UMD property for Banach spaces and operator spaces

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Résumé

Résumé

Cette thèse présente quelques résultats sur la théorie locale pour les espaces de Banach et d'opérateurs. La première partie consiste en l'étude de la propriété OUMD pour l'espace colonne C. La deuxième partie traite de la propriété UMD classique pour les espaces $L_p(L_q)$ itérés. Le résultat principal donne une construction nouvelle et très naturelle de treillis de Banach qui sont super-réflexifs et non-UMD : L'espace $L_p(L_q(L_p(L_q(\cdots$ itéré une infinité de fois est super-réflexif si $1 < p, q < \infty$ mais n'est pas UMD si $p \neq q$.

Mots-clefs

Transformation martingale, propriété UMD et UMD analytique, propriété OUMD, espaces d'opérateurs, espace de Hilbert en colonne, espaces L^p noncommutatifs à valeurs vectorielles.

Abstract

This thesis presents some results on the local theory of Banach spaces and operator spaces. The first part consists of the study of the OUMD property for the column Hilbert space C. In the second part we treat the classical UMD property for Banach spaces. We give estimates of the UMD constants for iterated $L_p(L_q)$ spaces. The main result yields a new and very natural construction of a family of super-reflexive and non-UMD Banach lattices: The space $L_p(L_q(L_p(L_q(\cdots$ iterated infinitely many times is super-reflexive if $1 < p, q < \infty$ but is not UMD if $p \neq q$.

Keywords

Martingale transformation, UMD and analytic UMD property, OUMD property, operator spaces, column Hilbert space, vector-valued noncommutative L^p spaces.

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Introduction

Cette thèse s'inscrit dans la théorie locale des espaces de Banach et d'opérateurs. Les sujets principaux concernent la propriété UMD (l'expression anglaise est “unconditional martingale difference”) pour les espaces de Banach et la généralisation de cette propriété dans la théorie des espaces d'opérateurs. Dans le cadre des espaces de Banach, cette propriété a été introduite par Maurey et Pisier pendant le fameux Séminaire Maurey-Schwartz, elle consiste à étudier les espaces de Banach tels que toutes les suites de différences de martingales sont inconditionnelles dans les espaces de Bochner correspondants, i.e. dans les espaces L^p à valeurs dans les espaces de Banach considérés. Burkholder et d'autres auteurs ont largement développé la théorie de la propriété UMD, les références classiques sont [Bur83, Bur01]. Dans [Pis98], Pisier a introduit la théorie des espaces L^p noncommutatifs à valeurs vectorielles (en fait, c'est à valeurs dans les espaces d'opérateurs). Dans cette théorie, en remplaçant les martingales classiques par les martingales noncommutatives et à valeurs dans un espace d'opérateurs, la propriété OUMD (pour l'expression “operator space unconditional martingale difference”) a été introduite. Beaucoup de difficultés pour généraliser les résultats classiques pour la propriété UMD à la propriété OUMD sont liées au manque d'outils de temps d'arrêt et on mentionnera un problème ouvert là-dessus.

Dans cette introduction, on commence par donner la définition de la propriété UMD pour les espaces de Banach, la théorie des espaces L^p noncommutatifs à valeurs vectorielles et la définition de la propriété OUMD, puis on va décrire les résultats principaux des différents chapitres de cette thèse.

0.1 Quelques rappels

0.1.1 UMD et AUMD pour les espaces de Banach

Soit $1 \leq p < \infty$. Pour un espace de Banach X et un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$, on peut construire l'espace de Bochner $L_p(\Omega, \mathcal{F}, \mathbb{P}; X)$, on va utiliser $L_p(X)$ pour simplifier la notation. La définition d'une martingale usuelle (ou scalaire) se généralise de façon évidente à la définition d'une martingale à valeurs dans X.

Définition 0.1. Soit $1 < p < \infty$. Un espace de Banach X est dit UMD_p s'il existe une constante c telle que pour toute martingale $(f_n)_{n \geq 0}$ convergente dans $L_p(X)$, on a pour tout choix de signs $\varepsilon_n = \pm 1$

$$\sup_n \Big\| \sum_0^n \varepsilon_k df_k \Big\|_{L_p(X)} \leq c \sup_n \|f_n\|_{L_p(X)} \tag{1}$$

où $df_n = f_n - f_{n-1}$ et par convention $f_{-1} = 0$. La meilleure constante c dans (1) est notée par $\beta_p(X)$, elle est dite la constante UMD_p de X. On dit X est UMD s'il est UMD_p pour un certain $1 < p < \infty$.

Remarque 0.2. En utilisant la décomposition de Gundy, Pisier a montré que la propriété UMD_p et la propriété UMD_q sont équivalentes pour tout $1 < p, q < \infty$. L'indépendance de p pour la propriété UMD_p est maintenant très connue.

Soit $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ le groupe unitaire de dimension 1, la mesure de Haar normalisée de $\mathbb{T}$ sera notée par m. Sur l'espace de probabilité $(\mathbb{T}^{\mathbb{N}}, m^{\otimes \mathbb{N}})$ on peut considérer la filtration canonique suivante :

$$\sigma(z_0) \subset \sigma(z_0, z_1) \subset \cdots \subset \sigma(z_0, z_1, \cdots, z_n) \subset \cdots .$$

Soit X un espace de Banach sur $\mathbb{C}$. Soit $1 \leq p < \infty$, par définition, une martingale de Hardy dans $L_p(\mathbb{T}^{\mathbb{N}}; X)$ est une martingale $(f_n)_{n \geq 0}$ par rapport à la filtration ci-dessus telle que

$$\sup_n \|f_n\|_{L_p(\mathbb{T}^{\mathbb{N}};X)} < \infty$$

et pour tout $n \geq 1$, la différence $df_n = f_n - f_{n-1}$ est analytique en la variable z_n, i.e., df_n est de la forme :

$$df_n(z_0, \cdots, z_{n-1}, z_n) = \sum_{k \geq 1} \phi_{n,k}(z_0, \cdots, z_{n-1}) z_n^k.$$

Par convention, on pose $df_0 := f_0$.

Définition 0.3. Soit $0 < p < \infty$, un espace de Banach X (sur $\mathbb{C}$) est dit AUMD_p (ou UMD_p analytique) s'il existe une constante c telle que

$$\sup_{\varepsilon_k \in \{-1,1\}} \|\sum_{k=0}^{n} \varepsilon_k df_k\|_{L_p(X)} \leq c \|\sum_{k=0}^{n} df_k\|_{L_p(X)},$$

pour tout $n \geq 0$ et tout martingale de Hardy $f = (f_k)_{k \geq 0}$ dans $L_p(\mathbb{T}^{\mathbb{N}}; X)$. La meilleure telle constante c sera notée par $\beta_p^a(X)$.

Remarque 0.4. Il est aussi connu que

$$\mathrm{AUMD}_p \Longleftrightarrow \mathrm{AUMD}_q$$

pour tout $0 < p, q < \infty$. Voir l'appendice au Chapitre 2 pour plus de détails.

0.1.2 Espaces d'opérateurs

Ici on rappelle un peu la théorie des espaces d'opérateurs. Le lecteur est renvoyé au livre [Pis03] pour une introduction complète à cette théorie.

Soit H un espace de Hilbert arbitraire, on note par $B(H)$ l'ensemble des opérateurs linéaires bornés sur H. Par définition, un espace d'opérateurs (concret) est simplement un sous-espace fermé de $B(H)$.

Soit $E \subset B(H)$ un espace d'opérateurs. On va noter par M_n et $M_n(E)$ les ensembles des matrices de taille $n \times n$ à coefficient dans $\mathbb{C}$ et dans E respectivement. Par l'identification

$$M_n \otimes B(H) = M_n(B(H)) \simeq B(\ell_2^n(H)),$$

on peut munir $M_n(B(H))$ et a fortiori le sous-espace

$$M_n(E) \subset M_n(B(H))$$

de la norme induite par $B(\ell_2^n(H))$. La norme sur $M_n(E)$ sera notée par $\|\cdot\|_n$. Pour tout $n \geq 1$, toute application linéaire $u : E \to F$ entre deux espaces d'opérateurs, elle est associée à une suite d'amplification

$$u_n : M_n(E) \to M_n(F)$$

définie par

$$u_n((x_{ij})_{1\leq i,j\leq n}) = (u(x_{ij})_{1\leq i,j\leq n}),$$

ceci correspond à l'application

$$u_n = Id_{M_n} \otimes u : M_n \otimes E \to M_n \otimes F.$$

Une application $u : E \to F$ est dite complètement bornée (c.b. en abrégé), si

$$\|u\|_{cb} := \sup_{n\geq 1} \|u_n\| < \infty,$$

dans ce cas, on note $\|u\|_{cb}$ la norme c.b. de u.

Dans la catégorie des espaces d'opérateurs, on a comme objets les espaces d'opérateurs et comme morphismes les applications linéaires complètement bornées. Deux espaces d'opérateurs E et F sont identifiés s'il existe $u : E \to F$ tel que $u_n : M_n \otimes E \to M_n \otimes F$ est un isomorphisme isométrique pour tout $n \geq 1$. Ainsi, la donnée d'un espace d'opérateurs abstrait est simplement la donnée d'un espace vectoriel E et une suite d'espaces normés $(M_n \otimes E, \|\cdot\|_n)$, on suppose toujours que muni de la norme $\|\cdot\|$, l'espace E est complet.

Le Théorème Fondamental de Ruan (see e.g. [Rua88]) caractérise de façon abstraite toutes les suites de normes sur $M_n \otimes E$ qui correspondent à une structure d'espace d'opérateurs concrète, i.e. toutes les suites de normes $(M_n \otimes E, \|\cdot\|_n)_{n\geq 1}$ telles qu'il existe $J : E \to B(H)$ pour lequel $J_n : M_n \otimes E \to M_n \otimes B(H)$ est une isométrie pour tout n. Ce théorème permet d'introduire dans la théorie des espaces d'opérateurs la notion de dualité, de quotient, de produit projectif et produit de Haagerup, d'interpolation complexe, etc...

Soit $\mathscr{K}$ l'ensemble des applications linéaires compactes sur ℓ_2. Une façon pour décrire le Théorème de Ruan est

Theorem 0.5. (Z.-J. Ruan) Soit E un espace vectoriel. Et soit α une norme sur $\mathscr{K} \otimes E$, le produit tensoriel algébrique de $\mathscr{K}$ et E. Par définition, si $a, b \in \mathscr{K}$ et si $x = \sum_j c_j \otimes y_j \in \mathscr{K} \otimes E$ avec $c_j \in \mathscr{K}$, $y_j \in E$, alors

$$a \cdot x \cdot b := \sum_j a c_j b \otimes y_j.$$

Supposons que α vérifie la condition suivante : Pour toutes les suites finies (a_i) et (b_i) dans $\mathscr{K}$ et toute suite finie (x_i) dans $\mathscr{K} \otimes E$, on a

$$\alpha(\sum_i a_i \cdot x_i \cdot b_i) \leq \|\sum_i a_i a_i^*\|^{1/2} \sup_i \alpha(x_i)\ \|\sum_i b_i^* b_i\|^{1/2}.$$

Alors il existe un espace de Hilbert H et une injection $J : E \to B(H)$ tels que

$$I_{\mathscr{K}} \otimes J : \mathscr{K} \otimes E \to \mathscr{K} \otimes B(H)$$

s'étend à une isométrie de $\mathscr{K} \otimes_\alpha E$ (la complétion de $\mathscr{K} \otimes E$ par rapport à la norme α) à $\mathscr{K} \otimes_{min} B(H)$ (on a un plongement canonique de $\mathscr{K} \otimes B(H)$ dans $B(\ell_2 \otimes_2 H)$, et $\mathscr{K} \otimes_{min} B(H)$ est juste sa complétion dans cet espace).

Pour donner une intuition de la donnée d'un espace d'opérateurs, rappellons que si B est un espace de Banach, $(a_i)_i$ est une suite finie dans B et $\lambda_i \in \mathbb{C}$, alors on connaît la norme de $\sum_i \lambda_i b_i$. Maintenant, si E est un espace d'opérateurs, $(x_i)_i$ est une suite finie dans E et $(\alpha_i)_i$ dans $\mathscr{K}$, alors on connaît la norme de $\sum_i \alpha_i \otimes x_i$. Donc la donnée d'un espace d'opérateurs E est simplement la donnée d'une façon cohérente (i.e. vérifiant la condition de Ruan) de calculer la norme d'un élément arbitraire dans $\mathscr{K} \otimes E$.

0.1.3 Espaces L^p noncommutatifs à valeurs vectorielles

La référence pour cette partie est le livre [Pis98].

On commence par la définition d'espaces L^p noncommutatifs, autrement dit, on commence par L^p noncommutatif à valeurs scalaires.

Définition 0.6. Soit $1 \leq p < \infty$, et soit $(\mathscr{M}, \tau)$ une algèbre de von Neumann munie d'une trace semi-finie, fidèle et normale. L'espace L^p noncommutatif associé est défini comme la complétion de l'espace vectoriel normé

$$\left\{x \in \mathscr{M} : \|x\|_p \stackrel{\text{def}}{=} (\tau(|x|^p))^{1/p}\right\}$$

par rapport à la norme $\|\cdot\|_p$.

La théorie des espaces L^p noncommutatifs à valeurs vectorielles est un analogue de la théorie des espaces de Bochner, dans cette dernière théorie on définit pour tout espace mesuré (Ω, μ) et tout espace de Banach X un espace de Banach $L_p(\Omega, \mu; X)$ (pour $1 \leq p \leq \infty$). Dans cette thèse, quand on parle les espaces L^p noncommutatifs à valeurs vectorielles, on restreint toujours au cas où les algèbres de von Neumann sont hyperfinies munies d'une trace semi-finie normale fidèle. Donc si $(\mathscr{M}, \tau)$ est une telle algèbre de von Neumann, et E est un espace d'opérateurs, on peut définir un nouvel espace d'opérateurs $L_p(\mathscr{M}, \tau; E)$ pour $1 \leq p < \infty$. Dans la terminologie usuelle, si l'algèbre de von Neumann est $(B(H), \mathrm{Tr})$, on dit c'est dans le cas "discret". Si $\dim H = n$, l'algèbre de von Neumann est donc (M_n, Tr).

Dans le cas discret, les espaces $L_p(B(H), \mathrm{Tr}; E)$ et $L_p(M_n, \mathrm{Tr}; E)$ seront notés par $S_p[H; E]$ et $S_p^n[E]$, et $S_p[\ell_2; E]$ sera noté par $S_p[E]$. En tant qu'espaces vectoriels, on a $S_p^n[E] \simeq S_p^n \otimes E$. Le fait remarquable (très utile) est : pour tout $1 \leq p < \infty$, si $u : E \to F$ est une application linéaire, alors

$$\|u\|_{cb} = \sup_{n \geq 1} \|u_n : S_p^n[E] \to S_p^n[F]\|,$$

où $u_n = Id_{S_p^n} \otimes u : S_p^n \otimes E \to S_p^n \otimes F$.

0.1.4 La propriété OUMD pour les espaces d'opérateurs

On rappelle ici la notion de martingale noncommutative.

La donnée $(\mathscr{M}, \tau)$ d'une algèbre de von Neumann hyperfinie munie d'une trace normale fidèle normalisée est considérée comme un espace de probabilité noncommutatif standard. Une filtration est une suite croissante $(\mathscr{M}_n)_{n \geq 0}$ de sous-algèbres de von Neumann, i.e. $\mathscr{M}_n \subset \mathscr{M}_{n+1}$ pour tout $n \geq 0$. Dans [Tak01], il est montré qu'il existe une unique projection normale de $\mathscr{M}$ à $\mathscr{M}_n$ pour tout n, on va noter cette projection par

$$\mathbb{E}^{\mathscr{M}_n} : \mathscr{M} \to \mathscr{M}_n.$$

Cette projection vérifie les conditions d'une espérance conditionnelle usuelles, i.e. $\mathbb{E}^{\mathscr{M}_n}$ est une application positive, $\mathbb{E}^{\mathscr{M}_n}(1) = 1$ et on a

$$\mathbb{E}^{\mathscr{M}_n}(axb) = a \cdot \mathbb{E}^{\mathscr{M}_n}(x) \cdot b$$

pour tout $a, b \in \mathscr{M}_n, x \in \mathscr{M}$. C'est pour cette raison qu'on appelle $\mathbb{E}^{\mathscr{M}_n}$ l'espérance conditionnelle de $\mathscr{M}$ relative à $\mathscr{M}_n$. Lorsque la filtration $(\mathscr{M}_n)$ est fixée, on utilise souvent la notation $\mathbb{E}_n = \mathbb{E}^{\mathscr{M}_n}$. La positivité de $\mathbb{E}_n$ permet en particulier d'étendre $\mathbb{E}_n$ à une projection de $L^p(\mathscr{M}, \tau)$ à $L^p(\mathscr{M}_n, \tau)$, on va noter cette extension encore par $\mathbb{E}_n$, et elle est appelée l'espérance conditionnelle de $L^p(\mathscr{M}, \tau)$ à $L^p(\mathscr{M}_n, \tau)$.

Définition 0.7. Étant donné un espace de probabilité noncommutatif $(\mathscr{M}, \tau)$ munie d'une filtration $(\mathscr{M}_n)_{n\geq 0}$, une suite $(x_n)_{n\geq 0}$ d'éléments dans $L^p(\mathscr{M}, \tau)$ est dite une martingale dans $L^p(\mathscr{M}, \tau)$ par rapport à $(\mathscr{M}_n)_{n\geq 0}$, si elle vérifie

$$x_n = \mathbb{E}_n(x_{n+1})$$

pour tout n.

Plus généralement, si E est un espace d'opérateurs, l'application

$$\mathbb{E}_n \otimes Id_E : L^p(\mathscr{M}, \tau) \otimes E \to L^p(\mathscr{M}_n, \tau) \otimes E$$

s'étend à une application complètement contractante de $L^p(\mathscr{M}, \tau; E)$ à $L^p(\mathscr{M}_n, \tau; E)$. Cette extension sera notée encore par

$$\mathbb{E}_n : L^p(\mathscr{M}, \tau; E) \to L^p(\mathscr{M}_n, \tau; E)$$

et elle sera appelée encore espérance conditionnelle.

En imitant la définition de la propriété pour les espaces de Banach, on peut introduire la notion de OUMD pour les espaces d'opérateurs.

Définition 0.8. Soit $1 < p < \infty$. Un espace d'opérateurs E est dit OUMD_p s'il existe une constante c telle que pour toute martingale $(x_n)_{n\geq 0}$ convergente dans $L_p(\mathscr{M}; E)$, on a pour tout choix de signs $\varepsilon_n = \pm 1$

$$\sup_n \|\sum_0^n \varepsilon_k dx_k\|_{L_p(\mathscr{M};E)} \leq c \sup_n \|x_n\|_{L_p(\mathscr{M};E)} \tag{2}$$

où $dx_n = x_n - x_{n-1}$ et par convention $x_{-1} = 0$. La meilleurs constante c dans (2) est notée par $\beta_p^{os}(E)$, elle est dite la constante OUMD_p de E.

Remarque 0.9. La propriété OUMD reste mystérieuse. Par exemple, le problème qui demande si OUMD_p et OUMD_q sont équivalents ou non reste ouvert depuis l'introduction de cette notion de OUMD.

0.2 Contenu de la thèse

Cette thèse contient deux chapitres, rédigés en anglais. Le chapitre 1 décrit un travail, intitulé *On the OUMD property for the column Hilbert space C*, qui a été accepté dans Indiana University Mathematics Journal. Dans le chapitre 2, le travail présenté est contenu dans un article intitulé *On the UMD constants for a class of iterated $L_p(L_q)$ spaces*, qui a été accepté dans Journal of Functional Analysis.

0.2.1 Chapitre 1

Le résultat principal de ce chapitre concerne un problème de Z.-J. Ruan qui demande si C l'espace de Hilbert en colonne a la propriété OUMD ou pas. L'auteur de cette thèse s'est intéressé à ce problème depuis le début de sa thèse et finalement a obtenu une réponse positive.

Par définition, l'espace de Hilbert en colonne C est l'espace d'opérateurs

$$C = \Big\{ a = (a_{i,j})_{i,j\geq 1} \in B(\ell_2) : a_{i,j} = 0, \text{ si } j \geq 2 \Big\}.$$

Donc C est un espace de Hilbert muni d'une structure d'une façon *en colonne* . Cet espace est assez fondamental et classique dans la théorie des espaces d'opérateurs. En pratique, C est compris par la formule

$$\| \sum_i a_i \otimes e_{i,1} \|_{B(H)\otimes_{min} C} = \| \sum_i a_i^* a_i \|_{B(H)}^{1/2}$$

pour toute suite finie (a_i) dans $B(H)$ (il suffit de prendre (a_i) les suites finies dans $\mathscr{K}$).

La question posée par Z.-J. Ruan est :

Question 0.10. Est-ce que l'espace C a la propriété OUMD$_p$ pour un $1 < p < \infty$ ou pour tout $1 < p < \infty$?

Le fait que même pour un espace si simple comme C, la question de demander s'il est OUMD ou pas n'est pas triviale donne une intuition de la difficulté pour étudier la propriété OUMD. Au moment où cette thèse est écrite, il semble qu'on connaisse très peu d'espaces d'opérateurs qui sont OUMD.

Les premiers résultats sur l'étude de la propriété OUMD ont été obtenu par Pisier et Xu dans leur article [PX97]. En fait, un résultat de cet article montre que pour tous $1 < p < \infty$, tous les espaces L^p noncommutatifs (scalaires) ont la propriété OUMD$_p$. Plus récemment, par montrer que l'espace d'opérateurs S_p a la propriété OUMD$_q$, Musat a montré que dans le cas où l'algèbre de von Neumann a la propriété QWEP au sens de Lance, l'espace L^p noncommutatif associé a la propriété OUMD$_q$ pour tout $1 < p, q < \infty$. Elle a aussi montré que $S_u[S_v]$ a la propriété OUMD$_p$ pour tout $1 < u, v, p < \infty$. Pour les détails, voir [Mus06].

Le résultat suivant non-publié de Musat donne un lien de la propriété OUMD$_p$ pour un espace d'opérateurs et la propriété UMD classique.

Théorème 0.11. (Musat) Soit $1 < p < \infty$, et soit E un espace d'opérateurs. Alors E a la propriété OUMD$_p$ si et seulement si $S_p[E]$ vu comme un espace de Banach a la propriété UMD.

En utilisant ce théorème, l'auteur a pu répondre la question de Z.-J. Ruan par le théorème suivant :

Théorème 0.12. Soit $1 < p < \infty$. Alors $S_p[C]$ vu comme un espace de Banach est UMD.

Avant ce résultat, il était déjà connu que $S_p[C]$ était super-réflexif pour tout $1 < p < \infty$. Une démonstration très courte de ce fait a été donnée par T. Oikhberg en réponse à une question de Musat.

La preuve du Théorème 0.12 utilise la description de $S_p[E]$ en produit de Haagerup

$$S_p[C] = C_p \otimes_h E \otimes_h R_p,$$

où C_p et R_p sont les sous-espaces en colonne et en ligne de S_p muni de la structure d'espace d'opérateurs standard. On va utiliser aussi un théorème de Kouba, qui dit essentiellement que le produit de Haagerup se comporte bien pour l'interpolation complexe, i.e. on a

Theorem 0.13. (Kouba) Soient (E_0, E_1) et (F_0, F_1) deux couples compatibles d'espaces d'opérateurs aux sens d'interpolation. Alors $(E_0 \otimes_h F_0, E_1 \otimes_h F_1)$ est un couple compatible, et pour tout $0 < \theta < 1$, on a

$$(E_0 \otimes_h F_0, E_1 \otimes_h F_1)_\theta = (E_0, E_1)_\theta \otimes_h (F_0, F_1)_\theta.$$

Pour démontrer le Théorème 0.12, on montrera d'abord que pour tout $1 < p < \infty$, il existe $1 < p_1, p_2, p_3 < \infty$ (qui ne sont pas uniques), tels qu'on a un plongement isométrique

$$S_p[C] \subset S_{p_1}[S_{p_2}[S_{p_3}]]. \tag{3}$$

L'auteur a montré aussi le théorème suivant qui généralise un résultat de Musat.

Théorème 0.14. Soit $1 < p, p_1, p_2, \cdots, p_n < \infty$. Alors $S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]$ est OUMD_p.

La preuve se fait par récurrence sur n et se décompose en plusieurs étapes selon les conditions différentes sur les indices $p, p_1, \cdots, p_n$. Cet argument est inspiré par la preuve de Musat pour la propriété OUMD_p de l'espace $S_u[S_v]$.

Le théorème 0.14 est utilisé pour montrer que le plongement mentionné dans (3) n'est pas complètement isométrique. Il n'est même pas complètement isomorphique. Précisément, on a obtenu

Corollaire 0.15. Soit $1 < p_1, p_2, \cdots, p_n < \infty$. Alors l'espace en colonne C ne se plonge pas complètement isomorphiquement dans $S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]$. Plus généralement, aucun sous-quotient (sous-espace de quotient) de $S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]$ n'est complètement isomorphe pas à C.

On peut considérer M_n comme une sous-algèbre sans unité de l'algèbre $M_\infty := B(\ell_2)$ en identifiant M_n et le sous-ensemble $\widetilde{M_n}$ de M_∞ défini par

$$\widetilde{M_n} := \Big\{ a = (a_{i,j})_{i,j \geq 1} \in B(\ell_2) : a_{i,j} = 0 \text{ si } i \geq n+1 \text{ ou } j \geq n+1 \Big\}.$$

La filtration canonique des algèbres de matrices est la filtration croissante $(M_n)_{n\geq 1}$ de sous-algèbre de M_∞. La projection canonique $E_n : M_\infty \to M_n$ est

$$E_n(a) = \sum_{\max(i,j)\leq n} a_{i,j} e_{i,j}, \text{ pour tout } a = (a_{i,j})_{i,j\geq 1} \in M_\infty.$$

Soit E un espace d'opérateurs, $1 \leq p \leq \infty$. La projection canonique de $S_p[E]$ à $S_p^n[E]$ est notée aussi $E_n : S_p[E] \to S_p^n[E]$. Si $x \in S_p[E]$, alors on définit

$$d_1 x = E_1(x) \text{ et } d_n x = E_n(x) - E_{n-1}(x), \text{ pour tout } n \geq 2.$$

Pour tout $1 < p < \infty$, l'espace E est dit OUMD_p par rapport à la filtration canonique des algèbres des matrices s'il existe une constante K telle que pour tout $N \geq 1$ et tout choix de signe $\varepsilon_n = \pm 1$, on a

$$\Big\| \sum_1^N \varepsilon_n d_n x \Big\|_{S_p[E]} \leq K \|x\|_{S_p[E]}.$$

En utilsant un théorème de S. Neuwirth et E. Ricard sur la projection triangulaire, on montre le théorème suivant :

Théorème 0.16. La propriété OUMD_p restreinte à la filtration canonique des algèbres des matrices est équivalente à la propriété OUMD_p.

A la fin du Chapitre 1, on donne une liste de problèmes sur la propriété OUMD.

0.2.2 Chapitre 2

L'objet de ce chapitre est l'étude des constantes UMD d'une suite d'espaces de Banach définis par récurrence. Par les résultats du Chapitre 1, on sait que si $1 < p_1, p_2, \cdots, p_n < \infty$, l'espace d'opérateurs $S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]$ est OUMD_p pour tout $1 < p < \infty$. Il est donc très naturel de demander comment les constantes OUMD_p de ces espaces dépendent des paramètres $p_1, p_2, \cdots, p_n$. En particulier, une question se pose :

Question 0.17. Soit $1 < p_0 < p_\infty < \infty$, et soit $1 < p < \infty$. Est-ce qu'il existe une constante K telle que pour tout n et tout $p_1, p_2, \cdots, p_n \in [p_0, p_\infty]$, on a

$$\beta_p^{os}(S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]) \leq K?$$

La Question 0.17 consiste à demander si ces espaces $S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]$ sont uniformément OUMD_p ou pas étant donné que $p_1, p_2, \cdots, p_n$ sont dans un intervalle raisonnable $[p_0, p_\infty]$.

En fait, même la question suivante n'avait pas été étudiée :

Question 0.18. Soit $1 < p_0 < p_\infty < \infty$, et soit $1 < p < \infty$. Est-ce qu'il existe une constante K telle que pour tout n et tout $p_1, p_2, \cdots, p_n \in [p_0, p_\infty]$, on a

$$\beta_p\Big(\ell_{p_1}(\ell_{p_2}(\cdots(\ell_{p_n})\cdots))\Big) \leq K?$$

Puisque $\ell_{p_1}(\ell_{p_2}(\cdots(\ell_{p_n})\cdots))$ est un sous-espace de $S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]$, une réponse négative pour la Question 0.18 implique aussi une réponse négative pour la Question 0.17.

Le résultat principal de ce chapitre est de répondre négativement à la Question 0.18 et donc aussi à la Question 0.17. En fait, une réponse négative à cette question donnerait plus de conséquences qu'une réponse positive. Ici, on mentionne que dans [Bou83, Bou84], Bourgain a construit les premiers exemples de treillis de Banach super-réflexifs mais non-UMD. Comme sous-produit de la réponse négative à la Question 0.18, on construit une famille de treillis de Banach super-réflexifs et non-UMD en itérant la construction $\ell_p(\ell_q)$, cette famille contient des échelles d'interpolation complexe qui ont aussi leur propre intérêt.

Le premier but de ce chapitre est de démontrer le théorème suivant :

Théorème 0.19. Supposons que $1 \leq p \neq q \leq \infty$. Soit $E_1 \overset{\text{def}}{=} \ell_p^{(2)}(\ell_q^{(2)})$, où $\ell_p^{(2)}$ et $\ell_q^{(2)}$ sont les espaces ℓ_p et ℓ_q de dimension 2. Définissons une suite d'espaces de Banach par récurrence :

$$E_{n+1} = \ell_p^{(2)}(\ell_q^{(2)}(E_n)).$$

Alors il existe une constante $c = c(p, q) > 1$, telle que pour tout $1 < s < \infty$,

$$\beta_s(E_n) \geq c^n.$$

En particulier,

$$\lim_{n\to\infty} \beta_s(E_n) = \infty.$$

La preuve est basée sur l'étude d'une constante associée à un espace de Banach et une famille de vecteurs introduite dans ce chapitre. Précisément, étant donné un espace de Banach X et une famille de vecteurs $\{x_i\}_{i\in I}$ dans X, on peut définir une constante $S(X; \{x_i\})$ à priori dans $[1, \infty]$ par

Définition 0.20. Soit X un espace de Banach. Supposons que $\{x_i\}_{i\in I}$ est une famille de vecteurs dans X. La constante $S(X;\{x_i\})$ est la meilleure constante c telle que

$$\Big\|\sum_{k=0}^{N}\mathbb{E}^{\mathcal{A}_k}(\theta_k)x_{i_k}\Big\|_{L_1(\Omega,\mathbb{P};X)}\leq c\Big\|\sum_{k=0}^{N}\theta_k x_{i_k}\Big\|_{L_\infty(\Omega,\mathbb{P};X)}$$

pour tout $N\geq 1$, tout espace de probabilité $(\Omega,\mathcal{F},\mathbb{P})$ muni d'une filtration $\mathcal{A}_0\subset\mathcal{A}_1\subset\cdots\subset\mathcal{A}_n\subset\cdots\subset\mathscr{F}$, tout choix de $N+1$ indices distincts $\{i_0,i_1,\cdots,i_N\}\subset I$ et tout choix de $N+1$ fonctions $\{\theta_0,\theta_1,\cdots,\theta_N\}\subset L_\infty(\Omega,\mathcal{F},\mathbb{P})$.

On va utiliser une inégalité de Stein assez connue pour les espaces UMD. La preuve de cette inégalité est due à Bourgain.

Théorème 0.21. (Bourgain) Soit X un espace de Banach qui est UMD. Alors pour tout $1<s<\infty$, toute suite finie de functions $(F_k)_{k\geq 0}$ dans $L_s(\Omega,\mathcal{F},\mathbb{P};X)$ et toute filtration $\mathcal{A}_0\subset\mathcal{A}_1\subset\cdots\subset\mathcal{A}_n\subset\cdots\subset\mathcal{F}$, on a

$$\Big\|\sum_k\varepsilon_k\mathbb{E}^{\mathcal{A}_k}(F_k)\Big\|_{L_s(\mu_\infty\times\mathbb{P};X)}\leq\beta_s(X)\Big\|\sum_k\varepsilon_k F_k\Big\|_{L_s(\mu_\infty\times\mathbb{P};X)},$$

où μ_∞ est la mesure de Haar normalisée sur le groupe de Cantor $\{-1,1\}^{\mathbb{N}}$, et $(\varepsilon_k)_{k\geq 0}$ est la suite de Rademacher sur $(\{-1,1\}^{\mathbb{N}},\mu_\infty)$.

En appliquant cette inégalité de Stein, on trouve que la constante $S(X;\{x_i\})$ est liée à la constante $\beta_s(X)$ par la proposition suivante :

Proposition 0.22. Lorsque la famille $\{x_i\}_{i\in I}$ est 1-inconditionnelle, on a

$$\beta_s(X)\geq S(X;\{x_i\})$$

pour tout $1<s<\infty$.

La raison d'introduire la constante $S(X;\{x_i\})$ apparaît clairement dans le théorème suivant :

Théorème 0.23. Soit E un espace de Banach avec une base 1-inconditionnelle notée par $\{e_i : i\in I\}$, et soit F n'importe quel espace de Banach. L'espace $E(F)$ est défini de façon évidente, i.e. $E(F)$ est la complétion du produit tensoriel algébrique $E\otimes F$ par rapport à la norme suivante : si $x=\sum_i e_i\otimes x_i\in E\otimes F$, alors

$$\|x\|_{E(F)}\overset{\text{def}}{=}\Big\|\sum_i e_i\|x_i\|_F\Big\|_E.$$

Pour toute famille $\{f_j : j\in J\}$ dans F, on a

$$S(E(F);\{e_i\otimes f_j\}_{i\in I,j\in J})\geq S(E;\{e_i\})S(F;\{f_j\}).$$

Remarque 0.24. Le phénomène du Théorème 0.23 n'apparaît pas pour la constante UMD$_s$. En effet, si $p\neq 2$ ou $s\neq 2$, on a $\beta_s(\ell_p)>1$. Puisqu'on a

$$\ell_p(\ell_p)\simeq\ell_p,$$

si on avait pour la constante UMD$_s$ le même phénomène comme décrit dans le théorème ci-dessus, alors

$$\beta_s(\ell_p)=\beta_s(\ell_p(\ell_p))\geq\beta_s(\ell_p)\beta_s(\ell_p),$$

donc on aurait $\beta_s(\ell_p)\leq 1$, d'où une contradiction.

Si les familles $\{e_i : i \in I\}$, $\{f_j : j \in J\}$ sont sans ambiguité dans le texte, on va simplifier la notation $S(E; \{e_i\}$, $S(F; \{f_j\})$ et $S(E(F); \{e_i \otimes f_j\})$ par $S(E)$, $S(F)$ et $S(E(F))$ respectivement. Le Théorème 0.23 affirme que sous certaines conditions, on a

$$S(E(F)) \geq S(E)S(F).$$

Remarque 0.25. Il est facile de montrer que par rapport à la base canonique de ℓ_p,

$$S(\ell_p) = 1.$$

Donc on a $S(\ell_p(\ell_p)) \geq S(\ell_p)S(\ell_p)$, comme prévu par le Théorème 0.23.

Maintenant, si E est un espace de dimension finie avec une base 1-inconditionnelle $\{e_i\}_{i=1}^d$, où $d = \dim E$, on peut donc voir E comme un treillis de Banach sur l'ensemble $\{1, 2, \cdots, d\}$ et on peut définir

$$E^{\otimes 1} := E \ \text{ et } \ E^{\otimes n+1} = E(E^{\otimes n}), \ \text{ pour } n \geq 1.$$

Puisque la base $\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} : 1 \leq i_1, i_2, \cdots, i_n \leq d\}$ pour $E^{\otimes n}$ est 1-inconditionnelle, en appliquant la Proposition 0.22 et le Théorème 0.23, pour tout $1 < s < \infty$,

$$\beta_s(E^{\otimes n}) \geq S(E^{\otimes n}) \geq S(E)^n.$$

Lorsqu'on a $S(E) > 1$, on obtient une minoration non triviale de $\beta_s(E^{\otimes n})$, i.e. on a

$$\beta_s(E^{\otimes n}) \xrightarrow{n \to \infty} \infty.$$

Par les observations précédentes, on doit étudier les treillis de Banach sur l'ensemble fini, dit $[d] = \{1, 2, \cdots, d\}$. Tout exemple explicite de treillis de Banach X sur l'ensemble $[d]$ tel que $S(X) = S(X; \{\delta_i\}_{i=1}^d) > 1$ nous donnera une information sur la propriété UMD pour les treillis de Banach.

On va se concentrer à étudier un exemple de treillis de Banach sur l'ensemble à 4 points. Pour tout $1 \leq p, q \leq \infty$, on peut définir un treillis de Banach $\ell_p^{(2)}(\ell_q^{(2)})$, i.e., si $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^4$(ou $\mathbb{C}^4$), alors

$$\|(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\|_{\ell_p^{(2)}(\ell_q^{(2)})} = \left\| \Big(\|(\lambda_1, \lambda_2)\|_{\ell_q}, \|(\lambda_3, \lambda_4)\|_{\ell_q} \Big) \right\|_{\ell_p}.$$

La constante $S(\ell_p^{(2)}(\ell_q^{(2)}))$ sera calculée par rapport à la base canonique de $\ell_p^{(2)}(\ell_q^{(2)})$, lorsque $p \neq q$, on trouve l'inégalité

$$S(\ell_p^{(2)}(\ell_q^{(2)})) > 1.$$

La preuve de cette inégalité est basée sur la proposition suivante :

Proposition 0.26. Soit $D = \{\pm 1\}$ muni de la probabilité uniforme $\mu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$. Soit P la projection définie sur $L_p(D, \mu; \ell_q^{(2)})$ par

$$\begin{array}{rccc} P : & L_p(D, \mu; \ell_q^{(2)}) & \to & L_p(D, \mu; \ell_q^{(2)}) \\ & (f, g) & \mapsto & (\mathbb{E}(f), g) \end{array}.$$

Lorsque $p \neq q$, la norme de P est strictement plus grande que 1.

Les résultats précédents sur la propriété UMD se généralisent à la propriété AUMD. Précisément, on va définir une constante $S^a(X; \{x_i\})$ pour toute famille $\{x_i\}_{i \in I}$ dans un espace de Banach X sur $\mathbb{C}$. Cette constante est un analogue de $S(X; \{x_i\})$, et elle vérifie

Proposition 0.27. Lorsque la famille $\{x_i\}_{i\in I}$ est 1-inconditionnelle, on a

$$\beta_s^a(X) \geq S(X; \{x_i\})$$

pour tout $1 \leq s < \infty$.

Théorème 0.28. Soit E un espace de Banach sur $\mathbb{C}$ avec une base 1-inconditionnelle notée par $\{e_i : i \in I\}$, et soit F n'importe quel espace de Banach sur $\mathbb{C}$. Alors, pour toute famille $\{f_j : j \in J\}$ dans F, on a

$$S^a(E(F); \{e_i \otimes f_j\}_{i\in I, j\in J}) \geq S^a(E; \{e_i\}) S^a(F; \{f_j\}).$$

La preuve que $S^a(\ell_p^{(2)}(\ell_q^{(2)}) > 1$ lorsque $1 \leq p \neq q < \infty$ est un peu compliquée par rapport à la preuve que $S(\ell_p^{(2)}(\ell_q^{(2)})) > 1$ lorsque $1 \leq p \neq q \leq \infty$, pour la première, on doit utiliser la moyenne géométrique $M(|f|)$ d'une fonction f sur $\mathbb{T}$ bornée inférieurement (il existe $\delta > 0$, telle que $|f| > \delta$) qui est donnée par la formule

$$\log M(|f|) = \int_{\mathbb{T}} \log |f(z)| dm(z).$$

La condition classique de Szegö sur les fonctions extérieures sera aussi utilisée.

Après ces résultats, on obtient

Théorème 0.29. Soit $1 \leq p \neq q < \infty$ et définit les espaces E_n sur $\mathbb{C}$ comme dans le Théorème 0.19. Alors il existe $\kappa = \kappa(p, q) > 1$ tel que pour tout $1 \leq s < \infty$, on a

$$\beta_s^a(E_n) \geq \kappa^n.$$

En particulier,

$$\lim_{n\to\infty} \beta_s^a(E_n) = \infty.$$

A la fin du Chapitre 2, on s'intéresse au problème ouvert sur l'ordre de croissance $\beta_p(n)$ défini par

$$\beta_p(n) = \sup \Big\{ \beta_p(X) : \dim X = n \Big\}.$$

On va définir

$$S(d) := \sup \Big\{ S(X; \{\delta_i\}_{i=1}^d) : \ X \text{ un treillis de Banach sur } \{1, 2, \cdots, d\} \Big\}.$$

Puisqu'on a $\beta_p(d^m) \geq S(d^m) \geq S(d)^m$, il est intéressant de savoir la valeur exacte de $S(d)$ pour d petit, par exemple, la valeur de $S(4)$.

0.2.3 Annexe

Dans la première partie de l'annexe, on va étudier les constantes UMD_s pour les espaces

$$C_{p_1} \otimes_h C_{p_2} \otimes_h \cdots \otimes_h C_{p_n}.$$

En particulier, on va définir les espaces $V_n(p, q)$ par

$$V_n(p, q) := \underbrace{C_p \otimes_h C_q \otimes_h C_p \otimes_h C_q \otimes_h \cdots \otimes_h C_p \otimes_h C_q}_{C_p \otimes_h C_q \text{ est répété } n \text{ fois}}.$$

On a le théorème suivant :

Théorème 0.30. Soit $1 \leq p \neq q \leq \infty$. Alors, pour tout $1 < s < \infty$, il existe une constante $w = w(p,q,s) > 1$ tel que on a
$$\beta_s(V_n(p,q)) \geq w^n.$$
A fortiori, on a
$$\beta_s^{os}(V_n(p,q)) \geq w^n.$$

La deuxième partie de l'annexe concerne la démonstration de l'équivalence suivante :
$$\mathrm{AUMD}_p \Longleftrightarrow \mathrm{AUMD}_q, \text{ pour tout } 1 \leq p,q < \infty.$$

Chapter 1

On the OUMD property for the column Hilbert space C

Introduction

In this chapter, we will study the OUMD property for operator spaces, i.e. the operator space version of the classical UMD property for Banach spaces. The first and main part of this chapter is focused on the study of the OUMD property for a particular operator space, namely, the so-called column Hilbert space, which is usually denoted by C. Based on some complex interpolation techniques adapted in the theory of operator spaces, we are able to relate an open problem proposed by Z.-J. Ruan to the OUMD_q property for non-commutative L_p-spaces. More precisely, we proved that

Theorem 1.1. Let $1 < p < \infty$, then there exist $1 < u, v < \infty$, such that we have an isometric (Banach) embedding

$$S_p[C] \hookrightarrow S_u[S_v].$$

Using the idea used for proving the above result, some non-embeddability results in the complete isomorphic sense concerning the operator space C and the operator spaces $S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]$ are also obtained.

The third part of this chapter contains some results on OUMD property for general operator spaces.

1.1 UMD property for Banach spaces

The UMD property for Banach spaces, which stands for "Unconditional Martingale Difference" property for Banach spaces, was first introduced by Maurey and Pisier in the study of vector valued martingale theory, Burkholder together with some other authors developed a rich theory on the UMD property.

Let us recall briefly the definition.

Definition 1.2. Suppose that $1 < p < \infty$. A Banach space B is UMD_p, if there exists a constant c, such that for all positive integers n, all sequences $\varepsilon = (\varepsilon_k)_{k=0}^n$ of numbers in $\{-1, 1\}$ and all B-valued martingale $(x_k)_{k=0}^n$ in a Bochner space $L_p(B) = L_p(\Omega, \mathcal{F}, \mathbb{P}; B)$, we have

$$\|\sum_{k=0}^n \varepsilon_k dx_k\|_{L_p(B)} \leq c\|\sum_{k=0}^n dx_k\|_{L_p(B)}, \tag{1.1}$$

where $dx_k = x_k - x_{k-1}$ and by convention $x_{-1} := 0$. By definition, a Banach space is UMD if it is UMD_p for some $1 < p < \infty$.

The best constant c satisfying inequality (1.1) is called the UMD_p constant of B and will be denoted by $\beta_p(B)$. In the languages of Banach space theory, inequality (1.1) means exactly that the martingale difference sequence $(dx_k)_{k=0}^n$ is an unconditional sequence in $L_p(B)$ and its unconditionality constant is dominated by a universal constant c. This explains why this property is called the "Unconditional Martingale Difference" property. For convenience, we define $\beta_p(B) = \infty$, if B is not UMD_p.

The fact that the UMD property is independent of p was first proved by Pisier (using the Burkholder-Gundy extrapolation techniques). More precisely, he proved that the finiteness of $\beta_p(B)$ for some $1 < p < \infty$ implies its finiteness for all $1 < p < \infty$. Thus, a Banach space is UMD if it is UMD_p for some $1 < p < \infty$ and equivalently for all $1 < p < \infty$.

The UMD property has very deep connections with the boundedness of certain singular integral operators such as the Hilbert transform, see e.g. Burkholder's article [Bur01]. Burkholder and McConnell [Bur83] proved that if a Banach space B is UMD_p, then the Hilbert transform is bounded on the Bochner space $L_p(\mathbb{T}, m; B)$. Bourgain [Bou83] showed that if the Hilbert transform is bounded on $L_p(\mathbb{T}, m; B)$, then B is UMD_p, in particular, by using this, he proved that the Schatten p-classes S_p are UMD spaces for $1 < p < \infty$. The classical examples of UMD spaces include all the finite dimensional Banach spaces, the Schatten p-classes S_p and more generally the noncommutative L_p-spaces associated to a von Neumann algebra $\mathscr{M}$, for all $1 < p < \infty$. Interested readers are referred to Burkholder's papers [Bur86, Bur01] for the details on UMD spaces.

1.2 Preliminaries on operator space theory

Some basic definitions from the operator space theory will be given in this section.

1.2.1 Operator spaces

The theory of operator spaces is a new developed theory after Ruan's thesis in 1988 by Effros, Ruan, Blecher, Paulsen, Pisier and many other authors. From some point of view, this theory can be described as a noncommutative Banach space theory, for instance, it contains also a fundamental extension theorem which plays the role of the Hahn-Banach Theorem in the Banach space theory, the Grothendieck program of studying various tensor products and factorizations of linear maps in the theory of Banach spaces has its analogue in the theory of operator spaces. The readers interested in the details on this theory are invited to read the monograph [Pis03].

The space of all bounded operators on a complex Hilbert space H, equipped with the operator norm will be denoted by $B(H)$.

Definition 1.3. An operator space is a closed subspace of $B(H)$.

Let $E \subset B(H)$ be an operator space. For any $n \geq 1$, let $M_n = M_n(\mathbb{C})$ be the space of $n \times n$ complex matrices and let $M_n(E)$ be the space of $n \times n$ matrices with entries in E. We may equip $M_n(B(H))$ with a norm by the natural identification

$$M_n(B(H)) \simeq M_n(\ell_2^n(H)),$$

where $\ell_2^n(H) := \underbrace{H \oplus H \oplus \cdots \oplus H}_{n \text{ times}}$. Thus, $M_n(E)$ is equipped with the norm induced by the inclusion $M_n(E) \subset M_n(B(H))$. The norm on $M_n(B(H))$ and the induced norm on $M_n(E)$ will all be called the matrix norm and they are all denoted by $\|\cdot\|_n$.

Let E and F be two operator spaces, and let $u : E \to F$ be an linear map. For any $n \geq 1$, we may define the n-amplification u_n of u by

$$\begin{array}{rccc} u_n : & M_n(E) & \to & M_n(F) \\ & (x_{ij})_{1\leq i,j\leq n} & \mapsto & (u(x_{ij}))_{1\leq i,j\leq n} \end{array}.$$

The map u is called completely bounded (in short c.b.) if

$$\|u\|_{cb} \overset{\text{def}}{=} \sup_{n\geq 1} \|u_n\|_{M_n(E)\to M_n(F)} < \infty.$$

In the case when $\|u\|_{cb} < \infty$, the quantity $\|u\|_{cb}$ will be called the c.b. norm of u. We will denote by

$$CB(E, F)$$

the space of all c.b. maps from E into F equipped with the c.b. norm.

A map $u : E \to F$ is called a complete isometry (or u is completely isometric) if

$$u_n : M_n(E) \to M_n(F)$$

is an isometry for all $n \geq 1$. Two operator spaces E and F will be identified if there exists a surjective complete isometry $u : E \to F$. Thus, an operator space is just a vector space E together with a sequence of matrix norms $\|\cdot\|_n$ on the spaces $M_n(E)$ which come from some embedding $E \subset B(H)$.

By the classical Gelfand and Naimark Theorem, we know that every C^*-algebra can be realized as a closed self-adjoint subalgebra of $B(H)$, such a realization will be called a representation. Moreover, if A is a C^*-algebra, let $A \subset B(H)$ and $A \subset B(K)$ be two representations, then they induce the same norm on $M_n(A)$ for any $n \geq 1$. It follows from this fact that any C^*-algebra (and a fortiori any von Neumann algebra) admits a natural operator space structure compatible with its structure as a C^*-algebra. By saying the operator space structure on a given C^*-algebra, we always mean the one given by any representation.

Z.-J. Ruan in [Rua88] gave an abstract characterization of operator spaces in terms of matrix norms. We describe briefly this characterization. Given a vector space E equipped with matrix norms $\|\cdot\|_m$ on $M_m(E)$ for each positive integer m (usually, we will suppose that E equipped with the norm $\|\cdot\|_1$ is complete), such that the matrix norms satisfies Ruan's axioms:

(R1) $\forall x \in M_m(E), y \in M_n(E)$,

$$\left\| \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\|_{m+n} = \max\Big\{ \|x\|_m, \|y\|_n \Big\},$$

(R2) $\forall x \in M_m(E), \forall \alpha, \beta \in M_m(\mathbb{C})$,

$$\|\alpha x \beta\|_m \leq \|\alpha\| \cdot \|x\|_m \cdot \|\beta\|.$$

Then there exists a Hilbert space H and a linear map

$$J : E \to B(H)$$

such that $J_n : M_n(E) \to M_n(B(H))$ is an isometry for all $n \geq 1$. Thus E can be identified with $J(E) \subset B(H)$, the image of J in $B(H)$.

Clearly, the matrix norms coming from an embedding $E \subset B(H)$ satisfy the axioms (R1) and (R2). Thus by an abstract operator space, we mean a vector space E equipped with a sequence of matrix norms $\| \cdot \|_n$ on $M_n(E)$ satisfying Ruan's axioms.

In particular, this abstract characterization allows us to define various important constructions of new operator spaces from the given ones. Among these are the projective tensor product, the quotient, the dual, complex interpolations etc.

Remark 1.4. Let E and F be two operator spaces. Then $CB(E, F)$ is equipped with an operator space structure such that we have isometrically

$$M_n(CB(E, F)) \simeq CB(E, M_n(F)).$$

By replacing M_n by $M_n \otimes M_m$, it is easy to see that the above isometry is indeed a complete isometry.

Remark 1.5. Let X be a Banach space. The operator space $\min(X)$ is given by the isometric embedding $X \subset C(K)$, where $K = (B_{X^*}, \sigma(X^*, X))$ is a compact space and $C(K)$ is the commutative C^*-algebra of all continuous complex functions on K. This operator space structure on X is called the minimal operator space structure, it is minimal in the sense that for any operator space $\widetilde{X}$ obtained by some isometric embedding $X \subset B(H)$, the identity map $Id_X : X \to X$ induce a complete contraction $Id_X : \widetilde{X} \to \min(X)$.

Note that if we have an isometric (resp. isomorphic) embedding $X \subset Y$ as Banach spaces, then we have completely isometrically (resp. isomorphically)

$$\min(X) \subset \min(Y).$$

There is also a maximal operator space structure on X. Let us recall its definition. Let I be the collection of all linear maps $u : X \to B(H_u)$ with $\|u\| \leq 1$. The operator space $\max(X)$ is given by the isometric embedding

$$\begin{array}{rcl} j : X & \to & \bigoplus_{u \in I} B(H_u) \subset B\Big(\bigoplus_{u \in I} H_u\Big) \\ x & \mapsto & \bigoplus_{u \in I} u(x) \end{array}.$$

For any Banach space X, we have completely isometrically

$$\min(X)^* = \max(X^*) \text{ and } \max(X)^* = \min(X^*),$$

where the dual in the operator space category will be explained below.

1.2.2 Minimal tensor product

Let $E \subset B(H)$ and $F \subset B(K)$ be two operator spaces. We have a natural embedding of the algebraic tensor product $E \otimes F$ into $B(H \otimes_2 K)$:

$$E \otimes F \subset B(H \otimes_2 K).$$

The minimal (or spatial) tensor product of E and F is defined as the completion of $E \otimes F$ with respect to the norm induced by $B(H \otimes_2 K)$. The resulting space will be denoted by $E \otimes_{min} F$. By definition, we have

$$E \otimes_{min} F \subset B(H \otimes_2 K),$$

thus $E \otimes_{min} F$ is equipped with the operator space structure via this embedding.

It is shown that the above definition for $E \otimes_{min} F$ does not depend on the particular embedding of E and F.

Some easy facts about the minimal tensor product are the following:

(i) Commutativity: We have completely isometrically

$$E \otimes_{min} F \simeq F \otimes_{min} E$$

via the natural flip map.

(ii) Injectivity: If $F_1 \subset E_1$ and $F_2 \subset E_2$ are operator spaces, then we have completely isometrically

$$F_1 \otimes_{min} F_2 \subset E_1 \otimes_{min} E_2.$$

1.2.3 Opposite

Let E be an operator space. The opposite of E, denoted by E^{op} is the same space E equipped with the operator space structure given by the matrix norms on $M_n(E)$ by the following: if $a = (a_{ij}) \in M_n(E)$, then

$$\|a\|_{M_n(E^{op})} := \|{}^t a\|_{M_n(E)},$$

where ${}^t a = (a_{ji})$ is the transpose of a.

1.2.4 Dual space, quotient space

Let E be an operator space. Let E^* be the Banach space dual of E. For any $n \geq 1$, we may equip $M_n(E^*)$ with a norm $\|\cdot\|_n$ such that we have isometrically

$$M_n(E^*) \simeq CB(E, M_n).$$

It can be easily checked that the so-defined sequence of matrix norms $\|\cdot\|_n$ on $M_n(E^*)$ verifies the axioms (R1) and (R2), thus it gives E^* an operator space structure. E^* equipped with this structure will be called the operator space dual of E. Unless otherwise stated, we will always equip E^* with this operator space dual structure.

It is shown (see. e.g. [BP91, ER91]) that if E is an operator space and let

$$E^{**} = (E^*)^*.$$

Then the canonical embedding $E \subset E^{**}$ is completely isometric.

It is well-known that the Banach space dual of $\mathscr{K} = \mathscr{K}(\ell_2)$ is the trace class S_1 by the duality

$$(a, b) = \sum_{i,j \geq 1} a_{ij} b_{ij}$$

for all $a \in \mathscr{K}, b \in S_1$. Thus we may equip with the trace class the standard operator space dual structure

$$S_1 = \mathscr{K}^*.$$

With the same duality defined as above, we have completely isometrically

$$B(\ell_2) = S_1^*.$$

Now let $F \subset E$ be operator spaces. The Banach space quotient E/F is defined as usual. By their operator space structures, $M_n(F) \subset M_n(E)$ are Banach spaces, thus we may equip $M_n(E/F)$ with a norm $\|\cdot\|_n$ by the algebraic isomorphism:

$$M_n(E/F) \simeq M_n(E)/M_n(F).$$

The obtained matrix norms satisfy the axioms (R1) and (R2), this gives an operator space structure on E/F.

1.2.5 Complex interpolation

Let E_0 and E_1 be two operator spaces such that (E_0, E_1) is a compatible couple of Banach spaces in the sense of interpolation. Let $0 < \theta < 1$. We can define the complex interpolation space (se. e. g. [BL76])

$$E_\theta = (E_0, E_1)_\theta.$$

For any $n \geq 1$, the couple $(M_n(E_0), M_n(E_1))$ is a compatible couple of Banach spaces, thus we may equip $M_n(E_\theta)$ with a norm $\|\cdot\|_n$ such that we have isometrically

$$M_n(E_\theta) \simeq (M_n(E_0), M_n(E_1))_\theta.$$

It can be checked that these matrix norms on $M_n(E_\theta)$ gives an abstract operator space structure on E_θ.

We introduce briefly an important example of operator space obtained by the complex interpolation method which will be used in the sequel. From now on, let us denote $\mathscr{K}$ by S_∞, which seems to be more appropriate in the following example.

Example 1.6. Let $(\Omega, \mathscr{F}, \nu)$ be a measure space. A classical result on complex interpolation is that

$$L_p(\nu) = (L_\infty(\nu), L_1(\nu))_{1/p}.$$

Since $L_\infty(\nu)$ is a C^*-algebra, it has a unique natural operator space structure. The standard operator space structure on $L_1(\nu)$ is the one such that its operator space dual is $L_\infty(\nu)$. Thus $L_p(\nu)$ may be equipped with an operator space structure by the above interpolation. The obtained structure on $L_p(\nu)$ will be called the standard operator space structure on $L_p(\nu)$.

Example 1.7. By the contractive inclusion $S_1 \subset S_\infty$, we see that (S_∞, S_1) is a compatible interpolation couple of Banach spaces. Let $1 < p < \infty$. It is well-known that the Schatten p-class can be obtained isometrically as

$$S_p = (S_\infty, S_1)_{1/p}.$$

Moreover, since S_∞ and S_1 are operator spaces, then S_p is equipped with an operator space structure by the interpolation method. This structure will be called the standard operator space structure on S_p.

Similarly, we can define the standard operator space structure on S_p^n. Obviously, we have completely isometrically $S_p^n \subset S_p^m \subset S_p$ if $n \leq m$.

1.2.6 Haagerup tensor product

Let E and F be two operator spaces. The Haagerup tensor norm $\|\cdot\|_h$ on the algebraic tensor product $E \otimes F$ is defined by: if $x \in E \otimes F$, then

$$\|x\|_h := \inf \left\{ \Big\| \sum_1^n a_i a_i^* \Big\| \cdot \Big\| \sum_1^n b_i^* b_i \Big\| : x = \sum_1^n a_i \otimes b_i, \text{ where } a_i \in E, b_i \in F, n \geq 1 \right\}.$$

Then the completion of $E \otimes F$ with respect to the norm $\|\cdot\|_h$ is denoted by $E \otimes_h F$.

More generally, for any $n \geq 1$, define $\|\cdot\|_{h,n}$ on $M_n(E \otimes F)$ by: if $x \in M_n(E \otimes F)$, then

$$\|x\|_{h,n} := \inf \left\{ \|a\|_{M_{n,m}(E)} \cdot \|b\|_{M_{m,n}(F)} : x = a \odot b, \text{ where } a \in M_{n,m}(E), b \in M_{m,n}(F), m \geq 1 \right\}.$$

We should explain the notation $a \odot b$ appears in the above definition. If

$$a = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq m} \in M_{n,m}(E) \text{ and } b = (b_{kl})_{1 \leq k \leq m, 1 \leq l \leq n} \in M_{m,n}(F),$$

then by definition,

$$a \odot b = (c_{pq})_{1 \leq p \leq n, 1 \leq q \leq n}, \quad \text{where } c_{pq} = \sum_{j=1}^m a_{pj} \otimes b_{jq}.$$

It is easy to check that $\|\cdot\|_h$ and $\|\cdot\|_{h,1}$ coincide on $E \otimes F$. It can also be checked that the norms $\|\cdot\|_{h,n}$ on $M_n(E \otimes F)$ verify the axioms (R1) and (R2). Moreover, it is clear that the completion of $M_n(E \otimes F)$ with respect to the norm $\|\cdot\|_{h,n}$ coincide with $M_n(E \otimes_h F)$. Thus the matrix norms $\|\cdot\|_{h,n}$ give an operator space structure on $E \otimes_h F$. This operator space will be called the Haagerup tensor product of E and F.

It is worthwhile to mention the noncommutativity of the Haagerup tensor product. More precisely, let E and F be any two operator spaces, then in general, we have

$$E \otimes_h F \not\simeq F \otimes_h E.$$

However, it is associative, i.e. if E_1, E_2, E_3 are operator spaces, then we have a natural complete isometric isomorphism:

$$(E_1 \otimes_h E_2) \otimes_h E_3 \simeq E_1 \otimes_h (E_2 \otimes_h E_3).$$

Thus we can define without ambiguity $E_1 \otimes_h E_2 \otimes_h \cdots \otimes_h E_n$ for any n-tuple of operator spaces $E_1, E_2, \cdots, E_n$.

It has be shown that the following feature never appears for tensor products between Banach spaces. The Haagerup tensor product is at the same time injective and projective.

(i) Injectivity: If $E_1 \subset E_2$ and $F_1 \subset F_2$, then we have complete isometric embedding

$$E_1 \otimes_h E_2 \subset F_1 \otimes_h F_2;$$

(ii) Projectivity: If $F_1 \subset E_1$ and $F_2 \subset E_2$, then the quotient maps $q_1 : E_1 \to E_1/F_1$ and $q_2 : E_2 \to E_2/F_2$ induce a quotient map

$$q_1 \otimes q_2 : E_1 \otimes_h E_2 \to (E_1/F_1) \otimes_h (E_2/F_2).$$

In other words, we have completely isometrically

$$(E_1/F_1) \otimes_h (E_2/F_2) \simeq (E_1 \otimes_h E_2)/\ker(q_1 \otimes q_2).$$

The Haagerup tensor product is self-dual, in the sense that if either E or F is finite dimensional, then we have completely isometrically

$$(E \otimes_h F)^* \simeq E^* \otimes_h F^*.$$

An important theorem of Kouba will be used frequently in the sequel of this chapter. In words, the theorem states that the functor of doing the complex interpolation and the functor of taking the Haagerup tensor product are commutative between them. We state the theorem below without proof. For its proof, see e.g.[Pis96].

Theorem 1.8 (Kouba)**.** Let (E_0, E_1) and (F_0, F_1) be two compatible interpolation couples of operator spaces. Then $(E_0 \otimes_h F_0, E_1 \otimes_h F_1)$ is a compatible interpolation couple. Moreover, for all $0 < \theta < 1$ we have completely isometrically

$$(E_0 \otimes_h F_0, E_1 \otimes_h F_1)_\theta = (E_0, E_1)_\theta \otimes_h (F_0, F_1)_\theta.$$

1.2.7 Projective tensor product

We have an identification (as C^*-algebras) between $M_l \otimes M_m$ and M_{lm} such that if

$$x = (x_{ij})_{i,j \leq l} \in M_l \text{ and } y \in (y_{pq})_{p,q \leq m} \in M_m,$$

then

$$x \otimes y = z = (z_{i,p;jq})_{i,j \leq l;p,q \leq m} \in M_{lm}, \text{ with } z_{i,p;j,q} = x_{ij} y_{pq}.$$

By this identification, $x \otimes y$ will be treated as a elementary tensor of $x \in M_l$ and $y \in M_m$ or it will be treated directly as a matrix in M_{lm} such that $(x \otimes y)_{i,p;j,q} = x_{ij} y_{pq}$.

More generally, if E and F are two operator spaces, let $x \in M_l(E)$ and $y \in M_m(F)$ then we define

$$x \otimes y = (z_{i,p;j,q})_{i,j \leq l;p,q \leq m} \in M_{lm}(E \otimes F)$$

by

$$(x \otimes y)_{i,p;j,q} = z_{i,p;j,q} = x_{ij} \otimes y_{pq}.$$

With the above notation, we can define a norm $\|\cdot\|_{\wedge,n}$ on $M_n(E \otimes F)$ by:

$$\|t\|_{\wedge,n} = \inf \left\{ \|\alpha\|_{M_{n,lm}} \|x\|_{M_l(E)} \|y\|_{M_m(F)} \|\beta\|_{M_{lm,n}} : t = \alpha \cdot (x \otimes y) \cdot \beta, \text{ where } l, m \geq 1 \right\}.$$

The matrix norms $\|\cdot\|_{\wedge,n}$ satisfy the axioms (R1) and (R2), thus it gives an operator space structure on the completion of $E \otimes F$ with respect to the norm $\|\cdot\|_\wedge = \|\cdot\|_{\wedge,1}$. This operator space will be denoted by $E \otimes^\wedge F$.

The connection of the projective tensor product and the minimal tensor product is given by the fact that we have completely isometrically

$$(E \otimes^\wedge F)^* = CB(E, F^*).$$

The projective tensor product is projective, i.e. if $F_1 \subset E_1$ and $F_2 \subset E_2$ are operator spaces, we have completely isometrically

$$(E_1/F_1) \otimes^\wedge (E_2/F_2) \simeq (E_1 \otimes^\wedge E_2)/\ker(q_1 \otimes q_2),$$

where $q_1 : E_1 \to E_1/F_1$ and $q_2 : E_2 \to E_2/F_2$ are the canonical quotient maps.

Remark 1.9. In the Banach space theory, the projective tensor product between Banach spaces is defined. In litterature, for two Banach spaces X and Y, their projective tensor product is denoted by $X \otimes_\pi Y$ or $X \hat{\otimes} Y$. As is indicated by the difference of notation, usually, the behind Banach space norm on the projective tensor product of two operator spaces E and F does not coincide with their Banach projective norm. That is, in general, we do not have isometrically $E \otimes^\wedge F = E \hat{\otimes} F$.

1.2.8 Some basic operator spaces

We should introduce the basic examples of operator spaces as the so-called "Column Hilbert Space C", the "Row Hilbert space R" and the corresponding column and row spaces in the Schatten p-classes, etc. All the facts mentioned are well-known for specialists, we omit their proofs.

Example 1.10. Let e_{ij} be the element of $B(\ell_2)$ corresponding to the matrix whose coefficients equal to one at the (i, j) entry and zero elsewhere. The column Hilbert space C is defined as

$$C = \overline{\text{span}}\{e_{i1}|i \geq 1\}$$

and the row Hilbert space R is defined as

$$R = \overline{\text{span}}\{e_{1j}|j \geq 1\}.$$

Their operator space structures are given by the embeddings $C \subset B(\ell_2)$ and $R \subset B(\ell_2)$.

For convenience, sometimes we will write $C_\infty = C$ and $R_\infty = R$. It can be shown that we have completely isometrically

$$C \simeq R^* \quad \text{and} \quad R \simeq C^*.$$

Example 1.11. In general, for $1 \leq p < \infty$, we define

$$C_p = \overline{\text{span}}\{e_{i1}|i \geq 1\} \subset S_p \ \text{ and } \ R_p = \overline{\text{span}}\{e_{1j}|j \geq 1\} \subset S_p.$$

We have

$$C_1 \simeq R = R_\infty$$

and

$$R_1 \simeq C = C_\infty.$$

More generally, if $1 \leq p \leq \infty$ and let p' be the conjugate exponent, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$C_p \simeq R_{p'}.$$

Concerning the dual spaces, we have completely isometrically

$$C_p^* \simeq C_{p'} \simeq R_p.$$

Moreover, the family $\{C_p : 1 \leq p \leq \infty\}$ forms an interpolation scale: if $1 \leq p_0, p_1 \leq \infty$ and

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

for $0 < \theta < 1$, then we have completely isometrically

$$C_{p_\theta} = (C_{p_0}, C_{p_1})_\theta.$$

It is well-known that we have completely isometrically

(i)
$$C \otimes_h E \otimes_h R = C \otimes_{min} E \otimes_{min} R = \mathscr{K} \otimes_{min} E;$$

(ii)
$$R \otimes_h E \otimes_h C = R \otimes^\wedge E \otimes^\wedge C = S_1 \otimes^\wedge E;$$

(iv)
$$C_p \otimes_h R_p = S_p.$$

Remark 1.12. For any interger $n \geq 1$, the n-dimensional analogue of C, R, C_p, R_p will be denoted by $C_\infty^n, R_\infty^n, C_p^n, R_p^n$ respectively.

1.2.9 Noncommutative Vector-valued L_p-spaces

In his monograph [Pis98], Pisier developed a theory of noncommutative vector-valued L_p-spaces. Let us first recall the noncommutative L_p spaces defined with a trace, i.e. the noncommutative L_p spaces in the scalar case.

Let $\mathscr{M}$ be a von Neumann algebra equipped with a semifinite normal faithful trace τ. Let $1 \leq p < \infty$. The map

$$x \mapsto \tau(|x|^p)^{1/p}$$

defines a norm on the following subset of $\mathscr{M}$:

$$\{x \in \mathscr{M} : \tau(|x|^p) < \infty\}.$$

Its completion with respect to this norm will be denoted by $L_p(\tau)$, this norm will be denoted by $\|\cdot\|_p$, i.e. we have $\|x\|_p = \tau(|x|^p)^{1/p}$.

It is well-known that $L_1(\tau)$ is the unique predual of $\mathscr{M}$. More precisely, we have

$$L_1(\tau)^* = \mathscr{M},$$

with respect to the duality given by

$$(x, y) = \tau(xy)$$

for all $x \in L_1(\tau), y \in \mathscr{M}$. By this duality, $L_1(\tau)$ is equipped with an operator space structure which will be considered as the standard one. Moreover, $(\mathscr{M}, L_1(\tau))$ is a natural compatible interpolation couple, for all $1 < p < \infty$, we have isometrically

$$L_p(\tau) = (\mathscr{M}, L_1(\tau))_{1/p}.$$

We will equip $L_p(\tau)$ with the operator space structure by

$$L_p(\tau) = (\mathscr{M}, L_1(\tau)^{op})_{1/p}.$$

For the vector-valued case, as done in [Pis98], the von Neumann algebra is supposed to be injective. Thus given $\mathscr{M}$ an injective von Neumann algebra equipped with a semifinite normal faithful trace τ and given an operator space E, we define

$$L^0_\infty(\tau; E) := \mathscr{M} \otimes_{min} E$$

and

$$L_1(\tau; E) := L_1(\tau)^{op} \otimes^\wedge E.$$

It is shown that

$$\Big(L^0_\infty(\tau; E),\ L_1(\tau; E)\Big) = \Big(\mathscr{M} \otimes_{min} E,\ L_1(\tau)^{op} \otimes^\wedge E\Big)$$

is a compatible interpolation couple. For $1 < p < \infty$, we define

$$L_p(\tau; E) := \Big(L^0_\infty(\tau; E),\ L_1(\tau; E)\Big)_{1/p}.$$

We describe briefly the norm in $L_p(\tau; E)$ without using the abstract interpolation method. If $1 \leq p < \infty$, then for an element $x \in L_p(\tau) \otimes E$, its norm in $L_p(\tau; E)$ is given by the following formula:

$$\|x\|_{L_p(\tau;E)} = \inf\Big\{\|a\|_{L_{2p}(\tau)}\|y\|_{M\otimes_{min}E}\|b\|_{L_{2p}(\tau)}\Big\},$$

where the infimum runs over all possible decomposition : $x = a \cdot y \cdot b$ with $a, b \in L_{2p}(\tau)$ and $y \in M \otimes_{min} E$.

Remark 1.13. Unlike the Bochner space $L_p(\mu; B)$ defined for a measure space $(\Omega, \mathscr{F}, \mu)$ and a Banach space B, when defining the noncommutative vector-valued L_p-spaces, the space used as the "vector value" has to admit an operator space structure. And the resulting space is again an operator space.

In particular, we can take the von Neumann algebra to be $B(\ell_2)$ equipped with the usual trace Tr. In §1.2.4, we have define the operator space structure on S_1 to be the operator dual of the C^*-algebra $\mathscr{K}$. Recall that in its definition, we used the duality between S_1 and $S_\infty = \mathscr{K}$ given by

$$(a, b) := \sum_{i,j\geq 1} a_{ij} b_{ij} = \mathrm{Tr}({}^t ab)$$

for any $a \in S_1$ and $b \in S_\infty$. We will always equip S_1 with this operator space structure (indeed, it is the "opposite" of the one defined using the procedure for a general von Neumann algebra, i.e. using the duality $(a, b) := \mathrm{Tr}(ab)$) which seems to be more appropriate for us. Thus we have

$$L_1(B(\ell_2), \mathrm{Tr}; E) = S_1 \otimes^\wedge E.$$

By using some density argument for interpolation method, we have for any $0 < \theta < 1$,

$$(B(\ell_2) \otimes_{min} E, S_1 \otimes^\wedge E)_\theta = (\mathscr{K} \otimes_{min} E, S_1 \otimes^\wedge E)_\theta.$$

We will use the notation

$$S_1[E] := S_1 \otimes^\wedge E$$

and

$$S_\infty[E] := \mathscr{K} \otimes_{min} E.$$

For all $1 < p < \infty$, let us denote

$$S_p[E] := L_p(B(\ell_2), \mathrm{Tr}; E).$$

Then we have

$$S_p[E] = \Big(S_\infty[E],\ S_1[E]\Big)_{1/p}.$$

Remark 1.14. Consider the von Neumann algebra M_n equipped with its usual trace Tr. Let E be an operator space. For any $1 \leq p \leq \infty$, we denote

$$S_p^n[E] := L_p(M_n, \mathrm{Tr}; E).$$

We have a natural identification of vector spaces:

$$S_p^n[E] = S_p^n \otimes E.$$

If $n \leq m$, then the canonical embedding $S_p^n \subset S_p^m$ induces a completely isometric embedding

$$S_p^n[E] \subset S_p^m[E].$$

It can be shown that the subset corresponding to the algebraic tensor product $S_p \otimes E$ is dense in $S_p[E]$. Hence $\bigcup_{n\geq 1} S_p^n[E]$ is a dense subset in $S_p[E]$.

The following proposition from [Pis98] is very useful for us, the proof is easy, we repeat its proof. .

Proposition 1.15. Let E be an operator space. For any $n \geq 1$, we have

$$S_p^n[E] = C_p^n \otimes_h E \otimes_h R_p^n$$

and

$$S_p[E] = C_p \otimes_h E \otimes_h R_p.$$

Proof. We only show the second identity, the first one then follows easily. If $p = \infty$, then the statement follows from the known identities:

$$S_\infty[E] = \mathscr{K} \otimes_{min} E = C \otimes_h E \otimes_h R = C_\infty \otimes_h E \otimes_h R_\infty.$$

For $p = 1$, note that $C_1 = R_\infty = R$ and $R_1 = C_\infty = C$, hence

$$S_1[E] = S_1 \otimes^\wedge E = R \otimes_h E \otimes_h C = C_1 \otimes_h E \otimes_h R_1.$$

For $1 < \ p < \ \infty$, we have

$$S_p[E] = (S_\infty[E], S_1[E])_{1/p} = (C_\infty \otimes_h E \otimes_h R_\infty, C_1 \otimes_h E \otimes_h R_1)_{1/p}.$$

Now we can apply Kouba's interpolation theorem, by the known identies $(C_\infty, C_1)_{1/p} = C_p$ and $(R_\infty, R_1)_{1/p} = R_p$, we have

$$S_p[E] = C_p \otimes_h E \otimes_h R_p.$$

□

The following results are very useful in applications, for their proofs, see [Pis98].

Theorem 1.16. Let E and F be two operator spaces and $u : E \to F$ be a linear map. For any $1 \leq p \leq \infty$, we have

$$\|u\|_{cb} = \sup_{n \geq 1} \big\|Id_{S_p^n} \otimes u : S_p^n[E] \to S_p^n[F]\big\| = \big\|Id_{S_p} \otimes u : S_p[E] \to S_p[F]\big\|.$$

Theorem 1.17. Let $1 < p \leq \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then we have completely isometrically

$$S_p[E]^* = S_{p'}[E^*].$$

The following theorem will be referred to as the noncommutative Fubini theorem.

Theorem 1.18. Let $(\mathscr{M}, \tau)$ and $(\mathscr{N}, \phi)$ be two injective von Neumann algebras equipped with semifinite normal faithful traces τ and ϕ respectively, and let $1 \leq p < \infty$. Then we have the following natural complete isometric isomorphisms:

$$L_p(\mathscr{M}, \tau; L_p(\mathscr{N}, \phi)) \simeq L_p(\mathscr{N}, \phi; L_p(\mathscr{M}, \tau)) \simeq L_p(\mathscr{M} \overline{\otimes} \mathscr{N}, \tau \otimes \phi),$$

where $\mathscr{M} \overline{\otimes} \mathscr{N}$ is the von Neumann tensor product of $\mathscr{M}$ and $\mathscr{N}$.

1.3 OUMD property for operator spaces

In this section, we will recall the noncommutative martingale theory and the definition of OUMD property for operator spaces.

1.3.1 Noncommutative martingales

Let $\mathscr{M}$ be a finite von Neumann algebra equipped with a normal faithful τ which is normalised in the sense that $\tau(1) = 1$. An increasing sequence $(\mathscr{M}_n)_{n\geq 0}$ of von Neumann subalgebras of $\mathscr{M}$ will be called a filtration. Here, by increasing sequence, we mean that for any $n \geq 0$,

$$\mathscr{M}_n \subset \mathscr{M}_{n+1}.$$

We will denote by $\mathscr{M}_\infty$ the closure of the space $\bigcup_{n\geq 0} \mathscr{M}_n$ in $\mathscr{M}$ with respect to the weak* topology $\sigma(\mathscr{M}, \mathscr{M}_*)$.

It is well-known from [Tak01] that for any $n \geq 0$, their exists a unique norm 1 positive projection from $\mathscr{M}$ onto $\mathscr{M}_n$. This projection will be denoted by

$$\mathbb{E}^{\mathscr{M}_n} : \mathscr{M} \to \mathscr{M}_n.$$

$\mathbb{E}^{\mathscr{M}_n}$ will be called the conditional expectation with respect to $\mathscr{M}_n$, it is characterised by the following conditional expectation properties:

$$\mathbb{E}^{\mathscr{M}_n}(1) = 1,$$

$$\mathbb{E}^{\mathscr{M}_n}(a \cdot x \cdot b) = a \cdot \mathbb{E}^{\mathscr{M}_n}(x) \cdot b, \forall a, b \in \mathscr{M}_n, \forall x \in \mathscr{M}.$$

When the filtration $(\mathscr{M}_n)_{n\geq 0}$ is fixed, then we will denote $\mathbb{E}^{\mathscr{M}_n}$ by $\mathbb{E}_n$ for simplicity.

In the above situation, let $1 \leq p < \infty$, then by the definition of $L_p(\mathscr{M}, \tau)$, it is clear that $\mathscr{M}$ is a dense subset of $L_p(\mathscr{M}, \tau)$. Clearly, we have (completely) isometrically:

$$L_p(\mathscr{M}_n, \tau) \subset L_p(\mathscr{M}, \tau).$$

It can be shown that for all $x \in \mathscr{M}$, we have

$$\|\mathbb{E}_n(x)\|_p \leq \|x\|_p.$$

Hence

$$\mathbb{E}_n : \mathscr{M} \to \mathscr{M}_n$$

can be uniquely extended to be a contractive projection from $L_p(\mathscr{M}, \tau)$ onto $L_p(\mathscr{M}_n, \tau)$. This projection will again be called the conditional expectation and will be denoted by

$$\mathbb{E}_n : L_p(\mathscr{M}, \tau) \to L_p(\mathscr{M}_n, \tau).$$

Remark 1.19. It is easy to check that the conditional expectation

$$\mathbb{E}_n : L_2(\mathscr{M}, \tau) \to L_2(\mathscr{M}_n, \tau)$$

is just the orthogonal projection from the Hilbert space $L_2(\mathscr{M}, \tau)$ onto its closed subspace $L_2(\mathscr{M}_n, \tau)$.

Definition 1.20. Let $1 \leq p < \infty$. A sequence of elements $(x_n)_{n\geq 0}$ in $L_p(\mathscr{M}, \tau)$ is said to be adapted to the filtration $(\mathscr{M}_n)_{n\geq 0}$, if $x_n \in L_p(\mathscr{M}_n, \tau)$ for any $n \geq 0$. An adapted sequence in $L_p(\mathscr{M}, \tau)$ is called a martingale if moreover, it satisfy the martingale condition:

$$\forall n \geq 0,\ x_n = \mathbb{E}_n(x_{n+1}).$$

Let $x = (x_n)_{n\geq 1}$ be a martingale in $L_p(\mathscr{M}, \tau)$, if

$$\|x\|_p := \sup_n \|x_n\|_p < \infty,$$

then we say that x is a L_p-bounded martingale.

The conditional expectation

$$\mathbb{E}_n : L_p(\mathscr{M}) \to L_p(\mathscr{M}_n)$$

is in fact completely contractive. Indeed, if we denote $\tau|_{\mathscr{M}_n}$ by τ_n. For any $m \geq 1$, by replacing $(\mathscr{M}, \tau)$ and $(\mathscr{M}_n, \tau_n)$ by $(M_m \otimes \mathscr{M}, \mathrm{Tr} \otimes \tau)$ and $(M_m \otimes \mathscr{M}_n, \mathrm{Tr} \otimes \tau_n)$ respectively, we see that the conditional expectation

$$\mathbb{E}^{M_m \otimes \mathscr{M}_n} : L_p(M_m \otimes \mathscr{M}, \mathrm{Tr} \otimes \tau) \to L_p(M_m \otimes \mathscr{M}_n, \mathrm{Tr} \otimes \tau)$$

is contractive. By the noncommutative Fubini theorem, this is equivalent to say that the following map

$$Id_{S_p^m} \otimes \mathbb{E}_n : S_p^m[L_p(\mathscr{M}, \tau)] \to S_p^m[L_p(\mathscr{M}_n, \tau_n)]$$

is contractive. Hence we have

$$\|\mathbb{E}_n : L_p(\mathscr{M} \to L_p(\mathscr{M}_n)\|_{cb} = \sup_{m \geq 1} \left\| Id_{S_p^m} \otimes \mathbb{E}_n : S_p^m[L_p(\mathscr{M}, \tau)] \to S_p^m[L_p(\mathscr{M}_n, \tau_n)] \right\| = 1.$$

If in additional, we assume that $\mathscr{M}$ in injective, then $\mathscr{M}_n$ is also injective, this follows from the injectivity of $\mathscr{M}$ and the fact that $\mathbb{E}_n : \mathscr{M} \to \mathscr{M}_n$ is completely contractive.

Now let E be any operator space and let $1 \leq p < \infty$. It is easy to show that

$$\mathbb{E}_n \otimes Id_E : L_p(\mathscr{M}, \tau) \otimes E \to L_p(\mathscr{M}_n, \tau) \otimes E$$

extends to a completely contractive projection from $L_p(\mathscr{M}, \tau; E)$ onto $L_p(\mathscr{M}_n, \tau; E)$, the resulting projection will still be called the conditional expectation and be denoted by $\mathbb{E}_n$, that is we have a completely contractive projection

$$\mathbb{E}_n : L_p(\mathscr{M}, \tau; E) \to L_p(\mathscr{M}_n, \tau; E).$$

Definition 1.21. In the above situation, a sequence $x = (x_n)_{n \geq 0}$ in $L_p(\mathscr{M}, \tau; E)$ will be called an E-valued martingale if

$$\forall n \geq 0, \;\; x_n = \mathbb{E}_n(x_{n+1}).$$

If

$$\|x\|_p := \sup_{n \geq 0} \|x_n\|_{L_p(\mathscr{M}, \tau; E)} < \infty,$$

we say that x is a bounded martingale in $L_p(\mathscr{M}, \tau; E)$.

1.3.2 OUMD property

After introducint the operator space-valued martingale, we can then introduce the OUMD$_p$ property after Pisier.

Definition 1.22. Let $1 < p < \infty$. An operator space E is OUMD$_p$, if there exists a universal constant c_p such that for any interger $n \geq 1$, any choice of signs $\varepsilon = (\varepsilon_k)_{k=0}^n$ such that $\varepsilon_k \in \{-1, 1\}$, and any E-valued noncommutative martingale $(x_k)_{k=0}^n$ in $L_p(\mathscr{M}, \tau; E)$ associated to any increasing filtration $(\mathscr{M}_k)_{k=0}^n$, we have

$$\| \sum_{k=0}^n \varepsilon_k dx_k \|_{L_p(\mathscr{M}, \tau; E)} \leq c_p \|x_n\|_{L_p(\mathscr{M}, \tau; E)},$$

where as usual, $dx_k := x_k - x_{k-1}$ and $x_1 := 0$. The best such constant c_p will be denoted by $\beta_p^{os}(E)$ and will be called the OUMD$_p$ constant of E. By convention, we will define $\beta_p^{os}(E) := \infty$ if E is not OUMD$_p$.

Remark 1.23. In the situation of the above definition, we can assume that $\mathscr{M}_n = \mathscr{M}$. Any $\varepsilon \in \{-1,1\}^n$ is associated with the operator $T_\varepsilon : L_p(\mathscr{M},\tau;E) \to L_p(\mathscr{M},\tau;E)$ which sends x to $\sum_0^n \varepsilon_k(x_k - x_{k-1})$, and we have

$$\beta_p^{os}(E) = \sup \|T_\varepsilon\|,$$

where the supremum runs over the set of all hyperfinite von Neumann algebras $(\mathscr{M},\tau)$, and all $n \geq 1$, all $\varepsilon \in \{-1,1\}^n$.

It is easy to deduce from the definition that if an operator space E is OUMD_p, then its operator dual space E^* is $\mathrm{OUMD}_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Let E be an operator space which is OUMD_p, for any given injective von Neumann algebra $(\mathscr{M},\tau)$, it is easy to see that the operator space $L_p(\mathscr{M},\tau;E)$ is again OUMD_p. It follows that the space $L_p(\mathscr{M},\tau;E)$, when considered as a Banach space, is UMD. In particular, $S_p[E]$ is UMD.

Remark 1.24. In the Banach space theory, it is well-known that UMD_p and UMD_q are equivalent properties for any $1 < p, q < \infty$. However, in the operator space theory, it remains open at this moment whether this still holds for the OUMD_p and OUMD_q properties.

An interesting example of operator space structure on ℓ_2 which does not provide an OUMD_p operator space for any $1 < p < \infty$ is given in the following remark.

Remark 1.25. Consider the operator space

$$\min(\ell_2).$$

Let $(\mathscr{R},\tau)$ be the hyperfinite II_1 factor, i.e. $(\mathscr{R},\tau)$ is the the infinite tensor product (in the von Neumann sense):

$$(\mathscr{R},\tau) = \bigotimes_{k=1}^{\infty}(M_2, \mathrm{tr}_2),$$

where (M_2, tr_2) is the algebra of 2×2 complex matrices equipped with the normalised trace

$$\mathrm{tr}_2 = \frac{1}{2}\mathrm{Tr}.$$

It is proven in [Pis98] that for all $1 \leq p < \infty$, the space $L_p(\mathscr{R},\tau;\min(\ell_2))$ contains a subspace which considered as a Banach space is isomorphic to c_0. This result implies that $L_p(\mathscr{R},\tau;\min(\ell_2))$ is not UMD and hence the operator space $\min(\ell_2)$ is not OUMD_p for any $1 < p < \infty$.

Recall the famous Dvoretzky Theorem: Every infinite-dimensional Banach space X contains ℓ_2^n uniformly with distortion $\leq 1+\varepsilon$ for each $\varepsilon > 0$. More precisely, for every $\varepsilon > 0$, there exists a sequence of subspace $X_n \subset X$ such that $\dim X_n = n$ and the Banach-Mazur distance

$$d_{BM}(X_n, \ell_2^n) := \inf\left\{\|u\| \cdot \|u^{-1}\| \,\middle|\, u : X_n \to \ell_2 \text{ is an isomorphism }\right\} \leq 1+\varepsilon.$$

It follows that $\min(X)$ contains $\min(\ell_2^n)$ uniformly with distortion $\leq 1+\varepsilon$ for each $\varepsilon > 0$, hence the Banach space $L_p(\mathscr{R},\tau;\min(X))$ contains uniformly ℓ_∞^n for every $1 < p < \infty$. This implies that the operator spaces $\min(X)$ is never OUMD_p for any $1 < p < \infty$. The same statement holds for $\max(X)$.

A question related to this remark will be given in the end of this chapter.

The first remarkable result concerning the OUMD property was proved in [PX96] and [PX97] by Pisier and Xu. The following theorem is an immediate consequence by their noncommutative analogue of the classical Burkholder-Gundy inequalities from martingale theory.

Theorem 1.26. (Pisier-Xu) Let $(\mathscr{M}, \tau)$ be a von Neumann algebra equipped with a normalized faithful trace τ. Then for all $1 < p < \infty$, the operator space $L_p(\mathscr{M}, \tau)$ is OUMD_p.

Later, Musat continued to study the OUMD property for noncommutative L_p-spaces. In [Mus06] she presented a proof of the following theorem:

Theorem 1.27. (Musat) Suppose that $\mathscr{M}$ is a QWEP von Neumann algebra equipped with a normal faithful tracial state τ. Then for any $1 < p, q < \infty$, the operator space $L_q(\mathscr{M}, \tau)$ is OUMD_p.

In particular, if $1 < p, q < \infty$. Then the Schatten q-class S_q is OUMD_p.

Remark 1.28. Very recently, a gap in the proof of the above result was found by Javier Parcet. Due to this fact, we will treat carefully in the sequel where we assume that the above theorem holds.

The following question is due to Z.-J. Ruan.

Question 1.29. Does the column Hilbert space C has OUMD_p for some or all $1 < p < \infty$?

Let us mention the following unpublished result of Musat.

Theorem 1.30 (Musat)**.** Let $1 < p < \infty$, and E be an operator space. Then E is OUMD_p if and only if $S_p[E]$ is UMD as a Banach space.

Note that one direction in the above theorem is quite easy: if E is OUMD_p, then $S_p[E]$ is UMD as a Banach space.

By this result, Question 1.29 is equivalent to the following

Question 1.31. Let $1 < p < \infty$. Does the Banach space $S_p[C]$ have UMD property?

1.4 The Banach space $S_p[C]$

In this section, the main result is the following theorem.

Theorem 1.32. Let $1 < p < \infty$. Then there exists $1 < u, v < \infty$, such that we have an isometric embedding

$$S_p[C] \hookrightarrow S_u[S_v].$$

Let us begin by a simple proposition.

Proposition 1.33. Let $1 \leq u, v \leq \infty$. Then $C_u \otimes_h C_v$ is isometric to a Schatten p-class S_p for certain $1 \leq p \leq \infty$. In particular, either $1 \leq u, v < \infty$ or $1 < u, v \leq \infty$, the space $C_u \otimes_h C_v$ is a UMD Banach space.

Proof. Define $\theta = \frac{1}{v}, \eta = \frac{1}{u} \in [0, 1]$, then we have

$$\frac{1}{v} = \frac{1-\theta}{\infty} + \frac{\theta}{1},$$

$$\frac{1}{u} = \frac{1-\eta}{\infty} + \frac{\eta}{1}.$$

Then by Kouba's interpolation result,

$$C_u \otimes_h C_\infty = (C_\infty \otimes_h C_\infty, C_1 \otimes_h C_\infty)_\eta.$$

By applying the isometric identities

$$C_\infty \otimes_h C_\infty = S_2, C_1 \otimes_h C_\infty = S_1,$$

we have

$$C_u \otimes_h C_\infty = (S_2, S_1)_\eta = S_{\frac{2}{1+\eta}}.$$

Similarly, we have isometric identity

$$C_u \otimes_h C_1 = S_{\frac{2}{\eta}}.$$

Finally, we obtain that

$$C_u \otimes_h C_v = (C_u \otimes_h C_\infty, C_u \otimes_h C_1)_\theta = (S_{\frac{2}{1+\eta}}, S_{\frac{2}{\eta}})_\theta = S_{\frac{2}{1-1/v+1/u}}.$$

The second assertion now follows easily. □

The following simple observation will be useful for us.

Remark 1.34. We have complete isometries

$$C \otimes_h C = C \otimes_{min} C \simeq C, \quad R \otimes_h R = R \otimes_{min} R \simeq R.$$

An application of Kouba's interpolation result yields complete isometry

$$C_p \otimes_h C_p \simeq C_p \tag{1.2}$$

for all $1 \leq p \leq \infty$.

More generally, for any integer $n \geq 1$ we have the following complete isometry

$$\underbrace{C_p \otimes_h C_p \otimes_h \cdots \otimes_h C_p}_{n \text{ times}} \simeq C_p.$$

In particular, we have the following isometry (in the Banach space category)

$$\underbrace{C_1 \otimes_h C_1 \otimes_h \cdots \otimes_h C_1}_{n \text{ times}} \simeq \underbrace{C_\infty \otimes_h C_\infty \otimes_h \cdots \otimes_h C_\infty}_{n \text{ times}} \simeq \ell_2.$$

Proof of Theorem 1.32. Let us first assume that $1 < p \leq 2$. Define $\theta = \frac{1+1/p}{2} \in (0,1)$, then $q = \theta p = \frac{p+1}{2} \in (1,\infty)$ and $r = \theta p' = \frac{p+1}{2(p-1)} \in (1,\infty)$. That is

$$\frac{1}{p} = \frac{1-\theta}{\infty} + \frac{\theta}{q},$$

$$\frac{1}{p'} = \frac{1-\theta}{\infty} + \frac{\theta}{r}.$$

By Proposition 1.15 and Kouba's interpolation result,

$$\begin{aligned}
S_p[C] &= C_p \otimes_h C_\infty \otimes_h C_{p'} \\
&= (C_\infty \otimes_h C_\infty \otimes_h C_\infty, C_q \otimes_h C_\infty \otimes_h C_r)_\theta \\
\text{By Remark 1.34} \quad &\overset{isometric}{=} (C_1 \otimes_h C_1 \otimes_h C_1, C_q \otimes_h C_\infty \otimes_h C_r)_\theta \\
&= C_{\frac{2p}{p+1}} \otimes_h C_{\frac{2p}{p-1}} \otimes_h C_{\frac{2p}{3p-3}} \\
&= C_{\frac{2p}{p+1}} \otimes_h R_{\frac{2p}{p+1}} \otimes_h R_{\frac{2p}{3-p}} \\
&= S_{\frac{2p}{p+1}} \otimes_h R_{\frac{2p}{3-p}}.
\end{aligned}$$

Hence we get the desired isometric embedding

$$S_p[C] \hookrightarrow C_{\frac{2p}{3-p}} \otimes_h S_{\frac{2p}{p+1}} \otimes_h R_{\frac{2p}{3-p}} = S_{\frac{2p}{3-p}}[S_{\frac{2p}{p+1}}].$$

Similar argument shows that if $1 < p \leq 2$, then isometrically, we have

$$S_{p'}[R] = R_p \otimes_h R_\infty \otimes_h R_{p'} = R_{\frac{2p}{p+1}} \otimes_h R_{\frac{2p}{p-1}} \otimes_h R_{\frac{2p}{3p-3}} = S_{\frac{2p}{p-1}} \otimes_h R_{\frac{2p}{3p-3}} \hookrightarrow S_{\frac{2p}{3p-3}}[S_{\frac{2p}{p-1}}].$$

By taking the opposite operator space, we have

$$(S_{p'}[R])^{op} = (R_p \otimes_h R_\infty \otimes_h R_{p'})^{op} = R_{p'}^{op} \otimes_h R_\infty^{op} \otimes_h R_p^{op} = C_{p'} \otimes_h C_\infty \otimes_h C_p = S_{p'}[C].$$

It follows that the Banach space $S_{p'}[C]$ embeds isometrically in $S_{\frac{2p}{3p-3}}[S_{\frac{2p}{p-1}}]$ whenever $1 < p \leq 2$. In other words, if $2 \leq p < \infty$, then

$$S_p[C] \hookrightarrow S_{2p/3}[S_{2p}].$$

□

Corollary 1.35. If the statement of Theorem 1.27 holds, then $S_p[C]$ is a UMD space.

Lemma 1.36. If $1 \leq p_1, p_2, \cdots, p_n < \infty$ or $1 < p_1, p_2, \cdots, p_n \leq \infty$. Then there exist $1 < q_1, q_2, \cdots, q_n < \infty$, such that we have

$$C_{p_1} \otimes_h C_{p_2} \otimes_h \cdots \otimes_h C_{p_n} \overset{isometric}{=} C_{q_1} \otimes_h C_{q_2} \otimes_h \cdots \otimes_h C_{q_n}.$$

Moreover, $C_{p_1} \otimes_h C_{p_2} \otimes_h \cdots \otimes_h C_{p_n}$ is a super-reflexive Banach space.

Proof. Assume first that $1 < p_1, p_2, \cdots, p_n \leq \infty$, then by choosing $\theta \in (0,1)$ such that $\theta > \max(1/p_1, 1/p_2, \cdots, 1/p_n)$, we can define $\widetilde{p}_1, \widetilde{p}_2, \cdots, \widetilde{p}_n \in (1, \infty]$ by $\widetilde{p}_1 = \theta p_1, \widetilde{p}_2 = \theta p_2, \cdots, \widetilde{p}_n = \theta p_n$ such that for $k = 1, 2, \cdots, n$,

$$\frac{1}{p_k} = \frac{1-\theta}{\infty} + \frac{\theta}{\widetilde{p}_k}.$$

It follows that

$$\begin{aligned}
& C_{p_1} \otimes_h C_{p_2} \otimes_h \cdots \otimes_h C_{p_n} \\
= \; & (C_\infty \otimes_h C_\infty \otimes_h \cdots \otimes_h C_\infty, C_{\widetilde{p}_1} \otimes_h C_{\widetilde{p}_2} \otimes_h \cdots \otimes_h C_{\widetilde{p}_n})_\theta \\
\overset{isometric}{=} \; & (C_1 \otimes_h C_1 \otimes_h \cdots \otimes_h C_1, C_{\widetilde{p}_1} \otimes_h C_{\widetilde{p}_2} \otimes_h \cdots \otimes_h C_{\widetilde{p}_n})_\theta \\
= \; & C_{q_1} \otimes_h C_{q_2} \otimes_h \cdots \otimes_h C_{q_n},
\end{aligned}$$

where $\frac{1}{q_k} = \frac{1-\theta}{1} + \frac{\theta}{\widetilde{p}_k} \in (0,1)$, and hence $1 < q_k < \infty$ for all $1 \leq k \leq n$.

Super-reflexivity of $C_{p_1} \otimes_h C_{p_2} \otimes_h \cdots \otimes_h C_{p_n}$ follows from the above first identity, which shows that it is a $(1-\theta)$-Hilbertian space.

The case when $1 < p_1, p_2, \cdots, p_n \leq \infty$ can be treated similarly or can be obtained by duality. □

The following proposition shows that the obtained embedding in Theorem 1.32 can not be completely isomorphic.

Proposition 1.37. Let $1 < p_1, p_2, \cdots, p_n < \infty$. The operator space C can never be embedded completely isomorphically into $S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]$. More generally, C can never be embedded completely isomorphically into any quotient of $S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]$.

Proof. Assume that we have a complete isomorphic embedding

$$j : C \to S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]].$$

By the injectivity of the Haagerup tensor product, we have a complete isomorphic embedding

$$j \otimes Id_R : C \otimes_h R \to S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]] \otimes_h R.$$

Since $1 < p_1, p_2, \cdots, p_n < \infty$ and $R = C_1$, hence by Lemma 1.36, $S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]\otimes_h R$ is a super-reflexive Banach space. This implies that $S_\infty = C \otimes_h R$ is also super-reflexive, which is a contradiction.

For any closed subspace $F \subset S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]$, we have

$$\frac{S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]}{F} \otimes_h R \simeq \frac{S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]] \otimes_h R}{F \otimes_h R}.$$

Indeed, by [Pis98], for any $1 \le p \le \infty$, if $E_2 \subset E_1$ is a closed subspace, then we have complete isometry

$$S_p[E_1/E_2] = S_p[E_1]/S_p[E_2].$$

Using the above fact, it is easy to see that

$$(E_1/E_2) \otimes_h R_p = \frac{E_1 \otimes_h R_p}{E_2 \otimes_h R_p}.$$

Note that the super-reflexive property is stable under taking the quotient, hence $\frac{S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]}{F} \otimes_h R$ is super-reflexive. Hence by using the same idea as above, assume that there is a completely isomorphic embedding:

$$i : C \to S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]/F,$$

then we have completely isomorphic embedding:

$$i \otimes Id_R : C \otimes_h R \to \frac{S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]}{F} \otimes_h R,$$

which leads to a contradiction. Hence C can not be embedded completely isomorphically into $S_{p_1}[S_{p_2}[\cdots[S_{p_n}]\cdots]]/F$. □

1.5 An equivalent definition of OUMD property

In this section, we give some equivalent conditions for $S_p[E]$ to be UMD, or equivalently, for E to be OUMD$_p$. We give the equivalence between the UMD property and the boundedness of the triangular projection on $S_p[E]$. Applying this equivalence, we prove that E is OUMD$_p$ if and only if E is OUMD$_p$ with respect to the so-called canonical filtration of matrix algebras.

We first give the following simple observation.

Proposition 1.38. Let $1 < p < \infty$, if we denote by $\mathcal{R}$ the Riesz projection $\mathcal{R} : L_p(\mathbb{T}, m) \to L_p(\mathbb{T}, m)$ defined by

$$\sum_{\text{finite}} x_n z^n \mapsto \sum_{n \geq 0} x_n z^n.$$

Then $S_p[E]$ is UMD if and only if

$$\mathcal{R}_E := Id_E \otimes \mathcal{R} : L_p(\mathbb{T}, m; E) \to L_p(\mathbb{T}, m; E)$$

is completely bounded.

Proof. By the classical results on UMD property, $S_p[E]$ is UMD if and only if the corresponding Riesz projection

$$\mathcal{R}_{S_p[E]} : L_p(\mathbb{T}, m; S_p[E]) \to L_p(\mathbb{T}, m; S_p[E])$$

is bounded. By the noncommutative Fubini theorem, the natural identification gives complete isometry

$$L_p(\mathbb{T}, m; S_p[E]) \simeq S_p[L_p(\mathbb{T}, m; E)].$$

In this identification, $\mathcal{R}_{S_p[E]}$ corresponds to

$$Id_{S_p} \otimes \mathcal{R}_E : S_p[L_p(\mathbb{T}, m; E)] \to S_p[L_p(\mathbb{T}, m; E)].$$

A very useful result in [Pis98] tells us that $\|\mathcal{R}_E\|_{cb} = \|Id_{S_p} \otimes \mathcal{R}_E\|$. Hence we have

$$\|\mathcal{R}_E\|_{cb} = \|\mathcal{R}_{S_p[E]}\|.$$

This ends our proof. □

The next theorem can be viewed as a special case of a result by Neuwirth and Ricard, see [NR11] Theorem 2.5 and Remark 3.1. Here we only give the easy side of inequality.

Theorem 1.39. Let T_E be the triangular projection on $S_p[E]$ defined by

$$(x_{ij}) \mapsto (x_{ij} 1_{j \geq i}).$$

Then $\|T_E\|_{cb} = \|T_E\| = \|\mathcal{R}_E\|_{cb}$.

Proof. We will show that

$$\|T_E\|_{cb} \leq \|\mathcal{R}_E\|_{cb}. \tag{1.3}$$

By transference method, we have

$$\|T_E\|_{B(S_p[E])} \leq \|\mathcal{R}_{S_p[E]}\|_{B(L_p(S_p[E]))}. \tag{1.4}$$

Indeed, any $x = (x_{ij})_{i,j \geq 1} \in S_p[E]$ is associated to a function $f_x : \mathbb{T} \to S_p[E]$ defined by

$$f_x(z) = D_{z^{-1}} x D_z,$$

where $D_z = \text{diag}(z, z^2, z^3, \cdots)$ is a diagonal unitary matrix, hence for any $z \in \mathbb{T}$,

$$\|f_x(z)\|_{S_p[E]} = \|x\|_{S_p[E]}.$$

Since

$$f_x(z) = (x_{ij} z^{j-i})_{i,j\geq 1},$$

we have

$$(\mathcal{R} f_x)(z) = (x_{ij} z^{j-i} 1_{j\geq i})_{i,j\geq 1} = D_{z^{-1}}(T_E x) D_z.$$

It follows that

$$\|(\mathcal{R} f_x)(z)\|_{S_p[E]} = \|T_E x\|_{S_p[E]}.$$

Then we have

$$\|T_E x\|_{S_p[E]} \leq \|\mathcal{R}_{S_p[E]}\|_{B(L_p(S_p[E]))} \|x\|_{S_p[E]}.$$

Thus we establish (1.4). By Proposition 1.38, the right hand side of (1.4) equals to $\|\mathcal{R}_E\|_{cb}$. For proving (1.3), it suffices to show that

$$\|T_E\|_{cb} \leq \|T_E\|.$$

We have

$$\|T_E\|_{cb} = \sup_n \|T_{S_p^n[E]}\|_{B(S_p[S_p^n[E]])}.$$

Thus it suffices to show that for any $n \geq 1$, we have

$$\|T_{S_p^n[E]}\| \leq \|T_E\|.$$

We can isometrically identify $S_p[S_p^n[E]]$ with $S_p[E]$ by the natural identification. We associate any $x = (x_{ij})_{i,j\geq 1} \in S_p[S_p^n[E]]$, with $x_{ij} \in S_p^n[E]$, to an element $\tilde{x} \in S_p[S_p^n[E]]$ given by

$$\tilde{x} = \begin{pmatrix} 0 & x_{11} & 0 & x_{12} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & x_{21} & 0 & x_{22} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then

$$T_E \tilde{x} = \widetilde{T_{S_p^n[E]} x}.$$

In the expression $T_E \tilde{x}$, we view $\tilde{x}$ as an element in $S_p[E]$ by the identification mentioned above. By observing the trivial equalities

$$\|\tilde{x}\|_{S_p[S_p^n[E]]} = \|x\|_{S_p[S_p^n[E]]} \text{ and } \|\widetilde{T_{S_p^n[E]} x}\|_{S_p[S_p^n[E]]} = \|T_{S_p^n[E]} x\|_{S_p[S_p^n[E]]},$$

we have

$$\begin{aligned} \|T_{S_p^n[E]} x\|_{S_p[S_p^n[E]]} &= \|\widetilde{T_{S_p^n[E]} x}\|_{S_p[S_p^n[E]]} = \|T_E \tilde{x}\|_{S_p[E]} \\ &\leq \|T_E\| \cdot \|\tilde{x}\|_{S_p[E]} = \|T_E\| \cdot \|\tilde{x}\|_{S_p[S_p^n[E]]} \\ &= \|T_E\| \cdot \|x\|_{S_p[S_p^n[E]]}. \end{aligned}$$

□

We refer to [JX05] and [JX08] for details on the canonical matrix filtration. As usual, we regard M_n as a non-unital subalgebra of $M_\infty = B(\ell_2)$ by viewing an $n \times n$ matrix as an infinite one whose left upper corner of size $n \times n$ is the given $n \times n$ matrix, and all other entries are zero. The unit of M_n is the projection $e_n \in M_\infty$ which projects a sequence in ℓ_2 into its first n coordinates. The canonical matrix filtration is the increasing filtration $(M_n)_{n\geq 1}$ of subalgebras of M_∞. We denote by $E_n : M_\infty \to M_n$ the corresponding conditional expectation. It is clear that

$$E_n(a) = e_n a e_n = \sum_{\max(i,j)\leq n} a_{ij} \otimes e_{ij}, \text{ for all } a = (a_{ij}) \in M_\infty.$$

Remark 1.40. Note that E_n is not faithful, thus the noncommutative martingales with respect to the filtration $(M_n)_{n\geq 1}$ are different from the usual ones. But this difference is not essential for what follows.

We can define the OUMD$_p$ property with respect to this canonical matrix filtration. Let $x \in S_p[E]$. Then

$$d_1 x = E_1(x), \quad d_n x = E_n(x) - E_{n-1}(x), \text{ for all } n \geq 2.$$

E is said to be OUMD$_p$ with respect to the canonical matrix filtration, if there exists a constant K depending only on p and E, such that for all positive integers N and all choices of signs $\varepsilon_n = \pm 1$, we have

$$\|\sum_{n=1}^{N} \varepsilon_n d_n x\|_{S_p[E]} \leq K\|x\|_{S_p[E]}.$$

Let $K_p(E)$ denote the best such constant.

Every choice of signs ε generates a transformation T_ε defined by

$$T_\varepsilon(x) = \sum_n \varepsilon_n d_n x.$$

An element $x \in S_p[E]$ is said to have finite support if the support of x defined by $\mathrm{supp}(x) = \{(i,j) \in \mathbb{N}^2 : x_{ij} \neq 0\}$ is finite. Note that T_ε is always well-defined on the subspace of finite supported elements.

An operator space E is OUMD$_p$ with respect to the canonical matrix filtration if for every choice of signs ε, we have

$$\|T_\varepsilon(x)\|_{S_p[E]} \leq K_p(E)\|x\|_{S_p[E]}, \quad |\mathrm{supp}(x)| < \infty.$$

Remark 1.41. The transformation T_ε is a Schur multiplication associated with the function $f_\varepsilon(i,j) = \varepsilon_{\max(i,j)}$. Indeed, pick up an arbitrary element $x = (x_{ij}) \in S_p^N[E]$, we have

$$d_n x = \sum_{\max(i,j)\leq n} x_{ij} \otimes e_{ij} - \sum_{\max(i,j)\leq n-1} x_{ij} \otimes e_{ij} = \sum_{\max(i,j)=n} x_{ij} \otimes e_{ij},$$

thus

$$T_\varepsilon(x) = \sum_{n=1}^{N} \varepsilon_n d_n x = \sum_{n=1}^{N} \varepsilon_n \sum_{\max(i,j)=n} x_{ij} \otimes e_{ij} = (\varepsilon_{\max(i,j)} x_{ij}).$$

.

Remark 1.42. Let $D_\varepsilon = \mathrm{diag}\{\varepsilon_1, \cdots, \varepsilon_n, \cdots\}$. Then $T_\varepsilon(x)$ multiplied on the left by the scalar matrix D_ε, we get $D_\varepsilon T_\varepsilon(x) = (\varepsilon_i \varepsilon_{\max(i,j)} x_{ij})$. After taking the average according to independent uniformly distributed choices of signs, we get the lower triangular projection of x, i.e, we have

$$\int D_\varepsilon T_\varepsilon(x) d\varepsilon = \int (\varepsilon_i \varepsilon_{\max(i,j)} x_{ij}) d\varepsilon = (x_{ij} 1_{i\geq j}).$$

The following result is inspired by [JX05] and [JX08]

Theorem 1.43. Let $1 < p < \infty$. Then E is OUMD_p if and only if it is OUMD_p with respect to the canonical matrix filtration. Moreover, we have:

$$\frac{1}{2}(K_p(E) - 1) \leq \|T_E\| \leq K_p(E).$$

Proof. Assume that E is OUMD_p. Then $S_p[E]$ is UMD and the triangular projection T_E is bounded. Let T_E^- be the triangular projection defined by $(x_{ij}) \mapsto (x_{ij}1_{j\leq i})$, it is clear that $\|T_E\| = \|T_E^-\|$. We have

$$d_n x = d_n T_E x + d_n T_E^- - D_n x,$$

where $D_n x = e_{nn} x e_{nn}$. Thus

$$\begin{aligned} \|\sum \varepsilon_n d_n x\|_{S_p[E]} &\leq \|\sum \varepsilon_n d_n T_E x\|_{S_p[E]} + \|\sum \varepsilon_n d_n T_E^- x\|_{S_p[E]} \\ &\quad + \|\sum \varepsilon_n D_n x\|_{S_p[E]}. \end{aligned}$$

Since $d_n T_E x$ is the n-th column of $T_E x$, it is easy to see

$$\|\sum \varepsilon_n d_n T_E x\|_{S_p[E]} = \|\sum d_n T_E x\|_{S_p[E]} = \|T_E x\|_{S_p[E]} \leq \|T_E\| \|x\|_{S_p[E]}.$$

The same reason shows that

$$\|\sum \varepsilon_n d_n T_E^- x\|_{S_p[E]} = \|\sum d_n T_E^- x\|_{S_p[E]} = \|T_E^- x\|_{S_p[E]} \leq \|T_E^-\| \|x\|_{S_p[E]}.$$

For the third term, we have obviously that

$$\|\sum \varepsilon_n D_n x\|_{S_p[E]} = \|\sum D_n x\|_{S_p[E]} \leq \|x\|_{S_p[E]}.$$

Combining these inequalities, we have

$$\|\sum \varepsilon_n d_n x\|_{S_p[E]} \leq (\|T_E\| + \|T_E^-\| + 1)\|x\|_{S_p[E]} = (2\|T_E\| + 1)\|x\|_{S_p[E]}.$$

So E is OUMD_p with respect to the canonical matrix filtration with $K_p(E) \leq 2\|T_E\| + 1$.

Conversely, assume that E is OUMD_p with respect to the canonical matrix filtration. We shall show that E is OUMD_p. It suffices to show that the triangular projection T_E is bounded. According to the remark 1.42, we have

$$\|(x_{ij}1_{i\geq j})\|_{S_p[E]} \leq \int \|D_\varepsilon T_\varepsilon(x)\|_{S_p[E]} d\varepsilon \leq K_p(E)\|x\|_{S_p[E]},$$

proving that $\|T_E^-\| \leq K_p(E)$, and hence $\|T_E\| \leq K_p(E)$. □

Remark 1.44. We have a slightly better estimation for $\|T_E\|$ as the following

$$\frac{1}{2}(K_p(E) - 1) \leq \|T_E\| \leq \frac{1}{2}(K_p(E) + 1).$$

We omit the proof here.

1.6 Some related questions

In this section, we will list some questions. The first one is a well known open problem asking whether the OUMD property depends on p or not.

Question 1.45. Do we have

$$\mathrm{OUMD}_p \Longleftrightarrow \mathrm{OUMD}_q$$

for all $1 < p, q < \infty$?

The following observation seems to be related to this question.

Remark 1.46. Let E be an operator space which is OUMD_p for some $1 < p < \infty$. Hence, the Banach space $S_p[E]$ is UMD. It follows that $S_p[E]$ is super-reflexive (for the definition of super-reflexive, see the next chapter). Now let $1 < q < \infty$. There exists $1 < u < \infty$ and $0 < \theta < 1$ such that $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{u}$. By Kouba's interpolation theorem, we have

$$S_q[E] = (S_p[E], S_u[E])_\theta.$$

Recall the following well-known result due to Pisier: Given a compatible interpolation couple of Banach spaces (X_0, X_1), if one of them is super-reflexive, then for any $0 < \theta < 1$, the interpolation space $(X_0, X_1)_\theta$ is super-reflexive. Applying this theorem to our situation, we see that the Banach space $S_q[E]$ must be super-reflexive.

Note that there exists an operator space E such that $S_p[E]$ is super-reflexive non-UMD. In [Bou83], Bourgain constructed super-reflexive non-UMD Banach lattices. Let X be one of such spaces. By results in [Pis79], each super-reflexive lattice can be obtained as complex interpolation space between a Hilbert space and some lattice. Thus, there exists $0 < \theta < 1$ and an interpolation couple (X_0, X_1) such that X_0 is a Hilbert space and X_1 is a lattice, $X = (X_0, X_1)_\theta$. Since X_0 is a Hilbert space, $X_0 = \ell_2(I)$, we may equip it with the operator space structure $OH(I)$. Let E be the operator space obtained by complex interpolation $E = (OH(I), \min(X_1))_\theta$. Then

$$S_p[E] = (S_p[OH(I)], S_p[\min(X_1)])_\theta.$$

Since $S_p[OH(I)]$ is UMD, a fortiori, it is super-reflexive, thus $S_p[E]$ is super-reflexive non-UMD.

Question 1.47. Let X be any UMD Banach space (of course, we suppose $\dim X = \infty$). Does there always exist an operator space E (Banach) isometric to X such that E is OUMD_p for some (or all) $1 < p < \infty$?

A naive idea that comes to mind is to consider the operator spaces $\min(X)$ and $\max(X)$. However, we know from Remark 1.25 that both of them are non-OUMD_p for any $1 < p < \infty$.

It seems rather difficult to decide whether the underlying Banach space structure of $C_{p_1} \otimes_h C_{p_2} \otimes_h \cdots \otimes_h C_{p_n}$ is UMD or not for $1 < p_1, p_2, \cdots, p_n < \infty$ and $n \geq 3$. And from the previous discussion, a whole description of the case for $n = 3$ is sufficient for solving Ruan's problem.

The following related question looks easier than the above question, and it is indeed the case. We will study it in the appendix.

Question 1.48. Let $1 < p_0 < p_\infty < \infty$. Does there exist a numerical constant K, such that (say for the OUMD_2 constants) for any n and any $p_1, p_2, \cdots, p_n \in [p_0, p_\infty]$, we have

$$\beta_2^{os}\Big(C_{p_1} \otimes_h C_{p_2} \otimes_h \cdots \otimes_h C_{p_n}\Big) \leq K?$$

Question 1.48 is in fact the original motivation of the next chapter. We will prove that there is a negative answer to Question 1.48.

Chapter 2

On the UMD constants for a class of iterated $L_p(L_q)$ spaces

Introduction

In this chapter, all the spaces are Banach spaces, so different from last chapter, here we only consider the classical UMD property. The main subject is to study the UMD constants for some finite-dimensional spaces. The "analytic-UMD" (in short AUMD) property for complex Banach spaces will also be studied.

We will study the UMD constants of Banach lattices like

$$\ell_{p_1}(\ell_{p_2}(\cdots(\ell_{p_n})\cdots)) \text{ or } L_{p_1}(L_{p_2}(\cdots(L_{p_n})\cdots)).$$

More precisely, let $1 \leq p, q \leq \infty$. Suppose that $E_0 = \mathbb{C}$ or $E_0 = \mathbb{R}$, if we have defined E_n for $n \geq 0$, then we can define

$$E_{n+1} = \ell_p^{(2)}(\ell_q^{(2)}(E_n)),$$

where $\ell_p^{(2)}$ (resp. $\ell_q^{(2)}$) is the 2-dimensional analogue of ℓ_p (resp. ℓ_q).

The main theorem in this chapter is

Theorem 2.1. Let $1 \leq p \neq q \leq \infty$. Then there exists $c = c(p, q) > 1$ depending only on p and q, such that for all $1 < s < \infty$, the UMD_s constants of the above defined spaces E_n satisfy

$$\beta_s(E_n) \geq c^n.$$

The result of this chapter has been published in [Qiu12].

2.1 Definitions

2.1.1 UMD and AUMD

For the definition of the classical UMD property, see §1.1. Here, we will briefly recall the definition of the AUMD property, which was introduced by Garling in [Gar88].

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be group of the unit circle equipped with its normalised Haar measure m. Consider the canonical filtration on the probability space $(\mathbb{T}^{\mathbb{N}}, m^{\otimes \mathbb{N}})$ defined by

$$\sigma(z_0) \subset \sigma(z_0, z_1) \subset \cdots \subset \sigma(z_0, z_1, \cdots, z_n) \subset \cdots.$$

Let X be a complex Banach space. By definition, an X-valued Hardy martingale in $L_s(\mathbb{T}^{\mathbb{N}}; X)$ is an X-valued martingale $f = (f_n)_{n\geq 0}$ with respect to the above filtration such that

$$\sup_n \|f_n\|_{L_s} < \infty,$$

and for $n \geq 1$, the martingale difference $df_n = f_n - f_{n-1}$ is analytic in the last variable z_n, i.e., df_n has the form:

$$df_n(z_0, \cdots, z_{n-1}, z_n) = \sum_{k\geq 1} \phi_{n,k}(z_0, \cdots, z_{n-1}) z_n^k.$$

In the sequel, we will set $df_0 := f_0$ by convention.

Definition 2.2. Let $0 < s < \infty$, a Banach space X is AUMD$_s$, if there is a constant $c > 0$, such that

$$\sup_{\varepsilon_k \in \{-1,1\}} \Big\| \sum_{k=0}^{n} \varepsilon_k df_k \Big\|_{L_s(X)} \leq c \Big\| \sum_{k=0}^{n} df_k \Big\|_{L_s(X)} \tag{2.1}$$

for all $n \geq 0$ and all X-valued Hardy martingale $f = (f_k)_{k\geq 0}$. The best such c is called the AUMD$_s$ constant of X and will be denoted by $\beta_s^a(X)$.

Remark 2.3. Note that in the definition of the property AUMD$_s$, we can have $0 < s \leq 1$, which is different from the property UMD$_s$.

Recall that an analytic martingale starting at $x \in X$ is any sequence f of the form

$$f_n(z) = x + \sum_{k=0}^{n} \varphi_k(z_0, \cdots, z_{k-1}) z_k,$$

where $z = (z_k)_{k\geq 0} \in \mathbb{T}^{\mathbb{N}}$, $\varphi_k(z_0, \cdots, z_{k-1}) \in X$.

It is known that in the definition of AUMD$_s$ property, in place of the Hardy martingales, we can only consider the analytic martingales. Using this, it was shown that AUMD$_p$ and AUMD$_q$ are equivalent properties, for an argument, see e.g. [Bur01] §8.

From now on, a space X is said to be AUMD if it is AUMD$_p$ for some or equivalently for all $0 < p < \infty$.

Note that UMD implies AUMD but not conversely, for instance, $L_1(\mathbb{T}, m)$ is an AUMD space which is not UMD. More generally, if X has AUMD property, then $L_1(X)$ also has (cf. [Gar88]).

2.1.2 Super-reflexivity

The notion of super-reflexivity involves the notion of finite representability. A Banach space Y is said to be λ-finitely representable in X, and is denoted as

$$Y \ \lambda\text{-f.r.}\ X,$$

if for every finite-dimensional subspace $E \subset X$ and every $\varepsilon > 0$, there exists a subspace $\widetilde{E} \subset X$ such that the Banach-Mazur distance $d_{BM}(E, \widetilde{E}) \leq \lambda + \varepsilon$. In the language of ultraproduct of Banach spaces, we have

$$Y \ \lambda\text{-f.r.}\ X \Longleftrightarrow Y \overset{\lambda}{\hookrightarrow} X^{\mathscr{U}} \text{ for some ultrafilter } \mathscr{U},$$

where $Y \overset{\lambda}{\hookrightarrow} X^{\mathscr{U}}$ means that there exists a subspace $Z \subset X^{\mathscr{U}}$ such that $d_{BM}(Y, Z) \leq \lambda$.

Definition 2.4. A Banach space X is said to be super-reflexive, if for every Y such that Y f.r. X, then Y is reflexive. Equivalently, X is super-reflexive if and only if for any ultrafilter $\mathscr{U}$, the ultrapower $X^{\mathscr{U}}$ is reflexive.

It is well-known that UMD implies super-reflexive but not conversely. The first super-reflexive non-UMD Banach space was constructed by Pisier in [Pis75]. Super-reflexive non-UMD Banach lattices were later constructed by Bourgain in [Bou83, Bou84]. We refer to Rubio de Francia's paper [RdF86] for some open problems related to the super-reflexive non-UMD Banach lattices.

2.2 Some elementary inequalities

2.2.1 Constant $c(p, q)$

We will use the following lemma.

Lemma 2.5. Let (Ω, ν) be a measure space such that ν is finite. Suppose that $\alpha \neq 1$ and $0 < \alpha < \infty$. If $F, f \in L_\alpha(\Omega, \nu) \bigcap L_1(\Omega, \nu)$ satisfy

$$\int (|F| + |g|)^\alpha d\nu \leq \int (|f| + |g|)^\alpha d\nu$$

for all $g \in L_\infty(\Omega, \nu)$. Then $|F| \leq |f|$ a.e.

Proof. Firstly, let us consider any function $g \in L_\infty(\Omega, \nu)$ such that there exists $\delta > 0$ and $|g| \geq \delta$ a.e. If F, f satisfy the conditions in the lemma, then for all $\varepsilon > 0$, we have

$$\int \Big(|F| + \frac{|g|}{\varepsilon}\Big)^\alpha d\nu \leq \int \Big(|f| + \frac{|g|}{\varepsilon}\Big)^\alpha d\nu,$$

and hence

$$\int (\varepsilon|F| + |g|)^\alpha d\nu \leq \int (\varepsilon|f| + |g|)^\alpha d\nu. \tag{2.2}$$

By the mean value theorem, there exists $\theta = \theta_\varepsilon \in (0, 1)$, such that

$$\frac{(\varepsilon|f| + |g|)^\alpha - |g|^\alpha}{\varepsilon} = \alpha(\theta\varepsilon|f| + |g|)^{\alpha-1}|f|.$$

If $\alpha < 1$, then

$$(\theta\varepsilon|f| + |g|)^{\alpha-1}|f| \leq |g|^{\alpha-1}|f| \in L_1(\Omega, \nu).$$

If $\alpha > 1$, then for $0 < \varepsilon < 1$, we have $0 < \theta\varepsilon < 1$, hence

$$(\theta\varepsilon|f| + |g|)^{\alpha-1}|f| \leq 2^{\alpha-1}(|f|^\alpha + |g|^{\alpha-1}|f|) \in L_1(\Omega, \nu).$$

By the dominated convergence theorem, we have

$$\lim_{\varepsilon \to 0^+} \frac{\int (\varepsilon|f| + |g|)^\alpha d\nu - \int |g|^\alpha d\nu}{\varepsilon} = \alpha \int |f||g|^{\alpha-1} d\nu.$$

The same equality holds for F. Combining this with (2.2), we get

$$\int |F||g|^{\alpha-1} d\nu \leq \int |f||g|^{\alpha-1} d\nu.$$

Replacing g by $|g|^{\frac{1}{\alpha-1}}$ yields

$$\int |F||g|d\nu \le \int |f||g|d\nu.$$

By approximation, the above inequality holds for all $g \in L_\infty(\Omega, \nu)$. Hence $|F| \le |f|$ a.e., as desired. $\square$

Proposition 2.6. Let (Ω, ν) be a measure space such that ν is finite. Suppose that $1 \le p \ne q < \infty$. If $F, f \in L_p(\Omega, \nu) \bigcap L_q(\Omega, \nu)$ satisfy

$$\int (|F|^q + |g|^q)^{p/q} d\nu \le \int (|f|^q + |g|^q)^{p/q} d\nu$$

for all $g \in L_\infty(\Omega, \nu)$. Then $|F| \le |f|$ a.e.

Proof. This is just a reformulation of Lemma 2.5. $\square$

Let $D = \{-1, 1\}$ be the Bernoulli probability space equipped with the measure $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$. For any $1 \le q \le \infty$, the 2-dimensional ℓ_q-space will be denoted by $\ell_q^{(2)}$.

Remark 2.7. Since (D, μ) is totally atomic, any statement with "a.e." is the same as the same statement without "a.e.". For instance, let F, f be functions on (D, μ), then $|F| \le |f|$ *a.e.* is the same as $|F| \le |f|$.

The following proposition is very useful for us, it will be used in §2.3.

Proposition 2.8. Suppose that $1 \le p \ne q \le \infty$. Let P be the projection on $L_p(\mu; \ell_q^{(2)})$ defined by

$$\begin{array}{cccc} P: & L_p(\mu; \ell_q^{(2)}) & \to & L_p(\mu; \ell_q^{(2)}) \\ & (f, g) & \mapsto & (\mathbb{E}f, g) \end{array},$$

where $\mathbb{E}$ is the expectation, i.e. $\mathbb{E}(f) = \int f d\nu$. Then the operator norm of P satisfies

$$\|P\|_{L_p(\mu;\ell_q^{(2)}) \to L_p(\mu;\ell_q^{(2)})} > 1$$

In words, P is not contractive.

Proof. Assume first that both p, q are finite. If P is contractive, then for any two functions f and g, we have

$$\int (|\mathbb{E}f|^q + |g|^q)^{p/q} d\mu \le \int (|f|^q + |g|^q)^{p/q} d\mu.$$

By Proposition 2.6, it follows that $|\mathbb{E}(f)| \le |f|$, which is a contradiction if we choose f to be $f = \delta_1$. Hence P is not contractive.

If $p = \infty$ and $1 < q < \infty$, then $p' = 1$ and $1 < q' < \infty$. Since the adjoint map P^* on $L_1(\mu; \ell_{q'}^{(2)})$ has the same form as P, the preceding argument shows that P^* and hence P is not contractive.

If $p = \infty$ and $q = 1$. Assume P is contractive, then we have

$$\big\| |\mathbb{E}f| + |g| \big\|_\infty \le \big\| |f| + |g| \big\|_\infty. \tag{2.3}$$

If we take $f = 1 + \varepsilon, g = 1 - \varepsilon$, where $\varepsilon : D \to D$ is the identity function. Then

$$\big\| |\mathbb{E}f| + |g| \big\|_\infty = \|1 + 1 - \varepsilon\|_\infty = 3$$

and

$$\big\||f|+|g|\big\|_\infty = \|1+\varepsilon+1-\varepsilon\| = 2,$$

Hence

$$\big\||\mathbb{E}f|+|g|\big\|_\infty > \big\||f|+|g|\big\|_\infty,$$

which contradicts to (2.3). This shows that P is not contractive.

If $1 \leq p < \infty$ and $q = \infty$, then $1 < p' \leq \infty$ and $q' = 1$, hence P^* is not contractive. It follows that P is not contractive. $\square$

Obviously, if $p = q$, the projection P as defined in Proposition 2.8 is contractive. The norm of P on $L_p(\mu;\ell_q^{(2)})$ will be denoted by $c(p,q)$:

$$c(p,q) := \|P\|_{L_p(\mu;\ell_q^{(2)})\to L_p(\mu;\ell_q^{(2)})}.$$

Thus if $p = q$, then

$$c(p,p) = 1.$$

If $1 \leq p \neq q \leq \infty$, then

$$c(p,q) > 1. \tag{2.4}$$

Remark 2.9. In Proposition 2.8, we do not make difference between the real space $\ell_q^{(2)}(\mathbb{R})$ and the complex $\ell_q^{(2)}(\mathbb{C})$. It is because the corresponding $c(p,q)$ in these two cases are the same. This follows from the following observation

$$\begin{aligned} c(p,q) &= \sup\left\{\left(\int (|\mathbb{E}f|^q + |g|^q)^{p/q}\,d\mu\right)^{1/p} : \int (|f|^q+|g|^q)^{p/q} d\mu \leq 1\right\} \\ &= \sup\left\{\left(\int \Big((\mathbb{E}|f|)^q + |g|^q\Big)^{p/q} d\mu\right)^{1/p} : \int (|f|^q+|g|^q)^{p/q} d\mu \leq 1\right\} \\ &= \sup\left\{\left(\int \Big((\mathbb{E}f)^q + g^q\Big)^{p/q} d\mu\right)^{1/p} : f\geq 0, g\geq 0, \int (f^q+g^q)^{p/q} d\mu \leq 1\right\} \end{aligned} \tag{2.5}$$

2.2.2 Some comments on $c(p,q)$

It is not difficult to check that

$$c(\infty,1) = c(1,\infty) = \frac{3}{2}.$$

By duality, we know that $c(1,\infty) = c(\infty,1)$, so it suffices to show that $c(\infty,1) = \frac{3}{2}$. By definition, we have

$$\begin{aligned} c(\infty,1) &= \sup\Big\{\big\||\mathbb{E}(f)|+|g|\big\|_\infty : \big\||f|+|g|\big\|_\infty \leq 1\Big\} \\ &= \sup\Big\{\|\mathbb{E}(f)+g\|_\infty : f\geq 0, g\geq 0, f+g\leq 1\Big\}. \end{aligned}$$

Then it is easy to see that

$$c(\infty,1) = \sup\Big\{\|\mathbb{E}(f)+1-f\|_\infty : 0\leq f\leq 1\Big\}.$$

Thus we can assume that

$$f = \alpha\chi_{\varepsilon=1} + \beta\chi_{\varepsilon=-1},$$

with $0 \le \alpha, \beta \le 1$. Then

$$\|\mathbb{E}(f) + 1 - f\|_\infty = 1 + \frac{|\alpha - \beta|}{2}.$$

Hence $c(\infty, 1) = \frac{3}{2}$.

We do not get the exact values of $c(p,q)$ for general $p \neq q$. However, we can study the function $c(p,q)$ on variables p, q. Let us define $k : [0,1]^2 \to \mathbb{R}$ by

$$k(\alpha, \beta) \stackrel{\text{def}}{=} \log\Big(c(\frac{1}{\alpha}, \frac{1}{\beta})\Big).$$

Then we have

Proposition 2.10. The function $k : [0,1]^2 \to \mathbb{R}$ defined above is convex and satisfies

$$k(\alpha, \alpha) = 0 \text{ and } k(\alpha, \beta) = k(1-\alpha, 1-\beta).$$

Proof. Since $c(p,p) = 1$ for any $1 \le p \le \infty$, we have

$$k(\alpha, \alpha) = 0.$$

By duality, we have $c(p,q) = c(p', q')$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. It follows that

$$k(\alpha, \beta) = k(1-\alpha, 1-\beta).$$

For the convexity of k, we will need the complex interpolation argument. Assume that $0 < \theta < 1$. Let $1 \le p_0, p_1, q_0, q_1 \le \infty$. Define p_θ, q_θ by

$$\Big(\frac{1}{p_\theta}, \frac{1}{q_\theta}\Big) = (1-\theta)\Big(\frac{1}{p_0}, \frac{1}{q_0}\Big) + \theta\Big(\frac{1}{p_1}, \frac{1}{q_1}\Big).$$

By complex interpolation, we have

$$\Big[L_{p_0}(\mu; \ell_{q_0}^{(2)}(\mathbb{C})), L_{p_0}(\mu; \ell_{q_0}^{(2)}(\mathbb{C}))\Big]_\theta = L_{p_0}(\mu; \ell_{q_0}^{(2)}(\mathbb{C})),$$

hence

$$c(p_\theta, q_\theta) \le c(p_0, q_0)^{1-\theta} c(p_1, q_1)^\theta.$$

It follows that

$$k\Big((1-\theta)(\alpha_0, \beta_0) + \theta(\alpha_1, \beta_1)\Big) \le (1-\theta)k(\alpha_0, \beta_0) + \theta k(\alpha_1, \beta_1).$$

Thus k is convex. □

It follows from Proposition 2.10 and the properties of convex functions that

$$\max_{1 \le p,q \le \infty} c(p,q) = \max\Big\{c(1,1), c(1,\infty), c(\infty,1), c(\infty,\infty)\Big\} = \frac{3}{2}.$$

By convexity of k, it is easy to show that

$$c(p,q) \le \Big(\frac{3}{2}\Big)^{|\frac{1}{p} - \frac{1}{q}|}. \tag{2.6}$$

Remark 2.11. By some computation, we can show that

$$c(\infty, q) \ge \Big(\frac{2+\sqrt{3}}{3}\Big)^{1/q} \text{ and } c(1,q) \ge \Big(\frac{2+\sqrt{3}}{3}\Big)^{1-1/q}.$$

And also

$$c(p, \infty) \ge \Big(\frac{3+\sqrt{3}}{4}\Big)^{1/p} \text{ and } c(p,1) \ge \Big(\frac{3+\sqrt{3}}{4}\Big)^{1-1/p}.$$

Remark 2.12. The behavior of the function $c(p,q)$ near the diagonal $p = q$ is useful for studying the conditions on the sequences (p_i) such that the family consists of the iterated spaces $L_{p_1}(L_{p_2}(\cdots(L_{p_n})\cdots))$ is uniformly UMD. For the detail, see Problem 2.5.

2.2.3 Constant $\kappa(p,q)$

As usual, we set

$$H_p(\mathbb{T}) = \Big\{ f \in L_p(\mathbb{T}, m) : \hat{f}(k) = 0, \forall k \in \mathbb{Z}_{<0} \Big\}.$$

We will say that a measurable function $f : \mathbb{T} \to \mathbb{C}$ is bounded from below, if there exists $\delta > 0$, such that $|f| \geq \delta$ a.e. on $\mathbb{T}$.

Definition 2.13. Let $f \in L_p(\mathbb{T})$ be a function bounded from below, then the geometric mean $M(|f|)$ of $|f|$ is defined by

$$\log M(|f|) = \int_{\mathbb{T}} \log |f(z)| dm(z).$$

In particular, if $f : \mathbb{D} \to \mathbb{C}$ is an outer function, then we have

$$M(|f|) = |f(0)| = |\mathbb{E}f|. \tag{2.7}$$

The following proposition will be used in §2.4 when we treat the analytic UMD property, where $k(p,q)$ plays the rôle of $c(p,q)$.

Proposition 2.14. Suppose that $1 \leq p \neq q \leq \infty$. Define $\kappa(p,q)$ to be the best constant C satisfying the property: For any measurable partition $\mathbb{T} = A\dot{\cup}B$ with $m(A) = m(B) = \frac{1}{2}$, for any function $f = f_1\chi_A + f_2\chi_B$ with $f_1 > 0, f_2 > 0$ and any function $g = g_1\chi_A + g_2\chi_B$, we have

$$\int_{\mathbb{T}} (M(|f|)^q + |g|^q)^{p/q} dm \leq C^p \int_{\mathbb{T}} (|f|^q + |g|^q)^{p/q} dm.$$

As usual, when $1 \leq p < \infty$ and $q = \infty$, the above inequality is understood as

$$\int_{\mathbb{T}} \max \Big\{ M(|f|), |g| \Big\}^p dm \leq C^p \int_{\mathbb{T}} \max \Big\{ |f|, |g| \Big\}^p dm.$$

When $p = \infty$ and $1 \leq q < \infty$, it is understood as

$$\sup_{z\in\mathbb{T}} \Big\{ M(|f|)^q + |g(z)|^q \Big\}^{1/q} \leq C \sup_{z\in\mathbb{T}} \Big\{ |f(z)|^q + |g(z)|^q \Big\}^{1/q}.$$

Then $\kappa(p,q) > 1$.

Proof. Firstly, let us assume that both p and q are finite. Assume by contradiction that $k(p,q) \leq 1$, that is, for any measurable partition $\mathbb{T} = A\dot{\cup}B$ such that $m(A) = m(B) = \frac{1}{2}$, if we consider the 2-valued functions $f = f_1\chi_A + f_2\chi_B$ and $g = g_1\chi_A + g_2\chi_B$ with $f_1 > 0, f_2 > 0$, then

$$\int_{\mathbb{T}} (M(|f|)^q + |g|^q)^{p/q} dm \leq \int_{\mathbb{T}} (|f|^q + |g|^q)^{p/q} dm.$$

It is easy to see that $f, g, M(|f|)$ are all functions measurable with respect to the σ-algebra $\mathscr{F} = \sigma(A, B)$ generated by the partition $A\dot{\cup}B$. Functions of the form $g = g_1\chi_A + g_2\chi_B$ run over the whole set $L_\infty(\mathbb{T}, \mathscr{F}, m)$. Hence we can apply Proposition 2.6 and get

$$M(f) \leq f.$$

By definition,

$$M(f) = f_1^{1/2} f_2^{1/2}.$$

Indeed, we have

$$\log M(f) = \int_{\mathbb{T}} \log(f)dm = \log(f_1)m(A) + \log(f_2)m(B) = \frac{\log(f_1) + \log(f_2)}{2} = \log \sqrt{f_1 f_2}.$$

If we take $f_1 > f_2$, then

$$M(f) > f_2^{1/2} f_2^{1/2} = f_2,$$

this contradicts to the inequality $M(f) \leq f$. Hence we must have $\kappa(p, q) > 1$.

If $q = \infty$ and $1 \leq p < \infty$. Assume that $\kappa(p, \infty) \leq 1$, then we have

$$\int_{\mathbb{T}} \max \left\{ M(|f|), |g| \right\}^p dm \leq \int_{\mathbb{T}} \max \left\{ |f|, |g| \right\}^p dm.$$

Choose $f = g = \chi_A + 4\chi_B$ with $\{A, B\}$ a measurable partition of $\mathbb{T}$ with the same measure $m(A) = m(B) = \frac{1}{2}$. Then $M(|f|) = 2$. Hence

$$\int_{\mathbb{T}} \max \left\{ M(|f|), |g| \right\}^p dm = 2^p m(A) + 4^p m(B) = \frac{2^p + 4^p}{2}.$$

It follows that

$$\int_{\mathbb{T}} \max \left\{ |f|, |g| \right\}^p dm = \int_{\mathbb{T}} |f|^p dm = \frac{1 + 4^p}{2} < \frac{2^p + 4^p}{2} = \int_{\mathbb{T}} \max \left\{ M(|f|), |g| \right\}^p dm.$$

This is a contradiction. Hence we must have $\kappa(p, \infty) > 1$.

Finally, if $p = \infty$ and $1 \leq q < \infty$. Assume that $\kappa(\infty, q) \leq 1$, then

$$\sup_{z \in \mathbb{T}} \left\{ M(|f|)^q + |g(z)|^q \right\}^{1/q} \leq \sup_{z \in \mathbb{T}} \left\{ |f(z)|^q + |g(z)|^q \right\}^{1/q},$$

or equivalently,

$$\sup_{z \in \mathbb{T}} \left\{ M(|f|)^q + |g(z)|^q \right\} \leq \sup_{z \in \mathbb{T}} \left\{ |f(z)|^q + |g(z)|^q \right\},$$

for any suitable f, g. Take $\mathbb{T} = A \dot{\cup} B$ any measurable partition of $\mathbb{T}$ as above. If we take $f = (2 - \chi_A + \chi_B)^{1/q}$ and $g = (1 + \chi_A - \chi_B)^{1/q}$. Then

$$|f(z)|^q + |g(z)|^q = 2 - \chi_A + \chi_B + 1 + \chi_A - \chi_B \equiv 3,$$

hence

$$\sup_{z \in \mathbb{T}} \left\{ |f(z)|^q + |g(z)|^q \right\} = 3.$$

Since $M(|f|) = \sqrt{1 \times 3^{1/q}} = \sqrt{3^{1/q}}$, we have

$$\sup_{z \in \mathbb{T}} \left\{ M(|f|)^q + |g(z)|^q \right\} = \sup_{z \in \mathbb{T}} \left\{ \sqrt{3} + 1 + \chi_A - \chi_B \right\} = 2 + \sqrt{3}$$

Hence we have

$$\sup_{z \in \mathbb{T}} \left\{ M(|f|)^q + |g(z)|^q \right\} > \sup_{z \in \mathbb{T}} \left\{ |f(z)|^q + |g(z)|^q \right\}.$$

This contradicts to the assumption. Hence $\kappa(\infty, q) > 1$.

□

2.3 UMD constants of iterated $L_p(L_q)$ spaces

The following definition is essential in the sequel.

Definition 2.15. Consider a Banach space X with a fixed family of vectors $\{x_i\}_{i\in I}$. We define $S(X;\{x_i\})$ to be the best constant C such that

$$\Big\|\sum_{k=0}^{N}\mathbb{E}^{\mathcal{A}_k}(\theta_k)x_{i_k}\Big\|_{L_1(\Omega,\mathbb{P};X)}\le C\Big\|\sum_{k=0}^{N}\theta_k x_{i_k}\Big\|_{L_\infty(\Omega,\mathbb{P};X)} \tag{2.8}$$

holds for any $N\in\mathbb{N}$, any probability space $(\Omega,\mathcal{F},\mathbb{P})$ equipped with a filtration $\mathcal{A}_0\subset\mathcal{A}_1\subset\cdots\subset\mathcal{A}_n\subset\cdots\subset\mathcal{F}$, any $N+1$ distinct indices $\{i_0,i_1,\cdots,i_N\}\subset I$ and any $N+1$ functions $\theta_0,\theta_1,\cdots,\theta_N$ in $L_\infty(\Omega,\mathcal{F},\mathbb{P})$.

If there does not exist such constant, we set $S(X;\{x_i\})=\infty$.

In what follows, we are mostly interested in the special case when $\{x_i\}$ is a 1-unconditional basic sequence, since in this case we can relate $S(X;\{x_i\})$ to the UMD constants of X. If $\{x_i\}$ is clear from the context and there is no confusion, we will use the simplified notation $S(X)$ for $S(X;\{x_i\})$. In particular, if X has a natural basis, then $S(X)$ will always mean to be calculated with this basis.

We will need the following well-known vector-valued Stein inequality in UMD spaces, which was originally proved by Bourgain [Bou86]. For the sake of completeness, we include the proof.

Theorem 2.16. Let X be a UMD space. Then for any $1<s<\infty$, any finite sequences of functions $(F_k)_{k\ge0}$ in $L_s(\Omega,\mathbb{P};X)$ and any filtration $\mathcal{A}_0\subset\mathcal{A}_1\subset\cdots\subset\mathcal{A}_n\subset\cdots$ on $(\Omega,\mathbb{P})$, we have

$$\Big\|\sum_k\varepsilon_k\mathbb{E}_k(F_k)\Big\|_{L_s(\mu_\infty\times\mathbb{P};X)}\le\beta_s(X)\Big\|\sum_k\varepsilon_kF_k\Big\|_{L_s(\mu_\infty\times\mathbb{P};X)}, \tag{2.9}$$

where $\mathbb{E}_k=\mathbb{E}^{\mathcal{A}_k}$ and $(\varepsilon_k)_{k\ge0}$ is the usual Rademacher sequence on $(D^{\mathbb{N}},\mu_\infty)$, $\mu_\infty=\mu^{\otimes\mathbb{N}}$.

Proof. We define a filtration $(\mathcal{C}_k)_{k\ge0}$ on $\Omega\times\{-1,1\}^{\mathbb{N}}$ by setting

$$\mathcal{C}_{2j}=\mathcal{A}_j\otimes\sigma(\varepsilon_0,\cdots,\varepsilon_j)$$

$$\mathcal{C}_{2j+1}=\mathcal{A}_{j+1}\otimes\sigma(\varepsilon_0,\cdots,\varepsilon_j).$$

Now consider $f\in L_s(\Omega\times\{\pm1\}^{\mathbb{N}};X)$ defined by

$$f=\sum_k\varepsilon_kF_k.$$

Note that we have

$$\mathbb{E}^{\mathcal{C}_{2j}}(f)=\sum_0^j\varepsilon_k\mathbb{E}_j(F_k)$$

and

$$\mathbb{E}^{\mathcal{C}_{2j-1}}(f)=\sum_0^{j-1}\varepsilon_k\mathbb{E}_j(F_k),$$

hence

$$(\mathbb{E}^{\mathcal{C}_{2j}}-\mathbb{E}^{\mathcal{C}_{2j-1}})(f)=\varepsilon_j\mathbb{E}_j(F_j).$$

It follows that

$$\sum_k \varepsilon_k \mathbb{E}_k(F_k) = \sum_j (\mathbb{E}^{\mathcal{C}_{2j}} - \mathbb{E}^{\mathcal{C}_{2j-1}})(f).$$

By the next remark, we see that

$$\|\sum_k \varepsilon_k \mathbb{E}_k(F_k)\|_{L_s(\mu_\infty \times \mathbb{P};X)} \leq \beta_s(X)\|f\|_{L_s(\mu_\infty \times \mathbb{P};X)} = \beta_s(X)\|\sum_k \varepsilon_k F_k\|_{L_s(\mu_\infty \times \mathbb{P};X)},$$

as desired. □

Remark 2.17. By an extreme point argument, we have

$$\sup_{-1\leq\alpha_k\leq 1} \|\sum_{k=0}^n \alpha_k df_k\|_{L_s(X)} = \sup_{\varepsilon_k\in\{-1,1\}} \|\sum_{k=0}^n \varepsilon_k df_k\|_{L_s(X)}.$$

Hence we have

$$\sup_{-1\leq\alpha_k\leq 1} \|\sum_{k=0}^n \alpha_k df_k\|_{L_s(X)} \leq \beta_s(X)\|\sum_{k=0}^n df_k\|_{L_s(X)}.$$

Proposition 2.18. Let X be a UMD space. Assume that $\{x_i\}_{i\in I}$ is a 1-unconditional basic sequence in X. Then for any $1 < s < \infty$, any finite sequence of functions $(\theta_k)_{k\geq 0}$ in $L_s(\Omega, \mathbb{P})$ and any filtration $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n \subset \cdots$ on $(\Omega, \mathbb{P})$, we have

$$\Big\|\sum_k \mathbb{E}_k(\theta_k)x_{i_k}\Big\|_{L_s(\Omega,\mathbb{P};X)} \leq \beta_s(X)\Big\|\sum_k \theta_k x_{i_k}\Big\|_{L_s(\Omega,\mathbb{P};X)}. \tag{2.10}$$

Proof. For any i_k's, consider the sequence $(F_k)_{k\geq 0}$ in $L_s(\Omega, \mathbb{P}; X)$ defined by

$$F_k(w) = \theta_k(w)x_{i_k}.$$

Then

$$\mathbb{E}_k(F_k) = \mathbb{E}_k(\theta_k)x_{i_k}.$$

By the 1-unconditionality of $\{x_i\}_{i\in I}$, for any fixed choice of signs $\varepsilon_k \in \{-1, 1\}$ and $w \in \Omega$, we have

$$\Big\|\sum_k \varepsilon_k F_k(w)\Big\|_X = \Big\|\sum_k \varepsilon_k \theta_k(w)x_{i_k}\Big\|_X = \Big\|\sum_k \theta_k(w)x_{i_k}\Big\|_X.$$

It follows that

$$\Big\|\sum_k \varepsilon_k F_k\Big\|_{L_s(\mu_\infty\times\mathbb{P};X)} = \Big\|\sum_k \theta_k x_{i_k}\Big\|_{L_s(\Omega,\mathbb{P};X)}.$$

Similarly, we have

$$\Big\|\sum_k \varepsilon_k \mathbb{E}_k(F_k)\Big\|_{L_s(\mu_\infty\times\mathbb{P};X)} = \Big\|\sum_k \mathbb{E}_k(\theta_k)x_{i_k}\Big\|_{L_s(\Omega,\mathbb{P};X)}.$$

By these equalities, (2.10) follows from (2.9). □

Let X be a UMD space. Assume that $\{x_i\}_{i\in I}$ is a 1-unconditional basic sequence in X and let $(\theta_k)_k$ be a sequence of functions in $L_\infty(\Omega, \mathbb{P})$. Then by Proposition 2.18 and the contractivity of the following inclusions

$$L_\infty(\Omega, \mathbb{P}; X) \subset L_s(\Omega, \mathbb{P}; X) \subset L_1(\Omega, \mathbb{P}; X),$$

we have

$$\Big\|\sum_k \mathbb{E}_k(\theta_k)x_{i_k}\Big\|_{L_1(\Omega,\mathbb{P};X)} \leq \beta_s(X)\Big\|\sum_k \theta_k x_{i_k}\Big\|_{L_\infty(\Omega,\mathbb{P};X)}.$$

Hence we obtain the following

Proposition 2.19. Let X be a UMD space. Assume that $\{x_i\}_{i\in I}$ is a 1-unconditional basic sequence in X. Then for all $1 < s < \infty$, we have

$$S(X;\{x_i\}) \le \beta_s(X).$$

The main purpose of introducing the constant $S(X)$ is explained by the following theorem, which says that this constant has some kind of multiplicative property. It is this property which will be very useful in studying the "iterated spaces".

Theorem 2.20. Let E be a Banach space with a 1-unconditional basis $\{e_i : i \in I\}$, let F be another Banach space. By definition, $E(F)$ is the completion of the algebraic tensor product $E \otimes F$ under the norm defined as follows: if

$$x = \sum_i e_i \otimes x_i \in E \otimes F,$$

where (x_i) is a finite supported sequence in F, then

$$\|x\|_{E(F)} = \big\|\sum_i e_i \otimes x_i\big\|_{E(F)} := \big\|\sum_i e_i \|x_i\|_F\big\|_E.$$

Given any fixed family of vectors $\{f_j : j \in J\}$ in F, let us consider the following family of vectors in $E(F)$:

$$\{e_i \otimes f_j : i \in I, j \in J\}.$$

We have

$$S(E(F)) \ge S(E)S(F),$$

where $S(E(F))$, $S(E)$ and $S(F)$ are defined with respect to the mentioned families of vectors respectively.

Proof. From the definition, for any $\varepsilon > 0$, there exist finite number of distinct indices $\{i_k : 1 \le k \le N_1\} \subset I$ and $\{j_n : 1 \le n \le N_2\} \subset J$, and there exist functions $\theta_k \in L_\infty(\Omega', \mathbb{P}'), 1 \le k \le N_1$ and functions $\xi_n \in L_\infty(\Omega_0, \mathbb{P}_0), 1 \le n \le N_2$ satisfying

$$\|\sum_k \theta_k e_{i_k}\|_{L_\infty(\Omega',\mathbb{P}';E)} \le 1$$

and

$$\|\sum_n \xi_n f_{j_n}\|_{L_\infty(\Omega_0,\mathbb{P}_0;F)} \le 1,$$

such that

$$\Big\|\sum_k \mathbb{E}^{\mathcal{A}_k}(\theta_k) e_{i_k}\Big\|_{L_1(\Omega',\mathbb{P}';E)} \ge S(E) - \varepsilon$$

and

$$\Big\|\sum_n \mathbb{E}^{\mathcal{B}_n}(\xi_n) f_{j_n}\Big\|_{L_1(\Omega_0,\mathbb{P}_0;F)} \ge S(F) - \varepsilon.$$

We can define a larger probability space

$$(\Omega, \mathbb{P}) = (\Omega' \times \Omega_0^{\mathbb{N}}, \mathbb{P}' \otimes \mathbb{P}_0^{\otimes \mathbb{N}}),$$

the general element in Ω will be denoted by $w = (w', (w_l)_{l\ge 0})$. Consider the σ-algebras $\mathcal{F}_{k,n}$ defined on $(\Omega, \mathbb{P})$ by

$$\mathcal{F}_{k,n} := \mathcal{A}_k \otimes \underbrace{\mathcal{B}_\infty \otimes \cdots \otimes \mathcal{B}_\infty}_{k-1 \text{ times}} \otimes \mathcal{B}_n \otimes \mathcal{C}_{\ge k+1},$$

where $\mathcal{B}_\infty = \sigma(\mathcal{B}_n : n \geq 0)$ is a σ-algebra on $(\Omega_0, \mathbb{P}_0)$, $\mathcal{B}_0$ is assumed to be trivial and $\mathcal{C}_{\geq k+1}$ is the trivial σ-algebra on $(\Omega_0^{\mathbb{N}_{\geq k+1}}, \mathbb{P}_0^{\mathbb{N}_{\geq k+1}})$. It is easy to check that $\mathcal{F}_{k,n}$ is a filtration with respect to the lexigraphic order, i.e. if $(k,n) < (k',n')$ (that is $k < k'$ or $k = k'$ but $n < n'$), then $\mathcal{F}_{k,n} \subset \mathcal{F}_{k',n'}$.

Now let us define $h : \Omega \to E(F)$ by

$$h(w) = h(w', (w_l)) = \sum_{k,n} \theta_k(w')\xi_n(w_k) e_{i_k} \otimes f_{j_n}.$$

Let

$$h_{k,n}(w) = \theta_k(w')\xi_n(w_k),$$

then

$$h = \sum_{k,n} h_{k,n} e_{i_k} \otimes f_{j_n}.$$

Clearly, we have

$$\mathbb{E}^{\mathcal{F}_{k,n}}(h_{k,n})(w) = \left[\mathbb{E}^{\mathcal{A}_k}(\theta_k)\right](w')\left[\mathbb{E}^{\mathcal{B}_n}(\xi_n)\right](w_k),\ a.e. \tag{2.11}$$

By the 1-unconditionality of $\{e_i : i \in I\}$, for a.e. $w \in \Omega$, we have

$$\begin{aligned}
\|h(w)\|_{E(F)} &= \Big\| \sum_{k,n} \theta_k(w')\xi_n(w_k) e_{i_k} \otimes f_{j_n} \Big\|_{E(F)} \\
&= \Big\| \sum_k e_{i_k} \| \sum_n \theta_k(w')\xi_n(w_k) f_{j_n} \|_F \Big\|_E \\
&= \Big\| \sum_k e_{i_k} |\theta_k(w')| \| \sum_n \xi_n(w_k) f_{j_n} \|_F \Big\|_E \\
&\leq \Big\| \sum_k e_{i_k} |\theta_k(w')| \Big\|_E = \Big\| \sum_k e_{i_k} \theta_k(w') \Big\|_E \leq 1.
\end{aligned}$$

Hence

$$\|h\|_{L_\infty(\Omega,\mathbb{P};E(F))} \leq 1.$$

If we denote

$$\widetilde{h} = \sum_{k,n} \mathbb{E}^{\mathcal{F}_{k,n}}(h_{k,n}) e_{i_k} \otimes f_{j_n},$$

then by (2.11),

$$\|\widetilde{h}(w)\|_{E(F)} = \Big\| \sum_k e_{i_k} |\mathbb{E}^{\mathcal{A}_k}(\theta_k)(w')| \| \sum_n \mathbb{E}^{\mathcal{B}_n}(\xi_n)(w_k) f_{j_n} \|_F \Big\|_E, a.e..$$

By Jensen's inequality, we have

$$\begin{aligned}
& \int \Big\| \sum_k e_{i_k} |\mathbb{E}^{\mathcal{A}_k}(\theta_k)(w')| \| \sum_n \mathbb{E}^{\mathcal{B}_n}(\xi_n)(w_k) f_{j_n} \|_F \Big\|_E d\mathbb{P}_0^{\otimes\mathbb{N}}((w_l)) \\
\geq\ & \Big\| \int \sum_k e_{i_k} |\mathbb{E}^{\mathcal{A}_k}(\theta_k)(w')| \| \sum_n \mathbb{E}^{\mathcal{B}_n}(\xi_n)(w_k) f_{j_n} \|_F d\mathbb{P}_0^{\otimes\mathbb{N}}((w_l)) \Big\|_E \\
=\ & \Big\| \sum_k e_{i_k} |\mathbb{E}^{\mathcal{A}_k}(\theta_k)(w')| \Big\|_E \cdot \Big\| \sum_n \mathbb{E}^{\mathcal{B}_n}(\xi_n) f_{j_n} \Big\|_{L_1(\Omega_0,\mathbb{P}_0;F)} \\
=\ & \Big\| \sum_k e_{i_k} \mathbb{E}^{\mathcal{A}_k}(\theta_k)(w') \Big\|_E \cdot \Big\| \sum_n \mathbb{E}^{\mathcal{B}_n}(\xi_n) f_{j_n} \Big\|_{L_1(\Omega_0,\mathbb{P}_0;F)}.
\end{aligned}$$

Note that in the last equality, we used the 1-unconditionality assumption on $\{e_i : i \in I\}$. By integrating both sides with respect to $\int d\mathbb{P}'(w')$, we get

$$\begin{aligned} &\Big\| \sum_{k,n} \mathbb{E}^{\mathcal{F}_{k,n}}(h_{k,n}) e_{i_k} \otimes f_{j_n} \Big\|_{L_1(\Omega,\mathbb{P};E(F))} \\ \geq\ & \Big\| \sum_k \mathbb{E}^{\mathcal{A}_k}(\theta_k) e_{i_k} \Big\|_{L_1(\Omega',\mathbb{P}';E)} \cdot \Big\| \sum_n \mathbb{E}^{\mathcal{B}_n}(\xi_n) f_{j_n} \Big\|_{L_1(\Omega_0,\mathbb{P}_0;F)} \\ \geq\ & (S(E)-\varepsilon)(S(F)-\varepsilon). \end{aligned}$$

Therefore

$$S(E(F)) \geq (S(E)-\varepsilon)(S(F)-\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$S(E(F)) \geq S(E)S(F),$$

as desired. □

To generalise the above theorem, let us recall the p-convexity and q-concavity for Banach lattices, interested readers are referred to [LT79] §1.d. for the details on the p-convexity and q-convavity.

Definition 2.21. Let X be a Banach lattice and let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$.

(i) X is called p-convex if there exists a constant $M < \infty$ such that

$$\|\Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p}\|_X \leq M\Big(\sum_{i=1}^n \|x_i\|_X^p\Big)^{1/p}, \quad \text{if } 1 \leq p < \infty$$

or

$$\| \vee_{i=1}^n |x_i| \|_X \leq M \max_{1\leq i\leq n} \|x_i\|_X, \quad \text{if } p = \infty,$$

for every choice of vectors $\{x_i\}_{i=1}^n$ in X. The smallest possible value of M is denoted by $M^{(p)}(X)$.

(ii) X is called q-concave if there exists a constant $M < \infty$ such that

$$\Big(\sum_{i}^n \|x_i\|_X^q\Big)^{1/q} \leq M\|\Big(\sum_{i=1}^n |x_i|^q\Big)^{1/q}\|_X, \quad \text{if } 1 \leq q < \infty$$

or

$$\max_{1\leq i\leq n} \|x_i\|_X \leq M\| \vee_{i=1}^n |x_i| \|_X, \quad \text{if } q = \infty,$$

for every choice of vectors $\{x_i\}_{i=1}^n$ in X. The smallest possible value of M is denoted by $M_{(q)}(X)$.

Note that every Banach lattice X is 1-convex and ∞-concave with

$$M^{(1)}(X) = M_{(\infty)}(X) = 1.$$

Remark 2.22. Assume that X is a Banach lattice, then it is easy to see that for any measurable function $x : (\Omega, \nu) \to X$, we have

$$\|\Big(\int_\Omega |x(w)|^p d\nu(w)\Big)^{1/p}\|_X \leq M^{(p)}(X) \cdot \Big(\int_\Omega \|x(w)\|_X^p d\nu(w)\Big)^{1/p}$$

and

$$\Big(\int_\Omega \|x(w)\|_X^q d\nu(w)\Big)^{1/q} \leq M_{(q)}(X) \cdot \|\Big(\int_\Omega |x(w)|^q d\nu(w)\Big)^{1/q}\|_X.$$

Definition 2.23. Let X be a Banach space, let $\{x_i\}_{i\in I}$ be any family of vectors in X and let $1 \le p \le q \le \infty$. Define $S_{q,p}(X;\{x_i\})$ to be the best constant M such that

$$\Big\| \sum_{k=0}^{N} \mathbb{E}^{\mathcal{A}_k}(\theta_k) x_{i_k} \Big\|_{L_p(\Omega,\mathbb{P};X)} \le M \Big\| \sum_{k=0}^{N} \theta_k x_{i_k} \Big\|_{L_q(\Omega,\mathbb{P};X)} \tag{2.12}$$

holds for any $N \in \mathbb{N}$, any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n \subset \cdots \subset \mathcal{F}$, any $N+1$ distinct indices $\{i_0, i_1, \cdots, i_N\} \subset I$ and any $N+1$ functions $\theta_0, \theta_1, \cdots, \theta_N$ in $L_q(\Omega, \mathcal{F}, \mathbb{P})$.

If there does not exist such constant, we define $S_{q,p}(X;\{x_i\}) = \infty$.

As we did previously, if the family $\{x_i\}$ is clear from the context, then we will use the notation $S_{q,p}(X)$ instead of $S_{q,p}(X;\{x_i\})$. Note that the previously defined $S(X;\{x_i\})$ is

$$S(X;\{x_i\}) = S_{\infty,1}(X;\{x_i\}).$$

The relation between $S_{q,p}(X)$ and the UMD constant of X is given by the following proposition.

Proposition 2.24. Let $1 < s < \infty$ and $1 \le p \le s \le q \le \infty$. Let X be a UMD space. Assume that $\{x_i\}_{i\in I}$ is a 1-unconditional basic sequence in X. Then we have

$$S_{q,p}(X;\{x_i\}) \le \beta_s(X).$$

Proof. The proof is similar to that of Proposition 2.19. □

Now we can state and prove the following theorem.

Theorem 2.25. Let E be a Banach space with a 1-unconditional basis $\{e_i : i \in I\}$, let F be another Banach space. Assume moreover that E is p-convex and q-concave when it is viewed as a Banach lattice over the I, the corresponding cosntants are denoted by $M^{(p)}(E)$ and $M_{(q)}(E)$. For any fixed family of vectors $\{f_j : j \in J\}$ in F, consider the family of vectors $\{e_i \otimes f_j : i \in I, j \in J\}$. Then we have

$$S_{q,p}(E(F)) \ge \frac{S_{q,p}(E) S_{q,p}(F)}{M^{(p)}(E) M_{(q)}(E)},$$

where $S_{q,p}(E(F))$, $S_{q,p}(E)$ and $S_{q,p}(F)$ are defined with respect to the mentioned families of vectors respectively.

Proof. We will use the notation that was used in the proof of Theorem 2.20. For any $\varepsilon > 0$, there exist finite number of distinct indices $\{i_k : 1 \le k \le N_1\} \subset I$ and $\{j_n : 1 \le n \le N_2\} \subset J$, and there exist functions $\theta_k \in L_q(\Omega', \mathbb{P}'), 1 \le k \le N_1$ and functions $\xi_n \in L_q(\Omega_0, \mathbb{P}_0), 1 \le n \le N_2$ satisfying

$$\| \sum_k \theta_k e_{i_k} \|_{L_q(\Omega',\mathbb{P}';E)} \le 1$$

and

$$\| \sum_n \xi_n f_{j_n} \|_{L_q(\Omega_0,\mathbb{P}_0;F)} \le 1,$$

such that

$$\| \sum_k \mathbb{E}^{\mathcal{A}_k}(\theta_k) e_{i_k} \|_{L_p(\Omega',\mathbb{P}';E)} \ge S_{q,p}(E) - \varepsilon$$

and

$$\|\sum_n \mathbb{E}^{\mathcal{B}_n}(\xi_n)f_{j_n}\|_{L_p(\Omega_0,\mathbb{P}_0;F)} \geq S_{q,p}(F)-\varepsilon.$$

As before, we define the larger probability space

$$(\Omega,\mathbb{P})=(\Omega'\times\Omega_0^{\mathbb{N}},\mathbb{P}'\otimes\mathbb{P}_0^{\otimes\mathbb{N}}),$$

and the general element in Ω will be denoted by $w=(w',(w_l)_{l\geq 0})$. The filtration $(\mathcal{F}_{k,n})$ on $(\Omega,\mathbb{P})$ is defined as before.

Define $h:\Omega\to E(F)$ by

$$h(w)=h(w',(w_l))=\sum_{k,n}\theta_k(w')\xi_n(w_k)e_{i_k}\otimes f_{j_n}=\sum_{k,n}h_{k,n}(w)e_{i_k}\otimes f_{j_n}.$$

Recall that

$$\mathbb{E}^{\mathcal{F}_{k,n}}(h_{k,n})(w)=\Big[\mathbb{E}^{\mathcal{A}_k}(\theta_k)\Big](w')\Big[\mathbb{E}^{\mathcal{B}_n}(\xi_n)\Big](w_k) \quad a.e.$$

For simplify the notation, let us denote

$$G(w_k)=\sum_n \xi_n(w_k)f_{j_n}.$$

We have

$$\begin{aligned}\|h\|_{L_q(\Omega,\mathbb{P};E(F))} &= \Big(\int_\Omega \|\sum_{k,n}\theta_k(w')\xi_n(w_k)e_{i_k}\otimes f_{j_n}\|^q_{E(F)}d\mathbb{P}(w)\Big)^{1/q}\\ &= \Big(\int_\Omega \|\sum_k e_{i_k}|\theta_k(w')|\cdot\|G(w_k)\|_F\|^q_E d\mathbb{P}(w)\Big)^{1/q}\end{aligned}$$

By the q-concavity of E, we have thus

$$\begin{aligned}\|h\|_{L_q(\Omega,\mathbb{P};E(F))} &\leq M_{(q)}(E)\left(\int_{\Omega'}\|\sum_k e_{i_k}\Big(\int_{\Omega_0^{\mathbb{N}}}|\theta_k(w')|^q\|G(w_k)\|^q_F d\mathbb{P}_0^{\otimes\mathbb{N}}((w_l))\Big)^{1/q}\|^q_E d\mathbb{P}(w')\right)^{1/q}\\ &= M_{(q)}(E)\left(\int_{\Omega'}\|\sum_k e_{i_k}|\theta_k(w')|\Big(\int_{\Omega_0^{\mathbb{N}}}\|G(w_k)\|^q_F d\mathbb{P}_0^{\otimes\mathbb{N}}((w_l))\Big)^{1/q}\|^q_E d\mathbb{P}(w')\right)^{1/q}.\end{aligned}$$

Note that for any k, we have

$$\Big(\int_{\Omega_0^{\mathbb{N}}}\|G(w_k)\|^q_F d\mathbb{P}_0^{\otimes\mathbb{N}}((w_l))\Big)^{1/q}=\Big(\int_{\Omega_0}\|G(w_k)\|^q_F d\mathbb{P}_0(w_k)\Big)^{1/q}=\|\sum_n\xi_n f_{j_n}\|_{L_q(\Omega_0,\mathbb{P}_0;F)}\leq 1,$$

hence by the 1-unconditionality of $\{e_i\}$, we get

$$\|h\|_{L_q(\Omega,\mathbb{P};E(F))} \leq M_{(q)}(E)\Big(\int_{\Omega'}\|\sum_k e_{i_k}\theta_k(w')\|^q_E d\mathbb{P}(w')\Big)^{1/q}\leq M_{(q)}(E).$$

Let us denote

$$\widetilde{h}=\sum_{k,n}\mathbb{E}^{\mathcal{F}_{k,n}}(h_{k,n})e_{i_k}\otimes f_{j_n},$$

and

$$\widetilde{G}(w_k)=\sum_n \mathbb{E}^{\mathcal{B}_n}(\xi_n)(w_k)f_{j_n},$$

then

$$\begin{aligned} \|\widetilde{h}(w)\|_{E(F)} &= \Big\| \sum_k e_{i_k} |\mathbb{E}^{\mathcal{A}_k}(\theta_k)(w')| \| \sum_n \mathbb{E}^{\mathcal{B}_n}(\xi_n)(w_k) f_{j_n} \|_F \Big\|_E \\ &= \| \sum_k e_{i_k} \mathbb{E}^{\mathcal{A}_k}(\theta_k)(w') \| \widetilde{G}(w_k) \|_F \big\|_E, \, a.e. \end{aligned}$$

By p-convexity of E, we have

$$\begin{aligned} & M^{(p)}(E) \cdot \Big(\int \| \sum_k e_{i_k} \mathbb{E}^{\mathcal{A}_k}(\theta_k)(w') \| \widetilde{G}(w_k) \|_F \|_E^p d\mathbb{P}_0^{\otimes \mathbb{N}}((w_l)) \Big)^{1/p} \\ \geq & \Big\| \sum_k e_{i_k} |\mathbb{E}^{\mathcal{A}_k}(\theta_k)(w')| \Big(\int \| \widetilde{G}(w_k) \|_F^p d\mathbb{P}_0^{\otimes \mathbb{N}}((w_l)) \Big)^{1/p} \Big\|_E \\ = & \Big\| \sum_k e_{i_k} |\mathbb{E}^{\mathcal{A}_k}(\theta_k)(w')| \Big\|_E \cdot \Big\| \sum_n \mathbb{E}^{\mathcal{B}_n}(\xi_n) f_{j_n} \Big\|_{L_p(\Omega_0, \mathbb{P}_0; F)} \\ = & \Big\| \sum_k e_{i_k} \mathbb{E}^{\mathcal{A}_k}(\theta_k)(w') \Big\|_E \cdot \Big\| \sum_n \mathbb{E}^{\mathcal{B}_n}(\xi_n) f_{j_n} \Big\|_{L_p(\Omega_0, \mathbb{P}_0; F)}. \end{aligned}$$

Note that in the last equality, we used the 1-unconditionality assumption on $\{e_i : i \in I\}$. By taking the L_p norms of both sides, we get

$$\begin{aligned} & M^{(p)}(E) \cdot \|\widetilde{h}\|_{L_p(\Omega, \mathbb{P}; E(F))} \\ = & M^{(p)}(E) \cdot \Big(\int \| \sum_k e_{i_k} \mathbb{E}^{\mathcal{A}_k}(\theta_k)(w') \| \widetilde{G}(w_k) \|_F \|_E^p d\mathbb{P}_0^{\otimes \mathbb{N}}((w_l)) d\mathbb{P}'(w') \Big)^{1/p} \\ \geq & \Big\| \sum_k \mathbb{E}^{\mathcal{A}_k}(\theta_k) e_{i_k} \Big\|_{L_p(\Omega', \mathbb{P}'; E)} \cdot \Big\| \sum_n \mathbb{E}^{\mathcal{B}_n}(\xi_n) f_{j_n} \Big\|_{L_p(\Omega_0, \mathbb{P}_0; F)} \\ \geq & (S_{q,p}(E) - \varepsilon)(S_{q,p}(F) - \varepsilon). \end{aligned}$$

Therefore

$$M^{(p)}(E) \cdot \|\widetilde{h}\|_{L_p(\Omega, \mathbb{P}; E(F))} \geq S_{q,p}(E) S_{q,p}(F).$$

It follows that

$$\frac{\|\widetilde{h}\|_{L_p(\Omega, \mathbb{P}; E(F))}}{\|h\|_{L_q(\Omega, \mathbb{P}; E(F))}} \geq \frac{S_{q,p}(E) S_{q,p}(F)}{M^{(p)}(E) M_{(q)}(E)}.$$

By definition of $S_{q,p}(E(F); \{e_i \otimes f_j\})$, we have

$$S_{q,p}(E(F)) \geq \frac{S_{q,p}(E) S_{q,p}(F)}{M^{(p)}(E) M_{(q)}(E)}.$$

as desired.

□

Lemma 2.26. Suppose that $1 \leq p \neq q \leq \infty$. If $E_1 = \ell_p^{(2)}(\ell_q^{(2)})$, then

$$S(E_1) \geq c(p, q) > 1.$$

Proof. Let us denote the canonical basis of $\ell_p^{(2)}$ and $\ell_q^{(2)}$ by $\{e_1^p, e_2^p\}$, $\{e_1^q, e_2^q\}$ respectively. Then

$$\Big\{ e_1^p \otimes e_1^q, e_1^p \otimes e_2^q, e_2^p \otimes e_1^q, e_2^p \otimes e_2^q \Big\}$$

is the canonical 1-unconditional basis of $\ell_p^{(2)}(\ell_q^{(2)})$. Consider the probability space (D,μ) equipped with the filtration

$$\{\emptyset, D\} \subset \sigma(\varepsilon),$$

where ε is the identity function on D. Note that the σ-algebra $\{\emptyset, D\}$ is the trivial one, so the conditional expectation with respect to $\{\emptyset, D\}$ is just the usual expectation. Define a linear map

$$T : L_\infty(D; E_1) \to L_1(D; E_1)$$

by setting

$$T\Big[a_{ij}(\varepsilon)e_i^p \otimes e_j^q\Big] = \begin{cases} \mathbb{E}(a_{ij})e_i^p \otimes e_j^q, \text{ if } j = 1, \\ \\ a_{ij}(\varepsilon)e_i^p \otimes e_j^q, \text{ if } j = 2. \end{cases}$$

By definition of $S(E_1)$ we have

$$S(E_1) \geq \|T\|_{L_\infty(D;E_1)\to L_1(D;E_1)}.$$

Now for any a, b two scalar functions on D , consider

$$f(\varepsilon) = e_1^p \otimes \Big[a(\varepsilon)e_1^q + b(\varepsilon)e_2^q\Big] + e_2^p \otimes \Big[a(-\varepsilon)e_1^q + b(-\varepsilon)e_2^q\Big].$$

Then

$$(Tf)(\varepsilon) = e_1^p \otimes \Big[\mathbb{E}(a)e_1^q + b(\varepsilon)e_2^q\Big] + e_2^p \otimes \Big[\mathbb{E}(a)e_1^q + b(-\varepsilon)e_2^q\Big].$$

If p, q are both finite, then for any fixed $\varepsilon \in D$, we have

$$\begin{aligned} \|f(\varepsilon)\|_{E_1} &= \Big\{(|a(\varepsilon)|^q + |b(\varepsilon)|^q)^{p/q} + (|a(-\varepsilon)|^q + |b(-\varepsilon)|^q)^{p/q}\Big\}^{1/p} \\ &= \Big\{(|a(1)|^q + |b(1)|^q)^{p/q} + (|a(-1)|^q + |b(-1)|^q)^{p/q}\Big\}^{1/p} \\ &= 2^{1/p}\Big\{\frac{1}{2}(|a(1)|^q + |b(1)|^q)^{p/q} + \frac{1}{2}(|a(-1)|^q + |b(-1)|^q)^{p/q}\Big\}^{1/p} \\ &= 2^{1/p}\Big\{\int(|a(\varepsilon)|^q + |b(\varepsilon)|^q)^{p/q}d\mu(\varepsilon)\Big\}^{1/p} \\ &= 2^{1/p}\big\|(a,b)\big\|_{L_p(\mu;\ell_q^{(2)})}. \end{aligned}$$

Similarly,

$$\|(Tf)(\varepsilon)\|_{E_1} = 2^{1/p}\big\|(\mathbb{E}a, b)\big\|_{L_p(\mu;\ell_q^{(2)})}.$$

It follows that

$$\|f\|_{L_\infty(D;E_1)} = 2^{1/p}\big\|(a,b)\big\|_{L_p(\mu;\ell_q^{(2)})}$$

and

$$\|Tf\|_{L_1(D;E_1)} = 2^{1/p}\big\|(\mathbb{E}a, b)\big\|_{L_p(\mu;\ell_q^{(2)})}.$$

Hence

$$\|T\|_{L_\infty(D;E_1)\to L_1(D;E_1)} \geq \frac{\|Tf\|_{L_1(D;E_1)}}{\|f\|_{L_\infty(D;E_1)}} = \frac{\big\|(\mathbb{E}a,b)\big\|_{L_p(\mu;\ell_q^{(2)})}}{\big\|(a,b)\big\|_{L_p(\mu;\ell_q^{(2)})}}. \tag{2.13}$$

Similarly, if $q = \infty$ and p is finite, then

$$\|f\|_{L_\infty(D;E_1)} = 2^{1/p}\big\|(a,b)\big\|_{L_p(\mu;\ell_\infty^{(2)})}$$

and

$$\|Tf\|_{L_1(D;E_1)} = 2^{1/p}\big\|(\mathbb{E}a, b)\big\|_{L_p(\mu;\ell_\infty^{(2)})}.$$

If $p = \infty$ and q is finite, then

$$\|f\|_{L_\infty(D;E_1)} = \big\|(a, b)\big\|_{L_\infty(\mu;\ell_q^{(2)})}$$

and

$$\|Tf\|_{L_1(D;E_1)} = \big\|(\mathbb{E}a, b)\big\|_{L_\infty(\mu;\ell_q^{(2)})}.$$

Therefore, (2.13) holds in full generality. By Proposition 2.8, we have

$$\|T\|_{L_\infty(D;E_1)\to L_1(D;E_1)} \geq \|P\|_{L_p(\mu;\ell_q^{(2)})\to L_p(\mu;\ell_q^{(2)})} = c(p, q).$$

Hence

$$S(E_1) \geq c(p, q) > 1,$$

as desired. □

Remark 2.27. Let $1 \leq p < \infty$ and let $(e_k)_{k\geq 0}$ be the canonical basis of $\ell_p = \ell_p(\mathbb{N})$, then

$$S(\ell_p) = 1.$$

Indeed, if $(\theta_k)_{k\geq 0}$ is a finite sequence of functions, then

$$\begin{aligned}
\Big\|\sum_k \mathbb{E}_k(\theta_k)e_k\Big\|_{L_1(\ell_p)} &\leq \Big\|\sum_k \mathbb{E}_k(\theta_k)e_k\Big\|_{L_p(\ell_p)} = \Big\|\Big(\sum_k |\mathbb{E}_k(\theta_k)|^p\Big)^{1/p}\Big\|_{L_p} \\
&= \Big\|\sum_k |\mathbb{E}_k(\theta_k)|^p\Big\|_{L_1}^{1/p} = \Big(\sum_k \|\mathbb{E}_k(\theta_k)\|_p^p\Big)^{1/p} \\
&\leq \Big(\sum_k \|\theta_k\|_p^p\Big)^{1/p} = \Big\|\sum_k \theta_k e_k\Big\|_{L_p(\ell_p)} \\
&\leq \Big\|\sum_k \theta_k e_k\Big\|_{L_\infty(\ell_p)}.
\end{aligned}$$

For $p = \infty$, let us consider c_0. We denote again the canonical basis of c_0 by $(e_k)_{k\geq 0}$. Then for any $(\theta_k)_{k\geq 0}$ finite sequence of functions, there exists a finite sequence of functions $(\eta_k)_{k\geq 0}$ such that

$$\sum_k |\eta_k| \leq 1 \, a.e.$$

and

$$\sup_k |\mathbb{E}_k(\theta_k)| = \sum_k \mathbb{E}_k(\theta_k)\eta_k.$$

Hence we have

$$\begin{aligned}
\Big\|\sum_k \mathbb{E}_k(\theta_k)e_k\Big\|_{L_1(c_0)} &= \Big\|\sup_k |\mathbb{E}_k(\theta_k)|\Big\|_{L_1} = \Big\|\sum_k \mathbb{E}_k(\theta_k)\eta_k\Big\|_{L_1} \\
&= \int \sum_k \mathbb{E}_k(\theta_k)\eta_k d\mathbb{P} = \int \sum_k \theta_k \mathbb{E}_k(\eta_k) d\mathbb{P} \\
&\leq \int \sup_k |\theta_k| \sum_k |\mathbb{E}_k(\eta_k)| d\mathbb{P} \leq \Big\|\sup_k |\theta_k|\Big\|_{L_\infty} \int \sum_k |\mathbb{E}_k(\eta_k)| d\mathbb{P} \\
&\leq \Big\|\sup_k |\theta_k|\Big\|_{L_\infty} \int \sum_k |\eta_k| d\mathbb{P} \leq \Big\|\sup_k |\theta_k|\Big\|_{L_\infty} \\
&= \Big\|\sum_k \theta_k e_k\Big\|_{L_\infty(c_0)},
\end{aligned}$$

hence

$$S(c_0) = 1.$$

We are ready for proving the main theorem in this section.

Theorem 2.28. Suppose that $1 \le p, q \le \infty$. Let $E_1 = \ell_p^{(2)}(\ell_q^{(2)})$ and define by recursion:

$$E_{n+1} = \ell_p^{(2)}(\ell_q^{(2)}(E_n)).$$

Then for any $1 < s < \infty$, we have

$$\beta_s(E_n) \ge S(E_n) \ge c(p,q)^n,$$

where $S(E_n)$ is computed with respect to the canonical basis of E_n. In particular, if $p \ne q$, then $\beta_s(E_n)$ has at least an exponential growth with respect to n and

$$\lim_n \beta_s(E_n) = \infty.$$

Proof. By Theorem 2.20,

$$S(E_{n+1}) \ge S(\ell_p^{(2)}(\ell_q^{(2)}))S(E_n).$$

By Lemma 2.26, we have

$$S(E_{n+1}) \ge c(p,q)S(E_n).$$

It follows that

$$S(E_n) \ge c(p,q)^n.$$

Since the canonical basis of E_n is 1-unconditional, by Proposition 2.19, for any $1 < s < \infty$, we have

$$\beta_s(E_n) \ge S(E_n) \ge c(p,q)^n.$$

□

The following simple observation shows that the exponential growth of $\beta_s(E_n)$ is optimal.

Proposition 2.29. Suppose $1 < p \ne q < \infty$. Let X be a Banach space. Define by recursion: $Y_0 = X$ and $Y_{n+1} = L_p(\mathbb{T}; L_q(\mathbb{T}; Y_n))$. Then for all $1 < s < \infty$, there exists $\chi = \chi(p,q,s)$, such that

$$\beta_s(Y_n) \le \chi^n \beta_s(X).$$

Proof. We will use the following well-known fact (see e.g. [Bur81, Bur83]) about UMD constants: for any $1 < r, s < \infty$, there exist $\alpha(r,s)$ and $\beta(r,s)$ such that for all Banach space X,

$$\alpha(r,s)\beta_s(X) \le \beta_r(X) \le \beta(r,s)\beta_s(X). \tag{2.14}$$

We will also use the elementary identity $\beta_s(L_s(X)) = \beta_s(X)$. Combining these, we have

$$\begin{aligned} \beta_s(Y_{n+1}) &= \beta_s(L_p(L_q(Y_n))) \le \beta(s,p)\beta_p(L_p(L_q(Y_n))) \\ &= \beta(s,p)\beta_p(L_q(Y_n)) \le \beta(s,p)\beta(p,q)\beta_q(L_q(Y_n)) \\ &= \beta(s,p)\beta(p,q)\beta_q(Y_n) \le \beta(s,p)\beta(p,q)\beta(q,s)\beta_s(Y_n). \end{aligned}$$

Let $\chi = \beta(s,p)\beta(p,q)\beta(q,s)$, then $\beta_s(Y_n) \le \chi^n \beta_s(Y_0) = \chi^n \beta_s(X)$. □

Remark 2.30. Even if one of p, q is infinite or equals to 1, then since $\dim(E_n) = 4^n$, we have $\beta_s(E_n) \lesssim \sqrt{\dim E_n} = 2^n$. Indeed, the Banach-Mazur distance between E_n and $\ell_2^{\dim E_n}$ is $\leq \sqrt{\dim E_n}$ (cf. e.g. [TJ89]).

Let us denote the E_n defined as above using $p = \infty, q = 1$ by $E_n(\infty, 1)$. Then we have

$$\dim E_n(\infty, 1) = 4^n.$$

Moreover, since $c(\infty, 1) = \frac{3}{2}$, by Theorem 2.28, we have

$$\beta_s(E_n(\infty, 1)) \geq \Big(\frac{3}{2}\Big)^n,$$

for all $1 < s < \infty$. Thus we reprove the following result originally proved Bourgain by different method.

Proposition 2.31. For each n, there exists an n-dimensional lattice such that (say) its UMD$_2$ constant is at least of the order n^θ with

$$\theta = \frac{\log(3/2)}{\log 4}.$$

Let θ_M be the largest possible θ appears in Proposition 2.31. Since for any n-dimensional space X, the Banach-Mazur distance $d_{BM}(X, \ell_2^n) \leq \sqrt{n}$. Necessarily, we have

$$\theta_M \leq \frac{1}{2}.$$

Related open problem due to Bourgain will formulated in the end of this chapter. Some ideas on this open problem will also be given.

2.4 Analytic UMD constants

The main idea in §2.3 can be easily adapted for treating the analytic UMD property. In this section, all spaces are over $\mathbb{C}$.

Denote the general element in $\mathbb{T}^{\mathbb{N}}$ be $z = (z_n)_{n \geq 0}$ and let $m_\infty = m^{\otimes \mathbb{N}}$ be the Haar measure on $\mathbb{T}^{\mathbb{N}}$. Recall the canonical filtration on $(\mathbb{T}^{\mathbb{N}}, m_\infty)$ defined by

$$\sigma(z_0) \subset \sigma(z_0, z_1) \subset \cdots \subset \sigma(z_0, z_1, \cdots, z_n) \subset \cdots.$$

From now on, we will denote $\mathcal{G}_n = \sigma(z_0, z_1, \cdots, z_n)$. Recall that $H_s(\mathbb{T}^{\mathbb{N}})$ is the subspace of $L_s(\mathbb{T}^{\mathbb{N}}, m_\infty)$ consisting of limit values of Hardy martingales, i.e. $f \in H_s(\mathbb{T}^{\mathbb{N}})$ if and only if $f \in L_s(\mathbb{T}^{\mathbb{N}}, m_\infty)$ and the associated martingale $(\mathbb{E}^{\mathcal{G}_n} f)_{n \geq 0}$ is a Hardy martingale. For convenience, we always assume $z_0 \equiv 1$ such that $\mathcal{G}_0$ is a trivial σ-algebra.

Definition 2.32. Let X be a Banach space and let $\{x_i\}_{i \in I}$ be a family of vectors in X. The number $S^a(X; \{x_i\})$ is defined to be the best constant C such that for any $N \in \mathbb{N}$ and any finite sequence of functions $(\theta_k)_{k=0}^N$ in $H_\infty(\mathbb{T}^{\mathbb{N}})$, we have

$$\Big\| \sum_k \mathbb{E}^{\mathcal{G}_k}(\theta_k) x_{i_k} \Big\|_{L_1(m_\infty; X)} \leq C \Big\| \sum_k \theta_k x_{i_k} \Big\|_{L_\infty(m_\infty; X)}.$$

If there does not exist such constant, we set $S^a(X; \{x_i\}) = \infty$.

If $\{x_i\}$ is clear from the context, then $S^a(X;\{x_i\})$ will be simplified as $S^a(X)$.

The Stein type inequality still holds in this setting, more precisely, we have

Proposition 2.33. Let X be an AUMD space. For any $1 \le s < \infty$, let $(F_k)_{k\ge 0}$ be an arbitrary finite sequence in $H_s(\mathbb{T}^{\mathbb{N}};X)$. Then we have

$$\Big\|\sum_k \zeta_k \mathbb{E}^{\mathcal{G}_k}(F_k)(z)\Big\|_{L_s(X)} \le \beta_s^a(X)\Big\|\sum_k \zeta_k F_k(z)\Big\|_{L_s(X)}, \tag{2.15}$$

where $\zeta=(\zeta_k)_{k\ge 0}$ is an independent copy of $z=(z_k)_{k\ge 0}$ and $L_s(X)=L_s(\mathbb{T}_z^{\mathbb{N}}\times\mathbb{T}_\zeta^{\mathbb{N}}, m_\infty\times m_\infty;X)$.

Proof. Consider the filtration on $\mathbb{T}_z^{\mathbb{N}}\times\mathbb{T}_\zeta^{\mathbb{N}}$ defined by

$$\mathcal{B}_{2j}=\sigma(z_0,\cdots,z_j)\otimes\sigma(\zeta_0,\cdots,\zeta_j),$$

$$\mathcal{B}_{2j-1}=\sigma(z_0,\cdots,z_j)\otimes\sigma(\zeta_0,\cdots,\zeta_{j-1}).$$

Then

$$f=\sum_k \zeta_k F_k(z)$$

is a Hardy martingale with respect to the above filtration. Let

$$f'=\sum_k \zeta_k\mathbb{E}^{\mathcal{G}_k}(F_k).$$

Then we have

$$f'=\sum_j(\mathbb{E}^{\mathcal{B}_{2j}}-\mathbb{E}^{\mathcal{B}_{2j-1}})(f).$$

It follows (see Remark 2.17) that

$$\|f'\|_{L_s(X)}\le\beta_s^a(X)\|f\|_{L_s(X)},$$

whence (2.15). $\square$

Proposition 2.34. Let X be an AUMD space. Assume that $\{x_i\}_{i\in I}$ is a 1-unconditional basic sequence in X. Then for any $1\le s<\infty$ and any finite sequence of functions $(\theta_k)_{k\ge0}$ in $H_s(\mathbb{T}^{\mathbb{N}})$,

$$\Big\|\sum_k \mathbb{E}^{\mathcal{G}_k}(\theta_k)x_{i_k}\Big\|_{L_s(m_\infty;X)}\le\beta_s^a(X)\Big\|\sum_k\theta_k x_{i_k}\Big\|_{L_s(m_\infty;X)}.$$

Proof. It follows verbatim the proof of Proposition 2.18. $\square$

Let X be as in Proposition 2.34, $\{x_i\}$ is a 1-unconditional basic sequence in X. Then for all $1\le s<\infty$, we have

$$\Big\|\sum_k \mathbb{E}^{\mathcal{G}_k}(\theta_k)x_{i_k}\Big\|_{L_1(m_\infty;X)}\le\beta_s^a(X)\Big\|\sum_k\theta_k x_{i_k}\Big\|_{L_\infty(m_\infty;X)}.$$

Hence we have proven the following proposition.

Proposition 2.35. Let X be a AUMD space. Assume that $\{x_i\}_{i\in I}$ is a 1-unconditional basic sequence in X. Then for all $1\le s<\ \infty$, we have

$$S^a(X;\{x_i\})\le\beta_s^a(X).$$

Theorem 2.36. Let E be a Banach space with a 1-unconditional basis $\{e_i : i \in I\}$, let F be another Banach space. Let $E(F)$ be defined as in Theorem 2.20. For any fixed family of vectors $\{f_j : j \in J\}$ in F, consider the family of vectors $\{e_i \otimes f_j : i \in I, j \in J\}$ in $E(F)$, then we have

$$S^a(E(F)) \geq S^a(E)S^a(F),$$

where $S^a(E(F))$, $S^a(E)$ and $S^a(F)$ are defined with respect to the mentioned families of vectors respectively.

Proof. The proof is similar to the proof of Theorem 2.20. We mention the slight difference concerning the filtration. Consider the infinite tensor product $L_\infty(\mathbb{T}^\mathbb{N}) \otimes L_\infty(\mathbb{T}^\mathbb{N}) \otimes \cdots$, define

$$z_{k,n} = \underbrace{1 \otimes \cdots \otimes 1}_{k \quad \text{times}} \otimes z_n \otimes 1 \otimes \cdots \text{, if } n \geq 1$$

and

$$z_{k,0} = z_k \otimes 1 \otimes 1 \otimes \cdots .$$

Then the filtration defined by $\mathcal{F}^a_{k,n} := \sigma(z_j : j \leq (k,n))$ is an analytic filtration, where the order on $\mathbb{N} \times \mathbb{N}$ is the lexigraphic order as defined in the proof of Theorem 2.20. This filtration plays the role similar to that of $(\mathcal{F}_{k,n})_{k,n}$ in the proof of Theorem 2.20. Note that we may restrict to the functions θ_k, ξ_n depending only on finitely many variables. Thus only a finite subset of $\mathbb{N} \times \mathbb{N}$ is used. □

The following lemma requires slightly more efforts than Lemma 2.26.

Lemma 2.37. Suppose that $1 \leq p \neq q \leq \infty$. If $E_1 = \ell_p^{(2)}(\ell_q^{(2)})$, then

$$S^a(E_1) \geq \kappa(p,q) > 1.$$

Proof. We will use the notation in the proof of Lemma 2.26. Define a linear map

$$U : H_\infty(\mathbb{T}, m; E_1) \to H_1(\mathbb{T}, m; E_1)$$

by

$$U\Big[a_{ij}(z)e_i^p \otimes e_j^q\Big] = \begin{cases} \mathbb{E}(a_{ij})e_i^p \otimes e_j^q, & \text{if } j = 1, \\ \\ a_{ij}(z)e_i^p \otimes e_j^q, & \text{if } j = 2. \end{cases}$$

We have

$$S^a(E_1) \geq \|U\|_{H_\infty(E_1) \to H_1(E_1)}.$$

For simplifying the notation, let us denote

$$C = \|U\|_{H_\infty(E_1) \to H_1(E_1)}.$$

By definition, for any a, b, c, d functions in $H_\infty(\mathbb{T})$, we have

$$\begin{aligned} &\int_\mathbb{T} \Big\{(|\mathbb{E}a|^q + |b(z)|^q)^{p/q} + (|\mathbb{E}c|^q + |d(z)|^q)^{p/q}\Big\}^{1/p} dm(z) \\ \leq\ & C \sup_{z \in \mathbb{T}} \Big\{(|a(z)|^q + |b(z)|^q)^{p/q} + (|c(z)|^q + |d(z)|^q)^{p/q}\Big\}^{1/p}. \end{aligned} \tag{2.16}$$

As usual, if $q = \infty$ and $1 \le p < \infty$, the above inequality is understood as

$$\begin{aligned}
&\int_{\mathbb{T}} \Big\{ \max\{|\mathbb{E}a|, |b(z)|\}^p + \max\{|\mathbb{E}c|, |d(z)|\}^p \Big\}^{1/p} dm(z) \\
\le\ & C \sup_{z\in\mathbb{T}} \Big\{ \max\{|a(z)|, |b(z)|\}^p + \max\{|c(z)|, |d(z)|\}^p \Big\}^{1/p}.
\end{aligned}$$

If $p = \infty$ and $1 \le q < \infty$, it is understood as

$$\begin{aligned}
&\int_{\mathbb{T}} \max\Big\{ (|\mathbb{E}a|^q + |b(z)|^q)^{1/q}, (|\mathbb{E}c|^q + |d(z)|^q)^{1/q} \Big\} dm(z) \\
\le\ & C \sup_{z\in\mathbb{T}} \max\Big\{ (|a(z)|^q + |b(z)|^q)^{1/q}, (|c(z)|^q + |d(z)|^q)^{1/q} \Big\}.
\end{aligned}$$

In the sequel, we treat only the case where both p and q are finite. The other cases can be treated similarly. Note that if a, c are outer functions, then by (2.7), we have

$$|\mathbb{E}a| = M(|a|) \quad \text{and} \quad |\mathbb{E}c| = M(|c|).$$

So for any functions $a, b, c, d \in H_\infty(\mathbb{T})$ such that a, c are outer, we have

$$\begin{aligned}
&\int_{\mathbb{T}} \Big\{ (M(|a|)^q + |b(z)|^q)^{p/q} + (M(|c|)^q + |d(z)|^q)^{p/q} \Big\}^{1/p} dm(z) \qquad (2.17) \\
\le\ & C \sup_{z\in\mathbb{T}} \Big\{ (|a(z)|^q + |b(z)|^q)^{p/q} + (|c(z)|^q + |d(z)|^q)^{p/q} \Big\}^{1/p}.
\end{aligned}$$

By the classical Szegö's condition, if a', b', c', d' are functions in $L_\infty(\mathbb{T})$ which are bounded from below, then there are outer functions $a, b, c, d \in H_\infty(\mathbb{T})$, such that

$$|a'| = |a|,\ |b'| = |b|,\ |c'| = |c|,\ |d'| = |d| \ a.e.$$

Hence (2.17) still holds for any 2-valued non-vanishing functions $a, b, c, d \in L_\infty(\mathbb{T})$ (note that for a function taking only two values, non-vanishing is the same as bounded from below). By approximation, we can further relax the non-vanishing condition on b, d. Now consider any measurable partition $\mathbb{T} = A \dot\cup B$, such that $m(A) = m(B) = \frac{1}{2}$. If $a = u\chi_A + v\chi_B$, $c = v\chi_A + u\chi_B$, $b = w\chi_A + t\chi_B$ and $d = t\chi_A + w\chi_B$, then

$$\begin{aligned}
&\Big\{ (|a(z)|^q + |b(z)|^q)^{p/q} + (|c(z)|^q + |d(z)|^q)^{p/q} \Big\}^{1/p} \\
=\ & \begin{cases} \Big\{ (|u|^q + |w|^q)^{p/q} + (|v|^q + |t|^q)^{p/q} \Big\}^{1/p}, & \text{if } z \in A, \\[2ex] \Big\{ (|v|^q + |t|^q)^{p/q} + (|u|^q + |w|^q)^{p/q} \Big\}^{1/p}, & \text{if } z \in B. \end{cases} \\
\equiv\ & \Big\{ (|u|^q + |w|^q)^{p/q} + (|v|^q + |t|^q)^{p/q} \Big\}^{1/p} \\
=\ & 2^{1/p} \Big\{ \int_{\mathbb{T}} (|a|^q + |b|^q)^{p/q} dm \Big\}^{1/p}.
\end{aligned}$$

That is $\Big\{ (|a(z)|^q + |b(z)|^q)^{p/q} + (|c(z)|^q + |d(z)|^q)^{p/q} \Big\}^{1/p}$ is a constant function. Hence

$$\begin{aligned}
&\sup_{z\in\mathbb{T}} \Big\{ (|a(z)|^q + |b(z)|^q)^{p/q} + (|c(z)|^q + |d(z)|^q)^{p/q} \Big\}^{1/p} \\
=\ & 2^{1/p} \Big\{ \int_{\mathbb{T}} (|a|^q + |b|^q)^{p/q} dm \Big\}^{1/p}.
\end{aligned}$$

Similarly $\Big\{(M(|a|)^q + |b(z)|^q)^{p/q} + (M(|c|)^q + |d(z)|^q)^{p/q}\Big\}^{1/p}$ is a constant function

$$\begin{aligned} &\Big\{(M(|a|)^q + |b(z)|^q)^{p/q} + (M(|c|)^q + |d(z)|^q)^{p/q}\Big\}^{1/p} \\ \equiv\ & 2^{1/p}\Big\{\int_{\mathbb{T}} (M(|a|)^q + |b|^q)^{p/q} dm\Big\}^{1/p}. \end{aligned}$$

Hence

$$\begin{aligned} &\int_{\mathbb{T}} \Big\{(M(|a|)^q + |b(z)|^q)^{p/q} + (M(|c|)^q + |d(z)|^q)^{p/q}\Big\}^{1/p} dm \\ =\ & 2^{1/p}\Big\{\int_{\mathbb{T}} (M(|a|)^q + |b|^q)^{p/q} dm\Big\}^{1/p}. \end{aligned}$$

Substituting these equalities to (2.17), we get

$$\Big\{\int_{\mathbb{T}} (M(|a|)^q + |b|^q)^{p/q} dm\Big\}^{1/p} \leq C\Big\{\int_{\mathbb{T}} (|a|^q + |b|^q)^{p/q} dm\Big\}^{1/p}.$$

By Proposition 2.14, we have

$$C \geq \kappa(p,q).$$

This completes the proof. □

Theorem 2.38. Suppose that $1 \leq p \neq q < \infty$. If E_n's are defined as in Theorem 2.28, then for any $1 \leq s < \infty$, we have

$$\beta_s^a(E_n) \geq S^a(E_n) \geq \kappa(p,q)^n.$$

Moreover, there exists $\kappa_2 = \kappa_2(p,q,s)$, such that

$$\beta_s^a(E_n) \leq \kappa_2^n.$$

Proof. The first part of proof is identical to the proof of Theorem 2.28. The second part of proof is identical to the proof of Proposition 2.29, where we use the equivalence between property AUMD$_s$ and property AUMD$_r$ for all $1 \leq s, r < \infty$, see the appendix for the details. □

2.5 Some constructions

For the sake of clearness, we introduce the family $X_n(p,q)$, which is defined as follows: Let $X_0(p,q) = \mathbb{R}$, and define by recursion that

$$X_{n+1}(p,q) = L_p(D,\mu; L_q(D,\mu; X_n(p,q))).$$

In the complex case, $X_n^{\mathbb{C}}(p,q)$ is defined similarly.

Obviously, $X_n(p,q)$ is isometric to E_n defined in the previous sections using p, q. Our main purpose for introducing X_n's is the existence of canonical isometric inclusion $X_n(p,q) \subset X_{n+1}(p,q)$. By these inclusions, the union $\cup_n X_n(p,q)$ is a normed space and its completion will be denoted by $X(p,q)$. We have

$$X(p,q) := \overline{\cup_n X_n(p,q)} \simeq \varinjlim X_n(p,q),$$

where the last term is the inductive limit of $X_n(p,q)$'s associated to the canonical inclusions. In the complex case, $X^{\mathbb{C}}(p,q)$ is defined similarly.

Remark 2.39. If $1 \leq p = q < \infty$, then $X(p,p)$ is the real space $L^p_{\mathbb{R}}(D^{\mathbb{N}}, \mu^{\otimes \mathbb{N}})$ and $X^{\mathbb{C}}(p,p)$ is the complex space $L^p_{\mathbb{C}}(D^{\mathbb{N}}, \mu^{\otimes \mathbb{N}})$.

We have the following complex interpolation result.

Proposition 2.40. Let $1 < p_0, p_1, q_0, q_1 < \infty$ and $0 < \theta < 1$. Then we have the following isometric isomorphism:

$$X^{\mathbb{C}}(p_\theta, q_\theta) = [X^{\mathbb{C}}(p_0, q_0), X^{\mathbb{C}}(p_1, q_1)]_\theta,$$

with $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_0}$ and $\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_0}$.

Proof. Note that $X(p,q)$ is a Banach lattice of functions on $(D^{\mathbb{N}}, \mu^{\otimes \mathbb{N}})$. Clearly, $X(p,q)$ is $\min(p,q)$-convex and $\max(p,q)$-concave in the sense of §1.d in [LT79], and hence by Theorem 1.f.1 (p. 80) and Proposition 1.e.3 (p. 61) in [LT79] it is reflexive. Then the above result is a particular case of a classical formula going back to Calderón ([Cal64], p. 125). □

Recall that a Banach space X over the complex field is θ-Hilbertian ($0 \leq \theta \leq 1$) if there exists an interpolation pair (X_0, X_1) of Banach spaces such that X is isometric with $[X_0, X_1]_\theta$ and X_1 is a Hilbert space.

Corollary 2.41. Let $1 < p \neq q < \infty$. Then $X(p,q)$ is non-UMD and $X^{\mathbb{C}}(p,q)$ is non-AUMD. Moreover, there exists $0 < \theta < 1$ such that $X^{\mathbb{C}}(p,q)$ is θ-Hilbertian. In particular, $X^{\mathbb{C}}(p,q)$ and a fortiori $X(p,q)$ is super-reflexive.

Proof. It follows easily from Theorem 2.28 and Theorem 2.38 that $X(p,q)$ is non-UMD and $X^{\mathbb{C}}(p,q)$ is non-AUMD.

For $0 < \theta < 1$ small enough, such that $\max(\frac{1/p-\theta/2}{1-\theta}, \frac{1/q-\theta/2}{1-\theta}) < 1$, we can find $1 < \tilde{p}, \tilde{q} < \infty$ satisfying the equalities:

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{\tilde{p}}, \quad \frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{\tilde{q}}.$$

By Proposition 2.40, we have

$$X^{\mathbb{C}}(p,q) = [X^{\mathbb{C}}(\tilde{p}, \tilde{q}), X^{\mathbb{C}}(2,2)]_\theta.$$

Since $X^{\mathbb{C}}(2,2) = L^2_{\mathbb{C}}(D^{\mathbb{N}}, \mu^{\otimes \mathbb{N}})$ is Hilbertian, $X^{\mathbb{C}}(p,q)$ is θ-Hilbertian. The super-reflexivity of $X^{\mathbb{C}}(p,q)$ follows from the well-known fact that any θ-Hilbertian space is super-reflexive for $\theta > 0$ (cf.[Pis79]). □

Remark 2.42. Let $1 < p \neq q < \infty$. For any $0 < \eta < 1$, let $\frac{1}{p_\eta} = \frac{1-\eta}{p} + \frac{\eta}{q}$ and $\frac{1}{q_\eta} = \frac{1-\eta}{q} + \frac{\eta}{p}$. By Proposition 2.40, we have

$$X^{\mathbb{C}}(p_\eta, q_\eta) = [X^{\mathbb{C}}(p,q), X^{\mathbb{C}}(q,p)]_\eta.$$

Note that in this interpolation scale, there is only one UMD space corresponding to $\eta = \frac{1}{2}$.

Remark 2.43. If $1 < p \neq q < \infty$, then $\ell_2(\{X_n(p,q)\})$, the ℓ_2-sum of the spaces $X_n(p,q)$, is also a natural example of super-reflexive non-UMD Banach lattice.

In the following, for simplifying our notation, we will write $L_{p_1}L_{p_2} = L_{p_1}(L_{p_2})$, $L_{p_1}L_{p_2}L_{p_3} = L_{p_1}(L_{p_2}(L_{p_3}))$, etc. Thus we have

$$X_n(p,q) = \underbrace{L_pL_q \cdots L_pL_q}_{n \text{ times } L_pL_q},$$

where $L_p = L_p(D,\mu)$ and $L_q = L_q(D,\mu)$ are two dimensional function spaces on (D,μ). The inclusions $X_n(p,q) \subset X_{n+1}(p,q)$ that are used for defining $X(p,q)$ are indicated by

$$X_{n+1}(p,q) = L_p(L_q(X_n(p,q))) = L_pL_q \underbrace{L_pL_q \cdots L_pL_q}_{X_n(p,q)}.$$

For futher discussions, let us now turn to the non-atomic case. Let $1 < p, q < \infty$, define

$$Z_n(p,q) = \underbrace{L_pL_q \cdots L_pL_q}_{n \text{ times } L_pL_q},$$

where $L_p = L_p(\mathbb{T}, m)$ and $L_q = L_q(\mathbb{T}, m)$. Clearly, $Z_n(p,q)$ can be embedded isometrically in different ways into $Z_{n+1}(p,q)$. However, in the sequel, we will only use the inclusions $Z_n(p,q) \subset Z_{n+1}(p,q)$ which are indicated by:

$$Z_{n+1}(p,q) = \underbrace{L_pL_q \cdots L_pL_q}_{Z_n(p,q)} L_pL_q.$$

More precisely, if $Z_{n+1}(p,q)$ is viewed as a function space over $\mathbb{T}^{2n+2}$, then $Z_n(p,q)$ is the subspace consisting of those functions depending only on the first $2n$ variables. We define the inductive limit

$$Z(p,q) := \varinjlim Z_n(p,q)$$

with respect to the above specified inclusions $Z_n(p,q) \subset Z_{n+1}(p,q)$.

Remark 2.44. The reason we turn to the non-atomic case is that we have

$$L_p(\mathbb{T}) \simeq L_p(\mathbb{T} \times \mathbb{T})$$

and hence for all n:

$$L_p(Z_n(p,q)) \simeq Z_n(p,q),$$

where the isomorphisms are in the sence of isometries between Banach lattices. Moreover, these isometries are compatible with the specified inclusions $Z_n(p,q) \subset Z_{n+1}(p,q)$ (the word "compatible" will be explained by a commutative diagram in Proposition 2.45). If however, we use the similar inclusions as $X_n(p,q) \subset X_{n+1}(p,q)$ for the spaces $Z_n(p,q)$, i.e. if we use the inclusions $Z_n(p,q) \subset Z_{n+1}(p,q)$ indicated by $Z_{n+1}(p,q) = L_pL_q(Z_n(p,q))$, then by applying $L_p(Z_n(p,q)) \simeq Z_n(p,q)$, we have the following commutative diagram

$$\begin{array}{ccc} Z_n(p,q) & \xrightarrow{\text{inclusion}} & Z_{n+1}(p,q) = L_pL_q(Z_n(p,q)) \\ \text{isometric} \big\downarrow \simeq & & \simeq \big\downarrow \text{isometric} \\ L_p(Z_n(p,q)) & \xrightarrow{\text{inclusion}} & L_pL_qL_p(Z_n(p,q)) \end{array}.$$

Note that the above commutative diagram is different from the one in Proposition 2.45, where by using the "specified inclusions" defined above this remark, we can replace $L_pL_qL_p(Z_n(p,q))$ by $L_pL_pL_q(Z_n(p,q)) = L_p(Z_{n+1}(p,q))$.

The $Z(p,q)$'s are Banach lattices of functions on the infinite torus $\mathbb{T}^{\mathbb{N}}$, they have the following properties.

Proposition 2.45. Let $1 < p, q < \infty$. We have isomorphisms

$$Z(p,q) \simeq Z(q,p)$$

and

$$L_p(Z(p,q)) \simeq L_q(Z(p,q)).$$

If $p \neq q$, then $Z(p,q)$ does not have unconditional basis.

Proof. Since $L_p(\mathbb{T})$ and $L_p(\mathbb{T} \times \mathbb{T})$ are isometric as Banach lattices, we have isometric isomorphisms which are compatible with the inclusions $Z_n(p,q) \subset Z_{n+1}(p,q)$, i.e. by using the isometry

$$Z_n(p,q) \simeq L_p(Z_n(p,q)$$

and

$$Z_{n+1}(p,q) = Z_n(p,q)L_pL_q,$$

we obtain

$$Z_{n+1}(p,q) \simeq L_pZ_n(p,q)L_pL_q,$$

and thus the commutative diagram

$$\begin{array}{ccc} Z_n(p,q) & \xrightarrow{\text{inclusion}} & Z_{n+1}(p,q) \\ {\scriptstyle\text{isometric}}\big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle\text{isometric}} \\ L_p(Z_n(p,q)) & \xrightarrow{\text{inclusion}} & L_p(Z_{n+1}(p,q)). \end{array}$$

By taking Banach space inductive limit, we have

$$Z(p,q) \xrightarrow[\text{isometric}]{\simeq} L_p(Z(p,q)).$$

If $p \neq q$, then $Z(p,q)$ and hence $L_p(Z(p,q))$ is non-UMD. By a result of D.J. Aldous (see [Ald79], Proposition 4), $Z(p,q)$ has no unconditional basis.

It is easy to see that $Z(p,q)$ and $Z(q,p)$ complementably embed into each other. Since $\ell_p^{(2)}(L_p) = L_p$ as Banach lattices, we have

$$\ell_p^{(2)}(L_p(Z(p,q))) = L_p(Z(p,q)).$$

Moreover, since $L_p(Z(p,q)) = Z(p,q)$, the above isometry implies that as Banach space

$$Z(p,q) = Z(p,q) \oplus Z(p,q).$$

Similarly,

$$Z(q,p) = Z(q,p) \oplus Z(q,p).$$

By the classical Pełcyński decomposition method, we have

$$Z(p,q) \simeq Z(q,p).$$

Hence

$$L_p(Z(p,q)) = Z(p,q) \simeq Z(q,p) = L_q(Z(q,p)) \simeq L_q(Z(p,q)).$$

$\square$

Let $(p_i)_{i\geq 1}$ be a sequence of real numbers such that $1 < p_i < \infty$. Define

$$X[(p_i)] = \varinjlim L_{p_n} \cdots L_{p_2} L_{p_1}$$

and

$$Z[(p_i)] = \varinjlim L_{p_1} L_{p_2} \cdots L_{p_n}.$$

Problem. Under which condition is $X[(p_i)]$ or $Z[(p_i)]$ in the UMD class ?

We have the following observations on the necessary condition:

(i) A trivial necessary condition is that there exist $1 < p_0, p_\infty < \infty$, such that for all $i \geq 1$:

$$p_0 \leq p_i \leq p_\infty$$

(ii) If the above condition is satisfied, then the sequence (p_i) has at least one cluster point $1 < p < \infty$. Then a necessary condition is that the sequence has only one cluster point, i.e.

$$\lim_{i\to\infty} p_i = p.$$

Indeed, assume that the sequence (p_i) has two cluster points $1 < p \neq q < \infty$, so that there exist two subsequences of (p_i) which tend to p, q respectively. Then one can easily show that by Theorem 2.28 and Theorem 2.38, both $X[(p_i)]$ and $Z[(p_i)]$ are non-UMD (they are in fact non-AUMD).

(iii) Now suppose that $1 < p_0 \leq p_i \leq p_\infty < \infty$ and moreover $\lim_{i\to\infty} p_i = p$. Since $\ell_{p_1}^{(2)}(\ell_{p_2}^{(2)}(\cdots(\ell_{p_n}^{(2)})\cdots))$ embeds isometrically into $L_{p_1} L_{p_2} \cdots L_{p_n}$, a necessary condition for $Z[(p_i)]$ to be UMD is

$$\prod_i c(p_{2i}, p_{2i+1}) < \infty.$$

Similarly, it is necessary that

$$\prod_i c(p_{2i+1}, p_{2i+2}) < \infty.$$

Combining these, a necessary condition for $Z[(p_i)]$ to be in the UMD class is

$$\prod_i c(p_i, p_{i+1}) < \infty. \tag{2.18}$$

The same statement remains true for $X[(p_i)]$. Note that by (2.4), $c(p_i, p_{i+1}) > 1$ if $p_i \neq p_{i+1}$.

Remark 2.46. Let $1 < p < q < \infty$. We have the following Banach lattices isometries

$$L_p L_q = L_p L_p L_q, \quad L_p L_q = L_p L_q L_q.$$

Since $L_p L_r L_q$ is an interpolation space between $L_p L_p L_q$ and $L_p L_q L_q$ for any $p \leq r \leq q$, the UMD$_s$ constant of $L_p L_r L_q$ is actually the same as that of $L_p(L_q)$. The same argument shows that $L_p L_u L_r L_v L_q$ has the same UMD$_s$ constant with $L_p L_q$, provided $p \leq u \leq r \leq v \leq q$. More generally, if $(p_i)_{i=1}^n$ is a finite sequence, assume that $(p_i)_{i=k}^l$ is consecutive monotone (non-increasing or non-decreasing) subsequence, then $L_{p_1} \cdots L_{p_k} \cdots L_{p_l} \cdots L_{p_n}$ and $L_{p_1} \cdots L_{p_k} L_{p_l} \cdots L_{p_n}$ have the same UMD$_s$ constant for all $1 < s < \infty$.

It follows from Remark 2.46 that for any monotone sequence $(p_i)_{i\geq 1}$ in some compact interval $[p_0, p_\infty] \subset (1, \infty)$. The space $X[(p_i)]$ or $Z[(p_i)]$ is UMD. Note that in the monotone case, the necessary condition in (2.18) can be verified directly. Indeed, if (p_i) is monotone, then we have

$$\sum_i |p_i - p_{i+1}| = \lim_i |p_1 - p_i| \leq |p_0 - p_\infty|,$$

hence by (2.6), we get

$$\prod_i c(p_i, p_{i+1}) \leq \Big(\frac{3}{2}\Big)^{\sum_i |p_i - p_{i+1}|} \leq \Big(\frac{3}{2}\Big)^{|p_0 - p_\infty|} < \infty.$$

Intuitively, if p_i tends to p sufficiently fast, then both $X[(p_i)]$ and $Z[(p_i)]$ are in the UMD class. More precisely, we have the following proposition.

Proposition 2.47. Let $1 < p_0 < p_\infty < \infty$ and let $(p_i)_{i\geq 1}$ be any sequence in $[p_0, p_\infty]$. Suppose that we have

$$\sum_i \Big|\frac{1}{p_i} - \frac{1}{p_{i+1}}\Big| < \infty \tag{2.19}$$

or equvalently

$$\sum_i |p_i - p_{i+1}| < \infty. \tag{2.20}$$

Then $X[(p_i)]$ and $Z[(p_i)]$ are both UMD spaces.

Note that (2.19) or (2.20) implies that $\lim_{i\to\infty} p_i$ exists.

Proof. By equality

$$\frac{1}{p_i} - \frac{1}{p_{i+1}} = \frac{p_{i+1} - p_i}{p_i p_{i+1}},$$

we have

$$\frac{|p_i - p_{i+1}|}{p_\infty^2} \leq \Big|\frac{1}{p_i} - \frac{1}{p_{i+1}}\Big| \leq \frac{|p_i - p_{i+1}|}{p_0^2}.$$

It follows that (2.19) and (2.20) are equivalent.

Now suppose that (2.19) (or (2.20)) holds. Recall that by interpolation method, if $0 \leq \theta \leq 1$ and $\frac{1}{r} = \frac{1-\theta}{s} + \frac{\theta}{t}$ with $1 < r, s, t < \infty$, then for any UMD space X, we have

$$\beta_r(X) \leq \beta_s(X)^{1-\theta} \beta_t(X)^\theta.$$

Now we introduce two auxiliary numbers q_0, q_∞ such that $1 < q_0 < p_0 < p_\infty < q_\infty < \infty$. Assume that $p_0 \leq u \neq v \leq p_\infty$. We have either $u < v$ or $v < u$. If $u < v$, then

$$\beta_u(X) \leq \beta_{q_0}(X)^\eta \beta_v(X)^{1-\eta}, \text{ with } \eta = \frac{\frac{1}{u} - \frac{1}{v}}{\frac{1}{q_0} - \frac{1}{v}}.$$

Since $q_0 < p_0 < v$, we have

$$\eta \leq \frac{\frac{1}{u} - \frac{1}{v}}{\frac{1}{q_0} - \frac{1}{p_0}}.$$

By the inequalities (2.14), we have then

$$\beta_u(X) \le \beta(q_0, v)^\eta \beta_v(X) \le \beta(q_0, v)^{\frac{\frac{1}{u}-\frac{1}{v}}{\frac{1}{q_0}-\frac{1}{p_0}}} \beta_v(X).$$

Similarly, we have

$$\beta_v(X) \le \beta(q_\infty, u)^{\frac{\frac{1}{u}-\frac{1}{v}}{\frac{1}{p_\infty}-\frac{1}{q_\infty}}} \beta_u(X).$$

Define

$$M_1 := \sup_{q_0 \le r, s \le q_\infty} \max \left\{ \beta(r,s)^{\frac{1}{\frac{1}{p_\infty}-\frac{1}{q_\infty}}}, \beta(r,s)^{\frac{1}{\frac{1}{p_\infty}-\frac{1}{q_\infty}}} \right\} < \infty.$$

Then we have

$$\frac{1}{M_1^{\frac{1}{u}-\frac{1}{v}}} \beta_u(X) \le \beta_v(X) \le M_1^{\frac{1}{u}-\frac{1}{v}} \beta_u(X).$$

Similar inequalities hold when $v < u$. In general, there exists $M < \infty$, such that for any $p_0 \le u, v \le p_\infty$, we have

$$\frac{1}{M^{|\frac{1}{u}-\frac{1}{v}|}} \beta_u(X) \le \beta_v(X) \le M^{|\frac{1}{u}-\frac{1}{v}|} \beta_u(X).$$

When applying the above inequalities to the spaces $L_{p_1} L_{p_2} \cdots L_{p_n}$, we get

$$\begin{aligned} \beta_{p_0}(L_{p_1} L_{p_2} \cdots L_{p_n}) &\le M^{|\frac{1}{p_0}-\frac{1}{p_1}|} \beta_{p_1}(L_{p_1} L_{p_2} \cdots L_{p_n}) = M^{|\frac{1}{p_0}-\frac{1}{p_1}|} \beta_{p_1}(L_{p_2} \cdots L_{p_n}) \\ &\le \beta_{p_n}(\mathbb{C}) \prod_{i=1}^{n} M^{|\frac{1}{p_{i-1}}-\frac{1}{p_i}|} \le \beta_{p_n}(\mathbb{C}) M^{\sum_{i=1}^{\infty} |\frac{1}{p_{i-1}}-\frac{1}{p_i}|} \end{aligned}$$

Since we have

$$\sup_{p_0 \le s \le p_\infty} \beta_{p_n}(\mathbb{C}) < \infty,$$

hence

$$\sup_n \beta_{p_0}(L_{p_1} L_{p_n} \cdots L_{p_n}) < \infty.$$

This implies that $Z[(p_i)]$ is UMD. Similar arguments show that $X[(p_i)]$ is also UMD. □

Remark 2.48. Let $1 < p < \infty$. Denote by γ_p the least constant $c \ge 1$ satisfying the following Stein's type inequality:

$$\Big\| \Big(\sum_{n=0}^{\infty} |\mathbb{E}^{\mathcal{F}_n} f_n|^2 \Big)^{1/2} \Big\|_p \le c \Big\| \Big(\sum_{n=0}^{\infty} |f_n|^2 \Big)^{1/2} \Big\|_p,$$

where $f_n (n \in \mathbb{N})$ are arbitrary measurable functions on $D^{\mathbb{N}}$ and $(\mathcal{F}_n)_{n \in \mathbb{N}}$ is the dyadic filtration.

If there exist $M > 0$ and $\varepsilon > 0$, such that

$$\gamma_p \ge 1 + M|p-2|, \ \text{ for all } \ |p-2| \le \varepsilon.$$

Then it can be shown that the sufficient condition in Proposition 2.47 is also necessary.

At the moment of this writing, this is not clear to the author.

2.6 Related questions and discussions

In this section, we mainly collect related open problems and describe some ideas which might be useful for future research in this direction.

2.6.1 An open problem on super-reflexive lattices

The following open problem was suggested by Rubio de Francia [RdF86].

Question 2.49. Let X be a super-reflexive Banach lattice. Do we have the following alternative: either the whole interpolation scale $X_\theta := [L_\infty, X]_\theta, 0 < \theta \leq 1$ belongs to the UMD class, or no X_θ is UMD ?

For the known examples of non-UMD super-reflexive lattices of Bourgain together with the examples provided in this chapter, the second possibility in Question 2.49 holds.

2.6.2 On the order of growth for UMD constants

Let $n \geq 1$ be an integer and $1 < p < \infty$. We define the constants $\beta_p(n)$ by

$$\beta_p(n) := \sup\Big\{\, \beta_p(X) : \ X \text{ is a } n\text{-dimensional normed space}\Big\}.$$

Question 2.50. What is the order of growth for $\beta_p(n)$? In particular, we can ask the following question: Do we have

$$\beta_p(n) \gtrsim \sqrt{n}?$$

We want to study this problem by introducing the constant $S(n)$ defined by

$$S(n) := \sup\Big\{ S(X; \{x_i\}_{i=1}^n) \Big|\ X \text{ is a Banach space, } \{x_i\}_{i=1}^n \text{ is 1-unconditional in } X \ \Big\}.$$

This constant is closely related to the constants $\beta_p(n)$ by the inequality:

$$\beta_p(n) \geq S(n).$$

So it is worth while to study the order of growth for $S(n)$. The following proposition can be deduced easily from Theorem 2.20.

Proposition 2.51. Let $n, m \geq 1$ be integers, then we have

$$S(n^m) \geq S(n)^m.$$

From the above proposition, we know that if for a single dimension k (say, a very low dimension like 2, 3, 4, etc.), $S(k) = \sqrt{k}$, then

$$S(n) \gtrsim \sqrt{n}$$

holds for general $n \geq 1$. For this reason, it is of interest to study the following question.

Question 2.52. What are the exact values of $S(2)$, $S(3)$ and $S(4)$?

A lower bound for $S(3)$ is given in the following proposition.

Proposition 2.53. We have

$$S(3) \geq \frac{5}{4}.$$

Proof. Consider the Banach lattice X on $\{1,2,3\}$ defined by

$$\|(\lambda_1,\lambda_2,\lambda_3)\|_X := \max\{|\lambda_1|+|\lambda_2|, |\lambda_3|\}, \text{ for } (\lambda_1,\lambda_2,\lambda_3) \in \mathbb{R}^3 \text{ (or } \mathbb{C}^3).$$

As usual, let $D = \{\pm 1\}$ equipped with the Bernoulli measure μ, let $\varepsilon : D \to D$ be the identity. Consider the functions $\theta_1 = \frac{1+\varepsilon}{2}, \theta_2 = \frac{1-\varepsilon}{2}, \theta_3 = 1$. We have

$$\|(\theta_1,\theta_2,\theta_3)\|_X = \|(\frac{1+\varepsilon}{2}, \frac{1-\varepsilon}{2}, 1)\|_X = \begin{cases} \|(1,0,1)\|_X = 1, & \text{if } \varepsilon = 1, \\ \|(0,1,1)\|_X = 1, & \text{if } \varepsilon = -1. \end{cases}$$

Hence

$$\|(\theta_1,\theta_2,\theta_3)\|_{L_\infty(D,\mu;X)} = 1.$$

We have

$$\|(\mathbb{E}(\theta_1),\theta_2,\theta_3)\| = \|(\frac{1}{2}, \frac{1-\varepsilon}{2}, 1)\|_X = \begin{cases} \|(\frac{1}{2},0,1)\|_X = 1, & \text{if } \varepsilon = 1, \\ \|(\frac{1}{2},1,1)\|_X = \frac{3}{2}, & \text{if } \varepsilon = -1. \end{cases}$$

Thus

$$\|(\mathbb{E}(\theta_1),\theta_2,\theta_3)\|_{L_1(D,\mu;X)} = \frac{1}{2}\times 1 + \frac{1}{2}\times\frac{3}{2} = \frac{5}{4}.$$

By the definition of $S(X;\{\delta_1,\delta_2,\delta_3\})$, we see that $S(X;\{\delta_1,\delta_2,\delta_3\}) \geq \frac{5}{4}$, hence

$$S(3) \geq S(X;\{\delta_1,\delta_2,\delta_3\}) \geq \frac{5}{4}.$$

□

The following result of Bourgain [Bou84] shows that we cannot only consider the symmetric spaces. Recall that a sequence $(e_n)_{n\geq 1}$ in a Banach space X is called 1-symmetric (see [LT77], §3. a), if we have

$$\|\sum_n \theta_n a_n e_{\pi(n)}\|_X = \|\sum_n a_n e_n\|_X$$

for any finitely supported sequence of scalars $(a_n)_{n\geq 1}$, any choice of signs $\theta_n = \pm 1$ and any permutation π of positive integers.

A Banach space X is called a symmetric space if it admits a symmetric Schauder basis.

Theorem 2.54. (Bourgain) There exists a numerical constant M such that for any $n \geq 1$ and any n-dimensional symmetric space X, we have

$$\beta_p(X) \leq M\frac{p^2}{p-1}(\log n)^2.$$

If we assume in addition that X is q-concave for some $q < \infty$ (note that X is a lattice), then we can improve the above estimate

$$\beta_p(X) \leq M_q\frac{p^2}{p-1}\log n,$$

where M_q only depends on the q-concave constant of X.

Let E be a Banach lattice on $[n]$ with the natural basis $\{\delta_i\}_{i=1}^n$. Then we define $E^{\otimes m}$ by the following induction

$$E^{\otimes 1} := E,$$
$$E^{\otimes m+1} := E(E^{\otimes m}).$$

Proposition 2.55. In the above situation, if we assume that for all m, the natural basis of $E^{\otimes m}$ is 1-symmetric, then

$$S(E) = 1.$$

Proof. Firstly, we have

$$S(E^{\otimes m}) \geq S(E)^m.$$

By Theorem 2.54, take $p = 2$,

$$S(E^{\otimes m}) \leq \beta_2(E^{\otimes m}) \leq 4M\Big(\log \dim E^{\otimes m}\Big)^2 \leq 4Mm^2\Big(\log \dim E\Big)^2.$$

Hence

$$S(E) \leq (4M)^{1/m} m^{2/m}\Big(\log \dim E\Big)^{2/m}.$$

The above inequality holds for all $m \geq 1$, this implies that

$$S(E) \leq \lim_{m\to\infty} (4M)^{1/m} m^{2/m}\Big(\log \dim E\Big)^{2/m} = 1.$$

Thus we have $S(E) = 1$, as desired. $\square$

Appendix

2.6.3 The spaces $C_{p_1} \otimes_h C_{p_2} \otimes_h \cdots \otimes_h C_{p_n}$

In the end of Chapter 1, we considered the spaces $C_{p_1} \otimes_h C_{p_2} \otimes_h \cdots \otimes_h C_{p_n}$ and we asked in Question 1.48 whether these operator spaces are in some sense uniformly OUMD$_p$ for all the sequence $(p_k)_{k=1}^n$ in a finite interval $[p_0, p_\infty] \subset (1, \infty)$.

For answering this question. We will define the space $V_n(p,q)$ by the following: Let $1 \leq p, q \leq \infty$, we define

$$V_n(p,q) := \underbrace{C_p \otimes_h C_q \otimes_h C_p \otimes_h C_q \otimes_h \cdots \otimes_h C_p \otimes_h C_q}_{\text{repeat } C_p \otimes_h C_q \text{ for } n \text{ times}}.$$

Define also the operator space

$$F_n(p,q) := \begin{cases} \underbrace{S_p[S_q[S_p[S_q \cdots [S_p] \cdots]]]}_{n \text{ times}}, & \text{if } n \text{ is odd} \\ \underbrace{S_p[S_q[S_p[S_q \cdots [S_q] \cdots]]]}_{n \text{ times}}, & \text{if } n \text{ is even} \end{cases}.$$

The Banach space $L_n(p,q)$ is by definition

$$L_n(p,q) := \begin{cases} \underbrace{\ell_p(\ell_q(\ell_p(\ell_q \cdots (\ell_p) \cdots)))}_{n \text{ times}}, & \text{if } n \text{ is odd} \\ \underbrace{\ell_p(\ell_q(\ell_p(\ell_q \cdots (\ell_q) \cdots)))}_{n \text{ times}}, & \text{if } n \text{ is even} \end{cases}.$$

It is easy to see that we have isometric embedding $L_n(p,q) \subset F_n(p,q)$.

A negative answer to Question 1.48 is given by the next theorem.

Theorem 2.56. Let $1 \leq p \neq q \leq \infty$. Then for any $1 < s < \infty$, there exists $w = w(p,q,s) > 1$ such that the UMD$_s$ constants $\beta_s(V_n(p,q))$ of the Banach space defined by $V_n(p,q)$ satisfy

$$\beta_s(V_n(p,q)) \geq w^n.$$

Proof. Firstly, consider the case when $\frac{1}{p} + \frac{1}{q} = 1$. In this case, we have completely isometrically $C_q = R_p$ and $C_p = R_q$. Hence

$$\begin{aligned} V_n(p,q) &= \underbrace{C_p \otimes_h C_q \otimes_h C_p \otimes_h C_q \otimes_h \cdots \otimes_h C_p \otimes_h C_q}_{\text{repeat } C_p \otimes_h C_q \text{ for } n \text{ times}} \\ &= C_p \otimes_h \Big(C_q \otimes_h C_p \otimes_h C_q \otimes_h \cdots \otimes_h C_p \Big) \otimes_h R_p \\ &= S_p[C_q \otimes_h C_p \otimes_h C_q \otimes_h \cdots \otimes_h C_p] \\ &= S_p[S_q[V_{n-2}(p,q)]]. \end{aligned}$$

Continuing this procedure, it is easy to see that we have completely isometrically

$$V_n(p,q) = F_n(p,q).$$

Since we have $L_n(p,q) \subset F_n(p,q) = V_n(p,q)$, it follows that

$$\beta_s(V_n(p,q)) \geq \beta_s(L_n(p,q)).$$

The last quantity $\beta_s(L_n(p,q)) \geq c(p,q)^{\lfloor n/2 \rfloor}$. Hence the claim in the theorem holds.

In general, by drawing the points $(\frac{1}{p}, \frac{1}{q})$ in the unit square $[0,1]^2$. It is easy to see that if $\frac{1}{p} + \frac{1}{q} \neq 1$ and $p \neq q$, then either there exists $0 < \theta < 1$ and $1 < p_\theta, q_\theta < \infty$ such that $\frac{1}{p_\theta} + \frac{1}{q_\theta} = 1$ and

$$\Big(\frac{1}{p_\theta}, \frac{1}{q_\theta}\Big) = (1-\theta)\Big(\frac{1}{\infty}, \frac{1}{\infty}\Big) + \theta\Big(\frac{1}{p}, \frac{1}{q}\Big)$$

or there exists $0 < \eta < 1$ and $1 < p_\eta, q_\eta < \infty$ such that $\frac{1}{p_\eta} + \frac{1}{q_\eta} = 1$ and

$$\Big(\frac{1}{p_\eta}, \frac{1}{q_\eta}\Big) = (1-\eta)\Big(\frac{1}{1}, \frac{1}{1}\Big) + \eta\Big(\frac{1}{p}, \frac{1}{q}\Big).$$

In the first case, we have

$$V_n(p_\theta, q_\theta) = \Big[V_n(\infty,\infty), V_n(p,q)\Big]_\theta.$$

Hence

$$\beta_s(V_n(p_\theta, q_\theta)) \leq \beta_s(V_n(\infty,\infty))^{1-\theta} \beta_s(V_n(p,q))^\theta.$$

Since $V_n(\infty,\infty) \simeq C_\infty$, we have thus $\beta_s(V_n(\infty,\infty)) = \beta_s(\ell_2) < \infty$. It follows that

$$\beta_s(V_n(p,q)) \geq \frac{w(p_\theta, q_\theta, s)^{n/\theta}}{\beta_s(\ell_2)^{\frac{1-\theta}{\theta}}}.$$

In the second case we have

$$V_n(p_\eta, q_\eta) = \Big[V_n(1,1), V_n(p,q)\Big]_\eta.$$

Similarly, we have

$$\beta_s(V_n(p,q)) \geq \frac{w(p_\eta, q_\eta, s)^{n/\eta}}{\beta_s(\ell_2)^{\frac{1-\eta}{\eta}}}.$$

□

2.6.4 Equivalence between AUMD$_p$ and AUMD$_q$

In this section, we will present the proof of the fact mentioned in the title. We only collect some known results from different works by many authors.

Given a complex Banach space X and given $0 < p < \infty$, a function $F \in L_p(\mathbb{T}^{\mathbb{N}*}; X)$ is represented by a Hardy martingale if

$$F(z) = f_0 + \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} f_{k-1,m}(z_1, z_2, \cdots, z_{k-1}) z_k^m,$$

where $f_0 \in X$, $f_{k-1,m}(z_1, z_2, \cdots, z_{k-1}) \in X$ and $z = (z_k)_{k\geq 1} \in \mathbb{N}^*$, by convention, $f_{0,m}$ are constant functions with values in X. In the following, we can assume without loss of generality that $f_0 = 0$, i.e. we can assume that $\mathbb{E}(F) = 0$.

A function $G \in L_p(\mathbb{T}^{\mathbb{N}*}; X)$ is represented by an analytic martingale starting from the origin if

$$G(z) = \sum_{k=1}^{\infty} g_{k-1}(z_1, z_2, \cdots, z_{k-1})z_k,$$

where $g_{k-1}(z_1, z_2, \cdots, z_{k-1}) \in X$ and $z = (z_k)_{k\geq 1} \in \mathbb{N}^*$. Let $G = (G_k)_{k\geq 1}$ be an analytic martingale as above, then

$$dG_k(z) = g_{k-1}(z_1, z_2, \cdots, z_{k-1})z_k,$$

hence

$$\|dG_n(z)\|_X = \|g_{k-1}(z_1, z_2, \cdots, z_{k-1})\|_X$$

is $\sigma(z_1, \cdots, z_{n-1})$-measurable. That is, the sequence $(\|dG_k\|)_{k\geq 1}$ is predictable.

Recall that by definition the AUMD$_p$ constant of X, denoted by $\beta_p^a(X)$, is defined as the best $c > 0$ appears in the following inequality: for all sequences $\eta = (\eta_k)$ of real numbers such that $|\eta_k| \leq 1$, the operator

$$F = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} f_{k-1,m}(z_1, z_2, \cdots, z_{k-1})z_k^m \xrightarrow{T} T(F) = \sum_{k=1}^{\infty}\eta_k \sum_{m=1}^{\infty} f_{k-1,m}(z_1, z_2, \cdots, z_{k-1})z_k^m$$

satisfies

$$\|T(F)\|_{L_p(\mathbb{T}^{\mathbb{N}^*};X)} \leq c\|F\|_{L_p(\mathbb{T}^{\mathbb{N}^*};X)}.$$

We have thus

$$\|T(F)\|_{L_p(\mathbb{T}^{\mathbb{N}^*};X)} \leq \beta_p^a(X)\|F\|_{L_p(\mathbb{T}^{\mathbb{N}^*};X)}.$$

Remark 2.57. Quanhua Xu has shown, using the approximation argument of Proposition 6 of [Edg86], that in the definition of AUMD$_p$, instead of using the Hardy martingales, we can use only the analytic martingales.

Let $f = (f_n)_n$ be an X-valued martingale, the maximum function will be denoted by $M(f)$, which is defined as

$$M(f) = \sup_n \|f_n\|_X.$$

The following theorem of Garling will be used.

Theorem 2.58. (Garling, [Gar88]) Let $0 < p < \infty$ and let $f = (f_n)$ be a Hardy martingale in $L_p(\mathbb{T}^{\mathbb{N}^*}; X)$, then we have

$$\|M(f)\|_p \leq e^{1/p}\|f\|_p.$$

Theorem 2.59. Let X be a complex Banach space which is AUMD$_p$ for some $0 < p < \infty$, then it is AUMD$_p$ for all $0 < p < \infty$.

Proof. ([Bur01]) Assume that X is AUMD$_p$ for some $0 < p < \infty$, we will show that it is AUMD$_q$ for any $0 < q < \infty$. By the definition and Theorem 2.58, there exists c_p such that for any analytic martingale f in $L_p(\mathbb{T}^{\mathbb{N}^*}; X)$, if g is a ± 1-transform of f, then

$$\|M(g)\|_p \leq c_p\|f\|_p.$$

We will assume without loss of generality that f is a finite analytic martingale

$$f(z) = \sum_{k=1}^{N} f_{k-1}(z_1, \cdots, z_{k-1}) z_k.$$

Assume also that f_{k-1} takes only finitely many values in X. We show that under the above assumption, for any $\delta > 0$, $\beta > 2\delta + 1$ and any $\lambda > 0$, we have

$$\mathbb{P}(M(g) > \lambda; M(f) \leq \delta\lambda) \leq \alpha \mathbb{P}(M(g) > \lambda),$$

where $\alpha = \frac{(4c_p\delta)^p}{(\beta-2\delta-1)^p}$. Let us define

$$T = \inf\Big\{n : \|g_n\|_X > \lambda\Big\},$$

$$S = \inf\Big\{n : \|g_n\|_X > \beta\lambda\Big\},$$

$$R = \inf\Big\{n : \|f_n\|_X > \delta\lambda \quad \text{or} \quad \|df_{n+1}\|_X > 2\delta\lambda\Big\}.$$

Note that since the sequence $(\|df_n)_{n\geq 1}$ is predictable, we see that T, S, R are all stopping times. Obviously, we have

$$T \leq S.$$

Consider the analytic martingales $F = (F_n)$ and $G = (G_n)$ defined by

$$F_n = f_{n\wedge S\wedge R} - f_{n\wedge T\wedge R},$$

$$G_n = g_{n\wedge S\wedge R} - g_{n\wedge T\wedge R}.$$

Since g is a ± 1-transform of f, hence G is also a ± 1-transform of F. We have

$$F_\infty = f_{S\wedge R} - f_{T\wedge R} = (f_{S\wedge R} - f_T)1_{R>T}.$$

On the set $\{R > T\}$, we have $\|f_T\|_X \leq \delta\lambda$, i.e.

$$\|f_T 1_{R>T}\|_X \leq \delta\lambda.$$

Since $S \wedge R \leq R$, by the definition of R, we have

$$\|f_{S\wedge R}\|_X \leq 3\delta\lambda.$$

Thus we have

$$\|F_\infty\|_X \leq 4\delta\lambda 1_{R>T}.$$

It follows that

$$\mathbb{E}(\|F_\infty\|_X^p) \leq (4\delta\lambda)^p \cdot \mathbb{P}(R > T) \leq (4\delta\lambda)^p \cdot \mathbb{P}(T < \infty) = (4\delta\lambda)^p \cdot \mathbb{P}(M(g) > \lambda) \quad (2.21)$$

Note that

$$\Big\{M(g) > \beta\lambda; M(f) \leq \delta\lambda\Big\} \subset \Big\{S < \infty; R = \infty\Big\}.$$

On the subset $\{S < \infty; R = \infty\}$, we have $G_\infty = g_S - g_T$. By definition of S, T, R, we have

$$\|g_S\|_X \geq \beta\lambda,$$

$$\|g_T 1_{R=\infty}\|_X \leq \lambda + 2\delta\lambda.$$

Hence on $\{S < \infty; R = \infty\}$, we have $\|G_\infty\|_X \geq (\beta - 2\delta - 1)\lambda$. It follows that

$$\begin{aligned} \mathbb{P}(M(g) > \beta\lambda; M(f) \leq \delta\lambda) &\leq \mathbb{P}(S < \infty; R = \infty) \\ &\leq \mathbb{P}(\|G_\infty\|_X \geq (\beta - 2\delta - 1)\lambda) \end{aligned}$$

By Chebychev's inequality

$$\begin{aligned} &\leq \frac{\mathbb{E}(\|G_\infty\|_X^p)}{(\beta - 2\delta - 1)^p\lambda^p} \leq \frac{\|M(G)\|_p^p}{(\beta - 2\delta - 1)^p\lambda^p} \\ &\leq \frac{c_p^p\|F_\infty\|_p^p}{(\beta - 2\delta - 1)^p\lambda^p} \\ &\leq \frac{c_p^p(4\delta\lambda)^p}{(\beta - 2\delta - 1)^p\lambda^p}\mathbb{P}(M(g) > \lambda) \\ &= \frac{(4c_p\delta)^p}{(\beta - 2\delta - 1)^p}\mathbb{P}(M(g) > \lambda) = \alpha\mathbb{P}(M(g) > \lambda). \end{aligned}$$

We can choose δ sufficiently small and choose suitable β such that $\alpha\beta^q < 1$, then

$$\|M(g)\|_q^q \leq \frac{\beta^q}{\delta^q(1 - \alpha\beta^q)}\|M(f)\|_q^q. \tag{2.22}$$

Indeed, we have

$$\begin{aligned} \|M(g)\|_q^q &= \beta^q\|M(g)/\beta\|_q^q = \beta^q\int_0^\infty \mathbb{P}(M(g) > \beta\lambda)q\lambda^{q-1}d\lambda \\ &\leq \alpha\beta^q\int_0^\infty \mathbb{P}(M(g) > \lambda)q\lambda^{q-1}d\lambda + \beta^q\int_0^\infty \mathbb{P}(M(f) > \delta\lambda)q\lambda^{q-1}d\lambda \\ &= \alpha\beta^q\|M(g)\|_q^q + \beta^q\delta^{-q}\|M(f)\|_q^q. \end{aligned}$$

Under the assumption of f, we know that $\|M(g)\|_q$ is finite, hence we have (2.22). After obtaining (2.22), then by Theorem 2.58 and the easy fact that $\|g\|_q \leq \|M(g)\|_q$, we find c_q such that

$$\|g\|_q \leq c_q\|f\|_q.$$

That is, X is AUMD$_q$.

□

F. X. Müller (personal communication to the author) gives an interesting proof of the fact AUMD$_p$ implies AUMD$_q$ for all $q \geq p$, his proof does not use Theorem 2.58 and Remark 2.57.

Definition 2.60. The p-sharp function of an X-valued martingale $f = (f_n)_{n\geq 1}$ (associated with a filtration of conditional expectations $(\mathbb{E}_n)_{n\geq 0}$ where $\mathbb{E} = \mathbb{E}_0$) is defined by

$$f_p^\sharp = \sup_n \left\{\mathbb{E}_n(\|f - f_{n-1}\|_X^p)\right\}^{1/p}$$

Note that for

$$F = \sum_{k=1}^\infty \sum_{m=1}^\infty f_{k-1,m}(z_1, z_2, \cdots, z_{k-1})z_k^m,$$

we have

$$F - F_{n-1} = \sum_{k=n}^\infty \sum_{m=1}^\infty f_{k-1,m}(z_1, z_2, \cdots, z_{k-1})z_k^m.$$

Using the above notation, we have

Proposition 2.61. The pointwise estimate for p-sharp functions

$$(TF)_p^\sharp(z) \leq \beta_p^a(X) \cdot F_p^\sharp(z)$$

holds.

Proof. Then for a fixed sequence $\eta_k = \pm 1$,

$$(TF)(z) - (TF)_{n-1}(z) = \sum_{k=n}^{\infty} \eta_k \sum_{m=1}^{\infty} f_{k-1,m}(z_1, z_2, \cdots, z_{k-1}) z_k^m.$$

We have

$$\begin{aligned}
& \mathbb{E}_n(\|TF - (TF)_{n-1}\|_X^p)(z) \\
= \quad & \int_{z_{n+1}} \int_{z_{n+2}} \cdots \Big\| \sum_{k=n}^{\infty} \eta_k \sum_{m=1}^{\infty} f_{k-1,m}(z_1, z_2, \cdots, z_{k-1}) z_k^m \Big\|^p dm(z_{n+1}) dm(z_{n+2}) \cdots \\
\leq \quad & \beta_p^a(X)^p \int_{z_{n+1}} \int_{z_{n+2}} \cdots \Big\| \sum_{k=n}^{\infty} \sum_{m=1}^{\infty} f_{k-1,m}(z_1, z_2, \cdots, z_{k-1}) z_k^m \Big\|^p dm(z_{n+1}) dm(z_{n+2}) \cdots .
\end{aligned}$$

That is

$$\mathbb{E}_n(\|TF - (TF)_{n-1}\|_X^p)(z) \leq \beta_p^a(X)^p \mathbb{E}_n(\|F - F_{n-1}\|_X^p)(z),$$

hence

$$(TF)_p^\sharp(z) \leq \beta_p^a(X) \cdot F_p^\sharp(z).$$

□

Recall that for any measurable function f on any measure space $(\Omega, \mathcal{F}, \nu)$, the decreasing rearrangement function of f is

$$f^*(t) = \inf\{c > 0 : \nu(|f| > c) \leq t\}, t > 0.$$

It is classical that for any $c > 0$, we have

$$|\{t : f^*(t) > c\}| = \nu(|f| > c),$$

where $|\{t : f^*(t) > c\}| =$ Lebesgue measure of the set $\{t : f^*(t) > c\}$. In short, we have

$$|f| \overset{d}{=} f^*.$$

The above notation should not be confused with the maximal function of a function f, which we recall here and denote it by $M(f)$. For any measurable function f on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration of condition expectations $(\mathbb{E}_n)$. We define

$$M(f) = \sup_n |\mathbb{E}_n(f)|.$$

More generally, for an X-valued martingale $f = (f_n)$ we define

$$M(f) = \sup_n \|f_n\|_X.$$

The following theorem makes a connection between $(Mf)^*$ and $(f_p^\sharp)^*$, the proof we give here is taken from [Lon93] p. 124, Theorem 3. 6. 8.

Theorem 2.62. Let $1 \le p < \infty$. Then for any X-valued martingale $f = (f_n)$, we have

$$(Mf)^*(t) \le 4(f_p^\sharp)^*(\frac{t}{2}) + (Mf)^*(2t).$$

Proof. Fix $t > 0$, define stopping times

$$S = \inf\{n : \|f_n\|_X > (Mf)^*(2t)\},$$
$$T = \inf\{n : \|f_n\|_X > \ 4(f_p^\sharp)^*(\frac{t}{2}) + (Mf)^*(2t)\},$$
$$R = \inf\{n : (\mathbb{E}_n(\|f - f_{n-1}\|_X^p))^{1/p} > (f_p^\sharp)^*(\frac{t}{2})\}.$$

Obviously, we have $S \le T$, and

$$\{T < \infty\} = \{Mf > 4(f_p^\sharp)^*(\frac{t}{2}) + (Mf)^*(2t)\},$$
$$\{S < \infty\} = \{Mf > (Mf)^*(2t)\},$$
$$\{R < \infty\} = \{f_p^\sharp > (f_p^\sharp)^*(\frac{t}{2})\},$$

it follows that

$$\mathbb{P}(S < \infty) \le 2t, \quad \mathbb{P}(R < \infty) \le \frac{t}{2}.$$

The inequality

$$(Mf)^*(t) \le 4(f_p^\sharp)^*(\frac{t}{2}) + (Mf)^*(2t)$$

is equivalent to

$$\mathbb{P}\Big(Mf > 4(f_p^\sharp)^*(\frac{t}{2}) + (Mf)^*(2t)\Big) \le t.$$

Thus what we need to show is

$$\mathbb{P}(T < \infty) \le t.$$

Note that

$$\begin{aligned}\{T < \infty\} &= \{T < \infty, S < R\} \cup \{T < \infty, R \le S\}\\ &\subset \{T < \infty, S < R\} \cup \{R < \infty\}\end{aligned}$$

and

$$\{T < \infty, S < R\} \subset \Big\{S < R, \|f_T - f_{S-1}\|_X > 4(f_p^\sharp)^*(\frac{t}{2})\Big\}.$$

Hence

$$\begin{aligned}\mathbb{P}(T < \ \infty, S < R) &\le \frac{1}{4(f_p^\sharp)^*(\frac{t}{2})}\int_{\{S<R\}} \|\mathbb{E}(f - f_{S-1}|\mathcal{F}_T)\|_X d\mathbb{P}\\ &\le \frac{1}{4(f_p^\sharp)^*(\frac{t}{2})}\int_{\{S<R\}} \mathbb{E}(\|f - f_{S-1}\|_X|\mathcal{F}_T) d\mathbb{P}\\ &\le \frac{1}{4(f_p^\sharp)^*(\frac{t}{2})}\int_{\{S<R\}} \mathbb{E}(\|f - f_{S-1}\|_X^p|\mathcal{F}_T)^{1/p} d\mathbb{P}\end{aligned}$$

since $\{S < R\} \in \mathcal{F}_S \subset \mathcal{F}_T$,

$$\begin{aligned}&= \frac{1}{4(f_p^\sharp)^*(\frac{t}{2})}\int_{\{S<R\}} \mathbb{E}(\|f - f_{S-1}\|_X^p|\mathcal{F}_S)^{1/p} d\mathbb{P}\\ &\le \frac{1}{4(f_p^\sharp)^*(\frac{t}{2})}\Big\|1_{\{S<R\}}\mathbb{E}(\|f - f_{S-1}\|_X^p|\mathcal{F}_S)^{1/p}\Big\|_\infty \mathbb{P}(S < \infty)\\ &\le \frac{t}{2}\end{aligned}$$

Hence we have indeed $\mathbb{P}(T < \infty) \le t$.

$\square$

Corollary 2.63. For any $1 \le q < \infty$, we have

$$\|Mf\|_{L_q} \le c_q \|f_p^\sharp\|_{L_q},$$

where c_q is a constant depends only on q.

Proof. By Theorem 2.62, we have

$$\begin{aligned}
\|Mf\|_{L_q} &= \|(Mf)^*\|_{L_q} \le \|4(f_p^\sharp)^*(\frac{\cdot}{2})\|_{L_q} + \|(Mf)^*(2\cdot)\|_{L_q} \\
&= 4 \cdot 2^{1/q} \|(f_p^\sharp)^*\|_{L_q} + \frac{1}{2^{1/q}} \|(Mf)^*\|_{L_q} \\
&= 4 \cdot 2^{1/q} \|f_p^\sharp\|_{L_q} + \frac{1}{2^{1/q}} \|Mf\|_{L_q}.
\end{aligned}$$

Hence

$$\|Mf\|_{L_q} \le \frac{4^{1+\frac{1}{q}}}{2^{\frac{1}{q}} - 1} \|f_p^\sharp\|_{L_q}.$$

$\square$

Combining the above results, we can prove again that
$\mathrm{AUMD}_p \Longrightarrow \mathrm{AUMD}_q$ for all $p \le q < \infty$.

Proof. As above, for any sequence of real numbers $\eta = (\eta_n)$ with $|\eta_n| \le 1$, we can define an operator

$$F = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} f_{k-1,m}(z_1, z_2, \cdots, z_{k-1}) z_k^m \xrightarrow{T_\eta} T(F) = \sum_{k=1}^{\infty} \eta_k \sum_{m=1}^{\infty} f_{k-1,m}(z_1, z_2, \cdots, z_{k-1}) z_k^m$$

such that

$$\|T_\eta(F)\|_{L_p(\mathbb{T}^{\mathbb{N}^*};X)} \le \beta_p^a(X) \|F\|_{L_p(\mathbb{T}^{\mathbb{N}^*};X)}.$$

By Proposition 2.61, we have a pointwise estimate

$$(T_\eta F)_p^\sharp(z) \le \beta_p^a(X) \cdot F_p^\sharp(z).$$

It follows that

$$\begin{aligned}
\|T_\eta F\|_{L_q(X)} &\le \|M(T_\eta F)\|_{L_q} \le c_q \|(T_\eta F)_p^\sharp\|_{L_q} \\
&\le c_q \cdot \beta_p^a(X) \cdot \|F_p^\sharp\|_{L_q}.
\end{aligned}$$

Since

$$\begin{aligned}
(\mathbb{E}_n \|F - F_{n-1}\|_X^p)^{1/p} &\le 2^{\frac{p-1}{p}} (\mathbb{E}_n \|F\|_X^p + \|F_{n-1}\|_X^p)^{1/p} \le 2(\mathbb{E}_n \|F\|_X^p)^{1/p} \\
&\le 2\Big\{\sup_n \mathbb{E}_n(\|F\|_X^p)\Big\}^{1/p} = 2M(\|F\|_X^p)^{1/p}.
\end{aligned}$$

Hence

$$F_p^\sharp \le 2\Big\{M(\|F\|_X^p)\Big\}^{1/p}.$$

Then we have

$$\|F_p^\sharp\|_{L_q} \leq 2\big\|M(\|F\|_X^p)^{1/p}\big\|_{L_q} = 2\big\|M(\|F\|_X^p)\big\|_{L_{q/p}}^{1/p}$$

Since $q/p > 1$, by Doob's inequality,

$$\begin{aligned} &\leq 2\Big(\frac{q}{q-p}\Big)^{1/p}\big\|\|F\|_X^p\big\|_{L_{q/p}}^{1/p} \\ &= 2\Big(\frac{q}{q-p}\Big)^{1/p}\big\|F\big\|_{L_q(X)}. \end{aligned}$$

That is, we have

$$\|T_\eta F\|_{L_q(X)} \leq 2\Big(\frac{q}{q-p}\Big)^{1/p} c_q \cdot \beta_p^a(X)\|F\|_{L_q(X)}.$$

In other words, we have

$$\beta_q^a(X) \leq 2\Big(\frac{q}{q-p}\Big)^{1/p} c_q \cdot \beta_p^a(X).$$

□

Bibliography

[Ald79] D. J. Aldous. Unconditional bases and martingales in $L_p(F)$. *Math. Proc. Cambridge Philos. Soc.*, 85(1):117–123, 1979.

[BL76] Jöran Bergh and Jörgen Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[Bou83] J. Bourgain. Some remarks on Banach spaces in which martingale difference sequences are unconditional. *Ark. Mat.*, 21(2):163–168, 1983.

[Bou84] J. Bourgain. On martingales transforms in finite-dimensional lattices with an appendix on the K-convexity constant. *Math. Nachr.*, 119:41–53, 1984.

[Bou86] J. Bourgain. Vector-valued singular integrals and the H^1-BMO duality. In *Probability theory and harmonic analysis (Cleveland, Ohio, 1983)*, volume 98 of *Monogr. Textbooks Pure Appl. Math.*, pages 1–19. Dekker, New York, 1986.

[BP91] David P. Blecher and Vern I. Paulsen. Tensor products of operator spaces. *J. Funct. Anal.*, 99(2):262–292, 1991.

[Bur81] D. L. Burkholder. A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional. *Ann. Probab.*, 9(6):997–1011, 1981.

[Bur83] D. L. Burkholder. A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions. In *Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981)*, Wadsworth Math. Ser., pages 270–286. Wadsworth, Belmont, CA, 1983.

[Bur86] Donald L. Burkholder. Martingales and Fourier analysis in Banach spaces. In *Probability and analysis (Varenna, 1985)*, volume 1206 of *Lecture Notes in Math.*, pages 61–108. Springer, Berlin, 1986.

[Bur01] Donald L. Burkholder. Martingales and singular integrals in Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. I*, pages 233–269. North-Holland, Amsterdam, 2001.

[Cal64] A.-P. Calderón. Intermediate spaces and interpolation, the complex method. *Studia Math.*, 24:113–190, 1964.

[Edg86] G. A. Edgar. Analytic martingale convergence. *J. Funct. Anal.*, 69(2):268–280, 1986.

[ER91] Edward G. Effros and Zhong-Jin Ruan. A new approach to operator spaces. *Canad. Math. Bull.*, 34(3):329–337, 1991.

[Gar88] D. J. H. Garling. On martingales with values in a complex Banach space. *Math. Proc. Cambridge Philos. Soc.*, 104(2):399–406, 1988.

[JX05] Marius Junge and Quanhua Xu. On the best constants in some non-commutative martingale inequalities. *Bull. London Math. Soc.*, 37(2):243–253, 2005.

[JX08] Marius Junge and Quanhua Xu. Noncommutative Burkholder/Rosenthal inequalities. II. Applications. *Israel J. Math.*, 167:227–282, 2008.

[Lon93] Rui Lin Long. *Martingale spaces and inequalities.* Peking University Press, Beijing, 1993.

[LT77] Joram Lindenstrauss and Lior Tzafriri. *Classical Banach spaces. I.* Springer-Verlag, Berlin, 1977. Sequence spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92.

[LT79] Joram Lindenstrauss and Lior Tzafriri. *Classical Banach spaces. II*, volume 97 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer-Verlag, Berlin, 1979. Function spaces.

[Mus06] Magdalena Musat. On the operator space UMD property for noncommutative L_p-spaces. *Indiana Univ. Math. J.*, 55(6):1857–1891, 2006.

[NR11] Stefan Neuwirth and Éric Ricard. Transfer of Fourier multipliers into Schur multipliers and sumsets in a discrete group. *Canad. J. Math.*, 63(5):1161–1187, 2011.

[Pis75] Gilles Pisier. Un exemple concernant la super-réflexivité. In *Séminaire Maurey-Schwartz 1974–1975: Espaces L^p applications radonifiantes et géométrie des espaces de Banach, Annexe No. 2*, page 12. Centre Math. École Polytech., Paris, 1975.

[Pis79] Gilles Pisier. Some applications of the complex interpolation method to Banach lattices. *J. Analyse Math.*, 35:264–281, 1979.

[Pis96] Gilles Pisier. The operator Hilbert space OH, complex interpolation and tensor norms. *Mem. Amer. Math. Soc.*, 122(585):viii+103, 1996.

[Pis98] Gilles Pisier. Non-commutative vector valued L_p-spaces and completely p-summing maps. *Astérisque*, 247:vi+131, 1998.

[Pis03] Gilles Pisier. *Introduction to operator space theory*, volume 294 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2003.

[PX96] Gilles Pisier and Quanhua Xu. Inégalités de martingales non commutatives. *C. R. Acad. Sci. Paris Sér. I Math.*, 323(7):817–822, 1996.

[PX97] Gilles Pisier and Quanhua Xu. Non-commutative martingale inequalities. *Comm.Math.Phy*, 189:667, 1997.

[Qiu12] Yanqi Qiu. On the UMD constants for a class of iterated $L_p(L_q)$ spaces. *J. Funct. Anal.*, 263(8):2409–2429, 2012.

[RdF86] José L. Rubio de Francia. Martingale and integral transforms of Banach space valued functions. In *Probability and Banach spaces (Zaragoza, 1985)*, volume 1221 of *Lecture Notes in Math.*, pages 195–222. Springer, Berlin, 1986.

[Rua88] Zhong-Jin Ruan. Subspaces of C^*-algebras. *J. Funct. Anal.*, 76(1):217–230, 1988.

[Tak01] M. Takesaki. *Theory of Operator Algebras I.* Springer-Verlag, New York, 2001.

[TJ89] Nicole Tomczak-Jaegermann. *Banach-Mazur distances and finite-dimensional operator ideals*, volume 38 of *Pitman Monographs and Surveys in Pure and Applied Mathematics.* Longman Scientific & Technical, Harlow, 1989.