Flots géométriques d'ordre quatre et pincement intégral de la courbure

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Introduction

Il est une quête interminable en géométrie riemannienne, qui consiste à tenter de démontrer que l'objet le plus naturel d'une variété riemannienne, sa courbure, est assez puissant pour pouvoir déterminer la structure de la variété sous-jacente, et en particulier sa topologie.

Cette quête pourrait avoir commencé par la découverte de la formule de Gauss-Bonnet, qui exprime la caractéristique d'Euler d'une variété riemannienne uniquement à l'aide de sa courbure, et a certainement été influencée par le théorème fondamental suivant :

Théorème 0.1. *Si* (M^n, g) *est une variété riemannienne complète à courbure sectionnelle constante égale à* 1, 0 *ou* -1, *alors elle est respectivement isométrique à un quotient de* $\mathbb{S}^n$, $\mathbb{R}^n$ *ou* $\mathbb{H}^n$ *munis de leur métrique standard.*

La “méthode de Bochner” a révélé de nombreux exemples de situations dans lesquelles la géométrie détermine la topologie des variétés riemanniennes. Elle est nommée d'après S. Bochner, qui a démontré en 1948 que la dimension du premier groupe de cohomologie d'une variété riemannienne, compacte, de dimension n, à courbure de Ricci positive ou nulle, est plus petite que n. Si, de plus, la courbure de Ricci est strictement positive en un point de la variété, alors son premier groupe de cohomologie s'annule.

Le résultat de S. Bochner a été étendu aux groupes de cohomologie d'ordre plus grand lorsqu'un certain opérateur de courbure est positif, et en 1975 S. Gallot et D. Meyer ont démontré que les nombres de Betti d'une variété riemannienne compacte à opérateur de courbure positif ou nul sont inférieurs à ceux du tore de même dimension ([GM75]). Si de plus l'opérateur de courbure est strictement positif en un point, alors ils s'annulent tous.

Le théorème de la sphère pour les variétés à courbure $1/4$-pincée, conjecturé par H. Rauch en 1951, constitue un autre exemple emblématique des liens qui existent entre la topologie et la géométrie des variétés riemanniennes. Il affirme qu'une variété riemannienne compacte ayant toutes ses courbures sectionnelles dans l'intervalle $]1/4, 1]$ est difféomorphe à un quotient de la sphère. W. Klingenberg, M. Berger et H. Rauch ont démontré qu'une telle variété est homéomorphe à un quotient de la sphère, et la conjecture complète est une conséquence d'un résultat de S. Brendle et R. Schoen ; en 2007, ils ont montré qu'une variété dont la plus petite courbure sectionnelle est, en chaque point, strictement positive et supérieure à un quart de la plus grande courbure sectionnelle, admet une métrique à courbure sectionnelle constante et strictement positive.

La preuve de ce dernier résultat utilise le flot de Ricci, introduit en 1982 par R. Hamilton. Dans [Ham82], R. Hamilton a montré que le flot de Ricci normalisé, partant d'une variété riemannienne compacte à courbure de Ricci strictement positive, existe pour tous temps, et converge vers une métrique à courbure sectionnelle constante et positive. Il s'ensuit que toute variété compacte de dimension trois qui admet une métrique à courbure de Ricci strictement positive est difféomorphe à un quotient de la sphère.

Le flot de Ricci est devenu un outil fondamental en géométrie riemannienne, et le résultat de [Ham82] a été étendu aux dimensions plus grandes, pour différentes notions de positivité de la courbure. Par exemple, R. Hamilton a démontré qu'en dimension quatre, le flot de Ricci normalisé partant d'une métrique à opérateur de courbure strictement positif converge vers une métrique à courbure sectionnelle constante et positive. Ce résultat a récemment été étendu à toutes les dimensions par C. Böhm et B. Wilking. En particulier, il complète le théorème de S. Gallot et D. Meyer lorsque l'opérateur de courbure est strictement positif ; la variété est alors non seulement une sphère homologique (c.-à-d. que tous les groupes de cohomologie sont triviaux), mais est de plus difféomorphe à un quotient de la sphère.

Pour démontrer de tels résultats de rigidité pour des variétés à courbure pincée avec le flot de Ricci, la stratégie consiste à prouver que le pincement est préservé le long du flot à l'aide du principe du maximum, et que dans un certain sens, le pincement s'améliore le long du flot. On montre alors que le flot existe pour tous temps positifs et converge vers une métrique à courbure sectionnelle constante. Cette stratégie a été appliquée avec succès pour différentes hypothèses de pincement, en invoquant des arguments de plus en plus élaborés.

Afin d'énoncer d'autres généralisations du théorème de R. Hamilton, on rappelle la décomposition orthogonale suivante de l'opérateur de courbure agissant sur les 2-formes d'une variété riemannienne (M^n, g) :

$$Rm_g = W_g + \mathcal{Z}_g + \mathcal{S}_g,$$

où W_g est la courbure de Weyl, $\mathcal{Z}_g$ ne dépend que de la partie sans trace de la courbure de Ricci :

$$\overset{\circ}{Ric}_g = Ric_g - \frac{1}{n} R_g \, g,$$

et $\mathcal{S}_g$ est un multiple de la courbure scalaire R_g que multiplie l'opérateur identité. On a de plus les identités suivantes :

$$|\mathcal{Z}_g|^2 = \frac{1}{n-2} \left| \overset{\circ}{Ric}_g \right|^2 \quad \text{et} \quad |\mathcal{S}_g|^2 = \frac{1}{2n(n-1)} R_g^2,$$

les normes étant prises en considérant les opérateurs de courbure comme des opérateurs symétriques sur les formes différentielles. On a par exemple, en utilisant la convention de sommation d'Einstein,

$$|W|^2 = \frac{1}{4} W_{ijkl} W^{ijkl} \quad \text{et} \quad |Ric|^2 = Ric_{ij} Ric^{ij}.$$

La partie sans aucune trace W_g s'annule systématiquement en dimensions deux et trois, et elle s'annule en dimension $n \geq 4$ si et seulement si la métrique est localement conformément plate. Une métrique pour laquelle la partie sans trace de la courbure de Ricci s'annule est appelée métrique d'Einstein. De plus, une métrique est à courbure sectionnelle constante si et seulement si elle est à la fois d'Einstein et localement conformément plate, donc si et seulement si elle vérifie

$$W_g = \mathcal{Z}_g = 0.$$

En 1985, G. Huisken a démontré le résultat de rigidité suivant ([Hui85]) :

Théorème 0.2. *Si (M^n, g) est une variété riemannienne compacte à courbure scalaire strictement positive telle que pour tout point $x \in M^n$,*

$$|W_g(x)|^2 + |\mathcal{Z}_g(x)|^2 < \delta_n \, |\mathcal{S}_g(x)|^2,$$

avec δ_n une constante explicite (par exemple $\delta_4 = \frac{1}{5}$), alors le flot de Ricci normalisé partant de g existe pour tous temps positifs et converge en topologie C^∞ vers une métrique à courbure sectionnelle constante et positive.

En particulier, M^n est difféomorphe à un quotient de la sphère $\mathbb{S}^n$.

En 1998, C. Margerin a amélioré la constante δ en dimension quatre, obtenant le théorème optimal suivant ([Mar98]) :

Théorème 0.3. *Si (M^4, g) est une variété riemannienne compacte à courbure scalaire strictement positive telle que pour tout point $x \in M^4$,*

$$|W_g(x)|^2 + |\mathcal{Z}_g(x)|^2 < |\mathcal{S}_g(x)|^2, \tag{0.1}$$

alors le flot de Ricci normalisé partant de g existe pour tous temps positifs et converge en topologie C^∞ vers une métrique à courbure sectionnelle constante et positive.

En particulier, M^4 est difféomorphe à la sphère $\mathbb{S}^4$ ou à l'espace projectif réel $\mathbb{R}\mathbf{P}^4$.

Ce théorème est optimal ; les seules variétés pour lesquelles l'égalité est atteinte dans (0.1) sont isométriques à $\mathbb{C}\mathbf{P}^2$ muni de la métrique de Fubini-Study, ou à un quotient du produit riemannien $\mathbb{S}^3 \times \mathbb{S}^1$.

Tous ces résultats on en commun d'apporter une information d'ordre topologique sur les variétés qui admettent une métrique dont la courbure satisfait à un certain pincement en chaque point. Certains de ces théorèmes sont en fait toujours vrais si l'on suppose seulement le pincement "en moyenne", c'est-à-dire en remplaçant le pincement ponctuel par un pincement intégral. On remplace alors l'hypothèse de courbure scalaire strictement positive par celle, plus faible, de constante de Yamabe strictement positive. La constante de Yamabe d'une variété riemannienne compacte (M^n, g) peut se définir de la façon suivante :

$$\mathrm{Y}(M,[g]) = \inf_{\tilde{g}\in[g]} \left\{ \frac{1}{\mathrm{Vol}(M,\tilde{g})^{1-\frac{2}{n}}} \int_M R_{\tilde{g}}\, dv_{\tilde{g}} \right\},$$

où $[g] = \left\{ e^{2f} g,\ f \in C^\infty(M) \right\}$ est la classe conforme de g. Elle est strictement positive si et seulement si il existe une métrique $\tilde{g} = e^{2f} g$ dans la classe conforme de g dont la courbure scalaire est strictement positive. De plus, on peut toujours trouver une métrique $\tilde{g} \in [g]$ telle que

$$\frac{1}{\mathrm{Vol}(M,\tilde{g})^{1-\frac{2}{n}}} \int_M R_{\tilde{g}}\, dv_{\tilde{g}} = \mathrm{Y}(M,[g]).$$

Une telle métrique est appelée *métrique de Yamabe*. Sa courbure scalaire est constante et égale à

$$R_{\tilde{g}} = \frac{\mathrm{Y}(M,[g])}{\mathrm{Vol}(M,\tilde{g})^{\frac{2}{n}}}.$$

En utilisant l'inégalité de Hölder, on remarque qu'on a toujours

$$\mathrm{Y}(M,[g]) \leq \| R_g \|_{L^{\frac{n}{2}}},$$

avec égalité si et seulement si g est une métrique de Yamabe.

Un exemple de ces résultats de pincement intégral est une généralisation du théorème de Bochner en dimension quatre, démontrée par M. Gursky dans [Gur98] et [Gur00], et qui peut s'énoncer de la manière suivante :

Théorème 0.4. *Soit (M^4, g) une variété riemannienne compacte et orientée à constante de Yamabe positive.*

i) Si la partie sans trace de la courbure de Ricci satisfait à

$$\| \mathcal{Z}_g \|^2_{L^2} \leq \frac{1}{24} \mathrm{Y}(M^4,[g])^2, \tag{0.2}$$

- *soit son premier nombre de Betti $b_1(M^4)$ s'annule,*
- *soit l'égalité est atteinte dans* (0.2)*, $b_1 = 1$, g est une métrique de Yamabe et (M^4, g) est conformément équivalente à un quotient du produit riemannien $\mathbb{S}^3 \times \mathbb{R}$.*

ii) Si la courbure de Weyl satisfait à

$$\| W_g \|^2_{L^2} \leq \frac{1}{24} \mathrm{Y}(M^4,[g])^2, \tag{0.3}$$

- *soit son second nombre de Betti $b_2(M^4)$ s'annule,*
- *soit l'égalité est atteinte dans* (0.3)*, $b_2 = 1$ et (M^4, g) est conformément difféomorphe à $\mathbb{C}\mathbf{P}^2$ muni de la métrique de Fubini-Study.*

En dimension quatre, sur une variété compacte (M^4, g), la formule de Gauss-Bonnet affirme que

$$\| W_g \|^2_{L^2} - \| \mathcal{Z}_g \|^2_{L^2} + \| \mathcal{S}_g \|^2_{L^2} = 8\pi^2 \chi(M^4),$$

où $\chi(M^4)$ est la caractéristique d'Euler de M^4. On peut donc réécrire le pincement intégral

$$\| W_g \|^2_{L^2} + \| \mathcal{Z}_g \|^2_{L^2} < \| \mathcal{S}_g \|^2_{L^2}, \tag{0.4}$$

et le remplacer par

$$\| W_g \|^2_{L^2} < 4\pi^2 \chi(M),$$

qui lui est équivalent et qui est invariant par changement conforme. Ainsi, si une variété compacte orientée de dimension quatre à constante de Yamabe strictement positive satisfait au pincement (0.4), on peut supposer que g est une métrique de Yamabe, et le Théorème 0.4 permet d'affirmer que ses groupes de cohomologie sont triviaux.

Dans [CGY03], A. Chang, M. Gursky et P. Yang ont démontré qu'une variété qui satisfait à ces conditions est en fait difféomorphe à un quotient de la sphère ; ils ont prouvé l'extension suivante du Théorème 0.3 :

Théorème 0.5. *Si (M^4, g) est une variété riemannienne compacte à constante de Yamabe strictement positive telle que*

$$\|W_g\|^2_{L^2} + \|\mathcal{Z}_g\|^2_{L^2} < \|\mathcal{S}_g\|^2_{L^2}, \tag{0.5}$$

alors elle est difféomorphe à la sphère $\mathbb{S}^4$ ou à l'espace projectif réel $\mathbb{R}\mathbf{P}^4$.

De plus, ce théorème est optimal, puisque les seules variétés pour lesquelles l'égalité est atteinte sont conformément équivalentes à $\mathbb{C}\mathbf{P}^2$ muni de la métrique de Fubini-Study, ou à un quotient du produit riemannien $\mathbb{S}^3 \times \mathbb{R}$.

Les preuves de ces résultats de pincement intégral reposent sur des arguments de géométrie conforme. L'idée générale est qu'en effectuant un bon changement conforme sur la métrique, on peut retrouver un pincement ponctuel de la courbure. On peut ensuite appliquer une technique de Bochner ponctuelle pour obtenir le Théorème 0.4, et le résultat ponctuel de C. Margerin (Théorème 0.3) permet d'obtenir le Théorème 0.5.

L'objet de cette thèse est d'explorer de nouvelles approches pour obtenir des résultats de rigidité sur les variétés à courbure intégralement pincée.

On étudie tout d'abord une classe de flots de courbure d'ordre quatre, qui contient les flots de gradient de fonctionnelles quadratiques en la courbure. Une propriété essentielle de ces flots de gradient est qu'ils préservent, et améliorent, des pincement intégraux de la courbure. Comme application de cette étude, nous présentons une nouvelle preuve d'une partie du Théorème 0.5 (c.-à-d. avec une hypothèse de pincement plus restrictive), qui repose entièrement sur l'étude d'un flot géométrique, et qui ne dépend pas de la version ponctuelle du théorème, due à C. Margerin.

On obtient ensuite des résultats de pincement intégral en combinant une formule de Weitzenböck avec l'inégalité de Sobolev induite par la positivité de la constante de Yamabe. Nous appliquons ce principe général pour démontrer un résultat de rigidité pour les singularités de nos flots d'ordre quatre. Grâce à cela, nous démontrons la convergence de ces flots sous une hypothèse de pincement intégral sur la métrique initiale.

Dans un travail en collaboration avec G. Carron, nous appliquons également cette méthode de Bochner intégrale pour généraliser les théorèmes classiques de Bochner-Weitzenböck aux variétés dont la courbure est intégralement pincée, et en particulier, nous redémontrons et étendons le Théorème 0.4 aux degrés et dimensions supérieurs.

1. From curvature pinching to topological information

There is a never ending quest in Riemannian geometry, which consists in trying to prove that the most natural object of a Riemannian manifold, its curvature, is powerful enough as to determine the structure of the underlying manifold, and especially its topology.

The quest may have been initiated by the discovery of the Gauss-Bonnet formula, which determines the Euler characteristic of a Riemannian manifold only by means of its curvature, and has certainly been marked by the following fundamental theorem:

Theorem 0.1. *If a complete Riemannian manifold (M^n, g) has constant sectional curvature 1, 0 or -1, then it is respectively isometric to a quotient of the standard $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{H}^n$.*

The so-called "Bochner technique" has led to several examples of situations where the curvature determines the topology of Riemannian manifolds. It is named after S. Bochner, who proved in 1948 that the dimension of the first cohomology group of a closed n-dimensional Riemannian manifold with nonnegative Ricci curvature is smaller than n. And if furthermore the Ricci curvature is positive somewhere, then its first cohomology group must vanish.

S. Bochner's result has been extended to higher-order cohomology groups when some curvature-related operator is nonnegative, and in 1975, S. Gallot and D. Meyer proved that the Betti numbers of a closed n-dimensional Riemannian manifold with nonnegative curvature operator are smaller than those of the torus of dimension n ([GM75]). If in addition the curvature operator is positive somewhere, then they must all vanish.

The sphere theorem for $1/4$-pinched Riemannian manifolds, conjectured by H. Rauch in 1951, is another emblematic example of the deep connections between the topology and the geometry of Riemannian manifolds. It asserts that a closed Riemannian manifold whose sectional curvatures lie in $(1/4, 1]$ is diffeomorphic to a quotient of the sphere. W. Klingenberg, M. Berger and H. Rauch proved that such a manifold is homeomorphic to a quotient of the sphere, and the full conjecture is a consequence of a result of S. Brendle and R. Schoen: in 2007, they proved that if at each point the lowest sectional curvature is positive and larger than one quarter of the largest sectional curvature, then the manifold carries a metric of positive constant sectional curvature.

Their proof uses the Ricci flow, introduced in 1982 by R. Hamilton. In [Ham82] he proved that the normalized Ricci flow starting on a closed three-dimensional Riemannian manifold with positive Ricci curvature exists for all time and converges to a metric of positive constant sectional curvature. Therefore, any closed three-dimensional manifold carrying a metric of positive Ricci curvature is diffeomorphic to a quotient of the sphere.

The Ricci flow has become a very powerful tool in Riemannian geometry, and the result of [Ham82] has been extended to higher dimensions and to various notions of positive curvature. For instance, R. Hamilton proved that in dimension four, the normalized Ricci flow starting from a metric of positive curvature operator converges to a metric of positive constant sectional curvature. This result has been recently extended to all dimensions by C. Böhm and B. Wilking. In particular, it completes the theorem of S. Gallot and D. Meyer when the curvature operator is positive: the manifold is not only an homological sphere (i.e. all its cohomology groups vanish), but is actually diffeomorphic to a quotient of the sphere.

The strategy to prove rigidity results for manifolds with pinched curvature by using the Ricci flow consists in proving with the maximum principle that the pinching is preserved along the flow, and that in some sense, it becomes better and better along the flow. Then one proves that the normalized flow exists for all time and converges to a metric of constant sectional curvature. This has been done for different pinching assumptions by using more and more elaborate arguments.

There have been other generalizations of R. Hamilton's result. To state them, we recall the following orthogonal decomposition of the curvature operator, acting on the 2-forms of a Riemannian manifold (M^n, g):

$$Rm_g = W_g + \mathcal{Z}_g + \mathcal{S}_g,$$

where W_g is the Weyl curvature, $\mathcal{Z}_g$ only depends on the traceless part of the Ricci curvature

$$\overset{\circ}{Ric}_g = Ric_g - \frac{1}{n} R_g\, g,$$

and $\mathcal{S}_g$ is a multiple of the scalar curvature R_g times the identity operator. Moreover, we have

$$|\mathcal{Z}_g|^2 = \frac{1}{n-2} \left|\overset{\circ}{Ric}_g\right|^2 \quad \text{and} \quad |\mathcal{S}_g|^2 = \frac{1}{2n(n-1)} R_g^2.$$

The norms are taken by considering curvature operators as symmetric operators on differential forms. For instance, with the Einstein summation convention, we have

$$|W|^2 = \frac{1}{4} W_{ijkl} W^{ijkl} \quad \text{and} \quad |Ric|^2 = Ric_{ij} Ric^{ij}.$$

The totally trace-free part W_g always vanishes in dimension 2 and 3, and in dimension $n \geq 4$, it vanishes if and only if the metric is locally conformally flat. A metric for which the traceless Ricci part $\mathcal{Z}_g$ vanishes is called an Einstein metric. Moreover, a metric has constant sectional curvature if and only if it is both Einstein and locally conformally flat, therefore if and only if

$$W_g = \mathcal{Z}_g = 0.$$

In 1985, G. Huisken proved the following rigidity result ([Hui85]):

Theorem 0.2. *If (M^n, g) is a closed Riemannian manifold with positive scalar curvature such that for all x in M^n,*

$$|W_g(x)|^2 + |\mathcal{Z}_g(x)|^2 < \delta_n\, |\mathcal{S}_g(x)|^2,$$

with δ_n an explicit constant, e.g. $\delta_4 = \frac{1}{5}$, then the solution of the normalized Ricci flow starting from g exists for all time and converges as t goes to $+\infty$ to a metric of positive constant curvature in the C^∞ topology.

In particular, M^n is diffeomorphic to a quotient of the sphere.

In 1998, C. Margerin improved the constant in dimension four, and obtained the following optimal theorem ([Mar98]):

Theorem 0.3. *If (M^4, g) is a closed Riemannian manifold with positive scalar curvature such that for all x in M^4,*

$$|W_g(x)|^2 + |\mathcal{Z}_g(x)|^2 < |\mathcal{S}_g(x)|^2, \tag{0.6}$$

then the solution of the normalized Ricci flow starting from g exists for all time and converges as t goes to $+\infty$ to a metric of positive constant curvature in the C^∞ topology.

In particular, M^4 is diffeomorphic to the sphere $\mathbb{S}^4$ or to the real projective space $\mathbb{R}\mathbf{P}^4$.

The theorem is optimal as the only manifolds for which equality holds in (0.6) are isometric either to $\mathbb{C}\mathbf{P}^2$ with the Fubini-Study metric or to a quotient of the Riemannian product $\mathbb{S}^3 \times \mathbb{S}^1$.

The common feature of all those results is to give topological information on a manifold that carry a metric whose curvature satisfy a certain pinching at each point. It has appeared that some of those theorems are still true if one only requires the curvature to be pinched in an "average" sense, that is when replacing the pointwise pinching by an integral pinching. In these results, the assumption of positive scalar curvature will be replaced by the weaker one of positive Yamabe constant. The Yamabe constant of a closed Riemannian manifold (M^n, g) can be defined by

$$\mathrm{Y}(M,[g]) = \inf_{\tilde g\in[g]}\left\{\frac{1}{\mathrm{Vol}(M,\tilde g)^{1-\frac{2}{n}}}\int_M R_{\tilde g}\,dv_{\tilde g}\right\},$$

where $[g] = \left\{e^{2f}g,\ f\in C^\infty(M)\right\}$ is the conformal class of g. It is positive if and only if there exists a metric $\tilde g = e^{2f}g$ in the conformal class of g which has positive scalar curvature.

Moreover, we can always find a metric $\tilde g\in[g]$ such that

$$\frac{1}{\mathrm{Vol}(M,\tilde g)^{1-\frac{2}{n}}}\int_M R_{\tilde g}\,dv_{\tilde g} = \mathrm{Y}(M,[g]).$$

Such a metric is called a *Yamabe minimizer*, and has a constant scalar curvature equal to

$$R_{\tilde g} = \frac{\mathrm{Y}(M,[g])}{\mathrm{Vol}(M,\tilde g)^{\frac{2}{n}}}.$$

Using the Hölder inequality, we see that we always have

$$\mathrm{Y}(M,[g]) \le \|R_g\|_{L^{\frac{n}{2}}},$$

with equality if and only if g is a Yamabe minimizer.

One example of these integral pinching results is a generalization of the Bochner theorem in dimension four proven by M. Gursky in [Gur98] and [Gur00], and can be stated as follows:

Theorem 0.4. *Assume that (M^4, g) is a closed oriented Riemannian manifold with positive Yamabe constant.*

i) If the traceless part of the Ricci curvature satisfies

$$\|\mathcal{Z}_g\|^2_{L^2} \le \frac{1}{24}\,\mathrm{Y}(M^4,[g])^2, \tag{0.7}$$

- *either its first Betti number $b_1(M^4)$ vanishes,*
- *or equality holds in* (0.7)*, $b_1 = 1$, g is a Yamabe minimizer and (M^4, g) is conformally equivalent to a quotient of the Riemannian product $\mathbb{S}^3\times\mathbb{R}$.*

ii) If the Weyl curvature satisfies

$$\|W_g\|^2_{L^2} \le \frac{1}{24}\,\mathrm{Y}(M^4,[g])^2, \tag{0.8}$$

- *either its second Betti number $b_2(M^4)$ vanishes,*
- *or equality holds in* (0.8)*, $b_2 = 1$ and (M^4, g) is conformally equivalent to $\mathbb{C}\mathbf{P}^2$ endowed with the Fubini-Study metric.*

On a four-dimensional closed manifold (M^4, g), the Gauss-Bonnet formula asserts that

$$\|W_g\|^2_{L^2} - \|\mathcal{Z}_g\|^2_{L^2} + \|\mathcal{S}_g\|^2_{L^2} = 8\pi^2\chi(M^4),$$

where $\chi(M^4)$ is the Euler characteristic of M^4. Consequently, the integral pinching

$$\|W_g\|^2_{L^2} + \|\mathcal{Z}_g\|^2_{L^2} < \|\mathcal{S}_g\|^2_{L^2}, \tag{0.9}$$

is equivalent to the conformally invariant one

$$\|W_g\|^2_{L^2} < 4\pi^2\chi(M).$$

Therefore, if a closed oriented four-dimensional manifold with positive Yamabe constant satisfies (0.9), we can suppose that g is a Yamabe minimizer, and it follows from Theorem 0.4 that its cohomology groups must vanish.

In [CGY03], A. Chang, M. Gursky and P. Yang proved that a manifold which satisfy these conditions is actually diffeomorphic to a quotient of the sphere; they gave the following extension of Theorem 0.3:

Theorem 0.5. *If (M^4, g) is a closed Riemannian manifold with positive Yamabe constant such that*

$$\|W_g\|_{L^2}^2 + \|\mathcal{Z}_g\|_{L^2}^2 < \|\mathcal{S}_g\|_{L^2}^2, \tag{0.10}$$

then it is diffeomorphic to the sphere $\mathbb{S}^4$ or to the real projective space $\mathbb{R}\mathbf{P}^4$.

Moreover, this theorem is optimal, as the only manifolds for which equality holds are conformally equivalent to $\mathbb{C}\mathbf{P}^2$ endowed with the Fubini-Study metric or to a quotient of the Riemannian product $\mathbb{S}^3 \times \mathbb{R}$.

The proofs of those integral pinching results are based on conformal geometry arguments. The general idea is to show that a good conformal change can be made on the metric in order to recover a pointwise pinching on the curvature. Then, one can apply a pointwise Bochner method to obtain Theorem 0.4, and the pointwise result of C. Margerin (Theorem 0.3) leads to Theorem 0.5.

The purpose of this thesis is to explore other approaches to obtain rigidity results for manifolds with integral pinched curvature.

We first study a class of fourth-order curvature flows, which contains the gradient flows of quadratic curvature functionals. An essential feature of those gradient flows is that they are defined in such a way that they preserve, and actually improve integral pinchings on the curvature. As an application of this study, we give a new proof of a part of Theorem 0.5 (i.e. with a stronger pinching assumption), which is entirely based on the study of a geometric flow, and doesn't rely on the pointwise version of the theorem, due to C. Margerin.

We then obtain integral pinching results by combining a Weitzenböck formula with the Sobolev inequality induced by the positivity of the Yamabe constant. We apply this general principle to prove a rigidity theorem for the singularities of our fourth-order flows. This is what allows us to prove the convergence of the flows under a pinching assumption on the initial metric.

In a joint work with G. Carron, we also applied this integral Bochner method to generalize classical Bochner-Weitzenböck theorems to manifold with integral pinched curvature, and in particular, we reprove and extend Theorem 0.4 to higher degrees and dimensions.

2. On some fourth-order curvature flows

According to [Per02], the Ricci flow can be seen as the gradient flow for a modified Einstein-Hilbert functional

$$\mathcal{F}^m(g) = \int_M R_g^m dm,$$

where m is a fixed measure on M and R_g is a scalar curvature modified by the measure. Moreover, the critical points of $\mathcal{F}^m$ are gradient Ricci solitons, which are the manifolds arising when performing a blow-up at a singular time. An essential property of the Ricci flow is that it is invariant (up to diffeomorphisms) by a change of measure m. This fact combined with the monotonicity of $\mathcal{F}^m$ leads to the noncollapsing result of G. Perelman.

In order to preserve integral pinchings on the curvature, we will consider gradient flows for quadratic curvature functionals, linear combination of:

$$\mathcal{F}_{Rm}(g) = \int_M |Rm_g|^2 \, dv_g, \quad \mathcal{F}_{Ric}(g) = \int_M |Ric_g|^2 \, dv_g \text{ and } \mathcal{F}_R(g) = \int_M R_g^2 dv_g, \tag{0.11}$$

or equivalently of

$$\mathcal{F}_W(g) = \int_M |W_g|^2 \, dv_g, \quad \mathcal{F}_{\overset{\circ}{Ric}}(g) = \int_M \left|\overset{\circ}{Ric}_g\right|^2 dv_g \text{ and } \mathcal{F}_R(g) = \int_M R_g^2 dv_g. \tag{0.12}$$

These gradient flows involve four space derivatives of the metric, and therefore do not share as much properties as an heat equation like the Ricci flow does. For example, the maximum principle doesn't apply.

Several other higher-order geometric gradient flows have been successfully carried out during the last decade, among which the Calabi flow on surfaces in [Chr91], general gradient flows on curves and surfaces in [Pol96, Pol97], the Willmore flow in [Sim01, KS01, KS02], flows of fourth and higher-order on immersed hypersurfaces in [Man02], a fourth-order equivalent of the Yamabe conformal flow introduced in [Bre03], the Calabi flow on Kähler manifolds in [CH08].

As for the gradient flows of the quadratic functionals we are interested in, Y. Zheng has considered the gradient flow of $\mathcal{F}_{Ric}$ in dimension three in [Zhe03], and J. Streets has studied the gradient flow of $\mathcal{F}_{Rm}$ in dimension four in [Str08]. They both proved short-time existence and integral estimates for their respective flows. In [Str09] and [Str10b], J. Streets proved long-time existence and convergence for his flow when it starts from a metric with a curvature very close to a constant one (in the L^2 sense). He proved similar results in dimension three in [Str12]. Unfortunately, the number ϵ which quantify how close the metric should start is non-explicit and results in a intricate way from a number of curvature estimates.

The first difficulty in the study of these fourth-order flows is that the metric could a priori collapse with bounded curvature at a singularity. It is a very classical fact about the Ricci flow that the curvature must blow up at a singular time. Indeed, if the curvature could stay bounded, then all the derivatives of the curvature would be bounded, and the solution would extend beyond the singular time. This fact results either from pointwise estimates on the curvature, obtained by the maximum principle, or as it was done in [Ham82], by remarking that a bound on the curvature induces a bound on the time derivative of the metric $\partial_t g$, which implies that the Sobolev constant stays bounded. Then, one can prove pointwise estimates from integral estimates by Sobolev inclusions.

For fourth-order curvature flows, neither does the maximum principle apply, nor does a curvature bound induce a control on the metric or on a Sobolev constant. Therefore, the fact that the curvature must blow-up at a singular time becomes non-trivial, and was only proven by J. Streets in [Str11], by showing pointwise curvature estimates using a contradiction argument.

The second difficulty arises when one wants to make a blow-up analysis at a singular time. Since the curvature blows up, one can define a sequence of normalized flows with bounded curvature. The existence of a limit for a subsequence of those manifolds comes from classical precompactness results when the injectivity radius is uniformly bounded from below by a positive constant. Whether such a lower bound exists or not depends on a noncollapsing property, which was proved for the Ricci flow by G. Perelman ([Per02]), but remains unattainable in general for fourth-order flows.

In this thesis, we deal with this issue by controlling the Yamabe constant. When we assume a positive lower bound on it, we can prove a noncollapsing property similar to the one of G. Perelman. As a consequence, we can always find a sequence of manifolds near a singular time which converges to a "singularity model".

Given a Riemannian manifold (M^n, g_0), the evolution equations we consider are given by:

$$\begin{cases} \partial_t g = P(g) \\ g(0) = g_0, \end{cases} \qquad E_P(g_0)$$

where $P : \mathcal{S}^2_+(M) \to \mathcal{S}^2(M)$ is a smooth map of the form

$$P(g) = \delta\tilde{\delta} Rm_g + a\Delta R_g g + b\nabla^2 R_g + Rm_g * Rm_g, \tag{0.13}$$

with a and b two real numbers.

The notation $\mathcal{S}^2(M)$ denotes the space of symmetric $(2,0)$-tensors, $\mathcal{S}^2_+(M)$ the space of metrics, and $S * T$ denotes any linear combination (with coefficients independent of the metric) of terms obtained from $S \otimes T$ by taking tensor products with g and g^{-1}, contracting and permuting indices. The operators δ and $\tilde{\delta}$ are defined in Appendix B. In particular we have

$$\delta\tilde{\delta}Rm_{ij} = \nabla^{\alpha}\nabla^{\beta}Rm_{i\alpha j\beta},$$

and we recall that

$$\delta\tilde{\delta}Rm_g = \nabla^*\nabla Ric_g + \frac{1}{2}\nabla^2 R_g + Rm_g * Rm_g.$$

The gradient flows for most of the quadratic functionals generated by those in (0.11) or (0.12) are of this type (see Section 2 of Chapter 2 for details).

We first prove short-time existence for this class of flows when $a < \frac{1}{2(n-1)}$. Since P is invariant by diffeomorphisms, it is not elliptic. We can proceed as for the Ricci flow, and apply the DeTurck trick to obtain an equation that doesn't possess that geometric invariance. We show in Section 1 of Chapter 2 that it is possible to make P become strongly elliptic if and only if $a < \frac{1}{2(n-1)}$. In that case, the flow exists and is unique on a small time interval, thus we obtain:

Theorem A (Short-time existence)**.** *Let (M, g_0) be a closed Riemannian manifold. When $a < \frac{1}{2(n-1)}$, there exists a unique maximal solution (g_t) of $E_P(g_0)$ defined on some time interval $[0, T)$, with T positive.*

In the sequel of the introduction, we will assume that P is a smooth map of the form (0.13), with $a < \frac{1}{2(n-1)}$. The next step is to show that the curvature must blow up at the time T of maximal existence, by proving curvature estimates. In [Bou10] we prove pointwise estimates on the curvature from integral one when the Yamabe constant remains uniformly positive along the flow. Here, we will use the method of [Str11], which doesn't need this hypothesis. Moreover, we improve the method of J. Streets with a point selection argument to prove local estimates when the curvature is locally bounded. In particular, we obtain the following property at a singular time:

Theorem B (Curvature blow-up)**.** *Let (M^n, g_0), $n \geq 3$, be a closed Riemannian manifold and let $g(t)$, $t \in [0, T)$ be the maximal solution of $E_P(g_0)$. If $T < \infty$, then the curvature blows up at T:*

$$\overline{\lim_{t\to T}} \; \|Rm_{g_t}\|_{L^\infty} = \infty.$$

Moreover, if $T < \infty$, the curvature blows up faster than $\frac{1}{(\sqrt{T-t})^q}$ for any $q < 1$: we can find sequences $t_k \to T$ and $x_k \in M$ such that

$$\left|Rm_{g(t_k)}\right|(x_k) \geq \frac{k}{\left(\sqrt{T-t_k}\right)^{1-\frac{1}{k}}}.$$

When in addition we have a uniform positive lower bound on the Yamabe constant, we obtain local estimates when the curvature is locally bounded in L^p with respect to the space variable, and in L^q with respect to the time variable, with $\frac{n}{2p} + \frac{2}{q} = 1$ (Theorem 2.7). In particular we obtain the following result when $p = \frac{n}{2}$ and $q = \infty$, which shows that the curvature must concentrate in the $L^{n/2}$ sense at a singularity:

Theorem C. *For all $\alpha > 0$, $k \in \mathbb{N}$ and $n \geq 3$, there exists a constant $\epsilon(\alpha, k, n, P)$ such that if $g(t)$, $t \in [0, T]$, are complete metrics on a manifold M^n which are solution to E_P, that satisfy*

$$\inf_{[0,T]} \mathrm{Y}(M, [g_t]) > 0,$$

and such that for some $x_0 \in M$ and $0 < r < T^{1/4}$

$$\sup_{[0,T]} \left(\int_{\mathrm{B}_g(x_0,r)} |Rm_g|^{\frac{n}{2}} \, dv_g \right) \leq \epsilon,$$

then for all $t \in (0,T]$ *and* $x \in \mathrm{B}_{g(t)}(x_0, \frac{r}{2})$

$$\sum_{j=0}^{k} \left|\nabla^j Rm_g\right|^{\frac{2}{2+j}} \leq \alpha\left(\frac{1}{r^2} + \frac{1}{\sqrt{t}}\right).$$

When the Yamabe constant remains uniformly positive along the flow, we can also obtain a model of the singularity by taking a limit of dilations of the metric near a singular time and a singular point. More precisely, if we take sequences of times $t_i \to T$ and points $x_i \in M$ such that the curvature at (x_i, t_i) blows up, and if we renormalize the metric at time t_i such that the curvature is bounded by 1, a subsequence of these dilated metrics converges. Hence, we obtain the following general behavior of the flows E_P:

Theorem D. *Let* (M^n, g_0), $n \geq 3$, *be a closed Riemannian manifold. Let* $g(t)$, $t \in [0,T)$ *be the maximal solution of* $E_P(g_0)$. *Suppose that* $\inf\limits_{[0,T)} \mathrm{Y}(M, [g_t]) > 0$. *Then one of the following situations occurs:*

1) *The flow exists for all time with uniform* C^k *curvature bounds and a uniform lower bound on the injectivity radius.*
2) *A finite or infinite time singularity occurs, where the curvature blows up:*

$$\overline{\lim_{t\to T}}\ \|Rm_{g_t}\|_{L^\infty} = \infty.$$

In that case, there exist sequences $t_i \to T$ *and* $x_i \in M$ *such that* $\left|Rm_{g(t_i)}(x_i)\right| \to \infty$ *and the renormalized metrics*

$$\left(M, \left|Rm_{g(t_i)}(x_i)\right| g(t_i),\, x_i\right)$$

converge to a non-flat complete Riemannian manifold in the pointed C^∞ *topology.*

The situation is particularly interesting in dimension four, which is the only dimension for which the functionals in (0.11) and (0.12) are scale-invariant. Moreover, in dimension four, thanks to the Gauss-Bonnet formula, a positive lower bound on the Yamabe constant exists as soon as the Yamabe constant is positive and the mean Q-curvature is positively bounded from below (where the Q-curvature is defined by $Q_g = \frac{1}{6}\Delta R_g + |\mathcal{S}_g|^2 - |\mathcal{Z}_g|^2$). We show that these conditions are satisfied for the gradient flows of a number of quadratic functionals, when assuming a bound on the initial energy. More precisely, for $\lambda \geq 0$, we consider the functionals

$$\mathcal{F}^\lambda(g) = (1-\lambda)\int_M |W_g|^2\, dv_g + \lambda \int_M |\mathcal{Z}_g|^2\, dv_g,$$

and the corresponding gradient flows:

$$\begin{cases} \partial_t g = -2\nabla\mathcal{F}^\lambda(g) \\ g(0) = g_0, \end{cases} \qquad E^\lambda(g_0)$$

on a closed Riemannian manifold (M^4, g_0). Short-time existence is supplied by Theorem A as soon as $\lambda > 0$, and according to the Gauss-Bonnet formula, the case $\lambda = 1$ corresponds to the gradient flow of $\frac{1}{2}\mathcal{F}_{Rm}$.

For that class of gradient flows, and when the initial value of the functional $\mathcal{F}^\lambda$ is less than $(1-\lambda)8\pi^2\chi(M)$, we prove that blow-up sequences possess a converging subsequence, and it happens that the limit is critical for the given functional, and is actually a complete Bach-flat (i.e. critical for the functional $\mathcal{F}^0$) manifold with zero scalar curvature:

Theorem E. *Let* λ *be in* $(0,1)$. *Let* (M^4, g_0) *be a closed Riemannian manifold with positive Yamabe constant such that*

$$\mathcal{F}^\lambda(g_0) \leq (1-\lambda)8\pi^2\chi(M). \tag{0.14}$$

If $g(t)$, $t \in [0,T)$ *is the maximal solution of* $E^\lambda(g_0)$, *then one of the following situations occurs:*

1) The flow exists for all time and for any sequence $t_i \to \infty$*, a subsequence of* $g(t_i)$ *converges in the* C^∞ *topology to a metric* g_∞*, which is critical for* $\mathcal{F}^\lambda$*.*

2) A finite or infinite time singularity occurs, where the curvature blows up:

$$\overline{\lim_{t \to T}} \ \|Rm_{g_t}\|_{L^\infty} = \infty.$$

In that case, there exist sequences $t_i \to T$ *and* $x_i \in M$ *such that* $\left|Rm_{g(t_i)}(x_i)\right| \to \infty$*, and the renormalized metrics*

$$\left(M, \left|Rm_{g(t_i)}(x_i)\right| g(t_i), x_i\right)$$

converge to a non-flat complete non-compact Riemannian manifold in the pointed C^∞ *topology, which is Bach-flat and scalar-flat.*

Remarks 0.6. 1) The condition (0.14) is equivalent to the following integral pinching between the scalar curvature and the traceless Ricci tensor:

$$\|\mathcal{Z}_{g_0}\|^2_{L^2} \leq (1-\lambda) \ \|\mathcal{S}_{g_0}\|^2_{L^2} \ .$$

2) When $\lambda = 1$, the theorem remains true but becomes useless, as the assumption (0.14) is only satisfied by Einstein metrics, which are critical for the functional.

To show long-time existence and convergence for the flow, we prove a rigidity result for those non-compact Bach-flat manifolds modeling the singularity. This is done by using an integral Bochner technique, and is explained in the next section. It allows us to rule out the formation of singularities when the initial energy is less than an explicit bound.

3. An integral Bochner technique

When some tensor satisfies an elliptic equation (an harmonic form, a Killing vector field, an harmonic curvature tensor, the curvature of critical metrics of a given functional), a Weitzenböck formula often implies that the norm of the tensor satisfy a certain elliptic inequality depending on a curvature term.

When the curvature term satisfies a pointwise pinching, the classical Bochner technique consists in applying the maximum principle to show that the tensor must vanish.

When the curvature term satisfies the right integral pinching with some Sobolev constant, we can integrate the inequality over the manifold to show that the tensor vanishes.

This technique has been used by E. Hebey and M. Vaugon in [HV96] to prove rigidity results for closed manifolds with harmonic Weyl curvature which satisfy an integral pinching with the Yamabe constant, or by G. Carron in [Car99] to prove the vanishing of L^2 cohomology groups on complete manifolds.

In Section 3 of Chapter 1, we present a general pinching theorem based on this idea. When we apply it to manifolds with harmonic curvature, we recover a number of results of [HV96], and extend them to non-compact manifolds. We also apply it to critical metrics for quadratic curvature functionals in dimension four, in order to rule out singularities of fourth-order flows, and to harmonic forms, in order to prove the vanishing of Betti numbers under optimal integral pinchings.

3.1. Application to critical metrics. We prove the following rigidity result for the singularity models of our flows:

Theorem F. *Let* (M^4, g) *be a complete Riemannian manifold with positive Yamabe constant and let* λ *be in* $[0,1]$*. Suppose that* R_g *is in* $L^2(M)$*. If* $\lambda = 0$*, suppose that* R_g *is constant.*

If g *is a critical metric of* $\mathcal{F}^\lambda$ *with*

$$\|W_g\|^2_{L^2} + \frac{1}{2} \ \|\mathcal{Z}_g\|^2_{L^2} < \frac{1}{8 \times 24} \ \mathrm{Y}(M,[g])^2, \tag{0.15}$$

then g *is of constant sectional curvature.*

As a consequence, no singularity can occur along the flow when the initial value of the functional is less than a given constant. Indeed, the bound on the functional is preserved along the flow, and correspond to an integral pinching on the curvature, satisfied by every metric in the flow. Since the singularity model is obtained as limit of dilated metrics at a singular time, this limit manifold also satisfies some integral pinching on the curvature. For the right value of the initial bound, this integral pinching is exactly (0.15), and Theorem F implies that the manifold modeling the singularity must be flat. This implies that there is actually no singularity. As a consequence, the flow exists for all time, and converges to a metric which is critical for the functional, and satisfies (0.15), hence to a spherical space form.

Theorem G. *Let λ be in $(0,1)$. If (M^4, g_0) is a closed Riemannian manifold with positive Yamabe constant such that*

$$\begin{cases} \lambda \leq \dfrac{4}{13} \\ \mathcal{F}^\lambda(g_0) < 2\lambda\pi^2\chi(M) \end{cases} \quad \textit{or} \quad \begin{cases} \lambda \geq \dfrac{4}{13} \\ \mathcal{F}^\lambda(g_0) < \dfrac{8}{9}(1-\lambda)\pi^2\chi(M), \end{cases}$$

then the solution of $E^\lambda(g_0)$ exists for all time and converges in the C^∞ topology to a metric of constant positive curvature. In particular, M^4 is diffeomorphic to the sphere $\mathbb{S}^4$ or the real projective space $\mathbb{R}\mathbf{P}^4$.

If we take $\lambda = \frac{4}{13}$ in the previous theorem, then according to the Gauss-Bonnet formula we obtain the following result:

Corollary H. *If (M^4, g_0) is a closed Riemannian manifold with positive Yamabe constant such that*

$$\|W_{g_0}\|^2_{L^2} + \frac{5}{8}\, \|\mathcal{Z}_{g_0}\|^2_{L^2} < \frac{1}{8}\, \|\mathcal{S}_{g_0}\|^2_{L^2},$$

then the flow of $\mathcal{F}^{4/13}$ exists for all time and converges to a metric of constant positive curvature. In particular, M^4 is diffeomorphic to the sphere $\mathbb{S}^4$ or the real projective space $\mathbb{R}\mathbf{P}^4$.

Under those stronger hypotheses, it provides an alternative proof of Theorem 0.5.

3.2. Application to harmonic forms (with G. Carron). This integral Bochner technique can also be applied to harmonic differential forms. With G. Carron, we prove in [BC12] an integral Bochner-Weitzenböck theorem. Thanks to this, we deduce the vanishing of Betti numbers under integral pinching assumptions in several situations, and we characterize the equality cases.

The first example of these integral pinching results is an integral version of the Bochner-Weitzenböck theorem of S. Gallot and D. Meyer ([GM75]). If we let $-\rho_g$ be the lowest eigenvalue of the traceless curvature operator $W_g + \mathcal{Z}_g$ of a Riemannian manifold (M^n, g), this Bochner-Weitzenböck theorem can be stated as follows:

Theorem 0.7. *If a closed Riemannian manifold (M^n, g) has a nonnegative curvature operator, i.e. if*

$$\rho_g \leq \frac{1}{n(n-1)} R_g, \tag{0.16}$$

then for all $1 \leq k \leq \frac{n}{2}$,

- *either its k^{th} Betti number $b_k(M^n)$ vanishes,*
- *or equality holds in (0.16), $1 \leq b_k \leq \binom{n}{k}$ and every harmonic k-form is parallel.*

We prove that a large part of the theorem remains true if we only make the assumption in an integral sense:

Theorem I. *If (M^n, g), $n \geq 4$, is a closed Riemannian manifold such that*

$$\|\rho_g\|_{L^{\frac{n}{2}}} \leq \frac{1}{n(n-1)} \mathrm{Y}(M^n, [g]), \tag{0.17}$$

then for all $1 \leq k \leq \frac{n-3}{2}$ or $k = \frac{n}{2}$,

- *either its k^{th} Betti number $b_k(M^n)$ vanishes,*
- *or equality holds in (0.17) and (up to a conformal change in the case $k = \frac{n}{2}$) the pointwise equality $\rho_g = \frac{1}{n(n-1)} R_g$ holds, $1 \leq b_k \leq \binom{n}{k}$, every harmonic k-form is parallel and g is a Yamabe minimizer.*

Remark 0.8. In Theorem I, as well as in the other theorems of this section, the two cases are not mutually exclusive, i.e. equality can hold in (0.17) while a number of Betti numbers vanish.

We also obtain an alternative proof of Theorem 0.4, and several generalizations of this result to higher dimensions and higher degrees. In particular, we prove the following extension to higher dimensions of the first part of Theorem 0.4:

Theorem J. *If (M^n, g), $n \geq 5$, is a closed Riemannian manifold with positive Yamabe constant such that*

$$\left\|\mathcal{Z}_g\right\|^2_{L^{\frac{n}{2}}} \leq \frac{1}{n(n-1)(n-2)} \mathrm{Y}(M^n, [g])^2, \tag{0.18}$$

- *either its first Betti number $b_1(M^n)$ vanishes,*
- *or equality holds in (0.18), $b_1 = 1$, and there exists an Einstein manifold (N^{n-1}, h) with positive scalar curvature such that (M^n, g) is isometric to a quotient of*

$$(N^{n-1} \times \mathbb{R}, h + (dt)^2).$$

We prove an analogue of the second part of Theorem 0.4 in dimension 6:

Theorem K. *If (M^6, g) is a closed Riemannian manifold with positive Yamabe constant such that*

$$\|W_g\|^2_{L^3} \leq \frac{1}{40} \mathrm{Y}(M^6, [g])^2, \tag{0.19}$$

- *either its third Betti number $b_3(M^6)$ vanishes,*
- *or equality holds in (0.19), $b_3 = 2$, and there exist two positive numbers a and b such that (M^6, g) is conformally equivalent to a quotient of*

$$\left(\mathbb{S}^3 \times \mathbb{S}^3, a\, g_{\mathbb{S}^3} + b\, g_{\mathbb{S}^3}\right).$$

These rigidity results with a pinching involving the norm of a curvature tensor are deduced from an integral Bochner-Weitzenböck theorem with a pinching on the lowest eigenvalue of the traceless Bochner-Weitzenböck curvature. We characterize the situations were such pinching theorems based on the norm of curvature tensors can be obtained, and prove the following result (the constants $a_{n,k}$ and $b_{n,k}$ are defined in Section 2.1 of Chapter 4):

Theorem L. *If (M^n, g) is a closed Riemannian manifold with positive Yamabe constant such that for some integer $1 \leq k \leq \frac{n}{2}$, $k \neq \frac{n-1}{2}$, the following pinching holds:*

$$\left(a_{n,k} \, \|W\|^2_{\frac{n}{2}} + b_{n,k} \, \left\|\overset{\circ}{Ric}\right\|^2_{\frac{n}{2}}\right)^{\frac{1}{2}} \leq \frac{k(n-k)}{n(n-1)} \mathrm{Y}(M, [g]), \tag{0.20}$$

- *either its k^{th} Betti number $b_k(M^n)$ vanishes,*
- *or $n = 4$ and equality holds in Theorem 0.4,*
- *or $k = 1$ and equality holds in Theorem J,*
- *or $k = 2$, $n \geq 7$ and (M^n, g) is isometric to a quotient of: a $\left(\mathbb{S}^2 \times \frac{1}{n-5}\mathbb{S}^{n-2}\right)$.*

– *or* $k = 3$*,* $n = 6$ *and equality holds in Theorem K.*

Finally, we prove an extension of Theorem J to non-compact manifolds:

Theorem M. *Let* (M^n, g)*,* $n \geq 4$*, be a complete non-compact Riemannian manifold with positive Yamabe constant. Assume that the lowest eigenvalue of the Ricci curvature satisfies* $Ric_- \in L^p$ *for some* $p > \frac{n}{2}$*, and assume that the scalar curvature is in* $L^{\frac{n}{2}}$*. If*

$$\left\| \overset{\circ}{Ric}_g \right\|_{L^{\frac{n}{2}}} + \frac{n-4}{4\sqrt{n(n-1)}} \left\| R_g \right\|_{L^{\frac{n}{2}}} \leq \frac{n}{4} \frac{1}{\sqrt{n(n-1)}} \, \Upsilon(M^n, [g]), \tag{0.21}$$

– *either* $H^1_c(M, \mathbb{Z}) = \{0\}$ *and in particular* M *has only one end,*
– *or equality holds in* (0.21)*, and there exists an Einstein manifold* (N^{n-1}, h) *with positive scalar curvature and* $\alpha > 0$ *such that* (M^n, g) *or one of its two-fold coverings is isometric to*

$$\left(N^{n-1} \times \mathbb{R}, \, \alpha \, \cosh^2(t) \left(h + (dt)^2\right)\right).$$

In particular, this theorem gives interesting information on the singularities of the gradient flows E^λ in dimension four. Indeed, it implies that a complete non-compact Bach-flat manifold (M^4, g) with zero scalar curvature which satisfies the pinching

$$\left\| \mathcal{Z}_g \right\|^2_{L^2} < \frac{1}{24} \, \Upsilon(M^4, [g])^2$$

has only one end. Under the additional assumption

$$\| W_g \|^2_{L^2} < 4\pi^2,$$

the Gauss-Bonnet formula shows that (M^4, g) is simply connected at infinity. Then, according to the work of G. Tian and J. Viaclovsky ([TV05]) and of J. Streets ([Str10a]), (M^4, g) can be conformally compactified to a smooth Bach-flat manifold by adding a point at infinity.

CHAPTER 1

The Yamabe constant and the Bochner technique

Dans ce chapitre, on montre un certain nombre de résultats liés à la constante de Yamabe, qui seront utilisés dans les chapitres suivants. En particulier, on introduit un invariant de Yamabe modifié, on explicite le lien entre la positivité de la constante de Yamabe, les inégalités de Sobolev et le non-effondrement des variétés riemanniennes, et on démontre une version abstraite de la méthode de Bochner intégrale utilisée dans les chapitres trois et quatre.

1. The Yamabe constant

The Yamabe constant can be defined by

$$\mathrm{Y}(M,g) = \inf_{\substack{u\in C_0^\infty(M)\\ u\neq 0}} \frac{\int_M \left(\frac{4(n-1)}{n-2}|du|^2 + R_g u^2\right) dv_g}{\left(\int_M u^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}}}.$$

It is conformally invariant: if f is a smooth function, then

$$\mathrm{Y}(M,g) = \mathrm{Y}\left(M, e^{2f} g\right),$$

thus it only depends on the conformal class $[g]$ of the metric g.

According to the work of H. Yamabe, N. Trudinger, T. Aubin and R. Schoen, when M is closed, there always exists a positive smooth function u such that

$$\int_M \left(\frac{4(n-1)}{n-2}|du|^2 + R_g u^2\right) dv_g = \mathrm{Y}(M,[g]), \text{ and } \int_M u^{\frac{2n}{n-2}} dv_g = 1. \tag{1.1}$$

Moreover, since C_0^∞ is dense in $H_1^2(M)$ (see [Aub98]), the infimum defining the Yamabe constant can also be taken over $H_1^2(M)$, and any function $u \in H_1^2(M)$ with $\|u\|_{L^{\frac{2n}{n-2}}} = 1$ attaining the infimum is smooth, positive and solution to the Yamabe equation

$$\frac{4(n-1)}{n-2}\Delta_g u + R_g u = \mathrm{Y}(M,[g]) u^{\frac{n+2}{n-2}} \tag{1.2}$$

(see [Aub98]). We can also write that equation

$$L_g(u) = \mathrm{Y}(M,[g]) u^{\frac{n+2}{n-2}},$$

where L_g denotes the conformal laplacian

$$L_g = \frac{4(n-1)}{n-2}\Delta_g + R_g,$$

and satisfies the following conformal covariance property: if $\tilde{g} = \varphi^{\frac{4}{n-2}} g$ with φ a smooth positive function, then

$$\varphi^{\frac{n+2}{n-2}} L_{\tilde{g}}(u) = L_g(\varphi u).$$

It follows in particular that

$$\varphi^{\frac{n+2}{n-2}} R_{\tilde{g}} = \frac{4(n-1)}{n-2}\Delta_g \varphi + R_g \varphi.$$

Therefore, if u is a positive smooth solution of (1.2), then the metric $\tilde{g} = u^{\frac{4}{n-2}} g$ is a Yamabe minimizer.

1.1. The modified Yamabe constant. For $\beta \geq 0$, we introduce the modified Yamabe constant

$$\mathrm{Y}_g(\beta) = \inf_{\substack{u\in C_0^\infty(M)\\ u\neq 0}} \frac{\int_M \left(\frac{4(n-1)}{n-2}\ |du|^2 + \beta R_g u^2\right) dv_g}{\left(\int_M u^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}}}. \tag{1.3}$$

In particular, for $\beta = 1$, this modified Yamabe constant is the Yamabe constant:

$$\mathrm{Y}_g(1) = \mathrm{Y}(M, [g]).$$

The function $\beta \to \mathrm{Y}_g(\beta)$ is an infimum of affine functions of β, hence it is concave and for all $\beta \in [0, 1]$ we obtain

$$(1 - \beta)\,\mathrm{Y}_g(0) + \beta\,\mathrm{Y}(M, [g]) \leq \mathrm{Y}_g(\beta).$$

When (M, g) is closed, $\mathrm{Y}_g(0) = 0$ and we have:

Proposition 1.1. *If (M^n, g) is a closed Riemannian manifold, then*

$$\beta\,\mathrm{Y}(M, [g]) \leq \mathrm{Y}_g(\beta). \tag{1.4}$$

If $\beta \in (0, 1)$, then equality holds in this inequality if and only if g is a Yamabe minimizer, and the only functions attaining the infimum in (1.3) *are constant functions.*

Proof. Since $\beta \to \mathrm{Y}_g(\beta)$ is concave, it is equal to its chord $\beta\,\mathrm{Y}(M, [g])$ at an interior point $\beta \in (0, 1)$ if and only if it is affine. Then, if for some u and some $\beta \in (0, 1)$, equality is attained in (1.3), the affine function of β corresponding to u is above the function Y_g and is equal to $\mathrm{Y}_g(\beta)$ at β, hence it must be equal to Y_g on $[0, 1]$. Therefore the function u realizes the infimum in (1.3) for all $\beta \in [0, 1]$. Taking $\beta = 0$ yields $\int_M |du|^2\, dv_g = 0$, hence u is constant. Then, taking $\beta = 1$ shows that g is a Yamabe minimizer. □

1.2. The Yamabe constant on complete non-compact manifolds. We still define the Yamabe constant by

$$\mathrm{Y}(M, g) = \inf_{\substack{u \in C_0^\infty(M)\\ u \neq 0}} \frac{\int_M \left(\frac{4(n-1)}{n-2}\, |du|^2 + R_g u^2\right) dv_g}{\left(\int_M u^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{n}}}$$

and we still have for any smooth function f:

$$\mathrm{Y}(M, g) = \mathrm{Y}(M, e^{2f} g).$$

The Yamabe functional

$$\mathcal{F}_g(u) = \frac{\int_M \left(\frac{4(n-1)}{n-2}\, |du|^2 + R_g u^2\right) dv_g}{\left(\int_M u^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{n}}}$$

is also well-defined when R_g is in $L^{\frac{n}{2}}(M, g)$, u is in $L^{\frac{2n}{n-2}}(M, g)$ and du is in $L^2(M, g)$. Moreover, when g is complete, $C_0^\infty(M)$ is dense in the space

$$H = \{u \in L^{\frac{2n}{n-2}}(M, g),\ |du| \in L^2(M, g)\}.$$

Therefore, we also have

$$\mathrm{Y}(M, [g]) = \inf_H \mathcal{F}_g,$$

and any function in H with $\|u\|_{L^{\frac{2n}{n-2}}} = 1$ attaining the infimum is a weak solution to the Yamabe equation (1.2). If in addition u is in $C^{0,\alpha}$, then by classical regularity theorems u is smooth, and by maximum principle it is positive (see [Aub98]).

2. Yamabe, Sobolev and noncollapsing

We recall the following definition:

Definition 1.2. We say that a Riemannian manifold (M^n, g) is κ-noncollapsed if every ball B of radius r such that $|Rm_g| \leq \frac{1}{r^2}$ on B, has volume at least κr^n.

We easily see that the property of being κ-noncollapsed is scale invariant. If we scale a κ-noncollapsed metric so that the curvature is bounded by 1, then according to the following lemma, its injectivity radius is bounded from below by some positive constant $\delta(n, \kappa)$:

Lemma 1.3 (Cheeger, see [Pet98], Lemma 4.5)**.** *For all $C > 0$ and $\kappa > 0$ there exists $\delta(n, C, \kappa) > 0$ such that the following property is true:*

If (M^n, g) is a complete Riemannian manifold such that $\mathrm{Vol}(\mathrm{B}(x, 1)) \geq \kappa$ for all x in M and $\sup_M |Rm_g| \leq C$, then $inj_g(M) \geq \delta$.

Therefore, the class of pointed k-noncollapsed manifolds with curvature bounded by 1 is precompact in the pointed C^α topology.

We now prove the following proposition:

Proposition 1.4. *Any complete Riemannian manifold (M^n, g), $n \geq 3$, with positive Yamabe constant is κ-noncollapsed, where*

$$\kappa = \left(\frac{\mathrm{Y}(M, [g])}{2^{n+5} n(n-1)} \right)^{\frac{n}{2}}.$$

If (M^n, g), $n \geq 3$, is a complete Riemannian manifold, and $U \subset M^n$ is an open set, we define the Sobolev constant of U by

$$\mathsf{s}_g(U) = \inf\{C \in \mathbb{R}, \|u\|_{L^{\frac{2n}{n-2}}} \leq C \left(\|du\|_{L^2} + \|u\|_{L^2}\right), \forall u \in C_0^\infty(U)\},$$

with the convention $\inf(\emptyset) = +\infty$. When M is closed, we always have the inclusion $H_1^2(M) \subset L^{\frac{2n}{n-2}}(M)$, hence $\mathsf{s}_g(M) < \infty$.

Lemma 1.5. *If (M^n, g), $n \geq 3$, is a complete Riemannian manifold with positive Yamabe constant, then for all open set $U \subset M^n$*

$$\mathsf{s}_g(U)^2 \leq \frac{1}{Y(M, [g])} \max\left(\sup_U |R_g|, \frac{4(n-1)}{n-2} \right).$$

Proof. Let u be in $C_0^\infty(U)$. By definition of the Yamabe constant, we have

$$\begin{aligned} \|u\|^2_{L^{\frac{2n}{n-2}}} &\leq \frac{1}{\mathrm{Y}(M, [g])} \left(\frac{4(n-1)}{n-2} \|du\|^2_{L^2} + \int_M R_g u^2 dv_g \right) \\ &\leq \frac{1}{\mathrm{Y}(M, [g])} \left(\frac{4(n-1)}{n-2} \|du\|^2_{L^2} + \|R_g\|_{L^\infty} \|u\|^2_{L^2} \right) \\ &\leq \frac{1}{\mathrm{Y}(M, [g])} \max\left(\frac{4(n-1)}{n-2}, \sup_U |R_g| \right) \left(\|du\|_{L^2} + \|u\|_{L^2}\right)^2. \end{aligned}$$

□

We recall the following lemma:

Lemma 1.6 (G. Carron, see [Heb96], Lemma 3.2)**.** *Let (M^n, g) be a complete Riemannian manifold. For any ball $B \subset M^n$ of radius 1,*

$$\mathrm{Vol}_g(B) \geq \left(2^{\frac{n}{2}+2}\, \mathsf{s}_g(B)\right)^{-n}.$$

We can now prove Proposition 1.4. Let (M, g) be a complete manifold with positive Yamabe constant and B a ball of radius r such that $|Rm_g| \leq \frac{1}{r^2}$ on B. If we consider the metric $\hat{g} = r^{-2} g$, which has the same Yamabe constant as g (since the Yamabe constant is conformally invariant, thus also scale invariant), we can suppose that $r = 1$, and that $|Rm_g| \leq 1$ on B. Then, according to Lemma 1.5, we have

$$\mathsf{s}_{\hat{g}}(B)^2 \leq \frac{1}{Y(M, [g])} \max\left(2n(n-1), \frac{4(n-1)}{n-2} \right) = \frac{2n(n-1)}{Y(M, [g])},$$

and according to Lemma 1.6, we obtain

$$\mathrm{Vol}_{\hat{g}}(B)^{\frac{2}{n}} \geq \frac{1}{2^{n+4}\, \mathsf{s}_g(B)^2} \geq \frac{\mathrm{Y}(M, [g])}{2^{n+5} n(n-1)}.$$

Therefore

$$\mathrm{Vol}_g(B) \geq \kappa r^n.$$

□

We finally recall the following fact:

Lemma 1.7. *If (M, g) is a complete Riemannian manifold such that the volume of unit balls is uniformly bounded from below:*

$$\inf_{x \in M} \mathrm{Vol}_g(\mathrm{B}(x, 1)) > 0,$$

then M is compact if and only if (M, g) has finite volume.

Proof. If M is compact, it has finite volume. Suppose that M is not compact and choose x in M. We can find a sequence of points (x_k) such that x_k is in $\mathrm{B}(x, k+1) \setminus \mathrm{B}(x, k)$. Then the balls $\mathrm{B}(x_{3k}, 1)$ are two by two disjoint, and thus

$$\mathrm{Vol}_g(M) \geq \sum_{k \geq 0} \mathrm{Vol}_g(\mathrm{B}(x_{3k}, 1)) = \infty.$$

□

And we prove the following Lemma, inspired by [CH02, Proposition 2.3]:

Lemma 1.8. *If (M, g) is a complete non-compact Riemannian manifold with positive Yamabe constant and scalar curvature in $L^{\frac{n}{2}}$, then it has infinite volume and satisfies the following Sobolev inequality:*

$$\|\varphi\|^2_{L^{\frac{2n}{n-2}}} \leq C \, \|d\varphi\|^2_{L^2}, \tag{1.5}$$

for some $C > 0$ and for all $\varphi \in C_0^\infty(M)$.

Proof. Let fix some ball $\mathrm{B}(x_0, r) \subset M$. Since $\mathrm{Y}(M, [g]) > 0$ and by using the Hölder inequality, for any smooth functions with support outside the ball $\mathrm{B}(x_0, r)$, we have

$$\|\varphi\|^2_{L^{\frac{2n}{n-2}}} \leq \frac{1}{\mathrm{Y}(M, [g])} \left(\frac{4(n-1)}{n-2} \, \|d\varphi\|^2_{L^2} + \left(\int_{M \setminus \mathrm{B}(x_0, r)} |R_g|^{n/2} \, dv_g \right)^{\frac{2}{n}} \|\varphi\|^2_{L^{\frac{2n}{n-2}}} \right).$$

Since R_g is in $L^{\frac{n}{2}}(M, g)$, we can take r such that

$$\left(\int_{M \setminus \mathrm{B}(x_0, r)} |R_g|^{n/2} \, dv_g \right)^{\frac{2}{n}} \leq \frac{\mathrm{Y}(M, [g])}{2},$$

and we obtain the Sobolev inequality on $M \setminus \mathrm{B}(x_0, r)$.

According to Lemma 1.6, there exists a uniform bound from below on the volume of any ball $\mathrm{B}(y, 1) \subset M \setminus \bar{\mathrm{B}}(x_0, r)$. And since $\bar{\mathrm{B}}(x_0, r)$ is compact, it is true for any unit ball in M. According to Lemma 1.7, the volume of (M, g) is infinite.

Therefore, according to [Car98, Proposition 2.5], there exists C' such that the Sobolev inequality (1.5) holds on M. □

3. Integral Bochner technique: a general setting

We consider a tensor T satisfying a condition of the following type:

$$\langle \nabla^* \nabla T \mid T \rangle + \lambda R_g \, |T|^2 \leq a \, |T|^2, \tag{1.6}$$

for some $\lambda \in \mathbb{R}$ and $a \in C^\infty(M)$.

When T is solution of a second-order elliptic equation, a Weitzenböck formula often provides a linear equation on T of the following form

$$\nabla^* \nabla T + A(T) = 0. \tag{1.7}$$

In that case, we get (1.6) with $-a$ the lowest eigenvalue of

$$A - \lambda R_g \,\mathrm{Id}\,.$$

The classical Kato inequality asserts that for any smooth tensor T on a Riemannian manifold (M^n, g),

$$|d\,|T||^2 \leq |\nabla T|^2\ .$$

When T is in the kernel of an elliptic operator of degree one, this Kato inequality can often be refined to

$$(1+\delta)\ |d\,|T||^2 \leq |\nabla T|^2, \tag{1.8}$$

for some $\delta > 0$.

By combining the Weitzenböck formula with the Sobolev inequality induced by the positivity of the Yamabe constant, we obtain:

Theorem 1.9. *Let (M^n, g) be a closed Riemannian manifold and let T be a tensor which satisfy* (1.6) *and* (1.8) *for some $a \in C^\infty(M)$, $\lambda > 0$ and $\delta \geq 0$. If $0 < \lambda(1-\delta) < \frac{n-2}{4(n-1)}$ and if*

$$\|a\|_{L^{\frac{n}{2}}} \leq \lambda\,\mathrm{Y}(M, [g]), \tag{1.9}$$

then

- *either T vanishes on M,*
- *or equality holds in* (1.9)*, g is a Yamabe minimizer, T is parallel and the pointwise inequality $a = \lambda R_g$ holds on M.*

In the proof, we use the modified Yamabe invariant $Y_g(\beta)$ with $\beta = \lambda(1-\delta)\frac{4(n-1)}{n-2}$. When $\beta = 1$, i.e. when $\lambda(1-\delta) = \frac{n-2}{4(n-1)}$, we cannot apply Proposition 1.1, and we don't get such a strong characterization of the equality case:

Theorem 1.10. *Let (M^n, g) be a closed Riemannian manifold and let T be a tensor which satisfy* (1.6) *and* (1.8) *for some $a \in C^\infty(M)$, $0 \leq \delta < 1$ and $\lambda = \frac{1}{1-\delta}\frac{n-2}{4(n-1)}$. If*

$$\|a\|_{L^{\frac{n}{2}}} \leq \lambda\,\mathrm{Y}(M, [g]), \tag{1.10}$$

then

- *either T vanishes on M,*
- *or equality holds in* (1.10) *and the metric $\tilde{g} = |T|^{\frac{1}{\lambda(n-1)}}\ g$ is a Yamabe minimizer.*

However, when $\beta = 1$, the vanishing result can be extended to non-compact manifolds:

Theorem 1.11. *Let (M^n, g) be a complete Riemannian manifold and let T be a tensor which satisfy* (1.6) *and* (1.8) *for some $a \in C^\infty(M)$, $0 \leq \delta < 1$ and $\lambda = \frac{1}{1-\delta}\frac{n-2}{4(n-1)}$. Suppose that for some $x_0 \in M$ we have $\mathrm{Vol}(x_0, R)^{\frac{\delta}{2}} = O(R)$ when $R \to \infty$. If*

$$\|a\|_{L^{\frac{n}{2}}} \leq \lambda\,\mathrm{Y}(M, [g]), \tag{1.11}$$

then

- *either T vanishes on M,*
- *or equality holds in* (1.11)*, and if furthermore R_g is either nonnegative or in $L^{\frac{n}{2}}$, then $u = |T|^{\frac{1}{\lambda(n-1)}}$ is in $L^{n/2}(M, g)$ and the metric $\tilde{g} = \frac{u}{\|u\|_{L^{n/2}}} g$ has constant scalar curvature equal to* $\mathrm{Y}(M, [g])$.

Remark 1.12. According to [Gal88, Theorem 1], the assumption on the growth of balls is in particular satisfied when the lowest eigenvalue Ric_- of the Ricci curvature is in L^p for some $\frac{n}{2} < p \leq \frac{1}{\delta}$.

When the Yamabe constant is positive, and when the curvature is in $L^{\frac{n}{2}}$ and satisfies an elliptic equation

$$\nabla^* \nabla Rm_g = Rm_g * Rm_g,$$

(for instance when the curvature is harmonic, i.e. when $\delta Rm_g = 0$), then according to Lemma 1.8 and by De Giorgi-Nash-Moser iterative scheme, the curvature is uniformly bounded. The curvature is therefore in $L^{\frac{n}{2}} \cap L^\infty$, hence it is in L^p for all $\frac{n}{2} \leq p \leq \infty$.

By a refinement of the method, the same is true if the curvature satisfies and elliptic equation

$$\nabla^* \nabla Ric_g = Rm_g * Ric_g,$$

for instance if the metric is critical for a quadratic curvature functional and has constant scalar curvature (see [And05, TV05]).

We now prove Theorems 1.9, 1.10 and 1.11. The general idea is the following: we multiply the Weitzenböck inequality (1.6) by some power of $|T|$ and integrate over the manifold. When the power of $|T|$ is well chosen, the left size of the inequation, which involves the derivative of $|T|$, can be interpreted as the value for the function $|T|^{1-\delta}$ of the functional involved in the definition of the modified Yamabe invariant $Y_g(\beta)$ (see (1.3)). By using this fact, we obtain an inequality between $Y_g(\beta)$ and $\|a\|_{L^{\frac{n}{2}}}$.

For $\varepsilon > 0$ we introduce

$$f_\varepsilon = \sqrt{|T|^2 + \varepsilon^2}.$$

Elementary computations lead to

$$f_\varepsilon \Delta f_\varepsilon - |df_\varepsilon|^2 = \langle \nabla^* \nabla T \mid T \rangle - |\nabla T|^2,$$

and for $0 < p \leq 2$ we get

$$\begin{aligned} \Delta f_\varepsilon^p &= p f_\varepsilon^{p-2} \left(f_\varepsilon \Delta f_\varepsilon - (p-1)|df_\varepsilon|^2 \right) \\ &= p f_\varepsilon^{p-2} \left(\langle \nabla^* \nabla T, T \rangle - |\nabla T|^2 + (2-p)|df_\varepsilon|^2 \right) \\ &\leq p f_\varepsilon^{p-2} \left(\langle \nabla^* \nabla T, T \rangle - (1+\delta) \left| d|T| \right|^2 + (2-p)|df_\varepsilon|^2 \right). \end{aligned}$$

Note that we have

$$|df_\varepsilon|^2 = \frac{|T|^2 \left| d|T| \right|^2}{|T|^2 + \varepsilon^2} \leq \left| d|T| \right|^2.$$

Therefore if $0 \leq \delta < 1$, by taking $p = 1 - \delta$, and $\beta = \lambda(1-\delta)\frac{4(n-1)}{n-2}$, we obtain

$$\Delta f_\varepsilon^{1-\delta} + \beta \frac{n-2}{4(n-1)} R_g f_\varepsilon^{-(1+\delta)} |T|^2 \leq (1-\delta) a f_\varepsilon^{-(1+\delta)} |T|^2. \tag{1.12}$$

If (M, g) is closed, by multiplying this inequality by $f_\varepsilon^{1-\delta}$ and integrating over M, we obtain

$$\int_M \left| df_\varepsilon^{1-\delta} \right|^2 dv_g + \beta \frac{n-2}{4(n-1)} \int_M R_g f_\varepsilon^{-2\delta} |T|^2 dv_g \leq (1-\delta) \int_M a\, f_\varepsilon^{-2\delta} |T|^2 dv_g.$$

We define $v = |T|^{1-\delta}$. Since $f_\varepsilon^{-2\delta} |T|^2 \leq |T|^{2(1-\delta)}$, by Fatou's Lemma we see that v is in $H_1^2(M)$, and by letting ε go to zero, we get by Lebesgue's dominated convergence theorem that

$$\int_M |dv|^2 dv_g + \beta \frac{n-2}{4(n-1)} \int_M R_g v^2 dv_g \leq (1-\delta) \int_M a\, v^2 dv_g.$$

Consequently, if $0 \leq \beta \leq 1$, i.e. if

$$0 \leq \lambda(1-\delta) \leq \frac{n-2}{4(n-1)},$$

we have

$$\frac{n-2}{4(n-1)} \mathrm{Y}_g(\beta) \|v\|^2_{L^{\frac{2n}{n-2}}} \leq (1-\delta) \int_M a v^2 dv_g,$$

then according to the Hölder inequality,

$$\frac{n-2}{4(n-1)} \mathrm{Y}_g(\beta) \|v\|^2_{L^{\frac{2n}{n-2}}} \leq (1-\delta) \|a\|_{L^{\frac{n}{2}}} \|v\|^2_{L^{\frac{2n}{n-2}}},$$

Therefore, either v vanishes on M, or

$$\frac{n-2}{4(n-1)}\,\mathrm{Y}_g(\beta) \leq (1-\delta)\,\|a\|_{L^{\frac{n}{2}}}\,.$$

According to Proposition 1.1, in that case we obtain

$$\beta\frac{n-2}{4(n-1)}\,\mathrm{Y}(M,[g]) \leq (1-\delta)\,\|a\|_{L^{\frac{n}{2}}}\,,$$

hence

$$\|a\|_{L^{\frac{n}{2}}} \geq \lambda\,\mathrm{Y}(M,[g]).$$

If inequality (1.9) holds and v doesn't vanish on M, then equality must hold everywhere. In particular, when $\lambda(1-\delta) < \frac{n-2}{4(n-1)}$, then $0<\beta<1$, we have $\mathrm{Y}_g(\beta) = \beta\,\mathrm{Y}(M,[g])$, and the function v attains the infimum in (1.4). According to Proposition 1.1, g is a Yamabe minimizer and v is constant, hence T has constant norm. Since equality must hold in the Kato inequality, T must be parallel. According to (1.6), the pointwise inequality $\lambda R_g \leq a$ holds on M, and as $\|a\|_{L^{\frac{n}{2}}} = \lambda\,\mathrm{Y}(M,[g])$, the equality $\lambda R_g = a$ holds on M.

When $\lambda = \frac{1}{1-\delta}\frac{n-2}{4(n-1)}$, i.e. $\beta = 1$, the function v is a minimizer for the Yamabe functional, hence it is a smooth positive solution of the Yamabe equation

$$\frac{4(n-1)}{n-2}\Delta_g v + R_g v = \mathrm{Y}(M,[g])v^{\frac{n+2}{n-2}},$$

and the metric $\tilde{g} = v^{\frac{4}{n-2}}g$ is a Yamabe minimizer.

If M is complete, but not compact, and if χ is a Lipschitz function with compact support and u a smooth function, we have the integration by parts formula

$$\int_M |d(\chi u)|^2\,dv_g = \int_M \left[|d\chi|^2\,u^2 + \chi^2 u\Delta u\right] dv_g.$$

By multiplying inequality (1.12) by $\chi^2 f_\varepsilon^{1-\delta}$ and integrating over M, we obtain

$$\int_M \left|d(\chi f_\varepsilon^{1-\delta})\right|^2\,dv_g + \beta\frac{n-2}{4(n-1)}\int_M R_g f_\varepsilon^{-2\delta}\,|T|^2\,\chi^2 dv_g$$
$$\leq (1-\delta)\int_M a\,f_\varepsilon^{-2\delta}\,|T|^2\,\chi^2 dv_g + \int_M |d\chi|^2\,f_\varepsilon^{2(1-\delta)}dv_g.$$

If we define $v = |T|^{1-\delta}$ and we let ε go to zero, we get by Fatou's Lemma and Lebesgue's dominated convergence theorem that

(1.13)
$$\int_M |d(\chi v)|^2\,dv_g + \frac{n-2}{4(n-1)}\int_M R_g(\chi v)^2 dv_g \leq (1-\delta)\int_M a\,(\chi v)^2 dv_g + \int_M |d\chi|^2\,v^2 dv_g.$$

hence

$$\frac{n-2}{4(n-1)}\,\mathrm{Y}(M,[g])\,\|\chi v\|^2_{L^{\frac{2n}{n-2}}} \leq (1-\delta)\int_M a(\chi v)^2 dv_g + \int_M |d\chi|^2\,v^2 dv_g,$$

and according to the Hölder inequality,

$$\frac{n-2}{4(n-1)}\,\mathrm{Y}(M,[g])\,\|\chi v\|^2_{L^{\frac{2n}{n-2}}} \leq (1-\delta)\left(\int_{\chi>0} |a|^{\frac{n}{2}}\,dv_g\right)^{\frac{2}{n}}\,\|\chi v\|^2_{L^{\frac{2n}{n-2}}}$$
$$+\,\|d\chi\|^2_{L^{\frac{2}{\delta}}}\,\|T\|^{\frac{2}{1-\delta}}_{L^2},$$

thus

$$\left(\lambda\,\mathrm{Y}(M,[g]) - \left(\int_{\chi>0} |a|^{\frac{n}{2}}\,dv_g\right)^{\frac{2}{n}}\right)\|\chi v\|^2_{L^{\frac{2n}{n-2}}} \leq (1-\delta)\,\|d\chi\|^2_{L^{\frac{2}{\delta}}}\,\|T\|^{\frac{2}{1-\delta}}_{L^2}, \tag{1.14}$$

For $R > 0$, we introduce the functions

$$\phi_R(t) = \begin{cases} 1 & \text{on } [0, R] \\ 2 - \frac{x}{R} & \text{on } [R, 2R] \\ 0 & \text{on } [2R, \infty) \end{cases}$$

and $\chi_R(x) = \phi_R(d(x_0, x))$ where $x_0 \in M$ is a fixed point. Then

$$\|d\chi_R\|^2_{L^{\frac{2}{\delta}}} \leq \frac{1}{R^2} \operatorname{Vol}(\mathrm{B}(x_0, 2R))^\delta.$$

If we let R go to ∞, we see that if v doesn't vanish, then

$$\left(\lambda \operatorname{Y}(M, [g]) - \|a\|_{L^{\frac{n}{2}}}\right) \leq 0.$$

We now characterize the equality case. We first prove that v is in $L^{\frac{2n}{n-2}}$. For $0 < 2r < R$, we introduce the functions

$$\phi_{r,R}(t) = \begin{cases} 0 & \text{on } [0, r] \\ \frac{t}{r} - 1 & \text{on } [r, 2r] \\ 1 & \text{on } [2r, R] \\ 2 - \frac{x}{R} & \text{on } [R, 2R] \\ 0 & \text{on } [2R, \infty) \end{cases}$$

and $\chi_{r,R}(x) = \phi_{r,R}(d(x_0, x))$. Then

$$\|d\chi_{r,R}\|^2_{L^{\frac{2}{\delta}}} \leq \frac{1}{r^2} \operatorname{Vol}(\mathrm{B}(x_0, 2r))^\delta + \frac{1}{R^2} \operatorname{Vol}(\mathrm{B}(x_0, 2R))^\delta.$$

Since a is in $L^{\frac{n}{2}}$, we can take $r > 0$ such that

$$\left(\int_{M \setminus \mathrm{B}(x_0, r)} |a|^{\frac{n}{2}}\ dv_g\right)^{\frac{2}{n}} < \lambda \operatorname{Y}(M, [g]).$$

From the assumption on the growth of the volume of balls, we get that

$$\overline{\lim_{R \to \infty}} \|d\chi_{r,R}\|^2_{L^{\frac{2}{\delta}}} \leq \frac{1}{r^2} \operatorname{Vol}(\mathrm{B}(x_0, 2r))^\delta.$$

Therefore, according to (1.14) and by Fatou's Lemma, v is in $L^{\frac{2n}{n-2}}$.

Then, $\int_M a(\chi_R v)^2 dv_g$ goes to $\int_M a v^2 dv_g$ according to Lebesgue's dominated convergence theorem. If we suppose that R_g is nonnegative or in $L^{\frac{n}{2}}$, then by Fatou's Lemma or Lebesgue's dominated convergence theorem, we obtain from (1.13) that the function v satisfies

$$\int_M |dv|^2\ dv_g + \frac{n-2}{4(n-1)} \int_M R_g v^2 dv_g \leq (1 - \delta) \int_M a\, v^2 dv_g.$$

We can now proceed as in the compact case. If equality holds and v doesn't vanish, then v is a minimizer of the Yamabe functional, hence $\frac{v}{\|v\|_{L^{\frac{2n}{n-2}}}}$ satisfies the Yamabe equation and since $v \in C^{0,1-\delta}$, it is smooth and positive. Then the metric $\tilde{g} = \left(\frac{v}{\|v\|_{L^{\frac{2n}{n-2}}}}\right)^{\frac{4}{n-2}} g$ has constant scalar curvature equal to $\operatorname{Y}(M, [g])$.

CHAPTER 2

Fourth-order curvature flows

Dans ce chapitre, on étudie une classe de flots de courbure d'ordre quatre. On montre en particulier l'existence en temps court et des estimées sur la courbure.

1. Short-time existence

On a Riemannian manifold (M^n, g_0), we consider the following class of evolution equations:

$$\begin{cases} \partial_t g = P(g) \\ g(0) = g_0, \end{cases} \qquad E_P(g_0)$$

where $P : \mathcal{S}^2_+(M) \to \mathcal{S}^2(M)$ is a smooth map of the form

$$P(g) = \delta\tilde{\delta}Rm_g + a\Delta R_g g + b\nabla^2 R_g + Rm_g * Rm_g,$$

with a and b two real numbers.

We show that we can prove short-time existence by using the DeTurck trick if and only if $a < \frac{1}{2(n-1)}$. In particular, we prove:

Theorem A. *Let (M, g_0) be a closed Riemannian manifold. When $a < \frac{1}{2(n-1)}$, there exists a unique maximal solution (g_t) of $E_P(g_0)$ defined on some time interval $[0, T)$, with T positive.*

As it is invariant by diffeomorphisms, the differential operator P is not elliptic. We use the DeTurck trick to fill the n-dimensional subspace in the kernel of $\sigma_\xi P'_g$ induced by this geometric invariance.

We use the notation $\mathcal{T}^{(p,\,q)}M$ for the space of (p, q)-tensors. We will sometimes raise or lower indices in the following way:

$$T_{i_1 \ldots i \ldots i_p} = g_{ij} T_{i_1 \ldots}{}^{j}{}_{\ldots i_p},$$

to identify $\mathcal{T}^{(p+1,\,q)}M$ and $\mathcal{T}^{(p,\,q+1)}M$.

We denote the Lie derivative along some vector field X by L_X. By extension, if V is a section of T^*M, we denote by L_V the Lie derivative along the vector field $V^\#$ associated to V. We recall that the Lie derivative of the metric g is given by

$$(L_V g)_{ij} = \nabla_i V_j + \nabla_j V_i.$$

We say that a differential operator $F : \mathcal{S}^2_+(M) \to \mathcal{T}^{(p,\,q)}M$ is geometric if it is invariant by diffeomorphisms, i.e. if for all metrics g and all diffeomorphisms $\phi : M \to M$,

$$F(\phi^* g) = \phi^* F(g).$$

This is in particular the case for the curvature operators and their derivatives with respect to the Levi-Civita connection.

We recall that if $L : h \mapsto L_k(\nabla^k h) + \cdots + L_0(h)$ is a linear differential operator of order k, its principal symbol $\sigma_\xi L$ is defined for all ξ in T^*M by

$$\sigma_\xi L(h) = L_k(\xi \otimes \cdots \otimes \xi \otimes h).$$

We say that L is elliptic if $\sigma_\xi L$ is an isomorphism for all $\xi \neq 0$.

We say that L is strongly elliptic if $k = 2k'$ and $(-1)^{k'+1}\sigma_\xi L$ is uniformly positive, i.e. if there exists $\alpha > 0$ such that for all h,

$$(-1)^{k'+1} \langle \sigma_\xi L(h) \mid h \rangle \geq \alpha \, |\xi|^k |h|^2 .$$

If g is a metric and ξ is in T^*M, let define

$$R_\xi(g) = \xi \otimes \xi - |\xi|^2 \, g.$$

If g and g_0 are two metrics, let define

$$(\gamma_{g,g_0})_i = \frac{1}{2} g_{i\delta} g^{\alpha\beta} (\Gamma^\delta_{\alpha\beta}(g) - \Gamma^\delta_{\alpha\beta}(g_0)).$$

Proposition 2.1. *Let $P : \mathcal{S}^2_+(M) \to \mathcal{S}^2(M)$ be smooth map of the form*

$$P(g) = \delta\tilde{\delta} Rm_g + a\Delta R_g g + b\nabla^2 R_g + Rm_g * Rm_g,$$

and let $V : \mathcal{S}^2_+(M) \to T^*M$ be defined by $V_g = -\nabla^*\nabla\gamma_{g,g_0} + \frac{2(b-a)-1}{4} dR_g$. Then

$$\sigma_\xi(P - L_V)'_g = -\frac{1}{2}\,|\xi|^4\,\mathrm{Id}_{\mathcal{S}^2(M)} + a\,\langle R_\xi \mid \cdot\rangle\,R_\xi,$$

and

- *If $a < \frac{1}{2(n-1)}$, then $P - L_V$ is strongly elliptic.*
- *If $a = \frac{1}{2(n-1)}$, then $P - L_W$ is not elliptic, for any $W : \mathcal{S}^2_+(M) \to T^*M$.*
- *If $a > \frac{1}{2(n-1)}$, then $P - L_W$ is not strongly elliptic, for any $W : \mathcal{S}^2_+(M) \to T^*M$.*

Proof. Since $\tilde{\delta} Rm = -\mathsf{D}Ric$ (Proposition B.1) and $\delta\mathsf{D}(R_g\, g) = \Delta R_g\, g + \tilde{\mathsf{D}}\mathsf{D}R_g$ (Proposition B.2), we see that

$$P - L_V = -\delta\mathsf{D}Ric + L_{\nabla^*\nabla\gamma_{\cdot,g_0} + \frac{1}{4}dR} + a\delta\mathsf{D}(R\,\cdot),$$

then Proposition A.3 shows that

$$\sigma_\xi(P - L_V)'_g = -\frac{1}{2}\,|\xi|^4\,\mathrm{Id}_{\mathcal{S}^2(M)} + a\,\langle R_\xi \mid \cdot\rangle\,R_\xi.$$

Let compute

$$|R_\xi|^2 = |\xi|^4 - 2\left\langle \xi\otimes\xi \mid |\xi|^2\, g\right\rangle + n\,|\xi|^4 = (n-1)\,|\xi|^4\,.$$

Moreover, for all $W : \mathcal{S}^2_+(M) \to T^*M$, the image of $\sigma_\xi(L_W)'_g$ lies in $R_\xi^\perp$:

$$\begin{aligned}\left\langle \sigma_\xi(L_W)'_g \mid R_\xi\right\rangle &= \left\langle \xi\otimes\sigma_\xi W'_g + \sigma_\xi W'_g\otimes\xi \mid \xi\otimes\xi - |\xi|^2\, g\right\rangle\\ &= 2\,|\xi|^2\langle\xi \mid \sigma_\xi W'_g\rangle - 2\,|\xi|^2\langle\xi \mid \sigma_\xi W'_g\rangle\\ &= 0.\end{aligned}$$

If $\mathbf{a < \frac{1}{2(n-1)}}$, then

$$\begin{aligned}-\left\langle\sigma_\xi(P - L_V)'_g(h) \mid h\right\rangle &= \frac{1}{2}\,|\xi|^4|h|^2 - a\,|\langle R_\xi \mid h\rangle|^2\\ &\geq \frac{1}{2}(1 - 2a_+(n-1))\,|\xi|^4|h|^2,\end{aligned}$$

and $P - L_V$ is strongly elliptic.
If $\mathbf{a = \frac{1}{2(n-1)}}$, then $\sigma_\xi(P - L_V)'_g$ is the orthogonal projection on $R_\xi^\perp$. In particular,

$$\begin{aligned}\left\langle\sigma_\xi(P - L_W)'_g(h) \mid R_\xi\right\rangle &= \left\langle\sigma_\xi(P - L_V)'_g(h) \mid R_\xi\right\rangle + \left\langle\sigma_\xi(L_{V-W})'_g(h) \mid R_\xi\right\rangle\\ &= \frac{1}{2}\,|\xi|^4\langle R_\xi \mid h\rangle - a\,|\xi|^2|R_\xi|^2\langle R_\xi \mid h\rangle\\ &= 0,\end{aligned}$$

i.e. the image of $\sigma_\xi(P - L_W)'_g$ is included in $R_\xi^\perp$, therefore $P - L_W$ is not elliptic.
If $\mathbf{a > \frac{1}{2(n-1)}}$, then for $\xi \neq 0$

$$\begin{aligned}-\left\langle\sigma_\xi(P - L_W)'_g(R_\xi) \mid R_\xi\right\rangle &= -\left\langle\sigma_\xi(P - L_V)'_g(R_\xi) \mid R_\xi\right\rangle + \left\langle\sigma_\xi(L_{V-W})'_g(R_\xi) \mid R_\xi\right\rangle\\ &= \frac{1}{2}(1 - 2a(n-1))\,|\xi|^4|R_\xi|^2\\ &< 0.\end{aligned}$$

Consequently, $P - L_W$ is not strongly elliptic. □

According to Proposition A.2, we obtain short-time existence and uniqueness of a solution of E_P as soon as $a < \frac{1}{2(n-1)}$.

2. Gradient flows for geometric functionals

If $T : \mathcal{S}^2_+(M) \to \mathcal{T}^{(p,q)}M$ is smooth map, we define the functional

$$\mathcal{F}_T(g) = \int_M |T(g)|^2_g \, dv_g,$$

and if $\mathcal{F} : \mathcal{S}^2_+(M) \to \mathbb{R}$ is a smooth functional, its gradient $\nabla\mathcal{F}$ is defined by

$$\mathcal{F}'_g(h) = (\nabla\mathcal{F}(g) \mid h)_{L^2} .$$

Then we define the gradient flow of $\mathcal{F}$ starting from g_0 by the following evolution equation:

$$\begin{cases} \partial_t g = -2\nabla\mathcal{F}(g) \\ g(0) = g_0. \end{cases}$$

As we immediately get $\partial_t\mathcal{F}(g_t) = -2\int_M |\nabla\mathcal{F}(g_t)|^2 \, dv_{g_t}$, we see that $\mathcal{F}$ decreases along the flow.

We recall that the curvature tensor has the following orthogonal decomposition:

$$Rm_g = W_g + \frac{1}{n-2} g . \overset{\circ}{Ric}_g + \frac{1}{2n(n-1)} R_g \, g . g,$$

where $u.v$ is the Kulkarni-Nomizu product of u and v in $\mathcal{S}^2(M)$ defined by

$$(u.v)_{ijkl} = u_{ik}v_{jl} + u_{jl}v_{ik} - u_{il}v_{jk} - u_{jk}v_{il}.$$

It follows that

$$\mathcal{F}_{Rm}(g) = \mathcal{F}_W(g) + \frac{1}{n-2}\mathcal{F}_{\overset{\circ}{Ric}}(g) + \frac{1}{2n(n-1)}\mathcal{F}_R(g).$$

The gradients of the quadratic curvature functionals are given by (see [Bes87], chapter 4.H)

$$\begin{aligned} \nabla\mathcal{F}_{Rm} &= -\delta\tilde{\delta}Rm - \frac{1}{2}Rm \vee Rm + \frac{1}{2}|Rm|^2 \, g, \\ \nabla(\mathcal{F}_{Ric} - \frac{1}{4}\mathcal{F}_R) &= -\delta\tilde{\delta}Rm - Ric \circ (Ric - \frac{1}{2}R\,g) - \overset{\circ}{Rm}(Ric) + \frac{1}{2}(|Ric|^2 - \frac{1}{4}R^2)g, \\ \nabla\mathcal{F}_W &= -\delta\tilde{\delta}W - \frac{1}{n-2}\overset{\circ}{W}(\overset{\circ}{Ric}) - \frac{1}{2}(W \vee W - |W|^2 \, g), \\ \nabla\mathcal{F}_R &= 2\delta\mathsf{D}(Rg) - 2R(Ric - \frac{1}{4}R\,g), \end{aligned}$$

where for a symmetric $(2,2)$ double-form T and an endomorphism u we wrote

$$(T \vee T)_{ij} = T_{\alpha\beta\gamma i}T^{\alpha\beta\gamma}{}_j \quad \text{and} \quad (\overset{\circ}{T}u)_{ij} = T_{\alpha i\beta j}u^{\alpha\beta}.$$

If we note $A_g = Ric_g - \frac{1}{2(n-1)}R_g \, g$ the Weyl-Schouten tensor, and

$$\sigma_2(A_g) = \frac{1}{2}(\mathsf{tr}(A_g)^2 - |A_g|^2)$$

the second symmetric function of the eigenvalues of A_g, we define the functional

$$\mathcal{F}_2(g) = \int_M \sigma_2(A_g) dv_g.$$

We have

$$\mathcal{F}_2(g) = \frac{n}{8(n-1)}\mathcal{F}_R(g) - \frac{1}{2}\mathcal{F}_{Ric}(g),$$

and its gradient is given by

$$\nabla\mathcal{F}_2 = -\frac{1}{2}\delta\mathsf{D}A + \frac{1}{2}\overset{\circ}{W}(\overset{\circ}{Ric}) + \frac{n-4}{2(n-2)}(A \circ (Ric - \frac{1}{2}R\,g) + \sigma_2 g).$$

From the relations between the derivatives of the curvature given in Propositions B.1 and B.2, it follows that for β in $[0,1]$, the gradient flow of the functional

$$\frac{1-\beta}{2}\mathcal{F}_{Rm} + \frac{\beta}{2}(\mathcal{F}_{Ric} - \frac{1}{4}\mathcal{F}_R) - \frac{a}{4}\mathcal{F}_R$$

is of the form E_P with

$$P(g) = \delta\tilde{\delta}Rm_g + a\Delta R_g g + a\nabla^2 R_g + Rm_g * Rm_g.$$

As a result, short-time existence is assured for these flows when $a < \frac{1}{2(n-1)}$ (Theorem A).

For $n \geq 4$, we can also write that for β in $[0,1]$, the gradient flow of the functional

$$\frac{n-2}{2(n-3)}\beta\mathcal{F}_W - (1-\beta)\mathcal{F}_2 + \frac{\alpha}{8(n-1)}\mathcal{F}_R$$

is of the form E_P with

$$P(g) = \delta\tilde{\delta}Rm_g + \frac{1-\alpha}{2(n-1)}(\Delta R_g\, g + \nabla^2 R_g) + Rm_g * Rm_g.$$

Short-time existence is assured when α is positive.

In low dimensions, additional relations between the curvature tensors allow us to write it in an easier way:

In dimension 3**:** We have

$$\mathcal{F}_W(g) = 0 \qquad \text{and} \qquad \mathcal{F}_{Rm} = \mathcal{F}_{Ric} - \frac{1}{4}\mathcal{F}_R.$$

The gradient flow of $-\mathcal{F}_2 + \frac{\alpha}{16}\mathcal{F}_R$ is of the form E_P with

$$P(g) = \delta\tilde{\delta}Rm_g + \frac{1-\alpha}{4}(\Delta R_g\, g + \nabla^2 R_g) + Rm_g * Rm_g,$$

and the flow exists for a short-time as soon as α is positive.

In dimension 4**:** We have

$$\mathcal{F}_2 = \frac{1}{24}\mathcal{F}_R - \frac{1}{2}\mathcal{F}_{\overset{\circ}{Ric}},$$

and the Gauss-Bonnet formula gives us the following relation between the functionals:

$$\mathcal{F}_{Rm}(g) - \mathcal{F}_{\overset{\circ}{Ric}}(g) = \mathcal{F}_W(g) + \mathcal{F}_2(g) = 8\pi^2\chi(M).$$

It follows that

$$\nabla\mathcal{F}_{Rm} = \nabla(\mathcal{F}_{Ric} - \frac{1}{4}\mathcal{F}_R) \qquad \text{and} \qquad \nabla\mathcal{F}_W(g) = -\nabla\mathcal{F}_2(g).$$

The gradient flow of $\mathcal{F}_W + \frac{\lambda}{24}\mathcal{F}_R$ is the same as the gradient flow of

$$\mathcal{F}^\lambda = (1-\lambda)\mathcal{F}_W + \frac{\lambda}{2}\mathcal{F}_{\overset{\circ}{Ric}},$$

and is of the form E_P with

$$P(g) = \delta\tilde{\delta}Rm_g + \frac{1-\lambda}{6}(\Delta R_g\, g + \nabla^2 R_g) + Rm_g * Rm_g.$$

Theorem A supplies short-time existence when λ is positive.

Moreover, since $W \vee W - \ |W|^2\ g = 0$ (see [Bes87]), the gradient of the Weyl functional $\mathcal{F}_W$, which is called the Bach tensor, takes the following shorter form:

$$\nabla\mathcal{F}_W = -\nabla\mathcal{F}_2 = -\delta\tilde{\delta}W - \frac{1}{2}\overset{\circ}{W}(\overset{\circ}{Ric}),$$

and can also be written

$$\nabla\mathcal{F}_W = 2\tilde{\delta}\delta W^+ - \overset{\circ}{W}{}^+_g(\overset{\circ}{Ric}_g) = 2\tilde{\delta}\delta W^- - \overset{\circ}{W}{}^-_g(\overset{\circ}{Ric}_g), \tag{2.1}$$

where $W = W^+ + W^-$ is the orthogonal decomposition of the Weyl tensor with respect to the splitting of $\Lambda^2 T^*M$ into self-dual forms and anti-self-dual forms.

On a four-dimensional manifold of positive Euler characteristic, we have the following control on the Yamabe constant when the functional $\mathcal{F}^\lambda = (1-\lambda)\mathcal{F}_W + \frac{\lambda}{2}\mathcal{F}_{\overset{\circ}{Ric}}$ is not too large:

Lemma 2.2 (M. Gursky, [Gur94], see also [Str10b])**.** *Let (M^4, g) be a closed Riemannian manifold. For all $\lambda \geq 0$*

$$\mathrm{Y}(M,[g])^2 \geq 24\left((1-\lambda)8\pi^2\chi(M) - \mathcal{F}^\lambda(g)\right).$$

Proof. If $\tilde{g} \in [g]$ is a Yamabe metric, it has constant scalar curvature and we get

$$\begin{aligned}\mathrm{Y}(M,[\tilde{g}])^2 &= \mathrm{Vol}_{\tilde{g}}(M)^{-1}\left(\int_M R_{\tilde{g}}dv_{\tilde{g}}\right)^2 = \int_M R_{\tilde{g}}^2 dv_{\tilde{g}} \\ &\geq 24\left(\frac{1}{24}\mathcal{F}_R(\tilde{g}) - \frac{1}{2}\mathcal{F}_{\overset{\circ}{Ric}}(\tilde{g})\right) = 24\,\mathcal{F}_2(\tilde{g}).\end{aligned}$$

Since $\mathrm{Y}(M,[g])$ and $\mathcal{F}_2$ are conformal invariants, it follows that the inequality is still true for g. Then

$$\begin{aligned}\mathrm{Y}(M,[g])^2 &\geq 24\,\mathcal{F}_2(g) \\ &= 24(\lambda\,\mathcal{F}_2(g) + (1-\lambda)(8\pi^2\chi(M) - \mathcal{F}_W(g))) \\ &\geq 24((1-\lambda)8\pi^2\chi(M) - \mathcal{F}^\lambda(g)).\end{aligned}$$

□

We finally prove the following proposition, which shows that we can use Theorem F when the energy $\mathcal{F}^\lambda$ is not too large:

Proposition 2.3. *Let (M^4, g) be a closed Riemannian manifold and let λ be in $(0,1)$. If there exists $\epsilon > 0$ such that*

$$\begin{cases} \lambda \leq \dfrac{4}{13} \\ \mathcal{F}^\lambda(g) \leq 2\lambda(\pi^2\chi(M) - \epsilon) \end{cases} \quad \textit{or} \quad \begin{cases} \lambda \geq \dfrac{4}{13} \\ \mathcal{F}^\lambda(g) \leq \dfrac{8}{9}(1-\lambda)(\pi^2\chi(M) - \epsilon), \end{cases}$$

then g satisfies

$$\mathcal{F}_W(g) + \frac{1}{4}\mathcal{F}_{\overset{\circ}{Ric}}(g) \leq \frac{1}{8\times 24}\,\mathrm{Y}(M,[g])^2 - \epsilon.$$

Proof. If $\lambda \leq \frac{4}{13}$, then $\frac{4}{13\lambda} \geq \frac{9}{13(1-\lambda)}$, so using the assumption on $\mathcal{F}^\lambda$, we can write:

$$\frac{9}{13}\mathcal{F}_W + \frac{2}{13}\mathcal{F}_{\overset{\circ}{Ric}} \leq \frac{4}{13\lambda}\mathcal{F}^\lambda \leq \frac{8}{13}(\pi^2\chi(M) - \epsilon).$$

In the same way, if $\lambda \geq \frac{4}{13}$, then $\frac{9}{13(1-\lambda)} \geq \frac{4}{13\lambda}$, and using the assumption on $\mathcal{F}^\lambda$, it follows that

$$\frac{9}{13}\mathcal{F}_W + \frac{2}{13}\mathcal{F}_{\overset{\circ}{Ric}} \leq \frac{9}{13(1-\lambda)}\mathcal{F}^\lambda \leq \frac{8}{13}(\pi^2\chi(M) - \epsilon).$$

Consequently,

$$\begin{aligned}\mathcal{F}_W(g) + \frac{1}{4}\mathcal{F}_{\overset{\circ}{Ric}}(g) &\leq \frac{1}{8}(8\pi^2\chi(M) - \mathcal{F}_W(g)) - \epsilon \\ &\leq \frac{1}{8\times 24}\,\mathrm{Y}(M,[g])^2 - \epsilon,\end{aligned}$$

according to Lemma 2.2 with $\lambda = 0$. □

3. Bando-Bernstein-Shi estimates

Let $P : \mathcal{S}^2_+(M) \to \mathcal{S}^2(M)$ be a smooth map of the form

$$P(g) = \delta\tilde{\delta} Rm_g + a\Delta R_g g + b\nabla^2 R_g + Rm_g * Rm_g,$$

with $a < \frac{1}{2(n-1)}$ and $b \in \mathbb{R}$.

In this section, we prove the following estimates:

Theorem 2.4. *Let $0 \leq s \leq 2$. If (g_t), $t \in [0,T)$, are solution of $E_P(g_0)$ on a manifold M^n and if $\varphi \in C_0^\infty(M,[0,1])$ is a function with compact support which satisfy*

$$\sup_{[0,T]} \left(\|d\varphi\|_\infty^2 + \|\nabla d\varphi\|_\infty \right) \leq \beta \quad \textit{and} \quad \sup_{[0,T]\times\{\varphi>0\}} \left(|Rm_g|\, \varphi^s \right) \leq \beta$$

then for all $k \in \mathbb{N}$, there exists a constant $c(n,k,\beta,P,T)$ such that for all $t \in (0,T]$,

$$\int_M \left|\nabla^k Rm_g\right|^2 \varphi^{2(k+2+s)} dv_g \leq \frac{c}{t^{\frac{k}{2}}} \sup_{[0,T]} \left(\int_{\varphi>0} |Rm_g|^2 \varphi^{2s} dv_g \right).$$

For tensors $T, T_1, \dots, T_j$ and nonnegative integers j and k, let write

$$\mathcal{P}_m(T_1,\dots,T_j) = \sum_{k_1+\dots+k_j=m} \nabla^{k_1} T_1 * \dots * \nabla^{k_j} T_j,$$
$$\mathcal{P}_{m,k}(T_1,\dots,T_j) = \sum_{\substack{k_1+\dots+k_j=m\\ k_1,\dots,k_j\leq k}} \nabla^{k_1} T_1 * \dots * \nabla^{k_j} T_j,$$

and

$$\mathcal{P}^{(j)}_m(T) = \sum_{k_1+\dots+k_j=m} \nabla^{k_1} T * \dots * \nabla^{k_j} T,$$
$$\mathcal{P}^{(j)}_{m,k}(T) = \sum_{\substack{k_1+\dots+k_j=m\\ k_1,\dots,k_j\leq k}} \nabla^{k_1} T * \dots * \nabla^{k_j} T.$$

We write $lot^{(k)}_g(\phi)$ for terms in the linear span of

$$\begin{gathered} \int_M \mathcal{P}^{(3)}_{2k+2,k+2}(Rm_g)\, \phi\, dv_g, \qquad \int_M \mathcal{P}^{(4)}_{2k,k}(Rm_g)\, \phi\, dv_g, \\ \int_M \mathcal{P}^{(2)}_{2k+3,k+2}(Rm_g) * \nabla\phi\, dv_g \quad \text{and} \quad \int_M \mathcal{P}^{(2)}_{2k+2,k+2}(Rm_g) * \nabla^2\phi\, dv_g. \end{gathered} \tag{2.2}$$

Those terms can be controlled by $\int_M \left|\nabla^{k+2} Rm_g\right|^2 \phi\, dv_g$ (see Proposition 2.6) and all the lower order terms appearing in the integral estimates on the curvature are of this form:

Proposition 2.5. *For all integers $k \geq 0$, we have*

$$\left(\int_M \left|\nabla^k Rm\right|^2 \phi\, dv \right)'_g (P_g) + \int_M \left(\left|\nabla^{k+2} Rm_g\right|^2 - \frac{a}{2} \left|\nabla^{k+2} R_g\right|^2 \right) \phi\, dv_g = lot^{(k)}_g(\phi),$$
$$\left(\int_M \left|\nabla^k R\right|^2 \phi\, dv \right)'_g (P_g) + (1 - 2a(n-1)) \int_M \left|\nabla^{k+2} R_g\right|^2 \phi\, dv_g = lot^{(k)}_g(\phi),$$

where the coefficients of the lower-order terms only depend on n, k and P.

Proof. We have

$$\left(\int_M \left|\nabla^k Rm\right|^2 \phi\, dv\right)'_g (P_g) = 2\int_M \left\langle (\nabla^k Rm)'_g(P_g) \mid \nabla^k Rm_g \right\rangle \phi\, dv_g$$
$$+ \int_M \left\langle \nabla^k Rm_g * \nabla^k Rm_g \mid P_g \right\rangle \phi\, dv_g + \frac{1}{2}\int_M \left|\nabla^k Rm_g\right|^2 tr(P_g)\phi\, dv_g,$$
$$\left(\int_M \left|\nabla^k Rm\right|^2 \phi\, dv\right)'_g (P_g) = 2\int_M \left\langle (\nabla^k Rm)'_g(P_g) \mid \nabla^k Rm_g \right\rangle \phi\, dv_g + lot_k(g).$$

According to Proposition B.2, we have

$$a\Delta R_g g + b\nabla^2 R_g = a\delta \mathsf{D}(R_g g) + (b-a)\tilde{\mathsf{D}}\mathsf{D} R_g.$$

According to Proposition C.7, we have

$$\int_M \left\langle (\nabla^k Rm)'_g(\delta\tilde{\delta} Rm_g) \mid \nabla^k Rm_g \right\rangle \phi\, dv_g$$
$$= -\int_M \left\langle \nabla^{*2}\nabla^{k+2} Rm_g \mid \nabla^k Rm_g \right\rangle \phi\, dv_g + lot_k(g),$$
$$= -\int_M \left|\nabla^{k+2} Rm_g\right| \phi\, dv_g - 2\int_M \left\langle \nabla^{k+2} Rm_g \mid \nabla\phi \otimes \nabla^{k+1} Rm_g \right\rangle\, dv_g$$
$$- \int_M \left\langle \nabla^{k+2} Rm_g \mid \nabla^2\phi \otimes \nabla^k Rm_g \right\rangle\, dv_g + lot_k(g),$$
$$= -\int_M \left|\nabla^{k+2} Rm_g\right|^2 \phi\, dv_g + lot_k(g).$$

We also have

$$\int_M \left\langle (\nabla^k Rm)'_g(\delta \mathsf{D}(R_g g)) \mid \nabla^k Rm_g \right\rangle \phi\, dv_g$$
$$= \int_M \left\langle g_\bullet (\mathsf{D}\nabla^*\nabla^{k+1}\tilde{\mathsf{D}} R_g) \mid \nabla^k Rm_g \right\rangle \phi\, dv_g + lot_k(g),$$
$$= \int_M \left\langle \mathsf{D}\nabla^*\nabla^{k+1}\tilde{\mathsf{D}} R_g \mid \nabla^k Ric_g \right\rangle \phi\, dv_g + lot_k(g),$$
$$= \int_M \left\langle \nabla^{k+1}\tilde{\mathsf{D}} R_g \mid \nabla\delta\nabla^k Ric_g \right\rangle \phi\, dv_g + lot_k(g),$$

By commuting ∇ and δ (see (C.12)), we have

$$\delta\nabla^k Ric_g = \nabla^k \delta Ric_g + \mathcal{P}^{(2)}_{k-1}(Rm_g) = -\frac{1}{2}\nabla^k \tilde{\mathsf{D}} R_g + \mathcal{P}^{(2)}_{k-1}(Rm_g)\, .$$

Therefore

$$\int_M \left\langle (\nabla^k Rm)'_g(\delta \mathsf{D}(R_g g)) \mid \nabla^k Rm_g \right\rangle \phi\, dv_g = -\frac{1}{2}\int_M \left|\nabla^{k+1}\tilde{\mathsf{D}} R_g\right|^2 \phi\, dv_g + lot_k(g),$$
$$= -\frac{1}{2}\int_M \left|\nabla^{k+2} R_g\right|^2 \phi\, dv_g + lot_k(g).$$

And we have

$$\int_M \left\langle (\nabla^k Rm)'_g(\tilde{\mathsf{D}}\mathsf{D} R_g) \mid \nabla^k Rm_g \right\rangle \phi\, dv_g = lot_k(g).$$

Finally, according to Proposition C.7,

$$\begin{aligned}\left(\int_M \left|\nabla^k R\right|^2 \phi\, dv\right)'_g (P_g) &= 2\int_M \left\langle (\nabla^k R)'_g(P_g) \mid \nabla^k R_g \right\rangle \phi\, dv_g + lot_k(g)\\ &= -(1-2a(n-1))\int_M \left\langle \nabla^{*2}\nabla^{k+2} R_g \mid \nabla^k R_g \right\rangle \phi\, dv_g + lot_k(g)\\ &= -(1-2a(n-1))\int_M \left|\nabla^{k+2} R_g\right|^2 \phi\, dv_g + lot_k(g).\end{aligned}$$

□

Proposition 2.6. *Let $0 \le s \le 2$. If (M^4, g) is a Riemannian manifold and $\varphi \in C_0^\infty(M, [0,1])$ is a function which satisfy*

$$\|d\varphi\|^2_{L^\infty} + \|\nabla d\varphi\|_{L^\infty} \le \beta \quad \textit{and} \quad \sup_{\{\varphi>0\}} \left(|Rm_g|\, \varphi^s\right) \le \beta$$

then for all $k \in \mathbb{N}$, there exist a constant c_k such that

$$lot_g^{(k)}(\varphi^{2(k+2+s)}) \le \frac{1}{2}\int_M \left|\nabla^{k+2} Rm_g\right|^2 \varphi^{2(k+2+s)} dv_g + c_k \beta^{k+2}\int_{\varphi>0} |Rm_g|^2\, \varphi^{2s} dv_g.$$

Moreover, the constant c_k only depends on the coefficients in the linear combination defining $lot_g^{(k)}(\varphi^{2(k+2+s)})$ and on the dimension n.

Proof. We use Corollary D.8 and Lemma D.6 to estimate each term in (2.2). There exist constants $C_1(n,k), \ldots, C_5(n,k)$ such that

$$\begin{aligned}&\int_M \left|\mathcal{P}^{(3)}_{2k+2,k+2}(Rm_g)\right| \varphi^{2(k+2+s)} dv_g\\ &\qquad \le \int_M \left|\mathcal{P}^{(3)}_{2k+2,k+2}(Rm_g)\right| \varphi^{2k+2+3s} dv_g\\ &\qquad \le \frac{1}{2}\int_M \left|\nabla^{k+2} Rm_g\right|^2 \varphi^{2(k+2+s)} dv_g + (1+C_1)\,\beta^{k+2}\int_{\varphi>0} |Rm_g|^2\, \varphi^s dv_g,\end{aligned}$$

$$\begin{aligned}&\int_M \left|\mathcal{P}^{(4)}_{2k,k}(Rm_g)\right| \varphi^{2(k+2+s)} dv_g\\ &\qquad \le \int_M \left|\mathcal{P}^{(4)}_{2k,k+2}(Rm_g)\right| \varphi^{2k+4s} dv_g\\ &\qquad \le \frac{1}{2}\int_M \left|\nabla^{k+2} Rm_g\right|^2 \varphi^{2(k+2+s)} dv_g + (1+C_2)\,\beta^{k+2}\int_{\varphi>0} |Rm_g|^2\, \varphi^s dv_g,\end{aligned}$$

$$\begin{aligned}&\|d\varphi\|_\infty \int_M \left|\mathcal{P}^{(2)}_{2k+3,k+2}(Rm_g)\right| \varphi^{2k+3+2s} dv_g\\ &\qquad \le \frac{1}{2}\int_M \left|\nabla^{k+2} Rm_g\right|^2 \varphi^{2(k+2+s)} dv_g + (1+C_3)\beta^{k+2}\int_{\varphi>0} |Rm_g|^2\, \varphi^s dv_g,\end{aligned}$$

$$\begin{aligned}&\|\nabla d\varphi\|_\infty \int_M \left|\mathcal{P}^{(2)}_{2k+2,k+2}(Rm_g)\right| \varphi^{2k+3+2s} dv_g\\ &\qquad \le \beta\int_M \left|\mathcal{P}^{(2)}_{2k+2,k+2}(Rm_g)\right| \varphi^{2k+2+2s} dv_g\\ &\qquad \le \frac{1}{2}\int_M \left|\nabla^{k+2} Rm_g\right|^2 \varphi^{2(k+2+s)} dv_g + (1+C_4)\beta^{k+2}\int_{\varphi>0} |Rm_g|^2\, \varphi^s dv_g,\end{aligned}$$

$$\begin{aligned}
&\|d\varphi\|_\infty^2 \int_M \left|\mathcal{P}^{(2)}_{2k+2,k+2}(Rm_g)\right| \varphi^{2(k+1+s)} dv_g \\
&\qquad \leq \frac{1}{2}\int_M \left|\nabla^{k+2} Rm_g\right|^2 \varphi^{2(k+2+s)} dv_g + (1+C_5)\beta^{k+2} \int_{\varphi>0} \left|Rm_g\right|^2 \varphi^s dv_g.
\end{aligned}$$

□

We can now prove Theorem 2.4. Let define a curvature term of order k, which involves the full Riemann tensor and an additional scalar curvature term:

$$\mathcal{A}_k(g) = \int_M \left|\nabla^k Rm_g\right|^2 \varphi^{2(k+2+s)} dv_g + \frac{a_+}{1-2a_+(n-1)} \int_M \left|\nabla^k R_g\right|^2 \varphi^{2(k+2+s)} dv_g,$$

where $a_+ = \max(a, 0)$, and let define the constant

$$c_a = \frac{1-2a_+(n-1)}{2}.$$

There exists $c'_k(n, P)$ such that for all integers $k \geq 0$,

$$(\mathcal{A}_k)'_g(P_g) + c_a \mathcal{A}_{k+2}(g) \leq c'_k \beta^{k+2} \int_{\varphi>0} \left|Rm_g\right|^2 \varphi^s dv_g.$$

Indeed, we can write

$$\begin{aligned}
&(\mathcal{A}_k)'_g(P_g) + c_a \mathcal{A}_{k+2}(g) \\
&\qquad = -(1-c_a)\int_M \left|\nabla^{k+2} Rm_g\right|^2 \varphi^{2(k+2+s)} dv_g \\
&\qquad\qquad - \frac{1}{2}(a_+ - a)\int_M \left|\nabla^{k+2} R_g\right|^2 \varphi^{2(k+2+s)} dv_g + lot_g^{(k)}(\varphi^{2(k+2+s)}) \\
&\qquad \leq -\frac{1}{2}\int_M \left|\nabla^{k+2} Rm_g\right|^2 \varphi^{2(k+2+s)} dv_g + lot_g^{(k)}(\varphi^{2(k+2+s)}) \\
&\qquad \leq c'_k \beta^{k+2} \int_{\varphi>0} \left|Rm_g\right|^2 \varphi^s dv_g \text{ according to Proposition 2.6.}
\end{aligned}$$

Now, if we define the polynomial

$$f_k(t) = \sum_{j=0}^{k} \frac{c_a^j t^j}{j!} \mathcal{A}_{2j}(g_t),$$

we see that it satisfies the following differential inequation:

$$\begin{aligned}
f'_k(t) &= \sum_{j=0}^{k-1} \frac{c_a^j t^j}{j!} \left((\mathcal{A}_{2j})'_{g_t}(P_{g_t}) + c_a \mathcal{A}_{2j+2}(g_t)\right) + \frac{c_a^k t^k}{k!} (\mathcal{A}_{2k})'_{g_t}(P_{g_t}) \\
&\leq \sum_{j=0}^{k} \frac{c_a^j t^j}{j!} c'_{2j} \beta^{2(j+1)} \int_{\varphi>0} \left|Rm_g\right|^2 \varphi^s dv_g \\
&\leq C'(1+t)^k \int_{\varphi>0} \left|Rm_g\right|^2 \varphi^s dv_g,
\end{aligned}$$

with $C' = C'(n, k, \beta, P)$. It follows by integration that

$$\begin{aligned}
\int_M \left|\nabla^{2k} Rm_g\right|^2 \varphi^{2(2k+2+s)} dv_g &\leq \mathcal{A}_{2k}(g_t) \\
&\leq \frac{k!}{c_a^k t^k} f_k(t) \\
&\leq \frac{c}{t^k} \int_{\varphi>0} \left|Rm_g\right|^2 \varphi^s dv_g.
\end{aligned}$$

And using Corollary D.3 with $q = 2$ and $\varepsilon = t^{\frac{1}{4}} \|\nabla\varphi\|_{L^\infty}$, we have

$$\left(\int_M \left|\nabla^k Rm_g\right|^2 \varphi^{2(k+2+s)} dv_g\right)^{\frac{1}{2}} \leq t^{\frac{1}{4}} \left(\int_M \left|\nabla^{k+1} Rm_g\right|^2 \varphi^{2(k+3+s)} dv_g\right)^{\frac{1}{2}} + \frac{C(1+T^{\frac{1}{4}} \|d\varphi\|_{L^\infty})}{t^{\frac{1}{4}}} \left(\int_M \left|\nabla^{k-1} Rm_g\right|^2 \varphi^{2(k+1+s)} dv_g\right)^{\frac{1}{2}},$$

hence the result is also true for k odd. □

4. Local estimates on the curvature

Let $P : \mathcal{S}^2_+(M) \to \mathcal{S}^2(M)$ be a smooth map of the form

$$P(g) = \delta\tilde{\delta} Rm_g + a\Delta R_g g + b\nabla^2 R_g + Rm_g * Rm_g,$$

with $a < \frac{1}{2(n-1)}$ and $b \in \mathbb{R}$.

In this section, we prove the following local estimates for solutions of E_P:

Theorem 2.7. *For all $\alpha > 0$, $k \in \mathbb{N}$ and $n \geq 3$, there exist a constant $\epsilon(\alpha, k, n, P)$ such that if $g(t)$, $t \in [0, r^4]$, are complete metrics on a manifold M^n which are solution of $E_P(g_0)$ and satisfy one of the following conditions:*

i) for some $x_0 \in M$ and $r > 0$

$$\int_0^{r^4} \left(\sup_{\mathrm{B}_{g_t}(x_0,r)} |Rm_{g_t}|^2\right) dt \leq \epsilon,$$

ii) $\inf\limits_{[0,r^4]} \mathrm{Y}(M, [g_t]) > 0$ and for some $x_0 \in M$ and $r > 0$,

$$\sup_{[0,r^4]} \left(\int_{\mathrm{B}_g(x_0,r)} |Rm_g|^{\frac{n}{2}} dv_g\right) \leq \epsilon,$$

iii) $\inf\limits_{[0,r^4]} \mathrm{Y}(M, [g_t]) > 0$ and for some $x_0 \in M$, $r > 0$ and $p > 1$,

$$\left(\int_0^{r^4} \left(\int_{\mathrm{B}_{g_t}(x_0,r)} |Rm_{g_t}|^{\frac{np}{2}} dv_{g_t}\right)^{\frac{4}{n(p-1)}} dt\right)^{\frac{p-1}{2p}} \leq \epsilon,$$

then for all $t \in (0, r^4]$ and $x \in \mathrm{B}_{g(t)}(x_0, \frac{r}{2})$,

$$\sum_{j=0}^k \left|\nabla^j Rm_g\right|^{\frac{2}{2+j}} \leq \frac{\alpha}{\sqrt{t}}.$$

It implies:

Theorem C. *For all $\alpha > 0$, $k \in \mathbb{N}$ and $n \geq 3$, there exists a constant $\epsilon(\alpha, k, n, P)$ such that if $g(t)$, $t \in [0, T]$, are complete metrics on a manifold M^n which are solution to E_P, that satisfy*

$$\inf_{[0,T]} \mathrm{Y}(M, [g_t]) > 0,$$

and such that for some $x_0 \in M$ and $0 < r < T^{1/4}$

$$\sup_{[0,T]} \left(\int_{\mathrm{B}_g(x_0,r)} |Rm_g|^{\frac{n}{2}} dv_g\right) \leq \epsilon, \tag{2.3}$$

then for all $t \in (0, T]$ and $x \in \mathrm{B}_{g(t)}(x_0, \frac{r}{2})$

$$\sum_{j=0}^k \left|\nabla^j Rm_g\right|^{\frac{2}{2+j}} \leq \alpha \left(\frac{1}{r^2} + \frac{1}{\sqrt{t}}\right). \tag{2.4}$$

Proof. The assumption (2.3) remains true on any subinterval of $[0,T]$. We can hence apply Theorem 2.7 on subintervals of $[0,T]$ of length r^4.

On $(0,r^4]$, the result is a direct consequence of Theorem 2.7, since the assumption ii) is satisfied and since

$$\frac{\alpha}{\sqrt{t}} \leq \alpha\left(\frac{1}{\sqrt{t}}+\frac{1}{r^2}\right).$$

Then, for any t_0 between 0 and $T-r^4$, we apply Theorem 2.7 to $g(t_0+t)$ for t in $[0,r^4]$. For x in $\mathrm{B}(x_0,r)$ and t in $(0,r^4]$, we obtain

$$\sum_{j=0}^{k}\left|\nabla^j Rm_{g(t_0+t)}\right|^{\frac{2}{2+j}} \leq \frac{\alpha}{\sqrt{t}},$$

which gives at time $t=r^4$

$$\sum_{j=0}^{k}\left|\nabla^j Rm_{g(t_0+r^4)}\right|^{\frac{2}{2+j}} \leq \frac{\alpha}{r^2}.$$

Since t_0 is arbitrary, we obtain (2.4) for t in $[r^4,T]$. □

It also implies the following global estimates:

Corollary 2.8. *For all $\alpha>0$, $k\in\mathbb{N}$ and $n\geq 3$, there exists a constant $\beta(\alpha,k,n,P)$ such that if $g(t)$, $t\in[0,T]$, are complete metrics on a manifold M^n which are solution of $E_P(g_0)$ and satisfy*

$$\sup_{[0,T]\times M}|Rm_g|\leq K, \tag{2.5}$$

then for all $t\in(0,T]$

$$\sup_M\left(\sum_{j=0}^{k}\left|\nabla^j Rm_g\right|^{\frac{2}{2+j}}\right) \leq \frac{\alpha}{\sqrt{t}}+\beta K.$$

Proof. We proceed in the same way: the curvature assumption (2.5) remains true on subsets of $[0,T]\times M$. We apply Theorem 2.7 i) on balls of radius r and time intervals of length r^4, with r well chosen.

Let take $r^4=\frac{\epsilon}{K^2}$ in Theorem 2.7. We have for all $x\in M$

$$\int_0^{r^4}\left(\sup_{\mathrm{B}_{g(t)}(x,r)}|Rm_g|^2\right)dt\leq\epsilon,$$

hence we have on $M\times[0,\frac{\epsilon}{K^2}]$:

$$\sum_{j=0}^{k}\left|\nabla^j Rm_g\right|^{\frac{2}{2+j}} \leq \frac{\alpha}{\sqrt{t}},$$

and in particular, if we take $\beta=\frac{\alpha}{\epsilon}$, we obtain at time $\frac{\epsilon}{K^2}$:

$$\sum_{j=0}^{k}\left|\nabla^j Rm_g\right|^{\frac{2}{2+j}}\left(\frac{\epsilon}{K^2}\right) \leq \beta K.$$

But this is also true for $g(t_0+t)$, solution of $E_P(g_{t_0})$ for t in $[0,\frac{\epsilon}{K^2}]$, thus we have on $M\times[\frac{\epsilon}{K^2},T]$:

$$\sum_{j=0}^{k}\left|\nabla^j Rm_g\right|^{\frac{2}{2+j}} \leq \beta K.$$

□

If $g(t)$ is an evolving metric defined on $M \times [0, r^4]$, we define for $1 < p < \infty$:

$$\mathcal{N}^p_{x,r}(Rm_g) = \left(\int_0^{r^4} \|Rm_g\|^{\frac{2p}{p-1}}_{L^{\frac{np}{2}}(\mathrm{B}_{g(t)}(x,r))} \, dt \right)^{\frac{p-1}{2p}},$$

for $p = 1$:

$$\mathcal{N}^1_{x,r}(Rm_g) = \sup_{[0,r^4]} \left(\|Rm_g\|_{L^{\frac{n}{2}}(\mathrm{B}_{g(t)}(x,r))} \right),$$

and for $p = \infty$:

$$\mathcal{N}^\infty_{x,r}(Rm_g) = \int_0^{r^4} \left(\sup_{\mathrm{B}_{g(t)}(x,r)} |Rm_g|^2 \right) dt.$$

Those quantities are invariant under a rescaling of the flow: if $\hat{g}(t) = \alpha\, g\left(\frac{t}{\alpha^2}\right)$, then

$$\mathcal{N}^p_{x,\sqrt{\alpha}\, r}(Rm_{\hat{g}}) = \mathcal{N}^p_{x,r}(Rm_g).$$

We will need the following Lemma:

Lemma 2.9. *Let M be a smooth manifold, let $x_0 \in M$ and $\rho > 0$. If $g(t)$, $t \in I$, is a smooth family of metrics on M such that*

$$\sup_{\substack{t \in I \\ x \in \mathrm{B}_{g(t)}(x_0,\rho)}} |\partial_t g_t|_{g_t} \leq A,$$

then for all $s, t \in I$ and $\rho \geq 0$,

$$\mathrm{B}_{g_s}\left(x_0, e^{-\frac{A}{2}|t-s|}\rho/2\right) \subset \mathrm{B}_{g_t}(x_0, \rho).$$

Proof. Let fix $s \in I$ and let consider

$$J = \{t \in I, \forall r \in [t,s],\ \mathrm{B}_{g_s}\left(x_0, e^{-\frac{A}{2}|t-s|}\rho/2\right) \subset \mathrm{B}_{g_r}(x_0,\rho)\}.$$

J is an interval containing s. Let prove that it is open and closed. If t belong to the closure of J, then for all $r \in (t, s]$,

$$\mathrm{B}_{g_s}\left(x_0, e^{-\frac{A}{2}|r-s|}\rho/2\right) \subset \mathrm{B}_{g_r}(x_0,\rho),$$

hence on $\mathrm{B}_{g_s}\left(x_0, e^{-\frac{A}{2}|t-s|}\rho/2\right) \times (t, s]$, we have $|\partial_t g_t|_{g_t} \leq A$. Using Lemma A.4, we can prove that

$$\mathrm{B}_{g_s}\left(x_0, e^{-\frac{A}{2}|t-s|}\rho/2\right) \subset \mathrm{B}_{g_t}(x_0,\rho/2).$$

Indeed, if x is in the first ball, and if γ is a geodesic from x_0 to x of length less than $e^{-\frac{A}{2}|t-s|}\rho/2$ for g_s, then it is of length less than $\rho/2$ for g_t. Therefore, we see that a neighborhood of t is in J. Then J is open and closed in I, thus $I = J$. □

We will now prove Theorem 2.7. We can suppose that $k \geq 3$. By taking

$$\breve{g}(t) = \frac{1}{r^2}\, g\left(r^4 t\right),$$

we can also suppose that $r = 1$. We define

$$\mathcal{C}^{(k)}_g = \sum_{j=0}^{k} \left|\nabla^j Rm_g\right|^{\frac{2}{2+j}}.$$

We will prove that for all $x \in \mathrm{B}_{g(t)}(x_0, 1)$,

$$\mathcal{C}^{(k)}_{g(t)}(x) \leq \frac{\alpha}{4(1 - r_{g(t)}(x))^2\sqrt{t}}$$

where $r_{g(t)}(x) = d_{g(t)}(x_0, x)$.

Suppose that there exists $g_i(t)$ complete solutions of E_P on manifolds M_i and points $x_i \in M_i$ such that

$$\mathcal{N}^p_{x_i,1}(Rm_{g_i}) \leq \frac{1}{i},$$

and that

$$\max_{\substack{t\in[0,1]\\ x\in\overline{\mathrm{B}}_{g_i(t)}(x_i,1)}} \left(\left(1-r_{g_i(t)}(x)\right)^2\sqrt{t}\,\mathcal{C}^{(k)}_{g_i(t)}(x)\right) > \frac{\alpha}{4}. \tag{2.6}$$

Let take $t_i \in (0,1]$ and $y_i \in \mathrm{B}_{g_i(t)}(x_i,1)$ for which the maximum is attained. Then for all $t\in[\frac{t_i}{2},t_i]$ and $x\in \mathrm{B}_{g_i(t)}\left(y_i,\frac{1-r_{g_i(t)}(y_i)}{2}\right)$,

$$\begin{aligned}\mathcal{C}^{(k)}_{g_i(t)}(x) &\leq \left(\frac{1-r_{g_i(t)}(y_i)}{1-r_{g_i(t)}(x)}\right)^2\sqrt{\frac{t_i}{t}}\,\mathcal{C}^{(k)}_{g_i(t_i)}(y_i)\\ &\leq 4\sqrt{2}\,\mathcal{C}^{(k)}_{g_i(t_i)}(y_i)\end{aligned}$$

We define $\lambda_i = \mathcal{C}^{(k)}_{g_i(t_i)}(y_i)$ and the renormalized flows

$$\hat{g}_i(t) = \lambda_i g_i\left(t_i + \frac{t}{\lambda_i^2}\right).$$

Then $\hat{g}_i$ are complete solutions of E_p, satisfy $\mathcal{C}^{(k)}_{\hat{g}_i(0)}(y_i)=1$ and for all $t\in[-\lambda_i^2 t_i,0]$ and for all $x\in\mathrm{B}_{\hat{g}_i(t)}\left(y_i,\sqrt{\lambda_i}\frac{1-r_{g_i(t)}(y_i)}{2}\right)$, we have the following control on the curvature:

$$\mathcal{C}^{(k)}_{\hat{g}_i(t)}(x) \leq 4\sqrt{2}. \tag{2.7}$$

Moreover, according to (2.6), we have

$$\lambda_i(1-r_{g_i(t)}(y_i))^2\sqrt{t_i} > \frac{\alpha}{4},$$

hence

$$\lambda_i(1-r_{g_i(t)}(y_i))^2 > \frac{\alpha}{4} \quad \text{and} \quad \lambda_i\sqrt{t_i} > \frac{\alpha}{4}.$$

Inequality (2.7) is in particular true for $t\in[-\frac{\alpha^2}{16},0]$ and $x\in\mathrm{B}_{\hat{g}_i(t)}\left(y_i,\frac{\sqrt{\alpha}}{2}\right)$. According to Lemma 2.9, we can find $\delta(\alpha)$ such that for all $t\in[-\frac{\alpha^2}{16},0]$,

$$\mathrm{B}_{\hat{g}_i(0)}\left(y_i,\delta(\alpha)\right) \subset \mathrm{B}_{\hat{g}_i(t)}\left(y_i,\frac{\sqrt{\alpha}}{2}\right).$$

Furthermore, we can take $\delta(\alpha)\leq\frac{\sqrt{n(n-1)}}{8\pi}$. Then, according to the Rauch comparison theorem, the exponential map $exp_{\hat{g}_i(0)}$ is a covering map from $\mathbf{B}=\mathrm{B}_{\mathbb{R}^n}(O,\delta(\alpha))$ to $\mathrm{B}_{\hat{g}_i(0)}\left(y_i,\delta(\alpha)\right)$. Hence we can lift the evolving metrics to $\mathbf{B}$ and obtain

$$\tilde{g}_i(t) = exp^*_{\hat{g}_i(0)}\hat{g}_i(t)$$

on $[-\frac{\alpha^2}{16},0]\times\mathbf{B}$ which are solution of E_P, satisfy $\mathcal{C}^{(k)}_{\tilde{g}_i(0)}(O)=1$ and the bound

$$\mathcal{C}^{(k)}_{\tilde{g}_i(t)}(x)\leq 4\sqrt{2}.$$

According to R. Hamilton's result [Ham95, Corollary 4.11], up to reducing δ, all the metrics $\tilde{g}_i(0)$ satisfy

$$\frac{1}{2}g_{eucl} \leq \tilde{g}_i(0) \leq 2g_{eucl}. \tag{2.8}$$

And according to Lemma A.4, there exists a constant $C(\alpha)$ such that for all $t\in[-\frac{\alpha^2}{16},0]$ and $i\in\mathbb{N}$,

$$e^{-C}g_{eucl} \leq \tilde{g}_i(t) \leq e^{C}g_{eucl}. \tag{2.9}$$

Now, if we take a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^n, [0,1])$ such that

$$\begin{cases} \varphi \equiv 0 & \text{on } \mathbb{R}^n \setminus \mathbf{B} \\ \varphi \equiv 1 & \text{on } \frac{1}{2}\mathbf{B}, \end{cases}$$

by Lemma A.5, there exists $\beta(\alpha)$ such that

$$\sup_{[-\frac{\alpha^2}{16},0]} \left(\|d\varphi\|_\infty^2 + \|\nabla d\varphi\|_\infty \right) \leq \beta.$$

Moreover, by the Bishop-Gromov comparison theorem, the volume of $\mathbf{B}$ is bounded by a constant depending only on n. Hence, according to the estimates of Theorem 2.4, the metrics $\tilde{g}_i(t)$ have their curvature bounded in $H_k^{2,s}(\varphi)$, uniformly on $[-\frac{\alpha^2}{32}, 0]$, for all $2 \leq s \leq 4$ and all $k \in \mathbb{N}$. And because of (2.9), the Sobolev constants s_g of all the metrics $\tilde{g}_i(t)$ are uniformly bounded. Hence, by the Sobolev inequalities Proposition E.1, all derivatives of the curvature of the metrics $\tilde{g}_i(t)$, $t \in [-\frac{\alpha^2}{32}, 0]$ are uniformly bounded on $\frac{1}{2}\mathbf{B}$.

Finally, according to [Ham95, Corollary 4.11], all space derivatives of the metrics $\tilde{g}_i(0)$ with respect to the euclidean metric are uniformly bounded for the euclidean metric, then by Lemma A.6, it is actually true on $[-\frac{\alpha^2}{32}, 0] \times \frac{1}{2}\mathbf{B}$. Since the metrics are solution of E_P, their time derivatives are uniformly bounded as well. Therefore, we can extract a subsequence converging in $C^\infty([-\frac{\alpha^2}{32}, 0] \times \frac{1}{2}\mathbf{B})$ to a limit $g_\infty(t)$.

If $p = \infty$, the lifted metrics $\tilde{g}_i(t)$ satisfy the bound

$$\mathcal{N}_{O,\delta(\alpha)}^\infty(Rm_{\tilde{g}_i}) \leq \frac{1}{i},$$

hence the limit metric must be flat, which is in contradiction with the fact that

$$\mathcal{C}_{\tilde{g}_i(0)}^{(k)}(O) = 1.$$

If $p \in [1, \infty)$ and if $\Upsilon(M, [g_i(t)]) \geq Y_0 > 0$, then the renormalized metrics also satisfy $\Upsilon(M, [\hat{g}_i(t)]) \geq Y_0 > 0$. According to Proposition 1.4 and Lemma 1.3, the injectivity radius of $\hat{g}_i(0)$ is uniformly bounded from below by some positive constant δ_0. We can assume that $\delta(\alpha) \leq \delta_0$, then $exp_{\hat{g}_i(0)}$ is an isometry on $\mathbf{B}$ and the lifted metrics $\tilde{g}_i(t)$ satisfy the bound

$$\mathcal{N}_{O,\delta(\alpha)}^p(Rm_{\tilde{g}_i}) \leq \frac{1}{i},$$

hence the limit metric must be flat, and we obtain the same contradiction. □

CHAPTER 3

Singularities and pinching results

Dans ce chapitre, on étudie les singularités pouvant apparaître le long des flots de courbure introduits dans le chapitre précédent. On prouve en particulier des résultats de rigidité pour les métriques modélisant ces singularités, et on en déduit la convergence de flots de gradients lorsque la courbure de la métrique initiale est intégralement pincée.

1. Compactness results

In order to perform a "blow-up" at a singular time, we need a compactness result for solutions of our equations. Once we have the estimates of Corollary 2.8, we can proceed as R. Hamilton in [Ham95] and use the following theorem:

Theorem 3.1 (R. Hamilton, [Ham95]). *Let $(M_i, g_i, x_i)_{i\in I}$ be a sequence of pointed complete Riemannian manifolds with uniform C^0 bounds on all the derivatives of the curvature. If the injectivity radius of g_i at x_i is uniformly bounded from below by a positive constant, we can find a converging subsequence in the pointed C^∞ topology.*

We can prove a version of [Ham95, Theorem 1.2] for solutions of E_P:

Theorem 3.2. *Let $(g_i(t), x_i)$ be pointed solutions of E_P on a Riemannian manifold M and a time interval $(-\alpha, \omega]$ containing 0. Suppose that the curvature of the metrics g_i are uniformly bounded on $(-\alpha, \omega] \times M$ and that the injectivity radius of $g_i(0)$ at x_i are uniformly bounded from below by a positive constant. Then there exists a subsequence of $(M, g_i(t), x_i)$ which converges in the pointed C^∞ topology to a pointed complete solution $(M_\infty, g_\infty(t), x_\infty)$ of E_P.*

Proof. We proceed exactly as in [Ham95]. By a diagonalization argument, we can suppose that $\alpha > -\infty$. Then according to Corollary 2.8, all the derivatives of the curvature are bounded on $(-\alpha+\varepsilon, \omega]$. According to Theorem 3.1 we can find a convergent subsequence of $(M_i, g_i(0), x_i)$ to a complete manifold $(M_\infty, g_\infty, x_\infty)$. Then, we can define the metrics g_t on M_∞ by the diffeomorphisms. All the derivatives of the curvature of g_i are bounded for g_i, hence also for a fixed metric g, according to Lemma A.6. Therefore we can find a subsequence converging smoothly to a limit $g_\infty(t)$ on $(-\alpha+\epsilon, \omega] \times M_\infty$. A diagonalization argument provides the converging subsequence on $(-\alpha, \omega] \times M_\infty$. The convergence being smooth, the limit metrics $g_\infty(t)$ satisfy the equation E_P. Finally, according to Lemma 2.9, we see that the balls at time $t \neq 0$ are precompact, hence the metrics $g_\infty(t)$ are complete. □

Remark 3.3. If we only assume a bound on the curvature, and no bound from below on the injectivity radius, we still have a weaker precompactness result. Indeed, as it was pointed out by J. Streets in [Str11], the precompactness results of D. Glickenstein [Gli03, Theorem 3] and J. Lott [Lot07, Theorem 1.4] only use the Ricci flow through the curvature estimates it brings, and thus extend to our flows. A sequence of solutions with such an assumption can collapse, and will subconverge to a metric space which is not necessarily a manifold.

2. Understanding singularities

In this section, we prove Theorems B, D and E.

2.1. Curvature blow-up and concentration of the curvature at a singular time. Theorem 2.7 implies that the curvature must blow up at a singular time, and that it must blow up at least at a certain rate: if in a neighbourhood of some time T, the curvature satisfies the following condition for some $0 < q < 1$:

$$\|Rm_g\|_{L^\infty} = O\left(\frac{1}{(T-t)^{\frac{q}{2}}}\right),$$

then for all $x \in M$ and sufficiently small r, the solution $g(T - r^4 + t)$ satisfy

$$\mathcal{N}^\infty_{x,r}(Rm_g) \leq \epsilon,$$

hence all the derivatives of the curvature of $g(t)$ are bounded on $[T - r^4/2, T]$. According to Lemma A.6, all the space and time derivatives of $g(t)$ are bounded with respect to a fixed metric. Hence the solution extend beyond T.

Consequently, if T is a singular time, then we can find sequences $t_k \to T$ and $x_k \in M$ such that

$$\left|Rm_{g(t_k)}\right| (x_k) \geq \frac{k}{\left(\sqrt{T - t_k}\right)^{1-\frac{1}{k}}}.$$

If the Yamabe constant remains uniformly positive along the flow, then the same is true if the curvature satisfies for some $\frac{n}{2} < p \leq \infty$ and $0 < q < 1 - \frac{n}{2p}$ the following condition:

$$\|Rm_g\|_{L^p} = O\left(\frac{1}{(T-t)^{\frac{q}{2}}}\right).$$

Moreover, according to Theorem C, there is a concentration phenomenon in the $L^{\frac{n}{2}}$ sense: there exist $\epsilon > 0$ and sequences of times $t_k \to T$ and $x_k \to x_\infty \in M$ such that

$$\int_{\mathrm{B}_{g(t_k)}\left(x_k, \frac{1}{k}\right)} \left|Rm_{g(t_k)}\right|^{\frac{n}{2}} dv_{g(t_k)} \geq \epsilon.$$

2.2. Blow-up at a singular time. If $g(t)$ is a solution of $E_P(g_0)$ on a compact manifold M and a time interval $[0, T)$, with $0 < T \leq \infty$, and if the curvature blows up at T, i.e. $\overline{\lim_{t\to T}} \left\|Rm_{g(t)}\right\|_{L^\infty} = \infty$, then we can choose a sequence $t_i \to T$ such that

$$\left\|Rm_{g(t_i)}\right\|_{L^\infty} = \sup_{t \leq t_i} \left\|Rm_{g(t)}\right\|_{L^\infty} \quad \text{and} \quad \left\|Rm_{g(t_i)}\right\|_{L^\infty} \to \infty.$$

Let define $\lambda_i = \left\|Rm_{g(t_i)}\right\|_{L^\infty}$ and the rescaled flows

$$g_i(t) = \lambda_i g\left(t_i + \frac{t}{\lambda_i^2}\right).$$

Then for all $i \in \mathbb{N}$, g_i is a solution of E_P on $\left[-\lambda_i^2 t_i, \lambda_i^2 (T - t_i)\right)$.

Let choose any $\alpha > 0$. For i big enough, g_i is a solution of E_P on $[-\alpha, 0]$. Moreover, it has been rescaled in such a way that its curvature is uniformly bounded by 1 on $M \times [-\alpha, 0]$, and that it satisfies $\left\|Rm_{g_i(0)}\right\|_{L^\infty} = 1$.

If the Yamabe constant of the initial flow is uniformly bounded from below by a positive constant:

$$\mathrm{Y}(M, [g_t]) \geq Y_0 > 0,$$

then as the Yamabe constant is scale invariant, the same is true for the rescaled flows $g_i(t)$. According to Proposition 1.4 and Lemma 1.3, the injectivity radii of $g_i(t)$ are uniformly bounded from below by a positive constant.

We take $x_i \in M$ such that $\left|Rm_{g(t_i)}(x_i)\right| = \lambda_i$, and we can apply the compactness theorem 3.2 to show that a subsequence of $(M, g_i(t), x_i)$ converges in the pointed C^∞ topology to a complete pointed solution $(M_\infty, g_\infty(t), x_\infty)$, $t \in [-\alpha, 0]$. By a diagonalization argument, $g_\infty(t)$ is actually defined on $(-\infty, 0]$, and if $T = \infty$, $g_\infty(t)$ is defined on $\mathbb{R}$. Moreover, since $\left|Rm_{g_\infty(0)}(x_\infty)\right| = 1$, the limit manifold $(M_\infty, g_\infty(0))$ is not flat.

If no singularity occurs, i.e. if the flow exists for all time with a uniform bound on the curvature, then according to Corollary 2.8, all the derivatives of the curvature are bounded on $M \times [0, \infty)$, and since the Yamabe constant is uniformly positive, the injectivity radius has a positive lower bound on $M \times [0, \infty)$. According Theorem 3.2 for any sequences $t_i \to \infty$ and $x_i \in M$, the sequence of pointed flows $(M, g_i(t), x_i)$ (where $g_i(t) = g(t_i + t)$) defined on $M \times (-t_i, \infty)$ has a subsequence converging to a limit flow $(M_\infty, g_\infty(t), x_\infty)$ defined on $M_\infty \times \mathbb{R}$.

2.3. Blow-up analysis for gradient flows in dimension four. For a closed Riemannian manifold (M, g_0), we consider $g(t)$, $t \in [0, T)$, the maximal solution of $E^\lambda(g_0)$.

If $\mathcal{F}^\lambda(g_0) < (1-\lambda)8\pi^2\chi(M)$ then the inequality remains true on $[0, T)$, since $\mathcal{F}^\lambda$ is decreasing along its negative gradient flow. If moreover $\mathrm{Y}(M, [g_0]) > 0$, then according to Lemma 2.2, $\mathrm{Y}(M, [g_t]) \geq Y_0$, with

$$Y_0 = \left(\frac{2}{3}\left((1-\lambda)8\pi^2\chi(M) - \mathcal{F}^\lambda(g_0)\right)\right)^{\frac{1}{2}} > 0.$$

If equality holds: $\mathcal{F}^\lambda(g_0) = (1-\lambda)8\pi^2\chi(M)$, since

$$\partial_t \mathcal{F}^\lambda(g_t) = -2\int_M \left|\nabla \mathcal{F}^\lambda(g_t)\right|^2 dv_{g_t},$$

either g_0 is a critical point of $\mathcal{F}^\lambda$, and the solution of $E^\lambda(g_0)$ is constant, or the inequality becomes immediately strict for $t > 0$ and we can replace g_0 by g_ε for a small ε.

For all t in $[0, T)$, we have

$$\int_0^t \left\|\nabla \mathcal{F}^\lambda(g_s)\right\|_{L^2}^2 ds = \mathcal{F}^\lambda(g_0) - \mathcal{F}^\lambda(g_t),$$

therefore

$$\int_0^T \left\|\nabla \mathcal{F}^\lambda(g_s)\right\|_{L^2}^2 ds \leq \mathcal{F}^\lambda(g_0) < \infty.$$

If g_i are rescaled flows:

$$g_i(t) = \lambda_i g\left(t_i + \frac{t}{\lambda_i^2}\right), \tag{3.1}$$

then by a change of variable, we obtain

$$\int_{-\alpha}^0 \left\|\nabla \mathcal{F}^\lambda(g_i(t))\right\|_{L^2}^2 dt = \int_{t_i - \frac{\alpha}{\lambda_i^2}}^{t_i} \left\|\nabla \mathcal{F}^\lambda(g(t))\right\|_{L^2}^2 dt,$$

and since $\left\|\nabla \mathcal{F}^\lambda(g(t))\right\|_{L^2}$ is in $L^2([0, T))$, if $\lambda_i \to \infty$ and $t_i \to T$, then

$$\int_{-\alpha}^0 \left\|\nabla \mathcal{F}^\lambda(g_i(t))\right\|_{L^2}^2 dt \to 0.$$

By Fatou's Lemma, this implies that the limit of rescaled flows at a singular time is critical for the functional $\mathcal{F}^\lambda$, and in particular is a constant flow.

Note that the volume remains constant along the flow:

$$\partial_t \operatorname{Vol}_g(M) = \int_M \frac{1}{2}\mathsf{tr}(P_g) dv_g = 0,$$

hence, the volume of rescaled flows satisfies $\operatorname{Vol}_{g_i}(M) = \lambda_i^{\frac{n}{2}} \operatorname{Vol}_g(M)$ and goes to infinity. Moreover, according to Lemma 1.4 there exists a positive constant $\kappa > 0$ such that for all $i \in \mathbb{N}$,

$$\inf_{x \in M} \operatorname{Vol}_{g_i}(\mathrm{B}(x, 1)) \geq \kappa,$$

hence the same is true for (M_∞, g_∞). According to Lemma 1.7, the volume of (M_∞, g_∞) is infinite. Therefore, the limit manifold cannot be compact, since it would be diffeomorphic to M by the definition of the pointed C^∞ topology, and of infinite volume.

Now, by taking the trace of $\nabla \mathcal{F}^\lambda(g_\infty(0)) = 0$, since $\mathsf{tr}(\nabla F^\lambda) = \frac{\lambda}{4}\Delta R$, we see that the scalar curvature of g_∞ is harmonic. Since it has a bounded L^2 norm by Fatou's Lemma, it has to be constant (see [Yau76, Theorem 3]), and as (M, g_∞) has infinite volume, the limit manifold is scalar-flat. Then $\nabla \mathcal{F}^\lambda(g_\infty(0)) = \nabla \mathcal{F}_W(g_\infty(0))$, so g_∞ is also Bach-flat.

On the other hand, if no singularity occurs, then for any sequence $t_i \to \infty$, let take $\lambda_i = 1$ in (3.1). Since $\left\|\nabla\mathcal{F}^\lambda(g(t))\right\|_{L^2}$ is in $L^2([0,\infty))$, we obtain

$$\int_{-\alpha}^{0} \left\|\nabla\mathcal{F}^\lambda(g_i(t))\right\|_{L^2}^2 \, dt \to 0.$$

Therefore, if we note $g_i(t) = g(t_i + t)$, the limit of the pointed flows $(M, g_i(t), x_i)$ is critical for the functional $\mathcal{F}^\lambda$.

Moreover, the limit manifold (M_∞, g_∞) has finite volume, thus is compact according to Lemma 1.7. This implies that M_∞ is diffeomorphic to M.

3. Rigidity results for critical metrics

In this section, we apply the results of Section 3 of Chapter 1 to prove pinching results for manifolds whose curvature satisfy an elliptic condition. In particular, we prove a rigidity result in dimension four for manifolds which are critical for a functional $\mathcal{F}^\lambda$ (Theorem F). We also obtain a number of pinching results for manifolds with harmonic curvature, that were already known in the compact case ([HV96, Gur00]).

Lemma 3.4. *If (M^n, g), $n \geq 3$, is a Riemannian manifold with constant scalar curvature, then*

$$\delta\mathsf{D}\overset{\circ}{Ric}_g = \nabla^*\nabla\overset{\circ}{Ric}_g + \frac{1}{n-1}R_g\overset{\circ}{Ric}_g - \overset{\circ}{\overline{\Big(W_g + \frac{n}{2}\mathcal{Z}_g\Big)}}(\overset{\circ}{Ric}_g) + \frac{1}{2}\left|\overset{\circ}{Ric}_g\right|^2 g.$$

Proof. According to Proposition B.2, we have

$$\begin{aligned}\delta\mathsf{D}\overset{\circ}{Ric}_g &= \nabla^*\nabla Ric_g + \frac{1}{2}\tilde{\mathsf{D}}\mathsf{D}R_g + Ric_g \circ \overset{\circ}{Ric}_g - \overset{\circ}{Rm}_g(\overset{\circ}{Ric}_g)\\ &= \nabla^*\nabla\overset{\circ}{Ric}_g + Ric_g \circ \overset{\circ}{Ric}_g - \overset{\circ}{Rm}_g(\overset{\circ}{Ric}_g),\end{aligned}$$

as we assumed R_g constant. Using the relation

$$\overset{\circ}{\overline{(g.u)}}\, v = \langle u \mid v\rangle\, g + (\mathsf{tr} v)\, u - u \circ v - v \circ u,$$

we easily get that

$$\overset{\circ}{Rm}(\overset{\circ}{Ric}_g) = \overset{\circ}{W}(\overset{\circ}{Ric}_g) - \frac{2}{n-2}\overset{\circ}{Ric}_g \circ \overset{\circ}{Ric}_g + \frac{1}{n-2}\left|\overset{\circ}{Ric}_g\right| g - \frac{1}{n(n-1)}R_g\overset{\circ}{Ric}_g,$$

and we obtain the result by writing

$$\overset{\circ}{\overline{(g.\overset{\circ}{Ric}_g)}}\,(\overset{\circ}{Ric}_g) = \left|\overset{\circ}{Ric}_g\right| g - 2\overset{\circ}{Ric}_g \circ \overset{\circ}{Ric}_g.$$

□

Corollary 3.5. *If (M^4, g) is a Riemannian manifold with constant scalar curvature, then*

$$\nabla\mathcal{F}^\lambda(g) = \frac{1}{2}\nabla^*\nabla\overset{\circ}{Ric}_g - (\overset{\circ}{\overline{W_g + \mathcal{Z}_g}})(\overset{\circ}{Ric}_g) + \frac{1}{4}\left|\overset{\circ}{Ric}_g\right|^2 g + \frac{2-\lambda}{12}R_g\overset{\circ}{Ric}_g.$$

Proof. We recall that $\mathcal{F}^\lambda = (1-\lambda)\mathcal{F}_W + \frac{\lambda}{2}\mathcal{F}_{\overset{\circ}{Ric}}$. According to the Gauss-Bonnet formula, we can also write $\mathcal{F}^\lambda = \mathcal{F}_W + \frac{\lambda}{24}\mathcal{F}_R - \lambda 8\pi^2\chi(M)$. Consequently (see Section 2 of Chapter 2),

$$\nabla\mathcal{F}^\lambda(g) = -\delta\tilde{\delta}W_g - \frac{1}{2}\overset{\circ}{W}_g(\overset{\circ}{Ric}_g) + \frac{\lambda}{12}\delta\mathsf{D}(R_g\, g) - \frac{\lambda}{12}R_g\overset{\circ}{Ric}_g,$$

and we obtain the formula since R_g is constant and $\tilde{\delta}W_g = -\frac{1}{2}\mathsf{D}A_g = -\frac{1}{2}\mathsf{D}\overset{\circ}{Ric}_g$ (Proposition B.1). □

We will use the following estimate, which is very close to Lemma 3.4 in [Hui85]:

Lemma 3.6. *On a Riemannian manifold (M^n, g), for $\epsilon \in \{+, -, \ \}$ and $\alpha \in \mathbb{R}$, we have*

$$\left|\left\langle W_g^\epsilon + \alpha g.\overset{\circ}{Ric}_g \mid \overset{\circ}{Ric}_g.\overset{\circ}{Ric}_g\right\rangle\right| \leq \sqrt{\frac{2(n-2)}{n-1}} \left|\overset{\circ}{Ric}_g\right|^2 \left(\left|W_g^\epsilon\right|^2 + \frac{2(n-2)}{n}\alpha^2 \left|\overset{\circ}{Ric}_g\right|^2\right)^{\frac{1}{2}}.$$

Proof. Let write the orthogonal decomposition

$$\overset{\circ}{Ric}_g.\overset{\circ}{Ric}_g = T + V + U,$$

where

$$\begin{aligned} U &= \frac{1}{2n(n-1)}\mathrm{tr}^2(\overset{\circ}{Ric}_g.\overset{\circ}{Ric}_g)g.g = -\frac{1}{n(n-1)} \left|\overset{\circ}{Ric}_g\right|^2 g.g \\ V &= \frac{1}{n-2}g.\left(\mathrm{tr}(\overset{\circ}{Ric}_g.\overset{\circ}{Ric}_g) - \frac{1}{n}\mathrm{tr}^2(\overset{\circ}{Ric}_g.\overset{\circ}{Ric}_g)g\right) \\ &= -\frac{2}{n-2}g.\left(\overset{\circ}{Ric}_g \circ \overset{\circ}{Ric}_g - \frac{1}{n}\left|\overset{\circ}{Ric}_g\right|^2 g\right). \end{aligned}$$

Then

$$\begin{aligned} \left|\left\langle W_g^\epsilon + \alpha\overset{\circ}{Ric}_g.g \mid \overset{\circ}{Ric}_g.\overset{\circ}{Ric}_g\right\rangle\right|^2 &= \left|\left\langle W_g^\epsilon + \alpha g.\overset{\circ}{Ric}_g \mid T + V\right\rangle\right|^2 \\ &= \left|\left\langle W_g^\epsilon + \alpha\sqrt{\frac{2}{n}}g.\overset{\circ}{Ric}_g \mid T + \sqrt{\frac{n}{2}}V\right\rangle\right|^2 \\ &\leq \left|W_g^\epsilon + \alpha\sqrt{\frac{2}{n}}g.\overset{\circ}{Ric}_g\right|^2 \left|T + \sqrt{\frac{n}{2}}V\right|^2 \\ &= \left(\left|W_g^\epsilon\right|^2 + \frac{2(n-2)\alpha^2}{n}\left|\overset{\circ}{Ric}_g\right|^2\right)\left(|T|^2 + \frac{n}{2}|V|^2\right). \end{aligned}$$

Using the fact that $|u.v|^2 = |u|^2|v|^2 + \langle u \mid v\rangle^2 - 2\langle u \circ u \mid v \circ v\rangle$, we obtain

$$\begin{aligned} \left|\overset{\circ}{Ric}_g.\overset{\circ}{Ric}_g\right|^2 &= 2\left|\overset{\circ}{Ric}_g\right|^4 - 2\left|\overset{\circ}{Ric}_g \circ \overset{\circ}{Ric}_g\right|^2, \\ |U|^2 &= \frac{2}{n(n-1)}\left|\overset{\circ}{Ric}_g\right|^4, \\ |V|^2 &= \frac{4}{n-2}\left(\left|\overset{\circ}{Ric}_g \circ \overset{\circ}{Ric}_g\right|^2 - \frac{1}{n}\left|\overset{\circ}{Ric}_g\right|^4\right). \end{aligned}$$

Therefore

$$|T|^2 + \frac{n}{2}|V|^2 = \left|\overset{\circ}{Ric}_g.\overset{\circ}{Ric}_g\right|^2 + \frac{n-2}{2}|V|^2 - |U|^2 = \frac{2(n-2)}{n-1}\left|\overset{\circ}{Ric}_g\right|^4.$$

□

If a four-dimensional manifold satisfies $\delta W_g^+ = 0$, it satisfies the following Weitzenböck formula:

Lemma 3.7 (Derdziński ([Der83])). *Let (M^4, g) be a complete oriented Riemannian manifold with $\delta W^+ = 0$. Then*

$$\left\langle \nabla^*\nabla W_g^+ \mid W_g^+\right\rangle + \frac{1}{2}R_g\left|W_g^+\right|^2 \leq \sqrt{6}\left|W_g^+\right|^3,$$

with equality if and only if the spectrum of W^+ is $\{-\nu, -\nu, 2\nu\}$.

Proof. We have the Weitzenböck formula (see [Bes87], 16.73):

$$-\frac{1}{2}\Delta \left|W_g^+\right|^2 = \left|\nabla W_g^+\right|^2 + \frac{1}{2}R_g \left|W_g^+\right|^2 - 18\det W_g^+.$$

And since the maximum of $\lambda_1\lambda_2\lambda_3$ under the constraints

$$\begin{cases} \lambda_1+\lambda_2+\lambda_3 = 0 \\ \lambda_1^2+\lambda_2^2+\lambda_3^2 = 1 \end{cases}$$

is obtained for $-\lambda_1 = -\lambda_2 = 2\lambda_3 = \frac{1}{\sqrt{6}}$ and is equal to $\frac{1}{3\sqrt{6}}$, we get

$$\langle \nabla^*\nabla W_g^+ \mid W_g^+\rangle = \frac{1}{2}\Delta \left|W_g^+\right|^2 + \left|\nabla W_g^+\right|^2 \leq -\frac{1}{2}R_g \left|W_g^+\right|^2 + \sqrt{6}\left|W_g^+\right|^3$$

□

We can hence apply Theorem 1.10 or Theorem 1.11 to W_g^+ and obtain:

Proposition 3.8. *Let (M^4, g) be a complete oriented Riemannian manifold with constant scalar curvature and $\delta W^+ = 0$. If*

$$\left\|W_g^+\right\|_{L^2}^2 < \frac{1}{24}\,\mathrm{Y}(M,[g])^2,$$

then $W_g^+ = 0$.

Remark 3.9. In the compact case, M. Gursky proved a slightly better result in [Gur00] by using the same technique, but with a modified Yamabe constant.

Proof. We have the refined Kato inequality (see [GL99]):

$$\frac{5}{3}\left|\nabla\left|W^+\right|\right|^2 \leq \left|\nabla W^+\right|^2$$

According to (2.1)

$$4\tilde{\delta}\delta W^+ = 2\tilde{\delta}\delta W + \overset{\circ}{W_g^+}(\overset{\circ}{Ric_g}) - \overset{\circ}{W_g^-}(\overset{\circ}{Ric_g}),$$

and if the scalar curvature is constant, then since $\delta W = -\frac{1}{2}\tilde{\mathsf{D}}A_g = -\frac{1}{2}\tilde{\mathsf{D}}Ric_g$ (Proposition B.1) and $\tilde{\delta}\tilde{\mathsf{D}}Ric_g = \delta\mathsf{D}Ric_g = \nabla^*\nabla Ric + Rm * Ric$ (Proposition B.2), we obtain

$$\nabla^*\nabla Ric + Rm * Ric = 0.$$

The result then comes from Theorem 1.10 and Theorem 1.11 with $\delta = \frac{2}{3}$, $\lambda = \frac{1}{2}$ and $a = \sqrt{6}\,|W^+|$. □

Using Lemma 3.4 and Lemma 3.6, we can apply the same technique to $\overset{\circ}{Ric_g}$ and obtain:

Proposition 3.10. *Let (M^4, g) be a complete oriented Riemannian manifold with constant scalar curvature and $\delta W^+ = 0$. If*

$$\frac{1}{2}\left\|W_g^+\right\|_{L^2}^2 + \left\|\mathcal{Z}_g\right\|_{L^2}^2 < \frac{1}{24}\,\mathrm{Y}(M,[g]),$$

then $\overset{\circ}{Ric_g} = 0$.

Proof. Since $\delta W_g^+ = 0$ and the scalar curvature is constant, we have the following refined Kato inequality (see [TV05, Lemma 5.1]):

$$\frac{3}{2}\left|\nabla\left|\overset{\circ}{Ric_g}\right|\right|^2 \leq \left|\nabla\overset{\circ}{Ric_g}\right|^2.$$

And we have

$$0 = 4\tilde{\delta}\delta W^+ = 2\tilde{\delta}\delta W + \overset{\circ}{W_g^+}(\overset{\circ}{Ric_g}) - \overset{\circ}{W_g^-}(\overset{\circ}{Ric_g}),$$

hence, writing that $\tilde{\delta}\delta W_g = \tilde{\delta}\tilde{\mathsf{D}} Ric_g = \delta \mathsf{D} Ric_g$ and with Lemma 3.4,

$$\begin{aligned} 0 &= \delta \mathsf{D} Ric_g - \overset{\circ}{W_g^+}(\overset{\circ}{Ric_g}) + \overset{\circ}{W_g^-}(\overset{\circ}{Ric_g}) \\ &= \nabla^*\nabla \overset{\circ}{Ric_g} + \frac{1}{3} R_g \overset{\circ}{Ric_g} - \overline{\overset{\circ}{\left(W_g^+ + g.\overset{\circ}{Ric_g}\right)}}(\overset{\circ}{Ric_g}) + \frac{1}{2}\left|\overset{\circ}{Ric_g}\right|^2 g. \end{aligned}$$

Finally, according to Lemma 3.6, and since $\langle \overset{\circ}{T} u \mid v\rangle = \langle T \mid u.v\rangle$, we have

$$\left\langle \nabla^*\nabla \overset{\circ}{Ric_g} \mid \overset{\circ}{Ric_g} \right\rangle + \frac{1}{3} R_g \left|\overset{\circ}{Ric_g}\right|^2 \leq \frac{2}{\sqrt{3}} \left(\left|W_g^+\right|^2 + \left|\overset{\circ}{Ric_g}\right|^2 \right)^{\frac{1}{2}} \left|\overset{\circ}{Ric_g}\right|^2 .$$

The result follows by taking $\delta = \frac{1}{2}$ and $\lambda = \frac{1}{3}$. □

By combining the two last propositions, and taking a two sheeted covering if (M, g) is not orientable, we obtain:

Proposition 3.11. *Let (M^4, g) be a complete Riemannian manifold with harmonic curvature. If*

$$\|W_g\|_{L^2}^2 + \left\|\mathcal{Z}_g\right\|_{L^2}^2 < \frac{1}{24} \mathrm{Y}(M, [g])^2,$$

then (M^4, g) is of constant nonnegative sectional curvature.

In dimension $n \geq 5$, we recover the following result ([HV96, Theorem 2]):

Proposition 3.12. *Let (M^n, g), $n \geq 5$, be a closed Riemannian manifold with harmonic curvature. If*

$$\|W_g\|_{L^{\frac{n}{2}}}^2 + \frac{n}{2} \left\|\mathcal{Z}_g\right\|_{L^{\frac{n}{2}}}^2 < \frac{1}{2(n-1)(n-2)} \mathrm{Y}(M, [g])^2,$$

then $\overset{\circ}{Ric_g} = 0$.

If equality is attained, then the metric is a Yamabe minimizer and the Ricci curvature is parallel.

Proof. The Weitzenböck formula comes from Lemma 3.4 and Lemma 3.6, and we can apply the refined Kato inequality for Codazzi tensors [HV96, Lemma] and take $\delta = \frac{2}{n}$. □

For non-compact manifolds, by taking $\delta = 0$ and $\lambda = \frac{n-2}{4(n-1)}$, we obtain:

Proposition 3.13. *Let (M^n, g), $n \geq 5$, be a complete Riemannian manifold with harmonic curvature and zero scalar curvature. If*

$$\|W_g\|_{L^{\frac{n}{2}}}^2 + \frac{n}{2} \|\mathcal{Z}_g\|_{L^{\frac{n}{2}}}^2 < \frac{1}{2(n-1)(n-2)} \mathrm{Y}(M, [g])^2,$$

then $\overset{\circ}{Ric_g} = 0$.

We finally prove:

Theorem F. *Let (M^4, g) be a complete Riemannian manifold with positive Yamabe constant and let λ be in $[0, 1]$. Suppose that R_g is in $L^2(M)$. If $\lambda = 0$, suppose that R_g is constant.*

If g is a critical metric of $\mathcal{F}^\lambda$ with

$$\|W_g\|_{L^2}^2 + \frac{1}{2} \|\mathcal{Z}_g\|_{L^2}^2 < \frac{1}{8 \times 24} \mathrm{Y}(M, [g])^2,$$

then g is of constant sectional curvature.

We know that according to the Gauss-Bonnet formula, $\nabla \mathcal{F}^\lambda = \nabla \mathcal{F}_W + \frac{\lambda}{24} \nabla \mathcal{F}_R$. Consequently,

$$\begin{aligned}
\mathsf{tr}(\nabla F^\lambda) &= \frac{\lambda}{24} \mathsf{tr}(\nabla \mathcal{F}_R) \\
&= \frac{\lambda}{12} \mathsf{tr}(\delta \mathsf{D}(R_g g)) \quad \text{(see Section 2 of Chapter 2)} \\
&= \frac{\lambda}{12} \mathsf{tr}(\Delta R_g g + \tilde{\mathsf{D}} \mathsf{D} R_g) \quad \text{(according to Proposition B.2)} \\
&= \frac{\lambda}{4} \Delta R_g
\end{aligned}$$

Hence, if $\lambda \neq 0$ and if g is a critical point of $\mathcal{F}^\lambda$, then R_g is harmonic. If M is compact, then R_g is a positive constant (since $\mathrm{Y}(M, [g]) > 0$). If M is not compact, since R_g is harmonic and in $L^2(M)$, it is also constant (see [Yau76], Theorem 3). As $\mathrm{Y}(M, [g]) > 0$, it is nonnegative.

According to Corollary 3.5,

$$\nabla \mathcal{F}^\lambda(g) = \frac{1}{2} \nabla^* \nabla \overset{\circ}{Ric}_g - (\overset{\circ}{\overline{W_g + \mathscr{Z}_g}})(\overset{\circ}{Ric}_g) + \frac{1}{4} \left| \overset{\circ}{Ric}_g \right|^2 g + \frac{2 - \lambda}{12} R_g \overset{\circ}{Ric}_g,$$

and since $\langle \overset{\circ}{T} u \mid v \rangle = \langle T \mid u.v \rangle$, we have

$$\begin{aligned}
0 =& \langle \nabla \mathcal{F}^\lambda(g) \mid \overset{\circ}{Ric}_g \rangle \\
=& \frac{1}{2} \langle \nabla^* \nabla \overset{\circ}{Ric}_g \mid \overset{\circ}{Ric}_g \rangle - \langle W_g + \frac{1}{2} \overset{\circ}{Ric}_g . g \mid \overset{\circ}{Ric}_g . \overset{\circ}{Ric}_g \rangle + \frac{2 - \lambda}{12} R_g \left| \overset{\circ}{Ric}_g \right|^2,
\end{aligned}$$

hence

$$\begin{aligned}
\langle \nabla^* \nabla \overset{\circ}{Ric}_g \mid \overset{\circ}{Ric}_g \rangle + \frac{1}{6} R_g \left| \overset{\circ}{Ric}_g \right|^2 &\leq 2 \langle W_g + \frac{1}{2} \overset{\circ}{Ric}_g . g \mid \overset{\circ}{Ric}_g . \overset{\circ}{Ric}_g \rangle \\
&\leq \frac{4}{\sqrt{3}} \left(\left| W_g^+ \right|^2 + \frac{1}{4} \left| \overset{\circ}{Ric}_g \right|^2 \right)^{\frac{1}{2}} \left| \overset{\circ}{Ric}_g \right|^2 .
\end{aligned}$$

Therefore, with $\delta = 0$ and $\lambda' = \frac{1}{6}$ in Theorems 1.10 and 1.11, we obtain that $\overset{\circ}{Ric}_g = 0$. Then the curvature is harmonic, thus of constant nonnegative sectional curvature according to Proposition 3.11.

4. Integral pinching results

In this section, we use the rigidity result for critical manifolds (Theorem F) to prove:

Theorem G. *Let λ be in $(0, 1)$. If (M^4, g_0) is a closed Riemannian manifold with positive Yamabe constant such that*

$$\begin{cases} \lambda \leq \dfrac{4}{13} \\ \mathcal{F}^\lambda(g_0) < 2\lambda\pi^2\chi(M) \end{cases} \quad \text{or} \quad \begin{cases} \lambda \geq \dfrac{4}{13} \\ \mathcal{F}^\lambda(g_0) < \dfrac{8}{9}(1 - \lambda)\pi^2\chi(M), \end{cases}$$

then the solution of $E^\lambda(g_0)$ exists for all time and converges in the C^∞ topology to a metric of constant positive curvature. In particular, M^4 is diffeomorphic to the sphere $\mathbb{S}^4$ or the real projective space $\mathbb{R}\mathbf{P}^4$.

We begin with the following lemma:

Lemma 3.14. *If (M_i, g_i, x_i) converges to $(M_\infty, g_\infty, x_\infty)$ in the pointed C^∞ topology, then*

$$\mathrm{Y}(M, [g_\infty]) \geq \overline{\lim_{i \to \infty}} \, \mathrm{Y}(M, [g_i]).$$

Proof. There exists diffeomorphisms $\phi_i : U_i \subset M_\infty \to V_i \subset M_i$, with $\{U_i\}$ an exhaustion of M_∞, such that $\phi_i^* g_i$ converges to g_∞.

Let u be in $C_0^\infty(M_\infty)$ with $\int_M |u|^{\frac{2n}{n-2}} dv_{g_\infty} = 1$. Since it has compact support, $\mathrm{supp}(u) \subset U_i$ for i big enough. Let define $u_i \in H_1^2(M)$ by $u_i = u \circ \phi_i^{-1}$ on V_i and 0 outside V_i.

Then,

$$\begin{aligned} &\int_{M_\infty} \frac{4(n-1)}{n-2} |du|^2_{\phi^* g_i} + R_{\phi^* g_i} u^2 dv_{\phi^* g_i} \\ &= \int_M \frac{4(n-1)}{n-2} |du_i|^2_{g_i} + R_{g_i} u_i^2 dv_{g_i} \\ &\geq \mathrm{Y}(M,[g_i]) \Big(\int_M |u_i|^{\frac{2n}{n-2}} dv_{g_i} \Big)^{\frac{n-2}{n}} \\ &= \mathrm{Y}(M,[g_i]) \Big(\int_M |u|^{\frac{2n}{n-2}} dv_{\phi^* g_i} \Big)^{\frac{n-2}{n}}, \end{aligned}$$

therefore

$$\int_{M_\infty} \frac{4(n-1)}{n-2} |du|^2_{g_\infty} + R_{g_\infty} u^2 dv_{g_\infty} \geq \overline{\lim_{i\to\infty}}\, \mathrm{Y}(M,[g_i]),$$

and since $C_0^\infty(M_\infty)$ is dense in $H_1^2(M_\infty, g_\infty)$, it follows that

$$\mathrm{Y}(M,[g_\infty]) \geq \overline{\lim_{i\to\infty}}\, \mathrm{Y}(M,[g_i]).$$

□

Since $\mathcal{F}^\lambda(g)$ is decreasing along its gradient flow, if the initial metric satisfies the hypotheses of Theorem G, then all the manifolds (M, g_t) satisfy the bound

$$\mathcal{F}^\lambda(g_t) \leq \mathcal{F}^\lambda(g_0) \leq 2\lambda(\pi^2\chi(M) - \epsilon) \qquad \text{if } \lambda \leq \frac{4}{13}$$

or

$$\mathcal{F}^\lambda(g_t) \leq \mathcal{F}^\lambda(g_0) \leq \frac{8}{9}\lambda(\pi^2\chi(M) - \epsilon) \qquad \text{if } \lambda \geq \frac{4}{13}$$

for some $\epsilon > 0$.

Then, according to Proposition 2.3, all the manifolds (M, g_t) satisfy the inequality

$$\mathcal{F}_W(g_t) + \frac{1}{4}\mathcal{F}_{\overset{\circ}{Ric}}(g_t) \leq \frac{1}{8\times 24}\mathrm{Y}(M,[g_t])^2 - \epsilon,$$

and since $\mathcal{F}_W$, $\mathcal{F}_{\overset{\circ}{Ric}}$ and Y are scale invariant, if T is a singular time, then the rescaled manifolds (M, g_i) satisfy the same inequality.

Consequently, according to Lemma 3.14, the limit manifold (M_∞, g_∞) satisfy

$$\begin{aligned} \mathcal{F}_W(g_\infty) + \frac{1}{4}\mathcal{F}_{\overset{\circ}{Ric}}(g_\infty) &\leq \underline{\lim_{i\to\infty}} \left(\mathcal{F}_W(g_i) + \frac{1}{4}\mathcal{F}_{\overset{\circ}{Ric}}(g_i) \right) \\ &\leq \overline{\lim_{i\to\infty}} \left(\frac{1}{8\times 24}\mathrm{Y}(M_i,[g_i])^2 - \epsilon \right) \\ &\leq \frac{1}{8\times 24}\mathrm{Y}(M,[g\infty])^2 - \epsilon. \end{aligned}$$

But (M_∞, g_∞) is non-flat, Bach-flat, scalar-flat, and satisfies

$$\mathrm{Y}(M,[g_\infty]) \geq \overline{\lim_{i\to\infty}}\, \mathrm{Y}(M,[g_i]) \geq Y_0 > 0,$$

and

$$\int_{M_\infty} R^2_{g_\infty} dv_{g_\infty} \leq 24(8\pi^2\chi(M) + \frac{1}{\lambda}\mathcal{F}^\lambda(g_0)),$$

so Theorem F with $\lambda = 0$ asserts that it is flat, a contradiction.

Therefore, the flow exists for all time, and for all sequences t_i, a subsequence of (M, g_{t_i}) converges to (M, g_∞), with g_∞ a critical metric for $\mathcal{F}^\lambda$ such that

$$\mathcal{F}_W(g_\infty) + \frac{1}{4}\mathcal{F}_{\overset{\circ}{Ric}}(g_\infty) \leq \frac{1}{8 \times 24}\,\mathrm{Y}(M_\infty, [g_\infty])^2 - \epsilon,$$

and

$$\mathrm{Y}(M, [g_\infty]) \geq \overline{\lim_{i\to\infty}}\,\mathrm{Y}(M, [g_i]) \geq Y_0 > 0.$$

According to Theorem F, g_∞ is a metric of positive constant curvature (since its Yamabe constant is positive). Let (N, h) be the the sphere of volume $\mathrm{Vol}_{g_0}(M)$ if $\chi(M) = 2$, or the real projective space of volume $\mathrm{Vol}_{g_0}(M)$ if $\chi(M) = 1$, endowed with its standard metric. Then (M, g_∞) is isometric to (N, h).

We have proven that for any sequence t_i, a subsequence of (M, g_{t_i}) converges to (N, h) in the C^∞ topology. Hence (M, g_t) converges to (N, h) as $t \to \infty$.

CHAPTER 4

The Bochner technique for harmonic forms

Dans ce chapitre, on applique la méthode de Bochner intégrale aux formes différentielles harmoniques des variétés riemanniennes. On en déduit plusieurs théorèmes de type Bochner-Weitzenböck, et on caractérise les cas d'égalité.

1. The Bochner-Weitzenböck formula

Let (M^n, g) be a Riemannian manifold, and ξ be a smooth k-form. If ξ is harmonic, i.e. closed and co-closed:

$$d\xi = d^*\xi = 0,$$

then it satisfies the Bochner-Weitzenböck formula

$$\nabla^*\nabla\xi + \mathcal{R}_k\xi = 0, \tag{4.1}$$

where the Bochner-Weitzenböck curvature

$$\mathcal{R}_k(x) \colon \Lambda^k T_x^* M \to \Lambda^k T_x^* M$$

is a symmetric operator which can be expressed by using the curvature operator. The trace of $\mathcal{R}_k$ is given by

$$\mathrm{tr}\,(\mathcal{R}_k) = \left(\dim \Lambda^k T_x^* M\right) \frac{k(n-k)}{n(n-1)} R_g.$$

We let $-r_k$ be the lowest eigenvalue of the traceless part of the Bochner-Weitzenböck curvature. Then, since the nonnegativity of $\mathcal{R}_k$ is equivalent to

$$r_k \leq \frac{k(n-k)}{n(n-1)} R_g,$$

the classical Bochner-Weitzenböck theorem can be stated as follows:

Theorem 4.1. *Let (M^n, g), $n \geq 2$, be a closed Riemannian manifold. If*

$$r_k \leq \frac{k(n-k)}{n(n-1)} R_g, \tag{4.2}$$

then

- *either its k^{th} Betti number $b_k(M^n)$ vanishes,*
- *or equality holds in (4.2), $1 \leq b_k \leq \binom{n}{k}$ and every harmonic k-form is parallel.*

We will prove the following integral version of Theorem 4.1:

Theorem 4.2. *If (M^n, g), $n \geq 4$, is a closed Riemannian manifold such that for some integer $1 \leq k \leq \frac{n-3}{2}$ or $k = \frac{n}{2}$ the following pinching holds:*

$$\|r_k\|_{L^{\frac{n}{2}}} \leq \frac{k(n-k)}{n(n-1)} \, \mathrm{Y}(M, [g]), \tag{4.3}$$

then

- *either its k^{th} Betti number $b_k(M^n)$ vanishes,*
- *or equality holds in* (4.3) *and (up to a conformal change in the case $k = \frac{n}{2}$) the pointwise equality $r_k = \frac{k(n-k)}{n(n-1)} R_g$ holds, $1 \leq b_k \leq \binom{n}{k}$, every harmonic k-form is parallel and g is a Yamabe minimizer.*

According to [GM75], for all $1 \leq k \leq n-1$, we have $r_k \leq k(n-k)\rho_g$, thus Theorem I is a direct consequence of this result.

In dimension four, if we let w_g^+ be the largest eigenvalue of the self-dual part W_g^+ of the Weyl curvature and b_2^+ be the dimension of the self-dual harmonic 2-forms, we obtain the following result, which was already obtained by M. Gursky in [Gur00]:

Theorem 4.3. *If (M^4, g) is a compact oriented Riemannian manifold such that*

$$\left\|w_g^+\right\|_{L^2} \leq \frac{1}{6} \, \mathrm{Y}(M^4, [g]), \tag{4.4}$$

then

– *either* $b_2^+(M^4) = 0$*,*
– *or equality holds in (4.4),* $1 \leq b_2^+ \leq 3$ *and for every self-dual harmonic 2-form* ω*, there is a Yamabe minimizer* $\tilde{g}$ *in* $[g]$ *such that* ω *is Kähler for* $\tilde{g}$*.*

Conversely, according to [Der83], for any metric conformally equivalent to one which is Yamabe and Kähler, equality holds in (4.4).

1.1. Examples of manifolds for which equality holds in (4.3). Equality holds in (4.3) for any metric with nonnegative $\mathcal{R}_k$ which is a positive Yamabe minimizer, as soon as $b_k \geq 1$. According to [GM75], we can construct examples of manifolds with nonnegative $\mathcal{R}_k$ by taking products of manifolds with nonnegative curvature operators. According to [BE87, IV.2], if the product is an Einstein manifold, it will be a Yamabe minimizer.

Let (M^n, g) be a product of round spheres and projective spaces

$$(\mathbb{S}^{n_1}, g_1) \times \cdots \times (\mathbb{S}^{n_p}, g_p) \times (\mathbb{CP}^{m_1}, h_1) \times \cdots \times (\mathbb{CP}^{m_q}, h_q),$$

with $n_i \geq 2$. Then (M, g) has a nonnegative curvature operator. For (M, g) to be Einstein, we have to take $R_{g_i} = \alpha \frac{n_i}{n}$ and $R_{h_i} = \alpha \frac{2m_i}{n}$ for some $\alpha > 0$.

If for some $0 \leq p' \leq p$ and $0 \leq m'_j \leq m_j$

$$\sum_{i=1}^{p'} n_i + 2 \sum_{j=1}^{q} m'_j = k,$$

then $b_k \geq 1$ and equality holds in (4.3). Hence, for all $k \geq 2$, there exist manifolds for which equality holds in (4.3).

For $k = 1$, according to [Sch89], the quotients of $\mathbb{S}^{n-1} \times \mathbb{R}$ by a group of transformations generated by isometries of S^{n-1} and a translation of parameter $T > 0$ are Yamabe minimizing if and only if $T^2 \leq \frac{4\pi^2}{n-2}$. For those manifolds, equality holds in (4.3) and in (0.18).

1.2. The integral Bochner-Weitzenböck Theorem. We will now prove Theorem 4.2 and Theorem 4.3. Let (M^n, g) be a Riemannian manifold. From (4.1), we get that any harmonic k-form ξ satisfies

$$\langle \nabla^* \nabla \xi \mid \xi \rangle + \frac{k(n-k)}{n(n-1)} R_g \, |\xi|^2 \leq r_k \, |\xi|^2 \, . \tag{4.5}$$

Moreover, for $k \in [0, n/2]$, the Kato inequality can be refined as

$$\frac{n+1-k}{n-k} \, |d|\xi||^2 \leq |\nabla \xi|^2 \, , \tag{4.6}$$

(see [Bou90], and [Bra00, CGH00] for the computation of the refined Kato constant).

We take $\delta = \frac{1}{n-k}$, $\lambda = \frac{k(n-k)}{n(n-1)}$ and $\beta = \frac{4k(n-1-k)}{n(n-2)}$, and we have

$$\begin{cases} 0 < \beta < 1 & \text{when} \quad 1 \leq k \leq \frac{n-3}{2}, \\ \beta = 1 & \text{when} \quad k = \frac{n-2}{2} \text{ or } k = \frac{n}{2}, \\ \beta > 1 & \text{when} \quad k = \frac{n-1}{2}. \end{cases}$$

hence if $k \neq \frac{n-1}{2}$, we can apply Theorem 1.9 or Theorem 1.10 and obtain that if ξ is a non-trivial harmonic k-form, then

$$\|r_k\|_{L^{n/2}} \geq \frac{k(n-k)}{n(n-1)} \, \mathrm{Y}(M, [g]). \tag{4.7}$$

If furthermore equality holds and $1 \leq k \leq \frac{n-3}{2}$, then g is a Yamabe minimizer and

$$r_k = \frac{k(n-k)}{n(n-1)} R_g.$$

By Theorem 4.1, every harmonic k-form is parallel and $b_k \leq \binom{n}{k}$.

If equality holds and $k = \frac{n}{2}$, then $\tilde{g} = |\xi|^{\frac{4}{n}} \, g$ is a Yamabe minimizer. Moreover, the form ξ is still harmonic for $\tilde{g}$ but has constant $\tilde{g}-$length

$$|\xi|_{\tilde{g}} = 1.$$

And since the traceless Bochner-Weitzenböck curvature $\mathcal{W}_{\frac{n}{2}}$ only depends on the Weyl curvature (see (B.1)), the pinching is conformally invariant and equality also holds for $\tilde{g}$.

Then, since equality must hold in the Kato inequality, ξ must be parallel. According to (4.5), the pointwise inequality $r_{n/2}(\tilde{g}) \leq \frac{n}{4(n-1)} \, \mathrm{Y}(M,[\tilde{g}])$ holds on M, and as

$$\left\| r_{n/2}(\tilde{g}) \right\|_{L^{n/2}} = \frac{n}{4(n-1)} \, \mathrm{Y}(M,[\tilde{g}]),$$

the equality $r_{n/2}(\tilde{g}) = \frac{n}{4(n-1)} R_{\tilde{g}}$ holds on M. By Theorem 4.1, every $\tilde{g}$-harmonic $n/2$-form is $\tilde{g}$-parallel and $b_{n/2} \leq \binom{n}{n/2}$.

For the middle degree $n/2$ when $n/2$ is even, the Hodge star operator $*$ induces a parallel decomposition $\Lambda^{\frac{n}{2}} T^*M = \Lambda_+^{\frac{n}{2}} T^*M \oplus \Lambda_-^{\frac{n}{2}} T^*M$. And since the traceless Bochner-Weitzenböck curvature $\mathcal{W}_{\frac{n}{2}}$ commutes with $*$, it admits a decomposition

$$\mathcal{W}_{\frac{n}{2}} = \mathcal{W}_{\frac{n}{2}}^+ \oplus \mathcal{W}_{\frac{n}{2}}^-.$$

If ξ is a non-trivial harmonic self-dual form, (i.e. $*\xi = \xi$), and if $-r_{n/2}^+$ is the lowest eigenvalue of $\mathcal{W}_{n/2}^+$, we get

$$\langle \nabla^* \nabla \xi \mid \xi \rangle + \frac{n}{4(n-1)} R_g \, |\xi|^2 \leq r_{n/2}^+ \, |\xi|^2 \, .$$

Hence Theorem 1.10 yields

$$\left\| r_{n/2}^+ \right\|_{L^{n/2}} \geq \frac{n}{4(n-1)} \, \mathrm{Y}(M,[g]).$$

In dimension 4, this inequality becomes

$$\left\| w^+ \right\|_{L^2} = \frac{1}{2} \, \left\| r_2^+ \right\|_{L^2} \geq \frac{1}{6} \, \mathrm{Y}(M,[g]), \tag{4.8}$$

and if equality holds, then there is a Yamabe minimizer $\tilde{g} \in [g]$ such that ξ is $\tilde{g}$-parallel with $|\xi|_{\tilde{g}}^2 = 2$. Consequently, ξ is a Kähler form on $(M,\tilde{g})$.

2. Pinching involving the norm of the curvature

In this section, we prove theorems 0.4, J, K and L.

2.1. Comparison between the first eigenvalue and the norm of curvature operators.

In order to obtain estimates on r_k, we will use the following lemma:

Lemma 4.4. *If $A : E \to E$ is a traceless self-adjoint endomorphism on a Euclidean space E of dimension d, then its lowest eigenvalue a satisfies*

$$a^2 \leq \frac{d-1}{d} \, |A|^2,$$

and equality holds if and only if the spectrum of A is $\{-\nu, \frac{1}{d-1}\nu\}$ with $\nu \geq 0$ and $\frac{1}{d-1}\nu$ of multiplicity $d-1$.

Proof. By a simple Lagrange multiplier argument, we see that

$$\left(\inf \Big\{ \lambda_1 \, , \, \sum_{i=1}^{d} \lambda_i^2 = 1 \ \text{ and } \ \sum_{i=1}^{d} \lambda_i = 0 \Big\} \right)^2 = \frac{d-1}{d}$$

□

For $1 \leq k \leq \frac{n-1}{2}$, let define the constants $a_{n,k}$ and $b_{n,k}$ by

$$a_{n,k} = \left(\binom{n}{k} - 1\right) \frac{k(n-k)}{n(n-1)} \frac{4(k-1)(n-k-1)}{(n-2)(n-3)},$$
$$b_{n,k} = \left(\binom{n}{k} - 1\right) \frac{k(n-k)}{n(n-1)} \frac{(n-2k)^2}{(n-2)^2}.$$

Lemma 4.5. *If $1 \leq k \leq \frac{n-1}{2}$, then*

$$r_k^2 \leq a_{n,k} \left|W\right|^2 + b_{n,k} \left|\overset{\circ}{Ric}\right|^2$$

and equality holds if and only if there exists a k-form u and a real number λ such that

$$\mathcal{R}_k = \lambda \operatorname{Id} - u \otimes u.$$

Proof. We apply Lemma 4.4 to $\mathcal{W}_k + \mathcal{Z}_k$, and use the fact that for a traceless operator T on k-forms

$$\left|\frac{g^j}{j!} . T\right| = \frac{1}{j!} \langle c^j \frac{g^j}{j!} . T | T \rangle = \binom{n-2k}{j} \left|T\right|^2 .$$

□

When $k = n/2$ we can refine this inequality by using the fact that the Hodge star operator commutes with $\mathcal{R}_{n/2}$ and the fact that the square of the Hodge star operator on $n/2$-forms is $(-1)^{n/2} \operatorname{Id}$.

Let $\Lambda_{\pm}^{n/2} T_x^* M$ be the eigenspaces of the Hodge star operator and $\mathcal{R}_{\pm,n/2}$ be the restriction of the Bochner-Weitzenböck curvature to $\Lambda_{\pm}^{n/2} T_x^* M$.

We define

$$a_{n,n/2} = \begin{cases} \frac{n(n-2)}{4(n-1)(n-3)} \left(\binom{n}{n/2} - 2\right) & \text{if } n/2 \text{ is even} \\ \frac{n(n-2)}{8(n-1)(n-3)} \left(\binom{n}{n/2} - 2\right) & \text{if } n/2 \text{ is odd.} \end{cases}$$

Lemma 4.6.

$$r_{n/2}^2 \leq a_{n,n/2} \left|W\right|^2 \tag{4.9}$$

and equality holds if and only if

- *when $n/2$ is odd: there exists a $n/2$-form u and a real number λ such that*

$$\mathcal{R}_{n/2} = \lambda \operatorname{Id} - u \otimes u - *u \otimes *u,$$

- *when $n/2$ is even: there is $\varepsilon \in \{-,+\}$ such that $W^{-\varepsilon} = 0$ and there exists a $n/2$-form u such that $*u = \varepsilon u$ and a real number λ such that*

$$\mathcal{R}_{\varepsilon,n/2} = \lambda \operatorname{Id} - u \otimes u.$$

Proof. When $n/2$ is odd, all the eigenspaces of the Bochner-Weitzenböck curvature are stable by the Hodge star operator hence they come with an even multiplicity. And when $n/2$ is even we obtain that $r_{n/2}$ is less than the lowest eigenvalue of $\mathcal{R}_{\varepsilon,n/2}$. □

2.2. Characterization of the equality case. An important feature of the Bochner-Weitzenböck curvature is that it satisfies the first Bianchi identity. Seeing once again $\mathcal{R}_k$ as a symmetric operator

$$\mathcal{R}_k \colon \Lambda^k T_x^* M \to \Lambda^k T_x^* M,$$

the first Bianchi identity asserts that if $(\theta_i)_i$ is an orthonormal basis of $(T_x^* M, g)$ then

$$\forall \alpha \in \Lambda^{k-1} T_x^* M, \sum_i \theta_i \wedge \mathcal{R}_k \left(\theta_i \wedge \alpha\right) = 0.$$

We now assume that there exists a real number λ and a k-form $u \in \Lambda^k T_x^* M$ such that

$$\mathcal{R}_k = \lambda \operatorname{Id} - u \otimes u.$$

We get that for any orthonormal basis $(e_i)_i$, if we let $(\theta_i)_i$ be its dual basis, then (see [Kul72])

$$\sum_i u \wedge \theta_i \otimes e_i \llcorner u = 0.$$

We introduce the orthogonal decomposition $T_xM = V \oplus V^\perp$ where

$$V^\perp = \{v, v \llcorner u = 0\},$$

and choose an orthonormal basis $(e_i)_i$ of T_xM diagonalizing the quadratic form

$$v \mapsto |v \llcorner u|^2,$$

and such that $(e_i)_{1 \le i \le \ell}$ is a basis of V. Then $\{e_i \llcorner u\}_{1 \le i \le \ell}$ is an orthogonal family of $\Lambda^{k-1}T^*M$.

From the identity

$$\sum_i u \wedge \theta_i \otimes e_i \llcorner u = 0,$$

we deduce that $i \in \{1, \dots, \ell\} \Rightarrow u \wedge \theta_i = 0$. Hence $\ell = k$ and

$$u = |u|\, \theta_1 \wedge \dots \theta_k = |u| dv_V.$$

We can go one step further. Indeed if $k \in [2, \frac{n-1}{2}]$, the curvature operator is uniquely determined by the Bochner-Weitzenböck curvature: the components of the curvature operator can be expressed by taking contractions of $\mathcal{R}_k$ (see [Lab06, Theorem 4.4]).

We first see that if $T_xM = V \oplus V^\perp$ and $u = dv_V$ then

$$\mathsf{tr}(u \otimes u) = *_V g_V,$$

where tr is the contraction operator defined in [Lab05], g_V is the metric on V viewed as a double $(1,1)$-form on V and

$$*_V : \Lambda^{(1,1)}V^* \to \Lambda^{(k-1,k-1)}V^*$$

is the Hodge star acting on double forms of V. The computations of [Lab06, theorem 4.4] imply that the traceless part of $\mathsf{tr}^{k-1}(\mathcal{R}_k)$ is proportional to the traceless part of the Ricci curvature, hence the Ricci curvature is a linear combination of g_V and $g_{V^\perp}$; we also get that $\mathsf{tr}^{k-2}(\mathcal{R}_k)$ is a linear combination of $g.Ric$, of g^2 and of the curvature operator. Hence in our case, we easily get that there are numbers $\alpha = \alpha(x)$, $\beta = \beta(x)$ and $\gamma = \gamma(x)$ such that the curvature operator at x is

$$\alpha \frac{g_V^2}{2} + \beta \frac{g_{V^\perp}^2}{2} + \gamma \frac{g^2}{2}.$$

Hence, using the orthogonal decomposition

$$\Lambda^k T^* \tilde{M} = \bigoplus_{j=0}^{k} \Lambda^{k-j} V^* \otimes \Lambda^j \left(V^\perp\right)^*,$$

we find that the eigenvalues of the Bochner-Weitzenböck curvature $\mathcal{R}_k$ are

$$\alpha j(k-j) + \beta j(n-k-j) + \gamma k(n-k),$$

with multiplicity $\binom{k}{j}\binom{n-k}{j}$, where $j \in \{0, \dots, k\}$. But the assumption asserts that $\mathcal{R}_k$ has only two eigenvalues and that the lowest one has multiplicity 1. The only possible case is $k = 2$ and $\alpha = (n-5)\beta$. Moreover, $\beta \ge 0$, since the lowest eigenvalue of the traceless part of $\mathcal{R}_k$ is a negative multiple of β. Consequently we have:

Proposition 4.7. *If there is a non-zero k-form u such that*

$$\mathcal{R}_k(x) = \lambda Id - u \otimes u$$

then $k = 2$ and T_xM has an orthogonal decomposition

$$T_xM = V \oplus V^\perp,$$

with $V^\perp = \{v, v \llcorner u = 0\}$ of codimension 2. Moreover, u is colinear to the volume form of V, and the curvature operator is of the form

$$(n-5)\beta \frac{g_V^2}{2} + \beta \frac{g_{V^\perp}^2}{2} + \gamma \frac{g^2}{2},$$

with $\beta \geq 0$.

When $n/2$ is odd, we use the complex structure given by the Hodge star operator on $n/2$-forms and obtain:

Proposition 4.8. *If $n/2$ is odd and if there is a non-zero $n/2$-form u such that*

$$\mathcal{R}_{n/2} = \lambda \operatorname{Id} - u \otimes u - *u \otimes *u,$$

then T_xM has an orthogonal decomposition

$$T_xM = V \oplus V^\perp,$$

with $V = \{v, v \wedge u = 0\}$ and $V^\perp = \{v, v \llcorner u = 0\}$ of dimension $n/2$.

Moreover, u is colinear to the volume form of V and $*u$ is colinear to the volume form of $V^\perp$.

Indeed, with the same orthogonal decomposition $T_xM = V \oplus V^\perp$ as before, with

$$V^\perp = \{v, v \llcorner u = 0\},$$

we get that for any vector $w \in V^\perp$,

$$w^\flat \wedge *u = *(w \llcorner u) = 0.$$

Hence there is a $(\ell - n/2)$-form $\psi \in \Lambda^{\ell - n/2} V^*$ such that $*u = \psi \wedge dv_{V^\perp}$ and $u = *_V \psi$.

The Bianchi identity implies that

$$0 = \sum_{i=1}^{\ell} *_V \psi \wedge \theta_i \otimes e_i \llcorner *_V \psi \pm \sum_{i=1}^{\ell} \psi \wedge dv_{V^\perp} \wedge \theta_i \otimes e_i \llcorner (\psi \wedge dv_{V^\perp}).$$

Because $(e_i \llcorner *_V \psi)_{1 \leq i \leq \ell} \cup \{e_i \llcorner (\psi \wedge dv_{V^\perp}))_{1 \leq i \leq \ell}$ is an orthogonal family, we conclude that $\ell = n/2$ and $\psi = 1$.

And when $n/2$ is even, we have:

Proposition 4.9. *Assume that $n/2$ is even, that for $\varepsilon \in \{-, +\}$ we have $W^{-\varepsilon} = 0$ and that there is a non-zero $n/2$-form u such that $*u = \varepsilon u$ and*

$$\mathcal{R}_{\varepsilon, n/2} = \lambda \operatorname{Id} - u \otimes u.$$

Then $n = 4$ and u is colinear to $g(J.,.)$ where J is an unitary complex structure on T_xM.

Indeed, we obtain that the Bianchi operator applied to $u \otimes u$ is a multiple of the Bianchi operator applied to the Hodge star operator. But the Bianchi operator applied to the Hodge star operator is a multiple of the Hodge star operator. Hence, if (e_i) is a orthonormal basis of T_xM then $(e_i \llcorner u)_i$ is a basis of $\Lambda^{n/2-1} T_x^* M$.

This can only occur when $n = 4$ and when $u = |u| g(J.,.)$, where J is an unitary complex structure on T_xM.

2.3. The pinching results. On a closed manifold, according to (4.7) and the inequalities of Section 2.1, if $b_k \neq 0$, we have

$$\left(a_{n,k} \, \|W\|^2_{\frac{n}{2}} + b_{n,k} \, \left\|\overset{\circ}{Ric}\right\|^2_{\frac{n}{2}} \right)^{\frac{1}{2}} \geq \|r_k\|_{L^{n/2}} \geq \frac{k(n-k)}{n(n-1)} \, \Upsilon(M, [g]). \tag{4.10}$$

We will now characterize the equality case in (4.10).

For one-forms in dimension greater than 5. If $b_1 \neq 0$ and if

$$\|\overset{\circ}{Ric}\|_{L^{\frac{n}{2}}} = \frac{1}{\sqrt{n(n-1)}}\,\mathrm{Y}(M,[g]),$$

then according to Theorem 4.2, Ric_g is nonnegative with b_1 zero eigenvalues which correspond to b_1 parallel vector fields. According to the DeRham splitting theorem, the universal cover of (M,g) splits as a Riemannian product $(N^{n-b_1} \times \mathbb{R}^{b_1}, h + (dt)^2)$. But according to Lemma 4.5, Ric_g has only two distinct eigenvalues, hence $b_1 = 1$ and (N,h) is Einstein with positive scalar curvature.

For one-forms in dimension 4. If equality holds, equality must also hold in the refined Kato inequality for ξ. Then according to Proposition F.1, M possess a normal cover

$$\widehat{M} = N^3 \times \mathbb{R}$$

with a warped product metric

$$\hat{g} = \eta^2(t)h + (dt)^2,$$

where for some $T > 0$, η is a T-periodic function and the deck transformation group is generated by

$$\gamma(x,t) = (\phi(x), t+T),$$

with $\phi\colon N \to N$ a h-isometry.

We can write that $\hat{g}$ is isometric to $\tilde{g} = e^{-2f(s)}(h + ds^2)$. Then,

$$\overset{\circ}{Ric}_{\tilde{g}} = \overset{\circ}{Ric}_h + \frac{1}{2}(1 - f'' - (f')^2)\left(h - 3ds^2\right).$$

Since equality holds in the inequality between the first eigenvalue and the norm of $\overset{\circ}{Ric}_{\tilde{g}}$, we have

$$\overset{\circ}{Ric}_{\tilde{g}} = r_1 e^{-2f}\left(ds^2 - \frac{1}{3}h\right),$$

then

$$\overset{\circ}{Ric}_h = \left(r_1 e^{-2f} + \frac{1}{2}(1 - f'' - (f')^2)\right)\left(3ds^2 - h\right)$$

and by taking the trace on $TN \subset T\widehat{M}$, we see that it must vanish. Thus (N^3,h) is Einstein hence of constant sectional curvature, and (M,g) is conformally equivalent to a quotient of $\mathbb{S}^3 \times \mathbb{R}$. We recover Theorem 0.4 i).

Remarks 4.10. i) If the translation parameter T is too large the the product metric cannot be a Yamabe minimizer. Indeed the second variation of the Yamabe functional has a negative eigenvalue at the product metric when

$$T^2 > \frac{4\pi^2(n-1)}{R_h}.$$

Conversely, on $\mathbb{S}^{n-1} \times \mathbb{S}^1$, the product metric is a Yamabe minimizer as soon as

$$T^2 \leq \frac{4\pi^2}{n-2} \tag{4.11}$$

(cf. [Sch89]). Therefore, in dimension 4, if $b_1 \neq 0$, equality holds in (0.7) if and only if (M,g) is conformally equivalent to a quotient of $\mathbb{S}^3 \times \mathbb{R}$ with translation parameter satisfying (4.11).

ii) If (M,g) satisfies the pinching

$$\int_M \left|\overset{\circ}{Ric}_g\right|^2 dv_g \leq \frac{1}{12}\int_M R_g^2 dv_g$$

which is conformally invariant according to the Gauss-Bonnet formula, we can suppose (up to a conformal change) that g is a Yamabe minimizer and satisfies (0.7).

For two-forms in dimension 4. If equality

$$\|W_g\|_{L^2}= \frac{1}{2\sqrt{6}}\,\mathrm{Y}(M,[g]),$$

holds and $b_2 \neq 0$, then by taking a two-fold covering if M is not orientable and choosing the right orientation, we have equality in (4.8), $b_2^+(M) = 0$ and $W_g^- = 0$. Hence (M,g) is conformally equivalent to a Kähler self-dual manifold with constant scalar curvature. According to [Bou81, Der83], (M,g) is conformally equivalent to $\mathbb{C}\mathbf{P}^2$ endowed with the Fubini-Study metric, and we recover Theorem 0.4 ii).

In degree $\mathbf{k} \in [\mathbf{2}, \frac{\mathbf{n-2}}{\mathbf{2}}]$ when $n \geq 7$. If equality holds in (4.10) and if there exists a non-trivial harmonic k-form, then according to Proposition 4.7 we must have $k=2$.

When $n \geq 7$, we have $k \leq \frac{n-3}{2}$, and according to Theorem 4.2, the metric g is a Yamabe minimizer and ξ is parallel. According to Proposition 4.7, we obtain a parallel decomposition $T^*M = V \oplus V^\perp$, and the universal cover of (M,g) splits as a Riemannian product

$$\pi\colon \tilde{M} = X_1 \times X_2 \to M$$

where X_1 has dimension 2 and $\pi^*\xi$ is colinear to λdv_{X_1}. Still from Proposition 4.7, we see that $\gamma = 0$, that X_2 has constant positive sectional curvature, that we can normalize to be 1, and that X_1 has constant sectional curvature, hence is a 2-sphere of curvature $n-5$.

For 2-forms in dimension 6. We consider a closed manifold (M^6, g) with $b_2 \neq 0$ which satisfies

$$\|r_2\|_{L^3}= \frac{4}{15}\,\mathrm{Y}(M,[g])$$

and

$$|r_2|^2 = a_{6,2}\,|W|^2 + b_{6,2}\left|\overset{\circ}{Ric}\right|^2.$$

In this case, there is an harmonic 2-form ξ for which equality holds in the refined Kato inequality, and the curvature operator is

$$\beta\left(\frac{g_V^2}{2} + \frac{g_{V^\perp}^2}{2}\right) + \gamma\frac{g^2}{2},$$

where at each point

$$T_xM = V \oplus V^\perp.$$

Following the computations done in [Car10], we introduce a local orthonormal frame $(e_1, e_2, e_3, \dots, e_6)$ and its dual frame $(\theta^1, \dots, \theta^6)$, with $V = \mathrm{Vect}(e_1, e_2)$. We can write that

$$d|\xi| = \rho|\xi|\theta^1 \quad \text{and} \quad \xi = |\xi|\theta^1 \wedge \theta^2.$$

The computation leads to

$$\nabla_{e_1}\xi = \rho\xi, \quad \nabla_{e_2}\xi = 0 \quad \text{and} \quad \nabla_{e_j}\xi = -\frac{1}{4}\rho|\xi|\theta^j \wedge \theta^2,$$

for $j \geq 3$. Hence, writing $\Omega = \frac{\xi}{|\xi|}$, we obtain $\nabla_{e_1}\Omega = \nabla_{e_2}\Omega = 0$,

$$R(e_1,e_2)\Omega = (\beta+\gamma)\Omega \quad \text{and} \quad R(e_1,e_2)\Omega = -\nabla_{[e_1,e_2]}\Omega.$$

This implies that

$$[e_1, e_2] \in V = \mathrm{Vect}(e_1, e_2).$$

Hence $\nabla_{[e_1,e_2]}\Omega = 0$ and thus $\beta + \gamma = 0$. However, the scalar curvature of g is

$$R_g = 14\beta + 30\gamma = -16\beta.$$

This is not possible, since $\beta \geq 0$ and since we have assumed that the Yamabe constant of (M, g) is positive.

The middle degree. We consider a closed manifold (M^n, g) with $b_{\frac{n}{2}} \neq 0$ and such that the following equality holds

$$a_{n,n/2} \, \|W_g\|_{L^{n/2}} = \frac{n}{4(n-1)} \, \mathrm{Y}(M, [g]).$$

If $n/2$ is even, by Proposition 4.9 we must have $n = 4$ and (M, g) conformally equivalent to $\mathbb{C}\mathbf{P}^2$ endowed with the Fubini-Study metric.

If $n/2$ is odd, then according to Section 1.2, up to a conformal change $\tilde{g} = |\xi|^{\frac{4}{n}} g$ on the metric, we can suppose that g is a Yamabe minimizer and that ξ is parallel.

According to Proposition 4.8, the universal cover of (M, g) splits as a Riemannian product $X_1 \times X_2$ where X_1 and X_2 have dimension $n/2$. Moreover, in the orthogonal decomposition

$$\Lambda^{\frac{n}{2}} T^*(X_1 \times X_2) = \bigoplus_{j=0}^{\frac{n}{2}} \Lambda^j T^* X_1 \otimes \Lambda^{\frac{n}{2}-j} T^* X_2,$$

the Bochner-Weitzenböck curvature has the decomposition

$$\mathcal{R}_{\frac{n}{2}} = \sum_{j=0}^{\frac{n}{2}} \left(\mathcal{R}^{X_1}_{\frac{n}{2}-j} \otimes \mathrm{Id}_{\Lambda^j T^* X_2} + \mathrm{Id}_{\Lambda^{\frac{n}{2}-j} T^* X_1} \otimes \mathcal{R}^{X_2}_j \right).$$

Hence for $j \in \{0, \dots, \frac{n}{2}\}$, $\mathcal{R}^{X_1}_{\frac{n}{2}-j}$ and $\mathcal{R}^{X_2}_j$ are multiple of the identity. In particular $\mathcal{R}^{X_1}_{\frac{n}{2}-2}$ and $\mathcal{R}^{X_2}_2$ are multiple of the identity, and by [Tac73] or [Lab06], it implies that X_1 and X_2 have constant sectional curvature.

Moreover, the eigenvalues of $\mathcal{R}_{\frac{n}{2}}$ are

$$j \left(\frac{n}{2} - j \right) \frac{R_g}{n(n-1)},$$

with multiplicity $\binom{n/2}{j}^2$, where $j \in \{0, \dots, \frac{n}{2}\}$. The only possibility to have 2 eigenvalues is when $n = 6$. Then X_1 and X_2 are two round spheres.

Remark 4.11. If X_1 and X_2 are two round spheres of the same radius, then the product is Einstein. According to [BE87] it is a Yamabe minimizer, and thus equality really holds in (0.19).

3. The non-compact case

We will prove the following result, which implies Theorem M:

Theorem 4.12. *Let (M^n, g), $n \geq 4$, be a complete non-compact Riemannian manifold with positive Yamabe constant. Assume that the lowest eigenvalue of the Ricci curvature satisfies $Ric_- \in L^p$ for some $p > \frac{n}{2}$, and assume that $R_g \in L^{\frac{n}{2}}$. If*

$$\|r_1\|_{L^{\frac{n}{2}}} + \frac{n-4}{4n} \|R_g\|_{L^{\frac{n}{2}}} \leq \frac{1}{4} \mathrm{Y}(M, [g]), \tag{4.12}$$

then

- *either $H^1_c(M, \mathbb{Z}) = \{0\}$ and in particular M has only one end.*
- *or equality holds in (4.12) and there exists an Einstein manifold (N^{n-1}, h) with positive scalar curvature and $\alpha > 0$ such that (M^n, g) or one of its two-fold covering is isometric to*

$$\left(N^{n-1} \times \mathbb{R}, \, \alpha \cosh^2(t) \left(h + (dt)^2 \right) \right).$$

According to Lemma 1.8, there exists C such that the following Sobolev inequality holds:

$$\forall \varphi \in C_0^\infty(M) \qquad \|\varphi\|^2_{L^{\frac{2n}{n-2}}} \leq C\, \|d\varphi\|^2_{L^2}\ . \tag{4.13}$$

Then, according to [CP04, Proposition 5.2], if $H^1_c(M,\mathbb{Z}) \neq \{0\}$, then M or one of its two-fold covering has at least two ends.

If M has at least two ends, then according to [CSZ97, Theorem 2], we can find a compact set $K \subset M$ with

$$M \setminus K = \Omega_- \cup \Omega_+,$$

and with both Ω_- and Ω_+ unbounded, and an harmonic function $\Phi\colon M \mapsto (-1,1)$ such that $d\Phi \in L^2$,

$$\lim_{\substack{x\to\infty\\ x\in\Omega_-}} \Phi(x) = -1 \quad \text{and} \quad \lim_{\substack{x\to\infty\\ x\in\Omega_+}} \Phi(x) = 1.$$

In particular $\xi = d\Phi$ is an L^2 harmonic 1-form on (M,g).

If M has only one end and $\check\pi\colon \check M \to M$ is a two-fold covering of M with at least two ends, then $(\check M, \check\pi^* g)$ satisfies the Sobolev inequality (4.13), and we can find a compact set $K \subset M$ such that $\Omega = M \setminus K$ is connected and $\check M \setminus \pi^{-1}(K) = \check\Omega_- \cup \check\Omega_+$ with $\check\Omega_-$ and $\check\Omega_+$ unbounded. Then we can find an harmonic function $\check\Phi\colon \check M \to (-1,1)$ such that $d\check\Phi \in L^2$,

$$\lim_{\substack{x\to\infty\\ x\in\check\Omega_-}} \check\Phi(x) = -1 \quad \text{and} \quad \lim_{\substack{x\to\infty\\ x\in\check\Omega_+}} \check\Phi(x) = 1.$$

Moreover this function is unique by maximum principle, hence the image of $\check\Phi$ by a deck transformation of $\check\pi\colon \check M \to M$ is either $\check\Phi$ or $-\check\Phi$. In particular, the function $|\xi|=|d\check\Phi|$ is well defined on M and is in $L^2(M,g)$.

The harmonic 1-form ξ satisfies

$$\langle \nabla^*\nabla\xi \mid \xi\rangle + \frac{1}{4} R_g\, |\xi|^2 \leq \left(r_1 + \frac{n-4}{4n} R_g\right) |\xi|^2$$

and

$$\frac{n}{n-1}\, |d\,|\xi||^2 \leq |\nabla\xi|^2\ .$$

Moreover, since $Ric_- = -r_1 + \frac{R_g}{n}$, Ric_- is in $L^{n/2}\cap L^p$ for some $p > n/2$, and according to [Gal88, Theorem 1], we have

$$\operatorname{Vol} B(x_0,R) = O\left(R^{2(n-1)}\right).$$

Therefore, we can apply Theorem 1.11 and we obtain

$$\|r_1\|_{L^{\frac{n}{2}}} + \frac{n-4}{4n}\, \|R\|_{L^{\frac{n}{2}}} \geq \left\| r_1 + \frac{n-4}{4n} R \right\|_{L^{\frac{n}{2}}} \geq \frac{1}{4}\,\mathrm{Y}(M,[g]). \tag{4.14}$$

If furthermore equality holds, then the function $v = |\xi|^{\frac{n-2}{n-1}}$ is in $L^{\frac{2n}{n-2}}(M,g)$ and we can suppose that $\|v\|_{L^{\frac{2n}{n-2}}} = 1$. Then v satisfies the Yamabe equation

$$\frac{4(n-1)}{n-2}\Delta_g v + R_g v = \mathrm{Y}(M,[g]) v^{\frac{n+2}{n-2}}.$$

Since equality must hold in the refined Kato inequality, according to Proposition F.1, M or one of its two-fold covering is isometric to $N \times \mathbb{R}$ endowed with a metric

$$\hat g = \eta^2(t) h + dt^2.$$

If we take the new coordinate $s = \int_0^t \eta^{-1}(\tau)d\tau$, we can write that

$$\hat g = e^{-2f(s)}(h + ds^2),$$

where s is in (s_-, s_+), with

$$s_+ = \int_0^{+\infty} \frac{dt}{\eta(t)} \quad \text{and} \quad s_- = \int_0^{-\infty} \frac{dt}{\eta(t)}.$$

Since $v = e^{(n-2)f}$ is a solution of the Yamabe equation for the metric $\hat{g}$, the function $w = e^{\frac{n-2}{2}f}$ is solution to the Yamabe equation for the metric $h + (ds)^2$, hence satisfies

$$-4\frac{n-1}{n-2}w''(s) + R_h w(s) = \Upsilon(M,[g])w(s)^{\frac{n+2}{n-2}}. \tag{4.15}$$

In particular, we see that R_h only depends on s, hence is constant.

Moreover, since Ric_- is in L^p for some $p > n/2$, and since

$$\Delta|\xi| \leq -Ric_-|\xi|,$$

we get by DeGiorgi-Nash-Moser iterative scheme (see for instance [Yan92, Theorem B.1]) that ξ is in L^∞, and that

$$\lim_{x\to\infty} |\xi| = 0. \tag{4.16}$$

We can now prove that $s_+ = +\infty$. Recall that

$$|\xi| = \eta^{1-n} = e^{(n-1)f} = w^{2\frac{n-1}{n-2}}.$$

If s_+ is finite, then because of (4.16), we get

$$\lim_{s\to s_+} w = 0.$$

The differential equation (4.15) implies that w' must have a non-zero limit when $s \to s_+$. Hence there exists $c > 0$ such that

$$w \underset{s\to s_+}{\sim} c(s_+ - s).$$

And since the metric $\hat{g} = w^{-\frac{4}{n-2}}(h + ds^2)$ is complete, we must have

$$\int_0^{s^+} w(s)^{-\frac{2}{n-2}} ds = +\infty,$$

hence $n = 4$. But according to (1), when $n = 4$, the scalar curvature of $\hat{g}$ satisfies

$$\frac{1}{w^3}R_{\hat{g}} = -6\left(\frac{1}{w}\right)'' + \frac{1}{w}R_h,$$

hence $R_{\hat{g}}$ goes to $-12c^2$ when $s \to s_+$, and therefore is not in $L^{n/2}(M, \hat{g})$. Consequently, $s_+ = +\infty$, and the same argument shows that $s_- = -\infty$.

From (4.15), we deduce that there is a constant c such that

$$-4\frac{n-1}{n-2}(w')^2 + R_h w^2 = \frac{n-2}{n}\Upsilon(M,[g])w^{\frac{2n}{n-2}} + c.$$

Since $\lim_{s\to\pm\infty} w = 0$, we must have $c = 0$. Moreover, since $\Upsilon(M, [g]$ is positive, and w is a positive function we must also have $R_h > 0$. Up to a change of time variable and a scaling on $\hat{g}$, we can suppose that

$$R_h = (n-2)(n-1).$$

Let $\varphi = e^{-f} = w^{-\frac{2}{n-2}}$. We obtain

$$-(\varphi')^2 + \varphi^2 = \frac{\Upsilon(M,[g])}{4n(n-1)}.$$

Therefore, for some s_0, we have

$$\varphi(s) = \sqrt{\frac{\Upsilon(M,[g])}{4n(n-1)}} \cosh(s - s_0).$$

Conversely, if (N^{n-1}, h) is a closed manifold with positive scalar curvature

$$R_h = (n-2)(n-1),$$

and if

$$(M, g) = \left(N^{n-1} \times \mathbb{R},\, \alpha \cosh^2(t) \left(h + (dt)^2\right)\right),$$

then

$$R_g = (n-1)(n-4)\frac{1}{\alpha \cosh^4(t)}$$

$$\overset{\circ}{Ric}_g = \overset{\circ}{Ric}_h + 2\frac{n-2}{n}\left(h - (n-1)ds^2\right)$$

We see that the lowest eigenvalue of $\overset{\circ}{Ric}_g$ satisfies

$$r_1(g) = \frac{1}{\alpha \cosh^4(t)}\left(r_1(h) + \frac{2(n-2)(n-1)}{n}\right),$$

and thus we obtain

$$\|r_1\|_{L^{\frac{n}{2}}} + \frac{n-4}{4n}\|R_g\|_{L^{\frac{n}{2}}} = \frac{C}{4}\,\mathrm{Y}(M, [g]),$$

with

$$\begin{aligned} C &= \frac{n(n-1) + 4r_1(h)}{\mathrm{Y}(M, [g])}\,\mathrm{Vol}((N, h))^{\frac{2}{n}}\left(\int_{\mathbb{R}} \frac{dt}{\cosh^n(t)}\right)^{\frac{2}{n}} \\ &= \left(1 + \frac{4r_1(h)}{n(n-1)}\right)\left(\frac{\mathrm{Vol}((N, h))}{\mathrm{Vol}(\mathbb{S}^{n-1})}\right)^{\frac{2}{n}}\frac{\mathrm{Y}(\mathbb{S}^n)}{\mathrm{Y}(N \times \mathbb{R}, [h + dt^2])}. \end{aligned}$$

According to [AB03, Proposition 2.12], we always have

$$\left(\frac{\mathrm{Vol}((N, h))}{\mathrm{Vol}(\mathbb{S}^{n-1})}\right)^{\frac{2}{n}}\frac{\mathrm{Y}(\mathbb{S}^n)}{\mathrm{Y}(N \times \mathbb{R}, [h + dt^2])} \geq 1.$$

Hence, for C to be equal to 1, $r_1(h)$ must vanish, i.e. h must be Einstein. Then, according to [Pet09, Corollary 1.3], we have

$$\left(\frac{\mathrm{Vol}((N, h))}{\mathrm{Vol}(\mathbb{S}^{n-1})}\right)^{\frac{2}{n}}\frac{\mathrm{Y}(\mathbb{S}^n)}{\mathrm{Y}(N \times \mathbb{R}, [h + dt^2])} = 1,$$

hence equality holds in (4.12). □

APPENDIX A

Classical results about geometric flows

We first recall the first variation formula for metrics perturbated by an evolving diffeomorphism, on which the DeTurck trick is based:

Lemma A.1. *Let (g_t) be a smooth family of metrics and let (ϕ_t) be a smooth family of diffeomorphisms. Then*

$$\partial_t(\phi_t^* g_t) = \phi_t^*(\partial_t g_t + L_{V_t} g_t),$$

where $V_t = \partial_t \phi_t \circ \phi_t^{-1}$.

Proof.

$$\begin{aligned}
\partial_t(\phi_t^* g_t)_{|t_0} &= \partial_t(\phi_{t_0}^* g_t)_{|t_0} + \partial_t(\phi_t^* g_{t_0})_{|t_0} \\
&= \phi_{t_0}^*\Big(\partial_t g_{t|t_0}\Big) + \phi_{t_0}^*\Big(\partial_t(\phi_t \circ \phi_{t_0}^{-1})^* g_{t_0}{}_{|t_0}\Big) \\
&= \phi_{t_0}^*\Big(\partial_t g_{t|t_0} + L_{\partial_t \phi_{t|t_0} \circ \phi_{t_0}^{-1}} g_t\Big).
\end{aligned}$$

□

Proposition A.2 (DeTurck trick)**.** *Let (M, g_0) be a closed Riemannian manifold.*

Let $P : \mathcal{S}^2_+(M) \to \mathcal{S}^2(M)$ and $V : \mathcal{S}^2_+(M) \to TM$ be geometric differential operators such that $(P - L_V)'_{g_0}$ is strongly elliptic. Then

$$\begin{cases} \partial_t g = P(g) \\ g(0) = g_0 \end{cases} \qquad E_P(g_0)$$

admits a unique maximal solution on an open interval $[0, T)$, T positive.

Proof. Since $(P - L_V)'_{g_0}$ is strongly elliptic, it follows from the theory of parabolic equations that

$$\begin{cases} \partial_t g = P(g) - L_{V_g} g \\ g(0) = g_0 \end{cases} \qquad (DT(g_0))$$

admits is a unique maximal solution $\tilde{g}(t)$, $t \in [0, T)$ with $T > 0$ (see [MM10]).

Let ϕ_t, $t \in [0, T)$ be the flow of $V(\tilde{g}_t)$:

$$\begin{cases} \partial_t \phi_t = V(\tilde{g}_t) \circ \phi_t, \\ \phi_0 = Id_M. \end{cases}$$

Let show that

$$(g_t)_{t\in[0,T_1)} \text{ is a solution of } E_P(g_0) \iff T_1 \le T \text{ and } \forall t \in [0, T_1)\ g_t = \phi_t^* \tilde{g}_t.$$

It will gives short-time existence and uniqueness for $E_P(g_0)$.

Let $g_t = \phi_t^* \tilde{g}_t$. Then $g(0) = g_0$ and by Lemma A.1, for all t in $[0, T)$,

$$\partial_t g = \phi_t^*(\partial_t \tilde{g} + L_{V_{\tilde{g}}} \tilde{g}) = \phi_t^* P(\tilde{g}) = P(g).$$

So g_t is solution of $E_P(g_0)$ on $[0, T)$.

Now, let g_t be a solution of $E_P(g_0)$ on $[0,T_1)$. Let ψ_t, $t\in[0,T_1)$ be the flow of $-V_{g_t}$. Then $\psi_t^* g_t$ is solution of $(DT(g_0))$ on $[0,T_1)$:

$$\begin{aligned}\partial_t(\psi_t^* g_t) &= \psi_t^*(\partial_t g - L_{V_g} g)\\ &= \psi_t^* P(g) - \psi_t^* L_{V_g} g)\\ &= P(\psi_t^* g) - L_{V_{\psi_t^* g}} \psi_t^* g.\end{aligned}$$

Therefore, $T_1 \leq T$ and for all t in $[0,T_1)$, $\psi_t^* g_t = \tilde{g}_t$.

Moreover, for all t in $[0,T_1)$, $\psi_t^{-1} = \phi_t$:

$$\partial_t(\psi_t^{-1}) = -\psi_t^*(-V_{g_t}) \circ \psi_t^{-1} = V_{\tilde{g}_t} \circ \psi_t^{-1},$$

and $\psi_0^{-1} = Id_M$, so ψ_t^{-1} is the flow of $V_{\tilde{g}_t}$. It follows that $g_t = \phi_t^* \tilde{g}_t$. □

We now compute the principal symbols of the operators we will use. With

$$R_\xi(g) = \xi\otimes\xi - |\xi|^2\, g,$$

and

$$(\gamma_{g,g_0})_i = \frac{1}{2} g_{i\delta} g^{\alpha\beta}(\Gamma^i_{\alpha\beta}(g) - \Gamma^i_{\alpha\beta}(g_0)),$$

we have:

Proposition A.3. *For all metrics g and all ξ in T^*M, we have*

$$\begin{aligned}&\sigma_\xi(L_V)'_g = \xi\otimes\sigma_\xi V'_g + \sigma_\xi V'_g \otimes \xi,\\ &\sigma_\xi R'_g = \langle R_\xi \mid \cdot\rangle,\\ &\sigma_\xi(Ric - L_{\gamma_{\cdot,g_0}})'_g = -\frac{1}{2}\,|\xi|^2\,\mathrm{Id}_{\mathcal{S}^2(M)},\\ &\sigma_\xi(\delta\mathsf{D}(R\,\cdot))'_g = \langle R_\xi \mid \cdot\rangle\; R_\xi,\\ &\sigma_\xi(\delta\mathsf{D}Ric - L_{\nabla^*\nabla\gamma_{\cdot,g_0}+\frac{1}{4}dR})'_g = \frac{1}{2}\,|\xi|^4\,\mathrm{Id}_{\mathcal{S}^2(M)}.\end{aligned}$$

Where the operators δ and D are defined in Appendix B and $V : \mathcal{S}^2_+(M) \to T^*M$ is any differential operator of degree at least one,

Proof. We recall that the Lie derivative of a metric is given by

$$(L_V g)_{ij} = \nabla_i V_j + \nabla_j V_i,$$

then by Lemma C.5,

$$(L_V)'_g(h)_{ij} = \nabla_i V'_g(h)_j + \nabla_j V'_g(h)_i + \nabla h * V,$$

and as V is of degree at least one,

$$\sigma_\xi(L_V)'_g(h)_{ij} = \xi_i \sigma_\xi V'_g(h)_j + \xi_j \sigma_\xi V'_g(h)_i.$$

By Proposition B.4,

$$R'_g(h) = \tilde{\delta}\delta h + \Delta\mathsf{tr} h - \langle Ric \mid h\rangle,$$

then

$$\sigma_\xi R'_g(h) = \langle \xi\otimes\xi \mid h\rangle - |\xi|^2\,\mathsf{tr}\,h = \langle R_\xi \mid h\rangle,$$

therefore, as $\delta\mathsf{D}(R_g\, g) = \Delta R_g\, g + \tilde{\mathsf{D}}\mathsf{D} R_g$ (Proposition B.2),

$$\sigma_\xi(\delta\mathsf{D}(R\,\cdot)'_g = \xi\otimes\xi\,\sigma_\xi R'_g - |\xi|^2\,\sigma_\xi R'_g\, g = \langle R_\xi \mid \cdot\rangle\; R_\xi.$$

It follows of Lemma B.3 that

$$\begin{aligned}(\gamma_{\cdot,g_0})'_g(h)_i &= \frac{1}{2}(\nabla^\alpha h_{\alpha i} - \frac{1}{2}\nabla_i \mathsf{tr}\,h) - \frac{1}{2} g_{i\delta} h^{\alpha\beta}(\Gamma^\delta_{\alpha\beta}(g) - \Gamma^\delta_{\alpha\beta}(g_0)),\\ (\gamma_{\cdot,g_0})'_g(h) &= -\frac{1}{2}(\delta h + \frac{1}{2}\tilde{\mathsf{D}}\mathsf{tr} h) + h * (\Gamma(g) - \Gamma(g_0)),\end{aligned}$$

and by Proposition B.4,

$$Ric'_g(h) = \frac{1}{2}(\nabla^*\nabla h - \mathsf{D}(\delta h + \frac{1}{2}\tilde{\mathsf{D}}\mathsf{tr}h)) - \tilde{\mathsf{D}}(\tilde{\delta} h + \frac{1}{2}\mathsf{D}\mathsf{tr}h) + h * Rm.$$

It follows that its principal symbol is

$$\sigma_\xi Ric'_g(h) = -\frac{1}{2}\,|\xi|^2\,h + \xi \otimes \sigma_\xi \gamma'_g(h) + \sigma_\xi \gamma'_g(h) \otimes \xi,$$

so

$$\sigma_\xi (Ric - L_{\gamma_{\cdot,g_0}})'_g = -\frac{1}{2}\,|\xi|^2\,Id_{\mathcal{S}^2(M)}.$$

Finally,

$$\begin{aligned}\sigma_\xi(\Delta Ric)'_g(h) &= -\,|\xi|^2\,\sigma_\xi Ric'_g(h)\\ &= -\frac{1}{2}\,|\xi|^4\,h + \xi \otimes (-\,|\xi|^2\,\sigma_\xi\gamma'_g(h)) + (-\,|\xi|^2\,\sigma_\xi\gamma'_g(h)) \otimes \xi,\end{aligned}$$

and then, since $\delta\mathsf{D}Ric_g = \nabla^*\nabla Ric_g + \frac{1}{2}\tilde{\mathsf{D}}\mathsf{D}R_g + Rm * Rm$ (Proposition B.2),

$$\sigma_\xi(\delta\mathsf{D}Ric - L_{\nabla^*\nabla\gamma + \frac{1}{4}dR})'_g = \sigma_\xi(\nabla^*\nabla Ric - L_{\nabla^*\nabla\gamma})'_g = -\frac{1}{2}\,|\xi|^4\,Id_{\mathcal{S}^2(M)}.$$

□

We recall the following basic estimate:

Lemma A.4. *Let $I \in \mathbb{R}$ be an interval and $g(t)$, $t \in I$ be a smooth family of metrics defined on some finite dimensional vector space.*

$$\sup_I |\partial_t g_t|_{g_t} \leq A,$$

then for all $s, t \in I$,

$$e^{-A|t-s|}g_s \leq g_t \leq e^{A|t-s|}g_s. \tag{A.1}$$

This inequality is easily proved by integration (see for instance [CK04, Lemma 6.49]). We can also estimate the C^2 norm of a fixed cut-off function φ:

Lemma A.5. *If $g(t)$, $t \in I$ is a smooth family of metrics on a manifold M^n such that*

$$\sup_{t\in I} \|\partial_t g_t\|_{C^1(M,g_t)} \leq A,$$

and if $\varphi \in C_0^\infty(M)$, then for all $s, t \in I$,

$$\|d\varphi\|_{C^1(M,g_s)} \leq \sqrt{2}\,e^{A|t-s|}\,\|d\varphi\|_{C^1(M,g_t)}$$

Proof. We have

$$\partial_t\left(|d\varphi|^2\right) \leq |\partial_t g||d\varphi|^2,$$

then $\nabla^2_{ij}\varphi = \partial_i\partial_j\varphi - \Gamma^k_{ij}\nabla_k\varphi$, hence according to Lemma B.3,

$$\left|\partial_t(\nabla^2\varphi)\right| \leq \frac{3}{2}\,|\nabla\partial_t g||d\varphi|$$

and

$$\partial_t\left(\left|\nabla^2\varphi\right|^2\right) \leq 2\,|\partial_t g||\nabla d\varphi|^2 + 3\,|\nabla\partial_t g||d\varphi|\left|\nabla^2\varphi\right|$$

therefore

$$\partial_t\left(|d\varphi|^2 + \left|\nabla^2\varphi\right|^2\right) \leq 2A\left(|d\varphi|^2 + \left|\nabla^2\varphi\right|^2\right),$$

and by integration, we obtain

$$\|d\varphi\|^2_{C^1(M,g_s)} \leq 2e^{2A|t-s|}\,\|d\varphi\|^2_{C^1(M,g_t)}\,.$$

□

The following Lemma is an adaptation of [Ham95, Lemma 2.4] to higher-order curvature flows:

Lemma A.6. *Let (M, g) be a Riemannian manifold and let $N, k \in \mathbb{N}$. Let K denote a compact subset of M and $I \subset \mathbb{R}$ a closed interval with $0 \in I$. Let g_i be a sequence of metrics defined on open neighborhoods of $I \times K$, and solutions to a $(k+2)^{th}$-order equation*

$$\partial_t g_t = \sum_{0 \le i_1 \le \cdots \le i_j \le k} \nabla_{g_t}^{i_1} Rm_{g_t} * \cdots * \nabla_{g_t}^{i_j} Rm_{g_t}, \tag{A.2}$$

If the following conditions are satisfied:

(i) There is a constant C so that the metrics $g_i(0)$ are uniformly equivalent to g on K, i.e.

$$e^{-C} g \le g_i(0) \le e^{C} g. \tag{A.3}$$

(ii) For all $1 \le j \le N$, the j^{th} g-derivative of $g_i(0)$ is uniformly bounded on K, i.e.

$$\sup_K \left|\nabla_g^j (g_i(0))\right|_g \le C_j. \tag{A.4}$$

(iii) For all $1 \le j \le N+k$, the j^{th} g_i-derivative of Rm_{g_i} is bounded with respect to g_i on $I \times K$, i.e.

$$\sup_{I \times K} \left|\nabla_{g_i}^j (Rm_{g_i})\right|_{g_i} \le C'_j. \tag{A.5}$$

Then the metrics g_i are uniformly equivalent to g on $I \times K$, and for all $1 \le j \le N$ the j^{th} g-derivative of g_i is uniformly bounded on $I \times K$, i.e.

$$\sup_{I \times K} \left|\nabla_g^j (g_i)\right|_g \le c_j,$$

and the constants c_j only depend on C_j, C'_j, the dimension, the length of I and the equation (A.2).

Proof. According to Lemma A.4, for any i, the metrics $g_i(t)$, $t \in I$ are uniformly equivalent to $g_i(0)$, hence to g by (A.3). The bounds will be taken for any of these equivalents metrics.

Since for any tensor T we have

$$\nabla_g T = \nabla_{g_i} T + (\Gamma_{g_i} - \Gamma_g) * T = \nabla_{g_i} T + g_i * \nabla_g (g_i) * T,$$

we obtain by induction on $j \le N$ that

$$\nabla_g^j (\partial_t g_i) = \sum_{p,q \ge 1} \sum_{\substack{0 \le i_1, \ldots, i_p \le j \\ 0 \le j_1, \ldots, j_q \le k+j}} \nabla_g^{i_1} g_i * \cdots * \nabla_g^{i_p} g_i * \nabla_{g_i}^{j_1} Rm_{g_i} * \cdots * \nabla_{g_i}^{j_q} Rm_{g_i}. \tag{A.6}$$

Because of the bounds (A.5), there exists a polynomial Q such that

$$\left|\partial_t \nabla_g^j (g_i)\right| = \left|\nabla_g^j (\partial_t g_i)\right| \le Q\left(|\nabla_g (g_i)|, \ldots, \left|\nabla_g^j (g_i)\right|\right).$$

Hence, by induction on j and integration on I, the derivatives $\nabla_g^j (g_i)$for $j \le N$ are uniformly bounded on $I \times K$. $\square$

APPENDIX B

Operators on double-forms

We use the formalism of double-forms of Labbi (see [Kul72] and [Lab05, Lab06, Lab08]). If for some r, $\mathcal{E} = (T^*M)^r$ is the bundle of $(r, 0)$ tensors on M, we denote by $\Lambda^{(p,\,q)}(\mathcal{E})$ the bundle of (p, q) forms with values in $\mathcal{E}$, i.e. $\Lambda^{(p,\,q)}(\mathcal{E})= \mathrm{End}(\Lambda^p T^*M \otimes \Lambda^q T^*M, \mathcal{E})$. For example, symmetric endomorphisms of T^*M can be seen as real valued $(1, 1)$-forms and curvature operators as real valued $(2, 2)$-forms.

We define the contraction and the multiplication of a double-form by the metric (which is a particular case of the Kulkarni-Nomizu product) as follows:

$$\mathsf{tr} : \Lambda^{(p+1,\,q+1)}(\mathcal{E}) \to \Lambda^{(p,\,q)}(\mathcal{E}) \quad \mathit{by} \quad (\mathsf{tr}T)_{i_1\dots i_p|j_1\dots j_q} = g^{\alpha\beta}T_{\alpha i_1\dots i_p|\beta j_1\dots j_q},$$

$$g_{\bullet} : \Lambda^{(p,\,q)}(\mathcal{E}) \to \Lambda^{(p+1,\,q+1)}(\mathcal{E}) \quad \mathit{by} \quad (g_{\bullet}T)_{i_0\dots i_p|j_0\dots j_q} = \sum_{k=0}^{p}\sum_{l=0}^{q} T_{i_0\dots\hat{i_k}\dots i_p|j_1\dots\hat{j_l}\dots j_q},$$

If S and T are in $\Lambda^{(p,\,q)}(\mathcal{E})$, we define their scalar product by

$$\langle S \mid T\rangle = \frac{1}{p!q!} g^{i_1k_1}\cdots g^{i_pk_p} g^{j_1l_1}\cdots g^{j_ql_q} \left\langle S_{i_1\dots i_p|j_1\dots j_q} \mid T_{k_1\dots k_p|l_1\dots l_q}\right\rangle_{\mathcal{E}}.$$

In the space of double-forms, $g_{\bullet}$ is the adjoint of tr for this scalar product.

The curvature tensor has the orthogonal decomposition

$$Rm_g = W_g + \mathcal{Z}_g + \mathcal{S}_g,$$

where

$$\mathcal{Z}_g = \frac{1}{n-2}\, g_{\bullet}\overset{\circ}{Ric}_g, \quad \text{and} \quad \mathcal{S}_g = \frac{R_g}{2n(n-1)}\, g_{\bullet}g.$$

The Bochner-Weitzenböck curvature $\mathcal{R}_k$ can be seen as a real valued (k, k)-form (see [Bou81, Lab06]) and has the following orthogonal decomposition:

$$\mathcal{R}_k = \mathcal{W}_k + \mathcal{Z}_k + \mathcal{S}_k \tag{B.1}$$

where for $k \in [2, n-2]$,

$$\mathcal{W}_k = -2\frac{g^{k-2}}{(k-2)!}{}_{\bullet} W_g, \quad \mathcal{Z}_k = \frac{n-2k}{n-2}\,\frac{g^{k-1}}{(k-1)!}{}_{\bullet}\overset{\circ}{Ric}_g, \quad \mathcal{S}_k = \frac{k(n-k)}{n(n-1)} R_g\, \mathrm{Id}_{\Lambda^k T^*M}.$$

On 1-forms, we have

$$\mathcal{R}_1 = Ric_g = \overset{\circ}{Ric}_g + \frac{R_g}{n} g.$$

We also define the following differential operators (we point out that our definition of D and $\tilde{D}$ differs by a sign from that of [Lab08]):

$$\delta : \Lambda^{(p+1,\,q)}(\mathcal{E}) \to \Lambda^{(p,\,q)}(\mathcal{E}) \quad \mathit{by} \quad (\delta T)_{i_1\dots i_p|j_1\dots j_q} = -\nabla^{\alpha} T_{\alpha i_1\dots i_p|j_1\dots j_q},$$

$$\tilde{\delta} : \Lambda^{(p,\,q+1)}(\mathcal{E}) \to \Lambda^{(p,\,q)}(\mathcal{E}) \quad \mathit{by} \quad (\tilde{\delta} T)_{i_1\dots i_p|j_1\dots j_q} = -\nabla^{\alpha} T_{i_1\dots i_p|\alpha j_1\dots j_q},$$

$$\mathsf{D} : \Lambda^{(p,\,q)}(\mathcal{E}) \to \Lambda^{(p+1,\,q)}(\mathcal{E}) \quad \mathit{by} \quad (\mathsf{D}T)_{i_0\dots i_p|j_1\dots j_q} = \sum_{k=0}^{p} (-1)^k \nabla_{i_k} T_{i_0\dots\hat{i_k}\dots i_p|j_1\dots j_q},$$

$$\tilde{\mathsf{D}} : \Lambda^{(p,\,q)}(\mathcal{E}) \to \Lambda^{(p,\,q+1)}(\mathcal{E}) \quad \mathit{by} \quad (\tilde{\mathsf{D}}T)_{i_1\dots i_p|j_0\dots j_q} = \sum_{k=0}^{p} (-1)^k \nabla_{j_k} T_{i_1\dots i_p|j_0\dots\hat{j_k}\dots j_q}.$$

In the space of double-forms, D is the formal adjoint of δ and $\tilde{\mathsf{D}}$ is the formal adjoint of $\tilde{\delta}$.

Moreover, we have:

(B.2) $$\mathrm{tr}\,\delta T = -\delta\,\mathrm{tr} T \qquad g\boldsymbol{.}\,\mathsf{D}T = -\mathsf{D}\,(g\boldsymbol{.}T)$$

(B.3) $$\mathrm{tr}\mathsf{D} + \mathsf{D}\mathrm{tr} = -\tilde{\delta} \qquad \mathrm{tr}\tilde{\mathsf{D}} + \tilde{\mathsf{D}}\mathrm{tr} = -\delta,$$

(B.4) $$g\boldsymbol{.}\delta + \delta g\boldsymbol{.} = -\tilde{\mathsf{D}} \qquad g\boldsymbol{.}\tilde{\delta} + \tilde{\delta} g\boldsymbol{.} = -\mathsf{D}.$$

The second Bianchi identity leads to:

Proposition B.1.

$$\mathsf{D}Rm_g = \tilde{\mathsf{D}}Rm_g = 0 \qquad \tilde{\delta}Ric_g = -\frac{1}{2}\mathsf{D}R_g,$$
$$\tilde{\delta}Rm_g = -\mathsf{D}Ric_g \qquad \tilde{\delta}W_g = -\frac{n-3}{n-2}\mathsf{D}A_g.$$

(with $A_g = Ric_g - \frac{1}{2(n-1)}R_g g$ the Weyl-Schouten tensor)

Proposition B.2.

$$\delta\mathsf{D}(R_g\, g) = \Delta R_g\, g + \tilde{\mathsf{D}}\mathsf{D}R_g,$$
$$\delta\mathsf{D}Ric_g = \nabla^*\nabla Ric_g + \frac{1}{2}\tilde{\mathsf{D}}\mathsf{D}R_g + Ric\circ Ric - \overset{\circ}{Rm}(Ric),$$
$$\delta\mathsf{D}Ric_g = \tilde{\delta}\tilde{\mathsf{D}}Ric_g = \delta\tilde{\delta}Rm_g = \tilde{\delta}\delta Rm_g.$$

Proof. For the first one, we have

$$\delta\mathsf{D}(g\boldsymbol{.}R_g) = -\delta(g\boldsymbol{.}(\mathsf{D}R_g)) = g\boldsymbol{.}(\delta\mathsf{D}R_g) + \tilde{\mathsf{D}}\mathsf{D}R_g,$$

For the second one, in coordinates,

$$(\mathsf{D}Ric)_{ij|k} = \nabla_i Ric_{j|k} - \nabla_j Ric_{i|k},$$

therefore,

$$\begin{aligned}(\delta\mathsf{D}Ric)_{j|k} &= -\nabla^\alpha\nabla_\alpha Ric_{j|k} + \nabla^\alpha\nabla_j Ric_{\alpha|k}\\ &= \nabla^*\nabla Ric_{j|k} - \nabla_j(\delta Ric)_k + Rm^\alpha{}_{j\alpha}{}^\beta Ric_{k|\beta} + Rm^\alpha{}_{jk}{}^\beta Ric_{\alpha|\beta}\\ &= \nabla^*\nabla Ric_{j|k} + \frac{1}{2}\nabla_j\nabla_k R + (Ric\circ Ric)_{j|k} - \overset{\circ}{Rm}(Ric)_{j|k}.\end{aligned}$$

Finally, $\delta\mathsf{D}Ric_g = \tilde{\delta}\tilde{\mathsf{D}}Ric_g$ since both Ric_g and $\delta\mathsf{D}Ric_g$ are symmetric according the above formula, and the other part results from Proposition B.1. □

We recall that:

Lemma B.3. *For all g in $\mathcal{S}^2_+(M)$ and h in $\mathcal{S}^2(M)$,*

$$dv_g'(h) = \frac{1}{2}\mathrm{tr}(h)dv_g,$$
$$(g^{ij})'(h) = -h^{ij},$$
$$\Gamma_g'(h)^k_{ij} = \frac{1}{2}g^{k\alpha}(\nabla_j\, h_{\alpha i} + \nabla_i\, h_{\alpha j} - \nabla_\alpha h_{ij}).$$

The first variation of the curvature tensors are given by:

Proposition B.4. *For all g in $\mathcal{S}^2_+(M)$ and h in $\mathcal{S}^2(M)$,*

(B.5) $$Rm_g'(h)_{ij|kl} = -\frac{1}{2}(\mathsf{D}\tilde{\mathsf{D}}h_{ij|kl} + Rm_{ijk}{}^\alpha h_{\alpha l} + Rm_{ij}{}^\alpha{}_l h_{k\alpha}),$$

(B.6) $$Ric_g'(h) = \frac{1}{2}(\tilde{\delta}\tilde{\mathsf{D}}h + \mathsf{D}\mathrm{tr}\tilde{\mathsf{D}}h - Ric\circ h - \overset{\circ}{Rm}(h)),$$

(B.7) $$R_g'(h) = \mathrm{tr}\tilde{\delta}\tilde{\mathsf{D}}h - \langle Ric \mid h\rangle = \tilde{\delta}\delta h + \Delta\mathrm{tr}\, h - \langle Ric \mid h\rangle\ .$$

We can also write

$$Ric'_g(h) = \frac{1}{2}(\nabla^*\nabla h - \mathsf{D}(\delta h + \frac{1}{2}\tilde{\mathsf{D}}\mathsf{tr}h)) - \tilde{\mathsf{D}}(\tilde{\delta}h + \frac{1}{2}\mathsf{D}\mathsf{tr}h) + h * Rm.$$

Proof. The first one is proved in [Lab08, Lemma 4.1]. Then, by using the formulas (B.3) and (B.4):

$$\begin{aligned} Ric'_g(h) &= \mathsf{tr}Rm'_g(h) - \overset{\circ}{Rm}(h) = -\frac{1}{2}(\mathsf{tr}\mathsf{D}\tilde{\mathsf{D}}h + Ric \circ h - \overset{\circ}{Rm}(h)) \\ &= \frac{1}{2}(\tilde{\delta}\tilde{\mathsf{D}}h + \mathsf{D}\mathsf{tr}\tilde{\mathsf{D}}h - Ric \circ h + \overset{\circ}{Rm}(h)) \\ &= \frac{1}{2}(\tilde{\delta}\tilde{\mathsf{D}}h - \mathsf{D}\delta h - \mathsf{D}\tilde{\mathsf{D}}\mathsf{tr}h - Ric \circ h + \overset{\circ}{Rm}(h)) \end{aligned}$$

and since $\mathsf{D}\tilde{\mathsf{D}}(\mathsf{tr}h) = \tilde{\mathsf{D}}\mathsf{D}(\mathsf{tr}h)$ and $\nabla^*\nabla = \tilde{\delta}\tilde{\mathsf{D}} + \tilde{\mathsf{D}}\tilde{\delta} + Rm * \cdot$ (see Lemma C.1),

$$Ric'_g(h) = \frac{1}{2}(\nabla^*\nabla h - \mathsf{D}(\delta h + \frac{1}{2}\tilde{\mathsf{D}}\mathsf{tr}h)) - \tilde{\mathsf{D}}(\tilde{\delta}h + \frac{1}{2}\mathsf{D}\mathsf{tr}h) + h * Rm.$$

By tracing (B.6), we obtain

$$\begin{aligned} R'_g(h) &= \mathsf{tr}\, Ric'_g(h) - \langle Ric \mid h \rangle = \frac{1}{2}(\mathsf{tr}\tilde{\delta}\tilde{\mathsf{D}}h + \mathsf{tr}\mathsf{D}\mathsf{tr}\tilde{\mathsf{D}}h) - \langle Ric \mid h \rangle \\ &= \frac{1}{2}(\mathsf{tr}\tilde{\delta}\tilde{\mathsf{D}}h - \tilde{\delta}\mathsf{tr}\tilde{\mathsf{D}}h) - \langle Ric \mid h \rangle \\ &= \mathsf{tr}\tilde{\delta}\tilde{\mathsf{D}}h - \langle Ric \mid h \rangle\ . \end{aligned}$$

And we also have

$$\begin{aligned} R'_g(h) &= -\tilde{\delta}\mathsf{tr}\tilde{\mathsf{D}}h - \langle Ric \mid h \rangle \\ &= \tilde{\delta}\tilde{\mathsf{D}}\mathsf{tr}h + \tilde{\delta}\delta h - \langle Ric \mid h \rangle \\ &= \Delta\mathsf{tr}\, h + \tilde{\delta}\delta h - \langle Ric \mid h \rangle\ . \end{aligned}$$

□

APPENDIX C

Derivative commuting

A double-form $T \in\Lambda^{(p,\,q)}(\mathcal{E})$ can be seen as a p-form with values in $\Lambda^q TM \otimes \mathcal{E}$, and the operators D and δ are the classical differential operators defined on p-forms. It can also be seen as a q-form with values in $\Lambda^q TM \otimes \mathcal{E}$ with the corresponding operators $\tilde{\mathsf{D}}$ and $\tilde{\delta}$. We recover the following commuting formulas for tensor-valued forms:

Lemma C.1. *On the space of double-forms, the following identities hold:*

(C.1) $$\mathsf{D}^2 = Rm * \cdot\,, \qquad \tilde{\mathsf{D}}^2 = Rm * \cdot\,,$$

(C.2) $$\delta^2 = Rm * \cdot\,, \qquad \tilde{\delta}^2 = Rm * \cdot\,,$$

(C.3) $$\nabla^*\nabla = \delta\mathsf{D} + \mathsf{D}\delta + Rm * \cdot\,, \qquad \nabla^*\nabla = \tilde{\delta}\tilde{\mathsf{D}} + \tilde{\mathsf{D}}\tilde{\delta} + Rm * \cdot\,.$$

By commuting derivatives, we obtain:

Lemma C.2. *For $j, k \in \mathbb{N}$ and all tensors T,*

(C.4) $$(\nabla)^k(\nabla^*\nabla)^{j+1}T = (\nabla^*\nabla)^{j+1}(\nabla)^k T + \mathcal{P}_{k+2j}(Rm, T)\ .$$

(C.5) $$(\nabla^*\nabla)^2 T = (\nabla^*)^2\nabla^2 T + \mathcal{P}_2(Rm, T)\ .$$

Proof. Let show that

$$\nabla(\nabla^*\nabla)T = (\nabla^*\nabla)\nabla T + \mathcal{P}_1(Rm, T)\ .$$

Indeed, we have

$$\begin{aligned}\nabla^i(\nabla^*\nabla)T &= \nabla^i\nabla^\alpha\nabla_\alpha T\\ &= \nabla^\alpha\nabla^i\nabla_\alpha T + Rm * \nabla T\\ &= \nabla^\alpha\nabla_\alpha\nabla^i T + \nabla(Rm * T) + Rm * \nabla T\\ &= (\nabla^*\nabla)\nabla^i T + \mathcal{P}_1(Rm, T)\ .\end{aligned}$$

Then (C.4) holds by induction.

By contracting the two first indices of (C.4) for $j = 0$ and $k = 1$, we obtain

$$\nabla^*(\nabla^*\nabla)T = (\nabla^*\nabla)\nabla^* T + \mathcal{P}_1(Rm),$$

then we get (C.5) by applying this equality to ∇T. ☐

By changing the order of derivatives, we also get the following formulas:

Lemma C.3. *On the space of double-forms, the following identities hold:*

(C.6) $$\mathsf{D}\tilde{\mathsf{D}} = \tilde{\mathsf{D}}\mathsf{D} + Rm * \cdot \qquad \delta\tilde{\delta} = \tilde{\delta}\delta + Rm * \cdot$$

(C.7) $$\tilde{\mathsf{D}}\delta = \delta\tilde{\mathsf{D}} + Rm * \cdot \qquad \mathsf{D}\tilde{\delta} = \tilde{\delta}\mathsf{D} + Rm * \cdot$$

(C.8) $$\nabla D = \mathsf{D}\nabla + Rm * \cdot \qquad \nabla\tilde{\mathsf{D}} = \tilde{\mathsf{D}}\nabla + Rm * \cdot$$

(C.9) $$\nabla\delta = \delta\nabla + Rm * \cdot \qquad \nabla\tilde{\delta} = \tilde{\delta}\nabla + Rm * \cdot$$

(C.10) $$(\nabla^*\nabla)\mathsf{D} = \mathsf{D}(\nabla^*\nabla) + \mathcal{P}_1(Rm, \cdot\,) \qquad (\nabla^*\nabla)\tilde{\mathsf{D}} = \tilde{\mathsf{D}}(\nabla^*\nabla) + \mathcal{P}_1(Rm, \cdot\,)$$

Then by induction we obtain:

Lemma C.4. *For all $k \in \mathbb{N}$ and all tensors T,*

(C.11)
$$(\nabla)^{k+1}\mathsf{D}T = \mathsf{D}(\nabla)^{k+1}T + \mathcal{P}_k(Rm,T)(\nabla)^{k+1}\tilde{\mathsf{D}}T = \tilde{\mathsf{D}}(\nabla)^{k+1}T + \mathcal{P}_k(Rm,T),$$

(C.12)
$$(\nabla)^{k+1}\delta T = \delta(\nabla)^{k+1}T + \mathcal{P}_k(Rm,T) \ \ (\nabla)^{k+1}\tilde{\delta}T = \tilde{\delta}(\nabla)^{k+1}T + \mathcal{P}_k(Rm,T)\ .$$

And for the first variation of tensors, we have:

Lemma C.5. *For all positive integers k and all tensors T,*

$$(\nabla^k T)'_g(h) = \nabla^k T'_g(h) + \mathcal{P}_{k-1}(\nabla h, T)\ . \tag{C.13}$$

Proof. In coordinates, we have

$$\nabla_i T^{j_1\ldots j_q}_{i_1\ldots i_p} = \partial_i T^{j_1\ldots j_q}_{i_1\ldots i_p} - \sum_{l=1}^{p} \Gamma^{\alpha}_{i i_l} T^{j_1\ldots j_q}_{i_1\ldots\alpha\ldots i_p} + \sum_{l=1}^{q} \Gamma^{j_l}_{i\alpha} T^{j_1\ldots\alpha\ldots j_q}_{i_1\ldots i_p},$$

therefore,

$$\begin{aligned}\nabla T'_g(h)^{j_1\ldots j_q}_{i\, i_1\ldots i_p} &= \nabla_i T'_g(h)^{j_1\ldots j_q}_{i_1\ldots i_p} - \sum_{l=1}^{p} \Gamma'_g(h)^{\alpha}_{i i_l} T^{j_1\ldots j_q}_{i_1\ldots\alpha\ldots i_p} + \sum_{l=1}^{q} \Gamma'_g(h)^{j_l}_{i\alpha} T^{j_1\ldots\alpha\ldots j_q}_{i_1\ldots i_p}\\ &= \nabla_i T'_g(h)^{j_1\ldots j_q}_{i_1\ldots i_p} + \nabla h * T,\end{aligned}$$

then by induction, if $(\nabla^k T)'_g(h) = \nabla^k T'_g(h) + \mathcal{P}_{k-1}(\nabla h, T)$,

$$\begin{aligned}(\nabla^{k+1}T)'_g(h) &= \nabla(\nabla^k T)'_g(h) + \nabla h * \nabla^k T\\ &= \nabla^{k+1}T'_g(h) + \nabla\ \mathcal{P}_k(h,T) + \nabla h * \nabla^k T\\ &= \nabla^{k+1}T'_g(h) + \mathcal{P}_k(\nabla h, T)\ .\end{aligned}$$

□

Proposition C.6.

$$\begin{aligned}Rm'_g(\delta\tilde{\delta}Rm_g) &= -\frac{1}{2}(\nabla^*\nabla)^2 Rm_g + \mathcal{P}^{(2)}_2(Rm_g),\\ Rm'_g(\delta\mathsf{D}(R_g g)) &= -\frac{1}{2}g.\mathsf{D}\tilde{\mathsf{D}}(\Delta R_g) + \mathcal{P}^{(2)}_2(Rm_g),\\ Rm'_g(\tilde{\mathsf{D}}\mathsf{D}R_g) &= \mathcal{P}^{(2)}_2(Rm_g),\end{aligned}$$

and

$$\begin{aligned}R'_g(\delta\tilde{\delta}Rm_g) &= -\frac{1}{2}\Delta^2 R_g + \mathcal{P}^{(2)}_2(Rm_g),\\ R'_g(\delta\mathsf{D}(R_g g)) &= (n-1)\Delta^2 R_g + \mathcal{P}^{(2)}_2(Rm_g),\\ R'_g(\tilde{\mathsf{D}}\mathsf{D}R_g) &= \mathcal{P}^{(2)}_2(Rm_g)\ .\end{aligned}$$

Proof. Since $Rm'_g = -\frac{1}{2}\mathsf{D}\tilde{\mathsf{D}} + Rm_g * \cdot$ (Proposition B.4), we obtain

$$\begin{aligned}Rm'_g(\delta\tilde{\delta}Rm_g) &= -\frac{1}{2}\mathsf{D}\tilde{\mathsf{D}}\delta\tilde{\delta}Rm_g + \mathcal{P}^{(2)}_2(Rm_g)\\ &= -\frac{1}{2}\mathsf{D}\delta\tilde{\mathsf{D}}\tilde{\delta}Rm_g + \mathcal{P}^{(2)}_2(Rm_g) \ \text{ by commuting } \tilde{\mathsf{D}} \text{ and } \delta\\ &= -\frac{1}{2}\nabla^*\nabla\tilde{\mathsf{D}}\tilde{\delta}Rm_g + \frac{1}{2}\delta\mathsf{D}\tilde{\mathsf{D}}\tilde{\delta}Rm_g + \mathcal{P}^{(2)}_2(Rm_g)\end{aligned}$$

(according to the Weitzenböck formula (C.3))

$$= -\frac{1}{2}(\nabla^*\nabla)^2 Rm_g + \frac{1}{2}\delta\tilde{\mathsf{D}}\mathsf{D}\tilde{\delta}Rm_g + \mathcal{P}_2^{(2)}(Rm_g) \text{ by (C.3) since } \tilde{\mathsf{D}}Rm_g = 0$$
$$= -\frac{1}{2}(\nabla^*\nabla)^2 Rm_g + \mathcal{P}_2^{(2)}(Rm_g) \text{ by commuting D and } \tilde{\delta}\text{, since } \mathsf{D}Rm_g = 0.$$

Then we have

$$Rm'_g(\delta\mathsf{D}(R_g g)) = -\frac{1}{2}\mathsf{D}\tilde{\mathsf{D}}\delta\mathsf{D}(R_g g) + \mathcal{P}_2^{(2)}(Rm_g)$$
$$= \frac{1}{2}\mathsf{D}\tilde{\mathsf{D}}\delta(g.\mathsf{D}R_g) + \mathcal{P}_2^{(2)}(Rm_g) \text{ by commuting } g. \text{ and D}$$
$$= -\frac{1}{2}\mathsf{D}\tilde{\mathsf{D}}(g.(\delta\mathsf{D}R_g)) - \frac{1}{2}\mathsf{D}\tilde{\mathsf{D}}\tilde{\mathsf{D}}\mathsf{D}R_g + \mathcal{P}_2^{(2)}(Rm_g)$$

(by using (B.4), since $\tilde{\mathsf{D}}^2 = Rm*\cdot$)

$$= -\frac{1}{2}g.\mathsf{D}\tilde{\mathsf{D}}(\Delta R_g g) + \mathcal{P}_2^{(2)}(Rm_g) \text{ by commuting } g. \text{ and D, } \tilde{\mathsf{D}}.$$

And

$$Rm'_g(\tilde{\mathsf{D}}\mathsf{D}R_g) = -\frac{1}{2}\mathsf{D}\tilde{\mathsf{D}}\tilde{\mathsf{D}}\mathsf{D}R_g + Rm_g * \tilde{\mathsf{D}}\mathsf{D}R_g = \mathcal{P}_2^{(2)}(Rm_g),$$

since $\tilde{\mathsf{D}}^2 = Rm_g*\cdot$.

Using that $R'_g = \tilde{\delta}\delta + \Delta\mathsf{tr} + Rm*\cdot$ (Proposition B.4), we find

$$R'_g(\delta\tilde{\delta}Rm_g) = \Delta\mathsf{tr}\delta\tilde{\delta}Rm_g + \mathcal{P}_2^{(2)}(Rm_g) \text{ since } \delta^2 = Rm*\cdot$$
$$= \Delta\delta\tilde{\delta}Ric_g + \mathcal{P}_2^{(2)}(Rm_g)$$
$$= -\frac{1}{2}\Delta^2 R_g + \mathcal{P}_2^{(2)}(Rm_g) \text{ by the Bianchi identity (Proposition B.1).}$$

and

$$R'_g(\delta\mathsf{D}(R_g g)) = \Delta\mathsf{tr}\delta\mathsf{D}(R_g g) + \mathcal{P}_2^{(2)}(Rm_g) \text{ since } \delta^2 = Rm*\cdot$$
$$= \Delta\delta\mathsf{D}\mathsf{tr}(R_g g) + \Delta\delta\tilde{\delta}(R_g g) + \mathcal{P}_2^{(2)}(Rm_g)$$

(by commuting tr and δ and using (B.3))

$$= (n-1)\Delta^2 R_g + \mathcal{P}_2^{(2)}(Rm_g)\ .$$

and

$$R'_g(\tilde{\mathsf{D}}\mathsf{D}R_g) = \mathsf{tr}\tilde{\delta}\tilde{\mathsf{D}}\tilde{\mathsf{D}}\mathsf{D}R_g + Rm_g * \tilde{\mathsf{D}}\mathsf{D}R_g = \mathcal{P}_2^{(2)}(Rm_g),$$

since $\tilde{\mathsf{D}}^2 = Rm_g*\cdot$. □

Proposition C.7. *For all $k \in \mathbb{N}$, we have*

$$(\nabla^k Rm)'_g(\delta\tilde{\delta}Rm_g) = -\frac{1}{2}\nabla^{*2}\nabla^{k+2}Rm_g + \mathcal{P}_{k+2}^{(2)}(Rm_g),$$
$$(\nabla^k Rm)'_g(\delta\mathsf{D}(R_g g)) = \frac{1}{2}g.(\mathsf{D}\nabla^*\nabla^{k+1}\tilde{\mathsf{D}}R_g) + \mathcal{P}_{k+2}^{(2)}(Rm_g),$$
$$(\nabla^k Rm)'_g(\tilde{\mathsf{D}}\mathsf{D}R_g) = \mathcal{P}_{k+2}^{(2)}(Rm_g),$$

and

$$(\nabla^k R)'_g(\delta\tilde{\delta}Rm_g) = -\frac{1}{2}\nabla^{*2}\nabla^{k+2}R_g + \mathcal{P}_{k+2}^{(2)}(Rm_g),$$
$$(\nabla^k R)'_g(\delta\mathsf{D}(R_g g)) = (n-1)\nabla^{*2}\nabla^{k+2}R_g + \mathcal{P}_{k+2}^{(2)}(Rm_g),$$
$$(\nabla^k R)'_g(\tilde{\mathsf{D}}\mathsf{D}R_g) = \mathcal{P}_{k+2}^{(2)}(Rm_g)\ .$$

Proof. According to Lemma C.5, we obtain

$$
\begin{aligned}
(\nabla^k Rm)'_g(\delta\tilde{\delta}Rm_g) &= \nabla^k Rm'_g(\delta\tilde{\delta}Rm_g) + \mathcal{P}^{(2)}_{k+2}(Rm_g) \\
&= -\frac{1}{2}\nabla^k(\nabla^*\nabla)^2 Rm_g + \mathcal{P}^{(2)}_{k+2}(Rm_g) \text{ according to Proposition C.6} \\
&= -\frac{1}{2}(\nabla^*\nabla)^2\nabla^k Rm_g + \mathcal{P}^{(2)}_{k+2}(Rm_g) \text{ by commuting } \nabla \text{ and } \nabla^*\nabla \text{ (C.4)} \\
&= -\frac{1}{2}(\nabla^*)^2\nabla^{k+2} Rm_g + \mathcal{P}^{(2)}_{k+2}(Rm_g) \text{ by (C.5),}
\end{aligned}
$$

and

$$
\begin{aligned}
(\nabla^k Rm)'_g(\delta\mathsf{D}(R_g g)) &= \nabla^k Rm'_g(\delta\mathsf{D}(R_g g)) + \mathcal{P}^{(2)}_{k+2}(Rm_g) \\
&= -\frac{1}{2}g.(\nabla^k\mathsf{D}\tilde{\mathsf{D}}\Delta R_g) + \mathcal{P}^{(2)}_{k+2}(Rm_g) \text{ according to Proposition C.6} \\
&= -\frac{1}{2}g.\nabla^k\mathsf{D}(\nabla^*\nabla)\tilde{\mathsf{D}}R_g + \mathcal{P}^{(2)}_{k+2}(Rm_g)
\end{aligned}
$$

(by commuting $\tilde{\mathsf{D}}$ and $\nabla^*\nabla$ (C.10))

$$
\begin{aligned}
&= -\frac{1}{2}g.\mathsf{D}\nabla^k(\nabla^*\nabla)\tilde{\mathsf{D}}R_g \text{ by commuting } \nabla \text{ and } \mathsf{D} \text{ (C.11)} \\
&= -\frac{1}{2}g.\mathsf{D}\nabla^*\nabla^{k+1}\tilde{\mathsf{D}}R_g \text{ by (C.5),}
\end{aligned}
$$

and also

$$
(\nabla^k Rm)'_g(\tilde{\mathsf{D}}\mathsf{D}(R_g g)) = \nabla^k Rm'_g(\tilde{\mathsf{D}}\mathsf{D}(R_g g)) + \mathcal{P}^{(2)}_{k+2}(Rm_g) = \mathcal{P}^{(2)}_{k+2}(Rm_g)\ .
$$

Then, we have

$$
\begin{aligned}
(\nabla^k R)'_g(\delta\tilde{\delta}Rm_g) &= \nabla^k(R'_g(\delta\tilde{\delta}Rm_g)) + \mathcal{P}^{(2)}_{k+2}(Rm_g) \\
&= -\frac{1}{2}\nabla^k(\nabla^*\nabla)^2 R_g \\
&= -\frac{1}{2}(\nabla^*\nabla)^2\nabla^k R_g + \mathcal{P}^{(2)}_{k+2}(Rm_g) \text{ by commuting } \nabla \text{ and } \nabla^*\nabla \text{ (C.4)} \\
&= -\frac{1}{2}(\nabla^*)^2\nabla^{k+2} R_g + \mathcal{P}^{(2)}_{k+2}(Rm_g) \text{ by (C.5).}
\end{aligned}
$$

and

$$
\begin{aligned}
(\nabla^k R)'_g(\delta\mathsf{D}(R_g g)) &= \nabla^k(R'_g(\delta\mathsf{D}(R_g g)) + \mathcal{P}^{(2)}_{k+2}(Rm_g) \\
&= (n-1)\nabla^k(\nabla^*\nabla)^2 R_g \\
&= (n-1)(\nabla^*)^2\nabla^{k+2} R_g + \mathcal{P}^{(2)}_{k+2}(Rm_g),
\end{aligned}
$$

and finally,

$$
(\nabla^k R)'_g(\tilde{\mathsf{D}}\mathsf{D}(R_g g)) = \nabla^k R'_g(\tilde{\mathsf{D}}\mathsf{D}(R_g g)) + \mathcal{P}^{(2)}_{k+2}(Rm_g) = \mathcal{P}^{(2)}_{k+2}(Rm_g)\ .
$$

□

APPENDIX D

Interpolation inequalities

For $\varphi \in C_0^\infty(M)$ a nonnegative function and T a tensor, we define the following weighted semi-norms:

$$\begin{aligned}\|T\|_{L^{p,s}(\varphi)} &= \|T\varphi^s\|_{L^p} = \left(\int_{\varphi>0} |T|^p\, \varphi^{sp}\, dv_g\right)^{\frac{1}{p}}\\ \|T\|_{\overset{\circ}{H}{}_k^{p,s}(\varphi)} &= \left\|\nabla^k T\right\|_{L^{p,s+k}} + \|d\varphi\|_\infty^k \|T\|_{L^{p,s}}\\ \|T\|_{H_k^{p,s}(\varphi)} &= \|T\|_{\overset{\circ}{H}{}_k^{p,s}(\varphi)} + \|T\|_{L^{p,s}}\ .\end{aligned}$$

We have the following interpolation inequalities:

Proposition D.1. *Let $k \in \mathbb{N}$, $p \in [2,\infty]$, $q \in [2,\infty)$ and $s \geq 0$. There exists a constant $C(n,k,p,q,s) > 0$ such that for all tensors T and for all $0 \leq j \leq k$,*

$$\begin{aligned}\|T\|_{\overset{\circ}{H}{}_j^{r_j,s}(\varphi)} &\leq C\ \|T\|_{L^{p,s}(\varphi)}^{1-\frac{j}{k}} \|T\|_{\overset{\circ}{H}{}_k^{q,s}(\varphi)}^{\frac{j}{k}}\\ \|T\|_{H_j^{r_j,s}(\varphi)} &\leq C\ \|T\|_{L^{p,s}(\varphi)}^{1-\frac{j}{k}} \|T\|_{H_k^{q,s}(\varphi)}^{\frac{j}{k}}\ .\end{aligned}$$

where

$$\frac{1}{r_j} = \frac{1-j/k}{p} + \frac{j/k}{q}.$$

The second one is a direct consequence of the first one and the usual Hölder inequality. We will now prove the first one by a method taken in [KS01].

Lemma D.2. *Let $p \in [1,\infty]$, $q \in [1,\infty)$, $r \in [2,\infty)$ with $\frac{1}{r} = \frac{1}{2p} + \frac{1}{2q}$ and let $s > 0$. There exists $C(n,r,s) > 0$ such that for all tensors T,*

$$\|\nabla T\|_{L^{r,s+1}}^2 \leq C\ \|T\|_{L^{p,s}(\varphi)} \left(\left\|\nabla^2 T\right\|_{L^{q,s+2}} + \|d\varphi\|_{L^\infty} \|\nabla T\|_{L^{q,s+1}}\right).$$

Proof. Integrating by part, we obtain

$$\begin{aligned}\|\nabla T\|_{L^{r,s+1}}^r &= \int_{\varphi>0} \left\langle \nabla T \mid |\nabla T|^{r-2}\, \nabla T \right\rangle \varphi^{(s+1)r} dv_g\\ &\leq C\Big(\int_{\varphi>0} |T|\left|\nabla^2 T\right||\nabla T|^{r-2}\ \varphi^{(s+1)r} dv_g\\ &\qquad + \|d\varphi\|_{L^\infty} \int_{\varphi>0} |T||\nabla T||\nabla T|^{r-2}\ \varphi^{(s+1)r-1} dv_g\Big)\\ &\leq C\Big(\|T\varphi^s\|_{L^p} \left\|\nabla^2 T\varphi^{s+2}\right\|_{L^q} \left\|\nabla T\varphi^{s+1}\right\|_{L^r}^{r-2}\\ &\qquad + \|d\varphi\|_{L^\infty} \|T\varphi^s\|_{L^p} \left\|\nabla T\varphi^{s+1}\right\|_{L^q} \left\|\nabla T\varphi^{s+1}\right\|_{L^r}^{r-2}\Big)\\ &= C\ \|T\|_{L^{p,s}(\varphi)} \|\nabla T\|_{L^{r,s+1}}^{r-2} \left(\left\|\nabla^2 T\right\|_{L^{q,s+2}} + \|d\varphi\|_{L^\infty} \|\nabla T\|_{L^{q,s+1}}\right).\end{aligned}$$

where we used the Hölder inequality with

$$\frac{1}{p} + \frac{1}{q} + \frac{r-2}{r} = 1.$$

□

Corollary D.3. *Let $k \in \mathbb{N}$, $q \in [2,\infty)$ and $s \geq 0$. There exists a constant $c(n,q,s) > 0$ such that for all tensors T and all $\varepsilon > 0$,*
(D.1)
$$\|d\varphi\|_{L^\infty} \left\|\nabla^k T\right\|_{L^{q,s+k}(\varphi)} \leq \varepsilon \left\|\nabla^{k+1} T\right\|_{L^{q,s+k+1}(\varphi)} + c^k \frac{1+\varepsilon}{\varepsilon} \|d\varphi\|_{L^\infty}^{k+1} \|T\|_{L^{q,s}(\varphi)} .$$

Proof. Applying Lemma D.2 with $p = q = r$, we obtain

$$\begin{aligned}
\|d\varphi\|_{L^\infty} \|\nabla T\|_{L^{q,s+1}} &\leq \sqrt{C}\, \|d\varphi\|_{L^\infty} \|T\|_{L^{q,s}(\varphi)}^{\frac{1}{2}} \left(\left\|\nabla^2 T\right\|_{L^{q,s+2}} + \|d\varphi\|_{L^\infty} \|\nabla T\|_{L^{q,s+1}}\right)^{\frac{1}{2}} \\
&\leq \sqrt{C}\, \|d\varphi\|_{L^\infty} \|T\|_{L^{q,s}(\varphi)}^{\frac{1}{2}} \left(\left\|\nabla^2 T\right\|_{L^{q,s+2}}^{\frac{1}{2}} + \|d\varphi\|_{L^\infty}^{\frac{1}{2}} \|\nabla T\|_{L^{q,s+1}}^{\frac{1}{2}}\right) \\
&\leq \frac{\varepsilon}{2} \left\|\nabla^2 T\right\|_{L^{q,s+2}} + \frac{1}{2} \|d\varphi\|_{L^\infty} \|\nabla T\|_{L^{q,s+1}} \\
&\qquad + \left(\frac{C}{2\varepsilon} + \frac{C}{2}\right) \|d\varphi\|_{L^\infty}^2 \|T\|_{L^{q,s}},
\end{aligned}$$

so we obtain (D.1) for $k = 1$:

$$\|d\varphi\|_{L^\infty} \|\nabla T\|_{L^{q,s+1}} \leq \varepsilon \left\|\nabla^2 T\right\|_{L^{q,s+2}} + \frac{C(1+\varepsilon)}{\varepsilon} \|d\varphi\|_{L^\infty}^2 \|T\|_{L^{q,s}} .$$

Let prove Corollary D.3 by induction on k. We apply the inequality above to $\nabla^k T$, $s+k$ and $\frac{\varepsilon}{2}$:

$$\|d\varphi\|_{L^\infty} \left\|\nabla^{k+1} T\right\|_{L^{q,s+k+1}} \leq \frac{\varepsilon}{2} \left\|\nabla^{k+2} T\right\|_{L^{q,s+k+2}} + \frac{2C(1+\varepsilon)}{\varepsilon} \|d\varphi\|_{L^\infty}^2 \left\|\nabla^k T\right\|_{L^{q,s+k}},$$

We take $c = 16C$ and we can suppose that $4C \geq 1$. By the induction assumption with $\frac{\varepsilon}{4C(1+\varepsilon)}$,

$$\begin{aligned}
\|d\varphi\|_{L^\infty}^2 \left\|\nabla^k T\right\|_{L^{q,s+k}} \leq{}& \frac{\varepsilon}{4C(1+\varepsilon)} \|d\varphi\|_{L^\infty} \left\|\nabla^{k+1} T\right\|_{L^{q,s+k+1}} \\
&+ 2c^k \frac{4C(1+\varepsilon)}{\varepsilon} \|d\varphi\|_{L^\infty}^{k+2} \|T\|_{L^{q,s}} .
\end{aligned}$$

Combining the two last inequalities, we obtain (D.1) for $k+1$. □

Lemma D.4. *Let $k \in \mathbb{N}$, $p \in [2,\infty]$, $q \in [2,\infty)$ and $s \geq 0$. There exists a constant $C(n,k,p,q,s) > 0$ such that for all tensors T,*

$$\|T\|_{\mathring{H}_{k+1}^{r,s}(\varphi)}^2 \leq C \|T\|_{\mathring{H}_k^{p,s}(\varphi)} \|T\|_{\mathring{H}_{k+2}^{q,s}(\varphi)},$$

where $\frac{1}{r} = \frac{1}{2p} + \frac{1}{2q}$.

Proof. According to Lemma D.3, we have

$$\|d\varphi\|_{L^\infty} \left\|\nabla^{k+1} T\right\|_{L^{q,s+k+1}} \leq 2c^{k+1} \|T\|_{\mathring{H}_{k+2}^{q,s}} .$$

Therefore, Lemma D.2 applied to $\nabla^k T$ gives

$$\left\|\nabla^{k+1} T\right\|_{L^{r,s}}^2 \leq C \left\|\nabla^k T\right\|_{L^{p,s}} \|T\|_{\mathring{H}_{k+2}^{q,s}} \leq C \|T\|_{\mathring{H}_k^{p,s}} \|T\|_{\mathring{H}_{k+2}^{q,s}} .$$

And by the Hölder inequality, we have

$$\|T\|_{L^{r,s}}^2 \leq \|T\|_{L^{p,s}} \|T\|_{L^{q,s}} \leq \|T\|_{\mathring{H}_k^{p,s}} \|T\|_{\mathring{H}_{k+2}^{q,s}} .$$

Combining the two inequalities gives the result.

□

We can now apply the following Lemma to prove Proposition D.1:

Lemma D.5 ([Ham82] Corollary 12.5)**.** *Let k be a positive integer.*

If $f : \{0, 1, \ldots, k\} \to \mathbb{R}$ satisfies

$$\forall\, 0 < j < k \quad f(j) \leq C f(j-1)^{1/2} f(j+1)^{1/2},$$

where C is a constant, then

$$\forall\, 0 \leq j \leq k \quad f(j) \leq C^{j(k-j)} f(0)^{1-\frac{j}{k}} f(k)^{\frac{j}{k}}.$$

Let define

$$f(j) = \|T\|_{\overset{\circ}{H}{}^{r_j,s}_j(\varphi)} .$$

Since $\frac{1}{r_j} = \frac{1}{2r_{j-1}} + \frac{1}{2r_{j+1}}$, Lemma D.4 shows that there exists $C(n,k,p,q,s)$ such that

$$f(j) \leq C f(j-1)^{1/2} f(j+1)^{1/2},$$

then Lemma D.5 gives Proposition D.1.

Thanks to this, we can know estimate the integral of tensor products, such as the lower-order terms appearing in the Bando-Bernstein-Shi estimates on the curvature. We begin with the following Lemma:

Lemma D.6. *For all tensors of the form $S * T$, there exists a constant C depending only on the dimension and the coefficients in the expression such that*

$$|S * T| \leq C\, |S||T| .$$

Proof. By Cauchy-Schwarz inequality, the norm of a tensor with contracted indices is not more than the norm of the tensor multiplied by a power of the dimension:

$$(g^{\alpha\beta} T_{\alpha\beta})^2 \leq n\, T_{\alpha\beta} T^{\alpha\beta}.$$

Then,

$$|S * T| \leq C(n) \left| S \otimes T \otimes g^{\otimes j} \otimes (g^{-1})^{\otimes k} \right| \leq C(n) n^{\frac{j+k}{2}}\, |S||T| .$$

□

Proposition D.7. *Let j, k and m be positive integers and let $F_g : \mathcal{T} \to \mathbb{R}$ be a map such that for all tensors T,*

$$F_g(T) = \mathcal{P}^{(j)}_{m,k}(T) .$$

Let $s \geq 0$, let $p \in [2, \infty]$ and $q \in [2, \infty)$ such that

$$\frac{j - m/k}{p} + \frac{m/k}{q} = 1,$$

There exists a constant $C(n,k,p,q,s,F)$ such that for all tensors T,

$$\int_M |F_g(T)|\, \varphi^{m+js} dv_g \leq C\, \|T\|^{j-\frac{m}{k}}_{L^{p,s}(\varphi)} \|T\|^{\frac{m}{k}}_{\overset{\circ}{H}{}^{q,s}_k(\varphi)} .$$

Proof. Let consider one term in $F_g(T)$, that can be written $\nabla^{k_1} T * \cdots * \nabla^{k_j} T$ with $k_i \leq k$ and $k_1 + \cdots + k_j = m$. By Lemma D.6 and the Hölder inequality, we have

$$\begin{aligned} \int_M \left|\nabla^{k_1} T * \cdots * \nabla^{k_j} T\right| \varphi^{m+js} dv_g &\leq C' \int_M \left|\nabla^{k_1} T\right| \cdots \left|\nabla^{k_j} T\right| \varphi^{m+js} dv_g \\ &\leq C' \left\|\nabla^{k_1} T\right\|_{L^{r_1, s+k_1}} \cdots \left\|\nabla^{k_j} T\right\|_{L^{r_j, s+k_j}} \\ &\leq C' \|T\|_{\overset{\circ}{H}{}^{r_1,s}_{k_1}} \cdots \|T\|_{\overset{\circ}{H}{}^{r_j,s}_{k_j}}, \end{aligned}$$

where

$$\frac{1}{r_i} = \frac{1 - k_i/k}{p} + \frac{k_i/k}{q},$$

with

$$\sum_{i=1}^{j} \frac{1}{r_i} = \frac{j - m/k}{p} + \frac{m/k}{q} = 1.$$

According to Proposition D.1, we get

$$\int_M \left|\nabla^{k_1} T * \cdots * \nabla^{k_j} T\right| \varphi^{m+js} dv_g \leq C \, \|T\|_{L^{p,s}(\varphi)}^{j-\frac{m}{k}} \|T\|_{\overset{\circ}{H}{}_k^{q,s}(\varphi)}^{\frac{m}{k}} .$$

The result follows since $F_g(T)$ is a linear combination of such terms. □

Corollary D.8. *Let j, k and m be positive integers such that $j \geq 2$ and $m \leq 2k$, let $s \geq 0$ and let $F_g : \mathcal{T} \to \mathbb{R}$ be a map such that for all tensors T,*

$$F_g(T) = \mathcal{P}_{m,k}^{(j)}(T) .$$

There exists a constant $C(n, k, s, F)$ such that for all tensors T and real number $\alpha > 0$,

$$\alpha \int_M |F_g(T)| \, \varphi^{m+js} dv_g \leq \frac{1}{2} \left\|\nabla^k T\right\|_{L^{2,k+s}}^2 + \left(\|d\varphi\|_{L^\infty}^{2k} + C \left(\alpha \, \|T\|_{L^{\infty,s}}^{j-2} \right)^{\frac{2k}{2k-m}} \right) \|T\|_{L^{2,s}}^2 .$$

Proof. According to Proposition D.7, we have

$$\alpha \int_M |F_g(T)| \, \varphi^{m+js} dv_g \leq C' \alpha \, \|T\|_{L^{p,s}(\varphi)}^{j-\frac{m}{k}} \|T\|_{\overset{\circ}{H}{}_k^{2,s}(\varphi)}^{\frac{m}{k}}$$

with $p = 2\frac{jk-m}{2k-m}$, and it follows that

$$\begin{aligned}
\alpha \int_M |F_g(T)| \, \varphi^{m+js} dv_g &\leq \frac{1}{4} \, \|T\|_{\overset{\circ}{H}{}_k^{2,s}}^2 + C\alpha^{\frac{kp}{kj-m}} \, \|T\|_{L^{p,s}}^p \\
&\leq \frac{1}{2} \left\|\nabla^k T\right\|_{L^{2,s}}^2 + \left(\|d\varphi\|_{L^\infty}^k + C\alpha^{\frac{kp}{kj-m}} \, \|T\|_{L^{\infty,s}}^{p-2} \right) \|T\|_{L^{2,s}}^2 \\
&= \frac{1}{2} \left\|\nabla^k T\right\|_{L^{2,s}}^2 + \left(\|d\varphi\|_{L^\infty}^k + C\alpha^{\frac{2k}{2k-m}} \, \|T\|_{L^{\infty,s}}^{\frac{2k(j-2)}{2k-m}} \right) \|T\|_{L^{2,s}}^2 .
\end{aligned}$$

□

APPENDIX E

Weighted Sobolev inequalities

In this chapter, we will prove the following Sobolev inequality:

Proposition E.1. *Let $(M^{n\geq 3}, g)$ be a Riemannian manifold. Let $k \in \mathbb{N}$ and $s > 0$. There exists $C(n,k,s)$ such that for all function $\varphi \in C_0^\infty(M,[0,1])$ and all tensors T,*

$$\left\|\nabla^k T\right\|_{L^{\infty,[\frac{n}{2}]+k+1+s}(\varphi)} \leq C\, \mathsf{s}_g^{n/2}\, \|T\|_{H^{2,s}_{[\frac{n}{2}]+k+1}(\varphi)}\ .$$

We first give a proof of the following classical multiplicative Sobolev inequalities:

Proposition E.2. *Let $(M^{n\geq 3}, g)$ be a complete Riemannian manifold and let $q \geq 2$. There exists C such that for all $u \in H_1^q(M)$,*

$$\|u\|_{L^p} \leq C\, \mathsf{s}_g^\alpha\, \|u\|_{L^m}^{1-\alpha}\, (\|du\|_{L^q} + \|u\|_{L^q})^\alpha,$$

with $2 \leq m \leq p$ and

$$\begin{aligned}
&\textit{if } q < n && p \leq \frac{nq}{n-q} \quad \textit{and} \quad C = C(n,q),\\
&\textit{if } q = n && p < \infty \quad \textit{and} \quad C = C(m,p),\\
&\textit{if } q > n && p \leq \infty \quad \textit{and} \quad C = C(n,m,q)
\end{aligned}$$

and

$$\alpha = \frac{\frac{1}{m} - \frac{1}{p}}{\frac{1}{m} - \frac{1}{q} + \frac{1}{n}}.$$

Proof. Let consider the function $u^{1+\tau}$ for $\tau > 0$. We have $du^{1+\tau} = (1+\tau)u^\tau du$, therefore $\left\|du^{1+\tau}\right\|_{L^2} \leq (1+\tau)\, \|u^\tau\|_{L^{\frac{2q}{q-2}}} \|du\|_{L^q}$. It follows that

$$\begin{aligned}
\left\|u^{1+\tau}\right\|_{L^{\frac{2n}{n-2}}} &\leq \mathsf{s}_g \left((1+\tau)\, \|u^\tau\|_{L^{\frac{2q}{q-2}}} \|du\|_{L^q} + \left\|u^{1+\tau}\right\|_{L^2}\right)\\
&\leq (1+\tau)\mathsf{s}_g \left(\|u\|^\tau_{L^{\frac{2q\tau}{q-2}}} \|du\|_{L^q} + \|u\|^{1+\tau}_{L^{2(1+\tau)}}\right)\\
&\leq (1+\tau)\mathsf{s}_g\, \|u\|^\tau_{L^{\frac{2q\tau}{q-2}}} \|u\|_{H_1^q}\ .
\end{aligned}$$

Hence we have

$$\|u\|^{1+\tau}_{L^{(1+\tau)\frac{2n}{n-2}}} \leq (1+\tau)\mathsf{s}_g\, \|u\|^\tau_{L^{\frac{2q\tau}{q-2}}} \|u\|_{H_1^q}\ . \tag{E.1}$$

For all $p \neq n$, let define $p^* = \frac{np}{n-p}$.

If $\mathbf{q < n}$: We first suppose that $q > 2$ and we choose

$$\tau = q^* \frac{q-2}{2q} = \frac{\frac{1}{2} - \frac{1}{q}}{\frac{1}{q} - \frac{1}{n}}.$$

Then

$$1 + \tau = \frac{\frac{1}{2} - \frac{1}{n}}{\frac{1}{q} - \frac{1}{n}} = \frac{q^*}{2^*},$$

and by (E.1),

$$\|u\|^{1+\tau}_{L^{q^*}} \leq (1+\tau)\, \mathsf{s}_g\, \|u\|^\tau_{L^{q^*}} \|u\|_{H_1^q}\ .$$

It follows that

$$\|u\|_{L^{q^*}} \leq \frac{q^*}{2^*} \mathsf{s}_g \ \|u\|_{H_1^q},$$

which is also true when $q = 2$. Then, as $\frac{1}{p} = \frac{1-\alpha}{m} + \alpha(\frac{1}{q} - \frac{1}{n})$, we have

$$\|u\|_{L^p} \leq \|u\|_{L^m}^{1-\alpha} \|u\|_{L^{q^*}}^{\alpha} \leq \frac{q^*}{2^*} \mathsf{s}_g^{\alpha} \ \|u\|_{L^m}^{1-\alpha} \|u\|_{H_1^q}^{\alpha} \ .$$

If q $\geq$ n: Let define

$$\gamma = 2^* \frac{q-2}{2q} = \frac{\frac{1}{2} - \frac{1}{q}}{\frac{1}{2} - \frac{1}{n}}.$$

We define a sequence (τ_k) by

$$\tau_0 = m \frac{q-2}{2q} \quad \text{and} \quad \tau_{k+1} = \gamma(1 + \tau_k).$$

Then, according to (E.1),

$$\|u\|_{L^{\frac{2q}{q-2}\tau_{k+1}}}^{\tau_{k+1}} \leq (1 + \tau_k)^{\gamma} \ \|u\|_{L^{\frac{2q}{q-2}\tau_k}}^{\gamma\tau_k} \ \mathsf{s}_g^{\gamma} \ \|u\|_{H_1^q}^{\gamma} \ .$$

It follows by induction that

$$\|u\|_{L^{\frac{2q}{q-2}\tau_{k+1}}}^{\tau_{k+1}} \leq (1 + \tau_k)^{\gamma} \cdots (1 + \tau_0)^{\gamma^{k+1}} \ \|u\|_{L^m}^{\gamma^{k+1}\tau_0} \left(\mathsf{s}_g \ \|u\|_{H_1^q} \right)^{\gamma(1+\gamma+\cdots\gamma^k)} .$$

If q $=$ n: Then $\gamma = 1$, $\tau_k = \tau_0 + k$, and it follows that

$$\|u\|_{L^{\frac{2q}{q-2}\tau_k}}^{\tau_k} \leq \tau_k \tau_{k-1} \cdots \tau_0 \ \|u\|_{L^m}^{\tau_0} \ \mathsf{s}_g^k \ \|u\|_{H_1^n}^k,$$

and thus

$$\|u\|_{L^{\frac{2q}{q-2}\tau_k}} \leq \tau_k^{\frac{k+1}{\tau_k}} \ \|u\|_{L^m}^{\frac{\tau_0}{\tau_k}} \ \mathsf{s}_g^{\frac{k}{\tau_k}} \ \|u\|_{H_1^n}^{\frac{k}{\tau_k}} \ .$$

Now, choose a positive integer k such that $p\frac{q-2}{2q} < \tau_0 + k \leq p$, and let $\theta = \frac{\tau_k}{k}\alpha$. Then $p \leq \frac{2q}{q-2}\tau_k$ and

$$\frac{1}{p} = \frac{1-\alpha}{m} = \frac{1-\theta}{m} + \frac{\tau_0}{m}\frac{\theta}{\tau_k} = \frac{1-\theta}{m} + \frac{\theta}{\frac{2q}{q-2}\tau_k},$$

thus

$$\begin{aligned} \|u\|_{L^p} &\leq \|u\|_{L^m}^{1-\frac{\tau_k}{k}\alpha} \|u\|_{L^{\frac{2q}{q-2}\tau_k}}^{\frac{\tau_k}{k}\alpha} \\ &\leq \tau_k^{\frac{k+1}{k}\alpha} \mathsf{s}_g^{\alpha} \ \|u\|_{L^m}^{1-\alpha} \|u\|_{H_1^n}^{\alpha} \\ &\leq p^{2\alpha} \mathsf{s}_g^{\alpha} \ \|u\|_{L^m}^{1-\alpha} \|u\|_{H_1^n}^{\alpha} \ . \end{aligned}$$

If q $>$ n: Then $\gamma > 1$, $\tau_k = \gamma^k(\tau_0 + \frac{\gamma}{\gamma-1}) - \frac{\gamma}{\gamma-1}$ and we can write

$$\|u\|_{L^{\frac{2q}{q-2}\tau_{k+1}}} \leq \left(\prod_{j=0}^{k} (1 + \tau_j)^{\frac{\gamma^{k-j}}{1+\tau_k}} \right) \|u\|_{L^m}^{\frac{\gamma^{k+1}\tau_0}{\tau_{k+1}}} \left(\mathsf{s}_g \ \|u\|_{H_1^q} \right)^{\frac{\gamma}{\gamma-1}\frac{\gamma^{k+1}-1}{\tau_{k+1}}} ,$$

and since $\tau_k \sim c\gamma^k$, the product is convergent:

$$\ln\Big(\prod_{j=0}^{k} (1 + \tau_j)^{\frac{\gamma^{k-j}}{1+\tau_k}} \Big) = \sum_{j=0}^{k} \frac{\gamma^{k-j}}{1+\tau_k} \ln(1 + \tau_j) \leq \sum_{j=0}^{\infty} \frac{1}{\tau_0 \gamma^j} \ln\big(\gamma^j(\tau_0 + \frac{\gamma}{\gamma-1})\big).$$

By letting $k \to \infty$, we obtain

$$\|u\|_{L^\infty} \leq C(n, m, q) \ \|u\|_{L^m}^{\frac{\tau_0}{\tau_0 + \frac{\gamma}{\gamma-1}}} \left(\mathsf{s}_g \ \|u\|_{H_1^q} \right)^{\frac{\frac{\gamma}{\gamma-1}}{\tau_0 + \frac{\gamma}{\gamma-1}}} ,$$

where $C(n,m,q) = \prod_{j=0}^{\infty}(1+\tau_j)^{\frac{\gamma^{k-j}}{1+\tau_k}}$ and we check that

$$\frac{\frac{\gamma}{\gamma-1}}{\tau_0 + \frac{\gamma}{\gamma-1}} = \frac{1}{1+m(\frac{1}{n}-\frac{1}{q})} = \frac{\frac{1}{m}}{\frac{1}{m}-\frac{1}{q}+\frac{1}{n}},$$

which gives the result when $p=\infty$.

Finally, by the Hôlder inequality, we obtain

$$\begin{aligned}\|u\|_{L^p} &\leq \|u\|_{L^m}^{\frac{m}{p}} \|u\|_{L^\infty}^{1-\frac{m}{p}} \\ &\leq C(n,m,q)\ \|u\|_{L^m}^{1-\frac{\frac{1}{m}-\frac{1}{p}}{\frac{1}{m}-\frac{1}{q}+\frac{1}{n}}} \left(\mathsf{s}_g\ \|u\|_{H_1^q}\right)^{\frac{\frac{1}{m}-\frac{1}{p}}{\frac{1}{m}-\frac{1}{q}+\frac{1}{n}}}.\end{aligned}$$

□

Lemma E.3. *Let $p \geq 2$, $s>0$ and $j,k \in \mathbb{N}$. There exists $C(n,p,k,j,s)$ such that for all tensor T and all nonnegative function $\varphi \in C_0^\infty(M)$,*

$$(1+\|d\varphi\|^k)\ \|T\|_{H_j^{p,s}(\varphi)} \leq C\ \|T\|_{H_{k+j}^{p,s}(\varphi)} \tag{E.2}$$

$$\left\|\nabla^j T\right\|_{H_k^{p,s+j}(\varphi)} \leq C\ \|T\|_{H_{k+j}^{p,s}(\varphi)} \tag{E.3}$$

Proof. According to Proposition D.1, we have

$$\|T\|_{H_j^{p,s}(\varphi)} \leq c\ \|T\|_{L^{p,s}(\varphi)}^{\frac{k}{j+k}} \|T\|_{H_{k+j}^{p,s}(\varphi)}^{\frac{j}{j+k}}$$

hence

$$\begin{aligned}(1+\|d\varphi\|^k)\ \left\|\nabla^j T\right\|_{L^{p,s+j}(\varphi)} &\leq c\left(\frac{k}{j+k}(1+\|d\varphi\|^k)^{\frac{j+k}{k}}\ \|T\|_{L^{p,s}(\varphi)} + \frac{j}{j+k}\ \|T\|_{H_{k+j}^{p,s}(\varphi)}\right) \\ &\leq 2c\ \|T\|_{H_{k+j}^{p,s}(\varphi)}\end{aligned}$$

To prove (E.3), we write

$$\left\|\nabla^j T\right\|_{H_k^{p,s+j}(\varphi)} = \left\|\nabla^{k+j} T\right\|_{L^{p,s+j+k}(\varphi)} + (1+\|d\varphi\|^k)\ \left\|\nabla^j T\right\|_{L^{p,s+j}(\varphi)},$$

According to (E.2),

$$(1+\|d\varphi\|^k)\ \left\|\nabla^j T\right\|_{L^{p,s+j}(\varphi)} \leq (1+\|d\varphi\|^k)\ \|T\|_{H_j^{p,s}(\varphi)} \leq C'\ \|T\|_{H_{k+j}^{p,s}(\varphi)},$$

hence

$$\left\|\nabla^j T\right\|_{H_k^{p,s+j}(\varphi)} \leq (C'+1)\ \|T\|_{H_{k+j}^{p,s}(\varphi)}\ .$$

□

We now take $\varphi \in C_0^\infty(M,[0,1])$. We have the following Sobolev inequality:

Lemma E.4. *Let $q \geq 2$ and $\varphi \in C_0^\infty(M,[0,1])$. There exists C such that for all tensor T,*

$$\|T\|_{L^{p,s+1}(\varphi)} \leq C\,\mathsf{s}_g^{\frac{n(p-q)}{pq}}\ \|T\|_{H_1^{q,s}(\varphi)}$$

with $p \geq 2$ and

$$\begin{aligned}&\textit{if } q<n \quad p \leq \frac{nq}{n-q} \quad \textit{and} \quad C = C(n,q,s),\\ &\textit{if } q=n \quad p<\infty \quad \textit{and} \quad C=C(n,p,s),\\ &\textit{if } q>n \quad p\leq\infty \quad \textit{and} \quad C=C(n,q,s)\end{aligned}$$

Proof. According to Proposition E.2 with $q = m$ and $\alpha = \frac{n(p-q)}{pq}$, we have

$$\|T\|_{L^{p,s+1}(\varphi)} = \left\|T\varphi^{s+1}\right\|_{L^p} \leq C\,\mathsf{s}_g^{\alpha}\left(\left\|\nabla(|T|\,\varphi^{s+1})\right\|_{L^q} + \left\|T\varphi^{s+1}\right\|_{L^q}\right),$$

and by the Kato inequality,

$$\begin{aligned}\left|\nabla(|T|\,\varphi^{s+1})\right| &\leq |\nabla\,|T||\,\varphi^{s+1} + (s+1)\,|T||d\varphi|\,\varphi^s\\ &\leq |\nabla T|\,\varphi^{s+1} + (s+1)\,|T||d\varphi|\,\varphi^s,\end{aligned}$$

hence we obtain

$$\begin{aligned}\|T\|_{L^{p,s+1}(\varphi)} &\leq C\,\mathsf{s}_g^{\alpha}\left(\left\||\nabla T|\,\varphi^{s+1}\right\|_{L^q} + (s+1)\,\|d\varphi\|_{L^\infty}\|T\varphi^s\|_{L^q} + \left\|T\varphi^{s+1}\right\|_{L^q}\right)\\ &\leq (s+1)C\,\mathsf{s}_g^{\alpha}\,\|T\|_{H_1^{q,s}(\varphi)}\end{aligned}$$

□

Lemma E.5. *Let $k, l \in \mathbb{N}$ and $q \geq 2$ such that $l < \frac{n}{q}$ and let $p = \frac{nq}{n-lq}$. There exists $C(n,k,q,s)$ such that for all tensors T,*

$$\|T\|_{H_k^{p,s+l}(\varphi)} \leq C\mathsf{s}_g^{l}\,\|T\|_{H_{k+l}^{q,s}(\varphi)}\ .$$

Proof. According to Lemma E.4 with $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ and (E.3), we have

$$\begin{aligned}\left\|\nabla^k T\right\|_{L^{p,s+k+1}(\varphi)} &\leq C\,\mathsf{s}_g\,\left\|\nabla^k T\right\|_{H_1^{q,s+k}(\varphi)}\\ &\leq C'\,\mathsf{s}_g\,\|T\|_{H_{k+1}^{q,s}(\varphi)}\end{aligned}$$

and we also have

$$\|T\|_{L^{p,s+1}(\varphi)} \leq C\,\mathsf{s}_g\,\|T\|_{H_1^{q,s}(\varphi)}$$

therefore, according to (E.2),

$$(1 + \|d\varphi\|^k)\,\|T\|_{L^{p,s+1}(\varphi)} \leq C'\mathsf{s}_g\,\|T\|_{H_{k+1}^{q,s}(\varphi)}$$

By adding the two inequalities we get the result when $l = 1$. But we also have

$$\|T\|_{H_{k+l}^{r,s+k+l}(\varphi)} \leq C\mathsf{s}_g\,\|T\|_{H_{k+l+1}^{q,s}(\varphi)}\ .$$

with $\frac{1}{r} = \frac{1}{q} - \frac{1}{n}$, hence, with $\frac{1}{p} = \frac{1}{r} - \frac{l}{n}$, we have $\frac{1}{p} = \frac{1}{q} - \frac{l+1}{n}$ and the result holds by induction on l. □

We can now prove Proposition E.1. From Lemma E.4, it follows that there exists $C_1(n,s)$ such that

$$\|T\|_{L^{\infty,s+[\frac{n}{2}]+1}} \leq C_1\mathsf{s}_g^{1/2}\,\|T\|_{H_1^{2n,s+[\frac{n}{2}]}}\ .$$

If n=2l+1: Applying Lemma E.5 with $q = 2$, we get $C_2(n,s)$ such that

$$\|T\|_{H_1^{2n,s+l}} \leq C_2\mathsf{s}_g^{l}\,\|T\|_{H_{l+1}^{2,s}},$$

and since $l = [\frac{n}{2}]$, we obtain

$$\|T\|_{L^{\infty,s+[\frac{n}{2}]+1}} \leq C\mathsf{s}_g^{n/2}\,\|T\|_{H_{[\frac{n}{2}]+1}^{2,s}}\ .$$

If n=2l+2: Applying Lemma E.4, we get

$$\begin{aligned}\|\nabla T\|_{L^{2n,s+l+2}} &\leq C_3\mathsf{s}_g^{1/2}\,\|\nabla T\|_{H_1^{n,s+l+1}}\\ &\leq C_3'\mathsf{s}_g^{1/2}\,\|T\|_{H_2^{n,s+l}}\ .\end{aligned}$$

by (E.3), and

$$(1+ \|d\varphi\|_{L^\infty}) \|T\|_{L^{2n,s+l+1}} \leq C_4 \mathsf{s}_g^{1/2}(1+ \|d\varphi\|_{L^\infty}) \|T\|_{H_1^{n,s+l}}$$
$$\leq C_4' \mathsf{s}_g^{1/2} \|T\|_{H_2^{n,s+l}} .$$

by (E.2). Hence we have

$$\|T\|_{H_1^{2n,s+l+1}} \leq C' \mathsf{s}_g^{1/2} \|T\|_{H_2^{n,s+l}} .$$

According to Lemma E.5 with $q = 2$, we have

$$\|T\|_{H_2^{n,s+l}} \leq C'' \mathsf{s}_g^{l} \|T\|_{H_{l+2}^{2,s}},$$

and since $l = [\frac{n}{2}] - 1$, we obtain

$$\|T\|_{L^{\infty,s+[\frac{n}{2}]+1}} \leq C \mathsf{s}_g^{n/2} \|T\|_{H_{[\frac{n}{2}]+1}^{2,s}} .$$

Finally, we get the result by applying this inequality to $\nabla^k T$ and by using (E.3). □

APPENDIX F

The refined Kato inequality for 1-forms.

Assume that (M^n, g) is a complete Riemannian manifold and that $\xi \in C^\infty(T^*M)$ is an harmonic 1-form. The refined Kato inequality asserts that

$$|d\,|\xi||^2 \leq \frac{n-1}{n}\,|\nabla\xi|^2\,. \tag{F.1}$$

Suppose that equality holds almost everywhere in (F.1). We can locally find a primitive Φ of ξ:

$$d\Phi = \xi.$$

Then Φ is an harmonic function and in this case, the refined Kato inequality is in fact the Yau inequality for harmonic functions ([Yau75, Lemma 2]). Moreover, passing to the normal covering $\pi\colon \widehat{M} \to M$ associated to the kernel of the homomorphism

$$\gamma \in \pi_1(M) \mapsto \int_\gamma \xi,$$

we have $\pi^*\xi = d\Phi$ for an harmonic function $\Phi \in C^\infty(\widehat{M})$ (see for instance [Hat02]). We will review the proof of a result of P. Li and J. Wang ([LW06]).

Proposition F.1. *Assume that (M^n, g) is a complete Riemannian manifold carrying a non-constant harmonic function Φ such that almost everywhere*

$$|d|d\Phi||^2 = \frac{n-1}{n}\,|\nabla d\Phi|^2\,.$$

Then there exists a complete Riemannian manifold (N^{n-1}, h) such that (M^n, g) is isometric to $N^{n-1} \times \mathbb{R}$ endowed with a warped product metric $\eta^2(t)h + (dt)^2$. Moreover, there are constants c_1, c_2 such that

$$\Phi(x,t) = c_1 + c_2 \int_0^t \frac{dr}{\eta(r)^{n-1}}.$$

Proof. We assume that on $U = \{x \in M^n,\ d\Phi(x) \neq 0\}$

$$|d|d\Phi||^2 = \frac{n-1}{n}\,|\nabla d\Phi|^2\,.$$

Moreover, we add a constant to Φ such that the set

$$N = \{x \in U, \Phi(x) = 0\}$$

is not empty.

The equality case in the Yau's inequality implies that there is a function $a\colon U \to \mathbb{R}$ such that if we let $\vec{\nu} = \frac{\nabla\Phi}{|\nabla\Phi|}$, then in the orthogonal decomposition $T_xM^n = \ker(d\Phi) \oplus \mathbb{R}\,\vec{\nu}$, we have

$$\nabla d\Phi = \begin{pmatrix} a\,\mathrm{Id} & 0 \\ 0 & -(n-1)a \end{pmatrix}.$$

Therefore, we see that

$$\vec{u} \in \ker(d\Phi) \ \Rightarrow\ d_{\vec{u}}\left(|d\Phi|^2\right) = 0.$$

Hence the length of $d\Phi$ is locally constant on the regular level sets of Φ. Moreover, we have

$$\nabla_{\vec{u}}(\vec{\nu}) = \frac{1}{|d\Phi|} \left(\nabla_{\vec{u}}(\nabla\Phi) - \langle \nabla_{\vec{u}}(\nabla\Phi) \mid \vec{\nu}\rangle \, \vec{\nu}\right),$$

and since $\nabla_{\vec{\nu}}(\nabla\Phi)$ is in $\mathbb{R}\,\vec{\nu}$, $\nabla_{\vec{\nu}}(\vec{\nu}) = 0$ and the integral curves of the vector field $\vec{\nu}$ are geodesics.

We consider the map $E\colon N \times \mathbb{R} \to M^n$ given by

$$E(x,t) = \exp_x(t\vec{\nu}(x)).$$

For $x \in N$, and $\vec{u} \in \vec{\nu}^{\perp}$, $\nabla_{\vec{u}} E(x,t)$ is a Jacobi field along $E(x,t)$, hence is orthogonal to $\vec{\nu}$ for all $t \in \mathbb{R}$. Consequently, $\Phi(E(x,t))$ only depends on t and there exists a function $\psi\colon \mathbb{R} \to \mathbb{R}$ such that $\psi(0) = 0$ and for all $(x,t) \in N \times \mathbb{R}$,

$$\Phi(E(x,t)) = \psi(t).$$

We fix $K \subset N$ a compact subset, and we let (α,ω) be the maximal open set containing 0 such that $E\colon K \times (\alpha,\omega) \to M^n$ is a local diffeomorphism. Then, on $K \times (\alpha,\omega)$, we have

$$E^*\left(\nabla d\Phi\right) = \psi'' dt \otimes dt + \psi' \nabla dt.$$

Consequently,

$$\psi'' = -(n-1)(a \circ E) \quad \text{and} \quad \psi' \nabla dt = (a \circ E)\left(E^* g\right).$$

Hence $a \circ E$ only depends on t, and the hypersurfaces $K \times \{t\} \subset (K \times (\alpha,\omega), E^* g)$ are totally umbilical. Therefore, we get that on $K \times (\alpha,\omega)$,

$$E^* g = \eta^2(t) h + (dt)^2,$$

with

$$a \circ E = \frac{\eta'}{\eta} \quad \text{and} \quad \psi(t) = c \int_0^t \frac{dr}{\eta(r)^{n-1}}. \tag{F.2}$$

Now, if ω is finite, then for some $x \in K$, $(E^* g)(x,w)$ is not invertible, hence we must have $\lim_{t\to\omega} \eta(t) = 0$. According to (F.2), we also have $\lim_{t\to\omega} \eta'(t) = 0$, thus $\eta(t) = o(w-t)$ and

$$\psi(w) = c \int_0^{\omega} \frac{dr}{\eta(r)^{n-1}} = +\infty,$$

which is not possible. Hence $\omega = +\infty$ and the same argument shows that $\alpha = -\infty$.

Therefore, $E\colon N \times \mathbb{R} \to M^n$ is an immersion. Since $d\Phi$ is locally constant on the level sets of Φ, N is a connected component of the closed set $\{x \in M, \phi(x) = 0\}$, thus is closed. Then, as E is a local isometry, $E(N \times \mathbb{R})$ is complete, hence closed in M^n, and open, thus E is a surjection.

Moreover, if $E(x,s) = E(y,t)$, then $\psi(s) = \psi(t)$ hence $s = t$, and following the flow of $-\vec{\nu}$ from $E(x,t)$ or $E(y,t)$ for a time t, we see that $x = y$. Therefore, E is also injective. $\square$

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