Modélisation asymptotique pour les problèmes de propagation d'ondes

Sébastien Tordeux

Habilitation à diriger les recherches

UPPA - Université de Pau et des Pays de l'Adour

Spécialité Mathématiques Appliquées

soutenue le 20 Janvier 2012 par

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Maître de Conférences, chaire d'excellence en Analyse numérique INRIA

Laboratoire de Mathématiques et de leurs Applications UMR CNRS 5142
et
Projet Magique 3D, INRIA Bordeaux Sud-Ouest

Modélisation asymptotique pour les problèmes de propagation d'ondes

Jury

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Rapporteurs :	M.	Grégoire Allaire	Pr.	Ecole Polytechnique
	M.	Oscar Bruno	Pr.	California Institute of Technologies
	M.	Peter Monk	Pr.	University of Delaware
Examinateurs :	Mme.	Hélène Barucq	Dr.	INRIA Bordeaux Sud-Ouest
	M.	Patrick Joly	Dr.	INRIA Rocquencourt
	Mme.	Isabelle Terrasse	Dr.	INRIA Bordeaux Sud-Ouest
	Mme.	Monique Madaune-Tort	Pr.	UPPA

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Première partie

Partie administrative

Curiculum Vitae de Sébastien Tordeux

Né le 18 septembre 1978	Tél. : 06-60-13-65-40
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Marié deux enfants	www : http ://stordeux.perso.univ-pau.fr

Activités Professionnelles

Depuis sept. 2010	**Maître de Conférences** en Mathématiques appliquées Chaire d'excellence INRIA-UPPA en Analyse numérique Projet Magique 3D INRIA Bordeaux Sud-Ouest LMA PAU, UMR CNRS 5142, Université de Pau et Pays de l'Adour.
Sept. 2006 à août 2010	**Maître de Conférences** en Mathématiques appliquées IMT, UMR CNRS 5219, INSA-Toulouse Détenteur de la PEDR de 2008 à 2010.
Jan. 2005 à août 2006	**Post-doc** à l'ETH Zurich (Polytechnicum) Département de Mathématiques, Seminar for Applied Mathematics, Responsable : **Ralf Hiptmair**.

Formation

Sept. 2001 à août 2004	**Doctorat de mathématiques appliquées** INRIA-Rocquencourt Bourse DGA, mention très honorable avec Félicitations du Jury, sous la direction de **Patrick Joly** au Projet ONDES.
Sept. 2000 à Août 2001	**DEA de mathématiques appliquées** M2SAP Université de Versailles-Saint Quentin, Mention Très Bien.
Sept. 1998 août 2001	**Ingénieur ENSTA** (Ecole Nationale Supérieure de Techniques Avancées) Option : mathématiques appliquées.
1998 :	DEUG de sciences de la matière Université d'Orsay Mention Très Bien.
1996–1998 :	Classes préparatoires aux grandes écoles au Lycée Lakanal

Activités de recherche

Publications dans des revues internationales

1. A mathematical analysis of the resonance of the finite thin slots, Marianne Clausel, Marc Durufle, Patrick Joly, Sébastien Tordeux, **Applied Numerical Mathematics**, Volume 56, Issues 10-11, (2006), p. 1432-1449.

2. Asymptotic analysis of an approximate model for time harmonic waves in media with thin slots, Patrick Joly, Sébastien Tordeux, **M2AN**, Volume 40, no 1, (2006), p. 63-97.

3. Matching of asymptotic expansions for wave propagation in media with thin slots I : The asymptotic expansion, Patrick Joly, Sébastien Tordeux, **Multiscale Modeling and Simulation : A SIAM Interdisciplinary Journal**, Volume 5, Issue 1, (2006) , p. 304-336.

4. Matching and multiscale expansions for a model singular perturbation problem, Sébastien Tordeux, Grégory Vial, Monique Dauge, **C. R. Math. Acad. Sci. Paris**, 343, no. 10, (2006), p. 637-642.

5. On moderately close inclusions for the Laplace equation, Virginie Bonnaillie-Noël, Marc Dambrine, Grégory Vial, Sébastien Tordeux, **C. R. Math. Acad. Sci. Paris**, 345, no. 11 (2007), p. 609-614.

6. Matching of asymptotic expansions for waves propagation in media with thin slots II : The error estimates, Patrick Joly, Sébastien Tordeux, **M2AN**, Volume 42, no2, (2008), p. 193-221.

7. Interactions between moderately close inclusions for the Laplace equation, Virginie Bonnaillie-Noël, Marc Dambrine, Sébastien Tordeux, Grégroy Vial, **M3AS**, Volume 10, no. 10 (2009), p. 1853-1882.

8. Asymptotic expansions of the eigenvalues of a 2-D boundary-value problem relative to two cavities linked by a hole of small size, Abderrahmane Bendali, Alain Huard, Abdelkader Tizaoui, Sébastien Tordeux, Jean-Paul Vila, **C. R. Math. Acad. Sci. Paris**, 347 (2009), p. 1147-1152.

9. Asymptotic modelling of conductive thin sheets, Kersten Schmidt, Sébastien Tordeux, **ZAMP**, Volume 61, Number 4, Pages 603-626 (2010).
10. Self-similar perturbation near corner : matching versus multiscale expansions for a model problem, with Monique Dauge, Sébastien Tordeux, Gregory Vial, Around the Research of Vladimir Maz'ya, **International Mathematical Series (Springer)** Volume 12, (2010) p. 95-134.
11. Justification of the Cavity Model in the Numerical Simulation of Patch Antennas by the Method of Matched Asymptotic Expansions, A. Bendali, A. Makhlouf and S. Tordeux, **Multiscale Modeling and Simulation : A SIAM Interdisciplinary Journal**, Volume 8, Issue 5, pp. 1902-1922 (2010)
12. Effect of micro-defects on structure failure : coupling asymptotic analysis and strong discontinuity approach, Virginie Bonnaillie-Noel, with Delphine Brancherie, Marc Dambrine, Sébastien Tordeux, Grégory Vial, accepted in **Eur. J. Comp. Mech**
13. Self-Adjoint curl operators, Ralf Hiptmair, Robert Kotiuga and Sébastien Tordeux, accepted in **Annali di Matematica Pura ed Applicata**
14. Field Behavior Near The Edge of a microstrip antenna by the method of matched asymptotic expansions, A. Bendali, A. Makhlouf and S. Tordeux, accepted in **Quaterly of Applied Mathematics**
15. Matched Asymptotic Expansions of the Eigenvalues of a 3-D boundary-value problem relative to two cavities linked by a hole of small size, MBarek Fares, Abdelkader Tizaoui, Sébastien Tordeux, **Communications in Computational Physics**, Volume 11, No. 2, pp. 456-471 (2012)
16. High order transmission conditions for thin conductive sheets in magneto-quasistatics, Kersten Schmidt, Sébastien Tordeux, **M2AN**, Volume 45 : pp 1115-1140 (2011)
17. Numerical study of acoustic multiperforated plates Abderrahmane Bendali, M'Barek Fares, Sophie Laurens, Sebastien Tordeux, **ESAIM Proc.**
18. Multiscale Expansion and Numerical Approximation for Surface Defects, Virginie Bonnaillie-Noel, Delphine Brancherie, Marc Dambrine, Fabrice Hérau, Sébastien Tordeux and Grégory Vial, accepted in **ESAIM Proc.**

Proceedings

1. Asymptotic expansion of highly conductive thin sheets, Proceedings in Applied Mathematics and Mechanics, Volume 7 Issue 1, (December 2007) Special Issue : Sixth International Congress on Industrial Applied Mathematics (ICIAM07) and GAMM Annual Meeting, Zurich 2007, with Kersten Schmidt.

2. Matched asymptotic expansions for the determination of the electromagnetic field near the edge of a patch antenna, Proceedings in Applied Mathematics and Mechanics, Volume 7 Issue 1 , (December 2007) Special Issue : Sixth International Congress on Industrial Applied Mathematics (ICIAM07) and GAMM Annual Meeting, Zurich 2007, with Abderrahmane Bendali, Abdelkader Makhlouf

3. Second order asymptotic expansion for an eigenvalue set in domain with small iris, Progress in Industrial Mathematics AT ECMI 2008 Mathematics in Industry, 2010, Volume 15, Part 3, p. 715-720.

Rapports de recherche et preprints

1. RR-5568 - Modèles asymptotiques pour la propagation des ondes dans des milieux comportant des fentes, with Patrick Joly and Marc Lenoir, Rapport de recherche de l'INRIA, 54 pages - Mai 2005

2. 2005-08 - Matching of asymptotic expansions for wave propagation in media with thin slots. (I) The asymptotic expansion, SAM report, with Patrick Joly, 30 pages, November 2005

3. RR-5799 - Un problème de Laplace non standard en milieu non borné, Rapport de recherche de l'INRIA, 10 pages - January 2006

4. 2006-04 - Matching of asymptotic expansions and multiscale expansion for the rounded corner problem, SAM report, with Grégory Vial, 15 pages, February 2006

5. 2008-28 - Asymptotic Modelling of Conductive thin Sheets, SAM report, with Kersten Schmidt, 26 pages, September 2008

6. 2009-17 - Matching of Asymptotic Expansions for a 2-D eigenvalue problem with two cavities linked by a narrow hole, with A. Bendali, A. Tizaoui and J. P. Vila, 78 pages, April 2009

Encadrements

- Co-encadrement du stage de Master 1 de Mathis Clayer à l'Université de Pau et Pays de l'Adour avec Victor Péron. Le stage portait sur un solveur de problèmes aux valeurs propres non linéaire.

- Co-encadrement du post-doctorat de Sophie Laurens à l'INSA-Toulouse avec Abderrahmane Bendali. Le post-doc portait sur la modélisation des parois multiperforées en acoustique.

- Co-encadrement du post-doctorat d'Abdelkader Tizaoui à l'INSA-Toulouse avec Abderrahmane Bendali. Le post-doc portait sur le calcul des fréquences de résonance de cavités reliées par un petit trou.

- Co-encadrement de la thèse de Kersten Schmidt à l'ETH-Zurich avec Ralf Hiptmair et Christoph Schwab. La thèse portait sur la modélisation des plaques conductrices minces.
  www : `http://e-collection.ethbib.ethz.ch/view/eth:41356`
  Actuellement **Assistant Professor à TU Berlin**

- Co-encadrement de la thèse d'Abdelkader Makhlouf à l'INSA-Toulouse avec Abderrahmane Bendali. La thèse portait sur la modélisation des antennes Patch.
  www : `http://eprint.insa-toulouse.fr/archive/00000277/`
  Actuellement **Maître de Conférences à l'Université d'Oran**

- Co-encadrement de la thèse de Vanessa Mattesi à INRIA Bordeaux Sud-Ouest avec Hélène Barucq.

Responsabilités pédagogique et scientifique

2008-2010	Organisation du séminaire de recherche Ondes et structures de l'IMT.
2008–2009	Administration du site web du département GMM à l'INSA-Toulouse.
2009	Co-organisateur du workshop Multiscale Asymptotics and Computational Approximation for surface Defects and Applications in Mechanics à l'ENS Cachan
2012	Co-organisateur du workshop First Russian-French Conference on Mathematical Geophysics, Mathematical Modeling in Continuum Mechanics and Inverse Problems Université de Pau INRIA Bordeaux Sud-Ouest Institute of Computational Mathematics and Mathematical Geophysics Université d'état de Novosibirsk (Russie)
2011-2012	Membre élu de la CNU 26 sur la liste SNESUP

Paticipations à des ANR

ANR MACADAM : Asymptotique multi-échelle et approximation numérique pour des défauts surfaciques et applications en mécanique, 48KE, de décembre 2006 à novembre 2009, en colaboration avec

- ENS Cachan Bretagne : Virginie Bonnaillie-Noël (porteuse du projet), Grégory Vial
- Université de Pau : Marc Dambrine
- UTC : Delphine Brancherie

ANR APAM : Acoustique et parois multiperforées, 450 KE de janvier 2009 à décembre 2011, en calaboration avec

- INSA : Abderrahmane Bendali (porteur du projet), Abdelkader Tizaoui, Jean-Paul Vila, Philippe Villedieu.
- CERFACS : Laurent Gicquel
- ONERA : Franck Simon, Estelle Piot
- SNECMA : Sébastien Roux

Conférences nationales et internationales

1. Asymptotical models for wave propagation in media including slots, JEE, Toulouse, France, 2002.
2. Asympotical models for wave propagation in media includings slots, WAVES 2003, Finland.
3. Mathematical justification of simplified models for acoustics wave in media including thin slots, WONAPDE 2004, Chile.
4. Matching of asymptotic expansions for the wave propagation in media with thin slot, WAVES 2005, Brown University, Providence.
5. Linear Force-Free Magnetic Field or Self-adjoint Extension of the curl Operator, invited, Advanced Computational Electromagnetism (ACE'06), Boston University, USA, 2006.
6. 2D-1D Coupling in a Problem related to the Scattering of Time-Harmonic Waves, Cinquièmes Journées Singulières, CIRM.
7. Matching of Asymptotic Expansions for a 2-D eigenvalue problem with two cavities linked by a narrow hole, WAVES 2009, Pau.
8. Matching of Asymptotic Expansions for eigenvalues problem with two cavities linked by a small hole, GDR Chant, Viennes, Autriche, 2011.
9. Parois perforées et multiperforées en acoustique, Polariton 2011, CIRM, Marseilles

10. Perforated and multiperforated plates in linear acoustic, Second International Workshop on Multiphysics, Multiscale and Optimization Problems 2011, University of the Basque Country, Bilbao

11. Self-adjoint curl operators, Anglet, journées Bordeaux-Pau-Toulouse, 2011

Exposés dans des séminaires

1. Méthodes asymptotiques pour la propagation des ondes acoustiques dans les milieux comportant des fentes, Crespo, ENSTA, Paris, France, 2002.

2. Justification mathématique de modèles simplifies pour la propagation des ondes dans les milieux comportant des fentes, Séminaire d'Analyse Numérique, Université de Rennes I, France, 2004.

3. Mathematical justification of simplified models for acoustics wave in media including thin slots, Seminar for Applied Mathematics, ETHZ, Zurich, Suisse, 2004.

4. Mathematical modeling for acoustic waves in media including thin slot, English-French Workshop, ENSTA, Paris, France, 2004.

5. Raccordement de développements asymptotiques pour la propagation des ondes dans les milieux comportant des fentes, Groupe de travail : Applications des Mathématiques, ENS Cachan, Rennes, France, 2004.

6. Matching of asymptotic expansions for the wave propagation in media with thin slot, AG Analysis und Numerik, Baal, Suisse, 2005.

7. Raccordement de développements asymptotiques pour la propagation des ondes dans les milieux comportant des fentes, Séminaire de Mathématiques et de Physique Appliquées, INSTN (CEA), Saclay, France, 2005.

8. Matching of asymptotic expansions for the wave propagation in media with thin slot, TiSCoPDE workshop (New Trends in Simulation and Control of PDEs), Berlin, Allemagne, 2005.

9. Raccordement de développements asymptotiques pour la propagation des ondes dans les milieux comportant des fentes, Groupe de travail, Laboratoire MIP, Toulouse, France, 2006

10. Comparaison des techniques de raccordements de développements asymptotiques et de développements multiéchelles, groupe de travail : Ondes et Structures, Laboratoire MIP, Toulouse, France, 2006

11. Matched Asymptotic Expansions for a Model of Patch Antenna, 20th birthday of CERFACS, Toulouse, 2007,

12. Justification et amélioration des modèles heuristiques d'antennes patch par développements asymptotiques raccordés, ANEDP, Orsay, 2008
13. Matching of Asymptotic Expansions for an eigenvalue problem with two cavities linked by a thin hole, ETH, Zurich, Suisse, 2009
14. Linear Force-Free Magnetic Field or Self-adjoint Extension of the curl Operator, Séminaire du projet INRIA POEMS, 2009
15. Raccordement de développements asymptotiques pour des problèmes aux valeurs propres comportant deux cavités reliées par un petit trou en dimensions 2 et 3, LJK, Grenoble, 2009
16. Raccordement de développements asymptotiques pour des problèmes aux valeurs propres comportant deux cavités reliées par un petit trou en dimensions 2 et 3, LMA, Pau, 2009
17. Raccordement de développements asymptotiques pour des problèmes aux valeurs propres comportant deux cavités reliées par un petit trou en dimensions 2 et 3, projet Poems, INRIA-Rocquencourt, 2009
18. Matching of Asymptotic Expansions for an eigenvalue problem with two cavities linked by a small hole, Institute of Computational Mathematics and Mathematical Geophysics, Novosibirsk State University, Russie, 2011.
19. Matching of Asymptotic Expansions for an eigenvalue problem with two cavities linked by a small hole, Sobolev Institute of Mathematics, Novosibirsk State University, Russie, 2011.
20. Perforated and multiperforated plates in linear acoustic, BCAM, Bilbao, Espagne, 2011.
21. Parois multiperforées en acoustique, Journée APAM, INSA-Toulouse, 2011

Activités d'enseignement

Université de Pau

Année	Intitulé	Filière	Cours	TD	TP
2011-2012	Propagation d'ondes	Master 2 Mathématique, Modélisation et Simulation	12	11	0
	Analyse numérique fondamentale	Master 1 Mathématique, Modélisation et Simulation	24	0	0
2010-2011	Propagation d'ondes	Master 2 Mathématique, Modélisation et Simulation	12	11	0
	Analyse numérique fondamentale	Master 1 Mathématique, Modélisation et Simulation	24	0	0

INSA-Toulouse (école publique d'ingénieurs en 5 ans)

Année	Intitulé	Filière et année	Cours	TD	TP
2006-2007	Optimisation Continue	Informatique et Mathématiques (2)	25	15	0
	Séries	Physique (2)	12,5	30	0
	Analyse numérique	Physique (3)	15	8,75	41,25
	Analyse numérique	Ingénierie civile (2)	0	7,5	7,5
	Eléments finis	Mathématiques (4)	8,75	10	8,25
2007-2008	Fonctions d'une variable	Premère année	0	23,75	0
	Optimisation Continue	Informatique et Mathématiques (2)	25	15	0
	Analyse numérique	Physique (3)	15	17,5	27,5
	Eléments finis	Mathématiques (4)	8,75	10	8,25
2008-2009	Fonctions de plusieurs variables	Chimie et Biologie (2)	15	17,5	0
	Algèbre bilinéaire	Chimie et Biologie (2)	0	15	0
	Fonctions de plusieurs variables	Ingénierie civile, Bac STI (2)	16,25	16,25	0
	Analyse numérique	Ingénierie civile (2)	13,75	7,5	0
	Analyse numérique	Ingénierie civile, Bac STI (2)	13,75	7,5	11
	Analyse numérique	Physique (3)	0	0	27,5
	Eléments finis	Mathématiques (4)	3,5	3,75	2,75
	Introduction à l'analyse asymptotique	M2R de Toulouse 3 (5)	12	0	0
2009-2010	Fonctions de plusieurs variables	Chimie et biologie (2)	15	17,5	0
	Analyse numérique	Ingénierie civile (2)	13,75	7,5	0
	Analyse numérique	Ingénierie civile, BAC STI (2)	13,75	7,5	11
	Optimisation numérique	Informatique et Mathématiques (3)	8,75	13,75	38,5
	Eléments finis	Mathématiques (4)	3,5	3,75	2,75
	Eléments finis	ENAC, électronique micro-ondes (5)	16	0	0

ETH Zurich (polytechnicum)

- Chargé de travaux dirigés multiscale iterative solvers, 2005/2006.
- Chargé de travaux dirigés Numerical methods for elliptic and parabolic partial differential equations, 2005/2006.
- Chargé de travaux dirigés Absorbing Boundary Conditions 2005/2006.

ENSTA (école publique d'ingénieurs en 3 ans après classes préparatoires)

- Chargé de travaux dirigés Introduction à Matlab, 2003/2004.
- Chargé de travaux dirigés Optimisation, 2003/2004.
- Chargé de travaux dirigés Introduction au calcul numérique, 2003/2004.

- Chargé de travaux dirigés Algèbre et Optimisation 2001/2003.
- Formation continue à l'ENSTA Introduction au calcul numérique et à l'analyse numérique, 2001/2002, 2002/2003.

Lycée Lakanal :

Kholles de physique en math'sup, 1998/1999 1999/2000.

Deuxième partie

Synthèse des travaux de recherche

Chapter 1

Asymptotic analysis of wave propagation problems

1.1 Introduction

Several physical problems involve multiscale features (phenomena at very different scales). The classical numerical methods, like Finite Differences or Finite Elements, require refined meshes to take into account the various scales characterizing the physical phenomena. This leads to costly computations due to the large number of degrees of freedom and in some circumstances to instabilities known as numerical locking phenomena. To overcome this difficulty, different methods have been proposed. They consist in

1. limiting the number of degrees of freedom by resorting to a local mesh refinement, see for example to [Schwab98] or [Rodriguez04].

2. performing an asymptotic analysis and deriving approximate model whose solution can be computed without mesh refinement. One can refer to the classical books [Il'in92] and [Maz'ya00].

3. combining an asymptotic analysis with a finite element method: The Galerkin space is augmented by shape functions reproducing the local properties of the solution of the exact model. A similar approach in singularity theory can be found in [Melenk96] or in [Chahine07].

These techniques become specially relevant in the context of wave propagation problems: A small geometrical detail with respect to the wavelength (thin wire, thin slot, thin hole, thin sheet) may have a significant impact and cannot be neglected. I have studied these phenomena in the context of scalar time-harmonic waves where the solutions are in the form

$$u(x,t) \;=\; u(x)\;\exp(-i\omega t). \tag{1.1}$$

More precisely, I was interested in

1. the coupling of a 1D thin slot with a 2D propagation domain, section 1.2;
2. the justification of the cavity model for Patch antennas, section 1.3;
3. the initialization of the crack phenomena by micro-defects, section 1.5;
4. the electromagnetic shielding by highly conducting thin sheets, section 1.6;
5. the effect of perforations on the acoustic resonances, section 1.7.

In the framework of these studies, I have developed with my coauthors numerical methods to compute the effect of small details with a small computational cost. These methods are based either on approximate models or on enriched Finite Element methods. I have either validated these approximate models by error estimates or I have compared their solutions to direct computations realized on refined meshes.

Moreover, I have tried to spread these asymptotic techniques in writing a pedagogical article comparing the matching of asymptotic expansions methods to a multi-scale corrector approach, section 1.4.

References

[Bendali96] A.Bendali, K. Lemrabet, The effect of a thin coating on the scattering of a time-harmonic wave for the Helmholtz equation. SIAM J. Appl. Math. 6 (1996), P. 1664-1693.

[Chahine07] E. Chahine, P. Laborde, J. Pommier, Y. Renard, M. Salaun, Study of some optimal xfem type methods, Computational Methods in Applied Sciences, vol. 5, p. 27-40, 2007.

[Il'in92] A. M. Il'in, Matching of Asymptotic Expansions of Solutions of Boundary Value Problems, volume 102 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 1992. Translated from the Russian by V. Minachin.

[Maz'ya00vol1] V. Maz'ya, S. Nazarov, B. Plamenevskij, Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains, Vol. I, Operator Theory: Advances and Applications, Vol. 111, Birkhauser Verlag, 2000.

[Maz'ya00vol2] V. Maz'ya, S. Nazarov, B. Plamenevskij, Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains, Vol. II, Operator Theory: Advances and Applications, Vol. 112, Birkhauser Verlag, 2000.

[Melenk96] J.M. Melenk, I. Babuska, The partition of unity finite element method : Basic theory and applications, Computer Methods in Applied Mechanics and Engineering, vol. 136, p. 289-314, 1996.

[Rodriguez04] J. Rodriguez. raffinement de maillage spatio-temporel pour les équations de l'élastodynamique. Doctorat, Université de Dauphine, décembre 2004.

[Schwab98] C. Schwab. p- and hp- Finite Element Methods: Theory and Applications in Solid and Fluid Mechanics. Oxford University Press, 1998.

1.2 Asymptotic modeling of thin slot

Context

This work was performed during my PhD thesis studies in the team-project Ondes from INRIA Rocquencourt under the supervision of Patrick Joly. The research articles were mostly written at ETH Zurich during my post-doc.

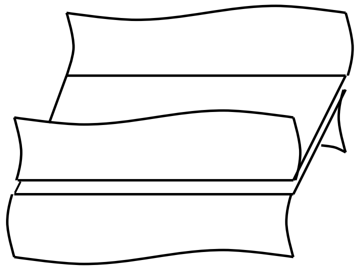

Figure 1.1: A 3D thin slot

Statement of the problem

It is well-known that resonance phenomena occur in thin slots, see Figure 1.1, illuminated by electromagnetic radiation. For a discrete set of frequencies, we observe

- an energy concentration in the vicinity of the thin slot,
- an important transmission through the thin slot,
- a polarization of the transmitted wave.

These effects must be taken into account for electromagnetic compatibility issues concerning population health or safety of electronic devices (black boxes, flying command, ...). Moreover, thin slots are frequently used as antennas and polarizers.

These series of articles focus on a model problem: The Helmholtz equation equipped with a Neumann boundary condition. The computational domain consists of two half spaces linked by a thin slot, see Figure 1.2. Approximate models are derived and justified. The small width of the thin slot is taken into account by

- approaching the solution inside the slot by a solution of the 1D Helmholtz equation posed on the segment linking A and B,

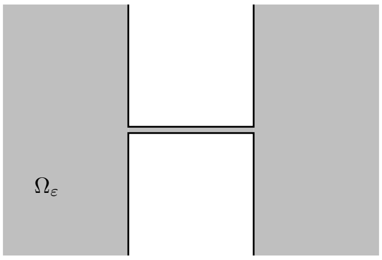

Figure 1.2: The 2D propagation domain

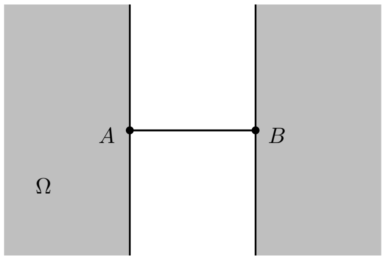

Figure 1.3: A 2D model and a 1D model

- keeping a 2D Helmholtz equation in the two half-spaces,
- coupling these two models by point-wise transmission conditions located at A and B.

Scientific achievements

- *Asymptotic analysis of an approximate model for time harmonic waves in media with thin slots, with Patrick Joly,* ***M2AN****, Volume 40, no 1, (2006), p. 63-97.*

  In this first article, we propose a low order approximate model by coupling a 2D Helmholtz equation and a 1D Helmholtz equation. Error estimates are obtained with standard variational technique. Our results are illustrated by numerical simulations.

- *Matching of asymptotic expansions for wave propagation in media with thin slots I: The asymptotic expansion, with Patrick Joly,* ***Multiscale Modeling and Simulation: A SIAM Interdisciplinary Journal****, Volume 5, Issue 1, (2006) , p. 304-336.*

- *Matching of asymptotic expansions for waves propagation in media with thin slots II: The error estimates, with Patrick Joly,* ***M2AN****, Volume 42, no2, (2008), p. 193-221.*

  In these two articles, high order coupling conditions are derived by the method of matching of asymptotic expansions. The first part presents the formal derivation of these models. The second part focuses on the error estimates which allow to justify such models.

- *A mathematical analysis of the resonance of the finite thin slots, with Marianne Clausel, Marc Durufle and Patrick Joly,* ***Applied Numerical Mathematics****, Volume 56, Issues 10-11, (2006), p. 1432-1449.*

  The technique of matching of asymptotic expansions makes it possible to understand the slot resonances. The results are illustrated by numerical simulations.

1.3 Asymptotic modeling of patch antennas

Context

This work was performed in the PhD. thesis of Abdelkader Makhlouf at INSA-Toulouse under the supervision of Abderrahmane Bendali.

The scientific objectives

The Patch or microstrip antennas are primarily used when it is important not to modify the geometrical shape of a structure, for example in aeronautic, space and defense industries. Typically a patch antenna is composed of a metallic plate located over a dielectric substrate with small width covering a metallic plane. This metallic plane is often straight and called ground plane.

Due to the small distance separating the patch from the ground plane, a direct computation of the radiation pattern can be rather time consuming. Indeed, the computations require a fine mesh in the neighborhood of the edge of the antenna. For simplicity, engineers often prefer to use the so called cavity model with magnetic walls. It consists in a three step procedure:

1. computing an approximation of the solution inside the cavity with a dimension reduction (the interior of the antenna closed by a magnetic wall),
2. extracting (from the field inside the cavity) a linear density of magnetic currents located at the edge of the antenna,
3. computing the field radiated by this line of currents.

We give a new perspective to the justification of the cavity model by using the technique of matching of asymptotic expansions. This work is rather different from the work addressed during my PhD due to the difficulty of handling the singularity in the neighborhood of the edges (which cannot be described by the technique of separation of variables).

Scientific achievements

Abdelkader Makhlouf has defended his PhD

- `http://eprint.insa-toulouse.fr/archive/00000277/`

and two research articles have been published:

- *Field Behavior Near The Edge of a Microstrip Antenna by the Method of Matched Asymptotic Expansions, A. Bendali, A. Makhlouf and S. Tordeux, accepted in* ***Quaterly of Applied Mathematics***

In the first article a mathematical point of view is adopted. A simplified two-dimensional problem incorporating the main features of the field behavior near the edge of the patch and inside the cavity is addressed. The method of matched asymptotic expansions is used to carry out a two-scale asymptotic analysis of the field relatively to the thickness of the cavity.

- *Justification of the Cavity Model in the Numerical Simulation of Patch Antennas by the Method of Matched Asymptotic Expansions, A. Bendali, A. Makhlouf and S. Tordeux,* ***Multiscale Modeling and Simulation: A SIAM Interdisciplinary Journal****, Volume 8, Issue 5, pp. 1902-1922 (2010)*

The second paper is dedicated to more modeling issues based on a rigorous mathematical approach. In particular, it is shown how the availability of a second-order asymptotic expansion yields an effective improvement of the usual cavity model. Numerical results assess the validity of the approach.

1.4 Comparison of the matching of asymptotic expansions technique with a corrector method

Context

This is a joint work with Monique Dauge and Grégory Vial. It was performed at ETH Zurich during my post-doc.

Main results

During his PhD thesis[1] supervized by Gabriel Caloz and Monique Dauge, Grégory Vial has used a corrector method to study the wave scattering by a thin layer with corner. A comparison between his PhD results and mine, obtained by matching of asymptotic expansions, shows numerous similarities.

Even if both approaches are very different, they lead to the same asymptotic expansion (up to a transformation). To illustrate this fact in a simple framework, we decided to write a paper dealing with the classical problem of the multi-scale rounded corner.

Scientific achievements

Two research articles have been written (a short and a long version)

- *Matching and multiscale expansions for a model singular perturbation problem, with Grégory Vial and Monique Dauge,* ***C. R. Math. Acad. Sci. Paris****, 343, no. 10, (2006), p. 637-642.*
- *Self-similar perturbation near corner : matching versus multiscale expansions for a model problem, with Monique Dauge and Gregory Vial, Around the Research of Vladimir Maz'ya,* ***International Mathematical Series*** *(Springer), Volume 12, (2010), p. 95-134.*

[1] http://w3.bretagne.ens-cachan.fr/math/people/gregory.vial/files/recherche/these_gvial.pdf

1.5 Effect of micro-defects on structure failure

Context

This work has been supported by the French Research Agency (ANR MACADAM `http://w3.bretagne.ens-cachan.fr/math/macadam/`). It was realized at INSA-Toulouse in collaboration with Virginie Bonnaillie-Noël (project leader), Delphine Brancherie, Marc Dambrine and Grégory Vial.

Statement of the problem

In fracture mechanics, the cracks are generally initiated by defects of small sizes, where energy concentrates.

In this work, a technique to initiate a crack propagation algorithm by an asymptotic analysis is proposed. The first step of the project consisted in computing with an enriched finite element method the constraint in the neighborhood of the defect. In the second part, it has been shown how to transfer this information to a crack propagation algorithm.

Statement of the problem

Three research articles have been written

- *On moderately close inclusions for the Laplace equation, with Bonnaillie-Noël Virginie, Dambrine Marc, Vial Grégory* ***C. R. Math. Acad. Sci. Paris****, 345, no. 11 (2007), p. 609-614.*

  This article is the short version of the following article.

- *Interactions between moderately close inclusions for the Laplace equation, with Virginie Bonnaillie-Noël, Marc Dambrine and Grégroy Vial,* ***M3AS****, Volume 10, no. 10 (2009), p. 1853-1882.*

  The effect of two very close small perturbations is studied for a Laplace problem. We derive the multi-scale asymptotic expansion at any order. We justify this expansion by error estimates. Based on the expansion at first order, we show how an enriched finite element method can be designed. Finally, the results are illustrated by numerical simulations

- *Effect of micro-defects on structure failure : coupling asymptotic analysis and strong discontinuity approach, Virginie Bonnaillie-Noël, with Delphine Brancherie, Marc Dambrine and Grégory Vial, accepted in* ***Eur. J. Comp. Mech***

  This article reviews the results obtained during the research program. We are interested in the elasticity system. We first show how to design an enriched finite element method based on the first order asymptotic expansion of the solution. The stress concentration generated by the presence of micro-defects is then transferred to a strong discontinuity approach to predict the damaged zones initiated by the geometrical perturbations.

1.6 Asymptotic modeling of thin sheets

Context

This work was performed at ETH Zurich in the PhD thesis of Kersten Schmidt (supervised with Christoph Schwab and Ralf Hiptmair). The articles were written when I was at INSA-Toulouse.

Industrial context

Kersten and the Seminar for Applied Mathematics was in contact with the group ABB (`http://www.abb.fr/`) which is an energy company. After his PhD thesis, Kersten has industrially implemented his work to deal with highly conductive sheets during a post-doc at ABB.

Statement of the problem

In many practical applications, electronic devices are surrounded by casings or other sheets of a highly conductive material to protect them from external electromagnetic fields (for example protection of the signal in data cables) or to protect the environment from the electromagnetic fields generated by devices (for example transformer or bushings). To reduce the costs in size and weight of the structure, these sheets have to be thin. This leads to a non-perfect shielding where the electromagnetic field partly penetrates the shields and have a small but significant effect in the protected region. The large ratio of characteristic lengths (width of the device against thickness of the sheet) leads to serious numerical problems. To overcome this difficulty, we have derived approximate impedance transmission conditions. This avoids to resort to a mesh refinement in the proximity of the sheet.

We have not considered the wave equation but an eddy-current model inside the conductive sheet and a magneto-quasistatic approximation outside the sheet. These two models are relevant for most of the industrial applications.

Scientific achievements

Kersten Schmidt defended his PhD thesis:

- `http://e-collection.ethbib.ethz.ch/view/eth:41356`

The results have also been reported in two papers:

- *Asymptotic modelling of conductive thin sheets, Kersten Schmidt and Sébastien Tordeux,* ***ZAMP****, Volume 61, Number 4, Pages 603-626 (2010)*

  In this article, a complete asymptotic expansion is derived. This approach is justified by error estimates. We illustrate our numerical results by high order finite element numerical simulations.

- *High order transmission conditions for thin conductive sheets in magneto-quasistatics, Kersten Schmidt and Sébastien Tordeux, accepted by* ***M2AN***

  Based on the last asymptotic expansion, approximate transmission conditions are obtained. We show that these transmission conditions define well-posed problems and we give estimates to characterize the quality of these approximations. We end the article with some numerical simulations.

1.7 Perforated plates in linear acoustic

Context

This work was performed in collaboration with Abderrahmane Bendali, Abdelkader Tizaoui who was post-doc at INSA-Toulouse, Sophie Laurens who is post-doc and M'Barek Fares of CERFACS. It was supported by the French Research Agency (ANR APAM). During this projet, we take benefits from numerous discussions with Estelle Piot from ONERA, Sébastien Roux from SNECMA and Laurent Gicquel from Cerfacs.

Statement of the problem

New environmental standards force the aeronautic industries to develop green technologies. One of the promising technique consists in using turbo-reactors with high air-fuel ratio which ensures a complete combustion of fuel. This improves the performance of the motors and reduces pollution. However, this mixture is less stable and interacts strongly with the acoustic wave.

In a turbo engine, the temperature of the combustion chamber can reach 2000 K. To protect the structure, small holes are perforated throw the wall linking the combustion chamber to the casing and fresh air (600 K) is injected. These multiperforated plates are one of the main sources of acoustic noise.

In real configurations, direct numerical computations are beyond reach. This is mostly due to the large number of perforations (approximately 2000) and to their small characteristic lengths (diameter of a perforation 0.5mm; spacing 5mm) with respect to the wave length (0.5m approximately). In this project, we are interested in giving a rigorous explanation to the approximate transmission classically used to model the multiperforated plates.

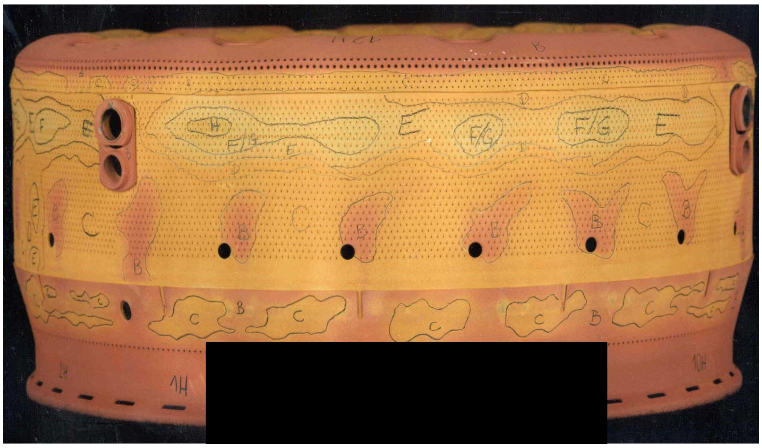

Figure 1.4: A combustion chamber of Turboméca

Scientific achievements

It consists in three research articles

- *Asymptotic expansions of the eigenvalues of a 2D boundary-value problem relative to two cavities linked by a hole of small size, with Abderrahmane Bendali, Alain Huard, Abdelkader Tizaoui, Jean-Paul Vila,* ***C. R. Math. Acad. Sci. Paris****, 347 (2009), p. 1147-1152.*

- *Matched asymptotic expansions of the eigenvalues of a 3D boundary-value problem relative to two cavities linked by a hole of small size, Abderrahmane Bendali, M'Barek Fares, Abdelkader Tizaoui and Sébastien Tordeux, accepted in* ***Communications in Computational Physics***

  In these two articles, is considered an elliptic operator with varying coefficients posed on two cavities linked by a narrow hole . In the first article, the computational domain is 2D and equipped with Dirichlet boundary conditions. In the second one, we consider 3D configuration with Neumann boundary conditions (which is the realistic case). We propose an asymptotic formula to compute the eigenvalues of this operator and we illustrate our results with numerical simulations.

- *Numerical study of acoustic multiperforated plates Abderrahmane Bendali, M'Barek Fares, Sophie Laurens, Sébastien Tordeux, submitted*

  It is rather classical to model multiperforated plates by approximate impedance boundary conditions. In this article, an instance of such boundary conditions obtained through a matched asymptotic expansion technique is compared to direct numerical computations based on a boundary element formulation.

1.8 Perspectives

During the last period, I was mainly interested in 2D Helmholtz problems with singular perturbations. This corresponds to physical problems which are invariant in one direction and harmonic in time. In order to improve the impact of these techniques, I will deal with the following cases:

Three dimensional problems

Three-dimensional costs are one order larger than the two-dimensional ones. Mesh refinement, even local, are time consuming and hard to implement. We wish to pursue to generalize our results to 3D like in [Bendali11]. For works on thin wire, one can cite [Fedoryuk81,Claeys09].

Time-domain problems

Most of the studies of time dependent problems assume a time-dependance of the solution in $\exp(-i\omega t)$. However in 3D, surprisingly enough the two-dimensional case seems to be more difficult, it would be possible to obtain similar results in the time-domain. For time-domain and regular perturbation problems, references [Chun09,Chun10] indicate some possible approaches.

Other elliptic systems

Only few mathematical papers deal with boundary-value problems involving elliptic operators other than the laplacian. In wave propagation, the elasticity system [Brancherie08] and the Maxwell system are especially relevant in the applications [Duruflé06,Caloz11].

Multiphysics phenomena

In numerous problems, the physic is not the same at the macroscopic and the microscopic scales. It is not possible to understand what is occurring in the neighborhood of a detail by performing an asymptotic analysis. One has to resort to different models at different scales. This is especially true in acoustic propagation. One can refer to [Sanchez82] for viscous effects and to [Howe79] for aeroacoustic effects.

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Chapter 2

Selfadjoint curl operators

2.1 Scientific context.

This work was realized during my post-doc at ETH Zurich with Ralf Hiptmair (ETH Professor) et Robert Kotiuga (Boston University Professor) who was invited professor.

Even if this problem finds its origin in the plasma theory, this work is mostly a theoretical functional analysis study. The paper was written in terms of differential forms to emphasize the metric questions. I summarize the results in the language of differential calculus to simplify a little bit the presentation. Details can be found in the paper:

- Self-Adjoint curl operators, Ralf Hiptmair, Robert Kotiuga and Sébastien Tordeux, accepted in **Annali di Matematica Pura ed Applicata**

2.2 Linear force-free field in plasma physics.

The magnetic field $\mathbf{B}$ and the current flux $\mathbf{j}$ are related in small plasmas by the quasi-static Maxwell-Ampère equation

$$\mathbf{curl}\,\mathbf{B} = \mu\,\mathbf{j}. \tag{2.1}$$

In absence of charge, the dynamic is governed by a Lorentz force which takes the form:

$$\mathbf{j} \wedge \mathbf{B}. \tag{2.2}$$

To be in equilibrium, the magnetic field should be parallel to its **curl**. Consequently, a force-free field satisfies:

$$\mathbf{curl}\,\mathbf{B} \;=\; \lambda(x)\,\mathbf{B}. \tag{2.3}$$

Moreover, a linear force-free field is a force-free field with λ constant. The magnetic field is then an eigenfunction of the **curl** operators

$$\mathbf{curl}\,\mathbf{B} \;=\; \lambda\,\mathbf{B} \quad \text{with } \lambda \in \mathbb{R} \tag{2.4}$$

and consequently an eigenfunction of the **curlcurl** operator

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{B} \;=\; \lambda^2\,\mathbf{B}. \tag{2.5}$$

In this article we tried to deal with the following issues:

- In spectral theory, the differential operators and their domains can not be distinguished. Is it possible to characterize the set of boundary conditions which are associated to self-adjoint **curl** operator?
- It is classical to find solvers for the eigenfunctions for the **curlcurl** operator. Is it possible to design a numerical code based on these solvers?

2.3 Characterization of the self-adjoint curl operators

Let D be a regular domain of $\mathbb{R}^3$ with boundary ∂D. We denote by $\mathcal{D}(D)$ the set of all compactly supported regular vector fields of D and by $\mathbf{L}^2(D)$ the set of square-integrable vector fields

$$\mathbf{H}(\mathbf{curl}, D) \;=\; \Big\{\mathbf{u} \in \mathbf{L}^2(D) : \mathbf{curl}\,\mathbf{u} \in \mathbf{L}^2(D)\Big\}. \tag{2.6}$$

We introduce the bilinear form $[\cdot,\cdot]$

$$\left\{\begin{array}{ccl} \mathbf{H}(\mathbf{curl}, D) \times \mathbf{H}(\mathbf{curl}, D) & \longrightarrow & \mathbb{R} \\ (\mathbf{u},\mathbf{v}) & \longmapsto & [\mathbf{u},\mathbf{v}] = \int_D \mathbf{curl}\,\mathbf{u}\cdot\mathbf{v} - \mathbf{curl}\,\mathbf{v}\cdot\mathbf{u} \end{array}\right. \tag{2.7}$$

which measures the defect of symmetry of the **curl** operator. The formal **curl** operator is defined as the operator with domain $\mathcal{D}(D)$ which associates $\mathbf{curl}\,\mathbf{u}$ to $\mathbf{u}$. This is a symmetric operator

$$[\mathbf{u},\mathbf{v}] \;=\; 0 \quad \text{for all } \mathbf{u} \text{ and } \mathbf{v} \text{ in } \mathcal{D}(D). \tag{2.8}$$

Indeed, due to the Stokes formula we can express the pairing $[\cdot,\cdot]$ as a surface pairing acting on the tangential traces $\gamma_t\mathbf{u}$ and $\gamma_t\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ on ∂D

$$[\mathbf{u},\mathbf{v}] = \int_{\partial D} \Big(\gamma_t\mathbf{u} \times \gamma_t\mathbf{v}\Big)\cdot\mathbf{n}. \tag{2.9}$$

To equip the **curl** operator with ad-hoc boundary conditions, we construct the self-adjoint extensions of the formal **curl** operator. The domain of its maxima self-adjoint extension is $\mathbf{H}(\mathbf{curl}, D)$ where as the domain of its minimum extension is

$$\mathbf{H}_0(\mathbf{curl}, D) \;=\; \Big\{\mathbf{u} \in \mathbf{H}(\mathbf{curl}, D) : \gamma_t\mathbf{u} = 0 \text{ on } \partial D\Big\}. \tag{2.10}$$

Every self-adjoint extension of the **curl** operator is an unbounded linear operator of $\mathbf{L}^2(D)$ whose domain $H_s(\mathbf{curl}, D)$ satisfies $\mathbf{H}_0(\mathbf{curl}, D) \subset \mathbf{H}_s(\mathbf{curl}, D) \subset \mathbf{H}(\mathbf{curl}, D)$ and

$$\mathbf{D}_s \;=\; \Big\{\mathbf{u} \in \mathbf{H}(\mathbf{curl}, D) : [\mathbf{u},\mathbf{v}] = 0 \quad \forall \mathbf{v} \in \mathbf{D}_s\Big\}. \tag{2.11}$$

This characterization of self-adjoint operators is not explicit. However it can be interpreted as a boundary condition via (2.9)

$$\mathbf{D}_s \;=\; \Big\{\mathbf{u} \in \mathbf{H}(\mathbf{curl}, D) : \gamma_t \mathbf{u} \in \mathbf{L}\Big\} \tag{2.12}$$

with $\mathbf{L}$ a complete Lagrangian[1] of the symplectic pairing $[\cdot,\cdot]_{\partial D}$ defined on $\mathbf{W} = \gamma_t \mathbf{H}(\mathbf{curl}, D)$

$$[\mathbf{w}, \mathbf{w}']_{\partial D} \;=\; \int_{\partial D} (\mathbf{w} \times \mathbf{w}') \cdot \mathbf{n}. \tag{2.14}$$

This result characterizes the set of all self-adjoint extension but remain abstract. However, it reveals that there exists an infinity of self-adjoint **curl** operator (as many as the complete Lagrangians of the symplectic pairing $[\mathbf{w}, \mathbf{w}']_{\partial D}$).

2.4 Some examples of self-adjoint curl based on the Hodge decomposition

In order to exhibit some particular self-adjoint realizations of the **curl** operator, we perform a Hodge decomposition on ∂D. For all $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, D)$ we have

$$\gamma_t \mathbf{u} = \mathbf{grad}_{\partial D} \phi_{\mathbf{u}} + (\mathbf{grad}_{\partial D} \psi_{\mathbf{u}}) \times \mathbf{n} + \mathbf{h}_{\mathbf{u}} \tag{2.15}$$

with $\phi_{\mathbf{u}}$, and $\psi_{\mathbf{u}}$ scalar potentials defined ∂D and $\mathbf{h}_{\mathbf{u}}$ a tangent harmonic field of ∂D. This decomposition makes it possible to write the pairing $[\cdot,\cdot]$

$$[\mathbf{u}, \mathbf{v}] \;=\; \int_{\partial D} \mathbf{grad}_{\partial D} \phi_{\mathbf{u}} \cdot \mathbf{grad}_{\partial D} \psi_{\mathbf{v}} \;-\; \mathbf{grad}_{\partial D} \phi_{\mathbf{v}} \cdot \mathbf{grad}_{\partial D} \psi_{\mathbf{u}} \;+\; (\mathbf{h}_{\mathbf{u}} \times \mathbf{h}_{\mathbf{v}}) \cdot \mathbf{n}. \tag{2.16}$$

For a simply connected domain D, the harmonic part is vanishing $\mathbf{h}_{\mathbf{u}} = \mathbf{h}_{\mathbf{v}} = 0$ and the last expression can be simplified in

$$[\mathbf{u}, \mathbf{v}] \;=\; \int_{\partial D} \mathbf{grad}_{\partial D} \phi_{\mathbf{u}} \cdot \mathbf{grad}_{\partial D} \psi_{\mathbf{v}} \;-\; \mathbf{grad}_{\partial D} \phi_{\mathbf{v}} \cdot \mathbf{grad}_{\partial D} \psi_{\mathbf{u}}. \tag{2.17}$$

We deduce two examples of self-adjoint **curl** operators whose domains are given by

$$\mathbf{D}_s^{\mathrm{I}} = \Big\{\mathbf{u} \in \mathbf{H}(\mathbf{curl}, D) : \phi_{\mathbf{u}} = 0\Big\} \quad \text{and} \quad \mathbf{D}_s^{\mathrm{II}} = \Big\{\mathbf{u} \in \mathbf{H}(\mathbf{curl}, D) : \psi_{\mathbf{u}} = 0\Big\} \tag{2.18}$$

or equivalently in terms of surface operators

$$\mathbf{D}_s^{\mathrm{I}} = \Big\{\mathbf{u} \in \mathbf{H}(\mathbf{curl}, D) : \mathbf{curl}_{\partial D}(\gamma_t \mathbf{u}) = 0\Big\}. \tag{2.19}$$

[1] A complete lagrangian is a set which satisfies

$$\mathbf{L} = \{\mathbf{w} \in \mathbf{W} : [\mathbf{w}, \mathbf{w}']_{\partial D} = 0 \quad \forall \mathbf{w}' \in \mathbf{L}\} \tag{2.13}$$

$$\mathbf{D}_s^{\mathrm{II}} = \Big\{ \mathbf{u} \in \mathbf{H}(\mathbf{curl}, D) : \mathrm{div}_{\partial D}(\gamma_t \mathbf{u}) = 0 \Big\}. \tag{2.20}$$

For a non trivial topology, the set of all tangent harmonic vector fields is a finite dimensional linear space which we denote by $\mathcal{H}^1(\partial D)$. Similarly, two examples of self-adjoint realizations of the **curl** operator are given by

$$\mathbf{D}_s^{\mathrm{I}} = \Big\{ \mathbf{u} \in \mathbf{H}(\mathbf{curl}, D) : \mathbf{curl}_{\partial D}(\gamma_t \mathbf{u}) = 0 \text{ et } \mathbf{h}_{\mathbf{u}} \in L_{\mathcal{H}} \Big\}, \tag{2.21}$$

$$\mathbf{D}_s^{\mathrm{II}} = \Big\{ \mathbf{u} \in \mathbf{H}(\mathbf{curl}, D) : \mathrm{div}_{\partial D}(\gamma_t \mathbf{u}) = 0 \text{ et } \mathbf{h}_{\mathbf{u}} \in L_{\mathcal{H}} \Big\}. \tag{2.22}$$

with $L_{\mathcal{H}}$ a complete Lagrangian of the symplectic pairing $[\cdot,\cdot]_{\partial D}$ restricted to $\mathcal{H}^1(\partial D)$. Finally, we explain what are the complete lagrangians of $\mathcal{H}^1(\partial D)$ with classical algebraic topology.

2.5 Relation between curl and curlcurl operators

It is well known that if R is a closed **curl** operator then $\mathsf{R}\mathsf{R}^*$ is a self-adjoint **curlcurl** operator. We show that conversely, there exists self-adjoint **curlcurl** operators which can not be put under the form $\mathsf{R}_1\mathsf{R}_2$ with R_1 and R_2 two closed **curl** operators. Therefore, the **curlcurl** eigenvalue solver classically used in electromagnetism could not be used to obtain the eigenfunctions of the **curl** operator.