x_{k+1}} t^{k+1}} \, dt^1 \cdots dt^{k+1}.$$

The case $k = 1$ is similar and even simpler. □

Definition 3.1.3 (**Doob's harmonic transform**). A non-negative function h is said to be excessive iff $-Lh \geq 0$. Define Doob's harmonic transform $((L^h)^x_y, \ x, y \in \text{supp}(h))$ of L as follows

$$(L^h)^x_y = \frac{L^x_y h(y)}{h(x)}.$$

As in [LJL12], the following proposition is a direct consequence of Proposition 3.1.1.

Proposition 3.1.2. *The based loop measure is invariant under the harmonic transform with respect to any strictly positive excessive function.*

Remark 3. Doob's h-transform with respect to a strictly positive function does not change the bridge measure.

Definition 3.1.4 (**Pointed loops and discrete pointed loops**). Using the same notation as before, set $\tau^*_1 = \tau_1 + \tau_{p(l)+1}$, $\tau^*_i = \tau_i$ for $1 < i < p(l) + 1$. Then $(\xi_1, \tau^*_1, \ldots, \xi_{p(l)}, \tau^*_{p(l)}) \in \bigcup\limits_{p \in \mathbb{N}_+} (S \times \mathbb{R}_+)^p$ is called the pointed loop obtained from the based loop $(\xi_1, \tau_1, \ldots, \xi_{p(l)+1} = \xi_1, \tau_{p(l)+1})$. Clearly, $\xi_1 \neq \xi_{p(l)}$ and $\xi_i \neq \xi_{i+1}$ for $i = 1, \ldots, p - 1$. The induced measure on pointed loops is denoted by μ^p. By removing the holding times from the pointed loop, we get a discrete based loop $\xi = (\xi_1, \ldots, \xi_{p(l)})$.

As a direct consequence of Proposition 3.1.1, we obtain the following by change of variables:

Proposition 3.1.3 (Expression of μ^p). *For $k \geq 2$,*

$$\mu^p(p(l) = k, \xi_1 = x_1, \ldots, \xi_k = x_k, \tau^*_1 \in dt^1, \ldots, \tau^*_k \in dt^k)$$
$$= Q^{x_1}_{x_2} \cdots Q^{x_k}_{x_1} \frac{t^1}{t^1 + \cdots + t^k} (-L^{x_1}_{x_1}) e^{L^{x_1}_{x_1} t^1} \cdots (-L^{x_k}_{x_k}) e^{L^{x_k}_{x_k} t^k} \, dt^1 \cdots dt^k.$$

For $k = 1$,

$$\mu^p(p(l) = 1, \xi_1 = x_1, \tau^*_1 \in dt^1) = \frac{1}{t^1} e^{L^{x_1}_{x_1} t^1} \, dt^1.$$

Definition 3.1.5 (Loops and loop measure). We define an equivalence relation between based loops. Two based loops are called equivalent iff they have the same time length and their periodical extensions are the same under a translation on $\mathbb{R}$. The equivalence class of a based loop l is called a loop and denoted l^o. Sometimes, for the simplicity of the notations, if there is no ambiguity, we will omit the superscript o and use the same notation l for a based loop and the associated loop. Moreover, the based loop measure induces a measure on loops, namely the loop measure μ. The loop measure is defined by the generator L. Sometimes, we will write $\mu(L, dl)$ instead of μ to stress this point.

Definition 3.1.6. For a pointed loop l, let $p(l)$ be the number of jumps made by l. For any pointed loop $(\xi_1, \tau_1, \ldots, \xi_n, \tau_n)$, define $N_y^x = \sum\limits_{i=1}^{p(l)} 1_{\{\xi_i = x, \xi_{i+1} = y\}}$ and $N^x = \sum\limits_{y \in S} N_y^x = \sum\limits_{i=1}^{p(l)} 1_{\{\xi_i = x\}}$. $p(l)$, $N_y^x(l)$ and $N^x(l)$ have the same value for equivalent pointed loops. Accordingly, they can be defined on the space of loops and denoted the same.

Definition 3.1.7 (Discrete loops and discrete loop measure). We define an equivalence relation $\sim$ on $\bigcup\limits_k S^k$ as follows: $(x_1, \ldots, x_n) \sim (y_1, \ldots, y_m)$ iff $m = n$ and $\exists j \in \mathbb{Z}$ such that $(x_1, \ldots, x_n) = (y_{1+j}, \ldots, y_{m+j})$. For any $(x_1, \ldots, x_n) \in \bigcup\limits_k S^k$, use $(x_1, \ldots, x_n)^o$ to stand for the equivalent class of $(x_1, \ldots, x_n)$. Then the space of discrete loops is $\{(x_1, \ldots, x_n)^o; (x_1, \ldots, x_n) \in \bigcup\limits_k S^k\}$. For any loop $l^o = (x_1, t^1, \ldots, x_k, t^k)^o$, use $l^{o,d}$ to stand for the discrete loop $(x_1, \ldots, x_k)^o$. The mapping from loops to discrete loops and the loop measure induces a measure on the space of discrete loops, namely the discrete loop measure μ^d.

Definition 3.1.8 (Powers). Let $l : [0, |l|] \to S$ be a based loop. Define the n-th power of $l^n : [0, n|l|] \to S$ as follows: for $k = 0, \ldots, n-1$ and $t \in [0, |l|]$, $l^n(t + k|l|) = l(t)$. The n-th powers of equivalent based loops are again equivalent. Consequently, the n-th powers of the loop is well-defined. The powers of the discrete loops are defined similarly.

Definition 3.1.9 (Multiplicity and primitive of the non-trivial loops). The multiplicity of a discrete loop is defined as follows:

$$n(l^{o,d}) = \max\{k \in \mathbb{N} : \exists \tilde{l}^{o,d}, l^{o,d} = (\tilde{l}^{o,d})^k\}$$

If $l^{o,d} = (\tilde{l}^{o,d})^{n(l^{o,d})}$, then $\tilde{l}^{o,d}$ is called a primitive of $l^{o,d}$. For a non-trivial loop l, the multiplicity is defined as follows:

$$n(l^o) = \max\{k \in \mathbb{N} : \exists \tilde{l}^o, l^o = (\tilde{l}^o)^k\}$$

For a trivial loop l, the multiplicity is defined to be 1. If $(\tilde{l}^o)^{n(l^o)} = l^o$, then $\tilde{l}^o$ will be called the primitive of l^o, as it is always unique. And we will use *prime* to stand for the mapping from a (discrete) loop to its primitive.

Definition 3.1.10 (Primitive (discrete) loops and (discrete) primitive loop measure). A (discrete) loop is called primitive iff its multiplicity is one. The mapping *prime* induces a measure on (discrete) primitive loops, namely the (discrete) primitive loop measure.

Proposition 3.1.4. *We have the following expression for the discrete loop measure:*

$$\mu^d((x_1,\ldots,x_k)^o) = \frac{1}{n((x_1,\ldots,x_k)^o)} Q^{x_1}_{x_2}\cdots Q^{x_k}_{x_1}.$$

Definition 3.1.11 (Pointed loop measure). We can define another measure μ^{p*} on the pointed loop space as follows:

- for $k \geq 2$,

$$\begin{aligned}\mu^{p*}&(p(l)=k,\xi_1=x_1,\tau_1\in dt^1,\ldots,\xi_k=x_k,\tau_k\in dt^k)\\ &= \frac{1}{k} Q^{x_1}_{x_2}\cdots Q^{x_k}_{x_1}(-L^{x_1}_{x_1})e^{L^{x_1}_{x_1}t^1}\,dt^1\cdots(-L^{x_k}_{x_k})e^{L^{x_k}_{x_k}t^k}\,dt^k.\end{aligned}$$

- for $k=1$, $\mu^{p*}(p(\xi)=1,\xi=x,\tau\in dt) = \frac{1}{t}e^{L^x_x t}\,dt$.

We call μ^{p*} the pointed loop measure.

Proposition 3.1.5. *μ^{p*} induces the same loop measure as μ^b and μ^p.*

Proof. It is obvious for the trivial loops. Let us focus on the non-trivial loops. For a non-trivial pointed loop $l=(\xi_1,\tau_1,\ldots,\xi_n,\tau_n)$, define $\theta(l)=(\xi_2,\tau_2,\ldots,\xi_n,\tau_n,\xi_1,\tau_1)$. Fix $n\geq 2$, $x_1,\ldots,x_n\in S$, $f:\mathbb{R}^n_+\to\mathbb{R}_+$ measurable, define

$$\Phi(l) = 1_{\{p(l)=n\}}1_{\{\xi_1=x_1,\ldots,\xi_n=x_n\}}f(\tau_1,\ldots,\tau_n)$$

and $\bar\Phi = \frac{1}{n}(\Phi+\Phi\circ\theta+\cdots+\Phi\circ\theta^{n-1})$. By Proposition 3.1.3,

$$\mu^p(\bar\Phi) = \frac{1}{n}Q^{x_1}_{x_2}\cdots Q^{x_n}_{x_1}\int\limits_{\mathbb{R}^n_+} f(t^1,\ldots,t^n)\Big(\prod_{i=1}^n(-L^{x_i}_{x_i})e^{L^{x_i}_{x_i}t^i}\,dt^i\Big).$$

From the definition of the pointed loop measure μ^{p*}, $\theta\circ\mu^{p*}=\mu^{p*}$,

$$\mu^{p*}(\bar\Phi) = \mu^{p*}(\Phi) = \frac{1}{n}Q^{x_1}_{x_2}\cdots Q^{x_n}_{x_1}\int\limits_{\mathbb{R}^n_+} f(t^1,\ldots,t^n)\Big(\prod_{i=1}^n(-L^{x_i}_{x_i})e^{L^{x_i}_{x_i}t^i}\,dt^i\Big).$$

We have $\mu^p(\bar{\Phi}) = \mu^{p*}(\bar{\Phi})$. For a positive functional Φ on the space of pointed loops, we have the following decomposition

$$\Phi = \sum_{n\geq 1}\sum_{x\in S^n} 1_{\{p(l)=n\}} 1_{\{\xi_1=x_1,\ldots,\xi_n=x_n\}} f^x(\tau_1,\ldots,\tau_n)$$

where $f^x(\tau_1,\ldots,\tau_n) = (\Phi|_{\{l:p(l)=n\}})(x_1,\tau_1,\ldots,x_n,\tau_n)$. Define

$$\bar{\Phi} = \sum_{n\geq 1}\sum_{x\in S^n} \overline{1_{\{p(l)=n\}} 1_{\{\xi_1=x_1,\ldots,\xi_n=x_n\}} f^x(\tau_1,\ldots,\tau_n)}.$$

It is clear that $^{-}: \Phi \to \bar{\Phi}$ is a well-defined linear map which preserves the positivity. By monotone convergence, $\mu^{p*}(\bar{\Phi}) = \mu^p(\bar{\Phi})$ for any positive measurable pointed loop functional. As a consequence, the loop measure induced by μ^{p*} is exactly μ. □

Definition 3.1.12. For a pointed loop $l = (\xi_1,\tau_1,\ldots,\xi_{p(l)},\tau_{p(l)})$, $\xi = (\xi_1,\ldots,\xi_{p(l)})$ is the corresponding discrete pointed loop. For any $F \subset S$, define $q(F,l) = \sum\limits_{x\in F} N^x(l)$ the number of times l visits F. Recursively define the i-th hitting time for F as follows ($i = 1,\ldots,q(F,l)$): $T_1^F(l) = T_1^F(\xi) = \inf\{m \leq p(l) : \xi_m \in F\}$ and $T_{i+1}^F(l) = T_{i+1}^F(\xi) = \inf\{m > T_i^F : m \leq p(l), \xi_m \in F\}$. Define $T = T_{q(F,l)}^F$ the last visiting time for F. Define $p(F,l) = \#\{i : \xi_{T_i^F} \neq \xi_{T_{i+1}^F}, i = 1,\ldots,q(F,l)\}$ with the convention that $\xi_{T_{q(F,l)+1}^F} = \xi_{T_1^F}$.
Define a pointed loop measure $\mu^{p*,F}$ as follows:

$$\mu^{p*,F} 1_{\{p(F,l)\neq 0\}} = 1_{\{\xi_1\in F,\xi_T\neq\xi_1\}} \frac{p(l)}{p(F,l)} \mu^{p*}$$
$$\mu^{p*,F} 1_{\{p(F,l)=0\}} = 1_{\{\xi_1\in F,p(F,l)=0\}} \frac{p(l)}{q(F,l)} \mu^{p*}.$$

Remark 4. $p(F,l) = 0$ iff. the intersection of the pointed loop l and the subset $F \subset S$ is a single element set: $|l \cap F| = 1$ (or $|\bigcup\limits_{i=1}^{q(F,l)} \{\xi_{T_i^F}\}| = 1$ equivalently). For a loop l with $p(F,l) \neq 0$ (or $p(F,l) = 2,\ldots,\infty$ equivalently), the term $1_{\{\xi_1\in F,\xi_T\neq\xi_1\}}$ in the above expression implies that $\mu^{p*,F}|_{\{l:p(F,l)\neq 0\}}$ is concentrated on the pointed loops satisfying the following two conditions:

1. the pointed loop starts from a point in F.

2. the trace of the pointed loop on F has an endpoint different from the starting point.

By an argument similar to remark 3.1.5, it can be showed that $\mu^{p*,F}$ induces a loop measure which is exactly the restriction of μ to the loops visiting F.

Definition 3.1.13 (Multi-occupation field). Define the circular permutation operator r_j as follows: $r_j(z^1, \dots, z^p) = (z^{1+j}, \dots, z^n, z^1, \dots, z^j)$. For any $f : S^n \to \mathbb{R}$ measurable, define the multi-occupation field of a based loop l of length t as

$$\langle l, f \rangle = \sum_{j=0}^{n-1} \int\limits_{0<s^1<\dots<s^n<t} f \circ r_j(l(s_1), \dots, l(s_n))\, ds^1 \cdots ds^n.$$

If l_1 and l_2 are two equivalent based loops, they correspond to the same multi-occupation field. Therefore, it is well-defined for loops. When $n = 1$, it is called the occupation time. For $x \in \mathbb{R}^m$ for some integer m, define $l^x = \langle l, \delta_x \rangle$ where $\delta_x(y) = 1_{\{x=y\}}$.

Definition 3.1.14 (Another bridge measure $\mu^{x,y}$). Another bridge measure $\mu^{x,y}$ can be defined on paths from x to y:

$$\mu^{x,y}(d\gamma) = \int\limits_0^\infty \mathbb{P}_t^{x,y}(d\gamma)dt.$$

For a path γ from x to y, let $p(\gamma)$ be the total number of jumps, T_i the i-th jumping time and T the time duration of γ. Then γ can be viewed as $(x, T_1, \gamma(T_1), T_2 - T_1, \gamma(T_2), \dots, T_{p(\gamma)} - T_{p(\gamma)-1}, y = \gamma(T_{p(\gamma)}), T - T_{p(\gamma)})$.

The bridge measure $\mu^{x,y}$ can be expressed as follows:

Proposition 3.1.6.

$$\begin{aligned}
&\mu^{x,y}(p(\gamma) = p, \gamma(T_1) = x_1, \dots, \gamma(T_{p-1}) = x_{p-1},\\
&T_1 \in dt^1, T_2 - T_1 \in dt^2, \dots, T_p - T_{p-1} \in dt^p, T - T^p \in dt^{p+1})\\
=&Q^x_{x_1} Q^{x_1}_{x_2} \cdots Q^{x_{p-1}}_y 1_{\{t^1,\dots,\, t^{p+1}>0\}} (-L^x_x) e^{L^x_x t^1} (-L^{x_1}_{x_1}) e^{L^{x_1}_{x_1} t^2} \cdots (-L^{x_{p-1}}_{x_{p-1}}) e^{L^{x_{p-1}}_{x_{p-1}} t^p} e^{L^y_y t^{p+1}} \prod_{j=1}^{p+1} dt^j
\end{aligned}$$

In the case $x = y$, $\gamma = (x, T_1, \gamma(T_1), T_2 - T_1, \gamma(T_2), \dots, T_{p(\gamma)} - T_{p(\gamma)-1}, y = T_{p(\gamma)}, T - T_{p(\gamma)})$ can be viewed as a based loop. Therefore, $\mu^{x,x}$ can be viewed as a measure on the based loop. Moreover, $\mu^{x,x}(dl) = 1_{\{l(0)=x\}}|l|\mu^b(dl)$. Consequently, the loop measure induced by $\mu^{x,x}$, which will be denoted by the same notation $\mu^{x,x}$, has the following relation with the loop measure μ.

Proposition 3.1.7.

$$\mu^{x,x}(dl) = l^x \mu(dl).$$

In the case $x \neq y$, $\gamma = (x, T_1, \gamma(T_1), T_2 - T_1, \gamma(T_2), \dots, T_{p(\gamma)} - T_{p(\gamma)-1}, y = \gamma(T_{p(\gamma)}), T - T_{p(\gamma)})$ can be viewed as a pointed loop. Similarly, $\mu^{x,y}$ can be viewed as a measure on the pointed loop. Moreover, $L_x^y \mu^{x,y}(dl) = 1_{\{l \text{ starts at } x \text{ and ends up at } y\}} p(l) \mu^{p*}(dl)$. Consequently, the loop measure induced by $\mu^{x,y}$, which will be denoted by the same notation $\mu^{x,y}$, has the following relation with the loop measure μ.

Proposition 3.1.8.

$$L_x^y \mu^{x,y}(dl) = N_x^y \mu(dl).$$

3.2 Compatibility of the loop measure with time change

Proposition 3.2.1. *Suppose $\lambda : S \to [0, \infty]$. Given a Markov process $(X_t, t \geq 0)$ in S, define $B_t = \int\limits_0^t \lambda(X_s)\, ds$. Let $(C_t, t \geq 0)$ be the right-continuous inverse of $(B_t, t \geq 0)$. Define $\zeta = \inf\{s \geq 0 : C_s = C_\infty\}$. Define $Y_t = X_{C_t}, t < \zeta$ (it will be called the time-changed process of X with respect to λ and denoted $\lambda(X)$). On the space of based (pointed) loops contained in $\{x \in S : \lambda(x) < \infty\}$, λ defines a similar operation. If l_1 and l_2 are two equivalent based (pointed) loops, $\lambda(l_1)$ and $\lambda(l_2)$ are equivalent again. Consequently, λ can be defined on the space of loops with the domain $D(\lambda) = \{$loops contained in $\{x \in S : \lambda(x) < \infty\}\}$. There are two Markovian loop measures μ_X, μ_Y defined by X, Y respectively. The following diagram commutes:*

$$\begin{array}{ccc} X & \overset{\lambda}{\to} & Y \\ \downarrow & & \downarrow \\ \mu_X & \overset{\lambda}{\to} & \mu_Y \end{array}$$

In particular, the loop measure is compatible with the notion of "trace on a set" (i.e. $\lambda = 1_A$) and "restriction" (i.e. $\lambda = 1_A + \infty \cdot 1_{A^c}$).

Proof. Let $\lambda \circ \mu$ be the image law of μ under the mapping λ. Denote by $\pi^{p \to o}$ the quotient map from pointed loops to loops. Then, we have to show that λ commutes with $\pi^{p \to o}$.
The holding times are almost surely different for μ_X, μ_Y and $\lambda \circ \mu_X$. So the same is true for the measures on pointed loops μ_X^{p*}, μ_Y^{p*} and $\lambda \circ \mu_X^{p*}$.
Every change of time can be done in three steps: i) Restriction, ii) trace, iii) time change with a function $0 < \lambda < \infty$. Accordingly, it is enough to deal with these three special cases separately:

i) $0 < \lambda < \infty$
   Let L and $\hat{L}$ represent the generator of X and Y. Then $\hat{L}_y^x = \frac{L_y^x}{\lambda_x}$.

By Definition 3.1.11 and its following remark,

$$\begin{aligned}\lambda\circ\mu_X^{p*}(p(\xi)=k,\,\xi_1=x_1,\cdots\xi_k=x_k,\tau_1\in dt^1,\cdots,\tau_k\in dt_k)\\ =&\mu_X^{p*}(p(\xi)=k,\xi_1=x_1,\cdots\xi_k=x_k,\lambda_{x_1}\tau_1\in dt^1,\cdots,\lambda_{x_k}\tau_k\in dt_k)\\ =&\frac{1}{k}L^{x_1}_{x_2}\cdots L^{x_k}_{x_1}e^{L^{x_1}_{x_1}t^1/\lambda_{x_1}}\cdots e^{L^{x_k}_{x_k}t^k/\lambda_{x_k}}\,dt^1\cdots dt^k\\ =&\frac{1}{k}\hat{L}^{x_1}_{x_2}\cdots\hat{L}^{x_k}_{x_1}e^{\hat{L}^{x_1}_{x_1}t^1}\cdots e^{\hat{L}^{x_k}_{x_k}t^k}\,dt^1\cdots dt^k\\ =&\mu_Y^{p*}(p(\xi)=k,\xi_1=x_1,\cdots\xi_k=x_k,\tau_1\in dt^1,\cdots,\tau_k\in dt_k)\end{aligned}$$

Therefore, $\lambda\circ\mu_X=\lambda\circ\pi^{p\to o}\circ\mu_X^{p*}=\pi^{p\to o}\circ\lambda\circ\mu_X^{p*}=\pi^{p\to o}\mu_Y^{p*}=\mu_Y$

ii) $\lambda=1_A+\infty\cdot 1_{A^c}$. In that case, $\lambda\circ\mu_X=\mu_X|_{D(\lambda)}=\mu_Y$.

iii) $\lambda=1_A+0\cdot 1_{A^c}$.

We needs to show that $\lambda\circ\mu_X=\mu_Y$. We will only prove this for the non-trivial loops. The trivial loop case can be proved in a similar way.

Use $\mathbb{P}^x$ to stand for the law of a minimal Markov process X starting from x. Let T_1 be the first jumping time, and set $T_{1,A}=\inf\{s\geq T_1,X_s\in A\}$. Let $(R^A)^x_y=\mathbb{P}^x[X_{T_{1,A}}=y]$ for $x,y\in S$. Obviously, $(R^A)^x_y=0$ for $y\in A^c$. By Proposition 2.3.1, the relation between the generator L of X and the generator $\hat{L}$ of Y is stated as follows: $\hat{L}^x_x=L^x_x(1-(R^A)^x_x)$, $\hat{L}^x_y=-(R^A)^x_yL^x_x$ for $x\neq y$.

Fix a non-trivial discrete pointed loop $(x_1,\ldots,x_n)$ where $x_i\in A$ for $i=1,\ldots,n$. Take $F=\bigcup\limits_{i=1}^n\{x_i\}$. Take n positive measurable functions $f_1,\ldots,f_n$ on S. By Definition 3.1.12 and its following remark, it is enough to show that

$$\lambda\circ\mu_X^{p*,F}(p(\xi)=n,\xi_1=x_1,\ldots,\xi_n=x_n,\prod_{i=1}^n f_i(\tau_i))=\mu_Y^{p*}(p(\xi)=n,\xi_1=x_1,\ldots,\xi_n=x_n,\prod_{i=1}^n f_i(\tau_i)).$$

In order that $\lambda(l)$, the image of the pointed loop l, equals $(p(\xi)=n,x_1,\tau_1,\ldots,x_n,\tau_n)$, the pointed loop l has to be of the following form $\mu_X^{p,*F}$-a.s.:

$$\begin{aligned}(&\xi_{111},\tau_{111},\ldots,\xi_{11M^1_1},\tau_{11M^1_1},\ldots,\xi_{1N_11},\tau_{1N_11},\ldots,\xi_{1N_1M^1_{N_1}},\tau_{1N_1M^1_{N_1}},\\ &\xi_{211},\tau_{211},\ldots,\xi_{21M^2_1},\tau_{21M^2_1},\ldots,\xi_{2N_21},\tau_{2N_21},\ldots,\xi_{2N_2M^2_{N_2}},\tau_{2N_2M^2_{N_2}},\\ &\cdots\\ &\xi_{n11},\tau_{n11},\ldots,\xi_{n1M^n_1},\tau_{n1M^n_1},\ldots,\xi_{nN_n1},\tau_{nN_n2},\ldots,\xi_{nN_nM^n_{N_n}},\tau_{nN_nM^n_{N_n}})\end{aligned}$$

with

- $\xi_{ij1} = x_i$ for all i, j;
- $\xi_{ijk} \in A^c$ for $k \neq 1$ and all i, j;
- $\tau_i = \sum_j \tau_{ij1}$.

Roughly speaking, $\xi_{ij1}, \tau_{ij1}, \ldots, \xi_{ijM^i_j}, \tau_{ijM^i_j}$ can be viewed as an excursion in A^c from x_i to x_i for $j \neq N_i$. And $\xi_{iN_i1}, \tau_{iN_i1}, \ldots, \xi_{iN_iM^i_{N_i}}, \tau_{iN_iM^i_{N_i}}$ can be viewed as an excursion in A^c from x_i to x_{i+1}. Accordingly,

$$\begin{aligned}&\lambda \circ \mu_X^{p*,F}(p(\xi) = n, \xi_1 = x_1, \ldots, \xi_n = x_n, \prod_{i=1}^n f_i(\tau_i))\\ &\qquad = \sum_\xi \mu_X^{p*,F}(\xi_{ij1} = x_i \text{ and for all } i, j, \xi_{ijk} \in A^c \text{ for } k \neq 1, \prod_{i=1}^n f_i(\sum_j \tau_{ij1})).\end{aligned}$$

Since $Q^x_y + \sum_{p=1}^{\infty} \sum_{z_1,\ldots,z_p \in A^c} Q^x_{z_1} Q^{z_1}_{z_2} \cdots Q^{z_{p-1}}_{z_p} Q^{z_p}_y = (R^A)^x_y$, the above quantity equals

$$\begin{aligned}&\frac{1}{n} \sum_{N_1,\ldots,N_n \geq 1} \prod_{i=1}^n ((R^A)^{x_i}_{x_i})^{N_i - 1} (R^A)^{x_i}_{x_{i+1}}\\ &\int f(t^{i11} + \cdots + t^{iN_i1})(-L^{x_i}_{x_i})^{N_i} e^{L^{x_i}_{x_i}(t^{i11}+\cdots+t^{iN_i1})}\, dt^{i11} \cdots dt^{iN_i1}\\ =&\frac{1}{n} \sum_{N_1,\ldots,N_k \geq 1} \prod_{i=1}^n \int ((R^A)^{x_i}_{x_i})^{N_i-1} (R^A)^{x_i}_{x_{i+1}} \frac{(t^i)^{N_i-1}}{(N_i-1)!} (-L^{x_i}_{x_i})^{N_i} e^{L^{x_i}_{x_i} t^i} f(t^i)\, dt^i\\ =&\frac{1}{n} \prod_{i=1}^n \int -L^{x_i}_{x_i} (R^A)^{x_i}_{x_{i+1}} e^{L^{x_i}_{x_i} t^i (1 - (R^A)^{x_i}_{x_i})} f(t^i)\, dt^i\\ =&\frac{1}{n} \prod_{i=1}^n \int (L_A)^{x_i}_{x_{i+1}} e^{(L_A)^{x_i}_{x_i} t^i} f(t^i)\, dt^i\\ =&\mu_Y^{p*}(p(\xi) = n, \xi_1 = x_1, \ldots, \xi_n = x_n, \prod_{i=1}^n f_i(\tau_i)) \text{ for } n \geq 2.\end{aligned}$$

For $n = 1$, it can be proved in a similar way. Finally, we conclude that $\lambda \circ \mu_X = \mu_Y$.

□

3.3 Decomposition of the loops and excursion theory

Fix some set $F \subset S$.

Definition 3.3.1 (excursion outside F). By non-empty excursion outside F, we mean a multiplet of the form $((\xi_1,\tau^1,\dots,\xi_k,\tau^k),A,B)$ for some $k\in\mathbb{N}_+,\xi_1,\dots,\xi_k\in F^c,A,B\in F$ and $\tau^1,\dots,\tau^k\in\mathbb{R}_+$. Let $T_0=0$ and $T_m=\tau^1+\cdots+\tau^m$ for $m=1,\dots,k$. Define $e:[0,T_k[\to F^c$ such that $e(u)=\xi_m$ for $u\in[T_{m-1},T_m[$. Therefore, the excursion can be viewed as a path e attached to starting point A and ending point B and it will also be denoted by (e,A,B). By empty excursion, we mean (ϕ,A,B).

Definition 3.3.2 (excursion measure outside F). Define a family of probability measure $\nu^{x,y}_{F,ex}$ indexed by $x,y\in F$ as follows:

$$\begin{aligned}\nu^{x,y}_{F,ex}&(\xi_1=x_1,\tau^1\in dt^1,\dots,\xi_k=x_k,\tau^k\in dt^k,A=u,B=v)\\&=\delta^{(x,y)}_{(u,v)}\frac{1}{(R^F)^x_y}1_{\{x_1,\dots,x_k\in F^c\}}Q^x_{x_1}L^{x_1}_{x_2}\cdots L^{x_{k-1}}_{x_k}L^{x_k}_y e^{L^{x_1}_{x_1}t^1}\cdots e^{L^{x_k}_{x_k}t^k}\,dt^1\cdots dt^k.\end{aligned}$$

and $\nu^{x,y}_{F,ex}[(\phi,A,B)=(\phi,u,v)]=\delta^{(x,y)}_{(u,v)}\frac{Q^x_y}{(R^F)^x_y}$. Recall that

$$(R^F)^x_y=\begin{cases}Q^x_y+\sum\limits_{k\geq1}\sum\limits_{x_1,\dots,x_k\in F^c}Q^x_{x_1}Q^{x_1}_{x_2}\cdots Q^{x_{k-1}}_{x_k}Q^{x_k}_y & \text{for } y\in F\\ 0 & \text{otherwise.}\end{cases}$$

Define a function $\phi^{br\to ex}$ from the space of bridges to the space of excursions as follows: Given a bridge γ from x to y, which is represented by

$$(x,T_1,\gamma(T_1),T_2-T_1,\dots,\gamma(T_{p(\gamma)-1}),T_{p(\gamma)}-T_{p(\gamma)-1},y=\gamma(T_{p(\gamma)}),T-T_{p(\gamma)}),$$

we represent $\phi^{br\to ex}(\gamma)$ by

$$((\gamma(T_1),T_2-T_1,\dots,\gamma(T_{p(\gamma)-1}),T_{p(\gamma)}-T_{p(\gamma)-1}),x,y).$$

The image measure of $\mu^{x,y}$ under $\phi^{br\to ex}$, namely $\phi^{br\to ex}\circ\mu^{x,y}$, has the following relation with the excursion measure $\nu^{x,y}_{F,ex}$:

Proposition 3.3.1.

$$\nu^{x,y}_{F,ex}(d\gamma)=\frac{1}{-L^y_yR^x_y}\phi^{br\to ex}\circ\mu^{x,y}(d\gamma,\gamma(T_1),\dots,\gamma(T_{p(\gamma)-1})\in F^c)$$

Define a function $\varphi^{ex\to po}_F$ from the space of non-empty excursions out of F to the space of pointed loops as follows:

$$\varphi^{ex\to po}_F:((\xi_1,\tau^1,\dots,\xi_k,\tau^k),A,B)\to(\xi_1,\tau^1,\dots,\xi_k,\tau^k)$$

Accordingly, $\nu^{x,y}_{F,ex}$ induces a pointed loop measure on the space of pointed loops outside of F, which is denoted by the same notation $\nu^{x,y}_{F,ex}$. The relation with the pointed loop measure is as follows:

Proposition 3.3.2. *Let* $C = \{(\xi_1, \tau^1, \ldots, \xi_n, \tau^n) \in \{pointed\ loops\} : Q^{\xi_n}_{\xi_1} > 0\}$. *Then,* $\varphi_F^{ex\to po} \circ \nu^{x,y}_{F,ex}$ *is absolutely continuous with respect to* μ^{p*}. *Moreover,*

$$Q^{\xi_n}_{\xi_1} \frac{d\varphi_F^{ex\to po} \circ \nu^{x,y}_{F,ex}}{d\mu^{p*}}((\xi_1, \tau^1, \ldots, \xi_n, \tau^n)) = 1_{\{R^x_y > 0, \xi_1, \ldots, \xi_n \in F^c\}} \frac{nQ^x_{\xi_1} Q^{\xi_n}_y}{R^x_y}$$

Definition 3.3.3 (Decomposition of a loop). Let $l = (\xi_1, \tau^1, \ldots, \xi_k, \tau^k)^o$ be a loop visiting F. The pre-trace of the loop l on F is obtained by removing all the ξ_m, τ^m such that $\xi_m \in F^c$ for $m = 1, \ldots, k$. We denote it by $Ptr_F(l)$. Suppose the pre-trace on F can be written as $(\mathfrak{x}_1, \mathfrak{s}^1, \ldots, \mathfrak{x}_q, \mathfrak{s}^q)^o$. Then we can write the loop l^o in the following form:

$$(\mathfrak{x}_1, \mathfrak{s}^1, y^1_1, t^1_1, \ldots, y^1_{m_1}, t^1_{m_1}, \mathfrak{x}_2, \mathfrak{s}^2, y^2_1, t^2_1, \ldots, y^2_{m_2}, t^2_{m_2}, \ldots, \mathfrak{x}_q, \mathfrak{s}^q, y^q_1, t^q_1, \ldots, y^q_{m_q}, t^q_{m_q})^o$$

with $\mathfrak{x}_i \in F$ for all i and $y^i_j \in F^c$ for all i, j (with the following convention: if $m_i = 0$ for some $i = 1, \ldots, q$, $y^i_1, t^i_1, \ldots, y^i_{m_i}, t^i_{m_i}$ does not appear in the above expression). We will use e_i to stand for $(y^i_1, t^i_1, \ldots, y^i_{m_i}, t^i_{m_i})$ with the convention that $e_i = \phi$ if $m_i = 0$. Define a point measure $\mathcal{E}_F(l) = \sum\limits_i \delta_{(e_i, \mathfrak{x}_i, \mathfrak{x}_{i+1})}$. Define $N^x_y(Ptr_F(l)) = \sum\limits_{i=1}^{q} 1_{\{\mathfrak{x}_i = x, \mathfrak{x}_{i+1} = y\}}$ with the convention that $\mathfrak{x}_{q+1} = \mathfrak{x}_1$. Set $q(Ptr_F(l)) = \sum\limits_{x,y} N^x_y(Ptr_F(l))$. In particular, in the case above, we have $q(Ptr_F(l)) = q$ if $q \geq 2$ and $q(Ptr_F(l)) = 0$ if $q = 1$.

Remark 5. The pre-trace $(\mathfrak{x}_1, \mathfrak{s}^1, \ldots, \mathfrak{x}_q, \mathfrak{s}^q)^o$ of a loop l on F is not necessarily a loop. We allow $\mathfrak{x}_i = \mathfrak{x}_{i+1}$ for some $i = 1, \ldots, q$ which is prohibited in the definition we gave of a loop.

Definition 3.3.4. The pre-trace of a loop l on F can always be written as follows:

$$(\mathfrak{x}_1, \mathfrak{s}^1_1, \ldots, \mathfrak{x}_1, \mathfrak{s}^1_{m_1}, \mathfrak{x}_2, \mathfrak{s}^2_1, \ldots, \mathfrak{x}_2, \mathfrak{s}^2_{m_2}, \ldots, \mathfrak{x}_k, \mathfrak{s}^k_1, \ldots, \mathfrak{x}_k, \mathfrak{s}^k_{m_k})^o$$

with $\mathfrak{x}_i \neq \mathfrak{x}_{i+1}$ for $i = 1, \ldots, k$ with the usual convention that $\mathfrak{x}_{k+1} = \mathfrak{x}_1$. Then, l_F, the trace of l on F is defined by

$$(\mathfrak{x}_1, \mathfrak{t}^1 = \mathfrak{s}^1_1 + \cdots + \mathfrak{s}^1_{m_1}, \mathfrak{x}_2, \mathfrak{s}^2_1 + \cdots + \mathfrak{s}^2_{m_2}, \ldots, \mathfrak{x}_k, \mathfrak{t}^k = \mathfrak{s}^k_1 + \cdots + \mathfrak{s}^k_{m_k})^o.$$

Formally, the trace of l on F is obtained by throwing away the parts out of F and then by gluing the rest in circular order.

By replacing μ by $\mu^{p*,F}$ and considering the pointed loops, we have the following propositions.

Proposition 3.3.3. *Let f be some measurable positive function on the space of excursions and g a positive measurable function on the space of pre-traces on F. Then,*

$$\mu(1_{\{l\ visits\ F\}} g(Ptr_F(l)) e^{-\langle \mathcal{E}_F(l), f\rangle}) = \mu(1_{\{l\ visits\ F\}} g(Ptr_F(l)) \prod_{x,y\in F} (\nu^{x,y}_{F,ex}(e^{-f}))^{N^x_y(Ptr_F(l))}).$$

Proposition 3.3.4. *The image measure $\mu^{p*}_{Ptr,F}$ of the pointed loop measure $\mu^{p*,F}$ under the map of the pre-trace on F can be described as follows:*

- *if $x_1,\ldots,x_q$ are not identical, then*

$$\mu^{p*}_{Ptr,F}(q(Ptr_F(l))=q,\mathfrak{x}_1=x_1,\mathfrak{s}^1\in ds^1,\ldots,\mathfrak{x}_q=x_q,\mathfrak{s}^q\in ds^q)$$
$$=\frac{1}{p(l_F)}\prod_{x,y}((R^F)^x_y)^{N^x_y(Ptr_F(l))}\prod_{i=1}^{q}(-L^{x_i}_{x_i})e^{L^{x_i}_{x_i}s^i}\,ds^i;$$

- *if $x_1=\ldots=x_q=x$ and $q>1$, then*

$$\mu^{p*}_{Ptr,F}(q(Ptr_F(l))=q,\mathfrak{x}_1=x_1=\cdots=\mathfrak{x}_q=x_q=x,\mathfrak{s}^1\in ds^1,\ldots,\mathfrak{s}^q\in ds^q)$$
$$=\frac{1}{q}((R^F)^x_x)^q\prod_{i=1}^{q}(-L^x_x)e^{L^x_x s^i}\,ds^i$$

- *if $q=1$ and $x_1=x$ and , then*

$$\mu^{p*}_{Ptr,F}(q(Ptr_F(l))=1,\mathfrak{x}=x,\mathfrak{s}\in ds)=\underbrace{(R^F)^x_x(-L^x_x)e^{L^x_x s}\,ds}_{\text{contribution of the non-trivial loops}}+\underbrace{\frac{1}{s}e^{L^x_x s}\,ds}_{\text{contribution of the trivial loops}}$$

where $\mu^{p*}_{Ptr,F}$ is the image measure of the pointed loop measure $\mu^{p*,F}|_{\{loops\ visiting\ F\}}$.

Proposition 3.3.5. *Under the same notation as Definition 3.3.4,*

- *for $k>1$,*

$$\mu^{p*}_{Ptr,F}(\mathfrak{x}_1=x_1,\ldots,\mathfrak{x}_k=x_k,m_1=q_1,\ldots,m_k=q_k,\mathfrak{t}^1\in dt^1,\ldots,\mathfrak{t}^k\in dt^k)$$
$$=\frac{1}{k}(L_F)^{x_1}_{x_2}\cdots(L_F)^{x_k}_{x_1}\prod_{i=1}^{k}e^{(L_F)^{x_i}_{x_i}t^i}\,dt^i\prod_{j=1}^{k}e^{(L^{x_j}_{x_j}-(L_F)^{x_j}_{x_j})t^j}\frac{((-L^{x_j}_{x_j}+(L_F)^{x_j}_{x_j})t^j)^{q_j-1}}{(q_j-1)!}$$

- *for $k=1$, $q_1=q>1$,*

$$\mu^{p*}_{Ptr,F}(\mathfrak{x}_1=x_1,m_1=q_1,\mathfrak{t}^1\in dt^1)$$
$$=\frac{1}{t_1}e^{(L_F)^{x_1}_{x_1}t^1}\,dt^1e^{(L^{x_1}_{x_1}-(L_F)^{x_1}_{x_1})t^1}\frac{((-L^{x_1}_{x_1}+(L_F)^{x_1}_{x_1})t^1)^{q_1}}{q_1!}$$

- *for $k=1$ and $q_1=1$,*

$$\mu^{p*}_{Ptr,F}(\mathfrak{x}_1=x_1,m_1=1,\mathfrak{t}^1\in dt^1)$$
$$=\frac{1}{t_1}e^{(L_F)^{x_1}_{x_1}t^1}\,dt^1e^{(L^{x_1}_{x_1}-(L_F)^{x_1}_{x_1})t^1}((-L^{x_1}_{x_1}+(L_F)^{x_1}_{x_1})t^1)+\frac{1}{t^1}e^{L^{x_1}_{x_1}t^1}\,dt^1$$

Proof. The result comes from Proposition 3.3.4 and Proposition 2.3.1. □

Combining Proposition 3.3.3 and Proposition 3.3.5, we have the following proposition:

Proposition 3.3.6.

$$\mu(1_{\{l\ visits\ F\}}g(l_F)e^{-\langle \mathcal{E}_F(l),f\rangle}) = \mu(1_{\{l\ visits\ F\}}g(l_F)\prod_{x\neq y\in F}\nu^{x,y}_{F,ex}(e^{-f})^{N^x_y(l_F)}e^{\sum\limits_{x\in F}(L^x_x-(L_F)^x_x)l^x_F\nu^{x,x}_{F,ex}(1-e^{-f})})$$

Corollary 3.3.7. *We see that $\nu^{x,y}_{F,ex}$ is a probability measure on the space of the excursions from x to y out of F. By mapping an excursion (e,x,y) into the Dirac measure $\delta_{(e,x,y)}$, $\nu^{x,y}_{F,ex}$ induces a probability measure on $\mathcal{M}^p(\{excursions\})$, the space of point measures over the space of excursions. We will adopt the same notation $\nu^{x,y}_{F,ex}$. Choose k samples of the excursions according to $\nu^{x,y}_{F,ex}$, namely $ex_1,\ldots,ex_k$, then $\sum\limits_i \delta_{ex_i}$ has the law $(\nu^{x,y}_{F,ex})^{\otimes k}$. For any $\beta=(\beta^x,x\in F)\in\mathbb{R}^F_+$, let $\mathcal{N}_F(\beta)$ be a Poisson random measure on the space of excursions with intensity $\sum\limits_x(-L^x_x+(L_F)^x_x)\beta^x\nu^{x,x}_{F,ex}$. Let $l_F\to K(l_F,\cdot)$ be a transition kernel from {the trace of the loop on F} to {point measure over the space of excursions} as follows:*

$$K(l_F,\cdot)=\bigotimes_{x\neq y\in F}(\nu^{x,y}_{F,ex})^{\otimes N^x_y(l_F)}\bigotimes \mathcal{N}_F((l^x_F,x\in F))$$

Then the joint measure of $(l_F,\mathcal{E}_F(l))$ is $\mu_F(dl_F)K(l_F,\cdot)$ where μ_F is the image measure of μ under $l\to l_F$. By Proposition 3.2.1, μ_F is actually the loop measure associated with the trace of the Markov process on F or with L_F equivalently.

Remark 6. $K(l_F,\cdot)$ can also be viewed as a Poisson random measure on the space of excursions with intensity $\sum\limits_x(-L^x_x+(L_F)^x_x)l^x_F\nu^{x,x}_{F,ex}+\sum\limits_{x\neq y\in F}\nu^{x,y}_{F,ex}$ conditioned to have exactly $N^x_y(l_F)$ excursions from x to y out of F for all $x\neq y\in F$.

Definition 3.3.5. Suppose χ is a non-negative function on S vanishing on F. For an excursion (e,A,B), define the real-valued function $\langle\chi,\cdot\rangle$ of the excursion as follows:

$$\langle\chi,(e,A,B)\rangle=\int\chi(e(t))\,dt.$$

Lemma 3.3.8. *We see that the excursion measure $\nu^{x,y}_{F,ex}$ varies as the generator changes. Let $\nu^{x,y,\chi}_{F,ex}$ be the excursion measure when L is replaced by $L-M_\chi$. Define $(R^F_\chi)^x_y$ as $(R^F)^x_y$ when L is replaced by $L-M_\chi$. Then,*

$$e^{-\langle\chi,\cdot\rangle}\cdot\nu^{x,y}_{F,ex}=\frac{(R^F_\chi)^x_y}{(R^F)^x_y}\nu^{x,y,\chi}_{F,ex}.$$

In particular,

$$\nu^{x,y}_{F,ex}[e^{-\langle\chi,\cdot\rangle}] = \frac{(R^F_\chi)^x_y}{(R^F)^x_y}.$$

Accordingly, we have the following corollary,

Corollary 3.3.9.

$$\mu(1_{\{l\ visits\ F\}} g(l_F) e^{-\sum\limits_{\varepsilon_F(l)} \langle\chi,\cdot\rangle})$$
$$= \mu\left(1_{\{l\ visits\ F\}} g(l_F) \prod_{x\neq y\in F} \left(\frac{(R^F_\chi)^x_y}{(R^F)^x_y}\right)^{N^x_y(l_F)} e^{\sum\limits_{x\in F}(L^x_x-(L_F)^x_x) l^x_F\left(1-\frac{(R^F_\chi)^x_x}{(R^F)^x_x}\right)}\right).$$

3.4 Further properties of the multi-occupation field

We know the loop measure varies as the generator varies. To emphasize this, we write $\mu(L,dl)$ instead of $\mu(dl)$.

Proposition 3.4.1. $e^{-\langle l,\chi\rangle}\mu(L,dl) = \mu(L-M_\chi,dl)$ *for positive measurable function* χ *on* S.

Proof. It is the direct consequence of the Feynman-Kac formula. To be more precise,

$$e^{-\langle l,\chi\rangle}\mu(L,dl) = \sum_{x\in S}\int \frac{1}{t}\mathbb{P}^x_{t,x}[e^{-\langle l,\chi\rangle}, dl]\,dt = \sum_{x\in S}\int \frac{1}{t}\mathbb{P}^x_t[e^{-\langle l,\chi\rangle}1_{\{l(t)=x\}}, dl]\,dt$$
$$= \sum_{x\in S}\int \frac{1}{t}\mathbb{P}^x_{(\chi)\,t}[1_{\{l(t)=x\}}, dl]\,dt = \mu(L-M_\chi,dl).$$

□

Proposition 3.4.2. *Suppose* $f: S^n\to\mathbb{R}$ *is positive measurable, then*

$$\mu(\langle l,f\rangle) = \sum_{(y_1,\ldots,y_n)\in S^n} V^{y_1}_{y_2}\cdots V^{y_n}_{y_1} f(y_1,\ldots,y_n).$$

Proof.

$$\mu(\langle l,f\rangle) = \int_0^\infty \frac{1}{t}\sum_{x\in S}\mathbb{P}^x_{t,x}[\sum_{j=0}^{n-1}\int_{0<s^1<\cdots<s^n<t} f\circ r_j(l(s^1),\ldots,l(s^n))\prod_{i=1}^n ds^i]\,dt$$
$$= \int_{0<s^1<\cdots<s^n<t<\infty} \sum_{j=1}^n \sum_{(x,x_1,\ldots,x_n)\in S^{n+1}} ds^1\cdots ds^n\,dt$$
$$\frac{1}{t} f\circ r_j(x_1,\ldots,x_n)(P_{s^1})^x_{x_1}(P_{s^2-s^1})^{x_2}_{x_1}\cdots(P_{t-s^n})^{x_n}_x$$

$$= \int_{0<s^1<\cdots<s^n<t<\infty} \sum_{j=1}^{n} \sum_{(x_1,\ldots,x_n)\in S^n} ds^1 \cdots ds^n \, dt$$
$$\frac{1}{t} f \circ r_j(x_1, \ldots, x_n) (P_{s^2-s^1})^{x_1}_{x_2} (P_{s^3-s^2})^{x_2}_{x_3} \cdots (P_{t-s^n+s^1})^{x_n}_{x_1}.$$

Performing the change of variables $a^0 = s^1, a^1 = s^2 - s^1, \ldots, a^{n-1} = s^n - s^{n-1}, a^n = t - s^n + s^1$,

$$\mu(\langle l, f\rangle) = \int_{a^0,\ldots,a^n>0, a^n>a^0} da^0 \cdots da^n \sum_{(x_1,\ldots,x_n)\in S^n} \sum_{j=0}^{n-1} \frac{1}{a^1 + \cdots + a^n}$$
$$f \circ r_j(x_1, \ldots, x_n)(P_{a^1})^{x_1}_{x_2} \cdots (P_{a^n})^{x_n}_{x_1}$$
$$= \int_{a^1,\ldots,a^n>0} da^1 \cdots da^n \sum_{(x_1,\ldots,x_n)\in S^n} \sum_{j=0}^{n-1} \frac{a_n}{a^1 + \cdots + a^n}$$
$$f \circ r_j(x_1, \ldots, x_n)(P_{a^1})^{x_1}_{x_2} \cdots (P_{a^n})^{x_n}_{x_1}.$$

Changing again variables with $b^{1+j} = a^1, \ldots, b^n = a^{n-j}, b^1 = a^{n-j+1}, \ldots, b^j = a^n$ and $y_{1+j} = x_1, \ldots, y_n = x_{n-j}, y_1 = x_{n-j+1}, \ldots, y_j = x_n$, and summing the integrals for all j,

$$\langle l, f\rangle = \int_{b^1,\ldots,b^n>0} \sum_{(y_1,\ldots,y_n)\in S^n} (P_{b^1})^{y_1}_{y_2} \cdots (P_{b^n})^{y_n}_{y_1} f(y_1, \ldots, y_n) \, db^1 \cdots db^n$$
$$= \sum_{(y_1,\ldots,y_n)\in S^n} V^{y_1}_{y_2} \cdots V^{y_n}_{y_1} f(y_1, \ldots, y_n).$$

□

Define $\tilde{\mathfrak{S}}_{n,m} \subset \mathfrak{S}_{n+m}$ to be the collection of permutations σ on $\{1, \ldots, n+m\}$ such that the order of $1, \ldots, n$ and $n+1, \ldots, n+m$ is preserved under the permutation σ respectively, i.e.

$$\tilde{\mathfrak{S}}_{n,m} = \{\sigma \in \mathfrak{S}_{n+m} : \sigma(1) < \cdots < \sigma(n) \text{ and } \sigma(n+1) < \cdots < \sigma(n+m)\}.$$

Define $\mathfrak{S}^1_{n,m} = \{\sigma \in \mathfrak{S}_{n,m}; \sigma(1) = 1\}$. Then, we have $\sigma(1) < \cdots < \sigma(n)$ for $\sigma \in \mathfrak{S}^1_{n,m}$.

Proposition 3.4.3 (Shuffle product)**.** *Suppose $f : S^n \to \mathbb{R}, g : S^m \to \mathbb{R}$ bounded or positive and measurable. Then,*

$$\langle l, f\rangle\langle l, g\rangle = \sum_{j=0}^{m-1} \sum_{\sigma\in\tilde{\mathfrak{S}}_{n,m}} \langle l, (f \otimes (g \circ r_j)) \circ \sigma^{-1}\rangle.$$

Proof. Let t be the length of l.

$$\langle l, f\rangle\langle l, g\rangle = \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} \int_{0<u^1<\cdots<u^n<t} f \circ r_j(l(u^1), \ldots, l(u^n)) \, du^1 \cdots du^n$$

$$\int_{0<v^1<\cdots<v^m<t} g\circ r_k(l(v^1),\dots,l(v^m))\,dv^1\cdots dv^m$$
$$=\sum_{j=0}^{n-1}\sum_{k=0}^{m-1}\int_{\substack{0<u^1<\cdots<u^n<t\\0<v^1<\cdots<v^m<t}} f\circ r_j(l(u^1),\dots,l(u^n))$$
$$g\circ r_k(l(v^1),\dots,l(v^m))\,du^1\cdots du^n\,dv^1\cdots dv^m.$$

Let $w=(u^1,\dots,u^n,v^1,\dots,v^m)$. Almost surely, $u^1<\cdots<u^n,v^1<\cdots<v^m$ are different from each other. Let $s=(s^1,\dots,s^{m+n})$ be the rearrangement of w in increasing order. Almost surely, for each w, there exists a unique $\sigma\in\tilde{\mathfrak{S}}_{n,m}$ such that $s=\sigma(w)$. We change w by $\sigma^{-1}(s)$,

$$\begin{aligned}\langle l,f\rangle\langle l,g\rangle &=\sum_{j=0}^{n-1}\sum_{k=0}^{m-1}\sum_{\sigma\in\tilde{\mathfrak{S}}_{n,m}}\int_{0<s^1<\cdots<s^{n+m}<t} ds^1\cdots ds^{n+m}\\ &\quad (f\circ r_j)\otimes(g\circ r_k)\circ\sigma^{-1}(l(s^1),\dots,l(s^{m+n}))\\ &=\sum_{\sigma\in\mathfrak{S}_{n,m}}\int_{0<s^1<\cdots<s^{n+m}<t}(f\otimes g)\circ\sigma^{-1}(l(s^1),\dots,l(s^{m+n}))\,ds^1\cdots ds^{n+m}\\ &=\sum_{\substack{\sigma\in\mathfrak{S}^1_{n,m}\\ r\in R}}\int_{0<s^1<\cdots<s^{n+m}<t}(f\otimes g)\circ r\circ\sigma^{-1}(l(s^1),\dots,l(s^{m+n}))\,ds^1\cdots ds^{n+m}\\ &=\sum_{\sigma\in\mathfrak{S}^1_{n,m}}\langle l,(f\otimes g)\circ\sigma^{-1}\rangle\\ &=\sum_{j=0}^{m-1}\sum_{\sigma\in\tilde{\mathfrak{S}}_{n,m}}\langle l,(f\otimes(g\circ r_j))\circ\sigma^{-1}\rangle.\end{aligned}$$

□

Corollary 3.4.4.

$$\mu(l^{x_1}\cdots l^{x_n})=\frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n}V^{x_{\sigma_1}}_{x_{\sigma_2}}\cdots V^{x_{\sigma_n}}_{x_{\sigma_1}}.$$

Proof.

$$l^{x_1}\cdots l^{x_n}=\prod_{i=1}^n\int_0^{|l|}1_{\{l(t_i)=x_i\}}\,dt_i=\int_0^{|l|}\prod_{i=1}^n 1_{\{l(t_i)=x_i\}}\,dt_i.$$

In the above expression, almost surely, one can write $t_1,\dots,t_n$ in increasing order, $s_1=t_{\sigma(1)}<\cdots<s_n=t_{\sigma(n)}$ for a unique $\sigma\in\mathfrak{S}_n$. Then,

$$\int_0^{|l|}\prod_{i=1}^n 1_{\{l(t_i)=x_i\}}\,dt_i=\sum_{\sigma\in\mathfrak{S}_n}\int_{0<t_{\sigma(1)}<\cdots<t_{\sigma(n)}<|l|}\prod_{i=1}^n 1_{\{l(t_{\sigma(i)})=x_{\sigma(i)}\}}\,dt_{\sigma(i)}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} \int_{0<s_1<\cdots<s_n<|l|} \prod_{i=1}^{n} 1_{\{l(s_i)=x_{\sigma(i)}\}}\, ds_i.$$

Since $\mathfrak{S}_n r_j = \mathfrak{S}_n$ for all $j = 1, \ldots, n$, the above expression equals to

$$\sum_{\sigma \in \mathfrak{S}_n} \int_{0<s_1<\cdots<s_n<|l|} \prod_{i=1}^{n} 1_{\{l(s_i)=x_{\sigma(i+j)}\}}\, ds_i.$$

for all $j = 1, \ldots, n$. Finally,

$$\begin{aligned} l^{x_1} \cdots l^{x_n} &= \frac{1}{n} \sum_{j=1}^{n} \sum_{\sigma \in \mathfrak{S}_n} \int_{0<s_1<\cdots<s_n<|l|} \prod_{i=1}^{n} 1_{\{l(s_i)=x_{\sigma(i+j)}\}}\, ds_i \\ &= \frac{1}{n} \sum_{\sigma \in \mathfrak{S}_n} l^{x_{\sigma(1)},\ldots,x_{\sigma(n)}}. \end{aligned}$$

Then, by Proposition 3.4.2, we are done. □

Corollary 3.4.5. *The linear space generated by all the multi-occupation fields is an algebra.*

Proof. By shuffle product, the operation of multiplication is closed. □

Theorem 3.4.6 (Blackwell's theorem, [DM78])**.** *Suppose $(E, \mathcal{E})$ is a Blackwell space, $\mathcal{S}, \mathcal{F}$ are sub-σ-field of $\mathcal{E}$ and $\mathcal{S}$ is separable. Then $\mathcal{F} \subset \mathcal{S}$ iff every atom of $\mathcal{F}$ is a union of atoms of $\mathcal{S}$.*

Theorem 3.4.7. *The family of all multi-occupation fields generates the Borel-σ-field on the loops.*

Lemma 3.4.8. *Suppose $(E, \mathcal{B}(E))$ is a Polish space with the Borel-σ-field. Let $\{f_i, i \in \mathbb{N}\}$ be measurable functions and denote $\mathcal{F} = \sigma(f_i, i \in \mathbb{N})$. Then, $\mathcal{F} = \mathcal{B}(E)$ iff for all $x \neq y \in E$, there exists f_i such that $f_i(x) \neq f_i(y)$.*

Proof. Since E is Polish, $\mathcal{B}(E)$ is separable and $(E, \mathcal{B}(E))$ is Blackwell space. The atoms of $\mathcal{B}(E)$ are all the one point sets. Obviously, $\mathcal{F} \subset \mathcal{B}(E)$ and $\mathcal{F}$ is separable. By Blackwell's theorem, $\mathcal{F} = \mathcal{B}(E)$ iff. the atoms of $\mathcal{F}$ are all the one point sets which is equivalent to the following: for all $x \neq y \in E$, there exists f_i such that $f_i(x) \neq f_i(y)$. □

Proof for Theorem 3.4.7. By Lemma 3.4.8 and the fact that

$$\{l^{x_1,\ldots,x_m} : m \in \mathbb{N}_+, (x_1, \ldots, x_m) \in S^m\}$$

is countable, it is sufficient to show that given all the multi-occupation fields of the loop l, the loop is uniquely determined.

Note first that the length of the loop can be recovered from the occupation field as $|l| = \sum_{x\in S} l^x$. Let $J(l) = \max\{n \in \mathbb{N} : \exists (x_1, \ldots, x_n) \in S^n$ such that $x_i \neq x_{i+1}$ for $i = 1, \ldots, n-1, x_1 \neq x_n$ and $l^{x_1,\ldots,x_n} > 0\}$, the total number of the jumps in the loop l. Define $D(l)$ to be the set of discrete pointed loop such that $l^{x_1,\ldots,x_{J(l)}} > 0$. As a discrete loop is viewed as an equivalent class of discrete pointed loop, it appears that $D(l)$ is actually the discrete loop l^d. A loop is defined by the discrete loop with the corresponding holding times. It remains to show that the corresponding holding times can be recovered from the multi-occupation field. Suppose we know that the multiplicity of the discrete loop $n(l^d) = n$, the length of the discrete loop $J(l) = qn$ and that $(x_1, \ldots, x_q, \ldots, x_1, \ldots, x_q) \in D(l)$ is a pointed loop representing l^d. Then the loop l can be written in the following form:

$$(x_1^1, \tau_1^1, \ldots, x_q^1, \tau_q^1, \ldots, x_1^n, \tau_1^n, \ldots, x_q^n, \tau_q^n)^o$$

with $x_j^i = x_j, i = 1, \ldots, q$ and $(\tau_1^1, \ldots, \tau_q^1) \geq \cdots \geq (\tau_1^n, \ldots, \tau_q^n)$ in the lexicographical order. For $k \in M_{n\times q}(\mathbb{N}_+)$ a n by q matrix, define $y(k) \in S^{\sum_{i,j} k_j^i}$ as follows

$$y(k) = (\underbrace{x_1^1, \ldots, x_1^1}_{k_1^1 \text{ times}}, \underbrace{x_2^1, \ldots, x_2^1}_{k_2^1 \text{ times}}, \ldots, \underbrace{x_q^n, \ldots, x_q^n}_{k_q^n \text{ times}}).$$

Define $k! = \prod_{i,j} k_j^i!$. Define $K^i = (k_1^i, \ldots, k_q^i)$ for $i = 1, \ldots, n$. Define $\tau^i = (\tau_1^i, \ldots, \tau_q^i)$ for $i = 1, \ldots, n$. For $K \in \mathbb{N}^q$ and $t \in \mathbb{R}^q$, define the polynomial $f^K(t) = \prod_{j=1}^q (t_i)^{K_j}$. We have the following expression,

$$l^{y(k)} = \frac{1}{k!}\sum_{i=1}^n (f^{K^1} \otimes \cdots \otimes f^{K^n}) \circ r_i(\tau^1, \ldots, \tau^n)$$

where $r_i(\tau^1, \ldots, \tau^n) = (\tau^{n-i+1}, \ldots, \tau^n, \tau^1, \ldots, \tau^{n-i})$. All the holding times are bounded by the length $|l|$ of the loop. By the theorem of Weierstrass, for any continuous function f on $(\mathbb{R}^q)^n$, the following quantity is determined by the family of occupation fields:

$$\sum_{i=1}^n f \circ r_i(\tau^1, \ldots, \tau^n).$$

As a consequence, $\sum_{i=1}^n \delta_{r_i(\tau^1,\ldots,\tau^n)}$ is uniquely determined. Since we order $\tau^1 \geq \cdots \geq \tau^n$ in the lexicographical order, $(\tau^1, \ldots, \tau^n)$ is uniquely determined. Finally, the loop l is determined by the family of the multi-occupation fields of l and we are done. $\square$

Proposition 3.4.9. *In the transient case, the Markovian loop measure is determined by the expectations of the multi-occupation field* $\{V^{x_1}_{x_2}\cdots V^{x_n}_{x_1}\}$.

Proof. Suppose F_n is a sequence of subset of S such that $\lim\limits_{n\to\infty} F_n = S$. For any loop l, let l_n be the trace of l on the subset F_n. Then, $\lim\limits_{n\to\infty} l_n = l$ in the sense of Skorokhod. Once the laws of l_n are determined, the Markovian loop measure is determined, too. Moreover, the law of l_n is Markovian loop measure, too. It is defined by the Markov process with potential $V|_{F_n\times F_n}$. Therefore, the expectations of the multi-occupation field for l_n is determined. As a consequence, it is enough to prove this proposition under the assumption that $|S|<\infty$. Now, suppose $|S|=N$. Recall that

$$\mu^{p*}(p(\xi)=k,\xi_1=x_1,\ldots,\xi_k=x_k,\tau_1\in dt^1,\ldots,\tau_k\in dt^k)$$
$$= Q^{x_1}_{x_2}\cdots Q^{x_k}_{x_1}L^{x_1}_{x_1}e^{-L^{x_1}_{x_1}t_1}\,dt^1\cdots L^{x_k}_{x_k}e^{-L^{x_k}_{x_k}t^k}\,dt^k.$$

So it is enough to prove $L^{x_1}_{x_2}\cdots L^{x_n}_{x_1}$ is uniquely determined.
Since one knows $\{V^{x_1}_{x_2}\cdots V^{x_n}_{x_1}\}$, one knows all the determinant of all the major sub-matrix $(V^x_y)_{x,y\in F\subset S}$. So one knows all the coefficient of the characteristic polynomial $p(\lambda)=\det(\lambda I - V)$. Cayley-Hamilton theorem ensures that $p(V)=0$. Let $q(\lambda)=\frac{p(0)-p(\lambda)}{p(0)\lambda}$, $p(0)=\det(-V)\neq 0$. Obviously, $q(0)=0$. $-L=V^{-1}=q(V)$ where q is determined by $\{V^{x_1}_{x_2}\cdots V^{x_n}_{x_1}, n\geq 1\}$. Besides, $(V^{k_1})^{x_1}_{x_2}\cdots(V^{k_n})^{x_n}_{x_1}(k_1,\ldots,k_n>0)$ is determined for $n\geq 1$ by the expectations of the multi-occupation field. Finally, $L^{x_1}_{x_2}\cdots L^{x_n}_{x_1}=(q(V))^{x_1}_{x_2}\cdots(q(V))^{x_n}_{x_1}$ is uniquely determined. □

3.5 The occupation field in the transient case

Assumption: Throughout this section, assume we are in the transient case.

Proposition 3.5.1. *Suppose χ is a non-negative function on S with compact support F. Let $\rho(M_{\sqrt{\chi}}VM_{\sqrt{\chi}})$ be the spectral radius of $M_{\sqrt{\chi}}VM_{\sqrt{\chi}}$. Then, for $z\in D=\{z\in\mathbb{C}:\mathrm{Re}(z)<\frac{1}{\rho(M_{\sqrt{\chi}}VM_{\sqrt{\chi}})}\}$, the following equation holds:*

$$\mu(e^{z\langle l,\chi\rangle}-1)=-\ln\det(I-zM_{\sqrt{\chi}}VM_{\sqrt{\chi}}).$$

Outside of D, $\mu(|e^{z\langle l,\chi\rangle}-1|)=\infty$.

Proof. Suppose $n=|\operatorname{supp}(\chi)|$ and $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of $M_{\sqrt{\chi}}VM_{\sqrt{\chi}}$ ordered in the sense of non-increasing module. Then, $|\lambda_1|=\rho(M_{\sqrt{\chi}}VM_{\sqrt{\chi}})$. By Corollary 3.4.4 and

Proposition 3.4.2,

$$\mu(\langle l,\chi\rangle^m) = (m-1)! \sum_{(x^1,\dots,x^m)\in S^m} V_{y_2}^{y_1}\cdots V_{y_1}^{y_m}\chi(y_1)\cdots\chi(y_m) = (m-1)!\operatorname{Tr}((M_{\sqrt{\chi}}VM_{\sqrt{\chi}})^m).$$

We have:

$$e^z = 1+z+\cdots+\frac{z^n}{n!}+z^{n+1}\int_{0<s_1<\cdots<s_{n+1}<1} e^{s_1 z}\,ds_1\cdots ds_{n+1}.$$

Therefore

$$|e^{z+h}-1-z-\cdots-\frac{z^n}{n!}| \le e^{\max(\operatorname{Re}(z),0)}\frac{|z|^{n+1}}{(n+1)!}.$$

In particular, $|e^x-1|\le e^{\max(\operatorname{Re}(x),0)}|x|$ and $|e^x-1-x|\le e^{\max(\operatorname{Re}(x),0)}\frac{|x|^2}{2}$.
For $z\in\mathbb{C}$ such that $\operatorname{Re}(z)<1/\rho(M_{\sqrt{\chi}}VM_{\sqrt{\chi}})=1/|\lambda_1|$, let $b=\max(\operatorname{Re}(z),0)$,

$$\begin{aligned}\mu(|e^{z\langle l,\chi\rangle}-1|) \le&\mu(e^{b\langle l,\chi\rangle}|z|\langle l,\chi\rangle) = \mu(\sum_{m=0}^{\infty}\frac{|z|b^m\langle l,\chi\rangle^{m+1}}{m!})\\ =&\sum_{m=0}^{\infty}\frac{|z|b^m\mu(\langle l,\chi\rangle^{m+1})}{m!} = \sum_{m=0}^{\infty}|z|b^m\operatorname{Tr}((M_{\sqrt{\chi}}VM_{\sqrt{\chi}})^{m+1})\\ \le&|z||\operatorname{supp}(\chi)|\sum_{m=0}^{\infty}b^m|(\rho(M_{\sqrt{\chi}}VM_{\sqrt{\chi}}))^{m+1}<\infty\end{aligned}$$

Consequently, $\Phi(z)=\mu(e^{z\langle l,\chi\rangle}-1)$ is well-defined for $z\in D$. Next, we will show that $\Phi(z)$ is analytic in D. Fix $z_0\in D$, take h small enough that $z_0+h\in D$. By an argument very similar to the above one, we have that $\mu(e^{z_0\langle l,\chi\rangle}\langle l,\chi\rangle)$ and $\mu(e^{(\operatorname{Re}(z_0)+\max(\operatorname{Re}(h),0))\langle l,\chi\rangle}\frac{\langle l,\chi\rangle^2}{2})$ are well-defined and finite.

$$\begin{aligned}&|\Phi(z_0+h)-\Phi(z_0)-h\mu(e^{z_0\langle l,\chi\rangle}\langle l,\chi\rangle)|\\ &\qquad=\mu(|e^{z_0\langle l,\chi\rangle}(e^{h\langle l,\chi\rangle}-1-h\langle l,\chi\rangle)|)\\ &\qquad\le\mu(e^{\operatorname{Re}(z_0)\langle l,\chi\rangle}e^{\max(\operatorname{Re}(h),0)\langle l,\chi\rangle}\frac{h^2\langle l,\chi\rangle^2}{2})=O(h^2).\end{aligned}$$

Finally, by dominated convergence, for $|z|<1/\rho(M_{\sqrt{\chi}}VM_{\sqrt{\chi}})$,

$$\Phi(z)=\sum_{n\ge1}\frac{\operatorname{Tr}((M_{\sqrt{\chi}}VM_{\sqrt{\chi}})^n)}{n} = -\ln\det(1-zM_{\sqrt{\chi}}VM_{\sqrt{\chi}}).$$

Since $\Phi(z)$ is analytic in $D=\{z\in\mathbb{C}:\operatorname{Re}(z)<1/\rho(M_{\sqrt{\chi}}VM_{\sqrt{\chi}})\}$, $\Phi(z)$ is the unique analytic continuation of $-\ln\det(1-zM_{\sqrt{\chi}}VM_{\sqrt{\chi}})$ in D.

$\ln\det(I-zM_{\sqrt{\chi}}VM_{\sqrt{\chi}})$ cannot be defined on $\mathbb{C}$ as an analytic function. Nevertheless, after cutting down several half lines starting from $1/\lambda_1,\dots,1/\lambda_n$, it is analytic and equals

$-\sum_{i=1}^{n}\ln(1-z\lambda_i)$. Moreover, when z converges to some λ_i, $|-\ln\det(I-zM_{\sqrt{\chi}}VM_{\sqrt{\chi}})|$ tends to infinity. But we have showed that $\mu(e^{z\langle l,\chi\rangle}-1) = -\ln\det(I-zM_{\sqrt{\chi}}VM_{\sqrt{\chi}})$ is well-defined as an analytic function on D. Consequently, $1/\lambda_1,\ldots,1/\lambda_n$ lie in D^c, i.e. $\mathrm{Re}(\frac{1}{\lambda_i}) \geq \frac{1}{\rho(M_{\sqrt{\chi}}GM_{\sqrt{\chi}})} = \frac{1}{|\lambda_1|}$. In particular, $\lambda_1 = \rho(M_{\sqrt{\chi}}GM_{\sqrt{\chi}})$. For $x \geq \frac{1}{\rho(M_{\sqrt{\chi}}GM_{\sqrt{\chi}})}$,

$$\mu(|e^{x\langle l,\chi\rangle}-1|) = \mu(e^{x\langle l,\chi\rangle}-1) \geq \mu(e^{\frac{1}{\rho(M_{\sqrt{\chi}}GM_{\sqrt{\chi}})}\langle l,\chi\rangle}-1).$$

By monotone convergence,

$$\begin{aligned}\mu(e^{\frac{1}{\rho(M_{\sqrt{\chi}}GM_{\sqrt{\chi}})}\langle l,\chi\rangle}-1) &= \lim_{y\uparrow\lambda_1}\mu(e^{y\langle\chi,l\rangle}-1) = \lim_{y\uparrow\lambda_1}-\ln\det(I-yM_\chi VM_\chi)\\ &= \lim_{y\uparrow\lambda_1}|-\ln\det(I-yM_\chi VM_\chi)| = \infty.\end{aligned}$$

Consequently, for $x \geq \frac{1}{\rho(M_{\sqrt{\chi}}GM_{\sqrt{\chi}})}$, $\mu(|e^{x\langle l,\chi\rangle}-1|) = \infty$. For all $y \in \mathbb{R}$, $\mu(|e^{iy\langle l,\chi\rangle}-1|) < \infty$. Therefore, by the triangular inequality, for $z = x+iy \notin D$,

$$\begin{aligned}\mu(|e^{z\langle l,\chi\rangle}-1|) \geq& |\mu(|e^{z\langle l,\chi\rangle}-e^{iy\langle l,\chi\rangle}|)-\mu(|e^{iy\langle l,\chi\rangle}-1|)|\\ =& |\mu(|e^{x\langle l,\chi\rangle}-1|)-\mu(|e^{iy\langle l,\chi\rangle}-1|)| = \infty.\end{aligned}$$

□

Lemma 3.5.2. *Suppose χ is a finitely supported non-negative function on S and F contains the support of χ. Then,*

$$\begin{aligned}\frac{\det(V_F)}{\det((V_\chi)_F)} &= \det(I+(M_\chi)_F V_F) = \det(I+M_{\sqrt{\chi}}VM_{\sqrt{\chi}})\\ &= \begin{vmatrix} I_F & V_F \\ -(M_\chi)_F & I_F\end{vmatrix} = 1+\sum_{A\subset F, A\neq\phi}\prod_{x\in A}\chi(x)V_A.\end{aligned}$$

Proof. By the resolvent equation, we have $V_F = (V_F)_\chi + (V_F)_\chi(M_\chi)_F V_F$. By Proposition 2.3.2, we have $(V_\chi)_F = (V_F)_\chi$. Combining these two results, we have $V_F = (V_\chi)_F + (V_\chi)_F(M_\chi)_F V_F$. Consequently,

$$\frac{\det(V_F)}{\det((V_\chi)_F)} = \det(I+(M_\chi)_F V_F).$$

The last equality follows from simple calculations in linear algebra. □

Corollary 3.5.3. *For non-negative χ not necessarily finitely supported,*

$$e^{\mu(1-e^{-\langle l,\chi\rangle})} = 1+\sum_{F\subset S, 0<|F|<\infty}\prod_{x\in F}\chi(x)\det(V_F)$$

Proof. For χ a non-negative finitely supported function, by Proposition 3.5.1 with Lemma 3.5.2,

$$\begin{aligned} e^{\mu(1-e^{-\langle l,\chi\rangle})} &= \det(I + M_{\sqrt{\chi}} V M_{\sqrt{\chi}}) = \det(I + (M_\chi)_{\mathrm{supp}(\chi)} V_{\mathrm{supp}(\chi)}) \\ &= 1 + \sum_{F \subset \mathrm{supp}(\chi), F \neq \phi} (\prod_{x \in F} \chi(x)) \det(V_F). \end{aligned}$$

The trace of the Markov process on F has the potential V_F and generator $\tilde{L}$. Since $\det(-\tilde{L}) > 0$ and $(-\tilde{L})V_F = Id$, $\det(V_F) > 0$. Finally, the result comes from monotone convergence theorem. □

Corollary 3.5.4. *For $a \geq 0$, let $\chi = a\delta_x$, then $\mu(1 - e^{-al^x}) = \ln(1 + aV^x_x)$. As a result, $\mu(l^x \in dt) = \frac{1}{t} e^{-t/V^x_x}\, dt$ for $t > 0$.*

Proposition 3.5.5. *For non-negative function χ,*

$$\mu(1_{\{l \text{ is trivial}\}}(1 - e^{-\langle l,\chi\rangle})) = \ln(\prod_{x \in S} \frac{\chi(x) - L^x_x}{-L^x_x})$$

$$\mu(1_{\{l \text{ is non-trivial}\}}(1 - e^{-\langle l,\chi\rangle})) = \ln(I + M_{\sqrt{\chi}} V M_{\sqrt{\chi}}) + \ln(\prod_{x \in S} \frac{-L^x_x}{\chi(x) - L^x_x}).$$

Proof. Since $\mu(p(\xi) = 1, \xi_1 = x, \tau_1 \in dt^1) = \frac{e^{L^x_x t^1}}{t^1} dt^1$,

$$\mu(1_{\{l \text{ is trivial}\}}(1 - e^{-\langle l,\chi\rangle})) = \sum_{x \in S} \int_0^\infty \frac{e^{L^x_x t^1}}{t^1}(1 - e^{-\chi(x)t^1}) dt^1 = \ln(\prod_{x \in S} \frac{\chi(x) - L^x_x}{-L^x_x}).$$

Combining with Proposition 3.5.1, we have

$$\begin{aligned} &\mu(1_{\{l \text{ is non-trivial}\}}(1 - e^{-\langle l,\chi\rangle})) \\ &\quad = \mu(1 - e^{-\langle l,\chi\rangle}) - \mu(1_{\{l \text{ is trivial}\}}(1 - e^{-\langle l,\chi\rangle})) \\ &\quad = \ln(\det(I + M_{\sqrt{\chi}} V M_{\sqrt{\chi}})) + \ln(\prod_{x \in S} \frac{-L^x_x}{\chi(x) - L^x_x}). \end{aligned}$$

□

Proposition 3.5.6. *If $\chi_1, \ldots, \chi_n$ are finitely supported non-negative functions on S, and for A a subset of $\{1, \ldots, n\}$ we set $\chi_A = \sum_{i \in A} \chi_i$, then for $n \geq 2$,*

$$\mu(\prod_{i=1}^n (1 - e^{-\langle l,\chi_i\rangle})) = -\sum_{A \subset \{1,\ldots,n\}} (-1)^{|A|} \ln\det(I + M_{\sqrt{\chi_A}} V M_{\sqrt{\chi_A}});$$

$$\mu(1_{\{l \text{ is trivial}\}} \prod_{i=1}^n (1 - e^{-\langle l,\chi_i\rangle})) = -\sum_{A \subset \{1,\ldots,n\}} (-1)^{|A|} \ln(\prod_{x \in F_A} \frac{-L^x_x + \chi_A(x)}{-L^x_x}).$$

Proof. We see that

$$\prod_{i=1}^{n}(1-e^{-\langle l,\chi_i\rangle}) = -\sum_{A\subset\{1,\ldots,n\}}(-1)^{|A|}(1-e^{-\langle l,\chi_A\rangle}).$$

Therefore,

$$\begin{aligned}\mu(\prod_{i=1}^{n}(1-e^{-\langle l,\chi_i\rangle})) &= -\sum_{A\subset\{1,\ldots,n\}}(-1)^{|A|}\mu(1-e^{-\langle l,\chi_A\rangle})\\ &= -\sum_{A\subset\{1,\ldots,n\}}(-1)^{|A|}\ln\det(I+M_{\sqrt{\chi_A}}VM_{\sqrt{\chi_A}}).\end{aligned}$$

The last equality is deduced from Proposition 3.5.1. By a similar method and Proposition 3.5.5, we get the following expression for the trivial loops:

$$\mu(1_{\{l \text{ is trivial}\}}\prod_{i=1}^{n}(1-e^{-\langle l,\chi_i\rangle})) = -\sum_{A\subset\{1,\ldots,n\}}(-1)^{|A|}\ln(\prod_{x\in F_A}\frac{-L_x^x+\chi_A(x)}{-L_x^x}).$$

□

Proposition 3.5.7. *For a finite subset $F\subset S$,*

$$\mu(l \textit{ is non-trivial and } l \textit{ visits } F) = \ln(\prod_{x\in F}(-L_x^x)\det(V_F)).$$

Proof. By Proposition 3.5.5,

$$\mu(1_{\{l \text{ is non-trivial}\}}(1-e^{-\langle l,t1_F\rangle})) = \ln\det(I+M_{\sqrt{t1_F}}VM_{\sqrt{t1_F}})+\ln(\prod_{x\in F}\frac{-L_x^x}{t-L_x^x}).$$

By Lemma 3.5.2, we have

$$\ln\det(I+M_{\sqrt{t1_F}}VM_{\sqrt{t1_F}}) = \ln(1+\sum_{A\subset F,A\neq\phi}t^{|A|}V_A).$$

Take $t\to\infty$, we have

$$\mu(1_{\{l \text{ is non-trivial and } l \text{ visits } F\}}) = \ln(\prod_{x\in F}(-L_x^x)\det(V_F)).$$

□

Similarly, one has the following property.

Proposition 3.5.8. *Suppose we are given $n\geq 2$ finite subset $F_1,\ldots,F_n$. For any subset $A\subset\{1,\ldots,n\}$, define $F_A=\bigcup_{i\in A}F_i$. Then,*

$$\begin{aligned}&\mu(l \textit{ is not trivial and it visits all } F_i \textit{ for } i=1,\ldots,n)\\ &\qquad = -\sum_{A\subset\{1,\ldots,n\},A\neq\phi}(-1)^{|A|}\ln\det(V_{F_A})+\sum_{x\in\bigcap_{i=1}^{n}F_i}\ln(-L_x^x).\end{aligned}$$

Proof. By Proposition 3.5.6, take $\chi_i = t1_{F_i}$:

$$\mu(1_{\{l \text{ is non-trivial}\}}\prod_{i=1}^{n}(1-e^{-\langle l,t1_{F_i}\rangle}))$$
$$= -\sum_{A\subset\{1,\ldots,n\},A\neq\phi}(-1)^{|A|}\ln\det(I+M_{\sqrt{\chi_A}}VM_{\sqrt{\chi_A}})$$
$$+\sum_{A\subset\{1,\ldots,n\},A\neq\phi}(-1)^{|A|}\ln(\prod_{x\in F_A}\frac{-L_x^x+\chi_A(x)}{-L_x^x}).$$

where $\chi_A = \sum_{i\in A} t1_{F_i}$. By Lemma 3.5.2, for A non-empty,

$$\begin{aligned}\det(I+M_{\sqrt{\chi_A}}VM_{\sqrt{\chi_A}}) =& 1+\sum_{B\subset F_A,B\neq\phi}t^{|B|}(\prod_{x\in B}(\sum_{i\in A}1_{\{x\in F_i\}}))\det(V_B)\\ \sim& t^{|F_A|}(\prod_{x\in F_A}(\sum_{i\in A}1_{\{x\in F_i\}}))\det(V_{F_A}) \text{ as } t\to\infty.\end{aligned}$$

And we have that

$$\prod_{x\in F_A}\frac{-L_x^x+\chi_A(x)}{-L_x^x}\sim t^{|F_A|}\prod_{x\in F_A}\frac{\sum_{i\in A}1_{\{x\in F_i\}}}{L_x^x} \text{ as } t\to\infty.$$

As a result,

$$\lim_{t\to\infty}(-1)^A\left(\ln\det(I+M_{\sqrt{\chi_A}}VM_{\sqrt{\chi_A}})+\ln\left(\prod_{x\in F_A}\frac{-L_x^x+\chi_A(x)}{-L_x^x}\right)\right)$$
$$= -\ln\det(V_{F_A})-\ln(\prod_{x\in F_A}(-L_x^x)).$$

Then,

$$\mu(l \text{ is not trivial and it visits all } F_i \text{ for } i=1,\ldots,n)$$
$$=\lim_{t\to\infty}\mu(1_{\{l \text{ is non-trivial}\}}\prod_{i=1}^{n}(1-e^{-\langle l,t1_{F_i}\rangle}))$$
$$= -\sum_{A\subset\{1,\ldots,n\},A\neq\phi}(-1)^{|A|}\ln\det(V_{F_A})-\sum_{A\subset\{1,\ldots,n\},A\neq\phi}(-1)^{|A|}\ln(\prod_{x\in F_A}(-L_x^x)).$$

Finally, by inclusion-exclusion principle, we have

$$\mu(l \text{ is not trivial and it visits all } F_i \text{ for } i=1,\ldots,n)$$
$$= -\sum_{A\subset\{1,\ldots,n\},A\neq\phi}(-1)^{|A|}\ln\det(V_{F_A})+\sum_{x\in\bigcap_{i=1}^{n}F_i}\ln(-L_x^x).$$

□

Corollary 3.5.9. *For $n \geq 2$ and n different states $x_1, \ldots, x_n$,*

$$\mu(l \textit{ visits each state of } \{x_1, \ldots, x_n\}) = -\sum_{A \subset \{x_1, \ldots, x_n\}, A \neq \phi} (-1)^{|A|} \ln \det(V_A).$$

Definition 3.5.1. For a loop l, let $N(l)$ be the number of different points visited by the loop. That is $N(l) = \sum_{x \in S} 1_{\{l^x > 0\}}$.

Corollary 3.5.10.

$$\mu(N 1_{\{N>1\}}) = \sum_{x \in S} \ln(-L_x^x V_x^x).$$

Proof.

$$\mu(N 1_{\{N>1\}}) = \sum_{x \in S} \mu(l \text{ is non-trivial and } l \text{ visits } x).$$

□

Corollary 3.5.11.

$$\mu(N^2 1_{\{N>1\}}) = \sum_{x,y \in S; x \neq y} \ln(\frac{V_x^x V_y^y}{V_x^x V_y^y - V_y^x V_x^y}) + \sum_{x \in S} \ln(-L_x^x V_x^x).$$

Proof.

$$\mu(N^2 1_{\{N>1\}}) = \sum_{x,y} \mu(l \text{ is non-trivial, } l \text{ visits } x \text{ and } l \text{ visits } y).$$

□

Consider the Laguerre-type polynomial L_k with generating function

$$e^{\frac{ut}{1+t}} - 1 = \sum_1^\infty t^k L_k(u).$$

Lemma 3.5.12.

$$\sum_1^\infty |t^k L_k(u)| \leq e^{\frac{|ut|}{1-|t|}} - 1.$$

Proof.

$$\sum_1^\infty t^k L_k(u) = e^{\frac{ut}{1+t}} - 1 = \sum_{k=1}^\infty \left(\frac{ut}{1+t}\right)^k = \sum_{k=1}^\infty u^k t^k (1 - t + t^2 \cdots)^k.$$

Therefore,

$$\sum_1^\infty |t^n L_n(u)| \leq \sum_{n=1}^\infty |u|^n |t|^n (1 + |t| + |t|^2 + \cdots)^n = e^{\frac{|ut|}{1-|t|}} - 1.$$

□

Proposition 3.5.13. *$(\sqrt{k}(V_x^x)^{k/2}L_k\left(\frac{l^x}{V_x^x}\right), k \geq 1)$ are orthonormal in $L^2(\mu)$. More generally,*

$$\mu\left(\sqrt{j}(V_x^x)^{j/2}L_j\left(\frac{l^x}{V_x^x}\right)\sqrt{k}(V_y^y)^{k/2}L_k\left(\frac{l^y}{V_y^y}\right)\right) = \delta_k^j.$$

Proof. $\forall s, t \leq 0$ with $|s|, |t|$ small enough with $\frac{V_x^x s}{1-sV_x^x}, \frac{V_y^y t}{1-tV_y^y} < 1/2$

$$\mu\left(\left(\sum_1^\infty (V_x^x s)^k L_k\left(\frac{l^x}{V_x^x}\right)\right)\left(\sum_1^\infty (V_y^y t)^k L_k\left(\frac{l^y}{V_y^y}\right)\right)\right) = \mu\left((e^{\frac{l^x s}{1+V_x^x s}} - 1)(e^{\frac{l^y t}{1+V_y^y t}} - 1)\right)$$
$$= \mu\left(1 - e^{\frac{l^x s}{1+V_x^x s}}\right) + \mu\left(1 - e^{\frac{l^y t}{1+V_y^y t}}\right) - \mu\left(1 - e^{\frac{l^x s}{1+V_x^x s}+\frac{l^y t}{1+V_y^y t}}\right) = -\ln(1 - stV_y^x V_x^y).$$

Recall that $\mu((l^x)^n) = (n-1)!(V_x^x)^n$. By Lemma 3.5.12,

$$\begin{aligned}\mu((\sum_{k=1}^\infty |(V_x^x s)^k L_k(l^x/V_x^x)|)^2) \leq& \mu\left(\left(e^{\frac{l^x s}{1-V_x^x s}} - 1\right)^2\right) = \sum_{n,m=1}^\infty \frac{1}{n!}\frac{1}{m!}\mu\left(\left(\frac{l^x s}{1-V_x^x s}\right)^{n+m}\right)\\ =& \sum_{n,m=1}^\infty \frac{1}{n+m}\frac{(n+m)!}{n!m!}\left(\frac{V_x^x s}{1-V_x^x s}\right)^{n+m}\\ \leq& \sum_{k\geq 1}\left(\frac{2V_x^x s}{1-V_x^x s}\right)^k < \infty.\end{aligned}$$

Therefore, $(\frac{1}{\sqrt{k}}(V_x^x)^k L_k(\frac{l^x}{V_x^x}), k \geq 1) \in L^2(\mu)$. Moreover, in the equation

$$\mu\left((\sum_1^\infty (V_x^x s)^k L_k(l^x/V_x^x))(\sum_1^\infty (V_y^y t)^k L_k(l^y/V_y^y))\right) = -\ln(1 - stV_y^x V_x^y).$$

we can expand both sides as series of s and t, compare the coefficients and deduce that

$$\mu\left((V_x^x)^j L_j(\frac{l^x}{V_x^x})(V_y^y)^k L_k(\frac{l^y}{V_y^y})\right) = \delta_k^j (V_y^x V_x^y)^k/k.$$

Therefore,

$$\mu\left(\sqrt{j}(V_x^x)^{j/2}L_j(\frac{l^x}{V_x^x})\sqrt{k}(V_y^y)^{k/2}L_k(\frac{l^y}{V_y^y})\right) = \delta_k^j.$$

□

3.6 The recurrent case

Proposition 3.6.1.

$$\mu(l^x \in ds, l^x > 0) = \frac{1}{s}\,ds.$$

Proof.

$$\mu(l^x e^{-pl^x}) = \mu(L - M_{p\delta_x}, l^x) = (V_{p\delta_x})^x_x = 1/p.$$

Therefore, $\mu(l^x \in ds, l^x > 0) = \frac{1}{s} 1_{\{s>0\}}\, ds.$ □

Lemma 3.6.2. *In the irreducible positive-recurrent case, there is a one-to-one correspondence between the semi-group of the Markov process and the Markovian loop measure.*

Proof. It is enough to show the loop measure determines the law of the Markov process. Let π be the invariant probability of the Markov process. Then it is positive everywhere. Define a based loop functional $\phi^b_{t,x,y}$ as follows: for any based loop l with length $|l|$, extend the function $(l(t), t \in [0, |l|])$ periodically, i.e. by setting $l(s + |l|) = l(s)$, and set:

$$\phi^b_{t,x,y}(l) = 1_{\{|l|>t\}} \int_0^{|l|} 1_{\{l(s)=x\}} 1_{\{l(s+t)=y\}}\, ds.$$

This rotation invariant functional defines a loop functional $\phi_{t,x,y}$ on the space of loops.

$$\frac{\mu(|l| \in du, \phi_{t,x,y})}{du} = \frac{1}{u} \sum_{z\in S} \mathbb{P}^z_{u,z}[\int_0^{|l|} 1_{\{l(s)=x\}} 1_{\{l(s+t)=y\}}\, ds] = (P_t)^x_y (P_{u-t})^y_x.$$

Taking u tends to infinity,

$$\lim_{u\to\infty} \frac{\mu(|l| \in du, \phi_{t,x,y})}{du} = \pi_x (P_t)^x_y.$$

Since $\mu(l^x e^{-p|l|}) = \mu(L-p, l^x) = (V_p)^x_x$, $\pi_x = \lim_{p\to 0} p\mu(l^x e^{-p|l|})$. Finally, we are able to determine the semi-group $(P_t)^x_y$ for all x, t, y. Accordingly, the law of the Markov process is uniquely determined. □

Remark 7. From the argument above, we see that an irreducible positive-recurrent semi-group cannot have the same loop measure as another irreducible transient or null-recurrent semi-group.

Finally, we can prove the following theorem.

Theorem 3.6.3. *In the irreducible recurrent case, there is a 1-1 correspondence between the semi-group of the Markov process and the Markovian loop measure.*

Proof. Given a minimal semi-group $(P_t, t \geq 0)$, we can always define the corresponding Markovian loop measure. It is left to show that we can recover the semi-group from the loop measure. Let the series of finite subset $F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots$ exhaust S. By Proposition

3.2.1, we know that the measure of the trace of the Markovian loop on F_i corresponds to the trace the Markov process on F_n. Since $|F_i| < \infty$, the trace of the Markov process on F_i is an irreducible and positive-recurrent Markov process. Let $(P_t^{(n)}, t \geq 0)$ be its semi-group. By Lemma 3.6.2, we can conclude that this trace of the Markov process is determined by the Markovian loop measure. Recall that $Y_t^{(n)}, t \geq 0$, the trace of the Markov process $X_t, t \geq 0$ on F_n is defined as follows:

$A_t^{(n)} = \int\limits_0^t 1_{\{X_s \in F_n\}} \, ds$, $\sigma_t^{(n)}$ is the right-continuous inverse of $A_t^{(n)}, t \geq 0$ and $Y_t^{(n)} = X_{\sigma_t^{(n)}}, t \geq 0$.

As n tends to infinity, $A_t^{(n)}$ increases to t and $\sigma_t^{(n)}$ decreases to t. Since $X_t, t \geq 0$ is right-continuous, $\lim\limits_{n\to\infty} Y_t^{(n)} = X_t$. As a consequence, for any bounded f, $P_t f(x) = \lim\limits_{n\to\infty} P_t^{(n)} f(x)$. Thus, we recover the semi-group P_t as the limit. $\square$

Chapter 4

Poisson process of loops

In this chapter, we study the Poisson point processes naturally defined on the set of Markov loops (which also known as "loop soups"). We mostly focus on the associated occupation fields and on the partitions defined by loop clusters.

4.1 Definitions and some basic properties

Definition 4.1.1. We denote by $\mathcal{L}$ the Poisson point process on $\mathbb{R}_+\times$loops with intensity Lebesgue $\otimes\,\mu$ and by $\mathcal{L}_\alpha$ the Poisson random measure on the space of loops, $\mathcal{L}_\alpha(B) = \mathcal{L}([0,\alpha]\times B)$. Its intensity is $\alpha\mu$.

The following proposition is taken from [Kin93].

Proposition 4.1.1. *Let $\mathcal{P}$ be a Poisson random measure on S with σ-finite intensity measure $\mu(dl)$.*

a) Suppose that Φ is a measurable complex valued function, with $\mu(|\operatorname{Im}(\Phi)|\wedge 1)<\infty$ and $\mu(|e^\Phi-1|)<\infty$, then

$$\mathbb{E}[\exp(\sum_{l\in\mathcal{P}}\Phi(l))] = e^{\alpha\int(e^{\Phi(l)}-1)\mu(dl)}$$

b) The above equation holds if Φ is non-negative measurable without further assumptions.

c) Suppose $F_1,\cdots,F_k$ are non-negative functions, then the following 'Campbell formula' holds

$$\mathbb{E}[\sum_{l_1,\ldots,l_k\in\mathcal{P}\ distinct}\prod_{i=1}^k F_i(l_i)] = \prod_{i=1}^k\mu(F_i)$$

d) Suppose that S,T are two measurable spaces and $\phi: S\to T$ is a measurable mapping. Let $\mathcal{P}$ be a Poisson random measure on S with intensity μ. Then $\phi\circ\mathcal{P}$ is the Poisson random measure on T with intensity $\phi\circ\mu$.

Proof. See [Kin93]. $\square$

From the expression of μ on trivial loops, we get the following:

Proposition 4.1.2. *Let* $\mathcal{L}_{\alpha,\,Trivial,x} = \{l \in \mathcal{L}_\alpha : l$ *is a trivial loop at* $x\}$. *Then,* $\{|l| : l \in \mathcal{L}_{\alpha,\,Trivial,x}\}$ *is a Poisson point measure on* $\mathbb{R}_+$ *with intensity* $\frac{\alpha}{t} e^{L^x_x t}\, dt$.

Recall that a Poisson-Dirichlet distribution has a representation by a Poisson point process, see section 9.4 in [Kin93].

Corollary 4.1.3.

$$\{\frac{\ell^x}{\sum\limits_{l \in \mathcal{L}_{\alpha,\,Trivial,x}} l^x}; \ell \in \mathcal{L}_{\alpha,\,Trivial,x}\}$$

follows a Poisson-Dirichlet $(0,\alpha)$ *distribution. Moreover, it is independent of* $\sum\limits_{l \in \mathcal{L}_{\alpha,\,Trivial,x}} l^x$ *which follows the* $\Gamma(\alpha, (-L^x_x)^{-1})$ *distribution.*

Recall that the density of $\Gamma(\alpha,\beta)$ distribution is $\dfrac{\beta^{-\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}$.

Proof. By Proposition 4.1.1,

$$\begin{aligned}
\mathbb{E}[\exp(-\lambda \sum_{l \in \mathcal{L}_{\alpha,Trivial,x}} l^x)] &= \exp(\int_0^\infty (e^{-\lambda t} - 1) \frac{\alpha}{t} e^{L^x_x t}\, dt) \\
&= \exp(\int_0^\infty \int_0^\infty \alpha (e^{-\lambda t} - 1) e^{-ts} e^{L^x_x t}\, ds\, dt) \\
&= \exp(\int_0^\infty \frac{\alpha}{\lambda + s - L^x_x} - \frac{\alpha}{s - L^x_x}\, ds) = (\frac{-L^x_x}{\lambda - L^x_x})^\alpha
\end{aligned}$$

which is exactly the Laplace transform of the $\Gamma(\alpha, (-L^x_x)^{-1})$ distribution. Therefore, $\sum\limits_{l \in \mathcal{L}_{\alpha,Trivial,x}} l^x$ follows the $\Gamma(\alpha, (-L^x_x)^{-1})$ distribution. $\square$

By taking the trace of the loops on $\{x\}$, we get a Poisson ensemble of Markov loops. To be more precise, we get a Poisson ensemble of trivial loops at x, but its intensity measure, (i.e. the loop measure), is associated with the generator $(L_{\{x\}})^x_x = -1/V^x_x$. As a consequence, we have the following proposition:

Proposition 4.1.4. $\hat{\mathcal{L}}^x_\alpha = \sum\limits_{l \in \mathcal{L}_\alpha} l^x$ *follows a* $\Gamma(\alpha, V^x_x)$ *distribution.* $\{\dfrac{l^x}{\hat{\mathcal{L}}^x_\alpha}, l \in \mathcal{L}_\alpha\}$ *follows a Poisson-Dirichlet distribution* $\Gamma(0,\alpha)$ *which is independent of* $\hat{\mathcal{L}}^x_\alpha$.

Definition 4.1.2. Define $\hat{\mathcal{L}}_\alpha^x = \sum_{l \in \mathcal{L}_\alpha} l^x$ and $\langle \hat{\mathcal{L}}_\alpha, \chi \rangle = \sum_{x \in S} \hat{\mathcal{L}}_\alpha^x \chi(x)$.

Proposition 4.1.5. *For any non-negative measurable χ on S,*

$$\mathbb{E}[e^{-\langle \hat{\mathcal{L}}_\alpha, \chi \rangle}] = (1 + \sum_{A \subset S, 0 < |A| < \infty} \prod_{x \in A} \chi(x) \det(V_A))^{-\alpha}.$$

For any non-negative finitely supported χ on S and $z \in D = \{z \in \mathbb{C} : \mathrm{Re}(z) < \frac{1}{\rho(M_{\sqrt{\chi}} V M_{\sqrt{\chi}})}\}$,

$$\mathbb{E}[e^{z \langle \hat{\mathcal{L}}_\alpha, \chi \rangle}] = (\det(I - z M_{\sqrt{\chi}} V M_{\sqrt{\chi}}))^{-\alpha}.$$

Outside of D, $\mathbb{E}[|e^{z \langle \hat{\mathcal{L}}_\alpha, \chi \rangle}|] = \infty$.

Proof. It is a direct consequence of Proposition 3.5.1, Corollary 3.5.3 and Proposition 4.1.1. $\square$

Proposition 4.1.6. *$\hat{\mathcal{L}}_1^x$ is exponentially distributed with parameter $1/V_x^x$.*

Proof. Since $\mathbb{E}[e^{-p \hat{\mathcal{L}}_1^x}] = \frac{1}{1 + p V_x^x}$ and $\hat{\mathcal{L}}_1^x \geq 0$, $\hat{\mathcal{L}}_1^x$ is exponentially distributed with parameter $1/V_x^x$. $\square$

Remark 8.

$$\mathbb{E}((1 - e^{-\frac{\hat{\mathcal{L}}_\alpha^x}{V_x^x}})^{-1}) = \zeta(\alpha), \alpha > 1.$$

Proof. By Proposition 4.1.5

$$\mathbb{E}((1 - e^{-\frac{\hat{\mathcal{L}}_\alpha^x}{V_x^x}})^{-1}) = \sum_{k=0}^{\infty} \mathbb{E}(e^{-\frac{k}{V_x^x} \hat{\mathcal{L}}_\alpha^x}) = \sum_{k=1}^{\infty} k^{-\alpha} = \zeta(\alpha).$$

$\square$

4.2 Moments and polynomials of the occupation field

Definition 4.2.1 (**α-permanent**)**.** Denote by $m(\sigma)$ the number of cycles in the decomposition of the permutation σ. For any square matrix $A = (A_j^i, i, j = 1, \ldots, n)$, define the α-permanent of A as

$$\mathrm{Per}_\alpha(A) = \sum_{\sigma \in S_n} \alpha^{m(\sigma)} A_{1\sigma(1)} \cdots A_{n\sigma(n)}.$$

Note that $\mathrm{Per}_{-1}(A) = \det(-A)$.

Proposition 4.2.1.

$$\mathbb{E}[\hat{\mathcal{L}}_\alpha^{x_1} \cdots \hat{\mathcal{L}}_\alpha^{x_n}] = \mathrm{Per}_\alpha((V_{x_m}^{x_l})_{1 \leq m, l \leq n}).$$

Proof. Let $F_i(l) = l^{x_i}$. By Corollary 3.4.4,

$$\mu(F_1 \cdots F_k) = \mu(l^{x_1} \cdots l^{x_n}) = \frac{1}{k} \sum_{\sigma \in \mathfrak{S}_k} V_{x_{\sigma(2)}}^{x_{\sigma(1)}} \cdots V_{x_{\sigma(1)}}^{x_{\sigma(k)}} = \sum_{\sigma \in \mathfrak{S}_k, m(\sigma)=1} V_{x_{\sigma(1)}}^{x_1} \cdots V_{x_{\sigma(k)}}^{x_k}.$$

Let $\mathcal{P}(\{1,\ldots,n\})$ be the collection of partitions of $\{1,\ldots,n\}$. For a partition π, we denote by $\#\pi$ the number of blocks in π, $\pi = (\pi_1, \ldots, \pi_{\#\pi})$.

$$\begin{aligned}
\mathbb{E}[\hat{\mathcal{L}}_\alpha^{x_1} \cdots \hat{\mathcal{L}}_\alpha^{x_n}] &= \mathbb{E}[(\sum_{l \in \mathcal{L}_\alpha} l^{x_1}) \cdots (\sum_{l \in \mathcal{L}_\alpha} l^{x_n})] \\
&= \sum_{\pi \in \mathcal{P}(\{1,\ldots,n\})} \mathbb{E}[\sum_{\substack{l_1,\ldots,l_{\#\pi} \in \mathcal{L}_\alpha \\ \text{distinct}}} \prod_{i=1}^{\#\pi} (\prod_{j \in \pi_i} F_j)(l_i)].
\end{aligned}$$

Define $G_i^\pi = \prod_{j \in \pi_i} F_j$ for $j = 1, \ldots, \#\pi$. Then,

$$\prod_{i=1}^{\#\pi} (\prod_{j \in \pi_i} F_j)(l_i) = \prod_{i=1}^{\#\pi} G_i(l_i).$$

By Campbell's formula,

$$\mathbb{E}[\sum_{\substack{l_1,\ldots,l_{\#\pi} \in \mathcal{L}_\alpha \\ \text{distinct}}} \prod_{i=1}^{\#\pi} (\prod_{j \in \pi_i} F_j)(l_i)] = \alpha^{\#\pi} \prod_{i=1}^{\#\pi} \mu(G_i).$$

Write π_i in decreasing order $p(i,1) < \cdots < p(i, \#\pi_i)$, then

$$\mu(G_i) = \sum_{\sigma \in \mathfrak{S}_{\#\pi_i}, m(\sigma)=1} \prod_{j=1}^{\#\pi_i} V_{x_{p(i,\sigma(j))}}^{x_{p(i,j)}} = \sum_{\substack{\sigma:\text{ circular} \\ \text{permutation} \\ \text{on } \pi_i}} \prod_{j \in \pi_i} V_{x_{\sigma(j)}}^{x_j}.$$

Clearly, there is a one-to-one correspondence between a permutation η on $\{1,\ldots,n\}$ and an $m(\eta)$-partition $\pi = (\pi_1, \ldots, \pi_{m(\eta)})$ together with these circular permutation on the blocks of π. Finally,

$$\begin{aligned}
\mathbb{E}[\hat{\mathcal{L}}_\alpha^{x_1} \cdots \hat{\mathcal{L}}_\alpha^{x_n}] &= \sum_{\pi \in \mathcal{P}(\{1,\ldots,n\})} \alpha^{\#\pi} \prod_{i=1}^{\#\pi} \mu(G_i) \\
&= \sum_{\pi \in \mathcal{P}(\{1,\ldots,n\})} \alpha^{\#\pi} \prod_{i=1}^{\#\pi} \sum_{\substack{\sigma:\text{ circular} \\ \text{permutation} \\ \text{on } \pi_i}} \prod_{j \in \pi_i} V_{x_{\sigma(j)}}^{x_j} \\
&= \sum_{\eta \in \mathfrak{S}_n} \alpha^{m(\eta)} \prod_{i=1}^{n} V_{x_{\eta(i)}}^{x_i} = \mathrm{Per}_\alpha(V_{x_j}^{x_i}, 1 \leq i, j \leq n)
\end{aligned}$$

$\square$

Definition 4.2.2. $\mu(l^x) = V_x^x$, define $\tilde{\mathcal{L}}_\alpha^x = \hat{\mathcal{L}}_\alpha^x - \alpha V_x^x$.

Note that $\mathbb{E}[\tilde{\mathcal{L}}_\alpha^x] = 0$.

Definition 4.2.3. For $A = (A_{ij})_{1\le i,j\le n}$, define

$$\mathrm{Per}_\alpha^0(A) = \sum_{\sigma\in S_n, \sigma(i)\neq i, i=1,\ldots,n} \alpha^{m(\sigma)} A_{1\sigma(1)}\cdots A_{n\sigma(n)}$$

with $m(\sigma)$ the number of cycles in σ.

Proposition 4.2.2.

$$\mathbb{E}[\tilde{\mathcal{L}}_\alpha^{x_1}\cdots\tilde{\mathcal{L}}_\alpha^{x_n}] = \mathrm{Per}_\alpha^0((V_{x_m}^{x_l})_{1\le m,l\le n}).$$

Proof. For $\sigma\in S_n$, let $n(k,\sigma)$ be the number of cycles of length k in σ. According to Proposition 4.2.1,

$$\mathbb{E}[\tilde{\mathcal{L}}_\alpha^{x_1}\cdots\tilde{\mathcal{L}}_\alpha^{x_n}] = \mathbb{E}[\prod_{i=1}^n(\hat{\mathcal{L}}_\alpha^{x_i} - \alpha V_{x_i}^{x_i})] = \sum_{A\subset\{1,\ldots,n\}} (-\alpha)^{|A|}(\prod_{j\in A} V_{x_j}^{x_j})\mathbb{E}[\prod_{j\in A^c}\hat{\mathcal{L}}_\alpha^{x_j}]$$
$$= \sum_{A\subset\{1,\ldots,n\}} (-\alpha)^{|A|}(\prod_{j\in A} V_{x_j}^{x_j})\,\mathrm{Per}_\alpha(V_{A^c})$$

where $A^c = \{1,\ldots,n\}\setminus A$. The above quantity equals

$$\sum_{A\subset\{1,\ldots,n\}} (-1)^{|A|} \sum_{\sigma\in\mathfrak{S}_n, \sigma|_A = Id} \alpha^{m(\sigma)} V_{x_{\sigma(1)}}^{x_1}\cdots V_{x_{\sigma(n)}}^{x_n}$$
$$= \sum_{\sigma\in\mathfrak{S}_n}\alpha^{m(\sigma)} V_{x_{\sigma(1)}}^{x_1}\cdots V_{x_{\sigma(n)}}^{x_n}(\sum_{A\subset\{1,\ldots,n\}} 1_{\{\sigma|_A = Id\}}(-1)^{|A|})$$
$$= \sum_{\sigma\in\mathfrak{S}_n}\alpha^{m(\sigma)} V_{x_{\sigma(1)}}^{x_1}\cdots V_{x_{\sigma(n)}}^{x_n} 1_{\{n(1,\sigma)=0\}} = \mathrm{Per}_\alpha^0((V_{x_m}^{x_l})_{1\le m,l\le n}).$$

□

It is well-known that the generalized Laguerre polynomials $(L_k^{\alpha-1}, k\in\mathbb{N}), \alpha > 0$ have the following generating function

$$\frac{e^{-\frac{xt}{1-t}}}{(1-t)^\alpha} = \sum_{k=0}^\infty t^k L_k^{\alpha-1}(x).$$

Moreover, $L_k^\alpha(x) = \displaystyle\sum_{i=0}^k (-1)^i \binom{n+\alpha}{n-i}\frac{x^i}{i!}$.

Definition 4.2.4. Define $P_k^{\alpha,\sigma}(x) = (-\sigma)^k L_k^{\alpha-1}\left(\dfrac{x}{\sigma}\right)$ and $Q_k^{\alpha,\sigma}(x) = P_k^{\alpha,\sigma}(x+\alpha\sigma)$.

These polynomials of the occupation field are related to Wick renormalisation in the symmetric case, when α is a half integer (see [LJ11]).

Proposition 4.2.3.

a) $$\frac{e^{\frac{xt}{1+t\sigma}}}{(1+t\sigma)^{\alpha}}=\sum_{k=0}^{\infty}t^kP_k^{\alpha-1,\sigma}(x) \text{ and } \frac{e^{\frac{xt+t\alpha\sigma}{1+t\sigma}}}{(1+t\sigma)^{\alpha}}=\sum_{k=0}^{\infty}t^kQ_k^{\alpha-1,\sigma}(x).$$

b) $P_k^{\alpha,V_x^x}(\hat{\mathcal{L}}_\alpha^x)=Q_k^{\alpha,V_x^x}(\tilde{\mathcal{L}}_\alpha^x)$.

c) $\mathbb{E}[P_k^{\alpha,V_x^x}(\hat{\mathcal{L}}_\alpha^x)P_l^{\alpha,V_x^x}(\hat{\mathcal{L}}_\alpha^x)]=\delta_l^k\binom{\alpha+k}{k}(V_y^xV_x^y)^k$.

Proof of c). By Proposition 4.1.5, for $|s|$ and $|t|$ small enough,

$$\mathbb{E}[\frac{e^{\frac{\hat{\mathcal{L}}_\alpha^x t}{1+tV_x^x}}}{(1+tV_x^x)^{\alpha}}\frac{e^{\frac{s\hat{\mathcal{L}}_\alpha^y}{1+sV_y^y}}}{(1+sV_y^y)^{\alpha}}]=(1-stV_y^xV_x^y)^{-\alpha}.$$

Since $L_k^{\alpha}(x)=\sum_{i=0}^{k}(-1)^i\binom{n+\alpha}{n-i}\frac{x^i}{i!}$, $|L_k^{\alpha}(x)|\leq\sum_{i=0}^{k}\binom{n+\alpha}{n-i}\frac{|x|^i}{i!}=L_k^{\alpha}(-|x|)$. Therefore,

$$\begin{aligned}\sum_{k,l\in\mathbb{N}}|t^kP_k^{\alpha,V_x^x}(\hat{\mathcal{L}}_\alpha^x)s^lP_l^{\alpha,V_y^y}(\hat{\mathcal{L}}_\alpha^y)| &\leq \sum_{k,l\in\mathbb{N}}(|t|V_x^x)^kL_k^{\alpha-1}(-\frac{\hat{\mathcal{L}}_\alpha^x}{V_x^x})(|s|V_y^y)^lL_l^{\alpha-1}(-\frac{\hat{\mathcal{L}}_\alpha^y}{V_x^x})\\ &= \frac{e^{\frac{\hat{\mathcal{L}}_\alpha^x|t|}{1-V_x^x|t|}}e^{\frac{\hat{\mathcal{L}}_\alpha^y|s|}{1-V_y^y|s|}}}{(1-|t|V_x^x)^{\alpha}(1-|s|V_y^y)^{\alpha}}\end{aligned}$$

By Proposition 4.1.5, for $|s|$ and $|t|$ small enough, $\mathbb{E}\left[\frac{e^{\frac{\hat{\mathcal{L}}_\alpha^x|t|}{1-V_x^x|t|}}e^{\frac{\hat{\mathcal{L}}_\alpha^y|s|}{1-V_y^y|s|}}}{(1-|t|V_x^x)^{\alpha}(1-|s|V_y^y)^{\alpha}}\right]<\infty$. Consequently,

$$\begin{aligned}\sum_{k,l\in\mathbb{N}}t^kP_k^{\alpha,V_x^x}(\hat{\mathcal{L}}_\alpha^x)s^lP_l^{\alpha,V_y^y}(\hat{\mathcal{L}}_\alpha^y) &= \mathbb{E}[\frac{e^{\frac{\hat{\mathcal{L}}_\alpha^x t}{1+tV_x^x}}}{(1+tV_x^x)^{\alpha}}\frac{e^{\frac{s\hat{\mathcal{L}}_\alpha^y}{1+sV_y^y}}}{(1+sV_y^y)^{\alpha}}]\\ &= (1-stV_y^xV_x^y)^{-\alpha}=\sum_{k\in\mathbb{N}}\binom{\alpha+k}{k}(stV_y^xV_x^y)^k.\end{aligned}$$

Finally, identifying the coefficients of s^kt^l, we obtain

$$\mathbb{E}[P_k^{\alpha,V_x^x}(\hat{\mathcal{L}}_\alpha^x)P_l^{\alpha,V_x^x}(\hat{\mathcal{L}}_\alpha^x)]=\delta_l^k\binom{\alpha+k}{k}(V_y^xV_x^y)^k.$$

□

Proposition 4.2.4. *Fix some $p\geq 1$, for $|t|$ small enough, $\alpha\to(1+tV_x^x)^{-\alpha}e^{\frac{\hat{\mathcal{L}}_\alpha^x t}{1+tV_x^x}}$ and $\alpha\to P_k^{\alpha,V_x^x}(\hat{\mathcal{L}}_\alpha^x)$ are continuous L^p-martingales indexed by $\alpha>0$.*

4.3 Limit behavior of the occupation field

Remark 9. $(X_\alpha = (\hat{\mathcal{L}}^{x_1}_\alpha, \dots, \hat{\mathcal{L}}^{x_n}_\alpha), \alpha \geq 0)$ is a multi-subordinator with respect to the increasing family of $\sigma-$fields $\mathcal{F}_\alpha = \sigma(\mathcal{L}_s, s \leq \alpha)$.
$\mathbb{E}[\hat{\mathcal{L}}^{x_1}_1, \dots, \hat{\mathcal{L}}^{x_n}_1] = (V^{x_1}_{x_1}, \dots, V^{x_n}_{x_n})$ and $\mathbb{E}[e^{-\lambda_1 \hat{\mathcal{L}}^{x_1}_\alpha - \dots - \lambda_n \hat{\mathcal{L}}^{x_n}_\alpha}] = e^{-\alpha \Phi(\lambda_1, \dots, \lambda_n)}$, where

$$\Phi(\lambda_1, \dots, \lambda_n) = \int\limits_{y^1, \dots, y^n \in \mathbb{R}^+} (1 - e^{-\sum\limits_{i=1}^n \lambda_i y^i}) \mu(l^{x_1} \in dy^1, \dots, l^{x_n} \in dy^n).$$

So $\lim\limits_{\alpha \to \infty} \frac{1}{\alpha}(\hat{\mathcal{L}}^{x_1}_\alpha, \dots, \hat{\mathcal{L}}^{x_n}_\alpha) = (V^{x_1}_{x_1}, \dots, V^{x_n}_{x_n})$. And $(\frac{\hat{\mathcal{L}}^{x_1}_\alpha - \alpha V^{x_1}_{x_1}}{\sqrt{\alpha}}, \dots, \frac{\hat{\mathcal{L}}^{x_n}_\alpha - \alpha V^{x_n}_{x_n}}{\sqrt{\alpha}})$ converges in law to a Gaussian variable with mean 0 and covariance $(C_{ij} = V^{x_i}_{x_j} V^{x_j}_{x_i}, i, j = 1, \dots, n)$.

The following result comes from [dA94]: the rescaled Lévy process $(\frac{1}{t}X(ts), s \geq 0), t > 0$ verifies the strong large deviation principle with a good rate function as $t \to \infty$ under the exponential integrability condition:

$$\exists \beta > 0, \mathbb{E}[e^{\beta ||X(1)||}] < \infty$$

This is true for the subordinator $((\hat{\mathcal{L}}^{x_1}_\alpha, \dots, \hat{\mathcal{L}}^{x_n}_\alpha), \alpha > 0)$ by Proposition 4.1.5. The proposition below follows by application of the contraction principle.

Proposition 4.3.1. $\frac{1}{\alpha}(\hat{\mathcal{L}}^{x_1}_\alpha, \dots, \hat{\mathcal{L}}^{x_n}_\alpha) \in \mathbb{R}^n$ *verifies a strong large derivation principle with good rate function* $\Lambda^* : \mathbb{R}^n \to [0, \infty]$ *when* α *tends to* ∞. *Here,* $\Lambda(u) = \ln \mathbb{E}[e^{\hat{\mathcal{L}}^{x_1}_1 u_1 + \dots + \hat{\mathcal{L}}^{x_n}_1 u_n}]$ *and* $\Lambda^*(y) = \sup\limits_{u \in \mathbb{R}^d} (\langle u, y \rangle - \Lambda(u))$.

To be more precise, for all open set $O \subset \mathbb{R}^n$,

$$\liminf_{\alpha \to \infty} \frac{1}{\alpha} \ln(\mathbb{P}[\frac{1}{\alpha}(\hat{\mathcal{L}}^{x_1}_\alpha, \dots, \hat{\mathcal{L}}^{x_n}_\alpha) \in O]) \geq - \inf_{y \in O} \Lambda^*(y)$$

and for all closed subset C of $\mathbb{R}^n$,

$$\limsup_{\alpha \to \infty} \frac{1}{\alpha} \ln(\mathbb{P}[\frac{1}{\alpha}(\hat{\mathcal{L}}^{x_1}_\alpha, \dots, \hat{\mathcal{L}}^{x_n}_\alpha) \in C]) \leq - \inf_{y \in C} \Lambda^*(y).$$

Remark 10. In particular, for $n = 1$,

$$\Lambda^*(y) = \begin{cases} \ln(\frac{V^x_x}{y}) - 1 + \frac{y}{V^x_x} & y > 0 \\ \infty & y \leq 0 \end{cases}$$

For $n = 2$,

$$\Lambda^*(y) = \begin{cases} \ln\Bigg(\frac{1 + \sqrt{1 + \frac{4y^1 y^2 V^{x_1}_{x_2} V^{x_2}_{x_1}}{\det(V|_{\{x_1, x_2\}})^2}}}{2y^1 y^2}\Bigg) + \ln\det(V|_{\{x_1, x_2\}}) & \\ + \frac{y^1 V^{x_2}_{x_2} + y^2 V^{x_1}_{x_1}}{\det(V|_{\{x_1, x_2\}})} - 1 - \sqrt{1 + \frac{4y^1 y^2 V^{x_1}_{x_2} V^{x_2}_{x_1}}{\det(V|_{\{x_1, x_2\}})^2}} & y_1, y_2 \geq 0 \\ \infty & \text{otherwise.} \end{cases}$$

Proof of the remark. For $n = 1$, by Proposition 4.1.5, $\Lambda(u) = -\ln(1 - uV_x^x)$ for $u < \dfrac{1}{V_x^x}$ and $\Lambda(u) = \infty$ otherwise. Then,

$$\Lambda^*(y) = \sup_{u \in \mathbb{R}}(uy - \Lambda(u)) = \begin{cases} \frac{y - V_x^x}{yV_x^x} y - \Lambda(\frac{y-V_x^x}{yV_x^x}) & y > 0 \\ \infty & y \leq 0 \end{cases} = \begin{cases} \ln(\frac{V_x^x}{y}) - 1 + \frac{y}{V_x^x} & y > 0 \\ \infty & y \leq 0. \end{cases}$$

Denote by $A(u_1, u_2)$ the matrix $\begin{bmatrix} V_{x_1}^{x_1} u_1 & V_{x_2}^{x_1}\sqrt{u_1 u_2} \\ V_{x_1}^{x_2}\sqrt{u_1 u_2} & V_{x_2}^{x_2} u_2 \end{bmatrix}$.

For $n = 2$, by Proposition 4.1.5, for $u_1 \geq 0, u_2 \geq 0$,

$$\mathbb{E}[e^{\hat{\mathcal{L}}_1^{x_1} u_1 + \hat{\mathcal{L}}_1^{x_1} u_2}] = \begin{cases} 1/\det(I - A(u_1, u_2)) & \text{if } 1 < 1/\rho(A(u_1, u_2)) \\ \infty & \text{otherwise} \end{cases}$$

where the spectral radius

$$\rho(A(u_1, u_2)) = \frac{V_{x_1}^{x_1} u_1 + V_{x_2}^{x_2} u_2 + \sqrt{(V_{x_1}^{x_1} u_1 - V_{x_2}^{x_2} u_2)^2 + 4V_{x_2}^{x_1} V_{x_1}^{x_2} u_1 u_2}}{2}$$

and

$$1/\det(I - A(u_1, u_2)) = \frac{1}{1 - V_{x_1}^{x_1} u_1 - V_{x_2}^{x_2} u_2 + \det(V|_{\{x_1, x_2\}}) u_1 u_2}.$$

Finally, for $u_1, u_2 \geq 0$,
if $1 - V_{x_1}^{x_1} u_1 - V_{x_2}^{x_2} u_2 + \det(V|_{\{x_1,x_2\}}) u_1 u_2 > 0$ and $V_{x_1}^{x_1} u_1 + V_{x_2}^{x_2} u_2 < 2$,

$$\Lambda(u_1, u_2) = -\ln(1 - V_{x_1}^{x_1} u_1 - V_{x_2}^{x_2} u_2 + \det(V|_{\{x_1,x_2\}}) u_1 u_2)$$

and $\Lambda(u_1, u_2) = \infty$ otherwise.
For $u_1, u_2 \leq 0$, $\Lambda(u_1, u_2) = -\ln(1 - V_{x_1}^{x_1} u_1 - V_{x_2}^{x_2} u_2 + \det(V|_{x_1,x_2}) u_1 u_2)$.
For $u_1 > 0, u_2 < 0$,

$$\mathbb{E}[e^{\hat{\mathcal{L}}_1^{x_1} u_1 + \hat{\mathcal{L}}_1^{x_1} u_2}] = \exp\left(\int (e^{u_1 l^{x_1}} - 1) e^{u_2 l^{x_2}} \mu(dl) + \int (e^{u_2 l^{x_2}} - 1)\mu(dl)\right).$$

By Proposition 3.5.1, $\int (e^{u_2 l^{x_2}} - 1)\mu(dl) = -\ln\det(1 - u^2 V_{x_2}^{x_2})$. By Theorem 3.4.1 and then Proposition 3.5.1,

$$\int (e^{u_1 l^{x_1}} - 1) e^{u_2 l^{x_2}} \mu(dl) = \int (e^{u_1 l^{x_1}} - 1)\mu(L - M_{-u_2 \delta_{x_2}}, dl)$$
$$= \begin{cases} -\ln(1 - u_1 (V_{-u_2\delta_{x_2}})_{x_1}^{x_1}) & \text{if } u_1 < 1/(V_{-u_2\delta_{x_2}})_{x_1}^{x_1} \\ \infty & \text{otherwise.} \end{cases}$$

By the resolvent equation, $V_{x_2}^{x_1} = (V_{-u_2\delta_{x_2}})_{x_2}^{x_1} + (V_{-u_2\delta_{x_2}})_{x_2}^{x_1}(-u_2)V_{x_2}^{x_2}$. Therefore, $(V_{-u_2\delta_{x_2}})_{x_2}^{x_1} = \frac{V_{x_2}^{x_1}}{1 - u_2 V_{x_2}^{x_2}}$. Again, by the resolvent equation, $V_{x_1}^{x_1} = (V_{-u_2\delta_{x_2}})_{x_1}^{x_1} + (V_{-u_2\delta_{x_2}})_{x_2}^{x_1}(-u_2)V_{x_1}^{x_2}$. We

deduce that $(V_{-u_2\delta_{x_2}})^{x_1}_{x_1} = V^{x_1}_{x_1} + \frac{u_2 V^{x_1}_{x_2} V^{x_2}_{x_1}}{1-u_2 V^{x_2}_{x_2}}$. Therefore, for $u_1 > 0, u_2 < 0$,
if $1 - V^{x_1}_{x_1} u_1 - V^{x_2}_{x_2} u_2 + \det(V|_{x_1,x_2}) u_1 u_2 > 0$,

$$\Lambda(u_1, u_2) = -\ln(1 - V^{x_1}_{x_1} u_1 - V^{x_2}_{x_2} u_2 + \det(V|_{x_1,x_2}) u_1 u_2)$$

and $\Lambda(u_1, u_2) = \infty$ otherwise. It is easy to check that $V^{x_1}_{x_1} u_1 + V^{x_2}_{x_2} u_2 < 2$ is implied by $1 - V^{x_1}_{x_1} u_1 - V^{x_2}_{x_2} u_2 + \det(V|_{x_1,x_2}) u_1 u_2 > 0$ for $u_1 > 0, u_2 < 0$. Similar results can be proved for $u_1 < 0, u_2 > 0$. In the end, for any $u_1, u_2 \in \mathbb{R}$,
if $1 - V^{x_1}_{x_1} u_1 - V^{x_2}_{x_2} u_2 + \det(V|_{\{x_1,x_2\}}) u_1 u_2 > 0$ and $V^{x_1}_{x_1} u_1 + V^{x_2}_{x_2} u_2 < 2$,

$$\Lambda(u_1, u_2) = -\ln(1 - V^{x_1}_{x_1} u_1 - V^{x_2}_{x_2} u_2 + \det(V|_{\{x_1,x_2\}}) u_1 u_2)$$

and $\Lambda(u_1, u_2) = \infty$ otherwise.
It is obvious that $\Lambda^*(y_1, y_2) = \infty$ for $y_1 \leq 0$ or $y_2 \leq 0$. Fixing $y_1, y_2 > 0$, we are able to solve $(\frac{\partial}{\partial u_1}\Lambda(u_1, u_2), \frac{\partial}{\partial u_2}\Lambda(u_1, u_2)) = (y_1, y_2)$. We find that the extreme value of $\langle u, y\rangle - \Lambda(u)$ is reached for

$$u_1 = \frac{V^{x_2}_{x_2}}{V^{x_1}_{x_1} V^{x_2}_{x_2} - V^{x_1}_{x_2} V^{x_2}_{x_1}} - \frac{1 + \sqrt{1 + \frac{4 y_1 y_2 V^{x_1}_{x_2} V^{x_2}_{x_1}}{(V^{x_1}_{x_1} V^{x_2}_{x_2} - V^{x_1}_{x_2} V^{x_2}_{x_1})^2}}}{2y_1}$$

$$u_2 = \frac{V^{x_1}_{x_1}}{V^{x_1}_{x_1} V^{x_2}_{x_2} - V^{x_1}_{x_2} V^{x_2}_{x_1}} - \frac{1 + \sqrt{1 + \frac{4 y_1 y_2 V^{x_1}_{x_2} V^{x_2}_{x_1}}{(V^{x_1}_{x_1} V^{x_2}_{x_2} - V^{x_1}_{x_2} V^{x_2}_{x_1})^2}}}{2y_2}$$

and then conclude that

$$\Lambda^*(y) = \ln\left(\frac{1 + \sqrt{1 + \frac{4 y^1 y^2 V^{x_1}_{x_1} V^{x_2}_{x_2}}{\det(V|_{\{x_1,x_2\}})^2}}}{2 y^1 y^2}) + \ln\det(V|_{\{x_1,x_2\}}\right)$$
$$+ \frac{y^1 V^{x_2}_{x_2} + y^2 V^{x_1}_{x_1}}{\det(V|_{\{x_1,x_2\}})} - 1 - \sqrt{1 + \frac{4 y^1 y^2 V^{x_1}_{x_1} V^{x_2}_{x_2}}{\det(V|_{\{x_1,x_2\}})^2}}.$$

□

4.4 Hitting probabilities

Definition 4.4.1. For $D \subset S$, define $loop^D = \{l; \langle l, 1_{\{S-D\}}\rangle = 0\}$, namely loops contained in D. Let $\mathcal{L}^D_\alpha = \mathcal{L}_\alpha \cap loop^D$ be the restriction of the Poisson ensemble on $loop^D$.

Since $\mu(\{l; l \text{ is a trivial loop at } x\}) = \infty$, $\mathcal{L}_\alpha$ contains infinitely many trivial loops at x $\mu - a.s.$.

Proposition 4.4.1. *For a finite subset F,*

$$\mathbb{P}[\nexists l \in \mathcal{L}_\alpha; l \textit{ is non-trivial and } l \textit{ visits } F] = (\prod_{x \in F}(-L_x^x)\det(V_F))^{-\alpha}.$$

Proof. It is a direct consequence of Proposition 3.5.7 and the definition of the Poisson random measure. □

Remark 11. For any subset F, we can find $F_1 \subset \cdots \subset F_n \subset \cdots$ a sequence of finite subsets of F increasing to F. Then,

$$\begin{aligned}
&\mathbb{P}[\nexists l \in \mathcal{L}_\alpha; l \text{ is non-trivial and } l \text{ visits F}] \\
&\qquad = \lim_{n\to\infty} \downarrow \mathbb{P}[\nexists l \in \mathcal{L}_\alpha; l \text{ is non-trivial and } l \text{ visits } F_n] \\
&\qquad = \lim_{n\to\infty} \downarrow (\prod_{x\in F_n}(-L_x^x)\det(V_{F_n}))^{-\alpha} \\
&\qquad = \inf_{A\subset F, |A|<\infty} (\prod_{x\in A}(-L_x^x)\det(V_A))^{-\alpha}.
\end{aligned}$$

4.5 Densities of the occupation field

A non-symmetric generalization of Dynkin's isomorphism was given in [LJ08]. Suppose L is the generator of a transient sub-Markovian process on $\{x_1, \ldots, x_n\}$, m is an excessive measure, and χ is a non-negative function on $\{x_1, \ldots, x_n\}$, then

$$\frac{1}{(2\pi)^n}\int e^{-\frac{1}{2}\langle z\bar{z},\chi\rangle_m} e^{\frac{1}{2}\langle Lz,z\rangle_m} \prod du^i\, dv^i = \det(-M_m L + M_{\chi m})^{-1}$$

where $z^j = u^j + \sqrt{-1}\cdot v^j$ for $j = 1, \ldots, n$ and $L^i_j = L^{x_i}_{x_j}$ for $i, j = 1, \ldots, n$. And it has been proved that if $\text{supp}(\chi) \subset F$, then $\mathbb{E}[e^{-\langle \hat{\mathcal{L}}_1, \chi\rangle}] = \frac{\det(V_F)_\chi}{\det(V_F)} = \frac{\det(-L_F)}{\det(-L_F+\chi)}$. So, we have the following representation.

Proposition 4.5.1. *Let $F = \{x_1, \ldots, x_n\} \subset S$ and $L_F = (-V_F)^{-1}$. For any bounded measurable function G,*

$$\mathbb{E}[G(\hat{\mathcal{L}}_1^{x_1}, \ldots, \hat{\mathcal{L}}_1^{x_n})] = \frac{\det(-M_m L_F)}{(2\pi)^n}\int G(\frac{1}{2}m_1|z^1|^2, \ldots, \frac{1}{2}m_n|z^n|^2) e^{\frac{1}{2}\langle L_F z, z\rangle_m} \prod du^i\, dv^i.$$

Remark 12. Recall that in the symmetric case, if ϕ is a Gaussian free field with covariance matrix given by the Green function, $\hat{\mathcal{L}}_{1/2}$ has the same law as $\frac{1}{2}\phi^2$. Moreover, if $\phi_1, \ldots, \phi_k$ are k i.i.d. copies of ϕ, then $\frac{1}{2}\sum_{j=1}^{k}\phi_k^2$ and $\hat{\mathcal{L}}_{k/2}$ have the same law. For details, see Chapter 5 in [LJ11] and Chapter 4 in [Szn12].

We can derive from this expression a formula for the joint densities of the occupation field, for $\alpha = 1$.

Proposition 4.5.2. *Let $F = \{x_1, \dots, x_n\} \subset S$ and $L_F = (-V_F)^{-1}$. Then, $f(\rho^1, \dots, \rho^n)$, the density of $(\hat{\mathcal{L}}_1^{x^1}, \dots, \hat{\mathcal{L}}_1^{x^n})$ with respect to the Lebesgue measure on $\mathbb{R}_+^n$ is*

$$\det(-L_F) \sum_{n_{ij}\in\mathbb{N}, i,j=1,\dots,n} 1_{\{\sum_j n_{ij} = \sum_j n_{ji},\, i=1,\dots,n\}} \left(\prod_{i,j=1,\dots,n} \frac{((L_F)_{x_j}^{x_i} \sqrt{\rho^i\rho^j})^{n_{ij}}}{n_{ij}!} \right).$$

Proof. The above Proposition shows that for any G bounded measurable

$$\mathbb{E}[G(\hat{\mathcal{L}}_1^{x_1}, \dots, \hat{\mathcal{L}}_1^{x_n})] = \frac{\det(-M_m L_F)}{(2\pi)^n} \int G(\frac{1}{2}m_1|z^1|^2, \dots, \frac{1}{2}m_n|z^n|^2) e^{\frac{1}{2}\langle L_F z, z\rangle_m} \prod du^i\, dv^i.$$

Using the polar coordinate, let $r^j = |z^j|$, $\theta_j \in [0, 2\pi[$, $u^j = r^j\cos(\theta_j)$ and $v^j = r^j\sin(\theta_j)$. Then, $\mathbb{E}[G(\hat{\mathcal{L}}_1^{x_1}, \dots, \hat{\mathcal{L}}_1^{x_n})]$ equals

$$\frac{\det(-M_m L_F)}{(2\pi)^n} \int G(\frac{1}{2}m_1(r^1)^2, \dots, \frac{1}{2}m_n(r^n)^2) e^{\sum_{i,j} \frac{1}{2}(L_F)_{x_j}^{x_i} r^i r^j m^j e^{i\theta_i} e^{-i\theta_j}} \prod dr^i\, d\theta_i.$$

Let $\rho^j = m_j(r^j)^2/2$ for $j = 1, \dots, n$, then $\mathbb{E}[G(\hat{\mathcal{L}}_1^{x_1}, \dots, \hat{\mathcal{L}}_1^{x_n})]$ equals

$$\frac{\det(-L_F)}{(2\pi)^n} \int G(\rho^1, \dots, \rho^n) e^{\sum_{i,j} (L_F)_{x_j}^{x_i} \left(\rho^i\rho^j \frac{m_i}{m_j}\right)^{1/2} e^{i\theta_i} e^{-i\theta_j}} \prod d\rho^i\, d\theta_i.$$

Therefore, the density of $\hat{\mathcal{L}}_1^{x_1}, \dots, \hat{\mathcal{L}}_1^{x_n}$, is

$$\begin{aligned} f(\rho^1, \dots, \rho^n) &= \int_{[0,2\pi]^n} \frac{\det(-L_F)}{(2\pi)^n} e^{\sum_{i,j} (L_F)_{x_j}^{x_i} \left(\rho^i\rho^j \frac{m_i}{m_j}\right)^{1/2} e^{i\theta_i} e^{-i\theta_j}} \prod d\theta_i \\ &= \int_{|z^i|=1, i=1,\dots,n} \frac{\det(-L_F)}{(2\pi i)^n} \frac{1}{z^1\cdots z^n} e^{\sum_{i,j} (L_F)_{x_j}^{x_i} \left(\rho^i\rho^j \frac{m_i}{m_j}\right)^{1/2} \frac{z^i}{z^j}} dz^1 \cdots dz^n. \end{aligned}$$

We expand $\exp\left(\sum_{i,j} (L_F)_{x_j}^{x_i} \left(\rho^i\rho^j \frac{m_i}{m_j}\right)^{1/2} \frac{z^i}{z^j}\right)$ into series, integrate it term by term and use Cauchy's formula. Only the constant terms in the expansion of

$$\exp\left(\sum_{i,j} (L_F)_{x_j}^{x_i} \left(\rho^i\rho^j \frac{m_i}{m_j}\right)^{1/2} \frac{z^i}{z^j}\right)$$

contribute. Accordingly, we have

$$f(\rho^1, \dots, \rho^n) = \det(-L_F) \sum_{\substack{n_{ij}\in\mathbb{N} \text{ for } i,j=1,\dots,n \\ \sum_j n_{ij} = \sum_j n_{ji} \text{ for } i=1,\dots,n}} \left(\prod_{i,j=1,\dots,n} \frac{((L_F)_{x_j}^{x_i} \sqrt{\rho^i\rho^j})^{n_{ij}}}{n_{ij}!} \right).$$

$\square$

Moreover, we have the follow expansions of the density of occupation field for general $\alpha > 0$:

Proposition 4.5.3. *Denote by* $\mathrm{Coeff}\left(\det(M_s + V_F)^{-\alpha}, s_1^{M_1} \cdots s_n^{M_n}\right)$[1] *the coefficient before the term* $s_1^{M_1} \cdots s_n^{M_n}$ *in the expansion of the function* $s \to \det(M_s + V_F)^{-\alpha}$ *for* s *small enough. Then the density* $(f^\alpha(\rho_1, \ldots, \rho_n), \rho_1, \ldots, \rho_n > 0)$ *of the occupation field* $(\mathcal{L}_\alpha^{x_1}, \ldots, \mathcal{L}_\alpha^{x_n})$ *has the following expression:*

$$f^\alpha(\rho_1, \ldots, \rho_n) = \sum_{M_1, \ldots, M_n \in \mathbb{N}} \frac{\mathrm{Coeff}\left(\det(M_s + V_F)^{-\alpha}, s_1^{M_1} \cdots s_n^{M_n}\right)}{\prod_{i=1}^n \Gamma(M_i + \alpha)} \prod_{i=1}^n \rho_i^{M_i + \alpha - 1}.$$

Proof. Let's calculate the Laplace transform of the function

$$(\rho_1, \ldots, \rho_n) \to f^\alpha(\rho_1, \ldots, \rho_n) e^{-c(\rho_1 + \cdots + \rho_n)}.$$

For c sufficient large, we have

$$\sum_{M_1, \ldots, M_n \in \mathbb{N}} \frac{|\,\mathrm{Coeff}\left(\det(M_s + V_F)^{-\alpha}, s_1^{M_1} \cdots s_n^{M_n}\right)|}{\prod_{i=1}^n \Gamma(M_i + \alpha)} \prod_{i=1}^n \rho_i^{M_i + \alpha - 1} e^{-c(\rho_1 + \cdots + \rho_n)} < \infty$$

$$\begin{aligned}
&\int d\rho_1 \cdots d\rho_n e^{-(\rho_1 \lambda_1 + \cdots + \rho_n \lambda_n)} f^\alpha(\rho_1, \ldots, \rho_n) e^{-c(\rho_1 + \cdots + \rho_n)} \\
&= \sum_{M_1, \ldots, M_n \in \mathbb{N}} \frac{\mathrm{Coeff}\left(\det(M_s + V_F)^{-\alpha}, s_1^{M_1} \cdots s_n^{M_n}\right)}{\prod_{i=1}^n \Gamma(M_i + \alpha)} \int \prod_{i=1}^n \rho_i^{M_i + \alpha - 1} e^{-\rho_i(\lambda_i + c)} d\rho_i \\
&= \sum_{M_1, \ldots, M_n \in \mathbb{N}} \frac{\mathrm{Coeff}\left(\det(M_s + V_F)^{-\alpha}, s_1^{M_1} \cdots s_n^{M_n}\right)}{\prod_{i=1}^n (\lambda_i + c)^{M_i + \alpha}} \\
&= \det(M_{\lambda + c}^{-1} + V_F)^{-\alpha} \prod_{i=1}^n (\lambda_i + c)^{-\alpha} \\
&= \det(I + M_{\lambda + c} V_F)^{-\alpha} = \mathbb{E}[e^{-\sum_{i=1}^n \hat{\mathcal{L}}_\alpha^{x_i} (\lambda_i + c)}].
\end{aligned}$$

Clearly, f^α is the density of $(\mathcal{L}_\alpha^{x_1}, \ldots, \mathcal{L}_\alpha^{x_n})$. □

4.6 Conditioned occupation field

Definition 4.6.1. For $F \subset S$, define $\mathcal{L}_\alpha|_F = \{l_F : l \in \mathcal{L}_\alpha\}$ where l_F is the trace of l on F, see Definition 3.3.4.

[1]For s sufficient close to $(0, \ldots, 0)$, $\det(M_s + V_F)^{-\alpha}$ is an analytic function.

Proposition 4.6.1. *Let X, Y be two Borel spaces. Let $\mathcal{P}$ be a Poisson random measure on $Z = X \times Y$ with $\sigma-$finite intensity measure $\mu(dx, dy) = m(dx)K(x, dy)$, K being a probability kernel. Let π_X and π_Y be the projection from $Z = X \times Y$ to X and Y respectively. Define $\mathcal{P}_X = \pi_X \circ \mathcal{P}$ and $\mathcal{P}_Y = \pi_Y \circ \mathcal{P}$. For all $\Phi : Y \to \mathbb{R}$ non-negative measurable, define $\phi : Y \to \mathbb{R}$ according to Φ by the following equation $e^{-\phi(x)} = \int\limits_Y e^{-\Phi(y)} K(x, dy)$. Then,*

$$\mathbb{E}[e^{-\langle \mathcal{P}_Y, \Phi \rangle} | \mathcal{F}_X] = e^{-\langle \mathcal{P}_X, \phi \rangle}.$$

Remark 13. The Poisson random measure $\mathcal{P}$ is the K-randomization[2] of the Poisson random measure $\pi_X \circ \mathcal{P}$.

Proof. Take $\Psi : X \to \mathbb{R}$ and $\Phi : Y \to \mathbb{R}$ non-negative measurable. Define ϕ by the following equation:

$$e^{-\phi(x)} = \int\limits_Y e^{-\Phi(y)} K(x, dy).$$

We have

$$\begin{aligned}
\mathbb{E}[e^{-\langle \mathcal{P}_X, \Psi \rangle} e^{-\langle \mathcal{P}_Y, \Phi \rangle}] &= \mathbb{E}[e^{-\langle \mathcal{P}, \Psi \otimes \Phi \rangle}] = e^{\mu(e^{-\Psi \otimes \Phi} - 1)} \\
&= \exp(\int\limits_{X \times Y} (e^{-\Psi(x)} e^{-\Phi(y)} - 1) m(dx) K(x, dy)) \\
&= \exp(\int\limits_X (e^{-\Psi(x)} \int\limits_Y e^{-\Phi(y)} K(x, dy) - 1) m(dx)) \\
&= \exp(\int\limits_X (e^{-\Psi(x) - \phi(x)} - 1) m(dx)) = \mathbb{E}[e^{-\langle \mathcal{P}_X, \Psi \rangle} e^{-\langle \mathcal{P}_Y, \phi \rangle}].
\end{aligned}$$

Since $\mathcal{F}_X = \sigma(\{e^{-\langle \mathcal{P}_X, \Psi \rangle} : \Psi$ is a non-negative measurable function on $X\})$,

$$\mathbb{E}[e^{-\langle \mathcal{P}_Y, \Phi \rangle} | \mathcal{F}_X] = e^{-\langle \mathcal{P}_X, \phi \rangle}.$$

□

Let f be a positive measurable function on the space of excursions. Recall that $\mathcal{E}_F(l)$ is the point measure of the excursions of the loop l outside of F (see Definition 3.3.3). As a consequence of Proposition 4.6.1 and Proposition 3.3.6 or Corollary 3.3.7, we have the following proposition.

[2] Please refer to Chapter 12 of [Kal02]

Proposition 4.6.2.

$$\mathbb{E}[e^{-\sum\limits_{l\in\mathcal{L}_\alpha}\langle\mathcal{E}_F(l),f\rangle}|\sigma(\mathcal{L}_\alpha|_F)] = (\prod_{x\neq y\in F}\nu_{F,ex}^{x,y}(e^{-f})^{N_y^x(\mathcal{L}_\alpha|_F)})\times e^{\sum\limits_{x\in F}(L_x^x-(L_F)_x^x)\widehat{(\mathcal{L}_\alpha|_F)}^x\nu_{F,ex}^{x,x}(1-e^{-f})}.$$

For an excursion (e,x,y) outside of F from x to y and χ any non-negative measurable function on S, set $\langle e,\chi\rangle = \int\chi(e(s))\,ds$. Then we have the following:

Proposition 4.6.3. *The conditional expectation* $\mathbb{E}[e^{-\langle\hat{\mathcal{L}}_\alpha,\chi\rangle}|\sigma(\mathcal{L}_\alpha|_F)]$ *equals*

$$\mathbb{E}[e^{-\langle\widehat{\mathcal{L}_\alpha^{F^c}},\chi\rangle}]e^{-\langle\hat{\mathcal{L}}_\alpha|_F,\chi\rangle}\exp\left(\sum_{x\in F}L_x^x\widehat{(\mathcal{L}_\alpha|_F)}^x((R^F)_x^x-(R_\chi^F)_x^x)\right)\prod_{x\neq y\in F}\left(\frac{(R_\chi^F)_y^x}{(R^F)_y^x}\right)^{N_y^x(\mathcal{L}_\alpha|_F)}.$$

Proof. The set of loops which do not intersect F, $\mathcal{L}_\alpha^{F^c}$, is independent of the set of loops which intersect F. Therefore,

$$\begin{aligned}
\mathbb{E}[e^{-\langle\hat{\mathcal{L}}_\alpha,\chi\rangle}|\sigma(\mathcal{L}_\alpha|_F)] =&\mathbb{E}[e^{-\langle\widehat{\mathcal{L}_\alpha^{F^c}},\chi\rangle}]\mathbb{E}[\exp(-\sum_{\substack{l\in\mathcal{L}_\alpha\\ l\text{ visits }F}}\langle l,\chi\rangle)|\sigma(\mathcal{L}_\alpha|_F)]\\
=&\mathbb{E}[e^{-\langle\widehat{\mathcal{L}_\alpha^{F^c}},\chi\rangle}]\mathbb{E}[\exp(-\sum_{\substack{l\in\mathcal{L}_\alpha\\ l\text{ visits }F}}(\langle l_F,\chi\rangle+\sum_{e\in\mathcal{E}_F(l)}\langle e,\chi\rangle))|\sigma(\mathcal{L}_\alpha|_F)]\\
=&\mathbb{E}[e^{-\langle\widehat{\mathcal{L}_\alpha^{F^c}},\chi\rangle}]\mathbb{E}[\exp(-\sum_{\substack{l\in\mathcal{L}_\alpha\\ l\text{ visits }F}}\sum_{e\in\mathcal{E}_F(l)}\langle e,\chi\rangle)\exp(-\langle\hat{\mathcal{L}}_\alpha|_F,\chi\rangle)|\sigma(\mathcal{L}_\alpha|_F)]\\
=&\mathbb{E}[e^{-\langle\widehat{\mathcal{L}_\alpha^{F^c}},\chi\rangle}]e^{-\langle\hat{\mathcal{L}}_\alpha|_F,\chi\rangle}\mathbb{E}[\exp(-\sum_{\substack{l\in\mathcal{L}_\alpha\\ l\text{ visits }F}}\sum_{e\in\mathcal{E}_F(l)}\langle e,\chi\rangle)|\sigma(\mathcal{L}_\alpha|_F)]\\
=&\mathbb{E}[e^{-\langle\widehat{\mathcal{L}_\alpha^{F^c}},\chi\rangle}]e^{-\langle\hat{\mathcal{L}}_\alpha|_F,\chi\rangle}\mathbb{E}[\exp(-\sum_{l\in\mathcal{L}_\alpha}\sum_{e\in\mathcal{E}_F(l)}\langle e,\chi\rangle|\sigma(\mathcal{L}_\alpha)|_F)].
\end{aligned}$$

By Proposition 4.6.2, taking the positive excursion function $f(\cdot)$ to be $\langle\cdot,\chi\rangle$,

$$\begin{aligned}
\mathbb{E}[e^{-\sum\limits_{l\in\mathcal{L}_\alpha}\sum\limits_{e\in\mathcal{E}_F(l)}\langle e,\chi\rangle}|\sigma(\mathcal{L}_\alpha|_F)] &= \mathbb{E}[e^{-\sum\limits_{l\in\mathcal{L}_\alpha}\langle\mathcal{E}_F(l),\langle\cdot,\chi\rangle\rangle}|\sigma(\mathcal{L}_\alpha|_F)]\\
&=(\prod_{x\neq y\in F}\nu_{F,ex}^{x,y}(e^{-\langle\cdot,\chi\rangle})^{N_y^x(\mathcal{L}_\alpha|_F)})\exp\left(\sum_{x\in F}(L_x^x-(L_F)_x^x)\widehat{(\mathcal{L}_\alpha|_F)}^x\nu_{F,ex}^{x,x}(1-e^{-\langle\cdot,\chi\rangle})\right).
\end{aligned}$$

By Lemma 3.3.8,

$$\nu_{F,ex}^{x,y}(e^{-\langle\cdot,\chi\rangle}) = \frac{(R_\chi^F)_y^x}{(R^F)_y^x}.$$

Then, by Proposition 2.3.1, $L_x^x-(L_F)_x^x = L_x^x(R^F)_x^x$. Then,

$$\mathbb{E}[e^{-\sum\limits_{l\in\mathcal{L}_\alpha}\sum\limits_{e\in\mathcal{E}_F(l)}\langle e,\chi\rangle}|\sigma(\mathcal{L}_\alpha|_F)]$$

$$= \left(\prod_{x \neq y \in F} \left(\frac{(R^F_\chi)^x_y}{(R^F)^x_y} \right)^{N^x_y(\mathcal{L}_\alpha|_F)} \right) \exp \left(\sum_{x \in F} (L^x_x - (L_F)^x_x) \widehat{(\mathcal{L}_\alpha|_F)}^x \left(1 - \frac{(R^F_\chi)^x_x}{(R^F)^x_x} \right) \right)$$

$$= \left(\prod_{x \neq y \in F} \left(\frac{(R^F_\chi)^x_y}{(R^F)^x_y} \right)^{N^x_y(\mathcal{L}_\alpha|_F)} \right) \exp \left(\sum_{x \in F} L^x_x \widehat{(\mathcal{L}_\alpha|_F)}^x ((R^F)^x_x - (R^F_\chi)^x_x) \right).$$

Finally, we get $\mathbb{E}[e^{-\langle \hat{\mathcal{L}}_\alpha, \chi \rangle} | \sigma(\mathcal{L}_\alpha|_F)]$ equals

$$\mathbb{E}[e^{-\langle \widehat{\mathcal{L}^{F^c}_\alpha}, \chi \rangle}] e^{-\langle \hat{\mathcal{L}}_\alpha|_F, \chi \rangle} \left(\prod_{x \neq y \in F} \left(\frac{(R^F_\chi)^x_y}{(R^F)^x_y} \right)^{N^x_y(\mathcal{L}_\alpha|_F)} \right) \exp \left(\sum_{x \in F} L^x_x \widehat{(\mathcal{L}_\alpha|_F)}^x ((R^F)^x_x - (R^F_\chi)^x_x) \right).$$

□

4.7 Loop clusters

Consider the space S as a graph (S, E) with S as the set of vertices and $E = \{\{x, y\} : N^x_y(l) > 0 \text{ or } N^y_x(l) > 0\}$ as the set of undirected edges. An edge $\{x, y\}$ is said to be open at time α if it is traversed by at least one loop of $\mathcal{L}_\alpha$, i.e. $N^x_y(\mathcal{L}_\alpha) + N^y_x(\mathcal{L}_\alpha) > 0$. The set of open edges defines a subgraph G_α with vertices S. The connected components of G_α define a partition of S denoted by $\mathcal{C}_\alpha$, namely the loop clusters at time α.
As in section 2 of [LJL12], we have the following proposition,

Proposition 4.7.1. *Given a collection of edges $F = \{e_1 = \{x_1, y_1\}, \ldots, e_k = \{x_k, y_k\}\}$, let $A = \bigcup_{i=1}^k \{x_i, y_i\}$. Then,*

$$\mathbb{P}[e_1, \ldots, e_k \textit{ are all closed}] = \det(I + (L|_F)|_{A \times A} V_A)^{-\alpha}$$

where $(L|_F)^x_y = \begin{cases} L^x_y & \textit{if } \{x, y\} \in F \\ 0 & \textit{otherwise.} \end{cases}$

Proof. Suppose S is finite,

$$\mathbb{P}[e_1, \ldots, e_k \text{ are all closed}] = \exp(-\alpha\mu(\sum_{i=1}^k N^{x_i}_{y_i}(l) + N^{y_i}_{x_i}(l) > 0))$$

$$= \exp(-\alpha\mu(\sum_{i=1}^k N^{x_i}_{y_i}(l) + N^{y_i}_{x_i}(l) > 0, \ l \text{ is non-trivial}))$$

$$= \exp(-\alpha\mu(l \text{ is non-trivial}) + \alpha\mu(\sum_{i=1}^k N^{x_i}_{y_i}(l) + N^{y_i}_{x_i}(l) = 0, \ l \text{ is non-trivial}))$$

Define $(L')^x_y = L^x_y$ if $\{x,y\} \notin F$ and $(L')^x_y = 0$ if $\{x,y\} \in F$. By Proposition 3.1.3,

$$\mu(\sum_{i=1}^k N^{x_i}_{y_i}(l) + N^{y_i}_{x_i}(l) = 0, l \text{ is non-trivial}) = \mu(L', l \text{ is non-trivial}).$$

(Recall that $\mu(L', dl)$ is the Markovian loop measure associated with the generator L'.) By Proposition 3.5.7, $\mu(L', l \text{ is non-trivial}) = -\ln(\prod_{x\in S}(-L')^x_x) + \ln\det(-L') = \ln(\prod_{x\in S}(-L)^x_x) - \ln\det(-L')$ and $\mu(l \text{ is non-trivial}) = \ln(\prod_{x\in S}(-L)^x_x) - \ln\det(-L)$. Therefore,

$$\mathbb{P}[e_1,\ldots,e_k \text{ are all closed}] = \left(\frac{\det(-L)}{\det(-L')}\right)^\alpha = \det(-L'V)^{-\alpha}.$$

Write as $-L' = -L + (L - L') = -L + L|_F$. Therefore, $\det(-L'V) = \det(I + (L|_F)|_{A\times A}V_A)$. Consequently,

$$\mathbb{P}[e_1,\ldots,e_k \text{ are all closed}] = \det(I + (L|_F)|_{A\times A}V_A)^{-\alpha}.$$

For S countable, let $A_1 \subset A_2 \subset \cdots$ exhausting S. Then we have

$$\mathbb{P}[e_1,\ldots,e_k \text{ are all closed}] = \exp(-\alpha\mu(\sum_{i=1}^k N^{x_i}_{y_i}(l) + N^{y_i}_{x_i}(l) > 0))$$

$$= \exp(-\alpha \lim_{n\to\infty} \mu(\sum_{i=1}^k N^{x_i}_{y_i}(l) + N^{y_i}_{x_i}(l) > 0, l \text{ is contained in } A_n)).$$

By Proposition 3.2.1,

$$\mu(\sum_{i=1}^k N^{x_i}_{y_i}(l) + N^{y_i}_{x_i}(l) > 0, l \text{ is contained in } A_n) = \mu(L|_{A_n\times A_n}, \sum_{i=1}^k N^{x_i}_{y_i}(l) + N^{y_i}_{x_i}(l) > 0).$$

By the calculation for the finite case,

$$\mu(L_{A_n\times A_n}, \sum_{i=1}^k N^{x_i}_{y_i}(l) + N^{y_i}_{x_i}(l) > 0) = \det(I + (L|_F)|_{A\times A}(-L|_{A_n\times A_n})^{-1}_A)^{-\alpha}.$$

It is not hard to check that $\lim_{n\to\infty}((-L|_{A_n\times A_n})^{-1})^x_y = V^x_y$ for $x,y\in S$. Finally,

$$\mathbb{P}[e_1,\ldots,e_k \text{ are all closed}] = \det(I + (L|_F)|_{A\times A}V_A)^{-\alpha}.$$

$\square$

As a corollary, we obtain another expression by using the Poisson kernel. For $X \subset S$, define the Poisson kernel $(H^X)^x_y = \mathbb{P}^x[X_{T_X} = y]$ the probability of hitting X at the position y for a process starting from x.

Proposition 4.7.2. *Given a partition* $\pi = \{S_1, \ldots, S_k\}$*, define* $\partial S_i = \{x \in S_i : \exists y \in S_i^c, Q_y^x + Q_x^y > 0\}$*,* $F = \bigcup_{i=1}^{k} \{\{x,y\} : Q_y^x + Q_x^y > 0, x \in S_i, y \in S_j\}$ *and* $A = \bigcup_{i=1}^{k} \partial S_i$*. Suppose* $|A| < \infty$*. Define* $H_{i,j} = H^{S_i^c}|_{\partial S_i \times \partial S_j}$ *and*

$$K = \begin{bmatrix} 0 & H_{1,2} & \cdots & H_{1,k} \\ H_{2,1} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & H_{k-1,k} \\ H_{k,1} & \cdots & H_{k,k-1} & 0 \end{bmatrix}.$$

Then,

$$\mathbb{P}[\mathcal{C}_\alpha \textit{ is finer than } \pi] = \mathbb{P}[\textit{all the edges in } F \textit{ are closed}] = (\det(I-K))^\alpha.$$

Proof. By taking the trace of the loops on A, we can suppose the state space S is finite and $\partial S_i = S_i$ for $i = 1, \ldots, k$. By an argument similar to the argument in the above proposition, we see that

$$\mathbb{P}[\mathcal{C}_\alpha \text{ is finer than } \pi] = \left(\frac{\det(L')}{\det(L)}\right)^{-\alpha}$$

where $(L')_y^x = L_y^x$ for $\{x,y\} \notin F$ and $(L')_y^x = 0$ for $\{x,y\} \in F$. To be more precise,

$$L' = \begin{bmatrix} L|_{S_1 \times S_1} & 0 & \cdots & 0 \\ 0 & L|_{S_2 \times S_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L|_{S_k \times S_k} \end{bmatrix}.$$

Therefore,

$$\mathbb{P}[\mathcal{C}_\alpha \text{ is finer than } \pi] = (\frac{\det(L')}{\det(L)})^{-\alpha} = (\det((-L')^{-1}(-L)))^\alpha$$

$$= \left(\det(-\begin{bmatrix} V^{S_1} & 0 & \cdots & 0 \\ 0 & V^{S_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & V^{S_k} \end{bmatrix} L)\right)^\alpha = \begin{vmatrix} I & -H_{1,2} & \cdots & -H_{1,k} \\ -H_{2,1} & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & -H_{k-1,k} \\ -H_{k,1} & \cdots & -H_{k,k-1} & I \end{vmatrix}^\alpha$$

$$= (\det(I-K))^\alpha.$$

(Note that $V^{S^i} L|_{S_i \times S_j} = H^{S_j^c}_{S_i, S_j} = H_{i,j}$ for $i \neq j \in \{1, \ldots, k\}$.) ∎

4.7.1 An example on the discrete circle

Consider a discrete circle G with n vertices $1,\dots,n$ and $2n$ oriented edges

$$E=\{(1,2),(2,3),\dots,(n-1,n),(n,1),(2,1),(3,2),\dots,(n,n-1),(1,n)\}$$

Define the clockwise edges set $E_+=\{(1,2),(2,3),\dots,(n-1,n),(n,1)\}$ and the counter clockwise edges $E_-=E-E_+$. Consider a Markovian generator L such that for any $e\in E_+$, $L_{e+}^{e-}=p, L_{e-}^{e+}=1-p, L_{e-}^{e-}=-(1+c)$ and L is null elsewhere. Then, we have a loop measure and Poissonian ensembles associated with L. The rest of this subsection is devoted to study the loop cluster $\mathcal{C}_\alpha$ in this example.

Lemma 4.7.3. *Let $T_{3,n}$ be a $n\times n$ tri-diagonal Toeplitz matrix of the following form:*

$$\begin{bmatrix} a & b & 0 & \cdots & 0\\ c & a & b & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & c & a & b\\ 0 & \cdots & 0 & c & a\end{bmatrix}_{n\times n}.$$

Let S_n be the following $n\times n$ matrix:

$$\begin{bmatrix} a & b & 0 & & c\\ c & a & b & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0\\ & \ddots & c & a & b\\ b & & 0 & c & a\end{bmatrix}_{n\times n}.$$

Let x_1,x_2 be the roots of $x^2-ax+bc=0$. Then,

- $\det(T_{3,n})=\dfrac{x_1^{n+1}-x_2^{n+1}}{x_1-x_2}$,
- $\det(S_n)=x_1^n+x_2^n+(-1)^{n+1}(b^n+c^n)$.

Proposition 4.7.4.

$$\begin{aligned}\textit{Set } x_1=&\frac{1}{2}(1+c+\sqrt{(1+c)^2-4p(1-p)}),\\ x_2=&\frac{1}{2}(1+c-\sqrt{(1+c)^2-4p(1-p)}).\end{aligned}$$

Then,

$$\mathbb{P}[\{1,n\}\textit{ is closed.}]=\left(\frac{(x_1^n-x_2^n)^2}{(x_1-x_2)(x_1^{n-1}-x_2^{n-1})(x_1^n+x_2^n-(p^n+(1-p)^n))}\right)^{-\alpha}.$$

Proof. By Proposition 3.2.1 and Proposition 3.5.8

$$\begin{aligned}\mathbb{P}[\{1,n\} \text{ is closed}] &= e^{-\alpha\mu(N_n^1(l)+N_1^n(l)>0)} = e^{-\alpha\mu(l \text{ visits } 1 \text{ and } n)} \\ &= \left(\frac{\det(V_{\{1,n\}})}{V_1^1 V_n^n}\right)^{\alpha} = \left(\frac{\det(-L|_{\{2,\ldots,n\}\times\{2,\ldots,n\}})\det(L|_{\{1,\ldots,n-1\}\times\{1,\ldots,n-1\}})}{\det(-L|_{\{2,\ldots,n-1\}\times\{2,\ldots,n-1\}})\det(-L)}\right)^{-\alpha} \\ &= \left(\frac{(x_1^n - x_2^n)^2}{(x_1-x_2)(x_1^{n-1}-x_2^{n-1})(x_1^n+x_2^n-(p^n+(1-p)^n))}\right)^{-\alpha}\end{aligned}$$

where $x_1 = \frac{1+c+\sqrt{(1+c)^2-4p(1-p)}}{2}$ and $x_2 = \frac{1+c-\sqrt{(1+c)^2-4p(1-p)}}{2}$. □

Proposition 4.7.5. *Conditionally on $\{1,n\}$ being closed, $\mathcal{C}_\alpha$ is a renewal process conditioned to jump at time n. To be more precise, by deleting edges $\{1,n\}$ and adding $\{0,1\},\{n,n+1\}$, we get a discrete segment with vertices $\{0,1,\ldots,n,n+1\}$ and edges $\{\{0,1\},\ldots,\{n,n+1\}\}$. Conditionally to $\{1,n\}$ being closed, C_α induces a partition on $\{1,\ldots,n\}$. The clusters of C_α are the intervals between the edges closed at time α (namely the edges which are not crossed by any loop of $\mathcal{L}_\alpha$). Then the left points of these closed edges, together with the left points of $\{0,1\}$ and $\{n,n+1\}$, form a renewal process conditioned to jump at n.*

Proof. Among the Poissonian loop ensembles, the ensemble of loops crossing $\{1,n\}$ and the rest are independent. Therefore, the conditional law $\mathcal{Q}$ of the loops not crossing $\{1,n\}$ conditioned on the event that no loop is crossing $\{1,n\}$ is exactly the same as the unconditioned law. Consider another Poissonian loop ensembles on $\mathbb{Z}$ driven by the following generator:

$$L_m^m = -(1+c), L_{m+1}^m = p, L_{m-1}^m = 1-p \text{ for all } m\in\mathbb{Z}, \text{ and } L \text{ is null elsewhere.}$$

Then, $\mathcal{Q}$ is the same as the conditional law of the loop ensembles contained in $\{1,\ldots,n\}$ given the condition that $\{0,1\}$ or $\{n,n+1\}$ are closed. By Proposition 3.1.2, after a harmonic transform, L is modified as follows:

$$L_m^m = -(1+c), L_{m+1}^m = L_{m-1}^m = \sqrt{p(1-p)} \text{ for all } m\in\mathbb{Z}, \text{ and } L \text{ is null elsewhere.}$$

According to Proposition 3.1 in [LJL12], in the case of $\mathbb{Z}$, conditionally to the event that $\{0,1\}$ is closed, the left points of the closed edges form a renewal process. There is an obvious one-to-one correspondence between the jumps of the renewal process and the closed edges. Finally, in the case of the circle, conditioning on $\{1,n\}$ being closed, we can identify $\mathcal{C}_\alpha$ to a renewal process conditioned to jump at time n. It is not hard to see the parameter κ in [LJL12] equals $\frac{1+c-2\sqrt{p(1-p)}}{\sqrt{p(1-p)}}$. □

We can go back to the symmetric model conditionally on $\{1,n\}$ being closed. Hence, we use the following modified model. Consider a pure-jump Markov process on $\{1,\dots,n,\dots\}$ with generator L: $L^m_{m+1}=L^{m+1}_m=1/2, L^m_m=-(1+\kappa/2)$ for $m\in\mathbb{N}_+$, $L^1_2=L^2_1=1/2$ and L is null elsewhere. Then, associated with this L, we have a loop measure μ and a Poisson point process of loops of intensity $\alpha\mu$. Let us treat the case $\alpha\in]0,1[$.

Hypothesis 4.7.1. Suppose $\alpha\in]0,1[$.

The corresponding loop probability depends on κ and we will denote it by $\mathbb{P}^{(\kappa)}$. It has been showed in [LJL12] that the left points of the closed edges form a renewal process $(S^{(\kappa)}_m,m\geq 0)(S^{\kappa}_0=0)$, see Proposition 3.1 in [LJL12]. Moreover, in Proposition 3.1 of [LJL12], it has been proved that $(\epsilon S^{(\kappa\epsilon^2)}_{\lfloor\epsilon^{\alpha-1}t\rfloor},t\geq 0)$ converges to a subordinator $(X^{(\kappa)}_t,t\geq 0)$ with potential density $U(x,y)=1_{\{y>x\}}\left(\frac{2\sqrt{\kappa}}{1-e^{-2\sqrt{\kappa}(y-x)}}\right)^{\alpha}$ in the sense of finite marginal distribution. Once we show the tightness in the sense of Skorokhod, we could replace the finite marginal convergence by the convergence in law in the sense of Skorokhod.

Lemma 4.7.6 (Tightness of $(\epsilon S^{(\kappa\epsilon^2)}_{\lfloor\epsilon^{\alpha-1}t\rfloor},t\geq 0)$)**.** *The distribution $(\epsilon S^{(\kappa\epsilon^2)}_{\lfloor\epsilon^{\alpha-1}t\rfloor},t\geq 0)$ is tight in the Skorokhod space. Therefore, $(\epsilon S^{(\kappa\epsilon^2)}_{\lfloor\epsilon^{\alpha-1}t\rfloor},t\geq 0)$ converges to a subordinator $(X^{(\kappa)}_t,t\geq 0)$ in the sense of Skorokhod.*

Proof. Define $\mathfrak{F}_n=\sigma(S^{(\kappa\epsilon^2)}_1,\dots,S^{(\kappa\epsilon^2)}_n)$ for $n\in\mathbb{N}$ and $\mathfrak{F}^{(\epsilon)}_t=\mathfrak{F}_{\lfloor\epsilon^{\alpha-1}t\rfloor}$ for $t\geq 0$. Then, $(\mathfrak{F}^{(\epsilon)}_t,t\geq 0)$ is a right-continuous filtration. As usual, by adding the negligible sets, we get a the complete filtration which are denoted by the same notation.
Let T be a $(\mathfrak{F}^{(\epsilon)}_t,t\geq 0)$ stopping time. Then, $\lfloor\epsilon^{\alpha-1}T\rfloor$ is a $(\mathfrak{F}_n,n\in\mathbb{N})$ stopping time.
In order to show the tightness, it is enough to verify the following Aldous' criteria, see [JS03].
For each $M>0,\delta>0$,

$$\lim_{K\to\infty}\lim_{\epsilon\to 0}\mathbb{P}^{(\epsilon)}[\epsilon S^{(\kappa\epsilon^2)}_{\lfloor\epsilon^{\alpha-1}M\rfloor}>K]=0 \tag{4.1}$$

$$\lim_{\theta\downarrow 0}\lim_{\epsilon\to 0}\sup_{T_1,T_2\in\mathfrak{F}^{(\epsilon)}_M,T_1\leq T_2\leq T_1+\theta}\mathbb{P}^{(\epsilon)}[|\epsilon S^{(\kappa\epsilon^2)}_{\lfloor\epsilon^{\alpha-1}T_2\rfloor}-\epsilon S^{(\kappa\epsilon^2)}_{\lfloor\epsilon^{\alpha-1}T_1\rfloor}|>\delta]=0 \tag{4.2}$$

Since we already know the finite marginal convergence, condition (4.1) reduces to $\mathbb{P}[X^{(\kappa)}_M=\infty]=0$. Since $S^{(\kappa\epsilon^2)}$ is a renewal process, for $T_1,T_2\in\mathfrak{F}^{(\epsilon)}_M$ such that $T_1\leq T_2\leq T_1+\theta$, $|\epsilon S^{(\kappa\epsilon^2)}_{\lfloor\epsilon^{\alpha-1}T_2\rfloor}-\epsilon S^{(\kappa\epsilon^2)}_{\lfloor\epsilon^{\alpha-1}T_1\rfloor}|\leq|\epsilon S^{(\kappa\epsilon^2)}_{\lfloor\epsilon^{\alpha-1}T_1\rfloor+\lceil\epsilon^{\alpha-1}\theta\rceil}-\epsilon S^{(\kappa\epsilon^2)}_{\lfloor\epsilon^{\alpha-1}T_1\rfloor}|\overset{\text{law}}{=}|\epsilon S^{(\kappa\epsilon^2)}_{\lceil\epsilon^{\alpha-1}\theta\rceil}|$. By the finite marginal convergence,

$$\lim_{\theta\downarrow 0}\lim_{\epsilon\to 0}\sup_{T_1,T_2\in\mathfrak{F}^{(\epsilon)}_M,T_1\leq T_2\leq T_1+\theta}\mathbb{P}^{(\epsilon)}[|\epsilon S^{(\kappa\epsilon^2)}_{\lfloor\epsilon^{\alpha-1}T_2\rfloor}-\epsilon S^{(\kappa\epsilon^2)}_{\lfloor\epsilon^{\alpha-1}T_1\rfloor}|>\delta]$$

$$\leq \lim_{\theta\downarrow 0}\lim_{\epsilon\to 0}\mathbb{P}^{(\epsilon)}[|\epsilon S^{(\kappa\epsilon^2)}_{\lceil \epsilon^{\alpha-1}\theta\rceil}| > \delta]$$

$$\leq \lim_{\theta\downarrow 0}\mathbb{P}[|X^{(\kappa)}_{2\theta}| > \delta] = 0$$

The proof is complete. □

Immediately, we get the convergence in the sense of the Skorokhod. Using this result, we will the convergence of the corresponding bridge processes in the following proposition.

Proposition 4.7.7. *Define $(Z^{(\kappa/n^2)}_m, m \geq 0)$ by $Z^{(\kappa/n^2)}_m = S^{(\kappa/n^2)}_m \wedge n$. The law of $Z^{(\kappa/n^2)}$ depends on n, κ and is are denoted by $\mathbb{Q}^{n,\kappa}$. Conditioned on $\{n, n+1\}$ being closed, the left points of the closed edges together with the left point of $\{0,1\}$ form a renewal process conditioned to jump at n. Define a conditioned loop probability as follows: $\tilde{\mathbb{P}}^{(\kappa/n^2)}[\cdot] = \mathbb{P}^{(\kappa/n^2)}[\cdot|\{n, n+1\}$ is closed$]$. Let $\tilde{\mathbb{Q}}^{n,\kappa}$ be the law of $Z^{n,\kappa}$ under $\tilde{\mathbb{P}}^{(\kappa/n^2)}$. As n tends to infinity, under $\tilde{\mathbb{P}}^{(\kappa/n^2)}$, $(Z^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor}/n, t \geq 0)$ converges in law to $(X^{(\kappa)}_t \wedge 1, t \geq 0)$ conditioned on $\{X^{(\kappa)}_{T_{]1,\infty[}-} = 1\}$, in the sense of finite marginal convergence.*

Before proving this, let us precise the law of $(X^{(\kappa)}_t, t < T_{]1,\infty[})$ conditioned on the event $\{X^{(\kappa)}_{T_{]1,\infty[}-} = 1\}$ in the following lemma. Recall that the potential density $U(x,y)$ of $X^{(\kappa)}$ equals $1_{\{y>x\}}\left(\frac{2\sqrt{\kappa}}{1-e^{-2\sqrt{\kappa}(y-x)}}\right)^{\alpha}$. Set $u(x) = U(0,x)$ and $h(x) = U(x,1) = u(1-x)$.

Lemma 4.7.8.

1. *For all positive functions f, we have*

$$\mathbb{E}^0[f(X^{(\kappa)}_s, s\in[0,t])1_{\{t<T_{]1,\infty[}\}}, X^{(\kappa)}_{T_{]1,\infty[}-} \in db]$$
$$= \mathbb{E}^0[X^{(\kappa)}_{T_{]1,\infty[}-} \in db]\mathbb{E}^0[f(X^{(\kappa)}_s, s\in[0,t])1_{\{t<T_{]1,\infty[}\}}\frac{u(b-X^{(\kappa)}_t)}{u(b)}].$$

2. *The conditioned process[3] is a h-transform of the original subordinator with respect to the excessive function $x \to u(1-x)$. To be more precise, for $y \in [x,1[$, its semi-group is given by*

$$Q^{(\kappa)}_t(x,dy) = \frac{u(1-y)}{u(1-x)}P^{(\kappa)}_t(x,dy).$$

The process is right-continuous on $[0,\zeta[$.

3. *Denote by ζ the lifetime of the conditioned process $Y^{(\kappa)}$. Then, $Y^{(\kappa)}_{\zeta-} = 1$.*

4. *The semi-group Q is a Feller semi-group.*

[3]More precisely, the process defined by the probability $\mathbb{E}^0[f(X^{(\kappa)}_s, s\in[0,t])1_{\{t<T_{]1,\infty[}\}}\frac{u(1-X^{(\kappa)}_t)}{u(1)}]$.

5. *The time reversal from the lifetime of the process $Y^{(\kappa)}$ is the left-continuous modification of $1 - Y^{(\kappa)}$ under $\mathbb{Q}^0$.*

Proof.

1. The subordinator $(X_t^{(\kappa)}, t \geq 0)$ has the potential density $U(x,y) = (\frac{2\sqrt{\kappa}}{1-e^{-2\sqrt{\kappa}(y-x)}})^\alpha$ for $y \geq x$. When y tends to x, $U(x,y)$ tends to ∞. As a consequence, the drift coefficient $d = 0$. It is proved by H. Kesten [Kes69] that for a fixed $x > 0$, x does not belong to the range of the subordinator with probability 1, see Proposition 1.9 in [Ber99]. By using the strong Markov property at stopping time S,

$$\mathbb{E}^0[f(X_s^\kappa, s \in [0,S])1_{\{S<T_{]1,\infty[}\}}, X^{(\kappa)}_{T_{]1,\infty[-}} \in db] \\ = \mathbb{E}^0\left[f(X_S^\kappa, S \in [0,t])1_{\{S<T_{]1,\infty[}\}}\mathbb{E}^{X_S^{(\kappa)}}[X^{(\kappa)}_{T_{]1,\infty[-}} \in db]\right].$$

Then, we use Lemma 1.10 in [Ber99]:

- $\mathbb{E}^{X_S^{(\kappa)}}[X^{(\kappa)}_{T_{]1,\infty[-}} \in db] = \bar{\Pi}(1-b)u(b - X_S^{(\kappa)})\,db,$[4]
- $\mathbb{E}^0[X^{(\kappa)}_{T_{]1,\infty[-}} \in db] = \bar{\Pi}(1-b)u(b)\,db.$

Immediately, we have

$$\mathbb{E}^0\left[f(X_s^\kappa, s \in [0,S])1_{\{S<T_{]1,\infty[}\}}\mathbb{E}^{X_S^{(\kappa)}}[X^{(\kappa)}_{T_{]1,\infty[-}} \in db]\right] \\ = \mathbb{E}^0[X^{(\kappa)}_{T_{]1,\infty[}} \in db]\mathbb{E}^0[f(X_s^\kappa, s \in [0,S])1_{\{S<T_{]1,\infty[}\}}\frac{u(b - X_S^{(\kappa)})}{u(b)}].$$

In particular, we take fixed time t,

$$\mathbb{E}^0[f(X_s^\kappa, s \in [0,t])1_{\{t<T_{]1,\infty[}\}}|X^{(\kappa)}_{T_{]1,\infty[-}} = 1] \\ = \mathbb{E}^0[f(X_s^\kappa, s \in [0,t])1_{\{t<T_{]1,\infty[}\}}\frac{u(1 - X_t^{(\kappa)})}{u(1)}]. \quad (*)$$

2. It is enough to show that $x \to u(1-x)$ is excessive. The rest follows from the classical results on the h-transform, see Chapter 11 of [CW05]. Take a positive function g, we have $P_tUg = \int_t^\infty P_s g\,ds$ and $Ug = \int_0^\infty P_s g\,ds$. Then, for all positive function g, we have $P_tUg \leq Ug$ and $\lim_{t\to 0} P_tUg = Ug$. As a consequence, except on a set $N(x)$ of zero Lebesgue measure, $y \to u(y,z)$ is an excessive function for all z, i.e.

   - $\int P_t(x,dy)u(y,z) \leq u(x,z)$,

[4] Here, $\bar{\Pi}$ represents the tail of the characteristic measure of the subordinator.

- $\lim_{t \to 0} P_t(x, dy) u(y, z) = u(x, z)$.

  Take a decreasing sequence $z_1 > \cdots$ with limit 1. As the increasing limit of a sequence of excessive functions $y \to u(y, z_n)$, $y \to u(y, 1)$ is excessive.

3. It is enough to show that $Q^x[T_{[1-\delta,\infty[} < \zeta] = 1$ for any $\delta > 0$. In fact, by Theorem 11.9 of [CW05],

$$\mathbb{Q}^x[T_{[1-\delta,\infty[} < \zeta] = \mathbb{P}^x \left[T_{[1-\delta,\infty[} < T_{]1,\infty[}, \frac{u(1 - X_{T_{[1-\delta,\infty[}})}{u(1-x)} \right].$$

$$\mathbb{P}^0[X_{T_{[1-\delta,\infty[}-} \in da, X_{T_{[1-\delta,\infty[}-} - X_{T_{[1-\delta,\infty[}-} \in db] = u(a)\, da \Pi(db).$$

Consequently,

$$\begin{aligned}\mathbb{Q}^x[T_{[1-\delta,\infty[} < \zeta] =& \mathbb{P}^0 \left[T_{[1-x-\delta,\infty[} < T_{]1-x,\infty[}, \frac{u(1 - x - X_{T_{[1-x-\delta,\infty[}})}{u(1-x)} \right] \\ =& \int\limits_{0<a<1-x-\delta<a+b<1-x} \frac{u(1-x-a-b)}{u(1-x)} u(a)\, da \Pi(db)\end{aligned}$$

Set $c = 1 - x - a - b$.

$$\begin{aligned}\mathbb{Q}^x[T_{[1-\delta,\infty[} < \zeta] =& \int\limits_{0<c<\delta<c+b<1-x} \frac{u(c)}{u(1-x)} u(1-x-c-b)\, dc \Pi(db) \\ =& \mathbb{P}^0 \left[T_{[\delta,\infty[} < T_{]1-x,\infty[}, \frac{u(1 - x - X_{T_{[\delta,\infty[}})}{u(1-x)} \right] \\ =& \mathbb{Q}^x[T_{[x+\delta,\infty[} < \zeta]\end{aligned}$$

By the right-continuity of the path,

$$\lim_{\delta \to 0} \mathbb{Q}^x[T_{[1-x-\delta,\infty[} < \zeta] = \lim_{\delta \to 0} \mathbb{Q}^x[T_{[x+\delta,\infty[} < \zeta] = 1.$$

But $\mathbb{Q}^x[T_{[x+a,\infty[} < \zeta]$ decreases as a increases. Then, we must have

$$\mathbb{Q}^x[T_{[y,\infty[} < \zeta] = 1 \text{ for } y \in [x, 1[.$$

4. We know that $P_t^{(\kappa)}$ is a Feller semi-group. For $f \in C_K([0,1[)$[5], $x \to Q_t f(x)$ belongs to $C_0([0,1[)$. By the Markov property of the semi-group Q_t, i.e. $||Q_t f||_\infty \le ||f||_\infty$, we have $Q_t f = \lim_{n \to \infty} Q_t(f|_{[0,1-1/n[}) \in C_0([0,1[)$. For fixed $x \in [0,1[$,

$$\lim_{t \to 0} Q_t f(x) = \lim_{t \to 0} \mathbb{P}^x[1_{\{t < T_{]1,\infty[}\}} f(X_t^{(\kappa)}) \frac{u(1 - X_t^{(\kappa)})}{u(1-x)}] = f(x).$$

Then we see that the semi-group $(Q_t, t \ge 0)$ is Feller.

[5] The collection of compact supported continuous function over $[0,1[$.

5. By a classical result about time reversal, the reversed process is a moderate Markov process, its semi-group $\hat{Q}_t(x, dy)$ is given by the following formula:

$$\langle g, Q_t f\rangle_G = \langle \hat{Q}_t g, f\rangle_G.$$

where $Q_t(x, dy) = \frac{U(y,1)}{U(x,1)} P_t(x, dy)$ and $G(dx) = \int\limits_0^\infty Q_t(0, dx)\, dt = \frac{U(0,x)U(x,1)}{U(0,1)}\, dx$. Denote by $(\hat{P}_t, t \geq 0)$ the dual semi-group of $(P_t, t \geq 0)$ or the semi-group of $-X^{(\kappa)}$ equivalently. Denote by $u(x)$ the function $U(0, x)$ and by $h(x)$ the function $U(x, 1)$.

$$\begin{aligned}\langle g, Q_t f\rangle_G =& \frac{P_t(hf)(x)}{h(x)} g(x) \frac{u(x)h(x)}{u(1)}\, dx \\ =& f(x) \frac{\hat{P}_t(ug)}{u(x)} \frac{u(x)h(x)}{u(1)}\, dx \\ =& \langle f, \frac{\hat{P}_t(ug)}{u}\rangle_G.\end{aligned}$$

This implies that the semi-group $(\hat{Q}_t, t \geq 0)$ associated with the reversed process of Y is given by

$$\hat{Q}_t(x, dy) = \hat{P}_t(x, dy) \frac{U(0, y)}{U(0, x)} = \hat{P}_t(x, dy) \frac{U(1 - y, 1)}{U(1 - x, 1)}.$$

Then, by a change of variable, we find it equals the semi-group of $1 - Y^{(\kappa)}$. By 3, the reversed process starts from 1. Then, it is exactly the left-continuous modification of $1 - Y^{(\kappa)}$ for $Y^{(\kappa)}$ starting from 0.

□

The above proposition gives the Radon-Nikodym derivative between the subordinator and its bridge on a sub-σ-field. We will prove Proposition 4.7.7 by showing the convergence of the Radon-Nikodym derivative from the discrete model to the continuous case.

Proof of Proposition 4.7.7. Define $f^{n,\kappa}(k) = \mathbb{P}^{\kappa/n^2}[S_1^{(\kappa/n^2)} = k]$, $g^{n,\kappa}(k) = \mathbb{P}^{\kappa/n^2}[S_1^{(\kappa/n^2)} \geq k]$ and $C^{\kappa/n^2}(k) = \mathbb{P}^{\kappa/n^2}[\{k, k+1\}$ is closed$]$. Let $W_i^{(\kappa)} = S_i^{(\kappa/n^2)} - S_{i-1}^{(\kappa/n^2)}$ for $i \in \mathbb{N}_+$. As mentioned above, $W_i, i \in \mathbb{N}_+$ is a sequence of i.i.d. variables. Then,

$$\frac{d\tilde{\mathbb{Q}}^{n,\kappa}}{d\mathbb{Q}^{n,\kappa}} = \frac{f^{n,\kappa}(n - S_{T_n - 1}^{(\kappa/n^2)})}{C^{\kappa/n^2}(n) g^{n,\kappa}(n - S_{T_n - 1}^{(\kappa/n^2)})}$$

where $T_p = \inf\{l \in \mathbb{N} : S_l^{(\kappa/n^2)} = p\}$. Let $\mathcal{F}_m = \sigma(S_0^{(\kappa/n^2)}, \dots, S_m^{(\kappa/n^2)})$ and $\mathcal{G}_m = \sigma(Z_0, \dots, Z_m)$. Then, $\mathcal{G}_m \cap \{m < T_n\} = \mathcal{F}_m \cap \{m < T_n\}$ and $\mathcal{G}_m \cap \{T_n \leq m\} = \mathcal{F}_{T_n - 1} \cap \{T_n \leq m\}$.[6]

$$\left.\frac{d\tilde{\mathbb{Q}}^{n,\kappa}}{d\mathbb{Q}^{n,\kappa}}\right|_{\mathcal{G}_m} = \mathbb{E}^{(\kappa/n^2)} \left[\left.\frac{d\tilde{\mathbb{Q}}^{n,\kappa}}{d\mathbb{Q}^{n,\kappa}} \right| \mathcal{G}_m \right]$$

[6] $T_n - 1$ is a $(\mathcal{F}_{m+1}, m \geq 0)$ stopping time.

$$
\begin{aligned}
=&1_{\{T_n\le m\}}\frac{f^{n,\kappa}(n-S^{(\kappa/n^2)}_{T_n-1})}{C^{\kappa/n^2}(n)g^{n,\kappa}(n-S^{(\kappa/n^2)}_{T_n-1})}\\
&+\mathbb{E}^{(\kappa/n^2)}\left[1_{\{T_n\ge m+1\}}\frac{f^{n,\kappa}(n-S^{(\kappa/n^2)}_{T_n-1})}{C^{\kappa/n^2}(n)g^{n,\kappa}(n-S^{(\kappa/n^2)}_{T_n-1})}\middle|\mathcal{G}_m\right]\\
=&1_{\{T_n\le m\}}\frac{f^{n,\kappa}(n-S^{(\kappa/n^2)}_{T_n-1})}{C^{\kappa/n^2}(n)g^{n,\kappa}(n-S^{(\kappa/n^2)}_{T_n-1})}\\
&+\mathbb{E}^{(\kappa/n^2)}\left[1_{\{T_n\ge m+1\}}\frac{f^{n,\kappa}(n-S^{(\kappa/n^2)}_{m}-(S^{(\kappa/n^2)}_{T_n-1}-S^{(\kappa/n^2)}_{m}))}{C^{\kappa/n^2}(n)g^{n,\kappa}(n-S^{(\kappa/n^2)}_{m}-(S^{(\kappa/n^2)}_{T_n-1}-S^{(\kappa/n^2)}_{m}))}\middle|\mathcal{G}_m\right]\\
=&1_{\{T_n\le m\}}\frac{f^{n,\kappa}(n-S^{(\kappa/n^2)}_{T_n-1})}{C^{\kappa/n^2}(n)g^{n,\kappa}(n-S^{(\kappa/n^2)}_{T_n-1})}+1_{\{T_n\ge m+1\}}\frac{C^{\kappa/n^2}(n-S^{(\kappa/n^2)}_{m})}{C^{\kappa/n^2}(n)}
\end{aligned}
$$

In the proof of Proposition 3.1 of [LJL12], it is been showed that

$$
C^{\kappa/n^2}(m)=\left(\frac{1-(1+\frac{\kappa}{2n^2}+\sqrt{\frac{\kappa}{n^2}+\frac{\kappa^2}{4n^4}})^{-2}}{1-(1+\frac{\kappa}{2n^2}+\sqrt{\frac{\kappa}{n^2}+\frac{\kappa^2}{4n^4}})^{-2(m+1)}}\right)^{\alpha}
$$

(In fact, $C^{\kappa/n^2}(m)$ is denoted $q^{\kappa/n^2}(m)$ there). We deduce the following estimation for $C^{\kappa/n^2}(\lfloor bn\rfloor)(b>0)$ as n tends to ∞:

$$
C^{\kappa/n^2}(\lfloor bn\rfloor)\sim\begin{cases}(\frac{2\sqrt{\kappa}}{1-e^{-2b\sqrt{\kappa}}})^{\alpha}n^{-\alpha} & \kappa>0\\ (bn)^{-\alpha} & \kappa=0\end{cases}
$$

Moreover, for any compact subset $K\subset]0,\infty[$, $(C^{\kappa/n^2}(\lfloor bn\rfloor)n^{\alpha},b\in K)$ converge uniformly. Define $\mathcal{G}^{(n)}_t=\mathcal{G}_{\lfloor n^{1-\alpha}t\rfloor}$ for $t\ge 0$. Then,

$$
\frac{d\tilde{\mathbb{Q}}^{n,\kappa}}{d\mathbb{Q}^{n,\kappa}}\bigg|_{\mathcal{G}^{(n)}_t}1_{\{Z^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor}/n<1\}}=\frac{d\tilde{\mathbb{Q}}^{n,\kappa}}{d\mathbb{Q}^{n,\kappa}}\bigg|_{\mathcal{G}^{(n)}_t}1_{\{S^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor}/n<1\}}=1_{\{S^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor}/n<1\}}\frac{C^{\kappa/n^2}(n-S^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor})}{C^{\kappa/n^2}(n)}.
$$

It is proved in [Ber96] that for any fixed $x>0$, $X^{(\kappa)}_{T_{]x,\infty[}}>x>X^{(\kappa)}_{T_{]x,\infty[}-}$ holds with probability 1 if $d=0$.

From Lemma 4.7.6, we know that the sequence of renewal process $S^{(\kappa/n^2)}$ converges towards the subordinator $X^{(\kappa)}$ in the sense of Skorokhod. By the coupling theorem of Skorokhod and Dudley, we can suppose that $S^{(\kappa/n^2)}$ converges to $X^{(\kappa)}$ almost surely as long as our result only depends on the law. Since for any $x>0$, $X^{(\kappa)}_{T_{]x,\infty[}}>x>X^{(\kappa)}_{T_{]x,\infty[}-}$ holds with probability 1, $\frac{S_{T_n-1}}{n}$ converges to $X^{(\kappa)}_{T_{]1,\infty[}-}$. Moreover, if we fix $t\ge 0$, $1_{\{S^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor}/n<1\}}\frac{C^{\kappa/n^2}(n-S^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor})}{C^{\kappa/n^2}(n)}$ converges to $1_{\{T_{]1,\infty[}>t\}}\frac{u(1-X^{(\kappa)}_t)}{u(1)}$ since $X^{(\kappa)}$ is continuous at time t and $X^{(\kappa)}_{T_{]1,\infty[}}>1>X^{(\kappa)}_{T_{]1,\infty[}-}$ almost surely. Consequently, $\frac{d\tilde{\mathbb{Q}}^{n,\kappa}}{d\mathbb{Q}^{n,\kappa}}|_{\mathcal{G}_t}1_{\{Z^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor}/n<1\}}$ converges to $1_{\{T_{]1,\infty[}>t\}}\frac{u(1-X^{(\kappa)}_t)}{u(1)}$ almost surely.

The Proposition 3.1 in [LJL12] gives the density of the renewal measure of the subordinator $(X_t^{(\kappa)}, t \geq 0)$:

$$u(s) = U(0,s) = (\frac{2\sqrt{\kappa}}{1-e^{-2\sqrt{\kappa}s}})^\alpha \text{ for } s > 0.$$

Let $\mathbb{Q}^x$ stand for the law of the Markov process with sub-Markovian semi-group $Q_t(x,dy) = \frac{u(1-y)}{u(1-x)} P_t(x,dy)$ and initial state x. By Lemma 4.7.8,

$$\mathbb{Q}^x[1_A, t < \zeta] = \mathbb{P}^x \left[1_A \frac{u(1-X_t^{(\kappa)})}{u(1-x)} 1_{\{t<\zeta\}} \right] \text{ for } A \in \mathcal{F}_t$$

where $\mathbb{P}^x$ is the law of the process $X^{(\kappa)}$ starting from x. By dominated convergence, for any $\delta > 0$,

$$\lim_{n\to\infty} \tilde{\mathbb{Q}}^{n,\kappa} \left[\frac{1}{n} Z^{(\kappa/n^2)}_{\lfloor n^{1-\alpha} t \rfloor} < 1-\delta \right] = \mathbb{Q}^0[X_t^{(\kappa)} < 1-\delta]$$

It is clear that $\tilde{\mathbb{Q}}^{n,\kappa}[\frac{1}{n} Z^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor} \leq 1] = 1 = \mathbb{Q}^0[X_t^{(\kappa)} \leq 1]$. Therefore, for any fixed t, $\frac{1}{n} Z^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor}$ converges in law towards $X_t^{(\kappa)}$ (under the law $\tilde{\mathbb{Q}}^{n,\kappa}$ and $\mathbb{Q}^0$ respectively) as n tends to infinity. In particular, we have

$$\begin{aligned}\lim_{n\to\infty} \mathbb{Q}^{n,\kappa}[1_{\{S^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor}/n<1\}} \frac{C^{\kappa/n^2}(n - S^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor})}{C^{\kappa/n^2}(n)}] &= \lim_{n\to\infty} \tilde{\mathbb{Q}}^{n,\kappa}[\frac{1}{n} Z^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor} < 1] \\ &= \mathbb{Q}^0[X_t^{(\kappa)} < 1] \\ &= \mathbb{P}^0[X_t^{(\kappa)} < 1, \frac{u(1-X_t)}{u(1)}].\end{aligned}$$

Taking any bounded continuous f, by the coupling assumption and dominated convergence[7],

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{Q}^{n,\kappa} \left[f\left(\frac{1}{n} S^{\kappa/n^2}_{\lfloor n^{1-\alpha}s\rfloor}, s \in [0,t]\right) 1_{\{S^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor}/n<1\}} \frac{C^{\kappa/n^2}(n - S^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor})}{C^{\kappa/n^2}(n)} \right] \\ &\qquad = \mathbb{P}^0 \left[f(X_s^{(\kappa)}, s \in [0,t]) \frac{u(1-X_t^{(\kappa)})}{u(1)}, X_t^{(\kappa)} < 1 \right]\end{aligned}$$

Equivalently,

$$\begin{aligned}&\lim_{n\to\infty} \tilde{\mathbb{Q}}^{n,\kappa} \left[f\left(\frac{1}{n} Z^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}s\rfloor}, s \in [0,t]\right), \frac{1}{n} Z^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor} < 1 \right] \\ &\qquad = \mathbb{Q}^0[f(X_s^{(\kappa)}, s \in [0,t]), X_t^{(\kappa)} < 1]\end{aligned}$$

Therefore, we have the finite marginals convergence. □

[7] The dominating sequence is $\left(1_{\{S^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor}/n<1\}} \frac{C^{\kappa/n^2}(n - S^{(\kappa/n^2)}_{\lfloor n^{1-\alpha}t\rfloor})}{C^{\kappa/n^2}(n)}, n > 0 \right)$.

Chapter 5

Loop erasure and spanning tree

In this section we will show that Poisson processes of loops appear naturally in the construction of random spanning trees.

5.1 Loop erasure

Suppose ω is the path of a minimal transient canonical Markov process, then its path can be expressed as a sequence $(x_0, t_0, x_1, t_1, \ldots)$. The corresponding discrete path $(x_0, x_1, \ldots)$ is the embedded Markov chain. From the transience assumption, $\sum\limits_{n\in\mathbb{N}} 1_{\{x_n=x\}} < \infty$ a.s..

Definition 5.1.1 (Loop erasure)**.** The loop erasure operation which maps a path ω to its loop erased path ω_{BE} is defined as: $\omega_{BE} = (y_0, \ldots)$ with $y_0 = x_0$. Define $T_0 = \inf\{n \in \mathbb{N} : \forall m \geq n, x_m \neq y_0\}$, then set $y_1 = x_{T_0}$. Similarly define $T_1 = \inf\{n \in \mathbb{N} : \forall m \geq n, x_m \neq y_1\}$, set $y_2 = x_{T_1}$ and so on. Let $\mathbb{P}^{\nu}_{BE}$ be the image measure of $\mathbb{P}^{\nu}$ where ν is the initial distribution of the Markov process.

Recall that ∂ is the cemetery point, that $Q^x_\partial = 1 - \sum\limits_{y\neq\partial} Q^x_y$ for $x \neq \partial$ and $Q^\partial_x = \delta^\partial_x$. Set $L^x_\partial = -\sum\limits_{y\neq\partial} L^x_y$ for $x \neq \partial$, $L^\partial_\partial = -1$ and $L^\partial_x = 0$ for $x \neq \partial$.

Proposition 5.1.1. *We have the following finite marginal distribution for the loop-erased random walk:*

$$\begin{aligned}\mathbb{P}^{\nu}_{BE}&[\omega_{BE} = (x_0, x_1, \ldots, x_n, \ldots)]\\ &= \nu_{x_0} \det(V_{\{x_0,\ldots,x_{n-1}\}}) L^{x_0}_{x_1} \cdots L^{x_{n-1}}_{x_n} \mathbb{P}^{x_n}[T_{\{x_0,\ldots,x_{n-1}\}} = \infty]\end{aligned}$$

$$= \nu_{x_0} L_{x_1}^{x_0} \cdots L_{x_n}^{x_{n-1}} \begin{vmatrix} V_{x_0}^{x_0} & \cdots & V_{x_{n-1}}^{x_0} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ V_{x_0}^{x_{n-1}} & \cdots & V_{x_{n-1}}^{x_{n-1}} & 1 \\ V_{x_0}^{x_n} & \cdots & V_{x_{n-1}}^{x_n} & 1 \end{vmatrix}.$$

Proof. Starting from x_n, the probability that the Markov process never reaches $x_0, \ldots, x_{n-1}$, $\mathbb{P}^{x_n}[T_{\{x_0,\ldots,x_{n-1}\}} = \infty]$ equals the same probability for the trace of the Markov process on $x_0, \ldots, x_n$, $\mathbb{P}^{x_n}_{\{x_0,\ldots,x_n\}}[T_{\{x_0,\ldots,x_{n-1}\}} = \infty]$. It equals the one step transition probability from x_n to ∂ for the trace of the process. Let $L_{\{x_0,\ldots,x_n\}}$ be the generator of the trace of the Markov process on $\{x_0, \ldots, x_n\}$. Then, the one step transition probability from x_n to ∂ equals $\frac{(L_{\{x_0,\ldots,x_n\}})_\partial^{x_n}}{-(L_{\{x_0,\ldots,x_n\}})_{x_n}^{x_n}}$. Since $(L_{\{x_0,\ldots,x_n\}})_\partial^{x_n} = -(L_{\{x_0,\ldots,x_n\}})_{x_n}^{x_n} - \sum_{i=0}^{n-1} (L_{\{x_0,\ldots,x_n\}})_{x_i}^{x_n}$ and $-(L_{\{x_0,\ldots,x_n\}})_{x_i}^{x_n} = (-1)^{i+1+n+1} \frac{\det(V|_{\{x_0,\ldots,x_n\}\setminus\{x_i\}\times\{x_0,\ldots,x_{n-1}\}})}{\det(V_{\{x_0,\ldots,x_n\}})}$, we have

$$(L_{\{x_0,\ldots,x_n\}})_\partial^{x_n} = \frac{1}{\det(V_{\{x_0,\ldots,x_n\}})} \begin{vmatrix} V_{x_0}^{x_0} & \cdots & V_{x_{n-1}}^{x_0} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ V_{x_0}^{x_{n-1}} & \cdots & V_{x_{n-1}}^{x_{n-1}} & 1 \\ V_{x_0}^{x_n} & \cdots & V_{x_{n-1}}^{x_n} & 1 \end{vmatrix}$$

$$\text{and } -(L_{\{x_0,\ldots,x_n\}})_{x_n}^{x_n} = \begin{vmatrix} V_{x_0}^{x_0} & \cdots & V_{x_{n-1}}^{x_0} \\ \vdots & \ddots & \vdots \\ V_{x_0}^{x_{n-1}} & \cdots & V_{x_{n-1}}^{x_{n-1}} \end{vmatrix}.$$

Therefore, $\mathbb{P}^{x_n}[T_{\{x_0,\ldots,x_{n-1}\}} = \infty] = \begin{vmatrix} V_{x_0}^{x_0} & \cdots & V_{x_{n-1}}^{x_0} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ V_{x_0}^{x_{n-1}} & \cdots & V_{x_{n-1}}^{x_{n-1}} & 1 \\ V_{x_0}^{x_n} & \cdots & V_{x_{n-1}}^{x_n} & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} V_{x_0}^{x_0} & \cdots & V_{x_{n-1}}^{x_0} \\ \vdots & \ddots & \vdots \\ V_{x_0}^{x_{n-1}} & \cdots & V_{x_{n-1}}^{x_{n-1}} \end{vmatrix}.$

Set $D_0 = \phi$ and $D_k = \{x_0, \ldots, x_{k-1}\}$ for $k \in \mathbb{N}_+$. Note that $Q|_{D_k^c \times D_k^c}$ is the transition probability for the process restricted in D_k^c. In order for the loop-erased path ω_{BE} to be $(x_0, x_1, \ldots, x_n, \ldots)$, the random walk must start from x_0. After some excursions back to x_0, it should jump to x_1 and never return to x_0. Next, after some excursions from x_1 to x_1, it jumps to x_2 and never returns to x_0, x_1, etc. Accordingly,

$$\begin{aligned} &\mathbb{P}^\nu_{BE}[\omega_{BE} = (x_0, x_1, \ldots, x_n, \ldots)] \\ &\qquad = \nu_{x_0} \prod_{k=0}^{n-1} \Big(\sum_{n \geq 0} ((Q|_{D_k^c \times D_k^c})^n)_{x_k}^{x_k}\Big) Q_{x_{k+1}}^{x_k} \mathbb{P}^{x_n}[T_{\{x_0,\ldots,x_{n-1}\}} = \infty] \end{aligned}$$

$$= \nu_{x_0} \prod_{k=0}^{n-1} (V^{D_k^c})_{x_k}^{x_k} L_{x_{k+1}}^{x_k} \mathbb{P}^{x_n}[T_{\{x_0,\dots,x_{n-1}\}} = \infty]$$

where $L^{D_k^c} = L|_{D_k^c \times D_k^c}$ is the generator of the Markov process restricted in D_k^c, and $V^{D_k^c}$ be the corresponding potential, see Definition 2.3.1.

Let V_F stands for the sub-matrix of V restricted to $F \times F$. It is also the potential of the trace of the Markov process on F and let $\mathbb{P}_F$ stand for its law. Then, for all $D \subset F$, we have $(V^{D^c})_F = (V_F)^{D^c}$. In particular, for $k < n$, we have $(V^{D_k^c})_{x_k}^{x_k} = ((V^{D_k^c})_{D_n})_{x_k}^{x_k} = ((V_{D_n})^{D_k^c})_{x_k}^{x_k}$. One can apply Jacobi's formula

$$\det(A|_{B\times B}) \det(A^{-1}) = \det(A^{-1}|_{B^c \times B^c})$$

for $A = (V_{D_n})^{D_k^c}$ and $B = \{x_k\}$. To be more precise, since $((V_{D_n})^{D_k^c})^{-1} = (-L_{D_n})|_{D_k^c \times D_k^c} = (-L_{D_n})|_{(D_n - D_k)\times(D_n - D_k)}$, we have

$$(V^{D_k^c})_{x_k}^{x_k} = ((V_{D_n})^{D_k^c})_{x_k}^{x_k} = \frac{\det(-L_{D_n}|_{(D_n - D_{k+1})\times(D_n - D_{k+1})})}{\det(-L_{D_n}|_{(D_n - D_k)\times(D_n - D_k)})}$$

with the convention that $\det(-L_{D_n}|_\phi) = 1$. Then,

$$\prod_{k=0}^{n-1} (V^{D_k^c})_{x_k}^{x_k} = \prod_{k=0}^{n-1} \frac{\det(-L_{D_n}|_{(D_n - D_{k+1})\times(D_n - D_{k+1})})}{\det(-L_{D_n}|_{(D_n - D_k)\times(D_n - D_k)})} = \frac{\det(-L_{D_n}|_{(D_n - D_n)\times(D_n - D_n)})}{\det(-L_{D_n}|_{(D_n - D_0)\times(D_n - D_0)})}$$
$$= \frac{1}{\det(-L_{D_n})} = \det((V_{D_n})_{D_n}) = \det(V_{\{x_0,\dots,x_{n-1}\}}).$$

Finally, by combining the results above, we conclude that

$$\begin{aligned}
&\mathbb{P}_{BE}^{\nu}[\omega_{BE} = (x_0, x_1, \dots, x_n, \dots)] \\
&\qquad = \nu_{x_0} \det(V_{\{x_0,\dots,x_{n-1}\}}) L_{x_1}^{x_0} \cdots L_{x_n}^{x_{n-1}} \mathbb{P}^{x_n}[T_{\{x_0,\dots,x_{n-1}\}} = \infty] \\
&\qquad = \nu_{x_0} L_{x_1}^{x_0} \cdots L_{x_n}^{x_{n-1}} \begin{vmatrix} V_{x_0}^{x_0} & \cdots & V_{x_{n-1}}^{x_0} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ V_{x_0}^{x_{n-1}} & \cdots & V_{x_{n-1}}^{x_{n-1}} & 1 \\ V_{x_0}^{x_n} & \cdots & V_{x_{n-1}}^{x_n} & 1 \end{vmatrix}.
\end{aligned}$$

□

Remark 14. Since a Markov chain in a countable space could be viewed as a pure-jump sub-Markov process with jumping rate 1, the above result holds for a sub-Markov chain if we replace L by the transition matrix $Q - Id$ and $V = (Id - Q)^{-1}$.

The following property was discovered by Omer Angel and Gady Kozma, see Lemma 1.2 in [Koz07]. Here, we give a different proof as an application of Proposition 5.1.1.

Proposition 5.1.2. *Let $(X_m, m \in [0,\zeta[)$ be a discrete Markov chain in a countable space S with time life ζ and initial point x_0. Fix some $w \in S \setminus \{x_0\}$, define $T_1 = \inf\{n > 0 : X_n = w\}$ and $T_N = \inf\{m > T_{N-1} : X_m = w\}$ with the convention that $\inf \phi = \infty$. We can perform loop-erasure for the path $(X_0, \ldots, X_{T_N})$, and let $LE[0,T_N]$ stand for the loop-erased self-avoiding path obtained in that way. If $T_N < \infty$ with positive probability, then the conditional law of $LE[0,T_N]$ given that $\{T_N < \infty\}$ is the same as the conditional law of $LE[0,T_1]$ given that $\{T_1 < \infty\}$.*

Proof. We suppose $T_1 < \infty$ with positive probability. By adding a small killing rate ϵ at all states and taking $\epsilon \downarrow 0$, we could suppose that we have a positive probability to jump to the cemetery point from any state. In particular, the Markov chain is transient.

Let ∂ be the cemetery point. Let $\tau(p)$ be a geometric variable with mean $1/p$, independent of the Markov chain. Let $(X_m^{(p)}, m \in [0,(\zeta-1)\wedge T_{\tau(p)}]$ be the sub-Markov chain X stopped after $T_{\tau(p)}$ which is again sub-Markov. Let $\mathbb{P}_p^{x_0}$ stand for the law of $X^{(p)}$ and let $\mathbb{P}_{p,BE}^{x_0}$ stand for the law of the loop-erased random walk associated to $(X_m^{(p)}, m \in [0,(\zeta-1)\wedge T_{\tau(p)}])$. Let $Q^{(p)}$ be the transition matrix of $X^{(p)}$ and use the notation Q for $Q^{(0)}$. Then, $(Q^{(p)})_i^w = (1-p)Q_i^w$ for $i \in S$ and $(Q^{(p)})_j^i = Q_j^i$ for $i \in S\setminus\{w\}$ and $j \in S$. Accordingly, $(Q^{(p)})_\partial^w = p + Q_\partial^w - pQ_\partial^w$. Define $V = (I-Q)^{-1}$, $V_{q\delta_w} = (M_{q\delta_w} + I - Q)^{-1}$ for $q \geq 0$ and $V^{(p)} = (I - Q^{(p)})^{-1} = (M_{(1-p)\delta_w}(I + \frac{p}{1-p} - Q))^{-1} = V_{\frac{p}{1-p}\delta_w} M_{\frac{1}{1-p}\delta_w}$. Set $C_\omega = \{$the loop-erased random walk stopped at $w\}$ Then,

$$\begin{aligned}
C_\Omega =& \{\text{the random walk stopped at } w\}\\
=& \bigcup_{n\geq 1}\{\text{the random walk stopped at } w \text{ at time } T_k\}\\
=& \bigcup_{k\geq 1}\{\tau(p) = k, T_k < \zeta\} \bigcup \bigcup_{k\geq 1}\{T_k = \zeta - 1, \tau(p) > k\}.
\end{aligned}$$

For $x_n = w$,

$$\begin{aligned}
&\mathbb{P}_{p,BE}^{x_0}[\omega_{BE} = (x_0, x_1, \ldots, x_n = w)]\\
&\qquad = (Q^{(p)})_{x_1}^{x_0} \cdots (Q^{(p)})_{x_n}^{x_{n-1}} (Q^{(p)})_\partial^{x_n} \begin{vmatrix} (V^{(p)})_{x_0}^{x_0} & \cdots & (V^{(p)})_{x_n}^{x_0} \\ \vdots & \ddots & \vdots \\ (V^{(p)})_{x_0}^{x_n} & \cdots & (V^{(p)})_{x_n}^{x_n} \end{vmatrix}\\
&\qquad\qquad = \frac{p + Q_\partial^w - pQ_\partial^w}{(1-p)Q_\partial^w} Q_{x_1}^{x_0} \cdots Q_{x_n}^{x_{n-1}} Q_\partial^{x_n} \begin{vmatrix} (V_{\frac{p}{1-p}\delta_w})_{x_0}^{x_0} & \cdots & (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_0} \\ \vdots & \ddots & \vdots \\ (V_{\frac{p}{1-p}\delta_w})_{x_0}^{x_n} & \cdots & (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_n} \end{vmatrix}.
\end{aligned}$$

By the resolvent equation, $V_j^i = (V_{\frac{p}{1-p}\delta_w})_j^i + \frac{p}{1-p}(V_{\frac{p}{1-p}\delta_w})_w^i V_j^w = (V_{\frac{p}{1-p}\delta_w})_j^i + \frac{p}{1-p}(V_{\frac{p}{1-p}\delta_w})_j^w V_w^i$. Therefore,

$$\begin{aligned}
\begin{vmatrix} V_{x_0}^{x_0} & \cdots & V_{x_n}^{x_0} \\ \vdots & \ddots & \vdots \\ V_{x_0}^{x_n} & \cdots & V_{x_n}^{x_n} \end{vmatrix} &= \begin{vmatrix} 1 & -\frac{p}{1-p}V_{x_0}^{x_n} & \cdots & -\frac{p}{1-p}V_{x_n}^{x_n} \\ (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_0} & (V_{\frac{p}{1-p}\delta_w})_{x_0}^{x_0} & \cdots & (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_0} \\ \vdots & \vdots & \ddots & \vdots \\ (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_0} & (V_{\frac{p}{1-p}\delta_w})_{x_0}^{x_n} & \cdots & (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_n} \end{vmatrix} \\
&= \begin{vmatrix} 1+\frac{p}{1-p}V_{x_n}^{x_n} & -\frac{p}{1-p}V_{x_0}^{x_n} & \cdots & -\frac{p}{1-p}V_{x_n}^{x_n} \\ 0 & (V_{\frac{p}{1-p}\delta_w})_{x_0}^{x_0} & \cdots & (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (V_{\frac{p}{1-p}\delta_w})_{x_0}^{x_n} & \cdots & (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_n} \end{vmatrix} \\
&= (1+\frac{p}{1-p}V_{x_n}^{x_n}) \begin{vmatrix} (V_{\frac{p}{1-p}\delta_w})_{x_0}^{x_0} & \cdots & (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_0} \\ \vdots & \ddots & \vdots \\ (V_{\frac{p}{1-p}\delta_w})_{x_0}^{x_n} & \cdots & (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_n} \end{vmatrix}.
\end{aligned}$$

Accordingly, $\dfrac{(1-p+pV_w^w)Q_\partial^w}{p+Q_\partial^w-pQ_\partial^w}\mathbb{P}_{p,BE}^{x_0}[\omega_{BE}=(x_0,x_1,\ldots,x_n=w)]$ does not depend on p. Consequently, it must be equal to $\mathbb{P}_{0,BE}^{x_0}[\omega_{BE}=(x_0,x_1,\ldots,x_n=w)]$. Equivalently,

$$\frac{(1-p+pV_w^w)Q_\partial^w}{p+Q_\partial^w-pQ_\partial^w}\mathbb{P}_{p,BE}^{x_0}[\cdot,C_w] = \mathbb{P}_{0,BE}^{x_0}[\cdot,C_w]$$

Therefore,

$$\mathbb{P}_{0,BE}^{x_0}[C_w] = \frac{(1-p+pV_w^w)Q_\partial^w}{p+Q_\partial^w-pQ_\partial^w}\mathbb{P}_{p,BE}^{x_0}[C_w].$$

Immediately, it implies that conditionally on C_w, the law of the loop-erased random walk does not depend on p:

$$\mathbb{P}_{p,BE}^{x_0}[\cdot|C_w] = \mathbb{P}_{0,BE}^{x_0}[\cdot|C_w].$$

Since

$$\begin{aligned}
\mathbb{P}_{p,BE}^{x_0}[\omega_{BE}\in\cdot,C_w] &= \sum_{k\geq 1}\mathbb{P}^{x_0}[\tau(p)=k,T_k<\zeta,LE[0,T_k]\in\cdot] \\
&\quad + \sum_{k\geq 1}\mathbb{P}^{x_0}[\tau(p)>k,T_k=\zeta-1,LE[0,T_k]\in\cdot] \\
&= \sum_{k\geq 1}(1-p)^{k-1}p\mathbb{P}^{x_0}[T_k<\infty,LE[0,T_k]\in\cdot] \\
&\quad + \sum_{k\geq 1}(1-p)^k\mathbb{P}^{x_0}[T_k<\infty,LE[0,T_k]\in\cdot]Q_\partial^w \\
&= \sum_{k\geq 1}(1-p)^{k-1}(p+Q_\partial^w-pQ_\partial^w)\mathbb{P}^{x_0}[T_k<\infty]\mathbb{P}^{x_0}[LE[0,T_k]\in\cdot|T_k<\infty],
\end{aligned}$$

we have

$$\mathbb{P}^{x_0}_{p,BE}[\omega_{BE} \in \cdot | C_w] = \frac{\sum\limits_{k\geq 1}(1-p)^{k-1}\mathbb{P}^{x_0}[T_k<\infty]\mathbb{P}^{x_0}[LE[0,T_k]\in\cdot|T_k<\infty]}{\sum\limits_{k\geq 1}(1-p)^{k-1}\mathbb{P}^{x_0}[T_k<\infty]}. \tag{*}$$

Since $\mathbb{P}^{x_0}_{p,BE}[\omega_{BE} \in \cdot | C_w]$ does not depend on $p \in [0,1]$, we will denote it by $\mathbb{Q}$. Then the equation (*) can be written as follows:

$$\mathbb{Q}[\cdot]\sum_{k\geq 1}(1-p)^{k-1}\mathbb{P}^{x_0}[T_k<\infty]$$
$$=\sum_{k\geq 1}(1-p)^{k-1}\mathbb{P}^{x_0}[T_k<\infty]\mathbb{P}^{x_0}[LE[0,T_k]\in\cdot|T_k<\infty].$$

Finally, by identifying the coefficients, we conclude that $\mathbb{P}^{x_0}[LE[0,T_k]\in\cdot|T_k<\infty]=\mathbb{Q}[\cdot]$ as long as $\mathbb{P}^{x_0}[T_k<\infty]$ for $k\geq 1$ and we are done. $\square$

Consider $(e_t, t\geq 0)$, a Poisson point process of excursions of finite lifetime at x with the intensity $Leb\otimes(-L^x_x-\frac{1}{V^x_x})\nu^{x\to x}_{\{x\},ex}$. (Recall that $\nu^{x\to x}_{\{x\},ex}$ is the normalized excursion measure at x, see Definition 3.3.2.) Let $(\gamma(t),t\geq 0)$ be an independent Gamma subordinator[1] with the Laplace exponent

$$\Phi(\lambda)=\int_0^\infty(1-e^{-\lambda s})s^{-1}e^{-s/V^x_x}\,ds.$$

Let $\mathfrak{R}_\alpha$ be the closure of the image of the subordinator γ up to time α, i.e. $\mathfrak{R}_\alpha=\overline{\{\gamma(t):t\in[0,\alpha]\}}$. Then, $[0,\gamma(\alpha)]\setminus\mathfrak{R}_\alpha$ is the union of countable disjoint open intervals, $\{]\gamma(t-),\gamma(t)[:\ t\in[0,\alpha],\gamma(t-)<\gamma(t)\}$. To such an open interval $]g,d[$, one can associate a based loop l as follows: During the time interval $]g,d[$, the Poisson point process $(e_t,t\geq 0)$ has finitely many excursions, namely $e_{t_1},\cdots,e_{t_n}, g<t_1<\cdots<t_n<d$. Each excursion e_{t_i} is viewed as a càdlàg path of lifetime ζ_{t_i}: $(e_{t_i}(s),s\in[0,\zeta_{t_i}[)$. Define $l:[0,d-g+\sum\limits_i\zeta_{t_i}]\to S$ as follows:

$$l(s)=\begin{cases} e_{t_i}(s-(\sum\limits_{j<i}\zeta_{t_j}+t_i)) & \text{if } s\in[\sum\limits_{j<i}\zeta_{t_j}+t_i,\sum\limits_{j\leq i}\zeta_{t_j}+t_i[\\ x & \text{otherwise.}\end{cases}$$

This mapping between an open interval $]g,d[$ and a based loop l depends on $]g,d[$ and $(e_t,t\in]g,d[)$ and we denote is by $\Psi^{]g,d[}$ ($l=\Psi^{]g,d[}(e)$). By mapping a based loop into a loop, we get a countable collection of loops for $\alpha\geq 0$, namely $\mathcal{O}_\alpha$.

Proposition 5.1.3. $(\mathcal{O}_\alpha,\alpha\geq 0)$ *has the same law as the Poisson point process of loops intersecting* $\{x\}$, *i.e.* $(\{l\in\mathcal{L}_\alpha: l^x>0\},\alpha>0)$.

[1] See Chapter III of [Ber96].

Proof. As both sides have independent stationary increment, it is enough to show $\mathcal{O}_1 = \{l \in \mathcal{L}_1 : l^x > 0\}$. It is well-known that $(\gamma(t) - \gamma(t-), t \in \mathbb{R})$ is a Poisson point process with characteristic measure $\frac{1}{s}e^{-s/V_x^x}\,ds$. Therefore, $\sum\limits_{l \in \mathcal{O}_\alpha} \delta_{l^x}$ is Poisson random measure with intensity $\frac{1}{s}e^{-s/V_x^x}\,ds$. On the other hand, for the Poisson ensemble of loops $\mathcal{L}_\alpha$, by taking the trace of the loops on x and dropping the empty ones, as a consequence of Proposition 3.2.1, we get a Poisson ensemble of trivial Markovian loops with intensity measure $\frac{1}{s}e^{s(L_{\{x\}})_x^x}\,ds$ where $(-L_{\{x\}})_x^x = 1/V_x^x$. Consequently, we have

$$\{l^x : l \in \mathcal{O}_1\} \text{ has the same law as } \{l^x : l \in \mathcal{L}_1, l^x > 0\}$$

In other words, by disregarding the excursions attached to each loop, the set of trivial loops in x obtained from $\mathcal{O}_1$ and $\{l \in \mathcal{L}_1 : l^x > 0\}$ is the same. In order to restore the loops, we need to insert the excursions into the trivial loops. Then, it remains to show that the excursions are inserted into the trivial loops in the same way. Finally, by using the independence between $(e_t, t \geq 0)$ and $(\gamma(t), t \geq 0)$ and the stationary independent increments property with respect to time t, it ends up in proving the following affirmation: $\Psi^{]0,T[}(e)$ induces the same probability on the loops with $l^x = T$ as the loop measure conditioned by $\{l^x = T\}$. By Proposition 3.1.7, we have $l^x\mu(dl) = \mu^{x,x}(dl)$. Hence, $\mu(dl|l^x = T) = \mu^{x,x}(dl|l^x = T)$ where $\mu^{x,x}$ is considered to be a loop measure. Let $\mathbb{P}^x$ be the law of the Markov process $(X_t, t \in [0, \zeta[)$ associated with the Markovian loop measure μ. Let $(L(x,t), t \in [0, \zeta[)$ be the local time process at x and $L^{-1}(x,t)$ be its right-continuous inverse. Let τ be an independent exponential variable with parameter 1. Define the process $X^{L^{-1}(x,\tau)}$ with lifetime $L^{-1}(x,\tau) \wedge \zeta$ as follows: $X^{L^{-1}(x,\tau)}(T) = X(T), T \in [0, L^{-1}(x,\tau) \wedge \zeta[$. Denote by $\mathbb{Q}[dl]$ the law of $X^{L^{-1}(x,\tau)}$. Then, $e^{-l^x}\mu^{x,x}(dl) = \mathbb{Q}[dl]$ where $\mu^{x,x}(dl)$ is considered to be a based loop measure. Therefore,

$$\mu^{x,x}(dl|l^x = T) = \mathbb{Q}[dl|l^x = T] = \mathbb{Q}[dl|\tau = T] = \text{the law of } \Psi^{]0,T[}(e)$$

in the sense of based loop measures. Then, the equality stills holds for loop measures and we are done. $\square$

Suppose $(X_t, t \in [0, \zeta[)$ is a transient Markov process on S. (Assume the process stays at the cemetery point after lifetime ζ.) Define the local time at x $L(x,t) = \int\limits_0^t 1_{\{X_s = x\}}\,ds$. Denote by $L^{-1}(x,t)$ its right-continuous inverse and by $L^{-1}(x,t-)$ its left-continuous inverse. The excursion process $(e_t, t \geq 0)$ is defined by $e_t(s) = X_{s+L^{-1}(x,t-)}, s \in [0, L^{-1}(x,t) - L^{-1}(x,t-)[$. Define a measure on the excursion which never returns to x by

$$\tilde{\nu}^{x\to}[dl] = \sum_{y \in S} Q_y^x \mathbb{P}^y[dl, \text{the process never hits } x].$$

We can calculate the total mass of $\tilde{\nu}^{x\rightarrow}$ as follows:

$$\tilde{\nu}^{x\rightarrow}[1] = \sum_{y\in S} Q^x_y \mathbb{P}^y[\text{the process never hits } x]$$
$$= 1 - \mathbb{E}^x[\{\text{after leaving } x, \text{ the process returns to } x\}]$$
$$= (1-(R^{\{x\}})^x) = \frac{(L_{\{x\}})^x_x}{L^x_x} = -\frac{1}{V^x_x L^x_x}.$$

After normalization, we get a probability measure $\nu^{x\rightarrow}$. The law of the first excursion is $-\frac{1}{V^x_x L^x_x}\nu^{x\rightarrow} + \left(1+\frac{1}{V^x_x L^x_x}\right)\nu^{x\rightarrow x}_{\{x\},ex}$. In particular, the first excursion is not an excursion from x back to x with probability $-\frac{1}{L^x_x V^x_x}$. According to the excursion theory, the excursion process is a Poisson point process stopped at the appearing of an excursion of infinite lifetime or an excursion that ends up at the cemetery. The characteristic measure is proportional to the law of the first excursion. By taking the trace of the process on x, we know that the total occupation time is an exponential variable with parameter $(-L_{\{x\}})^x_x = \frac{1}{V^x_x}$. According to the excursion theory, it is an exponential variable with parameter $\frac{-d}{V^x_x L^x_x}$, d being the mass of the characteristic measure. Immediately, we get $d = -L^x_x$. If we focus on the process of excursions from x back to x, it is a Poisson point process with characteristic measure $(-L^x_x - \frac{1}{V^x_x})\nu^{x\rightarrow x}_{\{x\},ex}$ stopped at an independent exponential time with parameter $\frac{1}{V^x_x}$. Let $(\gamma(t), t\geq 0)$ be a Gamma subordinator with Laplace exponent

$$\Phi(\lambda) = \int_0^\infty (1-e^{-\lambda s})s^{-1}e^{-s/V^x_x}\,ds.$$

Then, $\gamma(t)$ follows the $\Gamma(t, \frac{1}{V^x_x})$ distribution with density $\rho(y) = \frac{1}{\Gamma(t)(V^x_x)^t}y^{t-1}e^{-y/V^x_x}$. In particular, $\gamma(1)$ is an exponential variable of the parameter $1/V^x_x$. It is known that $(\frac{\gamma(t)}{\gamma(1)}, t\in[0,1])$ is independent of $\gamma(1)$, and that it is a Dirichlet process. (One can prove this by a direct calculation on the finite marginal distribution.) Moreover, the jumps of the process $(\frac{\gamma(t)}{\gamma(1)}, t\in[0,1])$ rearranged in decreasing order follow the Poisson-Dirichlet $(0,1)$ distribution. For $x\in S$, let Z_x be the last passage time in x: $Z_x = \sup\{t\in[0,\zeta[: X(t)=x\}$. Suppose the loop erased path ω_{BE} equals $(x_1,\ldots)$. Define $S_n = T_{x_n}$ for $n\geq 1$ and $S_0 = 0$. Let O_i be $(X_s, s\in[S_i, S_{i+1}[)$ i.e. the i-th loop erased from the process X. Then O_1 can be viewed as a Poisson point process $(e^{(1)}_t, t\in[0, L(x_1,\zeta)[)$ of excursions at x_1 killed at the arrival of an excursion with infinite lifetime or an excursion ending up at the cemetery. Conditionally on $\omega_{BE} = (x_1, x_2, \ldots)$, the shifted process $(X(s+T_1), s\in[0,\zeta[)$ is the Markov process restricted in $S\setminus\{x_1\}$ starting from $x_2 = X(T_1)$. Moreover, it is conditionally independent of the Poisson point process $e^{(1)}$. Once again, we can view O_2 as an killed Poisson point process of excursions at x_2 and denote it by $e^{(2)}$. Clearly, we have the independence between $e^{(1)}$ and $e^{(2)}$ conditionally on $\omega_{BE} = (x_1, x_2, \ldots)$. Repeating this procedure, we get a sequence of point process of excursions $e^{(1)}, \ldots$. Conditionally on $\omega_{BE} = (x_1, \ldots, x_n, \ldots)$, they are independent, and $e^{(n)}$ has the same law as the killed excursion process for the Markov process restricted in $D_n = S \setminus \{x_1, \ldots, x_{n-1}\}$. Let $O_i^{x_i}$ be the occupation time at x_i for the based loop O_i. Let $(\gamma_t^{(i)}, t \geq 0), i \geq 1$ be a sequence of independent Gamma subordinators with Laplace exponent

$$\Phi(\lambda) = \int_0^\infty (1 - e^{-\lambda s}) s^{-1} \exp\left(-\frac{s}{(V^{D_n})_{x_i}^{x_i}}\right) ds.$$

We suppose they are independent of the Markov process. Then, $O_i^{x_i}, i \geq 1$ has the same law as $\gamma^{(i)}(1), i \geq 1$ conditionally on ω_{BE}. In the spirit of Proposition 5.1.3 by cutting the excursion process according to the range of subordinator, if at time $\alpha \in [0,1]$, we cut the loop O_i according to the range of $(\frac{\gamma^{(i)}(s) O_i^{x_i}}{\gamma^{(i)}(1)}, s \in [0, \alpha])$, we get a point process of loops $(\mathcal{O}_\alpha^{(i)}, \alpha \in [0,1])$. Conditionally on ω_{BE}, it has the same law as the Poisson point process $(\mathcal{L}_\alpha^{D_i} \backslash \mathcal{L}_\alpha^{D_{i+1}}, \alpha \in [0,1])$. Moreover, conditionally on ω_{BE}, $(\mathcal{O}_\alpha^{(i)}, \alpha \in [0,1]), i \geq 1$ are independent. The definition of the Poisson random measure ensures independence among $(\mathcal{L}_\alpha^{D_i} \setminus \mathcal{L}_\alpha^{D_{i+1}}, \alpha \in [0,1]), i = 1, \ldots$. Consequently, we have the following proposition.

Proposition 5.1.4. *Conditionally on ω_{BE}, $(\mathcal{O}_\alpha, \alpha \in [0,1])$ has the same law as $(\{l \in \mathcal{L}_\alpha : l \text{ intersects } \omega_{BE}\}, \alpha \in [0,1])$.*

Remark 15. The jumps of the process $\dfrac{\gamma(t)}{\gamma(1)}$ rearranged in decreasing order follow the Poisson-Dirichlet $(0,1)$ distribution. Since a Poisson point process is always homogeneous in time, the following two cutting method gives the same loop ensemble in law:

- Cutting the loop according to the range of $\left(\dfrac{\gamma(t)}{\gamma(1)}, t \in [0,1]\right)$,
- Cutting the loop according to the Poisson-Dirichlet $(0,1)$ distribution.

As a result, a similar result holds for $\alpha = 1$ if we cut the loops according to the Poisson-Dirichlet $(0,1)$ distribution.

5.2 Rooted random spanning tree

Throughout this section, we consider a finite state space S with a transient Markov process $(X_t, t \geq 0)$ on it. Denote by Δ the cemetery point for X. As usual, denote by L the generator of X and by Q the transition matrix of the embedded Markov chain.

By the following algorithm, one can construct a random spanning tree of $S \cup \{\Delta\}$ rooted[2] at Δ. We give an orientation on the tree: each edge is directed towards the root.

Definition 5.2.1 (Wilson's algorithm)**.** Choose an arbitrary order on S: $S=\{v_1, \ldots, v_n\}$. Define $S_0 = \{\Delta\}$. Let T_0 be the tree with single vertex Δ. We recurrently construct a series of growing random trees $T_k, k \in \mathbb{N}$ as follows:
Suppose T_k is well-constructed with set of vertices S_k. If $S \cup \{\Delta\} \setminus S_k = \phi$, then we stop the procedure and set $\mathcal{T} = T_k$. Otherwise, there is a unique vertex in $S \cup \{\Delta\} \setminus S_k$ with the smallest sub-index and we denote it by y_{k+1}. Run a Markov chain from y_{k+1} with transition matrix Q. It will hit S_k in finitely many steps. We stop the Markov chain after it reaches S_k and erase progressively the loops according to the Definition 5.1.1. In this way, we get a loop-erased path η_{k+1} joining y_{k+1} to T_k. By adding this loop erased path η_{k+1} to T_k, we construct the random tree T_{k+1}. The procedure will stop after a finite number of steps and it produces a random spanning tree $\mathcal{T}$.

Remark 16. In Wilson's algorithm, the spanning tree is constructed by progressively adding new branches. For $k \in \mathbb{N}$, conditionally on the tree T_k that has been constructed at step k, the law of η_{k+1} is associated with the Markov process X stopped at the next jump after reaching T_k. At the same time, we remove $\#T_{k+1} - \#T_k$ loops based on each vertex in η_{k+1}. We cut those loops according to some independent Poisson-Dirichlet (0,1) distribution as in Proposition 5.1.4 and Remark 15 and we get an ensemble of loops $\mathcal{O}_{\eta_{k+1}}$. Conditional on $\mathcal{T}$, $\mathcal{O}_k$ is equal in law to $\mathcal{L}_1^{\{T_k\}^c} \setminus \mathcal{L}_1^{\{T_{k+1}\}^c}$. Those $(\mathcal{O}_{\eta_k}, k \geq 1)$ are independent as for $(\mathcal{L}_1^{\{T_k\}^c} \setminus \mathcal{L}_1^{\{T_{k+1}\}^c}, k \geq 1)$. It implies that $\bigcup\limits_{k\geq 1} \mathcal{O}_{\eta_k}$ has the same law as $\mathcal{L}_1 = \bigcup\limits_{k\geq 1} \mathcal{L}_1^{\{T_{k-1}\}^c} \setminus \mathcal{L}_1^{\{T_k\}^c}$. In summary, we have removed $\#S$ loops based on each vertex in S in Wilson's algorithm. By cutting all those loops according to some independent Poisson-Dirichlet (0,1) distribution as in Proposition 5.1.4 and Remark 15, we recover the Poisson ensemble of loops $\mathcal{L}_1$.

Proposition 5.2.1. *Denote by $\mu_{ST,\Delta}$ the distribution of the random spanning tree rooted at Δ given by Wilson's algorithm. Then,*

$$\mu_{ST,\Delta}(\mathcal{T} = T) = \det(V) 1_{\{T \textit{ is a spanning tree rooted at } \Delta\}} \prod_{\substack{(x,y) \textit{ is an edge in } T \\ \textit{directed towards the root } \Delta}} L_y^x \quad {}^3$$

where V is the potential of the process X [4]*.*

[2]By a random spanning tree rooted at Δ, we mean a random spanning tree with a special mark on the vertex Δ.

[3]Recall that $L_\delta^x = -\sum\limits_{y\in S} L_y^x$ for $x \in S$.

[4]Wilson's algorithm use the embedded Markov chain of X.

Proof. Suppose $|S| = n$. Choose an arbitrary order on S: $S=\{v_1, \dots, v_n\}$ and use Wilson's algorithm to construct a random spanning tree $\mathcal{T}$ rooted at Δ. Set $v_0 = \Delta$. For a rooted spanning tree T, let $A_m(T)$ be the set of vertices in $T_{\{v_0,\dots,v_m\}}$[5] for $m = 1, \dots, n$. Set $B_0(T) = \phi$. For $m = 1, \dots, n$, set $B_m(T) = \phi$ if v_m belongs $A_{m-1}(T)$. Otherwise, let $B_m(T)$ be the unique path joining v_m to $A_{m-1}(T)$ in T. We will calculate the conditional distribution of $B_m(\mathcal{T})$ given $A_{m-1}(\mathcal{T})$ for $m \geq 1$. Suppose that $v_m \notin A_{m-1}$. Let $(Y_t, t \geq 0)$ be the process $(X_t, t \geq 0)$ killed at the first jumping time after the process reaches the A_{m-1}. Then, Y is a transient Markov with generator

$$(L_Y)^x_y = \begin{cases} L^x_y & \text{for } x \text{ not contained in } \mathcal{T}_{\{v_0,\dots,v_{m-1}\}} \\ \delta^x_y L^x_x & \text{otherwise} \end{cases}$$

and potential V_Y such that

- $V_Y|_{A^c_{m-1}\times A^c_{m-1}} = V^{A^c_{m-1}}$;
- $(V_Y)^x_y = \sum\limits_{z\in A^c_{m-1}} (V^{A^c_{m-1}})^x_z L^z_y$ for $x \in A^c_{m-1}, y \in A_{m-1}$;
- $V_Y|_{A_{m-1}\times A^c_{m-1}} = 0$;
- $(V_Y)^x_y = \delta^x_y \frac{1}{-L^x_x}$ for $x, y \in A_{m-1}$.

Let ∂_Y stand for the cemetery point of Y. Then conditionally on $\mathcal{T}_{v_0,\dots,v_{m-1}}$, the probability $B_m = ((z_0, z_1), (z_1, z_2)\dots, (z_p, z_{p+1}))$ with $z_0 = v_m$, $z_{p+1} \in A_{m-1}$ and $z_0, \dots, z_p \in A^c_{m-1}$ equals the probability that the loop-erased path obtained by Y is $(z_0, z_1, \dots, z_p, z_{p+1}, \partial_Y)$. According to Proposition 5.1.1, that conditional probability equals

$$\det((V_Y)_{A_m\setminus A_{m-1}}) \prod_{(x,y) \text{ is contained in } B_m} L^x_y = \det((V^{A^c_{m-1}})_{A_m\setminus A_{m-1}}) \prod_{(x,y) \text{ is contained in } B_m} L^x_y.$$

By Jacobi's formula,

$$\det(-L|_{A^c_{m-1}\times A^c_{m-1}}) \det((V^{A^c_{m-1}})_{A_m\setminus A_{m-1}}) = \det(-L|_{A^c_m\times A^c_m}).$$

Accordingly,

$$\det((V^{A^c_{m-1}})_{A_m\setminus A_{m-1}}) = \frac{\det(V^{A^c_{m-1}})}{\det(V^{A^c_m})}.$$

Therefore, if $v_m \notin A_{m-1}$, i.e. $A_{m-1} \neq A_m$,

$$\mathbb{P}[B_m = ((z_0, z_1), \dots, (z_p, z_{p+1}))|A_{m-1}] = \frac{\det(V^{A^c_{m-1}})}{\det(V^{A^c_m})} \prod_{(x,y) \text{ is contained in } B_m} L^x_y.$$

[5]Here, $T_{\{v_0,\dots,v_m\}}$ is the smallest sub-tree of T containing the same root with the set of vertices $v_0, \dots, v_m$.

Trivially, if $v_m \in A_{m-1}$,

$$\mathbb{P}[B_m = \phi | A_{m-1}] = 1 = \frac{\det(V^{A^c_{m-1}})}{\det(V^{A^c_m})}$$

Finally, by multiplying all the conditional probability above, we find that

$$\mu_{ST,\Delta}(\mathcal{T} = T) = \det(V) 1_{\{T \text{ is a spanning tree rooted at } \Delta\}} \prod_{\substack{(x,y) \text{ is an edge in } T \\ \text{directed towards the root } \Delta}} L^x_y.$$

$\square$

Theorem 5.2.2 (Kirchhoff's theorem). *The probability of containing a certain edge is given by*

$$\mu_{ST,\Delta}(e = (e-, e+) \in \mathcal{T}) = L^{e-}_{e+}(V^{e-}_{e-} - V^{e+}_{e-})$$

with the convention that $V^\Delta_x = 0$ *and* $L^x_\Delta = -\sum_{y \in S} L^x_y$ *for* $x \in S$.

Proof. We list S by $v_1 = e-, v_2, \ldots,$. From Wilson's algorithm,

$$\begin{aligned} \mu_{ST,\Delta}(e = (e-, e+) \in \mathcal{T}) =& \mathbb{P}^{e-}_{BE}[\omega_{BE} = (e-, e+, \ldots)] \\ =& L^{e-}_{e+} \begin{vmatrix} V^{e-}_{e-} & 1 \\ V^{e+}_{e-} & 1 \end{vmatrix} \\ =& L^{e-}_{e+}(V^{e-}_{e-} - V^{e+}_{e-}). \end{aligned}$$

$\square$

Theorem 5.2.3 (Transfer current theorem). *For k edges* $e_i = (e_i-, e_i+), i = 1, \ldots, k$,

$$\mu_{ST,\Delta}(e_1, \ldots, e_k \in \mathcal{T}) = L^{e_1-}_{e_1+} \ldots L^{e_k-}_{e_k+} \det((K_{e_i,e_j})_{i,j=1,\ldots,k})$$

with $K_{(a,b),(c,d)} = V^a_c - V^b_c$ *for* $a, b, c, d \in S \cup \{\Delta\}$ *with the convention that* $V^\Delta_x = 0$ *and* $L^x_\Delta = -\sum_{y \in S} L^x_y$ *for* $x \in S$.

Proof. We could suppose $e_1-, \ldots, e_k-$ are k different vertices. Otherwise, both sides vanish.[6] Consider a modified Markov process X' with generator L' defined as follows[7]:

$$(L')^x_y = \begin{cases} L^x_y & \text{if } x \notin \{e_1-, \ldots, e_k-\}, y \in S \cup \{\Delta\}; \\ L^x_y & \text{if the edge } (x, y) \in \{e_1, \ldots, e_k\}; \\ -L^{e_i-}_{e_i+} & \text{if } x = y = e_i- \text{ for some } i = 1, \ldots, k; \\ 0 & \text{otherwise.} \end{cases}$$

[6] For example, if $e_1- = e_2-$, we have $\mu_{ST,\Delta}(e_1, \ldots, e_k \in \mathcal{T}) = 0$. For the right side, the determinant vanishes as $K_{e_j,e_1} = K_{e_j,e_2}$ for $j = 1, \ldots, k$.

[7] We define L' as a matrix labelled by $S \cup \{\Delta\}$. The process X' is killed at Δ.

Then we obtain another random spanning tree $\mathcal{T}'$ rooted at Δ with the law $\mu'_{ST,\Delta}$. Clearly, the random spanning tree $\mathcal{T}'$ has to contain $e_1, \dots, e_k$. Moreover, for any fixed spanning tree T rooted at Δ,

$$\mu'_{ST,\Delta}(\mathcal{T}' = T) = \frac{\det(-L)}{\det(-L')}\mu_{ST,\Delta}(\mathcal{T} = T).$$

Consequently,

$$\mu_{ST,\Delta}(e_1, \dots, e_k \in \mathcal{T}) = \frac{\det(-L')}{\det(-L)} = \det(-L'V).$$

For $x \notin \{e_1-, \dots, e_k-\}$, we have $(-L'V)^x_y = \delta^x_y$. For $i, j \in \{1, \dots, k\}$,

$$(-L'V)^{e_i-}_{e_j-} = -(L')^{e_i-}_{e_i-}V^{e_i-}_{e_j-} - (L')^{e_i-}_{e_i+}V^{e_i+}_{e_j-} = L^{e_i-}_{e_i+}V^{e_i-}_{e_j-} - L^{e_i-}_{e_i+}V^{e_i+}_{e_j-} = K_{e_i,e_j}.$$

Finally, $\mu_{ST,\Delta}(e_1, \dots, e_k \in \mathcal{T}) = \det(-L'V) = L^{e_1-}_{e_1+} \dots L^{e_k-}_{e_k+} \det(K_{e_i,e_j})_{i,j=1,\dots,k}$. $\square$

The following corollary is the analogue of the classical transfer current theorem for the rooted spanning tree with an elementary proof.

Corollary 5.2.4. *Define* $A_{e,\tilde{e}} = V^{e-}_{\tilde{e}-}L^{\tilde{e}-}_{\tilde{e}+} - V^{e+}_{\tilde{e}-}L^{\tilde{e}-}_{\tilde{e}+} - V^{e-}_{\tilde{e}+}L^{\tilde{e}+}_{\tilde{e}-} + V^{e+}_{\tilde{e}+}L^{\tilde{e}+}_{\tilde{e}-}$ *with the convention that* $V^\Delta_x = 0$ *and* $L^x_\Delta = -\sum_{y\in S} L^x_y$ *for* $x \in S$. *Then,*

$$\mu_{ST,\Delta}(\pm e_1, \dots, \pm e_k \in \mathcal{T}) = \det(A_{e_i,e_j})_{i,j=1,\dots,k}$$

Proof. Define $B_{e,\tilde{e}} = K_{e,\tilde{e}}L_{\tilde{e}} = V^{e-}_{\tilde{e}-}L^{\tilde{e}-}_{\tilde{e}+} - V^{e+}_{\tilde{e}-}L^{\tilde{e}-}_{\tilde{e}+}$. Immediately, we see that $B_{-e,\tilde{e}} = -B_{e,\tilde{e}}$. Theorem 5.2.3 gives that

$$\mu_{ST,\Delta}(e_1, \dots, e_k \in \mathcal{T}) = \det(B_{e_i,e_j})_{i,j=1,\dots,k}.$$

Fix k edges $e_1, \dots, e_k$, set $F^+ = \{e_1, \dots, e_k\}$, $F^- = \{-e_1, \dots, -e_k\}$ and $F = F^+ \cup F^-$. Define $M = B|_{F^+\times F^+}$ and $N = B|_{F^-\times F^-}$. Then, $B|_{F\times F} = \begin{bmatrix} M & -N \\ -M & N \end{bmatrix}$ For all $H \subset F$ with $\#H > k$,

$$0 = \mu_{ST,\Delta}(H \text{ is covered by the random spanning tree } \mathcal{T}) = \det(B_{e,\tilde{e}})_{e,\tilde{e}\in H}$$

Consequently,

$$\begin{aligned}
\det(B + \lambda I)_{F\times F} &= \sum_{H\subset F} \lambda^{2k-\#H} \det(B|_{H\times H}) \\
&= \sum_{H\subset F, \#H\leq k} \lambda^{2k-\#H} \det(B|_{H\times H}) \\
&= \lambda^k \mu_{ST,\Delta}(\pm e_1, \cdots, \pm e_k \in \mathcal{T}) + o(\lambda^k) \text{ as } \lambda \to 0.
\end{aligned}$$

We can compute $\det(B+\lambda I)_{F\times F}$ in another way:

$$\begin{aligned}
\det(B+\lambda I)_{F\times F} =& \begin{vmatrix} \lambda I + M & -N \\ -M & \lambda I + N \end{vmatrix} \\
=& \begin{vmatrix} \lambda I + M & -N \\ \lambda I & \lambda I \end{vmatrix} \\
=& \begin{vmatrix} \lambda I + M + N & -N \\ 0 & \lambda I \end{vmatrix} \\
=&\lambda^k \det(\lambda + M + N) \\
=&\lambda^k \det(\lambda + A_{e_i,e_j})_{i,j=1,\dots,k} \\
=&\lambda^k \det(A_{e_i,e_j})_{i,j=1,\dots,k} + o(\lambda^k) \text{ as } \lambda \to 0.
\end{aligned}$$

By comparing the dominant terms, the result is proven. □

Proposition 5.2.5. *In the special case where L and V are symmetric matrix, we have*

$$\mu_{ST,\Delta}(\pm e_1, \dots, \pm e_k \in \mathcal{T}) = \det(Z_{e_i,e_j})_{i,j=1,\dots,k}$$

where $Z_{(x,y),(u,v)} = \sqrt{L^x_y L^u_v}[V^x_u - V^y_u - V^x_v + V^y_v]$ is symmetric. Clearly, $Z_{-e,\tilde{e}} = Z_{e,-\tilde{e}} = -Z_{e,\tilde{e}}$. For any collection of edges $\eta_1, \dots, \eta_k$, $\det(Z_{\eta_i,\eta_j})_{i,j=1,\dots,k} = \mu_{ST,\Delta}(\pm\eta_1, \dots, \pm\eta_k \in \mathcal{T}) \geq 0$. As a consequence, Z is positive definite. There is a property of negative association between edges:

$$\begin{aligned}
&\mu_{ST,\Delta}[\pm\xi_1, \dots, \pm\xi_m, \pm\eta_1, \dots, \pm\eta_n \in \mathcal{T}] \\
&\qquad\qquad \leq \mu_{ST,\Delta}[\pm\xi_1, \dots, \pm\xi_m \in \mathcal{T}]\mu_{ST,\Delta}[\pm\eta_1, \dots, \pm\eta_n \in \mathcal{T}]
\end{aligned}$$

and the equality is obtained iff. one of the following three condition is fulfilled:

- $\det(Z_{\xi_i,\xi_j})_{i,j=1,\dots,m} = 0$, *i.e.* $\mu_{ST,\Delta}(\pm\xi_1, \dots, \pm\xi_m \in \mathcal{T}) = 0$,
- $\det(Z_{\eta_i,\eta_j})_{i,j=1,\dots,n} = 0$, *i.e.* $\mu_{ST,\Delta}(\pm\eta_1, \dots, \pm\eta_n \in \mathcal{T}) = 0$,
- $Z_{\xi_i,\eta_j} = 0$ *for* $i = 1, \dots, m$ *and* $j = 1, \dots, n$, *i.e* $\{\xi_i \in \mathcal{T}\}, \{\eta_j \in \mathcal{T}\}$ *are pairwise independent.*

Proof. Set $E_1 = \{\xi_1, \dots, \xi_m\}$ and $E_2 = \{\eta_1, \dots, \eta_n\}$. For $i, j \in \{1, 2\}$, write $Z_{i,j}$ in short for $Z|_{E_i \times E_j}$. We consider the non-trivial case: $\mu_{ST,\Delta}[\pm\xi_1, \dots, \pm\xi_m, \pm\eta_1, \dots, \pm\eta_n \in \mathcal{T}] > 0$.

$$\frac{\mu_{ST,\Delta}[\pm E_1, \pm E_2 \subset \mathcal{T}]}{\mu_{ST,\Delta}[\pm E_1 \subset \mathcal{T}]\mu_{ST,\Delta}[\pm E_2 \subset \mathcal{T}]} = \frac{\det(Z|_{(E_1\cup E_2)\times(E_1\cup E_2)})}{\det(Z_{1,1})\det(Z_{2,2})}$$

$$= \det(\begin{bmatrix} Z_{1,1}^{-1/2} & 0 \\ 0 & Z_{2,2}^{-1/2} \end{bmatrix} Z|_{(E_1\cup E_2)\times(E_1\cup E_2)} \begin{bmatrix} Z_{1,1}^{-1/2} & 0 \\ 0 & Z_{2,2}^{-1/2} \end{bmatrix})$$

$$= \det \begin{bmatrix} I & Z_{1,1}^{-1/2} Z_{1,2} Z_{2,2}^{-1/2} \\ Z_{1,1}^{-1/2} Z_{2,1} Z_{1,1}^{-1/2} & I \end{bmatrix}$$

Set $B = \begin{bmatrix} 0 & Z_{1,1}^{-1/2} Z_{1,2} Z_{2,2}^{-1/2} \\ Z_{1,1}^{-1/2} Z_{2,1} Z_{1,1}^{-1/2} & 0 \end{bmatrix}$, then

$$\frac{\mu_{ST,\Delta}[\pm E_1, \pm E_2 \subset \mathcal{T}]}{\mu_{ST,\Delta}[\pm E_1 \subset \mathcal{T}]\mu_{ST,\Delta}[\pm E_2 \subset \mathcal{T}]} = \det(I + B)$$

which is symmetric definite. Consequently, the eigenvalues of the symmetric matrix B are greater or equal to -1. If we replace E_1 by $-E_1 = \{-\xi_1, \dots, -\xi_m\}$ in the argument above, the probabilities do not change, the matrix $Z_{-E_1,-E_1} = Z_{E_1,E_1}$, but $Z_{-E_1,E_2} = Z_{E_2,-E_1} = -Z_{E_1,E_2} = -Z_{E_2,E_1}$. Finally, we see that

$$\frac{\mu_{ST,\Delta}[\pm E_1, \pm E_2 \subset \mathcal{T}]}{\mu_{ST,\Delta}[\pm E_1 \subset \mathcal{T}]\mu_{ST,\Delta}[\pm E_2 \subset \mathcal{T}]} = \det(I - B).$$

Similarly, the eigenvalues of the symmetric matrix $-B$ are greater or equal -1. Finally, $\lambda_1, \dots, \lambda_{m+n}$, the eigenvalues of B, must be contained in $[-1, 1]$. Consequently,

$$\frac{\mu_{ST,\Delta}[\pm E_1, \pm E_2 \subset \mathcal{T}]}{\mu_{ST,\Delta}[\pm E_1 \subset \mathcal{T}]\mu_{ST,\Delta}[\pm E_2 \subset \mathcal{T}]} = \det(I - B) = \det(I + B)$$

$$= \prod_i (1 - \lambda_i) = \prod_i (1 + \lambda_i) = \sqrt{\prod_i (1 - \lambda_i^2)} \leq 1$$

The equality is obtained iff. $\mu_{ST,\Delta}[\pm E_1 \subset \mathcal{T}] = \det(Z|_{E_1\times E_1}) = 0$ or $\mu_{ST,\Delta}[\pm E_2 \subset \mathcal{T}] = \det(Z|_{E_2\times E_2}) = 0$ or $Z|_{E_1\times E_2} = 0$. □

Remark 17. The negative association is not true in general. For example, $S = \{1, 2\}$ and $L = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$. Then,

$$\mu_{ST,\Delta}[\pm(1,2), \pm(2,\Delta) \in \mathcal{T}] = \mu_{ST,\Delta}[\pm(1,2) \in \mathcal{T}] = \mu_{ST,\Delta}[\pm(2,\Delta) \in \mathcal{T}] = 1/2.$$

In the end of this section, we would like to point out some possible generalization of the above results under the following assumption:

for any initial state, the process reaches the cemetery Δ after *finitely* many jumps.

Then, Wilson's procedure still works and it defines a random spanning tree $\mathcal{T}$ on the extended state space $S \cup \{\Delta\}$.

Proposition 5.2.6. *The distribution of $\mathcal{T}$ is characterized*[8] *by the following quantity:*

$$\mu_{ST,\Delta}(\mathcal{T}_{\{x_1,\ldots,x_k\}} = T)^9 = \prod_{e=(e-,e+) \textit{ is an edge in } T} L_{e+}^{e-} \det(V_T) \quad ^{10}$$

where V_T is short for $V_{\{vertices\ in\ T\ except\ for\ \Delta\}}$.

Proof. Suppose the state space S is enumerated as $v_1, \ldots, v_n, \ldots$. By an argument similar to Proposition 5.2.1, one can prove that

$$\mu_{ST,\Delta}(\mathcal{T}_{\{v_1,\ldots,v_n\}} = T) = \prod_{e=(e-,e+) \text{ is an edge in } T} L_{e+}^{e-} \det(V_T).$$

Note that the right hand side does not depend on the way in which we enumerate S. As a consequence, for any permutation $\sigma \in \mathfrak{S}_n$, if we list S in another way:

$$S = \{v_{\sigma(1)}, \ldots, v_{\sigma(n)}, v_{n+1}, \ldots\},$$

the law of $\mathcal{T}_{\{v_1,\ldots,v_n\}}$ is the same. In particular, for any $\{x_1, \ldots, x_k\}$ subset of $\{v_1, \ldots, v_n\}$, we choose the proper $\sigma \in \mathfrak{S}_n$ such that

$$v_{\sigma(i)} = x_i \text{ for } i = 1, \ldots, k \text{ and } \sigma(i) = i \text{ for } i = k+1, \ldots, n$$

Hence the following equality holds:

$$\mu_{ST,\Delta}(\mathcal{T}_{\{x_1,\ldots,x_k\}} = T) = \prod_{e=(e-,e+) \text{ is an edge in } T} L_{e+}^{e-} \det(V_T).$$

Notice that for fixed $x_1, \ldots, x_k$, we can always choose n large enough so that $\{v_1, \ldots, v_n\}$ includes $\{x_1, \ldots, x_k\}$. The proof is then complete. $\square$

Remark 18. Fix F a subset of S. Denote by $\mu_{ST,\Delta}^F$ the rooted spanning measure related to the restriction of the process X in F.

$$\begin{aligned}\mu_{ST,\Delta}(\mathcal{T}_{\{x_1,\ldots,x_k\}} = T \mid &\text{The vertex set of } \mathcal{T}_{\{x_1,\ldots,x_k\}} \text{ is exactly } F) \\ &= \mu_{ST,\Delta}^F(\mathcal{T} = T \mid \text{The leaves of } \mathcal{T} \text{ are contained in } \{x_1, \ldots, x_k\}).\end{aligned}$$

[8] In fact, one can deduce from it the marginal distribution $\mu_{ST,\Delta}(\text{The edges } e_1, \ldots, e_k \in \mathcal{T})$

[9] Given any oriented tree $\mathcal{T}$ rooted at Δ, and some vertices $x_1, \ldots, x_k$ in $\mathcal{T}$ different from Δ, define $\mathcal{T}_{\{x_1,\ldots,x_k\}}$ as the smallest sub-tree of $\mathcal{T}$ containing $x_1, \ldots, x_k$.

[10] This expression implies that the spanning tree distribution does not depends on the way in which we enumerate S.

Theorem 5.2.3 (transfer current theorem) remains valid as well as Corollary 5.2.4 remains true. We give the proof of the transfer current theorem as follows:

Proof. Take a sequence of sets exhausting S:

$$S_1 = \{e_1-, \dots, e_k-\} \subset \cdots \subset S_n \subset \text{ with } \bigcup_{n=1}^{\infty} S_n = S.$$

Consider a modified Markov process X' with generator L' given as follows[11]:

$$(L')^x_y = \begin{cases} L^x_y & \text{if } x \notin \{e_1-, \dots, e_k-\}, y \in S \cup \{\Delta\}; \\ L^x_y & \text{if the edge } (x,y) \in \{e_1, \dots, e_k\}; \\ -L^{e_i-}_{e_i+} & \text{if } x = y = e_i- \text{ for some } i = 1, \dots, k; \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $\mu'_{ST,\Delta}$ the corresponding random spanning tree measure. Set $F = \bigcup_i \{e_i-\}$. For any subset E such that $F \subset E \subset S$, one checks immediately that

$$((L')_E)|_{(E\setminus F)\times E} = (L_E)|_{(E\setminus F)\times E}. \tag{5.1}$$

Moreover, for a sequence of subsets $E_1 \subset \cdots$ which increases to S,

$$\lim_{n\to\infty} ((L')_{E_n})|_{F\times F} = L|_{F\times F}. \tag{5.2}$$

For a tree T, let $S(T)$ stand for the collection of the vertices in T except for Δ. Then, for any $n \in \mathbb{N}_+$

$$\begin{aligned}
\mu_{ST,\Delta}(e_1, \dots, e_k \in \mathcal{T}) =& \mu_{ST,\Delta}(e_1, \dots, e_k \in \mathcal{T}_{S_n}) \\
=& \sum_{T \text{ is a tree}} \mu_{ST,\Delta}(\mathcal{T}_{S_n} = T) 1_{\{T \text{ contains } e_1,\dots,e_k\}} \\
=& \sum_{T \text{ is a tree}} \frac{\det(V_{S(T)})}{\det(V'_{S(T)})} \mu'_{ST,\Delta}(\mathcal{T}_{S_n} = T) 1_{\{T \text{ contains } e_1,\dots,e_k\}} \\
=& \sum_{T \text{ is a tree}} \det(-(L')_{S(T)} V_{S(T)}) \mu'_{ST,\Delta}(\mathcal{T}_{S_n} = T) 1_{\{T \text{ contains } e_1,\dots,e_k\}}
\end{aligned}$$

By the equation (5.1) for $E = S(T)$, we have that

$$\det(-(L')_{S(T)} V_{S(T)}) = \begin{vmatrix} (-(L')_{S(T)})|_{F\times F} V_F & * \\ 0 & Id \end{vmatrix} = \det((-(L')_{S(T)})|_{F\times F} V_F).$$

[11] We define L' as a matrix labelled by $S \cup \{\Delta\}$. The process X' is killed at Δ.

Then by the equation (5.2), as $n \to \infty$, $(-(L')_{S(T)})|_{F\times F}V_F$ tends to $-L'|_{F\times F}V_F$ uniformly for all tree T containing S_n. Therefore, as $n \to \infty$,

$$\begin{aligned}\mu_{ST,\Delta}(e_1,\ldots,e_k \in \mathcal{T}) &= \sum_{T \text{ is a tree}} \det(-(L')_{S(T)}V_{S(T)})\mu'_{ST,\Delta}(\mathcal{T}_{S_n} = T)1_{\{T \text{ contains } e_1,\ldots,e_k\}} \\ &\sim \det(-L'|_{F\times F}V_F) \sum_{T \text{ is a tree}} \mu'_{ST,\Delta}(\mathcal{T}_{S_n} = T)1_{\{T \text{ contains } e_1,\ldots,e_k\}} \\ &= \det(-L'|_{F\times F}V_F)\end{aligned}$$

Finally, by the same calculation as in the proof of Theorem 5.2.3, we get that

$$\det(-L'|_{F\times F}V_F) = \det[(K_{e_i,e_j})_{i,j=1,\ldots,k}]\prod_{i=1}^{k} L^{e_i-}_{e_i+}.$$

$\square$

5.3 Spanning tree measure

Consider an irreducible recurrent Markov process X with generator L on a finite state space V. Fix a non-negative function $p : V \to \mathbb{R}_+, (p \neq 0)$. By killing the process at rate ϵp, we obtain a transient Markov process $X^{(\epsilon,p)}$ with the generator $L^{(\epsilon,p)} = L - M_{\epsilon p}$. The cemetery is denoted by Δ. Denote by $\mu_{ST,\Delta,\epsilon,p}$ the law of the random spanning tree rooted at Δ related to $X^{(\epsilon,p)}$.

Definition 5.3.1 (Spanning tree measure with weights function p on the roots)**.** We define the spanning tree measure $\mu_{ST,p}$ on V by $\lim_{\epsilon\to 0}\mu_{ST,\Delta,\epsilon,p}$ restricted to the edge set $V \times V$.[12]

It is not difficult to see that the spanning tree measure $\mu_{ST,p}$ is a mixture of spanning tree measure with fixed root. To be more precise, we have the following description:

Proposition 5.3.1. *For a spanning tree T, let $r(T)$ to stand for its root. Then,*

$$\mu_{ST,p}(\mathcal{T} = T) = \frac{p(r(T))\prod_{e \text{ is an edge in } T} L^{e-}_{e+}}{\sum_T p(r(T))\prod_{e \text{ is an edge in } T} L^{e-}_{e+}}.$$

In other words,

$$\mu_{ST,p} = \frac{p(v)m(v)}{\sum_{v\in V} p(v)m(v)}\mu_{ST,v}$$

[12]It is not difficult to see that the measure $\mu_{ST,p}$ is concentrated on trees since most of the mass $\mu_{ST,\Delta,\epsilon,p}$ is concentrated on the spanning tree rooted at Δ with only one edge towards the root. More precisely, under $\mu_{ST,\Delta,\epsilon,p}$, the chance that there exists at least 2 edges towards the root Δ is of order $o(\epsilon)$.

where m is the stationary distribution of the process X an $\mu_{ST,v}$ is the rooted spanning tree measure associated to the process X killed at the vertex v.

Proof. The first expression follows directly from the definition. For the second equation, it is enough to prove that $\mu_{ST,p}[v \text{ is the root.}] = \frac{p(v)m(v)}{\sum\limits_{v\in V} p(v)m(v)}$. By Kirchhoff's theorem (Theorem 5.2.2),

$$\begin{aligned}\mu_{ST,p}[v \text{ is the root.}] &= \lim_{\epsilon\to 0} \mu_{ST,\Delta,\epsilon,p}((v,\Delta) \text{ appears in the rooted spanning tree.}) \\ &= \lim_{\epsilon\to 0} \epsilon p(v)(V_{\epsilon p})_v^v.\end{aligned}$$

We shall prove in the following lemma that the above limit equals $\frac{p(v)m(v)}{\sum\limits_{v} p(v)m(v)}$. $\square$

As a consequence, we can express the probability of $\mu_{ST,p}(\pm e_1,\ldots,\pm e_k \in \mathcal{T})$ as a convex combination of $\mu_{ST,v}(\pm e_1,\ldots,\pm e_k \in \mathcal{T})$ for $v \in V$. But, it is not clear from this expression that the random spanning tree is a determinantal process. We look for an expression similar to Theorem 5.2.3. The idea is to use Theorem 5.2.3 for the probability $\mu_{ST,\Delta,\epsilon,p}$ and then let $\epsilon \to 0$. We need two limit properties for the potential proved in the following lemma.

Lemma 5.3.2. *Suppose $(X_t, t \geq 0)$ is a recurrent irreducible Markov process on $\{1,\ldots,n\}$. Denote by $\mathbb{E}^b$ its law with initial state b, by m its unique invariant distribution and by L its generator. Let p be a non-trivial non-negative function on $\{1,\ldots,n\}$, i.e. $\sum\limits_{i=1}^{n} p(i) > 0$. Then for any $a, b \in \{1,\ldots,n\}$*

i) $\lim\limits_{\epsilon\to 0} \epsilon (V_{\epsilon p})_a^b = \frac{m_a}{\langle m,p\rangle}$;

ii) $\lim\limits_{\epsilon\to 0} -(V_{\epsilon p})_a^b + (V_{\epsilon p})_a^a = \dfrac{\mathbb{E}^b[\int_0^{T_a} p(X_s)\,ds]}{\mathbb{E}^a[\int_0^{T_a+} p(X_s)\,ds]} = \dfrac{m_a V^{\{a\}^c} p(b)}{\langle m,p\rangle}$ *where* $T_a+ = \inf\{t > T_1 : X_t = a\}$ *and T_1 is the first jumping time.*

Proof.

i) **Firstly** Suppose p is strictly positive.

Define $A_t = \int_0^t \frac{1}{p(X_s)}\,ds, t \geq 0$ and its right-continuous inverse $\sigma_t = \inf\{s \geq 0, A_s > t\}, t \geq 0$. Define the time changed process $Y_t = (X_{\sigma_t}, t \geq 0)$. Then $(Y_t, t \geq 0)$ is a recurrent irreducible Markov process on $\{1,\ldots,n\}$ with generator $L^{(p)} = M_{1/p}L$

and resolvent $(V_r^{(p)}, r > 0)$. Its invariant probability is $\frac{mM_p}{\sum_{i=1}^{n} m_i p(i)}$. By the ergodic theorem,

$$\lim_{\epsilon\to 0} \epsilon((\epsilon - M_{1/p}L)^{-1})_a^b = \lim_{\epsilon\to 0} \epsilon((\epsilon - L^{(p)})^{-1})_a^b = \lim_{\epsilon\to 0} \epsilon(V_\epsilon^{(p)})_a^b = \frac{m_a p(a)}{\sum_{i=1}^{n} m_i p(i)}.$$

Since $V_{\epsilon p} = (M_{\epsilon p} - L)^{-1} = (\epsilon - M_{1/p}L)^{-1} M_p$,

$$\lim_{\epsilon\to 0} \epsilon(V_{\epsilon p})_a^b = \frac{m_a}{\langle m, p\rangle} \text{ for any } a, b \in \{1, \dots, n\}.$$

Second For p non-negative, take $h > 0$, then $p + h$ is strictly positive. Thus, we can apply the partial result for the strictly positive case:

$$\lim_{\epsilon\to 0} \epsilon(V_{\epsilon(p+h)})_a^b = \frac{m_a}{\langle m, p+h\rangle} \text{ for any } a, b \in \{1, \dots, n\}.$$

By the resolvent equation, $\epsilon V_{\epsilon(p+h)} - \epsilon V_{\epsilon p} = -h\epsilon^2 V_{\epsilon p} V_{\epsilon(p+h)}$. We will prove that $\sup_{\epsilon>0} ||\epsilon V_{\epsilon p}||_\infty < \infty$ in order to conclude

$$\lim_{\epsilon\to 0} \epsilon(V_{\epsilon p})_a^b = \frac{m_a}{\langle m, p\rangle} \text{ for any } a, b \in \{1, \dots, n\}.$$

By the resolvent equation, $V_{\epsilon p} = V_p + \frac{1-\epsilon}{\epsilon} V_{\epsilon p} M_{\epsilon p} V_p$. By Lemma 2.2.5,

$$||V_{\epsilon p}||_\infty \le ||V_p||_\infty + \frac{1-\epsilon}{\epsilon} ||V_p||_\infty = \frac{1}{\epsilon} ||V_p||_\infty.$$

By the assumption of irreducible recurrence, $||V_p||_\infty < \infty$ and the proof is complete.

ii) By the strong Markov property at the hitting time T_a,

$$\begin{aligned}
(V_{\epsilon p})_a^b =& \mathbb{E}^b[\int_0^\infty e^{-\int_0^t \epsilon p(X_s)\,ds} 1_{\{X_t=a\}}\,dt] \\
=& \mathbb{E}^b[\int_{T_a}^\infty e^{-\epsilon \int_0^t p(X_s)\,ds} 1_{\{X_t=a\}}\,dt] \\
=& \mathbb{E}^b[e^{-\epsilon \int_0^{T_a} p(X_s)\,ds}](V_{\epsilon p})_a^a.
\end{aligned}$$

By monotone convergence,

$$\frac{1}{\epsilon}\mathbb{E}^b[1 - \exp(-\epsilon \int_0^{T_a} p(X_s)\,ds)] = \mathbb{E}^b[\int_0^{T_a} p(X_s)\,ds] = V^{\{a\}^c} p(b),$$

where $V^{\{a\}^c}$ is the potential corresponding to the process X killed at $\{a\}$. Meanwhile,

$$\lim_{\epsilon\to 0}\epsilon(V_{\epsilon p})^a_a = \frac{m_a}{\langle m,p\rangle}.$$

Finally,

$$\lim_{\epsilon\to 0}(V_{\epsilon p})^b_a - (V_{\epsilon p})^a_a = \frac{m_a V^{\{a\}^c}p(b)}{\langle m,p\rangle}.$$

By the ergodic theorem and the strong Markov property,

$$\frac{m_a}{\langle m,p\rangle} = \lim_{t\to\infty}\frac{\mathbb{E}^a[\int_0^t 1_{\{X_s=a\}}\,ds]}{\mathbb{E}^a[\int_0^t p(X_s)\,ds]} = \frac{\mathbb{E}^a[\int_0^{T_a+} 1_{\{X_s=a\}}\,ds]}{\mathbb{E}^a[\int_0^{T_a+} p(X_s)\,ds]} = \frac{1}{-L^a_a\mathbb{E}^a[\int_0^{T_a+} p(X_s)\,ds]}.$$

Therefore we have another way to interpret the limit:

$$\lim_{\epsilon\to 0} -(V_{\epsilon p})^b_a + (V_{\epsilon p})^a_a = \frac{\mathbb{E}^b[\int_0^{T_a} p(X_s)\,ds]}{-L^a_a\mathbb{E}^a[\int_0^{T_a+} p(X_s)\,ds]}.$$

$\square$

Theorem 5.3.3 (Transfers current theorem)**.** *The finite marginal distribution is given by*

$$\mu_{ST,p}(e_1,\ldots,e_k\in\mathcal{T}) = L^{e_1-}_{e_1+}\cdots L^{e_k-}_{e_k+}\det(K^p_{e_i,e_j}, i,j=1,\ldots,k)$$

where $K^p_{e_i,e_j} = \dfrac{\mathbb{E}^{e_i+}[\int_0^{T_{e_j-}} p(X_s)\,ds] - \mathbb{E}^{e_i-}[\int_0^{T_{e_j-}} p(X_s)\,ds]}{-L^{e_j-}_{e_j-}\mathbb{E}^{e_j-}[\int_0^{T_{e_j-}+} p(X_s)\,ds]}.$

Proof. By Definition 5.3.1 and Theorem 5.2.3,

$$\begin{aligned}\mu_{ST,p}(e_1,\ldots,e_k\in\mathcal{T}) &= \lim_{\epsilon\to 0}\mu_{ST,\Delta,\epsilon,p}(e_1,\ldots,e_k\in\mathcal{T})\\ &= \lim_{\epsilon\to 0}\det((K^{(\epsilon,p)})_{e_i,e_j}, i,j=1,\ldots,k)\prod_{j=1}^k (L^{(\epsilon,p)})^{e_j-}_{e_j+}\end{aligned}$$

where

$$(K^{(\epsilon,p)})_{e_i,e_j} = (V_{\epsilon p})^{e_i-}_{e_j-} - (V_{\epsilon p})^{e_i+}_{e_j-}.$$

As ϵ tends to 0, $(L^{(\epsilon,p)})_{e_i}$ tends to $L^{e_i-}_{e_i+}$ for $i=1,\ldots,n$. By Proposition 5.3.2,

$$\lim_{\epsilon\to 0}(K^{(\epsilon,p)})_{e_i,e_j} = \frac{\mathbb{E}^{e_i+}[\int_0^{T_{e_j-}} p(X_s)\,ds] - \mathbb{E}^{e_i-}[\int_0^{T_{e_j-}} p(X_s)\,ds]}{-L^{e_j-}_{e_j-}\mathbb{E}^{e_j-}[\int_0^{T_{e_j-}+} p(X_s)\,ds]}.$$

Denote that limit by $K^p_{e_i,e_j}$. Then,

$$\mu_{ST,p}(e_1,\ldots,e_k \in \mathcal{T}) = L^{e_1-}_{e_1+}\cdots L^{e_k-}_{e_k+}\det(K^p_{e_i,e_j}, i,j=1,\ldots,k).$$

$\square$

An argument similar to Corollary 5.2.4 gives:

Corollary 5.3.4.

$$\mu_{ST,p}(\pm e_1,\ldots,\pm e_k \in \mathcal{T}) = \det(A^{(p)}_{e_i,e_j}, i,j=1,\ldots,k).$$

where $A^{(p)}_{e_i,e_j} = K^{(p)}_{e_i,e_j}L^{e_j-}_{e_j+} + K^{(p)}_{-e_i,-e_j}L^{e_j+}_{e_j-}$.

Chapter 6

Loop covering

6.1 Basic settings

Suppose $(G_n = (V_n, E_n, w_n), n \geq 1)$ is a sequence of undirected connected weighted graphs with maximum degrees D_n and minimum degrees d_n. Suppose the degrees are uniformly bounded from above and below, $D_n \leq D < +\infty$ and $d_n \geq d > 0$ for $n \geq 1$. Let V_n be the set of vertices and E_n the set of edges. Each edge $\{x, y\}$ is associated with a positive weight $(w_n)_{xy} = (w_n)_{yx}$. Let $(w_n)_x = \sum\limits_y (w_n)_{xy}$, $w_n = \sum\limits_x (w_n)_x$ and $(\pi_n)_x = (w_n)_x / w_n$[1]. Suppose $0 < r \leq 1/(w_n)_{xy} \leq R < \infty$ for all $n \geq 1, x, y \in V_n$. Use m_n to stand for the numbers of the vertices, $m_n = |V_n|$. Suppose $(X_m^{(n)}, m \in \mathbb{N})$ is the Markov chain associated to G_n with transition matrix Q_n where

$$(Q_n)_y^x = \begin{cases} w_{xy}/w_x & \text{if } \{x, y\} \in E_n, \\ (Q_n)_y^x = 0 & \text{otherwise.} \end{cases}$$

Given an additional killing parameter c_n, the non-trivial pointed loop measure μ_n^{p*} associated with the generator $-(1 + c_n)Id + Q_n$ has the following expression:

$$\mu_n^{p*}(\text{k jumps}, \xi_1 = x_1, \ldots, \xi_k = x_k) = \frac{1}{k} \left(\frac{1}{1 + c_n} \right)^k (Q_n)_{x_2}^{x_1} (Q_n)_{x_3}^{x_2} \cdots (Q_n)_{x_1}^{x_k}. \tag{6.1}$$

Let μ_n be the corresponding loop measure. We see that the non-trivial (pointed) loop measures μ_n^{p*} are finite. Set $\mathfrak{P}_n = \frac{\mu_n}{\mu_n(1)}$ and $\mathfrak{P}_n^{p*} = \frac{\mu_n^{p*}}{\mu_n^{p*}(1)}$.

Our main result is the determination of the limit of the probability under $\mathfrak{P}_n$ of the set of loops which cover V_n. We deduce from these results the probability of existence of such loops in a Poisson process of loops of intensity $\frac{\mu_n}{|V_n|}$. These limits show the existence of a phase

[1] In fact, π_n is the stationary distribution for the associated Markov chain on G_n.

transition according to the rate of increase of $-\ln c_n$. Our main assumptions, which will be checked in several examples are listed as follows:

(H1) *Denote by D_n the maximum degrees in G_n and by d_n the minimum degrees. Suppose the degrees are uniformly bounded from above and below: $D_n \leq D < +\infty$ and $d_n \geq d > 0$ for $n \geq 1$.*

(H2) *The weights are uniformly bounded from above and below: $\exists 0 < r < R$ such that $0 < r \leq 1/(w_n)_{xy} \leq R < \infty$ for all $n \geq 1, x, y \in V_n$.*

(H3) *The empirical distributions of the eigenvalues of the transition matrices Q_n converge in distribution to a probability measure ν as $n \to \infty$.*

For example, take $V_n = \mathbb{Z}^d/n\mathbb{Z}^d$. There is a map from $\mathbb{Z}^d$ to V_n which maps the vector v to $[v] \in V_n$ (the equivalence class of v). The edge set E_n is defined by

$$\{\{[u],[v]\} : u, v \in \mathbb{Z}^d \text{ and the distance between } u \text{ and } v \text{ is } 1\}.$$

We give each edge the same weight 1, i.e. $(w_n)_{xy} = 1$ for all $\{x, y\} \in E_n$. It is not hard to find that $D_n = d_n = d$ and we can take $R = r = 1$. We will show in section 4 that (H3) holds for this sequence of graph. The limit distribution is given by the self-convolutions of the semi-circle law.

For the sake of simplicity, we will use $\mathcal{C}$ to stand for the event $\{l \text{ covers every vertex}\}$. We can now state precisely the announced results:

Theorem 6.1.1. *We suppose (H1), (H2) and (H3).*

a) If $\lim\limits_{n\to\infty} -\dfrac{1}{m_n}\ln(c_n) \leq 0$, *then* $\lim\limits_{n\to\infty} \mathfrak{P}_n(\mathcal{C}) = 0$.

b) If $\lim\limits_{n\to\infty} -\dfrac{1}{m_n}\ln(c_n) = +\infty$, *then* $\lim\limits_{n\to\infty} \mathfrak{P}_n(\mathcal{C}) = 1$.

c) If $\lim\limits_{n\to\infty} -\dfrac{1}{m_n}\ln(c_n) = a \in]0, \infty[$, *then*

$$\lim_{n\to\infty} \mathfrak{P}_n(\mathcal{C}) = \frac{a}{a - \int \ln(1-x)\,\nu(dx)}$$

where $\int -\ln(1-x)\nu(dx) \in [\frac{r^2}{2R^2D^2}, \frac{28D^2R^2}{d^2r^2}]$.

Corollary 6.1.2. *We suppose (H1), (H2) and (H3). Let $\mathcal{L}^{(n)}$ be the Poisson collection of loops on G_n with intensity $\dfrac{1}{m_n}\mu_n$. Suppose $\lim\limits_{n\to\infty} -\dfrac{1}{m_n}\ln(c_n) = a$. Then, $\sum\limits_{l\in\mathcal{L}^{(n)}} 1_{\{l\in\mathcal{C}\}}$ converges in distribution to a Poisson random variable with parameter* $\max(a, 0)$ *as n tends to infinity.*

Remark 19. Suppose $\lim_{n\to\infty} -\frac{1}{m_n}\ln(c_n) = a$. We have a "phase transition" at $a = 0$ in the following sense:

- For $a \leq 0$, in the limit, there is no loops covering the whole space i.e. $\sum_{l\in\mathcal{L}^{(n)}} 1_{\{l\in\mathcal{C}\}}$ tends to 0 in probability.

- For $a > 0$, large loops covering the whole space appear in the limit.

The classical covering problem is about the mean covering time C at which a random walk on the weighted graph $G = (V, E, w)$ has visited every vertex. Often, one considers the covering-and-return time C^+, which is defined as the first return time to the initial point after the covering time. Using a spanning tree argument, Theorem 1 in Chapter 6 of [AF] shows that

$$\mathbb{E}^v(C^+) \leq \sum_{x,y} w_{xy} \min_{T \text{ is a spanning tree}} \sum_{\text{edge } e\in T} \frac{1}{w_e}. \tag{6.2}$$

Moreover, Lemma 25 in Chapter 6 of [AF] shows that $\max_v \mathbb{E}^v(C^+)$, $\min_v \mathbb{E}^v(C^+)$ and $\mathbb{E}^\pi(C)$ are equivalent up to some constants independent of the undirected weighted graphs.

In our problem, we need to use these classical results to deduce an upper bound of the covering time under the bridge measure $\mu_n(\cdot|p(l) = k)$. More precisely, we consider the conditional probability of the event $\mathcal{C}$ given the length of the loop. Trivially, it is zero for loops with length smaller than the size of the graph. As the size of the graph G_n grows to infinity, it tends to 1 for length larger than m_n^4 where m_n is the size of the graph, see Proposition 6.2.8. Besides, we need an estimation of the length of the loops, see Proposition 6.2.7. The proof is based on an upper bound on the transition functions of symmetric Markov processes. It is related to an estimate on Dirichlet forms, proved in [CKS87]. Proposition 18 of Chapter 6 in [AF] gives the result for regular case through an elementary argument which is used here to get the estimations for the traces of the transition matrices.

In the end, let us briefly present the plan of the paper. Except for section 6, we assume (H1), (H2) and (H3). Section 2 is devoted to proving Theorem 6.1.1 and Corollary 6.1.2. In section 3, we show a stability result for the limit distribution ν of the eigenvalues of Q_n by Cauchy's interlacing theorem, see Proposition 6.3.2 and its following remark. As a result, we obtain that the limit of $\mathfrak{P}_n(C)$ does not change if the weights are modified in a subgraph of G_n of size $o(m_n)$. As applications of Theorem 6.1.1, we analyse two examples of graphs: the case of discrete torus in section 4 and the case of balls in a regular tree in section 5. In the second example, we show that the empirical distributions of the eigenvalues of the transition matrices Q_n converge as $n \to \infty$ to a purely atomic distribution given by the roots of a class of polynomials, see Proposition 6.5.1. In section 6, we study the complete graph which does not satisfy (H1). By comparing a modified geometric variable with the covering time of the coupon collector problem, we get a result different from Theorem 6.1.1 and an equivalent of $\mathfrak{P}_n(C)$ if the killing rate is of order n^{-1}, see Theorem 6.6.3.

6.2 The limit of the percentage of non-trivial loops containing all the vertices

We suppose $|V_n| = n$ for all $n \in \mathbb{N}_+$, as the result for the case $|V_n| = m_n \overset{n\to\infty}{\to} \infty$ can be proved in the same way.

Write the eigenvalues of Q_n in a non-decreasing order $-1 \leq \lambda_{n,1} \leq \cdots \leq \lambda_{n,n} \leq 1$ (all the eigenvalues are real because the Markov chain is reversible) and set $\nu_n = \frac{1}{n}\sum\limits_{i=1}^{n} \delta_{\lambda_{n,i}}$. Then assumption (H3) can be written as follows:

ν_n converges in distribution to a probability measure ν on $[-1,1]$.

Immediately, we have $\int x\,\nu(dx) = 0$ since $\operatorname{Tr} Q_n = 0$ $\forall n \in \mathbb{N}_+$.

6.2.1 The d-regular aperiodic case

Suppose that all the graphs G_n are d-regular with weight 1 on every edge and that all the graphs G_n are aperiodic.

We use bounds for the second largest and the smallest eigenvalues of an irreducible aperiodic Markov transition matrix stated in [DS91]. To present these bounds, let us introduce some notations.

A weighted connected finite undirected graph $G = (V, E, w)$ is naturally associated with an irreducible reversible Markov chain with transition matrix Q^x_y. Denote by π its stationary distribution: $\pi_z = \frac{\sum\limits_{y} w_{zy}}{\sum\limits_{x,y} w_{xy}}$ for $z \in V$. Set the normalized weight $\bar{w}$ by:

$$\bar{w}_{xy} = \pi_x Q^x_y = \frac{w_{xy}}{\sum\limits_{x,y} w_{xy}}.$$

For any different $x, y \in V$, there exists at least a self-avoiding path from x to y. Choose one such path arbitrarily and denote it by γ_{xy}. Define the path length $|\gamma_{xy}|_w$ with respect to w

(which actually depends on $\bar{w}$):

$$|\gamma_{xy}|_w = \sum_{\{u,v\}\in\gamma_{xy}} 1/\bar{w}_{uv}.$$

Define

$$\kappa = \max_{u,v\in V} \sum_{\{x,y\in V:\{u,v\}\in\gamma_{xy}\}} |\gamma_{xy}|_w \pi_x \pi_y.$$

Proposition 6.2.1 (Poincaré inequality [DS91])**.** *The second largest eigenvalue β_1 of Q satisfies:*

$$\beta_1 \leq 1 - 1/\kappa$$

for any choice of $(\gamma_{xy}, x, y \in V)$.

Moreover, if Q is aperiodic, for each $x \in V$, there exists at least one path from x to x with odd number of edges. Choose one such path, namely σ_x. Define the length of the path related to w (which is determined by $\bar{w}$):

$$|\sigma_x|_w = \sum_{\{u,v\}\in\sigma_x} 1/\bar{w}_{uv}.$$

Define $\tau = \max_e \sum_{\{x:\sigma_x \text{ contains edge } e\}} \pi_x |\sigma_x|_w$.

Proposition 2 in [DS91] gives the following lower bound for the smallest eigenvalue:

Proposition 6.2.2 ([DS91])**.** *Suppose Q is aperiodic, then the smallest eigenvalue β_{min} of Q satisfies*

$$\beta_{min} \geq -1 + \frac{2}{\tau}$$

for any choice of σ_x for $x \in X$.

As an application in our sequence of graphs $(G_n, n \geq 1)$, we have the following estimates:

Corollary 6.2.3. *Let Q be the transition matrix associated with a regular connected graph with n vertices, degree d and weight 1.*

a) The second largest eigenvalue β_1 of Q satisfies:

$$\beta_1 \leq 1 - \frac{1}{dn^2}.$$

b) If Q is aperiodic, then the smallest eigenvalue β_{min} of Q satisfies:

$$\beta_{min} \geq -1 + \frac{2}{3dn^2}.$$

Proof. In the Poincaré's inequality, π is the stationary probability measure. Specially, in the case of regular graphs, it is uniformly distributed on vertices and

$$(\bar{w}_n)_{xy} = (\pi_n)_x (Q_n)^x_y = \frac{1}{dn}.$$

For the part a), one could choose γ_{xy} to be self-avoiding and consequently its length is no more than $n-1$. Thus, $|\gamma_{xy}|_w \leq dn^2$ and

$$\beta_1 \leq 1 - \frac{1}{dn^2}.$$

For the part b), among all the loop with odd number of edges, there is a loop with minimal number of edges, namely σ. Then σ is necessarily self-avoiding. Accordingly, the number of edges in σ is no more than n. Suppose the loop σ visits x_0. For any $x \in V$, there exists a self-avoiding path γ_{xx_0} from x to x_0 and its reverse γ_{x_0x}. The sum of γ_{xx_0}, γ_{x_0x} and σ is a loop containing x with a odd number of edges which is no more than $3n$. Thus, $|\sigma_x|_w \leq 3dn^2$ and $\tau \leq 3dn^2$. Therefore,

$$\beta_{min} \geq -1 + \frac{2}{3dn^2}.$$

□

As a consequence of Corollary 6.2.3, we have the following result:

Corollary 6.2.4. *If Q_n is the transition matrix associated with a regular connected aperiodic graph with degree d and n vertices, then*

$$\lim_{n\to\infty} \sup_{k \geq n^b} |\operatorname{Tr} Q_n^k - 1| = 0 \quad \forall b > 2.$$

Proof. By Corollary 6.2.3, for any eigenvalue λ different from 1, $1 - |\lambda| \geq \frac{2}{3dn^2}$. Denote by $\mathrm{Eig}(Q_n)$ the collection of eigenvalues with the multiplicities. Recall that the eigenvalue 1 is simple by Perron-Frobenius theorem. Then, we have

$$|\operatorname{Tr} Q_n^k - 1| = |\sum_{\lambda \in \mathrm{Eig}(Q_n)} \lambda^k - 1| = |\sum_{\lambda \in \mathrm{Eig}(Q_n), \lambda \neq 1} \lambda^k|$$

$$\leq \sum_{\lambda \in \mathrm{Eig}(Q_n), \lambda \neq 1} |\lambda|^k \leq n\left(1 - \frac{2}{3dn^2}\right)^k.$$

Consequently, $\lim_{n\to\infty} \sup_{k \geq n^b} |\operatorname{Tr} Q_n^k - 1| = 0$. □

We also need the following bound deduced from Proposition 18 of Chapter 6 in [AF]:

Proposition 6.2.5. *Let $X^{(n)}$ be the simple random walk on a regular n-vertex graph, $\mathbb{P}^x[X_k^{(n)} = x] \leq 10\max(\frac{1}{\sqrt{k}}, \frac{1}{n})$ for every vertex x.*

Proof. By Proposition 18 of Chapter 6 in [AF], for any x, y

$$\mathbb{P}^x[X_k^{(n)} = y] \leq 10k^{-1/2}, \quad k \leq n^2.$$

Conditioning with respect to $X_{k-n^2}^{(n)}$, we get

$$\mathbb{P}^x[X_k^{(n)} = y] \leq \frac{10}{n}, \quad k \geq n^2.$$

□

We can now prove the following estimates on the length of the loops on G_n.

Proposition 6.2.6. *Let μ_n be the loop measure on a d-regular connected aperiodic graph with n vertices defined by equation (6.1) in the section of basic settings of this chapter:*

a) $\mu_n(p(l) \in [n, n^2]) \leq \dfrac{20}{(1+c_n)^n}\left(\dfrac{n}{\sqrt{n-1}} - 1\right)$,

b) *For* $b > 2$, $\mu_n(p(l) \in [n^2, n^b]) \leq \dfrac{10b\ln n}{(1+c_n)^{n^2}}$,

c) *For* $b > 2$, $\mu_n(p(l) \geq n^b) \sim \displaystyle\sum_{k \geq n^b} \frac{1}{k}\left(\frac{1}{1+c_n}\right)^k$,

d) $\dfrac{1}{n}\mu_n(2 \leq p(l) < n) \in \left[\dfrac{1}{2(1+c_n)^2 d^2}, 20\right]$.

Proof. Let $(X_k^{(n)})_k$ denote the simple random walk on G_n and Q_n its transition matrix.

a),b) By Proposition 6.2.5, $\sup_{x \in V_n} \mathbb{P}^x[X_k^{(n)} = x] \leq 10\max(\frac{1}{n}, \frac{1}{\sqrt{k}})$. Consequently,

$$\mu_n(p(l) \in [n, n^2]) = \sum_{k=n}^{n^2} \frac{1}{k(1+c_n)^k}\left(\sum_{x \in V_n} \mathbb{P}^x[X_k^{(n)} = x]\right)$$
$$\leq \left(\frac{1}{1+c_n}\right)^n \sum_{k=n}^{n^2} \frac{10n}{k^{\frac{3}{2}}} \leq \frac{20}{(1+c_n)^n}\left(\frac{n}{\sqrt{n-1}} - 1\right),$$

$$\mu_n(p(l) \in [n^2, n^b]) = \sum_{k=n^2}^{n^b} \frac{1}{k}\left(\sum_{x \in V_n} \mathbb{P}^x[X_k^{(n)} = x]\right)\left(\frac{1}{1+c_n}\right)^k$$
$$\leq \left(\frac{1}{1+c_n}\right)^{n^2} \sum_{k=n^2}^{n^b} \frac{1}{k} n \frac{10}{n} \leq \frac{10b\ln n}{(1+c_n)^{n^2}}.$$

c) By Corollary 6.2.4, for $b > 2$, we have

$$\mu_n(p(l) \geq n^b) = \sum_{k \geq n^b} \frac{1}{k} \operatorname{Tr} Q_n^k \left(\frac{1}{1+c_n}\right)^k \sim \sum_{k \geq n^b} \frac{1}{k} \left(\frac{1}{1+c_n}\right)^k \quad \text{as } n \to \infty.$$

d) $\mu_n(2 \leq p(l) < n) = \sum_{k=2}^{n-1} \frac{1}{k} \left(\frac{1}{1+c_n}\right)^k \operatorname{Tr} Q_n^k \in \left[\frac{1}{2}\left(\frac{1}{1+c_n}\right)^2 \operatorname{Tr} Q_n^2, \sum_{k=2}^{n} \frac{1}{k} \operatorname{Tr} Q_n^k\right]$. In the case of the regular graph with degree d, $\operatorname{Tr} Q_n^2 \geq \frac{n}{d^2}$. While by Proposition 6.2.5, $\operatorname{Tr} Q_n^k \leq \frac{10n}{\sqrt{k}}$ $\forall k \in \mathbb{N}$. As a consequence,

$$\frac{1}{n}\mu_n(2 \leq p(l) < n) \in \left[\frac{1}{2(1+c_n)^2 d^2}, 20\right].$$

$\square$

Proposition 6.2.7. *Assume that for every $n \in \mathbb{N}_+$, G_n is a d-regular connected aperiodic graph with n-vertices and assume that (H3) holds.*

a.1) If $\liminf_{n\to\infty} c_n > 0$, then $\lim_{n\to\infty} \mathfrak{P}_n(p(l) < n) = 1$.

a.2) If $\lim_{n\to\infty} c_n = 0$ and $\lim_{n\to\infty} -\frac{1}{n}\ln(c_n) = 0$, then $\lim_{n\to\infty} \mathfrak{P}_n(p(l) < n) = 1$.

b) If $\lim_{n\to\infty} c_n = 0$ and $\lim_{n\to\infty} -\frac{1}{n}\ln(c_n) = \infty$, then $\lim_{n\to\infty} \mathfrak{P}_n(p(l) \geq n^4) = 1$.

c) If $\lim_{n\to\infty} c_n = 0$ and $\lim_{n\to\infty} -\frac{1}{n}\ln(c_n) = a \in]0,\infty[$, then

$$\lim_{n\to\infty} \mathfrak{P}_n(p(l) \geq n^4) = \lim_{n\to\infty} \mathfrak{P}_n(p(l) \geq n) = \frac{a}{a - \int \ln(1-x)\,\nu(dx)}.$$

Besides, $\int -\ln(1-x)\,\nu(dx) \in [\frac{1}{2d^2}, 20]$.

Proof.

a.1) and a.2) By the estimations in Proposition 6.2.6, we have

1) $\mu_n(p(l) \in [n, n^2]) \leq \frac{20n}{\sqrt{n-1}(1+c_n)^2}$,

2) $\mu_n(p(l) \in [n^2, n^4]) \leq \frac{40 \ln n}{(1+c_n)^2}$,

3) $\mu_n(p(l) > n^4) \sim \sum_{k \geq n^4} \frac{1}{k}\left(\frac{1}{1+c_n}\right)^k \leq \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1}{1+c_n}\right)^k = -\ln(c_n) + \ln(1+c_n)$,

4) $\mu_n(p(l) < n) \geq \frac{n}{2(1+c_n)^2 d^2}$.

As a summary, we have $\mu_n(p(l) \geq n) = o(\mu_n(p(l) < n))$. Therefore,

$$\lim_{n\to\infty} \mathfrak{P}_n(p(l) < n) = \lim_{n\to\infty} \frac{\mu_n(p(l) < n)}{\mu_n(p(l) < n) + \mu_n(p(l) \geq n)} = 1.$$

b) In this case, $\lim_{n\to\infty} \sup_{k \leq n^4} |(1+c_n)^k - 1| = 0$. By Proposition 6.2.6, we have

$$\begin{aligned}
\mu_n(p(l) \geq n^4) &\sim \sum_{k \geq n^4} \frac{1}{k} \left(\frac{1}{1+c_n}\right)^k \\
&= \sum_{k \geq 1} \frac{1}{k} \left(\frac{1}{1+c_n}\right)^k - \sum_{k < n^4} \frac{1}{k} \left(\frac{1}{1+c_n}\right)^k \\
&= -\ln(c_n) + \ln(1+c_n) - \sum_{k < n^4} \frac{1}{k} \left(\frac{1}{1+c_n}\right)^k.
\end{aligned}$$

Since $\left| \sum_{k<n^4} \frac{1}{k} \left(\frac{1}{1+c_n}\right)^k \right| \sim \sum_{k<n^4} \frac{1}{k} \sim 4\ln n = o(-\ln n)$, we have $\mu_n(p(l) \geq n^4) \sim -\ln(c_n)$. By Proposition 6.2.6, we know $\mu_n(p(l) < n^4) = O(n)$. Then,

$$\mu_n(p(l) < n^4) = o(-\ln(c_n)) = o(\mu_n(p(l) \geq n^4)).$$

Therefore,

$$\lim_{n\to\infty} \mathfrak{P}_n(p(l) \geq n^4) = \lim_{n\to\infty} \frac{\mu_n(p(l) \geq n^4)}{\mu_n(p(l) \geq n^4) + \mu_n(p(l) < n^4)} = 1.$$

c) If $\lim_{n\to\infty} c_n = 0$ and $\lim_{n\to\infty} -\frac{1}{n}\ln(c_n) = a \in]0, \infty[$, then

$$\lim_{n\to\infty} \sup_{k \leq n^4} \left| \left(\frac{1}{1+c_n}\right)^k - 1 \right| = 0.$$

We have assumed that $\frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_{n,i}}$ converges in distribution to the probability measure ν. Then for k fixed,

$$0 \leq \lim_{n\to\infty} \frac{\operatorname{Tr} Q_n^k}{n} = \lim_{n\to\infty} \frac{1}{n}(\lambda_{n,1}^k + \cdots + \lambda_{n,n}^k) = \int x^k \, \nu(dx).$$

Note that

$$\begin{aligned}
\lim_{A\to\infty} \limsup_{n\to\infty} \frac{1}{n} \mu_n(p(l) \in [A, n]) &= \lim_{A\to\infty} \limsup_{n\to\infty} \sum_{k=A}^{n} \frac{1}{k} \left(\frac{1}{1+c_n}\right)^k \frac{\operatorname{Tr} Q_n^k}{n} \\
&\leq \lim_{A\to\infty} \limsup_{n\to\infty} \sum_{k=A}^{n} \frac{1}{k} \frac{\operatorname{Tr} Q_n^k}{n}
\end{aligned}$$

By Proposition 6.2.5, $\dfrac{\operatorname{Tr} Q_n^k}{n} \leq \dfrac{10}{\sqrt{k}}$ for $k \leq n^2$. Therefore,

$$\lim_{A\to\infty} \limsup_{n\to\infty} \frac{1}{n}\mu_n(p(l) \in [A,n]) \leq \lim_{A\to\infty} \sum_{A}^{\infty} \frac{10}{k^{3/2}} = 0.$$

Consequently,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\mu_n(p(l) \in [2,n[)}{n} &= \lim_{A\to\infty} \lim_{n\to\infty} \frac{\mu_n(p(l) \in [2,A[)}{n} \\
&= \lim_{A\to\infty} \sum_{k=2}^{A} \frac{1}{k} \int x^k \, \nu(dx).
\end{aligned}$$

Since $\dfrac{1}{2d^2(1+c_n)^2} < \dfrac{1}{n}\mu_n(p(l) \in [2,n[) < 20$ for n large and $\int x^k \nu(dx) \geq 0$, we have that

$$\sum_{k=1}^{\infty} \frac{1}{2k} \int x^{2k} \, \nu(dx) \leq 20.$$

Note that $\left| \sum_{k=2}^{\infty} \frac{1}{k} x^k \right| \leq 2 \sum_{k=1}^{\infty} \frac{1}{2k} x^{2k}$ for $|x| \leq 1$ and $\operatorname{supp}(\nu) \subset [-1,1]$. By the dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\mu_n(p(l) \in [2,n[)}{n} &= \int (-\ln(1-x) - x)\nu(dx) \\
&= -\int \ln(1-x)\nu(dx) \in [\frac{1}{2d^2}, 20].
\end{aligned}$$

By Proposition 6.2.6, we have $\mu_n(p(l) \in [n, n^4]) = o(n)$ and

$$\begin{aligned}
\mu_n(p(l) \geq n^4) \sim \sum_{k \geq n^4} \frac{1}{k} \left(\frac{1}{1+c_n} \right)^k &= \sum_{k\geq 1} \frac{1}{k} \left(\frac{1}{1+c_n} \right)^k - \sum_{k=1}^{n^4-1} \frac{1}{k} \left(\frac{1}{1+c_n} \right)^k \\
&= -\ln c_n + \ln(1+c_n) - \sum_{k=1}^{n^4-1} \frac{1}{k} \left(\frac{1}{1+c_n} \right)^k \sim -an.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \mathfrak{P}_n(p(l) \geq n^4) = \lim_{n\to\infty} \mathfrak{P}_n(p(l) \geq n) = \frac{a}{a + \int -\ln(1-x)\, \nu(dx)}.$$

□

Proposition 6.2.8. *Take* $\epsilon > 0$, $\lim_{n\to\infty} \sup_{k \geq n^{3+\epsilon}} |\mathfrak{P}_n(\mathcal{C} \,|\, p(l) = k) - 1| = 0.$

Proof. According to Theorem 1 in Chapter 6 of the book [AF], the expectation of the "cover-and-retur" time is bounded from above by $dn(n-1)$. To be more precise, define the hitting time by $T_y^{(n)} = \inf\{m \geq 0, X_m^{(n)} = y\}$, define the covering time by $C^{(n)} = \max_{y \in V_n} T_y^{(n)}$ and define the cover-and-return time $C_+^{(n)} = \inf\{m \geq C^{(n)} : X_m^{(n)} = X_0^{(n)}\}$. Then, Theorem 1 in [AF] states that

$$\max_x \mathbb{E}^x[C_+^{(n)}] \leq dn(n-1). \tag{6.3}$$

By Markov's inequality, $\max_x \mathbb{P}^x[C_+^{(n)} \geq n^{3+\epsilon}] \leq \dfrac{dn(n-1)}{n^{3+\epsilon}}$.

$$\begin{aligned}
&\mathfrak{P}_n(l \text{ does not cover every vertex} \,|\, p(l) = k) \\
&\qquad = \frac{\mathfrak{P}_n(l \text{ does not cover every vertex}, p(l) = k)}{\mathfrak{P}_n(p(l) = k)} \\
&\qquad\qquad = \frac{\mu_n(l \text{ does not cover every vertex}, p(l) = k)}{\mu_n(p(l) = k)}.
\end{aligned}$$

By the definition[2] of μ_n, $\mu_n(p(l) = k) = \dfrac{1}{k}\left(\dfrac{1}{1+c_n}\right)^k \operatorname{Tr} Q_n^k$ and

$$\begin{aligned}
&\mu_n(l \text{ does not cover every vertex}, p(l) = k) \\
&\qquad = \frac{1}{k}\left(\frac{1}{1+c_n}\right)^k \sum_{x \in V_n} \mathbb{P}^x[X_k^{(n)} = x, X \text{ does not cover every vertex before time } k].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathfrak{P}_n(l \text{ does not cover every vertex} \,|\, p(l) = k) \\
&\qquad = \frac{1}{\operatorname{Tr} Q_n^k} \sum_{x \in V_n} \mathbb{P}^x[X_k^{(n)} = x, X \text{ does not cover every vertex before time } k] \\
&\qquad\qquad \leq \frac{1}{\operatorname{Tr} Q_n^k} \sum_{x \in V_n} \mathbb{P}^x[C_+^{(n)} > k].
\end{aligned}$$

We have that $\sum_{x \in V_n} \mathbb{P}^x[C_+^{(n)} > k] \leq \dfrac{dn^2(n-1)}{n^{3+\epsilon}} \leq \dfrac{d}{n^\epsilon}$. By Corollary 6.2.4, $\operatorname{Tr} Q_n^k$ tends to 1 uniformly for $k \geq n^{3+\epsilon}$.

Therefore, $\lim_{n\to\infty} \sup_{k \geq n^{3+\epsilon}} |\mathfrak{P}_n(\mathcal{C} \,|\, p(l) = k) - 1| = 0.$ □

Theorem 6.1.1 for a sequence of d-regular aperiodic connected graphs follows from Proposition 6.2.7 and Proposition 6.2.8. Indeed,

a) If $\lim_{n\to\infty} -\frac{\ln c_n}{n} \leq 0$, then $\lim_{n\to\infty} \mathfrak{P}_n[\mathcal{C}] \leq \lim_{n\to\infty} \mathfrak{P}_n[p(l) \geq n] = 0$.

[2]See equation (6.1) in the introduction.

b) If $\lim_{n\to\infty} -\frac{\ln c_n}{n} = \infty$, then

$$\lim_{n\to\infty} \mathfrak{P}_n[\mathcal{C}] \geq \lim_{n\to\infty} \mathfrak{P}_n[\mathcal{C}, p(l) > n^4] \overset{\text{By Proposition 6.2.8}}{=} \lim_{n\to\infty} \mathfrak{P}_n[p(l) > n^4] = 1.$$

c) Suppose $\lim_{n\to\infty} -\frac{\ln c_n}{n} = a \in]0, \infty[$. Then,

$$\lim_{n\to\infty} \mathfrak{P}_n\left[\mathcal{C}, p(l) \in [n, n^4[\right] \leq \lim_{n\to\infty} \mathfrak{P}_n\left[p(l) \in [n, n^4[\right] = 0.$$

Thus,

$$\lim_{n\to\infty} \mathfrak{P}_n[\mathcal{C}] = \lim_{n\to\infty} \mathfrak{P}_n[\mathcal{C}, p(l) > n^4]$$
$$\overset{\text{By Proposition 6.2.8}}{=} \lim_{n\to\infty} \mathfrak{P}_n[p(l) > n^4] = \frac{a}{a - \ln(1-x)\nu(dx)}.$$

We provide a proof of Corollary 6.1.2 in this setting as follows:

Proof of Corollary 6.1.2. The argument in Proposition 6.2.7, actually gives the following result:

$$\lim_{n\to\infty} \frac{1}{n}\mu_n(p(l) > n^4) = \lim_{n\to\infty} \frac{1}{n}\mu_n(p(l) \geq n) = \max(a, 0).$$

By Proposition 6.2.8, $\lim_{n\to\infty} \dfrac{\mu_n(\mathcal{C})}{n} = \lim_{n\to\infty} \dfrac{1}{n}\mu_n(p(l) > n^4) = \max(a, 0)$. Therefore, $\sum_{l\in\mathcal{L}^{(n)}} 1_{\{l\in C\}}$ converges in distribution to a Poisson random variable with parameter $\max(a, 0)$ as n tends to infinity. □

6.2.2 Non-regular aperiodic case, with unit weights

In this section, we still suppose that all the weights are 1 but the graph is not necessarily d-regular. We also assume that the graphs are aperiodic. Corollary 6.2.3, Corollary 6.2.4 and Proposition 6.2.8 remain valid except that one should replace the universal degree d by the maximum degree D everywhere.

The only statement in the previous section that has to be changed is Proposition 6.2.5 based on Proposition 18, Chapter 6 in [AF]. We use instead the following upper bound:

Proposition 6.2.9. *Given any unweighted (all the weights are* 1*) graph with* n *vertices, maximum degree* D *and minimum degree* d*, let* $(X_k, k \geq 0)$ *be the simple random walk on the graph. For* $k > 1$*, one has*

$$\sum_x \mathbb{P}^x[X_k = x] \leq \frac{14D^2}{d^2} \max\left(\frac{n}{\sqrt{k}}, 1\right).$$

Proof. Proposition 18 of Chapter 6 in [AF] states that $\mathbb{P}^x(X_k = y) \le \frac{10}{\sqrt{k}}$ for any vertex x, y in a regular graph and for any $k \le n^2$. Since our graph is not regular, we cannot directly apply this result. Nevertheless, the proof used there still works with a little modification and it is repeated here, for self containedness.
Use $N_i(A^c)$ to stand for the number of times the chain visits the vertex i before hitting A^c.

1) Suppose $0 < |A| < n$, we will show that $\mathbb{E}^i[N_i(A^c)] \le 5D|A|/d$:

$$\mathbb{E}^i[N_i(A^c)] = 1/\mathbb{P}^i[T_{A^c} < T_i^+] = d(i)r(i, A^c).$$

T_{A^c} is the hitting time for A^c and T_i^+ is the first return time for i. $d(i) \le D$ is the degree of the vertex i and $r(i, A^c)$ is the effective resistance between i and A^c which is bounded from above by $5|A|/d$. For the definition of the effective resistance and the relation between electrical network and reversible Markov chain, please refer to [LP].

In order to show $r(i, A^c) \le 5|A|/d$, let us choose a shortest path from i to A^c, namely $i = i_1 \to \cdots \to i_{k+1}$ such that $i_1, \ldots, i_k \in A$ and $i_{k+1} \in A^c$. If $k = 1$, $r(i, A^c) \le 1/d$. For $k > 1$, consider B the subset of A which consists of the vertices adjoint to some $\{i_j : j = 1, \ldots, k-1\}$, i.e.

$$B = \{y \in A : \exists 1 \le j \le k-1, y \text{ is adjoint to } i_j\}.$$

For each $b \in B$, b has at most 3 neighbours in $\{i_1, \ldots, i_{k+1}\}$. Otherwise, one could find a path from i to A^c containing b which is shorter than the path $i_1 \to \cdots \to i_{k+1}$. Therefore,

$$|B| \ge \frac{1}{3} \sum_{j=1}^{k-1} \sum_{y \in B} 1_{\{y \text{ is adjoint to } i_j\}} \ge \frac{1}{3} d(k-1)$$

where the second inequality comes from the fact that the neighbourhoods of $i_1, \ldots, i_{k-1}$ are all contained in B. Moreover, B is contained in A and hence $|A| \ge |B| \ge d(k-1)/3$. It implies that

$$r(i, A^c) \le k \le \frac{3|A|}{d} + 1 \le \frac{5|A|}{d}$$

as long as $d \le 2|A|$. For the case $d > 2|A|$, we know that there exists at least $d - |A|$ edges from i to A^c. Thus, $r(i, A^c) \le \dfrac{1}{d - |A|} \le \dfrac{2}{d} \le \dfrac{5|A|}{d}$.

2) We will show that $\displaystyle\sum_i \sum_{t \le k} \mathbb{P}^i[X_t = i] \le 1 + 6\frac{D^2}{d^2} \max\left(\frac{k}{n}, \sqrt{k}\right)$:
Let π stand for the invariant probability of the Markov chain and let $d(j)$ be the degree

of vertex j. Then $\pi_i = \frac{d(i)}{\sum_j d(j)} \in \left[\frac{d}{Dn}, \frac{D}{dn}\right]$. For that reason,

$$\sum_j \frac{d}{Dn} \mathbb{P}^j[X_k = i] \leq \mathbb{P}^\pi[X_k = i] = \pi_i \leq \frac{D}{dn}.$$

As a result, $\sum_j \mathbb{P}^j[X_k = i] \leq \frac{D^2}{d^2}$ and $\sum_{t \leq k} \sum_j \mathbb{P}^j[X_k = i] \leq \frac{kD^2}{d^2}$. Set $A = \{j : \sum_{t \leq k} \mathbb{P}^j[X_t = i] > s\}$. Then we have $|A| < \frac{k}{s} \frac{D^2}{d^2}$. For $|A| < n$, by splitting the chain at the hitting time T_{A^c}, we have

$$\sum_{t \leq k} \mathbb{P}^i[X_t = i] \leq s + \mathbb{E}^i[N_i(A^c)] \leq s + \frac{5k}{s} \frac{D^3}{d^3}.$$

Take $s = \lfloor \frac{\sqrt{k} D^2}{d^2} \rfloor + 1$ [3]. Then for $k \leq n^2$, we have $|A| < \frac{kD^2}{sd^2} \leq n$ and

$$\sum_{t \leq k} \mathbb{P}^i[X_t = i] \leq \sqrt{k}(\frac{D^2}{d^2} + 5\frac{D}{d}) + 1 \leq 1 + 6\sqrt{k}\frac{D^2}{d^2}.$$

For $k > n^2$, take $s = [\frac{kD^2}{nd^2}] + 1$, then $|A| < n$ and

$$\sum_{t \leq k} \mathbb{P}^i[X_t = i] \leq \frac{kD^2}{nd^2} + 1 + 5n\frac{D}{d} < 1 + 6\frac{D^2}{d^2}\frac{k}{n}.$$

Finally,

$$\sum_{t \leq k} \mathbb{P}^i[X_t = i] \leq 1 + 6\frac{D^2}{d^2} \max(\frac{k}{n}, \sqrt{k})$$

3) We will show that $S_k = \sum_i \mathbb{P}^i[X_k = i] \leq \frac{14D^2/d^2}{\sqrt{k}}$ for $k > 1$:
Let Q stand for the transition matrix and let $1 = \lambda_1 \geq \cdots \lambda_n \geq -1$ be the eigenvalues of Q. Then, $\sum_i \mathbb{P}^i[X_{2k} = i] + \sum_i \mathbb{P}^i[X_{2k+1} = i] = \operatorname{Tr} Q^{2k} + \operatorname{Tr} Q^{2k+1} = \sum_i \lambda_i^{2k}(1 + \lambda_i)$. As a result, S_k decreases when k increases. Therefore,

$$\begin{aligned}
&\sum_i \mathbb{P}^i[X_{2k} = i] + \mathbb{P}^i[X_{2k+1} = i] \\
&\qquad \leq \frac{1}{k+1} \sum_i \sum_{t \leq 2k+1} \mathbb{P}^i[X_t = i] \\
&\qquad \leq \frac{n}{k+1}\left(1 + 6\frac{D^2}{d^2} \max\left(\frac{2k+1}{n}, \sqrt{2k+1}\right)\right).
\end{aligned}$$

Finally, for $k > 1$, $\sum_i \mathbb{P}^i[X_k = i] \leq \frac{14D^2}{d^2} \max\left(\frac{n}{\sqrt{k}}, 1\right)$.

[3] $\lfloor x \rfloor$ is the largest integer not greater than x.

$\square$

Finally, Theorem 6.1.1 and Corollary 6.1.2 remain valid.

6.2.3 General case

We consider weighted graphs satisfying (H1), (H2) and (H3).

Weighted aperiodic case

Let us first consider weighted aperiodic graphs satisfying (H1), (H2) and (H3). Compared to the uniform 1−weight case, the proof is exactly the same with a few changes in the coefficients. The main idea of the proof of Corollary 6.2.3 remains and the spectral gap of Q_n is still of order $O(n^2)$. Consequently, Corollary 6.2.4 and its proof remain the same.
For Proposition 6.2.9, replace D by D/r and d by d/R. The proof is similar.
For Proposition 6.2.6, there are a few changes in the coefficients:

Proposition 6.2.10. *Consider a sequence of connect n-vertex graphs G_n satisfying (H1), (H2) and (H3).*

a) $\mu_n(p(l) \in [n, n^2]) \leq \dfrac{D^2R^2}{d^2r^2} \dfrac{28n/\sqrt{n-1} - 28}{(1+c_n)^n}$,

b) *For $b > 2$ fixed,* $\mu_n(p(l) \in [n^2, n^b]) \leq \dfrac{14D^2R^2}{d^2r^2} \dfrac{b \ln n}{(1+c_n)^{n^2}}$,

c) *For $b > 2$ fixed,* $\mu_n(p(l) \geq n^b) \sim \displaystyle\sum_{k \geq n^b} \frac{1}{k} \left(\frac{1}{1+c_n}\right)^k$,

d) $\dfrac{1}{n}\mu_n(2 \leq p(l) < n) \in \left[\dfrac{r^2}{2(1+c_n)^2R^2D^2}, 28\dfrac{D^2R^2}{d^2r^2}\right]$.

For Proposition 6.2.7, just replace $\int -\ln(1-x)\nu(dx) \in [\frac{1}{2d^2}, 20]$ by

$$\int -\ln(1-x)\nu(dx) \in \left[\frac{r^2}{2R^2D^2}, \frac{28D^2R^2}{d^2r^2}\right].$$

For Proposition 6.2.8, one should use the upper bound of the expectation of the covering-and-return time for a general reversible Markov process stated in Theorem 1 in Chapter 6 of the book [AF] and recalled in the introduction (inequality (6.2)). By the assumptions (H1) and (H2), we have $\frac{1}{w_{xy}} \leq R$, $|E_n| \leq Dn$ and $\sum\limits_{x,y} w_{xy} \leq Dn/r$. Therefore,

$$\mathbb{E}^v(C^+) \leq \sum_{x,y} w_{xy} \min_{T \text{ is a spanning tree}} \sum_{\text{edge } \{x,y\} \in T} \frac{1}{w_{xy}} \leq \frac{DRn^2}{r}.$$

Thus, the result of Proposition 6.2.8 is still valid. Finally, for Theorem 6.1.1 and Corollary 6.1.2, nothing needs to be changed.

Weighted Periodic case

Under the assumption (H1), (H2) and (H3), we consider periodic graphs. Then the period must be 2 since $\operatorname{Tr} Q_n^2 > 0$. The largest eigenvalue of Q_n is 1 and the smallest one is -1. In this case, one can divide the vertices into two parts as follows: fix a vertex x, set $A_n = \{y \in V_n : \sum_{k\geq 0} \mathbb{P}_n^x[X_{2k} = y] > 0\}$ and set $B_n = V_n - A_n$. This partition $V_n = \{A_n, B_n\}$ does not depend on the choice of x. Moreover, if the initial distribution is supported on A_n (resp. B_n), then the chain $(X_{2m}^{(n)}, m \in \mathbb{N})$ is a reversible aperiodic Markov chain on A_n (resp. B_n) with the transition matrix $Q_n^2|_{A_n}$ (resp. $Q_n^2|_{B_n}$) and the stationary distribution $\frac{1}{\pi(A_n)}\pi|_{A_n}$ (resp. $\frac{1}{\pi(B_n)}\pi|_{B_n}$). A direct consequence is that $Q_n^2|_{A_n\times B_n}$ and $Q_n^2|_{B_n\times A_n}$ are zero matrices. If one puts the eigenvalues of $Q_n^2|_{A_n}$ and $Q_n^2|_{B_n}$ together, one gets exactly the squares of the eigenvalues of Q_n. Since $1 - |\lambda| \geq \dfrac{1-|\lambda|^2}{2}$ for $|\lambda| \leq 1$, one can still get a similar spectrum gap proposition as Corollary 6.2.3 by considering the aperiodic Markov chains on A_n and B_n. (In fact, as aperiodic reversible Markov chains, they are related to weighted undirected aperiodic graphs. Moreover, one can check that the associated graphs satisfy (H1) and (H2).) Then, instead of Corollary 6.2.4, we have

$$\lim_{n\to\infty} \sup_{k\geq n^b \text{ and } k \text{ is even}} |\operatorname{Tr} Q_n^k - 2| = 0 \text{ and } \operatorname{Tr} Q_n^k = 0 \text{ for } k \text{ odd}.$$

For Proposition 6.2.8, one considers only the loops with even length and the proof remains the same. For the statement of Proposition 6.2.7 and Theorem 6.1.1, just replace

$$\int -\ln(1-x)\nu(dx) \in \left[\frac{1}{2d^2}, 20\right] \text{ by } \int -\ln(1-x)\nu(dx) \in \left]\frac{r^2}{2R^2D^2}, \frac{28D^2R^2}{d^2r^2}\right[.$$

The proof is the same except for little changes in the constants. Finally, Corollary 6.1.2 is unchanged.

6.3 A stability result

Proposition 6.3.1. *Assume that Q_n is a $m_n \times m_n$ transition matrix, the following two statements are equivalent:*

a) The empirical distributions ν_n of the eigenvalues of the transition matrices Q_n converge to ν as $n \to \infty$.

b) For all $0 < \rho < 1$, $-\dfrac{1}{m_n} \ln\det(1-\rho Q_n)$ *converges to* $\int -\ln(1-\rho x)\nu(dx)$ *as* $n \to \infty$.

Proof.

- a)$\Longrightarrow$b):

$$\lim_{n\to\infty} \frac{-\ln\det(1-\rho Q_n)}{m_n} = \lim_{n\to\infty} -\frac{1}{m_n} \operatorname{Tr}\ln(1-\rho Q_n) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{\operatorname{Tr}(Q_n^k)\rho^k}{m_n k}$$
$$\overset{\text{dominated convergence}}{=} \sum_{k=1}^{\infty} \lim_{n\to\infty} \frac{\operatorname{Tr}(Q_n^k)\rho^k}{m_n k} = \sum_{k=1}^{\infty} \int \frac{x^k\rho^k}{k}\nu(dx) = \int -\ln(1-\rho x)\nu(dx).$$

- b)$\Longrightarrow$a): The distributions of the eigenvalues of the transition matrices Q_n form a tight sequence of probability measures on $[-1,1]$. In order to show its convergence, it is enough to show that the limits are the same for all convergent subsequences. Finally, for two probabilities ν and $\tilde{\nu}$ on [-1,1], by comparing the derivatives of the two parts with respect to ρ, it can be showed that

  $$\text{“}\int -\ln(1-\rho x)\nu(dx) = \int -\ln(1-\rho x)\tilde{\nu}(dx) \text{ for all } \rho \in [0,1[.\text{”}$$

  implies that

  $$\text{“}\int x^k\nu(dx) = \int x^k\tilde{\nu}(dx) \text{ for all } k \geq 1.\text{”}.$$

  Therefore, $\nu = \tilde{\nu}$.

$\square$

Recall (Cauchy's interlacing theorem)**.** *Let* S *be a* $n \times n$ *Hilbert matrix, i.e.* $\bar{S}^t = S$. *Let* $\pi : \{1, \ldots, n\} \to \mathbb{R}^+$ *strictly positive and* M_π *be the diagonal matrix such that* $(M_\pi)_i^i = \pi(i)$ *for* $i = 1, \ldots, n$. *Let* $A = M_\pi S$. *For each principal minor of* A, *its eigenvalues are real. Let* $F \subset \{1, \ldots, n\}$, $|F| = m$ *and* $B = A|_{F\times F}$. *If the eigenvalues of* A *are* $\alpha_1 \leq \cdots \leq \alpha_n$, *and those of* B *are* $\beta_1 \leq \cdots \leq \beta_m$, *then for all* $j = 1, \ldots, m$,

$$\alpha_j \leq \beta_j \leq \alpha_{n-m+j}.$$

One can find the interlacing theorem as Theorem 4.3.15 in [HJ90].[4]

We are ready to state the following stability result.

[4]The theorem is stated for $\pi \equiv 1$. By using Theorem 4.3.15 in [HJ90] for $M_{\pi^{-1/2}} A M_{\pi^{1/2}}$, we get this trivial generalization.

Proposition 6.3.2. *Suppose $(G_n = (V_n, E_n, w_n), n \geq 1)$ is a sequence of undirected weighted graphs. We assume G_n has m_n vertices V_n, E_n is the set of edges and w_n the weights of the edges (if $e = (e-, e+)$ is not an edge in the graph, we put $w_{e-,e+} = 0$). Let ω_n be a measure on V_n defined by $(\omega_n)_x = \sum_y (w_n)_{xy}$. Define $(Q_n)^x_y = \begin{cases} (w_n)_{xy}/(\omega_n)_x & \text{if } \{x,y\} \in E_n \\ 0 & \text{otherwise.} \end{cases}$ Suppose $(G'_n = (V'_n, E'_n, w'_n), n \geq 1)$ is a sequence of undirected weighted sub-graphs such that $V'_n \subset V_n, E'_n \subset E_n, W'_n = W_n|_{E'_n}$ for all n. Let $a_n = |E_n| - |E'_n|$. Write the eigenvalues of Q_n (resp. Q'_n) in non-decreasing order $\lambda_{n,1} \leq \cdots \leq \lambda_{n,m_n}$ (resp. $\lambda'_{n,1} \leq \cdots \leq \lambda'_{n,m'_n}$). Define*

$$\nu_n = \frac{1}{m_n}(\delta_{\lambda_{n,1}} + \cdots + \delta_{\lambda_{n,m_n}}) \text{ and } \nu'_n = \frac{1}{m'_n}(\delta_{\lambda'_{n,1}} + \cdots + \delta_{\lambda'_{n,m'_n}}).$$

Suppose: $a_n = o(m_n)$. Then

$$\lim_{n\to\infty} \nu_n = \nu \Leftrightarrow \lim_{n\to\infty} \nu'_n = \nu.$$

We will explain the meaning of "stability" in the following remark.

Remark 20. In the above proposition, we consider the case that $G'_n = (V'_n, E'_n, w'_n)$ is a sub-graph of $G_n = (V_n, E_n, w_n)$ for all $n \geq 1$. If G'_n is not a sub-graph of G_n for some $n \geq 1$, we consider the biggest common sub-graph $G''_n = (V''_n, E''_n, w''_n)$ of G_n and G'_n for all $n \geq 1$. By applying Proposition 6.3.2 to the pair of sequences $((G_n, n \geq 1), (G''_n, n \geq 1))$ and to the pair of sequences $((G'_n, n \geq 1), (G''_n, n \geq 1))$, we see that a similar result as Proposition 6.3.2 holds for the pair of sequences $((G_n, n \geq 1), (G'_n, n \geq 1))$ provided that

$$|E_n| - |E''_n| + |E'_n| - |E''_n| = o(|V_n|). \tag{$*$}$$

Then suppose we have two sequences of graphs $(G_n, n \geq 1)$ and $(G'_n, n \geq 1)$ such that the condition $(*)$ holds for both of them. If the sequence of graphs $(G_n, n \geq 1)$ satisfies (H3), then $(G'_n, n \geq 1)$ also satisfies (H3). Moreover, the corresponding sequences of empirical distributions of eigenvalues have the same limit distribution ν. The word "stability" means the above result.

Proof of Proposition 6.3.2. Since $a_n = o(m_n)$, $\lim_{n\to\infty} \frac{m_n}{m'_n} = 1$. By Proposition 6.3.1, it is enough to prove that for all $\rho > 0$,

$$\lim_{n\to\infty} \frac{1}{m_n}(\ln\det(1 - \rho Q_n) - \ln\det(1 - \rho Q'_n)) = 0.$$

Since $a_n = o(m_n)$, for n large enough, there exists $A_n \subset V_n$ such that

$$Q_n|_{A_n} = Q'_n|_{A_n} \text{ and } |V_n| - |A_n| \leq a_n.$$

It is enough to show that

$$\begin{cases} \lim\limits_{n\to\infty} \frac{1}{m_n}(\ln\det(1-\rho Q_n) - \ln\det(1-\rho Q_n|_{A_n})) = 0 \\ \lim\limits_{n\to\infty} \frac{1}{m_n}(\ln\det(1-\rho Q'_n) - \ln\det(1-\rho Q'_n|_{A_n})) = 0. \end{cases}$$

In the following, we will give the proof for the first limit as the second can be proved in the same way. Let $\beta_1 \leq \cdots \leq \beta_{|A_n|}$ be the eigenvalues of $Q_n|_{A_n}$. Then,

$$\begin{aligned} &\ln\det(1-\rho Q_n) - \ln\det(1-\rho Q_n|_{A_n}) \\ &\qquad = \ln(1-\rho\lambda_{n,1}) + \cdots + \ln(1-\rho\lambda_{n,m_n}) - \ln(1-\rho\beta_1) - \cdots - \ln(1-\rho\beta_{|A_n|}) \end{aligned}$$

By Cauchy's interlacing theorem, we have

- $\sum\limits_{i=1}^{|A_n|} \ln(1-\rho\lambda_{n,i}) - \ln(1-\rho\beta_i) \geq 0;$
- $\sum\limits_{i=1}^{|A_n|} \ln(1-\rho\lambda_{n,i+|V_n|-|A_n|}) - \ln(1-\rho\beta_i) \leq 0.$

Consequently,

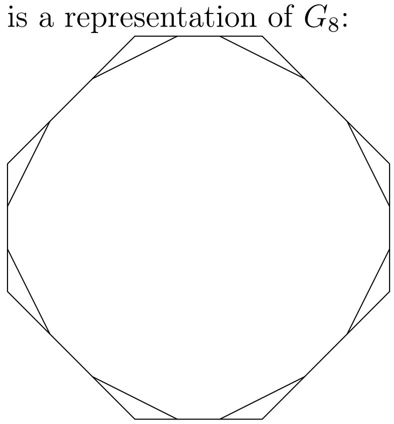
$$\ln\det(1-\rho Q_n) - \ln\det(1-\rho Q_n|_{A_n}) \in [\sum_{i=1+|A_n|}^{|V_n|} \ln(1-\rho\lambda_{n,i}), \sum_{i=1}^{|V_n|-|A_n|} \ln(1-\rho\lambda_{n,i})].$$

Since $-1 \leq \lambda_{n,1} \leq \cdots \leq \lambda_{n,m_n} \leq 1$,

$$\begin{aligned} &\ln\det(1-\rho Q_n) - \ln\det(1-\rho Q_n|_{A_n}) \\ &\qquad \in (|V_n| - |A_n|)[\ln(1-\rho), \ln(1+\rho)] \subset [\ln(1-\rho)a_n, \ln(1+\rho)a_n] \end{aligned}$$

Since $a_n = o(m_n)$, $\lim\limits_{n\to\infty} \dfrac{1}{m_n}(\ln\det(1-\rho Q_n) - \ln\det(1-\rho Q_n|_{A_n})) = 0.$ $\square$

Remark 21. If $\liminf\limits_{n\to\infty} \dfrac{a_n}{m_n} > 0$ and $m_n \sim m'_n$, $(\nu_n)_n$ and $(\nu'_n)_n$ can converge towards two different measures. For example, let $(G_n = (V_n, E_n, w_n), n \geq 2)$ be a sequence of graphs with equal edge weight 1. Here, $V_n = \{1, \ldots, 3n\}$, $E_{n,1} = \{\{1,2\},\{2,3\}\ldots,\{3n-1,3n\},\{3n,1\}\}$ and $E_{n,2} = \{\{1,3\},\{4,6\},\ldots,\{3n-2,3n\}\}$. Take $E_n = E_{n,1} \cup E_{n,2}$. The following picture is a representation of G_8:



Let $(G'_n = (V'_n, E'_n, w'_n), n \geq 2)$ be another sequence of graphs with weights 1 on each edge such that $V'_n = V_n$ and $E'_n = E_{n,1}$, i.e. G'_n is the discrete circle with $3n$ vertices. By Proposition 6.2.2, $\lambda_{n,1} \geq -\frac{11}{12}$. To prove this lower bound, choose $\sigma_{3i-2} = \sigma_{3i-1} = \sigma_{3i}$ to be the cycle $3i-2 \to 3i-1 \to 3i \to 3i-2$ for $i = 1, \ldots, n$. Then, $(\pi_n)_{3i-1} = \frac{1}{4n}, (\pi_n)_{3i-2} = (\pi_n)_{3i} = \frac{3}{8n}$ for $i = 1, \ldots, n$. Moreover, $|\sigma_x|_{w_n} = 24n$ and $\tau = 24$. Therefore, $\lambda_{n,1} \geq -\frac{11}{12}$. It implies that $\nu = \lim_{n\to\infty} \nu_n$ satisfies $\nu[-1, -\frac{11}{12}[= 0$. While for $(G'_n, n \geq 2)$, $\lim_{n\to\infty} \nu'_n$ exists and $\nu'_n(dy) = 1_{\{y \in]-1,1[\}} \dfrac{1}{\pi\sqrt{1-y^2}}\, dy$.

6.4 Example: discrete torus

Let V_n be the discrete torus $\mathbb{Z}^d/n\mathbb{Z}^d$. There is a map from $\mathbb{Z}^d$ to V_n which maps the vector v to $[v] \in V_n$(the equivalence class of v). The edge set E_n is defined by

$$\{\{[u],[v]\} : u, v \in \mathbb{Z}^d \text{ and the distance between } u \text{ and } v \text{ is } 1\}.$$

Finally, give each edge the same weight 1. Let $Q_{d,n}$ be the transition matrix. We will find the limit distribution of the eigenvalues as follows:
Let $(P_{d,n,t}, t \geq 0)$ stand for the semi-group of a simple random walk on $\mathbb{Z}^d/n\mathbb{Z}^d$ with jumping rate d. Then $P_{d,n,t} = e^{-dt(I-Q_{d,n})}$. Since it can be viewed as d independent simple random walks on $\mathbb{Z}^d/n\mathbb{Z}^d$ with jumping rate 1, $P_{d,n,t} = P_{1,n,t}^{\bigotimes d}$. As $-\dfrac{dP_{d,n,t}}{dt} = d(I - Q_{d,n})$,

$$Q_{d,n} = \frac{1}{d}(Q_{1,n} \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes Q_{1,n}).$$

For $d = 1$, the eigenvalues of $Q_{d,n}$ are $\cos\left(\dfrac{2\pi}{n}p_1\right)$ for $p_1 = 0, \ldots, n-1$. Therefore, in general, the eigenvalues of $Q_{d,n}$ are $\dfrac{1}{d}\left(\cos\left(\dfrac{2\pi}{n}p_1\right) + \cdots + \cos\left(\dfrac{2\pi}{n}p_d\right)\right)$ for $p_1, \ldots, p_d \in \{0, \ldots, n-1\}$. In fact, the eigenvectors are $\{f_{p_1,\ldots,p_d} : p_1, \ldots, p_d = 0, \ldots, n-1\}$ where $f_{p_1,\ldots,p_d} : \{0, \ldots, n-1\}^d \to \mathbb{C}$ such that $f_{p_1,\ldots,p_d}(x_1, \ldots, x_d) = \exp\left(\frac{2i\pi}{n}(p_1x_1 + \cdots + p_dx_d)\right)$. Rewrite the eigenvalues in non-increasing order $\lambda_1, \ldots, \lambda_{n^d}$. Define $\nu_n^d = \dfrac{1}{n^d}\sum \delta_{\lambda_i}$ and $\tilde{\nu}_n^d = \dfrac{1}{n^d}\sum \delta_{d\lambda_i}$. Then $\tilde{\nu}_n^d = (\nu_n^1)^{*d}$ ($*$ stands for the convolution). For $f \in C([-1,1])$,

$$\lim_{n\to\infty} \int f(x)\nu_n^1(dx) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f\left(\cos\left(\frac{2\pi}{n}i\right)\right)$$

$$= \int_0^1 f(\cos(2\pi x))\, dx = \int_{]-1,1[} f(y)\frac{1}{\pi\sqrt{1-y^2}}\, dy.$$

Therefore, as $n \to \infty$, v_n^1 converges to $1_{\{y \in]-1,1[\}} \dfrac{1}{\pi\sqrt{1-y^2}}\, dy$. Consequently,

$$\nu = \lim_{n\to\infty} \nu_n^d = m^{*d} \text{ where } m(dy) = 1_{\{y\in[-1/d,1/d]\}} \frac{d}{\pi\sqrt{1-d^2y^2}}\, dy.$$

The same argument as in the case $d = 1$ shows that:

$$\int -\ln(1-x)\,\nu(dx) = \int\limits_{[0,1]^d} -\ln\left(1 - \frac{\cos(2\pi x_1) + \cdots + \cos(2\pi x_d)}{d}\right) dx_1 \cdots dx_d.$$

By Theorem 6.1.1, we obtain the following description of the proportion of loops covering $\mathbb{Z}^d/n\mathbb{Z}^d$ as n tends to $+\infty$:

Proposition 6.4.1.

a) If $\lim\limits_{n\to\infty} \dfrac{-\ln c_n}{n^d} \leq 0$, *then* $\lim\limits_{n\to\infty} \mathfrak{P}_n(\mathcal{C}) = 0$.

b) If $\lim\limits_{n\to\infty} \dfrac{-\ln c_n}{n^d} = \infty$, *then* $\lim\limits_{n\to\infty} \mathfrak{P}_n(\mathcal{C}) = 1$.

c) If $\lim\limits_{n\to\infty} \dfrac{-\ln c_n}{n^d} = a \in]0,\infty[$, *then*

$$\lim_{n\to\infty} \mathfrak{P}_n(\mathcal{C}) = \frac{a}{a + \int\limits_{[0,1]^d} -\ln(1 - \frac{\cos(2\pi x_1)+\cdots+\cos(2\pi x_d)}{d})\, dx_1 \cdots dx_d}$$

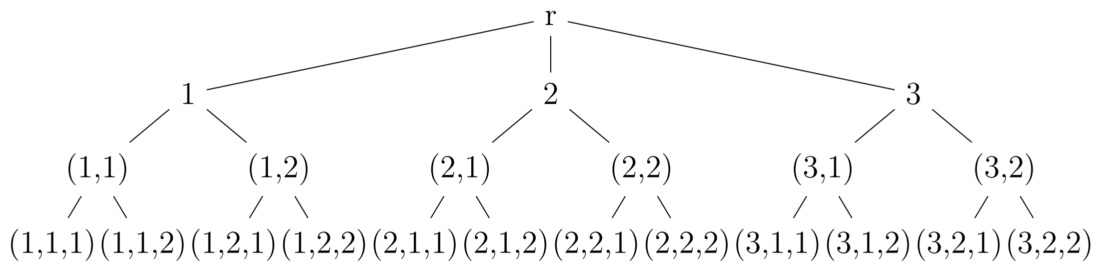
For $d = 1$, *the above limit equals* $\dfrac{a}{a + \ln 2}$.

6.5 Example: the balls in a regular tree

Let T be an infinite regular tree with degree d. For any pair of vertices x, y in the tree, there is a unique path $\gamma_{x,y}$ joining x and y. The graph distance between x and y is the number of edges in that path. Fix a vertex r, let G_n be the balls with radius n centered at r. We give uniform unit weight on each edge. To be more precise, let us fix some notations in the following definition.

Definition 6.5.1. Define $G_n = (V_n, E_n, w_n)$ with depth n by recurrence. For all n, w_n gives equal weight 1 for every edge in E_n. Define $V_0 = \{r\}$, $E_0 = \phi$. Define $V_1 = \{r, 1, \ldots, d\}$, $E_1 = \{\{r,1\}, \ldots, \{r,d\}\}$. Once G_n is well-defined for $n \leq k$, define G_{k+1} as follows: $V_{k+1} = V_k \bigcup((V_k \backslash V_{k-1}) \times \{1, \ldots, d-1\})$ and $E_{k+1} = E_k \bigcup\{\{v,(v,j)\} : v \in V_k \backslash V_{k-1}, j = 1, \ldots, d-1\}$.

The following picture is for $d = 3, n = 3$.



Let $m_n = |V_n|$, then

$$m_0 = 1 \text{ and } m_n = 1 + d(d-1)^0 + \cdots + d(d-1)^{n-1} = \begin{cases} \frac{d(d-1)^n - 2}{d-2} & d > 2 \\ 1 + 2n & d = 2. \end{cases}$$

Let $Q_{d,n}$ be the transition matrix for the graph G_n. Let $\nu_{d,n}$ be the distribution of the eigenvalues of $Q_{d,n}$. The following two propositions describe the limit distribution of the eigenvalues of $Q_{d,n}$.

Proposition 6.5.1. *For $d \geq 3$, the distribution of the eigenvalues of $Q_{d,n}$ converges to a purely atomic distribution ν supported on $[-2\frac{\sqrt{d-1}}{d}, 2\frac{\sqrt{d-1}}{d}]$ as $n \to \infty$. To be more precise,*

- *Let $\theta(M, 0), \ldots, \theta(M, M)$ be the roots of the equation $(d-1)\sin((M+2)\theta) = \sin(M\theta)$ in $]0, \pi[$ and $\lambda(M, i) = 2\frac{\sqrt{d-1}}{d}\cos(\theta(M, i))$ for $i = 0, \ldots, M$. Then,*

$$\nu = \sum_{M=0}^{\infty} \frac{(d-2)^2}{(d-1)^{2+M}} \sum_{i=0}^{M} \delta_{\lambda(M,i)}.$$

- $\displaystyle\int -\ln(1-x)\nu(dx) = \frac{1}{d-1}\ln\left(\frac{d}{d-1}\right).$

Proof. To prove the convergence, it is enough to show that the moments of the probability distribution $(\nu_{d,n})_n$ converges as $n \to \infty$, i.e. $\frac{1}{|V_n|}\operatorname{Tr}(Q_{d,n})^k$ converges as $n \to \infty$ for all $k \geq 0$. Denote by W_i the collection of vertices at distance i away from the root r. The total number of vertices in W_i is $d(d-1)^{i-1}$ for $i \geq 1$ and $|W_0| = 1$. Suppose $x \in W_{n-l}$ with a fixed $l \in \mathbb{N}$, $((Q_{d,n})^k)_x^x$ does not depend on x or n as long as $n \geq l + k/2$. It implies the uniform convergence of $((Q_{d,n})^k)_x^x$ as $n \to \infty$ for $x \in W_{n-l}$. Denote the limit by $P(k, l)$. As $|W_{n-l}| = d(d-1)^{n-l-1}$ and $|V_n| = \frac{d(d-1)^n - 2}{d-2}$,

$$\lim_{n\to\infty} \frac{|W_{n-l}|}{|V_n|} = \frac{d-2}{(d-1)^{l+1}}.$$

As a consequence,

$$\lim_{n\to\infty} \frac{1}{|V_n|}\operatorname{Tr}(Q_{d,n})^k = \sum_{l\in\mathbb{N}} \frac{d-2}{(d-1)^{l+1}} P(k, l).$$

Consequently, for $d \geq 3$, the distribution of the eigenvalues of $Q_{d,n}$ converges to ν on $[-1,1]$ as $n \to \infty$. Then, our next task is to identify those $P(k,l)$. We know that $P(k,l) = ((Q_{d,n})^k)^x_x$ for $n \geq l + k/2$ and $x \in W_{n-l}$. Let $(X^{(n)}_m)_m$ be the associated Markov chain on G_n. For $n \geq l + k/2$ and $x \in W_{n-l}$, $P(k,l) = \mathbb{P}^x[X^{(n)}_k = x]$. Define $U_k = \min_{q=0,\dots,k} d(X^{(n)}_q, r)$[5]. By using the symmetry, a direct calculation gives that

$$\mathbb{P}^x[X^{(n)}_k \in W_{n-l} | U_k = u] = (d-1)^{n-l-u} \mathbb{P}^x[X^{(n)}_k = x | U_k = u].$$

Therefore,

$$\mathbb{P}^x[X^{(n)}_k = x] = \mathbb{E}^x\left[1_{\{X^{(n)}_k \in W_{n-l}\}} \frac{1}{(d-1)^{n-l-U_k}}\right].$$

Define $Y^{(n)}_m = n - d(X^{(n)}_m, r)$ for $m \in \mathbb{N}$. Then, $Y^{(n)}$ is a Markov chain on $\{0,1,\dots,n\}$ with transition matrix $((T^{(n)})^i_j, i,j = 0,\dots,n)$:

$$T^{(n)} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \frac{1}{d} & 0 & \frac{d-1}{d} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \frac{1}{d} & \ddots & \frac{d-1}{d} \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}.$$

Define $M_k = \max_{i=1,\dots,k} Y^{(n)}_i$. We can express $P(k,l)$ in terms of $Y^{(n)}$ for $n \geq l + k/2$:

$$P(k,l) = \mathbb{P}^x[X^{(n)}_k = x] = \mathbb{P}^l\left[Y^{(n)}_k = l, \frac{1}{(d-1)^{M_k - l}}\right].$$

Consider the Markov chain Y on $\mathbb{N}$ with the transition matrix $(T^i_j, i,j \in \mathbb{N})$:

$$T = \begin{bmatrix} 0 & 1 & 0 & \dots \\ \frac{1}{d} & 0 & \frac{d-1}{d} & \ddots \\ 0 & \frac{1}{d} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Conditionally on $Y^{(n)}_k = l$ for $n > l + k/2$, we have $M_k < n$ and $(Y^{(n)}_j, j = 1,\dots,k)$ has the same law as $(Y_j, j = 1,\dots,k)$. Therefore,

$$P(k,l) = \mathbb{P}^l[Y_k = l, \frac{1}{(d-1)^{M_k - l}}]$$

[5] Recall that the graph distance $d(y,r)$ is the length of the shortest path joining y to the root r. In other words, if $y \in W_m$, then $d(y,r) = m$. Here, U_k is the highest level reached by the Markov chain up to time k.

$$
\begin{aligned}
&= \sum_{M\geq l} \frac{1}{(d-1)^{M-l}} \mathbb{P}^l[Y_k = l, M_k = M] \\
&= \sum_{M\geq l} \frac{1}{(d-1)^{M-l}} (\mathbb{P}^l[Y_k = l, M_k \leq M] - \mathbb{P}^l[Y_k = l, M_k \leq M-1]) \\
&= \sum_{M\geq l} \mathbb{P}^l[Y_k = l, M_k \leq M] \left(\frac{1}{(d-1)^{M-l}} - \frac{1}{(d-1)^{M+1-l}} \right) \\
&= \frac{d-2}{d-1} \sum_{M\geq l} \mathbb{P}^l[Y_k = l, Y \text{ is killed at } [M+1, n]] \left(\frac{1}{(d-1)^{M-l}} \right) \\
&= \frac{d-2}{d-1} \sum_{M\geq l} ((T|_{\{0,\dots,M\}\times\{0,\dots,M\}})^k)_l^l \frac{1}{(d-1)^{M-l}} \\
&= \frac{d-2}{d-1} \sum_{M\geq l} ((S|_{\{0,\dots,M\}\times\{0,\dots,M\}})^k)_l^l \frac{1}{(d-1)^{M-l}}
\end{aligned}
$$

where

$$
S = \begin{bmatrix} 0 & \frac{1}{\sqrt{d}} & 0 & \cdots \\ \frac{1}{\sqrt{d}} & 0 & \frac{\sqrt{d-1}}{d} & \ddots \\ 0 & \frac{\sqrt{d-1}}{d} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.
$$

As a symmetric transition matrix, $S|_{\{0,\dots,M\}\times\{0,\dots,M\}} = \Omega_M^t \Lambda_M \Omega_M$ where $\Omega^t\Omega = Id$ and

$$
\Lambda_M = \begin{bmatrix} \lambda(M,0) & & \\ & \ddots & \\ & & \lambda(M,M) \end{bmatrix}.
$$

Then, $(S|_{\{0,\dots,M\}\times\{0,\dots,M\}})^k = \Omega_M^t \begin{bmatrix} (\lambda(M,0))^k & & \\ & \ddots & \\ & & (\lambda(M,M))^k \end{bmatrix} \Omega_M$. Define the atomic distribution: $\nu_{M,l} = \sum_{i=0}^{M} ((\Omega_M)_l^i)^2 \delta_{\lambda(M,i)}$. Then, $((S|_{\{0,\dots,M\}\times\{0,\dots,M\}})^k)_l^l = \int x^k \nu_{M,l}(dx)$ and

$$
\begin{aligned}
\nu &= \sum_{l\in\mathbb{N}} \sum_{M\geq l} \frac{d-2}{(d-1)^{l+1}} \frac{d-2}{(d-1)^{M+1-l}} \nu_{M,l} \\
&= \sum_{M\in\mathbb{N}} \frac{(d-2)^2}{(d-1)^{2+M}} \sum_{l\leq M} \nu_{M,l} \\
&= \sum_{M\in\mathbb{N}} \frac{(d-2)^2}{(d-1)^{2+M}} \sum_{i=0}^{M} \delta_{\lambda(M,i)}.
\end{aligned}
$$

Next, we will give a more precise description of those $\lambda(M,i)$ for $0 \leq i \leq M$. Define $B_M(\lambda) = \det(\lambda \cdot Id - S|_{\{0,\dots,M\}\times\{0,\dots,M\}})$. Then, $B_0(\lambda) = \lambda$, $B_1(\lambda) = \lambda^2 - \frac{1}{d}$ and

$B_{M+2}(\lambda) = \lambda B_{M+1}(\lambda) - \dfrac{d-1}{d^2} B_M(\lambda)$. Therefore, $B_M(\lambda) = C_1(\lambda)(x_1(\lambda))^M + C_2(\lambda)(x_2(\lambda))^M$ where

$$x_1(\lambda) = \begin{cases} \frac{1}{2}\left(\lambda + \sqrt{\lambda^2 - \frac{4(d-1)}{d^2}}\right) & \text{if } \lambda^2 \geq \frac{4(d-1)}{d^2}, \\ \frac{1}{2}\left(\lambda + i\cdot\sqrt{-\lambda^2 + \frac{4(d-1)}{d^2}}\right) & \text{if } \lambda^2 < \frac{4(d-1)}{d^2}; \end{cases}$$

$$x_2(\lambda) = \begin{cases} \frac{1}{2}\left(\lambda - \sqrt{\lambda^2 - \frac{4(d-1)}{d^2}}\right) & \text{if } \lambda^2 \geq \frac{4(d-1)}{d^2}, \\ \frac{1}{2}\left(\lambda - i\cdot\sqrt{-\lambda^2 + \frac{4(d-1)}{d^2}}\right) & \text{if } \lambda^2 < \frac{4(d-1)}{d^2}; \end{cases}$$

$$C_1(\lambda) = \begin{cases} \frac{\lambda}{2} + \frac{\frac{\lambda^2}{2} - \frac{1}{d}}{\sqrt{\lambda^2 - \frac{4(d-1)}{d^2}}} & \text{if } \lambda^2 > \frac{4(d-1)}{d^2}, \\ \frac{\lambda}{2} & \text{if } \lambda^2 = \frac{4(d-1)}{d}, \\ \frac{\lambda}{2} - i\cdot\frac{\frac{\lambda^2}{2} - \frac{1}{d}}{\sqrt{-\lambda^2 + \frac{4(d-1)}{d^2}}} & \text{if } \lambda^2 < \frac{4(d-1)}{d^2}; \end{cases}$$

$$C_2(\lambda) = \begin{cases} \frac{\lambda}{2} - \frac{\frac{\lambda^2}{2} - \frac{1}{d}}{\sqrt{\lambda^2 - \frac{4(d-1)}{d^2}}} & \text{if } \lambda^2 > \frac{4(d-1)}{d^2}, \\ \frac{\lambda}{2} & \text{if } \lambda^2 = \frac{4(d-1)}{d}, \\ \frac{\lambda}{2} + i\cdot\frac{\frac{\lambda^2}{2} - \frac{1}{d}}{\sqrt{-\lambda^2 + \frac{4(d-1)}{d^2}}} & \text{if } \lambda^2 < \frac{4(d-1)}{d^2}. \end{cases}$$

For $\lambda^2 \geq \frac{4(d-1)}{d^2}$, $x_1, x_2, C_1, C_2 > 0$ and $B_M(\lambda) > 0$. As a result, the eigenvalues of $S|_{\{0,\dots,M\}\times\{0,\dots,M\}}$ are contained in $]-\frac{2}{d}\sqrt{d-1}, \frac{2}{d}\sqrt{d-1}[$. For $\lambda^2 < \frac{4(d-1)}{d^2}$, $|x_1| = |x_2| = \frac{\sqrt{d-1}}{d}$. In polar coordinates $x_1(\lambda) = \frac{\sqrt{d-1}}{d}e^{i\theta(\lambda)}$ and $x_2(\lambda) = \frac{\sqrt{d-1}}{d}e^{-i\theta(\lambda)}$ where $\theta(\lambda) \in]0,\pi[$. We have $\lambda = x_1 + x_2 = 2\frac{\sqrt{d-1}}{d}\cos(\theta)$. In particular, there is a bijection between λ and θ if $\theta \in]0,\pi[$ or $\lambda^2 < \frac{4(d-1)}{d^2}$ equivalently. As a function of θ,

$$B_M = \frac{(d-1)\sin((M+2)\theta) - \sin(M\theta)}{d\sqrt{d-1}\sin(\theta)}\left(\frac{\sqrt{d-1}}{d}\right)^M, \theta \in]0,\pi[.$$

Hence, $B_M = 0$ iff. $F_M(\theta) \stackrel{def}{=} (d-1)\sin((M+2)\theta) - \sin(M\theta) = 0$. Note that B_M has at most $M+1$ roots. Consequently, F_M has at most $M+1$ roots in $]0,\pi[$. Since $F_M(\frac{1}{2M+4}) > 0, F_M(\frac{3}{2M+4}) < 0, F_M(\frac{5}{2M+4}) > 0, F_M(\frac{7}{2M+4}) < 0, \dots$, the $M+1$ zeros of $B_M(\theta)$ are located in the following $M+1$ open intervals:

$$]\frac{1}{2M+4}, \frac{3}{2M+4}[,]\frac{3}{2M+4}, \frac{5}{2M+4}[, \dots,]\frac{2M+1}{2M+4}, \frac{2M+3}{2M+4}[.$$

As a consequence, we get that $\text{supp}(\nu) = [-\frac{2\sqrt{d-1}}{d}, \frac{2\sqrt{d-1}}{d}]$. Finally, we calculate

$$\int -\ln(1-x)\nu(dx):$$

$$
\begin{aligned}
\int -\ln(1-x)\nu(x) &= -\sum_{M\in\mathbb{N}} \frac{(d-2)^2}{(d-1)^{2+M}} \sum_{i=0}^{M} \ln(1-\lambda(M,i)) \\
&= -\sum_{M\in\mathbb{N}} \frac{(d-2)^2}{(d-1)^{2+M}} \ln\det(I - S|_{\{0,\dots,M\}\times\{0,\dots,M\}}) \\
&= -\sum_{M\in\mathbb{N}} \frac{(d-2)^2}{(d-1)^{2+M}} \ln B_M(1) \\
&= -\sum_{M\in\mathbb{N}} \frac{(d-2)^2}{(d-1)^{2+M}} \ln\left(\left(\frac{d-1}{d}\right)^M\right) \\
&= -\sum_{M\in\mathbb{N}} \frac{M(d-2)^2}{(d-1)^{2+M}} \ln\left(\frac{d-1}{d}\right) \\
&= \frac{1}{d-1}\ln\left(\frac{d}{d-1}\right).
\end{aligned}
$$

□

Proposition 6.5.2. *For $d=2$, the distribution of the eigenvalues of $Q_{2,n}$ converges to the semi-circle law ν on $[-1,1]$ as $n\to\infty$. As a result, $\int -\ln(1-x)\nu(dx) = \ln(2)$.*

Proof. In the discrete circle, the empirical distribution of the eigenvalues converges to the circular law ν on $[-1,1]$ defined by $\nu(dx) = 1_{\{x\in[-1,1]\}}\frac{1}{\pi\sqrt{1-x^2}}\,dx$. As a consequence of Proposition 6.3.2, the distribution of the eigenvalues of $Q_{2,n}$ converges to the circular law as the difference between these two graphs is small. □

By Theorem 6.1.1, we obtain the following result for the proportion of loops covering the ball with radius n of the regular tree T as n tends to ∞:

Proposition 6.5.3.

- *For $d\geq 3$:*

  a) If $\lim_{n\to\infty} -\frac{1}{(d-1)^n}\ln c_n \leq 0$, $\lim_{n\to\infty}\mathfrak{P}_n(\mathcal{C}) = 0$.

  b) If $\lim_{n\to\infty} -\frac{1}{(d-1)^n}\ln c_n = \infty$, $\lim_{n\to\infty}\mathfrak{P}_n(\mathcal{C}) = 1$.

  c) If $\lim_{n\to\infty} -\frac{1}{(d-1)^n}\ln c_n = a \in]0,\infty[$,

$$\lim_{n\to\infty}\mathfrak{P}_n(\mathcal{C}) = \frac{a^{\frac{d-2}{d}}}{\frac{1}{d-1}\ln\left(\frac{d}{d-1}\right) + a^{\frac{d-2}{d}}}.$$

- *For $d=2$:*

a) If $\lim_{n\to\infty} \frac{-\ln c_n}{2n} \leq 0$, $\lim_{n\to\infty} \mathfrak{P}_n(\mathcal{C}) = 0$.

b) If $\lim_{n\to\infty} \frac{-\ln c_n}{2n} = \infty$, $\lim_{n\to\infty} \mathfrak{P}_n(\mathcal{C}) = 1$.

c) If $\lim_{n\to\infty} \frac{-\ln c_n}{2n} = a \in]0,\infty[$,

$$\lim_{n\to\infty} \mathfrak{P}_n(\mathcal{C}) = \frac{a}{\ln 2 + a}.$$

6.6 The case of the complete graph

Let $(G_n = (V_n, E_n, w_n))$ be the complete graph of n vertices and weights 1 on each edge. Recall that we study the asymptotic behavior of $\mathfrak{P}_n(\mathcal{C})$, the proportion of loops covering G_n. Since there is no universal degree bound, the result could not be derived from Theorem 6.1.1. In fact, the distribution of the length of the loops is quite different: the loops of length between m_n and m_n^4 is no longer negligible.

Instead, we use a different method to analyse this problem: we compare the covering time of the coupon collector problem and an independent modified geometric variable. The later one is very close to the distribution of the length of the loop.

Let us first explain the reason for which we can reduce our problem to the classical coupon collector problem as follows: Let $(X_k^{(n)}, k \geq 1)$ be the simple random walk on the complete graph with uniform initial distribution. We denote by $\mathbb{P}_k^{(n)}$ the law of $(X_j^{(n)}, j = 1, \ldots, k)$. Then, for our covering problem, these two models are very close to each other in the following sense:

Lemma 6.6.1. *Let* $\mathbb{P}_k^{(n)}(\mathcal{C})$ *be the probability that* $(X_1^{(n)}, \ldots, X_k^{(n)})$ *covers the graph* G_n*. Then,* $\mathbb{P}_k^{(n)}(\mathcal{C})$ *and* $\mathfrak{P}_n(\mathcal{C}|p(l) = k)$ *are equivalent. More precisely,*

$$\mathbb{P}_k^{(n)}(\mathcal{C}) \geq \frac{n-2}{n} \mathfrak{P}_n(\mathcal{C}|p(l) = k).$$
$$\mathbb{P}_k^{(n)}(\mathcal{C}) \leq \frac{n-1}{n-2} \mathfrak{P}_n(\mathcal{C}|p(l) = k)$$

Proof. Define $S_1^{(k)} = \{(x_1, \ldots, x_k) \in \{1, \ldots, n\}^k : x_2 \neq x_1, x_3 \neq x_2, \ldots, x_k \neq x_{k-1}\}$ and $S_2^{(k)} = \{(x_1, \ldots, x_k) \in \{1, \ldots, n\}^k : x_1 \neq x_k, x_2 \neq x_1, x_3 \neq x_2, \ldots, x_k \neq x_{k-1}\}$. Then, $|S_1^{(k)}| = n(n-1)^{k-1}$ and $|S_2^{(k)}| = (n-1)^k + (-1)^k(n-1)$. Notice that $\mathfrak{P}_n(\cdot|p(l) = k)$ is the uniform distribution on $S_2^{(k)}$ and $\mathbb{P}_k^{(n)}$ is the uniform distribution on $S_1^{(k)}$. Consequently,

$$\mathbb{P}_k^{(n)}(\mathcal{C}, X_k^{(n)} \neq X_1^{(n)}) = \frac{|S_2^{(k)}|}{|S_1^{(k)}|} \mathfrak{P}_n(\mathcal{C}|p(l) = k). \tag{6.4}$$

Moreover,

$$\begin{aligned}
\mathbb{P}_k^{(n)}(\mathcal{C}, X_k^{(n)} = X_1^{(n)}) =& \mathbb{P}_{k-1}^{(n)}(\mathbb{P}_k^{(n)}(\mathcal{C}, X_k^{(n)} = X_1^{(n)} | \sigma(X_1^{(n)}, \dots, X_{k-1}^{(n)}))) \\
=& \mathbb{E}_{k-1}^{(n)}\left(\frac{1}{n-1}\mathbb{P}_{k-1}^{(n)}(\mathcal{C}) 1_{\{X_{k-1}^{(n)} \neq X_1^{(n)}\}}\right) \\
\leq& \frac{1}{n-1}\mathbb{P}_{k-1}^{(n)}(\mathcal{C}) \\
\leq& \frac{1}{n-1}\mathbb{P}_k^{(n)}(\mathcal{C}).
\end{aligned} \tag{6.5}$$

From equation (6.4), we have

$$\mathbb{P}_k^{(n)}(\mathcal{C}) \geq \frac{|S_2^{(k)}|}{|S_1^{(k)}|}\mathfrak{P}_n(\mathcal{C}|p(l) = k) \geq \frac{n-2}{n}\mathfrak{P}_n(\mathcal{C}|p(l) = k).$$

By combining equation (6.4) with inequality (6.5), we have

$$\mathbb{P}_k^{(n)}(\mathcal{C}) \leq \frac{|S_2^{(k)}|}{|S_1^{(k)}|}\mathfrak{P}_n(\mathcal{C}|p(l) = k) + \frac{1}{n-1}\mathbb{P}_k^{(n)}(\mathcal{C}).$$

It implies that

$$\mathbb{P}_k^{(n)}(\mathcal{C}) \leq \frac{n-1}{n-2}\frac{|S_2^{(k)}|}{|S_1^{(k)}|}\mathfrak{P}_n(\mathcal{C}|p(l) = k) \leq \frac{n-1}{n-2}\mathfrak{P}_n(\mathcal{C}|p(l) = k).$$

$\square$

By using the fact that $|S_2| = (n-1)^k + (-1)^k(n-1)$, we have the following formula for $\operatorname{Tr} Q_n^k$:

Lemma 6.6.2. *We have* $\operatorname{Tr} Q_n^k = 1 + \frac{(-1)^k}{(n-1)^{k-1}}$.

We state the main result in the following theorem.

Theorem 6.6.3.

1. *a) If* $\lim_{n\to\infty} \frac{-\ln c_n}{\ln n} \leq 1$, $\lim_{n\to\infty} \mathfrak{P}_n(\mathcal{C}) = 0$.

   b) If $\lim_{n\to\infty} \frac{-\ln c_n}{\ln n} = \infty$, $\lim_{n\to\infty} \mathfrak{P}_n(\mathcal{C}) = 1$.

   c) If $\lim_{n\to\infty} \frac{-\ln c_n}{\ln n} = d \in]1, \infty[$, $\lim_{n\to\infty} \mathfrak{P}_n(\mathcal{C}) = 1 - 1/d$.

2. *Fix* $\beta > 0$, *set* $c_n = \frac{\beta}{n}$. *Let* ξ *be a random variable that follows the Gumbel distribution*[6]. *Then,*

$$\mathfrak{P}_n[\mathcal{C}] \sim \frac{1}{\beta n^\beta (\ln n)^2}\mathbb{E}[e^{-\beta\xi}] = \frac{\Gamma(1+\beta)}{\beta n^\beta (\ln n)^2}.$$

[6] $\mathbb{P}[\xi \leq t] = e^{-e^{-t}}$ for $t \in \mathbb{R}$.

We will prove this theorem after two lemmas.

Lemma 6.6.4. *Let $C^{(n)}$ be the covering time of the simple random walk on the complete graph K_n.*

1. *The sequence of $(\frac{C^{(n)}-n\ln n}{n})_n$ converges in law to the Gumbel distribution, see section 2 of Chapter 6 in [AF].*
2. *Fix $\lambda \in [0,1[$, $\mathbb{P}\left[\frac{1}{n-1}(C^{(n)}-(n-1)\ln(n-1)) \leq \gamma\right] \leq \frac{1}{1-\lambda}e^{-\lambda e^{-\gamma}}$ for all $n \geq 1$ and $\gamma < -1$.*

To our best knowledge, the second estimation is new.

Proof. It is known that $\frac{1}{n}(C^{(n)}-n\ln n) \xrightarrow{d} \xi$ as $n \to \infty$ where ξ follows the Gumbel distribution, see section 2 of Chapter 6 in [AF]. This classical result is well-known for the coupon collector problem. As mentioned in [AF], the complete graph case is a little variation of this problem: Let $\tilde{T}_i^{(n)}$ be the first time that we have collect i coupons. Similar notation $T_i^{(n)}$ for the first time visiting i different vertices in the complete graph with n vertices. For the coupon collector problem, $\tilde{T}_{i+1}^{(n)} - \tilde{T}_i^{(n)}, i \in 1,\dots,n-1$ is a sequence of independent geometric variables with corresponding expectations $\frac{n}{n-i}$. In the complete graph case, $T_{i+1}^{(n)} - T_i^{(n)}, i = 1,\dots,n-1$ is a sequence of independent geometric variables with corresponding expectations $\frac{n-1}{n-i}$. So that $C^{(n)} = \sum_{i=1}^{n-1} Z_i^{(n)}$ where $Z_i^{(n)}$ are independent geometric variables with expectations $\frac{n-1}{n-i}$ for $i = 1,\dots,n-1$. For the second part of this lemma, it is enough to prove that

$$\mathbb{E}\left[\exp\left(\lambda(n-1)\exp\left(-\frac{1}{n-1}\sum_{i=1}^{n-1} Z_i^{(n)}\right)\right)\right] \leq \frac{1}{1-\lambda}. \tag{6.6}$$

Then, we get the desired estimation by Markov inequality.

Proof of inequality (6.6): We expand the exponential and use the independence between $(Z_i^{(n)}, i = 1,\dots,n-1)$. Then,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\lambda(n-1)\exp\left(-\frac{1}{n-1}\sum_{i=1}^{n-1} Z_i^{(n)}\right)\right)\right] \\
&\qquad = \sum_{\ell\geq 0}\lambda^\ell\frac{(n-1)^\ell}{\ell!}\mathbb{E}[\exp(-\frac{\ell}{n-1}\sum_{i=1}^{n-1} Z_i^{(n)})] \\
&\qquad\qquad = \sum_{\ell\geq 0}\lambda^\ell\frac{(n-1)^\ell}{\ell!}\prod_{i=1}^{n-1}\mathbb{E}[\exp(-\frac{\ell}{n-1}Z_i^{(n)})].
\end{aligned}$$

Recall that the Laplace transform $s \to \Phi(s)$ of a geometric variable G with expectation $1/p$ is given by

$$\mathbb{E}[e^{-sG}] = \frac{pe^{-s}}{1-(1-p)e^{-s}} = \frac{p}{e^s - (1-p)}.$$

Applying this to all $Z_i^{(n)}$, we have

$$\begin{aligned}
\mathbb{E}&\left[\exp\left(\lambda(n-1)\exp\left(-\frac{1}{n-1}\sum_{i=1}^{n-1} Z_i^{(n)}\right)\right)\right] \\
&= \sum_{\ell\geq 0} \lambda^\ell \frac{(n-1)^\ell}{\ell!} \prod_{i=1}^{n-1} \frac{\frac{n-i}{n-1}}{e^{\frac{\ell}{n-1}} - (1-\frac{n-i}{n-1})} \\
&= \sum_{\ell\geq 0} \lambda^\ell \frac{(n-1)^\ell}{\ell!} \prod_{i=1}^{n-1} \frac{\frac{n-i}{n-1}}{e^{\frac{\ell}{n-1}} - \frac{i-1}{n-1}} \\
&= \sum_{\ell\geq 0} \lambda^\ell \frac{(n-1)^\ell (n-1)!}{\ell!} \prod_{i=1}^{n-1} \frac{1}{(n-1)e^{\frac{\ell}{n-1}} - i + 1}.
\end{aligned}$$

Since $e^{\ell/(n-1)} \geq 1 + \ell/(n-1)$,

$$\begin{aligned}
\mathbb{E}&\left[\exp\left(\lambda(n-1)\exp\left(-\frac{1}{n-1}\sum_{i=1}^{n-1} Z_i^{(n)}\right)\right)\right] \\
&\leq \sum_{\ell\geq 0} \lambda^\ell \frac{(n-1)^\ell (n-1)!}{\ell!} \prod_{i=1}^{n-1} \frac{1}{(n-1)(1+\ell/(n-1)) - i + 1} \\
&= \sum_{\ell\geq 0} \lambda^\ell \frac{(n-1)^\ell (n-1)!}{\ell!} \prod_{i=1}^{n-1} \frac{1}{n+\ell-i} \\
&= \sum_{\ell\geq 0} \lambda^\ell \frac{(n-1)^\ell (n-1)!}{(n+\ell-1)!} \leq \sum_{\ell\geq 0} \lambda^\ell = \frac{1}{1-\lambda}.
\end{aligned}$$

$\square$

Lemma 6.6.5. *Consider a sequence of random variable η_n with the following distribution*

$$\mathbb{P}(\eta_n = p) = \frac{\frac{1}{p}(1+c_n)^{-p}}{\sum_{k=2}^{\infty} \frac{1}{k}(1+c_n)^{-k}} = \frac{\frac{1}{p}(1+c_n)^{-p}}{-\ln c_n + \ln(1+c_n) - (1+c_n)^{-1}} \ \textit{for } p \geq 2.$$

Then,

a) If $\lim\limits_{n\to\infty} \dfrac{-\ln c_n}{\ln n} \leq 1$, *then* $\lim\limits_{n\to\infty} \mathbb{P}[\eta_n < n] = 1$;

b) If $\lim\limits_{n\to\infty} \dfrac{-\ln c_n}{\ln n} = \infty$, *then* $\lim\limits_{n\to\infty} \mathbb{P}[\eta_n > n^2] = 1$;

c) If $\lim_{n\to\infty} \frac{-\ln c_n}{\ln n} = d \in]1,\infty[$*, then*

$$\lim_{n\to\infty} \mathbb{P}(\eta_n > n(\ln n)^2) = 1 - 1/d \text{ and } \lim_{n\to\infty} \mathbb{P}(\eta_n < n) = 1/d.$$

Proof.

a) We have that

$$\begin{aligned}
\mathbb{P}[\eta_n \geq n] =& \frac{1}{-\ln c_n + \ln(1+c_n) - (1+c_n)^{-1}} \sum_{p\geq n} \frac{1}{p}(1+c_n)^{-p} \\
\leq& \frac{(1+c_n)^{-n+1}}{-\ln c_n + \ln(1+c_n) - (1+c_n)^{-1}} \sum_{k\geq 1} \frac{1}{k}(1+c_n)^{-k} \\
=& \frac{(1+c_n)^{-n+1}(-\ln(c_n) + \ln(1+c_n))}{-\ln c_n + \ln(1+c_n) - (1+c_n)^{-1}}.
\end{aligned}$$

As a result, $\lim_{n\to\infty} \mathbb{P}[\eta_n \geq n] = 0$ if $\lim_{n\to\infty} \frac{-\ln c_n}{\ln n} < 1$. For the case $\lim_{n\to\infty} \frac{-\ln c_n}{\ln n} = 1$, we will prove that $\lim_{n\to\infty} \mathbb{P}[\eta_n < n] \geq 1 - \delta$ for any $\delta > 0$. Fix $\delta > 0$,

$$\begin{aligned}
\mathbb{P}[\eta_n < n] \geq& \frac{1}{-\ln c_n + \ln(1+c_n) - (1+c_n)^{-1}} \sum_{p=2}^{n^{1-\delta}} \frac{1}{p}(1+c_n)^{-p} \\
\geq& \frac{\min\{(1+c_n)^{-p} : p \in [2, n^{1-\delta}]\}}{-\ln c_n + \ln(1+c_n) - (1+c_n)^{-1}} \sum_{p=2}^{n^{1-\delta}} \frac{1}{p} \\
\geq& \frac{\min\{(1+c_n)^{-p} : p \in [2, n^{1-\delta}]\}}{-\ln c_n + \ln(1+c_n) - (1+c_n)^{-1}} \ln(n^{1-\delta}).
\end{aligned}$$

Under the condition $\lim_{n\to\infty} \frac{-\ln c_n}{\ln n} = 1$, we have $\lim_{n\to\infty} \min\{(1+c_n)^{-p} : p \in [2, n^{1-\delta}]\} = 1$. Therefore, $\lim_{n\to\infty} \mathbb{P}[\eta_n < n] \geq 1 - \delta$ for any fixed $\delta > 0$.

b) If $\lim_{n\to\infty} \frac{-\ln c_n}{\ln n} = \infty$,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}[\eta_n < n^2] \leq& \lim_{n\to\infty} \frac{1}{-\ln c_n + \ln(1+c_n) - (1+c_n)^{-1}} \sum_{p=2}^{n^2} \frac{1}{p} \\
\leq& \lim_{n\to\infty} \frac{\ln(n^2)}{-\ln c_n + \ln(1+c_n) - (1+c_n)^{-1}} = 0.
\end{aligned}$$

c) Suppose $\lim_{n\to\infty} \frac{-\ln c_n}{\ln n} = d \in]1,\infty[$. Similar to a), we can prove $\lim_{n\to\infty} \mathbb{P}(\eta_n < n) = 1/d$. Similar to b), we can prove $\lim_{n\to\infty} \mathbb{P}(\eta_n \in [n, n(\ln n)^2]) = 0$. Consequently, we have $\lim_{n\to\infty} \mathbb{P}(\eta_n > n(\ln n)^2) = 1 - 1/d$.

$\square$

Proof of Theorem 6.6.3.

$$\mathfrak{P}_n[\mathcal{C}] = \sum_{k=2}^{\infty} \mathfrak{P}_n[\mathcal{C}|p(l) = k] \frac{\mu_n(p(l) = k)}{\sum\limits_{j=2}^{\infty} \mu_n(p(l) = j)}$$

with $\mu_n(p(l) = j) = \frac{1}{j}(1 + c_n)^{-j} \operatorname{Tr} Q_n^j$. By Lemma 6.6.1,

$$\mathfrak{P}_n[\mathcal{C}|p(l) = k] = \mathbb{P}_k^{(n)}(\mathcal{C})(1 + O(1/n)).$$

Therefore,

$$\mathfrak{P}_n[\mathcal{C}] = \frac{1 + O(\frac{1}{n})}{\sum\limits_{k=2}^{\infty} \frac{1}{k} \operatorname{Tr} Q_n^k (1 + c_n)^{-k}} (\sum_{k=2}^{\infty} \frac{1}{k} \operatorname{Tr} Q_n^k (1 + c_n)^{-k} \mathbb{P}_k^{(n)}[\mathcal{C}]).$$

By Lemma 6.6.2, we replace $\operatorname{Tr} Q_n^k$ by 1:

$$\begin{aligned}\mathfrak{P}_n[\mathcal{C}] =& \frac{1 + O(\frac{1}{n})}{\sum\limits_{k=2}^{\infty} \frac{1}{k}(1 + c_n)^{-k}} (\sum_{k=2}^{\infty} \frac{1}{k}(1 + c_n)^{-k} \mathbb{P}_k^{(n)}[\mathcal{C}]) \\ =& \frac{1 + O(\frac{1}{n})}{-\ln(c_n) + \ln(1 + c_n) - (1 + c_n)^{-1}} (\sum_{k=2}^{\infty} \frac{1}{k}(1 + c_n)^{-k} \mathbb{P}_k^{(n)}[\mathcal{C}]).\end{aligned}$$

Consider an independent random variable η_n with the following distribution

$$\mathbb{P}(\eta_n = p) = \frac{\frac{1}{p}(1 + c_n)^{-p}}{\sum\limits_{k=2}^{\infty} \frac{1}{k}(1 + c_n)^{-k}} \text{ for } p = 2, 3, \ldots.$$

Then, $\mathfrak{P}_n[\mathcal{C}] = (1 + O(\frac{1}{n}))\mathbb{P}^{(n)}[C^{(n)} \leq \eta_n]$ where $C^{(n)}$ is the covering time of the simple random walk on the complete graph K_n. Recall that $\frac{C^{(n)} - n \ln n}{n}$ converges in law to a Gumbel distribution, see Lemma 6.6.4.

Then, the part 1) follows from an estimate for η_n in Lemma 6.6.5.

For the part 2), let us fix $\beta > 0$ and $c_n = \beta/n$. Then,

$$\begin{aligned}\mathbb{P}\left(\frac{\eta_n - n \ln n}{n} \in [\gamma, \gamma + \epsilon[\right) \sim& \frac{1}{\ln n} \sum_{p = \lfloor n(\ln n + \gamma) \rfloor}^{\lceil n(\ln n + \gamma + \epsilon) \rceil} \frac{1}{p}(1 + \beta/n)^{-n \ln n}(1 + \beta/n)^{-(p - \ln n)} \\ \sim& \frac{1}{\ln n} \sum_{p = \lfloor n(\ln n + \gamma) \rfloor}^{\lceil n(\ln n + \gamma + \epsilon) \rceil} \frac{1}{p}(1 + \beta/n)^{-n \ln n} e^{-\beta \frac{p - n \ln n}{n}}.\end{aligned}$$

Since $(1+\beta/n)^{-n\ln n} = e^{-\ln n(n\ln(1+\beta/n))} = e^{-\ln n(\beta+o(1/n))} = n^{-\beta}e^{-o(\ln n/n)} \sim n^{-\beta}$, we have

$$\mathbb{P}\left(\frac{\eta_n - n\ln n}{n} \in [\gamma,\gamma+\epsilon[\right) \sim \frac{1}{\ln n}n^{-\beta}e^{-\beta\gamma}\sum_{p=\lfloor n(\ln n+\gamma)\rfloor}^{\lceil n(\ln n+\gamma+\epsilon)\rceil}\frac{1}{p}\exp\left(-\beta\frac{p-n\ln n-n\gamma}{n}\right).$$

Notice that

$$\exp\left(-\frac{p-n\ln n-n\gamma}{n}\right) \in [e^{-\epsilon},1] \text{ for } p \in \left[\lfloor n(\ln n+\gamma)\rfloor, \lceil n(\ln n+\gamma+\epsilon)\rceil\right];$$
$$\sum_{p=\lfloor n(\ln n+\gamma)\rfloor}^{\lceil n(\ln n+\gamma+\epsilon)\rceil}\frac{1}{p} \sim \ln\left(\frac{n(\ln n+\gamma+\epsilon)}{n(\ln n+\gamma)}\right) \sim \frac{\epsilon}{\ln n}.$$

We can conclude that for fixed γ,

$$\limsup_{n\to\infty} n^{\beta}(\ln n)^2\mathbb{P}\left(\frac{\eta_n-n\ln n}{n}\in[\gamma,\gamma+\epsilon[\right) \leq e^{-\beta\gamma}\epsilon;$$
$$\liminf_{n\to\infty} n^{\beta}(\ln n)^2\mathbb{P}\left(\frac{\eta_n-n\ln n}{n}\in[\gamma,\gamma+\epsilon[\right) \geq e^{-\beta\gamma-\beta\epsilon}\epsilon.$$

Moreover, we have the following estimate for η_n:

- for $\gamma \geq 0$, $\sup\limits_{n\geq 2} n^{\beta}(\ln n)^2\mathbb{P}\left(\frac{\eta_n-n\ln n}{n}\in[\gamma,\gamma+\epsilon[\right) \leq cst\cdot e^{-\beta\gamma}\epsilon$,
- for $\gamma \leq 0$, $\sup\limits_{n\geq 2} n^{\beta}(\ln n)^2\mathbb{P}\left(\frac{\eta_n-n\ln n}{n}\in[\gamma,\gamma+\epsilon[\right) \leq cst\cdot e^{-(\beta+1)\gamma}\epsilon$.

where the constant cst does not depend on γ. By the second part of Lemma 6.6.4, the following estimate on the covering time $C^{(n)}$ holds for $\gamma < -1$:

$$\mathbb{P}\left[\frac{1}{n}(C^{(n)}-n\ln n)\leq\gamma\right] \leq cst\cdot e^{(2+\beta)\gamma}$$

where the constant cst does not depend on γ, n. (We replace $\frac{1}{n-1}(C^{(n)}-(n-1)\ln(n-1))$ by $\frac{1}{n}(C^{(n)}-n\ln n)$ since the difference tends 0 as $n\to\infty$.) Finally, the integrability condition is fulfilled and we have

$$\lim_{n\to\infty} n^{\beta}(\ln n)^2\mathfrak{P}[\mathcal{C}] = \int_{\mathbb{R}}\mathbb{E}[\xi\leq\gamma]e^{-\beta\gamma}\,d\gamma = \frac{1}{\beta}\mathbb{E}[e^{-\beta\xi}] = \frac{\Gamma(1+\beta)}{\beta}.$$

□

We deduce from Theorem 6.6.3 the following result on the asymptotic distribution of the number of loops covering G_n.

Corollary 6.6.6. *Let $\mathcal{L}^{(n)}$ be the Poisson collection of loops with intensity $\frac{\mu_n}{\ln n}$. Suppose $\lim\limits_{n\to\infty}\frac{-\ln c_n}{\ln n} = d$. Then, $\sum\limits_{l\in\mathcal{L}^{(n)}} 1_{\{l\in\mathcal{C}\}}$ converges in distribution to a Poisson random variable with parameter $(d-1)_+$ as n tends to infinity.*

Proof. The total mass of μ_n, namely $||\mu_n||$, is equivalent to $\sum_{k\geq 2}\frac{1}{k}(\frac{1}{1+c_n})^k = -\ln(1-\frac{1}{1+c_n}) - \frac{1}{1+c_n}$. By Theorem 6.6.3, if $\lim_{n\to\infty}\frac{-\ln c_n}{\ln n}\leq 1$, $\lim_{n\to\infty}\mathfrak{P}_n(\mathcal{C})=0$, i.e. $\mu_n(\mathcal{C})=o(||\mu_n||)=o(\ln n)$. Therefore, $\lim_{n\to\infty}\frac{\mu_n(\mathcal{C})}{\ln n}=0$. If $\lim_{n\to\infty}\frac{-\ln c_n}{\ln n}=\infty$, $\lim_{n\to\infty}\mathfrak{P}_n(\mathcal{C})=1$, i.e. $\mu_n(\mathcal{C})\sim||\mu_n||\sim-\ln c_n$. Therefore, $\lim_{n\to\infty}\frac{\mu_n(\mathcal{C})}{\ln n}=\infty$. If $\lim_{n\to\infty}\frac{-\ln c_n}{\ln n}=d\in]1,\infty[$, $\lim_{n\to\infty}\mathfrak{P}_n(\mathcal{C})=1-1/d$, i.e. $\mu_n(\mathcal{C})=(1-1/d)||\mu_n||\sim-(1-1/d)\ln c_n\sim(d-1)\ln n$. Therefore, $\lim_{n\to\infty}\frac{\mu_n(\mathcal{C})}{\ln n}=d-1$. In summary, suppose $\lim_{n\to\infty}\frac{-\ln c_n}{\ln n}=d$, then $\lim_{n\to\infty}\frac{\mu_n(\mathcal{C})}{\ln n}=(d-1)_+$. Consequently, $\sum_{l\in\mathcal{L}^{(n)}}1_{\{l\in\mathcal{C}\}}$ converges in distribution to a Poisson random variable with parameter $(d-1)_+$ as n tends to infinity. □

Similarly, we have the following corollary:

Corollary 6.6.7. *Suppose that $c_n=\beta/n$. Let $\mathcal{L}^{(n)}$ be the Poisson collection of loops with intensity $n^\beta(\ln n)^2\mathfrak{P}_n$ (or $n^\beta(\ln n)\mu_n$ equivalently). Then, $\sum_{l\in\mathcal{L}^{(n)}}1_{\{l\in\mathcal{C}\}}$ converges in distribution to a Poisson random variable with parameter $\frac{\Gamma(1+\beta)}{\beta}$ as n tends to infinity.*

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