Analysis, estimation and control of nonlinear oscillations

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Analyse, estimation et contrôle des phénomènes oscillatoires non linéaires

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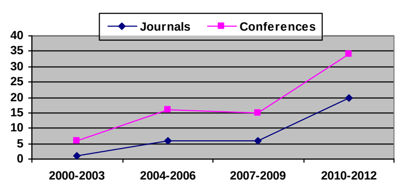
Summary

The thesis is presented in order to obtain the "Habilitation à diriger des recherches" degree in automatics and industrial informatics.

The author received the Ph.D. degree in Automatic Control from the Saint-Petersburg State Electrical Engineering University (Russia) in 2001, and the Dr.Sc. degree in Automatic control in 2006 from Institute for Problems of Mechanical Engineering RAS (Saint-Petersburg, Russia). From 2000 to 2009 he was research fellow of the Institute for Problems of Mechanical Engineering RAS, Control of Complex Systems Laboratory. From 2006 to 2011 he was working with the LSS (Supelec, France), the Montefiore Institute (University of Liege, Belgium) and the Automatic control group at IMS lab (University of Bordeaux I, France). Since 2011 he joined the Non-A team at INRIA-LNE centre in Lille, France. He is a member of the IFAC TC on Adaptive and Learning Systems and a Senior member of IEEE. His main research interests include nonlinear oscillation analysis, observation and control, switched and nonlinear system stability. During his scientific carrier he published over 140 papers, books and technical reports, their distribution in time is given in the following chart:



The subject of the dissertation deals with the main research direction of the author: analysis, estimation and control in nonlinear oscillating systems. It is an emerging area of research touching many applicative domains. This field strives for new estimation and control algorithms since frequently, due to peculiarities of this type of systems, the conventional approaches do not provide solutions with a satisfactory performance. Some control and estimation solutions for oscillating systems proposed by the author are given in the thesis. The hybrid or/and supervisory systems method is selected as the basement of design of new tools for analysis, observation and control of nonlinear oscillations.

The first chapter of the thesis deals with presentation of scientific background and experience of the author. In the second chapter an approach to analysis of existence of oscillations is presented, an adaptive control algorithm for bifurcation control is briefly described, and an approach for entrainment of periodical systems based on PRC is introduced. In the third chapter the main planned future directions of research of the author are given. Appendices contain the full list of publications and the texts of four selected papers.

Acknowledgments

I would like to thank my *family*, for being tolerant during all these years of my "scientific work" and for permanent important assistance, and my *colleagues* and *friends* from Saint-Petersburg, Paris, Liege, Bordeaux, Lille and abroad, for the strong support, for fruitful ideas, suggestions, criticism and discussions, it is a great pleasure to share the workspace with You!

Contents

Notations

- The set of real numbers is denoted by $\mathbb{R}$ and the set of nonnegative real numbers by $\mathbb{R}_+$.
- A Lebesgue measurable signal $d : \mathbb{R}_+ \to \mathbb{R}^n$, $\mathbb{R}_+ = \{\tau \in \mathbb{R}, \tau \geq 0\}$ is an essentially bounded function of time $t \geq 0$ if $\|d\| = \text{ess sup}\{|d(t)|, t \geq 0\} < +\infty$, where $|\cdot|$ denotes Euclidean vector norm. The set of all such inputs d with the property $\|d\| < +\infty$ we will denote as L_∞.
- Norm of matrix A is calculated as the sum of absolute values of all its elements $|A|_1 = \sum_{i,j} |a_{ij}|$.
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. Continuous function $\chi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is from class $\mathcal{KL}$ if it is positive definite ($\chi(0, \cdot) = 0$) and non-decreasing in the first argument for any fixed second one, and it is strictly decreasing to zero in the second argument for any fixed first one.
- The notation $\overline{1, n}$ denotes the sequence of integers $1, ..., n$.
- For a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ the notation $DV(x)f(x)$, $x \in \mathbb{R}^n$ stands for directional derivative with respect to a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$. If the function V is Lipschitz continuous, then $DV(x)f(x)$ is stated for upper directional Dini derivative:
$$DV(x)f(x) = \lim_{t \to 0^+} \sup \frac{V(x + tf(x)) - V(x)}{t}.$$

CHAPTER 1

SCIENTIFIC BACKGROUND AND EXPERIENCE

In this chapter, my scientific background, the main achievements and scientific carrier are expressed shortly. Next, in the second chapter, the main results obtained in the direction of analysis and control of nonlinear oscillations are presented with more details. Some planned directions of research are briefly described in the third chapter.

1.1. Diplomas and grades

I have obtained three scientific degrees in Russia, the last one is the full doctor of science degree, which is someway related with the "thèse d'état" (the subjects of that dissertation and the present HDR thesis are interconnected). The complete list of my theses is as follows:

- June **2006**: Doctor of engineering sciences
  Institute for problems of mechanical engineering, Saint-Petersburg, Russia
  Title: "Robust and adaptive control of nonlinear oscillations"
  Director: Prof. A.L. Fradkov
  Jury: Prof. G.A. Leonov, Prof. I.M. Ananievskiy, Prof. A.V. Timofeev
- September **2001**: PhD
  Saint-Petersburg State Electrical Engineering University, Saint- Petersburg, Russia
  Title: "Structure design of control systems with artificial neural networks"
  Director: Prof. V.A. Terekhov
  Jury: Prof. N.D. Polyahov, Prof. V.O. Nikiforov, Prof. A.L. Fradkov
- February **1998**: Engineer in systems control
  Saint-Petersburg State Electrical Engineering University, Saint- Petersburg, Russia
  Title: "Adaptive control with artificial neural networks"
  Director: Prof. V.A. Terekhov

1.2. Activity of research

During my scientific career I was working in 6 scientific centers. The years and the subjects are indicated below in the reverse order:

- **2011-present**: INRIA-LNE, Non-A project (supervisors Prof. J.-P. Richard and Prof. W. Perruquetti), France. CR1.
  Subject: Nonlinear and hybrid observation/estimation.
- **2009-2011**: CNRS-IMS, Université Bordeaux 1, France. Post-doc.
  Subject: Nonlinear observation/estimation, model based fault detection, fault tolerant control for nonlinear systems, aerospace applications.
  Supervisor: Prof. A. Zolghadri
- **2007–2009**: Systems and Control, Department of Electrical Engineering and Computer Science, Université de Liège, Belgium. Post-doc.
  Subject: Control of nonlinear oscillations.
  Supervisor: Prof. R. Sepulchre
- **2006–2007**: LSS, SUPELEC, Paris, France. Post-doc.
  Subject: Adaptive and robust synchronization.
  Supervisor: Prof. E. Panteley
- **2000–2006**: Laboratory of Control of Complex Systems, Institute for Problems of Mechanical Engineering, Saint-Petersburg. Senior researcher.
  Subject: Robust and adaptive control of nonlinear oscillations.
  Supervisor: Prof. A.L. Fradkov
- **1995–2001**: Department of Automation and Control Processes, Saint-Petersburg State Electrical Engineering University. Researcher.
  Subject: Control laws with adaptive neural networks.
  Supervisor: Prof. V.A. Terekhov

1.3. Industry contacts, development activities

In my career I have participating in four projects dealing with industrial collaboration and development. Two of them were related with aerospace applications, one with regulation of car engines, and one with adjustment and training of artificial networks for recognition of characters:

(1) **2009–2012**: Participant in FP7 ADDSAFE project
Subject: Advanced Fault Diagnosis for Safer Flight Guidance and Control
Partners:
* AIRBUS France SAS (France), DEIMOS Space (Spain)
* University of Leicester, Delft University of Technology, Deutsches Zentrum für Luft- und-Raumfahrt E.V., University of Hull, Computer and Automation Research Institute
Supervisor: Prof. A. Zolghadri

The state-of-practice for aircraft manufacturers is to diagnose guidance & control faults and to obtain the full flight envelope protection at all operation times. It is required to provide a high level of hardware redundancy (in order to perform coherency tests) and to ensure a sufficient available control action. This approach for Fault Detection and Diagnosis (FDD) based on hardware-redundancy fits also into current aircraft certification processes while ensuring the highest level of safety standards. However, these FDD solutions increase the aircraft weight and complexity and, as a consequence, its manufacturing and maintenance costs. In addition, its applicability becomes increasingly problematic being in conjunction with the many innovative solutions developed by the aeronautical sector towards achieving the future "sustainable" (more affordable, safer, cleaner and quieter) aircrafts.

This applicability gap has resulted in a "fault diagnosis bottleneck", i.e. a technological barrier constraining the full realization of the next generation of air transport. Since for the latter one needs to ensure the highest levels of aircraft safety when implementing novel green and efficient technologies.

The ADDSAFE project addressed the FDD challenges arising from this "fault diagnosis bottleneck". The overall aim was to research and develop model-based FDD methods for faults in aircraft flight control systems (sensor and actuator malfunctions). Highlighting the link between aircraft sustainability and FDD, it can be demonstrated that improving the fault diagnosis performance in flight control systems optimizes the aircraft structural design, for example, which results in weight saving and in its turn helps to progress the aircraft performance and to decrease its environmental footprint. The ADDSAFE project tried to overcome the technological gap in aircraft FDD by facing the following three challenges:

1. Helping the scientific community to develop the best suited FDD methods capable of handling the real world problems faced in the aircraft diagnosis.

2. Ensuring acceptance and widespread use of these advanced theoretical methods by the aircraft industry.

3. Contribute towards reducing the costs of aircraft development and maintenance by using model-based diagnostic systems in conjunction with reliable software verification & validation tools.

I was participating in the part dealing with the Oscillatory Failure Case (OFC) detection (see the papers [**EZ12, EZR11**]). Some failures located in the electrical flight control system may result in an unwanted control surface oscillation. This particular kind of failure is called OFC. OFCs have an influence on structural loads, aeroelasticity and controllability when located within the actuator bandwidth. OFC amplitude must be contained, by system design, within an envelope function of the frequency. So, the capability to detect these failures is very important because it has an impact on the structural design of the aircraft. By proper design the OFC amplitude must be maintained within an envelope function of the frequency. Usual monitoring techniques cannot always guarantee staying within an envelope with acceptable robustness, thus a specific OFC detection algorithm has to be designed [**Gou10**].

(2) **2009–2010**: Participant in SIRASAS project
Subject: Robust and Innovative Strategies for Autonomy of Aeronautics and Space Systems.
Supervisor: Prof. A. Zolghadri
The overall objective of the SIRASAS project was to increase autonomy and operational effectiveness of aerospace systems. In the spatial domain, the goal was to reduce the need for regular monitoring by operators on the ground, and to equip the space systems with capabilities for an autonomous FDD. In aviation, the objective was in the context of global optimization for a new generation of civil aircrafts: to reduce the cognitive load of piloting and to improve performance while maintaining robustness compatible with the operational constraints. The obtained results are given in [**EZ12, EZ11, EZS10, EZS11**], they were also related to OFC.

(3) **2002–2012**: Project executive with GENERAL MOTORS.
Subject: Hybrid, robust and adaptive control of internal combustion engines.
Supervisor: Prof. V.O. Nikiforov
The project is devoted to application of different theoretical methods for control and estimation in different spark ignition engines. The problems of simultaneous regulation of the air-to-fuel ratio (AFR) and the engine torque moment tracking are considered. It is required to design a model starting from a dataset of real measurements (model can be analytical and/or approximated), next to build a corresponding observer for measurements of internal variables and a controller. The field of AFR regulation constitutes one of the main engine control problems and is originated by the growing ecological requirements on engine characteristics. The ecological cleanliness of engines is maintained by the three-way catalytic converter, which oxidizes HC and CO and reduces NOx species. However,

efficiency of the converter is guaranteed if AFR is close to the stoichiometric value (and the conversion efficiencies of the converter are significantly reduced away from the stoichiometry). This is why the primary objective of the AFR control system is to track the fuel injection in stoichiometric proportion to the ingested air flow.

Each year the obtained solutions are verified in real experiments for different mass production vehicles (Chevrolet Corvette, Chevrolet Tahoe, GMC Yukon and Chevrolet Equinox). Among theoretical methods used for design in this project it is necessary to mention: the hybrid and supervisory systems approach, the adaptive control method, the robust control approaches, the interval estimation approach and the iterative learning techniques. All these results have been published in technical reports and some of them can be found in [**EJN10b, EJN10a, GJEN10a, GJEN10b, KEJ+12**].

(4) **2001–2006**: Engineer in A2iA Corp., Paris.
Subject: Intelligent adaptation of character recognizers, adaptive tuning of neural networks.
Supervisor: Prof. N.D. Gorski
The A2iA Corporation is a worldwide leading developer of natural handwriting recognition, Intelligent Word Recognition and Intelligent Character Recognition technologies and products for the payment, mail, document and forms processing (for a detailed description visit http://www.a2ia.com). I was responsible for retraining of Intelligent Character Recognizers, composed by several artificial neural networks, their adaptation for new countries and new symbols.

1.4. Teaching activities

I performed a limited teaching activity in Saint-Petersburg State Electrical Engineering University (during the PhD thesis work) and in Bordeaux 1:

- **1999**: Adaptive control in technical systems, Saint-Petersburg State Electrical Engineering University, 26 hours, course, laboratory and practical works.
- **1999-2000**: Application of artificial neural networks for control problems, Saint-Petersburg State Electrical Engineering University, 32 hours, course, laboratory and practical works.
- **2000**: Local control systems, Saint-Petersburg State Electrical Engineering University, 17 hours, laboratory and practical works.
- **2010-2011**: C language, IUT Bordeaux, 48 hours, laboratory works.

- **2011**: Adaptive control course for the Ecole Doctorale Systèmes - EDSYS – Toulouse, 5 hours.

1.5. Supervising activities

I have been participating in several juries for PhD defense and in supervision of visiting master/PhD students and post-docs:

- Kremlev A.S. Development of adaptive and robust approaches for disturbances compensation of a finite dimension. ITMO University, Saint-Petersburg, Official opponent, 2005.
- Migush S. Adaptive control of internal combustion engines. ITMO University, Saint-Petersburg, Official opponent, 2005.
- supervision of two students (master level, Stanislav Chebotarev and Peter Semenov) and two researchers (post-doc level, Sergey Chepinskiy and Anton Pyrkin) from ITMO University, Saint-Petersburg during their visits of IMS, Bordeaux and INRIA-LNE, Lille in October 2010, May 2011 and June 2012.
- supervision of PhD student Hector Rios from Universidad Nacional Autónoma de México (UNAM), Mexico, September - November 2011 and 2012.
- supervision of two students "Master EEA 2 recherche", direction "Automatique, Productique, Signal", record "Automatique et Mécatronique, Automobile, Aéronautique et Spatial (AM2AS)", IMS, Université Bordeaux 1, 2010.
- a supervisor (with Prof. W. Perruquetti and Dr. E. Moulay) of PhD student Emmanuel Bernuau (LAGIS, EC Lille), 2011-2013.
- a supervisor (with Prof. W. Perruquetti) of post-doc Andrei Polyakov (INRIA-LNE, Chaslim ANR project), 2012-2013.
- a supervisor (with Prof. W. Perruquetti) of post-doc Antonio Estrada (INRIA-LNE), 2012-2014.
- a supervisor (with Prof. W. Perruquetti and Dr. G. Zheng) of PhD student Matteo Guerra (LAGIS, EC Lille), 2012-2015.
- a supervisor (with Prof. W. Perruquetti and Dr. G. Zheng) of PhD student Zilong Shao (LAGIS, EC Lille), 2012-2015.

1.6. Prizes and distinctions

Several personal grants have been collected in Russia:

- Russian Science Support Foundation: 2006, 2005, 2004.
- Presidium of Russian Academy of Science Program № 19: 2005, 2004.
- Russian Fond of Basic Research: 2003.

- Grants of Saint-Petersburg government, Ministry of Common and Professional Education, rector of "Saint-Petersburg State Electr. Engineering University" and Russian Academy of Sciences in the framework of support of young researchers: 1999, 2000, 2001.

I also won several personal scientific competitions for young scientists:

- prize of "Academia Europaea", Moscow, 2007.
- prize of "Main Scientific Research Institute *Electropribor*", Saint-Petersburg. Section "Theory and Control Systems", 2000, 2001.
- The Baltic Olympiad of Automation Control (BOAC). Saint-Petersburg, 1996.

I have participated in IPCs for several IEEE and IFAC conferences:

- 2011, IFAC World Congress, Milan, Italy
- 2010, IFAC ALCOSP, Antalya, Turkey
- 2009, IEEE Conference on Control Applications, Saint-Petersburg, Russia
- 2008-2012, IASTED Conference on Control Applications, Vancouver, Canada; Cambridge, UK

I am a permanent reviewer for many major international journals in the Control domain, like IEEE Trans. Automatic Control, Automatica, Systems&Control Letters, SIAM Journal on Optimization and Control, AMS Review, Int. Journal of Robust and Nonlinear Control, Int. Journal of Adaptive Control and Signal Processing. I also was involved in the reviewing process for all main control conferences the last years (like IFAC World Congress 2005, 2008, 2011; IEEE CDC 2006-2012; ECC 2003, 2005, 2007, 2009; ACC 2008-2012). Participating in the conferences I was chairing the sessions on CDC 2007, 2006; IFAC World Congress 2011, 2008, 2005; IFAC Symposium on Nonlinear Control Systems 2010, 2004; ECC 2009, 2007, 2003.

Finally, I organized 2 invited sessions on IEEE CDC 2012 (Robust Estimation of Uncertain Systems I, Robust Estimation of Uncertain Systems II).

I was invited with seminars to several European and American scientific centers:

- University of Bremen, Germany, 2010
- Ecole Centrale de Lille, France, 2009
- GM R&D Center, Warren, USA, 2008, 2011, 2012
- University of Liege, Belgium, 2007
- University of Stuttgart, Germany, 2007
- IPME RAS, Saint-Petersburg, Russia, 2006
- IPM RAS, Moscow, Russia, 2006
- SUPELEC, France, 2005.

I am a Senior member of IEEE and a member of IFAC Technical Committee 1.2 on Adaptive and Learning Systems.

1.7. Scientific publications

The complete list of my publications is given in Appendix A. Some selected papers dealing with the subject of HDR work are included in appendices B-E. Below a classification by subjects of the papers published in the peer reviewed international journals is given (the numbers of works correspond to the *subsection A.1*):

Interval observers:	[1, 2, 6, 11]
Oscillations:	[4, 5, 15, 22, 25, 26, 27, 31]
Fault detection, isolation and compensation:	[3, 8, 9, 14]
Hybrid/supervisory estimation and control:	[7, 13, 16, 17, 18, 29]
Adaptive/robust estimation and control:	[21, 23, 24, 28, 32]
Spark ignition engine control:	[10, 19, 20]

Applications to oscillatory systems are considered in the papers [7, 14, 21, 23, 24, 28, 32]. The list of the subjects selected for the classification also describes my main scientific interests. The papers [12, 30, 33] are not included in this classification since they deal with another subjects.

Thus analysis, observation and regulation of oscillating systems are the main directions of my scientific work, that is why this subject is selected for the present HDR thesis. The papers on oscillating systems can be partitioned as follows:

Analysis of oscillations:	[4, 22, 25, 26, 31]
Estimation:	[7, 14, 27, 28, 32]
Control:	[5, 21, 23, 24]
Phase resetting:	[15]

This classification and contents of the corresponding papers mainly defines the structure of the next chapter.

CHAPTER 2

ANALYSIS, ESTIMATION AND CONTROL OF OSCILLATIONS

This chapter is devoted to introduction of some of my past research activities dealing with oscillating systems.

On the present stage of development of the theory of automatic control the role of control and estimation problems for nonlinear oscillatory processes grows. It is connected, first of all, with an active development of new areas of practical applications, like active vibration control (not only vibration cancellation, but also excitation), control of engineering systems in bifurcation modes, control and observation of open/interacting physical and biological systems, robotic control and estimation. Increase of requirements to quality of transients in the traditional fields of application of the theory of synthesis of nonlinear oscillatory systems (such as electrical engineering and robotics) results in necessity of development of new methods of design of oscillatory systems. The solutions are complicated since it is required to take into account the uncertain conditions of functioning and parametric incertitude.

For example, such a situation arises in the construction of resonant vibrating machines [**EF07a, EF07b, EF07c**], or in resonance entrainment of robotic mechanical systems [**EFI12b**], which operate on a natural frequency in order to minimize the energy consumption (see Fig. (1) for an illustration).

The overall performance of vibrating machines are largely defined by the frequency of oscillations of the excitation unit. Therefore, the most effective are resonant machines, in which the excitation unit operates on a resonant frequency of the

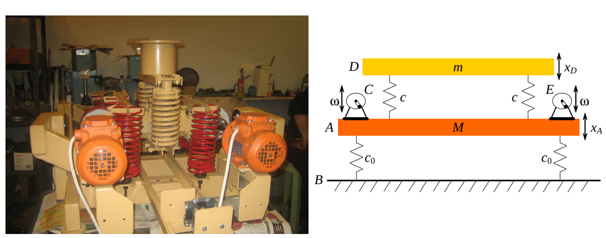


FIGURE 1. Vibrating platform setup in IPME

machine. The principle of work of such machines is based on the phenomenon of resonance for oscillatory system, when the periodic influence on a properly selected frequency maximizes the amplitude of the system response. On the contrary, in this mode any given oscillation amplitude is reached at a minimal power consumed by the actuator. At resonance, the amplitude of oscillations is very sensitive to the exciting force frequency, while the resonant frequency itself is a complex function of the system parameters. Therefore, any external disturbance or a deviation of the values of parameters from the nominal result in a divergence of the system from the resonant operating mode, which increases the energy losses by the excitation unit. Since external disturbances or parameter variations are usual conditions of functioning for real world applications, the problem of resonant control of vibration machines (or robot locomotion) under such conditions are important theoretical challenges.

The phenomena of non-linear oscillations covers a wide plenty of possible behaviors of dynamic systems: from periodic or harmonious oscillations to recurrent and random movements. There exist a lot of theories proposed for definition, description and analysis of nonlinear oscillations [**AVK66, FP98, LBS95, MCB03**]. An important and practically useful approach to studying complex oscillatory modes of movement is based on the concept of oscillations introduced in 1973 by V.A. Yakubovich [**Yak73**]. This approach allows him to receive the frequency domain conditions of existence of oscillations for the class of Lurie systems (which consist of the nominal linear part closed by a nonlinear output feedback) and some conditions of oscillations in discontinuous systems [**Yak75, YT89**]. Extensions of that theory on completely nonlinear systems and time-delay nonlinear systems have been developed in [**EF09a, EF07d**] (the full text of the paper [**EF09a**] is given in Appendix B), some interesting applications of the Yakubovich's theory to biological system has been presented in [**EF08a**].

An attractive extension of the Yakubovich's conditions of oscillations has been recently proposed in [**EP10**], which is based on the homogeneity theory [**BB05, Ros92**]. The homogeneity is a propriety of nonlinear dynamical systems introduced more than thirty years ago [**RS76**] meaning that the state vector rescaling does not change the system behavior. Thus, the behavior of the system trajectories on a suitably defined sphere around the origin can be extended to the whole state space. The global behavior of homogeneous systems simplifies their analysis, however this property restricts the approach applicability since the most nonlinear systems (by definition) have different types of behavior depending on the state space region. To overcome this issue, the papers [**APA08, EP10**] introduce into consideration the local homogeneity notion (illustrated by the bi-limit homogeneity in [**APA08**]), which is the existence of a homogeneous approximating (dynamical) system that coincides with the original nonlinear system on a compact set or a sphere. It has been shown that locally around the sphere the stability/instability property of the approximating dynamics is inherited by the original system (and *vice versa* for a sphere with a finite and nonzero radius). A universal formula to design of such homogeneous approximating dynamics has been proposed in [**EP10**]. Since the approximating dynamics is homogeneous its stability/instability can be checked using the first order approximation at the origin or using a homogeneous Lyapunov/Chetaev function [**Ros92, APA08, EP10**]. Therefore, to analyze the stability behavior of a nonlinear system around a sphere one can apply the linearization approach at the origin to an auxiliary approximating dynamical system. Such an ability becomes very useful for analysis of oscillations, then establishing for the corresponding homogeneous approximations their stability/instability it is possible to detect an oscillating regime presence [**EP10**]. This technique can be also applied for control design in order to create the desired oscillating or chaotic trajectories in the system [**EP11**]. A short introduction to this theory is given in section 2.1.

In general, for nonlinear systems the questions of analysis and synthesis of oscillatory modes are incompletely investigated. Appearance of oscillatory modes is usually connected with approaching of values of the system parameters to a bifurcation point, which results in creation of oscillations or instability. The problem of bifurcation control is a modern direction of the control theory [**AF86, CHY03**]. One of the main problems in this area consists in a complex dependence of the control law coefficients on parameters of the plant. In actual practice, the values of these parameters are different from ones used at analytical calculation of the control law. The bifurcation or resonant property of a nonlinear system are very sensitive to small changes of parameters (even small error at calculation of parameters of the control unit may result in a significant deviation in the system behavior from the desired one). Moreover, a nonlinear system in bifurcation point is on its stability limit, and a small error in the values of control coefficients can lead to an unstable behavior of the system. To overcome this issue it is possible to use methods of adaptive control for on-line adjustment of control parameters, for tuning the system to a bifurcation or resonant mode with desired properties. This problem has been solved in [**EF06**], it is based on a special adaptive observer design. Next, these adaptive observers have been applied in [**Efi06, EF09c**] for dynamical synchronization and stabilization of nonlinear oscillating systems (the text of paper [**EF09c**] is included in Appendix C). The main facts about the adaptive control of bifurcations from [**EF06**] are given in section 2.2.

The problems of adaptive and robust stabilization of oscillating nonlinear systems in the presence of exogenous disturbances and uncertain parameters are studied in [**EF08b, EF09b**]. These results are based on the Input-to-State Stability (ISS) theory (see a recent survey [**DES11**] for its introduction) and, in particular, on the input-to-output stability property, which enters in the family of ISS properties. The idea is that if a system has an attracting oscillating mode, then the oscillating trajectories belong to a compact forward invariant set in the state space of the system. It is possible to introduce an auxiliary output to the system indicating this set. In this case the problem of the oscillating regime stabilization can be formalized as the problem of this output stabilization. To analyze robustness with respect to external disturbances (for stabilization of the oscillating mode) it is possible to use the input-to-output stability property. The stability property in the system can be provided using different design approaches (like backstepping, forwarding, method of Control Lyapunov Function (CLF), passivation, for example). A development of the backstepping method for input-to-output stabilization of nonlinear dynamical systems is given in [**EF09b**] (the CLF method has been extended in [**Efi02b**]). An extension of the direct adaptive control method to nonlinear systems containing a parametric incertitude for input-to-output stabilization is presented in [**EF08b**]. A further development of these control methods (and the passivation approach) to the lattices of oscillators is presented in a recent work [**EF12**], which text is included in Appendix D.

There exists a branch of methods dealing with control and estimation of periodical systems [**BEPZ12, PBEZ11, Efi08**]. Some particular control problems arise in biological applications. Many biological and technical systems perform periodical movements (circadian oscillations in nature or satellite on an orbit, for instance). Trajectories of such systems form a limit cycle in its state space. The problem of phase resetting in this case is equivalent to a controlled position shift on the limit cycle on period of oscillations (for circadian oscillators this corresponds to organism adaptation to a new light environmental conditions, or to the rendezvous problem solution for satellites). In [**ESS09**] this problem has been solved applying Phase Response Curve (PRC) approach (well known modeling technique in chronobiology) with infinitesimal controls. A short introduction in the PRC method application for phase dynamics modeling and control in nonlinear oscillators is given in section 2.3. In [**Efi11**] the PRC approach has been extended for modeling and control with inputs of arbitrary amplitude (the text of that work is added in Appendix E).

The relevance of the topic selected for the dissertation work is confirmed by necessity of development of methods of robust/adaptive control and estimation for oscillatory plants in the conditions of incompleteness of *a priori* and on-line information about the plant model and the values of internal variables, and external environmental conditions of functioning as well. The objective of the developed methods consists in quality improvement of transient processes, simplicity of design and in minimalism of applicability conditions.

For achievement of this objective the following problems have been considered:

(1) The conditions of existence of oscillations in the sense of Yakubovich for nonlinear dynamic systems [**EF09a, EF08a, EF07d, EP10**].
(2) The methods of robust control of nonlinear oscillating systems:
   (a) method of design of oscillating modes based on the concept of oscillations in the sense of Yakubovich and the homogeneity theory [**EF09a, EP11**];
   (b) method of CLF [**Efi02b**];
   (c) backstepping method [**EF09b**];
   (d) wave regulation for the lattices of oscillators [**EF12**];
   (e) phase resetting [**ESS09, Efi11**].
(3) The method of direct adaptive control [**EF08b**].
(4) Method of adaptive tuning to bifurcations, adaptive observer design for oscillating systems and adaptive synchronization [**EF06, Efi06, EF09c**].

Application of the developed apparatus guarantees realization of a given control goal in the presence of parametric and signal uncertainties with partial noisy measurements.

2.1. Yakubovich's conditions of oscillation existence via homogeneity approach

In this section the problem of oscillation detection for nonlinear systems is addressed. The notions of homogeneity in the bi-limit [**APA08**] and local homogeneity [**EP10**] (the homogeneity in the *multi-limit*) are introduced. Some sufficient conditions of oscillation existence for systems homogeneous in multi-limit are presented. The proposed approach allows one to estimate the number of oscillating modes and the regions of their location. Efficiency of the technique is demonstrated on several examples.

2.1.1. Introduction. The homogeneity is a propriety of nonlinear dynamical systems introduced more than thirty years ago [**RS76**] meaning that the state vector rescaling does not change the system behavior. Thus, the behavior of the system trajectories on a suitably defined sphere around the origin can be extended to the whole state space. This property is used for stability analysis [**Ros92, APA08, BR01, Her91a, Hon02**], systems approximation [**Her91b**], stabilization [**BB05, Gu00, Kaw91, MP06, Pra97, SA96**] and observation

[**APA08**]. In the work [**APA08**] the homogeneity in the bi-limit has been introduced, which is homogeneity with different weights and approximating functions at a vicinity of the origin and far outside. This notion has been extended to local homogeneity in [**EP10**], the definition of this notion will be introduced in the next subsection.

The conditions of oscillations in the sense of Yakubovich proposed in [**EF09a**] are based on existence of two Lyapunov functions. The first Lyapunov function ensures local instability of the origin, while the second Lyapunov function provides global boundedness of the system trajectories, which under some additional mild conditions implies existence of oscillations. Such existence of two Lyapunov functions nicely interacts with homogeneity in the bi-limit: in both cases two subspaces of the system operation are considered separately. This observation served as a motivation for development of [**EP10**], which proposes conditions of stability and instability for homogeneous systems (peculiarity of oscillating systems is that the instability around the origin is required). In addition, the conditions are developed establishing the connection between stability/instability properties of the original nonlinear system and its local approximating dynamics. The obtained stability/instability conditions have been used to formulate conditions of oscillating trajectories existence (the regions of oscillations in the state space are also estimated by the approach). These new conditions of oscillation existence relax the conservatism of conditions from [**EF09a**] extending them to the case of existence of several oscillating zones, when the system may be asymptotically stable around the origin and at infinity with instability regions among them. As a side of results, the necessary and sufficient instability conditions in terms of existence of Lyapunov or Chetaev functions have been proposed for homogeneous systems. Formulation of these results is given in the next subsections.

To highlight importance of local homogeneity recall that analysis (global) of nonlinear dynamical systems is a hardly solving problem, that is why a local or approximate analysis is very useful and appreciated in applications. The linearization approach allows one to make a conclusion on the system behavior around a trajectory. The local homogeneity gives the same conclusion, but for a sphere with specified radius. It is shown that if the system is locally homogeneous and the approximating dynamics is stable/unstable, then the original system on the sphere has the same property. For nonlinear systems there is no method to choose Lyapunov functions. Using local homogeneity notion this problem can be seriously simplified. Indeed, if the approximating dynamics is homogeneous, then it is well known that the homogeneous systems possess homogeneous Lyapunov functions. The homogeneous Lyapunov function has the same shape on the homogeneous norm contours (a polynomial function of this norm, for example). The proposed results show that the original system admits locally this Lyapunov function.

The outline of this section is as follows. The homogeneity and the oscillatory properties are introduced in subsection 2.1.2. Formulation of the main results of the homogeneity approach application for oscillation detection from [**EP10**] are presented in subsection 2.13. Some applications of the proposed approach are discussed in subsection 2.1.4.

2.1.2. Preliminaries. Consider the nonlinear dynamical system:

$$\dot{x} = f(x), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$ is a nonlinear function ensuring existence and uniqueness of the system (1) solutions (for any initial conditions $x_0 \in \mathbb{R}^n$ the solution $x(t, x_0)$ of the system (1) is defined at least locally for $t \leq T$, further we will simply write $x(t)$ if origin of initial conditions is clear from the context). If for all initial conditions $x_0 \in \mathbb{R}^n$ the solutions are defined for all $t \geq 0$ then the system (1) is called **forward complete**.

A set $A \subset \mathbb{R}^n$ is called **forward invariant** for the forward complete system (1) if for all $x_0 \in A$ the property $x(t, x_0) \in A$ holds for all $t \geq 0$; the set $A \subset \mathbb{R}^n$ is called **backward invariant** if for all $x_0 \in A$ the property $x(t, x_0) \in A$ holds for all $t \leq 0$; this set A is called **invariant** if it is simultaneously forward and backward invariant.

The system (1) is called locally or globally asymptotically stable (at the origin) if the standard conditions are satisfied [**Kha02**]. The asymptotic stability of the system (1) with respect to an invariant set is treated in the sense of [**LSW96**].

2.1.2.1. *Homogeneity.* For any $r_i > 0$, $i = \overline{1, n}$ and define the dilation matrix and the vector of weights $r = [r_1 ... r_n]^T$. For any $r_i > 0$, $i = \overline{1, n}$ a homogeneous norm can be defined as follows

$$|x|_r = \left(\sum_{i=1}^{n} |x_i|^{\rho / r_i} \right)^{1/\rho}, \rho > 0.$$

For any $x \in \mathbb{R}^n$ a homogeneous norm has to be positive definite and to admit an important property that is: $|\Lambda_r x|_r = \lambda |x|_r$. For all $x \in \mathbb{R}^n$, its Euclidean norm $|x|$ is related to the homogeneous norm through two functions $\underline{\sigma}_r, \bar{\sigma}_r \in K_\infty$:

$$\underline{\sigma}_r(|x|_r) \leq |x| \leq \bar{\sigma}_r(|x|_r),$$

the functions $\underline{\sigma}_r$, $\bar{\sigma}_r$ define the Euclidean norm deviations with respect to the homogeneous norm. Define

$$S_r = \{x \in \mathbb{R}^n : |x|_r = 1\}.$$

Definition 1. *The function* $g : \mathbb{R}^n \to \mathbb{R}$ *is called* r*-homogeneous* ($r_i > 0$, $i = \overline{1,n}$) *if for any* $x \in \mathbb{R}^n$

$$g(\Lambda_r x) = \lambda^d g(x)$$

or some $d \geq 0$ *and all* $\lambda \geq 0$.

The system (1) *is called* r*-homogeneous* ($r_i > 0$, $i = \overline{1,n}$) *if for any* $x \in \mathbb{R}^n$

$$f(\Lambda_r x) = \lambda^d \Lambda_r f(x)$$

or some $d \geq -\min_{1\leq i\leq n} r_i$ *and all* $\lambda \geq 0$. □

Theorem 1. [**Ros92**] *For the system* (1) *with* r*-homogeneous and continuous function* $f : \mathbb{R}^n \to \mathbb{R}^n$ *the following properties are equivalent:*

- the system (1) *is (locally) asymptotically stable;*

- there exists continuously differentiable homogeneous Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ *such that for all* $x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), DV(x)f(x) \leq -\alpha(|x|), V(\Lambda_r x) = \lambda^k V(x), k \geq \max_{1\leq i\leq n} r_i,$$

for some $\alpha_1, \alpha_2 \in K_\infty$, $\alpha \in K$. □

Note, that the continuity of the function f is required for the necessary (converse) part only. The r-homogeneity property used in Definition 1 and Theorem 1 is introduced for some $r > 0$ and all $\lambda \geq 0$. Restricting the set of admissible values for λ we can introduce **local homogeneity** [**EP10**] (similarly to the 0-limit or the ∞-limit homogeneity of [**APA08**]).

Definition 2. *The function* $g : \mathbb{R}^n \to \mathbb{R}$ *is called* (r,λ_0,g_0)*-homogeneous* ($r_i > 0$, $i = \overline{1,n}$; $\lambda_0 \in \mathbb{R}_+ \cup \{+\infty\}$; $g_0 : \mathbb{R}^n \to \mathbb{R}$, $g_0(0) = 0$) *if for any* $x \in S_r$

$$\lim_{\lambda\to\lambda_0} \lambda^{-d_0} g(\Lambda_r x) - g_0(x) = 0$$

for some $d_0 \geq 0$ *(uniformly on* S_r *for* $\lambda_0 \in \{0, +\infty\}$).

The system (1) *is called* (r,λ_0,f_0)*-homogeneous* ($r_i > 0$, $i = \overline{1,n}$; $\lambda_0 \in \mathbb{R}_+ \cup \{+\infty\}$; $f_0 : \mathbb{R}^n \to \mathbb{R}^n$, $f_0(0) = 0$) *if for any* $x \in S_r$

$$\lim_{\lambda\to\lambda_0} \lambda^{-d_0} \Lambda_r^{-1} f(\Lambda_r x) - f_0(x) = 0$$

for some $d_0 \geq -\min_{1\leq i\leq n} r_i$ *(uniformly on* S_r *for* $\lambda_0 \in \{0, +\infty\}$). □

In the paper [**APA08**] this definition has been introduced for $\lambda_0 = 0$ and $\lambda_0 = +\infty$ (the function g is called homogeneous in the bi-limit if it is simultaneously (r_0,0,g_0)-homogeneous and (r_∞,$+\infty$,g_∞)-homogeneous), the case $\lambda_0 = 0$ has been also treated in [**Ros92, BR01**]. Note, that the system (1) can be also homogeneous in more than two limits (some examples are presented in subsection 2.1.4). In the following a function g (respectively system (1)) is **homogeneous in**

the multi-limit if there exist a finite number of triplets (r_i,λ_i,g_i (respectively f_i)) for which the function (respectively the system (1)) is (r_i,λ_i,g_i (respectively f_i)) locally homogeneous for each index i [**EP10**].

If the pairs of functions g, g_0 and f, f_0 are continuous, then for any compact set $X \subset \mathbb{R}^n$ and any $\varepsilon > 0$ there exist $\underline{\lambda}_\varepsilon \leq \lambda_0 \leq \bar{\lambda}_\varepsilon$ such that for all $\lambda \in (\underline{\lambda}_\varepsilon, \bar{\lambda}_\varepsilon)$:

$$\sup_{x \in S_r} |\lambda^{-d_0} g(\Lambda_r x) - g_0(x)| \leq \varepsilon, \sup_{x \in S_r} |\lambda^{-d_0} \Lambda_r^{-1} f(\Lambda_r x) - f_0(x)| \leq \varepsilon.$$

The coefficients $r_i > 0$, $i = \overline{1,n}$ are called weights, d_0 is the degree of homogeneity, f_0 or g_0 are the approximating functions.

The following formulas give an example for choice of locally approximating functions for any $0 < \lambda_0 < +\infty$ and $x \in S_r$:

$$g_0(x) = \lambda_0^{-d_0} g(\Lambda_{r,0} x), f_0(x) = \lambda_0^{-d_0} \Lambda_{r,0}^{-1} f(\Lambda_{r,0} x), \Lambda_{r,0} = \operatorname{diag}\{\lambda_0^{r_i}\}_{i=1}^n.$$

By construction, the limit relations from Definition 2 are satisfied for any $0 < \lambda_0 < +\infty$:

$$\lim_{\lambda \to \lambda_0} \lambda^{-d_0} g(\Lambda_r x) - g_0(x) = 0, \lim_{\lambda \to \lambda_0} \lambda^{-d_0} \Lambda_r^{-1} f(\Lambda_r x) - f_0(x) = 0.$$

Moreover, the approximating functions can be chosen homogeneous:

$$g_0(x) = |x|_r^d \lambda_0^{-d_0} g(\Lambda_{r,0} \Lambda_{|x|}^{-1} x), f_0(x) = |x|_r^d \lambda_0^{-d_0} \Lambda_{r,0}^{-1} f(\Lambda_{r,0} \Lambda_{|x|}^{-1} x),$$

where $\Lambda_{|x|} = \operatorname{diag}\{|x|_r^{r_i}\}_{i=1}^n$, provided that $g_0(0)$ and $f_0(0)$ are well defined. Straightforward calculations show that

$$g_0(\Lambda_r x) = \lambda^d |x|_r^d \lambda_0^{-d_0} g(\Lambda_{r,0} \Lambda_r^{-1} \Lambda_{|x|}^{-1} \Lambda_r x) = \lambda^d g_0(x),$$

$$f_0(\Lambda_r x) = \lambda^d |x|_r^d \lambda_0^{-d_0} \Lambda_{r,0}^{-1} f(\Lambda_{r,0} \Lambda_r^{-1} \Lambda_{|x|}^{-1} \Lambda_r x) = \lambda^d f_0(x). \tag{2}$$

The proposed formulas do not cover two limit cases of a great importance with $\lambda_0 = 0$ and $\lambda_0 = +\infty$. For the case $\lambda_0 = 0$ at least one variant of the approximating dynamics can be pointed out for differentiable functions g and f with the property $g(0) = 0$, $f(0) = 0$: $g_0(x) = g'(0)x$, $f_0(x) = f'(0)x$ for $r_i = 1$, $i = \overline{1,n}$ and $d_0 = 1$ (due to linearity the approximating functions are homogeneous). Indeed, in this case

$$g(\Lambda_r x) = \lambda\{\lambda^{-1}[g(0 + \lambda x) - g(0)]\}, f(\Lambda_r x) = \lambda\{\lambda^{-1}[f(0 + \lambda x) - f(0)]\}$$

and the required in Definition 2 limit relation holds for $\lambda \to 0$ since

$$\lim_{\lambda \to 0} \lambda^{-1}[g(0 + \lambda x) - g(0)] = g'(0)x, \lim_{\lambda \to 0} \lambda^{-1}[f(0 + \lambda x) - f(0)] = f'(0)x.$$

Thus in the equilibriums the local homogeneity approach always may provide results similar to the linearization technique. However, opposite to linearization (that is unique) the system may have several local homogeneous approximations, some of them may provide better information about the system properties. To illustrate this claim consider the system $\dot{x} = -x^3 + x^4 - x^5$. This system has an equilibrium at the origin, its linearization has form $\dot{x} = 0$, that does not give an inside look at the system behavior around the point $x = 0$. However, for $r = 1$ the system also has local homogeneous approximations $\dot{x} = -x^3$ for $d_0 = 3$, $\lambda_0 = 0$ and $\dot{x} = -x^5$ for $d_1 = 5$, $\lambda_1 = +\infty$. These systems are asymptotically stable. Recalling the results of two theorems presented below, we can conclude that the original system is also stable at the origin and far outside.

THEOREM 2. [**Ros92**] *Let the system* (1) *be* $(r,0,f_0)$*-homogeneous with the continuous functions* $f : \mathbb{R}^n \to \mathbb{R}^n$ *and* $f_0 : \mathbb{R}^n \to \mathbb{R}^n$. *If the system* $\dot{x} = f_0(x)$ *is (locally) asymptotically stable, then the system* (1) *is also locally asymptotically stable.* □

THEOREM 3. [**APA08**] *Let the system* (1) *be* $(r,+\infty,f_\infty)$*-homogeneous with the continuous functions* $f : \mathbb{R}^n \to \mathbb{R}^n$ *and* $f_\infty : \mathbb{R}^n \to \mathbb{R}^n$. *If the system* $\dot{x} = f_\infty(x)$ *is globally asymptotically stable, then there exists a compact invariant set* $X_\infty \subset \mathbb{R}^n$ *containing the origin such that the system* (1) *is globally asymptotically stable with respect to the set* X_∞. □

The theorems 2 and 3 present results on the system (1) stability derived from the corresponding properties of the approximating systems for $\lambda_0 = 0$ or $\lambda_0 = +\infty$. The converse Lyapunov theorem similar to Theorem 1 for the homogeneous in the bi-limit systems can be also found in [**APA08**].

2.1.2.2. *Conditions of oscillations.* The function $g : \mathbb{R}^n \to \mathbb{R}$ is called monotone if the conditions $x_1 \le x_1', \ldots, x_n \le x_n'$ imply that everywhere either $g(x_1, ..., x_n) \le g(x_1', ..., x_n')$ or $g(x_1, ..., x_n) \ge g(x_1', ..., x_n')$.

DEFINITION 3. [**Yak73, EF09a**] *The solution* $x(t, x_0)$ *with* $x_0 \in \mathbb{R}^n$ *of the system* (1) *is called* $[\pi^-, \pi^+]$*-oscillation with respect to the output* $\psi = \eta(x)$ *(where* $\eta : \mathbb{R}^n \to \mathbb{R}$ *is a continuous monotone function) if the solution is defined for all* $t \ge 0$ *and*

$$\underline{\lim}_{t\to+\infty} \psi(t) = \pi^-; \overline{\lim}_{t\to+\infty} \psi(t) = \pi^+; -\infty < \pi^- < \pi^+ < +\infty.$$

The solution $x(t, x_0)$ *with* $x_0 \in \mathbb{R}^n$ *of the system* (1) *is called* ***oscillating****, if there exist some output* ψ *and constants* π^-, π^+ *such that* $x(t, x_0)$ *is* $[\pi^-, \pi^+]$*-oscillation with respect to the output* ψ. *A forward complete system* (1) *is called* ***oscillatory****, if for almost all* $x_0 \in \mathbb{R}^n$ *the solutions* $x(t, x_0)$ *of the system are oscillating. An oscillatory system* (1) *is called* ***uniformly oscillatory****, if for almost all* $x_0 \in \mathbb{R}^n$

for corresponding solutions $x(t, x_0)$ there exist output ψ and constants π^-, π^+ not depending on initial conditions. □

In other words the solution $x(t, x_0)$ is oscillating if the output $\psi(t) = \eta(x(t, x_0))$ is asymptotically bounded and there is no single limit value of $\psi(t)$ for $t \to +\infty$. Note that the term "almost all solutions" is used to emphasize that generally the system (1) has a nonempty set of equilibrium points, thus, there exists a set of initial conditions with zero measure such that the corresponding solutions are not oscillating. It is worth to stress that the notion of oscillations in the sense of Yakubovich is rather generic. It includes periodical oscillations (limit cycles), quasi-periodical, recurrent and chaotic trajectories. The oscillating trajectories could be repelling being oscillating. The trajectories also could be unbounded, it is required to find a function of the state vector, that is bounded and admits certain requirements introduced in Definition 3. Despite its complexity this notion has Lyapunov characterization for a general nonlinear system.

THEOREM 4. [**EF09a**] *Let the system* (1) *have two locally Lipschitz continuous Lyapunov functions V_1 and V_2 fulfilling the following inequalities for all $x \in \mathbb{R}^n$:*

$$\upsilon_1(|x|) \le V_1(x) \le \upsilon_2(|x|), \upsilon_3(|x|) \le V_2(x) \le \upsilon_4(|x|), \upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4 \in \mathcal{K}_\infty,$$

and for some $0 < X_1 < \upsilon_1^{-1} \circ \upsilon_2 \circ \upsilon_3^{-1} \circ \upsilon_4(X_2) < +\infty$:

$$DV_1(x)f(x) > 0 \text{ for all } 0 < |x| < X_1 \text{ and } x \notin \Xi;$$
$$DV_2(x)f(x) < 0 \text{ for all } |x| > X_2 \text{ and } x \notin \Xi,$$

where $\Xi \subset \mathbb{R}^n$ is a set with zero Lebesgue measure containing all equilibriums of the system, and

$$\Omega \cap \Xi = \emptyset, \Omega = \{x : \upsilon_2^{-1} \circ \upsilon_1(X_1) \le |x| \le \upsilon_3^{-1} \circ \upsilon_4(X_2)\}.$$

Then the system (1) *is oscillatory.* □

The Lyapunov function for the linearized system (1) at the origin is a candidate for the function V_1 [**Yak75**]. Instead of the existence of the function V_2 one can require just boundedness of the system (1) solutions with known upper bound (if this fact could be verified using another approach not dealing with Lyapunov functions analysis).

THEOREM 5. [**EF09a**] *Let the system* (1) *be uniformly oscillatory with respect to the output $\psi = \eta(x)$ (where $\eta : \mathbb{R}^n \to \mathbb{R}$ is a continuous function) with some $-\infty < \pi^- < \pi^+ < +\infty$, and for all $x \in \mathbb{R}^n$ the relations $\chi_1(|x|) \le \eta(x) \le \chi_2(|x|)$, $\chi_1, \chi_2 \in \mathcal{K}_\infty$ are satisfied. Let the set of initial conditions, for which the system is not oscillating, consist in just one point $\Xi = \{0\}$. Then there exist two locally Lipschitz continuous Lyapunov functions $V_1 : \mathbb{R}^n \to \mathbb{R}_+$ and $V_2 : \mathbb{R}^n \to \mathbb{R}_+$ such* *that for all* $x \in \mathbb{R}^n$ *the inequalities hold:*

$$\upsilon_1(|x|) \leq V_1(x) \leq \upsilon_2(|x|),\ \upsilon_3(|x|) \leq V_2(x) \leq \upsilon_4(|x|),\ \upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4 \in \mathcal{K}_\infty;$$

$$DV_1(x)f(x) > 0 \text{ for all } 0 < |x| < \chi_2^{-1}(\pi^-);$$
$$DV_2(x)f(x) < 0 \text{ for all } |x| > \chi_1^{-1}(\pi^+).$$

□

The theorems 4 and 5 present the sufficient and necessary conditions for the system (1) to be oscillatory. Being rather simple these conditions can be useful in different applications [**EF09a, EF07d, EF08a**]. However, in some situations these conditions could be restrictive. For example, in sufficient part they need the knowledge of two Lyapunov functions for the system (1), that can be an ambitious requirement. Additionally, the conditions are oriented to the locally unstable origin case, however, a system with several limit cycles can have a locally stable origin. These shortages can be resolved applying homogeneity approach in [**EP10**], as it is shown below.

2.1.3. Homogeneity approach for oscillation detection. For the systems homogeneous in the bi-limit, the functions V_1 and V_2 can be chosen according to the corresponding approximations at the origin or at infinity. For this purpose the Lyapunov theorems for locally homogeneous unstable/stable systems can be developed and applied to detect presence of oscillations next.

Recall, that a *cone* in $\mathbb{R}^n$ is a set consisting of half-lines emanating from some point called the vertex of the cone, in other words the set $K \subset \mathbb{R}^n$ is a cone if $\lambda K \subset K$ for any $\lambda > 0$; denote by $\mathcal{L}b(K)$ the lateral borders of the cone K.

2.1.3.1. *Unstable homogeneous systems.* There exist three basic setups for instability analysis [**Che61, Shn07**]. First, the Lyapunov case or the case with anti-stable (strongly unstable) equilibrium, when the system linearization has all roots with positive real parts (this case is studied for homogeneous systems in Lemma 1 below). Second, the case when there is a cone with all trajectories exiting from the cone basement, this situation is considered applying Chetaev function in Lemma 2. Third, the case when the cone is repulsing for all trajectories, again this case is covered by Chetaev functions approach, the corresponding extension to homogeneous systems is presented in Lemma 3 below.

LEMMA 1. [**EP10**] *For the system* (1) *with* r*-homogeneous and continuous function* $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$ *the following properties are equivalent:*
- the system (1) *is (locally) strongly unstable, i.e. there exists* $\delta > 0$ *such that for any* $0 < |x_0| < \delta$ *there exists* $T_{x_0} > 0$ *such that* $|x(t, x_0)| > \delta$ *for all* $t \geq T_{x_0}$*;*
- there exists continuously differentiable r*-homogeneous Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *such that for all* $x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|),\ DV(x)f(x) \ge \alpha(|x|),$$
$$V(\Lambda_r x) = \lambda^k V(x),\ k \ge \max_{1\le i\le n} r_i,$$

for some $\alpha_1, \alpha_2 \in K_\infty$, $\alpha \in K$. □

This lemma can be viewed as an analogue of Theorem 1 for the case of local instability of the homogeneous system (1) at the origin. It deals with the Lyapunov instability theorem. The following results present a similar extension of Chetaev theorems.

LEMMA 2. [**EP10**] *Consider the forward complete system* (1) *with a Lipschitz continuous function* $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$ *and the set* $B_\delta = \{x \in \mathbb{R}^n : 0 < |x| < \delta, x \in K\}$ *for some* $\delta > 0$ *with all points* $\mathcal{L}b(K) \cap B_\delta$ *being the points of entry of trajectories into* B_δ. *Then the following properties are equivalent:*
- for any $x_0 \in B_\delta$ *there exists* $T_{x_0} > 0$ *such that* $|x(t, x_0)| > \delta$ *for* $t \ge T_{x_0}$ *and* $x(t, x_0) \in K$ *for all* $t \in [0, T_{x_0}]$ *(i.e. the system* (1) *is locally unstable);*
- there exists a continuously differentiable Chetaev function $V : \mathbb{R}^n \to \mathbb{R}_+$ *such that for all* $x \in B_\delta \cup \{0\}$*:*

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|), DV(x)f(x) \ge \alpha_3(|x|),$$

for some $\alpha_1, \alpha_2 \in K_\infty$, $\alpha_3 \in K$.

For the system (1) *with* r*-homogeneous and continuous function* f *the Chetaev function* V *has the same property:* $V(\Lambda_r x) = \lambda^k V(x)$, $k \ge \max_{1\le i\le n} r_i$. □

LEMMA 3. [**EP10**] *Consider the forward complete system* (1) *with a Lipschitz continuous function* $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$ *and the set* $B_\delta = \{x \in \mathbb{R}^n : 0 < |x| < \delta, x \in K\}$ *for some* $\delta > 0$ *with all points* $\mathcal{L}b(K) \cap B_\delta$ *being the points of exit of trajectories from* B_δ. *Then the following properties are equivalent:*
- the system (1) *is (locally) unstable into the backward invariant set* B_δ*;*
- there exists a continuously differentiable Chetaev function $V : \mathbb{R}^n \to \mathbb{R}_+$ *such that for all* $x \in B_\delta \cup \{0\}$*:*

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|), DV(x)f(x) \ge \alpha(|x|),$$

for some $\alpha_1, \alpha_2 \in K_\infty$, $\alpha \in K$.

For the system (1) *with* r*-homogeneous function* f *the Chetaev function* V *has the same property:* $V(\Lambda_r x) = \lambda^k V(x)$, $k \ge \max_{1\le i\le n} r_i$. □

The first lemma allows for consideration strongly unstable or anti-stable systems (i.e. the systems with linearization having all eigenvalues with strictly positive real parts), lemmas 2 and 3 oriented on analysis of unstable systems, when linearization has some eigenvalues with positive real parts (hyperbolic equilibriums, for instance). In all cases existence of positive definite Chetaev functions with positive derivatives on appropriate regions are proven (the functions can be chosen homogeneous for homogeneous systems (1)). The function V established in Lemma 1 is called Lyapunov function, by the name of the author its proposed, that is a particular case of Chetaev functions presented in lemmas 2 and 3 [**Che61**].

2.1.3.2. *Stability/instability conditions for locally homogeneous systems.* An advantage of the r-homogeneous system (1) is that the global behavior of the system can be completely characterized by the behavior of the system on the sphere with the unit radius S_r. To explain this property let us introduce the coordinate transformation $x = \Lambda_r y$ that connects any $x \in \mathbb{R}^n$ with $y \in S_r$ for suitably chosen $\lambda \geq 0$. Let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a continuously differentiable homogeneous Lyapunov function (as used in Theorem 1 and Lemma 1), then

$$DV(x)f(x) = DV(\Lambda_r y)f(\Lambda_r y) = \lambda^{d+k} DV(y)f(y) = |x|_r^{d+k} DV(y)f(y),$$

where d is the homogeneity degree of the function f and k is the degree of the Lyapunov function V. Therefore, sign definiteness of the function V derivative can be checked on the sphere S_r only.

For the (r,λ_0,f_0)-homogeneous system (1) this technique establishes the relation between the global stability properties of the approximating dynamics

$$\dot{x} = f_0(x) \tag{3}$$

and the local ones of the original system (1). The conditions of such a relation are established below.

Proposition 1. [**EP10**] *Let the system* (1) *be* (r,λ_0,f_0)*-homogeneous, the functions* $f : \mathbb{R}^n \to \mathbb{R}^n$ *and* $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ *be continuous and the approximating dynamics* (3) *have* r*-homogeneous and continuously differentiable Lyapunov function* $V_0 : \mathbb{R}^n \to \mathbb{R}_+$, $\alpha_1(|x|) \leq V_0(x) \leq \alpha_2(|x|)$, $\alpha_1, \alpha_2 \in K_\infty$ *for all* $x \in \mathbb{R}^n$.

(i): *Let* $a_s = -\sup_{y \in S_r} DV_0(y)f_0(y)$, $a_s > 0$, *then*

(1) *if* $\lambda_0 = 0$, *then there exists* $0 < \bar{\lambda}_\varepsilon$ *such that the system* (1) *is locally asymptotically stable with the domain of asymptotic stability containing the set*
$$X_0 = \{x \in \mathbb{R}^n : |x| \leq \alpha_1^{-1} \circ \alpha_2 \circ \underline{\sigma}_r(\bar{\lambda}_\varepsilon)\};$$
(2) *if* $\lambda_0 = +\infty$, *then there exists* $0 < \underline{\lambda}_\varepsilon < +\infty$ *such that the system* (1) *is globally asymptotically stable with respect to the forward invariant set* $X_\infty = \{x \in \mathbb{R}^n : |x| \leq \alpha_1^{-1} \circ \alpha_2 \circ \bar{\sigma}_r(\underline{\lambda}_\varepsilon)\}$;
(3) *if* $0 < \lambda_0 < +\infty$, *then there exist* $0 < \underline{\lambda}_\varepsilon \leq \lambda_0 \leq \bar{\lambda}_\varepsilon < +\infty$ *such that the system* (1) *is finite time stable with respect to the forward invariant set* X_∞ *with the region of attraction*
$$X = \{x \in \mathbb{R}^n : \alpha_1^{-1} \circ \alpha_2 \circ \bar{\sigma}_r(\underline{\lambda}_\varepsilon) < |x| < \alpha_1^{-1} \circ \alpha_2 \circ \underline{\sigma}_r(\bar{\lambda}_\varepsilon)\}$$

provided that the set X is connected and nonempty.

(ii): *Let $a_u = \inf_{y\in S_r} DV_0(y)f_0(y)$, $a_u > 0$, then*

(1) *if $\lambda_0 = 0$, then there exists $0 < \bar{\lambda}_\varepsilon$ such that the system (1) is asymptotically stable with respect to the forward invariant set $\mathbb{R}^n\backslash X_0$ with the region of attraction $X_0\backslash\{0\}$;*

(2) *if $\lambda_0 = +\infty$, then there exists $0 < \underline{\lambda}_\varepsilon < +\infty$ such that the set $\mathbb{R}^n\backslash X_\infty$ is forward invariant for the system (1);*

(3) *if $0 < \lambda_0 < +\infty$, then there exist $0 < \underline{\lambda}_\varepsilon \le \lambda_0 \le \bar{\lambda}_\varepsilon < +\infty$ such that the system (1) is finite time stable with respect to the forward invariant set $\mathbb{R}^n\backslash X_0$ with the region of attraction X provided that the set X is connected and nonempty.* □

In other words the result of Proposition 1 means that the behavior of the system (1) is inherited after (3) into the set $X_r = \{x \in \mathbb{R}^n : \underline{\lambda}_\varepsilon < |x|_r < \bar{\lambda}_\varepsilon\}$ provided that it contains a contour of the function V_0 (the set X is connected and nonempty). The first two parts of the case (i) correspond to theorems 2 and 3. If we assume that the approximating vector field f_0 is r-homogeneous, then the requirement on existence of the r-homogeneous Lyapunov function V_0 follows by Theorem 1 and Lemma 1 results. The conditions of the proposition can be relaxed skipping homogeneity of f, f_0, using the continuity assumption as follows.

COROLLARY 1. [**EP10**] *Let $r > 0$, $\lambda_0 \ge 0$ and $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ be given and the approximating dynamics (3) have r-homogeneous and continuously differentiable Lyapunov function $V_0 : \mathbb{R}^n \to \mathbb{R}_+$, $\alpha_1(|x|) \le V_0(x) \le \alpha_2(|x|)$, $\alpha_1, \alpha_2 \in K_\infty$ for all $x \in \mathbb{R}^n$. Let one of the following properties hold*

(i) $a = -\sup_{y\in S_r} DV_0(y)f_0(y)$, $a > 0$;

(ii) $a = \inf_{y\in S_r} DV_0(y)f_0(y)$, $a > 0$,

and there exist $\underline{\lambda}_\varepsilon \le \lambda_0 \le \bar{\lambda}_\varepsilon$ such that $\sup_{y\in S_r} |DV_0(y)[\lambda^{-d}\Lambda_r^{-1}f(\Lambda_r y) - f_0(y)]| < a$ for all $\lambda \in (\underline{\lambda}_\varepsilon, \bar{\lambda}_\varepsilon)$, then all claims (i),1-(i),3 and (ii),1-(ii),3 of Proposition 1 are valid. □

Another way to relax the conditions of Proposition 1 consists in application of results of lemmas 2 and 3 for instability detection at the origin (for $\lambda_0 = 0$).

PROPOSITION 2. [**EP10**] *Let the system (1) be $(r,0,f_0)$-homogeneous, the functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and the approximating dynamics (3) have r-homogeneous and continuously differentiable Chetaev function $V_0 : \mathbb{R}^n \to \mathbb{R}_+$, $\alpha_1(|x|) \le V_0(x) \le \alpha_2(|x|)$, $\alpha_1, \alpha_2 \in K_\infty$ for all $x \in B_\delta$, $B_\delta = \{x \in \mathbb{R}^n : 0 < |x|_r < \delta, x \in K\}$, $\delta > 1$ and K be a closed cone with the vertex at the origin. Let $a = \inf_{y\in S_r\cap K} DV_0(y)f_0(y)$, $a > 0$, then the system (1) is unstable at the origin.* □

Application of these results for oscillation detection is discussed below.

2.1.3.3. *Oscillations in locally homogeneous systems.* Let $0 \le \lambda_0 \le ... \le \lambda_N \le +\infty$ be an ordered sequence for a given finite integer $N > 0$.

THEOREM 6. [**EP10**] *Let the system* (1) *be* (r_j,λ_j,f_j)*-homogeneous for* $j = \overline{1,N}$*, the functions* $f : \mathbb{R}^n \to \mathbb{R}^n$ *and* $f_j : \mathbb{R}^n \to \mathbb{R}^n$*,* $j = \overline{1,N}$ *be continuous and the locally approximating dynamical systems* $\dot{x} = f_j(x)$*,* $j = \overline{1,N}$ *have* r_j*-homogeneous and continuously differentiable Lyapunov functions* $V_j : \mathbb{R}^n \to \mathbb{R}_+$*,* $\alpha_{1,j}(|x|) \le V_j(x) \le \alpha_{2,j}(|x|)$*,* $\alpha_{1,j}, \alpha_{2,j} \in K_\infty$ *for all* $x \in \mathbb{R}^n$ *and* $j = \overline{1,N}$*. Let* $\Xi \subset \mathbb{R}^n$ *be the set containing all equilibriums of the system* (1)*.*
Let one of the following conditions hold.

(i): *There exists* $1 \le j* < N$ *such that*

$$a_{j*} = \inf_{y \in S_{r_{j*}}} DV_{j*}(y) f_{j*}(y) > 0,$$

$$a_{j*+1} = -\sup_{y \in S_{r_{j*+1}}} DV_{j*+1}(y) f_{j*+1}(y) > 0$$

and the sets

$$\begin{aligned} X_k &= \{x \in \mathbb{R}^n : \alpha_{1,k}^{-1} \circ \alpha_{2,k} \circ \bar{\sigma}_{r_k}(\underline{\lambda}_k) < |x| \\ &< \alpha_{1,k}^{-1} \circ \alpha_{2,k} \circ \underline{\sigma}_{r_k}(\bar{\lambda}_k)\}, k = j*, j*+1 \end{aligned}$$

are connected and nonempty where

$$\sup_{y \in S_{r_k}} |DV_k(y)[\lambda^{-d} \Lambda_{r_k}^{-1} f(\Lambda_{r_k} y) - f_k(y)]| < a_k$$

for all $\lambda \in (\underline{\lambda}_k, \bar{\lambda}_k)$*,* $\underline{\lambda}_k \le \lambda_k \le \bar{\lambda}_k$ *(such constants* $\underline{\lambda}_k$*,* $\bar{\lambda}_k$ *exist due to homogeneity assumption),* $k = j*, j*+1$*, and*

$$\Omega \cap \Xi = \emptyset, \Omega = \Omega_{j*} \cap \Omega_{j*+1},$$

$$\Omega_{j*} = \mathbb{R}^n \backslash \{x \in \mathbb{R}^n : |x| \le \alpha_{1,j*}^{-1} \circ \alpha_{2,j*} \circ \underline{\sigma}_{r_{j*}}(\bar{\lambda}_{j*})\},$$

$$\Omega_{j*+1} = \{x \in \mathbb{R}^n : |x| \le \alpha_{1,j*+1}^{-1} \circ \alpha_{2,j*+1} \circ \bar{\sigma}_{r_{j*+1}}(\underline{\lambda}_{j*+1})\};$$

(ii): *There exists* $1 \le j* < N$ *such that*

$$\begin{aligned} a_{j*} &= -\sup_{y \in S_{r_{j*}}} DV_{j*}(y) f_{j*}(y) > 0, \\ a_{j*+1} &= \inf_{y \in S_{r_{j*+1}}} DV_{j*+1}(y) f_{j*+1}(y) > 0 \end{aligned}$$

and the sets X_k, $k = j*, j*+1$ *are connected and nonempty where* $\underline{\lambda}_k \leq \lambda_k \leq \bar{\lambda}_k$ *are defined as for the case* (i), *and*

$$\begin{aligned} \Omega \cap \Xi &= \emptyset, \\ \Omega &= \{x \in \mathbb{R}^n : \alpha_{1,j*}^{-1} \circ \alpha_{2,j*} \circ \underline{\sigma}_{r_{j*}}(\bar{\lambda}_{j*}) \leq |x| \\ &\leq \alpha_{1,j*+1}^{-1} \circ \alpha_{2,j*+1} \circ \bar{\sigma}_{r_{j*+1}}(\underline{\lambda}_{j*+1})\}. \end{aligned}$$

Then the system (1) *has oscillating trajectories into the set* Ω. □

The result of the last theorem implies that if the system (1) is locally homogeneous and unstable in an inner (outer) subset, and locally homogeneous and stable in an outer (inner) subset, then between these subsets should exist an invariant set containing oscillating trajectory providing that the equilibriums are excluded from this region. The conditions of Theorem 6 can be relaxed taking in mind the results of Corollary 1 or Proposition 2 (these reformulations are omitted here for brevity of presentation and due to its triviality). The set Ω can be used to estimate the constants π^-, π^+, i.e. to estimate the amplitude of oscillation.

It is worth to note that Theorem 4 deals with one oscillating zone only, it is also assumed that the origin is strongly unstable. Theorem 6 relaxes these constraints, it allows one to detect multiple oscillating zones presence, the origin can be stable or unstable. The choice of Lyapunov functions could be simplified in the homogeneous case.

2.1.3.4. *Procedure for oscillations detection.* In this subsection we are going to comment on the proposed conditions of oscillating trajectories analysis and present the procedure for their applications [**EP10**].

The analysis of nonlinear system for stability, instability or existence of oscillations is rather complex problem that lacks for constructive applied approaches. The class of linear dynamic systems, on the contrary, has a complete list of methods for their analysis. The class of homogeneous systems being rather generic includes the linear systems. Considering homogeneous systems it seems possible to propose constructive (applicable in practice) approaches for nonlinear system analysis, or at least reduce the complexity of investigation. The class of locally homogeneous systems introduced here is much more larger than the class of homogeneous ones. Any system (1) with polynomial vector fields is locally homogeneous at least in two limits, for instance.

Two type of conditions are established, the first ones deal with stability/instability analysis for locally homogeneous systems on the basis of their locally approximating dynamics. It is shown that if the approximating system (3) has corresponding Lyapunov or Chetaev functions, then there exists a domain in the state space of the system (1), where the system inherits the same stability/instability properties.

The domain location depends on the value λ_0. If the approximating dynamics f_0 is homogeneous, then existence of homogeneous Lyapunov or Chetaev functions has been proven (Theorem 1 and lemmas 1-3). The second set of conditions is devoted to exposure of oscillating in the sense of Yakubovich trajectories for locally homogeneous systems. It is shown that if the system has two locally approximating dynamics, one is stable and another one is unstable, in the corresponding domains and the region between these domains does not contain equilibriums, then the system is oscillating in the sense of Yakubovich. The formal procedure for application of these conditions can be described as follows:

(1) Find the coordinates of all equilibriums of the system (define the set Ξ).
(2) Calculate the partition $0 \leq \lambda_0 \leq ... \leq \lambda_N \leq +\infty$, $N > 0$ defining the regions of local homogeneity.
(3) For each locally approximating dynamics f_k, $k = \overline{0, N}$ it is necessary to find Lyapunov or Chetaev functions establishing local stability/instability of the system.
(4) Verify the conditions (i) or (ii) of Theorem 6 (taking in mind Corollary 1 or Proposition 2).

The most complex steps of the procedure are 2 and 3. There exist no common recommendations for the partition $0 \leq \lambda_0 \leq ... \leq \lambda_N \leq +\infty$ calculation for a given nonlinear system (1). The partition is strongly related with the shape of f. The formulas for approximating functions f_0 for any $0 \leq \lambda_0 < +\infty$ are given after Definition 2, however the issue is to find λ_0 and the corresponding weights r such that the approximating function f_0 generates stable or unstable dynamics in (3). As it was mentioned above, for a polynomial or monotone function f at least two values $\lambda_0 = 0$ and $\lambda_N = +\infty$ can be tested. On the step 3 it is required to find Lyapunov or Chetaev functions. If the approximating system f_k is homogeneous, then according to results presented in this work these functions can be chosen homogeneous also.

The procedure can be also applied in a reverse way for design of complex oscillating systems, as well as, for the control synthesis providing oscillating behavior in a nonlinear system [**EP11**]. This development will be presented later, now let us test this procedure for several academic examples.

2.1.4. Examples of oscillating systems. Consider the system

$$\dot{x}_1 = -x_1 + x_2 + 2\tanh(x_1);$$

$$\dot{x}_2 = -2x_1 + x_3; \tag{4}$$

$$\dot{x}_3 = -1.5x_1 + 2\tanh(x_1),$$

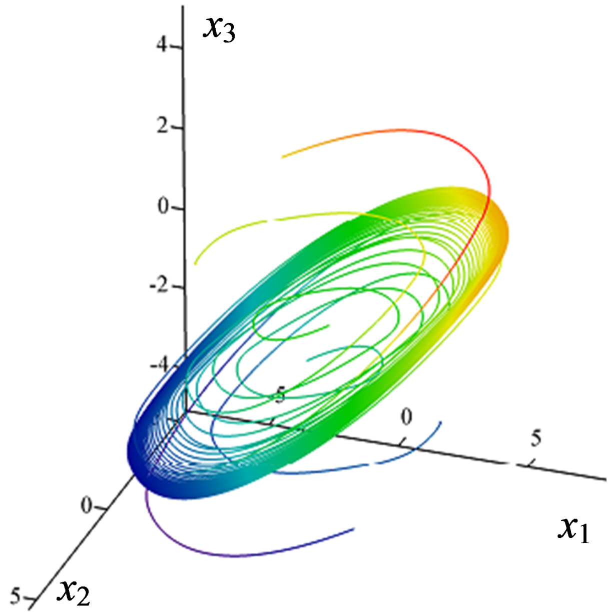


FIGURE 2. The results of the system (4) simulation

where $x = [x_1\ x_2\ x_3]^T \in \mathbb{R}^3$ is the state vector, and the system (4) is in Lurie form (linear asymptotically stable system closed by nonlinear feedback). The system (4) has the single equilibrium at the origin, it is homogeneous in the bi-limit, namely (r_j,λ_j,f_j)-homogeneous with $j = 1, 2$ and

$$r_1 = [0.5\ 0.5\ 0.5], \lambda_1 = 0,\ f_1(x) = A_1 x;\ r_2 = [0.5\ 0.5\ 0.5],\ \lambda_2 = +\infty,\ f_2(x) = A_2 x;$$

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 0.5 & 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ -1.5 & 0 & 0 \end{bmatrix}$$

with zero degree. Indeed, let us compute

$$\begin{aligned}
\lim_{\lambda\to\lambda_1}\Lambda_{r_1}^{-1}f(\Lambda_{r_1}x) &= \lim_{\lambda\to 0}\begin{bmatrix}\lambda^{0.5} & 0 & 0\\ 0 & \lambda^{0.5} & 0\\ 0 & 0 & \lambda^{0.5}\end{bmatrix}^{-1}\begin{bmatrix}-\lambda^{0.5}x_1+\lambda^{0.5}x_2+2\tanh(\lambda^{0.5}x_1)\\ -2\lambda^{0.5}x_1+\lambda^{0.5}x_3\\ -1.5\lambda^{0.5}x_1+2\tanh(\lambda^{0.5}x_1)\end{bmatrix}\\
&= \lim_{\lambda\to 0}\begin{bmatrix}-x_1+x_2+2\lambda^{-0.5}\tanh(\lambda^{0.5}x_1)\\ -2x_1+x_3\\ -1.5x_1+2\lambda^{-0.5}\tanh(\lambda^{0.5}x_1)\end{bmatrix}\\
&= \begin{bmatrix}-x_1+x_2\\ -2x_1+x_3\\ -1.5x_1\end{bmatrix}+2\lim_{\lambda\to 0}\begin{bmatrix}\lambda^{-0.5}\tanh(\lambda^{0.5}x_1)\\ 0\\ \lambda^{-0.5}\tanh(\lambda^{0.5}x_1)\end{bmatrix}=\begin{bmatrix}-x_1+x_2\\ -2x_1+x_3\\ -1.5x_1\end{bmatrix}+2\begin{bmatrix}x_1\\ 0\\ x_1\end{bmatrix}\\
&= \begin{bmatrix}x_1+x_2\\ -2x_1+x_3\\ 0.5x_1\end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\lim_{\lambda\to\lambda_2}\Lambda_{r_2}^{-1}f(\Lambda_{r_2}x) &= \lim_{\lambda\to +\infty}\begin{bmatrix}\lambda^{0.5} & 0 & 0\\ 0 & \lambda^{0.5} & 0\\ 0 & 0 & \lambda^{0.5}\end{bmatrix}^{-1}\begin{bmatrix}-\lambda^{0.5}x_1+\lambda^{0.5}x_2+2\tanh(\lambda^{0.5}x_1)\\ -2\lambda^{0.5}x_1+\lambda^{0.5}x_3\\ -1.5\lambda^{0.5}x_1+2\tanh(\lambda^{0.5}x_1)\end{bmatrix}\\
&= \lim_{\lambda\to +\infty}\begin{bmatrix}-x_1+x_2+2\lambda^{-0.5}\tanh(\lambda^{0.5}x_1)\\ -2x_1+x_3\\ -1.5x_1+2\lambda^{-0.5}\tanh(\lambda^{0.5}x_1)\end{bmatrix}\\
&= \begin{bmatrix}-x_1+x_2\\ -2x_1+x_3\\ -1.5x_1\end{bmatrix}+2\lim_{\lambda\to +\infty}\begin{bmatrix}\lambda^{-0.5}\tanh(\lambda^{0.5}x_1)\\ 0\\ \lambda^{-0.5}\tanh(\lambda^{0.5}x_1)\end{bmatrix}=\begin{bmatrix}-x_1+x_2\\ -2x_1+x_3\\ -1.5x_1\end{bmatrix}.
\end{aligned}$$

All eigenvalues of the matrix A_1 have positive real parts, and all eigenvalues of the matrix A_2 have negative real parts. Therefore, the conditions of Theorem 6 hold and the system (4) is oscillating, actually the results of its simulation presented in Fig. (2) show, that it has the stable limit cycle.

Next, consider the system

$$\begin{aligned}
\dot{x}_1 &= 2x_1-|0.5\pi-x_1^2-x_2^2-x_3^2|x_1^3-3(x_1-x_2-x_3)\cos(x_1^2+x_2^2+x_3^2);\\
(5)\quad \dot{x}_2 &= x_2-|0.4\pi-x_1^2-x_2^2-x_3^2|x_2^3-2(2x_2+x_1-0.5x_3)\cos(x_1^2+x_2^2+x_3^2);\\
\dot{x}_3 &= -x_3-x_1-|0.6\pi-x_1^2-x_2^2-x_3^3|x_3^3+(2x_3+x_1)\sin(x_1^2+x_2^2+x_3^3),
\end{aligned}$$

where $x=[x_1\ x_2\ x_3]^T\in\mathbb{R}^3$. The origin is the only equilibrium of the system (5). This system is homogeneous in three limits with

$$r_1=[0.5\ 0.5],\ \lambda_1=0;\ r_2=[0.5\ 0.5],\ \lambda_2=+\infty;\ r_3=[0.5\ 0.5],\ \lambda_3=0.5\pi;$$

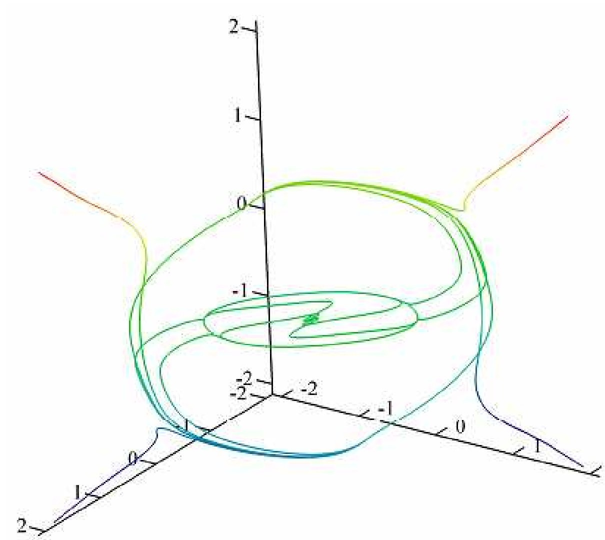


FIGURE 3. The results of the system (5) simulation

$$f_1(x) = \begin{bmatrix} -x_1 + 3x_2 + 3x_3 \\ -3x_2 - 2x_1 + x_3 \\ -x_1 - x_3 \end{bmatrix},\ f_2(x) = -|x|^2 \begin{bmatrix} x_1^3 \\ x_2^3 \\ x_3^3 \end{bmatrix},$$

$$f_3(x) = \begin{bmatrix} 2x_1 \\ \{1 - 0.05\pi^2[x_2^2/(x_1^2 + x_2^2 + x_3^2)]\}x_2 \\ \{1 - 0.05\pi^2[x_3^2/(x_1^2 + x_2^2 + x_3^2)]\}x_3 \end{bmatrix}$$

with degree $d = 0$ for the first and the third cases, $d = 2$ in the second case. In all modes the approximating dynamics has the Lyapunov function $V(x) = x^T x$, and it is easy to verify that the vector fields f_1, f_2 are asymptotically stable and f_3 is unstable. Thus, the conditions of Theorem 6 can be verified twice signalizing that the system has two zones with oscillating trajectories. The results of this system simulation presented in Fig. (3) confirm this conclusion, actually the system has two limit cycles (the inner is unstable, and the outer is stable).

Further, consider the system

$$\text{(6)} \qquad \begin{aligned} \dot{x}_1 &= -0.5\text{sign}(x_1)\sqrt{|x_1|} + x_1 - 0.5x_1^3 - x_2 - x_2^3; \\ \dot{x}_2 &= -0.5\text{sign}(x_2)\sqrt{|x_2|} + x_2 - 0.5x_2^3 + x_1 + x_1^3, \end{aligned}$$

where $x = [x_1\ x_2]^T \in \mathbb{R}^2$. The system (6) has the single equilibrium at the origin, it is homogeneous in two limits and additionally the conditions of Corollary 1 are satisfied for the third limit, i.e.

$$r_1 = [0.5\ 0.5],\ \lambda_1 = 0;\ r_2 = [0.5\ 0.5],\ \lambda_2 = +\infty;$$

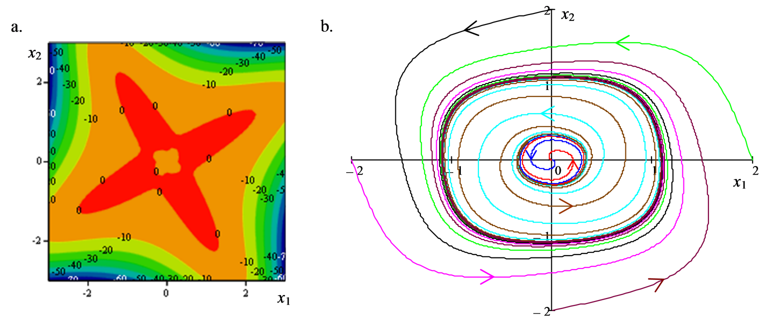


FIGURE 4. The illustrations for the system (6)

$$f_1(x) = -0.5 \begin{bmatrix} \sqrt{|x_1|}\mathrm{sign}(x_1) \\ \sqrt{|x_2|}\mathrm{sign}(x_2) \end{bmatrix},\ f_2(x) = \begin{bmatrix} -0.5x_1^3 - x_2^3 \\ -0.5x_2^3 + x_1^3 \end{bmatrix},$$

$d = -0.25$ and $d = 1$ correspondingly. The vector fields f_1, f_2 are asymptotically stable with Lyapunov functions

$$V_1(x) = 0.5(x_1^2 + x_2^2),\ V_2(x) = 0.25(x_1^4 + x_2^4).$$

As the third case from Corollary 1 we propose

$$r_3 = [0.5\ 0.5],\ \lambda_3 = 0.5,\ f_3(x) = \begin{bmatrix} x_1 - x_2 \\ x_2 + x_1 \end{bmatrix}.$$

The approximating vector field f_3 is unstable with Lyapunov function $V_3(x) = 0.5(x_1^2 + x_2^2)$. Then

$$DV_3(y)[\lambda^{-d}\Lambda_{r_3}^{-1}f(\Lambda_{r_3}y) - f_3(y)] = -0.5\lambda^{-0.25}(|y_1|^{1.5} + |y_2|^{1.5}) + \lambda[y_1^3y_2 - y_2^3y_1 - 0.5(y_1^4 + y_2^4)],$$

$a = 1$ and

$$\sup_{y \in S_{r_3}} |DV_3(y)[\lambda^{-d}\Lambda_{r_3}^{-1}f(\Lambda_{r_3}y) - f_3(y)]| < 1$$

for all $\lambda \in (\underline{\lambda}_3, \bar{\lambda}_3)$ where $\underline{\lambda}_3 = 0.16$ and $\bar{\lambda}_3 = 0.55$, $\underline{\lambda}_3 \leq \lambda_3 \leq \bar{\lambda}_3$. Thus, according to Corollary 1, the system (6) is locally unstable into the set

$$X_{r_3} = \{x \in \mathbb{R}^3 : \underline{\lambda}_3 < |x_1|^2 + |x_2|^2 < \bar{\lambda}_3\} = \{x \in \mathbb{R}^3 : \sqrt{\underline{\lambda}_3} < |x| < \sqrt{\bar{\lambda}_3}\}.$$

This conclusion can be illustrated by the expression $DV_3(x)f(x)$, that levels are plotted in Fig. (4),a. The expression represents the time derivative of the Lyapunov function V_3 calculated for the system (6), analysis of Fig. (4),a shows that this derivative is strictly positive on some ring around the origin. Consequently, Theorem 6 conditions are satisfied twice for the system (6), and it has two limit cycles as it is confirmed by the results of simulation presented in Fig. (4),b.

Finally, consider the system

$$\dot{x}_1 = -x_1 + 1.5x_2 - 0.3x_1^3 + x_1x_2^2; \tag{7}$$

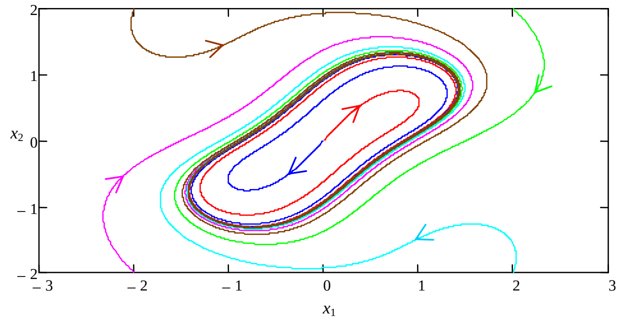


FIGURE 5. The trajectories of the system (7)

$$\dot{x}_2 = -x_1 + 2x_2 - 0.5x_2^3 - x_2 x_1^2,$$

where $x = [x_1\ x_2]^T \in \mathbb{R}^2$. The system (7) has the single equilibrium at the origin, it is homogeneous in two limits:

$$r_1 = [0.5\ 0.5],\ \lambda_1 = 0;\ r_2 = [0.5\ 0.5],\ \lambda_2 = +\infty;$$

$$f_1(x) = Ax, A = \begin{bmatrix} -1 & 1.5 \\ -1 & 2 \end{bmatrix},\ f_2(x) = \begin{bmatrix} -0.2x_1^3 + x_1 x_2^2 \\ -0.5x_2^3 - x_1^2 x_2 \end{bmatrix},$$

and $d = 1$ in both cases respectively. The vector field f_1, f_2 is linear and unstable with Chetaev function

$$V_1(x) = 0.5(x_1^2 + x_2^2)$$

and the cone $K = \{x \in \mathbb{R}^2 : x_2 \geq |x_1|\}$ (the case of Proposition 2), f_2 is asymptotically stable with Lyapunov function

$$V_2(x) = 0.25(x_1^2 + x_2^2).$$

Therefore, applying arguments similar to the ones in Theorem 6 we can substantiate existence of the limit cycle (the trajectories are globally bounded and the single equilibrium is unstable, therefore, there exists a compact set attracting almost all trajectories, the system is oscillating in the sense of Yakubovich, that for planar systems is equivalent to a limit cycle existence). The trajectories of the system are plotted in Fig. (5).

2.1.5. Control design. Following [**EP11**] consider an affine in control nonlinear system

$$\dot{x} = f(x) + G(x)u,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m, \tag{8}$$

where f and the columns of G are locally Lipschitz continuous vector fields, $f(0) = 0$. There exist a lot of approaches devoting to the stabilizing control u construction [**KKK95, SJK97**] for (8), and some methods of anti-control design

[**Che99, CS06, VC96**] (the controls granting the closed loop system with instability property).

2.1.5.1. *CLF approach.* Among approaches for stabilizing control design it is worth to mention Control Lyapunov Function (CLF) method [**Art83, Son89, Efi02b, Efi02a**] that gives a universal formula for the control laws. For the homogeneous system (8) this approach has been developed in [**FP00, Mou08, MP06, NNYN09**].

Definition 4. [**Son89**] A continuously differentiable and positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called a CLF for the system (8) if for all $x \in \mathbb{R}^n \setminus \{0\}$,

$$\inf_{u \in \mathbb{R}^m} \{a(x) + B(x)^T u\} < 0,$$

where $a(x) = DV(x)f(x)$, $B(x) = [DV(x)G(x)]^T$. Such a CLF satisfies the Small Control Property (SCP) if for each $\epsilon > 0$ there is a $\delta > 0$ such that, if $x \neq 0$ satisfies $||x|| < \delta$, then there is some $||u|| < \epsilon$ such that

$$a(x) + B(x)^T u < 0.$$

□

It is possible to show [**Son89**] that a continuously differentiable and positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ is a CLF for the system (8) if for all $x \in \mathbb{R}^n \setminus \{0\}$ the property

$$a(x) < 0 \quad \text{if} \quad ||B(x)|| = 0$$

holds. The SCP property is equivalent to the following one:

$$\lim_{||x|| \to 0} \frac{a(x)}{||B(x)||} \leq 0.$$

Now we are in position to introduce the new anti-control (destabilizing) Lyapunov function.

Definition 5. [**EP11**] A continuously differentiable and positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an Anti-control Lyapunov Function (ALF) for the system (8) if for all $x \in \mathbb{R}^n \setminus \{0\}$,

$$\sup_{u \in \mathbb{R}^m} \{a(x) + B(x)^T u\} > 0.$$

Such an ALF satisfies the Small Control Property (SCP) if for each $\epsilon > 0$ there is a $\delta > 0$ such that, if $x \neq 0$ satisfies $||x|| < \delta$, then there is some $||u|| < \epsilon$ such that

$$a(x) + B(x)^T u > 0.$$

□

Similarly, it is possible to conclude that a continuously differentiable and positive definite function $V:\mathbb{R}^n\to\mathbb{R}_+$ is an ALF, if for all $x\in\mathbb{R}^n\setminus\{0\}$ the property

$$a(x)>0 \quad \text{if} \quad ||B(x)||=0$$

holds. The SCP property for ALF is equivalent to the limit one:

$$\lim_{||x||\to 0}\frac{a(x)}{||B(x)||}\geq 0.$$

LEMMA 4. [**Mou08**] *If for the system (8) there exists a CLF $V:\mathbb{R}^n\to\mathbb{R}_+$, then the control*

$$u(x)=-\phi_1[a(x),||B(x)||]B(x),$$
$$\phi_1(a,b)=\begin{cases}\frac{a+\sqrt[p]{|a|^p+b^{2q}}}{b^2} & if\ b\neq 0;\\ 0 & if\ b=0\end{cases} \tag{9}$$

for any $2q\geq p>1$, $q>1$ is continuous for all $x\in\mathbb{R}^n\setminus\{0\}$ and ensures the system stabilization. If the function a and all elements of the vector function B are r–homogeneous with degrees d_a and d_B respectively:

$$a(\Lambda_r x)=\lambda^{d_a}a(x);\ B_i(\Lambda_r x)=\lambda^{d_B}B_i(x),\ i=\overline{1,m},$$

then the control (9) is elementwise r–homogeneous with degree d_a-d_B provided that it is possible to choose $2d_Bq=d_ap$. If the CLF V is r–homogeneous with degree d_V and the vector field f and all columns of the matrix function G are r–homogeneous with degrees d_f and d_G respectively:

$$f(\Lambda_r x)=\lambda^{d_f}\Lambda_r f(x);\ G^i(\Lambda_r x)=\lambda^{d_G}\Lambda_r G^i(x),\ i=\overline{1,m},$$

then the control (9) is elementwise r–homogeneous with degree d_f-d_G and the closed loop system (8), (9) is r–homogeneous with degree d_f provided that $2(d_G+d_V)q=(d_f+d_V)p$.

If furthermore V satisfies the SCP, then the feedback control (9) is also continuous at the origin. □

The last part of this lemma has been proven in [**Mou08**] for the case $m=1$ only.

LEMMA 5. [**EP11**] *If for the system (8) there exists an ALF $V:\mathbb{R}^n\to\mathbb{R}_+$, then the control*

$$u(x)=-\phi_2[a(x),||B(x)||]B(x),$$
$$\phi_2(a,b)=\begin{cases}\frac{a-\sqrt[p]{|a|^p+b^{2q}}}{b^2} & if\ b\neq 0;\\ 0 & if\ b=0\end{cases} \tag{10}$$

for any $2q \geq p > 1$, $q > 1$ is continuous for all $x \in \mathbb{R}^n \setminus \{0\}$ and ensures the system instability. If the function a and all elements of the vector function B are r–homogeneous with degrees d_a and d_B respectively:

$$a(\Lambda_r x) = \lambda^{d_a} a(x);\ B_i(\Lambda_r x) = \lambda^{d_B} B_i(x),\ i = \overline{1,m},$$

then the control (10) is elementwise r–homogeneous with degree $d_a - d_B$ provided that it is possible to choose $2d_B q = d_a p$. If the ALF V is r–homogeneous with degree d_V and the vector field f and all columns of the matrix function G are r–homogeneous with degrees d_f and d_G respectively:

$$f(\Lambda_r x) = \lambda^{d_f} \Lambda_r f(x);\ G^i(\Lambda_r x) = \lambda^{d_G} \Lambda_r G^i(x),\ i = \overline{1,m},$$

then the control (10) is elementwise r–homogeneous with degree $d_f - d_G$ and the closed loop system (8), (10) is r–homogeneous with degree d_f provided that $2(d_G + d_V)q = (d_f + d_V)p$.

If furthermore V satisfies the SCP, then the feedback control (10) is also continuous at the origin. □

Together with the local homogeneity concept, these two lemmas may be used to propose a universal control formula for the system (8) stabilization/destabilization at a specified sphere around the origin.

THEOREM 7. [**EP11**] *Assume that for the system (8):*

(i) there exists a r–homogeneous CLF (ALF) $V_0 : \mathbb{R}^n \to \mathbb{R}_+$, $d_V = \deg_r(V_0)$;

(ii) the function f is (r, λ_0, f_0)–homogeneous with degree d_f;

(iii) the columns G^i, $i = \overline{1,m}$ of the matrix function G are (r, λ_0, G_0^i)–homogeneous with degree d_G;

(iv) there exist $2q \geq p > 1$, $q > 1$ such that $2(d_G + d_V)q = (d_f + d_V)p$.

Then the system (8) with the control (9) (the control (10)) for $a(x) = DV_0(x)f_0(x)$, $B(x) = [DV_0(x)G_0(x)]^T$ is (r, λ_0, F_0)–homogeneous with degree d_f, where the vector field $F_0(x) = f_0(x) + G_0(x)u(x)$ is stable (unstable). □

For $0 < \lambda_0 < +\infty$, a variant of homogeneous approximating functions f_0 and G_0^i, $i = \overline{1,m}$ is given in (2). Any other approach (backstepping, forwarding or feedback linearization) generating a r–homogeneous control may substitute the CLF/ALF controls (9), (10) in this theorem under conditions (i)–(iii). Owing the framework of homogeneity in the multi-limit, Theorem 7 provides an approach to design oscillating systems.

COROLLARY 2. [**EP11**]*Let for some r_k, $k = \overline{0,N}$, $0 < N < +\infty$ and $0 \leq \lambda_0 < \ldots < \lambda_N \leq +\infty$ for the system (8) the following properties be true:*

(i) there exist r_{2j}–homogeneous CLFs (ALFs) $V_{2j} : \mathbb{R}^n \to \mathbb{R}_+$ and r_{2j+1}–homogeneous ALFs (CLFs) $V_{2j+1} : \mathbb{R}^n \to \mathbb{R}_+$ for $j = \overline{0, N/2}$, define $h_{2j} = 1$, $h_{2j+1} = 2$ ($h_{2j} = 2$, $h_{2j+1} = 1$) and $d_{V,k} = \deg_{r_k}(V_k)$ for $k = \overline{0, N}$;

(ii) the function f is (r_k, λ_k, f_k)-–homogeneous with degree $d_{f,k}$, $k = \overline{0, N}$;

(iii) the columns G^i, $i = \overline{1, m}$ of the matrix function G are (r_k, λ_k, G^i_k)–homogeneous with degree $d_{G,k}$, $k = \overline{0, N}$;

(iv) for all $k = \overline{0, N}$ there exist $2q_k \geq p_k > 1$, $q_k > 1$ such that $2(d_{G,k} + d_{V,k})q_k = (d_{f,k} + d_{V,k})p_k$;

(v) the vector fields f and G^i, $i = \overline{1, m}$ are linearly independent for all $x \in \mathbb{R}^n \setminus \{0\}$.

Then the system (8) with the control

$$u(x) = -\sum_{k=0}^{N} \left[\varrho_k(||x||_k, \lambda_k)\phi_{h_k}[a_k(x), ||B_k(x)||]B_k(x)\right], \tag{11}$$

$$a_k(x) = DV_k(x)f_k(x),\ B_k(x) = [DV_k(x)G_k(x)]^T,$$

has different oscillating trajectories into the sets $X_k = \{x \in \mathbb{R}^n : \lambda_k < ||x|| < \lambda_{k+1}\}$, $k = \overline{0, N-1}$, where the continuous weighting functions $\varrho_k(\lambda_k, \lambda_k) = 1$ and $\sum_{k=0}^{N} \varrho_k(||x||_k, \lambda_k) = 1$ for any $x \in \mathbb{R}^n$. □

Note that there exist constants $0 < \chi_k < +\infty$, $k = \overline{0, N}$ such that the choice

$$\varrho_k(||x||_k, \lambda_k) = e^{-\chi_k \nu_k(||x||_{r_k}, \lambda_k)^2},\ k = \overline{0, N} \tag{12}$$

is admissible, where $\nu_k(s, \lambda) = s - \lambda$, $k = \overline{0, N}$ and if $\lambda_N = +\infty$ then

$$\nu_N(s, +\infty) = \begin{cases} 0 & \text{if } s - \lambda'_N \geq 0; \\ s - \lambda'_N & \text{if } s - \lambda'_N < 0 \end{cases}$$

for some $\lambda_{N-1} < \lambda'_N < +\infty$. The term $e^{-\chi_k \nu_k(||x||_{r_k}, \lambda_k)^2}$ can be replaced with any other type of weighting functions (polynomial, for instance). For this choice of the functions ϱ_k, $k = \overline{0, N}$, the normalization condition holds with some error (dependent on χ_k), however the arguments of Theorem 6 remain valid.

2.1.5.2. *Example.* Consider a bilinear planar system:

$$\dot{x} = f(x) + g(x)u,\ x \in \mathbb{R}^2,\ u \in \mathbb{R},$$

$$f(x) = [x_2\ \ -2x_1]^T,\ g(x) = [x_1\ x_2]^T.$$

The vector fields f and g are linearly independent for all $x \in \mathbb{R}^2 \setminus \{0\}$. Choose $\lambda_0 = 0$ and $r_0 = [1\ \ 1]$, then due to linearity of f and g we obtain $f_0(x) = f(x)$, $g_0(x) = g(x)$. Take $V_0(x) = (x_1 + x_2)^2 + x_2^2$, then $a_0(x) = 2(x_1 + x_2)(x_2 - 4x_1) - 4x_1x_2$, $b_0(x) = 2(x_1 + x_2)^2 + 2x_1x_2$ and V_0 is a CLF. Since $d_{f,0} = 0$, $d_{g,0} = 0$ and $d_{V,0} = 2$, pick $q_0 = 1$ and $p_0 = 2$. Next, choose $\lambda_1 = 2$ and $r_1 = [1\ 2]$. According to (2) we

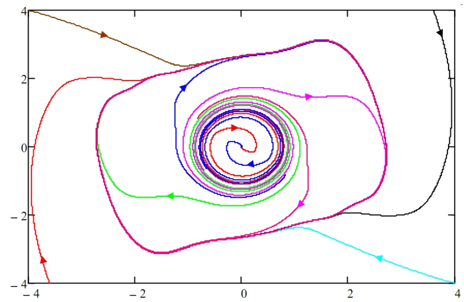


FIGURE 6. Trajectories of the closed-loop system

obtain

$$f_1(x) = [x_2 \ \ -0.5||x||_{r_1}^2 x_1]^T,\ g_1(x) = g(x)$$

for $d_{f,1} = 1$ and $d_{g,1} = 0$. For $d_{V,1} = 4$ take $V_1(x) = x_2^2 + (x_1|x_1| + x_2)^2$, then

$$a_1(x) = 2(x_1|x_1| + x_2)(2|x_1|x_2 - 0.5||x||_{r_1}^2 x_1) - ||x||_{r_1}^2 x_1 x_2,$$
$$b_1(x) = 2(x_1|x_1| + x_2)(2|x_1|x_1 + x_2) + 2x_2^2$$

and V_1 is an ALF. Therefore, $q_1 = 1.25$ and $p_1 = 2$ is an admissible choice. Finally, let $\lambda_2 = +\infty$ and $r_2 = [1\ 1]$, then again $f_2(x) = f(x)$, $g_2(x) = g(x)$, $V_2(x) = (x_1 + x_2)^2 + x_2^2$ is a CLF, $a_2(x) = a_0(x)$, $b_2(x) = b_0(x)$, $q_2 = 1$ and $p_2 = 2$. In the control (11) we put (12) with $\chi_k = 5$, $k = 0, 1, 2$. The results of the system simulation are shown in Fig. 6, the system has two limit cycles (the inner is unstable and the outer one is stable).

2.1.6. Conclusion. The notion of local homogeneity is a rather weak property (the formulas are given providing examples of approximating homogeneous functions in common case). To apply this property it is required to find a stable/unstable approximating dynamics, being homogeneous the choice of Lyapunov or Chetaev functions is straightforward for them (they have to be functions of the corresponding homogeneous norms). The original system (1) inherits locally stability or instability from the approximating systems (3), as well as it locally accedes to the Lyapunov or Chetaev functions.

The presented in [**EP10**] conditions of oscillation existence for systems homogeneous in the multi-limit allows one to detect multiple oscillating modes presence for homogeneous vector fields in several limits. Efficiency of the proposed conditions is demonstrated on several examples. To detect oscillations it is necessary to divide the state space on subsets where the system is stable or unstable. For a generic nonlinear system there exists only one approach to prove stability/instability, it is the second Lyapunov method. The application of this method is difficult, since there is no technique for the Lyapunov (Chetaev) functions choice in a particular

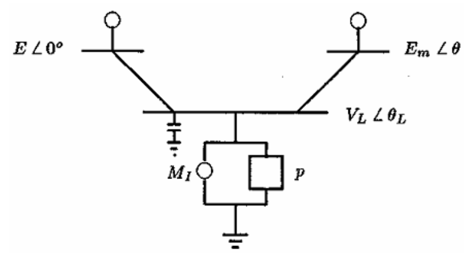


FIGURE 7. Power system structure scheme

example. The proposed approach, based on homogeneity in the multi-limit, allows one to decompose this complex unstructured problem to several simpler ones. It is proposed to find local approximating homogeneous systems, if they stable or unstable, then the procedure for detection of oscillations is described in subsection 2.1.3.

Partly the proposed conditions interact with the second part of Hilbert's 16th problem that addresses the issue of evaluation of an upper bound for the number of limit cycles in polynomial vector fields of degree n and investigation their relative positions (for the planar system (1)). Homogeneity is naturally satisfied for the polynomial vector fields, verifying local homogeneity conditions proposed in this work and checking conditions of Theorem 6 it is possible to estimate the maximum number of limit cycles and localize their positions.

2.2. Adaptive control and estimation of bifurcations

In this section the adaptive output feedback control algorithm from [**EF06**] is presented, which provides an exact tuning of adjustable parameters to unknown values ensuring the desired bifurcation properties of a nonlinear system in the output canonical form. Design of the algorithm is based on passification and an adaptive observer. Several examples, like neural integrator, resonant pendulum and three dimensional oscillator, are presented and illustrated by simulations.

2.2.1. Introduction. Bifurcation control is a relatively new area of active research, which deals with design of a controller providing desired bifurcation properties for a given nonlinear system [**AF86, CHY03, CMW00, WA95**]. For an illustration of the bifurcation control problem consider a simple electric power system model (its structure scheme is given in Fig. (7)) [**CHY03**]:

$$\begin{aligned}
\dot{\theta} &= \omega, \\
\dot{\omega} &= 16.6667\sin(\theta_L - \theta + 0.0873)V_L - 0.1667\omega + 1.8807, \\
\dot{\theta}_L &= 496.8718V_L^2 - 166.6667\cos(\theta_L - \theta + 0.0873)V_L \\
&\quad -666.6667\cos(\theta_L - 0.2094)V_L - 93.3333V_L + 33.3333p + 43.333, \\
\dot{V}_L &= -78.7638V_L^2 + 26.2172\cos(\theta_L - \theta + 0.0124)V_L \\
&\quad +104.8689\cos(\theta_L - 0.1346)V_L + 14.5229V_L - 5.2288p - 7.0327,
\end{aligned}$$

where θ, ω are the rotation angle and angular velocity of the generator, V_L, θ_L are the load voltage and angle respectively. The load is represented by an induction motor M_I (see Fig. (7)) in parallel with a constant active-reactive load, where p is the variable reactive power demand, it is the primary system parameter. In Fig. (7), $E\angle 0^o$ and $E_m\angle\theta$ are the phasor and generator terminal voltages respectively. For a small gradual increase of p a sequence of bifurcations appears: a periodic orbit occurs for $p = 10.818$, the first period-doubling bifurcation for $p = 10.873$, the second one for $p = 10.946$ and a saddle-node bifurcation for $p = 11.410$ (actually the bifurcation diagram even much more complex, see [**CHY03**] for details). The following control problems can be associated with this system: suppression or avoiding of the period-doubling bifurcations, delaying the bifurcations. Practically this leads to avoidance of the voltage collapse and limiting of chaotic behavior of the system.

Therefore, conventionally, bifurcation control is aimed at shifting position of the bifurcation point or changing its type, i.e. changing qualitatively the behavior of the system (bifurcation point is understood as the value of the system parameter such that in the vicinity the system has qualitatively different behavior). Such a significant change of the system properties requires good knowledge of its model. Perhaps, that is the reason why frequently the system model is assumed to be completely known in the existing papers. Moreover, in most papers it is assumed that the whole state vector of the system is available for measurement. In practice, however, in most cases the system model has uncertainties and only a part of the system state is available for measurement.

In this section another version of bifurcation control problem is considered, which allows one to take into account both uncertainty of the system parameters and incompleteness of measurements. To simplify the problem statement, first assume that the whole state vector of the system is available for measurement. Consider a controlled system

$$\dot{x} = f(x, p, u, t), t \geq 0, \tag{13}$$

where $x \in \mathbb{R}^n$ is state vector, $p \in \mathbb{R}^k$ is a parameter vector, $u \in \mathbb{R}^m$ is control; $f : \mathbb{R}^{n+k+m+1} \to \mathbb{R}^n$ is a vector function. The problem is to design a control law $u = U(x,t)$ such that the closed loop system

$$\dot{x} = f(x, p, U(x,t), t)$$

admits a bifurcation with desired properties for some nominal value $p*$ of the vector p, which is unknown to the control designer.

One example of such a situation is the resonance control [**BS03, LCC04**], where for $p = p*$ the system excited with a external periodic signal may have a desired resonance regime. In practice, however, some parameters of the system can differ from the nominal values. Since bifurcation or resonance properties of the system (13) may depend on p in a complicated and sensitive manner, small changes in parameter values may result in significant changes of the system behavior. Moreover, since the system at a bifurcation point lies on the border of stability, small changes in p may lead to instability of the system. Another similar problem was studied recently in the papers [**MS03, MSA03**] motivated by biological applications. Particularly, in [**MSA03**] a state feedback solution of the above problem was proposed for systems modeled with the first and second order linear differential equations without external inputs.

The approach considered in this section is aimed at bifurcation control of the systems with uncertainties and incomplete measurements, when the state feedback $u = U(x,t)$ cannot be implemented and should be replaced by an output feedback. To solve the problem it is proposed to use the adaptive control approach. Adaptive controller consists of two loops: main loop and adaptation one. The main loop is modeled as

$$u = \widehat{U}\,(y, \mu, t), \tag{14}$$

where $y = h(x) \in \mathbb{R}^p$ is the vector of measurable output variables and $\mu \in \mathbb{R}^q$ is the vector of adjustable parameters. We assume that the system (13), (14) possesses a bifurcation with the desired properties for unknown value $\mu_0 = \mu_0(p)$ of vector μ. The adaptation loop is has the form

$$\dot{\nu} = H(y, \nu, t), \mu = \chi(\nu),$$

where $\nu \in \mathbb{R}^r$ is the state vector of adaptive controller and vector-function H describes an adaptation algorithm. The design objective is to provide boundedness of all the trajectories of the closed loop system (13), (14), and convergence of the variable $\mu(t)$ to its desired value μ_0 with $t \to +\infty$. Such a problem was called *tuning to bifurcation* [**EF06**].

Many of the standard adaptive nonlinear control schemes [**FMN99, KKK95**] are not applicable to the above problem. The reason is that asymptotic stability (or partial asymptotic stability) of the adaptive control system often holds without convergence of adjustable parameters to their desired values. Such an assumption is not suitable for the above problem, where for an exactly tuned system a boundedness is guaranteed only, and the asymptotic stability may be absent. Moreover, the controlled system near the bifurcation point may have both stable and unstable trajectories.

A solution for the above posed problem for a class of nonlinear system in Lurie form with persistently exciting measured input signal has been proposed in [**EF06**]. Subsection 2.2.2 contains some auxiliary results and definitions for the result of [**EF06**] presentation. In subsection 2.2.3 the main results of [**EF06**] are formulated. Application examples are presented and illustrated by simulations in subsection 2.2.4.

2.2.2. Preliminaries. Consider a linear dynamical system

$$\dot{\xi} = A\xi + Bu, \varsigma = C\xi \tag{15}$$

with state $\xi \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ and output $\varsigma \in \mathbb{R}^l$. Introduce the following notations:

$$\delta(s) = \det(sI_n - A), W(s) = C(sI_n - A)^{-1}B,$$
$$\varphi(s) = \delta(s)\det(GW(s)), \Gamma = \lim_{s\to+\infty} sGW(s),$$

where I_n is identity matrix of size $n \times n$ and G is some matrix of size $m \times l$. Matrix inequalities are understood in sense of quadratic forms.

DEFINITION 6. [**Fra03**] *System* (15) *is called G-minimum phase if the polynomial* $\varphi(s)$ *is Hurwitz (its zeros belong to the open left complex half-plane). System* (15) *is called G-hyper minimum phase if it is G-minimum phase and* $\Gamma = \Gamma^T > 0$. □

For $l = m$ and $G = I_m$ the terms *minimum phase* and *hyper minimum phase* are used.

DEFINITION 7. [**Fra03**] *System* (15) *is called strictly G-passive if there exist positive definite quadratic forms* $V, S : \mathbb{R}^n \to \mathbb{R}_+$ *such, that* $V(\xi(t)) - V(\xi(0)) \leq \int_0^t \varsigma(\tau)^T Gu(\tau)d\tau - S(\xi)$, $t \geq 0$. □

In what follows an important role will be played by the so called *passification lemma.* It is formulated below for the case $l = m$, $G = I_m$ and can be easily extended to non-square systems.

LEMMA 6. [**EF06**] *Let rank*$(B) = m$. *Then the following properties are equivalent.*

(1) *System* (15) *is hyper minimum phase.*
(2) *There exist matrices* $P = P^T > 0$ *and* K *of sizes* $n \times n$ *and* $m \times m$ *respectively such that*

$$P(A + BKC) + (A + BKC)^T P < 0, PB = C^T.$$

(3) *There exist matrices* $P = P^T > 0$ *and* K *of sizes* $n \times n$ *and* $m \times n$ *respectively such that*

$$P(A + BK) + (A + BK)^T P < 0, PB = C^T.$$

(4) *There exist matrices* $P = P^T > 0$ *and* K *of sizes* $n \times n$ *and* $n \times m$ *respectively such that*

$$P(A + KC) + (A + KC)^T P < 0, PB = C^T.$$

(5) *There exists* $m \times m$ *matrix* K *such, that system* (15) *with feedback* $u = K\varsigma + v$, *where* $v \in \mathbb{R}^m$ *is a vector of new input, is strictly passive.*
(6) *There exists* $m \times n$ *matrix* K *such, that system* (15) *with feedback* $u = K\xi + v$, *where* $v \in \mathbb{R}^m$ *is a vector of new input, is strictly passive.* □

REMARK 1. Note that, though G-passivity of $W(s)$ coincides with passivity of $GW(s)$ or with passivity of (15) with respect to new output $G\varsigma$, passification problems by state and output feedback are not equivalent since the numbers of unknowns in the gain matrix K are different. However, solvability conditions for all problems coincide. □

REMARK 2. To find matrix K in the relations 2, 3, 4 the "high-gain" arguments can be employed. Namely, to satisfy 2 one can choose $K = -\kappa I_m$, to satisfy 3 one can choose $K = -\kappa C$ while to satisfy 4 one can choose $K = -\kappa B$, where $\kappa > 0$ is sufficiently large [**Fra03**]. □

DEFINITION 8. [**FMN99, LPPT02**] *Essentially bounded matrix function* $B(t)$ *is called persistently excited (PE) if there exist positive constants* L *and* σ *such that*

$$\int_t^{t+L} B(s)B(s)^T ds \geq \sigma I_n$$

for any $t \geq 0$. □

LEMMA 7. [**YW77**] *Consider vector-functions* f, $\tilde{\theta} : [0, +\infty) \to \mathbb{R}^m$. *Assume that* $\tilde{\theta}(t)$ *is continuously differentiable,* $\dot{\tilde{\theta}}(t) \to 0$ *as* $t \to +\infty$ *and* f *is PE. Then* $\tilde{\theta}(t) \to 0$ *as* $t \to +\infty$ *provided that* $\tilde{\theta}(t)^T f(t) \to 0$ *as* $t \to +\infty$. □

The following statement is a straightforward consequence of the results of [**RO87**], see also [**SW01**].

LEMMA 8. *A system*

$$\dot{z} = f_1(z, y); \dot{y} = f_2(z, y); z \in \mathbb{R}^{l_z}, y \in \mathbb{R}^{l_y},$$

where f_1, f_2 are smooth enough, is asymptotically stable with respect to the part of variables y:

$$|y(t)| \leq \beta(|(y(0), z(0))|, t), \beta \in \mathcal{KL},$$

if there exists a differentiable function V such that

$$\alpha_1(|y|) \leq V(y, z) \leq \alpha_1(|(y, z)|), \dot{V} \leq -\alpha_3(|y|),$$

where $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$, $\alpha_i(0) = 0$ are continuous functions and $\lim_{s \to +\infty} \alpha_i(s) = +\infty$ for $i = \overline{1, 3}$. □

2.2.3. Robust tuning to bifurcation. Consider a subclass of systems (13) with control (14) in the so called Lurie form with input:

$$\dot{x} = Ax + \varphi(y) + B(y)(\mu - \mu_0) + d, y = Cx, \tag{16}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $d \in \mathbb{R}^n$ are state, measured output and exciting input vectors respectively; $\mu \in \mathbb{R}^q$ is vector of adjustable parameters serving as an estimate of unknown constant vector $\mu_0 \in \mathbb{R}^q$ (the system is in the output canonical form). Let the function φ be continuous and globally Lipschitz, $B(y)$ be continuously differentiable. The signal $d : \mathbb{R}_+ \to \mathbb{R}^n$ is assumed to be Lebesgue measurable and essentially bounded function of time $t \geq 0$.

The problem is to find an algorithm of adjusting $\mu(t)$, ensuring boundedness of trajectories of the closed loop system and the limit relation

$$\lim_{t \to +\infty} \mu(t) = \mu_0, \tag{17}$$

where system (16) exhibits a desired bifurcation for the case $d(t) = 0$, $t \geq 0$.

The posed problem differs from the standard adaptive observer design problem due to presence of the feedback μ in equation (16), i.e. it can be classified as adaptive observer based controller design. An additional difficulty is in that the solutions of the system (16) may not be assumed bounded for any values of μ since it is not the case near the bifurcation point $\mu = \mu_0$. It is supposed that the signal $d(t)$ is directly measured, which is a realistic assumption in some applications. To consider the systems, which are on the border of stability, let us introduce the set $\mathcal{D}$ of all input functions $d : \mathbb{R}_+ \to \mathbb{R}^n$ such that all solutions of the system (16) for $\mu = \mu_0$ are bounded. Boundedness of the system solutions in the bifurcation point is a critical restriction imposed on properties of system (16). For example, for the case of integrator system the set $\mathcal{D}$ includes any scalar Lebesgue measurable and essentially bounded functions having bounded integral. Thus, in this case constant inputs are excluded from the set $\mathcal{D}$.

ASSUMPTION 1. *For any Lebesgue measurable and essentially bounded inputs* $\mu(t)$, $d(t)$ *and any initial conditions* $x(0) \in \mathbb{R}^n$ *system* (16) *has well defined solution* $x(t)$ *for all* $t \geq 0$ *(forward completeness property).* □

ASSUMPTION 2. *For given signal* $d \in \mathcal{D}$ *solutions of the system*

$$\dot{x} = Ax + \varphi(y) + d(t) + \delta(t)$$

are bounded for any initial conditions $x(0) \in \mathbb{R}^n$ *and any Lebesgue measurable essentially bounded input* $\delta(t)$ *provided that there exists a function* $\chi \in \mathcal{KL}$ *such that* $|\delta(t)| \leq \chi\left(|\delta(0)|, t\right)$, $t \geq 0$. □

ASSUMPTION 3. *Let* $B(y) = B_0 G(y)$, *where* $G(y)$ *is* $p \times q$ *matrix,* B_0 *is a* $n \times p$ *matrix and the system*

$$\dot{\xi} = A^T \xi + C^T u, \varsigma = B_0^T \xi \tag{18}$$

with state $\xi \in \mathbb{R}^n$, *input* $u \in \mathbb{R}^p$ *and output* $\varsigma \in \mathbb{R}^p$ *be hyper minimum phase.* □

ASSUMPTION 4. *For given input* $d \in \mathcal{D}$ *signal* $B\left(y(t)\right)^T$ *from equation* (16) *is PE.* □

Let us discuss the above assumptions. The Assumption 1 ensures existence of original system solutions for all $t \geq 0$, see also [**AS99**] for necessary and sufficient conditions of forward completeness. Assumption 2 claims that the system (16) has bounded solutions in the bifurcation point $\mu = \mu_0$ for the pointed out class of inputs from $\mathcal{D}$, and this property is robust with respect to additive converging to zero disturbance δ. Assumption 3, according to Lemma 6, means that there exist some matrices K, $P = P^T > 0$ of sizes $(n \times p)$ and $(n \times n)$ respectively, such that for some $\alpha > 0$:

$$P(A + KC) + (A + KC)^T P < -\alpha I_n, PB_0 = C^T.$$

Thus, Assumption 3 provides conditions to design an adaptive observer for (16):

$$\dot{\psi} = A\psi + \varphi(y) - KC(x - \psi) + d, \tag{19}$$

where $\psi \in \mathbb{R}^n$ is an estimate of x. Note that in view of equivalence between statements 2 and 4 in Lemma 6 one may choose the gain matrix K in the form $K = B_0 K_0$, where K_0 satisfies 2.

To solve the posed problem it is suggested to adjust the estimates μ of unknown parameters μ_0 according to the following speed gradient algorithm:

$$\dot{\mu} = -\gamma G(y)^T (y - C\psi), \gamma > 0. \tag{20}$$

Therefore, the proposed adaptive observer based controller is described by equations (19), (20). The main result of [**EF06**] is as follows.

THEOREM 8. *Let for the system* (16) *Assumptions 1–4 hold and the matrix K is chosen to satisfy property 4 of Lemma 6. Then solutions of the system* (16), (19), (20) *are bounded and the limit relation* (17) *holds.* □

The proof is based on the analysis of the dynamics of the observer estimation error $e = x - \psi$:

$$\dot{e} = (A + KC)\, e + B\,(y(t))\,(\mu - \mu_0). \tag{21}$$

Continue with a more complex subclass of systems (13), which includes the system (16) in feedback cascade with a non asymptotically stabilizable by output feedback system:

$$\dot{x} = A_1 x + A_2 z + \varphi_1(y) + B(y)(\mu - \mu_0) + d_1(t), \tag{22}$$

$$\dot{z} = A_3 z + \varphi_2(y) + d_2(t), y = Cx, \tag{23}$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^l$ are state vectors of systems (22) and (23); $y \in \mathbb{R}^p$ as before is available for measurements output; $d = (d_1, d_2) \in \mathbb{R}^{n+l}$ is vector of external excitation input. The signal $d : \mathbb{R}_+ \to \mathbb{R}^{n+l}$ is Lebesgue measurable and essentially bounded function of time $t \geq 0$. Functions φ_1, φ_2 are continuous and globally Lipschitz, $B(y)$ is continuously differentiable. Set $\mathcal{D}$ for the systems (22) and (23) is defined as before.

ASSUMPTION 5. *For any Lebesgue measurable and essentially bounded inputs $\mu(t)$, $d(t)$ and any initial conditions $x(0) \in \mathbb{R}^n$, $z(0) \in \mathbb{R}^l$ the system* (22), (23) *is forward complete.* □

ASSUMPTION 6. *For $d \in \mathcal{D}$ the solutions of the system*

$$\dot{x} = A_1 x + A_2 z + \varphi_1(y) + d_1(t) + \delta_1(t);$$

$$\dot{z} = A_3 z + \varphi_2(y) + d_2(t) + \delta_2(t),$$

are bounded for any initial conditions $x(0) \in \mathbb{R}^n$, $z(0) \in \mathbb{R}^l$ and any Lebesgue measurable essentially bounded input $\delta(t) = (\delta_1(t), \delta_2(t))$ provided that there exists a function $\chi \in \mathcal{KL}$ with the property $|\delta(t)| \leq \chi\,(|\delta(0)|, t)$, $t \geq 0$. □

ASSUMPTION 7. *Let $B(y) = B_0 G(y)$ and there exist matrices K_1, K_2 of sizes $(n \times p)$, $(l \times p)$ and positive definite symmetric matrices P_1, P_2 with dimensions $(n \times n)$, $(l \times l)$ such that the inequalities*

$$P_1(A_1 + K_1 C) + (A_1 + K_1 C)^T P_1 < -k_1 I_n, P_1 B_0 = C^T,$$

$$A_3^T P_2 + P_2 A_3 \leq -k_2 I_l, P_1 A_2 = C^T K_2^T P_2$$

hold for some constants $k_1 > 0$, $k_2 \geq 0$. □

The above assumption for system (22) is equivalent to the hyper minimum phase property of system (18) with the minimum phase property of matrix A_3 under auxiliary constraint $P_1 A_2 = C^T K_2^T P_2$.

ASSUMPTION 8. *For a given input $d \in \mathcal{D}$, the signal $B(y(t))^T$ from equation (22) is PE.* □

Assumptions 5, 6 and 8 for the systems (22), (23) are analogues of Assumptions 1, 2 and 4 for the system (16). Assumption 7, additionally to properties introduced in Assumption 3 for the system (22), fixes the Lyapunov stability property of linear part of the system (23). The observer based controller for (22), (23) has form:

$$\dot{\psi} = A_1 \psi + A_2 \zeta + \varphi_1(y(t)) - K_1(y(t) - C\psi) + d_1(t); \tag{24}$$

$$\dot{\zeta} = A_3 \zeta + \varphi_2(y(t)) + K_2(y(t) - C\psi) + d_2(t), \tag{25}$$

where $\psi \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^l$ are estimates of x, z. Adaptation algorithm for (24), (25) preserves its form (20).

THEOREM 9. [**EF06**] *Let for the system* (22), (23) *Assumptions 5–8 hold. Then solutions of the system* (20), (22)–(25) *are bounded. Furthermore, let at least one of the following conditions hold:*

(1) $k_2 > 0$;
(2) $A_2 = 0$;
(3) $A_3 A_2 p = 0$ *for any* $p \in X^0 \subset \mathbb{R}^l$, *where* X^0 *is subspace of the system* $\dot{s} = A_3 s$ *solutions corresponding to pure imaginary eigenvalues of matrix* A_3 *and* $B(y(t))$ *is a non constant signal of time* $t \geq 0$ *for any given* $d \in \mathcal{D}$;
(4) $A_3 A_2 p \neq 0$, $p \in X^0$ *and* $B(y(t)) = B$.

Then the limit relation (17) *holds.*

(5) *Additionally, if* $A_3 A_2 p \neq 0$, $p \in X^0$, *then the relation* (17) *holds for almost all* $d \in \mathcal{D}$. □

Let us discuss how to verify auxiliary conditions of the Theorem 9. The first condition ($k_2 > 0$) can be established when applying Assumption 7. The second

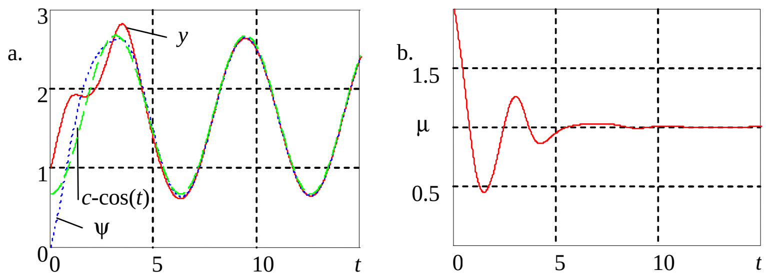


FIGURE 8. Trajectories for neural integrator

condition ($A_2 = 0$) follows from the structure of the system equations. In fact the first two conditions of the Theorem 9 correspond to situation considered in Theorem 8. To verify the last three conditions it is necessary to compute all eigenvalues of matrix A_3 and determine existence of a pure imaginary eigenvalue. If system (23) has not pure imaginary eigenvalues (or they are present, but their influence is annihilated by the matrix A_2) and $B(y(t))$ is not a constant matrix, then the third condition is true. If for time varying $B(y(t))$ the system (23) has pure imaginary eigenvalues, which govern $A_2\varepsilon(t)$, then it is necessary to compare frequencies of the exciting input signal d and the known frequencies of the system (23). If frequencies are different, then the fifth condition holds. If $B(y(t))$ is a constant matrix, then the fourth condition should be applied.

2.2.4. Applications.

2.2.4.1. *Neural integrator.* In work [**MSA03**] the following model of neural integrator was used

$$\dot{x} = (\mu - \mu_0)x + d$$

to design a dynamical feedback μ providing bifurcation tuning for the case $d = 0$. For this system with $y = x$, the adaptive observer equations (19), (20) take form:

$$\dot{\psi} = k_1(y - \psi) + d(t);\ \dot{\mu} = -\gamma y(y - \psi), \gamma > 0.$$

Assumption 1 is satisfied as for a linear autonomous system, Assumption 2 holds for $d(t) = \sin(\omega t)$, $\omega > 0$, which forms the set of admissible inputs $\mathcal{D}$ during simulation. Assumption 4 also follows for such class of inputs. Theorem 8 can be applied. The examples of system trajectories are shown in Fig. (8) for $\mu_0 = 1$, $\omega = 1$, $\gamma = 1$ and $k_1 = 1$.

2.2.4.2. *Resonance tuning of a pendulum.* Consider the pendulum equations

$$\dot{x}_1 = x_2;\ \dot{x}_2 = (\mu - \omega^2)\sin(x_1) + d(t),$$

where $\omega > 0$ is an unknown natural frequency of the pendulum; $\mu \in \mathbb{R}$ is the adjusted parameter, as before, that is introduced to tune pendulum frequency to the desired value; $d(t) = \sin(\omega_d t)$ is a sinusoidal signal with known exciting frequency $\omega_d > 0$. The problem is to entrain the resonance regime of the pendulum by diminishing the tuning error $\mu_0 = \omega^2 - \omega_d^2$ **[BS03]**. If the frequency of the pendulum and the frequency of the external input d coincide, then pendulum solution exhibits oscillations with an infinitely growing amplitude. Input d can be generated by the designer with known frequency ω_d, while the natural frequency of the pendulum ω may depend on uncertain and unpredictable external factors, and its exact value is unmeasured. Let $y = (x_1 x_2)^T$ and the pendulum equations can be rewritten in the form (16) as follows

$$\dot{x}_1 = x_2;\ \dot{x}_2 = -\omega_d^2 \sin(x_1) + (\mu - \mu_0)\sin(x_1) + d(t),$$

where $\mu_0 = \omega^2 - \omega_d^2$. Note, that since the state space vector of the plant is available for measurements we can build the observer based controller tacking into account dynamics of x_2 variable only. Observer equations (19), (20) take the form

$$\begin{aligned} \dot{\psi}_2 &= -\omega_d^2 \sin(y_1) + k_1 e_2 + d(t); \\ \dot{\mu} &= -\gamma \sin(y_1) e_2, e_2 = y_2 - \psi_2. \end{aligned}$$

It is possible to show that conditions of Theorem 8 hold in this example. Particularly PE condition is satisfied for $B(y) = [0 \sin(y_1)]^T$:

$$\int_t^{t+L} B(y)^T B(y) ds = \int_t^{t+L} \sin(y_1(t))^2 ds \geq \sigma > 0.$$

The last inequality holds because $d(t)$ is not constant and, therefore, $y_1(t)$ does not tend to a constant. A trajectory of this system simulation is shown in Fig. (9) for $\omega = \sqrt{3}$, $\omega_d = 1$, $\gamma = k_1 = 1$.

2.2.4.3. *Oscillator with changing zero dynamics.* Consider a dynamical system of the form (22), (23):

$$\begin{aligned} \dot{y} &= z_1 + (\mu - \mu_0) y; \\ \dot{z}_1 &= -\sin(y) + z_2; \dot{z}_2 = -2z_1 + d(t), \end{aligned} \tag{26}$$

which in bifurcation point $\mu = \mu_0$ is described as a series connection of nonlinear pendulum subsystem (y, z_1) and linear oscillating second order subsystem (z_1, z_2) forced by external input d. For bounded inputs $\mu(t)$ and $d(t)$ this system has the right hand side bounded by the norm of space vector, the last fact means that the system admits Assumption 5. Using mentioned decomposition of the system on series connection of subsystems it is possible to show that for $d(t) = a\sin(\omega t)$ with $a \in \mathbb{R}$, $\omega \in \mathbb{R}_+ \setminus \{\sqrt{2}\}$ this system possesses conditions of Assumptions 6 and 8. Assumption 7 also holds for adaptive observer:

$$\begin{aligned} \dot{\psi} &= \zeta_1 + k_1(y - \psi); & \dot{\zeta}_2 &= -2\zeta_1 + d(t); \\ \dot{\zeta}_1 &= -\sin(y) + (y - \psi) + \zeta_2; & \dot{\mu} &= -\gamma y(y - \psi). \end{aligned}$$

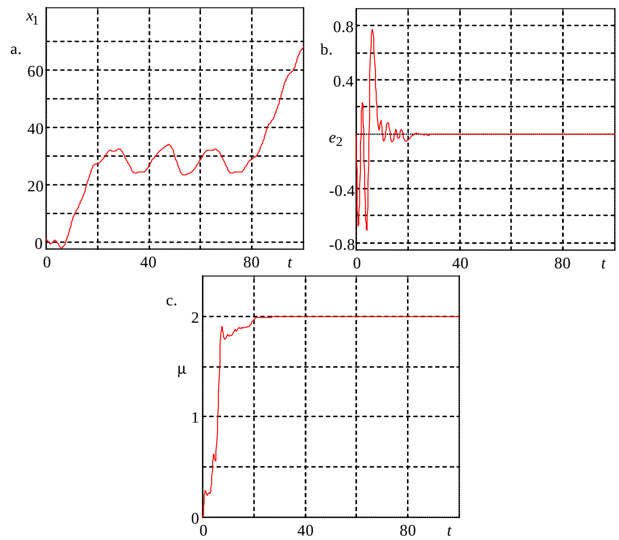


FIGURE 9. Resonance regime tuning for the pendulum trajectories

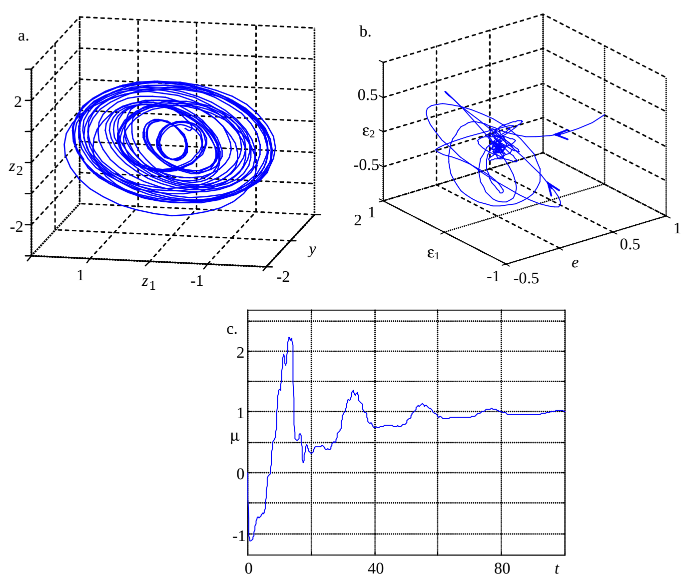


FIGURE 10. Trajectories for the complex oscillator (26)

Here $A_2 \neq 0$ while solutions of y-subsystem for the pointed class of inputs d perform nonlinear oscillations with a different frequency. Thus fifth condition of Theorem 9 is satisfied. The system (26) trajectories are presented in Fig. (10) for $\mu_0 = 1$, $k_1 = 0.5$, $\gamma = 5$, $a = 1$ and $\omega = 2$.

2.2.5. Conclusions. In this section an adaptive control problem is discussed, which is oriented on adaptive tuning to bifurcation regimes of nonlinear systems. This problem formulation differs from the one proposed in [**MS03, MSA03**], since:

- **(A):** the presence of an external exciting input is taken into account,
- **(B):** more general form of nonlinear controlled system is considered,
- **(C):** only part of the system variables is assumed to be available for measurements.

An adaptive output feedback controller is proposed in [**EF06**], which tunes a nonlinear uncertain dynamical system to its bifurcation point under some mild conditions. The solution is based on the design theory of passification-based adaptive observers. Proposed approach can be applied for any type of bifurcations and especially in the case when the plant possesses unstable behavior near the bifurcation point. Rather restrictive assumption on relative degree of the system has been relaxed in the paper [**EF05**].

2.3. Phase resetting control

The problem of phase resetting for oscillating systems is considered in this section. An elementary control strategy is reviewed, proposed in [**ESS09**], which is based on the phase response curve (PRC) model (the first order reduced model of originally nonlinear system obtained for the inputs with small amplitude). Performance of the obtained solution is illustrated on a popular model of circadian oscillations (a biological system).

2.3.1. Introduction. Any periodical oscillating mode can be characterized by its amplitude, frequency (or frequencies spectrum), and phase. Each of these characteristics can be controlled in various ways [**ABK01, Kur00**]. One of the problems, associated with control of periodical oscillations, consists in *phase* or *frequency resetting* by an external (periodical) input, i.e. in the assignment of desired values for phase and frequency applying some (may be periodical) control input [**Ble88, GGS81, Win80**]. Frequently, when it is necessary to reset both frequency and phase, this problem is called *entrainment* [**Izh07, PRK01**]. This problem is also usually addressed in synchronization framework for oscillators, when external input is just an output of another oscillator, which phase and frequency become desired for resetting [**PRK01**]. In this section we will focus our attention on phase resetting problem only.

Despite this problem is rather old and practically important [**Tas99**] it was not widely addressed for nonlinear oscillators in control theory literature. There exist a few approaches to solve the problem. The first one is based on master-slave synchronization theory, when master oscillator attempts to provide its own phase to slave

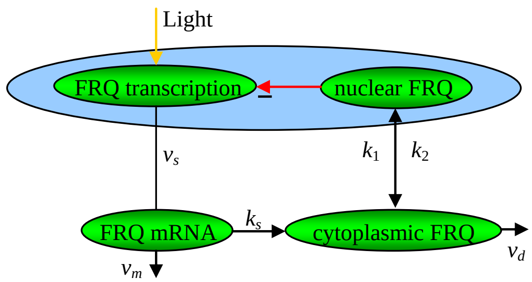


FIGURE 11. Scheme of the model for circadian oscillation in Neurospora

one. This approach is very promising, but it assumes rather serious modifications of slave equations, similarity of both systems and, frequently, measurements of all states. The approach is suitable for a technical system synthesis [**Kur00**], but it meets serious obstructions for application in biology, physics or chemistry. Another line of researches deals with optimal or predictive control application for phase resetting [**BSD07, BSD08, FP04**]. These methods require availability of full exact information on the model of oscillators and its coefficients that also makes hard its application. The third approach uses assumption on weak coupling/excitation, i.e. it imposes relatively small amplitude for external input [**PRK01**], that is a mild or natural restriction in some applications. The last approach is based on PRC application and *Poincaré phase map* approach [**ESS09, GG82, GNH+02, PRK01**]. The advantages of this approach are that PRC is a very simple tool commonly recognized as one of the main tools for phase resetting dynamics investigation in biology [**Izh07**] and that it is a scalar map of scalar argument, which completely describes phase resetting caused by a disturbing finite-time input (even for high dimension systems). Moreover, PRC can be measured experimentally even for oscillators, which have not well investigated detailed models.

In this section, following [**ESS09**] we are going to study the last approach: PRC method for phase resetting. A control strategy for timing of input "pulses" based on PRC is presented.

2.3.2. Motivation and problem statement. A common illustration of phase resetting is the jet-lag that most scientists experience when traveling to a conference. The organism needs some time to "reset" the phase of its initial circadian rhythm to shifted environmental light conditions. This problem prompted biologists to study phase resetting and entrainment mechanisms in simple models of circadian oscillations. Let us consider a simple biological model of circadian oscillations in Neurospora [**LGG99**] often used as a reference example and presented in Fig. (11). The mathematical model writes

$$\begin{aligned}
\dot{M} &= (v_s + u)\tfrac{K_I^n}{K_I^n + F_N^n} - v_m \tfrac{M}{K_m + M};\\
\dot{F}_C &= k_s M - v_d \tfrac{F_C}{K_d + F_C} - k_1 F_C + k_2 F_N;\\
\dot{F}_N &= k_1 F_C - k_2 F_N,
\end{aligned} \tag{27}$$

where the variables $M > 0$, $F_C > 0$, $F_N > 0$ denote the concentrations (defined with respect to the total cell volume) of the FRQ mRNA, and of the cytoplasmic and nuclear forms of FRQ respectively. Parameter v_s defines the rate of FRQ transcription. The control u models influence of light, which increases rate of FRQ transcription. Description of other parameters appearing in these equations can be found in [**LGG99**]; the used values are the following (in nM and h):

$$v_m = 0.505, v_d = 1.4, k_s = 0.5, k_1 = 0.5, k_2 = 0.6,$$

$$K_M = 0.5, K_I = 1, K_d = 0.13, n = 4, 1 \le v_s + u \le 2.5.$$

For all these values the system (27) possesses a unique unstable equilibrium and an asymptotically stable limit cycle. For different values of the parameter $1 \le v_s \le 2.5$ period of the limit cycle lies in the range $18.87 \le T \le 25.2$.

It is shown in [**LGG99, Tas99**] that periodical excitation by light input results in phase and frequency entrainment of the natural circadian oscillations. This means that application of a suitable input u over a periodic time window close to the natural limit cycle period T may entrain and/or shift the phase of the periodic solution of (27). In the works [**BSD07, BSD08, LGG99**] the input has been modeled as a sequence of pulses of limited duration and amplitude (for instance, one unique pulse of duration $T_w = 1$ hr and amplitude $\Delta = v_s/10$ could be applied every 24 hrs). In mathematical biology the steady state phase shift $\Delta\phi$ that results from a particular (brief) input is commonly studied via the PRC [**Izh07, PRK01**].

The goal of this section is to design a control for phase resetting based on PRC. In the subsection 2.3.3, the PRC map is introduced and phase model is derived. In subsection 2.3.4 two control algorithms are presented. Application of these controls for motivating example is demonstrated in section 2.3.5.

2.3.3. Deriving a phase model in the vicinity of a stable limit cycle. In this subsection the derivation of a phase model starting from differential equation that has a stable limit cycle is summarized. Details of the standard procedure can be found in [**Izh07, PRK01**].

2.3.3.1. *Linearized model.* Consider a (smooth) dynamical system

$$\dot{x} = f(x, u), x \in \mathbb{R}^n, u \in \mathbb{R} \tag{28}$$

and assume that for $u(t) \equiv 0$, $t \ge 0$ the system (28) has (non-constant) T-periodical solution $\gamma(t) = \gamma(t + T) \in \mathbb{R}^n$, $t \ge 0$. This means that the corresponding limit cycle is described by the set $\Gamma = \{x \in \mathbb{R}^n : x = \gamma(t), 0 \le t < T\}$, that attracts a non-empty open set A of initial conditions in $\mathbb{R}^n$ and that the linearized system

$$\delta\dot{x}(t) = A(t)\delta x(t) + b(t)u, \tag{29}$$

$$A(t) = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=\gamma(t)}, \; b(t) = \left.\frac{\partial f(\gamma(t),u)}{\partial u}\right|_{u=0}$$

has $n-1$ multipliers strictly inside the unit cycle and one multiplier equals to one [**YS75**], where $\delta x(t) = x(t) - \gamma(t)$ and the matrix function A and the vector function b are T-periodical due to the properties of γ. Multipliers are the eigenvalues of the monodromy matrix $M = \Phi(T)$ defined via the *fundamental matrix function* Φ of the system (29) and the solution of *adjoin* system Ψ:

$$\dot{\Phi}(t) = A(t)\Phi(t), \Phi(0) = I;$$

$$\dot{\Psi}(t) = -A(t)^T\Psi(t), \Psi(0) = I,$$

I is the identity matrix, $\Phi(t)^T\Psi(t) = I$.

2.3.3.2. *Phase and isochron variables.* Any point $x_0 \in \Gamma$ can be characterized by a scalar *phase* $\phi_0 \in [0, 2\pi)$, which uniquely determines the position of the point x_0 on the limit cycle Γ no matter how big is the dimension of the state space $\mathbb{R}^n$ [**Izh07, PRK01**]. One can define a smooth bijective *phase map* $\vartheta : \Gamma \to [0, 2\pi)$ assigning the corresponding phase ϕ_0 to any point x_0 on the limit cycle, i.e. $\phi_0 = \vartheta(x_0)$. Any solution of the system (28) on the cycle $x(t, x_0, 0)$, $x_0 \in \Gamma$ can be related to $\gamma(t)$ via phase ϕ_0 due to $x(t, x_0, 0) = \gamma(t + \phi_0\omega^{-1})$, $\omega = 2\pi/T$. Thus $\phi_0 = \vartheta(x_0) = \vartheta[\gamma(\phi_0\omega^{-1})]$ for any $x_0 \in \Gamma$, then $\vartheta^{-1}(\phi) = \gamma(\phi\omega^{-1})$. The *phase variable* $\phi : \mathbb{R}_+ \to [0, 2\pi)$ is defined for trajectories $x(t, x_0, 0)$, $x_0 \in \Gamma$ as $\phi(t) = \vartheta[x(t, x_0, 0)] = \vartheta[\gamma(t + \phi_0\omega^{-1})]$. Due to periodical nature of $\gamma(t)$ the function $\phi(t)$ is also periodical, moreover the function ϑ can be defined in a particular way providing that $\phi(t) = \omega t + \phi_0$, $\dot{\phi}(t) = \omega$ [**Izh07, PRK01**].

Phase notion can be extended to any solution $x(t, x_0, 0)$ starting in the attracted set $x_0 \in A$. By definition of the attraction, for all $x_0 \in A$ there exists $\theta_0 \in [0, 2\pi)$ such that $\lim\limits_{t\to+\infty} |x(t, x_0, 0) - \gamma(t + \theta_0\omega^{-1})| = 0$, where θ_0 is the asymptotic phase of the point x_0. Then there exists the *isochron map* $\upsilon : A \to [0, 2\pi)$ connecting a point $x_0 \in A$ and the corresponding asymptotic phase θ_0, i.e. $\theta_0 = \upsilon(x_0)$ and $\upsilon(x_0) = \vartheta\left[\lim\limits_{t\to 0} \gamma(t + \theta_0\omega^{-1})\right] = \lim\limits_{t\to 0} \vartheta[\gamma(t + \theta_0\omega^{-1})] = \theta_0$. The *asymptotic phase variable* $\theta : \mathbb{R}_+ \to [0, 2\pi)$ is derived as $\theta(t) = \upsilon[x(t, x_0, 0)]$, $t \geq 0$ (it is supposed that $x(t, x_0, 0) \in A$ for all $t \geq 0$). Locally around Γ the property $\dot{\theta}(t) = \omega$ is satisfied since by definition the map υ coincides with the smooth map ϑ for all $x \in \Gamma$.

The notion of asymptotic phase variable can be extended to a generic $u(t) \neq 0$, $t \geq 0$ providing that the corresponding trajectory $x(t, x_0, u)$ stays into the set A for all $t \geq 0$. In this case the asymptotic phase variable can be defined in a trivial way as $\theta(t) = \upsilon[x(t, x_0, u)]$, $t \geq 0$. Then the variable $\theta(t')$ at an instant $t' \geq 0$

evaluates the asymptotic phase of the point $x(t', x_0, u)$ if one would pose $u(t) = 0$ for $t \geq t'$. Dynamics of the asymptotic phase variable $\theta(t)$ in the generic case for $u(t) \neq 0$, $t \geq 0$ is hard to derive. A local model obtained in a small neighborhood of the limit cycle for infinitesimal inputs is presented below [**Izh07, PRK01**].

2.3.3.3. *Asymptotic phase dynamics.* A phase model can be defined for the variable θ from the linearized model (29) along the limit cycle: we only consider solutions $x(t, x_0, u)$ with initial conditions $x_0 \in \Gamma$ and that asymptotically converge to Γ.

To derive the model note that for $u(t) = 0$, $t \geq 0$ by definition $\dot{\gamma}(t) = f(\gamma(t), 0)$ for all $t \geq 0$, then $\ddot{\gamma}(t) = A(t)\dot{\gamma}(t)$ and $\dot{\gamma}(t) = \Phi(t)\dot{\gamma}(0)$. Therefore, $\dot{\gamma}(0) = f(\gamma(0), 0)$ is the left eigenvector of the matrix M for the eigenvalue $\lambda_1(M) = 1$. There exists the right eigenvector $m \in \mathbb{R}^n$ such that $m^T M = m^T$ and $m^T \dot{\gamma}(0) = \omega$, $\omega = 2\pi/T$. Finally, define $Q(t) = m^T \Psi(t)^T$ then

$$Q(t) f(\gamma(t), 0) = m^T \Psi(t)^T f(\gamma(t), 0) = m^T \Psi(t)^T \Phi(t) \dot{\gamma}(0) = m^T \dot{\gamma}(0) = \omega.$$

Therefore $Q(t) = \partial v(x)/\partial x|_{x=\gamma(t)} + \zeta(t)$, where $\zeta(t)$ is a row-vector orthogonal to $f(\gamma(t), 0)$ (for example, $\zeta(t) = \tilde{m}_i^T \Psi(t)$ for some right eigenvector $\tilde{m}_i \in \mathbb{R}^n$ such that $\tilde{m}_i^T M = \lambda_i(M) \tilde{m}_i^T$ for the eigenvalue $\lambda_i(M) \neq 1$, $i = 2, ..., n$). The map $\partial v(x)/\partial x|_{x=\gamma(t)}$ is independent on perturbations orthogonal to the limit cycle flow $f(\gamma(t), 0)$ (only shifts in the direction of the limit cycle are tabulated). Since m is the eigenvector corresponding to movement on the limit cycle, then by the same reason $Q(t) = m^T \Psi(t)^T$ is also independent of perturbations orthogonal to the limit cycle flow $f(\gamma(t), 0)$. Therefore, the convention $Q(t) = \partial v(x)/\partial x|_{x=\gamma(t)} = m^T \Psi(t)^T$ is adopted. The first equality explains the physical meaning of $Q(t)$, while the last equality used for numerical calculation. The function $Q(t)$ is T-periodic by construction. The function $Q(\phi\omega^{-1})$ for phase $\phi \in [0, 2\pi)$ is called *infinitesimal* PRC [**Izh07**]. Infinitesimal PRC Q serves as a delta-impulse response characteristics in the direction of the limit cycle.

The linearized model along the trajectory $x(t)$ satisfies:

$$\dot{\theta}(x(t)) = \dot{\theta}(\gamma(t) + \delta x(t)) = \partial\theta(x)/\partial x|_{x=\gamma(t)+\delta x(t)} \left[f(\gamma(t), 0) + A(t)\delta x(t) + b(t)u\right]$$

and the first approximation at the point $\delta x = 0$ has form

$$\dot{\theta} = \omega + Q(t)b(t)u(t). \tag{30}$$

The model (30) is derived around the solution $\gamma(t)$, due to periodicity for any other solution $\gamma(t + \phi\omega^{-1})$ and u the model has similar form [**Izh07, PRK01, Win80**]:

$$\dot{\theta} = \omega + Q(t + \phi\omega^{-1})b(t + \phi\omega^{-1})u(t). \tag{31}$$

2.3.4. PRC-based control design. This subsection presents the main results of the paper [**ESS09**] and it starts with the PRC-based model introduction and PRC control map definition. Next, two control algorithms are given.

2.3.4.1. *PRC-based model.* Let $\theta_d(t) = \omega t$ be the reference for the variable θ, then $\chi(t) = \theta(t) - \theta_d(t) \in [0, 2\pi)$ is the resetting error and according to (31) we have:

$$\dot{\chi} = Q(t + \phi\omega^{-1})b(t + \phi\omega^{-1})u(t). \tag{32}$$

Assume that u is a finite-time input, i.e. $u(t) \neq 0$ for all $0 < t < T_w$ and $u(t) = 0$ for all $t \geq T_w$, then integration of (32) yields for $t \geq T_w$ (note that $\chi(0) = \theta(0) = \phi(0) = \vartheta(x(0))$ for all $x(0) \in \Gamma$):

$$\begin{aligned} \chi(t) &= \chi(0) + \int_0^t Q(\tau + \chi(0)\omega^{-1})b(\tau + \chi(0)\omega^{-1})u(\tau)d\tau \\ &= \chi(0) + PRC(\chi(0)), \end{aligned} \tag{33}$$

where

$$PRC(\phi) = \int_0^{T_w} Q(\tau + \phi\omega^{-1})b(\tau + \phi\omega^{-1})u(\tau)d\tau.$$

In (33) the map PRC is defined for particular input $u \in L_\infty$, such kind of PRC definition is rather common [**Izh07, GS06**]. The map (33) coincides with infinitesimal PRC Q in the case when input u is a delta-impulse and $b(t) = b$.

From (33) we conclude that for a finite-time input the corresponding phase changes become apparent after the window T_w in the first order approximation model (32). In the original nonlinear system (28), the convergence of the phase resetting error to the value $\chi(0) + PRC(\chi(0))$ may be delayed, the length of this shift depends on the accuracy of the first order approximation. Since (33) defines the value of $\chi(t)$ for all $t \geq T_w$ it is proposed to choose some $T_s > T_w$ and consider $\chi(t) = \chi(0) + PRC(\chi(0))$ for $t \geq T_s$ only.

Assume now, that a train of "pulses" is given, i.e. there exists a series of time instants t_i, $i \geq 0$, $t_0 \geq 0$, $t_{i+1} - t_i \geq T_s$ such that the input u is activated at time instants t_i for all $i \geq 0$. Denote $\chi_i = \chi(t_i)$, $i \geq 0$. Let $\theta_0 \in [0, 2\pi)$ be the initial phase value, then $\theta(t) = \omega t + \theta_0$ for $0 \leq t \leq t_0$ and $\chi_0 = \theta_0$. At the time instant of the input activation we have $\theta(t_0) = \omega t_0 + \theta_0$, then from (33) after the first finite-time input

$$\chi_1 = \chi_0 + PRC(\theta(t_0)) = \chi_0 + PRC(\omega t_0 + \chi_0).$$

The error at time instants t_0 and $t_0 + T_s$ equals χ_0 and χ_1 respectively, and $\theta(t) = \omega t + \chi_1$ for $t \geq t_0 + T_s$. Therefore, $\theta(t_1) = \omega t_1 + \chi_1$ and linearization of the system (28) dynamics has to be carried out around the new trajectory $\gamma(t + \theta(t_1)\omega^{-1})$, then according to (33) the new phase resetting error value after the second "pulse" is

$$\chi_2 = \chi_1 + PRC(\theta(t_1)) = \chi_1 + PRC(\omega t_1 + \chi_1).$$

Again $\theta(t) = \omega t + \chi_2$ for $t \geq t_1 + T_s$, the phase $\theta(t_2) = \omega t_2 + \chi_2$ and from (33):

$$\chi_3 = \chi_2 + PRC(\theta(t_2)) = \chi_2 + PRC(\omega t_2 + \chi_2).$$

Repeating these calculations for all $i \geq 0$ we obtain:

$$\chi_{i+1} = \chi_i + PRC(\theta(t_i)) = \chi_i + PRC(\omega t_i + \chi_i), \tag{34}$$

where we assume that all summation operations in the right-hand side of (34) are done by modulo 2π. If $t_i = iT$, then the formula (34) is reduced to

$$\chi_{i+1} = \chi_i + PRC(\chi_i),$$

which is the Poincaré phase map [**Izh07, PRK01**]. The equation (34) describes phase resetting evolution originated by a train of "pulses" under condition of the first approximation model validity for the system (28).

There exists one "free" parameter t_i in the model (34) available for adjustment (the time instant when the next input u is introduced). Assigning t_i, $i \geq 0$ one may ensure desired phase resetting for the system (28). Let $\varpi_i = \theta(t_i) \in [0, 2\pi)$, $i \geq 0$ be the controlled phase of the "pulse" u activation in (28), then the model (34) can be rewritten as follows:

$$\chi_{i+1} = \chi_i + PRC(\varpi_i), i \geq 0. \tag{35}$$

The problem is to design sequences of ϖ_i, $i \geq 0$ providing phase resetting from any initial phase $\chi_0 \in [0, 2\pi)$ to the zero. The model (35) is the *first* order discrete nonlinear system, such class of systems is well investigated in the control theory literature [**Oga06**] (that is an advantage of the model (35) comparing it with (28)).

In the work [**ESS09**] two strategies for ϖ_i design have been proposed, one is *open-loop* control and another is *feedback* based control algorithm. Both are described below.

2.3.4.2. *Open-loop PRC-based control.* This strategy is based on the model (35) and it does not require any additional measured information about actual current phase of the system. A peculiarity of the system (35) and the problem of phase resetting consists in that $\theta \in [0, 2\pi)$, thus shift of the phase in both directions is possible for the resetting. To choose the direction one has to analyze which strategy (decreasing or increasing of the phase) leads to fastest resetting (of course this has sense only if PRC map takes negative and positive values).

In this section for brevity of exposition we assume that PRC map has particular properties (it is similar to type II PRC from [**HMM95**] or type 1 PRC from

[**Win80**]). The corresponding control strategies for other types of PRC maps can be easily deduced from this main case.

ASSUMPTION 9. *The map PRC is continuous and it has one zero $\phi_s^0 \in [0, 2\pi)$ with negative slope and another $\phi_u^0 \in [0, 2\pi)$ with positive slope, $\phi_s^0 < \phi_u^0$.* □

Since the map PRC is 2π-periodical from (33), the zeros can be arranged in the required order $\phi_s^0 < \phi_u^0$ changing the initial point on the limit cycle. Assumption 9 completely describes the form of PRC, in this case

$$\phi_{\max} = \arg \sup_{\phi \in [0,2\pi)} PRC(\phi), PRC_{\max} = PRC(\phi_{\max});$$

$$\phi_{\min} = \arg \inf_{\phi \in [0,2\pi)} PRC(\phi), PRC_{\min} = PRC(\phi_{\min}),$$

and $\phi_s^0 < \phi_{\min} < \phi_u^0 < \phi_{\max}$, $PRC_{\max} > 0$, $PRC_{\min} < 0$. Obviously, ϕ_s^0 corresponds to the stable equilibrium of the system (35) (for "zero" controls $\varpi_i = \chi_i$, $i \geq 0$) and ϕ_u^0 is the unstable one.

Define

$$n_{inc} = (2\pi - \chi_0)/PRC_{\max}, n_{dec} = -\chi_0/PRC_{\min},$$

where integer parts of the numbers n_{inc} and n_{dec} determine the number of steps required for resetting of the initial phase χ_0 into a neighborhood of the zero applying increasing or decreasing strategy. These numbers are minimal since for their calculation we use the maximum amplitudes of the shifts $PRC_{\max}$, $PRC_{\min}$ achievable in both directions. Next, in this neighborhood the phase can be resettled to the desired one applying one step shift with the same strategy due to assumed continuity of the map PRC. Thus the resetting requires $N+1$ "pulses", $N = round[\min\{n_{inc}, n_{dec}\}]$, where the function $round[n]$ returns the greatest integer not bigger than n. The following control is proposed to solve the problem:

$$\varpi_i = \begin{cases} \phi_{\max} & \text{if } n_{inc} \leq n_{dec}; \\ \phi_{\min} & \text{if } n_{inc} > n_{dec}, \end{cases} \quad 0 \leq i < N; \tag{36}$$

$$PRC(\varpi_N) + \chi_N = 0; \tag{37}$$

$$\chi_i = \begin{cases} \chi_0 + PRC_{\max} i & \text{if } n_{inc} \leq n_{dec}; \\ \chi_0 + PRC_{\min} i & \text{if } n_{inc} > n_{dec}, \end{cases} \quad 0 < i \leq N, \tag{38}$$

where the last step control ϖ_{N+1} is calculated as a solution of the equation (37), where it is assumed that χ_{i+1}, $0 \leq i < N$ are derived via (35) with the control (36) substitution (the formula (38)) and $\chi_{N+1} = 0$ due to (37). This strategy has been called "open-loop" since it does not establish any relations with the real values of phase variable.

Finally, the values of time instants t_i, $0 \leq i \leq N$ of inputs u activation should be calculated using the values ϖ_i, χ_i, $0 \leq i \leq N$ from (36), (37) and keeping in mind that $t_{i+1} - t_i \geq T_s$, $0 \leq i \leq N$ (the condition of the models (34) and (35) validity), and that all variables live on the cycle, i.e. $\theta \in [0, 2\pi)$. Calculating the values of phase at the end of the "pulse" window we obtain:

$$\theta_{i+1} = \theta(t_i + T_s) = [\omega(t_i + T_s) + \chi_{i+1}] \bmod 2\pi$$

for all $0 \leq i < N$, then we have

$$t_0 = g[(\varpi_0 - \chi_0)\omega^{-1}],$$

$$t_{i+1} = t_i + T_s + g[(\varpi_{i+1} - \theta_{i+1})\omega^{-1}], 0 \leq i < N, \tag{39}$$

$$g(\tau) = \begin{cases} \tau & \text{if}\, \tau \geq 0, \\ \tau + T & \text{otherwise.} \end{cases}$$

The formula (39) realizes inverse operation after summations by modulo 2π implemented for values of variables θ_i, χ_i and ϖ_i computation, $-T < (\varpi_i - \theta_i)\omega^{-1} < T$. Map g takes into account relation between $\theta(t_i + T_s)$ and ϖ_{i+1}, if $\theta(t_i + T_s) \leq \varpi_{i+1}$ then the next "pulse" can be activated immediately, if $\theta(t_i + T_s) > \varpi_{i+1}$ then it is necessary to wait one "turn" and the input has to be applied on the next period T only.

2.3.4.3. *Feedback PRC-based control.* This strategy assumes on-line measurements of the current phase value after each "pulse" application that increases accuracy of the resetting. To realize this strategy it is enough to replace in (36), (37) the values χ_{i+1} generated by (38) with available for measurements values $\chi_{i+1} = \chi(t_i + T_s)$, $i \geq 0$. By measurements we mean its calculation based on the available measurements for the state vector $x(t_i + T_s)$ (or a component).

The overall strategy for control design is similar to (36)-(38) for $i \geq 0$:

$$n_{inc}^i = (2\pi - \chi_i)/PRC_{\max}, n_{dec}^i = -\chi_i/PRC_{\min}, \tag{40}$$

$$\varpi_i = \begin{cases} \phi_{\max} & \text{if } 1 \leq n_{inc}^i \leq n_{dec}^i; \\ \phi_{\min} & \text{if } n_{inc}^i > n_{dec}^i \geq 1; \\ \ell(\chi_i) & \text{otherwise,} \end{cases} \tag{41}$$

where the function $\ell(\chi)$ represents a solution of the equation $PRC(\ell(\chi)) + \chi = 0$, then

$$t_0 = g[(\varpi_0 - \chi_0)\omega^{-1}],$$

and for $i \geq 0$,

$$\theta_{i+1} = \theta(t_i + T_s) = [\omega(t_i + T_s) + \chi_{i+1}] \bmod 2\pi,$$

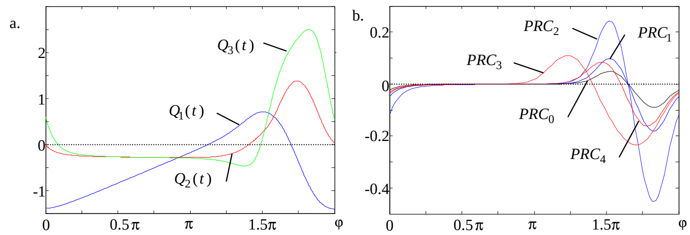


FIGURE 12. PRC curves of the system (27)

$$t_{i+1} = t_i + T_s + g[(\varpi_{i+1} - \theta_{i+1})\omega^{-1}]. \tag{42}$$

The feedback control strategy persists under convergent perturbations, that is its advantage with respect to the open-loop controls. However, the feedback approach requires more measurement information and it has more computational complexity. Application of the open-loop strategy becomes more reliable if on-line measurements are not realizable or too noisy.

Both strategies optimize the number of phase resetting steps, it can be naturally modified to guarantee uniform error χ decreasing choosing direction with minimal distance to zero. Let us demonstrate efficacy of the proposed controls on the motivating example.

2.3.5. Application to motivating example. In this subsection we apply the presented control strategies to the circadian oscillator in Neurospora (27), which is characterized by a limit cycle and its infinitesimal PRCs (in Fig. (12),a the functions Q_i, $1 \le i \le 3$ are plotted).

We use a very simple pulse input defined as

$$u(t) = \begin{cases} \Delta & \text{if } t < T_w; \\ 0 & \text{otherwise} \end{cases}$$

with $\Delta = 0.1$ and $T_w = 1$. Such an input can induce up to 1.04h of phase advance and 1.95h of phase delay (in Fig. (12),b the PRC maps (33) are presented, curve PRC_0 corresponds to the nominal case). By varying the strength and the duration of the input pulses, we can generate a family of PRCs (the curves PRC_1-PRC_4 in Fig. (12),b). An increase of Δ does not modify the shape of the PRC but enlarges its amplitude (in Fig. (12),b compare PRC_0 with PRC_1 for $\Delta = 0.2$, $T_w = 1$ and PRC_2 for $\Delta = 0.5$, $T_w = 1$). An increase of T_w modifies amplitude, zeros and slopes of the PRC (see PRC_3 for $\Delta = 0.1$, $T_w = 5$ and PRC_4 for $\Delta = 0.1$, $T_w = 2$ in Fig. (12),b).

In Fig. (13) we demonstrate the results of simulation for the reference input. This figure represents evolution of the resetting error χ for three cases: χ_a for the

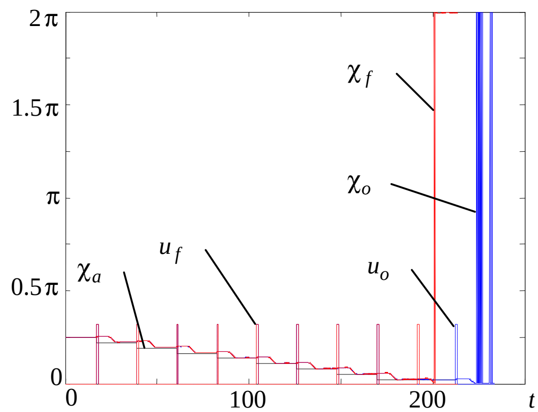


FIGURE 13. Results of simulation without disturbances

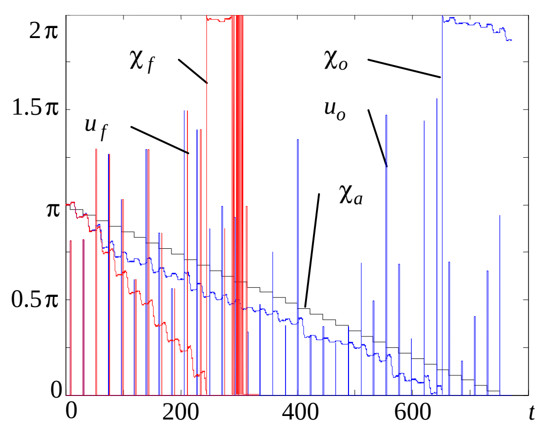


FIGURE 14. Results of simulation with disturbances

open-loop discrete model (36)-(38), χ_o presents resetting error for the open-loop-controlled system (27), and χ_f shows the error of the feedback control for the system (27). Curves u_o and u_f correspond to the control in open-loop and feedback cases (in scale multiplied by 10). The curve χ_a indicates the reference behavior for the variable χ_o according to (36)-(39). If T_s is sufficient, the discrete model (36)-(38) captures the main behavior of the nonlinear model and χ_o accurately follows χ_a (during simulation $T_s = T$). The feedback control is a little more efficient than the open-loop one.

In Fig. (14) we introduce multiplicative perturbations in the amplitude of our control inputs (χ_a, χ_o, χ_f, u_o and u_f denote the same variables). As expected, the feedback loop strategy (40)-(42) is much more robust than the open-loop one (36)-(39).

2.3.6. Conclusion. The idea of [**ESS09**] for a phase resetting control design based on PRC is presented. Two control algorithms are discussed performing open-loop and feedback strategies. Efficacy of these controls is demonstrated on simulations.

Many questions remain open after [**ESS09**], and may determine possible directions of future researches. The first one is accuracy of the proposed approach based on the first approximation of dynamics of the nonlinear system (28). The approach may be developed to the case of inputs with "arbitrary" amplitudes as in [**Efi11**]. Robustness of these controls with respect to disturbances, delays and model mismatches is uncovered. The proposed method can be extended for the case of table of "pulses" of different forms and vector controls.

CHAPTER 3

FUTURE DIRECTIONS OF RESEARCH

As a future direction of researches it is proposed to apply the hybrid systems methods to the problems of estimation and parametric identification of oscillating nonlinear systems. The main idea consists in extension of existing approaches applying supervisory/hybrid techniques in order to obtain new algorithms for estimation and control of nonlinear systems with improved performance (time of convergence, robustness, accuracy etc.).

3.1. Background

The main idea for future development belongs to intersection of two different areas of theory and applications: hybrid systems and oscillatory processes.

Oscillatory systems constitute a broad class of plants subjected by uncertainties and operating in different modes. Many promising examples of such systems can be found in biology (circadian rhythms, cells differentiation, locomotion) or techniques (vibrating machines, avionic systems, robot locomotion). New areas of applications (like systems biology, computer physics) and growing performance requirements in existent fields (like robotics and aerospace) emerge the task of oscillatory systems estimation and control. Oscillatory systems typically require special techniques for their treatment (some of them have been discussed before in chapter 2). For example, to identify a periodical system it is necessary to evaluate its frequency (or frequencies spectrum), amplitude and phase [**LGP09**]. Due to nonlinear dependence of the system variables on these parameters, the conventional estimation technique cannot provide a satisfactory accuracy and performance of observation, and some particular estimation schemes have to be designed. Another example is the class of chaotic systems, which can be considered as a subclass of oscillating ones (oscillating in the sense of Yakubovich), these systems are very popular in secured information transmission and encoding [**Kur00**]. The field of observers design for synchronization of chaotic systems is one of the most quickly growing during the last decade.

Hybrid or switched systems theory can be applied to design complex oscillating systems extending conventional approaches. Frequently, a complex system dynamics can be satisfactorily described, from a practical viewpoint, by using several models, each corresponding to an operating mode of the system. In such a case, it is common practice to implement a group of “local” controllers or observers depending on the operating mode. In such a scenario the natural question which arises next is how to make estimation of a multi-model system (see [**FJP08**] for an example). In addition to the problem of the hybrid systems estimation, application of supervisory (hybrid) algorithms may seriously improve performance of the conventional estimation algorithms. The appeal to hybrid systems looks natural dealing with multi-goals or multi-models systems.

Therefore, from one side there exists a growing area of applications dealing with analysis and synthesis of oscillating systems, which strives for development of new estimation approaches. From another side there exists a branch of methods based on hybrid/supervisory systems theory, that can be applied for development of new estimation and control algorithms for complex nonlinear systems. It is proposed to fill this gap proposing new estimation algorithms for oscillating systems. My previous research experience is concentrated on the intersection of these two areas of theory and practice. The Non-A team of INRIA-LNE performs research in similar directions providing me a good possibility to accomplish the project in the forthcoming years.

3.2. Research direction

The research directions can be divided on the short term and long term projects.

3.2.1. Short term. First, it is suggested to concentrate attention on theoretical developments for analysis and synthesis of oscillating systems using the homogeneity framework. This line of research is now realized in the PhD thesis work of Emmanuel Bernuau, in cooperation with Prof. W. Perruquetti and Prof. E. Moulay. The problems of stability and robust stability of the systems possessing not a unique equilibrium point or invariant set can be also analyzed in continuation of [**Efi12a**].

Second, the problem of design of observers for nonlinear oscillating systems can be investigated. There exist a lot of open issues in this area. The comprehensive solutions exist for linear systems, there are many extensions to nonlinear systems, but all of them are oriented on particular form of the system model and use a similarity with linear one. The oscillating systems frequently have a complex nonlinear model, which complicates design of observers, especially in the presence of external disturbances, parametric uncertainties and measurement noise. Applications of algebraic estimation technique, supervisory control, sliding-mode differentiators or interval approaches for design of observers as in

[**EZR11, BEPZ12, EF11a, EF11b, REZ12, BEP11**] indicate some promising ways to the problem solution. Hybrid algorithms (containing continuous-time and discrete-time parts) improve accuracy of observation and ensure finite-time estimation. Sometimes they also simplify analysis, despite the overall system becomes more complex (hybrid). Posterior application of these approaches for control design in the sense of [**EF09c**] seems to be also promising.

3.2.2. Long term. In long term perspective it is proposed to concentrate attention on biological and physical applications for estimation and control. The biology and physics are the most promising areas for cybernetics and informatics appliance. These areas are included in the INRIA scientific priorities for the upcoming years. The problems of entrainment and phase resetting controls design [**ESS09, Efi11**], wave regulation in lattices of oscillators [**EF12**] need special solutions for estimation and supervision. A lot of biomedical applications suffer from lack of adequate estimation and identification algorithms, like in the heart position estimation based on acceleration measurements [**GGdM+05**], for example (*a post-doc on this subject is launched in Non-A group of INRIA-LNE in October 2012*).

Another direction of a future research may be related with formation/swarming control of a group of mobile robots. During the last decade this topic of research has attracted a lot of attention [**GF07**]. Investigation of formation mechanisms (flocking or aggregating) in biology [**MCS03**] attracts researchers over the world. Mechanism for formation creation and maintenance is one of the main topics of research in control community. In this project it is proposed to use the supervisory control approach to design the algorithms of synchronous motion. In this case independently designed controls for aggregation, flocking or collision avoidance, for instance, will be combined in one algorithm applying the supervisor, which has to switch on the controls depending on the current operating conditions. The advantages of this approach consists in simplicity of independent controls design for various modes of collective motion, flexibility with respect to inclusion of additional subtasks, possibility of complex hierarchical systems construction [**ELP11, EPLF08, EPL09**] (*two PhD positions are created in Non-A group of INRIA-LNE in October 2012 dealing with design of robotic systems (in mobile robots and manipulators) applying the supervisory approach, an ANR project proposal is under preparation*).

Continuously increasing requirements on safety and reliability of technical systems lead to design of more sophisticated fault detection (FD) algorithms and fault tolerant control (FTC) systems. The main objective of FD and FTC is to maintain the specified performance of a system in the presence of faults. Especial attention to these problems is paid in flight and aeronautic applications [**ZJ03**]. The active

FTC is characterized by on-line FD with posterior faults compensation via a control reconfiguration mechanism. Appearance of the reconfiguration naturally leads to supervisory/hybrid systems framework application for FD and FTC. Despite the FTC problem is well addressed in the literature, typically, the fault isolation and the fault compensation problems are studied independently (under assumption on persistent fault detection). In reality the FTC systems possess multiple switches that dramatically influences on the performance. Analysis of the FTC from the switched systems theory positions [**ECH12**], development of the applicability conditions guaranteeing the minimum time of fault detection and stability of the system [**EZR11**] are perspective problems, which solution may be oriented on the flight control safety improvement [**EZ11**]. The use of the hybrid systems method for fault detection and compensation could be an interesting direction of research with applications in aerospace.

3.3. Applications and industrial transfer

It is important when a theoretical work is inspired or supported by some real world applications. A proper treating of applications helps to better formulate a theoretical problem and to find its solution taking into account all important constraints.

Some of my previous experience of industrial collaborations is presented in the section 1.3. In my plans is to continue to work with these industrial partners, as well as to try to find some other potential areas of practical development. This correlates with INRIA's policy to generate innovative scientific products and to transfer them to industrial partners. Below several possible directions are highlighted.

3.3.1. Biological systems. Bioinformatics and applied mathematics applications for biological and medical systems is a priority direction of research for INRIA. Development of control and estimation algorithms for biological systems is a domain of research consistent with the INRIA's interests. In this framework we are planning to launch a collaborative project with biologists working on circadian rhythm investigation of *Picoeucaryote Ostreococcus tauri* (a simple one-cell organism, with a well studied circadian rhythm behavior, which is an oscillating process) [**TPM$^+$10**]. The idea is to estimate and to control phase entrainment processes for this organism using similar ideas to [**ESS09, Efi11**].

3.3.2. Robotic swarming. The Non-A team is performing a joint work with POPS team of INRIA-LNE on routing of a network of robots and sensors. Application of supervisory techniques for estimation and control in this area is rather potential leading to many industrial collaborations.

The problem of robotic swarming or synchronization can be associated/transformed to the problem of a certain set stabilization. As it was already discussed in Chapter 2, the analysis and design of oscillating systems can be interpreted as a set stability or stabilization problem. Thus in the core, the robotic swarming problem can be solved using the same branch of methods as used for oscillations. Another direct interpretation is that oscillations are presented in many operating modes of robotic systems.

3.3.3. Automotive applications. During the last ten years I participated in collaboration with GM R&D Center at Warren, MI, US [**EJN10b, EJN10a, GJEN10a, GJEN10b, KEJ$^+$12**]. This collaboration is still active and, I hope, will bring us a lot of new statements of theoretical problems to solve and the corresponding solutions, dealing with control, estimation and modeling of spark ignition engines in uncertain environment.

3.4. Conclusion

The goal of my future researches is to develop new solutions oriented on a finite-time reliable estimation of state vector and uncertain parameters/inputs for complex oscillatory systems. An important aspect of research is that the proposed algorithms for control and observation have to be suitable for application in practice and, next, for industrial transfer. My previously performed research forms a basis for this project execution.

The completing of this project is planned under an intensive collaboration with PhD students, masters and post-docs, therefore it will include the knowledge dissemination and education. Due to a rich experience of international scientific collaborations (and many stays performed at different universities) several international collaborative projects are planned for the upcoming years (Russia, Mexico etc.).

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APPENDICES

A. Publications

Theses and dissertations.

(1) Efimov D.V. Robust and adaptive control of nonlinear oscillations. Thesis for degree of Doctor of Engineering Sciences, Saint-Petersburg, IPME RAS, 2006.
(2) Efimov D.V. Structural design of control systems with artificial neural networks. PhD thesis. Saint-Petersburg State Electrical Engineering University, 2001.
(3) Efimov D.V. Control laws with adaptive neural networks. Engineer degree thesis, Saint-Petersburg State Electrical Engineering University, 1998.

A.1. Peer reviewed international journals.

(1) Efimov D., Raïssi T., Chebotarev S., Zolghadri A. Interval State Observer for Nonlinear Time Varying Systems, **Automatica**, 2012, *DOI*: 10.1016/j.automatica.2012.07.004.
(2) Efimov D., Raïssi T., Zolghadri A. Control of nonlinear and LPV systems: interval observer-based framework. **IEEE Trans. Automatic Control**, 2012, *accepted*.
(3) Seydou R., Raïssi T., Zolghadri A., Efimov D. Actuator fault detection and diagnosis for flat systems: A constraint satisfaction technique. **Int. J. Applied Mathematics and Computer Science**, 2012, *accepted*.
(4) Efimov D. Global Lyapunov analysis of multi-stable nonlinear systems. **SIAM Journal on Optimization and Control**, 2012, *accepted*.
(5) Efimov D.V., Fradkov A.L. Natural Wave Control in Lattices of Linear Oscillators. **Systems & Control Letters**, 61, 2012, pp. 887–893.
(6) Efimov D., Fridman L.M., Raïssi T., Zolghadri A., Seydou R. Interval Estimation for LPV Systems Applying High Order Sliding Mode Techniques. **Automatica**, 48, 2012, pp. 2365–2371.
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OSCILLATORITY OF NONLINEAR SYSTEMS WITH STATIC FEEDBACK*

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Abstract. New Lyapunov-like conditions for oscillatority of dynamical systems in the sense of Yakubovich are proposed. Unlike previous results these conditions are applicable to nonlinear systems and allow for consideration of nonperiodic, e.g., chaotic modes. Upper and lower bounds for oscillations amplitude are obtained. The relation between the oscillatority bounds and excitability indices for the systems with the input are established. Control design procedure providing nonlinear systems with oscillatority property is proposed. Examples illustrating proposed results for Van der Pol system, Lorenz system, and Hindmarsh–Rose neuron model as well as computer simulation results are given.

Key words. analysis of oscillations, control of oscillations

AMS subject classifications. 34C15, 93B52

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1. Introduction. Most works on analysis or synthesis of nonlinear systems are devoted to studying stability-like behavior. Their typical results show that the motions of a system are close to a certain limit motion (limit mode) that either exists in the system or it is created by a controller. Evaluating deflection of the system trajectory from the limit mode, one may obtain quantitative information about system behavior [10, 27].

During recent years an interest in studying more complex dynamical systems behavior including oscillatory and, particularly, chaotic modes has grown significantly. Most authors deal with relaxed stability properties (orbital stability, Zhukovsky stability, partial stability) of some periodic limit modes [16, 19]. However, in order to study irregular, chaotic behavior the development of analysis and design methods for nonperiodical oscillations is needed. One such method based on the concept of excitability index (limit oscillation amplitude) for the systems excited with a bounded control was proposed in [7, 8].

It is worth noting that there exist many definitions for the term "oscillation" [11, 16]. For example, oscillation is understood as "any effect that varies in a back-and-forth or reciprocating manner" [6]. Otherwise, oscillation is the behavior of a sequence or a function, that does not converge, but also does not diverge to $+\infty$ or $-\infty$; that is, oscillation is the failure to have a limit [29]. Geometrically, an oscillating function of real numbers follows some path in a space, without settling into ever-smaller regions. In more simple cases the path might look like a loop coming back on itself, that is, periodic behavior; in more complex cases it may be a quite irregular movement covering a whole region [29]. Existing approaches based on Lyapunov stability theory [17, 23] or relaxed stability properties (orbital stability, Zhukovsky

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stability, partial stability) [16, 19, 24] are not completely suitable for study of complex oscillations. Indeed, these approaches require information on some limit modes, which stability should be investigated (that is not suitable for chaotic or irregular oscillations, for example). Besides, these approaches are not suitable for distinguishing between simple bounded behavior and oscillating one (a trajectory can converge to a steady-state solution that is a stable behavior from any kind of stability definition, but it is not an oscillation). Despite significant success in study of regular oscillations [4, 5, 12, 18, 20], comprehensive solutions for generic irregular oscillations have not been obtained yet.

An important and useful concept for studying irregular oscillations is that of "oscillatority" introduced by V.A.Yakubovich in 1973 [31]. Frequency domain conditions for oscillatority were obtained for Lurie systems, and split in linear and nonlinear parts [16, 31, 32]. However, when studying physical and biological systems in many cases it is hard to decompose the system into two parts: Linear nominal system plus nonlinear feddback. Mechanical systems (where energy plays a role of Lyapunov function) serve as a widespread example of such systems. Extension of analysis and design methods to oscillations in such class of systems is still to appear.

In this paper an approach to detection of oscillations and design of oscillatory systems for a class of nonlinear systems is suggested. New conditions for oscillatority of dynamical systems in the sense of Yakubovich are proposed. These conditions are applicable to nonlinear systems, and they are formulated in terms of Lyapunov functions existence. As a result upper and lower bounds for oscillations amplitude are obtained. A variant of converse Lyapunov theorem for strictly unstable systems is proposed. The relation between the oscillatority bounds and excitability indices for the systems with input are established. Design procedure for oscillations excitation is presented. Potentiality of the proposed technique is illustrated by four examples of analytical computations and computer simulations.

The main advantage of the obtained solution consists in possibility of application to a wide range of oscillation analysis and design problems. The proposed conditions are applicable even in the cases when other existing solutions cannot be used due to complexity of oscillations or system models [5, 18, 20].

Section 2 contains auxiliary statements and definitions (two preliminary results are placed in Appendix). Main definitions and oscillation existence conditions are presented in section 3. Section 4 deals with the task of static feedback design, which ensures oscillations appearance in closed loop system with desired bounds on amplitude. Conclusion is given in section 5. Examples illustrating proposed results for Van der Pol system, Lorenz system, and Hindmarsh–Rose neuron model as well as computer simulation results are presented in the text.

2. Preliminaries.

Let us consider a general model of nonlinear dynamical system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}); \quad \mathbf{y} = \mathbf{h}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} \in R^n$ is the state space vector; $\mathbf{u} \in R^m$ is the input vector; $\mathbf{y} \in R^p$ is the output vector; $\mathbf{f}$ and $\mathbf{h}$ are locally Lipschitz continuous functions on R^n, $\mathbf{h}(0) = 0$, and $\mathbf{f}(0, 0) = 0$. For initial condition $\mathbf{x}_0 \in R^n$ and Lebesgue measurable input $\mathbf{u}$ the solution $\mathbf{x}(\mathbf{x}_0, \mathbf{u}, t)$ of the system (1) is defined at least locally for $t \leq T$, $\mathbf{y}(\mathbf{x}_0, \mathbf{u}, t) = \mathbf{h}(\mathbf{x}(\mathbf{x}_0, \mathbf{u}, t))$ (further we will simply write $\mathbf{x}(t)$ or $\mathbf{y}(t)$ if all other arguments are clear from the context). If for all initial conditions $\mathbf{x}_0 \in R^n$ and inputs $\mathbf{u}$ the solutions are defined for all $t \geq 0$, then such system is called forward complete.

In this work we will consider feedback connection of system (1) with static system $\mathbf{u} = \mathbf{k}(\mathbf{y})$.

As usual, it is said that a continuous function $\rho : R_+ \to R_+$ belongs to class K, if it is strictly increasing and $\rho(0) = 0$; $\rho \in K_\infty$ if $\rho \in K$ and $\rho(s) \to \infty$ for $s \to \infty$; Lebesgue measurable function $\mathbf{x} : R_+ \to R^n$ is essentially bounded, if $\|\mathbf{x}\| = ess \sup\{|\mathbf{x}(t)|, t \geq 0\} < +\infty$, where $|\cdot|$ denotes usual Euclidean norm, $R_+ = \{\tau \in R : \tau \geq 0\}$. Notation $DV(\mathbf{x})\mathbf{F}(\cdot)$ stands for directional derivative of function V with respect to vector field $\mathbf{F}$ if function V is differentiable and for Dini derivative in the direction of $\mathbf{F}$

$$DV(\mathbf{x})\mathbf{F}(\cdot) = \lim_{t \to 0^+} \inf \frac{V(\mathbf{x} + t\mathbf{F}(\cdot)) - V(\mathbf{x})}{t}$$

if function V is Lipschitz continuous. In what follows we need the standard dissipativity property [30] and some its modifications. Function $f(x_1, \ldots, x_n)$ defined on R^n is called monotone if the condition $x_1 \leq x_1', \ldots, x_n \leq x_n'$ implies that everywhere either $f(x_1, \ldots, x_n) \leq f(x_1', \ldots, x_n')$ or $f(x_1, \ldots, x_n) \geq f(x_1', \ldots, x_n')$ everywhere.

DEFINITION 1. *The system* (1) *is dissipative if there exists continuous function* $V : R^n \to R_+$ *and a function* $\varpi : R^{n+m+p} \to R$ *such that for all* $\mathbf{x}_0 \in R^n$ *and Lebesgue measurable and locally essentially bounded* $\mathbf{u} : R_+ \to R^m$ *the following inequality is satisfied:*

$$V(\mathbf{x}(t)) \leq V(\mathbf{x}_0) + \int_0^t \varpi(\mathbf{x}(\tau), \mathbf{y}(\tau), \mathbf{u}(\tau))\, d\tau, \quad t \geq 0. \tag{2}$$

The functions ϖ *and* V *are called supply rate and storage functions of the system* (1).

In the case when storage function is continuously differentiable, inequality (2) can be rewritten in a simple form:

$$\dot{V}(\mathbf{x}, \mathbf{u}) = L_{\mathbf{f}(\mathbf{x},\mathbf{u})} V(\mathbf{x}) \leq \varpi(\mathbf{x}, \mathbf{u}, \mathbf{y}).$$

DEFINITION 2. *Dissipative system* (1) *is called*

– passive if $\varpi(\mathbf{x}, \mathbf{y}, \mathbf{u}) = \mathbf{y}^T\mathbf{u} - \beta(\mathbf{x})$, *where* β *is a continuous function reflecting the dissipation rate in the system; if* $\beta(\mathbf{x}) \geq \widehat{\beta}(|\mathbf{x}|), \widehat{\beta} \in K$, *then system* (1) *is called strictly passive* [13]*;*

– h-dissipative, if it has continuously differentiable storage function V *and*

$$\underline{\alpha}(|\mathbf{y}|) \leq V(\mathbf{x}) \leq \overline{\alpha}(|\mathbf{x}|), \quad \omega(\mathbf{y}, \mathbf{u}) = -\alpha(|\mathbf{y}|) + \sigma(|\mathbf{u}|),$$
$$\sigma \in K, \quad \alpha, \underline{\alpha}, \overline{\alpha} \in K_\infty;$$

– input-output-to-state stable (IOSS), if it has continuously differentiable storage function W *and* [26]

$$\alpha_1(|\mathbf{x}|) \leq W(\mathbf{x}) \leq \alpha_2(|\mathbf{x}|), \quad \alpha_1, \alpha_2 \in K_\infty,$$
$$\omega(\mathbf{x}, \mathbf{y}, \mathbf{u}) = -\alpha_3(|\mathbf{x}|) + \sigma_1(|\mathbf{u}|) + \sigma_2(|\mathbf{y}|),$$

$\alpha_3 \in K_\infty, \quad \sigma_1, \sigma_2 \in K$ [26]*;*

– input-to-state stable (ISS), if it has continuously differentiable storage function U *and* [21]

$$\alpha_4(|\mathbf{x}|) \leq U(\mathbf{x}) \leq \alpha_5(|\mathbf{x}|), \quad \alpha_4, \alpha_5 \in K_\infty;$$

$$\omega(\mathbf{x}, \mathbf{y}, \mathbf{u}) = -\alpha_6(|\mathbf{x}|) + \delta(|\mathbf{u}|), \quad \alpha_6 \in K_\infty, \quad \delta \in K.$$

If inequality sign in (2) for the case $\varpi(\mathbf{x}, \mathbf{y}, \mathbf{u}) = \mathbf{y}^T\mathbf{u} - \beta(\mathbf{x})$ can be replaced with equality, then it is said that the system possesses passivity property with known dissipation rate β.

Term h-dissipativity was introduced with minor differences in [2]. An important example of such kind of systems is $\mathbf{y}$-strictly passive systems [13]. Also, passive system (1) can be transformed to h-dissipative under suitable feedback transformation.

Storage functions for IOSS and ISS systems are called Lyapunov functions [23, 26]. Existence of corresponding Lyapunov functions is the equivalent characterization of ISS and IOSS properties [21, 26].

The interrelations of the properties introduced in Definition 2 are established in the Lemma A.1 (see Appendix), which was proved in [1] with a more restrictive requirement for h-dissipativity storage function:

$$\alpha_7(|\mathbf{x}|) \le V(\mathbf{x}) \le \alpha_8(|\mathbf{x}|), \quad \alpha_7, \alpha_8 \in K_\infty.$$

General result in this direction was obtained in [15], where it was proven that input-to-output stability (this property is closely connected with h-dissipativity; see also [24] for more details) and IOSS are equivalent to ISS property for the system (1).

3. Oscillatority conditions. At first it is necessary to give a precise definition of the term "oscillatority" placed in the title of this section and the paper. There are several approaches to define oscillation phenomena for nonlinear dynamical systems [16]. Perhaps, the most general one is the concept introduced by Yakubovich [31, 32]. Here we recover definitions from [31, 32] with some mild modifications [11, 16] dealing with high dimension and general form of the system.

DEFINITION 3. *Solution $\mathbf{x}(\mathbf{x}_0, 0, t)$ with $\mathbf{x}_0 \in R^n$ of system (1) is called $[\pi^-, \pi^+]$-oscillation with respect to output $\psi = \eta(\mathbf{x})$ (where $\eta : R^n \to R$ is a continuous monotone function) if the solution is defined for all $t \ge 0$ and*

$$\underline{\lim}_{t \to +\infty} \psi(t) = \pi^-; \quad \overline{\lim}_{t \to +\infty} \psi(t) = \pi^+; \quad -\infty < \pi^- < \pi^+ < +\infty.$$

*Solution $\mathbf{x}(\mathbf{x}_0, 0, t)$ with $\mathbf{x}_0 \in R^n$ of system (1) is called **oscillating**, if there exist some output ψ and constants π^-, π^+ such that $\mathbf{x}(\mathbf{x}_0, 0, t)$ is $[\pi^-, \pi^+]$-oscillation with respect to the output ψ. Forward complete system (1) with $\mathbf{u}(t) \equiv 0, t \ge 0$ is called **oscillatory**, if for almost all $\mathbf{x}_0 \in R^n$ solutions of the system $\mathbf{x}(\mathbf{x}_0, 0, t)$ are oscillating. Oscillatory system (1) is called **uniformly oscillatory**, if for almost all $\mathbf{x}_0 \in R^n$ for corresponding solutions $\mathbf{x}(\mathbf{x}_0, 0, t)$ there exist output ψ and constants π^-, π^+ not depending on initial conditions.*

In other words, the solution $\mathbf{x}(\mathbf{x}_0, 0, t)$ is oscillating if output $\psi(t) = \eta(\mathbf{x}(\mathbf{x}_0, 0, t))$ is asymptotically bounded and there is no single limit value of $\psi(t)$ for $t \to +\infty$ that is close to definition of oscillatority from [29].

Note that the term "almost all solutions" is used to emphasize that generally system (1) for $\mathbf{u}(t) \equiv 0$, $t \ge 0$ has a nonempty set of equilibrium points; thus, there exists a set of initial conditions with zero measure such that corresponding solutions are not oscillations. It is worth stressing that constants π^- and π^+ are exact asymptotic bounds for output ψ. Therefore, in order to compute these values the exact estimates for the system solutions should be known, which is a hard task for general nonlinear system (1). Fortunately, information on approximate estimates of constants π^- and π^+ is sufficient to obtain estimates on system amplitude oscillations. The oscillation property introduced in Definition 3 is defined for zero input and any initial conditions of system (1). The following property is a closely related characterization of the system behavior, which develops the proposed above property for the case of nonzero input but for specified initial conditions [8].

DEFINITION 4. *Let* $\mathbf{u} : R_+ \to R^m$ *be Lebesgue measurable and essentially bounded function and* $\mathbf{x}_0 \in R^n$ *be given such that* $\mathbf{x}(\mathbf{x}_0, \mathbf{u}, t)$ *be defined for all* $t \geq 0$. *The functions* $\chi^-_{\psi,\mathbf{x}_0}(\gamma), \chi^+_{\psi,\mathbf{x}_0}(\gamma)$ *defined for* $||\mathbf{u}|| \leq \gamma, \gamma \in R_+$ *are called lower and upper excitation indices of system* (1) *in point* $\mathbf{x}_0$ *with respect to the output* $\psi = \eta(\mathbf{x})$ *(where* $\eta : R^n \to R$ *is a continuous monotone function), if*

$$\left(\chi^-_{\psi,\mathbf{x}_0}(\gamma), \chi^+_{\psi,\mathbf{x}_0}(\gamma)\right) = \arg\max_{(a,b)\in E(\gamma)} \{b - a\},$$

$$E(\gamma) = \left\{ (a,b) : \left(\begin{array}{c} a = \underline{\lim}_{t\to+\infty}\, \eta\,(\mathbf{x}(\mathbf{x}_0, \mathbf{u}, t)), \\ b = \overline{\lim}_{t\to+\infty}\, \eta\,(\mathbf{x}(\mathbf{x}_0, \mathbf{u}, t)) \end{array} \right) \right\}_{\|\mathbf{u}\|\leq\gamma}.$$

Lower and upper excitation indices of a forward complete system (1) *with respect to the output* ψ *are*

$$\chi^-_{\psi}(\gamma) = \inf_{\mathbf{x}_0\in R^n} \chi^-_{\psi,\mathbf{x}_0}(\gamma), \quad \chi^+_{\psi}(\gamma) = \sup_{\mathbf{x}_0\in R^n} \chi^+_{\psi,\mathbf{x}_0}(\gamma).$$

In the same way it is possible to introduce indices for a vector output $\psi = \eta(\mathbf{x})$, in this case indices would be vectors of the same dimension as the output ψ.

Excitation indices characterize ability of system (1) to exhibit forced or controllable oscillations caused by bounded inputs. It is clear that properties $\pi^- = \chi^-_\psi(0)$ and $\pi^+ = \chi^+_\psi(0)$ are satisfied. For nonzero inputs the excitability indices characterize maximum (over specified set of inputs $\|\mathbf{u}\| \leq \gamma$) asymptotic amplitudes $\chi^+_\psi(\gamma) - \chi^-_\psi(\gamma)$ of ψ.

Note that it is useful to calculate or estimate values of $\chi^-_\psi(\gamma)$ and $\chi^+_\psi(\gamma)$ for all $0 \leq \gamma < +\infty$ due to the following reason. Let oscillation amplitude be an inverse function of input amplitude, then the maximum oscillation amplitude be reached for some $\gamma*$ and for all $\gamma \geq \gamma*$ the amplitude decreases. The indices $\chi^-_\psi(\gamma)$ and $\chi^+_\psi(\gamma)$ preserve their values for $\gamma \geq \gamma*$. Hence, to catch the critical value $\gamma*$ of input amplitude providing maximum output amplitude for ψ, it is necessary to build full graphics of functions $\chi^-_\psi(\gamma)$ and $\chi^+_\psi(\gamma)$. The obtained characteristics will be closely related with the Cauchy gain recently investigated in [22] (in fact, $\pi^+ - \pi^-$ or $\chi^+_{\psi,\mathbf{x}_0}(\gamma) - \chi^-_{\psi,\mathbf{x}_0}(\gamma)$ are asymptotic amplitudes of $\psi(t)$ in the sense of [22] for zero or nonzero input $\mathbf{u}$, while $\chi^+_\psi(\gamma)$ reflects the Cauchy gain of the system (1)).

On the other hand, excitation indices from Definition 4 describe robustness of the oscillations property proposed in Definition 3. Conditions of oscillations existence in the system are summarized in the following theorem.

THEOREM 1. *Let system* (1) *with* $\mathbf{u}(t) \equiv 0, t \in R_+,$ *i.e.,*

$$\dot{\mathbf{x}} = \mathbf{f}\,(\mathbf{x}, 0)\,, \tag{3}$$

have two continuous and locally Lipschitz Lyapunov functions V_1 *and* V_2 *satisfying for all* $\mathbf{x} \in R^n$ *the following inequalities:*

$$\upsilon_1(|\mathbf{x}|) \leq V_1(\mathbf{x}) \leq \upsilon_2(|\mathbf{x}|), \quad \upsilon_3(|\mathbf{x}|) \leq V_2(\mathbf{x}) \leq \upsilon_4(|\mathbf{x}|), \quad \upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4 \in K_\infty,$$

and for some $0 < X_1 < \upsilon_1^{-1} \circ \upsilon_2 \circ \upsilon_3^{-1} \circ \upsilon_4(X_2) < +\infty$*:*

$DV_1(\mathbf{x})\,\mathbf{f}(\mathbf{x}, 0) > 0$ *for* $0 < |\mathbf{x}| < X_1$ *and* $\mathbf{x} \notin \Xi$,

$DV_2(\mathbf{x})\,\mathbf{f}(\mathbf{x}, 0) < 0$ *for* $|\mathbf{x}| > X_2$ *and* $\mathbf{x} \notin \Xi$, *where $\Xi \subset R^n$ is a set with zero Lebesgue measure, which contain all equilibriums of the system, and*

$$\Omega \cap \Xi = \emptyset,$$

where $\Omega = \left\{ \mathbf{x} : \ \upsilon_2^{-1} \circ \upsilon_1(X_1) < |\, \mathbf{x} \,| < \upsilon_3^{-1} \circ \upsilon_4(X_2) \right\}$.

Then the system (3) *is oscillatory.*

Proof. Consider set $\Xi_0 \subset R^n$ of initial conditions not containing equilibrium points (which belong to set Ξ) of system (3). Then the solutions of the system starting from Ξ_0 are globally bounded, due to $\dot{V}_2 < 0$ for $|\, \mathbf{x} \,| > X_2$, and defined for all $t \geq 0$. Since the trajectory $\mathbf{x}(\mathbf{x}_0, 0, t)$, $\mathbf{x}_0 \in \Xi_0$, $t \geq 0$ is bounded, it has a nonempty closed, invariant, and compact ω-limit set, which belongs to the set Ω. Indeed, $V_2(t)$ asymptotically enters into the set where $V_2(t) < \upsilon_4(X_2)$, then $|\, \mathbf{x}(t) \,| < \upsilon_3^{-1} \circ \upsilon_4(X_2)$. In the same way function $V_1(t)$ is upper bounded and its limit values fall into the set where $V_1(t) > \upsilon_1(X_1)$; i.e., again $|\, \mathbf{x}(t) \,| > \upsilon_2^{-1} \circ \upsilon_1(X_1)$.

As it was supposed, Ω does not contain equilibrium points of the system. Hence, ω-limit set also does not include such invariant solutions. Then for each $\mathbf{x}_0 \in \Xi_0$ there exists an index i, $1 \leq i \leq n$ such that the solution is $[\, \pi^-, \pi^+ \,]$-oscillation with respect to output x_i with $-\upsilon_3^{-1} \circ \upsilon_4(X_2) \leq \pi^- < \pi^+ < \upsilon_3^{-1} \circ \upsilon_4(X_2)$. Suppose that there is no such output. It means that for all $1 \leq i \leq n$ for output x_i equality $\pi^- = \pi^+$ holds. However, the latter could be true only in equilibrium points, which are excluded from the set Ω by the theorem conditions. Therefore, for almost all initial conditions the system solutions have such oscillating output and system (3) is oscillatory by Definition 3. Note that for different $\mathbf{x}_0 \in \Xi_0$ oscillating outputs x_i may exist for different i, $1 \leq i \leq n$. ∎

Remark 1. The set Ω determines lower and upper bounds for the values of π^- and π^+.

Like in [32] one can consider the Lyapunov function candidate for linearized near the origin system (3) as a function V_1 to prove local instability of the system. Instead of existence of storage function V_2, one can require just boundedness of the system solution $\mathbf{x}(t)$ with a known upper bound. It can be obtained using another approach not dealing with time derivative of Lyapunov function analysis. In this case Theorem 1 is transforming into Theorem 3.4 from [11]; see also [33].

COROLLARY 1. *Define Ξ as the set of the system* (3) *equilibriums, i.e.,* $\Xi = \{ \mathbf{x} \in R^n : \mathbf{f}(\mathbf{x}, 0) = 0 \}$, *which consists in isolated points, and* $\mathbf{A}(\mathbf{x}_0) = d\, \mathbf{f}(\mathbf{x}, 0) / d\, \mathbf{x}|_{\mathbf{x}=\mathbf{x}_0}$ *is the matrix of the system* (3) *linearization in point $\mathbf{x}_0 \in R^n$. Let the following conditions be valid:*

1. *For all $\mathbf{x}_0 \in \Xi$ the matrices of the system* (3) *linearization $\mathbf{A}(\mathbf{x}_0)$ have eigenvalues with positive real parts.*
2. *There exists $R > 0$ such that for almost all initial conditions $\mathbf{x}_0 \in R^n$:*

$$\lim_{t \to +\infty} |\mathbf{x}(\mathbf{x}_0, 0, t)| \ \leq R.$$

Then the system (3) *is oscillatory.*

Proof. By conditions of the corollary for almost all initial conditions the ω-limit set is compact and it does not contain the equilibriums of the system. Further the proof is similar to the proof of Theorem 1. ∎

Conditions of Theorem 1 are rather general and define the class of systems, which oscillatory behavior can be investigated by the approach, namely systems which have an attracting compact set in state space containing oscillatory movements of the systems. For such systems Theorem 1 or Corollary 1 give useful tools for testing their oscillating behavior and obtaining estimates for amplitude of oscillations.

Theorem 1 presents the sufficient conditions for system (1) to be oscillating in the sense of Yakubovich. It is possible to show that for a subclass of uniformly oscillating systems these conditions are also necessary. To prove this result we need the following two lemmas.

LEMMA 1. *Let there exist constant* $r > 0$ *such that for solutions of systems* (3) *the following property is satisfied:*

$$0 < |\mathbf{x}_0| < r \quad \Rightarrow \quad |\mathbf{x}(\mathbf{x}_0, 0, t)| > r$$

for all $t \geq T_{\mathbf{x}_0}$, *where* $0 < T_{\mathbf{x}_0} < +\infty$. *Then there exists a continuous and locally Lipschitz–Lyapunov function* $V_1(\mathbf{x})$ *such that for all* $\mathbf{x} \in R^n$

$$\upsilon_1(|\mathbf{x}|) \leq V_1(\mathbf{x}) \leq \upsilon_2(|\mathbf{x}|), \quad \upsilon_1, \upsilon_2 \in K_\infty,$$

additionally for all $0 < |\mathbf{x}| < r$ *it holds:*

$$D V_1(\mathbf{x}) \mathbf{f}(\mathbf{x}, 0) > 0.$$

Proof. For $|\mathbf{x}_0| < r$ let us introduce the function:

$$v(\mathbf{x}_0) = \inf_{0 \leq t \leq T_{\mathbf{x}_0}} |\mathbf{x}(\mathbf{x}_0, 0, t)|.$$

According to conditions of the lemma this function admits the following properties:

(i) $v(0) = 0$ and $v(\mathbf{x}) > 0$ for $0 < |\mathbf{x}| < r$;

(ii) $v(\mathbf{x}_0) = \inf_{0 \leq t \leq T_{\mathbf{x}_0} + \Delta} |\mathbf{x}(\mathbf{x}_0, 0, t)|$ for any $\Delta \geq 0$.

Additionally for $0 < |\mathbf{x}| < r$ the property $|v(0) - v(\mathbf{x})| = v(\mathbf{x}) \leq |\mathbf{x}| = |0 - \mathbf{x}|$ holds, which means continuity of function v at the origin. In the set $|\mathbf{x}| < r$ the relation $\delta(|\mathbf{x}|) \leq v(\mathbf{x}) \leq |\mathbf{x}|$ holds, where $\delta(s) = s(1+s)^{-1} \inf_{|\mathbf{x}| = s} v(\mathbf{x})$ is a continuous and strictly increasing function, $\delta(0) = 0$. The locally Lipschitz property of function v in the set $0 < |\mathbf{x}| < r$ follows from the following series of inequalities satisfied for any $\mathbf{x}_1$, $\mathbf{x}_2$ belonging to this set and some constants $L > 0$, $M > 0$, $T = \max\{T_{\mathbf{x}_1}, T_{\mathbf{x}_2}\}$:

$$|\mathbf{x}(\mathbf{x}_1, 0, t) - \mathbf{x}(\mathbf{x}_2, 0, t)| \leq M|\mathbf{x}_1 - \mathbf{x}_2|, \quad t \leq T;$$

$$||\mathbf{x}(\mathbf{x}_1, 0, t)| - |\mathbf{x}(\mathbf{x}_2, 0, t)|| \leq L|\mathbf{x}_1 - \mathbf{x}_2|, \quad t \leq T;$$

$$|v(\mathbf{x}_1) - v(\mathbf{x}_2)| = |\inf_{0 \leq t \leq T} |\mathbf{x}(\mathbf{x}_1, 0, t)| - \inf_{0 \leq t \leq T} |\mathbf{x}(\mathbf{x}_2, 0, t)||$$

$$\leq \sup_{0 \leq t \leq T} ||\mathbf{x}(\mathbf{x}_1, 0, t)| - |\mathbf{x}(\mathbf{x}_2, 0, t)|| \leq L|\mathbf{x}_1 - \mathbf{x}_2|.$$

By construction for initial conditions $|\mathbf{x}_0| < r$ the relation $v(\mathbf{x}(\mathbf{x}_0, 0, t)) \geq v(\mathbf{x}(\mathbf{x}_0, 0, 0))$, $t \leq T_{\mathbf{x}_0}$ holds, then $D v(\mathbf{x}) \mathbf{f}(\mathbf{x}, 0) \geq 0$ for all $|\mathbf{x}| < r$ and function $v(t)$ is not decreasing. To design a strictly increasing function let us introduce for $|\mathbf{x}_0| < r$ the function:

$$V_1(\mathbf{x}_0) = \inf_{0 \leq t \leq T_{\mathbf{x}_0}} k(t) v(\mathbf{x}(\mathbf{x}_0, 0, t)),$$

where $k : R_+ \to R_+$ is a continuously differentiable function with the following properties for all $t \in R_+$:

$$\kappa_1 \leq k(t) \leq \kappa_2, \quad 0 < \kappa_1 < \kappa_2 < +\infty; \quad \partial k / \partial t < 0.$$

As an example of such function k it is possible to choose the following one:

$$k(t) = \kappa_1 + (\kappa_2 - \kappa_1)\, e^{-t}, \quad \dot{k}(t) = (\kappa_1 - \kappa_2)\, e^{-t}.$$

By construction $V_1(0) = 0$ and $V_1(\mathbf{x}) > 0$ for $0 < |\mathbf{x}| < r$. In the set $|\mathbf{x}| < r$ the relation $\kappa_1\, \delta(|\mathbf{x}|) \le v(\mathbf{x}) \le \kappa_2 |\mathbf{x}|$ holds. The locally Lipschitz continuity of function V_1 in the set $0 < |\mathbf{x}| < r$ follows from the same arguments, since the following series of inequalities are satisfied for any $\mathbf{x}_1,\ \mathbf{x}_2$ belonging to this set and some constants $L > 0$, $M > 0$, $T = \max\{T_{\mathbf{x}_1}, T_{\mathbf{x}_2}\}$:

$$|\mathbf{x}(\mathbf{x}_1, 0, t) - \mathbf{x}(\mathbf{x}_2, 0, t)| \le M|\mathbf{x}_1 - \mathbf{x}_2|, \quad t \le T;$$

$$|v(\mathbf{x}_1) - v(\mathbf{x}_2)| \le L\,|\mathbf{x}_1 - \mathbf{x}_2|;$$

$$|v(\mathbf{x}(\mathbf{x}_1, 0, t)) - v(\mathbf{x}(\mathbf{x}_2, 0, t))| \le M\,L\,|\mathbf{x}_1 - \mathbf{x}_2|, \quad t \le T;$$

$$\begin{aligned}|V_1(\mathbf{x}_1) - V_1(\mathbf{x}_2)| = &\ |\inf\nolimits_{0\le t\le T_{\mathbf{x}_1}} k(t)\, v(\mathbf{x}(\mathbf{x}_1, 0, t)) - \inf\nolimits_{0\le t\le T_{\mathbf{x}_2}} k(t)\, v(\mathbf{x}(\mathbf{x}_2, 0, t))| \\ &\le \sup\nolimits_{0\le t\le T} k(t)|v(\mathbf{x}(\mathbf{x}_1, 0, t) - v(\mathbf{x}(\mathbf{x}_2, 0, t))| \le \kappa_2 M\,L\,|\mathbf{x}_1 - \mathbf{x}_2|.\end{aligned}$$

For $|\mathbf{x}| \ge r$ extend function $V_1 : R^n \to R_+$ in such a way that for all $\mathbf{x} \in R^n$ function V_1 is continuous and locally Lipschitz and there exist two functions $\upsilon_1, \upsilon_2 \in K_\infty$ such that for all $\mathbf{x} \in R^n$:

$$\upsilon_1(|\mathbf{x}|) \le V_1(\mathbf{x}) \le \upsilon_2(|\mathbf{x}|),$$

where $\upsilon_1(s) \le \kappa_1\, \delta(s)$, $\kappa_2\, s \le \upsilon_2(s)$ for $s < r$. By construction for initial conditions $0 < |\mathbf{x}_0| < r$ the following relations hold:

$$\begin{aligned}V_1(\mathbf{x}(\mathbf{x}_0, 0, t)) &= \inf_{0\le\tau\le T_{\mathbf{x}(\mathbf{x}_0,0,t)}} k(\tau)\, v(\mathbf{x}[\mathbf{x}(\mathbf{x}_0, 0, t), 0, \tau]) \\ &> \inf_{0\le\tau\le T_{\mathbf{x}_0}} k(\tau)\, v(\mathbf{x}[\mathbf{x}_0, 0, \tau]) = V_1(\mathbf{x}_0), 0 < t \le T_{\mathbf{x}_0}, T_{\mathbf{x}(\mathbf{x}_0,0,t)} < T_{\mathbf{x}_0},\end{aligned}$$

then $D\,V_1(\mathbf{x})\,\mathbf{f}(\mathbf{x}, 0) > 0$ for all $0 < |\mathbf{x}| < r$. ☐

Under conditions of Lemma 1 solutions $\mathbf{x}(\mathbf{x}_0, 0, t)$ of the system (3) are locally unstable for initial conditions $\mathbf{x}_0$ which belong to the sphere $0 < |\mathbf{x}_0| < r$. According to the result of the lemma in this case the system (3) has corresponding Lyapunov function with positive time derivative for $0 < |\mathbf{x}| < r$. It is possible to say that Lemma 1 presents a variant of necessary conditions of a Lyapunov function existence for a subclass of strictly unstable systems, which is a new result.

LEMMA 2. *Let there exist constants $R > 0$ and $0 < T_{R,\mathbf{x}_0} < +\infty$ such that for solutions of the system* (3) *the following property is satisfied:*

$$|\mathbf{x}_0| > R \quad \Rightarrow \quad |\mathbf{x}(\mathbf{x}_0, 0, t)| < R, t \ge T_{R,\mathbf{x}_0}.$$

Then there exists a continuous and locally Lipschitz–Lyapunov function $V_2(\mathbf{x})$ such that for all $\mathbf{x} \in R^n$

$$\upsilon_3(|\mathbf{x}|) \le V_2(\mathbf{x}) \le \upsilon_4(|\mathbf{x}|), \quad \upsilon_3, \upsilon_4 \in K_\infty,$$

and for all $|\mathbf{x}| > R$ it holds that

$$D\,V_2(\mathbf{x})\,\mathbf{f}(\mathbf{x}, 0) < 0.$$

Proof. For $|\mathbf{x}_0| > R$ let us introduce the function

$$v(\mathbf{x}_0) = \sup_{t\geq 0} |\mathbf{x}(\mathbf{x}_0, 0, t)| = \sup_{T_{R,\mathbf{x}_0}\geq t\geq 0} |\mathbf{x}(\mathbf{x}_0, 0, t)|.$$

Under conditions of the lemma the property $v(\mathbf{x}) > R$ for $|\mathbf{x}| > R$ is satisfied. Additionally due to continuity of solutions of the system (3) with respect to initial conditions for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{array}{l} \mathbf{x}_1 \in R^n, \quad \mathbf{x}_2 \in R^n, \\ |\mathbf{x}_1 - \mathbf{x}_2| \leq \delta \quad \Rightarrow |\mathbf{x}(\mathbf{x}_2, 0, t) - \mathbf{x}(\mathbf{x}_1, 0, t)| \leq \varepsilon, t \leq t_{\max}, t_{\max} = \max\{T_{R,\mathbf{x}_1}, T_{R,\mathbf{x}_2}\}. \end{array}$$

Note that for solutions of the system the equality $\sup_{t_{\max}\geq t\geq 0} |\mathbf{x}(\mathbf{x}_i, 0, t)| = \sup_{t\geq 0} |\mathbf{x}(\mathbf{x}_i, 0, t)|$, $i = 1, 2$ is satisfied. Then for any initial conditions under constrain $|\mathbf{x}_1 - \mathbf{x}_2| \leq \delta$, $|\mathbf{x}_1| > R$, $|\mathbf{x}_2| > R$ it holds that

$$\begin{aligned} &|v(\mathbf{x}_1) - v(\mathbf{x}_2)| \\ &\quad = \left| \sup_{t_{\max}\geq t\geq 0} |\mathbf{x}(\mathbf{x}_1, 0, t)| - \sup_{t_{\max}\geq t\geq 0} |\mathbf{x}(\mathbf{x}_2, 0, t)| \right| \\ &\quad \leq \sup_{t_{\max}\geq t\geq 0} ||\mathbf{x}(\mathbf{x}_1, 0, t)| - |\mathbf{x}(\mathbf{x}_2, 0, t)|| \leq \varepsilon, \end{aligned}$$

which means continuity of function v for $|\mathbf{x}| > R$. In the set $|\mathbf{x}| > R$ for function v the following relation also holds:

$$|\mathbf{x}| \leq v(\mathbf{x}) \leq \delta(|\mathbf{x}|),$$

where $\delta(s) = s + \sup_{|\mathbf{x}| = s} v(\mathbf{x})$ is a continuous and strictly increasing function. The locally Lipschitz continuity of function v into set $|\mathbf{x}| > R$ follows from the series of inequalities satisfied for any $\mathbf{x}_1,\ \mathbf{x}_2$ from the set and some $L > 0$:

$$||\mathbf{x}(\mathbf{x}_1, 0, t)| - |\mathbf{x}(\mathbf{x}_2, 0, t)|| \leq L|\mathbf{x}_1 - \mathbf{x}_2|, \quad t \leq t_{\max},$$

$$\begin{aligned} &|v(\mathbf{x}_1) - v(\mathbf{x}_2)| \\ &\quad = \left| \sup_{t_{\max}\geq t\geq 0} |\mathbf{x}(\mathbf{x}_1, 0, t)| - \sup_{t_{\max}\geq t\geq 0} |\mathbf{x}(\mathbf{x}_2, 0, t)| \right| \\ &\quad \leq \sup_{t_{\max}\geq t\geq 0} ||\mathbf{x}(\mathbf{x}_1, 0, t)| - |\mathbf{x}(\mathbf{x}_2, 0, t)|| \leq L\,|\mathbf{x}_1 - \mathbf{x}_2|. \end{aligned}$$

By construction for all initial conditions with $|\mathbf{x}_0| > R$ it holds that

$$v(t) = v(\mathbf{x}(\mathbf{x}_0, 0, t)) \leq v(\mathbf{x}(\mathbf{x}_0, 0, 0)) = v(0),$$

then $D\,v(\mathbf{x})\,\mathbf{f}(\mathbf{x}, 0) \leq 0$ for $|\mathbf{x}| > R$ and function v is not increasing. To design a strictly decreasing function, consider the following one for $|\mathbf{x}_0| > R$:

$$V_2(\mathbf{x}_0) = \sup_{T_{R,\mathbf{x}_0}\geq t\geq 0} k(t)\,v(\mathbf{x}(\mathbf{x}_0, 0, t)),$$

where $k : R_+ \to R_+$ is a continuously differentiable function with properties for all $t \in R_+$:

$$\kappa_3 \leq k(t) \leq \kappa_4, \quad 0 < \kappa_3 < \kappa_4 < +\infty; \quad \partial\,k/\partial\,t > 0.$$

For example, it is possible to choose as a function $k(t)$ the following one:

$$k(t) = \frac{\kappa_3 + \kappa_4 t}{1+t}, \quad \dot{k}(t) = \frac{\kappa_4 - \kappa_3}{(1+t)^2}.$$

Under conditions of the lemma in the set $|\mathbf{x}| > R$ for function V_2 the relation $\kappa_3 |\mathbf{x}| \leq V_2(\mathbf{x}) \leq \kappa_4 \delta(|\mathbf{x}|)$ holds. For any initial conditions under constrain $|\mathbf{x}_1 - \mathbf{x}_2| \leq \delta$, $|\mathbf{x}_1| > R$, $|\mathbf{x}_2| > R$ it holds that

$$\begin{aligned} &|V_2(\mathbf{x}_1) - V_2(\mathbf{x}_2)| \\ &\quad = \left| \sup_{T_{R,\mathbf{x}_1} \geq t \geq 0} k(t)\, v(\mathbf{x}(\mathbf{x}_1, 0, t)) - \sup_{T_{R,\mathbf{x}_2} \geq t \geq 0} k(t)\, v(\mathbf{x}(\mathbf{x}_2, 0, t)) \right| \\ &\quad \leq \sup_{t_{\max} \geq t \geq 0} k(t) ||\mathbf{x}(\mathbf{x}_1, 0, t)| - |\mathbf{x}(\mathbf{x}_2, 0, t)|| \leq \kappa_4 \varepsilon, \end{aligned}$$

which means continuity of function V_2 for $|\mathbf{x}| > R$. The locally Lipschitz continuity of function V_2 into set $|\mathbf{x}| > R$ follows from the same inequalities satisfied for any $\mathbf{x}_1$, $\mathbf{x}_2$ from the set and some $L > 0$:

$$\begin{aligned} &|V_2(\mathbf{x}_1) - V_2(\mathbf{x}_2)| \\ &\quad = \left| \sup_{T_{R,\mathbf{x}_1} \geq t \geq 0} k(t)\, v(\mathbf{x}(\mathbf{x}_1, 0, t)) - \sup_{T_{R,\mathbf{x}_2} \geq t \geq 0} k(t)\, v(\mathbf{x}(\mathbf{x}_2, 0, t)) \right| \\ &\quad \leq \sup_{t_{\max} \geq t \geq 0} k(t) ||\mathbf{x}(\mathbf{x}_1, 0, t)| - |\mathbf{x}(\mathbf{x}_2, 0, t)|| \leq \kappa_4 L |\mathbf{x}_1 - \mathbf{x}_2|. \end{aligned}$$

For $|\mathbf{x}| \leq R$ we extend the definition of function V_2 such that for all $\mathbf{x} \in R^n$ function $V_2 : R^n \to R_+$ would be continuous and locally Lipschitz and for all $\mathbf{x} \in R^n$:

$$\upsilon_3(|\mathbf{x}|) \leq V_2(\mathbf{x}, t) \leq \upsilon_4(|\mathbf{x}|),$$

where $\upsilon_3, \upsilon_4 \in K_\infty$ and $\kappa_4 s \geq \upsilon_3(s)$, $\upsilon_4(s) \geq \kappa_3 \delta(s)$ for $s > R$. By construction for all initial conditions with $|\mathbf{x}_0| > R$, it holds that

$$\begin{aligned} V_2(t) = V_2(\mathbf{x}(\mathbf{x}_0, 0, t)) &= \sup_{T_{R,\mathbf{x}(\mathbf{x}_0,0,t)} \geq \tau \geq 0} k(\tau)\, v(\mathbf{x}[\mathbf{x}(\mathbf{x}_0, 0, t), 0, \tau]) \\ < \sup_{T_{R,\mathbf{x}_0} \geq \tau \geq 0} k(\tau) v(\mathbf{x}[\mathbf{x}_0, 0, \tau]) &= V_2(\mathbf{x}_0) = V_2(0), 0 < t \leq T_{R,\mathbf{x}_0}, T_{R,\mathbf{x}(\mathbf{x}_0,0,t)} < T_{R,\mathbf{x}_0}, \end{aligned}$$

and then $D V_2(\mathbf{x}) \mathbf{f}(\mathbf{x}, 0) < 0$ for $|\mathbf{x}| > R$. ∎

Under conditions of the lemma set, $A = \{\mathbf{x} : |\mathbf{x}| < R\}$ is a globally attractive invariant set for solutions of system (3) with zero input; see also [17] for other converse Lyapunov theorems for set stability. Contrarily to the case considered in this paper, the Lyapunov functions $W : R^n \to R_+$ proposed in [17] possess for all $\mathbf{x} \in R^n$ the properties

$$\alpha_1(|\mathbf{x}|_A) \leq W(\mathbf{x}) \leq \alpha_2(|\mathbf{x}|_A), \quad \alpha_1, \alpha_2 \in K_\infty,$$

where $|\mathbf{x}|_A$ is the distance from point $\mathbf{x}$ to the set A, which stability is investigated.

Now we are ready to substantiate the necessary conditions of oscillatority.

THEOREM 2. *Let system* (3) *be uniformly oscillatory with respect to the output* $\psi = \eta(\mathbf{x})$ *(where* $\eta : R^n \to R$ *is a continuous function), and for all* $\mathbf{x} \in R^n$ *the following relations are satisfied:*

$$\chi_1(|\mathbf{x}|) \leq \eta(\mathbf{x}) \leq \chi_2(|\mathbf{x}|), \quad \chi_1, \chi_2 \in K_\infty;$$

the set of initial conditions for which the system is not oscillating consists in just one point $\Xi = \{ \mathbf{x} : \mathbf{x} = 0 \}$. *Then there exist two continuous and locally Lipschitz Lyapunov functions* $V_1 : R^n \to R_+$ *and* $V_2 : R^n \to R_+$ *such that for all* $\mathbf{x} \in R^n$ *the inequalities hold:*

$$\upsilon_1 (|\mathbf{x}|) \le V_1(\mathbf{x}) \le \upsilon_2 (|\mathbf{x}|), \quad \upsilon_3 (|\mathbf{x}|) \le V_2(\mathbf{x}) \le \upsilon_4 (|\mathbf{x}|), \quad \upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4 \in K_\infty;$$

$$DV_1(\mathbf{x})\mathbf{f}(\mathbf{x}, 0) > 0 \text{ for } 0 < |\mathbf{x}| < \chi_2^{-1}(\pi^-);$$

$$DV_2(\mathbf{x})\mathbf{f}(\mathbf{x}, 0) < 0 \text{ for } |\mathbf{x}| > \chi_1^{-1}(\pi^+).$$

Proof. Since system (3) is uniformly oscillatory with respect to output $\psi = \eta(\mathbf{x})$, then for almost all initial conditions (except the origin) there exists constants $-\infty < \pi^- < \pi^+ < +\infty$ such that

$$\begin{aligned} \underline{\lim}_{t \to +\infty} \eta(\mathbf{x}(\mathbf{x}_0, 0, t)) &= \underline{\lim}_{t \to +\infty} \psi(t) = \pi^-; \\ \overline{\lim}_{t \to +\infty} \eta(\mathbf{x}(\mathbf{x}_0, 0, t)) &= \overline{\lim}_{t \to +\infty} \psi(t) = \pi^+. \end{aligned}$$

By radial unboundedness and positive definiteness of function η it means that all solutions of the system converge to the invariant set $\Omega = \{ \mathbf{x} : \chi_2^{-1}(\pi^-) \le \mathbf{x} \le \chi_1^{-1}(\pi^+) \}$. Then there exist constants $X_1 < \chi_2^{-1}(\pi^-)$ and $X_2 > \chi_1^{-1}(\pi^+)$ such that conditions of Lemmas 1 and 2 hold for $r = X_1$ and $R = X_2$. Based on these facts, the existence of Lyapunov functions V_1 and V_2 follows. $\square$

For uniformly oscillatory systems with single equilibrium point at the origin, Theorems 1 and 2 give necessary and sufficient conditions of oscillations existence (Van der Pol or Hindmarsh and Rose systems (see below) are examples of uniformly oscillatory systems). The oscillatority concept introduced by Yakubovich covers situations of periodic and chaotic oscillations. That allows one to analyze behavior of wide spectrum of oscillating dynamical systems using common approach. Note that for chaotic systems constants π^- and π^+ evaluate geometrical size of strange attractor. Let us demonstrate on examples the efficiency of the proposed approach for analysis of oscillation phenomena in nonlinear systems.

Example 1. Consider the Van der Pol system:

$$\dot{x}_1 = x_2; \quad \dot{x}_2 = -x_1 + \varepsilon (1 - x_1^2) x_2,$$

where $\varepsilon > 0$ some parameter. To detect presence of oscillations in this system, it is required (according to Theorem 1) to find two Lyapunov functions, which establish local instability of equilibrium $(0, 0)$ and global boundedness of the system solutions. Since the system has only one equilibrium point in the origin, the set Ω from the theorem does not contain the point $(0, 0)$. Let us consider the following Lyapunov functions for $0 < \varepsilon \le 1$:

$$\begin{aligned} V_1(\mathbf{x}) &= 0.5 \left((1 - \varepsilon + \varepsilon^{-1}) x_1^2 + (1 + \varepsilon^{-1}) x_2^2 + \varepsilon (x_2 - \varepsilon x_1)^2 \right); \\ V_2(\mathbf{x}) &= 0.5 \left(\varepsilon^{-1} x_2 - 2 x_1 + 1/3 x_1^3 \right)^2 + 1/12 x_1^4, \\ \dot{V}_1 &= \varepsilon x_2^2 + (x_2 - \varepsilon x_1)^2 + \left[\varepsilon^3 x_1 - (1 + \varepsilon + \varepsilon^2) x_2 \right] x_1^2 x_2; \\ \dot{V}_2 &= - \left[0.5 \sqrt{\varepsilon} \left(2 - \varepsilon^{-2} \right) x_1 - \varepsilon^{-0.5} x_2 \right]^2 - 1/3 \varepsilon^{-1} x_1^4 \\ &\quad + \left[0.25 \varepsilon \left(2 - \varepsilon^{-2} \right)^2 + 2 \varepsilon^{-1} \right] x_1^2. \end{aligned}$$

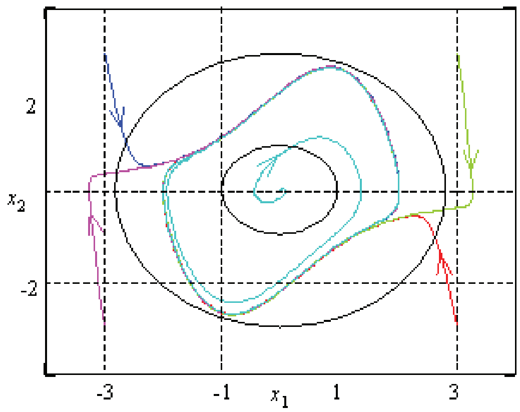

FIG. 1. *Trajectories and set Ω for Van der Pol system.*

Function $\dot{V}_1$ is strictly positive in the set $0 < |\,\mathbf{x}\,| < X_1$, where $X_1 = X_1(\,\varepsilon\,) > 0$ (the same conclusion was obtained in [12] for $\varepsilon = 1$, $X_1 = \sqrt{3}$). Instability of the system also can be verified for a linearized version of the system, which eigenvalues $\lambda_{1,2} = 0.5\,\left(\,\varepsilon \pm \sqrt{\varepsilon^2 - 4}\,\right)$ are always positive for $\varepsilon > 0$. Analyzing function $\dot{V}_2$ it is possible to obtain $X_2 \le \sqrt{3\,[\,0.25\,\varepsilon^2\,(\,2 - \varepsilon^{-2}\,)^2 + 2\,]}$. Results of the set Ω calculation and computer simulation of the system for $\varepsilon = 1$ are presented in Figure 1, where the set Ω is bounded by solid ellipses.

Example 2. Let us consider Lorenz model:

$$\begin{aligned}\dot{x} &= \sigma\,(\,y - x\,),\\ \dot{y} &= r\,x - y - x\,z,\\ \dot{z} &= -b\,z + x\,y\,,\end{aligned}$$

where parameters $\sigma = 10$, $r = 28$, and $b = 8/3$. With such choice of parameter values the system is chaotic, which is a good example of complex nonlinear oscillation processes. To apply the result of Theorem 1 here let us note that the system has three equilibriums with coordinates

$$\mathbf{x}_e^1 = (\,0\;0\;0\,)^T, \quad \mathbf{x}_e^2 = (\,\sqrt{72}\;\sqrt{72}\;27\,)^T, \quad \mathbf{x}_e^3 = (\,-\sqrt{72}\;\;-\sqrt{72}\;27\,)^T.$$

The matrix of linear approximation of this system at the equilibriums

$$A(\,\mathbf{x}_e\,) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - \mathbf{x}_{e,3} & -1 & -\mathbf{x}_{e,1} \\ \mathbf{x}_{e,2} & \mathbf{x}_{e,1} & -b \end{bmatrix}$$

has for the given values of parameters eigenvalues with positive real parts for all equilibriums. Therefore the system is locally unstable. Lyapunov function

$$V(\,x, y, z\,) = 0.5\,\left(\,\sigma^{-1}x^2 + y^2 + (\,z - r\,)^2\,\right)$$

for this system has the following time derivative:

$$\begin{aligned}\dot{V} &= -x^2 + x\,y\, - y^2 - b\,z^2 + r\,b\,z\\ &\le -0.5\,x^2 - 0.5\,y^2 - 0.5\,b\,z^2 + 0.5\,b\,r^2\,,\end{aligned}$$

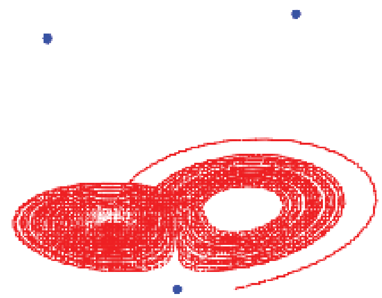

FIG. 2. *Trajectory of Lorenz system.*

which implies global boundedness of all trajectories of Lorenz system. All conditions of Corollary 1 are satisfied and system is oscillatory in the sense of Definition 3. An example of state space trajectory of the system is presented in Figure 2 (blue dots correspond to coordinates of equilibriums $\mathbf{x}_e^i$).

Example 3. A Hindmarsh and Rose model neuron is defined by the following system of differential equations [14]:

$$\begin{aligned}
&\dot{x} = -a\,x^3 + b\,x^2 + y - z + u,\\
&\dot{y} = c - d\,x^2 - y,\\
&\dot{z} = \varepsilon\,[\,s\,(\,x - x_0\,) - z\,]\,,
\end{aligned}$$

where $x \in R_+$ is the membrane potential, $y \in R_+$ is recovery variable, and $z \in R_+$ is adaptation variable. External stimulation is given by input $u \in R$. It is a well-known fact that this model demonstrates complex oscillatory behavior for the following values of the model parameters $a = 1$, $b = 3$, $c = 1$, $d = 5$, $s = 4$, $x_0 = 0.795$, $\varepsilon = 0.001$ with input $u = 0$. Let us investigate oscillatority property of the model for the case $u = 0$ applying the proposed approach.

As the first let us compute the number of equilibriums in the system which coordinates are solutions of the following system of nonlinear equations:

$$\begin{aligned}
&-a\,x_e^3 + (\,b - d\,)\,x_e^2 - s\,x_e + s\,x_0 + c = 0\,;\\
&y_e = c - d\,x_e^2\,;\\
&z_e = s\,(\,x_e - x_0\,)\,.
\end{aligned}$$

As in the first example we are interested in a situation when the model has a single equilibrium. This is the case when the first cubic equation above has only one real solution and two complex solutions. Under conditions

$$n \geq 0, \quad \frac{m}{6\,a} + \frac{2}{3}\,\frac{3\,s\,a - (\,b - d\,)^2}{a\,u} \neq 0,$$

$$n = 4\,s^3\,a - s^2\,(\,b - d\,)^2 + \left[\,27\,a^2\,(\,s\,x_0 + c\,) - 18\,s\,a\,(\,b - d\,) + 4\,(\,b - d\,)^3\,\right]\,(\,s\,x_0 + c\,),$$

$$m = \sqrt[3]{12\,a\,\sqrt{3\,n} - 36\,s\,a\,(\,b - d\,) + 108\,a^2\,(\,s\,x_0 + c\,) + 8\,(\,b - d\,)^3},$$

the model has the following single equilibrium

$$\begin{aligned}
&x_e = a^{-1}\,(\,m/6 - 2/3\,[\,3\,s\,a - (\,b - d\,)^2\,]/m + (\,b - d\,)/3\,)\,;\\
&y_e = c - d\,x_e^2\,;\\
&z_e = s\,(\,x_e - x_0\,)\,.
\end{aligned}$$

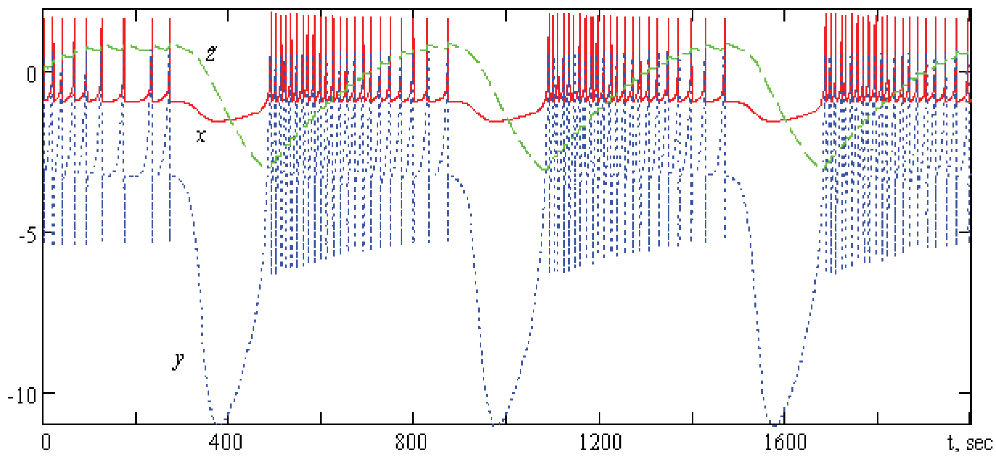

FIG. 3. *Trajectories of Hindmarsh and Rose neuron model.*

To prove global boundedness of the system solutions, it is possible to use the following Lyapunov function:

$$V_2 = 0.5\ \left(s\,x^2 + \varepsilon^{-1} z^2 + s\,a\,y^2/d^2 \right),$$

in which the time derivative for the model admits inequality:

$$\dot{V}_2 \le s\,x\ \left(-0.5\,a\,x^3 + b\,x^2 + 8\,d^2 x/a \right) - 0.25\,s\,a\,y^2/d^2 - 0.5\,z^2 + 8\,s\,a\,c^2/d^2 + 0.5\,s^2\,x_0^2.$$

To prove local instability of the equilibrium, consider linearization of the system with matrix

$$A(\,x_e, y_e, z_e\,) = \begin{bmatrix} -3\,a\,x_e^2 + 2\,b\,x_e & 1 & -1 \\ -2\,d\,x_e & -1 & 0 \\ \varepsilon\,s & 0 & -\varepsilon \end{bmatrix}.$$

According to Hurwitz criteria matrix A has eigenvalues with positive real parts if at least one from the following inequalities is satisfied:

$$3\,a\,x_e^2 - 2\,b\,x_e + 1 + \varepsilon \le 0, \quad 3\,a\,x_e^2 + 2\,(\,d - b\,)\,x_e + s \le 0,$$

$$3\,a\,(\,\varepsilon + 1\,)\,x_e^2 + 2\,(\,d - (\,\varepsilon + 1\,)\,b\,)\,x_e + \varepsilon\,(\,s + 1\,) \le 0,$$

$$\begin{aligned} 9a^2(\varepsilon+1)x_e^4 &+ a[6d - 12(\varepsilon+1)b]x_e^3 + \left[4b[(\varepsilon+1)b - d] + 3a[\varepsilon^2 + (2+s)\varepsilon + 1]\right] x_e^2 \\ &+ 2\left[d - [\varepsilon^2 + (s+2)\varepsilon + 1]b\right] x_e + (s+1)\varepsilon^2 + \varepsilon \le 0. \end{aligned}$$

Thus we obtain all set of restrictions on admissible values of the model parameters under which the system is uniformly oscillatory. The proposed values in [14] of the model parameters admit all these conditions (there exists single unstable equilibrium with globally bounded solutions). The result of the model simulation is shown in Figure 3, where $\tilde{z} = 10\,z$ is a scaled adaptation variable.

A link between oscillatority and excitation indices is established in the following corollary.

COROLLARY 2. *Let for initial condition* $\mathbf{x}_0 \in R^n$ *the solution* $\mathbf{x}(\mathbf{x}_0, \mathbf{k}(\mathbf{x}), t)$ *of system* (1) *with control* $\mathbf{u} = \mathbf{k}(\mathbf{x})$, $\mathbf{k}(0) = 0$ *be* $[\pi^-, \pi^+]$*-oscillation with respect to output*

$$\psi = \eta(\mathbf{x}), \quad \alpha_1(|\mathbf{x}|) \le \eta(\mathbf{x}), \quad \alpha_1 \in K_\infty.$$

Then excitation indices of system (1) *satisfy inequality*

$$\pi^+ - \pi^- \le \chi^+_{\psi,\mathbf{x}_0}(\gamma) - \chi^-_{\psi,\mathbf{x}_0}(\gamma),$$

for $\gamma \ge \gamma*$*, where* $\gamma* = \sup_{|\mathbf{x}| \le \alpha_1^{-1}(\pi^+)} |\mathbf{k}(\mathbf{x})|$.

Proof. From oscillatority property with respect to output ψ, the solutions of the closed by feedback $\mathbf{k}$ system (1) are asymptotically bounded:

$$|\mathbf{x}(t)| \le \alpha_1^{-1}(\pi^+), \quad t \ge 0.$$

Therefore input $\mathbf{u} = \mathbf{k}(\mathbf{x})$ is upper bounded by $\gamma \ge \gamma*$ and the statement follows from Definitions 3 and 4 (excitation indices are not decreasing functions of γ). ∎

Hence, to compute estimates on excitation indices it is enough to find some control $\mathbf{k}$ for system (1), which ensures oscillations existence in closed loop system.

In the proof of Theorem 1 a component of state space vector was proposed as an oscillating output. However, such output does not discover all features of oscillation processes in the system and it does not restrict the possible set of oscillating variables of the system. To avoid this obstacle we formulate the same conclusion for output oscillations of system (3) rewriting conditions of the theorem with respect to $\mathbf{y}$:

$$\upsilon_1(|\mathbf{y}|) \le V_1(\mathbf{x}) \le \upsilon_2(|\mathbf{y}|), \upsilon_3(|\mathbf{y}|) \le V_2(\mathbf{x}) \le \upsilon_4(|\mathbf{y}|),$$

$$DV_1(\mathbf{x})\mathbf{f}(\mathbf{x}, 0) > 0 \text{ for } 0 < |\mathbf{y}| < Y_1;$$

$$DV_1(\mathbf{x})\mathbf{f}(\mathbf{x}, 0) > 0 \text{ for } |\mathbf{y}| > Y_2,$$

$$Y_1 < \upsilon_1^{-1} \circ \upsilon_2 \circ \upsilon_3^{-1} \circ \upsilon_4(Y_2).$$

Then the set $\Omega = \left\{ \mathbf{y} : \upsilon_2^{-1} \circ \upsilon_1(Y_1) < |\mathbf{y}| < \upsilon_3^{-1} \circ \upsilon_4(Y_2) \right\}$ and the system is oscillatory if set Ω does not contain equilibrium points of closed loop system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, 0)$. A more constructive result, which points out on oscillating variables, can be presented as follows.

LEMMA 3. *Let system* (1) *have IOSS Lyapunov function* W *and* h*-dissipative storage function* V *as in Definition* 2 *and* $\lim_{s \to +\infty} \alpha(s)^{-1}\sigma_2(s) < +\infty$ *(conditions of Lemma* A.1 *hold). Suppose that* $\mathbf{u} = \mathbf{k}(\mathbf{x})$ *and*

(i) $\alpha_6(|\mathbf{x}|) > \delta(|\mathbf{k}(\mathbf{x})|)$ *for* $|\mathbf{x}| > X \ge 0$ *and* $\mathbf{x} \notin \Xi$,

(ii) $L_{\mathbf{f}(\mathbf{x},\mathbf{k}(\mathbf{x}))}V(\mathbf{x}) > 0$ *for* $0 < |\mathbf{h}(\mathbf{x})| \le Y$ *and* $\mathbf{x} \notin \Xi$,

for some positive constants X *and* Y *with* $Y < \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5(X)$ *(where functions* $\alpha_4, \alpha_5, \alpha_6$ *and* δ *defined in Lemma* A.1*), set* Ξ *has zero Lebesgue measure. If set* $\Omega = \left\{ V(\mathbf{x}) : \underline{\alpha}(Y) \le V(\mathbf{x}) \le \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5(X) \right\}$ *does not contain equilibrium points of closed loop system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}))$*, then the system is oscillatory.*

Proof. First of all note that from point (i) the system satisfies all conditions from Lemma A.1 to be ISS with respect to input $\mathbf{u}$ and it also has bounded (i.e., defined for all $t \ge 0$) solutions due to property (i). As before, $\mathbf{x}(t)$ and $\mathbf{y}(t)$ have nonempty closed and compact ω-limit sets, which are upper bounded by estimate $|\mathbf{x}| \le \alpha_4^{-1} \circ \alpha_5(X)$.

From point (ii) of the lemma it is possible to conclude that $\dot{V} > 0$ for small enough $0 < |\mathbf{y}| \le Y$. Then the set of ω-limit trajectories for function $V(t)$ belongs to the set Ω. Now the result immediately follows similarly to the final steps of Theorem 1 proof. $\square$

Generically function V depends on part of variables only, which helps to define a subset of oscillating variables in the system. Additionally, Lemma 3 points out a way to find functions V_1 and V_2 ($V_1(\mathbf{x}) = V(\mathbf{x})$ and $V_2(\mathbf{x}) = U(\mathbf{x})$ from Appendix). Results of proposed theorems and Lemma 3 do not deal with feedback $\mathbf{k}$ design problem. Now let us continue with the task of control design that ensures desired oscillation parameters for passive systems.

4. Stabilization of oscillation regimes. In this section the problem of feedback design for passive system is considered, and the proposed feedback ensures oscillatority of closed loop system. Section 4 is based on result of Lemma A.2, although conditions imposed on feedback $\mathbf{k}$ in the Lemma A.2 look complex and hardly verified, they are very natural and can be easily resolved. For example, if σ_1 and σ_2 are quadratic functions of their arguments, then control $\mathbf{k}$ with linear growth rate with respect to $\mathbf{y}$ satisfies all proposed conditions.

THEOREM 3. *Let system* (1) *be passive with known dissipation rate* β *and IOSS in the sense of Definition* 2 *and*

$$\underline{\alpha}(|\mathbf{y}|) \le V(\mathbf{x}) \le \overline{\alpha}(|\mathbf{x}|), \quad \underline{\alpha}, \overline{\alpha} \in K_\infty.$$

Consider control $\mathbf{u} = \mathbf{k}(\mathbf{x}) + \mathbf{d}$, *which possesses the following properties for all* $\mathbf{x} \in R^n$:

(1) *for some* $0 < K < +\infty$,

$$|\mathbf{k}(\mathbf{x})| \le \lambda(|\mathbf{y}|) + K;$$

(2) *decreasing of storage function* V *for large values of the output, i.e., inequality holds*

$$\beta(\mathbf{x}) - \mathbf{y}^T\mathbf{k}(\mathbf{x}) + \mu(|\mathbf{d}|) + \mu(K) \ge \kappa(|\mathbf{y}|) + \mathbf{y}^T\mathbf{d};$$

(3) $\mathbf{y}^T\mathbf{k}(\mathbf{x}) > \beta(\mathbf{x})$ *for* $0 < |\mathbf{y}| < Y < +\infty$, $Y < \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5 \circ \alpha_6^{-1} \circ \delta(K)$, $\lim_{s \to +\infty} \frac{\sigma_2(s) + \sigma_1 \circ \lambda(s)}{\kappa(s)} < +\infty$, *where* $\lambda \in K$, $\kappa \in K_\infty$, $\mu \in K$ *(functions* $\alpha_4, \alpha_5, \alpha_6$ *and* δ *obtained in Lemma* A.2*) and* $\mathbf{d} \in R^m$ *is new input (Lebesgue measurable and essentially bounded function of time). Then*

(i) *system solutions are bounded;*

(ii) *if set* $\Omega = \{ V(\mathbf{x}) : \underline{\alpha}(Y) \le V(\mathbf{x}) \le \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5 \circ \alpha_6^{-1} \circ \delta(K) \}$ *does not contain equilibrium points of system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}))$ *then for* $\mathbf{d}(t) \equiv 0, t \ge 0$ *closed loop system is an oscillatory one.*

Proof. Introduce partition of control input:

$$\mathbf{u} = \mathbf{k}(\mathbf{x}) = -\mathbf{k}_1(\mathbf{x}) + \mathbf{k}_2(\mathbf{x}),$$

such that

$$|\mathbf{k}_1(\mathbf{x})| \le \lambda(|\mathbf{y}|), \quad |\mathbf{k}_2(\mathbf{x})| \le K;$$

$$\mathbf{y}^T\mathbf{k}_1(\mathbf{x}) + \beta(\mathbf{x}) + \mu(|\mathbf{d}|) \ge \kappa(|\mathbf{y}|) + \mathbf{y}^T\mathbf{d};$$

$$\mathbf{y}^T\mathbf{k}_2(\mathbf{x}) > \beta(\mathbf{x}) + \mathbf{y}^T\mathbf{k}_1(\mathbf{x}) \text{ for } 0 < |\mathbf{y}| < Y < +\infty.$$

This separation is possible due to conditions of Theorem 3. Introduce auxiliary input $\tilde{\mathbf{d}} = \mathbf{d} + \mathbf{k}_2(\mathbf{x})$ (essentially bounded by conditions of the theorem $\|\tilde{\mathbf{d}}\| \leq K + \|\mathbf{d}\|$). For system (1) all conditions of Lemma A.2 are satisfied for the feedback $\mathbf{u} = -\mathbf{k}_1(\mathbf{x}) + \tilde{\mathbf{d}}$ and system is ISS with respect to input $\tilde{\mathbf{d}}$. According to ISS property [21] and boundedness of $\tilde{\mathbf{d}}$, boundedness of system solution immediately follows and statement (i) of Theorem 3 is proven. To justify statement (ii) note that the conditions of Lemma 3 also hold. $\square$

Theorem 3 extends the result from [3] and [28] to the case of general nonlinear dynamical systems. Additional special attention is given to the lower estimate of the oscillation amplitude for $\mathbf{d}(t) \equiv 0$, $t \geq 0$.

Exciting part $\mathbf{k}_2$ of feedback $\mathbf{k}$ defines the size of set Ω (due to constants Y and K are prescribed by $\mathbf{k}_2$) and, hence, it regulates the gap between values of π^- and π^+.

Remark 2. It is worth stressing that the control in Theorem 3 is proposed to satisfy some sector condition with respect to output $\mathbf{y}$. For design of such controls in practical application it is possible to use speed-gradient approach [9, 10], e.g., choose $\mathbf{u} = \varphi(\mathbf{y})$, where $\varphi(\mathbf{y})^T\mathbf{y} > 0$ for $0 < |\mathbf{y}| < Y_1$ and $\varphi(\mathbf{y})^T\mathbf{y} < 0$ for $|\mathbf{y}| > Y_2 > Y_1$.

Example 4. Let us consider controlled linear oscillator:

$$\dot{x}_1 = x_2; \quad \dot{x}_2 = -x_1 + u,$$

which is passive with storage function

$$V(\mathbf{x}) = 0.5\left(x_1^2 + x_2^2\right), \quad \dot{V} = x_2\, u,$$

and IOSS with corresponding Lyapunov function

$$W(\mathbf{x}) = 0.5\left(x_1^2 + (x_1 + x_2)^2\right),$$
$$\dot{W} \leq -0.5\left(x_1^2 + x_2^2\right) + x_2^2 + u^2$$

with output $y = x_2$ ($\sigma_1(s) = \sigma_2(s) = s^2$). Then control $u = -k_1(\mathbf{x}) + k_2(\mathbf{x})$ with $k_1(\mathbf{x}) = a\,x_2$, $a > 0.5$ and $k_2(\mathbf{x}) = K\,sign(x_2)$ admits all condition of Theorem 3 with $\lambda(s) = a\,s$, $\kappa(s) = (a - 0.5)\,s^2$, $\mu(s) = 0.5\,s^2$. All functions σ_2, $\sigma_1 \circ \lambda$ and κ are square-law and, hence,

$$\lim_{s \to +\infty} \frac{\sigma_2(s) + \sigma_1 \circ \lambda(s)}{\kappa(s)} < +\infty;$$

inequality $x_2\,k_2(\mathbf{x}) > x_2\,k_1(\mathbf{x})$ holds for $0 < |x_2| < Y$, $Y = K/a$. This system is ISS for control $u = -k_1(\mathbf{x}) + d$ with ISS Lyapunov function:

$$U(\mathbf{x}) = W(\mathbf{x}) + \frac{1 + 2\,a^2}{a - 0.5}\,V(\mathbf{x}),$$
$$\dot{U} \leq -0.5\left(x_1^2 + x_2^2\right) + \left(2 + \frac{0.5 + a^2}{a - 0.5}\right)d^2.$$

Then set

$$\Omega = \left\{\mathbf{x} : \ K/a \leq |\mathbf{x}| \leq \sqrt{1 + \frac{1.5\,a - 0.75}{a^2 + 0.5}}\sqrt{4 + \frac{1 + 2\,a^2}{a - 0.5}}\;K\right\}$$

is always nonempty. Simulation results and bounds of set Ω are shown in Figure 4 for $a = 1$ and $K = 1/3$.

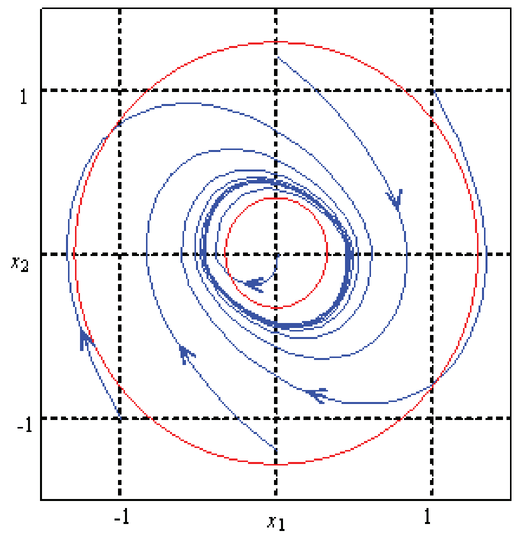

FIG. 4. *Trajectories of linear oscillator under nonlinear feedback.*

Based on the results of Theorem 3 and Corollary 2 it is possible to obtain the estimates of excitation indices of closed loop system for the case of nonvanishing signal $\mathbf{d}$.

COROLLARY 3. *Let all conditions of Theorem 3 hold. Then for* $\|\,\mathbf{d}\,\| \le \gamma < +\infty$

$$0 \le \chi_V^-(\gamma) \le \chi_V^+(\gamma) \le \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5 \circ \alpha_6^{-1} \circ \delta(K+\gamma),$$

if additionally

$$\mathbf{y}(t)^T\mathbf{d}(t) \ge 0 \textit{ for all } t \ge 0, (3),$$

then

$$\underline{\alpha}(Y) \le \chi_V^-(\gamma) < \chi_V^+(\gamma) \le \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5 \circ \alpha_6^{-1} \circ \delta(K+\gamma).$$

Proof. Upper estimate on excitation indices follows from ISS property of the system with respect to input $\tilde{\mathbf{d}}$ (asymptotic gain property in [25]). Now let us consider time derivative of storage function V:

$$\begin{aligned}\dot{V} &= \mathbf{y}^T(-\mathbf{k}_1(\mathbf{x}) + \mathbf{k}_2(\mathbf{x}) + \mathbf{d}) - \beta(\mathbf{x}) \\ &\ge \left[\mathbf{y}^T(-\mathbf{k}_1(\mathbf{x}) + \mathbf{k}_2(\mathbf{x})) - \beta(\mathbf{x})\right] + \mathbf{y}^T\mathbf{d}\,.\end{aligned}$$

From conditions of Theorem 3, the expression in square brackets is positive for $0 < |\,\mathbf{y}\,| < Y < +\infty$, but the presence of sign-varying term $\mathbf{y}^T\mathbf{d}$ allows one to claim only $0 \le \chi_V^-(\gamma) \le \chi_V^+(\gamma)$ in common case. But if $\mathbf{y}(t)^T\mathbf{d}(t) \ge 0$ for all $t \ge 0$, then

$$\begin{aligned}&[\mathbf{y}^T(-\mathbf{k}_1(\mathbf{x}) + \mathbf{k}_2(\mathbf{x})) - \beta(\mathbf{x})] + \mathbf{y}^T\mathbf{d} \\ &\quad \ge \mathbf{y}^T(-\mathbf{k}_1(\mathbf{x}) + \mathbf{k}_2(\mathbf{x})) - \beta(\mathbf{x}),\end{aligned}$$

and the desired result follows by the same line of consideration as in Theorem 3. Further let us suppose that it is possible a situation $\chi_V^-(\gamma) = \chi_V^+(\gamma)$ for some γ. But according to Definition 4, excitation indices admit conditions:

$$\gamma_1 \le \gamma_2 \quad \Rightarrow \quad \chi_V^-(\gamma_2) \le \chi_V^-(\gamma_1) \text{ and } \chi_V^+(\gamma_1) \le \chi_V^+(\gamma_2).$$

Applying the same arguments as in Corollary 2 for the results of Theorem 3 it is possible to obtain

$$0 < \chi_V^+(0) - \chi_V^-(0) \leq \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5 \circ \alpha_6^{-1} \circ \delta(K) - \underline{\alpha}(Y),$$

therefore, $\chi_V^+(\gamma) - \chi_V^-(\gamma) > 0$ for any $\gamma \geq 0$. ∎

According to the corollary index $\chi_V^+(\gamma)$ is always bounded, that is more, it can not be equal to $\chi_V^-(\gamma)$ for any $\gamma \in R_+$ with (3). Thus, system can not lose its oscillation ability for any large enough input disturbance possessing "coordination" condition (3) and such input $\mathbf{d}$ does not provide new equilibrium points into set $\Omega = \left\{ V(\mathbf{x}) : \ \underline{\alpha}(Y) \leq V(\mathbf{x}) \leq \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5(K + \gamma) \right\}$ for system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{k}_1(\mathbf{x}) + \mathbf{k}_2(\mathbf{x}) + \mathbf{d})$. Also it is worth to note, that the requirement (3) can be satisfied for $t \geq T$ only, where $0 \leq T < +\infty$.

5. Conclusion. In this paper conditions for oscillatority in the sense of Yakubovich applicable to nonlinear systems are proposed. Upper and lower bounds for oscillation amplitude are evaluated. Presented conditions are also necessary for some special class of uniformly oscillating systems. Relation between the oscillatority bounds and excitability indices for the systems with input is established. An important advantage of the results of the paper is their applicability to complex nonperiodic (e.g., chaotic) oscillations. Such an advantage is achieved due to using the concept of oscillatority in the sense of Yakubovich as the starting point of the whole study. The results are illustrated by examples: Evaluation of oscillations for Van der Pol and Hindmarsh–Rose neuron systems. As a side result a smooth nonquadratic Lyapunov function providing boundedness of Van der Pol system solutions has been found.

Appendix.

LEMMA A.1. *Let system* (1) *have IOSS Lyapunov function W and h-dissipative storage function V as in Definition* 2. *If*

$$\lim_{s \to +\infty} \frac{\sigma_2(s)}{\alpha(s)} < +\infty,$$

then system (1) *is ISS with ISS Lyapunov function*

$$U(\mathbf{x}) = V(\mathbf{x}) + \tilde{W}(\mathbf{x}), \quad \tilde{W}(\mathbf{x}) = \rho(W(\mathbf{x})),$$
$$\rho(r) = \int_0^r q(s)\, ds, \quad q(s) = \frac{\alpha \circ \sigma_2^{-1}\left(0.25\, \alpha_3 \circ \alpha_2^{-1}(s)\right)}{1 + 0.5\, \alpha_3 \circ \alpha_2^{-1}(s)},$$
$$\alpha_4(s) = \rho \circ \alpha_1(s), \quad \alpha_5(s) = \overline{\alpha}(s) + \rho \circ \alpha_2(s), \quad \alpha_6(s) = 0.5\, q(\alpha_1(s))\, \alpha_3(s),$$
$$L_{\mathbf{f}(\mathbf{x},\mathbf{u})} U(\mathbf{x}) \leq -\alpha_6(|\mathbf{x}|) + \delta(|\mathbf{u}|), \quad \delta(s) = \sigma(s) + 2\chi(2\sigma_1(s))\,\sigma_1(s),$$
$$\chi(2\sigma_2(s)) = \alpha(s)\,[1 + 2\sigma_2(s)]^{-1}.$$

Proof. According to conditions of the lemma and Definition 2, the following series of inequalities holds for all $\mathbf{x} \in R^n$ and $\mathbf{u} \in R^m$:

$$\alpha_1(|\mathbf{x}|) \leq W(\mathbf{x}) \leq \alpha_2(|\mathbf{x}|); \quad L_{\mathbf{f}(\mathbf{x},\mathbf{u})} W(\mathbf{x}) \leq -\alpha_3(|\mathbf{x}|) + \sigma_1(|\mathbf{u}|) + \sigma_2(|\mathbf{y}|);$$
$$\underline{\alpha}(|\mathbf{y}|) \leq V(\mathbf{x}) \leq \overline{\alpha}(|\mathbf{x}|); \quad L_{\mathbf{f}(\mathbf{x},\mathbf{u})} V(\mathbf{x}) \leq -\alpha(|\mathbf{y}|) + \sigma(|\mathbf{u}|),$$

where $\alpha, \alpha_1, \alpha_2, \alpha_3, \underline{\alpha}, \overline{\alpha} \in K_\infty$ and $\sigma, \sigma_1, \sigma_2 \in K$. Let us consider a new IOSS Lyapunov function

$$\tilde{W}(\mathbf{x}) = \rho(W(\mathbf{x})), \quad \rho(r) = \int_0^r q(s)\, ds,$$

where q is some function from class K (that will be defined later). Clearly function $\tilde{W}$ is again continuously differentiable, positive definite, and radially unbounded provided that $\rho \in K_\infty$. Its time derivative admits an estimate:

$$L_{\mathbf{f}(\mathbf{x},\mathbf{u})}\tilde{W}(\mathbf{x}) \le q(W(\mathbf{x}))\,[-\alpha_3(|\mathbf{x}|) + \sigma_1(|\mathbf{u}|) + \sigma_2(|\mathbf{y}|)]\,.$$

To disclose the above inequality let us analyze consequently three situations:

(a) If $0.5\,\alpha_3(|\mathbf{x}|) \ge \sigma_1(|\mathbf{u}|) + \sigma_2(|\mathbf{y}|)$, then

$$L_{\mathbf{f}(\mathbf{x},\mathbf{u})}\tilde{W}(\mathbf{x}) \le -0.5\,q(W(\mathbf{x}))\,\alpha_3(|\mathbf{x}|)\,;$$

(b) If $0.5\,\alpha_3(|\mathbf{x}|) < \sigma_1(|\mathbf{u}|) + \sigma_2(|\mathbf{y}|)$ and $\sigma_1(|\mathbf{u}|) \le \sigma_2(|\mathbf{y}|)$, then

$$\begin{aligned} L_{\mathbf{f}(\mathbf{x},\mathbf{u})}\tilde{W}(\mathbf{x}) &\le -q(W(\mathbf{x}))\,\alpha_3(|\mathbf{x}|) + 2\,q(W(\mathbf{x}))\,\sigma_2(|\mathbf{y}|) \\ &\le -q(W(\mathbf{x}))\,\alpha_3(|\mathbf{x}|) + 2\,\chi(2\,\sigma_2(|\mathbf{y}|))\,\sigma_2(|\mathbf{y}|)\,, \end{aligned}$$

where $\chi(s) = q \circ \alpha_2 \circ \alpha_3^{-1}(2\,s)$;

(c) If $0.5\,\alpha_3(|\mathbf{x}|) < \sigma_1(|\mathbf{u}|) + \sigma_2(|\mathbf{y}|)$ and $\sigma_1(|\mathbf{u}|) > \sigma_2(|\mathbf{y}|)$, then

$$\begin{aligned} L_{\mathbf{f}(\mathbf{x},\mathbf{u})}\tilde{W}(\mathbf{x}) &\le -q(W(\mathbf{x}))\,\alpha_3(|\mathbf{x}|) + 2\,q(W(\mathbf{x}))\,\sigma_1(|\mathbf{u}|) \\ &\le -q(W(\mathbf{x}))\,\alpha_3(|\mathbf{x}|) + 2\,\chi(2\,\sigma_1(|\mathbf{u}|))\,\sigma_1(|\mathbf{u}|)\,. \end{aligned}$$

Thus, the time derivative of function $\tilde{W}$ calculated for system (1) can be rewritten in the form:

$$\begin{aligned} L_{\mathbf{f}(\mathbf{x},\mathbf{u})}\tilde{W}(\mathbf{x}) &\le -0.5\,q(W(\mathbf{x}))\,\alpha_3(|\mathbf{x}|) \\ &\quad + 2\,\chi(2\,\sigma_2(|\mathbf{y}|))\,\sigma_2(|\mathbf{y}|) + 2\,\chi(2\,\sigma_1(|\mathbf{u}|))\,\sigma_1(|\mathbf{u}|)\,. \end{aligned}$$

Let function χ be taken to possess the following equality:

$$\chi(2\,\sigma_2(s)) = \frac{\alpha(s)}{1 + 2\,\sigma_2(s)},$$

such choice of χ is possible due to

$$\lim_{s\to+\infty} \frac{\sigma_2(s)}{\alpha(s)} < +\infty$$

with $q(s) = \frac{\alpha\circ\sigma_2^{-1}(0.25\,\alpha_3\circ\alpha_2^{-1}(s))}{1+0.5\,\alpha_3\circ\alpha_2^{-1}(s)}$ from class K. Then system (1) is ISS with ISS Lyapunov function $U(\mathbf{x}) = V(\mathbf{x}) + \tilde{W}(\mathbf{x})$ ($\alpha_4(s) = \rho\circ\alpha_1(s)$, $\alpha_5(s) = \overline{\alpha}(s) + \rho\circ\alpha_2(s)$), indeed:

$$\begin{aligned} L_{\mathbf{f}(\mathbf{x},\mathbf{u})}U(\mathbf{x}) &\le -0.5\,q(W(\mathbf{x}))\,\alpha_3(|\mathbf{x}|) + \sigma(|\mathbf{u}|) \\ &\quad + 2\,\chi(2\,\sigma_1(|\mathbf{u}|))\,\sigma_1(|\mathbf{u}|) \le -\alpha_6(|\mathbf{x}|) + \delta(|\mathbf{u}|)\,, \end{aligned}$$

where $\alpha_6(s) = 0.5\,q(\alpha_1(s))\,\alpha_3(s)$ and $\delta(s) = \sigma(s) + 2\,\chi(2\,\sigma_1(s))\,\sigma_1(s)$. $\square$

The next lemma is a corollary of Lemma A.1 presenting a variant of ISS stabilizing control law for a passive system.

LEMMA A.2. *Let system* (1) *be passive and IOSS in the sense of Definition* 2 *and*

$$\underline{\alpha}(|\mathbf{y}|) \leq V(\mathbf{x}) \leq \overline{\alpha}(|\mathbf{x}|), \quad \underline{\alpha}, \overline{\alpha} \in K_\infty.$$

Then control

$$\begin{gathered}\mathbf{u} = -\mathbf{k}(\mathbf{x}) + \mathbf{d}, \quad |\mathbf{k}(\mathbf{x})| \leq \lambda(|\mathbf{y}|), \quad \lambda \in K; \\ \mathbf{y}^T\mathbf{k}(\mathbf{x}) + \beta(\mathbf{x}) \geq \kappa(|\mathbf{y}|) + 0.5\,|\mathbf{y}|^2, \quad \kappa \in K_\infty; \\ \lim_{s\to+\infty} \frac{\sigma_2(s) + \sigma_1 \circ \lambda(s)}{\kappa(s)} < +\infty,\end{gathered}$$

where $\mathbf{d} \in R^m$ *is new input (Lebesgue measurable and essentially bounded function of time), and provides for the system ISS property with ISS Lyapunov function:*

$$\begin{gathered}U(\mathbf{x}) = V(\mathbf{x}) + \tilde{W}(\mathbf{x}), \quad \tilde{W}(\mathbf{x}) = \rho(W(\mathbf{x})), \quad \rho(r) = \int_0^r q(s)\,ds, \\ q(s) = \frac{\kappa \circ \tilde{\sigma}_2^{-1}\left(0.25\,\alpha_3 \circ \alpha_2^{-1}(s)\right)}{1 + 0.5\,\alpha_3 \circ \alpha_2^{-1}(s)}, \quad \alpha_4(s) = \rho \circ \alpha_1(s), \\ \alpha_5(s) = \overline{\alpha}(s) + \rho \circ \alpha_2(s), \quad \alpha_6(s) = 0.5\,q(\alpha_1(s))\,\alpha_3(s), \\ \delta(s) = 0.5\,s^2 + 2\,\chi(2\,\sigma_1(2\,s))\,\sigma_1(2\,s).\end{gathered}$$

Proof. From Definition 2 the following conditions hold for all $\mathbf{x} \in R^n$ and $\mathbf{u} \in R^m$:

$$\begin{gathered}\alpha_1(|\mathbf{x}|) \leq W(\mathbf{x}) \leq \alpha_2(|\mathbf{x}|); \\ L_{\mathbf{f}(\mathbf{x},\mathbf{u})}W(\mathbf{x}) \leq -\alpha_3(|\mathbf{x}|) + \sigma_1(|\mathbf{u}|) + \sigma_2(|\mathbf{y}|); \\ \underline{\alpha}(|\mathbf{y}|) \leq V(\mathbf{x}) \leq \overline{\alpha}(|\mathbf{x}|); \quad L_{\mathbf{f}(\mathbf{x},\mathbf{u})}V(\mathbf{x}) \leq -\beta(|\mathbf{x}|) + \mathbf{y}^T\mathbf{u}\end{gathered}$$

with $\alpha_1, \alpha_2, \alpha_3, \underline{\alpha}, \overline{\alpha} \in K_\infty$, $\sigma_1, \sigma_2 \in K$ and β some nonnegative definite function. Substituting control in these inequalities, it is possible to obtain

$$\begin{gathered}L_{\mathbf{f}(\mathbf{x},\mathbf{u})}W(\mathbf{x}) \leq -\alpha_3(|\mathbf{x}|) + \sigma_1(|\mathbf{d} - \mathbf{k}(\mathbf{x})|) + \sigma_2(|\mathbf{y}|) \\ \leq -\alpha_3(|\mathbf{x}|) + \sigma_1(2\,|\mathbf{d}|) + \sigma_1(2\,\lambda(|\mathbf{y}|)) + \sigma_2(|\mathbf{y}|); \\ L_{\mathbf{f}(\mathbf{x},\mathbf{u})}V(\mathbf{x}) \leq -\beta(|\mathbf{x}|) + \mathbf{y}^T(\mathbf{d} - \mathbf{k}(\mathbf{x})) \leq -\kappa(|\mathbf{y}|) + 0.5\,|\mathbf{d}|^2.\end{gathered}$$

Thus, such control provides for closed loop system IOSS property and h-dissipativity with respect to new input $\mathbf{d}$. If

$$\lim_{s\to+\infty} \frac{\tilde{\sigma}_2(s)}{\kappa(s)} < +\infty, \quad \tilde{\sigma}_2(s) = \sigma_2(s) + \sigma_1 \circ \lambda(s),$$

then all conditions of Lemma A1 are satisfied and the system is ISS with ISS Lyapunov function

$$\begin{gathered}U(\mathbf{x}) = V(\mathbf{x}) + \tilde{W}(\mathbf{x}), \quad \tilde{W}(\mathbf{x}) = \rho(W(\mathbf{x})), \quad \rho(r) = \int_0^r q(s)\,ds, \\ q(s) = \frac{\kappa \circ \tilde{\sigma}_2^{-1}\left(0.25\,\alpha_3 \circ \alpha_2^{-1}(s)\right)}{1 + 0.5\,\alpha_3 \circ \alpha_2^{-1}(s)}, \quad \alpha_4(s) = \rho \circ \alpha_1(s), \\ \alpha_5(s) = \overline{\alpha}(s) + \rho \circ \alpha_2(s), \quad \alpha_6(s) = 0.5\,q(\alpha_1(s))\,\alpha_3(s), \\ \delta(s) = 0.5\,s^2 + 2\,\chi(2\,\sigma_1(2\,s))\,\sigma_1(2\,s). \qquad \square\end{gathered}$$

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Robust and Adaptive Observer-Based Partial Stabilization for a Class of Nonlinear Systems

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***Abstract*—The problem of adaptive stabilization with respect to a set for a class of nonlinear systems in the presence of external disturbances is considered. A novel adaptive observer-based solution for the case of noisy measurements is proposed. The efficiency of proposed solution is demonstrated via example of swinging a pendulum with unknown parameters.**

***Index Terms*—Adaptive control, nonlinearity, observers.**

I. Introduction

The problem of nonlinear adaptive control got a number of solutions during the last decade [2], [11], [14], [15], [17], [20]–[22], [24]. Most of the existing solutions are tailored to achieve such goals as regulation or tracking, where the system trajectory converges to a point or to a curve. In these cases the goal functionals possess radial unboundedness with respect to the whole state vector of the controlled system. However, in a number of applications the plant stabilization with respect to *a part of variables* (i.e. with respect to a set) is needed. For example, such problems arise when stabilizing the desired energy level for physical or mechanical systems, synchronization, etc. [10], [11], [23].

An additional requirement may consist in boundedness of control signal [3], [11]. In the presence of parametric uncertainty, the dependence of bounded control law on adjusted parameters leads to the problem of adaptation with nonlinear parameterization of the controller. This fact prevents from applying previously mentioned results. The problem is to design an output feedback control for an unknown plant, providing stabilization of the given set or its vicinity in the presence of disturbances and measurement errors (size of the vicinity should be proportional to the level of disturbances). Such a statement of the problem looks natural when the level of disturbances is unknown, though bounded. Among numerous examples of such situations are energy-control problems and synchronization problems.

A number of problems of the above class were solved previously by the speed-gradient method under assumption of passifiability [10], [11], [27], [28]. However, many systems of interest, e.g. those having relative degree greater than one cannot be made passive. Solutions for nonpassifiable systems were suggested in [12] based on a special nonlinear observer structure proposed by Nikiforov [11], [12], [24].

In the previous works of the authors [5]–[9] solutions for such sort of problems were proposed for output synchronization, observation, I-O stabilization. This technical note is devoted to the robust and adaptive partial stabilization. Partial stabilization is considered with respect to a function and the goal set is a surface in the state space. We stress, that consideration of partial stability as the set stability is only one of the possible notions of partial stability; under some circumstances, indeed, more than one measure is requested to formulate the property in a suitable way.

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II. Preliminaries

Let us consider dynamical systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad \mathbf{y} = \mathbf{h}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} \in R^n$ is the state vector; $\mathbf{u} \in R^m$ is the input vector; $\mathbf{y} \in R^p$ is the output vector; $\mathbf{f}$ and $\mathbf{h}$ are locally Lipschitz continuous vector functions, $\mathbf{h}(0) = 0$, $\mathbf{f}(0, 0) = 0$. Euclidean norm will be denoted as $|\mathbf{x}|$, and $\|\mathbf{u}\|_{[t_0,t]}$ denotes the L_∞^m norm of the input ($\mathbf{u}(t)$ is Lebesgue measurable and locally essentially bounded function $\mathbf{u} : R_+ \to R^m$, $R_+ = \{\tau \in R : \tau \geq 0\}$)

$$\|\mathbf{u}\|_{[t_0,T]} = ess\sup\left\{|\mathbf{u}(t)|, t \in [t_0, T]\right\}.$$

If $T = +\infty$ then we will simply write $\|\mathbf{u}\|$. We will denote as M_{R^m} the set of all such Lebesgue measurable inputs $\mathbf{u}$ with property $\|\mathbf{u}\| < +\infty$. For initial state $\mathbf{x}_0$ and input $\mathbf{u} \in \mathsf{M}_{R^m}$, let $\mathbf{x}(t, \mathbf{x}_0, \mathbf{u})$ be the unique maximal solution of (1) (we will use notation $\mathbf{x}(t)$ if all other arguments are clear from the context; $\mathbf{y}(t, \mathbf{x}_0, \mathbf{u}) = \mathbf{h}(\mathbf{x}(t, \mathbf{x}_0, \mathbf{u}))$), which is defined on some finite interval $[0, T)$; if for every initial state $\mathbf{x}_0 \in R^n$ and $\mathbf{u} \in \mathsf{M}_{R^m}$ the solutions are defined for all $t \geq 0$, then system is called forward complete. It is said that system (1) has unboundedness observability (UO) property, if for each state $\mathbf{x}_0 \in R^n$ and input $\mathbf{u} \in \mathsf{M}_{R^m}$ such that $T < +\infty$ necessarily

$$\limsup_{t \to T} |\mathbf{y}(t, \mathbf{x}_0, \mathbf{u})| = +\infty.$$

Characterization of forward completeness and UO properties were investigated in [1]. Distance in R^n from given point $\mathbf{x}$ to set A is denoted as $|\mathbf{x}|_{\mathsf{A}} = \mathrm{dist}(\mathbf{x}, \mathsf{A}) = \inf_{\boldsymbol{\eta} \in \mathsf{A}} |\mathbf{x} - \boldsymbol{\eta}|$. As usual, a continuous function $\sigma : R_+ \to R_+$ belongs to class K if it is strictly increasing and $\sigma(0) = 0$; additionally, it belongs to class K_∞ if it is also radially unbounded; a continuous function $\beta : R_+ \times R_+ \to R_+$ is from class KL, if $\beta(\cdot, t) \in \mathsf{K}$ for any $t \in R_+$, and $\beta(s, \cdot)$ is strictly decreasing to zero for each $s \in R_+$.

System (1) is called bounded-input-bounded-state (BIBS) stable if for all $\mathbf{x}_0 \in R^n$, $\mathbf{u} \in \mathsf{M}_{R^m}$, $t \geq 0$ the property $|\mathbf{x}(t, \mathbf{x}_0, \mathbf{u})| \leq \max\{\mu(|\mathbf{x}_0|), \mu(\|\mathbf{u}\|)\}$ holds for $\mu \in \mathsf{K}$.

Definition 1 [13], [29]: UO system (1) is input-to-output stable (IOS), if there exist $\beta \in \mathsf{KL}$ and $\gamma \in \mathsf{K}$ such, that inequality $|\mathbf{y}(t, \mathbf{x}_0, \mathbf{u})| \leq \beta(|\mathbf{x}(t_0)|, t - t_0) + \gamma(\|\mathbf{u}\|_{[t_0,t)})$, $t \geq t_0 \geq 0$ holds for all $\mathbf{x}(t_0) \in R^n$ and $\mathbf{u} \in M_{R^m}$. □

Definition 2 [9]: Forward complete system (1) is called integral input-to-state stable (iISS) with respect to closed invariant set A if there exist functions $\alpha \in \mathsf{K}_\infty$, $\gamma \in \mathsf{K}$ and $\beta \in \mathsf{KL}$ such that, for any $\mathbf{x}_0 \in R^n$, $\mathbf{u} \in \mathsf{M}_{R^m}$

$$\alpha\left(|\mathbf{x}(t, \mathbf{x}_0, \mathbf{u})|_A\right) \leq \beta\left(|\mathbf{x}_0|_A, t\right) + \int_0^t \gamma\left(|\mathbf{u}(\tau)|\right) d\tau, \quad t \geq 0.$$

□

A. Robust Stabilization With Respect to a Set via Passification Approach

Let us consider a system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})[\mathbf{u} + \mathbf{v}], \quad \mathbf{y} = \mathbf{h}(\mathbf{x}) \tag{2}$$

where $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$, $\mathbf{y} \in R^m$ are state, input, and output vectors correspondingly; $\mathbf{v} \in R^m$ is an external disturbances vector; $\mathbf{f}$, $\mathbf{h}$ and columns of matrix $\mathbf{G}$ are locally Lipschitz continuous vector functions, $\mathbf{h}(0) = 0$, $\mathbf{f}(0) = 0$.

Definition 3 [4], [11], [31]: It is said that system (2) is passive with continuously differentiable storage function $V : R^n \to R_+$ if for all $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$, $\mathbf{v} \in R^m$ it holds that $\dot{V} \leq \mathbf{y}^T[\mathbf{u} + \mathbf{v}]$. □

The passification method [11], [27], [28] is based on a feedback design making the closed-loop system passive. It allows one to solve partial stabilization problem for system (2) with respect to the zero level set of storage function. The key property for this approach to partial stabilization is detectability assumption [26]–[28] described as follows.

Definition 4: It is said that passive system (2) with storage function $V : R^n \to R_+$ is V-detectable with respect to output $\mathbf{y}$ if for all $\mathbf{x}_0 \in R^n$ it holds

$$\mathbf{y}(t, \mathbf{x}_0, 0) \equiv 0, \quad t \geq 0 \Rightarrow \lim_{t \to +\infty} V(\mathbf{x}(t, \mathbf{x}_0, 0)) = 0.$$

□

The following result [7], [9] gives conditions of iISS with respect to set stabilization by passification.

Theorem 1: Let the system (2) be passive with continuously differentiable storage function $V : R^n \to R_+$ and a non decreasing function $\varphi : R^m \to R^m$, $\varphi(0) = 0$ have the property $\mathbf{y}^T\varphi(\mathbf{y}) > 0$ for $\mathbf{y} \in R^m/\{0\}$, $\lim_{|\mathbf{x}|_{\mathsf{V}_0} \to \infty} |\mathbf{h}(\mathbf{x})|/V(\mathbf{x}) < \infty$. Additionally, let there exist functions $\alpha_1, \alpha_2 \in \mathsf{K}_\infty$ such that for all $\mathbf{x} \in R^n$ inequalities $\alpha_1(|\mathbf{x}|_{\mathsf{V}_0}) \leq V(\mathbf{x}) \leq \alpha_2(|\mathbf{x}|_{\mathsf{V}_0})$ are satisfied, where $\mathsf{V}_0 = \{\mathbf{x} \in R^n : V(\mathbf{x}) = 0\}$ is a compact set. Then the system (2) with control $\mathbf{u} = -\varphi(\mathbf{y})$ has iISS property with respect to set V_0 if the system is V-detectable with respect to the output $\mathbf{y}$. □

B. Positivity in the Average

Identification ability of adaptation algorithms is one of the most attractive problems in the adaptive control theory. The solution of this problem is closely connected with persistent excitation (PE) property. There exist several closely related definitions of PE property [10], [18], [19], [22]. Here we will use the following one.

Definition 5: Function $a : R_+ \to R$ is called positive in the average (PA) if there exists some $\Delta > 0$ and $r > 0$ such that for all $t \geq 0$ and $\delta \geq \Delta > 0$

$$\int_t^{t+\delta} a(\tau) d\tau \geq r\delta.$$

□

The importance of the PA property is explained in the following lemma, for which a slightly modified version was proven in [8].

Lemma 1: Let us consider time-varying linear dynamical system $\dot{p} = -a(t)p + b(t)$, $t_0 \geq 0$, where $p \in R$, $p(t_0) \in R$ and functions $a : R_+ \to R$, $b : R_+ \to R$ are Lebesgue measurable, b is locally essentially bounded, function a is PA for some $r > 0$, $\Delta > 0$ and essentially bounded from below, i.e. there exists $A \in R_+$ such, that $ess\inf\{a(t), t \geq t_0\} \geq -A$. Then the solution are defined for all $t \geq t_0$ and

$$|p(t)| \leq \begin{cases} |p(t_0)|\, e^{-r(t-t_0)+(A+r)\Delta} & \\ \quad +\|b\| \max\{A^{-1}e^{At_0}, r^{-1}e^{-rt_0}\}, & A \neq 0; \\ |p(t_0)|\, e^{-r(t-t_0)+r\Delta} & \\ \quad +\|b\| \max\{\Delta, r^{-1}e^{-rt_0}\}, & A = 0. \end{cases}$$

□

It is possible to show that PA property is *equivalent* to some versions of PE property. However, PA is more convenient for quantitative analysis. Standard sufficient conditions for PE that can be interpreted for PA can be found, e.g. in [22].

C. Adaptive Observer Design

Let us consider the following uncertain system:

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{y})\mathbf{x} + \varphi(\mathbf{y}) + \mathbf{B}(\mathbf{y})\boldsymbol{\theta} + \mathbf{d}_1, \quad \mathbf{y} = \mathbf{C}\mathbf{x}, \quad \mathbf{y}_d = \mathbf{y} + \mathbf{d}_2 \tag{3}$$

where $\mathbf{x} \in R^n$ is a state vector; $\mathbf{y} \in R^m$ is an output vector; $\boldsymbol{\theta} \in \Omega_\theta \subset R^p$ is a vector of uncertain parameters, which values belong to compact set Ω_θ; $\mathbf{d}_1 \in \mathsf{M}_{R^n}$, $\mathbf{d}_2 \in \mathsf{M}_{R^m}$ are vector signals of external disturbances and measurement noise, $\mathbf{d} = [\mathbf{d}_1^T\ \mathbf{d}_2^T]^T$; $\mathbf{y}_d$ is vector of noisy measurements of the system (3) output. Vector function φ and columns of matrix functions $\mathbf{A}$ and $\mathbf{B}$ are locally Lipschitz continuous, and $\mathbf{C}$ is some constant matrix of appropriate dimension.

The problem is to design an adaptive observer, which in the absence of disturbances provides partial estimates of unmeasured components of vector $\mathbf{x}$ and estimates of unknown vector $\boldsymbol{\theta}$. For any $\mathbf{d} \in \mathsf{M}_{R^{n+m}}$, the observer should ensure boundedness of the system solutions. In works [6], [8], [12] a solution is proposed under the following suppositions.

Assumption 1: For all $\mathbf{x}_0 \in \Omega_x$, $\boldsymbol{\theta} \in \Omega_\theta$, $\mathbf{d} \in \mathsf{M}_{R^{n+m}}$ system (3) is BIBS

$$|\mathbf{x}(t, \mathbf{x}_0, \boldsymbol{\theta}, \mathbf{d})| \leq \sigma_0(|\mathbf{x}_0|) + \sigma_0(\|\mathbf{d}\|), \quad \sigma_0 \in K, \quad t \geq 0.$$

□

The rest suppositions deal with stabilizability by output feedback of the linear part of system (3).

Assumption 2: There exist matrix $\mathbf{L}$, locally Lipschitz continuous matrix function $\mathbf{K} : R^m \to R^{n\times m}$ and continuously differentiable function $V : R^n \to R_+$ satisfying relations

$$\alpha_1(|\mathbf{x}|) \leq V(\mathbf{x}) \leq \alpha_2(|\mathbf{x}|)$$
$$\partial V(\mathbf{x})/\partial \mathbf{x} \mathbf{G}(\mathbf{y}_d)\mathbf{x} \leq -\alpha_3|\mathbf{L}\mathbf{x}|^2, \quad |\mathbf{C}\mathbf{x}| \leq |\mathbf{L}\mathbf{x}|$$

for all $\mathbf{y}_d \in R^m$, $\mathbf{x} \in R^n$, where α_1, α_2 are from class K_∞ and $\alpha_3 > 0$, $\mathbf{G}(\mathbf{y}_d) = \mathbf{A}(\mathbf{y}_d) - \mathbf{K}(\mathbf{y}_d)\mathbf{C}$. □

Assumption 2 ensures uniform asymptotic stability with respect to variable $\mathbf{Ls}$ [11], [25] for the system

$$\dot{\mathbf{s}} = \mathbf{G}(\mathbf{y}_d)\mathbf{s} + \mathbf{r} \tag{4}$$

coupled with the system (3) and uniform stability property with respect to variable $\mathbf{s}$ for the case $\mathbf{r} = 0$.

The next assumption requires bounded input-bounded state stability of the auxiliary system (4).

Assumption 3: For all initial conditions $\mathbf{s}_0 \in R^n$ and inputs $\mathbf{r} \in \mathsf{M}_{R^n}$, $\mathbf{y}_d \in \mathsf{M}_{R^m}$ the system (4) is BIBS uniformly with respect to signal $\mathbf{y}_d$

$$|\mathbf{s}(t, \mathbf{s}_0, \mathbf{r}, \mathbf{y}_d)| \leq \sigma_1(|\mathbf{s}_0|) + \sigma_1(\|\mathbf{r}\|), \quad \sigma_1 \in \mathsf{K}, \quad t \geq 0.$$

□

Consider the following equations of adaptive observer:

$$\dot{\mathbf{z}} = \mathbf{A}(\mathbf{y}_d)\mathbf{z} + \varphi(\mathbf{y}_d) + \mathbf{B}(\mathbf{y}_d)\widehat{\boldsymbol{\theta}} + \mathbf{K}(\mathbf{y}_d)(\mathbf{y}_d - \widehat{\mathbf{y}}) \quad (5)$$

$$\dot{\boldsymbol{\eta}} = \mathbf{G}(\mathbf{y}_d)\boldsymbol{\eta} - \boldsymbol{\Omega}\dot{\widehat{\boldsymbol{\theta}}} \quad (6)$$

$$\dot{\boldsymbol{\Omega}} = \mathbf{G}(\mathbf{y}_d)\boldsymbol{\Omega} + \mathbf{B}(\mathbf{y}_d) \quad (7)$$

$$\dot{\widehat{\boldsymbol{\theta}}} = \gamma\boldsymbol{\Omega}^T\mathbf{C}^T(\mathbf{y}_d - \widehat{\mathbf{y}} + \mathbf{C}\boldsymbol{\eta}), \quad \widehat{\mathbf{y}} = \mathbf{C}\mathbf{z}, \quad \gamma > 0 \quad (8)$$

where $\mathbf{z} \in R^n$ is the vector of variable $\mathbf{x}$ estimates; vector $\boldsymbol{\eta} \in R^n$ and matrix $\boldsymbol{\Omega} \in R^{n\times p}$ are auxiliary variables, which help to overcome high relative degree obstruction for system (3); $\widehat{\boldsymbol{\theta}} \in R^p$ is vector of $\boldsymbol{\theta}$ estimates.

Theorem 2 [6]: Let assumptions 1–3 hold and minimum singular value $a(t)$ of matrix function $\mathbf{C}^T\boldsymbol{\Omega}^T(t)$ be PA. Then solutions of system (3) and (5)–(8) are bounded for any initial conditions and $\mathbf{d} \in \mathsf{M}_{R^{n+m}}$ and any $\gamma > 0$, in the absence of disturbance $\mathbf{d}$ the following relations hold:

$$\lim_{t\to+\infty}\widehat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}, \quad \lim_{t\to+\infty}\mathbf{L}\mathbf{x}(t) - \mathbf{L}\mathbf{z}(t) = 0.$$

□

III. Main Result

Let us consider uncertain nonlinear system

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{y})\mathbf{x} + \varphi(\mathbf{y}) + \mathbf{B}(\mathbf{y})\boldsymbol{\theta} + \mathbf{R}(\mathbf{y})[\mathbf{u} + \mathbf{d}_3] + \mathbf{d}_1$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}, \quad \mathbf{y}_d = \mathbf{y} + \mathbf{d}_2 \quad (9)$$

where (as for the system (3) previously) $\mathbf{x} \in R^n$ is the state vector; $\mathbf{y} \in R^m$ is the output vector; $\boldsymbol{\theta} \in \Omega_\theta \subset R^p$ is the vector of unknown parameters with values from Ω_θ; $\mathbf{u} \in R^q$ is the control; $\mathbf{d}_1 \in \mathsf{M}_{R^n}$, $\mathbf{d}_2 \in \mathsf{M}_{R^m}$, $\mathbf{d}_3 \in \mathsf{M}_{R^q}$ are vector signals of external disturbances and measurement noise, $\mathbf{d} = [\mathbf{d}_1^T\ \mathbf{d}_2^T\ \mathbf{d}_3^T]^T$; $\mathbf{y}_d$ is noisy output vector of the system (9). Vector function φ and matrix functions $\mathbf{A}, \mathbf{B}, \mathbf{R}$ are continuous and locally Lipschitz.

Assumption 4: There exist locally Lipschitz continuous functions $\mathbf{u} : R^{m+k+p} \to R^q$, $\boldsymbol{\psi} : R^n \to R^r$ and matrix $\mathbf{L}$ with dimension $(k \times n)$ such, that control

$$\mathbf{u} = \mathbf{u}(\mathbf{y}, \mathbf{L}\mathbf{x}, \boldsymbol{\theta}) \quad (10)$$

guarantees for system (9) forward completeness and one of the following properties.

A. IOS from input $\mathbf{d}$ to output $\boldsymbol{\psi}$.

B. iISS with respect to set $\mathsf{Z} = \{\mathbf{x} : \boldsymbol{\psi}(\mathbf{x}) = 0\}$ for input $\mathbf{d}$. □

Starting from control (10), depending on unmeasured variables $\mathbf{L}\mathbf{x}$ and vector of uncertain parameters of the system $\boldsymbol{\theta}$, it is necessary to design a new control using only measured signal $\mathbf{y}_d$. The control should provide boundedness of the closed-loop system solutions for $\mathbf{d} \in \mathsf{M}_{R^{n+m+q}}$ and for the case $\mathbf{d} = 0$ it should ensure asymptotic convergence to zero of output $\boldsymbol{\psi}$ or attractiveness of the set Z.

It is worth to stress, that two outputs have been introduced, $\mathbf{y}$ defines the measured variables of the system (9), $\boldsymbol{\psi}$ characterizes the distance to the goal set. Although the vector of unknown parameters $\boldsymbol{\theta}$ appears in a linear fashion in the right-hand side of system (9), the right-hand side of the closed-loop system (9), (10) may nonlinearly depend on $\boldsymbol{\theta}$ since Assumption 4 does not specify the form of function $\mathbf{u}$ dependence on its arguments.

The form of the system (9) is similar to the system (3) (observer canonical form) for which it is possible to design adaptive observer (5)–(8). Substituting in control (10) the estimates of vectors $\mathbf{L}\mathbf{x}$ and $\boldsymbol{\theta}$ provided by the observer, it is possible to solve the posed problem (we assume that matrixes $\mathbf{L}$ in assumptions 2 and 4 are identical). The principal difference of the solved problem from the problem of adaptive observer design as in Theorem 2 consists in appearance of control $\mathbf{u}$ in the right-hand side of system (9). Generally speaking in the absence of control (10) system can possess unbounded solutions (Assumption 1 fails). Fortunately, this obstacle does not prevent from the design of the observer similarly to (5)–(8)

$$\dot{\mathbf{z}} = \mathbf{A}(\mathbf{y}_d)\mathbf{z} + \varphi(\mathbf{y}_d) + \mathbf{B}(\mathbf{y}_d)\widehat{\boldsymbol{\theta}} + \mathbf{R}(\mathbf{y}_d)\mathbf{u} + \mathbf{K}(\mathbf{y}_d)(\mathbf{y}_d - \widehat{\mathbf{y}}) \quad (11)$$

$$\dot{\boldsymbol{\eta}} = \mathbf{G}(\mathbf{y}_d)\boldsymbol{\eta} - \boldsymbol{\Omega}\dot{\widehat{\boldsymbol{\theta}}} \quad (12)$$

$$\dot{\boldsymbol{\Omega}} = \mathbf{G}(\mathbf{y}_d)\boldsymbol{\Omega} + \mathbf{B}(\mathbf{y}_d) \quad (13)$$

$$\dot{\widehat{\boldsymbol{\theta}}} = \gamma\boldsymbol{\Omega}^T\mathbf{C}^T(\mathbf{y}_d - \widehat{\mathbf{y}} + \mathbf{C}\boldsymbol{\eta}), \quad \widehat{\mathbf{y}} = \mathbf{C}\mathbf{z}, \quad \gamma > 0 \quad (14)$$

where all symbols have the same meaning, and $\gamma > 0$ is adaptation gain. Since matrix function $\mathbf{R}$ depends on the output vector only and control $\mathbf{u}$ is produced by the controller, their appearance does not change dynamics of state estimation error $\mathbf{e} = \mathbf{x} - \mathbf{z}$ and auxiliary error $\boldsymbol{\delta} = \mathbf{e} + \boldsymbol{\eta} - \boldsymbol{\Omega}(\boldsymbol{\theta} - \widehat{\boldsymbol{\theta}})$

$$\begin{aligned}\dot{\mathbf{e}} = {} & \mathbf{G}(\mathbf{y}_d)\mathbf{e} + \varphi(\mathbf{y}) - \varphi(\mathbf{y}_d) + \mathbf{B}(\mathbf{y})\boldsymbol{\theta} - \mathbf{B}(\mathbf{y}_d)\widehat{\boldsymbol{\theta}} \\ & - \mathbf{K}(\mathbf{y}_d)\mathbf{d}_2(t) + \mathbf{R}(\mathbf{y})\mathbf{d}_3 + [\mathbf{R}(\mathbf{y}) - \mathbf{R}(\mathbf{y}_d)]\,\mathbf{u} \\ & + \mathbf{d}_1(t) + [\mathbf{A}(\mathbf{y}) - \mathbf{A}(\mathbf{y}_d)]\,\mathbf{x}, \end{aligned} \quad (15)$$

$$\begin{aligned}\dot{\boldsymbol{\delta}} = {} & \mathbf{G}(\mathbf{y}_d)\boldsymbol{\delta} + \varphi(\mathbf{y}) - \varphi(\mathbf{y}_d) + [\mathbf{B}(\mathbf{y}) - \mathbf{B}(\mathbf{y}_d)]\,\boldsymbol{\theta} \\ & - \mathbf{K}(\mathbf{y}_d)\mathbf{d}_2(t) + \mathbf{R}(\mathbf{y})\mathbf{d}_3 + [\mathbf{R}(\mathbf{y}) - \mathbf{R}(\mathbf{y}_d)]\,\mathbf{u} \\ & + \mathbf{d}_1(t) + [\mathbf{A}(\mathbf{y}) - \mathbf{A}(\mathbf{y}_d)]\,\mathbf{x}. \end{aligned} \quad (16)$$

For the case of the absence of the disturbances $\mathbf{d} = 0$, systems (15) and (16) can be rewritten as follows:

$$\dot{\mathbf{e}} = \mathbf{G}(\mathbf{y})\mathbf{e} + \mathbf{B}(\mathbf{y})[\boldsymbol{\theta} - \widehat{\boldsymbol{\theta}}], \quad (17)$$

$$\dot{\boldsymbol{\delta}} = \mathbf{G}(\mathbf{y})\boldsymbol{\delta}. \quad (18)$$

Forms of (17) and (18) are similar to the observer (5)–(8) and, therefore, the convergence proof for the observer (11)–(14) follows from Theorem 2 with minimal modifications dealing with a prior absence of Assumption 1 for system (9). In the presence of noise $\mathbf{d}_2$, the dependence of right-hand sides of (15) and (16) on vectors $\mathbf{u}$ and $\mathbf{x}$ makes difficulties for employing of the proof of Theorem 2. This is the reason why this case will be considered under special conditions below.

Theorem 3: For system (9), let Assumption 2 hold and Assumption 3 be satisfied for any Lebesgue measurable signal $\mathbf{y}$; minimum singular value $a(t)$ of matrix function $\mathbf{C}^T\boldsymbol{\Omega}^T(t)$ be PA; $|\mathbf{B}(\mathbf{y}_d(t))| \le B$, $|\mathbf{R}(\mathbf{y}_d(t))| \le R$ for all $t \ge 0$, $B, R \in R_+$. Then the control law $\mathbf{u} = \mathbf{u}(\mathbf{y}_d, \mathbf{L}\mathbf{z}, \widehat{\boldsymbol{\theta}})$ ensures forward completeness of system (9), boundedness of the system (11)–(14) solutions, and boundedness of variable $\boldsymbol{\psi}(\mathbf{x}(t))$ for all initial conditions, $\mathbf{d} \in M_{R^{n+m+q}}$ and any $\gamma > 0$ provided that at least one of the following additional suppositions is valid.

1) Assumption 4.A holds, control $\mathbf{u} = \mathbf{u}(\mathbf{y}_d, \mathbf{L}\mathbf{z}, \widehat{\theta})$ is globally Lipschitz function with respect to the last two arguments and $\mathbf{d}_2(t) \equiv 0$ for all $t \ge 0$;
2) Assumption 4.A holds control $\mathbf{u} = \mathbf{u}(\mathbf{y}_d, \mathbf{L}\mathbf{z}, \widehat{\boldsymbol{\theta}})$ is a globally Lipschitz function, function φ is globally Lipschitz, $\mathbf{A}(\mathbf{y}) = \mathbf{A}$, $\mathbf{B}(\mathbf{y}) = \mathbf{B}$ and $\mathbf{R}(\mathbf{y}) = \mathbf{R}$;
3) Assumption 4.B holds and $\mathbf{d}(t) \equiv 0$ for all $t \ge 0$.

Additionally, if $\mathbf{d}(t) \equiv 0$ for all $t \ge 0$, then limit relations $\lim_{t\to+\infty}\boldsymbol{\psi}(t) = 0$ (Assumption 4.A) or $\lim_{t\to+\infty}|\mathbf{x}(t)|_{\mathsf{Z}} = 0$ (Assumption 4.B) hold.

Proof: At first let us consider the case $\mathbf{d}_2(t) \equiv 0$, $t \ge 0$ under Assumption 4.A. Then (15) and (16) take the form

$$\dot{\mathbf{e}} = \mathbf{G}(\mathbf{y})\mathbf{e} + \mathbf{B}(\mathbf{y})[\boldsymbol{\theta} - \widehat{\boldsymbol{\theta}}] + \mathbf{R}(\mathbf{y})\mathbf{d}_3 + \mathbf{d}_1(t) \quad (19)$$

$$\dot{\boldsymbol{\delta}} = \mathbf{G}(\mathbf{y})\boldsymbol{\delta} + \mathbf{R}(\mathbf{y})\mathbf{d}_3 + \mathbf{d}_1(t). \quad (20)$$

Equations (20) and (13) have form (4) with bounded inputs, thus, according to Assumption 3 variables $\boldsymbol{\delta}$ and $\boldsymbol{\Omega}$ are bounded. Let us consider the time derivative of Lyapunov function $W(\widehat{\boldsymbol{\theta}}) = \gamma^{-1}(\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}})^T(\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}})$ as follows:

$$\begin{aligned}\dot{W} =& -2(\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}})^T\boldsymbol{\Omega}^T\mathbf{C}^T\left[\mathbf{C}\left(\boldsymbol{\delta}+\boldsymbol{\Omega}(\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}})\right)+\mathbf{d}_2\right]\\ \leq& -\gamma a(t)W+|\mathbf{C}\boldsymbol{\delta}+\mathbf{d}_2|^2. \qquad (21)\end{aligned}$$

For bounded input $\mathbf{C}\boldsymbol{\delta}+\mathbf{d}_2$, the boundedness of error $\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}$ follows from Lemma 1 and PA property of signal a. Having in mind this conclusion, transform equation (19) to the form (4) with bounded inputs. Applying again Assumption 3 one can substantiate boundedness of $\mathbf{e}$. Variable $\boldsymbol{\eta}$ is a part of error $\boldsymbol{\delta}$, where all other parts and $\boldsymbol{\delta}$ are bounded. Hence $\boldsymbol{\eta}$ is also bounded. Let us substitute the control into (9), as

$$\begin{aligned}\dot{\mathbf{x}} = \mathbf{A}(\mathbf{y})\mathbf{x}+\varphi(\mathbf{y})+\mathbf{B}(\mathbf{y})\boldsymbol{\theta}+\mathbf{R}(\mathbf{y})\left[\mathbf{u}(\mathbf{y},\mathbf{L}\mathbf{x},\boldsymbol{\theta})+\mathbf{d}_3\right]\\ +\mathbf{R}(\mathbf{y})\mathbf{e}_u+\mathbf{d}_1 \qquad (22)\end{aligned}$$

where $\mathbf{e}_u = \mathbf{u}(\mathbf{y},\mathbf{L}\mathbf{z},\widehat{\boldsymbol{\theta}})-\mathbf{u}(\mathbf{y},\mathbf{L}\mathbf{x},\boldsymbol{\theta})$ is the error of control (10) realization. By conditions control $\mathbf{u}=\mathbf{u}(\mathbf{y}_d,\mathbf{L}\mathbf{z},\widehat{\boldsymbol{\theta}})$ is globally Lipschitz continuous and errors $\mathbf{e}$ and $\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}$ are bounded. Therefore, there exists a constant $L_u>0$ such, that for all $t\geq 0$ inequality $|\mathbf{e}_u(t)|\leq L_u[|\mathbf{L}\mathbf{e}(t)|+|\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}(t)|]$ holds and error $\mathbf{e}_u$ is bounded. According to Assumption 4, control (10) ensures boundedness of function $\boldsymbol{\psi}$.

Assume now the presence of noise $\mathbf{d}_2$ under structure restrictions $\mathbf{A}(\mathbf{y})=\mathbf{A}$, $\mathbf{B}(\mathbf{y})=\mathbf{B}$ and $\mathbf{R}(\mathbf{y})=\mathbf{R}$ (in this case $\mathbf{K}(\mathbf{y})=\mathbf{K}$). Equations (15) and (16) can be rewritten as follows:

$$\begin{aligned}\dot{\mathbf{e}} &= \mathbf{G}(\mathbf{y}_d)\mathbf{e}+\varphi(\mathbf{y})-\varphi(\mathbf{y}_d)+\mathbf{B}[\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}]-\mathbf{K}\mathbf{d}_2(t)\\ &\quad +\mathbf{R}\mathbf{d}_3+\mathbf{d}_1(t),\\ \dot{\boldsymbol{\delta}} &= \mathbf{G}(\mathbf{y}_d)\boldsymbol{\delta}+\varphi(\mathbf{y})-\varphi(\mathbf{y}_d)-\mathbf{K}\mathbf{d}_2(t)+\mathbf{R}\mathbf{d}_3+\mathbf{d}_1(t).\end{aligned}$$

Since φ is globally Lipschitz continuous, applying Assumption 3 we justify the boundedness of variables $\boldsymbol{\delta}$, $\boldsymbol{e}$, and $\boldsymbol{\Omega}$. Analyzing properties of function W we obtain boundedness of variable $\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}$. Boundedness of all other variables of the system can be proven in the same way as in the previous case.

In the absence of disturbances ($\mathbf{d}=0$), system (15) and (16) take the form of (17) and (18). It follows from Assumption 2 that the variable $\boldsymbol{\delta}$ is bounded and system is asymptotically stable with respect to the part of variables $\mathbf{L}\boldsymbol{\delta}$ [25]. Assumption 3 gives boundedness of variable $\boldsymbol{\Omega}$. In this case for time derivative of function W Lemma 1 provides asymptotic convergence to zero of variable $\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}(t)$. According to assumptions 2 and 3, the system (17) is asymptotically stable with respect to variable $\mathbf{L}\mathbf{e}$ and has bounded solutions. Since signals $\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}(t)$ and $\mathbf{L}\mathbf{e}(t)$ converge to zero, the error $\mathbf{e}_u(t)$ also converges to zero. Having in mind the properties of control (10) from Assumption 4.A, we obtain convergence to zero of variable $\boldsymbol{\psi}(t)$. In such case error $\mathbf{e}_u(t)$ is integrally bounded and we can apply Assumption 4.B. Proof is completed. ■

Remark: Globally Lipschitz property requirement for control (9) naturally holds for bounded controls. □

For Assumption 4.A, the theorem provides convergence conditions in the presence of external disturbances and noise. The noisy case needs additional structural restrictions.

For Assumption 4.B, Theorem 3 does not propose constructive conditions for the case of disturbances $\mathbf{d}$ presence. Robust properties of control (10) in this case are oriented on parametric uncertainty and partial state measurements compensation. It is possible to weaken requirements of Theorem 3 for the case of Assumption 4.B, supposing boundedness and asymptotic convergence to zero of disturbance $\mathbf{d}$.

Theorem 1 presents results for iISS stabilization of passive systems with respect to a set. Dependence of Hamiltonian on vector of uncertain parameters $\boldsymbol{\theta}$ in nonlinear fashion prevents application of conventional convex adaptation techniques. Combining results of theorems 1 and 3, it is possible to propose a solution of this problem.

Corollary 1: Assume that:

1) System (9) for $\mathbf{d}=0$ is passive with respect to output $\boldsymbol{\psi}=L_{\mathbf{R}(\mathbf{y})}W(\mathbf{x})^T$ and input $\mathbf{u}$ with smooth storage function $W: R^n\to R_+$, $\alpha_1(|\mathbf{x}|_{\mathsf{W}_0})\leq W(\mathbf{x})\leq\alpha_2(|\mathbf{x}|_{\mathsf{W}_0})$, $\alpha_1,\alpha_2\in\mathsf{K}_\infty$, where $\mathsf{W}_0=\{\mathbf{x}:W(\mathbf{x})=0\}$ is a compact set; system (9) is W-detectable for output $\boldsymbol{\psi}$, $\lim_{|\mathbf{x}|_{\mathsf{V}_0}\to\infty}|\boldsymbol{\psi}(\mathbf{x})|/W(\mathbf{x})<\infty$.
2) For system (9), Assumption 2 holds and Assumption 3 is satisfied for any Lebesgue measurable signal $\mathbf{y}$; minimum singular value $a(t)$ of matrix function $\mathbf{C}^T\boldsymbol{\Omega}^T(t)$ is PA; $|\mathbf{B}(\mathbf{y}_d(t))|\leq B$, $|\mathbf{R}(\mathbf{y}_d(t))|\leq R$ for all $t\geq 0$, $B,R\in R_+$.
3) Smooth function $\varphi: R^q\to R^q$ for all $\boldsymbol{\psi}\in R^q/\{0\}$ possesses inequality $\boldsymbol{\psi}^T\varphi(\boldsymbol{\psi})>0$ and $\mathbf{u}=-\varphi(\boldsymbol{\psi}(\mathbf{x}))=\mathbf{u}(\mathbf{y},\mathbf{L}\mathbf{x},\boldsymbol{\theta})$.

Then control law $\mathbf{u}=\mathbf{u}(\mathbf{y},\mathbf{L}\mathbf{z},\widehat{\boldsymbol{\theta}})$ provides for system (9) and (11)–(14) global boundedness of solutions for the case $\mathbf{d}=0$ and any $\gamma>0$, additionally $\lim_{t\to+\infty}|\mathbf{x}(t)|_{\mathsf{W}_0}=0$.

Proof: The first and the third parts of conditions provide implementation of Theorem 1 in this case. In such situation control $\mathbf{u}=-\varphi(\boldsymbol{\psi}(\mathbf{x}))=\mathbf{u}(\mathbf{y},\mathbf{L}\mathbf{x},\boldsymbol{\theta})$ ensures iISS property with respect to compact set W_0 and input $\mathbf{d}_3$ for system (9). Due to compactness property of the set, the system is also forward complete. Therefore, all conditions of Assumption 4.B are satisfied and taking in mind other conditions of the corollary, the result of Theorem 3 holds. ■

IV. Adaptive Swinging a Pendulum

Consider the problem of energy stabilization for a pendulum with partial observations and parametric uncertainty

$$\dot{x}_1=x_2,\quad y=x_1,\quad \dot{x}_2=-\omega^2\sin(x_1)+u$$

where $\mathbf{x}=[x_1x_2]^T$ is a state vector; ω is unknown frequency, $\theta=\omega^2$. It is required to stabilize the desired value H^* of energy $H(x_1,x_2)=0.5x_2^2+\omega^2(1-\cos(x_1))$. The system is passive with respect to output $\psi=x_2[H(x_1,x_2)-H^*]$ with smooth storage function $W(x_1,x_2)=0.5[H(x_1,x_2)-H^*]^2$. The system is W-detectable with respect to the output [26]. If $H^*\leq 2\omega^2$, then the zero level set of the storage function is compact. The value $H^*=2\omega^2$ corresponds to stabilization of the upper equilibrium of the pendulum.

In [11], the energy control law $u=-\varphi(\psi)$ was proposed, and successfully tested by simulation for $\varphi(\psi)=\tanh(\psi)$. Let us show that such control law and storage function satisfy conditions of Corollary 1. The equations (11)–(14) take the form

$$\begin{aligned}\dot{z}_1 &= z_2+K(x_1-z_1);\quad K>0;\\ \dot{z}_2 &= K(x_1-z_1)-\widehat{\theta}\sin(x_1)\\ &\quad -\varphi\left(z_2\left[0.5z_2^2+\widehat{\theta}(1-\cos(x_1))-2\widehat{\theta}\right]\right)\\ \dot{\Omega}_1 &= -K\Omega_1+\Omega_2;\quad \dot{\eta}_1=-K\eta_1+\eta_2-\Omega_1\dot{\widehat{\theta}}\\ \dot{\Omega}_2 &= -K\Omega_1-\sin(x_1);\quad \dot{\eta}_2=-K\eta_1-\Omega_2\dot{\widehat{\theta}}\\ \dot{\widehat{\theta}} &= \gamma\Omega_1(x_1-z_1+\eta_1).\end{aligned}$$

To test the PA property of signal $\mathbf{C}^T\boldsymbol{\Omega}^T(t)=\Omega_1(t)$, it is enough to establish PE property of signal $v(t)=\sin(x_1(t))$ or PA property of $v'(t)=\sin^2(x_1(t))$. Indeed, $v(t)$ is the single input of stable linear filter (13). Clearly, that forced part of solution (proportional to

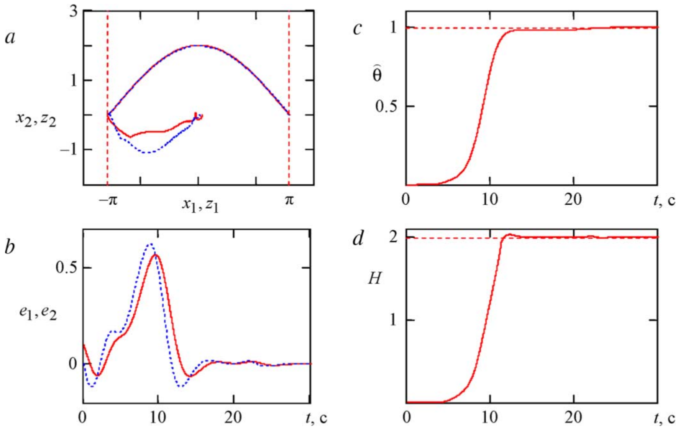

Fig. 1. Adaptive swinging up of the pendulum.

$v(t)$) defines properties of signal $\Omega_1(t)$ (transient motions converge to zero asymptotically). The PA property of signal $v'(t)$ implies, that the system trajectories do not converge and do not stay into the points $x_1 = \pm n\pi, n = 0, 1, 2, \ldots$. This convergence is possible only in the equilibriums of the system $(\pm n\pi, 0), n = 0, 1, 2, \ldots$, but linearization of the pendulum dynamics closed by the proposed control is unstable in these equilibriums for $0 < H^* < 2\omega^2$, since these equilibriums are not the desired final positions of the system. Moreover, the simulation below show, that even for the case $H^* = 2\omega^2$ the algorithm keeps its identification abilities.

The proposed observer with control

$$u = -\varphi\left(z_2\left[0.5z_2^2 + \widehat{\theta}(1-\cos(x_1)) - 2\widehat{\theta}\right]\right)$$

provides stabilization of the upper equilibrium of the pendulum (in this case $H^* = 2\omega^2 = 2\widehat{\theta}$). The simulation results are shown in Fig. 1 for $\omega = K = \gamma = 1$ and zero initial conditions (except $x_1 0) = 0.1$). Trajectories in the state space of the pendulum (solid line) and the adaptive observer (z_1, z_2) (dotted line) are shown in Fig. 1(a). The observation error is presented in Fig. 1(b) separately. In Fig. 1(c) and 1(d), plots of variables $\widehat{\theta}(t)$ and $H(t)$ are shown.

Note that solutions from papers, [10], [16], [23], and [30] cannot be applied in this example due to boundedness of control or since output stabilization is required here.

V. Conclusion

In this technical note, the previous results of the authors [5]–[9] obtained for output synchronization, observation, I-O stabilization are extended to the robust and adaptive partial stabilization problems for a class of nonlinear systems affine in control and disturbances. Applicability conditions of the algorithms are established in the presence of external disturbances and partial observations with measurement noise.

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Natural wave control in lattices of linear oscillators

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ABSTRACT

The problem of natural wave control involves steering a lattice of oscillators towards a desired natural (i.e. zero-input) assignment of energy and phase across the lattice. This problem is formulated and solved for lattices of linear oscillators via a passivity-based approach. Numerical simulations of 1D and 2D linear lattices and a 1D lattice of nonlinear oscillators confirm the effectiveness of the proposed controls.

1. Introduction

Analysis and control of a complex motion in the interconnected and spatially distributed systems have been an area of intensively growing research during the last decades. It has numerous applications in many disciplines, especially in physics [1,2], biology [3] and electrical engineering [4–6]. The problems of power system control, traffic control, and control of communication networks can be solved in the framework of regulation of spatially distributed or networked systems (see special issues in the engineering journals [7–11]). In physics, applications of control theory for studying dynamics of complex systems led to the appearance of a new interdisciplinary area of science gradually becoming known as "Cybernetical Physics" [12].

Design and application of control strategies to the manipulation of complex oscillatory and spatiotemporal patterns have become a central issue of nonlinear dynamics and physics [13,14]. Wave propagation [15,4,16,13,17–19], Belousov–Zhabotinsky reaction supervision [20,21], excitable biological tissue regulation [22–25], and Frenkel–Kontorova models [12] have been controlled by using feedback methods that point out the possibility of dynamical pattern manipulation in excitable media. Such sorts of problems arise in a variety of engineering applications ranging from the macroscopic (e.g., cross directional paper machine processes, automated highway systems, unmanned aerial vehicle or mobile robot formations, satellite constellations), to the microscopic (e.g., arrays of micro-cantilevers or nanostructures) [26]. For the spatially distributed control of a system one needs to design a stabilizing algorithm taking into account the problem of a proper control propagation over time and space. Design and analysis of an active device capable to control the mechanical waves is considered in [27], where the framework of a "mechanical wave diode" is presented. All previously mentioned works are oriented on particular control strategies synthesis and none of them studies spatially distributed systems in a systematic way. A promising approach to systematic development of control methods for complex networks or continuous distributed systems is to start with the systems of a low complexity. In this work the lattice of linear oscillators is considered as such a simple spatially distributed system.

Dynamics of many spatially distributed systems can be described by partial differential equations or, after discretization, by lattices of nonlinear oscillators [28]. Though stabilization-like control goals for lattices are well studied [29,26], excitation of waves was considered previously only for some special cases [12,30,31]. In a steady state vicinity, the nonlinear lattice can be reduced to the considered lattice of linear oscillators. Such a reduction leads to many important properties being lost, for example, the breather/soliton existence phenomenon can be analyzed in a nonlinear framework only [32] (that is an important area of research in physics initiated by the Fermi–Pasta–Ulam (FPU) numerical experiments [1,33]). However, as the first step for the problem formulation and a preliminary solution presentation, the lattices of linear oscillators are very useful, as we are going to show below.

In this work the problem of wave control is addressed. The existence conditions of solitons or breathers were intensively

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studied before; here we attempt to solve the problem of a given wave excitation starting from any initial state of a lattice. Such a control can be important for researchers working experimentally with different waves for their analysis, as well as for understanding how these waves arise in nature.

The proposed solution to the problem is based on transformation of the system to a canonical form followed by spectrum localization. The idea is to excite the desired modes in order to ensure a required oscillation amplitude and phase resetting for the prespecified vertices. Our approach to control design uses the speed-gradient method proposed in [34,35] and extended to control of oscillations and partial stabilization in [36,37]. The essence of the speed-gradient method is in evaluation of the speed $\dot{Q}$ of variation of the given goal function Q along plant trajectories, and changing the control variables in the direction of gradient of $\dot{Q}$ with respect to the control.

The outline of the paper is as follows. The lattice equations and the problem statement are given in Section 2. The control design is performed in Section 3. The results of applications for 1D and 2D lattices, as well as for a 1D nonlinear FPU lattice, are presented in Section 4.

2. Problem statement

Consider a 1D lattice of linear oscillators (the symbol $\overline{1,n}$ denotes the sequence of integers $1,\ldots,n$):

$$\ddot{x}_i = \Omega^2(x_{i+1} + x_{i-1} - 2x_i) + b_i u, \quad i = \overline{1,n}, x_0 = x_{n+1} = 0 \quad (1)$$

or a 2D lattice

$$\ddot{x}_{i,j} = \Omega^2(x_{i,j+1} + x_{i,j-1} + x_{i-1,j} + x_{i+1,j} - 4x_{i,j}) + \mathbf{b}_{i,j}^T\mathbf{u}, \quad (2)$$
$$x_{i,m+1} = x_{i,0} = x_{0,j} = x_{n+1,j} = 0, \quad i = \overline{1,n}, j = \overline{1,m},$$

where $\Omega \in R$ is the lattice's "frequency", $n > 1, m > 1$ define the lattice's dimension, x_i or $x_{i,j}$ correspond to 1D or 2D vertices' positions, $\dot{x}_i, \dot{x}_{i,j}$ and $\ddot{x}_i, \ddot{x}_{i,j}$ stand for velocities and accelerations correspondingly, and the scalar b_i or the vector $\mathbf{b}_{i,j}$ determine the appearance of the scalar control u in the 1D case or the vector one $\mathbf{u} \in R^m$ in 2D. Typically the controls influence the lattices from one side, i.e.

$$b_1 \neq 0, \qquad b_k = 0, \quad k = \overline{2,n};$$
$$\mathbf{b}_{1,j} \neq 0, \qquad \mathbf{b}_{k,j} = 0, \quad k = \overline{2,n}, j = \overline{1,m}$$

in 1D and 2D cases respectively. The ND case ($N > 2$) can be treated similarly.

The systems (1), (2) are linear and by enumeration of the states they can be written in the form

$$\dot{\mathbf{s}} = \mathbf{A}\mathbf{s} + \mathbf{B}\mathbf{u}, \quad \mathbf{s} \in R^l$$

for suitably defined matrices $\mathbf{A}$ and $\mathbf{B}$, where $l = n$ for a 1D lattice and $l = nm$ in the 2D case (the 1D lattice is already in this form with $\mathbf{s} = [x_1,\ldots,x_n]$). The spectrum of the systems (1) or (2) is pure imaginary $\lambda_r = \pm\omega_r i, r = \overline{1,R}$ with multiplicities p_r (the matrix $\mathbf{A}$ has eigenvalues $-\omega_r^2, r = \overline{1,R}$) and $\sum_{r=1}^R p_r = l$ (in the 1D case all $p_r = 1, r = \overline{1,R}$, but in the 2D case $p_1 = \min\{n,m\}$). Then there exists a linear transformation of coordinates $\mathbf{z} = \mathbf{Rs}$ with nonsingular $\mathbf{R}$ composed by right eigenvectors of the matrix $\mathbf{A}$ (this matrix is symmetric from (1), and (2) and has l independent eigenvectors) such that the systems (1), and (2) can be represented in the *canonical* form:

$$\dot{z}_{k,1} = z_{k,2}, \qquad \dot{z}_{k,2} = -\omega_k^2 z_{k,1} + \boldsymbol{\beta}_k^T\mathbf{u}, \quad k = \overline{1,l}, \quad (3)$$
$$\mathbf{u} \in R^m, \quad B = [\boldsymbol{\beta}_1 \ldots \boldsymbol{\beta}_l]^T = \mathbf{R}^{-1}\mathbf{B}.$$

For each $k = \overline{1,l}$ the system (3) is called the normal mode of (1), and (2). Actually the system (1), and (2) has the same form as (3) with an additional coupling of neighbors and the control acting just on one layer in the lattices. In (3) all oscillators are uncoupled with the control influencing on all modes directly. The representation of lattices (3) indicates that in the uncontrolled case ($\mathbf{u} = 0$) the systems (1) and (2) solutions are purely oscillating:

$$x_i(t) = \sum_{k=1}^n \alpha_k \sin(\omega_k t + \phi_k), \quad i = \overline{1,n}; \quad (4a)$$
$$x_{k,j}(t) = \sum_{j=1}^m \sum_{k=1}^n \alpha_{k,j} \sin(\omega_{k,j} t + \phi_{k,j}),$$
$$i = \overline{1,n}, j = \overline{1,m}, \quad (4b)$$

where $\alpha_i, \omega_i, \phi_i$ and $\alpha_{i,j}, \omega_{i,j}, \phi_{i,j}$ are real constant parameters determined by initial conditions for the 1D and 2D cases respectively. Any such solution (4) for all admissible sets of parameters we will further call a *natural wave* for the lattices. Motivation of the term "natural" is that such waves exist in (1), and (2) without an external influence. According to the lattice structure not all real values α_i, ϕ_i and $\alpha_{i,j}, \phi_{i,j}$ for $i = \overline{1,n}, j = \overline{1,m}$ can be admissible (the frequencies $\omega_i, \omega_{i,j}$ are predefined by the lattice dimension and Ω).

The problem is to create a predefined natural wave in the lattice. To this end, m vertices are chosen (m is the number of the controls available) and their indices are collected in the set $\mathcal{V}$, $card(\mathcal{V}) = m$ (the symbol $card(\cdot)$ is stated for a set cardinality). The number $d \le m$ of the frequencies is specified (these frequencies define the stabilized wave spectrum), and their indexes are given in the set $\mathcal{F}$, $card(\mathcal{F}) = d$. The reference behavior for all vertices in $\mathcal{V}$ is defined in the form

$$x_i^*(t) = \sum_{k\in\mathcal{F}} A_{i,k} \sin(\omega_k t + \varphi_{i,k}), \quad i \in \mathcal{V}; \quad (5a)$$
$$x_{i,j}^*(t) = \sum_{k\in\mathcal{F}} A_{i,j,k} \sin(\omega_k t + \varphi_{i,j,k}), \quad (i,j) \in \mathcal{V} \quad (5b)$$

for all $t \ge 0$ in the 1D and 2D cases respectively, where $A_{i,k}, \varphi_{i,k}$ and $A_{i,j,k}, \varphi_{i,j,k}$ are given constants. It is assumed that $x_i(t) = x_i^*(t)$ for all $i \in \mathcal{V}$ for the system (1) or $x_{i,j}(t) = x_{i,j}^*(t)$ for all $(i,j) \in \mathcal{V}$ for the system (2) are admissible invariant solutions with $\mathbf{u}(t) \equiv 0$ for all $t \ge 0$ (not all triples $A_{(\cdot)}, \omega_{(\cdot)}, \varphi_{(\cdot)}$ are admissible for the chosen sets $\mathcal{V}$ and $\mathcal{F}$, that follows from the lattice topology). In other words, the solutions (5) belong to the natural waves for (1), and (2). From the physical point of view, it is necessary to ensure the lattice oscillations of a predefined spectrum with a given profile of the wave front. Then, it is required to design a control $\mathbf{u}$ such that

$$\lim_{t\to+\infty}[x_i(t) - x_i^*(t)] = 0, \quad i \in \mathcal{V};$$
$$\lim_{t\to+\infty}[x_{i,j}(t) - x_{i,j}^*(t)] = 0, \quad (i,j) \in \mathcal{V}.$$

Thus we are looking for the control that excites a predefined natural wave for the lattices (1) or (2) with the fixed behavior of the vertices in the set $\mathcal{V}$, i.e. a wave of a given spectrum with predefined shape. Since the wave is natural, in this case it may be $\mathbf{u}(t) \to 0$ for $t \to +\infty$. It is assumed that the state of the lattice is accessible for measurements, i.e. $z_{k,1}, z_{k,2}$ for all $k = \overline{1,l}$.

3. Main result

To solve the posed problem, a decomposition to two subproblems with two different subgoals is proposed. The *first* one consists in spectrum $\mathcal{F}$ localization, i.e. it is necessary to eliminate all oscillating modes in the system corresponding to the frequencies not included in $\mathcal{F}$ and to excite the desired modes $k \in \mathcal{F}$ in order to ensure the required amplitudes of oscillation $A_{(\cdot)}$ for the vertices given in $\mathcal{V}$. The *second* subgoal is the phase resetting for the vertices $\mathcal{V}$ (the desired phases $\varphi_{(\cdot)}$ have to be assigned); this objective deals with the wave coordination in time and space. Having the same spectrum the waves can have different forms ("standing" or "running"), the wave form can be assigned by coordination of phases for the nodes in $\mathcal{V}$. These subgoals are *independent*, and in the general case achievement of one of them does not necessarily imply achievement of another goal. Let us consider two solutions of these subproblems utilizing the special form of (3).

A. *Spectrum localization*

This problem can be stated and solved for the canonical representation of the lattices (3). Each normal mode in (3) has the energy or Hamiltonian function

$$H_k(z_{k,1}, z_{k,2}) = 0.5\,(z_{k,2}^2 + \omega_k^2 z_{k,1}^2), \quad k = \overline{1, l},$$

with the time derivative

$$\dot{H}_k = z_{k,2}(-\omega_k^2 z_{k,1} + \boldsymbol{\beta}_k^T \mathbf{u}) + \omega_k^2 z_{k,1} z_{k,2} = z_{k,2}\,\boldsymbol{\beta}_k^T \mathbf{u}.$$

Then the problem of the spectrum localization can be solved by stabilization of the desired values for the energies H_k^*, $k = \overline{1, l}$, which we further assume given as follows: $H_i^* = 0$ for all $i \notin \mathcal{F}$, and some $H_i^* \neq 0$ for all $i \in \mathcal{F}$. For brevity of presentation let $\mathbf{H} = [H_1, \ldots, H_l]^T$, $\mathbf{H}^* = [H_1^*, \ldots, H_l^*]^T$ and $S_k = \{i = \overline{1, l} : \omega_k = \omega_i\}$, $k = \overline{1, l}$ be the set of all normal mode indices with the same frequency. The proposed control design is based on the speed-gradient approach [34–37] with the energy-based goal function $Q(\mathbf{z}) = [\mathbf{H}(\mathbf{z}) - \mathbf{H}^*]^T [\mathbf{H}(\mathbf{z}) - \mathbf{H}^*]$.

Theorem 1. *For any $\mathbf{H}^* \in R_+^l$ consider the control*

$$\mathbf{u} = -\chi(\mathbf{y}), \quad \mathbf{y} = \sum_{k=1}^{l} (H_k - H_k^*) z_{k,2}\,\boldsymbol{\beta}_k, \tag{6}$$

where $\mathbf{y}^T \chi(\mathbf{y}) > 0$ for any $\mathbf{y} \in R^m \setminus \{0\}$ is differentiable and $\chi(0) = 0$, $\chi'(0) \neq 0$. Let for all $k = \overline{1, l}$ the vectors $\boldsymbol{\beta}_s$, $s \in S_k$ be linearly independent, then (3), and (6) are partially stable with respect to the variable $\mathbf{H}(t) - \mathbf{H}^*$ and for all initial conditions $\mathbf{H}(0) \in H = \{\mathbf{H} \in R_+^l : H_i > 0,\ i \in \mathcal{F}\}$ the relation

$$\lim_{t \to +\infty} \mathbf{H}(t) = \mathbf{H}^*$$

holds, provided that the matrix $\mathbf{W}$, defined for $k = \overline{1, d}$:

$$W_{2k-1,2k} = 1; \qquad W_{2k-1,i} = 0, \quad i = \overline{1, 2d} \text{ and } i \neq 2k;$$
$$W_{2k,2k-1} = -\omega_k^2; \qquad W_{2k,2h-1} = 0, \quad h = \overline{1, d} \text{ and } h \neq k;$$
$$W_{2k,2h} = \chi'(0)\,\boldsymbol{\beta}_k^T \boldsymbol{\beta}_h H_h^*, \quad h = \overline{1, d},$$

has all eigenvalues with positive real parts.

Proof. Consider the following Lyapunov function (proof of the theorem utilizes the passivity property of the speed-gradient algorithms for conservative systems [36,37])

$$V(\mathbf{H}) = 0.5\,(\mathbf{H} - \mathbf{H}^*)^T (\mathbf{H} - \mathbf{H}^*)$$

that for the system (3) has the time derivative $\dot{V} = \mathbf{y}^T \mathbf{u}$. Substitution of (6) gets $\dot{V} = -\mathbf{y}^T \chi(\mathbf{y}) \leq 0$, that ensures for the system (3), and (6) partial stability with respect to the variable $\mathbf{H} - \mathbf{H}^*$ [38]. For any finite $\mathbf{H}^*$ this property implies boundedness of the variable $\mathbf{H}$, that in its turn guarantees the states $z_{k,1}, z_{k,2}$, $k = \overline{1, l}$ definiteness for all $t \geq 0$ and their boundedness.

Note that if $\mathbf{H}(0) = 0$ then $\mathbf{u}(t) = 0$ for all $t \geq 0$ and the lattices (1), and (2) have no oscillations ($x_i(t) = 0$, $x_{i,j}(t) = 0$ for all $t \geq 0$, $i = \overline{1, n}$, $j = \overline{1, m}$). To prove that in the case $\mathbf{H}(0) \neq 0$ the oscillations never die under (6) consider the system (3), and (6) linearization at the origin (the origin corresponds to the case $\mathbf{H} = 0$):

$$\dot{\zeta}_{k,1} = \zeta_{k,2}, \qquad \dot{\zeta}_{k,2} = -\omega_k^2 \zeta_{k,1} + \boldsymbol{\beta}_k^T \chi'(0) \sum_{i \in \mathcal{F}} H_i^* \zeta_{i,2} \boldsymbol{\beta}_i,$$

$k = \overline{1, l}$; $\zeta_{k,1}, \zeta_{k,2}$ are the corresponding coordinates of the linearized system. Enumerating the states it is possible to put all modes from the set $\mathcal{F}$ as the first $2d$ coordinates, then the matrix of the system will be low-triangular. The first block is described by the matrix $\mathbf{W}$ having $2d$ eigenvalues with positive real parts, for $i \notin \mathcal{F}$ the blocks on the main diagonal have pure imaginary eigenvalues. Therefore, if $\mathbf{H}(0) \in H$, then $H_i(t) \neq 0$ for all $t \geq 0$ for all $i \in \mathcal{F}$.

By construction $\dot{V} < 0$ for all $\mathbf{y} \neq 0$, and $\dot{V} = 0$ if and only if $\mathbf{y} \equiv 0$ and $\mathbf{u} \equiv 0$. Next, we would like to prove the system (3) (strong) *observability* with respect to the output $\mathbf{y}$ for the set of initial conditions $\mathbf{H}(0) \in H$, i.e.

$$\mathbf{y}(t) \equiv 0, \qquad \mathbf{u}(t) \equiv 0 \quad \text{for all } t \geq 0 \Rightarrow \mathbf{H}(t) = \mathbf{H}^*,\ t \geq 0$$

(owing (6) the converse trivially holds and $\mathbf{H}(t) = \mathbf{H}^*$ for all $t \geq 0 \Rightarrow \mathbf{y}(t) \equiv 0$, $\mathbf{u}(t) \equiv 0$, $t \geq 0$). If this property is satisfied, then by standard arguments $\mathbf{H}(t) \to \mathbf{H}^*$ with $t \to +\infty$.

Assume that $\mathbf{y}(t) \equiv 0$, $\mathbf{u}(t) \equiv 0$ for all $t \geq 0$. Since $\mathbf{u}(t) \equiv 0$, $t \geq 0$ all normal modes are isolated in (3) and for all $t \geq 0$ and for $r = \overline{1, l}$ we have

$$H_r(t) = \eta_r; \qquad z_{r,1}(t) = a_r \sin(\omega_r t + \upsilon_r),$$

$$z_{r,2}(t) = a_r \omega_r \cos(\omega_r t + \upsilon_r), \quad a_r = \sqrt{2\,\eta_r \omega_r^{-2}},$$

where η_r and υ_r are the constants dependent on initial conditions. By contrast, take a set of indices $R = \{k = \overline{1, l} : \eta_k \neq H_k^*\}$. From the output $\mathbf{y}$ definition we obtain

$$0 = \sum_{r \in R} [\eta_r - H_r^*]\, a_r \omega_r \cos(\omega_r t + \upsilon_r)\,\boldsymbol{\beta}_r^T, \quad t \geq 0.$$

It could be the case that $H_r^* \neq 0$ and $\eta_r = 0$ for all $r \in R$, then $a_r = 0$ and the observability fails. However, from the consideration above if $H_i(0) \neq 0$ for all $i \in \mathcal{F}$, then it should be $\eta_r \neq 0$ for all $r \in R$ with $H_r^* \neq 0$. Therefore for $\mathbf{H}(0) \in H$ we may restrict our consideration to the case $a_r \neq 0$, $r \in R$, then

$$\sum_{r \in R} \gamma_r \cos(\omega_r t + \upsilon_r)\,\boldsymbol{\beta}_r^T = 0, \quad \gamma_r = a_r \omega_r [\eta_r - H_r^*]$$

and the last equation can hold for all $t \geq 0$ if for all $r \in R$ the equalities

$$\sum_{s \in S_r} \gamma_s \boldsymbol{\beta}_s = 0, \quad \upsilon = 1, 2$$

are satisfied. Since γ_s are arbitrary real constants the equalities above cannot be true if the vectors $\boldsymbol{\beta}_s$, $s \in S_r$, $r \in R$ are linearly independent as it is claimed in the theorem conditions. We arrive at the contradiction, the set R is empty and the system (3) is (strongly) observable with respect to the output $\mathbf{y}$ for $\mathbf{H}(0) \in H$. $\square$

Under conditions of Theorem 1 the control (6) solves the problem of spectrum localization for the lattices. The conditions include the restriction on initial conditions $\mathbf{H}(0) \in H$, that can be easily satisfied (if it is not the case and $\mathbf{H}(0) = 0$, then a pulse $\mathbf{u}$ has to be applied exciting the lattice; next at the instant of time $t' \geq 0$ when $\mathbf{H}(t') \in H$ the control (6) can be activated), and the requirement on linear independence of $\boldsymbol{\beta}_s$, $s \in S_k$, $k = \overline{1, l}$. This condition is always true for the 1D case, since in this case all sets S_k, $k = \overline{1, l}$ contain just one element (all frequencies are distinct).

B. *Phase resetting and wave control*

This problem can be solved for the canonical representation (3) and for the systems (1), and (2) directly. In this work we will focus our attention on the case (3) only.

The desired phases $\varphi_{i,k}$, $i \in \mathcal{V}$ or $\varphi_{i,j,k}$, $(i,j) \in \mathcal{V}$ correspond to the frequency ω_k, $k \in \mathcal{F}$, the normal modes into the set S_k all have the same frequency ω_k and different phases υ_p, $p \in S_k$. The variables x_i and $x_{i,j}$ for all $i = \overline{1,n}$, $j = \overline{1,m}$ are linear combinations of the variables $z_{k,\upsilon}$, $k = \overline{1,l}$, $\upsilon = 1, 2$, thus for given H_k^*, $k = \overline{1,l}$ the desired values υ_k^* for the phases υ_k, $k \in \mathcal{F}$ can be derived from the phases $\varphi_{i,k}$ and $\varphi_{i,j,k}$. In this case the problem of phase resetting for x_i, $x_{i,j}$ is reduced to the problem of phase resetting for the modes $k \in \mathcal{F}$ in (3).

Each normal mode in (3) for $k = \overline{1,l}$ can be rewritten in action-angle coordinates [39]

$$I_k = H_k(z_{k,1}, z_{k,2}),$$
$$\upsilon_k = \pi/2 - \operatorname{atan}(\omega_k z_{k,1}/z_{k,2}) \Leftrightarrow z_{k,1} = \omega_k^{-1}\sqrt{2 I_k}\cos(\upsilon_k),$$
$$z_{k,2} = \sqrt{2 I_k}\sin(\upsilon_k)$$

as follows:

$$\dot I_k = \sqrt{2 I_k}\sin(\upsilon_k)\,\boldsymbol{\beta}_k^T\mathbf{u};$$
$$\dot\upsilon_k = \frac{\cos(\upsilon_k)\boldsymbol{\beta}_k^T}{\sqrt{2 I_k}}\mathbf{u} - \omega_k, \quad k = \overline{1,l}, \tag{7}$$

where $I_k \in R_+$ and $\upsilon_k \in [0, 2\pi)$ (furthermore all additions or subtractions with the phase variables υ_k are understood by modulus 2π). Let us stress that the action-angle representation (7) of the normal form (3) is correct for the case $\mathbf{H} \neq 0$ only. We will further accept that for the case $I_k = 0$ always $\upsilon_k = 0$ (formally, if $I_k = 0$ then there are no oscillations and the phase can be defined artificially). From these equations we see that for $\mathbf{u}(t) \equiv 0$ for all $t \geq 0$ we have $\upsilon_k(t) = \upsilon_k(0) - \omega_k t$, $k = \overline{1,l}$ and the phase resetting consists in replacement of the initial conditions $\upsilon_r(0)$ with some predefined values υ_r^*, $r \in \mathcal{F}$ which characterize the required profile of the wave in the lattice. Therefore, the objective of the phase resetting is

$$\lim_{t\to+\infty} |\upsilon_r(t) - (\upsilon_r^* - \omega_r t)| = 0, \quad r \in \mathcal{F}.$$

The problem of phase resetting can be easily solved for (7) applying an approach similar to that presented in Theorem 1, but we are more interested in simultaneous solution of the spectrum localization and phase resetting problems, i.e. we will assume that the desired values H_k^* for the action variable I_k, $k = \overline{1,l}$ ($H_i^* = 0$ for all $i \notin \mathcal{F}$ and $H_i^* \neq 0$ for all $i \in \mathcal{F}$) with the desired values υ_r^* for υ_r, $r \in \mathcal{F}$ are given. Then Theorem 1 admits the following extension.

Theorem 2. *For any $\mathbf{H}^* \in R_+^l$ consider the control*

$$\mathbf{u} = -\chi(\boldsymbol{\psi}), \tag{8}$$
$$\boldsymbol{\psi} = \sum_{k=1}^{l}(I_k - H_k^*)\sqrt{2 I_k}\sin(\upsilon_k)\,\boldsymbol{\beta}_k + \sum_{r\in\mathcal{F}}(\upsilon_r - \upsilon_r^* + \omega_r t)\cos(\upsilon_r)\boldsymbol{\beta}_r \Big/ \sqrt{2 I_r},$$

where $\boldsymbol{\psi}^T\chi(\boldsymbol{\psi}) > 0$ for any $\boldsymbol{\psi} \in R^m \setminus \{0\}$, $\chi(0) = 0$, $\sup_{\boldsymbol{\psi}\in R^m} |\chi(\boldsymbol{\psi})| < +\infty$. Let for all $k = \overline{1,l}$ the vectors $\boldsymbol{\beta}_s$, $s \in S_k$ be linearly independent, then the system (7), and (8) is partially stable with respect to the variables $\mathbf{H}(t) - \mathbf{H}^*$, $\upsilon_r(t) - \upsilon_r^* + \omega_r t$, $r \in \mathcal{F}$ and for all initial conditions the relations hold:

$$\lim_{t\to+\infty}\mathbf{H}(t) = \mathbf{H}^*, \qquad \lim_{t\to+\infty}[\upsilon_r(t) - \upsilon_r^* + \omega_r t] = 0, \quad r \in \mathcal{F}.$$

Proof. The control (8) is always finite and even for the case $\mathbf{H} = 0$ the solutions of the system (7), and (8) are well defined at least locally. Consider the following Lyapunov function

$$V = 0.5\left[\sum_{k=1}^{l}(I_k - H_k^*)^2 + \sum_{r\in\mathcal{F}}(\upsilon_r - \upsilon_r^* + \omega_r t)^2\right],$$

in this case the function V is explicitly time-varying and its time derivative has the form $\dot V = \boldsymbol{\psi}^T\mathbf{u}$. Substitution of (8) gives $\dot V = -\boldsymbol{\psi}^T\chi(\boldsymbol{\psi}) \leq 0$, that ensures the system (7), and (8) has partial stability with respect to the variables $\mathbf{H} - \mathbf{H}^*$ and $\upsilon_r(t) - \upsilon_r^* + \omega_r t$, $r \in \mathcal{F}$ [38]. Boundedness of $\mathbf{H}$ ensures that the system (7) state trajectories are well defined and bounded for all $t \geq 0$.

By construction $\dot V < 0$ for all $\boldsymbol{\psi} \neq 0$, and $\dot V = 0$ if and only if $\boldsymbol{\psi} \equiv 0$ and $\mathbf{u} \equiv 0$ (since the control (8) is time-varying, even $\mathbf{H} = 0$ does not imply $\mathbf{u} = 0$). Next, the last thing to prove is the system (7) (strong) observability with respect to the output $\boldsymbol{\psi}$, i.e.

$$\boldsymbol{\psi}(t) \equiv 0, \qquad \mathbf{u}(t) \equiv 0 \quad \text{for all } t \geq 0 \Rightarrow I_k(t) = H_k^*,\ k = \overline{1,l},$$
$$\upsilon_r(t) = \upsilon_r^* - \omega_r t, \quad r \in \mathcal{F} \text{ for all } t \geq 0$$

(the converse is obviously satisfied). If the observability is true, then the desired attractivity is substantiated.

Let $\boldsymbol{\psi}(t) \equiv 0$, $\mathbf{u}(t) \equiv 0$ for all $t \geq 0$. Since $\mathbf{u}(t) \equiv 0$, $t \geq 0$ all systems in (7) are isolated and for all $t \geq 0$ and $r = \overline{1,l}$ the relations $I_r(t) = \eta_r$, $\upsilon_r = \kappa_r - \omega_r t$ hold, where η_r and κ_r are the constants dependent on initial conditions. By contrast, assume that there exist some sets of indices $R = \{k = \overline{1,l} : \eta_k \neq H_k^*\}$ and $P = \{r \in \mathcal{F} : \kappa_r \neq \upsilon_r^*\}$, then from the output $\boldsymbol{\psi}$ definition for all $t \geq 0$ we obtain

$$\sum_{k\in R}\gamma_k\sin(\kappa_k - \omega_k t)\,\boldsymbol{\beta}_k + \sum_{r\in P}\alpha_r\cos(\kappa_r - \omega_r t)\,\boldsymbol{\beta}_r = 0,$$
$$\gamma_k = (\eta_k - H_k^*)\sqrt{2\eta_k},\ \alpha_r = (\kappa_r - \upsilon_r^*)\Big/\sqrt{2\eta_r},\ t \geq 0.$$

The last equation can hold for all $t \geq 0$ if for all $r \in R\cup P$ the equalities $\sum_{s\in S_r}\gamma_s\boldsymbol{\beta}_s = 0$, $\sum_{s\in S_r}\alpha_s\boldsymbol{\beta}_s = 0$ are satisfied. Since γ_s, α_s are arbitrary real constants the equalities above cannot be true if the vectors $\boldsymbol{\beta}_s$, $s \in S_r$, $r \in R\cup P$ are linearly independent as it is claimed in the theorem conditions. We arrive at the contradiction; the sets R, P are empty and the system (7) is (strongly) observable with respect to the output $\boldsymbol{\psi}$. □

Under conditions of Theorem 2 the control (8) solves the problems of spectrum localization and phase resetting for the lattices (1), and (2). Therefore, a natural wave stabilization with desired parameters is ensured. The conditions include the requirement on linear independence of $\boldsymbol{\beta}_s$, $s \in S_k$, $k = \overline{1,l}$ only. Since the control (8) is time-varying the problem of oscillation death is naturally solved: at the point $\mathbf{H} = 0$ the bounded control (8) automatically generates an exciting pulse. Again the vectors $\boldsymbol{\beta}_s$ are always linearly independent in the 1D case.

4. Examples

A. 1D *lattice*

The results of the control (8) application for the 1D lattice with $n = 20$ is presented in Fig. 1 with $\chi(y) = 5\tanh(y)$, $b_1 = 1$, $b_k = 0$, $k = \overline{2,n}$ and $\Omega = 1$. One mode with the frequency $\omega = 2.14$ is chosen into the set $\mathcal{F}$; the desired value of the phase is $\pi/2$. In Fig. 1(a) the energies $H_r(t)$, $r = \overline{1,l}$ are plotted, in Fig. 1(b) the variables $\varphi_r(t) = \upsilon_r(t) + \omega_r t$, $r = \overline{1,l}$ are presented (these variables have to converge to constant values, in particular $\varphi_r(t) \to \upsilon_r^* + \omega_r t$ for $r \in \mathcal{F}$) and the control is shown in Fig. 1(c). The red solid curves correspond to regulated variables H_r, φ_r, while the red dashed lines represent the corresponding desired values H_r^*, υ_r^*.

B. 2D *lattice*

The results of the control (8) application for the 2D lattice with $n = m = 10$ is presented in Fig. 2 with $\chi(y) = 5\tanh(y)$. The vectors $\mathbf{b}_{1,j}$, $j = \overline{1,m}$ form the identity matrix (each node in the input layer is regulated by its own control) and $\mathbf{b}_{k,j} = 0$, $k = \overline{2,n}$,

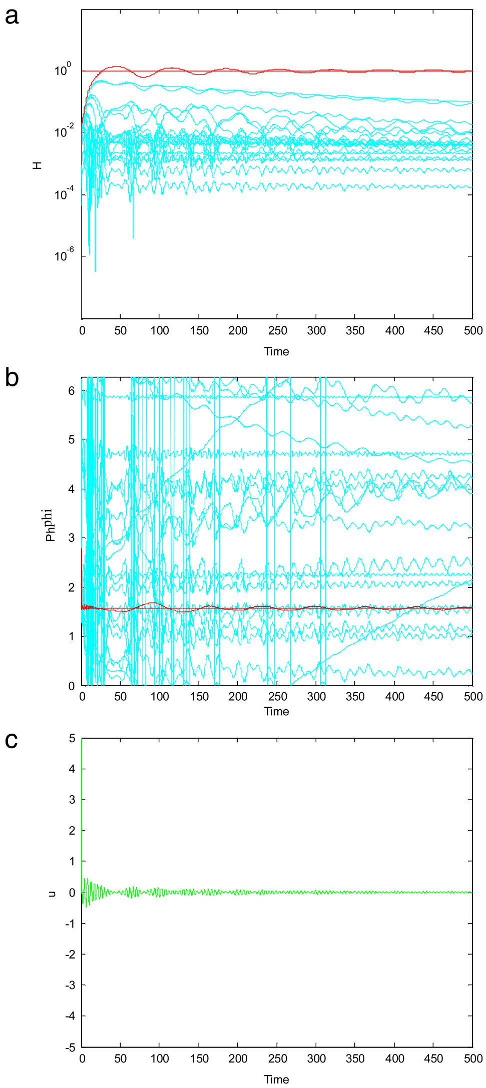

Fig. 1. Results of simulation for 1D case. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$j = \overline{1, m}$ and $\Omega = 1$. The frequency $\omega_1 = 2$ has maximal multiplicity $p_1 = n = 10$ in this case, and all normal modes with this frequency are chosen to reach the level of energy equal to 1 (the corresponding desired phases are uniformly distributed from 0 and π). In Fig. 2(a) the energies $H_r(t)$, $r = \overline{1, l}$ are presented, in Fig. 2(b) the variables $\varphi_r(t) = \upsilon_r(t) + \omega_r t$, $r = \overline{1, l}$ are shown (these variables have to converge to constant values, in particular $\varphi_r(t) \to \upsilon_r^* + \omega_r t$ for $r \in \mathcal{F}$) and the controls are plotted in Fig. 2(c). The red solid curves correspond to regulated variables H_r, φ_r, while the red dashed lines represent the corresponding desired values H_r^*, υ_r^*.

C. *FPU nonlinear lattice*

Finally consider a nonlinear 1D lattice from the FPU experiment [40,33]:

$$\ddot{x}_i = \Omega^2(x_{i+1} + x_{i-1} - 2x_i) + \alpha[(x_{i+1} - x_i)^2 - (x_{i-1} - x_i)^2] + b_i u, \quad i = \overline{1, n}, \; x_0 = x_{n+1} = 0, \tag{9}$$

where $\alpha > 0$ is a parameter, and all symbols have the same meaning as previously. The difference between (1) and (9) consists in the appearance of the nonlinear coupling proportional to α. The system (9) admits soliton or breather solutions.

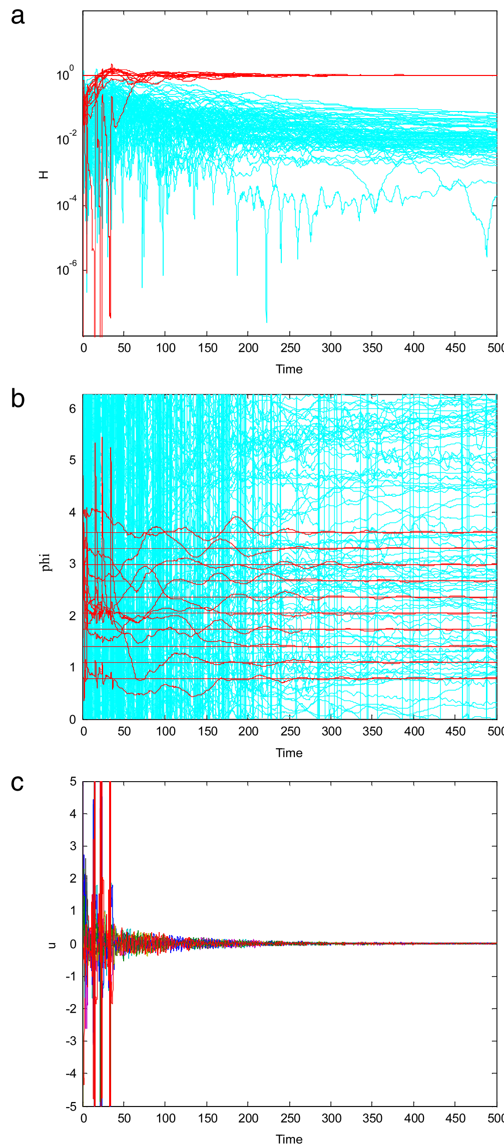

Fig. 2. Results of simulation for 2D case. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Let us apply to (9) the controls developed for the system (1) to demonstrate that the linear approximation of lattice oscillations is rather reliable. For this purpose, we chose the same parameters of the system (9) as for (1) in the first simulation: $n = 20$; $\chi(y) = 5\tanh(y)$; $b_1 = 1$, $b_k = 0$, $k = \overline{2, n}$; $\Omega = 1$ and $\alpha = 0.333$. The mode with the frequency $\omega = 2.14$ is chosen into the set $\mathcal{F}$; the desired value of the corresponding phase is $\pi/2$. The results of the lattice (9) simulation are shown in Fig. 3. In Fig. 3(a) the energies $H_r(t)$, $r = \overline{1, l}$ are plotted, in Fig. 3(b) the variables $\varphi_r(t) = \upsilon_r(t) + \omega_r t$, $r = \overline{1, l}$ are presented and the control is shown in Fig. 3(c). The red solid curves correspond to regulated variables H_r, φ_r, while the red dashed lines represent the corresponding desired values H_r^*, υ_r^*.

Comparing the results presented in Figs. 1 and 3 note that the convergence of H_r in the FPU case is less monotone but still takes

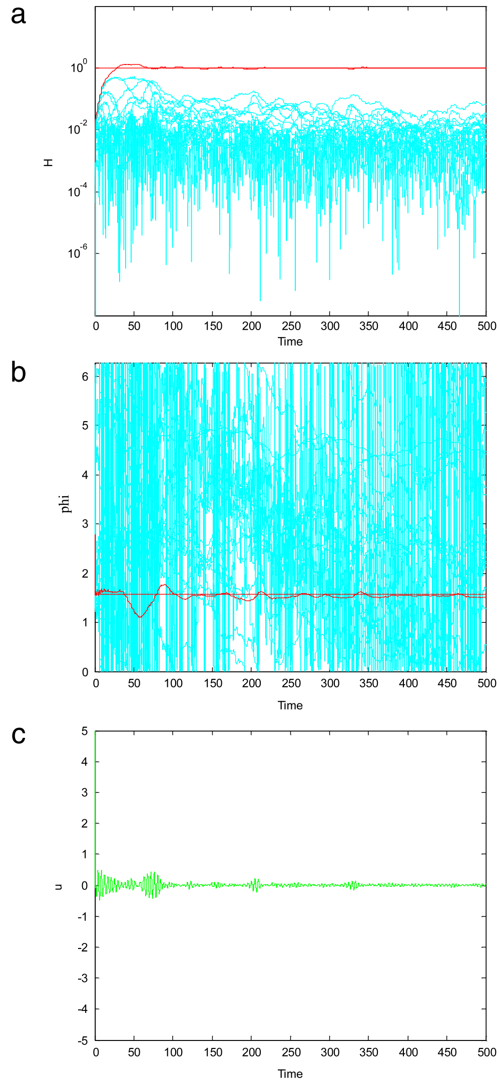

Fig. 3. Results of simulation for the FPU case. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

place, while the phases φ_r do not approach a constant value except the one from the set $\mathcal{F}$ (for the nonlinear case the phase definition and its dynamics are more complicated: formally the frequency of oscillations in (9) depend on the system current energy). Nevertheless, the control converges to zero asymptotically. These results demonstrate that the proposed wave regulation approach is rather robust, and it is also relevant for weakly nonlinear lattices (even if the deviations from the steady state have significant amplitudes).

5. Conclusion

The problem of natural wave stabilization at the desired energy levels for lattices of linear oscillators is posed and solved. A passivity-based approach is used to derive the control algorithms. Analytical applicability conditions of the method are established. Simulation results confirm the efficiency of the proposed methodology.

It is interesting to note that it is hard to solve the posed problem for the original linear systems (1) or (2). However, applying the nonlinear transformation we obtain a straightforward solution for the nonlinear representation (7). The proposed control law is also nonlinear. The phase resetting problem is addressed.

Some preliminary results of the application of the proposed method to a nonlinear FPU lattice are reported. A theoretical extension of the proposed approach to a nonlinear case is left for future work.

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Phase resetting control based on direct phase response curve

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Abstract The problem of controlled phase adjustment (resetting) for models of biological oscillators is considered. The proposed approach is based on oscillators excitation by a pulse, that results in the phase advancement or delay. Design procedure is presented for a series of pulses generation ensuring the required phase resetting. The solution is based on the direct phase response curve (PRC) approach. The notion of direct PRC is developed and non-local PRC model is proposed for oscillators. This model is more suitable for phase dynamics description under inputs excitation with sufficiently high amplitudes. The proposed model is used for controls design. Two control strategies are tested, the open-loop control (that generates a predefined table of instants of the pulses activation ensuring the resetting) and the feedback control (that utilizes information about the current phase value measured once per pulse application). The open-loop control is easier for implementation, the feedback control needs the estimation of the actual phase in the oscillating system. The algorithm of phase estimation is also presented. The conditions of the model and the controls validity and accuracy are determined. Performance of the obtained solution is demonstrated via computer simulation for two models of circadian oscillations and a model of heart muscle contraction. It is shown that in the absence of disturbances the open-loop and the feedback controls have similar performance. Additionally, the feedback control is insensitive to external disturbances influence. In these examples the presented scheme for phase values estimation demonstrates better accuracy than the conventional one.

Keywords Phase response curve · Entrainment · Control theory · Circadian rhythm

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1 Introduction

Oscillations occur in many systems in biology, physics, chemistry and engineering (Blekhman 1971; Izhikevich 2007; Kuramoto 1984; Mosekilde et al. 2002; Pikovsky et al. 2001; Winfree 2001). Any periodic (limit-cycle) oscillating mode can be characterized by its frequency spectrum, phase and amplitude. There exist many approaches dealing with regulation of these characteristics (Astashev et al. 2001; Belykh et al. 2005; Kovaleva 2004; Kurths 2000; Efimov 2005). A well-known property of a limit-cycle oscillation is that it can be phase-reset by a brief stimulus, with the perturbation causing a change in the phase of the oscillation with respect to that of an ongoing unperturbed control oscillation (Blekhman 1971; Guevara 1981; Winfree 2001). Another property of a limit-cycle oscillation is that it can be entrained by periodic stimulation, with a fixed phase relationship between the stimulation waveform and the entrained waveform (Izhikevich 2007; Pikovsky et al. 2001). The "phase-resetting problem" centers around developing a stimulation protocol that can be made to yield any desired value of phase.

One example of the phase-resetting problem involves the circadian rhythm. Many organisms, from bacteria to mammals, display circadian rhythms. These are sustained oscillations with a period close to 24 h that can typically be reset by a light stimulus (Leloup and Goldbeter 1998; Leloup et al. 1999). In addition, periodic excitation by light can result in phase and frequency entrainment of the natural circadian rhythm (Tass 1999), resulting in a fixed phase relationship between the light stimulus and some marker event in the circadian rhythm (e.g., the point in time of the daily temperature minimum of the organism). One important practical problem in the case of the circadian rhythm in human beings is the "jet-lag" produced by long-distance flights. In this case, the internal circadian rhythm can take several days to adapt its phase to the new timing of the environmental light conditions. Our task then is to come up with a control strategy that would decrease the length of the transient in adjusting from the old timing of the periodic light input to the new timing of that input, as well as increase the accuracy of the resulting phase under the new environmental conditions. The goal of the work presented below is to propose a solution of the phase-resetting problem. In the case of jet-lag, this would correspond to developing an algorithm that would specify a light administration protocol that would ensure the fastest possible phase resetting (i.e., minimize the length of the transient).

There exist a few approaches for solution of the phase resetting problem. The first one is based on master-slave synchronization theory. This approach is very well developed, but it assumes similarity of both systems and, frequently, synchronizations of the systems states. The approach is suitable for technical systems synthesis (Belykh et al. 2005; Kurths 2000), but it meets serious obstructions for application in biology, physics or chemistry. Another line of researches deals with optimal or predictive control application for phase resetting (Bagheri 2007; Bagheri et al. 2008; Forger and Paydarfar 2004). These methods require availability of full exact information about the oscillator model and its coefficients, that makes hard its application in some cases.

The third approach uses assumption on weak coupling/excitation, i.e., it assumes relatively small amplitude for external input (Pikovsky et al. 2001), that sometimes may be a mild restriction. The last approach is based on PRC application and *Poincaré phase map* approach (Glass et al. 2002; Guevara and Glass 1982; Izhikevich 2007). The advantage of this approach (Danzl and Moehlis 2008; Efimov et al. 2009) consists in low dimension of PRC (it is a scalar map of scalar argument, that completely describes phase resetting caused by "pulse" input) and that PRC can be measured experimentally even for oscillators which have not well investigated detailed models.

In this work we are going to extend the approach for phase resetting from (Efimov et al. 2009) where two control strategies (open-loop and feedback) are proposed based on analytical PRC derived for the linearised model of the system on the limit cycle. The phase model based on such analytical PRC is local and may accurately describe the phase resetting phenomenon for sufficiently small inputs only. In this work the direct PRC (the PRC measured for nonlinear systems) model is introduced and used for control design. The advantage of the direct PRC is that it can be computed for an input with any desired amplitude. Applicability conditions and accuracy of this approach are investigated. In the next section some preliminary results and definitions are introduced. Sections 3 and 4 contain the main results dealing with PRC formalism and control design. Simulations are presented in Sect. 5.

2 Preliminaries

Let $R_+ = \{t \in R : t \geq 0\}$ and the norm of Lebesgue measurable and essentially bounded function $u : R_+ \to R$ be defined as $||u||_{[t_0,t)} = ess \sup\{|u(t)|, t \in [t_0, t)\}$. The set of all such functions with property $||u||_{[0,+\infty)} = ||u|| < +\infty$ we denote as $\mathcal{L}_\infty$.

Consider the following system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u), \tag{1}$$

where $\mathbf{x} \in R^n$ is the state, $u \in \mathcal{L}_\infty$ is the input, $\mathbf{f} : R^{n+1} \to R^n$ is a smooth function. Let $\mathbf{x}(t, \mathbf{x}_0, u)$ be the solution of the system (1) with initial condition $\mathbf{x}_0 \in R^n$ and input $u \in \mathcal{L}_\infty$ (we use the short notation $\mathbf{x}(t)$ if all arguments are clear from the context), by standard arguments this solution is unique, continuous and defined at least locally (the solutions inherit smoothness property after the input $u(t)$). It is assumed that for $u(t) \equiv 0, t \geq 0$ the system (1) has a non-constant T-periodic solution $\gamma(t) = \gamma(t + T) \in R^n, t \geq 0$. The image of this solution in the state space is invariant and closed, if the trajectory is isolated, then its image is called *limit cycle*, i.e., $\Gamma = \{\mathbf{x} \in R^n : \mathbf{x} = \gamma(t), 0 \leq t < T\}$ (Γ is a compact set). The limit cycle is *attracting* if there exists non empty set $\mathcal{A} \subset R^n$ (the set of the limit cycle attraction) such that $\lim_{t\to+\infty} dist[\mathbf{x}(t, \mathbf{x}_0, 0), \Gamma] = 0$ for all $\mathbf{x}_0 \in \mathcal{A}$, where $dist[\mathbf{x}, \Gamma]$ denotes the distance to the set Γ from the point $\mathbf{x} \in R^n$. The set Γ is called *stable* for the system (1) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\mathbf{x}_0 \in \Gamma_\delta = \{\mathbf{x} \in R^n : dist[\mathbf{x}, \Gamma] \leq \delta\}$ it holds $\mathbf{x}(t, \mathbf{x}_0, 0) \in \Gamma_\varepsilon = \{\mathbf{x} \in R^n : dist[\mathbf{x}, \Gamma] \leq \varepsilon\}, t \geq 0$ (the orbital stability of limit cycle). If the set Γ is stable and attractive, then it is *asymptotically stable* for the system (1) (Andronov et al. 1987; Lin et al. 1996). Asymptotic stability of the limit cycle can be established analyzing linearization of the system (1) around Γ

(Andronov et al. 1987; Guckenheimer and Holmes 1990). Further we assume that (1) for $u(t) \equiv 0$, $t \geq 0$ has asymptotically stable limit cycle Γ with the attraction set $\mathcal{A}$.

2.1 Phase

Any point $\mathbf{x}_0 \in \Gamma$ can be characterized by a scalar *phase* $\varphi_0 \in [0, 2\pi)$, that determines position of the point $\mathbf{x}_0$ on Γ (the limit cycle is one-dimensional closed curve in R^n (Blekhman 1971; Izhikevich 2007; Pikovsky et al. 2001)). The smooth bijective *phase map* $\vartheta : \Gamma \to [0, 2\pi)$ assigns the corresponding phase φ_0 to each point $\mathbf{x}_0$ on the limit cycle, i.e. $\varphi_0 = \vartheta(\mathbf{x}_0)$. Any solution of the system (1) on the cycle $\mathbf{x}(t, \mathbf{x}_0, 0)$, $\mathbf{x}_0 \in \Gamma$ satisfies $\mathbf{x}(t, \mathbf{x}_0, 0) = \gamma(t + \varphi_0\omega^{-1})$, $\omega = 2\pi/T$, provided we choose the convention $\gamma(t) = \mathbf{x}(t, \vartheta^{-1}(0), 0)$. The *phase variable* $\varphi : R_+ \to [0, 2\pi)$ is defined for trajectories $\mathbf{x}(t, \mathbf{x}_0, 0)$, $\mathbf{x}_0 \in \Gamma$ as $\varphi(t) = \vartheta[\mathbf{x}(t, \mathbf{x}_0, 0)] = \vartheta[\gamma(t + \varphi_0\omega^{-1})]$. Due to periodic nature of $\gamma(t)$ the function $\varphi(t)$ is also periodic, moreover the function ϑ can be defined in the particular way providing $\varphi(t) = \omega t + \varphi_0$, $\dot{\varphi}(t) = \omega$ (Izhikevich 2007; Pikovsky et al. 2001).

Phase notion can be extended to any solution $\mathbf{x}(t, \mathbf{x}_0, 0)$ starting in the attracting set $\mathbf{x}_0 \in \mathcal{A}$. By definition of the attracting set, for all $\mathbf{x}_0 \in \mathcal{A}$ there exists $\theta_0 \in [0, 2\pi)$ such that $\lim_{t\to+\infty} |\mathbf{x}(t, \mathbf{x}_0, 0) - \gamma(t+\theta_0\omega^{-1})| = 0$, where θ_0 is the *asymptotic phase* of the point $\mathbf{x}_0$. There exists the *asymptotic phase map* $\upsilon : \mathcal{A} \to [0, 2\pi)$ connecting a point $\mathbf{x}_0 \in \mathcal{A}$ and the corresponding phase θ_0, i.e. $\theta_0 = \upsilon(\mathbf{x}_0)$. The *asymptotic phase variable* $\theta : R_+ \to [0, 2\pi)$ is derived as $\theta(t) = \upsilon[\mathbf{x}(t, \mathbf{x}_0, 0)]$, $t \geq 0$ for $\mathbf{x}_0 \in \mathcal{A}$. In the case $\varphi(t) = \omega t + \varphi_0$ we have $\theta(t) = \omega t + \theta_0$ and $\dot{\theta}(t) = \omega$, that implies invariance of this map: if $\upsilon(\mathbf{x}_1) = \upsilon(\mathbf{x}_2)$, then $\upsilon[\mathbf{x}(t, \mathbf{x}_1, 0)] = \upsilon[\mathbf{x}(t, \mathbf{x}_2, 0)]$ for all $t \geq 0$ and $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{A}$ (Izhikevich 2007). Locally around Γ the map υ coincides with the smooth map ϑ (then in some vicinity of the set Γ the map υ may be also smooth).

The notion of asymptotic phase variable can be extended to a generic case $u \in \mathcal{L}_\infty$ providing that the corresponding trajectory $\mathbf{x}(t, \mathbf{x}_0, u)$ with $\mathbf{x}_0 \in \Gamma_\varepsilon = \{\mathbf{x} \in R^n : dist[\mathbf{x}, \Gamma] \leq \varepsilon\} \subset \mathcal{A}$ stays into the set $\mathcal{A}$ for all $t \geq 0$ (denote the subset of inputs $u \in \mathcal{L}_\infty$ preserving invariance of the set $\mathcal{A}$ as $\mathcal{M} \subset \mathcal{L}_\infty$, then $\mathbf{x}(t, \mathbf{x}_0, u) \in \mathcal{A}$ for all $t \geq 0$ with $\mathbf{x}_0 \in \Gamma_\varepsilon$ and $u \in \mathcal{M}$). In this case the asymptotic phase variable can be defined in a trivial way as $\theta(t) = \upsilon[\mathbf{x}(t, \mathbf{x}_0, u)]$, $t \geq 0$ for any $\mathbf{x}_0 \in \Gamma_\varepsilon$ and $u \in \mathcal{M}$. Then the variable $\theta(t')$, $t' \geq 0$ evaluates the asymptotic phase of the trajectory $\mathbf{x}(t', \mathbf{x}_0, u)$ if one would pose $u(t) = 0$ for $t \geq t'$. The dynamics of the asymptotic phase variable $\theta(t)$ in the generic case $u \in \mathcal{M}$ is hard to derive. Some local models obtained in a small neighborhood of the limit cycle for infinitesimal inputs are presented in (Efimov et al. 2009; Izhikevich 2007; Pikovsky et al. 2001), less conservative model is proposed in Sect. 3.

The phase map ϑ can be easily assigned for a given initial point $\vartheta^{-1}(0)$, while evaluation of the asymptotic phase map υ is more challenging. On the set Γ both maps coincide, but outside the limit cycle mainly numerical methods are used for the map υ calculation (Izhikevich 2007; Pikovsky et al. 2001). In (Demongeot and Francoise 2006) the first approximation for the variable $\theta(t)$ is obtained for perturbed Hamiltonian systems. The following simple lemma provides a method for $\theta(t)$ approximation in some vicinity of the limit cycle.

Lemma 1 *Let* $|\mathbf{x}(t) - \gamma(t + \varphi\omega^{-1})| \leq \varepsilon, t \in [t_1, t_2]$ *for some* $\varepsilon > 0, \varphi \in [0, 2\pi)$ *and* υ *be smooth into the set* Γ_ε*, then*

$$\theta(t) = \upsilon[\mathbf{x}(t)] = \omega t + \varphi + Q(t + \varphi\omega^{-1})[\mathbf{x}(t) - \gamma(t + \varphi\omega^{-1})] + o(\varepsilon^2), \quad t \in [t_1, t_2],$$
$$Q(t) = \partial\upsilon(\mathbf{x})/\partial\mathbf{x}|_{\mathbf{x}=\gamma(t)},$$

where $o(\varepsilon^2)$ *denotes second order and higher terms with respect to* ε*.*

Proof Let $\mathbf{x}(t) - \gamma(t + \varphi\omega^{-1}) = \varepsilon\mathbf{e}(t)$, then $|\mathbf{e}(t)| \leq 1$ for all $t \in [t_1, t_2]$. Consider expansion of the function $\upsilon[\mathbf{x}(t)] = \upsilon[\gamma(t + \varphi\omega^{-1}) + \varepsilon\mathbf{e}(t)]$ in Taylor series with respect to ε into the set Γ_ε:

$$\upsilon[\mathbf{x}(t)] = \upsilon[\gamma(t + \varphi\omega^{-1})] + \partial\upsilon(\mathbf{x})/\partial\mathbf{x}|_{\mathbf{x}=\gamma(t+\varphi\omega^{-1})}\, \varepsilon\mathbf{e}(t) + o(\varepsilon^2),$$

that implies the desired result. □

The T-periodic function $Q(t)$ is called *infinitesimal PRC* and it can be computed based on linearised around Γ equations of the system (1) (Izhikevich 2007; Govaerts and Sautois 2006). In the same way other high order approximations of the asymptotic phase map υ can be established.

2.2 Phase response curve

Phase response curve is used to describe changes in the phase caused by external "pulse"-like input u (by "pulse"-like we mean that $u(t) \neq 0$, $0 < t < \mathcal{T} < +\infty$ with $u(t) \equiv 0$ for all $t \leq 0$ and $t \geq \mathcal{T}$). Denote the set of such inputs as $\mathcal{U} \subset \mathcal{M}$, then $\lim_{t\to+\infty} dist[\mathbf{x}(t, \mathbf{x}_0, u), \Gamma] = 0$ for all $\mathbf{x}_0 \in \Gamma_\varepsilon$ and $u \in \mathcal{U}$.

Definition 1 *(Izhikevich 2007; Pikovsky et al. 2001)*. For all $\mathbf{x}_0 \in \Gamma$ and given $u \in \mathcal{U}$,

$$PRC(\varphi_{old}) = \varphi_{new} - \varphi_{old}, \varphi_{old} = \vartheta(\mathbf{x}_0),$$
$$\lim_{t\to+\infty} |\mathbf{x}(t, \mathbf{x}_0, u) - \gamma(t + \varphi_{new}\omega^{-1})| = 0.$$

□

Thus, PRC tabulates the difference between the initial phase φ_{old} and the shifted one φ_{new}, PRC is a function of the phase when the "pulse" starts to influence on the system dynamics, φ_{old} in our case. To preserve continuity of PRC map it is typically defined as $PRC : [0, 2\pi) \to [-\pi, \pi)$ (definition $PRC : [0, 2\pi) \to [0, 2\pi)$ is also possible), however, even in this case the map PRC can be discontinuous in general for large enough inputs (type 0 PRC from (Winfree 2001)). Another equivalent definition of PRC is

$$\begin{aligned} PRC(\varphi_{old}) &= \lim_{t\to+\infty} (\upsilon[\mathbf{x}(t, \mathbf{x}_0, u)] - \vartheta[\mathbf{x}(t, \mathbf{x}_0, 0)]) \\ &= \lim_{t\to+\infty} (\upsilon[\mathbf{x}(t, \mathbf{x}_0, u)] - \upsilon[\mathbf{x}(t, \mathbf{x}_0, 0)]), \\ &\qquad \varphi_{old} = \vartheta(\mathbf{x}_0), \end{aligned}$$

that uses the asymptotic phase map.

The shortage of the above definitions of PRC consists in computational complexity of this map evaluation. For instance, Definition 1 requires information on asymptotic estimates of the system (1) solutions (for all $t \geq 0$), that is infeasible in practice. For particular type of inputs, like delta-function impulses, PRC can be computed based on linearization of the system (1) around Γ (Izhikevich 2007; Govaerts and Sautois 2006), in this case the PRC map equals $Q(\varphi\omega^{-1})$. This approach also provides good estimates on the PRC shape for generic inputs with small amplitudes (Izhikevich 2007; Govaerts and Sautois 2006). To overcome this shortage and to simplify computations of PRC we equivalently reformulate Definition 1 as follows.

Definition 2 For all $\mathbf{x}_0 \in \Gamma$, given $u \in \mathcal{U}$ and $\varepsilon > 0$, ε-transient response curve (ε-TRC) is the map $TRC_\varepsilon : [0, 2\pi) \to R_+$ defined as

$$TRC_\varepsilon(\varphi_{old}) = \arg\min_{t \geq \mathcal{T}}\{dist[\mathbf{x}(\tau, \mathbf{x}_0, u), \Gamma] \leq \varepsilon \text{ for all } \tau \geq t\}, \quad \varphi_{old} = \vartheta(\mathbf{x}_0),$$

and PRC is the map $PRC : [0, 2\pi) \to [-\pi, \pi)$ defined as

$$PRC(\varphi_{old}) = \theta_{new} - \varphi_{old}, \theta_{new} = \upsilon[\mathbf{x}(TRC_\varepsilon(\varphi_{old}), \mathbf{x}_0, u)] - \omega TRC_\varepsilon(\varphi_{old}).$$

□

From the map υ definition we get that $\lim_{t\to+\infty} |\mathbf{x}(t, \mathbf{x}_0, u) - \gamma(t + \theta_{new}\omega^{-1})| = 0$. This definition utilizes the fact that for $t \geq \mathcal{T}$ (when the input is zero) the asymptotic phase and the phase variables have similar rate ω. Computation of the PRC in accordance with Definition 2 requires information on the state $\mathbf{x}$ behavior on a finite interval and the asymptotic map υ. For small ε the value $\upsilon[\mathbf{x}(TRC_\varepsilon(\varphi_{old}), \mathbf{x}_0), u)]$ can be estimated applying the result of Lemma 1, that gives an approximation of PRC.

Definition 3 *For all* $\mathbf{x}_0 \in \Gamma$, *given* $u \in \mathcal{U}$ *and* $\varepsilon > 0$ *the map* $PRC_\varepsilon : [0, 2\pi) \to [-\pi, \pi)$ *defined as*

$$\begin{aligned}
PRC_\varepsilon(\varphi_{old}) &= \widehat{\theta}_{new} - \varphi_{old},\\
\widehat{\theta}_{new} &= \widehat{\upsilon}[\mathbf{x}(TRC_\varepsilon(\varphi_{old}), \mathbf{x}_0, u)] - \omega TRC_\varepsilon(\varphi_{old}),\\
\widehat{\upsilon}[\mathbf{x}(TRC_\varepsilon(\varphi_{old}), \mathbf{x}_0, u)] &= \widehat{\varphi}_{new} + Q(\widehat{\varphi}_{new}\omega^{-1})[\mathbf{x}(TRC_\varepsilon(\varphi_{old})) - \gamma(\widehat{\varphi}_{new}\omega^{-1})],\\
\widehat{\varphi}_{new} &= \vartheta[\arg\min_{\mathbf{z}\in\Gamma}\{|\mathbf{x}(TRC_\varepsilon(\varphi_{old})) - \mathbf{z}|\}].
\end{aligned}$$

□

The map PRC_ε is an estimate of the “reference” map PRC from Definition 2, the advantage of PRC_ε consists in simplicity of computation. Both maps TRC_ε and PRC_ε can be computed or measured for the system (1) for given $\varepsilon > 0$ and the “pulse” $u \in \mathcal{U}$ on finite time intervals.

An accuracy of the map PRC_ε can be evaluated noting that the set Γ is asymptotically stable ($u(t) \equiv 0$ for $t \geq TRC_\varepsilon(\varphi_{old})$ and asymptotic stability property holds), then there exists $\rho = \rho(\varepsilon) > 0$ such that $|\mathbf{x}(t, \mathbf{x}_0, u) - \gamma(t + \widehat{\theta}_{new}\omega^{-1})| \leq \rho$ for all $t \geq TRC_\varepsilon(\varphi_{old})$ and all $\varphi_{old} \in [0, 2\pi)$, $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Lemma 2 *Let υ be smooth into the set Γ_ε. Then for all $\varphi \in [0, 2\pi)$ and the given $u \in \mathcal{U}$ there exists $\rho = \rho(\varepsilon) > 0 (\rho(\varepsilon) \to 0$ as $\varepsilon \to 0)$ such that*

$$|PRC_\varepsilon(\varphi) - PRC(\varphi)| \le q\rho + o(\rho^2), \quad q = \sup_{0\le t\le T} |Q(t)|.$$

Proof According to the definitions for all $\varphi_{old} \in [0, 2\pi)$

$$|PRC_\varepsilon(\varphi_{old}) - PRC(\varphi_{old})| = |\widehat{\theta}_{new} - \varphi_{old} - \theta_{new} + \varphi_{old}| = |\widehat{\theta}_{new} - \theta_{new}|$$
$$= |\widehat{\upsilon}[\mathbf{x}(TRC_\varepsilon(\varphi_{old}), \mathbf{x}_0, u)] - \upsilon[\mathbf{x}(TRC_\varepsilon(\varphi_{old}), \mathbf{x}_0, u)]|.$$

Since $|\mathbf{x}(t, \mathbf{x}_0, u) - \gamma(t + \widehat{\theta}_{new}\omega^{-1})| \le \rho$ for all $t \ge TRC_\varepsilon(\varphi_{old})$ and all $\varphi_{old} \in [0, 2\pi)$, applying the result of Lemma 1 we obtain:

$$\upsilon[\mathbf{x}(TRC_\varepsilon(\varphi_{old}))] = \omega TRC_\varepsilon(\varphi_{old}) + \widehat{\theta}_{new} + Q(TRC_\varepsilon(\varphi_{old}) + \widehat{\theta}_{new}\omega^{-1})$$
$$\times[\mathbf{x}(TRC_\varepsilon(\varphi_{old})) - \gamma(TRC_\varepsilon(\varphi_{old}) + \widehat{\theta}_{new}\omega^{-1})] + o(\rho^2)$$

and by the definition $\widehat{\upsilon}[\mathbf{x}(TRC_\varepsilon(\varphi_{old}))] = \omega TRC_\varepsilon(\varphi_{old}) + \widehat{\theta}_{new}$. Therefore,

$$|PRC_\varepsilon(\varphi_{old}) - PRC(\varphi_{old})| = |Q(TRC_\varepsilon(\varphi_{old}) + \widehat{\theta}_{new}\omega^{-1})[\mathbf{x}(TRC_\varepsilon(\varphi_{old}))$$
$$-\gamma(TRC_\varepsilon(\varphi_{old}) + \widehat{\theta}_{new}\omega^{-1})] + o(\rho^2)| \le q\rho + o(\rho^2),$$

where q is a limit cycle characteristics of the system (1) (independent on u). □

The result of this lemma implies that choosing value of ε sufficiently small it is possible to reduce the gap between the maps PRC and PRC_ε (the constant q is independent on ε).

3 ε-PRC based phase model

Now, assume that a series of "pulses" $w \in \mathcal{U}$ is given and

$$u(t) = \sum_{i\ge 0} w(t - t_i), \quad t_0 \ge 0; \quad t_{i+1} - t_i > \mathcal{T}, \quad i \ge 0.$$

The goal is to derive a phase model for such type of inputs using PRC_ε. If w is a delta-impulse and $t_i = iT, i \ge 0$ then the model of phase behavior is defined by *Poincaré phase map* (Ermentrout and Kopell 1991; Mirollo and Strogatz 1990)

$$\varphi(t_{i+1}) = \varphi(t_i) + Q[\varphi(t_i)\omega^{-1}], \tag{2}$$

where the summation operation is taken by modulo 2π (the same convention is assumed for the phase models presented below). If this system has equilibriums, i.e., there exist solutions $\varphi^0 \in [0, 2\pi)$ of the equation $Q(\varphi^0\omega^{-1}) = 0$ (infinitesimal PRC has zeros), then trajectories will converge to stable ones. The Poincaré phase map approach also has its extension for the case of small amplitude inputs (Hansel et al. 1995). The similar to (2) model can be obtained for "pulses" with period different from T (a mild modification is required (Izhikevich 2007)).

The scalar-valued phase model (2) completely describes the phase dynamics of the system (1) under impulsive inputs. Let us develop this approach for the case of more general ε-PRC proposing similar to (2) model for a "pulse"-like input from the set $\mathcal{U}$.

To derive the phase model for a series of "pulses" we have to take care about the time of trajectories returning into the set Γ_ε after excitation by a "pulse" $w \in \mathcal{U}$ (this time is estimated by ε-TRC, see Definition 2). While a trajectory stays out of the set Γ_ε we can not apply the next "pulse" since PRC_ε defines phase resetting only after the trajectory entrance into Γ_ε.

The next issue which has to be taking into account is that ε-PRC and ε-TRC maps both are calculated for the trajectories initiated into the set Γ (on the limit cycle), while in the case under analysis after the first "pulse" the trajectory in general lives outside the set. Thus, two important cases of the "pulse" $w \in \mathcal{U}$ activation should be investigated, one for the trajectories with initial conditions into the set Γ and another for initial conditions in some neighborhood of Γ. These cases are considered separately in two propositions below.

Proposition 1 *For any* $\mathbf{x}_0 \in \Gamma$ *and* $w \in \mathcal{U}$, *there exists* $\rho = \rho(\varepsilon) > 0 \quad (\rho(\varepsilon) \to 0$ *as* $\varepsilon \to 0)$ *such that for all* $t \geq TRC_\varepsilon(\theta_0)$:

$$|\mathbf{x}(t, \mathbf{x}_0, w) - \gamma(t + \theta_1\omega^{-1})| \leq \rho, \quad \theta(t) = \upsilon[\mathbf{x}(t, \mathbf{x}_0, w)] = \omega t + \theta_1 + e(t),$$
$$|e(t)| \leq \eta(\rho) = q\rho + o(\rho^2),$$

where $\theta_1 = \theta_0 + PRC_\varepsilon(\theta_0)$, $\theta_0 = \upsilon(\mathbf{x}_0)$, *providing that the asymptotic phase map* υ *is smooth into the set* Γ_ε.

Proof Let for $\mathbf{x}_0 \in \Gamma$, $\theta_0 = \upsilon(\mathbf{x}_0)$ the "pulse" $w \in \mathcal{U}$ be applied, then the next phase θ_1 at the time instant $t = TRC_\varepsilon(\theta_0)$ is defined in accordance with the definition:

$$\theta_1 = \theta_0 + PRC_\varepsilon(\theta_0).$$

Due to asymptotic stability of the set Γ there exists $\rho = \rho(\varepsilon) > 0$ such that $|\mathbf{x}(t, \mathbf{x}_0, w) - \gamma(t + \theta_1\omega^{-1})| \leq \rho$ for all $t \geq TRC_\varepsilon(\theta_0)$, $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$. According to Lemma 1, under assumption on smoothness of the map υ, $\theta(t) = \omega t + \theta_1 + e(t)$ for $t \geq TRC_\varepsilon(\theta_0)$, $e(t) = Q(t + \theta_1\omega^{-1})[\mathbf{x}(t, \mathbf{x}_0, w) - \gamma(t + \theta_1\omega^{-1})] + o(\rho^2)$ and $|e(t)| \leq \eta(\rho)$ for $t \geq TRC_\varepsilon(\theta_0)$, where $q = \sup_{0 \leq t \leq T} |Q(t)|$. □

The proposition means that ε-PRC and ε-TRC maps, substituted in the model (2), provide an estimation of the asymptotic phase variable $\upsilon[\mathbf{x}(t, \mathbf{x}_0, w)]$ with accuracy proportional to ε. Decreasing value of ε it is possible to ensure desired quality of the phase variable estimation.

Let the next "pulse" $w \in \mathcal{U}$ be generated at time instant $t_1 \geq TRC_\varepsilon(\theta_0)$, then $\mathbf{x}_1 = \mathbf{x}(t_1, \mathbf{x}_0, w)$ and the initial phase for this "pulse" $\upsilon(\mathbf{x}_1) \in [\theta_1 - \eta(\rho), \theta_1 + \eta(\rho)]$.

The peculiarity of this case is that $\mathbf{x}_1 \notin \Gamma$, thus, we can not directly apply the maps PRC or PRC_ε to calculate the next phase θ_2.

Proposition 2 *For any $\mathbf{x}_1 \in \Gamma_\varepsilon$ and $w \in \mathcal{U}$, with property $|\mathbf{x}_1 - \gamma(\theta_1\omega^{-1})| \leq \nu$ for some $\theta_1 \in [0, 2\pi)$ and $\nu > 0$, there exist $L \in R_+$ and $\rho = \rho(\varepsilon) > 0$ $(\rho(\varepsilon) \to 0$ as $\varepsilon \to 0)$ such that for all $t \geq TRC_\varepsilon(\theta_1)$:*

$$|\mathbf{x}(t, \mathbf{x}_1, w) - \gamma(t + \theta_2\omega^{-1})| \leq L\nu + \rho, \quad \theta(t) = \upsilon[\mathbf{x}(t, \mathbf{x}_1, w)] = \omega t + \theta_2 + e(t),$$
$$|e(t)| \leq \eta(L\nu + \rho),$$

where $\theta_2 = \theta_1 + PRC_\varepsilon(\theta_1)$, providing that the asymptotic phase map υ is smooth into the set $\Gamma_{L\nu+\varepsilon}$.

Proof Define $\mathbf{x}'_1 = \vartheta^{-1}(\theta_1)(\mathbf{x}'_1 = \gamma(\theta_1\omega^{-1}))$ as the point on the limit cycle with the phase of interest θ_1. For this nominal trajectory $\mathbf{x}(t, \mathbf{x}'_1, w)$ the next phase θ_2 at time instant $t = TRC_\varepsilon(\theta_1)$ is defined as

$$\theta_2 = \theta_1 + PRC_\varepsilon(\theta_1)$$

and, according to Proposition 1, $|\mathbf{x}(t, \mathbf{x}'_1, w) - \gamma(t + \theta_2\omega^{-1})| \leq \rho$, $\theta'(t) = \omega t + \theta_2 + e'(t)$, $|e'(t)| \leq \eta(\rho)$ for all $t \geq TRC_\varepsilon(\theta_1)$. By conditions $|\mathbf{x}_1 - \mathbf{x}'_1| \leq \nu$, then due to continuity property of the system (1) solutions and asymptotic stability of Γ, for all $\mathbf{x}_1, \mathbf{x}'_1 \in \Gamma_\varepsilon$ and $w \in \mathcal{U}$ there exists $L \in R_+$ such that $|\mathbf{x}(t, \mathbf{x}_1, w) - \mathbf{x}(t, \mathbf{x}'_1, w)| \leq L|\mathbf{x}_1 - \mathbf{x}'_1| \leq L\nu$ for $t \geq TRC_\varepsilon(\theta_1)$. Therefore, $|\mathbf{x}(t, \mathbf{x}_1, w) - \gamma(t + \theta_2\omega^{-1})| \leq L\nu + \rho$ for $t \geq TRC_\varepsilon(\theta_1)$. Assuming smoothness of the asymptotic phase map υ into the set $\Gamma_{L\nu+\varepsilon}$ with lemma 1 we obtain the desired estimates on the real phase of the trajectory.

□

Thus, Proposition 2 extends the result of Proposition 1 to initial conditions $\mathbf{x}_1 \in \Gamma_\varepsilon$. In this case ε-PRC and ε-TRC maps provide an estimation of the asymptotic phase variable $\upsilon[\mathbf{x}(t, \mathbf{x}_1, w)]$ with accuracy proportional to ε and ν. However, if ν comes from Proposition 1, then it is also proportional to ε. Therefore, tacking the value of ε sufficiently small we again can ensure desired quality of the asymptotic phase variable estimation. Both θ_1 and θ_2 (calculated in Propositions 1 and 2 correspondingly via (2)-like phase model) serve as "average" values for the real $\theta(t)$.

Finally, define

$$TRC_{\max} = \max_{\mathbf{x}_0 \in \Gamma_\varepsilon} TRC^{\mathbf{x}_0}_{\max},$$
$$TRC^{\mathbf{x}_0}_{\max} = \arg\min_{t \geq T}\{dist[\mathbf{x}(\tau, \mathbf{x}_0, w), \Gamma] \leq \varepsilon \mathrm{forall} \tau \geq t\},$$

then $TRC_{\max} \geq \sup_{\theta \in [0, 2\pi)} TRC_\varepsilon(\theta)$ and applying "pulses" $w \in \mathcal{U}$ with minimum sampling time $TRC_{\max}$ one ensures that the next step trajectory starts into the set Γ_ε. Therefore, for the input

$$u(t) = \sum_{i \geq 0} w(t - t_i), \ t_0 \geq 0; \quad t_{i+1} - t_i > \mathcal{T} \geq TRC_{\max}, \ i \geq 0 \tag{3}$$

result of Proposition 2 can be applied iteratively to calculate the "average" phase and its estimation accuracy on the next step. Summarizing the discussion above, for any number of "pulses" the following result can be proven.

Theorem 1 *For any* $\mathbf{x}_0 \in \Gamma$ *and the input* $u \in \mathcal{L}_\infty$ *defined in* (3)*, there exist* $\rho = \rho(\varepsilon) > 0$ $(\rho(\varepsilon) \to 0$ *as* $\varepsilon \to 0)$ *and* $L \in R_+$ *such that for the solution* $\mathbf{x}(t, \mathbf{x}_0, u)$ *of the system* (1)*,* (3) *the following inequalities are satisfied for all* $i \geq 0$*:*

$$\begin{aligned}&|\mathbf{x}(t) - \gamma(t + \theta_{i+1}\omega^{-1})| \leq \nu_{i+1}, \quad \theta(t) = \omega t + \theta_{i+1} + e(t), \quad |e(t)| \leq \eta(\nu_{i+1}),\\ &t_i + TRC_\varepsilon(\omega t_i + \theta_i) \leq t \leq t_{i+1},\end{aligned} \tag{4}$$

where

$$\theta_{i+1} = \omega t_i + \theta_i + PRC_\varepsilon(\omega t_i + \theta_i), \quad \theta_0 = \upsilon(\mathbf{x}_0); \tag{5}$$

$$\nu_{i+1} = L\nu_i + \rho, \nu_0 = 0, \tag{6}$$

providing that the asymptotic phase map υ *is smooth into the set* $\Gamma_\kappa, \kappa = (L+1)\varepsilon$*.*

Proof Since $t_{i+1} - t_i \geq TRC_{\max}, i \geq 0$ and $u \in \mathcal{L}_\infty$, then $\mathbf{x}(t_i) \in \Gamma_\varepsilon, i \geq 0$ by definitions. On the interval $[0, t_0)$ the input (3) equals zero and $\theta(t) = \omega t + \theta_0$. For the interval $[t_0, t_1)$ all conditions of the proposition 1 hold and the input $w \in \mathcal{U}$ is applied at the phase $\theta(t_0) = \omega t_0 + \theta_0$, then for $t_0 + TRC_\varepsilon(\omega t_0 + \theta_0) \leq t \leq t_1$:

$$|\mathbf{x}(t) - \gamma(t + \theta_1\omega^{-1})| \leq \rho, \quad \upsilon[\mathbf{x}(t)] = \omega t + \theta_1 + e(t), \quad |e(t)| \leq \eta(\rho).$$

At the instant of the second "impulse" t_1 the following properties are satisfied:

$$\mathbf{x}(t_1) \in \Gamma_\varepsilon, \quad |\mathbf{x}(t_1) - \gamma(t_1 + \theta_1\omega^{-1})| \leq \rho.$$

For t_1 with the initial phase $\theta(t_1) = \omega t_1 + \theta_1$ all conditions of Proposition 2 become true with $\nu = \rho$, then for $t_1 + TRC_\varepsilon(\omega t_1 + \theta_1) \leq t \leq t_2$:

$$\begin{aligned}&|\mathbf{x}(t) - \gamma(t + \theta_2\omega^{-1})| \leq (L+1)\rho, \quad \upsilon[\mathbf{x}(t)] = \omega t + \theta_2 + e(t),\\ &|e(t)| \leq \eta((L+1)\rho).\end{aligned}$$

At the instant of the third "impulse" t_2 it holds

$$\mathbf{x}(t_2) \in \Gamma_\varepsilon, \quad |\mathbf{x}(t_2) - \gamma(t_2 + \theta_2\omega^{-1})| \leq (L+1)\rho,$$

then again allying Proposition 2 for the initial phase $\theta(t_2) = \omega t_2 + \theta_2$ and $\nu = (L+1)\rho$ we obtain for $t_2 + TRC_\varepsilon(\omega t_2 + \theta_2) \leq t \leq t_3$:

$$\begin{aligned}&|\mathbf{x}(t) - \gamma(t + \theta_3\omega^{-1})| \leq L(L+1)\rho + \rho, \quad \upsilon[\mathbf{x}(t)] = \omega t + \theta_3 + e(t),\\ &|e(t)| \leq \eta(L(L+1)\rho + \rho).\end{aligned}$$

Validity of the estimates (4)–(6) can be substantiated repeating these steps.

By the definition of $TRC_{\max}$ there exist some $\tilde{\mathbf{x}}_i \in \Gamma, i \geq 0$ such that $|\tilde{\mathbf{x}}_i - \mathbf{x}(t_i)| \leq \varepsilon$, then by the same arguments as in the proof of Proposition 2 we have $|\mathbf{x}(t, \mathbf{x}(t_i), u) - \mathbf{x}(t, \tilde{\mathbf{x}}_i, u)| \leq L|\tilde{\mathbf{x}}_i - \mathbf{x}(t_i)| \leq L\varepsilon$ for $t_i + TRC_\varepsilon(\theta_i') \leq t \leq t_{i+1}$. Therefore, the smoothness property of the map υ has to be imposed on the set Γ_κ only. □

The phase model (5) (the analogue of the model (2)) describes dynamics of θ_i, that defines the mean value of the interval, where the asymptotic phase variable $\theta(t)$ is located for $t_i + TRC_\varepsilon(\omega t_i + \theta_i) \leq t \leq t_{i+1}$. The radius of the interval around θ_i is defined by the discrete system (6). If $L < 1$, then the accuracy of the variable $\theta(t)$ estimation by θ_i stays bounded, if $L \geq 1$ then the estimation error increases. Tacking sufficiently small value of the ε it is possible to ensure desired quality of approximation with model (5) for any given *finite* number of "pulses" $w \in \mathcal{U}$.

The most hardly verified condition in Theorem 1 deals with smoothness requirement for the asymptotic phase map υ. Decreasing value of ε the set Γ_κ may be reduced to the set Γ, where the map υ coincides with the smooth ϑ.

If we assume that $T \geq TRC_{\max}$, then the "pulse" $w \in \mathcal{U}$ can be applied with the fixed period T. In this case the model (5) reduces to (2), moreover the following extension of Poincaré phase map approach can be obtained for ε-PRC map.

Corollary 1 *Let all conditions of Theorem 1 hold, $T \geq TRC_{\max}$ and $t_i = iT, i \geq 0$. Let φ^0 be a solution of the equation $PRC_\varepsilon(\varphi^0) = 0$, PRC_ε be locally continuously differentiable and $\partial PRC_\varepsilon(\varphi)/\partial\varphi < 0, |\partial PRC_\varepsilon(\varphi)/\partial\varphi| < 1$ for $\varphi \in [\varphi^0 - \Delta, \varphi^0 + \Delta], \Delta > 0$. Then there exists $\varepsilon_{\max} > 0$ and for all $0 < \varepsilon \leq \varepsilon_{\max}$ there exists $\mu = \mu(\varepsilon) > 0$ $(\mu(\varepsilon) \to 0$ as $\varepsilon \to 0)$ such that for all $\theta_0 = \vartheta(\mathbf{x}_0) \in [\varphi^0 - \Delta, \varphi^0 + \Delta], \mathbf{x}_0 \in \Gamma$ the corresponding solution $\mathbf{x}(t, \mathbf{x}_0, u)$ of the system (1), (3) possesses the following properties for all $i \geq 0$:*

$$|\mathbf{x}(t) - \gamma(t + \theta_{i+1}\omega^{-1})| \leq \mu, \upsilon[\mathbf{x}(t)] = \omega t + \theta_{i+1} + e(t), |e(t)| \leq \eta(\mu)$$
$$for\ all\ iT + TRC_\varepsilon(\theta_i) \leq t \leq (i+1)T,$$

where

$$\theta_{i+1} = \theta_i + PRC_\varepsilon(\theta_i), \tag{7}$$

providing that the asymptotic phase map υ is smooth into the set Γ_μ.

Proof If $t_i = iT, i \geq 0$ then $\omega t_i = 2\pi i$ and since PRC_ε map is 2π-periodic the model (5) reduces to (7). Under conditions of the corollary, φ^0 is a stable equilibrium of the system (7) and the initial phase θ_0 belongs to the region of the equilibrium attraction. That implies $\theta_i \to \varphi^0$ as $i \to +\infty$ and in this case there exists some sufficiently small $\varepsilon_{\max} > 0$ such that for all $\varepsilon \leq \varepsilon_{\max}$ in (6) $L < 1$ (additionally to orbital stability (or stability of the set Γ) in this case the system has local asymptotic stability property for the trajectory $\gamma(t + \varphi^0\omega^{-1})$), then there exists $\mu = \sup_{i\geq 0}\{\nu_i\} \leq \rho/(1 - L)$ which majorizes the maximum error of the phase estimation for (6). □

Requirement on differentiability of the map PRC_ε can be relaxed replacing the differential above with any other condition of the equilibrium stability (slope calculation, for instance).

According to the corollary, existence of stable equilibriums of the system (7) ensures finite accuracy of the asymptotic phase variable approximation by the model (7) with periodic inputs, that correlates with (Ermentrout and Kopell 1991; Mirollo and Strogatz 1990). Summarizing the results of Theorem 1 and Corollary 1 it is worth stressing that based on ε-PRC and ε-TRC maps the phase models (5) or (7) possess similar to (2) properties ensuring reasonable quality of the phase approximation. This justifies applicability of these models for control design.

4 PRC-based control design

There exists one "free" parameter t_i in the model (5) available for adjustment (the time instant when the next "pulse" $w \in \mathcal{U}$ is introduced). Assigning $t_i, i \geq 0$ one may ensure desired phase resetting for the system (1). Let $\varpi_i = \omega t_i + \theta_i \in [0, 2\pi), i \geq 0$ be the controlled phase of the "pulse" $w \in \mathcal{U}$ feeding on the input of the system (1), then the model (5) can be rewritten as follows:

$$\theta_{i+1} = \varpi_i + PRC_\varepsilon(\varpi_i), \quad i \geq 0. \tag{8}$$

The problem is to design sequences of $\varpi_i, i \geq 0$ providing phase resetting from any initial phase $\theta_0 \in [0, 2\pi)$ to the desired one $\theta_d \in [0, 2\pi)$. The model (8) is the *first* order discrete nonlinear system, such class of systems is well investigated in the control theory literature (Ogata 2006) (that is an advantage of the model (8) comparing it with (1)).

Following (Efimov et al. 2009), in this paper we examine two strategies for ϖ_i design, one is *open-loop* or *feedforward* control and another is proportional *feedback* control algorithm described below.

4.1 Open-loop PRC-based control

This strategy is based on the model (8) and it does not require any additional measured information about actual current phase of the system. A peculiarity of the system (8) and the problem of phase resetting consists in that $\theta \in [0, 2\pi)$, thus shift of the phase in both directions is possible for the resetting. To choose the direction one has to analyze which strategy (decreasing or increasing phase) leads to the fastest phase resetting. Since accuracy of the model (5) and (8) depends on the number of pulses applied, the choice of the resetting direction is rather important. Of course this has sense only if ε-PRC map has significantly different negative and positive values (see the ε-PRC map in Fig. 2, for an example).

In this work for brevity of exposition we assume that the ε-PRC map has particular properties (it is similar to type II PRC from (Hansel et al. 1995) or type 1 PRC from (Winfree 2001)). The corresponding control strategies for other types of PRC maps can be easily deduced from this main case.

Assumption 1 The map PRC_ε is continuous and it has one zero $\varphi_s^0 \in [0, 2\pi)$ with negative slope and another $\varphi_u^0 \in [0, 2\pi)$ with positive slope, $\varphi_s^0 < \varphi_u^0$. □

Since the map PRC_ε is 2π-periodic, the zeros can be arranged in the required order $\varphi_s^0 < \varphi_u^0$ changing the initial point on the limit cycle. Assumption 1 completely describes the shape of PRC_ε, in this case it is possible to define

$$\varphi_{\max} = \arg \sup_{\varphi \in [0,2\pi)} PRC_\varepsilon(\varphi),\ PRC_{\max} = PRC_\varepsilon(\varphi_{\max});$$
$$\varphi_{\min} = \arg \inf_{\varphi \in [0,2\pi)} PRC_\varepsilon(\varphi),\ PRC_{\min} = PRC_\varepsilon(\varphi_{\min}),$$

and $\varphi_s^0 < \varphi_{\min} < \varphi_u^0 < \varphi_{\max}$, $PRC_{\max} > 0$, $PRC_{\min} < 0$. Obviously, φ_s^0 corresponds to the stable equilibrium of the system (5) or (8) and φ_u^0 is the unstable one.

Let the initial phase $\theta_0 \in [0, 2\pi)$ and the desired one $\theta_d \in [0, 2\pi)$ be given, $\theta_0 \neq \theta_d$. Define

$$n_{inc} = \begin{cases} (\theta_d + 2\pi - \theta_0)/PRC_{\max} & \text{if } \theta_0 > \theta_d; \\ (\theta_d - \theta_0)/PRC_{\max} & \text{if } \theta_0 < \theta_d, \end{cases}$$
$$n_{dec} = \begin{cases} (\theta_d - \theta_0)/PRC_{\min} & \text{if } \theta_0 > \theta_d; \\ (\theta_d - 2\pi - \theta_0)/PRC_{\min} & \text{if } \theta_0 < \theta_d, \end{cases}$$

where integer parts of the numbers n_{inc} and n_{dec} determine the number of steps required for resetting of the initial phase θ_0 into a neighborhood of the desired θ_d applying increasing or decreasing strategy. These numbers are minimal since for their calculation we use the maximum amplitudes of the shift $PRC_{\max}$, $PRC_{\min}$ in both directions. Next, in this neighborhood the phase can be resettled to the desired value applying one step shift with the same strategy due to assumed continuity of the map PRC_ε. Therefore, the resetting requires $N + 1$ "pulses" $w \in \mathcal{U}$, $N = floor[\min\{n_{inc}, n_{dec}\}]$, where function $floor[n]$ returns the greatest integer not bigger than n. The following control is proposed to solve the problem:

$$\varpi_i = \begin{cases} \varphi_{\max} & \text{if } n_{inc} \leq n_{dec}; \\ \varphi_{\min} & \text{if } n_{inc} > n_{dec}, \end{cases} \quad 0 \leq i < N; \tag{9}$$

$$PRC_\varepsilon(\varpi_N) = \theta_d - \theta_N, \tag{10}$$

$$\theta_i = \begin{cases} \varphi_{\max} + PRC_{\max} i & \text{if } n_{inc} \leq n_{dec}; \\ \varphi_{\min} + PRC_{\min} i & \text{if } n_{inc} > n_{dec}, \end{cases} \quad 0 < i \leq N, \tag{11}$$

where the last step control ϖ_N is calculated as a solution of the Eq. (10). In control (9), (10) it is assumed that θ_i, $i \geq 0$ are derived via (8) under the control (9) substitution (the formula (11)) and $\theta_{N+1} = \theta_d$ due to (10). This strategy has been called "feedforward" or "open-loop" since it does not establish a relation with real values of the asymptotic phase variable.

Finally, the values of the time instants t_i, $0 \le i \le N$ of "pulses" $w \in \mathcal{U}$ activation should be calculated based on the values ϖ_i, $0 \le i \le N$ from (9), (10) tacking in mind that $t_{i+1} - t_i \ge TRC_{\max}$, $i \ge 0$ (the condition of the models (5) and (8) validity):

$$t_0 = g[(\varpi_0 - \theta_0)\omega^{-1}],\ t_{i+1} = t_i + TRC_{\max} + g[\{\varpi_{i+1} - \theta(t_i + TRC_{\max})\}\omega^{-1}], \tag{12}$$

$$\theta(t_i + TRC_{\max}) = \omega(t_i + TRC_{\max}) + \theta_{i+1}, \quad 0 \le i < N;$$
$$g(\tau) = \begin{cases} \tau & \text{if } \tau \ge 0, \\ \tau + T & \text{otherwise.} \end{cases}$$

Theorem 2 *Let Assumption 1 hold for given $w \in \mathcal{U}$ and some $\varepsilon > 0$ such that $PRC_{\max} \gg q\rho + o(\rho^2)$, $-PRC_{\min} \gg q\rho + o(\rho^2)$ ($0 < \rho = \rho(\varepsilon) < 1$ is from Lemma 2). Then for any $\lambda > 0$ there exist $0 < \varepsilon' \le \varepsilon$ and $T_\lambda > 0$ such that for all $\mathbf{x}_0 \in \Gamma$ and given $\theta_d \in [0, 2\pi)$ in the system (1), (8)–(12) with $PRC_{\varepsilon'}, TRC_{\varepsilon'}$:*

$$|\upsilon[\mathbf{x}(t)] - \omega t - \theta_d| \le \lambda \quad for all \quad t \ge T_\lambda,$$

providing that the asymptotic phase map υ is smooth into the set Γ_κ.

Proof All conditions of Theorem 1 hold, that implies existence of $T' > 0$ such that in the system (1), (9)–(12) with $PRC_\varepsilon, TRC_\varepsilon$:

$$|\upsilon[\mathbf{x}(t)] - \omega t - \theta_d| \le \eta(\nu_{N+1}) \quad \text{for all } t \ge T',$$

where ν_{N+1} is the corresponding solution of the Eq. (6). Decreasing ε one can ensure that $\eta(\nu_{N+1}(\varepsilon)) \le \lambda$ for any $\lambda > 0$. According to Lemma 2 the accuracy of approximation of PRC map by PRC_ε is upper bounded by $q\rho + o(\rho^2)$, $\rho < 1$. Thus, ε decreasing should preserve assumption 1 conditions, then there exists $\varepsilon' > 0$ such that for new $PRC_{\varepsilon'}, TRC_{\varepsilon'}$ the system (1), (9)–(12) solutions admit the desired estimate for some $T_\lambda > 0$ and any $\lambda > 0$. □

In other words the theorem claims that, if there exists some $\varepsilon > 0$ sufficiently small such that it is possible to ensure resetting to the desired phase with some accuracy in finite number of steps (Assumption 1 is satisfied), then decreasing the value of ε provides resetting with arbitrary accuracy.

4.2 Feedback PRC-based control

This strategy assumes on-line measurements of the current phase variable after each "pulse" application that increases accuracy of resetting. To realize this strategy it is enough to replace in (9), (10) the values θ_i generated by (11) with available for measurements variable $\theta(t_i)$.

By measurements of $\theta(t)$ we understand an estimation of this variable based on the available for measurements state vector $\mathbf{x}(t)$. Two approaches can be mentioned, one is based on the closest point on the limit cycle:

$$\widehat{\theta}(t) \stackrel{\Delta}{=} \arg \inf_{\varphi \in [0, 2\pi)} |\mathbf{x}(t) - \gamma(t + \varphi\omega^{-1})|. \tag{13}$$

Another approach for $\widehat{\theta}(t)$ calculation utilizes the result of Lemma 1. Both approaches can be extended to the case of a state component measurements only.

The overall strategy for control design is similar to (9)–(12). For the given phases $\theta_0 \in [0, 2\pi)$ and $\theta_d \in [0, 2\pi)$, $\theta_0 \neq \theta_d$, assumes that $\widehat{\theta}(t_i + TRC_{\max})$ is on-line measured/estimated phase with accuracy $\nu > 0$:

$$|\widehat{\theta}(\tau_{i+1}) - \upsilon[\mathbf{x}(\tau_{i+1})]| \leq \nu, \quad \widehat{\theta}(\tau_{i+1}) = \omega\tau_{i+1} + \theta_{i+1}, \tag{14}$$

$$\tau_{i+1} = t_i + TRC_{\max}, \quad i \geq 0,$$

then the controls ϖ_i can be computed as follows for $i \geq 0$:

$$\begin{aligned} n_{inc}^i &= \begin{cases} (\theta_d + 2\pi - \theta_i)/PRC_{\max} & \text{if } \theta_i > \theta_d; \\ (\theta_d - \theta_i)/PRC_{\max} & \text{if } \theta_i < \theta_d, \end{cases} \\ n_{dec}^i &= \begin{cases} (\theta_d - \theta_i)/PRC_{\min} & \text{if } \theta_i > \theta_d; \\ (\theta_d - 2\pi - \theta_i)/PRC_{\min} & \text{if } \theta_i < \theta_d, \end{cases} \\ \varpi_i &= \begin{cases} \varphi_{\max} & \text{if } 1 \leq n_{inc}^i \leq n_{dec}^i; \\ \varphi_{\min} & \text{if } n_{inc}^i > n_{dec}^i \geq 1; \\ \ell(\theta_i) & \text{otherwise,} \end{cases} \end{aligned} \tag{15}$$

$$t_0 = g[(\varpi_0 - \theta_0)\omega^{-1}], \quad t_{i+1} = \tau_{i+1} + g[\{\varpi_{i+1} - \widehat{\theta}(\tau_{i+1})\}\omega^{-1}], \tag{16}$$

where the function $\ell(\theta)$ represents a solution of the equation $PRC_\varepsilon[\ell(\theta)] = \theta_d - \theta$. Then the following result can be proven.

Theorem 3 *Let assumption* 1 *hold for given* $w \in \mathcal{U}$, $\varepsilon > 0$. *Then there exist* $L \in R_+$, $T_{end} > 0$, $\chi = \chi(\nu) > 0$ *and* $\rho = \rho(\varepsilon) > 0$ ($\chi(\nu) \to 0$ *and* $\rho(\varepsilon) \to 0$ *as* $\nu \to 0$, $\varepsilon \to 0$) *such that for all* $\mathbf{x}_0 \in \Gamma$ *and given* $\theta_d \in [0, 2\pi)$ *in the system* (1), (14)–(16)*:*

- *for all* $i \geq 0$:

$$\mathbf{x}(t) \in \Gamma_\varepsilon, \quad |\mathbf{x}(t) - \gamma(t + \theta_{i+1}\omega^{-1})| \leq \chi \quad for\ t_i + TRC_{\max} \leq t \leq t_{i+1};$$

- $|\upsilon[\mathbf{x}(t)] - \omega t - \theta_d| \leq \eta(L\chi + \rho)$ *for all* $t \geq T_{end}$,

providing that the asymptotic phase map υ *is smooth into the set* $\Gamma_{L\chi+\rho}$.

Proof For $\mathbf{x}_0 \in \Gamma$, $\theta_0 = \vartheta(\mathbf{x}_0)$ let $\varpi_0 \in [0, 2\pi)$ be the corresponding control from (14), (15), $t_0 \geq 0$ be the time instant of the first "pulse" $w \in \mathcal{U}$ activation from (16). Then $dist[\mathbf{x}(t), \Gamma] \leq \varepsilon$, $\tau_1 \leq t < t_1$ and the value $\widehat{\theta}(\tau_1) = \omega\tau_1 + \theta_1$ is calculated satisfying $|\widehat{\theta}(\tau_1) - \upsilon[\mathbf{x}(\tau_1)]| \leq \nu$. Due to asymptotic stability of the set Γ there exists $\chi = \chi(\nu) > 0$ such that $|\mathbf{x}(t) - \gamma(t + \theta_1\omega^{-1})| \leq \chi$ for all $\tau_1 \leq t < t_1$ ($u(t) \equiv 0$ for $t \geq \tau_1$ and asymptotic stability property holds); $\chi(\nu) \to 0$ as $\nu \to 0$. Let ϖ_1, t_1 be derived from (14)–(16) for obtained $\widehat{\theta}(\tau_1)$. Then again $dist[\mathbf{x}(t), \Gamma] \leq \varepsilon$, $\tau_2 \leq t < t_2$ and $|\widehat{\theta}(\tau_2) - \upsilon[\mathbf{x}(\tau_2)]| \leq \nu$, by the same arguments $|\mathbf{x}(t) - \gamma(t + \theta_2\omega^{-1})| \leq \chi$ for all $\tau_2 \leq t < t_2$. Recursively repeating these steps we substantiate the first part of the theorem.

To prove the second part we use Proposition 2. Since $\mathbf{x}(t_i) \in \Gamma_\varepsilon$ and $|\mathbf{x}(t_i) - \gamma(t_i + \theta_i\omega^{-1})| \leq \chi$ for all $i \geq 0$ the conditions of Proposition 2 hold and there exist $L \in R_+$ and $\rho = \rho(\varepsilon) > 0$ ($\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$) such that for all $t \geq t_i + TRC_\varepsilon(\omega t_i + \theta_i)$:

$$|\mathbf{x}(t) - \gamma(t + \tilde{\theta}_{i+1}\omega^{-1})| \leq L\chi + \rho, \quad \upsilon[\mathbf{x}(t)] = \omega t + \tilde{\theta}_{i+1} + e(t),$$
$$|e(t)| \leq \eta(L\chi + \rho),$$

where $\tilde{\theta}_{i+1} = \omega t_i + \theta_i + PRC_\varepsilon(\omega t_i + \theta_i)]$. Therefore, under Assumption 1 the control (12), (14)–(16) provides $\tilde{\theta}_N = \theta_d$ on some step $N > 0$. Finally, due to compactness of the set Γ there exists the common finite time for resetting T_{end}. □

If it is possible to estimate value of the variable θ into the set Γ_ε with some accuracy $\nu > 0$, then the control (14)–(16) ensures for the system (1) phase resetting with accuracy proportional to ν and ε. If the result of Lemma 1 is used for $\widehat{\theta}(\tau_i)$ calculation, then $\nu = \nu(\varepsilon)$ and $\nu(\varepsilon) \to 0$ as $\varepsilon \to 0$. In this case decreasing the value of ε it is possible to ensure finite time practical phase resetting with any desired accuracy.

According to (15)–(16), substitution of this feedback control in the model (8) results in convergence of $\theta(t)$ to the desired trajectory $\omega t + \theta_d$ with the fastest rate. This optimality is not robust, the presence of uncertainties originated by measurement error in (14) or accuracy of approximation ε may lead to the performance loss. For uncertain systems the robust exponential stabilization gives better performance even being slowly. Let for all $i \geq 0$ the noisy measurements of $\widehat{\theta}(\tau_{i+1}) = \omega\tau_{i+1} + \theta_{i+1}$ are given by (14), then the linear proportional feedback has form:

$$\varpi_i = -k(\theta_i - \theta_d), \quad k = \min\{-PRC_{\min}, PRC_{\max}\}/\pi, \tag{17}$$

the time instants t_i, $i \geq 0$ of "pulses" activation are calculated in accordance with (16). Substitution of the control (17) in the model (8) gives $\theta_{i+1} - \theta_d = (1 - k)(\theta_i - \theta_d)$ ensuring exponential convergence of θ_i to θ_d. The result of Theorem 3 can be easily rewritten for the case of the control (14), (16), (17).

The feedback control strategy persists under convergent perturbations, that is its advantage with respect to open-loop controls. However, feedback requires more measurement information and it has more computational complexity. Let us demonstrate efficiency of the proposed controls on several examples of biological systems.

5 Examples

In this section three examples of models of biological oscillators are considered. The first one is the heart muscle model from (Karreman 1949; Karreman and Prood 1995), that is a version of the Liénard equation. Two models of circadian rhythm from works (Leloup and Goldbeter 1998; Leloup et al. 1999) of third and tenth order respectively are the last two examples. For all examples we will consider the square pulses of the form:

$$w(t) = \begin{cases} \Delta & \text{if } t > 0 \wedge t < \delta; \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

where the parameters $\delta > 0$, $\Delta > 0$ values will be specified later depending on the application.

5.1 Heart muscle oscillations

In the paper (Karreman and Prood 1995), it has been found by computer analysis that the equation

$$\ddot{x} + 0.1(x+1)(x-1)(x+3)(x-2.2)\dot{x} + x = u(t), \tag{19}$$

where $x \in R$ is the muscle displacement from a reference point and $u \in R$ is an external force, can describe the papillary muscle asymmetric in time contraction. For the case $u = 0$ the model (19) has stable equilibrium in the origin and two limit cycles, the internal limit cycle is unstable and the external one is locally asymptotically stable. The phase resetting problem solution will be applied to the latter one (Γ and $\gamma(t)$ corresponds to the trajectory on the external limit cycle, $T = 6.584$).

Firstly, let us compare the accuracy of the asymptotic phase estimation $\widehat{\theta}(t)$ in the formula (13) and its improvement provided by Lemma 1:

$$\breve{\theta}(t) = \widehat{\theta}(t) + Q(\widehat{\theta}(t)\omega^{-1})[\mathbf{x}(t) - \gamma(\widehat{\theta}(t)\omega^{-1})], \tag{20}$$

where infinitesimal PRC map Q is computed for the linearised on the limit cycle Γ model applying the conventional approach (Izhikevich 2007; Govaerts and Sautois 2006). For this purpose the model (19) is excited by the pulse (18) with $\delta = 1$ and two amplitudes $\Delta_1 = 0.1$ and $\Delta_2 = 0.5$. Next, the asymptotic phase estimates $\widehat{\theta}(T)$, $\widehat{\theta}(5T)$ and $\breve{\theta}(T)$, $\breve{\theta}(5T)$ are computed in accordance with the formulas (13) and (20) respectively. These estimates are plotted in Fig. 1a for the amplitude Δ_1 and in Fig. 1b for the amplitude Δ_2. Analysis of the figures shows that the estimates $\widehat{\theta}(5T)$ and $\breve{\theta}(5T)$ coincide in both cases, thus the values $\widehat{\theta}(5T)$, $\breve{\theta}(5T)$ can be chosen as the true one for the variable $\theta(t)$. The estimate $\breve{\theta}(T)$ has better accuracy than $\widehat{\theta}(T)$, its superiority becomes more evident with the amplitude of the pulse growth. Further we will use the expression (20) in this example for the phase $\theta(t)$ estimation.

Let $\delta = 1$ and $\Delta = 0.3$, the corresponding phase response maps are plotted in Fig. 2. The map PRC is computed in accordance with the conventional approach (Izhikevich

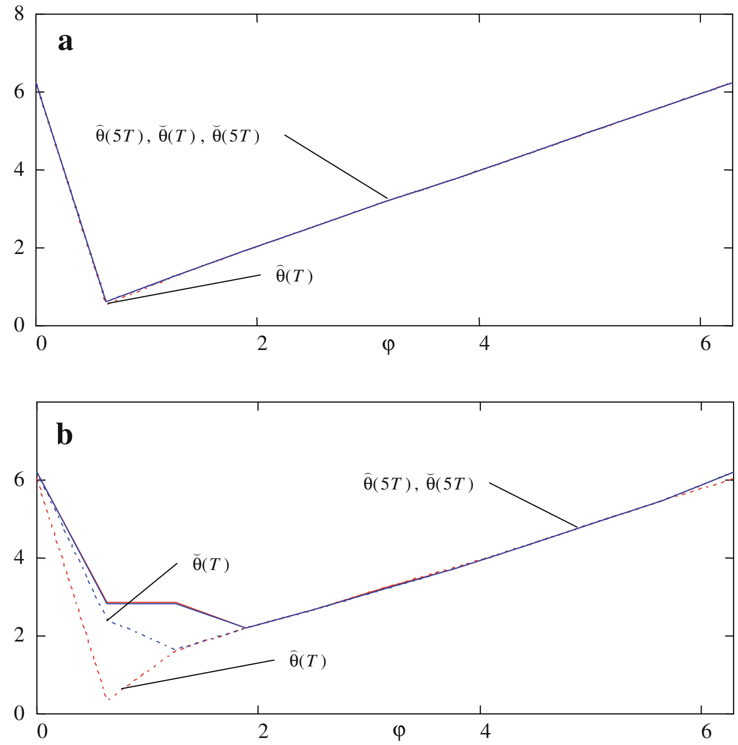

Fig. 1 The asymptotic phases estimates comparison for the model (19)

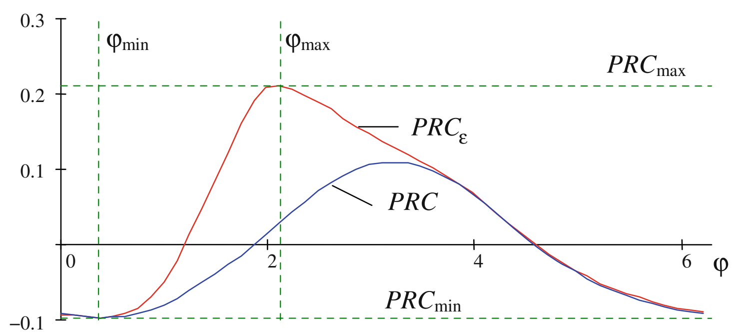

Fig. 2 The phase response maps for the model (19)

2007; Govaerts and Sautois 2006) using the map Q, the map PRC_{ε} is calculated as it was described in definition 3 for $\varepsilon = 0.01$, $T > TRC_{\max}$ in this case. For small amplitudes Δ the maps PRC and PRC_{ε} should coincide, but if the pulse amplitude is sufficiently big (as in our case) the difference is remarkable since the linearised model of the system (19) used for PRC calculation is no more valid in such situation. The Assumption 1 is satisfied for the map PRC_{ε}. Trajectories of the model (19) with the open-loop control (9)–(12) and the feedback control (15), (16), (20) are shown in Fig. 3 (the case without disturbances in Fig. 3a, and with a stochastic additive disturbance in the control channel in Fig. 3b), the time axis is scaled in the periods number, the indexes o and f denote the phase variables for the open-loop and the feedback cases respectively. Without disturbances the trajectories coincide [the chosen value of ε ensures perfect prediction of the asymptotic phase variable behavior

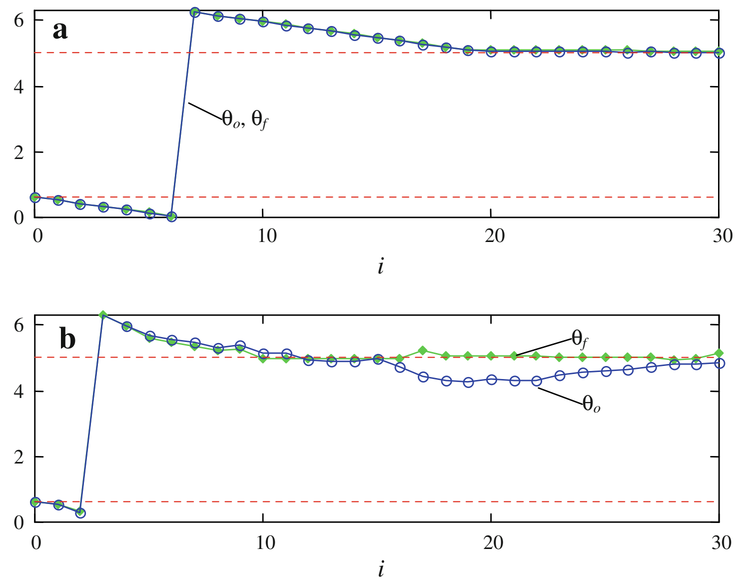

Fig. 3 The trajectories of the model (19)

by (5)], disturbances demonstrate efficiency of the feedback control (it is assumed that the disturbance corrupts the pulse amplitude with the deviation ±50%).

5.2 Circadian oscillations in neurospora

Following (Leloup et al. 1999) let us consider the following model:

$$\begin{aligned}\dot{M} &= [v_s + u(t)]\frac{K_I^n}{K_I^n + F_N^n} - v_m\frac{M}{K_m + M};\\ \dot{F}_c &= k_s M - v_d\frac{F_c}{K_d + F_c} - k_1 F_c + k_2 F_N;\\ \dot{F}_N &= k_1 F_c - k_2 F_N,\end{aligned} \tag{21}$$

where variables M, F_c and F_N denote, respectively, the concentrations (defined with respect to the total cell volume) of the *frq* mRNA and of the cytosolic and nuclear forms of FRQ, $u \in R$ is the control input representing light excitation of the circadian rhythm. In (Leloup et al. 1999) the model (21) was considered with the following values of parameters:

$$v_m = 0.505, \quad v_d = 1.4, \quad v_s = 1.6, \quad k_s = 0.5, \quad k_1 = 0.5, \quad k_2 = 0.6, \quad K_I = 1,$$
$$K_m = 0.5, \quad K_d = 0.13, \quad n = 4.$$

For $u = 0$ the system (21) has the single equilibrium and one stable limit cycle, $T = 21.5$.

We again start with the estimates (13) and (20) comparison. As before, the model (21) is excited by the pulse (18) with $\delta = 1$ and two amplitudes $\Delta_1 = 3$ and $\Delta_2 = 6$. Next, the asymptotic phase estimates $\widehat{\theta}(T)$, $\widehat{\theta}(5T)$ and $\breve{\theta}(T)$, $\breve{\theta}(5T)$ are computed in

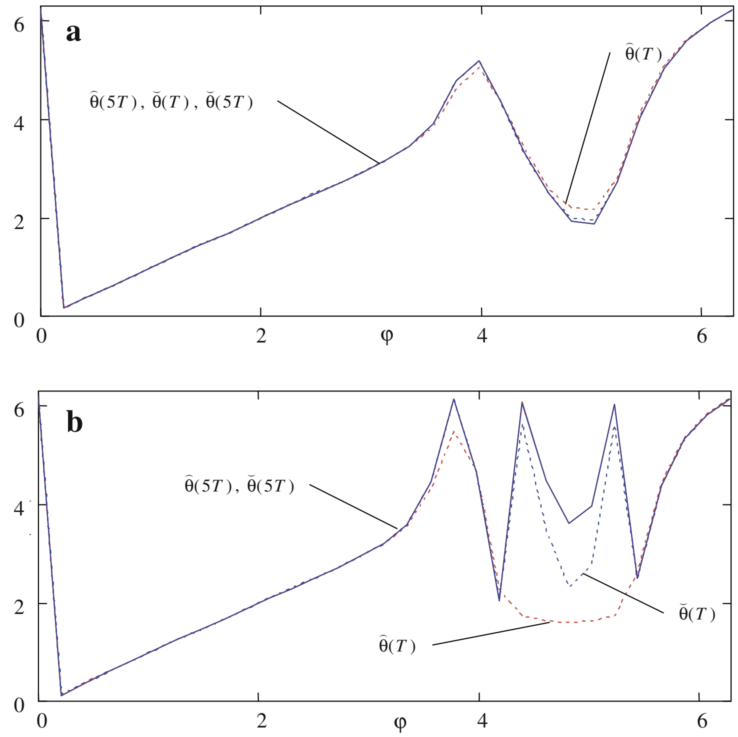

Fig. 4 The asymptotic phases estimates comparison for the model (21)

accordance with (13) and (20) respectively. The estimates are presented in Fig. 4a for the amplitude Δ_1 and in Fig. 4b for the amplitude Δ_2. Again, the estimates $\widehat{\theta}(5T)$ and $\breve{\theta}(5T)$ coincide in both cases and the values $\widehat{\theta}(5T) = \breve{\theta}(5T)$ are chosen as the true one for the variable $\theta(t)$ values. The estimate $\breve{\theta}(T)$ has better accuracy than $\widehat{\theta}(T)$, especially for bigger amplitude of the pulse.

Let $\delta = 1$ and $\Delta = 3$, the corresponding phase response maps are plotted in Fig. 5. The map PRC is computed analytically (Izhikevich 2007; Govaerts and Sautois 2006) and the map PRC_ε is calculated in accordance with the definition for $\varepsilon = 0.15$, $T > TRC_{\max}$ in this case. Since the chosen pulse amplitude is rather big the maps PRC and PRC_ε do not coincide. The assumption 1 is satisfied for the map PRC_ε. Trajectories of the model (21) with the open-loop control (9)–(12) and the feedback control (15), (16), (20) are shown in Fig. 6, the time axis is scaled in the periods number, the indexes o and f denote the phase variables for the open-loop and the feedback cases respectively. Since the chosen value of ε does not guarantee a good estimation of the asymptotic phase variable dynamics in (5), the feedback control demonstrates better performance in this case.

5.3 Circadian oscillations in drosophila

Let us consider the model from the paper (Leloup and Goldbeter 1998):

$$\dot{M}_P = [v_{sp} + u(t)]\frac{K_{IP}^n}{K_{IP}^n + C_N^n} - v_{mP}\frac{M_P}{K_{mP} + M_P} - k_d M_P; \qquad (22)$$

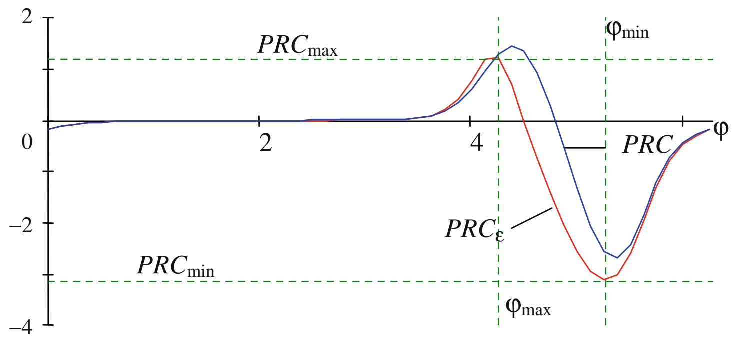

Fig. 5 The phase response maps for the model (21)

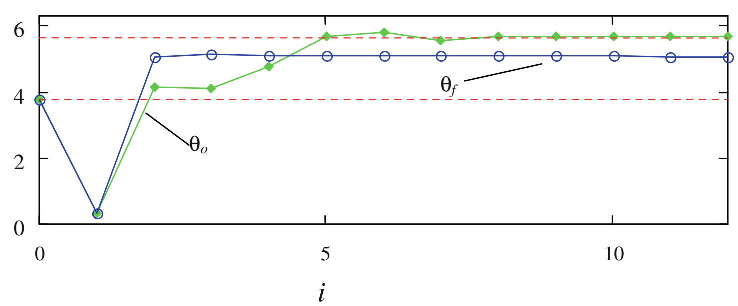

Fig. 6 The trajectories of the model (21)

$$\dot{P}_0 = k_{sP} M_P - V_{1P} \frac{P_0}{K_{1P} + P_0} + V_{2P} \frac{P_1}{K_{2P} + P_1} - k_d P_0; \tag{23}$$

$$\dot{P}_1 = V_{1P} \frac{P_0}{K_{1P} + P_0} - V_{2P} \frac{P_1}{K_{2P} + P_1} - V_{3P} \frac{P_1}{K_{3P} + P_1} + V_{4P} \frac{P_2}{K_{4P} + P_2} - k_d P_1; \tag{24}$$

$$\dot{P}_2 = V_{3P} \frac{P_1}{K_{3P} + P_1} - V_{4P} \frac{P_2}{K_{4P} + P_2} - k_3 P_2 T_2 + k_4 C - v_{dP} \frac{P_2}{K_{dP} + P_2} - k_d P_2; \tag{25}$$

$$\dot{M}_T = v_{sp} \frac{K_{IT}^n}{K_{IT}^n + C_N^n} - v_{mT} \frac{M_T}{K_{mT} + M_T} - k_d M_T; \tag{26}$$

$$\dot{T}_0 = k_{sT} M_T - V_{1T} \frac{T_0}{K_{1T} + T_0} + V_{2T} \frac{T_1}{K_{2T} + T_1} - k_d T_0; \tag{27}$$

$$\dot{T}_1 = V_{1T} \frac{T_0}{K_{1T} + T_0} - V_{2T} \frac{T_1}{K_{2T} + T_1} - V_{3T} \frac{T_1}{K_{3T} + T_1} + V_{4T} \frac{T_2}{K_{4T} + T_2} - k_d T_1; \tag{28}$$

$$\dot{T}_2 = V_{3T} \frac{T_1}{K_{3T} + T_1} - V_{4T} \frac{T_2}{K_{4T} + T_2} - k_3 P_2 T_2 + k_4 C - v_{dT} \frac{T_2}{K_{dT} + T_2} - k_d T_2; \tag{29}$$

$$\dot{C} = k_3 P_2 T_2 - k_4 C - k_1 C + k_2 C_N - k_{dC} C; \tag{30}$$

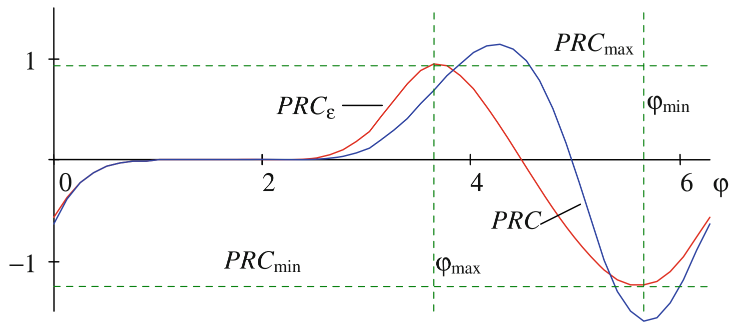

Fig. 7 The phase response maps for the model (22)–(31)

$$\dot{C}_N = k_1 C - k_2 C_N - k_{dN} C_N, \tag{31}$$

where M_P is cytosolic concentration of *per* mRNA; P_0, P_1, P_2 are unphosphorylated, monophosphorylated and bisphosphorylated concentrations of PER protein correspondingly; M_T is cytosolic concentration of *tim* mRNA; T_0, T_1, T_2 are unphosphorylated, monophosphorylated and bisphosphorylated concentrations of TIM protein correspondingly; C is PER-TIM complex concentration and C_N is nuclear form of PER-TIM complex. As in work (Leloup and Goldbeter 1998) we will consider the following values of the model (22)–(31) parameters:

$$\begin{aligned}
&K_{IP} = K_{IT} = v_{sT} = v_{sP} = 1, \quad v_{mP} = v_{mT} = 0.7,\\
&K_{dP} = K_{dT} = K_{mP} = K_{mT} = 0.2,\\
&k_{sP} = k_{sT} = 0.9, \quad v_{dP} = v_{dT} = 2,\\
&K_{1P} = K_{1T} = K_{2P} = K_{2T} = K_{3P} = K_{3T} = K_{4P} = K_{4T} = 2,\\
&V_{1P} = V_{1T} = V_{3P} = V_{3T} = 8, \quad k_1 = 0.6, \quad k_2 = 0.2, \quad k_3 = 1.2, \quad k_4 = 0.6, n = 4,\\
&k_d = k_{dC} = k_{dN} = 0.01, \quad V_{2P} = V_{2T} = V_{4P} = V_{4T} = 1.
\end{aligned}$$

This model was regulated in the papers (Bagheri 2007; Bagheri et al. 2008) via optimal control approach. For the case $u = 0$ this model has the single equilibrium and one stable limit cycle, $T = 24.13$.

Let $\delta = 1$ and $\Delta = 2$, the corresponding phase response maps are plotted in Fig. 7. The map PRC is computed in accordance with the standard approach (Izhikevich 2007; Govaerts and Sautois 2006), the map PRC_ε is derived for $\varepsilon = 0.01, T > TRC_{\max}$. For chosen amplitude Δ the maps PRC and PRC_ε do not coincide. The assumption 1 is satisfied for the map PRC_ε. Trajectories of the model (22)–(31) with the open-loop control (9)–(12) and the feedback control (15), (16), (20) are shown in Fig. 8 (the case without disturbances in Fig. 8a, and with a stochastic additive disturbance in the control channel in Fig. 8b), the time axis is scaled in the periods number, the indexes o and f denote the phase variables for the open-loop and the feedback cases respectively. Without disturbances the trajectories are similar, disturbances demonstrate efficiency of the feedback control (the disturbance modifies the pulse amplitude with the maximal deviation $\pm 100\%$).

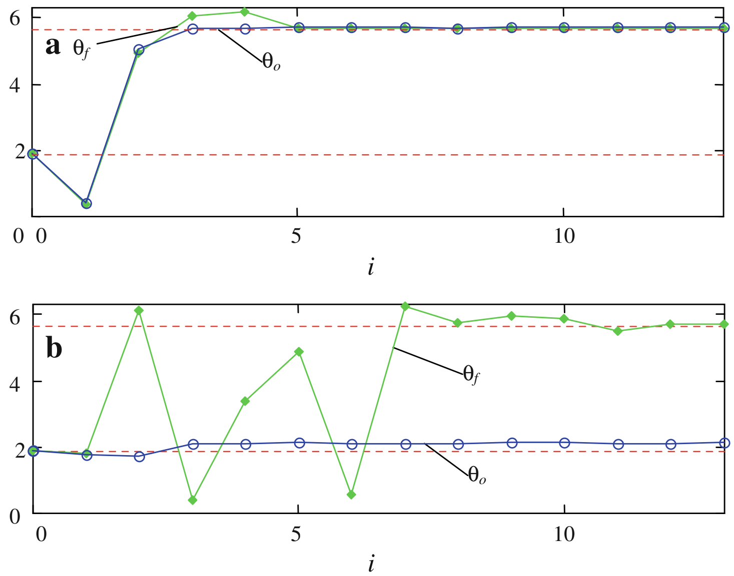

Fig. 8 The trajectories of the model (22)–(31)

6 Conclusion

The paper presents a generic approach for solution of the phase resetting problem for biological oscillators. The proposed solution is based on the PRC concept. According to the conventional definition of PRC and the related phase models, they are mainly developed for the phase dynamics description in the case of infinitesimal inputs (Izhikevich 2007). To overcome this restriction, a new definition of the phase response map is presented. The proposed ε-PRC and ε-TRC maps describe the phase response on any desired amplitude of the pulse control. They also can be easily computed for a mathematical model or measured experimentally. Moreover, the approach develops a mathematical setup for the application of PRC measured experimentally in biological oscillators. The accuracy and relation of the ε-PRC and ε-TRC maps with respect to the conventional PRC are characterized in Lemma 2. New phase model based on the ε-PRC and ε-TRC maps is derived and its properties are analyzed (Theorem 1 and Corollary 1).

Based on this model open-loop and feedback controls for phase resetting are designed. These controls ensure phase adjustment from any initial value to the desired one (Theorems 2 and 3). The controls compute the instants of pulses activation for the fastest resetting. The open-loop strategy is based on the PRC map and the initial phase only. The feedback control is based on the current estimated value of the phase. The estimation is required once per pulse application. The algorithm of the phase estimation is derived based on the result of Lemma 1.

Performance of the proposed model and controls is demonstrated by computer simulations for three examples: the model of heart muscle contraction and two models of circadian rhythm. In all models the proposed algorithm for the phase values estimation demonstrates its superiority over conventional one, especially for big amplitudes of exciting pulses. For the case without disturbances the open-loop and the feedback controls show a similar performance. Appearance of the disturbances diminishes the open-loop control accuracy, while the feedback one demonstrates a certain degree of robustness.

These results allows us to conclude that, if the pulse amplitude is sufficiently high, then the proposed ε-PRC and ε-TRC maps provide better results for modeling and control of oscillators. If the input amplitude is restricted, then the conventional approaches based on infinitesimal PRC (Danzl and Moehlis 2008; Efimov et al. 2009) can effectively solve the problem.

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