Comportement asymptotique de marches aléatoires de branchement dans R^d et dimension de Hausdorff

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Résumé

Nous calculons presque sûrement, simultanément, les dimensions de Hausdorff des ensembles de branches infinies de la frontière d'un arbre de Galton-Watson super-critique (muni d'une métrique aléatoire) le long desquelles les moyennes empiriques d'une marche aléatoire de branchement vectorielle admettent un ensemble donné de points limites. Cela va au-delà de l'analyse multifractale, question pour laquelle nous complétons les travaux antérieurs en considérant les ensembles associés à des niveaux situés dans la frontière du domaine d'étude. Nous utilisons une méthode originale dans ce contexte, consistant à construire des mesures de Mandelbrot inhomogènes appropriées. Cette méthode est inspirée de l'approche utilisée pour résoudre des questions similaires dans le contexte de la dynamique hyperboliques pour les moyennes de Birkhoff de potentiels continus. Elle exploite des idées provenant du chaos multiplicatif et de la théorie de la percolation pour estimer la dimension inférieure de Hausdorff des mesures de Mandelbrot inhomogènes. Cette méthode permet de renforcer l'analyse multifractale en raffinant les ensembles de niveaux de telle sorte qu'ils contiennent des branches infinies le long desquels on observe une version quantifiée de la loi des grands nombres d'Erdös Renyi ; de plus elle permet d'obtenir une loi de type 0-∞ pour les mesures de Hausdorff de ces ensembles. Nos résultats donnent naturellement des informations géométriques et de grandes déviations sur l'hétérogénéité du processus de naissance le long des différentes branches infinies de l'arbre de Galton-Watson.

Mots-clefs : Marches aléatoires de branchement vectorielles, chaos multiplicatif, percolation, dimension de Hausdorff, analyse multifractale.

Asymptotic behavior of branching random walks in $\mathbb{R}^d$ and Hausdorff dimension.

Abstract

We compute almost surely (simultaneaously) the Hausdorff dimensions of the sets of infinite branches of the boundary of a super-critical Galton-Watson tree (endowed with a random metric)along which the averages of a vector valued branching random walk have a given set of limit points. This goes beyond multifractal analysis, for which we complete the previous works on the subject by considering the sets associated with levels in the boundary of the domain of study. Our method is inspired by some approach used to solve similar questions in the different context of hyperbolic dynamics for the Birkhoff averages of continuous potentials. It also exploits ideas from multiplicative chaos and percolation theories, which are used to estimate the lower Hausdorff dimension of a family of inhomogeneous Mandelbrot measures. This method also makes it possible to strengthen the multifractal analysis of the branching random walk averages by refining the level sets so that they contain branches over which a quantified version of the Erdös Renyi law of large numbers holds, and yields a 0-∞ law for the Hausdorff measures of these sets. Our results naturally give geometric and large deviations information on the heterogeneity of the birth process along different infinite branches of the Galton-Watson tree.

Keywords : Branching random walks, multiplicative chaos, percolation, Hausdorff dimension, multifractal analysis.

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Chapitre 1

Introduction

1.1 Introduction and mains results

My PhD thesis was founded via the french National Research Agency project "DMASC", whose goal was both to develop theoretical results on multifractal analysis and large deviations theory, and develop tools based on these theories to study the dynamics of cardiac electro-mecanical activity, starting with cardiac signals processing. This manuscript is devoted to theoretical results. I also participated to the elaboration of an applied paper published this year by DMASC project in the journal Physica A (P. Loiseau, C. Médigue, P. Gonçalvès, N. Attia, S. Seuret, F. Cottin, D. Chemla, M. Sorine, and J. Barral, Physica A, **391** (2012) 5658–5671) about the application to the so-called R-R signals of some new signal processing tools dedicated to "multifractal" signals developed in [5].

The theoretical part of this thesis deals with the natural question of measuring, through their Hausdorff dimensions, the sizes of the sets of infinite branches of the boundary of a supercritical Galton-Watson tree (endowed with a random ultrametric) along which the averages of a vector valued branching random walk have a given set of limit points ; this goes beyond the multifractal analysis, which consists in computing the Hausdorff dimension of the level sets of infinite branches over which the averages of the walk has a given limit, and for which we solve a delicate question left open in previous works, namely the case of levels belonging to the boundary of the set of possible levels. The corresponding questions have been studied extensively in the context of Birkhoff averages of continuous potentials on conformal repellers, a situation which, though different, is inspiring in our context (we will give some references making it possible to compare our results with the corresponding litterature). The new method we develop, combined with recent almost sure large deviations results for random walks also yields a refinement of the multifractal analysis by considering level sets over the branch of which large deviations with respect to the average behavior are measured through a quantified version of the Erdös Renyi law of large numbers, and giving a 0-∞ law for the Hausdorff measures of these sets. Before going to the detail of our results, we need some definitions and notations.

Let $(N, X_1, X_2, \dots)$ be a random vector taking values in $\mathbb{N}_+ \times (\mathbb{R}^d)^{\mathbb{N}_+}$.

Let $\{(N_{u0}, X_{u1}, X_{u2}), \ldots)\}_u$ be a family of independent copies of $(N, X_1, X_2, \ldots)$ indexed by the finite sequences $u = u_1 \cdots u_n$, $n \geq 0$, $u_i \in \mathbb{N}_+$ ($n = 0$ corresponds to the empty sequence denoted $\emptyset$), and let T be the Galton-Watson tree with defining elements $\{N_u\}$: we have $\emptyset \in \mathsf{T}$ and, if $u \in \mathsf{T}$ and $i \in \mathbb{N}_+$ then ui, the concatenation of u and i, belongs to T if and only if $1 \leq i \leq N_u$. Similarly, for each $u \in \bigcup_{n\geq 0} \mathbb{N}_+^n$, denote by $\mathsf{T}(u)$ the Galton-Watson tree rooted at u and defined by the $\{N_{uv}\}$, $v \in \bigcup_{n\geq 0} \mathbb{N}_+^n$.

The probability space over which the previous random variables are built is denoted $(\Omega, \mathcal{A}, \mathbb{P})$, and the expectation with respect to $\mathbb{P}$ is denoted $\mathbb{E}$.

We assume that $\mathbb{E}(N) > 1$ so that the Galton-Watson tree is supercritical. Without loss of generality, we also assume that the probability of extinction equals 0, so that $\mathbb{P}(N \geq 1) = 1$.

Also, we will assume without loss of generality the following property about X :

$$\nexists\ (q, c) \in (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R},\ \langle q|X_i\rangle = c \quad \forall\ 1 \leq i \leq N \text{ a.s.} \tag{1.1}$$

Otherwise, if $d = 1$, then the X_i, $1 \leq i \leq N$, are equal to the same constant almost surely, and the situation is trivial, and if $d \geq 2$, they belong to the same hyperplane so that we can reduce our study to the case of $\mathbb{R}^{d-1}$ valued random variables.

For each $u \in \bigcup_{n\geq 0} \mathbb{N}_+^n$, we denote by $|u|$ its length, i.e. the number of letters of u, and $[u]$ the cylinder $u \cdot {\mathbb{N}_+}^{\mathbb{N}_+}$, i.e. the set of those $t \in {\mathbb{N}_+}^{\mathbb{N}_+}$ such that $t_1 t_2 \cdots t_{|u|} = u$. If $t \in {\mathbb{N}_+}^{\mathbb{N}_+}$, we set $|t| = \infty$, and the set of prefixes of t consists of $\{\emptyset\} \cup \{t_1 t_2 \cdots t_n : n \geq 1\} \cup \{t\}$.

Our results will depend on the metric under which we will work. Let us recall that given a subset K of $\mathbb{N}_+^{\mathbb{N}_+}$ endowed with a metric d making it σ-compact, $g : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous non-decreasing function near 0 and such that $g(0) = 0$, and E a subset of K, the Hausdorff measure of E with respect to the gauge function g is defined as

$$\mathcal{H}^g(E) = \lim_{\delta\to 0^+} \inf \Big\{ \sum_{i\in\mathbb{N}} g(\operatorname{diam}(U_i)) \Big\},$$

the infimum being taken over all the countable coverings $(U_i)_{i\in\mathbb{N}}$ of E by subsets of K of diameters less than or equal to δ.

If $s \in \mathbb{R}_+^*$ and $g(u) = u^s$, then $\mathcal{H}^g(E)$ is also denoted $\mathcal{H}^s(E)$ and called the s-dimensional Hausdorff measure of E. Then, the Hausdorff dimension of E is defined as

$$\dim E = \sup\{s > 0 : \mathcal{H}^s(E) = \infty\} = \inf\{s > 0 : \mathcal{H}^s(E) = 0\},$$

with the convention $\sup \emptyset = 0$ and $\inf \emptyset = \infty$. Moreover, if E is a Borel set and μ is a measure supported on E, then its lower Hausdorff dimension is defined as

$$\underline{\dim}(\mu) = \inf \Big\{ \dim F :\ F \text{ Borel},\ \mu(F) > 0 \Big\},$$

and we have

$$\underline{\dim}(\mu) = \operatorname{ess\,inf}_\mu\ \liminf_{r\to 0^+} \frac{\log \mu(B(t,r))}{\log(r)},$$

where $B(t,r)$ stands for the closed ball of radius r centered at t ; (see [28] for instance, in which upper Hausdorff dimensions and lower and upper packing dimensions are also considered).

Let us assume for the moment that $\mathbb{N}_+^{\mathbb{N}_+}$ is endowed with the standard ultrametric distance

$$d_1 : (s,t) \mapsto \exp(-|s \wedge t|),$$

where $s \wedge t$ stands for the longest common prefix of s and t, and with the convention that $\exp(-\infty) = 0$.

The boundary of T is the subset of $\mathbb{N}_+^{\mathbb{N}_+}$ defined as

$$\partial \mathsf{T} = \bigcap_{n \geq 1} \bigcup_{u \in \mathsf{T}_n} [u],$$

where $\mathsf{T}_n = \mathsf{T} \cap \mathbb{N}_+^n$. With probability 1, the metric space $(\partial \mathsf{T}, d_1)$ is compact.

The vector space $\mathbb{R}^d$ is endowed with the canonical scalar product and the associated Euclidean norm respectively denoted $\langle \cdot | \cdot \rangle$ and $\| \cdot \|$. For $q \in \mathbb{R}^d$ let

$$S(q) = \sum_{i=1}^{N} \exp(\langle q | X_i \rangle).$$

Let us assume that

$$\forall\, q \in \mathbb{R}^d, \ \mathbb{E}(S(q)) < \infty \tag{1.2}$$

(we will see how to relax this assumption). In particular, the logarithmic moment generating function

$$\widetilde{P} : q \in \mathbb{R}^d \mapsto \log \mathbb{E}(S(q)) \tag{1.3}$$

is finite over $\mathbb{R}^d$, and under (1.1) it is strictly convex.

We are interested in the asymptotic behavior of the branching random walk

$$S_n X(t) = \sum_{k=1}^{n} X_{t_1 \cdots t_k} \quad (t \in \partial \mathsf{T}).$$

Since this quantity depends on $t_1 \cdots t_n$ only, we also denote by $S_n X(u)$ the constant value of $S_n X(\cdot)$ over $[u]$ whenever $u \in \mathsf{T}_n$.

The multifractal analysis of $S_n X$ is a first natural consideration. It consists in computing the Hausdorff dimensions of the sets

$$E_X(\alpha) = \Big\{ t \in \partial \mathsf{T} : \lim_{n \to \infty} \frac{S_n X(t)}{n} = \alpha \Big\}, \quad (\alpha \in \mathbb{R}^d).$$

Indeed, considering the branching measure on $\partial \mathsf{T}$ (see Section 6 for the definition) makes it possible to show that $E_X(\alpha_0)$ is of full Hausdorff dimension in $\partial \mathsf{T}$, where $\alpha_0 = \nabla \widetilde{P}(0) = \mathbb{E}\Big(\sum_{i=1}^N X_i \Big) / \mathbb{E}(N)$, and it is worth investigating the existence of other branches over which such a law of large numbers holds, with different values of α.

Such deviations phenomena with respect to a typical one emerge in many sufficiently random, or chaotic, contexts. Quantifying these phenomena has always generated strong, reach and new interplays between probability or dynamical systems theory, and geometric measure theory. As important examples, let us mention the problem of measuring how often on a Brownian motion the law of the iterated logarithm fails, by measuring the Hausdorff or packing dimension of sets of rapid points [65, 50] and slow points [44, 67] ; we must also mention the works of Jarnik and his followers about the sets of irrational numbers with a given rate of approximation by rational numbers [42] and the works of Besicovitch and Eggleston around the Hausdorff dimension of the sets of real numbers with prescribed frequencies of their digits in a given basis [23] ; also, and this is closely related to the subject of this thesis (and Besicovitch sets), after the concept of multifractals, i.e. the notion of iso-Hölder sets of a measure or a function, was pointed out by physicits in turbulence and statistical physics [36, 38, 21] as a convenient way to describe the heterogeneity of energy distribution by a hierarchy of fractal sets, a considerable mathematical literature has developed around the multifractal analysis of measures generated by multiplicative procedures, like Mandelbrot measures defined below, self-conformal measures and Gibbs measures on hyperbolic repellers (see [68, 4] and references therein), the multifractal analysis of the harmonic measures on the Brownian frontier [52] and more generally on the boundary of simply connected planar domains [59], and the multifractal analysis of functions [40, 41, 10, 7, 6].

Let us come back to our problem. We will see that the domain of those α for which $E_X(\alpha) \neq \emptyset$ is the convex compact set

$$I = I_X = \{\alpha \in \mathbb{R}^d : \widetilde{P}^*(\alpha) \geq 0\},$$

where

$$f^*(\alpha) = \inf\{f(q) - \langle q|\alpha\rangle : q \in \mathbb{R}^d\}$$

for any function $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ and any $\alpha \in \mathbb{R}^d$.

A direct approach to the Hausdorff dimension of $E_X(\alpha)$ consists in taking $\alpha \in I$ such that $\alpha = \nabla\widetilde{P}(q)$ for some $q \in \mathbb{R}^d$ and $\widetilde{P}^*(\alpha) > 0$ (we will see in the proof of Proposition 3.1 that this is equivalent to taking $\alpha \in \mathring{I}$, the interior of I). Then, one considers the associated Mandelbrot measure on $\partial\mathsf{T}$ defined as

$$\mu_q([u]) = \exp(\langle q|S_nX(u)\rangle - n\widetilde{P}(q))Z(q,u), \quad (u \in \mathsf{T}_n),$$

where

$$Z(q,u) = \lim_{p\to\infty} \sum_{v\in\mathsf{T}_p(u)} \exp(\langle q|(S_{n+p}X(u\cdot v) - S_nX(u))\rangle - p\widetilde{P}(q))$$

and here we simply denoted $[u] \cap \partial\mathsf{T}$ by $[u]$ (see Section 6.1 for a general description of Mandelbrot measures). Under our assumptions, the fact that $\widetilde{P}^*(\alpha) > 0$ implies that this measure is almost surely positive [45, 13, 54, 56] (while $\mu_q = 0$ almost surely if $\widetilde{P}^*(\alpha) \leq 0$). Moreover, μ_q is carried by $E_X(\alpha)$ and its Hausdorff dimension is $\widetilde{P}^*(\alpha)$, so that $\dim E_X(\alpha) \geq \widetilde{P}^*(\alpha)$. Then, a simple covering argument yields $\dim E_X(\alpha) = \widetilde{P}^*(\alpha)$. This approach holds for each $\alpha \in \mathring{I}$ almost surely, and it has been followed in the case $d = 1$ for the multifractal analysis of Mandelbrot measures, especially in [39, 62, 27, 60, 1, 17] (under always weaker assumptions), where more general metric than d_1 are considered to get results for geometric realizations of these measures in

$\mathbb{R}^n$. Nevertheless, it is possible to strengthen the result to get, with probability 1, $\dim E_X(\alpha) = \widetilde{P}^*(\alpha)$ for all $\alpha \in \mathring{I}$. This is done in [2] for the case $d = 1$ on homogeneous Galton Watson trees with bounded branching number. Let us assume in addition to (1.2) the following property.

$$\exists\, \gamma : \ q \in J \mapsto \gamma_q \in (1, \infty), \ C^0 \text{ and such that } \mathbb{E}\big(S(q)^{\gamma_q}\big) < \infty \text{ for all } q \in J. \qquad (1.4)$$

We obtain the following result, which does not follow from a direct adaptation of the techniques used in [3], as we explain in Section 2.3, and is proved in Chapter 2.

Theorem 1.1 *Assume* (1.2) *and* (1.4) *hold. Suppose that* $\partial\mathsf{T}$ *is endowed with the metric* d_1. *With probability* 1, *for all* $\alpha \in \mathring{I}$, $\dim E_X(\alpha) = \widetilde{P}^*(\alpha)$ *; in particular,* $E_X(\alpha) \neq \emptyset$.

It is worth mentioning that a direct consequence of the previous result is the following large deviation property, which could also be deduced from the study achieved in [16] : With probability 1, for all $\alpha \in \mathring{I}$,

$$\begin{aligned} &\lim_{\epsilon\to 0^+} \liminf_{n\to\infty} n^{-1} \log \#\big\{u \in \mathsf{T}_n : n^{-1} S_n X(u) \in B(\alpha,\epsilon)\big\} \\ &= \lim_{\epsilon\to 0^+} \limsup_{n\to\infty} n^{-1} \log \#\big\{u \in \mathsf{T}_n : n^{-1} S_n X(u) \in B(\alpha,\epsilon)\big\} = \widetilde{P}^*(\alpha). \end{aligned}$$

The method we use to prove Theorem 1.1 requires to simultaneously building the measures μ_q and computing their Hausdorff dimension ; it will extensively use techniques combining analytic functions in several variables theory and large deviations estimates. However, this approach is unable to cover the levels $\alpha \in \partial I$. When $d = 1$, this boundary consists of two points, and the question has been solved partially in [2] and completely in [7] in the case of homogeneous trees : when $\alpha \in \partial I$ takes the form $\widetilde{P}'(q)$ for $q \in \mathbb{R}$ such that $\widetilde{P}^*(\widetilde{P}'(q)) = 0$, one substitutes to μ_q a measure that is naturally deduced from the fixed points of the associated smoothing transformation in the "boundary case" (see [18, 2]) ; when $\alpha \in \partial I$ and $\alpha = \lim_{q\to\infty} \widetilde{P}'(q)$ (resp. $\lim_{q\to-\infty} \widetilde{P}'(q)$), a "concatenation" method is used in [7] to build an inhomogeneous Mandelbrot measure carried by $E_X(\alpha)$ and with the right dimension (see Section 6.2 for a foreword about inhomogeneous Mandelbrot measures).

It turns out that these methods are not sufficient to deal with the points of ∂I when $d \geq 2$. In collaboration with J. Barral, we adopted an approach, based on a more sofisticated construction of inhomogeneous Mandelbrot measures than that considered in [7], that makes it possible to deal with the cases $\alpha \in \mathring{I}$ and $\alpha \in \partial I$ in a unified way. This approach is inspired by the study of vector Birkhoff averages on mixing subshift of finite type and their geometric realizations on conformal repellers [12, 30, 35, 34, 11, 63, 33]. It makes it possible to conduct the calculation of the Hausdorff dimension of far more general sets and obtain in the context of branching random walks the counterpart of the main results obtained in the papers mentioned above about the asymptotic behavior of Birkhoff averages.

Let us assume

$$\forall\, q \in \mathbb{R}^d,\ \mathbb{E}(S(q)) < \infty, \quad \text{and} \quad \forall q \in J,\ \exists \gamma > 1,\ \mathbb{E}(S(q)^\gamma) < \infty. \qquad (1.5)$$

If K is a compact connected subset of $\mathbb{R}^d$, let

$$E_X(K) = \Big\{t \in \partial\mathsf{T} : \bigcap_{N\geq 1} \overline{\Big\{\frac{S_nX(t)}{n} : n \geq N\Big\}} = K\Big\},$$

the set of those $t \in \partial\mathsf{T}$ such that the set of limit points of $(S_nX(t)/n)_{n\geq 1}$ is equal to K.

Recall that $I = I_X = \{\alpha \in \mathbb{R}^d : \widetilde{P}^*(\alpha) \geq 0\}$.

Theorem 1.2 *Assume* (1.5). *Suppose that $\partial\mathsf{T}$ is endowed with the metric d_1. With probability 1, for all compact connected subset K of $\mathbb{R}^d$, we have $E_X(K) \neq \emptyset$ if and only if $K \subset I$, and in this case*

$$\dim E_X(K) = \inf_{\alpha\in K} \widetilde{P}^*(\alpha).$$

Section 6.2 aims at giving basic properties of inhomogeneous measures that should convince the reader of the pertinence of this family of measures to control the sets $E_X(K)$.

The first part of the next result about multifractal analysis is a corollary of the previous result. The second one provides an information to be compared with the information provided by the approach consisting in putting on $E_X(\nabla\widetilde{P}(q))$ the Mandelbrot measure μ_q to get the dimension of $E_X(\nabla\widetilde{P}(q))$. The third one adds a precision about the level set of maximal dimension which will be useful to state Theorem 1.8.

Theorem 1.3 *(Multifractal analysis) Assume* (1.5). *With probability 1,*

1. *for all $\alpha \in \mathbb{R}^d$, $E_X(\alpha) \neq \emptyset$ if and only if $\alpha \in I$, and in this case* $\dim E_X(\alpha) = \widetilde{P}^*(\alpha)$ *;*
2. *if $\alpha \in I$, then $E_X(\alpha)$ carries uncountably many mutually singular inhomogeneous Mandelbrot measures of Hausdorff dimension $\widetilde{P}^*(\alpha)$ (in particular $E_X(\alpha)$ is not countable when $\widetilde{P}^*(\alpha) = 0$). Moreover, if $\alpha \in \mathring{I}$ then $E_X(\alpha)$ carries a unique Mandelbrot measure of maximal Hausdorff dimension, namely μ_q if α is written $\nabla\widetilde{P}(q)$.*
3. *$E_X(\nabla\widetilde{P}(0))$ is the unique level set of maximal Hausdorff dimension* $\dim\partial\mathsf{T} = \log\mathbb{E}(N)$.

The part on the uniqueness of the Mandelbrot measure should be compared to the uniqueness of the ergodic measure carried by such a set in the multifractal analysis of Hölder potentials on symbolic spaces, especially when this potential only depends on the first digit. As a corollary of Theorem 1.3, we get the following large deviations result, which completes what can be deduced from the study achieved by Biggins in [15], by including the points of ∂I.

Corollary 1.1 *(Large deviations) Assume* (1.5). *With probability 1, for all $\alpha \in I$,*

$$\begin{aligned}&\lim_{\epsilon\to 0^+}\liminf_{n\to\infty} n^{-1}\log\#\big\{u \in \mathsf{T}_n : n^{-1}S_nX(u) \in B(\alpha,\epsilon)\big\}\\ &= \lim_{\epsilon\to 0^+}\limsup_{n\to\infty} n^{-1}\log\#\big\{u \in \mathsf{T}_n : n^{-1}S_nX(u) \in B(\alpha,\epsilon)\big\} = \widetilde{P}^*(\alpha).\end{aligned}$$

The upper bounds estimates for the Hausdorff dimensions of the sets $E_X(K)$ are quite easy. Lower bounds estimates are much more delicate and will use a uniform version of the percolation argument introduced by Kahane in [46] to calculate the Hausdorff dimension of a non-degenerate Mandelbrot measure μ on a homogeneous tree, without using assumptions assuring that $\mathbb{E}(\|\mu\| \log^+(\|\mu\|)) < \infty$ as it was done by Peyrière in [64, 45]. Specifically, the approach developed in this paper extends Kahane's result by making it possible to control simultaneously the lower Hausdorff dimension of uncountably many non degenerate limits of inhomogeneous Mandelbrot martingales on $\partial\mathsf{T}$, and under the random metrics described below. The simplest extension of Kahane's result consisting in dealing with a single Mandelbrot measure on $\partial\mathsf{T}$ is described in Section 6.1. Let us also mention that our uniform approach is based, in an essential way, on vector martingales theory (see [61] for instance), whose first using in the context of the branching random walk seem to go back to [43], in which the authors consider the case where $N = 2$, X_1 and X_2 are two independent real valued random variables of characteristic function φ, and study the continuous functions valued martingale $t \in I \mapsto Z_n(t) = (2\varphi(t))^{-n} \sum_{u\in\{0,1\}^n} e^{itS_nX(u)}$, in the domain $\{t : 2|\varphi(t)|^2 > 1\}$ (they use the L^2 criterion for convergence) ; they also prove that when X_1 and X_2 are Bernoulli variables taking values 0 and 1 with probability $1/2$, $\#\{u \in \{0,1\}^n : S_nX(u) = 0\} = o(n)$.

Remark 1.1 Theorem 1.2 should be compared to the results of [35] and [63], and Theorem 1.3(1) and Corollary 1.1 to the results of [30], obtained in the context of Birkhoff averages of continuous potentials over a symbolic space endowed with the standard metric.

The space $\mathbb{N}_+^{\mathbb{N}_+}$ can be endowed with other ultrametric distances than d_1. Here we consider the following natural, and classical, generalization of d_1. Instead of $(N, X_1, X_2, \ldots)$ we consider $(N, (X_1, \phi_1), (X_2, \phi_2), \ldots)$ a random vector taking values in $\mathbb{N}_+ \times (\mathbb{R}^d \times \mathbb{R}_+^*)^{\mathbb{N}_+}$.

Then we consider $\{(N_{u0}, (X_{u1}, \phi_{u1}, (X_{u2}, \phi_{u2}), \ldots)\}_{u\in\bigcup_{n\geq 0}\mathbb{N}_+^n}$, a family of independent copies of $(N, (X_1, \phi_1), (X_2, \phi_2), \ldots)$. This yields two branching random walks S_nX and $S_n\phi$. Moreover, if there exists $\gamma > 0$ such that $\mathbb{E}\Big(\sum_{i=1}^N \exp(-\gamma\phi_i)\Big) < 1$ (which happens for instance if $\mathbb{E}\Big(\sum_{i=1}^N \exp(-\gamma\phi_i)\Big) < \infty$ for all $\gamma > 0$ by dominated convergence) then (see Lemma 4.1), with probability 1, $S_n\phi(u)$ tends to ∞ uniformly in $u \in \mathsf{T}_n$, as $n \to \infty$, so that we get the random ultrametric distance

$$d_\phi : (s,t) \mapsto \exp(-S_{|s\wedge t|}\phi(s \wedge t)),$$

on $\partial\mathsf{T}$, and $(\partial\mathsf{T}, d_\phi)$ is compact. Such metrics are used to obtain geometric realization of Mandelbrot measures on random self-similar sets satisafying some separation conditions [27, 62, 2, 17].

For all $(q,t) \in \mathbb{R}^d \times \mathbb{R}$, let

$$S(q,t) = \sum_{i=1}^N \exp(\langle q|X_i\rangle - t\phi_i).$$

Then, since the ϕ_i are positive, for each $q \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}^d$ there exists a unique $t = \widetilde{P}_\alpha(q) \in \mathbb{R}$ such that

$$\mathbb{E}\Big(\sum_{i=1}^{N} \exp(\langle q|X_i - \alpha\rangle - t\phi_i)\Big) = 1.$$

Moreover, it is direct to see that $(\alpha, q) \mapsto P_\alpha(q)$ is analytic by using the implicit function theorem and the analyticity of $(\alpha, q, t) \mapsto \mathbb{E}\Big(\sum_{i=1}^{N} \exp(\langle q|X_i - \alpha\rangle - t\phi_i)\Big)$.

Notice that $\widetilde{P}_\alpha(0)$ does not depend on α. Notice also that when $\phi_i = 1$ for all $i \geq 1$, we have $d_\phi = d_1$, and $\widetilde{P}_\alpha(q) = \widetilde{P}(q) - \langle q|\alpha\rangle$, hence $\widetilde{P}_\alpha^*(0) = \widetilde{P}^*(\alpha)$.

We assume that

$$\forall\ (q, \alpha) \in \mathbb{R}^d \times \mathbb{R}^d \text{ such that } \widetilde{P}_\alpha^*(\nabla \widetilde{P}_\alpha(q)) > 0,\ \exists\ \gamma > 1,\ \mathbb{E}(S(q, \widetilde{P}_\alpha(q))^\gamma) < \infty. \quad (1.6)$$

Theorem 1.2 now takes the following form.

Theorem 1.4 *Suppose that $\partial\mathsf{T}$ is endowed with the distance d_ϕ almost surely. Assume* (1.6).

With probability 1, for all compact connected subset K of $\mathbb{R}^d$, we have $E_X(K) \neq \emptyset$ if and only if $K \subset I$, and in this case

$$\dim E_X(K) = \inf_{\alpha \in K} \widetilde{P}_\alpha^*(0).$$

The analogues, under d_ϕ, of some parts of Theorem 1.3 hold under the same assumptions as in Theorem 1.4, but we need additional assumptions to extend the other ones : we need to assume the following property, which under (1.6) holds for instance if the ϕ_i are uniformly bounded, or if they are independent of N and the X_i and $\sup_{i\geq 1} \mathbb{E}(\phi_i) < \infty$, or if $\sup_{q\in\mathbb{R}^d} \mathbb{E}\Big(\sum_{i=1}^{N} \exp\big(2(\langle q|X_i\rangle - \widetilde{P}(q))\big)\Big) < \infty$:

$$\sup_{q\in\mathbb{R}^d} \mathbb{E}\Big(\sum_{i=1}^{N} \phi_i \exp(\langle q|X_i\rangle - \widetilde{P}(q))\Big) < \infty. \quad (1.7)$$

Under this property (that we have not been able to relax), we have the following fact (the proof is given in Section 4.1) :

Proposition 1.1 *Assume* (1.7). *For each $\alpha \in \mathring{I}$, there exists a unique $q = q_\alpha$ such that $\nabla \widetilde{P}_\alpha(q) = 0$. Moreover, $\alpha \in \mathring{I} \mapsto q_\alpha$ is analytic.*

Theorem 1.3 has the following extension, whose proof is given in Section 5.1.

Theorem 1.5 *Suppose that $\partial\mathsf{T}$ is endowed with the distance d_ϕ almost surely. Assume* (1.6). *With probability 1,*

1. *for all $\alpha \in \mathbb{R}^d$, $E_X(\alpha) \neq \emptyset$ if and only if $\alpha \in I$, and in this case* $\dim E_X(\alpha) = \widetilde{P}_\alpha^*(0)$ *;*

2. *if $\alpha \in I$, then $E_X(\alpha)$ carries uncountably many mutually singular inhomogeneous Mandelbrot measures of Hausdorff dimension $\widetilde{P}^*_\alpha(0)$; if (1.7) holds and $\alpha \in \mathring{I}$, then $E_X(\alpha)$ carries a unique Mandelbrot measure of maximal Hausdorff dimension (namely the measure $\mu_{q_\alpha,\alpha}$ defined in the beginning of Section 4.3) ;*

3. *recall that $\widetilde{P}_\alpha(0)$ does not depend on α. Denote this number by t_0. Let $\alpha_0 = \mathbb{E}\Big(\sum_{i=1}^N X_i \exp(-t_0\phi_i)\Big)$. The level set $E_X(\alpha_0)$ is the unique level set of maximal dimension* $\dim \partial\mathsf{T} = t_0$.

Corollary 1.3 has the following analogue under d_ϕ :

Corollary 1.2 *(Large deviations)Assume* (1.6). *With probability 1, for all $\alpha \in I$,*

$$\lim_{\epsilon\to 0^+} \liminf_{n\to\infty} n^{-1} \log \#\big\{u \in \bigcup_{k\geq 1} \mathsf{T}_k : \mathrm{diam}([u]) = e^{-n}; |u|^{-1} S_{|u|}X(u) \in B(\alpha,\epsilon)\big\}$$
$$= \lim_{\epsilon\to 0^+} \limsup_{n\to\infty} n^{-1} \log \#\big\{u \in \bigcup_{k\geq 1} \mathsf{T}_k : \mathrm{diam}([u]) = e^{-n}; |u|^{-1} S_{|u|}X(u) \in B(\alpha,\epsilon)\big\} = \widetilde{P}^*_\alpha(0).$$

Notice that contrarily to what happens when $\partial\mathsf{T}$ is endowed with d_1, in general the mapping $\alpha \in I \mapsto \dim E_X(\alpha) = \widetilde{P}^*_\alpha(0)$ is not concave when $\partial\mathsf{T}$ is endowed with d_ϕ. For instance, when $X_i = \phi_i$, the distortion induced by d_ϕ can be observed by noting that in this case $\widetilde{P}_\alpha(q) = q - \widetilde{P}^{-1}(\alpha q)$, which implies $\widetilde{P}^*_\alpha(0) = \widetilde{P}^*(\alpha)/\alpha$ for $\alpha \in I$.

Remark 1.2 When $\partial\mathsf{T}$ is endowed with a random metric, our results should be compared to those obtained in [63] (for Theorem 1.4) and [11, 34] in the context of Birkhoff averages on conformal repellers (for Theorem 1.5).

Theorem 1.4 has the following generalization. Instead of $(N,(X_1,\phi_1),(X_2,\phi_2),\ldots)$, consider a random $(N,(X_1,\widetilde{X}_1,\phi_1),(X_2,\widetilde{X}_2,\phi_2),\ldots)$ vector taking values in $\mathbb{N}_+ \times (\mathbb{R}^d \times (\mathbb{R}_+^*)^d \times \mathbb{R}_+^*)^{\mathbb{N}_+}$.

Then we consider $\{(N_{u0},(X_{u1},\widetilde{X}_{u1},\phi_{u1},(X_{u2},\widetilde{X}_{u2},\phi_{u2}),\ldots)\}_{u\in\bigcup_{n\geq 0}\mathbb{N}_+^n}$ a family of independent copies of $(N,(X_1,\widetilde{X}_1,\phi_1),(X_2,\widetilde{X}_2,\phi_2),\ldots)$. If we denote by X^j (resp. $\widetilde{X}^j$) the j-th component of X (resp. $\widetilde{X}$), we can consider the asymptotic behavior of the vector $\left(\frac{S_nX^j(t)}{S_n\widetilde{X}^j(t)}\right)_{1\leq j\leq d}$. The previous situation corresponds to $\widetilde{X}_i^j = 1$ almost surely for all $1 \leq i \leq N$ and $1 \leq j \leq d$.

For all $(q,t) \in \mathbb{R}^d \times \mathbb{R}$, let

$$\widetilde{S}(q,t) = \sum_{i=1}^N \exp(\langle q|\widetilde{X}_i\rangle - t\phi_i).$$

We assume that

$$\forall\ (q,t) \in \mathbb{R}^d \times \mathbb{R},\ \exists\ \gamma > 1,\ \mathbb{E}(S(q,t)^\gamma) + \mathbb{E}(\widetilde{S}(q,t)^\gamma) < \infty. \tag{1.8}$$

If $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ and $u \in \mathbb{R}^d$, we define

$$\alpha \cdot u = (\alpha_1 u_1, \dots, \alpha_d u_d).$$

For each $q \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}^d$ there exists a unique $t = \widetilde{P}_\alpha(q) \in \mathbb{R}$ such that

$$\mathbb{E}\Big(\sum_{i=1}^N \exp(\langle q|X_i - \alpha \cdot \widetilde{X}_i\rangle - t\phi_i)\Big) = 1.$$

Theorem 1.6 *Assume* (1.8). *Suppose that $\partial\mathsf{T}$ is endowed with the distance d_ϕ almost surely. Let $\mathcal{K} = \{\alpha \in \mathbb{R}^d : \widetilde{P}^*_\alpha(0) \geq 0\}$.*

1. *The set $\mathcal{K}$ is a non-empty compact connected set.*

   If K is a compact connected subset of $\mathbb{R}^d$ set

$$E_{X,\widetilde{X}}(K) = \Big\{t \in \partial\mathsf{T} : \bigcap_{N \geq 1} \overline{\Big\{\Big(\frac{S_n X^j(t)}{S_n \widetilde{X}^j(t)}\Big)_{1 \leq j \leq d} : n \geq N\Big\}} = K\Big\}.$$

2. *With probability 1, for every compact connected subset K of $\mathbb{R}^d$, we have $E_{X,\widetilde{X}}(K) \neq \emptyset$ if and only if $K \subset \mathcal{K}$, and in this case $\dim E_{X,\widetilde{X}}(K) = \inf_{\alpha \in K} \widetilde{P}^*_\alpha(0)$.*

Remark 1.3 (1) When K is a singleton, Theorem 1.6 should be compared to the results obtained in [11, 33, 9] for ratios of Birkhoff averages.

(2) Suppose that $d = 1$ to simplify the discussion. The special case $\widetilde{X}_i = \phi_i$ corresponds to study the level sets associated with the averages $S_nX(t)/S_n\phi(t)$, and it is closely related to the multifractal analysis of Mandelbrot measures under d_ϕ. A direct application of the definitions yields $\widetilde{P}_\alpha(q) = \tau(q) - q\alpha$ where $\tau(q)$ is the convex function defined by $\mathbb{E}\Big(\sum_{i=1}^N \exp(qX_i - \tau(q)\phi_i)\Big) = 1$, so $\widetilde{P}^*_\alpha(0) = \widetilde{\tau}^*(\alpha)$, and this situation appears as, and indeed is, similar to the case $\widetilde{X}_i = \phi_i = 1$ considered in Theorem 1.2.

In the general case considered in Theorems 1.4 and 1.5, the introduction of $\widetilde{P}_\alpha(\cdot)$ comes from the remark that the points t such that $S_nX(t)/n - \alpha$ tends to 0, are also those for which $S_n(X - \alpha)(t)/S_n\phi(t)$ tends to 0, hence the study of $E_X(\alpha)$ is reducible to that of the level 0 $S_n(X - \alpha)/S_n\phi$, which, as we just said, is similar to that of the level 0 of S_nX/n when working under the metric d_1. Such an idea seems to come from [34] in the study of Birkhoff averages on conformal repellers.

Theorems 1.2 and 1.4 follow from Theorem 1.6. However, it turns out that the proof of Theorem 1.2 simplifies the understanding of the proof of the other results. So, we will first give the proof of Theorem 1.2 in Chapter 3, then explain in Chapter 4 the (non direct) modifications needed to prove Theorem 1.4, and finally Theorem 1.6.

Now let us come to the refinement of the multifractal analysis announced in the beginning of the introduction. It consists in a better description of the asymptotic behavior $\lim_{n\to\infty} S_nX(t)/n = \alpha$ along the branches distinguished when considering the set $E_X(\alpha)$. Specifically, our goal is, given a sequence $k(n)$ increasing to ∞, to quantify along the walk $S_nX(t)$, how many blocks

$$\Delta S_n X(j,t) = S_{jn}X(t) - S_{(j-1)n}X(t)$$

behave like $n\beta$, with $\beta \neq \alpha$.

Recall the definition of the Mandelbrot measures μ_q given above, which is the most natural measure carried by $E_X(\nabla\widetilde{P}(q))$ with full Hausdorff dimension when $\partial\mathsf{T}$ is endowed with d_1. For each $q \in J$, let

$$\Lambda_q : \lambda \in \mathbb{R}^d \mapsto \log \mathbb{E}\Big(\sum_{i=1}^{N} \exp\left(\langle (q+\lambda|X_i\rangle) - \widetilde{P}(q)\right)\Big) = \widetilde{P}(q+\lambda) - \widetilde{P}(q). \tag{1.9}$$

According to the quantified version of the Erdös-Renyi law of large numbers [25] established in [8] (see specifically [8, section 3.4]) if we define for $t \in \partial\mathsf{T}$, B any Borel subset of $\mathbb{R}^d$ and any $\lambda \in \mathbb{R}^d$

$$\mu_n^t(B) = \frac{\#\Big\{1 \leq j \leq k(n) : \dfrac{\Delta S_n X(j,t)}{n} \in B\Big\}}{k(n)}$$

and

$$\Lambda_n^t(\lambda) = \log \int_{\mathbb{R}^d} \exp(n\langle \lambda | x\rangle) \mathrm{d}\mu_n^t(x), \tag{1.10}$$

then, for all $q \in J$, with probability 1, for μ_q-almost every t, $\lim_{n\to\infty} S_nX(t)/n = \nabla\widetilde{P}(q) = \nabla\Lambda_q(0)$, i.e. $t \in E_X(\nabla\Lambda_q(0))$, and the following large deviations properties hold (notice that by convention the concave Legendre transform defined in this paper, which is convenient to express Hausdorff dimensions, is the opposite of the more standard convex convention used in [8]) :

LD(q) :

(1) for all $\lambda \in \mathbb{R}^d$ such that $\liminf_{n\to\infty} \frac{\log(k(n))}{n} > -\Lambda_q^*(\nabla\Lambda_q(\lambda))$, we have

$$\lim_{n\to\infty} \frac{1}{n}\Lambda_n^t(\lambda) = \Lambda_q(\lambda);$$

due to the Gartner-Ellis theorem [24], this implies that for all $\lambda \in \mathbb{R}^d$ such that $\liminf_{n\to\infty} \frac{\log(k(n))}{n} > -\Lambda_q^*(\nabla\Lambda_q(\lambda))$,

$$\lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \log \mu_n^t(B(\nabla\Lambda(\lambda),\epsilon)) = \Lambda_q^*(\nabla\Lambda_q(\lambda)).$$

(2) For all $\lambda \in \mathbb{R}^d$ such that $\limsup_{n\to\infty} \frac{\log(k(n))}{n} < -\Lambda_q^*(\nabla\Lambda_q(\lambda))$, there exists $\epsilon > 0$ such that for n large enough, $\Big\{1 \leq j \leq k(n) : \frac{\Delta S_n X(j,t)}{n} \in B(\nabla\Lambda_q(\lambda),\epsilon)\Big\} = \emptyset$.

In particular, for all $q \in J$, if we define

$$E_X^{\mathrm{LD}(q)}(\nabla\widetilde{P}(q)) = \{t \in E_X(\nabla\widetilde{P}(q)) : \mathrm{LD}(q) \text{ holds}\},$$

with probability 1, $\mu_q(E_X^{\mathrm{LD}(q)}(\nabla\widetilde{P}(q)) > 0$, hence we have the following fact.

Proposition 1.2 *Suppose that $\partial\mathsf{T}$ is endowed with the distance d_1 almost surely. Assume* (1.5)*. For all $q \in J$, with probability 1,* $\dim E_X^{\mathrm{LD}}(\nabla\widetilde{P}(q)) = \widetilde{P}^*(\nabla\widetilde{P}(q))$.

It turns out that to strengthen this result into a result valid simultaneously for all $q \in J$, the simultaneous consideration of the measures μ_q, together with a combination of the approach developed in Chapter 2 for the simultaneous study of the sets $E_X(\nabla\widetilde{P}(q))$ and the approach of [8] to get the previous proposition seems to be not sufficient. However, the family of inhomogeneous Mandelbrot measures that we will consider is adapted to get the strong version. Notice that the previous result only deals with the level sets $E_X(\alpha)$ with $\alpha \in \mathring{I}$. Indeed, when $\alpha \in \partial I \cap \nabla\widetilde{P}(\mathbb{R}^d)$, the function Λ_q is well defined as well as $\mathrm{LD}(q)$, but our approach fails. When $\alpha \in \partial I \setminus \nabla\widetilde{P}(\mathbb{R}^d)$, there is no natural substitute to $\mathrm{LD}(q)$.

We also want such a strong version for any random metric d_ϕ. For this we assume (1.7). In this case, according to Proposition 1.1, if $\alpha \in \mathring{I}$ we can build the measure $\mu_{q_\alpha,\alpha}$ of Theorem 1.5(2), which is carried by $E_X(\alpha)$ and of maximal Hausdorff dimension.

For $\alpha \in \mathring{I}$ let

$$\Lambda_{q_\alpha,\alpha} : \lambda \in \mathbb{R}^d \mapsto \log \mathbb{E}\Big(\sum_{i=1}^N \exp(\langle\lambda|X_i\rangle + \langle q_\alpha|X_i - \alpha\rangle - \widetilde{P}_\alpha(q_\alpha)\phi_i)\Big). \tag{1.11}$$

When $\phi_i = 1$ for all $i \geq 1$, this corresponds to (1.9). Given an increasing sequence of positive integers $(k(n))_{n\geq 1}$, for any $\alpha \in \mathring{I}$, define for any $t \in \mathbb{N}_+^{\mathbb{N}_+}$ the large deviation property (which given α, due to [8] applied to our context, is pointed out almost surely, $\mu_{q_\alpha,\alpha}$-almost everywhere)

$\mathrm{LD}(q_\alpha, \alpha)$:

(1) for all $\lambda \in \mathbb{R}^d$ such that $\liminf_{n\to\infty} \frac{\log(k(n))}{n} > -\Lambda^*_{q_\alpha,\alpha}(\nabla\Lambda_{q_\alpha,\alpha}(\lambda))$, we have

$$\lim_{n\to\infty} \frac{1}{n}\Lambda_n^t(\lambda) = \Lambda_{q_\alpha,\alpha}(\lambda),$$

so that for all $\lambda \in \mathbb{R}^d$ such that $\liminf_{n\to\infty} \frac{\log(k(n))}{n} > -\Lambda^*_{q_\alpha,\alpha}(\nabla\Lambda_{q_\alpha,\alpha}(\lambda))$,

$$\lim_{\epsilon\to 0}\lim_{n\to\infty} \frac{1}{n}\log\mu_n^t(B(\nabla\Lambda(\lambda),\epsilon)) = \Lambda^*_{q_\alpha,\alpha}(\nabla\Lambda_{q_\alpha,\alpha}(\lambda)).$$

(2) For all $\lambda \in \mathbb{R}^d$ such that $\limsup_{n\to\infty} \frac{\log(k(n))}{n} < -\Lambda^*_{q_\alpha,\alpha}(\nabla\Lambda_{q_\alpha,\alpha}(\lambda))$, there exists $\epsilon > 0$ such that for n large enough, $\Big\{1 \leq j \leq k(n) : \frac{\Delta S_n X(j,t)}{n} \in B(\nabla\Lambda_{q_\alpha,\alpha}(\lambda),\epsilon)\Big\} = \emptyset$.

Now define

$$E_X^{\mathrm{LD}(q_\alpha,\alpha)}(\alpha) = \{t \in E_X(\alpha) : \mathrm{LD}(q_\alpha,\alpha)) \text{ holds}\}.$$

We will prove in Section 5.2 the following result.

Theorem 1.7 *Assume* (1.6) *and* (1.7)*. Suppose that $\partial\mathsf{T}$ is endowed with the distance d_ϕ almost surely. With probability 1, for all $\alpha \in \mathring{I}$,* $\dim E_X^{\mathrm{LD}(q_\alpha,\alpha)}(\alpha) = \widetilde{P}^*_\alpha(0)$.

It is worth mentioning that changing the metric has changed the large deviations principle $\mathrm{LD}(q)$ into $\mathrm{LD}(q_\alpha,\alpha)$, so that the previous result says nothing about $\dim E_X^{\mathrm{LD}(q)}(\alpha)$ under d_ϕ when $\alpha=\nabla\widetilde{P}(q)$ if ϕ is not a multiple of $(1)_{i\geq 1}$. Then a natural related question is : is it possible to find in $E_X(\alpha)$ other subsets of full Hausdorff dimension $\widetilde{P}_\alpha^*(0)$, each corresponding to a different large deviations principle for $(\mu_n^t)_{n\geq 1}$ holding over its infinite branches ?

It is now interesting to try precising the information we obtained about Hausdorff dimension of the sets $E_X^{\mathrm{LD}(q_\alpha,\alpha)}(\alpha)$ by information concerning their Hausdorff measure when the gauge function varies. Our approach provides a 0-∞ law for the sets $E_X^{\mathrm{LD}(q_\alpha,\alpha)}(\alpha)$ subject to $\dim E_X^{\mathrm{LD}(q_\alpha,\alpha)}(\alpha)<\dim\partial\mathsf{T}$. Indeed the level set $E_X(\alpha_0)$ of maximal Hausdorff dimension has a particular status because it carries a Mandelbrot measure of maximal Hausdorff dimension on $(\partial\mathsf{T},d_\phi)$, and the behavior of its Hausdorff measures turns out to differ from that of the other sets $E_X(\alpha)$, because it is closely related to that of $\partial\mathsf{T}$. We refer the reader to [37, 53, 73] for the study of the Hausdorff measures of $\partial\mathsf{T}$. We notice that in the deterministic case, such a 0-∞ law has been obtained in [58] when $d=1$ for the sets $E_X(\alpha)$ seen as Besicovich subsets of the attractor of an IFS of contractive similtudes of $\mathbb{R}$ satisfying the open set condition. Our result, which is proved in Section 5.3, is the following.

Theorem 1.8 *Assume* (1.6) *and* (1.7)*. Suppose that $\partial\mathsf{T}$ is endowed with the distance d_ϕ almost surely. With probability 1, for all $\alpha\in\mathring{I}$ such that $\dim E<\dim\partial\mathsf{T}$, if $E\in\{E_X(\alpha),E_X^{\mathrm{LD}(q_\alpha,\alpha)}(\alpha)\}$, for all gauge functions g, $\mathcal{H}^g(E)=\infty$ if $\limsup_{t\to 0^+}\log(g(t))/\log(t)\leq\dim E$ and $\mathcal{H}^g(E)=0$ otherwise.*

Let us give some simple consequences of our study. The first one concerns the branching process itself. Assuming that the moment generating function of the random variable N is finite over $\mathbb{R}$, the previous results apply to the natural branching random walk associated with the branching numbers, namely $S_nN(t)=N_{t_1}+N_{t_1t_2}+\cdots+N_{t_1\cdots t_n}$ and provide, if N is not constant, geometric and large deviations information on the heterogeneity of the birth process along different infinite branches. The same kind of information can be derived for the branching random walk obtained from an homogeneous percolation process : the X_u, $u\in\bigcup_{n\geq 1}\mathbb{N}_+^n$ are independent copies of the same Bernoulli variable, and are independent of $\partial\mathsf{T}$. The branching random walk $S_nX(t)$ must be interpreted as the covering number of t by the family of balls $[u]$ of generation not greater than n such that $X_u=1$ (see [31] for related results concerning some subsets of the sets $E_X(\alpha)$ on a dyadic tree). Then, our results also provide for instance a joint multifractal analysis of $S_nN(t)$ and $S_nX(t)$.

1.2 Some comments and perspectives

1.2.1 Possible relaxation of the assumptions

We have deliberately assumed strong assumptions like (1.6) on the moment generating functions of the random walks to be sure to have a whole compact domain I of study at our disposal, and be in a situation comparable to that of the context of

Birkhoff averages of continuous potentials on conformal repellers, where this domain is always compact, the pressure functions (analogues of $\widetilde{P}$), being always defined everywhere, and with bounded subgradiant, because the potentials are bounded. In our context, the analogous property would amount to assume that the random variables X_i and ϕ_i, $1 \le i \le N$ are bounded, which is rather restrictive, and in this sense the assumptions of the introduction are not so unsatisfactory. However they can be relaxed.

For Theorem 1.1, see Section 2.3.

For Theorem 1.2, the point is to consider the domain of the convex function $\widetilde{P}$ as possibly smaller than $\mathbb{R}^d$. We will assume that it has a non-empty interior. This makes it possible to define $J = \{q \in \mathrm{dom}(\widetilde{P}) :\ \nabla\widetilde{P}(q) \text{ exists and } \widetilde{P}^*(\nabla\widetilde{P}(q)) > 0\}$. Then, if we assume that for all $q \in J$ there exists $\gamma > 1$ such that $\mathbb{E}(S(q)^\gamma) < \infty$, the conclusions of Theorem 1.2 hold, except that I_X can cease to be compact (the other conclusions of Proposition 3.1 about I_X remain valid). Otherwise, we can consider $\widetilde{J} = \{q \in J :\ \exists\, \gamma > 1 : \mathbb{E}(S(q)^\gamma) < \infty\}$ (it equals J if N is bounded), and then the conclusions of Theorem 1.2 hold for the compact connected sets of points α of I_X for which there exists a sequence $(q_n)_{n\ge1}$ of points in $\widetilde{J}$ such that $\nabla\widetilde{P}(q_n) \to \alpha$, $\widetilde{P}^*(\nabla\widetilde{P}(q_n)) > 0$ and $\widetilde{P}^*(\nabla\widetilde{P}(q_n)) > 0$ as $n \to \infty$. Theorem 1.3 is also valid under these assumptions.

For instance, if we consider for the X_i i.i.d positive vectors with d i.i.d. α-stable components ($\alpha \in (0,1)$), and independent of N, there exists $c > 0$ such that $\widetilde{P}(q) = \log(\mathbb{E}(N)) - c\sum_{i=1}^d |q_i|^\alpha$ for $q \in (-\infty, 0]^d$ and $\widetilde{P}(q) = +\infty$ for $q \in \mathbb{R}^d \setminus (-\infty, 0]^d$, and we find $I_X = \overline{\{c\alpha(|q_i|^{\alpha-1})_{1\le i\le d} : \log(\mathbb{E}(N)) - c(1-\alpha)\sum_{i=1}^d |q_i|^\alpha > 0 \text{ and } q \in (-\infty,0]^d\}}$.

For Theorems 1.4 the following condition is a bit technical because closely connected to the heart of the proof, but in the spirit of the previous lines. In general, the results are valid for the compact connected sets of points $\alpha \in I$ for which we can find a sequence $(q_n, \alpha_n)_{n\ge1}$ in the interior of $\Big\{(q,\alpha) \in \mathbb{R}^d \times \mathbb{R}^d : \widetilde{P}_\alpha(q) \text{ and } \nabla\widetilde{P}_\alpha(q) \text{ are defined, } \widetilde{P}_\alpha(q) - \langle q|\nabla\widetilde{P}_\alpha(q)\rangle > 0 \text{ and } \exists\, \gamma > 1 : \mathbb{E}\Big(\Big(\sum_{i=1}^N \exp(\langle q|X_i - \alpha\cdot\widetilde{X}_i\rangle - \widetilde{P}_\alpha(q)\phi_i)\Big)^\gamma\Big) < \infty\Big\}$, such that $\alpha_n \to \alpha$, $\nabla\widetilde{P}_{\alpha_n}(q_n) \to 0$, and $\widetilde{P}_\alpha(q_n) - \langle q_n|\nabla\widetilde{P}_\alpha(q_n)\rangle \to \widetilde{P}^*_\alpha(0)$, as $n \to \infty$. Similar conditions can be stated in relation with Theorem 1.8.

The validity of Theorems 1.7 and 1.8 is based on Proposition 1.1, which already requires the strong assumption (1.7). Nevertheless, under our assumptions and without assuming (1.7), it is true that $\alpha \mapsto q_\alpha$ is well defined and analytic in a neighborhood of α_0, because q_{α_0} is clearly identified as being 0, and one can apply the implicit function theorem. However, we were unable to prove that this neighborhood can be extended to $\mathring{I}$ in general.

1.2.2 Possible analogous results for the packing dimension

For all the sets $E_X(\alpha)$ and $E_X^{\mathrm{LD}(q_\alpha,\alpha)}(\alpha)$, it is clear that packing and Hausdorff dimension coincide, due to the self-similarity properties.

For the connected compact sets K of I_X, one expects $\dim_P K = \sup_{\alpha\in K} \widetilde{P}^*_\alpha(0)$. In fact, our approach as it is developed may provide this result for K in the interior of I_X, and also an extension of Theorem 1.8 for packing measures, with a 0-∞ law related to $\liminf_{r\to 0^+} \log(g(r))/\log(r)$. However, we meet some difficulties to control what happens when K touches the boundary of I_X (also for the Hausdorff measures of the sets $E_X(\alpha)$ when $\alpha \in \partial I_X$). This is due to an not sufficient control of the right tail at of the supremum of the total masses of the collection of inhomogeneous martingales we build. However, we realized these last days that the solution may come from the adaptation of some ideas in [14], in which case we would also have an alternative to the uniform Kahane's argument.

1.2.3 Some other perspectives

We briefly mention here that we strongly believe that our method is suitable to solve related questions like (1) the multifractal analysis of Lyapunov exponents of products of independent matrices on Galton-Watson trees ; (2) strengthening the multifractal analysis of statistically self-similar measures obtained in [17] under the open set condition, which does not obtain almost surely the whole spectrum but only its value for each fixed singularity almost surely. We think inhomogeneous Mandelbrot measures may introduce flexibility to control some delicate geometric aspects of the problem. Finally, it would be interesting to study the possibility of combining our approach with that of [51], to improve the partial results obtained there about the joint multifractal analysis of the branching and the visibility measures.

Chapitre 2

Uniform result for the sets $E_X(\alpha)$ when $\alpha \in \mathring{I}_X$, and under the metric d_1

Recall that $J = \{q \in \mathbb{R}^d : \widetilde{P}^*(\nabla\widetilde{P}(q)) > 0\}$. We prove Theorem 1.1 by exploiting the simultaneous construction of the Mandelbrot measures μ_q, $q \in J$. We will focus on the lower bounds for the Hausdorff dimensions of the sets $E_X(\alpha)$, $\alpha \in \nabla\widetilde{P}(J)$, only, since the upper bounds estimates follow the same lines as in Chapter 3. At the end of the chapter, we will give some partial answers to the calculation of $\dim E_X(\alpha)$ for $\alpha \in \partial I$, and we also give some information about the pressure like function

$$P(q) = \limsup_{n\to\infty} \frac{1}{n} \log\Big(\sum_{u\in\mathsf{T}_n} \exp(\langle q|S_nX(u)\rangle)\Big) \quad (q \in \mathbb{R}^d)$$

used in Section 3.2 to get general upper bounds for the sets $E_X(K)$. The main information we collect from Section 3.2 is that, with probability 1, for all $q \in \mathbb{R}^d$ we have $P(q) \leq \widetilde{P}(q)$, and for all $\alpha \in \mathbb{R}^d$, $\dim E_X(\alpha) \leq P^*(\alpha) \leq \widetilde{P}^*(\alpha)$.

2.1 Lower bounds for the Hausdorff dimensions

For $(q,p) \in J \times [1,\infty)$, we define the function

$$\phi(p,q) = \exp(\widetilde{P}(pq) - p\widetilde{P}(q)).$$

and for $q \in J$ and $u \in T$, we define the sequence

$$Y_n(u,q) = \mathbb{E}\Big(\sum_{i=1}^{N} e^{\langle q|X_i\rangle}\Big)^{-n} \sum_{v\in\mathsf{T}_n(u)} e^{\big\langle q|S_{|u|+n}X(uv)-S_{|u|}(u)\big\rangle}, \quad (n \geq 1).$$

When $u = \emptyset$, $Y_n(\emptyset, q)$ will be denoted by $Y_n(q)$.
The sequence $\big(Y_n(u,q)\big)_{n\geq 1}$ is a positive martingale with expectation 1, which converges almost surely and in L^1 norm to a positive random variable $Y(u,q)$ (see [45, 13] or [16, Theorem 1]). However, our study will need the almost sure simultaneous convergence of these martingales to positive limits (see Proposition 2.1(1)).

Let us start by stating two propositions, the proof of which is postponed to the end of this section. The uniform convergence part of Proposition 2.1 is essentially Theorem 2 of [16], with slightly different assumptions. However, for the reader convenience, and since the method used by Biggins will be used also in proving Propositions 2.2 and 2.3, we will include its proof. The second part of Proposition 2.1 defines the family of Mandelbrot measures built simultaneously to control the Hausdorff dimensions of the sets $E_X(\nabla\widetilde{P}(q))$, $q \in J$, from below. Then Proposition 2.2 introduces suitable logarithmic moment generating functions associated with these measures to get the desired lower bounds via large deviations inequalities.

Proposition 2.1 *1. Let K be a compact subset of J. There exists $p_K \in (1,2]$ such that for all $u \in \bigcup_{n\geq 0}\mathbb{N}_+^n$, the continuous functions $q \in K \mapsto Y_n(u,q)$ converge uniformly, almost surely and in L_{p_K} norm, to a limit $q \in K \mapsto Y(u,q)$. In particular, $\mathbb{E}(\sup_{q\in K} Y(u,q)^{p_K}) < \infty$. Moreover, $Y(u,\cdot)$ is positive almost surely.*

In addition, for all $n \geq 0$, $\sigma\big(\{(X_{u1},\cdots,X_{uN(u)}), u \in \mathsf{T}_n\}\big)$ and $\sigma\big(\{Y(u,\cdot), u \in \mathsf{T}_{n+1}\}\big)$ are independent, and the random functions $Y(u,\cdot), u \in \mathsf{T}_{n+1}$, are a independent copies of $Y(\cdot)$.

2. With probability 1, for all $q \in J$, the function

$$\mu_q([u]) = \mathbb{E}\Big(\sum_{i=1}^{N} e^{\langle q|X_i\rangle}\Big)^{-|u|} e^{\langle q|S_{|u|}(u)\rangle} Y(u,q)$$

defines a measure on $\partial\mathsf{T}$.

For $q \in J$, let

$$L_n(q,\lambda) = \frac{1}{n}\log\int_{\partial\mathsf{T}} \exp\big(\langle\lambda|S_nX(t)\rangle\big)d\mu_q(t), \quad (\lambda \in \mathbb{R}^d),$$

and

$$L(q,\lambda) = \limsup_{n\to\infty} L_n(q,\lambda).$$

Proposition 2.2 *Let K be a compact subset of J. There exists a compact neighborhood Λ of the origin such that, with probability 1,*

$$\lim_{n\to\infty}\sup_{\lambda\in\Lambda}\sup_{q\in K}|L_n(q,\lambda) - (\widetilde{P}(q+\lambda) - \widetilde{P}(q))| = 0, \tag{2.1}$$

in particular $L(q,\lambda) = \widetilde{P}(q+\lambda) - \widetilde{P}(q)$ for $(q,\lambda) \in K \times \Lambda$.

Corollary 2.1 *With probability 1, for all $q \in J$, for μ_q-almost every $t \in \partial\mathsf{T}$,*

$$\lim_{n\to\infty}\frac{S_nX(t)}{n} = \nabla\widetilde{P}(q).$$

Proof It follows from Proposition 2.2 that there exists $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$, and such that for all $\omega \in \Omega'$, for all $q \in J$, there exists a neighborhood of 0 over which $L_n(q,\lambda)$ converges uniformly in λ towards $L(q,\lambda) = \widetilde{P}(q+\lambda) - \widetilde{P}(q)$.

For each $\omega \in \Omega'$, let us define for each $q \in J$ the sequence of measures $\{\mu^\omega_{q,n}\}_{n\geq 1}$ as $\mu^\omega_{q,n}(B) = \mu^\omega_q(\{t \in \partial\mathsf{T} : \frac{1}{n}S_nX(t) \in B\})$ for all Borel set $B \subset \mathbb{R}^d$. We denote $L(q,\lambda)$ by $L_q(\lambda)$. Since

$$L_n(q,\lambda) = \frac{1}{n}\log\int_{\mathbb{R}^d}\exp(n\langle\lambda|u\rangle)\,d\mu^\omega_{q,n}(u),$$

it is known that for all closed subset Γ of $\mathbb{R}^d$, we have for all $q \in J$

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu^\omega_{q,n}(\Gamma) \leq \sup_{\alpha\in\Gamma} L^*_q(\alpha).$$

Let $\epsilon > 0$, and for each $q \in J$ let $A_{q,\epsilon} = \big\{\alpha \in \mathbb{R}^d : d(\alpha, \nabla L_q(0)) > \epsilon)\big\}$. In particular we have $\limsup_{n\to\infty}\frac{1}{n}\log\mu^\omega_{q,n}(A_{q,\epsilon}) \leq \sup_{\alpha\in A_{q,\epsilon}} L^*_q(\alpha)$. In addition, since $L_q(\lambda) = \widetilde{P}(q+\lambda) - \widetilde{P}(q)$ in a neighborhood of 0, we have $\nabla L_q(0) = \nabla\widetilde{P}(q)$ and $L^*_q(\nabla L_q(0)) = 0 = \max L^*_q$. Moreover, since L_q is differentiable at 0, we have $L^*_q(\alpha) < L^*_q(\nabla L_q(0))$ for all $\alpha \neq \nabla L_q(0)$. Indeed, suppose that $L^*_q(\alpha) = 0$; then it follows from the definition of the Legendre transformation and the fact that $L_q(0) = 0$, that

$$\forall\lambda \in \mathbb{R}^d, \ \ L_q(\lambda) \geq L_q(0) + \langle\lambda|\alpha\rangle\,,$$

hence α belongs to the subgradient of L_q at 0, which reduces to $\{\nabla L_q(0)\}$ since L_q is differentiable at 0.

Now, due to the upper semi-continuity of the concave function L^*_q, we have $\gamma_{q,\epsilon} = \sup_{\alpha\in A_{q,\epsilon}} L^*_q(\alpha) < 0$.

Consequently, for all $q \in J$, for n large enough, $\mu^\omega_{q,n}(A_{q,\epsilon}) \leq e^{n\gamma_{q,\epsilon}/2}$, i.e.

$$\mu^\omega_q\big(\big\{t \in \partial\mathsf{T} : \frac{1}{n}S_nX(t) \in A_{q,\epsilon}\big\}\big) \leq e^{n\gamma_{q,\epsilon}/2}.$$

Then it follows from the Borel-Cantelli Lemma (applied with respect to μ^ω_q) that for all $q \in J$, for μ^ω_q-almost every $t \in \partial\mathsf{T}$, we have $\frac{1}{n}S_nX(t) \in B(\nabla\widetilde{P}(q),\epsilon)$ for n large enough. Letting ϵ tend to 0 along a countable sequence yields the desired conclusion.

The next corollary uses the notations of the previous proof.

Corollary 2.2 *With probability 1, for all $q \in J$, the sequence of random measure $(\mu^\omega_{q,n})_{n\geq 1}$ satisfies the following large deviation property : for all λ in a neighborhood of* 0*,*

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\log\mu^\omega_{q,n}\big(B(\nabla L_q(\lambda),\epsilon)\big) = L^*_q(\nabla L_q(\lambda)).$$

Proof It is a consequence of Gartner Ellis theorem (see [22]).

We need a last proposition to get the lower bounds in Theorem 1.1. Its proof will end the section.

Proposition 2.3 *With probability 1, for all $q \in J$, for μ_q-almost every $t \in \partial\mathsf{T}$,*

$$\lim_{n\to\infty} \frac{\log Y(t_{|n}, q)}{n} = 0.$$

Proof of the lower bounds in Theorem 1.1 : From Corollary 2.1, we have with probability 1, $\mu_q\big(E_X(\nabla\widetilde{P}(q))\big) = 1$. In addition, with probability 1, for μ_q-almost every $t \in E(\nabla\widetilde{P}(q))$, from the same corollary and Proposition 2.3, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log(\mu_q[t_{|n}])}{\log(\mathrm{diam}([t_{|n}]))} &= \lim_{n\to\infty} \frac{-1}{n} \log\Big(\exp\big(\langle q|S_nX(t)\rangle - n\widetilde{P}(q)\big) Y(t_{|n}, q)\Big)\\
&= \widetilde{P}(q) + \lim_{n\to\infty} \frac{\langle q|S_nX(t)\rangle}{-n} + \lim_{n\to\infty} \frac{\log Y(t_{|n}, q)}{n}\\
&= \widetilde{P}(q) - \big\langle q|\nabla\widetilde{P}(q)\big\rangle = \widetilde{P}^*(\nabla\widetilde{P}(q)).
\end{aligned}$$

This implies that the lower Hausdorff dimension of μ_q is equal to $\widetilde{P}^*(\nabla\widetilde{P}(q))$, hence $\dim E_X(\nabla\widetilde{P}(q) \geq E(\nabla\widetilde{P}(q)$.

Before giving the proofs of the previous propositions, we will first recall the Cauchy formula for holomorphic functions in several variables.

Definition 2.1 *Let $d \geq 1$. Asubset D of $\mathbb{C}^d$ is an open polydisc if there exist open discs $D_1, ..., D_d$ of $\mathbb{C}$ such that $D = D_1 \times ... \times D_d$. If we denote by ζ_j the centre of D_j, then $\zeta = (\zeta_1, ..., \zeta_d)$ is the centre of D and if r_j is the radius of D_j then $r = (r_1, ..., r_d)$ is the multiradius of D. The set $\partial D = \partial D_1 \times ... \times \partial D_d$ is the distinguished boundary of D. We denote by $D(\zeta, r)$ the polydisc with center ζ and radius r.*

Let $D = D(\zeta, r)$ be a polydisc of $\mathbb{C}^d$ and $g \in C(\partial D)$ a continuous function on ∂D. We define the integral of g on ∂D as

$$\int_{\partial D} g(\zeta) d\zeta_1...d\zeta_d = (2i\pi)^d r_1...r_d \int_{[0,1]^d} g(\zeta(\theta)) e^{i2\pi\theta_1}...e^{i2\pi\theta_d} d\theta_1...d\theta_d,$$

where $\zeta(\theta) = (\zeta_1(\theta), ..., \zeta_d(\theta))$ and $\zeta_j(\theta) = \zeta_j + r_j e^{i2\pi\theta_j}$ for $j = 1, ..., d$.

Theorem 2.1 *Let $D = D(a, r)$ be polydisc in $\mathbb{C}^d$ with a multiradius whose components are positive, and f be a holomorphic function in a neiborhood of D. Then, for all $z \in D$*

$$f(z) = \frac{1}{(2i\pi)^d} \int_{\partial D} \frac{f(\zeta) d\zeta_1...d\zeta_d}{(\zeta_1 - z_1)...(\zeta_d - z_d)}.$$

It follows that

$$\sup_{z\in D(a,r/2)} |f(z)| \leq 2^d \int_{[0,1]^d} |f(\zeta(\theta))| \, d\theta_1...d\theta_d \tag{2.2}$$

Now, we give the proofs of the previous propositions. We start by reminding that we assumed (1.4) :

$$\exists \;\; q \in J \mapsto p_q \in (1, \infty), \;\; C^0 \text{ and such that } \mathbb{E}\big(S(q)^{p_q}\big) < \infty \text{ for all } q \in J.$$

Next we establish several lemmas.

Lemma 2.1 *For all nontrivial compact $K \subset J$ there exists a real number $1 < p_K < 2$ such that for all $1 < p \le p_K$ we have*

$$\sup_{q\in K} \phi(p_K, q) < 1.$$

Proof Let $q \in J$, since $\widetilde{P}^*(\nabla \widetilde{P}(q)) > 0$ one has $\frac{\partial \phi}{\partial p}(1^+, q) < 0$ and there exists $p_q > 1$ such that $\phi(p_q, q) < 1$. Therefore, in a neighborhood V_q of q, one has $\phi(p, q') < 1$ for all $q' \in V_q$. If K is a nontrivial compact of J, it is covered by a finite number of such V_{q_i} and for all $q \in K$ one has $\phi(p_K, q) < 0$ with $p_K = \inf_i p_{q_i}$. Now, if $1 < p \le p_K$ and $\sup_{q\in K} \phi(p, q) = 1$, there exists $q \in K$ tel que $\varphi(p, q) = 1$. By log-convexity of the mapping $p \mapsto \phi(p, q)$ and the fact that $\phi(1, q) = 1$, there is a contradiction.

Lemma 2.2 *For all compact $K \subset J$, there exists $\widetilde{p}_K > 1$ such that,*

$$\sup_{q\in K} \mathbb{E}\Big(\big(\sum_{i=1}^{N} e^{\langle q|X_i\rangle}\big)^{\widetilde{p}_K}\Big) < \infty.$$

Proof Since $q \mapsto p_q$ is continuous, there exists $\widetilde{p}_K > 1$ such that we have

$$\mathbb{E}\Big(\big(\sum_{i=1}^{N} e^{\langle q|X_i\rangle}\big)^{\widetilde{p}_K}\Big) < \infty \text{ for all } q \in K.$$

Then the result follows from the same continuity argument as in [16, p. 141].

The next lemma comes from [16].

Lemma 2.3 *If $\{X_i\}_{i\in I}$ is a finite family of integrable and independent complex random variables with $\mathbb{E}(X_i) = 0$, then $\mathbb{E}|\sum_{i\in I} X_i|^p \le 2^p \sum \mathbb{E}|X_i|^p$ for $1 \le p \le 2$.*

Lemma 2.4 *Let $(N, V_1, V_2, \cdots)$ be a random vector taking values in $\mathbb{N}_+ \times \mathbb{C}^{\mathbb{N}_+}$ and such that $\sum_{i=1}^N V_i$ is integrable and $\mathbb{E}\big(\sum_{i=1}^N V_i\big) = 1$. Let M be an integrable complex random variable. Consider $\big\{(N_u, V_{u1}, V_{u2}, \ldots)\big\}_{u\in\bigcup_{n\ge 0}\mathbb{N}_+^n}$ a sequence of independent copies of $(N, V_1, \cdots, V_N)$ and $\{M_u\}_{u\in\bigcup_{n\ge0}\mathbb{N}_+^n}$ a sequence of copies of M such that for all $n \ge 1$, the random variables $M(u)$, $u \in \mathbb{N}_+^n$, are independent, and independent of $\big\{(N_u, V_{u1}, V_{u2}, \ldots)\big\}_{u\in\bigcup_{k=0}^{n-1}\mathbb{N}_+^k}$. We define the sequence $(Z_n)_{n\ge0}$ by $Z_0 = \mathbb{E}(M)$ and for $n \ge 1$*

$$Z_n = \sum_{u\in \mathsf{T}_n} \big(\prod_{k=1}^{n} V_{u_{|k}}\big) M(u).$$

Let $p \in (1, 2]$. There exists a constant C_p depending on p only such that for all $n \ge 1$

$$\mathbb{E}(|Z_n - Z_{n-1}|^p) \le C_p \mathbb{E}(|M|^p)\Big(\mathbb{E}\big(\sum_{i=1}^{N}|V_i|^p\big)\Big)^{n-1}\Big(\mathbb{E}\big(\sum_{i=1}^{N}|V_i|^p\big) + \mathbb{E}\big(|\sum_{i=1}^{N} V_i|^p\big) + 1\Big).$$

Proof The definition of the process Z_n gives immediately

$$Z_n - Z_{n-1} = \sum_{u\in \mathsf{T}_{n-1}} \prod_{k=1}^{n-1} V_{u_{|k}} \Big(\sum_{i=1}^{N_u} V_{ui} M(ui) - M(u) \Big). \tag{2.3}$$

For each $n \geq 1$ let $\mathcal{F}_n = \sigma\big\{(N_u, V_{u1}, \dots) : |u| \leq n-1\big\}$ and let $\mathcal{F}_0$ be the trivial sigma-field. The random variable $Z_n - Z_{n-1}$ is a weighted sum of independent and identically distributed random variables with zero mean, namely the random variables $\sum_{i=1}^{N_u} V_{ui} M(ui) - M(u)$, which are independent of $\mathcal{F}_{n-1}$. Applying the Lemma 2.3 with

$$X_u = \prod_{k=1}^{n-1} V_{u_{|k}} \Big(\sum_{i=1}^{N_u} V_{ui} M(ui) - M(u) \Big), \ \ (u \in \mathsf{T}_n),$$

given $\mathcal{F}_{n-1}$, and noticing that the weights $\prod_{k=1}^{n-1} V_{u_{|k}}$, $u \in \mathsf{T}_{n-1}$, are $\mathcal{F}_{n-1}$-measurable, we get

$$\begin{aligned} \mathbb{E}\big(|Z_n - Z_{n-1}|^p\big) &= \mathbb{E}\Big(\mathbb{E}\big(|Z_n - Z_{n-1}|^p\big) \mid \mathcal{F}_{n-1}\Big) \\ &\leq \mathbb{E}\Big(2^p \sum_{u\in \mathsf{T}_{n-1}} \prod_{k=1}^{n-1} |V_{u_{|k}}|^p \mathbb{E}\Big| \sum_{i=1}^{N_u} V_{ui} M(ui) - M(u)\Big|^p\Big). \end{aligned}$$

It is easy to see that $\mathbb{E}\big(\sum_{u\in \mathsf{T}_{n-1}} \prod_{k=1}^{n-1} |V_{u_{|k}}|^p\big) = \prod_{k=1}^{n-1} \mathbb{E}\big(\sum_{i=1}^{N} |V_i|^p\big)$. Using the inequality

$$|x+y|^r \leq 2^{r-1}(|x|^r + |y|^r), \quad (r > 1), \tag{2.4}$$

we get

$$\mathbb{E}\Big(\Big|\sum_{i=1}^{N_u} V_{ui} M(ui) - M(u)\Big|^p\Big) \ \leq \ 2^{p-1}\mathbb{E}\Big(\Big|\sum_{1=1}^{N_u} V_{ui} M(ui)\Big|^p + \mathbb{E}(|M|)^p\Big).$$

Write $M(ui) = M(ui) - \mathbb{E}(M(ui)) + \mathbb{E}(M(ui))$. Then from the inequality (2.4), we get

$$\begin{aligned} &\mathbb{E}\Big(\Big|\sum_{i=1}^{N_u} V_{ui} M(ui)\Big|^p\Big) = \mathbb{E}\Big(\Big|\sum_{1=1}^{N_u} V_{ui}(M(ui) - \mathbb{E}(M(ui))) + V_{ui}\mathbb{E}(M(ui))\Big|^p\Big) \\ &\leq 2^{p-1}\mathbb{E}\Big(\Big|\sum_{i=1}^{N_u} V_{ui}(M(ui) - \mathbb{E}(M(ui)))\Big|^p\Big) + 2^{p-1}\mathbb{E}(|M|^p)\mathbb{E}\Big(\Big|\sum_{1=1}^{N_u} V_{ui}\Big|^p\Big). \end{aligned}$$

It follows, from the Lemma 2.3 applied with $X_i = V_{ui}(M(ui) - \mathbb{E}(M(ui)))$, and from the independence of $M(ui)$ and $(N_u, V_{u1}, \cdots, V_{uN_u})$, that

$$\begin{aligned} \mathbb{E}\Big(\Big|\sum_{1=1}^{N_u} V_{ui}(M(ui) - \mathbb{E}(M(ui)))\Big|^p\Big) &\leq 2^p\mathbb{E}\Big(\sum_{i=1}^{N_u} \Big|V_{ui}(M(ui) - \mathbb{E}(M(ui)))\Big|^p\Big) \\ &\leq 2^p\mathbb{E}\Big(\Big|M(u) - \mathbb{E}(M(u))\Big|^p\Big)\mathbb{E}\Big(\sum_{i=1}^{N_u} |V_{ui}|^p\Big) \\ &\leq 2^{2p}\mathbb{E}\big(|M|^p\big)\mathbb{E}\Big(\sum_{i=1}^{N} |V_i|^p\Big). \end{aligned}$$

Finally, we have

$$\mathbb{E}\Big(\Big|\sum_{i=1}^{N_u} V_{ui} M(ui)\Big|^p\Big) \leq C_p \mathbb{E}|M|^p\big)\Big(\mathbb{E}\big(\sum_{i=1}^{N} |V_i|^p\big) + \mathbb{E}\big(|\sum_{i=1}^{N} V_i|^p\big) + 1\Big).$$

Now we prove Propositions 2.1 and 2.2.

Proof of the Proposition 2.1 : (1) Recall that the uniform convergence result uses an argument developed in [17]. Fix a compact $K \subset J$. By Lemma 2.2 we can fix a compact neighborhood K' of K and $\widetilde{p}_{K'} > 1$ such that

$$\sup_{q\in K'} \mathbb{E}\Big(\big(\sum_{i=1}^{N} e^{\langle q|X_i\rangle}\big)^{\widetilde{p}_{K'}}\Big) < \infty.$$

By Lemma 2.1, we can fix $1 < p_K \le \min(2, \widetilde{p}_{K'})$ such that $\sup_{q\in K}\phi(p_K, q) < 1$. Then for each $q \in K$, there exists a neighborhood $V_q \subset \mathbb{C}^d$ of q, whose projection to $\mathbb{R}^d$ is contained in K', and such that for all $u \in T$ and $z \in V_q$, the random variable

$$W_z(u) = \frac{e^{\langle z|X_u\rangle}}{\mathbb{E}\Big(\sum_{i=1}^{N} e^{\langle z|X_i\rangle}\Big)}$$

is well defined, and we have

$$\sup_{z\in V_q} \phi(p_K, z) < 1,$$

where for all $z, z' \in \mathbb{C}^d$ we set $\langle z|z'\rangle = \sum_{i=1}^{d} z_i \bar{z}_i$, and

$$\phi(p_K, z) = \frac{\mathbb{E}\Big(\sum_{i=1}^{N} |e^{\langle z|X_i\rangle}|^{p_K}\Big)}{\Big|\mathbb{E}\big(\sum_{i=1}^{N} e^{\langle z|X_i\rangle}\big)\Big|^{p_K}}.$$

By extracting a finite covering of K from $\bigcup_{q\in K} V_q$, we find a neighborhood $V \subset \mathbb{C}^d$ of K such that

$$\sup_{z\in V} \phi(p_K, z) < 1.$$

Since the projection of V to $\mathbb{R}^d$ is included in K' and the mapping $z \mapsto \mathbb{E}\big(\sum_{i=1}^{N} e^{\langle z|X_i\rangle}\big)$ is continuous and does not vanish on V, by considering a smaller neighborhood of K included in V if necessary, we can assume that

$$A_V = \sup_{z\in V} \mathbb{E}\Big(\Big|\sum_{i=1}^{N} e^{\langle z|X_i\rangle}\Big|^{p_K}\Big)\Big|\mathbb{E}\big(\sum_{i=1}^{N} e^{\langle z|X_i\rangle}\big)\Big|^{-p_K} + 1 < \infty.$$

Now, for $u \in T$, we define the analytic extension to V of $Y_n(u, q)$ given by

$$\begin{aligned} Y_n(u, z) &= \sum_{v\in \mathsf{T}_n(u)} W_z(u\cdot v_1)\cdots W_z(u\cdot v_1\cdots v_n) \\ &= \mathbb{E}\big(\sum_{i=1}^{N} e^{\langle z|X_i\rangle}\big)^{-n} \sum_{v\in \mathsf{T}_n(u)} e^{\left\langle z|S_{|u|+n}X(uv) - S_{|u|}(u)\right\rangle}. \end{aligned}$$

We denote also $Y_n(\emptyset, z)$ by $Y_n(z)$. Now, applying Lemma 2.4, with

$$V_i = e^{\langle z|X_i\rangle} / \mathbb{E}\big(\sum_{j=1}^{N} e^{\langle z|X_j\rangle}\big), \text{ and } M = 1,$$

we get

$$\mathbb{E}\big(|Y_n(z) - Y_{n-1}(z)|^{p_K} \big) \leq C_{p_K}\Big(\mathbb{E}\big(\sum_{i=1}^N |V_i|^{p_K}\big)\Big)^{n-1}\Big(\mathbb{E}\big(\sum_{i=1}^N |V_i|^{p_K}\big) + \mathbb{E}\big(|\sum_{i=1}^N V_i|^{p_K}\big) + 1\Big).$$

Notice that $\mathbb{E}\Big(\sum_{i=1}^N |V_i|^{p_K}\Big) = \phi(p_K, z)$. Then,

$$\mathbb{E}\big(|Y_n(z) - Y_{n-1}(z)|^{p_K} \big) \leq C_{p_K} \sup_{z\in V} \phi(p_K, z)^n + C_{p_K} A_V \sup_{z\in V} \phi(p_K, z)^{n-1}.$$

With probability 1, the functions $z \in V \mapsto Y_n(z), n \geq 0$, are analytic. Fix a closed polydisc $D(z_0, 2\rho) \subset V$. Equation (2.2) gives

$$\sup_{z\in D(z_0,\rho)} |Y_n(z) - Y_{n-1}(z)| \leq 2^d \int_{[0,1]^d} \Big|Y_n(z_0 + 2\rho e^{i2\pi t}) - Y_{n-1}(z_0 + 2\rho e^{i2\pi t})\Big|\, dt.$$

Furthermore Jensen's inequality and Fubini's Theorem give

$$\begin{aligned}
\mathbb{E}\big(\sup_{z\in D(z_0,\rho)} & |Y_n(z) - Y_{n-1}(z)|^{p_K} \big) \\
&\leq \mathbb{E}\Big((2^d \int_{[0,1]^d} \Big|Y_n(z_0 + 2\rho e^{i2\pi t}) - Y_{n-1}(z_0 + 2\rho e^{i2\pi t})\Big|\, dt)^{p_K}\Big) \\
&\leq 2^{dp_K}\mathbb{E}\Big(\int_{[0,1]^d} \Big|Y_n(z_0 + 2\rho e^{i2\pi t}) - Y_{n-1}(z_0 + 2\rho e^{i2\pi t})\Big|^{p_K} dt\Big) \\
&\leq 2^{dp_K} \int_{[0,1]^d} \mathbb{E}\Big|Y_n(z_0 + 2\rho e^{i2\pi t}) - Y_{n-1}(z_0 + 2\rho e^{i2\pi t})\Big|^{p_K} dt \\
&\leq 2^{dp_K} C_{p_K} \sup_{z\in V} \phi(p_K, z)^n + C_{p_K} \sup_{z\in V} \phi(p_K, z)^{n-1} A.
\end{aligned}$$

Since $\sup_{z\in V} \phi(p_K, z) < 1$, it follows that $\sum_{n\geq 1} \Big\| \sup_{z\in D(z_0,\rho)} |Y_n(z) - Y_{n-1}(z)| \Big\|_{p_K} < \infty$. This implies, $z \mapsto Y_n(z)$ converge uniformly, almost surely and in L^{p_K} norm over the compact $D(z_0, \rho)$, to a limit $Y(z)$. This also implies that

$$\Big\| \sup_{z\in D(z_0,\rho)} Y(z)\Big\|_{p_K} < \infty.$$

Since K can be covered by finitely many such polydiscs $D(z_0, \rho)$ we get the uniform convergence, almost surely and in L^{p_K} norm, of the sequence $(q \in K \mapsto Y_n(q))_{n\geq 1}$ to $q \in K \mapsto Y(q)$. Moreover, since J can be covered by a countable union of such compact K we get the simultaneous convergence for all $q \in J$. The same holds simultaneously for all the function $q \in J \mapsto Y_n(u, q)$, $u \in \bigcup_{n\geq 0} \mathbb{N}_+^n$, because $\bigcup_{n\geq 0} \mathbb{N}_+^n$ is countable.

To finish the proof of Proposition 2.2(1), we must show that with probability 1, $q \in K \mapsto Y(q)$ does not vanish. Without loss of generality we can suppose that $K = [0, 1]^d$. If I is a dyadic closed subcube of $[0, 1]^d$, we denote by E_I the event $\{\exists\ q \in I : Y(q) = 0\}$. Let $I_0, I_1, \cdots, I_{2^d-1}$ stand for the descendants of I. The event E_I being a tail event of probability 0 or 1, if we suppose that $P(E_I) = 1$, there exists

$j \in \{0, 1, \cdots, 2^d - 1\}$ such that $P(E_{I_j}) = 1$. Suppose now that $P(E_K) = 1$. The previous remark allows to construct a decreasing sequence $(I(n))_{n\geq 0}$ of dyadic subscubes of K such that $P(E_{I(n)}) = 1$. Let q_0 be the unique element of $\cap_{n\geq 0} I(n)$. Since $q \mapsto Y(q)$ is continuous we have $P(Y(q_0) = 0) = 1$, which contradicts the fact that $(Y_n(q_0))_{n\geq 1}$ converge to $Y(q_0)$ in L^1.

(2) It is a consequence of the branching property

$$Y_{n+1}(u, q) = \sum_{i=0}^{N} e^{\langle q | X_{ui}\rangle - \widetilde{P}(q)} Y_n(ui, q).$$

Proof of Proposition 2.2 : Let K be a compact subset of J. Since for all $q \in K$, we have $q + 0 \in J$, there exists a compact neighborhood Λ of the origin such that $\{q + \lambda : q \in K, \lambda \in \Lambda\} \subset J$. Let $R = \{q + \lambda : q \in K, \lambda \in \Lambda\}$. For $q \in K$ and $\lambda \in \Lambda$ we define

$$F_n(q, \lambda) = \sum_{u \in \mathsf{T}_n} e^{\langle q+\lambda | S_n X(u)\rangle - n\widetilde{P}(q)} Y(q, u)$$

and

$$Z_n(q, \lambda) = \frac{F_n(q, \lambda)}{\mathbb{E}(F_n(q, \lambda))} = \sum_{u \in \mathsf{T}_n} e^{\langle q+\lambda | S_n X(u)\rangle - n\widetilde{P}(q+\lambda)} Y(q, u).$$

As in the proof of Proposition 2.1, we can find $p_R \in (1, 2]$ and a neighborhood $V \times V_\Lambda \subset \mathbb{C}^d \times \mathbb{C}^d$ of $K \times \Lambda$ such that the functions

$$F_n(z, z') = \Big(\mathbb{E}\big(\sum_{i=1}^{N} e^{\langle z+z' | X_i\rangle}\big)\Big)^{-n} \sum_{u \in \mathsf{T}_n} e^{\langle z+z' | S_n X(u)\rangle} Y(z, u),$$

and

$$Z_n(z, z') = \frac{F_n(z, z')}{\mathbb{E}(F_n(z, z'))}$$

are well defined on $V \times V_\Lambda$, and

$$\begin{cases} \sup_{z' \in V_\Lambda} \sup_{z \in V} \phi(p_R, z + z') < 1, \\ A_{V \times V_\Lambda} = \displaystyle\sup_{(z,z') \in V \times V_\Lambda} \mathbb{E}\Big(\big|\sum_{i=1}^{N} e^{\langle z+z' | X_i\rangle}\big|^{p_R}\Big)\Big|\mathbb{E}\big(\sum_{i=1}^{N} e^{\langle z+z' | X_i\rangle}\big)\Big|^{-p_R} + 1 < \infty \end{cases}.$$

Suppose that for each $(z_0, z_0') \in V \times V_\Lambda$ and $\rho > 0$ such that $D(z_0, 2\rho) \times D(z_0', 2\rho) \subset V \times V_\Lambda$ we have

$$\sum_{n\geq 1} \mathbb{E}\Big(\sup_{(z,z') \in D(z_0,\rho) \times D(z_0',\rho)} |Z_n(z, z') - Z_{n-1}(z, z')|^{p_R}\Big) < \infty. \tag{2.5}$$

Then, with probability 1, $(z, z') \mapsto Z_n(z, z')$ converges uniformly on $D(z_0, \rho) \times D(z_0', \rho)$ to a limit $Z(z, z')$, whose restriction to $K \times \Lambda$ can be shown to be positive, in the same way as $Y(\cdot)$ was shown to be positive. Since $K \times \Lambda$ can be covered by finitely many polydiscs of the previous form $D(z_0, \rho) \times D(z_0', \rho)$, we get the almost sure uniform convergence of $Z_n(q, \lambda)$ over $K \times \Lambda$ to $Z(q, \lambda) > 0$, hence the almost sure uniform convergence of $\frac{1}{n} \log(Z_n(q, \lambda))$ to 0 over $K \times \Lambda$. This yields the conclusion.

Now we prove (2.5). Given $(z,z') \in V \times V_\Lambda$, applying Lemma 2.4 with

$$V_i = e^{\langle z+z'|X_i\rangle}/\mathbb{E}\Big(\sum_{j=1}^N e^{\langle z+z'|X_j\rangle}\Big) \text{ and } M = Y(z),$$

we get

$$\begin{aligned}&\mathbb{E}\big(\,|Z_n(z,z') - Z_{n-1}(z,z')|^{p_R}\,\big)\\ &\qquad\qquad \le C_{p_R}\mathbb{E}(|Y(z)|^{p_R})(\phi(p_R, z+z')^n + A_{V\times V_\Lambda}\phi(p_R,z+z')^{n-1}).\end{aligned}$$

For $\widetilde{z} = (z,z') \in V \times V_\Lambda$ and $n \ge 1$ let $M_n(\widetilde{z}) = Z_n(z,z') - Z_{n-1}(z,z')$. With probability 1 the functions $\widetilde{z} \in V \times V_\Lambda \mapsto M_n(\widetilde{z})$, $n \ge 1$, are analytic. Fix a closed polydisc $D(\widetilde{z}_0, 2\rho) \subset V \times V_\Lambda$ with $\rho > 0$. The Cauchy formula gives

$$\sup_{Z \in D(\widetilde{z}_0,\rho)} |M_n(\widetilde{z})| \le 2^{2d}\int_{[0,1]^{2d}} |M_n(\widetilde{z}_0 + 2\rho e^{i2\pi t})|dt.$$

Furthermore Jensen's inequality and Fubini's Theorem give

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{\widetilde{z}\in D(Z_0,\rho)} |M_n(\widetilde{z})|^{p_R}\Big)\\
\le\;& \mathbb{E}\Big((2^{2d}\int_{[0,1]^{2d}} \Big|M_n(\widetilde{z}_0 + 2\rho e^{i2\pi t})\Big|\,dt)^{p_R}\Big)\\
\le\;& 2^{2dp_R}\mathbb{E}\Big(\int_{[0,1]^{2d}} \Big|M_n(\widetilde{z}_0 + 2\rho e^{i2\pi t})\Big|^{p_R}\,dt\Big)\\
\le\;& 2^{2dp_R}\int_{[0,1]^{2d}} \mathbb{E}\Big|M_n(\widetilde{z}_0 + 2\rho e^{i2\pi t})\Big|^{p_R}\,dt\\
\le\;& 2^{2dp_R}C_{p_R}\mathbb{E}\Big(\sup_{z\in V}|Y(z)|^{p_R}\Big)\\
&\cdot\Big(\sup_{(z,z')\in V\times V_\Lambda}\phi(p_R, z+z')^n + A_{V\times V_\Lambda}\sup_{(z,z')\in V\times V_\Lambda}\phi(p_R,z+z')^{n-1}\Big)\Big).
\end{aligned}$$

Since $\sup_{(z,z')\in V\times V_\Lambda}\phi(p_R, z+z') < 1$, we get the conclusion.

Proof of the Proposition 2.3 Let K be a compact subset of J and $\lambda \in \{-1,1\}$. For $a > 0$, $q \in K$ and $n \ge 1$, we set

$$E^\lambda_{n,a} = \big\{t \in \partial\mathsf{T} : Y(t_{|n}, q)^\lambda > a^{n\lambda}\big\}.$$

It is sufficient to show that for $\lambda \in \{-1,1\}$,

$$\mathbb{E}\Big(\sup_{q\in K}\sum_{n\ge1}\mu_q(E^\lambda_{n,a})\Big) < \infty. \tag{2.6}$$

Indeed, if this holds, then with probability 1, for each $q \in K$, $\sum_{n\ge1}\mu_q(E^\lambda_{n,a}) < \infty$, for all $\lambda \in \{-1,1\}$, hence by the Borel-Cantelli lemma, for μ_q-almost every $t \in \partial\mathsf{T}$, if n is big enough we have

$$-\log a \le \liminf_{n\to\infty}\frac{1}{n}\log Y(t_{|n},q) \le \limsup_{n\to\infty}\frac{1}{n}\log Y(t_{|n},q) \le \log a.$$

Letting a tend to 1 along a countable sequence yields the result.

Let us prove (2.6) for $\lambda = 1$ (the case $\lambda = -1$ is similar). At first we have,

$$\begin{aligned}\sup_{q\in K}\mu_q(E^1_{n,a}) &= \sup_{q\in K}\sum_{u\in\mathsf{T}_n}\mu_q([u])\mathbf{1}_{\{Y(u,q)>a^n\}}\\ &= \sup_{q\in K}\sum_{u\in\mathsf{T}_n} e^{\langle q|S_nX(u)\rangle}e^{-n\widetilde{P}(q)}Y(u,q)\mathbf{1}_{\{Y(u,q)>a^n\}}\\ &\le \sup_{q\in K}\sum_{u\in\mathsf{T}_n} e^{\langle q|S_nX(u)\rangle}e^{-n\widetilde{P}(q)}(Y(u,q))^{1+\nu}a^{-n\nu}, \quad \nu>0.\end{aligned}$$

For $q \in K$ and $\nu > 0$, we set $H_n(q,\nu) = \sum_{u\in\mathsf{T}_n} e^{\langle q|S_nX(u)\rangle}e^{-n\widetilde{P}(q)}a^{-n\nu}$, and we have

$$\mathbb{E}\big(\sup_{q\in K}\mu_q(E^1_{n,a})\big) \le \mathbb{E}\big(\sup_{q\in K}Y(q)^{1+\nu}\big)\mathbb{E}\big(\sup_{q\in K}H_n(q,\nu)\big). \tag{2.7}$$

Lemma 2.5 *Let* $H_n(z,\nu) = \mathbb{E}\Big(\sum_{i=1}^N e^{\langle z|X_i\rangle}\Big)^{-n}\sum_{u\in\mathsf{T}_n} e^{\langle z|S_nX(u)\rangle}a^{-n\nu}$. *There exists a neighborhood* $V\subset\mathbb{C}^d$ *of* K *such that, for all* $z\in V$*, for all* $n\in\mathbb{N}^*$*,*

$$\mathbb{E}\Big(\big|H_n(z,\nu)\big|\Big) \le a^{-n\nu/2}. \tag{2.8}$$

Proof For $z\in\mathbb{C}^d$ and close to K,

$$\widetilde{H}_1(z,\nu) = \Big|\mathbb{E}\big(\sum_{i=1}^N e^{\langle z|X_i\rangle}\big)\Big|^{-1}\mathbb{E}\big(\sum_{i=1}^N\big|e^{\langle z|X_i\rangle}\big|\big)\,a^{-\nu}$$

is well defined since $\mathbb{E}\big(\sum_{i=1}^N e^{\langle z|X_i\rangle}\big)$ does not vanish. Let $q\in K$. Since $\mathbb{E}(\widetilde{H}_1(q,\nu) = H_1(q,\nu)) = a^{-\nu}$, there exists a neighborhood $V_q\subset\mathbb{C}^d$ of q such that for all $z\in V_q$ we have $\mathbb{E}\Big(\big|\widetilde{H}_1(z,\nu)\big|\Big)\le a^{-\nu/2}$. By extracting a finite covering of K from $\bigcup_{q\in K}V_q$, we find a neighborhood $V\subset\mathbb{C}^d$ of K such that $\mathbb{E}\Big(\big|\widetilde{H}_1(z,\nu)\big|\Big)\le a^{-\nu/2}$ for all $z\in V$. Therefore,

$$\begin{aligned}\mathbb{E}\Big(\big|H_n(z,\nu)\big|\Big) &= \Big|\mathbb{E}\big(\sum_{i=1}^N e^{\langle z|X_i\rangle}\big)\Big|^{-n}\mathbb{E}\Big(\Big|\sum_{u\in\mathsf{T}_n} e^{\langle z|S_nX(u)\rangle}\Big|\Big)\,a^{-n\nu}\\ &\le \Big|\mathbb{E}\big(\sum_{i=1}^N e^{\langle z|X_i\rangle}\big)\Big|^{-n}\mathbb{E}\Big(\sum_{u\in\mathsf{T}_n}\big|e^{\langle z|S_nX(u)\rangle}\big|\Big)\,a^{-n\nu}\\ &= \Big|\mathbb{E}\big(\sum_{i=1}^N e^{\langle z|X_i\rangle}\big)\Big|^{-n}\mathbb{E}\Big(\sum_{i=1}^N\big|e^{\langle z|X_i\rangle}\big|\Big)^n\,a^{-n\nu}\\ &\le \mathbb{E}\Big(\big|\widetilde{H}_1(z,\nu)\big|\Big)^n \le a^{-n\nu/2}.\end{aligned}$$

With probability 1, the functions $z\in V\longmapsto H_n(z,\nu)$ are analytic. Fix a closed polydisc $D(z_0,2\rho)\subset V$, $\rho>0$. Cauchy's formula gives

$$\sup_{z\in D(z_0,\rho)}\big|H_n(z,\nu)\big| \le 2^d\int_{[0,1]^d}\big|H_n(z_0+2\rho e^{i2\pi t},\nu)\big|dt.$$

Furthermore Fubini's Theorem gives

$$\begin{aligned}
\mathbb{E}\big(\sup_{z\in D(z_0,\rho)} |H_n(z,\nu)|\big) &\le \mathbb{E}\big(2^d \int_{[0,1]^d} \big|H_n(z_0+2\rho e^{i2\pi t},\nu)\big|\, dt\big) \\
&\le 2^d \int_{[0,1]^d} \mathbb{E}\,\big|H_n(z_0+2\rho e^{i2\pi t},\nu)\big|\, dt \\
&\le 2^d a^{-n\nu/2}.
\end{aligned}$$

Taking $\nu = p_K - 1$ in (2.7), we get

$$\mathbb{E}\big(\sup_{q\in K} \mu_q(E^1_{n,a})\big) \le 2^d \mathbb{E}\big(\sup_{q\in K} Y(q)^{p_K}\big) a^{-n(p_K-1)/2}.$$

Since $\mathbb{E}\big(\sup_{q\in K} Y(q)^{p_K}\big) < \infty$, we get (2.6).

2.2 Behavior of $P(q)$ outside J and partial results for $\alpha \in \partial I$

We have proved that, with probability 1, for all $\alpha \in \mathring{I}$, we have

$$\dim E_X(\alpha) = \widetilde{P}^*(\alpha) = P^*(\alpha).$$

Also, as a consequence of Proposition 2.1, by considering $\log(Y_n(q))/n$ we have $\widetilde{P}(\cdot) = P(\cdot)$ almost surely on J. For $\alpha \notin I$, we have $\widetilde{P}^*(\alpha) < 0$, then the set $E_X(\alpha) = \emptyset$. So, it is naturally to ask :

1. What is the value of P outside J
2. Why the previous method no longer works when $\alpha \in \partial I$?
3. Is, for $\alpha \in \partial I$, the set $E_X(\alpha)$ non-empty, and if so what is its Hausdorff dimension ?

The last question is solved in Theorem 1.3, but we will give in this section some weak results in this direction.

2.2.1 Answer to the question 1.

Let $\mathbb{S}^d = \{v \in \mathbb{R}^d : \|x\| = 1\}$, and consider, for $v \in \mathbb{S}^d$, the mapping $\widetilde{P}^*_v : \lambda \ge 0 \mapsto \widetilde{P}^*(\nabla \widetilde{P}(\lambda v))$. Since the concave function $\widetilde{P}^*$ reaches its supremum at $\lambda = 0$, we deduce that $\widetilde{P}^*_v$ is a decreasing function. Then, we can define for all $v \in \mathbb{S}^d$

$$\lambda_v = \sup\{\lambda > 0 : [0, \lambda v[\subset J\},$$

and we have the following proposition.

Proposition 2.4 *With probability 1, for $v \in \mathbb{S}^d$, we have*

1. *If $\lambda_v = \infty$, then $\forall\, \lambda \ge 0$, $\widetilde{P}(\lambda v) = P(\lambda v)$.*

2. *If* $\lambda_v < \infty$, *then,* $\forall\ 0 \leq \lambda \leq \lambda_v$, $\widetilde{P}(\lambda v) = P(\lambda v)$ *and* $\forall\ \lambda \geq \lambda_v$, $P(\lambda v) = \lambda\langle \nabla \widetilde{P}(\lambda_v v)|v\rangle$.
3. *If* $\lambda_v < \infty$, *then* $\forall \lambda > \lambda_v$, $\widetilde{P}^*(\nabla \widetilde{P}(\lambda v)) < 0$ *and* $P^*(\nabla P(\lambda v)) = 0$.

Proof. (1) and (2). Here we follow the approach used in [66] to obtain the L^q spectrum of Mandelbrot measures.

Recall that, with probability 1, for all $q \in J$, we have $\widetilde{P}(q) = P(q)$. Then, with probability 1, $\forall v \in \mathbb{S}^d$ and $\forall\ 0 \leq \lambda \leq \lambda_u$,

$$P(\lambda u) = \widetilde{P}(\lambda u).$$

Let $v \in \mathbb{S}^d$ such that $\lambda_v < \infty$. In this case, we have $\lambda_v v \in \partial J$ and $\widetilde{P}^*(\nabla \widetilde{P}(\lambda_v v)) = 0$, so that

$$\widetilde{P}(\lambda_v v) = \lambda_v \langle \nabla \widetilde{P}(\lambda_v v)|v\rangle.$$

Thus, the line $\Delta\ :\ y(\lambda v) = \lambda\langle \nabla \widetilde{P}(\lambda_v v)|v\rangle$ is the tangent to the graph of P at $(\lambda_v v, \widetilde{P}(\lambda_v v))$. Since P is convex, $\forall \lambda \in \mathbb{R}_+$, $P(\lambda v) \geq \lambda\langle v, \nabla \widetilde{P}(\lambda_v v)\rangle$. On the other hand, for each $\lambda > \lambda_v$

$$\begin{aligned} \sum_{u \in \mathsf{T}_n} \exp(\langle \lambda v|S_n X(u)\rangle) &= \sum_{u \in \mathsf{T}_n} \exp(\left\langle \frac{\lambda}{\lambda_v}\lambda_v v|S_n X(u)\right\rangle) \\ &\leq [\sum_{u \in \mathsf{T}_n} \exp(\langle \lambda_v v|S_n X(u)\rangle)]^{\frac{\lambda}{\lambda_v}}, \end{aligned}$$

which implies that $P(\lambda v) \leq \frac{\lambda}{\lambda_v} P(\lambda_v v) = \frac{\lambda}{\lambda_v}\widetilde{P}(\lambda_v v) = \lambda_v\langle \nabla \widetilde{P}(\lambda_v v)|v\rangle = \lambda\langle \nabla \widetilde{P}(\lambda_v v)|v\rangle$.

(3) Let $v \in \mathbb{S}^d$ such that $\lambda_v < \infty$. From (2), we have for all $\lambda > \lambda_v$ that $\langle \nabla P(\lambda_v v)|v\rangle = \langle \nabla P(\lambda v)|v\rangle$. Moreover, $P(\lambda v) = \lambda\langle \nabla \widetilde{P}(\lambda_v v)|v\rangle$, so that $P^*(\nabla P(\lambda v)) = 0$. And since $\lambda_v v \in \partial J$, we have $\lambda v \notin J$, so that $\nabla \widetilde{P}(\lambda v) \notin I$, hence $\widetilde{P}^*(\nabla \widetilde{P}(\lambda v)) < 0$.

2.2.2 Partial answer to the second question

Let $\alpha \in \partial I$, then two cases can occur.

1. There exists $q \in \partial J$, such that $\alpha = \nabla \widetilde{P}(q)$. Then $\widetilde{P}^*(\alpha) = 0$. Also, $\frac{\partial \phi}{\partial p}(1, q) = 0$, so we cannot find $p > 1$ such that $\phi(p, q) < 1$ and Lemma 2.1 is not verified.
2. There does not exist $q \in \partial J$ such that $\alpha = \nabla \widetilde{P}(q)$. In this case, $\widetilde{P}^*(\alpha) \geq 0$ and there exists a sequence $(q_n)_{n\geq 1}$ converging in norm to ∞ and $\alpha = \lim_{n\to\infty} \nabla \widetilde{P}(q_n)$. In this situation, there is no q to construct the Mandelbrot measure μ_q.

This problem was solved in dimension 1 in [7], and when $d \geq 1$ we have the following result.

Theorem 2.2 *Let* $\alpha \in \partial I$. *Assume that*

$$\mathcal{R}(\alpha) :\ \exists v \in \mathbb{S}^d \ \ \textit{such that}\ \ \alpha = \lim_{\lambda \to \lambda_v} \nabla \widetilde{P}(\lambda v)$$

holds. Then with probability 1, $\dim E_X(\alpha) = \widetilde{P}^*(\alpha)$.

Proof Notice first that if $\alpha \in \partial I$ and $\alpha = \nabla \widetilde{P}(q)$ for some $q \in \partial J$ then condition $\mathcal{R}$ holds true automatically. But, $\mathcal{R}$ is difficult to check when $\lambda_v = \infty$.
Under $\mathcal{R}$, we can go back to dimension 1, and the proof is given in [7].

2.3 Remarks

1. To estimate the dimension of the measure μ_q, we could have introduced, the logarithmic generating functions

$$\widetilde{L}_n(q,s) = \frac{1}{n}\log\int_{\partial\mathsf{T}} \mu_q(x_{|n})^s d\mu_q(x), \quad (q \in J,\ s \in \mathbb{R}),$$

and studied their convergence in the same way as $L_n(q,s)$ was studied in Proposition 2.2. However, we would have had to find an analytic extension of the mapping $q \mapsto Y(q)^{1+s}$, almost surely in a deterministic neighborhood of any compact subset of J in order to apply the technique using Cauchy formula. It turns out that the existence of such an extension is not clear, but assuming its existence, the same approach as in the proof of Corollary 2.1 would give the Hausdorff dimension of μ_q. If we only seek for a result valid for each $q \in J$ almost surely, then it is not hard to get the almost sure uniform convergence of $s \mapsto \widetilde{L}_n(q,s)$ in a compact neighborhood of 0 towards $s \mapsto \widetilde{P}(q(1+s)) - (1+s)\widetilde{P}(q)$, and the same approach as that of Corollary 2.1 yields the dimension of μ_q.

2. The method used in this paper is not a direct extension of that used in [2] for the case $d = 1$ on homogeneous trees. Indeed, in [2] the complex extension is used to build simultaneously the measures μ_q, but the proof that, uniformly in q, μ_q is carried by $E_X(P'(q))$ and has a Hausdorff dimension $P(q) - qP'(q)$ uses a real analysis method, which seems hard to extend when $d \geq 2$.

3. Our assumptions can be relaxed as follows. We could assume that $\widetilde{P}$ is finite over a neighborhood V of 0, consider $J_V = \{q \in V : \widetilde{P}(q) - \langle q|\nabla\widetilde{P}(q)\rangle > 0\}$, and suppose that there exists a continuous function $q \in J_V \mapsto p_q \in (1,\infty)$ such that $\mathbb{E}\Big(\big(\sum_{i=1}^N e^{\langle q|X_i\rangle}\big)^{p_q}\Big) < \infty$ for all $q \in J_V$. Then the same conclusions as in Theorem 1.1 hold with $I = \{\nabla\widetilde{P}(q) : q \in J_V\}$.

Chapitre 3

The Hausdorff dimensions of the sets $E_X(K)$ under the metric d_1

This chapter is devoted to the proof of Theorem 1.2.

3.1 The domain of study

Recall that $I_X = I = \{\alpha \in \mathbb{R}^d : \widetilde{P}^*(\alpha) \geq 0\}$. Define also $J = \{q \in \mathbb{R}^d : \widetilde{P}^*(\nabla \widetilde{P}(q)) > 0\}$.

Proposition 3.1

1. *I is convex, compact and non-empty.*
2. *$I = \overline{\{\nabla \widetilde{P}(q) : q \in J\}}$, and $\mathring{I} = \nabla \widetilde{P}(J)$.*

Proof Recall that we can assumed (1.1).

(1) At first notice that I contains $\nabla \widetilde{P}(0)$ ($\widetilde{P}^*(\nabla \widetilde{P}(0)) = \widetilde{P}(0) = \log(\mathbb{E}(N)) > 0$). The convexity of I comes from the concavity of the function $\widetilde{P}^*$. The fact that I is closed results from the upper semi-continuity of $\widetilde{P}^*$. It remains to show that I is bounded. Suppose that this is not the case. Let $(\alpha_n)_{n\geq 1} \in I^{\mathbb{N}_+}$ which tends to ∞ as n tends to ∞. Since $\mathbb{S}^{d-1}$ is compact, without loss of generality we can assume that $(\frac{\alpha_n}{\|\alpha_n\|})_{n\geq 1}$ converges to a limit $u \in \mathbb{S}^{d-1}$ as $n \to \infty$. Let $\lambda > 0$. From the definition of $\widetilde{P}^*$, since $\widetilde{P}^*(\alpha) \geq 0$, we have

$$\begin{aligned} 0 \leq \widetilde{P}(\lambda u) - \lambda\langle u|\alpha_n\rangle &= \widetilde{P}(\lambda u) - \lambda\langle u|\|\alpha_n\|u\rangle + \lambda\langle u|\|\alpha_n\|u - \alpha_n\rangle \\ &= \widetilde{P}(\lambda u) - \lambda\|\alpha_n\|\Big(1 + \Big\langle u\Big|u - \frac{\alpha_n}{\|\alpha_n\|}\Big\rangle\Big). \end{aligned}$$

Since $\frac{\alpha_n}{\|\alpha_n\|}$ converge to u as $n \to \infty$, this yields

$$\widetilde{P}(\lambda u) \geq \lambda\|\alpha_{n_k}\|(1 + o(1)) \quad \text{as } n \to \infty,$$

hence $\widetilde{P}(\lambda u) = \infty$ for all $\lambda > 0$. This contradics the finiteness of $\widetilde{P}$ over $\mathbb{R}^d$.

(2) We first notice that (1.1) implies the strict convexity of $\widetilde{P}$, hence the second differential of $\widetilde{P}$ is positive definite, so $\nabla\widetilde{P}$ is a diffeomorphism from J onto its image. Indeed, suppose that there exists $q \neq q' \in \mathbb{R}^d$ and $\lambda \in (0,1)$ such that $\widetilde{P}(\lambda q+(1-\lambda)q') = \lambda\widetilde{P}(q)+(1-\lambda)\widetilde{P}(q')$. Applying successively the Hölder inequality to $\sum_{i=1}^N \exp(\langle q|X_i\rangle)\exp(\langle q'|X_i\rangle)^{1-\lambda}$ in $\mathbb{R}^N$ and $\mathbb{E}(Z^\lambda Z'^{1-\lambda})$ with $Z = \Big(\sum_{i=1}^N \exp(\langle q|X_i\rangle)\Big)^\lambda$ and $Z = \Big(\sum_{i=1}^N \exp(\langle q'|X_i\rangle)\Big)^{1-\lambda}$, we see that this forces the existence of a deterministic $c \in \mathbb{R}_+^*$ such that $\exp(\langle q|X_i\rangle) = c\exp(\langle q'|X_i\rangle)$ almost surely for all $1 \leq i \leq N$, in contradiction with (1.1).

The previous lines imply that $\mathring{I} \neq \emptyset$ since it must contain $\nabla\widetilde{P}(J)$, which is an open set. Now, we use the general facts about the concave function $\widetilde{P}^*$: its domain, i.e. $\{\alpha \in \mathbb{R}^d : \widetilde{P}^*(\alpha) > -\infty\}$, is included in the closure of the range of $\nabla\widetilde{P}$, and its interior is included in the image of $\nabla\widetilde{P}$ (see [71, Sec. 24, p. 227]).

Suppose that $\alpha \in \mathring{I}$ and there exists a sequence $(q_n)_{n\geq 1}$ of vectors in $\mathbb{R}^d$ such that $\alpha = \lim_{n\to\infty} \nabla\widetilde{P}(q_n)$ and $\widetilde{P}^*(\nabla\widetilde{P}(q_n)) \leq 0$ for all $n \geq 1$. Then the concave function $\widetilde{P}^*$ being continuous at α, we have $\widetilde{P}^*(\alpha) = 0$. Since $\nabla\widetilde{P}(0) \in \mathring{I}$ and $\widetilde{P}^*(\nabla\widetilde{P}(0)) = \widetilde{P}(0) > 0$ (or more generally since $J \neq \emptyset$), the concavity of $\widetilde{P}^*$ implies that $\widetilde{P}^*$ takes negative values over $\mathring{I}$, which is excluded by definition of I. Thus $\mathring{I} \subset \nabla\widetilde{P}(J)$. Consequently, $\mathring{I} = \nabla\widetilde{P}(J)$ and $I = \overline{\nabla\widetilde{P}(J)}$.

Corollary 3.1 *Let D be a dense subset of J. For all $\alpha \in I$, there exists a sequence $(q_n)_{n\geq 1}$ of elements of D such that $\lim_{n\to\infty} \nabla\widetilde{P}(q_n) = \alpha$ and $\lim_{n\to\infty} \widetilde{P}^*(\nabla\widetilde{P}(q_n)) = \widetilde{P}^*(\alpha)$.*

Proof Fix a point $\beta \in \mathring{I} \setminus \{\alpha\}$. The restriction of $\widetilde{P}^*$ to $[\beta,\alpha]$ is continuous, since $\widetilde{P}^*$ is concave and upper semi-continuous. Thus $\lim_{t\to 0^+} \widetilde{P}^*(t\beta + (1-t)\alpha) = \widetilde{P}^*(\alpha)$. For each integer $n \geq 1$ let $\alpha_n = n^{-1}\beta + (1-n^{-1})\alpha$. This point is in $\mathring{I}$, so it takes the form $\nabla\widetilde{P}(\lambda_n)$ with $\lambda_n \in J$. Since D is dense in J and $\widetilde{P}^*(\nabla\widetilde{P}(\cdot))$ is continuous, we can find $q_n \in D$ such that $\|\lambda_n - q_n\| \leq 1/n$ and $|\widetilde{P}^*(\alpha_n) - \widetilde{P}^*(\nabla\widetilde{P}(q_n))| \leq 1/n$. By construction the sequence $(q_n)_{n\geq 1}$ is as desired.

3.2 Upper bounds for the Hausdorff dimensions

Let us define the pressure like function

$$P(q) = \limsup_{n\to\infty} \frac{1}{n} \log\Big(\sum_{u\in\mathsf{T}_n} \exp(\langle q|S_nX(u)\rangle)\Big) \quad (q \in \mathbb{R}^d). \tag{3.1}$$

Proposition 3.2 *With probability 1, $P(q) \leq \widetilde{P}(q)$ for all $q \in \mathbb{R}^d$.*

Proof The fonctions $\widetilde{P}$ and P being convex, we only need to prove the inequality for each $q \in \mathbb{R}^d$ almost surely. Fix $q \in \mathbb{R}^d$. For $s > \widetilde{P}(q)$ we have

$$\begin{aligned} \mathbb{E}(\sum_{n\geq 1} e^{-ns} \sum_{u\in \mathsf{T}_n} \exp(\langle q|S_nX(u)\rangle) &= \sum_{n\geq 1} e^{-ns}\mathbb{E}(\sum_{i=1}^{N} \exp(\langle q|X_i\rangle))^n \\ &= \sum_{n\geq 1} e^{n(\widetilde{P}(q)-s)}. \end{aligned}$$

Consequently, $\sum_{n\geq 1} e^{-ns} \sum_{u\in \mathsf{T}_n} \exp(\langle q|S_nX(u)\rangle) < \infty$ almost surely, so that $\sum_{u\in \mathsf{T}_n} \exp(\langle q|S_nX(u)\rangle) = O(e^{ns})$ and $P(q) < s$. Since $s > \widetilde{P}(q)$ is arbitrary, we have the conclusion.

For $\alpha \in \mathbb{R}^d$ let

$$\widehat{E}_X(\alpha) = \Big\{t \in \partial\mathsf{T} : \alpha \in \bigcap_{N\geq 1} \overline{\Big\{\frac{S_nX(t)}{n} : n \geq N\Big\}}\Big\}.$$

Proposition 3.3 *With probability 1, for all $\alpha \in \mathbb{R}^d$, $\dim \widehat{E}_X(\alpha) \leq P^*(\alpha)$, where $P^*(\alpha) = \inf_{q\in\mathbb{R}^d} P(q) - \langle q|\alpha\rangle$, and a negative dimension means that $\widehat{E}_X(\alpha)$ is empty.*

Proof We have

$$\begin{aligned} \widehat{E}_X(\alpha) &= \bigcap_{\epsilon>0} \bigcap_{N\geq 1} \bigcup_{n\geq N} \{t \in \partial\mathsf{T} : \|S_nX(t) - n\alpha\| \leq n\epsilon\} \\ &\subset \bigcap_{q\in\mathbb{R}^d} \bigcap_{\epsilon>0} \bigcap_{N\geq 1} \bigcup_{n\geq N} \{t \in \partial\mathsf{T} : |\langle q|S_nX(t) - n\alpha\rangle| \leq n\|q\|\epsilon\}. \end{aligned}$$

Fix $q \in \mathbb{R}^d$ and $\epsilon > 0$. For $N \geq 1$, the set $E(q, N, \epsilon, \alpha) = \bigcup_{n\geq N}\{t \in \partial\mathsf{T} : |\langle q|S_nX(t) - n\alpha\rangle| \leq n\|q\|\epsilon\}$ is covered by the union of those $[u]$ such that $u \in \mathsf{T}_n$ and $\langle q|S_nX(u) - n\alpha\rangle + n\|q\|\epsilon \geq 0$. Consequently, for $s \geq 0$,

$$\mathcal{H}^s_{e^{-N}}(E(q, N, \epsilon, \alpha)) \leq \sum_{n\geq N} \sum_{u\in\mathsf{T}_n} e^{-ns} \exp(\langle q|S_nX(u) - n\alpha\rangle + n\|q\|\epsilon).$$

Thus, if $\eta > 0$ and $s > P(q) + \eta - \langle q|\alpha\rangle + \|q\|\epsilon$, by definition of $P(q)$, for N large enough we have

$$\mathcal{H}^s_{e^{-N}}(E(q, N, \epsilon, \alpha)) \leq \sum_{n\geq N} e^{-n\eta/2}.$$

This yields $\mathcal{H}^s(E(q, N, \epsilon, \alpha)) = 0$, hence $\dim E(q, N, \epsilon, \alpha) \leq s$. Since this holds for all $\eta > 0$, we get $\dim E(q, N, \epsilon, \alpha) \leq P(q) - \langle q|\alpha\rangle + \|q\|\epsilon$. It follows that

$$\dim \widehat{E}_X(\alpha) \leq \inf_{q\in\mathbb{R}} \inf_{\epsilon>0} P(q) - \langle q|\alpha\rangle + \|q\|\epsilon = P^*(\alpha).$$

If $P^*(\alpha) < 0$, we necessarily have $\widehat{E}_X(\alpha) = \emptyset$.

Corollary 3.2 *With probability 1, for all compact connected subset K of $\mathbb{R}^d$, we have $E_X(K) = \emptyset$ if $K \not\subset I$, and $\dim E_X(K) \leq \inf_{\alpha\in K} \widetilde{P}^*(\alpha)$ otherwise.*

Proof We have $E_X(K) = \bigcap_{\alpha\in K} \widehat{E}_X(\alpha)$. Consequently, due to the previous proposition, if $K \not\subset I$, $E_X(K) = \emptyset$. Otherwise, $\dim E_X(K) \leq \inf_{\alpha\in K} \dim \widehat{E}_X(\alpha) \leq \inf_{\alpha\in K} P^*(\alpha) \leq \inf_{\alpha\in K} \widetilde{P}^*(\alpha)$.

3.3 Construction of inhomogeneous Mandelbrot measures and lower bounds for the Hausdorff dimensions of the sets $E_X(K)$

3.3.1 A family of inhomogeneous Mandelbrot martingales

The set of parameters

Recall that $J = \{q \in \mathbb{R}^d : \widetilde{P}^*(\nabla \widetilde{P}(q)) > 0\}$. For $(q,p) \in J \times [1,\infty)$, let

$$\varphi(p,q) = \widetilde{P}(pq) - p\widetilde{P}(q).$$

Let $(D_j)_{j\geq 1}$ be an increasing sequence of non-empty set of J, such that D_j has cardinality j and $D = \bigcup_{j\geq 1} D_j$ is a dense subset of J. Without loss of generality we assume that $\nabla\widetilde{P}$ does not vanish on D (this will be used in (3.14)).

Let $(N_j)_{j\geq 0}$ be a sequence of integers such that $N_0 = 0$, and that we will specify at the end of this section.

Then let $(M_j)_{j\geq 0}$ be the increasing sequence defined as

$$M_j = \sum_{k=1}^{j} N_k \text{ for all } j \geq 0. \tag{3.2}$$

For $n \in \mathbb{N}$, let j_n denote the unique integer satisfying

$$M_{j_n} + 1 \leq n \leq M_{j_n+1}.$$

We will build a random family of measures indexed by the set of sequences

$$\mathcal{J} = \{\varrho = (q_k)_{k\geq 1} : \forall j \geq 0, q_{M_j+1} = q_{M_j+2} = \cdots = q_{M_{j+1}} \in D_{j+1}\}. \tag{3.3}$$

Since each D_j is finite, so compact, for all $j \geq 1$, the set $\mathcal{J}$ is compact for the metric

$$d(\varrho = (q_k)_{k\geq 1}, \varrho' = (q'_k)_{k\geq 1}) = \sum_{k\geq 1} 2^{-k} \frac{|q_k - q'_k|}{1 + |q_k - q'_k|}.$$

For $\varrho = (q_k)_{k\geq 1} \in \mathcal{J}$ and $n \geq 1$ we will denote by $\varrho_{|n}$ the sequence $(q_k)_{1\leq k\leq n}$.

Inhomogeneous Mandelbrot martingales indexed by $\mathcal{J}$

For each $\beta \in (0,1]$, let W_β be a random variable taking the value $1/\beta$ with probability β and the value 0 with probability $1-\beta$. Then let $\{W_{\beta,u}\}_{u\in\bigcup_{n\geq 0}\mathbb{N}_+^n}$ be a family of independent copies of W_β. Denote by $(\Omega_\beta, \mathcal{A}_\beta, \mathbb{P}_\beta)$ the probability space on which this family is defined.

We naturally extend to $(\Omega_\beta \times \Omega, \mathcal{A}_\beta \otimes \mathcal{A}, \mathbb{P}_\beta \otimes \mathbb{P})$ the random variables $W_{\beta,u}$ and the random vectors $(N_{u0}, X_{u1}, X_{u2}), \dots)$ as

$$W_{\beta,u}(\omega_\beta, \omega) = W_{\beta,u}(\omega_\beta)$$
$$(N_{u0}(\omega_\beta, \omega), X_{u1}(\omega_\beta, \omega), X_{u2}(\omega_\beta, \omega), \dots) = (N_{u0}(\omega), X_{u1}(\omega), X_{u2}(\omega), \dots),$$

so that the families $\{W_{\beta,u}\}_{u\in\bigcup_{n\geq 0}\mathbb{N}_+^n}$ and $\{(N_{u0}, X_{u1}, X_{u2}, \dots)\}_{u\in\bigcup_{n\geq 0}\mathbb{N}_+^n}$ are independent.

The expectation with respect to $\mathbb{P}_\beta \otimes \mathbb{P}$ will be also denoted by $\mathbb{E}$.

For $n \geq 1$ and $\beta \in (0,1]$, we set $\mathcal{F}_n = \sigma\Big((N_u, X_{u1}, X_{u2}, \dots) : u \in \bigcup_{k=0}^n \mathbb{N}_+^{n-1}\Big)$ and $\mathcal{F}_{\beta,n} = \sigma\Big(W_{\beta,u1}, (W_{\beta,u2}, \dots) : u \in \bigcup_{k=0}^n \mathbb{N}_+^{n-1}\Big)$. We also denote by $\mathcal{F}_0$ and $\mathcal{F}_{\beta,0}$ the trivial σ-field.

If $\beta\mathbb{E}(N) > 1$, the random variables $N_{\beta,u}(\omega_\beta, \omega) = \sum_{i=1}^{N_u(\omega)} \mathbf{1}_{\{\beta^{-1}\}}(W_{\beta,ui}(\omega_\beta))$ define a new supercritical Galton-Watson process to which are associated the trees $\mathsf{T}_{\beta,n} \subset \mathsf{T}_n$ and $\mathsf{T}_{\beta,n}(u) \subset \mathsf{T}_n(u)$, $u \in \bigcup_{n\geq 0}\mathbb{N}_+^n$, $n \geq 1$, as well as the infinite tree $\mathsf{T}_\beta \subset \mathsf{T}$ and the boundary $\partial\mathsf{T}_\beta \subset \partial\mathsf{T}$ conditionally on non extinction.

For $u \in \bigcup_{n\geq 0}\mathbb{N}_+^n$, $1 \leq i \leq N(u)$ and $\varrho \in \mathcal{J}$ we define

$$W_{\varrho,ui} = \frac{\exp(\langle q_{|u|+1}|X_{ui}\rangle)}{\mathbb{E}(\sum_{i=1}^{N}\exp(\langle q_{|u|+1}|X_i\rangle)} = \exp(\langle q_{|u|+1}|X_{ui}\rangle - \widetilde{P}(q_{|u|+1})),$$

and

$$W_{\beta,\varrho,ui} = \frac{W_{\beta,ui}\exp(\langle q_{|u|+1}|X_{ui}\rangle)}{\mathbb{E}(\sum_{i=1}^{N} W_{\beta,i}\exp(\langle q_{|u|+1}|X_i\rangle)} = W_{\beta,ui}W_{\varrho,ui}$$

(since $\mathbb{E}(W_{\beta,i}) = 1$ and $(\mathbb{E}(W_{\beta,i}))_{i\geq 1}$ and $(N, (X_i)_{i\geq 1})$ are independent).

For $\varrho = (q_k)_{k\geq 1} \in \mathcal{J}$, $u \in \bigcup_{n\geq 0}\mathbb{N}_+^n$ and $n \geq 0$ we define

$$\begin{cases} Y_n(\varrho, u) = \displaystyle\sum_{v_1\cdots v_n \in \mathsf{T}_n(u)} \prod_{k=1}^n W_{\varrho, u\cdot v_1\cdots v_k} \\ Y_n(\beta, \varrho, u) = \displaystyle\sum_{v_1\cdots v_n \in \mathsf{T}_n(u)} \prod_{k=1}^n W_{\beta,\varrho, u\cdot v_1\cdots v_k} \end{cases}$$

When $u = \emptyset$ those quantities will be denoted by $Y_n(\varrho)$ and $Y_n(\beta,\varrho)$ respectively, and when $n = 0$, their values equal 1.

For $\beta \in (\mathbb{E}(N)^{-1}, 1]$, $L \geq 1$ and $\epsilon > 0$ we set

$$\mathcal{J}_{\beta,L,\epsilon} = \Big\{\varrho \in \mathcal{J} : \frac{1}{n}\sum_{k=1}^n \widetilde{P}^*(\nabla\widetilde{P}(q_k)) \geq -\log\beta + \epsilon, \forall n \geq L\Big\},$$

which is a compact subset of $\mathcal{J}$.

Notice that since $\widetilde{P}^*$ takes values between 0 and $\widetilde{P}(0) = \log(E(N))$ over I, we have

$$\Big\{\varrho \in \mathcal{J} : \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \widetilde{P}^*(\nabla\widetilde{P}(q_k) > 0\Big\} = \bigcup_{\beta\in(\mathbb{E}(N)^{-1},1],L\geq 1,\epsilon>0} \mathcal{J}_{\beta,L,\epsilon}. \tag{3.4}$$

Specification of the sequence $(N_j)_{j\geq 1}$

The function $\widetilde{P}$ is analytic. We denote by H its Hessian matrix. For each $j \geq 1$,

$$m_j = \sup_{t\in[0,1]} \sup_{v\in\mathbb{S}^{d-1}} \sup_{q\in D_j} {}^t vH(q+tv)v \tag{3.5}$$

and

$$\widetilde{m}_j = \sup_{t\in[0,1]} \sup_{p\in[1,2]} \sup_{q\in D_j} {}^t qH(q+t(p-1)q)q \tag{3.6}$$

are finite.

Next we notice that due to (1.1), two applications of the Cauchy-Schwartz inequality as in the proof of Proposition 3.1 yield

$$c(q,q') = \mathbb{E}\Big(\sum_{i=1}^{N} \exp\Big[\frac{1}{2}(\langle q|X_i\rangle - \widetilde{P}(q))\Big] \exp\Big[\frac{1}{2}(\langle q'|X_i\rangle - \widetilde{P}(q'))\Big]\Big) < 1 \tag{3.7}$$

if $q \neq q' \in \mathbb{R}^d$. For $j \geq 2$ let

$$c_j = \sup_{q\neq q'\in D_j} c(q,q') < 1. \tag{3.8}$$

Let $(\gamma_j)_{j\geq1} \in (0,1]^{\mathbb{N}_+}$ be a positive sequence such that $\gamma_j^2 m_j$ converges to 0 as $j\to\infty$ (in particular $\lim_{j\to\infty}\gamma_j = 0$) and $\gamma_{j+1}^2 m_{j+1} = o(\log c_j)$ as $j\to\infty$.

Let $(\widetilde{p}_j)_{j\geq1}$ be a sequence in $(1,2)$ such that $(\widetilde{p}_j - 1)\widetilde{m}_j$ converges to 0 as j tends to ∞.

We can also suppose that $\widetilde{p}_j$ is small enough so that we also have

$$\sup_{q\in D_j} \mathbb{E}(S(q)^{\widetilde{p}_j}) < \infty.$$

For each $q \in J$ there exists a real number $1 < p_q < 2$ such that $\varphi(p,q) < 0$ for all $p \in (1,p_q)$. Indeed, since $\widetilde{P}^*(\nabla\widetilde{P}(q)) > 0$ one has $\frac{\partial\phi}{\partial p}(q,1^+) < 0$.

For all $j \geq 1$ we set

$$p_j = \min(\widetilde{p}_j, \inf_{q\in D_{j+1}} p_q), \quad \text{and} \quad a_j = \sup_{q\in D_j} \varphi(p_j,q).$$

By construction, we have $a_j < 0$. Then let

$$s_j = \max\left\{\frac{\|S(q)\|_{p_j}}{\|S(q)\|_1} : q\in D_j\right\} \quad \text{and} \quad r_j = \max\big(a_j/p_j, (2jp_j)^{-1}(1-p_j)\big). \tag{3.9}$$

Recall that $N_0 = 0$. For $j \geq 1$ choose an integer N_j big enough so that

$$\frac{(j+1)!s_{j+1}}{1-\exp(r_{j+1})}\exp(N_j r_{j+1}) \leq j^{-2} \tag{3.10}$$

(which is possible since $r_{j+1} < 0$),

$$\frac{(j+1)!s_{j+1}}{1-\exp(r_{j+1})} + \frac{(j+2)!s_{j+2}}{1-\exp(r_{j+2})} \leq C_0 \exp(N_j \gamma_{j+1}^2 m_{j+1}) \tag{3.11}$$

with $C_0 = \dfrac{s_1}{1-\exp(r_1)} + \dfrac{2s_2}{1-\exp(r_2)}$,

$$N_j \geq \max(\gamma_{j+1}^{-2}, 3\log((j+1)!)); \tag{3.12}$$

if $j \geq 2$

$$(j!)^2 c_j^{N_j/2} \leq j^{-2} \tag{3.13}$$

and

$$\Big(\sum_{k=1}^{j-1} N_k\Big) \max(1, \max\{\|\nabla\widetilde{P}(q)\| : q \in D_{j-1}\}) \leq j^{-1} N_j \min(1, \{\|\nabla\widetilde{P}(q)\| : q \in D_j\}). \tag{3.14}$$

3.3.2 A family of measures indexed by $\mathcal{J}$

Proposition 3.4

1. *For all $u \in \bigcup_{n\geq 0}\mathbb{N}_+^n$, the sequence of continuous functions $Y_n(\cdot, u)$ converges uniformly on $\mathcal{J}$, almost surely and in L^1 norm, to a positive limit $Y(\cdot, u)$.*
2. *With probability 1, for all $\varrho \in \mathcal{J}$, the mapping*

$$\mu_\varrho([u]) = (\prod_{k=1}^{n} W_{\varrho,u_1\cdots u_k})Y(\varrho, u), \quad u \in \mathsf{T}_n.$$

defines a positive measure on $\partial\mathsf{T}$.

3. *With probability 1, for all $(\varrho, \varrho') \in \mathcal{J}^2$, the measures μ_ϱ and $\mu_{\varrho'}$ are absolutely continuous with respect to each other or mutually singular according to whether ϱ and ϱ' coincide ultimately or not.*

The measures μ_ϱ will be used to approximate from below the Hausdorff dimensions of the sets $E_X(K)$ in the next section.

Lemma 3.1 *[72] Let $(X_j)_{j\geq 1}$ be a sequence of centered independent real valued random variables. For every finite $I \subset \mathbb{N}_+$ and $p \in (1,2)$*

$$\mathbb{E}\Big(\Big|\sum_{i\in I} X_i\Big|^p\Big) \leq 2^{p-1}\sum_{i\in I}\mathbb{E}(|X_i|^p).$$

Lemma 3.2 *Let* $\varrho \in \mathcal{J}$ *and* $\beta \in (0,1]$. *Define* $Z_n(\beta,\varrho) = Y_n(\beta,\varrho) - Y_{n-1}(\beta,\varrho)$ *for* $n \geq 0$. *For every* $p \in (1,2)$ *we have*

$$\mathbb{E}(|Z_n(\beta,\varrho)|^p) \leq (2\beta^{-1})^p \frac{\mathbb{E}(S(q_n)^p)}{\mathbb{E}(S(q_n))^p} \prod_{k=1}^{n-1} \beta^{1-p} \exp\left(\widetilde{P}(pq_k) - p\widetilde{P}(q_k)\right). \tag{3.15}$$

Proof Fix $p \in (1,2]$. By using the branching property we can write

$$Z_n(\beta,\varrho) = \sum_{u \in T_{n-1}} \prod_{k=1}^{n-1} W_{\beta,u_1\cdots u_k} W_{\varrho,u_1\cdots u_k} \Big(\sum_{i=1}^{N(u)} W_{\beta,ui} W_{\varrho,ui} - 1 \Big).$$

Let $A_u = \sum_{i=1}^{N(u)} W_{\beta,ui} W_{\varrho,ui}$. By construction, the random variables $(A_u - 1)$, $u \in \mathsf{T}_{n-1}$, are centered and i.i.d., and independent of $\mathcal{F}_{\beta,n-1} \otimes \mathcal{F}_{n-1}$. Consequently, conditionally on $\mathcal{F}_{\beta,n-1} \otimes \mathcal{F}_{n-1}$, we can apply Lemma 3.1 to the family $\{A_u \prod_{k=1}^{n-1} W_{\beta,u_1\cdots u_k} W_{\varrho,u_1\cdots u_k}\}_{u \in T_{n-1}}$. Noticing that the A_u, $u \in T_{n-1}$, have the same distribution, this yields

$$\mathbb{E}(|Z_n(\beta,\varrho)|^p \, |\mathcal{F}_{n-1} \otimes \mathcal{F}_{\beta,n-1}) \leq 2^{p-1} \mathbb{E}(|A-1|^p) \sum_{u \in T_{n-1}} \prod_{k=1}^{n-1} W^p_{\beta,u_1\cdots u_k} W^p_{\varrho,u_1\cdots u_k}.$$

Since $\mathbb{E}(A) = 1$ and $A \geq 0$, convexity inequalities yield $\mathbb{E}(|A-1|^p) \leq 2\mathbb{E}(A^p)$. Moreover, since $0 \leq W_{\beta,i} \leq \beta^{-1}$, we have $A \leq \beta^{-1} S(q_n)/\mathbb{E}(S(q_n))$, and then $2^{p-1}\mathbb{E}(|A-1|^p) \leq (2\beta^{-1})^p \frac{\mathbb{E}(S(q_n)^p)}{\mathbb{E}(S(q_n))^p}$. Moreover, a recursive using of the branching property and the independence of the random vectors $(N_u, X_{u1}, \ldots)$ and random variables $W_{\beta,u}$ used in the constructions yields, setting $W_{q_k,i} = \exp(\langle q_k | X_i \rangle)/\mathbb{E}(S(q_k))$:

$$\begin{aligned}\mathbb{E}\Big(\sum_{u \in T_{n-1}} \prod_{k=1}^{n-1} W^p_{\beta,u_1\cdots u_k} W^p_{\varrho,u_1\cdots u_k} \Big) &= \prod_{k=1}^{n-1} \mathbb{E}(W_\beta^p) \mathbb{E}\Big(\sum_{i=1}^{N} W^p_{q_k,i} \Big) \\ &= \prod_{k=1}^{n-1} \beta^{1-p} \frac{\mathbb{E}(S(pq_k))}{\mathbb{E}(S(q_k))^p} = \prod_{k=1}^{n-1} \beta^{1-p} \exp\left(\widetilde{P}(pq_k) - p\widetilde{P}(q_k)\right).\end{aligned}$$

Collecting the previous estimates yields the conclusion.

Proof of Proposition 3.4. (1) Recall the definitions of the paragraph of section 3.3.1 in which the parameter set $\mathcal{J}$ is defined.

Let us first assume that $u = \emptyset$. First observe that if $n \geq 1$, it is easily seen from its construction that $Y_n(\cdot) = Y_n(\cdot,\emptyset)$ is a continuous function, constant over the set of those ϱ sharing the same n first components.

For $n \geq 1$ and $\varrho \in \mathcal{J}$, we have $M_{j_n} + 1 \leq n \leq M_{j_n+1}$, and Lemma 3.2 applied with $p = p_{j_n+1}$ and $\beta = 1$ provides us with the inequality

$$\begin{aligned}\|Y_n(\varrho) - Y_{n-1}(\varrho)\|^{p_{j_n+1}}_{p_{j_n+1}} &\leq 2^{p_{j_n+1}} \frac{\mathbb{E}(S(q_n)^{p_{j_n+1}})}{\mathbb{E}(S(q_n))^{p_{j_n+1}}} \prod_{k=1}^{n-1} \exp\left(\widetilde{P}(p_{j_n+1}q_k) - p_{j_n+1}\widetilde{P}(q_k)\right) \\ &= 2^{p_{j_n+1}} \frac{\mathbb{E}(S(q_n)^{p_{j_n+1}})}{\mathbb{E}(S(q_n))^{p_{j_n+1}}} \prod_{k=1}^{n-1} \exp\left(\varphi(p_{j_n+1}, q_k)\right)\end{aligned}$$

$$\begin{aligned} &\le 2^{p_{j_n+1}} s_{j_n+1}^{p_{j_n+1}} \prod_{k=1}^{n-1} \exp\big(\sup_{q\in D_{j_n+1}} \varphi(p_{j_n+1}, q)\big) \\ &\quad (\text{since } \{q_k : 1\le k\le n\} \subset D_{j_n+1}) \\ &\le 2^{p_{j_n+1}} s_{j_n+1}^{p_{j_n+1}} \exp((n-1)p_{j_n+1} r_{j_n+1}) \\ &\qquad (\text{due (3.9)}\,;\text{ this bound is independent of } \varrho). \end{aligned}$$

Notice that by the definition of $\mathcal{J}$, the cardinality of $\{\varrho_{|n} : \varrho\in\mathcal{J}\}$ is equal to that of $\prod_{j=1}^{j_n+1} D_j$, i.e. $(j_n+1)!$, and $Y_n(\varrho)-Y_{n-1}(\varrho)$ only depends on $(q_1,\cdots,q_n)$. Consequently,

$$\|\sup_{\varrho\in\mathcal{J}} |Y_n(\varrho)-Y_{n-1}(\varrho)|\|_1 \le \sum_{\{\varrho_{|n}:\varrho\in\mathcal{J}\}} \|Y_n(\varrho)-Y_{n-1}(\varrho)\|_{p_{j_n+1}} \le 2(j_n+1)!s_{j_n+1}\exp((n-1)r_{j_n+1}).$$

We deduce from this that

$$\begin{aligned} \sum_{n\ge1} \|\sup_{\varrho\in\mathcal{J}} |Y_n(\varrho)-Y_{n-1}(\varrho)|\|_1 &\le \sum_{j\ge0}\sum_{M_j+1\le n\le M_{j+1}} 2(j+1)!s_{j+1}\exp((n-1)r_{j+1}) \\ &\le \sum_{j\ge0} 2(j+1)!s_{j+1}\frac{\exp(M_j r_{j+1})}{1-\exp(r_{j+1})} \\ &\le \frac{2s_1}{1-\exp(r_1)} + \sum_{j\ge1} 2(j+1)!s_{j+1}\frac{\exp(N_j r_{j+1})}{1-\exp(r_{j+1})} \\ &\le \frac{2s_1}{1-\exp(r_1)} + 2\sum_{j\ge1} j^{-2} < \infty, \end{aligned}$$

where we have used (3.10). The convergence of the above series gives the desired uniform convergence, almost surely and in L^1 norm, of Y_n to a function Y.

Let us show that Y does not vanish on $\mathcal{J}$ almost surely. For each $n\ge1$ let $\mathcal{J}_n = \{\varrho_{|n} : \varrho\in\mathcal{J}\}$, and for $\gamma=(\gamma_1,\dots,\gamma_n)\in\mathcal{J}_n$ define the event $E_\gamma = \{\omega\in\Omega : \exists\varrho\in\mathcal{J},\ Y(\varrho)=0,\ \varrho_{|n}=\gamma\}$. Let $E=\{\omega\in\Omega:\exists\varrho\in\mathcal{J},\ Y(\varrho)=0\}$. Since the functions Y_n are almost surely positive, this event is a tail event, and it has probability 0 or 1. The same property holds for the events E_γ, $\gamma\in\bigcup_{n\ge1}\mathcal{J}_n$.

Suppose that E has probability 1. Since $E=\bigcup_{\varrho_1\in\mathcal{J}_1} E_{\gamma_1}$, necessarily, there exists $\gamma_1\in\mathcal{J}_1$ such that $\mathbb{P}(E_{\gamma_1})>0$, and so $\mathbb{P}(E_{\gamma_1})=1$. Iterating this remark we can build an infinite deterministic sequence $\gamma=(\gamma_k)_{k\ge1}\in\mathcal{J}$ such that $\mathbb{P}(E_{(\gamma_1,\dots,\gamma_n)})=1$ for all $n\ge1$. This means that almost surely, for all $n\ge1$, there exists $\varrho^{(n)}\in\mathcal{J}$ such that $\varrho^{(n)}_{|n}=(\gamma_1,\dots,\gamma_n)$ and $Y(\varrho^{(n)})=0$. But $\varrho^{(n)}_{|n}=(\gamma_1,\dots,\gamma_n)$ implies that $\varrho^{(n)}$ converges to γ. Hence, by continuity of Y at γ, we get $Y(\gamma)=0$ almost surely. However, a consequence of our convergence result for Y_n is that the martingale $Y_n(\gamma)$ converges in L^1 to $Y(\gamma)$, so that $\mathbb{E}(Y(\gamma))=1$. This is a contradiction. Thus $\mathbb{P}(E)=0$.

Now let $u\in\bigcup_{n\ge1}\mathbb{N}_+^n$. By using the same calculations as above, for all $n\ge1$ we get

$$\|\sup_{\varrho\in\mathcal{J}} |Y_n(\varrho,u)-Y_{n-1}(\varrho,u)|\|_1 \le 2(j_{|u|+n}+1)!s_{j_{|u|+n}+1}\exp((n-1)r_{j_{|u|+n}+1}).$$

Thus

$$\sum_{n\ge1} \|\sup_{\varrho\in\mathcal{J}} |Y_n(\varrho,u)-Y_{n-1}(\varrho,u)|\|_1$$

$$\begin{aligned}
&\leq \sum_{n=1}^{M_{j_{|u|}+1}-|u|} 2(j_{|u|}+1)!s_{j_{|u|}+1}\exp((n-1)r_{j_{|u|}+1})\\
&\quad+\sum_{j\geq j_{|u|}+1}\ \sum_{M_j+1\leq |u|+n\leq M_{j+1}} 2(j+1)!s_{j+1}\exp((n-1)r_{j+1})\\
&\leq \sum_{j_{|u|}\leq j\leq j_{|u|}+1}\frac{2(j+1)!s_{j+1}}{1-\exp(r_{j+1})}+\sum_{j\geq j_{|u|}+2}2(j+1)!s_{j+1}\frac{\exp((M_j-|u|)r_{j+1})}{1-\exp(r_{j+1})}.
\end{aligned}$$

Now, since $M_{j_{|u|}}+1\leq |u|\leq M_{j_{|u|}+1}$, for $j\geq j_{|u|}+2$ we have and $M_j-|u|\geq N_j$ and

$$\begin{aligned}
&\sum_{n\geq 1}\|\sup_{\varrho\in\mathcal{J}}|Y_n(\varrho,u)-Y_{n-1}(\varrho,u)|\|_1\\
&\leq \sum_{j_{|u|}\leq j\leq j_{|u|}+1}\frac{2(j+1)!s_{j+1}}{1-\exp(r_{j+1})}+\sum_{j\geq j_{|u|}+2}2(j+1)!s_{j+1}\frac{\exp(N_j r_{j+1})}{1-\exp(r_{j+1})} \qquad (3.16)\\
&\leq 2C_0\exp(N_{j_{|u|}}\gamma^2_{j_{|u|}+1}m_{j_{|u|}+1})+2\sum_{j\geq j_{|u|}+2}j^{-2},
\end{aligned}$$

where we have used (3.10) and (3.11). This yields the desired convergence to a limit $Y(\cdot,u)$. Moreover, since $\bigcup_{k\geq 0}\mathbb{N}_+^k$ is countable, the convergence holds almost surely, simultaneously for all u. Then the proof finishes as for $u=\emptyset$.

Let us put the previous upper bound in a form that will be useful. Let $\epsilon_k=\gamma^2_{j_k+1}m_{j_k+1}$ for all $k\geq 0$. It follows from the above calculations, the fact that $Y_0(\cdot,u)=1$ for all $u\in\bigcup_{k\geq 0}\mathbb{N}_+^k$, and the fact that $|u|\geq N_{j_{|u|}}$ that there exists a constant $C_{\mathcal{J}}$ such that :

$$\|\sup_{\varrho\in\mathcal{J}}Y(\varrho,u)\|_1\leq C_{\mathcal{J}}\exp(\epsilon_{|u|}N_{j_{|u|}})\leq C_{\mathcal{J}}\exp(\epsilon_{|u|}|u|)\quad (\forall\, u\in\bigcup_{k\geq 0}\mathbb{N}_+^k). \qquad (3.17)$$

Remark 3.1 *Let K be a compact subset of J containing the unique element of D_1. Then, it is not difficult to see that there exists $p_K\in(1,2)$ such that $\sup_{j\geq 1}\sup_{q\in D_j\cap K}\varphi(p_K,q)<0$ and $\sup_{j\geq 1}\sup_{q\in D_j\cap K}\frac{\|S(q)\|_{p_K}}{\|S(q)\|_1}<\infty$. Then it follows from the previous calculations that if we define $\mathcal{J}(K)=\{\varrho\in\mathcal{J}:\ \forall\, k\geq 1,\ q_k\in K\}$, then*

$$\|\sup_{\varrho\in\mathcal{J}(K)}Y(\varrho,u)\|_{p_K}=O((j_{|u|}+2)!).$$

(2) This is a direct consequence of the branching property.

(3) Postponed to Section 3.3.5.

□

3.3.3 Lower bounds for the Hausdorff dimensions of the measures $\{\mu_\varrho\}_{\varrho\in\mathcal{J}}$

The estimation of the Hausdorff dimensions of the measures μ_ϱ, $\varrho\in\mathcal{J}$, will use the second part of the next proposition, together with a uniform version of the percolation-covering argument introduced by Kahane in [47, 46] in order to remove a technical assumption made in [45] to estimate the Hausdorff dimension of Mandelbrot measures on the boundary of a homogeneous tree.

Proposition 3.5 *Let $\beta \in (0,1]$ such that $\beta\mathbb{E}(N) > 1$. Conditionally on non extinction of $(\mathsf{T}_{\beta,n}(u))_{n\geq 1}$, for all $N \geq 1$ and $\epsilon \in \mathbb{Q}_+^*$,*

1. *the sequence of continuous functions $Y_n(\cdot,\beta)$ converges uniformly, almost surely and in L^1 norm, to a positive limit $Y(\beta,\cdot)$ on $\mathcal{J}_{\beta,L,\epsilon}$.*
2. *the sequence of continuous functions*

$$\varrho \mapsto \widetilde{Y}_n(\beta,\varrho) = \sum_{u\in\mathsf{T}_n} (\prod_{k=1}^n W_{\beta,u_1\cdots u_k})\mu_\varrho([u])$$

converges uniformly, almost surely and in L^1-norm, towards $Y(\beta,\cdot)$ on $\mathcal{J}_{\beta,L,\epsilon}$.

Proof (1) Let $L \geq 1$ and $\epsilon > 0$. For $\varrho \in \mathcal{J}_{\beta,L,\epsilon}$ and $n \geq 1$, Lemma 3.2 applied with $p = p_{j_n+1}$ provides us with the inequality

$$\begin{aligned} & \|Y_n(\beta,\varrho) - Y_{n-1}(\beta,\varrho)\|_{p_{j_n+1}}^{p_{j_n+1}} \\ \leq\ & (2\beta^{-1})^{p_{j_n+1}} \frac{\mathbb{E}(S(q_n)^{p_{j_n+1}})}{\mathbb{E}(S(q_n))^{p_{j_n+1}}} \prod_{k=1}^{n-1} \beta^{1-p_{j_n+1}} \exp\left(\widetilde{P}(p_{j_n+1}q_k) - p_{j_n+1}\widetilde{P}(q_k)\right). \end{aligned}$$

Let $q \in D_{j_n+1}$ and set $g : \lambda \in \mathbb{R} \mapsto \widetilde{P}(\lambda q)$. For $p \in [1,2]$ we have

$$g(p) = g(1) + (p-1)g'(1) + (p-1)^2 \int_0^1 (1-t)g''(1+t(p-1))\,\mathrm{d}t,$$

with

$$g''(1+t(p-1)) = q^t H(q+t(p-1)q)q \leq \sup_{q\in D_{j_n+1}} q^t H(q+t(p-1)q)q \leq \widetilde{m}_{j_n+1},$$

where $(\widetilde{m}_j)_{j\geq 1}$ is defined in (3.6). Let $\eta_j = 2(p_j-1)\widetilde{m}_j$ for $j \geq 1$. By construction of $(p_j)_{j\geq 1}$ we have $\lim_{j\to\infty} \eta_j = 0$. Specifying $p = p_{j_n+1}$ we have now

$$\begin{aligned} \widetilde{P}(p_{j_n+1}q) - p_{j_n+1}\widetilde{P}(q) &= g(p) - pg(1) \\ &\leq (1-p_{j_n+1})(g(1) - g'(1)) + \eta_{j_n+1}(p_{j_n+1}-1) \\ &= (1-p_{j_n+1})\widetilde{P}^*(\nabla\widetilde{P}(q)) + \eta_{j_n+1}(p_{j_n+1}-1). \end{aligned}$$

We can insert this upper bound in our estimation of $\|Y_n(\beta,\varrho) - Y_{n-1}(\beta,\varrho)\|_{p_{j_n}+1}^{p_{j_n}+1}$ and get, remembering that $\varrho \in \mathcal{J}_{\beta,L,\epsilon}$, for $n \geq L+1$

$$\begin{aligned} & \|Y_n(\beta,\varrho) - Y_{n-1}(\beta,\varrho)\|_{p_{j_n+1}}^{p_{j_n+1}} \\ \leq\ & (2\beta^{-1})^{p_{j_n+1}} s_{j_n+1}^{p_{j_n+1}} \exp\Big((1-p_{j_n+1}) \sum_{k=1}^{n-1} \big(\log(\beta) + \widetilde{P}^*(\nabla\widetilde{P}(q_k)) - \eta_{j_n+1}\big)\Big) \\ \leq\ & (2\beta^{-1})^{p_{j_n+1}} s_{j_n+1}^{p_{j_n+1}} \exp\big((n-1)(1-p_{j_n+1})(\epsilon - \eta_{j_n+1})\big). \end{aligned}$$

Let $j(\epsilon)=\min\{j\geq\lfloor\epsilon^{-1}\rfloor+1:\eta_j\leq\epsilon/2\}$ and $n_\epsilon=\min\{n\geq L+1:j_{n+1}\geq j(\epsilon)\}$. For $n\geq n_\epsilon$ we have, remembering (3.9)

$$\begin{aligned}
&\|Y_n(\beta,\varrho)-Y_{n-1}(\beta,\varrho)\|_{p_{j_n+1}}^{p_{j_n+1}}\\
\leq\ & (2\beta^{-1})^{p_{j_n+1}}s_{j_n+1}^{p_{j_n+1}}\exp\big((n-1)(1-p_{j_n+1})\epsilon/2)\big)\\
\leq\ & (2\beta^{-1})^{p_{j_n+1}}s_{j_n+1}^{p_{j_n+1}}\exp\big((n-1)(1-p_{j_n+1})/2(j_n+1))\big)\\
=\ & (2\beta^{-1})^{p_{j_n+1}}s_{j_n+1}^{p_{j_n+1}}\exp\big((n-1)p_{j_n+1}r_{j_n+1})\big)
\end{aligned}$$

Consequently, using the same inequalities as in the proof of Proposition 3.4 we get

$$\begin{aligned}
\sum_{n\geq n_\epsilon}\|\sup_{\varrho\in\mathcal{J}_{\beta,L,\epsilon}}|Y_n(\beta,\varrho)-Y_{n-1}(\beta,\varrho)|\|_1 &\leq \sum_{j\geq j(\epsilon)}\sum_{M_j+1\leq n\leq M_{j+1}}2(j+1)!s_{j+1}\exp((n-1)r_{j+1})\\
&\leq \sum_{j\geq 0}2\beta^{-1}(j+1)!s_{j+1}\frac{\exp(M_jr_{j+1})}{1-\exp(r_{j+1})}\\
&\leq \frac{2\beta^{-1}s_1}{1-\exp(r_1)}+\sum_{j\geq 1}2\beta^{-1}(j+1)!s_{j+1}\frac{\exp(N_jr_{j+1})}{1-\exp(r_j)}\\
&\leq \frac{2\beta^{-1}s_1}{1-\exp(r_1)}+2\beta^{-1}\sum_{j\geq 1}j^{-2}<\infty.
\end{aligned}$$

This yields the conclusion on uniform convergence. The fact that the limit $Y(\beta,\cdot)$ does not vanish almost surely, conditionally on non extinction of $(\mathsf{T}_{\beta,n})_{n\geq 1}$, follows the same lines as in the study of $Y(\cdot)$, combined with the fact that for a fixed $\varrho\in\mathcal{J}_{\beta,L,\epsilon}$, the probability that the limit of $Y_n(\beta,\varrho)$ be 0 equals that of the extinction of $(\mathsf{T}_{\beta,n})_{n\geq 1}$. This comes from the fact that conditionally on non extinction, the event $\{Y(\beta,\varrho)=0\}$ is asymptotic so has probability 0 or 1, and it has probability 0 since the convergence of $Y_n(\beta,\varrho)$ to $Y(\beta,\varrho)$ holds in L^1.

Thus, we have the desired result for a given couple (L,ϵ) ; but it holds simultaneously for all $L\geq 1$ and $\epsilon\in\mathbb{Q}_+^*$ since $\mathbb{N}_+\times\mathbb{Q}_+^*$ is countable.

(2) Here we develop, in the context of the boundary of a supercritical Galton-Watson tree, a uniform version of the argument used by Kahane in [46] on homogeneous trees, and written in complete rigor in [75] and in [32] (for general multiplicative chaos). Our uniform approach can be generalized to uncountable families of multiplicative chaos on general σ-compact sets.

Fix $L\geq 1$ and $\epsilon>0$. Denote by E the separable Banach space of the real valued continuous functions over the compact set $\mathcal{J}_{\beta,L,\epsilon}$ endowed with the supremum norm $\|\ \|_\infty$.

For $n\geq m\geq 1$ and $\varrho\in\mathcal{J}_{\beta,L,\epsilon}$ let

$$Z_{m,n}(\varrho)=\sum_{u\in T_m}Y_{n-m}(\varrho,u)\prod_{k=1}^m W_{\beta,u_{|k}}W_{\varrho,u_{|k}}.$$

Notice that $Z_{n,n}(\varrho)=Y_n(\beta,\varrho)$. Moreover, since $Y_n(\beta,\cdot)$ converges almost surely and in L^1 norm to $Y(\beta,\cdot)$ as $n\to\infty$, $Y_n(\beta,\cdot)$ belongs to $L_E^1=L_E^1(\Omega_\beta\times\Omega,\mathcal{A}_\beta\times\mathcal{A},\mathbb{P}_\beta\times\mathbb{P})$ (where we use the notations of [61, Section V-2]), so that the continuous random function $\mathbb{E}(Z_{n,n}(\varrho)|\mathcal{F}_{\beta,m}\otimes\mathcal{F}_n)$ is well defined by [61, Proposition V-2-5] ; also, for any fixed $\varrho \in \mathcal{J}_{\beta,L,\epsilon}$, we can deduce from the definitions and the independence assumptions that

$$Z_{m,n}(\varrho) = \mathbb{E}(Z_{n,n}(\varrho)|\mathcal{F}_{\beta,m} \otimes \mathcal{F}_n)$$

almost surely. By [61, Proposition V-2-5] again, since $g \in E \mapsto g(\varrho)$ is a continuous linear form over E, we thus have

$$Z_{m,n}(\varrho) = \mathbb{E}(Z_{n,n}(\cdot)|\mathcal{F}_{\beta,m} \otimes \mathcal{F}_n)(\varrho)$$

almost surely. By considering a dense countable set of ϱ in $\mathcal{J}_{\beta,L,\epsilon}$, we can conclude that the random continuous functions $Z_{m,n}(\cdot)$ and $\mathbb{E}(Z_{n,n}(\cdot)|\mathcal{F}_{\beta,m} \otimes \mathcal{F}_n)$ are equal almost surely.

Similarly, since for each $\varrho \in \mathcal{J}_{\beta,L,\epsilon}$ the martingale $(Y_n(\beta,\varrho), \mathcal{F}_{\beta,n} \otimes \mathcal{F}_n)$ converges to $Y(\beta,\varrho)$ almost surely and in L^1, and $Y(\beta,\cdot) \in L^1_E$, by using [61, Proposition V-2-5] again we can get

$$Z_{n,n}(\cdot) = \mathbb{E}(Y(\beta,\cdot)|\mathcal{F}_{\beta,n} \otimes \mathcal{F}_n), \text{ hence } Z_{m,n}(\cdot) = \mathbb{E}(Y(\beta,\cdot)|\mathcal{F}_{\beta,m} \otimes \mathcal{F}_n), \qquad (3.18)$$

almost surely. Moreover, it follows from Proposition 3.4(1) and the definition of $\mu_\varrho([u])$ that $Z_{m,n}(\cdot)$ converges almost surely uniformly and in L^1 norm, as $n \to \infty$, to $\widetilde{Y}_m(\beta,\cdot)$. This and (3.18) yield, using [61, Proposition V-2-6],

$$\widetilde{Y}_m(\beta,\cdot) = \lim_{n\to\infty} Z_{m,n}(\cdot) = \mathbb{E}\big(Y(\beta,\cdot)|\mathcal{F}_{\beta,m} \otimes \sigma(\bigcup_{n\geq 1} \mathcal{F}_n)\big),$$

and finally

$$\lim_{m\to\infty} \widetilde{Y}_m(\beta,\cdot) = \mathbb{E}\big(Y(\beta,\cdot)|\sigma(\bigcup_{m\geq 1} \mathcal{F}_{\beta,m}) \otimes \sigma(\bigcup_{n\geq 1} \mathcal{F}_n)\big) = Y(\beta,\cdot).$$

almost surely (since by construction $Y(\beta,\cdot)$ is $\sigma(\bigcup_{m\geq 1} \mathcal{F}_{\beta,m}) \otimes \sigma(\bigcup_{n\geq 1} \mathcal{F}_n)$-measurable), where the convergences hold in the uniform norm.

Proposition 3.6 *With probability 1, for all $\varrho \in \mathcal{J}$,*

$$\underline{\dim}(\mu_\varrho) \geq \liminf_{n\to\infty} n^{-1} \sum_{k=1}^n \widetilde{P}^*(\nabla \widetilde{P}(q_k)).$$

Proof Let $\beta \in (0,1]$ such that $\beta\mathbb{E}(N) > 1$. Let $L \geq 1$ and $\epsilon \in \mathbb{Q}_+^*$.

For every $t \in \partial\mathsf{T}$ and $\omega_\beta \in \Omega_\beta$ set

$$Q_{\beta,n}(t,\omega_\beta) = \prod_{k=1}^n W_{\beta,t_{|k}},$$

so that for $\varrho \in \mathcal{J}_{\beta,L,\epsilon}$, $\widetilde{Y}_n(\beta,\varrho)$ is the total mass of the measure $Q_{\beta,n}(t,\omega_\beta) \cdot \mathrm{d}\mu^\omega_\varrho(t)$.

Now, Proposition 3.5 claims that there exists a measurable subset A of $\Omega \times \Omega_\beta$ of full probability in the set of those $(\omega, \omega_\beta,)$ such that $(\mathsf{T}_{\beta,n})_{n\geq 1}$ survives such that for all $(\omega,\omega_\beta) \in A$, for all $\varrho \in \mathcal{J}_{\beta,L,\epsilon}$, $\widetilde{Y}_n(\beta,\varrho)$ does not converge to 0. Moreover, since the branching number of the tree T is $\mathbb{P}$-almost surely equal to the constant $\mathbb{E}(N)$ and $\beta\mathbb{E}(N) > 1$, conditionally on T, the $\mathbb{P}_\beta$-probability of non extinction of

$(\mathsf{T}_{\beta,n})_{n\geq 1}$ is positive ([57, Th. 6.2]). Thus, the projection of A to Ω has $\mathbb{P}$-probability 1, and there exists a measurable subset $\Omega(\beta, L, \epsilon)$ of Ω, such that $\mathbb{P}(\Omega(\beta, L, \epsilon)) = 1$ and for all $\omega \in \Omega(\beta, L, \epsilon)$, there exists $\Omega_\beta^\omega \subset \Omega_\beta$ of positive probability such that for all $\omega \in \Omega(\beta, L, \epsilon)$, for all $\varrho \in \mathcal{J}_{\beta,L,\epsilon}$, for all $\omega_\beta \in \Omega_\beta^\omega$, $\widetilde{Y}_n(\beta, \varrho)$ does not converge to 0. In terms of the multiplicative chaos theory developed in [47], this means, that for all $\omega \in \Omega(\beta, L, \epsilon)$ and $\varrho \in \mathcal{J}_{\beta,L,\epsilon}$, the set of those ω_β such that the multiplicative chaos $(Q_{\beta,n}(\cdot, \omega))_{n\geq 1}$ has not killed the measure μ_ϱ on the compact set $\partial\mathsf{T}$ has a positive $\mathbb{P}_\beta$-probability. Now, the good property of $(Q_{\beta,n}(\cdot, \omega))_{n\geq 1}$ is that $\mathbb{E}_\beta\big(\sup_{t\in B}(Q_{\beta,n}(t))^h\big) = e^{n(1-h)\log(\beta)} = (|B|)^{-(1-h)\log(\beta)}$ for any $h \in (0, 1)$ and any ball B of generation n in $\partial\mathsf{T}$, where $|B|$ stands for the diameter of B and $\mathbb{E}_\beta$ stands for the expecttion with respect to $\mathbb{P}_\beta$. Thus, we can apply Theorem 3 of [47] and claim that for all $\omega \in \Omega(\beta, L, \epsilon)$ and all $\varrho \in \mathcal{J}_{\beta,L,\epsilon}$, the measure μ_ϱ is not carried by a Borel set of Hausdorff dimension less than $-\log(\beta)$.

Let $\Omega' = \bigcap_{\beta\in(\mathbb{E}(N)^{-1},1]\cap\mathbb{Q}_+^*, L\geq 1, \epsilon\in\mathbb{Q}^*} \Omega(\beta, L, \epsilon)$. This set is of $\mathbb{P}$-probability 1. Let $\varrho \in \mathcal{J}$. If $D := \liminf_{n\to\infty} n^{-1} \sum_{k=1}^n \widetilde{P}^*(\nabla\widetilde{P}(q_k)) = 0$, then it is clear that $\underline{\dim}(\mu_\varrho) \geq D$. Otherwise, by (3.4), there exists a sequence of points $(\beta_n, L_n, \epsilon_n) \in (\mathbb{E}(N)^{-1}, 1] \times \mathbb{N}_+ \times \mathbb{Q}_+^*$ such that $D \geq -\log(\beta_n) + \epsilon_n/2$ for all $n \geq 1$, $\lim_{n\to\infty} -\log(\beta_n) = D$, $\lim_{n\to\infty} \epsilon_n = 0$, and $\varrho \in \bigcap_{n\geq 1} \mathcal{J}_{\beta_n,L_n,\epsilon_n}$. Consequently, the previous paragraph implies that for all $\omega \in \Omega'$, $\underline{\dim}(\mu_\varrho^\omega) \geq \limsup_{n\to\infty} -\log(\beta_n) = D$.

3.3.4 Lower bounds for the Hausdorff dimensions of the set $E_X(K)$

The sharp lower bound estimates for the Hausdorff dimensions of the set $E_X(K)$ are direct consequences of Proposition 3.6 and the following last two propositions.

Proposition 3.7 *With probability* 1, *for all* $\varrho = (q_k)_{k\geq 1} \in \mathcal{J}$, *for* μ_ϱ*-almost all* $t \in \partial\mathsf{T}$ *we have*

$$\lim_{n\to\infty} n^{-1}\Big(S_nX(t) - \sum_{k=1}^n \nabla\widetilde{P}(q_k)\Big) = 0.$$

Proof Let v a be vector of the canonical basis $\mathcal{B}$ of $\mathbb{R}^d$. For $\varrho \in \mathcal{J}$, $n \geq 1$ and $\epsilon > 0$, we set :

$$E^1_{\varrho,n,\epsilon}(v) = \Big\{t \in \partial\mathsf{T} : \Big\langle v\Big|S_nX(t) - \sum_{k=1}^n \nabla\widetilde{P}(q_k)\Big\rangle \geq n\epsilon\Big\}$$
$$E^{-1}_{\varrho,n,\epsilon}(v) = \Big\{t \in \partial\mathsf{T} : \Big\langle v\Big|S_nX(t) - \sum_{k=1}^n \nabla\widetilde{P}(q_k)\Big\rangle \leq -n\epsilon\Big\}$$

Suppose we have shown that for all $\epsilon > 0$, $\lambda \in \{-1, 1\}$ and $v \in \mathcal{B}$ we have

$$\mathbb{E}\Big(\sup_{\varrho\in\mathcal{J}} \sum_{n\geq 1} \mu_\varrho(E^\lambda_{\varrho,n,\epsilon}(v))\Big) < \infty. \tag{3.19}$$

Then, with probability 1, for all $\varrho \in \mathcal{J}$, $\lambda \in \{-1, 1\}$, $\epsilon \in \mathbb{Q}_+^*$ and $v \in \mathcal{B}$, $\sum_{n\geq 1} \mu_\varrho(E^\lambda_{\varrho,n,\epsilon}(v)) < \infty$, consequently, by the Borel-Cantelli lemma, for μ_ϱ-almost every

t, for all $v \in \mathcal{B}$, we have

$$\lim_{n\to\infty} \Big\langle v \Big| n^{-1}\Big(S_nX(t) - \sum_{k=1}^n \nabla\widetilde{P}(q_k)\Big)\Big\rangle = 0,$$

which yields the desired result.

Now we prove (3.19) when $\lambda = 1$ (the case $\lambda = -1$ is similar). Let $\varrho \in \mathcal{J}$. For every $\gamma > 0$ have

$$\mu_\varrho(E^1_{\varrho,n,\epsilon}) \quad\le\quad \sum_{u\in\mathsf{T}_n} \mu_\varrho([u]) \prod_{k=1}^n \exp\big(\gamma\langle v|X_{u_{|k}} - \gamma\langle v|\nabla\widetilde{P}(q_k)\rangle - \gamma\epsilon\big) = f_{n,\gamma}(\varrho),$$

and due to the definition of μ_ϱ, $f_{n,\gamma}(\varrho)$ can be written

$$f_{n,\gamma}(\varrho) = \sum_{u\in\mathsf{T}_n} Y(\varrho,u) \prod_{k=1}^n \exp\big(\langle q_k+\gamma v|X_{u_{|k}}\rangle - \widetilde{P}(q_k) - \langle\gamma v|\nabla\widetilde{P}(q_k)\rangle - \gamma\epsilon\big).$$

We have

$$\sup_{\varrho\in\mathcal{J}} f_{n,\gamma}(\varrho) \le \sum_{u\in\mathsf{T}_n} \sup_{\varrho\in\mathcal{J}} Y(\varrho,u) \sup_{\varrho_{|n}:\varrho\in\mathcal{J}} \prod_{k=1}^n \exp\big(\langle q_k+\gamma v|X_{u_{|k}}\rangle - \widetilde{P}(q_k) - \langle\gamma v|\nabla\widetilde{P}(q_k)\rangle - \gamma\epsilon\big).$$

Consequently, since $\mathbb{E}(\sup_{\varrho\in\mathcal{J}} Y(\varrho,u)) \le C_{\mathcal{J}}\exp(\epsilon_{|u|}|u|)$ by (3.17), we have (taking into account the independences)

$$\begin{aligned}
&\mathbb{E}(\sup_{\varrho\in\mathcal{J}} f_{n,\gamma}(\varrho))\\
\le\;& C_{\mathcal{J}}\exp(n\epsilon_n)\mathbb{E}\Big(\sum_{u\in\mathsf{T}_n} \sup_{\varrho_{|n}:\varrho\in\mathcal{J}} \prod_{k=1}^n \exp\big(\langle q_k+\gamma v|X_{u_{|k}}\rangle - \widetilde{P}(q_k) - \langle\gamma v|\nabla\widetilde{P}(q_k)\rangle - \gamma\epsilon\big)\Big)\\
\le\;& C_{\mathcal{J}}\exp(n\epsilon_n)\mathbb{E}\Big(\sum_{u\in T_n} \sum_{\varrho_{|n}:\varrho\in\mathcal{J}} \prod_{k=1}^n \exp\big(\langle q_k+\gamma v|X_{u_{|k}}\rangle - \widetilde{P}(q_k) - \langle\gamma v|\nabla\widetilde{P}(q_k)\rangle - \gamma\epsilon\big)\Big)\\
=\;& C_{\mathcal{J}}\exp(n\epsilon_n) \sum_{\varrho_{|n}:\varrho\in\mathcal{J}} \prod_{k=1}^n \exp\big(\widetilde{P}(q_k+\gamma v) - \widetilde{P}(q_k) - \langle\gamma v|\nabla\widetilde{P}(q_k)\rangle - \gamma\epsilon\big).
\end{aligned}$$

For each $\varrho \in \mathcal{J}$, we have $q_k \in D_{j_n+1}$ for all $1 \le k \le n$. Thus, writing for each $1 \le k \le n$ the Taylor expansion with integral rest of order 2 of $\gamma \mapsto \widetilde{P}(q_k+\gamma v) - \widetilde{P}(q_k) - \langle\gamma v|\nabla\widetilde{P}(q_k)\rangle$ at 0, taking $\gamma = \gamma_{j_n+1}$, and using (3.5) we get

$$\sum_{k=1}^n \widetilde{P}(q_k+\gamma v) - \widetilde{P}(q_k) - \langle\gamma v|\nabla\widetilde{P}(q_k)\rangle \le n\gamma_{j_n+1}^2 m_{j_n+1}$$

uniformly in $\varrho \in \mathcal{J}$. Consequently, using that $\epsilon_n = 2\gamma_{j_n+1}^2 m_{j_n+1}$ and $\mathrm{card}(\{\varrho_{|n} : \varrho \in \mathcal{J}\}) = (j_n+1)!$, we get

$$\mathbb{E}(\sup_{\varrho\in\mathcal{J}} f_{n,\varrho,\gamma_{j_n+1}}) \le C_{\mathcal{J}}(j_n+1)!\exp\big((-n\gamma_{j_n+1}(\epsilon - 3\gamma_{j_n+1}m_{j_n+1})\big).$$

Now we use (3.12) : $(j_n+1)! \le \exp(N_{j_n}^{1/3}) \le \exp(n^{1/3})$ and $\gamma_{j_n+1} \ge N_{j_n}^{-1/2} \ge n^{-1/2}$. Thus

$$\mathbb{E}\Big(\sup_{\varrho\in\mathcal{J}} \mu_\varrho(E^1_{\varrho,n,\epsilon}(v))\Big) \le \mathbb{E}(\sup_{\varrho\in\mathcal{J}} f_{n,\varrho,\gamma_{j_n+1}}) \le C_{\mathcal{J}}\exp(n^{1/3})\exp\big(-n^{1/2}(\epsilon - 3\gamma_{j_n+1}m_{j_n+1})\big).$$

Since $\gamma_{j_n+1}m_{j_n+1}$ tends to 0 as n tends to ∞, we get $\sum_{n\ge1}\mathbb{E}\big(\sup_{\varrho\in\mathcal{J}} \mu_\varrho(E^1_{\varrho,n,\epsilon}(v))\big) < \infty$, as desired.

Proposition 3.8 *For every compact connected subset K of I there exists $\varrho \in \mathcal{J}$ such that*

$$\begin{cases} \bigcap_{N\geq 1} \overline{\left\{n^{-1}\sum_{k=1}^{n} \nabla\widetilde{P}(q_k) : n \geq N\right\}} = K \\ \liminf_{n\to\infty} n^{-1}\sum_{k=1}^{n} \widetilde{P}^*(\nabla\widetilde{P}(q_k)) \geq \inf\{\widetilde{P}^*(\alpha) : \alpha \in K\} \end{cases}.$$

Proof For every integer $m \geq 1$, let $B(\alpha_{m,\ell}, 1/m)_{1\leq\ell\leq L_m}$ be a finite covering of K by balls centered on K, with $L_m \geq 2$. Since K is connected, without loss of generality we can assume that $B(\alpha_{m,\ell}, 1/m) \cap B(\alpha_{m,\ell+1}, 1/m) \neq \emptyset$ for all $1 \leq \ell \leq L_m - 1$, and $B(\alpha_{m+1,1}, 1/(m+1)) \cap B(\alpha_{m,L_m}, 1/m) \neq \emptyset$.

Now, applying Corollary 3.1, for each $\alpha_{m,\ell}$ let $q_{m,\ell} \in D$ such that $\|\nabla\widetilde{P}(q_{m,\ell}) - \alpha_{m,\ell}\| \leq 1/m$ and $|\widetilde{P}^*(\nabla\widetilde{P}(q_{m,\ell})) - \widetilde{P}^*(\alpha_{m,\ell})| \leq 1/m$. Let $j_{1,1} = \min\{j \geq 1 : q_{1,1} \in D_j\}$. Then for each $\alpha_{m,\ell}$, $1 \leq L_m - 1$, let $j_{m,\ell+1} = \min\{j > j_{m,\ell} : q_{m,\ell+1} \in D_j\}$, and let $j_{m+1,1} = \min\{j > j_{m,L_m} : q_{m+1,1} \in D_j\}$.

We build ϱ as follows. We pick $q \in D_1$ and let q_k be equal to q for $1 \leq k \leq M_{j_{1,1}-1}$. Then, we let $q_k = q_{1,j_{1,1}}$ for $k \in [M_{j_{1,1}-1}+1, M_{j_{1,2}-1}]$, and so on recursively $q_k = q_{m,\ell}$ for $k \in [M_{j_{m,\ell}-1}+1, M_{j_{m,\ell+1}-1}]$ if $1 \leq \ell \leq L_m - 1$ and $q_k = q_{m,L_m}$ for $k \in [M_{j_{m,L_m}-1}+1, M_{j_{m+1,1}-1}]$.

Now let $n \geq M_{j_{2,1}}+1$. There is an integer $m_n \geq 2$ such that either $n \in [M_{j_{m_n,\ell_n}-1}+1, M_{j_{m_n,\ell_n+1}-1}]$ for some $1 \leq \ell_n \leq L_{m_n} - 1$ or $n \in [M_{j_{m_n,L_{m_n}}-1}+1, M_{j_{m_n+1,1}-1}]$.

In the first case, let us write $\sum_{k=1}^{n} \nabla\widetilde{P}(q_k) = S_1 + S_2 + S_3$, where

$$S_1 = \sum_{k=1}^{M_{j_{m_n,\ell_n}-2}} \nabla\widetilde{P}(q_k),\ S_2 = \sum_{k=M_{j_{m_n,\ell_n}-2}+1}^{M_{j_{m_n,\ell_n}-1}} \nabla\widetilde{P}(q_k),\ S_3 = \sum_{k=M_{j_{m_n,\ell_n}-1}+1}^{n} \nabla\widetilde{P}(q_k).$$

We have $S_2 = (M_{j_{m_n,\ell_n}-1} - M_{j_{m_n,\ell_n}-2})\nabla\widetilde{P}(q)$, where $q = q_{m_n,\ell_n-1}$ if $\ell_n \geq 2$ and $q = q_{m_n-1,L_{m_n-1}}$ otherwise, so by construction of $\alpha_{m,\ell}$ and $q_{m,\ell}$, we have $\|S_2 - (M_{j_{m_n,\ell_n}-1} - M_{j_{m_n,\ell_n}-2})\alpha_1\| \leq (M_{j_{m_n,\ell_n}-1} - M_{j_{m_n,\ell_n}-2})/(m_n-1)$, where $\alpha_1 = \alpha_{m_n,\ell_n-1}$ if $\ell_n \geq 2$ and $\alpha_1 = \alpha_{m_n-1,L_{m_n-1}}$ otherwise. Also, we have $S_3 = (n - M_{j_{m_n,\ell_n}-1})\nabla\widetilde{P}(q)$ with $q = q_{m_n,\ell_n}$, so $\|S_3 - (n - M_{j_{m_n,\ell_n}-1})\alpha_2\| \leq (n - M_{j_{m_n,\ell_n}-1})/m_n$, where $\alpha_2 = \alpha_{m_n,\ell_n}$. Moreover, due to (3.14), we have $\|S_1\| \leq (j_{m_n,\ell_n}-1)^{-1}N_{j_{m_n,\ell_n}-1}\|\nabla\widetilde{P}(q))\| \leq (j_{m_n,\ell_n}-1)^{-1}n\|\nabla\widetilde{P}(q)\|$ with $q = q_{m_n,\ell_n}$, so $S_1 \leq (j_{m_n,\ell_n}-1)^{-1}n(\|\alpha_2\| + 1/m_n)$; also due to (3.14) we have $\|M_{j_{m_n,\ell_n}-2}\alpha_{m_n,\ell_n}\| \leq (j_{m_n,\ell_n}-1)^{-1}n\|\alpha_{m_n,\ell_n}\|$. Moreover, our construction of the balls $B(\alpha_{m,\ell}, 1/m)$ implies that $\|\alpha_1 - \alpha_2\| \leq 1/(m_n - 1)$. Consequently, putting the previous estimates together we get

$$\Big\|\sum_{k=1}^{n} \nabla\widetilde{P}(q_k) - n\alpha_{m_n,\ell_n}\Big\| \leq n\Big(\frac{3}{m_n-1} + \frac{2\|\alpha_{m_n,\ell_n}\| + 1/m_n}{j_{m_n,\ell_n}-1}\Big).$$

The same estimate holds if $n \in [M_{j_{m_n,L_{m_n}}-1}+1, M_{j_{m_n+1,1}-1}]$. Consequently, since as n tends to ∞ the sequence α_{m_n,ℓ_n} describes all the $\alpha_{m,\ell}$, the set of limit points of $n^{-1}\sum_{k=1}^{n} \nabla\widetilde{P}(q_k)$ is the same as that of the sequence $((\alpha_{m,\ell})_{1\leq\ell\leq L_m})_{m\geq 1}$, that is K by construction.

The fact that $\liminf_{n\to\infty} n^{-1}\sum_{k=1}^{n} \widetilde{P}^*(\nabla\widetilde{P}(q_k)) \geq \inf\{\widetilde{P}^*(\alpha) : \alpha \in K\}$ is a direct consequence of the choice of the vectors $q_{m,\ell}$, since $P^*(\nabla\widetilde{P}(q_{m,\ell})) \geq \inf\{\widetilde{P}^*(\alpha) : \alpha \in K\} - 1/m$.

3.3.5 Mutual singularity of the measures μ_ϱ and application

Proposition 3.9 *With probability 1, for all $\varrho,\varrho'\in\mathcal{J}$ such that $q_j\neq q'_j$ for infinitely many $j\geq 1$, the measures μ_ϱ and $\mu_{\varrho'}$ are mutually singular.*

Proof We will use the notion of Hellinger distance between probability measures (it was already used in the context of Mandelbrot martingales in [55] to prove the mutual singularity of the branching and visibility measures on $\partial\mathsf{T}$).

For $j\geq 1$ let

$$\widetilde{\mathcal{J}}_j=\{(\varrho,\varrho')\in\mathcal{J}\times\mathcal{J}: q_k\neq q'_k,\ \forall\ M_{j-1}+1\leq k\leq M_j\}.$$

and let

$$\widetilde{\mathcal{J}}=\bigcap_{\ell\geq 1}\bigcup_{j\geq\ell}\widetilde{\mathcal{J}}_j=\{(\varrho,\varrho')\in\mathcal{J}\times\mathcal{J}: q_k\neq q'_k \text{ for infinitely many } k\}.$$

For $n\geq 1$ and $(\varrho,\varrho')\in\widetilde{\mathcal{J}}$ let

$$A_n(\varrho,\varrho')=\sum_{u\in\mathsf{T}_n}\mu_\varrho([u])^{1/2}\mu_{\varrho'}([u])^{1/2}.$$

Notice that $(A_n(\varrho,\varrho'))_{n\geq 1}$ is non increasing. Let $A(\varrho,\varrho')$ denote its limit. If we show that $A(\varrho,\varrho')=0$ almost surely, then by definition almost surely the Hellinger distance between $\mu_\varrho/\|\mu_\varrho\|$ and $\mu_{\varrho'}/\|\mu_{\varrho'}\|$ is 1, i.e. μ_ϱ and $\mu_{\varrho'}$ are mutually singular. Of course, we want $A(\varrho,\varrho')=0$ almost surely, simultaneously for all $(\varrho,\varrho')\in\mathcal{J}$. We notice that for every $j\geq\ell\geq 1$ and $(\varrho,\varrho')\in\widetilde{\mathcal{J}}_j$, we have

$$A(\varrho,\varrho')\leq\sum_{u\in\mathsf{T}_{M_j}}\mu_\varrho([u])^{1/2}\mu_{\varrho'}([u])^{1/2}.$$

Consequently,

$$A=\sup_{(\varrho,\varrho')\in\widetilde{\mathcal{J}}}A(\varrho,\varrho')\leq\sum_{j\geq\ell}\sup_{(\varrho,\varrho')\in\widetilde{\mathcal{J}}_j}A(\varrho,\varrho')\leq\sum_{j\geq\ell}\sup_{(\varrho,\varrho')\in\widetilde{\mathcal{J}}_j}\sum_{u\in\mathsf{T}_{M_j}}\mu_\varrho([u])^{1/2}\mu_{\varrho'}([u])^{1/2},$$

so

$$\begin{aligned}
&\mathbb{E}(A)\\
\leq\ &\sum_{j\geq\ell}\mathbb{E}\Big(\sup_{(\varrho,\varrho')\in\widetilde{\mathcal{J}}_j}\sum_{u\in\mathsf{T}_{M_j}}\mu_\varrho([u])^{1/2}\mu_{\varrho'}([u])^{1/2}\Big)\\
=\ &\sum_{j\geq\ell}\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{M_j}}\sup_{(\varrho,\varrho')\in\widetilde{\mathcal{J}}_j}\Big(\prod_{k=1}^{M_j}W^{1/2}_{\varrho,u_1\cdots u_k}W^{1/2}_{\varrho',u_1\cdots u_k}\Big)Y(\varrho,u)^{1/2}Y(\varrho',u)^{1/2}\Big)\\
\leq\ &\sum_{j\geq\ell}\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{M_j}}\sup_{(\varrho,\varrho')\in\widetilde{\mathcal{J}}_j}\prod_{k=1}^{M_j}W^{1/2}_{\varrho,u_1\cdots u_k}W^{1/2}_{\varrho',u_1\cdots u_k}\Big)\mathbb{E}(\sup_{\varrho\in\mathcal{J}}Y(\varrho,u(j)))\\
\leq\ &\sum_{j\geq\ell}(\#\{(\varrho_{|M_j},\varrho'_{|M_j}):(\varrho,\varrho')\in\widetilde{\mathcal{J}}_j\})\sup_{(\varrho,\varrho')\in\widetilde{\mathcal{J}}_j}\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{M_j}}\prod_{k=1}^{M_j}W^{1/2}_{\varrho,u_1\cdots u_k}W^{1/2}_{\varrho',u_1\cdots u_k}\Big)\mathbb{E}(\sup_{\varrho\in\mathcal{J}}Y(\varrho,u(j)))
\end{aligned}$$

$$\leq \sum_{j\geq \ell}(j!)^2\Big(\sup_{(\varrho,\varrho')\in\widetilde{\mathcal{J}}_j}\prod_{k=1}^{M_j} c(q_k,q'_k)\Big)\mathbb{E}\big(\sup_{\varrho\in\mathcal{J}} Y(\varrho,u(j))\big),$$

where $u(j)$ is any word in $\mathbb{N}_+^{M_j}$, $c(q_k,q'_k)$ is defined in (3.7), and we used the fact that

$$\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{M_j}}\prod_{k=1}^{M_j} W^{1/2}_{\varrho,u_1\cdots u(j)}W^{1/2}_{\varrho',u_1\cdots u_k}\Big)=\prod_{k=1}^{M_j} c(q_k,q'_k).$$

Moreover,

$$\prod_{k=1}^{M_j} c(q_k,q'_k)\leq \prod_{k=M_{j-1}+1}^{M_j} c(q_k,q'_k)$$

since $c(q_k,q'_k)$ is always bounded by 1. Consequently, due to the definition of $\widetilde{\mathcal{J}}_j$ and c_j in (3.8), and recalling (3.17) we have

$$\mathbb{E}(A)\leq \sum_{j\geq\ell}\mathcal{C}_{\mathcal{J}}(j!)^2\exp(N_j\gamma_{j+1}^2 m_{j+1})c_j^{N_j}.$$

Remember that we required $\gamma_{j+1}^2 m_{j+1}=o(\log c_j)$ as $j\to\infty$ in the definition of $(\gamma_j)_{j\geq 1}$. Consequently, for ℓ large enough, we have $\mathbb{E}(A)\leq\sum_{j\geq\ell}\mathcal{C}_{\mathcal{J}}/j^2$ due to (3.13). Consequently $A=0$ almost surely.

Application : $E_X(\alpha)$ carries uncountably many mutually singular inhomogeneous Mandelbrot measures of Hausdorff dimension $\widetilde{P}^*(\alpha)$

Indeed, with probability 1, for all $\alpha\in I$ simultaneously, we can find uncountably many $\varrho\in\mathcal{J}$ such that μ_ϱ is carried by $E_X(\alpha)$ and these measures are all mutually singular ; moreover they can be taken to have Hausdorff dimension equal to $\widetilde{P}^*(\alpha)$. To see this, given $\alpha\in I$, for each $j\geq 1$, fix in D_j points $q^{(0)}_{\alpha,j}$ and $q^{(1)}_{\alpha,j}$, distinct if $j\geq 2$, among those q such that $(\nabla\widetilde{P}(q),\widetilde{P}^*(\nabla\widetilde{P}(q)))$ is as close as possible to $(\alpha,\widetilde{P}^*(\alpha))$. Now the set of those sequences ϱ of the form $(\underbrace{q^{(\mathbf{1}_C(j))}_{\alpha,j},\cdots,q^{(\mathbf{1}_C(j))}_{\alpha,j}}_{N_j})_{j\geq 1}$, where C runs in the set of equivalent classes of subsets of $\mathbb{N}_+$ under the relation $S\sim S'$ if $\mathbf{1}_S(j)=\mathbf{1}_{S'}(j)$ for j large enough, is as desired.

Chapitre 4

The Hausdorff dimensions of the sets $E_X(K)$ and $E_{X,\widetilde{X}}(K)$ under the metric d_ϕ

We prove Theorems 1.4 and 1.6.

4.1 Justification of some claims of the introduction

We start with the justification of the fact that d_ϕ is a metric, and then we prove Proposition 1.1. The fact that d_ϕ is a metric is a direct consequence of the following lemma, which tells us a little more about $(S_n\phi)_{n\geq 1}$.

Lemma 4.1 *Assume* $\mathbb{E}(S(0,\gamma)) < \infty$ *for all* $\gamma \in \mathbb{R}$. *There exist* $0 < \beta_1 < \beta_2 < 1$ *such that, with probability 1, under the metric* d_ϕ, *for* n *large enough,*

$$\beta_1^n \leq \min\{\exp(-S_n\phi(u)) = \operatorname{diam}([u]) : u \in \mathsf{T}_n\} \leq \max\{\exp(-S_n\phi(u)) : u \in \mathsf{T}_n\} \leq \beta_2^n.$$

We notice that the same result with $\beta_1 = 0$ as close as desired of 0^- can be deduced from Corollary 1.1 applied to the positive random branching random walk $(S_n\phi)_{n\geq 1}$ since under our assumptions I_ϕ must be a compact subset of $\mathbb{R}_+$.

Proof For every $n \geq 1$ let us denote $r_n = \max\{\operatorname{diam}([u]) : u \in \mathsf{T}_n\}$ and $r'_n = \min\{\operatorname{diam}([u]) : u \in \mathsf{T}_n\}$.

For each $\beta \in \mathbb{R}_+^*$ and $\gamma > 0$, we have $\mathbb{P}(r_n \geq \beta^n) = \mathbb{P}(\{\exists\ u \in \mathsf{T}_n,\ \exp(-S_n\phi(u))\beta^{-n} \geq 1\}) \leq \mathbb{E}(\sum_{u\in\mathsf{T}_n} \beta^{-n\gamma}\exp(-\gamma S_n\phi(u))) = \beta^{-n\gamma}\big(\mathbb{E}(S(0,\gamma))\big)^n$. Since the ϕ_i are positive and we assumed that $\mathbb{E}(S(0,\gamma)) < \infty$ for all $\gamma \in \mathbb{R}$, there exists $\gamma_0 > 0$ such that $\mathbb{E}(S(0,\gamma_0)) < 1$. Consequently, if $\beta \in (\mathbb{E}(S(0,\gamma_0))^{1/\gamma_0}, 1)$, we have $\sum_{n\geq 1}\mathbb{P}(r_n \geq \beta^n) < \infty$, hence by the Borel-Cantelli Lemma, with probability 1, for n lage enough, $r_n \leq \beta^n$.

Similarly, for each $\beta \in \mathbb{R}_+^*$ and $\gamma > 0$, we may write the inequality $\mathbb{P}(r'_n \leq \beta^n) = \mathbb{P}(\{\exists\ u \in \mathsf{T}_n,\ \exp(-S_n\phi(u))\beta^{-n} \leq 1\}) \leq \mathbb{E}(\sum_{u\in\mathsf{T}_n} \beta^{n\gamma}\exp(\gamma S_n\phi(u))) =$

$\beta^{n\gamma}\Big(\mathbb{E}(S(0,-\gamma))\Big)^n$. Since $1 < \mathbb{E}(S(0,-\gamma)) < \infty$ choosing $\beta \in (0,1)$ small enough so that $\beta^\gamma\mathbb{E}(S(0,-\gamma)) < 1$, yields that, with probability 1, for n lage enough, $r'_n \geq \beta^n$.

Proof of Proposition 1.1. Denote by λ the supremum in (1.7). Fix $q \in \mathbb{R}^d$ and $\alpha \in \mathring{I}$. Then for $t \in \mathbb{R}$ write

$$h(t) = \log \mathbb{E}\Big(\sum_{i=1}^N \exp\big(\langle q|X_i-\alpha\rangle - t\phi_i\big)\Big) = \log\Big[\mathbb{E}\Big(\sum_{i=1}^N \exp\big(\langle q|X_i\rangle - \widetilde{P}(q) - t\phi_i\big)\Big)\Big] + \widetilde{P}(q) - \langle q|\alpha\rangle.$$

We have $h(0) = \widetilde{P}(q) - \langle q|\alpha\rangle$ and $h'(0) = -\mathbb{E}\Big(\sum_{i=1}^N \phi_i \exp\big(\langle q|X_i\rangle - \widetilde{P}(q)\big)\Big) \geq -\lambda$. Moreover h is convex, so for all $t \geq 0$ we have $h(t) \geq h(0) - \lambda t$. By definition, we have $h(\widetilde{P}_\alpha(q)) = 0$, hence $\widetilde{P}_\alpha(q) \geq \lambda^{-1}(\widetilde{P}(q) - \langle q|\alpha\rangle)$. Now it follows from the convexity in q of $\widetilde{P}_\alpha(q)$, the strict convexity of $\widetilde{P}(q) - \langle q|\alpha\rangle$, and the fact that $\widetilde{P}(q) - \langle q|\alpha\rangle$ reaches its infimum that $\widetilde{P}_\alpha(q)$ reaches its infimum at some $q_\alpha \in \mathbb{R}^d$. If there are two distinct such q_α and q'_α, then $\nabla\widetilde{P}_\alpha(q) = 0$ over $[q_\alpha, q'_\alpha]$, i.e. due to (4.1) below, $\mathbb{E}\Big(\sum_{i=1}^N (X_i - \alpha)\exp(\langle q|X_i - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_i)\Big) = 0$. Differentiating in the direction of $v = q_\alpha - q'_\alpha$ and taking the scalar product with v yields $\mathbb{E}\Big(\sum_{i=1}^N \langle v|X_i - \alpha\rangle^2 \exp(\langle q|X_i - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_i)\Big) = 0$ over $[q_\alpha, q'_\alpha]$, hence $\langle q_\alpha|X_i - \alpha\rangle = \langle q'_\alpha|X_i - \alpha\rangle$ almost surely for all $1 \leq i \leq N$. But this contradicts (1.1).

To see that $\alpha \mapsto q_\alpha$ is analytic, an examination of the differential of $q \mapsto \nabla\widetilde{P}_\alpha(q)$ at q_α shows that it is invertible, except if there exists $q_0 \in \mathbb{R}^d\setminus\{0\}$ such that $\langle q_0|X_i - \alpha\rangle = 0$ for all $1 \leq i \leq N$ almost surely, which is forbidden by (1.1). Then the invertibility of $q \mapsto \nabla\widetilde{P}_\alpha(q)$ at q_α for each $\alpha \in \mathring{I}$ makes it possible to apply the implicit function theorem to $(\alpha, q) \mapsto \nabla\widetilde{P}_\alpha(q)$ at (α, q_α).

□

Now we explain the modifications to make with respect to the proof of Theorem 1.2 to get Theorem 1.4, by following the same structure of paragraphs. Then, we explain how to get Theorem 1.6.

4.2 Upper bounds for the Hausdorff dimensions

For each $(q, \alpha, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ let us define

$$P_\alpha(q) = \inf\Big\{t \in \mathbb{R} : \limsup_{n\to\infty} \frac{1}{n} \log\Big(\sum_{u\in\mathsf{T}_n} \exp(\langle q|S_n(X-\alpha)(u)\rangle - tS_n\phi(u))\Big) \leq 0\Big\}.$$

The following proposition is a direct consequence of the log-convexity of the mappings $(q,t) \mapsto \sum_{u\in\mathsf{T}_n} \exp(\langle q|S_n(X-\alpha)(u)\rangle - tS_n\phi(u))$ and $(\alpha, t) \mapsto \sum_{u\in\mathsf{T}_n} \exp(\langle q|S_n(X-\alpha)(u)\rangle - tS_n\phi(u))$ given $\alpha \in \mathbb{R}^d$ and $q \in \mathbb{R}^d$ respectively.

Proposition 4.1 *The mappings $q \mapsto P_\alpha(q)$ and $\alpha \mapsto P_\alpha(q)$ are convex.*

Proposition 4.2 *With probability 1,* $P_\alpha(q) \le \widetilde{P}_\alpha(q)$ *for all* $(q,\alpha) \in \mathbb{R}^d \times \mathbb{R}^d$.

Proof Due to Proposition 4.1, we only need to prove the inequality for each $(q,\alpha) \in \mathbb{R}^d \times \mathbb{R}^d$ almost surely. Fix $(q,\alpha) \in \mathbb{R}^d \times \mathbb{R}^d$. For $t > \widetilde{P}_\alpha(q)$ we have

$$\mathbb{E}(\sum_{n\ge 1}\sum_{u\in \mathsf{T}_n} \exp(\langle q|S_n(X-\alpha)(u)\rangle - tS_n\phi(u)) \;=\; \sum_{n\ge1}\mathbb{E}(\sum_{i=1}^{N}\exp(\langle q|X_i-\alpha\rangle - t\phi_i))^n < \infty.$$

Consequently, $\sum_{u\in\mathsf{T}_n}\exp\big(\langle q|S_n(X-\alpha)(u)\rangle - tS_n\phi(u)\big)$ is bounded almost surely, so $t \ge P_\alpha(q)$ almost surely. Since $t > \widetilde{P}_\alpha(q)$ is arbitrary, we have the conclusion.

Recall that for $\alpha\in\mathbb{R}^d$ we defined

$$\widehat{E}_X(\alpha) = \Big\{t\in\partial\mathsf{T} : \alpha \in \bigcap_{N\ge1}\overline{\Big\{\frac{S_nX(t)}{n} : n\ge N\Big\}}\Big\}.$$

Proposition 4.3 *With probability 1, for all* $\alpha\in\mathbb{R}^d$, $\dim \widehat{E}_X(\alpha) \le P_\alpha^*(0)$*, a negative dimension meaning that* $\widehat{E}_X(\alpha)$ *is empty.*

Proof For every $n\ge1$ let us denote $r_n = \max\{\mathrm{diam}([u]) : u\in\mathsf{T}_n\}$. Recall that

$$\begin{aligned}\widehat{E}_X(\alpha) &= \bigcap_{\epsilon>0}\bigcap_{N\ge1}\bigcup_{n\ge N}\{t\in\partial\mathsf{T} : \|S_nX(t)-n\alpha\|\le n\epsilon\}\\ &\subset \bigcap_{q\in\mathbb{R}^d}\bigcap_{\epsilon>0}\bigcap_{N\ge1}\bigcup_{n\ge N}\{t\in\partial\mathsf{T} : |\langle q|S_nX(t)-n\alpha\rangle|\le n\|q\|\epsilon\}.\end{aligned}$$

Fix $q\in\mathbb{R}^d$ and $\epsilon>0$. For $N\ge1$, the set $E(q,N,\epsilon,\alpha) = \bigcup_{n\ge N}\{t\in\partial\mathsf{T} : |\langle q|S_nX(t) - n\alpha\rangle| \le n\|q\|\epsilon\}$ is covered by the union of those $[u]$ such that $u\in\mathsf{T}_n$ and $\langle q|S_nX(u) - n\alpha\rangle + n\|q\|\epsilon \ge 0$. Consequently, for $s\ge0$,

$$\begin{aligned}\mathcal{H}^s_{r_N}(E(q,N,\epsilon,\alpha)) &\le \sum_{n\ge N}\sum_{u\in\mathsf{T}_n}\mathrm{diam}([u])^s\exp(\langle q|S_nX(u)-n\alpha\rangle + n\|q\|\epsilon)\\ &= \sum_{n\ge N}\sum_{u\in\mathsf{T}_n}\exp(\langle q|S_n(X-\alpha)(u)\rangle - sS_n\phi(u) + n\|q\|\epsilon).\end{aligned}$$

Hence, if $\eta>0$ and $s > P_\alpha(q)+\eta+\|q\|\epsilon$, by definition of $P_\alpha(q)$, for N large enough we have

$$\mathcal{H}^s_{r_N}(E(q,N,\epsilon,\alpha)) \le \sum_{n\ge N} e^{-n\eta/2}.$$

Since r_N tends to 0 almost surely as N tends to ∞, we thus have $\mathcal{H}^s(E(q,N,\epsilon,\alpha)) = 0$, hence $\dim E(q,N,\epsilon,\alpha)\le s$. Since this holds for all $\eta>0$, we get $\dim E(q,N,\epsilon,\alpha) \le P_\alpha(q)+\|q\|\epsilon$. It follows that

$$\dim\widehat{E}_X(\alpha) \le \inf_{q\in\mathbb{R}}\inf_{\epsilon>0} P_\alpha(q)+\|q\|\epsilon = \inf_{q\in\mathbb{R}} P_\alpha(q).$$

If $\inf_{q\in\mathbb{R}} P_\alpha(q) < 0$, we necessarily have $\widehat{E}_X(\alpha)=\emptyset$.

Corollary 4.1 *With probability 1, for all compact connected subset* K *of* $\mathbb{R}^d$*, we have* $E_X(K)=\emptyset$ *if* $K\not\subset I$*, and* $\dim E_X(K) \le \inf_{\alpha\in K}\widetilde{P}_\alpha^*(0)$ *otherwise.*

4.3 Construction of inhomogeneous Mandelbrot measures and lower bounds for the Hausdorff dimensions of the sets $E_X(K)$

Preliminary facts

A calculation shows that

$$\nabla \widetilde{P}_\alpha(q) = \frac{\mathbb{E}\Big(\sum_{i=1}^N X_i \exp(\langle q|X_i - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_i)\Big) - \alpha}{\mathbb{E}\Big(\sum_{i=1}^N \phi_i \exp(\langle q|X_i - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_i)\Big)}. \tag{4.1}$$

By construction, for each $(q, \alpha) \in \mathbb{R}^d \times \mathbb{R}^d$, a Mandelbrot measure $\mu_{q,\alpha}$ on $\partial\mathsf{T}$ is associated with the vectors $(N_u, \exp(\langle q|X_{u1} - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_{u1}), \exp(\langle q|X_{u2} - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_{u2}), \dots)$, $u \in \bigcup_{n\geq 0} \mathbb{N}_+^n$, and this measure is non degenerate (see section 6) if and only if the "entropy"

$$h(q,\alpha) = -\mathbb{E}\Big(\sum_{i=1}^N \big(\langle q|X_i - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_i\big) \exp(\langle q|X_i - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_i)\Big) > 0, \tag{4.2}$$

Define the "Lyapounov exponent"

$$\lambda(q,\alpha) := \mathbb{E}\Big(\sum_{i=1}^N \phi_i \exp(\langle q|X_i - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_i)\Big) \in (0,\infty). \tag{4.3}$$

An identification shows that

$$\widetilde{P}_\alpha^*(\nabla \widetilde{P}_\alpha(q)) = \widetilde{P}_\alpha(q) - \langle q|\nabla \widetilde{P}_\alpha(q)\rangle = \frac{h(q,\alpha)}{\lambda(q,\alpha)},$$

hence $\mu_{q,\alpha}$ is non degenerate if and only if $\widetilde{P}_\alpha^*(\nabla \widetilde{P}_\alpha(q)) > 0$. Moreover (see section 6 again), with probability 1, we have

$$\lim_{n\to\infty} \frac{S_n X(t)}{n} = \alpha_X(q,\alpha) \quad \mu_{q,\alpha}\text{-a.e.}, \tag{4.4}$$

where

$$\alpha_X(q,\alpha) = \mathbb{E}\Big(\sum_{i=1}^N X_i \exp(\langle q|X_i - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_i)\Big),$$

Notice that when the assumption of Proposition 1.1 holds, $\alpha \in \mathring{I}$ and $q = q_\alpha$, we have $\nabla \widetilde{P}_\alpha(q) = 0$, hence $\alpha_X(q,\alpha) = \alpha$ and $\widetilde{P}_\alpha^*(\nabla \widetilde{P}_\alpha(q)) = \widetilde{P}_\alpha(q)$.

We will exploit the previous facts to build the new family of inhomogeneous Mandelbrot measures adapted to the context associated with the random metric d_ϕ. Let

$$J_\phi = \Big\{(q,\alpha) \in \mathbb{R}^d \times I : \widetilde{P}_\alpha(q) - \langle q|\nabla \widetilde{P}_\alpha(q)\rangle > 0\Big\}.$$

We will need the following generalization of Corollary 3.1.

Proposition 4.4 *Let D be a dense subset of J_ϕ. For all $\alpha \in I$, there exists a sequence $(q_n, \alpha_n)_{n\geq 1}$ of elements of D such that* $\lim_{n\to\infty} \alpha_X(q_n, \alpha_n) = \alpha$ *and* $\lim_{n\to\infty} \widetilde{P}_{\alpha_n}(q_n) - \langle q_n|\nabla \widetilde{P}_{\alpha_n}(q_n)\rangle = \widetilde{P}_\alpha^*(0)$.

Moreover, if (1.7) *holds, and D contains a sequence $(q_{\alpha_m}, \alpha_m)_{m\geq 1}$ such that $\{\alpha_m : m \geq 1\}$ is dense in $\mathring{I}$, then for $\alpha \in \mathring{I}$ the previous sequence can be chosen so that $q_n = q_{\alpha_n}$.*

Proof Let $\alpha \in I$. Since $E_X(\alpha) \neq \emptyset$, we have $0 \leq \dim E_X(\alpha) \leq \inf_{q\in\mathbb{R}^d} \widetilde{P}_\alpha(q) = \widetilde{P}_\alpha^*(0)$. The upper-semi-continuous concave function $\widetilde{P}_\alpha^*$ possesses the same properties as $\widetilde{P}^*$ in Corollary 3.1. Consider a sequence $(q_n)_{n\geq 1}$ such that $\nabla \widetilde{P}_\alpha(q_n)$ tends to 0, $\widetilde{P}_\alpha(q_n) - \langle q_n|\nabla \widetilde{P}_\alpha(q_n)\rangle > 0$ so that $(q_n, \alpha) \in J_\phi$, and $\lim_{n\to\infty} \widetilde{P}_\alpha(q_n) - \langle q_n|\nabla \widetilde{P}_\alpha(q_n)\rangle = \widetilde{P}_\alpha^*(0)$.

Since $\mathbb{E}\Big(\sum_{i=1}^N \phi_i \exp(\langle q|X_i - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_i)\Big) \in (0, \infty)$, $\lim_{n\to\infty} \nabla \widetilde{P}_\alpha(q_n) = 0$ implies $\lim_{n\to\infty} \alpha_X(q_n, \alpha) = \alpha$. Now, the mappings $(q, \alpha) \mapsto \alpha_X(q, \alpha)$ and $(q, \alpha) \mapsto \widetilde{P}_\alpha(q) - \langle q|\nabla \widetilde{P}_\alpha(q)\rangle$ being continuous, the first property follows. The second one is obvious due to Proposition 1.1.

4.3.1 Parametrized family of inhomogeneous Mandelbrot martingales

At first we need to make the following observation. Applications of the Cauchy-Schwartz inequality as of the Hölder inequality in the proof of Proposition 3.1 yield (remembering the definition of $\widetilde{P}_\alpha(\cdot)$), for $(q, \alpha) \neq (q', \alpha') \in \mathbb{R}^d \times \mathbb{R}^d$

$$c((q, \alpha), (q', \alpha')) = \mathbb{E}\Big(\sum_{i=1}^N \exp\Big[\frac{1}{2}(\langle q|X_i - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_i)\Big] \exp\Big[\frac{1}{2}(\langle q'|X_i - \alpha'\rangle - \widetilde{P}_{\alpha'}(q')\phi_i)\Big]\Big) < 1, \tag{4.5}$$

except in the case that $\langle q|X_i - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_i = \langle q'|X_i - \alpha'\rangle - \widetilde{P}_{\alpha'}(q')\phi_i$ almost surely for all $1 \leq i \leq N$. This situation holds if and only if the ϕ_i depend linearly on the X_i. Moreover, if this equality holds for all the distinct couples (q, α) and (q', α') of an open set, which can be taken in the form $U \times V$, differentiating with respect to q yields, for all $1 \leq i \leq N$, for all $(q, \alpha) \in U \times V$, $X_i - \alpha - \nabla \widetilde{P}_\alpha(q)\phi_i = 0$ almost surely, hence $\nabla \widetilde{P}_\alpha(q)$ must be constant. Consequently, up to a simple transformation, the situation reduces to $X_i = \phi_i u$ for a fixed vector $u \in \mathbb{R}^d$, so that we can assume that $d = 1$ and $X_i = \phi_i$ almost surely for all $1 \leq i \leq N$. Then, coming back to $\langle q|X_i - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_i = \langle q'|X_i - \alpha'\rangle - \widetilde{P}_{\alpha'}(q')\phi_i$, we see that since $X_i = \phi_i$, $1 \leq i \leq N$, we have $(q - \widetilde{P}_\alpha(q) - (q' - \widetilde{P}_{\alpha'}(q')))X_i = \alpha q - \alpha' q$, so either $q - \widetilde{P}_\alpha(q)$ is constant over $U \times V$, which is impossible, or there exists $c \in \mathbb{R}$ such that $X_i = c$, almost surely for all $1 \leq i \leq N$, which contradicts (1.1).

Consequently, we can find a dense countable subset D of $\mathbb{R}^d \times \mathbb{R}^d$ such that $c((q, \alpha), (q', \alpha')) < 1$ for all $(q, \alpha) \neq (q', \alpha') \in D$. Also, under (1.7) we can find a dense countable subset of $\mathring{I}$ such that for all $\alpha \neq \alpha' \in D$ we have $c((q_\alpha, \alpha), (q'_\alpha, \alpha')) < 1$. Indeed, if $\alpha \neq \alpha'$ in $\mathring{I}$, the equality $\langle q_\alpha|X_i - \alpha\rangle - \widetilde{P}_\alpha(q_\alpha)\phi_i = \langle q_{\alpha'}|X_i - \alpha'\rangle - \widetilde{P}_{\alpha'}(q_{\alpha'})\phi_i$ almost surely for all $1 \leq i \leq N$ implies that the Mandelbrot measures $\mu_{q_\alpha,\alpha}$ and $\mu_{q_{\alpha'},\alpha'}$

coincide, so that by Proposition 6.1 we have $\alpha_X(q_\alpha, \alpha) = \alpha_X(q_{\alpha'}, \alpha')$, i.e. $\alpha = \alpha'$, which is a contradiction.

The set of parameters

For $(q, \alpha, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ let

$$S_\alpha(q,t) = \sum_{i=1}^{N} \exp(\langle q|X_i - \alpha\rangle - t\phi_i),$$

and

$$\psi_\alpha(q,t) = \log \mathbb{E}\big(S_\alpha(q,t)\big).$$

Useful in some calculations will be the equalities :

$$\frac{\partial \psi_\alpha}{\partial q}(q, \widetilde{P}_\alpha(q)) = \alpha_X(q,\alpha) - \alpha \quad \text{and} \quad \frac{\partial \psi_\alpha}{\partial t}(q, \widetilde{P}_\alpha(q)) = -\lambda(q,\alpha), \tag{4.6}$$

and

$$\frac{\mathrm{d}\psi_\alpha((1+u)q, (1+u)\widetilde{P}_\alpha(q))}{\mathrm{d}u}(0) = -h(q,\alpha) = \langle q|\alpha_X(q,\alpha) - \alpha\rangle - \widetilde{P}_\alpha(q)\lambda(q,\alpha). \tag{4.7}$$

Let $(D_j)_{j\geq 1}$ be an increasing sequence of non-empty subsets of J_ϕ such that D_j has cardinality j, $D = \bigcup_{j\geq 1} D_j$ is a dense subset of J_ϕ, and $c((q,\alpha),(q',\alpha')) < 1$ for all $(q,\alpha) \neq (q',\alpha') \in D$. Moreover, assume without loss of generality that $\alpha_X(q,\alpha)$ does not vanish on D. If, moreover, (1.7) holds, according to Proposition 4.4 and the discussion of the beginning of this section, we can choose D so that it contains only pairs of the form (q_α, α), $\alpha \in \mathring{I}$ (this will be used in the proof of Theorem 1.8).

Let $(N_j)_{j\geq 0}$ be a sequence of integers such that $N_0 = 0$, and that we will specify at the end of this section. Then let $(M_j)_{j\geq 0}$ be the increasing sequence defined as

$$M_j = \sum_{k=1}^{j} N_k \text{ for all } j \geq 0. \tag{4.8}$$

For $n \in \mathbb{N}$, let j_n denote the unique integer satisfying

$$M_{j_n} + 1 \leq n \leq M_{j_n+1}.$$

We will build a random family of measures indexed by the set

$$\mathcal{J}_\phi = \{\varrho = ((q_k,\alpha_k)_{k\geq 1} : \forall j \geq 0, (q_{M_j+1}, \alpha_{M_j+1}) = (q_{M_j+2}, \alpha_{M_j+2}) = \cdots = (q_{M_{j+1}}, \alpha_{M_{j+1}}) \in D_{j+1}\}.$$

Since each D_j is finite, so compact, for all $j \geq 1$, the set $\mathcal{J}_\phi$ is compact for the metric

$$d(\varrho, \varrho') = \sum_{k\geq 1} 2^{-k} \frac{|q_k - q'_k| + |\alpha'_k - \alpha_k|}{1 + |q_k - q'_k| + |\alpha'_k - \alpha_k|}.$$

For $\varrho = (q_k, \alpha_k)_{k\geq 1} \in \mathcal{J}_\phi$ and $n \geq 1$ we will denote by $\varrho_{|n}$ the sequence $(q_k, \alpha_k)_{1\leq k\leq n}$.

Inhomogeneous Mandelbrot martingales indexed by $\mathcal{J}_\phi$

For $u \in \bigcup_{n\geq 0} \mathbb{N}_+^n$, $1 \leq i \leq N(u)$, $\beta\mathbb{E}(N) > 1$, and $\varrho \in \mathcal{J}_\phi$ we define

$$W_{\varrho,ui} = \exp(\langle q_{|u|+1} | X_{ui} - \alpha_{|u|+1}\rangle - \widetilde{P}_{\alpha_{|u|+1}}(q_{|u|+1})\phi_{ui}),$$

and

$$W_{\beta,\varrho,ui} = W_{\beta,ui} W_{\varrho,ui}.$$

For $\varrho = (q_k, \alpha_k)_{k\geq 1} \in \mathcal{J}_\phi$, $u \in \bigcup_{n\geq 0} \mathbb{N}_+^n$, $\beta\mathbb{E}(N) > 1$, and $n \geq 0$ we define

$$\begin{cases} Y_n(\varrho, u) = \sum_{v_1\cdots v_n \in \mathsf{T}_n(u)} \prod_{k=1}^n W_{\varrho, u\cdot v_1\cdots v_k} \\ Y_n(\beta, \varrho, u) = \sum_{v_1\cdots v_n \in \mathsf{T}_n(u)} \prod_{k=1}^n W_{\beta,\varrho, u\cdot v_1\cdots v_k} \end{cases}$$

When $u = \emptyset$ those quantities will be denoted by $Y_n(\varrho)$ and $Y_n(\beta, \varrho)$ respectively, and when $n = 0$, their values equal 1.

Recall the definition of $h(q, \alpha)$ given in (4.2). For $\beta \in (\mathbb{E}(N)^{-1}, 1]$, $L \geq 1$ and $\epsilon > 0$ we set

$$\mathcal{J}_{\phi,\beta,L,\epsilon} = \Big\{\varrho \in \mathcal{J}_\phi : \frac{1}{n}\sum_{k=1}^n h(q_k, \alpha_k) \geq -\log\beta + \epsilon, \forall n \geq L\Big\},$$

which is a compact subset of $\mathcal{J}_\phi$.

Notice that $h(q_k, \alpha_k)>0$, and this number is the opposite of the derivative at 1 of the convex function $f : \lambda \geq 0 \mapsto \log\mathbb{E}(\sum_{i=1}^N W_i^\lambda)$, with $W_i = \exp(\langle q_k | X_i - \alpha_k\rangle - \widetilde{P}_{\alpha_k}(q_k)\phi_i)$, so that $f(1) = 0$ and $f(0) = \log\mathbb{E}(N) > 0$. Thus $h(q_k, \alpha_k) \in (0, \log\mathbb{E}(N)]$. Consequently,

$$\Big\{\varrho \in \mathcal{J}_\phi : \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^n h(q_k, \alpha_k) > 0\Big\} = \bigcup_{\beta\in(\mathbb{E}(N)^{-1},1], L\geq 1, \epsilon>0} \mathcal{J}_{\phi,\beta,L,\epsilon}. \tag{4.9}$$

For $n \geq 1$ and $\beta \in (0, 1]$, we set $\mathcal{F}_n = \sigma\Big((N_u, (X_{u1}, \phi_{u1}, (X_{u2}, \phi_{u2}), \ldots) : u \in \bigcup_{k=0}^n \mathbb{N}_+^{n-1}\Big)$ and $\mathcal{F}_{\beta,n} = \sigma\Big(W_{\beta,u1}, (W_{\beta,u2}, \ldots) : u \in \bigcup_{k=0}^n \mathbb{N}_+^{n-1}\Big)$. We also denote by $\mathcal{F}_0$ and $\mathcal{F}_{\beta,0}$ the trivial σ-field.

Specification of the sequence $(N_j)_{j\geq 1}$

For each $\alpha \in I$ the function ψ_α is analytic. We denote by H_{ψ_α} its Hessian matrix. For each $j \geq 1$,

$$m_j = \max_{t\in[0,1]} \max_{v\in\mathbb{S}^{d-1}} \max_{(q,\alpha)\in D_j} {}^t\begin{pmatrix} v \\ 0 \end{pmatrix} H_{\psi_\alpha}(q + tv, \widetilde{P}_\alpha(q)) \begin{pmatrix} v \\ 0 \end{pmatrix} + \max_{t\in[0,1]} \max_{v\in\{-1,1\}} \max_{(q,\alpha)\in D_j} \frac{\partial^2}{\partial t^2}\psi_\alpha(q, \widetilde{P}_\alpha(q) + tv) \tag{4.10}$$

and

$$\widetilde{m}_j = \max_{t\in[0,1]} \max_{p\in[1,2]} \sup_{q\in D_j} {}^t\begin{pmatrix} q \\ \widetilde{P}_\alpha(q)\end{pmatrix} H_{\psi_\alpha}\big(q+t(p-1)q, \widetilde{P}_\alpha(q)+t(p-1)\widetilde{P}_\alpha(q)\big)\begin{pmatrix} q \\ \widetilde{P}_\alpha(q)\end{pmatrix} \quad (4.11)$$

are finite. Let

$$\widehat{m}_j = \max(m_j, \widetilde{m}_j).$$

For $j \geq 2$ let

$$c_j = \sup_{(q,\alpha)\neq(q',\alpha')\in D_j} c((q,\alpha),(q',\alpha')) < 1$$

(recall the definition (4.5)). Let $(\gamma_j)_{j\geq 1} \in (0,1]^{\mathbb{N}_+}$ be a positive sequence such that $\gamma_j^2\widehat{m}_j$ converges to 0 as $j \to \infty$ (in particular $\lim_{j\to\infty}\gamma_j = 0$) and $\gamma_{j+1}^2\widehat{m}_{j+1} = o(\log c_j)$ as $j \to \infty$.

Let $(\widetilde{p}_j)_{j\geq 1}$ be a sequence in $(1,2)$ such that $(\widetilde{p}_j - 1)\widetilde{m}_j$ converges to 0 as j tends to ∞.

We can also suppose that $\widetilde{p}_j$ is small enough so that we also have

$$\sup_{(q,\alpha)\in D_j} \mathbb{E}(S_\alpha(q,\widetilde{P}_\alpha(q))^{\widetilde{p}_j}) < \infty.$$

For each $(q,\alpha) \in \mathcal{J}_\phi$ there exists a real number $1 < p_q < 2$ such that $\psi_\alpha(pq, p\widetilde{P}_\alpha(q)) < 0$ for all $p \in (1,p_q)$. Indeed, $\widetilde{P}_\alpha^*(\nabla\widetilde{P}_\alpha(q)) > 0$ is equivalent to $\frac{\mathrm{d}}{\mathrm{d}p}(\psi_\alpha(pq, p\widetilde{P}_\alpha(q)))(1^+) < 0$.

For all $j \geq 1$ we set

$$p_j = \min(\widetilde{p}_j, \inf_{(q,\alpha)\in D_{j+1}} p_q) \quad \text{and} \quad a_j = \sup_{(q,\alpha)\in D_j} \psi_\alpha(p_j q, p_j\widetilde{P}_\alpha(q)).$$

By construction, we have $a_j < 0$. Then let

$$s_j = \max\left\{\|S_\alpha(q,\widetilde{P}_\alpha(q)))\|_{p_j} : (q,\alpha)\in D_j\right\} \quad \text{and} \quad r_j = \max\left(a_j/p_j, (2jp_j)^{-1}(1-p_j)\right) \quad (4.12)$$

Now set $N_0 = 0$ and for $j \geq 1$ choose an integer N_j big enough so that

$$\frac{(j+1)!s_{j+1}}{1-\exp(r_{j+1})}\exp(N_j r_{j+1}) \leq j^{-2}, \quad (4.13)$$

$$\frac{(j+1)!s_{j+1}}{(1-\exp(r_{j+1})} + \frac{(j+2)!s_{j+2}}{(1-\exp(r_{j+2})} \leq C_0\exp(N_j\gamma_{j+1}^2 m_{j+1}), \quad (4.14)$$

with $C_0 = \dfrac{s_1}{1-\exp(r_1)} + \dfrac{2s_2}{1-\exp(r_2)}$,

$$N_j \geq \max\left((\gamma_{j+1}^2\widehat{m}_{j+1})^{-2}, 5\log((j+1)!\right); \quad (4.15)$$

if $j \geq 2$

$$(j!)^2 c_j^{N_j/2} \leq j^{-2}$$

and

$$\Big(\sum_{k=1}^{j-1} N_k\Big)\max(1,\max\{\|\alpha_X(q,\alpha)\| : (q,\alpha)\in D_{j-1}\}) \leq j^{-1}N_j\min(1,\{\|\alpha_X(q,\alpha)\| : (q,\alpha)\in D_j\}). \quad (4.16)$$

4.3.2 A family of measures indexed by $\mathcal{J}_\phi$

Proposition 4.5

1. *For all $u \in \bigcup_{n\geq 0}\mathbb{N}_+^n$, the sequence of continuous functions $Y_n(\cdot,u)$ converges uniformly, almost surely and in L^1 norm, to a positive limit $Y(\cdot,u)$ on $\mathcal{J}_\phi$.*
2. *With probability 1, for all $\varrho \in \mathcal{J}_\phi$, the mapping*
$$\mu_\varrho([u]) = (\prod_{k=1}^n W_{\varrho,u_1\cdots u_k})Y(\varrho,u).$$
defines a positive measure on $\partial\mathsf{T}$.
3. *With probability 1, for all $(\varrho,\varrho') \in \mathcal{J}^2$, the measures μ_ϱ and $\mu_{\varrho'}$ are absolutely continuous with respect to each other or mutually singular according to whether ϱ and ϱ' coincide ultimately or not.*

The following lemma is a direct extension of Lemma 3.2.

Lemma 4.2 *Let $\varrho \in \mathcal{J}$ and $\beta \in (0,1]$. Define $Z_n(\beta,\varrho) = Y_n(\beta,\varrho) - Y_{n-1}(\beta,\varrho)$ for $n \geq 0$. For every $p \in (1,2)$ we have*

$$\mathbb{E}(|Z_n(\beta,\varrho)|^p) \leq (2\beta^{-1})^p\mathbb{E}\big(S_{\alpha_n}(q_n,\widetilde{P}_{\alpha_n}(q_n))^p\big)\prod_{k=1}^{n-1}\beta^{1-p}\exp\big(\psi_{\alpha_k}(pq_k,p\widetilde{P}_{\alpha_k}(q_k))\big). \tag{4.17}$$

Proof of Proposition 4.5. The proof is formally the same as that of Proposition 3.4, but uses Lemma 4.2 instead of Lemma 3.2.

If we set $\epsilon_k = \gamma_{j_k+1}^2\widehat{m}_{j_k+1}$ for all $k \geq 0$, we still get $C_{\mathcal{J}_\phi} > 0$ such that :

$$\|\sup_{\varrho\in\mathcal{J}_\phi} Y(\varrho,u)\|_1 \leq C_{\mathcal{J}_\phi}\exp(\epsilon_{|u|}N_{j_{|u|}}) \leq C_{\mathcal{J}_\phi}\exp(\epsilon_{|u|}|u|) \quad (\forall\ u \in \bigcup_{k\geq 0}\mathbb{N}_+^k). \tag{4.18}$$

□

Remark 4.1 *This is the analogue of Remark 3.1.*

Let K be a compact subset of J_ϕ containing the unique element of D_1. Then, there exists $p_K \in (1,2)$ such that

$$\sup_{j\geq 1}\sup_{(q,\alpha)\in D_j\cap K}\psi_\alpha(p_Kq,p_K\widetilde{P}_\alpha(q)) < 0 \text{ and } \sup_{j\geq 1}\sup_{(q,\alpha)\in D_j\cap K}\mathbb{E}\big(S_\alpha(q,\widetilde{P}_\alpha(q))^{p_K}\big) < \infty.$$

Then if we define $\mathcal{J}(K) = \{\varrho \in \mathcal{J} :\ \forall\ k \geq 1,\ q_k \in K\}$, we have

$$\|\sup_{\varrho\in\mathcal{J}(K)} Y(\varrho,u)\|_{p_K} = O((j_{|u|}+2)!).$$

4.3.3 Lower bounds for the Hausdorff dimensions of the measures $\{\mu_\varrho\}_{\varrho\in\mathcal{J}}$

Proposition 4.6 *Let* $\beta \in (0,1]$ *such that* $\beta\mathbb{E}(N) > 1$. *Conditionally on non extinction of* $(\mathsf{T}_{\beta,n}(u))_{n\geq 1}$, *for all* $N \geq 1$ *and* $\epsilon \in \mathbb{Q}_+^*$,

1. *the sequence of continuous functions* $Y_n(\cdot,\beta)$ *converges uniformly, almost surely and in* L^1 *norm, to a positive limit* $Y(\beta,\cdot)$ *on* $\mathcal{J}_{\phi,\beta,L,\epsilon}$,
2. *the sequence of continuous functions*

$$\varrho \mapsto \widetilde{Y}_n(\beta,\varrho) = \sum_{u\in\mathsf{T}_n} (\prod_{k=1}^{n} W_{\beta,u_1\cdots u_k})\mu_\varrho([u])$$

converges uniformly, almost surely and in L^1 *norm, towards* $Y(\beta,\cdot)$ *on* $\mathcal{J}_{\phi,\beta,L,\epsilon}$.

Proof (1) Let $L \geq 1$ and $\epsilon > 0$. For $\varrho \in \mathcal{J}_{\phi,\beta,L,\epsilon}$ and $n \geq 1$, Lemma 4.2 applied with $p = p_{j_n+1}$ provides us with the inequality

$$\begin{aligned} &\|Y_n(\beta,\varrho) - Y_{n-1}(\beta,\varrho)\|_{p_{j_n+1}}^{p_{j_n+1}} \\ \leq\ & (2\beta^{-1})^{p_{j_n+1}}\mathbb{E}\big(S_{\alpha_n}(q_n,\widetilde{P}_{\alpha_n}(q_n))^{p_{j_n+1}}\big)\prod_{k=1}^{n-1}\beta^{1-p_{j_n+1}}\exp\big(\psi_{\alpha_k}(p_{j_n+1}q_k, p_{j_n+1}\widetilde{P}_{\alpha_k}(q_k))\big)\end{aligned}$$

Let $(\alpha,q) \in D_{j_n+1}$ and set $g : \lambda \in \mathbb{R} \mapsto \psi_\alpha(pq, p\widetilde{P}_\alpha(q)$. By construction we have $g(1) = 0$ so for $p \in [1,2]$

$$g(p) = (p-1)g'(1) + (p-1)^2\int_0^1 (1-t)g''(1+t(p-1))\,\mathrm{d}t,$$

with $g'(1) = -h(q,\alpha)$ (see (4.2) for the definition) and

$$\begin{aligned} g''(1+t(p-1)) &= {}^t\begin{pmatrix} q \\ \widetilde{P}_\alpha(q)\end{pmatrix} H_{\psi_\alpha}\big(q + t(p-1)q, \widetilde{P}_\alpha(q) + t(p-1)\widetilde{P}_\alpha(q)\big)\begin{pmatrix} q \\ \widetilde{P}_\alpha(q)\end{pmatrix} \\ &\leq \widetilde{m}_{j_n+1},\end{aligned}$$

where $(\widetilde{m}_j)_{j\geq 1}$ is defined in (4.11). Let $\eta_j = 2(p_j - 1)\widetilde{m}_j$ for $j \geq 1$. By construction of $(p_j)_{j\geq 1}$ we have $\lim_{j\to\infty}\eta_j = 0$. Specifying $p = p_{j_n+1}$ we have now

$$\psi_\alpha(p_{j_n+1}q, p_{j_n+1}\widetilde{P}_\alpha(q)) \leq (1-p_{j_n+1})h(q,\alpha) + \eta_{j_n+1}(p_{j_n+1}-1).$$

We can insert this upper bound in our estimation of $Y_n(\beta,\varrho) - Y_{n-1}(\beta,\varrho)$ and get, remembering that $\varrho \in \mathcal{J}_{\phi,\beta,L,\epsilon}$, for $n \geq L+1$

$$\begin{aligned} &\|Y_n(\beta,\varrho) - Y_{n-1}(\beta,\varrho)\|_{p_{j_n+1}}^{p_{j_n+1}} \\ \leq\ & (2\beta^{-1})^{p_{j_n+1}} s_{j_n+1}^{p_{j_n+1}} \exp\Big((1-p_{j_n+1})\sum_{k=1}^{n-1}\log(\beta) + h(q_k,\alpha_k) - \eta_{j_n+1}\Big) \\ \leq\ & (2\beta^{-1})^{p_{j_n+1}} s_{j_n+1}^{p_{j_n+1}} \exp\big((n-1)(1-p_{j_n+1})(\epsilon - \eta_{j_n+1})\big).\end{aligned}$$

The proof ends like that of Proposition 3.5.

Recall definitions (4.2) and (4.3). The proof of the next proposition follows the same lines as that of Proposition 4.9 in the next section.

Proposition 4.7 *There exists a positive sequence $(\delta_n)_{n\geq 1}$ converging to 0 such that, with probability 1, for all $\varrho=(q_k,\alpha_k)_{k\geq 1}\in\mathcal{J}_\phi$, for μ_ϱ-almost all $t\in\partial\mathsf{T}$, for n large enough, we have*

$$n^{-1}\Big|S_n\phi(t)-\sum_{k=1}^n\lambda(q_k,\alpha_k)\Big|\leq\delta_n.$$

Proposition 4.8 *With probability 1, for all $\varrho\in\mathcal{J}_\phi$,*

$$\underline{\dim}(\mu_\varrho):=\inf\Big\{\dim\,E:\ E\text{ Borel},\ \mu_\varrho(E)>0\Big\}\geq\liminf_{n\to\infty}\Big(\frac{h(q_n,\alpha_n)}{\lambda(q_n,\alpha_n)}=\widetilde{P}^*_{\alpha_n}(\nabla\widetilde{P}_{\alpha_n}(q_n))\Big).$$

Proof Let $\beta\in(0,1]$ such that $\beta\mathbb{E}(N)>1$. Let $L\geq 1$ and $\epsilon\in\mathbb{Q}_+^*$.

For every $t\in\partial\mathsf{T}$ and $\omega_\beta\in\Omega_\beta$ set

$$Q_{\beta,n}(t,\omega_\beta)=\prod_{k=1}^n W_{\beta,t_{|k}},$$

so that for $\varrho\in\mathcal{J}_{\beta,L,\epsilon}$, $\widetilde{Y}_n(\beta,\varrho)$ is the total mass of the measure $Q_{\beta,n}(t,\omega_\beta)\cdot\mathrm{d}\mu^\omega_\varrho(t)$.

There exists a measurable subset $\Omega(\beta,L,\epsilon)$ of Ω, such that $\mathbb{P}(\Omega(\beta,L,\epsilon))=1$ and for all $\omega\in\Omega(\beta,L,\epsilon)$, there exists $\Omega^\omega_\beta\subset\Omega_\beta$ of positive probability such that for all $\omega\in\Omega(\beta,L,\epsilon)$, for all $\varrho\in\mathcal{J}_{\phi,\beta,L,\epsilon}$, for all $\omega_\beta\in\Omega_\beta^\omega$, $\widetilde{Y}_n(\beta,\varrho)$ does not converge to 0. In terms of the multiplicative chaos theory developed in [47], this means, that for all $\omega\in\Omega(\beta,L,\epsilon)$ and $\varrho\in\mathcal{J}_{\beta,L,\epsilon}$, the set of those ω_β such that the multiplicative chaos $(Q_{\beta,n}(\cdot,\omega))_{n\geq 1}$ has not killed the measure μ_ϱ on the compact set $\partial\mathsf{T}$ has a positive $\mathbb{P}_\beta$-probability.

Let $\Omega'=\bigcap_{\beta\in(\mathbb{E}(N)^{-1},1]\cap\mathbb{Q}_+^*,L\geq 1,\epsilon\in\mathbb{Q}^*}\Omega(\beta,L,\epsilon)$. This set is of $\mathbb{P}$-probability 1. Let $\varrho\in\mathcal{J}_\phi$. The same argument as in the proof of Proposition 3.6 shows that setting $D=\liminf_{n\to\infty}n^{-1}\sum_{k=1}^n h(q_k,\alpha_k)$, the lower Hausdorff dimension of μ_ϱ with respect to the metric d_1 it larger than or equal to D; in particular, we have $\liminf_{n\to\infty}\dfrac{\log\mu_\varrho([t_{|n}])}{-n}\geq D$, μ_ϱ-almost everywhere. Then, since by definition of d_ϕ the diameter $|[t_{|n}]|$ of $[t_{|n}]$ equals $\exp(-S_n\phi(t))$, Proposition 4.7 together with Lemma 4.1 yield that for all $\eta\in(0,1)$, for μ_ϱ-almost every t, for n large enough,

$$\frac{\log\mu_\varrho([t_{|n}])}{\log|[t_{|n}]|}\geq(1-\eta)\frac{nD}{\sum_{k=1}^n\lambda(q_k,\alpha_k)}.$$

Let $D_\phi=\liminf_{n\to\infty}\dfrac{h(q_n,\alpha_n)}{\lambda(q_n,\alpha_n)}$. By definition of D we have $\liminf_{n\to\infty}\dfrac{Dn}{\sum_{k=1}^n\lambda(q_k,\alpha_k)}\geq D_\phi$: Indeed, let $(n_j)_{j\geq 1}$ be an increasing sequence of integers such that $\lim_{j\to\infty}n_j^{-1}\sum_{k=1}^{n_j}h(q_k,\alpha_k)=D$. For any $\delta>0$, for k large enough we have $h(q_k,\alpha_k)\geq(D_\phi-\delta)\lambda(q_k,\alpha_k)$, from which we deduce that $\liminf_{j\to\infty}\dfrac{n_jD}{\sum_{k=1}^{n_j}\lambda(q_k,\alpha_k)}\geq D_\phi-\delta$. Consequently, $\underline{\dim}(\mu_\varrho)\geq D_\phi$ in the metric d_ϕ.

4.3.4 Lower bounds for the Hausdorff dimensions of the set $E_X(K)$

The sharp lower bound estimates for the Hausdorff dimensions of the set $E_X(K)$ are direct consequences of the following two propositions.

Proposition 4.9 *There exists a positive sequence $(\delta_n)_{n\geq 1}$ converging to 0 such that, with probability 1, for all $\varrho = (q_k)_{k\geq 1} \in \mathcal{J}_\phi$, for μ_ϱ-almost all $t \in \partial\mathsf{T}$, for n large enough we have*

$$n^{-1}\Big\|S_nX(t) - \sum_{k=1}^{n} \alpha_X(q_k,\alpha_k)\Big\| \leq \delta_n.$$

Proof Fix a positive sequence $(\delta_n)_{n\geq 1}$ converging to 0. Let v be a vector of the canonical basis $\mathcal{B}$ of $\mathbb{R}^d$. For $\varrho \in \mathcal{J}_\phi$ and $n \geq 1$, we set :

$$E^1_{\varrho,n,\delta_n}(v) = \Big\{t \in \partial\mathsf{T} : \Big\langle v\Big|S_nX(t) - \sum_{k=1}^{n}\alpha_X(q_k,\alpha_k)\Big\rangle \geq n\delta_n\Big\}$$
$$E^{-1}_{\varrho,n,\delta_n}(v) = \Big\{t \in \partial\mathsf{T} : \Big\langle v\Big|S_nX(t) - \sum_{k=1}^{n}\alpha_X(q_k,\alpha_k)\Big\rangle \leq -n\delta_n\Big\}$$

It is enough to specify $(\delta_n)_{n\geq 1}$ such that for $\lambda \in \{-1,1\}$ and $v \in \mathcal{B}$ we have

$$\mathbb{E}\Big(\sup_{\varrho\in\mathcal{J}_\phi}\sum_{n\geq 1}\mu_\varrho(E^\lambda_{\varrho,n,\delta_n}(v))\Big) < \infty. \tag{4.19}$$

Then, with probability 1, for all $\varrho \in \mathcal{J}_\phi$, $\lambda \in \{-1,1\}$ and $v \in \mathcal{B}$, $\sum_{n\geq 1}\mu_\varrho(E^\lambda_{\varrho,n,\delta_n}(v)) < \infty$, and consequently, by the Borel-Cantelli lemma, for μ_ϱ-almost every t for all $v \in \mathcal{B}$, for n large enough we have

$$\Big|\Big\langle v\Big|n^{-1}\Big(S_nX(t) - \sum_{k=1}^{n}\alpha_X(q_k,\alpha_k)\Big)\Big\rangle\Big| \leq \delta_n,$$

which yields the desired result.

Now we prove (4.19) when $\lambda = 1$ (the case $\lambda = -1$ is similar). Let $\varrho \in \mathcal{J}_\phi$. For every $\gamma > 0$ we have

$$\mu_\varrho(E^1_{\varrho,n,\delta_n}(v)) \leq f_{n,\gamma}(\varrho),$$

where

$$\begin{aligned} f_{n,\gamma}(\varrho) &= \sum_{u\in\mathsf{T}_n}\mu_\varrho([u])\prod_{k=1}^{n}\exp\big(\gamma\langle v|X_{u_{|k}} - \alpha_X(q_k,\alpha_k)\rangle - \gamma\delta_n\big) \\ &= \sum_{u\in\mathsf{T}_n}\Pi_{n,\gamma}(\varrho,u)Y(\varrho,u), \end{aligned}$$

where we used the definition of μ_ϱ and set

$$\Pi_{n,\gamma}(\varrho,u) = \prod_{k=1}^{n}\exp\big(\langle q_k + \gamma v|X_{u_{|k}} - \alpha_k\rangle - \widetilde{P}_{\alpha_k}(q_k)\phi_{u_{|k}} - \langle\gamma v|\alpha_X(q_k,\alpha_k) - \alpha_k\rangle - \gamma\delta_n\big).$$

Since $\Pi_n(\varrho,u)$ only depends on $\varrho_{|n}$ and u, we have

$$\sup_{\varrho\in\mathcal{J}_\phi} f_{n,\gamma}(\varrho) \le \sum_{u\in\mathsf{T}_n}\ \sup_{\varrho_{|n}:\varrho\in\mathcal{J}_\phi} \Pi_n(\varrho,u)\cdot \sup_{\varrho\in\mathcal{J}_\phi} Y(\varrho,u).$$

Consequently, since $\mathbb{E}(\sup_{\varrho\in\mathcal{J}} Y(\varrho,u)) \le C_{\mathcal{J}_\phi}\exp(\epsilon_{|u|}|u|)$ by (4.18), we have (taking into account the independences)

$$\begin{aligned}
&\mathbb{E}(\sup_{\varrho\in\mathcal{J}_\phi} f_{n,\gamma}(\varrho))\\
\le\ & C_{\mathcal{J}_\phi}\exp(n\epsilon_n)\mathbb{E}\Big(\sum_{u\in\mathsf{T}_n}\ \sup_{\varrho_{|n}:\varrho\in\mathcal{J}_\phi}\Pi_n(\varrho,u)\Big)\\
\le\ & C_{\mathcal{J}_\phi}\exp(n\epsilon_n)\mathbb{E}\Big(\sum_{u\in\mathsf{T}_n}\ \sum_{\varrho_{|n}:\varrho\in\mathcal{J}_\phi}\Pi_n(\varrho,u)\Big)\\
=\ & C_{\mathcal{J}_\phi}\exp(n\epsilon_n)\sum_{\varrho_{|n}:\varrho\in\mathcal{J}_\phi}\prod_{k=1}^n \exp\big(\psi_{\alpha_k}(q_k+\gamma v,\widetilde{P}_{\alpha_k}(q_k)) - \langle\gamma v|\alpha_X(q_k,\alpha_k)-\alpha_k\rangle - \gamma\delta_n\big).
\end{aligned}$$

For each $\varrho\in\mathcal{J}_\phi$, we have $q_k\in D_{j_n+1}$ for all $1\le k\le n$. Thus, writing for each $1\le k\le n$ the Taylor expansion with integral rest of order 2 of $\gamma\mapsto\psi_{\alpha_k}(q_k+\gamma v,\widetilde{P}_{\alpha_k}(q_k)) - \langle\gamma v|\alpha_X(q_k,\alpha_k)-\alpha_k\rangle$ at 0, taking $\gamma=\gamma_{j_n+1}$, and using (4.6) and (4.10) we get

$$\sum_{k=1}^n \psi_{\alpha_k}(q_k+\gamma_{j_n+1}v,\widetilde{P}_{\alpha_k}(q_k)) - \langle\gamma_{j_n+1}v|\alpha_X(q_k,\alpha_k)-\alpha_k\rangle - \gamma_{j_n+1}\delta_n \le n\gamma_{j_n+1}^2 m_{j_n+1} - n\gamma_{j_n+1}\delta_n$$

uniformly in $\varrho\in\mathcal{J}_\phi$. Consequently, using that $\epsilon_n = 2\gamma_{j_n+1}^2\widehat{m}_{j_n+1}$, $m_{j_n+1}\le\widehat{m}_{j_n+1}$, and $\mathrm{card}(\{\varrho_{|n}:\varrho\in\mathcal{J}\}) = (j_n+1)!$, we get

$$\mathbb{E}(\sup_{\varrho\in\mathcal{J}_\phi} f_{n,\gamma_{j_n+1}}(\varrho)) \le C_{\mathcal{J}_\phi}(j_n+1)!\exp\big((-n\gamma_{j_n+1}(\delta_n - 3\gamma_{j_n+1}^2\widehat{m}_{j_n+1})\big).$$

Let $\delta_n = 4\gamma_{j_n+1}^2\widehat{m}_{j_n+1}$. Now we use (4.15) : $(j_n+1)! \le \exp(N_{j_n}^{1/5}) \le \exp(n^{1/5})$ and $\gamma_{j_{n+1}}^2\widehat{m}_{j_n+1} \ge N_{j_n}^{-1/2} \ge n^{-1/2}$. Thus

$$\mathbb{E}\big(\sup_{\varrho\in\mathcal{J}_\phi}\mu_\varrho(E^1_{\varrho,n,\delta_n}(v))\big) \le \mathbb{E}(\sup_{\varrho\in\mathcal{J}_\phi} f_{n,\gamma_{j_n+1}}(\varrho)) \le C_{\mathcal{J}_\phi}\exp(n^{1/5})\exp(-n^{1/2}).$$

Since $\delta_n = 4\gamma_{j_n+1}^2\widehat{m}_{j_n+1}$ tends to 0 as n tends to ∞, and $\sum_{n\ge1}\mathbb{E}\big(\sup_{\varrho\in\mathcal{J}_\phi}\mu_\varrho(E^1_{\varrho,n,\delta_n})\big) < \infty$, we get the desired conclusion.

Proposition 4.10 *For every compact connected subset K of I there exists $\varrho\in\mathcal{J}_\phi$ such that*

$$\begin{cases}\bigcap_{N\ge1}\overline{\Big\{n^{-1}\sum_{k=1}^n\alpha_X(q_k,\alpha_k) : n\ge N\Big\}} = K\\ \liminf_{n\to\infty}\widetilde{P}^*_{\alpha_n}(\nabla\widetilde{P}_{\alpha_n}(q_n)) \ge \inf\{P^*(\alpha):\alpha\in K\}\end{cases}.$$

Proof It is similar to that of Proposition 3.8 but uses Propositions 4.4 and 4.8 , as well as (4.16) instead of Corollary 3.1, Proposition 3.6 and (3.14).

4.4 Proof of Theorem 1.6

(1) We will interpret $\mathcal{K}$ via the set of the possible accumulating points of $(S_nX, S_n\widetilde{X})$. Consider

$$\widetilde{P}_{(X,\widetilde{X})}(q,\widetilde{q}) = \log \mathbb{E}\Big(\sum_{i=1}^{N} \exp(\langle q|X\rangle + \langle \widetilde{q}|\widetilde{X}\rangle)\Big)$$

and the compact set

$$I_{(X,\widetilde{X})} = \Big\{(\alpha,\widetilde{\alpha}) \in \mathbb{R}^d \times \mathbb{R}^d : \widetilde{P}^*_{(X,\widetilde{X})}(\alpha,\widetilde{\alpha}) \geq 0\Big\}.$$

Due to Theorem 1.3 applied to $(X,\widetilde{X})$ and $\widetilde{X}$, the projection on the second coordinate of $I_{(X,\widetilde{X})}$ equals $I_{\widetilde{X}}$. Moreover using arguments similar to those use to prove Lemma 4.1, we can get that $I_{\widetilde{X}} \subset {\mathbb{R}_+^*}^d$.

Let $\Phi : \mathbb{R}^d \times {\mathbb{R}_+^*}^d \to \mathbb{R}^d$ be the mapping defined as

$$\Phi(x_1,\ldots,x_d,y_1,\ldots y_d) = \Big(\frac{x_1}{y_1},\ldots,\frac{x_d}{y_d}\Big).$$

Proposition 4.11 *We have*

$$\mathcal{K} = \Phi(I_{(X,\widetilde{X})}).$$

Notice that this proposition implies the compactness and connectedness of $\mathcal{K}$ since Φ is continuous and $I_{(X,\widetilde{X})}$ is convex.

Proof First for $\alpha \in \mathbb{R}^d$ let us define

$$\widehat{E}_{X,\widetilde{X}}(\alpha) = \left\{t : \alpha \in \bigcap_{N\geq 1} \overline{\Big\{\Big(\frac{S_nX^j(t)}{S_n\widetilde{X}^j(t)}\Big)_{1\leq j\leq d} : n \geq N\Big\}}\right\}.$$

For every $t \in \partial\mathsf{T}$ and $\alpha \in \mathbb{R}^d$,

$$\alpha \in \bigcap_{N\geq 1} \overline{\Big\{\Big(\frac{S_nX^j(t)}{S_n\widetilde{X}^j(t)}\Big)_{1\leq j\leq d} : n \geq N\Big\}} \quad \text{iff} \quad 0 \in \bigcap_{N\geq 1} \overline{\Big\{\Big(\frac{S_n(X^j-\alpha_j\widetilde{X}^j)(t)}{n}\Big)_{1\leq j\leq d} : n \geq N\Big\}},$$

in other words $\widehat{E}_{X,\widetilde{X}}(\alpha) = \widehat{E}_{X-\alpha\cdot\widetilde{X}}(0)$.

Now suppose that $\alpha \in \mathcal{K}$, i.e. $\inf_{q\in\mathbb{R}^d} \widetilde{P}_\alpha(q) \geq 0$. Like in Proposition 4.4 we can find a sequence $(q_k)_{k\geq 1}$ in $\mathbb{R}^d$ such that $\widetilde{P}^*_\alpha(\nabla\widetilde{P}_\alpha(q_k)) > 0$ and $\nabla\widetilde{P}_\alpha(q_k)$ converges to 0. For each q_k in the sequence, let $\mu_{q_k,\alpha}$ be the Mandelbrot measure associated with the family of vectors $(N_u, X'_{u1} = \langle q_k|X_{u1} - \alpha\cdot\widetilde{X}_{u1}\rangle - \widetilde{P}_\alpha(q_k)\phi_{u1}, X'_{u2} = \langle q_k|X_{u2} - \alpha\cdot\widetilde{X}_{u2}\rangle - \widetilde{P}_\alpha(q_k)\phi_{u2},\ldots)$, $u \in \bigcup_{n\geq 0}\mathbb{N}_+^n$ (see Section 6). For the same reasons as those invoked in Section 4.3 for the measure $\mu_{q,\alpha}$, this measure is non degenerate. Also, here

$$\nabla\widetilde{P}_\alpha(q_k) = \frac{\mathbb{E}\Big(\sum_{i=1}^N (X_i - \alpha\cdot\widetilde{X}_i)\exp(\langle q_k|X_i - \alpha\cdot\widetilde{X}_i\rangle - \widetilde{P}_\alpha(q_k)\phi_i)\Big)}{\mathbb{E}\Big(\sum_{i=1}^N \phi_i\exp(\langle q_k|X_i - \alpha\cdot\widetilde{X}_i\rangle - \widetilde{P}_\alpha(q_k)\phi_i)\Big)}.$$

Consequently, if we set for $q \in \mathbb{R}^d$

$$\begin{aligned}\alpha_X(q,\alpha) &= \mathbb{E}\Big(\sum_{i=1}^N X_i \exp(\langle q|X_i - \alpha\cdot\widetilde{X}_i\rangle - \widetilde{P}_\alpha(q)\phi_i)\Big),\\ \alpha_{\widetilde{X}}(q,\alpha) &= \mathbb{E}\Big(\sum_{i=1}^N \widetilde{X}_i \exp(\langle q|X_i - \alpha\cdot\widetilde{X}_i\rangle - \widetilde{P}_\alpha(q)\phi_i)\Big).\end{aligned}$$

by Proposition 6.1, with probability 1, for $\mu_{q_k,\alpha}$-almost every $t\in\partial\mathsf{T}$,

$$\begin{aligned}\lim_{n\to\infty}\frac{S_nX(t)-\alpha\cdot S_n\widetilde{X}(t)}{n} &= \mathbb{E}\Big(\sum_{i=1}^N \phi_i\exp(X_i')\Big)\nabla\widetilde{P}_\alpha(q_k) \qquad (4.20)\\ &= \alpha_X(q_k,\alpha)-\alpha\cdot\alpha_{\widetilde{X}}(q_k,\alpha)\\ \lim_{n\to\infty}\frac{S_nX(t)}{n} &= \alpha_X(q_k,\alpha),\\ \lim_{n\to\infty}\frac{S_n\widetilde{X}(t)}{n} &= \alpha_{\widetilde{X}}(q_k,\alpha),\\ \lim_{n\to\infty}\frac{S_n\phi(t)}{n} &= \mathbb{E}\Big(\sum_{i=1}^N \phi_i\exp(X_i')\Big)\end{aligned}$$

We notice that $\mathbb{E}\Big(\sum_{i=1}^N \phi_i\exp(X_i')\Big)$, which depends on k, is a limit point of $\frac{S_n\phi(t)}{n}$, so it belongs to a deterministic compact set isolated from 0. Also, $(\alpha_X(q_k,\alpha),\alpha_{\widetilde{X}}(q_k,\alpha))$ being a limit point of $\frac{S_n(X,\widetilde{X})(t)}{n}$, it belongs to $I_{(X,\widetilde{X})}$. Moreover, since $\nabla\widetilde{P}_\alpha(q_k)$ converges to 0 as $k\to\infty$, the above limit show that $\alpha=\lim_{k\to\infty}\Phi(\alpha_X(q_k,\alpha),\alpha_{\widetilde{X}}(q_k,\alpha))$, and since Φ is continuous, α belongs to the closed set $\Phi(I_{X,\widetilde{X}})$. This yields $\mathcal{K}\subset\Phi(I_{X,\widetilde{X}})$.

Now consider $(\gamma,\widetilde{\gamma})\in I_{X,\widetilde{X}}$. Our study of the level sets of the limit points of $S_n(X,\widetilde{X})/n$ shows that $\lim_{n\to\infty}S_n(X,\widetilde{X})(t)/n=(\gamma,\widetilde{\gamma})$ a.s. for some $t\in\partial\mathsf{T}$. It follows that if $\alpha=\Phi(\gamma,\gamma)$, we have $\widehat{E}_{X,\widetilde{X}}(\alpha)\neq\emptyset$. Moreover, our study in Section 4.2 implies that $\dim\widehat{E}_{X,\widetilde{X}}(\alpha)\le\inf_{q\in\mathbb{R}^d}\widetilde{P}_\alpha(q)$ almost surely for all α simultanously, a negative dimension meaning that $\widehat{E}_{X,\widetilde{X}}(\alpha)=\emptyset$. We thus have $\inf_{q\in\mathbb{R}^d}\widetilde{P}_\alpha(q)\ge 0$ and $\alpha\in\mathcal{K}$. Thus we have proved that $\Phi(I_{X,\widetilde{X}})\subset\mathcal{K}$.

(2) The upper bound for the Hausdorff dimension comes from the previous lines. For the lower bound, the strategy is exactly the same as the one developed to prove Theorem 1.4. The essential changes are as follow. We again set $J_\phi=\big\{(q,\alpha)\in\mathbb{R}^d\times\mathbb{R}^d:\widetilde{P}_\alpha^*(\nabla\widetilde{P}_\alpha(q))>0\big\}$, but remind that the definition of $\widetilde{P}_\alpha$ has changed.

The sets D_j, $j\ge1$, and the set of sequences $\mathcal{J}_\phi$ are defined formally in the same way as in Section 4.3, and for $u\in\bigcup_{n\ge0}\mathbb{N}_+^n$, $1\le i\le N(u)$, and $\varrho\in\mathcal{J}_\phi$ we define

$$W_{\varrho,ui}=\exp(\langle q_{|u|+1}|X_{ui}-\alpha_{|u|+1}\cdot\widetilde{X}_{ui}\rangle-\widetilde{P}_{\alpha_{|u|+1}}(q_{|u|+1})\phi_{ui})$$

to build the associated inhomogeneous Mandelbrot measure μ_ϱ. Then, choosing the sequence $(N_j)_{j\ge1}$ suitably, with probability 1, for all $\varrho=((q_k,\alpha_k))_{k\ge1}\in\mathcal{J}_\varrho$, μ_ϱ has $\partial\mathsf{T}$ as closed support, and for μ_ϱ-almost every t we have

$$\lim_{n\to\infty}\frac{1}{n}\Big(S_nX(t)-\sum_{k=1}^n\alpha_X(q_k,\alpha_k)\Big) = 0,$$

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n}\Big(S_n\widetilde{X}(t) - \sum_{k=1}^{n} \alpha_{\widetilde{X}}(q_k,\alpha_k)\Big) &= 0, \\ \text{so } \lim_{n\to\infty} \frac{1}{n}\Big(S_n(X-\alpha\cdot\widetilde{X})(t) - \sum_{k=1}^{n}(\alpha_X(q_k,\alpha_k) - \alpha\cdot\alpha_{\widetilde{X}}(q_k,\alpha_k)\Big) &= 0 \quad (\forall \alpha\in\mathbb{R}^d), \end{aligned}$$

moreover $\underline{\dim}\mu_\varrho \geq \liminf_{n\to\infty} \widetilde{P}^*_{\alpha_k}(\nabla\widetilde{P}_{\alpha_k}(q_k))$.

Also, given a compact connected subset K of $\mathcal{K}$, for each point α of K we can find $(q,\alpha')\in J_\phi$ such that α' is arbitrarily close to α, $\nabla\widetilde{P}_{\alpha'}(q)$ is arbitrarily close to 0 and $\widetilde{P}^*_{\alpha'}(\nabla\widetilde{P}_{\alpha'}(q))$ is arbitrarily close to $\inf_{q\in\mathbb{R}^d}\widetilde{P}_\alpha(q)$. Notice that assuming that $\nabla\widetilde{P}_{\alpha'}(q)$ is close to 0 implies, after (4.20), that $\alpha_X(q'_\alpha) - \alpha'\cdot\alpha_{\widetilde{X}}(q,\alpha')$ is close to 0, and so $\alpha_X(q'_\alpha) - \alpha\cdot\alpha_{\widetilde{X}}(q,\alpha')$ is. This and the previous properties valid for any μ_ϱ can be used to build, in the spirit of Propositions 3.8 and 4.10, a sequence $\varrho\in\mathcal{J}_\phi$ such that $\liminf_{n\to\infty}\widetilde{P}^*_{\alpha_k}(\nabla\widetilde{P}_{\alpha_k}(q_k)) \geq \inf_{\alpha\in K}\inf_{q\in\mathbb{R}^d}\widetilde{P}_\alpha(q)$ and

$$K = \bigcap_{N\geq 1}\overline{\Big\{\Phi\Big(n^{-1}\sum_{k=1}^{n}\alpha_X(q_k,\alpha_k), n^{-1}\sum_{k=1}^{n}\alpha_{\widetilde{X}}(q_k,\alpha_k)\Big) : n\geq N\Big\}},$$

so that $\mu_\varrho(E_{X,\widetilde{X}}(K)) > 0$.

Chapitre 5

Results about the level sets $E_X(\alpha)$ and $E_X^{\mathrm{LD}(q_\alpha,\alpha)}$ under the metric d_ϕ

We prove Theorems 1.5, 1.7 and 1.8

5.1 Proof of Theorem 1.5

(1) It is a consequence of Theorem 1.4.

(2) The part concerning inhomogeneous measures can be proved by using the same approach as in Section 3.3.5.

The point concerning the fact that almost surely, for all $\alpha \in \mathring{I}$ simultaneously, the measures $\mu_{q_\alpha,\alpha}$ is supported on $E_X(\alpha)$ and of maximal Hausdorff dimension requires first to prove that the associated homogeneous Mandelbrot martingales converge almost surely simultaneously to non trivial limits $\mu_{q_\alpha,\alpha}$, which are indeed carried by the sets $E_X(\alpha)$ and have lower Hausdorff dimension at least $\widetilde{P}^*_\alpha(0)$. This can be done by mimicking the approach of Chapter 2, since we know that $\alpha \in \mathring{I} \mapsto q_\alpha$ is analytic (Proposition 1.1). The point concerning the uniqueness of the Mandelbrot measure carried by $E_X(\alpha)$ follows from the following general fact.

Proposition 5.1 *Let μ' be a non degenerate Mandelbrot measure built simultaneously with $(S_nX, S_n\phi)_{n\geq 1}$ as in Section 6. Fix $(q,\alpha) \in \mathbb{R}^{2d}$. For $s \in [0,1]$ let*

$$\Phi(s) = \log \mathbb{E}\Big(\sum_{i=1}^{N} \exp(X_i') \exp(s(\langle q|X_i - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_i - X_i'))\Big).$$

Suppose that $\Phi(1) = 0$. Then

$$\mathbb{E}\Big(\sum_{i=1}^{N}(\langle q|X_i - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_i) \exp(X_i')\Big) - \mathbb{E}\Big(\sum_{i=1}^{N} X_i' \exp(X_i')\Big) \leq 0 \qquad (5.1)$$

with equality if and only if $X_i' = \langle q|X_i - \alpha\rangle - \widetilde{P}_\alpha(q)\phi_i$ for all $1 \leq i \leq N$ almost surely.

We assume this proposition and apply it with $q = q_\alpha$, where $\alpha \in \nabla\widetilde{P}(J)$ (this is possible since we assumed (1.7)). Suppose that μ' is a non degenerate Mandelbrot measure built simultaneously with $(S_nX, S_n\phi)_{n\ge1}$, such that $\mathbb{E}\Big(\sum_{i=1}^N \|X_i\|\exp(X_i')\Big) < \infty$, and $\mathbb{E}\Big(\sum_{i=1}^N X_i\exp(X_i')\Big) = \alpha$, i.e. μ' is supported on $E_X(\alpha)$, and $\dim\mu' = \widetilde{P}_\alpha^*(0) = \dim\mu_{q_\alpha,\alpha}$, where we recall that $\mu_{q_\alpha,\alpha}$ is the Mandelbrot measure generated by the vectors $(N_u, (\langle q|X_{ui}-\alpha\rangle - \widetilde{P}_\alpha(q)\phi_{ui})_{i\ge1})$, $u\in\bigcup_{n\ge0}\mathbb{N}_+^n$. By using Theorem 6.1 we get

$$\widetilde{P}_\alpha^*(0) = -\frac{\mathbb{E}\Big(\sum_{i=1}^N X_i'\exp(X_i')\Big)}{\mathbb{E}\Big(\sum_{i=1}^N \phi_i\exp(X_i')\Big)} \tag{5.2}$$

(notice that since $\alpha\in\mathring{I}$ we have $\widetilde{P}_\alpha^*(0)>0$ hence $\mathbb{E}\Big(\sum_{i=1}^N\phi_i\exp(X_i')\Big)<\infty$). On the other hand, the equality $\mathbb{E}\Big(\sum_{i=1}^N X_i\exp(X_i')\Big)=\alpha$ implies that $\mathbb{E}\Big(\sum_{i=1}^N(\langle q_\alpha|X_i-\alpha\rangle - \widetilde{P}_\alpha(q_\alpha)\phi_i)\exp(X_i')\Big) = -\widetilde{P}_\alpha(q_\alpha)\mathbb{E}\Big(\sum_{i=1}^N\phi_i\exp(X_i')\Big) = -\widetilde{P}_\alpha^*(0)\mathbb{E}\Big(\sum_{i=1}^N\phi_i\exp(X_i')\Big)$, and by (5.2) this equals $\mathbb{E}\Big(\sum_{i=1}^N X_i'\exp(X_i')\Big)$. Thus equality holds in (5.1), hence the conclusion.

Proof of Proposition 5.1. It is enough to notice that Φ is convex and $\Phi(0)=\Phi(1)=0$, and equality in (5.1) means that $\Phi'(0^+)=0$, so that $\Phi(s)=0$ for all $s\in[0,1]$. Using Hölder's inequality yields the conclusion. □

(3) The fact that $E_X(\alpha_0)$ and $\partial\mathsf{T}$ have Hausdorff dimension t_0 comes from the fact that by using the natural covering of $\partial\mathsf{T}$ by the cylinders of generation n and the definition of t_0 one finds $\dim\partial\mathsf{T}\le t_0$ almost surely, and the Mandelbrot measure $\mu_{0,\alpha}$, which does not depend on α, is by construction of dimension t_0 and is carried by $E_X(\alpha_0)$ almost surely. Suppose that another level set $E_X(\alpha)$ has Hausdorff dimension $t_0=\widetilde{P}_\alpha(0)$. This means $\widetilde{P}_\alpha^*(0)=t_0$, so that $\widetilde{P}_\alpha(q)$ reaches its minimum at $q=0$. Then, due to (4.1), we have $\alpha=\alpha_0$.

5.2 Proof of Theorem 1.7

Additional conditions on $(N_j)_{j\ge1}$ and definitions

We need to slightly modify the set $\mathcal{J}_\phi$ by requiring, in addition to the initial conditions on $(N_j)_{j\ge0}$, that for all $j\ge1$

$$N_{j+1} > M_jk(M_j) \quad\text{and}\quad ((j+3)!)^2\exp(-N_j/j)\le j^{-2}. \tag{5.3}$$

Since the sequence $(k(n))_{n\ge1}$ is increasing, writing $M_{j_n}+1\le n\le M_{j_n+1}$ we have $nk(n)\le M_{j_n+1}k(M_{j_n+1}) < N_{j_n+2} < M_{j_n+2}$, so $j_{nk(n)}\le j_n+1$.

For each integer $m\ge1$, define the compact set

$$K_m = \{(q,\alpha)\in J_\phi\cap B(0,m) : d((q,\alpha),\partial J_\phi)\ge 1/m\}\cup\{(q_1,\alpha_1)\},$$

where $B(0,m)$ is the ball of radius m centered at 0 in $\mathbb{R}^{2d}$ and (q_1,α_1) is the unique element of D_1. Then, for $\ell, m \geq 1$ let

$$\mathcal{J}_{\phi,m} = \Big\{\varrho = (q_k,\alpha_k)_{k\geq 1} \in \mathcal{J}_\phi \cap K_m^{\mathbb{N}_+} : \exists\ \alpha \in \nabla\widetilde{P}(J),\ \lim_{k\to\infty}(q_k,\alpha_k) = (q_\alpha,\alpha)\Big\} \quad (5.4)$$

and

$$I_m = \Big\{\alpha \in \nabla\widetilde{P}(J) : (q_\alpha,\alpha) \in \{\lim_{k\to\infty}\varrho_k = (q_k,\alpha_k) : \varrho \in \mathcal{J}_{\phi,m}\}\Big\}.$$

By construction we have $\nabla\widetilde{P}(J) = \bigcup_{m\geq 1} I_m$. If $(q_\alpha,\alpha) = \lim_{k\to\infty}\varrho_k$, set $\alpha = \alpha_\varrho$.

Let $\kappa = \liminf_{n\to\infty}\log(k(n))/n$ and $\kappa' = \limsup_{n\to\infty}\log(k(n))/n$. For all integers $\ell, m \geq 1$ and closed dyadic cube Q in $\mathbb{R}^d$, define the sets

$$\mathcal{J}_{\phi,m,\ell,Q} = \Big\{\varrho \in \mathcal{J}_{\phi,m} : \forall\ \lambda \in \widetilde{Q},\ -\Lambda^*_{q_{\alpha_\varrho},\alpha_\varrho}(\nabla\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda)) < \min(\ell,\kappa - 1/\ell)\Big\},$$
$$\mathcal{J}'_{\phi,m,\ell,Q} = \Big\{\varrho \in \mathcal{J}_{\phi,m} : \forall\ \lambda \in Q,\ -\Lambda^*_{q_{\alpha_\varrho},\alpha_\varrho}(\nabla\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda)) > \kappa' + 1/\ell\Big\},$$

where $\widetilde{Q}$ stands for the union of Q and the closed dyadic cubes of the same generation as Q and neighboring Q.

Proposition 5.2 *With probability 1, for all integers $\ell, m \geq 1$ and all dyadic cubes Q,*

1. *for all $\varrho \in \mathcal{J}_{\phi,m,\ell,Q}$, for μ_ϱ-almost every t, we have $\lim_{n\to\infty} n^{-1}\Lambda_n^t(\lambda) = \Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda)$ for all $\lambda \in Q$.*
2. *For all $\varrho \in \mathcal{J}'_{\phi,m,\ell,Q}$, for μ_ϱ-almost every t, for all $\lambda \in Q$, there exists $\epsilon > 0$ such that for n large enough, $\Big\{1 \leq j \leq k(n) : \dfrac{\Delta S_n X(j,t)}{n} \in B(\nabla\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda),\epsilon)\Big\} = \emptyset$.*

Suppose the proposition has been proved. Let us prove the theorem. We have

$$\begin{aligned}
&\{(\alpha,\lambda) \in \nabla\widetilde{P}(J)\times\mathbb{R}^d :\ -\Lambda^*_{q_\alpha,\alpha}(\nabla\Lambda_{q_\alpha,\alpha}(\lambda)) < \kappa\}\\
=\ &\bigcup_{m\geq 1}\bigcup_{\ell\geq 1}\{(\alpha,\lambda) \in \nabla\widetilde{P}(J)\times\mathbb{R}^d : \alpha \in I_m,\ -\Lambda^*_{q_\alpha,\alpha}(\nabla\Lambda_{q_\alpha,\alpha}(\lambda)) < \min(\ell,\kappa-1/\ell)\}\\
=\ &\bigcup_{m\geq 1}\bigcup_{\ell\geq 1}\bigcup_{\substack{Q,\\ \text{dyadic cube}}}\{\alpha \in I_m : \forall\,\lambda \in \widetilde{Q},\ -\Lambda^*_{q_\alpha,\alpha}(\nabla\Lambda_{q_\alpha,\alpha}(\lambda)) < \min(\ell,\kappa-1/\ell)\}\times\widetilde{Q},
\end{aligned}$$

where we have used the continuity in (α,λ) of $-\Lambda^*_{q_\alpha,\alpha}(\nabla\Lambda_{q_\alpha,\alpha}(\lambda))$. Consequently, due to Proposition (5.2)(1), with probability 1, for all $\alpha \in \nabla\widetilde{P}(J)$, if m is large enough so that $\alpha \in I_m$, for all $\varrho \in \mathcal{J}_{\phi,m}$ such that $\lim_{k\to\infty}\varrho_k = (q_\alpha,\alpha)$, since each $\lambda \in \mathbb{R}^d$ such that $-\Lambda^*_{q_\alpha,\alpha}(\nabla\Lambda_{q_\alpha,\alpha}(\lambda)) < \kappa$ belongs, for ℓ large enough, to a dyadic cube Q such that $-\Lambda^*_{q_\alpha,\alpha}(\nabla\Lambda_{q_\alpha,\alpha}) < \min(\ell,\kappa-1/\ell)$ over $\widetilde{Q}$, we have that μ_ϱ-almost everywhere, $\lim_{n\to\infty} n^{-1}\Lambda_n^t(\lambda) = \Lambda_{q_\alpha,\alpha}$ for all $\lambda \in \mathbb{R}^d$ such that $-\Lambda^*_{q_\alpha,\alpha}(\nabla\Lambda_{q_\alpha,\alpha}(\lambda)) < \kappa$, i.e. the first part of the large deviations principle $LD(q_\alpha,\alpha)$. For the second point, one uses a similar argument. Now, to get the desired lower bound for $\dim E_X^{\mathrm{LD}(q_\alpha,\alpha)}(\alpha)$, it is enough to pick ϱ such that $\lim_{k\to\infty}\varrho_k = (q_\alpha,\alpha)$ and $\lim_{k\to\infty}\widetilde{P}_{\alpha_k}(q_k) - \langle q_k|\nabla\widetilde{P}_{\alpha_k}(q_k)\rangle = \widetilde{P}^*_\alpha(0)$, as in Proposition 4.4 (then Proposition(4.8) yields the result).

Proof of Proposition 5.2. Fix m,ℓ,Q. We need to cut the sets $\mathcal{J}_{\phi,m,\ell,Q}$ and $\mathcal{J}'_{\phi,m,\ell,Q}$ as follows : for every integer $L \geq 1$ let

$$\mathcal{J}_{\phi,m,\ell,L,Q} = \Big\{\varrho \in \mathcal{J}_{\phi,m,\ell,Q} : \forall\ k \geq L,\ \forall\ \lambda \in \widetilde{Q},\ -\Lambda^*_{q_k,\alpha_k}(\nabla\Lambda_{q_k,\alpha_k}(\lambda)) < \min(2\ell,\kappa-1/2\ell)\Big\},$$

and

$$\begin{aligned}&\mathcal{J}'_{\phi,m,\ell,L,Q}\\ =\;&\left\{\varrho\in\mathcal{J}_{\phi,m,\ell,Q}:\forall\,k\geq L,\ \forall\,\lambda\in Q,\ |\Lambda^*_{q_k,\alpha_k}(\nabla\Lambda_{q_k,\alpha_k}(\lambda))-\Lambda^*_{q_{\alpha_\varrho},\alpha_\varrho}(\nabla\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda))|<1/2\ell\right\}.\end{aligned}$$

Due to the continuity of $\Lambda_{q,\alpha}$ as a function of (q,α) taking values in the set of continuous functions over $\mathbb{R}^d\times\mathbb{R}^d$ endowed with the topology of the uniform convergence over compact sets and the compactness of $\widetilde{Q}$, we have $\mathcal{J}_{\phi,m,\ell,Q}=\bigcup_{L\geq 1}\mathcal{J}_{\phi,m,\ell,L,Q}$ and $\mathcal{J}'_{\phi,m,\ell,Q}=\bigcup_{L\geq 1}\mathcal{J}'_{\phi,m,\ell,L,Q}$.

(1) Fix $L\geq 1$. For $\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}$, $\lambda\in\widetilde{Q}$, $n\geq 1$, $1\leq j\leq k(n)$ and $t\in\partial\mathsf{T}$, set

$$s_{n,j}(\varrho,\lambda)=\sum_{k=(j-1)n+1}^{jn}\Lambda_{q_k,\alpha_k}(\lambda)$$

and

$$Z_{n,j}(\varrho,\lambda,t)=\exp(\langle\lambda|\Delta S_nX(j,t)\rangle-s_{n,j}(\varrho,\lambda)).\tag{5.5}$$

It is enough that we prove that for every $\lambda\in\widetilde{Q}$ and $\epsilon>0$ we have

$$\mathbb{E}\left(\sum_{n\geq 1}\sup\left\{\mu_\varrho\Big(\Big\{t\in\partial\mathsf{T}:\Big|k(n)^{-1}\sum_{j=1}^{k(n)}(Z_{n,j}(\varrho,\lambda,t)-1)\Big|>\epsilon\Big):\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}\right\}\right)<\infty.\tag{5.6}$$

Indeed, suppose that (5.6) holds true. Then, for every $\lambda\in\widetilde{Q}$ and $\epsilon\in(0,1)$, with probability 1, for all $\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}$, applying the Borel-Cantelli lemma to μ_ϱ yields, for μ_ϱ-almost every t, an integer $n_\varrho\geq 1$ such that for all $n\geq n_\varrho$,

$$1-\epsilon\leq k(n)^{-1}\sum_{j=1}^{k(n)}Z_{n,j}(\varrho,\lambda,t)\leq 1+\epsilon.$$

Moreover, given $\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}$, there exists $k_\varrho>1$ such that for all $k\geq k_\varrho$, one has $|\Lambda_{q_k,\alpha_k}(\lambda)-\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda)|\leq\epsilon$, hence for $n\geq k_\varrho$ we have

$$\begin{cases}|s_{n,j}(\varrho,\lambda)-n\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda)|\leq n\epsilon & \text{if } j\geq 2\\ |s_{n,1}(\varrho,\lambda)-n\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda)|\leq n\epsilon+C_\varrho(\lambda)\end{cases},$$

where $C_\varrho(\lambda)=\sum_{k=1}^{k_\varrho}|\Lambda_{q_k,\alpha_k}(\lambda)-\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda)|$.

Consequently, for $n\geq\max(k_\varrho,n_\varrho)$, recalling the definition (1.10) of $\Lambda_n^t(\lambda)$ we have

$$(1-\epsilon)\exp(-n\epsilon-C_\varrho(\lambda))\leq\exp\left(\Lambda_n^t(\lambda)-n\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda)\right)\leq(1+\epsilon)\exp(n\epsilon+C_\varrho(\lambda)).$$

Letting ϵ go to 0 along a discrete family, this gives that almost surely, for all $\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}$, for μ_ϱ-almost every t, $\lim_{n\to\infty}n^{-1}\Lambda_n^t(\lambda)=\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda)$. Then this convergence holds almost surely for a countable and dense subset of elements λ of $\widetilde{Q}$, and finally the convexity of the functions Λ_n^t gives the convergence for all $\lambda\in Q$, since Q is included in the interior of $\widetilde{Q}$. Then the desired result comes from the Gartner-Ellis theorem.

To prove (5.6), we need the following lemma, in which $\mathcal{Q}_\varrho$ stands for the Peyrière measure associated with μ_ϱ (see section 6.2).

Lemma 5.1 *1. Let $\varrho \in \mathcal{J}_{\phi,m,\ell,L,Q}$ and $\lambda \in \widetilde{Q}$. The random variables $(\omega, t) \mapsto Z_{n,j}(\varrho, \lambda, t) - 1$, $1 \le j \le k(n)$, defined on $\Omega \times \mathbb{N}_+^{\mathbb{N}_+}$ are centered and independent with respect to $\mathcal{Q}_\varrho$.*

2. There exists $p(m, \ell, L, Q) \in (1, 2]$ and $C(m, \ell, L, Q) > 0$ such that for all $p \in (1, p(m, \ell, L, c)]$, for all $\epsilon > 0$,

$$\mathcal{Q}_\varrho\Big(\Big|k(n)^{-1}\sum_{j=1}^{k(n)}(Z_{n,j}(\varrho,\lambda,t)-1)\Big| > \epsilon\Big) \;\le\; C(m,\ell,L,Q)\exp(-n(p-1)\ell/4)$$

independently of $\varrho \in \mathcal{J}_{\phi,m,\ell,L,Q}$ and $\lambda \in \widetilde{Q}$.

We postpone the proof of this lemma to the end of the section.

Now, for $\varrho \in \mathcal{J}_{\phi,m,\ell,L,Q}$, $\lambda \in \widetilde{Q}$, and $t \in \partial\mathsf{T}$ let

$$V_n(\varrho,\lambda,t) = \Big|k(n)^{-1}\sum_{j=1}^{k(n)}(Z_{n,j}(\varrho,\lambda,t)-1)\Big|,$$

and notice that by construction $V_n(\varrho, \lambda, t)$ is constant over each cylinder $[u]$ of generation $nk(n)$, so that we also denote it by $V_n(\varrho, \lambda, u)$. Now we can write

$$\mu_\varrho(\{t \in \partial\mathsf{T} :\ V_n(\varrho,\lambda,t) > \epsilon\}) = \sum_{u\in\mathsf{T}_{nk(n)}} \mathbf{1}_{\{V_n(\varrho,\lambda,u)>\epsilon\}}\mu_\varrho([u]).$$

Recall that by definition we have $\mu_\varrho([u])$ as $(\prod_{k=1}^{n} W_{\varrho,u_1\cdots u_k})Y(\varrho, u)$, with $\mathbb{E}(Y(\varrho, u)) = 1$, and due to Remark 4.1, since $\mathcal{J}_{\phi,m,\ell,L,Q} \subset \mathcal{J}(K_m)$, we have $\|\sup_{\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}} Y(\varrho, u)\|_1 \le O((j_{|u|} + 2)!)$.

Consequently, using the independence between $(\prod_{k=1}^{n} W_{\varrho,u_1\cdots u_k})_{\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}}$ and $Y(\cdot, u)$ for all $u \in \mathsf{T}_{nk(n)}$, we get

$$\begin{aligned}
&\mathbb{E}\Big(\sup\Big\{\mu_\varrho(\{t\in\partial\mathsf{T}:\ V_n(\varrho,\lambda,t)>\epsilon\}) : \varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}\Big\}\Big)\\
\le\;& \mathbb{E}\Big(\sum_{u\in\mathsf{T}_{nk(n)}}\sup_{\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}}\{\mathbf{1}_{\{V_n(\varrho,\lambda,u)>\epsilon\}}\prod_{k=1}^{nk(n)}W_{\varrho,u_1\cdots u_k}\Big)\ \|\sup_{\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}}Y(\varrho,u)\|_1\\
\le\;& \sum_{\varrho_{|nk(n)}:\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}}\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{nk(n)}}\{\mathbf{1}_{\{V_n(\varrho,\lambda,u)>\epsilon\}}\prod_{k=1}^{nk(n)}W_{\varrho,u_1\cdots u_k}\Big)\ \|\sup_{\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}}Y(\varrho,u)\|_1\\
=\;& \sum_{\varrho_{|nk(n)}:\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}}\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{nk(n)}}\{\mathbf{1}_{\{V_n(\varrho,\lambda,u)>\epsilon\}}\Big(\prod_{k=1}^{nk(n)}W_{\varrho,u_1\cdots u_k}\Big)Y(\varrho,u)\Big)\ \|\sup_{\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}}Y(\varrho,u)\|_1\\
=\;& \sum_{\varrho_{|nk(n)}:\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}}\mathcal{Q}_\varrho(V_n(\varrho,\lambda,t)>\epsilon)\ \|\sup_{\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}}Y(\varrho,u)\|_1\\
\le\;& (\#\varrho_{|nk(n)} : \varrho\in\mathcal{J}_\phi)\Big(\sup_{\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}}\mathcal{Q}_\varrho(V_n(\varrho,\lambda,t)>\epsilon)\Big)\ \|\sup_{\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}}Y(\varrho,u)\|_1
\end{aligned}$$

$$\leq \quad (j_{nk(n)}!) C(m,\ell,L,Q)\exp(-n(p-1)\ell/4) O((j_{nk(n)}+2)!),$$

where we have used Lemma 5.1(2). Now recall that due to (5.3) we have $j_{nk(n)} \leq j_n+1$, hence

$$\mathbb{E}\Big(\sup\Big\{\mu_\varrho(\{t\in\partial\mathsf{T}:\ V_n(\varrho,\lambda,t)>\epsilon\}) : \varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}\Big\}\Big) \\ \leq O(1)C(m,\ell,L,Q)((j_n+3)!)^2\exp(-n(p-1)\ell/4).$$

It follows that

$$\begin{aligned} &\sum_{n\geq 1}\mathbb{E}\Big(\sup\Big\{\mu_\varrho(\{t\in\partial\mathsf{T}:\ V_n(\varrho,\lambda,t)>\epsilon\}) : \varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}\Big\}\Big)\\ =\ & O\Big(\sum_{j\geq 0}\sum_{M_j+1\leq n\leq M_{j+1}}((j+3)!)^2\exp(-n(p-1)\ell/4)\Big)\\ =\ & O\Big(\sum_{j\geq 0}\frac{((j+3)!)^2}{1-\exp(-(p-1)\ell/4)}\exp(-M_j(p-1)\ell/4)\Big)\\ =\ & O\Big(\sum_{j\geq 0}((j+3)!)^2\exp(-N_j(p-1)\ell/4)\Big).\end{aligned}$$

Due to (5.3) the above series converges.

(2) This situation is not empty only if $\kappa'<\infty$. Fix $L>0$. Given $\varrho\in\mathcal{J}'_{\phi,m,\ell,L,Q}$, $\lambda\in Q$, $\epsilon>0$, $n\geq 1$ and $1\leq j\leq k(n)$, we set $V_j(\varrho,\lambda,t)=\mathbf{1}_{\{\Delta S_nX(j,t)-n\nabla\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda)\in B(0,n\epsilon)\}}$. Mimicking what was done above, we can get

$$\begin{aligned} &\mathbb{E}\Big(\sup_{\varrho\in\mathcal{J}'_{\phi,m,\ell,L,Q}}\mu_\varrho\Big(\Big\{t:\sum_{j=1}^{k(n)}V_j(\varrho,\lambda,t)\geq 1\Big\}\Big)\Big)\\ \leq\ & (\#\varrho_{|nk(n)}:\varrho\in\mathcal{J}_\phi)\Big(\sup_{\varrho\in\mathcal{J}'_{\phi,m,\ell,L,Q}}\mathcal{Q}_\varrho\Big(\Big\{\sum_{j=1}^{k(n)}V_j(\varrho,\lambda,t)\geq 1\Big\}\Big)\Big)\ \|\sup_{\varrho\in\mathcal{J}'_{\phi,m,\ell,L,Q}}Y(\varrho,u)\|_1\\ =\ & O((j_n+3)!^2)\sup_{\varrho\in\mathcal{J}'_{\phi,m,\ell,L,Q}}\mathcal{Q}_\varrho\Big(\Big\{\sum_{j=1}^{k(n)}V_j(\varrho,\lambda,t)\geq 1\Big\}\Big).\end{aligned}$$

We have

$$\begin{aligned} &\mathcal{Q}_\varrho\Big(\Big\{\sum_{j=1}^{k(n)}V_j(\varrho,\lambda,t)\geq 1\Big\}\Big)\\ \leq\ & \sum_{j=1}^{k(n)}\mathcal{Q}_\varrho(\{V_j(\varrho,\lambda,t)\geq 1\})\leq\sum_{j=1}^{k(n)}\mathcal{Q}_\varrho(\langle\lambda|\Delta S_nX(j,t)-n\nabla\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda)\rangle\geq -n\epsilon\|\lambda\|)\\ \leq\ & \sum_{j=1}^{k(n)}\exp\big(n\epsilon\|\lambda\|-n\langle\lambda|\nabla\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda)\rangle\big)\mathbb{E}_{\mathcal{Q}_\varrho}(\exp(\langle\lambda|\Delta S_nX(j,t)\rangle))\\ =\ & \sum_{j=1}^{k(n)}\exp\big(n\epsilon\|\lambda\|-n\langle\lambda|\nabla\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda)\rangle\big)\exp\Big(\sum_{k=(j-1)n+1}^{jn}\Lambda_{q_k,\alpha_k}(\lambda)\Big)\end{aligned}$$

$$\begin{aligned}&\leq A\sum_{j=1}^{k(n)}\exp\big(n\epsilon\|\lambda\|-n\langle\lambda|\nabla\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda)\rangle\big)\exp\Big(n/2\ell+n\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda)\Big)\\&= A\sum_{j=1}^{k(n)}\exp\big(n\epsilon\|\lambda\|+n/2\ell+n\Lambda^*_{q_{\alpha_\varrho},\alpha_\varrho}(\nabla\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda))\big)\leq A\,k(n)\exp\big(n\epsilon\|\lambda\|-n/2\ell-n\kappa'\big),\end{aligned}$$

where we have used the definition of $\mathcal{J}'_{\phi,m,\ell,L,Q}$ and

$$\log(A)=\sup_{\varrho\in\mathcal{J}'_{\phi,m,\ell,L,Q}}\sup_{\lambda\in Q}\sum_{k=1}^{L}|\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda)-\Lambda_{q_k,\alpha_k}(\lambda)|.$$

Thus, taking $0<\epsilon=\epsilon_\lambda$ small enough so that $\epsilon\|\lambda\|\leq\ell/8$, since $\log(k(n))<n(\kappa'+\ell/8)$ for n large enough, we get that

$$\sup_{\varrho\in\mathcal{J}'_{\phi,m,\ell,L,Q}}\mathcal{Q}_\varrho\Big(\Big\{\sum_{j=1}^{k(n)}V_j(\varrho,\lambda,t)\geq 1\Big\}\Big)=O(\exp(-n/4\ell)).$$

Mimicking the end of the proof of part (1) of this proposition, we can get that given $\lambda\in Q$, with $\epsilon=\epsilon_\lambda$

$$\mathbb{E}\Big(\sum_{n\geq 1}\sup_{\varrho\in\mathcal{J}'_{\phi,m,\ell,L,Q}}\mu_\varrho\Big(\Big\{t:\sum_{j=1}^{k(n)}V_j(\varrho,\lambda,t)\geq 1\Big\}\Big)\Big)<\infty,$$

hence, with probability 1, by the Borel Cantelli-Lemma applied to each μ_ϱ, we have that for all $\varrho\in\mathcal{J}'_{\phi,m,\ell,L,Q}$, for μ_ϱ-almost every t, for n large enough, $\{1\leq j\leq k(n): n^{-1}\Delta S_n(j,t)\in B(\nabla\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda),\epsilon_\lambda)\}=\emptyset$.

Now, for each $\lambda\in Q$, there exists $\eta_\lambda>0$ such that for all $\lambda'\in B(\lambda,\eta_\lambda)$, for all $\varrho\in\mathcal{J}'_{\phi,m,\ell,L,Q}$ we have $B(\nabla\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda'),\epsilon_\lambda/2)\subset B(\nabla\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda),\epsilon_\lambda)$. We can extract from $(B(\lambda,\eta_\lambda))_{\lambda\in Q}$ a finite subfamily $(B(\lambda_i,\eta_{\lambda_i}))_{1\leq i\leq r}$ which covers Q. Since this family is finite, with probability 1, for all $\varrho\in\mathcal{J}'_{\phi,m,\ell,L,Q}$, for μ_ϱ-almost every t, for n large enough, for all $1\leq i\leq r$, $\{1\leq j\leq k(n): n^{-1}\Delta S_n(j,t)\in B(\nabla\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda_i),\epsilon_{\lambda_i})\}=\emptyset$; consequently, by construction of $(B(\lambda_i,\eta_{\lambda_i}))_{1\leq i\leq r}$, for all $\lambda'\in Q$, $\{1\leq j\leq k(n): n^{-1}\Delta S_n(j,t)\in B(\nabla\Lambda_{q_{\alpha_\varrho},\alpha_\varrho}(\lambda'),\inf_{1\leq i\leq r}\epsilon_{\lambda_i}/2)\}=\emptyset$. This finishes the proof of the proposition. □

Proof of Lemma 5.1. (1) Let $n\geq 1$ and $(f_1,\dots,f_{k(n)})$ be $k(n)$ Borel functions defined on $\mathbb{R}$. Using the fact that $Z_{n,j}(\varrho,\lambda,t)$ is $\sigma(N(u),X_{ui}:u\in\bigcup_{k=(j-1)n}^{jn-1}\mathbb{N}_+^k,i\in\mathbb{N}_+)\otimes\mathcal{C}$-measurable, using the definition of μ_ϱ as well as the independence between generations and the branching property yields

$$\begin{aligned}&\mathbb{E}_{\mathcal{Q}_\varrho}\Big(\prod_{j=1}^{k(n)}f_j(Z_{n,j}(\varrho,\lambda,t))\Big)\\&=\mathbb{E}\Big(\prod_{j=1}^{k(n)}f_j(Z_{n,j}(\varrho,\lambda,u))\,\mathrm{d}\mu_\varrho(t)\Big)\\&=\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{nk(n)}}Y(\varrho,u))\prod_{j=1}^{k(n)}\Big(f_j(Z_{n,j}(\varrho,\lambda,u))\prod_{k=(j-1)n+1}^{jn}W_{\varrho,u_1\cdots u_k}\Big)\Big)\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\Big(\sum_{u\in\mathsf{T}_{nk(n)}}\prod_{j=1}^{k(n)}\Big(f_j(Z_{n,j}(\varrho,\lambda,u))\prod_{k=(j-1)n+1}^{jn} W_{\varrho,u_1\cdots u_k}\Big)\Big)\\
&= \mathbb{E}\Big(\sum_{u\in\mathsf{T}_{nk(n)}}\prod_{j=1}^{k(n)}\Big(f_j(Z_{n,j}(\varrho,\lambda,u))\prod_{k=(j-1)n+1}^{jn} \exp(\langle q_k|X_{u_1\cdots u_k}-\alpha_k\rangle-\widetilde{P}_{\alpha_k}(q_k)\phi_{u_1\cdots u_k})\Big)\Big).
\end{aligned}
$$

Set

$$
U_{n,j}(u)=f_j(Z_{n,j}(\varrho,\lambda,u))\prod_{k=(j-1)n+1}^{jn}\exp(\langle q_k|X_{u_1\cdots u_k}-\alpha_k\rangle-\widetilde{P}_{\alpha_k}(q_k)\phi_{u_1\cdots u_k})
$$

and notice that this random variable is measurable with respect to $\mathcal{G}_{n,j}=\sigma\big((N_u,(X_{u1},\phi_{u1}),\dots):u\in\bigcup_{k=(j-1)n}^{jn-1}\mathbb{N}_+^k\big)$. Now, the above equality can be rewritten

$$
\mathbb{E}_{\mathcal{Q}_\varrho}\Big(\prod_{j=1}^{k(n)} f_j(Z_{n,j}(\varrho,\lambda,t))\Big)=\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{n(k(n)-1)}}\sum_{v\in T_n(u)}\prod_{j=1}^{k(n)}U_{n,j}(uv)\Big).
$$

Conditioning on $\mathcal{G}_{n,k(n)}$ and using the independences and identity in distribution between the random variables of the construction we get

$$
\mathbb{E}_{\mathcal{Q}_\varrho}\Big(\prod_{j=1}^{k(n)} f_j(Z_{n,j}(\varrho,\lambda,t))\Big)=\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{n(k(n)-1)}}\prod_{j=1}^{k(n)-1}U_{n,j}(uv)\Big)\widetilde{U}_{n,k(n)},
$$

where

$$
\begin{aligned}
\widetilde{U}_{n,j}=\mathbb{E}\Big(\sum_{u\in\mathsf{T}_n} & f_j\big(\exp(\langle\lambda|S_nX(u)\rangle-s_{n,j}(\varrho,\lambda)\big)\\
&\cdot\prod_{k=1}^{n}\exp(\langle q_{(j-1)n+k}|X_{u_1\cdots u_k}-\alpha_{(j-1)n+k}\rangle-\widetilde{P}_{\alpha_{(j-1)n+k}}(q_{k+(j-1)n})\phi_{u_1\cdots u_k})\Big)
\end{aligned}
$$

for $1\le j\le k(n)$. Iterating this yields

$$
\mathbb{E}_{\mathcal{Q}_\varrho}\Big(\prod_{j=1}^{k(n)} f_j(Z_{n,j}(\varrho,\lambda,t))\Big)=\prod_{j=1}^{k(n)}\widetilde{U}_{n,j}.
$$

Applying this with $f_{j'}=1$ for $j'\neq j$ yields

$$
\mathbb{E}_{\mathcal{Q}_\varrho}\Big(\prod_{j=1}^{k(n)} f_j(Z_{n,j}(\varrho,\lambda,t))\Big)=\prod_{j=1}^{k(n)}\mathbb{E}_{\mathcal{Q}_\varrho}\big(f_j(Z_{n,j}(\varrho,\lambda,t))\big),
$$

hence the desired independence. Then taking $f_j(z)=z$ and $f_{j'}(z)=1$ for $j'\neq j$ yields, writing k' for $(j-1)n+k$

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{Q}_\varrho}\big(Z_{n,j}(\varrho,\lambda,t)\big)\\
&= \mathbb{E}\Big(\sum_{u\in\mathsf{T}_n}\prod_{k=1}^{n}\exp\big(\langle\lambda|X_{u_1\cdots u_k}\rangle-\Lambda_{q_{k'},\alpha_{k'}}(\lambda)+\langle q_{k'}|X_{u_1\cdots u_k}-\alpha_{k'}\rangle-\widetilde{P}_{\alpha_{k'}}(q_{k'})\phi_{u_1\cdots u_k}\big)\Big)\\
&= \prod_{k=1}^{n}\mathbb{E}\Big(\sum_{i=1}^{N}\exp\big((\langle\lambda|X_i\rangle+\langle q_{k'}|X_i-\alpha_{k'}\rangle-\widetilde{P}_{\alpha_{k'}}(q_{k'})\phi_i-\Lambda_{q_{k'},\alpha_{k'}}(\lambda)\big)\Big)
\end{aligned}
$$

$$= \; 1$$

by definition of $\Lambda_{q_{k'},\alpha_{k'}}(\lambda)$. Finally, the random variables $Z_{n,j}(\varrho,\lambda,t)-1$ are $\mathcal{Q}_\varrho$-independent and centered.

(2) Thanks to (1), we can apply Lemma 3.1 and get

$$\mathcal{Q}_\varrho\Big(\Big|k(n)^{-1}\sum_{j=1}^{k(n)}(Z_{n,j}(\varrho,\lambda,t)-1)\Big|>\epsilon\Big)\le(\epsilon k(n))^{-p}\mathbb{E}_{\mathcal{Q}_\varrho}\Big(\Big|\sum_{j=1}^{k(n)}(Z_{n,j}(\varrho,\lambda,t)-1)\Big|^p\Big)$$
$$\le\; 2^{p-1}(\epsilon k(n))^{-p}\sum_{j=1}^{k(n)}\mathbb{E}_{\mathcal{Q}_\varrho}(|Z_{n,j}(\varrho,\lambda,t)-1|^p)\le 2^{2p-1}(\epsilon k(n))^{-p}\sum_{j=1}^{k(n)}\mathbb{E}_{\mathcal{Q}_\varrho}(Z_{n,j}(\varrho,\lambda,t)^p)$$

since $\mathbb{E}_{\mathcal{Q}_\varrho}(Z_{n,j}(\varrho,\lambda,t))=1$. Moreover, calculations similar to those used to establish part (1) of this lemma yield

$$\mathbb{E}_{\mathcal{Q}_\varrho}(Z_{n,j}(\varrho,\lambda,t)^p)=\exp\Big(\sum_{k=(j-1)n+1}^{jn}\Lambda_{q_k,\alpha_k}(p\lambda)-p\Lambda_{q_k,\alpha_k}(\lambda)\Big).$$

Since $\mathcal{J}_{\phi,m,\ell,L,Q}\subset\mathcal{J}(K_m)$, with K_m a compact subset of J_ϕ, using Taylor's expansion we have $\Lambda_{q_k,\alpha_k}(p\lambda)-p\Lambda_{q_k,\alpha_k}(\lambda)=(1-p)\Lambda^*_{q_k,\alpha_k}(\nabla\Lambda_{q_k,\alpha_k}\lambda)+O((p-1)^2)$ uniformly in $\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}$, $\lambda\in\widetilde{Q}$ and $p-1$ small enough. Consequently, using the definition of $\mathcal{J}_{\phi,m,\ell,L,Q}$ for $k\ge L$, we have

$$\Lambda_{q_k,\alpha_k}(p\lambda)-p\Lambda_{q_k,\alpha_k}(\lambda)\le(p-1)\min(2\ell,\kappa-1/2\ell)+O((p-1)^2),$$

hence for all $1\le j\le k(n)$

$$\sum_{k=(j-1)n+1}^{jn}\Lambda_{q_k,\alpha_k}(p\lambda)-p\Lambda_{q_k,\alpha_k}(\lambda)\le A+n\big((p-1)\min(2\ell,\kappa-1/2\ell)+O((p-1)^2)\big)$$

uniformly in $\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}$, $\lambda\in\widetilde{Q}$ and $p-1$ small enough, where

$$A=\sup_{p\in[1,2]}\;\sup_{\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}}\;\sup_{\lambda\in\widetilde{Q}}\sum_{k=1}^{L}|\Lambda_{q_k,\alpha_k}(p\lambda)-p\Lambda_{q_k,\alpha_k}(\lambda)|.$$

The previous estimates yield

$$\mathcal{Q}_\varrho\Big(\Big|k(n)^{-1}\sum_{j=1}^{k(n)}(Z_{n,j}(\varrho,\lambda,t)-1)\Big|>\epsilon\Big)\le e^A\epsilon^{-p}k(n))^{1-p}\exp\big(n\big((p-1)\min(2\ell,\kappa-1/2\ell)+O((p-1)^2)\big)\big)$$

in the same uniform manner as above. Take p close enough to 1 so that $O((p-1)^2)\le(p-1)/8\ell$.

Now, for n large enough, we have $k(n)\ge\exp(n(\min(2\ell,\kappa-1/8\ell)))$, so that

$$\mathcal{Q}_\varrho\Big(\Big|k(n)^{-1}\sum_{j=1}^{k(n)}(Z_{n,j}(\varrho,\lambda,t)-1)\Big|>\epsilon\Big)\le e^A\epsilon^{-p}\exp\big(n\big(1-p)\ell/4\big)$$

uniformly in $\varrho\in\mathcal{J}_{\phi,m,\ell,L,Q}$ and $\lambda\in\widetilde{Q}$. □

5.3 Proof of Theorem 1.8

We need two new propositions. Let $\widetilde{\mathcal{J}_\phi} = \big\{\varrho \in \mathcal{J}_\phi : \overline{\{q_k, \alpha_k\} : k \geq 1\}} \subset J_\phi\big\}$.

Proposition 5.3 *There exists a positive sequence $(\delta_n)_{n\geq 1}$ converging to 0 such that, with probability 1, for all $\varrho \in \widetilde{\mathcal{J}_\phi}$, for μ_ϱ-almost every t, for n large enough, $n^{-1}|\log Y(\varrho, t_{|n})| \leq \delta_n$.*

Proof For each $m \geq 1$, let $\mathcal{J}_{\phi,m}$ be defined as in (5.4). Fix a positive sequence $(\delta_n)_{n\geq 1}$ converging to 0, to be specified later, and $m \geq 1$. For every $\varrho \in \mathcal{J}_{\phi,m}$, $\eta \in \{-1, 1\}$ and $n \geq 1$ let

$$E(\varrho, n, \eta) = \{t \in \partial\mathsf{T} : Y(\varrho, t_{|n})^\eta \geq e^{n\delta_n}\},$$

and notice that for any $\gamma > 0$

$$\mu_\varrho(E(\varrho, n, \eta)) \leq \sum_{u\in\mathsf{T}_n} \mu_\varrho([u])Y(\varrho, u)^{\gamma\eta}e^{-n\gamma\delta_n} = \sum_{v_1\cdots v_n\in\mathsf{T}_n} \Big(\prod_{k=1}^n W_{\varrho,u_1\cdots u_k}\Big)Y(\varrho, u)^{1+\gamma\eta}e^{-n\gamma\delta_n}.$$

According to Remark 4.1, we can choose $\gamma < 1$ small enough so that there exists $C_m > 0$ for which $\mathbb{E}\big(\sup_{\varrho\in\mathcal{J}_{\phi,m}} Y(\varrho, u)^{1+\gamma}\big) \leq C_m(j_n + 2)!)^{1+\gamma}$ for all $n \geq 1$ and $u \in \mathbb{N}_+^n$. Then, using an approach similar to the one used to establish (4.19), we can get

$$\mathbb{E}\big(\sup_{\varrho\in\mathcal{J}_{\phi,m}} \mu_\varrho(E(\varrho, n, \eta)) \leq C_m(j_n + 2)!)^{1+\gamma}e^{-n\gamma\delta_n} \leq C_m e^{n^{2/5}}e^{-n\gamma\delta_n}.$$

Now, taking $\delta_n = \gamma^{-1}\gamma_{j_n+1}^2 m_{j_n+1}$ and using (4.15) we get

$$\mathbb{E}\big(\sup_{\varrho\in\mathcal{J}_{\phi,m}} \mu_\varrho(E(\varrho, n, \eta)) \leq C_m e^{n^{2/5}}e^{-n^{1/2}N_{j_n}^{1/2}\gamma_{j_n+1}^2 m_{j_n+1}} \leq C_m e^{n^{2/5}}e^{-n^{1/2}}.$$

The previous inequality yields

$$\mathbb{E}\Big(\sum_{n\geq 1}\sum_{\eta\in\{-1,1\}}\sup_{\varrho\in\mathcal{J}_{\phi,m}} \mu_\varrho(E(\varrho, n, \eta))\Big) < \infty,$$

hence the desired result when $\varrho \in \mathcal{J}_{\phi,m}$. Since $\widetilde{\mathcal{J}_\phi}$ is the countable union of the sets $\mathcal{J}_{\phi,m}$, the result holds with $\widetilde{\mathcal{J}_\phi}$.

Proposition 5.4 *There exists a positive sequence $(\delta_n)_{n\geq 1}$ converging to 0 such that, with probability 1, for all $\varrho \in \mathcal{J}_\phi$, for μ_ϱ-almost every t, for n large enough,*

$$n^{-1}\Big|\sum_{k=1}^n \langle q_k|X_{t_1\cdots t_k}-\alpha_k\rangle - \widetilde{P}_{\alpha_k}(q_k)\phi_{t_1\cdots t_k} - \sum_{k=1}^n \langle q_k|\alpha_X(q_k,\alpha_k)-\alpha_k\rangle - \widetilde{P}_{\alpha_k}(q_k)\lambda(q_k,\alpha_k)\Big| \leq \delta_n.$$

Proof It is similar to that of Proposition 4.9. Fix $(\delta_n)_{n\geq 1}$ a positive sequence converging to 0.

For $\varrho \in \mathcal{J}_\phi$ $n \geq 1$, and $t \in \partial\mathsf{T}$ we set :

$$s_n(\varrho) = \sum_{k=1}^n \langle q_k|\alpha_X(q_k, \alpha_k) - \alpha_k\rangle - \widetilde{P}_{\alpha_k}(q_k)\lambda(q_k, \alpha_k),$$

$$s_n(\varrho, t_{|n}) = \sum_{k=1}^{n} \langle q_k | X_{t_1\cdots t_k} - \alpha_k\rangle - \widetilde{P}_{\alpha_k}(q_k)\phi_{t_1\cdots t_k}.$$

$$E^1_{\varrho,n,\delta_n} = \Big\{ t\in\partial\mathsf{T} : s_n(\varrho,t_{|n}) - s_n(\varrho) \geq n\delta_n \Big\}$$

$$E^{-1}_{\varrho,n,\delta_n} = \Big\{ t\in\partial\mathsf{T} : s_n(\varrho,t_{|n}) - s_n(\varrho) \leq -n\delta_n \Big\}$$

It is enough to specify $(\delta_n)_{n\geq 1}$ such that for $\lambda\in\{-1,1\}$ we have

$$\mathbb{E}\Big(\sup_{\varrho\in\mathcal{J}_\phi} \sum_{n\geq 1} \mu_\varrho(E^\lambda_{\varrho,n,\delta_n})\Big) < \infty. \tag{5.7}$$

We prove (5.7) when $\lambda = 1$ (the case $\lambda = -1$ is similar). Let $\varrho\in\mathcal{J}_\phi$. For every $\gamma > 0$ we have

$$\mu_\varrho(E^1_{\varrho,n,\delta_n}(v)) \leq f_{n,\gamma}(\varrho),$$

where

$$\begin{aligned} f_{n,\gamma}(\varrho) &= \sum_{u\in\mathsf{T}_n} \mu_\varrho([u] \exp\big(\gamma(s_n(\varrho,u) - s_n(\varrho)) - \gamma n\delta_n\big) \\ &= \sum_{u\in\mathsf{T}_n} \Pi_{n,\gamma}(\varrho,u) Y(\varrho,u), \end{aligned}$$

where we used the definition of μ_ϱ and

$$\begin{aligned} \Pi_{n,\gamma}(\varrho,u) &= \prod_{k=1}^{n} \exp\big((1+\gamma)\langle q_k|X_{u_{|k}} - \alpha_k\rangle - (1+\gamma)\widetilde{P}_{\alpha_k}(q_k)\phi_{u_{|k}}\big) \\ &\qquad \cdot \exp\big(-(\gamma\langle q_k|\alpha_X(q_k,\alpha_k) - \alpha_k\rangle - \gamma\widetilde{P}_{\alpha_k}(q_k)\lambda(q_k,\alpha_k)) - \gamma\delta_n\big). \end{aligned}$$

The same arguments as in the proof of Proposition 4.9 yields

$$\begin{aligned} \mathbb{E}(\sup_{\varrho\in\mathcal{J}_\phi} f_{n,\gamma}(\varrho)) &\leq C_{\mathcal{J}_\phi} \exp(n\epsilon_n)\mathbb{E}\Big(\sum_{u\in\mathsf{T}_n} \sum_{\varrho_{|n}:\varrho\in\mathcal{J}_\phi} \Pi_n(\varrho,u)\Big) \\ &= C_{\mathcal{J}_\phi} \exp(n\epsilon_n) \sum_{\varrho_{|n}:\varrho\in\mathcal{J}_\phi} \prod_{k=1}^{n} e_k(\varrho,\gamma), \end{aligned}$$

where

$$e_k(\varrho,\gamma) = \exp\Big(\psi_{\alpha_k}((1+\gamma)q_k, (1+\gamma)\widetilde{P}_{\alpha_k}(q_k)) - \gamma\frac{\mathrm{d}\psi_{\alpha_k}((1+u)q_k,(1+u)\widetilde{P}_\alpha(q_k))}{\mathrm{d}u}(0) - \gamma\delta_n\Big),$$

since by (4.7) $\langle q_k|\alpha_X(q_k,\alpha_k) - \alpha_k\rangle - \widetilde{P}_{\alpha_k}(q_k)\lambda(q_k,\alpha_k)) = \dfrac{\mathrm{d}\psi_{\alpha_k}((1+u)q_k,(1+u)\widetilde{P}_\alpha(q_k))}{\mathrm{d}u}(0)$. For each $\varrho\in\mathcal{J}_\phi$, we have $q_k\in D_{j_n+1}$ for all $1\leq k\leq n$. Thus, by writing for each $1\leq k\leq n$ the Taylor expansion with integral rest of order 2 of $\gamma\mapsto\psi_{\alpha_k}\big((1+\gamma)q_k,(1+\gamma)\widetilde{P}_{\alpha_k}(q_k)\big)$ at 0 as in the proof of Proposition 4.6, and taking $\gamma = \gamma_{j_n+1}$, we get

$$\sum_{k=1}^{n} \psi_{\alpha_k}\big((1+\gamma)q_k,(1+\gamma)\widetilde{P}_{\alpha_k}(q_k)\big) - \gamma_{j_n+1}\delta_n \leq n\gamma^2_{j_n+1}\widetilde{m}_{j_n+1} - n\gamma_{j_n+1}\delta_n$$

uniformly in $\varrho \in \mathcal{J}_\phi$. Consequently, using that $\epsilon_n = 2\gamma_{j_n+1}^2 \widehat{m}_{j_n+1}$, $\widetilde{m}_j \le \widehat{m}_{j_n+1}$, and $\mathrm{card}(\{\varrho_{|n} : \varrho \in \mathcal{J}\}) = (j_n+1)!$, we get

$$\mathbb{E}(\sup_{\varrho\in\mathcal{J}} f_{n,\gamma_{j_n+1}}(\varrho)) \le C_{\mathcal{J}}(j_n+1)! \exp\big((-n\gamma_{j_n+1}(\delta_n - 3\gamma_{j_n+1}^2\widehat{m}_{j_n+1})\big).$$

Let $\delta_n = 4\gamma_{j_n+1}^2\widehat{m}_{j_n+1}$. Now we use (4.15) : $(j_n+1)! \le \exp(N_{j_n}^{1/5}) \le \exp(n^{1/5})$ and $\gamma_{j_{n+1}}^2\widehat{m}_{j_n+1} \ge N_{j_n}^{-1/2} \ge n^{-1/2}$. Thus

$$\mathbb{E}\big(\sup_{\varrho\in\mathcal{J}_\phi} \mu_\varrho(E^1_{\varrho,n,\delta_n}(v))\big) \le \mathbb{E}(\sup_{\varrho\in\mathcal{J}_\phi} f_{n,\gamma_{j_n+1}}(\varrho)) \le C_{\mathcal{J}_\phi}\exp(n^{1/5})\exp(-n^{1/2}).$$

Since $\delta_n = 4\gamma_{j_n+1}^2\widehat{m}_{j_n+1}$ tends to 0 as n tends to ∞, we get $\sum_{n\ge1}\mathbb{E}\big(\sup_{\varrho\in\mathcal{J}_\phi}\mu_\varrho(E^1_{\varrho,n,\delta_n})\big) < \infty$, as desired.

Now we come to the proof of Theorem 1.8. We fix a set Ω' of probability 1 over which the conclusions of Theorem 1.7 hold, as well as those of Propositions 4.7, 5.3 and 5.4 with the same sequence $(\delta_n)_{n\ge1}$.

Let $\alpha \in \mathring{I}$ such that $\widetilde{P}^*_\alpha(0) < \dim\partial\mathsf{T}$, as well as g, a gauge function satisfying the property $\limsup_{r\to0^+}\log(g(r))/\log(r) \le \widetilde{P}^*_\alpha(0)$ (the fact that $\mathcal{H}^g(E_X^{\mathrm{LD}(q_\alpha,\alpha)}(\alpha)) = 0$ if this does not hold is obvious). The conclusion will come if for all $\omega\in\Omega'$ there exists $\varrho\in\widetilde{\mathcal{J}}_\phi$ such that $\mu_\varrho(E_X^{\mathrm{LD}(q_\alpha,\alpha)}(\alpha)) > 0$ and there is a positive sequence $(\delta'_n)_{n\ge1}$ such that $\lim_{n\to\infty} n\delta'_n = +\infty$ and for μ_ϱ-almost every t, for n large enough, we have $g(\mathrm{diam}([t_{|n}]) \ge C\mu_\varrho([t_{|n}])^{1-\delta'_n}$.

We can find a non decreasing function θ such that $g(r) \ge r^{\widetilde{P}^*_\alpha(0)+\theta(r)}$, with $\lim_{r\to0^+}\theta(r) = 0$. Let $\theta_n = \theta(\beta_2^n)$, where β_2 is chosen as in Lemma 4.1.

Let $(\eta_k)_{k\ge1}$ be decreasing sequence converging to 0, and such that $\lim_{n\to\infty} n\eta_n = \infty$ and $n\delta_n = o\Big(\sum_{k=1}^n \eta_k\Big)$ as $n\to\infty$. Now fix $\varrho\in\widetilde{J}_\phi$ such that $q_k = q_{\alpha_k}$ for all $k\ge1$, α_k converges to α, so that $\widetilde{P}_{\alpha_k}(q_k) = \widetilde{P}^*_{\alpha_k}(0)$ converges to $\widetilde{P}^*_\alpha(0)$, but enough slowly so that for k large enough we have $\widetilde{P}_{\alpha_k}(q_k) \ge (1+\eta_k)^2(\widetilde{P}^*_\alpha(0)+\theta_k)$. We recall that due to the relation between α_k and q_k we have $\alpha_X(q_k,\alpha_k) = \alpha_k$. Proposition 5.4 applied with μ_ϱ then takes a particularly simple form, and writing $\mu_\varrho([t_n]) = Y(\varrho, t_1\cdots t_n)\prod_{k=1}^n \exp(\langle q_{\alpha_k}|X_{t_{|k}} - \alpha_k\rangle - \widetilde{P}_{\alpha_k}(q_{\alpha_k})\phi_{t_{|k}})$ and using simultaneously Propositions 5.3 and 5.4 we get, for μ_ϱ- almost every t, for n large enough,

$$\begin{aligned}\mu_\varrho([t_n]) &\le \exp\Big(2n\delta_n - \sum_{k=1}^n (1+\eta_k)^2(\widetilde{P}^*_\alpha(0)+\theta_k)\lambda(q_{\alpha_k},\alpha_k)\Big)\\ &\le \exp\Big(2n\delta_n - (1+\eta_n)(\widetilde{P}^*_\alpha(0)+\theta_n)\sum_{k=1}^n(1+\eta_k)\lambda(q_{\alpha_k},\alpha_k)\Big).\end{aligned}$$

Now, we notice that since each Lyapounov exponent $\lambda(q_{\alpha_k},\alpha_k)$ belongs to the compact interval I_ϕ, it is larger than or equal to $\log(1/\beta_2)$. Consequently, since $n\delta_n = o\Big(\sum_{k=1}^n\eta_k\Big)$, for n large enough we have $2n\delta_n - (1+\eta_n)(\widetilde{P}^*_\alpha(0)+\theta_n)\sum_{k=1}^n\eta_k\lambda(q_{\alpha_k},\alpha_k) \le -(1+\eta_n)(\widetilde{P}^*_\alpha(0)+\theta_n)n\delta_n$, so that

$$\mu_\varrho([t_n]) \le \exp\Big(-(1+\eta_n)(\widetilde{P}^*_\alpha(0)+\theta_n)\Big(n\delta_n + \sum_{k=1}^n\lambda(q_{\alpha_k},\alpha_k)\Big)\Big).$$

On the other hand, Proposition 4.7 yields, for n large enough,

$$\operatorname{diam}([t_{|n}]) = \exp(-S_n\phi(t)) \geq \exp\Big(- n\delta_n - \sum_{k=1}^{n} \lambda(q_{\alpha_k}, \alpha_k)\Big)\Big).$$

Thus

$$\begin{aligned}\mu_\varrho([t_n]) &\leq (\operatorname{diam}([t_{|n}]))^{(1+\eta_n)(\widetilde{P}_\alpha^*(0)+\theta_n)} \\ &\leq (\operatorname{diam}([t_{|n}]))^{(1+\eta_n)(\widetilde{P}_\alpha^*(0)+\theta(\operatorname{diam}([t_{|n}]))} \leq g(\operatorname{diam}([t_{|n}]))^{(1+\eta_n)}.\end{aligned}$$

Finally, the sequence $\delta'_n = 1 - 1/(1+\eta_n)$ is as desired.

Chapitre 6

Homogeneous and inhomogeneous Mandelbrot measures ; some basic properties

6.1 Mandelbrot measures

Let $(N,(X_1,X_1',\phi_1),(X_2,X_2',\phi_2),\ldots)$ be a random vector taking values in $\mathbb{N}_+\times(\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}_+^*)^{\mathbb{N}_+}$. Suppose that N satisfies the assumptions of the previous sections and, moreover,

$$\begin{cases}\mathbb{E}\Big(\sum_{i=1}^N \exp(X_i')\Big)=1,\\ \mathbb{E}\Big(\sum_{i=1}^N X_i'\exp(X_i')\Big)<0,\\ \quad\mathbb{E}\Big(\Big(\sum_{i=1}^N \exp(X_i')\Big)\log^+\Big(\sum_{i=1}^N \exp(X_i')\Big)\Big)<\infty\end{cases}, \tag{6.1}$$

and

$$\mathbb{E}\Big(\sum_{i=1}^N \|X_i\|\exp(X_i')\Big)<\infty. \tag{6.2}$$

Let $\{(N_{u0},(X_{u1},X_{u1}',\phi_{u1}),(X_{u2},X_{u2}',\phi_{u2}),\ldots)\}_u$ be a family of independent copies of $(N,(X_1,X_1',\phi_1),(X_2,X_2',\phi_2),\ldots)$ indexed by the finite sequences $u=u_1\cdots u_n$, $n\geq 0$, $u_i\in\mathbb{N}_+$ and defined on a probability space $(\Omega',\mathcal{A}',\mathbb{P}')$

Condition (6.1) implies that with probability 1, for all $n\geq 1$ and $u\in\mathbb{N}_+^n$, the sequence

$$Y_p'(u)=\sum_{v\in\mathsf{T}_p(u)}\exp(S_{n+p}X'(uv)-S_nX'(u))\quad(p\geq 1)$$

converges to a positive limit $Y'(u)$, while the limit exists and vanishes if the condition is violated. This fact was proven by Kahane in [45] when N is constant and Biggins in [13] in general. Then with the family $\{(N_{u0},(X_{u1},X_{u1}'),(X_{u2},X_{u2}'),\ldots)\}_u$, we can associate the Mandelbrot measure defined on the σ-field $\mathcal{C}$ generated by the cylinders of $\mathbb{N}_+^{\mathbb{N}_+}$ as

$$\mu'([u])=\begin{cases}\exp(S_nX'(u))Y'(u) & \text{if } u\in\mathsf{T}_n\\ 0 & \text{otherwise}\end{cases},$$

and supported on $\partial\mathsf{T}$. The *branching measure* corresponds to the choice $X'_i = -\log\mathbb{E}(N)$ for $1 \le i \le N$.

Using 6.2 to apply the strong law of large numbers to $S_nX(t)$ with respect to the so-called Peyrière's measure $\mathcal{Q}'$ defined on $\mathcal{A}'\otimes\mathcal{C}$ as $\mathcal{Q}'(E) = \mathbb{E}_{\mathbb{P}'}\Big(\int \mathbf{1}_E(\omega',t)\,\mu'(\mathrm{d}t)\Big)$, we get the well known following fact (see [45, 55, 3] for instance) :

Proposition 6.1 *With probability 1, for μ'-almost every t,*

$$\lim_{n\to\infty}\frac{S_nX(t)}{n} = \mathbb{E}\Big(\sum_{i=1}^N X_i\exp(X'_i)\Big) \quad \text{and} \quad \lim_{n\to\infty}\frac{S_n\phi(t)}{n} = \mathbb{E}\Big(\sum_{i=1}^N \phi_i\exp(X'_i)\Big)\in\mathbb{R}_+\cup\{+\infty\}.$$

Moreover, still following either of [45, 55, 3], since $E(Y') < \infty$ we have :

Proposition 6.2 *With probability 1, for μ'-almost every t,*

$$\limsup_{n\to\infty}\frac{\log Y'(t_{|n})}{-n} \le 0.$$

Corollary 6.1 *With probability 1, for μ'-almost every t,*

$$\limsup_{n\to\infty}\frac{\log\mu'([t_{|n}])}{-n} \le -\mathbb{E}\Big(\sum_{i=1}^N X'_i\exp(X'_i)\Big).$$

The next result holds thanks to a similar argument to that yielding Corollary 6.1 under the property $E(Y'\log^+ Y') < \infty$, hence in particular when $E({Y'}^h) < \infty$ for some $h > 1$ (see [45, 55, 3]). However this property necessitates to add assumptions on $(N, X'_1, X'_2, \dots)$, while the approach developed by Kahane in [46] for Mandelbrot canonical cascades on homogeneous trees, and extended in this paper to simultaneously control the lower Hausdorff dimensions of uncountably many inhomogeneous Mandelbrot measures, requires no additional information :

Proposition 6.3 *With probability 1, for μ'-almost every t,*

$$\liminf_{n\to\infty}\frac{\log\mu'([t_{|n}])}{-n} \ge -\mathbb{E}\Big(\sum_{i=1}^N X'_i\exp(X'_i)\Big).$$

We deduce the following theorem.

Theorem 6.1 *With probability 1, for μ'-almost every t,*

$$\lim_{n\to\infty}\frac{\log\mu'([t_{|n}])}{-S_n\phi(t)} = -\frac{\mathbb{E}\Big(\sum_{i=1}^N X'_i\exp(X'_i)\Big)}{\mathbb{E}\Big(\sum_{i=1}^N \phi_i\exp(X'_i)\Big)}.$$

Consequently, if d_ϕ defines a metric on $\partial\mathsf{T}$ (see the introduction), then μ' is exact dimensional with dimension $D = -\mathbb{E}\Big(\sum_{i=1}^N X'_i\exp(X'_i)\Big)/\mathbb{E}\Big(\sum_{i=1}^N \phi_i\exp(X'_i)\Big)$, in the sense that $D = \inf\big\{\dim F:\ F \text{ Borel},\ \mu(F)>0\big\} = \inf\big\{\dim_P F:\ F \text{ Borel},\ \mu(F) = \|\mu\|\big\}$, where $\dim_P$ stands for the packing dimension.

6.2 Inhomogeneous Mandelbrot measures

We consider an integer valued random variable N and a family $(N, X'_{k,1}, X'_{k,2}, \ldots)$, $k \geq 1$, of random vectors taking values in $\mathbb{N}_+ \times \mathbb{R}^{\mathbb{N}_+}$, and such that the normalization $\mathbb{E}\Big(\sum_{i=1}^N \exp(X'_{k,i})\Big) = 1$ holds.

For each $k \geq 1$, let $\{(N_u, X'_{k,u1}, X'_{k,u2}, \ldots)\}_{u\in\mathbb{N}_+^{k-1}}$ be a family of copies of $(N, X'_{k,1}, X'_{k,2}, \ldots)$, so that all the random vectors so obtained as u runs in $\bigcup_{k\geq 0} \mathbb{N}_+^{k-1}$ are independent.

With probability 1, for all $n \geq 1$ and $u \in \mathbb{N}_+^n$, the sequence

$$Y'_n(u) = \sum_{v\in\mathsf{T}_p(u)} \exp(X'_{n+1,uv_1} + X'_{n+2,uv_1v_2} \cdots X'_{n+p,uv_1\cdots v_p}) \quad (p \geq 1)$$

converges to a non negative limit $Y'(u)$. Then with the family $\{(N_u, X'_{k,u1}, X'_{k,u2}, \ldots)\}_{k\geq 1, u\in\mathbb{N}_+^{k-1}}$, we can associate the inhomogeneous Mandelbrot measure defined on the σ-field $\mathcal{C}$ generated by the cylinders of $\mathbb{N}_+^{\mathbb{N}_+}$ as

$$\mu'([u]) = \begin{cases} \exp(X'_{1,u_1} + X'_{2,u_1u_2} + \cdots X'_{n,u_1\cdots u_n}) Y'(u) & \text{if } u \in \mathsf{T}_n \\ 0 & \text{otherwise} \end{cases},$$

and supported by $\partial\mathsf{T}$. It is proved in [3] (and this also follows from computations similar to those achieved in the previous chapters) that if there exists $\gamma > 1$

$$\sup_{k\geq 1} \mathbb{E}\Big(\sum_{i=1}^N \exp(\gamma X'_{k,i})\Big) + \sup_{k\geq 1} \mathbb{E}\Big(\Big(\sum_{i=1}^N \exp(X'_{k,i})\Big)^\gamma \Big) < \infty,$$

then the limit measure μ' is positive almost surely (in fact [3] assumes weaker assumptions). Suppose, moreover, that

$$\sum_{k\geq 1} k^{-2} \mathbb{E}\Big(\sum_{i=1}^N {X'_{k,i}}^2 \exp(X'_{k,i})\Big) < \infty.$$

Proposition 6.4 *Under the above assumptions, with probability 1, for μ'-almost every t, ve have*

$$\liminf_{n\to\infty} \frac{-1}{n} \sum_{k=1}^n \mathbb{E}\Big(\sum_{i=1}^N X'_{k,i} \exp(X'_{k,i})\Big) = \liminf_{n\to\infty} \frac{\log \mu'([t_{|n}])}{-n}$$
$$\leq \limsup_{n\to\infty} \frac{\log \mu'([t_{|n}])}{-n} = \limsup_{n\to\infty} \frac{-1}{n} \sum_{k=1}^n \mathbb{E}\Big(\sum_{i=1}^N X'_{k,i} \exp(X'_{k,i})\Big).$$

This yields the Hausdorff and packing dimensions of μ' (see [26, 28] for precise definitions). Following the ideas of [3], which itself followed the L^2-martingale approach to the law of large numbers used by Peyrière in [45] to compute the Hausdorff dimension of Mandelbrot measures on homogeneous trees, one can easily prove the following result about the asymptotic behavior of the branching random walk S_nX, if it is built simultanously with μ'.

Proposition 6.5 *Suppose, in addition to the above assumptions, that*

$$\sum_{k\geq 1} k^{-2}\mathbb{E}\Big(\sum_{i=1}^{N}\|X_{k,i}\|^2\exp(X'_{k,i})\Big) < \infty.$$

With probability 1, for μ'-almost every t, we have

$$\lim_{n\to\infty} n^{-1}\Big(S_nX(t) - \sum_{k=1}^{n}\mathbb{E}\Big(\sum_{i=1}^{N}X_i\exp(X'_{k,i})\Big)\Big) = 0.$$

The previous propositions entail inhomogeneous Mandelbrot measures as a useful tool to study level sets of infinite branches over which $S_nX(t)$ possesses a given set of limit points. This thesis provides for all these level sets simultaneously, such a measure of maximal lower Hausdorff dimension.

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