Mouvement brownien branchant avec sélection

Soutenance de thèse de Pascal MAILLARD

effectuée sous la direction de Zhan SHI

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Université Pierre et Marie Curie
11 octobre 2012
Thesis structure

Introduction + 3 chapters:

1. The number of absorbed individuals in branching Brownian motion with a barrier
2. Branching Brownian motion with selection of the $N$ right-most particles
3. A note on stable point processes occurring in branching Brownian motion
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1. The number of absorbed individuals in branching Brownian motion with a barrier
2. Branching Brownian motion with selection of the $N$ right-most particles
3. A note on stable point processes occurring in branching Brownian motion

In this presentation: Chapters 1 and 2.
Branching Brownian motion (BBM)

Definition

- A particle performs **standard Brownian motion** started at a point \( x \in \mathbb{R} \).
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Branching Brownian motion (BBM)

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- Each offspring repeats this process independently of the others.
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- Each offspring repeats this process independently of the others.

→ **A Brownian motion** indexed by a **tree**.
Introduction

Branching Brownian motion (BBM) (2)

Context

- An example of a multitype branching process (type space: $\mathbb{R}$)

\[ x \sim \exp(\beta) \]

\[ \text{time} \downarrow \]

\[ \text{position} \rightarrow \]

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Branching Brownian motion (BBM) (2)

Context

- An example of a multitype branching process (type space: $\mathbb{R}$)
- Discrete counterpart: branching random walk

$$x \sim \exp(\beta).$$
**Introduction**

**Branching Brownian motion (BBM) (2)**

**Context**

- An example of a multitype branching process (type space: \( \mathbb{R} \))
- Discrete counterpart: branching random walk
- Interpretations:
  - Model for an asexual population undergoing mutation (position = fitness)
  - Spin glass (with infinitely deep hierarchy)
  - Directed polymer on a tree
  - Prototype of a travelling wave

\[ \sim \exp(\beta) \]
We always suppose $m := \mathbb{E}[L] - 1 > 0$.

**Right-most particle**

Let $R_t$ be the position of the right-most particle. Then, as $t \to \infty$, almost surely on the event of survival,

$$\frac{R_t}{t} \to \sqrt{2\beta m}.$$
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\]

Convention
We will henceforth set 
\[ \beta = 1/(2m). \]
Let $g : \mathbb{R} \rightarrow [0, 1]$ be measurable. Define

$$u(t, x) = \mathbb{E}_x \left[ \prod_{u \in \mathcal{N}_t} g(X_u(t)) \right].$$

Then $u$ satisfies the following partial differential equation:

**Fisher–Kolmogorov–Petrovskii–Piskunov (FKPP) equation**

\[
\begin{aligned}
\partial_t u &= \frac{1}{2} \partial_x^2 u + \beta (\mathbb{E}[u^L] - u) \\
\quad u(0, x) &= g(x) \quad \text{(initial condition)}
\end{aligned}
\]

The prototype of a parabolic PDE admitting travelling wave solutions.
Two models of BBM with selection:

1. **BBM with absorption**: Let \( f(t) \) be a continuous function (the barrier). Kill an individual as soon as its position is less than \( f(t) \) (one-sided FKPP).

2. **BBM with constant population size (N-BBM)**: Fix \( N \in \mathbb{N} \). As soon as the number of individuals exceeds \( N \), kill the left-most individuals until the population size equals \( N \) (noisy FKPP).

\[ y = -x + ct \]
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Outline

1. Introduction

2. Branching Brownian motion with absorption
   - Results
   - Proof idea

3. BBM with constant population size

4. Perspectives
We take $f(t) = -x + ct$ (linear barrier).

Vast literature, known results (sample):

- almost sure extinction $\Leftrightarrow c \geq 1$
  ($c = 1$: critical case
  $c > 1$: supercritical case)

- growth rates for $c < 1$.

- asymptotics for extinction
  probability for $c = 1 - \varepsilon$, $\varepsilon$ small

We are interested in the number of absorbed individuals in the case $c \geq 1$
(question raised by D. Aldous).
Our results (critical case)

Let $Z_x$ denote the number of individuals absorbed at the line $-x + ct$.

**Theorem**

Assume that $c = 1$ and that $\mathbb{E}[L(\log L)^2] < \infty$. For each $x > 0$,

$$
\mathbb{P}(Z_x > n) \sim \frac{x e^x}{n(\log n)^2}, \quad \text{as } n \to \infty.
$$

If, furthermore, $\mathbb{E}s^L < \infty$ for some $s > 1$, then

$$
\mathbb{P}(Z_x = \delta n + 1) \sim \frac{x e^x}{\delta n^2(\log n)^2}, \quad \text{as } n \to \infty,
$$

where $\delta$ is the span of $L - 1$. 
Our results (supercritical case)

Theorem

Assume that $c > 1$ and that $\mathbb{E}[s^L] < \infty$ for some $s > 1$. Let $\lambda_c < \bar{\lambda}_c$ be the roots of the equation $\lambda^2 - 2c\lambda + 1 = 0$ and define $d = \frac{\bar{\lambda}_c}{\lambda_c}$. There exists $K = K(c, L) > 0$, such that for all $x > 0$,

$$
\mathbb{P}(Z_x = \delta n + 1) \sim \frac{K(e^{\bar{\lambda}_c x} - e^{\lambda_c x})}{n^{d+1}} \quad \text{as } n \to \infty.
$$

Aïdékon, Hu and Zindy (2012+): Similar results for branching random walk ($c \geq 1$), with more explicit $K$. 
Other studies


Aïdékon, Hu and Zindy (2012+): Similar results for branching random walk ($c \geq 1$), with more explicit $K$.

In contrast to the above papers, our proofs are entirely analytic. Strategy: derive asymptotics on the generating function of $Z_x$ near its singularity 1 (following an idea of R. Pemantle’s).
The number of absorbed individuals

Theorem (Neveu, 1988)

\((Z_x)_{x \geq 0}\) is a continuous-time Galton–Watson process. The infinitesimal generating function

\[ a(s) = \frac{d}{dx} \mathbb{E}[s^{Z_x}] \]

admits the decomposition

\[ a = -\psi' \circ \psi^{-1}, \]

where \(\psi\) is an FKPP travelling wave of speed \(c\), i.e.

\[ \frac{1}{2} \psi''(s) - c\psi'(s) + \beta(\mathbb{E}[s^{L}] - s) = 0, \]

and \(\psi(x) \uparrow 1\), as \(x \to \infty\).
Tail asymptotics $c = 1$

Follow from a Tauberian theorem and the following lemma:

Lemma

$$a''(1 - s) \sim \frac{1}{s \log^2 s}, \quad s \downarrow 0.$$
Tail asymptotics $c = 1$

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**Lemma**

$$a''(1 - s) \sim \frac{1}{s \log^2 s}, \quad s \downarrow 0.$$ 

**Proof of lemma:**

- Solve two-dimensional ODE satisfied by $(\psi', \psi)$
- Use known asymptotic: $1 - \psi(x) \sim Cxe^{-x}$ as $x \to \infty$. 
Asymptotics on density \((c \geq 1)\)

Derive asymptotics of \(a(s)\) near \(s = 1\) in the complex plane and use transfer theorems by Flajolet and Odlyzko.
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Derive asymptotics of \(a(s)\) near \(s = 1\) in the complex plane and use transfer theorems by Flajolet and Odlyzko.

To this end,

- show that \(a(s)\) can be analytically extended to a region \(\Delta(r, \varphi)\),
- analyse its asymptotic behaviour near the point \(s = 1\) inside \(\Delta(r, \varphi)\).
Asymptotics on $a(s)$ near $s = 1$

**Theorem**

For every $\varphi \in (0, \pi)$ there exists $r > 1$, such that $a(s)$ possesses an analytical extension to $\Delta(\varphi, r)$. Moreover, as $1 - s \to 1$ in $\Delta(\varphi, r)$, the following holds.

If $c = 1$, then there exists $K = K(L)$, such that

$$a(1 - s) = -s + s \log \frac{1}{s} - s \log \log \frac{1}{s} \left(\log \frac{1}{s}\right)^2 + Ks\left(\log \frac{1}{s}\right)^2 + o\left(s\left(\log \frac{1}{s}\right)^2\right).$$

If $c > 1$, then there exists $K = K(c, L) \neq 0$ and a polynomial $h(s)$, such that

$$a(1 - s) = -\lambda cs + h(s) + Ksd + o(s^d),$$

where $d \in \mathbb{N}$. If $d \in \mathbb{N}$,

$$a(1 - s) = -\lambda cs + h(s) + Ksd \log s + o(s^d).$$
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- *If $c = 1$, then $\exists K = K(L)$, such that*

$$a(1-s) = -s + \frac{s}{\log \frac{1}{s}} - s \frac{\log \log \frac{1}{s}}{(\log \frac{1}{s})^2} + \frac{Ks}{(\log \frac{1}{s})^2} + o \left( \frac{s}{(\log \frac{1}{s})^2} \right).$$
Asymptotics on $a(s)$ near $s = 1$

**Theorem**

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  $$

- If $c > 1$, then $\exists K = K(c, L) \neq 0$ and a polynomial $h(s)$, such that
  
  - if $d \notin \mathbb{N}$: $a(1 - s) = -\lambda_c s + h(s) + Ks^d + o(s^d)$,
  - if $d \in \mathbb{N}$: $a(1 - s) = -\lambda_c s + h(s) + Ks^d \log s + o(s^d)$. 

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Proof: Main idea

As before, write two-dimensional ODE satisfied by \((\psi', \psi)\) in a subset of the complex plane. Changing coordinates leads to the classic
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\[
z f'(z) = \lambda f(z) + pz + \ldots, \quad \lambda, p \in \mathbb{C}.
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The set of solutions to this equation is known explicitly.
Proof: Main idea

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The set of solutions to this equation is known explicitly.

Note. Major technical difficulty in the proofs: justifying the coordinate changes.
Outline

1. Introduction

2. Branching Brownian motion with absorption

3. BBM with constant population size
   - Introduction
   - Results

4. Perspectives
BBM with constant population size

Recall: Fix $N \in \mathbb{N}$. As soon as the number of individuals exceeds $N$, kill the left-most individuals until the population size equals $N$. Much harder than BBM with absorption:

- strong interaction between particles
- no exact description through differential equations
BBM with constant population size

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- no exact description through differential equations

Heuristic picture of $N$-BBM (BDMM 06)

- Meta-stable state: speed $c_N^{\text{det}} = \sqrt{1 - \pi^2 / \log^2 N}$, empirical measure seen from the left-most particle approximately proportional to $\sin(\pi x / \log N)e^{-x}1_{(0,\log N)}(x)$, diameter $\approx \log N$. After a time of order $\log^3 N$, a particle “breaks out” and goes far to the right (close to $N = \log N + 3 \log \log N$), spawning $O(N)$ descendants. This leads to a shift ($O(1)$) of the whole system to the right. Relaxation time of order $\log^2 N$, then process repeats.
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Real speed of the system is approximately

$$c_N = \sqrt{1 - \frac{\pi^2}{a_N^2}} = c_N^{\text{det}} + \frac{3\pi^2 \log \log N + o(1)}{\log^3 N},$$

and $O(1 / \log^3 N)$ fluctuations.
Main result

Order the individuals according to position: $X_1(t) > X_2(t) > \ldots$
Define $x_\alpha$ by $(1 + x_\alpha)e^{-x_\alpha} = \alpha$. 

Theorem
Suppose $E[L_2] < \infty$ and at time $0$, there are $N$ particles distributed independently in $(0, a_N)$ according to density proportional to $\sin(\pi x / a_N)e^{-x}$. Then, for every $\alpha \in (0, 1)$, 

$$(X_\alpha N(t \log 3 N) - c_N t \log 3 N)_{t \geq 0} \Rightarrow (L_t^\alpha)_{t \geq 0}$$

Here, $(L_t^\alpha)_{t \geq 0}$ is a (pure-jump) Lévy process with $L_0 = 0$ and Lévy measure the image of $\pi^2 x - 2 1_{x > 0} dx$ by the map $x \mapsto \log(1 + x)$. 

Proof idea: Approximate the $N$-BBM by BBM with a certain (random) absorbing barrier, called the B-BBM.
Main result

Order the individuals according to position: \( X_1(t) > X_2(t) > \ldots \)
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\[ \text{Suppose } \mathbb{E}[L^2] < \infty \text{ and at time } 0, \text{ there are } N \text{ particles distributed independently in } (0, a_N) \text{ according to density proportional to } \sin(\pi x / a_N)e^{-x}. \text{ Then, for every } \alpha \in (0, 1), \]

\[ (X_{\alpha N}(t \log^3 N) - c_N t \log^3 N) \overset{\text{fidi}}{\rightarrow} (L_t + x_\alpha)_{t \geq 0}. \]

Here, \( (L_t)_{t \geq 0} \) is a (pure-jump) \textbf{Lévy process} with \( L_0 = 0 \) and \textbf{Lévy measure the image of } \( \pi^2 x^{-2}1_{x > 0} \text{ dx by the map } x \mapsto \log(1 + x). \)
Main result

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The B-BBM

$a$: Position of a second barrier (idea from BBS (2010)).
Add drift $-c$, with $c = \sqrt{1 - \pi^2/a^2}$.
$A$: Determines number of particles ($N \approx 2\pi e^{A+a}/a^3$).
Let first $a$, then $A$ go to $\infty$.
The B-BBM

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Let first \(a\), then \(A\) go to \(\infty\).

When particle hits \(a\), it will create \(\approx WN\) descendants, where
\(\mathbb{P}(W > x) \sim x^{-1}\) (BBS (2010)).
Breakout when \(W > \varepsilon e^A\), \(\varepsilon\) small.
The B-BBM

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Breakout when \(W > \varepsilon e^A\), \(\varepsilon\) small.

After breakout, move barrier smoothly by random amount \(\Delta\).
The B-BBM (continued)

Three details:

1. Particles that hit $a$ and have few descendants are important: compensator for the limiting Lévy process.

2. B-BBM until the first breakout $= \text{spine} + \text{BBM (weakly)}$ conditioned not to hit $a$ (Doob transform of BBM).

3. Shape of barrier given by a family $(f_\Delta)$ $\Delta \geq 0$ of explicitly given, smooth, increasing functions with $f_0 = 0$ and $f_{+\infty} = \Delta$.

$\Delta \approx a^3$

$\approx 1$

$\approx a^2$
The B-BBM (continued)

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The B-BBM (continued)

Three details:

1. Particles that hit \( a \) and have few descendants are important: compensator for the limiting Lévy process.

2. B-BBM until the first breakout = spine + BBM (weakly) conditioned not to hit \( a \) (Doob transform of BBM).

3. Shape of barrier given by a family \( (f_\Delta)_{\Delta \geq 0} \) of explicitly given, smooth, increasing functions with \( f_\Delta(0) = 0 \) and \( f_\Delta(+\infty) = \Delta \).
**B-BBM ↔ N-BBM**

First idea: couple both processes.

- **black** particles: present in B-BBM and *N*-BBM,
- **red** particles: present in B-BBM but *not* in *N*-BBM,
- **blue** particles: present in *N*-BBM but *not* in B-BBM.

**Problem**

Dependencies between particles too difficult to handle.
Introduce two auxiliary particle systems: The $B^b$-BBM and the $B^\#$-BBM (stochastically) bound the $N$-BBM (and the $B$-BBM) from below and above (in the sense of stochastic order on the empirical measures).
Bounding the $N$-BBM from below: The $B^b$-BBM

Kill a particle
- whenever it hits 0 or
- whenever it has $N$ particles to its right (red particles).

$\implies$ more particles are being killed than in $N$-BBM.
Bounding the $N$-BBM from below: The $B^b$-BBM

Kill a particle
- whenever it hits 0 or
- whenever it has $N$ particles to its right (red particles).

$\implies$ more particles are being killed than in $N$-BBM.

At timescale $\log^3 N$, number of red particles stays negligible.
Bounding the $N$-BBM from above: The $B^\#$-BBM

Kill a particle whenever it (at the same time)
- hits 0 and
- has $N$ particles to its right.

A particle survives temporarily (blue particles) if it has less than $N$ particles to its right the moment it hits 0.
Outline

1. Introduction
2. Branching Brownian motion with absorption
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4. Perspectives
$N$-BBM $\leftrightarrow$ noisy FKPP

**Noisy FKPP equation**

\[
\begin{cases}
    u(t, x) : \mathbb{R}_+ \times \mathbb{R} \to [0, 1] \\
    \partial_t u = \partial_x^2 u + u(1 - u) + \sqrt{\varepsilon u(1 - u)} \dot{W} \\
    u(0, x) = 1_{(x < 0)} \quad \text{(IC)}
\end{cases}
\]
$N$-BBM $\leftrightarrow$ noisy FKPP

Noisy FKPP equation

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- Admits travelling wave solutions with same phenomenology as $N$-BBM ($N \approx \varepsilon^{-1}$), cf Mueller, Mytnik and Quastel (2010)
Perspectives

\( N\)-BBM ←→ noisy FKPP

**Noisy FKPP equation**

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\end{aligned}
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- Admits travelling wave solutions with same phenomenology as \(N\)-BBM \((N \simeq \varepsilon^{-1})\), cf Mueller, Mytnik and Quastel (2010)
- Dual to BBM with particles coalescing at rate \(\varepsilon\).
  \(\rightarrow\) density-dependent selection
Known: Empirical measure of $N$-BBM seen from the left-most particle is an ergodic Markov process.
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Open problem
Show that stationary probability converges as $N \to \infty$ to the Dirac-measure in $xe^{-x} \, dx$. 
Known: Empirical measure of $N$-BBM seen from the left-most particle is an ergodic Markov process.

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→ ongoing work with J. Berestycki and M. Jonckheere.
Varying displacement

Q: What changes if one replaces BBM by BRW (or, equivalently, by branching Lévy process)?

A: Depends on the right tail of the jump distribution.

Ongoing work joint with Jean Bérard: Consider $N$-BRW where at each time step, particles split into two and children jump according to the law of a random variable $X \geq 0$, with $P(X > x) \sim x^{-\alpha}$, $\alpha > 0$. Keep only the $N$ right-most particles at every time step. Right scaling: space by $\left( \frac{N \log N}{1/\alpha} \right)$, time by $\log N$. 
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Right scaling: space by $(N \log N)^{1/\alpha}$, time by $\log N$. 
Other open questions

- Speed of the system
- Genealogy
- Inhomogeneous media
- ...

Perspectives