Analyse par ondelettes du mouvement multifractionnaire stable linéaire

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Analyse par ondelettes du mouvement multifractionnaire stable linéaire

Résumé

Le *mouvement brownien fractionnaire* (mbf) constitue un important outil de modélisation utilisé dans plusieurs domaines (biologie, économie, finance, géologie, hydrologie, télécommunications,...) ; toutefois, ce modèle ne parvient pas toujours à donner une description suffisamment fidèle de la réalité, à cause, entre autres, des deux limitations suivantes : d'une part le mbf est un processus gaussien, et d'autre part, sa rugosité locale (mesurée par un exposant de Hölder) reste la même tout le long de sa trajectoire, puisque cette rugosité est partout égale au paramètre de Hurst H qui est une constante. En vue d'y remédier, S. Stoev et M.S. Taqqu (voir [28, 30]) ont introduit le *mouvement multifractionnaire stable linéaire* (mmsl) ; ce processus stochastique strictement α-stable ($\mathcal{S}t\alpha\mathcal{S}$), désigné par $Y = \{Y(t) : t \in \mathbb{R}\}$, est obtenu en modifiant, la représentation du mbf sous forme de moyenne mobile non anticipative, de la façon suivante : la mesure brownienne est remplacée par une mesure $\mathcal{S}t\alpha\mathcal{S}$, et le paramètre de Hurst H par une fonction $H(\cdot)$ dépendant de t. Dans la thèse, on se place systématiquement dans le cas où cette fonction est continue et à valeurs dans l'intervalle ouvert $]1/\alpha, 1[$. Il convient aussi de noter que l'on a pour tout $t \in \mathbb{R}$, $Y(t) := X(t, H(t))$, où $X = \{X(u, v) : (u, v) \in \mathbb{R}\times]1/\alpha, 1[\}$ est le champ stochastique $\mathcal{S}t\alpha\mathcal{S}$, tel que pour tout v fixé, le processus $X(\cdot, v) := \{X(u, v) : u \in \mathbb{R}\}$ est un *mouvement fractionnaire stable linéaire* (mfsl). La thèse comporte principalement trois parties.

(1) L'objectif principal de la première partie, est de déterminer de fins modules de continuité, globaux et locaux, du mmsl $\{Y(t) : t \in \mathbb{R}\}$; pour ce faire, la stratégie consiste à étudier avec précision le comportement trajectoriel du champ $\{X(u, v) : (u, v) \in \mathbb{R}\times]1/\alpha, 1[\}$, ainsi que celui de sa dérivée partielle de tout ordre par rapport à v. L'étude repose essentiellement sur une nouvelle représentation de $\{X(u, v) : (u, v) \in \mathbb{R}\times]1/\alpha, 1[\}$, sous la forme d'une série aléatoire dont on montre la convergence presque sûre dans certains espaces de Hölder ; signalons que cette série est obtenue en décomposant le noyau associé à $\{X(u, v) : (u, v) \in \mathbb{R}\times]1/\alpha, 1[\}$ dans une base d'ondelettes de Daubechies de classe $\mathcal{C}^3(\mathbb{R})$. La méthodologie de cette première partie, s'inspire, dans une certaine mesure, de celle déjà utilisée par Ayache, Roueff et Xiao (voir [5]) dans le cadre du *drap fractionnaire stable linéaire*.

(2) Comme nous venons de l'indiquer, la représentation du champ $\{X(u, v) : (u, v) \in \mathbb{R}\times]1/\alpha, 1[\}$ sous la forme d'une série aléatoire via une base d'ondelettes de Daubechies de classe $\mathcal{C}^3(\mathbb{R})$, est un outil très utile, lorsqu'il s'agit de mener une étude précise du comportement trajectoriel de ce champ. Cependant cette représentation a deux inconvénients : d'une part, elle n'est pas suffisamment explicite et ne peut donc permettre une simulation du mmsl en un temps de calcul raisonnable, d'autre part, elle ne sépare pas complètement les parties hautes et basses fréquences de $\{X(u, v) : (u, v) \in \mathbb{R}\times]1/\alpha, 1[\}$. Afin d'y remédier, l'objectif principal de la deuxième partie de la thèse consiste d'abord à introduire, via la base de Haar, une autre représentation en série aléatoire de ce champ ; il consiste ensuite à établir, au moyen de transformées d'Abel, la convergence presque sûre de cette série uniformément en (u, v) dans un rectangle compact ; il consiste enfin à estimer finement la vitesse de convergence, il s'avère que cette vitesse est raisonnable.

(3) L'objectif principal de la troisième partie de la thèse, est de construire au moyen des coefficients d'ondelettes $\{d_{j,k}\}_{(j,k)\in\mathbb{Z}^2}$ du mmsl, des estimateurs statistiques de trois importants paramètres reliés à ce processus, à savoir : le minimum de la fonction $H(\cdot)$ sur un intervalle compact arbitraire, $H(t_0)$ la valeur de celle-ci en un point t_0 choisi arbitrairement, et la valeur du paramètre de stabilité α. On montre qu'à une échelle donnée j assez grande, des estimateurs des deux premiers paramètres sont obtenus grâce à des moyennes empiriques d'ordre $\beta \leq 1/4$,

issues de certains des $\{d_{j,k}\}_{(j,k)\in\mathbb{Z}^2}$; de plus, le maximum des amplitudes de ces coefficients, fournit un estimateur du troisième paramètre.

Wavelet analysis of Linear Multifractional Stable Motion

Abstract

Fractional Brownian Motion (FBM) is an important tool in modeling used in several areas (biology, economics, finance, geology, hydrology, telecommunications, and so on); however, this model does not always give a sufficiently accurate description of reality, two important ones among its limitations, are the following: on one hand, FBM is a Gaussian process, and on the other hand, its local roughness (measured through a Hölder exponent) remains the same all along its path, since this roughness is everywhere equal to the Hurst parameter H which is a constant. In order to overcome the latter two limitations, S. Stoev and M.S. Taqqu (see [28, 30]) introduced *Linear Multifractional Stable Motion* (LMSM); this strictly α-stable ($\mathcal{S}t\alpha\mathcal{S}$) stochastic process, denoted by $Y = \{Y(t) : t \in \mathbb{R}\}$, is obtained by modifying, the non anticipative moving average representation of FBM, in the following way: the Brownian measure is replaced by a $\mathcal{S}t\alpha\mathcal{S}$ one, and the Hurst parameter H by a function $H(\cdot)$ depending on t. Throughout the thesis, one assumes the latter function to be continuous and with values in the open interval $(1/\alpha, 1)$. Also, it is worth noticing that one has for all $t \in \mathbb{R}$, $Y(t) := X(t, H(t))$, where $X = \{X(u,v) : (u,v) \in \mathbb{R} \times (1/\alpha, 1)\}$ is the $\mathcal{S}t\alpha\mathcal{S}$ stochastic field, such that for all fixed v, the process $X(\cdot, v) := \{X(u,v) : u \in \mathbb{R}\}$ is a *Linear Fractional Stable Motion* (LFSM). The thesis mainly consists into three parts.

(1) The main goal of the first part, is to determine, sharp global and local moduli of continuity for the LMSM $\{Y(t) : t \in \mathbb{R}\}$; to do so, the strategy consists in studying with precision path behavior of the field $\{X(u,v) : (u,v) \in \mathbb{R} \times (1/\alpha, 1)\}$, as well as that of its partial derivative of any order with respect to v. The study mainly relies on a new representation of $\{X(u,v) : (u,v) \in \mathbb{R} \times (1/\alpha, 1)\}$, as a random series which almost surely converges in some Hölder spaces; let us mention that this series is obtained by decomposing the kernel associated to $\{X(u,v) : (u,v) \in \mathbb{R} \times (1/\alpha, 1)\}$ in a $\mathcal{C}^3(\mathbb{R})$ Daubechies wavelet basis. The methodology, in this first part, is, to a certain extent, inspired by that already used by Ayache, Roueff and Xiao (see [5]) in the setting of *linear fractional stable sheet*.

(2) As we have just indicated, the representation of the field $\{X(u,v) : (u,v) \in \mathbb{R} \times (1/\alpha, 1)\}$ as a random series via a $\mathcal{C}^3(\mathbb{R})$ Daubechies wavelet basis, is a quite useful tool for conducting a precise study on path behavior of this field. However, this representation has two drawbacks: on one hand it is not explicit enough therefore it does not allow for a simulation of LMSM in a reasonable calculation time, on the other hand, it does not completely separate the high frequency part of $\{X(u,v) : (u,v) \in \mathbb{R} \times (1/\alpha, 1)\}$ from the low one. In order to overcome the latter two drawbacks, the main goal of the second part of the thesis, first consists in introducing, via the Haar basis, another random series representation of this field; then it consists in establishing, thanks to Abel transforms, the almost sure convergence of this series uniformly in (u,v) belonging to a compact rectangle; last but not least, it consists in deriving sharp estimates of the rate of convergence, this rate turns out to be reasonable.

(3) The main goal of the third part of the thesis, is to construct, by making use of wavelet coefficients $\{d_{j,k}\}_{(j,k)\in\mathbb{Z}^2}$ of LMSM, statistical estimators of three important parameters related to this process, namely: the minimum of the function $H(\cdot)$ over an arbitrary compact interval, $H(t_0)$ its value at an arbitrarily chosen point t_0, and the value of the stability parameter α. One shows that at a given scale j big enough, estimators of the first two parameters can be obtained thanks to empirical means of order $\beta \leq 1/4$, derived from some of the $d_{j,k}$'s; moreover, the maximum of the amplitudes of these coefficients, provides an estimator for the third parameter.

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CHAPITRE 1

Introduction

Rappelons que le *mouvement brownien fractionnaire* (mbf en abrégé) de paramètre de Hurst $H \in]0,1[$ (voir [26, 17]), noté par $B_H = \{B_H(t) : t \in \mathbb{R}\}$, est l'unique processus gaussien centré, H-auto-similaire et à accroissements stationnaires ; la représentation sous forme de moyenne mobile non anticipative de ce processus est donnée pour tout $t \in \mathbb{R}$, par l'intégrale de Wiener,

$$B_H(t) := \int_{\mathbb{R}} \Big\{ (t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \Big\} W(ds), \tag{0.1}$$

où, pour tous nombres réels x et κ,

$$(x)_+^{\kappa} := \begin{cases} x^{\kappa}, \text{si } x \in]0,+\infty[, \\ \\ 0, \text{si } x \in]-\infty, 0]. \end{cases}$$

Bien que le mbf soit un important outil de modélisation utilisé dans plusieurs domaines (biologie, économie, finance, géologie, hydrologie, télécommunications, ...) ; ce modèle n'est quand même pas toujours très réaliste, à cause, entre autres, des deux limitations suivantes, qui l'empêchent de pouvoir décrire fidèlement de multiples signaux de la vie réelle :

(1) le mbf est gaussien ;

(2) sa rugosité locale, mesurée par exemple au moyen de l'exposant de Hölder local (voir (8.3) dans le chapitre 2 pour la définition précise de cet exposant), reste la même tout le long de sa trajectoire et vaut presque sûrement H en tout point t_0.

Dans l'objectif de disposer de modèles qui soient en meilleure adéquation avec la réalité, de nombreuses extensions du mbf ont été introduites au cours de ces deux ou trois dernières décennies ; parmi les plus importantes figurent le *mouvement fractionnaire stable linéaire* (mfsl en abrégé) (voir [26, 17]) et le *mouvement brownien multifractionnaire* gaussien (mbm en abrégé) (voir [22, 9]).

Le mfsl permet de s'éloigner de façon considérable du cadre gaussien ; en effet, ce processus est défini dans un contexte de lois de probabilité à queues épaisses (on dit aussi à queues lourdes), en remplaçant dans (0.1), $H - 1/2$ par $H - 1/\alpha$ et la mesure brownienne $W(\cdot)$ par $\mathrm{Z}_\alpha(\cdot)$ une mesure strictement α-stable [1] ($\mathcal{S}t\alpha\mathcal{S}$) ; signalons que dans toute la thèse on se restreindra à $\alpha \in]1,2[$, parce que cette condition permet d'avoir la continuité des trajectoires des processus et des champs étudiés.

Le mbm permet d'avoir une rugosité locale qui varie d'un point à un autre ; il est défini en remplaçant dans (0.1), le paramètre de Hurst constant H par une fonction $H(\cdot)$ qui dépend de la variable t ; lorsque $H(\cdot)$ est suffisamment régulière, l'exposant de Hölder local du mbm vaut, presque sûrement $H(t_0)$, en tout point t_0 (voir [22, 9, 6, 3]).

Ainsi, pour être dans un cadre de lois de probabilité à queues épaisses et avoir de plus une rugosité locale qui varie d'un point à un autre tout en restant contrôlable au moyen d'un paramètre fonctionnel déterministe noté par $H(\cdot)$, il semble naturel de considérer le processus $\mathcal{S}t\alpha\mathcal{S}$, $Y = \{Y(t) : t \in \mathbb{R}\}$

1. Dans toute la thèse, on considère que cette mesure est à valeurs réelles, « à accroissements indépendants » (« independently scattered » en anglais) et que la mesure de contrôle associée est la mesure de Lebesgue.

défini pour tout $t \in \mathbb{R}$, de la façon suivante :

$$Y(t) := \int_{\mathbb{R}} \Big\{ (t-s)_+^{H(t)-1/\alpha} - (-s)_+^{H(t)-1/\alpha} \Big\} \mathrm{Z}_\alpha(ds)\,. \tag{0.2}$$

Ce processus $Y = \{Y(t) : t \in \mathbb{R}\}$ s'appelle le *mouvement multifractionnaire stable linéaire* (mmsl en abrégé) de paramètre (fonctionnel de Hurst) $H(\cdot)$ et de paramètre de stabilité α, il a été introduit par Stoev et Taqqu dans [**28, 30**] ; selon ces deux auteurs il serait un bon candidat pour modéliser certains phénomènes non gaussiens liés à des changements de régimes, lors d'échanges de paquets d'informations sur des réseaux de télécommunications. Dans toute cette thèse, on suppose que $H(\cdot)$ est une fonction continue sur la droite réelle et à valeurs dans un intervalle compact $[\underline{H}, \overline{H}] \subset]1/\alpha, 1[$. Il est important de garder toujours à l'esprit que le mmsl est intimement lié au champ stochastique $\mathcal{S}t\alpha\mathcal{S}$ $X = \{X(u,v) : (u,v) \in \mathbb{R}\times]1/\alpha, 1[\}$, défini pour tout (u,v) par l'intégrale stochastique,

$$X(u,v) := \int_{\mathbb{R}} \Big\{ (u-s)_+^{v-1/\alpha} - (-s)_+^{v-1/\alpha} \Big\} \mathrm{Z}_\alpha(ds)\,; \tag{0.3}$$

en effet, à cause (0.2), on a pour tout $t \in \mathbb{R}$,

$$Y(t) := X(t, H(t)). \tag{0.4}$$

Dans toute la thèse, ce champ X sera appelé *le champ qui engendre les mmsl* ; on se gardera de le confondre avec le mfsl : en fait, pour tout $v \in]1/\alpha, 1[$ fixé, le processus stochastique $X(\cdot, v) := \{X(u,v) : u \in \mathbb{R}\}$ est un mfsl de paramètre de Hurst v.

Nous allons désormais décrire de façon succincte, chacun des quatre chapitres (les chapitres 2 à 5) qui constituent le corps de cette thèse ; au préalable signalons que, au chapitre 2, on se place dans le cas où la mesure $\mathrm{Z}_\alpha(\cdot)$ est strictement α-stable ($\mathcal{S}t\alpha\mathcal{S}$), alors que dans les trois autres chapitres on se restreint au cas où cette mesure est symétrique α-stable ($\mathcal{S}\alpha\mathcal{S}$).

L'objectif du chapitre 2, consiste à faire une fine étude du comportement global et local du champ $\partial_v^q X$, où $q \in \mathbb{Z}_+$, ainsi que du processus mmsl Y associé à X ; pour ce faire nous utilisons des méthodes d'ondelettes qui s'inspirent, dans une certaine mesure, de celles qui ont déjà été mises en œuvre par Ayache, Roueff et Xiao dans le cadre du drap fractionnaire stable linéaire (voir [**5**]).

Considérons ψ une ondelette « mère » de Daubechies à support compact, qui est trois fois continûment dérivable sur $\mathbb{R}$. En développant, pour tout $(u,v) \in \mathbb{R}\times]1/\alpha, 1[$ fixé, le noyau déterministe associé à la variable aléatoire $X(u,v)$ voir (0.3), i.e. la fonction $s \mapsto (u-s)_+^{v-1/\alpha} - (-s)_+^{v-1/\alpha}$, dans la base $\big\{2^{j/\alpha}\psi(2^j \cdot -k) : (j,k) \in \mathbb{Z}^2\big\}$, on obtient la représentation suivante de $X(u,v)$,

$$X(u,v) = \sum_{(j,k)\in\mathbb{Z}^2} 2^{-jv} \epsilon_{j,k} \big(\Psi(2^j u - k, v) - \Psi(-k, v) \big), \tag{0.5}$$

où :

- Ψ désigne la fonction déterministe régulière, définie pour tout $(x,v) \in \mathbb{R}\times]1/\alpha, 1[$ par,

$$\Psi(x,v) := \int_{\mathbb{R}} (x-s)_+^{v-1/\alpha} \psi(s) ds\,; \tag{0.6}$$

- $\big\{\epsilon_{j,k} : (j,k) \in \mathbb{Z}^2\big\}$ est la suite de variables aléatoires $\mathcal{S}t\alpha\mathcal{S}$ définies par

$$\epsilon_{j,k} := 2^{j/\alpha} \int_{\mathbb{R}} \psi(2^j s - k) \mathrm{Z}_\alpha(ds)\,. \tag{0.7}$$

Il convient de souligner que Ψ ainsi que ses dérivées partielles de tous ordres, sont des fonctions bien localisées en la variable $x \in \mathbb{R}$, uniformément en la variable $v \in [a,b]$, où a et b désignent deux réels arbitraires et fixés vérifiant $1/\alpha < a < b < 1$; plus précisément, pour tout $(p,q) \in \{0,1,2,3\} \times \mathbb{Z}_+$, on a,

$$\sup_{(x,v)\in\mathbb{R}\times[a,b]} (3+|x|)^2 \big|(\partial_x^p \partial_v^q \Psi)(x,v)\big| < \infty. \tag{0.8}$$

Soulignons aussi que l'on dispose, presque sûrement, d'un contrôle « déterministe » du comportement asymptotique de la suite $\{\epsilon_{j,k} : (j,k) \in \mathbb{Z}^2\}$; plus précisément, pour tout réel $\eta > 0$ fixé et arbitrairement petit, on a, presque sûrement,

$$|\epsilon_{j,k}| \le C \left(\big(1+|j|\big)^{1/\alpha+\eta} \big(1+|k|\big)^{1/\alpha} \log^{1/\alpha+\eta}\big(2+|k|\big) \right), \tag{0.9}$$

où C désigne une variable aléatoire finie presque sûrement et qui ne dépend pas de (j,k).

Au moyen de (0.8) et (0.9), on montre que, presque sûrement, pour tous réels fixés $M > 0$ et $\gamma \in [0, a-1/\alpha)$, la série définie en (0.5) et sa dérivée partielle par rapport à v d'ordre arbitraire $q \in \mathbb{Z}_+$, sont convergentes dans l'espace $\mathcal{E}_\gamma(a,b,M) := \mathcal{C}^1\big([a,b], \mathcal{C}^\gamma([-M,M],\mathbb{R})\big)$ (notons que $\mathcal{C}^\lambda(I,\mathbb{B})$ désigne l'espace des fonctions λ-höldériennes définies sur un intervalle I et à valeurs dans un espace de Banach $\mathbb{B}$). Ensuite, en combinant ce dernier résultat avec (0.8) et (0.9), on obtient de fines propriétés trajectorielles du champ $\partial_v^q X$: on a, presque sûrement,

$$\sup_{(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2} \left\{ \frac{\big|\big(\partial_v^q X\big)(u_1,v_1) - \big(\partial_v^q X\big)(u_2,v_2)\big|}{|u_1-u_2|^{v_1\vee v_2 - 1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta} + |v_1-v_2|} \right\} < \infty, \tag{0.10}$$

et

$$\sup_{(u,v)\in\mathbb{R}\times[a,b]} \left\{ \frac{\big|\big(\partial_v^q X\big)(u,v)\big|}{|u|^v\big(1+\big|\log|u|\big|\big)^{q+1/\alpha+\eta}} \right\} < \infty. \tag{0.11}$$

Ainsi, en imposant à $H(\cdot)$ des conditions de Hölder assez faibles, les relations (0.10) et (0.11) (dans lesquelles on prend $q=0$), nous permettent d'obtenir des modules de continuité globaux et locaux du mmsl Y ; les plus importants d'entre eux, sont :

$$\sup_{(t,s)\in[M_1,M_2]^2} \left\{ \frac{\big|Y(t)-Y(s)\big|}{|t-s|^{\min_{x\in[M_1,M_2]} H(x)-1/\alpha}\big(1+\big|\log|t-s|\big|\big)^{2/\alpha+\eta}} \right\} < \infty \tag{0.12}$$

et

$$\sup_{t\in[M_1,M_2]} \left\{ \frac{\big|Y(t)-Y(t_0)\big|}{|t-t_0|^{H(t_0)}\big(1+\big|\log|t-t_0|\big|\big)^{1/\alpha+\eta}} \right\} < \infty\,; \tag{0.13}$$

signalons que $M_1 < M_2$ et t_0 sont des réels arbitraires et fixés, signalons aussi que la finitude de ces deux derniers supremas, a lieu presque sûrement.

Finalement, on montre que le module de continuité global (0.12) est quasi-optimal, c'est à dire que le supremum devient presque sûrement infini lorsque le facteur logarithmique est élevé à une certaine puissance négative. De plus, on prouve que le module de continuité local (0.13) est optimal, c'est à dire que le supremum devient presque sûrement infini lorsque $\eta = 0$.

Résumons maintenant le contenu du chapitre 3. Il convient d'abord de préciser que les parties hautes et basses fréquences du champ $X = \{X(u,v) : (u,v) \in [0,1]\times]1/\alpha,1[\}$, sont respectivement les champs stochastiques $\mathcal{S}\alpha\mathcal{S}$, notés par $X_1 = \{X_1(u,v) : (u,v) \in [0,1]\times]1/\alpha, 1[\}$ et $X_2 = \{X_2(u,v) : (u,v) \in [0,1]\times]1/\alpha, 1[\}$, et définis pour tout $(u,v) \in [0,1]\times]1/\alpha, 1[$, par,

$$X_1(u,v) := \int_0^1 (u-s)_+^{v-1/\alpha} Z_\alpha(ds)$$

et

$$X_2(u,v) := \int_{-\infty}^0 \Big\{(u-s)_+^{v-1/\alpha} - (-s)_+^{v-1/\alpha}\Big\} Z_\alpha(ds)\,;$$

notons que les propriétés de X_1 sont loin d'être tout à fait semblables à celles de X_2.

Le chapitre 3 s'inspire dans une certaine mesure de l'article [4] ; sa motivation de départ est d'introduire des représentations de X_1 et X_2 en séries aléatoires suffisamment simples et explicites pour permettre la simulation de $Y_1 = \{Y_1(t) : t \in [0,1]\} = \{X_1(t, H(t)) : t \in [0,1]\}$ et $Y_2 = \{Y_2(t) : t \in [0,1]\} = \{X_2(t, H(t)) : t \in [0,1]\}$, les parties hautes et basses fréquences du mmsl $Y = \{Y(t) : t \in [0,1]\} = \{X(t, H(t)) : t \in [0,1]\} = \{Y_1(t) + Y_2(t) : t \in [0,1]\}$, en un temps de calcul raisonnable.

Les preuves des résultats présentés dans le chapitre 2, témoignent que la représentation en série (0.5), est un outil puissant pour une étude fine des propriétés des trajectoires de X et Y ; mais, cette représentation, par l'intermédiaire de l'ondelette de Daubechies régulière ψ, souffre des deux inconvénients suivants :

- La fonction Ψ (voir (0.6)) et les variables aléatoires $\epsilon_{j,k}$ (voir (0.7)) ne peuvent pas être définies au moyen de formules simples et explicites, puisque ce n'est pas le cas pour l'ondelette de Daubechies ψ elle-même ; donc (0.5) ne peut guère fournir une méthode efficace de simulation de X et Y.
- Dans (0.5), X_1 et X_2 les parties hautes et basses fréquences de X, ne sont pas complètement séparées ; cela provient essentiellement du fait que le diamètre du support de ψ, est strictement plus grand que 1 (ce diamètre est même nettement plus grand que 1).

Afin de remédier à ces deux inconvénients, dans ce chapitre, nous remplaçons ψ par la fonction de Haar $h := \mathbb{1}_{[0,1/2[} - \mathbb{1}_{[1/2,1[}$, où $\mathbb{1}_S$ désigne la fonction indicatrice d'un sous-ensemble arbitraire S de $\mathbb{R}$. La fonction continûment différentiable $\theta : \mathbb{R}\times]1/\alpha, 1[\to \mathbb{R}$ est définie par (0.6) où ψ est remplacée par h ; même si θ joue un rôle assez semblable à celui de Ψ, il existe une différence considérable entre ces deux fonctions, en effet :

- d'une part, θ présente l'avantage d'être donnée par la formule simple et explicite, pour tout $(x,v) \in \mathbb{R}\times]1/\alpha, 1[$,

$$\theta(x,v) = \big(1+v-1/\alpha\big)^{-1}\Big\{(x-1)_+^{1+v-1/\alpha} - 2(x-1/2)_+^{1+v-1/\alpha} + (x)_+^{1+v-1/\alpha}\Big\};$$

- mais, d'autre part, θ souffre de l'inconvénient d'être moins régulière que Ψ et, chose plus importante, elle ne vérifie pas la propriété de bonne localisation (0.8).

Pour tout $(j,k) \in \mathbb{Z}^2$, nous désignons par $\zeta_{j,k}$ la variable aléatoire $\mathcal{S}\alpha\mathcal{S}$ définie au moyen de (0.7) où ψ est remplacée par h ; contrairement à $\epsilon_{j,k}$, la variable aléatoire $\zeta_{j,k}$ est donnée explicitement par la formule simple,

$$\zeta_{j,k} = -2^{j/\alpha}\bigg(\mathrm{Z}_\alpha\left(\frac{k}{2^j}\right) - 2\mathrm{Z}_\alpha\left(\frac{k+1/2}{2^j}\right) + \mathrm{Z}_\alpha\left(\frac{k+1}{2^j}\right)\bigg),$$

où $\{\mathrm{Z}_\alpha(t) : t \in \mathbb{R}\}$ est le processus de Lévy $\mathcal{S}\alpha\mathcal{S}$ qui est induit par la mesure $\mathrm{Z}_\alpha(\cdot)$.

Ainsi, en reprenant la même méthode qui a permi d'obtenir, pour tout $(u,v) \in [0,1]\times]1/\alpha, 1[$ fixé, la représentation (0.5) de la variable aléatoire $X(u,v)$, on aboutit aux deux représentations suivantes pour les variables aléatoires $X_1(u,v)$ et $X_2(u,v)$:

$$X_1(u,v) = \frac{u^{1+v-1/\alpha}}{1+v-1/\alpha} \mathrm{Z}_\alpha(1) + \sum_{j=0}^{+\infty} 2^{-jv} \sum_{k=0}^{2^j-1} \zeta_{j,k} \theta(2^j u - k, v), \tag{0.14}$$

et

$$X_2(u,v) = \sum_{j=-\infty}^{+\infty} 2^{-jv} \sum_{k=1}^{+\infty} \zeta_{j,-k} \big(\theta(2^j u + k, v) - \theta(k,v)\big). \tag{0.15}$$

Il s'agit d'un problème délicat de montrer que les séries (0.14) et (0.15), sont presque sûrement convergentes dans l'espace des fonctions continues $\mathcal{C}([0,1] \times [a,b], \mathbb{R})$; la principale difficulté dans ce problème vient du fait que θ est une fonction mal localisée en la variable x : en effet, lorsque $v \in [a,b]$ est fixé et que x tend vers $+\infty$, alors $\theta(x,v)$ converge vers 0, à la même vitesse que $x^{v-1-1/\alpha}$. Afin de contourner cette difficulté, nous utilisons des transformées d'Abel appropriées, ce qui nous permet de prouver que l'on a, presque sûrement, pour tout réel $\eta > 0$ arbitrairement petit, et tout entier positif J suffisamment grand,

$$\|X_1(\cdot,\cdot) - X_1^J(\cdot,\cdot)\|_{\mathcal{C}([0,1]\times[a,b],\mathbb{R})} = \mathcal{O}\left(2^{-J(a-1/\alpha)} J^{2/\alpha+\eta}\right)$$

et

$$\|X_2(\cdot,\cdot) - X_2^J(\cdot,\cdot)\|_{\mathcal{C}([0,1]\times[a,b],\mathbb{R})} = \mathcal{O}\left(2^{-J(1-b)} J^{1/\alpha+\eta}\right);$$

X_1^J et X_2^J étant, respectivement les sommes partielles de X_1 et X_2, définies pour tout $(u,v) \in [0,1] \times [a,b]$ par,

$$X_1^J(u,v) = \frac{u^{1+v-1/\alpha}}{1+v-1/\alpha} \mathrm{Z}_\alpha(1) + \sum_{j=0}^{J-1} 2^{-jv} \sum_{k=0}^{2^j-1} \zeta_{j,k} \theta(2^j u - k, v)$$

et

$$X_2^J(u,v) = \sum_{j=1-J}^{J-1} 2^{-jv} \sum_{k=1}^{2^{J-|j|}} \zeta_{j,-k} \big(\theta(2^j u + k, v) - \theta(k,v)\big).$$

Résumons maintenant le contenu du chapitre 4. Dans le cas du mfsl, le problème de l'estimation du paramètre de Hurst H a déjà été étudié dans plusieurs travaux (voir entre autres [**1**, **13**, **14**, **24**, **27**]) et des estimateurs fortement consistants, obtenus au moyen de transformées par ondelettes de ce processus, ont été proposés ; notons au passage que ces estimateurs ne nécessitent pas la connaissance du paramètre de stabilité α.

Par ailleurs, dans le cas du mbm, le problème de l'estimation de $H(t_0)$ (t_0 est un point arbitraire et fixé), a lui aussi déjà été étudié dans plusieurs travaux (voir entre autres [**8**, **10**, **11**, **7**, **19**, **23**]) et des estimateurs fortement consistants, obtenus en « localisant autour de t_0 »des variations quadratiques généralisées de ce processus, ont été proposés.

En revanche, dans le cas du mmsl aucun travail sur l'estimation de $H(t_0)$ (t_0 est un point arbitraire et fixé) ou encore de $\min_{t\in I} H(t)$ (I est un intervalle compact d'intérieur non vide arbitraire et fixé), n'a encore été entrepris ; c'est ce que nous nous proposons de faire dans le chapitre 4, au moyen de coefficients d'ondelettes de ce processus.

Dans ce chapitre, on suppose que le paramètre fonctionnel $H(\cdot)$ du mmsl $\{Y(t) : t \in [0,1]\}$, vérifie sur l'intervalle $[0,1]$, une condition de Hölder uniforme d'ordre strictement plus grand que

$\max_{t\in[0,1]} H(t)$. Pour tout $j \in \mathbb{Z}_+$ et $k \in \{0,\ldots,2^j-1\}$, le coefficient d'ondelette $d_{j,k}$ du mmsl $\{Y(t)\}_{t\in[0,1]}$, est défini par,

$$d_{j,k} := 2^j \int_{\mathbb{R}} Y(t)\psi(2^j t - k)dt\,; \tag{0.16}$$

signalons que l'on impose à l'ondelette ψ (qui ne doit pas être confondue avec l'ondelette de Daubechies dont il était question précédemment) que de faibles hypothèses : ψ est une fonction arbitraire, à valeurs réelles, continue sur $\mathbb{R}$, à support compact non vide inclus dans l'intervalle $[0,1]$, de plus ψ possède au moins deux moments nuls, ce qui signifie que

$$\int_{\mathbb{R}} \psi(s)ds = \int_{\mathbb{R}} s\psi(s)ds = 0\,;$$

il est à noter qu'il n'y a pas besoin que $\left\{2^{j/2}\psi(2^j\cdot -k) : (j,k)\in\mathbb{Z}^2\right\}$ soit une base orthonormale de $L^2(\mathbb{R})$.

Nous allons maintenant présenter la procédure qui permet l'estimation de $\min_{t\in I} H(t)$ et de $H(t_0)$, où $I\subset[0,1]$ est un intervalle arbitraire compact d'intérieur non vide, et $t_0\in[0,1]$ est un point arbitraire et fixé.

On désigne par $(I_j)_{j\in\mathbb{Z}_+}$ une suite arbitraire d'intervalles compacts de $[0,1]$ dont les diamètres $|I_j|$ vérifient pour tout entier j, assez grand, l'inégalité,

$$|I_j| \geq 2^{-j/2}.$$

Signalons que,

(a) lorsqu'on cherche à estimer $H(t_0)$, $(I_j)_{j\in\mathbb{Z}_+}$ est alors typiquement une suite décroissante au sens de l'inclusion et qui « converge » vers $\{t_0\}$ c'est à dire que

$$\{t_0\} = \bigcap_{j\in\mathbb{Z}_+} I_j\ ;$$

(b) lorsqu'on cherche à estimer $\min_{t\in I} H(t)$, $(I_j)_{j\in\mathbb{Z}_+}$ est alors typiquement la suite constante qui vaut I.

En toute généralité, à chacun de ces intervalles I_j est associé un ensemble d'indices k, noté par ν_j et défini par,

$$\nu_j := \left\{k\in\{0,\ldots,2^j-1\} : 2^{-j}k \in I_j\right\}.$$

Soit $\beta\in[0,\alpha/4[$ un réel arbitraire et fixé ; signalons que lorsqu'on ne connait pas α on choisit alors β dans l'intervalle $]0,1/4]$. Désignons par V_j la moyenne empirique définie par,

$$V_j := \frac{1}{\mathrm{card}(\nu_j)} \sum_{k\in\nu_j} |d_{j,k}|^\beta.$$

Le principal résultat du chapitre 4 est, que l'on a, presque sûrement,

$$\left|\frac{\log_2(V_j)}{-j\beta} - \min_{t\in I_j} H(t)\right| \xrightarrow[j\to+\infty]{p.s.} 0. \tag{0.17}$$

Ce résultat repose essentiellement sur le fait que,

$$\frac{V_j}{\mathbb{E}(V_j)} \xrightarrow[j\to+\infty]{p.s.} 1. \tag{0.18}$$

Les trois principaux ingrédients de la preuve de (0.18) sont : l'inégalité de Markov, le lemme de Borel-Cantelli, et un résultat, issu de [24], qui donne une « sympathique » estimation de $\left|\mathrm{cov}(|\xi_1|^\beta, |\xi_2|^\beta)\right|$, où ξ_1 et ξ_2 sont deux variables aléatoires $\mathcal{S}\alpha\mathcal{S}$ vérifiant certaines conditions.

Signalons enfin qu'il résulte de (0.17) et de la continuité de $H(\cdot)$ lorsqu'on se place dans la situation (a) (voir ci-dessus), la statistique

$$\frac{\log_2(V_j)}{-j\beta},$$

fournit un estimateur fortement consistant de $H(t_0)$, et lorsqu'on se place dans la situation (b) cette statistique fournit, cette fois, un estimateur fortement consistant de $\min_{t\in I} H(t)$.

Résumons enfin le contenu du chapitre 5. L'objectif de ce chapitre est de construire un estimateur par ondelettes fortement consistant du paramètre de stabilité α du mmsl $\{Y(t) : t \in [0,1]\}$. On impose au paramètre fonctionnel $H(\cdot)$ de processus et à l'ondelette analysante ψ de vérifier les hypothèses du chapitre précédent ; de plus, tout comme dans le chapitre précédent le coefficient d'ondelette $d_{j,k}$ de $\{Y(t) : t \in [0,1]\}$, est défini par (0.16). Posons,

$$D_j := \max_{0\le k<2^j} |d_{j,k}|.$$

À la vue des hypothèses faites sur ψ, notamment la compacité de son support et la nullité de son premier moment, on montre facilement, grâce au module de continuité global (0.12), que, l'on a presque sûrement, pour tout réel $\eta > 0$ arbitrairement petit,

$$\limsup_{j\to+\infty} \left\{ 2^{j(\min_{x\in[0,1]} H(x)-1/\alpha-\eta)} D_j \right\} < \infty. \tag{0.19}$$

À l'inverse, compte tenu de la quasi-optimalité de ce module de continuité, il semble naturel de se demander si, l'on a, presque sûrement,

$$\limsup_{j\to+\infty} \left\{ 2^{j(\min_{x\in[0,1]} H(x)-1/\alpha+\eta)} D_j \right\} = \infty. \tag{0.20}$$

Il est à noter, que dans notre cadre, on n'est pas autorisé à faire appel à la caractérisation de la régularité de Hölder globale au moyen des coefficients d'ondelettes (voir [**20**, **21**, **12**]), pour établir (0.20) ; en effet pour pouvoir y faire appel, il aurait au moins fallu imposer à ψ d'être une fonction continûment dérivable et à $\left\{2^{j/2}\psi(2^j \cdot -k) : (j,k) \in \mathbb{Z}^2\right\}$ de former une base.

Afin de contourner cette difficulté, nous utilisons certaines propriétés spécifiques au mmsl, ainsi que la compacité du support de ψ. Nous montrons alors qu'un résultat plus fort que (0.20) est même vrai ; à savoir, on a presque sûrement, pour tout $\eta > 0$ arbitrairement petit,

$$\liminf_{j\to+\infty} \left\{ 2^{j(\min_{x\in[0,1]} H(x)-1/\alpha+\eta)} D_j \right\} = \infty. \tag{0.21}$$

Enfin, posons,

$$\frac{1}{\widehat{\alpha}_j} = \widehat{H}_j + \frac{\log_2(D_j)}{j},$$

où $\widehat{H}_j$ est l'estimateur fortement consistant de $\min_{x\in[0,1]} H(x)$ construit au chapitre 4 ; alors, en combinant (0.19) avec (0.21), on peut montrer que $\widehat{\alpha}_j$ est un estimateur fortement consistant de α.

Avant de clore cette introduction, signalons que l'estimation du paramètre α dans le cas du mfsl (rappelons que dans ce cas la fonction $H(\cdot)$ devient une constante) a fait l'objet de l'article [**2**], que nous avons décidé d'inclure tel quel dans le mémoire de cette thèse.

CHAPITRE 2

Représentation via des ondelettes de Daubechies et étude fine du comportement trajectoriel

1. Introduction

One of the most natural extension of the well-known Gaussian Fractional Brownian Motion (FBM, for brevity) to the setting of heavy-tailed stable distributions, is the so called Linear Fractional Stable Motion (LFSM, for brevity); we refer to [26, 31, 17] for a detailed presentation of the latter process, as well as other classical examples of stable processes, it is worth noticing that the path behavior of LFSM is much more complex than that of FBM. Stoev and Taqqu [28, 30] have introduced a generalization of LFSM, called Linear Multifractional Stable Motion (LMSM, for brevity); the latter denomination comes from the fact that the constant Hurst parameter of LFSM is replaced by a deterministic function depending on the time variable. According to [30], a LMSM model is a good candidate to adequately describe some features of traffic traces on telecommunication networks, typically changes in operating regimes and burstiness (the presence of rare but extremely busy periods of activity). Before ending this paragraph, we note in passing that, a Real Harmonisable Multifractional Stable Process, has been recently introduced in [15].

In order to precisely define LMSM, first, we need to fix some notations to be used throughout the article.

- Recall that heaviness of the tail of a stable distribution is governed by a parameter belonging to $(0,2)$, usually denoted by α; the smaller α is the more heavy is the tail. In this article, we always assume that $\alpha \in (1,2)$, since it has been shown in [28], that the latter assumption is actually a necessary condition for the paths of LMSM to be, with probability 1, continuous functions.
- $H(\cdot)$ denotes an arbitrary deterministic continuous function defined on the real line and with values in a compact interval $[\underline{H},\overline{H}] \subset (1/\alpha,1)$; similarly to the Hurst parameter of LFSM, this function will be an essential parameter for LMSM.
- $Z_\alpha(ds)$ is an independently scattered strictly α-stable ($\mathcal{S}t\alpha\mathcal{S}$) random measure on $\mathbb{R}$, with Lebesgue measure as its control measure and an arbitrary Borel function $\beta(\cdot):\mathbb{R}\to[-1,1]$ as its skewness intensity. Many information on such random measures and the corresponding stochastic integrals can be found in the book [26].

LMSM's are generated by the $\mathcal{S}t\alpha\mathcal{S}$ random field $\widetilde{X}=\{\widetilde{X}(u,v):(u,v)\in\mathbb{R}\times(1/\alpha,1)\}$, defined for all (u,v) as the stochastic integral,

$$\widetilde{X}(u,v)=\int_{\mathbb{R}}\Big\{(u-s)_+^{v-1/\alpha}-(-s)_+^{v-1/\alpha}\Big\}Z_\alpha(ds), \tag{1.1}$$

where, for each real numbers x and κ,

$$(x)_+^{\kappa}:=\begin{cases} x^{\kappa}, \text{if } x\in(0,+\infty),\\ 0, \text{if } x\in(-\infty,0].\end{cases} \tag{1.2}$$

Actually, $\widetilde{Y} = \{\widetilde{Y}(t) : t \in \mathbb{R}\}$, the LMSM of functional Hurst parameter $H(\cdot)$, is defined for all $t \in \mathbb{R}$, as

$$\widetilde{Y}(t) = \widetilde{X}(t, H(t)). \tag{1.3}$$

Observe that, assuming $\beta(\cdot)$ to be a constant, then for each fixed $v \in (1/\alpha, 1)$, the process $\widetilde{X}(\cdot, v) := \{\widetilde{X}(u, v) : u \in \mathbb{R}\}$ is the usual LFSM of Hurst parameter v; therefore LMSM reduces to LFSM when one also assumes $H(\cdot)$ to be a constant.

In this article, we construct via Daubechies wavelets, a "nice" modification of the field $\{\widetilde{X}(u, v) : (u, v) \in \mathbb{R} \times (1/\alpha, 1)\}$ denoted by $\{X(u, v) : (u, v) \in \mathbb{R} \times (1/\alpha, 1)\}$ (see Theorem 2.1), also, we denote by $\{Y(t) : t \in \mathbb{R}\}$ the modification of LMSM defined for each $t \in \mathbb{R}$, as $Y(t) := X(t, H(t))$. Our main goal is to make a comprehensive study of the local and asymptotic behavior of $\{Y(t) : t \in \mathbb{R}\}$, to this end one needs to derive fine path properties of $\{X(u, v) : (u, v) \in \mathbb{R} \times (1/\alpha, 1)\}$, this is done by using wavelet methods, reminiscent of those in [5].

The remaining of the paper is structured in the following way. Section 2 is devoted to the construction of $\{X(u, v) : (u, v) \in \mathbb{R} \times (1/\alpha, 1)\}$ which is obtained as a random series of functions, resulting from the decomposition of the kernel in (1.1) into a Daubechies wavelet basis. In Section 3, we show that this random series and all its term by term pathwise partial derivatives of any order with respect to v, are convergent in a strong sense: with probability 1, in the space $\mathcal{E}_\gamma(a, b, M) := \mathcal{C}^1\big([a, b], \mathcal{C}^\gamma([-M, M], \mathbb{R})\big)$, where the real numbers $M > 0$, $0 < 1/\alpha < a < b < 1$ and $0 \leq \gamma < a - 1/\alpha$ are arbitrary and fixed, and where $\mathcal{C}^\lambda(I, \mathbb{B})$ denotes the space of the λ-Hölder functions defined on an interval I and with values in a Banach space $\mathbb{B}$; notice that an important consequence of the latter result is that, for each $q \in \mathbb{Z}_+$, a typical path of the field $\{(\partial_v^q X)(u, v) : (u, v) \in \mathbb{R} \times (1/\alpha, 1)\}$ belongs to $\mathcal{E}_\gamma(a, b, M)$. In Section 4, fine path properties of the latter field, are derived thanks to its wavelet series representation; namely we determine a global modulus of continuity for it on the rectangle $[-M, M] \times [a, b]$, also we give, when $(u, v) \in \mathbb{R} \times [a, b]$, an upper bound for $\big|(\partial_v^q X)(u, v)\big|$. The latter two results are used in Section 5, in order to obtain global and local moduli of continuity for the LMSM $\{Y(t) : t \in \mathbb{R}\}$. The optimality of some of these moduli of continuity is discussed in Sections 6 and 7; under some Hölder conditions on $H(\cdot)$, it turns out that the global one is quasi-optimal (it provides, up to a logarithmic factor, a sharp estimate of the behavior of $\{Y(t) : t \in \mathbb{R}\}$, on an arbitrary fixed compact interval) and the local one is optimal (it provides, without any logarithmic gap, a sharp estimate of the behavior of $\{Y(t) : t \in \mathbb{R}\}$ on a neighborhood of an arbitrary fixed point). In Section 8, by making use of the quasi-optimality of the global modulus of continuity of LMSM, we show that its local Hölder exponent is almost surely, at a point t_0, equal to $H(t_0) - 1/\alpha$ (the exceptional negligible event on which the equality is not satisfied, does not depend on t_0). Finally, some technical lemmas as well as their proofs are given in Section 9 (the Appendix).

2. Wavelet series representation of the field generating LMSM's

Let $\widetilde{X} = \{\widetilde{X}(u, v) : (u, v) \in \mathbb{R} \times (1/\alpha, 1)\}$ be the $\mathcal{S}t\alpha\mathcal{S}$ stochastic field introduced in (1.1), the goal of this section is to construct a modification of $\widetilde{X}$, denoted by X, which is defined as a random wavelet series. We note in passing that, random wavelet series representations of LFSM and other self-similar stable fields with stationary increments, have been introduced in [16].

First, we need to fix some notations related to wavelets that will be extensively used throughout the article.

– The real-valued function ψ defined on the real line, denotes a 3 times continuously differentiable compactly supported Daubechies mother wavelet [12, 20, 21]; observe that ψ has $Q \geq 15$ vanishing moments i.e.:

$$\int_{\mathbb{R}} t^m \psi(t)dt = 0, \text{ for all } m = 0, \ldots, Q-1, \text{ and } \int_{\mathbb{R}} t^Q \psi(t)dt \neq 0. \tag{2.1}$$

The fact that ψ is a compactly supported function will play a crucial role; for the sake of convenience, we assume that R is a fixed real number strictly bigger than 1, such that

$$\operatorname{supp}\psi \subseteq [-R,R]. \tag{2.2}$$

– The real-valued function Ψ is defined for all $(x,v)\in\mathbb{R}\times(1/\alpha,1)$ as,

$$\Psi(x,v) := \int_{\mathbb{R}} (s)_+^{v-1/\alpha}\psi(x-s)ds = \int_{\mathbb{R}} (x-s)_+^{v-1/\alpha}\psi(s)ds; \tag{2.3}$$

recall that the definition of $(\cdot)_+^{v-1/\alpha}$ is given in (1.2). Denoting by Γ the usual Gamma function:

$$\Gamma(u) := \int_0^{+\infty} t^{u-1}\, e^{-t}dt, \text{ for all } u\in(0,+\infty),$$

and denoting, for each fixed v, by $\widehat{\Psi}(\cdot,v)$ the Fourier transform of the function $\Psi(\cdot,v)$:

$$\widehat{\Psi}(\xi,v) := \int_{\mathbb{R}} e^{-i\xi x}\Psi(x,v)dx, \text{ for all } \xi\in\mathbb{R},$$

one has,

$$\widehat{\Psi}(\xi,v) = \Gamma(v+1-1/\alpha)\frac{e^{-i\operatorname{sgn}(\xi)(v+1-1/\alpha)\frac{\pi}{2}}\widehat{\psi}(\xi)}{|\xi|^{v+1-1/\alpha}}, \text{ for all } \xi\in\mathbb{R}\setminus\{0\}; \tag{2.4}$$

the latter equality can be obtained by using a result in [25] concerning Fourier transforms of left-sided fractional derivatives.

– $\{\epsilon_{j,k} : (j,k)\in\mathbb{Z}^2\}$ is the sequence of the real-valued $\mathcal{S}t\alpha\mathcal{S}$ random variables defined as,

$$\epsilon_{j,k} := 2^{j/\alpha}\int_{\mathbb{R}} \psi(2^j s-k)Z_\alpha(ds). \tag{2.5}$$

Now we are in position to state the main result of this section.

Theorem 2.1. *Let Ψ be the function defined in (2.3), let $\{\epsilon_{j,k} : (j,k)\in\mathbb{Z}^2\}$ be the sequence of the real-valued $\mathcal{S}t\alpha\mathcal{S}$ random variables defined in (2.5), and let Ω_0^* be the event of probability 1 introduced in Lemma 2.1 below. The following two results hold.*

(i) For all fixed $\omega\in\Omega_0^*$ and $(u,v)\in\mathbb{R}\times(1/\alpha,1)$, one has

$$\sum_{(j,k)\in\mathbb{Z}^2} 2^{-jv}\big|\epsilon_{j,k}(\omega)\big|\big|\Psi(2^ju-k,v)-\Psi(-k,v)\big| < \infty. \tag{2.6}$$

Therefore, the series of real numbers:

$$\sum_{(j,k)\in\mathbb{Z}^2} 2^{-jv}\epsilon_{j,k}(\omega)\big(\Psi(2^ju-k,v)-\Psi(-k,v)\big), \tag{2.7}$$

converges to a finite limit which does not depend on the way the terms of the series are ordered; this limit is denoted by $X(u,v,\omega)$. Moreover for each $\omega\notin\Omega_0^*$ and every $(u,v)\in\mathbb{R}\times(1/\alpha,1)$, one sets $X(u,v,\omega)=0$.

(ii) The field $\{X(u,v) : (u,v)\in\mathbb{R}\times(1/\alpha,1)\}$ is a modification of the $\mathcal{S}t\alpha\mathcal{S}$ field $\{\widetilde{X}(u,v) : (u,v)\in\mathbb{R}\times(1/\alpha,1)\}$ defined in (1.1).

In order to prove Theorem 2.1, we need some preliminary results.

Remark 2.1. *(i) $\|\epsilon_{j,k}\|_\alpha$, the scale parameter of $\epsilon_{j,k}$, does not depend on (j,k), since classical computations, allow to show that,*

$$\|\epsilon_{j,k}\|_\alpha = \|\epsilon_{0,0}\|_\alpha = \left\{\int_{\mathbb{R}} |\psi(t)|^\alpha dt\right\}^{1/\alpha}. \tag{2.8}$$

(ii) The skewness parameter of $\epsilon_{j,k}$, is denoted by $\beta_{j,k}$ and is given by,

$$\beta_{j,k} = \|\epsilon_{0,0}\|_\alpha^{-\alpha} \int_{\mathbb{R}} \psi^{<\alpha>}(x)\beta(2^{-j}x + 2^{-j}k)dx,$$

where $z^{<\alpha>} := |z|^\alpha sgn(z)$ for all $z \in \mathbb{R}$, and where $\beta(\cdot)$ is the skewness intensity function of the $\mathcal{S}t\alpha\mathcal{S}$ measure $Z_\alpha(ds)$; notice that, when the latter function is a constant, then the random variables $\epsilon_{j,k}$ become identically distributed, since, not only they have the same scale parameter, but also the same skewness parameter.

(iii) Property 1.2.15 on page 16 in [26], as well as the fact that $\|\epsilon_{j,k}\|_\alpha$ does not vanish and does not depend on (j,k), imply that there exist two constants $0 < c' \le c''$ non depending on (j,k), such that, one has for all real number $x \ge 1$,

$$c'x^{-\alpha} \le \mathbb{P}\big(|\epsilon_{j,k}| > x\big) \le c''x^{-\alpha}. \tag{2.9}$$

(iv) In view of (2.5), (2.2) and the fact that $Z_\alpha(ds)$ is independently scattered, for each fixed integers $p > 2R$ and $j \in \mathbb{Z}$, one has that $\{\epsilon_{j,pq} : q \in \mathbb{Z}\}$ is a sequence of independent random variables.

The following lemma, which has been derived in [5], gives rather sharp estimates of the asymptotic behavior of the sequence $\big\{|\epsilon_{j,k}| : (j,k) \in \mathbb{Z}^2\big\}$. It can be proved by showing that for every fixed real number $\eta > 0$, one has,

$$\mathbb{E}\left(\sum_{(j,k)\in\mathbb{Z}^2} \mathbb{1}_{\left\{|\epsilon_{j,k}|>(1+|j|)^{1/\alpha+\eta}(1+|k|)^{1/\alpha}\log^{1/\alpha+\eta}(2+|k|)\right\}}\right) < \infty;$$

the latter fact, easily results from the second inequality in (2.9).

Lemma 2.1. [5] *There exists an event of probability 1, denoted by Ω_0^*, such that for every fixed real number $\eta > 0$, one has, for all $\omega \in \Omega_0^*$ and for each $(j,k) \in \mathbb{Z}^2$,*

$$\big|\epsilon_{j,k}(\omega)\big| \le C(\omega)\big(1+|j|\big)^{1/\alpha+\eta}\big(1+|k|\big)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+|k|\big) \le C'(\omega)\big(3+|j|\big)^{1/\alpha+\eta}\big(3+|k|\big)^{1/\alpha+\eta}, \tag{2.10}$$

where C and C' are two positive and finite random variables only depending on η.

The following proposition, which shows that the function Ψ and its partial derivatives of any order, have nice smoothness and localization properties, will also play an important role throughout our article.

Proposition 2.1. *The function Ψ satisfies the following two properties.*

(i) For all $(p,q) \in \{0,1,2,3\} \times \mathbb{Z}_+$ and $(x,v) \in \mathbb{R} \times (1/\alpha, 1)$, the partial derivative $(\partial_x^p\partial_v^q\Psi)(x,v)$ exists and is given by,

$$(\partial_x^p\partial_v^q\Psi)(x,v) = \int_{\mathbb{R}} (s)_+^{v-1/\alpha}\log^q((s)_+)\psi^{(p)}(x-s)ds = \int_{\mathbb{R}} (x-s)_+^{v-1/\alpha}\log^q((x-s)_+)\psi^{(p)}(s)ds, \tag{2.11}$$

where $\psi^{(p)}$ is the derivative of ψ of order p and $0\log^q(0) := 0$. Moreover the function $\partial_x^p\partial_v^q\Psi$ is continuous on $\mathbb{R} \times (1/\alpha, 1)$.

(ii) For each $(p,q) \in \{0,1,2,3\} \times \mathbb{Z}_+$ and for every real numbers a, b satisfying $1 > b > a > 1/\alpha$, the function $\partial_x^p\partial_v^q\Psi$ is well-localized in the variable x uniformly in the variable $v \in [a,b]$; namely one has

$$\sup_{(x,v)\in\mathbb{R}\times[a,b]} (3+|x|)^2\big|(\partial_x^p\partial_v^q\Psi)(x,v)\big| < \infty. \tag{2.12}$$

Proof of Proposition 2.1. Let us first show that Part (i) holds. In view of (2.3), the function Ψ can be expressed, for all $(x,v)\in\mathbb{R}\times(1/\alpha,1)$ as,

$$\Psi(x,v)=\int_{\mathbb{R}} L(x,v,s)\,ds. \tag{2.13}$$

where $L(x,v,s):=(s)_+^{v-1/\alpha}\psi(x-s)$. Also observe that for all $(p,q)\in\{0,\ldots,3\}\times\mathbb{Z}_+$ and $(x,v,s)\in\mathbb{R}\times(1/\alpha,1)\times\mathbb{R}$, the partial derivative $(\partial_x^p\partial_v^q L)(x,v,s)$ exists and is given by

$$(\partial_x^p\partial_v^q L)(x,v,s)=(s)_+^{v-1/\alpha}\log^q((s)_+)\psi^{(p)}(x-s). \tag{2.14}$$

Therefore, to show that the partial derivative $(\partial_x^p\partial_v^q\Psi)(x,v)$ exists and is given by (2.11), it is sufficient to prove that for all real numbers M, a and b, satisfying

$$M>0 \text{ and } 1/\alpha<a<b<1, \tag{2.15}$$

one has

$$\int_{\mathbb{R}}\sup_{(x,v)\in[-M,M]\times[a,b]}\big|(\partial_x^p\partial_v^q L)(x,v,s)\big|\,ds<\infty. \tag{2.16}$$

This is true, since Relations (2.14), (2.2) and (2.15), imply that

$$\int_{\mathbb{R}}\sup_{(x,v)\in[-M,M]\times[a,b]}\big|(\partial_x^p\partial_v^q L)(x,v,s)\big|ds\le\|\psi^{(p)}\|_{L^\infty(\mathbb{R})}\int_{-M-R}^{M+R}\Big((s)_+^{a-1/\alpha}+(s)_+^{b-1/\alpha}\Big)\big|\log((s)_+)\big|^q ds<\infty.$$

Finally, observe that it follows from (2.11), (2.14), (2.16) and the dominated convergence Therorem, that for all $(p,q)\in\{0,\ldots,3\}\times\mathbb{Z}_+$, the function $\partial_x^p\partial_v^q\Psi$ is continuous over $\mathbb{R}\times(1/\alpha,1)$.

Let us show that Part (ii) of the proposition holds. Relations (2.2) and (2.11), imply that for all $(p,q)\in\{0,\ldots,3\}\times\mathbb{Z}_+$ and for each $(x,v)\in(-\infty,-R)\times(1/\alpha,1)$, one has

$$(\partial_x^p\partial_v^q\Psi)(x,v)=0. \tag{2.17}$$

Combining (2.17) with the fact $\partial_x^p\partial_v^q\Psi$ is a continuous function over the compact set $[-R,2R]\times[a,b]$, it follows that

$$\sup_{(x,v)\in(-\infty,2R]\times[a,b]}(3+|x|)^2\big|(\partial_x^p\partial_v^q\Psi)(x,v)\big|<\infty.$$

Therefore, it remains to show that

$$\sup_{(x,v)\in(2R,+\infty)\times[a,b]}(3+x)^2\big|(\partial_x^p\partial_v^q\Psi)(x,v)\big|<\infty. \tag{2.18}$$

In view of (2.11) and (2.2), one has for each $(x,v)\in(2R,+\infty)\times[a,b]$,

$$(\partial_x^p\partial_v^q\Psi)(x,v)=\int_{-R}^{R}K_q(x,v,s)\psi^{(p)}(s)\,ds,$$

where $K_q(x,v,s):=(x-s)^{v-1/\alpha}\log^q(x-s)$. For each $l\in\{1,2,3\}$ and real number s, one sets $\psi^{(p-l)}(s)=\int_{-\infty}^{s}\psi^{(p+1-l)}(t)\,dt$; observe that, in view of (2.1) and (2.2), the supports of the latter three functions are included in $[-R,R]$. Thus integrating three times by parts, one gets that,

$$(\partial_x^p\partial_v^q\Psi)(x,v)=-\int_{-R}^{R}(\partial_s^3K_q)(x,v,s)\psi^{(p-3)}(s)\,ds. \tag{2.19}$$

Next standard computations, allow to show that there is a constant $c_{q,\alpha}>0$, only depending on q and α, such that for all $(x,v,s)\in(2R,+\infty)\times[a,b]\times[-R,R]$, one has,

$$\big|(\partial_s^3K_q)(x,v,s)\big|\le c_{q,\alpha}(x-s)^{-2}\le 4c_{q,\alpha}x^{-2}. \tag{2.20}$$

Finally, putting together (2.19) and (2.20), one obtains (2.18). $\square$

Now we are in position to prove Theorem 2.1.

Proof of Theorem 2.1 Part (i). Let $\omega \in \Omega_0^*$ and $(u,v) \in \mathbb{R} \times (1/\alpha, 1)$ be arbitrary and fixed. Also, we assume that η is an arbitrarily small fixed positive real number. By using the triangle inequality, (2.12) (in which one takes $p=q=0$ and a,b such that $v \in [a,b]$), and (2.10), it follows that for all fixed $j \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \big|\epsilon_{j,k}(\omega)\big|\big|\Psi(2^j u-k,v)-\Psi(-k,v)\big| \\
&\le C_1(\omega)\big(3+j\big)^{1/\alpha+\eta}\sum_{k\in\mathbb{Z}}\left(\frac{\big(3+|k|\big)^{1/\alpha+\eta}}{\big(3+|2^j u-k|\big)^2}+\frac{\big(3+|k|\big)^{1/\alpha+\eta}}{\big(3+|k|\big)^2}\right) \\
&\le C_2(\omega)\big(3+j\big)^{1/\alpha+\eta}\big(3+2^j|u|\big)^{1/\alpha+\eta}\sum_{k\in\mathbb{Z}}\left(\frac{\big(3+|k|\big)^{1/\alpha+\eta}}{\big(3+|2^j u-[2^j u]-k|\big)^2}+\frac{\big(3+|k|\big)^{1/\alpha+\eta}}{\big(3+|k|\big)^2}\right),
\end{aligned} \tag{2.21}$$

where $[2^j u]$ denotes the integer part of $2^j u$ and where $C_1(\omega)$ and $C_2(\omega)$ are two finite constants non depending on j and u. Then, noticing that,

$$\sup_{x\in[0,1]}\left\{\sum_{k\in\mathbb{Z}}\frac{\big(3+|k|\big)^{1/\alpha+\eta}}{\big(3+|x-k|\big)^2}\right\} \le \sum_{k\in\mathbb{Z}}\frac{\big(3+|k|\big)^{1/\alpha+\eta}}{\big(2+|k|\big)^2} < \infty, \tag{2.22}$$

it follows from (2.21) that

$$\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} 2^{-jv}\big|\epsilon_{j,k}(\omega)\big|\big|\Psi(2^j u-k,v)-\Psi(-k,v)\big| < \infty. \tag{2.23}$$

Let us now prove that,

$$\sum_{j\in\mathbb{Z}_-}\sum_{k\in\mathbb{Z}} 2^{-jv}\big|\epsilon_{j,k}(\omega)\big|\big|\Psi(2^j u-k,v)-\Psi(-k,v)\big| < \infty. \tag{2.24}$$

Applying the Mean Value Theorem, one has for all $(j,k) \in \mathbb{Z}_- \times \mathbb{Z}$,

$$\Psi(2^j u-k,v)-\Psi(-k,v) = 2^j u(\partial_x\Psi)(\nu-k,v), \tag{2.25}$$

where $\nu \in [-2^j|u|, 2^j|u|] \subseteq [-|u|,|u|]$. Then putting together (2.25), (2.10) and (2.12) (in which one takes $p=1$, $q=0$ and a,b such that $v \in [a,b]$), one obtains that,

$$\begin{aligned}
&\sum_{|k|\le|u|} \big|\epsilon_{j,k}(\omega)\big|\big|\Psi(2^j u-k,v)-\Psi(-k,v)\big| \\
&\le C_3(\omega)|u|\big(2|u|+1\big)\big(3+|u|\big)^{1/\alpha+\eta}\left(\sup_{x\in\mathbb{R}}\big|(\partial_x\Psi)(x,v)\big|\right)2^j\big(3+|j|\big)^{1/\alpha+\eta}
\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}
&\sum_{|k|>|u|} \big|\epsilon_{j,k}(\omega)\big|\big|\Psi(2^j u-k,v)-\Psi(-k,v)\big| \\
&\le C_4(\omega)|u|\big(3+|u|\big)^{1/\alpha+\eta}\left(\sum_{k\in\mathbb{Z}}\big(3+|k|\big)^{1/\alpha+\eta-2}\right)2^j\big(3+|j|\big)^{1/\alpha+\eta},
\end{aligned} \tag{2.27}$$

where $C_3(\omega)$ and $C_4(\omega)$ are two positive finite constants non depending on j and u. Next combining (2.26) and (2.27), with the fact that $v \in (1/\alpha, 1)$, one gets (2.24). Finally (2.23) and (2.24) show that (2.6) holds. $\square$

Proof of Theorem 2.1 Part (ii). For all $(j,k)\in\mathbb{Z}^2$ and any $s\in\mathbb{R}$, we set

$$\psi_{j,k}(s)=2^{j/\alpha}\psi(2^js-k), \tag{2.28}$$

where ψ is the Daubechies mother wavelet introduced at the very beginning of this section; observe that the sequence $\{\psi_{j,k}:(j,k)\in\mathbb{Z}^2\}$ forms an unconditional basis of $L^\alpha(\mathbb{R})$ and the sequence $\{2^{j(1/2-1/\alpha)}\psi_{j,k}:(j,k)\in\mathbb{Z}^2\}$ is an orthonormal basis of $L^2(\mathbb{R})$ (see [20, 21]). Therefore, noticing that for any fixed $(u,v)\in\mathbb{R}\times(1/\alpha,1)$, the function $s\mapsto(u-s)_+^{v-1/\alpha}-(-s)_+^{v-1/\alpha}$ belongs to $L^\alpha(\mathbb{R})\cap L^2(\mathbb{R})$, it follows that,

$$(u-s)_+^{v-1/\alpha}-(-s)_+^{v-1/\alpha}=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}w_{j,k}(u,v)\psi_{j,k}(s), \tag{2.29}$$

where

$$\begin{aligned} w_{j,k}(u,v)&:=2^{j(1-1/\alpha)}\int_{\mathbb{R}}\big\{(u-s)_+^{v-1/\alpha}-(-s)_+^{v-1/\alpha}\big\}\psi(2^js-k)ds\\ &=2^{-jv}\big\{\Psi(2^ju-k,v)-\Psi(-k,v)\big\}, \end{aligned} \tag{2.30}$$

and where the convergence of the series, as a function of s, holds in $L^\alpha(\mathbb{R})$ as well as in $L^2(\mathbb{R})$; observe that the limit of the series does not depend on the way its terms are ordered. Next, using (2.29), (2.30), (1.1), a classical property of the stochastic integral $\int_{\mathbb{R}}(\,\cdot\,)Z_\alpha(ds)$, (2.28) and (2.5), we get that the random series

$$\sum_{(j,k)\in\mathbb{Z}^2}2^{-jv}\epsilon_{j,k}\big(\Psi(2^ju-k,v)-\Psi(-k,v)\big),$$

converges in probability to the random variable $\widetilde{X}(u,v)$; observe that the terms of the latter series can be ordered in an arbitrary way. Finally, combining the latter result with Part (i) of Theorem 2.1, we obtain that the random variables $\widetilde{X}(u,v)$ and $X(u,v)$ are equal almost surely. □

3. Convergence of the wavelet series in Hölder spaces

The goal of this section is to show that when the terms of the series in (2.7), viewed as a random series of functions of the variable (u,v), are ordered in an appropriate way, then not only this series converges almost surely for every fixed $(u,v)\in\mathbb{R}\times(1/\alpha,1)$, but also, it is, as well as all its term by term pathwise partial derivatives of any order with respect to v, almost surely convergent in some Hölder spaces. Let us first precisely define these spaces.

Definition 3.1. *Let $(\mathbb{B},\|\cdot\|)$ be a Banach space and $\mathcal{K}$ a subset of $\mathbb{R}$. For every $\gamma\in[0,1]$, the Banach space of γ-Hölder functions from $\mathcal{K}$ to $\mathbb{B}$, is denoted by $\mathcal{C}^\gamma(\mathcal{K},\mathbb{B})$ and defined as,*

$$\mathcal{C}^\gamma(\mathcal{K},\mathbb{B}):=\Big\{f:\mathcal{K}\to\mathbb{B}:\mathcal{N}_\gamma(f)<\infty\Big\},$$

where

$$\mathcal{N}_\gamma(f):=\sup_{x\in\mathcal{K}}\|f(x)\|+\sup_{x,y\in\mathcal{K}}\frac{\|f(x)-f(y)\|}{|x-y|^\gamma},$$

is the natural norm on this space. Notice that in the definition of $\mathcal{N}_\gamma(f)$, we assume that $0/0=0$. Also notice that $\mathcal{C}^1(\mathcal{K},\mathbb{B})$ is usually called the space of the Lipschitz functions from $\mathcal{K}$ to $\mathbb{B}$.

Definition 3.2. *Let γ, M, a and b be arbitrary and fixed real numbers satisfying $\gamma\in[0,1]$, $M>0$ and $a<b$. We denote by $\mathcal{E}_\gamma(a,b,M)$, the Banach space*

$$\mathcal{E}_\gamma(a,b,M):=\mathcal{C}^1\big([a,b],\mathcal{C}^\gamma([-M,M],\mathbb{R})\big),$$

of the Lipschitz functions defined on $[a,b]$ and with values in the Hölder space $\mathcal{C}^\gamma([-M,M],\mathbb{R})$. Observe that each function f belonging to $\mathcal{E}_\gamma(a,b,M)$, can be viewed as a bivariate real-valued function

$(u,v) \mapsto f(u,v) := \big(f(v)\big)(u)$ *on the rectangle* $[-M,M]\times[a,b]$*; moreover, the natural norm on* $\mathcal{E}_\gamma(a,b,M)$*, is equivalent to the norm* $|||\,.\,|||$ *defined as,*

$$\begin{aligned}|||f||| := \sup_{(u,v)\in[-M,M]\times[a,b]} |f(u,v)| &+ \sup_{(u_1,u_2,v)\in[-M,M]^2\times[a,b]} \frac{\big|(\Delta^{1,\cdot}_{u_1-u_2} f)(u_2,v)\big|}{|u_1-u_2|^\gamma}\\ &+ \sup_{(u,v_1,v_2)\in[-M,M]\times[a,b]^2} \frac{\big|(\Delta^{\cdot,1}_{v_1-v_2} f)(u,v_2)\big|}{|v_1-v_2|}\\ &+ \sup_{(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2} \frac{\big|(\Delta^{1,1}_{(u_1-u_2,v_1-v_2)} f)(u_2,v_2)\big|}{|u_1-u_2|^\gamma|v_1-v_2|},\end{aligned} \tag{3.1}$$

where,

$$\begin{aligned}&(\Delta^{1,\cdot}_{u_1-u_2} f)(u_2,v) := f(u_1,v)-f(u_2,v),\\ &(\Delta^{\cdot,1}_{v_1-v_2} f)(u,v_2) := f(u,v_1)-f(u,v_2),\\ &(\Delta^{1,1}_{(u_1-u_2,v_1-v_2)} f)(u_2,v_2) := f(u_1,v_1)-f(u_1,v_2)-f(u_2,v_1)+f(u_2,v_2).\end{aligned} \tag{3.2}$$

Notice that in (3.1), we assume that $0/0=0$.

Now we are in position to state the main result of this section.

Theorem 3.1. *We use the same notations as in Theorem 2.1. The following two results hold for all* $\omega\in\Omega_0^*$*, the event of probability* 1 *introduced in Lemma 2.1.*

(i) For each fixed $u\in\mathbb{R}$*, the function* $X(u,\cdot,\omega): v\mapsto X(u,v,\omega)$ *is infinitely differentiable over* $(1/\alpha,1)$*; its derivative of any order* $q\in\mathbb{Z}_+$ *at all* $v\in(1/\alpha,1)$*, is given by*

$$(\partial_v^q X)(u,v,\omega) = \sum_{p=0}^{q}\binom{q}{p}\big(-\log 2\big)^p \sum_{(j,k)\in\mathbb{Z}^2} j^p 2^{-jv}\epsilon_{j,k}(\omega)\Big(\big(\partial_v^{q-p}\Psi\big)(2^j u-k,v)-\big(\partial_v^{q-p}\Psi\big)(-k,v)\Big), \tag{3.3}$$

where, $0^0:=1$*, for every fixed* (u,v) *the series is absolutely convergent (its terms can therefore be ordered in an arbitrary way), and* $\binom{q}{p}$ *denotes the binomial coefficient* $\frac{q!}{p!(q-p)!}$.

(ii) For each fixed $q\in\mathbb{Z}_+$ *and* $M,a,b\in\mathbb{R}$ *satisfying* $M>0$ *and* $1/\alpha<a<b<1$*, the function* $(\partial_v^q X)(\cdot,\cdot,\omega):(u,v)\mapsto(\partial_v^q X)(u,v,\omega)$ *belongs to the space* $\mathcal{E}_\gamma(a,b,M)$ *for all* $\gamma\in[0,a-1/\alpha)$.

The proof of Theorem 3.1 mainly relies on the following proposition.

Proposition 3.1. *Let* M *be an arbitrary and fixed positive real number. For every* $n\in\mathbb{Z}_+$*, denote by* $X_{M,n}=\big\{X_{M,n}(u,v)\,:\,(u,v)\in\mathbb{R}\times(1/\alpha,1)\big\}$ *the* $\mathcal{S}t\alpha\mathcal{S}$ *random field defined for every* $(u,v)\in\mathbb{R}\times(1/\alpha,1)$*, as the finite sum,*

$$X_{M,n}(u,v) = \sum_{(j,k)\in D_{M,n}} 2^{-jv}\epsilon_{j,k}\big(\Psi(2^j u-k,v)-\Psi(-k,v)\big), \tag{3.4}$$

where

$$D_{M,n} := \big\{(j,k)\in\mathbb{Z}^2 : |j|\le n \text{ and } |k|\le M2^{n+1}\big\}. \tag{3.5}$$

Then, the following three results hold.

(i) For all fixed $\omega\in\Omega$ *(the underlying probability space) and* $u\in\mathbb{R}$*, the function* $X_{M,n}(u,\cdot,\omega):$ $v\mapsto X_{M,n}(u,v,\omega)$ *is infinitely differentiable over* $(1/\alpha,1)$*; its derivative of any order* $q\in\mathbb{Z}_+$ *at any point* $v\in(1/\alpha,1)$*, is denoted by* $(\partial_v^q X_{M,n})(u,v,\omega)$.

(ii) For all fixed $\omega\in\Omega$*,* $q,n\in\mathbb{Z}_+$ *and* $a,b\in\mathbb{R}$ *satisfying* $1/\alpha<a<b<1$*, the function* $(\partial_v^q X_{M,n})(\cdot,\cdot,\omega)$ *belongs to the Banach space* $\mathcal{E}_1(a,b,M)$.

(iii) For each fixed $\omega \in \Omega_0^*$, $q \in \mathbb{Z}_+$, *and* $a, b, \gamma \in \mathbb{R}$ *satisfying* $1/\alpha < a < b < 1$ *and* $0 \leq \gamma < a - 1/\alpha$, $\big((\partial_v^q X_{M,n})(\cdot,\cdot,\omega)\big)_{n\in\mathbb{Z}_+}$ *is a Cauchy sequence in the Banach space* $\mathcal{E}_\gamma(a,b,M)$.

Proof of Proposition 3.1. Parts (i) and (ii) of Proposition 3.1 are more or less straightforward consequences of Proposition 2.1. In view of Definition 3.2, Part (iii) of Proposition 3.1 results from the following four lemmas. □

Lemma 3.1. *Let* M, a *and* b *be fixed real numbers satisfying* $M > 0$ *and* $1/\alpha < a < b < 1$. *For all fixed* $q \in \mathbb{Z}_+$ *and* $\omega \in \Omega_0^*$, *when* n *goes to infinity,*

$$\Big|\big(\partial_v^q X_{M,n+l}\big)(u,v,\omega) - \big(\partial_v^q X_{M,n}\big)(u,v,\omega)\Big| \tag{3.6}$$

converges to 0, *uniformly in* $(u,v) \in [-M,M]\times[a,b]$ *and in* $l \in \mathbb{Z}_+$.

Lemma 3.2. *Let* M, a, b *and* γ *be fixed real numbers satisfying* $M > 0$, $1/\alpha < a < b < 1$ *and* $\gamma < a - 1/\alpha$. *For all fixed* $q \in \mathbb{Z}_+$ *and* $\omega \in \Omega_0^*$, *when* n *goes to infinity,*

$$\frac{\Big|\big(\Delta^{1,\cdot}_{u_1-u_2}\big(\partial_v^q X_{M,n+l}\big)\big)(u_2,v,\omega) - \big(\Delta^{1,\cdot}_{u_1-u_2}\big(\partial_v^q X_{M,n}\big)\big)(u_2,v,\omega)\Big|}{|u_1-u_2|^\gamma} \tag{3.7}$$

converges to 0 *uniformly in* $(u_1,u_2,v) \in [-M,M]^2\times[a,b]$ *and in* $l \in \mathbb{Z}_+$.

Lemma 3.3. *Let* M, a *and* b *be fixed real numbers satisfying* $M > 0$ *and* $1/\alpha < a < b < 1$. *For all fixed* $q \in \mathbb{Z}_+$ *and* $\omega \in \Omega_0^*$, *when* n *goes to infinity,*

$$\frac{\Big|\big(\Delta^{\cdot,1}_{v_1-v_2}\big(\partial_v^q X_{M,n+l}\big)\big)(u,v_2,\omega) - \big(\Delta^{\cdot,1}_{v_1-v_2}\big(\partial_v^q X_{M,n}\big)\big)(u,v_2,\omega)\Big|}{|v_1-v_2|} \tag{3.8}$$

converges to 0 *uniformly in* $(u,v_1,v_2) \in [-M,M]\times[a,b]^2$ *and in* $l \in \mathbb{Z}_+$.

Lemma 3.4. *Let* M, a, b *and* γ *be fixed real numbers satisfying* $M > 0$, $1/\alpha < a < b < 1$ *and* $\gamma < a - 1/\alpha$. *For all fixed* $q \in \mathbb{Z}_+$ *and* $\omega \in \Omega_0^*$, *when* n *goes to infinity,*

$$\frac{\Big|\Big(\Delta^{1,1}_{(u_1-u_2,v_1-v_2)}\big(\partial_v^q X_{M,n+l}\big)\Big)(u_2,v_2,\omega) - \Big(\Delta^{1,1}_{(u_1-u_2,v_1-v_2)}\big(\partial_v^q X_{M,n}\big)\Big)(u_2,v_2,\omega)\Big|}{|u_1-u_2|^\gamma|v_1-v_2|} \tag{3.9}$$

converges to 0 *uniformly in* $(u_1,u_2,v_1,v_2) \in [-M,M]^2\times[a,b]^2$ *and in* $l \in \mathbb{Z}_+$.

The proofs of the previous four lemmas are quite similar, so we will only give that of Lemma 3.4.

Proof of Lemma 3.4. In view of the convention that $0/0 = 0$, there is no restriction to assume that $u_1 \neq u_2$ and $v_1 \neq v_2$. By using (3.4), (3.2) and Leibniz formula, one can rewrite (3.9) as,

$$\frac{\Big|\sum_{p=0}^q \binom{q}{p}\big(-\log 2\big)^p \sum_{(j,k)\in D_{M,n+l}\setminus D_{M,n}} j^p \epsilon_{j,k}(\omega)\Big(\Delta^{1,1}_{(u_1-u_2,v_1-v_2)}\Theta^{q-p}_{j,k}\Big)(u_2,v_2)\Big|}{|u_1-u_2|^\gamma|v_1-v_2|}, \tag{3.10}$$

where for all $(u,v) \in \mathbb{R}\times(1/\alpha,1)$,

$$\Theta^{q-p}_{j,k}(u,v) = 2^{-jv}\big(\partial_v^{q-p}\Psi\big)(2^j u - k, v). \tag{3.11}$$

In the sequel, we denote by $D^c_{M,n}$ the set defined as $D^c_{M,n} = \big\{(j,k) \in \mathbb{Z}^2 : (j,k) \notin D_{M,n}\big\}$; recall that $D_{M,n}$ has been introduced in (3.5). Using (3.10), Taylor formula with respect to the variable v, (3.2), and the triangle inequality, one obtains that

$$\frac{\Big|\Big(\Delta^{1,1}_{(u_1-u_2,v_1-v_2)}\big(\partial_v^q X_{n+l}\big)\Big)(u_2,v_2,\omega)-\Big(\Delta^{1,1}_{(u_1-u_2,v_1-v_2)}\big(\partial_v^q X_n\big)\Big)(u_2,v_2,\omega)\Big|}{|u_1-u_2|^\gamma|v_1-v_2|}$$
$$\leq G^{1,q}_{M,n}(u_1,u_2,v_2,\omega)+|v_1-v_2|G^{2,q}_{M,n}(u_1,u_2,v_1,v_2,\omega),$$

where

$$G^{1,q}_{M,n}(u_1,u_2,v_2,\omega):=\sum_{p=0}^{q}\binom{q}{p}(\log 2)^p\sum_{(j,k)\in D^c_{M,n}}|j|^p|\epsilon_{j,k}(\omega)|\frac{\Big|\Big(\Delta^{1,\cdot}_{u_1-u_2}\big(\partial_v\Theta^{q-p}_{j,k}\big)\Big)(u_2,v_2)\Big|}{|u_1-u_2|^\gamma} \tag{3.12}$$

and

$$G^{2,q}_{M,n}(u_1,u_2,v_1,v_2,\omega)$$
$$:=\sum_{p=0}^{q}\binom{q}{p}(\log 2)^p\sum_{(j,k)\in D^c_{M,n}}|j|^p|\epsilon_{j,k}(\omega)|\frac{\Big|\int_0^1(1-s)\Big(\Delta^{1,\cdot}_{u_1-u_2}\big(\partial_v^2\Theta^{q-p}_{j,k}\big)\Big)(u_2,v_2+s(v_1-v_2))ds\Big|}{|u_1-u_2|^\gamma}. \tag{3.13}$$

Thus, for proving the lemma, it is sufficient to show that, when $n\to+\infty$, $G^{1,q}_{M,n}(u_1,u_2,v_2,\omega)$ and $G^{2,q}_{M,n}(u_1,u_2,v_1,v_2,\omega)$ converge to 0, uniformly in $(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2$.

First, let us study $G^{1,q}_{M,n}(u_1,u_2,v_2,\omega)$. It follows from (3.11), that,

$$(\partial_v\Theta^{q-p}_{j,k})(u,v)=2^{-jv}\big(\partial_v^{q+1-p}\Psi\big)(2^ju-k,v)-(\log 2)j2^{-jv}\big(\partial_v^{q-p}\Psi\big)(2^ju-k,v). \tag{3.14}$$

Next, putting together (3.14), (3.2), the triangle inequality, Lemma 2.1, (9.1) and (9.2), one has,

$$\sum_{(j,k)\in D^c_{M,n}}|j|^p|\epsilon_{j,k}(\omega)|\frac{\Big|\Big(\Delta^{1,\cdot}_{u_1-u_2}\big(\partial_v\Theta^{q-p}_{j,k}\big)\Big)(u_2,v_2)\Big|}{|u_1-u_2|^\gamma}$$
$$\leq\sum_{(j,k)\in D^c_{M,n}}2^{-jv_2}|j|^p|\epsilon_{j,k}(\omega)|\frac{\Big|\big(\partial_v^{q+1-p}\Psi\big)(2^ju_1-k,v_2)-\big(\partial_v^{q+1-p}\Psi\big)(2^ju_2-k,v_2)\Big|}{|u_1-u_2|^\gamma}$$
$$+(\log 2)\sum_{(j,k)\in D^c_{M,n}}2^{-jv_2}|j|^{p+1}|\epsilon_{j,k}(\omega)|\frac{\Big|\big(\partial_v^{q-p}\Psi\big)(2^ju_1-k,v_2)-\big(\partial_v^{q-p}\Psi\big)(2^ju_2-k,v_2)\Big|}{|u_1-u_2|^\gamma}$$
$$\leq C_1(\omega)\Big(A_n\big(u_1,u_2,v_2;M,\gamma,\eta,p,\partial_v^{q+1-p}\Psi\big)+B_n\big(u_1,u_2,v_2;M,\gamma,\eta,p,\partial_v^{q+1-p}\Psi\big)\Big)$$
$$+C_1(\omega)(\log 2)\Big(A_n\big(u_1,u_2,v_2;M,\gamma,\eta,p+1,\partial_v^{q-p}\Psi\big)+B_n\big(u_1,u_2,v_2;M,\gamma,\eta,p+1,\partial_v^{q-p}\Psi\big)\Big),$$

where C_1 denotes the random variable C' introduced in Lemma 2.1. Then Lemma 9.2 and (3.12) imply that, when $n\to+\infty$, $G^{1,q}_{M,n}(u_1,u_2,v_2,\omega)$ converges to 0, uniformly in $(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2$.

Let us now study $G^{2,q}_{M,n}(u_1,u_2,v_1,v_2,\omega)$. It follows from (3.14) that,

$$(\partial_v^2\Theta^{q-p}_{j,k})(u,v)=2^{-jv}\big(\partial_v^{q+2-p}\Psi\big)(2^ju-k,v)-2(\log 2)j2^{-jv}\big(\partial_v^{q+1-p}\Psi\big)(2^ju-k,v)$$
$$+(\log 2)^2j^22^{-jv}\big(\partial_v^{q-p}\Psi\big)(2^ju-k,v). \tag{3.15}$$

Next, putting together (3.15), (3.2), the triangle inequality, Lemma 2.1, (9.1) and (9.2), one has,

$$
\begin{aligned}
&\sum_{(j,k)\in D_{M,n}^c} |j|^p|\epsilon_{j,k}(\omega)| \frac{\Big|\int_0^1 (1-s)\Big(\Delta^{1,\cdot}_{u_1-u_2}\big(\partial_v^2\Theta_{j,k}^{q-p}\big)\Big)(u_2,v_2+s(v_1-v_2))ds\Big|}{|u_1-u_2|^\gamma} \\
&\le C_1(\omega)\int_0^1 \Big(A_n\big(u_1,u_2,v_2+s(v_1-v_2);M,\gamma,\eta,p,\partial_v^{q+2-p}\Psi\big)+ \\
&\qquad\qquad B_n\big(u_1,u_2,v_2+s(v_1-v_2);M,\gamma,\eta,p,\partial_v^{q+2-p}\Psi\big)\Big)ds \\
&+C_2(\omega)\int_0^1 \Big(A_n\big(u_1,u_2,v_2+s(v_1-v_2);M,\gamma,\eta,p+1,\partial_v^{q+1-p}\Psi\big)+ \\
&\qquad\qquad B_n\big(u_1,u_2,v_2+s(v_1-v_2);M,\gamma,\eta,p+1,\partial_v^{q+1-p}\big)\Big)ds \\
&+C_2(\omega)\int_0^1 \Big(A_n\big(u_1,u_2,v_2+s(v_1-v_2);M,\gamma,\eta,p+2,\partial_v^{q-p}\Psi\big)+ \\
&\qquad\qquad B_n\big(u_1,u_2,v_2+s(v_1-v_2);M,\gamma,\eta,p+2,\partial_v^{q-p}\Psi\big)\Big)ds,
\end{aligned}
$$

where $C_2(\omega) = (2\log 2)C_1(\omega)$. Then Lemma 9.2 and (3.13) imply that, when $n \to +\infty$, $G_{M,n}^{2,q}(u_1,u_2,v_1,v_2,\omega)$ converges to 0, uniformly in $(u_1,u_2,v_1,v_2) \in [-M,M]^2 \times [a,b]^2$. $\square$

Now we are in position to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\omega \in \Omega_0^*$ be arbitrary and fixed. First we show that Part (i) of the theorem holds. By using Lemma 2.1, Proposition 2.1 and a method similar to the one which allowed to derive (2.6), we can prove that, for all fixed $q \in \mathbb{N}$ and $(u,v) \in \mathbb{R}\times(1/\alpha,1)$, one has,

$$
\sum_{p=0}^{q}\binom{q}{p}\big(\log 2\big)^p \sum_{(j,k)\in\mathbb{Z}^2} |j|^p 2^{-jv}\big|\epsilon_{j,k}(\omega)\big|\,\big|\big(\partial_v^{q-p}\Psi\big)(2^j u-k,v)-\big(\partial_v^{q-p}\Psi\big)(-k,v)\big| < \infty.
$$

Therefore, the series of real numbers,

$$
\sum_{p=0}^{q}\binom{q}{p}\big(-\log 2\big)^p \sum_{(j,k)\in\mathbb{Z}^2} j^p 2^{-jv}\epsilon_{j,k}(\omega)\left(\big(\partial_v^{q-p}\Psi\big)(2^j u-k,v)-\big(\partial_v^{q-p}\Psi\big)(-k,v)\right),
$$

is convergent, and its finite limit, denoted by $\check{X}^{(q)}(u,v,\omega)$, does not depend on the way the terms of the series are ordered. Let us now assume that $u \in \mathbb{R}$ is arbitrary and fixed and that the variable v belongs to an arbitrary fixed compact interval $[a,b]$ contained in $(1/\alpha,1)$. We denote by M an arbitrary fixed positive real number such that $u \in [-M,M]$. In view of Theorem 2.1 Part (i), Proposition 3.1 Part (iii), and (3.1), when n goes to infinity, the following two results are satisfied:

- the function $v \mapsto X_{M,n}(u,v,\omega)$ converges to the function $v \mapsto X(u,v,\omega)$, uniformly in $v \in [a,b]$;
- for each fixed $q \in \mathbb{N}$, the function $v \mapsto (\partial_v^q X_{M,n})(u,v,\omega)$ converges to the function $v \mapsto \check{X}^{(q)}(u,v,\omega)$, uniformly in $v \in [a,b]$.

The latter two results imply that $v \mapsto X(u,v,\omega)$ is an infinitly differentiable function over $[a,b]$ and one has, for all $q \in \mathbb{N}$ and $v \in [a,b]$,

$$
(\partial_v^q X)(u,v,\omega) = \lim_{n\to+\infty}(\partial_v^q X_{M,n})(u,v,\omega) = \check{X}^{(q)}(u,v,\omega); \tag{3.16}
$$

these equalities mean that (3.3) is satisfied. Thus, it remains to show that Part (ii) of the theorem holds. In fact, the equality $X(u,v,\omega) = \lim_{n\to+\infty} X_{M,n}(u,v,\omega)$, (3.16), and Proposition 3.1 Part (iii), imply that this is indeed the case. $\square$

Before ending this section, let us stress that for each fixed $\omega \in \Omega_0^*$, $q \in \mathbb{Z}_+$ and $M, a, b \in \mathbb{R}$ satisfying $M > 0$ and $1/\alpha < a < b < 1$, Theorem 3.1 Part (ii), allows to derive, uniformly in $v \in [a,b]$, a global modulus of continuity of the function $u \mapsto (\partial_v^q X)(u,v,\omega)$, on the interval $[-M, M]$; also, it allows to derive, uniformly in $u \in [-M,M]$, a global modulus of continuity of the function $v \mapsto (\partial_v^q X)(u,v,\omega)$, on the interval $[a,b]$. More precisely, in view of Definition 3.2, a straightforward consequence of Theorem 3.1 Part (ii), is the following:

Corollary 3.1. *For each fixed $\omega \in \Omega_0^*$, $q \in \mathbb{Z}_+$ and $M, a, b, \eta \in \mathbb{R}$ satisfying $M > 0$, $1/\alpha < a < b < 1$ and $\eta > 0$, one has,*

$$\sup_{(u_1,u_2,v)\in[-M,M]^2\times[a,b]} \left\{ \frac{\big|\big(\partial_v^q X\big)(u_1,v,\omega) - \big(\partial_v^q X\big)(u_2,v,\omega)\big|}{|u_1-u_2|^{a-1/\alpha-\eta}} \right\} < \infty, \tag{3.17}$$

and

$$\sup_{(u,v_1,v_2)\in[-M,M]\times[a,b]^2} \left\{ \frac{\big|\big(\partial_v^q X\big)(u,v_1,\omega) - \big(\partial_v^q X\big)(u,v_2,\omega)\big|}{|v_1-v_2|} \right\} < \infty. \tag{3.18}$$

4. Fine path properties of the field generating LMSM's

The main two goals of this section are the following:

– to give an improved version of the global modulus of continuity (3.17);

– to derive, an upper bound of $\big|\big(\partial_v^q X\big)(u,v,\omega)\big|$, for all $\omega \in \Omega_0^*$, $q \in \mathbb{Z}_+$, $v \in [a,b] \subset (1/\alpha, 1)$ and $u \in \mathbb{R}$.

More precisely, we will show that the following two results hold.

Proposition 4.1. *For each fixed $\omega \in \Omega_0^*$, $q \in \mathbb{Z}_+$ and $M, a, b, \eta \in \mathbb{R}$ satisfying $M > 0$, $1/\alpha < a < b < 1$ and $\eta > 0$, one has,*

$$\begin{aligned} &\sup_{(u_1,u_2,v)\in[-M,M]^2\times[a,b]} \left\{ \frac{\big|\big(\partial_v^q X\big)(u_1,v,\omega) - \big(\partial_v^q X\big)(u_2,v,\omega)\big|}{|u_1-u_2|^{v-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta}} \right\} \\ &\le \sup_{(u_1,u_2,v)\in[-M,M]^2\times[a,b]} \\ &\left\{ \frac{\sum_{p=0}^q \binom{q}{p} \big(\log 2\big)^p \sum_{(j,k)\in\mathbb{Z}^2} |j|^p 2^{-jv} |\epsilon_{j,k}(\omega)| \Big| \big(\partial_v^{q-p}\Psi\big)(2^j u_1 - k, v) - \big(\partial_v^{q-p}\Psi\big)(2^j u_2 - k, v)\Big|}{|u_1-u_2|^{v-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta}} \right\} \\ &< \infty. \end{aligned} \tag{4.1}$$

Proposition 4.2. *For each fixed $\omega \in \Omega_0^*$, $q \in \mathbb{Z}_+$ and $a, b, \eta \in \mathbb{R}$ satisfying $1/\alpha < a < b < 1$ and $\eta > 0$, one has,*

$$\begin{aligned} &\sup_{(u,v)\in\mathbb{R}\times[a,b]} \left\{ \frac{\big|\big(\partial_v^q X\big)(u,v,\omega)\big|}{|u|^v\big(1+\big|\log|u|\big|\big)^{q+1/\alpha+\eta}} \right\} \\ &\le \sup_{(u,v)\in\mathbb{R}\times[a,b]} \\ &\left\{ \frac{\sum_{p=0}^q \binom{q}{p} \big(\log 2\big)^p \sum_{(j,k)\in\mathbb{Z}^2} |j|^p 2^{-jv} |\epsilon_{j,k}(\omega)| \Big| \big(\partial_v^{q-p}\Psi\big)(2^j u - k, v) - \big(\partial_v^{q-p}\Psi\big)(-k, v)\Big|}{|u|^v\big(1+\big|\log|u|\big|\big)^{q+1/\alpha+\eta}} \right\} \\ &< \infty. \end{aligned} \tag{4.2}$$

The proofs of Propositions 4.1 and 4.2 are, to a certain extent, inspired by that of Theorem 1 in [5].

PROOF OF PROPOSITION 4.1. Let $(u_1,u_2,v)\in[-M,M]^2\times[a,b]$ be arbitrary and fixed; in all the sequel we assume that $u_1\neq u_2$. Observe that, in view of (2.12), there is a constant $c_1>0$, non depending on (u_1,u_2,v), such that for all $p\in\{0,\dots,q\}$ and $(j,k)\in\mathbb{Z}^2$, one has,

$$\big|\big(\partial_v^{q-p}\Psi\big)(2^ju_1-k,v)-\big(\partial_v^{q-p}\Psi\big)(2^ju_2-k,v)\big|\le c_1\left(\big(3+|2^ju_1-k|\big)^{-2}+\big(3+|2^ju_2-k|\big)^{-2}\right). \quad (4.3)$$

Also notice that $\big|\big(\partial_v^{q-p}\Psi\big)(2^ju_1-k,v)-\big(\partial_v^{q-p}\Psi\big)(2^ju_2-k,v)\big|$ can be bounded more sharply when the condition

$$2^j|u_1-u_2|\le 1 \quad (4.4)$$

holds, namely using the Mean Value Theorem and (2.12), one has,

$$\begin{aligned}&\big|\big(\partial_v^{q-p}\Psi\big)(2^ju_1-k,v)-\big(\partial_v^{q-p}\Psi\big)(2^ju_2-k,v)\big|\\&\le 2^j|u_1-u_2|\sup_{(u,v)\in[u_1\wedge u_2,u_1\vee u_2]\times[a,b]}\big|\big(\partial_x\partial_v^{q-p}\Psi\big)(2^ju-k,v)\big|\\&\le c_12^j|u_1-u_2|\sup_{u\in[u_1\wedge u_2,u_1\vee u_2]}\big(3+|2^ju-k|\big)^{-2}\\&\le c_12^j|u_1-u_2|\big(2+|2^ju_1-k|\big)^{-2},\end{aligned} \quad (4.5)$$

where the last inequality results from the triangle inequality and (4.4). Denote by $j_0>-\log_2(4M)$ the unique integer satisfying

$$2^{-1}<2^{j_0}|u_1-u_2|\le 1. \quad (4.6)$$

Then, the first inequality in (2.10), (4.3) and (4.5), entail that, for all $\eta>0$ and $\omega\in\Omega_0^*$,

$$\begin{aligned}&\sum_{(j,k)\in\mathbb{Z}^2}|j|^p2^{-jv}\big|\epsilon_{j,k}(\omega)\big|\,\big|\big(\partial_v^{q-p}\Psi\big)(2^ju_1-k,v)-\big(\partial_v^{q-p}\Psi\big)(u_2-k,v)\big|\le\\&C(\omega)\sum_{(j,k)\in\mathbb{Z}^2}2^{-jv}(1+|j|)^{p+1/\alpha+\eta}(1+|k|)^{1/\alpha}\log^{1/\alpha+\eta}(2+|k|)\big|\big(\partial_v^{q-p}\Psi\big)(2^ju_1-k,v)-\big(\partial_v^{q-p}\Psi\big)(2^ju_2-k,v)\big|\\&\le C(\omega)c_1\Big(\breve{A}_{j_0}(u_1,v)|u_1-u_2|+\breve{B}_{j_0}(u_1,u_2,v)\Big),\end{aligned} \quad (4.7)$$

where the random variable C has been introduced in Lemma 2.1 and where for each $J\in\mathbb{Z}$, $(y_1,y_2)\in\mathbb{R}^2$, and $v\in[a,b]$,

$$\breve{A}_J(y_1,v):=\sum_{j\le J}\sum_{k\in\mathbb{Z}}2^{j(1-v)}(1+|j|)^{p+1/\alpha+\eta}(1+|k|)^{1/\alpha}\log^{1/\alpha+\eta}(2+|k|)\big(2+|2^jy_1-k|\big)^{-2} \quad (4.8)$$

and

$$\begin{aligned}&\breve{B}_J(y_1,y_2,v)\\&:=\sum_{j>J}\sum_{k\in\mathbb{Z}}2^{-jv}(1+|j|)^{p+1/\alpha+\eta}(1+|k|)^{1/\alpha}\log^{1/\alpha+\eta}(2+|k|)\left(\big(3+|2^jy_1-k|\big)^{-2}+\big(3+|2^jy_2-k|\big)^{-2}\right).\end{aligned} \quad (4.9)$$

Let us now give an appropriate upper bound for $\breve{A}_{j_0}(u_1,v)$. Assume that $j\le j_0$; using Lemma 9.5 (in which one takes $\theta=1/\alpha$, $\zeta=1/\alpha+\eta$ and $u=2^ju_1$) and the inequality $|u_1|\le M$, one obtains that,

$$\sum_{k\in\mathbb{Z}}\frac{(1+|k|)^{1/\alpha}\log^{1/\alpha+\eta}(2+|k|)}{\big(2+|2^ju_1-k|\big)^2}\le c_22^{j_0/\alpha}(1+|j_0|)^{1/\alpha+\eta},$$

where c_2 is a constant only depending on M, α and η. Next, it follows from the latter inequality, (4.8) and Lemma 9.4 (in which one takes $\theta = 1 - v$, $\theta_0 = 1 - b$, $\lambda = p + 1/\alpha + \eta$, $n_0 = -\infty$ and $n_1 = j_0$) that,

$$\begin{aligned}\check{A}_{j_0}(u_1, v) &\leq c_2 2^{j_0/\alpha}(1 + |j_0|)^{1/\alpha+\eta} \sum_{j \leq j_0} 2^{j(1-v)}(1 + |j|)^{p+1/\alpha+\eta} \leq c_3 2^{j_0(1-v+1/\alpha)}(1 + |j_0|)^{p+2/\alpha+2\eta} \\ &\leq c_4 |u_1 - u_2|^{v-1/\alpha-1}\big(1 + \big|\log|u_1 - u_2|\big|\big)^{p+2/\alpha+2\eta}, \end{aligned} \tag{4.10}$$

where the last inequality results from (4.6), and where c_3 and c_4 are two constants non depending on (u_1, u_2, v). Let us now give an appropriate upper bound for $\check{B}_{j_0}(u_1, u_2, v)$. In view of (4.9), this quantity can be expressed as,

$$\check{B}_{j_0}(u_1, u_2, v) = T_{j_0}(u_1, v) + T_{j_0}(u_2, v), \tag{4.11}$$

where, for each $J \in \mathbb{Z}$, $y \in \mathbb{R}$ and $v \in [a, b]$,

$$T_J(y, v) := \sum_{j>J} \sum_{k\in\mathbb{Z}} 2^{-jv} \frac{(1 + |j|)^{p+1/\alpha+\eta}(1 + |k|)^{1/\alpha} \log^{1/\alpha+\eta}(2 + |k|)}{\big(3 + |2^j y - k|\big)^2}. \tag{4.12}$$

Assume that $j > j_0$ and that $x \in \{u_1, u_2\}$; using Lemma 9.5 (in which one takes $\theta = 1/\alpha$, $\zeta = 1/\alpha+\eta$ and $u = 2^j x$) and the inequality $|x| \leq M$, one gets that,

$$\sum_{k\in\mathbb{Z}} \frac{(1 + |k|)^{1/\alpha} \log^{1/\alpha+\eta}(2 + |k|)}{\big(2 + |2^j x - k|\big)^2} \leq c_2 2^{j/\alpha}(1 + |j|)^{1/\alpha+\eta}.$$

Next, in view of (4.12), it follows from the latter inequality and Lemma 9.4 (in which one takes $\theta = v - 1/\alpha$, $\theta_0 = a - 1/\alpha$, $\lambda = p + 2/\alpha + 2\eta$, $n_0 = j_0 + 1$ and $n_1 = +\infty$) that,

$$\begin{aligned}T_{j_0}(x, v) &\leq c_2 \sum_{j>j_0} 2^{-j(v-1/\alpha)}(1 + |j|)^{p+2/\alpha+2\eta} \leq c_5 2^{-j_0(v-1/\alpha)}(1 + |j_0|)^{p+2/\alpha+2\eta} \\ &\leq c_6 |u_1 - u_2|^{v-1/\alpha}\big(1 + \big|\log|u_1 - u_2|\big|\big)^{p+2/\alpha+2\eta}, \end{aligned} \tag{4.13}$$

where the last inequality results from (4.6), and where c_5 and c_6 are two constants non depending on (x, v). Next, (4.13) and (4.11) imply that

$$\check{B}_{j_0}(u_1, u_2, v) \leq 2c_6 |u_1 - u_2|^{v-1/\alpha}\big(1 + \big|\log|u_1 - u_2|\big|\big)^{p+2/\alpha+2\eta}. \tag{4.14}$$

Next putting together, (4.10), (4.14) and (4.7), one obtains that, for all $\eta > 0$ and $\omega \in \Omega_0^*$,

$$\begin{aligned}&\sum_{(j,k)\in\mathbb{Z}^2} |j|^p 2^{-jv}\big|\epsilon_{j,k}(\omega)\big|\,\big|\big(\partial_v^{q-p}\Psi\big)(2^j u_1 - k, v) - \big(\partial_v^{q-p}\Psi\big)(u_2 - k, v)\big| \\ &\leq C(\omega) c_7 |u_1 - u_2|^{v-1/\alpha}\big(1 + \big|\log|u_1 - u_2|\big|\big)^{p+2/\alpha+2\eta}, \end{aligned} \tag{4.15}$$

where c_7 is a constant non depending on (u_1, u_2, v). Finally, (3.3), the triangle inequality and (4.15) entail that (4.1) holds. $\square$

Proof of Proposition 4.2. Let $(u, v) \in \mathbb{R} \times [a, b]$ be arbitrary and fixed, in all the sequel we assume that $u \neq 0$. Observe that, in view of (2.12), there is a constant $c_1 > 0$, non depending on (u, v), such that for all $p \in \{0, \ldots, q\}$ and $(j, k) \in \mathbb{Z}^2$, one has,

$$\big|\big(\partial_v^{q-p}\Psi\big)(2^j u - k, v) - \big(\partial_v^{q-p}\Psi\big)(-k, v)\big| \leq c_1\Big(\big(3 + |2^j u - k|\big)^{-2} + \big(3 + |k|\big)^{-2}\Big). \tag{4.16}$$

Also notice that $|(\partial_v^{q-p}\Psi)(2^j u - k, v) - (\partial_v^{q-p}\Psi)(-k, v)|$ can be bounded more sharply when the condition

$$2^j|u| \leq 1 \tag{4.17}$$

holds, namely, using the Mean Value Theorem and (2.12), one has,

$$\begin{aligned}\big|(\partial_v^{q-p}\Psi)(2^j u-k,v)-(\partial_v^{q-p}\Psi)(-k,v)\big| &\le 2^j|u|\sup_{y\in[u\wedge 0,u\vee 0]}\big|(\partial_x\partial_v^{q-p}\Psi)(2^j y-k,v)\big| \\ &\le c_1 2^j|u|\sup_{y\in[u\wedge 0,u\vee 0]}\big(3+|2^j y-k|\big)^{-2} \\ &\le c_1 2^j|u|\big(2+|k|\big)^{-2}, \qquad (4.18)\end{aligned}$$

where the last inequality results from the triangle inequality and (4.17). Denote by $j_1\in\mathbb{Z}$ the unique integer satisfying

$$2^{-1}<2^{j_1}|u|\le 1. \qquad (4.19)$$

Then the first inequality in (2.10), (4.16) and (4.18) entail that, for all $\eta>0$ and $\omega\in\Omega_0^*$,

$$\begin{aligned}&\sum_{(j,k)\in\mathbb{Z}^2}|j|^p 2^{-jv}\big|\epsilon_{j,k}(\omega)\big|\,\big|(\partial_v^{q-p}\Psi)(2^j u-k,v)-(\partial_v^{q-p}\Psi)(-k,v)\big| \\ &\le C(\omega)\sum_{(j,k)\in\mathbb{Z}^2}2^{-jv}\big(1+|j|\big)^{p+1/\alpha+\eta}\big(1+|k|\big)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+|k|\big)\big|(\partial_v^{q-p}\Psi)(2^j u-k,v)-(\partial_v^{q-p}\Psi)(-k,v)\big| \\ &\le C(\omega)c_1\Big(|u|\check{A}_{j_1}(0,v)+\check{B}_{j_1}(u,0,v)\Big), \qquad (4.20)\end{aligned}$$

where the random variable C has been introduced in Lemma 2.1 and where $\check{A}_{j_1}(0,v)$ and $\check{B}_{j_1}(u,0,v)$ are defined respectively by (4.8) and (4.9). Let us now give an appropriate upper bound for $\check{A}_{j_1}(0,v)$. Observe that

$$c_2:=\sum_{k\in\mathbb{Z}}\frac{(1+|k|)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+|k|\big)}{\big(2+|k|\big)^2}<\infty.$$

Thus, (4.8) and Lemma 9.4 (in which one takes $\theta=1-v$, $\theta_0=1-b$, $\lambda=p+1/\alpha+\eta$, $n_0=-\infty$ and $n_1=j_1$) imply that,

$$\begin{aligned}\check{A}_{j_1}(0,v)=c_2\sum_{j\le j_1}2^{j(1-v)}\big(1+|j|\big)^{p+1/\alpha+\eta}&\le c_3 2^{j_1(1-v)}\big(1+|j_1|\big)^{p+1/\alpha+\eta} \\ &\le c_4|u|^{v-1}\big(1+\big|\log|u|\big|\big)^{p+1/\alpha+\eta}, \qquad (4.21)\end{aligned}$$

where the last inequality results from (4.19) and where c_3 and c_4 are two constants non depending on (u,v). Let us now give an appropriate upper bound for $\check{B}_{j_1}(u,0,v)$. In view of (4.9), this quantity can be expressed as,

$$\check{B}_{j_1}(u,0,v):=T_{j_1}(u,v)+T_{j_1}(0,v), \qquad (4.22)$$

where $T_{j_1}(u,v)$ and $T_{j_1}(0,v)$ are defined by (4.12). Assume that $j>j_1$ and that $x\in\{u,0\}$; it follows from Lemma 9.5 in which one takes $\theta=1/\alpha$ and $\zeta=1/\alpha+\eta$, that,

$$\sum_{k\in\mathbb{Z}}\frac{(1+|k|)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+|k|\big)}{\big(3+|2^j x-k|\big)^2}\le c_5\big(1+2^j|x|\big)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+2^j|x|\big)\le c_6 2^{(j-j_1)/\alpha}(1+j-j_1)^{1/\alpha+\eta},$$

where the last inequality results from (4.19) and where c_5 and c_6 are two constants non depending on x, v, j and j_1. Therefore, in view of (4.12), one obtains that

$$T_{j_1}(x,v)\le c_6\sum_{j>j_1}2^{-jv}2^{(j-j_1)/\alpha}\big(1+|j|\big)^{p+1/\alpha+\eta}\big(1+j-j_1\big)^{1/\alpha+\eta}. \qquad (4.23)$$

Next, setting $l = j - j_1$ in the right-hand side of (4.23) and using Lemma 9.1, it follows that,

$$
\begin{aligned}
T_{j_1}(x,v) &\le c_6 \sum_{l=1}^{+\infty} 2^{-j_1 v} 2^{-l(v-1/\alpha)} (1+l)^{1/\alpha+\eta} \big(1+|l+j_1|\big)^{p+1/\alpha+\eta} \\
&\le c_7 2^{-j_1 v} \sum_{l=1}^{+\infty} 2^{-l(v-1/\alpha)} (1+l)^{1/\alpha+\eta} \Big(\big(1+l\big)^{p+1/\alpha+\eta} + \big(1+|j_1|\big)^{p+1/\alpha+\eta} \Big) \\
&\le c_7 2^{-j_1 v} \sum_{l=1}^{+\infty} 2^{-l(a-1/\alpha)} (1+l)^{1/\alpha+\eta} \Big(\big(1+l\big)^{p+1/\alpha+\eta} + \big(1+|j_1|\big)^{p+1/\alpha+\eta} \Big) \\
&\le c_8 2^{-j_1 v} \big(1+|j_1|\big)^{p+1/\alpha+\eta}, \\
&\le c_9 |u|^v \left(1+\big|\log|u|\big|\right)^{p+1/\alpha+\eta}, \qquad (4.24)
\end{aligned}
$$

where the last inequality results from (4.19) and where the constants c_7, c_8 and c_9 do not depend on x, v and j_1. Next, (4.22) and (4.24) imply that,

$$
\check{B}_{j_1}(u,0,v) \le 2c_9 |u|^v \left(1+\big|\log|u|\big|\right)^{p+1/\alpha+\eta}. \tag{4.25}
$$

Next, putting together (4.20), (4.21) and (4.25), one gets that,

$$
\begin{aligned}
&\sum_{(j,k)\in\mathbb{Z}^2} |j|^p 2^{-jv} \big|\epsilon_{j,k}(\omega)\big| \Big| \big(\partial_v^{q-p}\Psi\big)(2^j u - k, v) - \big(\partial_v^{q-p}\Psi\big)(-k,v)\Big| \\
&\le C(\omega) c_{10} |u|^v \left(1+\big|\log|u|\big|\right)^{p+1/\alpha+\eta}, \qquad (4.26)
\end{aligned}
$$

where c_{10} is a constant non depending on (u,v). Finally, (3.3), the triangle inequality and (4.26) entail that (4.2) holds. □

Before ending this section, let us stress that, thanks to (3.18) and (4.1), for each fixed $\omega \in \Omega_0^*$, $q \in \mathbb{Z}_+$ and $M, a, b \in \mathbb{R}$ satisfying $M > 0$ and $1/\alpha < a < b < 1$, one can derive, a global modulus of continuity of the function $(u,v) \mapsto (\partial_v^q X)(u,v,\omega)$, on the rectangle $[-M,M] \times [a,b]$. More precisely, the following result holds.

Corollary 4.1. *For each fixed $\omega \in \Omega_0^*$, $q \in \mathbb{Z}_+$ and $M, a, b, \eta \in \mathbb{R}$ satisfying $M > 0$, $1/\alpha < a < b < 1$ and $\eta > 0$, one has,*

$$
\sup_{(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2} \left\{ \frac{\big|\big(\partial_v^q X\big)(u_1,v_1,\omega) - \big(\partial_v^q X\big)(u_2,v_2,\omega)\big|}{|u_1-u_2|^{v_1\vee v_2 - 1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta} + |v_1-v_2|} \right\} < \infty. \tag{4.27}
$$

Proof of Corollary 4.1. For each $(u_1,u_2,v_1,v_2) \in [-M,M]^2 \times [a,b]^2$, one sets,

$$
f(u_1,u_2,v_1,v_2) := \frac{\big|\big(\partial_v^q X\big)(u_1,v_1,\omega) - \big(\partial_v^q X\big)(u_2,v_2,\omega)\big|}{|u_1-u_2|^{v_1\vee v_2 - 1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta} + |v_1-v_2|},
$$

with the convention that $0/0 = 0$. Using the fact that $f(u_1,u_2,v_1,v_2) = f(u_2,u_1,v_2,v_1)$, it follows that,

$$
\begin{aligned}
&\sup_{(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2} \left\{ \frac{\big|\big(\partial_v^q X\big)(u_1,v_1,\omega) - \big(\partial_v^q X\big)(u_2,v_2,\omega)\big|}{|u_1-u_2|^{v_1\vee v_2 - 1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta} + |v_1-v_2|} \right\} \qquad (4.28) \\
&= \sup_{(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2} \left\{ \frac{\big|\big(\partial_v^q X\big)(u_1,v_1\vee v_2,\omega) - \big(\partial_v^q X\big)(u_2,v_1\wedge v_2,\omega)\big|}{|u_1-u_2|^{v_1\vee v_2 - 1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta} + |v_1-v_2|} \right\}.
\end{aligned}
$$

Moreover, using the triangle inequality, and the inequality for all $(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2$,

$$\max\Big\{|u_1-u_2|^{v_1\vee v_2-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta},|v_1-v_2|\Big\}$$
$$\le|u_1-u_2|^{v_1\vee v_2-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta}+|v_1-v_2|,$$

one gets that,

$$\begin{aligned}
&\sup_{(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2}\left\{\frac{\big|(\partial_v^q X)(u_1,v_1\vee v_2,\omega)-(\partial_v^q X)(u_2,v_1\wedge v_2,\omega)\big|}{|u_1-u_2|^{v_1\vee v_2-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta}+|v_1-v_2|}\right\}\\
&\le\sup_{(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2}\left\{\frac{\big|(\partial_v^q X)(u_1,v_1\vee v_2,\omega)-(\partial_v^q X)(u_2,v_1\vee v_2,\omega)\big|}{|u_1-u_2|^{v_1\vee v_2-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta}+|v_1-v_2|}\right\}\\
&\quad+\sup_{(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2}\left\{\frac{\big|(\partial_v^q X)(u_2,v_1\vee v_2,\omega)-(\partial_v^q X)(u_2,v_1\wedge v_2,\omega)\big|}{|u_1-u_2|^{v_1\vee v_2-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta}+|v_1-v_2|}\right\}\\
&\le\sup_{(u_1,u_2,v)\in[-M,M]^2\times[a,b]}\left\{\frac{\big|(\partial_v^q X)(u_1,v,\omega)-(\partial_v^q X)(u_2,v,\omega)\big|}{|u_1-u_2|^{v-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta}}\right\}\\
&\quad+\sup_{(u,v_1,v_2)\in[-M,M]\times[a,b]^2}\left\{\frac{\big|(\partial_v^q X)(u,v_1,\omega)-(\partial_v^q X)(u,v_2,\omega)\big|}{|v_1-v_2|}\right\}.
\end{aligned}\tag{4.29}$$

Finally, putting together, (4.28), (4.29), (3.18) and (4.1), one obtains (4.27).

□

5. Global and local moduli of continuity of LMSM

From now on and till the end of the article, LMSM is identified with its modification $\{Y(t):t\in\mathbb{R}\}$, defined for all $t\in\mathbb{R}$, by,

$$Y(t)=X(t,H(t)),\tag{5.1}$$

where $\{X(u,v):(u,v)\in\mathbb{R}\times(1/\alpha,1)\}$ is the $\mathcal{S}t\alpha\mathcal{S}$ field introduced in Theorem 2.1; recall that $H(\cdot)$ denotes an arbitrary continuous function defined on the real line and with values in a compact interval $[\underline{H},\overline{H}]\subset(1/\alpha,1)$.

First we provide a global modulus of continuity for $\{Y(t):t\in\mathbb{R}\}$ on an arbitrary nonempty compact interval; there is no restriction to assume the latter interval of the form $[-M,M]$ where M is an arbitrary positive real number.

Theorem 5.1. *Let Ω_0^* be the event of probability 1 introduced in Lemma 2.1. Then for each $\omega\in\Omega_0^*$ and for all positive real numbers M and η, one has,*

$$\sup_{(t,s)\in[-M,M]^2}\left\{\frac{\big|(Y(t,\omega)-Y(s,\omega)\big|}{|t-s|^{H(t)\vee H(s)-1/\alpha}\big(1+\big|\log|t-s|\big|\big)^{2/\alpha+\eta}+\big|H(t)-H(s)\big|}\right\}<\infty.\tag{5.2}$$

Proof of Theorem 5.1. The theorem easily results from (5.1) and Corollary 4.1 in which one takes $q=0$, $a=\min_{x\in[-M,M]}H(x)$ and $b=\max_{x\in[-M,M]}H(x)$. □

Remark 5.1. *(i) Theorem 5.1 remains valid under the weaker condition that $H(\cdot)$ is a continuous function on the real line with values in the open interval $(1/\alpha,1)$; indeed, even in this case, $H\big([-M,M]\big)$ is still a compact interval included in $(1/\alpha,1)$.*

(ii) A straightforward consequence of Theorem 5.1 is that: LMSM has a modification with almost surely continuous paths, as soon as its functional Hurst parameter $H(\cdot)$ is a continuous function

with values in $(1/\alpha, 1)$*; this solves and provides a positive answer to the conjecture made by Stoev and Taqqu in Remark 1 at page 166 of* [30].

The following corollary easily follows from Theorem 5.1.

Corollary 5.1. *(i) Assume that for some real numbers* $M_1 < M_2$*, one has for each* $\eta > 0$*,*

$$\sup_{(t,s)\in[M_1,M_2]^2} \frac{|H(t)-H(s)|}{|t-s|^{H(t)\vee H(s)-1/\alpha}\big(1+\big|\log|t-s|\big|\big)^{2/\alpha+\eta}} < \infty, \tag{5.3}$$

then it follows that, for all $\omega \in \Omega_0^*$ *and* $\eta > 0$*,*

$$\sup_{(t,s)\in[M_1,M_2]^2} \left\{ \frac{|Y(t,\omega)-Y(s,\omega)|}{|t-s|^{H(t)\vee H(s)-1/\alpha}\big(1+\big|\log|t-s|\big|\big)^{2/\alpha+\eta}} \right\} < \infty. \tag{5.4}$$

(ii) Assume that for some real numbers $M_1 < M_2$*, one has for each* $\eta > 0$*,*

$$\sup_{(t,s)\in[M_1,M_2]^2} \frac{|H(t)-H(s)|}{|t-s|^{\min_{x\in[M_1,M_2]} H(x)-1/\alpha}\big(1+\big|\log|t-s|\big|\big)^{2/\alpha+\eta}} < \infty, \tag{5.5}$$

then it follows that, for all $\omega \in \Omega_0^*$ *and* $\eta > 0$*,*

$$\sup_{(t,s)\in[M_1,M_2]^2} \left\{ \frac{|Y(t,\omega)-Y(s,\omega)|}{|t-s|^{\min_{x\in[M_1,M_2]} H(x)-1/\alpha}\big(1+\big|\log|t-s|\big|\big)^{2/\alpha+\eta}} \right\} < \infty. \tag{5.6}$$

Remark 5.2. *(i) The Condition (5.3) is satisfied as soon as*

$$H(\cdot) \in \mathcal{C}^{\max_{x\in[M_1,M_2]} H(x)-1/\alpha}\big([M_1,M_2],\mathbb{R}\big).$$

(ii) The Condition (5.5) is satisfied as soon as

$$H(\cdot) \in \mathcal{C}^{\min_{x\in[M_1,M_2]} H(x)-1/\alpha}\big([M_1,M_2],\mathbb{R}\big).$$

Let us now provide a local modulus of continuity for $\{Y(t) : t \in \mathbb{R}\}$.

Theorem 5.2. *Assume that the skewness intensity function* $\beta(\cdot)$ *of the* $\mathcal{S}\alpha\mathcal{S}$ *measure* $Z_\alpha(ds)$ *is a constant. Let* $t_0 \in \mathbb{R}$ *be arbitrary and fixed. Then, one has almost surely, for all positive real numbers* M *and* η*,*

$$\sup_{t\in[-M,M]} \left\{ \frac{|Y(t)-Y(t_0)|}{|t-t_0|^{H(t_0)}\big(1+\big|\log|t-t_0|\big|\big)^{1/\alpha+\eta} + |H(t)-H(t_0)|} \right\} < \infty. \tag{5.7}$$

Proof of Theorem 5.2. First observe that for any fixed $t_0 \in \mathbb{R}$, the process $\{X(t,H(t_0)) : t \in \mathbb{R}\}$ has stationary increments since it is a Linear Fractional Stable Motion of Hurst parameter $H(t_0)$; hence, the processes $\{X(t,H(t_0)) - X(t_0,H(t_0)) : t \in \mathbb{R}\}$ and $\{X(t-t_0,H(t_0)) : t \in \mathbb{R}\}$ have the same finite dimensional distributions. Therefore, using their path continuity, and the fact that the set of the dyadic numbers in $[-M,M]$ is dense in $[-M,M]$, it follows that the random variables,

$$\sup_{t\in[-M,M]} \left\{ \frac{|X(t,H(t_0))-X(t_0,H(t_0))|}{|t-t_0|^{H(t_0)}\big(1+\big|\log|t-t_0|\big|\big)^{1/\alpha+\eta}} \right\}$$

and

$$\sup_{t\in[-M,M]} \left\{ \frac{|X(t-t_0,H(t_0))|}{|t-t_0|^{H(t_0)}\big(1+\big|\log|t-t_0|\big|\big)^{1/\alpha+\eta}} \right\},$$

are equal in law; thus, taking in Proposition 4.2, $q = 0$ and a, b such that $H(t_0) \in [a, b]$, one gets that, almost surely,

$$\sup_{t\in[-M,M]} \left\{ \frac{\big|X(t, H(t_0)) - X(t_0, H(t_0))\big|}{|t-t_0|^{H(t_0)}\big(1+\big|\log|t-t_0|\big|\big)^{1/\alpha+\eta}} \right\} < \infty. \tag{5.8}$$

On the other hand, taking in (3.18), $q = 0$, $a = \underline{H} := \inf_{x\in\mathbb{R}} H(x)$ and $b = \overline{H} := \sup_{x\in\mathbb{R}} H(x)$, one obtains that,

$$\sup_{t\in[-M,M]} \left\{ \frac{\big|X(t, H(t)) - X(t, H(t_0))\big|}{\big|H(t) - H(t_0)\big|} \right\} < \infty. \tag{5.9}$$

Finally putting together, (5.1), (5.8) and (5.9), it follows that (5.7) holds. □

The following result is a straightforward consequence of Theorem 5.2.

Corollary 5.2. *Assume that the skewness intensity function $\beta(\cdot)$ of the $\mathcal{S}t\alpha\mathcal{S}$ measure $Z_\alpha(ds)$ is a constant. Also assume that $t_0 \in \mathbb{R}$ is such that, for each $\eta > 0$, one has for all $t \in \mathbb{R}$,*

$$\big|H(t) - H(t_0)\big| \le c|t-t_0|^{H(t_0)}\big(1+\big|\log|t-t_0|\big|\big)^{1/\alpha+\eta}, \tag{5.10}$$

where $c > 0$ is a constant only depending on t_0 and η. Then, one has almost surely, for each positive real numbers M and η,

$$\sup_{t\in[-M,M]} \left\{ \frac{\big|Y(t) - Y(t_0)\big|}{|t-t_0|^{H(t_0)}\big(1+\big|\log|t-t_0|\big|\big)^{1/\alpha+\eta}} \right\} < \infty. \tag{5.11}$$

6. Quasi-optimality of global modulus of continuity of LMSM

The goal of this section is to show that, under some conditions, a bit stronger than (5.5), the global modulus of continuity, given in (5.6), is quasi-optimal, more precisely:

Theorem 6.1. *Assume that $M_1 < M_2$ are two arbitrary fixed real numbers such that the condition,*

$(\mathcal{A})$: $H(\cdot)$ belongs to the Hölder space $\mathcal{C}^{\gamma_*}\big([M_1, M_2], \mathbb{R}\big)$ for some $\gamma_* \in \big(\min_{x\in[M_1,M_2]} H(x) - 1/\alpha, 1\big]$,

is satisfied. Let us set

$$\rho := \sup\left\{ \theta \in \mathbb{R}_+ : \exists t_0 \in [M_1, M_2] \text{ s.t. } H(t_0) = \min_{x\in[M_1,M_2]} H(x) \text{ and } \sup_{t\in[M_1,M_2]} \frac{|H(t) - H(t_0)|}{|t-t_0|^\theta} < \infty \right\} \tag{6.1}$$

and

$$\tau := \frac{1 + 2\alpha^{-1}}{\alpha\rho - 1}, \tag{6.2}$$

with the convention that $\tau := 0$ when $\rho = +\infty$. Assume that

$$\alpha\rho > 1, \tag{6.3}$$

then, τ is a well-defined nonnegative real number, and one has, almost surely, for all $\eta > 0$,

$$\sup_{(t,s)\in[M_1,M_2]^2} \left\{ \frac{|Y(t) - Y(s)|}{|t-s|^{\min_{x\in[M_1,M_2]} H(x) - 1/\alpha}\big(1+\big|\log|t-s|\big|\big)^{-\tau-\eta}} \right\} = \infty. \tag{6.4}$$

Remark 6.1. *Notice that the Conditions $(\mathcal{A})$ and (6.3) are satisfied when $H(\cdot)$ belongs to the Hölder space $\mathcal{C}^\gamma\big([M_1, M_2], \mathbb{R}\big)$, for some $\gamma > 1/\alpha$.*

In order to prove Theorem 6.1, we need some preliminary results. Let us first introduce $\widetilde{\Psi}$ the real-valued deterministic continuous function defined, for all $(x,v) \in \mathbb{R} \times (1/\alpha, 1)$, as,

$$\widetilde{\Psi}(x,v) := \frac{1}{\Gamma(v+1-1/\alpha)\Gamma(1/\alpha - v+1)} \int_{\mathbb{R}} (s-x)_+^{1/\alpha - v} \psi^{(2)}(s) ds, \tag{6.5}$$

where $\psi^{(2)}$ is the second derivative of the Daubechies mother wavelet ψ introduced at the very beginning of Section 2, and where Γ is the usual Gamma function; also, recall that the definition of $(\cdot)_+^{1/\alpha - v}$ is given in (1.2). By using a result in [25] concerning Fourier transforms of right-sided fractional derivatives, one has for each $(\xi, v) \in \mathbb{R} \times (1/\alpha, 1)$,

$$\widehat{\widetilde{\Psi}}(\xi, v) = \frac{1}{\Gamma(v+1-1/\alpha)} |\xi|^{v+1-1/\alpha} e^{-i\mathrm{sgn}(\xi)(v+1-1/\alpha)\frac{\pi}{2}} \widehat{\psi}(\xi), \tag{6.6}$$

where $\widehat{\widetilde{\Psi}}(\cdot, v)$ denotes the Fourier transform of the function $\widetilde{\Psi}(\cdot, v)$. Let us now give some useful properties of the function $\widetilde{\Psi}$.

Proposition 6.1. *The function $\widetilde{\Psi}$ satisfies the following three properties.*

(i) For all real numbers a, b such that $1 > b > a > 1/\alpha$, the function $\widetilde{\Psi}$ is well-localized in the variable x uniformly in the variable $v \in [a,b]$; namely one has,

$$\sup_{(x,v) \in \mathbb{R} \times [a,b]} (3+|x|)^2 \big|\widetilde{\Psi}(x,v)\big| < \infty. \tag{6.7}$$

(ii) For any fixed $v \in \big(1/\alpha, 1\big)$, the first moment of the function $\widetilde{\Psi}(\cdot, v)$ vanishes, which means that

$$\int_{\mathbb{R}} \widetilde{\Psi}(x,v) dx = 0. \tag{6.8}$$

(iii) Let Ψ be the function introduced in (2.3) then, for each fixed $v \in \big(1/\alpha, 1\big)$, the system of functions $\big\{2^{j/2}\Psi(2^j \cdot -k, v) : (j,k) \in \mathbb{Z}^2\big\}$ and $\big\{2^{j/2}\widetilde{\Psi}(2^j \cdot -k, v) : (j,k) \in \mathbb{Z}^2\big\}$ is biorthogonal; this means that for any $j \in \mathbb{Z}$, $j' \in \mathbb{Z}$, $k \in \mathbb{Z}$ and $k' \in \mathbb{Z}$, one has,

$$2^{(j+j')/2} \int_{\mathbb{R}} \Psi(2^j t - k, v) \widetilde{\Psi}(2^{j'} t - k', v) dt = \delta_{(j,k;j',k')}, \tag{6.9}$$

where $\delta_{(j,k;j',k')} = 1$ if $(j,k) = (j',k')$ and 0 otherwise.

Proof of Proposition 6.1. Part (i) can be obtained by using the fact that

$$\sup_{v \in [a,b]} \left\{ \frac{1}{\Gamma(v+1-1/\alpha)\Gamma(1/\alpha - v+1)} \right\} < \infty$$

and a method similar to the one used in the proof of Proposition 2.1 Part (ii). In view of the definition of a Fourier transform, taking in (6.6) $\xi = 0$ one gets Part (ii). Let us now give the proof of Part (iii). Using Parseval formula, (2.4) and (6.6), one obtains for all $(j,k) \in \mathbb{Z}^2$,

$$\begin{aligned}
&\int_{\mathbb{R}} 2^{j/2} \Psi(2^j t - k, v) 2^{j'/2} \widetilde{\Psi}(2^{j'} t - k', v) dt \\
&= 2^{-(j+j')/2} (2\pi)^{-1} \int_{\mathbb{R}} e^{-i\xi(k/2^j - k'/2^{j'})} \widehat{\Psi}(2^{-j}\xi, v) \overline{\widehat{\widetilde{\Psi}}(2^{-j'}\xi, v)} d\xi \\
&= 2^{-(j+j')/2 + (j-j')(v+1-1/\alpha)} (2\pi)^{-1} \int_{\mathbb{R}} e^{-i\xi(k/2^j - k'/2^{j'})} \widehat{\psi}(2^{-j}\xi) \overline{\widehat{\psi}(2^{-j'}\xi)} d\xi \\
&= 2^{(j-j')(v+1-1/\alpha)} \int_{\mathbb{R}} 2^{j/2} \psi(2^j t - k) 2^{j'/2} \psi(2^{j'} t - k') dt \\
&= \delta_{(j,k;j',k')},
\end{aligned}$$

where the last equality results from the fact that $\big\{2^{j/2}\psi(2^j\cdot -k) : (j,k)\in\mathbb{Z}^2\big\}$ is an orthonormal basis for $L^2(\mathbb{R})$. $\square$

In all the remaining of this section, $M_1 < M_2$ denote two arbitrary real numbers such that the Conditions $(\mathcal{A})$ and (6.3) hold. For the sake of simplicity, we set,

$$H_* := \min_{x\in[M_1,M_2]} H(x). \tag{6.10}$$

Lemma 6.1. *Let Ω_0^* be the event of probability 1 introduced in Lemma 2.1 and let $\big\{g_{j,k} : (j,k)\in\mathbb{N}\times\mathbb{Z}\big\}$ be the sequence of the random variables defined on Ω_0^* as,*

$$g_{j,k} = 2^{j(1+H_*)}\int_{\mathbb{R}} Y(t)\widetilde{\Psi}(2^j t-k,H_*)dt. \tag{6.11}$$

Assume that there exists $\omega_0\in\Omega_0^*$, $\tau_0>\tau$ and $\eta_0>0$ such that

$$\sup_{(t,s)\in[M_1,M_2]^2} \frac{\big|Y(t,\omega_0)-Y(s,\omega_0)\big|}{|t-s|^{H_*-1/\alpha}\big(1+\big|\log|t-s|\big|\big)^{-\tau_0-\eta_0}} < \infty. \tag{6.12}$$

Then one has

$$\limsup_{j\to+\infty} j^{\tau_0}2^{-j/\alpha}\max\Big\{\big|g_{j,k}(\omega_0)\big| : k\in\mathbb{Z} \text{ and } M_1+2^{-\frac{j}{2\alpha}} \le k/2^j \le M_2-2^{-\frac{j}{2\alpha}}\Big\} = 0. \tag{6.13}$$

Remark 6.2. *Notice that (6.7) (in which one takes a,b such that $H_*\in[a,b]$), Proposition 4.2 (in which one takes $q=0$, $a=\underline{H}:=\inf_{x\in\mathbb{R}}H(x)$, $b=\overline{H}:=\sup_{x\in\mathbb{R}}H(x)$ and η an arbitrary positive real number) and Relation (5.1), imply that the random variables $g_{j,k}$ are well-defined and finite on Ω_0^*.*

Proof of Lemma 6.1. In all the sequel, we assume that $j\in\mathbb{N}$ and $k\in\mathbb{Z}$ are arbitrary and satisfy

$$M_1+2^{-\frac{j}{2\alpha}} \le \frac{k}{2^j} \le M_2-2^{-\frac{j}{2\alpha}}. \tag{6.14}$$

It follows from (6.11) and (6.8) in which one takes $v=H_*$, that,

$$g_{j,k}(\omega_0) = 2^{j(1+H_*)}\int_{\mathbb{R}} \big(Y(t,\omega_0)-Y(k2^{-j},\omega_0)\big)\widetilde{\Psi}(2^j t-k,H_*)dt. \tag{6.15}$$

In order to conveniently bound $\big|g_{j,k}(\omega_0)\big|$, we split the integration domain $\mathbb{R}$ into the following three disjoint subdomains:

$$\mathcal{B}_1 := [M_1,M_2],\ \mathcal{B}_2 := [-2M_0,2M_0]\setminus[M_1,M_2] \text{ and } \mathcal{B}_3 := \mathbb{R}\setminus[-2M_0,2M_0], \text{where } M_0 := |M_1|+|M_2|. \tag{6.16}$$

Therefore, (6.15) implies that,

$$\big|g_{j,k}(\omega_0)\big| \le \sum_{l=1}^{3} A_{j,k}^l(\omega_0), \tag{6.17}$$

where, for each $l\in\{1,2,3\}$, one has set,

$$A_{j,k}^l(\omega_0) = 2^{j(1+H_*)}\int_{\mathcal{B}_l} \big|Y(t,\omega_0)-Y(k2^{-j},\omega_0)\big|\big|\widetilde{\Psi}(2^j t-k,H_*)\big|dt. \tag{6.18}$$

First, we show that (6.13) holds when $\big|g_{j,k}(\omega_0)\big|$ is replaced by $A^1_{j,k}(\omega_0)$. Relation (6.12) and the change of variable $u = 2^j t - k$, yield

$$\begin{aligned}
A^1_{j,k}(\omega_0) &\le C_1(\omega_0) 2^{j(1+H_*)} \int_{\mathcal{B}_1} |t-k2^{-j}|^{H_*-1/\alpha}\Big(1+\big|\log|t-k2^{-j}|\big|\Big)^{-\tau_0-\eta_0} |\widetilde{\Psi}(2^j t-k,H_*)|dt \\
&\le C_1(\omega_0) 2^{j(1+H_*)} \int_{\mathbb{R}} |t-k2^{-j}|^{H_*-1/\alpha}\Big(1+\big|\log|t-k2^{-j}|\big|\Big)^{-\tau_0-\eta_0} |\widetilde{\Psi}(2^j t-k,H_*)|dt \\
&= C_1(\omega_0) 2^{j/\alpha} \int_{\mathbb{R}} |u|^{H_*-1/\alpha}\Big(1+\big|\log|2^{-j}u|\big|\Big)^{-\tau_0-\eta_0} |\widetilde{\Psi}(u,H_*)|du \\
&= C_1(\omega_0) j^{-\tau_0-\eta_0} 2^{j/\alpha} \int_{\mathbb{R}} |u|^{H_*-1/\alpha}\left(\frac{1}{j}+\left|\log(2)-\frac{\log|u|}{j}\right|\right)^{-\tau_0-\eta_0} |\widetilde{\Psi}(u,H_*)|du, \qquad (6.19)
\end{aligned}$$

where

$$C_1(\omega_0) := \sup_{(t,s)\in\mathcal{B}_1^2} \frac{\big|Y(t,\omega_0)-Y(s,\omega_0)\big|}{|t-s|^{H_*-1/\alpha}\Big(1+\big|\log|t-s|\big|\Big)^{-\tau_0-\eta_0}} < \infty.$$

Let us now show that,

$$\sup_{j\ge 1} \int_{\mathbb{R}} |u|^{H_*-1/\alpha}\left(\frac{1}{j}+\left|\log(2)-\frac{\log|u|}{j}\right|\right)^{-\tau_0-\eta_0} |\widetilde{\Psi}(u,H_*)|du < \infty. \qquad (6.20)$$

In view of (6.7) and the inequality,

$$\left(\frac{1}{j}+\left|\log(2)-\frac{\log|u|}{j}\right|\right)^{-\tau_0-\eta_0} \le \Big(\frac{\log 2}{2}\Big)^{-\tau_0-\eta_0},$$

which holds for all real number u satisfying $|u|\le 2^{j/2}$, one gets, for some constants $c_2,\dots,c_5$ and all integer $j\ge 1$, that,

$$\begin{aligned}
&\int_{\mathbb{R}} |u|^{H_*-1/\alpha}\left(\frac{1}{j}+\left|\log(2)-\frac{\log|u|}{j}\right|\right)^{-\tau_0-\eta_0} |\widetilde{\Psi}(u,H_*)|du \\
&\le c_2 j^{\tau_0+\eta_0} \int_{|u|>2^{j/2}} \frac{|u|^{H_*-1/\alpha}}{(3+|u|)^2}du + c_2\Big(\frac{\log 2}{2}\Big)^{-\tau_0-\eta_0} \int_{|u|\le 2^{j/2}} \frac{|u|^{H_*-1/\alpha}}{(3+|u|)^2}du \\
&\le 2c_2 j^{\tau_0+\eta_0} \int_{u>2^{j/2}} \frac{u^{H_*-1/\alpha}}{(3+u)^2}du + c_3 \int_{\mathbb{R}} \frac{|u|^{H_*-1/\alpha}}{(3+|u|)^2}du \\
&\le c_4 j^{\tau_0+\eta_0} \frac{2^{-j/2(1+1/\alpha-H_*)}}{1+1/\alpha-H_*} + c_5,
\end{aligned}$$

which shows that (6.20) is satisfied. Next, (6.19) and (6.20) entail that

$$\limsup_{j\to+\infty} j^{\tau_0} 2^{-j/\alpha} \max\left\{A^1_{j,k}(\omega_0) : k\in\mathbb{Z} \text{ and } M_1+2^{-\frac{j}{2\alpha}} \le k/2^j \le M_2-2^{-\frac{j}{2\alpha}}\right\} = 0. \qquad (6.21)$$

Next, we prove that (6.13) holds when $\big|g_{j,k}(\omega_0)\big|$ is replaced by $A^2_{j,k}(\omega_0)$. Let us set,

$$C_6(\omega_0) := \sup_{t\in[-2M_0,2M_0]} \big|Y(t,\omega_0)\big| < \infty; \qquad (6.22)$$

observe that $C_6(\omega_0)$ is finite, since the function $t\mapsto Y(t,\omega_0)$ is continuous over the compact interval $[-2M_0,2M_0]$. Also, observe that, in view of (6.14) and (6.16), one has that for all $t\in\mathcal{B}_2$,

$$|2^j t-k| > 2^{j(1-\frac{1}{2\alpha})}.$$

Therefore, it follows from (6.7), that for each $t \in \mathcal{B}_2$,

$$|\widetilde{\Psi}(2^j t - k, H_*)| \le c_7 2^{-j(2-\frac{1}{\alpha})}, \tag{6.23}$$

where c_7 is a constant non depending on t, j and k. Putting together, (6.18), (6.22) and (6.23), one gets that,

$$A^2_{j,k}(\omega_0) \le C_8(\omega_0) 2^{-j(1-H_*-\frac{1}{\alpha})},$$

where $C_8(\omega_0)$ is a constant non depending on j and k. The latter inequality and the inequality $H_* < 1$, imply that,

$$\limsup_{j\to+\infty} j^{\tau_0} 2^{-j/\alpha} \max\left\{A^2_{j,k}(\omega_0) : k \in \mathbb{Z} \text{ and } M_1 + 2^{-\frac{j}{2\alpha}} \le k/2^j \le M_2 - 2^{-\frac{j}{2\alpha}}\right\} = 0. \tag{6.24}$$

Next, we prove that (6.13) holds when $\left|g_{j,k}(\omega_0)\right|$ is replaced by $A^3_{j,k}(\omega_0)$. Observe that by using the triangle inequality, (6.14) and (6.16), one has, for each $t \in \mathcal{B}_3$,

$$|2^j t - k| = 2^j \left|t - \frac{k}{2^j}\right| \ge 2^j \left(|t| - \frac{|k|}{2^j}\right) > 2^j \left(|t| - M_0\right) > 2^{j-1}|t|.$$

Therefore, it follows from (6.7), that for each $t \in \mathcal{B}_3$,

$$|\widetilde{\Psi}(2^j t - k, H_*)| \le c_9 2^{-2j} |t|^{-2}, \tag{6.25}$$

where c_9 is a constant non depending on t, j and k. On the other hand, using (5.1) and Proposition 4.2, in the case where $q = 0$, $a = \underline{H} := \inf_{x\in\mathbb{R}} H(x)$ and $b = \overline{H} := \sup_{x\in\mathbb{R}} H(x)$, one obtains that for any fixed $\eta > 0$, and for each $t \in \mathcal{B}_3$,

$$|Y(t,\omega_0)| \le C_{10}(\omega_0)|t|^{\overline{H}}\big(1 + \big|\log|t|\big|\big)^{1/\alpha+\eta},$$

where $C_{10}(\omega_0)$ is a positive finite constant non depending on t. Next, combining the latter inequality with (6.14) and (6.22), one gets that, for all $j \in \mathbb{N}$ and $k \in \mathbb{Z}$ satisfying (6.14), and for each $t \in \mathcal{B}_3$, one has,

$$|Y(t,\omega_0) - Y(k2^{-j},\omega_0)| \le C_{11}(\omega_0)|t|^{\overline{H}}\big(1 + \big|\log|t|\big|\big)^{1/\alpha+\eta}, \tag{6.26}$$

where $C_{11}(\omega_0)$ is a constant non depending on j, k and t. Next, (6.18), (6.25) and (6.26), yield

$$A^3_{j,k}(\omega_0) \le C_{12}(\omega_0) 2^{-(1-H_*)j},$$

where $C_{12}(\omega_0)$ is a constant non depending on j and k. Moreover, the latter inequality implies that,

$$\limsup_{j\to+\infty} j^{\tau_0} 2^{-j/\alpha} \max\left\{A^3_{j,k}(\omega_0) : k \in \mathbb{Z} \text{ and } M_1 + 2^{-\frac{j}{2\alpha}} \le k/2^j \le M_2 - 2^{-\frac{j}{2\alpha}}\right\} = 0. \tag{6.27}$$

Finally, putting together, (6.17), (6.21), (6.24) and (6.27), it follows that (6.13) holds. □

Lemma 6.2. *Let Ω_0^* be the event of probability 1 introduced in Lemma 2.1 and let $\{\widetilde{g}_{j,k} : (j,k) \in \mathbb{N}\times\mathbb{Z}\}$ be the sequence of the random variables defined on Ω_0^* as,*

$$\widetilde{g}_{j,k} = 2^{j(1+H_*)} \int_{\mathbb{R}} X(t, H(k2^{-j}))\widetilde{\Psi}(2^j t - k, H_*)dt. \tag{6.28}$$

Assume that $H(\cdot)$ satisfies the Condition $(\mathcal{A})$. Then, for each $\omega \in \Omega_0^*$ and all $\theta \in [0, \min\{\gamma_* + 1/\alpha - H_*, 1 - H_*\})$, one has,

$$\limsup_{j\to+\infty} 2^{j(\theta-1/\alpha)} \max\left\{\left|g_{j,k}(\omega) - \widetilde{g}_{j,k}(\omega)\right| : k \in \mathbb{Z} \textit{ and } M_1 + 2^{-\frac{j}{2\alpha}} \le k/2^j \le M_2 - 2^{-\frac{j}{2\alpha}}\right\} = 0. \tag{6.29}$$

Remark 6.3. *Notice that (6.7) (in which one takes a, b such that $H_* \in [a,b]$) and Proposition 4.2 (in which one takes $q = 0$, $a = \underline{H}$, $b = \overline{H}$ and η an arbitrary positive real number), imply that the random variables $\widetilde{g}_{j,k}$ are well-defined and finite on Ω_0^*.*

Proof of Lemma 6.2. In all the sequel, we assume that $j \in \mathbb{N}$ and $k \in \mathbb{Z}$ are arbitrary and satisfy (6.14). Using (5.1), (6.11) and (6.28), one has,

$$|g_{j,k}(\omega) - \widetilde{g}_{j,k}(\omega)| \le \sum_{l=1}^{3} L_{j,k}^{l}(\omega), \tag{6.30}$$

where for all $l \in \{1,2,3\}$,

$$L_{j,k}^{l}(\omega) = 2^{j(1+H_*)} \int_{\mathcal{B}_l} |X(t,H(t),\omega) - X(t,H(k2^{-j}),\omega)||\widetilde{\Psi}(2^j t - k, H_*)|dt; \tag{6.31}$$

recall that the sets $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ have been defined in (6.16). Let us now prove that (6.29) holds, when $\left|g_{j,k}(\omega) - \widetilde{g}_{j,k}(\omega)\right|$ is replaced by $L_{j,k}^{1}(\omega)$. It follows from the definition of $\mathcal{B}_1$, (3.18) (in which one takes $q=0$, $M = M_0$, $a = \underline{H}$ and $b = \overline{H}$), (6.14), the Condition $(\mathcal{A})$, and the change of variable $u = 2^j t - k$, that,

$$\begin{aligned}
L_{j,k}^{1}(\omega) &\le C_1(\omega) 2^{j(1+H_*)} \int_{\mathcal{B}_1} |H(t) - H(k2^{-j})||\widetilde{\Psi}(2^j t - k, H_*)|dt \\
&\le C_2(\omega) 2^{j(1+H_*)} \int_{\mathcal{B}_1} |t - k2^{-j}|^{\gamma_*}|\widetilde{\Psi}(2^j t - k, H_*)|dt \\
&\le C_2(\omega) 2^{j(1+H_*)} \int_{\mathbb{R}} |t - k2^{-j}|^{\gamma_*}|\widetilde{\Psi}(2^j t - k, H_*)|dt \\
&= C_2(\omega) 2^{jH_*} \int_{\mathbb{R}} |2^{-j}u|^{\gamma_*}|\widetilde{\Psi}(u, H_*)|du \\
&\le C_3(\omega) 2^{j(H_* - \gamma_*)},
\end{aligned} \tag{6.32}$$

where the positive and finite constants $C_1(\omega)$, $C_2(\omega)$ and $C_3(\omega)$, do not depend on j and k. Then, using (6.32) and the inequality $\theta < \gamma_* + 1/\alpha - H_*$, one gets that,

$$\limsup_{j\to+\infty} 2^{j(\theta-1/\alpha)} \max\left\{L_{j,k}^{1}(\omega) : k \in \mathbb{Z} \text{ and } M_1 + 2^{-\frac{j}{2\alpha}} \le k/2^j \le M_2 - 2^{-\frac{j}{2\alpha}}\right\} = 0. \tag{6.33}$$

Next, let us prove that (6.29) holds, when $\left|g_{j,k}(\omega) - \widetilde{g}_{j,k}(\omega)\right|$ is replaced by $L_{j,k}^{2}(\omega)$. Let us set,

$$C_4(\omega) := \sup_{(u,v)\in[-2M_0,2M_0]\times[\underline{H},\overline{H}]} \left|X(u,v,\omega)\right| < \infty; \tag{6.34}$$

observe that $C_4(\omega)$ is finite, since the function $(u,v) \mapsto X(u,v,\omega)$ is continuous over the compact rectangle $[-2M_0, 2M_0] \times [\underline{H}, \overline{H}]$. Putting together, (6.31), (6.34), and (6.23), one obtains that,

$$L_{j,k}^{2}(\omega) \le C_5(\omega) 2^{-j(1-H_*-\frac{1}{\alpha})}, \tag{6.35}$$

where $C_5(\omega)$ is a constant non depending on j and k. Then, using (6.35) and the inequality $\theta < 1 - H_*$, it follows that,

$$\limsup_{j\to+\infty} 2^{j(\theta-1/\alpha)} \max\left\{L_{j,k}^{2}(\omega) : k \in \mathbb{Z} \text{ and } M_1 + 2^{-\frac{j}{2\alpha}} \le k/2^j \le M_2 - 2^{-\frac{j}{2\alpha}}\right\} = 0. \tag{6.36}$$

Next, let us prove that (6.29) holds, when $\left|g_{j,k}(\omega) - \widetilde{g}_{j,k}(\omega)\right|$ is replaced by $L_{j,k}^{3}(\omega)$. Setting in Proposition 4.2, $q = 0$, $a = \underline{H}$ and $b = \overline{H}$, one gets that for any fixed $\eta > 0$ and for each $t \in \mathcal{B}_3$,

$$|X(t,H(t),\omega) - X(t,H(k2^{-j}),\omega)| \le C_6(\omega)|t|^{\overline{H}}\big(1 + \left|\log|t|\right|\big)^{1/\alpha+\eta},$$

where $C_6(\omega)$ is a constant non depending on t and (j,k). Next combining the latter inequality with (6.31) and (6.25), it follows that,

$$L_{j,k}^{3}(\omega) \le C_7(\omega) 2^{-(1-H_*)j}, \tag{6.37}$$

where $C_7(\omega)$ is a constant non depending on j and k. Then, using (6.37) and the inequality $\theta < 1-H_*$, it follows that,

$$\limsup_{j\to+\infty} 2^{j(\theta-1/\alpha)} \max\Big\{L^3_{j,k}(\omega) : k\in\mathbb{Z} \text{ and } M_1+2^{-\frac{j}{2\alpha}} \le k/2^j \le M_2-2^{-\frac{j}{2\alpha}}\Big\} = 0. \tag{6.38}$$

Finally, putting together, (6.30), (6.33), (6.36) and (6.38), it follows that (6.29) holds. $\square$

Proposition 6.2. *Let Ω_0^* be the event of probability 1 which has been introduced in Lemma 2.1. Then for all $\omega\in\Omega_0^*$, $v\in(1/\alpha,1)$ and $(j,k)\in\mathbb{Z}^2$, one has,*

$$2^{j(1+v)}\int_{\mathbb{R}} X(t,v,\omega)\widetilde{\Psi}(2^jt-k,v)dt = \epsilon_{j,k}(\omega), \tag{6.39}$$

where $\epsilon_{j,k}$ is the random variable defined in (2.5).

Proof of the Proposition 6.2. First observe that by using (6.7) and (4.2) in which one takes $q=0$ and a,b such that $v\in[a,b]$, it follows that, for all $\omega\in\Omega_0^*$ and $(j,k)\in\mathbb{Z}^2$,

$$2^{j(1+v)}\left(\sum_{(j',k')\in\mathbb{Z}^2} 2^{-j'v}|\epsilon_{j',k'}(\omega)|\big|\Psi(2^{j'}\cdot-k',v)-\Psi(-k',v)\big|\right)\big|\widetilde{\Psi}(2^j\cdot-k,v)\big| \in L^1_t(\mathbb{R});$$

therefore we are allowed to apply the dominated convergence Theorem, and we obtain, in view of Theorem 2.1 Part (i), that

$$\begin{aligned}&2^{j(1+v)}\int_{\mathbb{R}} X(t,v,\omega)\widetilde{\Psi}(2^jt-k,v)dt\\ &= 2^{j(1+v)}\sum_{(j',k')\in\mathbb{Z}^2} 2^{-j'v}\epsilon_{j',k'}(\omega)\int_{\mathbb{R}}\big(\Psi(2^{j'}t-k',v)-\Psi(-k',v)\big)\widetilde{\Psi}(2^jt-k,v)dt.\end{aligned}$$

Finally, combining the latter equality with Proposition 6.1 Parts (ii) and (iii), one gets (6.39). $\square$

Remark 6.4. *Let τ and ρ be as in Theorem 6.1, also we suppose that (6.3) holds. We denote by τ_0 an arbitrary real number such that $\tau_0>\tau\ge 0$.*

(i) One has,

$$\frac{1+2\alpha^{-1}+\tau_0}{\rho} < \alpha\tau_0. \tag{6.40}$$

(ii) Denote by $d(\tau_0)$ and $e(\tau_0)$ the positive real numbers defined as,

$$d(\tau_0) := \frac{2}{3}\left(\frac{1+2\alpha^{-1}+\tau_0}{\rho}\right)+\frac{1}{3}(\alpha\tau_0) \quad\text{and}\quad e(\tau_0) := \frac{1}{3}\left(\frac{1+2\alpha^{-1}+\tau_0}{\rho}\right)+\frac{2}{3}(\alpha\tau_0), \tag{6.41}$$

then,

$$\frac{1+2\alpha^{-1}+\tau_0}{\rho} < d(\tau_0) < e(\tau_0) < \alpha\tau_0. \tag{6.42}$$

(iii) For any fixed $t_0\in[M_1,M_2]$ and $j\in\mathbb{N}$, denote by $D_j(t_0,\tau_0)$ the set of indices, defined as,

$$D_j(t_0,\tau_0) := \Big\{k\in\mathbb{Z} \,:\, k2^{-j}\in[M_1,M_2] \text{ and } j^{-e(\tau_0)} \le |t_0-k2^{-j}| \le j^{-d(\tau_0)}\Big\}, \tag{6.43}$$

then, for all j big enough, the set $D_j(t_0,\tau_0)$ is nonempty and satisfies,

$$D_j(t_0,\tau_0) \subseteq \Big\{k\in\mathbb{Z} : M_1+2^{-\frac{j}{2\alpha}} \le k/2^j \le M_2-2^{-\frac{j}{2\alpha}}\Big\}. \tag{6.44}$$

Proof of Remark 6.4. Observe that, in view of (6.2), one has

$$\frac{1+2\alpha^{-1}+\tau}{\rho}=\alpha\tau;$$

therefore, (6.3), implies that Part (i) holds. Part (ii) easily follows from (6.40) and (6.41). Let us give the proof of Part (iii), for the sake of simplicity we set $d=d(\tau_0)$ and $e=e(\tau_0)$. Observe that the set $D_j(t_0,\tau_0)$ is nonempty for all j big enough, since $\lim_{j\to+\infty}2^j(j^{-d}-j^{-e})=+\infty$. Let $j\geq 1$ and k be two arbitrary integers such that j is big enough and $k\in D_j(t_0,\tau_0)$. In order to show that, they satisfy (6.14), we will study three cases: $t_0\in(M_1,M_2)$, $t_0=M_1$ and $t_0=M_2$.

Let us first suppose that $M_1<t_0<M_2$ i.e. $\min\{t_0-M_1,M_2-t_0\}>0$, then in view of the fact that j is big enough, one can assume that $j^{-d}+2^{-\frac{j}{2\alpha}}\leq\min\{t_0-M_1,M_2-t_0\}$; the latter inequality and the inequality $|t_0-k2^{-j}|\leq j^{-d}$ imply that (6.14) holds.

Let us now assume that $t_0=M_1$. It follows from the equality $|t_0-k2^{-j}|=k2^{-j}-M_1$ and the inequalities $j^{-e}\leq|t_0-k2^{-j}|\leq j^{-d}$, that, $M_1+j^{-e}\leq k2^{-j}\leq M_1+j^{-d}$. Moreover, in view of the fact that j is big enough, one can assume $M_1+j^{-e}\geq M_1+2^{-\frac{j}{2\alpha}}$ and $M_1+j^{-d}\leq M_2-2^{-\frac{j}{2\alpha}}$; thus (6.14) holds. At last, the case where $t_0=M_2$, can be treated similarly to the case where $t_0=M_1$. □

Lemma 6.3. *Let τ be as in Theorem 6.1, also we suppose that (6.3) holds. We denote by τ_0 an arbitrary fixed real number such that $\tau_0>\tau\geq 0$. Then, for all $t_0\in[M_1,M_2]$, there exists $\Omega^*_{1,\tau_0}(t_0)$ an event of probability 1 (which a priori depends on τ_0 and t_0) included in Ω^*_0 (recall that the latter event has been introduced in Lemma 2.1), such that, for each $\omega\in\Omega^*_{1,\tau_0}(t_0)$, one has,*

$$\liminf_{j\to+\infty} j^{\tau_0}2^{-j/\alpha}\max\Big\{\big|\epsilon_{j,k}(\omega)\big|:k\in D_j(t_0,\tau_0)\Big\}>0,\tag{6.45}$$

where the $\epsilon_{j,k}$'s are the random variables defined in (2.5) and where $D_j(t_0,\tau_0)$ is the set introduced in (6.43).

Proof of Lemma 6.3. Let p be a fixed integer such that $p>2R$ (see (2.2) for the definition of R). We assume that j is an arbitrary big enough integer, so that the set,

$$\overline{D_j}(t_0,\tau_0):=\big\{q\in\mathbb{Z}\,:\,pq\in D_j(t_0,\tau_0)\big\}=\left\{q\in\mathbb{Z}\,:\,\frac{pq}{2^j}\in[M_1,M_2]\text{ and }j^{-e(\tau_0)}\leq|pq2^{-j}-t_0|\leq j^{-d(\tau_0)}\right\}.\tag{6.46}$$

is nonempty. From now on, for the sake of simplicity, $d(\tau_0)$ and $e(\tau_0)$ are respectively denoted by d and e. Notice that, since j is big enough, the cardinality of the set $\overline{D_j}(t_0,\tau_0)$ satisfies,

$$c_1j^{-d}2^j\leq\operatorname{card}\big(\overline{D_j}(t_0,\tau_0)\big)\leq c_2j^{-d}2^j,\tag{6.47}$$

where c_1 and c_2 are two positive constants non depending on j. We denote by Γ_j the event defined as,

$$\Gamma_j:=\Big\{\omega\in\Omega_0^*\,:\,\max\Big\{\big|\epsilon_{j,k}(\omega)\big|:k\in D_j(t_0,\tau_0)\Big\}\leq j^{-\tau_0}2^{j/\alpha}\Big\}.\tag{6.48}$$

Let us provide an appropriate upper bound for the probability $\mathbb{P}(\Gamma_j)$; since j is big enough, there is no restriction to suppose that $j^{-\tau_0}2^{j/\alpha}\geq 1$ and that $c_3j^{\alpha\tau_0}2^{-j}<1$, where c_3 is the positive constant c' in (2.9). Next, using, (6.48), (6.46), Part (iv) of Remark 2.1, (2.9), and the first inequality in

(6.47), one obtains that,

$$\begin{aligned}\mathbb{P}(\Gamma_j) &\leq \mathbb{P}\Bigg(\bigcap_{q\in\overline{D_j}(t_0,\tau_0)}\Big\{\big|\epsilon_{j,pq}\big|\leq j^{-\tau_0}2^{j/\alpha}\Big\}\Bigg) = \prod_{q\in\overline{D_j}(t_0,\tau_0)}\mathbb{P}\Big(\big|\epsilon_{j,pq}\big|\leq j^{-\tau_0}2^{j/\alpha}\Big)\\ &= \prod_{q\in\overline{D_j}(t_0,\tau_0)}\Big(1-\mathbb{P}\Big(\big|\epsilon_{j,pq}\big|> j^{-\tau_0}2^{j/\alpha}\Big)\Big)\\ &\leq \Big(1-c_3 j^{\alpha\tau_0}2^{-j}\Big)^{c_1 j^{-d}2^j};\end{aligned}\tag{6.49}$$

moreover, the inequality $\log(1-x)\leq -x$ for all $x\in[0,1)$, allows to prove that,

$$\Big(1-c_3 j^{\alpha\tau_0}2^{-j}\Big)^{c_1 j^{-d}2^j} := \exp\Big(c_1 j^{-d}2^j\log(1-c_3 j^{\alpha\tau_0}2^{-j})\Big)\leq \exp\big(-c_1c_3 j^{\alpha\tau_0-d}\big).\tag{6.50}$$

Finally putting together (6.42), (6.49) and (6.50), one gets that,

$$\sum_{j\in\mathbb{N}}\mathbb{P}(\Gamma_j)<\infty;$$

thus Borel-Cantelli Lemma implies that (6.45) holds. □

Lemma 6.4. *Let τ be as in Theorem 6.1 also we suppose that (6.3) holds. We denote by τ_0 an arbitrary fixed real number such that $\tau_0>\tau\geq 0$. Then there exists $t_0\in[M_1,M_2]$ (a priori t_0 depends on τ_0) such that, for all $\omega\in\Omega_0^*$ (the event of probability 1 introduced in Lemma 2.1), one has,*

$$\limsup_{j\to+\infty} j^{\tau_0}2^{-j/\alpha}\max\Big\{\big|\widetilde{g}_{j,k}(\omega)-\epsilon_{j,k}(\omega)\big|: k\in D_j(t_0,\tau_0)\Big\}=0;\tag{6.51}$$

recall that the random variables $\widetilde{g}_{j,k}$ and $\epsilon_{j,k}$ have been defined respectively in (6.28) and (2.5), also recall that the set $D_j(t_0,\tau_0)$ has been introduced in (6.43).

Proof of Lemma 6.4. Let ρ be as in (6.1). Assume that $\rho_0\in(1/\alpha,\rho)$ is arbitrary and such that,

$$\frac{1+2\alpha^{-1}+\tau_0}{\rho}<\frac{1+2\alpha^{-1}+\tau_0}{\rho_0}<d(\tau_0)<e(\tau_0)<\alpha\tau_0,\tag{6.52}$$

where $d(\tau_0)$ and $e(\tau_0)$ are defined in (6.41). Then, in view of (6.1) and (6.10), there exists $t_0\in[M_1,M_2]$, which satisfies,

$$\begin{cases} H(t_0)=H_*\\ \sup_{t\in[M_1,M_2]}\frac{|H(t)-H(t_0)|}{|t-t_0|^{\rho_0}}<\infty.\end{cases}\tag{6.53}$$

In all the sequel, we suppose that j is an arbitrary big enough integer, thus the set $D_j(t_0,\tau_0)$ is nonempty and (6.44) holds; also we suppose that $k\in D_j(t_0,\tau_0)$ is arbitrary. Using (6.39) in which one takes $v=H_*$, (6.28), and the equality, for each fixed $t\in\mathbb{R}$,

$$X(t,H(k2^{-j}),\omega)-X(t,H_*,\omega)=\big(H(k2^{-j})-H_*\big)\int_0^1\big(\partial_v X\big)\big(t,H_*+\theta(H(k2^{-j})-H_*),\omega\big)d\theta,$$

one gets that

$$\begin{aligned}\widetilde{g}_{j,k}(\omega)-\epsilon_{j,k}(\omega) &= 2^{j(1+H_*)}\int_{\mathbb{R}}\big(X(t,H(k2^{-j}),\omega)-X(t,H_*,\omega)\big)\widetilde{\Psi}(2^jt-k,H_*)dt\\ &= 2^{j(1+H_*)}\big(H(k2^{-j})-H_*\big)\int_{\mathbb{R}}\int_0^1\big(\partial_v X\big)\big(t,H_*+\theta(H(k2^{-j})-H_*),\omega\big)\widetilde{\Psi}(2^jt-k,H_*)d\theta dt.\end{aligned}$$

Therefore, it follows from (6.8) in which one takes $v = H_*$, that,

$$|\epsilon_{j,k}(\omega) - \widetilde{g}_{j,k}(\omega)| \le |H(k2^{-j}) - H_*| \sum_{l=1}^{3} F_{j,k}^{l}(\omega), \tag{6.54}$$

where, for each $l \in \{1,2,3\}$

$$\begin{aligned} F_{j,k}^{l}(\omega) = 2^{j(1+H_*)} \int_{\mathcal{B}_l} \int_0^1 |\widetilde{\Psi}(2^j t - k, H_*)| \\ \times \big| \big(\partial_v X\big)\big(t, H_* + \theta(H(k2^{-j}) - H_*), \omega\big) - \big(\partial_v X\big)\big(k2^{-j}, H_* + \theta(H(k2^{-j}) - H_*), \omega\big)\big| d\theta dt; \end{aligned} \tag{6.55}$$

recall that the sets $\mathcal{B}_l$ are defined in (6.16). Observe that, in view of (6.53) and (6.43), one has,

$$|H(k2^{-j}) - H_*| \le c_1 j^{-d(\tau_0)\rho_0}, \tag{6.56}$$

where c_1 is a constant non depending on j and k. Also, observe that in view of (6.52), there exists η_1, an arbitrarily small positive real number such that

$$\frac{1 + 2\alpha^{-1} + \tau_0 + \eta_1}{\rho_0} < d(\tau_0). \tag{6.57}$$

Let us now, prove that (6.51) holds when $\big|\widetilde{g}_{j,k}(\omega) - \epsilon_{j,k}(\omega)\big|$ is replaced by $|H(k2^{-j}) - H_*|F_{j,k}^1(\omega)$. Using Proposition 4.1 (in which one takes $q=1$, $M = M_0$, $a = \underline{H}$, $b = \overline{H}$ and $\eta = \eta_1$), the inequality $H(k2^{-j}) \ge H_*$ and the fact that $k2^{-j} \in \mathcal{B}_1 \subset [-M_0, M_0]$, one gets that,

$$\begin{aligned} F_{j,k}^1(\omega) &\le C_2(\omega) 2^{j(1+H_*)} \int_{\mathcal{B}_1} \int_0^1 |\widetilde{\Psi}(2^j t - k, H_*)| \\ &\qquad \times |t - k2^{-j}|^{H_* - 1/\alpha + \theta(H(k2^{-j}) - H_*)} \Big(1 + \big|\log|t - k2^{-j}|\big|\Big)^{1+2/\alpha+\eta_1} d\theta dt \\ &\le C_2(\omega) 2^{j(1+H_*)} \int_{\mathcal{B}_1} |\widetilde{\Psi}(2^j t - k, H_*)| |t - k2^{-j}|^{H_* - 1/\alpha} \Big(1 + \big|\log|t - k2^{-j}|\big|\Big)^{1+2/\alpha+\eta_1} \\ &\qquad \times \Big\{ \int_0^1 |t - k2^{-j}|^{\theta(H(k2^{-j}) - H_*)} d\theta \Big\} dt \\ &\le C_3(\omega) 2^{j(1+H_*)} \int_{\mathbb{R}} |\widetilde{\Psi}(2^j t - k, H_*)| |t - k2^{-j}|^{H_* - 1/\alpha} \Big(1 + \big|\log|t - k2^{-j}|\big|\Big)^{1+2/\alpha+\eta_1} dt, \end{aligned}$$

where $C_2(\omega)$ is a constant non depending on j and k and where $C_3(\omega) = \big(1 + 2M_0\big)^{\overline{H} - H_*} C_2(\omega)$. Then, setting $u = 2^j t - k$ in the last integral, and using Lemma 9.1, one obtains that,

$$\begin{aligned} F_{j,k}^1(\omega) &\le C_3(\omega) 2^{j/\alpha} \int_{\mathbb{R}} |u|^{H_* - 1/\alpha} \Big(1 + \big|\log|2^{-j} u|\big|\Big)^{1+2/\alpha+\eta_1} |\widetilde{\Psi}(u, H_*)| du \\ &\le C_4(\omega) 2^{j/\alpha} \int_{\mathbb{R}} |u|^{H_* - 1/\alpha} \Big(j^{1+2/\alpha+\eta_1} + \big(1 + \big|\log|u|\big|\big)^{1+2/\alpha+\eta_1}\Big) |\widetilde{\Psi}(u, H_*)| du \\ &\le C_5(\omega) j^{1+2/\alpha+\eta_1} 2^{j/\alpha}, \end{aligned} \tag{6.58}$$

where $C_4(\omega)$ and $C_5(\omega)$ are two constants non depending on j and k. Putting together, (6.56), (6.57) and (6.58), it follows that

$$\limsup_{j \to +\infty} j^{\tau_0} 2^{-j/\alpha} \max\Big\{ |H(k2^{-j}) - H_*| F_{j,k}^1(\omega) : k \in D_j(t_0, \tau_0) \Big\} = 0. \tag{6.59}$$

Let us now prove that (6.51) holds when $\big|\widetilde{g}_{j,k}(\omega)-\epsilon_{j,k}(\omega)\big|$ is replaced by $|H(k2^{-j})-H_*|F^2_{j,k}(\omega)$. We set,

$$C_6(\omega) := \sup_{(u,v)\in[-2M_0,2M_0]\times[\underline{H},\overline{H}]} \big|(\partial_v X)(u,v,\omega)\big| < \infty; \tag{6.60}$$

observe that $C_6(\omega)$ is finite, since the function $(u,v)\mapsto(\partial_v X)(u,v,\omega)$ is continuous over the compact rectangle $[-2M_0,2M_0]\times[\underline{H},\overline{H}]$. Putting together, (6.55), (6.60), (6.44) and (6.23), one obtains that,

$$F^2_{j,k}(\omega)\le C_7(\omega)2^{-j(1-H_*-\frac{1}{\alpha})}, \tag{6.61}$$

where $C_7(\omega)$ is a constant non depending on j and k. Then, using (6.61), the fact that $H(\cdot)$ is a bounded function, and the inequality $0<1-H_*$, it follows that,

$$\limsup_{j\to+\infty} j^{\tau_0}2^{-j/\alpha}\max\Big\{|H(k2^{-j})-H_*|F^2_{j,k}(\omega) : k\in D_j(t_0,\tau_0)\Big\}=0. \tag{6.62}$$

Let us now prove that (6.51) holds when $\big|\widetilde{g}_{j,k}(\omega)-\epsilon_{j,k}(\omega)\big|$ is replaced by $|H(k2^{-j})-H_*|F^3_{j,k}(\omega)$. Setting in Proposition 4.2, $q=1$, $a=\underline{H}$ and $b=\overline{H}$, one gets , in view of (6.60), that for any fixed $\eta>0$, for each $t\in\mathcal{B}_3$ and for all $\theta\in[0,1]$,

$$\big|\big(\partial_v X\big)\big(t,H_*+\theta(H(k2^{-j})-H_*),\omega\big)-\big(\partial_v X\big)\big(k2^{-j},H_*+\theta(H(k2^{-j})-H_*),\omega\big)\big|\le C_8(\omega)|t|^{\overline{H}}\big(1+\big|\log|t|\big|\big)^{1+1/\alpha+\eta},$$

where $C_8(\omega)$ is a constant non depending on t, θ and (j,k). Next combining the latter inequality with (6.55) and (6.25), it follows that,

$$F^3_{j,k}(\omega_0)\le C_9(\omega)2^{-(1-H_*)j}, \tag{6.63}$$

where $C_9(\omega)$ is a constant non depending on j and k. Then, using (6.63), the fact that $H(\cdot)$ is a bounded function, and the inequality $0<1-H_*$, it follows that,

$$\limsup_{j\to+\infty} j^{\tau_0}2^{-j/\alpha}\max\Big\{|H(k2^{-j})-H_*|F^3_{j,k}(\omega) : k\in D_j(t_0,\tau_0)\Big\}=0. \tag{6.64}$$

Finally, putting together, (6.54), (6.59) (6.62) and (6.64), it follows that (6.51) holds. □

Lemma 6.5. *Let τ be as in Theorem 6.1, also we suppose that the Conditions $(\mathcal{A})$ and (6.3) hold. We denote by τ_0 an arbitrary fixed real number such that $\tau_0>\tau\ge 0$. Then there exists Ω^*_{2,τ_0} an event of probability 1 (which a priori depends on τ_0) included in Ω^*_0 (recall that the latter event has been introduced in Lemma 2.1), such that, for each $\omega\in\Omega^*_{2,\tau_0}$, one has,*

$$\liminf_{j\to+\infty} j^{\tau_0}2^{-j/\alpha}\max\Big\{\big|g_{j,k}(\omega)\big| : k\in\mathbb{Z} \text{ and } M_1+2^{-\frac{j}{2\alpha}}\le k/2^j\le M_2-2^{-\frac{j}{2\alpha}}\Big\}>0, \tag{6.65}$$

where the $g_{j,k}$'s are the random variables defined in (6.11).

Proof of Lemma 6.5. Putting together (6.44) and Lemmas 6.4, 6.3, 6.2; one gets the lemma. □

Now, we are in position to prove Theorem 6.1.

Proof of Theorem 6.1. Denote by Ω^*_3, the event of probability 1, defined as,

$$\Omega^*_3 := \bigcap_{\tau_0\in\mathbb{Q}\text{ and }\tau_0>\tau}\Omega^*_{2,\tau_0};$$

recall that the events Ω^*_{2,τ_0} have been introduced in Lemma 6.5. It is clear that (6.65) holds for all $\omega\in\Omega^*_3$ and for all real number $\tau_0>\tau\ge 0$; therefore, it follows from Lemma 6.1, that for each $\omega\in\Omega^*_3$, $\tau_0>\tau$ and $\eta_0>0$,

$$\sup_{(t,s)\in[M_1,M_2]^2}\frac{\big|Y(t,\omega)-Y(s,\omega)\big|}{|t-s|^{H_*-1/\alpha}\big(1+\big|\log|t-s|\big|\big)^{-\tau_0-\eta_0}}=\infty.$$

Then, in view of (6.10), one gets the theorem. □

7. Optimality of local modulus of continuity of LMSM

The goal of this section is to show that under a condition a bit stronger than (5.10), the local modulus of continuity given in (5.11), is optimal, more precisely:

Theorem 7.1. *Let M be an arbitrary positive real number. Assume that $t_0 \in (-M, M)$ satisfies for some constant $c > 0$ and all $t \in \mathbb{R}$,*

$$\big|H(t) - H(t_0)\big| \le c|t - t_0|^{H(t_0)}\big(1 + \big|\log|t - t_0|\big|\big)^{1/\alpha}. \tag{7.1}$$

Then, one has almost surely,

$$\sup_{t \in [-M,M]} \left\{ \frac{\big|Y(t) - Y(t_0)\big|}{|t - t_0|^{H(t_0)}\big(1 + \big|\log|t - t_0|\big|\big)^{1/\alpha}} \right\} = \infty. \tag{7.2}$$

The proof of Theorem 7.1 relies on (3.18) in which one takes $q = 0$, also, more importantly, it relies on the following proposition.

Proposition 7.1. *Let M be an arbitrary positive real number. For all $t_0 \in (-M, M)$, one has almost surely,*

$$\sup_{t \in [-M,M]} \left\{ \frac{\big|X(t, H(t_0)) - X(t_0, H(t_0))\big|}{|t - t_0|^{H(t_0)}\big(1 + \big|\log|t - t_0|\big|\big)^{1/\alpha}} \right\} = \infty. \tag{7.3}$$

In order to show that Proposition 7.1 holds, we need to introduce some additional notations, also we need to derive some preliminary results. Let m_0 be the positive integer defined as,

$$m_0 := \big[\log_2(3R + 2)\big] + 1; \tag{7.4}$$

recall that, R is a fixed real number strictly bigger than 1, such that (2.2) holds. For all $j \in \mathbb{N}$, one sets,

$$r(j, m_0) := j m_0 \text{ and } l(j, m_0) := \big[2^{r(j,m_0)} t_0 + R + 2\big]; \tag{7.5}$$

observe that, the inequalities,

$$(R + 1)2^{-r(j,m_0)} < l(j, m_0)2^{-r(j,m_0)} - t_0 < (R + 1)2^{1-r(j,m_0)} < 4/5, \tag{7.6}$$

hold. One denotes by $\check{\epsilon}_j$ the $\mathcal{S}\alpha\mathcal{S}$ random variable,

$$\check{\epsilon}_j := \epsilon_{r(j,m_0), l(j,m_0)}\,, \tag{7.7}$$

in other words, $\check{\epsilon}_j$ is defined through (2.5) in which j and k are replaced, respectively by $r(j, m_0)$ and $l(j, m_0)$.

Lemma 7.1. *The $\mathcal{S}\alpha\mathcal{S}$ random variables $\check{\epsilon}_j$, $j \in \mathbb{N}$ are independent and they all have the same scale parameter, namely, for each $j \in \mathbb{N}$,*

$$\|\check{\epsilon}_j\|_\alpha = \left\{ \int_{\mathbb{R}} |\psi(t)|^\alpha dt \right\}^{1/\alpha}. \tag{7.8}$$

Proof of Lemma 7.1. First observe that (7.8) is a straightforward consequence of (7.7) and (2.8). Let us now prove that the random variables $\check{\epsilon}_j$, $j \in \mathbb{N}$ are independent. Notice that (2.2), entails that,

$$\operatorname{supp} \psi\big(2^{r(j,m_0)} \cdot - l(j, m_0)\big) \subseteq \big[l(j, m_0)2^{-r(j,m_0)} - R2^{-r(j,m_0)}, l(j, m_0)2^{-r(j,m_0)} + R2^{-r(j,m_0)}\big];$$

therefore, in view of (2.5) and the fact that the $\mathcal{S}t\alpha\mathcal{S}$ random measure $Z_\alpha(ds)$ is independently scattered, it is sufficient to show that the intervals $\big[l(j,m_0)2^{-r(j,m_0)} - R2^{-r(j,m_0)}, l(j,m_0)2^{-r(j,m_0)} + R2^{-r(j,m_0)}\big]$, $j \in \mathbb{N}$ are disjoint. The latter result can be obtained by proving that, the inequality,

$$R2^{-r(j,m_0)} + R2^{-r(j+p,m_0)} < \big|l(j,m_0)2^{-r(j,m_0)} - l(j+p,m_0)2^{-r(j+p,m_0)}\big|, \tag{7.9}$$

holds for all $(j,p) \in \mathbb{N}^2$. By using the triangle inequality, (7.6) and the first equality in (7.5), one has,

$$\begin{aligned}&\big|l(j,m_0)2^{-r(j,m_0)} - l(j+p,m_0)2^{-r(j+p,m_0)}\big| \geq \big|l(j,m_0)2^{-r(j,m_0)} - t_0\big| - \big|l(j+p,m_0)2^{-r(j+p,m_0)} - t_0\big| \\ &> (R+1)2^{-r(j,m_0)} - (R+1)2^{1-r(j+p,m_0)} = (R+1)2^{-jm_0}\big(1-2^{1-pm_0}\big) \geq (R+1)2^{-jm_0}\big(1-2^{1-m_0}\big).\end{aligned} \tag{7.10}$$

On the other hand, the first equality in (7.5), imply that

$$R2^{-r(j,m_0)} + R2^{-r(j+p,m_0)} = R2^{-jm_0}\big(1+2^{-pm_0}\big) \leq R2^{-jm_0}\big(1+2^{-m_0}\big). \tag{7.11}$$

Next, notice that (7.4), implies that $2^{-m_0} < (3R+2)^{-1}$ and consequently that,

$$R\big(1+2^{-m_0}\big) < \frac{3R(R+1)}{3R+2} < (R+1)\big(1-2^{1-m_0}\big). \tag{7.12}$$

Finally, putting together (7.10), (7.11) and (7.12), one gets (7.9) $\square$

Lemma 7.2. *One has, almost surely,*

$$\limsup_{j\to+\infty} \frac{|\check{\epsilon}_j|}{j^{1/\alpha}\log^{1/\alpha}(j)} \geq 1. \tag{7.13}$$

Proof of Lemma 7.2. Notice that, in view of Lemma 7.1, the events $\Big\{|\check{\epsilon}_j| > j^{1/\alpha}\log^{1/\alpha}(j)\Big\}$, $j \in \mathbb{N}$, are independent; moreover, (7.7) and the first inequality in (2.9), imply that,

$$\sum_{j=2}^{+\infty} \mathbb{P}\Big(|\check{\epsilon}_j| > j^{1/\alpha}\log^{1/\alpha}(j)\Big) \geq c' \sum_{j=2}^{+\infty} j^{-1}\log^{-1}(j) = +\infty.$$

Thus, applying the second Borel-Cantelli Lemma, one gets (7.13). $\square$

Lemma 7.3. *Let Ω_0^* be the event of probability 1 introduced in Lemma 2.1. Assume that for some $t_0 \in (-M,M)$ and $\omega_0 \in \Omega_0^*$, one has,*

$$\sup_{t\in[-M,M]} \left\{ \frac{\big|X(t,H(t_0),\omega_0) - X(t_0,H(t_0),\omega_0)\big|}{|t-t_0|^{H(t_0)}\big(1+\big|\log|t-t_0|\big|\big)^{1/\alpha}} \right\} < \infty. \tag{7.14}$$

Then, it follows that,

$$\limsup_{j\to+\infty} \frac{|\check{\epsilon}_j(\omega_0)|}{j^{1/\alpha}} < \infty. \tag{7.15}$$

Proof of Lemma 7.3. First notice that, (7.7), (6.39) in which one takes $v = H(t_0)$, (6.8), and the change of variable $x = t - l_j 2^{-r_j}$, imply that,

$$\begin{aligned}\check{\epsilon}_j(\omega_0) &= 2^{r_j(1+H(t_0))} \int_{\mathbb{R}} \Big(X(t,H(t_0),\omega_0) - X(t_0,H(t_0),\omega_0)\Big)\widetilde{\Psi}(2^{r_j}t - l_j, H(t_0))dt \\ &= 2^{r_j(1+H(t_0))} \int_{\mathbb{R}} \Big(X\big(x + l_j 2^{-r_j}, H(t_0),\omega_0\big) - X\big(t_0,H(t_0),\omega_0\big)\Big)\widetilde{\Psi}(2^{r_j}x, H(t_0))dx,\end{aligned} \tag{7.16}$$

where, for the sake of simplicity, we have set $r_j = r(j,m_0)$ and $l_j = l(j,m_0)$. Let $s_* := |t_0| + 2$, observe that, in view of (7.6), one has,

$$\forall\, x \in \mathbb{R},\ |x| \geq s_* \Longrightarrow \big|x + l_j 2^{-r_j}\big| \geq 1. \tag{7.17}$$

Also, observe that (7.16) entails that,

$$\left|\check{\epsilon}_j(\omega_0)\right| \le S_j + Z_j, \tag{7.18}$$

where

$$S_j = 2^{r_j(1+H(t_0))} \int_{|x|<s_*} \left| X\big(x + l_j 2^{-r_j}, H(t_0), \omega_0\big) - X\big(t_0, H(t_0), \omega_0\big)\right| \left|\widetilde{\Psi}(2^{r_j}x, H(t_0))\right| dx \tag{7.19}$$

and

$$Z_j = 2^{r_j(1+H(t_0))} \int_{|x|\ge s_*} \left| X\big(x + l_j 2^{-r_j}, H(t_0), \omega_0\big) - X\big(t_0, H(t_0), \omega_0\big)\right| \left|\widetilde{\Psi}(2^{r_j}x, H(t_0))\right| dx. \tag{7.20}$$

Let us now give an appropriate upper bound for S_j. Notice that, the fact that $t \mapsto X(t, H(t_0), \omega_0)$ is a continuous function over $\mathbb{R}$, entails that, (7.14) remains valid, when $[-M, M]$ is replaced by any other compact interval; also notice that, in view of (7.6), when $|x| < s_*$, then $x + l_j 2^{-r_j}$ belongs to the compact interval $\big[-s_* - |t_0| - 4/5, s_* + |t_0| + 4/5\big]$. Thus, using (7.14) in which M is replaced by $s_* + |t_0| + 4/5$, one gets that,

$$\begin{aligned} S_j &\le C_1(\omega_0) 2^{r_j(1+H(t_0))} \int_{|x|<s_*} |\nu_j + x|^{H(t_0)} \Big(1 + \big|\log|\nu_j + x|\big|\Big)^{1/\alpha} \left|\widetilde{\Psi}(2^{r_j}x, H(t_0))\right| dx \\ &\le C_1(\omega_0) 2^{r_j(1+H(t_0))} \int_{\mathbb{R}} |\nu_j + x|^{H(t_0)} \Big(1 + \big|\log|\nu_j + x|\big|\Big)^{1/\alpha} \left|\widetilde{\Psi}(2^{r_j}x, H(t_0))\right| dx, \end{aligned} \tag{7.21}$$

where, $C_1(\omega_0)$ is a constant non depending on j, and

$$\nu_j := l_j 2^{-r_j} - t_0; \tag{7.22}$$

observe that (7.6) implies that,

$$R + 1 < 2^{r_j}\nu_j < 2R + 2. \tag{7.23}$$

For the sake of convenience, let us set,

$$c_2 := \sup_{y\in\mathbb{R}} (3 + |y|)^2 \left|\widetilde{\Psi}(y, H(t_0))\right| < \infty; \tag{7.24}$$

observe that the inequality in (7.24) results from (6.7). Next, making in (7.21) the change of variable $u = x/\nu_j$, and using the triangle inequality, (7.23), (7.24), (7.22), the last two inequalities in (7.6), and the the first equality in (7.5), it follows that,

$$\begin{aligned} S_j &\le C_1(\omega_0)\, 2^{r_j(1+H(t_0))}\nu_j \int_{\mathbb{R}} |\nu_j + \nu_j u|^{H(t_0)} \Big(1 + \big|\log|\nu_j + \nu_j u|\big|\Big)^{1/\alpha} \left|\widetilde{\Psi}(2^{r_j}\nu_j u, H(t_0))\right| du \\ &= C_1(\omega_0)\, 2^{r_j(1+H(t_0))}\nu_j^{1+H(t_0)} \int_{\mathbb{R}} |1 + u|^{H(t_0)} \Big(1 + \big|\log(\nu_j) + \log|1+u|\big|\Big)^{1/\alpha} \left|\widetilde{\Psi}(2^{r_j}\nu_j u, H(t_0))\right| du \\ &\le C_3(\omega_0) \big(2^{r_j}\nu_j\big)^{1+H(t_0)} \left|\log(\nu_j)\right|^{1/\alpha}, \\ &\le C_4(\omega_0)\, j^{1/\alpha} \end{aligned} \tag{7.25}$$

where, the constant,

$$C_3(\omega_0) := c_2\big(\log(5/4)\big)^{-1/\alpha} C_1(\omega_0) \int_{\mathbb{R}} \frac{\big|2 + |\log|1+u||\big|^{1/\alpha}}{(3 + |u|)^{2-H(t_0)}} du < \infty,$$

and the constant $C_4(\omega_0) = C_3(\omega_0)\big(2R+2\big)^{1+H(t_0)} m_0^{1/\alpha}$. Let us now give an appropriate upper bound for Z_j. Using (7.20), (7.24) and the triangle inequality, one obtains that,

$$Z_j \le c_2 2^{-r_j(1-H(t_0))} \int_{|x|\ge s_*} \left|X\big(x + l_j 2^{-r_j}, H(t_0), \omega_0\big)\right| x^{-2} dx + C_5(\omega_0) 2^{-r_j(1-H(t_0))}, \tag{7.26}$$

where, the constant

$$C_5(\omega_0) := c_2\big|X\big(t_0, H(t_0), \omega_0\big)\big| \int_{|x|\geq s_*} x^{-2}dx < \infty.$$

Next, observe that, (7.17) and (7.6) imply that for all real number x which satisfies $|x| \geq s_*$, and for each $j \in \mathbb{N}$, one has,

$$1 \leq \big|x + l_j 2^{-r_j}\big| \leq |x| + |t_0| + 1;$$

thus, taking in (4.2), $q = 0$, a, b such that $H(t_0) \in [a, b]$, and η an arbitrary fixed positive real number, it follows that,

$$\big|X\big(x + l_j 2^{-r_j}, H(t_0), \omega_0\big)\big| \leq C_6(\omega_0)\big(|x| + |t_0| + 1\big)^{H(t_0)}\Big(1 + \log\big(|x| + |t_0| + 1\big)\Big)^{1/\alpha+\eta}, \tag{7.27}$$

where the finite constant $C_6(\omega_0)$ does not depend on x and j. Next, combining (7.26) with (7.27), one gets, that,

$$Z_j \leq C_7(\omega_0) 2^{-r_j(1-H(t_0))}, \tag{7.28}$$

where the finite constant

$$C_7(\omega_0) := C_5(\omega_0) + c_2 \int_{|x|\geq s_*} \big(|x| + |t_0| + 1\big)^{H(t_0)}\Big(1 + \log\big(|x| + |t_0| + 1\big)\Big)^{1/\alpha+\eta} x^{-2}dx.$$

Finally, putting together, (7.18), (7.25), (7.28) and the first equality in (7.5), one obtains (7.15) □

Now, we are in position to prove Proposition 7.1 and Theorem 7.1.

PROOF OF PROPOSITION 7.1. The proposition is a straightforward consequence of Lemmas 7.2 and 7.3. □

PROOF OF THEOREM 7.1. Using (5.1) and the triangle inequality, one has, for all $t \in [-M, M]$,

$$\big|X(t, H(t_0)) - X(t_0, H(t_0))\big| \leq \big|Y(t) - Y(t_0)\big| + \big|X(t, H(t)) - X(t, H(t_0))\big|,$$

and, as a consequence,

$$\begin{aligned}
&\sup_{t\in[-M,M]} \left\{ \frac{\big|X(t, H(t_0)) - X(t_0, H(t_0))\big|}{|t - t_0|^{H(t_0)}\big(1 + \big|\log|t - t_0|\big|\big)^{1/\alpha}} \right\} \\
&\leq \sup_{t\in[-M,M]} \left\{ \frac{\big|Y(t) - Y(t_0)\big|}{|t - t_0|^{H(t_0)}\big(1 + \big|\log|t - t_0|\big|\big)^{1/\alpha}} \right\} + \sup_{t\in[-M,M]} \left\{ \frac{\big|X(t, H(t)) - X(t, H(t_0))\big|}{|t - t_0|^{H(t_0)}\big(1 + \big|\log|t - t_0|\big|\big)^{1/\alpha}} \right\}.
\end{aligned}$$

Thus, in view of (7.3), in order to show that (7.2) holds, it is sufficient to prove that,

$$\sup_{t\in[-M,M]} \left\{ \frac{\big|X(t, H(t)) - X(t, H(t_0))\big|}{|t - t_0|^{H(t_0)}\big(1 + \big|\log|t - t_0|\big|\big)^{1/\alpha}} \right\} < \infty. \tag{7.29}$$

Taking in (3.18) $q = 0$, $a = \underline{H} := \inf_{x\in\mathbb{R}} H(x)$ and $b := \overline{H} := \sup_{x\in\mathbb{R}} H(x)$, one gets that,

$$\sup_{t\in[-M,M]} \left\{ \frac{\big|X(t, H(t)) - X(t, H(t_0))\big|}{|H(t) - H(t_0)|} \right\} < \infty. \tag{7.30}$$

Finally, combining (7.1) with (7.30), it follows that (7.29) holds. □

8. Local Hölder exponent of LMSM

The goal of this section is to determine the local Hölder exponent of a typical path of LMSM. Let us first recall, in a general framework, the definition of this exponent.

Denote by f an arbitrary deterministic real-valued continuous function defined on the real line. The critical global Hölder regularity of f, over an arbitrary nonempty compact interval $[M_1, M_2]$, can be measured through,

$$\rho_f^{\mathrm{unif}}\big([M_1, M_2]\big) := \sup\left\{\rho \geq 0 : \sup_{s', s'' \in [M_1, M_2]} \frac{|f(s') - f(s'')|}{|s' - s''|^{\rho}} < \infty\right\}, \tag{8.1}$$

the uniform (or global) Hölder exponent of f over $[M_1, M_2]$; observe that one has

$$\rho_f^{\mathrm{unif}}\big([M_1', M_2']\big) \geq \rho_f^{\mathrm{unif}}\big([M_1, M_2]\big), \tag{8.2}$$

when $[M_1', M_2'] \subseteq [M_1, M_2]$. The local Hölder regularity of f in a neighborhood of some point $t_0 \in \mathbb{R}$, can be measured through,

$$\rho_f^{\mathrm{unif}}(t_0) := \sup\left\{\rho_f^{\mathrm{unif}}\big([M_1, M_2]\big) : M_1 \in \mathbb{R},\ M_2 \in \mathbb{R} \text{ and } M_1 < t_0 < M_2\right\}, \tag{8.3}$$

the local Hölder exponent of f at t_0; notice that the latter exponent is sometime called the uniform pointwise Hölder exponent of f at t_0 (see [30]).

Let $t \mapsto Y(t,\omega)$ be a continuous path of the LMSM $\{Y(t) : t \in \mathbb{R}\}$. The uniform Hölder exponent of $t \mapsto Y(t,\omega)$ over $[M_1, M_2]$, is denoted by $\rho_Y^{\mathrm{unif}}\big([M_1, M_2], \omega\big)$; the local Hölder exponent of $t \mapsto Y(t,\omega)$ at t_0, is denoted by $\rho_Y^{\mathrm{unif}}(t_0, \omega)$.

Thanks to Part (ii) of Corollary 5.1 and thanks to Theorem 6.1, under some Hölder condition on $H(\cdot)$, one can, almost surely for all $t_0 \in \mathbb{R}$, completely determine $\rho_Y^{\mathrm{unif}}(t_0, \omega)$, more precisely:

Theorem 8.1. *There is Ω_4^* an event of probability 1 (non depending on t_0), such that for all $\omega \in \Omega_4^*$ and for each $t_0 \in \mathbb{R}$ satisfying,*

$$\rho_H^{unif}(t_0) > 1/\alpha, \tag{8.4}$$

one has,

$$\rho_Y^{unif}(t_0, \omega) = H(t_0) - 1/\alpha. \tag{8.5}$$

Notice that the latter theorem is a more precise result than Theorem 4.1 in [30].

Proof of Theorem 8.1. The theorem does not make sense if there is no $t_0 \in \mathbb{R}$ which satisfies (8.4), so in all the sequel, we assume that (8.4) is satisfied for some $t_0 \in \mathbb{R}$. In view of (8.2) and (8.3), this assumption implies that the set,

$$\Lambda := \left\{(\mu_1, \mu_2) \in \mathbb{Q}^2 : \mu_1 < \mu_2 \text{ and } \rho_H^{\mathrm{unif}}\big([\mu_1, \mu_2]\big) > 1/\alpha\right\},$$

is nonempty. Next, observe that, (8.1), Part (ii) of Corollary 5.1, Theorem 6.1 and Remark 6.1, implies that, for all $(\mu_1, \mu_2) \in \Lambda$, one has, almost surely,

$$\rho_Y^{\mathrm{unif}}\big([\mu_1, \mu_2]\big) = \min_{x \in [\mu_1, \mu_2]} H(x) - 1/\alpha; \tag{8.6}$$

moreover, the fact that Λ is a countable set, entails that (8.6) even holds on Ω_4^*, an event of probability 1 which does not depend on (μ_1, μ_2). Also, observe that, for each $t_0 \in \mathbb{R}$ which satisfies (8.4), one has, for all $\omega \in \Omega_4^*$,

$$\rho_Y^{\mathrm{unif}}(t_0, \omega) = \sup\left\{\rho_Y^{\mathrm{unif}}\big([\mu_1, \mu_2]\big) : (\mu_1, \mu_2) \in \Lambda \text{ and } \mu_1 < t_0 < \mu_2\right\}; \tag{8.7}$$

the latter equality can be obtained by using (8.2), (8.3) and the fact that the set of the rational numbers is dense in the set of the real numbers. Finally, since $H(\cdot)$ is a continuous function, combining (8.6) with (8.7), one gets (8.5). $\square$

9. Appendix

The following lemma is a standard result.

Lemma 9.1. *For all fixed positive real number λ, there exists a finite constant c which only depends on λ, such that for each nonnegative real numbers x and y, one has,*

$$(x+y)^\lambda \le c(x^\lambda + y^\lambda),$$

with the convention that $0^\lambda = 0$.

The following technical lemma plays a crucial role in the proof of Part (iii) of Proposition 3.1 as well as in those of other important results in our article.

Lemma 9.2. *Let $(p,q) \in \{0,1,2\} \times \mathbb{Z}_+$ be arbitrary and fixed. We set $\phi := \partial_x^p \partial_v^q \Psi$, where Ψ is the function introduced in (2.3). Let M, ν, a, b and κ be arbitrary and fixed real numbers satisfying $M > 0$, $1 > b > a > 1/\alpha$, $a - 1/\alpha > \kappa$ and $a - 1/\alpha - \kappa > \nu \ge 0$. At last, let i be an arbitrary and fixed nonnegative integer. For all $n \in \mathbb{Z}_+$ and $(t,s,v) \in \mathbb{R}^2 \times (1/\alpha, 1)$ we set,*

$$A_n(t,s,v;M,\kappa,\nu,i,\phi) := \sum_{|j|\le n} \sum_{|k|>M2^{n+1}} 2^{-jv} \frac{\big|\phi(2^j t - k, v) - \phi(2^j s - k, v)\big|}{|t-s|^\kappa} (3+|j|)^{i+1/\alpha+\nu} (3+|k|)^{1/\alpha+\nu} \tag{9.1}$$

and

$$B_n(t,s,v;M,\kappa,\nu,i,\phi) := \sum_{|j|\ge n+1} \sum_{k\in\mathbb{Z}} 2^{-jv} \frac{\big|\phi(2^j t - k, v) - \phi(2^j s - k, v)\big|}{|t-s|^\kappa} (3+|j|)^{i+1/\alpha+\nu} (3+|k|)^{1/\alpha+\nu}, \tag{9.2}$$

with the convention that $A_n(t,t,v;M,\kappa,\nu,i,\phi) = B_n(t,t,v;M,\kappa,\nu,i,\phi) = 0$ for any $t \in \mathbb{R}$. Then, when n goes to $+\infty$, $A_n(t,s,v;M,\kappa,\nu,i,\phi)$ and $B_n(t,s,v;M,\kappa,\nu,i,\phi)$ converge to 0, uniformly in $(t,s,v) \in [-M,M]^2 \times [a,b]$.

In order to prove Lemma 9.2, we need some preliminary results.

Lemma 9.3. *For all fixed real numbers $\xi > 0$ and $M > 0$, there exists a constant $c > 0$ such that for each integer $n \ge 0$,*

$$\sum_{k>M2^{n+1}} (1+k)^{-1-\xi} \le c2^{-n\xi}.$$

Proof of Lemma 9.3. Clearly, one has for all integer $k \ge 1$, $(1+k)^{-1-\xi} \le \int_{k-1}^k (1+x)^{-1-\xi} dx$. Therefore,

$$\sum_{k>M2^{n+1}} (1+k)^{-1-\xi} \le \int_{M2^{n+1}-1}^{+\infty} (1+x)^{-1-\xi} dx = \xi^{-1} M^{-\xi} 2^{-(n+1)\xi}.$$

□

Lemma 9.4. *Let $\lambda \in \mathbb{R}$ and $\theta_0 > 0$ be fixed. Set $c := \sum_{m=0}^{+\infty} 2^{-m\theta_0}(1+m)^{|\lambda|} < +\infty$. Then for all real number θ such that $|\theta| \ge \theta_0$ and each $n_0, n_1 \in \{0, \pm 1, \dots, \pm\infty\}$ satisfying $n_0 < n_1$, one has,*

$$\sum_{n=n_0}^{n_1} 2^{n\theta}(1+|n|)^\lambda \le c \begin{cases} 2^{n_0\theta}(1+|n_0|)^\lambda & \text{if } \theta < 0 \\ 2^{n_1\theta}(1+|n_1|)^\lambda & \text{if } \theta > 0, \end{cases} \tag{9.3}$$

with the convention that $2^{-\infty}(1+\infty)^\lambda = 0$ and $2^{+\infty}(1+\infty)^\lambda = +\infty$

Proof of Lemma 9.4. First, notice that the lemma clearly holds in the following three cases:

- $n_0 = -\infty$ and $n_1 = +\infty$;
- $n_0 = -\infty$ and $\theta < 0$;
- $n_1 = +\infty$ and $\theta > 0$.

Indeed, in the latter three cases, (9.3) becomes $+\infty \le +\infty$.

Let us study the case where $\theta < 0$ and $-\infty < n_0 < n_1 \le +\infty$, the case where $\theta > 0$ and $-\infty \le n_0 < n_1 < +\infty$ can be treated similarly. One has

$$\sum_{n=n_0}^{n_1} 2^{n\theta}(1+|n|)^{\lambda} \le \sum_{m=0}^{+\infty} 2^{(m+n_0)\theta}(1+|m+n_0|)^{\lambda} = 2^{n_0\theta}(1+|n_0|)^{\lambda} \sum_{m=0}^{+\infty} 2^{m\theta}\left(\frac{1+|m+n_0|}{1+|n_0|}\right)^{\lambda}$$
$$\le 2^{n_0\theta}(1+|n_0|)^{\lambda} \sum_{m=0}^{+\infty} 2^{-m\theta_0}\left(\frac{1+|m+n_0|}{1+|n_0|}\right)^{\lambda}.$$

Thus, it remains to show that

$$\sum_{m=0}^{+\infty} 2^{-m\theta_0}\left(\frac{1+|m+n_0|}{1+|n_0|}\right)^{\lambda} \le c := \sum_{m=0}^{+\infty} 2^{-m\theta_0}(1+m)^{|\lambda|}. \tag{9.4}$$

In fact, (9.4) can be obtained by proving the following: for every integer $m \ge 0$, one has,

$$\frac{1}{1+m} \le \frac{1+|m+n_0|}{1+|n_0|} \le 1+m. \tag{9.5}$$

Clearly the second inequality in (9.5) is satisfied. To prove that the first one holds, we argue by cases:

- if $n_0 \ge 0$, one gets $\frac{1+|m+n_0|}{1+|n_0|} = \frac{1+m+n_0}{1+n_0} = 1 + \frac{m}{1+n_0} \ge 1 \ge \frac{1}{1+m}$;
- if $n_0 < 0$ and $m \ge -n_0 = |n_0|$, then $\frac{1+|m+n_0|}{1+|n_0|} \ge \frac{1}{1+|n_0|} \ge \frac{1}{1+m}$;
- if $n_0 < 0$ and $m < -n_0 = |n_0|$, then

$$\frac{1+|m+n_0|}{1+|n_0|} = \frac{1-m+|n_0|}{1+|n_0|} = 1 - \frac{m}{1+|n_0|} \ge 1 - \frac{m}{1+m} = \frac{1}{1+m}.$$

□

The following lemma is a more or less classical result, we refer for instance to [5] for its proof.

Lemma 9.5. [5] *For all fixed real numbers $\theta \in [0,1)$ and $\zeta \ge 0$, there exists a constant $c > 0$ such that for any $u \in \mathbb{R}$, one has,*

$$\sum_{k\in\mathbb{Z}} \frac{(1+|k|)^{\theta}\log^{\zeta}(2+|k|)}{(2+|u-k|)^2} \le c(1+|u|)^{\theta}\log^{\zeta}(2+|u|).$$

Now, we are in position to prove Lemma 9.2.

Proof of Lemma 9.2. Let $t, s \in [-M, M]$ be arbitrary and fixed; there is no restriction to assume that $s \ne t$. We denote by $j_0 > -\log_2(2M) - 1$ the unique integer such that

$$2^{-j_0-1} < |t-s| \le 2^{-j_0}. \tag{9.6}$$

From now on, for the sake of simplicity we set:

$$A_n(t,s,v) := A_n(t,s,v; M, \kappa, \nu, i, \phi) \text{ and } B_n(t,s,v) := B_n(t,s,v; M, \kappa, \nu, i, \phi).$$

Let us first prove that, when $n \to +\infty$, $A_n(t,s,v)$ converges to 0, uniformly in $(t,s,v) \in [-M,M]^2 \times [a,b]$. So, in the sequel, we assume that j is an arbitrary integer satisfying $|j| \le n$. We need to derive suitable upper bounds for the quantity

$$A_n^j(t,s,v) := \sum_{|k|>M2^{n+1}} \frac{\left|\phi(2^j t-k,v)-\phi(2^j s-k,v)\right|}{|t-s|^\kappa}(3+|k|)^{1/\alpha+\nu}. \tag{9.7}$$

For this purpose, we consider two cases $j \le j_0$ and $j \ge j_0+1$ separately. First, we suppose that

$$j \le j_0. \tag{9.8}$$

Using the Mean Value Theorem, (2.12), (9.6) and (9.8), one obtains that

$$\begin{aligned}\left|\phi(2^j t-k,v)-\phi(2^j s-k,v)\right| &\le c_1 2^j |t-s| \sup_{u\in I}(3+|u|)^{-2}\\ &\le c_1 2^j|t-s|(2+|2^j t-k|)^{-2},\end{aligned} \tag{9.9}$$

where I denotes the compact interval with end-points $2^j t-k$ and $2^j s-k$. It is worth noticing that, in view of (9.6) and (9.8), the length of I is at most 1; this is why the last inequality holds. Next, (9.9) and (9.7) entail that

$$A_n^j(t,s,v) \le c_1 2^j |t-s|^{1-\kappa} \sum_{|k|>M2^{n+1}} (3+|k|)^{1/\alpha+\nu}(2+|2^j t-k|)^{-2}. \tag{9.10}$$

Moreover, using the inequalities $|t| \le M, |j| \le n$ and $|k| > M2^{n+1}$, one gets

$$(3+|k|)^{1/\alpha+\nu}(2+|2^j t-k|)^{-2} \le (3+|k|)^{1/\alpha+\nu}(2+|k|-2^j M)^{-2} \le c_2(1+|k|)^{-(2-1/\alpha-\nu)}. \tag{9.11}$$

Putting together (9.10) and (9.11), one obtains that

$$A_n^j(t,s,v) \le c_3 2^j |t-s|^{1-\kappa} \sum_{|k|>M2^{n+1}} (1+|k|)^{-(2-1/\alpha-\nu)}.$$

Then Lemma 9.3 (in which one takes $\xi = 1-1/\alpha-\nu$) and Relation (9.6), imply that

$$A_n^j(t,s,v) \le c_4 2^{j_0(\kappa-1)+j-n(1-1/\alpha-\nu)}. \tag{9.12}$$

Let us now study the second case where

$$j_0+1 \le j. \tag{9.13}$$

It follows from (9.7), (9.6) and (9.13) that

$$A_n^j(t,s,v) \le 2^{j\kappa} \sum_{|k|>M2^{n+1}} \left\{\left|\phi(2^j t-k,v)\right| + \left|\phi(2^j s-k,v)\right|\right\}(3+|k|)^{1/\alpha+\nu}. \tag{9.14}$$

Moreover, using (2.12) and the fact that $|j| \le n$, one has for all $(u,v) \in [-M,M]\times[a,b]$ and $k \in \mathbb{Z}$ satisfying $|k| > M2^{n+1}$,

$$\begin{aligned}\left|\phi(2^j u-k,v)\right| &\le c_5(3+|2^j u-k|)^{-2} \le c_5(3+|k|-2^j|u|)^{-2}\\ &\le c_5(3+|k|-2^n M)^{-2} \le c_6(3+|k|)^{-2}.\end{aligned} \tag{9.15}$$

Combining (9.15) with (9.14), one gets that

$$A_n^j(t,s,v) \le c_6 2^{j\kappa+1} \sum_{|k|>M2^{n+1}} (3+|k|)^{-(2-1/\alpha-\nu)}.$$

Thus, it follows from Lemma 9.3 (in which one takes $\xi = 1-1/\alpha-\nu$), that

$$A_n^j(t,s,v) \le c_7 2^{j\kappa-n(1-1/\alpha-\nu)}. \tag{9.16}$$

Putting together (9.1), (9.7), (9.12) and (9.16), one obtains that

$$A_n(t,s,v) \le c_8 2^{-n(1-1/\alpha-\nu)} \left[2^{j_0(\kappa-1)} \sum_{j=-\infty}^{j_0} 2^{j(1-v)} (3+|j|)^{i+1/\alpha+\nu} + \sum_{j=j_0+1}^{+\infty} 2^{j(\kappa-v)} (3+|j|)^{i+1/\alpha+\nu} \right]. \tag{9.17}$$

Next, using Lemma 9.4 with $n_0 = -\infty$, $n_1 = j_0$, $\theta = 1-v > 0$, $\theta_0 = 1-b$ and $\lambda = i+1/\alpha+\nu$, one gets that

$$\sum_{j=-\infty}^{j_0} 2^{j(1-v)} (3+|j|)^{i+1/\alpha+\nu} \le c_9 2^{j_0(1-v)} (1+|j_0|)^{i+1/\alpha+\nu} \tag{9.18}$$

and using again the same lemma with $n_0 = j_0+1$, $n_1 = +\infty$, $\theta = \kappa - v < 0$, $\theta_0 = 1/\alpha$ and $\lambda = i+1/\alpha+\nu$, one obtains that

$$\sum_{j=j_0+1}^{+\infty} 2^{j(\kappa-v)} (3+|j|)^{i+1/\alpha+\nu} \le c_{10} 2^{j_0(\kappa-v)} (1+|j_0|)^{i+1/\alpha+\nu}. \tag{9.19}$$

Putting together (9.17), (9.18), (9.19), the inequality $v-\kappa \ge 1/\alpha$ and the inequality $j_0 > -\log_2(2M)-1$, one obtains that

$$A_n(s,t,v) \le c_{11} 2^{-j_0(v-\kappa)} (1+|j_0|)^{i+1/\alpha+\nu} 2^{-n(1-1/\alpha-\nu)} \le c_{12} 2^{-n(1-1/\alpha-\nu)}, \tag{9.20}$$

where

$$c_{12} := c_{11} \sup \left\{ 2^{-j/\alpha} (1+|j|)^{i+1/\alpha+\nu} \ :\ j \in \mathbb{Z} \text{ and } j > -\log_2(2M)-1 \right\} < +\infty.$$

The last inequality in (9.20) implies that when $n \to +\infty$, $A_n(t,s,v)$ converges to 0, uniformly in $(t,s,v) \in [-M,M]^2 \times [a,b]$.

From now on, our goal is to prove that $B_n(t,s,v)$ converges to 0 uniformly in t,s,v, when n goes to infinity. So, in all the sequel j denotes an arbitrary integer satisfying $|j| \ge n+1$. First we derive a suitable upper bound for the quantity

$$B^j(t,s,v) := \sum_{k\in\mathbb{Z}} \frac{|\phi(2^j t - k, v) - \phi(2^j s - k, v)|}{|t-s|^\kappa} (3+|k|)^{1/\alpha+\nu}. \tag{9.21}$$

As above, we distinguish two cases : $j \le j_0$ and $j \ge j_0+1$. First, we suppose that (9.8) is verified. Similarly to (9.10), one has that

$$B^j(t,s,v) \le c_{13} 2^j |t-s|^{1-\kappa} \sum_{k\in\mathbb{Z}} (3+|k|)^{1/\alpha+\nu} (2+|2^j t - k|)^{-2}.$$

Then, using (9.6), Lemma 9.5 (in which we take $\theta = 1/\alpha+\nu$ and $\zeta = 0$), and the fact that $|t| \le M$, one obtains that,

$$B^j(t,s,v) \le c_{14} 2^{j+j_0(\kappa-1)} (1+2^j)^{1/\alpha+\nu}. \tag{9.22}$$

Now let us suppose that (9.13) is satisfied. By using this relation, (9.6), the triangle inequality, (2.12), Lemma 9.5 (in which one takes $\theta = 1/\alpha+\nu$ and $\zeta = 0$) and the fact that $t,s \in [-M,M]$, one gets that,

$$B^j(t,s,v) \le 2^{j\kappa}\sum_{k\in\mathbb{Z}}(3+|k|)^{1/\alpha+\nu}\left\{\left|\phi(2^jt-k,v)\right|+\left|\phi(2^js-k,v)\right|\right\}$$
$$\le c_{15}2^{j\kappa}\sum_{k\in\mathbb{Z}}(1+|k|)^{1/\alpha+\nu}\left\{(2+|2^jt-k|)^{-2}+(2+|2^js-k|)^{-2}\right\}$$
$$\le c_{16}2^{j\kappa}\left\{(1+2^j|t|)^{1/\alpha+\nu}+(1+2^j|s|)^{1/\alpha+\nu}\right\}$$
$$\le c_{17}2^{j(\kappa+1/\alpha+\nu)}. \tag{9.23}$$

There is no restriction to assume that $n \ge \log_2(M)+2$, then in view of the inequality $j_0 > -\log_2(M)-2$, one has that $-n-1 < j_0$ and thus (9.22) entails that,

$$\sum_{j=-\infty}^{-n-1} 2^{-jv}(3+|j|)^{i+1/\alpha+\nu}B^j(t,s,v) \le c_{14}\sum_{j=-\infty}^{-n-1} 2^{j(1-v)+j_0(\kappa-1)}(1+2^j)^{1/\alpha+\nu}(3+|j|)^{i+1/\alpha+\nu}$$
$$\le c_{18}2^{j_0(\kappa-1)}\sum_{j=-\infty}^{-n-1} 2^{j(1-v)}(3+|j|)^{i+1/\alpha+\nu}.$$

Next, using Lemma 9.4 (in which one takes $n_0=-\infty$, $n_1=-n-1$, $\theta=1-v$, $\theta_0=1-b$ and $\lambda=i+1/\alpha+\nu$), the inequality $2^{j_0(\kappa-1)} < (4M)^{1-\kappa}$ and the inequality $v\le b$, one gets that

$$\sum_{j=-\infty}^{-n-1} 2^{-jv}(3+|j|)^{i+1/\alpha+\nu}B^j(t,s,v) \le c_{19}2^{-n(1-b)}(4+n)^{i+1/\alpha+\nu}. \tag{9.24}$$

Let us now give a suitable upper bound for $\sum_{j\ge n+1} 2^{-jv}(3+|j|)^{i+1/\alpha+\nu}B^j(t,s,v)$. First we assume that $j_0 \ge n+1$; then, using (9.22), one has that,

$$\sum_{j=n+1}^{j_0} 2^{-jv}(3+|j|)^{i+1/\alpha+\nu}B^j(t,s,v) \le c_{14}2^{j_0(\kappa-1)}\sum_{j=-\infty}^{j_0} 2^{j(1-v)}(3+|j|)^{i+1/\alpha+\nu}(1+2^j)^{1/\alpha+\nu}$$
$$\le c_{20}2^{j_0(\kappa-1+1/\alpha+\nu)}\sum_{j=-\infty}^{j_0} 2^{j(1-v)}(3+|j|)^{i+1/\alpha+\nu}$$
$$\le c_{21}2^{-j_0(a-1/\alpha-\kappa-\nu)}(3+|j_0|)^{i+1/\alpha+\nu}$$
$$\le c_{22}2^{-n(a-1/\alpha-\kappa-\nu)}(3+n)^{i+1/\alpha+\nu}. \tag{9.25}$$

Observe that the third inequality in (9.25) follows from Lemma 9.4 (in which we take $n_0=-\infty$, $n_1=j_0$, $\theta=1-v$, $\theta_0=1-b$ and $\lambda=i+1/\alpha+\nu$) as well as from the inequality $v\ge a$. Also observe that the last inequality in (9.25), results from the fact that the function $x\mapsto 2^{-x(a-1/\alpha-\kappa-\nu)}(3+x)^{i+1/\alpha+\nu}$ is continuous over $\mathbb{R}_+$ and decreasing for x big enough.

On the other hand, by making use of (9.23), one has that

$$\sum_{j=j_0+1}^{+\infty} 2^{-jv}(3+|j|)^{i+1/\alpha+\nu}B^j(t,s,v) \le c_{17}\sum_{j=j_0+1}^{+\infty} 2^{j(\kappa+1/\alpha+\nu-v)}(3+|j|)^{i+1/\alpha+\nu}$$
$$\le c_{23}2^{-j_0(a-1/\alpha-\kappa-\nu)}(3+|j_0|)^{i+1/\alpha+\nu}$$
$$\le c_{24}2^{-n(a-1/\alpha-\kappa-\nu)}(3+n)^{i+1/\alpha+\nu}. \tag{9.26}$$

Observe that the second inequality in (9.26) follows from Lemma 9.4 (in which we take $n_0=j_0+1$, $n_1=+\infty$, $\theta=\kappa+1/\alpha+\nu-v$, $\theta_0=a-1/\alpha-\kappa-\nu$ and $\lambda=i+1/\alpha+\nu$) as well as from the inequality $v \geq a$. Also observe that the last inequality in (9.26), results from the fact that the function $x \mapsto 2^{-x(a-1/\alpha-\kappa-\nu)}(3+x)^{i+1/\alpha+\nu}$ is continuous over $\mathbb{R}_+$ and decreasing for x big enough.

Combining (9.25) with (9.26), it follows that one has, in the case where $j_0 \geq n+1$,

$$\sum_{j=n+1}^{+\infty} 2^{-jv}(3+|j|)^{i+1/\alpha+\nu} B^j(t,s,v) \leq c_{25} 2^{-n(a-1/\alpha-\kappa-\nu)}(3+n)^{i+1/\alpha+\nu}. \tag{9.27}$$

Let us now assume that $j_0 < n+1$, then, by making use of (9.23), one has that

$$\begin{aligned}\sum_{j=n+1}^{+\infty} 2^{-jv}(3+|j|)^{i+1/\alpha+\nu} B^j(t,s,v) &\leq c_{17} \sum_{j=n+1}^{+\infty} 2^{j(\kappa+1/\alpha+\nu-v)}(3+|j|)^{i+1/\alpha+\nu} \\ &\leq c_{26} 2^{-n(a-1/\alpha-\kappa-\nu)}(3+n)^{i+1/\alpha+\nu},\end{aligned} \tag{9.28}$$

where the last inequality follows from Lemma 9.4 (in which we take $n_0 = n+1$, $n_1 = +\infty$, $\theta = \kappa + 1/\alpha + \nu - v$, $\theta_0 = a - 1/\alpha - \kappa - \nu$ and $\lambda = i + 1/\alpha + \nu$) as well as from the inequality $v \geq a$.

Finally, (9.2), (9.21), (9.24), (9.27) and (9.28) imply that, for all $n \geq \log_2(M) + 2$,

$$B_n(t,s,v) \leq c_{27}\big(2^{-n(1-b)} + 2^{-n(a-1/\alpha-\kappa-\nu)}\big)(4+n)^{i+1/\alpha+\nu}; \tag{9.29}$$

which in turn entails that when $n \to +\infty$, $B_n(t,s,v)$ converges to 0, uniformly in $(t,s,v) \in [-M,M]^2 \times [a,b]$.

□

CHAPITRE 3

Représentations des parties hautes et basses fréquences via des ondelettes de Haar : vitesses de convergence des séries

1. Introduction

Let $X = \{X(u,v) : (u,v) \in \mathbb{R} \times (1/\alpha, 1)\}$ be the modification of the stochastic field generating LMSM's whose paths are, with probability 1, continuous functions on $\mathbb{R} \times (1/\alpha, 1)$ (see Theorems 2.1 and 3.1 of chapter 2); by restricting the underlying probability space denoted by Ω (more precisely by supposing that $\Omega := \Omega_0^*$, where Ω_0^* is the event of probability 1 introduced in Lemma 2.1 of chapter 2), one can assume, without any loss of generality, that this continuity property holds, not only for almost all the paths of X, but also for all of them. Recall that, for every $(u,v) \in \mathbb{R} \times (1/\alpha, 1)$, one has, almost surely,

$$X(u,v) = \int_{\mathbb{R}} \Big\{ (u-s)_+^{v-1/\alpha} - (-s)_+^{v-1/\alpha} \Big\} \mathrm{Z}_\alpha(ds)\,, \tag{1.1}$$

where, for each real numbers x and κ,

$$(x)_+^\kappa := \begin{cases} x^\kappa, \text{if } x \in (0,+\infty), \\ \\ 0, \text{if } x \in (-\infty, 0]. \end{cases} \tag{1.2}$$

Throughout this chapter, we assume that $Z_\alpha(ds)$ is an independently scattered symmetric α-stable ($\mathcal{S}\alpha\mathcal{S}$) random measure on $\mathbb{R}$; thus setting, for any real number t, $Z_\alpha(t) = Z_\alpha([0,t])$ when $t \geq 0$ and $Z_\alpha(t) = -Z_\alpha([t,0])$ when $t < 0$, it follows that $\{Z_\alpha(t) : t \in \mathbb{R}\}$ is a usual $\mathcal{S}\alpha\mathcal{S}$ Lévy process, moreover we always suppose that its paths are càdlàg functions on $\mathbb{R}$. A modification of the high frequency part of X, is the $\mathcal{S}\alpha\mathcal{S}$ stochastic field $\widetilde{X}_1 = \{\widetilde{X}_1(u,v) : (u,v) \in \mathbb{R} \times (1/\alpha, 1)\}$ defined for each $(u,v) \in \mathbb{R} \times (1/\alpha, 1)$, as:

$$\widetilde{X}_1(u,v) := \int_0^{+\infty} (u-s)_+^{v-1/\alpha} Z_\alpha(ds)\,; \tag{1.3}$$

a modification of the low frequency part of X, is the $\mathcal{S}\alpha\mathcal{S}$ stochastic field $\widetilde{X}_2 = \{\widetilde{X}_2(u,v) : (u,v) \in \mathbb{R} \times (1/\alpha, 1)\}$ defined for each $(u,v) \in \mathbb{R} \times (1/\alpha, 1)$, as:

$$\widetilde{X}_2(u,v) := \int_{-\infty}^0 \Big\{ (u-s)_+^{v-1/\alpha} - (-s)_+^{v-1/\alpha} \Big\} Z_\alpha(ds)\,. \tag{1.4}$$

It is worth noticing that the properties of these two fields are far from being completely similar. Also, observe that, in view of (1.1), (1.3) and (1.4), one has for all $(u,v) \in \mathbb{R} \times (1/\alpha, 1)$, almost surely,

$$X(u,v) = \widetilde{X}_1(u,v) + \widetilde{X}_2(u,v).$$

In order to motivate the present chapter, one needs to briefly recall the main result in Section 3 of chapter 2. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a 3 times continuously differentiable compactly supported Daubechies mother wavelet [12, 20, 21] and let $\mathcal{E}_\gamma(a,b,M) := \mathcal{C}^1\big([a,b], \mathcal{C}^\gamma([-M,M],\mathbb{R})\big)$ be the Banach space of the Lipschitz functions defined on $[a,b]$ and with values in the Hölder space $\mathcal{C}^\gamma([-M,M],\mathbb{R})$ (see Definition 3.2 in chapter 2), where γ, a, b and M denote four arbitrary real numbers satisfying $M > 0$,

$1/\alpha < a < b < 1$ and $0 \le \gamma < a - 1/\alpha$. According to Proposition 3.1 and Theorem 3.1 of chapter 2, one has, for all $\omega \in \Omega := \Omega_0^*$, in $\mathcal{E}_\gamma(a,b,M)$,

$$X(u,v,\omega) = \lim_{n\to+\infty} \sum_{(j,k)\in D_{M,n}} 2^{-jv}\epsilon_{j,k}(\omega)\big(\Psi(2^j u - k, v) - \Psi(-k,v)\big), \tag{1.5}$$

where:

- $D_{M,n}$ is the set of indices defined as,

$$D_{M,n} := \left\{(j,k) \in \mathbb{Z}^2 : |j| \le n \text{ and } |k| \le M2^{n+1}\right\}, \tag{1.6}$$

- Ψ is the smooth deterministic function (see in chapter 2, (2.3) and Part (i) of Proposition 2.1), defined for all $(x,v) \in \mathbb{R} \times (1/\alpha, 1)$ as,

$$\Psi(x,v) := \int_{\mathbb{R}} (x-s)_+^{v-1/\alpha}\psi(s)ds\,, \tag{1.7}$$

- $\{\epsilon_{j,k} : (j,k) \in \mathbb{Z}^2\}$ is the sequence of the real-valued random variables (see (2.5) in chapter 2) defined as,

$$\epsilon_{j,k} := 2^{j/\alpha}\int_{\mathbb{R}} \psi(2^j s - k)Z_\alpha(ds)\,; \tag{1.8}$$

  observe that, our assumption that $Z_\alpha(ds)$ is a $\mathcal{S}\alpha\mathcal{S}$ random measure, implies that all the $\epsilon_{j,k}$'s have the same $\mathcal{S}\alpha\mathcal{S}$ distribution of scale parameter equals $\|\psi\|_{L^\alpha(\mathbb{R})}$.

Also recall that the following two results, are the two main ingredients of the proof of the fact that the convergence in (1.5) holds in the space $\mathcal{E}_\gamma(a,b,M)$.

(1) The sequence of real numbers $\{\epsilon_{j,k}(\omega) : (j,k) \in \mathbb{Z}^2\}$ satisfies (see Lemma 2.1 in chapter 2), for every fixed arbitrarily small $\eta > 0$,

$$\big|\epsilon_{j,k}(\omega)\big| \le C'(\omega)\big(3+|j|\big)^{1/\alpha+\eta}\big(3+|k|\big)^{1/\alpha+\eta}, \tag{1.9}$$

where C' is a positive and finite random variable only depending on η.

(2) The function Ψ as well as all its partial derivatives of any order, are well-localized in the variable x uniformly in the variable $v \in [a,b]$ (see Part (ii) of Proposition 2.1 in chapter 2), namely, for each $(p,q) \in \{0,1,2,3\} \times \mathbb{Z}_+$, one has

$$\sup_{(x,v)\in\mathbb{R}\times[a,b]} (3+|x|)^2\big|(\partial_x^p\partial_v^q\Psi)(x,v)\big| < \infty. \tag{1.10}$$

The proofs of several results in chapter 2 testify that, the series representation in (1.5) as well as its pathwise partial derivative with respect to v, are powerful tools for a fine study of path properties of the field $X = \{X(u,v) : (u,v) \in \mathbb{R} \times (1/\alpha,1)\}$ and a corresponding LMSM $Y = \{Y(t) : t \in \mathbb{R}\}$ defined as:

$$Y(t) := X(t, H(t))\,, \text{ for all } t \in \mathbb{R}, \tag{1.11}$$

where $H(\cdot)$ denotes an arbitrary deterministic continuous function on the real line with values in a compact interval $[\underline{H}, \overline{H}] \subset (1/\alpha, 1)$. However, this representation via smooth Daubechies wavelets, has the following two drawbacks.

- The function Ψ (see (1.7)) and the random variables $\epsilon_{j,k}$ (see (1.8)) cannot be defined by simple explicit formulas, since this is not the case for the Daubechies wavelet ψ itself; therefore (1.5) can hardly provide an efficient simulation method of the field X and Y a corresponding LMSM.
- In (1.5), the high and the low frequency parts of X (namely the fields $\widetilde{X}_1$ and $\widetilde{X}_2$ defined through (1.3) and (1.4)) are not completely separated; basically, this comes from the fact that the diameter of the support of ψ, is strictly larger than 1 (it is even much larger than 1).

In order to avoid the latter two drawbacks, in the present chapter, we replace ψ by the Haar mother wavelet h defined for all $s \in \mathbb{R}$, as:

$$h(s) := \mathbb{1}_{[0,1/2)}(s) - \mathbb{1}_{[1/2,1)}(s), \tag{1.12}$$

where $\mathbb{1}_S$ is the indicator function of an arbitrary subset S of $\mathbb{R}$. The continuously differentiable function $\theta : \mathbb{R} \times (1/\alpha, 1) \to \mathbb{R}$ is defined through (1.7) in which ψ is replaced by h; it is worth noticing that, despite the fact that θ will basically play the same role as Ψ, there is a considerable difference between both functions, indeed:

- on one hand, θ has the advantage to be explicitly given by a simple formula, namely, in view of (1.12), for all $(x, v) \in \mathbb{R} \times (1/\alpha, 1)$, one has,

$$\begin{aligned} \theta(x,v) \ &:= \ \int_{\mathbb{R}} (x-s)_+^{v-1/\alpha} h(s) ds \\ &= \ \big(1+v-1/\alpha\big)^{-1} \Big\{ (x-1)_+^{1+v-1/\alpha} - 2(x-1/2)_+^{1+v-1/\alpha} + (x)_+^{1+v-1/\alpha} \Big\}; \end{aligned} \tag{1.13}$$

- but, on the other hand, θ is less regular than Ψ and more importantly it fails to satisfy the "nice" localization property (1.10).

For each $(j,k) \in \mathbb{Z}^2$, we denote by $\zeta_{j,k}$ the $\mathcal{S}\alpha\mathcal{S}$ random variable defined through (1.8) in which ψ is replaced by h; in contrast with $\epsilon_{j,k}$, the random variable $\zeta_{j,k}$ is explicitly given by a simple formula, namely, in view of (1.12), one has:

$$\begin{aligned} \zeta_{j,k} \ &:= \ 2^{j/\alpha} \int_{\mathbb{R}} h(2^j s - k) \mathrm{Z}_\alpha(\mathrm{ds}) \\ &= \ -2^{j/\alpha} \bigg(\mathrm{Z}_\alpha\left(\frac{k}{2^j}\right) - 2\mathrm{Z}_\alpha\left(\frac{k+1/2}{2^j}\right) + \mathrm{Z}_\alpha\left(\frac{k+1}{2^j}\right) \bigg), \end{aligned} \tag{1.14}$$

where $\{Z_\alpha(t) : t \in \mathbb{R}\}$ is the $\mathcal{S}\alpha\mathcal{S}$ Lévy process with càdlàg paths which has been introduced at the very beginning of this introduction. Observe that the $\zeta_{j,k}$'s are identically distributed with a scale parameter equals 1; also observe that for every fixed $j \in \mathbb{Z}$, $\{\zeta_{j,k} : k \in \mathbb{Z}\}$ is a sequence of independent random variables.

In all the remaining of this chapter, we always assume that (u, v) belongs to the compact rectangle $[0,1] \times [a,b]$, where a and b are two arbitrary fixed real numbers satisfying $1/\alpha < a < b < 1$; typically one has $a = \min_{x\in[0,1]} H(t)$ and $b = \max_{x\in[0,1]} H(t)$ where $H(\cdot)$ is the continuous functional parameter of Y, the LMSM defined through (1.11). The stochastic $\mathcal{S}\alpha\mathcal{S}$ fields X, $\widetilde{X}_1$ and $\widetilde{X}_2$ are identified with their restrictions to $[0,1] \times [a,b]$ (notice that in the integral in (1.3), one can then replace $+\infty$ by 1).

Let us now introduce random series representations of $\widetilde{X}_1$ and $\widetilde{X}_2$ via Haar functions. On one hand, (1.3), the fact that the sequence of functions:

$$\Big\{\mathbb{1}_{[0,1]}(\cdot)\Big\} \cup \Big\{2^{j/2} h(2^j \cdot -k) : j \in \mathbb{Z}_+ \text{ and } k \in \{0,\ldots,2^j-1\}\Big\},$$

is an orthonormal basis of the Lebesgue Hilbert space $L^2\big([0,1]\big)$ (see [12, 20, 21]), standard computations, Hölder inequality, and a classical property of the stochastic integral with respect to Z_α, imply (similarly to the the proof of Part (*ii*) of Theorem 2.1 in chapter 2) that for all fixed $(u,v) \in [0,1] \times [a,b]$, the sequence of the $\mathcal{S}\alpha\mathcal{S}$ random variables $\big(X_1^J(u,v)\big)_{J\in\mathbb{N}}$, defined as:

$$X_1^J(u,v) = \frac{u^{1+v-1/\alpha}}{1+v-1/\alpha} Z_\alpha(1) + \sum_{j=0}^{J-1} 2^{-jv} \sum_{k=0}^{2^j-1} \zeta_{j,k} \theta(2^j u - k, v), \tag{1.15}$$

converges in probability to the random variable $\widetilde{X}_1(u,v)$, when J goes to $+\infty$. On the other hand, (1.4), the fact that the sequence of functions:

$$\left\{2^{j/\alpha}h(2^j\cdot+k) : (j,k)\in\mathbb{Z}\times\mathbb{N}\right\},$$

is an unconditional basis of the Lebesgue space $L^\alpha\big((-\infty,0]\big)$ (see [12, 20, 21]), standard computations, and a classical property of the stochastic integral with respect to Z_α, imply (similarly to the the proof of Part (*ii*) of Theorem 2.1 in chapter 2) that for all fixed $(u,v)\in[0,1]\times[a,b]$, the sequence of the $\mathcal{S}\alpha\mathcal{S}$ random variables $\big(X_2^J(u,v)\big)_{J\in\mathbb{N}}$, defined as:

$$X_2^J(u,v)=\sum_{j=1-J}^{J-1}2^{-jv}\sum_{k=1}^{2^{J-|j|}}\zeta_{j,-k}\big(\theta(2^ju+k,v)-\theta(k,v)\big), \tag{1.16}$$

converges in probability to the random variable $\widetilde{X}_2(u,v)$, when J goes to $+\infty$. It is clear that for every $J\in\mathbb{N}$, the paths of the $\mathcal{S}\alpha\mathcal{S}$ fields $X_1^J=\{X_1^J(u,v):(u,v)\in[0,1]\times[a,b]\}$ and $X_2^J=\{X_2^J(u,v):(u,v)\in[0,1]\times[a,b]\}$ belong to $\mathcal{C}:=\mathcal{C}([0,1]\times[a,b],\mathbb{R})$, the Banach space of the real-valued continuous functions on the rectangle $[0,1]\times[a,b]$ equipped with the usual supremum norm, denoted by $\|\cdot\|_{\mathcal{C}}$. A natural question one can address is that, whether or not, the sequences of continuous random functions $(X_1^J)_{J\in\mathbb{N}}$ and $(X_2^J)_{J\in\mathbb{N}}$, almost surely converge in the space $\mathcal{C}$; assume for a while that the answer to the question is positive and denote by X_1 and X_2 the limits of these two sequences, then the $\mathcal{S}\alpha\mathcal{S}$ fields $X_1=\{X_1(u,v):(u,v)\in[0,1]\times[a,b]\}$ and $X_2=\{X_2(u,v):(u,v)\in[0,1]\times[a,b]\}$ are modifications with almost surely continuous paths, respectively of the high and the low frequency parts of the field $X=\{X(u,v):(u,v)\in[0,1]\times[a,b]\}$.

To show that the sequences $(X_1^J)_{J\in\mathbb{N}}$ and $(X_2^J)_{J\in\mathbb{N}}$ are almost surely convergent in the space $\mathcal{C}$, is a tricky problem ; the main difficulty in it comes from the fact that θ (see (1.13)) is a badly localized function in the variable x; actually when $v\in[a,b]$ is fixed and x goes to $+\infty$, then $\theta(x,v)$ vanishes at the same rate as $x^{v-1/\alpha-1}$. In order to overcome this difficulty, we will use Abel transfroms.

The two main results of this chapter are the following.

Theorem 1.1. *Let Ω_0^{**} be the event of probability 1 introduced in Proposition 2.1. Then, for each $\omega\in\Omega_0^{**}$, the sequence of the continuous functions $(X_1^J(\cdot,\cdot,\omega))_{J\in\mathbb{N}}$ defined through (1.15), converges in the space $\mathcal{C}:=\mathcal{C}([0,1]\times[a,b],\mathbb{R})$ to a limit denoted by $X_1(\cdot,\cdot,\omega)$; moreover, one has for all fixed $\eta>0$,*

$$\big\|X_1(\cdot,\cdot,\omega)-X_1^J(\cdot,\cdot,\omega)\big\|_{\mathcal{C}}=\mathcal{O}\left(2^{-J(a-1/\alpha)}J^{2/\alpha+\eta}\right). \tag{1.17}$$

Notice that the $\mathcal{S}\alpha\mathcal{S}$ field $X_1=\{X_1(u,v):(u,v)\in[0,1]\times[a,b]\}$ is a modification with almost surely continuous paths, of the high frequency part of the $\mathcal{S}\alpha\mathcal{S}$ field $X=\{X(u,v):(u,v)\in[0,1]\times[a,b]\}$ which generates LMSM's.

Theorem 1.2. *Let Ω_1^{**} and Ω_2^{**} be the events of probability 1 introduced in Propositions 3.1 and 3.3. Then, for each $\omega\in\Omega_1^{**}\cap\Omega_2^{**}$ (notice that the event $\Omega_1^{**}\cap\Omega_2^{**}$ is of probability 1), the sequence of the continuous functions $(X_2^J(\cdot,\cdot,\omega))_{J\in\mathbb{N}}$ defined through (1.16), converges in the space $\mathcal{C}:=\mathcal{C}([0,1]\times[a,b],\mathbb{R})$ to a limit denoted by $X_2(\cdot,\cdot,\omega)$; moreover, one has for all fixed $\eta>0$,*

$$\big\|X_2(\cdot,\cdot,\omega)-X_2^J(\cdot,\cdot,\omega)\big\|_{\mathcal{C}}=\mathcal{O}\left(2^{-J(1-b)}J^{1/\alpha+\eta}\right). \tag{1.18}$$

Notice that the $\mathcal{S}\alpha\mathcal{S}$ field $X_2=\{X_2(u,v):(u,v)\in[0,1]\times[a,b]\}$ is a modification with almost surely continuous paths, of the low frequency part of the $\mathcal{S}\alpha\mathcal{S}$ field $X=\{X(u,v):(u,v)\in[0,1]\times[a,b]\}$ which generates LMSM's.

2. Proof of Theorem 1.1

Let $\Theta : \mathbb{R} \times (1/\alpha, 1) \to \mathbb{R}$ be the continuously differentiable function defined for every $(x, v) \in \mathbb{R} \times (1/\alpha, 1)$ as:

$$\Theta(x, v) := \theta(x, v) - \theta(x - 1, v). \tag{2.1}$$

For all $(j, k) \in \mathbb{Z}_+^2$, let $\lambda_{j,k}$ be the $\mathcal{S}\alpha\mathcal{S}$ random variable of scale parameter $(k+1)^{1/\alpha}$ defined as:

$$\lambda_{j,k} := \sum_{m=0}^{k} \zeta_{j,m}. \tag{2.2}$$

Lemma 2.1. *One has for each $j \in \mathbb{Z}_+$ and $(u, v) \in [0,1] \times [a,b]$,*

$$\sum_{k=0}^{2^j-1} \zeta_{j,k}\theta(2^j u - k, v) = \lambda_{j,2^j-1}\theta(2^j u - 2^j + 1, v) + \sum_{k=0}^{2^j-2} \lambda_{j,k}\Theta(2^j u - k, v), \tag{2.3}$$

with the convention that $\sum_{k=0}^{-1} \lambda_{0,k}\Theta(u - k, v) = 0$.

Proof of Lemma 2.1. It is clear that (2.3) holds when $j = 0$, so let us assume that $j \geq 1$. Using (2.2) and (2.1), one obtains that,

$$\begin{aligned}\sum_{k=0}^{2^j-1} \zeta_{j,k}\theta(2^j u - k, v) &= \lambda_{j,0}\theta(2^j u, v) + \sum_{k=1}^{2^j-1} (\lambda_{j,k} - \lambda_{j,k-1})\theta(2^j u - k, v) \\ &= \sum_{k=0}^{2^j-1} \lambda_{j,k}\theta(2^j u - k, v) - \sum_{k=0}^{2^j-2} \lambda_{j,k}\theta(2^j u - k - 1, v) \\ &= \lambda_{j,2^j-1}\theta(2^j u - 2^j + 1, v) + \sum_{k=0}^{2^j-2} \lambda_{j,k}\Theta(2^j u - k, v).\end{aligned}$$

□

Let us now provide a rather sharp estimate of the asymptotic behavior of the sequence of random variables $\{\lambda_{j,k} : (j,k) \in \mathbb{Z}_+^2\}$.

Proposition 2.1. *There exists an event of probability 1, denoted by Ω_0^{**}, such that for every fixed real number $\eta > 0$, one has, for all $\omega \in \Omega_0^{**}$ and for each $(j,k) \in \mathbb{Z}_+^2$,*

$$\big|\lambda_{j,k}(\omega)\big| \leq C(\omega)\big(1 + j\big)^{1/\alpha} \log^{1/\alpha+\eta}\big(3 + j\big)\,\big(1 + k\big)^{1/\alpha} \log^{1/\alpha+\eta}\big(3 + k\big), \tag{2.4}$$

where C is a positive and finite random variable only depending on η.

In order to prove Proposition 2.1, we need two preliminary results.

Lemma 2.2. *For each fixed $j \in \mathbb{Z}_+$, the $\mathcal{S}\alpha\mathcal{S}$ process $\{\lambda_{j,k} : k \in \mathbb{Z}_+\}$ has the same finite dimensional distributions as the process $\{Z_\alpha(k+1) : k \in \mathbb{Z}_+\}$; recall that $\{Z_\alpha(t) : t \in \mathbb{R}_+\}$ is a $\mathcal{S}\alpha\mathcal{S}$ Lévy process with càdlàg paths.*

Proof of Lemma 2.2. Let $\{\delta_m : m \in \mathbb{Z}_+\}$ be the sequence of the independent and identically distributed $\mathcal{S}\alpha\mathcal{S}$ random variables with a scale parameter equals 1, defined for all $m \in \mathbb{Z}_+$ as:

$$\delta_m = Z_\alpha(m+1) - Z_\alpha(m). \tag{2.5}$$

It follows from (2.5) and the equality $Z_\alpha(0) = 0$, that one has for each $k \in \mathbb{Z}_+$,

$$Z_\alpha(k+1) = \sum_{m=0}^{k} \delta_m. \tag{2.6}$$

Then, combining (2.6) with (2.2), and using the fact that for each fixed $j \in \mathbb{Z}_+$, $\{\zeta_{j,m} : m \in \mathbb{Z}_+\}$ is a sequence of independent and identically distributed $\mathcal{S}\alpha\mathcal{S}$ random variables with a scale parameter equals 1, one gets the lemma. □

Lemma 2.3. *Let $\eta > 0$ be arbitrary and fixed. We set,*

$$M^*_{\eta,\alpha} := \sup\left\{ \frac{|Z_\alpha(t)|}{t^{1/\alpha} \log^{1/\alpha+\eta}(2+t)} : t \in [1, +\infty) \right\}. \tag{2.7}$$

Then $M^*_{\eta,\alpha}$ is an almost surely finite random variable; moreover there is a constant $c > 0$ such that for all real number $y \geq 1$, one has,

$$\mathbb{P}\left(M^*_{\eta,\alpha} > y\right) \leq c y^{-\alpha}. \tag{2.8}$$

Proof of Lemma 2.3. The fact that $M^*_{\eta,\alpha}$ is an almost surely finite random variable, has been derived in [18]. The inequality (2.8) can be obtained by applying Theorem 10.5.1 in [26], to the almost surely bounded $\mathcal{S}\alpha\mathcal{S}$ processes $\left\{\frac{Z_\alpha(t)}{t^{1/\alpha}\log^{1/\alpha+\eta}(2+t)} : t \in [1,+\infty)\right\}$ and $\left\{-\frac{Z_\alpha(t)}{t^{1/\alpha}\log^{1/\alpha+\eta}(2+t)} : t \in [1,+\infty)\right\}$. □

Now, we are in position to prove Proposition 2.1.

Proof of Proposition 2.1. For each fixed $j \in \mathbb{Z}_+$, let $\mu^{j,*}_{\eta,\alpha}$ be the random variable defined as:

$$\mu^{j,*}_{\eta,\alpha} := \sup\left\{ \frac{|\lambda_{j,k}|}{(1+k)^{1/\alpha} \log^{1/\alpha+\eta}(3+k)} : k \in \mathbb{Z}_+ \right\}. \tag{2.9}$$

In view of Lemma 2.2, one has, for all $j \in \mathbb{Z}_+$,

$$\mu^{j,*}_{\eta,\alpha} \overset{(d)}{=} \nu, \tag{2.10}$$

where $\overset{(d)}{=}$ means equality in distribution, and ν is the random variable, defined as:

$$\nu := \sup\left\{ \frac{|Z_\alpha(k+1)|}{(1+k)^{1/\alpha} \log^{1/\alpha+\eta}(3+k)} : k \in \mathbb{Z}_+ \right\}. \tag{2.11}$$

Notice that Lemma 2.3 implies that ν is almost surely finite; moreover, thanks to (2.10), the latter property is also satisfied by $\mu^{j,*}_{\eta,\alpha}$, for any arbitrary $j \in \mathbb{Z}_+$. Next, using (2.10), (2.11), (2.7) and (2.8), one gets that,

$$\begin{aligned}
\sum_{j=0}^{+\infty} \mathbb{P}\left(\mu^{j,*}_{\eta,\alpha} > (1+j)^{1/\alpha} \log^{1/\alpha+\eta}(3+j)\right) &= \sum_{j=0}^{+\infty} \mathbb{P}\left(\nu > (1+j)^{1/\alpha} \log^{1/\alpha+\eta}(3+j)\right) \\
&\leq \sum_{j=0}^{+\infty} \mathbb{P}\left(M^*_{\eta,\alpha} > (1+j)^{1/\alpha} \log^{1/\alpha+\eta}(3+j)\right) \\
&\leq c\sum_{j=0}^{+\infty} (1+j)^{-1} \log^{-1-\alpha\eta}(3+j) < \infty;
\end{aligned}$$

thus the proposition results from Borel-Cantelli Lemma as well as from the fact that $\mu^{j,*}_{\eta,\alpha}$ is almost surely finite for each $j \in \mathbb{Z}_+$. □

The following proposition provides sharp estimates of the rate of vanishing of $\theta(x,v)$ and $\Theta(x,v)$ (see (1.13) and (2.1)) when x goes to infinity.

Proposition 2.2. *(i) For each* $(x,v)\in(-\infty,0]\times(1/\alpha,1)$*, one has,*
$$\theta(x,v)=\Theta(x,v)=0.$$
(ii) There exists a constant $c>0$*, such that for all* $(x,v)\in(0,+\infty)\times[a,b]$*, one has,*
$$|\theta(x,v)|\le c(1+x)^{v-1/\alpha-1}.$$
(iii) There is a constant $c'>0$*, such that for all* $(x,v)\in(0,+\infty)\times[a,b]$*, one has,*
$$|\Theta(x,v)|\le c'(1+x)^{v-1/\alpha-2}.$$

Proof of Proposition 2.2. In view of (1.13), (2.1) and (1.2), it is clear that Part (i) of the proposition holds. Let us prove the two other parts of it. Observe that the fact that $(x,v)\mapsto(1+x)^{1-v-1/\alpha}\,\theta(x,v)$ and $(x,v)\mapsto(1+x)^{2-v-1/\alpha}\,\Theta(x,v)$ are continuous functions on the compact rectangle $[0,4]\times[a,b]$, implies that,
$$c_1:=\sup\Big\{(1+x)^{1-v-1/\alpha}|\theta(x,v)|:(x,v)\in[0,4]\times[a,b]\Big\}<\infty \tag{2.12}$$
and
$$c_2:=\sup\Big\{(1+x)^{2-v-1/\alpha}|\Theta(x,v)|:(x,v)\in[0,4]\times[a,b]\Big\}<\infty. \tag{2.13}$$
From now on, we assume that $(x,v)\in(4,+\infty)\times[a,b]$. Observe that, in view of (1.13) and (1.2), one has,
$$\theta(x,v)=\frac{x^{1+v-1/\alpha}}{1+v-1/\alpha}\Big\{\big(1-x^{-1}\big)^{1+v-1/\alpha}-2\big(1-(2x)^{-1}\big)^{1+v-1/\alpha}+1\Big\}. \tag{2.14}$$
Next, let us show that there are two constants $c_3>0$ and $c_4>0$, such that for all $(z,v)\in[0,2^{-1}]\times[a,b]$, one has,
$$\Big|(1-z)^{1+v-1/\alpha}-1+(v-1/\alpha+1)z\Big|\le c_3z^2 \tag{2.15}$$
and
$$\Big|(1-z)^{1+v-1/\alpha}-1+(v-1/\alpha+1)z-\frac{(v-1/\alpha+1)(v-1/\alpha)}{2}z^2\Big|\le c_4z^3. \tag{2.16}$$
Observe that (2.15) easily results from (2.16), so we only need to prove that the latter inequality holds. Applying, for each fixed $v\in[a,b]$, Taylor-Lagrange formula, to the function $y\mapsto(1-y)^{1+v-1/\alpha}$, on the interval $[0,z]$, one gets that
$$\begin{aligned}&(1-z)^{1+v-1/\alpha}-1+(v-1/\alpha+1)z-\frac{(v-1/\alpha+1)(v-1/\alpha)}{2}z^2\\&=-\frac{(v-1/\alpha+1)(v-1/\alpha)(v-1/\alpha-1)}{6}(1-\xi)^{v-1/\alpha-2}\,z^3,\end{aligned} \tag{2.17}$$
where $\xi\in[0,z]\subset[0,2^{-1}]$, then (2.16) easily follows from (2.17). Next, using the triangle inequality and (2.15) (in the case where $z=x^{-1}$ and also in the case where $z=(2x)^{-1}$), one gets that,
$$\begin{aligned}&\Big|\big(1-x^{-1}\big)^{1+v-1/\alpha}-2\big(1-(2x)^{-1}\big)^{1+v-1/\alpha}+1\Big|\\&=\Big|\big(1-x^{-1}\big)^{1+v-1/\alpha}-1+(v-1/\alpha+1)x^{-1}\\&\qquad-2\big(1-(2x)^{-1}\big)^{1+v-1/\alpha}+2-2(v-1/\alpha+1)(2x)^{-1}\Big|\\&\le\Big|\big(1-x^{-1}\big)^{1+v-1/\alpha}-1+(v-1/\alpha+1)x^{-1}\Big|\\&\qquad+2\Big|\big(1-(2x)^{-1}\big)^{1+v-1/\alpha}-1+(v-1/\alpha+1)(2x)^{-1}\Big|\\&\le c_5x^{-2},\end{aligned} \tag{2.18}$$

where $c_5 := (3/2)c_3$. Next, putting together (2.12), (2.14) and (2.18), one obtains Part (ii) of the proposition. Let us now prove that Part (iii) of it, holds; to this end, we set

$$d_0 = 1,\ d_1 = -2,\ d_2 = 0,\ d_3 = 2 \text{ and } d_4 = -1. \tag{2.19}$$

Standard computations allow to show that, for all $m \in \{0,1,2\}$,

$$\sum_{l=0}^{4} l^m d_l = 0, \tag{2.20}$$

with the convention that $0^0 := 1$; moreover, in view of (2.1), (1.13) and (1.2), for each $(x,v) \in (4,+\infty) \times [a,b]$, one has,

$$\begin{aligned} \Theta(x,v) &= \big(1+v-1/\alpha\big)^{-1} \left\{ \sum_{l=0}^{4} d_l \big(x - l/2\big)^{1+v-1/\alpha} \right\} \\ &= \frac{x^{1+v-1/\alpha}}{\big(1+v-1/\alpha\big)} \left\{ \sum_{l=0}^{4} d_l \Big(1 - \frac{l}{2x}\Big)^{1+v-1/\alpha} \right\}. \end{aligned} \tag{2.21}$$

Next, using (2.20), the triangle inequality, and (2.16) in which one takes $z = l/(2x)$, it follows that,

$$\begin{aligned} &\left| \sum_{l=0}^{4} d_l \Big(1 - \frac{l}{2x}\Big)^{1+v-1/\alpha} \right| \\ &= \left| \sum_{l=0}^{4} d_l \left\{ \Big(1 - \frac{l}{2x}\Big)^{1+v-1/\alpha} \right.\right. \\ &\qquad\qquad \left.\left. -1 + (v-1/\alpha+1)\frac{l}{2x} - \frac{(v-1/\alpha+1)(v-1/\alpha)}{2}\Big(\frac{l}{2x}\Big)^2 \right\} \right| \\ &\le c_6 x^{-3}, \end{aligned} \tag{2.22}$$

where $c_6 := 8^{-1} c_4 \big(\sum_{l=1}^{4} l^3 |d_l| \big)$. Finally, putting together (2.13), (2.21) and (2.22), one obtains Part (iii) of the proposition. □

Lemma 2.4. *One has*

$$M := \sup \left\{ \sum_{k\in\mathbb{Z}} \big|\Theta(x-k,v)\big| : (x,v) \in \mathbb{R} \times [a,b] \right\} < \infty. \tag{2.23}$$

PROOF OF LEMMA 2.4. Observe that, for each fixed $v \in [a,b]$, the function

$$x \mapsto \sum_{k\in\mathbb{Z}} \big|\Theta(x-k,v)\big|,$$

defined on $\mathbb{R}$ and a priori taking its values in $\mathbb{R} \cup \{+\infty\}$, is 1-periodic; therefore it is sufficicent to show that (2.23) holds when $(x,v) \in [-1/2,1/2] \times [a,b]$. Using Parts (i) and (iii) of Proposition 2.2, as well as, the triangle inequality, one gets that,

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \big|\Theta(x-k,v)\big| &\le c' \sum_{k\in\mathbb{Z}} \big(1+|x-k|\big)^{b-1/\alpha-2} \\ &\le c' \sum_{k\in\mathbb{Z}} \big(1+|k|-|x|\big)^{b-1/\alpha-2} \\ &\le c' \sum_{k\in\mathbb{Z}} \big(2^{-1}+|k|\big)^{b-1/\alpha-2} < \infty. \end{aligned}$$

□

Now, we are in position to show that Theorem 1.1 holds.

Proof of Theorem 1.1. Let $\omega \in \Omega_0^{**}$ be arbitrary and fixed; recall that Ω_0^{**} is the event of probability 1 introduced in Proposition 2.1. Let us first show that the sequence of the continuous functions $(X_1^J(\cdot,\cdot,\omega))_{J\in\mathbb{N}}$ defined through (1.15), is a Cauchy sequence in $\mathcal{C} := \mathcal{C}([0,1]\times[a,b],\mathbb{R})$, the space of the real-valued continuous functions over $[0,1]\times[a,b]$, equipped with the usual supremum norm, denoted by $\|\cdot\|_{\mathcal{C}}$. Let $\eta > 0$ be arbitrary and fixed, using, (1.15), the triangle inequality, (2.3), (2.4), Parts (i) and (ii) of Proposition 2.2, and (2.23), one gets that, for all $(J,Q) \in \mathbb{N}^2$,

$$\begin{aligned}
&\big\|X_1^{J+Q}(\cdot,\cdot,\omega) - X_1^J(\cdot,\cdot,\omega)\big\|_{\mathcal{C}}\\
&= \Big\| \sum_{j=J}^{J+Q-1} 2^{-j\cdot} \sum_{k=0}^{2^j-1} \zeta_{j,k}\theta(2^j\cdot -k,\cdot)\Big\|_{\mathcal{C}}\\
&\leq \sum_{j=J}^{J+Q-1} 2^{-ja} \Big\| \sum_{k=0}^{2^j-1} \zeta_{j,k}\theta(2^j\cdot -k,\cdot)\Big\|_{\mathcal{C}}\\
&= \sum_{j=J}^{J+Q-1} 2^{-ja} \Big\| \lambda_{j,2^j-1}\theta(2^j\cdot -2^j+1,\cdot) + \sum_{k=0}^{2^j-2} \lambda_{j,k}\Theta(2^j\cdot -k,\cdot)\Big\|_{\mathcal{C}}\\
&\leq \sum_{j=J}^{J+Q-1} 2^{-ja} \left(|\lambda_{j,2^j-1}| \big\|\theta(2^j\cdot -2^j+1,\cdot)\big\|_{\mathcal{C}} + \left(\max_{0\leq l\leq 2^j-2} |\lambda_{j,k}|\right) \Big\| \sum_{k=0}^{2^j-2} |\Theta(2^j\cdot -k,\cdot)| \Big\|_{\mathcal{C}} \right)\\
&\leq C(\omega)(c+M) \sum_{j=J}^{J+Q-1} 2^{-j(a-1/\alpha)}(2+j)^{2/\alpha+\eta}\log^{1/\alpha+\eta}(3+j)\\
&\leq C(\omega)(c+M) \sum_{j=J}^{+\infty} 2^{-j(a-1/\alpha)}(2+j)^{2/\alpha+\eta}\log^{1/\alpha+\eta}(3+j).
\end{aligned} \tag{2.24}$$

It follows from (2.24) that $(X_1^J(\cdot,\cdot,\omega))_{J\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}$, and that $X_1(\cdot,\cdot,\omega)$, its limit, satisfies for all $J \in \mathbb{N}$,

$$\big\|X_1(\cdot,\cdot,\omega) - X_1^J(\cdot,\cdot,\omega)\big\|_{\mathcal{C}} \leq C(\omega)(c+M) \sum_{j=J}^{+\infty} 2^{-j(a-1/\alpha)}(2+j)^{2/\alpha+\eta}\log^{1/\alpha+\eta}(3+j). \tag{2.25}$$

Next, let us show that there exists a constant $c_1 > 0$, such that for all $J \in \mathbb{N}$, one has,

$$\sum_{j=J}^{+\infty} 2^{-j(a-1/\alpha)}(2+j)^{2/\alpha+\eta}\log^{1/\alpha+\eta}(3+j) \leq c_1 2^{-J(a-1/\alpha)}(2+J)^{2/\alpha+\eta}\log^{1/\alpha+\eta}(3+J). \tag{2.26}$$

This is the case since,

$$\begin{aligned}
&\sum_{j=J}^{+\infty} 2^{-j(a-1/\alpha)}(2+j)^{2/\alpha+\eta}\log^{1/\alpha+\eta}(3+j)\\
&=\sum_{j=0}^{+\infty} 2^{-(j+J)(a-1/\alpha)}(2+j+J)^{2/\alpha+\eta}\log^{1/\alpha+\eta}(3+j+J)\\
&\le \sum_{j=0}^{+\infty} 2^{-(j+J)(a-1/\alpha)}(2+j)^{2/\alpha+\eta}(2+J)^{2/\alpha+\eta}\log^{1/\alpha+\eta}\{(3+j)(3+J)\}\\
&\le 2^{-J(a-1/\alpha)}(2+J)^{2/\alpha+\eta}\sum_{j=0}^{+\infty} 2^{-j(a-1/\alpha)}(2+j)^{2/\alpha+\eta}\Big(\log(3+j)+\log(3+J)\Big)^{1/\alpha+\eta}\\
&\le c_1 2^{-J(a-1/\alpha)}(2+J)^{2/\alpha+\eta}\log^{1/\alpha+\eta}(3+J),
\end{aligned}$$

where

$$c_1 := \sum_{j=0}^{+\infty} 2^{-j(a-1/\alpha)}(2+j)^{2/\alpha+\eta}\Big(\log(3+j)+1\Big)^{1/\alpha+\eta}.$$

Finally, combining (2.25) with (2.26), one gets (1.17) in which η is replaced by 2η. □

3. Proof of Theorem 1.2

3.1. Study of the part $j \ge 0$ of the series.

Theorem 3.1. *For each $J \in \mathbb{N}$, we denote by $X_{2,+}^J = \{X_{2,+}^J(u,v) : (u,v) \in [0,1]\times[a,b]\}$ the $\mathcal{S}\alpha\mathcal{S}$ field with paths in $\mathcal{C}$, defined, for every $(u,v)\in[0,1]\times[a,b]$, as:*

$$X_{2,+}^J(u,v) = \sum_{j=0}^{J-1} 2^{-jv}\sum_{k=1}^{2^{J-j}} \zeta_{j,-k}\big(\theta(2^j u+k,v)-\theta(k,v)\big). \tag{3.1}$$

Let Ω_1^{**} be the event of probability 1 introduced in Proposition 3.1. Then for all $\omega \in \Omega_1^{**}$, $\big(X_{2,+}^J(\cdot,\cdot,\omega)\big)_{J\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}$, moreover, its limit $X_{2,+}(\cdot,\cdot,\omega)$, satisfies, for each fixed $\eta>0$,

$$\big\|X_{2,+}(\cdot,\cdot,\omega)-X_{2,+}^J(\cdot,\cdot,\omega)\big\|_{\mathcal{C}} = \mathcal{O}\left(2^{-J(1-b)}J^{1/\alpha+\eta}\right) \tag{3.2}$$

In order to show that $\big(X_{2,+}^J(\cdot,\cdot,\omega)\big)_{J\in\mathbb{N}}$ is a Cauchy sequence, one needs to appropriately bound the quantity $\big\|X_{2,+}^{J+Q}(\cdot,\cdot,\omega)-X_{2,+}^J(\cdot,\cdot,\omega)\big\|_{\mathcal{C}}$, for all $(J,Q)\in\mathbb{N}^2$. Observe that, in view of (3.1), one has for every $(u,v)\in[0,1]\times[a,b]$,

$$X_{2,+}^{J+Q}(u,v,\omega)-X_{2,+}^J(u,v,\omega) = A_{2,+}^{J,Q}(u,v,\omega)+B_{2,+}^{J,Q}(u,v,\omega), \tag{3.3}$$

where:

$$A_{2,+}^{J,Q}(u,v,\omega) := \sum_{j=0}^{J-1} 2^{-jv}\sum_{k=2^{J-j}+1}^{2^{J+Q-j}} \zeta_{j,-k}(\omega)\big(\theta(2^j u+k,v)-\theta(k,v)\big), \tag{3.4}$$

and

$$B_{2,+}^{J,Q}(u,v,\omega) := \sum_{j=J}^{J+Q-1} 2^{-jv}\sum_{k=1}^{2^{J+Q-j}} \zeta_{j,-k}(\omega)\big(\theta(2^j u+k,v)-\theta(k,v)\big). \tag{3.5}$$

For all $(j,k) \in \mathbb{Z}_+ \times \mathbb{N}$, let $\lambda_{j,-k}$ be the $\mathcal{S}\alpha\mathcal{S}$ random variable of scale parameter $k^{1/\alpha}$ defined as:

$$\lambda_{j,-k} := \sum_{m=1}^{k} \zeta_{j,-m}. \tag{3.6}$$

The proof of the following lemma is similar to that of Lemma 2.1.

Lemma 3.1. *Recall that the function Θ has been introduced in (2.1). Let $(J,Q) \in \mathbb{N}^2$ and $(u,v) \in [0,1] \times [a,b]$ be arbitrary and fixed.*

(i) For each $j \in \{0,\dots,J-1\}$, one has,

$$\begin{aligned}
&\sum_{k=2^{J-j}+1}^{2^{J+Q-j}} \zeta_{j,-k}\theta(2^j u + k, v) \qquad (3.7)\\
&= \lambda_{j,-2^{J+Q-j}}\theta(2^j u + 2^{J+Q-j}, v) - \lambda_{j,-2^{J-j}}\theta(2^j u + 2^{J-j}+1, v) - \sum_{k=2^{J-j}+2}^{2^{J+Q-j}} \lambda_{j,-(k-1)}\Theta(2^j u + k, v).
\end{aligned}$$

(ii) For each $j \in \{J,\dots,J+Q-1\}$, one has

$$\begin{aligned}
&\sum_{k=1}^{2^{J+Q-j}} \zeta_{j,-k}\theta(2^j u + k, v) \qquad (3.8)\\
&= \lambda_{j,-2^{J+Q-j}}\theta(2^j u + 2^{J+Q-j}, v) - \sum_{k=2}^{2^{J+Q-j}} \lambda_{j,-(k-1)}\Theta(2^j u + k, v).
\end{aligned}$$

The proof of the following proposition is similar to that of Proposition 2.1.

Proposition 3.1. *There exists an event of probability 1, denoted by Ω_1^{**}, such that for every fixed real number $\eta > 0$, one has, for all $\omega \in \Omega_1^{**}$ and for each $(j,k) \in \mathbb{Z}_+ \times \mathbb{N}$,*

$$\big|\lambda_{j,-k}(\omega)\big| \le C(\omega)\big(1+j\big)^{1/\alpha}\log^{1/\alpha+\eta}\big(3+j\big)\, k^{1/\alpha}\log^{1/\alpha+\eta}\big(2+k\big), \tag{3.9}$$

where C is a positive and finite random variable only depending on η.

The following propostion easily results from Proposition 2.2.

Proposition 3.2. *There exist two constants $c > 0$ and $c' > 0$, such that for all $(j,k) \in \mathbb{Z}_+ \times \mathbb{N}$ and $(u,v) \in [0,1] \times [a,b]$, one has,*

$$\big|\theta(2^j u + k, v)\big| \le c\big(1+k\big)^{v-1/\alpha-1} \tag{3.10}$$

and

$$\big|\Theta(2^j u + k, v)\big| \le c'\big(1+k\big)^{v-1/\alpha-2}. \tag{3.11}$$

The following lemma is a straightforward consequence of Lemma 3.1 as well as Propositions 3.1 and 3.2.

Lemma 3.2. *Let $\eta > 0$ be arbitrary and fixed. There exists a positive and finite random variable C only depending on $\eta > 0$, such that any $\omega \in \Omega_1^{**}$, $(J,Q) \in \mathbb{N}^2$ and $v \in [a,b]$, satisfy the following two properties:*

(i) for each $j \in \{0, \ldots, J-1\}$, one has,

$$\sup_{u\in[0,1]} \Big| \sum_{k=2^{J-j}+1}^{2^{J+Q-j}} \zeta_{j,-k}(\omega)\theta(2^j u + k, v) \Big| \tag{3.12}$$
$$\le C(\omega)(1+j)^{1/\alpha} \log^{1/\alpha+\eta}(2+j) \Big\{ 2^{-(1-v)(J+Q-j)}(J+Q-j)^{1/\alpha+\eta}$$
$$+2^{-(1-v)(J-j)}(J-j)^{1/\alpha+\eta} + \sum_{k=2^{J-j}+2}^{2^{J+Q-j}} k^{-(2-v)} \log^{1/\alpha+\eta}(k) \Big\};$$

(ii) for each $j \in \{J, \ldots, J+Q-1\}$, one has

$$\sup_{u\in[0,1]} \Big| \sum_{k=1}^{2^{J+Q-j}} \zeta_{j,-k}(\omega)\theta(2^j u + k, v) \Big| \tag{3.13}$$
$$\le C(\omega)(1+j)^{1/\alpha} \log^{1/\alpha+\eta}(2+j) \Big\{ 2^{-(1-v)(J+Q-j)}(J+Q-j)^{1/\alpha+\eta}$$
$$+ \sum_{k=2}^{2^{J+Q-j}} k^{-(2-v)} \log^{1/\alpha+\eta}(k) \Big\}.$$

Lemma 3.3. *Let $\eta > 0$ be an arbitrarily small fixed real number. There is a constant $c > 0$, such that, for all $J \in \mathbb{N}$, for each $j \in \{0, \ldots, J-1\}$ and for any $v \in [a, b]$, one has,*

$$\sum_{k=2^{J-j}+2}^{+\infty} k^{-(2-v)} \log^{1/\alpha+\eta}(k) \le c\, 2^{-(1-v)(J-j)}(J-j)^{1/\alpha+\eta}. \tag{3.14}$$

Proof of Lemma 3.3. By using the fact that $y \mapsto y^{-(2-v)} \log^{1/\alpha+\eta}(y)$ is a decreasing function on the interval $[3, +\infty)$, one has that,

$$\sum_{k=2^{J-j}+2}^{+\infty} k^{-(2-v)} \log^{1/\alpha+\eta}(k) \le \int_{2^{J-j}}^{+\infty} y^{-(2-v)} \log^{1/\alpha+\eta}(y)\, dy. \tag{3.15}$$

Moreover, setting in the last integral $z = 2^{-(J-j)}y$, one obtains that,

$$\int_{2^{J-j}}^{+\infty} y^{-(2-v)} \log^{1/\alpha+\eta}(y)\, dy = 2^{J-j} \int_1^{+\infty} \left(2^{J-j}z\right)^{-(2-v)} \log^{1/\alpha+\eta}\left(2^{J-j}z\right) dz$$
$$= 2^{-(1-v)(J-j)} \int_1^{+\infty} z^{-(2-v)} \Big((J-j)\log(2) + \log(z) \Big)^{1/\alpha+\eta} dz$$
$$\le c\, 2^{-(1-v)(J-j)}(J-j)^{1/\alpha+\eta}, \tag{3.16}$$

where

$$c = \int_1^{+\infty} z^{-(2-b)} \big(1+\log(z)\big)^{1/\alpha+\eta} dz.$$

Finally, combining (3.15) with (3.16), one gets the lemma. □

Lemma 3.4. *Let $\eta > 0$ be an arbitrarily small fixed real number. There exists a positive and finite random variable C only depending on b and η, such that any $\omega \in \Omega_1^{**}$, $(J, Q) \in \mathbb{N}^2$ and $j \in \{0, \ldots, J-$*

$1\}$, *satisfy:*

$$\left\|2^{-j\cdot}\sum_{k=2^{J-j}+1}^{2^{J+Q-j}}\zeta_{j,-k}(\omega)\theta(2^j\cdot+k,\cdot)\right\|_{\mathcal{C}}:=\sup_{(u,v)\in[0,1]\times[a,b]}\left|2^{-jv}\sum_{k=2^{J-j}+1}^{2^{J+Q-j}}\zeta_{j,-k}(\omega)\theta(2^ju+k,v)\right|$$
$$\le C(\omega)2^{-(1-b)J}J^{1/\alpha+\eta}2^{-(2a-1)j}(1+j)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+j\big). \tag{3.17}$$

Proof of Lemma 3.4. Let us set $c_1:=\sup_{n\in\mathbb{N}}\left\{2^{-(1-b)n}(1+n)^{1/\alpha+\eta}\right\}<\infty$, then one has,

$$2^{-(1-v)(J+Q-j)}(J+Q-j)^{1/\alpha+\eta}\le c_1 2^{-(1-v)(J-j)}(J-j)^{1/\alpha+\eta}; \tag{3.18}$$

the inequality (3.18) follows from the fact that,

$$\begin{aligned}2^{-(1-v)(J+Q-j)}(J+Q-j)^{1/\alpha+\eta} &= 2^{-(1-v)Q}\Big(1+\frac{Q}{J-j}\Big)^{1/\alpha+\eta}2^{-(1-v)(J-j)}(J-j)^{1/\alpha+\eta}\\ &\le 2^{-(1-b)Q}(1+Q)^{1/\alpha+\eta}2^{-(1-v)(J-j)}(J-j)^{1/\alpha+\eta}.\end{aligned}$$

Next putting together, (3.12), (3.18) and (3.14), one gets that for any arbitrary $v\in[a,b]$,

$$\begin{aligned}&\sup_{u\in[0,1]}\left|2^{-jv}\sum_{k=2^{J-j}+1}^{2^{J+Q-j}}\zeta_{j,-k}(\omega)\theta(2^ju+k,v)\right|\\ &\le C(\omega)2^{-(1-v)(J-j)-jv}(J-j)^{1/\alpha+\eta}(1+j)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+j\big)\\ &\le C(\omega)2^{-(1-v)J-(2v-1)j}(J-j)^{1/\alpha+\eta}(1+j)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+j\big)\\ &\le C(\omega)2^{-(1-b)J}J^{1/\alpha+\eta}2^{-(2a-1)j}(1+j)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+j\big),\end{aligned} \tag{3.19}$$

where C is a positive and finite random variable only depending on b and η; thus (3.19) implies that (3.17) holds. □

Lemma 3.5. *Let $\eta>0$ be an arbitrarily small fixed real number. There exists a positive and finite random variable C only depending on a, b and η, such that any $\omega\in\Omega_1^{**}$ and $(J,Q)\in\mathbb{N}^2$, satisfy*

$$\left\|A_{2,+}^{J,Q}(\cdot,\cdot,\omega)\right\|_{\mathcal{C}}\le C(\omega)2^{-(1-b)J}J^{1/\alpha+\eta}; \tag{3.20}$$

recall that $A_{2,+}^{J,Q}(\cdot,\cdot,\omega)$ has been defined in (3.4).

Proof of Lemma 3.5. The lemma can be obtained by using (3.4), the triangle inequality, (3.17) and the fact that

$$\sum_{j=0}^{+\infty}2^{-(2a-1)j}(1+j)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+j\big)<\infty; \tag{3.21}$$

notice that (3.21) results from our assumption that $a>1/\alpha>1/2$. □

Lemma 3.6. *Let $\eta>0$ be an arbitrarily small fixed real number. There exists a positive and finite random variable C only depending on a, b and η, such that any $\omega\in\Omega_1^{**}$ and $(J,Q)\in\mathbb{N}^2$, satisfy*

$$\left\|B_{2,+}^{J,Q}(\cdot,\cdot,\omega)\right\|_{\mathcal{C}}\le C(\omega)\,2^{-aJ}J^{1/\alpha}\log^{1/\alpha+\eta}\big(1+J\big); \tag{3.22}$$

recall that $B_{2,+}^{J,Q}(\cdot,\cdot,\omega)$ has been defined in (3.5).

Proof of Lemma 3.6. First notice that by using the fact that,

$$2^{-(1-v)(J+Q-j)}(J+Q-j)^{1/\alpha+\eta}+\sum_{k=2}^{2^{J+Q-j}}k^{-(2-v)}\log^{1/\alpha+\eta}(k)\le c_1,$$

where the finite constant

$$c_1 := \Big(\sup_{n\in\mathbb{N}} 2^{-(1-b)n} n^{1/\alpha+\eta}\Big) + \sum_{k=2}^{+\infty} k^{-(2-b)} \log^{1/\alpha+\eta}(k);$$

it follows from (3.13), that for any arbitrary $v \in [a,b]$ and $j \in \{J,\dots,J+Q-1\}$, one has

$$\sup_{u\in[0,1]} \left| 2^{-jv} \sum_{k=1}^{2^{J+Q-j}} \zeta_{j,-k}(\omega)\theta(2^j u+k,v) \right| \le C_2(\omega) 2^{-ja}(1+j)^{1/\alpha} \log^{1/\alpha+\eta}\big(2+j\big), \tag{3.23}$$

where C_2 is a positive and finite random variable only depending on b and η. Next, using (3.5), the triangle inequality, and (3.23), one gets that,

$$\big\|B_{2,+}^{J,Q}(\cdot,\cdot,\omega)\big\|_{\mathcal{C}} \le 2C_2(\omega)\sum_{j=J}^{+\infty} 2^{-ja}(1+j)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+j\big); \tag{3.24}$$

moreover, similarly to (2.26), one can show that,

$$\sum_{j=J}^{+\infty} 2^{-ja}(1+j)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+j\big) \le c_3 2^{-Ja}(1+J)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+J\big), \tag{3.25}$$

where c_3 is a finite constant only depending on a. Finally combining (3.24) with (3.25) one obtains the lemma. □

Now we are in position to prove Theorem 3.1.

Proof of Theorem 3.1. The theorem results from (3.3), the triangle inequality, Lemma 3.5, Lemma 3.6 and the inequalities $0 < 1-b < a$; these inequalities are consequences of our assumption that $[a,b] \subset (1/\alpha,1)$. □

3.2. Study of the part $j<0$ of the series.

Theorem 3.2. *For each integer $J \ge 2$, we denote by $X_{2,-}^J = \{X_{2,-}^J(u,v) : (u,v)\in[0,1]\times[a,b]\}$ the $\mathcal{S}\alpha\mathcal{S}$ field with paths in $\mathcal{C}$, defined, for every $(u,v)\in[0,1]\times[a,b]$, as:*

$$X_{2,-}^J(u,v) = \sum_{j=1}^{J-1} 2^{jv} \sum_{k=1}^{2^{J-j}} \zeta_{-j,-k}\big(\theta(2^{-j}u+k,v)-\theta(k,v)\big). \tag{3.26}$$

Let Ω_2^{**} be the event of probability 1 introduced in Proposition 3.3. Then for all $\omega \in \Omega_2^{**}$, $\big(X_{2,-}^J(\cdot,\cdot,\omega)\big)_{J\ge 2}$ is a Cauchy sequence in $\mathcal{C}$, moreover, its limit $X_{2,-}(\cdot,\cdot,\omega)$, satisfies, for each fixed $\eta>0$,

$$\big\|X_{2,-}(\cdot,\cdot,\omega) - X_{2,-}^J(\cdot,\cdot,\omega)\big\|_{\mathcal{C}} = \mathcal{O}\Big(2^{-J(1-b)}J^{1/\alpha}\log^{1/\alpha+\eta}\big(J\big)\Big) \tag{3.27}$$

In order to show that $\big(X_{2,-}^J(\cdot,\cdot,\omega)\big)_{J\ge2}$ is a Cauchy sequence, one needs to appropriately bound the quantity $\big\|X_{2,-}^{J+Q}(\cdot,\cdot,\omega) - X_{2,-}^J(\cdot,\cdot,\omega)\big\|_{\mathcal{C}}$, for all integer $J\ge 2$, and for all $Q\ge 1$. Observe that, in view of (3.26), one has for every $(u,v)\in[0,1]\times[a,b]$,

$$X_{2,-}^{J+Q}(\cdot,\cdot,\omega) - X_{2,-}^J(\cdot,\cdot,\omega) = A_{2,-}^{J,Q}(u,v,\omega) + B_{2,-}^{J,Q}(u,v,\omega), \tag{3.28}$$

where:

$$A_{2,-}^{J,Q}(u,v,\omega) := \sum_{j=1}^{J-1} 2^{jv} \sum_{k=2^{J-j}+1}^{2^{J+Q-j}} \zeta_{-j,-k}(\omega)\big(\theta(2^{-j}u+k,v)-\theta(k,v)\big), \tag{3.29}$$

and

$$B_{2,-}^{J,Q}(u,v,\omega) := \sum_{j=J}^{J+Q-1} 2^{jv} \sum_{k=1}^{2^{J+Q-j}} \zeta_{-j,-k}(\omega)\big(\theta(2^{-j}u+k,v)-\theta(k,v)\big). \tag{3.30}$$

For all $(j,k)\in\mathbb{N}^2$, let $\lambda_{-j,-k}$ be the $\mathcal{S}\alpha\mathcal{S}$ random variable of scale parameter $k^{1/\alpha}$ defined as:

$$\lambda_{-j,-k} := \sum_{m=1}^{k} \zeta_{-j,-m}. \tag{3.31}$$

The proof of the following lemma is similar to that of Lemma 2.1.

Lemma 3.7. *Recall that the function Θ has been introduced in (2.1).*
Let $(J,Q)\in(\mathbb{N}\setminus\{1\})\times\mathbb{N}$ and $(u,v)\in[0,1]\times[a,b]$ be arbitrary and fixed.
(i) For each $j\in\{1,\dots,J-1\}$, one has,

$$\begin{aligned}
&\sum_{k=2^{J-j}+1}^{2^{J+Q-j}} \zeta_{-j,-k}\big(\theta(2^{-j}u+k,v)-\theta(k,v)\big) \\
&= \lambda_{-j,-2^{J+Q-j}}\left(\theta(2^{-j}u+2^{J+Q-j},v)-\theta(2^{J+Q-j},v)\right) \\
&\qquad -\lambda_{-j,-2^{J-j}}\left(\theta(2^{-j}u+2^{J-j}+1,v)-\theta(2^{J-j}+1,v)\right) \\
&\qquad -\sum_{k=2^{J-j}+2}^{2^{J+Q-j}} \lambda_{-j,-(k-1)}\big(\Theta(2^{-j}u+k,v)-\Theta(k,v)\big).
\end{aligned} \tag{3.32}$$

(ii) For each $j\in\{J,\dots,J+Q-1\}$, one has

$$\begin{aligned}
&\sum_{k=1}^{2^{J+Q-j}} \zeta_{-j,-k}\big(\theta(2^{-j}u+k,v)-\theta(k,v)\big) \\
&= \lambda_{-j,-2^{J+Q-j}}\left(\theta(2^{-j}u+2^{J+Q-j},v)-\theta(2^{J+Q-j},v)\right) \\
&\qquad -\sum_{k=2}^{2^{J+Q-j}} \lambda_{-j,-(k-1)}\big(\Theta(2^{-j}u+k,v)-\Theta(k,v)\big).
\end{aligned} \tag{3.33}$$

The proof of the following proposition is similar to that of Proposition 2.1.

Proposition 3.3. *There exists an event of probability 1, denoted by Ω_2^{**}, such that for every fixed real number $\eta>0$, one has, for all $\omega\in\Omega_2^{**}$ and for each $(j,k)\in\mathbb{Z}_+\times\mathbb{N}$,*

$$\big|\lambda_{-j,-k}(\omega)\big| \le C(\omega)\big(1+j\big)^{1/\alpha}\log^{1/\alpha+\eta}\left(3+j\right)k^{1/\alpha}\log^{1/\alpha+\eta}\left(2+k\right), \tag{3.34}$$

where C is a positive and finite random variable only depending on η.

We denote by $\partial_x\theta$ and $\partial_x\Theta$, the partial derivatives of order 1 with respect to x of the functions θ and Θ; observe that in view of (1.13), (1.2) and (2.1), one has for all $(x,v)\in\mathbb{R}\times(1/\alpha,1)$,

$$(\partial_x\theta)(x,v) = (x-1)_+^{v-1/\alpha} - 2(x-1/2)_+^{v-1/\alpha} + (x)_+^{v-1/\alpha} \tag{3.35}$$

and

$$(\partial_x\Theta)(x,v) = (\partial_x\theta)(x,v) - (\partial_x\theta)(x-1,v) = \sum_{l=0}^{4} d_l(x-l/2)_+^{v-1/\alpha}, \tag{3.36}$$

where $d_0,\dots,d_4$ have been defined in (2.19). The proof of the following proposition relies on (3.35) and (3.36); we will not give it since it is very similar to that of Proposition 2.2.

Proposition 3.4. *(i) For each $(x, v) \in (-\infty, 0] \times (1/\alpha, 1)$, one has,*

$$(\partial_x \theta)(x, v) = (\partial_x \Theta)(x, v) = 0.$$

(ii) There exists a constant $c > 0$, such that for all $(x, v) \in (0, +\infty) \times [a, b]$, one has,

$$|(\partial_x \theta)(x, v)| \le c(1+x)^{v-1/\alpha-2}.$$

(iii) There is a constant $c' > 0$, such that for all $(x, v) \in (0, +\infty) \times [a, b]$, one has,

$$|(\partial_x \Theta)(x, v)| \le c'(1+x)^{v-1/\alpha-3}.$$

The following proposition easily results from the Mean Value Theorem and from Proposition 3.4.

Proposition 3.5. *There exist two constants $c > 0$ and $c' > 0$, such that for all $(j, k) \in \mathbb{N}^2$ and $(u, v) \in [0, 1] \times [a, b]$, one has,*

$$\left|\theta(2^{-j}u + k, v) - \theta(k, v)\right| \le c2^{-j}\big(1+k\big)^{v-1/\alpha-2} \tag{3.37}$$

and

$$\left|\Theta(2^{-j}u + k, v) - \Theta(k, v)\right| \le c'2^{-j}\big(1+k\big)^{v-1/\alpha-3}. \tag{3.38}$$

The following lemma is a straightforward consequence of Lemma 3.7 as well as Propositions 3.3 and 3.5.

Lemma 3.8. *Let $\eta > 0$ be arbitrary and fixed. There exists a positive and finite random variable C only depending on $\eta > 0$, such that any $\omega \in \Omega_2^{**}$, $(J, Q) \in (\mathbb{N} \setminus \{1\}) \times \mathbb{N}$ and $v \in [a, b]$, satisfy the following two properties:*

(i) for each $j \in \{1, \ldots, J-1\}$, one has,

$$\sup_{u \in [0,1]} \left| \sum_{k=2^{J-j}+1}^{2^{J+Q-j}} \zeta_{-j,-k}(\omega)\big(\theta(2^{-j}u + k, v) - \theta(k, v)\big) \right| \tag{3.39}$$

$$\le C(\omega) 2^{-j}(1+j)^{1/\alpha} \log^{1/\alpha+\eta}(2+j) \Bigg\{ 2^{-(2-v)(J+Q-j)}(J+Q-j)^{1/\alpha+\eta}$$

$$+ 2^{-(2-v)(J-j)}(J-j)^{1/\alpha+\eta} + \sum_{k=2^{J-j}+2}^{2^{J+Q-j}} k^{-(3-v)} \log^{1/\alpha+\eta}(k) \Bigg\};$$

(ii) for each $j \in \{J, \ldots, J+Q-1\}$, one has

$$\sup_{u \in [0,1]} \left| \sum_{k=1}^{2^{J+Q-j}} \zeta_{-j,-k}(\omega)\big(\theta(2^{-j}u + k, v) - \theta(k, v)\big) \right| \tag{3.40}$$

$$\le C(\omega) 2^{-j}(1+j)^{1/\alpha} \log^{1/\alpha+\eta}(2+j) \Bigg\{ 2^{-(2-v)(J+Q-j)}(J+Q-j)^{1/\alpha+\eta}$$

$$+ \sum_{k=2}^{2^{J+Q-j}} k^{-(3-v)} \log^{1/\alpha+\eta}(k) \Bigg\}.$$

The proof of the following lemma mainly relies on (3.39), it can be done similarly to that of Lemma 3.4.

Lemma 3.9. *Let $\eta > 0$ be an arbitrarily small fixed real number. There exists a positive and finite random variable C only depending on b and η, such that any $\omega \in \Omega_2^{**}$, $(J,Q) \in (\mathbb{N} \setminus \{1\}) \times \mathbb{N}$ and $j \in \{1, \dots, J-1\}$, satisfy:*

$$\begin{aligned}
&\left\| 2^{j\cdot} \sum_{k=2^{J-j}+1}^{2^{J+Q-j}} \zeta_{-j,-k}(\omega)\big(\theta(2^{-j}\cdot + k, \cdot) - \theta(k,\cdot)\big) \right\|_{\mathcal{C}} \\
&:= \sup_{(u,v)\in[0,1]\times[a,b]} \left| 2^{jv} \sum_{k=2^{J-j}+1}^{2^{J+Q-j}} \zeta_{-j,-k}(\omega)\big(\theta(2^{-j}\cdot + k, \cdot) - \theta(k,\cdot)\big) \right| \\
&\le C(\omega) 2^{-(1-b)J} 2^{-(J-j)} (J-j)^{1/\alpha+\eta} (1+j)^{1/\alpha} \log^{1/\alpha+\eta}\big(2+j\big). \qquad (3.41)
\end{aligned}$$

Lemma 3.10. *Let $\eta > 0$ be an arbitrarily small fixed real number. There exists a positive and finite random variable C only depending on b and η, such that any $\omega \in \Omega_2^{**}$ and $(J,Q) \in (\mathbb{N} \setminus \{1\}) \times \mathbb{N}$, satisfy*

$$\big\| A_{2,-}^{J,Q}(\cdot,\cdot,\omega) \big\|_{\mathcal{C}} \le C(\omega) 2^{-(1-b)J} J^{1/\alpha} \log^{1/\alpha+\eta}\big(1+J\big); \qquad (3.42)$$

recall that $A_{2,-}^{J,Q}(\cdot,\cdot,\omega)$ has been defined in (3.29).

Proof of Lemma 3.10. Using (3.4), the triangle inequality, and (3.17), one obtains that,

$$\begin{aligned}
&\big\| A_{2,-}^{J,Q}(\cdot,\cdot,\omega) \big\|_{\mathcal{C}} \qquad (3.43) \\
&\le C(\omega) 2^{-(1-b)J} \sum_{j=1}^{J-1} 2^{-(J-j)} (J-j)^{1/\alpha+\eta} (1+j)^{1/\alpha} \log^{1/\alpha+\eta}\big(2+j\big) \\
&\le C(\omega) 2^{-(1-b)J} J^{1/\alpha} \log^{1/\alpha+\eta}\big(1+J\big) \sum_{j=1}^{J-1} 2^{-(J-j)} (J-j)^{1/\alpha+\eta};
\end{aligned}$$

moreover, observe that,

$$\sum_{j=1}^{J-1} 2^{-(J-j)} (J-j)^{1/\alpha+\eta} \le c_1, \qquad (3.44)$$

where c_1 is the finite constant non depending on J, defined as

$$c_1 = \sum_{n=1}^{+\infty} 2^{-n} n^{1/\alpha+\eta}.$$

Finally, combining (3.43) with (3.44) one gets the lemmma. □

Lemma 3.11. *Let $\eta > 0$ be an arbitrarily small fixed real number. There exists a positive and finite random variable C only depending on b and η, such that any $\omega \in \Omega_2^{**}$ and $(J,Q) \in (\mathbb{N} \setminus \{1\}) \times \mathbb{N}$, satisfy*

$$\big\| B_{2,-}^{J,Q}(\cdot,\cdot,\omega) \big\|_{\mathcal{C}} \le C(\omega)\, 2^{-(1-b)J} J^{1/\alpha} \log^{1/\alpha+\eta}\big(1+J\big); \qquad (3.45)$$

recall that $B_{2,-}^{J,Q}(\cdot,\cdot,\omega)$ has been defined in (3.30).

Proof of Lemma 3.11. First notice that by using the fact that,

$$2^{-(2-v)(J+Q-j)} (J+Q-j)^{1/\alpha+\eta} + \sum_{k=2}^{2^{J+Q-j}} k^{-(3-v)} \log^{1/\alpha+\eta}(k) \le c_1,$$

where the finite constant

$$c_1 := \Big(\sup_{n\in\mathbb{N}} 2^{-(2-b)n} n^{1/\alpha+\eta}\Big) + \sum_{k=2}^{+\infty} k^{-(3-b)} \log^{1/\alpha+\eta}(k);$$

it follows from (3.40), that for any arbitrary $v \in [a,b]$ and $j \in \{J,\dots,J+Q-1\}$, one has

$$\sup_{u\in[0,1]} \Big| 2^{jv} \sum_{k=1}^{2^{J+Q-j}} \zeta_{-j,-k}(\omega)\big(\theta(2^{-j}u+k,v)-\theta(k,v)\big)\Big| \le C_2(\omega) 2^{-j(1-b)}(1+j)^{1/\alpha} \log^{1/\alpha+\eta}\big(2+j\big), \quad (3.46)$$

where, C_2 is a positive and finite random variable only depending on b and η. Next, using (3.30), the triangle inequality, and (3.46), one gets that,

$$\big\|B_{2,-}^{J,Q}(\cdot,\cdot,\omega)\big\|_{\mathcal{C}} \le C_2(\omega) \sum_{j=J}^{+\infty} 2^{-j(1-b)}(1+j)^{1/\alpha} \log^{1/\alpha+\eta}\big(2+j\big); \quad (3.47)$$

Moreover, similarly to (2.26), one can show that,

$$\sum_{j=J}^{+\infty} 2^{-j(1-b)}(1+j)^{1/\alpha} \log^{1/\alpha+\eta}\big(2+j\big) \le c_3 2^{-J(1-b)}(1+J)^{1/\alpha} \log^{1/\alpha+\eta}\big(2+J\big), \quad (3.48)$$

where c_3 is a finite constant only depending on b. Finally combining (3.47) with (3.48) one obtains the lemma. □

Now we are in position to prove Theorems 3.2 and 1.2.

Proof of Theorem 3.2. The theorem results from (3.28) as well as Lemmas 3.10 and 3.11. □

Proof of Theorem 1.2. Putting together (1.16), (3.1), (3.26), Theorem 3.1 and Theorem 3.2, one gets the theorem.

□

4. Simulations

Let us stress that Theorem 1.1 and Theorem 1.2 provide an efficient method for simulating paths of the high frequency part and the low frequency part of LMSM, namely of the $\mathcal{S}\alpha\mathcal{S}$ processes $\{Y_1(t) : t \in [0,1]\} := \{X_1(t, H(t)) : t \in [0,1]\}$ and $\{Y_2(t) : t \in [0,1]\} := \{X_2(t, H(t)) : t \in [0,1]\}$. The following four simulations have been performed by using (1.15) and (1.16) in which $J = 12$.

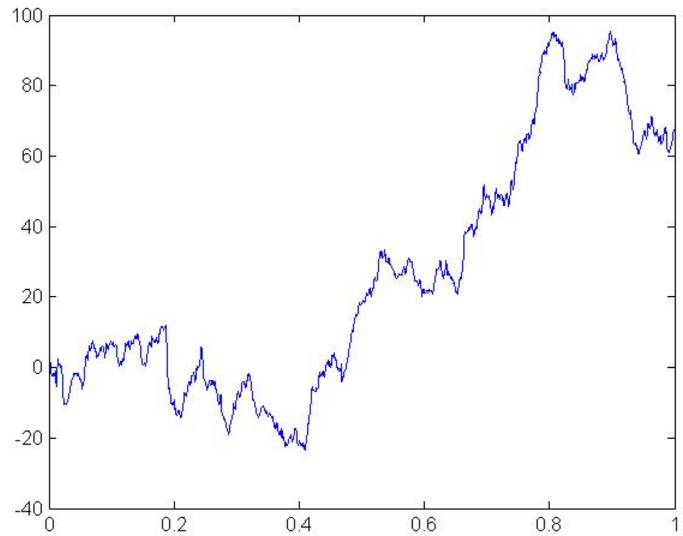

FIGURE 1. Simulation of a path of the process Y_1 when $\alpha = 1.8$ and $H(t) = 0.7$ for all $t \in [0,1]$.

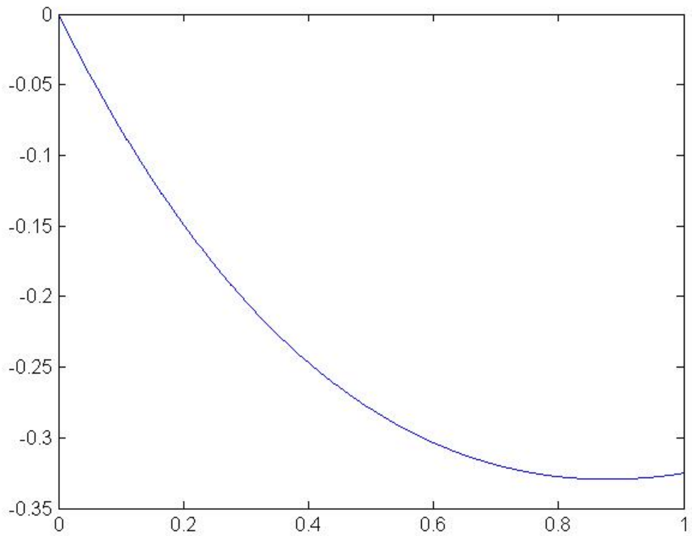

FIGURE 2. Simulation of a path of the process Y_2 when $\alpha = 1.8$ and $H(t) = 0.7$ for all $t \in [0,1]$.

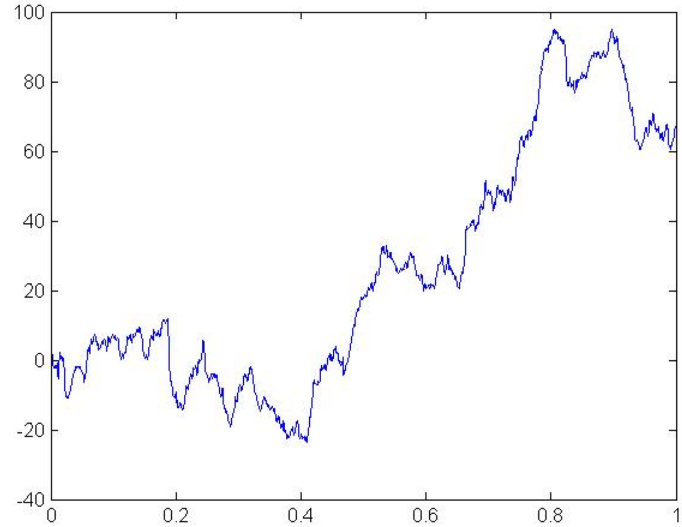

FIGURE 3. Simulation of a path of the LMSM when $\alpha = 1.8$ and $H(t) = 0.7$ for all $t \in [0, 1]$.

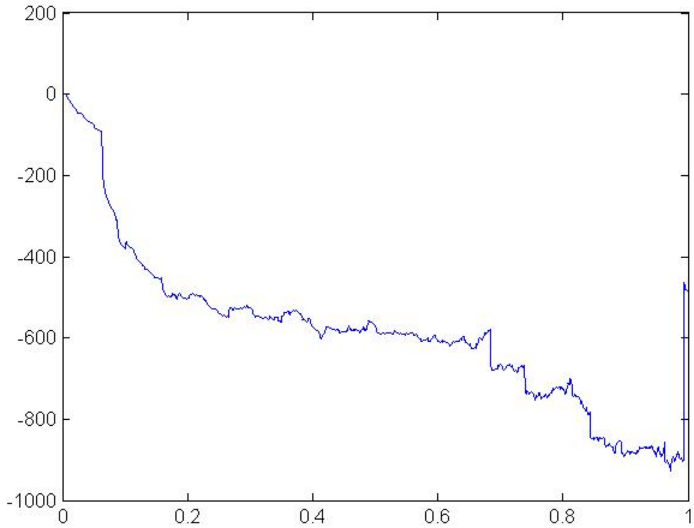

FIGURE 4. Simulation of a path of the process Y_1 when $\alpha = 1.4$ and $H(t) = 0.9 - 0.2t$ for all $t \in [0, 1]$.

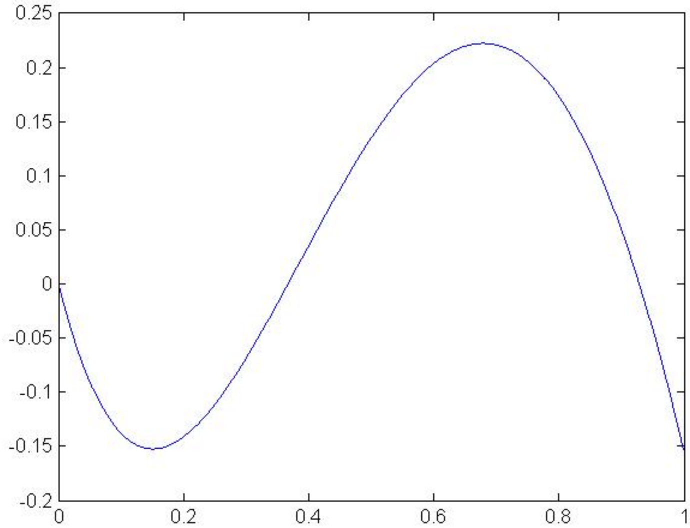

FIGURE 5. Simulation of a path of the process Y_2 when $\alpha = 1.4$ and $H(t) = 0.9 - 0.2t$ for all $t \in [0, 1]$.

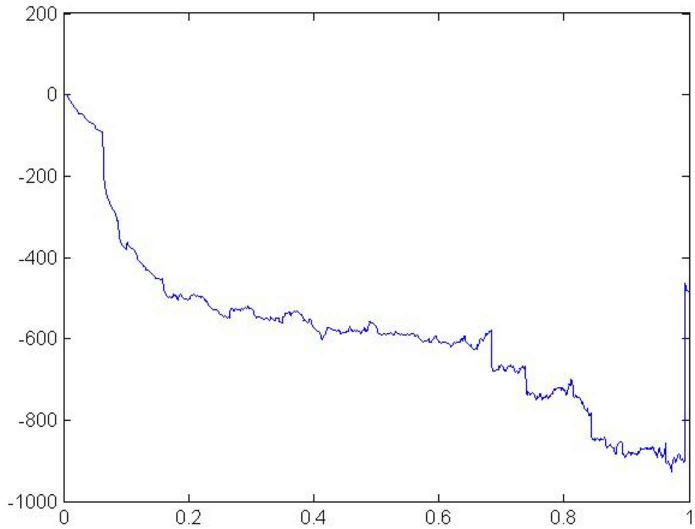

FIGURE 6. Simulation of a path of the LMSM when $\alpha = 1.4$ and $H(t) = 0.9 - 0.2t$ for all $t \in [0, 1]$.

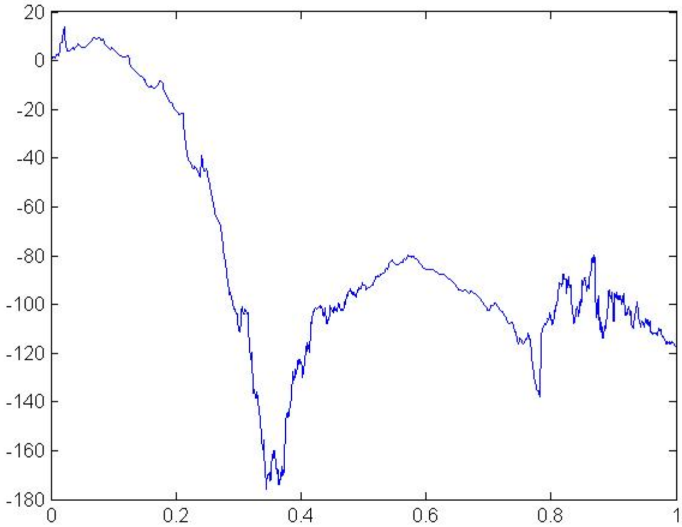

FIGURE 7. Simulation of a path of the process Y_1 when $\alpha = 1.7$ and $H(t) = 0.2\sin(4\pi t) + 0.8$ for all $t \in [0, 1]$.

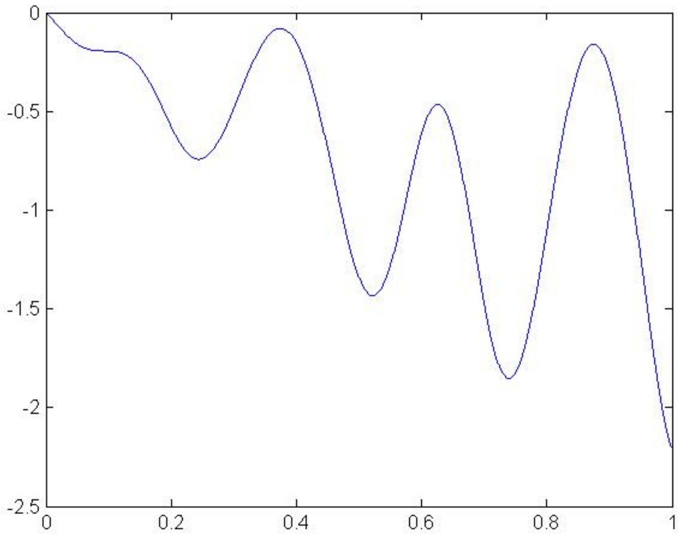

FIGURE 8. Simulation of a path of the process Y_2 when $\alpha = 1.7$ and $H(t) = 0.2\sin(4\pi t) + 0.8$ for all $t \in [0, 1]$.

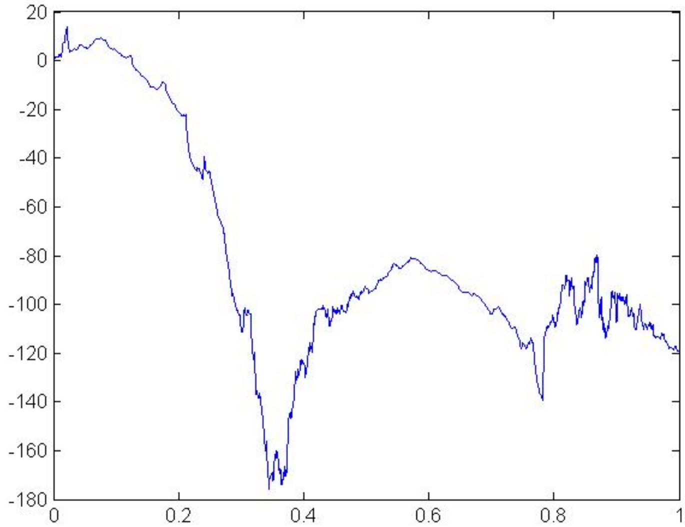

FIGURE 9. Simulation of a path of the LMSM when $\alpha = 1.7$ and $H(t) = 0.2\sin(4\pi t) + 0.8$ for all $t \in [0, 1]$.

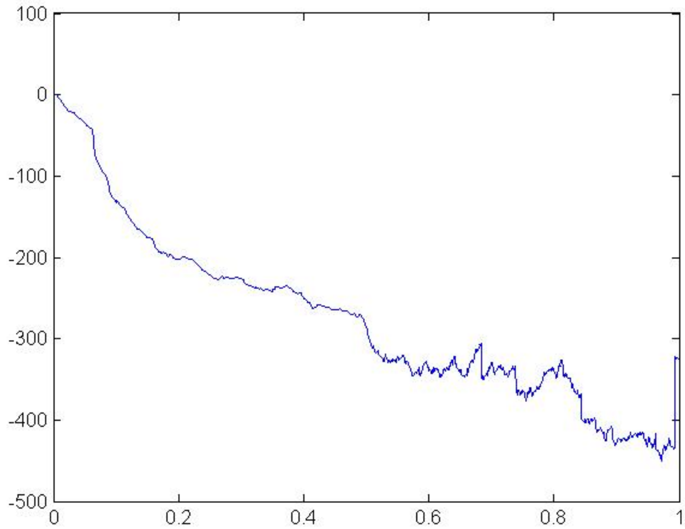

FIGURE 10. Simulation of a path of the process Y_1 when $\alpha = 1.6$ and $H(t) = 0.65 + 0.25/(1 + \exp(100(t - 0.5)))$ for all $t \in [0, 1]$.

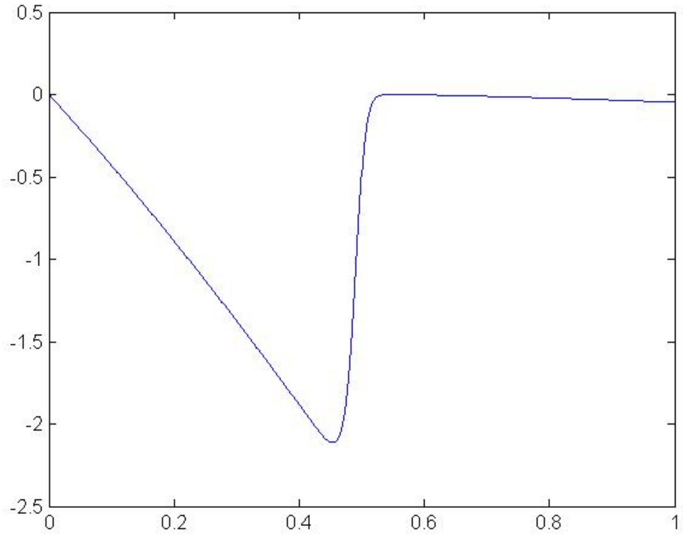

FIGURE 11. Simulation of a path of the process Y_2 when $\alpha = 1.6$ and $H(t) = 0.65 + 0.25/(1 + \exp(100(t - 0.5)))$ for all $t \in [0, 1]$.

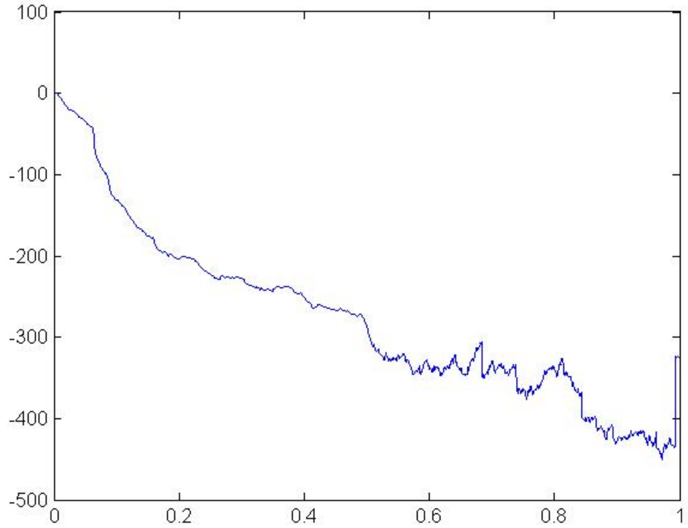

FIGURE 12. Simulation of a path of the LMSM when $\alpha = 1.6$ and $H(t) = 0.65 + 0.25/(1 + \exp(100(t - 0.5)))$ for all $t \in [0, 1]$.

CHAPITRE 4

Inférence statistique concernant le paramètre fonctionnel de Hurst $H(\cdot)$

1. Introduction

Soit α un nombre réel fixé une fois pour toute vérifiant $1 < \alpha < 2$. On note par $\{X(u,v) : (u,v) \in [0,1]\times]1/\alpha,1[\}$ la restriction à $[0,1]\times]1/\alpha,1[$ de la modification à trajectoires continues (voir les Théorèmes 2.1 et 3.1 du chapitre 2) du champ stochastique qui engendre les mouvements multifractionnaires stables linéaires (mmsl en abrégé en français, LMSM en abrégé en anglais) ; de plus, tout au long du présent chapitre, on restreint l'espace de probabilité Ω, à Ω_0^* l'évènement de probabilité 1 qui a été introduit dans le Lemme 2.1 du chapitre 2. Rappelons que, pour tout $(u,v) \in \mathbb{R}\times(1/\alpha,1)$, l'on a, presque sûrement,

$$X(u,v) = \int_{\mathbb{R}} \Big\{(u-s)_+^{v-1/\alpha} - (-s)_+^{v-1/\alpha}\Big\} \mathrm{Z}_\alpha(ds)\,, \tag{1.1}$$

où, pour tous nombres réels x et κ,

$$(x)_+^\kappa := \begin{cases} x^\kappa, \text{si } x \in (0,+\infty), \\ \\ 0, \text{si } x \in (-\infty,0]. \end{cases} \tag{1.2}$$

Dans tout ce chapitre, on suppose que $\{Z_\alpha(s) : s \in \mathbb{R}\}$ est un processus de Lévy symétrique $\alpha-$stable ($\mathcal{S}\alpha\mathcal{S}$ en abrégé) dont les trajectoires sont des fonctions càdlàg.

On note par $H(\cdot)$ une fonction déterministe définie sur $[0,1]$ et à valeurs dans un intervalle compact $[\underline{H},\overline{H}] \subset]1/\alpha,1[$, où $\underline{H} := \min_{x\in[0,1]} H(x)$ et $\overline{H} := \max_{x\in[0,1]} H(x)$. Dans tout ce chapitre, on suppose que $H(\cdot)$ est höldérienne, ce qui signifie qu'il existe deux constantes $c_1 > 0$ et $\rho_H > 0$, telles que

$$\forall t_1,t_2 \in [0,1];\ |H(t_1) - H(t_2)| \le c_1|t_1 - t_2|^{\rho_H}\ ; \tag{1.3}$$

de plus, on impose à ρ_H de vérifier les inégalités :

$$1 \ge \rho_H > \overline{H} := \max_{x\in[0,1]} H(x). \tag{1.4}$$

Rappelons que $\{Y(t) : t \in [0,1]\}$ le mouvement multifractionnaire stable linéaire (mmsl) de paramètre de stabilité α et de paramètre fonctionnel (de Hurst) $H(\cdot)$, introduit par Stoev et Taqqu dans [28, 30], est défini, pour tout $t \in \mathbb{R}$, par

$$Y(t) = X(t,H(t)). \tag{1.5}$$

Rappelons aussi que, d'une part le mmsl généralise le mouvement fractionnaire stable linéaire (mfsl en abrégé en français, LFSM en abrégé en anglais) qui est l'un des processus stables les plus importants (voir par exemple [26, 31, 17]), et que d'autre part il généralise le mouvement brownien multifractionnaire gaussien (mbm en abrégé en français, MBM en abrégé en anglais) qui a été introduit dans [22, 9]. Plus précisément, lorsque la fonction $H(\cdot)$ est une constante noté H, alors le mmsl se réduit au mfsl de paramètre de Hurst H, par ailleurs lorsque (1.1) la mesure stable $\mathrm{Z}_\alpha(ds)$ est remplacée par une mesure brownienne et α par 2, alors le processus gaussien défini par (1.5) est un mbm.

Dans le cas du mfsl, le problème de l'estimation de H a déjà été étudié dans plusieurs travaux (voir entre autres [**1**, **13**, **14**, **24**, **27**]) et des estimateurs fortement consistants, obtenus au moyen de transformées par ondelettes de ce processus, ont été proposés ; notons au passage que ces estimateurs ne nécessitent pas la connaissance de α.

Par ailleurs, dans le cas du mbm, le problème de l'estimation de $H(t_0)$ (t_0 est un point arbitraire et fixé), a lui aussi déjà été étudié dans plusieurs travaux (voir entre autres [**8**, **10**, **11**, **7**, **19**, **23**]) et des estimateurs fortement consistants, obtenus en "localisant autour de t_0", des variations quadratiques généralisées de ce processus ont été proposés.

En revanche, dans le cas du mmsl aucun travail sur l'estimation de $H(t_0)$ (t_0 est un point arbitraire et fixé) ou encore de $\min_{t\in I} H(t)$ (I est un intervalle compact d'intérieur non vide arbitraire et fixé), n'a encore été entrepris ; c'est ce que nous nous proposons de faire dans le présent chapitre, au moyen de coefficients d'ondelettes de ce processus.

La suite $\{d_{j,k} \,:\, (j,k)\in\mathbb{Z}^2\}$, des coefficients d'ondelettes du mmsl $\{Y(t) \,:\, t\in\mathbb{R}\}$, est définie par,

$$d_{j,k} = 2^j \int_{\mathbb{R}} Y(t)\psi(2^j t - k)dt. \tag{1.6}$$

De plus, on impose seulement à l'ondelette analysante ψ de vérifier les hypothèses suivantes :

(1) ψ est une fonction non identiquement nulle continue sur $\mathbb{R}$ et à valeurs réelles,

(2) son support est un compact inclus dans $[0,1]$,

(3) elle admet, au moins, $N\geq 2$ moments nuls, c'est à dire

$$\forall\, Q = 0,\dots,N-1 \,:\, \int_{\mathbb{R}} s^Q\psi(s)ds = 0 \tag{1.7}$$

Il est à noter que l'on n'a pas besoin de supposer que la suite de fonctions $\{2^{j/2}\psi(2^j\cdot - k) : (j,k)\in\mathbb{Z}^2\}$ soit une base orthonormale de $\mathrm{L}^2(\mathbb{R})$.

Fixons des notations qu'on utilisera assez souvent dans ce chapitre.

- On désigne par $\{I_j\,;\, j\in\mathbb{Z}_+\}$ une suite arbitraire d'intervalles compacts d'intérieur non vide, inclus dans $[0,1]$ et vérifiant la condition suivante : pour tout $j\in\mathbb{Z}_+$,

$$|I_j| = \mathrm{diam}(I_j) := \sup\left\{|x_1 - x_2| \,:\, (x_1,x_2)\in I_j^2\right\} \geq 2^{-j/2}\,; \tag{1.8}$$

  il est à noter que :
  - lorsqu'on cherchera à estimer $H(t_0)$, $\{I_j\,;\, j\in\mathbb{Z}_+\}$ sera alors typiquement une suite décroissante au sens de l'inclusion et qui "converge" vers $\{t_0\}$ c'est à dire que

$$\{t_0\} = \bigcap_{j\in\mathbb{Z}_+} I_j\,;$$

  - lorsqu'on cherchera à estimer $\min_{t\in I} H(t)$, $\{I_j\,;\, j\in\mathbb{Z}_+\}$ sera alors typiquement une suite qui devient constante à partir d'un certain rang et vaut I.
- Puisque la fonction $H(\cdot)$ est continue sur $[0,1]$, elle l'est aussi sur chacun des intervalles I_j introduit ci-dessus. On pose alors

$$\underline{H}_j := \min_{t\in I_j} H(t)\,; \tag{1.9}$$

  En outre, puisque pour tout $j\in\mathbb{Z}_+$, les intervalles I_j sont compacts, on sait qu'il existe, au moins, un réel $\mu_j\in I_j$ tel que

$$\underline{H}_j = H(\mu_j). \tag{1.10}$$

- Enfin, on note par $\{\nu_j\,;\, j \in \mathbb{Z}_+\}$ la suite des ensembles d'indices k, définis par
$$\nu_j = \Big\{k \in \{0, 1, \dots, 2^j - 1\} : k2^{-j} \in I_j\Big\}. \tag{1.11}$$
Constatons que (1.8) implique que $\nu_j \neq \emptyset$. Pour tout $j \in \mathbb{Z}_+$, le cardinal de ν_j est noté par
$$\operatorname{card}(\nu_j) = n_j. \tag{1.12}$$

L'objectif de ce chapitre consiste à établir le théorème suivant :

Théorème 1.1. *Supposons que $\beta \in]0, \alpha/4[$ est un réel arbitraire fixé. Pour tout $j \in \mathbb{Z}_+$, on définit la statistique V_j par*
$$V_j = \frac{1}{n_j} \sum_{k \in \nu_j} |d_{j,k}|^\beta. \tag{1.13}$$
On suppose également que, la Condition (1.8) est vérifiée, autrement dit que,
$$2^{j/2} = \mathcal{O}(n_j). \tag{1.14}$$
Alors, on a, presque sûrement,
$$\lim_{j \to +\infty} \left| \frac{\log_2(V_j)}{-j\beta} - \underline{H}_j \right| = 0. \tag{1.15}$$

Remarque 1.1. (1) *Lorsque le paramètre $\alpha \in]1, 2[$ est inconnu, on choisira alors β dans l'intervalle $]0, 1/4]$.*

(2) *Désignons par $I \subset [0,1]$ un intervalle compact arbitraire d'intérieur non vide ; il est clair qu'il existe $j_0 \in \mathbb{Z}_+$, tel que pour tout entier $j > j_0$, on a $|I| \geq 2^{-j/2}$. Ainsi, en prenant $I_j = [0,1]$ lorsque $j \leq j_0$ et $I_j = I$ sinon, Il résulte alors de (1.15) que, $\frac{\log_2(V_j)}{-j\beta}$, est un estimateur fortement consistant (i.e. qui converge presque sûrement) de $\min_{t \in I} H(t)$.*

(3) *Désignons par $t_0 \in [0,1]$ un point arbitraire et fixé. En supposant que $\{I_j\,;\, j \in \mathbb{Z}_+\}$ est une suite arbitraire d'intervalles d'intérieur non vide, décroissante et telle que $\{t_0\} = \bigcap_{j \in \mathbb{Z}_+} I_j$; il résulte alors de (1.15) et de la continuité de la fonction $H(\cdot)$, que, $\frac{\log_2(V_j)}{-j\beta}$, est un estimateur fortement consistant (i.e. qui converge presque sûrement) de $H(t_0)$.*

2. Preuve du Théorème 1.1

La preuve du Théorème 1.1 repose sur plusieurs lemmes et plus particulièrement sur le suivant :

Lemme 2.1. *On a,*
$$\lim_{j \to +\infty} \left| \frac{\log_2\Big(n_j^{-1} \sum_{k \in \nu_j} 2^{-j\beta H(k2^{-j})}\Big)}{-j\beta} - \underline{H}_j \right| = 0. \tag{2.1}$$

Preuve du Lemme 2.1. Il vient de (1.9) et de (1.11) que
$$n_j^{-1} \sum_{k \in \nu_j} 2^{-j\beta H(k2^{-j})} \leq 2^{-j\beta \underline{H}_j} \tag{2.2}$$
Pour tout entier $j \geq 1$, notons par $\widetilde{\nu}_j(\mu_j)$ l'ensemble d'indices k défini par
$$\widetilde{\nu}_j(\mu_j) = \Big\{k \in \{0, 1, \dots, 2^j - 1\} \,:\, k2^{-j} \in I_j \cap B(\mu_j, j^{-1/\rho_H})\Big\}, \tag{2.3}$$
où
$$B(\mu_j, j^{-1/\rho_H}) = \{x \in \mathbb{R} \,:\, |x - \mu_j| \leq j^{-1/\rho_H}\}. \tag{2.4}$$

Remarquons que $I_j \cap B(\mu_j, j^{-1/\rho_H})$ est un intervalle compact, non-vide (μ_j y appartient), inclus dans $[0,1]$, et que son diamètre vérifie :

$$|I_j \cap B(\mu_j, j^{-1/\rho_H})| \geq \min\left\{\frac{|I_j|}{2}, j^{-1/\rho_H}\right\}, \tag{2.5}$$

pour tout entier $j \geq 1$. Pour établir (2.5), désignons par r_j et z_j les deux nombres réels de l'intervalle $[0,1]$, tels que $I_j = [r_j, z_j]$. Supposons, par exemple, que

$$z_j - \mu_j = \min\{\mu_j - r_j, z_j - \mu_j\}; \tag{2.6}$$

le cas où le minimum est atteint en $\mu_j - r_j$ se traite de la même manière. De (2.6), on obtient

$$\mu_j - r_j \geq z_j - \mu_j \iff 2(\mu_j - r_j) \geq z_j - r_j \iff \mu_j - r_j \geq \frac{z_j - r_j}{2} = \frac{|I_j|}{2}.$$

Par conséquent, l'inégalité précédente nous donne

$$\left[\mu_j - \frac{|I_j|}{2}, \mu_j\right] \subset I_j. \tag{2.7}$$

Ceci implique que

$$\left[\mu_j - \min\left\{\frac{|I_j|}{2}, j^{-1/\rho_H}\right\}, \mu_j\right] \subset I_j \cap B(\mu_j, j^{-1/\rho_H}),$$

alors on a (2.5).

Des relations (1.8) et (2.5), on en tire l'existence d'un entier $j_0 \geq 1$ tel que pour tout $j \geq j_0$, on a

$$|I_j \cap B(\mu_j, j^{-1/\rho_H})| \geq 2^{-j}. \tag{2.8}$$

Cela montre que, pour tout entier $j \geq j_0$, $\widetilde{\nu}_j(\mu_j)$ est un ensemble non-vide. De plus, on peut supposer que j_0 soit suffisamment grand de telle manière que

$$\mathrm{card}(\widetilde{\nu}_j(\mu_j)) \geq c_5 2^j \min\left\{\frac{|I_j|}{2}, j^{-1/\rho_H}\right\}, \tag{2.9}$$

et

$$c_6 2^j |I_j| \leq n_j \leq c_7 2^j |I_j|, \tag{2.10}$$

où c_5, c_6, c_7 sont des constantes strictement positives. Notons que (2.9) résulte de (2.3) et (2.5), et que (2.10) résulte de (1.11). En combinant ensemble (2.9) et (2.10), il vient que pour tout entier $j \geq j_0$

$$\frac{\mathrm{card}(\widetilde{\nu}_j(\mu_j))}{n_j} \geq \frac{c_5 2^j \min\left\{\frac{|I_j|}{2}, j^{-1/\rho_H}\right\}}{c_7 2^j |I_j|} = c_8 \min\left\{\frac{1}{2}; \frac{j^{-1/\rho_H}}{|I_j|}\right\}.$$

En utilisant le fait que $|I_j| \leq 1$ et $1/\rho_H \geq 1$, on a, pour tout entier $j \geq 2$

$$j^{-1/\rho_H} \leq j^{-1} \leq 1/2.$$

Ainsi, on en déduit que, pour tout entier $j \geq j_0 \geq 2$

$$\frac{\mathrm{card}(\widetilde{\nu}_j(\mu_j))}{n_j} \geq c_8 j^{-1/\rho_H}. \tag{2.11}$$

Montrons maintenant qu'il existe une constante strictement positive, noté c_9, qui ne dépend pas de j et de k, telle que pour tout $j \geq j_0$ et tout $k \in \widetilde{\nu}_j(\mu_j)$, on a

$$2^{-j\beta H(k2^{-j})} \geq c_9 2^{-j\beta \underline{H}_j}\ ; \tag{2.12}$$

en employant (1.10), (1.3) et (2.3), on a, pour tout entier $j \geq 1$ et tout $k \in \widetilde{\nu}_j(\mu_j)$, que

$$2^{-j\beta(H(k2^{-j}) - \underline{H}_j)} = 2^{-j\beta|H(k2^{-j}) - \underline{H}_j|} \geq 2^{-j\beta c_1 |k2^{-j} - \mu_j|^{\rho_H}} \geq 2^{-\beta c_1} > 0,$$

ce qui prouve (2.12). En remarquant que $\widetilde{\nu}_j(\mu_j) \subset \nu_j$ et en utilisant les relations (2.11) et (2.12), on obtient que

$$n_j^{-1} \sum_{k \in \nu_j} 2^{-j\beta H(k2^{-j})} \geq n_j^{-1} \sum_{k \in \widetilde{\nu}_j(\mu_j)} 2^{-j\beta H(k2^{-j})} \geq c_{10} j^{-1/\rho_H} 2^{-j\beta \underline{H}_j}; \tag{2.13}$$

d'où,

$$\log_2 \Big(n_j^{-1} \sum_{k \in \nu_j} 2^{-j\beta H(k2^{-j})} \Big) \geq -j\beta \underline{H}_j + \log_2(c_{10} j^{-1/\rho_H}). \tag{2.14}$$

Par ailleurs, il résulte de (2.2), que

$$\log_2 \Big(n_j^{-1} \sum_{k \in \nu_j} 2^{-j\beta H(k2^{-j})} \Big) \leq -j\beta \underline{H}_j. \tag{2.15}$$

Enfin, en utilisant (2.14), (2.15) et $\lim_{j \to +\infty} \frac{\log_2(c_{10} j^{-1/\rho_H})}{j} = 0$, il en résulte (2.1). □

Définition 2.1. *On pose pour tout $j \in \mathbb{Z}_+$ et tout $k \in \{0, 1, \dots, 2^j - 1\}$*

$$\widetilde{d}_{j,k} = 2^j \int_{\mathbb{R}} X(t, H(k2^{-j})) \psi(2^j t - k) dt. \tag{2.16}$$

Rappelons tout d'abord un résultat de Delbeke et Abry [14, Corollaire 1], qui permet de réécrire (2.16) sous la forme d'une intégrale stochastique. Plus précisément, on a, presque sûrement, pour tout $(j,k) \in \mathbb{Z}^2$,

$$\widetilde{d}_{j,k} \overset{p.s.}{=} 2^{-j(H(k2^{-j}) - 1/\alpha)} \int_{\mathbb{R}} \Phi_\alpha(2^j s - k, H(k2^{-j})) \mathrm{Z}_\alpha(ds), \tag{2.17}$$

où la fonction Φ_α est la fonction à valeurs réelles, définie pour tout $(s,v) \in \mathbb{R} \times]1/\alpha, 1[$, par

$$\Phi_\alpha(s,v) = \int_{\mathbb{R}} (y-s)_+^{v-1/\alpha} \psi(y) dy. \tag{2.18}$$

Proposition 2.1. *La fonction Φ_α satisfait les trois propriétés suivantes :*

(i) La fonction Φ_α est continue sur $\mathbb{R} \times]1/\alpha, 1[$;

(ii) La fonction Φ_α est bien localisée en s uniformément en $v \in [\underline{H}, \overline{H}]$, c'est à dire qu'il existe une constante strictement positive c_2 telle que pour tout $(s,v) \in \mathbb{R} \times [\underline{H}, \overline{H}]$, on a

$$\big|\Phi_\alpha(s,v)\big| \leq c_2 \big(1 + |s|\big)^{-(N+1/\alpha - \overline{H})} \tag{2.19}$$

(iii) La fonction

$$\begin{array}{ll} [\underline{H}, \overline{H}] & \to \mathbb{R}_+ \\ v & \mapsto \|\Phi_\alpha(\cdot, v)\|_{\mathrm{L}^\alpha(\mathbb{R})}, \end{array}$$

est continue. De plus, on a

$$c_3 := \min_{v \in [\underline{H}, \overline{H}]} \|\Phi_\alpha(\cdot, v)\|_{\mathrm{L}^\alpha(\mathbb{R})} > 0. \tag{2.20}$$

Preuve de la Proposition 2.1. Premièrement, montrons le point (i). On fixe $(\tilde{u}, \tilde{v}) \in \mathbb{R} \times]1/\alpha, 1[$ et on note par $\{(u_n, v_n) : n \in \mathbb{Z}_+\}$ une suite d'éléments de $\mathbb{R} \times]1/\alpha, 1[$ qui converge vers $(\tilde{u}, \tilde{v})$. Puisque, pour tout $y \in \mathbb{R}$ fixé, la fonction

$$\begin{array}{ll} \mathbb{R} \times]1/\alpha, 1[& \to \mathbb{R} \\ (s,v) & \mapsto (y-s)_+^{v-1/\alpha} \psi(y) \end{array}$$

est continue, on en déduit que

$$\lim_{n \to +\infty} (y - u_n)_+^{v_n - 1/\alpha} \psi(y) = (y - \tilde{u})_+^{\tilde{v} - 1/\alpha} \psi(y). \tag{2.21}$$

Par ailleurs, en utilisant l'inégalité triangulaire et le fait que pour tout entier positif n, $0 < v_n - 1/\alpha < 1/2$, on a

$$|(y-u_n)_+^{v_n-1/\alpha}\psi(y)| \leq \Big(1+|y|+\sup_{n\in\mathbb{Z}_+}|u_n|\Big)|\psi(y)|. \tag{2.22}$$

De plus, la fonction $y \mapsto \Big(1+|y|+\sup_{n\in\mathbb{Z}_+}|u_n|\Big)|\psi(y)|$ appartient à $\mathrm{L}^1(\mathbb{R})$. Ainsi, grâce à (2.21) et (2.22), nous pouvons utiliser le Théorème de convergence dominée et il en résulte que

$$\lim_{n\to+\infty}\Phi_\alpha(u_n,v_n) = \lim_{n\to+\infty}\int_{\mathbb{R}}(y-u_n)_+^{v_n-1/\alpha}\psi(y)dy$$
$$= \int_{\mathbb{R}}(y-\tilde{u})_+^{\tilde{v}-1/\alpha}\psi(y)dy = \Phi_\alpha(\tilde{u},\tilde{v}).$$

Montrons le point (ii). En utilisant le fait que $\mathrm{supp}(\psi) \subset [0,1]$, la fonction Φ_α, définie en (2.18) se réécrit, pour tout $(s,v) \in \mathbb{R}\times]1/\alpha,1[$, de la manière suivante :

$$\Phi_\alpha(s,v) = \int_0^1 (y-s)_+^{v-1/\alpha}\psi(y)dy. \tag{2.23}$$

En combinant (1.2) et (2.23), on en déduit que, pour tout $v \in]1/\alpha,1[$,

$$\mathrm{supp}(\Phi_\alpha(\cdot,v)) \subset]-\infty,1]. \tag{2.24}$$

Notons que (2.23) implique que, pour tout $(s,v) \in [-1,1]\times[\underline{H},\overline{H}]$, on a

$$\Big\{(1+|s|)^{N+1/\alpha-\overline{H}}|\Phi_\alpha(s,v)|\Big\} \leq 2^{N+1/\alpha-\overline{H}}\int_0^1\Big(|y|+|s|\Big)^{v-1/\alpha}|\psi(y)|dy$$
$$\leq 2^{N+1/\alpha-\overline{H}}\|\psi\|_{\mathrm{L}^\infty(\mathbb{R})}\int_0^1\Big(1+|s|\Big)^{\overline{H}-1/\alpha}dy \leq 2^N\|\psi\|_{\mathrm{L}^\infty(\mathbb{R})} < +\infty. \tag{2.25}$$

Désormais, supposons que $(s,v) \in]-\infty,-1[\times[\underline{H},\overline{H}]$. On note par $\psi^{(-1)}$ la primitive de ψ, définie pour tout $z\in\mathbb{R}$, par

$$\psi^{(-1)}(z) = \int_{-\infty}^z \psi(y)dy.$$

En utilisant (1.7) et $\mathrm{supp}(\psi) \subset [0,1]$, on en déduit que $\psi^{(-1)}$ est une fonction à support compact dans $[0,1]$. Par récurrence, on définit pour tout $m \in \{2,\ldots,N\}$,

$$\psi^{(-m)}(z) = \int_{-\infty}^z \psi^{(-m-1)}(y)dy.$$

Remarquons qu'à cause de (1.7) et de $\mathrm{supp}(\psi^{(-m-1)}) \subset [0,1]$, la fonction $\psi^{(-m)}$ est à support compact dans $[0,1]$ pour tout $m\in\{2,\ldots,N\}$. Par conséquent, en faisant N intégrations par parties, il vient que

$$\Phi_\alpha(s,v) = (-1)^N\Big(\prod_{i=0}^{N-1}(v-1/\alpha-i)\Big)\int_0^1(y-s)^{v-1/\alpha-N}\psi^{(-N)}(y)dy.$$

Ainsi, en employant l'inégalité : pour tout $y\in[0,1]$, $y-s \geq |s| \geq 2^{-1}(1+|s|)$, il en résulte que,

$$|\Phi_\alpha(s,v)| \leq \Big(\prod_{i=0}^{N-1}|v-1/\alpha-i|\Big)2^{N+1/\alpha-v}\|\psi^{(-N)}\|_{\mathrm{L}^\infty(\mathbb{R})}(1+|s|)^{v-1/\alpha-N}$$
$$\leq \Big(\prod_{i=0}^{N-1}(\overline{H}+1/\alpha+i)\Big)2^{N+1/\alpha-\overline{H}}\|\psi^{(-N)}\|_{\mathrm{L}^\infty(\mathbb{R})}(1+|s|)^{\overline{H}-1/\alpha-N} \tag{2.26}$$

Par conséquent, lorsque l'on combine les relations (2.25) et (2.26), on obtient (2.19).

Finalement, établissons le point (iii). Afin de montrer que $v \mapsto \|\Phi_\alpha(\cdot, v)\|_{\mathrm{L}^\alpha(\mathbb{R})}$ est continue sur $[\underline{H}, \overline{H}]$, il suffit de prouver que $v \mapsto \int_{\mathbb{R}} |\Phi_\alpha(u, v)|^\alpha du$ est continue sur $[\underline{H}, \overline{H}]$.

Pour ce faire, on considère une suite $\{v_n : n \in \mathbb{Z}_+\}$ de réels appartenant à $[\underline{H}, \overline{H}]$ qui converge vers $\tilde{v}$. Du point (i), nous savons que pour tout $u \in \mathbb{R}$, nous avons

$$\lim_{n \to +\infty} |\Phi_\alpha(u, v_n)|^\alpha = |\Phi(u, \tilde{v})|^\alpha. \tag{2.27}$$

De (2.19), on en déduit que, pour tout $u \in \mathbb{R}$ et tout $n \in \mathbb{N}$,

$$|\Phi_\alpha(u, v_n)|^\alpha \le c_2^\alpha (1 + |u|)^{-\alpha(N+1/\alpha-\overline{H})}. \tag{2.28}$$

En outre, l'application $u \mapsto (1+|u|)^{-\alpha(N+1/\alpha-\overline{H})}$ appartient à $\mathrm{L}^1(\mathbb{R})$ puisque le nombre N de moments nuls, est supérieur ou égal à 2. En employant (2.27) et (2.28), nous sommes en mesure d'appliquer le Théorème de convergence dominée et l'on a

$$\lim_{n \to +\infty} \|\Phi_\alpha(\cdot, v_n)\|_\alpha^\alpha = \lim_{n \to +\infty} \int_{\mathbb{R}} |\Phi_\alpha(u, v_n)|^\alpha du = \int_{\mathbb{R}} |\Phi_\alpha(u, \tilde{v})|^\alpha du = \|\Phi_\alpha(\cdot, \tilde{v})\|_\alpha^\alpha.$$

Prouvons maintenant (2.20). En employant la continuité de $v \mapsto \|\Phi_\alpha(\cdot, v)\|_{\mathrm{L}^\alpha(\mathbb{R})}$ et la compacité de $[\underline{H}, \overline{H}]$, on sait qu'il existe un réel $v_0 \in [\underline{H}, \overline{H}]$ tel que $c_3 = \|\Phi_\alpha(\cdot, v_0)\|_{\mathrm{L}^\alpha(\mathbb{R})}$. Supposons que $c_3 = 0$, alors on a, pour tout $u \in \mathbb{R}$,

$$\Phi_\alpha(u, v_0) = 0,$$

ce qui revient à dire que pour tout $\xi \in \mathbb{R}$,

$$\widehat{\Phi}_\alpha(\xi, v_0) = 0, \tag{2.29}$$

où $\widehat{\Phi}_\alpha(\cdot, v_0)$ désigne la transformée de Fourier de $\Phi_\alpha(\cdot, v_0)$, définie pour tout $\xi \in \mathbb{R}$, par,

$$\widehat{\Phi}_\alpha(\xi, v_0) = \int_{\mathbb{R}} \Phi_\alpha(x, v_0) e^{-i\xi x} dx.$$

Par ailleurs, en notant par Γ la fonction Gamma définie, pour tout $x \in]0, +\infty[$, par $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$, l'on a que la fonction $\frac{1}{\Gamma(1+v_0-1/\alpha)} \Phi_\alpha(\cdot, v_0)$ est la primitive fractionnaire à droite à l'ordre $1 + v_0 - 1/\alpha$ de l'ondelette ψ ; il en résulte que (voir [25]), pour tout $\xi \in \mathbb{R} \setminus \{0\}$,

$$\widehat{\Phi}_\alpha(\xi, v_0) = \Gamma(1 + v_0 - 1/\alpha) \frac{e^{i\mathrm{sign}(\xi)(1+v_0-1/\alpha)} \widehat{\psi}(\xi)}{|\xi|^{1+v_0-1/\alpha}}. \tag{2.30}$$

En combinant (2.29) et (2.30) on trouve que $\widehat{\psi}$ est la fonction identiquement nulle ; cela est impossible, en effet par hypothèse ψ n'est pas identiquement nulle. □

Lemme 2.2. *Il existe une constante strictement positive, noté c_{12}, telle que pour tout $j \in \mathbb{Z}_+$ et tout $k \in \{0, 1, \ldots, 2^j - 1\}$, on a*

$$c_3 2^{-jH(k2^{-j})} \le \|\widetilde{d}_{j,k}\|_\alpha \le c_{12} 2^{-jH(k2^{-j})}, \tag{2.31}$$

où $\|\widetilde{d}_{j,k}\|_\alpha$ désigne le paramètre d'échelle de la variable aléatoire $\mathcal{S}\alpha\mathcal{S}$ $\widetilde{d}_{j,k}$, et où la constante c_3 a été définie en (2.20).

La preuve de ce lemme repose sur l'utilisation du point (iii) de la Proposition 2.1.

Preuve du Lemme 2.2. En combinant (2.17) avec une propriété usuelle de l'intégrale stochastique stable, on a

$$\begin{aligned} \|\widetilde{d}_{j,k}\|_\alpha^\alpha &= 2^{-j\alpha H(k2^{-j})+j} \int_{\mathbb{R}} |\Phi_\alpha(2^j s - k, H(k2^{-j}))|^\alpha ds \\ &= 2^{-j\alpha H(k2^{-j})} \int_{\mathbb{R}} |\Phi_\alpha(u, H(k2^{-j}))|^\alpha du. \end{aligned}$$

Ainsi, il vient que

$$\|\widetilde{d}_{j,k}\|_\alpha = 2^{-jH(k2^{-j})}\|\Phi_\alpha(\cdot, H(k2^{-j}))\|_{\mathrm{L}^\alpha(\mathbb{R})}. \tag{2.32}$$

En utilisant (2.20) et en posant

$$c_{12} := \max_{v\in[\underline{H},\overline{H}]} \|\Phi_\alpha(\cdot, v)\|_{\mathrm{L}^\alpha(\mathbb{R})},$$

on en déduit (2.31). □

Lemme 2.3. *Il existe une constante $c_{13} > 0$ telle que pour tout $j \in \mathbb{Z}_+$ et tout $(k,l) \in \{0,1,\ldots,2^j - 1\}^2$, on obtient*

$$|\mathrm{cov}(|\widetilde{d}_{j,k}|^\beta, |\widetilde{d}_{j,l}|^\beta)| \leq c_{13}2^{-j\beta(H(k2^{-j})+H(l2^{-j}))}. \tag{2.33}$$

Preuve du Lemme 2.3. De l'inégalité de Cauchy-Schwarz et d'une propriété usuelle des variables aléatoires $\mathcal{S}\alpha\mathcal{S}$, on a

$$\begin{aligned}\left(\mathrm{cov}(|\widetilde{d}_{j,k}|^\beta, |\widetilde{d}_{j,l}|^\beta)\right)^2 &\leq \mathrm{Var}(|\widetilde{d}_{j,k}|^\beta)\mathrm{Var}(|\widetilde{d}_{j,l}|^\beta),\\ &= \left(\mathrm{Var}(|X_0|^\beta)\right)^2\|\widetilde{d}_{j,k}\|_\alpha^{2\beta}\|\widetilde{d}_{j,l}\|_\alpha^{2\beta},\\ &= c_{14}\|\widetilde{d}_{j,k}\|_\alpha^{2\beta}\|\widetilde{d}_{j,l}\|_\alpha^{2\beta}, \end{aligned} \tag{2.34}$$

où X_0 est une variable aléatoire $\mathcal{S}\alpha\mathcal{S}$ de paramètre d'échelle égal à 1. En combinant (2.34) avec (2.31), puis en posant $c_{13} = \sqrt{c_{14}}$, on obtient (2.33). □

Pour tout $j \in \mathbb{Z}_+$, tout $k \in \{0,1,\ldots,2^j - 1\}$, on pose

$$\widetilde{d}_{j,k}^{\mathrm{nor}} = \frac{\widetilde{d}_{j,k}}{\|\widetilde{d}_{j,k}\|_\alpha}. \tag{2.35}$$

Remarquons tout d'abord que ces variables aléatoires sont bien définies puisque pour tout $j \in \mathbb{Z}_+$ et tout $k \in \{0,1,\ldots,2^j - 1\}$, le coefficient d'échelle $\|\widetilde{d}_{j,k}\|_\alpha$ est non-nul. De plus, par construction, les variables aléatoires $\widetilde{d}_{j,k}^{\mathrm{nor}}$ sont toutes des variables aléatoires $\mathcal{S}\alpha\mathcal{S}$ de même paramètre d'échelle égal à 1. Ensuite, posons, pour tout $j \in \mathbb{Z}_+$ et tout $k \in \{0,1,\ldots,2^j - 1\}$,

$$\theta_{j,k} = \|\widetilde{d}_{j,k}\|_\alpha^{-1}. \tag{2.36}$$

Il résulte de (2.17), (2.35) et (2.36) que les variables aléatoires $\widetilde{d}_{j,k}^{\mathrm{nor}}$ peuvent s'écrire sous la forme de l'intégrale stochastique stable suivante :

$$\widetilde{d}_{j,k}^{\mathrm{nor}} = \int_{\mathbb{R}} f_{j,k}(s)\mathrm{Z}_\alpha(ds) \tag{2.37}$$

$$= \theta_{j,k}2^{-j(H(k2^{-j})-1/\alpha)}\int_{\mathbb{R}} \Phi_\alpha(2^js - k, H(k2^{-j}))\mathrm{Z}_\alpha(ds). \tag{2.38}$$

Proposition 2.2. *Soit $\epsilon > 0$ un réel arbitrairement petit. Posons*

$$\lambda := \min\Big\{\frac{\alpha}{2}(N - \overline{H}) - \frac{1}{2} - \epsilon; \Big(\frac{\alpha-1}{2}\Big)(N + \frac{1}{\alpha} - \overline{H})\Big\}. \tag{2.39}$$

Il existe une constante $c_{11} > 0$ telle que pour tout $j \in \mathbb{Z}_+$ et tout $(k,l) \in \{0,1,\ldots,2^j - 1\}^2$, on a

$$\Big|\mathrm{cov}(|\widetilde{d}_{j,k}|^\beta, |\widetilde{d}_{j,l}|^\beta)\Big| \leq c_{11}2^{-j\beta(H(k2^{-j})+H(l2^{-j}))}\big(1 + |k-l|\big)^{-1-\lambda}. \tag{2.40}$$

Afin d'établir cette proposition, nous allons utiliser le Théorème 2.4 de [24] qui repose sur les deux hypothèses (A_1) et (A_2) données à la page 1094 de [24] ; les deux lemmes suivants permettent de montrer que (A_1) et (A_2) sont bien vérifiées dans le cas qui nous intéresse.

Lemme 2.4. *Il existe deux constantes $0 < c_{15} < 1$ et $Q_1 \in \mathbb{Z}_+$ qui ne dépendent pas de j,k et l, telles que pour tout $j \in \mathbb{Z}_+$ et tout $(k,l) \in \{0,1,\dots,2^j-1\}^2$ satisfaisant $|k-l| \geq Q_1$, on a*

$$\|f_{j,k}\|^{\alpha/2}_{\mathrm{L}^\alpha(\mathbb{R})}\|f_{j,l}\|^{\alpha/2}_{\mathrm{L}^\alpha(\mathbb{R})} - \int_{\mathbb{R}} |f_{j,k}(s)f_{j,l}(s)|^{\alpha/2}ds \geq c_{15}\|f_{j,k}\|^{\alpha/2}_{\mathrm{L}^\alpha(\mathbb{R})}\|f_{j,l}\|^{\alpha/2}_{\mathrm{L}^\alpha(\mathbb{R})}. \tag{2.41}$$

Lemme 2.5. *Il existe deux constantes $c_{16} > 0$, $Q_2 \in \mathbb{Z}_+$ qui ne dépendent pas de j,k et l, ni de x,y telles que pour tout $(x,y) \in \mathbb{R}^2$, tout $j \in \mathbb{Z}_+$ et tout $(k,l) \in \{0,1,\dots,2^j-1\}^2$ satisfaisant $|k-l| \geq Q_2$, on a*

$$\|xf_{j,k} + yf_{j,l}\|^\alpha_{\mathrm{L}^\alpha(\mathbb{R})} \geq c_{16}\Big(\|xf_{j,k}\|^\alpha_{\mathrm{L}^\alpha(\mathbb{R})} + \|yf_{j,l}\|^\alpha_{\mathrm{L}^\alpha(\mathbb{R})}\Big). \tag{2.42}$$

Ces deux lemmes sont des conséquences du point (ii) de la Propostion 2.1 et du résultat suivant.

Lemme 2.6. *Pour tout réel $\epsilon > 0$ arbitrairement petit, il existe une constante $c_{17} > 0$ qui ne dépend pas de j, k et l, telle que pour tout $j \in \mathbb{Z}_+$ et tout $(k,l) \in \{0,1,\dots,2^j-1\}^2$, on obtient*

$$\varphi_2(j,k,l) := \int_{\mathbb{R}} \big|\Phi_\alpha(u-k, H(k2^{-j}))\Phi_\alpha(u-l, H(l2^{-j}))\big|^{\alpha/2}du \tag{2.43}$$

$$\leq c_{17}\big(1+|k-l|\big)^{-(\frac{\alpha}{2}(N-\overline{H})-1/2-\epsilon)}. \tag{2.44}$$

Preuve du Lemme 2.6. Effectuons le changement de variable $u' = u - l$, puis en employant (2.19), on a

$$\begin{aligned}\varphi_2(j,k,l) &= \int_{\mathbb{R}} |\Phi_\alpha(u-(k-l), H(k2^{-j}))\Phi_\alpha(u, H(l2^{-j}))|^{\alpha/2}du \\ &\leq c_2^2 \int_{\mathbb{R}} (1+|u-(k-l)|)^{-\alpha/2(N+1/\alpha-\overline{H})}(1+|u|)^{-\alpha/2(N+1/\alpha-\overline{H})}du \\ &\leq c_2^2 \int_{\mathbb{R}} (1+|u-(k-l)|)^{-(\alpha/2(N+1/\alpha-\overline{H})-1-\epsilon)}(1+|u|)^{-\alpha/2(N+1/\alpha-\overline{H})}du.\end{aligned}$$

Ainsi, en prenant $\delta = \frac{\alpha}{2}(N+1/\alpha-\overline{H}) - 1 - \epsilon = \frac{\alpha}{2}(N-\overline{H}) - \frac{1}{2} - \epsilon > 0$ si ϵ est suffisamment petit, $\gamma = \frac{\alpha}{2}(N+1/\alpha-\overline{H})$ et $q = k-l$ dans le Lemme 3.1, on a (2.44). □

Preuve du Lemme 2.4. Supposons que $j \in \mathbb{Z}_+$ et $(k,l) \in \{0,1,\dots,2^j-1\}^2$ sont arbitraires et fixés. Grâce à (2.35) et (2.37), on a

$$\|f_{j,k}\|_{\mathrm{L}^\alpha(\mathbb{R})} = \|f_{j,l}\|_{\mathrm{L}^\alpha(\mathbb{R})} = 1. \tag{2.45}$$

Nous allons maintenant borner de façon convenable, l'intégrale

$$\int_{\mathbb{R}} |f_{j,k}(s)f_{j,l}(s)|^{\alpha/2}ds.$$

De (2.38) et (2.36), on a

$$\begin{aligned}\int_{\mathbb{R}} |f_{j,k}(s)f_{j,l}(s)|^{\alpha/2}ds &= \left(\frac{2^{-jH(k2^{-j})}}{\|\widetilde{d}_{j,k}\|_\alpha}\frac{2^{-jH(l2^{-j})}}{\|\widetilde{d}_{j,l}\|_\alpha}\right)2^j \\ &\times \int_{\mathbb{R}} |\Phi_\alpha(2^js-k, H(k2^{-j}))\Phi_\alpha(2^js-l, H(l2^{-j}))|^{\alpha/2}ds.\end{aligned}$$

En utilisant (2.31), le changement de variable $u = 2^js$, (2.43) et (2.44), on a

$$\begin{aligned}\int_{\mathbb{R}} |f_{j,k}(s)f_{j,l}(s)|^{\alpha/2}ds &\leq c_3^{-\alpha}\int_{\mathbb{R}} \big|\Phi_\alpha(u-k, H(k2^{-j}))\Phi_\alpha(u-l, H(l2^{-j}))\big|^{\alpha/2}du \\ &= c_3^{-\alpha}\varphi_2(j,k,l) \\ &\leq c_{18}\big(1+|k-l|\big)^{-(\alpha/2(N-\overline{H})-1/2-\epsilon)},\end{aligned} \tag{2.46}$$

où $c_{18} = c_3^{-\alpha} c_{17}$ est une constante qui ne dépend pas de j, k et l. Ainsi, étant donné que $\alpha/2(N - \overline{H}) - 1/2 - \epsilon > 0$, il existe une constante $Q_1 \in \mathbb{Z}_+$, qui ne dépend pas de j, k et l, telle que pour tout $j \in \mathbb{Z}_+$, tout $(k,l) \in \{0, 1, \dots, 2^j - 1\}^2$ vérifiant $|k - l| \geq Q_1$, on obtient

$$\int_{\mathbb{R}} |f_{j,k}(s) f_{j,l}(s)|^{\alpha/2} ds \leq c_{18}(1 + Q_1)^{-(\alpha/2(N - \overline{H}) - 1/2 - \epsilon)} \leq \frac{1}{2}. \tag{2.47}$$

Par conséquent, en posant $c_{15} = 1/2$ et en utilisant (2.45) et (2.47), on a (2.41). □

Preuve du Lemme 2.5. Notons d'abord que dans (2.42), on peut supposer, sans perte de généralité, que la constante c_{16} appartient à l'intervalle $]0, 1]$ et que $0 < |y| \leq |x|$ (le cas où $0 < |x| \leq |y|$ peut être traité de la même manière). Posons alors $z = y/x$, on a

$$|z| \leq 1. \tag{2.48}$$

En multipliant les deux membres de l'inégalité (2.42) par $|x|^{-\alpha}$ et en utilisant (2.45), on trouve que cette inégalité est équivalente à

$$\|f_{j,k} + z f_{j,l}\|^{\alpha}_{\mathrm{L}^{\alpha}(\mathbb{R})} \geq c_{16}(1 + |z|^{\alpha}). \tag{2.49}$$

Concentrons nous désormais sur la preuve de (2.49). On a, en utilisant l'inégalité triangulaire,

$$\begin{aligned} \|f_{j,k} + z f_{j,l}\|^{\alpha}_{\mathrm{L}^{\alpha}(\mathbb{R})} &= \int_{\mathbb{R}} \left| \Big(f_{j,k}(s) + z f_{j,l}(s) \Big)^2 \right|^{\alpha/2} ds \\ &\geq \int_{\mathbb{R}} \left| \Big(|f_{j,k}(s)| - |z||f_{j,l}(s)| \Big)^2 \right|^{\alpha/2} ds. \end{aligned} \tag{2.50}$$

En développant le carré et en utilisant l'inégalité $|x - y|^{\alpha/2} \geq |x^{\alpha/2} - y^{\alpha/2}| \geq x^{\alpha/2} - y^{\alpha/2}$ pour tout $(x, y) \in \mathbb{R}_+^2$, nous trouvons que

$$\begin{aligned} \|f_{j,k} + z f_{j,l}\|^{\alpha}_{\mathrm{L}^{\alpha}(\mathbb{R})} &= \int_{\mathbb{R}} \Big| |f_{j,k}(s)|^2 + z^2 |f_{j,l}(s)|^2 - 2|z| |f_{j,k}(s) f_{j,l}(s)| \Big|^{\alpha/2} ds \\ &\geq \int_{\mathbb{R}} \Big(|f_{j,k}(s)|^2 + z^2 |f_{j,l}(s)|^2 \Big)^{\alpha/2} ds - 2^{\alpha/2} |z|^{\alpha/2} \int_{\mathbb{R}} |f_{j,k}(s) f_{j,l}(s)|^{\alpha/2} ds. \end{aligned} \tag{2.51}$$

Puisque la fonction $x \mapsto x^{\alpha/2}$ est concave sur $\mathbb{R}_+$, on obtient que, pour tout $s \in \mathbb{R}$

$$\begin{aligned} \Big(|f_{j,k}(s)|^2 + z^2 |f_{j,l}(s)|^2 \Big)^{\alpha/2} &= 2^{\alpha/2} \Big(\frac{1}{2} |f_{j,k}(s)|^2 + \frac{z^2}{2} |f_{j,l}(s)|^2 \Big)^{\alpha/2} \\ &\geq 2^{\alpha/2 - 1} \Big(|f_{j,k}(s)|^{\alpha} + |z|^{\alpha} |f_{j,l}(s)|^{\alpha} \Big). \end{aligned} \tag{2.52}$$

En combinant ensemble (2.45), (2.48), (2.51) et (2.52), on a

$$\|f_{j,k} + z f_{j,l}\|^{\alpha}_{\mathrm{L}^{\alpha}(\mathbb{R})} \geq 2^{\alpha/2 - 1}(1 + |z|^{\alpha}) - 2^{\alpha/2} \int_{\mathbb{R}} |f_{j,k}(s) f_{j,l}(s)|^{\alpha/2} ds. \tag{2.53}$$

Notons par Q_2 un entier positif satisfaisant

$$c_{18} \big(1 + Q_2\big)^{1/2 + \epsilon - \alpha/2(N - \overline{H})} \leq 2^{-2}, \tag{2.54}$$

où c_{18} est la constante introduite en (2.46) qui ne dépend pas de j, k et l. Posons alors $c_{16} = 2^{\alpha/2 - 2}$, ainsi il résulte de (2.53) (2.54) et (2.46) que pour tout $j \in \mathbb{Z}_+$, tout $(k,l) \in \{0, 1, \dots, 2^j - 1\}^2$, on a

$$\begin{aligned} \|f_{j,k} + z f_{j,l}\|^{\alpha}_{\mathrm{L}^{\alpha}(\mathbb{R})} &\geq 2^{\alpha/2 - 1} \big(1 + |z|^{\alpha}\big) - 2^{\alpha/2 - 2} \geq 2^{\alpha/2 - 2} \big(1 + 2|z|^{\alpha}\big) \\ &\geq c_{16} \big(1 + |z|^{\alpha}\big). \end{aligned}$$

Par conséquent, on obtient (2.49). □

Lemme 2.7. *Il existe une constante $c_{21} > 0$, qui ne dépend pas de j, k et l, telle que pour tout $j \in \mathbb{Z}_+$, tout $(k,l) \in \{0,1,\dots,2^j-1\}^2$, on a*

$$\varphi_1(j,k,l) := \int_{\mathbb{R}} \big|\Phi_\alpha(u-k, H(k2^{-j}))\big|^{\alpha-1} \big|\Phi_\alpha(u-l, H(l2^{-j}))\big| du \tag{2.55}$$

$$\leq c_{21}\big(1+|k-l|\big)^{-(\frac{\alpha-1}{2})(N+1/\alpha-\overline{H})}. \tag{2.56}$$

Preuve du Lemme 2.7. En effectuant le changement de variable $u' = u - l$, puis en utilisant (2.19), il vient que

$$\begin{aligned}\varphi_1(j,k,l) &= \int_{\mathbb{R}} |\Phi_\alpha(u-(k-l), H(k2^{-j}))|^{\alpha-1} |\Phi_\alpha(u, H(l2^{-j}))| du \\ &\leq c_2^2 \int_{\mathbb{R}} (1+|u-(k-l)|)^{-(\alpha-1)(N+1/\alpha-\overline{H})}(1+|u|)^{-(N+1/\alpha-\overline{H})} du.\end{aligned} \tag{2.57}$$

Rappelons que le nombre de moments de la fonction ψ, noté N, est supérieur ou égal à 2. Posons

$$\delta = (\alpha-1)(N+1/\alpha-\overline{H}),\ \gamma = N+1/\alpha-\overline{H} \text{ et } \lambda = \frac{(\alpha-1)}{2}(N+1/\alpha-\overline{H}).$$

Compte tenu de (2.57) et du Lemme 3.2, pour montrer que la relation (2.56) est vérifiée, il suffit de prouver que δ, γ et λ sont trois réels vérifiant les quatre conditions (3.5) :

(i) $\lambda \leq \gamma$, c'est vrai car $\alpha \in]1,2[$;

(ii) $\delta > \frac{\alpha-1}{\alpha} \iff N+1/\alpha-\overline{H} > 1/\alpha \iff N-\overline{H} > 0$, ce qui est vrai car $N \geq 2$ et $\overline{H} < 1$,

(iii) $\gamma = N+1/\alpha-\overline{H} > N-\frac{1}{2} \geq \frac{3}{2} > 1$,

(iv) $\gamma - \lambda = (\frac{3}{2}-\frac{\alpha}{2})(N+1/\alpha-\overline{H}) > \frac{1}{2}(N+1/\alpha-\overline{H}) = \frac{1}{2}(N-1/\alpha-\overline{H})+1/\alpha > 1/\alpha$, car $N \geq 2$.

□

Nous sommes maintenant en mesure d'établir la Proposition 2.2.

Preuve de la Proposition 2.2. On définit l'entier Q_0 par $Q_0 = \max\{Q_1, Q_2\}$. Rappelons que Q_1 et Q_2 sont deux entiers introduit respectivement au Lemme 2.4 et au Lemme 2.5.
Tout d'abord, étudions le cas où $j \in \mathbb{Z}_+$ et $(k,l) \in \{0,1,\dots,2^j-1\}$ sont arbitraires et vérifient $|k-l| \geq Q_0$. Remarquons que (2.35) entraîne que

$$|\mathrm{cov}\big(|\widetilde{d}_{j,k}|^\beta, |\widetilde{d}_{j,l}|^\beta\big)| = \|\widetilde{d}_{j,k}\|_\alpha^\beta \|\widetilde{d}_{j,l}\|_\alpha^\beta |\mathrm{cov}\big(|\widetilde{d}_{j,k}^{\mathrm{nor}}|^\beta, |\widetilde{d}_{j,l}^{\mathrm{nor}}|^\beta\big)|. \tag{2.58}$$

De plus, il résulte de [24, Théorème 2.4] et de (2.37) que

$$|\mathrm{cov}\big(|\widetilde{d}_{j,k}^{\mathrm{nor}}|^\beta, |\widetilde{d}_{j,l}^{\mathrm{nor}}|^\beta\big)| \leq c_{28}\Big([f_{j,k}, f_{j,l}]_1 + [f_{j,k}, f_{j,l}]_2\Big); \tag{2.59}$$

où c_{28} est une constante qui dépend de c_{15} (voir Lemme 2.4), de c_{16} (voir Lemme 2.5) et de β ; toutefois c_{28} ne dépend pas de j, k et l, car $\|\widetilde{d}_{j,k}^{\mathrm{nor}}\|_\alpha = \|\widetilde{d}_{j,k}^{\mathrm{nor}}\|_\alpha = 1$.

Par définition, on a

$$[f_{j,k}, f_{j,l}]_1 := [f_{j,k}, f_{j,l}]_1^* + [f_{j,l}, f_{j,k}]_1^*, \tag{2.60}$$

avec

$$[f_{j,k}, f_{j,l}]_1^* := \int_{\mathbb{R}} |f_{j,k}(s)|^{\alpha-1} |f_{j,l}(s)| ds, \tag{2.61}$$

et

$$[f_{j,k}, f_{j,l}]_2 := \int_{\mathbb{R}} |f_{j,k}(s) f_{j,l}(s)|^{\alpha/2} ds. \tag{2.62}$$

Nous allons maintenant majorer $[f_{j,k}, f_{j,l}]_1$. Il résulte de (2.60), (2.38) et (2.36), que

$$[f_{j,k}, f_{j,l}]_1^* = \left(\frac{2^{-jH(k2^{-j})}}{\|\widetilde{d}_{j,k}\|_\alpha}\right)^{\alpha-1}\left(\frac{2^{-jH(l2^{-j})}}{\|\widetilde{d}_{j,l}\|_\alpha}\right)2^j$$
$$\times\int_{\mathbb{R}}\left|\Phi_\alpha(2^js-k, H(k2^{-j}))\right|^{\alpha-1}\left|\Phi_\alpha(2^js-l, H(l2^{-j}))\right|ds\,;$$

ainsi, en utilisant (2.31), le changement de variable $u = 2^js$, et (2.55), on a

$$[f_{j,k}, f_{j,l}]_1^* \le c_3^{-\alpha}\varphi_1(j,k,l). \tag{2.63}$$

De manière analogue, on montre, pour tout $j \in \mathbb{Z}_+$ et tout $(k,l) \in \{0,1,\dots,2^j-1\}$, que

$$[f_{j,l}, f_{j,k}]_1^* \le c_3^{-\alpha}\varphi_1(j,l,k). \tag{2.64}$$

Il résulte de (2.60), (2.63) et (2.64) que

$$[f_{j,k}, f_{j,l}]_1 \le c_3^{-\alpha}\Big(\varphi_1(j,k,l)+\varphi_1(j,l,k)\Big). \tag{2.65}$$

Majorons maintenant $[f_{j,k}, f_{j,l}]_2$. Il résulte de (2.62), (2.38) et (2.36), que

$$[f_{j,k}, f_{j,l}]_2 = \left(\frac{2^{-jH(k2^{-j})}}{\|\widetilde{d}_{j,k}\|_\alpha}\right)^{\frac{\alpha}{2}}\left(\frac{2^{-jH(l2^{-j})}}{\|\widetilde{d}_{j,l}\|_\alpha}\right)^{\frac{\alpha}{2}}2^j$$
$$\times\int_{\mathbb{R}}\left|\Phi_\alpha(2^js-k, H(k2^{-j}))\Phi_\alpha(2^js-l, H(l2^{-j}))\right|^{\frac{\alpha}{2}}ds\,;$$

ainsi, en utilisant (2.31), le changement de variable $u = 2^js$, et (2.43), on obtient

$$[f_{j,k}, f_{j,l}]_2 \le c_3^{-\alpha}\varphi_2(j,k,l). \tag{2.66}$$

En combinant (2.58) avec (2.31), (2.59), (2.65) et (2.66), on a

$$|\mathrm{cov}\big(|\widetilde{d}_{j,k}|^\beta, |\widetilde{d}_{j,l}|^\beta\big)| \le \Big(c_{12}^{2\beta}c_3^{-\alpha}\Big)2^{-j\beta(H(k2^{-j})+H(l2^{-j}))}$$
$$\times\Big(\varphi_1(j,k,l)+\varphi_1(j,l,k)+\varphi_2(j,k,l)\Big).$$

Ensuite, en employant (2.56) et (2.44), on montre que (2.40) est vraie pour tout $j \in \mathbb{Z}_+$, tout $(k,l) \in \{0,1,\dots,2^j-1\}^2$ vérifiant $|k-l| \ge Q_0$; nous désignons par c_{15}' la constante qui apparait alors dans le membre de droite de l'inégalité (2.40). Étudions maintenant le cas où $j \in \mathbb{Z}_+$ et $(k,l) \in \{0,1,\dots,2^j-1\}^2$ sont arbitraires et vérifient $|k-l| < Q_0$. Posons $c_{15}'' = (1+Q_0)^\lambda c_{13}$ où c_{13} est la constante qui a été introduite dans (2.33). En utilisant (2.33), on obtient que

$$|\mathrm{cov}\big(|\widetilde{d}_{j,k}|^\beta, |\widetilde{d}_{j,l}|^\beta\big)| \le c_{13}2^{-j\beta(H(k2^{-j})+H(l2^{-j}))}$$
$$= c_{13}(1+|k-l|)^\lambda\frac{2^{-j\beta(H(k2^{-j})+H(l2^{-j}))}}{(1+|k-l|)^\lambda}$$
$$\le c_{15}''2^{-j\beta(H(k2^{-j})+H(l2^{-j}))}(1+|k-l|)^{-\lambda}.$$

Finalement, en prenant $c_{15} = \max\{c_{15}', c_{15}''\}$, il en résulte que (2.40) est vraie pour tout $j \in \mathbb{Z}_+$, tout $(k,l) \in \{0,1,\dots,2^j-1\}^2$. □

Lemme 2.8. *Soit λ un réel strictement positif fixé. Pour tout entier $j \ge 1$, posons*

$$B_j^\lambda = \left(\sum_{(k,l)\in\nu_j\times\nu_j}2^{-j\beta(H(k2^{-j})+H(l2^{-j}))}\Big(1+|k-l|\Big)^{-\lambda}\right)\left(\sum_{k\in\nu_j}2^{-j\beta H(k2^{-j})}\right)^{-2}. \tag{2.67}$$

Alors, pour tout $\widetilde{\lambda} \in]0,\lambda[$, *il existe une constante* $c_{30} > 0$, *qui ne dépend pas de* j, *telle que pour tout* $j \geq 1$, *on a*

$$B_j^{\lambda} \leq c_{30} \left\{ j^{1/\widetilde{\lambda}}\, n_j^{-1}\, 2^{j2\beta(\overline{H}-\underline{H})} + j^{-\lambda/\widetilde{\lambda}} \right\}; \tag{2.68}$$

rappelons que n_j *désigne le cardinal de* ν_j.

Preuve du Lemme 2.8. Posons, pour tout $j \geq 1$,

$$\Delta_j = \left\{ (k,l) \in \nu_j \times \nu_j \,:\, |k-l| \leq j^{1/\widetilde{\lambda}} \right\} \tag{2.69}$$

et

$$\Delta_j^c = \left\{ (k,l) \in \nu_j \times \nu_j \,:\, |k-l| > j^{1/\widetilde{\lambda}} \right\}; \tag{2.70}$$

de plus, pour tout $k \in \nu_j$, on pose

$$\Delta_j(k) = \left\{ l \in \nu_j \,:\, (k,l) \in \Delta_j \right\}. \tag{2.71}$$

Notons que, pour tout $j \geq 1$,

$$\nu_j \times \nu_j = \Delta_j \cup \Delta_j^c \tag{2.72}$$

et que pour tout $k \in \nu_j$,

$$\mathrm{card}\big(\Delta_j(k)\big) \leq 2j^{1/\widetilde{\lambda}} + 1 \leq 3j^{1/\widetilde{\lambda}}; \tag{2.73}$$

la première inégalité résulte du fait que dans tout intervalle compact de diamètre arbitraire d, il y a au plus $[d]+1$ entiers.

On a, d'après (2.70), que

$$\begin{aligned}
&\sum_{(k,l)\in\Delta_j^c} 2^{-j\beta(H(k2^{-j})+H(l2^{-j}))}(1+|k-l|)^{-\lambda} \leq j^{-\lambda/\widetilde{\lambda}} \sum_{(j,k)\in\Delta_j^c} 2^{-j\beta(H(k2^{-j})+H(l2^{-j}))} \\
&\leq j^{-\lambda/\widetilde{\lambda}} \sum_{(j,k)\in\nu_j\times\nu_j} 2^{-j\beta(H(k2^{-j})+H(l2^{-j}))} \leq j^{-\lambda/\widetilde{\lambda}} \left(\sum_{k\in\nu_j} 2^{-j\beta H(k2^{-j})} \right)^2.
\end{aligned} \tag{2.74}$$

Montrons maintenant qu'il existe $c_{31} > 0$ une constante qui ne dépend ni de j ni de (k,l), telle que pour tout $j \geq 1$ et tout $(k,l) \in \Delta_j$,

$$2^{-j\beta(H(k2^{-j})+H(l2^{-j}))} \leq c_{31} 2^{-j2\beta H(k2^{-j})}. \tag{2.75}$$

En combinant (1.3) et (2.69), il vient que

$$\begin{aligned}
&2^{-j\beta(H(k2^{-j})+H(l2^{-j}))} = 2^{-j2\beta H(k2^{-j})} 2^{j\beta(H(k2^{-j})-H(l2^{-j}))} \\
&\leq 2^{-j2\beta H(k2^{-j})} 2^{j\beta|H(k2^{-j})-H(l2^{-j})|} \leq 2^{-j2\beta H(k2^{-j})} 2^{j\beta c_1 |k2^{-j}-l2^{-j}|^{\rho_H}} \\
&\leq 2^{-j2\beta H(k2^{-j})} 2^{\beta c_1 \left(j^{1+\rho_H/\widetilde{\lambda}}\right) 2^{-j\rho_H}};
\end{aligned}$$

ainsi, en prenant,

$$c_{31} := \sup_{j\geq 1} \left\{ 2^{\beta c_1 \left(j^{1+\rho_H/\widetilde{\lambda}}\right) 2^{-j\rho_H}} \right\} < +\infty,$$

on obtient (2.75). Ensuite, au moyen de (2.69), (2.71), (2.73) et (2.75), on a

$$\sum_{(k,l)\in\Delta_j} 2^{-j\beta(H(k2^{-j})+H(l2^{-j}))}\big(1+|k-l|\big)^{-\lambda} \le c_{31} \sum_{(k,l)\in\Delta_j} 2^{-j2\beta H(k2^{-j})}\big(1+|k-l|\big)^{-\lambda}$$
$$\le c_{31}\sum_{k\in\nu_j} 2^{-j2\beta H(k2^{-j})}\Bigg(\sum_{l\in\Delta_j(k)}\big(1+|k-l|\big)^{-\lambda}\Bigg) \le 3c_{31}j^{1/\widetilde{\lambda}}\sum_{k\in\nu_j}2^{-j2\beta H(k2^{-j})}$$
$$\le 3c_{31}j^{1/\widetilde{\lambda}}\,n_j\,2^{-j2\beta\underline{H}}. \tag{2.76}$$

Par ailleurs,

$$\Bigg(\sum_{k\in\nu_j}2^{-j\beta H(k2^{-j})}\Bigg)^2 \ge n_j^2 2^{-j2\beta\overline{H}}. \tag{2.77}$$

Finalement, en combinant (2.67), (2.72), (2.74), (2.76) et (2.77), on obtient (2.68). □

Proposition 2.3. *Pour tout $j\in\mathbb{Z}_+$, posons*

$$\widetilde{V}_j = n_j^{-1}\sum_{k\in\nu_j}|\widetilde{d}_{j,k}|^{\beta}. \tag{2.78}$$

Supposons que n_j vérifie (1.14). On a, presque sûrement,

$$\frac{\widetilde{V}_j}{\mathbb{E}(\widetilde{V}_j)} \xrightarrow[j\to+\infty]{p.s.} 1 \tag{2.79}$$

Pour établir cette proposition, nous avons besoin de la Proposition 2.2, du Lemme 2.8 ainsi que du lemme suivant.

Lemme 2.9. *Il existe deux constantes c_{32} et c_{33} strictement positives telles que pour tout entier $j\ge 1$,*

$$c_{32}n_j^{-1}\sum_{k\in\nu_j}2^{-j\beta H(k2^{-j})} \le \mathbb{E}\big(\widetilde{V}_j\big) \le c_{33}n_j^{-1}\sum_{k\in\nu_j}2^{-j\beta H(k2^{-j})} \tag{2.80}$$

Preuve du Lemme 2.9. En utilisant (2.78) et la linéarité de l'espérance, on a

$$\mathbb{E}\big(\widetilde{V}_j\big) = n_j^{-1}\sum_{k\in\nu_j} = c(\alpha,\beta)n_j^{-1}\sum_{k\in\nu_j}\|\widetilde{d}_{j,k}\|_{\alpha}^{\beta},$$

où la constante $c(\alpha,\beta)$ désigne le moment absolu d'ordre β d'une variable aléatoire $\mathcal{S}\alpha\mathcal{S}$ de paramètre d'échelle 1. Il résulte de l'égalité précédente et de (2.31), que

$$c(\alpha,\beta)c_3^{\beta}n_j^{-1}\sum_{k\in\nu_j}2^{-j\beta H(k2^{-j})} \le \mathbb{E}\big(\widetilde{V}_j\big) \le c(\alpha,\beta)c_{12}^{\beta}n_j^{-1}\sum_{k\in\nu_j}2^{-j\beta H(k2^{-j})},$$

ainsi en posant $c_{32}=c(\alpha,\beta)c_3^{\beta}$ et $c_{33}=c(\alpha,\beta)c_{12}^{\beta}$. □

Preuve de la Proposition 2.3. Pour tout $\eta>0$, tout entier $j\ge 1$, on a, d'après l'inégalité de Markov

$$\mathbb{P}\Bigg(\Big|\frac{\widetilde{V}_j}{\mathbb{E}(\widetilde{V}_j)}-1\Big|>\eta\Bigg) \le \eta^{-2}\frac{\mathrm{Var}\big(\widetilde{V}_j\big)}{\Big(\mathbb{E}\big(\widetilde{V}_j\big)\Big)^2}. \tag{2.81}$$

Ensuite, d'après (2.78) et la Proposition 2.2, il vient que

$$\begin{aligned}\operatorname{Var}(\widetilde{V}_j) &= n_j^{-2} \sum_{(k,l)\in\nu_j\times\nu_j} \operatorname{cov}\Big(|\widetilde{d}_{j,k}|^{\beta}, |\widetilde{d}_{j,l}|^{\beta}\Big)\\ &\le c_{15} n_j^{-2} \sum_{(k,l)\in\nu_j\times\nu_j} 2^{-j\beta(H(k2^{-j})+H(l2^{-j}))}\big(1+|k-l|\big)^{-\lambda}\,;\end{aligned} \tag{2.82}$$

ainsi, en utilisant (2.82), (2.80), le Lemme 2.8 et (1.14), on a

$$\frac{\operatorname{Var}(\widetilde{V}_j)}{\Big(\mathbb{E}(\widetilde{V}_j)\Big)^2} \le c_{34}\left\{j^{1/\widetilde{\lambda}}\, 2^{-j(\frac{1}{2}-2\beta(\overline{H}-\underline{H}))} + j^{-\lambda/\widetilde{\lambda}}\right\},$$

où $\widetilde{\lambda} \in]0,\lambda[$ est arbitraire et fixé, et où c_{34} est une constante qui ne dépend pas de j. Finalement, grâce à cette dernière inégalité, à (2.81) et à l'inégalité $2\beta(\overline{H}-\underline{H}) < 1/2$, en appliquant le Lemme de Borel-Cantelli, on obtient (2.79). □

Il convient maintenant de rappeler que dans tout le présent chapitre l'espace de probabilité sous-jacent Ω est restreint à Ω_0^*, l'évènement de probabilité qui a été introduit dans le Lemme 2.1 du chapitre 2.

Lemme 2.10. *Il existe une variable aléatoire C finie et positive telle que pour tout $\omega \in \Omega = \Omega_0^*$, tout $j \in \mathbb{Z}_+$ et tout $k \in \{0, 1, \dots, 2^j - 1\}$, on a*

$$\big|d_{j,k}(\omega) - \widetilde{d}_{j,k}(\omega)\big| \le C(\omega) 2^{-j\rho_H}. \tag{2.83}$$

Preuve du Lemme 2.10. En utilisant (1.5), (1.6), (2.16), le changement de variables $t = 2^{-j}k + 2^{-j}x$, et le fait que le support de la fonction continue ψ est inclus dans $[0, 1]$, il vient que

$$\begin{aligned}&\big|d_{j,k}(\omega) - \widetilde{d}_{j,k}(\omega)\big|\\ &\le \int_{\mathbb{R}} \Big|X\big(2^{-j}k + 2^{-j}x, H(2^{-j}k + 2^{-j}x), \omega\big) - X\big(2^{-j}k + 2^{-j}x, H(2^{-j}k), \omega\big)\Big|\big|\psi(x)\big|dx\\ &\le \int_0^1 \sup_{u\in[0,2]} \Big|X\big(u, H(2^{-j}k + 2^{-j}x), \omega\big) - X\big(u, H(k2^{-j}), \omega\big)\Big|\big|\psi(x)\big|dx\,;\end{aligned}$$

ainsi, au moyen de la relation (3.18) du chapitre 2 (dans laquelle on prend $q = 0$, $M = 2$, $a = \underline{H}$ et $b = \overline{H}$), et au moyen de la relation (1.3) du présent chapitre, on obtient

$$\begin{aligned}\big|d_{j,k}(\omega) - \widetilde{d}_{j,k}(\omega)\big| &\le C'(\omega)\|\psi\|_{\mathrm{L}^\infty(\mathbb{R})} \int_0^1 \big|H(2^{-j}k + 2^{-j}x) - H(2^{-j}k)\big|dx\\ &\le c_1 C'(\omega)\|\psi\|_{\mathrm{L}^\infty(\mathbb{R})} \int_0^1 \big(2^{-j}x\big)^{\rho_H} dx = \frac{c_1 C'(\omega)}{1+\rho_H}\|\psi\|_{\mathrm{L}^\infty(\mathbb{R})} 2^{-j\rho_H},\end{aligned}$$

où C' est une variable aléatoire finie et positive. Enfin, grâce à la dernière inégalité, en posant

$$C(\omega) = \frac{c_1 C'(\omega)}{1+\rho_H}\|\psi\|_{\mathrm{L}^\infty(\mathbb{R})},$$

on aboutit à (2.83). □

Proposition 2.4. *Rappelons que V_j et $\widetilde{V}_j$ ont été introduits respectivement en (1.13) et (2.78). Supposons que n_j, le cardinal de ν_j, vérifie (1.14), alors, on a, presque sûrement,*

$$\frac{V_j}{\mathbb{E}(\widetilde{V}_j)} \xrightarrow[j\to+\infty]{p.s.} 1. \tag{2.84}$$

Preuve de la Proposition 2.4. Soit j un entier supérieur ou égal à 1. Compte tenu de (2.79), pour montrer (2.84), il suffit d'établir que, presque sûrement,

$$\frac{|V_j - \widetilde{V}_j|}{\mathbb{E}(\widetilde{V}_j)} \xrightarrow[j\to+\infty]{p.s.} 0. \tag{2.85}$$

Tout d'abord, majorons $\left|V_j(\omega) - \widetilde{V}_j(\omega)\right|$, pour tout $\omega \in \Omega$. Il résulte de (1.13), (2.78), et de l'inégalité $\left||x|^\beta - |y|^\beta\right| \le |x-y|^\beta$ pour tout $(x,y) \in \mathbb{R}^2$, que

$$\left|V_j(\omega) - \widetilde{V}_j(\omega)\right| \le n_j^{-1} \sum_{k\in\nu_j} \left|d_{j,k}(\omega) - \widetilde{d}_{j,k}(\omega)\right|^\beta ;$$

puis, en employant (2.83), il vient

$$\left|V_j(\omega) - \widetilde{V}_j(\omega)\right| \le C^\beta(\omega) 2^{-j\beta\rho_H}. \tag{2.86}$$

Par ailleurs, grâce à la minoration fournie par (2.80), on a

$$\mathbb{E}(\widetilde{V}_j) \ge c_{32} 2^{-j\beta\overline{H}}. \tag{2.87}$$

Enfin, en combinant (2.86), (2.87) et (1.4), on en déduit (2.85). $\square$

Lemme 2.11. *Pour tout entier $j \ge 1$, on pose*

$$\widetilde{W}_j = -\frac{1}{j\beta} \log_2 \left(\frac{\mathbb{E}(\widetilde{V}_j)}{n_j^{-1} \sum_{k\in\nu_j} 2^{-j\beta H(k2^{-j})}} \right). \tag{2.88}$$

Alors, on a

$$\lim_{j\to+\infty} \widetilde{W}_j = 0. \tag{2.89}$$

Preuve du Lemme 2.11. D'après (2.80), pour tout entier $j \ge 1$, on a

$$0 < c_{32} \le \frac{\mathbb{E}(\widetilde{V}_j)}{n_j^{-1} \sum_{k\in\nu_j} 2^{-j\beta H(k2^{-j})}} \le c_{33},$$

puis en utilisant la croissance de la fonction $x \mapsto \log_2(x)$, on obtient,

$$\log_2(c_{32}) \le \log_2 \left(\frac{\mathbb{E}(\widetilde{V}_j)}{n_j^{-1} \sum_{k\in\nu_j} 2^{-j\beta H(k2^{-j})}} \right) \le \log_2(c_{33})$$

et par conséquent (voir (2.88)),

$$-\frac{\log_2(c_{33})}{j\beta} \le \widetilde{W}_j \le -\frac{\log_2(c_{32})}{j\beta}.$$

Ainsi, en faisant tendre j vers $+\infty$, on aboutit à (2.89). $\square$

Nous sommes maintenant en mesure de prouver le Théorème 1.1.

Preuve du Théorème 1.1. Remarquons que, en utilisant (2.88) et l'inégalité triangulaire, on a pour tout entier $j \geq 1$,

$$\begin{aligned}
&\left| \frac{\log_2(V_j)}{-j\beta} - \underline{H}_j \right| \\
&= \left| \frac{1}{-j\beta} \log_2 \left(\frac{V_j}{\mathbb{E}(\widetilde{V}_j)} \right) - \frac{\log_2 \left(\mathbb{E}(\widetilde{V}_j) \right)}{j\beta} - \underline{H}_j \right| \\
&= \left| \frac{1}{-j\beta} \log_2 \left(\frac{V_j}{\mathbb{E}(\widetilde{V}_j)} \right) + \widetilde{W}_j - \frac{1}{j\beta} \log_2 \left(n_j^{-1} \sum_{k \in \nu_j} 2^{-j\beta H(k2^{-j})} \right) - \underline{H}_j \right| \\
&\leq \left| \frac{1}{-j\beta} \log_2 \left(\frac{V_j}{\mathbb{E}(\widetilde{V}_j)} \right) \right| + \left| \widetilde{W}_j \right| + \left| \frac{1}{-j\beta} \log_2 \left(n_j^{-1} \sum_{k \in \nu_j} 2^{-j\beta H(k2^{-j})} \right) - \underline{H}_j \right| ;
\end{aligned}$$

ainsi (2.84), la continuité en 1 de la fonction $x \mapsto \log_2(x)$, (2.89) et (2.1), impliquent que (1.15) est vraie. $\square$

3. Lemmes techniques

Lemme 3.1. *Soient δ et γ deux réels strictement positifs satisfaisant*

$$\gamma > 1 + \delta. \tag{3.1}$$

Pour tout $q \in \mathbb{Z}$, on pose

$$r_q(\delta, \gamma) = \int_{\mathbb{R}} (1 + |u - q|)^{-\delta} (1 + |u|)^{-\gamma} du. \tag{3.2}$$

Alors, on a

$$\sup_{q \in \mathbb{Z}} \{(1 + |q|)^{\delta} r_q(\delta, \gamma)\} < +\infty. \tag{3.3}$$

Preuve du Lemme 3.1. Premièrement, notons que pour tout $u \in \mathbb{R}$, tout $q \in \mathbb{Z}$, on a

$$\begin{aligned}
(1 + |q|)^{\delta} &= (1 + |u - q - u|)^{\delta} \leq (1 + |u - q| + |u|)^{\delta} \\
&\leq (1 + |u - q|)^{\delta} (1 + |u|)^{\delta}.
\end{aligned} \tag{3.4}$$

Il résulte de (3.2), (3.4) et (3.1) que

$$(1 + |q|)^{\delta} r_q(\delta, \gamma) \leq \int_{\mathbb{R}} (1 + |u|)^{-(\gamma - \delta)} du < +\infty.$$

$\square$

Lemme 3.2. *Soient γ, δ et λ trois réels strictement positifs satisfaisant les quatres conditions suivantes :*

$$\begin{aligned}
\lambda \leq \delta, &\quad \gamma - \lambda > 1/\alpha, \\
\delta > \frac{\alpha - 1}{\alpha}, &\qquad \gamma > 1.
\end{aligned} \tag{3.5}$$

Alors, on a

$$c_{26} := \sup_{q \in \mathbb{Z}} \left\{ (1 + |q|)^{\lambda} r_q(\delta, \gamma) \right\} < +\infty, \tag{3.6}$$

où $r_q(\delta, \gamma)$ a été définie en (3.2).

Preuve du Lemme 3.2. En utilisant (3.2), l'inégalité triangulaire et l'inégalité suivante

$$\forall\,(x,y)\in\mathbb{R}_+^2\,;\,(x+y)^\lambda\le c_{25}(x^\lambda+y^\lambda),$$

on obtient

$$\begin{aligned}(1+|q|)^\lambda r_q(\delta,\gamma)&=\int_{\mathbb{R}}\frac{(1+|q|)^\lambda}{(1+|u-q|)^\delta}(1+|u|)^{-\gamma}du\\&\le c_{25}\int_{\mathbb{R}}(1+|u-q|)^{-(\delta-\lambda)}(1+|u|)^{-\gamma}du\\&+c_{25}\int_{\mathbb{R}}(1+|u-q|)^{-\delta}(1+|u|)^{-(\gamma-\lambda)}du.\end{aligned}\tag{3.7}$$

Finalement, (3.5), (3.7) et l'inégalité de Hölder impliquent que

$$\begin{aligned}&(1+|q|)^\lambda r_q(\delta,\gamma)\le c_{25}\int_{\mathbb{R}}(1+|u|)^{-\gamma}du\\&+c_{25}\Big(\int_{\mathbb{R}}(1+|u|)^{-\delta\frac{\alpha}{\alpha-1}}\Big)^{\frac{\alpha-1}{\alpha}}\Big(\int_{\mathbb{R}}(1+|u|)^{-(\gamma-\lambda)\alpha}\Big)^{\frac{1}{\alpha}}<+\infty.\end{aligned}\tag{3.8}$$

$\square$

CHAPITRE 5

Estimation du paramètre de stabilité α

1. Estimation du paramètre de stabilité α dans le cas fractionnaire

Cette section correspond à notre article [2], que nous avons décidé d'incorporer tel quel dans le mémoire de cette thèse.

1.1. Introduction and statement of the main results. Let H and α be two parameters such that $\alpha \in (1,2)$ and $1/\alpha < H < 1$. We denote by $\{X_{H,\alpha}(t)\}_{t\in\mathbb{R}}$ the symmetric α-stable linear fractional stable motion (lfsm for brevity) (see e.g. [26, 17]), defined, for all $t \in \mathbb{R}$, as,

$$X_{H,\alpha}(t) := \int_{\mathbb{R}} \Big\{ (t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha} \Big\} \mathrm{Z}_\alpha\,(ds)\,, \tag{1.1}$$

where $\mathrm{Z}_\alpha\,(\cdot)$ is a symmetric α-stable random measure and, for each $z \in \mathbb{R}$

$$(z)_+ := \max\{z, 0\}. \tag{1.2}$$

The parameter H characterizes the self-similarity property of lfsm; namely, for all fixed positive real-number a, the processes $\{X_{H,\alpha}(at)\}_{t\in\mathbb{R}}$ and $\{a^H X_{H,\alpha}(t)\}_{t\in\mathbb{R}}$ have the same finite dimensional distributions. The parameter α governs the tail heaviness of the latter distributions. The process $\{X_{H,\alpha}(t)\}_{t\in\mathbb{R}}$ has a modification with continuous nowhere differentiable sample paths; it is identified with this modification in all the sequel.

The statistical problem of the estimation of H has already been studied in several articles: [27, 1, 29, 24], and strongly consistent estimators (i.e. convergent almost surely), based on $(d_{j,k})_{(j,k)\in\mathbb{Z}^2}$, the discrete wavelet transform of lfsm, have been proposed; notice that the latter estimators of H do not require that α to be known. Throughout our paper, for all $(j,k) \in \mathbb{Z}^2$, the wavelet coefficient $d_{j,k}$ is defined as,

$$d_{j,k} = 2^j \int_{\mathbb{R}} X_{H,\alpha}(t)\psi(2^j t - k)dt; \tag{1.3}$$

moreover, we only impose to the analyzing wavelet ψ a very weak assumption: *ψ is an arbitrary real-valued non-vanishing continuous function with a compact support in $[0,1]$ and it has 2 vanishing moments i.e.*

$$\int_{\mathbb{R}} \psi(s)ds = \int_{\mathbb{R}} s\psi(s)ds = 0. \tag{1.4}$$

It is worth noticing that we do not need that $\left\{2^{j/2}\psi(2^j\cdot -k) : (j,k) \in \mathbb{Z}^2\right\}$ be an orthonormal wavelet basis for $L^2(\mathbb{R})$.

In view of the fact that the problem of the estimation of H is now well understood, from now on we assume the latter parameter to be known. Our goal is to construct, by using the wavelet coefficients $(d_{j,k})_{0\le k<2^j}$, a strongly consistent (i.e. almost surely convergent when $j \to +\infty$) estimator $\widehat{\alpha}_j$ of the parameter α. Let us outline the main ideas which lead to this estimator.

– The starting point, is a result of [31], according to which, with probability 1, the quantity $H - 1/\alpha$, is the critical uniform Hölder exponent of the sample paths of $X_{H,\alpha}$ over any arbitrary compact interval and in particular the interval $[0,1]$; more precisely, one has, almost surely for all arbitrarily small $\eta > 0$,

$$\sup_{t_1,t_2\in[0,1]} \left\{ \frac{\left|X_{H,\alpha}(t_1) - X_{H,\alpha}(t_2)\right|}{|t_1-t_2|^{H-1/\alpha-\eta}} \right\} < \infty \tag{1.5}$$

and

$$\sup_{t_1,t_2\in[0,1]} \left\{ \frac{\left|X_{H,\alpha}(t_1) - X_{H,\alpha}(t_2)\right|}{|t_1-t_2|^{H-1/\alpha+\eta}} \right\} = \infty. \tag{1.6}$$

– Next, let us set,

$$D_j = \max_{0\le k<2^j} |d_{j,k}|. \tag{1.7}$$

In view of the fact that the wavelet ψ has a first vanishing moment, one can derive from (1.5), that, almost surely, for all arbitrarily small $\epsilon > 0$,

$$\limsup_{j\to+\infty} \left\{ 2^{j(H-1/\alpha-\epsilon)} D_j \right\} < \infty. \tag{1.8}$$

– Notice that, since we do not impose to ψ to be a continuously differentiable function and to $\left\{2^{j/2}\psi(2^j\cdot -k) : (j,k)\in\mathbb{Z}^2\right\}$ to form an orthonormal wavelet basis for $L^2(\mathbb{R})$, a priori it is not at all clear that (1.6), implies that, almost surely for all arbitrarily small $\epsilon > 0$,

$$\limsup_{j\to+\infty} \left\{ 2^{j(H-1/\alpha+\epsilon)} D_j \right\} = \infty. \tag{1.9}$$

Yet, by making use of some specific properties of lfsm as well as the fact that ψ is compactly supported, we will be able to show that, a result stronger than (1.9) holds; namely, one has almost surely, for all arbitrarily small $\epsilon > 0$,

$$\liminf_{j\to+\infty} \left\{ 2^{j(H-1/\alpha+\epsilon)} D_j \right\} = \infty. \tag{1.10}$$

– Finally, combining (1.8) with (1.10), one can get the following theorem, which is our main result.

Theorem 1.1. *For each $j \in \mathbb{N}$, one set,*

$$\frac{1}{\widehat{\alpha}_j} = H + \frac{\log(D_j)}{j\log(2)},$$

where D_j is defined in (1.7). Then, one has almost surely,

$$\widehat{\alpha}_j \xrightarrow[j\to+\infty]{a.s.} \alpha.$$

1.2. Proofs.

1.2.1. *Proof of Relation 1.8.* The proof is standard in the wavelet setting, we give it for the sake of completeness. Let $\check{\Omega}$ be an event of probability 1 on which Relation (1.5) holds and let $\omega \in \check{\Omega}$ be arbitrary and fixed. Assume that $\epsilon > 0$ is arbitrary and fixed and denote by $C(\omega)$ the finite quantity defined as,

$$C(\omega) := \sup_{t_1,t_2\in[0,1]} \left\{ \frac{\left|X_{H,\alpha}(t_1,\omega) - X_{H,\alpha}(t_2,\omega)\right|}{|t_1-t_2|^{H-1/\alpha-\epsilon}} \right\}. \tag{1.11}$$

On the other hand, notice that (1.3), (1.4) and the fact that,

$$\operatorname{supp}\psi \subseteq [0,1], \tag{1.12}$$

imply that, for all $(j,k) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ satisfying $0 \le k < 2^j$, one has,

$$d_{j,k}(\omega) = 2^j \int_{k2^{-j}}^{(k+1)2^{-j}} \Big\{ X_{H,\alpha}(t,\omega) - X_{H,\alpha}(k2^{-j},\omega) \Big\} \psi(2^j t - k) dt. \tag{1.13}$$

Next, combining (1.13) with (1.11), one gets

$$\begin{aligned} |d_{j,k}(\omega)| &\le 2^j \int_{k2^{-j}}^{(k+1)2^{-j}} \Big| X_{H,\alpha}(t,\omega) - X_{H,\alpha}(k2^{-j},\omega) \Big| \big|\psi(2^j t - k)\big| dt \\ &\le \|\psi\|_{L^\infty(\mathbb{R})} C(\omega) 2^j \int_{k2^{-j}}^{(k+1)2^{-j}} \big| t - k2^{-j} \big|^{H-1/\alpha-\epsilon} dt \\ &\le \|\psi\|_{L^\infty(\mathbb{R})} C(\omega) 2^{-j(H-1/\alpha-\epsilon)}, \end{aligned}$$

which proves that (1.8) is satisfied. □

1.2.2. *Proof of Relation 1.10.* Let us first recall that in [14], a nice stochastic integral representation of the wavelet coefficients $d_{j,k}$ has been obtained, namely one has almost surely that

$$d_{j,k} = 2^{-j(H-1/\alpha)} \int_{\mathbb{R}} \Phi_{H,\alpha}(2^j s - k) \mathrm{Z}_\alpha\,(ds)\,, \tag{1.14}$$

where $\Phi_{H,\alpha}$ is the real-valued continuous function defined for each $x \in \mathbb{R}$, as,

$$\Phi_{H,\alpha}(x) = \int_{\mathbb{R}} (y-x)_+^{H-1/\alpha} \psi(y) dy = \int_0^1 (y-x)_+^{H-1/\alpha} \psi(y) dy; \tag{1.15}$$

notice that the last equality results from (1.12).

Proposition 1.1. *The function $\Phi_{H,\alpha}$ satisfies the following two nice properties:*

(i) one has,

$$\operatorname{supp} \Phi_{H,\alpha} \subseteq (-\infty, 1]; \tag{1.16}$$

(ii) there is a constant $c_1 > 0$ such for all $x \in (-\infty, 1]$,

$$\big|\Phi_{H,\alpha}(x)\big| \le c_1 \big(1 + |x|\big)^{-(2+1/\alpha-H)}. \tag{1.17}$$

Proof of Proposition 1.1. Part (i) is a consequence of (1.15) and (1.2). Let us show that Part (ii) holds. First observe that, (1.15) easily implies that,

$$\sup_{x \in [-1,1]} \Big\{ \big(1+|x|\big)^{2+1/\alpha-H} \big|\Phi_{H,\alpha}(x)\big| \Big\} \le 4 \|\psi\|_{L^\infty(\mathbb{R})} < \infty. \tag{1.18}$$

Let us now suppose that $x < -1$. We denote by $\psi^{(-1)}$ the primitive of ψ, defined for all $z \in \mathbb{R}$, as

$$\psi^{(-1)}(z) = \int_{-\infty}^z \psi(y) dy.$$

Observe that (1.12) and (1.4) entail that the continuous function $\psi^{(-1)}$ has a compact support included in $[0,1]$. We denote by $\psi^{(-2)}$ the primitive of $\psi^{(-1)}$, defined for all $z \in \mathbb{R}$, as

$$\psi^{(-2)}(z) = \int_{-\infty}^z \psi^{(-1)}(y) dy.$$

Observe that $\operatorname{supp} \psi^{(-1)} \subseteq [0,1]$ and (1.4) entail that the continuous function $\psi^{(-2)}$ has a compact support included in $[0,1]$; therefore integrating two times by parts in (1.15), we obtain

$$\Phi_{H,\alpha}(x) = (H-1/\alpha)(H-1/\alpha-1) \int_0^1 (y-x)^{H-1/\alpha-2} \psi^{(-2)}(y) dy. \tag{1.19}$$

Next, using (1.19) and the inequalities: for all $y \in [0,1]$, $y - x \geq |x| \geq 2^{-1}\big(1+|x|\big)$, it follows that,

$$\big|\Phi_{H,\alpha}(x)\big| \leq 2^{2+1/\alpha-H} \|\psi^{(-2)}\|_{L^\infty(\mathbb{R})} \big(1+|x|\big)^{H-1/\alpha-2}. \tag{1.20}$$

Finally, combining (1.18) with (1.20), we get Part (ii) of the proposition. □

A straightforward consequence of (1.14) and Part (i) of Proposition 1.1, is that,

$$d_{j,k} = 2^{-j(H-1/\alpha)} \int_{-\infty}^{(k+1)2^{-j}} \Phi_{H,\alpha}(2^j s - k) \mathrm{Z}_\alpha\,(ds)\,. \tag{1.21}$$

Let us now introduce some additional notations. We assume that $\delta \in (0, 1/3)$ is arbitrary and fixed. For all $j \in \mathbb{Z}_+$, we define the positive integer e_j as,

$$e_j := [2^{j\delta}], \tag{1.22}$$

where $[\cdot]$ is the integer part function. Then, for any integer l such that

$$0 \leq l \leq [2^{j(1-\delta)}] - 1, \tag{1.23}$$

we set

$$G_{j,le_j} := \int_{((l-1)e_j+1)2^{-j}}^{(le_j+1)2^{-j}} \Phi_{H,\alpha}(2^j s - le_j) \mathrm{Z}_\alpha\,(ds)\,, \tag{1.24}$$

and

$$R_{j,le_j} := \int_{-\infty}^{((l-1)e_j+1)2^{-j}} \Phi_{H,\alpha}(2^j s - le_j) \mathrm{Z}_\alpha\,(ds)\,. \tag{1.25}$$

Thus, in view of (1.21), the wavelet coefficient d_{j,le_j} can be expressed as,

$$d_{j,le_j} = 2^{-j(H-1/\alpha)} \Big(G_{j,le_j} + R_{j,le_j}\Big). \tag{1.26}$$

Now, our goal will be to derive the following two lemmas which respectively provide lower and upper asymptotic estimates for $\max_{0\leq l<[2^{j(1-\delta)}]} |G_{j,le_j}|$ and $\max_{0\leq l<[2^{j(1-\delta)}]} |R_{j,le_j}|$.

Lemma 1.1. *One has, almost surely*

$$\liminf_{j\to+\infty} \Big\{ 2^{j\frac{2\delta}{\alpha}} \max_{0\leq l<[2^{j(1-\delta)}]} |G_{j,le_j}| \Big\} \geq 1. \tag{1.27}$$

Lemma 1.2. *One has, almost surely*

$$\limsup_{j\to+\infty} \Big\{ 2^{j\frac{2\delta}{\alpha}} \max_{0\leq l<[2^{j(1-\delta)}]} |R_{j,le_j}| \Big\} = 0. \tag{1.28}$$

The proof of Lemma 1.1 mainly relies on the following two results.

Lemma 1.3. *(see e.g. [26]) Let Y be an arbitrary symmetric α-stable random variable with a non-vanishing scale parameter $\|Y\|_\alpha$, then for any real number $t \geq \|Y\|_\alpha$, one has,*

$$c_3 \|Y\|_\alpha^\alpha t^{-\alpha} \leq \mathbb{P}(|Y| > t) \leq c_2 \|Y\|_\alpha^\alpha t^{-\alpha}, \tag{1.29}$$

where c_2 and c_3 are two positive constants only depending on α.

Lemma 1.4. *For each fixed $j \in \mathbb{Z}_+$, $\{G_{j,le_j} : 0 \leq l \leq [2^{j(1-\delta)}] - 1\}$ is a sequence of identically distributed independent symmetric α-stable random variables whose scale parameters, denoted $\|G_{j,le_j}\|_\alpha$, satisfy for all l,*

$$\|G_{j,le_j}\|_\alpha^\alpha = 2^{-j} \int_{1-e_j}^{1} |\Phi_{H,\alpha}(x)|^\alpha dx. \tag{1.30}$$

Proof of Lemma 1.4. The independence of these symmetric α-stable random variables is a straightforward consequence of the fact that they are defined (see (1.24)) through stable stochastic integrals over disjoint intervals. In order to show that they are identically distributed it is sufficient to prove that (1.30) holds for each l. Using a standard property of stable stochastic integrals (see e.g. [26]) and (1.24), one gets

$$\|G_{j,le_j}\|_\alpha^\alpha = \int_{((l-1)e_j+1)2^{-j}}^{(le_j+1)2^{-j}} |\Phi_{H,\alpha}(2^j s - le_j)|^\alpha ds;$$

then the change of variable $u = 2^j s - le_j$ allows to obtain (1.30). □

Now, we are in position to prove Lemma 1.1.

Proof of Lemma 1.1. Let $j \in \mathbb{Z}_+$ be arbitrary and fixed. Using the fact that $\{G_{j,le_j} : 0 \le l \le [2^{j(1-\delta)}] - 1\}$ is a sequence of independent identically distributed random variables (see Lemma 1.4), one gets,

$$\begin{aligned} \mathbb{P}\Big(\max_{0\le l<[2^{j(1-\delta)}]} |G_{j,le_j}| \le 2^{-j\frac{2\delta}{\alpha}}\Big) &= \prod_{l=0}^{[2^{j(1-\delta)}]-1} \mathbb{P}\Big(|G_{j,le_j}| \le 2^{-j\frac{2\delta}{\alpha}}\Big) \\ = \mathbb{P}\Big(|G_{j,0}| \le 2^{-j\frac{2\delta}{\alpha}}\Big)^{[2^{j(1-\delta)}]} &= \Big(1 - \mathbb{P}\Big(|G_{j,0}| > 2^{-j\frac{2\delta}{\alpha}}\Big)\Big)^{[2^{j(1-\delta)}]}. \end{aligned} \tag{1.31}$$

Observe that, in view of (1.30) in which one takes $l = 0$ and in view of the assumption that $\delta \in (0, 1/3)$, there exist a positive constant c_4 and a positive integer j_0 such that, one has,

$$2^{-j\frac{2\delta}{\alpha}} \ge \|G_{j,le_j}\|_\alpha = 2^{-j/\alpha}\left(\int_{1-e_j}^{1} |\Phi_{H,\alpha}(x)|^\alpha dx\right)^{1/\alpha} \ge c_4^{1/\alpha} 2^{-j/\alpha}, \tag{1.32}$$

for all integers j and l satisfying $j \ge j_0$ and $0 \le l < [2^{j(1-\delta)}]$; notice that the last inequality in (1.32), follows from the fact that we have chosen j_0, such that for every $j \ge j_0$,

$$\int_{1-e_j}^{1} |\Phi_{H,\alpha}(x)|^\alpha dx \ge 2^{-1}\int_{-\infty}^{1} |\Phi_{H,\alpha}(x)|^\alpha dx,$$

and the last integral is positive since $\Phi_{H,\alpha}$ is a non-vanishing function (this is a consequence of our assumptions on ψ). Also notice that, one can suppose that $c_4 \in \big(0, c_3^{-1}\big)$ (the positive constant c_3 has been introduced in Lemma 1.3). Next, it follows from (1.31), from the first inequality in (1.29) in which one $t = 2^{-j\frac{2\delta}{\alpha}}$, and from (1.32), that, for all integer $j \ge j_0$,

$$\mathbb{P}\Big(\max_{0\le l<[2^{j(1-\delta)}]} |G_{j,le_j}| \le 2^{-j\frac{2\delta}{\alpha}}\Big) \le \Big(1 - c_5 2^{-j(1-2\delta)}\Big)^{[2^{j(1-\delta)}]}, \tag{1.33}$$

where the constant $c_5 := c_3 c_4 \in (0,1)$. Then, (1.33), the fact that $\delta \in (0, 1/3)$, and standard computations, allow to show that,

$$\sum_{j=j_0}^{+\infty} \mathbb{P}\Big(\max_{0\le l<[2^{j(1-\delta)}]} |G_{j,le_j}| \le 2^{-j\frac{2\delta}{\alpha}}\Big) < \infty;$$

thus, applying the Borel-Cantelli Lemma, one gets (1.27). □

The proof of Lemma 1.2 mainly relies on the following result as well as on Lemma 1.3.

Lemma 1.5. *For all non-negative integers j and l such that $l < [2^{j\delta}]$, the scale parameter $\|R_{j,le_j}\|_\alpha$ of the symmetric α-stable random variable R_{j,le_j} (see (1.25)) satisfies,*

$$\|R_{j,le_j}\|_\alpha^\alpha = 2^{-j}\int_{-\infty}^{1-e_j} |\Phi_{H,\alpha}(x)|^\alpha dx \le c_6 2^{-j\alpha(2\delta+1/\alpha-\delta H)}, \tag{1.34}$$

where c_6 is a positive constant non depending on j and l.

Proof of Lemma 1.5. The equality in (1.34) can be obtained by using (1.25) and the arguments which have allowed to derive (1.30). Let us show that the inequality in (1.34) holds; there is no restriction to assume that $j \ge \delta^{-1}$. Using (1.17) and (1.22), one has,

$$\begin{aligned} 2^{-j}\int_{-\infty}^{1-e_j} |\Phi_{H,\alpha}(x)|^\alpha dx &\le c_1^\alpha 2^{-j}\int_{-\infty}^{1-e_j} \big(1-x\big)^{-2\alpha-1+\alpha H} dx \\ \le c_1^\alpha 2^{-j}\int_{2^{j\delta}-2}^{+\infty} \big(1+x\big)^{-2\alpha-1+\alpha H} dx &= c_1^\alpha \frac{2^{-j}\big(2^{j\delta}-1\big)^{-\alpha(2-H)}}{\alpha(2-H)} \le c_6 2^{-j\alpha(2\delta+1/\alpha-\delta H)}, \end{aligned}$$

where the constant

$$c_6 := c_1^\alpha \frac{2^{\alpha(2-H)}}{\alpha(2-H)}.$$

□

Now, we are in position to prove Lemma 1.2.

Proof of Lemma 1.2. First, observe that in view of the assumption that $\delta \in (0, 1/3)$, one has for a fixed arbitrarily small $\eta > 0$,

$$\frac{2\delta+\eta}{\alpha} < 2\delta + 1/\alpha - \delta H;$$

therefore, it follows from Lemma 1.5, that there exists a positive integer j_1, such that for all integers j and l, satisfying $j \ge j_1$ and $0 \le l < [2^{j(1-\delta)}]$, one has,

$$\|R_{j,le_j}\|_\alpha \le 2^{-j\left(\frac{2\delta+\eta}{\alpha}\right)}.$$

Thus, we are allowed to apply the second inequality in (1.29), in the case where $Y = R_{j,le_j}$ and $t = 2^{-j\left(\frac{2\delta+\eta}{\alpha}\right)}$. As a consequence, we obtain that, for all $j \ge j_1$,

$$\begin{aligned} \mathbb{P}\Big(\max_{0\le l<[2^{j(1-\delta)}]} |R_{j,le_j}| > 2^{-j\left(\frac{2\delta+\eta}{\alpha}\right)}\Big) &\le \sum_{l=0}^{[2^{j(1-\delta)}]-1} \mathbb{P}\Big(|R_{j,le_j}| > 2^{-j\left(\frac{2\delta+\eta}{\alpha}\right)}\Big) \\ \le c_2 2^{j(2\delta+\eta)} \sum_{l=0}^{[2^{j(1-\delta)}]-1} \|R_{j,le_j}\|_\alpha^\alpha &\le c_7 2^{-j\alpha(2\delta+1/\alpha-\delta H)+j(1+\delta+\eta)}, \end{aligned} \tag{1.35}$$

where the last inequality results from (1.34) and the constant $c_7 := c_2 c_6$. Assume that $\delta(\alpha-1) > \eta$, then one has,

$$\alpha(2\delta + 1/\alpha - \delta H) > \alpha(\delta + 1/\alpha) = \alpha\delta + 1 > 1 + \delta + \eta.$$

Therefore, it follows from (1.35) that,

$$\sum_{j=j_1}^{+\infty} \mathbb{P}\Big(\max_{0\le l<[2^{j(1-\delta)}]} |R_{j,le_j}| > 2^{-j\left(\frac{2\delta+\eta}{\alpha}\right)}\Big) < \infty;$$

thus, applying the Borel-Cantelli Lemma, one gets (1.28). □

Remark 1.1. *Our proofs of Lemmas 1.1 and 1.2, only allow to derive that Relations (1.27) and (1.28) hold on some event of probability* 1*, denoted by* $\widetilde{\Omega}_\delta$*, since it a priori depends on* $\delta \in (0, 1/3)$*. Yet, one can easily show that these two relations also hold, for every real number* $\delta \in (0, 1/3)$*, on an event of probability* 1 *which does not depend on* δ*, namely the event* $\bigcap_{\delta \in \mathbb{Q} \cap (0,1/3)} \widetilde{\Omega}_\delta$*.*

Now, we are in position to prove Relation (1.10).
Assume that ϵ is a fixed arbitrarily small positive real number and that $\delta \in (0, 1/3)$ is such that,

$$\epsilon/2 = 2\delta/\alpha. \tag{1.36}$$

Next observe that (1.36), (1.7), (1.26) and the triangle inequality, imply that for all $j \in \mathbb{Z}_+$,

$$\begin{aligned} 2^{j(H-1/\alpha+\epsilon/2)} D_j &\geq 2^{j\frac{2\delta}{\alpha}} \max_{0 \leq l < [2^{j(1-\delta)}]} |G_{j,le_j} + R_{j,lej}| \\ &\geq 2^{j\frac{2\delta}{\alpha}} \max_{0 \leq l < [2^{j(1-\delta)}]} |G_{j,le_j}| - 2^{j\frac{2\delta}{\alpha}} \max_{0 \leq l < [2^{j(1-\delta)}]} |R_{j,le_j}|; \end{aligned}$$

therefore, one has that,

$$\begin{aligned} &\liminf_{j \to +\infty} \left\{ 2^{j(H-1/\alpha+\epsilon/2)} D_j \right\} \\ &\geq \liminf_{j \to +\infty} \left\{ 2^{j\frac{2\delta}{\alpha}} \max_{0 \leq l < [2^{j(1-\delta)}]} |G_{j,le_j}| \right\} - \limsup_{j \to +\infty} \left\{ 2^{j\frac{2\delta}{\alpha}} \max_{0 \leq l < [2^{j(1-\delta)}]} |R_{j,le_j}| \right\}. \end{aligned} \tag{1.37}$$

Finally putting together, (1.27), (1.28), (1.37) and (1.36), one gets (1.10).

1.2.3. *Proof of Theorem 1.1.* Relations (1.8) and (1.10) imply that there is Ω^* an event of probability 1 such that each $\omega \in \Omega^*$ satisfies the following property: for all arbitrarily small $\epsilon > 0$, there are two finite positive constants $A = A(\omega, \epsilon)$ and $B = B(\omega, \epsilon)$, and there exists $j_2 = j_2(\omega, \epsilon) \in \mathbb{Z}_+$, such that, one has for all integer $j \geq j_2$,

$$A 2^{-j(H-1/\alpha+\epsilon)} \leq D_j(\omega) \leq B 2^{-j(H-1/\alpha-\epsilon)}.$$

This entails that,

$$-H + 1/\alpha - \epsilon \leq \liminf_{j \to +\infty} \left\{ \frac{\log(D_j(\omega))}{j \log(2)} \right\} \leq \limsup_{j \to +\infty} \left\{ \frac{\log(D_j(\omega))}{j \log(2)} \right\} \leq -H + 1/\alpha + \epsilon.$$

Then letting ϵ goes to zero, one gets that,

$$\lim_{j \to +\infty} \left\{ \frac{\log(D_j(\omega))}{j \log(2)} \right\} = -H + 1/\alpha.$$

□

2. Simulations

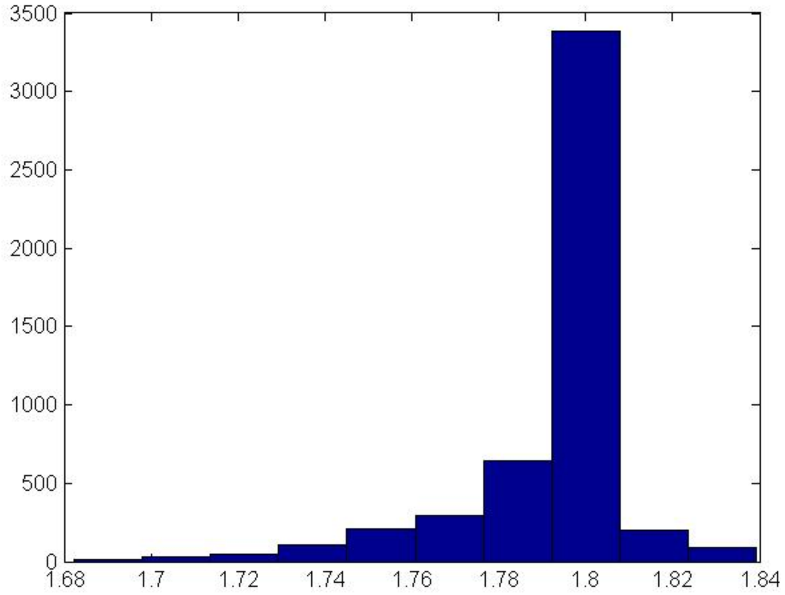

Figure 1. l'histogramme suivant représente les $\widehat{\alpha}$ obtenues à partir de 5000 trajectoires simulées sur $[0,1]$, d'un mfsl de paramètres $\alpha = 1.8$ et $H = 0.6$; ici $j = 11$ et ψ est l'ondelette mère de Daubechies avec 8 moments nuls.

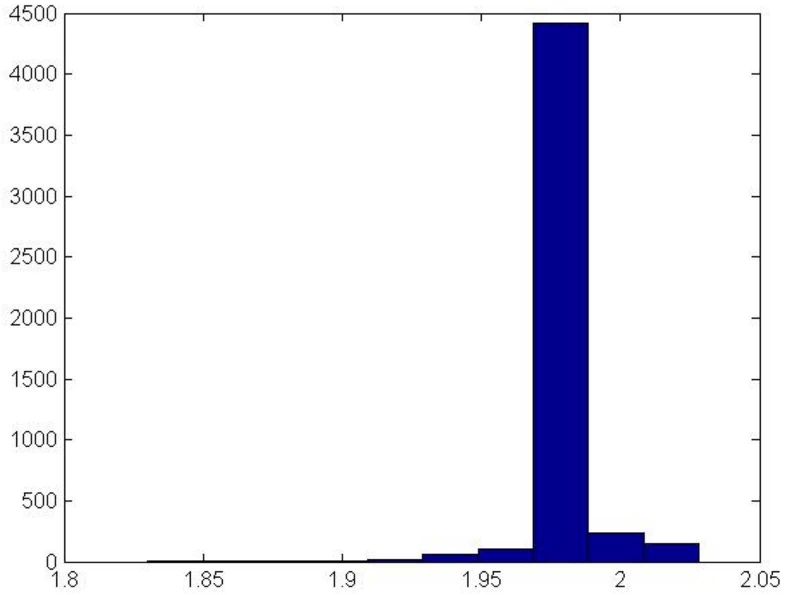

Figure 2. l'histogramme suivant représente les $\widehat{\alpha}$ obtenues à partir de 5000 trajectoires simulées sur $[0,1]$, d'un mfsl de paramètres $\alpha = 1.9$ et $H = 0.55$; ici $j = 11$ et ψ est l'ondelette mère de Daubechies avec 8 moments nuls.

3. Estimation du paramètre de stabilité α dans le cas multifractionnaire

3.1. Introduction et énoncé du résultat principal. Soit α un nombre réel fixé une fois pour toute vérifiant $1 < \alpha < 2$. On note par $\{X(u,v) : (u,v) \in [0,1]\times]1/\alpha, 1[\}$ la restriction à $[0,1]\times]1/\alpha, 1[$ de la modification à trajectoires continues (voir les Théorèmes 2.1 et 3.1 du chapitre 2) du champ stochastique qui engendre les mouvements multifractionnaires stables linéaires (mmsl en abrégé en français, LMSM en abrégé en anglais) ; de plus, tout au long de la présente section, on restreint l'espace de probabilité Ω, à Ω_0^* l'évènement de probabilité 1 qui a été introduit dans le Lemme 2.1 du chapitre 2. Rappelons que, pour tout $(u,v) \in \mathbb{R}\times]1/\alpha, 1[$, l'on a, presque sûrement,

$$X(u,v) = \int_{\mathbb{R}} \Big\{(u-s)_+^{v-1/\alpha} - (-s)_+^{v-1/\alpha}\Big\} \mathrm{Z}_\alpha(ds), \tag{3.1}$$

où, pour tous nombres réels x et κ,

$$(x)_+^\kappa := \begin{cases} x^\kappa, \text{si } x \in]0,+\infty[, \\ \\ 0, \text{si } x \in]-\infty, 0]. \end{cases} \tag{3.2}$$

Dans toute cette section, on suppose que $\{\mathrm{Z}_\alpha(s) : s \in \mathbb{R}\}$ est un processus de Lévy symétrique $\alpha-$stable ($\mathcal{S}\alpha\mathcal{S}$ en abrégé) dont les trajectoires sont des fonctions càdlàg.

On note par $H(\cdot)$ une fonction déterministe définie sur $[0,1]$ et à valeurs dans un intervalle compact $[\underline{H}, \overline{H}] \subset]1/\alpha, 1[$, où $\underline{H} := \min_{x\in[0,1]} H(x)$ et $\overline{H} := \max_{x\in[0,1]} H(x)$. Dans toute cette section, on suppose que $H(\cdot)$ est höldérienne, ce qui signifie qu'il existe deux constantes $c_1 > 0$ et $\rho_H > 0$, telles que

$$\forall t_1, t_2 \in [0,1];\ |H(t_1) - H(t_2)| \le c_1 |t_1 - t_2|^{\rho_H}\ ; \tag{3.3}$$

de plus, on impose à ρ_H de vérifier les inégalités :

$$1 \ge \rho_H > \overline{H} := \max_{x\in[0,1]} H(x). \tag{3.4}$$

Rappelons que $\{Y(t) : t \in [0,1]\}$ le mouvement multifractionnaire stable linéaire (mmsl) de paramètre de stabilité α et de paramètre fonctionnel (de Hurst) $H(\cdot)$, introduit par Stoev et Taqqu dans [**28**, **30**], est défini, pour tout $t \in \mathbb{R}$, par

$$Y(t) = X(t, H(t)). \tag{3.5}$$

L'objectif de cette section est de montrer que la méthode qui nous a permis précédemment d'estimer le paramètre α du mouvement fractionnaire stable linéaire (mfsl en abrégé en français, LFSM en abrégé en anglais) peut être étendue au mmsl. La suite $\{d_{j,k} : (j,k) \in \mathbb{Z}^2\}$, des coefficients d'ondelettes du mmsl $\{Y(t) : t \in \mathbb{R}\}$, est définie par,

$$d_{j,k} = 2^j \int_{\mathbb{R}} Y(t)\psi(2^j t - k)dt. \tag{3.6}$$

De plus, on impose seulement à l'ondelette analysante ψ de vérifier les hypothèses suivantes :

(1) ψ est une fonction non identiquement nulle, continue sur $\mathbb{R}$ et à valeurs réelles,

(2) son support est un compact inclus dans $[0,1]$,

(3) elle admet, au moins, 2 moments nuls, c'est à dire

$$\int_{\mathbb{R}} \psi(s)ds = \int_{\mathbb{R}} s\psi(s)ds = 0. \tag{3.7}$$

Il est à noter que l'on n'a pas besoin de supposer que la suite de fonctions $\{2^{j/2}\psi(2^j\cdot -k) : (j,k) \in \mathbb{Z}^2\}$ soit une base orthonormale de $\mathrm{L}^2(\mathbb{R})$.

Puisque dans le chapitre 4 (voir dans ce chapitre, le Théorème 1.1 et les parties (2) et (1) de la Remarque 1.1), nous avons introduit une procédure d'estimation de $\underline{H} := \min_{x\in[0,1]} H(x)$ qui ne nécessite pas la connaissance de α ; dans le présente section nous supposons que $\underline{H}$ est connu et notre objectif consiste à construire, au moyen des coefficients d'ondelettes $\{d_{j,k} : 0 \le k < 2^j\}$ où $j \ge 0$, un estimateur fortement consistant (c'est à dire qui converge presque sûrement lorsque j tend vers $+\infty$) du paramètre α ; cet estimateur est noté par $\widehat{\alpha}_j$. Donnons d'abord les principales idées sur lesquelles repose la construction de $\widehat{\alpha}_j$.

- Tout d'abord, notons que $\rho_Y^{\mathrm{unif}}\big([0,1]\big)$ l'exposant de Hölder uniforme du mmsl sur $[0,1]$ (voir (8.1) du chapitre 2 pour la définition de cet exposant) vaut presque sûrement $\underline{H} - 1/\alpha$; cela découle des relations (3.3) et (3.4) dans la présente section combinées avec les trois résultats suivants du chapitre 2 : la Partie (ii) du Corollaire 5.1, le Théorème 6.1 et la Remarque 6.1. On a donc presque sûrement, pour tout réel arbitrairement petit $\eta > 0$,
$$\sup_{(t_1,t_2)\in[0,1]^2} \left\{ \frac{\big|Y(t_1) - Y(t_2)\big|}{|t_1 - t_2|^{\underline{H}-1/\alpha-\eta}} \right\} < +\infty, \tag{3.8}$$
et
$$\sup_{(t_1,t_2)\in[0,1]^2} \left\{ \frac{\big|Y(t_1) - Y(t_2)\big|}{|t_1 - t_2|^{\underline{H}-1/\alpha+\eta}} \right\} = +\infty. \tag{3.9}$$
- Ensuite, on pose
$$D_j := \max_{0\le k<2^j} |d_{j,k}|. \tag{3.10}$$
Grâce à la nullité du premier moment de l'ondelette ψ, on montre en utilisant (3.8), que, presque sûrement, on a, pour tout réel $\epsilon > 0$ arbitrairement petit
$$\limsup_{j\to+\infty} \left\{ 2^{j(\underline{H}-1/\alpha-\epsilon)} D_j \right\} < \infty. \tag{3.11}$$
- Comme dans le cas fractionnaire, il n'est pas du tout évident que (3.9) implique que pour tout $\epsilon > 0$ arbitrairement petit, on a, presque sûrement,
$$\limsup_{j\to+\infty} \left\{ 2^{j(\underline{H}-1/\alpha+\epsilon)} D_j \right\} = +\infty. \tag{3.12}$$
Néanmoins, en utilisant des propriétés spécifiques au mmsl ainsi que le fait que le support de ψ est compact, dans la sous-section 3.2.2 on établit un résultat plus fort que (3.12) ; plus précisément, on montre que pour tout $\epsilon > 0$ arbitrairement petit, on a, presque sûrement,
$$\liminf_{j\to+\infty} \left\{ 2^{j(\underline{H}-1/\alpha+\epsilon)} D_j \right\} = +\infty. \tag{3.13}$$
Enfin, en assemblant (3.11) et (3.13), on obtient le théorème suivant, qui est le résultat principal de cette section.

Théorème 3.1. *Pour tout $j \in \mathbb{N}$, posons,*
$$\frac{1}{\widehat{\alpha}_j} = \left(\min_{x\in[0,1]} H(x) \right) + \frac{\log(D_j)}{j\log(2)}, \tag{3.14}$$
où D_j a été définie en (3.10). Alors, on a, presque sûrement,
$$\widehat{\alpha}_j \xrightarrow[j\to+\infty]{p.s.} \alpha. \tag{3.15}$$

3.2. Preuves.

3.2.1. *Preuve de la relation (3.11).* La preuve de cette relation est similaire à celle effectuée dans le cadre fractionnaire. Soit $\breve{\Omega}$ l'évènement de probabilité 1 sur lequel la relation (3.8) a lieu et considérons un $\omega \in \breve{\Omega}$ arbitraire et fixé. Supposons que $\epsilon > 0$ soit un réel arbitraire fixé et notons par $C(\omega)$ la variable aléatoire finie, définie par

$$C(\omega) := \sup_{(t_1,t_2)\in[0,1]^2} \left\{ \frac{|Y(t_1,\omega) - Y(t_2,\omega)|}{|t_1 - t_2|^{\underline{H}-1/\alpha-\epsilon}} \right\} < +\infty. \tag{3.16}$$

Ensuite, remarquons que (3.6), (3.7) et le fait que

$$\operatorname{supp}\psi \subseteq [0,1] \tag{3.17}$$

impliquent que pour tout $(j,k) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ satisfaisant $0 \le k < 2^j$, on a

$$d_{j,k} = 2^j \int_{k2^{-j}}^{(k+1)2^{-j}} \left\{ Y(t) - Y(k2^{-j}) \right\} \psi(2^j t - k) dt.$$

Ensuite, en combinant (3.6) avec (3.16), on a

$$\begin{aligned} |d_{j,k}(\omega)| &\le 2^j \int_{k2^{-j}}^{(k+1)2^{-j}} \left| Y(t,\omega) - Y(k2^{-j},\omega) \right| |\psi(2^j t - k)| dt \\ &\le C(\omega)\|\psi\|_{\mathrm{L}^\infty(\mathbb{R})} 2^j \int_{k2^{-j}}^{(k+1)2^{-j}} |t - k2^{-j}|^{\underline{H}-1/\alpha-\epsilon} dt. \\ &\le C(\omega)\|\psi\|_{\mathrm{L}^\infty(\mathbb{R})} 2^{-j(\underline{H}-1/\alpha-\epsilon)}, \end{aligned}$$

ce qui prouve que (3.11) est satisfaite. □

3.2.2. *Preuve de la relation (3.13).* Nous désignons par $\{\widetilde{d}_{j,k} : (j,k) \in \mathbb{Z}^2\}$ la suite de variables aléatoires $\mathcal{S}\alpha\mathcal{S}$ définies par

$$\widetilde{d}_{j,k} = 2^j \int_{\mathbb{R}} X(t, H(k2^{-j}))\psi(2^j t - k) dt; \tag{3.18}$$

de plus, pour tout $j \in \mathbb{Z}_+$, nous posons,

$$\widetilde{D}_j := \max_{0\le k<2^j} |\widetilde{d}_{j,k}|. \tag{3.19}$$

Grâce à un résultat de [14], on peut presque sûrement, pour tout $(j,k) \in \mathbb{Z}^2$, représenter $\widetilde{d}_{j,k}$, sous la forme de l'intégrale stochastique suivante :

$$\widetilde{d}_{j,k} = 2^{-j(H(k2^{-j})-1/\alpha)} \int_{\mathbb{R}} \Phi_\alpha(2^j s - k, H(k2^{-j}))\mathrm{Z}_\alpha(ds), \tag{3.20}$$

où Φ_α est la fonction continue à valeurs réelles, définie pour tout $(x,v) \in \mathbb{R}\times]1/\alpha, 1[$, par

$$\Phi_\alpha(x,v) = \int_{\mathbb{R}} (y-x)_+^{v-1/\alpha}\psi(y)dy = \int_0^1 (y-x)_+^{v-1/\alpha}\psi(y)dy; \tag{3.21}$$

remarquons que la dernière inégalité résulte de (3.17) et que cette fonction Φ_α a déjà été introduite en (2.18) dans le chapitre 4.

Proposition 3.1. *La fonction Φ_α satisfait les trois propriétés suivantes :*

(i) on a, pour tout $v \in [\underline{H}, \overline{H}]$,

$$\operatorname{supp}\Phi_\alpha(\cdot, v) \subseteq]-\infty, 1]; \tag{3.22}$$

(ii) il existe une constante $c_1 > 0$ telle que pour tout $x \in \mathbb{R}$ et tout $v \in [\underline{H}, \overline{H}]$, on a

$$\left|\Phi_\alpha(x,v)\right| \le c_1\big(1+|x|\big)^{-(2+1/\alpha-\overline{H})}; \tag{3.23}$$

(iii) La fonction

$$\begin{array}{ll} [\underline{H},\overline{H}] & \to \mathbb{R}_+ \\ v & \mapsto \left\|\Phi_\alpha(\cdot,v)\right\|_{\mathrm{L}^\alpha(\mathbb{R})}, \end{array} \tag{3.24}$$

est continue, de plus

$$c_2 := \max_{v\in[\underline{H},\overline{H}]} \left\|\Phi_\alpha(\cdot,v)\right\|_{\mathrm{L}^\alpha(\mathbb{R})} < \infty \text{ et } c_3 := \min_{v\in[\underline{H},\overline{H}]} \left\|\Phi_\alpha(\cdot,v)\right\|_{\mathrm{L}^\alpha(\mathbb{R})} > 0. \tag{3.25}$$

Le point (i) de la Proposition 3.1 est une conséquence directe de (3.2), les deux autres points ne sont rien d'autre que les points (ii) et (iii) de la Proposition 2.1 du chapitre 4 ; signalons au passage que la finitude de la constante c_2 est une conséquence de la continuité sur l'intervalle compact $[\underline{H},\overline{H}]$ de la fonction $v \mapsto \left\|\Phi_\alpha(\cdot,v)\right\|_{\mathrm{L}^\alpha(\mathbb{R})}$. Il est important de noter que (3.20) et (3.22) impliquent que

$$\widetilde{d}_{j,k} = 2^{-j(H(k2^{-j})-1/\alpha)} \int_{-\infty}^{(k+1)2^{-j}} \Phi_\alpha(2^j s-k, H(k2^{-j}))\mathrm{Z}_\alpha\,(ds)\,. \tag{3.26}$$

Il convient maintenant d'introduire quelques notations supplémentaires :

- $t_0 \in [0,1]$ désigne un réel tel que

$$H(t_0) = \underline{H}; \tag{3.27}$$

- Pour tout entier $j \ge 1$, $I_j(t_0)$ désigne l'intervalle compact inclus dans $[0,1]$, d'intérieur non vide et défini par

$$I_j(t_0) = [0,1] \cap [t_0 - j^{-\frac{1}{\rho_H}}, t_0 + j^{-\frac{1}{\rho_H}}]; \tag{3.28}$$

  signalons que son diamètre, noté par $|I_j(t_0)|$, vérifie

$$j^{-\frac{1}{\rho_H}} \le |I_j(t_0)| \le 2j^{-\frac{1}{\rho_H}}. \tag{3.29}$$

- On associe à l'intervalle $I_j(t_0)$, l'ensemble $\nu_j(t_0)$ d'indices k, défini pour tout entier $j \ge 1$ par

$$\nu_j(t_0) = \big\{k \in \{0,1,\dots,2^j-1\}\,;\, k2^{-j} \in I_j(t_0)\big\}\,; \tag{3.30}$$

cet ensemble d'entiers vérifie le lemme suivant qui donne, pour j assez grand, un fin encadrement du nombre d'entiers contenus dans $\nu_j(t_0)$.

Lemme 3.1. *Il existe $c_5 > 0$ et $c_6 > 0$ deux constantes, telles que pour tout entier j assez grand, on a*

$$c_5 2^j j^{-\frac{1}{\rho_H}} \le \operatorname{card}(\nu_j(t_0)) \le c_6 2^j j^{-\frac{1}{\rho_H}}. \tag{3.31}$$

Preuve du Lemme 3.1. Cela résulte du fait que le nombre d'entiers contenus dans un intervalle arbitraire de diamètre d, est compris entre $[d]$ et $[d]+1$ où $[\cdot]$ désigne la fonction partie entière. $\square$

- On suppose que $\delta \in]0,1/3[$ est un réel arbitraire fixé.
- Pour tout $j \in \mathbb{Z}_+$, nous définissons l'entier $e_j \ge 1$ par,

$$e_j := [2^{j\delta}]. \tag{3.32}$$

- Pour tout $j \ge 1$, nous posons

$$\widetilde{\nu}_j(t_0) = \big\{k \in \nu_j(t_0)\,;\, e_j \text{ divise } k\big\} = \big\{k \in \nu_j(t_0)\,;\, \exists\, l \in \mathbb{Z}_+ \text{ tel que } k = le_j\big\} \tag{3.33}$$

  et

$$\widetilde{\lambda}_j(t_0) = \big\{l \in \mathbb{Z}_+\,;\, le_j \in \nu_j(t_0)\big\}; \tag{3.34}$$

il est clair que les ensembles $\widetilde{\nu}_j(t_0)$ et $\widetilde{\lambda}_j(t_0)$ sont en bijection, ainsi,

$$\mathrm{card}(\widetilde{\nu}_j(t_0)) = \mathrm{card}(\widetilde{\lambda}_j(t_0)). \tag{3.35}$$

Lemme 3.2. *Il existe $c_7 > 0$ et $c_8 > 0$ et il existe un entier $j_0 \geq 1$ tels que pour tout $j \geq j_0$, on a*

$$c_7 2^{j(1-\delta)} j^{-\frac{1}{\rho_H}} \leq \mathrm{card}(\widetilde{\lambda}_j(t_0)) \leq c_8 2^{j(1-\delta)} j^{-\frac{1}{\rho_H}}. \tag{3.36}$$

Preuve du Lemme 3.2. D'après (3.33) et (3.35), il existe $c'_7 > 0$ et $c'_8 > 0$ deux constantes telles que pour tout j assez grand, on a,

$$c'_7\Big(\frac{\mathrm{card}(\nu_j(t_0))}{e_j}\Big) \leq \mathrm{card}(\widetilde{\lambda}_j(t_0)) \leq c'_8\Big(\frac{\mathrm{card}(\nu_j(t_0))}{e_j}\Big),$$

ainsi, en utilisant le Lemme 3.1 et (3.32), on obtient (3.36). □

Pour tout entier $j \geq 1$ et tout $l \in \widetilde{\lambda}_j(t_0)$, on pose

$$G_{j,le_j} := \int_{((l-1)e_j+1)2^{-j}}^{(le_j+1)2^{-j}} \Phi_\alpha(2^j s - le_j, H(le_j 2^{-j})) Z_\alpha(ds)\,, \tag{3.37}$$

et

$$R_{j,le_j} := \int_{-\infty}^{((l-1)e_j+1)2^{-j}} \Phi_\alpha(2^j s - le_j, H(le_j 2^{-j})) Z_\alpha(ds)\,; \tag{3.38}$$

il résulte alors de (3.26) que,

$$\widetilde{d}_{j,le_j} = 2^{-j(H(le_j 2^{-j})-1/\alpha)}\Big(G_{j,le_j} + R_{j,le_j}\Big). \tag{3.39}$$

Maintenant, notre objectif est d'établir les deux lemmes suivant qui établissent respectivement des estimations asymptotiques inférieures et supérieures de $\max_{l\in\widetilde{\lambda}_j(t_0)} |G_{j,le_j}|$ et de $\max_{l\in\widetilde{\lambda}_j(t_0)} |R_{j,le_j}|$.

Lemme 3.3. *On a, presque sûrement,*

$$\liminf_{j\to+\infty} \left\{ 2^{j\frac{2\delta}{\alpha}} \max_{l\in\widetilde{\lambda}_j(t_0)} |G_{j,le_j}| \right\} \geq 1. \tag{3.40}$$

Lemme 3.4. *On a, presque sûrement,*

$$\limsup_{j\to+\infty} \left\{ 2^{j\frac{2\delta}{\alpha}} \max_{l\in\widetilde{\lambda}_j(t_0)} |R_{j,le_j}| \right\} = 0. \tag{3.41}$$

Concentrons-nous d'abord sur la preuve du Lemme 3.4, celle-ci repose sur les deux lemmes suivant :

Lemme 3.5. [26] *Soit Y une variable aléatoire symétrique α-stable dont le paramètre d'échelle, $\|Y\|_\alpha$, ne s'annule pas. Alors pour tout réel $t \geq \|Y\|_\alpha$, on a,*

$$c_{13}\|Y\|_\alpha^\alpha t^{-\alpha} \leq \mathbb{P}(|Y| > t) \leq c_{14}\|Y\|_\alpha^\alpha t^{-\alpha}, \tag{3.42}$$

où c_{13} et c_{14} sont deux constantes positives qui ne dépendent que de α.

Lemme 3.6. *Pour tout entier $j \geq 1$ et pour tout $l \in \widetilde{\lambda}_j(t_0)$, le paramètre d'échelle $\|R_{j,le_j}\|_\alpha$ de R_{j,le_j}, la variable aléatoire $S\alpha S$ introduite en (3.38), satisfait*

$$\|R_{j,le_j}\|_\alpha^\alpha = 2^{-j}\int_{-\infty}^{1-e_j} \big|\Phi_\alpha(x, H(le_j 2^{-j}))\big|^\alpha dx \leq c_9 2^{-j\alpha(2\delta+1/\alpha-\delta\overline{H})}, \tag{3.43}$$

où c_9 est une constante strictement positive qui ne dépend pas de j et l.

Preuve du Lemme 3.6. D'après (3.38), il vient que

$$\|R_{j,le_j}\|_\alpha^\alpha = \int_{-\infty}^{((l-1)e_j+1)2^{-j}} \left|\Phi_\alpha(2^j s - le_j, H(le_j 2^{-j}))\right|^\alpha ds.$$

En effectuant alors le changement de variable $u = 2^j s - le_j$, on obtient,

$$\|R_{j,le_j}\|_\alpha^\alpha = 2^{-j}\int_{-\infty}^{1-e_j} \left|\Phi_\alpha(x, H(le_j 2^{-j}))\right|^\alpha dx,$$

ce qui prouve l'égalité (3.43). Afin d'établir l'inégalité dans (3.43), il n'y a aucune restriction à supposer que $j \geq \delta^{-1}$. En employant (3.23) et (3.32), on a

$$\begin{aligned}
\|R_{j,le_j}\|_\alpha^\alpha &\leq c_1^\alpha 2^{-j}\int_{-\infty}^{1-e_j} (1+|u|)^{-(2\alpha+1-\alpha\overline{H})}du \\
&\leq c_1^\alpha 2^{-j}\int_{2^{j\delta}-2}^{+\infty} (1+u)^{-(2\alpha+1-\alpha\overline{H})}du = c_1^\alpha \frac{2^{-j}\left(2^{j\delta}-1\right)^{-\alpha(2-\overline{H})}}{\alpha(2-\overline{H})} \\
&\leq c_{10} 2^{-j\alpha(2\delta+1/\alpha-\delta\overline{H})},
\end{aligned}$$

avec

$$c_{10} = c_1^\alpha \frac{2^{\alpha(2-\overline{H})}}{\alpha(2-\overline{H})}.$$

□

On est maintenant en mesure de prouver le Lemme 3.4.

Preuve du Lemme 3.4. Puisque $\delta \in]0, 1/3[$, on a pour tout réel $\eta > 0$ arbitrairement petit fixé

$$\frac{2\delta+\eta}{\alpha} < 2\delta + 1/\alpha - \delta\overline{H};$$

ainsi, étant donné que $\alpha > 1$, du Lemme 3.6, on en tire l'existence d'un entier $j_1 > 0$ tel que pour tout $j \geq j_1$ et tout $l \in \widetilde{\lambda}_j(t_0)$, on a

$$\|R_{j,le_j}\|_\alpha \leq 2^{-j\left(\frac{2\delta+\eta}{\alpha}\right)}.$$

Nous pouvons donc appliquer la seconde inégalité de (3.42) dans le cas où $Y = R_{j,le_j}$ et $t = 2^{-j\left(\frac{2\delta+\eta}{\alpha}\right)}$; il en résulte que pour tout $j \geq j_1$,

$$\begin{aligned}
\mathbb{P}\left(\max_{l\in\widetilde{\lambda}_j(t_0)} |R_{j,le_j}| > 2^{-j\left(\frac{2\delta+\eta}{\alpha}\right)}\right) &\leq \sum_{l\in\widetilde{\lambda}_j(t_0)} \mathbb{P}\left(|R_{j,le_j}| > 2^{-j\left(\frac{2\delta+\eta}{\alpha}\right)}\right) \\
&\leq c_{14} \sum_{l\in\widetilde{\lambda}_j(t_0)} \|R_{j,le_j}\|_\alpha^\alpha 2^{j(2\delta+\eta)},
\end{aligned} \tag{3.44}$$

où $c_{14} > 0$ est une constante qui ne dépend pas de j. Ensuite, en utilisant (3.43) et (3.36), on a

$$\mathbb{P}\left(\max_{l\in\widetilde{\lambda}_j(t_0)} |R_{j,le_j}| > 2^{-j\left(\frac{2\delta+\eta}{\alpha}\right)}\right) \leq c_8 c_9 c_{14} j^{-\frac{1}{\rho_H}} 2^{-j\alpha(2\delta+1/\alpha-\delta\overline{H})} 2^{j(1+\delta+\eta)}. \tag{3.45}$$

De plus, étant donné que η est arbitrairement petit, on peut supposer que $\delta(\alpha-1) > \eta$, et ainsi, on obtient

$$\alpha(2\delta + 1/\alpha - \delta\overline{H}) > \alpha(\delta + 1/\alpha) = \alpha\delta + 1 > 1 + \delta + \eta.$$

Il résulte alors de (3.45) que

$$\sum_{j \geq j_1} \mathbb{P}\left(\max_{l \in \widetilde{\lambda}_j(t_0)} |R_{j,le_j}| > 2^{-j\left(\frac{2\delta+\eta}{\alpha}\right)} \right) < +\infty.$$

Finalement, en employant le Lemme de Borel-Cantelli, on obtient (3.41). □

Concentrons-nous maintenant sur la preuve du Lemme 3.3, celle-ci repose principalement sur le Lemme 3.5 et le lemme suivant :

Lemme 3.7. *Pour tout entier $j \geq 1$, $\{G_{j,le_j} : l \in \widetilde{\lambda}_j(t_0)\}$ est une suite de variables aléatoires $\mathcal{S}\alpha\mathcal{S}$ indépendantes dont les paramètres d'échelles sont donnés par*

$$\|G_{j,le_j}\|_\alpha^\alpha = 2^{-j} \int_{1-e_j}^{1} \left|\Phi_\alpha(x, H(le_j 2^{-j}))\right|^\alpha dx; \tag{3.46}$$

de plus, il existe deux constantes $c_{11} > 0$ et $c_{12} > 0$ et il existe un entier $j_0 \geq 1$, tels que pour tout $j \geq j_0$ et tout $l \in \widetilde{\lambda}_j(t_0)$, on a

$$c_{11} 2^{-j} \leq \left\|G_{j,le_j}\right\|_\alpha^\alpha \leq c_{12} 2^{-j}. \tag{3.47}$$

Preuve du Lemme 3.7. L'indépendance, pour tout $j \geq 1$ fixé, des variables aléatoires $\mathcal{S}\alpha\mathcal{S}$ G_{j,le_j} où $l \in \widetilde{\lambda}_j(t_0)$, est une conséquence du fait qu'elles sont définies par des intégrales stables sur des intervalles disjoints (à un ensemble Lebesgue négligeable près).

Au moyen de (3.37) et du changement de variable $u = 2^j s - le_j$, on obtient (3.46). La deuxième inégalité dans (3.47) est bien vérifiée ; en effet, en posant $c_{12} = c_2^\alpha$, il résulte de (3.46) et (3.25) que

$$\left\|G_{j,le_j}\right\|_\alpha^\alpha \leq 2^{-j} \left\|\Phi_\alpha(\cdot, H(le_j 2^{-j}))\right\|_{\mathrm{L}^\alpha(\mathbb{R})}^\alpha \leq c_{12} 2^{-j}.$$

Pour établir la première inégalité dans (3.47), remarquons que, (3.46), (3.25) et (3.23), impliquent que pour tout $j \geq 1$ et tout $l \in \widetilde{\lambda}_j(t_0)$

$$\begin{aligned}
\|G_{j,le_j}\|_\alpha^\alpha &\geq 2^{-j} \left(\left\|\Phi_\alpha(\cdot, H(le_j 2^{-j}))\right\|_{\mathrm{L}^\alpha(\mathbb{R})}^\alpha - \int_{-\infty}^{1-e_j} \left|\Phi_\alpha(x, H(le_j 2^{-j}))\right|^\alpha dx \right) \\
&\geq 2^{-j} \left(c_3^\alpha - c_1^\alpha \int_{-\infty}^{1-e_j} (1-x)^{-(2\alpha+1-\alpha\overline{H})} \right) \\
&\geq 2^{-j} \left(c_3^\alpha - c_1^\alpha (2\alpha - \alpha\overline{H})^{-1} e_j^{-(2\alpha-\alpha\overline{H})} \right);
\end{aligned}$$

ainsi $c_{11} = 2^{-1} c_3^\alpha$ et en prenant j_0 tel que pour tout $j \geq j_0$, on a,

$$c_1^\alpha (2\alpha - \alpha\overline{H})^{-1} e_j^{-(2\alpha-\alpha\overline{H})} \leq 2^{-1} c_3^\alpha,$$

on obtient la première inégalité dans (3.47). □

Nous sommes maintenant en mesure de prouver le Lemme 3.3.

preuve du Lemme 3.3. Soit j un entier arbitraire assez grand de sorte que $\lambda_j(t_0)$ soit non vide et que $c_{12} 2^{-j} \leq 2^{-j\frac{2\delta}{\alpha}}$; en utilisant le fait que $\{G_{j,le_j}\,;\, l \in \widetilde{\lambda}_j(t_0)\}$ est une suite finie de variables aléatoires indépendantes, on a

$$\begin{aligned}
\mathbb{P}\left(\max_{l \in \widetilde{\lambda}_j(t_0)} |G_{j,le_j}| \leq 2^{-j\frac{2\delta}{\alpha}} \right) &= \prod_{l \in \widetilde{\lambda}_j(t_0)} \mathbb{P}\left(|G_{j,le_j}| \leq 2^{-j\frac{2\delta}{\alpha}} \right) \\
&= \prod_{l \in \widetilde{\lambda}_j(t_0)} \left(1 - \mathbb{P}\left(|G_{j,le_j}| > 2^{-j\frac{2\delta}{\alpha}} \right) \right).
\end{aligned}$$

D'après (3.47), (3.42) et (3.36), il vient que

$$\mathbb{P}\left(\max_{l\in\widetilde{\lambda}_j(t_0)}|G_{j,le_j}|\leq 2^{-j\frac{2\delta}{\alpha}}\right)\leq\prod_{l\in\widetilde{\lambda}_j(t_0)}\left(1-c_{11}c_{13}2^{-j(1-2\delta)}\right)$$
$$\leq\left(1-c_{11}c_{13}2^{-j(1-2\delta)}\right)^{c_7 2^{j(1-\delta)}\left(j^{-\frac{1}{\rho_H}}\right)}. \tag{3.48}$$

Ainsi, on obtient,

$$\sum_{j=j_2}^{+\infty}\mathbb{P}\left(\max_{l\in\widetilde{\lambda}_j(t_0)}|G_{j,le_j}|\leq 2^{-j\frac{2\delta}{\alpha}}\right)<+\infty,$$

où j_2 désigne un entier assez grand ; en appliquant alors le lemme de Borel-Cantelli, on aboutit à (3.40). □

Remarque 3.1. *Les preuves du Lemme 3.3 et du Lemme 3.4 nous assurent que les relations (3.40) et (3.41) sont vraies sur un évènement de probabilité 1, noté $\widetilde{\Omega}_\delta$ parce qu'il dépend a priori de $\delta\in]0,1/3[$. Cependant, on peut facilement prouver qu'elles sont également vraies sur l'évènement de probabilité 1, $\bigcap_{\delta\in\mathbb{Q}\cap]0,1/3[}\widetilde{\Omega}_\delta$, qui, quant à lui, ne dépend pas de δ.*

Proposition 3.2. *Pour tout réel $\epsilon>0$ arbitrairement petit, on a, presque sûrement,*

$$\liminf_{j\to+\infty}\left\{2^{j(\underline{H}-1/\alpha+\epsilon)}\max_{k\in\nu_j(t_0)}|\widetilde{d}_{j,k}|\right\}=+\infty. \tag{3.49}$$

Preuve du Lemme 3.2. Nous désignons par ϵ un réel fixé strictement positif et arbitrairement petit, de plus nous supposons que $\delta\in]0,1/3[$ est tel que,

$$\epsilon/2=2\delta/\alpha. \tag{3.50}$$

On a alors, en utilisant (3.34), (3.39), (3.27), (3.3), (3.30), (3.28), et l'inégalité triangulaire, pour tout entier j assez grand,

$$2^{j(\underline{H}-1/\alpha+\epsilon/2)}\max_{k\in\nu_j(t_0)}|\widetilde{d}_{j,k}|\geq 2^{j(\underline{H}-1/\alpha+\epsilon)}\max_{l\in\widetilde{\lambda}_j(t_0)}|\widetilde{d}_{j,le_j}|$$
$$\geq 2^{j\frac{2\delta}{\alpha}}\max_{l\in\widetilde{\lambda}_j(t_0)}\left|2^{-j(H(le_j2^{-j})-H(t_0))}\left\{G_{j,le_j}+R_{j,le_j}\right\}\right|$$
$$\geq 2^{-c_1}2^{j\frac{2\delta}{\alpha}}\max_{l\in\widetilde{\lambda}_j(t_0)}\left|G_{j,le_j}+R_{j,le_j}\right|$$
$$\geq 2^{-c_1}2^{j\frac{2\delta}{\alpha}}\max_{l\in\widetilde{\lambda}_j(t_0)}\left|G_{j,le_j}\right|-2^{-c_1}2^{j\frac{2\delta}{\alpha}}\max_{l\in\widetilde{\lambda}_j(t_0)}\left|R_{j,le_j}\right|; \tag{3.51}$$

ainsi, (3.40) et (3.41) impliquent que

$$\liminf_{j\to+\infty}\left\{2^{j(\underline{H}-1/\alpha+\epsilon/2)}\max_{k\in\nu_j(t_0)}|\widetilde{d}_{j,k}|\right\}$$
$$\geq 2^{-c_1}\liminf_{j\to+\infty}\left\{2^{j\frac{2\delta}{\alpha}}\max_{l\in\widetilde{\lambda}_j(t_0)}\left|G_{j,le_j}\right|\right\}-2^{-c_1}\limsup_{j\to+\infty}\left\{2^{j\frac{2\delta}{\alpha}}\max_{l\in\widetilde{\lambda}_j(t_0)}\left|R_{j,le_j}\right|\right\}=2^{-c_1}>0,$$

ce qui prouve (3.49). □

Lemme 3.8. *Pour tout $\epsilon>0$ arbitrairement petit, on a, presque sûrement,*

$$\liminf_{j\to+\infty}\left\{2^{j(\underline{H}-1/\alpha+\epsilon)}\widetilde{D}_j\right\}=+\infty. \tag{3.52}$$

Preuve du Lemme 3.8. Soit $j \geq 1$ un entier, on a, d'après (3.19) et (3.30) que

$$2^{j(\underline{H}-1/\alpha+\epsilon)} \widetilde{D}_j \geq 2^{j(\underline{H}-1/\alpha+\epsilon)} \max_{k \in \nu_j(t_0)} |\widetilde{d}_{j,k}|.$$

Ainsi, il vient que

$$\liminf_{j \to +\infty} \left\{ 2^{j(\underline{H}-1/\alpha+\epsilon)} \widetilde{D}_j \right\} \geq \liminf_{j \to +\infty} \left\{ 2^{j(\underline{H}-1/\alpha+\epsilon)} \max_{k \in \nu_j(t_0)} |\widetilde{d}_{j,k}| \right\}.$$

Donc de (3.49), on obtient (3.52). □

Il convient maintenant de rappeler que dans tout le présent chapitre l'espace de probabilité sous-jacent Ω est restreint à Ω_0^*, l'évènement de probabilité qui a été introduit dans le Lemme 2.1 du chapitre 2. Le lemme suivant n'est rien d'autre que le Lemme 2.10 du chapitre 4.

Lemme 3.9. *Il existe une variable aléatoire C finie et positive telle que pour tout $\omega \in \Omega = \Omega_0^*$, tout $j \in \mathbb{Z}_+$ et tout $k \in \{0, 1, \dots, 2^j - 1\}$, on a*

$$\left| d_{j,k}(\omega) - \widetilde{d}_{j,k}(\omega) \right| \leq C(\omega) 2^{-j\rho_H}. \tag{3.53}$$

Nous sommes maintenant en mesure de prouver la relation (3.13).

Preuve de la relation (3.13). Soit ϵ un réel fixé strictement positif et arbitrairement petit. En utilisant (3.10), l'inégalité triangulaire, en utilisant (3.19), et (3.53), on a, presque sûrement, pour tout entier $j \in \mathbb{Z}_+$,

$$\begin{aligned} 2^{j(\underline{H}-1/\alpha+\epsilon)} D_j &= 2^{j(\underline{H}-1/\alpha+\epsilon)} \max_{0 \leq k < 2^j} |\widetilde{d}_{j,k} - (\widetilde{d}_{j,k} - d_{j,k})| \\ &\geq \left\{ 2^{j(\underline{H}-1/\alpha+\epsilon)} \max_{0 \leq k < 2^j} |\widetilde{d}_{j,k}| \right\} - \left\{ 2^{j(\underline{H}-1/\alpha+\epsilon)} \max_{0 \leq k < 2^j} |\widetilde{d}_{j,k} - d_{j,k}| \right\}. \\ &\geq 2^{j(\underline{H}-1/\alpha+\epsilon)} \widetilde{D}_j - C 2^{j(\underline{H}-1/\alpha+\epsilon-\rho_H)}; \end{aligned}$$

ainsi, il résulte de (3.52) et (3.4) que l'on a presque sûrement,

$$\begin{aligned} &\liminf_{j \to +\infty} \left\{ 2^{j(\underline{H}-1/\alpha+\epsilon)} D_j \right\} \\ &\geq \liminf_{j \to +\infty} \left\{ 2^{j(\underline{H}-1/\alpha+\epsilon)} \widetilde{D}_j \right\} - C \limsup_{j \to +\infty} 2^{j(\underline{H}-1/\alpha+\epsilon-\rho_H)} = +\infty - 0 = +\infty, \end{aligned}$$

ce qui prouve (3.13). □

3.2.3. *Preuve du Théorème 3.1.* (3.11) et (3.13) entraînent l'existence d'un évènement de probabilité 1, noté $\check{\Omega}_0$, tel que pour tout $\omega \in \check{\Omega}_0$ et tout $\epsilon > 0$ arbitrairement petit, il existe deux constantes finies et strictement positives notées par $A(\omega, \epsilon)$ et $B(\omega, \epsilon)$, et il existe un entier positif $j_3(\omega, \epsilon)$ vérifiant la propriété suivante : pour entier $j \geq j_3(\omega, \epsilon)$, on a,

$$A(\omega, \epsilon) 2^{-j(\underline{H}-1/\alpha+\epsilon)} \leq D_j(\omega) \leq B(\omega, \epsilon) 2^{-j(\underline{H}-1/\alpha-\epsilon)}.$$

Ceci entraîne que

$$-\underline{H} + 1/\alpha - \epsilon \leq \liminf_{j \to +\infty} \left\{ \frac{\log(D_j(\omega))}{j \log(2)} \right\} \leq \limsup_{j \to +\infty} \left\{ \frac{\log(D_j(\omega))}{j \log(2)} \right\} \leq -\underline{H} + 1/\alpha + \epsilon. \tag{3.54}$$

enfin, en faisant tendre ϵ vers 0, on obtient,

$$\lim_{j \to +\infty} \left\{ \frac{\log(D_j(\omega))}{j \log(2)} \right\} = -\underline{H} + 1/\alpha.$$

□

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