On Minkowski dimension of quasicircles

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Chapter 1

Introduction

1.1 General introduction

Let $\Omega \subsetneqq \mathbb{C}$ be a simply connected domain containing 0: by the Riemann Mapping theorem, there is a unique conformal map f from the unit disk $\mathbb{D} = \{|z| < 1\}$ onto Ω such that $f(0) = 0, f'(0) > 0$. In this thesis we are interested in domains with fractal boundary and more precisely in the Hausdorff dimension of these boundaries. Well-known examples of fractal curves which have deserved a lot of investigations and attentions are the Julia sets and the limit sets of quasifuchsian groups because of their dynamical properties.
For instance, let us consider the family of quadratic polynomials

$$P_t(z) = z^2 + t,\ t \in \mathbb{C}$$

in the neighborhood of $t = 0$. There is a smooth family of conformal maps ϕ_t from $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ onto the basin of infinity of the polynomial $P_t(z)$ (the component containing ∞ of its Fatou set) with $\phi_0(z) = z$ and conjugating P_0 to P_t on their basins of infinity. We thus have:

$$\phi_t(P_0) = P_t(\phi_t(z)),\ z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}. \tag{1.1}$$

Each ϕ_t extends to a quasiconformal map on the sphere $\overline{\mathbb{C}}$. Taking the derivative of the equation (1.1) with respect to t, we obtain the equation:

$$\dot{\phi}_t(z^2) = 2\phi_t(z)\dot{\phi}_t(z) + 1,$$

where $\dot{\phi}_t = \dfrac{\partial \phi}{\partial t}$. Let $V(z)$ denote the holomorphic vector field of $V(z) = \dfrac{\partial \phi_t}{\partial t}\Big|_{t=0}$. Using thermodynamic formalism, Ruelle [Rue82] (see also [Zin96] and [MM08]) proved that

$$\frac{d^2}{dt^2}\mathrm{H.dim}(J(P_t))\Big|_{t=0} = \lim_{r \longrightarrow 1} \frac{1}{4\pi} \frac{1}{\log \frac{1}{1-r}} \int_{|z|=r} |V'(z)|^2 |dz|. \tag{1.2}$$

Using then the explicit formula of V, he could proved that

$$\mathrm{H.dim}(J(P_t)) = 1 + \frac{|t|^2}{4\log 2} + o(|t|^2). \tag{1.3}$$

for this particular family. For more details, see Chapter 2.

Passing to the disc instead of its complement, in [MM08], Mc Mullen asked the following question: Under what general circumstances does a smooth family of conformal maps $\phi_t : \mathbb{D} \longrightarrow \overline{\mathbb{C}}$ with $\phi_0 = id$ satisfy

$$\frac{d^2}{dt^2}\text{H.dim}\,(\phi_t(\partial\mathbb{D}))\Bigg|_{t=0} = \lim_{r\longrightarrow 1}\frac{1}{4\pi|\log(1-r)|}\int_{|z|=r}|\dot{\phi}_0'(z)|^2|dz| \quad ? \tag{1.4}$$

The question addresses the problem of how much formula (1.4) owes to dynamical properties. In [MM08], again Mc Mullen confirmed that this formula holds for all the smooth family of polynomial $F_t = z^d + t(b_2 z^{d-2} + ... + b_d)$. In more details, the Julia set $\Gamma_t = J(F_t)$ is a Jordan curve, with $\Gamma_0 = J(F_0) = \mathbb{T}$ and there is a unique smooth family of conformal maps $\phi_t : \overline{\mathbb{C}} \setminus \mathbb{D} \longrightarrow \mathbb{C}$ conjugating the action of F_0 to F_t, satisfying $\phi_0 = id$ and extending quasiconformal on the whole plane $\overline{\mathbb{C}}$. The Hausdorff dimension δ_t of Γ_t is real analytic and moreover by Bowen formula [Bow79] δ_t is the unique zero of the pressure function $P(-\delta_t \log|\phi_t'(z)|)$. Then (1.4) could be derived from the equation

$$P(-\delta_t \log|\phi_t'(z)|) = 0.$$

1.2 The setting of the problem

Let us consider a general analytic one-parameter family (ϕ_t), $t \in U$ (a neighborhood of $t = 0$), conformal maps with $\phi_0 = id$ and $\phi_t(0) = 0$, $\forall t \in U$. Then

$$\phi_t(z) = \int_0^z e^{\log\phi_t'(u)}du$$

and

$$\frac{\partial}{\partial t}\phi_t(z) = \int_0^z \frac{\partial}{\partial t}\Big(\log\phi_t'(u)\Big)e^{\log\phi_t'(u)}du.$$

From which follows that

$$V(z) = \frac{\partial}{\partial t}\phi_t(z)\Bigg|_{t=0} = \int_0^z \frac{\partial}{\partial t}\Big(\log\phi_t'(u)\Big)\Bigg|_{t=0}du$$

and $b(z) = V'(z) = \frac{\partial}{\partial t}\Big(\log\phi_t'(z)\Big)\Big|_{t=0}$ belongs to the Bloch space $\mathcal{B}$ which is defined as follows:

$$\mathcal{B} = \Bigg\{ b \text{ holomorphic in } \mathbb{D};\ \sup_{\mathbb{D}}(1-|z|)|b'(z)| < \infty \Bigg\}.$$

A subspace of Bloch space is the little Bloch space $\mathcal{B}_0$ which is defined as

$$\mathcal{B}_0 = \Bigg\{ b \text{ holomorphic in } \mathbb{D};\ \lim_{|z|\longrightarrow 1}(1-|z|)|b'(z)| = 0 \Bigg\}.$$

It follows from Manẽ-Sad-Sullivan's theorem (see [GL00]) ϕ_t has a quasiconformal extension to the plane if t is small enough. In particular $\Gamma_t = \phi_t(\partial\mathbb{D})$ is well-defined.

Conversely, starting from a function $b \in \mathcal{B}$, it is known that if we put

$$\phi_t(z) = \int_0^z e^{tb(u)}du, \quad b \in \mathcal{B}, \tag{1.5}$$

is an analytic family and there exists a neighborhood U of 0 such that if $t \in U$ then ϕ_t is a conformal map with quasiconformal extension and we denote by Γ_t the image of the unit circle by ϕ_t.

The natural framework for this study is the universal Teichmüller space. Let us defined a version of its as

$$\mathcal{T} = \left\{ \log \phi', \phi : \mathbb{D} \longrightarrow \mathbb{C} \text{ holomorphic and injective with quasiconformal extension to } \mathbb{C} \right\}.$$

It is known since Ahlfors-Beurling's work that $\mathcal{T}$ is an open set of $\mathcal{B}$ and in particular a neighborhood of 0. Given $b \in \mathcal{B}$, seen as a vector of the tangent space of $\mathcal{T}$ at 0, the present work studies the asymptotic properties of $\Lambda(tb)$ as t goes to 0, where $\Lambda(b)$ stands for Minkowski dimension of Γ_t where $\Gamma_t = \phi_t(\partial\mathbb{D}), \phi_t' = e^{tb}$.

In this situation, although one can not use thermodynamic formalism to treat the problem anymore, it does not mean that Ruelle's result is unhelpful in this case. On the contrary, Ruelle's formula still takes an important role as a prediction that (1.4) may work for some other cases. The problem we first meet when we leave the dynamical context is that the Hausdorff dimension of the quasicirle Γ_t is quite difficult to handle analytically. We use another notion of fractal dimension: the Minkowski dimension, that can be derived from the spectrum of the integral means of $\phi_t'(z)$ and thereby we obtain some positive answers for Mc Mullen's open question. More precisely, we first point out a condition which is stated in the term of square function of the dyadic martingale of the Bloch function $b(z)$ for which the smooth family of conformal map

$$\phi_t(z) = \int_0^z e^{tb(u)} du, \quad z \in \mathbb{D}, t \in U$$

satisfies (1.4) where the Hausdorff dimension replaced by the Minkowski dimension. In other words, we prove, using a probability argument, that for a relatively large class of functions in $\mathcal{B}$

$$\text{M.dim}(\Gamma_t) = 1 + \limsup_{r \longrightarrow 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta}{4\pi \log \frac{1}{1-r}} \frac{t^2}{2} + o(t^2). \tag{1.6}$$

In dynamical context, the two formulas coincide. We also show that a similar result can be derived for the case of b being a lacunary series by using a classical analytic argument.

On the other hand, we prove that (1.6) cannot hold for all $b \in \mathcal{B}$ by constructing a counterexample. This construction is reminiscent of Kahane's construction of a non Smirnov domain.

For the reader's convenience, let us describe briefly the content of each chapter of this thesis. In this first chapter we have already given motivation and the principal points of this thesis.

The purpose of Chapter 2 is to reproduce the calculation due to Ruelle [Rue82] (see also [Zin96], [MM08]) of the Hausdorff dimension of the Julia set of the family of quadratic polynomial $P_t(z) = z^2 + t$ with t in the principal cardioid $\mathcal{C}$. Concretely, we'll show that in the neighborhood of $t = 0$, Hausdorff dimension of the Julia set has the development:

$$\text{H.dim}(J(P_t)) = 1 + \frac{|t|^2}{4 \log 2} + o(|t|^2).$$

Although this result is well-known, we still would like to write it down here so as to introduce to the reader the original problem and how the above formula is obtained by using thermodynamic formalism from which in [MM08] Mc Mullen could generalize this result to the Julia sets of the family of hyperbolic rational maps and the limit sets of the family of quasifuchsian groups and then he asked the above question.

The third chapter consists in the first principal result of this thesis. We will describe a large family of Bloch function b for which if $\phi_t(z) = \int_0^z e^{tb(u)} du$, $z \in \mathbb{D}, t \in U$, then (1.6) is true. This class will be defined in term of the square function of the associated of dyadic martingale of Reb.

In the following chapter, we study a particular case of Bloch function: we show that if b is given by a lacunary series, then (1.6) holds for the conformal map ϕ_t (defined above). The method that we use in this part is based on Kayumov's work ([Kay01]). In the next section of this chapter, we give an example about the Bloch series b_{RS} which is constructed from the Rudin-Shapiro polynomials. By using similar technique as for lacunary series, it give us an upper bounded for the spectrum of integral means of $\phi' = \exp b_{RS}$.

The last chapter will be reserved to the second principal result of this thesis. It consists in a counterexample for the formula (1.6). The starting point is the construction by Kahane and Piranian of a so-called "non-Smirnov" rectifiable domain. These authors have constructed a Bloch function b such that if we consider the associated family (ϕ_t) as above the $\phi_t(\partial\mathbb{D})$ is rectifiable for $t < 0$. This function is very singular in the sense that

$$b(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta),$$

where μ is singular with respect to Lebesgue measure on the circle. We use this feature to prove that there exists $c > 0$ such that

$$\text{M.dim}(\Gamma_t) \geq 1 + ct^2, \quad t > 0 \text{ small}$$

which contradicts $\lim_{t \longrightarrow 0} \frac{\text{M.dim}(\Gamma_t) - 1}{t^2} = 0$ by (1.6).

Chapter 2

Thermodynamic formalism and holomorphic dynamical systems of quadratic polynomials

In this chapter, we first introduce the holomorphic dynamical systems of the quadratic polynomials $P_t(z) = z^2 + t,\ \ z, t \in \mathbb{C}$. If t stays in the principal cardioid $\mathcal{C}$ of the Mandelbrot set $\mathcal{M}$ then the Julia set $J(P_t)$ is a Jordan curve on which the polynomial P_t acts as an expanding conformal repeller. Base on this fact, we make use of thermodynamic formalism to compute the Hausdorff dimension of the Julia set $J(P_t)$. More precisely, in the neighborhood of $t = 0$, the Hausdorff dimension of the Julia set $J(P_t)$ has the development:

$$\text{H.dim}(J(P_t)) = 1 + \frac{|t|^2}{4\log 2} + o(|t|^2).$$

2.1 Holomorphic dynamical systems of quadratic polynomials.

Let

$$P_t(z) = z^2 + t, \quad (z, t \in \overline{\mathbb{C}}).$$

We will study the behaviour of the sequence of the iterations

$$P_t^{\circ n}(z) = \underbrace{(P_t \circ P_t \circ \ldots \circ P_t)}_{\text{n times}}(z).$$

First, let us recall some notions. The *forward orbit* of $z \in \overline{\mathbb{C}}$ is the finite or infinite sequence $(z, P_t(z), P_t^{\circ 2}(z), \ldots)$. A point $z \in \overline{\mathbb{C}}$ is called a *fixed point* of the polynomial P_t if $P_t(z) = z$. And a point $z \in \overline{\mathbb{C}}$ is called a *periodic point* of period k of P_t if there exists a positive integer k such that $P_t^{\circ k}(z) = z$ and $P_t^{\circ j}(z) \neq z$ for $1 \leq j \leq k-1$. This periodic point z of period k is *attracting*; *repelling* or *indifferent* if its multiplier $\lambda = |(P_t^k(z))'|$ is strictly less than 1; strictly bigger than 1 or equal to 1. The point 0 which satisfies $P_t'(z) = 0$ is called the *critical point*.

2.1.1 The Fatou set and the Julia set.

We define the *basin of infinity* as the set of points escaping to infinity:

$$D(\infty) = \{z \in \mathbb{C} : P_t^{\circ n}(z) \longrightarrow \infty\}.$$

We call its complement the *filled Julia set* and denote it $K(P_t) = \mathbb{C} \setminus D(\infty)$. The filled Julia set is never void since P_t always have a repelling cycle. We define the *Julia set* as the common boundary of $K_t(P)$ and $D(\infty)$: $J(P_t) = \partial K(P_t) = \partial D(\infty)$. The *Fatou set* which is denoted by $\Omega(P_t)$ is defined as the complement of Julia set: $\Omega(P_t) = \overline{\mathbb{C}} \setminus J(P_t)$.

Proposition 2.1. *Let $r = 1 + |t|$. Then $K(P_t) = \bigcap_{n\geq 0} P_t^{\circ -n}(\overline{D_r})$, where $\overline{D_r} = \{|z| \leq r\}$.*

Proof: First let us recall the fact that for $|z| \geq r = 1 + |t|$, we have

$$|P_t(z)| \geq \frac{|z|^2}{r}. \tag{2.1}$$

Because

$$|z^2 + t| = |z|^2 \left| 1 + \frac{t}{z^2} \right| \geq |z|^2 \left(1 - \frac{|t|}{|z|^2} \right) \geq |z|^2 \left(\frac{1 + |t| + |t|^2}{(1 + |t|)^2} \right) \geq \frac{|z|^2}{1 + |t|} = \frac{|z|^2}{r}.$$

We observe that $K(P_t) \supset \bigcap_{n\geq 0} P_t^{\circ -n}(\overline{D_r})$ follows from the definition.

Moreover, if we assume that $K(P_t) \setminus (\bigcap_{n\geq 0} P_t^{\circ -n}(\overline{D_r})) \neq$, then let $z \in K(P_t) \setminus \bigcap_{n\geq 0} P_t^{\circ -n}(\overline{D_r})$. It means that there exists a positive integer n_0 such that $z \notin P_t^{\circ -n_0}(\overline{D_r})$ or in order words, $|P_t^{\circ n_0}(z)| > r$. This implies from (2.1) that for all $n \geq n_0$ we have $|P_t^{\circ n}(z)| > \left(\frac{|P_t^{\circ n_0}|^2}{r} \right)^{n-n_0}$. It follows that $P_t^{\circ n}(z) \longrightarrow \infty$ as $n \longrightarrow \infty$. This yields to $z \notin K(P_t)$ which contradicts the assumption. Thus, $K(P_t) = \bigcap_{n\geq 0} P_t^{\circ -n}(\overline{D_r})$.

Because in this thesis we are just interested in connected Julia sets, we will restrict our attention to the case where the Julia set of P_t is connected. The following important theorem due to Fatou and Julia will show us how the connectedness of the Julia set $J(P_t)$ depends on the parameter t.

Theorem 2.2. *(*Julia, Fatou*) If the critical point 0 stays in the filled-in Julia set $K(P_t)$ then $K(P_t)$ is connected. Otherwise, $K(P_t)$ is homeomorphic to a Cantor set.*

Proof: [DH] Let $r > 1 + |t|$. Denote $V_n = P_t^{\circ -n}(D_r)$ and $V_0 = D_r$. It is easy to see that $\overline{V_{n+1}} \subset V_n$.

In the case 0 stays in $K(P_t)$, if we consider one ramification of the inverse function P_t^{-1} of P_t, then $P_t : V_{n+1} \longrightarrow V_n$ is a homeomorphism. Therefore, V_n is homeomorphic to a disk D_r and then $K(P_t) = \bigcap_{n\geq 0} \overline{V_n}$ is connected.

In the case 0 doesn't stay in $K(P_t)$, there exists a integer number m such that $0 \in V_m$ and $t = P_t(0) \notin V_m$. Then, V_m is homeomorphic to a disk, but for $n \geq m$, $P_t : V_{n+1} \longrightarrow V_n$ is a double ramification. It follows that for k, the open set V_{m+k} has 2^k connected components which is homeomorphic to a disk. Let δ_k be the maximum diameter (with respect to the Poincaré metric on the sphere) of these components. We know that the two non ramification of P_t^{-1} g_0^{-1} et g_1^{-1} is the $\lambda-$Lipschitz continuous functions (with respect to the Poincaré metric on the sphere). Thus, $\delta_k \leq \lambda^{k-1}\delta_1$. In particular, the fact that δ_k tends to 0 as $k \longrightarrow \infty$ implies that $K(P_t)$ is non-connected and moreover it is homeomorphic to a Cantor set.

If $K(P_t)$ is connected then the basin of infinity $D(\infty)$ is simply connected. Therefore, $J(P_t) = \partial K(P_t) = \partial D(\infty)$ is connected. We define *Mandelbrot set* $\mathcal{M}$ as a set of all the parameters t such that the Julia set $J(P_t)$ is connected.

2.1.2 Conformal representation of $\mathbb{C} \setminus K(P_t)$.

Suppose that $K(P_t)$ is connected: there exists then a conformal map $\varphi_t : D(\infty) \longrightarrow \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ of the form $\varphi_t(z) = z + ...$ which conjugates P_t to P_0. Indeed, if we chose one branch of (2^n)th roots of $P_t^{\circ n}(z) = z^{2^n}(1 + ...)$ and denote it by $\varphi_n(z) = z(1 + ...)^{2^{-n}}$, it follows that $\varphi_n^2 = \varphi_{n-1} \circ P_t$. The fact that for $|z| \geq r = 1 + |t|$, $|P_t(z)| \geq r$ implies that $P_t^{\circ n}(z)$ doesn't vanish on $\{|z| > r\}$. Moreover, we have

$$\frac{\varphi_{n+1}}{\varphi_n} = \left(\frac{\phi_1 \circ P^{\circ n}}{P^{\circ n}} \right)^{2^{-n}} = \left(1 + \frac{t}{(P^{\circ n})^2} \right)^{2^{-(n+1)}}.$$

Put $\varphi_t(z) = z \prod_{n=0}^{\infty} \left(1 + \frac{t}{(P^n(z))^2} \right)^{2^{-(n+1)}}$. Then,

$$\varphi_t(z) = z \exp \left\{ \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \log \left(1 + \frac{t}{(P^{\circ n}(z))^2} \right) \right\}.$$

Since the sum $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \log \left(1 + \frac{t}{(P^{\circ n}(z))^2} \right)$ converges on $D(\infty)$, then φ_t is well-defined. We call φ_t Böttcher map. This map is conformal and satisfies the functional equation $\varphi_t(P_t) = (\varphi_t)^2$.

In the following, we will consider the hyperbolic component of $\mathcal{M}$ consisting in the set of parameters t such that P_t has an attracting fixed pointed.

Principal Cardioid.

Fatou and Julia observed that if P_t admits a attracting fixed point z_0, then it attracts the critical point 0 of P_t. Indeed, let U be the connected component of Fatou set which contains the fixed point z_0 . Immediately, we see that U contains the attraction basin of z_0: $D(z_0) = \{z \in \mathbb{C} : P^{\circ n}(z) \longrightarrow z_0, n \longrightarrow \infty\}$. Let V be a closed disk for the Poincaré metric on U centered at z_0. The map $P_t^{\circ n}$ induces a holomorphic map from U into itself, which is not a isomorphism, so which is λ-Lipschitz on V with $\lambda < 1$; it follows that every point in V is attracted by z_0. Hence, $U = D(z_0)$. In other words, the connected component of Fatou set which contains the fixed point is the attracting basin of the fixed point. Moreover, as P_t is a proper map on the Fatou components; $P_t(U) \subset U$ and both two inverse images of $z \in D(z_0)$ stay in $D(z_0)$, then $P_t(U) = U$. Assume that if the critical point $0 \notin U$, then P_t is a covering map and hence is an isometry map (respect to hyperbolic metric) of the open simply connected set U because $P_t^{\circ -1}$ admits z_0 as a fixed point, then by applying the Schwartz lemma to one branch of $P_t^{-1} : U \longrightarrow U$ we deduce that for $z \in U$, $|(P_t^{\circ -1})'(z)| = 1$ which contradicts the existence of the attracting fixed point z_0. Thus, the critical point $0 \in U$.

If z_0 is a attracting fixed point then $|P_t'(z_0)| = |2z_0| < 1$. Thus the parameter t stays in the cardioid denoted by $\mathcal{C} = \{t : t = z(1-z), z \in \{|u| < \frac{1}{2}\}\}$. We call this component $\mathcal{C}$ of Mandelbrot set $\mathcal{M}$ the principal cardioid. Conversely, if $t \in \mathcal{C}$, then the polynomial P_t has a attracting fixed point and it can be shown that $J(P_t)$ is a Jordan curve, actually a quasicircle, as we now explain:

Let $t \in \mathcal{C}$ and ϕ_t be the smooth family of conformal map ϕ_t from $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ onto the basin of infinity of the polynomial $P_t(z) = z^2 + t$ (the component containing ∞ of its Fatou set) defined as $\phi_t = \varphi_t^{-1}$, where φ_t is the Böttcher map. Then $\phi_0(z) = z$ and conjugates P_0 to P_t on their basins of infinity. We thus have:

$$\phi_t(z^2) = (\phi_t(z))^2 + t, z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}. \tag{2.2}$$

By construction, for each $z \in \{|z| \geq 1\}$, the function $t \longrightarrow \phi_t(z)$ is holomorphic in $\mathcal{C}$. Then, $\phi_t(z)$ is a holomorphic motion on $\{|z| > 1\} \times \mathcal{C}$. By Manẽ-Sad-Sullivan's theorem, each ϕ_t extends to a quasiconformal map on the whole plane $\overline{\mathbb{C}}$. If we take the derivative of the equation (2.2) with t, we obtain the equation:

$$\dot{\phi}_t(z^2) = 2\phi_t(z)\dot{\phi}_t(z) + 1, \tag{2.3}$$

where $\dot{\phi}_t = \dfrac{\partial \phi}{\partial t}$. Let $V(z)$ denote the holomorphic vector field of (ϕ_t) $V(z) = \dfrac{\partial \phi_t}{\partial t}\Big|_{t=0}$. Letting $t = 0$ in the equation (2.3), we get that the holomorphic vector field V satisfies the functional equation:

$$V(z^2) = 2zV(z) + 1. \tag{2.4}$$

If we replace z by z^2 in the preceding equation, we obtain that

$$V(z^4) = 2z^2V(z^2) + 1. \tag{2.5}$$

Injecting $V(z^2)$ in (2.4) into (2.5), one gets $V(z) = -\left(\dfrac{1}{2z} + \dfrac{1}{2z2z^2}\right) + \dfrac{V(z^4)}{2z2z^2}$. And by induction we can obtain $V(z) = -\displaystyle\sum_{k=1}^{n-1} \frac{1}{2z2z^2...2z^{2^k}} + \frac{V(z^{2^n})}{2z2z^2...2z^{2^{n-1}}}$. The term $\dfrac{V(z^{2^n})}{2^{n+1}z^{2^n-1}}$ tends to 0 as n tends to ∞. Therefore $V(z)$ can be written as an infinite sum

$$V(z) = -z \sum_{k=0}^{\infty} \frac{1}{2^{k+1}z^{2^{k+1}}}.$$

In the next paragraph we show how thermodynamic formalism allows to compute Hausdorff dimension of $J(P_t)$.

2.2 Thermodynamic formalism.

2.2.1 Expanding map

Definition: Let (X,ρ) be a compact metric space. A continuous mapping $f : X \longrightarrow X$ is said to be expanding on X (with respect to the metric ρ) if there exist constants $\lambda > 1, \nu > 0$ and $n \geq 0$ such that for all $x, y \in X$

$$\rho(x,y) \leq 2\nu \Longrightarrow \rho(f(x), f(y)) \geq \lambda\rho(x,y).$$

Example: The map $f(z) = z^2$ is expanding on $\mathbb{T}$ with the usual metric ρ on the circle.

2.2.2 Topological entropy and topological pressure

Let $A = \{1, 2, ..., d\}$ and $X = A^{\mathbb{N}}$.We call A the alphabet and X the set of infinite words on A. If $x_1, x_2, ..., x_n \in A$ we denote by $x_1x_2...x_n$ the set of all words starting with these n letters. We call it a cylinder of the order n. Let σ be the shift map on X which is defined as $\sigma((x_i)) = (x_{i+1})$.

Denote by $\mathcal{A}_n$ be the set of all the cylinders of order n . Let $\mathcal{B}$ be the $\sigma-$algebra generated by the cylinders. We can define a natural filtration of $\mathcal{B}$ by defining $\mathcal{B}_n$ as a $\sigma-$algebra generated by $\mathcal{A}_n$.

A measure μ is said to be $\sigma-$invariant on X if $A \in \mathcal{B}$ we have $\mu(\sigma^{-1}(A)) = \mu(A)$.

For example, if $A = \{0, 1\}$ then the measure $\mu = \mu_0^{\otimes\mathbb{N}}$ is $\sigma-$invariant, where

$$\mu_0(0) = \mu_0(1) = \frac{1}{2}.$$

Denote by $M(X,\sigma) \neq \emptyset$ the space of $\sigma-$invariant measure on X. The above example says that $M(X,\sigma) \neq$. Then we can define the *topological entropy* of the shift map σ with respect to a measure $\mu \in M(X,\sigma)$ by

$$h_\mu(\sigma) = \lim_{n\longrightarrow\infty} \frac{-\sum_{A\in\mathcal{A}_n} \mu(A)\log(\mu(A))}{n}.$$

$h_\mu(\sigma)$ is well-defined because the sequence $u_n = -\sum\limits_{A\in\mathcal{A}_n} \mu(A)\log(\mu(A))$ satisfies the property that $u_{n+p} \leq u_n + u_p$ and therefore the sequence $\dfrac{u_n}{n}$ converges as $n \longrightarrow \infty$.

For each continuous function φ on X, we define the *topological pressure* $P(\varphi,\sigma)$ is the limit:

$$P(\varphi,\sigma) = \lim_{n\longrightarrow\infty} \frac{1}{n} \log\left(\sum_{C\in\mathcal{A}_n} e^{S_n(\varphi)(C)}\right),$$

where $S_n(\varphi)(C) = \sup\limits_{z\in C}\left\{\sum\limits_{k=0}^{n-1}\varphi(\sigma^{\circ k}(z))\right\}$. The topological pressure of $\varphi = 0$ is simply the topological entropy of μ with respect to σ.

We recall the following important result about the topological pressure.

Theorem 2.3. *(Variational principle)*

$$P(\varphi,\sigma) = \sup_{\mu \in M(X,\sigma)} \{h_\mu(\sigma) + \int_X \varphi d\mu\}$$

Proof: see [Zin96].

2.2.3 Ruelle operator

Let M with $M(i,j) \in \{0;1\}$ $\forall i,j$ be a aperiodic matrix (i.e there exists $n > 0$ such that $\forall i,j$ $M^n(i,j) > 0$). Let $A = \{1,2,...d\}$. The associated 1-sided shift space of type finite is defined as

$$\Sigma = \{(x_0,x_1,...); \forall i \geq 0, x_i \in A \text{ and } M(x_i,x_{i+1}) = 1\}.$$

We equip Σ with the metric $d((x_i),(y_i)) = 1/2^n$, where n is the smallest index such that $x_n \neq y_n$. The shift map σ is such that $\sigma(\Sigma) \subset \Sigma$. (Σ, d) is a Cantor set and the map σ is locally expanding on Σ by a factor of 2.

For $\alpha > 0$, let $\mathcal{C}^\alpha(\Sigma)$ denote the Hölder space with exponent α (the space of functions f such that: $\exists C, \forall x,y \in \Sigma, \quad |f(x) - f(y)| \leq C d(x,y)^\alpha$). Then $\mathcal{C}^\alpha(\Sigma)$ is a Banach space with the norm

$$\|f\|_\alpha = \sup_\Sigma |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}.$$

Given $\varphi \in \mathcal{C}^\alpha(\Sigma)$, we define the transfer operator (or Ruelle operator) on $\mathcal{C}^\alpha(\Sigma)$ by

$$\mathcal{L}_\varphi(f)(y) = \sum_{x \in \sigma^{-1}(y)} e^{\varphi(x)} f(x) = \sum_{i \in A; iy \in \Sigma} e^{\varphi(iy)} f(iy).$$

The function φ is called a potential function. It is a positive linear operator (i.e it maps a positive function to a positive function). Its adjoint operator denoted by $\mathcal{L}^*$ acts linearly on the space of positive measure . For all $n \geq 1$,

$$\mathcal{L}^n_\varphi(f)(y) = \sum_{x \in \sigma^{-n}(y)} e^{S_n(\varphi)(x)} f(x).$$

Theorem 2.4. *(Perron-Frobenius-Ruelle) Let φ be a Hölder continuous function with exponent α and $\mathcal{L}_\varphi$ be the associated Ruelle operator.*

(i) The operator $\mathcal{L}_\varphi$ as acting on $\mathcal{C}^\alpha$ admits a strictly positive eigenvalue β_φ with eigenspace of dimension 1 generated by $g_\varphi > 0$.

(ii) There exists a probability measure μ_φ; eigenvector of $\mathcal{L}^*_\varphi$ with eigenvalue β_φ such that $\forall h \in \mathcal{C}^\alpha(\Sigma)$ $\dfrac{\mathcal{L}^n_\varphi(h)}{\beta^n_\varphi}$ converges uniformly on Σ to $g_\varphi \dfrac{\int h d\mu_\varphi}{\int g_\varphi d\mu_\varphi}$. Moreover the topological pressure of the potential φ is : $P(\varphi,\sigma) = \log(\beta_\varphi)$.

Proof: see [Zin96]. Suppose that $P(\varphi) = 0$ i.e $\beta_\varphi = 1$. Let g be the associated eigenfunction of the eigenvalue 1 of the Ruelle operator $\mathcal{L}_\varphi$, then there exists a unique positive probability measure μ on Σ satisfying

$$\int_\Sigma \mathcal{L}_\varphi d\mu = \int_\Sigma f d\mu$$

for all $f \in \mathcal{C}^\alpha(\Sigma)$ and $\int g d\mu = 1$. Moreover this measure has the following property.

Definition: Let $\varphi \in \mathcal{C}(\Sigma)$ (the space of continuous function). A probability measure μ is called a Gibbs measure with respect to φ if there exist constants $A, B > 0$ and $C \in \mathbb{R}$ such that

$$\forall x \in X, \forall n \geq 0, A \leq \frac{\mu(x_1...x_n)}{e^{S_n(\varphi)(x)+Cn}} \leq B.$$

Proposition 2.5. *The measure μ is the unique $\sigma-$invariant Gibbs measure satisfies*

$$P(\varphi) = h_\mu(\sigma) + \int_\Sigma \varphi d\mu = 0.$$

Proof: see [Zin96]. Now, return to the general case with the same notation $g_\varphi, \mu, \beta_\varphi$ in as Theorem 2.4, we then have $m_\varphi = g_\varphi \mu_\varphi$ is the unique $\sigma-$ invariant measure such that

$$P(\varphi) = h_{m_\varphi}(\sigma) + \int_\Sigma \varphi dm_\varphi.$$

See [Zin96]. The measure which satisfies the last equality is called *equilibrium measure*. In particular, if $P(\varphi) = 0$ then $m_\varphi = g_\varphi \mu_\varphi$ is the unique $\sigma-$ invariant equilibrium measure on Σ.

2.2.4 Conformal repeller.

Definition: Let f be a holomorphic map in the neighborhood of the compact set J. (J, f) is a conformal repeller if there exists an open set V such that $J \subset V \subset \mathbb{C}$ and:

(i) There exists $C > 0$ and $\alpha > 1$ such that $|(f^n)'(z)| \geq C\alpha^n$ for all $z \in J$; $n \geq 1$;

(ii) $J = \bigcap_{n \geq 1} f^{-n}(V)$;

(iii) For any open set U such that $U \cap J \neq \emptyset$, there exists $n > 0$ such that $J \subset f^n(U \cup J)$.

Note that the condition (i) implies that the conformal map f is expanding on J with respect to the hyperbolic metric on $V \supset J$. Sometime, we call (f, J) by conformal expanding repeller.

Example: $(P(z) = z^d, \mathbb{T})$ is a conformal repeller with the open set $V = \{\frac{1}{2} < |z| < 2\}$.

2.2.5 Markov partition.

Definition: A Markov partition of J is a finite covering of J by the sets $J_j, 1 \leq j \leq k$ with the associated Markov map $f : J \longrightarrow J$ verified the following conditions:

(i) $\overline{int(J_j)} = J_j$ for all $1 \leq j \leq k$

(ii) $int(J_j) \cap int(J_i) = \emptyset$ for $1 \leq j, i \leq k$

(iii) If $z \in int(J_j)$ and $f(z) \in int(J_i)$ then $J_i \subset f(J_j)$

(iv) $f|_{J_j}$ is injective, and extends to a conformal map f_j on the neighborhood of J_j.

Let (f, J) be a conformal repeller and assume that J has a Markov partition $J_1, ..., J_k$. Let $M(i, j) = 1$ if $f(J_i) \supset J_j$ and 0 otherwise. Since (f, J) is repeller i.e there exists a positive integer such that for every non-empty open set $U \subset J$ such that $f^m(U) = J$. In other words, there exists an $m > 0$ such that $M^m(i, j) > 0$ or M is aperiodic. The associated shift space Σ admits a Hölder continuous projection

$$\pi : \Sigma \longrightarrow J$$

where each $x = (x_0, x_1, x_2, ..., x_n, ...) \in \Sigma$ gives the sequence of tiles $\{J_0, J_1, ..., J_k\}$ visited by the forward orbit $(z, f(z), f^{\circ 2}(z), ..., f^{\circ n}(z), ...)$ of $z = \pi_0(x)$, see [PU]. We define cylinders $x_0x_1...x_n$ as the set of all $x \in J$ such that $f^{\circ j}(x) \in J_j$ for $j = 0, ..., n$. Denote by $\mathcal{A}_n$ the set of cylinders of the order n of the Markov partition.

Put $\Pi(p) = \sum_{C \in \mathcal{A}_n} \sup_{z \in C} |(f^{\circ n})'|^{-p}(z)$. Applying the Koebe distortion theorem to all the cylinders of order n, we then have: there exists a constant $K \geq 1$ (independent of z and n) such that:

$$\frac{1}{K}\mathrm{diam}(C_n)(z) \leq |(f^{\circ n})'(z)|^{-1} \leq K\mathrm{diam}(C_n)(z),$$

where C_n is the cylinder of order n containing z .

As a consequence, the limit

$$\Pi(p) = \lim_{n \longrightarrow \infty} \frac{\log \Pi_n(p)}{n}$$

exists. In addition, the function $\Pi(p)$ is a convex on $\mathbb{R}$ strictly decreasing from $-\infty$ to ∞, therefore there exists a unique real number denoted by δ such that $\Pi(\delta) = 0$. It has been shown by Bowen (in [Bow79]) that δ is the Hausdorff dimension of the set J.

As an application of the whole theory above, we will give in the next section the computation of the Hausdorff dimension of the Julia set $J(P_t)$, $t \in \mathcal{C}$.

2.3 The computation of Hausdorff dimension of Julia set $J(P_t)$

The Julia set of the of polynomial $P_0(z) = z^2$ has a natural Markov partition $J(P_0) = J_0 \cup J_1$ where J_0, J_1 are the upper and the lower semi unit circle. Let A be a matrix of rank 2×2 with $A(i, j) = 1$ if $P_0(J_i) \supset J_j$ and 0 otherwise, then $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and A is a aperiodic matrix. Then $\Sigma \cong (\mathbb{Z}/2)^{\mathbb{N}}$. We recall the Hölder continuous projection (for some exponent $\alpha > 0$)

$$\pi_0 : \Sigma \longrightarrow J(P_0)$$

where each $x = (x_0, x_1, x_2, ...) \in \Sigma$ gives the sequences of tiles $\{J_0, J_1\}$ visited by the forward orbit $(z, z^2, z^4, ..., z^{2^k}, ...)$ of $z = \pi_0(x)$.

This projection allows us to define a family of projection $\pi_t : \Sigma \longrightarrow J(P_t)$ satisfying $\pi_t(x) = \phi_t(\pi_0(x))$. According to theory of holomorphic motion, for $|t|$ small, ϕ_t has a

homeomorphic extension from $J(P_0)$ to $J(P_t)$, which allows us to define a Markov partition for all $J(P_t)$ and moreover it is Hölder continuous on J_0 then it implies that the function $\varphi_t(x) = \log|P_t'(\pi_t(x))| = \log|2\pi_t(x)|$ is a Hölder continuous function with some exponent α on Σ. Denote $\Gamma_n(t)$ the set of all cylinders of order n for $J(P_t)$.
In this case $\Pi_n(p) = \sum_{\gamma\in\Gamma_n(t)} |(P_t^{\circ n})'|^{-p}(\gamma)$ becomes

$$\Pi_n(p) = \sum_{\gamma\in\Gamma_n(0)} e^{S_n(-p\log|2\pi_t|)(\gamma)}.$$

Then the function $\Pi(p)$ turns out to be the pressure of the potential $-p\log|2\pi_t(x)|$ on Σ. By Bowen's formula, the Hausdorff dimension $\delta_t = \text{H.dim}(J(P_t))$ is the unique real number such that

$$P(-\delta_t\log|P_t'(\pi_t(x))|) = P(-\delta_t\log|2\phi_t(z)|) = 0.$$

Apply Perron-Frobenious-Ruelle's theorem to the potential $-\delta_t\log|2\phi_t(\pi_0(x))|$ on Σ, there exist a positive eigenfunction g_t and a unique positive measure μ_t on Σ satisfying

$$\mathcal{L}_\varphi(g_t) = g_t;$$

$$\int_\Sigma \mathcal{L}_{\varphi_t}(f)d\mu_t = \int_\Sigma f d\mu_t, \quad \forall f\in\mathcal{C}^\alpha(\Sigma)$$

and $\int_\Sigma g_{\varphi_t}d\mu_t = 1$. We define the associated equilibrium measure on Σ by $m(\varphi_t) = g_{\varphi_t}\mu_t$. Note that in the case of $t=0$, the potential $\varphi = -\log 2$, then the measure $\nu = \pi_0(m_0)$ on the circle (the pushforward of equilibrium measure $m_0 = m(\varphi_0)$ of φ_0 by π_0) is the normalized Lebesgue measure on the unit circle $\dfrac{|dz|}{2\pi}$.

Theorem 2.6. *Let φ_t be a smooth path in $\mathcal{C}^\alpha(\Sigma)$ and let $\dot\varphi_0 = d\varphi_t/dt|_{t=0}$. We then have*

$$\left.\frac{dP(\varphi_t)}{dt}\right|_{t=0} = \int_\Sigma \dot\varphi_0 dm_0$$

and if the first derivative at $t=0$ is zero, then

$$\left.\frac{d^2P(\varphi_t)}{dt^2}\right|_{t=0} = Var(\dot\varphi_0, m_0) + \int_\Sigma \ddot\varphi_0 dm_0,$$

where m_0 is the equilibrium measure for φ_0 and $Var(\dot\varphi, m_0) = \lim_{n\longrightarrow\infty}\dfrac{1}{n}\|\sum_{k=0}^{n-1}\dot\varphi_0(\sigma^i(x))\|_2^2$.

Proof: See [PP90]. Note that [PP90] treats the second derivative in the case where $\int_\Sigma \varphi_t dm_0$ is constant. Here we can obtain the general formula above by using the fact that $P(\varphi - \int_\Sigma \varphi_t dm_0) = P(\varphi) - \int_\Sigma \varphi_t dm_0$.

Put $\ddot\delta_0 = \left.\dfrac{d^2\nu_t}{dt^2}\right|_{t=0}$. We change the variable $z = \pi_0(x),\ x\in\Sigma$ and without misunderstanding, we write $\text{Var}(\dot\varphi, m_0)$ by $\text{Var}(\dot\varphi)$. Since $P(-\delta_t\log|2\phi_t|) = 0$, then $\left.\dfrac{dP(-\delta_t\log|2\phi_t|)}{dt}\right|_{t=0} =$

0 and $\dfrac{d^2P(-\delta_t \log|2\phi_t|)}{dt^2}\Bigg|_{t=0} = 0$. If we replace φ_t by $-\delta_t \log|2\phi_t|$ in the above theorem and compute the quantity $\dfrac{d^2P(-\delta_t \log|2\phi_t|)}{dt^2}\Bigg|_{t=0}$, we obtain that:

$$\begin{aligned}\frac{d^2P(-\delta_t \log|2\phi_t|)}{dt^2}\Bigg|_{t=0} &= \mathrm{Var}\left(\frac{d(-\log|2\phi_t|)}{dt}\Bigg|_{t=0}\right) + \ddot{\delta}_0 \int_{\mathbb{T}} -\log|2z|\frac{|dz|}{2\pi} \\ &\quad + \int_{\mathbb{T}} \left(\frac{d^2(-\log|2\phi_t|)}{dt^2}|_{t=0}\right)\frac{|dz|}{2\pi}.\end{aligned}$$

Since $J(P_t)$ is homeomorphic to the unit circle, then $\delta_t \geq \delta_0 = 1$ and therefore $\dot{\delta}_0 = 0$. And since $\phi_t(z)$ is holomorphic on $\mathbb{C} \setminus \overline{\mathbb{D}}$ and continuous up to the boundary $\partial\mathbb{D}$, by the mean value's theorem we have

$$\frac{1}{2\pi}\int_{\mathbb{T}} \log(2|\phi_t(e^{i\theta})|)d\theta = \log 2 \quad \forall t.$$

This implies that $\displaystyle\int_{\mathbb{T}} \left(\frac{d^2(-\log|2\phi_t|)}{dt^2}|_{t=0}\right)\frac{|dz|}{2\pi} = 0$. Moreover, we have

$$\int_{\mathbb{T}} \log(2|e^{i\theta}|)d\theta = \log 2$$

and

$$\frac{d(\log|2\phi_t|)}{dt} = \mathrm{Re}\left(\frac{\dot{\phi}_t}{\phi_t}\right).$$

Therefore

$$\ddot{\delta}_0 = \frac{d^2\mathrm{H.dim}(J(P_t))}{dt^2}\Bigg|_{t=0} = \frac{\mathrm{Var}(\mathrm{Re}(V(z)/z))}{\log 2}, \tag{2.6}$$

where

$$\mathrm{Var}\left(\frac{\mathrm{Re}(V(z))}{z}\right) = \frac{1}{2}\mathrm{Var}\left(\frac{V(z)}{z}\right) = \frac{1}{4\pi}\lim_{n\longrightarrow\infty}\frac{1}{n}\|\sum_{k=0}^{n-1}\frac{V(z^{2^k})}{z^{2^k}}\|_2^2.$$

In addition, the implicit formula of $V(z)$: $V(z) = -z\displaystyle\sum_{k=0}^{\infty}\frac{1}{2^{k+1}z^{2^{k+1}}}$ implies that $V(z)/z = V'(z) - V'(z^2)$. This fact helps us to deduce that

$$\sum_{k=0}^{n-1}\frac{V(z^{2^k})}{z^{2^k}} = \sum_{k=0}^{n-1}(V'(z^{2^k}) - V'(z^{2^{k+1}})) = V'(z) - V'(z^{2^n}).$$

This yields to

$$\begin{aligned}\mathrm{Var}(V(z)/z) &= \lim_{n\longrightarrow\infty}\frac{1}{2\pi}\int_0^{2\pi}|V'(e^{i\theta}) - V'(e^{i2^n\theta})|^2 d\theta \\ &= \lim_{n\longrightarrow\infty}\frac{\|V'_{n-1}(e^{i\theta})\|_2^2}{2\pi n} + \lim_{n\longrightarrow\infty}\frac{1}{2\pi}\frac{1}{n}\sum_{k=0}^{\infty}((1-2^{-k}) - (1-2^{-(k+n)})^2 \\ &= \lim_{n\longrightarrow\infty}\frac{\|V'_{n-1}(e^{i\theta})\|_2^2}{2\pi n}.\end{aligned}$$

Because $\frac{1}{n}\sum_{k=0}^{\infty}((1-2^{-k})-(1-2^{-(k+n)}))^2 = \frac{1}{n}(1-\frac{1}{2^n})^2 \longrightarrow 0$ as $n \longrightarrow \infty$.

Moreover, if we put $r = 1-2^n$, we have

$$\lim_{n\longrightarrow\infty} \frac{\|V'_{n-1}(e^{i\theta})\|_2^2}{n} = \lim_{r\longrightarrow 1} \frac{\|V'(re^{i\theta})\|_2^2 \log 2}{|\log(1-r)|}. \tag{2.7}$$

Indeed,

$$\begin{aligned}\Big| V'_{n-1}(e^{i\theta}) - V'(re^{i\theta}) \Big| &\le \sum_{k=0}^{n-1}(1-2^{-k})(1-r^{2^k}) + \sum_{k=n}^{\infty}(1-2^{-k})r^{2^k} \\ &\le \sum_{k=1}^{n-1} 2^{k-n+1} + \sum_{k=n}^{\infty} e^{-2^{k-n}} \le 2 + \sum_{k=0}^{\infty}\exp(-2^n) < C+\infty.\end{aligned}$$

By triangle's inequality, we have

$$\Big| \|V'_n(e^{i\theta})\|_2 - \|V'(re^{i\theta})\|_2 \Big| \le \|V'_n(e^{i\theta}) - V'(re^{i\theta})\|_2 \le C.$$

Thus, if we divide both sides by $\sqrt{n}$ then

$$\lim_{n\longrightarrow\infty} \frac{\|V'_{n-1}(e^{i\theta})\|_2 - \|V'(re^{i\theta})\|_2}{\sqrt{n}} = 0.$$

Since

$$\lim_{r\longrightarrow 1} \frac{1}{4\pi|\log(1-r)|}\int_{|z|=r} |V'(z)|^2|dz| = \lim_{r\longrightarrow 1}\frac{1}{2|\log(1-r)|}\sum_{k\ge 0}(1-\frac{1}{2^k})^2 r^{-2^{k+1}} = \frac{1}{2\log 2} < +\infty,$$

then $\dfrac{\|V'_{n-1}(e^{i\theta})\|_2^2}{n}$ is also bounded. Using the fact that the function x^2 is uniformly continuous on some compact set of $[0,+\infty)$ we deduce (2.7). Therefore we have

$$\frac{\mathrm{Var}(\mathrm{Re}(V(z)/z))}{\log 2} = \lim_{r\longrightarrow 1}\frac{1}{4\pi|\log(1-r)|}\int_{|z|=r}|V'(z)|^2|dz| = \frac{1}{2\log 2}.$$

The above equality and (2.6) imply that

$$\frac{d^2}{dt^2}\mathrm{H.dim}\,(\phi_t(\mathbb{T})) = \lim_{r\longrightarrow 1}\frac{1}{4\pi|\log(1-r)|}\int_{|z|=r}|V'(z)|^2|dz| = \frac{1}{2\log 2}.$$

This yields to Ruelle's formula ([Rue82])

$$\mathrm{H.dim}(J(P_t)) = 1 + \frac{|t|^2}{4\log 2} + o(|t|^2).$$

The general similar result for the family of polynomial $F_t = z^d + t(b_2z^{d-2} + b_3z^{d-3} + ... + b_d)$, $t \in \mathbb{C}$ and near zero can be found in [MM08].

Chapter 3

Martingale condition

As introduced in Chapter 1, the main result of this chapter is Theorem 3.7. We aim to prove this theorem by using a probability argument, namely dyadic martingale. In this chapter, first we will give the construction of the dyadic martingale of a Bloch function, then by the means of dyadic martingale we prove Theorem 3.7. The proof of this theorem will be separated into two steps: in the first one we derive the Minkowski dimension from the spectrum of integral means $\beta(p, \phi')$ for p small; in the second one by using the exponential transformation of a dyadic martingale, we obtain the spectrum of the integral means $\beta(p, \phi')$ for p small. In the end of this chapter, we'll point out a non-trivial application of Theorem 3.7.

3.1 Bloch function and dyadic martingale

3.1.1 Preliminaries on Bloch function

Proposition 3.1. *If $b \in \mathcal{B}$ and $b(0) = 0$ then*

$$\frac{1}{2\pi} \int_{\mathbb{T}} |b(r\xi)|^{2n} |d\xi| \leq n! \|b\|_{\mathcal{B}}^{2n} \left(\log \frac{1}{1-r^2} \right)^n \tag{3.1}$$

for $0 < r < 1$ and $n = 0, 1, ...$

Proof: [Pom92] The case $n = 0$ is trivial. Suppose that (3.1) holds for some n. Hardy's identity shows that

$$\frac{d}{dr}\left(r\frac{d}{dr} \right)\left(\frac{1}{2\pi} \int_{\mathbb{T}} |b(r\xi)|^{2n+2} |d\xi| \right) = \frac{4(n+1)^2 r}{2\pi} \int_{\mathbb{T}} |b(r\xi)|^{2n} |b'(r\xi)|^2 |d\xi|.$$

Put $\lambda(r) = \log\left(\dfrac{1}{1-r^2} \right)$. Since $b \in \mathcal{B}$, then

$$\begin{aligned}\frac{d}{dr}\left(r\frac{d}{dr} \right)\left(\frac{1}{2\pi} \int_{\mathbb{T}} |b(r\xi)|^{2n+2} |d\xi| \right) &\leq 4(n+1)^2 r n! \parallel b \parallel_{\mathcal{B}}^{2n} .(1-r^2)^{-2} \parallel b \parallel_{\mathcal{B}}^2 \\ &\leq (n+1)! \parallel b \parallel_{\mathcal{B}}^{2n+2} \frac{d}{dr}\left(r\frac{d}{dr}\lambda(r)^{n+1} \right).\end{aligned}$$

Hence we obtain by integration that

$$\frac{d}{dr}\left(\frac{1}{2\pi}\int_{\mathbb{T}}|b(r\xi)|^{2n}|d\xi|\right)\leq (n+1)!\parallel b\parallel_{\mathcal{B}}^{2n+2}\frac{d}{dr}\left(\lambda(r)^{n+1}\right)$$

and then (3.1) for the case $n+1$ follows by another integration because both sides vanish for $r=0$.

This proposition implies that if $b\in\mathcal{B}, b(0)=0$,

$$\limsup_{r\longrightarrow 1}\frac{\int_0^{2\pi}|b(re^{i\theta})|^2d\theta}{2\pi\log(\frac{1}{1-r})}\leq\|b\|_{\mathcal{B}}^2<+\infty. \tag{3.2}$$

This proposition can be generalized as follows.

Corollary 3.2. *If $b\in\mathcal{B}$ and $b(0)=0$ then there exists a constant C such that*

$$\int_{\mathbb{T}}|b(r\xi)|^p|d\xi|\leq C\left(\log\frac{1}{1-r^2}\right)^{p/2}$$

for $0<r<1$ and $p>0$.

Proof: For $p>0$, there exists a positive integer n such that $0<\dfrac{p}{2n}<1$. Applying Hölder's inequality for $\alpha=\dfrac{p}{2n}<1$,

$$\frac{1}{2\pi}\int_{\mathbb{T}}|b(r\xi)|^{2n}|d\xi|\geq\left(\frac{1}{2\pi}\int_{\mathbb{T}}|b(r\xi)|^{2n\alpha}|d\xi|\right)^{1/\alpha}.$$

Then Proposition 3.1 implies that

$$\left(\frac{1}{2\pi}\int_{\mathbb{T}}|b(r\xi)|^p|d\xi|\right)=\left(\frac{1}{2\pi}\int_{\mathbb{T}}|b(r\xi)|^{2n\alpha}|d\xi|\right)\leq\left(\frac{1}{2\pi}\int_{\mathbb{T}}|b(r\xi)|^{2n}|d\xi|\right)^{\alpha}\leq C\left(\log\frac{1}{1-r^2}\right)^{p/2},$$

where $C=(n!\|b\|_{\mathcal{B}}^{2n})^{\alpha}$.

A complex-valued continuous function on the unit circle $\mathbb{T}$ is called a **Zygmund function** if there exists a constant C such that

$$\sup_{|z|=1}|h(e^{i\theta}z)-2h(z)+h(e^{-i\theta}z)|\leq C\theta,\quad \text{for } \theta>0.$$

Let Λ_* denote the **Zygmund class** which consists of all the Zygmund functions.

Theorem 3.3. ***(Zygmund)*** *Let b be analytic on the disk $\mathbb{D}$ and let $h(z)$ be a primitive function of b. Then b belongs to Bloch space $\mathcal{B}$ if and only if h is continuous in the closed disk $\overline{\mathbb{D}}$ and h is a Zygmund function.*

Proof: [Dur70] If $h(z)$ is continuous in $|z| \leq 1$, it can be represented as a Poisson integral:

$$h(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - t) h(e^{it}) dt, \quad z = re^{i\theta}. \tag{3.3}$$

Since the second derivative $P_{\theta\theta}(r, \theta)$ is an even function of θ and

$$\int_0^{\pi} P_{\theta\theta}(r, t) dt = P_\theta(r, \pi) - P_\theta(r, 0) = 0,$$

it follows that

$$h_{\theta\theta}(z) = \frac{1}{2\pi} \int_0^{2\pi} P_{\theta\theta}(r, t) \left(h(e^{i(\theta+t)}) - 2h(e^{\theta}) + h(e^{i(\theta-t)}) \right) dt.$$

The hypothesis $h(e^{i\theta}) \in \Lambda_*$ therefore implies

$$|h_{\theta\theta}| \leq A \int_0^{\pi} t P_{\theta\theta}(r, t) dt = A(P(r, 0) - P(r, \pi)) = \mathcal{O}\left(\frac{1}{1-r} \right),$$

since $P_{\theta\theta}(r, \theta) \geq 0$ for $0 \leq \theta \leq \pi$ and $r \leq 2 - \sqrt{3}$, as a calculation shows. On the other hand, (3.3) and the boundedness of $h(e^{it})$ easily show

$$h_\theta(z) = \mathcal{O}\left(\frac{1}{1-r} \right).$$

Thus

$$h''(z) = r^{-2} e^{-2i\theta} (i h_\theta(z) - h_{\theta\theta}(z)) = \mathcal{O}\left(\frac{1}{1-r} \right).$$

In other words, $b(z) = h'(z)$ is a Bloch function.

Conversely, by Hardy-Littlewood's theorem (see Theorem 5.1 in [Dur70]), the primitive of the Bloch function is continuous in the closed disk $\mathbb{D}$. We need to show that $h \in \Lambda_*$. For $0 < t < 1$, let us use the notation

$$\Delta_t = G(\theta + t) - G(\theta),$$

$$\Delta_t^2 = G(\theta + 2t) - 2G(\theta + t) + G(\theta).$$

We are required to show that $\Delta_t^2 = \mathcal{O}(t)$, uniformly in θ, as $t \longrightarrow 0$. Our strategy is to write

$$\Delta_t^2 h(e^{i\theta}) = \Delta_t^2 (h(e^{i\theta}) - h(\rho e^{i\theta})) + \Delta_t^2 h(\rho e^{i\theta}) \tag{3.4}$$

$(0 < \rho < 1)$, to set $\rho = 1 - t$, and to show that as $t \longrightarrow 0$, each of the two in terms (3.4) is uniformly $\mathcal{O}(t)$. The identity

$$h(e^{i\theta}) - h(\rho e^{i\theta}) = (1 - \rho) e^{i\theta} h'(\rho e^{i\theta}) + e^{2i\theta} \int_\rho^1 (1 - r) h''(re^{i\theta}) dr \tag{3.5}$$

is easily verified through integration by parts. Now set $\rho = 1 - t$. Under the hypothesis that $h''(re^{i\theta}) = \mathcal{O}\left(\frac{1}{1-r} \right)$, the integral in (3.5) is then uniformly $\mathcal{O}(h)$. Thus

$$\Delta_t^2 (h(e^{i\theta}) - h(\rho e^{i\theta})) = t \Delta_t^2 (e^{i\theta} h'(\rho e^{i\theta})) + \mathcal{O}(t). \tag{3.6}$$

But

$$\Delta_t(e^{i\theta}h'(\rho e^{i\theta})) = \Delta_t(e^{i\theta})h'(\rho e^{i(\theta+t)}) + e^{i\theta}\Delta(h'(\rho e^{i\theta}))$$
$$= \mathcal{O}\left(t\log\frac{1}{t} \right) + i\rho e^{i\theta}\int_0^t e^{i(\theta+x)}h''(\rho e^{i(\theta+x)})dx = \mathcal{O}(1) \qquad (3.7)$$

uniformly in θ. Hence the expression (3.6) is uniformly $\mathcal{O}(t)$. Finally,

$$\Delta_t^2 h(\rho e^{i(\theta-t)}) = i\rho\int_0^t \left(e^{i(\theta+x)}h'(\rho e^{i(\theta+x)}) - e^{i(\theta-x)}h'(\rho e^{i(\theta-x)}) \right) dx.$$

Analyzing the integrand as in (3.7), we see that

$$e^{i(\theta+x)}h'(\rho e^{i(\theta+x)}) - e^{i(\theta-x)}h'(\rho e^{i(\theta-x)}) = \Delta_x(e^{i\theta}h'(\rho e^{i(\theta)})) + \Delta_x(e^{i(\theta-x)}h'(\rho e^{i(\theta-x)})).$$

Similarly as (3.7), we have

$$\left| \Delta_x(e^{i\theta}h'(\rho e^{i(\theta)})) + \Delta_x(e^{i\theta-x}h'(\rho e^{i(\theta-x)})) \right| \leq Cx\log\frac{1}{x} + C' \leq Ct\log\frac{1}{t} + C',$$

where C, C' are independent of θ and $0 < x < t < 1$ small. Hence

$$|\Delta_t^2 h(\rho e^{i\theta})| \leq \left(Ct\log\frac{1}{t} + C' \right) \int_0^t dx = \mathcal{O}(t),$$

and the proof is complete.

Let $I = (e^{i\theta_1}, e^{i\theta_2})$ be a subarc of $\partial\mathbb{D}$. We can define b_I, the mean value of b on the arc $I \subset \partial\mathbb{D}$, as the limit $\lim\limits_{r\longrightarrow 1}(b_r)_I$, where $b_r(z) = b(rz), z \in \mathbb{D}$. Integration by parts shows that

$$b_I = \lim_{r\longrightarrow 1}\frac{1}{|I|}\int_I b(re^{i\theta})d\theta = \frac{-ie^{-i\theta_2}h(e^{i\theta_2}) + ie^{-i\theta_1}h(e^{i\theta_1})}{|I|} + \frac{1}{|I|}\int_I e^{-i\theta}h(e^{i\theta})d\theta$$

and by the property of continuity up to the boundary of the primitive function $h(z)$, the limit exists. Hence the mean value of Bloch function is well-defined.

Let $h(z)$ be a primitive function of $z^{-1}b(z)$, $z \in \mathbb{D}$, where $b(0) = 0$ and $b \in \mathcal{B}$.

Proposition 3.4. *If $b \in \mathcal{B}, b(0) = 0$, for $|z| \leq 1$ and $0 < |t| \leq \pi$, then*

$$\left| \frac{h(e^{it}z) - h(z)}{it} - b\left(\left(1 - \frac{|t|}{2\pi} \right) z \right) \right| \leq C \parallel b \parallel_{\mathcal{B}}, \qquad (3.8)$$

where C is absolute constant.

Proof: [Pom92] Without loss of generality, we assume that $\frac{1}{2} < |z| < 1$ and $0 < |t| \leq \frac{\pi}{2}$. Now let $0 < t < \frac{\pi}{2}$, let $|\tau| \leq t$ and choose σ such that $t = \pi(1 - e^{-\sigma})$. Integration by parts shows that

$$\int_{e^{-\sigma}z}^{e^{i\tau}z} b'(\xi)\log\left(\frac{e^{i\tau}z}{\xi} \right) d\xi = h(e^{i\tau}z) - h(e^{-\sigma}z) - (\sigma + i\tau)b(e^{-\sigma}z).$$

Also,

$$\left| \log\left(\frac{e^{i\tau}z}{\xi}\right)\right| \leq \log\frac{1}{|\xi|} + \left|\arg\left(\frac{e^{i\tau}z}{\xi}\right)\right| \leq C'(1-|\xi|),$$

for C' absolute constant and for $\xi \in [e^{-\sigma}z, e^{i\tau}z]$. Thus the integrand is bounded by $C' \parallel b \parallel_{\mathcal{B}}$. Hence,

$$|h(e^{i\tau}z) - h(e^{-\sigma}z) - (\sigma + i\tau)b(e^{-\sigma}z)| \leq C''\sigma \parallel b \parallel_{\mathcal{B}}, \tag{3.9}$$

and we obtain (3.8) if we choose $\tau = t$ and $\tau = 0$ and then subtract.

Corollary 3.5. *If $b \in \mathcal{B}, b(0) = 0$, for $|z| \leq 1$, $0 < |t| \leq \pi$ and for $0 \leq \alpha \leq |t|$, then*

$$\left| \frac{h(e^{it}z) - h(z)}{it} - b\left(\left(1 - \frac{|t|}{2\pi}\right)e^{i\alpha}z\right)\right| \leq C \parallel b \parallel_{\mathcal{B}}, \tag{3.10}$$

where C is absolute constant.

Proof: We use the same assumption of the proof of Proposition 1. Now let $0 < t < \frac{\pi}{2}$, let $|\tau| \leq t$ and choose σ such that $t = \pi(1 - e^{-\sigma})$. First, we fix z_0 and put $z_1 = e^{i\alpha}z_0$ then we will obtain (3.10) by choosing a suitable value of τ in the formula (3.9) and subtracting as follows:
Choose $z = z_1, \tau = t - \alpha$, (3.9) implies that:

$$|h(e^{i(t-\alpha)}z_1) - h(e^{-\sigma}z_1) - (\sigma + i(t-\alpha))b(e^{-\sigma}z_1)| \leq C''\sigma \parallel b \parallel_{\mathcal{B}},$$

or,

$$|h(e^{it}z_0) - h(e^{-\sigma}z_1) - (\sigma + i(t-\alpha))b(e^{-\sigma}z_1)| \leq C''\sigma \parallel b \parallel_{\mathcal{B}}, \tag{3.11}$$

and choose $z = z_1, \tau = 0$, (3.9) implies that:

$$|h(z_1) - h(e^{-\sigma}z_1) - \sigma b(e^{-\sigma}z_1)| \leq C''\sigma \parallel b \parallel_{\mathcal{B}} . \tag{3.12}$$

Then subtracting (3.11) and (3.12) implies that:

$$|h(e^{it}z_0) - h(z_1) + i(\alpha - t)b(e^{-\sigma}z_1)| \leq 2C''\sigma \parallel b \parallel_{\mathcal{B}} . \tag{3.13}$$

Choose $z = z_1, \tau = -\alpha$, (3.9) implies that:

$$|h(e^{-i\alpha}z_1) - h(e^{-\sigma}z_1) + (\sigma - i\alpha)b(e^{-\sigma}z_1)| \leq C''\sigma \parallel b \parallel_{\mathcal{B}},$$

or

$$|h(z_0) - h(e^{-\sigma}z_1) - (\sigma - i\alpha)b(e^{-\sigma}z_1)| \leq C''\sigma \parallel b \parallel_{\mathcal{B}}, \tag{3.14}$$

then subtracting (3.12) and (3.14) implies that:

$$|h(z_1) - h(z_0) + i\alpha b(e^{-\sigma}z_1)| \leq 2C''\sigma \parallel b \parallel_{\mathcal{B}} . \tag{3.15}$$

Finally, subtracting (3.13) and (3.15) implies that:

$$|h(e^{it}z_0) - h(z_0) - itb(e^{-\sigma}z_1)| \leq 4C''\sigma \parallel b \parallel_{\mathcal{B}},$$

or

$$\left| \frac{h(e^{it}z_0) - h(z_0)}{it} - b\left(\left(1 - \frac{|t|}{2\pi}\right)e^{i\alpha}z_0\right)\right| \leq C \parallel b \parallel_{\mathcal{B}}, \tag{3.16}$$

We recall now the notion of dyadic martingale of a Bloch function.

3.1.2 Dyadic martingale.

On the probability space $(\partial\mathbb{D}, |.|)$ $(|d\xi| = d\theta/2\pi,\ \ \xi = e^{i\theta} \in \partial\mathbb{D})$, we consider the increasing sequence of σ-algebras $\{\mathcal{F}_n, n \geq 0\}$ generated by the partitions of the unit circle by the intervals bounded by the (2^n)th roots of the unity.

Let b be a Bloch function, $b(0) = 0$. We defined $S = (S_n, \mathcal{F}_n)$ by setting $S_n|I = b_I$ on each dyadic interval I of rank n. In other words $S_n = \mathbf{E}(b|\mathcal{F}_n)$. Then

$$\forall \xi \in \partial\mathbb{D},\ \ S_n(\xi) = \sum_{I\in\mathcal{F}_n} b_I \chi_I(\xi).$$

This sequence is a martingale in the sense that $\mathbf{E}(S_{n+1}|\mathcal{F}_n) = S_n$. And it has the property:

$$\forall n, \forall \xi \in \partial\mathbb{D},\qquad \Bigg|\ S_n(\xi) - b((1-2^{-n})\xi)\ \Bigg| \leq C\|b\|_{\mathcal{B}}, \tag{3.17}$$

where C is an absolute constant. The property (3.17) follows from Corollary 3.5.

We consider the increasing sequence $\langle S\rangle_n^2 = \sum_{j=1}^n \mathbf{E}((\Delta S_j)^2|\mathcal{F}_{j-1})$, where $\Delta S_j = S_j - S_{j-1}$. In the dyadic case ΔS_j^2 is $\mathcal{F}_{j-1}-$measurable. Indeed, in order to show that $\Delta S_j^2 \in \mathcal{F}_{j-1}$, we observe that each dyadic interval $I = \Bigg[\ e^{\frac{m2\pi i}{2^{j-1}}}, e^{\frac{(m+1)2\pi i}{2^{j-1}}}\ \Bigg) \in \mathcal{F}_{j-1}, m = 1, ..., 2^{j-1} - 1$, can be split into two intervals $I_L = \Bigg[\ e^{\frac{m2\pi i}{2^{j-1}}}, e^{\frac{(m+\frac{1}{2})2\pi i}{2^{j-1}}}\ \Bigg)$ and $I_R = \Bigg[\ e^{\frac{(m+\frac{1}{2})2\pi}{2^{j-1}}}, e^{\frac{(m+1)2\pi}{2^{j-1}}}\ \Bigg)$ which are the left and right part of I respectively and $I_L, I_R \in \mathcal{F}_j$. We regard that every interval $J \in \mathcal{F}_j$ is either an I_L or I_R for some unique $I \in \mathcal{F}_{j-1}$. Thus we can write

$$\begin{aligned} S_j(\xi) &= \sum_{J\in\mathcal{F}_j} b_I \chi_I(\xi) \\ &= \sum_{I\in\mathcal{F}_{j-1}} \Bigg(\ b_{I_L}\chi_{I_L}(\xi) + b_{I_R}\chi_{I_R}(\xi)\ \Bigg). \end{aligned}$$

And as $b_I = \dfrac{1}{2}(b_{I_L} + b_{I_R})$, then we can write S_{j-1} in form

$$\begin{aligned} S_{j-1}(\xi) &= \sum_{I\in\mathcal{F}_{j-1}} b_I \chi_I(\xi) \\ &= \frac{1}{2}\sum_{I\in\mathcal{F}_{j-1}} \Bigg(\ b_{I_L} + b_{I_R}\ \Bigg)\Bigg(\ \chi_{I_L}(\xi) + \chi_{I_R}(\xi)\ \Bigg). \end{aligned}$$

The substraction implies that

$$\Delta S_j(\xi) = \frac{1}{2}\sum_{I\in\mathcal{F}_{j-1}} \Bigg[\ (b_{I_L} - b_{I_R})\chi_{I_L}(\xi) + (b_{I_R} - b_{I_L})\chi_{I_R}(\xi)\ \Bigg].$$

From this we deduce that $\Delta S_j^2(\xi) = \dfrac{1}{4}\displaystyle\sum_{I\in\mathcal{F}_{j-1}} \Bigg(\ b_{I_L} - b_{I_R}\ \Bigg)^2 \chi_I(\xi)$, so that $\Delta S_j^2 \in \mathcal{F}_{j-1}$.

Then, $\langle S\rangle_n^2 = \sum_{j=1}^n (\Delta S_j)^2$, and we call $\langle S\rangle_\infty^2 = \displaystyle\sum_{j\geq 1} (\Delta S_j)^2$ the square function.

The following theorem characterizes the orthogonality property in $L^2(\mathbb{T})$ of the dyadic martingale S_n.

Theorem 3.6. *Let b be a Bloch function and S_n be its dyadic martingale, then we have*

$$\| S_n \|^2_{L^2(\mathbb{T})} = \sum_{j=1}^{n} \| \Delta S_j(b) \|^2_{L^2(\mathbb{T})} . \tag{3.18}$$

Proof: [Gra08] We first observe that we can rewrite $\Delta S_j(b)$ as

$$\begin{aligned}
\Delta S_j(b) &= \sum_{I\in\mathcal{F}_j} (b)_I \chi_I - \sum_{J\in\mathcal{F}_{j-1}} (b)_J \chi_J \\
&= \sum_{J\in\mathcal{F}_{j-1}} \left(\sum_{I\in\mathcal{F}_j, I\subset J} (b)_I \chi_I - (b)_J \chi_J \right) \\
&= \sum_{J\in\mathcal{F}_{j-1}} \left(\sum_{I\in\mathcal{F}_j, I\subset J} (b)_I \chi_I - \frac{1}{2} \sum_{I\in\mathcal{F}_j, I\subset J} (b)_J \chi_J \right) \\
&= \sum_{J\in\mathcal{F}_{j-1}} \sum_{I\in\mathcal{F}_j, I\subset J} (b)_I (\chi_I - \frac{1}{2}\chi_J).
\end{aligned}$$

Using this identity we obtain that for given integer $j > j'$ we have

$$\int_{\mathbb{T}} |\Delta S_j(b)(\xi)\Delta S_{j'}(b)(\xi)||d\xi| = \sum_{J\in\mathcal{F}_{j-1}} \sum_{I\in\mathcal{F}_j, I\subset J} (b)_I \sum_{J'\in\mathcal{F}_{j'-1}} \sum_{I'\in\mathcal{F}_{j'}, I'\subset J'} (b)_{I'} \int_{\mathbb{T}} (\chi_I - \frac{1}{2}\chi_J)(\chi_{I'} - \frac{1}{2}\chi_{J'})|d\xi|$$

Based on the property of dyadic partition, if $j \neq j'$, then $J \subset J'$ or $J \subset J'$ so we can assume that $J \subset J'$. If $J \subset I'$ then $\int_{\mathbb{T}} (\chi_I - \frac{1}{2}\chi_J)(\chi_{I'} - \frac{1}{2}\chi_{J'})|d\xi| = 0$ and if $J \cap I' =$ we also deduce that $\int_{\mathbb{T}} (\chi_I - \frac{1}{2}\chi_J)(\chi_{I'} - \frac{1}{2}\chi_{J'})|d\xi| = 0$. Thus, we conclude that $\langle \Delta S_j(b), \Delta S'_j(b)\rangle = 0$ whenever $j \neq j'$. Now, we can obtain (3.18).

The principal result of this chapter is based on the computation of the integral means $\int_{|z|=r} e^{t\mathrm{Re}(b(z))}|dz|,\ \ b \in \mathcal{B}$ in which there is only the real part of a Bloch function b that appears, so that we just need the dyadic martingale which arises from a real part of the Bloch function. Let us state this result.

3.2 The main Theorem.

3.2.1 Statement of Theorem 3.7.

Let b be a Bloch function and b_n be the dyadic martingale of $\mathrm{Re}(b)$. Let us assume the following condition for its square function $\langle S\rangle^2_n$:

$$\forall\theta \in [0, 2\pi], \qquad \left| \langle S\rangle^2_n(e^{i\theta}) - \frac{1}{2\pi}\int_0^{2\pi} \langle S\rangle^2_n(e^{i\theta})d\theta \right| \leq n\delta(n), \tag{$*$}$$

where $\delta(n)$ is a positive function which depends only on n and which tends to zero as n tends to ∞. Let us also write $d(t) = \mathrm{M.dim}(\Gamma_t)$.

Theorem 3.7. *If b belongs to $\mathcal{B}$ and satisfies the condition $(*)$ then the Minkowski dimension of Γ_t has the following development at zero:*

$$M.dim(\Gamma_t) = 1 + \limsup_{r \longrightarrow 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta}{4\pi \log \frac{1}{1-r}} \frac{t^2}{2} + o(t^2), \tag{3.19}$$

$\left(\textit{By Proposition (3.1)}, \limsup_{r \longrightarrow 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta}{\log \frac{1}{1-r}} \textit{ exists} \right)$.

Put $\Omega_t = \phi_t(\mathbb{D})$.

3.2.2 The first step of the proof.

Let f be a conformal map from $\mathbb{D}$ into $\mathbb{C}$. For $p \in \mathbb{R}$ we define

$$\beta(p, f') = \limsup_{r \longrightarrow 1} \frac{\log \left(\int_0^{2\pi} |f'(re^{i\theta})|^p d\theta \right)}{|\log(1-r)|}$$

be the spectrum of integral means of $f'(z)$. It means that $\beta(p, f')$ be the smallest number such that, for every $\epsilon > 0$

$$\int_0^{2\pi} |f'(re^{i\theta})|^p d\theta = \mathcal{O}\left(\frac{1}{(1-r)^{\beta(p,f')+\epsilon}} \right) \quad \text{as } r \longrightarrow 1.$$

Recall the family of conformal maps of $\mathbb{D}$ into $\mathbb{C}$ (ϕ_t): $\phi_t(z) = \displaystyle\int_0^z e^{tb(u)} du, t \in U$, a neighborhood of 0. The spectrum of integral means of ϕ_t' satisfies $\beta(p, \phi_t') = \beta(tp, \phi')$, where $\phi' = \exp b(z)$ $(z \in \mathbb{D}; b \in \mathcal{B})$, because

$$\beta(p, \phi_t') = \limsup_{r \longrightarrow 1} \frac{\log \left(\int_0^{2\pi} (\exp\{t\mathrm{Re} b(re^{i\theta})\})^p d\theta \right)}{|\log(1-r)|} = \limsup_{r \longrightarrow 1} \frac{\log \left(\int_0^{2\pi} \exp\{tp\mathrm{Re} b(re^{i\theta})\} d\theta \right)}{|\log(1-r)|}.$$

This implies that for p small and $\phi'(z) = \exp(b(z)), b \in \mathcal{B}$

$$\beta(p, \phi') = \beta(1, \phi_p') = \limsup_{r \longrightarrow 1} \frac{\log \left(\int_0^{2\pi} \exp\{p\mathrm{Re} b(re^{i\theta})\} d\theta \right)}{|\log(1-r)|}.$$

In other words, $\beta(p, \phi')$ is well-defined for p small.

Next, we reproduce the crucial result of the Minkowski dimension of quasicircles (boundaries of quasidisks).

Theorem 3.8. *Let f map $\mathbb{D}$ conformally onto a bounded domain Ω and let $1 \le p \le 2$. If*

$$N(\epsilon, \partial\Omega) = \mathcal{O}(\epsilon^{-p}) \quad as \quad \epsilon \longrightarrow 0 \tag{3.20}$$

then

$$\int_{\mathbb{T}} |f'(r\xi)|^p |d\xi| = \mathcal{O}\left(\frac{1}{(1-r)^{p-1}} \log \frac{1}{1-r} \right) \quad as \quad r \longrightarrow 1. \tag{3.21}$$

If (3.21) holds and if Ω is a quasidisk then

$$N(\epsilon, \partial\Omega) = \mathcal{O}\left(\frac{1}{\epsilon^p} \left(\log \frac{1}{\epsilon} \right)^2 \right) \quad as \quad \epsilon \longrightarrow 0. \tag{3.22}$$

Proof: [Pom92] Let $M, M_1, ...$ be suitable constants. For $n = 1, 2, ...$ let

$$z_{n\nu} = \left(1 - \frac{1}{2^n} \right) \exp \frac{2i\pi(\nu - 1)}{2^n} (\nu = 1, ..., 2^n; n = 1, 2, ...)$$

denote the dyadic points and $I_{n\nu}$ the corresponding dyadic arcs

$$I_{n\nu} = I(z_{n\nu}) = \left\{ e^{it} : \frac{2\pi(\nu-1)}{2^n} \ge t \ge \frac{2n\nu}{2^n} \right\}.$$

We first assume that (3.20) holds. By Corollary 1.6 [Pom92] it suffices to prove (3.21) for the case that $r = r_n = 1 - 2^{-n}$. For fixed $n = 1, 2, ...$ we write $\epsilon_{-1} = 0$ and

$$\epsilon_k = 2^k (1 - r_n)^{1/p} = 2^{k - n/p} \quad \text{for} \quad k = 0, 1, ... \tag{3.23}$$

Let m_k denote the number of points $z_{n\nu}$ $(\nu = 1, ..., 2^n)$ such that

$$\epsilon_{k-1} d_f(z_{n\nu}) \equiv \text{dist}(f(z_{n\nu}), \partial\Omega) < \epsilon_k. \tag{3.24}$$

Since $d_f(z)$ is bounded we have $m_k = 0$ for $k \le Mn$. It follows from Corollary 1.6 and the fact that $(1 - |z|)|f'(z)| \le 4 d_f(z)$ that

$$\begin{aligned} \int_0^{2\pi} |f'(r_n e^{it})|^p dt \le \frac{M_1}{2^n} \sum_{\nu=0}^{2^n} |f'(z_{n\nu})|^p \le \sum_{k<Mn} \frac{M_2 \epsilon_k^p m_k}{2^n (1 - r_n)^p} \\ = M_2 2^{(p-2)n} \sum_{k<Mn} 2^{kp} m_k \end{aligned} \tag{3.25}$$

$$\tag{3.26}$$

by (3.23). Now let $k \ge 1$. We see from (3.20) that $\partial\Omega$ can be covered by

$$N_k = N(\epsilon, \partial\Omega) < M_3 \epsilon^{-p} \tag{3.27}$$

disks of diameter $\epsilon_k/3$; we may assume that each disk contains a point of $\partial\Omega$. Let V_k be the union of these disks.

Consider an index ν such that (3.24) holds. Then $f(z_{n\nu}) \notin \overline{V}_k$. Hence we can find a connected set $A_{k\nu} \subset \mathbb{D}$ connecting $[0, e^{2i\pi(\nu-1)/2^n}]$ with $[0, e^{2i\pi\nu/2^n}]$ such that $f(A_{k\nu})\partial V_k$.

It follows from Corollary 4.19 [Pom92] that $d_f(z_{n\nu}) < M_4\Lambda(f(A_{k\nu})) + \epsilon/3$. Summing over all ν such that (3.24) holds we see that

$$m_k\epsilon/6 = m_k(\epsilon_{k-1} - \epsilon/3) < M_4 \sum_\nu \Lambda(f(A_{k\nu})) \leq M_4\Lambda(\partial V_k) \leq \pi M4N_k\epsilon_k/3 \qquad (3.28)$$

because the sets $A_{k\nu}$ are essentially disjoint. Hence $m_k < M_5\epsilon_k^{-p}M_5 2^{-kp+n}$ by (3.27) and (3.23) so that, by (3.25),

$$\int_0^{2\pi} |f'(r_n e^{it})|^p dt < M_7 2^{(p-2)n}\left(2^n + \sum_{1\leq k\leq Mn} 2^{kp2}2^{-kp+n}\right) < M_8 n 2^{(p-1)n}$$

which proves (3.21) because $1 - r_n = 2^{-n}$.

Conversely we assume that (3.21) holds and that Ω is a quasidisk. Let $0 < \epsilon < d_f(0)$ be given and let q_n denote the number of $\nu \in \{1, 2, ..., 2^n\}$ such that

$$\epsilon/c < d_f(z_{n\nu}) < \epsilon c \qquad (3.29)$$

where c will be chosen below. Since Ω is a quasidisk we see from Theorem 5.2 [Pom92] (iii) with $z = 0$ that $q_n = 0$ for $n > M_9\log(c/\epsilon)$. Furthermore

$$\frac{2\pi}{2^n}\sum_{\nu=1}^{2^n} d_f(z_{n\nu})^p \leq M_{10}\int_0^{2\pi}(1-r_n)^p|f'(r_n e^{it})|^p dt < \frac{M_{11}}{2^n}\log\frac{1}{1-r_n}$$

by Corollary 1.6 and by (3.21) so that $q_n < M_{12}(c/\epsilon)^p$. It follows that

$$\sum_{n=1}^{\infty} q_n < M_13(c/\epsilon)^p[\log(c/\epsilon)]^2. \qquad (3.30)$$

Let $\xi \in \mathbb{T}$. Since $\epsilon < d_f(0)$ there exists $r = r(\xi)$ such that $d_f(r\xi) = \epsilon$ and thus $n = n(\xi)$ and $\nu = \nu(\xi)$ such that

$$\xi \in I_{n\nu},\ \ M_{14}^{-1}\epsilon < d_f(z_{n\nu}) < M_14\epsilon,$$

where we have again used Corollary 1.6 [Pom92]. Hence (3.29) holds if we choose $c = M_{14}$. Since Ω is a quasidisk it follows from corollary 5.3 [Pom92] that $\mathrm{diam}(f(I_{n\nu})) < M_{15}d_f(z_{n\nu})$ so that $f(I_{n\nu})$ lies in at most M_{16} disks of diameter ϵ. If we consider all $\xi \in \mathbb{T}$ we deduce that $N(\epsilon, \partial\Omega) \leq M_{16}\sum_n q_n$ and (3.22) follows from (3.30).

Corollary 3.9. *If f maps $\mathbb{D}$ conformally onto a quasidisk Ω then*

$$M.dim\partial\Omega = p$$

where p is the unique solution of $\beta(p, f') = p - 1$.

Proof: [Pom92] Since Ω is a quasidisk it follows from Theorem 5.2 (iii)[Pom92] that $\beta(p + \delta, f') \leq \beta(p, f') + q\delta(\delta > 0)$ for some $q < 1$. Hence $\beta(p, f') = p - 1$ has a unique solution p and

$$\int_{\mathbb{T}} |f'(r\xi)|^{p+\delta}|d\xi| = \mathcal{O}((1-r)^{1-p-q\delta-\eta}) = \mathcal{O}\left((1-r)^{1-p-\delta}\log\frac{1}{1-r}\right)$$

for suitable $\eta > 0$ so that (3.21) holds with p replaced by $p+\delta$. Therefore $\mathrm{M.dim}\partial\Omega \leq p+\delta$ by (3.22) and thus $\mathrm{M.dim}\partial\Omega \leq p$. The converse follows similarly from the fact that (3.20) implies (3.21).

As a consequence of Corollary 3.9, we deduce the next proposition.

Proposition 3.10. *Let b be a Bloch function. If the spectrum of integral means of $\phi'(z) = \exp b(z)$ $(z \in \mathbb{D})$ has the development at $p = 0$:*

$$\beta(p, \phi') = ap^2 + o(p^2)$$

then the Minkowski dimension of Γ_t has the development at $t = 0$:

$$d(t) = 1 + at^2 + o(t^2).$$

Proof: We observe that $d(t) \longrightarrow 1$, as $t \longrightarrow 0$. Put $x(t) = d(t) - 1$. Corollary 3.9 implies that

$$\beta(d(t), \phi_t') = d(t) - 1.$$

Since $\beta(d(t), \phi_t') = \beta(td(t), \phi')$, we get

$$\beta(t(1 + x(t)), \phi') = x(t). \tag{3.31}$$

And by the assumption we have $\beta(t(1+x(t)), \phi') = at^2(1+x(t))^2 + o(t^2(1+x(t))^2)$. Since $x(t) \longrightarrow 0$ as $t \longrightarrow 0$, then $t^2(1 + x(t))^2 = t^2 + o(t^2)$. This implies that

$$\beta(t(1 + x(t)), \phi') = at^2 + o(t^2) \tag{3.32}$$

From (3.31) and (3.32), we obtain $x(t) = at^2 + o(t^2)$. The result follows.

Next we proceed to the second step of this proof.

3.2.3 The second step of the proof.

According to Proposition 3.10, in order to finish the proof of Theorem 3.7, we need to show that the family of conformal maps $\phi_t(z) = \int_0^z e^{tb(u)} du$, where the Bloch function $b(z)$ satisfies the condition $(*)$ is such that the spectrum of integral means of $\phi'(z) = \exp b(z)$ has the development: $\beta(p, \phi') = ap^2 + o(p^2)$ at $p = 0$. This will be shown in the following theorem.

Theorem 3.11. *If b belongs to $\mathcal{B}$ and satisfies the condition $(*)$ then the spectrum of the integral means of function $\phi'(z) = \exp b(z)$ has the following development at $p = 0$:*

$$\beta(p, \phi') = \frac{1}{4} \limsup_{r \longrightarrow 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta}{2\pi \log(\frac{1}{1-r})} p^2 + \mathcal{O}(p^4).$$

Proof of Theorem 3.11: Let us give some remarks and the strategy for the proof of this theorem. First, we note that if $\gamma = \mathrm{Re}(b(0)) \neq 0$, then put $b_1(z) = b(z) - b(0)$ and we have

$$\beta(p,\phi') = \limsup_{r\longrightarrow 1} \frac{\log\int_0^{2\pi} e^{p\gamma+p\mathrm{Re}b_1(re^{i\theta})}d\theta}{\log\frac{1}{1-r}} = \limsup_{r\longrightarrow 1}\left\{\frac{\log\int_0^{2\pi} e^{p\mathrm{Re}b_1(re^{i\theta})}d\theta}{\log\frac{1}{1-r}} + \frac{p\gamma}{\log\frac{1}{1-r}}\right\}$$
$$= \limsup_{r\longrightarrow 1} \frac{\log\int_0^{2\pi} e^{p\mathrm{Re}b_1(re^{i\theta})}d\theta}{\log\frac{1}{1-r}}.$$

This says that we do not loose generality if we assume that $b(0) = 0$. Moreover, we observe that for each $r \in (0,1)$, there exists n such that $1/2^{n+1} \leq 1-r \leq 1/2^n$ and from (3.17) $\left(|b_n(e^{i\theta}) - \mathrm{Re}(b(re^{i\theta}))| \leq C\|b\|_{\mathcal{B}},\ (r = 1-2^{-n}) \right)$, we deduce that

$$\beta(p,\phi') = \limsup_{r\longrightarrow 1} \frac{\log(\int_0^{2\pi} e^{pb(re^{i\theta})}d\theta)}{\log(\frac{1}{1-r})} = \limsup_{n\longrightarrow\infty} \frac{\log(\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta)}{n\log 2}.$$

Then, Theorem 3.11 will follow from the estimation of the integral $\displaystyle\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta$.

The principal idea of this estimation is to make use of the exponential transformation of dyadic martingale b_n (the dyadic martingale of Reb) which is defined as a sequence

$$\begin{cases} Z_0 = \exp(pb_0); \\ Z_n = \dfrac{\exp(pb_n)}{\prod_{k=1}^n \cosh(p\Delta b_k)}, n \geq 1. \end{cases}$$

Checking the condition $\mathbf{E}(Z_n|\mathcal{F}_{n-1}) = Z_{n-1}$, we see that $Z = (Z_n, \mathcal{F}_n)$ is a positive martingale. The integral $\displaystyle\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta$ will be derived from the following equality which follows from the martingale's property that

$$\forall n \in \mathbb{N}, \quad \mathbf{E}(Z_n) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\exp pb_n(e^{i\theta})}{\prod_{k=1}^n \cosh(p\Delta b_k(e^{i\theta}))}d\theta = \mathbf{E}(Z_0) = 1.$$

In other words,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{pb_n(e^{i\theta}) - \log(\prod_{k=1}^n \cosh(p\Delta b_n(e^{i\theta})))}d\theta = 1. \tag{3.33}$$

The rest part of the estimation of $\displaystyle\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta$ is quite simply. We just apply the following inequalities and the condition $(*)$ to (3.33).

$$\left| \log(\prod_{k=1}^n \cosh(p\Delta b_n)) - \frac{p^2}{2}\langle S\rangle_n^2 \right| \leq \frac{p^4}{12}\sum_{k=1}^n (\Delta b_k)^4 \leq C'p^4\|b\|_{\mathcal{B}}^2\langle S\rangle_n^2, \tag{3.34}$$

where C' is an absolute constant. The first inequality of (3.34) follows from the estimate

$$\left| \log(\cosh(x)) - \frac{x^2}{2} \right| \leq \frac{x^4}{12}, \quad (x \in \mathbb{R}).$$

Indeed, put $g(x) = \log(\cosh(x)) - \frac{x^2}{2} - \frac{x^4}{12}$. We see that $g''(x) = -(\tanh x)^2 - x^2 \leq 0, \ \forall x \in \mathbb{R}$. Hence, $g'(x) = \int_0^x g''(u)du \leq 0, \ \forall x > 0$. Therefore, $g(x) = \int_0^x g'(u)du \leq 0, \ \forall x > 0$ and since $g(x)$ is a even function, then $g(x) \leq 0, \ \forall x \in \mathbb{R}$. Similarly, put $h(x) = \log(\cosh(x)) - \frac{x^2}{2} + \frac{x^4}{12}$. We observe that $h''(x) = -(\tanh x)^2 + x^2 \geq 0, \quad \forall x \in \mathbb{R}$ because $|\tanh x| \leq |x|, \quad \forall x \in \mathbb{R}$. Analogously, we obtain that $\forall x \in \mathbb{R}, \quad h(x) \geq 0$.
Besides, (3.17) ($\forall \xi \in \mathbb{T}, \ |\Delta b_n(\xi)| \leq C\|b\|_{\mathcal{B}}$) implies that

$$\sum_{k=1}^{n} (\Delta b_k)^4 = \sum_{k=1}^{n} (\Delta b_k)^2 (\Delta b_k)^2 \leq C^2 \|b\|_{\mathcal{B}}^2 \sum_{k=1}^{n} (\Delta b_k)^2 = C^2 \|b\|_{\mathcal{B}}^2 \langle S \rangle_n^2.$$

Then the second inequality of (3.34) follows.

Finally, we apply the following lemma to conclude that for p small

$$\beta(p, \phi') = \limsup_{n \longrightarrow \infty} \frac{\log\left(\int_0^{2\pi} e^{p b_n(e^{i\theta})} d\theta \right)}{n \log 2} = \frac{p^2}{4} \limsup_{r \longrightarrow 1} \frac{\int_0^2 |b(re^{i\theta})|^2 d\theta}{2\pi \log \frac{1}{1-r}} + \mathcal{O}(p^4)$$

Lemma 3.12. *Let b be a Bloch function and $\langle S \rangle_n^2$ be the square function of the dyadic martingale b_n of $Re(b)$. Then*

$$\limsup_{n \longrightarrow \infty} \frac{\int_0^{2\pi} \langle S \rangle_n^2 (e^{i\theta}) d\theta}{n \log 2} = \limsup_{r \longrightarrow 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta}{2 \log \frac{1}{1-r}} \leq \pi \|b\|_{\mathcal{B}}^2.$$

Proof: Recall $\tilde{b} = \mathrm{Re} b$ and b_n is a dyadic martingale of $\tilde{b}$. We have:

$$\|b_n\|_2^2 = \int_0^{2\pi} b_n^2(\theta) d\theta = \int_0^{2\pi} \sum_{k=1}^{n} (\Delta b_k(\theta))^2 d\theta = \int_0^{2\pi} \langle S \rangle_n^2 (\theta) d\theta.$$

The second equality follows from Theorem 3.6 and the third one follows from the definition of the square function of the dyadic martingale b_n. Moreover, the fact that $|b_n(e^{i\theta}) - \tilde{b}(re^{i\theta})| \leq C\|b\|_{\mathcal{B}}$ if $r = 1 - 2^{-n}$ (see (3.17)) implies that:

$$\left| \, \|b_n(e^{i\theta})\|_2 - \|\tilde{b}(re^{i\theta})\|_2 \, \right| \leq \|b_n(e^{i\theta}) - \tilde{b}(re^{i\theta})\|_2 \leq 2\pi (C\|b\|_{\mathcal{B}}).$$

Therefore if we divide both sides by $(n \log 2)^{1/2}$ of the above inequalities and take the limit as $n \longrightarrow \infty$, then we obtain:

$$\lim_{n \longrightarrow \infty} \left(\frac{\int_0^{2\pi} (b_n(e^{i\theta}))^2 d\theta}{n \log 2} \right)^{1/2} - \left(\frac{\int_0^{2\pi} (\tilde{b}((1-2^{-n})e^{i\theta}))^2 d\theta}{n \log 2} \right)^{1/2} = 0. \tag{3.35}$$

By Proposition 3.1, $\frac{\int_0^{2\pi} (\tilde{b}((1-2^{-n})e^{i\theta}))^2 d\theta}{n \log 2}$ is bounded and then by (3.35) $\frac{\int_0^{2\pi} (b_n(e^{i\theta}))^2 d\theta}{n \log 2}$ is also bounded. Moreover since the function x^2 is continuous uniformly on some compact set of $[0, +\infty)$, then (3.35) implies that

$$\lim_{n \longrightarrow \infty} \frac{\int_0^{2\pi} (b_n(e^{i\theta}))^2 d\theta}{n \log 2} - \frac{\int_0^{2\pi} (\tilde{b}((1-2^{-n})e^{i\theta}))^2 d\theta}{n \log 2} = 0.$$

Thus,

$$\limsup_{n\longrightarrow\infty}\frac{\int_0^{2\pi}(b_n(e^{i\theta}))^2d\theta}{n\log 2}=\limsup_{n\longrightarrow\infty}\frac{\int_0^{2\pi}(\tilde{b}((1-2^{-n})e^{i\theta}))^2d\theta}{n\log 2}.$$

Then,

$$\limsup_{n\longrightarrow\infty}\frac{\int_0^{2\pi}\langle S\rangle_n^2(\theta)d\theta}{n\log 2}=\limsup_{r\longrightarrow 1}\frac{\int_0^{2\pi}(\tilde{b}(re^{i\theta}))^2d\theta}{\log(\frac{1}{1-r})}\quad (r=1-2^{-n}).$$

Furthermore, since b is holomorphic in the unit disk $\mathbb{D}$ and by Proposition 3.1, we have:

$$\limsup_{r\longrightarrow 1}\frac{\int_0^{2\pi}(\mathrm{Re}b(re^{i\theta}))^2d\theta}{\log(\frac{1}{1-r})}=\limsup_{r\longrightarrow 1}\frac{\int_0^{2\pi}|b(re^{i\theta})|^2d\theta}{2\log(\frac{1}{1-r})}\leq\pi\|b\|_{\mathcal{B}}^2.$$

The lemma is proven.

The proof of Theorem 3.11 remains the main step: that is to estimate the integral $\displaystyle\int_0^{2\pi}e^{pb_n(e^{i\theta})}d\theta$.

The main step of the proof.

Put $\epsilon_n(\theta)=\begin{cases}\dfrac{\log(\prod_{k=1}^n\cosh(p\Delta b_n(e^{i\theta})))-\frac{p^2}{2}\langle S\rangle_n^2(e^{i\theta})}{\frac{p^2}{2}\langle S\rangle_n^2(e^{i\theta})}, & \text{if } \langle S\rangle_n^2(e^{i\theta})\neq 0\\ 0, & \text{otherwise}\end{cases}\quad ,(\theta\in[0,2\pi]).$

It follows that

$$\log\left(\prod_{k=1}^n\cosh(p\Delta b_n(e^{i\theta}))\right)=\frac{p^2}{2}\langle S\rangle_n^2(e^{i\theta})\left(1+\epsilon_n(\theta)\right),\tag{3.36}$$

where $|\epsilon_n(\theta)|\leq C'p^2\|b\|_{\mathcal{B}}^2$ by (3.34).

Put $I_n=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\langle S\rangle_n^2(\theta)d\theta$. By (3.36), (3.33) is equivalent to

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left\{pb_n(e^{i\theta})-\frac{p^2}{2}\langle S\rangle_n^2(e^{i\theta})(1+\epsilon_n(\theta))\right\}d\theta=1.$$

By subtraction and adding the term $\dfrac{p^2}{2}I_n(1+\epsilon_n(\theta))$, we can rewrite the preceding equality as follows

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left\{pb_n(e^{i\theta})-\frac{p^2}{2}I_n(1+\epsilon_n(\theta))-\frac{p^2}{2}(\langle S\rangle_n^2(e^{i\theta})-I_n)(1+\epsilon_n(\theta))\right\}d\theta=1.$$

Remark that I_n is a number, so we can take the term $\exp\left(\dfrac{p^2}{2}I_n\right)$ out of the above integral and then the equality turns out to be

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left\{pb_n(e^{i\theta})-\frac{\epsilon_n(\theta)p^2}{2}I_n-\frac{p^2}{2}(\langle S\rangle_n^2(e^{i\theta})-I_n)(1+\epsilon_n(\theta))\right\}d\theta=\exp\left(\frac{p^2}{2}I_n\right).$$

Put $I = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \exp\left\{ pb_n(e^{i\theta}) - \frac{\epsilon_n(\theta)p^2}{2} I_n - \frac{p^2}{2}(\langle S\rangle_n^2(e^{i\theta}) - I_n)(1+\epsilon_n(\theta)) \right\} d\theta$. Next, we will estimate the integral I. Combining the condition $(*)$ $\left| \langle S\rangle_n^2(e^{i\theta}) - I_n \right| \leq n\delta(n)$ with the fact that $|\epsilon_n(\theta)| \leq C'p^2\|b\|_{\mathcal{B}}^2$, it follows that $\left| (1+\epsilon_n(\theta))(\langle S\rangle_n^2(e^{i\theta}) - I_n) \right| \leq (1 + C'p^2\|b\|_{\mathcal{B}}^2)n\delta(n)$. Then, this implies that:

$$\exp\left\{ -C'p^4\|b\|_{\mathcal{B}}^2 I_n - \frac{n\delta(n)}{2}p^2(1 + C'p^2\|b\|_{\mathcal{B}}^2) \right\} \frac{1}{2\pi}\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta \leq I$$

and

$$I \leq \exp\left\{ C'p^4\|b\|_{\mathcal{B}}^2 I_n + \frac{n\delta(n)}{2}p^2(1 + C'p^2\|b\|_{\mathcal{B}}^2) \right\} \frac{1}{2\pi}\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta.$$

Replacing I by $\exp\left(\frac{p^2}{2} I_n \right)$ and then taking logarithm of two sides of the above inequalities, we deduce that

$$\log\left(\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta \right) - \frac{n\delta(n)}{2}p^2(1 + C'p^2\|b\|_{\mathcal{B}}) - C'p^4\|b\|_{\mathcal{B}}^2 I_n - \log(2\pi) \leq \frac{p^2}{2} I_n$$

and

$$\frac{p^2}{2} I_n \leq \log\left(\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta \right) + \frac{n\delta(n)}{2}p^2(1 + C'p^2\|b\|_{\mathcal{B}}) + C'p^4\|b\|_{\mathcal{B}}^2 I_n - \log(2\pi).$$

In the following, if we divide both sides of the inequalities by $n\log 2$, we obtain the inequalities

$$\frac{p^2}{2}\frac{I_n}{n\log 2} \geq \frac{\log\left(\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta \right)}{n\log 2} - \left(p^2(1+C'p^2\|b\|_{\mathcal{B}}^2)\frac{n\delta(n)}{2n\log 2} + C'p^4\|b\|_{\mathcal{B}}^2\frac{I_n}{n\log 2} + \frac{\log(2\pi)}{n\log 2} \right)$$

and

$$\frac{p^2}{2}\frac{I_n}{n\log 2} \leq \frac{\log\left(\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta \right)}{n\log 2} + \left(p^2(1+C'p^2\|b\|_{\mathcal{B}}^2)\frac{n\delta(n)}{2n\log 2} + C'p^4\|b\|_{\mathcal{B}}^2\frac{I_n}{n\log 2} - \frac{\log(2\pi)}{n\log 2} \right).$$

Taking the lim sup as n tends to ∞ of these inequalities, we get

$$\left(\frac{p^2}{2} - C'p^4\|b\|_{\mathcal{B}}^2 \right) \limsup_{n\longrightarrow\infty} \frac{I_n}{n\log 2} \leq \limsup_{n\longrightarrow\infty} \frac{\log\left(\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta \right)}{n\log 2} \leq \left(\frac{p^2}{2} + C'p^4\|b\|_{\mathcal{B}}^2 \right) \limsup_{n\longrightarrow\infty} \frac{I_n}{n\log 2}.$$

Finally, we obtain the estimation

$$\left| \limsup_{n\longrightarrow\infty} \frac{\log\left(\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta \right)}{n\log 2} - \frac{p^2}{2}\limsup_{n\longrightarrow\infty}\frac{I_n}{n\log 2} \right| \leq C'p^4\|b\|_{\mathcal{B}}^2 \limsup_{n\longrightarrow\infty}\frac{I_n}{n\log 2}, \quad (3.37)$$

where $\limsup\limits_{n\longrightarrow\infty}\dfrac{I_n}{n\log 2}=\limsup\limits_{n\longrightarrow\infty}\dfrac{\int_0^{2\pi}\langle S\rangle_n^2(\theta)d\theta}{2\pi n\log 2}=\limsup\limits_{r\longrightarrow 1}\dfrac{\int_0^{2\pi}|b(re^{i\theta})|^2d\theta}{4\pi\log(\frac{1}{1-r})}\leq\dfrac{\|b\|_{\mathcal{B}}^2}{2}<+\infty$ by Lemma 3.12. Thus, the estimation (3.37) gives us the desired formula for the spectrum of integral means

$$\beta(p,\phi')=\limsup_{r\longrightarrow 1}\frac{\log\left(\int_0^{2\pi}e^{pb(e^{i\theta})}d\theta\right)}{\log(\frac{1}{1-r})}=\frac{p^2}{4}\limsup_{r\longrightarrow 1}\frac{\int_0^{2\pi}|b(re^{i\theta})|^2d\theta}{2\pi\log(\frac{1}{1-r})}+\mathcal{O}(p^4),$$

as p tends to zero. This finishes the proof of Theorem 3.11.

From Theorem 3.11 and Proposition 3.10, we conclude Theorem 3.7.

Corollary 3.13. *Let b belong to little Bloch space $\mathcal{B}_0$ then $H.dim(\Gamma_t)=M.dim(\Gamma_t)\equiv 1$*

Remark: More information about Bloch function. Let b belong to little Bloch space $\mathcal{B}_0$ then there exists a positive decreasing function $\gamma(\delta)$ depending on b, tending to zero as δ tends to zero and satisfying

$$|b'(z)|\leq(1-|z|)^{-1}\gamma(1-|z|)). \tag{3.38}$$

We observe that the property

$$\forall n,\quad \xi\in\partial\mathbb{D},\quad \Big|\, S_n-b((1-2^n)\xi)\,\Big|\leq C\|b\|_{\mathcal{B}}$$

can be improved more precisely that

$$\forall n,\quad \xi\in\partial\mathbb{D},\quad \Big|\, S_n-b((1-2^n)\xi)\,\Big|\leq C\gamma(2^{-n})\|b\|_{\mathcal{B}},$$

see [Mak90]. And as a consequence of the preceding inequality, the increments ΔS_k of martingale S are bounded by $C\gamma(2^{-k})$, $k=1,2,...$ Then the square function $\langle S\rangle_n^2$ is equal to the sum $\sum_{k=1}^n(\Delta S_k)^2$ which is less than $C\sum_{k=1}^n[\gamma(2^{-k})]^2\leq C'\displaystyle\int_{2^{-n}}^1\frac{\gamma^2(s)}{s}ds$. So we have

$$\frac{\langle S\rangle_n}{n}\leq C'\frac{\Phi(\delta)\log 2}{-\log\delta},$$

where $\Phi(\delta)=\displaystyle\int_\delta^1\frac{\gamma(s)^2}{s}ds$ and $\delta=2^{-n}$. We observe that

$$\lim_{\delta\longrightarrow 0}\frac{\Phi(\delta)\log 2}{-\log\delta}=\lim_{\delta\longrightarrow 0}\frac{\int_\delta^1\frac{\gamma(s)^2}{s}ds\log 2}{-\log\delta}.$$

Applying L'Hôpital's rule, we deduce that

$$\lim_{\delta\longrightarrow 0}\frac{\int_\delta^1\frac{\gamma(s)^2}{s}ds\log 2}{\log\delta}=\lim_{\delta\longrightarrow 0}\gamma(\delta)^2\log 2=0.$$

It follows that $\langle S\rangle_n \leq o(n)$.

Proof of Corollary 3.13: Since b belongs to little Bloch space $\mathcal{B}_0$, then $\langle S\rangle_n \leq o(n)$. It means that $\langle S\rangle_n$ satisfies the condition $(*)$ and the corollary follows from the proof of Theorem 3.11 and the fact that $1 \leq \text{H.dim}(\Gamma_t) \leq \text{M.dim}(\Gamma_t)$.

If Corollary 3.13 gives us a trivial example of Bloch function which satisfies the condition $(*)$, then in the following we will give a non-trivial example for this condition.

3.2.4 An example with constant square function.

First, we define the independent Bernoullian random variables ε_n on $\partial\mathbb{D}$ by the formula

$$\varepsilon_n(e^{2\pi i x}) = \begin{cases} -1, & x_n = 0 \text{ or } 3, \\ 1, & x_n = 1 \text{ or } 2, \end{cases} \quad (n = 1, 2, ...)$$

where x_n denotes the 4-adic nth digit of $x \in [0, 1]$.

Proposition 3.14. *For any bounded sequence of a real numbers $\{a_n\}$, the 4-adic martingale $S_n = \sum_{k=1}^{n} a_k \varepsilon_k$ is a dyadic martingale (if considered as dyadic) of some Bloch function.*

Proof: See [Mak90].

Let $\{a_k\}$ be a bounded sequence of real number, then $\limsup_{n\longrightarrow\infty} \frac{\sum_{k=1}^n a_k^2}{n} = \alpha < +\infty$. By Proposition 3.14, there exists a Bloch function b which generates the dyadic martingale S_n.

Let $\phi_t(z) = \int_0^z e^{tb(u)} du$: these are conformal mappings from $\mathbb{D}$ onto Ω_t. The Minkowski dimension of $\Gamma_t = \partial\Omega_t$ has the following development at 0:

$$\begin{aligned} \text{M.dim}(\Gamma_t) &= 1 + \limsup_{r\longrightarrow 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta}{4\pi \log \frac{1}{1-r}} \frac{t^2}{2} + o(t^2) \\ &= 1 + \frac{\alpha}{2\log 2} t^2 + o(t^2). \end{aligned} \tag{3.39}$$

Indeed, since $\Delta S_k(e^{i\theta}) = a_k \varepsilon_k(e^{i\theta})$ then $\langle S\rangle_n^2(e^{i\theta}) = \sum_{k=1}^n a_k^2$ is a constant square function. Therefore

$$\left| \langle S\rangle_n^2(e^{i\theta}) - \frac{1}{2\pi} \int_0^{2\pi} \langle S\rangle_n^2(e^{i\theta}) d\theta \right| = 0.$$

Thus, certainly the square function $\langle S\rangle_n^2$ satisfies the condition $(*)$. Besides, we have

$$\limsup_{r\longrightarrow 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta}{2\pi \log(\frac{1}{1-r})} = 2 \limsup_{n\longrightarrow\infty} \frac{\int_0^{2\pi} \langle S\rangle_n^2(\theta) d\theta}{2\pi n \log 2} = 2 \limsup_{n\longrightarrow\infty} \frac{\sum_{k=1}^n a_k^2}{n \log 2} = \frac{2\alpha}{\log 2}, \quad (r = 1 - 2^n).$$

Then, (3.39) follows from Theorem 3.7.

Chapter 4

Bloch series

In this chapter we consider some special functions b in Bloch space giving rise to conformal map and we address the question of Minkowski dimension of the related quasicircles.

4.1 Lacunary series

4.1.1 Lacunary series

Definition: A lacunary series is a series which has form:

$$b(re^{i\theta}) = \sum_{k=1}^{\infty} a_k r^{n_k} e^{in_k\theta},$$

where $\{n_k\}$ is a gap sequence. A series is called by Hadamard lacunary series if the sequence $\{n_k\}$ has the property $\dfrac{n_{k+1}}{n_k} \geq q \quad (k = 1, 2, ...)$ for some real $q > 1$. The following proposition shows that lacunary series naturally give rise to Bloch function.

Proposition 4.1. *If b is a lacunary series then b belongs to Bloch space $\mathcal{B}$ if and only if $\sup_k |a_k|$ is bounded. Also b belongs to little Bloch space $\mathcal{B}_0$ if and only if a_k tends to zero as k tends to ∞.*

Proof: [Pom92] The implications $\Rightarrow$ hold for all Bloch functions. Any Bloch function can be written down as a Fourier series $b(z) = \sum_{n=0}^{\infty} a_n z^n$. By Cauchy integral formula, we have

$$|na_n| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|b'(re^{i\theta})|}{r^n} r d\theta.$$

Let $r = 1 - 1/n$, we deduce that $r^{-(n-1)} \leq \dfrac{1}{\sqrt{e}}$. Then by the assumption that $b \in \mathcal{B}$:

$$|b'(re^{i\theta})| \leq \frac{C}{1-r}$$

implies that $|a_n| \leq M$, where M chosen to be absolute constant.

For the implications $\Leftarrow$, let $|a_k| \le M$ and $|z| = r < 1$. It follows from the definition of b that

$$\frac{r|b'(z)|}{1-r} \le \frac{M}{1-r}\sum_{k=1}^{\infty} n_k r^{n_k} = M \sum_{m=0}^{\infty}\left(\sum_{n_k \le m} n_k\right) r^m,$$

furthermore that $n_k \le q^{k-j} n_j$ for $k \le j$ and $n_k \ge q^{k-j} n_j$ for $k \ge j$. We conclude that

$$\frac{r|b'(z)|}{1-r} \le M \sum_{m=0}^{\infty} \frac{qm}{q-1} r^m = \frac{Mq}{q-1}\frac{r}{(1-r)^2}$$

so that $(1-r^2)|b'(z)| \le 2M\dfrac{q}{q-1}$. Hence $b \in \mathcal{B}$.

If $a_k \longrightarrow 0$ then we choose N so large that $|a_k| \le \varepsilon$ for $k \ge N$ and write

$$b(z) = p(z) + \sum_{k=N}^{\infty} a_k z^{n_k},$$

where p is a polynomial. As above we see that

$$\limsup_{r\longrightarrow 1}(1-r^2)|b'(z)| \le \lim_{r\longrightarrow 1}(1-r^2)|p'(z)| + \frac{2\varepsilon q}{q-1}$$

for every $\varepsilon > 0$. Hence the limes superior is equal to zero and thus $b \in \mathcal{B}_0$.

As introduced in this section we will compute the spectrum of integral means of $\phi'(z) = \exp b(z), z \in \mathbb{D}$:

$$\beta(p, \phi') = \limsup_{r\longrightarrow 1} \frac{\log\left(\int_{\mathbb{T}} |e^{b(r\xi)}|^p |d\xi|\right)}{\log \frac{1}{1-r}}, \quad p \text{ small}$$

where $b(re^{i\theta}) = \sum_{k=1}^{\infty} a_k r^{n_k} e^{in_k\theta}$ and $\dfrac{n_{k+1}}{n_k} \ge q > 1, k = 1, 2, \ldots$

This work was almost done by Kayumov in [Kay01]. In more details, he proved that for a general lacunary series $b(re^{i\theta}) = \sum_{k=1}^{\infty} a_k r^{n_k} e^{in_k\theta}$ the spectrum of integral means $\beta(p, \phi')$ has the development at $p = 0$:

$$\beta(p, \phi') = \frac{p^2}{4}\limsup_{r\longrightarrow 1}\frac{\sum_{k=1}^{\infty}|a_k|^2 r^{2n_k}}{\log\frac{1}{1-r}} + \mathcal{O}(p^{5/2}).$$

Furthermore, he also pointed out that the better estimation could be obtained in the particular case $n_k = q^k$, $q \ge 2$ integer:

$$\beta(p, \phi') = \frac{p^2}{4}\limsup_{r\longrightarrow 1}\frac{\sum_{k=1}^{\infty}\log I_0(p|a_k|r^{q^k})}{\log\frac{1}{1-r}} + \mathcal{O}(p^q), \quad q \ge 2$$

as $p \longrightarrow 0$. As a consequence of this result, we have

$$\beta(p, \phi') = \limsup_{r\longrightarrow 1}\frac{p^2}{4}\frac{\sum_{k=1}^{\infty}|a_k|^2 r^{2q^k}}{\log\frac{1}{1-r}} + \mathcal{O}(p^q),$$

as $p \longrightarrow 0$. However, the latter result does not work for the case: $n_k = 2^k$. In this section we first reproduce the Kayumov's work [Kay01] in details, then by inheriting Kayumov's method, we will give the expansion of $\beta(p, \phi')$ at $p = 0$ up to the cube term p^3 for the case of $q = 2$.

4.1.2 The general lacunary series.

Theorem 4.2. ***(Kayumov)[Kay01]*** *Let $b(u) = \sum_{k=1}^{\infty} a_k z^{n_k}$ with the sequence $\{a_k\}$ bounded and the sequence $\{n_k\}$ satisfying $\dfrac{n_{k+1}}{n_k} \geq q \geq 2$. Then the spectrum of integral means $\beta(p, \phi')$ has the development at $p = 0$:*

$$\beta(p, \phi') = \limsup_{r \longrightarrow 1} \frac{\log \int_0^{2\pi} |\phi'(re^{i\theta})|^p d\theta}{\log(\frac{1}{1-r})} = \frac{p^2}{4} \limsup_{r \longrightarrow 1} \frac{\sum_{k=1}^{\infty} |a_k|^2 r^{2n_k}}{\log \frac{1}{1-r}} + \mathcal{O}(p^{5/2}).$$

To prove this theorem, we need the following lemmas. Let $\varphi : \mathbb{D} \longrightarrow \mathbb{C}$ be a conformal map and let

$$\kappa = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| z \frac{\varphi''(z)}{\varphi'(z)} \right| .$$

By Becker univalence criterion, $\kappa < 6$, see [Pom75].

Lemma 4.3. *Let φ be a conformal map from $\mathbb{D}$ into $\mathbb{C}$ and κ defined as above. Then the spectrum of integral means $\beta(p, \varphi')$ satisfies the following inequality*

$$\beta(p, \varphi') \leq \kappa^2 \frac{p^2}{4}, \quad p \in \mathbb{R}.$$

Proof: We consider that $u(r) = \displaystyle\int_{|\xi|=1} |\varphi'(r\xi)|^p r |d\xi|$, for $0 \leq r \leq 1$. If we take derivative of function $u(r)$ by r i.e differentiating under the integral sign with variable r, then we have

$$u'(r) = p \int_{|\xi|=1} |\varphi'(r\xi)|^p \mathrm{Re} \left(\xi \frac{\varphi''(r\xi)}{\varphi'(r\xi)} \right) r |d\xi|.$$

Applying Hardy's identity for an arbitrary analytic function g and $p \in \mathbb{R}$

$$\frac{d}{dr} \left(r \frac{d}{dr} \int_{|\xi|=1} |g(r\xi)|^p |d\xi| \right) = p^2 r \int_{|\xi|=1} |g(r\xi)|^{p-2} |g'(r\xi)| d\xi|$$

to the function φ', we obtain

$$u''(r) + \frac{u'(r)}{r} = \frac{1}{r} \frac{d}{dr} [r u'(r)] = p^2 \int_{|\xi|=1} |\varphi'(r\xi)|^p \left| \xi \frac{\varphi''(r\xi)}{\varphi'(r\xi)} \right|^2 |d\xi|.$$

We observe that the right hand side of the equality above is less than $p^2 \dfrac{\kappa^2 u}{r^2 (1 - r^2)^2}$ which is equal to $p^2 \dfrac{\kappa^2 u}{(4 - \eta)(1 - r)^2}$, for some η. In addition that the left hand side of the equality above is positive thus it implies that the function $ru'(r)$ is increasing function and therefore $u'(r)$ is also positive. We observe that the equation

$$v''(r) = \frac{p^2\kappa^2 v}{(4-\eta)(1-r)^2} + \frac{p^2\kappa^2 v'}{(4-\eta)(1-r)}$$

has a solution $v(r) = (1-r)^{-\frac{p^2\kappa^2}{4-\eta}}$. Since $u'(r)$ is positive , then

$$u''(r) \leq \frac{p^2\kappa^2 u}{(4-\eta)(1-r)} + \frac{p^2\kappa^2 u'}{(4-\eta)(1-r)}.$$

Moreover since $u'(0) \leq v'(0), u(0) \leq v(0)$, then by differential's inequality we deduce that $u(r) \leq v(r)$ for $0 \leq r \leq 1$. Let r tend to 1, then η tends to zero, so we get the inequality

$$\beta(p, \varphi') \leq \frac{p^2\kappa^2}{4}.$$

Remark: Let $b \in \mathcal{B}$ and assume that $b(0) = 0$ then

$$|b(z)| = \left| \xi \int_0^r b'(s\xi)ds \right| \leq \int_0^r |b'(s\xi)|ds \leq \int_0^r \frac{\|b\|_{\mathcal{B}}}{1-s^2}ds = \frac{\|b\|_{\mathcal{B}}}{2}\log\frac{1+r}{1-r}.$$

In general for $b \in \mathcal{B}$, then there exists $C > 0$ such that

$$|b(z)| \leq C \log \frac{2}{1-|z|}.$$

Lemma 4.4. *Let $\log\phi', \log f'$ belong to Bloch space $\mathcal{B}$. Then there exists a positive constant C such that*

$$\frac{1}{C}(1-r)^{Cp^{5/2}} \int_0^{2\pi} |\phi'|^p d\theta \leq \int_0^{2\pi} |g'| d\theta \leq C \frac{1}{(1-r)^{Cp^{5/2}}} \int_0^{2\pi} |\phi'|^p d\theta,$$

where $\phi'^p = g' f'^{p^2}$ and $0 < p < 1$.

Proof: Since $p\log\phi', p^2\log f', \log g' \in \mathcal{B}$, then by the above remark and Lemma 4.3 there exists a positive constant C such that

$$|\phi'|^p, |g'|^p \leq C \left(\frac{1}{1-r}\right)^{Cp}$$

and

$$\int_0^{2\pi} |\phi'|^p d\theta, \int_0^{2\pi} |f'|^{\pm p} d\theta, \int_0^{2\pi} |g'| d\theta \leq C \left(\frac{1}{1-r}\right)^{Cp^2}.$$

Now we estimate the integral $\int_0^{2\pi} |\phi'|^p d\theta$ by using the Hölder's inequality as follows

$$\begin{aligned}
\int_0^{2\pi} |\phi'|^p d\theta &= \int_0^{2\pi} |g'||f'|^{p^2} d\theta \\
&\leq \left(\int_0^{2\pi} |g'|^{\frac{1}{1-p^{\frac{3}{2}}}} d\theta \right)^{1-p^{\frac{3}{2}}} \left(\int_0^{2\pi} |f'|^{p^{\frac{1}{2}}} d\theta \right)^{p^{\frac{3}{2}}} \\
&\leq \sup_{|z|\leq 1} (|g'|^{p^{\frac{3}{2}}}) \left(\int_0^{2\pi} |g'| d\theta \right)^{1-p^{\frac{3}{2}}} C \left(\frac{1}{1-r} \right)^{Cp^{\frac{5}{2}}} \\
&\leq C^2 \left(\frac{1}{1-r} \right)^{2Cp^{\frac{5}{2}}} \left(\int_0^{2\pi} |g'| d\theta \right) \left(\int_0^{2\pi} |g'| d\theta \right)^{p^{\frac{3}{2}}} \\
&\leq C^3 \left(\frac{1}{1-r} \right)^{3Cp^{\frac{5}{2}}} \int_0^{2\pi} |g'| d\theta.
\end{aligned}$$

Analogously, applying the Hölder inequality to $\int_0^{2\pi} |\phi'|^p |f'|^{-p^2} d\theta$ we obtain

$$\int_0^{2\pi} |g'| d\theta = \int_0^{2\pi} |\phi'|^p |f'|^{-p^2} d\theta \leq C^3 \left(\frac{1}{1-r} \right)^{3Cp^{\frac{5}{2}}} \int_0^{2\pi} |\phi'|^p d\theta.$$

Corollary 4.5. *Let* $\log \phi' = \sum_{k=1}^{\infty} a_k z^{n_k} \in \mathcal{B}$. *Then there exists a constant* $C > 0$ *such that*

$$\frac{1}{C}(1-r)^{Cp^{\frac{5}{2}}} I(r,p) \leq \int_0^{2\pi} |\phi'|^p d\theta \leq C \left(\frac{1}{1-r} \right)^{Cp^{\frac{5}{2}}} I(r,p)$$

where $I(r,p) = \int_0^{2\pi} \prod_{k=1}^{\infty} |1 + \frac{p}{2} a_k z^{n_k}|^2 d\theta.$

Proof: By the inequality that for x small $\left| \log(1+x) - x + \frac{x^2}{2} \right| \leq |x|^3$, the function $p \log \phi'$ can be rewritten as

$$p \log \phi'(z) = \sum_{k=1}^{\infty} p a_k z^{n_k} = \sum_{k=1}^{\infty} 2 \log(1 + \frac{p}{2} a_k z^{n_k}) + \frac{p^2}{4} \sum_{k=1}^{\infty} a_k^2 z^{2n_k} + R_p(z),$$

where $|R_p(z)| \leq \sum_{k=1}^{\infty} |p a_k r^{n_k}|^3$. Put $\log f'(z) = \sum_{k=1}^{\infty} a_k^2 z^{2n_k}$, then $\log f' \in \mathcal{B}$. It is easy to see that $\sum_{k=1}^{\infty} (|a_k| r^{n_k})^3 = \mathcal{O}\left(\log \frac{1}{1-r} \right)$. Then the corollary follows by Lemma 4.4.

Proof of Theorem 4.2: If we apply Corollary 4.5, then the only task remains for the proof is to compute the integral $I(r,p)$. We observe that

$$\int_0^{2\pi}\prod_{k=1}^{\infty}\left|\,1+\frac{p}{2}a_k z^{n_k}\,\right|^2 d\theta = \int_0^{2\pi}\prod_{k=1}^{\infty}\left(\,1+\frac{p^2}{4}|a_k|r^{2n_k}+p|a_k|r^{n_k}cos(n_k\theta+\theta_k)\,\right)d\theta,$$

where θ_k is the argument of coefficient a_k. We multiply out and integrate term-by-term. Using the properties of lacunary series that $\sum_{k=1}^{j-1} n_{p_k} \leq \sum_{k=1}^{p_j-1} n_k < \frac{n_{p_j}}{q-1} \leq n_{p_j}$, we deduce that the integral of each term

$$\cos(n_{p_1}+\theta_{p_1})\cos(n_{p_2}+\theta_{p_2})...\cos(n_{p_j}+\theta_{p_j}) = \frac{1}{2^j}\sum\cos((n_{p_1}\pm n_{p_2}\pm...\pm n_{p_j})\theta+\gamma_p)$$

on the interval $[0,2\pi]$ vanishes. Therefore $I(r,p) = 2\pi\prod_{k=1}^{\infty}(1+\frac{p^2}{4}|a_k|^2 r^{2n_k})$. Furthermore for p small this product equals to

$$2\pi\left(\,\frac{1}{1-r}\,\right)^{\mathcal{O}(p^3)}\exp\left\{\,\frac{p^2}{4}\sum_{k=1}^{\infty}|a_k|r^{2n_k}\,\right\}.$$

This concludes the proof.

4.1.3 The lacunary series with gap sequence $\{n_k\}=\{q^k\}$, $q\geq 3$ integer.

Theorem 4.6. *With the same assumption of Theorem 4.2, the spectrum of integral means $\beta(p,\phi')$ has the development at $p=0$:*

$$\beta(p,\phi') = \limsup_{r\longrightarrow 1}\frac{\sum_{k=1}^{\infty}\log I_0(p|a_k|r^{q^k})}{\log\frac{1}{1-r}} + \mathcal{O}(p^q), \quad q\geq 2$$

as t tends to zero, where I_0 is a modified Bessel function (see definition below).

Proof: Let

$$I_n(x) = \left(\,\frac{x}{2}\,\right)^n\sum_{\nu=0}^{\infty}\frac{x^{2\nu}}{2^{2\nu}\nu!(\nu+n)!}, \quad (x\in\mathbb{R}; n=0,1,2,3,...)$$

be modified Bessel functions. They appear in the Fourier series

$$\exp(x\cos(\theta)) = I_0(x)+2\sum_{n=1}^{\infty}I_n(x)\cos(n\theta).$$

The definition of I_n implies that $I_n(x)\leq\frac{|x|^n}{2^n}\frac{\exp\frac{|x|^2}{4}}{n!}$. Using this inequality, we obtain that

$$\left|\,x\cos(\theta)-\log(I_0(x)+2\sum_{n=1}^{q-1}I_n(x)\cos(n\theta))\,\right|\leq C|x|^q. \tag{4.1}$$

We have
$$\int_0^{2\pi} |\phi'|^p d\theta = \int_0^{2\pi} \prod_{k=1}^{\infty} \exp\left\{ p|a_k| r^{q^k} \cos(q^k\theta + \theta_k) \right\} d\theta,$$
where θ_k is the argument of coefficient a_k. Combining the inequality (4.1) with the fact that $\sum_{k=1}^{\infty} r^{q^k} = \mathcal{O}\left(\log \frac{1}{1-r} \right)$, we deduce that

$$\int_0^{2\pi} |\phi'|^p \theta = \left(\frac{1}{1-r} \right)^{\mathcal{O}(p^q)} \int_0^{2\pi} \prod_{k=1}^{\infty} \left(I_0(p|a_k|r^{q^k}) + 2\sum_{j=1}^{q-1} I_j(t|a_k|r^{q^k}) \cos j(q^k\theta + \theta_k) \right) d\theta.$$

Using the analogue argument as Theorem 4.2, after multiplying and integrating term-by-term, the integral above equals to $2\pi \prod_{k=1}^{\infty} I_0(t|a_k|r^{q^k})$. The theorem follows.

Corollary 4.7.
$$\beta(p, \phi') = \limsup_{r \longrightarrow 1} \frac{p^2}{4} \frac{\sum_{k=1}^{\infty} |a_k|^2 r^{2q^k}}{\log \frac{1}{1-r}} + \mathcal{O}(p^{\alpha}),$$
as t tends to zero, where $\alpha = \min\{q, 4\} \geq 3$.

Proof: By the following inequality for $I_0(x)$
$$\left| \log(I_0(x)) - \frac{x^2}{4} \right| \leq x^4, \quad x \text{ small},$$
we obtain
$$\left| \sum_{k=1}^{\infty} \log(I_0(p|a_k|r^{q^k})) - \frac{p^2}{4} \sum_{k=1}^{\infty} |a_k|^2 r^{2q^k} \right| \leq p^4 \sum_{k=1}^{\infty} |a_k|^4 r^{4q^k} \leq C p^4 \sum_{k=1}^{\infty} r^{2q^k}.$$
It follows that
$$\beta(p, \phi') = \frac{p^2}{4} \limsup_{r \longrightarrow 1} \frac{\sum_{k=1}^{\infty} |a_k|^2 r^{q^k}}{\log \frac{1}{1-r}} + \mathcal{O}(p^{\alpha}),$$
where $\alpha = \min\{q, 4\} \geq 3$.

4.1.4 The lacunary series with gap sequence $\{n_k\} = \{2^k\}$.

Recall Γ_t be the image of the unit circle by the conformal map $\phi_t = \int_0^z e^{tb(u)} du$, where $b(z) = \sum_{k=1}^{\infty} z^{2^k}$.

Theorem 4.8. *With the same assumption as above, the spectrum $\beta(p, \phi')$ has the development at $p = 0$:*
$$\beta(p, \phi') = \frac{p^2}{4} \limsup_{n \longrightarrow \infty} \frac{\sum_{k=1}^{n} |a_k|^2}{n \log 2} + \frac{p^3}{8} \limsup_{n \longrightarrow \infty} \frac{\sum_{k=1}^{n-1} |a_k|^2 |a_{k+1}| \cos(\theta_k - \theta_{k+1})}{n \log 2} + \mathcal{O}(p^4),$$
where $\theta_k = \arg(a_k)$.

In order to prove this theorem, we need the following proposition.

Proposition 4.9. *Let b be a lacunary series. If $r_j = 1 - \dfrac{1}{n_j}$ for $j = 1, 2, \ldots$ then*

$$\Bigg| \; a_0 + \sum_{k=1}^{j} a_k \xi^{n_k} - b(r\xi) \; \Bigg| \leq K(q) \sup_k |a_k| \quad (\xi \in \mathbb{T}) \tag{4.2}$$

for $r_j \leq r \leq r_{j+1}$ where $\dfrac{n_{k+1}}{n_k} \geq q \quad (k = 1, 2, \ldots)$ and $K(q)$ depends only on $q > 1$.

Proof: [Pom92] We may assume that $|a_k| \leq 1$. We have

$$\Bigg| \; a_0 + \sum_{k=1}^{j} a_k \xi^{n_k} - b(r\xi) \; \Bigg| \leq \sum_{k=1}^{j} (1 - r^{n_k}) + \sum_{k=j+1}^{\infty} r^{n_k}.$$

Since $1 - r^{n_k} \leq n_k(1 - r_j) = \dfrac{n_k}{n_j}$ and $r^{n_k} \leq e^{\frac{n_k}{n_{j+1}}}$, we see from $n_k \leq q^{k-j} n_j$ for $k \leq j$ and $n_k \geq q^{k-j} n_j$ for $k \geq j$ that the right hand side is less than

$$\sum_{k=1}^{j} q^{k-j} + \sum_{k=j+1}^{\infty} \exp(-q^{k-j-1}) < \frac{q}{q-1} + \sum_{\nu=0}^{\infty} \exp(-q^{\nu}).$$

It concludes the proof.

Proof of Theorem 4.8: The reason that Theorem 4.6 doesn't work for $q = 2$ is in its proof the inequality (4.1) is not a good estimation for $q = 2$. Thus in this one, instead of using (4.1) for $q = 2$, we apply it for $q = 4$. A difficulty in the computation of the integral $\displaystyle\int_0^{2\pi} e^{pb(re^{i\theta})} d\theta$ is $b(re^{i\theta})$ being an infinite sum. Accordingly, this integrand turns out to be incontrollable while we multiply out and integrate term by term as above. However we can handle this difficulty by making use of Proposition 4.9: $|b(r\xi) - b_n(\xi)| \leq K$, where $b_n(\xi) = \sum_{k=1}^{n} a_k \xi^{2^k}$, $r = 1 - 2^{-n}$ and K is an absolute constant. We then have

$$\beta(p, \phi') \; = \; \limsup_{r \longrightarrow 1} \frac{\log \left(\int_0^{2\pi} |e^{pb(re^{i\theta})}| d\theta \right)}{\log \frac{1}{1-r}} \; = \; \limsup_{n \longrightarrow \infty} \frac{\log \left(\int_0^{2\pi} |e^{pb_n(e^{i\theta})}| d\theta \right)}{n \log 2}.$$ By (4.1), the integral $\displaystyle\int_0^{2\pi} |e^{pb_n(e^{i\theta})}| d\theta$ equals to

$$\frac{1}{(1-r)^{\mathcal{O}(p^4)}} \int_0^{2\pi} \prod_{k=1}^{n} \Bigg(\; I_0(p|a_k|) + 2I_1(p|a_k|) \cos(2^k \theta + \theta_k)$$
$$+ 2I_2(p|a_k|) \cos(2.2^k \theta + \theta_k) + 2I_3(p|a_k|) \cos(3.2^k \theta + \theta_k) \Bigg) \, d\theta.$$

From now on, we will use the similar argument as Theorem 4.6 to conclude the proof. Using the property of this lacunary series while we multiply out and integrate term-by-term the integrand, we then deduce that the above integral is equal to

$$
\begin{aligned}
&\prod_{k=1}^{n} I_0(p|a_k|) + \sum_{k=1}^{n} \Bigg\{ 4I_2(p|a_k|)I_1(p|a_{k+1}|) \prod_{l\neq\{k,k+1\}}^{n} I_0(p|a_l|) \int_0^{2\pi} \cos(2^{k+1}\theta+\theta_k)\cos(2^{k+1}\theta+\theta_{k+1})d\theta \\
&+ 4I_2(p|a_k|)I_1(p|a_{k+1}|) \prod_{j\leq k} I_0(p|a_j|) \\
&\times \int_0^{2\pi} \cos(2^{k+1}\theta+\theta_k)\cos(2^{k+1}\theta+\theta_{k+1}) \prod_{n\geq l>k+1} 2I_1(p|a_l|)\cos(2^l\theta+\theta_l)d\theta \\
&+ 8I_2(p|a_k|)I_3(p|a_{k+1}|)I_1(p|a_{k+2}|) \prod_{l\neq\{k,k+1,k+2\}}^{n} I_0(p|a_l|) \\
&\times \int_0^{2\pi} \cos(2.2^k\theta+\theta_k)\cos(3.2^{k+1}\theta+\theta_{k+1})\cos(2^{k+2}\theta+\theta_{k+2})d\theta \\
&+ 4I_2(p|a_k|)I_3(p|a_{k+1}|) \prod_{j\leq k-1} I_0(p|a_j|) \\
&\times \int_0^{2\pi} \cos(2.2^k\theta+\theta_k)\cos(3.2^{k+1}\theta+\theta_{k+1}) \prod_{n\geq l\geq k+2} 2I_1(p|a_l|)\cos(2^l\theta+\theta_l)d\theta \\
&+ 8I_2(p|a_k|)I_3(p|a_{k+1}|)I_1(p|a_{k+3}|) \prod_{l\neq\{k,k+1,k+3\}}^{n} I_0(p|a_l|) \\
&\times \int_0^{2\pi} \cos(2.2^k\theta+\theta_k)\cos(3.2^{k+1}\theta+\theta_{k+1})\cos(2^{k+3}\theta+\theta_{k+3})d\theta \\
&+ 4I_2(p|a_k|)I_3(p|a_{k+1}|)I_0(p|a_{k+2}|) \prod_{j\leq k-1} I_0(p|a_j|) \\
&\times \int_0^{2\pi} \cos(2.2^k\theta+\theta_k)\cos(3.2^{k+1}\theta+\theta_{k+1}) \prod_{n\geq l>k+2} 2I_1(p|a_l|)\cos(2^l\theta+\theta_l)d\theta \Bigg\}
\end{aligned}
$$

Put

$$
\begin{aligned}
A_{1k} &= 4I_2(p|a_k|)I_1(p|a_{k+1}|) \prod_{l\neq\{k,k+1\}}^{n} I_0(p|a_l|); \\
B_{1k} &= \int_0^{2\pi} \cos(2^{k+1}\theta+\theta_k)\cos(2^{k+1}\theta+\theta_{k+1})d\theta; \\
A_{2k} &= 4I_2(p|a_k|)I_1(p|a_{k+1}|) \prod_{j\leq k} I_0(p|a_j|) \prod_{n\geq l>k+1} I_1(p|a_l|);
\end{aligned}
$$

$$B_{2k} = \int_0^{2\pi} \cos(2^{k+1}\theta + \theta_k)\cos(2^{k+1}\theta + \theta_{k+1}) \prod_{n \geq l > k+1} 2\cos(2^l\theta + \theta_l)d\theta;$$

$$A_{3k} = 8I_2(p|a_k|)I_3(p|a_{k+1}|)I_1(p|a_{k+2}|) \prod_{l \neq \{k,k+1,k+2\}}^{n} I_0(p|a_l|);$$

$$B_{3k} = \int_0^{2\pi} \cos(2.2^k\theta + \theta_k)\cos(3.2^{k+1}\theta + \theta_{k+1})\cos(2^{k+2}\theta + \theta_{k+2})d\theta;$$

$$A_{4k} = 4I_2(p|a_k|)I_3(p|a_{k+1}|) \prod_{j \leq k-1} I_0(p|a_j|) \prod_{n \geq l \geq k+2} I_1(p|a_l|);$$

$$B_{4k} = \int_0^{2\pi} \cos(2.2^k\theta + \theta_k)\cos(3.2^{k+1}\theta + \theta_{k+1}) \prod_{n \geq l \geq k+2} 2\cos(2^l\theta + \theta_l)d\theta;$$

$$A_{5k} = 8I_2(p|a_k|)I_3(p|a_{k+1}|)I_1(p|a_{k+3}|) \prod_{l \neq \{k,k+1,k+3\}}^{n} I_0(p|a_l|);$$

$$B_{5k} = \int_0^{2\pi} \cos(2.2^k\theta + \theta_k)\cos(3.2^{k+1}\theta + \theta_{k+1})\cos(2^{k+3}\theta + \theta_{k+3})d\theta;$$

$$A_{6k} = 4I_2(p|a_k|)I_3(p|a_{k+1}|)I_0(p|a_{k+2}|) \prod_{j \leq k-1} I_0(p|a_j|) \prod_{n \geq l > k+2} I_1(p|a_l|);$$

$$B_{6k} = \int_0^{2\pi} \cos(2.2^k\theta + \theta_k)\cos(3.2^{k+1}\theta + \theta_{k+1}) \prod_{n \geq l > k+2} 2\cos(2^l\theta + \theta_l)d\theta.$$

By a simple computation, we have

$$B_{1k} = \int_0^{2\pi} \cos(2^{k+1}\theta + \theta_k)\cos(2^{k+1}\theta + \theta_{k+1})d\theta = \pi\cos(\theta_k - \theta_{k+1});$$

$$B_{3k} = \int_0^{2\pi} \cos(2.2^k\theta + \theta_k)\cos(3.2^{k+1}\theta + \theta_{k+1})\cos(2^{k+2}\theta + \theta_{k+2})d\theta = \frac{\pi}{2}\cos(\theta_{k+2} - \theta_{k+1} + \theta_k)$$

and

$$B_{5k} = \int_0^{2\pi} \cos(2.2^k\theta + \theta_k)\cos(3.2^{k+1}\theta + \theta_{k+1})\cos(2^{k+3}\theta + \theta_{k+3})d\theta = \frac{\pi}{2}\cos(\theta_k + \theta_{k+1} - \theta_{k+3}).$$

Moreover, $|B_{2k}| \leq \pi$. Indeed, we have

$$\begin{aligned} B_{2k} &= \int_0^{2\pi} \cos(2^{k+1}\theta + \theta_k)\cos(2^{k+1}\theta + \theta_{k+1}) \prod_{n \geq l \geq k+2} 2\cos(2^l\theta + \theta_l)d\theta \\ &= \int_0^{2\pi} \frac{(\cos(\theta_k - \theta_{k+1}) + \cos(2^{k+2}\theta + \theta_k + \theta_{k+1})}{2} \prod_{n \geq l \geq k+2} 2\cos(2^l\theta + \theta_l)d\theta \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \frac{\cos(\theta_k - \theta_{k+1})}{2} \prod_{n\geq l\geq k+2} 2\cos(2^l\theta + \theta_l)d\theta \\
&+ \frac{1}{2}\int_0^{2\pi} \cos(2^{k+2}\theta + \theta_k + \theta_{k+1}) 2\cos(2^{k+2}\theta + \theta_{k+2}) \prod_{n\geq l\geq k+3} 2\cos(2^l\theta + \theta_l)d\theta \\
&= \int_0^{2\pi} \cos(2^{k+2}\theta + \theta_k + \theta_{k+1})\cos(2^{k+2}\theta + \theta_{k+2}) \prod_{n\geq l\geq k+3} 2\cos(2^l\theta + \theta_l)d\theta.
\end{aligned}
$$

By induction, we can see that $|B_{2k}| \leq \pi$. Similarly, $|B_{4k}| \leq \pi$ and $|B_{6k}| \leq \pi$. The above result can be simplified to be

$$
\begin{aligned}
\int_0^{2\pi} |e^{pb_n(e^{i\theta})}|d\theta = 2\pi \Bigg\{ \prod_{k=1}^{n} I_0(p|a_k|) + 2\sum_{k=1}^{n-1} \Bigg\{ \Bigg(\prod_{l\neq\{k,k+1\}} I_0(p|a_l|) \Bigg) I_2(p|a_k|)I_1(p|a_{k+1}|)\cos(\theta_k - \theta_{k+1}) \\
+ \frac{1}{2\pi}(A_{2k}B_{2k} + ... + A_{6k}B_{6k}) \Bigg\}\Bigg\}.
\end{aligned}
$$

If we take logarithm of two both sides, we then have

$$
\begin{aligned}
\log \int_0^{2\pi} |e^{pb_n(e^{i\theta})}|d\theta = \log(2\pi) + \log \Bigg\{ \prod_{k=1}^{n} I_0(p|a_k|) \\
+ 2\sum_{k=1}^{n-1} \Bigg\{ \Bigg(\prod_{l\neq\{k,k+1\}}^{n} I_0(p|a_l|) \Bigg) I_2(t|a_k|)I_1(p|a_{k+1}|)\cos(\theta_k - \theta_{k+1}) \\
+ \frac{1}{2\pi}(A_{2k}B_{2k} + ... + A_{6k}B_{6k}) \Bigg\}\Bigg\}.
\end{aligned}
$$

Put

$$
\begin{aligned}
J = \log \Bigg\{ \prod_{k=1}^{n} I_0(p|a_k|) + 2\sum_{k=1}^{n-1} \Bigg\{ \Bigg(\prod_{l\neq\{k,k+1\}} I_0(p|a_l|) \Bigg) I_2(p|a_k|)I_1(p|a_{k+1}|)\cos(\theta_k - \theta_{k+1}) \\
+ \frac{1}{2\pi}(A_{2k}B_{2k} + ... + A_{6k}B_{6k}) \Bigg\}\Bigg\}.
\end{aligned}
$$

Put $C = \prod_{k=1}^{n} I_0(p|a_k|)$. Then J can be rewritten as follows:

$$
\begin{aligned}
J = \Bigg\{ \sum_{k=1}^{n} \log I_0(p|a_k|) \Bigg\} + \log \Bigg\{ 1 + 2\sum_{k=1}^{n-1} \Bigg\{ \frac{I_2(p|a_k|)I_1(p|a_{k+1}|)}{I_0(p|a_k|)I_0(p|a_k|)}\cos(\theta_k - \theta_{k+1}) \\
+ \frac{A_{2k}B_{2k}}{2\pi C} + ... + \frac{A_{6k}B_{6k}}{2\pi C} \Bigg\}\Bigg\} = J_1 + J_2.
\end{aligned}
$$

Let $M_1 = \sup_k |a_k|$. Then $M_1 < +\infty$ because (a_k) is bounded. By the inequality

$$
\left| \log I_0(x) - \frac{1}{4}x^2 \right| \leq x^4,
$$

we can easily estimate $J_1 = \sum_{k=1}^{n} \log I_0(p|a_k|)$ for p small as follows

$$\Bigg| \sum_{k=1}^{n} \log I_0(p|a_k|) - \frac{p^2}{4} \sum_{k=1}^{n} |a_k|^2 \Bigg| \leq p^4 \sum_{k=1}^{n} |a_k|^4 \leq M_1^4 p^4 n.$$

In order to estimate J_2, we need the following inequalities.

First, it follows from the definition of Bessel functions that for small p $\Bigg| \dfrac{A_{ik}B_{ik}}{2\pi C} \Bigg| \leq Mp^4$, where $i = 2, .., 6$ and M is an absolute constant.

Second, for $a, b \leq M_1$

$$\Bigg| \frac{I_2(ax)}{I_0(ax)} \frac{I_1(bx)}{I_0(bx)} - \frac{a^2bx^3}{16} \Bigg| \leq M_1^5 |x|^5. \tag{4.3}$$

(4.3) follows from the two following inequalities. For a small real x,

$$\Bigg| \frac{I_1(x)}{I_0(x)} - \frac{x}{2} \Bigg| \leq |x|^3$$

and

$$\Bigg| \frac{I_2(x)}{I_0(x)} - \frac{x^2}{8} \Bigg| \leq x^4.$$

These two inequalities follow from the differential inequality.

Third, for x small $\Bigg| \log(1+x) - x \Bigg| \leq x^2$. Next, by using triangle inequality and these above inequalities we deduce that

$$\Bigg| J_2 - \frac{p^3}{8} \sum_{k=1}^{n-1} |a_k|^2 |a_{k+1}| \cos(\theta_k - \theta_{k+1}) \Bigg| \leq Cp^4(n-1),$$

Then we obtain the following inequality

$$\Bigg| J - \frac{p^2}{4} \sum_{k=1}^{n} |a_k|^2 - \frac{p^3}{8} \sum_{k=1}^{n-1} |a_k|^2 |a_{k+1}| \cos(\theta_k - \theta_{k+1}) \Bigg| \leq Kp^4 n,$$

where K is absolute constant.

Now if we divide $\log \left(\int_0^{2\pi} |e^{pb_n(e^{i\theta})}| d\theta \right)$ by $n \log 2$ and take the lim sup of $\dfrac{\log \left(\int_0^{2\pi} |e^{pb_n(e^{i\theta})}| d\theta \right)}{n \log 2}$ as n tends to ∞, then we deduce that the spectrum of integral means $\beta(p, \phi')$ has the development at $p = 0$:

$$\beta(p, \phi') = \frac{p^2}{4} \limsup_{n \longrightarrow \infty} \frac{\sum_{k=1}^{n} |a_k|^2}{n \log 2} + \frac{p^3}{8} \limsup_{n \longrightarrow \infty} \frac{\sum_{k=1}^{n-1} |a_k|^2 |a_{k+1}| \cos(\theta_k - \theta_{k+1})}{n \log 2} + \mathcal{O}(p^4).$$

Remarks: First, let $\phi'(z) = \exp(b(z))$, where b is a Bloch function satisfying the condition $(*)$. By Theorem 3.7 the spectrum of integral means $\beta(p, \phi')$ has the development at $p = 0$:

$$\beta(p, \phi') = ap^2 + \mathcal{O}(p^4),$$

where $a = \dfrac{1}{4} \limsup_{r \longrightarrow 1} \dfrac{\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta}{\log \frac{1}{1-r}}$. This says that the cube term p^3 in the above expansion at $p = 0$ of $\beta(p, \phi')$ vanishes while if $b(z) = \sum_{k=1}^{\infty} z^{2^k}$ then by Theorem 4.8, the spectrum $\beta(p, \phi')$ has the development at $p = 0$:

$$\beta(p, \phi') = \frac{p^2}{4 \log 2} + \frac{p^3}{8 \log 2} + \mathcal{O}(p^4).$$

It follows that the lacunary series $b(z) = \sum_{k=1}^{\infty} z^{2^k}$ does not satisfy the condition $(*)$. Because if we assume that the lacunary series $b(z) = \sum_{k=1}^{\infty} z^{2^k}$ satisfies condition $(*)$ then it follows from the above argument that

$$\frac{p^3}{8 \log 2} = \mathcal{O}(p^4).$$

This yields to the contradiction if we divide the above equality by p^3 and let $p \longrightarrow 0$.

The second remark is concerned about the connection between the dynamics of quadratic polynomial $P(z) = z^2 + t$, (t stays in the principal cardioid $\mathcal{C}$) and the lacunary series $b(z) = \sum_{k=1}^{\infty} a_k z^{2^k}$. As discussed in Chapter 2, there is a conformal map ϕ_t from $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ onto the basin of infinity of the polynomial $P_t(z)$ with $\phi_0(z) = z$ and conjugating P_0 to P_t on their basin of infinity and ϕ_t has a quasiconformal extension to the whole plane $\overline{\mathbb{C}}$. The derivative of the holomorphic vector field $V(z) = \left.\dfrac{\partial \phi_t}{\partial t}\right|_{t=0}$ on $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is a lacunary series

$$V'(z) = \sum_{k=0}^{\infty} \left(1 - \frac{1}{2^k} \right) z^{-2^k}$$

on $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ in the sense that by changing variable $z \longrightarrow \dfrac{1}{z}$: $b(z) = V'(\dfrac{1}{z}) = \sum_{k=0}^{\infty} \left(1 - \dfrac{1}{2^k} \right) z^{2^k}$ being a lacunary series on the unit disk $\mathbb{D}$.

4.2 Bloch series generated by Rudin-Shapiro polynomials

As talked before, this section is reserved for an interesting Bloch series which is generated by Rudin-Shapiro polynomial

4.2.1 Rudin-Shapiro polynomials

Definition: We define the trigonometric polynomials P_m and Q_m inductively as follow:

$$\begin{aligned} P_0 &= Q_0 = 1 \\ P_{m+1}(t) &= P_m(t) + e^{i2^m t} Q_m(t) \\ Q_{m+1}(t) &= P_m(t) - e^{i2^m t} Q_m(t). \end{aligned}$$

After finding the construction of these polynomials in 1959 [Rud59], Walter Rudin discovered that it was also constructed before by H. S. Shapiro in 1951 in his PhD's thesis, see [Sha51]. Therefore we call them Rudin-Shapiro polynomials. These polynomials have the properties: for $m = 0, 1, 2, ...$

(a)

$$|P_{m+1}(t)|^2 + |Q_{m+1}(t)|^2 = 2(|P_m(t)|^2 + |Q_m(t)|^2),$$

(b)

$$\begin{aligned} |P_m(t)|^2 + |P_m(t-\pi)|^2 &= 2^{m+1}, \\ |Q_m(t)|^2 + |Q_m(t-\pi)|^2 &= 2^{m+1}. \end{aligned}$$

Consequences of (a) by induction are

$$|P_m(t)|^2 + |Q_m(t)|^2 = 2^{m+1}$$

and

$$\sup_{t\in[0,2\pi]} |P_m(t)|, \sup_{t\in[0,2\pi]} |Q_m(t)| \le 2^{\frac{m+1}{2}}.$$

We observe that (a) easily follows from the parallelogram law that

$$\begin{aligned} |P_{m+1}(t)|^2 + |Q_{m+1}(t)|^2 &= |P_m(t) + e^{i2^m t} Q_m(t)|^2 + |P_m(t) - e^{i2^m t} Q_m(t)|^2 \\ &= 2|P_m(t)|^2 + 2|Q_m(t)|^2. \end{aligned}$$

The property (b) obviously holds for $n = 0$ and then it follows from induction as follows

$$\begin{aligned} |P_{m+1}(t)|^2 &= |P_m(t)|^2 + |Q_m(t)|^2 + e^{i2^m t}\overline{P_m(t)} Q_m(t) + e^{i2^m t} P_m(t)\overline{Q_m(t)} \\ &= 2^{m+1} - |P_m(t-\pi)|^2 + 2^{m+1} - |Q_m(t-\pi)|^2 + 2\mathrm{Re}\{e^{i2^m t}\overline{P_m(t)} Q_m(t)\} \\ &= 2^{m+2} - |P_m(t-\pi)|^2 - |Q_m(t-\pi)|^2 - 2\mathrm{Re}\{e^{i2^m (t-\pi)}\overline{P_m(t-\pi)} Q_m(t-\pi)\} \\ &= 2^{m+2} - |P_{m+1}(t-\pi)|^2. \end{aligned}$$

The third equality above is given by a calculation that involves the equality $\cos(t) = -\cos(t-\pi)$, see [CH08].

Now we will construct a Bloch function from the Rudin-Shapiro polynomials. This construction is similar to the one of the function which belongs to Lipschitz space $Lips_{\frac{1}{2}}(\mathbb{T})$ but does not belong to the space of absolute convergence Fourier series in [Kat68].

4.2.2 Bloch function generated by Rudin-Shapiro polynomials.

Let

$$b_{RS}(z) = \sum_{m=0}^{\infty} a_m 2^{-\frac{(m+1)}{2}} z^{2^m} Q_m(z),$$

where Q_m is a Rudin-Shapiro polynomial. Then b_{RS} is a Bloch function. This follows from the next proposition.

Proposition 4.10. *For $(a_m) \in l^\infty$, the series*

$$b_{RS}(z) = \sum_{m=0}^{\infty} a_m 2^{-\frac{(m+1)}{2}} z^{2^m} Q_m(z)$$

belongs to Bloch space $\mathcal{B}$.

Proof: If we denote each term $a_m 2^{-\frac{(m+1)}{2}} z^{2^m} Q_m(z)$ of the series by $b_m(z)$, then the function $b_{RS}(z) = \sum_{m=1}^{\infty} b_m(z)$. In this proof, we'll use the Bernstein's inequality of polynomial whose proof is omitted here.

Bernstein's inequality: If $P(z)$ is a trigonometric polynomial of degree n, then it satisfies the inequality

$$\|P'\|_\infty \leq n\|P\|_\infty.$$

Let $z \in \mathbb{D}$, and $n \in \mathbb{N}$ such that $2^{-(n+1)} \leq 1 - |z| \leq 2^{-n}$. By the assumption, we have $|a_m| \leq M$. In order to estimate the derivative of function $b_{RS}(z)$, we separate it into two sums

$$b'_{RS}(z) = \sum_{m=0}^{n-1} b'_m(z) + \sum_{m=n}^{\infty} b'_m(z),$$

then we handle each one.

For the first one, Bernstein's inequality gives that $|b'_m(z)| \leq 2^{m+1}|b_m(z)|$ because $b_m(z)$ is a polynomial of degree 2^{m+1}. Moreover, a consequence of property (a) above gives us the estimation that for all m, $|Q_m(z)| \leq 2^{\frac{m+1}{2}}$, then $|b_m(z)| \leq M$. Therefore, the first sum

$$\sum_{m=0}^{n-1} |b'_m(z)| \leq M \sum_{m=0}^{n-1} 2^{m+1} \leq M(2^n - 1).$$

The estimation of the second one can be obtained from the observation that

$$b'_m(z) = a_m 2^{-\frac{(m+1)}{2}} (2^m z^{2^m - 1} Q_m(z) + Q'_m(z) z^{2^m}).$$

Since $|Q_m(z)| \leq 2^{\frac{(m+1)}{2}}$ and $|Q'_m(z)| \leq 2^m \|Q_m(z)\|_\infty \leq 2^m 2^{\frac{m+1}{2}}$, then

$$|b'_m(z)| = |a_m| 2^m (|z|^{2^m - 1} + |z|^{2^m}) \leq |a_m| 2^{m+1} |z|^{2^m - 1}.$$

Moreover, for $m \geq n$ where $1 - 2^{-n+1} \leq |z| \leq 1 - 2^{-n}$, we have

$$|b'_m(z)| \leq M 2^{m+1} (1 - 2^{-n})^{2^m - 1} \leq M 2^{m+1} e^{-2^{-n}(2^m - 1)}$$

Thus

$$\sum_{m=n}^{\infty}|b'_m(z)| \le M\sum_{m=n}^{\infty}2^{m+1}e^{-2^{m-n}+2^{-n}} = M2^{n+1}e^{2^{-n}}\sum_{m=0}^{\infty}2^m e^{-2^m} \le e2^{n+1}MC,$$

where C is a constant obtained from the following

$$\sum_{m=0}^{\infty}2^m e^{-2^m} \simeq \int_1^{\infty}e^{-2^x}2^x dx = \int_1^{\infty}e^{-u}du = C < +\infty,$$

(by changing variable $2^x = u$). It follows that

$$(1-|z|)|b'(z)| \le 2^{-n}(M(2^n-1)+eM2^{n+1}C) < +\infty.$$

In other word, b_{RS} belongs to Bloch space.

4.2.3 The spectrum of integral means of $\phi' = \exp b_{RS}(z)$.

With the same spirit of the preceding section, we will compute the spectrum of integral means of $\phi'(z) = \exp(b_{RS}(z))$, where $b_{RS} = \sum_{m=0}^{\infty}c_m 2^{\frac{-(m+1)}{2}}z^{2^m}Q_m(z)$. Let $b_m(z) = c_m 2^{\frac{-(m+1)}{2}}z^{2^m}Q_m(z)$. It follows from the proof of Proposition 4.10 that

$$(1-|z|)\left|\sum_{m=0}^{\infty}(b_m^2(z))'\right| = (1-|z|)\sum_{m=0}^{\infty}2|b'_m(z)||b_m(z))| \le (1-|z|)\sum_{m=0}^{\infty}2M|b'_m(z)| \le +\infty.$$

It means that $\sum_{m=0}^{\infty}b_m^2(z)$ belongs to Bloch space. Similarly, the series $\sum_{m=0}^{\infty}b_m^3(z)$ also belongs to Bloch space. By the inequality that for x small $\left|\log(1+x)-x+\dfrac{x^2}{2}\right| \le |x|^3$, we can write the function $pb_{RS}(z)$ as

$$pb_{RS}(z) = \sum_{m=0}^{\infty}pb_m(z) = \sum_{m=0}^{\infty}2\log\left(1+\frac{p}{2}b_m(z)\right)+\frac{p^2}{4}\sum_{m=0}^{\infty}b_m^2(z)+R_p(z),$$

where $|R_p(z)| \le C\sum_{m=0}^{\infty}|pb_m(z)|^3$. Now if we apply Lemma 4.4 to the function $\phi'(z)$ then we obtain that

$$\int_0^{2\pi}\left|\phi'(z)\right|^p d\theta = \left(\frac{1}{1-r}\right)^{\mathcal{O}(p^{\frac{5}{2}})}\int_0^{2\pi}\prod_{m=0}^{\infty}\left|1+\frac{p}{2}\frac{c_m}{2^{\frac{m+1}{2}}}z^{2^m}Q_m(z)\right|^2 d\theta.$$

We observe that

$$\begin{aligned}
\left| 1 + \frac{p}{2} \frac{c_m}{2^{\frac{m+1}{2}}} z^{2^m} Q_m(z) \right|^2 &= 1 + \frac{p^2}{4} \frac{|c_m|^2}{2^{m+1}} |z|^{2^m} |Q_m(z)|^2 + p \frac{|c_m|}{2^{\frac{m+1}{2}}} \sum_{k=2^m+1}^{2^{m+1}} \epsilon_k \cos(k\theta + \theta_m) \\
&\leq 1 + \frac{p^2}{4} \frac{|c_m|^2}{2^{m+1}} |z|^{2^m} (|Q_m(\theta)|^2 + |Q_m(\pi - \theta)|^2) \\
&+ p \frac{|c_m|}{2^{\frac{m+1}{2}}} \sum_{k=2^m+1}^{2^{m+1}} \epsilon_k \cos(k\theta + \theta_m) \\
&= 1 + \frac{p^2}{4} |c_m|^2 |z|^{2^m} + p \frac{|c_m|}{2^{\frac{m+1}{2}}} \sum_{k=2^m+1}^{2^{m+1}} \epsilon_k \cos(k\theta + \theta_m),
\end{aligned}$$

where ϵ_k is the coefficient of polynomial $Q_m(z)$ and θ_m is the argument of c_m. The third equality above is obtained by using the property that $|Q_m(\theta)|^2 + |Q_m(\theta - \pi)|^2 = 2^{m+1}$. We note that the sum $\sum_{j=0}^n \pm k_j \leq \sum_{j=0}^n 2^j = 2^{n+1} - 1 < 2^{n+1}$, thus the integral of all the products $\prod_{j=0}^n \cos(k_j \theta)$ vanishes on $[0, 2\pi]$. So by the similar argument as lacunary series, if we multiply out and integrate term-by-term, then we obtain the upper bound for the integral

$$\int_0^{2\pi} \prod_{m=0}^{\infty} \left| 1 + \frac{p}{2} \frac{c_m}{2^{\frac{m+1}{2}}} z^{2^m} Q_m(z) \right|^2 d\theta \leq 2\pi \prod_{m=0}^{\infty} \left(1 + \frac{p^2}{4} |c_m| r^{2^{m+1}} \right),$$

where

$$\prod_{m=0}^{\infty} \left(1 + \frac{p^2}{4} |c_m| r^{2^{m+1}} \right) = \left(\frac{1}{1-r} \right)^{\mathcal{O}(p^3)} \exp \left\{ \frac{p^2}{4} \sum_{k=1}^{\infty} |c_m| r^{2^{m+1}} \right\}.$$

Theorem 4.11. *Let $(c_m) \in l_\infty$. Then the spectrum of integral means of $\phi' = exp(b_{RS})$ satisfies the following inequalities*

$$\beta(p, \phi') \leq \frac{p^2}{2} \limsup_{r \longrightarrow 1} \frac{\int_0^{2\pi} |b_{RS}(re^{i\theta})|^2 d\theta}{2\pi \log \frac{1}{1-r}} + \mathcal{O}(p^{\frac{5}{2}}),$$

where $b_{RS}(z) = \sum_{m=0}^{\infty} \frac{c_m}{2^{\frac{m+1}{2}}} z^{2^m} Q_m(z)$.

Proof: By the above argument in order to conclude the proof, we need to show that

$$\int_0^{2\pi} |b_{RS}(re^{i\theta})|^2 d\theta = \int_0^{2\pi} \sum_{k=1}^{\infty} \frac{|c_m|^2}{2^{m+1}} r^{2^{m+1}} |Q_m|^2 d\theta = \pi \sum_{k=1}^{\infty} |c_m| r^{2^{m+1}}.$$

If we change variable θ by $(\theta - \pi)$ of the integral $\int_\pi^{2\pi} \sum_{k=1}^{\infty} \frac{|c_m|^2}{2^{m+1}} r^{2^{m+1}} |Q_m(\theta)|^2$ then it turns out to be

$$\int_0^{\pi} \sum_{k=1}^{\infty} \frac{|c_m|^2}{2^{m+1}} r^{2^{m+1}} |Q_m(\theta - \pi)|^2 d\theta.$$

Using the property that $|Q_m(\theta)|^2 + |Q_m(\theta - \pi)|^2 = 2^{m+1}$ of Rudin-Shapiro polynomials, we deduce that

$$\int_0^{2\pi} |b_{RS}(re^{i\theta})|^2 d\theta = \int_0^{\pi} \sum_{k=1}^{\infty} \frac{|c_m|^2}{2^{m+1}} r^{2^{m+1}} \left(|Q_m(\theta)|^2 + |Q_m(\theta-\pi)|^2 \right) d\theta = \pi \sum_{k=1}^{\infty} |c_m| r^{2^{m+1}}.$$

It finishes the proof.

Recall Γ_t be the image of the circle by the conformal map $\phi_t = \int_0^z e^{tb_{RS}(u)} du$. Then the Minkowski dimension of Γ_t satisfies the following inequality.

Corollary 4.12.

$$M.dim(\Gamma_t) \leq 1 + \frac{t^2}{2} \limsup_{r \longrightarrow 1} \frac{\int_0^{2\pi} |b_{RS}(re^{i\theta})|^2 d\theta}{2\pi \log \frac{1}{1-r}} + o(t^2).$$

Proof: It follows from the analogous argument as the proof of Proposition 3.10.

Chapter 5

Counterexample

The aim of this chapter is to construct a counterexample to the formula (1.6) (mentioned in Chapter 1). This construction is reminiscent of Kahane's construction of a non-Smirnov domain. The first section of this chapter will be devoted to Kahane measure and its Herglotz transform. Then, based on this construction, in the next section we build a family of conformal maps (ϕ_t) (t being real and small) for which the formula (1.6) still holds on the left ($t < 0$) and turns out to be a counterexample to the formula (1.6) on the right ($t > 0$). These facts will be proved in the two following sections. More precisely, in the case of t negative, by applying the theory of Hardy space H^1, we'll prove that $\Gamma_t = \phi_t(\partial\mathbb{D})$ is a rectifiable curve, therefore the $\mathrm{H.dim}(\Gamma_t) = \mathrm{M.dim}(\Gamma_t) = 1$. In the case of t positive by making use of the random walk argument, we'll prove that there exists a constant $c > 0$ such that $\mathrm{M.dim}(\Gamma_t) \geq 1 + ct^2$ from which we obtain the contradiction to the formula (1.6).

5.1 Kahane measure and its Herglotz transform.

First of all, let us recall the construction of Kahane measure.

5.1.1 Kahane measure.

Denote by ω_0 the interval $[0, 1]$ and by ω_j one of intervals of form 4-adic $[p4^{-j}, (p+1)4^{-j}]$ contained in ω_0. We construct simultaneously a sequence of measure μ_j and their supports E_j as follow:

μ_0 is the Lebesgue measure on interval ω_0;

μ_j is proportional to the Lebesgue measure on each ω_j.

We denote by $D_j(\omega_j)$ its density on a given interval ω_j and its support E_j is the union of intervals ω_j where $D_j(\omega_j) \neq 0$. In order to obtain μ_{j+1} from μ_j, we divide each interval $\omega = \omega_j$ of rank j contained in E_j into four equal subintervals $\omega^1, \omega^2, \omega^3, \omega^4$ of rank $j+1$ and put

$$D_{j+1}(\omega^1) = D_{j+1}(\omega^4) = D_j(\omega) - 1,$$
$$D_{j+1}(\omega^2) = D_{j+1}(\omega^3) = D_j(\omega) + 1.$$

Put $\mu = \lim_{j \longrightarrow \infty} \mu_j$ and $E = \bigcap_{j=0}^{\infty} E_j$. We call this measure μ Kahane's measure.

There is another way to define the set E. Recall the independent Bernoullian random variables ε_k on $\partial\mathbb{D}$ (defined in 3.2.4): put $\Sigma_j(e^{2\pi ix}) = \sum_{k=1}^{j} \varepsilon_k(e^{2\pi ix})$ and let N be the first number such that $1 + \sum_{k=1}^{j} \varepsilon_k(e^{2\pi ix}) = 0$ $(x \in [0,1])$. By the definition of D_k, we have :

$$\forall x \in [0,1], \;\; D_0(x) = 1; \;\; D_k(x) = (D_{k-1}(x) + \varepsilon_k(e^{2\pi ix}))1_{E_{k-1}}(x).$$

Therefore for $x \in [0,1]$

$$D_k(x) = \left(\left(\left(\left(1 + \varepsilon_1(e^{2\pi ix}) \right) 1_{E_0}(x) + \varepsilon_2(e^{2\pi ix}) \right) 1_{E_1}(x) + ... + \right) 1_{E_{k-2}}(x) + \varepsilon_k(e^{2\pi ix}) \right) 1_{E_{k-1}}(x).$$

Since $E_0 \supset E_1 \supset ... \supset E_{k-1}$ then $1_{E_0} ... 1_{E_{k-1}} = 1_{E_{k-1}}$, therefore

$$D_k(x) = (1 + \Sigma_k(e^{2\pi ix}))1_{E_{k-1}}(x).$$

This implies that the support of D_k: $E_k = E_{k-1} \cap \{1 + \Sigma_k > 0\}$. Then,

$$E_k = \{1 + \Sigma_1 > 0, ..., 1 + \Sigma_k > 0\}, \; (k = 1, 2, ...).$$

Moreover, for $x \in [0,1]$,

$$\begin{aligned} D_k(x) &= (1 + \Sigma_k(e^{2\pi ix}))1_{E_{k-1}}(x) \\ &= (1 + \Sigma_k(e^{2\pi ix}))1_{E_k}(x) + (1 + \Sigma_k(e^{2\pi ix}))1_{E_{k-1} \setminus E_k}(x) \\ &= (1 + \Sigma_k(e^{2\pi ix}))1_{E_k}(x). \end{aligned}$$

Because on the set $E_{k-1} \setminus E_k$ we have $1 + \Sigma_k(x) = 0$.

In his paper [Kah69], Kahane showed that the set $E = \bigcap_{k=0}^{\infty} E_k$ (support of the measure μ) has a null Lebesgue measure. Therefore this measure is totally singular.

5.1.2 Herglotz transform of Kahane measure

Let $b(z)$ be *Herglotz transform* of Kahane measure μ: that is

$$b(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\frac{\theta}{2\pi}).$$

Kahane proved that $b \in \mathcal{B}$. Put $\Lambda_j(e^{2\pi ix}) = 1 + \Sigma_j(e^{2\pi ix})$ and

$$S_j(e^{2\pi ix}) = \Lambda_{j \wedge N}(e^{2\pi ix}) = \begin{cases} 1 + \Sigma_j(e^{2\pi ix}), & \text{if } x \in \{N > j\} = E_j \\ 0, & \text{otherwise} \end{cases}, \;\; (x \in [0,1]).$$

Similarly to the example of the square constant function in 3.2.4, Λ_j is a dyadic martingale (if considered as dyadic). By the construction of μ, $\{N = j\} = \bigcup \omega_j = E_{j-1} \setminus E_j \in \mathcal{F}_j$, where ω_j is an interval 4-adic of rank $j-1$ i.e dyadic of rank j. Therefore N is a stopping time with respect to the σ-algebra $\{\mathcal{F}_j, j \geq 0\}$ (defined in 3.1.2). Thus, $S_j(e^{2\pi ix}) = \Lambda_{N\wedge j}(e^{2\pi ix}) = D_j(x), (x \in [0,1])$ is a dyadic martingale as well. Moreover, we have the following lemma.

Lemma 5.1. *S_j is the dyadic martingale of the Bloch function $Re(b)$.*

Proof: Indeed, let $h(\theta)$ be the cumulative distribution function of Kahane measure μ, (i.e. $h(\varphi) = \mu(\{\frac{\varphi}{2\pi} > 0\})$ $(\varphi \in [0, 2\pi])$ and $h(0) = 0$. We observe that for $z \in \mathbb{D}$

$$b(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} h'(\varphi)d\varphi = \frac{1}{2\pi}\int_0^{2\pi}\left(1 + 2\sum_{n=1}^{\infty} e^{-in\varphi}z^n\right) h'(\varphi)d\varphi. \tag{5.1}$$

By Schwartz integral formula and $\text{Im}b(0) = 0$, we have

$$b(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \text{Re}b(e^{i\varphi})d\varphi = \frac{1}{2\pi}\int_0^{2\pi}\left(1 + 2\sum_{n=1}^{\infty} e^{-in\varphi}z^n\right) \text{Re}b(e^{i\varphi})d\varphi. \tag{5.2}$$

From (5.1),(5.2) we obtain $\displaystyle\int_0^{2\pi} e^{-in\varphi}(\text{Re}b(e^{i\varphi}) - h'(\varphi))d\varphi = 0$ $(n = 0, 1, 2, ...)$. Since the sequence $\{e^{in\theta}\}(n = 0, 1, 2, ...)$ is a basis of $L^2([0, 2\pi])$, then

$$\text{Re}b(e^{i\varphi}) - h'(\varphi) = 0 \quad \text{ in } L^2([0, 2\pi]).$$

Thus, $\text{Re}b(e^{i\varphi}) = h'(\varphi)$ a.e in $[0, 2\pi]$. We observe that for each subarc 4-adic $\omega_j = [\frac{\varphi_0}{2\pi}, \frac{\varphi_0}{2\pi} + \frac{\varphi}{2\pi}]$ of rank j of the interval $[0, 1]$, since D_j is a dyadic martingle then

$$\begin{aligned}\frac{\mu(\omega_j)}{|\omega_j|} &= \frac{1}{|\omega_j|}\int_{\omega_j} D_j(x)dx \\ &= \frac{1}{|\omega_j|}\int_{\omega_j} S_j(e^{2\pi ix})dx \\ &= S_j(e^{2\pi ix})|_{\omega_j},\end{aligned}$$

while $\dfrac{\mu(\omega_j)}{|\omega_j|} = \dfrac{h(\varphi+\varphi_0) - h(\varphi_0)}{|\varphi|}$. Therefore,

$$S_j(e^{2\pi ix})|_{\omega_j} = \frac{h(\varphi+\varphi_0) - h(\varphi_0)}{|\varphi|} = \frac{1}{|\omega_j|}\int_{\omega_j} \text{Re}b(e^{2\pi ix})dx = (\text{Re}b)_{\omega_j}.$$

It means that S_j is the dyadic martingale of the Bloch function Reb. Now, let us state concretely the second principal result of this thesis.

5.2 Statement of Theorem 5.2.

Let μ be Kahane's measure and $b(z)$ its Herglotz transform. We recall that Γ_t is the image of the unit circle $\mathbb{T}$ by the conformal map $\phi_t(z)$ which is defined as $\phi_t'(z) = e^{tb(z)}$, t small enough. We recall again (1.6) that if a family of conformal maps $\phi_t(z) = \int_0^z e^{tb(u)}du$, $(z \in \mathbb{D}; b \in \mathcal{B})$ satisfies (1.4) with Hausdorff dimension replaced by Minkowski dimension, then

$$\text{M.dim}(\Gamma_t) = 1 + \limsup_{r\longrightarrow 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta}{4\pi \log\frac{1}{1-r}} \frac{t^2}{2} + o(t^2).$$

Theorem 5.2. *The behaviour of the curve Γ_t differs with the sign of t:*
In the case of negative t, the singular property of the Kahane's measure μ (the density function of the probability measure μ is non negative and zero almost everywhere) makes $\phi_t' \in H^1$. This is equivalent to the rectifiability of Γ_t and then H.dim(Γ_t) = M.dim$(\Gamma_t) \equiv 1$.
On the other hand, in the case of positive t, Γ_t is a fractal curve and its Minkowski dimension satisfies the following inequality:

$$d(t) \geq 1 + \frac{t^2}{8\log 2}, \quad \forall t > 0 \text{ small enough,}$$

as a consequence the family of conformal map (ϕ_t) gives a counterexample to (1.6).

Proof: First of all, we use the singularity of Kahane measure to show that in the case of small negative t, H.dim(Γ_t) = M.dim$(\Gamma_t) \equiv 1$.

5.3 Negative t.

We recall now the two theorems on $H^p(p > 0)$ functions and then we'll show how they imply the first part of Theorem 5.2. Let us introduce some notions. First of all, we give the definition of *Hardy space* $H^p(p > 0)$.

5.3.1 Hardy space

Definition: Let f be analytic in the unit disk $\mathbb{D}$ and f is said to be of class $H^p(p > 0)$ if the *integral means*

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}$$

remains bounded as $r \longrightarrow 1$. Thus, H^∞ is the class of bounded analytic functions in the unit disk. The class $H^p(p > 0)$ forms a vector space. We call $H^p(p > 0)$ by Hardy space. Given a function $f(z) \not\equiv 0$ of class $H^p(p > 0)$. Let (a_n) (may be finite, or even empty) be the sequence zeroes of the function f. Then the associated *Blaschke product* of f is

$$B(z) = z^m \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}.$$

And the associated *outer function* of class H^p of f is defined as

$$F(z) = e^{i\gamma} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |f(e^{i\theta})| d\theta \right\},$$

where γ is a real number. Let $S(z) = f(z)/(F(z)B(z))$. Then $S(z)$ is the *singular inner function* of f with the form

$$S(z) = \exp\left\{ -\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(t) \right\},$$

where $\mu(t)$ be a bounded non-decreasing singular function ($\mu'(t)$ is non-negative and equal to zero a.e), see [Dur70]. Furthermore, we have the following theorem.

Theorem 5.3. *(Canonical factorization theorem). Every function $f(z) \not\equiv 0$ of class $H^p (p > 0)$ has a unique factorization of the form $f(z) = B(z)S(z)F(z)$ where $B(z)$ is a Blaschke product, $S(z)$ is a singular inner function and $F(z)$ is an outer function of class H^p. Conversely, every such product $B(z)S(z)F(z)$ belongs to H^p.*

Proof: See [Dur70].

5.3.2 Rectifiable curve

A continuous complex-valued function $w = w(t)(0 \le t \le 2\pi)$ such that $w(0) = w(2\pi)$ and $w(t_1) \neq w(t_2)$ for $0 \le t_1 < t_2 < 2\pi$ is said of *bounded variation* on $[0, 2\pi]$ if the total variation

$$V(w) = \sup \left\{ \sum_{k=1}^{n} |w(t_k) - w(t_{k-1})| \right\} < +\infty,$$

where the supremum is taken over all the finite partitions $0 = t_0 < t_1 < ... < t_n = 2\pi$ of $[0, 2\pi]$.

A Jordan curve Γ is the image of a continuous complex-valued function $w = w(t)(0 \le t \le 2\pi)$ such that $w(0) = w(2\pi)$ and $w(t_1) \neq w(t_2)$ for $0 \le t_1 < t_2 < 2\pi$. The curve Γ is said to be *rectifiable* if $w(t)$ is of *bounded variation*.

A complex-valued function w on $[0, 2\pi]$ is said to be *absolutely continuous* on $[0, 2\pi]$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for every finite collection $\{(x_i, x_i')\}$ of non-overlapping sub-intervals of $[0, 2\pi]$ with

$$\sum_{i=1}^{m} |x_i' - x_i| < \delta$$

then

$$\sum_{i=1}^{m} |w(x_i') - w(x_i)| < \epsilon.$$

Assume that $w(t)$ is absolutely continuous on $[0, 2\pi]$. Let $\epsilon = 1$ and then there exists $\delta > 0$. Let K be the biggest integer less than $1 + 2\pi/\delta$. Then each subdivision $[t_k, t_{k+1})$ of a partition of $[0, 2\pi]$ can be split into K sets of intervals with total length less than δ. By the definition of absolutely continuous, the total variation $V(w)$ is less than K. Therefore $w(t)$ is of bounded variation. See [Roy87]. Furthermore, let f is conformal mapping the unit disk $\mathbb{D}$ onto a Jordan domain Ω. Then we have two following theorems

Theorem 5.4. *If $f \in H^1$ and its boundary function $f(e^{i\theta})$ is equal almost everywhere to a function of bounded variation, then $f(z)$ is continuous in the closed unit disk $\overline{\mathbb{D}}$ and $f(e^{i\theta})$ is absolutely continuous.*

Proof: See [Dur70].

Theorem 5.5. *A function $f(z)$ analytic in the unit disk $\mathbb{D}$ and continuous upto boundary is absolutely continuous on the circle $|z| = 1$ if and only if $f' \in H^1$.*

Proof: See [Dur70].

Theorem 5.4 and Theorem 5.5 imply the following theorem.

Theorem 5.6. *Let $f(z)$ maps the unit disk $\mathbb{D}$ conformally onto a Jordan domain Ω. Then the boundary $\partial\Omega$ is rectifiable if and only if $f' \in H^1$.*

5.3.3 The first part of the proof

Since t small enough and $b(z)$ is a Bloch function, then by Becker univalence criterion $\phi_t(z)$ maps conformally the unit disk $\mathbb{D}$ onto a quasidisk Ω_t. And its derivative has the form

$$\phi_t'(z) = \exp\left\{ t\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta) \right\},$$

where $t < 0$ and μ is a positive singular measure i.e the density function $h'(\theta)$ of Kahane measure μ is non negative and zero almost everywhere on $[0, 2\pi]$ (mentioned above). Then, Theorem 5.3 yields $\phi_t' \in H^1$.

Since $\phi_t' \in H^1$ is equivalent to the rectifiability of the boundary Γ_t by Theorem 5.6, then obviously by the definition of Hausdorff dimension

$$\text{H.dim}(\Gamma_t) = \text{M.dim}(\Gamma_t) \equiv 1. \tag{5.3}$$

The first part of Theorem 5.2 follows.

Now, we go to the second part of the proof of Theorem 5.2: the case of small positive t.

5.4 Positive t.

We want to show that $d(t) \geq 1 + \dfrac{t^2}{8\log 2}$, $t > 0$ small. Analogously to part 2, in order to prove this, we need to show that the spectrum of integral means $\beta(p, \phi')$ where $\phi' = \exp b(z)$ satisfies the following inequality

$$\beta(p, \phi') \geq \frac{p^2}{8\log 2}, \quad p > 0 \text{ small}. \tag{5.4}$$

First, from the fact that $|S_j(e^{i\theta}) - \text{Re}(b(re^{i\theta}))| \leq C\|b\|_{\mathcal{B}}$, $(r = 1 - 2^{-j})$ (see (3.17)), we deduce that

$$\beta(p, \phi') = \limsup_{r \longrightarrow 1} \frac{\log\left(\int_0^{2\pi} e^{p\text{Re}b(re^{i\theta})} d\theta \right)}{\log \frac{1}{1-r}} = \limsup_{j \longrightarrow \infty} \frac{\log\left(\int_0^{2\pi} e^{pS_j(e^{i\theta})} d\theta \right)}{j\log 2}.$$

This leads us to estimate the integral $\displaystyle\int_{\mathbb{T}} e^{pS_j(e^{i\theta})} d\theta$, $(S_j = \Lambda_{j\wedge N})$, $p > 0$ small. The difficult point is that S_j is not a sum of independent random variables. However, we'll go around this difficulty by using the stopping time of the random walk argument of the dyadic martingale S_j which will be introduced in the following.

5.4.1 Random walk argument.

Let us describe this random walk on graph. On the lattice $\mathbb{Z}^+ \times \mathbb{Z}$, we consider that a particle moves in the direction parallel to two diagonals of the unit square. We denote the individual steps generically by $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ with the probability $p = \dfrac{1}{2}$ (defined in 3.2.4) and the position of the particle by $\Sigma_1, \Sigma_2, ..., \Sigma_n$.

Theorem 5.7. *(The ballot theorem) Let n and r be positive integers. There are exactly* $\frac{r}{n}\binom{n}{\frac{n+r}{2}}$ *paths* $(\Sigma_1, ..., \Sigma_n = r)$ *from the orgin to the point* (n, r) *such that* $\Sigma_1 > 0, ..., \Sigma_n > 0$, *where*

$$\binom{n}{\frac{n+r}{2}} = \begin{cases} \dfrac{n!}{(\frac{n+r}{2})!(n-\frac{n+r}{2})!} & \textit{if } \left(\dfrac{n+r}{2}\right) \textit{ is a positive integer less than } n \\ 0 \quad \textit{otherwise.} \end{cases}$$

Proof: See [Fel67].

Denote the event at epoch n the particle is at the point r by $\{\Sigma_n = r\}$. We observe that there are 2^n paths with the length n, thus the probability for the particle moves from the origin to the position (n, r) on the graph is $P(\{\Sigma_n = r\}) = \binom{n}{\frac{n+r}{2}} 2^{-n}$.

Denote $p_{n,r} = P(\{\Sigma_n = r\})$. We have the following lemma.

Lemma 5.8. *The probability that no return to the origin occurs up to and including epoch $2n$ is the same as the probability that a return occurs at epoch $2n$*

$$P(\{\Sigma_1 \neq 0, ..., \Sigma_{2n} \neq 0\}) = P(\{\Sigma_{2n} = 0\}) = \frac{C_{2n}^n}{2^{2n}}.$$

Proof: Remark that the event on the left occurs either all the Σ_j are positive, or all are negative. The two events are equality probable, then we just need to show that

$$P(\{\Sigma_1 > 0, ..., \Sigma_{2n} > 0\}) = \frac{1}{2}\frac{C_{2n}^n}{2^{2n}}.$$

It is clear that

$$P(\{\Sigma_1 > 0, ..., \Sigma_{2n} > 0\}) = \sum_{r=1}^{n} P(\{\Sigma_1 > 0, ..., \Sigma_{2n-1} > 0, \Sigma_{2n} = 2r\}) \tag{5.5}$$

By ballot theorem the number of paths satisfying the condition indicated on the right side equals to $\binom{2n-1}{n-r-1} - \binom{2n-1}{n-r}$, and so the rth terms of the sum equals to $\frac{1}{2}(p_{2n-1,2r-1} - p_{2n-1,2r+1})$. The negative part of the rth term cancels against the positive part of the $(r+1)$st term with the result that the sum in (5.5) reduces to $\frac{1}{2}p_{2n-1,1}$. It is easily verified that $p_{2n-1,1} = \dfrac{C_{2n}^n}{2^{2n}}$ and this concludes the proof.

For a path of length $2n$ with all verticals strictly above the horizontal axis $y = 0$ passes through the point $(1, 1)$. Taking this point as new origin we obtain a path of length $2n - 1$ with all verticals above or on the new axis $y = 1$. It follows that

$$P(\{\Sigma_1 > 0, ..., \Sigma_{2n} > 0\}) = \frac{1}{2}P(\{\Sigma_1 \geq 0, ..., \Sigma_{2n-1} \geq 0\}).$$

But Σ_{2n-1} is an odd number, and hence $\Sigma_{2n-1} \geq 0$ implies that also $\Sigma_{2n} \geq 0$. This concludes the following lemma.

Lemma 5.9. *For a random walk $\Sigma_n = \varepsilon_1+\varepsilon_2+...+\varepsilon_n$, where ε_k are Bernoulli independent random variable with the probability $p=\frac{1}{2}$, we have:*

$$P(\Sigma_1 \geq 0, \Sigma_2 \geq 0, ..., \Sigma_{2n} \geq 0) = P(\Sigma_{2n} = 0) = \frac{C_{2n}^n}{2^{2n}}.$$

Moreover by Stirling's formula $P(N > 2n) \simeq \dfrac{1}{\sqrt{2n}}$.

According to the assumption of this dyadic martingale, the particle will stop as it reaches to the horizontal axis $y=0$ on the lattice. We can write the event $\{N > k\}$ by $\{1+\Sigma_1 > 0, ..., 1+\Sigma_k > 0\}$ and then by $\{\Sigma_1 \geq 0, ..., \Sigma_k \geq 0\}$. We note that for k, $\{N > 2k+1\} = \{N > 2k\}$. Indeed, $\{\Sigma_1 \geq 0, ..., \Sigma_{2k} \geq 0\} = \{\Sigma_1 \geq 0, ..., \Sigma_{2k} \geq 0, \Sigma_{2k+1} \geq 0\} \cap \{\Sigma_1 \geq 0, ..., \Sigma_{2k} \geq 0, \Sigma_{2k+1} < 0\}$. By the assumption of the stopping time, the particle will stop as it reaches to the axis $y=0$, hence the particle cannot move to the position $\{\Sigma_1 \geq 0, ..., \Sigma_{2k} \geq 0, \Sigma_{2k+1} < 0\}$. Thus, $\{N > 2k+1\} = \{N > 2k\}$.
Now we proceed to the main step of the proof of Theorem 5.2.

5.4.2 The main step of the proof.

We estimate the integral $\displaystyle\int_{\mathbb{T}} e^{pS_j(e^{i\theta})} d\theta$. First we note that on the set $\{N \leq j\}$ $S_j(e^{i\theta}) = \Lambda_{j\wedge N}(e^{i\theta}) = 0$, then

$$\begin{aligned} \frac{1}{2\pi}\int_{\mathbb{T}} e^{pS_j(e^{i\theta})} d\theta &= \frac{1}{2\pi}\int_{\{N>j\}} e^{pS_j(e^{i\theta})} d\theta + \frac{1}{2\pi}\int_{\{N\leq j\}} e^{pS_j(e^{i\theta})} d\theta \\ &= \frac{1}{2\pi}\int_{\{N>j\}} e^{pS_j(e^{i\theta})} d\theta + P(\{N \leq j\}), \end{aligned} \tag{5.6}$$

where $\displaystyle\frac{1}{2\pi}\int_{\{N>j\}} e^{pS_j(e^{i\theta})} d\theta = \frac{1}{2\pi}\int_{\{N>j\}} e^{p(1+\Sigma_j(e^{i\theta}))} d\theta$. We observe that

$$\frac{1}{2\pi}\int_{\{N>j\}} e^{p(1+\Sigma_j(e^{i\theta}))} d\theta = \frac{1}{2\pi}\int_{\mathbb{T}} e^{p(1+\Sigma_j(e^{i\theta}))} d\theta - \frac{1}{2\pi}\int_{\{N\leq j\}} e^{p(1+\Sigma_j(e^{i\theta}))} d\theta.$$

Since $\Sigma_j = \sum_{k=1}^{j} \varepsilon_k$, where ε_k with $k=1,2,...$ are the independent random variables, then the integral

$$\frac{1}{2\pi}\int_{\mathbb{T}} e^{p(1+\Sigma_j(e^{i\theta}))} d\theta = e^p \prod_{k=1}^{j} \mathbf{E}(e^{p\varepsilon_k}) = e^p \prod_{k=1}^{j} \cosh p = e^p (\cosh p)^j.$$

Besides, the integral $\int_{\{N\leq j\}} e^{p(1+\Sigma_j(e^{i\theta}))} d\theta$ can be rewritten as:

$$\int_{\{N\leq j\}} e^{p(1+\Sigma_j(e^{i\theta}))} d\theta = \sum_{k=1}^{j} \int_{\{N=k\}} e^{p(1+\Sigma_j(e^{i\theta}))} d\theta.$$

The fact that $1+\Sigma_k(e^{i\theta})$ is equal to zero on each set $\{N=k\}$ makes the value of the integral $\sum_{k=1}^{j} \int_{\{N=k\}} e^{p(1+\Sigma_j(e^{i\theta}))} d\theta$ unchanged if we divide the integrand $e^{p(1+\Sigma_j(e^{i\theta}))}$ by the term $e^{1+\Sigma_k(e^{i\theta})}$. Thus we have:

$$\begin{aligned}\sum_{k=1}^{j}\int_{\{N=k\}} e^{p(1+\Sigma_j(e^{i\theta}))}d\theta &= \sum_{k=1}^{j}\int_{\{N=k\}} e^{p(1+\Sigma_j(e^{i\theta})-1-\Sigma_k(e^{i\theta}))}d\theta \\ &= \sum_{k=1}^{j}\int_{\{N=k\}} e^{p(\Sigma_j(e^{i\theta})-\Sigma_k(e^{i\theta}))}d\theta.\end{aligned}$$

In addition, if we rewrite the integral $\int_{\{N=k\}} e^{p(\Sigma_j(e^{i\theta})-\Sigma_k(e^{i\theta}))}d\theta$ as $\int_{\mathbb{T}} 1_{\{N=k\}}e^{p(\Sigma_j(e^{i\theta})-\Sigma_k(e^{i\theta}))}d\theta$, then by the independence of two random variables $1_{\{N=k\}}$ and $e^{p(\Sigma_j-\Sigma_k)}$ it follows that

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{T}} 1_{\{N=k\}}e^{p(\Sigma_j(e^{i\theta})-\Sigma_k(e^{i\theta}))}d\theta &= P(\{N=k\})\mathbf{E}(e^{p(\Sigma_j(e^{i\theta})-\Sigma_k(e^{i\theta}))}) \\ &= P(\{N=k\})(\cosh p)^{j-k}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{\{N>j\}} e^{pS_j(e^{i\theta})}d\theta &= e^p\cosh(p)^j\left(1-\sum_{k=1}^{j}\frac{P(\{N=k\})}{(\cosh p)^k e^p}\right) \\ &\geq e^p(\cosh p)^j\left(1-\sum_{k=1}^{j}P(\{N=k\})\right), \quad (p>0) \\ &= e^p(\cosh p)^j P(\{N>j\}), \quad (p>0). \end{aligned} \tag{5.7}$$

The inequality above follows from the fact that for $p>0$ $(\cosh p)^k e^p \geq 1$, $k=1,2,..,j$.

From (5.6), (5.7) and Jensen's inequality, we deduce that

$$\begin{aligned}\log\left(\int_{\mathbb{T}} e^{pS_j(e^{i\theta})}d\theta\right) &\geq \frac{1}{2}\log\left(2\int_{\{N>j\}} e^{pS_j(e^{i\theta})}d\theta\right)+\frac{1}{2}\log\left(4\pi P(\{N\leq j\})\right) \\ &\geq \frac{p}{2}+\log(4\pi)+\frac{1}{2}\log(\cosh(p)^j)+\frac{1}{2}\log(P(\{N>j\}))+\frac{1}{2}\log(P(\{N\leq j\})).\end{aligned}$$

By Lemma 5.9: $\log(P(N>j)) \simeq -\dfrac{\log j}{2}$ and $\log(P(N\leq j)) \simeq -\dfrac{1}{\sqrt{j}}$ as $j\longrightarrow\infty$, thus when we divide the above inequality by $j\log 2$ and take the lim sup as $j\longrightarrow\infty$, we obtain

$$\beta(p,\phi') \geq \limsup_{j\longrightarrow\infty}\frac{\log(\cosh(p)^j)}{2j\log 2} = \frac{\log\cosh(p)}{2\log 2}, \quad p>0.$$

Moreover, the inequality $\log\cosh(x) \geq \dfrac{x^2}{2}-\dfrac{x^4}{12}$, $(x>0)$ (proved in 3.2.3) implies that $\log\cosh(x) \geq \dfrac{x^2}{4}$ for $x>0$ small enough, which implies (5.4): $\beta(p,\phi') \geq \dfrac{p^2}{8\log 2}$, $p>0$ small. As a consequence of (5.4), the spectrum of integral means $\beta(d(t),\phi'_t)$ of the family of the conformal maps $\phi'_t(z)=\exp tb(z)$ satisfies the following inequality:

$$\beta(d(t),\phi'_t) = \beta(td(t),\phi') \geq \frac{t^2d(t)^2}{8\log 2}, \quad t>0 \text{ small},$$

where $d(t) = \mathrm{M.dim}(\Gamma_t) \geq 1$.

Finally, by Corollary 3.9: $d(t) = \beta(d(t), \phi_t') + 1$, we deduce that :

$$d(t) \geq 1 + \frac{t^2}{8 \log 2}, \quad t > 0 \text{ small.} \tag{5.8}$$

This means that (1.6) fails for the family of conformal map (ϕ_t), $t > 0$ because if this family holds for (1.6) then the fact that

$$\limsup_{r \longrightarrow 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta}{\log \frac{1}{1-r}} = 2 \limsup_{r \longrightarrow 1} \frac{\int_0^{2\pi} |\mathrm{Re} b(re^{i\theta})|^2 d\theta}{\log \frac{1}{1-r}} = 0 \tag{5.9}$$

which follows from the following results would contradict (5.8). Theorem 5.2 is proven.

Theorem 5.10. *Let $\langle S \rangle_j^2$ be the square function of the dyadic martingale S_j of Reb (b defined above) and p be a real positive. Then there exist positive constants $M_1, M_2, K_1, K_2, T_1, T_2$ do not depend on j such that:*

if $p > 1$,

$$M_1 j^{(p-1)/2} \leq \frac{1}{2\pi} \int_0^{2\pi} (\langle S \rangle_j^2(\theta))^{p/2} d\theta \leq M_2 j^{(p-1)/2};$$

if $p = 1$,

$$K_1 \log j \leq \frac{1}{2\pi} \int_0^{2\pi} (\langle S \rangle_j^2(\theta))^{p/2} d\theta \leq K_2 \log j;$$

if $p < 1$,

$$T_1 \leq \frac{1}{2\pi} \int_0^{2\pi} (\langle S \rangle_j^2(\theta))^{p/2} d\theta \leq T_2.$$

Proof: First we'll show that

$$\frac{1}{2\pi} \int_0^{2\pi} (\langle S \rangle_j^2(\theta))^{p/2} d\theta = \sum_{k=1}^{j-1} ((k+1)^{p/2} - k^{p/2}) P(\{N > k\}) \tag{5.10}$$

and then we'll prove that there exist positive constants A_1, A_2 do not depend on j such that

$$A_1 \sum_{n=1}^{l} \frac{1}{(2n)^{(3-p)/2}} \leq \int_0^{2\pi} (\langle S \rangle_j^2(\theta))^{p/2} d\theta \leq A_2 \sum_{n=1}^{l} \frac{1}{(2n)^{(3-p)/2}}. \tag{5.11}$$

The proof will follow from the estimation of the sum $\displaystyle\sum_{n=1}^{l} \frac{1}{(2n)^{(3-p)/2}}$. That is the strategy of the proof.

Now, let us begin with the first step. We separate the unit circle into two sets $\{N > j\}$ and $\{N \leq j\}$, then

$$\frac{1}{2\pi} \int_0^{2\pi} (\langle S \rangle_j^2(\theta))^{p/2} d\theta = \frac{1}{2\pi} \int_{\{N > j\}} (\langle S \rangle_j^2(\theta))^{p/2} d\theta + \frac{1}{2\pi} \int_{\{N \leq j\}} (\langle S \rangle_j^2)^{p/2} d\theta.$$

We observe that on the set $\{N > j\}$, $\langle S\rangle_j^2 = \sum_{l=1}^{j}(S_l - S_{l-1})^2 = j$, hence $\frac{1}{2\pi}\int_{\{N>j\}}(\langle S\rangle_j^2(\theta))^{p/2}d\theta = j^{p/2}P(\{N > j\})$. Besides,

$$\int_{\{N\le j\}}(\langle S\rangle_j^2(\theta))^{p/2}d\theta = \sum_{k=1}^{j}\int_{\{N=k\}}(\langle S\rangle_j^2(\theta))^{p/2}d\theta.$$

Since $S_l = 0, \forall l \ge k$ on the set $\{N = k\}$, then

$$\langle S\rangle_j^2 = \sum_{l=1}^{k}(S_l - S_{l-1})^2 = k.$$

This implies that

$$\frac{1}{2\pi}\int_{\{N\le j\}}(\langle S\rangle_j^2(\theta))^{p/2}d\theta = \sum_{k=1}^{j}\frac{1}{2\pi}\int_{\{N=k\}}(\langle S\rangle_j^2(\theta))^{p/2}d\theta = \sum_{k=1}^{j}k^{p/2}P(\{N = k\}).$$

By using summation by parts, we have

$$\sum_{k=1}^{j}k^{p/2}P(\{N = k\}) = \sum_{k=1}^{j-1}((k+1)^{p/2} - k^{p/2})P(\{N > k\}) - j^{p/2}P(\{N > j\}).$$

This implies (5.10).

We see that if $p \ge 2$ then

$$\frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2(\theta))d\theta \ge \sum_{k=1}^{j-1}(p/2)k^{(p-2)/2}P(\{N > k\})$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2(\theta))d\theta \le \sum_{k=1}^{j-1}(p/2)(k+1)^{(p-2)/2}P(\{N > k\}),$$

and since $(k+1)^{(p-2)/2} \le e^{(p-2)/2}k^{(p-2)/2}, k = 1, 2, \ldots$ then

$$\frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2(\theta))d\theta \le e^{(p-2)/2}\sum_{k=1}^{j-1}(p/2)k^{(p-2)/2}P(\{N > k\}).$$

Thus,

$$\frac{p}{2}\sum_{k=1}^{j-1}k^{(p-2)/2}P(\{N > k\}) \le \frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2(\theta))^{p/2}d\theta \le \frac{pe^{(p-2)/2}}{2}\sum_{k=1}^{j-1}k^{(p-2)/2}P(\{N > k\}).$$

If $p < 2$ we have the inverse inequality

$$\frac{p}{2}\sum_{k=1}^{j-1}k^{(p-2)/2}P(\{N > k\}) \ge \frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2(\theta))^{p/2}d\theta \ge \frac{pe^{(p-2)/2}}{2}\sum_{k=1}^{j-1}k^{(p-2)/2}P(\{N > k\}).$$

Using the remark in 3.3.3 that $P(\{N > 2n+1\}) = P(\{N > 2n\})$, therefore without loose generality we assume that $j = 2(l+1)$. If $p \geq 2$ then

$$\begin{aligned} p/2 + p\sum_{n=1}^{l}(2n)^{(p-2)/2}P(\{N > 2n\}) &\leq \frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2(\theta))^{p/2}d\theta \\ &\leq \frac{pe^{(p-2)/2}}{2}\left((1+e^{(p-2)/2})\sum_{n=1}^{l}(2n)^{(p-2)/2}P(\{N > 2n\}) + 1/2 \right). \end{aligned}$$

If $p < 2$ then

$$\begin{aligned} p/2 + p\sum_{n=1}^{l}(2n)^{(p-2)/2}P(\{N > 2n\}) &\geq \frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2(\theta))^{p/2}d\theta \\ &\geq \frac{pe^{(p-2)/2}}{2}\left((1+e^{(p-2)/2})\sum_{n=1}^{l}(2n)^{(p-2)/2}P(\{N > 2n\}) + 1/2 \right). \end{aligned}$$

By Lemma 5.9, there exist absolute positive constants C_1, C_2 such that

$$C_1\frac{1}{\sqrt{2n}} \leq P(N > 2n) = P(\{\Sigma_{2n} = 0\}) \leq C_2\frac{1}{\sqrt{2n}}, \;\; \forall n.$$

This implies that

$$C_1\sum_{n=1}^{l}\frac{1}{(2n)^{(3-p)/2}} \leq \sum_{n=0}^{l}(2n)^{(p-2)/2}P(\{N > 2n\}) \leq C_2\sum_{n=1}^{l}\frac{1}{(2n)^{(3-p)/2}}.$$

This implies (5.11).

Now, we have if $p \geq 3$ the function $f(x) = \dfrac{1}{(2x)^{(3-p)/2}}$ is increasing on $[1, \infty)$, then

$$2^{(p-3)/2} + \int_1^l \frac{1}{(2x)^{(3-p)/2}}dx \leq \sum_{n=1}^{l+1}\frac{1}{(2n)^{(3-p)/2}} \leq \int_1^{l+1}\frac{1}{(2x)^{(3-p)/2}}dx,$$

where $\displaystyle\int_1^l \frac{1}{(2x)^{(3-p)/2}}dx = \frac{1}{p-1}\left((j-2)^{(p-1)/2} - 2^{(p-1)/2} \right)$ and $\displaystyle\int_1^{l+1}\frac{1}{(2x)^{(3-p)/2}}dx = \frac{1}{p-1}\left((j)^{(p-1)/2} - 2^{(p-1)/2} \right)$.

If $p < 3$ the function $f(x) = \dfrac{1}{2n^{(3-p)/2}}$ is decreasing on $[1, \infty)$, then :

if $\dfrac{3-p}{2} < 1 \Longleftrightarrow p > 1$ then

$$\int_1^{l+1}\frac{1}{(2x)^{(3-p)/2}}dx \leq \sum_{n=1}^{l}\frac{1}{(2n)^{(3-p)/2}} \leq \int_1^{l}\frac{1}{(2x)^{(3-p)/2}}dx + \frac{1}{2^{(3-p)/2}},$$

where $\displaystyle\int_1^l \frac{1}{(2x)^{(3-p)/2}}dx = \frac{1}{p-1}\left((j-2)^{(p-1)/2} - 2^{(p-1)/2} \right)$ and $\displaystyle\int_1^{l+1} \frac{1}{(2x)^{(3-p)/2}}dx = \frac{1}{p-1}\left((j)^{(p-1)/2} - 2^{(p-1)/2} \right)$;

if $\dfrac{3-p}{2} = 1 \Longleftrightarrow p = 1$ then

$$\frac{1}{2}\int_1^{l+1} \frac{1}{x}dx \leq \sum_{n=1}^{l} \frac{1}{2n} \leq \frac{1}{2}\int_1^{l} \frac{1}{x}dx + \frac{1}{2},$$

where $\displaystyle\int_1^{l+1} \frac{1}{x}dx = \log(j) - \log 2$ and $\displaystyle\int_1^{l} \frac{1}{x}dx = \log(j-2) - \log 2$;

if $\dfrac{3-p}{2} > 1 \Longleftrightarrow p < 1$ then $\displaystyle\sum_{n=1}^{l} \frac{1}{(2n)^{(3-p)/2}}$ converges as $j \longrightarrow \infty$ because

$$\int_1^{l+1} \frac{1}{(2x)^{(3-p)/2}}dx \leq \sum_{n=1}^{l} \frac{1}{(2n)^{(3-p)/2}} \leq \int_1^{l} \frac{1}{(2x)^{(3-p)/2}}dx + \frac{1}{2^{(3-p)/2}},$$

where $\displaystyle\int_1^l \frac{1}{(2x)^{(3-p)/2}}dx = \frac{1}{p-1}\left((j-2)^{(p-1)/2} - 2^{(p-1)/2} \right)$ and $\displaystyle\int_1^{l+1} \frac{1}{(2x)^{(3-p)/2}}dx = \frac{1}{p-1}\left((j)^{(p-1)/2} - 2^{(p-1)/2} \right)$ converges as $j \longrightarrow \infty$.

Now we can go to the conclusion that there exist positive constants $M_1, M_2, K_1, K_2, T_1, T_2$ do not depend on j such that:

if $p > 1$,

$$M_1 j^{(p-1)/2} \leq \frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2(\theta))^{p/2} d\theta \leq M_2 j^{(p-1)/2};$$

if $p = 1$,

$$K_1 \log j \leq \frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2(\theta))^{p/2} d\theta \leq K_2 \log j;$$

if $p < 1$,

$$T_1 \leq \frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2(\theta))^{p/2} d\theta \leq T_2.$$

The theorem is proven.

Corollary 5.11. *Let b be the Bloch function (defined above) and p be positive. Then*

$$\limsup_{r\longrightarrow 1} \frac{\int_0^{2\pi} |Reb(re^{i\theta})|^p d\theta}{(\log \frac{1}{1-r})^{p/2}} = 0$$

Proof: The proof will be given as follows. First of all, we'll show that for $p > 0$

$$\limsup_{j\longrightarrow\infty} \frac{\int_0^{2\pi} |\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta}{(j\log 2)^{p/2}} = \limsup_{j\longrightarrow\infty} \frac{\int_0^{2\pi} |S_j|^p d\theta}{(j\log 2)^{p/2}}. \tag{5.12}$$

Then we estimate $\dfrac{\int_0^{2\pi}|S_j|^p d\theta}{(j\log 2)^{p/2}}$ by using the fact that: for $1<p<\infty$ (see [Bur66]) there exist absolute positive constants b_p and B_p such that

$$b_p\|\langle S\rangle_j\|_p \le \|S_j\|_p \le B_p\|\langle S\rangle_j\|_p$$

and for $0<p\le 1$ (see [Gan91]) there also exists a positive absolute constant ν_p such that $\|S_j\|_p \le \nu_p\|\langle S\rangle_j\|_p$, where $\langle S\rangle_j = \sqrt{\sum_{k=1}^n(\Delta S_k)^2}$. Then the proof will follow by Theorem 5.10. That is the main idea of the proof.

First, let us prove (5.12). The fact that $|S_j(\theta)-\mathrm{Re}b(re^{i\theta})|\le C\|b\|_{\mathcal{B}}$ if $r=1-2^{-j}$ (see (3.17)) implies that for $p\ge 1$

$$\left|\ \|S_j\|_p - \|\mathrm{Re}b(re^{i\theta})\|_p\ \right| \le \|S_j-\mathrm{Re}b(re^{i\theta})\|_p \le 2\pi(C\|b\|_{\mathcal{B}}).$$

Therefore if we divide both sides by $(j\log 2)^{1/2}$ of the above inequalities and take the limit as j tends to ∞, then we obtain

$$\lim_{j\longrightarrow\infty}\left(\frac{\int_0^{2\pi}|S_j|^p d\theta}{(j\log 2)^{p/2}}\right)^{1/p} - \left(\frac{\int_0^{2\pi}|\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta}{(j\log 2)^{p/2}}\right)^{1/p} = 0. \qquad (5.13)$$

According to Corollary 3.2 $\dfrac{\int_0^{2\pi}|\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta}{(j\log 2)^{p/2}}$ is bounded and then by (5.13) $\dfrac{\int_0^{2\pi}|S_j|^p d\theta}{(j\log 2)^{p/2}}$ is also bounded. Moreover since the function x^p is continuous uniformly on some compact set of $[0,+\infty)$, then (5.13) implies that

$$\lim_{j\longrightarrow\infty}\frac{\int_0^{2\pi}|S_j|^p d\theta}{(j\log 2)^{p/2}} - \frac{\int_0^{2\pi}|\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta}{(j\log 2)^{p/2}} = 0$$

Thus,

$$\limsup_{j\longrightarrow\infty}\frac{\int_0^{2\pi}|S_j|^p d\theta}{(j\log 2)^{p/2}} = \limsup_{j\longrightarrow\infty}\frac{\int_0^{2\pi}|\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta}{(j\log 2)^{p/2}}.$$

In the case of $0<p\le 1$, again the fact that $|S_j(\theta)-\mathrm{Re}b(re^{i\theta})|\le C\|b\|_{\mathcal{B}}$ if $r=1-2^{-j}$ (see (3.17)) implies that

$$\left|\int_0^{2\pi}|S_j|^p d\theta - \int_0^{2\pi}|\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta\right| \le \int_0^{2\pi}|S_j-\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta \le 2\pi(C\|b\|_{\mathcal{B}})^p.$$

Analogously, if we divide the above inequalities by $(j\log 2)^{p/2}$ and take the limit as j tends to ∞, then we have

$$\lim_{j\longrightarrow\infty}\frac{\int_0^{2\pi}|S_j|^p d\theta}{(j\log 2)^{p/2}} - \frac{\int_0^{2\pi}|\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta}{(j\log 2)^{p/2}} = 0$$

which implies that

$$\limsup_{j\longrightarrow\infty}\frac{\int_0^{2\pi}|S_j|^p d\theta}{(j\log 2)^{p/2}} = \limsup_{j\longrightarrow\infty}\frac{\int_0^{2\pi}|\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta}{(j\log 2)^{p/2}}.$$

Then (5.12) follows.

According to Theorem 5.10, if we divide the integral $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}(\langle S\rangle_j^2(\theta))^{p/2}d\theta$ by $(j\log 2)^{p/2}$ and let $j\longrightarrow\infty$, then we have

$$\limsup_{j\longrightarrow\infty}\frac{\int_0^{2\pi}(\langle S\rangle_j^2(\theta))^{p/2}d\theta}{(j\log 2)^{p/2}}=0.$$

This finishes the proof.

Appendix A

Hausdorff dimension and Minkowski dimension.

Let $\alpha > 0$. The *α-dimensional Hausdorff measure* of a Borel set $E \subset \mathbb{C}$ is defined by

$$\Lambda_\alpha(E) = \lim_{\epsilon \longrightarrow 0} \inf_{(B_k)} \sum_k (\mathrm{diam} B_k)^\alpha,$$

where the infimum is taken over the covers (B_k) of E with $\mathrm{diam} B_k \leq \epsilon$ for all k. The **Haudorff dimension** is defined by

$$\mathrm{H.dim} E = \inf\{\alpha : \Lambda_\alpha(E) = 0\}.$$

sets of non-integer number dimension are called "fractals".

Proposition A.1. *The Hausdorff dimension $H.dim(E)$ is the unique real $d \geq 0$ such that*

$$\Lambda_\alpha(E) = +\infty \quad \text{if} \quad 0 < \alpha < H.dim(E)$$

and

$$\Lambda_\alpha(E) = 0 \quad \text{if} \quad \alpha > H.dim(E).$$

Proof: see [Pom92].

Let E be a bounded set in $\mathbb{C}$ and let $N(\varepsilon, E)$ denote the minimal numbers of disks of diameter ε that are needed to cover E. Up to bounded multiplies it is the same as the number of squares of grid of mesh size ε that intersect E. We defined the **Minkowski dimension** of E by

$$\mathrm{M.dim} E = \limsup_{\varepsilon \longrightarrow 0} \frac{\log N(\varepsilon, E)}{\log(1/\varepsilon)}. \tag{A.1}$$

Proposition A.2. *If E is any bounded set in $\mathbb{C}$ then*

$$H.dim E \leq \liminf_{\varepsilon \longrightarrow 0} \frac{\log N(\varepsilon, E)}{\log(1/\varepsilon)} \leq M.dim E. \tag{A.2}$$

Proof: Let β be any number greater than the limes inferior in (A.2). Then there are $\varepsilon_n \longrightarrow 0$ such that $N_n = N(\varepsilon_n, E) < \varepsilon_n^{-\beta}$. If E is covered by the discs $D_1, D_2, ...D_{N_n}$ of diameter ε_n then, for $\alpha > \beta$,

$$\sum_{k=1}^{N_n} (diam D_k)^{\alpha} = N_n \varepsilon_n^{\alpha} < \varepsilon_n^{\alpha-\beta} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

and thus H.dim$E \leq \alpha$ which implies (A.2).

The two following examples will illustrate for Proposition A.2.

Example 1: Let $E = \{\frac{1}{n}, n \leq 1\} \cup 0$. Then H.dim$(E) = 0$ while M.dim$(E) = \frac{1}{2}$.

Proof. Indeed, for all $\epsilon > 0$, there exist n such that $\frac{1}{n} - \frac{1}{n+1} \leq \varepsilon \leq \frac{1}{n-1} - \frac{1}{n}$. Then, we cover the set $\{\frac{1}{m}, m \geq n(n+1)\}$ by a ball with diameter ε; the set $\{\frac{1}{m}, n \leq m \leq n(n+1)\}$ by $\left[\frac{1}{n\varepsilon} - 1\right]$ balls with diameter ε and each points of the set $\{\frac{1}{m}, m = 1, 2, ..., n-1\}$ by balls with diameter ε. Thus, $N(\varepsilon, E) = 1 + \left[\frac{1}{n\varepsilon} - 1\right] + n - 1 = \left[\frac{1}{n\varepsilon}\right] + n - 1$. Since, $n(n-1) \leq \left[\frac{1}{n\varepsilon}\right] \leq n(n+1)$. Therefore, $2(n-1) \leq N(\varepsilon, E) \leq 2n$ and,

$$\frac{\log 2(n-1)}{\log n(n+1)} \leq \frac{\log N(\varepsilon, E)}{\log \frac{1}{\varepsilon}} \leq \frac{\log 2n}{\log n(n-1)}.$$

Hence, let $n \longrightarrow \infty$, we deduce that M.dim$(E) = \dfrac{1}{2}$ and since E is countable set thus H.dim$(E) = 0$.

Example 2: Let K be a self-similar curve with $0 < \Lambda_{\text{H.dim}(K)} < +\infty$, then H.dim$(K) =$ M.dim(K).

Proof. Since K is a self-similar curve, then there exist finite similitude maps $(S_i, i = 1, ..., n)$ such that we have the partition $K = \bigcup_{i=1}^{n} S_i(K)$. Put $K_i = S_i(K)$.

Let $\beta =$ M.dim(K). Then there exists a positive sequence (ε_m) such that: $N(\varepsilon_m, K) \simeq \varepsilon_m^{-\beta}$, as m tends to $+\infty$. This implies $\sum_{i=1}^{n} N(\varepsilon_m, K_i) \simeq \varepsilon_m^{-\beta}$, as m tends to $+\infty$. Cover K_i by the balls with diameter $\varepsilon_{im} = r_i \varepsilon_m$, where $r_i = Lip(S)$ called similitude ratio of the map S_i. It follows from the self-similarity of K that $r_i < 1$ and

$$N(\varepsilon_{im}, K_i) = N(\varepsilon_m, K).$$

Moreover,

$$N(\varepsilon_{im}, K_i) \simeq \varepsilon_{im}^{-\beta}$$

and

$$N(\varepsilon_m, K_i) \simeq \varepsilon_m^{-\beta}.$$

Thus,

$$\frac{N(\varepsilon_m, K_i)}{N(\varepsilon_{im}, K_i)} \simeq \left(\frac{\varepsilon_m}{\varepsilon_{im}}\right)^{-\beta} = r_i^{\beta},$$

as m tends to $+\infty$. It follows that

$$\sum_{i=1}^{n} N(\varepsilon_m, K_i) \simeq \sum_{i=1}^{n} r_i^{\beta} N(\varepsilon_m, K) = N(\varepsilon_m, K) \sum_{i=1}^{n} r_i^{\beta}.$$

Divide two sides by $N(\varepsilon_m, K)$ and let m tend to $+\infty$. Then we obtain

$$\sum_{i=1}^{n} r_i^{\beta} = 1. \tag{A.3}$$

Let α = H.dim(K). We recall $\Lambda_\alpha(K) = \lim\limits_{\varepsilon \longrightarrow 0} \inf\limits_{B_k} \sum\limits_{k} (diam(B_k))^{\alpha}$. Since Λ_α is a measure, then for disjoint sets $K_i : K = \cup_{i=1}^{n} S_i(K)$, we have

$$\Lambda_\alpha(K) = \sum_{i=1}^{n} \Lambda_\alpha(K_i).$$

Moreover,

$$\begin{aligned} \Lambda_\alpha(K_i) &= \lim_{\varepsilon \longrightarrow 0} \inf_{B_k} \sum_{k} (diam(r_i B_k))^{\alpha} \\ &= r_i^{\alpha} \lim_{\varepsilon \longrightarrow 0} \inf_{B_k} \sum_{k} (diam(B_k))^{\alpha} = r_i^{\alpha} \Lambda_\alpha(K). \end{aligned}$$

This implies that

$$\Lambda_\alpha(K) = \sum_{i=1}^{n} \Lambda_\alpha(K_i) = \sum_{i=1}^{n} r_i^{\alpha} \Lambda_\alpha(K) = \Lambda_\alpha(K) \sum_{i=1}^{n} r_i^{\alpha}.$$

By the assumption that $0 < \Lambda_\alpha(K) < +\infty$, we deduce that

$$\sum_{i=1}^{n} r_i^{\alpha} = 1. \tag{A.4}$$

From (A.3) and (A.4), we obtain $\alpha = \beta$. Since the function $f(t) = \sum_{i=1}^{n} r_i^{t} - 1$ is continuous and $f(0) = n - 1 > 0$ and

$$\lim_{t \longrightarrow \infty} f(t) = 0 - 1 = -1 < 0,$$

then the equation $\sum\limits_{i=1}^{n} r_i^{t} = 1$ has the unique solution. It follows that H.dim(K) = M.dim(K).

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Thanh Hoang Nhat LE : Sur la dimension de Minkowski des quasicercles

Résumé: Cette thèse a pour origine une question de Mc Mullen: A quelle(s) condition(s) est-ce suive famille lisse d'applications conformes $\phi_t : \mathbb{D} \longrightarrow \mathbb{C}$ avec $\phi_0 = id$ vérifie

$$\frac{d^2}{dt^2}\text{H.dim}\,(\phi_t(\partial\mathbb{D}))\Big|_{t=0} = \lim_{r\longrightarrow 1}\frac{1}{4\pi|\log(1-r)|}\int_{|z|=r}|\dot{\phi}_0'(z)|^2|dz| \qquad (a)?$$

Mc Mullen a montré (a) pour des familles (ϕ_t) provenent d'un systeme dynamique. Afin de répondre à cette question, on considère une famille à 1-paramètre analytique générale (ϕ_t), $t \in U$, un voisinage de 0, d'applications conformes avec $\phi_0 = id$ et $\phi_t(0) = 0, \forall t \in U$ définies par $\phi_t(z) = \int_0^z e^{tb(u)}du, \quad b \in \mathcal{B}$, où $\mathcal{B}$ est l'espace de Bloch. En utilisant un argument de probabilité, tout d'abord on exhibe une classe relativement grande des fonctions de Bloch pour lesquelles $(\phi_t)_{t\in U}$ satisfait (a) où la dimension de Hausdorff est remplacée par la dimension de Minkowski. Cette classe est définie en terme de la fonction carée de la martingale dyadique associée à Re(b). En utilisant un argument d'analyse classique, on démontre aussi qu'un résultat similaire peut être obtenu dans le cas où b est une série lacunaire. Le second résultat principal de cette thèse est un contre-exemple. Le point de départ est la construction par Kahane et Piranian de ce qui est appelé un domaine rectifiable "non-Smirnov". Ces auteurs ont construit une fonction de Bloch singulière b telle que si l'on considère la famille associée (ϕ_t) comme ci-dessus, alors $\phi_t(\partial\mathbb{D})$ est rectifiable avec $t < 0$. En utilisant les propriétés de cette fonction de Bloch b, on prouve qu'il existe une constante $c > 0$ telle que M.dim$(\phi_t(\partial\mathbb{D})) \geq 1 + ct^2$, $(t > 0$ petit) ce qui contredit (a), où la dimension de Hausdorff est remplacée par la dimension de Minkowski.

Mots clés: la dimension de Minkowski, la dimension de Hausdorff, fonction de Bloch, la question ouverte de Mc Mullen, martingale dyadique, mesure de Kahane, serie lacunaire.

On Minkowski dimension of quasicircles

Abstract: This thesis has its origin in a question raised by the Mc Mullen's open question: Under what general circumstances does a smooth family of conformal maps $\phi_t : \mathbb{D} \longrightarrow \overline{\mathbb{C}}$ with $\phi_0 = id$ satisfy

$$\frac{d^2}{dt^2}\text{H.dim}\,(\phi_t(\partial\mathbb{D}))\Big|_{t=0} = \lim_{r\longrightarrow 1}\frac{1}{4\pi|\log(1-r)|}\int_{|z|=r}|\dot{\phi}_0'(z)|^2|dz| \qquad (a)?$$

Mc Mullen has shown that (a) is true for some families (ϕ_t) arising from some dynamical systems. In order to answer this question, we consider a general analytic 1-parameter family (ϕ_t), $t \in U$, a neighborhood of 0, conformal maps with $\phi_0 = id$ and $\phi_t(0) = 0, \forall t \in U$ defined as $\phi_t(z) = \int_0^z e^{tb(u)}du, \quad b \in \mathcal{B}$, where $\mathcal{B}$ is the Bloch space. By using a probability argument, we first describe a relatively large class of functions in $\mathcal{B}$ for which $(\phi_t)_{t\in U}$ satisfies (a), where Hausdorff dimension is replaced by Minkowski dimension. This class is defined in terms of the square function of the associated dyadic martingale of Re(b). We also show that a similar result can be derived for lacunary series by using a classical analytic argument. The second principal result of this thesis is a counter-example. The starting point is the construction by Kahane and Piranian of a so-called "non-Smirnov" rectifiable domain. These authors have constructed a singular Bloch function b such that if we consider the associated family (ϕ_t) as above then $\phi_t(\partial\mathbb{D})$ is rectifiable for $t < 0$. Using the properties of this Bloch function b, we prove that there exists $c > 0$ such that M.dim$(\phi_t(\partial\mathbb{D})) \geq 1 + ct^2$, $(t > 0$ small) thus contradicting (a), where the Hausdorff dimension replaced by the Minkowski dimension.

Keywords : Bloch function, dyadic martingale, Hausdorff dimension, Kahane measure, lacunary series, Mc Mullen's open question, Minkowski dimension.

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