Extension of the canonical trace and associated determinants

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N° d'ordre: D.U. 1964

UNIVERSITÉ BLAISE PASCAL

U.F.R Sciences et Technologies

ÉCOLE DOCTORALE DES SCIENCES FONDAMENTALES N° 617

UNIVERSITÉ DE OUAGADOUGOU

U.F.R Sciences Exactes et Appliquées

THÈSE

présentée pour obtenir le grade de

DOCTEUR D'UNIVERSITÉ

Spécialité: Mathématiques

Par **OUEDRAOGO Marie Françoise**

TITRE :

EXTENSION OF THE CANONICAL TRACE AND ASSOCIATED DETERMINANTS

Soutenue publiquement le 22 octobre 2009 devant la commission d'examen

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Remerciements

Le travail que je présente ici doit sa réalisation à un certain nombre d'institutions et de personnes que je tiens à remercier:

- L'Université de Ouagadougou (U.F.R. Sciences Exactes et Appliquées)
- L'Université Blaise Pascal de Clermont Ferrand (le Laboratoire de Mathématiques et l'École Doctorale Sciences Fondamentales)
- L'Université de Corse Pascal Paoli (Faculté des Sciences et Techniques)
- L'Agence Universitaire de la Francophonie (AUF)

Toutes ces institutions qui m'ont aidée et accompagnée tout au long de cette thèse, grâce entre autres à une bourse AUF et un poste ATER voudront bien trouver dans ces quelques lignes mes remerciements: notamment pour leur accueil, car sans elles, ce travail n'aurait pas progressé au rythme soutenu qu'il a eu.

Je tiens à remercier mes deux directeurs de thèse: Akry Koulibaly, pour avoir guidé mes premiers pas dans la recherche en m'initiant à des techniques algébriques qui ont abouti à la soutenance d'une thèse de troisième cycle à l'Université de Ouagadougou en 1999. Je lui suis reconnaissante de son soutien, sa disponibilité et ses multiples conseils, notamment pour les aspects administratifs de la préparation de cette deuxième thèse; Sylvie Paycha, qui m'a guidée sur le plan scientifique durant cette thèse. Elle m'a introduite dans l'univers des opérateurs pseudodifférentiels. Son intérêt, son enthousiasme et ses questions ont été une aide précieuse et une source de motivation pour moi. En dehors des mathématiques, elle m'a accueillie dans sa famille et m'a fait faire l'ascension du Puy de Dôme (une prouesse pour une Sahélienne comme moi!). Grâce à elle, j'ai pu participer à des écoles d'été et faire des séjours de recherche dans des instituts de recherche comme: Max Planck Institute for Mathematics (Bonn), Hausdorff Research Institute for Mathematics (Bonn), Erwin Schrödinger International Institute for Mathematical Physics (Vienne).

Je suis très sensible à l'honneur que m'ont fait Maxim Braverman et Simon Scott en acceptant d'être les rapporteurs de cette thèse. Je suis très reconnaissante à Saad Baaj, Moulay Benameur, Denis Perrot et Elmar Schrohe pour leurs présences dans le jury de cette thèse.

Je voudrais adresser un remerciement aux membres du Laboratoire de Mathématique de l'Université Blaise Pascal et plus particulièrement aux secrétaires Noelle Rougane qui a pris sa retraite, Marie-Paule Bressassouly et Valérie Sourlier qui me facilitaient la tâche dans les multiples démarches administratives et pratiques durant mes séjours.
Je remercie également mes camarades thésards passés ou présents en particulier Jacqueline et Jérémy avec qui j'ai commencé la première année de thèse et partagé le bureau 2210.

Un grand merci à tous mes collègues du département de Mathématiques de l'Université de Ouagadougou et en particulier à mes complices de tous les jours Béré, MiKaïlou et Patricia, qui me permettent d'oublier les mathématiques le temps d'une discussion (ou autre...).

Merci à tous mes collègues du département de mathématiques de l'Université de Corse pour leur acceuil chaleureux, leur disponibilité. Je remercie tout particulièrement Catherine qui n'a eu de cesse de me soutenir durant la dernière année de thèse. Je me souviendrai toujours de mes discussions avec elle et Jean-Louis sur une question mathématique (ou autre!) devant leur cheminée, entourés de leurs chats, pendant que la neige habille de blanc le paysage alentour.

Pour finir, un grand merci à toute ma famille, mes amis et connaissances et à tous ceux qui d'une façon ou d'une autre m'ont soutenue, encouragée, supportée et motivée tout au long de ce travail.

Contents

Introduction

In this thesis we study the canonical trace on certain classes of pseudodifferential operators and associated multiplicative determinants on the one hand, regularized traces on classical pseudodifferential operators and the multiplicative anomaly of related determinants such as the zeta determinant on the other hand.
The canonical trace is the unique extension [MSS] of the L^2-trace to classical pseudodifferential operators with non-integer order which vanishes on non-integer order brackets.[1]
It was introduced by M. Kontsevich and S. Vishik in [KV1], [KV2] as a tool to study properties of determinants of elliptic pseudodifferential operators. We consider pseudodifferential operators acting on smooth sections of a finite rank hermitian vector bundle E over a smooth closed riemannian manifold M of dimension n.
The L^2-trace Tr is defined on classical pseudodifferential operators of order with real part $< -n$. Naturally associated with this trace are Fredholm determinants [ReSi]

$$\det(I + A) = \exp(\mathrm{Tr}(\log(I + A)))$$

defined for operators A of order with real part $< -n$; they are multiplicative:

$$\det((I + A)(I + B)) = \det(I + A) \det(I + B).$$

Since Seeley's seminal work [Se] it is well known that the generalized zeta function $\zeta(A, Q, z) = \mathrm{Tr}(AQ^{-z})$ is holomorphic on the half plane $\mathrm{Re}(z) > \frac{n+a}{q}$, where Q is an elliptic operator with appropriate spectral properties and positive order q, and A is a classical operator of order a. The canonical trace TR provides a meromorphic extension $\zeta^{mer}(A, Q, z) = \mathrm{TR}(AQ^{-z})$ (which we denote by the same symbol $\zeta(A, Q, z)$) to the whole complex plane with simple poles. If a is not an integer or if A is a differential operator, there is no pole at $z = 0$ and $\mathrm{TR(A)} = \zeta(A, Q, 0)$ is independent of Q. In particular, $\mathrm{TR}(Q^{-z}) = \zeta(I, Q, z)$ is holomorphic at zero; its derivative at zero gives rise to the famous zeta determinant

$$\det_\zeta(Q) = \exp(-\partial_z \zeta(I, Q, z)_{z=0})$$

[1] This uniqueness result actually follows from the description of classical pseudodifferential operators in terms of brackets derived in [L] in Proposition 4.7.

introduced by D. B. Ray and I. M. Singer [RaSi] in the mathematics literature and by S. Hawkings [Haw] in the physics literature. For $Q = I + A$ with A of order with real part $< -n$, the zeta determinant coincides with the Fredholm determinant; this holds in particular for operators of the type $I + A$ with A a smoothing operator, i.e. defined by a smooth Schwartz kernel K_A via the identity

$$Au(x) = \int_M K_A(x, y)u(y)dy, \quad \forall x \in M.$$

For such an operator

$$\mathrm{Tr}(A) = \int_M \mathrm{tr}_x(K_A(x, x))dx = \int_M \int_{T_x^*M} \mathrm{tr}_x(\sigma(A)(x, \xi))\, đ\xi\, dx$$

where tr_x stands for the fibrewise trace and $\sigma(A)$ is the local symbol of A. Since the kernel of a general classical pseudodifferential operator A presents singularities along the diagonal, or equivalently since its symbol does not lie in L^1 as a function of the variable ξ, to define its trace one needs to regularize the local Schwartz kernel restricted to the diagonal $K_A(x, x) = \int_{T_x^*M} \sigma(A)(x, \xi)đ\xi$, by extracting a finite part of a divergent expression, using Hadamard finite parts. For any $x \in M$, the integral of the fibrewise trace $\mathrm{tr}_x\sigma(A)$ over the ball $B_x^*(0, R)$ of radius R in the cotangent bundle T_x^*M has an asymptotic expansion in decreasing powers of R; furthermore this integral is polynomial in $\log R$ so that the cut-off integral

$$-\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}_x\left(\sigma(A)(x, \xi)\right)\, đ\xi := \mathrm{fp}_{R\to\infty} \int_{B_x^*(0,R)} \mathrm{tr}_x\left(\sigma(A)(x, \xi)\right)\, đ\xi$$

is well defined. It coincides with the ordinary integral whenever the latter converges. Whenever the operator A has non-integer order or has order with real part $< -n$, the expression $\left(-\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}_x\left(\sigma_A(x, \xi)\right)\, đ\xi\right) dx$ defines a global density on M so that its canonical trace

$$\mathrm{TR}(A) := \int_M \left(-\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}_x\left(\sigma_A(x, \xi)\right)\, đ\xi\right) dx$$

is well defined. In particular if A has order with real part $< -n$, it coincides with $\mathrm{Tr}(A)$. M. Kontsevich and S. Vishik extended the canonical trace to odd-class classical operators in odd dimensions and G. Grubb in [Gr] to even-class operators in even dimensions. The canonical trace was actually proved to be the unique extension to a linear form on the algebra of odd-class operators in odd dimensions, which vanishes on brackets [MSS]. M. Lesch in [L] further extended the canonical trace to log-polyhomogeneous operators of non-integer order. Here, we use both extensions, to odd-class classical operators in odd dimensions and to log-polyhomogeneous operators; this last extension is useful in view of determinants which involve traces of logarithms.

The goal of this thesis is to investigate, on the grounds of a careful study of the underlying linear forms, two types of determinants, the first ones multiplicative, the others not, which both extend Fredholm determinants on operators of the type $I + A$ with A a smoothing operator:

1) The first class of determinants we consider in odd dimensions, are multiplicative determinants of the type

$$\mathrm{DET}(A) = \exp(\mathrm{TR}(\log A))$$

defined from the canonical trace for operators in the odd-class with appropriate spectral cut.

2) The second type of determinants we consider, now in any dimension, are the zeta-determinant mentioned above and the related weighted determinants

$$\mathrm{Det}^Q(A) = \exp(\mathrm{Tr}^Q(\log A))$$

for any classical pseudodifferential operator with appropriate spectral cut. Here $\mathrm{Tr}^Q(A)$ is a regularized (or Q-weighted) trace corresponding to the constant term in the Laurent expansion of the map $\mathrm{TR}(AQ^{-z})$ at $z = 0$, Q being as before an elliptic operator with positive order and appropriate spectral properties.
In the odd-dimensional case, and for A and Q in the odd-class with Q of even order, $\mathrm{Det}^Q(A) = \mathrm{DET}(A)$ so that the two types of determinants coincide. But in general, neither the zeta determinant $\det_\zeta$ nor weighted determinants Det^Q are multiplicative; in particular, the zeta determinant presents a by now well-known multiplicative anomaly first investigated simultaneously by K. Okikiolu [Ok2] and M. Kontsevich and S. Vishik [KV1].

1) Going back to the first type of determinant, let us describe our approach to multiplicative determinants in the odd-class in odd dimensions. Their classification requires classifying traces on the algebra $C\ell^0_{odd}(M, E)$ of zero order odd-class operators in odd dimensions acting on smooth sections of the bundle E. Whereas traces on the algebra of odd-class classical operators $C\ell_{odd}(M, E)$ acting on smooth sections of E in odd dimensions are proportional to the canonical trace, since $C\ell^0_{odd}(M, E)$ is a subalgebra of $C\ell_{odd}(M, E)$, we can expect to find other traces. The leading symbol traces used by S. Paycha and S. Rosenberg in [PR] and given by

$$\mathrm{Tr}^\lambda_0(A) = \lambda(\mathrm{tr}_x(\sigma_0(A))),$$

where λ is a distribution in $\mathcal{D}'(S^*M)$ indeed give rises to traces on the algebra $C\ell^0(M, E)$, which induce traces on $C\ell^0_{odd}(M, E)$. We prove the following characterization:

Theorem 3.3.4 *If the dimension of the underlying manifold M is odd, any trace on $C\ell^0_{odd}(M,E)$ is a linear combination of the canonical trace and a leading symbol trace.*

This is reminiscent of a similar result by J.-M. Lescure and S. Paycha [LP] who showed that any trace on $C\ell^0(M,E)$ is linear combination of the noncommutative residue and leading symbol trace. In order to define multiplicative determinants corresponding to these traces, following the same line of proof as in [LP] where the authors studied multiplicative determinants associated with the noncommutative residue and the leading symbol traces, we first extend the traces. Since the leading symbol trace has been taken care of in [LP], we focus here on the canonical trace. In Chapter 3, we actually extend the canonical trace to the whole algebra of odd-class log-polyhomogeneous operators in odd dimensions and prove the cyclicity of the canonical trace on this algebra (Corollary 3.5.9). In Chapter 2, we provide an alternative description of this algebra in terms of powers of the (symmetrized) logarithm of a reference elliptic odd-class operator Q (Theorem 2.2.4), which gives further insight on the operators in that class. Since $\left(C\ell^0_{odd}(M,E)\right)^*$ is a Fréchet Lie group with exponential mapping and Lie algebra $C\ell^0_{odd}(M,E)$, (Proposition 1.3.4 and Proposition 6.1.4), the above classification of traces on $C\ell^0_{odd}(M,E)$ induces a classification of multiplicative determinants given in Chapter 6:

Proposition 6.1.5 *Any multiplicative map on the range of the exponential mapping in $\left(C\ell^0_{odd}(M,E)\right)^*$ is of the form:*

$$\mathrm{Det}(A) = \exp\left(\alpha \mathrm{TR}(\mathrm{Log}(A)) + \mathrm{Tr}_0^\tau(\mathrm{Log}(A))\right),$$

where α is a real number and τ is a distribution in the cotangent unit sphere S^*M.

Leading symbol determinants studied in [LP] vanish on operators of the type $I+$ smoothing. With the idea in mind of extending Fredholm determinants on operators of the type $I+$ smoothing, we focus on determinants associated with the canonical trace of the form

$$\mathrm{DET} = \exp \circ \mathrm{TR} \circ \log$$

on zero order odd-class operators. For zero order operators, this makes sense since their logarithms also lie in the odd-class. But, for an odd-class operator with positive order, the logarithm is no longer odd-class and we introduce for this purpose the symmetrized logarithm, a notion introduced by M. Braverman in [B]. For an admissible operator A with positive order a and spectral cuts θ and $\theta - a\pi$, the symmetrized logarithm of A is

$$\log_\theta^{\mathrm{sym}} A := \frac{1}{2}\left(\log_\theta A + \log_{\theta - a\pi} A\right)$$

where $\log_\theta A$ stands for a determination of the logarithm corresponding to the choice of a spectral cut θ. Unlike the logarithm, the symmetrized logarithm of an odd-class operator is odd-class, a conservation property which enables us to define the symmetrized determinant

$$\mathrm{DET}_\theta^{\mathrm{sym}}(A) := \exp\left(\mathrm{TR}(\log_\theta^{\mathrm{sym}} A)\right).$$

For an even order operator, we have $\log_\theta^{\mathrm{sym}} A = \log_\theta A - ik\pi I$ for some integer k so that $\mathrm{DET}_\theta^{\mathrm{sym}}(A)$ reduces to the ζ-determinant:

$$\mathrm{DET}_\theta^{\mathrm{sym}}(A) = \exp(\mathrm{TR}(\log A)) = \det_\zeta(A).$$

The symmetrized determinant $\mathrm{DET}_\theta^{\mathrm{sym}}(A)$ coincides with the symmetrized determinant defined by M. Braverman [B], but the originality here is that the symmetrized trace is replaced by the canonical trace. Under suitable assumptions on the spectral cut, the symmetrized determinant is multiplicative. In Chapter 6, we derive its multiplicativity from the cyclicity of the canonical trace.

Theorem 6.3.8 *Let M be an odd-dimensional manifold. Suppose that A is an odd-class admissible operator with positive order a and spectral cuts θ and $\theta - a\pi$ and that B is an odd-class admissible operator with positive order b and spectral cuts ϕ and $\phi - b\pi$. Let us assume that for each t in $[0,1]$, $A_\theta^t B$ has principal angle $\psi(t)$, depending on the choice of θ and ϕ, where $t \to \psi(t)$ is continuous. Set $\psi(0) = \phi$ and $\psi(1) = \psi$. Then*

$$\mathrm{DET}_{\widetilde{\psi}}^{\mathrm{sym}}(AB) = \mathrm{DET}_\theta^{\mathrm{sym}}(A)\,\mathrm{DET}_\phi^{\mathrm{sym}}(B),$$

where $\widetilde{\psi}$ is an angle sufficiently close to ψ.

2) Let us now turn to the second type of determinant, namely the zeta determinant and the weighted determinant which present a multiplicative anomaly. In Chapter 5, we derive the multiplicative anomaly for zeta determinants from the multiplicative anomaly for weighted determinants. Our approach therefore differs from that of previous authors who computed the zeta determinant anomaly:

- M. Kontsevich and S. Vishik [KV2] for pseudodifferential operators with leading symbols sufficiently close to positive definite self-adjoint ones.
- M. Wodzicki [W1] for positive definite commuting elliptic differential operators,
- L. Friedlander [Fr] for positive definite elliptic pseudodifferential operators,
- K. Okikiolu [Ok2] for pseudodifferential operators with scalar leading symbols.

It is close to C. Ducourtioux's approach [Du1] who relates the two anomalies, however, without deriving one from the other as we do here. Our approach to the study of the multiplicative anomaly of the zeta determinant is essentially based on the vanishing of the noncommutative residue of an operator

$$L(A,B) = \log(AB) - \log A - \log B.$$

The multiplicative anomaly

$$\mathcal{M}^Q(A,B) = \frac{\mathrm{Det}^Q(AB)}{\mathrm{Det}^Q(A)\,\mathrm{Det}^Q(B)}$$

of a weighted determinant Det^Q reads:

$$\mathcal{M}^Q(A,B) = \exp(\mathrm{Tr}^Q(L(A,B))).$$

By results of S. Scott [Sc], the noncommutative residue $\mathrm{res}(L(A,B))$ vanishes as a consequence of the cyclicity of the noncommutative residue, leading to the multiplicativity of the residue determinant $\det_{\mathrm{res}}(A) = \exp(\mathrm{res}(\log A))$. Hence $L(A,B)$ is a finite sum of commutators as a result of which the weighted trace of $L(A,B)$ is local as a sum of noncommutative residues (see Section 4.4). Thus, the locality of the multiplicative anomaly $\mathcal{M}^Q(A,B)$ is closely related to the multiplicativity of the residue determinant.
In Chapter 4 which is dedicated to the weighted trace of $L(A,B)$, we prove an explicit local formula for $\mathrm{Tr}^Q(L(A,B))$ and hence for the logarithm of the multiplicative anomaly:

Theorem 4.5.2 *Let A and B be two admissible operators in $C\ell(M,E)$ with positive orders a and b. Assume that there is some positive ϵ such that A^tB is admissible for any $t \in]-\epsilon,\epsilon[$. Then we have*

$$\mathrm{res}(L(A,B)) = 0.$$

Moreover, there is an operator $W(\tau)(A,B) := \frac{d}{dt}_{|t=0} L(A^t, A^\tau B)$ in $C\ell^0(M,E)$ depending continuously on τ such that

$$\mathrm{Tr}^Q(L(A,B)) = \int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau+b} - \frac{\log Q}{q}\right)\right) d\tau$$

where Q is any weight of positive order q.

Here res stands for the noncommutative residue of a classical pseudodifferential operator, which is a local expression since it involves the integral over the base manifold of a finite number of homogeneous components of the symbol of the operator.
The weighted determinant Det^Q is related to the zeta determinant by a local expression

$$\frac{\det_\zeta(A)}{\mathrm{Det}^Q(A)} = \exp\left(-\frac{1}{2a}\mathrm{res}\left[\left(\log A - \frac{a}{q}\log Q\right)^2\right]\right).$$

The multiplicative anomaly

$$\mathcal{M}_\zeta(A,B) := \frac{\det_\zeta(AB)}{\det_\zeta(A)\det_\zeta(B)}$$

of the zeta determinant therefore relates to that of the weighted determinant by the local formula which involves the residue of a classical operator:

$$\log \mathcal{M}_\zeta(A,B) = \log \mathcal{M}^Q(A,B) + \mathrm{res}\left(\frac{L(A,B)\log Q}{q} - \frac{\log^2 AB}{2(a+b)} + \frac{\log^2 A}{2a} + \frac{\log^2 B}{2b}\right).$$

Since we previously argued that $\mathcal{M}^Q(A,B)$ is also local, we a priori know that $\mathcal{M}_\zeta(A,B)$ which differs from it by a residue, is local. We then infer from Theorem 5.3.2 the following explicit local formula for the zeta determinant anomaly.

Theorem 5.3.2 *Let A and B be two admissible operators in $C\ell(M,E)$ with positive orders a,b and with spectral cuts θ and ϕ in $[0,2\pi[$ such that there is a cone delimited by the rays L_θ and L_ϕ which does not intersect the spectra of the leading symbols of A, B and AB. Then the product AB is admissible with a spectral cut ψ inside that cone. A local formula of the multiplicative anomaly $\mathcal{M}_\zeta(A,B)$ reads:*

$$\begin{aligned}\log \mathcal{M}_\zeta(A,B) &= \int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau+b} - \frac{\log B}{b}\right)\right)d\tau \\ &\quad + \mathrm{res}\left(\frac{L(A,B)\,\log B}{b} - \frac{\log^2 A\,B}{2(a+b)} + \frac{\log^2 A}{2a} + \frac{\log^2 B}{2b}\right)\end{aligned}$$

and similarly replacing $\frac{\log B}{b}$ by $\frac{\log A}{a}$. When A and B commute the multiplicative anomaly reduces to:

$$\log \mathcal{M}_\zeta(A,B) = \frac{ab}{2(a+b)}\mathrm{res}\left[\left(\frac{\log A}{a} - \frac{\log B}{b}\right)^2\right].$$

This thesis is organized around six chapters, the first one of which provides prerequisites on classical pseudodifferential operators. The second chapter reviews properties of logarithms of classical pseudodifferential operators and introduces the notion of symmetrized logarithm together with its properties. Chapter 3 characterizes, in odd dimensions, the canonical trace on odd-class operators of order zero and extends this canonical trace to odd-class log-polyhomogeneous operators. Chapter 4 investigates regularized traces of the difference $L(A,B) = \log AB - \log A - \log B$ providing a local formula in terms of the noncommutative residue. Chapter 5 is devoted to regularized determinants for elliptic operators, for which an explicit formula of the multiplicative anomaly is derived on the grounds of the results in Chapter 4. On the basis of the results of Chapter 3, in Chapter 6 we classify multiplicative determinants for zero order odd-class elliptic operators in odd dimensions, and extend them to symmetrized canonical determinants for odd-class operators of positive order.

Conventions and Notations

- $S^m(U)$: the space of symbols of order m on a open subset U of $\mathbb{R}^n$
- $S^{-\infty}(U) := \bigcap_{m\in\mathbb{R}} S^m(U)$: the algebra of smoothing symbols on U
- $S(U) := \langle \bigcup_{m\in\mathbb{R}} S^m(U)\rangle$: the algebra generated by all symbols on U
- $CS^m(U)$: the space of classical symbols of order m on U
- $CS(U) := \langle \bigcup_{m\in\mathbb{C}} CS^m(U)\rangle$: the algebra generated by all classical symbols on U
- $CS^{m,k}(U)$: the set of log-polyhomogeneous symbols of order m and log degree k on U
- $CS^{\star,k}(U) = \langle \bigcup_{m\in\mathbb{C}} CS^{m,k}(U)\rangle$; $\qquad CS^{\star,\star}(U) = \bigcup_{k\in\mathbb{N}} CS^{\star,k}(U)$
- $CS^m_{odd}(U)$: the space of odd-class classical symbols of order m on U
- $CS_{odd}(U) = \bigcup_{m\in\mathbb{Z}} CS^m_{odd}(U)$: the subalgebra of odd-class classical symbols on U
- $Op(\sigma)$: a pseudodifferential operator with symbol σ
- $K_{Op(\sigma)}$: the Schwartz kernel of the pseudodifferential operator $Op(\sigma)$
- $\Gamma(M,E)$: the vector space of smooth sections of the bundle E
- $\Psi DO^a(M,E)$: the set of pseudodifferential operators of order a acting on $\Gamma(M,E)$
- $C\ell^a(M,E)$: the set of classical operators of order a
- $C\ell(M,E) := \langle \bigcup_{a\in\mathbb{C}} C\ell^a(M,E)\rangle$: the algebra generated by all classical operators
- $C\ell^{a,k}(M,E)$: the set of log-polyhomogeneous operators of order a and log degree k
- $C\ell^{\star,k}(M,E) = \bigcup_{a\in\mathbb{C}} C\ell^{a,k}(M,E)$; $\quad C\ell^{\star,\star}(M,E) = \bigcup_{k\geq 0} C\ell^{\star,k}(M,E)$
- $\sigma(A)$: a local symbol of the operator A
- $\sigma_{a-j}(A)$: the homogeneous component of degree $a-j$ of the symbol of the classical operator A

- $\sigma^L(A)$: the leading symbol of the classical operator A
- $\sigma_{a-j,l}(A)$: the homogeneous component of degree $a-j$ of the symbol of the log-polyhomogeneous operator A
- $C\ell^{a,k}_{odd}(M,E)$: the set of odd-class log-polyhomogeneous operators of order a and log degree k
- $C\ell^{\star,\star}_{odd}(M,E)$: the algebra of odd-class log-polyhomogeneous operators
- $C\ell_{odd}(M,E) = \bigcup\limits_{a\in\mathbb{Z}} C\ell^{a}_{odd}(M,E)$: the algebra of odd-class classical operators
- $\left(C\ell^{0}_{odd}(M,E)\right)^*$: the set of invertible odd-class operators of zero order
- $Sp(A)$: the spectrum of A
- A^z_θ : complex powers of the admissible operator A with spectral cut θ
- $\Pi_{\theta,\phi}(A)$: spectral projection of the admissible operator A with spectral cuts θ and ϕ
- $\log_\theta A$: logarithm of the admissible operator A with spectral cut θ
- $\log^{\mathrm{sym}}_\theta A$: symmetrized logarithm of the admissible operator A
- $\mathrm{Tr}(A)$: the L^2-trace or usual trace of A
- $\mathrm{res}(A)$: the noncommutative residue of A
- $\mathrm{TR}(A)$: the canonical trace of A
- $L(A,B) = \log(AB) - \log A - \log B$
- $\mathrm{Tr}^Q_\alpha(A)$: the Q-weighted trace of A for weight Q with spectral cut α
- $\det_{\zeta,\theta}(A)$: the ζ-determinant of A with spectral cut θ
- $\mathrm{Det}^Q_\theta(A)$: the weighted determinant of A with spectral cut θ
- $\mathcal{M}^Q(A,B)$: the multiplicative anomaly for weighted determinant of A and B
- $\mathcal{M}_\zeta(A,B)$: the multiplicative anomaly for ζ-determinant of A and B
- $\mathrm{DET}^{\mathrm{sym}}_\theta(A)$: the symmetrized determinant of A with spectral cut θ

CHAPTER 1

Chapter 1

Prerequisites on pseudodifferential operators

In this chapter we introduce various sets of symbols and corresponding pseudodifferential operators. In particular we show that the group of zero order odd-class invertible pseudodifferential operators acting on smooth sections of a vector bundle over a closed manifold, forms a Fréchet Lie group (Proposition 1.3.4) which will be used later on in this work. To do so, we use basic results in the theory of pseudodifferential operators and their symbols on closed manifolds and vectors bundles and introduce relevant definitions which we recall in this chapter using the following monographs on pseudodifferential operators [Sh], [Gi], [Di], [T].

1.1 Spaces of symbols on an open subset of $\mathbb{R}^n$

1.1.1 The algebra of symbols

Let U be an open subset of $\mathbb{R}^n$. Given a real number m, the *space of symbols* $S^m(U)$ consists of complex valued functions $\sigma(x,\xi)$ in $C^\infty(U \times \mathbb{R}^n)$ such that for any compact subset K of U and any two multiindices $\alpha = (\alpha_1, \cdots, \alpha_n)$ in $\mathbb{N}^n, \beta = (\beta_1, \cdots, \beta_n)$ in $\mathbb{N}^n$, there exists a constant $C_{K,\alpha,\beta}$ satisfying for all (x,ξ) in $K \times \mathbb{R}^n$

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{K,\alpha,\beta}(1+|\xi|)^{m-|\beta|}, \tag{1.1}$$

where $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ and $|\beta| = \beta_1 + \cdots + \beta_n$. The real m is called the *order* of the symbol σ. Notice that if $m_1 < m_2$, then $S^{m_1}(U) \subset S^{m_2}(U)$.
Let $(K_i)_{i\in I}$ be an increasing family of compacts of U verifying $U = \bigcup_{i\in I} K_i$ and for any

compact K on U, there exist $i \in I$ such that $K \subset K_i$. From the above expression we define the following family of semi-norms:

$$\sup_{x \in K_i} \sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{|\beta| - m} \parallel \partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi) \parallel$$

which makes $S^m(U)$ a Fréchet space.

The product $\star$ on symbols is defined as follows: if σ_1 lies in $S^{m_1}(U)$ and σ_2 lies in $S^{m_2}(U)$,

$$\sigma_1 \star \sigma_2(x, \xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_1(x, \xi) \partial_x^\alpha \sigma_2(x, \xi) \tag{1.2}$$

i.e. for any integer $N \geq 1$ we have

$$\sigma_1 \star \sigma_2(x, \xi) - \sum_{|\alpha| < N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_1(x, \xi) \partial_x^\alpha \sigma_2(x, \xi) \in S^{m_1 + m_2 - N}(U).$$

In particular, $\sigma_1 \star \sigma_2$ belongs to $S^{m_1+m_2}(U)$.
We denote by $S^{-\infty}(U) := \bigcap_{m \in \mathbb{R}} S^m(U)$ the algebra of smoothing symbols on U, by $S(U) := \langle \bigcup_{m \in \mathbb{R}} S^m(U) \rangle$ the algebra generated by all symbols on U. The relation $\sigma_1 \simeq \sigma_2$ defined by $\sigma_1 - \sigma_2 \in S^{-\infty}(U)$ is an equivalence relation on $S(U)$.

Example 1.1.1. *Here are two classical examples of symbols.*

1. *Smooth functions in $C^\infty(U \times \mathbb{R}^n)$ with compact support in ξ are smoothing symbols. Indeed, let $\sigma(x, \xi)$ be a smooth function in $C^\infty(U \times \mathbb{R}^n)$ with compact support in ξ. For any compact subset K of U, there exists $r_K > 0$ such that for all x in K and for all ξ, $|\xi| \geq r_K$, $\sigma(x, \xi) = 0$. Then $\sigma(x, \xi)$ is a symbol with order m for all real m.*

2. *Any smooth function $\sigma(x, \xi)$ in $C^\infty(U \times \mathbb{R}^n)$ which is positively homogeneous of degree m for $|\xi|$ large enough is a symbol of order m. Indeed, for any compact subset K of U, there exist $r_K > 0$ such that for x in K and $|\xi| \geq r_K$, $\sigma(x, t\xi) = t^m \sigma(x, \xi)$ for $t > 0$. Since $\partial_x^\alpha \partial_\xi^\beta \sigma(x, t\xi) = t^{m-|\beta|} \partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)$, then $\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi) = |\xi|^{m-|\beta|} \partial_x^\alpha \partial_\xi^\beta \sigma(x, \frac{\xi}{|\xi|})$ which by the compactness of the unit sphere S^{n-1} yields a uniform upper bound of $|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)|$ on a compact subset K of U.*
   In particular, the product $f\chi$ of a positively homogeneous function f in ξ of degree m with a function χ in $C^\infty(\mathbb{R}^n)$ which vanishes for $|\xi| \leq \frac{1}{2}$ and such that $\chi(\xi) = 1$ for $|\xi| \geq 1$, is a symbol of order m.

Example 1.1.2. *The product of any symbol $\sigma(x,\xi)$ by a Schwartz function $u(\xi)$ i.e. u belongs to*

$$\mathcal{S}(\mathbb{R}^n) := \{\varphi \in C^\infty(\mathbb{R}^n),\, \forall\, \alpha, \beta \in \mathbb{N}^n, \sup_{y\in\mathbb{R}^n} |y^\alpha \left(\partial^\beta \varphi\right)(y)| < \infty\}$$

is a smoothing symbol. Indeed, for a compact subset K of U and for any multiindices α, β in $\mathbb{N}^n$, setting $\tau(x,\xi) := \sigma(x,\xi)u(\xi)$ we have:

$$|\partial_x^\alpha \partial_\xi^\beta \tau(x,\xi)| \leq \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \left|\partial_x^\alpha \partial_\xi^{\beta-\gamma}\sigma(x,\xi)\partial_\xi^\gamma u(\xi)\right| \leq \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} C_{K,\gamma} \left|(1+|\xi|)^{m-|\beta-\gamma|}\partial_\xi^\gamma u(\xi)\right|$$

using the upper bound

$$|\partial_x^\alpha \partial_\xi^{\beta-\gamma}\sigma(x,\xi)| \leq C_{K,\gamma}(1+|\xi|)^{m-|\beta-\gamma|}$$

since $\sigma(x,\xi)$ is a symbol of order m. For any real value s we have

$$\left|(1+|\xi|)^{m-|\beta-\gamma|}\partial_\xi^\gamma u(\xi)\right| = \left|(1+|\xi|)^{m-s+|\gamma|}\partial_\xi^\gamma u(\xi)\right| (1+|\xi|)^{s-|\beta|};$$

Since u belongs to $\mathcal{S}(U)$, $\sup_{\xi\in\mathbb{R}^n} \left|(1+|\xi|)^{m-s+|\gamma|}\partial_\xi^\gamma u(\xi)\right|$ is finite. Setting

$$C := \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \sup_{\xi\in\mathbb{R}^n} \left|(1+|\xi|)^{m-s+|\gamma|}\partial_\xi^\gamma u(\xi)\right|$$

we obtain the following upper bound:

$$|\partial_x^\alpha \partial_\xi^\beta \tau(x,\xi)| \leq C(1+|\xi|)^{s-|\beta|}$$

so that $\tau(x,\xi)$ is a smoothing symbol. In particular, $\sigma(1-\chi)$, where σ lies in $\mathcal{S}(U)$ and χ is as in the above example, is a smoothing symbol.

1.1.2 Classical symbols

A symbol σ in $S^m(U)$ is called *classical* of real order m if there is an asymptotic expansion

$$\sigma(x,\xi) \sim \sum_{j=0}^\infty \chi(\xi)\, \sigma_{m-j}(x,\xi), \tag{1.3}$$

where χ is a smooth cut-off function which vanishes for $|\xi| \leq \frac{1}{2}$ and is identically one outside the unit ball $B(0,1)$, such that for any integer $N \geq 1$, we have

$$\sigma(x,\xi) - \sum_{j=0}^{N-1} \chi(\xi)\sigma_{m-j}(x,\xi) \in S^{m-N}(U).$$

Here $\sigma_{m-j}(x,\xi)$ is a positively homogeneous function in $C^\infty(U \times (\mathbb{R}^n - \{0\}))$ of degree $m-j$, i.e. for all positive real number t,

$$\sigma_{m-j}(x,t\xi) = t^{m-j}\sigma_{m-j}(x,\xi).$$

This definition is independent of the choice of the cut-off function χ which only modifies the asymptotic expansion by a smoothing symbol.
The components $\sigma_{m-j}(x,\xi)$ are uniquely determined for $|\xi| > 1$ by the following recursive formulae:

$$\sigma_m(x,\xi) = \lim_{\lambda\to\infty} \frac{\sigma(x,\lambda\xi)}{\lambda^m}$$

and for $N \geq 1$,

$$\sigma_{m-N}(x,\xi) = \lim_{\lambda\to\infty} \frac{(\sigma - \sum_{j=0}^{N-1}\sigma_{m-j})(x,\lambda\xi)}{\lambda^{m-N}}.$$

Let $CS^m(U)$ denotes the subset of $S^m(U)$ of classical symbols of order m. The subset $CS^m(U)$ can be endowed with a structure of Fréchet space equipped with the countable family of semi-norms defined as follows: for any family (K_i) of compact subsets of U such that $U = \bigcup_i K_i$, for any $j \geq 0$ and $N \geq 1$, for any multiindices α, β :

$$\begin{aligned}
&\sup_{x\in K_i}\sup_{\xi\in\mathbb{R}^n}(1+|\xi|)^{|\beta|-m} \parallel \partial_x^\alpha\partial_\xi^\beta\sigma(x,\xi) \parallel;\\
&\sup_{x\in K_i}\sup_{\xi\in\mathbb{R}^n}(1+|\xi|)^{|\beta|-m+N} \parallel \partial_x^\alpha\partial_\xi^\beta(\sigma - \sum_{j=0}^{N-1}\chi(\xi)\,\sigma_{m-j})(x,\xi) \parallel;\\
&\sup_{x\in K_i}\sup_{|\xi|=1} \parallel \partial_x^\alpha\partial_\xi^\beta\sigma_{m-j}(x,\xi) \parallel . \qquad (1.4)
\end{aligned}$$

These definitions extend to complex powers replacing the real number m by the real part of a complex number z. A symbol σ is classical of complex order z with real part $\mathrm{Re}(z) = m$ if for a cut-off function χ and if for all non negative integers j, for any integer $N \geq 1$, there are positively homogeneous functions $\sigma_{z-j}(x,\xi)$ of degree $z-j$ such that

$$\sigma(x,\xi) - \sum_{j=0}^{N-1}\chi(\xi)\sigma_{z-j}(x,\xi) \in S^{m-N}(U).$$

If a, b are two classical symbols with formal expansions

$$a \sim \sum_{j=0}^{\infty} a_{m-j}, \quad b \sim \sum_{j=0}^{\infty} b_{p-j}$$

then their star product $a \star b$ is a classical symbol of order $m+p$ with homogeneous component of degree $m+p-j$ given by:

$$(a \star b)_{m+p-j} = a_m b_{p-j} + \sum_{k+l+|\alpha|=j, l<j} \frac{1}{\alpha!} \partial_\xi^\alpha a_{m-k} D_x^\alpha b_{p-l}$$

where $D_x = -i\partial_x$. We denote by

$$CS(U) = \langle \bigcup_{m \in \mathbb{C}} CS^m(U) \rangle$$

the algebra generated by all classical symbols on U.

1.1.3 Log-polyhomogeneous symbols

Log-polyhomogeneous symbols and associated pseudodifferential operators were used by E. Schrohe [Schr] for the construction of complex powers of elliptic pseudodifferential operators in that class and were developed by M. Lesch in [L]. A log-polyhomogeneous symbol is a finite linear combination of non negative integer powers of $\log |\xi|$ with classical symbols as coefficients. More precisely, a symbol σ in $S^m(U)$ is called *log-polyhomogeneous* of real order m and log degree k (see [L], Definition 3.1) if it has an asymptotic expansion of the form (1.3) but where now

$$\sigma_{m-j}(x,\xi) = \sum_{l=0}^{k} \sigma_{m-j,l}(x,\xi) \, \log^l |\xi|, \, \forall \, (x,\xi) \in U \times \mathbb{R}^n.$$

Here k is a non negative integer and every $\sigma_{m-j,l}, l = 0, \cdots, k$ is positively homogeneous of degree $m-j$. With these conventions we have for a cut-off function χ:

$$\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \chi(\xi) \, \sigma_{m-j}(x,\xi) = \sum_{j=0}^{\infty} \sum_{l=0}^{k} \chi(\xi) \, \sigma_{m-j,l}(x,\xi) \, \log^l |\xi|, \;\; \forall \, (x,\xi) \in U \times \mathbb{R}^n. \quad (1.5)$$

We denote the set of log-polyhomogeneous symbols of order m and log degree k by $CS^{m,k}(U)$. As for classical symbols the set $CS^{m,k}(U)$ can be equipped with a Fréchet topology replacing in formula (1.4) for classical symbols, the homogeneous components by $\sigma_{m-j}(x,\xi) = \sum_{l=0}^{k} \sigma_{m-j,l}(x,\xi) \, \log^l |\xi|$ in the case of log-polyhomogeneous symbols.

Log-polyhomogeneous symbols extend to complex orders. A symbol σ is log-polyhomogeneous of complex order z with real part $\mathrm{Re}(z) = m$ and log degree k if for a cut-off function χ and for all non negative integers j, for any integer $N \geq 1$, there are positively homogeneous functions $\sigma_{z-j,l}(x,\xi)$ of degree $z-j$ such that

$$\sigma(x,\xi) - \sum_{j=0}^{N-1} \sum_{l=0}^{k} \chi(\xi)\sigma_{z-j,l}(x,\xi)\, \log^l |\xi| \in S^{\mathrm{Re}(z)-N}(U).$$

If a, b are two log-polyhomogeneous symbols with formal expansions

$$a \sim \sum_{j=0}^{\infty} a_{m-j} = \sum_{j=0}^{\infty} \sum_{l=0}^{k} a_{m-j,l}\, \log^l |\xi|, \quad b \sim \sum_{j=0}^{\infty} b_{p-j} = \sum_{j=0}^{\infty} \sum_{l'=0}^{k'} b_{p-j,l'}\, \log^{l'} |\xi|$$

then the star product $a \star b$ is a log-polyhomogeneous symbol of order $m+p$ and log degree $k+k'$ and its component of order $m+p-j$ is given by:

$$\begin{aligned}
(a \star b)_{m+p-j} &= \sum_{|\alpha|+s+t=j} \frac{1}{\alpha!}\partial_\xi^\alpha \left(\sum_{l=0}^{k} a_{m-s,l} \log^l |\xi| \right) D_x^\alpha \left(\sum_{l'=0}^{k'} b_{p-t,l'} \log^{l'} |\xi| \right) \qquad (1.6)\\
&= \sum_{|\alpha|+s+t=j} \frac{1}{\alpha!} \left(\sum_{l=0}^{k} \left(\partial_\xi^\alpha a_{m-s,l}\right) \log^l |\xi| + \sum_{l=0}^{k} a_{m-s,l} \left(\partial_\xi^\alpha \log^l |\xi|\right) \right) \\
&\quad \left(\sum_{l'=0}^{k'} (D_x^\alpha b_{p-t,l'}) \log^{l'} |\xi| \right).
\end{aligned}$$

Here as before $D_x = -i\partial_x$.

Let us set:

$$CS^{\star,k}(U) = \langle \bigcup_{m\in\mathbb{C}} CS^{m,k}(U) \rangle, \quad CS^{\star,\star}(U) = \bigcup_{k\in\mathbb{N}} CS^{\star,k}(U),$$

where as before $\langle S \rangle$ stands for the algebra generated by the set S.
Note that $CS^{\star,0}(U) = CS(U)$ i.e. a classical symbol is a particular log-polyhomogeneous symbol.
For a vector space V we set

$$CS^m(U,V) = CS^m(U) \otimes \mathrm{End}(V),\ CS^{\star,k}(U,V) = CS^{\star,k}(U) \otimes \mathrm{End}(V).$$

Here $\mathrm{End}(V)$ denotes the set of all endomorphisms of the vector space V.

Let us now define the notion of holomorphic family of log-polyhomogeneous symbols. Holomorphic families of classical symbols were first introduced by V. Guillemin [Gu2] under the name of gauged symbols and later popularized by M. Kontsevich and S. Vishik

[KV1]. The notion of holomorphic family of classical symbols was generalized in [PS] to the log-polyhomogeneous case.
Let us first recall the notion of holomorphic family for functions. A function $f : \Omega \to E$ on complex domain Ω with values in a topological vector space E is holomorphic at $z_0 \in \Omega$ if there is a vector $f'(z_0)$ in E such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right|$$

tends to zero as z tends to z_0; it is holomorphic on Ω if this holds at each point z_0 in Ω. Known results for Banach space valued holomorphic functions (see e.g. [Hi] Chapter 8) generalize to (sequentially) complete Hausdorff locally convex topological vector space valued functions, i.e. to E-valued functions with the topology of the complete Hausdorff space E defined by a family of semi-norms $\|\cdot\|_\alpha, \alpha \in \mathcal{A}$. Inductive limits of Fréchet spaces (known as LF spaces) of interest to us fall in this class of spaces.

In particular, holomorphicity implies analyticity. Starting from a holomorphic function f at $z_0 \in \Omega$ one first observes that convergence as z tends to z_0 holds uniformly on compact subsets of Ω in a neighborhood of z_0 as a result of the Cauchy formula (see [Hi] Theorem 8.1.1 in the Banach case)

$$f(z_0) = \frac{1}{2i\pi} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

where r is a positive number such that the disk centered at z_0 of radius r is contained in Ω. Thus $f : \Omega \to E$ is uniformly complex-differentiable on compact subsets in a neighborhood of z_0; by induction one shows that it is infinitely (uniformly) complex-differentiable (see e.g. [Hi] Theorem 8.1.5 in the Banach case) on (compact subsets of) Ω in a neighborhood of z_0 with derivative given by

$$f^{(k)}(z_0) = \frac{k!}{2i\pi} \int_{|z - z_0| = r} \frac{f(\zeta)}{(\zeta - z)^{k+1}} \, d\zeta \quad \forall \quad k \in \mathbb{N}.$$

It follows that (see [Hi] Theorem 8.1.6 in the Banach case)

$$\frac{\|f^{(k)}(z_0)\|_\alpha}{k!} \leq \max_{|z - z_0| = r} \frac{\|f(z)\|_\alpha}{r^k} \quad \forall k \in \mathbb{N}, \quad \forall \alpha \in \mathcal{A}. \tag{1.7}$$

For any complex number z such that $|z - z_0| < r$, we write

$$\begin{aligned} f(z) &= \frac{1}{2i\pi} \int_{|u-z_0|=r} f(u)\,(u-z)^{-1}\,du \\ &= \frac{1}{2i\pi} \int_{|u-z_0|=r} \frac{f(u)}{1-\frac{z-z_0}{u-z_0}}(u-z_0)^{-1}\,du \\ &= \frac{1}{2i\pi} \sum_{k=0}^{\infty}(z-z_0)^k \int_{|u-z_0|=r} f(u)\,(u-z_0)^{-(k+1)}\,du \end{aligned}$$

since $\left|\frac{z-z_0}{u-z_0}\right| < 1$. Thus

$$f(z) = \sum_{k=0}^{\infty} f^{(k)}(z_0)\,\frac{(z-z_0)^k}{k!}. \tag{1.8}$$

By (1.7) this series converges uniformly on any disk $|z - z_0| < r' < r$ so that f is (uniformly) analytic in (compact subsets) of a neighborhood of z_0. Note that the radius of convergence is independent of α.

This applies to the space $E := \mathcal{S}(U)$ (or more generally to $E := \mathcal{S}(U) \otimes \mathrm{End}(V)$ where V is some finite dimensional vector space) of all symbols on an open subset U of $\mathbb{R}^n$ considered here, seen as the inductive limit of the Fréchet spaces $F_\nu := \mathcal{S}^{\leq\nu}(U)$ of symbols whose order has real part non larger than $\nu \in \mathbb{R}$. The Fréchet structure on F_ν is given by the following semi-norms labelled by multiindices α, β and positive integers i (see [Hi]):

$$\sup_{x\in K_i, \xi\in \mathbb{R}^n} (1+|\xi|)^{-\nu+|\beta|}\,\|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\|,$$

where $K_i, i \in \mathbb{N}$ is a countable sequence of compact sets covering U.

Definition 1.1.3. *Let k be a non negative integer and let Ω be a domain of $\mathbb{C}$. A family $(\sigma(z))_{z\in\Omega} \subset CS^{m(z),k}(U)$ of log-polyhomogeneous symbols is holomorphic when*

1. *The function $z \to m(z)$ with $m(z)$ the order of $\sigma(z)$ is holomorphic in z.*

2. *For (x, ξ) in $U \times \mathbb{R}^n$, the function $z \to \sigma(z)(x,\xi) := \sigma(z, x, \xi)$ is holomorphic as a function in $C^\infty(\Omega \times U \times \mathbb{R}^n)$ and for each z in Ω,*

$$\sigma(z)(x,\xi) \sim \sum_{j=0}^{\infty} \chi(\xi)\sigma_{m(z)-j}(z)(x,\xi)$$

lies in $CS^{m(z),k}(U)$ for some cut-off function χ.

3. For any integer $N \geq 1$, the remainder term

$$\sigma_N(z)(x,\xi) := \sigma(z)(x,\xi) - \sum_{j=0}^{N-1} \chi(\xi)\sigma_{m(z)-j}(z)(x,\xi)$$

is holomorphic in $z \in \Omega$ as an element of $C^\infty(\Omega \times U \times \mathbb{R}^n)$ and its l-th derivative

$$\sigma_N^{(l)}(z)(x,\xi) := \partial_z^l \left(\sigma_N(z)(x,\xi)\right)$$

lies in $S^{\mathrm{Re}(m(z))-N+\epsilon}(U)$ for all $\epsilon > 0$ locally uniformly on Ω, i.e the l-th derivative $\partial_z^k \sigma_{(N)}(z)$ satisfies a uniform estimate (1.1) w.r. to z on compact subsets in Ω.

In particular, for any integer $j \geq 0$, the (positively) homogeneous component $\sigma_{m(z)-j}(z)$ of degree $m(z)-j$ of the symbol is holomorphic on Ω as an element of $C^\infty(\Omega \times U \times \mathbb{R}^n)$.

Lemma 1.1.4. *The derivative of a holomorphic family of log-polyhomogeneous symbols $\sigma(z)$ in $CS^{m(z),k}(U)$ defines a holomorphic family of log-polyhomogeneous symbols in $CS^{m(z),k+1}(U)$.*

Proof: (see [PS]) For a holomorphic family $\sigma(z)$ in $CS^{m(z),k}(U)$ of log-polyhomogeneous symbols, we have

$$\sigma(z)(x,\xi) \sim \sum_{j=0}^{\infty} \chi(\xi)\sigma_{m(z)-j}(z)(x,\xi) = \sum_{j=0}^{\infty}\sum_{l=0}^{k} \chi(\xi)\, \sigma_{m(z)-j,l}(z)(x,\xi)\, \log^l|\xi|.$$

We want to show that

$$\partial_z\left(\sigma(z)(x,\xi)\right) \sim \sum_{j=0}^{\infty} \partial_z\left(\sigma_{m(z)-j}(z)\right).$$

Indeed we have

$$\sigma(z)(x,\xi) = \sum_{j=0}^{N-1} \sigma_{m(z)-j}(z)(x,\xi) + \sigma_N(z)(x,\xi).$$

By definition 1.1.3,

$$\sigma_N'(z)(x,\xi) := \partial_z\left(\sigma_N(z)(x,\xi)\right) = \partial_z\left(\sigma(z)(x,\xi)\right) - \sum_{j=0}^{N-1} \chi(\xi)\partial_z\left(\sigma(z)_{m(z)-j}(x,\xi)\right).$$

Taking different values of N shows that each term $\chi(\xi)\sigma(z)_{m(z)-j}(x,\xi)$ which lies in $S^{m(z)-j}(U)$ is holomorphic and hence that

$$\partial_z\left(\sigma(z)(x,\xi)\right) \sim \sum_{j=0}^{\infty} \chi(\xi)\partial_z\left(\sigma_{m(z)-j}(z)\right).$$

To evaluate the derivative $\partial_z\left(\sigma(z)_{m(z)-j}(x,\xi)\right)$, we compute the derivative of each homogeneous component $\partial_z\left(\sigma(z)_{m(z)-j,l}(x,\xi)\right)$. Using the positive homogeneity of the component $\sigma(z)_{m(z)-j,l}(x,\xi)$, we have for $|\xi|\neq 0$

$$\begin{aligned}
& \partial_z\left(\sigma(z)_{m(z)-j,l}(x,\xi)\right) \\
= \; & \partial_z\left(|\xi|^{m(z)-j}\sigma(z)_{m(z)-j,l}(x,\frac{\xi}{|\xi|})\right) \\
= \; & \left(m'(z)|\xi|^{m(z)-j}\sigma(z)_{m(z)-j,l}(x,\frac{\xi}{|\xi|})\right)\log|\xi| + |\xi|^{m(z)-j}\partial_z\left(\sigma(z)_{m(z)-j,l}(x,\frac{\xi}{|\xi|})\right) \\
= \; & m'(z)\sigma(z)_{m(z)-j,l}(x,\xi)\log|\xi| + |\xi|^{m(z)-j}\partial_z\left(\sigma(z)_{m(z)-j,l}(x,\frac{\xi}{|\xi|})\right).
\end{aligned}$$

Since $\sigma(z)_{m(z)-j,l}(x,\frac{\xi}{|\xi|})$ is a symbol of constant order zero, so is its derivative. Hence $\partial_z\left(\sigma(z)_{m(z)-j,l}(x,\xi)\right)$ lies in $CS^{m(z),k+1}(U)$ and

$$\begin{aligned}
& \partial_z\left(\sigma_{m(z)-j}(z)(x,\xi)\right) \\
= \; & \sum_{l=0}^{k}\partial_z\left(\sigma_{m(z)-j,l}(z)(x,\xi)\ \log^l|\xi|\right) \\
= \; & \sum_{l=0}^{k}\left(m'(z)\sigma(z)_{m(z)-j,l}(x,\xi)\log|\xi| + |\xi|^{m(z)-j}\partial_z\left(\sigma(z)_{m(z)-j,l}(x,\frac{\xi}{|\xi|})\right)\right)\log^l|\xi| \\
= \; & m'(z)\sigma_{m(z)-j}(z)(x,\xi)\log|\xi| + \sum_{l=0}^{k}|\xi|^{m(z)-j}\partial_z\left(\sigma(z)_{m(z)-j,l}(x,\frac{\xi}{|\xi|})\right)\log^l|\xi|.
\end{aligned}$$

□

Thus, differentiating w.r. to z introduces a logarithmic term to each term $\sigma_{m(z)-j}(z)(x,\xi)$. In particular, if $\sigma(z)$ is a holomorphic family of classical symbols then $\partial_z(\sigma(z)_{m(z)-j})$ is not classical any more since it involves a $\log|\xi|$ term. It follows that the derivative of a holomorphic family $\sigma(z)$ of classical symbols yields a holomorphic family of symbols $\sigma'(z) := \partial_z\sigma(z)$ of order $m(z)$, whose asymptotic expansion involves a logarithmic term and reads:

$$\sigma'(z)(x,\xi)\sim\sum_{j=0}^{\infty}\chi(\xi)\left(\log|\xi|\,m'(z)\,\sigma_{m(z)-j}(z)(x,\xi)+\sigma'_{m-j}(z)(x,\xi)\right),\ \forall\,(x,\xi)\in U\times\mathbb{R}^n$$

for some smooth cut-off function χ and some positively homogeneous symbol

$$\sigma'_{m(z)-j}(z)(x,\xi)=\partial_z(\sigma(z)_{m(z)-j}(x,\xi))=|\xi|^{\alpha(z)-j}\,\partial_z\left(\sigma_{m(z)-j}(z)(x,\frac{\xi}{|\xi|})\right)$$

of degree $m(z)-j$.

Iterating the lemma leads to the following proposition.

Proposition 1.1.5 ([PS]). *If $\sigma(z)$ is a holomorphic family of log-polyhomogeneous symbols, then so is each derivative*

$$\sigma^{(l)}(z)(x,\xi) := \partial_z^l(\sigma(z)(x,\xi)) \in CS^{m(z),k+l}(U)$$

Precisely, $\sigma^{(l)}(z)(x,\xi)$ has an asymptotic expansion

$$\sigma^{(l)}(z)(x,\xi) \sim \sum_{j\geq 0} \sigma^{(l)}(z)_{m(z)-j}(x,\xi)$$

where

$$\sigma^{(l)}(z)_{m(z)-j}(x,\xi) = \partial_z^l\left(\sigma(z)_{m(z)-j}(x,\xi)\right).$$

1.1.4 Odd-class symbols

The following definition is a straightforward extension of the notion of classical odd and even class operator introduced by M. Kontsevich and S. Vishik in [KV1].

Definition 1.1.6 ([PS]). *A log-polyhomogeneous symbol σ in $CS^{m,k}(U)$ with integer order m is said to be odd-class if in the asymptotic expansion (1.5), for each $j \geq 0$, for $l = 0, \cdots, k$ we have*

$$\sigma_{m-j,l}(x,-\xi) = (-1)^{m-j}\sigma_{m-j,l}(x,\xi), \quad \text{for} \quad |\xi| \geq 1 \tag{1.9}$$

and is said to be even class if for each $j \geq 0$ we have

$$\sigma_{m-j,l}(x,-\xi) = (-1)^{m-j+1}\sigma_{m-j,l}(x,\xi), \text{ for } \quad |\xi| \geq 1.$$

Example 1.1.7.

1. *All polynomial symbols of the form $p(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$ are odd-class symbols.*

2. *The symbol $\sigma(x,\xi) = \sigma(\xi) = (1+|\xi|^2)^{-1}$ is a classical odd-class symbol. Indeed it has the asymptotic expansion*

$$\sigma(x,\xi) \sim \sum_{k=0}^{\infty}(-1)^k|\xi|^{-2k-2}$$

and is a classical symbol of order -2 *with positively homogeneous components:*

$$\sigma_{-2-j}(x,\xi) = \begin{cases} (-1)^{\frac{j}{2}}|\xi|^{-2-j} & \text{if } j \text{ even} \\ 0 & \text{if } j \text{ odd} \end{cases}$$

$$\text{i.e.} \quad \sigma_{-2-j}(x,-\xi) = (-1)^{-2-j}\sigma_{-2-j}(x,\xi).$$

3. *The symbol* $\sigma(x,\xi) = \sigma(\xi) = \sqrt{1+|\xi|^2}$ *is a classical even-class symbol. Indeed this symbol has the asymptotic expansion*

$$\sigma(x,\xi) \sim \sum_{k=0}^{\infty} \alpha_k |\xi|^{-2k+1}.$$

It is therefore a classical symbol of order 1 *with positively homogeneous components:*

$$\sigma_{1-j}(x,\xi) = \begin{cases} \alpha_{\frac{j}{2}}|\xi|^{1-j} & \text{if } j \text{ even} \\ 0 & \text{if } j \text{ odd.} \end{cases}$$

Hence $\sigma_{1-j}(x,-\xi) = \alpha_{\frac{j}{2}}|\xi|^{1-j}$ *if* j *is even and* 0 *otherwise,*

$$\text{i.e.} \quad \sigma_{1-j}(x,-\xi) = \sigma_{1-j}(x,\xi) = (-1)^{2-j}\sigma_{1-j}(x,\xi).$$

Lemma 1.1.8.

1. *The star product defined in (1.2) of two log-polyhomogeneous odd-class symbols of log type* k *and* k' *is an odd-class symbol of log type* $k+k'$.

2. *The star product of two log-polyhomogeneous even-class symbols of log type* k *and* k' *is an odd-class symbol of log type* $k+k'$.

3. *The star product of a log-polyhomogeneous odd-class symbol of log type* k *by a log-polyhomogeneous even-class symbol of log type* k' *is a log-polyhomogeneous even-class symbol of log type* $k+k'$.

Proof: Assume that a, b are two log-polyhomogeneous symbols with formal expansions

$$a \sim \sum_{j=0}^{\infty} a_{m-j} = \sum_{j=0}^{\infty}\sum_{l=0}^{k} a_{m-j,l} \log^l|\xi|, \quad b \sim \sum_{j=0}^{\infty} b_{p-j} = \sum_{j=0}^{\infty}\sum_{l'=0}^{k'} b_{p-j,l'} \log^{l'}|\xi|.$$

Using the notations of (1.6), for two log-polyhomogeneous symbols a and b, we have

$$\begin{aligned}&(a\star b)_{m+p-j}\\ =\ &\sum_{|\alpha|+s+t=j}\frac{1}{\alpha!}\left(\sum_{l=0}^{k}(\partial_\xi^\alpha a_{m-s,l})\log^l|\xi|+\sum_{l=0}^{k}a_{m-s,l}(\partial_\xi^\alpha\log^l|\xi|)\right)\left(\sum_{l'=0}^{k'}(D_x^\alpha b_{p-t,l'})\log^{l'}|\xi|\right).\end{aligned}$$

Hence its homogeneous components are

$$\begin{aligned}&(a\star b)_{m+p-j,l}\\ =\ &\sum_{|\alpha|+s+t=j}\frac{1}{\alpha!}\sum_{i=0}^{l}\Big((\partial_\xi^\alpha a_{m-s,i})\log^i|\xi|+a_{m-s,|\alpha|+i}(\partial_\xi^\alpha\log^{|\alpha|+i}|\xi|)\Big)\left(D_x^\alpha(b_{p-t,l-i})\log^{l-i}|\xi|\right).\end{aligned}$$

Note that if $f(-\xi)=(-1)^a f(\xi),\quad\forall\xi\in\mathbb{R}-\{0\}$ for some smooth function f on $\mathbb{R}-\{0\}$ and some integer a then, for any multiindex α,

$$(\partial^\alpha f)(-\xi)=(-1)^{a+|\alpha|}\partial^\alpha f(\xi)\quad\forall\xi\in\mathbb{R}-\{0\}.\tag{1.10}$$

1. If a and b are odd-class symbols then by definition

$$a_{m-j,l}(x,-\xi)=(-1)^{m-j}a_{m-j,l}(x,\xi)\quad\text{and}\quad b_{p-j,l'}(x,-\xi)=(-1)^{p-j}b_{p-j,l'}(x,\xi).$$

Notice that since $\partial_\xi^\alpha a_{m-j,l}(x,-\xi)=(-1)^{|\alpha|}(\partial_\xi^\alpha a_{m-j,l})(x,-\xi)$, then by (1.10)

$$(\partial_\xi^\alpha a_{m-j,l})(x,-\xi)=(-1)^{m-j-|\alpha|}\partial_\xi^\alpha a_{m-j,l}(x,\xi).$$

On the other hand, applying (1.10) to $f(\xi)=\log|\xi|$ in which case $a=0$ we have:

$$\partial_\xi^\alpha\log^l|-\xi|=(-1)^{|\alpha|}(\partial_\xi^\alpha\log^l)|\xi|.$$

It follows that

$$\begin{aligned}&(a\star b)_{m+p-j,l}(x,-\xi)\\ =\ &\sum_{|\alpha|+s+t=j}\frac{1}{\alpha!}\sum_{i=0}^{l}\Big((\partial_\xi^\alpha a_{m-s,i})(x,-\xi)\log^i|\xi|+a_{m-s,|\alpha|+i}(x,-\xi)(\partial_\xi^\alpha\log^{|\alpha|+i})|-\xi|\Big)\\ &\Big((D_x^\alpha b_{p-t,l-i})(x,-\xi)\log^{l-i}|\xi|\Big)\\ =\ &(-1)^{m+p-j}(a\star b)_{m+p-j,l}(x,\xi)\end{aligned}$$

so that $a\star b$ is odd-class.

2. If a and b are even-class symbols then

$$a_{m-j,l}(x,-\xi) = (-1)^{m-j+1} a_{m-j,l}(x,\xi) \quad \text{and} \quad b_{p-j,l'}(x,-\xi) = (-1)^{p-j+1} b_{p-j,l'}(x,\xi).$$

Hence

$$(a \star b)_{m+p-j,l}(x,-\xi) = (-1)^{m+p-j} (a \star b)_{m+p-j,l}(x,\xi)$$

so that $a \star b$ is odd-class.

3. If a is odd-class and b is even-class then

$$a_{m-j,l}(x,-\xi) = (-1)^{m-j} a_{m-j,l}(x,\xi) \quad \text{and} \quad b_{p-j,l'}(x,-\xi) = (-1)^{p-j+1} b_{p-j,l'}(x,\xi).$$

Hence

$$(a \star b)_{m+p-j,l}(x,-\xi) = (-1)^{m+p-j+1} (a \star b)_{m+p-j,l}(x,\xi)$$

so that $a \star b$ is even-class.

□

Let us now consider the case of classical symbols: we call $CS^m_{odd}(U)$ the set of odd-class symbols of integer order m and we set

$$CS_{odd}(U) = \bigcup_{m\in\mathbb{Z}} CS^m_{odd}(U).$$

By Lemma 1.1.8 applied to $k = k' = 0$, the product of two classical odd-class symbols is a classical odd-class symbol. Thus $CS_{odd}(U)$ equipped with the star product (1.2) is an algebra.

1.2 Pseudodifferential operators

1.2.1 Pseudodifferential operators on an open subset of $\mathbb{R}^n$

Let U be an open subset of $\mathbb{R}^n$. To a symbol σ in $S(U)$, we can associate the continuous operator $Op(\sigma) : C^\infty_c(U) \to C^\infty(U)$ defined for u in $C^\infty_c(U)$ by

$$(Op(\sigma)u)(x) = \int_{\mathbb{R}^n} e^{ix.\xi} \sigma(x,\xi) \widehat{u}(\xi) đ\xi.$$

Here $C^\infty(U)$ (resp. $C^\infty_c(U)$) denotes the space of smooth (resp. compactly supported) complex valued functions on U, $đ\xi := \frac{1}{(2\pi)^n}\, d\xi$ with $d\xi$ the ordinary Lebesgue measure on $\mathbb{R}^n$ and

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-iy.\xi} u(y) dy$$

is the Fourier transform of u. Indeed let β be a multiindex and let u be in $C_c^\infty(U)$. We want to show that $\partial_x^\beta Op(\sigma)u$ exists i.e. $Op(\sigma)u$ belongs to $C^\infty(U)$. Let us fix x_0 in U and K a compact neighborhood of x_0. Let m be the order of the symbol σ. For any fixed $\xi \in \mathbb{R}^n$, the function $x \mapsto e^{ix.\xi}\sigma(x,\xi)\widehat{u}(\xi)$ is smooth as a product of smooth functions. By the properties of the Fourier transform $\widehat{u}$, for N in $\mathbb{N}$ with $N > m+n+1$ there exists a real constant C_1 such that

$$|\widehat{u}(\xi)| < C_1(1+|\xi|)^{-N}$$

for any ξ in $\mathbb{R}^n$. On the other hand, there exists a real constant C_2 such that

$$|\partial_x^\beta \sigma(x,\xi)| < C_2(1+|\xi|)^m$$

for any ξ in $\mathbb{R}^n$ and any x in K. It follows that

$$|\partial_x^\beta \sigma(x,\xi)\widehat{u}(\xi)| < C_1C_2(1+|\xi|)^{m-N} \leq C_1C_2(1+|\xi|)^{-n-1}.$$

Since $|\partial_x^\beta \sigma(x,\xi)\widehat{u}(\xi)|$ is bounded from above by the L^1-function $\xi \mapsto C_1C_2(1+|\xi|)^{-n-1}$, by Lebesgue's dominated convergence theorem, the derivative $\partial_x^\beta(Op(\sigma)u)$ exists and for x in a neighborhood of x_0,

$$\partial_x^\beta(Op(\sigma)u)(x) = \int_{\mathbb{R}^n} \partial_x^\beta \left(e^{ix.\xi}\sigma(x,\xi)\widehat{u}(\xi)\right) đ\xi.$$

Using the expression of the Fourier transform of u, we can write $Op(\sigma)$ as an operator with kernel. Indeed,

$$(Op(\sigma)u)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y).\xi}\sigma(x,\xi)u(y)dyđ\xi = \int_{\mathbb{R}^n} K_{Op(\sigma)}(x,y)u(y)dy,$$

where

$$K_{Op(\sigma)}(x,y) = \int_{\mathbb{R}^n} e^{i(x-y).\xi}\sigma(x,\xi)đ\xi.$$

$Op(\sigma)$ is called a *pseudodifferential operator* (ΨDO) on U with Schwartz kernel $K_{Op(\sigma)}(x,y)$, which is a distribution on $U \times U$ smooth off the diagonal.
For a multiindex α, let us compute $(x-y)^\alpha K_{Op(\sigma)}(x,y)$. Integrating by parts and using the fact that

$$(x-y)^\alpha e^{i(x-y).\xi} = \partial_\xi^\alpha e^{i(x-y).\xi}$$

we have

$$(x-y)^\alpha K_{Op(\sigma)}(x,y) = \int_{\mathbb{R}^n}(x-y)^\alpha e^{i(x-y).\xi}\sigma(x,\xi)đ\xi = \int_{\mathbb{R}^n} e^{i(x-y).\xi}\partial_\xi^\alpha\sigma(x,\xi)đ\xi.$$

This integral is absolutely convergent for $|\alpha| > m+n$. Similarly, for $|\alpha| > m+n+|\beta|$, we can permute ∂_ξ^β with the integral since we have an estimate

$$|\partial_\xi^\beta\partial_\xi^\alpha\sigma(x,\xi)| \leq C(1+|\xi|)^{m-|\alpha|-|\beta|}.$$

We infer that the map $(x, y) \mapsto (x-y)^\alpha K_{Op(\sigma)}(x, y)$ lies in $C^j(U \times U)$ for all $j < |\alpha| - m - n$ and in particular that $K_{Op(\sigma)}(x, y)$ is smooth off the diagonal.

The order of the pseudodifferential operator $Op(\sigma)$ is the order of the symbol σ.

From now on, the symbol of a pseudodifferential operator A is denoted by $\sigma(A)$ and its order by a.

If $\sigma(A)$ is a classical symbol of order a with the asymptotic expansion (1.3), then A is called a classical ΨDO of order a on U. The first homogeneous component in the asymptotic expansion of the symbol $\sigma(A)$ of A not identically equal to zero is called *leading symbol* of the operator A. We denote it by $\sigma^L(A)$.

If $\sigma(A)$ is a log-polyhomogeneous symbol of order a and log degree k, then A is called log-polyhomogeneous ΨDO of order a and log degree k.

If $\sigma(A)$ is a smoothing symbol, A is called a smoothing operator on U. This is equivalent to say that A has a smooth Schwartz kernel.

Example 1.2.1. *An important example which can be seen as a motivation to introduce the notion of ΨDOs is that of polynomials $\sigma(x, \xi) = p(x, \xi)$. Here*

$$p(x, \xi) = \sum_{|\alpha| \leq a} p_\alpha(x) \xi^\alpha$$

is a polynomial with respect to ξ of degree a and coefficients $p_\alpha(x)$ in $C^\infty(\mathbb{R}^n)$. The corresponding linear operator $Op(\sigma)$, namely a differential operator, is defined on a function u in $C_c^\infty(\mathbb{R}^n)$ by

$$(Op(\sigma)u)(x) = \int_{\mathbb{R}^n} e^{ix.\xi} \sigma(x, \xi) \widehat{u}(\xi) đ\xi = \int_{\mathbb{R}^n} e^{ix.\xi} \sum_{|\alpha| \leq a} p_\alpha(x) \xi^\alpha \widehat{u}(\xi) đ\xi.$$

Clearly, the space $C_c^\infty(\mathbb{R}^n)$ is contained in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and for u in $C_c^\infty(\mathbb{R}^n)$, $D^\alpha u$ belongs to $\mathcal{S}(\mathbb{R}^n)$, for all α. Using the properties of the Fourier transform

$$\widehat{D_x^\alpha u}(\xi) = \xi^\alpha \widehat{u}(\xi)$$

and the Fourier inversion formula, we obtain

$$(Op(\sigma)u)(x) = \int_{\mathbb{R}^n} e^{ix.\xi} \sum_{|\alpha| \leq a} p_\alpha(x) \widehat{D_x^\alpha u}(\xi) đ\xi = \sum_{|\alpha| \leq a} p_\alpha(x) D_x^\alpha u(\xi).$$

Example 1.2.2. *Let f be a smooth function in $C^\infty(U)$. The multiplication operator $u \mapsto fu$ on smooth functions u of U is a zero order classical ΨDO on U. Indeed, it reads $P : u \mapsto fu$ where $(fu)(x) = \int_{\mathbb{R}^n} e^{ix.\xi}\sigma(x,\xi)\widehat{u}(\xi)đ\xi$ with $\sigma(x,\xi) = f(x)$.*

A direct computation shows that the composition of two differential operators $P = \sum_{|\alpha|\leq p} a_\alpha(x) D_x^\alpha$ and $Q = \sum_{|\beta|\leq q} b_\beta(x) D_x^\beta$ with symbols $\sigma(P)$ and $\sigma(Q)$ has symbol $\sigma(P)\star\sigma(Q)$. This follows from Leibniz's rule which yields:

$$D_x^\alpha(fg) = \sum_{\gamma+\mu=\alpha} \frac{\alpha!}{\beta!\mu!} D_x^\gamma(f)\, D_x^\mu(g), \quad \partial_\xi^\beta(\xi^{\beta+\gamma}) = \frac{(\beta+\gamma)!}{\gamma!}\xi^\gamma$$

from which we infer that for any u in $C_c^\infty(\mathbb{R}^n)$:

$$\begin{aligned} ((PQ)u)(x) &= \sum_{|\alpha|\leq p} \sum_{|\beta|\leq q} a_\alpha(x) D_x^\alpha \left(b_\beta(x) D_x^\beta u(x)\right) \\ &= \sum_{|\alpha|\leq p} \sum_{|\beta|\leq q} a_\alpha(x) \sum_{\gamma+\mu=\alpha} \frac{\alpha!}{\beta!\mu!} (D_x^\gamma b_\beta(x)) D_x^{\beta+\mu} u(x). \end{aligned}$$

The symbol of the product is

$$\begin{aligned} \sigma(x,\xi) &= \sum_{|\alpha|\leq p} \sum_{|\beta|\leq q} a_\alpha(x) \sum_{\gamma+\mu=\alpha} \frac{\alpha!}{\beta!\mu!} (D_x^\gamma b_\beta(x)) \xi^{\beta+\mu} \\ &= \sum_{|\alpha|\leq p} \sum_{|\beta|\leq q} a_\alpha(x) \sum_{\gamma+\mu=\alpha} \frac{\alpha!}{\beta!\mu!} (D_x^\gamma b_\beta(x)) \xi^{\beta+\alpha-\gamma} \\ &= \sum_{|\alpha|\leq p} \sum_{|\beta|\leq q} a_\alpha(x) \sum_{\gamma} \frac{1}{\gamma!} (D_x^\gamma b_\beta(x)) \xi^\beta \partial_\xi^\gamma \xi^\alpha \\ &= \sum_{\gamma} \frac{1}{\gamma} \partial_\xi^\gamma \sigma(P)(x,\xi) D_x^\gamma \sigma(Q)(x,\xi). \end{aligned}$$

It follows that the symbol of the product PQ is $\sigma(P) \star \sigma(Q)$.

Composing general ΨDOs is not always possible; however it is for properly supported operators. Also, the formulae obtained for differential operators remain true for ΨDOs up to a smoothing operator and up to the fact that the sum becomes infinite.

The following definitions and properties are contained in [Sh], Section 3.

Definition 1.2.3. *A continuous map $f : X \to Y$ between topological spaces is called proper if for any compact $K \subset Y$ the inverse image $f^{-1}(K)$ is a compact in X.*

Definition 1.2.4. *Let A be a ΨDO on U and let* $\mathrm{Supp}(K_A)$ *be the support of the kernel K_A of A. The ΨDO A is called properly supported if both the canonical projections $\pi_1, \pi_2 :$* $\mathrm{Supp}(K_A) \to U$ *are proper maps.*

Proposition 1.2.5.

1. *Let A be a properly supported ΨDO on U. Then A define a continuous map*

$$A : C_c^\infty(U) \to C_c^\infty(U).$$

2. *Any ΨDO A on U can be written in the form $A = A_0 + A_1$ where A_0 is a properly supported ΨDO on U and A_1 is a smoothing operator on U i.e. has kernel K_{A_1} in $C^\infty(U \times U)$.*

A properly supported ΨDO on U admits a symbol given by $\sigma(A)(x,\xi) = e^{-ix.\xi} A e^{ix.\xi}$. Since any ΨDO A can be written as a sum $A_0 + A_1$ with A_0 properly supported and A_1 smoothing, one defines the symbol $\sigma(A)$ of A to be that of A_0. Hence, with the notations of Proposition 1.2.5 we have

$$A = Op(\sigma(A)) + A_1. \tag{1.11}$$

The product on symbols induces a composition on properly supported operators. The composition AB of two properly supported ΨDOs A and B on U is a properly supported ΨDO on U and its symbol is $\sigma(AB) = \sigma(A) \star \sigma(B)$. Moreover if A is a ΨDO on U and R is a smoothing operator on U then the products AR and RA are both smoothing operators on U. It follows that the product on properly supported operators extends to ΨDOs. Furthermore the sets of ΨDOs, classical ΨDOs, log-polyhomogeneous ΨDOs are algebras.

Change of variables on pseudodifferential operators:
Let U_1, U_2 be two open subsets of $\mathbb{R}^n$ and $\Phi : U_1 \to U_2$ a diffeomorphism, with $\Phi^\star : C^\infty(U_2) \to C^\infty(U_1)$ the induced diffeomorphism i.e. $\Phi^\star$ is defined by $\Phi^\star f(x) = f(\Phi(x))$. For any ΨDO A on U_1 we define a linear operator $\Phi^\sharp A : C_c^\infty(U_2) \to C^\infty(U_2)$ by the following commutative diagram:

$$\begin{tikzcd}
C_c^\infty(U_2) \arrow[r, "\Phi^\sharp A"] \arrow[d, "\Phi^\star"'] & C^\infty(U_2) \arrow[d, "\Phi^\star"] \\
C_c^\infty(U_1) \arrow[r, "A"] & C^\infty(U_1)
\end{tikzcd}$$

If u belongs to $C_c^\infty(U_2)$ then

$$(\Phi^\sharp A)u = A(u \circ \Phi) \circ \Phi^{-1}$$

so that $\Phi^\sharp A$ is a pseudodifferential operator on U_2. A symbol of $\Phi^\sharp A$ is given by

$$\sigma(\Phi^\sharp A)(y,\eta)_{|_{y=\Phi(x)}} \sim \sum_\alpha \frac{1}{\alpha!} D_\xi^\alpha \sigma(A)(x,{}^t\,\Phi'(x)\eta).D_z^\alpha e^{i\Phi''_x(z).\eta}{}_{|_{z=x}},$$

where

$$\Phi''_x(z) = \Phi(z) - \Phi(x) - \Phi'(x)(z-x)$$

and Φ' is the differential of Φ. Notice that $D_z^\alpha e^{i\Phi''_x(z).\eta}{}_{|_{z=x}}$ is a polynomial in η. It follows that a ΨDO transforms into another ΨDO under the action of a diffeomorphism. Moreover if A is a classical ΨDO of order a on U, its leading symbol $\sigma^L(A)$ transforms by change of variable as a function on the cotangent bundle T^*U and is invariant under the action of the diffeomorphism. Indeed, if we set $\eta = {}^t\,\Phi'^{-1}\xi$ and $y = \Phi(x)$

$$\sigma_a(\Phi^\sharp A)(y,\eta) = \lim_{\lambda\to\infty} \frac{\sigma(\Phi^\sharp A)(y,\lambda\eta)}{\lambda^a} = \lim_{\lambda\to\infty} \frac{\sigma(A)(x,\lambda\,\xi)}{\lambda^a} = \sigma_a(A)(x,\xi).$$

These results easily extend to $\mathrm{End}(V)$-valued pseudodifferential operators (resp. classical, resp. log-polyhomogeneous) on U acting on $C_c^\infty(U)\otimes V$ with values in $C^\infty(U)\otimes V$.

1.2.2 Pseudodifferential operators acting on sections of a vector bundle

Let M be a smooth closed manifold of dimension n equipped with an atlas. Recall that an atlas is a collection $\{(U_i,\phi_i,\varphi_i), i \in I\}$ where $\{U_i, i \in I\}$ is an open cover of M, (U_i,ϕ_i) is a coordinate chart for each i i.e. an open subset U_i of M and a diffeomorphism $\phi_i : U_i \to \phi_i(U_i) \subset \mathbb{R}^n$ and $\{\varphi_i, i \in I\}$ is a partition of unity subordinated to the covering. Let $C^\infty(M)$ be the space of smooth complex-valued functions on M. Let $A : C^\infty(M) \to C^\infty(M)$ be a linear operator. A localization of A subordinated to the chart (U,ϕ) of A is any map $\varphi A \widetilde{\varphi}$ where φ and $\widetilde{\varphi}$ are smooth functions with compact support in U. Here we identify φ with the multiplication operator by φ.

Definition 1.2.6. *A linear map $A : C^\infty(M) \to C^\infty(M)$ is called a ΨDO on M if given any local chart (U,ϕ) on M, any localization $A_U := \varphi A \widetilde{\varphi}$ subordinated to this chart, the induced localized operator $A_{\phi(U)} := \phi_* A_U = A_U \circ \phi^{-1}$ is a ΨDO on $\phi(U)$. The symbol $\sigma_\phi(A)(x,.)$ in a given local chart (U,ϕ) around x in U is defined by the symbol of $A_{\phi(U)}$.*

The definition of ΨDOs extends to linear operators acting on smooth sections of a vector bundle replacing local charts (U, ϕ) by local trivializations (U, ϕ, u). Let M be a smooth closed riemannian manifold of dimension n. Let $\pi : E \to M$ be a hermitian vector bundle of finite rank k. We denotes by $\Gamma(M, E)$ the vector space of smooth sections of E. The space $\Gamma(M, E)$ is equipped with the hermitian product

$$< \mu, \nu >= \int_M < \mu(x), \nu(x) >_{E_x} dx,$$

where dx is the riemannian volume measure on M and $<,>_{E_x}$ is the hermitian scalar product on the fibre E_x of E over M.

Let us recall that a local trivialization of the vector bundle is a triple (U, ϕ, u) where (U, ϕ) is a local chart of M and u gives rise to a diffeomorphism

$$\begin{aligned} \pi^{-1}(U) &\to \phi(U) \times \mathbb{C}^k \\ z &\mapsto (\phi(\pi(z)), u(z)) \,. \end{aligned}$$

We denote by $\Gamma(U, E)$ the vector space of smooth sections of E restricted to U. The maps ϕ and u induce two maps:

$$\begin{aligned} \phi_u^* &: C^\infty\left(\phi(U), \mathbb{C}^k\right) \to \Gamma(U, E) \\ \phi_{u_*} &: \Gamma(U, E) \to C^\infty\left(\phi(U), \mathbb{C}^k\right) \end{aligned}$$

defined by: $(\phi_u^* s)(x) = u^{-1}(s(\phi(x)))$ and $(\phi_{u_*} s)(x) = u(s(\phi^{-1}(x)))$.
For a linear map $A : \Gamma(M, E) \to \Gamma(M, E)$ we define the induced map $A_U := r_U A i_U$ in $\Gamma(U, E)$ where $r_U : \Gamma(M, E) \to \Gamma(U, E)$ is the natural restriction and $i_U : \Gamma(U, E) \to \Gamma(M, E)$ is the natural embedding. We call the operator $\phi^\sharp A_U := \phi_{u_*} A_U \phi_u^*$ the operator A read in the local trivialization (U, ϕ, u).
Let $\{(U_i, \phi_i, u_i), i \in I\}$ be a finite trivializing covering of M for E where $\{U_i, i \in I\}$ is an open cover of M, (U_i, ϕ_i, u_i) is a local trivialization of E for each i and let $\{\varphi_i, i \in I\}$ be a partition of unity subordinated to the covering. Let $A : \Gamma(M, E) \to \Gamma(M, E)$ be a linear map. We can write A on the form $A = \sum_{j,l \in I} \varphi_j A \varphi_l$.

Definition 1.2.7. *A linear map $A : \Gamma(M, E) \to \Gamma(M, E)$ is a ΨDO of order a if each operator $\varphi_j A \varphi_l$ read in the local trivialization (U_i, ϕ_i, u_i) i.e. $\phi_i^\sharp(\varphi_j A \varphi_l)_{U_i}$ is a ΨDO of order a on $\phi_i(U_i)$ with values in* $\mathrm{End}(V)$ *where V is the model space of the fibre of E.*

It follows from the behavior under change of variables introduced in the previous section that all these definitions and properties are independent of the choice of finite trivializing covering and partition of unity.

We denote by $\Psi DO^a(M,E)$ the set of ΨDOs of order a acting on $\Gamma(M,E)$. On the space $\Gamma(M,E)$ endowed with its Fréchet topology, a ΨDO A is continuous but in general a ΨDO with non negative order does not extend to a continuous operator on $L^2(M,E)$, the L^2-closure of the space $\Gamma(M,E)$.

Pseudodifferential operators act naturally on Sobolev spaces (see e.g. [Gi] Section 1.3 and [Sh] Section 6 and 7) of the space $\Gamma(M,E)$ for the hermitian product $\langle\cdot,\cdot\rangle$. For any real s, $H^s(\mathbb{R}^n)$ is defined as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm

$$\| u \|_s^2 = \| u \|_{s,\mathbb{R}^n}^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 đ\xi.$$

The space $C_c^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ and the Plancherel theorem shows that $H^0(\mathbb{R}^n)$ is isomorphic to $L^2(\mathbb{R}^n)$. Now we use covering of the manifold M by local trivializations and a partition of unity subordinated to the covering to define Sobolev spaces $H^s(M,E)$ on the vector bundle E. Indeed let (U_i,ϕ_i,u_i) be local trivializations and $\{\varphi_i, i\in I\}$ a partition of unity subordinated to the covering. The space $H^s(M,E)$ is the completion of the space $\Gamma(M,E)$ with respect to the norm

$$\| u \|_s^2 = \| u \|_{s,M,E}^2 = \sum_{i=1}^{I} \| \phi_i^\sharp(\varphi_i u)_{U_i} \|_{s,\mathbb{R}^n}^2 .$$

Since the various norms defined above for some fixed real s are equivalents, the space $H^s(M,E)$ is defined independent of the choices made. We have the following properties ([Gi], [Sh]):

Proposition 1.2.8.

1. *The natural inclusion $H^s(M,E)\to H^t(M,E)$ is compact for $s>t$.*
2. *If $A\in\Psi DO^a(M,E)$ then $A: H^s(M,E)\to H^{s-a}(M,E)$ is continuous for all s.*

We deduce the following property for operators of order no larger than zero and operators of order smaller than zero.

Corollary 1.2.9.

1. *Any linear operator in $\Psi DO^0(M,E)$ is continuous.*
2. *Any operator in $\Psi DO^a(M,E)$ with order a smaller than zero is a compact operator on any $H^s(M,E)$ with s in $\mathbb{R}$ and in particular on $L^2(M,E)$.*

Proof:

1. By item 2 of the above proposition if A lies in $\Psi DO^0(M,E)$, $A : H^s(M,E) \to H^s(M,E)$ is continuous for all s and in particular in $L^2(M,E) = H^0(M,E)$.

2. If A lies in $\Psi DO^a(M,E)$ with a smaller than zero then by item 1 of the above proposition $A : H^s(M,E) \to H^a(M,E)$ is compact for all $s > a$ and in particular for $s = 0$.

□

A pseudodifferential operator $A : \Gamma(M,E) \to \Gamma(M,E)$ is classical if each operator $\phi_i^\sharp(\varphi_j A \varphi_i)_{U_i}$ is classical, a property which holds independently of the choice of local trivialization. Let $C\ell^a(M,E)$ denote the set of classical ΨDOs of order a. If A_1 lies in $C\ell^{a_1}(M,E)$ and A_2 lies in $C\ell^{a_2}(M,E)$, then the product A_1A_2 belongs to $C\ell^{a_1+a_2}(M,E)$ and we denote by

$$C\ell(M,E) := \langle \bigcup_{a\in\mathbb{C}} C\ell^a(M,E)\rangle$$

the algebra generated by all classical ΨDOs acting on smooth sections of E. If A lies in $C\ell^a(M,E)$ its symbol $\sigma(A)$ is only locally defined but its leading symbol $\sigma^L(A)$, as already mentioned, is independent of the choice of local chart and hence is globally defined. Let T^*M be the cotangent bundle of M, $p : T^*M - \{0\} \to M$ the canonical projection and p^*E the induced vector bundle over T^*M. The leading symbol $\sigma^L(A)$ is a section of $\mathrm{End}(p^*E)$ so that for any x in M and any ξ in $T_x^*M - \{0\}$, $\sigma^L(A)(x,\xi)$ is an endomorphism of the fibre E_x of E.

We denote the set of log-polyhomogeneous operators of order a and log degree k by $C\ell^{a,k}(M,E)$ and we set:

$$C\ell^{\star,k}(M,E) = \bigcup_{a\in\mathbb{C}} C\ell^{a,k}(M,E), \quad C\ell^{\star,\star}(M,E) = \langle \bigcup_{k\geq 0} C\ell^{\star,k}(M,E)\rangle$$

the latter corresponding to the algebra generated by log-polyhomogeneous operators. Note that $C\ell^a(M,E) = C\ell^{a,0}(M,E)$ i.e. a classical ΨDO is a particular case of log-polyhomogeneous operator.

The definition of a holomorphic family of log-polyhomogeneous symbols extends to a holomorphic family of ψDOs.

Definition 1.2.10. *A family $(A(z))_{z\in\Omega}$ in $C\ell^{\star,\star}(M,E)$ with distribution kernels $(x,y) \mapsto K_{A(z)}(x,y)$ is holomorphic if*

1. *the order $a(z)$ of $A(z)$ is holomorphic in z.*

2. *In any local trivialization of E, we can write $A(z)$ in the form $A(z) = Op(\sigma(z)) + R(z)$, for some holomorphic family of symbols $(\sigma(z))_{z\in\Omega}$ and some holomorphic family $(R(z))_{z\in\Omega}$ of smoothing operators i.e. given by a holomorphic family of smooth Schwartz kernels.*

3. *The (smooth) restrictions of the distribution kernels $K_{A(z)}$ to the complement of the diagonal $\Delta \subset M \times M$, form a holomorphic family with respect to the topology given by the uniform convergence in all derivatives on compact subsets of $M \times M - \Delta$.*

1.2.3 Topology on pseudodifferential operators

For a real number a, we can equip the vector spaces $\Psi DO^a(M, E)$, $C\ell^a(M, E)$ and $C\ell^{a,k}(M, E)$ with Fréchet topologies via the Fréchet topology on symbols. Indeed, let $\{U_i, \phi_i, u_i, i \in I\}$ be a finite trivializing covering of M for E where as before $\{U_i, i \in I\}$ is an open cover of M, (U_i, ϕ_i, u_i) is a local trivialization of E for each i and let $\{\varphi_i, i \in I\}$ be a partition of unity subordinated to the covering. A pseudodifferential operator A in $\Psi DO^a(M, E)$ can be written $A = \sum_{i\in I}(A_i + R_i)$ where R_i is a smoothing operator with smooth kernel K_i with compact support in $U_i \times U_i$ and the operators A_i are properly supported in U_i. Let us denote by $\sigma^{(i)}(A)(x, \xi)$ the local symbol of A in the local trivialization (U_i, ϕ_i, u_i). Recall that with the notation of Subsection 1.2.1, this is the symbol of the properly supported A_i which can be written $A_i = Op(\sigma^{(i)}(A)(x, \xi))$.

In the local trivialization (U_i, ϕ_i, u_i), we equip $\Psi DO^a(M, E)$ with the following countable set of semi-norms labelled by multiindices α, β: for any compact subset $K \subset \phi_i(U_i)$,

$$\sup_{x\in K}\sup_{\xi\in\mathbb{R}^n}(1 + |\xi|)^{|\beta|-a} \parallel \partial_x^\alpha\partial_\xi^\beta\sigma^{(i)}(A)(x, \xi)(x, \xi) \parallel;$$

$$\sup_{x,y\in K} \parallel \partial_x^\alpha\partial_y^\beta K_i(x, y) \parallel .$$

Let us specialize to $C\ell^{a,k}(M, E)$. In the local trivialization (U_i, ϕ_i, u_i), we equip $C\ell^{a,k}(M, E)$ with the following countable set of semi-norms: for any compact subset $K \in \phi_i(U_i)$ for any $j \geq 0$ and $N \geq 1$, for any multiindices α, β

$$\begin{aligned}
&\sup_{x\in K}\sup_{\xi\in\mathbb{R}^n}(1 + |\xi|)^{|\beta|-a} \parallel \partial_x^\alpha\partial_\xi^\beta\sigma^{(i)}(A)(x, \xi) \parallel;\\
&\sup_{x\in K}\sup_{\xi\in\mathbb{R}^n}(1 + |\xi|)^{|\beta|-a+N} \parallel \partial_x^\alpha\partial_\xi^\beta(\sigma^{(i)}(A) - \sum_{j=0}^{N-1}\chi(\xi)\,\sigma_{a-j}^{(i)}(A))(x, \xi) \parallel;\\
&\sup_{x\in K}\sup_{|\xi|=1} \parallel \partial_x^\alpha\partial_\xi^\beta\sigma_{a-j}^{(i)}(A)(x, \xi) \parallel;\\
&\sup_{x,y\in K} \parallel \partial_x^\alpha\partial_y^\beta K_i(x, y) \parallel .
\end{aligned} \tag{1.12}$$

1.2.4 Elliptic classical pseudodifferential operators

Let U be an open subset of $\mathbb{R}^n$ and let A be a classical ΨDO of order a on U. The operator A is said to be *elliptic* of order a if its leading symbol $\sigma_a(P)(x,\xi) = \sigma^L(P)(x,\xi)$ is never zero for $|\xi| \neq 0$.

Example 1.2.11.

1. *The Laplacian* $\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial^2_{x_j}}$ *has leading symbol* $\sigma_2(\Delta)(x,\xi) = \sum_{j=1}^{n} \xi_j^2$ *and is therefore elliptic.*

2. *The Neumann operator* $A = D_x + iD_y$ *has principal symbol* $\sigma_1(A) = \xi + i\eta$ *and is therefore elliptic.*

Let us recall from Proposition 1.2.5 that any ΨDO A on U can be written in the form $A = A_0 + R$ where A_0 is a properly supported operator on U and R is a smoothing operator on U. It follows that a classical ΨDO A is elliptic if and only if A_0 is elliptic.

Proposition 1.2.12. *Let A be a classical elliptic operator of order a on U. If A is properly supported then there exists a classical properly supported operator B of order $-a$ on U such that $AB - I$ and $BA - I$ are smoothing operators.*

Proof: Assume that such an operator B exist. This operator has a symbol $\sigma(B)$ such that $\sigma(A) \star \sigma(B) = 1$. Hence the $\star$ product of symbols gives for $|\xi| \geq 1$

$$1 = \sigma(A)(x,\xi) \star \sigma(B)(x,\xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma(A)(x,\xi) D_x^\alpha \sigma(B)(x,\xi).$$

It follows that

$$\begin{aligned} \sigma_a(A)\sigma_{-a}(B) &= 1 \\ \sigma_a(A)\sigma_{-a-j}(B) + \sum_{k+l+|\alpha|=j, l<j} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_{a-k}(A) D_x^\alpha \sigma_{-a-l}(B) &= 0, \quad j = 1, 2, \cdots \end{aligned}$$

so that we can compute $\sigma_{-a}(B)$ and obtain the homogeneous components $\sigma_{-a-j}(B)$ recursively. Thus if the symbol of B has the formal expansion $\sigma_{-a}(B) \sim \sum_{j=0}^{\infty} \sigma_{-a-j}(B)$ with $\sigma_{-a-j}(B)$ obtained above, then $AB - I = R$, where R is a smoothing operator. In the same way there exist an operator B' and a smoothing operator R' such that $B'A - I = R'$. Now $B'AB = B' + B'R = B + R'B$. Since $B'R$ and $R'B$ are smoothing operators, the result follows. □

Remark 1.2.13. *The operator B is called a parametrix of the operator A.*

The definition of ellipticity extended to a classical operator $A \in C\ell^a(M, E)$ where $\pi : E \to M$ is a hermitian vector bundle of finite rank k over M. Recall that its leading symbol $\sigma^L(A)$ can be defined as a section of $\mathrm{End}(p^*E)$ where as before p is the canonical projection $p : T_x^*M - \{0\} \to M$. For x in M and ξ in $T_x^*M - \{0\}$, $\sigma^L(A)(x, \xi)$ is an endomorphism of the fibre E_x of E. Hence A is elliptic if $\sigma^L(A)(x, \xi)$ is invertible for any x in M and any ξ in $T_x^*M - \{0\}$.

1.3 Zero order odd-class pseudodifferential operators

1.3.1 Odd-class pseudodifferential operators

Let a be an integer and k a non negative integer. A log-polyhomogeneous operator A in $C\ell^{a,k}(M, E)$ is odd-class (resp. even-class) if in each local trivialization, the local symbol $\sigma(A)(x, \xi)$ is odd-class (resp. even-class). This property of being odd-class for the local symbol is invariant under a change of local coordinates so that it is enough to verify this property for a fixed finite covering of M by local trivializations.
We denote by $C\ell^{a,k}_{odd}(M, E)$ (resp. $C\ell^{\star,\star}_{odd}(M, E)$) the set of odd-class (a, k) log-polyhomogeneous (resp. odd-class log-polyhomogeneous) pseudo-differential operators. Since a change of coordinates keeps an odd-class log-polyhomogeneous symbol odd-class, the set $C\ell^{\star,\star}_{odd}(M, E)$ is an algebra.

Let us now consider the case of classical operators: we call $C\ell^a_{odd}(M, E)$ the set of odd-class operators of integer order a and we set

$$C\ell_{odd}(M, E) = \bigcup_{a\in\mathbb{Z}} C\ell^a_{odd}(M, E).$$

The following lemma can be seen as a particular case of Lemma 1.1.8 with $k = k' = 0$, adding the fact that the class of the composition of odd-class (resp. even-class) of symbols remains unchanged under a change of coordinates. We then recover known properties of classical odd-class operators [KV1, Du1].

Lemma 1.3.1.

1. *The composition of two classical odd-class operators is an odd-class operator i.e. $C\ell_{odd}(M, E)$ is an algebra.*

2. *The composition of two classical even-class operators is a classical odd-class operator and the composition of a classical odd-class operator by a classical even-class operator is a classical even-class operator.*

3. *If B is an invertible classical odd-class (resp. even-class) operator, then B^{-1} is an classical odd-class (resp. even-class) operator.*

Example 1.3.2. *This example is the analogous of Example 1.1.7 on odd-class (resp. even-class) symbols for classical pseudodifferential operators.*

1. *All differential operators and their parametrices belong to* $C\ell_{odd}(M, E)$.

2. *Let* Δ *be the Laplacian in* $\mathbb{R}^n$. *The operator* $(1+\Delta)^{-1}$ *is odd-class. Indeed its symbol has the asymptotic expansion*

$$\sigma((1+\Delta)^{-1})(x,\xi)) = \sigma((1+\Delta)^{-1})(\xi)) = \left(1+|\xi|^2\right)^{-1} \sim \sum_{k=0}^{\infty}(-1)^k|\xi|^{-2k-2}.$$

3. $\sqrt{1+\Delta}$ *is an even-class operator. Indeed its symbol has the asymptotic expansion*

$$\sigma(\sqrt{1+\Delta})(\xi) = \sqrt{1+|\xi|^2} \sim \sum_{k=0}^{\infty}\alpha_k|\xi|^{-2k+1}.$$

1.3.2 Lie group of zero order odd-class classical ΨDOs

The space $C\ell^a(M, E)$ is a Fréchet space equipped with the Fréchet topology introduced in Section 1.2.3, hence $C\ell^0(M, E)$ is a Fréchet space. It has been shown by M. Kontsevich and S. Vishik [KV1] (see also [LP]) that $\left(C\ell^0(M, E)\right)^*$ is a Fréchet Lie group with exponential mapping and with Lie algebra $C\ell^0(M, E)$. For these notion of Lie group and Lie algebra with exponential mapping in the infinite dimensional case, we refer to A. Kriegel and P. W. Michor in [KM], who claim that all known smooth Fréchet Lie groups admit exponential mapping.

Definition 1.3.3 ([KM] Definition 36.8.)**.** *A Lie group* $\mathcal{G}$ *with Lie algebra* $Lie(\mathcal{G})$ *admits an exponential mapping if there exists a smooth mapping* $\mathrm{Exp} : Lie(\mathcal{G}) \to \mathcal{G}$ *such that* $t \mapsto \mathrm{Exp}(tX)$ *is a one-parameter subgroup i.e. a Lie group homomorphism* $(\mathbb{R}, +) \to \mathcal{G}$ *with tangent vector* X *at* 0.

Proposition 1.3.4. $\left(C\ell^0_{odd}(M, E)\right)^*$ *is a Fréchet Lie group with Lie algebra* $C\ell^0_{odd}(M, E)$.

Remark 1.3.5. *We give here an exhaustive proof of this result. We will show later (Proposition 6.1.4) that there is an exponential mapping.*

Proof: By Lemma 1.3.1, the composition of two operators in $\left(C\ell^0_{odd}(M, E)\right)^*$ belongs to $\left(C\ell^0_{odd}(M, E)\right)^*$ and the same holds for the inverse so that the set $\left(C\ell^0_{odd}(M, E)\right)^*$ is a group in the Fréchet algebra $C\ell^0_{odd}(M, E)$. Let us show that it is an open subset of $C\ell^0_{odd}(M, E)$. For that, let A be an operator in $\left(C\ell^0_{odd}(M, E)\right)^*$. We want to build an open neighborhood of A in $\left(C\ell^0_{odd}(M, E)\right)^*$. The algebra $C\ell^0_{odd}(M, E)$ is contained in $C\ell^0(M, E)$ and by Corollary 1.2.9, $C\ell^0(M, E)$ corresponds to all bounded ΨDOs on

$L^2(M,E) = H^0(M,E)$ and hence it is contained in the Banach algebra $\mathcal{L}(L^2(M,E))$. But by the inverse local theorem, the set of invertible operators on $\mathcal{L}(L^2(M,E))$ is an open set. Hence A admits an open neighborhood V in the set of invertible operators in $\mathcal{L}(L^2(M,E))$.

For any pseudodifferential operator $A : \Gamma(M,E) \to \Gamma(M,E)$, let us consider it as an operator $A_s : H^s(M,E) \to H^{s-a}(M,E)$ with a the order of A and $H^s(M,E)$ as defined in section 1.2.2. We have

$$\dim kerA_s = \dim kerA$$

and

$$\operatorname{codim} ImA_s = \operatorname{codim} ImA$$

so that

$$\begin{aligned} A \quad \text{invertible} &\Leftrightarrow \dim kerA = \operatorname{codim} ImA = \dim kerA_s = \operatorname{codim} ImA_s = 0 \\ &\Leftrightarrow kerA_s = \{0\} \quad \text{and} \quad ImA_s = H^{s-a}(M,E) \\ &\Leftrightarrow A_s \quad \text{invertible.} \end{aligned}$$

On the other hand, by Corollary 1.2.9, the inclusion $C\ell^0(M,E) \to \mathcal{L}(L^2(M,E))$ is continuous so that the inclusion $i : C\ell^0_{odd}(M,E) \to C\ell^0(M,E) \to \mathcal{L}(L^2(M,E))$ is also continuous and the inverse image $i^{-1}(V)$ yields an open neighborhood of A in $\left(C\ell^0_{odd}(M,E)\right)^*$. It follows that $\left(C\ell^0_{odd}(M,E)\right)^*$ is canonically equipped with a structure of manifold which makes it a Lie group. $\square$

CHAPTER 2

Chapter 2

Logarithms of elliptic pseudodifferential operators

In this chapter, we prove (see Theorem 2.2.4) that any operator in the algebra of odd-class log-polyhomogeneous operators is a finite linear combinations of products of classical operators and symmetrized logarithms of elliptic operators, all taken in the odd-class. A similar property already observed in [Du1] holds for all log-polyhomogeneous operators since these can be written as finite linear combination of products of classical operators and logarithms of elliptic operators, a result for which we provide a detailed proof (see Proposition 2.1.19). Whereas the logarithm of an odd-class operator is not generally odd-class, the symmetrized logarithm introduced by M. Braverman [B] does lie in the odd-class (Proposition 2.2.1).

2.1 Complex powers and logarithms of admissible pseudodifferential operators

Complex powers were first introduced by R. T. Seeley in [Se]. He proved that under appropriate conditions, an elliptic classical operator A admits a family of complex powers. This notion has been further developed by M. Shubin [Sh] for elliptic differential operators whose leading symbols do not take values in a closed angle Λ of the complex plane. This notion was extended to pseudodifferential operators with the help of the notion of admissibility. In this section, we introduce the notion of admissible elliptic operators and recall the construction and the properties of complex powers. We then define the logarithms and establish some of their properties.

We consider as before a finite rank hermitian vector bundle $\pi : E \to M$ over a closed riemannian manifold M of dimension n. To define the notion of admissibility, we need the definition of the spectrum of an operator. Let us recall this notion. Let as before $H^s(M,E)$ denote the Sobolev closure of $\Gamma(M,E)$ for the norm $\| \cdot \|_s$ defined in Section 1.2.2 and let A be a ΨDO of order a considered as an operator in $H^s(M,E)$ with domain $H^{s+a}(M,E)$. The *spectrum* $Sp(A)$ of A is defined as follows: a complex number λ lies outside $Sp(A)$ if and only if the operator $A-\lambda I$ has bounded inverse $R_A(\lambda) := (A-\lambda I)^{-1}$ in $L^2(M,E)$, called the *resolvent* of A at the point λ. The spectrum of A is always a closed subset of $\mathbb{C}$. The point λ in $Sp(A)$ is called an eigenvalue of A if $\ker(A-\lambda I) \neq \{0\}$. If A is an elliptic operator in $C\ell(M,E)$, then (see e.g. [Sh] Section 8.3) every point in the spectrum $Sp(A)$ is an eigenvalue of A. Furthermore the spectrum $Sp(A)$ is either a discrete subset of $\mathbb{C}$ or the whole space $\mathbb{C}$. Moreover, resolvents of elliptic operators on closed manifolds have the following property (see e.g. [Se] Corollary 1):

Proposition 2.1.1. *Let A be an elliptic operator in $C\ell(M,E)$ with positive order a. Assume that there exists an open angle Λ in the complex plane $\mathbb{C}$ with vertex 0 which does not intersect the spectrum of the leading symbol $\sigma^L(A)$ of A. Then*

1. *There exists $R > 0$ such that the operator $R_A(\lambda) = (A-\lambda I)^{-1}$ is invertible for $\lambda \in \Lambda$ and $|\lambda| > R$.*

2. *For any real numbers s, p with $0 \leq p \leq a$ the following norm estimate hold*

$$\| (A-\lambda I)^{-1} \|_{s,s+p} \leq C_{s,p} |\lambda|^{\frac{p}{a}-1}, \quad \lambda \in \Lambda, \quad |\lambda| > R$$

where $\| \, . \, \|_{s,s+p}$ denotes the norm on bounded operators from $H^s(M,E)$ to $H^{s+p}(M,E)$.

2.1.1 Admissible pseudodifferential operators

Definition 2.1.2. *Let A be an operator in $C\ell(M,E)$ with positive order. The operator A has principal angle θ if for every (x,ξ) in $T^*M - \{0\}$, its leading symbol $\sigma^L(A)(x,\xi)$ has no eigenvalues on the ray $L_\theta = \{re^{i\theta}, r \geq 0\}$.*

If an operator A admits a principal angle θ, then ([Se], [Sh]) there exists $\epsilon > 0$ with the property that the conical neighborhood $\Lambda_\epsilon = \{\rho\, e^{i\phi}, 0 < \rho < \infty, \theta - \epsilon < \phi < \theta + \epsilon\}$ of L_θ is such that any ray contained in Λ_ϵ contains no eigenvalue of $\sigma^L(A)(x,\xi)$. Moreover since M is compact, the spectrum $Sp(A)$ of A which consists of eigenvalues of A is discrete and $Sp(A) \cap \Lambda_\epsilon$ is a finite set.

Definition 2.1.3. *We call an operator A in $C\ell(M,E)$ admissible with spectral cut (or Agmon angle) θ if A has principal angle θ and its spectrum $Sp(A)$ does not meet L_θ.*

Remark 2.1.4. *If an operator A is admissible, A is elliptic and invertible. Indeed, this easily follows from the fact that A is elliptic if the leading symbol $\sigma^L(A)(x,\xi)$ is invertible for all x in M, ξ in $T^*_xM, \xi \neq 0$ and the fact that $Sp(A) \cap L_\theta = \emptyset$ i.e. $0 \notin Sp(A)$.*

2.1.2 Complex powers of admissible pseudodifferential operators

Let A be an admissible operator in $C\ell(M,E)$ with spectral cut θ and positive order a. Let Γ_θ denote the contour in $\mathbb{C}$, along the ray L_θ around the spectrum of A, defined by $\Gamma_\theta = \Gamma_\theta^1 \cup \Gamma_\theta^2 \cup \Gamma_\theta^3$ where

$$\begin{aligned} \Gamma_\theta^1 &= \{\rho\, e^{i\theta}, +\infty > \rho \geq r\} \\ \Gamma_\theta^2 &= \{r\, e^{it}, \theta \geq t \geq \theta - 2\pi\} \\ \Gamma_\theta^3 &= \{\rho\, e^{i(\theta-2\pi)}, r \leq \rho \leq +\infty\} \end{aligned} \tag{2.1}$$

and where r is any small positive real number such that $\Gamma_\theta \cap Sp(A) = \emptyset$. The contour Γ_θ is shown in Figure 1.

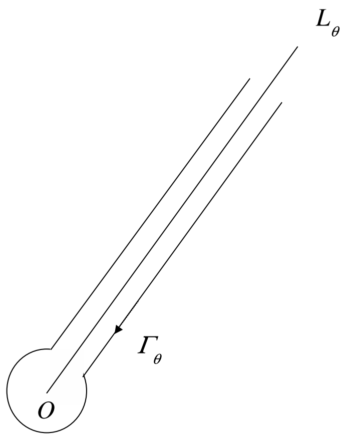

Figure 1

For $\mathrm{Re}(z) < 0$, the complex power A_θ^z of A is defined by the Cauchy integral [Se]

$$A_\theta^z = \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda_\theta^z \,(A-\lambda)^{-1}\, d\lambda \tag{2.2}$$

where λ_θ^z is defined as $\exp(z \log_\theta \lambda)$ with

$$\log_\theta \lambda = \log|\lambda| + i\mathrm{Arg}\lambda, \quad \theta - 2\pi \leq \mathrm{Arg}\lambda \leq \theta,$$

i.e. $\mathrm{arg}\lambda = \theta$ on Γ_θ^1 and $\mathrm{arg}\lambda = \theta - 2\pi$ on Γ_θ^3. Here $(A-\lambda)^{-1}$ is the resolvent of A. The above integral makes sense since by proposition 2.1.1, $\| (A-\lambda)^{-1} \| = O\left(\frac{1}{|\lambda|}\right)$ when

$|\lambda| \to +\infty$ and since $\int_\rho^{+\infty} \frac{\lambda^z}{|\lambda|} < +\infty$ for $\mathrm{Re}(z) < 0$. Here $\|\ .\ \|$ denotes the norm on bounded operators of $L^2(M, E)$.
Let us compute the symbol of the operator A_θ^z. Since this symbol is defined through the symbol of the resolvent $(A - \lambda I)^{-1}$, we denote by b_{-a-j} the components of the symbol of $(A - \lambda I)^{-1}$. These components are defined by the recursive system of equalities ([Sh] Paragraph 11.1, see also [KV1] Paragraph 2)

$$\begin{aligned}
b_{-a} &:= (\sigma_a(A) - \lambda)^{-1}, \\
b_{-a-1} &:= -b_{-a}\left(\sigma_{a-1}(A)b_{-a} + \sum_i \partial_{\xi_i}\sigma_a(A)D_{x_i}b_{-a}\right), \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
b_{-a-j} &:= -b_{-a}\left(\sum_{k+l+|\alpha|=j, l<j} \frac{1}{\alpha}\partial_\xi^\alpha \sigma_{a-k}(A)D_x^\alpha b_{-a-l}\right).
\end{aligned}$$

It is easy to see that $b_{-a-j}(x, \xi, \lambda)$ is positively homogeneous in $(\xi, \lambda^{\frac{1}{a}})$ of degree $-a-j$ i.e.

$$b_{-a-j}(x, t\xi, t^a\lambda) = t^{-a-j} b_{-a-j}(x, \xi, \lambda), \quad \text{for} \quad t > 0. \tag{2.3}$$

Indeed, for $t > 0$, $b_{-a}(x, t\xi, t^a\lambda) = (\sigma_a(A) - \lambda)^{-1}(x, t\xi, t^a\lambda)$ and since $\sigma_a(A)$ is positively homogeneous of degree a, then $b_{-a}(x, t\xi, t^a\lambda) = t^{-a} b_{-a}(x, \xi, \lambda)$. Recursively,

$$\begin{aligned}
& b_{-a-j}(x, t\xi, t^a\lambda) \\
= \;& -b_{-a}(x, t\xi, t^a\lambda) \sum_{k+l+|\alpha|=j, l<j} \frac{1}{\alpha}\partial_\xi^\alpha \sigma_{a-k}(A)(x, t\xi) D_x^\alpha b_{-a-l}(x, t\xi, t^a\lambda) \\
= \;& -t^{-a}\, b_{-a}(x, \xi, \lambda) \sum_{k+l+|\alpha|=j, l<j} \frac{1}{\alpha} t^{a-k-|\alpha|}\partial_\xi^\alpha \sigma_{a-k}(A)(x, \xi) t^{-a-l} D_x^\alpha b_{-a-l}(x, \xi, \lambda) \\
= \;& t^{-a-j} b_{-a-j}(x, \xi, \lambda).
\end{aligned}$$

For $\mathrm{Re}(z) < 0$, let us define the functions

$$b^{(z)}_{az-j}(x, \xi) = \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda_\theta^z\, b_{-a-j}(x, \xi, \lambda)\, d\lambda. \tag{2.4}$$

These functions are positively homogeneous of degree $az - j$. Indeed, for $\xi \neq 0$ and $t > 1$, by a change of variable

$$b^{(z)}_{az-j}(x, t\xi) = \frac{i}{2\pi} \int_{\Gamma(t)} (t^a\mu)^z\, b_{-a-j}(x, t\xi, t^a\mu) t^a\, d\mu.$$

Here the integration along the contour $\Gamma(t)$ is equal to the integration around the contour Γ_θ since the functions $b^{(z)}_{az-j}$ do not have singularities on a small enough disc around the origin on ξ. Now, using (2.3) we obtain

$$\begin{aligned} b^{(z)}_{az-j}(x,t\xi) &= \frac{i}{2\pi}\int_{\Gamma(t)} (t^a\mu)^z t^{-a-j}\, b_{-a-j}(x,\xi,\mu) t^a\, d\mu \\ &= t^{az-j}\frac{i}{2\pi}\int_{\Gamma(t)} \mu^z\, b_{-a-j}(x,\xi,\mu)\, d\mu \\ &= t^{az-j}\, b^{(z)}_{az-j}(x,\xi). \end{aligned}$$

Moreover, the result is obtained for $t>1$ and $\xi \neq 0$ and hence holds for $t \neq 0$.
We therefore infer that the operator A^z_θ is a classical ΨDO of order az; its leading symbol is $\sigma_{az}(A^z_\theta)(x,\xi) = (\sigma_a(A_\theta))^z(x,\xi)$ so that A^z_θ is elliptic. The homogeneous components of the symbol of A^z_θ are

$$\sigma_{az-j}(A^z_\theta)(x,\xi) = b^{(z)}_{az-j}(x,\xi) = \frac{i}{2\pi}\int_{\Gamma_\theta} \lambda^z_\theta\, b_{-a-j}(x,\xi,\lambda)\, d\lambda. \tag{2.5}$$

The definition of complex powers can be extended to the whole complex plane by setting $A^z_\theta := A^k A^{z-k}_\theta$ for k in $\mathbb{N}$ and $\mathrm{Re}(z) < k$; this definition is independent of the choice of the positive integer k and preserves the usual properties, i.e.

$$A^{z_1}_\theta A^{z_2}_\theta = A^{z_1+z_2}_\theta, \quad A^k_\theta = A^k, \quad \text{for } k \in \mathbb{Z}.$$

Complex powers of operators depend on the choice of spectral cut. Let L_θ and L_ϕ be two spectral cuts for A outside an angle Λ which contains the spectrum of $\sigma^L(A)(x,\xi)$. We note that there is only a finite number of points of the spectrum $Sp(A)$ of A outside Λ. Assume that $0 \le \theta < \phi < 2\pi$; let us denote by $\lambda_1, \cdots, \lambda_k$ the points of $Sp(A)$ contained in the angle $\{z \in \mathbb{C}, \theta < \mathrm{Arg} z < \phi\}$, and $\Gamma_{\theta,\phi}$ a contour surrounding the $\lambda'_\nu s$ for all $1 \le \nu \le k$ and contained in the angle $\{z \in \mathbb{C}, \theta < argz < \phi\}$. In fact, $\Gamma_{\theta,\phi}$ is a contour around the cone

$$\Lambda_{\theta,\phi} := \{\rho\, e^{it}, \infty > \rho \ge r, \quad \theta < t < \phi\} \tag{2.6}$$

delimited by the angles θ and ϕ. Let us consider a contour $\Gamma = \Gamma^1_\theta \cup \Gamma^1_\phi \cup \Gamma^3_{\theta,\phi}$ around the remaining spectrum of A, with Γ^1_θ, Γ^1_ϕ defined in (2.1) and $\Gamma^3_{\theta,\phi} = \{r\, e^{i\lambda}, \phi - 2\pi \le \lambda \le \theta\}$, where r is any small positive real number such that $\Gamma \cap Sp(A) = \emptyset$.
For $\mathrm{Re}(z) < 0$ we have:

$$A^z_\theta = \frac{i}{2\pi}\left(\int_{\Gamma_{\theta,\phi}} \lambda^z_\theta\, (A-\lambda)^{-1}\, d\lambda + \int_\Gamma \lambda^z_\theta\, (A-\lambda)^{-1}\, d\lambda\right)$$

and likewise for the spectral cut ϕ,

$$A_\phi^z = \frac{i}{2\pi}\left(\int_{\Gamma_{\theta,\phi}} \lambda_\phi^z\,(A-\lambda)^{-1}\,d\lambda + \int_\Gamma \lambda_\phi^z\,(A-\lambda)^{-1}\,d\lambda\right).$$

Since by definition, $\int_\Gamma \lambda_\theta^z\,(A-\lambda)^{-1}\,d\lambda = \int_\Gamma \lambda_\phi^z\,(A-\lambda)^{-1}\,d\lambda$, it follows that

$$\begin{aligned} A_\theta^z - A_\phi^z &= \frac{i}{2\pi}\int_{\Gamma_{\theta,\phi}} (\lambda_\theta^z - \lambda_\phi^z)(A-\lambda)^{-1}\,d\lambda \\ &= \frac{e^{2i\pi z}-1}{2i\pi}\int_{\Gamma_{\theta,\phi}} \lambda_\theta^z\,(A-\lambda)^{-1}\,d\lambda \\ &= \left(1-e^{2i\pi z}\right)\,\Pi_{\theta,\phi}(A)A_\theta^z, \end{aligned}$$

where we have set ([W2], [Po1])

$$\Pi_{\theta,\phi}(A) = \frac{1}{2i\pi}\int_{\Gamma_{\theta,\phi}} \lambda^{-1}A(A-\lambda)^{-1}\,d\lambda.$$

The following lemma shows that $\Pi_{\theta,\phi}(A)$ is a projection.

Lemma 2.1.5. *Let A be an admissible operator in $C\ell(M,E)$ with spectral cuts θ,ϕ and positive order a. If $0\leq\theta<\phi<2\pi$, then the operator $\Pi_{\theta,\phi}(A)$ is a projection.*

Proof: A proof of this result can be found in [Po1], Proposition 3.2. Let θ_1 and ϕ_1 be two spectral cuts of A with $\theta_1<\theta<\phi<\phi_1<\theta+2\pi$ and such that there are no eigenvalues of A and $\sigma^L(A)$ in the cones $\Lambda_{\theta_1,\theta}$ and $\Lambda_{\phi_1,\phi}$. Hence, the integration of $\Pi_{\theta,\phi}(A)$ over $\Gamma_{\theta,\phi}$ is equal to the integration over Γ_{θ_1,ϕ_1}. It follows that

$$\Pi_{\theta,\phi}(A)^2 = \frac{-1}{4\pi^2}\int_{\Gamma_{\theta,\phi}}\int_{\Gamma_{\theta_1,\phi_1}} \lambda^{-1}\mu^{-1}A^2(A-\lambda)^{-1}(A-\mu)^{-1}\,d\lambda\,d\mu$$

By the Hilbert identity $(A-\lambda)^{-1}(A-\mu)^{-1} = (\lambda-\mu)^{-1}[(A-\lambda)^{-1}-(A-\mu)^{-1}]$, we obtain

$$\begin{aligned} &4\pi^2\Pi_{\theta,\phi}(A)^2 \\ =\ & \int_{\Gamma_{\theta,\phi}} \frac{1}{\lambda}A^2(A-\lambda)^{-1}\left(\int_{\Gamma_{\theta_1,\phi_1}} \frac{\mu^{-1}d\mu}{\mu-\lambda}\right)d\lambda + \int_{\Gamma_{\theta_1,\phi_1}} \frac{1}{\mu}A^2(A-\mu)^{-1}\left(\int_{\Gamma_{\theta,\phi}} \frac{\lambda^{-1}d\lambda}{\lambda-\mu}\right)d\mu \\ =\ & 2i\pi\int_{\Gamma_{\theta,\phi}} \lambda^{-2}A^2(A-\lambda)^{-1}d\lambda. \end{aligned}$$

Here we have used the Cauchy formula and the fact that μ lies outside $\Gamma_{\theta,\phi}$ in the second integral. Now using the fact that $A(A-\lambda)^{-1} = 1+\lambda(A-\lambda)^{-1}$ we obtain

$$\Pi_{\theta,\phi}(A)^2 = \frac{i}{2\pi}\int_{\Gamma_{\theta,\phi}} \lambda^{-2}A\,d\lambda + \frac{i}{2\pi}\int_{\Gamma_{\theta,\phi}} \lambda^{-1}A(A-\lambda)^{-1}d\lambda = \Pi_{\theta,\phi}(A).$$

□

When the cone $\Lambda_{\theta,\phi}$ does not intersect the spectrum of the leading symbol of A, as previously observed, it only contains a finite number of eigenvalues of A and $\Pi_{\theta,\phi}(A)$ is a finite rank projection and hence a smoothing operator. In general, $\Pi_{\theta,\phi}(A)$, which is a pseudodifferential projection, is a zero order operator with leading symbol given by $\pi_{\theta,\phi}(\sigma^L(A))$ defined similarly to $\Pi_{\theta,\phi}$ replacing A by the leading symbol of A so that:

$$\sigma^L(\Pi_{\theta,\phi}(A)) = \pi_{\theta,\phi}(\sigma^L(A)) := \sigma^L(A) \left(\frac{1}{2i\pi} \int_{\Gamma_{\theta,\phi}} \lambda^{-1} (\sigma^L_A - \lambda)^{-1} \, d\lambda \right).$$

To sum up, we have the following proposition:

Proposition 2.1.6. *Let θ and ϕ be two spectral cuts for an admissible operator A such that $0 \leq \theta < \phi + 2k\pi < 2\pi$ for some integer k. Then*

$$A^z_\theta - A^z_\phi = e^{2ikz\pi} I + \left(1 - e^{2i\pi\, z}\right) \Pi_{\theta,\phi}(A) A^z_\theta. \tag{2.7}$$

Remark 2.1.7. *Complex powers can also be defined for zero order elliptic operators. Indeed if A is an operator in $C\ell(M,E)$ of order 0 with spectral cut θ, then A is bounded on $H^s(M,E)$ and hence in $L^2(M,E)$. In that case its spectrum $Sp(A)$ lies inside the circle $\{\lambda : |\lambda| \leq \| A \|\}$. It follows that the operators A^z_θ can be defined directly using a Cauchy integral formula*

$$A^z_\theta = \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda^z_\theta \, (A - \lambda)^{-1} \, d\lambda$$

where Γ_θ is a contour around the spectrum of A and avoiding the spectral cut L_θ, shown in Figure 2.

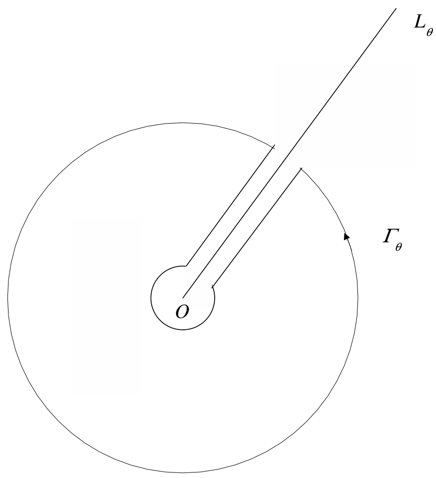

Figure 2

2.1.3 Logarithms of admissible pseudodifferential operators

Let A be an admissible operator in $C\ell(M,E)$ with non negative order a. The complex powers $(A_\theta^z)_{z\in\mathbb{C}}$ is a holomorphic family of classical ΨDOs. The notion of holomorphic family of classical ΨDOs was introduced in Section 1.2.2 using the related notion of holomorphic family of log-polyhomogeneous symbols given in Section 1.1.3. With the notation of Definition 1.2.10 the derivative of a family $A(z) = Op(\sigma(z)) + R(z)$, for some holomorphic family of symbols $\sigma(z)$ and some holomorphic family $R(z)$ of smoothing operators is $A'(z) = Op(\sigma\prime(z)) + R'(z)$. Here $\sigma\prime(z)$ is the derivative of holomorphic function and $R'(z)$ is the derivative in Schwartz space.

Let us now define the logarithm of an admissible operator. Given a Banach (unital) algebra $\mathcal{A}$, and a in $\mathcal{A}$, let $\theta \in \mathbb{R}$ be such that the spectrum of a, i.e. the set of complex scalars λ such that $a - \lambda 1$ is not invertible in $\mathcal{A}$, does not meet the ray $R_\theta = \{re^{i\theta}, r \geq 0\}$. Then (see e.g. [Hi]) the map

$$z \mapsto a_\theta^z := \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda_\theta^z \, (a - \lambda I)^{-1} \, d\lambda$$

defines a holomorphic function on the complex plane with values in $\mathcal{A}$, where we have set as before

$$\lambda_\theta^z = |\lambda|^z \, e^{iz\,\mathrm{Arg}(\lambda)} \quad \text{for} \quad \theta - 2\pi \leq \mathrm{Arg}\lambda < \theta.$$

Here Γ_θ is any bounded contour around the spectrum of a which does not intersect the ray R_θ. Hence, the logarithm of a

$$\begin{aligned} \log_\theta a &:= (\partial_z a_\theta^z)_{|z=0} \\ &= \frac{i}{2\pi} \left(\partial_z \int_{\Gamma_\theta} \lambda_\theta^z \, (a - \lambda I)^{-1} \, d\lambda \right)_{z=0} \\ &= \frac{i}{2\pi} \int_{\Gamma_\theta} \log_\theta \lambda \, (a - \lambda I)^{-1} \, d\lambda \end{aligned}$$

lies in $\mathcal{A}$, where we have set as before

$$\log_\theta \lambda = \log|\lambda| + i\,\mathrm{Arg}(\lambda) \quad \text{for} \quad \theta - 2\pi \leq \mathrm{Arg}\lambda < \theta.$$

This applies to the Banach algebra $\mathcal{A} = \mathcal{B}(H)$ of bounded linear operators on a Hilbert space H equipped with the operator norm $\|A\| = \sup_{\|x\|=1} \|Ax\|$ where $\|\cdot\|$ stands for the norm on H. Thus, given an admissible operator A in $C\ell(M,E)$ with zero order and spectral cut θ, its complex powers give rise to a holomorphic map $z \mapsto A_\theta^z$ on the complex plane with values in $\mathcal{B}(H^s(M,E))$ for any real number s, where $H^s(M,E)$ stands for the H^s-closure of the space $\Gamma(M,E)$ of smooth sections of E (see Section 1.2.2). The logarithm of A is the bounded operator on $H^s(M,E)$ defined in terms of the derivative at $z=0$ of this complex power:

$$\begin{aligned}\log_\theta A &:= (\partial_z A_\theta^z)_{|z=0}\\ &= \frac{i}{2\pi}\left(\partial_z \int_{\Gamma_\theta} \lambda_\theta^z (A-\lambda I)^{-1}\, d\lambda\right)_{|z=0}\\ &= \frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\, (A-\lambda I)^{-1}\, d\lambda\end{aligned}$$

with the notation of (2.2).

The notion of logarithm extends to an admissible operator A with positive order a and spectral cut θ in the following way. For any positive ϵ, the map $z \mapsto A_\theta^{z-\epsilon}$ of order $a(z-\epsilon)$ defines a holomorphic function on the half plane $\mathrm{Re}(z) < \epsilon$ with values in $\mathcal{B}(H^s(M,E))$ for any real number s. Thus we can set

$$\log_\theta A = A_\theta^\epsilon \left(\partial_z \left(A_\theta^{z-\epsilon}\right)\right)_{|z=0} = A_\theta^\epsilon \left(\partial_z \left(\frac{i}{2\pi}\int_{\Gamma_\theta} \lambda_\theta^{z-\epsilon} (A-\lambda)^{-1}\, d\lambda\right)\right)_{|z=0}. \tag{2.8}$$

for any positive ϵ the operator $\log_\theta A\, A^{-\epsilon} = A^{-\epsilon} \log_\theta A$ lies in $\mathcal{B}(H^s(M,E))$ for any real number s. It follows that $\log_\theta A$, which is clearly independent of the choice of $\epsilon > 0$, defines a bounded linear operator from $H^s(M,E)$ to $H^{s-\epsilon}(M,E)$ for any positive ϵ.

Just as complex powers, the logarithm depends on the choice of spectral cut. Indeed, differentiating (2.7) w.r. to z at $z=0$ yields for two spectral cuts θ, ϕ such that $0 \leq \theta < \phi + 2k\pi < 2\pi$, for some integer k:

$$\log_\theta A - \log_\phi A = 2ik\pi\, I - 2i\pi \Pi_{\theta,\phi}(A). \tag{2.9}$$

As a result of the above discussion and as already observed in [Ok1], when the leading symbol $\sigma^L(A)$ has no eigenvalue inside the cone $\Lambda_{\theta,\phi}$ delimited by $\Gamma_{\theta,\phi}$ then $\Pi_{\theta,\phi}$ which is a finite rank projection, is smoothing. When θ and ϕ differ by a multiple of 2π, $\Pi_{\theta,\phi}$ vanishes.

Lemma 2.1.8 ([KV1, Ok1]). *Let A be an admissible operator in $C\ell(M,E)$ with spectral cut θ. Then $\log_\theta(A)$ is a ΨDO of order ϵ for any $\epsilon > 0$. In some local chart, the symbol of $\log_\theta A$ reads:*

$$\sigma(\log_\theta A)(x,\xi) = a \log|\xi| I + \sigma_\theta^A(x,\xi) \tag{2.10}$$

where a denotes the order of A and σ_θ^A is a symbol of order zero.
Moreover, the leading symbol of σ_θ^A is given by

$$(\sigma_\theta^A)^L(x,\xi) = \log_\theta\left(\sigma^L(A)(x, \frac{\xi}{|\xi|})\right), \ \forall (x,\xi) \in T^*M - \{0\}.$$

Remark 2.1.9. *In particular, if $\sigma(A)$ has scalar leading symbol then so that has σ_θ^A. Notice that $\sigma(\Pi_{\theta,\phi}(A))$ then also has scalar leading symbol, which confirms the independence of this property on the choice of spectral cut.*

Proof: Given a local trivialization over a local chart, the symbol of the operator A_θ^z has the formal expansion $\sigma(A_\theta^z) \sim \sum_{j\geq 0} b_{az-j}^{(z)}$ where a is the order of A and $b_{az-j}^{(z)}$, given by (2.4), is a positively homogeneous function of degree $az-j$. Since $\log_\theta A = A\,\left(\partial_z A_\theta^{z-1}\right)_{|z=0}$, we have

$$\sigma(\log_\theta A) \sim \sigma(A) \star \sigma\left(\partial_z A_\theta^{z-1}\right)_{|z=0}.$$

Suppose that $\xi \neq 0$; using the positive homogeneity of the components, we have for $j \geq 0$,

$$b_{az-a-j}^{(z-1)}(x,\xi) = |\xi|^{az-a-j} b_{az-a-j}^{(z-1)}\left(x, \frac{\xi}{|\xi|}\right)$$

and hence

$$\partial_z b_{az-a-j}^{(z-1)}(x,\xi) = a\log|\xi||\xi|^{az-a-j} b_{az-a-j}^{(z-1)}\left(x, \frac{\xi}{|\xi|}\right) + |\xi|^{az-a-j}\partial_z b_{az-a-j}^{(z-1)}\left(x, \frac{\xi}{|\xi|}\right).$$

It follows that

$$\left(\partial_z b_{az-a-j}^{(z-1)}(x,\xi)\right)_{|z=0} = a\log|\xi|\, b_{-a-j}^{(-1)}(x,\xi) + |\xi|^{-a-j}\left(\partial_z b_{az-a-j}^{(z-1)}\left(x, \frac{\xi}{|\xi|}\right)\right)_{|z=0}.$$

Hence $\left(\partial_z A_\theta^{z-1}\right)_{|z=0}$ has symbol $\left(\partial_z b^{(z-1)}(x,\xi)\right)_{|z=0}$ of the form

$$a\,\log|\xi|\,\sigma(A^{-1})(x,\xi) + \tau(A)(x,\xi)$$

with $\tau(A)$ a classical symbol of order $-a$ whose homogeneous component of degree $-a-j$ reads:

$$\tau(A)_{-a-j}(x,\xi) = |\xi|^{-a-j}\left(\partial_z b_{az-a-j}^{(z-1)}\left(x, \frac{\xi}{|\xi|}\right)\right)_{|z=0}.$$

Thus the operator $\log_\theta A = A\,\left(\partial_z A_\theta^{z-1}\right)_{|z=0}$ has a symbol of the form

$$a\,\log|\xi| + \sigma_\theta^A(x,\xi),$$

where

$$\sigma_\theta^A(x,\xi) \sim \sum_{k=0}^{\infty}\sum_{i+j+|\alpha|=k}\frac{1}{\alpha!}\,\partial_\xi^\alpha\sigma(A)_{a-i}(x,\xi)\,D_x^\alpha\tau(A)_{-a-j}(x,\xi) \tag{2.11}$$

is a classical symbol of order zero. Its leading symbol reads

$$\begin{aligned}(\sigma_\theta^A)^L(x,\xi) &= \sigma^L(A)(x,\xi)\,|\xi|^{-a}\left(\partial_z b_{az-a}^{(z-1)}\left(x,\frac{\xi}{|\xi|}\right)\right)_{|z=0}\\ &= \sigma^L(A)(x,\xi)\left(\partial_z\left(\sigma^L(A)\left(x,\frac{\xi}{|\xi|}\right)\right)_\theta^{z-1}\right)_{|z=0}\\ &= \log_\theta \sigma^L(A)\left(x,\frac{\xi}{|\xi|}\right)\end{aligned}$$

for any (x,ξ) in $T^*M-\{0\}$ $\square$

As it can be seen from (2.10), logarithms of classical pseudo-differential operators are not classical anymore; powers of the logarithm of a given admissible operator combined with all classical pseudodifferential operators generate the algebra of log-polyhomogenous operators.

Lemma 2.1.10. *Let A and B be admissible operators in $C\ell(M,E)$ with spectral cuts θ and ϕ respectively. Then*

$$\frac{\log_\theta A}{a}-\frac{\log_\phi B}{b}\in C\ell^0(M,E), \tag{2.12}$$

where a is the order of A and b the order of B.

Remark 2.1.11. *Let us point out that this statement does not depend on the choice of spectral cuts since by formula (2.9) a modification of the spectral cut only modifies the logarithm by zero order classical ΨDOs.*

Proof: In some local chart, using formula (2.10), we have

$$\sigma\left(\frac{\log_\theta A}{a}\right)(x,\xi)=\log|\xi|I+\frac{1}{a}\,\sigma_\theta^A(x,\xi)\quad\text{and}\quad\sigma\left(\frac{\log_\phi B}{b}\right)(x,\xi)=\log|\xi|I+\frac{1}{b}\,\sigma_\phi^B(x,\xi)$$

where σ_θ^A and σ_ϕ^B are classical symbols of zero order. Hence

$$\sigma\left(\frac{\log_\theta A}{a}-\frac{\log_\phi B}{b}\right)(x,\xi)=\sigma(\frac{\log_\theta A}{a})(x,\xi)-\sigma(\frac{\log_\phi B}{b})(x,\xi)=\frac{1}{a}\,\sigma_\theta^A(x,\xi)-\frac{1}{b}\,\sigma_\phi^B(x,\xi)$$

and the result follows. $\square$

Lemma 2.1.12. *Let A be an operator in $C\ell(M,E)$ and let B be an admissible operator in $C\ell(M,E)$ with spectral cut ϕ; then*

$$[A,\log_\phi B]\in C\ell(M,E)$$

where $[R,S]$ stands for the commutator $RS-SR$.

Proof: Let us evaluate $\sigma([A, \log_\phi B])$. As already pointed out in Remark 2.1.11, the result does not depend on the choice of the spectral cut. We know that

$$\sigma([A, \log_\phi B]) \sim [\sigma(A), \sigma(\log_\phi B)]_\star$$

where for two symbols σ, δ the star bracket is given by $[\sigma, \delta]_\star := \sigma \star \delta - \delta \star \sigma$.
Since by (2.10), $\sigma(\log_\phi B)(x,\xi) = b \log|\xi| I + \sigma_\phi^B(x,\xi)$ where $\sigma_\phi^B(x,\xi)$ is a classical symbol of zero order,

$$[\sigma(A), \sigma(\log_\phi B)]_\star(x,\xi) = [\sigma(A)(x,\xi), b\log|\xi| I + \sigma_\phi^B(x,\xi)] = [\sigma(A)(x,\xi), \sigma_\phi^B(x,\xi)]$$

and the result follows. □

Corollary 2.1.13. *Let A be an admissible operator in $C\ell(M,E)$ with spectral cut θ; then $\mathrm{ad}_{\log_\theta A}$ is a derivation on $C\ell(M,E)$ and $C\ell^{\star,\star}(M,E)$.*

Proof: For any log-polyhomogeneous operators B, C in $C\ell^{\star,\star}(M,E)$,

$$\mathrm{ad}_{\log_\theta A}(BC) = [\log_\theta A, BC] = [\log_\theta A, B]C + B[\log_\theta A, C] = \mathrm{ad}_{\log_\theta A}(B)C + B\mathrm{ad}_{\log_\theta A}(C).$$

Since $C\ell^{\star,\star}(M,E)$ is an algebra, it follows that $[\log_\theta A, BC]$, $[\log_\theta A, B]C$ and $B[\log_\theta A, C]$ belong to $C\ell^{\star,\star}(M,E)$ and the result follows. If B and C are classical operators, since the three commutators belong to $C\ell(M,E)$ so does $\mathrm{ad}_{\log_\theta A}$. □

For further use let us investigate the differentiability of a logarithm of a differentiable family of admissible ΨDOs. A family A_t of classical operators in $C\ell(M,E)$ is said to be differentiable if it satisfies the requirements of Definition 1.2.10 replacing holomorphic by differentiable.

Proposition 2.1.14 ([OP]). *Let A_t be a differentiable family of admissible operators in $C\ell(M,E)$ with constant spectral cut θ. Then for any positive integer K, we have*

$$\frac{d}{dt}\log_\theta A_t = \dot{A}_t A_t^{-1} + \sum_{k=1}^{K} \frac{(-1)^k}{k+1} \mathrm{ad}_{A_t}^k(\dot{A}_t)\, A_t^{-(k+1)} + R_K(A_t, \dot{A}_t)$$

where we have set $\dot{A}_t := \frac{d}{dt}A_t$ and

$$R_K(A_t, \dot{A}_t) := -\frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda \left[(\lambda - A_t)^{-1}, \mathrm{ad}_{A_t}^K(\dot{A}_t)\right](\lambda - A_t)^{-K-1}\, d\lambda,$$

since A_t commutes with $(\lambda - A_t)^{-k}$.
If A_t commutes with $\dot{A}_t$, then $\frac{d}{dt}\log_\theta A_t = \dot{A}_t A_t^{-1}$. If A_t has constant order a, then $\frac{d}{dt}\log_\theta A_t$ lies in $C\ell(M,E)$.

Proof: By (4.3) in [Ok1] we first observe that for any t in a compact neighborhood K_{t_0} of t_0, one can bound the modulus of the order $\alpha(t)$ from above by some integer k, in which case

$$\exists C > 0, \quad \forall t \in K_{t_0}, \quad \|(A_t - \lambda)^{-1}\|_{s,s-k} \leq |\lambda^{-1}|,$$

where $\|\cdot\|_{s,s'}$ stands for the operator norm of bounded operators from the Sobolev closure $H^s(M,E)$ to the Sobolev closure $H^{s'}(M,E)$ of $\Gamma(M,E)$. Moreover, $(A_t - \lambda)^{-1}$ is differentiable at t_0 with derivative given by:

$$\frac{d}{dt}_{|t=t_0} (A_t - \lambda)^{-1} = -(A_{t_0} - \lambda)^{-1} \dot{A}_{t_0} (A_{t_0} - \lambda)^{-1}$$

as a consequence of the identity

$$(\lambda - A_t)^{-1} - (\lambda - A_{t_0})^{-1} = (t - t_0)(\lambda - A_{t_0})^{-1} \Delta_t (\lambda - A_t)^{-1},$$

where we have set $\Delta_t := \frac{A_t - A_{t_0}}{t - t_0}$.
For operators A_t of zero order this leads to

$$\begin{aligned} \frac{d}{dt}_{|t=t_0} \log_\theta A_t &= \frac{i}{2\pi} \int_{\Gamma_\theta} \log_\theta \lambda \frac{d}{dt}_{|t=t_0} (A_t - \lambda)^{-1} d\lambda \\ &= -\frac{i}{2\pi} \int_{\Gamma_\theta} \log_\theta \lambda (A_{t_0} - \lambda)^{-1} \dot{A}_{t_0} (A_{t_0} - \lambda)^{-1} d\lambda, \end{aligned}$$

where we have set $\dot{A}_{t_0} = \frac{d}{dt}_{|t=t_0} A_t$.
In order to generalize this to higher order operators, we need to consider the family (see (2.8)):

$$\log_\theta A_t A_t^{-1} = \frac{i}{2\pi} \int_{\Gamma_\theta} \log_\theta \lambda \lambda^{-1} (A_t - \lambda)^{-1} d\lambda$$

for which we can also write:

$$\begin{aligned} \frac{d}{dt}_{|t=t_0} \left(\log_\theta A_t A_t^{-1}\right) &= \frac{i}{2\pi} \int_{\Gamma_\theta} \log_\theta \lambda \lambda^{-1} \frac{d}{dt}_{|t=t_0} (A_t - \lambda)^{-1} d\lambda \\ &= -\frac{i}{2\pi} \int_{\Gamma_\theta} \log_\theta \lambda \lambda^{-1} (A_{t_0} - \lambda)^{-1} \dot{A}_{t_0} (A_{t_0} - \lambda)^{-1} d\lambda. \end{aligned}$$

This leads to the expected formula

$$
\begin{aligned}
\frac{d}{dt}_{|t=t_0} \log_\theta A_t &= \frac{d}{dt}_{|t=t_0} \left(\log_\theta A_t\, A_t^{-1}\right)\, A_t + \left(\log_\theta A_{t_0}\, A_{t_0}^{-1}\right)\, \dot{A}_{t_0} \\
&= -\frac{i}{2\pi} \int_{\Gamma_\theta} \log_\theta \lambda\, \lambda^{-1} \, (A_{t_0} - \lambda)^{-1}\, \dot{A}_{t_0}\, (A_{t_0} - \lambda)^{-1}\, A_{t_0}\, d\lambda \\
&\quad + \frac{i}{2\pi} \int_{\Gamma_\theta} \log_\theta \lambda\, \lambda^{-1}\, (A_{t_0} - \lambda)^{-1} \dot{A}_{t_0}\, d\lambda \\
&= -\frac{i}{2\pi} \int_{\Gamma_\theta} \log_\theta \lambda\, \lambda^{-1}\, (A_{t_0} - \lambda)^{-1}\, \dot{A}_{t_0}\, (A_t - \lambda)^{-1}\, (A_{t_0} - (A_{t_0} - \lambda))\, d\lambda \\
&= -\frac{i}{2\pi} \int_{\Gamma_\theta} \log_\theta \lambda\, (A_{t_0} - \lambda)^{-1}\, \dot{A}_{t_0}\, (A_{t_0} - \lambda)^{-1}\, d\lambda.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& \left[(\lambda - A_{t_0})^{-1}, \dot{A}_{t_0}\right] \\
= \;& (\lambda - A_{t_0})^{-1} \left(\dot{A}_{t_0}(\lambda - A_{t_0}) - (\lambda - A_{t_0})\dot{A}_{t_0}\right) (\lambda - A_{t_0})^{-1} \\
= \;& (\lambda - A_{t_0})^{-1}\, [A_{t_0}, \dot{A}_{t_0}]\, (\lambda - A_{t_0})^{-1} \\
= \;& [A_{t_0}, \dot{A}_{t_0}](\lambda - A_{t_0})^{-2} + \left[(\lambda - A_{t_0})^{-1}, [A_{t_0}, \dot{A}_{t_0}]\right] (\lambda - A_{t_0})^{-1} \\
= \;& [A_{t_0}, \dot{A}_{t_0}](\lambda - A_{t_0})^{-2} + (\lambda - A_{t_0})^{-1} \left[A_{t_0}, [A_{t_0}, \dot{A}_{t_0}]\right] (\lambda - A_{t_0})^{-2} \\
= \;& [A_{t_0}, \dot{A}_{t_0}](\lambda - A_{t_0})^{-2} + [A_{t_0}, [A_{t_0}, \dot{A}_{t_0}]](\lambda - A_{t_0})^{-3} \\
& + \left[(\lambda - A_{t_0})^{-1}, [A_{t_0}, [A_{t_0}, \dot{A}_{t_0}]]\right] (\lambda - A_{t_0})^{-2} \\
= \;& [A_{t_0}, \dot{A}_{t_0}](\lambda - A_{t_0})^{-2} + \mathrm{ad}^2_{A_{t_0}}(\dot{A}_{t_0})\, (\lambda - A_{t_0})^{-3} \\
& + \left[(\lambda - A_{t_0})^{-1}, \mathrm{ad}^2_{A_{t_0}}(\dot{A}_{t_0})\right] (\lambda - A_{t_0})^{-2} \\
= \;& \sum_{k=1}^{K} \mathrm{ad}^k_{A_{t_0}}(\dot{A}_{t_0})(\lambda - A_{t_0})^{-(k+1)} + \left[(\lambda - A_{t_0})^{-1}, \mathrm{ad}^K_{A_{t_0}}(\dot{A}_{t_0})\right] (\lambda - A_{t_0})^{-K}.
\end{aligned}
$$

Hence,

$$\begin{aligned}
\frac{d}{dt}_{|t=t_0} \log_\theta A_t &= -\frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda \left[(A_{t_0}-\lambda)^{-1}, \dot{A}_{t_0}\right](A_{t_0}-\lambda)^{-1}\, d\lambda \\
&\quad -\frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\, \dot{A}_{t_0}\,(A_{t_0}-\lambda)^{-1}\, d\lambda \\
&= \sum_{k=0}^{K} \mathrm{ad}^k_{A_{t_0}}(\dot{A}_{t_0}) \frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\,(\lambda - A_{t_0})^{-(k+2)}\, d\lambda \\
&\quad +\frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\, \left[(\lambda - A_{t_0})^{-1}, \mathrm{ad}^K_{A_{t_0}}(\dot{A}_{t_0})\right](\lambda - A_{t_0})^{-K-1}\, d\lambda.
\end{aligned}$$

A Cauchy integral yields

$$\begin{aligned}
\frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\,(\lambda - A_{t_0})^{-2}\, d\lambda &= \frac{i}{2\pi}\int_{\Gamma_\theta} \lambda_\theta^{-1}\,(\lambda - A_{t_0})^{-1}\, d\lambda \\
&= -A_{t_0}^{-1}.
\end{aligned}$$

Similarly, by integration by parts,

$$\begin{aligned}
\frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\,(\lambda - A_{t_0})^{-3}\, d\lambda &= \frac{1}{2}\frac{i}{2\pi}\int_{\Gamma_\theta} \lambda_\theta^{-1}\,(\lambda - A_0)^{-2}\, d\lambda \\
&= -\frac{1}{2}\frac{i}{2\pi}\int_{\Gamma_\theta} \lambda_\theta^{-2}\,(\lambda - A_{t_0})^{-1}\, d\lambda \\
&= \frac{1}{2}A_{t_0}^{-2}.
\end{aligned}$$

Iterating this procedure yields

$$\frac{d}{dt}_{|t=t_0} \log_\theta A_t = \sum_{k=0}^{K} \frac{(-1)^k}{k+1} \mathrm{ad}^k_{A_{t_0}}(\dot{A}_{t_0}) A_{t_0}^{-(k+1)} + R_K(A_{t_0}, \dot{A}_{t_0}),$$

where we have set

$$R_K(A_{t_0}, \dot{A}_{t_0}) = -\frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\, \left[(\lambda - A_{t_0})^{-1}, \mathrm{ad}^K_{A_{t_0}}(\dot{A}_{t_0})\right](\lambda - A_{t_0})^{-K-1}\, d\lambda.$$

□

Corollary 2.1.15. *Let A and B be admissible operators in $C\ell(M,E)$ with positive orders a, b and spectral cuts θ and ϕ respectively and such that AB is also admissible with spectral cut ψ depending on the choice of θ and ϕ. Let us assume that for each t in $[0,1]$, the operator $A^t_\theta B$ has spectral cut $\psi(t)$, where $t \to \psi(t)$ is continuous. Set $\psi(0) = \phi$ and $\psi(1) = \psi$. If A commutes with B, then*

$$\log_\psi AB = \log_\theta A + \log_\phi B. \tag{2.13}$$

Proof: Let us consider the family $A_t = A^t_\theta B$. It is a differentiable family of admissible operators. Following arguments similar to Okikiolu's (see [Ok2]), we can build a finite partition $\bigcup_{k=1}^K J_k$ of $[0,1]$ in such a way that we can choose on each of the intervals $J_k = [t_k, t_{k+1}]$ a common fixed spectral cut ψ_k of $A^t_\theta B$ when t varies in J_k. Indeed, the angle ψ_k is close to each angle $\psi_k(t)$ in the sense that there is ϵ such that there are no eigenvalues of $A^t_\theta B$ in the cone $\Lambda_{\psi_k-\epsilon,\psi_k(t)-\epsilon}$. We have $\dot{A}_t = \frac{d}{dt} A_t = \log_\theta A\, A^t B$. Since A commutes with B, A_t commutes with $\dot{A}_t$. Then applying the above proposition to each interval J_k, we obtain

$$\frac{d}{dt} \log_{\psi(t)} A_t = \dot{A}_t A_t^{-1} = \log_\theta A.$$

Integrating from $t = 0$ to $t = 1$, it follows that

$$\int_0^1 \left(\frac{d}{dt} \log_{\psi_k} A_t \right) dt = \log_\psi AB - \log_\phi B = \log_\theta A.$$

□

We introduce the following class of Ψ*DOs* defined in [Ok2] and further used in [B]. A pseudodifferential operator $A : \Gamma(M,E) \to \Gamma(M,E)$ is logarithmic if its symbol in local chart has an asymptotic expansion of the form

$$\sigma(A)(x,\xi) \sim \gamma \log|\xi| + \sum_{j=0}^{\infty} \chi(\xi)\sigma_{a-j}(A)(x,\xi), \qquad (x,\xi) \in T^\star M, \quad a, \gamma \in \mathbb{C}, \tag{2.14}$$

where each term $\sigma_{a-j}(A)(x,\xi)$ is positively homogeneous in ξ of degree $a-j$. The number a is called the degree of the logarithmic Ψ*DO* A and the number γ is called the type of this operator. We denote the set of logarithmic Ψ*DOs* of degree a and type γ ((a,γ)-logarithmic) by $\widetilde{C\ell}^{a,\gamma}(M,E)$. Notice that

- A logarithm is a logarithmic operator of zero degree i.e. $a = 0$.
- $\widetilde{C\ell}^{a,\gamma}(M,E) \subset C\ell^{a,1}(M,E)$ i.e. a logarithmic Ψ*DO* is a log-polyhomogeneous operator.

- $\widetilde{C\ell}^{a,0}(M,E) = C\ell^a(M,E)$ i.e. a classical operator is a logarithmic operator.

We set:

$$\widetilde{C\ell}^{a}(M,E) = \bigcup_{\gamma\in\mathbb{C}} \widetilde{C\ell}^{a,\gamma}(M,E), \quad \widetilde{C\ell}(M,E) = \bigcup_{a\in\mathbb{C}} \widetilde{C\ell}^{a}(M,E)$$

A logarithmic operator A in $\widetilde{C\ell}^{a}(M,E)$ is odd-class if in the asymptotic expansion (2.14), each term satisfies

$$\sigma_{a-j}(A)(x,-\xi) = (-1)^{a-j}\sigma_{a-j}(A)(x,\xi), \quad \text{for} \quad |\xi|\geq 1.$$

Denote by $\widetilde{C\ell}^{a,\gamma}_{odd}(M,E)$ (resp. $\widetilde{C\ell}_{odd}(M,E)$) the set of odd-class (a,γ)-logarithmic (resp. odd-class logarithmic) pseudodifferential operators.

Lemma 2.1.16.

1. *If $A,B \in \widetilde{C\ell}(M,E)$, then $[A,B]\in C\ell(M,E)$.*

2. *If $A,B \in \widetilde{C\ell}_{odd}(M,E)$, then $[A,B]\in C\ell_{odd}(M,E)$.*

3. *If A in $C\ell(M,E)$ is admissible with spectral cut θ, then $\mathrm{ad}_{\log_\theta A}$ is a derivation on $\widetilde{C\ell}(M,E)$.*

Proof: Let us prove the first item. Suppose that A belongs to $\widetilde{C\ell}^{a,\gamma}(M,E)$ and B belongs to $\widetilde{C\ell}^{p,\delta}(M,E)$. Then we have $\sigma(A)(x,\xi)=\gamma\log|\xi|+\sigma(A_1)(x,\xi)$ and $\sigma(B)(x,\xi)=\delta\log|\xi|+\sigma(B_1)(x,\xi)$ where A_1,B_1 are classical ΨDOs of order a,b respectively. Hence

$$\begin{aligned}
&\sigma(AB)(x,\xi)\\
\sim\ & \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha\sigma(A)(x,\xi)D_x^\alpha\sigma(B)(x,\xi)\\
\sim\ & \gamma\delta\log^2|\xi| + \gamma\log|\xi|\sigma(B_1)(x,\xi) + \delta\log|\xi|\sigma(A_1)(x,\xi) + \sigma(A_1)(x,\xi)\sigma(B_1)(x,\xi)\\
& +\gamma\sum_{\alpha\neq 0}\frac{1}{\alpha!}\partial_\xi^\alpha\log|\xi|D_x^\alpha\sigma(B_1)(x,\xi) + \delta\sum_{\alpha\neq 0}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma(A_1)(x,\xi)D_x^\alpha\sigma(B_1)(x,\xi).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sigma(BA)(x,\xi)\\
\sim\ & \gamma\delta\log^2|\xi| + \delta\log|\xi|\sigma(A_1)(x,\xi) + \gamma\log|\xi|\sigma(B_1)(x,\xi) + \sigma(B_1)(x,\xi)\sigma(A_1)(x,\xi)\\
& +\delta\sum_{\alpha\neq 0}\frac{1}{\alpha!}\partial_\xi^\alpha\log|\xi|D_x^\alpha\sigma(A_1)(x,\xi) + \gamma\sum_{\alpha\neq 0}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma(B_1)(x,\xi)D_x^\alpha\sigma(A_1)(x,\xi).
\end{aligned}$$

It follows that

$$\begin{aligned}
& \sigma([A,B])(x,\xi) \\
\sim\ & [\sigma(A_1)(x,\xi),\sigma(B_1)(x,\xi)] \\
& +\gamma\sum_{\alpha\neq 0}\frac{1}{\alpha!}\partial_\xi^\alpha \log|\xi| D_x^\alpha \sigma(B_1)(x,\xi)+\delta\sum_{\alpha\neq 0}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma(A_1)(x,\xi)D_x^\alpha\sigma(B_1)(x,\xi) \\
& -\delta\sum_{\alpha\neq 0}\frac{1}{\alpha!}\partial_\xi^\alpha \log|\xi| D_x^\alpha \sigma(A_1)(x,\xi)-\gamma\sum_{\alpha\neq 0}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma(B_1)(x,\xi)D_x^\alpha\sigma(A_1)(x,\xi).
\end{aligned}$$

Since $\partial_\xi^\alpha \log|\xi|$ is homogeneous of degree $-|\alpha|$, $[A,B]$ lies in $C\ell(M,E)$.
Furthermore

$$\begin{aligned}
\sigma([A,B])_{a+b-j}(x,\xi) &= \sigma_a(A)(x,\xi)\sigma_{b-j}(B)(x,\xi)-\sigma_{a-j}(A)(x,\xi)\sigma_b(B)(x,\xi) \\
&\quad +\sum_{k+l+|\alpha|=j,l<j}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma_{a-k}(A)(x,\xi)D_x^\alpha\sigma_{b-l}(B)(x,\xi)
\end{aligned}$$

and the second item follows. Moreover, if A is an admissible operator, $\log_\theta A$ belongs to $\widetilde{C\ell}(M,E)$ and hence the commutator $[\log_\theta A, B]$ belongs to $C\ell(M,E)$ which is include on $\widetilde{C\ell}(M,E)$. The third item follows. $\square$

Let us introduce some notations. In what follows $\mathcal{L}^{(k)}(P,Q)$ denotes a Lie monomial of degree k, i.e. a linear combination of expressions of the form $[R_k,[R_{k-1},\cdots[R_3,[R_2,R_1]]\cdots]]$ where each of the elements R_i is either P or Q.

Lemma 2.1.17.

1. *Let A and B be admissible operators in $C\ell(M,E)$ with spectral cuts θ and ϕ respectively. Then*
   - *(a)* $\mathrm{ad}^k_{\log_\theta A}(\log_\phi B)\in C\ell^0(M,E),\quad \forall k>0.$
   - *(b)* $\mathcal{L}^{(k)}(\log_\theta A,\log_\phi B)\in C\ell^0(M,E),\quad \forall k>1.$
2. *Moreover if A and B have scalar leading symbols, then*
   - *(a)* $\mathrm{ad}^k_{\log_\theta A}(\log_\phi B)\in C\ell^{-k}(M,E),\quad \forall k>0.$
   - *(b)* $\mathcal{L}^{(k)}(\log A,\log B)\in C\ell^{-k+1}(M,E),\quad \forall k>1.$

Remark 2.1.18. *Here again, as already pointed out in Remark 2.1.11, the result does not depend on the choice of the spectral cut.*

Proof: For simplicity, we drop the subscripts which specify the spectral cuts.

1. We prove the results by induction on k.

   (a) For $k=1$, since

   $$\mathrm{ad}_{\log A}(\log B) = [\log A, \log B] = \left[\log A, \log B - \frac{b}{a}\log A\right]$$

   where a is the order of A and b is the order of B, by Lemma 2.1.10 $\log B - \frac{b}{a}\log A$ is a classical zero order operator and by Lemma 2.1.12, the operator bracket $[\log A, \log B]$ is a classical operator. Let us prove that it is zero order.
   By (2.10), $\sigma(\log_\theta A)(x,\xi) = a\log|\xi| I + \sigma_\theta^A(x,\xi)$ and $\sigma(\log_\phi B)(x,\xi) = b\log|\xi| I + \sigma_\phi^B(x,\xi)$. As before we define the bracket of symbols $[\sigma,\tau]_\star$ to be $\sigma\star\tau - \tau\star\sigma$. We have

   $$\begin{aligned}[]
   [\sigma(\log A), \sigma(\log B)]_\star (x,\xi) &\sim [\sigma(\log A)(x,\xi), \sigma(\log B)(x,\xi)] \\
   &\sim \left[\sigma_\theta^A(x,\xi), \sigma_\phi^B(x,\xi)\right].
   \end{aligned}$$

   Since σ_ϕ^B and σ_ϕ^B are classical symbols of zero order, $[\log A, \log B]$ is a classical zero order operator.
   Let us now assume that the property holds for a given positive integer k, i.e. $\mathrm{ad}^k_{\log A}(\log B)$ belongs to $C\ell^0(M,E)$. Then

   $$\mathrm{ad}^{k+1}_{\log A}(\log B) = \left[\log A, \mathrm{ad}^k_{\log A}(\log B)\right]$$

   and

   $$\sigma(\mathrm{ad}^{k+1}_{\log A}(\log B)) \sim \left[\sigma(\log A), \sigma(\mathrm{ad}^k_{\log A}(\log B))\right]_\star \sim \left[\sigma_\theta^A, \sigma(\mathrm{ad}^k_{\log A}(\log B))\right]_\star.$$

   Since σ_0^A and $\sigma(\mathrm{ad}^k_{\log A}(\log B))$ are classical symbols of zero order, the result follows.

   (b) For $k=2$, $\mathcal{L}^{(2)}(\log A, \log B)$ is a linear combination of expressions of the type $[\log A, \log B] = \mathrm{ad}_{\log A}(\log B)$ so that $\mathcal{L}^{(2)}(\log A, \log B)$ lies in $C\ell^0(M,E)$. Now, assuming that $\mathcal{L}^{(k)}(\log A, \log B)$ is a linear combination of expressions of the type $\left[\log G, \mathcal{L}^{(k-1)}(\log A, \log B)\right] = \mathrm{ad}_{\log G}(\mathcal{L}^{(k-1)}(\log A, \log B))$ where $\log G = \log A$ or $\log G = \log B$ we have

   $$\mathcal{L}^{(k+1)}(\log A, \log B) = \left[\log G, \mathcal{L}^{(k)}(\log A, \log B)\right] = \mathrm{ad}_{\log G}(\mathcal{L}^{(k)}(\log A, \log B)).$$

   By assumption $\mathcal{L}^{(k)}(\log A, \log B)$ lies in $C\ell^0(M,E)$ and hence $\mathcal{L}^{(k+1)}(\log A, \log B)$ lies in $C\ell^0(M,E)$.

2. Here we will prove the result for step (a). The step (b) can be proved using the arguments of step (b) of 1). We prove the result by induction on k. For $k = 1$, by item 1, the bracket $[\log A, \log B]$ has zero order. We have

$$[\sigma(\log A), \sigma(\log B)]_\star(x,\xi) \sim \left[\sigma_\theta^A(x,\xi), \sigma_\phi^B(x,\xi)\right].$$

Since A and B have scalar leading symbols, by Remark 2.1.9, so have σ_θ^A and σ_ϕ^B and hence $[\sigma_\theta^A, \sigma_\phi^B]^L = [(\sigma_\theta^A)^L, (\sigma_\phi^B)^L] = 0$. It follows that $[\log A, \log B]$ has order -1. If we assume that $\mathrm{ad}^k_{\log A}(\log B)$ lies in $C\ell^{-1}(M,E)$, then since

$$\sigma(\mathrm{ad}^{k+1}_{\log A}(\log B)) \sim \left[\sigma_\theta^A, \sigma(\mathrm{ad}^k_{\log A}(\log B))\right]_\star$$

and

$$\left(\sigma(\mathrm{ad}^{k+1}_{\log A}(\log B))\right)^L = \left[(\sigma_0^A)^L, (\sigma(\mathrm{ad}^k_{\log A}(\log B))^L\right] = 0.$$

The result follows.

□

In the following proposition, we recall well-known results about log-polyhomogeneous operators i.e. all log-polyhomogeneous operators can be written as finite linear combination of products of classical operators and logarithms.

Proposition 2.1.19. *Let Q be an admissible operator in $C\ell(M,E)$ with positive order q and spectral cut α. Then for $k \geq 0$,*

1. $(\log_\alpha Q)^k C\ell(M,E) \subset \bigoplus_{l=0}^{k} C\ell(M,E)(\log_\alpha Q)^l$.

2. $C\ell^{\star,k}(M,E) = \bigoplus_{l=0}^{k} C\ell(M,E)\,(\log_\alpha Q)^l = \bigoplus_{l=0}^{k} (\log_\alpha Q)^l\, C\ell(M,E)$.

Proof: We proceed by induction on k; let A be a classical operator in $C\ell(M,E)$.

1. For $k = 1$, $\log_\alpha Q\, A = A \log_\alpha Q + [\log_\alpha Q, A]$. Since $[\log_\alpha Q, A]$ is a classical operator, $(\log_\alpha Q)\, A$ lies in $C\ell(M,E) \bigoplus C\ell(M,E) \log_\alpha Q$.

   Assume that $(\log_\alpha Q)^{k-1} A$ lies in $\bigoplus_{l=0}^{k-1} C\ell(M,E)(\log_\alpha Q)^l$. Then

$$\begin{array}{rcl}(\log_\alpha Q)^k A &=& (\log_\alpha Q)^{k-1} (\log_\alpha Q)\, A \\ &=& (\log_\alpha Q)^{k-1} A \log_\alpha Q + (\log_\alpha Q)^{k-1} [\log_\alpha Q, A]\end{array}$$

   Since by the induction assumption, $(\log_\alpha Q)^{k-1}[\log_\alpha Q, A]$ and $(\log_\alpha Q)^{k-1} A$ belong to $\bigoplus_{l=0}^{k-1} C\ell(M,E)(\log_\alpha Q)^l$, the result follows.

2. From 1) it follows that one can put the powers of the logarithms either on the l.h.s. or on the r.h.s.. Indeed, $(\log_\alpha Q)^k C\ell(M,E) \subset \bigoplus_{l=0}^k C\ell(M,E)(\log_\alpha Q)^l$ implies that

$$\bigoplus_{l=0}^k (\log_\alpha Q)^l C\ell(M,E) \subset \bigoplus_{l=0}^k C\ell(M,E)(\log_\alpha Q)^l.$$

On the other hand, by the same argument, one can write

$$\bigoplus_{l=0}^k C\ell(M,E)\,(\log_\alpha Q)^l \subset \bigoplus_{l=0}^k (\log_\alpha Q)^l\, C\ell(M,E).$$

The second identity follows.
Let us check that $\bigoplus_{l=0}^k C\ell(M,E)\,(\log_\alpha Q)^l = C\ell^{\star,k}(M,E)$.
The inclusion from left to right is straightforward. Indeed, let $A = \sum_{l=0}^k A_l (\log_\alpha Q)^l$ be an element of $\bigoplus_{l=0}^k C\ell(M,E)\,(\log_\alpha Q)^l$. Since $(\log_\alpha Q)^l$ lies in $C\ell^{\star,k}(M,E)$ for any $l \le k$, it follows that A belongs to $C\ell^{\star,k}(M,E)$.
To show the inclusion from right to left, let us show by induction on k that any element A of $C\ell^{\star,k}(M,E)$ can be written on the form $A = \sum_{l=0}^k A_l (\log_\alpha Q)^l$.
This clearly holds for $k = 0$. Let us assume it holds for k. Since $\log_\alpha Q$ has a symbol of the form $q \log|\xi| + \sigma_Q^0$ with σ_Q^0 a classical symbol, $(\log_\alpha Q)^k$ has a symbol of the form $q^k \log^k|\xi| + \sigma_Q^{k-1}$ with σ_Q^{k-1} a log-polyhomogeneous symbol of log-type $k-1$. It follows that to any A in $C\ell^{\star,k}(M,E)$ with symbol $\sigma(A) = \sum_{l=0}^k \sigma_l(A) \log^l|\xi|$, using a partition of unity adapted to a finite trivializing covering of M for E,we can associate a classical operator

$$A_k := Op(\sigma_k(A)).$$

Then

$$B_1 := A - \frac{1}{q^k} A_k (\log_\alpha Q)^k \in C\ell^{\star,k-1}(M,E).$$

Similarly, to the operator B_1 we can associate a classical operator

$$A_{k-1} := Op(\sigma_{k-1}(B_1)).$$

Hence

$$B_2 := B_1 - \frac{1}{q^{k-1}} A_{k-1}(\log_\alpha Q)^{k-1} = A - \frac{1}{q^k} A_k(\log_\alpha Q)^k - \frac{1}{q^{k-1}} A_{k-1}(\log_\alpha Q)^{k-1}$$

lies in $C\ell^{\star,k-2}(M,E)$. Iterating this procedure, we build a sequence of classical operators $A_l, l = 1, \cdots, k$ such that

$$B_k := A - \sum_{l=1}^{k} \frac{1}{q^l} A_l(\log_\alpha Q)^l$$

is a classical operator.

□

This yields back well-known results (see [L]) concerning the structure of $C\ell^{\star,\star}(M,E)$.

Proposition 2.1.20. *The algebra* $C\ell^{\star,\star}(M,E) := \bigoplus_{k=0}^{\infty} C\ell^{\star,k}(M,E)$ *is a* $\mathbb{Z}$*-graded algebra.*

2.2 Symmetrized logarithms and odd-class pseudodifferential operators

Inspired by M. Braverman [B], we introduce the symmetrized logarithm of a odd-class classical admissible ΨDOs. We show that this symmetrized logarithm is also odd-class and we characterize the odd-class log-polyhomogeneous operators in terms of a finite linear combination of products of classical operators and symmetrized logarithms of admissible odd-class operators.

Let A be an admissible operator in $C\ell^a_{odd}(M,E)$ which admits spectral cuts θ and $\theta - a\pi$. The symmetrized logarithm of A is defined by the formula

$$\log_\theta^{\mathrm{sym}} A := \frac{1}{2}\left(\log_\theta A + \log_{\theta - a\pi} A\right). \tag{2.15}$$

Proposition 2.2.1. *The symmetrized logarithm of* A *in* $C\ell^a_{odd}(M,E)$ *with spectral cuts* θ *and* $\theta - a\pi$ *is an odd-class log-polyhomogeneous operator of log degree 1.*

Proof: The proof is similar to proofs in [B] (see also [Pa2]).
Assume that A is an odd-class classical operator of order a and θ, $\theta - a\pi$ are spectral cuts for A. By formulae (2.10), (2.11), since σ_θ^A is a classical ΨDO of zero order,

$$\sigma(\log_\theta(A))(x,\xi) = a\,\log|\xi| I + \sigma_\theta^A(x,\xi) \sim a\,\log|\xi| I + \sum_{j=0}^{\infty} \sigma_{-j}(\log_\theta(A))(x,\xi)$$

$$\sigma_{-j}(\log_\theta(A))(x,\xi) = |\xi|^{-j}\partial_z \sigma_{az-j}(A^z_\theta)(x,\frac{\xi}{|\xi|})_{|z=0}.$$

Since the operator $(A^z_\theta)_{|z=0}$ is equal to I, the identity of matrices, we have

$$\sigma_{az-j}(A^z_\theta)(x,\xi)_{|z=0} = \delta_{0,j} I.$$

We need to evaluate

$$\sigma_{-j}(\log^{\mathrm{sym}}_\theta A)(x,-\xi) = \frac{1}{2}\left(\sigma_{-j}(\log_\theta A)(x,-\xi) + \sigma_{-j}(\log_{\theta-a\pi} A)(x,-\xi)\right).$$

By formula (2.5) the homogeneous components of the symbol of A^z_θ are

$$\sigma_{az-j}(A^z_\theta)(x,\xi) = \frac{i}{2\pi}\int_{\Gamma_\theta} \lambda^z_\theta\, b_{-a-j}(x,\xi,\lambda)\, d\lambda.$$

with b_{-a-j}, the homogeneous components of the resolvent $(A-\lambda I)^{-1}$ defined in Section 2.1.2. These components $b_{-a-j}(x,\xi,\lambda)$ are positively homogeneous in $(\xi, \lambda^{\frac{1}{a}})$ i.e.

$$b_k(x, t\xi, t^{\frac{1}{a}}\lambda) = t^k b_k(x,\xi,\lambda) \quad \forall t>0, \quad \forall (x,\xi)\in T^\star M.$$

Moreover, if A belongs to $C\ell^a_{odd}(M,E)$, using the explicit formulae of b_{-a-j}, this extends to any real number t since we have

$$b_k(x,-\xi,(-1)^a\lambda) = (-1)^k b_k(x,\xi,\lambda). \tag{2.16}$$

Assume that $\operatorname{Re} z<0$. Then using formula (2.16) we can write

$$\begin{aligned}\sigma_{az-j}(A^z_\theta)(x,-\xi) &= \frac{i}{2\pi}\int_{\Gamma_\theta} \lambda^z_\theta\, b_{-a-j}(x,-\xi,\lambda)\, d\lambda \\ &= (-1)^{a+j}\frac{i}{2\pi}\int_{\Gamma_\theta} \lambda^z_\theta\, b_{-a-j}(x,\xi,(-1)^a\lambda)\, d\lambda.\end{aligned}$$

By a change of variable $\mu = e^{-ia\pi z}$, we obtain

$$\begin{aligned}\sigma_{az-j}(A^z_\theta)(x,-\xi) &= (-1)^{a+j}\frac{i}{2\pi}\int_{\Gamma_{\theta-a\pi}} (e^{ia\pi}\mu)^z_\theta\, b_{-a-j}(x,\xi,\mu)\, d(e^{ia\pi}\mu) \\ &= (-1)^{a+j} e^{ia\pi}\frac{i}{2\pi}\int_{\Gamma_{\theta-a\pi}} e^{iaz\pi}\mu^z_{\theta-a\pi}\, b_{-a-j}(x,\xi,\mu)\, d\mu \\ &= (-1)^j e^{iaz\pi}\sigma_{az-j}(A^z_{\theta-a\pi})(x,\xi).\end{aligned}$$

Thus

$$\sigma_{az-j}(A^z_\theta)(x,-\xi) = e^{i\pi(az+j)}\sigma_{az-j}(A^z_{\theta-a\pi})(x,\xi). \tag{2.17}$$

Since both the left and the right hand side of this equality are analytic in z, we conclude that the equality holds for all complex z in $\mathbb{C}$. Similarly,

$$\begin{aligned}\sigma_{az-j}(A^z_{\theta-a\pi})(x,-\xi) &= (-1)^{a+j}\frac{i}{2\pi}\int_{\Gamma_{\theta-a\pi}}(e^{-ia\pi}\mu)^z_\theta\, b_{-a-j}(x,\xi,\mu)\, d(e^{-ia\pi}\mu)\\ &= (-1)^{a+j}e^{-ia\pi}\frac{i}{2\pi}\int_{\Gamma_\theta} e^{-iaz\pi}\mu^z_{\theta-a\pi}\, b_{-a-j}(x,\xi,\mu)\, d\mu\\ &= (-1)^j e^{-iaz\pi}\sigma_{az-j}(A^z_\theta)(x,\xi).\end{aligned}$$

We therefore infer that

$$\sigma_{az-j}(A^z_{\theta-a\pi})(x,-\xi) = e^{-i\pi(az+j)}\sigma_{az-j}(A^z_\theta)(x,\xi). \tag{2.18}$$

Differentiating equation (2.17) w.r. to z yields:

$$\begin{aligned}\sigma_{az-j}(\partial_z A^z_\theta)(x,-\xi) &= \partial_z\left(\sigma_{az-j}(A^z_\theta)(x,-\xi)\right)\\ &= \partial_z\left(e^{i\pi(az+j)}\sigma_{az-j}(A^z_{\theta-a\pi})(x,\xi)\right)\\ &= e^{i\pi(az+j)}\left(ia\pi\sigma_{az-j}(A^z_{\theta-a\pi})(x,\xi)+\sigma_{az-j}(\partial_z A^z_{\theta-a\pi})(x,\xi)\right)\end{aligned}$$

i.e.

$$\sigma_{az-j}(\partial_z A^z_\theta)(x,-\xi) = e^{i\pi(az+j)}\left(ia\pi\sigma_{az-j}(A^z_{\theta-a\pi})(x,\xi)+\sigma_{az-j}(\partial_z A^z_{\theta-a\pi})(x,\xi)\right). \tag{2.19}$$

Similarly, differentiating equation (2.18) w.r. to z yields:

$$\begin{aligned}\sigma_{az-j}(\partial_z A^z_{\theta-a\pi})(x,-\xi) &= \partial_z\left(\sigma_{az-j}(A^z_{\theta-a\pi})(x,-\xi)\right)\\ &= \partial_z\left(e^{-i\pi(az+j)}\sigma_{az-j}(A^z_\theta)(x,\xi)\right)\\ &= e^{-i\pi(az+j)}\left(-ia\pi\sigma_{az-j}(A^z_\theta)(x,\xi)+\sigma_{az-j}(\partial_z A^z_\theta)(x,\xi)\right)\end{aligned}$$

i.e.

$$\sigma_{az-j}(\partial_z A^z_{\theta-a\pi})(x,-\xi) = e^{-i\pi(az+j)}\left(-ia\pi\sigma_{az-j}(A^z_\theta)(x,\xi)+\sigma_{az-j}(\partial_z A^z_\theta)(x,\xi)\right). \tag{2.20}$$

Now, combining equations (2.19) and (2.20) yields at $z=0$:

$$\begin{aligned}\sigma_{-j}(\log^{\mathrm{sym}}_\theta A)(x,-\xi) &= \frac{1}{2}\left(\sigma_{-j}(\log_\theta A)(x,-\xi)+\sigma_{-j}(\log_{\theta-a\pi} A)(x,-\xi)\right)\\ &= \frac{1}{2}\left(\sigma_{-j}((\partial_z A^z_\theta)_{|z=0})(x,-\xi)+\sigma_{-j}((\partial_z A^z_{\theta-a\pi})_{|z=0})(x,-\xi)\right)\\ &= \frac{1}{2}(-1)^j\left(ia\pi\sigma_{-j}(I)(x,\xi)+\sigma_{-j}(\log_{\theta-a\pi} A)(x,\xi)\right)\\ &\quad+\frac{1}{2}(-1)^j\left(-ia\pi\sigma_{-j}(I)(x,\xi)+\sigma_{-j}(\log_\theta A)(x,\xi)\right)\\ &= (-1)^j\sigma_{-j}(\log^{\mathrm{sym}}_\theta A)(x,\xi)\end{aligned}$$

and $\log^{\mathrm{sym}}_\theta A = \frac{1}{2}\left(\log_\theta A + \log_{\theta-a\pi}\right)$ is an odd-class log-polyhomogeneous operator. □

Corollary 2.2.2. *If A in $C\ell^a_{odd}(M,E)$ admits a spectral cut θ and a is even, then $\log_\theta(A)$ is an odd-class log-polyhomogeneous operator.*

Proof: If a is even i.e. $a = 2k$ for some integer k, then $\log_{\theta-a\pi} A = \log_{\theta-2k\pi} A$ so that

$$\log_{\theta-2k\pi} A - \log_\theta A = -2ik\pi I.$$

We obtain $\log^{\rm sym}_\theta A = \log_\theta A - ik\pi I$. Since I is an odd-class operator we deduce that $\log_\theta A$ is an odd-class operator.
Another way to obtain the result is to compute directly homogeneous components de the symbol of $\log_\theta A$ by evaluating formula (2.19) at $z = 0$. This gives

$$\sigma_{-j}(\log_\theta A)(x,-\xi) = (-1)^j ia\pi\sigma_{-j}(I)(x,\xi) + (-1)^j\sigma_{-j}(\log_{\theta-a\pi} A)(x,\xi).$$

But $(-1)^j ia\pi\sigma_{-j}(I)(x,\xi) = 0$ for $j > 0$. Hence

$$\text{For} \quad j>0, \quad \sigma_{-j}(\log_\theta A)(x,-\xi) = (-1)^j\sigma_{-j}(\log_{\theta-a\pi} A)(x,\xi).$$

Now, if $a = 2k$ is even then using formula $\log_{\theta-2k\pi} A - \log_\theta A = -2ik\pi I$, we have for $j > 0$,

$$\sigma_{-j}(\log_\theta A)(x,-\xi) = (-1)^j\sigma_{-j}(\log_\theta A)(x,\xi) \quad \text{and} \quad \sigma_0(\log_\theta A)(x,-\xi) = \sigma_0(\log_\theta A)(x,\xi).$$

The result follows. □

In this section we show the equivalent of Proposition 2.1.19 for the case of odd-class operators, namely that all log-polyhomogeneous operators in the odd-class algebra can be written as finite linear combinations of products of odd-class classical operators and symmetrized logarithms of odd-class elliptic operators. To do so, let us first show that in Proposition 2.1.19, we can replace the logarithm of the admissible operator Q by the symmetrized logarithm of an admissible odd-class operator.

Proposition 2.2.3. *Let Q be any odd-class admissible operator in $Cl_{odd}(M,E)$ with positive order q and spectral cuts α and $\alpha - q\pi$. Then for $k \geq 0$,*

1. $(\log^{\rm sym}_\alpha Q)^k C\ell_{odd}(M,E) \subset \bigoplus_{l=0}^{k} C\ell_{odd}(M,E)(\log^{\rm sym}_\alpha Q)^l$.

2. $C\ell^{\star,k}(M,E) = \bigoplus_{l=0}^{k} C\ell(M,E)\,(\log^{\rm sym}_\alpha Q)^l = \bigoplus_{l=0}^{k}(\log^{\rm sym}_\alpha Q)^l\, C\ell(M,E)$.

Proof: Since the proposition is the odd-class case of Proposition 2.1.19, the proof goes similarly. We have just to verify some points using the odd-class property.

1. If Q is odd-class, the same argument used to prove item 1 of Proposition 2.1.19 holds, replacing $C\ell(M,E)$ by $C\ell_{odd}(M,E)$, since $\log_\alpha^{\text{sym}} Q$ lies in the odd class and the odd-class is stable under products.

2. The first identity in 2) can be derived as in the proof of item 2 of Proposition 2.1.19. All we need to show is that

$$\bigoplus_{l=0}^{k} C\ell(M,E)(\log_\alpha Q)^l = \bigoplus_{l=0}^{k} C\ell(M,E)(\log_\alpha^{\text{sym}} Q)^l.$$

   Let us prove the inclusion form left to right. To do so let us write

$$\log_\alpha^{\text{asym}} Q := \frac{1}{2}\left(\log_\theta A - \log_{\theta-a\pi} A\right).$$

   It is easy to check (formula 2.12) that $\log_\alpha^{\text{asym}} Q$ is a classical operator of zero order and

$$\log_\alpha Q = \log_\alpha^{\text{sym}} Q + \log_\alpha^{\text{asym}} Q.$$

   This last equality combined with the fact that $\log_\alpha^{\text{asym}} Q$ is a classical operator implies that

$$(\log_\alpha Q)^l = (\log_\alpha^{\text{sym}} Q + \log_\alpha^{\text{asym}} Q)^l$$

   is a finite linear combination of products of the type

$$C_1(\log_\alpha^{\text{sym}} Q)^{k_1} \cdots C_p(\log_\alpha^{\text{sym}} Q)^{k_p}$$

   with $k_1 + \cdots + k_p = l$. This clearly lies in $\bigoplus_{j=0}^{l} C\ell(M,E)(\log_\alpha^{\text{sym}} Q)^j$. Hence

$$\bigoplus_{l=0}^{k} C\ell(M,E)(\log_\alpha Q)^l \subset \bigoplus_{l=0}^{k} C\ell(M,E)(\log_\alpha^{\text{sym}} Q)^l.$$

   The inclusion from right to left can be shown similarly writing

$$(\log_\alpha^{\text{sym}} Q)^l = (\log_\alpha Q - \log_\alpha^{\text{asym}} Q)^l.$$

□

We provide a description of the algebra of odd-class log-polyhomogeneous pseudodifferential operators in terms of symmetrized logarithms.

Theorem 2.2.4. *Let Q be any odd-class admissible operator in $C\ell_{odd}(M,E)$ with positive order q and spectral cuts α and $\alpha - q\pi$. Then,*

$$C\ell_{odd}^{\star,k}(M,E) = \bigoplus_{l=0}^{k} (\log_\alpha^{\mathrm{sym}} Q)^l \, C\ell_{odd}(M,E) = \bigoplus_{l=0}^{k} C\ell_{odd}(M,E)\, (\log_\alpha^{\mathrm{sym}} Q)^l.$$

Proof: Let us check the independence on the choice of Q. Assume that Q_i, $i=1,2$ are odd-class operators with spectral cuts $\alpha_i, \alpha_i - q_i\pi$. Let $A = \sum_{l=0}^{k} A_l (\log_{\alpha_1}^{\mathrm{sym}} Q_1)^l$. Since

$$\log_{\alpha_1} Q_1 = \log_{\alpha_1}^{\mathrm{sym}} Q_1 + \log_{\alpha_1}^{\mathrm{asym}} Q_1 \quad \text{and} \quad \log_{\alpha_2} Q_2 = \log_{\alpha_2}^{\mathrm{sym}} Q_2 + \log_{\alpha_2}^{\mathrm{asym}} Q_2,$$

we can write

$$\begin{aligned} \log_{\alpha_1}^{\mathrm{sym}} Q_1 &= \frac{q_1}{q_2} \log_{\alpha_2}^{\mathrm{sym}} Q_2 + \left(\log_{\alpha_1} Q_1 - \frac{q_1}{q_2} \log_{\alpha_2} Q_2 \right) - \left(\log_{\alpha_1}^{\mathrm{asym}} Q_1 - \frac{q_1}{q_2} \log_{\alpha_2}^{\mathrm{asym}} Q_2 \right) \\ &= \frac{q_1}{q_2} \log_{\alpha_2}^{\mathrm{sym}} Q_2 + B_{12}, \end{aligned}$$

where

$$B_{12} = \left(\log_{\alpha_1} Q_1 - \frac{q_1}{q_2} \log_{\alpha_2} Q_2 \right) - \left(\log_{\alpha_1}^{\mathrm{asym}} Q_1 - \frac{q_1}{q_2} \log_{\alpha_2}^{\mathrm{asym}} Q_2 \right) \in C\ell_{odd}(M,E).$$

By the first part of Proposition 2.2.3, $(\log_{\alpha_1}^{\mathrm{sym}} Q_1)^l = \left(\frac{q_1}{q_2} \log_{\alpha_2}^{\mathrm{sym}} Q_2 + B_{12} \right)^l$ which is a finite linear combination of products of the type $C_1 (\log_{\alpha_2}^{\mathrm{sym}} Q_2)^{k_1} \cdots C_p (\log_{\alpha_2}^{\mathrm{sym}} Q_2)^{k_p}$ with $k_1 + \cdots + k_p = l$, lies in $\bigoplus_{j=0}^{l} C\ell_{odd}(M,E) (\log_{\alpha_2}^{\mathrm{sym}} Q_2)^j$.

Hence $\bigoplus_{l=0}^{k} C\ell_{odd}(M,E)\, (\log_\alpha^{\mathrm{sym}} Q)^l$ is independent of the odd-class operator Q.

We are now left to show that

$$\bigoplus_{l=0}^{k} C\ell_{odd}(M,E)\, (\log_\alpha^{\mathrm{sym}} Q)^l = C\ell_{odd}^{\star,k}(M,E).$$

The inclusion from left to right follows from item 2 of Proposition 2.2.3 and the fact that an operator of $\bigoplus_{l=0}^{k} C\ell_{odd}(M,E)\, (\log_\alpha^{\mathrm{sym}} Q)^l$ is a linear combination of products of odd-class operators and therefore lies in the odd class i.e. lies in $C\ell_{odd}^{\star,k}(M,E)$.

Let us prove the inclusion from left to right.
Let A be an operator in $Cl_{odd}^{\star,k}(M,E)$ with order a and symbol of the form

$$\sigma(A) \sim \sum_{j=0}^{\infty}\sum_{l=0}^{k} \sigma_{a-j,l}(A)\log^l|\xi|.$$

Since A is odd-class, each $\sigma_{a-j,l}(A)$ verify $\sigma_{a-j,l}(A)(x,-\xi) = (-1)^{a-j}\sigma_{a-j,l}(A)(x,\xi)$.
Let us write

$$\sigma(A) \sim \sum_{l=0}^{k} \sigma_l(A)\log^l|\xi|$$

with $\sigma_l(A) \sim \sum_{j=0}^{\infty}\sigma_{a-j,l}(A)$; each $\sigma_l(A)$ is an odd-class symbol.

On the other hand, by item 2 of Proposition 2.2.3, we can write $A = \sum_{l=0}^{k} A_l(\log_\alpha^{\mathrm{sym}} Q)^l$.
In order to show that A_l is an odd-class operator for $l = 0,\cdots,k$ we compute the components of $\sigma(A)$. Since the odd-class operator $\log_\alpha^{\mathrm{sym}} Q$ has a symbol of the form $q\log|\xi| + \sigma^0$ with σ^0 a classical odd class symbol, $(\log_\alpha^{\mathrm{sym}} Q)^l$ has a symbol of the form $q^l\log^l|\xi| + \sigma^{l-1}$ with σ^{l-1} an odd class log-polyhomogeneous symbol of log-type $l-1$. The odd-class log-polyhomogeneous symbol σ^{l-1} has an expression of the form $\sum_{j=0}^{l-1}\sigma_{j,l-1}\log^j|\xi|$ with each $\sigma_{j,l-1}$ an odd-class symbol. We have

$$\begin{aligned}
\sigma(A) &\sim \sigma(A_0) + \sum_{l=1}^{k}\sum_{\alpha}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma(A_l)D_x^\alpha\sigma\left((\log_\alpha^{\mathrm{sym}} Q)^l\right)\\
&\sim \sigma(A_0) + \sum_{l=1}^{k}\sigma(A_l)\sigma\left((\log_\alpha^{\mathrm{sym}} Q)^l\right) + \sum_{l=1}^{k}\sum_{\alpha\neq 0}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma(A_l)D_x^\alpha\sigma\left((\log_\alpha^{\mathrm{sym}} Q)^l\right)\\
&\sim \sigma(A_0) + \sum_{l=1}^{k}q^l\sigma(A_l)\log^l|\xi| + \sum_{l=1}^{k}\sigma(A_l)\sigma^{l-1} + \sum_{l=1}^{k}\sum_{\alpha\neq 0}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma(A_l)D_x^\alpha\sigma^{l-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sigma(A) \sim\ & \sigma(A_0) + \sum_{l=1}^{k}q^l\sigma(A_l)\log^l|\xi| + \sum_{l=1}^{k}\sum_{j=1}^{l-1}\sigma(A_l)\sigma_{j,l-1}\log^j|\xi|\\
&+ \sum_{l=1}^{k}\sum_{\alpha\neq 0}\sum_{j=1}^{l-1}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma(A_l)\left(D_x^\alpha\sigma_{j,l-1}\right)\log^j|\xi|.
\end{aligned}$$

Comparing with $\sigma(A) \sim \sum_{l=0}^{k} \sigma_l(A) \log^l |\xi|$, we obtain:

$$\begin{aligned}
\sigma_k(A) &= q^k \sigma(A_k), \\
\sigma_{k-1}(A) &= q^{k-1}\sigma(A_{k-1}) + \sigma(A_k)\sigma_{k-1,k-1} + \sum_{\alpha\neq 0} \frac{1}{\alpha!}\partial_\xi^\alpha \sigma(A_k) D_x^\alpha \sigma_{k-1,k-1} \log^j |\xi|,
\end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sigma_i(A) = q^i\sigma(A_i) + \sum_{l=i+1}^{k} \sigma(A_l)\sigma_{i,l-1} + \sum_{l=i+1}^{k}\sum_{\alpha\neq 0} \frac{1}{\alpha!}\partial_\xi^\alpha \sigma(A_l) D_x^\alpha \sigma_{i,l-1} \log^j |\xi|$$

for $i = k-2, \cdots, 0$. Since A is an odd-class operator, $\sigma_k(A)$ lies in the odd-class which implies that $\sigma(A_k)$ is odd-class symbol and A_k is odd-class operator. But A_k odd-class operator implies that A_{k-1} is odd-class operator. We show this way, inductively on l, that A_l lies in the odd-class for $l = 0, \cdots, k$. Finally,

$$\bigoplus_{l=0}^{k} C\ell_{odd}(M,E)\,(\log_\alpha^{\mathrm{sym}} Q)^l = C\ell_{odd}^{\star,k}(M,E).$$

□

Here again, this provides similar information on the structure of $C\ell_{odd}^{\star,\star}(M,E)$.

Proposition 2.2.5. *The algebra* $C\ell_{odd}^{\star,\star}(M,E) := \bigoplus_{k=0}^{\infty} C\ell_{odd}^{\star,k}(M,E)$ *is a* $\mathbb{Z}$*-graded algebra.*

CHAPTER 3

Chapter 3

The canonical trace on odd-class pseudodifferential operators

The aim of this chapter is twofold. We first characterize traces on $C\ell^0_{odd}(M,E)$ (Theorem 3.3.4) when the underlying manifold M is odd dimensional; these are linear combinations of leading symbol traces, which involve the leading symbol, and the canonical trace which involves the whole symbol of the operator. The canonical trace is extended to odd-class operators in odd-dimensions after which we express regularized traces of log-polyhomogeneous odd-class operators in terms of this canonical trace (Theorem 3.5.7) and derive from there the cyclicity of the canonical trace on odd-class log-polyhomogeneous operators in odd dimensions (Corollary 3.5.9).

3.1 The L^2-trace on smoothing operators

The L^2-trace (or usual trace)

$$\begin{aligned}\mathrm{Tr} : C\ell^{-\infty}(M,E) &\to \mathbb{R}\\ A \mapsto \mathrm{Tr}(A) &:= \int_M \mathrm{tr}_x\left(K_A(x,x)\right) dx = \int_M \int_{T_x^*M} \mathrm{tr}_x\left(\sigma(A)(x,\xi)\right)\, đ\xi\, dx.\end{aligned}$$

where tr_x is the fibrewise trace and where, as before, $đ\xi := \frac{1}{(2\pi)^n}\, d\,\xi$ with $d\xi$ the ordinary Lebesgue measure on the cotangent bundle $T_x^*M \simeq \mathbb{R}^n$, is up to a multiplicative factor the unique trace on the algebra of smoothing operators $C\ell^{-\infty}(M,E)$. Indeed, in [Gu3] V. Guillemin showed the following proposition:

Proposition 3.1.1.

1. *Any operator R in $C\ell^{-\infty}(M,E)$ such that $\mathrm{Tr}(R) = 0$ is sum of commutators in $C\ell^{-\infty}(M,E)$.*

2. Any trace in $C\ell^{-\infty}(M,E)$ is proportional to the usual trace.

Let R be a smoothing operator in $C\ell^{-\infty}(M,E)$ such that $\mathrm{Tr}(R)\neq 0$. For any pseudodifferential projection J in $C\ell^{-\infty}(M,E)$ of rank 1, the smoothing operator $R-\mathrm{Tr}(R)J$ verifies $\mathrm{Tr}(R-\mathrm{Tr}(R)J)=0$. As a consequence of the above proposition it follows that we can express R as a sum of commutators and a smoothing operator

$$R=\mathrm{Tr}(R)J+\sum_{j=1}^{N}[S_j,T_j]. \tag{3.1}$$

where J is as above and $S_j,\ T_j$ are smoothing operators.

The trace Tr further extends by the same formula to a trace on the algebra of classical operators of order with real part $<-n$ (where n is the dimension of the underlying manifold) since these are trace-class. However,

Proposition 3.1.2. *The trace* Tr *does not extend to a trace functional on the whole algebra $C\ell(M,E)$.*

Proof: This follows from Wodzicki's characterization of traces on $C\ell(M,E)$ [W1], but we give here a more simple and direct proof which can be found in [L]. Assume λ is a trace on $C\ell(M,E)$ such that for any A in $C\ell(M,E)$ with order a, if $\mathrm{Re}(a)<-n$, then $\lambda(A)=\mathrm{Tr}(A)$. We may choose an elliptic operator A in $C\ell(M,E)$ with non-vanishing Fredholm index. Let B be a parametrix in $C\ell(M,E)$ of A. Then

$$I-BA,\ I-AB\in C\ell^{-\infty}(M,E)$$

and we arrive at the contradiction

$$0\neq \mathrm{ind}(A)=\mathrm{Tr}(I-BA)-\mathrm{Tr}(I-AB)=\mathrm{Tr}([A,B])=\lambda([A,B])=0.$$

□

There is therefore no trace on $C\ell(M,E)$ which extends the L^2-trace. M. Wodzicki in [W1] (see also [K]) proved that on a connected closed manifold of dimension >1, any trace on $C\ell(M,E)$ is proportional to the noncommutative residue defined as follows: for any classical operator A of order a which symbol has the asymptotic expansion $\sigma(A)(x,\xi)\sim \sum_{j=0}^{\infty}\chi(\xi)\sigma(A)_{a-j}(x,\xi)$ in local coordinates, the noncommutative residue of A is

$$\mathrm{res}(A)=\int_M \mathrm{res}_x(A)dx=\int_M\int_{S^*M}\mathrm{tr}_x(\sigma_{-n}(A)(x,\xi))d\!\!{}^-\xi\,dx.$$

Here n is the dimension of the manifold M and $\sigma(A)_{-n}(x,\xi)$ is the homogeneous component of degree $-n$ of the symbol of A. This definition is independent of the chosen local chart. From this expression, it is easy to see that the noncommutative residue is local in the sense that it depends on a finite number of homogeneous components of the symbol of A. We deduce the following result.

Lemma 3.1.3. *Let A be a classical ΨDO.*

1. *If $ord(A) < -n$, then* $\mathrm{res}(A) = 0$.
2. *If A lies in $C\ell_{odd}(M, E)$ and n is odd, then* $\mathrm{res}(A) = 0$.

Proof: For the first item, if A is of order $< -n$ then $\sigma_{-n}(A) = 0$.
For the second item, if A lies in $C\ell_{odd}(M, E)$ and n is odd, then $\sigma_{-n}(A)(x, -\xi) = -\sigma_{-n}(A)(x, \xi)$. Integrating over S^*M the odd-function $\sigma_{-n}(A)(x, \xi)$ we infer that $\mathrm{res}_x(A) = -\mathrm{res}_x(A) = 0$. □
Consequently, if the dimension of M is odd, the algebra of odd-class operators is contained in the kernel of the noncommutative residue i.e. $C\ell_{odd}(M, E) \subset Ker(\mathrm{res})$.

3.2 Classification of traces on $C\ell_{odd}(M, E)$

In odd dimensions $C\ell_{odd}(M, E)$ is contained in $Ker(\mathrm{res})$ so that it is natural to look for other traces than res on this subalgebra.

3.2.1 Linear forms on odd-class symbols and Stokes' property

Let us recall that a smooth function $f(\xi)$ on $\mathbb{R}^n - \{0\}$ is called positively homogeneous of degree m if for any $t > 0$, $f(t\xi) = t^m f(\xi)$. Euler's identity for homogeneous function of degree m is given by

$$\sum_{i=1}^{n} \xi_i \partial_{\xi_i} f = mf.$$

This follows directly from the fact that

$$\sum_{i=1}^{n} \xi_i \partial_{\xi_i}(f(\xi)) = \partial_t(f(t\xi))_{|t=1} = \partial_t(t^m f(\xi))_{|t=1} = mf(\xi).$$

Let us consider the $n-1$ form

$$\sigma = \sum_{i=1}^{n} (-1)^{i+1} \xi_j d\xi_1 \wedge \cdots \wedge d\xi_{i-1} \wedge d\xi_{i+1} \wedge \cdots \wedge d\xi_n.$$

We have $d\sigma = nd\xi_1 \wedge \cdots \wedge d\xi_n$ and restricted to the unit sphere S^{n-1}, σ is the volume form on S^{n-1}. If the degree of f is $-n$ we define the integral $\mathrm{res} f = \int_{S^{n-1}} f\sigma$.

Lemma 3.2.1 ([FGLS]). *Let f be a homogeneous function on $\mathbb{R}^n - \{0\}$. Each of the following conditions is sufficient for f to be a sum of derivatives:*

1. *$deg(f) \neq -n$.*
2. *$deg(f) = -n$ and $\mathrm{res} f = 0$.*

Proof:

1. If $deg(f) = m \neq -n$ let us consider the homogeneous function $g_i(\xi) := \frac{1}{m+n}\xi_i f(\xi)$. By Euler's identity we have
$$\sum_{i=1}^{n} \partial_{\xi_i} g_i(\xi) = \frac{1}{m+n}\left(\sum_{i=1}^{n} \xi_i \partial_{\xi_i} f + nf\right) = f.$$
2. If $deg(f) = -n$ and $\mathrm{res} f = \int_{S^{n-1}} f\sigma = 0$, let us write $S := S^{n-1}$ and consider the equation
$$\Delta_S g = f_{|S}$$
where Δ_S is the restriction of the Laplacian to the unit sphere S and $f_{|S}$ is the restriction of f to S. Since $\int_{S^{n-1}} f\sigma = 0$, $f_{|S}$ is orthogonal to the constants which form the kernel $\ker(\Delta_S)$. It follows that the equation above has a unique solution. In polar coordinates $(r, \omega) \in \mathbb{R}_0^+ \times S$, the Laplacian reads
$$\Delta = -\sum_{i=1}^{n} \partial_{\xi_i}^2 = -r^{1-n}\partial_r(r^{n-1}\partial_r) - r^{-2}\Delta_S.$$
Therefore, for any function $g \in C^\infty(S)$,
$$\Delta(g(\omega)r^{2-n}) = r^{-n}\Delta_S g(\omega) = r^{-n} f_{|S} = f.$$
It follows that f is a sum of derivatives.

□

Proposition 3.2.2. *Let σ be a classical symbol in $CS^m(\mathbb{R}^n)$. If σ belongs to $Ker(\mathrm{res})$ then there exists τ_i in $CS^{m+1}(\mathbb{R}^n)$ such that*
$$\sigma \sim \sum_{i=1}^{n} \partial_{\xi_i}\tau_i.$$
Moreover if σ is odd-class then the τ_i can be chosen in $CS_{odd}^{m+1}(\mathbb{R}^n)$.

Proof: The proof is similar to the one in [MSS]. Let σ in $CS^m(\mathbb{R}^n)$ be such that $\mathrm{res}(\sigma) = \int_{S^{n-1}} \sigma_{-n}(x,\xi) d\!\!\bar{}\,\xi = 0$. For a cut-off function χ we write $\sigma \sim \sum_{j=0}^{\infty} \chi\sigma_{m-j}$, with σ_{m-j} in $C^\infty(\mathbb{R}^n - \{0\})$ a positively homogeneous function of degree $m-j$ in ξ i.e.

$$\sigma_{m-j}(x, t\xi) = t^{m-j}\sigma_{m-j}(x,\xi), \quad \forall t > 0.$$

Applying Lemma 3.2.1 to each positively homogeneous component σ_{m-j}, we build positively homogeneous components $\tau_{i,m-j+1}$ such that $\sigma_{m-j} = \sum_i \tau_{i,m-j+1}$ so that

$$\sigma \sim \sum_{i=1}^{n}\sum_{j=0}^{\infty} \chi\partial_{\xi_i}(\tau_{i,m-j+1}) \sim \sum_{i=1}^{n} \partial_{\xi_i}\tau_i,$$

where we have set $\tau_i \sim \sum_{j=1}^{\infty} \chi\tau_{i,m-j+1}$. Since $\partial_{\xi_i}\chi$ has compact support, the difference $\sigma - \sum_{i=1}^{n} \partial_{\xi_i}\tau_i$ is smoothing. Let us show that the symbols τ_i can be chosen odd-class if σ is odd-class. A close look at the proof of Lemma 3.2.1 shows that f is odd-class implies that $f = \sum_{i=1}^{n} \partial_i f_i$ with f_i odd-class. Indeed, if f has order different from $-n$ then $f_i = \frac{1}{m+n}\xi_i f(\xi)$. If f has order $-n$ with $\mathrm{res}(f) = 0$ then $f = \Delta(g(\omega)r^{2-n})$ where $(r,\omega) \in \mathbb{R}_0^+ \times S$ and S is the unit sphere. It follows that τ_i lies in $CS^{m+1}_{odd}(\mathbb{R}^n)$. □

3.2.2 Characterization of traces on $C\ell_{odd}(M,E)$

In this paragraph, we show that any trace on the algebra $C\ell_{odd}(M,E)$ is proportional to the canonical trace when the dimension of the underlying manifold M is odd. Although this uniqueness result had already been proved in [MSS] by a similar method, we provide a proof for completeness in order to adapt it later to the case of zero order operators. We start with the following decomposition result.

Proposition 3.2.3. *Assume that M is an odd-dimensional manifold. Let A be an odd-class operator in $C\ell_{odd}(M,E)$. Then for any odd-class pseudodifferential projection J of rank 1, there exist C_k, D_k in $C\ell_{odd}(M,E)$ and a smoothing operator R_A such that*

$$A = \sum_{k=1}^{N} [C_k, D_k] + \mathrm{Tr}(R_A)J. \tag{3.2}$$

Proof: The proof is inspired by that of [MSS]. Let A be a classical ΨDO of order a in $C\ell(M,E)$. Let us consider a finite trivializing covering $\{(U_j, \phi_j, u_j), j \in I\}$ of M for E

and a finite subordinate partition of unity $\{\varphi_j,\, j \in I\}$. For each index j let $\psi_j \in C_c^\infty(U_j)$ be a real function such that $\psi_j = 1$ near $\mathrm{Supp}(\varphi_j)$. We localize A writing

$$A = \sum_{j\in I} \varphi_j A \psi_j + R.$$

Each operator $\varphi_j A \psi_j$ may be considered as a ΨDO on $\mathbb{R}^n$ with symbol $\sigma_j = \sigma$ in $CS(\mathbb{R}^n)$. Since we are interested in odd-class operators, to simplify, we can assume that σ is an odd-class symbol of order a. By Proposition 3.2.2, we know that there exist odd-class symbols τ_l of order $a+1$ such that

$$\sigma \sim \sum_{l=1}^{n} \partial_{\xi_l} \tau_l.$$

For any symbol τ we have,

$$Op(\partial_{\xi_l}\tau) = -i[x_l, Op(\tau)]$$

up to a smoothing operator since

$$\sigma([x_l, Op(\tau)]) = x_l \cdot \tau - \tau \cdot x_l - i^{-1}\partial_{\xi_l}\tau = i\partial_{\xi_l}\tau.$$

It follows that

$$Op\left(\sum_{l=1}^{n} \partial_{\xi_l}\tau_l\right) = \sum_{l=1}^{n} Op\left(\partial_{\xi_l}\tau_l\right) = -i\sum_{l=1}^{n}[x_l, Op(\tau_l)].$$

Since $\sigma \sim \sum\limits_{l=1}^{n} \partial_{\xi_l}\tau_l$, there exists a smoothing operator R' such that

$$Op(\sigma) = Op\left(\sum_{l=1}^{n} \partial_{\xi_l}\tau_l\right) + R' = -i\sum_{l=1}^{n}[x_l, Op(\tau_l)] + R'.$$

Then for χ in $C_c^\infty(\mathbb{R}^n)$ such that $\chi\varphi_j = \varphi_j$, $\chi\psi_j = \psi_j$, we have

$$\varphi_j Op(\sigma)\psi_j = -i\sum_{l=1}^{n} \varphi_j[x_l, Op(\tau_l)]\psi_j + \varphi_j R'\psi_j = -i\sum_{l=1}^{n}[\chi x_l, \varphi_j Op(\tau_l)\psi_j] + \varphi_j R'\psi_j.$$

Using formula (1.11) to write $\varphi_j A\psi_j = Op(\sigma_j) + R_j$ with R_j smoothing operators, we have $A = \sum\limits_j Op(\sigma_j) + \sum\limits_j R_j + R$ so that A can be written in the form

$$A = \sum_{k=1}^{N'} [\alpha_k, B_k] + R_A, \tag{3.3}$$

where α_k is a smooth function in M, B_k lies in $C\ell^{a+1}_{odd}(M,E)$, and R_A is a smoothing operator. Let us recall that by formula (3.1), the smoothing operator R_A can be decomposed in the form

$$R_A = \mathrm{Tr}(R_A)J + \sum_{j=1}^{N''}[S_j, T_j]$$

where J is any odd-class pseudodifferential projection of rank 1 and $S_j,\ T_j$ are smoothing operators. Summing up, the expression for A becomes

$$A = \sum_{k=1}^{N}[C_k, D_k] + \mathrm{Tr}(R_A)J.$$

□

Let us reformulate Proposition 3.2.3 in the following way:

Proposition 3.2.4. *Assume that M is an odd-dimensional manifold. All traces on $C\ell_{odd}(M,E)$ are proportional to one another and uniquely determined by their restriction to smoothing operators.*
In particular, the L^2-trace Tr *on smoothing operators uniquely extends to a trace* $\widetilde{\mathrm{Tr}}$ *on $C\ell_{odd}(M,E)$.*

Proof: Let Λ be a linear form on $C\ell_{odd}(M,E)$ which vanishes on brackets. By (3.2) we have $\Lambda(A) = \mathrm{Tr}(R_A)\,\Lambda(J)$ which shows that Λ is determined by its restriction to smoothing operators since R_A is smoothing. It moreover shows that all such traces are proportional. □

3.2.3 Explicit construction of the canonical trace

In [KV1], [KV2] M. Kontsevich and S. Vishik introduced the *canonical trace* TR for any non-integer order classical ΨDO. We recall the construction of this functional and its properties to show that any trace on $C\ell_{odd}(M,E)$ is proportional to TR. For that let us first recall the definition of the cut-off integral: for a trace-class operator A the expression $\mathrm{Tr}(A) = \int_M \int_{T_x^*M} \mathrm{tr}_x(\sigma(A)(x,\xi))đ\xi\, dx$ makes sense since the symbol of A has order $< -n$. But in general this expression does not make sense. We need to extract a finite part from a divergent expression of this type using Hadamard finite parts (see e.g. [H, Schw]).
Let A be a classical ΨDO of order a with local symbol given by formula (1.3):

$$\sigma(A)(x,\xi) = \sigma(x,\xi) = \sum_{j=0}^{N-1}\chi(\xi)\sigma_{a-j}(x,\xi) + \sigma_N(x,\xi)$$

for a fixed positive integer N, with positively homogeneous components $\sigma_{a-j}(x,\xi)$ of degree $a-j$, $\sigma_N(x,\xi)$ a symbol of order $a-N$ and χ a cut-off function which vanishes for $|\xi| \leq \frac{1}{2}$ and such that $\chi(\xi) = 1$ for $|\xi| \geq 1$. Let $B_x^*(0, R)$ be the ball of radius R in the cotangent space T_x^*M at point x in M and S_x^*M the unit cosphere at point x. For N sufficiently large, the integral $\int_{B_x^*(0,R)} \sigma_N(x,\xi)d\xi$ is well defined. We write

$$\int_{B_x^*(0,R)} \chi(\xi)\sigma_{a-j}(x,\xi)d\xi = \int_{B_x^*(0,1)} \chi(\xi)\sigma_{a-j}(x,\xi)d\xi + \int_{B_x^*(0,R)\setminus B_x^*(0,1)} \chi(\xi)\sigma_{a-j}(x,\xi)d\xi.$$

Using the fact that σ_{a-j} is positively homogeneous of degree $a - j$ we have

$$\int_{B_x^*(0,R)\setminus B_x^*(0,1)} \chi(\xi)\sigma_{a-j}(x,\xi)d\xi = \int_1^R \int_{|\omega|=1} r^{a-j+n-1}\sigma_{a-j}(x,\omega)d\omega dr.$$

If a is an integer, then there exits an integer j_0 such that $a - j_0 + n = 0$ and hence for $N - 1 > j_0$,

$$\begin{aligned}
& \sum_{j=0}^{N-1} \int_{B_x^*(0,R)\setminus B_x^*(0,1)} \chi(\xi)\sigma_{a-j}(x,\xi)d\xi \\
= \quad & \sum_{j=0}^{N-1} \int_1^R \int_{|\omega|=1} r^{a-j+n-1}\sigma_{a-j}(x,\omega)d\omega dr \\
\underset{R\to\infty}{\sim} \quad & \sum_{\substack{j=0 \\ a-j+n\neq 0}}^{N-1} \frac{1}{a+n-j} R^{a+n-j} \int_{|\omega|=1} \sigma_{a-j}(x,\omega)d\omega + \log R \int_{|\omega|=1} \sigma_{-n}(x,\omega)d\omega + c_x.
\end{aligned}$$

where c_x is a constant term. It follows that the integral $\int_{B_x^*(0,R)} \sigma(x,\xi)đ\xi$ admits the asymptotic expansion

$$\begin{aligned}
& \int_{B_x^*(0,R)} \sigma(x,\xi)đ\xi \\
\underset{R\to\infty}{\sim} \quad & \sum_{\substack{j=0 \\ a-j+n\neq 0}}^{N-1} \frac{1}{a+n-j} R^{a+n-j} \int_{S_x^*M} \sigma_{a-j}(x,\xi)đ\xi + \log R \int_{S_x^*M} \sigma_{-n}(x,\xi)đ\xi + c_x(\sigma),
\end{aligned}$$

with the constant term given by

$$\begin{aligned}
c_x(\sigma) \quad = \quad & \int_{T_x^*M} \sigma_N(x,\xi)đ\xi + \sum_{j=0}^{N-1} \int_{B_x^*(0,1)} \chi(\xi)\sigma_{a-j}(x,\xi)đ\xi \\
& - \sum_{\substack{j=0 \\ a-j+n\neq 0}}^{N-1} \frac{1}{a-j+n} \int_{S_x^*M} \sigma_{a-j}(x,\xi)đ\xi. \qquad (3.4)
\end{aligned}$$

We define the finite part integral to be the constant term in the asymptotic expansion:

$$\,⨍_{T_x^*M} \sigma(x,\xi) đ\xi := \underset{R\to\infty}{\mathrm{LIM}} \int_{B_x^*(0,R)} \sigma(x,\xi) đ\xi = c_x(\sigma).$$

Remark 3.2.5. *Because of the logarithm term in R, we cannot expect the finite part to be invariant under a change of coordinates. However, as we shall see in further details in the more general case of log-polyhomogeneous operators (see Lemma 3.5.2), if the order of A is not an integer, there is no longer a logarithmic term and $⨍_{T_x^*M} \sigma(A)(x,\xi) đ\xi$ is independent of the local representation of $\sigma(A)(x,\xi)$.*

We are now able to introduce the canonical trace:

Proposition 3.2.6 ([KV1])**.** *Let A be a classical operator in $C\ell(M,E)$ with non-integer order. Then*

$$\mathrm{TR}(A) := \int_M \mathrm{TR}_x(A) dx = \int_M ⨍_{T_x^*M} \mathrm{tr}_x\left(\sigma(A)(x,\xi)\right) đ\xi dx$$

is well defined and satisfies the following elementary properties:

1. *For any operator A in $C\ell(M,E)$ with order a such that $a < -n$, $\mathrm{TR}(A) = \mathrm{Tr}(A)$.*
2. *For A, B in $\bigcup_{a\in\mathbb{R}\setminus\mathbb{Z}} C\ell^a(M,E)$ and for any real α in $\mathbb{R}$, such that $ord(\alpha A + B)$ is not an integer, $\mathrm{TR}(\alpha A + B) = \alpha \mathrm{TR}(A) + \mathrm{TR}(B)$.*
3. *For A, B in $C\ell(M,E)$ such that $ord(A) + ord(B)$ is not an integer, $\mathrm{TR}([A,B]) = 0$.*

Example 3.2.7. *Any differential operator A has a well-defined canonical trace which vanishes:*

$$\mathrm{TR}(A) = \int_M ⨍_{\mathbb{R}^n} \mathrm{tr}_x\left(\sigma(A)(x,\xi)\right) đ\xi dx = 0.$$

Indeed, in local coordinates the symbol of A reads $\sigma(A)(x,\xi) = \sum_{|\alpha|\le a} p_\alpha(x)\xi^\alpha$ with homogeneous components $\sigma_{a-j}(A)(x,\xi) = p_\alpha(x)\xi^\alpha$ and $\sigma_N(A)(x,\xi) = 0$ for $N \ge a$. It follows that

$$\begin{aligned} ⨍_{\mathbb{R}^n} \sigma(A)(x,\xi) đ\xi &= \sum_{|\alpha|\le a} p_\alpha(x) \underset{R\to\infty}{\mathrm{LIM}} \int_{B_x^*(0,R)} \xi^\alpha đ\xi \\ &= \sum_{|\alpha|\le a} p_\alpha(x) \underset{R\to\infty}{\mathrm{LIM}} \left(\frac{R^{|\alpha|+N}}{|\alpha|+N}\right) \int_{S_x^*M} \xi^\alpha đ\xi \\ &= 0. \end{aligned}$$

A fundamental property of the canonical trace is given in the following:

Theorem 3.2.8 ([KV1]). *Let $a(z)$ be a holomorphic function on $\mathbb{C}$ such that $a'(z) \neq 0$ for z in $a^{-1}\{j-n : j \in \mathbb{N}\}$. Let $A(z)$ be a holomorphic family of classical operators of order $a(z)$. Then the function $z \to \mathrm{TR}(A(z))$ is holomorphic in the domain $\{z \in \mathbb{C} : \mathrm{Re}(a(z)) < -n\}$ and can be extended to a meromorphic function $\mathrm{TR}(A(z))$ with simple poles at $z_j = a^{-1}(j-n), j \in \mathbb{N}$ and the complex residues read:*

$$Res_{z=z_j}\mathrm{TR}(A(z)) = -\frac{1}{a'(z_j)}\mathrm{res}(A(z_j)).$$

Here res *denotes the noncommutative residue.*

The canonical trace can therefore be extended by continuity to $C\ell_{odd}(M,E)$ when M is an odd-dimensional manifold. Notice that M. Kontsevich and S. Vishik [KV1] have already extended the canonical trace to odd-class operators in odd dimension manifolds using an even order positive definite odd-class operator Q and a holomorphic family AQ^{-z} where A is an odd-class operator.

Proposition 3.2.9. *Assume that the dimension of M is odd. Let A be in $C\ell_{odd}(M,E)$ and let Q be an admissible operator in $C\ell_{odd}(M,E)$ with positive order and spectral cut α. The function $\mathrm{TR}(AQ_\alpha^{-z})$ is holomorphic at $z=0$ and $\lim_{z\to 0}\mathrm{TR}(AQ_\alpha^{-z}) := \overline{\mathrm{TR}}(A)$ satisfies the following properties*

1. *$\overline{\mathrm{TR}}(A)$ is independent of the choice of Q.*
2. *For $A, B \in C\ell_{odd}(M,E)$, $\overline{\mathrm{TR}}([A,B]) = 0$.*
3. *$\overline{\mathrm{TR}}$ extends the L^2-trace on smoothing operators.*

Proof: Since the dimension n of M is odd, if A lies in the odd-class $C\ell_{odd}(M,E)$, $\mathrm{res}(A) = \int_M \int_{S^*M} \mathrm{tr}_x(\sigma(A)_{-n}(x,\xi)) đ\xi\, dx = 0$. It follows from the previous theorem that $\mathrm{TR}(AQ_\alpha^{-z})$ is holomorphic at $z=0$. Hence $\overline{\mathrm{TR}}$ is well defined on operators in $Cl_{odd}(M)$ and is independent of Q. $\overline{\mathrm{TR}}$ extends the L^2-trace on smoothing operators since it extends TR which coincides with the L^2-trace on smoothing operators. By applying the previous theorem to the holomorphic family $\mathrm{TR}([AQ_\alpha^{-z}, BQ_\alpha^{-z}]$, we obtain $\overline{\mathrm{TR}}([A,B]) = 0$. $\square$

By Proposition 3.2.4 the extensions $\widetilde{\mathrm{Tr}}$ and $\overline{\mathrm{TR}}$ coincide so that we have:

$$\mathrm{TR} = \overline{\mathrm{TR}} = \widetilde{\mathrm{Tr}}.$$

Proposition 3.2.4 can therefore be reformulated as follows:

Theorem 3.2.10. *Assume that the dimension of the underlying manifold M is odd. Any trace on $C\ell_{odd}(M,E)$ is a constant multiple of the canonical trace* TR.

Remark 3.2.11. *This was first proved by L. Maniccia, E. Schrohe and J. Seiler in [MSS]. In [Pa2], S. Paycha proved the uniqueness by proving the equivalence between Stokes' property for linear forms on symbols and the vanishing of linear forms on operator brackets. R. Ponge in [Po2] classified traces using the fact that any non-integer order operator or odd-class operators in odd dimensions or even-class operators in even dimensions is a sum of commutators up to a smoothing operator.*

3.3 Classification of traces on $C\ell^0_{odd}(M,E)$

Since $C\ell^0_{odd}(M,E)$ is a subalgebra of $C\ell_{odd}(M,E)$, we can expect other traces to arise in this subalgebra. For that, let us reformulate Proposition 3.2.2 and formula (3.3) to the context of $C\ell^0_{odd}(M,E)$. We obtain the following consequences:

Proposition 3.3.1 ([NO]). *If σ lies in $CS^0_{odd}(\mathbb{R}^n)$ and has the asymptotic expansion $\sigma \sim \sum_{j=0}^{\infty} \chi\sigma_{-j}$, then there is a finite set $\{\tau_i, i=1,\cdots,n\}$ of symbols in $CS^0_{odd}(\mathbb{R}^n)$ such that*

$$\sigma - \sigma_0 \sim \sum_{i=1}^{n} \partial_{\xi_i}\tau_i.$$

Proof: Apply Proposition 3.2.2 to the symbol $\sigma-\sigma_0$ of $CS^{-1}_{odd}(\mathbb{R}^n)$. $\square$

Let us consider a finite trivializing covering $\{(U_j,\phi_j,u_j), j\in I\}$ of M for E and a finite subordinate partition of unity $\{\psi_j,\, j\in I\}$. Let A be an operator in $C\ell^0_{odd}(M,E)$. Let $\sigma^j(A)$ be its symbol read in the local chart (U_j,ϕ_j). The leading symbol $\sigma_0(A)$ of A is globally defined as a section of $\mathrm{End}(p^*E)$ where p is the canonical projection $p: T^*_xM - \{0\} \to M$. Let $\sigma^j_0(A)$ denotes the leading symbol read in the local chart (U_j,ϕ_j). With the notations of Section 1.2.2, the local operator $Op(\sigma^j_0(A))$'s patched up to an operator $A' = \sum_{j\in I} \psi_j\phi_{u_*}Op(\sigma^j_0(A))\phi^*_u\psi_j$ which has order 0. By abuse of notation, we write $Op(\sigma_0(A))$ for A'.

Proposition 3.3.2 ([NO]). *If A lies in $C\ell^0_{odd}(M,E)$, then there exist operators B_k in $C\ell^0_{odd}(M,E)$, smooth functions α_k, k in $\{1,\cdots,n\}$ on M and a smoothing operator R_A such that*

$$A - Op(\sigma_0(A)) = \sum_{k=1}^{n} [\alpha_k, B_k] + R_A.$$

Proof: It follows from formula (3.3) applied to $A-Op(\sigma_0(A))$ of $C\ell^{-1}_{odd}(M,E)$. $\square$

As for classical odd-class operators, let us use formula (3.1) to decompose the smoothing operator R_A in the form

$$R_A = \mathrm{TR}(R_A)J + \sum_{j=1}^{N''}[S_j, T_j]$$

where J is any pseudodifferential projection of rank 1 and $S_j,\ T_j$ are smoothing operators. Summing up, we obtain

$$A - Op(\sigma_0(A)) = \sum_{k=1}^{N}[C_k, D_k] + \mathrm{TR}(R_A)J. \tag{3.5}$$

where J, C_k, D_k lie in $C\ell^0_{odd}(M,E)$.

We now introduce another type of trace on the algebra $C\ell^0(M,E)$, defined by S. Paycha and S. Rosenberg in [PR], which had actually already been considered by Wodzicki in an unpublished manuscript.

Lemma 3.3.3 (Lemma 3.1 in [PR]). *For any distribution λ in $\mathcal{D}'(S^*M)$, the map Tr_0^λ : $C\ell^0(M,E) \to \mathbb{R}$ given by $\mathrm{Tr}_0^\lambda(A) = \lambda(\mathrm{tr}_x(\sigma_0(A))$ is a trace.*

Proof: This follows from the multiplicativity of the leading symbol:

$$\mathrm{tr}_x(\sigma_0(AB)) = \mathrm{tr}_x(\sigma_0(A)\sigma_0(B)) = \mathrm{tr}_x(\sigma_0(B)\sigma_0(A)) = \mathrm{tr}_x(\sigma_0(BA)).$$

□

Using formula (3.5), we observe that:

Theorem 3.3.4 ([NO]). *If the dimension of the underlying manifold M is odd, any trace on $C\ell^0_{odd}(M,E)$ is a linear combination of the canonical trace and a leading symbol trace.*

Proof: This is a straightforward application of formula (3.5). If A lies in $C\ell^0_{odd}(M,E)$ then $A - Op(\sigma_0(A)) = \sum_{k=1}^{N}[C_k, D_k] + \mathrm{TR}(R_A)J$. If λ is a trace on $C\ell^0_{odd}(M,E)$, applying λ to both sides of the previous expression for A we have

$$\lambda(A) = \lambda(Op(\sigma_0(A))) + \mathrm{TR}(R_A)\lambda(J).$$

By construction, $Op(\sigma_0(A))$ is an operator with symbol σ_0, the leading symbol of A. Hence, there exists a distribution τ on $C^\infty(S^*M)$ such that

$$\lambda(A) = \tau(\sigma_0(A)) + \mathrm{TR}(R_A)\lambda(J).$$

□

3.4 The local residue density extended to $C\ell^{\star,\star}_{odd}(M,E)$

The noncommutative residue does not extend to a trace on $C\ell^{\star,\star}(M,E)$, where it defines a $\mathbb{Z}$-graded trace, but the local residue density does extend locally and this extension turns out to be a useful tool for our purposes.

Let A be a log-polyhomogeneous operator in $C\ell^{a,k}(M,E)$. Recall that in local coordinates its symbol $\sigma(A)$ has the asymptotic expansion (1.5) given by:

$$\sigma(A)(x,\xi) \sim \sum_{j=0}^{\infty} \chi(\xi)\,\sigma_{a-j}(A)(x,\xi) = \sum_{j=0}^{\infty}\sum_{l=0}^{k} \chi(\xi)\,\sigma_{a-j,l}(A)(x,\xi)\,\log^l|\xi|.$$

In [L], M. Lesch extended the residue trace to a log-polyhomogeneous operator A in $C\ell^{a,k}(M,E)$ with $k>0$ by setting:

$$\mathrm{res}_k(A) := (k+1)!\int_M \mathrm{res}_{x,k}(A)\,dx = (k+1)!\int_M \left(\int_{S_x^*M} \mathrm{tr}_x\left(\sigma_{-n,k}(A)(x,\xi)\right)\,đ\xi\right)\,dx \quad (3.6)$$

where

$$\mathrm{res}_{x,k}(A) := \int_{S_x^*M} \mathrm{tr}_x\left(\sigma_{-n,k}(A)(x,\xi)\right)\,đ\xi.$$

Indeed, for a log-polyhomogeneous operator A in $C\ell^{a,k}(M,E)$ with $k>0$, the form $\sigma_{-n,k}(A)(x,\xi)dx$ defines a global density on M. This is not the case for the lower densities $\sigma_{-n,0}(A)(x,\xi)dx,\cdots,\sigma_{-n,k-1}(A)(x,\xi)dx$. Nevertheless we set by extension [PS]

$$\mathrm{res}_{x,l}(A) := \int_{S_x^*M} \mathrm{tr}_x\left(\sigma_{-n,l}(A)(x,\xi)\right)\,đ\xi, \quad \text{for } l\le k. \quad (3.7)$$

In [Ok2], K. Okikiolu extended the noncommutative residue to logarithms of admissible operators. Let A be an admissible operator of order a with spectral cut θ. The symbol of $\log_\theta A$ has an asymptotic expansion in local coordinates of the form (2.10)

$$\sigma(\log_\theta A)(x,\xi) = a\,\log|\xi| I + \sigma_\theta^A(x,\xi) \sim a\log|\xi| + \sum_{j=0}^{\infty}\chi(\xi)(\sigma_\theta^A)_{-j}(x,\xi).$$

K. Okikiolu proved that in that case, the local density denoted by

$$\mathrm{res}_x(\log_\theta A)dx := \int_{S_x^*M} \mathrm{tr}_x\left((\sigma_\theta^A)_{-n}(x,\xi)\right)\,đ\xi dx$$

defines a global density so that $\mathrm{res}(\log_\theta A)$ is called the noncommutative residue of $\log_\theta A$. In [PS], Definition 1.3, the local noncommutative residue is extended to log-polyhomogeneous operators by setting $\mathrm{res}_{x,0}(B) = \int_{S_x^*M} \mathrm{tr}_x \left(\sigma_{-n}(B)(x,\xi)\right) \, đ\xi$. Indeed,

$$\begin{aligned}
\mathrm{res}_{x,0}(B) &= \int_{S_x^*M} \mathrm{tr}_x \left(\sigma_{-n}(B)(x,\xi)\right) \, đ\xi \\
&= \sum_{l=0}^{k} \int_{S_x^*M} \mathrm{tr}_x \left(\sigma_{-n,l}(B)(x,\xi)\right) \log^l |\xi| \, đ\xi \\
&= \int_{S_x^*M} \mathrm{tr}_x \left(\sigma_{-n,0}(B)(x,\xi)\right) \, đ\xi
\end{aligned}$$

since the logarithmic terms in $|\xi|$ are cancelled in the integration over the unit cosphere. It follows that this coincides with the case $k = 0$ of formula (3.7). We will denote $\mathrm{res}_{x,0}(A) := \mathrm{res}_x(A)$. For operators in the odd-class in odd dimensions, the local residue vanishes and hence local residues patch up to a globally defined residue.

Proposition 3.4.1. *Assume that M is an odd-dimensional manifold. Let A be a log-polyhomogeneous operator in $C\ell^{a,k}_{odd}(M,E)$. Then $\mathrm{res}_{x,l}(A)$ vanishes for $l = 0, \cdots, k$, hence $\mathrm{res}_l(A) := \int_M \mathrm{res}_{x,l}(A)(x,\xi)\, dx = 0$.*
In particular A has well-defined noncommutative residue and

$$\mathrm{res}(A) = \int_M \left(\int_{S_x^*M} \mathrm{tr}_x \left((\sigma_{-n}(A)(x,\xi)) \right) đ\xi \right) dx = \mathrm{res}_0(A) = 0.$$

Proof: The operator A in $C\ell^{a,k}_{odd}(M,E)$ is an odd-class operator, so

$$\sigma_{-n,l}(A)(x,-\xi) = (-1)^n \sigma_{-n,l}(A)(x,\xi).$$

The dimension of M is odd and we have to integrate over the unit cosphere S_x^*M the odd function $\sigma_{-n,l}(A)(x,\xi)$. It follows that $\mathrm{res}_{x,l}(A) = (-1)^n \mathrm{res}_{x,l}(A) = 0$. Hence $\mathrm{res}_l(A) := \int_M \mathrm{res}_{x,l}(A)(x,\xi)\, dx$ is well defined and vanishes for any $l = 0, \cdots, k$. In particular, $\mathrm{res}(A) = \int_M \mathrm{res}_{x,0}(A)(x,\xi)\, dx = 0$ so that A has well-defined noncommutative residue which vanishes. □

3.5 The canonical trace extended to $C\ell^{\star,\star}_{odd}(M,E)$

In this section, we extend the canonical trace previously defined on classical operators with non-integer order and odd-class classical operators to odd-class log-polyhomogeneous ones. Although such an extension had already been built in [PS], we adopt here a slightly different point of view and therefore explicitly construct this extension for completeness.

We use the finite part of a holomorphic family of log-polyhomogeneous operators to prove that the canonical trace extends by continuity to $C\ell_{odd}^{\star,\star}(M,E)$.

In section 3.2.3, we recalled the construction of the canonical trace introduced by M. Kontsevich and S. Vishik [KV1] on classical pseudodifferential operators with non-integer order and proved that it uniquely extends to a trace on classical odd-class operators in odd dimensions. On the other hand, M. Lesch [L] further extended the canonical trace to log-polyhomogeneous operators with non-integer order in the following way: let $\sigma \sim \sum_{j=0}^{\infty}\sum_{l=0}^{k} \sigma_{a-j,l} \log^l |\xi|$ be a log-polyhomogeneous symbol on an open subset U of $\mathbb{R}^n$ with order a and log degree k. The integral $\int_{B_x^*(0,R)} \sigma(x,\xi)\, đ\xi$ has an asymptotic expansion for $R \to \infty$ (see [L] formula (5.5), [PS] Lemma 1.6)

$$\int_{B_x^*(0,R)} \sigma(x,\xi)\, đ\xi$$
$$\underset{R\to\infty}{\sim} \quad c_x(\sigma) + \sum_{\substack{j=0\\ a-j+n\neq 0}}^{N-1} \sum_{l=0}^{k} P_l(\sigma_{a-j,l})(\log R) R^{a+n-j} + \sum_{l=0}^{k} \frac{\log^{l+1} R}{l+1} \int_{S_x^*U} \sigma_{-n,l}(x,\xi) đ\xi$$

where $P_l(\sigma_{a-j,l})(X)$ is a polynomial of degree l with coefficients depending on $\sigma_{a-j,l}$. The finite part integral is defined by the constant term in the asymptotic expansion:

$$⨍_{\mathbb{R}^n} \sigma(x,\xi) đ\xi := \underset{R\to\infty}{\mathrm{LIM}} \int_{B_x^*(0,R)} \sigma(x,\xi) đ\xi = c_x(\sigma).$$

Remark 3.5.1. *This method of extraction of a finite part from a divergent expression is already used in the classical case in order to define the canonical trace. In fact we obtain*

$$\begin{aligned} c_x(\sigma) &= \int_{T_x^*U} \sigma_N(x,\xi) đ\xi + \sum_{j=0}^{N-1} \int_{B_x^*(0,1)} \chi(\xi) \sigma_{a-j}(x,\xi) đ\xi \\ &\quad + \sum_{\substack{j=0\\ a-j+n\neq 0}}^{N-1} \sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(a-j+n)^{l+1}} \int_{S_x^*U} \sigma_{a-j,l}(x,\xi) đ\xi. \end{aligned}$$

As in the classical case, the finite part integral is not invariant under a change of coordinates of $\mathbb{R}^n$. Indeed, the transformation rule is given in the following lemma:

Lemma 3.5.2 ([L] Proposition 5.2)**.** *Let P be a regular matrix in* $\mathrm{GL}(n,\mathbb{R})$ *and let σ be a log-polyhomogeneous symbol on an open subset U of $\mathbb{R}^n$ with order a and log degree k. We have the transformation rule*

$$⨍_{\mathbb{R}^n} \sigma(x,P\xi) d\xi = |\det P|^{-1} \left(⨍_{\mathbb{R}^n} \sigma(x,\xi) d\xi + \sum_{l=0}^{k} \frac{(-1)^{l+1}}{l+1} \int_{S_x^*U} \sigma_{-n,l}(x,\xi) \log^{l+1} |P^{-1}\xi| d\xi \right).$$

We refer to [L] for the proof.
Let us now apply these results to odd-class operators $C\ell^{\star,\star}_{odd}(M,E)$.

Proposition 3.5.3. *If the dimension of M is odd, the canonical trace extends to the algebra $C\ell^{\star,\star}_{odd}(M,E)$ of odd-class log-polyhomogeneous operators.*

Proof: Let us assume that A in $C\ell^{a,k}(M,E)$ is odd-class. Its symbol in local coordinates reads $\sigma(A) \sim \sum_{j=0}^{\infty}\sum_{l=0}^{k} \sigma(A)_{a-j,l} \log^l|\xi|$ with $\sigma_{a-j,l}(A)(x,-\xi) = (-1)^{a-j}\sigma_{a-j,l}(A)(x,\xi)$. It follows from Lemma 3.5.2 that $\mathrm{TR}_x(A)dx = \left(⨍_{T^*_xM} \mathrm{tr}_x\left(\sigma(A)(x,\xi)\right) đ\xi\right) dx$ defines a global density on M since we have to integrate over the unit cosphere S^*_xM the odd function $\sigma_{-n,l}(A)(x,\xi)$, and for a change of coordinates, the odd function $\sigma_{-n,l}(A)(x,\xi)\log^{l+1}|P^{-1}\xi|$. Hence $\mathrm{TR}(A) = \int_M \mathrm{TR}_x(A)dx$ is well defined. $\square$

Independently of the dimension of the underlying manifold, as in the case of classical pseudodifferential operators, it has been shown by M. Lesch that the canonical trace extends to non-integer order log-polyhomogeneous operators. We recall in the following theorem the analogous of Proposition 3.2.6 and Theorem 3.2.8 for log-polyhomogeneous operators.

Theorem 3.5.4 (Section 5 in [L])**.** *For a in $\mathbb{C}\setminus\mathbb{Z}$, for any positive integer k, there exists a linear functional $\mathrm{TR} : C\ell^{a,k}(M,E) \to \mathbb{C}$ with the following properties:*

1. *For A in $C\ell^{a,k}(M,E)$, $\mathrm{TR}(A) = \int_M \mathrm{TR}_x(A)dx = \int_M ⨍_{\mathbb{R}^n} \mathrm{tr}_x\left(\sigma(A)(x,\xi)\right) đ\xi dx$*

2. *$\mathrm{TR}[A,B] = 0$ if A lies in $C\ell^{a,k}(M,E)$, B lies in $C\ell^{p,l}(M,E)$ and $a+b \notin \mathbb{Z}$.*

3. *Let $a(z)$ be a holomorphic function on $\mathbb{C}$ such that $a'(z) \neq 0$ for z in $a^{-1}\{j-n : j \in \mathbb{N}\}$. Let $A(z)$ be a holomorphic family of log-polyhomogeneous operators in $C\ell^{a(z),k}(M,E)$. Then the function $z \mapsto \mathrm{TR}(A(z))$ is meromorphic with poles at $z_j = a^{-1}(j-n)$, j in $\mathbb{N}$ of order smaller than $k+1$ and:*

$$Res_{k+1}\mathrm{TR}(A(z))_{|_{z=z_j}} = \frac{(-1)^{k+1}}{(k+1)a'(z_j)}\mathrm{res}_k(A(z_j))$$

where Res_{k+1} is the coefficient of $(z-z_j)^{-k-1}$ in the Laurent expansion of the meromorphic function $\mathrm{TR}(A(z))$ and res_k is the noncommutative residue defined by formula (3.6).

Let us recall a result which extends results of [PS] to the log-polyhomogeneous case. This is unpublished work by the authors of [PS] communicated to me by S. Paycha.

Theorem 3.5.5. *Let $A(z)$ be a holomorphic family of log-polyhomogeneous operators in $C\ell^{a(z),k}(M,E)$ parametrized by z in Ω, a domain of $\mathbb{C}$ with non constant order $a(z) = -qz+a$. Then for any z_0 in Ω we have*

$$\mathrm{fp}_{z=z_0}\mathrm{TR}(A(z)) = \int_M dx \left(\mathrm{TR}_x(A(z_0)) + \sum_{l=0}^{k} \frac{(-1)^{l+1}}{(a'(z_0))^{l+1}} \mathrm{res}_{x,l}(A^{(l+1)}(z_0)) \right), \qquad (3.8)$$

where $\mathrm{fp}_{z=z_0}$ *stands for the constant term in the Laurent expansion.*

Applying this theorem to $A(z) = AQ_\alpha^{-z}$ where A lies in $C\ell^{\star,\star}(M,E)$, Q is admissible with spectral cut α, $z_0 = 0$ and $A(0) = A$, we obtain the following formula for a log-polyhomogeneous operators [PS]:

$$\mathrm{fp}_{z=0}\mathrm{TR}(AQ_\alpha^{-z}) = \int_M dx \left(\mathrm{TR}_x(A) + \sum_{l=0}^{k} \frac{1}{(-q)^{l+1}} \mathrm{res}_{x,l}\left(A(\log_\alpha Q)^{l+1}\right) \right). \qquad (3.9)$$

Remark 3.5.6.

1. *If A is a classical ΨDO, formula (3.9) gives back the defect formula of [PS]*

$$\mathrm{fp}_{z=0}\mathrm{TR}(AQ_\alpha^{-z}) = \int_M dx \left(\mathrm{TR}_x(A) - \frac{1}{q}\mathrm{res}_x(A\log_\alpha Q) \right). \qquad (3.10)$$

2. *If A is a logarithmic operator i.e. if the symbol of A is of the form*

$$\sigma(A)(x,\xi) \sim \gamma \log|\xi| + \sum_{j=0}^{\infty} \chi(\xi)\sigma_{a-j}(A)(x,\xi),$$

by formula (3.9) we get

$$\mathrm{fp}_{z=0}\mathrm{TR}(AQ_\alpha^{-z}) = \int_M dx \left(\mathrm{TR}_x(A) - \frac{1}{q}\mathrm{res}_{x,0}(A\log_\alpha Q) + \frac{1}{q^2}\mathrm{res}_{x,1}\left(A(\log_\alpha Q)^2\right) \right). \qquad (3.11)$$

The following theorem compares regularized traces with the canonical trace, thus generalizing a result of [Pa2] established in the classical case.

Theorem 3.5.7. *Assume that the dimension of M is odd. Let $A(z)$ be a holomorphic family of log-polyhomogeneous operators in $C\ell^{a(z),k}(M,E)$ with non constant order $a(z) = -qz+a$.*

1. *If $A(0) = A$ lies in the odd-class then,*

$$\mathrm{fp}_{z=0}\mathrm{TR}(A(z)) = \mathrm{TR}(A) + \sum_{l=0}^{k} \frac{(-1)^{l+1}}{(a'(0))^{l+1}} \int_M dx \left(\mathrm{res}_{x,l}(A^{(l+1)}(0))\right).$$

2. *If $A(0) = A$ and if for all positive integers j, $A^{(j)}(0)$ lies in the odd-class then,*

$$\mathrm{fp}_{z=0}\mathrm{TR}(A(z)) = \lim_{z\to 0} \mathrm{TR}(A(z)) = \mathrm{TR}(A).$$

Proof:

1. Let us assume that $A(z)$ lies in $C\ell^{a(z),k}(M,E)$. Since $A(0) = A$ lies in the odd-class, by Proposition 3.5.3 $\mathrm{TR}(A)$ is well defined $\mathrm{TR}(A) = \int_M \mathrm{TR}_x(A)dx$. Applying formula (3.8) to the holomorphic family $A(z)$ at $z = 0$, we have

$$\mathrm{fp}_{z=0}\mathrm{TR}(A(z)) = \mathrm{TR}(A) + \sum_{l=0}^{k} \frac{(-1)^{l+1}}{(a'(0))^{l+1}} \int_M dx \left(\mathrm{res}_{x,l}(A^{(l+1)}(0))\right).$$

2. If $A(0) = A$ and for all positive integers j, $A^{(j)}(0)$ lies in the odd-class using Proposition 3.4.1, we get $\mathrm{res}_{x,l}\left(A^{(j)}(0)\right) = 0$ so that $\mathrm{fp}_{z=0}\mathrm{TR}(A(z)) = \lim_{z\to 0} \mathrm{TR}(A(z)) = \mathrm{TR}(A)$.

$\square$

Example 3.5.8. *Let A be a classical odd-class operator and let Q be an admissible odd-class operator with positive order q and spectral cuts α and $\alpha - q\pi$. Consider the holomorphic family $A(z) = A\frac{Q_\alpha^z + Q_{\alpha-q\pi}^z}{2}$. For $l \geq 0$, $A^{(l)}(0)$ lie in the odd-class. We have $A(0) = A$ and for $l > 1$, $A^{(l)}(0) = A(\log_\alpha^{\mathrm{sym}} Q)^l$. In odd dimension, the family $A(z)$ fulfills the assumptions of the theorem so that:*

$$\mathrm{fp}_{z=0}\mathrm{TR}\left(A\frac{Q_\alpha^z + Q_{\alpha-q\pi}^z}{2}\right) = \lim_{z\to 0} \mathrm{TR}\left(A\frac{Q_\alpha^z + Q_{\alpha-q\pi}^z}{2}\right) = \mathrm{TR}(A).$$

When Q has even order, the same result holds with $A(z) = A\, Q_\alpha^z$.

We are ready to check the expected cyclicity property of TR on $C\ell^{\star,\star}_{odd}(M,E)$ thus extending the result of M. Kontsevich and S. Vishik [KV1].

Corollary 3.5.9. *Assume that the dimension of M is odd. For any odd-class operators A, B in $C\ell^{\star,\star}_{odd}(M,E)$,*

$$\mathrm{TR}[A,B] = 0.$$

Proof: This follows from applying Theorem 3.5.7 to the family $A(z) = [AQ_\alpha^z,\, BQ_\alpha^z]$. Here Q is an odd-class admissible operator with even order q and spectral cut α. The operator $A(z)$ is a holomorphic approximation of the bracket $[A,\, B]$. Using Leibniz's rule, we have for $j \geq 0$:

$$A^{(j)}(z) = \sum_{l=0}^{j} \mathrm{C}_j^l \left[A\left(\log_\alpha Q\right)^l Q_\alpha^z,\, B\left(\log_\alpha Q\right)^{j-l} Q_\alpha^z\right]$$

and at $z = 0$,

$$A^{(j)}(0) = \sum_{l=0}^{j} \mathrm{C}_j^l \left[A\left(\log_\alpha Q\right)^l,\, B\left(\log_\alpha Q\right)^{j-l}\right].$$

The operator $A^{(j)}(0)$ lies in the odd class since this class is stable under products and $\log_\alpha Q$ lies in the odd-class. Thus, by Theorem 3.5.7, $\mathrm{fp}_{z=0}\mathrm{TR}(A(z)) = \mathrm{TR}([A, B])$. Now using the fact that the canonical trace vanishes on non-integer order brackets and taking finite parts as $z \to 0$ we get $\mathrm{TR}[A, B] = 0$. $\square$

CHAPTER 4

Chapter 4

The regularized trace of the logarithm of a product

In this chapter we compare regularized traces of the logarithm of a product of classical pseudodifferential operators with the sum of the regularized traces of the logarithms of the operators involved in the product. We therefore investigate regularized traces of the difference:

$$L(A, B) := \log(AB) - \log A - \log B.$$

Since $L(A, B)$ has vanishing residue [Sc], it can be expressed as a finite sum of operator brackets (Proposition 4.4.2), a property from which we then infer that a regularized trace of this difference $L(A, B)$ is local as a finite sum of noncommutative residues (Theorem 4.4.3). Theorem 4.5.2 provides an explicit local formula for a regularized trace of L(A,B) in terms of the noncommutative residue. We first recall known properties of regularized traces.

4.1 Weighted traces of classical pseudodifferential operators

Since traces on $C\ell(M, E)$ are proportional to the noncommutative residue which vanishes on smoothing operators, the L^2-trace on smoothing operators does not extend to the whole algebra $C\ell(M, E)$. Instead we use linear extensions called weighted traces, of the ordinary L^2-trace on smoothing operators to the whole algebra $C\ell(M, E)$. Weighted traces studied in [MN] are defined via meromorphic extensions of generalized zeta functions.

Given an admissible operator Q in $C\ell(M, E)$ with positive order q and spectral cut

α, and given an operator A in $C\ell(M,E)$ with real order a, we can approximate A by a holomorphic family $A(z) = A\,Q_\alpha^{-z}$ where Q_α^{-z} is the complex power defined in (2.2). Recall (Theorem 3.2.8) that the map $z \mapsto \mathrm{Tr}\,(A\,Q_\alpha^{-z})$ is holomorphic on the domain $\{z \in \mathbb{C}, \mathrm{Re}(z) > \frac{n+a}{q}\}$ and has a meromorphic extension $\mathrm{TR}\,(A\,Q_\alpha^{-z})$ to $\mathbb{C}$ with simple poles at $z_j = \frac{n+a-j}{q}$, j in $\mathbb{N}$ and the complex residues read:

$$Res_{z=z_j}\mathrm{TR}(A\,Q_\alpha^{-z}) = \frac{1}{q}\,\mathrm{res}\left(A\,Q_\alpha^{-z_j}\right).$$

Using this meromorphic extension, one can defined the Q-weighted trace of any classical operator A by [MN]:

$$\mathrm{Tr}_\alpha^Q(A) := \mathrm{fp}_{z=0}\mathrm{TR}\left(A\,Q_\alpha^{-z}\right) = \lim_{z\to 0}\left(\mathrm{TR}\left(A\,Q_\alpha^{-z}\right) - \frac{1}{qz}\,\mathrm{res}(A)\right).$$

The admissible operator Q with positive order q and spectral cut α is called a *weight.*

Remark 4.1.1.

1. *If the classical operator A is trace-class, then AQ_α^{-z} is also trace-class in a neighborhood of $z = 0$ so that $\mathrm{Tr}_\alpha^Q(A) = \mathrm{Tr}(A)$ is the usual trace of A.*

2. *If the dimension of M is odd and A lies in $C\ell_{odd}(M,E)$ then $\mathrm{res}(A) = 0$ and by Theorem 3.5.7, $\mathrm{Tr}_\alpha^Q(A) = \mathrm{TR}(A)$ is independent of the choice of the weight Q as long as $\log_\alpha Q$ lies in the odd-class, a property which holds in particular if Q lies in the odd-class and has even order.*

Weighted traces are not cyclic on $C\ell(M,E)$ in spite of their names but they are interesting because they do not vanish on trace-class operators for which they coincide with the ordinary trace. Let us recall basic properties of weighted traces.

Proposition 4.1.2 ([CDMP], [MN])**.** *Let A and B be two operators in $C\ell(M,E)$. For any weight Q with positive order q and spectral cut α, the operators $[A, \log_\alpha Q]$ and $[B, \log_\alpha Q]$ lie in $C\ell(M,E)$ and*

$$\mathrm{Tr}_\alpha^Q\left([A,B]\right) = -\frac{1}{q}\,\mathrm{res}\left(A\left[B, \log_\alpha Q\right]\right) = \frac{1}{q}\,\mathrm{res}\left(B\left[A, \log_\alpha Q\right]\right). \tag{4.1}$$

In particular, if $Q = A$ or $Q = B$ then $\mathrm{Tr}_\alpha^Q\left([A,B]\right) = 0$.

Proof: By Lemma 2.1.12, the operators $[A, \log_\alpha Q]$ and $[B, \log_\alpha Q]$ are classical. Let a be the order of A and b the order of B. For $\mathrm{Re}(z) > \frac{a+b+n}{q}$, the map $z \mapsto \mathrm{Tr}\left([A,B]Q_\alpha^{-z}\right)$ is holomorphic and

$$\begin{aligned}
\mathrm{Tr}\left([A,B]Q_\alpha^{-z}\right) &= \mathrm{Tr}\left(A[B, Q_\alpha^{-z}] + AQ_\alpha^{-z}B - BAQ_\alpha^{-z}\right) \\
&= \mathrm{Tr}\left(A[B, Q_\alpha^{-z}] + [AQ_\alpha^{-\frac{z}{2}}, Q_\alpha^{-\frac{z}{2}}B]\right) \\
&= \mathrm{Tr}\left(A[B, Q_\alpha^{-z}]\right).
\end{aligned}$$

Moreover, since the noncommutative residue vanishes on brackets of classical operators, we have $0 = \mathrm{res}([A,B]) = q Res_{z=0} \mathrm{TR}([A,B]\, Q_\alpha^{-z})$. Hence, the map $z \mapsto \mathrm{TR}\left([A,B] Q_\alpha^{-z}\right)$ is holomorphic at $z=0$ and so is the map $z \mapsto \mathrm{TR}\left(A[B, Q_\alpha^{-z}]\right)$. It follows that

$$\begin{aligned} \mathrm{Tr}_\alpha^Q\left([A,B]\right) &= \mathrm{fp}_{z=0} \mathrm{TR}\left([A,B]\, Q_\alpha^{-z}\right) \\ &= Res_{z=0} \mathrm{TR}(z^{-1} A[B, Q_\alpha^{-z}]) = \frac{1}{q} \mathrm{res}\left(\frac{d}{dz}(A[B, Q_\alpha^{-z}])_{|z=0}\right) \\ &= -\frac{1}{q} \mathrm{res}\left(A[B, \log_\alpha Q]\right). \end{aligned}$$

The second equality immediately follows from the trace property of the noncommutative residue. □

4.2 Weighted traces involving logarithms

Recall from Lemma 2.1.12 that, whereas a logarithm is not classical, the bracket $[A, \log B]$ is classical for A classical and B admissible, and recall from Lemma 2.1.10 that if A is also admissible then $\frac{\log A}{a} - \frac{\log B}{b}$ is classical where a is the order of A, b the order of B.

Corollary 4.2.1. *Let A and B be admissible operators in $C\ell(M,E)$ with spectral cuts θ and ϕ respectively. For any weight Q with positive order q and spectral cut α,*

$$\mathrm{Tr}_\alpha^Q\left([\log_\theta A, B]\right) = -\frac{1}{q} \mathrm{res}\left((\log_\theta A - \frac{a}{b}\log_\phi B)[B, \log_\alpha Q]\right) \tag{4.2}$$

where $a > 0$ is the order of A and $b > 0$ is the order of B.
In particular,

$$\mathrm{Tr}_\theta^A\left([\log_\theta A, B]\right) = \mathrm{Tr}_\phi^B\left([\log_\theta A, B]\right) = \mathrm{Tr}_\theta^A\left([\log_\theta A, \log_\phi B]\right) = 0. \tag{4.3}$$

Proof: Since $[\log_\theta A, B] = [\log_\theta A - \frac{a}{b}\log_\phi B, B]$ and $\log_\theta A - \frac{a}{b}\log_\phi B$ is a classical operator (formula (2.12)), by the above proposition we have

$$\mathrm{Tr}_\alpha^Q\left([\log_\theta A, B]\right) = -\frac{1}{q} \mathrm{res}\left((\log_\theta A - \frac{a}{b}\log_\phi B)[B, \log_\alpha Q]\right).$$

If $Q = A$, it follows that

$$\begin{aligned} \mathrm{Tr}_\theta^A\left([\log_\theta A, B]\right) &= -\frac{1}{a} \mathrm{res}\left((\log_\theta A - \frac{a}{b}\log_\phi B)[B, \log_\theta A]\right) \\ &= -\frac{1}{a} \mathrm{res}\left((\log_\theta A - \frac{a}{b}\log_\phi B)[B, \log_\theta A - \frac{a}{b}\log_\phi B]\right) \\ &= -\frac{1}{a} \mathrm{res}\left(\left[(\log_\theta A - \frac{a}{b}\log_\phi B)B, \log_\theta A - \frac{a}{b}\log_\phi B\right]\right) \\ &= 0 \end{aligned}$$

since $\left[(\log_\theta A - \frac{a}{b}\log_\phi B)B, \log_\theta A - \frac{a}{b}\log_\phi B\right]$ is a bracket of classical ΨDOs and the residue is tracial.
If $Q = B$, since B and $\log_\phi B$ commute i. e. $[B, \log_\phi B] = 0$, then

$$\mathrm{Tr}_\phi^B\left([\log_\theta A, B]\right) = -\frac{1}{b}\mathrm{res}\left((\log_\theta A - \frac{a}{b}\log_\phi B)[B, \log_\phi B]\right) = 0.$$

Now $\mathrm{Tr}_\theta^A\left([\log_\theta A, \log_\phi B]\right) = \mathrm{Tr}_\theta^A\left([\log_\theta A, \log_\phi B - \frac{b}{a}\log_\theta A]\right) = 0$, with B replaced by $\log_\phi B - \frac{b}{a}\log_\theta A$.
Let us provide an alternative direct proof of this last statement. Since the map $z \mapsto \mathrm{TR}([B, \log_\theta A]A^{-z})$ is meromorphic by Theorem 3.2.8 and since TR is cyclic on non integer order operators, we have the following identity of meromorphic functions:

$$\mathrm{TR}\left([B, \log_\theta A]A_\theta^{-z}\right) = \mathrm{TR}\left(B[\log_\theta A, A_\theta^{-z}]\right) = 0$$

using the fact that $\log_\theta A$ commutes with A_θ^{-z}. It follows that the constant term $\mathrm{Tr}_\theta^A\left([\log_\theta A, B]\right)$ in the Laurent expansion vanishes. Similarly we prove that $\mathrm{Tr}_\phi^B([\log_\theta A, B]) = 0$. □

Remark 4.2.2. *Using the definition of the weighted trace,*

$$\mathrm{Tr}_\theta^A([\log_\theta A, B]) = \lim_{z \to 0}\left(\mathrm{TR}\left([\log_\theta A, B]\, A_\theta^{-z}\right) - \frac{1}{az}\mathrm{res}([\log_\theta A, B])\right).$$

Since $\mathrm{TR}\left([B, \log_\theta A]A_\theta^{-z}\right) = \mathrm{Tr}_\theta^A\left([\log_\theta A, B]\right) = 0$, *we immediately deduce that*

$$\mathrm{res}\left([\log_\theta A, B]\right) = 0$$

from which we infer (using the formula $[\log_\theta A, \log_\phi B] = [\log_\theta A, \log_\phi B - \frac{b}{a}\log_\theta A]$*) that*

$$\mathrm{res}\left([\log_\theta A, \log_\phi B]\right) = 0. \tag{4.4}$$

Replacing in formula (3.10) the l.h.s. $\mathrm{fp}_{z=0}\mathrm{TR}\left(A\, Q_\alpha^{-z}\right)$ by $\mathrm{Tr}_\alpha^Q(A)$ yields the following defect formula for weighted traces:

$$\mathrm{Tr}_\alpha^Q(A) = \int_M dx\left(\mathrm{TR}_x(A) - \frac{1}{q}\mathrm{res}_x(A\,\log_\alpha Q)\right). \tag{4.5}$$

When A is a differential operator, then $\mathrm{TR}_x(A)$ vanishes for any x in M (Example 3.2.7) and $\mathrm{res}_x\left(A\,\log_\alpha Q\right)\,dx$ defines a global density in which case

$$\mathrm{Tr}_\alpha^Q(A) = -\frac{1}{q}\mathrm{res}(A\,\log_\alpha Q)$$

is a local expression. If $\mathrm{res}(A\log_\alpha Q)$ vanishes, e.g. if A has non integer order or in odd dimensions if both A and Q lie in the odd-class and Q has even order, then $\mathrm{TR}_x(A)\,dx$ defines a global density so that $\mathrm{TR}(A)$ is well-defined and

$$\mathrm{Tr}_\alpha^Q(A) = \mathrm{TR}(A).$$

Generally speaking, weighted traces are not expected to be local. However the difference of two weighted traces is local. Indeed, given two weights Q_1 and Q_2 with spectral cuts α_1, α_2 and positive orders q_1, q_2, substracting the above expression (4.5) obtained with Q_2 from the one obtained with Q_1 we get back the well-known formula [MN], [CDMP]:

$$\mathrm{Tr}_{\alpha_1}^{Q_1}(A) - \mathrm{Tr}_{\alpha_2}^{Q_2}(A) = \mathrm{res}\left(A\left(\frac{\log_{\alpha_2} Q_2}{q_2} - \frac{\log_{\alpha_1} Q_1}{q_1}\right)\right), \tag{4.6}$$

which is local keeping in mind that the difference $\frac{\log_{\alpha_2} Q_2}{q_2} - \frac{\log_{\alpha_1} Q_1}{q_1}$ lies in $C\ell(M,E)$.

Weighted traces can be extended to logarithms: let A be an admissible operator in $C\ell(M,E)$ with positive order a and spectral cut θ. Given an admissible operator Q in $C\ell(M,E)$ with positive order q and spectral cut α, the map $z \mapsto \mathrm{TR}(\log_\theta A\, Q_\alpha^{-z})$ is meromorphic with simple pole at $z=0$ and the complex residue reads ([Du1], Lemma II.4.2):

$$Res_{z=0}\mathrm{TR}(\log_\theta A\, Q_\alpha^{-z}) = \frac{1}{q}\,\mathrm{res}\left(\log_\theta A - \frac{a}{q}\log_\alpha Q\right).$$

The Q-weighted trace $\mathrm{Tr}_\alpha^Q(\log_\theta A)$ is defined as before, picking out the constant term of the meromorphic map $z \mapsto \mathrm{TR}(\log_\theta A\, Q_\alpha^{-z})$ i.e.

$$\mathrm{Tr}_\alpha^Q(\log_\theta A) := \mathrm{fp}_{z=0}\mathrm{TR}\left(\log_\theta A\, Q_\alpha^{-z}\right).$$

With these notations we have $\zeta'_{Q,\alpha}(0) = -\mathrm{Tr}_\alpha^Q(\log_\alpha Q) = -\lim_{z\to 0}\mathrm{TR}(\log_\alpha Q\, Q_\alpha^{-z})$.

Let us recall that weighted traces of logarithms depend on the choice of the weight in the following way ([Ok2] Lemma 0.1, [Du1] Proposition II.4.6): Let A, Q_1, Q_2 be admissible operators with orders a, q_1, q_2 respectively and spectral cuts θ, α_1, α_2 respectively. Then

$$\begin{aligned} &\mathrm{Tr}_{\alpha_1}^{Q_1}(\log_\theta A) - \mathrm{Tr}_{\alpha_2}^{Q_2}(\log_\theta A) \\ = \; & -\frac{1}{2}\mathrm{res}\left[\left(\log_\theta A - \frac{a}{q_1}\log_{\alpha_1} Q_1\right)\left(\frac{\log_{\alpha_1} Q_1}{q_1} - \frac{\log_{\alpha_2} Q_2}{q_2}\right)\right] \\ & -\frac{1}{2}\mathrm{res}\left[\left(\log_\theta A - \frac{a}{q_2}\log_{\alpha_2} Q_2\right)\left(\frac{\log_{\alpha_1} Q_1}{q_1} - \frac{\log_{\alpha_2} Q_2}{q_2}\right)\right]. \end{aligned} \tag{4.7}$$

4.3 Weighted traces of differentiable families of operators

For further use, we prove in the following property by which the canonical and weighted traces as well as the noncommutative residue commute with differentiation on differentiable families of operators with constant order. Differentiable families of symbols and operators are defined in the same way as holomorphic families in Definitions 1.1.3 and 1.2.10 replacing holomorphic by differentiable.

Proposition 4.3.1 ([OP]). *Let A_t be a differentiable family of $C\ell(M,E)$ of constant order a.*

1. *The noncommutative residue commutes with differentiation*

$$\frac{d}{dt}\mathrm{res}(A_t) = \mathrm{res}(\dot{A}_t). \tag{4.8}$$

where $\dot{A}_t = \frac{d}{dt}A_t$.

2. *If the order a is non integer, the canonical trace commutes with differentiation*

$$\frac{d}{dt}\mathrm{TR}(A_t) = \mathrm{TR}(\dot{A}_t). \tag{4.9}$$

3. *For any weight Q with positive order q and spectral cut α,*

$$\frac{d}{dt}\mathrm{Tr}_\alpha^Q(A_t) = \mathrm{Tr}_\alpha^Q(\dot{A}_t). \tag{4.10}$$

Proof: By formula (1.3) we write

$$\sigma(A_t)(x,\xi) = \sum_{j=0}^{N-1} \chi(\xi)\,\sigma_{a-j}(A_t)(x,\xi) + \sigma_N(A_t)(x,\xi).$$

1. By assumption, the map $t \mapsto \mathrm{tr}_x\left(\sigma_{-n}(A_t)(x,\cdot)\right)$ is differentiable leading to a differentiable map $t \mapsto \int_{S_x^*M} \mathrm{tr}_x\left(\sigma_{-n}(A_t)(x,\cdot)\right)$ after integration over the compact set S_x^*M with derivative: $t \mapsto \int_{S_x^*M} \mathrm{tr}_x\left(\dot{\sigma}_{-n}(A_t)(x,\cdot)\right)$, where $\dot{\sigma}(A_t) = \sigma(\dot{A}_t)$ stands for the derivative of $\sigma(A_t)$ at t. Thus, the map $t \mapsto \mathrm{res}(A_t)$ is differentiable with derivative

$$\frac{d}{dt}\mathrm{res}(A_t) = \mathrm{res}(\dot{A}_t).$$

2. Since $\mathrm{TR}(A_t) = \int_M ⨍_{T_x^*M} \mathrm{tr}_x\left(\sigma(A_t)(x,\xi)\right) đ\xi dx$, to prove formula (4.9) we need to check the differentiability of the map $t \mapsto ⨍_{T_x^*M} \mathrm{tr}_x(\sigma(A_t)(x,\cdot))$ and to prove that

$$\frac{d}{dt} ⨍_{T_x^*M} \mathrm{tr}_x(\sigma(A_t)(x,\cdot)) = ⨍_{T_x^*M} \mathrm{tr}_x(\dot\sigma(A_t)(x,\cdot)),$$

where as before $\dot\sigma(A_t) = \sigma(\dot A_t)$ stands for the derivative of $\sigma(A_t)$ at t.
The cut-off integral involves the whole symbol and we set $\sigma_t := \sigma(A_t)$ in order to simplify notations. Since the family σ_t has constant order, N can be chosen independently of t in the asymptotic expansion. The corresponding cut-off integral is given explicitly by (3.4):

$$\begin{aligned} & ⨍_{T_x^*M} \mathrm{tr}_x(\sigma_t(x,\xi))\, đ\xi \\ = & \int_{T_x^*M} \mathrm{tr}_x\left((\sigma_t)_N(x,\xi)\right) đ\xi + \sum_{j=0}^{N-1} \int_{|\xi|\le 1} \chi(\xi)\, \mathrm{tr}_x\left((\sigma_t)_{a-j}(x,\xi)\right) đ\xi \\ & - \sum_{j=0, a-j+n\neq 0}^{N-1} \frac{1}{a-j+n} \int_{S_x^*M} \mathrm{tr}_x\left((\sigma_t)_{a-j}(x,\omega)\right) đ\omega. \end{aligned}$$

The map $t \mapsto \int_{T_x^*M} \mathrm{tr}_x\left((\sigma_t)_N(x,\xi)\right) đ\xi$ is differentiable at any point t_0 since by assumption the maps $t \mapsto \mathrm{tr}_x\left((\sigma_t)_N\right)$ are differentiable with modulus bounded from above $|\mathrm{tr}_x\left((\dot\sigma_t)_N\right)| \le C|\xi|^{a-N}$ by an L^1 function provided N is chosen large enough, where the constant C can be chosen independently of t in a compact neighborhood of t_0. Its derivative is given by $t \mapsto \int_{T_x^*M} \mathrm{tr}_x\left((\dot\sigma_t)_N(x,\xi)\right) đ\xi$. The remaining integrals $\int_{|\xi|\le 1} \chi(\xi)\, \mathrm{tr}_x\left((\sigma_t)_{a-j}(x,\xi)\right) đ\xi$ and $\int_{S_x^*M} \mathrm{tr}_x\left((\sigma_t)_{a-j}(x,\omega)\right) đ\omega$ are also differentiable as integrals over compact sets of integrands involving differentiable maps $t \mapsto \mathrm{tr}_x\left((\sigma_t)_{a-j}\right)$ with derivatives given by $\int_{S_x^*M} \chi(\xi)\, \mathrm{tr}_x\left((\dot\sigma_t)_{a-j}(x,\xi)\right) đ\xi$ and $\int_{|\xi|=1} \mathrm{tr}_x\left((\dot\sigma_t)_{a-j}(x,\omega)\right) đ\omega$. Thus, the map $t \mapsto \mathrm{TR}(A_t)$ is differentiable with derivative

$$\frac{d}{dt}\mathrm{TR}(A_t) = \mathrm{TR}(\dot A_t).$$

3. By formula (4.5) we have

$$\mathrm{Tr}_\alpha^Q(A_t) = \int_M dx \left(⨍_{T_x^*M} \mathrm{tr}_x(\sigma(A_t)(x,\cdot)) đ\xi - \frac{1}{q} \int_{S_x^*M} \mathrm{res}_x\left(\sigma_{-n}(A_t \log_\alpha Q)(x,\cdot)\right) đ\xi \right)$$

which reduces the proof of the differentiability of $t \mapsto \mathrm{Tr}_\alpha^Q(A_t)$ to that of the two maps $t \mapsto ⨍_{T_x^*M} \mathrm{tr}_x \sigma(A_t)(x,\cdot)$ and $t \mapsto \int_{S_x^*M} \mathrm{res}_x\left(\sigma_{-n}(A_t \log_\alpha Q)\right)(x,\cdot)$.

Differentiability of the first map was shown in the second item of the proof. Let us therefore investigate the second map. By (1.2) we have

$$\sigma_{-n}(A_t \log_\alpha Q) = \sum_{|\alpha|+a-j-k=-n} \frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha \sigma_{a-j}(A_t)\, \partial_x^\alpha \sigma_{-k}(\log_\alpha Q).$$

By assumption, the maps $t \mapsto \sigma_{a-j}(A_t)$ are differentiable any non negative integer j so that the map $t \mapsto \int_{S_x^*M} \mathrm{res}_x\left(\sigma_{-n}(A_t \log_\alpha Q)(x,\cdot)\right)$ is differentiable with derivative given by the map $t \mapsto \int_{S_x^*M} \mathrm{res}_x\left(\dot\sigma_{-n}(A_t \log_\alpha Q)(x,\cdot)\right)$. Integrating over the compact manifold M then yields that the map $t \mapsto \mathrm{Tr}_\alpha^Q(A_t)$ is differentiable with derivative

$$\frac{d}{dt}\mathrm{Tr}_\alpha^Q(A_t) = \mathrm{Tr}_\alpha^Q(\dot A_t).$$

□

4.4 Locality of the weighted trace of $L(A, B)$

Let A and B be two admissible operators in $C\ell(M,E)$ with positive orders a and b and spectral cuts θ and ϕ respectively. Assume that their product AB is also admissible with spectral cut ψ. We consider the following expression

$$L(A,B) := \log_\psi(AB) - \log_\theta A - \log_\phi B.$$

In [Sc], S. Scott showed the multiplicativity of the associated residue determinant

$$\det_{\mathrm{res}}(A) := \exp(\mathrm{res}(\log A)).$$

He actually showed more, namely that given two admissible operators A and B such that their product AB is also admissible,

$$\mathrm{res}(L(A,B)) = \mathrm{res}(\log_\psi(AB) - \log_\theta A - \log_\phi B) = 0. \tag{4.11}$$

Remark 4.4.1. *Strictly speaking, we should specify the spectral cuts θ of A, ϕ of B and ψ of AB in the expression $L(A,B)$ setting instead*

$$L^{\theta,\phi,\psi}(A,B) := \log_\psi(AB) - \log_\theta A - \log_\phi B.$$

Then by formula (2.9)

$$L^{\theta,\phi,\psi}(A,B) - L^{\theta',\phi',\psi'}(A,B) = -2i\pi\,\left(\Pi_{\psi,\psi'}(AB) - \Pi_{\theta,\theta'}(A) - \Pi_{\phi,\phi'}(B)\right)$$

so that a change of spectral cut introduces pseudodifferential projections which have vanishing residue by results of M. Wodzicki [W2]. Thus, a change of spectral cut does not affect the residues.
More generally, we can choose fixed spectral cuts θ and ϕ by the following argument of K. Okikiolu [Ok1]:

$$L^{\theta,\phi,\psi}(A,B) = L^{\pi,\pi,\psi-(\theta+\phi)}(e^{i(\pi-\theta)}A, e^{i(\pi-\phi)}B).$$

Indeed, if A, B, AB have spectral cut θ, ϕ, ψ respectively, then $A' = e^{i(\pi-\theta)}A$ and $B' = e^{i(\pi-\phi)}B$ have spectral cut π and $A'B'$ has spectral cut $\psi + 2\pi - \theta - \phi$. So we can assume that $\theta = \phi = \pi$ without loss of generality. To simplify notations, we drop the explicit mention of the spectral cuts.

Any trace on $C\ell(M,E)$ i.e. any linear form on $C\ell(M,E)$ which vanishes on commutators $[C\ell(M,E), C\ell(M,E)]$ is proportional to the noncommutative residue [W1] (see also [K]); in other words:

$$\forall A \in C\ell(M,E) \quad (\mathrm{res}(A) = 0 \Longrightarrow A \in [C\ell(M,E), C\ell(M,E)])\,.$$

It follows from (4.11) that

$$L(A,B) \in [C\ell(M,E), C\ell(M,E)]$$

so that $L(A,B)$ is a finite sum of commutators. The following proposition provides a refinement of this statement.

Proposition 4.4.2 ([OP])**.** *Let A and B be two admissible operators, which w.l.o.g. are assumed to have π as spectral cut, such that their product AB is also admissible with spectral cut π. Then $L(A,B)$ is a finite sum of Lie brackets of operators in $C\ell^0(M,E)$:*

$$L(A,B) \in [C\ell^0(M,E), C\ell^0(M,E)].$$

Proof: Up to a pseudodifferential projection, let us check that $L(A,B)$ lies in $C\ell^0(M,E)$. Since AB has order $a+b$, by Lemma 2.1.8 we have

$$\begin{aligned}
&\sigma(L(A,B))(x,\xi)\\
= \;& \sigma(\log AB)(x,\xi) - \sigma(\log A)(x,\xi) - \sigma(\log B)(x,\xi)\\
= \;& (a+b)\,\log|\xi|\,I + \sigma_0^{AB}(x,\xi) - a\,\log|\xi| I - \sigma_0^{A}(x,\xi) - b\,\log|\xi|\,I - \sigma_0^{B}(x,\xi)\\
= \;& \sigma_0^{AB}(x,\xi) - \sigma_0^{A}(x,\xi) - \sigma_0^{B}(x,\xi)
\end{aligned}$$

where σ_0^C denotes the symbol of order zero associated to the symbol of $\log C$. It follows that the operator $L(A,B)$ is indeed classical of order 0 with leading symbol given for any $(x,\xi) \in T^*M - \{0\}$ by

$$\begin{aligned}
\sigma^L(L(A,B))(x,\xi) &= \log\sigma^L(AB)(x,\frac{\xi}{|\xi|}) - \log\sigma^L(A)(x,\frac{\xi}{|\xi|}) - \log\sigma^L(B)(x,\frac{\xi}{|\xi|})\\
&=: L(\sigma^L(A),\sigma^L(B))(x,\frac{\xi}{|\xi|}).
\end{aligned}$$

Here as before, $\sigma^L(C)$ stands for the leading symbol of the operator C.
Let us apply the usual Campbell-Hausdorff formula to the matrices $\sigma^L(A)(x, \frac{\xi}{|\xi|})$ and $\sigma^L(B)(x, \frac{\xi}{|\xi|})$ and implement the fibrewise trace tr_x. This yields:

$$\begin{aligned} & \mathrm{tr}_x \left(L(\sigma^L(A), \sigma^L(B))(x, \frac{\xi}{|\xi|}) \right) \\ = \ & \mathrm{tr}_x \left(\log \sigma^L(AB)(x, \frac{\xi}{|\xi|}) - \log \sigma^L(A)(x, \frac{\xi}{|\xi|}) - \log \sigma^L_B(x, \frac{\xi}{|\xi|}) \right) = 0. \end{aligned}$$

It follows that any leading symbol trace $\mathrm{Tr}_0^\lambda(C) := \lambda\left(\mathrm{tr}_x(\sigma_0(C))\right)$ (Lemma 3.3.3) on the algebra $C\ell^0(M,E)$ where λ is a current in $C^\infty(S^*M)'$, vanishes on $L(A,B)$:

$$\mathrm{Tr}_0^\lambda(L(A,B)) = \lambda\left(\mathrm{tr}_x\left(\sigma_0(L(A,B))\right)\right) = 0.$$

Thus both the noncommutative residue and leading symbol traces vanish on $L(A,B)$. But by the results of [LP], any trace on $C\ell^0(M,E)$, i.e. any linear form on $C\ell^0(M,E)$ which vanishes on $[C\ell^0(M,E), C\ell^0(M,E)]$, is a linear combination of the noncommutative residue and a leading symbol trace. Consequently all traces on $C\ell^0(M,E)$ vanish on the operator $L(A,B)$ which therefore lies in $[C\ell^0(M,E), C\ell^0(M,E)]$. $\square$

Since the operator $L(A,B)$ is classical we can compute its weighted trace. The following result is reminiscent of an observation made in [Ok1] (see also [Sc]), namely that only the first n homogeneous components of the symbols come into play for the derivation of the Campbell-Hausdorff formula for operators with scalar leading symbols; the weighted trace of $L(A,B)$ presents a similar feature in our more general situation.

In the following, to simplify notations, we drop the mention of the spectral cut α of the weight Q.

Theorem 4.4.3 ([OP]). *Given a weight Q and two admissible operators A and B in $C\ell(M,E)$, the weighted trace $\mathrm{Tr}^Q(L(A,B))$ is a local expression as a finite sum of non-commutative residues.*
If both operators A and B have non negative order, then for any operator S in $C\ell(M,E)$ with order whose real part is $< -n$,

$$\frac{d}{dt}\mathrm{Tr}^Q(L(A(1+tS), B)) = \frac{d}{dt}\mathrm{Tr}^Q(L(A, B(1+tS))) = 0, \tag{4.12}$$

so that $\mathrm{Tr}^Q(L(A,B))$ only depends on the first n homogeneous components of the symbols of A and B.

Proof: By Proposition 4.4.2, $L(A,B)$ is a finite sum of commutators of zero order classical pseudodifferential operators $[P_j, Q_j]$. By formula (4.1), each weighted trace $\mathrm{Tr}^Q([P_j, Q_j]$ is proportional to $\mathrm{res}\,(Q_j\,[P_j, \log_\alpha Q])$. Since P_j and Q_j are of order zero, so is $Q_j[P_j, \log_\alpha Q]$ of order zero so that $\mathrm{Tr}^Q(L(A,B))$ is indeed a finite sum of noncommutative residues of zero order operators.
Let us check that requirement (4.12) is equivalent to the fact that $\mathrm{Tr}^Q(L(A,B))$ only depends on the first n homogeneous components of the symbols of A and B.
Given an operator S in $C\ell(M,E)$ of order $< -n$ and an operator A in $C\ell(M,E)$ of order a, we first observe that in any local trivialization the first n homogeneous components of the symbols of A and $A(1+S)$ coincide since AS has order $a-n$. Conversely, if the first n homogeneous components of the symbols of two classical operators A and B of orders a and b coincide, then $a=b$ and if B is invertible, the first n homogeneous components of the symbol of B^{-1} defined inductively using formula (1.2) by:

$$\begin{aligned}\sigma_{-b}(B^{-1}) &= (\sigma_b(B))^{-1}\,,\\ \sigma_{-b-j}(B^{-1}) &= -\,(\sigma_b(B))^{-1}\sum_{k+l+|\alpha|=j, l<j}\frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha\sigma_{b-k}(B)\,\partial_x^\alpha\sigma_{-b-l}(B^{-1}),\end{aligned}$$

coincide with that of the symbol of A^{-1} since the terms corresponding to $j\leq n$ only involve homogeneous components $\sigma_{b-k}(B)=\sigma_{a-k}(A)$ and $\sigma_{-b-l}(B^{-1})$ with k and l no larger than n. Consequently, by (1.2) it follows that $S=A^{-1}\,B$ has order $< -n$. Thus, showing that the expression $\mathrm{Tr}^Q(L(A,B))$ only depends on the first n homogeneous components of A amounts to showing that $\mathrm{Tr}^Q(L(A+S,B))=\mathrm{Tr}^Q(L(A,B))$ for any classical operator S of order $< -n$.
This part of the proof is inspired by steps of Okikiolu's proof of the Campbell-Hausdorff formula [Ok1]. Let us further observe that the proof of (4.12) reduces to the proof at $t_0=0$. Indeed for any real number t_0, for any S,T in $C\ell(M,E)$ of order $< -n$ and for any admissible operators A,B,C,D, we have

$$\frac{d}{dt}_{|t=0}\mathrm{Tr}^Q(L(C(1+tT),D))=0 \Longrightarrow \frac{d}{dt}_{|t=t_0}\mathrm{Tr}^Q(L(A(1+tS),B))=0. \tag{4.13}$$

To check this implication, we set $u=t-t_0$ so that

$$1+tS=1+t_0S+uS=(1+uS(1+t_0S)^{-1})(1+t_0S).$$

Setting $T=S(1+t_0S)^{-1}$ which also has order $< -n$, we have

$$A(1+tS)=A(1+uS(1+t_0S)^{-1})(1+t_0S)=A(1+uT)(1+t_0S)$$

and

$$A(1+tS)B=A(1+uT)(1+t_0S)B.$$

It follows that

$$\begin{aligned}
& L(A(1+tS),B)-L(A(1+uT),(1+t_0S)B)\\
= \;& \log(A(1+tS)B)-\log(A(1+tS))-\log(B)-\log(A(1+uT)(1+t_0S)B)\\
& +\log(A(1+uT))+\log((1+t_0S)B)\\
= \;& -\log(A(1+uT)(1+t_0S))-\log(B)+\log(A(1+uT))+\log((1+t_0S)B)\\
= \;& L(1+t_0S,B)-L(A(1+uT),1+t_0S),
\end{aligned}$$

and hence

$$L(A(1+tS),B)=L(A(1+uT),(1+t_0S)B)+L(1+t_0S,B)-L(A(1+uT),1+t_0S).$$

Differentiating w.r. to t at $t=t_0$ on the l.h.s boils down to differentiating the r.h.s. at $u=0$ and the implication (4.13) then easily follows.
We are therefore left to prove that $\frac{d}{dt}_{|t=0}\mathrm{Tr}^Q(L(A(1+tS),B)=0$. Applying (4.10) to the operator $A_t:=L(A(1+tS),B)$ we have

$$\frac{d}{dt}_{|t=0}\mathrm{Tr}^Q(L(A(1+tS),B))=\mathrm{Tr}^Q\left(\frac{d}{dt}_{|t=0}(L(A(1+tS),B))\right).$$

We therefore need to investigate the behaviour of $\frac{L(A(1+tS),B)-L(A,B)}{t}$ as $t\to 0$. Since

$$L(A(1+tS),B)-L(A,B)=\log(A(1+tS)B)-\log(AB)-(\log(A(1+tS))-\log A)\,,$$

let us study the difference $\log(A(1+tS)C)-\log(AC)$ with C equal to either B or the identity operator. Applying Proposition 2.1.14 to $A_t:=A(1+tS)C$ so that $A_0=AC$ and $\dot{A}_0=ASC$, and then implementing the weighted trace Tr^Q yields

$$\begin{aligned}
& \frac{d}{dt}_{|t=0}\mathrm{Tr}^Q\left(\log(A(1+tS)C)\right)\\
= \;& \mathrm{Tr}^Q(ASC\,(AC)^{-1})+\sum_{k=1}^{K}\frac{(-1)^k}{k+1}\mathrm{Tr}^Q\left(\mathrm{ad}_{AC}^k(ASC)\,(AC)^{-(k+1)}\right)+\mathrm{Tr}^Q(R_K(AC,ASC))
\end{aligned}$$

for arbitrary large K, with remainder term

$$R_K(A_t,\dot{A}_t)=-\frac{i}{2\pi}\int_{\Gamma_\theta}\log_\theta\lambda\,\left[(\lambda-AC)^{-1},\mathrm{ad}_{AC}^K(ASC)\right](\lambda-AC)^{-K-1}\,d\lambda.$$

But for any positive integer k, by (4.1) we have

$$\begin{aligned}
\mathrm{Tr}^Q\left(\mathrm{ad}_{AC}^k(ASC)\,(AC)^{-(k+1)}\right) &= \mathrm{Tr}^Q\left(\mathrm{ad}_{AC}(\mathrm{ad}_{AC}^{k-1}(ASC))\,(AC)^{-(k+1)}\right)\\
&= \mathrm{Tr}^Q\left(\mathrm{ad}_{AC}\left(\mathrm{ad}_{AC}^{k-1}(ASC)\,(AC)^{-(k+1)}\right)\right)\\
&= \frac{1}{q}\mathrm{res}\left(\mathrm{ad}_{AC}^{k-1}(ASC)\,(AC)^{-(k+1)}\,[AC,\log Q]\right)\\
&= 0,
\end{aligned}$$

since the operator $\mathrm{ad}_{AC}^{k-1}(ASC)\,(AC)^{-(k+1)}\,[AC, \log Q]$ has order $(k-1)(a+c)+a+c+s-(k+1)(a+c) = s-a-c$ (here s is the order of S, a the order of A, c the order of C) and hence, is smaller than $-n$. Thus

$$\frac{d}{dt}_{|t=0} \mathrm{Tr}^Q \left(\log(A(1+tS)C)\right) = \mathrm{Tr}^Q \left(ASC\ (AC)^{-1}\right) + \mathrm{Tr}^Q \left(R_K(AC, ASC)\right),$$

independently of the choice of the integer K. The remainder term

$$\mathrm{Tr}^Q(R_K(AC, ASC)) = -\mathrm{Tr}^Q \left(\frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\ \left[(\lambda - AC)^{-1}, \mathrm{ad}_{AC}^K(ASC)\right](\lambda - AC)^{-K-1}\,d\lambda\right)$$

depends on S via the iterated brackets $\mathrm{ad}_{AC}^K(ASC)$ and hence via K. Since it is independent of K, it is also independent of S. Setting $S=0$ which has order $< -n$, we infer that $\mathrm{Tr}^Q(R_K(AC, ASC))$ vanishes for all positive integers K. Thus

$$\frac{d}{dt}_{|t=0} \mathrm{Tr}^Q \left(\log(A(1+tS)C)\right) = \mathrm{Tr}^Q \left(ASC\,(AC)^{-1}\right) = \mathrm{Tr}^Q \left(ASA^{-1}\right)$$

independently of C. Setting back $C=B$ and $C=I$ yields

$$\begin{aligned} & \frac{d}{dt}_{|t=0} \mathrm{Tr}^Q(L(A(1+tS), B)) \\ = \ & \mathrm{Tr}^Q\left(\frac{d}{dt}_{|t=0}\log(A(1+tS)B\right) - \mathrm{Tr}^Q\left(\frac{d}{dt}_{|t=0}\log(A(1+tS))\right) \\ = \ & 0. \end{aligned}$$

□

4.5 A local formula for the weighted trace of $L(A, B)$

We derive an explicit local expression for the weighted traces $\mathrm{Tr}^Q(L(A,B))$ of $L(A,B)$ (Theorem 4.5.2). Our approach is inspired by the proof of K. Okikiolu for the Campbell-Hausdorff formula for operators with scalar leading symbols. In the case of operators with scalar leading symbols, as it was noticed and used by K. Okikiolu, as from a certain order in the Campbell-Hausdorff expansion, one can implement ordinary traces since the iterated brackets have decreasing order. In our more general situation, such a phenomenon does not accour so that we use weighted traces instead.

Proposition 4.5.1 ([OP]). *Let A and B be two admissible operators in $C\ell(M,E)$ with positive orders a and b. We have the following identities for weighted traces: for any real $\mu > 0$,*

$$\frac{d}{dt}_{|t=0} \mathrm{Tr}^B(L(A^t, B^\mu)) = 0, \quad \frac{d}{dt}_{|t=0} \mathrm{Tr}^A(L(A^t, B^\mu)) = 0$$

as well as for the noncommutative residue:

$$\frac{d}{dt}_{|t=0} \mathrm{res}(L(A^t, B^\mu)) = 0,$$

provided there is some positive ϵ such that $A^t B^\mu$ is admissible for any $t \in]-\epsilon, \epsilon[$.

Proof: Let us prove the result for the B-weighted trace; a similar proof yields the result for the A-weighted trace. By Proposition 4.3.1, weighted traces and the residue commute with differentiation on constant order operators so that

$$\frac{d}{dt}_{|t=0} \mathrm{Tr}^B \left(L(A^t, B^\mu)\right) = \mathrm{Tr}^B \left(\frac{d}{dt}_{|t=0} L(A^t, B^\mu)\right)$$

resp.

$$\frac{d}{dt}_{|t=0} \mathrm{res} \left(L(A^t, B^\mu)\right) = \mathrm{res} \left(\frac{d}{dt}_{|t=0} (L(A^t, B^\mu)\right).$$

But

$$\frac{d}{dt}_{|t=0} L(A^t, B^\mu) = \frac{d}{dt}_{|t=0} \log(A^t B^\mu) - \frac{d}{dt}_{|t=0} \left(\log A^t\right).$$

We therefore apply Proposition 2.1.14 to $A_t := A^t B^\mu$ so that $A_0 = B^\mu$, including the case $\mu = 0$ for which $A_t = A^t$ and $A_0 = I$. Since $\dot{A}_0 = \log A\, B^\mu$ and $\dot{A}_0\, A_0^{-1} = \log A$, implementing the weighted trace Tr^B yields

$$\begin{aligned} &\frac{d}{dt}_{|t=0} \mathrm{Tr}^B \left(\log(A^t B^\mu)\right) \\ = \;& \mathrm{Tr}^B(\log A) + \sum_{k=1}^{K} \frac{(-1)^k}{k+1} \mathrm{Tr}^B \left(\mathrm{ad}^k_{B^\mu}(\log A\, B^\mu)\, B^{-\mu(k+1)}\right) + \mathrm{Tr}^B(R_K(B^\mu, \log A\, B^\mu)) \end{aligned}$$

for arbitrary large K, with remainder term

$$R_K(B^\mu, \log A\, B^\mu) = -\frac{i}{2\pi} \int_{\Gamma_\theta} \log_\theta \lambda \, \left[(\lambda - B^\mu)^{-1}, \mathrm{ad}^K_{B^\mu}(\log A\, B^\mu)\right] (\lambda - B^\mu)^{-K-1}\, d\lambda.$$

But for any positive integer k, by formula (4.1) we have

$$\begin{aligned} \mathrm{Tr}^B \left(\mathrm{ad}^k_{B^\mu}(A\, B^\mu)\, B^{-\mu(k+1)}\right) &= \mathrm{Tr}^B \left(\mathrm{ad}_{B^\mu}(\mathrm{ad}^{k-1}_{B^\mu}(A\, B^\mu))\, B^{-\mu(k+1)}\right) \\ &= \mathrm{Tr}^B \left(\mathrm{ad}_{B^\mu} \left(\mathrm{ad}^{k-1}_{B^\mu}(A\, B^\mu)\, B^{-\mu(k+1)}\right)\right) \\ &= \frac{1}{b} \mathrm{res} \left(\mathrm{ad}^{k-1}_{B^\mu}(A\, B^\mu)\, B^{-\mu(k+1)}\, [B^\mu, \log B]\right) \\ &= 0, \end{aligned}$$

since B commutes with $\log B$. A similar computation shows that

$$\mathrm{Tr}^B(R_K(B^\mu, \log A\, B^\mu)) = 0.$$

Thus

$$\frac{d}{dt}_{|t=0} \mathrm{Tr}^B \left(\log(A^t B^\mu)\right) = \mathrm{Tr}^B \left(\log A\right).$$

independently of the choice of the integer K. It follows that $\frac{d}{dt}_{|t=0} \mathrm{Tr}^B \left(\log(A^t B^\mu)\right) = \mathrm{Tr}^B \left(\log A\right)$ independently of μ so that

$$\frac{d}{dt}_{|t=0} \mathrm{Tr}^B \left(L(A^t, B^\mu)\right) = 0.$$

Similarly, replacing the weighted trace Tr^B by the noncommutative residue res and using the cyclicity of the noncommutative residue, yields

$$\frac{d}{dt}_{|t=0} \mathrm{res} \left(L(A^t, B^\mu)\right) = 0.$$

□

The following statement provides a local formula for the weighted trace of $L(A, B)$. It also shows that the residue of $L(A, B)$ vanishes and therefore yields back the multiplicativity of the residue determinant derived in [Sc].

Theorem 4.5.2 ([OP]). *Let A and B be two admissible operators in $C\ell(M, E)$ with positive orders a and b. Assume that there is some positive ϵ such that $A^t B$ is admissible for any $t \in]-\epsilon, \epsilon[$. Then we have*

$$\mathrm{res}(L(A, B)) = 0.$$

Moreover, there is an operator

$$W(\tau)(A, B) := \frac{d}{dt}_{|t=0} L(A^t, A^\tau B) \tag{4.14}$$

in $C\ell^0(M, E)$ depending continuously on τ such that

$$\mathrm{Tr}^Q(L(A, B)) = \int_0^1 \mathrm{res} \left(W(\tau)(A, B) \left(\frac{\log(A^\tau B)}{a\tau + b} - \frac{\log Q}{q} \right) \right) d\tau \tag{4.15}$$

where Q is any weight of positive order q.

Proof: By Proposition 4.5.1, we know that

$$\frac{d}{dt}_{|t=0}\mathrm{res}(L(A^t,B)) = \frac{d}{dt}_{|t=0}\mathrm{Tr}^A(L(A^t,B)) = \frac{d}{dt}_{|t=0}\mathrm{Tr}^B(L(A^t,B) = 0.$$

We want to compute

$$\frac{d}{dt}_{|t=\tau}\mathrm{res}(L(A^t,B)) = \frac{d}{dt}_{|t=0}\mathrm{res}(L(A^{t+\tau},B))$$

and

$$\frac{d}{dt}_{|t=\tau}\mathrm{Tr}^Q(L(A^t,B)) = \frac{d}{dt}_{|t=0}\mathrm{Tr}^Q(L(A^{t+\tau},B)).$$

For this we observe that

$$L(AB,D) - L(A,BD) = -\log(AB) - \log(D) + \log A + \log(BD) = L(B,D) - L(A,B)$$

Replacing A by A^t, B by A^τ and D by B, we get

$$L(A^{t+\tau},B) - L(A^t,A^\tau B) = L(A^\tau,B) - L(A^t,A^\tau) = L(A^\tau,B).$$

Implementing the noncommutative residue, by Proposition 4.5.1 we have:

$$\begin{aligned}\frac{d}{dt}_{|t=\tau}\mathrm{res}(L(A^t,B)) &= \frac{d}{dt}_{|t=0}\mathrm{res}(L(A^{t+\tau},B))\\ &= \frac{d}{dt}_{|t=0}\mathrm{res}(L(A^t,A^\tau B))\\ &= 0.\end{aligned}$$

Hence since $\mathrm{res}(L(I,B)) = 0$.

$$\mathrm{res}(L(A,B)) = \int_0^1 \frac{d}{dt}_{|t=\tau}\mathrm{res}(L(A^t,B))\,d\tau + \mathrm{res}(L(I,B)) = 0. \tag{4.16}$$

If instead we implement the weighted trace Tr^Q, we have:

$$\frac{d}{dt}_{|t=\tau}\mathrm{Tr}^Q(L(A^t,B)) = \frac{d}{dt}_{|t=0}\mathrm{Tr}^Q(L(A^{t+\tau},B)) = \frac{d}{dt}_{|t=0}\mathrm{Tr}^Q(L(A^t,A^\tau B)).$$

Since A and B have positive order so has $A^\tau\, B$ so that applying Proposition 4.5.1 with weighted traces $\mathrm{Tr}^{A^\tau\, B}$ yields:

$$\begin{aligned}\frac{d}{dt}_{|t=\tau}\mathrm{Tr}^Q(L(A^t,B) &= \frac{d}{dt}_{|t=0}\mathrm{Tr}^Q(L(A^t,A^\tau B))\\ &= \frac{d}{dt}_{|t=0}\mathrm{Tr}^{A^\tau B}(L(A^t,A^\tau B))\\ &\quad+\frac{d}{dt}_{|t=0}\left(\mathrm{Tr}^Q(L(A^t,A^\tau B)) - \mathrm{Tr}^{A^\tau B}(L(A^t,A^\tau B))\right)\\ &= \frac{d}{dt}_{|t=0}\left(\mathrm{Tr}^Q(L(A^t,A^\tau B)) - \mathrm{Tr}^{A^\tau B}(L(A^t,A^\tau B))\right).\end{aligned}$$

Applying (4.6) to $Q_1 = Q$ and $Q_2 = A^\tau B$, we infer that

$$\begin{aligned} & \frac{d}{dt}_{|t=0} \left(\mathrm{Tr}^Q(L(A^t, A^\tau B)) - \mathrm{Tr}^{A^\tau B}(L(A^t, A^\tau B))\right) \\ = \; & \frac{d}{dt}_{|t=0} \mathrm{res}\left(L(A^t, A^\tau B)\left(\frac{\log(A^\tau B)}{a\tau + b} - \frac{\log Q}{q}\right)\right) \\ = \; & \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau + b} - \frac{\log Q}{q}\right)\right), \end{aligned}$$

where q is the order of Q and where we have set $W(\tau)(A,B) := \frac{d}{dt}_{|t=0} L(A^t, A^\tau B)$. Since $\mathrm{Tr}^Q(L(A^0, B)) = 0$, we finally find that

$$\begin{aligned} \mathrm{Tr}^Q(L(A,B)) &= \mathrm{Tr}^Q(L(A^1, B)) - \mathrm{Tr}^Q(L(A^0, B)) \\ &= \int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau + b} - \frac{\log Q}{q}\right)\right) d\tau. \qquad (4.17) \end{aligned}$$

□

CHAPTER 5

Chapter 5

The multiplicative anomaly for regularized determinants

In this chapter we investigate the multiplicative anomaly for regularized determinants of elliptic operators. The local formula for the multiplicative anomaly of weighted determinants (Proposition 5.2.1) corresponds to an exponentiated weighted trace of $L(A, B)$ studied in the previous chapter. It compares with the multiplicative anomaly for the ζ-determinant by a local term which, combined with the explicit formula for regularized traces of $L(A, B)$, provides an explicit local formula for the multiplicative anomaly of the zeta determinant (Theorem 5.3.2).

5.1 The ζ-determinant and the weighted determinant

The determinant on the linear group $Gl(\mathbb{R}^n)$ reads

$$\det A = \exp\left(\mathrm{tr}(\log A)\right)$$

where tr is the matrix trace. It is independent of the choice of spectral cut used to define the logarithm and is multiplicative as a result of the Campbell-Hausdorff formula and the cyclicity of the trace, namely:

$$\det(AB) = \exp\left(\mathrm{tr}(\log AB)\right) = \exp\left(\mathrm{tr}(\log A + \log B)\right) = \det A \det B.$$

This determinant extends to admissible operators by means of the ζ-determinant: an admissible operator A in $C\ell(M, E)$ with spectral cut θ and positive order has well-defined ζ-determinant:

$$\det_{\zeta,\theta}(A) := \exp\left(-\zeta'_{A,\theta}(0)\right) = \exp\left(\mathrm{Tr}_\theta^A(\log_\theta A)\right)$$

since the ζ-function $\zeta_{A,\theta}(z) := \mathrm{TR}(A_\theta^{-z})$ is holomorphic at $z = 0$. In the second equality, the weighted trace has been extended to logarithms as before, picking out the constant term of the meromorphic map $z \mapsto \mathrm{TR}(\log_\theta A\, Q^{-z})$.

The ζ-determinant were first introduced by D. B. Ray and M. I. Singer [RaSi] in relation with the R-torsion. K. Okikiolu investigated the ζ-determinant on elliptic classical ΨDOs [Ok2]. She used the Campbell-Hausdorff formula for classical ΨDOs [Ok1] she established to prove that the ζ-determinant is not multiplicative and hence presents a multiplicative anomaly studied independently by M. Kontsevich and S. Vishik in [KV1]. Before investigating the multiplicative anomaly of the ζ-determinant and proving its locality in Section 5.3, let us point out that the ζ-determinant generally depends on the choice of spectral cut. However, it is invariant under mild changes of spectral cut in the following sense.

Lemma 5.1.1. *Let $0 \leq \theta < \phi < 2\pi$ be two spectral cuts of the admissible operator A. If there is a cone $\Lambda_{\theta,\phi}$ (formula (2.6)) which does not intersect the spectrum of the leading symbol of A then*

$$\det_{\zeta,\theta}(A) = \det_{\zeta,\phi}(A).$$

Proof: The classical proof of this result starts from the definition

$$\det_{\zeta,\theta}(A) = \exp\left(-\zeta'_{A,\theta}(0)\right)$$

of the determinant in terms of the zeta function and uses Lidskii's theorem ([ReSi]) which says that the trace of a trace-class operator is equal to the sum of its eigenvalues. Since for $\mathrm{Re}(z)$ large enough, the operator A_θ^{-z} is trace-class then

$$\mathrm{Tr}(A_\theta^{-z}) = \mathrm{TR}(A_\theta^{-z}) = \sum_{\lambda \in Sp(A)} \lambda_\theta^{-z}$$

where $Sp(A)$ is the spectrum of A and each eigenvalue λ is counted with multiplicities. Let us denote by $\lambda_1, \cdots, \lambda_k$ the finite number of eigenvalues of A contained in the cone $\Lambda_{\theta,\phi}$. It follows that

$$\mathrm{TR}(A_\theta^{-z}) - \mathrm{TR}(A_\phi^{-z}) = \mathrm{Tr}(A_\theta^{-z}) - \mathrm{Tr}(A_\phi^{-z}) = \sum_{i=1}^{k} \left(\lambda_\theta^{-z} - \lambda_\phi^{-z}\right)$$

from which the result follows differentiating with respect to z and applying the exponential map.
We give an alternative proof which starts from the definition of the determinant $\det_{\zeta,\theta}(A) = \exp\left(\mathrm{Tr}_\theta^A(\log_\theta A)\right)$ in terms of a weighted trace of the logarithm of the operator using a

formula proved in [PS]: for an admissible operator A in $C\ell(M,E)$ with spectral cut θ and positive order a, the logarithm of $\det_{\zeta,\theta}(A)$ is given by

$$\log\det_{\zeta,\theta}(A) = \int_M dx\,\left[\mathrm{TR}_x(\log_\theta A) - \frac{1}{2a}\mathrm{res}_x(\log^2_\theta A)\right] \tag{5.1}$$

where res_x is the noncommutative residue density extended to log-polyhomogeneous operators defined previously. It follows from formula (5.1) that

$$\frac{\det_{\zeta,\phi}(A)}{\det_{\zeta,\theta}(A)} = \exp\left(\int_M dx\,\left[\mathrm{TR}_x(\log_\phi A - \log_\theta A) - \frac{1}{2a}\mathrm{res}_x(\log^2_\phi A - \log^2_\theta A)\right]\right).$$

By formula (2.9), $\log_\phi A - \log_\theta A = 2i\pi\Pi_{\theta,\phi}(A)$ is a finite rank operator and hence a smoothing operator under the assumptions of the proposition so that

$$\begin{aligned}
\frac{\det_{\zeta,\phi}(A)}{\det_{\zeta,\theta}(A)} &= \exp\left(\int_M dx\,\left[\mathrm{TR}_x(2i\pi\,\Pi_{\theta,\phi}(A)) - \frac{1}{2a}\mathrm{res}_x\left(2i\pi\,\Pi_{\theta,\phi}(A)(\log_\phi A + \log_\theta A)\right)\right]\right)\\
&= \exp\left(2i\pi\,\mathrm{Tr}(\Pi_{\theta,\phi}(A)) - \frac{2i\pi}{2a}\mathrm{res}\left(\Pi_{\theta,\phi}(A)(\log_\phi A + \log_\theta A)\right)\right)\\
&= \exp\left(2i\pi\,\mathrm{rk}(\Pi_{\theta,\phi}(A))\right)\\
&= 1.
\end{aligned}$$

Here rk stands for the rank and we have used the fact that the noncommutative residue vanishes on smoothing operators on which the canonical trace coincides with the usual trace on smoothing operators. □

Remark 5.1.2. *If there are infinitely many eigenvalues of A in the cone $\Lambda_{\phi,\theta}$, $\det_{\zeta,\phi}(A)$ and $\det_{\zeta,\theta}(A)$ might differ.*

Lemma 5.1.3. *Let A be an admissible operator in $C\ell(M,E)$ with spectral cut θ and positive order a. Then, for any integer k*

$$\frac{\det_{\zeta,\theta+2k\pi}(A)}{\det_{\zeta,\theta}(A)} = \exp\left(-\frac{2ik\pi}{a}\mathrm{res}(\log_\theta A)\right). \tag{5.2}$$

Proof: We first derive this formula using the description of the zeta determinant in terms of the zeta function. It is easy to see that for any integer k, $A^{-z}_{\theta+2k\pi} = e^{2ik\pi}A^{-z}_\theta$ and then

$$\zeta_{A,\theta+2k\pi}(z) = \mathrm{TR}(A^{-z}_{\theta+2k\pi}) = e^{2ik\pi}\zeta_{A,\theta}(z).$$

Differentiating this expression w.r. to z at z=0 gives the result.
An alternative proof uses the formula of the zeta determinant in terms of a weighted trace of the logarithm of the operator. Since by definition $\det_{\zeta,\theta}(A) = \exp\left(\mathrm{Tr}^A_\theta(\log_\theta A)\right)$ and

$\det_{\zeta,\theta+2k\pi}(A) = \exp\left(\mathrm{Tr}^A_{\theta+2k\pi}(\log_{\theta+2k\pi} A)\right)$, let us compute $\mathrm{Tr}^A_{\theta+2k\pi}(\log_{\theta+2k\pi} A) - \mathrm{Tr}^A_\theta(\log_\theta A)$. Applying formula (4.7) and formula (2.9) we get

$$\begin{aligned}
& \mathrm{Tr}^A_{\theta+2k\pi}(\log_{\theta+2k\pi} A) - \mathrm{Tr}^A_\theta(\log_\theta A) \\
= \; & \mathrm{Tr}^A_{\theta+2k\pi}\left(\log_{\theta+2k\pi} A - \log_\theta A\right) + \left(\mathrm{Tr}^A_{\theta+2k\pi} - \mathrm{Tr}^A_\theta\right)(\log_\theta A) \\
= \; & \mathrm{Tr}^A_{\theta+2k\pi}(2ik\pi I) - \frac{1}{2a}\mathrm{res}\left[(\log_\theta A - \log_{\theta+2k\pi} A)(\log_{\theta+2k\pi} A - \log_\theta A)\right] \\
& -\frac{1}{2a}\mathrm{res}\left[(\log_\theta A - \log_\theta A)(\log_{\theta+2k\pi} A - \log_\theta A)\right] \\
= \; & \mathrm{Tr}^A_{\theta+2k\pi}(2ik\pi I) + \frac{1}{2a}\mathrm{res}\left[(\log_\theta A - \log_{\theta+2k\pi} A)^2\right] \\
= \; & \mathrm{Tr}^A_{\theta+2k\pi}(2ik\pi I) + \frac{1}{2a}\mathrm{res}\left[(2ik\pi I)^2\right] \\
= \; & -\frac{2ik\pi}{a}\mathrm{res}(\log_\theta A).
\end{aligned}$$

Here again we use the fact that the noncommutative residue vanishes on differential operators and the fact that $\mathrm{TR}(I) = 0$ by Example 3.2.7 combined with formula (4.5) which reduces to

$$\mathrm{Tr}^A_\theta(I) = -\frac{1}{a}\mathrm{res}(\log_\theta A) = \zeta_{A,\theta}(0). \tag{5.3}$$

□

Our approach to the multiplicative anomaly of the ζ-determinant will be based on the locality of the regularized trace of the operator $L(A,B)$ studied in the previous chapter. In order to relate these two expressions, let us introduce another type of regularized determinant, namely the weighted determinant: given an admissible operator A in $C\ell(M,E)$ with spectral cut θ and positive order, for a weight Q with spectral cut α, the Q-weighted determinant of A [Du1] (see also [FrG]) reads:

$$\mathrm{Det}^Q_\theta(A) := \exp(\mathrm{Tr}^Q_\alpha(\log_\theta A)).$$

Since the Q-weighted trace restricts to the ordinary trace on trace-class operators, this determinant, as the ζ-determinant, extends the ordinary determinant on operators in the determinant class. The Q-weighted determinant, as well as being dependent on the choice of spectral cut θ, also depends on the choice of spectral cut α. Nevertheless, as for the ζ-determinant, it is invariant under mild changes of the spectral cut of A.

Lemma 5.1.4. *Let $0 \leq \theta < \phi < 2\pi$ be two spectral cuts of the admissible operator A. If there is a cone $\Lambda_{\theta,\phi}$ (see formula (2.6)) which does not intersect the spectrum of the leading symbol of A then*

$$\mathrm{Det}^Q_\theta(A) = \mathrm{Det}^Q_\phi(A).$$

Proof: Under the assumptions of the proposition, the cone $\Lambda_{\phi,\theta}$ contains only a finite number of points in the spectrum of A so that $\log_\phi A - \log_\theta A = 2i\pi\Pi_{\theta,\phi}(A)$ is a finite rank operator and hence a smoothing operator. Hence,

$$\begin{aligned}\frac{\mathrm{Det}_\phi^Q(A)}{\mathrm{Det}_\theta^Q(A)} &= \exp\left(\mathrm{Tr}^Q\left(\log_\phi A - \log_\theta A\right)\right) = \exp\left(\mathrm{Tr}^Q\left(2i\pi\,\Pi_{\theta,\phi}(A)\right)\right)\\ &= \exp\left(2i\pi\,\mathrm{Tr}\left(\Pi_{\theta,\phi}(A)\right)\right) = \exp(2i\pi\,\mathrm{rk}(\Pi_{\theta,\phi}(A)))\\ &= 1,\end{aligned}$$

where as before rk stands for the rank. □

The Q-weighted determinant and the ζ-determinant differ by local expression.

Proposition 5.1.5 ([Du1])**.** *Let A be an admissible operator in $C\ell(M,E)$ with spectral cut θ and positive order a. For a weight Q in $C\ell(M,E)$ with spectral cut α and positive order q,*

$$\frac{\det_{\zeta,\theta}(A)}{\mathrm{Det}_\theta^Q(A)} = \exp\left(-\frac{1}{2a}\mathrm{res}\left[\left(\log_\theta A - \frac{a}{q}\log_\alpha Q\right)^2\right]\right). \tag{5.4}$$

Proof: Indeed by formula (4.7) we have

$$\mathrm{Tr}_\theta^A(\log_\theta A) - \mathrm{Tr}_\alpha^Q(\log_\theta A) = -\frac{1}{2a}\mathrm{res}\left[\left(\log_\theta A - \frac{a}{q}\log_\alpha Q\right)^2\right].$$

Recall from Corollary 2.1.10 that the operator $\log_\theta A - \frac{a}{q}\log_\alpha Q$ is classical. It follows that

$$\frac{\det_{\zeta,\theta}(A)}{\mathrm{Det}_\theta^Q(A)} = \frac{\exp\left(\mathrm{Tr}_\theta^A(\log_\theta A)\right)}{\exp(\mathrm{Tr}_\alpha^Q(\log_\theta A))} = \exp\left(-\frac{1}{2a}\mathrm{res}\left[\left(\log_\theta A - \frac{a}{q}\log_\alpha Q\right)^2\right]\right).$$

□

This relation will allow us in the next sections to relate the multiplicative anomalies for the two determinants. Concretely since the r.h.s. of formula (5.4) is a local expression, the locality of one of them implies the locality of the other one. Let us start with the multiplicative anomaly of the Q-weighted determinant which is easier to compute.

5.2 Multiplicative anomaly for the weighted determinant

As ζ-determinant, the Q-weighted determinant is not multiplicative. The multiplicative anomaly for Q-weighted determinants of two admissible operators A and B with spectral cuts θ, ϕ such that AB has spectral cut ψ is defined by:

$$\mathcal{M}^Q_{\theta,\phi,\psi}(A,B) := \frac{\mathrm{Det}^Q_\psi(AB)}{\mathrm{Det}^Q_\theta(A)\,\mathrm{Det}^Q_\phi(B)},$$

which we write $\mathcal{M}^Q(A,B)$ for simplicity when there is no ambiguity for the choice of ψ, θ, ϕ. It follows that

$$\begin{aligned}\log \mathcal{M}^Q(A,B) &= \log \mathrm{Det}^Q(AB) - \log \mathrm{Det}^Q(A) - \log \mathrm{Det}^Q(B)\\ &= \mathrm{Tr}^Q(\log AB) - \mathrm{Tr}^Q(\log A) - \mathrm{Tr}^Q(\log B)\\ &= \mathrm{Tr}^Q(L(A,B))\end{aligned}$$

so that the multiplicative anomaly for Q-weighted determinants studied in [Du1] has logarithm given by the Q-weighted trace of $L(A,B)$, as a result of which it is local.

Proposition 5.2.1. *Let A and B be two admissible operators with positive orders a, b and with spectral cuts θ and ϕ in $[0, 2\pi[$ such that there is a cone delimited by the rays L_θ and L_ϕ which does not intersect the spectra of the leading symbols of A, B and AB. Then the product AB is admissible with a spectral cut ψ inside that cone and for any weight Q with spectral cut α, dropping the explicit mention of the spectral cuts we have:*

$$\log \mathcal{M}^Q(A,B) = \int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau + b} - \frac{\log Q}{q}\right)\right) d\tau. \tag{5.5}$$

Q-weighted determinants are multiplicative on commuting operators.

Proof: Since the leading symbol of the product AB has spectrum which does not intersect the cone delimited by L_θ and L_ϕ, the operator AB only has a finite number of eigenvalues inside that cone. We can therefore choose a ray ψ which avoids both the spectrum of the leading symbol of AB and the eigenvalues of AB. By Lemma 5.1.4, the Q-weighted determinants $\det^Q_\theta(A)$, $\det^Q_\phi(B)$ and $\det^Q_\psi(AB)$ do not depend on the choices of spectral cuts satisfying the requirements of the proposition. Since

$$\log \mathcal{M}^Q(A,B) = \mathrm{Tr}^Q(L(A,B)),$$

the logarithm of the multiplicative anomaly for Q-weighted determinants is a local quantity as a finite sum of noncommutative residues as a consequence of Theorem 4.5.2.
To prove the second part of the statement we observe that

$$[A,B] = 0 \Longrightarrow L(A,B) = 0. \tag{5.6}$$

Even though this property is equivalent to formula 2.13, let us prove directly that $L(A,B)$ vanishes. Indeed, let Γ be a contour along a spectral ray around the spectrum of $A^{t_0}B$ for some fixed t_0, then

$$\begin{aligned}\frac{d}{dt}_{|t=t_0}\log(A^tB) &= \frac{i}{2\pi}\int_\Gamma \log\lambda\,\frac{d}{dt}_{|t=t_0}(A^tB-\lambda)^{-1}\,d\lambda\\ &= \frac{i}{2\pi}\int_\Gamma \log\lambda\,(A^{t_0}B-\lambda)^{-1}\,\log A\,A^{t_0}B\,(A^{t_0}B-\lambda)^{-1}\,d\lambda.\end{aligned}$$

Since $[A,B]=0$, by integration by parts we obtain

$$\begin{aligned}\frac{d}{dt}_{|t=t_0}\log(A^tB) &= \log A\,A^{t_0}B\,\frac{i}{2\pi}\int_\Gamma \log\lambda\,(A^{t_0}B-\lambda)^{-2}\,d\lambda\\ &= -\log A\,A^{t_0}B\,\frac{i}{2\pi}\int_\Gamma \lambda^{-1}(A^{t_0}B-\lambda)^{-1}\,d\lambda\\ &= -\log A\;A^{t_0}B\,(A^{t_0}B)^{-1}\\ &= -\log A.\end{aligned}$$

Similarly, we have $\frac{d}{dt}_{|t=t_0}\log(A^t)=-\log A$ so that finally

$$\frac{d}{dt}_{|t=t_0}L(A^t,B)=\frac{d}{dt}_{|t=t_0}\log(A^tB)-\frac{d}{dt}_{|t=t_0}\log(A^t)$$

vanishes. It follows that

$$L(A,B)=\int_0^1\frac{d}{dt}_{|t=\tau}L(A^t,B)\,d\tau=0.$$

Since $L(A,B)$ vanishes when A and B commute, Q-weighted determinants are multiplicative on commuting operators. $\square$

5.3 Multiplicative anomaly for the ζ-determinant

As already mentioned the ζ-determinant is not multiplicative. Let A and B be two admissible operators with positive order and spectral cuts θ and ϕ and such that AB is also admissible with spectral cut ψ. The multiplicative anomaly

$$\mathcal{M}_\zeta^{\theta,\phi,\psi}(A,B):=\frac{\det_{\zeta,\psi}(AB)}{\det_{\zeta,\theta}(A)\det_{\zeta,\phi}(B)},$$

was proved to be local, independently by M. Wodzicki [W1], for positive definite commuting elliptic differential operators, by L. Friedlander [Fr] for positive definite elliptic pseudodifferential operators, by K. Okikiolu [Ok2] for pseudodifferential operators with scalar leading symbol and by M. Kontsevich and S. Vishik [KV1] for pseudodifferential operators with leading symbols "sufficiently close to positive definite self-adjoint ones". For simplicity, we drop the explicit mention of θ, ϕ, ψ and write $\mathcal{M}_\zeta(A, B)$.

Since the ζ-determinant is related to the Q-weight determinant by formula (5.4) the multiplicative anomaly of the ζ-determinant is related to the multiplicative anomaly of the Q-weighted determinant.

Proposition 5.3.1. *Let A and B be two admissible operators with positive orders a, b and with spectral cuts θ and ϕ in $[0, 2\pi[$ such that there is a cone delimited by the rays L_θ and L_ϕ which does not intersect the spectra of the leading symbols of A, B and AB. Then the product AB is admissible with a spectral cut ψ inside that cone and for any weight Q with spectral cut α, dropping the explicit mention of the spectral cuts we have:*

$$\log \mathcal{M}_\zeta(A, B) = \log \mathcal{M}^Q(A, B) + \mathrm{res}\left(\frac{L(A, B)\log Q}{q} - \frac{\log^2 AB}{2(a+b)} + \frac{\log^2 A}{2a} + \frac{\log^2 B}{2b}\right) \tag{5.7}$$

so that the multiplicative anomaly of the ζ-determinant is local.

Proof: By formula (5.4), we have

$$\begin{aligned}\log \mathcal{M}_\zeta(A, B) \quad &= \quad \log \mathcal{M}^Q(A, B) - \frac{1}{2(a+b)}\mathrm{res}\left[\left(\log AB - \frac{a+b}{q}\log Q\right)^2\right] \\ &\quad + \frac{1}{2a}\mathrm{res}\left[\left(\log A - \frac{a}{q}\log Q\right)^2\right] + \frac{1}{2b}\mathrm{res}\left[\left(\log B - \frac{b}{q}\log Q\right)^2\right].\end{aligned}$$

Let us recall, as before, from Corollary 2.1.10 that each operator $\log C - \frac{c}{q}\log Q$ where c is the positive order of C is classical. Using formula (4.4) to simplify we obtain

$$\log \mathcal{M}_\zeta(A, B) = \log \mathcal{M}^Q(A, B) + \mathrm{res}\left(\frac{L(A, B)\log Q}{q} - \frac{\log^2 AB}{2(a+b)} + \frac{\log^2 A}{2a} + \frac{\log^2 B}{2b}\right).$$

Since the logarithm of the multiplicative anomaly of the Q-weighted determinant is given by a local residue, we recover the fact that the multiplicative anomaly for the ζ-determinants has logarithm given by a finite sum of noncommutative residues as a result of which it is local. □

Let us now compute the explicit local formula of the multiplicative anomaly of the ζ-determinant.

Theorem 5.3.2. *Let A and B be two admissible operators in $C\ell(M,E)$ with positive orders a,b and with spectral cuts θ and ϕ in $[0,2\pi[$ such that there is a cone delimited by the rays L_θ and L_ϕ which does not intersect the spectra of the leading symbols of A, B and AB. Then the product AB is admissible with a spectral cut ψ inside that cone. A local formula of the multiplicative anomaly $\mathcal{M}_\zeta(A,B)$ reads:*

$$\begin{aligned}\log \mathcal{M}_\zeta(A,B) &= \int_0^1 \operatorname{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau+b}-\frac{\log B}{b}\right)\right)d\tau \\ &\quad +\operatorname{res}\left(\frac{L(A,B)\,\log B}{b}-\frac{\log^2 A\,B}{2(a+b)}+\frac{\log^2 A}{2a}+\frac{\log^2 B}{2b}\right) \\ &= \int_0^1 \operatorname{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau+b}-\frac{\log A}{a}\right)\right)d\tau \\ &\quad +\operatorname{res}\left(\frac{L(A,B)\,\log A}{a}-\frac{\log^2 A\,B}{2(a+b)}+\frac{\log^2 A}{2a}+\frac{\log^2 B}{2b}\right) \qquad (5.8)\end{aligned}$$

where $W(\tau)(A,B)$ is a classical operator of order zero depending continuously on τ given by formula (4.14). When A and B commute the multiplicative anomaly reduces to:

$$\log \mathcal{M}_\zeta(A,B) = \frac{ab}{2(a+b)}\operatorname{res}\left[\left(\frac{\log A}{a}-\frac{\log B}{b}\right)^2\right]. \qquad (5.9)$$

Remark 5.3.3. *For commuting operators, (5.9) gives back the results of M. Wodzicki as well as formula (III.3) in [Du1]:*

$$\log \mathcal{M}_\zeta(A,B) = \frac{\operatorname{res}\left(\log^2(A^a B^{-b})\right)}{2ab(a+b)}.$$

Proof: By formula (5.7) we have

$$\log \mathcal{M}_\zeta(A,B) = \log \mathcal{M}^Q(A,B) + \operatorname{res}\left(\frac{L(A,B)\log Q}{q}-\frac{\log^2 AB}{2(a+b)}+\frac{\log^2 A}{2a}+\frac{\log^2 B}{2b}\right).$$

Using formula (5.5) and formula (4.6) (since $L(A,B)$ is a classical operator) applied to $Q_1=Q$ and $Q_2=B$, we can express the local formula of $\log \mathcal{M}_\zeta(A,B)$.

$$\begin{aligned}\log \mathcal{M}_\zeta(A,B) &= \operatorname{Tr}^Q(L(A,B)) + \operatorname{res}\left(\frac{L(A,B)\log Q}{q}-\frac{\log^2 AB}{2(a+b)}+\frac{\log^2 A}{2a}+\frac{\log^2 B}{2b}\right) \\ &= \operatorname{Tr}^B(L(A,B)) + \operatorname{res}\left(\frac{L(A,B)\log B}{b}-\frac{\log^2 AB}{2(a+b)}+\frac{\log^2 A}{2a}+\frac{\log^2 B}{2b}\right) \\ &= \int_0^1 \operatorname{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau+b}-\frac{\log B}{b}\right)\right)d\tau \\ &\quad +\operatorname{res}\left(L(A,B)\,\frac{\log B}{b}-\frac{\log^2 AB}{2(a+b)}+\frac{\log^2 A}{2a}+\frac{\log^2 B}{2b}\right).\end{aligned}$$

which proves the first equality in (5.8). The second one can be derived similarly exchanging the roles of A and B.
When A and B commute, by formula (5.6) $L(A,B)$ vanishes and by Proposition 5.2.1 $\log \mathcal{M}^Q(A,B)$ vanishes so that formula (5.7) reduces to:

$$\begin{aligned}
\log \mathcal{M}_\zeta(A,B) &= \log \mathcal{M}^Q(A,B) + \operatorname{res}\left(\frac{L(A,B)\log Q}{q} - \frac{\log^2 AB}{2(a+b)} + \frac{\log^2 A}{2a} + \frac{\log^2 B}{2b}\right) \\
&= -\operatorname{res}\left(\frac{\log^2 A\,B}{2(a+b)} - \frac{\log^2 A}{2a} - \frac{\log^2 B}{2b}\right) \\
&= \frac{ab}{2(a+b)}\operatorname{res}\left[\left(\frac{\log A}{a} - \frac{\log B}{b}\right)^2\right].
\end{aligned}$$

□

We can also obtain independently the local formula of the multiplicative anomaly of the ζ-determinant relating it to the locality of the weighted trace of the operator $L(A,B)$. To do so let us establish the following useful Lemma.

Lemma 5.3.4. *Let A and B be admissible operators in $C\ell(M,E)$ with positive orders a,b and spectral cuts θ and ϕ respectively and such that AB is also admissible with spectral cut ψ. Then*

$$K(A,B) := \frac{\log_\psi^2 A\,B}{2(a+b)} - \frac{\log_\theta^2 A}{2a} - \frac{\log_\phi^2 B}{2b}$$

has a symbol of the form

$$\sigma(K) \sim \log|\xi|(\sigma_0^{AB} - \sigma_0^A - \sigma_0^B) + \sigma_0^K$$

for some zero order classical symbol σ_0^K and where we have written $\sigma(\log A)(x,\xi) = a\log|\xi| I + \sigma_0^A(x,\xi)$ for an admissible operator A of order a.
In particular, both operators $L(A,B)\,\frac{\log_\theta A}{a} - K(A,B)$ and $L(A,B)\,\frac{\log_\phi B}{b} - K(A,B)$ are classical operators of zero order.

Proof: Recall that by formula (2.9), another choice of spectral cut only changes the logarithms by adding an operator in $C\ell^0(M,E)$ so that it will not affect the statement. As usual, we drop the explicit mention of spectral cut assuming the operators have common spectral cuts. An explicit computation on symbols shows the result.
Indeed, since $\sigma(\log A)(x,\xi) = a\log|\xi| + \sigma_0^A(x,\xi)$, we have

$$\begin{aligned}
\sigma(\log^2 A)(x,\xi) &= \sigma(\log A) \star \sigma(\log A)(x,\xi) \\
&\sim a^2 \log^2|\xi| I + 2a\log|\xi|\,\sigma_0^A(x,\xi) + \sigma_0^A(x,\xi)\cdot\sigma_0^A(x,\xi) \\
&\quad + \sum_{\alpha\neq 0} \frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha \sigma_0^A(x,\xi)\,\partial_x^\alpha \sigma_0^A(x,\xi).
\end{aligned}$$

This yields:

$$\begin{aligned}
& \sigma(K)(x,\xi) \\
\sim\; & \log|\xi|\left(\sigma_0^{AB}-\sigma_0^{A}-\sigma_0^{B}\right)(x,\xi)+\frac{1}{2(a+b)}\sigma_0^{AB}(x,\xi)\sigma_0^{AB}(x,\xi) \\
+\; & \sum_{\alpha\neq 0}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma_0^{AB}(x,\xi)D_x^\alpha\sigma_0^{AB}(x,\xi)-\frac{1}{2a}\sigma_0^{A}(x,\xi)\sigma_0^{A}(x,\xi)-\sum_{\alpha\neq 0}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma_0^{A}(x,\xi)D_x^\alpha\sigma_0^{A}(x,\xi) \\
-\; & \frac{1}{2b}\sigma_0^{B}(x,\xi)\sigma_0^{B}(x,\xi)-\sum_{\alpha\neq 0}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma_0^{B}(x,\xi)D_x^\alpha\sigma_0^{B}(x,\xi)
\end{aligned}$$

from which the first part of the statement follows. On the other hand, it follows from (4.12) combined with (2.10) that $L(A,B)\,\frac{\log A}{a}$ and $L(A,B)\,\frac{\log B}{b}$ both have symbols which differ from $\log|\xi|\left(\sigma_0^{AB}-\sigma_0^{A}-\sigma_0^{B}\right)(x,\xi)$ by a classical symbol of order zero, from which we infer the second part of the statement. $\square$

Now, using equation (5.1) we have

$$\begin{aligned}
\log\mathcal{M}_\zeta(A,B) &= \log\det{}_\zeta(AB)-\log\det{}_\zeta(A)-\log\det{}_\zeta(B) \\
&= \int_M dx\left[\mathrm{TR}_x(L(A,B))-\mathrm{res}_x\left(\frac{\log^2 AB}{2(a+b)}-\frac{\log^2 A}{2a}-\frac{\log^2 B}{2b}\right)\right].
\end{aligned}$$

Substracting the defect formula (4.5) applied to the operator $L(A,B)$ and weight B to $\log\mathcal{M}_\zeta(A,B)$ and combining with equation (4.15) applied to $Q=B$ we write:

$$\begin{aligned}
& \log\mathcal{M}_\zeta(A,B) \\
=\; & \mathrm{Tr}^B(L(A,B))+\int_M dx\left[\mathrm{res}_x\left(\frac{L(A,B)\,\log B}{b}-\frac{\log^2 A\,B}{2(a+b)}+\frac{\log^2 A}{2a}+\frac{\log^2 B}{2b}\right)\right] \\
=\; & \int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau+b}-\frac{\log B}{b}\right)\right)d\tau \\
& +\mathrm{res}\left(\frac{L(A,B)\,\log B}{b}-\frac{\log^2 A\,B}{2(a+b)}+\frac{\log^2 A}{2a}+\frac{\log^2 B}{2b}\right),
\end{aligned}$$

which proves the first equality in (5.8).

CHAPTER 6

Chapter 6

Determinants on odd-class pseudodifferential operators

In the present chapter we investigate determinants on odd-class elliptic operators which in contrast to the ζ-determinant on elliptic operators studied in the previous chapter, are multiplicative and hence do not present a multiplicative anomaly. Multiplicative determinants are associated to traces on zero order odd-class classical operators, which by the results of Chapter 3, are linear combinations of the leading symbol trace and the canonical trace. Since multiplicative determinants associated with leading symbol traces were studied elsewhere [LP], we focus here on the only remaining multiplicative determinants on odd-class operators, those associated with the canonical trace, which unlike the leading symbol trace determinants, extend Fredholm determinants on operators of the type I+ smoothing. They are expected to be of the type $\mathrm{DET} := \exp \circ \mathrm{TR} \circ \log$, which is indeed the case for zero order odd-class operators since their logarithms also lie in the odd-class. However, this does not hold any longer for positive order operators whose logarithms are not expected to lie in the odd-class in general. Nevertheless, the logarithm extends to a symmetrized logarithm $\log^{\mathrm{sym}}$ for elliptic operators with positive order which fulfill certain technical assumptions on the spectral cut. We show that $\exp \circ \mathrm{TR} \circ \log^{\mathrm{sym}}$ indeed yields a multiplicative determinant under some natural restrictions on the spectral cuts (Theorem 6.3.8) on such odd-class elliptic operators, which coincides with Braverman's determinant. Our approach via the canonical trace shows that the origin of its multiplicativity lies in the cyclicity of the canonical trace on odd-class operators and stresses the fact that it is a natural extension of the canonical determinant DET on zero order operators associated with the canonical trace TR.

6.1 Classification of infinitesimal multiplicative determinants on $\left(C\ell^0_{odd}(M,E)\right)^*$

Well-known general results in the finite dimensional context concerning determinants associated with traces generalize to the context of Banach spaces (see [HS]) and further to Fréchet spaces as in [LP]. Let $\mathcal{G}$ be a Fréchet Lie group with exponential mapping Exp (see Definition 1.3.3) and Lie algebra $Lie(\mathcal{G})$. In [KM] Remark 36.9, A. Kriegel and P. W. Michor showed that if $\mathcal{G}$ admits an exponential mapping, it follows that Exp is a diffeomorphism from a neighborhood of 0 in $Lie(\mathcal{G})$ to a neighborhood of 1 in $\mathcal{G}$ if a suitable inverse function theorem is applicable. This is the case for smooth Banach Lie groups. A generalization of the inverse function theorem on Banach spaces to a class of tame Fréchet spaces is developed in [Ha] under the name of Nash-Moser theorem (Part III, Theorem 1.1.1.). In contrast to the Banach space setup for which the invertibility of the derivative at a point is sufficient for a function to be locally invertible, for the Nash-Moser theorem to hold, it is necessary that the derivative is invertible in a neighborhood of the point. For further details see [Ha].
The existence of a smooth exponential mapping for a Lie group is insured by a notion of regularity ([Mi], [KM]) on this group. For J. Milnor [Mi], a Lie group $\mathcal{G}$ modelled on a locally convex space is a regular Lie group if for each smooth curve $u : [0,1] \to Lie(\mathcal{G})$, there exists a smooth curve $\gamma_u : [0,1] \to \mathcal{G}$ (which is unique, Lemma 38.3 in [KM]) which solves the initial value problem $\dot{\gamma} = \gamma.u$ with $\gamma(0) = 1_{\mathcal{G}}$, where $1_{\mathcal{G}}$ is the identity of $\mathcal{G}$, with smooth evolution map

$$\begin{aligned} C^\infty([0,1], Lie(\mathcal{G})) &\to \mathcal{G} \\ u &\mapsto \gamma_u(1). \end{aligned}$$

For example, Banach Lie groups and finite dimensional Lie groups are regular. For our purpose in this section, we assume that the Lie group $\mathcal{G}$ is regular.

In the following Lemma we give the construction of a locally defined determinant from a trace on $Lie(\mathcal{G})$ i.e. a linear form on $Lie(\mathcal{G})$ which vanishes on brackets.

Lemma 6.1.1. *A continuous linear map $\lambda : Lie(\mathcal{G}) \to \mathbb{C}$ gives rise to a multiplicative map $\Lambda : R(\mathrm{Exp}) \subset \mathcal{G} \to \mathbb{C}^*$ defined on the range of the exponential mapping by $\Lambda(g) = \exp(\lambda(\mathrm{Log}(g)))$ where $\mathrm{Log} = \mathrm{Exp}^{-1}$, making the following diagram commutative: for any small enough neighborhood U_0 of zero in $Lie(\mathcal{G})$.*

$$\begin{tikzcd}
U_0 \subset Lie(\mathcal{G}) \arrow[r, "\lambda"] \arrow[d, "\mathrm{Exp}"'] & \mathbb{C} \arrow[d, "\exp"] \\
\mathrm{Exp}(U_0) \subset \mathcal{G} \arrow[r, "\Lambda"] & \mathbb{C}^*
\end{tikzcd}$$

Proof: We first observe that $\log(\Lambda(g)) = \lambda(\mathrm{Log}(g))$ where g belongs to $\mathrm{Exp}(U_0)$ i.e. there exists u in $U_0 \subset Lie(\mathcal{G})$ such that $g = \mathrm{Exp}(u)$. Since $\mathcal{G}$ is a regular Fréchet Lie group i.e. admits an exponential mapping Exp, let us consider the C^1-path $\gamma(t) = \mathrm{Exp}(tu)$ going from $1_{\mathcal{G}}$ to $\mathrm{Exp}(u) = g$. We have $\gamma^{-1}(t)\dot{\gamma}(t) = u$ and hence

$$\lambda\left(\int_0^1 \gamma^{-1}(t)\dot{\gamma}(t)dt\right) = \int_0^1 \lambda(\gamma^{-1}(t)\dot{\gamma}(t))dt = \lambda(u) = \lambda(\mathrm{Log}(g))$$

using the continuity of λ. It follows that if γ_1, γ_2 are two C^1-paths going from $1_{\mathcal{G}}$ to g_1 and g_2 respectively, then $\gamma_1\gamma_2$ is a C^1-path going from $1_{\mathcal{G}}$ to $g_1 g_2$ and we have

$$\begin{aligned}
\lambda\left((\gamma_1(t)\gamma_2(t))^{-1}\dot{\overbrace{\gamma_1(t)\gamma_2(t)}}\right) &= \lambda\left(\gamma_2(t)^{-1}\gamma_1(t)^{-1}\dot{\gamma_1}(t)\gamma_2(t) + \gamma_2(t)^{-1}\dot{\gamma_2}(t)\right) \\
&= \lambda\left(\gamma_1(t)^{-1}\dot{\gamma_1}(t)\right) + \lambda\left(\gamma_2(t)^{-1}\dot{\gamma_2}(t)\right)
\end{aligned}$$

where we have use the tracial property of λ. Now, for g_1, $g_2 \in R(\mathrm{Exp}) \subset \mathcal{G}$,

$$\log(\Lambda(g_1.g_2)) = \lambda(\mathrm{Log}(g_1.g_2))) = \lambda\left(\mathrm{Log}(g_1)\right) + \lambda\left(\mathrm{Log}(g_2)\right).$$

□

Conversely, following [LP] we give a construction of a trace from a determinant.

Lemma 6.1.2. *A multiplicative map $\Lambda : U_1 \to \mathbb{C}^*$ defined on a neighborhood U_1 of 1 in $\mathcal{G}$ which is of class C^1 on $\mathcal{G}$ yields a continuous linear form $\lambda : Lie(\mathcal{G}) \to \mathbb{C}$ which makes the following diagram commutative:*

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\Lambda"] & \mathbb{C}^* \\
Lie(\mathcal{G}) \arrow[r, "\lambda"] \arrow[u, "\mathrm{Exp}"] & \mathbb{C} \arrow[u, "\exp"']
\end{tikzcd}$$

i.e. for all $u \in Lie(\mathcal{G})$, $\lambda(u) = D_e\Lambda(u)) = \frac{d}{dt}_{|t=0}\Lambda(Exp(tu))$.

Proof: Indeed, for $u_1,\, u_2 \in Lie(\mathcal{G})$,

$$\begin{aligned}\lambda([u_1,u_2]) &= \frac{d}{ds}_{|s=0}\frac{d}{dt}_{|t=0}\Lambda\left(\mathrm{Exp}(tu_1).\mathrm{Exp}(su_2).\mathrm{Exp}(-tu_1)\right)\\ &= \frac{d}{ds}_{|s=0}\frac{d}{dt}_{|t=0}\Lambda(\mathrm{Exp}(tu_1))\Lambda(\mathrm{Exp}(su_2))\Lambda(\mathrm{Exp}(-tu_1))\\ &= \frac{d}{ds}_{|s=0}\frac{d}{dt}_{|t=0}\Lambda(\mathrm{Exp}(su_2))\\ &= 0.\end{aligned}$$

Here we use the fact that Λ is multiplicative, which implies that $\Lambda(g^{-1}) = \Lambda(g)^{-1}$. □

Remark 6.1.3. *The two Lemmata imply that continuous traces on $Lie(\mathcal{G})$ are in one to one correspondence with C^1-multiplicative maps on the open subset of $\mathcal{G}$ corresponding to the range of the exponential mapping.*

As we saw in Chapter 1, Proposition 1.3.4, $\left(C\ell^0_{odd}(M,E)\right)^*$ is a Fréchet Lie group. The following proposition provides more information on its Lie structure i.e. $\left(C\ell^0_{odd}(M,E)\right)^*$ is a regular Fréchet Lie group which admits an exponential mapping and its Lie algebra is $C\ell^0_{odd}(M,E)$.

Proposition 6.1.4. *$\left(C\ell^0_{odd}(M,E)\right)^*$ is a regular Fréchet Lie group with exponential mapping and its Lie algebra is $C\ell^0_{odd}(M,E)$.*

Proof: By Proposition 1.3.4, we already know that $\left(C\ell^0_{odd}(M,E)\right)^*$ is a Fréchet Lie group and its Lie algebra is $C\ell^0_{odd}(M,E)$. Let us construct an exponential mapping. Given any operator B in $C\ell^0_{odd}(M,E)$, the differential equation

$$A_t^{-1}\dot{A}_t = B, \quad A_0 = I$$

has a unique solution in $\left(C\ell^0_{odd}(M,E)\right)^*$ given by:

$$A_t = \frac{i}{2\pi}\int_\Gamma \exp(t\lambda)(B-\lambda)^{-1}\,d\lambda,$$

where Γ is a contour around the spectrum of B which is bounded since B has zero order. Let us check that A_t belongs to $\left(C\ell^0_{odd}(M,E)\right)^*$. The homogeneous component of A_t are:

$$\sigma(A_t)_{-j} = \frac{i}{2\pi}\int_\Gamma \exp(t\lambda)b_{-j}(B)\,d\lambda$$

where b_{-j} denote the components of the resolvent $(B-\lambda)^{-1}$ of B at the point λ. They are given by:

$$\begin{aligned}b_0 &:= (\sigma_0(B)-\lambda)^{-1},\\ b_{-j} &:= -b_0\sum_{k+l+|\alpha|=j,l<j}\frac{1}{\alpha}\partial_\xi^\alpha\sigma_{-k}(B)D_x^\alpha b_{-l}.\end{aligned}$$

Since B lies in $C\ell^0_{odd}(M,E)$ it follows that A_t lies in $\left(C\ell^0_{odd}(M,E)\right)^*$. This defines an exponential mapping

$$\mathrm{Exp} : C\ell^0_{odd}(M,E) \to \left(C\ell^0_{odd}(M,E)\right)^*.$$

Moreover, it follows that for any smooth curve $u : [0,1] \to C\ell^0_{odd}(M,E)$, there exists a unique smooth curve $\gamma_u : [0,1] \to \left(C\ell^0_{odd}(M,E)\right)^*$ defined by the following diagram

$$[0,1] \xrightarrow{\quad u \quad} C\ell^0_{odd}(M,E) \xrightarrow{\quad \mathrm{Exp} \quad} \left(C\ell^0_{odd}(M,E)\right)^*$$

which solves the initial value problem $\gamma^{-1}\dot{\gamma} = u$. □

On the basis of Remark 6.1.3, we infer from the classification of traces on $C\ell^0_{odd}(M,E)$ derived in Section 3.3 a description of multiplicative maps defined on the range of the exponential mapping in $\left(C\ell^0_{odd}(M,E)\right)^*$.

Proposition 6.1.5. *Any multiplicative map on the range of the exponential mapping in $\left(C\ell^0_{odd}(M,E)\right)^*$ is of the form:*

$$\mathrm{Det}(A) = \exp\left(\alpha \mathrm{TR}(\mathrm{Log}(A)) + \mathrm{Tr}_0^\tau(\mathrm{Log}(A))\right),$$

where α is a real number and τ is a distribution in the cotangent unit sphere S^*M.

Proof: By Theorem 3.3.4, we know that any trace on $C\ell^0_{odd}(M,E)$ is a linear combination of the canonical trace and a leading symbol trace. By Proposition 6.1.4, we can apply Lemma 6.1.1 to $\mathcal{G} = \left(C\ell^0_{odd}(M,E)\right)^*$ and $Lie(\mathcal{G}) = C\ell^0_{odd}(M,E)$. It follows that a multiplicative determinant is of the form

$$\mathrm{Det}(A) = \exp\left(\alpha \mathrm{TR}(\mathrm{Log}(A)) + \mathrm{Tr}_0^\tau(\mathrm{Log}(A))\right).$$

□

6.2 Determinants on zero order odd-class operators

In this paragraph we extend the multiplicative maps defined in the previous section beyond the range of the exponential mapping, namely to the pathwise connected component of the identity. There are two possible ways to do so. The first one is to use a C^1-path and to define a determinant of the form

$$\Lambda(g) = \exp\left(\int_0^1 \lambda(\gamma(t)^{-1}\dot{\gamma}(t))\, dt\right) \tag{6.1}$$

where $\gamma : [0,1] \to \mathcal{G}$ is a C^1-path with $\gamma(1) = g$. Such an approach was adopted in [HS] by P. de la Harpe and G. Skandalis in the case of a Banach Lie group. In her thesis [Du1], C. Ducourtioux applied this method with slight modifications to construct a determinant associated to a weighted trace with associated Lie algebras $C\ell^0(M,E)$ and $C\ell(M,E)$. In [LP], J. M. Lescure and S. Paycha showed that such a construction extends to Fréchet Lie groups with exponential mapping.
Let $\widetilde{\mathcal{G}}$ denote the pathwise connected component of the identity $1_{\mathcal{G}}$ of $\mathcal{G}$ and $\mathcal{P}(G)$ the set of C^1-paths $\gamma : [0,1] \to \mathcal{G}$ starting at $1_{\mathcal{G}}$ so that $\gamma(0) = 1_{\mathcal{G}}$ in $\widetilde{\mathcal{G}}$. On $\mathcal{P}(G)$ we introduce the map: $Det_\lambda : \mathcal{P}(G) \to \mathbb{C}^*$ defined by

$$Det_\lambda(\gamma) = \exp\left(\int_\gamma \lambda(\omega)\right) = \exp\left(\int_0^1 \lambda(\gamma^*\omega)\right)$$

where $\omega = g^{-1}dg$ is the Maurer-Cartan form on $\mathcal{G}$. Note that since λ satisfies the tracial property, we have the multiplicative property:

Lemma 6.2.1. *Let γ_1, γ_2 be two C^1-paths in $\mathcal{P}(G)$. Then*

$$Det_\lambda(\gamma_1\gamma_2) = Det_\lambda(\gamma_1)Det_\lambda(\gamma_2).$$

Proof: The same proof applies as in Lemma 6.1.1. □

In general the Maurer-Cartan form $\omega = g^{-1}dg$ is not exact on $\mathcal{G}$ so that for a C^1-path $c : [0,1] \to \mathcal{G}$ with $c(0) = c(1)$, the integral $\int_c \omega = \int_0^1 c^*\omega$ does not vanish.

Proposition 6.2.2.

1. *The map*

$$\begin{aligned}\Phi : L(\mathcal{G}) &\to Lie(\mathcal{G})\\ c &\mapsto \int_c \omega = \int_0^1 c^*\omega\end{aligned}$$

defined on the space $L(\mathcal{G})$ of C^1-loops in $\mathcal{G}$ i.e. C^1-paths such that $c(0) = c(1)$, induces a map $\Phi : \Pi_1(\mathcal{G}) \to Lie(\mathcal{G})$ on the fundamental group $\Pi_1(\mathcal{G})$ of $\mathcal{G}$.

2. *Consequently, the map Det_λ only depends on the homotopy class of the path γ. If $Det_\lambda(\Pi_1(\mathcal{G})) = 1$, then it induces a multiplicative map:*

$$\begin{aligned}Det_\lambda : \mathcal{G} &\to \mathbb{C}^*\\ g &\mapsto \exp\left(\int_0^1 \lambda(\gamma^*\omega)\right)\end{aligned}$$

independently of the choice of path γ.

3. *If g lies in the range of the exponential mapping* Exp *then*

$$Det_\lambda(g) = \exp\left(\lambda(\mathrm{Log}(g)\right)$$

where $\mathrm{Log} = \mathrm{Exp}^{-1}$ *is the inverse of the exponential mapping.*

Proof:

1. We want to show that two homotopic loops c_1 and c_2 have common primitive. Let us first recall the following general construction of a primitive: for ω a differential form on $\mathcal{G}$, let $\gamma : [0,1] \to \mathcal{G}$ be a C^1-path and $F : [0,1] \to \mathcal{G}$ be such that for any $t \in [0,1]$ $F'(t) = \omega((\gamma(t))\gamma'(t)$. If ω is an exact form i.e. $w = df$ then $F(t) = f(\gamma(t))$ is a primitive of F'. If the form ω is closed, then ω is locally exact. Let $0 = t_0 < t_1 < \cdots < t_k = 1$ be a subdivision of the interval $[0,1]$ such that $\gamma([t_{i-1}, t_i])$ is a subset of $\mathcal{G}$. There exists f_i defined on $[t_{i-1}, t_i]$ such that $df_i = \omega$. We can construct a function $F(t)$ on $[0,1]$ in the following manner: $F(t) = f_0(\gamma(t))$ on $[t_0, t_1]$, $F(t) = f_1(\gamma(t)) - h_1$ on $[t_1, t_2]$ where $h_1 = f_1(\gamma(t_1)) - f_0(\gamma(t_1))$ and for $i = 3, \cdots, k$, $F(t) = f_i(\gamma(t)) - h_i$ on $[t_{i-1}, t_i]$ where $h_i = f_i(\gamma(t_i)) - f_{i-1}(\gamma(t_i)) + h_{i-1}$. Now let $F(t)$ and $G(t)$ be two primitives of c_1 and c_2 respectively. Since c_1 and c_2 are homotopic, there exists a family of C^1-paths $(\alpha_i)_{0 \le i \le k}$ defined in a neighborhood of 1 such that $c_1 = c_2 \prod_{i=0}^{k} \alpha_i$. Each path α_i is closed so that $\int_c \omega$ vanishes on α_i. It follows that $F(t) = G(t)$ i.e. the map Φ is well-defined on $\Pi_1(\mathcal{G})$.

2. The multiplicativity of Det_λ on $\mathcal{G}$ follows from Lemma 6.2.1. Indeed, let g_1, g_2 be two elements of $\mathcal{G}$ and γ_1, γ_2 two C^1-paths in $\mathcal{P}(G)$ such that $\gamma_1(1) = g_1$ and $\gamma_2(1) = g_2$. Then $Det_\lambda(g_1 g_2) = \exp\left(\int_0^1 \lambda(\gamma_1\gamma_2)^*\omega)\right) = Det_\lambda(g_1)Det_\lambda(g_2)$

3. For g in the range of the exponential mapping, $\mathrm{Log} = \mathrm{Exp}^{-1}$ is well-defined so that $\lambda(\mathrm{Log}(\gamma(t)))$ is a primitive of $\lambda(\gamma(t)^{-1}\dot\gamma(t))\,dt$. It follows that

$$\int_0^1 \lambda(\gamma(t)^{-1}\dot\gamma(t))\,dt = \lambda(\mathrm{Log}(g)).$$

$\square$

In the case of $\mathcal{G} = \left(C\ell^0_{odd}(M,E)\right)^*$ the logarithm locally defined as inverse of the exponential mapping extends beyond a neighborhood of the identity provided one chooses a spectral cut θ thereby to fix a determination $\log_\theta$ of the logarithm.

An alternative way to extend local determinants beyond a neighborhood of the identity is therefore to set

$$Det_\theta^\lambda(A) = \exp\left(\lambda(\log_\theta A)\right).$$

Recall from Corollary 2.2.2 that if the operator A lies in the odd-class and has even order, then the logarithm $\log_\theta A$ is also odd-class. If ϕ is another spectral cut of A such that $0 \leq \theta < \phi < 2\pi$, by formula (2.9) we have $\log_\theta A - \log_\phi A = -2i\pi\Pi_{\theta,\phi}(A)$ where $\Pi_{\theta,\phi}(A)$ is the odd-class projection defined in Lemma 2.1.5. In other words, the fundamental group of $\left(C\ell^0_{odd}(M,E)\right)^*$ is generated by the homotopy class of loops $\exp(2i\pi tP)$ where P is a projector in $C\ell^0_{odd}(M,E)$ (see [KV1] Section 4, and in [Du1] Lemma A.5).
The following Proposition provides a way to build maps which send the fundamental group to 1.

Proposition 6.2.3. *Any continuous trace λ on $C\ell^0_{odd}(M,E)$ which takes integer values on the image of $\Pi_{\theta,\phi}(A)$ for all θ,ϕ and A gives rise to a multiplicative determinant $\mathrm{Det}^\lambda(A) = \exp(\lambda(\log_\theta A))$, on admissible operators, independent of the choice of the spectral cut θ for an operator A with spectral cut θ.*

Example 6.2.4. *For an admissible operator A in $C\ell^0_{odd}(M,E)$ with spectral cut θ, the determinant associated to the leading symbol trace is defined by*

$$\mathrm{Det}^\lambda_0(A) := \exp\left(\mathrm{Tr}^\lambda_0(\log_\theta A)\right).$$

In [LP] Example 2, it is shown that if P is a zero order pseudodifferential idempotent, then its leading symbol p is also an idempotent so that the fibrewise trace $\mathrm{tr}_x(p(x,.))$ is the rank $rk(p(x,.))$. Hence

$$\mathrm{Tr}^\lambda_0(\Pi_{\theta,\phi}(A)) = \lambda(\mathrm{tr}_x(\sigma_0(\Pi_{\theta,\phi}(A)(x,\xi))) = rk(\Pi_{\theta,\phi}(A))\mathrm{Tr}^\lambda_0(I).$$

It follows that $\mathrm{Det}^\lambda_0(A)$ is independent of the choice of the spectral cut θ.

In contrast, the canonical trace does not satisfy the requirement of Proposition 6.2.3 so that a determinant associated to the canonical trace will depend on the choice of spectral cut. To build such a determinant, we first observe that if A is a zero order odd-class operator so is $\log_\theta A$ in odd-class. Hence the canonical trace extends to logarithms of admissible odd-class operators of zero order with its property of cyclicity in odd dimensions. Let us set for an admissible operator A in $C\ell^0_{odd}(M,E)$

$$\mathrm{DET}_\theta(A) := \exp(\mathrm{TR}(\log_\theta A)).$$

6.3 A symmetrized canonical determinant on $(C\ell_{odd}(M,E))^*$

Our aim in this section is to further extend the determinant DET_θ beyond operators of zero order. But in general, the logarithms of admissible odd-class operator with odd order is no longer odd-class. Instead, we carry out the extension with the help of symmetrized logarithms since the symmetrized logarithm of admissible odd-class operators belong to the odd-class. This section is inspired from Braverman's work [B] in which the author introduced a symmetrized determinant using symmetrized regularized traces (see section 3.6 in [B]). In contrast to his approach which uses regularized traces applied to symmetrized logarithms, we define a symmetrized determinant with the help of the canonical trace applied to symmetrized logarithms, thus clarifying the presentation and simplifying the proofs.

Definition 6.3.1. *Suppose that M is an odd-dimensional manifold. Let A be an odd-class admissible operator with positive order a which admits spectral cuts θ and $\theta - a\pi$. A determinant associated to* TR *is defined by setting:*

$$\mathrm{DET}_\theta^{\mathrm{sym}}(A) := \exp\left(\mathrm{TR}(\log_\theta^{\mathrm{sym}} A)\right). \tag{6.2}$$

Remark 6.3.2. *If A has even order, then $\log_\theta^{\mathrm{sym}} A = \log_\theta A - ik\pi I$ and $\mathrm{DET}_\theta^{\mathrm{sym}}$ coincides with the determinant defined in [PS], which in turn coincides with the ζ-determinant:*

$$\log\, \mathrm{DET}_\theta^{\mathrm{sym}}(A) = \log\, \det_{\zeta,\theta}(A) = \mathrm{TR}(\log_\theta A).$$

Indeed, using formula (5.3) and Proposition 3.4.1, $\mathrm{Tr}_\theta^A(I) = -\frac{1}{a}\mathrm{res}(\log_\theta A) = 0$. Hence $\mathrm{Tr}_\theta^A(\log_\theta^{\mathrm{sym}} A) = \mathrm{Tr}_\theta^A(\log_\theta A)$ and with item 2 of Theorem 3.5.7 we have

$$\mathrm{Tr}_\theta^A(\log_\theta^{\mathrm{sym}} A) = \mathrm{fp}_{z=0}\mathrm{TR}\left(\log_\theta^{\mathrm{sym}} A(A_\theta^z)\right) = \mathrm{TR}\left(\log_\theta^{\mathrm{sym}} A\right) = \mathrm{TR}\left(\log_\theta A\right).$$

Proposition 6.3.3. *Under the assumptions of Definition 6.3.1, $\mathrm{DET}_\theta^{\mathrm{sym}}(A)$ coincides with the symmetrized determinant introduced in [B]:*

$$\mathrm{Det}_\theta^{\mathrm{sym}} A := \exp\left(\frac{1}{2}\mathrm{Tr}^{\mathrm{sym}}\left(\log_\theta A + \log_{\theta - a\pi} A\right)\right) = \exp\left(\mathrm{Tr}^{\mathrm{sym}}\left(\log_\theta^{\mathrm{sym}} A\right)\right)$$

where $\mathrm{Tr}^{\mathrm{sym}} A := \mathrm{Tr}_\alpha^{Q,sym} A = \frac{1}{2}\left(\mathrm{Tr}_\alpha^Q A + \mathrm{Tr}_{\alpha - q\pi}^Q A\right)$. Here Q is any odd-class admissible operator with positive order q and spectral cuts α, $\alpha - q\pi$.

Proof: It is easy to see that

$$\mathrm{TR}(\log_\theta^{\mathrm{sym}} A) = \mathrm{fp}_{z=0}\mathrm{TR}\left(\frac{1}{2}(\log_\theta^{\mathrm{sym}} A)(A_\theta^z + A_{\theta - a\pi}^z)\right) = \mathrm{Tr}^{\mathrm{sym}}(\log_\theta^{\mathrm{sym}} A)$$

when apply Theorem 3.5.7 to the family $A(z) = \frac{1}{2}(\log_\theta^{\mathrm{sym}} A)(A_\theta^z + A_{\theta - a\pi}^z)$ since all derivatives $A^{(j)}(0)$ lie in the odd class as powers of symmetrized logarithms. $\square$

Remark 6.3.4. *Though our symmetrized determinant coincides with the one defined by M. Braverman, its expression is more simple since it involves the canonical trace instead of a symmetrized regularized trace.*

If the order of A is even, by Remark 6.3.2, the symmetrized determinant coincides with the ζ-determinant. As we saw for zero order operators, the ζ-determinant generally depends on the choice of spectral cut but it is invariant under mild changes of spectral cut. The same property holds for the symmetrized determinant since in that case, they coincide.

Let us examine now the case of odd order odd-class operators. As already observed by M. Braverman, if the order of A is odd, the symmetrized determinant generally depends on the choice of spectral cut since infinitely many eigenvalues of A might lie in the cone $\Lambda_{\theta-a\pi,\phi-a\pi}$. Nevertheless, we have the following proposition, proved in [B]:

Proposition 6.3.5. *Let M be an odd-dimensional manifold and let A be an odd-class admissible operator with odd positive order a which admits spectral cuts θ, $\theta-\pi$ and ϕ, $\phi-\pi$. Suppose that $0 \leq \phi-\theta < \pi$. We have*

$$\mathrm{DET}^{\mathrm{sym}}_{\theta}(A) = \pm \mathrm{DET}^{\mathrm{sym}}_{\phi}(A)$$

in the following cases:

1. *if only a finite number of eigenvalues of A lie in $\Lambda_{\theta,\phi} \cup \Lambda_{\theta-\pi,\phi-\pi}$,*
2. *if all but finitely many eigenvalues of A lie in $\Lambda_{\theta,\phi} \cup \Lambda_{\theta-\pi,\phi-\pi}$.*

Proof: By formula (2.9) we write $\log_\phi A = \log_\theta A + 2i\pi\Pi_{\theta,\phi}(A)$ and $\log_{\phi-\pi} A = \log_{\theta-\pi} A + 2i\pi\Pi_{\theta-\pi,\phi-\pi}(A)$, where $\Pi_{\theta,\phi}(A)$ and $\Pi_{\theta-\pi,\phi-\pi}(A)$ denote respectively the spectral projections corresponding to the eigenvalues of A which lie in the cones $\Lambda_{\theta,\phi}$ and $\Lambda_{\theta-\pi,\phi-\pi}$. It follows that $\log^{\mathrm{sym}}_\phi A = \log^{\mathrm{sym}}_\theta A + i\pi\left(\Pi_{\theta,\phi}(A) + \Pi_{\theta-\pi,\phi-\pi}(A)\right)$. In the first case $\Pi_{\theta,\phi}(A)$ and $\Pi_{\theta-\pi,\phi-\pi}(A)$ are finite rank projectors and in the second case, $I-\Pi_{\theta,\phi}(A)$ and $I-\Pi_{\theta-\pi,\phi-\pi}(A)$ or one of them are of finite rank. Recall that I is a differential operator and hence by Example 3.2.7 $\mathrm{TR}(I) = 0$. It follows that $\mathrm{TR}(\log^{\mathrm{sym}}_\phi A) = \mathrm{TR}(\log^{\mathrm{sym}}_\theta A) + i\alpha\pi$ for some integer α in $\mathbb{Z}$ and the result follows. $\square$

From Corollary 2.1.15 we infer the following multiplicative property of the symmetrized determinant for commuting operators.

Proposition 6.3.6. *Suppose that M is an odd-dimensional manifold. Let A be an odd-class admissible operator with positive order a and spectral cuts θ and $\theta-a\pi$ and let B be an odd-class admissible operator with positive order b and spectral cuts ϕ and $\phi-b\pi$ such that AB is also admissible with spectral cuts ψ and $\psi-(a+b)\pi$ depending on the choice of θ and ϕ. If $[A,B]=0$ then*

$$\mathrm{DET}^{\mathrm{sym}}_{\psi}(AB) = \mathrm{DET}^{\mathrm{sym}}_{\theta}(A)\,\mathrm{DET}^{\mathrm{sym}}_{\phi}(B).$$

Proof: Indeed,

$$\begin{aligned}\log \mathrm{DET}^{\mathrm{sym}}_{\psi}(AB) &= \mathrm{TR}(\log^{\mathrm{sym}}_{\psi} AB)\\ &= \mathrm{TR}(\log^{\mathrm{sym}}_{\theta} A + \log^{\mathrm{sym}}_{\phi} B)\\ &= \log \mathrm{DET}^{\mathrm{sym}}_{\theta}(A) + \log \mathrm{DET}^{\mathrm{sym}}_{\phi}(B)\end{aligned}$$

since by Corollary 2.1.15,

$$\log_{\psi}(AB) = \log_{\theta} A + \log_{\phi} B$$

and

$$\log_{\psi-(a+b)\pi}(AB) = \log_{\theta-a\pi} A + \log_{\phi-b\pi} B$$

whenever $[A, B] = 0$. $\square$

This result generalizes to non commuting operators as it was shown by M. Braverman [B] using, under suitable assumptions, the formula for the multiplicative anomaly established by K. Okikiolu [Ok2]. Our proof of this result is based on the cyclicity of the canonical trace on odd-class operators in odd dimensions. For that, we first recall the following definition [B].

Definition 6.3.7. *Let θ be a principal angle for an elliptic operator A in $C\ell^{a}_{odd}(M, E)$. A spectral cut $\phi \geq \theta$ is sufficiently close to θ if there are no eigenvalues of A in the cones $\Lambda_{(\theta,\phi]}$ and $\Lambda_{(\theta-a\pi,\phi-a\pi]}$. We shall denote by $\log_{\widetilde{\theta}} A$, $\log^{\mathrm{sym}}_{\widetilde{\theta}} A$, $\mathrm{DET}^{\mathrm{sym}}_{\widetilde{\theta}} A$ the corresponding numbers obtained using a spectral cut sufficiently close to θ. Clearly, those numbers are independent of the choice of $\widetilde{\theta}$.*

Theorem 6.3.8. *Let M be an odd-dimensional manifold. Suppose that A is an odd-class admissible operator with positive order a and spectral cuts θ and $\theta - a\pi$ and that B is an odd-class admissible operator with positive order b and spectral cuts ϕ and $\phi - b\pi$. Let us assume that for each t in $[0, 1]$, $A^{t}_{\theta}B$ has principal angle $\psi(t)$, depending on the choice of θ and ϕ, where $t \to \psi(t)$ is continuous. Set $\psi(0) = \phi$ and $\psi(1) = \psi$. Then*

$$\mathrm{DET}^{\mathrm{sym}}_{\widetilde{\psi}}(AB) = \mathrm{DET}^{\mathrm{sym}}_{\theta}(A)\, \mathrm{DET}^{\mathrm{sym}}_{\phi}(B),$$

where $\widetilde{\psi}$ is an angle sufficiently close to ψ.

Let us give the following two alternative proofs of this proposition. The first one is based on the cyclicity of the canonical trace on odd-class operators in odd dimensions.
Proof 1: Using formula (2.17) we know that

$$\sigma_{az-j}(A^{z}_{\theta})(x, -\xi) = (-1)^{j} e^{iaz\pi} \sigma_{az-j}(A^{z}_{\theta-a\pi})(x, \xi).$$

For a fixed t, the operator $A_\theta^t B$ is classical with order $at+b$. Since A and B are odd-class

$$\begin{aligned}
&\sigma_{at+b-j}(A_\theta^t B)(x,-\xi)\\
=&\sum_{|\alpha|+k+l=j}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma_{at-k}(A_\theta^t)(x,-\xi)D_x^\alpha\sigma_{b-l}(A_\theta^z)(x,-\xi)\\
=&\sum_{|\alpha|+k+l=j}\frac{1}{\alpha!}(-1)^{|\alpha|+k}e^{iat\pi}\partial_\xi^\alpha\sigma_{at-k}(A_{\theta-a\pi}^t)(x,\xi)(-1)^l e^{ib\pi}D_x^\alpha\sigma_{b-l}(A_\theta^z)(x,\xi).
\end{aligned}$$

Hence

$$\sigma_{at+b-j}(A_\theta^t B)(x,-\xi)=(-1)^j e^{i(at+b)\pi}\sigma_{at+b-j}(A_{\theta-a\pi}^t B)(x,\xi). \tag{6.3}$$

It follows that if ψ_t is a principal angle for $A_\theta^t B$, then $\psi_t-(at+b)\pi$ is a principal angle for $A_{\theta-a\pi}^t B$. Let $R(\lambda)$ and $\hat{R}(\lambda)$ denote the resolvents of $A_\theta^t B$ and $A_{\theta-a\pi}^t B$ respectively. Using formula (6.3) and the standard formulae for the parametrix ([KV1, B]) we deduce that the symbols $r(x,\xi;\lambda)$ and $\hat{r}(x,\xi;\lambda)$ verify

$$r_{-at-b-j}(x,-\xi;e^{i(at+b)\pi}\lambda)=(-1)^j e^{i(at+b)\pi}\hat{r}_{-at-b-j}(x,\xi;\lambda).$$

Then by the same computation used to prove Proposition 2.2.1 we obtain

$$\sigma_{s(at+b)-j}(A_\theta^t B)^s_{\psi(t)}(x,-\xi)=(-1)^j e^{is(at+b)\pi}\sigma_{s(at+b)-j}(A_{\theta-a\pi}^t B)^s_{\psi(t)-(at+b)\pi}(x,\xi)$$

and hence $\log^{\mathrm{sym}}_{\psi(t)}(A_\theta^t B)$ is an odd-class operator. Let us set

$$\log\mathcal{M}^{\mathrm{sym}}(A_\theta^t,B):=\log\mathrm{DET}^{\mathrm{sym}}_{\psi(t)}(A_\theta^t B)-\log\mathrm{DET}^{\mathrm{sym}}_\theta(A_\theta^t)-\log\mathrm{DET}^{\mathrm{sym}}_\phi(B).$$

Following the same arguments similar to Okikiolu's (see [Ok2]), already used in the proof of Corollary 2.1.15, we can build a finite partition $\bigcup_{k=1}^K J_k$ of $[0,1]$ so as to choose on each of the intervals $J_k=[t_k,t_{k+1}]$ a common fixed principal angle ψ_k of $A_\theta^t B$ when t varies in J_k. Then there exists ϵ such that there are no eigenvalues of $A_\theta^t B$ on the cone $\Lambda_{[\psi_k-\epsilon,\psi_k+\epsilon]}$. Since ψ_k is a spectral cut for $A_\theta^t B$, $\psi_k-m\pi$ where $m=at+\pi$ is also a spectral cut for $A_\theta^t B$; hence there exist ϵ' such that there are no eigenvalues of $A_\theta^t B$ on the cone $\Lambda_{[\psi_k-m\pi-\epsilon',\psi_k-m\pi+\epsilon']}$. Let us choose an angle $\widetilde{\psi}_k$ sufficiently close to ψ by the following manner: let η be such that $0<\eta\le Min(\epsilon,\epsilon')$ and $\psi-\eta\le\widetilde{\psi}_k\le\psi_k+\eta$. It is easy to see that there are no eigenvalues of $A_\theta^t B$ inside the cone $\Lambda_{\psi_k,\widetilde{\psi}_k}$ since $\psi_k-\epsilon\le\psi_k-\eta\le\psi_k+\eta\le\psi_k+\epsilon$. Again, there are no eigenvalues of $A_\theta^t B$ on the cone $\Lambda_{\psi_k-m\pi,\widetilde{\psi}_k-m\pi}$ since $\psi_k-m\pi-\epsilon'\le\psi_k-m\pi-\eta\le\psi_k-m\pi+\eta\le\psi_k-m\pi+\epsilon'$. It follows that $\widetilde{\psi}_k$ is an angle sufficiently close to ψ_k.

We want to show that for all t in $[0,1]$, $\frac{d}{dt}\left(\log\mathcal{M}^{\mathrm{sym}}(A_\theta^t,B)\right)=0$ i.e. for all τ in $[0,1]$,

$$\frac{d}{dt}_{|t=0}\left(\log\mathcal{M}^{\mathrm{sym}}(A_\theta^{t+\tau},B)\right)=0.$$

Let us start by proving the result at $\tau = 0$. In practice we work on each of the intervals J_k with the spectral cut $\widetilde{\psi}_k$; to simplify notations, we just write $\widetilde{\psi}$ instead of $\widetilde{\psi}_k$.

$$\begin{aligned}
& \frac{d}{dt}_{|t=0}\left(\log \mathcal{M}^{\mathrm{sym}}(A_\theta^t, B)\right) \\
= & \frac{d}{dt}_{|t=0}\mathrm{TR}\left(\log_{\widetilde{\psi}}^{\mathrm{sym}}(A_\theta^t B) - \log_\theta^{\mathrm{sym}}(A_\theta^t) - \log_\phi^{\mathrm{sym}}(B)\right) \\
= & \frac{1}{2}\frac{d}{dt}_{|t=0}\mathrm{TR}\left(\log_{\widetilde{\psi}}(A_\theta^t B)_{\widetilde{\psi}} + \log_{\widetilde{\psi}-m\pi}(A_{\theta-a\pi}^t B)_{\widetilde{\psi}-m\pi} - \log_\theta A_\theta^t - \log_{\theta-a\pi} A_{\theta-a\pi}^t\right) \\
= & \frac{1}{2}\mathrm{TR}\left[(\dot{\overline{A_\theta^t B}})_{\widetilde{\psi}}(A_\theta^t B)_{\widetilde{\psi}}^{-1} + (\dot{\overline{A_{\theta-a\pi}^t B}})_{\widetilde{\psi}-m\pi}(A_{\theta-a\pi}^t B)_{\widetilde{\psi}}^{-1}\right]_{|t=0} \\
& -\frac{1}{2}\mathrm{TR}\left[(\dot{\overline{A_\theta^t}})_\theta (A_\theta^t)_\theta^{-1} + (\dot{\overline{A_{\theta-a\pi}^t}})_{\theta-a\pi}(A_{\theta-a\pi}^t)_{\theta-a\pi}^{-1}\right]_{|t=0} \\
= & 0,
\end{aligned}$$

where we have used the formula of Proposition 2.1.14

$$\frac{d}{dt}\log C_t = \dot{C}_t C_t^{-1} + \sum_{k=1}^{K}\frac{(-1)^k}{k+1}\mathrm{ad}_{C_t}^k(\dot{C}_t)\, C_t^{-(k+1)} + R_K(C_t, \dot{C}_t)$$

combined with the traciality of TR in the third identity. Now, replacing B by $A_\theta^\tau B$ yields

$$\frac{d}{dt}_{|t=0}\left(\log \mathcal{M}^{\mathrm{sym}}(A_\theta^t, A_\theta^\tau B)\right) = 0.$$

An easy computation shows that

$$\log \mathcal{M}^{\mathrm{sym}}(A_\theta^{t+\tau}, B) - \log \mathcal{M}^{\mathrm{sym}}(A_\theta^t, A_\theta^\tau B) = \mathrm{TR}\left(-\log_\theta^{\mathrm{sym}}(A_\theta^\tau) - \log_{\widetilde{\psi}}^{\mathrm{sym}} B - \log_{\widetilde{\psi}}^{\mathrm{sym}}(A_\theta^\tau B)\right).$$

Since the r.h.s. of the previous equation in independent of t, it follows that for all $\tau \in [0, 1]$

$$\frac{d}{dt}_{|t=\tau}\left(\log \mathcal{M}^{\mathrm{sym}}(A_\theta^t, B)\right) = 0.$$

□

The second proof taken in [B] uses a formula proved in [KV1, Ok2] which expressed the multiplicative anomaly of the ζ-determinant in terms of an integral of noncommutative residues.

Proof 2: It is shown in [B] (Proposition 4.3) that

$$\log \mathrm{Det}_\theta^{\mathrm{sym}} A = \frac{1}{2}(\log \det_{\zeta,\theta} A + \log \det_{\zeta,\theta-a\pi} A)$$

Since the multiplicative anomaly for the ζ-determinant is given [KV1, Ok2] modulo $2i\pi$ by the formula:

$$\log \det_{\zeta,\psi} AB - \log \det_{\zeta,\theta} A - \log \det_{\zeta,\phi} B = \frac{a}{2} \int_0^1 \operatorname{res}\left(\left(\frac{\log_{\psi(t)} A_\theta^t B}{at+b} - \frac{\log_\theta A}{a}\right)^2\right) dt,$$

using Remark 6.3.2 item 2, we obtain modulo $2i\pi$ [B]:

$$\log \mathrm{DET}_\psi^{\mathrm{sym}}(AB) - \log \mathrm{DET}_\theta^{\mathrm{sym}}(A) - \log \mathrm{DET}_\phi^{\mathrm{sym}}(B) = \frac{a}{4} \int_0^1 \operatorname{res}\left(U(t)^2 + V(t)^2\right) dt$$

where

$$U(t) = \frac{\log_{\psi(t)} A_\theta^t B}{at+b} - \frac{\log_\theta A}{a}$$

and

$$V(t) = \frac{\log_{\psi(t)-(a+b)\pi} A_{\theta-a\pi}^t B}{at+b} - \frac{\log_{\theta-a\pi} A}{a}.$$

We want to prove that $U(t)^2+V(t)^2$ is an odd-class operator and hence its noncommutative residue vanishes on odd dimensions. Differentiating equation (2.17) at $z=0$ yields

$$\sigma_{-j}(\log_\theta A)(x,-\xi) = (-1)^j \left(ia\pi\delta_{j,0} + \sigma_{-j}\left(\log_{\theta-a\pi} A\right)(x,\xi)\right). \tag{6.4}$$

Applying this equation to $\log_\theta A$ and $\log_{\psi(t)-(a+b)\pi} A_{\theta-a\pi}^t B$, we get

$$\frac{1}{a}\sigma_{-j}(\log_\theta A)(x,-\xi) = (-1)^j \left(i\pi\delta_{j,0} + \frac{1}{a}\sigma_{-j}\left(\log_{\theta-a\pi} A\right)(x,\xi)\right)$$

and

$$\frac{1}{at+b}\sigma_{-j}(\log_{\psi(t)} A_\theta^t B)(x,-\xi) = (-1)^j \left(i\pi\delta_{j,0} + \frac{1}{at+b}\sigma_{-j}\left(\log_{\psi(t)-(a+b)\pi} A_{\theta-a\pi}^t B\right)(x,\xi)\right).$$

Combining these two equations yields:

$$\sigma_{-j}(U(t))(x,-\xi) = (-1)^j \sigma_{-j}(V(t)(x,\xi). \tag{6.5}$$

Similarly,

$$\sigma_{-j}(V(t))(x,-\xi) = (-1)^j \sigma_{-j}(U(t)(x,\xi). \tag{6.6}$$

We deduce therefore that

$$\sigma_{-j}(U(t)+V(t))(x,-\xi) = (-1)^j \sigma_{-j}(U(t)+V(t)(x,\xi).$$

i.e. $U(t)+V(t)$ is an odd-class operator.
Let us show now that $U(t)^2+V(t)^2$ is also an odd-class operator. Using formula (1.2) which gives the composition of the symbols of ΨDOs we know that

$$\begin{aligned}\sigma_{-j}(U(t)^2)(x,\xi) &\sim \sigma_0(U(t))(x,\xi)\sigma_{-j}(U(t))(x,\xi)\\ &+ \sum_{k+l+|\alpha|=j}\frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha\sigma_{-k}(U(t))(x,\xi)\partial_x^\alpha\sigma_{-l}(U(t))(x,\xi).\end{aligned}$$

This implies using Equation 6.5 that

$$\begin{aligned}&\sigma_{-j}(U(t)^2)(x,-\xi)\\ \sim\ &\sigma_0(V(t))(x,\xi)(-1)^j\sigma_{-j}(V(t))(x,\xi)\\ +\ &\sum_{k+l+|\alpha|=j}(-1)^{k+|\alpha|}\frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha\sigma_{-k}(V(t))(x,\xi)\partial_x^\alpha(-1)^l\sigma_{-l}(V(t))(x,\xi).\end{aligned}$$

As before, it follows that $\sigma_{-j}(U(t)^2)(x,-\xi)=(-1)^j\sigma_{-j}(V(t)^2)(x,\xi)$ from which we deduce that $U(t)^2+V(t)^2$ is an odd-class operator. $\square$

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ABSTRACT

This thesis is devoted to the study of the canonical trace and two types of determinants: on the one hand a determinant associated with the canonical trace on a class of pseudodifferential operators and on the other hand determinants associated with regularized traces.

In the first part, in odd dimension, we revisit the uniqueness of the canonical trace on the space of classical pseudodifferential operators of odd class before extending it to log-polyhomogeneous operators of odd class. We classify the traces on the algebra of classical pseudodifferential operators of odd class and order zero.

In the second part, we establish the locality of the multiplicative anomaly of the weighted determinant and the zeta determinant. These results are obtained thanks to the study of the locality of the weighted trace of the operator $L(A, B)$. We then derive from these results the local expression of the multiplicative anomalies in terms of the noncommutative residue.

In the third part, we classify multiplicative determinants on the grounds of the classification of traces on classical pseudodifferential operators of odd class and order zero in odd dimension. We also define the symmetrized determinant obtained from the canonical trace applied to the symmetrized logarithm of an odd class operator in odd dimension. We show the multiplicativy of this determinant under some restrictions on the spectral cuts of the operators.

RÉSUMÉ

Cette thèse est consacrée à l'étude de la trace canonique et de deux types de déterminants: d'une part un déterminant associé à la trace canonique sur une classe d'opérateurs pseudodifférentiels et d'autre part des déterminants associés à des traces régularisées.

Dans la première partie, en dimension impaire, nous revisitons l'unicité de la trace canonique sur l'espace des opérateurs pseudodifférentiels classiques de classe impaire avant de l'étendre aux opérateurs log-polyhomogènes de classe impaire. Nous classifions les traces sur l'algèbre des opérateurs pseudodifférentiels classiques de classe impaire d'ordre zéro.

Dans la deuxième partie, nous établissons la localité de l'anomalie multiplicative du déterminant pondéré et du déterminant zeta. Ces résultats sont obtenus grâce à l'étude de la localité de la trace pondérée de l'opérateur $L(A, B)$. Nous dduisons alors de ces résultats l'expression locale de ces anomalies multiplicatives en fonction du résidu noncommutatif.

Dans la troisième partie, nous classifions les déterminants multiplicatifs en utilisant la classification des traces sur les opérateurs pseudodifférentiels de classe impaire et d'ordre zéro en dimension impaire. Nous définissons aussi le déterminant symétrisé obtenu de la trace canonique appliquée au logarithme symétrisé en dimension impaire. Nous montrons la multiplicativité de ce déterminant sous certaines restrictions sur les coupures spectrales des opérateurs.