Valorisation financière sur les marchés d'électricité

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Ach ! Wie qualvoll ist die Zeit zwischen einem grossen Plan und der Ausführung !
Wievel grundlose Angst ! Wievel Unentschlossenheit ! Es geht um das Leben.
-Es geht um weit mehr : um die Ehre !
Friedrich Von Schiller

Il y a toujours quelque émerveillement au calcul rétrospectif des chances d'aboutissement de certains projets. Cette thèse ne déroge pas à la règle et fut une étonnante série de coïncidences et d'évènements singuliers [1], bien avant même son commencement officiel. On ne peut, à l'issue d'un tel chemin, se faire l'apôtre du hasard froid et n'éprouver aucune gratitude pour les coups du destin. On ne peut non plus refaire l'histoire de ces trois années rétrospectivement sans perdre certes la candeur, la naïveté ou parfois le désespoir qui a pu m'habiter à un moment ou un autre, mais surtout l'intime sentiment d'une nécessité de continuer. Cette nécessité, mêlée aux heureux hasards saupoudrant la narration, tient aux personnes qui m'ont entouré en cette période et que je tiens à remercier ici.

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1. on oserait presque emprunter le terme de synchronicité acausale à Carl Gustav Jung.

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Pour finir, je souhaite dédier cette thèse à mon père, le premier esprit mathématicien que j'ai admiré.

Résumé

Cette thèse traite de la valorisation de produits dérivés du prix de l'électricité, et s'incrit dans le domaine des mathématiques financières. Dans la première partie, nous nous intéressons à la valorisation par absence d'opportunité d'arbitrage de portefeuilles incluant la possibilité de transformation d'actifs par le biais d'un système de production, cela sur des marchés présentant des coûts de transaction proportionnels. Faisant appel à cette théorie spécifique, nous proposons un concept alternatif d'absence d'opportunité d'arbitrage pour une fonction de production et étendons certains résultats de Rásonyi [Rásonyi 10] à temps discret. Cela nous permet de démontrer la propriété fondamentale de fermeture pour l'ensemble des portefeuilles atteignables de ce type, ainsi que des corollaires comme l'existence d'un portefeuille optimal ou un théorème de sur-réplication. Nous continuons l'approche avec fonction de production en temps discret en étendant la modélisation à un marché en temps continu avec ou sans frictions. Cette approche s'inspire grandement du cadre théorique proposé par Denis et Kabanov [Denis 11b]. Cela permet aussi de déduire la propriété de fermeture et la caractérisation des actifs réplicables.

Dans le seconde partie, nous nous concentrons sur des problématiques de valorisation de produits dérivés sur électricité. Dans un premier temps nous présentons une classe de modèles faisant apparaitre un lien structurel entre le coût de production d'électricité et les matières premières nécessaires à sa production. En formulant spécifiquement la fonction des prix des commodités et du niveau de la demande, on obtient une formule explicite pour le prix de l'électricité spot. Le passage par une mesure martingale spécifique au traitement des incomplétudes de marchés, permet d'obtenir un certain prix d'absence d'opportunité d'arbitrage pour les contrats futures sur électricité minimisant le risque quadratique de couverture. Nous spécifions alors le modèle pour obtenir des formules analytiques et proposons des méthodes de calibration et d'estimation statistique des paramètres dans le cas où le prix spot dépend de deux combustibles. Dans un second temps, nous abordons par des méthodes de contrôle stochastique initiées par Bouchard, Elie et Touzi [Bouchard 09] le problème de la prime de risque associée à un produit dérivé de contrat futures non disponible. Utilisant des résultats de dualité déjà existant, nous étendons l'application numérique au cas d'un marché semi-complet. Notre point de vue se concentre essentiellement sur la représentation sous forme d'espérance de la fonction valeur du problème.

Abstract

This Ph.D. dissertation deals with the pricing of derivatives on electricity price. It belongs to the field of Arbitrage Pricing Theory and financial mathematics.

The first part is a theoretical extension of Arbitrage Pricing Theory : we assess the problem of pricing European contingent claims when the financial agent has the possibility to transform assets by means of production possibilities. We propose a specific concept of arbitrage for such portfolios and extend some results of Rásonyi [Rásonyi 10] in discrete time for markets with proportional transaction costs. This allows to show the closedness property and corollaries such as portfolio optimization problem or a super-hedging theorem. We then study such portfolios with financial possibilities in continuous time, with or without frictions. This framework is mostly motivated by Denis & Kabanov [Denis 11b]. We prove here the same closedness result and the super-replication theorem as corollary. We apply these results to the pricing of futures contract on electricity.

The second part is dedicated to applications of expectation representation in the treatment of incompleteness of financial markets, with a focus on electricity derivative pricing. We start with the presentation of a class of models allowing to link the electricity spot price with its production cost by a structural relationship. We specify a two combustibles model with possible breakdown. It provides explicit formulae allowing to fit several pattern of electricity spot prices, by means of the demand level and commodity prices. Using the minimal martingale method of Föllmer and Schweizer [Föllmer 91], we are able to explicit an arbitrage price and a hedging strategy for futures contracts minimizing a quadratic risk criterion. We then specify the model to obtain explicit formulae and propose calibration and statistical estimation of parameters in a two combustibles model. We address in a second time the question of the risk premium associated to the holding of a European option upon a non-yet available electricity futures contract. We essentially apply the ideas of Bouchard and al. [Bouchard 09] to the semi-complete market framework defined by Becherer [Becherer 01] and propose numerical procedures to obtain the risk premium associated to a contingent claim and a given loss function.

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Introduction

Cette thèse a pour objectif de présenter quelques problématiques de finance concernant les marchés dérégulés d'électricité traitées par les mathématiques financières. Elle est composée de deux volets relativement indépendants concernant les techniques utilisées, précédés d'un chapitre introductif. La première partie est à caractère théorique et représente un développement particulier de la théorie de valorisation par absence d'opportunité d'arbitrage pour un agent ayant des capacités de production. La seconde partie regroupe deux applications du calcul stochastique à la valorisation de contrat futures et d'option sur ces contrats respectivement. Bien que faisant appel à des approches différentes, ces deux applications, et la première partie, ont en commun l'utilisation de mesures martingales équivalentes pour la valorisation dans le cadre de marché incomplet. Pour introduire ces deux parties, nous commençons par décrire les marchés d'électricité et leurs particularité, ainsi que les différentes approches utilisées jusque là pour résoudre les problèmes abordés ou similaires.

Les marchés dérégulés d'électricité

Un risque financier récent

La dérégulation des marchés d'électricité est un phénomène récent et mondialement observé, démarré par le Chili au début des années 1980 sous la forme d'un système compétitif de production d'électricité basé sur les prix marginaux. A l'instar du Chili, un mouvement général de dissolution des monopoles industriels apparait, avec notamment la séparation des activités de transmission et de distribution d'une part et des activités de production ou de vente d'autre part. Aujourd'hui, il existe un éventail considérable de marchés de gros organisés autour de regroupements. Citons les plus connus : PJM (Pennsylvani-New Jersey-Maryland) aux États-Unis d'Amérique, Nordpool en Scandinavie (Danemark, Suède, Finlande, Estonie et Norvège) ou EPEX en centre Europe (France, Allemagne, Autriche et Suisse). L'objectif prioritaire de la dérégulation est d'atteindre un équilibre compétitif permettant une optimalité économique de l'allocation des ressources.

L'introduction de la compétition dans ce secteur introduit pour ses acteurs de nouvelles sources de risque. Les prix sont désormais variables et *fixés par le marché*. La complexité de l'industrie électrique impliquant déjà de nombreuses sources de risque (endogènes et exogènes), les marchés de l'électricité ont rapidement fait émergé un besoin de gestion globale des risques financiers et industriels. Cette gestion des risques est dévenue prioritaire pour les participants à un marché dérégulé. Aujourd'hui, les participants historiques (les producteurs) ont été rejoints par de nouveaux investisseurs. En effet, le risque prix de l'électricité est devenu une opportunité de diversification des risques pour des investisseurs institutionnels. La possession d'une centrale électrique apparait alors comme un actif avantageux au sein d'un portefeuille financier [2]. Les investissements dans le domaine énergétique croissant rapidement, la place centrale de l'électricité dans les marchés d'énergie attire irrémédiablement l'intérêt des banques et fonds d'investissements pour les produits financiers ou les entreprises du domaine électrique.

Les marchés d'électricité ont de nombreuses différences avec les marchés de commodités habituels (matières premières) ou les marchés actions. C'est pourtant pour ces derniers que les mathématiques financières ont permis des avancées techniques importantes. Nous allons voir ici quelles sont ces différences et ce qu'elles impliquent sur les moyens applicables aux marchés pour la gestion des risques.

Les caractéristiques de l'électricité

L'électricité désigne dans le langage courant l'énergie électrique. Sans rentrer dans les détails du processus de production physique, rappelons que l'électricité est un phénomène physique utilisé comme transporteur d'énergie électromagnétique. Celle-ci peut être produite par la transformation de différentes sources initiales d'énergie :

- l'énergie potentielle mécanique, que nous obtenons par la retenue des eaux d'un barage (centrales hydro-électriques),
- l'énergie cinétique (éoliennes),
- l'énergie potentielle chimique, qui est *contenue* dans diverses matières premières et transformée en énergie thermique (centrales nucléaires et thermiques). Cette dernière ressource retiendra toute notre attention dans ce qui suit.

L'électricité est le moyen le plus rapide et le plus pratique pour transporter de l'énergie dans les pays ayant développé les infrastructures nécessaires. Le stockage de l'énergie

2. Entre 2003 et 2004, Goldman Sachs a fait l'acquisition de plus de 30 centrales pour un montant supérieur à 4 milliards de dollars dans deux états des États-Unis d'Amérique.

électro-magnétique nécessite des moyens technologiques sans commune mesure avec les quantités produites à l'échelle des marchés. Elle peut être reconvertie en une autre forme d'énergie, mais à très fort coût et faible rendement. Cela implique que l'électricité ne peut être, une fois produite, considérée comme un bien consommable ou revendable à loisir dans un futur proche ou éloigné. Ce fait implique un changement de paradigme si nous souhaitons appliquer les méthodes de finance mathématique car la possession d'un actif est nécessaire pour la gestion du risque de son prix, cf. [Vehvilainen 02].

Un second corollaire découle de ce qui précède. De la quasi-immédiateté et de la commodité de sa fourniture, il apparait que la demande d'électricité est essentiellement motivée par son usage. Le prix d'achat devient alors un critère secondaire de consommation : la demande est localement inélastique aux variations de prix. La considérer comme une source exogène de risque dans des modèles de prix permet alors parfois des calculs explicites, voir notamment [Barlow 02]. La demande d'électricité procède par un soutirage d'énergie disponible sur un réseau ouvert aux consommateurs. Comme l'électricité n'est pas stockable, la production doit correspondre à la demande de manière continue. D'autre part les réseaux de distribution d'électricité étant le seul moyen de transport de cette énergie, les marchés d'électricité sont des regroupements géographiquement localisés. La fourniture dépend donc de la répartition des noeuds de distribution physiques. Les prix peuvent ainsi varier d'une aire géographique à une autre selon les moyens de productions localement existant.

Les prix d'électricité sont ainsi influencés par de nombreux facteurs physiques de production, mais également par l'organisation du marché. Dans une économie régulée ou sur un marché, le prix de l'électricité est fixé en fonction du coût de production, et d'une règle de prix marginal local : Le fournisseur d'électricité calcule un prix virtuel pour une unité supplémentaire d'énergie à un noeud du réseau et utilise cet incrément du coût de production comme prix de vente. Ainsi, le coût de production est normalement amorti. Cette règle initiera au Chapitre 4 la construction d'un modèle structurel de prix.

L'impact sur les prix

Ces préliminaires sur le sous-jacent permettent de mettre en lumière les particularités des marchés d'électricité, par opposition aux autres marchés. En premier lieu, l'équilibre entre offre et demande ne pouvant être parfaitement et instantanément ajustés par la production, le marché spot désigne en réalité la fixation du prix pour le lendemain. Comme la fourniture d'électricité se fait sous la forme d'une puissance et se vend sous forme d'énergie, l'électricité est vendue sur des périodes de temps, dont la granularité minimale est l'heure. Le prix spot de l'électricité est donc un prix fixé pour le lendemain sur des tranches horaires. La continuité du prix spot est donc une construction conceptuelle.

La fixation des prix dépend de la demande et du coût de production marginal au niveau de cette demande. La demande elle même dépendant de facteurs qu'il est raisonnable de considérer comme saisonniers et stationnaires (température, activité économique), les prix Spot reflètent ces propriétés. Le coût de production pouvant varier grandement d'une centrale de production à une autre, il n'est pas impossible d'observer sur le marché des phénomènes de très grande variation temporaire des prix. Ces sauts dénommés pics de prix, ainsi que la grande volatilité du prix Spot, représentent des difficultés supplémentaires pour une représentation mathématique fidèle du processus de prix.

La liste de ces particularités peut être retrouvée dans [Burger 04] ou [Coulon 09a]. Ces caractéristiques ont fait l'objet de nombreuses tentatives de modélisation stochastique. L'introduction d'une composante déterministe pour appréhender la saisonnalité des prix a été faite entre autres par [Lucia 02] ou [Cartea 05]. La modélisation des pics de prix ou la stationnarité à long terme ont été étudiées dans [Burger 04, Cartea 05] et [Benth 07a] à titre d'exemple. Ces caractéristiques sont désormais des critères de pertinence des modèles proposés dans la littérature. Nous jugerons de la pertinence du modèle du Chapitre 4 à l'aune de ces critères.

Les contrats financiers assurant la livraison d'une quantité fixée d'énergie pour une date future obéissent également à la règle de la granularité. Ces contrats dit futures ou forward (si négociés de gré à gré) sont en fait des contrats de type swap [Deng 06] couvrant des périodes fixées de temps. Ces périodes sont liées naturellement au calendrier qui guide l'activité économique : elles décrivent les semaines, les mois, les trimestres ou les années du calendrier grégorien. Afin d'assurer une liquidité suffisante pour ses participants, le marché est organisé pour ne proposer qu'une partie des périodes possibles, en fonction de la taille de celle-ci et de son éloignement dans le temps. Encore une fois, l'impossibilité de stocker le sous-jacent empêche de reconstituer une structure par terme par des arguments d'arbitrage usuels. L'étude de ce problème fait l'objet du chapitre 5.

Enfin, le besoin de contrats couvrant des risques spécifiques a fait apparaitre des produits dérivés de toute sorte, cf. [Deng 06] ou les monographies [Clewlow 00, Pilipovic 97]. Citons les options d'achat et de vente sur contrat Futures, dont la valorisation et la couverture partielle sont abordées dans le chapitre 5. Notons aussi l'exemple des contrats spread, dont les contributions de cette thèse ont en partie permis la valorisation, cf. [Aid 10].

Valorisation des produits dérivés sur le marché d'électricité

La théorie de valorisation par absence d'opportunité d'arbitrage

La théorie d'évaluation des actifs contingents par absence d'opportunité d'arbitrage, ou APT (pour *Arbitrage Pricing Theory*), permet, à partir d'une règle économique interdisant les profils sans risques, de déduire le ou les prix justes d'un produit dérivé. Celle-ci est développée de façon sommaire dans le chapitre 1 pour en introduire les résultats essentiels. Une des conditions fondamentales à l'application de cette théorie est la description de l'ensemble des portefeuilles réalisables. Les résultats de cette théorie sont en effet valables, et permettent la valorisation de produits dérivés, seulement si le sous-jacent est échangeable. Ce n'est toutefois pas le cas concernant le prix spot de l'électricité.

De façon informelle, il n'existe pas d'arbitrage s'il existe une probabilité sous laquelle le processus de prix est une martingale. En marché incomplet, cette probabilité peut ne pas être unique. Ici, quand bien même le prix de l'électricité n'autoriserait aucune probabilité équivalente martingale, aucun arbitrage n'est possible à partir d'un portefeuille constitué d'électricité. Le marché n'est donc même pas incomplet au sens habituel du terme !

Il n'est toutefois pas vrai qu'aucun contrôle n'est possible sur l'électricité. Un producteur peut en effet mettre en place une stratégie financière avec un portefeuille de matières premières nécessaires à la production d'électricité d'une part, et par le contrôle de sa production d'autre part. Dans la première partie de cette thèse, nous proposons une extension paramétrique de l'APT pour les producteurs d'électricité afin de leur permettre la valorisation de produits financiers sur l'électricité. Cette condition est paramétrique parce qu'il n'existe pas de condition économique *naturelle* équivalente à l'absence d'opportunité d'arbitrage pour un producteur. Il faut donc interdire d'une certaine manière, *a priori*, la possibilité de certains profits.

Dans la première partie, nous introduisons donc différentes formes d'une condition économique imposée au producteur d'électricité, dont le portefeuille est préalablement défini comme un ensemble d'actifs financiers échangeables sur un marché ou transformables selon une fonction de production donnée. Dans le chapitre deux, nous explorons de manière assez exhaustive la modélisation en temps discret avec coûts de transaction proportionnels. Dans le troisième chapitre, nous proposons un critère de valorisation pour une classe générale de modèles de marché financier.

Notre contribution est donc la suivante. Nous montrons mathématiquement que l'ensemble des richesses terminales atteignables par les portefeuilles précédement décrits est un ensemble fermé si la condition économique additionnelle est supposée. Cette propriété permet de proposer, dans le cas discret, une formulation duale de la condition économique. Dans les cas discrets et continus, nous obtenons une formulation duale pour l'appartenance d'une variable aléatoire à l'ensemble des richesses terminales atteignables. Ce théorème permet de déterminer quelle est la richesse initiale avec laquelle faire un portefeuille d'investissement-production qui permet de couvrir une option donnée. Nous illustrons ce résultat par des exemples. Nous définissons un système de production électrique contrôlable et proposons le prix de couverture d'un contrat futures sur l'électricité pour le possesseur de ce système de production.

Modèles structurels de valorisation

La littérature en mathématiques financières portant sur la valorisation de produits dérivés de l'électricité est vaste, répondant à des besoins spécifiques : prévision, couverture de risque, gestion optimale de production. Il est courant, cf. [Ventosa 05, Coulon 09b] ou [Carmona 11], de diviser la littérature en trois directions distinctes : la modélisation stochastique des prix sous forme réduite, la modélisation par fondamentaux et la modélisation structurelle.

La première direction s'attache à une modélisation endogène et synthétique du prix de l'électricité spot [Lucia 02, Benth 04, Cartea 05, Benth 07a, Benth 07b, Geman 02] ou des contrats futures [Lucia 02, Fleten 03, Kiesel 09]. Ces modèles permettent par l'estimation historique des paramètres ou par la calibration selon des données de marché d'extraire l'information des observations de prix pour simuler des trajectoires avec un réalisme certain et détecter des tendances. L'étude porte éventuellement sur la corrélation des prix d'électricité avec d'autres commodités, cf. [Frikhal 10].

La deuxième direction, à l'opposé, a pour objet la modélisation des moyens de productions et des contraintes physiques sur le système. Cette direction est moins répandue dans la littérature mais beaucoup plus utilisée dans l'industrie énergétique, qui dispose de nombreuses données sur la production. On citera la monographie [Kallrath 09] sur le sujet. Le réalisme fonctionnel de ces modèles contrebalance une complexité qui empêche bien souvent les calculs explicites. L'objectif de ces modèle est en effet la simulation du système électrique étudié pour des visées prédictives en gestion de production.

La troisième direction est une synthèse des deux premières. A l'instar de Barlow [Barlow 02], un grand nombre de modèles a été proposé, utilisant la demande d'électricité comme facteur de risque exogène et l'introduisant dans des modèles de production plus ou moins complexes, cf. [Eydeland 99], [Burger 04] ou [Cartea 08]. En introduisant des actifs échangeables dans le processus de production, voir notamment [Coulon 09b] ou très récemment

[Carmona 11], il est possible d'obtenir des formules explicites de relation entre le prix d'électricité et les prix des commodités nécessaires à la production.

C'est cette approche que nous développons dans le premier chapitre de la seconde partie de cette thèse. En partant d'un modèle structurel qui utilise des informations publiques sur les capacités de production pour un marché donné, des prix de marché des commodités et la demande d'électricité, nous proposons un modèle de prix Spot de l'électricité possédant quelques particularités recherchées (pics, périodicité, clusters de volatilité).

L'ensemble des directions de modélisation ont pour point commun de permettre la simulation des prix d'électricité à partir de facteurs exogènes ou non. La valorisation et la couverture des produits dérivés par le biais de ces modèles n'est toutefois pas l'objectif principal. Seuls les modèles faisant apparaitre des relations simples entre le prix de l'électricité et ceux d'autres actifs financiers permettent éventuellement d'inférer des stratégies de couverture.

L'approche proposée dans le chapitre 4 est d'utiliser une mesure de valorisation spécifique introduite dans le cadre de marché incomplet par Föllmer et Schweizer [Föllmer 91]. Nous réintroduisons alors la valorisation par espérance, et calculons le prix des contrats futures en relation avec le prix Spot. Cette méthode réutilisée dans [Aid 10] permet alors la valorisation de produits dérivés sur électricité non pas dans le cadre d'une couverture parfaite, mais celle donnée par la mesure de valorisation qui correspond à la minimisation du risque quadratique local.

Dans ce chapitre, notre contribution est la spécification de ce modèle sur l'exemple du marché français. Nous proposons des méthodes d'estimation statistique usuelles pour le modèle proposé, puis de calibration à partir des prix de contrat futures.

Incomplétude du marché à terme

Comme nous l'apercevons, l'impossibilité de stocker l'électricité empêche d'utiliser les méthodes classiques de valorisation financière. Les relations d'arbitrage supprimées, l'étude de la structure par terme de l'électricité pose également des barrières qu'il n'est pas envisageable de franchir avec les méthodes usuelles.

Du au manque de finesse dans l'information sur la structure par terme, nous appelons ce problème celui de la granularité de la courbe de prix futures. Ce problème a été assez peu étudié dans la littérature, bien que sa considération soit précoce dans l'industrie électrique. Citons [Verschuere 03] dans le cas qui nous intéresse, à savoir le problème de couverture sur le marché à terme, et [Lindell 09] pour la considération de ce problème à des fins de reconstitution de la structure par terme à granularité horaire.

Notre problématique dans le chapitre 5 est la gestion du risque lié à la possession d'une option sur un contrat futures non encore apparu. Pour traiter ce cadre de marché incomplet, nous proposons d'aborder le problème en terme de prime de risque liée à une fonction de perte. C'est ainsi l'occasion d'utiliser l'approche de cible stochastique en espérance introduite par [Bouchard 09]. Nous reprenons notamment avec une très légère généralisation l'application proposée dans cet article afin de proposer une stratégie de couverture du risque utilisant le contrat futures de granularité supérieure disponible.

Dans la modélisation proposée, la forme d'incomplétude du marché est très spécifique. Elle correspond fortement à la définition donnée par Becherer [Becherer 01] de marché semi-complet : le marché composé des actifs disponibles est complet, c'est à dire qu'il est possible de couvrir parfaitement toute option ayant pour sous-jacent un actif disponible sur le marché. L'approche par cible stochastique étant une approche directe, nous montrons que s'il est possible de se ramener par une espérance conditionnelle à un problème en marché complet, alors le problème peut être traité ensuite par une méthode de dualité exhibant la probabilité équivalente martingale. Le cadre de marché complet a effectivement été exploré de cette façon dans [Bouchard 09]. Par cette procédure, on exhibe de manière non arbitraire une mesure de probabilité équivalente martingale *de marché*, laissant toutefois le risque extérieur au marché évalué sous la probabilité historique. Dans l'idée, nous faisons alors le lien avec la mesure minimale de Föllmer et Schweizer introduite dans le chapitre précédent.

Pour finir, notre approche nous conduit à étudier une cible stochastique intermédiaire qui peut être non-explicite et nécessiter une résolution numérique. Grâce au principe de programmation dynamique, nous conservons un problème sous forme d'EDP non-linéaire. Nous proposons alors une résolution numérique de cette EDP par des méthodes de Monte-Carlo et des processus tangents. La représentation de Feynman-Kac de l'EDP linéaire est associée à une méthode de point fixe utilisant les processus tangents pour le calcul des dérivés et du contrôle optimal. Bien que non formalisée, cette méthode s'avère efficace et ouvre une nouvelle piste de recherche dans la résolution numérique d'équations de type HJB.

Technical introduction

This thesis intends to treat some financial pricing problems on deregulated electricity markets. By means of the theory of financial mathematics, we attempt to formulate various approaches of electricity futures contracts pricing. This thesis is divided in two parts. The first part investigates Arbitrage Pricing Theory with an emphasis on the mathematical development of financial markets with proportional transaction costs. In this part, we propose an economical condition allowing an investor with production possibilities to price and hedge derivatives on his production and the financial market. We extend the fundamental results of Arbitrage Pricing Theory to that case. The second part of this thesis is composed of two chapters developing specific models of electricity futures prices for hedging purposes. The first one proposes a structural model of electricity spot prices. This allows to evaluate futures prices formation and hedging by alternative assets. The second one treats the incompleteness of the term structure of electricity prices. We focus on the control of loss on a derivative product upon unknown futures prices. The common ground is the exhibition of a specific equivalent martingale measure for pricing purposes. We also use the related expectation operators for explicit or numerical resolution.

Arbitrage pricing with production possibilities

In the first part, we consider the situation of an investor with production possibilities. This is essentially motivated by the economical assumption that electricity is a non-storable good. It consequently forbids to consider financial portfolios based on electricity spot price. Since electricity markets are still mostly constituted of electricity producers, it is viable to take the approach of an electricity provider. This is a micro economical point of view where the electricity spot price is exogenous to the agent.

We consider the set $\mathfrak{X}$ of portfolio strategies under a general form

$$V_t = \xi_t + R_t(\beta_t)$$

where ξ_t will denote a usual self-financing portfolio composed of financial assets, and $R_t(\beta_t)$ is the net return of production controlled by a process β. The return function R_t will thus transform a consumed quantity of assets β into a new position $R_t(\beta_t)$. It is formally a generalization to general orders of financial (selling or buying) orders, when they are introduced in an elementary way (see [Bouchard 06] and [De Vallière 07] for a useful formulation in the incomplete information case in markets with proportional transaction costs). We oppose here the linear structure of financial orders to the non-linear general structure of industrial transformation. A direct problem appears immediately : production returns are not bounded with an economical assumption such as the absence of arbitrage on a financial market. This raises two natural questions. The first one is how to define an economical assumption similar to the no-arbitrage condition and the second one questions the possible assumptions on the production function in order to have fundamental properties for $\mathfrak{X}$ under this new condition.

After introducing the fundamental results of Arbitrage Pricing Theory in chapter 1, we consider in the second chapter of this part the latter questions in a specific market setting. The material dimension of production incites us to express the manipulated quantity of assets in units. This is indeed done in the particular treatment of markets with proportional transaction costs. It started with [Kabanov 02] and has been repeatedly used after that, see the monograph [Kabanov 09] for a complete presentation. This costs are widespread on every type of financial market. Moreover, the linearity of all the considered objects in this framework underlines the mathematical treatment of non-linearity we introduce with production possibilities. The introduction of non-linearity actually follows [Bouchard 05] where the authors introduce a non-linear industrial asset. In the latter, the robust no-arbitrage condition of [Schachermayer 04] is extended to non-linear assets in order to prove the closedness property of the set of attainable terminal wealth, and then to have existence in the portfolio optimization problem. The dual characterization of the robust no-arbitrage condition in the non-linear context has recently been done in [Pennanen 10] for illiquidity matters. All these studies were done in the discrete time setting, allowing to handle very general conditions on the non-linear framework. See also Kabanov and Kijima [Kabanov 06] and the references therein for the particular consideration of industrial investment.

The main distinction between our work and the above research is that we do not consider industrial assets in the latter sense. Contrary to pure financial assets, industrial assets cannot be short-sold. Moreover, they produce at each period a (random) return, labelled in terms of pure financial assets, which depend on the current inventory in industrial assets. This model is well-adapted to industrial investment problems but not to production issues, since the production regime does not appear as a control. What we propose is a general framework of investment-production possibilities. The investment possibilities are given by a model of a financial market and/or possible financial strategies. The production possibilities are given by an endomorphism R on the set of assets in the financial market. The production asset can transform some asset into others, with possible random factors (prices, failures).

The contribution of the second chapter is then a re-edition of the closedness property and its corollaries for this class of models. The main novelty is that we do not use the robust no-arbitrage condition any more, as in [Pennanen 10]. As we said, there is no economical justification for the absence of sure profits for a producer selling its production on a market. We thus introduce an extended version of the *no sure profit* condition of [Rásonyi 10] for linear production function, which can be used to allow limited profits for a general production function. This condition of absence of arbitrage of the second kind is particularly well suited to our extension, and avoid to prove the closedness property at first. We thus propose a dual characterization of this condition (a fundamental theorem of asset pricing) with a direct proof, and then we prove the key property of closedness for the set of terminal attainable wealth. We then explore, as corollaries, the super-hedging theorem under many additional assumptions and the portfolio optimization problem.

The extension of this class of models to continuous time or frictionless market is the object of Chapter 3. In this chapter, we want to propose a very general and flexible condition for investor-producers such as before. For this purpose, we propose an abstract financial setting which includes the main classes of financial models : frictionless markets with general semimartingales [Schachermayer 04], càdlàg price processes subject to strictly positive proportional transaction costs [Campi 06] and discrete time markets with convex transaction costs [Pennanen 10]. The attempt to model a great variety of situations draws its inspiration from [Denis 11b] and [Denis 11c]. By focusing on the production condition only, we can propose a general financial setting where the no-arbitrage condition of the financial market is expressed by its dual formulation, namely, the existence of a martingale deflator. We provide examples of applications in order to ensure that the general model suits to applications.

The counterpart of a general financial setting is that we have to impose strong conditions on the production. First of all, we reduce to the case of a discrete time control on the portfolio process. Although it keeps a realistic value, we were not able to extend the discrete time framework to a continuous or impulse-control setting of production possibilities. We comment this question in the chapter. Then, the production function has to be concave and bounded. The concavity assumption is a direct consequence of the available convergence theorems for sequences of random objects in the continuous time setting, see Theorems 1.2.3 and 1.2.4 in chapter 1. It ensures the convexity of the set, which is a fundamental assumption in the theory. The boundedness assumption is introduced in order to keep the admissibility property of portfolios. In continuous time, this property is essential as we rely on the concept of Fatou-closure of considered sets, from which it is possible to obtain the weak* topology closure, see Theorem 1.2.7 below.

In chapter 3, we propose a flexible parametric condition on production profits. It is also based on the *no sure profit* condition fashion, and allows for variations in its expression. Under this condition, we extend the closedness property of the set of possible financial terminal positions to positions allowing production profits. The corollary, which is of central interest here, is the super-hedging theorem. As in the previous chapter, we provide an application to an electricity producer willing to price an electricity futures contract .

Specific pricing measures for electricity derivatives hedging

The second part of this thesis includes two chapters, both being application of financial mathematics to electricity futures contracts. Chapter 4 presents a structural model of electricity Spot price depending on storable assets used in the production in order to obtain futures prices under some specific risk neutral measure. Chapter 5 is an application of the stochastic target approach to the risk premium associated to the holding of an option upon a non-tradable futures contract.

A structural model of electricity prices

The objective of this chapter is to present a model for electricity spot prices and the corresponding forward contracts, which relies on the underlying market of fuels, thus avoiding the electricity non-storability restriction. The structural aspect of our model comes from the fact that the electricity spot prices depend on the dynamics of the electricity demand at any instant, and on the random available capacity of each production means. Our model explains, in a stylized fact, how the prices of different fuels together with the demand combine to produce electricity prices. This modelling methodology allows one to transfer to electricity prices the risk-neutral probabilities of the market of fuels and, under the hypothesis of independence between demand and outages on one hand, and prices of fuels on the other hand, it provides a regression-type relation between electricity forward prices and fuels forward prices. Moreover, the model produces, by nature, the well-known peaks observed on electricity market data. In our model, spikes occur when the producer has to switch from one technology to another. Numerical tests performed on a very crude approximation of the French electricity market using only two fuels (gas and oil) provide an illustration of the potential interest of this model.

Considering the electricity spot market, we start from an aggregated bid-ask equilibrium of a competitive market. As a fundamental assumption, see Barlow's model [Barlow 02] for a complete explanation, we will assume that the demand is inelastic. The direct consequence is that the electricity spot price on the market is only related to the aggregated offer function where the level of production is fixed by the demand variable. The electricity spot price P_t will depend on several other random variables S_t (commodity prices, generation capacity, failure,...) and will be seen as a general function of the demand :

$$P_t = f(D_t, S_t)$$

Since this demand is a non-tradable risk factor, the considered market will be incomplete. There are two well-known consequences. First of all, this implies that the perfect replication of contingent claims is not possible (at least at a reasonable price). Secondly, standard results in Arbitrage Pricing Theory assess that the set of equivalent martingale measures is not reduced to a singleton. This implies an infinite number of no-arbitrage prices for a claim. This is where we introduce the so-called *minimal martingale measure*. This specific measure was first introduced by Föllmer and Schweizer [Föllmer 91] to partially hedge any claim in incomplete market. In our context, it will be used to price widespread contracts on electricity : forward and futures contracts.

Under this probability measure, the asset price process for fuels is a martingale, whereas the dynamics for Demand and the failure probabilities remain the same. This is a specific incomplete market setting where the market composed of tradable assets is supposed to be complete. This market is then augmented to satisfy the representation of risks induced by the model.

In this chapter, our contribution stands in the explicitness of a simple structural model with two combustibles, estimation of production parameters with public data and parameters calibration with futures prices. In this fashion, we exhaust the exploitation of the model and provide different guidelines for further enhancement. Indeed, the proposed model obtains interesting new results and offers many perspectives for further developments. We see three different areas to explore. First, the supposed competitive equilibrium on the spot market could be changed to take into account possible strategic bidding. This feature could provide a measure to the possible deviation of forward electricity prices from their equilibrium due to frictions on the spot. Second, the spot market could be extended to a multizonal framework to take into account the fact that electricity is exchanged between different countries and that a spot price is formed in each country. Finally, the relation linking forward electricity prices to forward fuels prices could be extended to a wider class of contingent claims. This point has been investigated recently in [Aid 10] for the pricing of spread options. We hope to develop these other points in future papers.

Controlling loss with a cascading strategy on electricity Futures contracts

In chapter 5, we face a specific source of unhedgeable risk, given by the apparition of a futures price at an intermediary date between the present and the term of an option based upon this precise contract. We decide to adopt here the approach of [Bouchard 09] : the stochastic target with a target in expectation. By this bias, we try to control in expectation a risk criterion given by a threshold.

The stochastic target is a control problem where the terminal condition T is given and the objective is to find the viability set before T. It has been initiated by Soner and Touzi [Soner 02a, Soner 02b] for target reaching in the almost sure sense. In [Bouchard 09], Bouchard, Elie and Touzi generalized the approach to targets in expectation, in order to provide a stochastic control formulation of the quantile hedging problem. It also has been extended in several directions : for jump-diffusion processes in [Bouchard 02] and [Moreau 11], for the obstacle version in [Bouchard 10] and the general semimartingale framework with constraints in [Bouchard 11a]. Significantly, the equivalence with a standard control problem has been noticed in [Bouchard 12]. We actually use this property in our context.

In chapter 5, we follow the application provided in [Bouchard 09], with minor modifications. We indeed use a specific model for the apparition of the new futures contract that comes close to the definition of semi-complete market, see [Becherer 01]. This setting allows, by using a conditional expectation, to retrieve a complete market setting and thus use the approach of [Bouchard 09] with full power. This allows to provide an intermediary target and a new problem under the standard form. However, the main drawback is that this condition is not explicit in most cases. This is why we propose a heuristic method for solving numerically non-linear PDEs based on probabilistic methods. Since we try to fully exploit the expectation formulation of the value function of the problem, we propose a mixed method based on the Feynman-Kac representation of a linear PDE and tangent processes in order to obtain partial derivatives. A fixed point algorithm is used to find the optimal control. The method appears to be very efficient since it avoids the curse of dimensionality of the PDE.

The contributions are thus the following. Theoretically, we provide the extension of the complete market solution of loss control first given in [Bouchard 09] to specific incomplete markets based on risk factors independent to the market and arriving at deterministic times. In practice, we provide a numerical method for the general resolution of the non-linear PDE associated to the stochastic target problem. We also apply this method to the initial problem of controlling loss on a portfolio endowed with an option on a non-existing contract.

Chapitre 1

Arbitrage Pricing Theory and fundamental results

This introductory chapter is motivated by self countenance of the thesis. We first recall the purpose of Arbitrage Pricing Theory. We follow two monographs of reference, which are [Delbaen 06] and [Kabanov 09]. One can find a good historical introduction of this theory in Part 1 of the first book and Chapter 2 of the second one. We focus on the special case of markets with proportional transaction costs, which underlies the first part of the thesis, and a specific martingale measure in incomplete markets which is of use in Chapter 4 and in relation with semi-complete markets introduced in Chapter 5. In a second time, we introduce the fundamental results of Measure Theory, Probability and Arbitrage Pricing Theory on which we rely repeatedly in the Thesis.

Specific notations

These notations concern the whole thesis. Specific notations are introduced in the context if needed.

Unless otherwise specified, any element $x \in \mathbb{R}^d$ will be viewed as a column vector with entries x^i, $i \leq d$, and transposition is denoted by x' so that $x'y$ stands for the natural scalar product. We write $\mathbb{M}^d$ to denote the set of square matrices M of dimension d with entries M^{ij}, $i, j \leq d$. The identity matrix is denoted by I_d. As usual, $\mathbb{R}^d_+$ and $\mathbb{R}^d_-$ stand for $[0,\infty)^d$ and $(-\infty, 0]^d$. The closure of a set $\Theta \subset \mathbb{R}^n$ is denoted by $\bar{\Theta}$, $n \geq 1$. We write cone(Θ) (resp. conv(Θ)) to denote the cone (resp. convex cone) generated by Θ. Given a filtration $\mathbb{F}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a set-valued $\mathcal{F}$-measurable family $A = (A_t)_{t \leq T}$, we denote by $L^0(A, \mathbb{F})$ the set of adapted processes $X = (X_t)_{t \leq T}$ such that

$X_t \in A_t$ $\mathbb{P}-$a.s. for all $t \leq T$. For a σ-algebra $\mathcal{G}$ and a $\mathcal{G}$-measurable random set A, we write $L^0(A, \mathcal{G})$ for the collection of $\mathcal{G}$-measurable random variables that take values in A $\mathbb{P}-$a.s. We similarly define the notations $L^p(A, \mathcal{G})$ for $p \in \mathbb{N} \cup \{\infty\}$, and simply write L^p if A and $\mathcal{G}$ are clearly given by the context. Unless otherwise specified, inequalities between random variables or inclusion between random sets have to be understood in the a.s. sense.

1.1 Arbitrage Pricing Theory

Arbitrage Pricing Theory has for purpose to seek pricing rules for financial instruments based on an economical assumption made on the financial market. *In nuce*, it intends to derive the existence of a fair pricing rule from a mathematical formulation of the absence of arbitrage on the financial market. Formally, when the financial market prices are represented by a process S, the no-arbitrage property for this market holds if and only if there exists a stochastic deflator, i.e., a strictly positive martingale ρ such that the process $Z := \rho S$ is a martingale. The process Z can then be seen as the market price of assets with which agents shall price derivative products. It is the core of mathematical applications to finance. This result is commonly expressed by the existence of a measure equivalent to the historical probability under which price processes are (local) martingales. It allows an incredible amount of applications to derivative pricing and risk hedging, the most commonly known being the seminal and pathbreaking paper of Black and Scholes [Black 76].

The theoretical side of this branch of applied mathematics is focused on such a rule, trying to link martingale theory to no-arbitrage arguments. It is almost all contained in one result known as the Fundamental Theorem of Asset Pricing (FTAP). By introducing several imperfections in the market, or portfolios constraints, in order to improve the model representation of the economical reality, several variants of the FTAP can be expressed. This was first established for the discrete time and finite probability space framework by Harrison & Pliska [Harrison 81]. Starting from results of Harrisson & Kreps [Harrison 79], extension to the infinite probability space is proved by Dalang, Morton and Willinger [Dalang 90]. Then follows a long line of contributions, see [Delbaen 06] and the references therein.

Let us denote by $\mathfrak{X}(T)$ the set of possible terminal wealth that is attainable with self-financing portfolios starting with a zero wealth. A no arbitrage condition expresses a condition on the possible outcomes of $\mathfrak{X}(T)$. For example, if $\mathfrak{X}(T)$ is composed of real outcomes, the no-arbitrage condition of the first kind can informally be written as :

$$\mathbf{NA} \ : \ \mathfrak{X}(T) \cap \mathbb{R}_+ = \{0\} \, .$$

The FTAP thus expresses an equivalence between such a mathematical expression and a dual condition providing the martingale deflator ρ of the previous paragraph. When there exists a unique process ρ, the market is said to be complete. In general, the martingale deflator ρ is not unique, due to some frictions. It encompass the case of transaction costs, or unavailable assets or information.

1.1.1 The proportional transaction costs framework

A specific and recent branch of the theory is the study of financial markets subject to transaction costs. Transaction costs are market frictions that can be observed on all financial markets. The difference between a bid price and the ask price, which can be indifferently credited to transaction costs or liquidity matters, fundamentally changes the way to model financial strategies. Take the case of proportional transaction costs. Commonly, a financial portfolio V is represented by a stochastic integral with respect to the asset price S, where the integrand ν represents the strategy (the amount of money put in the risky asset). When the agent is subject to proportional transaction costs λ on buying and selling orders, the portfolio shall be written

$$V_t = x + \int_0^t \nu_s dS_s - \int_0^t \lambda S_s d|\nu|_s \, .$$

Therefore, strictly positive proportional transaction costs force the strategy to be a finite variation process, whereas frictionless markets allow for a quadratic variation process ν. The set of portfolio processes is totally different from the frictionless case, and so is $\mathfrak{X}(T)$.

A geometrical representation, introduced by Kabanov [Kabanov 99] for currency markets, has emerged as consequence. It follows from the observation that with proportional transaction costs, the expression of the wealth is sensitive to the numéraire in which it is expressed. Therefore, it is more convenient to express exchange rates between currencies, or assets, and holdings in quantity of assets, than to reduce to a single wealth value, which is virtual if the exchange rates evolve through time. By directing the exchange rate from an asset to another, we are also able to make a distinction between bid ans ask prices, and to introduce random proportional transaction costs. We introduce then the following notations. If the market contains d assets, we denote by π^{ij} the quantity of asset i necessary to obtain one unit of asset j. π^{ij} is an adapted random process. It will be reintroduced in the introduction of the next chapter. Formally, it allows to define a random region of the space $\mathbb{R}^d$:

$$K_t(\omega) := \mathrm{conv}\left(\pi^{ij}(\omega)e_i - e_j, e_i \; ; \; i,j \leq d\right) \; , \tag{1.1.1}$$

where e_i stands for the i-th unit vector of $\mathbb{R}^d$ defined by $e_i^k = 1_{i=k}$. This region is a random closed convex cone indexed by t. It denotes the solvency region, i.e., the set of possible portfolio positions that can be modified by an allowed transaction in order to be non-negative in every component (every asset holding). In the literature, this geometrical object has almost replaced the notion of price. Indeed, portfolio modifications are made by transfers of assets which are represented by vectors in $-K_t$. It induces a geometrical vision that allows to use tools from convex analysis that we present in the Section 1.2. We refer to Kabanov and Safarian [Kabanov 09] for a wide overview of models with proportional transaction costs.

In markets with transaction costs, there are two possible expressions of arbitrage. One is the possibility to reach a solvent wealth non equivalent to zero (i.e. in $\int K_T$) with a portfolio starting with a null wealth. The other is the possibility to reach a solvent position (i.e. in K_T) with a portfolio starting from an insolvent position (not in K_0). Whereas the first one is a direct adaptation of the no-arbitrage condition in the frictionless case, the second one is more specific. This condition has been introduced by [Rásonyi 10] and is the object of study of the first part of the Thesis.
Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ be a discrete time filtered stochastic basis, with $\mathbb{T} := \{0, 1, \ldots, T\}$. We introduce a $\mathbb{F}$-adapted process K_t which values are closed subsets of $\mathbb{R}^d$, and which is defined by equation 1.1.1 above for all $t \in \mathbb{T}$. We also define its polar cone

$$K_t^*(\omega) := \left\{ y \in \mathbb{R}_+^d \;\; : \;\; xy \geq 0 \;\; \forall x \in K_t(\omega) \right\}$$

and assume that $\mathrm{int}K_t^* \neq \emptyset$ $\mathbb{P}$-almost surely. This condition is called *efficient frictions* and is assumed in the largest part of Chapter 2. It means that there are strictly positive transaction costs on every possible transfer. We then define

$$\mathfrak{X}_t(T) := \left\{ \sum_{s=t}^{T} \xi_s \;\; : \;\; \xi_s \in L^0(-K_s, \mathcal{F}_s) \text{ for } t \leq s \leq T \right\}$$

the set of terminal attainable wealth with a self-financing portfolio starting with a null wealth at time t. The no-arbitrage condition of second kind (called no sure gain in liquiditation value in [Rásonyi 10]) reads as follows.

Definition 1.1.1. *There is no arbitrage of the second kind if for all* $0 \leq t \leq T$, $\xi \in L^0(\mathbb{R}^d, \mathcal{F}_t)$ *and* $V \in \mathfrak{X}_t(T)$,

$$\xi + V \in L^0(K_T, \mathcal{F}_T) \Longrightarrow \xi \in L^0(K_t, \mathcal{F}_t) \ .$$

This definition is partially recalled in Chapter 2. Part 1 relies heavily on this definition of Arbitrage and we will see that it is possible to extend this definition or transform it in the study of investment-production portfolios.

1.1.2 Equivalent martingale measures and incomplete market

As said before, the set of equivalent martingale measures is not unique in incomplete market. There is not a unique martingale deflator ρ such that $Z := \rho S$ is a martingale. Therefore, several pricing rules implies several no-arbitrage prices. In a frictionless market, the process ρ takes the form of a change of probability measure. Thus, in incomplete market, there are several probability measures $\mathbb{Q}$, equivalent to the initial measure of the model, under which the process S is a (local) martingale. The following introductory sections recall implicit assumptions in Chapters 4 and 5.

The minimal martingale measure

This section intends to introduce the minimal martingale measure of Föllmer & Schweizer [Föllmer 91]. This measure is central in Chapter 4. We will also make the link with the semi-complete market setting, which is a special case of incomplete market. This paragraph is inspired from Schweizer [Schweizer 95, Schweizer 01].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual assumptions (right-continuity and completeness), with $T > 0$ finite. Let S be a $\mathbb{F}$-adapted $\mathbb{R}^d$-valued càdlàg process.

Definition 1.1.2. *A real-valued process* ρ *is a* martingale density *for* S *if* ρ *is a local* $\mathbb{P}$*-martingale with* $\rho_0 = 1$ $\mathbb{P} - a.s.$ *and such that* ρS *is a local* $\mathbb{P}$*-martingale.*

In the above definition (taken from [Schweizer 95]), it is always possible to take a càdlàg version of ρ. If S admits a martingale density ρ which is strictly positive (which is called a *strict martingale density*), then S is a $\mathbb{P}$-semimartingale.

Definition 1.1.3. *A* $\mathbb{R}^d$*-valued* $\mathbb{P}$*-semimartingale* S *satisfies the* structure condition *(SC) if it admits a canonical decomposition*

$$S = S_0 + M + A$$

with $M \in \mathcal{M}^2_{loc}(\mathbb{P})$, $A^i \ll \langle M^i \rangle \ll B$ with A^i having predictable densities θ^i for $i = 1 \dots d$ and some given càdlàg increasing process B null at 0, and there exists $\lambda \in L^2_{loc}(M)$ such that

$$\left[\frac{d\left\langle M^i, M^j \right\rangle_t}{dB_t}\right]_{1 \le i,j \le d} \cdot \lambda_t = \left[\theta^i_t \frac{d\left\langle M^i \right\rangle_t}{dB_t}\right]_{1 \le i \le d}.$$

It is always possible to find a process B as above. Schweizer [Schweizer 95] showed the following characterization of martingale densities.

Theorem 1.1.1. *Assume that S satisfies (SC). Then $\rho \in \mathcal{M}^2_{loc}(\mathbb{P})$ is a martingale density for S if and only if ρ satisfies the stochastic differential equation*

$$\rho_t = 1 - \int_0^t \rho_{s^-} \lambda_s dM_s + R_t \quad 0 \le t \le T$$

for some $R \in \mathcal{M}^2_{0,loc}(\mathbb{P})$ strongly orthogonal to M^i for $i = 1 \dots d$.

The natural interpretation is to see ρ as a change of measure. In the above theorem, one can take in particular $R = 0$, and $\widehat{\rho} := \mathcal{E}\left(-\int \lambda \cdot dM\right)$ is thus the density of a measure $\mathbb{Q}^{min} \ll \mathbb{P}$ with respect to $\mathbb{P}$. If $\widehat{\rho}$ is a martingale, then Q^{min} is called the *minimal local martingale measure* for S. If in addition we suppose that $\widehat{\rho}$ is square-integrable, Q^{min} is called the *minimal martingale measure* for S. This probability measure satisfies several criteria which are not detailed here. We provide here a sufficient condition for the uniqueness of $\mathbb{Q}^{min}$ (Theorem 7 in [Schweizer 95]). It also justifies the appellation of minimal measure.

Theorem 1.1.2. *Assume that S is continuous. Then it admits a strict martingale density if and only if S satisfies (SC). In that case, if*

$$\mathcal{H}(\mathbb{Q}^{min}|\mathbb{P}) := \mathbb{E}^{\mathbb{Q}^{min}}\left[\log \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}}\right] < +\infty,$$

then $\mathbb{Q}^{min}$ is the unique minimizer of $\left\{H(\mathbb{Q}|\mathbb{P}) - \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left[\left\langle \int \lambda \cdot dM \right\rangle_T\right]\right\}$ over all non negative $\mathbb{Q} \ll \mathbb{P}$ such that $\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}$ is a martingale density for S satisfying $\mathbb{E}^{\mathbb{Q}}\left[\left\langle \int \lambda \cdot dM \right\rangle_T\right] < +\infty$.

It is thus possible to identify the minimal (local) martingale measure given this criterion. In Chapter 4, we directly propose the equivalent martingale measure $\mathbb{Q}^{min}$ for the asset prices S. It appears from the above construction that in the financial setting, the minimal martingale measure affects the dynamics of the price S (which depends on the process

M) but let unchanged the orthogonal part in $\mathcal{F}$. This property is commented in Chapter 4. Note that if $\langle \int \lambda \cdot dM \rangle_T$ is deterministic, $\mathbb{Q}^{min}$ is also the unique minimizer of

$$D(\mathbb{Q}, \mathbb{P}) := \left(\text{Var} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \right] \right)^{1/2}$$

over all equivalent local martingale measures $\mathbb{Q}$ of $\mathbb{P}$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^2(\mathbb{R}_+^*, \mathbb{P})$. This is the case in the proposed model in Chapter 4.

The semi-complete market framework

Now, we benefit from the above notations to introduce the semi-complete market framework. This concept is used in Chapter 5. In his Thesis, Becherer [Becherer 01] defines a semi-complete market model as a complete financial sub-market and additional independent sources of risk. In what follows, we take the definitions from [Bouchard 11b]. Let us define the filtration $\mathcal{F}_t^S := \sigma \{S_s \; : \; 0 \leq s \leq t\}$. We assume without loss of generality that $(\mathcal{F}_t^S)_{0 \leq t \leq T}$ is completed and right-continuous. The filtration $(\mathcal{F}_t^S)_t$ is thus a subfiltration of $\mathbb{F}$, representing the information coming from the financial market.

Definition 1.1.4. *The financial market is complete if*

$$\mathbb{E}^{\mathbb{Q}}[H] = \mathbb{E}^{\mathbb{Q}'}[H] \text{ for all } \mathbb{Q}, \mathbb{Q}' \in \mathcal{M}(\mathbb{P}) \text{ and all } H \in L^\infty(\mathbb{R}, \mathcal{F}_T^S) \, .$$

Considering the above definitions, Definition 1.1.4 implies that for any martingale densities ρ and ρ' for S being true martingales, we have that

$$\mathbb{E}\left[\rho_t | \mathcal{F}_t^S\right] = \mathbb{E}\left[\rho'_t | \mathcal{F}_t^S\right] \, .$$

We finish this section by quickly saying that the minimal martingale measure appears in the semi-complete market setting in portfolio optimization problems, see [Becherer 01] and [Bouchard 11b].

1.2 Fundamental results

Arbitrage Pricing Theory has offered great improvements in the general theory of stochastic processes. The questions it raises involves many tools from convex analysis and topology. We start with the leading example of the theory, which is that the FTAP often relies on the Hahn-Banach selection theorem. We propose here the geometrical version of Hahn-Banach theorem one can find in [Brezis 83]. It will be used in Part 1.

Theorem 1.2.1 (Hahn-Banach selection theorem). *Let $A \subset E$ and $B \subset E$ be two non-empty disjoint convex subset of E, a topological vector space. Suppose that A is closed and B is compact. Then there exists a closed hyperplane which separate A and B in the strict sense.*

In the theory, the martingale deflator Z_T will play the role of the hyperplane generator, and A will denote the set of terminal attainable wealth of self financing portfolios. It will be associated to Theorem 1.2.8 to be applied to random sets. Nevertheless, we need the closedness property of this set to apply the theorem. This is where Arbitrage Pricing Theory provides enhancements.

1.2.1 Convergence lemmata

Let us first introduce a fundamental result [Komlós 67].

Theorem 1.2.2 (Komlos theorem). *Let $(\xi^n)_{n\geq 1}$ be a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ bounded in L^1, i.e., with $\sup_n \mathbb{E}\left[|\xi^n|\right] < \infty$. Then there exists a random variable $\xi \in L^1$ and a subsequence $(\xi^{n_k})_{k\geq 1}$ Césaro convergent to ξ a.s., that is, $k^{-1}\sum_{i=1}^{k} \xi^{n_i} \to \xi$ a.s. Moreover, the subsequence (ξ^n) can be chosen in such a way that any further subsequence is also Césaro convergent to ξ a.s.*

A fundamental result as a generalization of Komlos Theorem is due to Delbaen and Schachermayer [Delbaen 94].

Theorem 1.2.3. *Let $(\xi^n)_{n\geq 1}$ be a sequence of positive random variables. Then there exists a sequence $\eta^n \in \operatorname{conv}\{\xi^m, m \geq n\}$ and a random variable η with values in $[0, \infty]$ such that $\eta^n \to \eta$ a.s.*

The notation conv is used to define a closed convex set generated by the given elements. This theorem has recently been extended by Campi and Schachermayer [Campi 06] to be applied to finite variation predictable processes defined on a finite time interval $[0, T]$ and a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Theorem 1.2.4 (Campi-Schachermayer theorem). *Let V^n be a sequence of finite variation, predictable processes such that the corresponding sequence $(\mathrm{Var_T(V^n)})_{n\geq 1}$ is bounded in L^1 under some probability $\mathbb{Q} \sim \mathbb{P}$. then there exists a sequence $W^n \in \operatorname{conv}\{V^m, m \geq n\}$ such that W^n converges for a.e. ω for every $t \in [0, T]$ to a finite variation predictable process W^0.*

Here, $\mathrm{Var_T(V)}$ denotes the total absolute variation of the process V on $[0, T]$. This theorem is the essence to prove the closedness property of the set of attainable claims.

Let us introduce a last convergence result which is a simple property one can find in [Kabanov 04].

Theorem 1.2.5. *Let $\eta^n \in L^0$ taking values in $\mathbb{R}^d$ be such that $\bar{\eta} := \liminf_n |\eta^n| < \infty$. Then there are $\eta^{n(k)} \in L^0$ such that for all ω, the sequence $\eta^{n(k)(\omega)}(\omega)$ is a convergent subsequence of the sequence $\eta^n(\omega)$.*

This result can be turned over in order to prove that for an unbounded sequence, we can find a subsequence converging to infinity almost surely.

1.2.2 Fatou-convergence

All these convergence results are used to demonstrate a certain type of closedness property for a set of random variables. In Arbitrage Pricing Theory in continuous time, we use the specific notion of Fatou-convergence.

Definition 1.2.1. *A sequence in L^0 is said to be Fatou-convergent if it is uniformly bounded by below (in some specific sense if it is in a multidimensional space) and convergent almost surely.*

This concept allows to define naturally the Fatou-closedness concept. This concept is useful for a proper definition of closure for subset of L^0. The last set being infinite dimensional, it is not locally convex, which is the basis assumption for bipolar theorems. Fatou-closedness allows to easily obtain a closedness result in L^∞. This comes from the so-called Krein-Smulian Theorem (see Proposition 5.5.1 in [Kabanov 09]) :

Theorem 1.2.6 (Krein-Smulian theorem)**.** *Let $A \subset L^\infty$ be a convex set. Then*

$$A \text{ is weak* closed} \Leftrightarrow A \cap \{\xi \ : \ \|\xi\|_\infty < \kappa\} \text{ is closed in probability for every } \kappa.$$

If A is a subset of L^0 taking values in $\mathbb{R}^d$ and such that any elements of A are bounded by below, we have the following link between A and L^∞ :

Theorem 1.2.7. *If A is Fatou-closed, then the set $A \cap L^\infty$ is weak* closed.*

This theorem is the central tool to have applications of the FTAP, such as the super-replication theorem. It allows to apply Theorem 1.2.1.

1.2.3 Measurable selection

All the above results would only belong to the theory of convex analysis and topology if there was no random part in the manipulated objects. The central result we need to cite here is a measurable selection argument, which can be found in a raw form in ([Dellacherie 78], III-45) and under a more convenient form in [Kabanov 09]. It is a special case of the Jankov-von Neumann Theorem, see Th 18.22 in [Aliprantis 06].

Theorem 1.2.8. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, let $(E, \mathcal{E})$ be a Borel space and let $\Gamma \subset \Omega \times \mathbb{E}$ be an element of the σ-algebra $\mathcal{F} \otimes \mathcal{E}$. Then the projection Pr_Γ of Γ onto Ω is an element of $\mathcal{F}$, and there exists an E-valued random variable ξ such that $\xi(\omega) \in \Gamma_\omega$ for all non-empty ω-sections Γ_ω of Γ.*

Another lemma will be of great interest. It is has been rediscovered by Rásonyi [Rásonyi 08] to develop the concept of arbitrage of second kind in markets with proportional transaction costs with new tools. We recall this lemma since the results of Rásonyi [Rásonyi 08, Rásonyi 10] are central to this part of the thesis. In the following, B_1 denotes the unit ball of $\mathbb{R}^d$.

Lemma 1.2.1. *Let $\mathcal{G} \subset \mathcal{H} \subset \mathcal{F}$ be σ-algebras. Let $C \subset B_1$ be a $\mathcal{H}$-measurable random convex compact set. Then, there exists a $\mathcal{G}$-measurable random convex compact set $\mathbb{E}[C|\mathcal{G}] \subset B_1$ satisfying*

$$L^0(\mathbb{E}[C|\mathcal{G}], \mathcal{G}) = \{\mathbb{E}[\vartheta|\mathcal{G}] : \vartheta \in L^0(C, \mathcal{H})\}.$$

In the above definition, $L^0(\mathbb{E}[C|\mathcal{G}], \mathcal{G})$ denotes the set of $\mathcal{G}$-measurable random variables in L^0 taking values in $\mathbb{E}[C|\mathcal{G}]$ almost surely.

In Chapter 5, we also use a measurable selection theorem for optimization problems. Let $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ be two Borel spaces and let u be a bounded real-valued function on $X \times Y$. We are interested in a measurable map $f : X \mapsto Y$ such that

$$u(x, f(x)) \geq \sup_{y \in D(x)} u(x, y) - \varepsilon$$

for some $\varepsilon > 0$, where $D \subset X \times Y$ and $D(x)$ is the x-section of D. We appeal here to Theorem 3.1 and Corollary 3.1 in [Rieder 78] in the case of Example 2.4 in the latter. For a definition of a selection class, see [Rieder 78].

Theorem 1.2.9. *Let $\mathcal{L}$ be a selection class for $(\mathcal{B}(X), \mathcal{B}(Y)$. Assume that $D \in \mathcal{L}$ and $\{(x, y) \in D \ : \ u(x, y) \geq c\} \in \mathcal{L}$ for all $c \in \mathbb{R}$. Then for all $\varepsilon > 0$, there exists a measurable map $f : X \mapsto Y$ such that for all $x \in pD$, $f(x) \in D(x)$ and*

$$u(x, f(x)) \geq \begin{cases} \sup_{y \in D(x)} u(x, y) - \varepsilon & \text{if } \sup_{y \in D(x)} u(x, y) < +\infty \\ 1/\varepsilon & \text{if } \sup_{y \in D(x)} u(x, y) = +\infty \end{cases}.$$

Moreover, the map $x \mapsto \sup_{y \in D(x)} u(x, y)$ is measurable.

Chapitre 2

No marginal arbitrage for high production regime in discrete time

2.1 Introduction

As explained in the introduction, we are motivated by applications in optimal hedging of electricity derivatives for electricity producers. Electricity producers sell derivative contracts that allow them to buy electricity at different periods and at a price fixed in advance. In practice, the producer can deliver the required quantities of electricity either by producing it or by buying it on the spot market. He can also try to cover himself through future contracts, but the granularity of the available maturities on the market is in general insufficient.

It is a typical situation where a financial agent can manage a portfolio by either trading on a financial market or by producing a good himself. Such models have already been studied in the literature, in particular by Bouchard and Pham [Bouchard 05] who discussed the questions of no-arbitrage, super-hedging and expected utility maximization in a discrete time model with proportional transaction costs, see also Kabanov and Kijima [Kabanov 06] and the references therein.

As in Bouchard and Pham [Bouchard 05], we work in a discrete time model with proportional transaction costs. Although it does not need to be explicit in the model, we have in mind that the assets are divided in two classes : the pure financial assets and the ones that are used for production purposes. Both can be traded in the market but some of them can be consumed in order to produce other assets. For instance, coal can be traded on the market but is also used to produce electricity that can then be sold so as to provide currencies. The quantity used for production on the time period $[t, t+1]$ is chosen at time t. It gets out of the portfolio and enters a production process. Depending on the quantity used, a (random) return enters the portfolio at time $t+1$. Therefore, the main difference with Bouchard and Pham [Bouchard 05] is that we explicitly decide at each time what should be the regime of production, rather than letting it be determined just by inventories.

Obviously, both approaches could be combined. We refrain from doing this in this chapter in order to isolate the effect of our production model and to avoid too many unnecessary complexities.

As in [Bouchard 05], we first discuss the absence of arbitrage opportunity and its dual characterization. In [Bouchard 05], the authors adapt the notion of robust no-arbitrage introduced by Schachermayer [Schachermayer 04]. It essentially means that there is still no-arbitrage even if transaction costs are slightly reduced and production returns are slightly increased. In the last section of this chapter, we adapt the arguments of [Bouchard 05] to our context, and prove that there is no difficulty to do so. However, we prefer to adopt along the chapter the (more natural) notion of no-arbitrage of second kind, which was recently introduced in the context of financial markets with transactions costs by Rásonyi [Rásonyi 10] under the name of *no-sure gain in liquidation value*, see also Denis and Kabanov [Denis 11b] for a continuous time version. It says that we cannot turn a position which is not solvent at time t into a position which is a.s. solvent at a later time T by trading on the market. In models without transaction costs, this corresponds to the usual notion of no-arbitrage.

Another difference with Bouchard and Pham [Bouchard 05] is that we allow for reasonable arbitrages due to the production possibilities. Here, reasonable means that it may be possible to have a.s. *positive* net returns for low production regimes. However, they should be limited in the sense that *marginal* arbitrages for high production regimes are not possible. The way we model this consists in assuming that the production function $\beta \to R(\beta)$ admits an affine upper bound $\beta \to c + L\beta$, which is somehow sharp for large values of β, and that the linear model in which R is replaced by L admits no arbitrage of second kind. In the case where each component of R is concave, we have in mind that it should hold for L such $\lim_{\alpha\to\infty} R(\alpha\beta)/\alpha = L\beta$ (whenever it makes sense), i.e. no-arbitrage holds in a marginal way for large regimes β. From the economic point of view, this means that gains can be made from the production in reasonable situations, but that it becomes (marginally) risky when the regime of production is pushed too high.

Note that our approach is different from the notions of no *marginal* and no *scalable* arbitrage studied in Pennanen and Penner [Pennanen 10] in the context of market models with convex trading cost functions, see also Pennanen [Pennanen 11] and the references therein. The differences will be highlighted in Remark 2.2.4 below.

From the mathematical point of view, it allows to reduce at first to a linear model for which a nice dual formulation of the no-arbitrage condition is available, in the sense that the set of dual variables can be fully described in terms of martingales evolving in appropriate sets. This is not the case for non-linear models, compare with [Bouchard 05]. They are constructed by following the arguments of Rásonyi [Rásonyi 10] which do not require to prove the closedness of the set of attainable claims a-priori. Once they are constructed, one can then show that the set of attainable claims is indeed closed in probability in the linear and in the original models. As usual this leads to a dual formulation of these sets, and can also be used to prove existence for expected utility maximization problems, which, in particular, opens the door to the study of indifference prices.

We refer to Kabanov and Safarian [Kabanov 09] for a wide overview of models with proportional transaction costs. See also Pennanen and Penner [Pennanen 10] and Rásonyi [Rásonyi 08] for some more recent results in discrete time.

The rest of the chapter is organized as follows. We first describe our model, state the dual characterization of our no-arbitrage condition and important closedness properties in Section 2.2. Section 2.3 discusses applications to super-hedging and utility maximization problems. We then develop a model corresponding to this framework directly inspired from the first part of the thesis. The proofs are collected in Section 2.5. In order to ensure exhaustion, we propose as an additional section to study the robust no-arbitrage condition for our model and the necessary closedness property of the set of attainable claims under this condition.

2.2 Definitions and main results

2.2.1 Model description

From now on we denote by $T \in \mathbb{N}\setminus\{0\}$ a fixed time horizon and set $\mathbb{T} := \{0, 1, \ldots, T\}$. The complete filtration of the investor, $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$, is supported by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $\mathcal{F}_T = \mathcal{F}$ and that $\mathcal{F}_0$ is trivial.

As in [Schachermayer 04], we model exchange prices by an adapted process $\pi = (\pi_t)_{t\in\mathbb{T}}$ taking values in the set $\mathbb{M}^d$ of square d-dimensional matrices, for some $d \geq 1$, satisfying the following conditions for all $t \le T$ and $i, j, k \le d$:

$$\text{(i) } \pi_t^{ij} > 0, \ \text{(ii) } \pi_t^{ii} = 1, \ \text{(iii) } \pi_t^{ij}\pi_t^{jk} \ge \pi_t^{ik} \,. \tag{2.2.1}$$

Here, π_t^{ij} should be interpreted as the number of units of asset i required to obtain one unit of asset j at time t. The conditions (i) and (ii) need no comment. The third condition is also natural, it means that it is always cheaper to buy directly units of asset k from units of asset i rather then going through the asset j. Note that, combined with (ii), it implies that $\pi_t^{ij}\pi_t^{ji} \ge 1$, which means that the ask price is always greater than the bid price. The case where $\pi_t^{ij}\pi_t^{ji} = 1$ corresponds to the situation where the ask and bid prices are the same, i.e. there is no friction.

All over this paper, we shall consider the so-called *efficient friction* case :

Assumption 2.2.1. $\pi_t^{ij}\pi_t^{ji} > 1$ *for all* $i \ne j \le d$ *and* $t \in \mathbb{T}$.

It means that ask prices are always strictly greater than bid prices.

As in [Kabanov 02] and [Kabanov 03], we model portfolios as d-dimensional processes, each component i corresponding to the number of units of asset i held. The composition of a portfolio holding V_t at time t can be changed by acting on the financial market. If ξ_t denotes the net number of additional units of each asset in the portfolio after trading at time t, it should satisfy the standard self-financing condition. In our context, this means that $\xi_t \in -K_t$, whenever we allow to throw away a non-negative number of the holdings, where, for each $\omega \in \Omega$,

$$K_t(\omega) := \operatorname{conv}\left(\pi^{ij}(\omega)e_i - e_j, e_i \,;\, i, j \le d\right) \,, \tag{2.2.2}$$

where e_i stands for the i-th unit vector of $\mathbb{R}^d$ defined by $e_i^k = 1_{i=k}$.

Note that $V_t \in K_t$ means that there exists $\xi_t \in -K_t$ such that $V_t + \xi_t = 0$. This explains why K_t is usually referred to as the *solvency cone*, i.e. the set of positions that can be turned into positions with non-negative entries by immediately trading on the market.

As in Bouchard and Pham [Bouchard 05], we also allow for production. In [Bouchard 05], the production regime depends only on the inventories in some production assets. Here, we consider a different approach based on a full control of the production regimes. Namely, we consider a family of random maps $(R_t)_{t\in\mathbb{T}}$ from $\mathbb{R}_+^d$ into $\mathbb{R}^d$ which corresponds to production functions. It turns β_t units of assets taken from the portfolio at time t into $R_{t+1}(\beta_t)$ additional units of assets in the portfolio at time $t+1$. For the moment, we only assume that R_{t+1} is $\mathcal{F}_{t+1}$ measurable, in the sense that $R_{t+1}(\beta) \in L^0(\mathbb{R}^d, \mathcal{F}_{t+1})$

for all $\beta \in L^0(\mathbb{R}^d_+, \mathcal{F}_t)$. The control β_t can be associated to a *regime of production*. Componentwise, the greater β_t gets, the more the producer is putting into the production system.

All together, a strategy is a pair of adapted processes

$$(\xi, \beta) \in \mathcal{A}_0 := L^0((-K) \times \mathbb{R}^d_+, \mathbb{F}),$$

i.e. such that $(\xi_t, \beta_t) \in L^0((-K_t) \times \mathbb{R}^d_+, \mathcal{F}_t)$ for all $0 \leq t \leq T$. The corresponding portfolio process, starting from 0, can be written as $V^{\xi,\beta} = (V^{\xi,\beta}_t)_{t\in\mathbb{T}}$ where

$$V^{\xi,\beta}_t := \sum_{s=0}^{t} (\xi_s - \beta_s + R_s(\beta_{s-1})1_{s\geq 1}) \ . \tag{2.2.3}$$

Remark 2.2.1. Observe that we do not impose constraints on portfolio processes. In particular, one can consume some asset for production purposes although we do not hold them. This means that one can borrow some units of assets to use them in the production system. As usual additional convex constraints could be introduced without much difficulty.

In the following, we shall denote by

$$\mathfrak{X}^R_t(T) := \left\{ \sum_{s=t}^{T} \xi_s - \beta_s + R_s(\beta_{s-1})1_{s\geq t+1} \ , \ (\xi, \beta) \in \mathcal{A}_0 \right\} \ , \ t \leq T \ , \tag{2.2.4}$$

the set of portfolio holdings that are attainable at time T by trading from time t with a zero initial holding.

Remark 2.2.2. The sequence of random cones $K = (K_t)_{t\in\mathbb{T}}$ is defined here through the bid-ask process π. However, it should be clear that all our analysis would remain true in a more abstract framework. Namely, one could only consider that K is a sequence of closed convex cones such that K_t is $\mathcal{F}_t$-measurable, $\mathbb{R}^d_+ \subset K_t$ and $K_t \cap (-K_t) = \{0\}$ for all $t \leq T$.

2.2.2 The no-arbitrage condition

In a model without production, i.e. $R \equiv 0$, it was recently proposed by Rásonyi [Rásonyi 10] to consider the following *no-arbitrage of second kind condition*, also called *no-sure gain in liquidation value*, **NGV** in short :

$\mathbf{NA2}^0$: $(\zeta + \mathfrak{X}^0_t(T)) \cap L^0(K_T, \mathcal{F}) \neq \{0\} \Rightarrow \zeta \in L^0(K_t, \mathcal{F})$, for all $\zeta \in L^0(\mathbb{R}^d, \mathcal{F}_t)$ and $t \leq T$.

It means that we cannot end-up at time T with a solvable position without taking any risk if the initial position was not already solvable.

In this paper, we shall impose a similar condition on the pure financial part of the model, i.e. there is no-arbitrage of second kind for strategies of the form $(\xi, 0) \in \mathcal{A}_0$. Contrary to [Bouchard 05], we do not exclude arbitrages coming from the production whenever the production regime is small. We only exclude marginal arbitrages for high regimes of production in the following sense :

Definition 2.2.1. 1. *Given* $L \in L^0(\mathbb{M}^d, \mathbb{F})$*, we say that there is* no arbitrage of second kind *for the linear production map* L*, in short* $\mathbf{NA2}^L$ *holds, if*
(i) $\zeta - \beta + L_{t+1}\beta \in L^0(K_{t+1}, \mathcal{F}_{t+1}) \Rightarrow \zeta \in K_t$,
(ii) $-\beta + L_{t+1}\beta \in L^0(K_{t+1}, \mathcal{F}_{t+1}) \Rightarrow \beta = 0$,
for all $(\zeta, \beta) \in L^0(\mathbb{R}^d \times \mathbb{R}^d_+, \mathcal{F}_t)$ *and* $t < T$.
2. *We say that there is* no marginal arbitrage of second kind for high production regimes*, in short* **NMA2** *holds, if there exists* $(c, L) \in L^0(\mathbb{R}^d, \mathbb{F}) \times L^0(\mathbb{M}^d, \mathbb{F})$ *such that* $\mathbf{NA2}^L$ *holds and*

$$c_{t+1} + L_{t+1}\beta - R_{t+1}(\beta) \in L^0(K_{t+1}, \mathcal{F}_{t+1}) \text{ for all } \beta \in L^0(\mathbb{R}^d_+, \mathcal{F}_t) \text{ and } t < T. \quad (2.2.5)$$

The condition (2.2.5) means that the production function R_t admits an affine upper-bound. In most production models, the map R_t is concave (component by component) and therefore typically admits such a bound. In (i) and (ii), we focus on the production model where R is replaced with the linear map associated to L. The fact that we consider the production map $\beta \mapsto L_{t+1}\beta$ instead of $\beta \mapsto c_{t+1} + L_{t+1}\beta$ coincides with the idea that we only want to avoid arbitrages for high production regimes : for large values of $|L_{t+1}\beta|$, $|c_{t+1}|$ becomes negligible.
For $L \equiv 0$, the condition (i) is equivalent to the **NGV** condition of [Rásonyi 10], this follows from a simple induction under the standing Assumption 2.2.1 above. Our version is a simple extension to the production-investement model. The condition (i) means that, even if we produce, we cannot have for sure a solvable position at time $t+1$ if the position was not already solvable at time t. The condition (ii) means that producing may lead to net *losses*.

In the following, unless otherwise specified, we shall consider (c, L) has given once for all, and such that (2.2.5) is satisfied (whenever **NMA2** holds). We shall refer to the linear model as the one where R is replaced by $\beta \mapsto L\beta$.

Remark 2.2.3. If $\operatorname{esssup}\{|R_{t+1}(\beta)|, \beta \in L^0(\mathbb{R}^d_+, \mathcal{F})\} \in L^\infty$ for all $t < T$, then one can choose $L \equiv 0$. In this case, **NMA2** coincides with the **NGV** condition of [Rásonyi 10] on the pure financial part, i.e. the no-arbitrage condition is set only on strategies of the form $(\xi, 0)$. This will have some consequence in Chapter 3.

We conclude this section with a remark that highlights the differences between the notion of *no marginal arbitrage for high production regimes* introduced here and the (seemingly close) notions of *no marginal arbitrage* and *no scalable arbitrage* discussed in [Pennanen 11].

Remark 2.2.4. 1. In [Pennanen 11], see also the references therein, the author discusses the notion of *no marginal arbitrage* in the context of discrete time models with stock prices depending in a convex way of the quantity to buy/sell. In the terminology of this paper, a *marginal arbitrage* has to be understood as an arbitrage obtained when trading the marginal price process associated to infinitesimal trades. In our context, where the non-linearity only comes from the production map R, this would (essentially) correspond to an arbitrage obtained for infinitesimal values of β, i.e. marginally around $\beta = 0$. Here, we also consider arbitrages that can happen marginally, but, as explained above, as a "surplus" around large regimes/values of β and not around 0. This explain why we use the terminology of marginal arbitrage *for high production regimes*. This clearly differentiate the two (very) different notions.
2. In [Pennanen 11], the author also discusses the notion of *no scalable arbitrage*. It expresses the fact that an arbitrage cannot be arbitrarily scaled by a positive scalar. In our setting, the *no scalable arbitrage* condition would read :

$$\bigcap_{\alpha>0} \alpha \mathfrak{X}_0^R(T) \cap L^0(\mathbb{R}_+^d, \mathcal{F}) = \{0\}.$$

For real valued concave maps R satisfying $R(0) = 0$, the *no scalable arbitrage* condition (essentially) means that the usual no-arbitrage condition holds when considering the production map $\beta \mapsto \nabla R(\infty)\beta$, whenever we can give a sense to the gradient ∇R and it admits a limit at infinity. In this case, with $L := \nabla R(\infty)$ in **NMA2** , we see that (at least formally) our no marginal arbitrage of second kind condition for high production regimes, could be viewed as a no scalable arbitrage of second kind condition.
This is not the case in general. Apart from technicalities (for instance, we do not assume here any concavity, except for the super-hedging theorems of Section 2.3.1), the main reason is that we are not interested by arbitrages that are scalable but by arbitrages that can appear marginally as a "surplus" given that the production regime is already high. To illustrate this, let us consider a very simple (degenerate) two dimensional model with two periods $t = 0, 1$. We take $\pi_t^{12} = 2$ and $\pi_t^{21} = 1$ for $t = 0, 1$, $R_1^1(\beta) = -\bar{c} + \bar{L}_1\beta^1$ and $R^2 = 0$ where $\bar{c} > 0$ is a constant and $\mathbb{P}\left[\bar{L}_1 = 1\right] = 0$. This model satisfies (2.2.5) with

$c_1 = (-\bar{c}, 0)$, $L_1^{11} = \bar{L}_1$, $L_1^{ij} = 0$ for $(i,j) \neq (1,1)$. In this model, direct computations show that a claim of the form $g = (\lambda_g(\bar{L}_1 - 1), 0)$, with $\lambda_g > 0$, is scalable, i.e. belongs to $\cap_{\alpha>0}\alpha\mathfrak{X}_0^R(T)$, if and only if, for each $\alpha > 0$, one can find $\beta_0^{1\alpha} \in \mathbb{R}_+$ and $\gamma^\alpha \in L^0(\mathbb{R}_+, \mathcal{F}_1)$ such that $\beta_0^{1\alpha} = \lambda_g/\alpha + (\bar{c}+\gamma^\alpha)/(\bar{L}_1 - 1)$. Because $\bar{c} > 0$ and γ^α has to take non-negative values, this is not possible, except in the case where $\bar{L}_1$ is not random (otherwise $\beta_0^{1,\alpha}$ would be a random variable as opposed to a real number). This shows that such claims are not scalable (in general) in the sense that they do not belong to $\cap_{\alpha>0}\alpha\mathfrak{X}_0^R(T)$. Hence, the *no scalable arbitrage* condition does not (in general) say anything on such claims, while our **NMA2** condition says exactly that they cannot belong to $L^0(\mathbb{R}_+^2, \mathcal{F}_1) \setminus \{0\}$.

2.2.3 Dual characterization of the no-arbitrage condition and closedness properties

Before we state our main results, let us introduce some additional notations and definitions.

We first define the positive dual cone process $K^* = (K_t^*)_{t\in\mathbb{T}}$ associated to K by

$$K_t^*(\omega) := \left\{ z \in \mathbb{R}^d \;:\; x'z \geq 0 \;\text{ for all } x \in K_t(\omega) \right\} \;,\; \omega \in \Omega \,.$$

For $t \leq \tau \leq T$, we denote by $\mathcal{M}_t^\tau(\text{int}K^*)$ the set of martingales Z with positive components satisfying $Z_s \in L^0(\text{int}K_s^*, \mathcal{F}_s)$ for all $t \leq s \leq \tau$.

Elements of $\mathcal{M}_t^T(\text{int}K^*)$ were called *strictly consistent price systems*, on $[t,T]$, in [Schachermayer 04]. They have the standard interpretation to be associated to a system of prices in a fictitious market without transaction costs that admits a martingale measure, and such that the relative prices evolve in the interior of the corresponding bid-ask intervals of the original model induced by π, i.e. are more favorable for the financial agent. Indeed, one easily checks that

$$K_t^*(\omega) := \left\{ z \in \mathbb{R}_+^d \;:\; z^j \leq z^i \pi_t^{ij}(\omega) \;\text{ for all } i \neq j \leq d \right\} \,. \tag{2.2.6}$$

Otherwise stated, given $Z \in \mathcal{M}_t^T(\text{int}K^*)$, the process $\bar{Z}$, defined by $\bar{Z}_s^i := Z_s^i/Z_s^1$ for $t \leq s \leq T$, i.e. where the first asset is taken as a numéraire, is a martingale on $[t,T]$ under the measure $\mathbb{Q}$ induced by the conditional density process $(Z_s^1/Z_t^1)_{t\leq s\leq T}$ and satisfies $\bar{Z}_s^j/\bar{Z}_s^i < \pi_s^{ij}$ for $t \leq s \leq T$.

Remark 2.2.5. Note that the Assumption 2.2.1 above implies that, and is actually equivalent to, $\text{int}K_t^* \neq \emptyset$ for all $t \leq T$. This follows from (2.2.6).

Altogether, elements of $\mathcal{M}_0^T(\text{int}K^*)$ play a similar role as equivalent martingale measures in frictionless markets, see e.g. [Schachermayer 04] and the references therein. In particular, it was shown in [Rásonyi 10] that, for $L \equiv 0$, the no-arbitrage condition $\mathbf{NA2}^0$ is equivalent to :

$\mathbf{PCE}^0$: for each $0 \leq t \leq T$ and $X \in L^1(\text{int}K_t^*, \mathcal{F}_t)$, there exists a process $Z \in \mathcal{M}_t^T(\text{int}K^*)$ satisfying $Z_t = X$.

This not only means that the no-arbitrage condition $\mathbf{NA2}^0$ implies the existence of a strictly consistent price system, but that strictly consistent price systems defined on any subinterval $[t, \tau]$ can also be extended consistently on $[t, T]$: for $Z \in \mathcal{M}_t^\tau(\text{int}K^*)$, one can find a strictly consistent price system $\tilde{Z} \in \mathcal{M}_t^T(\text{int}K^*)$ such that $\tilde{Z} = Z$ on $[t, \tau]$. Such a property is obvious in frictionless markets but in general not true in our multivariate setting where the geometry of the cones $(K_t^*)_{t\in\mathbb{T}}$ is non-trivial.

In our production-investment setting, such price systems should also take into account the production function. When it is linear, given by the random matrix process L, the cost in units at time t of a return (in units) $L_{t+1}\beta$ at time $t{+}1$ is $\beta \in L^0(\mathbb{R}_+^d, \mathcal{F}_t)$. Otherwise stated, one can build the position $(L_{t+1} - I_d)\beta$ at time $t+1$ from a zero holding at time t. For the price system $\bar{Z}$ and the associated pricing measure $\mathbb{Q}$, see the discussion above, the value at time t of this return is $\mathbb{E}^{\mathbb{Q}}[\bar{Z}'_{t+1}(L_{t+1} - I_d)\beta \mid \mathcal{F}_t]$. If the fictitious price system is strictly more favorable than the original one, one should actually be able to choose it in such a way that $\mathbb{E}^{\mathbb{Q}}[\bar{Z}'_{t+1}(L_{t+1} - I_d)\beta \mid \mathcal{F}_t] < 0$ for all $\beta \in L^0(\mathbb{R}_+^d, \mathcal{F}_t) \setminus \{0\}$.

The above discussion leads to the introduction of the set $\mathcal{L}_t^\tau(\text{int}\mathbb{R}_-^d)$ of martingales Z on $[t, \tau]$ with positive components satisfying $\mathbb{E}\left[|Z'_{s+1}(L_{s+1} - I_d)| \mid \mathcal{F}_s\right] < \infty$ as well as $\mathbb{E}\left[Z'_{s+1}(L_{s+1} - I_d) \mid \mathcal{F}_s\right] \in \text{int}\mathbb{R}_-^d$ for all $t \leq s < \tau$, $t \leq \tau \leq T$ $\mathbb{P}-$a.s.

Our first main result extends the property $\mathbf{NA2}^0 \Leftrightarrow \mathbf{PCE}^0$ to $\mathbf{NA2}^L \Leftrightarrow \mathbf{PCE}^L$ where $\mathbf{PCE}^L$: for each $0 \leq t \leq T$ and $X \in L^1(\text{int}K_t^*, \mathcal{F}_t)$, there exists a process $Z \in \mathcal{M}_t^T(\text{int}K^*) \cap \mathcal{L}_t^T(\text{int}\mathbb{R}_-^d)$ satisfying $Z_t = X$.

Theorem 2.2.1. $\mathbf{NA2}^L \Leftrightarrow \mathbf{PCE}^L$.

Remark 2.2.6. Note that the property $\mathbf{PCE}^L$ allows one to construct (in theory) all the elements of $\mathcal{M}_0^T(\text{int}K^*) \cap \mathcal{L}_0^T(\text{int}\mathbb{R}_-^d)$ by a simple forward induction. First, one can start with any $Z_0 \in \text{int}K_0^*$. Assuming that a given $Z \in \mathcal{M}_0^t(\text{int}K^*) \cap \mathcal{L}_0^t(\text{int}\mathbb{R}_-^d)$ has been constructed, one can then choose any random variables $Z_{t+1} \in L^0(\text{int}K_{t+1}^*, \mathcal{F}_{t+1})$ such that $\mathbb{E}\left[Z_{t+1} \mid \mathcal{F}_t\right] = Z_t$ and $\mathbb{E}\left[Z'_{t+1}(L_{t+1} - I_d) \mid \mathcal{F}_t\right] \in \text{int}\mathbb{R}_-^d$. This corresponds to simple linear inequalities. When Ω is finite, the set of such random variables can be described explicitly.

By similar arguments as developed in Lemma 12 in [Campi 06], the existence of $Z \in$

$\mathcal{M}_0^T(\text{int}K^*)\cap\mathcal{L}_0^T(\text{int}\mathbb{R}^d_-)$ then allows to provide a L^1 upper-bound on strategies $(\xi,\beta)\in\mathcal{A}_0$ satisfying $V_T^{\xi,\beta}+\kappa\in K_T$ for some $\kappa\in\mathbb{R}^d$. However, because no integrability condition is imposed a-priori on c, it requires the additional assumption :

$$\exists\,\check{Z}\in\mathcal{M}_0^T(\text{int}K^*)\cap\mathcal{L}_0^T(\text{int}\mathbb{R}^d_-)\text{ s.t. }\mathbb{E}\left[|\check{Z}'_Tc_t|\right]<\infty\;\forall\,0<t\leq T\;. \tag{2.2.7}$$

Lemma 2.2.1. *Assume that* (2.2.7) *holds. Then, there exists* $\mathbb{Q}\sim\mathbb{P}$ *and a constant* $\alpha\geq 0$*, such that, for all* $\kappa\in\mathbb{R}^d$ *and* $(\xi,\beta)\in\mathcal{A}_0$ *satisfying* $V_T^{\xi,\beta}+\kappa\in K_T$*, one has :*

$$\mathbb{E}^{\mathbb{Q}}\left[\sum_{0\leq t\leq T}(|\xi_t|+|\beta_t|)\right]\leq\alpha\left(\mathbb{E}\left[\check{Z}_TC_0^T\right]+\check{Z}'_0\kappa\right)$$

where

$$C_t^T:=\sum_{s=t+1}^{T}c_s\;\;,\;\;t<T\;. \tag{2.2.8}$$

Remark 2.2.7. Given $(\xi,\beta)\in\mathcal{A}_0$, let us denote by

$$\mathcal{V}_t^{\xi,\beta}:=\sum_{s=0}^{t}\left(\xi_s-\beta_s+L_s\beta_{s-1}1_{s\geq1}\right)\;. \tag{2.2.9}$$

In view of Theorem 2.2.1, applying Lemma 2.2.1 to the case $R(\beta)=0+L\beta$, i.e. $c=0$, leads to the following corollary : Assume that $\mathbf{NA2}^L$ holds. Then, there exists $\mathbb{Q}\sim\mathbb{P}$, $Z_0\in\text{int}K_0^*$ and a constant $\alpha\geq0$ such that, for all $\kappa\in\mathbb{R}^d$ and $(\xi,\beta)\in\mathcal{A}_0$ satisfying $\mathcal{V}_T^{\xi,\beta}+\kappa\in K_T$, one has :

$$\mathbb{E}^{\mathbb{Q}}\left[\sum_{0\leq t\leq T}(|\xi_t|+|\beta_t|)\right]\leq\alpha Z'_0\kappa\;.$$

The last remark combined with Komlos Lemma readily implies that the sets

$$\mathfrak{X}_t^L(T):=\left\{\sum_{s=t}^{T}\left(\xi_s-\beta_s+L_s(\beta_{s-1})1_{s\geq t+1}\right),\;(\xi,\beta)\in\mathcal{A}_0\right\},$$

are Fatou-closed, in the sense that the limit in probability of sequences of elements $(g_n)_{n\geq1}\subset\mathfrak{X}_t^L(T)$ satisfying $g_n+\kappa\in K_T$ for all $n\geq1$ belongs to $\mathfrak{X}_t^L(T)$ as well. Under (2.2.7), a similar result could be easily proved by appealing to Lemma 2.2.1 for the sets $\mathfrak{X}_t^R(T)$, recall (2.2.4), under the following upper-semicontinuity assumption :

Assumption 2.2.2. *We assume that for all* $\beta^0\in\mathbb{R}^d_+$ *andt* $\leq T$*,*

$$\limsup_{\beta\in\mathbb{R}^d_+,\beta\to\beta^0}\;R_t(\beta)-R_t(\beta^0)\in-K_t\;.$$

where the limsup is taken component by component. Such Fatou-closure properties are well-enough for applications, however it requires (2.2.7). In order to deal with the general case, i.e. when (2.2.7) may not hold, we shall need to use more sophisticated arguments, which actually allow to show the following stronger closedness property.

Theorem 2.2.2. *$\mathfrak{X}_0^L(T)$ is closed in probability under* **NA2**^{L}. *The same holds for $\mathfrak{X}_0^R(T)$ under* **NMA2** *and Assumption 2.2.2.*

2.3 Applications

2.3.1 Super-hedging theorems

As usual, the closedness property allows to provide dual formulations for the set of attainable claims. We first formulate it in the linear model. In this section, we denote by $\mathcal{M}_0^T(K^*)$ the set of martingales Z satisfying $Z_s \in L^0(K_s^*, \mathcal{F}_s)$ for all $s \leq T$, and by $\mathcal{L}_0^T(\mathbb{R}_-^d)$ the set of martingales Z with non-negative components satisfying $\mathbb{E}\left[|Z'_{s+1}(L_{s+1} - I_d)| \mid \mathcal{F}_s\right] < \infty$ and $\mathbb{E}\left[Z'_{s+1}(L_{s+1} - I_d) \mid \mathcal{F}_s\right] \in \mathbb{R}_-^d$ for all $s < T$.

Proposition 2.3.1. *Assume that* **NA2**^{L} *holds and let $V \in L^0(\mathbb{R}^d, \mathcal{F})$ be such that $V + \kappa \in L^0(K_T, \mathcal{F})$ for some $\kappa \in \mathbb{R}^d$. Then the following assertions are equivalent :*
(i) $V \in \mathfrak{X}_0^L(T)$,
(ii) $\mathbb{E}\left[Z'_T V\right] \leq 0$ *for all* $Z \in \mathcal{M}_0^T(K^*) \cap \mathcal{L}_0^T(\mathbb{R}_-^d)$,
(iii) $\mathbb{E}\left[Z'_T V\right] \leq 0$ *for all* $Z \in \mathcal{M}_0^T(\text{int}K^*) \cap \mathcal{L}_0^T(\text{int}\mathbb{R}_-^d)$.

In the original non-linear model, an abstract dual formulation is also available. However, due to the non-linearity of the set of attainable terminal claims, it requires the introduction of the following support function :

$$\alpha^R(Z) := \sup\left\{\mathbb{E}\left[Z'_T V\right],\ V \in \mathfrak{X}_{0b}^R(T)\right\}\ ,\ Z \in \mathcal{M}_0^T(K^*)\ ,$$

where

$$\mathfrak{X}_{0b}^R(T) := \left\{V \in \mathfrak{X}_0^R(T) \text{ s.t. } V + \kappa \in K_T \ \text{ for some } \ \kappa \in \mathbb{R}^d\right\}.$$

Remark 2.3.1. 1. It will be clear from the proof in Section 2.5.2, see (2.5.6) with $\varepsilon = 0$, that $\alpha^R(Z) \leq \mathbb{E}\left[Z'_T C_0^T\right]$ for all $Z \in \mathcal{M}_0^T(K^*) \cap \mathcal{L}_0^T(\mathbb{R}_-^d)$, whenever the last term is well-defined, which is in particular the case if c_t is essentially bounded from below, component by component, for each $t \leq T$.
2. Let α^L be defined as α^R in the case $R(\beta) = 0 + L\beta$. Since $0 \in \mathfrak{X}_0^L(T)$, we have $\alpha^L \geq 0$. On the other hand, 1. applied to $R(\beta) = 0 + L\beta$, i.e. $c = 0$, implies that $\alpha^L(Z) \leq 0$ for all $Z \in \mathcal{M}_0^T(K^*) \cap \mathcal{L}_0^T(\mathbb{R}_-^d)$. Hence, $\alpha^L(Z) = 0$ for all $Z \in \mathcal{M}_0^T(K^*) \cap \mathcal{L}_0^T(\mathbb{R}_-^d)$.

Moreover, as usual, we shall need the set $\mathfrak{X}_0^R(T)$ to be convex, which is easily checked under the additional assumption **(R)**(a) below. We will also need that bounded strategies lead to L^1-bounded from below terminal wealth values. We therefore impose the following conditions.

Assumption 2.3.1. *We assume the following :*
(a) For all $\alpha \in L^0([0,1],\mathcal{F})$, $\beta_1, \beta_2 \in L^0(\mathbb{R}^d_+,\mathcal{F})$ *and* $t \le T$, *we have*

$$\alpha R_t(\beta_1) + (1-\alpha)R_t(\beta_2) - R_t(\alpha\beta_1 + (1-\alpha)\beta_2) \in -K_t \ .$$

(b) For all $t \le T$ *and* $\beta \in L^\infty(\mathbb{R}^d_+,\mathcal{F})$, $R_t^-(\beta) \in L^1(\mathbb{R}^d,\mathcal{F})$ *where we used the notation* $R^- := (\max\{-R^i, 0\})_{i\le d}$.

Remark 2.3.2. The technical Assumption 2.3.1(b) is by no means restrictive. One can for instance reduce to it whenever there exists a deterministic map $\psi : \mathbb{R}^d_+ \mapsto [1,\infty)$ such that $\text{esssup}\{|R_t^-(\beta)|/\psi(\beta),\ t \le T,\ \beta \in \mathbb{R}^d_+\} =: \eta \in L^0(\mathbb{R}_+,\mathcal{F})$. Indeed, in this case, it suffices to replace the original probability measure $\mathbb{P}$ by $\tilde{\mathbb{P}} \sim \mathbb{P}$ defined by $d\tilde{\mathbb{P}}/d\mathbb{P} = e^{-\eta}/\mathbb{E}\left[e^{-\eta}\right]$. Since $\tilde{\mathbb{P}} \sim \mathbb{P}$, this does not affect the conditions $\mathbf{NA2}^L$, Assumption 2.2.2 and Assumption 2.3.1(a).

Proposition 2.3.2. *Assume that* **NMA2** , *Assumption 2.2.2 and Assumption 2.3.1 hold. Fix* $V \in L^0(\mathbb{R}^d,\mathcal{F})$ *such that* $V + \kappa \in L^0(K_T,\mathcal{F})$, *for some* $\kappa \in \mathbb{R}^d$, *and consider the following assertions :*

(i) $V \in \mathfrak{X}_0^R(T)$,
(ii) $\mathbb{E}\left[Z_T' V\right] \le \alpha^R(Z)$ *for all* $Z \in \mathcal{M}_0^T(K^*)$,
(iii) $\mathbb{E}\left[Z_T' V\right] \le \alpha^R(Z)$ *for all* $Z \in \mathcal{M}_0^T(\text{int}K^*)$.

Then, (i)$\Leftrightarrow$ (ii)$\Rightarrow$ (iii). *If moreover there exists* $Z \in \mathcal{M}_0^T(\text{int}K^*)$ *such that* $\alpha^R(Z) < \infty$, *then* (iii)$\Rightarrow$ (ii).

In the case where the linear map L coincides with the asymptotic behavior of R.

Assumption 2.3.2. *We assume that for all* $\beta \in \mathbb{R}^d_+$ *and* $t \le T$,

$$\lim_{\eta\to\infty} R_t(\eta\beta)/\eta = L_t\beta \ .$$

one can restrict to elements in $\mathcal{L}_0^T(\mathbb{R}^d_-)$ (resp. $\mathcal{L}_0^T(\text{int}\mathbb{R}^d_-)$) in the above dual formulations.

Proposition 2.3.3. *Let the conditions of Proposition 2.3.2 hold. Assume further that Assumption 2.3.2 is satisfied. Fix* $V \in L^0(\mathbb{R}^d,\mathcal{F})$ *such that* $V + \kappa \in L^0(K_T,\mathcal{F})$, *for some* $\kappa \in \mathbb{R}^d$, *and consider the following assertions :*

(i) $V \in \mathfrak{X}_0^R(T)$,

(ii) $\mathbb{E}[Z_T' V] \le \alpha^R(Z)$ *for all* $Z \in \mathcal{M}_0^T(K^*) \cap \mathcal{L}_0^T(\mathbb{R}_-^d)$,

(iii) $\mathbb{E}[Z_T' V] \le \alpha^R(Z)$ *for all* $Z \in \mathcal{M}_0^T(\text{int}K^*) \cap \mathcal{L}_0^T(\text{int}\mathbb{R}_-^d)$.

Then, (i)$\Leftrightarrow$ (ii)$\Rightarrow$ (iii). *If moreover there exists* $Z \in \mathcal{M}_0^T(\text{int}K^*) \cap \mathcal{L}_0^T(\text{int}\mathbb{R}_-^d)$ *such that* $\alpha^R(Z) < \infty$, *then* (iii)$\Rightarrow$ (ii).

Remark 2.3.3. It follows from Remark 2.3.1 that (i)$\Leftrightarrow$ (ii)$\Leftrightarrow$ (iii) in Propositions 2.3.2 and 2.3.3 whenever assumption (2.2.7) holds. It is the case under **NMA2** whenever c is essentially bounded.

2.3.2 Utility maximization

In order to avoid technical difficulties, we shall only discuss here the case of a (possibly) random utility function defined on $\mathbb{R}^d$ and essentially bounded from above. More general cases could be discussed by following the line of arguments of [Bouchard 05].

We therefore let U be a $\mathbb{P}$ – a.s.-upper semi-continuous concave random map from $\mathbb{R}^d$ to $[-\infty, 1]$ such that $U(V) = -\infty$ on $\{V \notin K_T\}$ for $V \in L^0(\mathbb{R}^d, \mathcal{F})$. Given an initial holding $x_0 \in \mathbb{R}^d$, we assume that

$$\mathcal{U}(x_0) := \left\{V \in \mathfrak{X}_0^R(T) \ : \ \mathbb{E}[|U(x_0+V)|] < \infty\right\} \neq \emptyset.$$

Then, existence holds for the associated expected utility maximization problem whenever Assumption 2.2.2, Assumption 2.3.1 and **NMA2** hold, and there exists $Z \in \mathcal{M}_0^T(\text{int}K^*)$ such that $\alpha^R(Z) < \infty$. The latter being a consequence of **NMA2** when c is essentially bounded, recall Remark 2.3.1 and Theorem 2.2.1.

Proposition 2.3.4. *Assume that Assumption 2.2.2, Assumption 2.3.1 and* **NMA2** *hold, and that* $\alpha^R(Z) < \infty$ *for some* $Z \in \mathcal{M}_0^T(\text{int}K^*)$. *Assume further that* $\mathcal{U}(x_0) \neq \emptyset$. *Then, there exists* $V(x_0) \in \mathfrak{X}_0^R(T)$ *such that*

$$\mathbb{E}[U(x_0+V(x_0))] = \sup_{V \in \mathcal{U}(x_0)} \mathbb{E}[U(x_0+V)] \ .$$

2.4 Example : an electricity generation pricing and hedging model

Let us consider a market model where the agent produces electricity which can then be sold on the spot market. For ease of presentation, we only consider the case where the production takes place in a single monetary zone, say Euro, but the model might be extended to several currencies. The market consists in three assets : the first one is cash, the second one is coal and the last one is fuel. We assume througouth this section that conditions (2.2.1) and Assumption 2.2.1 hold. Allowed self-financed strategies ξ are described by the bid-ask process $(\pi^{ij})_{1\le i,j\le 3}$ for that market. The agent can use coal or fuel for production purpose, but can also buy a one period ahead delivery contract to small local electricity producers. Given a regime β_t, the producer obtains a return $r^1_{t+1}(\beta_t)$ labeled in cash at time $t+1$, depending on the electricity spot price. Since he does not produce coal or fuel, there is no return in these two assets. As a consequence, the production function R_{t+1} has the form $(r^1_{t+1}, 0, 0)$, and is a random $\mathcal{F}_{t+1}$-measurable function.

Remark 2.4.1. *If r^1_t is $\mathbb{P}-a.s.$ concave and non-decreasing, then $r^1_t(\alpha\beta)/\alpha$ admits $\mathbb{P}-a.s.$ a limit $L^1_t(\beta)$ as $\alpha\to\infty$, where the map $\beta\mapsto L^1_t(\beta)$ is $\mathbb{P}-a.s.$ linear. It follows that $R_t(\alpha\beta)/\alpha$ admits a limit as $\alpha\to\infty$ with can be associated to a random matrix L_t of dimension 3. Moreover, we clearly can find $c_t\in L^0(\mathbb{R}^d,\mathcal{F}_t)$ such that (2.2.5) holds.*

2.4.1 The model of electricity generation

We consider now a specific model of such a situation. We denote by β^2_t (resp. β^3_t) denotes the number of units of coal (resp. fuel) sent to power plants using coal (resp. fuel) at time t. Hereafter coal and fuel are called technologies 2 and 3. The agent has $n_i\ge 1$ power plants that use the technology $i=2,3$. The k-th power plant that uses the technology i has a maximal capacity $\Delta^{ik}_{t+1}\in L^0(\mathbb{R}_+\cup\{\infty\},\mathcal{F}_{t+1})$ for the time period $[t,t+1]$, $i=2,3$ and $k=1,\ldots,n_i$. The case $\Delta^{ik}_{t+1}=\infty$ means that there is no limit on the number of quantities that can be treated. Each of them convert one unit of raw material sent to the plant at time t into $q^{ik}_{t+1}\in L^0(\mathbb{R}_+,\mathcal{F}_{t+1})$ MWh of energy that are sold on the spot market at a price $P_{t+1}\in L^0(\mathbb{R},\mathcal{F}_{t+1})$. The factor q^{ik}_{t+1} is called the heat rate of the k-th power plant, which uses the technology i. The randomness of Δ^{ik}_{t+1} and q^{ik}_{t+1} allows for instance to model possible break-downs or specific unexpected problems in the production process. For ease of presentation, we assume that the producer has an idea on which power plant is more efficient than the other and uses in priority the ones that are more efficient. Without loss of generality, we can assume that the power plant 1 is the more efficient, the power plant 2 is the second more efficient one and so on, namely

$$q^{ik}_{t+1}\ge q^{i(k+1)}_{t+1}\quad \mathbb{P}-\text{a.s. for all } k\in[1,n_i-1],\ i=2,3 \text{ and } t<T. \tag{2.4.1}$$

The production function r^{1i} associated to the technology $i=1,2$ is thus given by

$$r^{1i}_{t+1}(\beta^i)=P_{t+1}\sum_{k=1}^{n_i}\left(q^{ik}_{t+1}\min\{\beta^i-\bar\Delta^{ik}_{t+1};\Delta^{ik}_{t+1}\}^+\right)-\sum_{k=1}^{n_i}\gamma^{ik}_{t+1}\mathbb{1}_{\{\{\beta^i\ge\bar\Delta^{ik}_{t+1}\}\}}$$

where $\bar{\Delta}^{ik}_{t+1} := \mathbb{1}_{\{k\geq 2\}} \sum_{1=\ell<k} \Delta^{i\ell}_{t+1}$ denotes the maximal capacity of the best $k-1$ plants, y^+ denotes the positive part of a real number y, and $\gamma^{ik}_{t+1} \in L^0(\mathbb{R}_+, \mathcal{F}_{t+1})$ stands for a (possibly random) fixed cost associated to the k-th power plant (e.g. a starting costs for power plants that need to be switched on).
We denote by β^1_t the amount of cash used at time t to buy one period ahead delivery contracts to small local electricity producers. The price of these contracts at time t is $f_t \in L^0((0,\infty), \mathcal{F}_t)$ per MWh. Thus, consuming β^1_t units of cash at time t produces

$$r^{11}_{t+1}(\beta^1_t) := \frac{s_{t+1}}{f_t}\beta^1_t$$

units of cash at time $t+1$, once MWh have been sold on the spot market at the spot price P_{t+1}.
Altogether, the production map is given by

$$R_{t+1}(\beta_t) = \left(r^1_{t+1}(\beta_t) := \sum_{i=1}^{3} r^{1i}_{t+1}(\beta^i_t), 0, 0 \right) . \tag{2.4.2}$$

Note that r^1_{t+1} is not concave, except if $\gamma^{ik} = 0$ for all i, k, and $s_{t+1} \geq 0$, which may not be the case on the electricity spot market. However, R_{t+1} satisfies (2.2.5) with L defined by

$$L^{11}_{t+1} := s_{t+1}/f_t \ , \ L^{1i}_{t+1} := \mathbb{1}_{\{\{k^i_t<\infty\}\}} P_{t+1} l^{ik^i_t}_{t+1} \ \text{ for } i = 2,3, \text{ and } \ L^{ji}_{t+1} := 0 \text{ for } j \neq 1 \ ,$$

where

$$k^i_t := \min\{k \leq n_i \ : \ \Delta^{ik}_{t+1} = \infty\} \ ,$$

with the usual convention $\min \emptyset = \infty$. The above choice of L is the smallest possible one (component by component) under (2.4.1). As for the minimal possible c (component by component) such that (2.2.5) holds, it takes the form $c_{t+1} = (c^1_{t+1}, 0, 0)$ with

$$c^1_{t+1} = \max_{\beta \in \mathbb{R}^3_+} \left(r^1_{t+1}(\beta) - \sum_{i=1}^{3} L^{1i}_{t+1} \beta^i \right) \ ,$$

which is $\mathbb{P}-$a.s. finite.

2.4.2 The no-arbitrage condition

By the previous results, condition (ii) of $\mathbf{NA2}^L$ is satisfied if and only if, for all $t \leq T-1$ and $\beta_t \in L^0(\mathbb{R}^3_+, \mathcal{F}_t)$,

$$\sum_{i=1}^{3} \left(L^{1i}_{t+1} - \pi^{1i}_{t+1} \right) \beta^i_t \geq 0 \ \Rightarrow \ \beta_t = 0$$

which is equivalent to

$$\mathbb{P}\left[P_{t+1} < f_t|\mathcal{F}_t\right] > 0 \text{ and } \mathbb{P}\left[\mathbb{1}_{\{\{k_t^i<\infty\}\}} P_{t+1} q_{t+1}^{ik_t^i} < \pi_{t+1}^{1i}|\mathcal{F}_t\right] > 0 \text{ for } i = 2, 3.$$

Assuming that the above condition is satisfied, then (i) of $\mathbf{NA2}^L$ is equivalent to the existence of an element $Z \in \mathcal{M}_0^T(\mathrm{int}K^*) \cap \mathcal{L}_0^T(\mathrm{int}\mathbb{R}_-^d)$. Let $\mathbb{Q} \sim \mathbb{P}$ be defined by $d\mathbb{Q}/d\mathbb{P} = Z_T^1$ and $\bar{Z} := Z/Z^1$. As in [Kabanov 02], [Schachermayer 04] and [Rásonyi 10], the fact that $Z \in \mathcal{M}_0^T(\mathrm{int}K^*)$ is equivalent to $\bar{Z}^i/\bar{Z}^j < \pi^{ji}$ for all $i \neq j$, and each $\bar{Z}^i$ is a $\mathbb{Q}$-martingale, $i = 2, 3$. The new condition $Z \in \mathcal{L}_0^T(\mathrm{int}\mathbb{R}_-^d)$ is equivalent to $\mathbb{E}^{\mathbb{Q}}[s_{t+1} \mid \mathcal{F}_t] < f_t$ and $\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{\{k_t^i<\infty\}\}} s_{t+1} l_{t+1}^{ik_t^i} - \bar{Z}_{t+1}^i \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{\{k_t^i<\infty\}\}} s_{t+1} l_{t+1}^{ik_t^i} \mid \mathcal{F}_t] - \bar{Z}_t^i < 0$ for $i = 2, 3$. Note that Assumption 2.2.2 trivially holds in this example, so that Theorem 2.2.2 implies that $\mathfrak{X}_0^R(T)$ is closed in probability whenever the above conditions are satisfied.

2.5 Proofs

2.5.1 No-arbitrage of second kind in the linear model and (K, L)-strictly consistent price systems

In this section, we first prove that the no-arbitrage of second kind assumption $\mathbf{NA2}^L$ implies the existence of an element $Z \in \mathcal{M}_0^T(\mathrm{int}K^*) \cap \mathcal{L}_0^T(\mathrm{int}\mathbb{R}_-^d)$, which we call (K, L)-*strictly consistent price system*.

The arguments used in the proof of Proposition 2.5.1 below are inspired by [Rásonyi 10], up to non-trivial modifications. This proposition readily implies that $\mathbf{NA2}^L \Rightarrow \mathbf{PCE}^L$ up to an obvious induction argument.

Proposition 2.5.1. *Assume that* $\mathbf{NA2}^L$ *holds. Then, for all* $t < T$ *and* $X \in L^1(\mathrm{int}K_t^*, \mathcal{F}_t)$, *there exists* $Z \in L^1(\mathrm{int}K_{t+1}^*, \mathcal{F}_{t+1})$ *such that* $X = \mathbb{E}[Z \mid \mathcal{F}_t]$,
$\mathbb{E}[|Z'(L_{t+1} - I_d)| \mid \mathcal{F}_t] < \infty$ *and* $\mathbb{E}[Z'(L_{t+1} - I_d) \mid \mathcal{F}_t] \in \mathrm{int}\mathbb{R}_-^d$.

Proof We fix $t < T$. For ease of notation, we set $M_{t+1} := L_{t+1} - I_d$. We next define $\gamma_{t+1} := e^{-\sum_{i,j\le d}|M_{t+1}^{ij}|}$ and $\bar{M}_{t+1} := \gamma_{t+1} M_{t+1}$. Clearly, $\bar{M}_{t+1}$ is essentially bounded.
1. We first show that $\mathrm{int}\mathbb{R}_-^d \subset \mathrm{cone}(\mathrm{int}\mathbb{E}[\Theta|\mathcal{F}_t]) =: H$, where

$$\Theta := \left\{\bar{M}_{t+1}' y + r,\ (y, r) \in (K_{t+1}^* \cap B_1) \times [0, 1]^d\right\},$$

recall that B_1 is the unit ball of $\mathbb{R}^d$. For later use, observe that, since $\bar{M}_{t+1}$ is essentially bounded, Lemma 1.2.1 in Chapter 1 applies to Θ up to an obvious scaling argument.
If $\mathrm{int}\mathbb{R}_-^d \not\subset H$, then $\mathbb{R}_-^d \not\subset \overline{H}$ on a set $A \in \mathcal{F}_t$ with $\mathbb{P}[A] > 0$. For each $\omega \in A$, $\bar{H}(\omega)$ being a closed convex cone, we can then find $p(\omega) \in \mathbb{R}_-^d$ and $\beta(\omega) \in \mathbb{R}^d$ such that

$$p(\omega)'\beta(\omega) < 0 \le q'\beta(\omega) \text{ for all } q \in \bar{H}(\omega) \quad \text{for } \omega \in A\ . \tag{2.5.1}$$

By the measurable selection argument of Theorem 1.2.8, one can assume that p and β are $\mathcal{F}_t$-measurable. The right-hand side of (2.5.1), Lemma 1.2.1 and the fact that K^*_{t+1} is a cone then imply that

$$(Y'\bar{M}_{t+1}+\rho')\beta\mathbf{1}_A \geq 0 \text{ for all } (Y,\rho) \in L^\infty(K^*_{t+1}\times\mathbb{R}^d_+,\mathcal{F}_{t+1}),$$

which leads to $\beta\mathbf{1}_A \in \mathbb{R}^d_+$ and $\bar{M}_{t+1}\beta\mathbf{1}_A \in K_{t+1}$. Since K_{t+1} is a cone, the later implies $M_{t+1}\beta\mathbf{1}_A \in K_{t+1}$. In view of $\mathbf{NA2}^L$, this implies that $\beta\mathbf{1}_A = 0$, which contradicts the left-hand side of (2.5.1).

2. We next show that there exists $\tilde{Y} \in L^\infty(\text{int}K^*_{t+1},\mathcal{F}_{t+1})$ such that $\mathbb{E}\left[\tilde{Y}'\bar{M}_{t+1} \mid \mathcal{F}_t\right] \in \text{int}\mathbb{R}^d_-$.

To see this, fix $\eta \in L^\infty(\text{int}\mathbb{R}^d_-,\mathcal{F}_t)$ and $Z \in L^\infty(\text{int}K^*_{t+1},\mathcal{F}_{t+1})$. Set $\bar{Z} := \mathbb{E}\left[Z'\bar{M}_{t+1} \mid \mathcal{F}_t\right]$. We can then find $\varepsilon \in L^\infty((0,1],\mathcal{F}_t)$ such that $\eta - \varepsilon\bar{Z} \in L^\infty(\text{int}\mathbb{R}^d_-,\mathcal{F}_t)$. In view of step 1 and Lemma 1.2.1, there exists $(Y,\rho) \in L^\infty(K^*_{t+1}\times\mathbb{R}^d_+,\mathcal{F}_{t+1})$ and $\alpha \in L^0(\text{int}\mathbb{R}_+,\mathcal{F}_t)$ such that $\eta - \varepsilon\bar{Z} = \alpha\mathbb{E}\left[Y'\bar{M}_{t+1}+\rho \mid \mathcal{F}_t\right]$ or, equivalently,

$$\eta - \alpha\mathbb{E}\left[\rho \mid \mathcal{F}_t\right] = \mathbb{E}\left[(\alpha Y+\varepsilon Z)'\bar{M}_{t+1} \mid \mathcal{F}_t\right] .$$

Clearly, $\eta - \alpha\mathbb{E}\left[\rho \mid \mathcal{F}_t\right] \in \text{int}\mathbb{R}^d_-$ and $\alpha Y+\varepsilon Z \in L^0(\text{int}K^*_{t+1},\mathcal{F}_{t+1})$. The required result is thus obtained for $\tilde{Y} := (\alpha Y+\varepsilon Z)/(1+\alpha)$.

3. We now show that $K^*_t \times \{0\} \subset \text{cone}(\mathbb{E}\left[\Gamma|\mathcal{F}_t\right]) =: E$ where

$$\Gamma := \left\{(\gamma_{t+1}y, \bar{M}'_{t+1}y+r),\ (y,r) \in (K^*_{t+1}\cap B_1)\times[0,1]^d\right\} .$$

Since $\mathbb{E}\left[\Gamma|\mathcal{F}_t\right]$ is a.s. convex and compact, see Lemma 1.2.1, it follows that E is $\mathbb{P}-$a.s. convex and closed. Thus, if $K^*_t \times \{0\} \not\subset E$ on a set $A \in \mathcal{F}_t$, with $\mathbb{P}\left[A\right] > 0$, the same arguments as in step 1 imply that we can find $(p,0) \in L^0(K^*_t\times\{0\},\mathcal{F}_t)$ and $(\zeta,\beta) \in L^0(\mathbb{R}^d\times\mathbb{R}^d,\mathcal{F}_t)$ such that

$$p'\zeta < 0 \text{ on } A \quad\text{and}\quad 0 \leq Y'(\gamma_{t+1}\zeta+\bar{M}_{t+1}\beta)+\rho'\beta \text{ for all } (Y,\rho) \in L^\infty(K^*_{t+1}\times\mathbb{R}^d_+,\mathcal{F}_{t+1}) .$$

The right-hand side implies that $\beta \in \mathbb{R}^d_+$ and $\gamma_{t+1}\zeta+\bar{M}_{t+1}\beta = \gamma_{t+1}\left(\zeta+M_{t+1}\beta\right) \in K_{t+1}$, and therefore $\zeta+M_{t+1}\beta \in K_{t+1}$. In view of $\mathbf{NA2}^L$, this implies that $\zeta \in K_t$, thus leading to a contradiction with the left-hand side, since $p \in K^*_t$.

4. We can now conclude the proof. Fix $X \in L^1(\text{int}K^*_t,\mathcal{F}_t)$, let $\tilde{Y}$ be as in step 2 and fix $\varepsilon \in L^1((0,1],\mathcal{F}_t)$ such that $\tilde{X} := X - \varepsilon\mathbb{E}\left[\gamma_{t+1}\tilde{Y} \mid \mathcal{F}_t\right] \in L^1(K^*_t,\mathcal{F}_t)$. It then follows from step 3 and Lemma 1.2.1 that we can find $Y \in L^\infty(K^*_{t+1},\mathcal{F}_{t+1})$ and $\alpha \in L^0(\mathbb{R}_+,\mathcal{F}_t)$ such that $\tilde{X} = \mathbb{E}\left[\gamma_{t+1}\alpha Y \mid \mathcal{F}_t\right]$ and $\mathbb{E}\left[\gamma_{t+1}\alpha Y'M_{t+1} \mid \mathcal{F}_t\right] \in \mathbb{R}^d_-$. This implies that $X = \mathbb{E}\left[Z \mid \mathcal{F}_t\right]$ and $\mathbb{E}\left[Z'M_{t+1} \mid \mathcal{F}_t\right] \in \text{int}\mathbb{R}^d_-$ where $Z := \gamma_{t+1}(\alpha Y+\varepsilon\tilde{Y}) \in \text{int}K^*_{t+1}$. Since

$X \in L^1$ and $K^* \subset \mathbb{R}^d_+$, we must have $Z \in L^1$. Moreover, $\tilde{Y}, Y$ and $\gamma_{t+1}M_{t+1} = \bar{M}_{t+1}$ are essentially bounded, while α and ε are $\mathcal{F}_t$-measurable, so that $\mathbb{E}\left[|Z'M_{t+1}| \mid \mathcal{F}_t\right] < \infty$ $\mathbb{P}-$ a.s. This shows the required result. □

It remains to prove the opposite inclusion of Theorem 2.2.1.

Proposition 2.5.2. $\mathbf{PCE}^L \Rightarrow \mathbf{NA2}^L$.

Proof We fix $t < T$.
1. We first assume that we can find $(\zeta, \beta) \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d_+, \mathcal{F}_t)$ satisfying

$$\zeta - \beta + L_{t+1}\beta \in K_{t+1}, \tag{2.5.2}$$

and such that $\zeta \notin K_t$ on a set $A \in \mathcal{F}_t$ of positive measure. This implies that we can find $Z_t \in L^1(\text{int}K_t^*, \mathcal{F}_t)$ such that

$$Z_t'\zeta < 0 \quad \text{on } A\,. \tag{2.5.3}$$

In view of $\mathbf{PCE}^L$, we can then find $Z_{t+1} \in L^1(\text{int}K_{t+1}^*, \mathcal{F}_{t+1})$ such that $\mathbb{E}\left[Z_{t+1}|\mathcal{F}_t\right] = Z_t$, $\mathbb{E}\left[|Z_{t+1}'(L_{t+1} - I_d)| \mid \mathcal{F}_t\right] < \infty$ and $\mathbb{E}\left[Z_{t+1}'(L_{t+1} - I_d)|\mathcal{F}_t\right] \in \text{int}\mathbb{R}^d_-$. By (2.5.2), we have $Z_{t+1}'\zeta + Z_{t+1}'(L_{t+1} - I_d)\beta \geq 0$ which, by taking conditional expectations, leads to $Z_t'\zeta + \mathbb{E}\left[Z_{t+1}'(L_{t+1} - I_d)|\mathcal{F}_t\right]\beta \geq 0$. Since $\mathbb{E}\left[Z_{t+1}'(L_{t+1} - I_d)|\mathcal{F}_t\right] \in \text{int}\mathbb{R}^d_-$ and $\beta \in \mathbb{R}^d_+$, this leads to a contradiction with (2.5.3).
2. We now assume that $\beta \in L^0(\mathbb{R}^d_+, \mathcal{F}_t)$ is such that $(L_{t+1} - I_d)\beta \in K_{t+1}$. For Z_{t+1} defined as above, we obtain $Z_{t+1}'(L_{t+1} - I_d)\beta \geq 0$ while $\mathbb{E}\left[Z_{t+1}'(L_{t+1} - I_d)|\mathcal{F}_t\right] \in \text{int}\mathbb{R}^d_-$. This implies that $\beta = 0$. □

2.5.2 The closedness properties

In this section, we prove that the set $\mathfrak{X}_0^L(T)$ is closed in probability whenever there exists a (K, L)-strictly consistent price system, i.e. $\mathcal{M}_0^T(\text{int}K^*) \cap \mathcal{L}_0^T(\text{int}\mathbb{R}^d_-) \neq \emptyset$, and that the same holds for $\mathfrak{X}_0^R(T)$ under the additional Assumption 2.2.2. In view of Theorem 2.2.1, Theorem 2.2.2 is a direct consequence of Corollary 2.5.1 below. We start with the proof of the key Lemma 2.2.1 which will be later applied to the linear case $R(\beta) = 0 + L\beta$.

Proof of Lemma 2.2.1. Fix $\check{Z}$ such that (2.2.7) holds. In this proof, we set $M_{t+1} := L_{t+1} - I_d$ and $\bar{Z}_t := \mathbb{E}\left[\check{Z}_{t+1}'M_{t+1}|\mathcal{F}_t\right]$, for $t < T$, in order to alleviate notations. We first observe that $(\check{Z}_t, \bar{Z}_t) \in \text{int}K_t^* \times \text{int}\mathbb{R}^d_-$ implies :

$$\check{Z}_t'\xi \leq -\varepsilon|\xi| \text{ and } \bar{Z}_t'\beta \leq -\varepsilon|\beta| \text{ for all } (\xi, \beta) \in L^0((-K_t) \times \mathbb{R}^d_+, \mathcal{F}_t)\ ,\ t \leq T\ , \tag{2.5.4}$$

for some $\varepsilon \in L^0((0,1), \mathcal{F})$, compare with Lemma 11 in [Campi 06].

We next deduce from (2.2.3)-(2.2.5) that

$$V_T^{\xi,\beta} = X_T \text{ where } X_t := \sum_{s\le t} \xi_s+\zeta_s+(c_s+M_s\beta_{s-1})\mathbb{1}_{\{s\ge 1\}} \text{ for some } \zeta \in L^0(-K,\mathbb{F}). \tag{2.5.5}$$

Since $X_T+\kappa = V_T^{\xi,\beta}+\kappa \in K_T$, we have $\check{Z}'_T X_T \ge -\check{Z}'_T\kappa$ so that $\mathbb{E}\left[\check{Z}'_T X_T|\mathcal{F}_{T-1}\right]$ is well-defined since $\check{Z}_T \in L^1$. It then follows from the martingale property of $\check{Z}$, (2.5.4), (2.2.7) and (2.5.5) that

$$-\check{Z}'_{T-1}\kappa \;\; \le \;\; \mathbb{E}\left[\check{Z}'_T X_T|\mathcal{F}_{T-1}\right] \le \check{Z}'_{T-1}X_{T-1}+\mathbb{E}\left[\check{Z}'_T C^T_{T-1} - \varepsilon\left(|\xi_T|+|\zeta_T|+|\beta_{T-1}|\right)|\mathcal{F}_{T-1}\right] ,$$

where C^T_{T-1} is defined in (2.2.8). Iterating this procedure leads to

$$-\check{Z}'_0\kappa \le \mathbb{E}\left[\check{Z}'_T X_T\right] \le \mathbb{E}\left[\check{Z}_T C_0^T - \varepsilon \sum_{0\le t\le T}\left(|\xi_t|+|\zeta_t|+|\beta_{t-1}|\mathbb{1}_{\{t\ge 1\}}\right)\right] \tag{2.5.6}$$

which implies the required result for $\mathbb{Q}\sim\mathbb{P}$ defined by $d\mathbb{Q}/d\mathbb{P} := \varepsilon\alpha$ with $\alpha := 1/\mathbb{E}[\varepsilon]$. □

We can now prove the closedness properties.

Corollary 2.5.1. *Assume that there exists $Z \in \mathcal{M}_0^T(\mathrm{int}K^*) \cap \mathcal{L}_0^T(\mathrm{int}\mathbb{R}^d_-)$. Then, $\mathfrak{X}_0^L(T)$ is closed in probability. If moreover Assumption 2.2.2 is satisfied, then the same holds for $\mathfrak{X}_0^R(T)$.*

Proof We use an induction argument which combines technics first introduced in [Kabanov 04] and Lemma 2.2.1 applied to the linear case $R(\beta) = 0 + L\beta$.
1. We first check that $\mathfrak{X}_T^R(T)$ is closed in probability, recall (2.2.4). Indeed, let $(g_n)_{n\ge1} \subset \mathfrak{X}_T^R(T)$ be such that $g_n \to g \in L^0(\mathbb{R}^d,\mathcal{F})$ $\mathbb{P}-$a.s. as $n\to\infty$. Let $(\xi_T^n,\beta_T^n)_{n\ge1} \in L^0((-K_T)\times\mathbb{R}^d_+,\mathcal{F}_T)$ be such that $\xi^n_T - \beta^n_T = g_n$ for all $n\ge 1$ and set

$$E := \{\liminf_{n\to\infty} |\beta^n_T| < \infty\} .$$

We claim that $E=\Omega$. Indeed, letting $(\bar{\xi}^n_T,\bar{\beta}^n_T) := (\xi^n_T,\beta^n_T)/(1+|\beta^n_T|)\mathbb{1}_{\{E^c\}}$, we obtain $\bar{\xi}^n_T = \mathbb{1}_{\{E^c\}}g_n/(1+|\beta^n_T|)+\bar{\beta}^n_T$. In view of Lemma 1.2.5 in Chapter 1 we can assume, after possibly passing to an $\mathcal{F}_T$-measurable subsequence, that $\mathbb{1}_{\{E^c\}}g_n/(1+|\beta^n_T|)+\bar{\beta}^n_T \to \bar{\beta}_T \in L^0(\mathbb{R}^d_+,\mathcal{F}_T)$ $\mathbb{P}-$a.s. as $n\to\infty$, with $|\bar{\beta}_T| = 1$ on E^c. On the other hand $\bar{\xi}^n_T\mathbb{1}_{\{E^c\}} \in -K_T\mathbb{1}_{\{E^c\}}$ $\mathbb{P}-$a.s. Since $-K_T\cap\mathbb{R}^d_+ = \{0\}$, this leads to a contradiction. It follows that $\liminf_{n\to\infty}|\beta^n_T| < \infty$ $\mathbb{P}-$a.s. The closedness property of $\mathfrak{X}_T^R(T)$ then follows from Lemma 1.2.5 again. The fact that $\mathfrak{X}_T^L(T)$ is closed in probability follows from the same arguments.

2. We now fix $t < T$, assume that $\mathfrak{X}^R_{t+1}(T)$ and $\mathfrak{X}^L_{t+1}(T)$ are closed in probability and deduce that the same holds for $\mathfrak{X}^R_t(T)$. The corresponding result for $\mathfrak{X}^L_t(T)$ is obviously obtained by considering the special case where $R(\beta) = 0 + L\beta$.
Let $(g_n)_{n\geq 1} \subset \mathfrak{X}^R_t(T)$ and $(\xi^n, \beta^n)_{n\geq 1} \subset \mathcal{A}_0$ be such that

$$V_T^{\xi^n,\beta^n} = g_n \ \text{ for all } \ n \geq 1 \ . \tag{2.5.7}$$

We assume that

$$g_n \to g \in L^0(\mathbb{R}^d, \mathcal{F}) \ \ \mathbb{P}-\text{a.s. as } n \to \infty \ .$$

In view of (2.2.5), we can find $(V^n)_{n\geq 1} \subset \mathfrak{X}^L_{t+1}(T)$ such that

$$\xi^n_t + (L_{t+1} - I_d)\beta^n_t + C^T_t + V^n = g_n \ ,$$

where C^T_t has been defined in (2.2.8). Set $\alpha_n := 1 + |\xi^n_t| + |\beta^n_t|$. We claim that $E := \{\liminf_{n\to\infty} \alpha_n < \infty\}$ has probability one. Indeed, the previous equality implies that

$$\bar\xi^n_t + (L_{t+1} - I_d)\bar\beta^n_t + \bar V^n = \mathbb{1}_{\{E^c\}} \left(g_n - C^T_t\right)/\alpha_n$$

where $(\bar\xi^n_t, \bar\beta^n_t) := \mathbb{1}_{\{E^c\}}(\xi^n_t, \beta^n_t)/\alpha_n \in L^0((-K_t) \times \mathbb{R}^d_+, \mathcal{F}_t)$ and $\bar V^n := \mathbb{1}_{\{E^c\}} V^n/\alpha_n \in \mathfrak{X}^L_{t+1}(T)$. Moreover, Lemma 1.2.5 implies that, after possibly passing to an $\mathcal{F}_t$-measurable subsequence, $(\bar\xi^n_t, \bar\beta^n_t) \to (\bar\xi_t, \bar\beta_t)$ $\mathbb{P}-$a.s. as $n \to \infty$ for some $(\bar\xi_t, \bar\beta_t) \in L^0((-K_t)\times\mathbb{R}^d_+, \mathcal{F}_t)$ such that $(\bar\xi_t, \bar\beta_t) \neq 0$ on E^c. Since $\mathfrak{X}^L_{t+1}(T)$ is closed in probability, it follows that

$$\bar V^n = \mathbb{1}_{\{E^c\}} \left(g_n - C^T_t\right)/\alpha_n - \bar\xi^n_t - (L_{t+1} - I_d)\bar\beta^n_t \to -\bar\xi_t - (L_{t+1} - I_d)\bar\beta_t \in \mathfrak{X}^L_{t+1}(T) \text{ as } n \to \infty \ .$$

We can then find $(\xi, \beta) \in \mathcal{A}_0$ such that

$$\bar\xi_t + (L_{t+1} - I_d)\bar\beta_t + \sum_{t+1\leq s\leq T} \xi_s + (L_{s+1}\mathbb{1}_{\{s+1\leq T\}} - I_d)\beta_s = 0 \ .$$

We can now appeal to Lemma 1.2.5 applied to the case $R(\beta) = 0 + L\beta$ to deduce that $\mathbb{E}^{\mathbb{Q}}\left[|\bar\xi_t| + |\bar\beta_t|\right] \leq 0$, for some $\mathbb{Q} \sim \mathbb{P}$. Since $(\bar\xi_t, \bar\beta_t) \neq 0$ on E^c, this implies that $\mathbb{P}[E^c] = 0$, and therefore $\liminf_{n\to\infty} \alpha_n < \infty$ $\mathbb{P}-$a.s. Using Theorem 1.2.5, one can then assume, after possibly passing to an $\mathcal{F}_t$-measurable random subsequence, that $(\xi^n_t, \beta^n_t)_{n\geq 1}$ converges $\mathbb{P}-$a.s. to some $(\xi_t, \beta_t) \in L^0((-K_t)\times\mathbb{R}^d_+, \mathcal{F}_t)$, for all $t \leq T$. Using Assumption 2.2.2 and d iterative applications of Lemma 1.2.5, we can then find an $\mathcal{F}_{t+1}$-measurable subsequence $(\sigma(n))_{n\geq 1}$ such that $R_{t+1}(\beta^{\sigma(n)}_t) \to R_{t+1}(\beta_t) + \zeta_{t+1}$ $\mathbb{P}-$a.s. as $n \to \infty$ with $\zeta_{t+1} \in L^0(-K_{t+1}, \mathcal{F}_{t+1})$. It then follows from (2.5.7) that

$$\sum_{t+1\leq s\leq T} \left(\xi^{\sigma(n)}_s + R_{s+1}(\beta^{\sigma(n)}_s)\mathbb{1}_{\{s+1\leq T\}} - \beta^{\sigma(n)}_s\right) \to g - \zeta_{t+1} - (\xi_t + R_{t+1}(\beta_t) - \beta_t) \ \mathbb{P}-\text{a.s.}$$

We conclude by using the fact that the left-hand side term belongs to $\mathfrak{X}^R_{t+1}(T)$ which is closed in probability by assumption. □

2.5.3 Super-hedging theorems

We now turn to the proof of the super-hedging theorems, Propositions 2.3.1, 2.3.2 and 2.3.3. The result of Proposition 2.3.1 is a consequence of Proposition 2.3.3 and Remark 2.3.1. The fact that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) in Propositions 2.3.2 and 2.3.3 is obvious. In the following, we prove that (iii) $\Rightarrow$ (i) in Propositions 2.3.2 and 2.3.3 under the corresponding additional assumptions. The fact that (ii) $\Rightarrow$ (i) is obtained by similar, actually shorter, arguments which are fully contained in what follows.

Proof of (iii) $\Rightarrow$ (i) **in Proposition 2.3.2 :** For ease of notations, we write M for $L - I_d$.
Fix $V \in L^0(\mathbb{R}^d, \mathcal{F})$ such that $V + \kappa \in K_T$ for some $\kappa \in \mathbb{R}^d$, and assume that $\mathbb{E}\left[Z_T' V\right] \leq \alpha^R(Z)$ for all $Z \in \mathcal{M}_0^T(\text{int}K^*)$, but that $V \notin \mathfrak{X}_{0b}^R(T)$. Then, $\mathfrak{X}_{0b}^R(T)$ being closed in probability by Theorem 2.2.2, it follows that, for k large enough (after possibly passing to a subsequence), $V^k := V\mathbb{1}_{\{|V|\leq k\}} - \kappa \mathbb{1}_{\{|V|>k\}}$ does not belong to $\mathfrak{X}_{0b}^R(T)$ either but satisfies

$$\mathbb{E}\left[Z_T' V^k\right] \leq \mathbb{E}\left[Z_T' V\right] \leq \alpha^R(Z) \ \text{ for all } Z \in \mathcal{M}_0^T(\text{int}K^*). \tag{2.5.8}$$

Since $\mathfrak{X}_0^R(T)$ is closed in probability, $\mathfrak{X}_0^R(T) \cap L^1(\mathbb{R}^d, \mathcal{F})$ is closed in $L^1(\mathbb{R}^d, \mathcal{F})$. The later being convex under Assumption 2.3.1(a), we deduce from the Hahn-Banach theorem (see Theorem 1.2.1 in Chapter 1) that we can find $Y \in L^\infty(\mathbb{R}^d, \mathcal{F})$ and $r \in \mathbb{R}$ such that

$$\mathbb{E}\left[Y' X\right] \leq r < \mathbb{E}\left[Y' V^k\right] \ \text{ for all } X \in \mathfrak{X}_0^R(T) \cap L^1(\mathbb{R}^d, \mathcal{F})\ .$$

Set $Z_t^Y := \mathbb{E}\left[Y | \mathcal{F}_t\right]$. Recalling that $R(0)^- \in L^1$ under Assumption 2.3.1(b), we deduce that any element of the form

$$X = \xi + \sum_{0<t\leq T} (R_t^i(0) \wedge 1)_{i\leq d}\ ,\ \xi \in L^1(-K_s, \mathcal{F}_s) \text{ for some } s \leq T,$$

belongs to $\mathfrak{X}_0^R(T) \cap L^1(\mathbb{R}^d, \mathcal{F})$. This easily leads to $Z_s^Y \in K_s^*$ for $s \leq T$. Fix $\tilde{Z} \in \mathcal{M}_0^T(\text{int}K^*)$, such that $\alpha^R(\tilde{Z}) < \infty$, which is possible by assumption, and $\varepsilon \in (0, 1)$, so that $\check{Z} := \varepsilon \tilde{Z} + (1-\varepsilon) Z^Y \in \mathcal{M}_0^T(\text{int}K^*)$ and

$$\mathbb{E}\left[\check{Z}_T' X\right] \leq (1-\varepsilon) r + \varepsilon \alpha^R(\tilde{Z}) < \mathbb{E}\left[\check{Z}_T' V^k\right] \ \ \forall\, X \in \mathfrak{X}_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}). \tag{2.5.9}$$

In order to conclude the proof, it suffices to show that

$$\alpha^R(Z) = \sup\left\{\mathbb{E}\left[Z_T' X\right],\ X \in \mathfrak{X}_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F})\right\}\ ,\ Z \in \mathcal{M}_0^T(K^*)\ , \tag{2.5.10}$$

which, combined with (2.5.9), would imply that $\alpha^R(\check{Z}) < \mathbb{E}\left[\check{Z}_T' V^k\right]$, thus leading to a contradiction to (2.5.8) since $\check{Z} \in \mathcal{M}_0^T(\text{int}K^*)$.

To see that the above claim holds, first observe that, for $X \in \mathfrak{X}_0^R(T)$ such that $X+\rho \in K_T$ for some $\rho \in \mathbb{R}^d$, one can always construct an essentially bounded sequence, $X^n := X\mathbb{1}_{\{|X| \le n\}} - \rho\mathbb{1}_{\{|X|>n\}}$ for $n \ge 1$, which converges $\mathbb{P}-$a.s. to X. Using Fatou's Lemma, one then obtains $\liminf_{n\to\infty} \mathbb{E}\left[Z_T' X^n\right] \ge \mathbb{E}\left[Z_T' X\right]$ for all $Z \in \mathcal{M}_0^T(K^*)$. Moreover, $X+\rho \in K_T$ implies $X - X^n \in K_T$ so that $X^n \in \mathfrak{X}_0^R(T)$ for all $n \ge 1$. This proves (2.5.10). □

Proof of (iii) $\Rightarrow$ (i) **in Proposition 2.3.3 :** It suffices to repeat the argument of the above proof with $\tilde{Z} \in \mathcal{L}_0^T(\text{int}\mathbb{R}_-^d)$, which is possible by assumption, and to show that one can choose Z^Y such that $\mathbb{E}\left[Z_t^{Y\prime}(L_{t+1} - I_d)|\mathcal{F}_t\right] \in \mathbb{R}_-^d$ for all $t \le T$. To see this, recall from the above arguments that Z^Y is a martingale and that it satisfies

$$\mathbb{E}\left[Z_T^{Y\prime} X\right] \le r \ \text{ for all } X \in \mathfrak{X}_0^R(T) \cap L^1(\mathbb{R}^d, \mathcal{F}) ,$$

for some $r \in \mathbb{R}$. It then follows from Assumption 2.3.1(b) that

$$\mathbb{E}\left[Z_T^{Y\prime} \sum_{t<T} (R_{t+1}^i(\beta_t) \wedge n - \beta_t^i)_{i \le d}\right] \le r \ \text{ for all } \beta \in L^\infty(\mathbb{R}_+^d, \mathbb{F}) \text{ and } n \ge 1 .$$

Since Z_T^Y has non-negative components, as an element of $K_T^* \subset \mathbb{R}_+^d$ $\mathbb{P}-$a.s., the monotone convergence theorem implies that

$$\mathbb{E}\left[Z_T^{Y\prime} \sum_{t<T} (R_{t+1}(\beta_t) - \beta_t)\right] \le r \ \text{ for all } \beta \in L^\infty(\mathbb{R}_+^d, \mathbb{F}) .$$

In particular, Assumption 2.3.1(b) and the above imply that $Z_T^{Y\prime} \sum_{1 \le t \le T} R_t(0) \in L^1$ and that for any $s \le T-1$

$$\mathbb{E}\left[{Z_{s+1}^Y}'(R_{s+1}^0(\beta_s) - \beta_s)\right] + \ell \le r \quad \text{for all } \beta_s \in L^\infty(\mathbb{R}_+^d, \mathcal{F}_s) , \tag{2.5.11}$$

where

$$R^0 := R - R(0) \ \text{ and } \ \ell := \mathbb{E}\left[Z_T^{Y\prime} \sum_{1 \le t \le T} R_t(0)\right] .$$

Using the first assertion in Assumption 2.3.1, we then deduce that, for $\eta \ge 1$ and $\beta_s \in L^\infty(\mathbb{R}_+^d, \mathcal{F}_s)$,

$$\begin{aligned} &R_{s+1}(\beta_s) - \eta^{-1} R_{s+1}(\eta\beta) - (1-\eta^{-1})R_{s+1}(0) \\ &= R_{s+1}(\eta^{-1}\eta\beta_s + (1-\eta^{-1})0) - \eta^{-1}R_{s+1}(\eta\beta) - (1-\eta^{-1})R_{s+1}(0) \in K_{s+1} . \end{aligned}$$

This shows that, for all $\beta_s \in L^\infty(\mathbb{R}^d_+, \mathcal{F}_s)$, the sequence $({Z^Y_{s+1}}' R^0_{s+1}(n\beta_s)/n)_{n\geq 1}$ is non-increasing and that, by (2.5.11),

$$\mathbb{E}\left[{Z^Y_{s+1}}'(R^0_{s+1}(n\beta_s)/n - \beta_s)\right] \leq (r-\ell)/n\ .$$

Sending $n \to \infty$, using the monotone convergence theorem and recalling Assumption 2.3.2 leads to

$$\mathbb{E}\left[{Z^Y_{s+1}}'(L_{s+1}\beta_s - \beta_s)\right] \leq 0\ .$$

By arbitrariness of $\beta_s \in L^\infty(\mathbb{R}^d_+, \mathcal{F}_s)$, this readily implies that $\mathbb{E}\left[{Z^Y_{s+1}}'(L_{s+1} - I_d)|\mathcal{F}_s\right] \in \mathbb{R}^d_-$. □

2.5.4 Utility maximization

Proof of Proposition 2.3.4. Let $(V^n)_{n\geq 1}$ be a maximizing sequence. Since $U(V) = -\infty$ on $\{V \notin K_T\}$, it must satisfy $V^n + x_0 \in K_T$ for all $n \geq 1$. It then follows from the definition of α^R and our assumptions that there exists $Z \in \mathcal{M}^T_0(\mathrm{int}K^*)$ such that $\mathbb{E}\left[Z'_T(V^n + x_0)\right] \leq \alpha^R(Z) + Z'_0 x_0 < \infty$ for all $n \geq 1$.
Since $Z_T \in \mathrm{int}K^*_T$ and $V^n + x_0 \in K_T$, for all $n \geq 1$, we can find $\varepsilon \in L^0((0,1], \mathcal{F}_T)$ such that

$$\mathbb{E}\left[\varepsilon|V^n + x_0|\right] \leq \alpha^R(Z) + Z'_0 x_0 < \infty \text{ for all } n \geq 1\ ,$$

compare with Lemma 11 in [Campi 06]. By Komlos Lemma, see Theorem 1.2.2 in Chapter 1, one can then find a sequence $(\tilde{V}^n)_{n\geq 1}$ such that $\tilde{V}^n \in \mathrm{conv}(V^k,\ k \geq n)$ for all $n \geq 1$, and $(\tilde{V}^n)_{n\geq 1}$ converges $\mathbb{P}-$ a.s. to some $V(x_0) \in L^0(\mathbb{R}^d, \mathcal{F})$. Since $\mathfrak{X}^R_0(T)$ is convex under Assumption 2.3.1(a), $(\tilde{V}^n)_{n\geq 1} \subset \mathfrak{X}^R_0(T)$. Since $\mathfrak{X}^R_0(T)$ is closed in probability, see Theorem 2.2.2, we have $V(x_0) \in \mathfrak{X}^R_0(T)$. Moreover, the random map U being $\mathbb{P}-$ a.s. concave, $(\tilde{V}^n)_{n\geq 1}$ is also a maximizing sequence. Since $U(x_0 + \tilde{V}^n)^+ \leq 1$ for each $n \geq 1$, we finally deduce from Fatou's Lemma and the $\mathbb{P}-$ a.s. upper semi-continuity of U that

$$\sup_{V \in \mathcal{U}(x_0)} \mathbb{E}\left[U(x_0 + V)\right] = \limsup_{n\to\infty} \mathbb{E}\left[U(x_0 + \tilde{V}^n)\right] \leq \mathbb{E}\left[U(x_0 + V(x_0))\right]\ .$$

□

2.6 Absence of arbitrage of the first kind

2.6.1 Additional notations and fundamental theorem of asset pricing

As explained in the introduction, there is a strong distinction between arbitrage of the first kind and of the second kind in markets with transaction costs. For a pure financial market with proportional transaction costs, the intuitive condition of absence of arbitrages of the first kind was widely studied. It takes a rather standard form, close to the one used on markets without transaction costs :

$$\mathfrak{X}_0^0(T) \cap L^0(\mathbb{R}_+^d, \mathcal{F}) = \{0\} \ .$$

The *robust no-arbitrage condition*, introduced by W. schachermayer in [Schachermayer 04], is the only one to imply, without additional assumption, the closedness of the set of attainable claims. The concept has been extended to industrial investment possibilities in [Bouchard 05] in a natural way : there is no arbitrage if we slightly reduce transaction costs and if we increase the return on production. In this section, we will apply our concept of marginal arbitrage for high production regimes within the framework of the robust no-arbitrage condition of [Bouchard 05]. However, our condition will be slightly weaker, see Remark 2.6.1 below.

We justify this extension for the following reasons. The robust no-arbitrage condition, $\mathbf{NA}^r(K, L)$ for short, is equivalent to a weaker condition than $\mathbf{PCE}^L$. It is only equivalent to the existence of a (K, L)-strictly consistent price system, compare with $\mathbf{PCE}^L$. Another strong difference with the previous condition is that we can relax Assumption 2.2.1, see section 2.1 above, which allows to incorporate frictionless markets as well.

For sake of clarity, we introduce additional notations. We use the same model as the one described in section 2.1 except that we do not need Assumption 2.2.1 to hold. For later use, let us denote K^0 the set-valued process defined by $K_t^0 := K_t \cap (-K_t)$, $t \leq T$. We now emphasize the dependence of $\mathfrak{X}_t^L(T)$ on the process π of exchange prices by writing $\mathfrak{X}_t^{\pi,L}(T)$ instead of $\mathfrak{X}_t^L(T)$. From now on, K (resp. $\widetilde{K}$) will be the set-valued process generated by π (resp. $\widetilde{\pi}$) as in definition (2.2.2) above. Wet set Π the set of exchange prices satisfying (2.2.1).

Definition 2.6.1. 1. *We say that there is* no marginal robust arbitrage for high production regimes*, in short* **NMAr** *holds, if there exists* $(c, L) \in L^\infty(\mathbb{R}^d \times \mathbb{M}^d, \mathbb{F})$ *such that* $\mathbf{NA}^r(\pi, L)$ *and equation* (2.2.5) *hold.*
2. *We say that* $\mathbf{NA}^r(\pi, L)$ *holds if* $(\pi, L) \in \Pi \times \mathbb{M}^d$ *is dominated by some* $(\widetilde{\pi}, \widetilde{L}) \in \Pi \times \mathbb{M}^d$ *such that*

$$\mathfrak{X}_0^{\widetilde{\pi},\widetilde{L}}(T) \cap L^0(\mathbb{R}_+^d) = \{0\} \tag{2.6.1}$$

3. *A sequence* $(\pi, L) \in \Pi \times \mathbb{M}^d$ *is dominated by* $(\widetilde{\pi}, \widetilde{L}) \in \Pi \times \mathbb{M}^d$ *if for each* $t \in \mathbb{T}$ *:*
(a) $K_t \backslash K_t^0 \subset ri(\widetilde{K}_t)$ *and* $K_t \subset \widetilde{K}_t$,
(b) $(\widetilde{L}_t - L_t)\beta \in ri(K_t)$ *for all* $\beta \in \mathbb{R}_+^d \backslash \{0\}$.

Remark 2.6.1. 1. *In the above definition, condition* (a) *can be replaced by the equivalent following condition :*
(a') $\widetilde{\pi}^{ij} \leq \pi^{ij}$ *for all* $i, j = 1, \cdots, d$ *and* $\widetilde{\pi}^{ij} = \pi^{ij}$ *on* $\{\pi^{ij}\pi^{ji} = 1\}$.
The latter means that the classical condition (2.6.1) *is verified for a model where we slightly reduce transaction costs which are not already null. The relaxation of Assumption 2.2.1 allows that possibility.*
2. *Note that in [Bouchard 05], the above domination is directly applied to the non-linear production function* R *but the robust no-arbitrage definition is quite the same. In our definition, contrary to [Bouchard 05], we forbid arbitrages for the asymptotic linear model, but we allow for reasonable arbitrages for low production regimes, see comments in section 2.2 above.*

The result of this section is that the above no-arbitrage condition can be characterized by the existence of a (K, L)-strictly consistent price system.

Theorem 2.6.1. $\mathbf{NA}^r(K, L) \;\Leftrightarrow\; \mathcal{M}_0^T(ri(K^*)) \cap \mathcal{L}_0^T(\mathrm{int}\mathbb{R}^d_-) \neq \emptyset$.

As a by-product, we shall show that the set $\mathfrak{X}_0^{\pi,L}(T)$ is closed in probability and therefore Fatou-closed. By the same arguments as those of sections 4.3 and 4.4, we have the super-hedging theorem and existence in the utility maximization problem, see section 3 above.

2.6.2 Proof of the theorem

In order to prove the theorem, we first show the following closedness property.

Proposition 2.6.1. $\mathfrak{X}_0^{K,L}(T)$ *is closed in probability under* $\mathbf{NA}^r(K, L)$.

The proof follows closely the argument of Lemma 5.4 in [Bouchard 05]. Theorem 2.6.1 then derives from similar arguments to those of [Kabanov 03]. We basically recall ideas of the proof of Corollary 2.5.1 in the previous section.

Proof of Proposition 2.6.1 It is obvious that $\mathfrak{X}_T^{\pi,L}(T)$ is closed since K_T is a.s. closed. We use an induction argument and show that $\mathfrak{X}_t^{\pi,L}(T)$ is closed in probability whenever $\mathfrak{X}_{t+1}^{\pi,L}(T)$ is, $t < T$. Let $(V_T^{\xi^n,\beta^n})_{n\geq 1} \subset \mathfrak{X}_t^{\pi,L}(T)$ be a sequence that converges $\mathbb{P}-$a.s. for $t < T$, with $(\xi^n, \beta^n) \in \mathcal{A}_0$ for all $n \geq 1$. Set $\eta^n := |\xi_t^n| + |\beta_t^n|$. Since $\{\liminf_{n\to\infty} \eta^n < +\infty\} \in \mathcal{F}_t$ we can argue separately on that set and its complementary by considering a strategy conditionally to that partition.
1. On $\{\liminf_{n\to\infty} \eta^n < +\infty\}$, we can assume, by possibly passing to a $\mathcal{F}_t$-measurable random subsequence (see Lemma 1.2.5 in Chapter 1), that $(\xi_t^n, \beta_t^n)_{n\geq 1}$ converges $\mathbb{P}-$a.s.

in $L^0((-K_t) \times \mathbb{R}^d_+, \mathcal{F}_t)$. By definition, we can also write

$$V_T^{\xi^n,\beta^n} \in \xi_t^n - \beta_t^n + L_{t+1}\beta_t^n + \mathfrak{X}_{t+1}^{\pi,L}(T) \ .$$

Since $\mathfrak{X}_{t+1}^{K,L}(T)$ is closed in probability, the required result follows.

2. On $\{\lim_{n\to\infty} \eta^n = +\infty\}$, we set $(\widetilde{\xi}^n, \widetilde{\beta}^n) := (1+\eta^n)^{-1}(\xi^n, \beta^n)$. The new sequence is essentially bounded so that one can assume again that $(\widetilde{\xi}_t^n, \widetilde{\beta}_t^n)_{n\geq 1}$ converges $\mathbb{P}-$a.s. to some $(\widetilde{\xi}_t, \widetilde{\beta}_t) \in L^0((-K_t) \times \mathbb{R}^d_+, \mathcal{F}_t)$. Since $V_T^{\xi^n,\beta^n}$ converges $\mathbb{P}-$a.s. and the dynamics is linear in the control (ξ^n, β^n), we obtain

$$V_T^{\widetilde{\xi},\widetilde{\beta}} = 0 \tag{2.6.2}$$

We claim that $\widetilde{\xi}_t \in K_t^0$ and $\widetilde{\beta}_t = 0$. Indeed, assume to the contrary that $\widetilde{\xi}_t \notin K_t^0$ on some set $B \in \mathcal{F}_t$ with $\mathbb{P}[B] > 0$. Then, we can find $\varepsilon \in L^0(\mathbb{R}^d_+ \setminus \{0\}, \mathcal{F}_t)$ such that $\varepsilon \neq 0$ on B and $\varepsilon = 0$ elsewhere and such that $\xi_t + \varepsilon \in -\widetilde{K}_t$, see condition 3.(a) of Definition 2.6.1 above. By defining $\widetilde{\xi}'$ through $\widetilde{\xi}'_s = \widetilde{\xi}_s + \varepsilon \mathbf{1}_{s=t}$, $t \leq s \leq T$, we get $V_T^{\widetilde{\xi}',\widetilde{\beta}} = \varepsilon$ which contradicts condition (2.6.1). Since $(\widetilde{L}_{t+1} - L_{t+1})\mathbb{R}^d_+ \setminus \{0\} \subset \mathrm{ri}(\widetilde{K}_{t+1})$, the latter property is also violated if $\widetilde{\beta}_t \neq 0$. Eventually, we have $\widetilde{\xi}_t \in K_t^0$ and $\widetilde{\beta}_t = 0$. Then, we can find a partition of d disjoint sets $\Gamma_i \in \mathcal{F}_t$ with $\Gamma_i \subset \left\{(\widetilde{\xi}_t)^i \neq 0\right\}$ for $i = 1 \cdots d$. We set

$$(\check{\xi}^n, \check{\beta}^n) := (\xi^n, \beta^n) - \sum_{i=1}^d \mathbf{1}_{\Gamma_i} \frac{\xi_t^{n,i}}{\widetilde{\xi}_t^i} (\widetilde{\xi}, \widetilde{\beta}) \ .$$

Since $\widetilde{\xi}_s \in K_s$, one can easily check that $\check{\xi}_s^n \in -K_s$ for $t \leq s \leq T$. Moreover equation (2.6.2) implies that $V_T^{\check{\xi}^n,\beta^n} = V_T^{\xi^n,\beta^n}$. Note that $\check{\xi}^{n,i} = 0$ on Γ_i. By repeating this argument a finite number of times, we finally end up with the situation where $(\check{\xi}^n, \beta^n)$ converges $\mathbb{P}-$a.s. □

We now turn to the proof of Theorem 2.6.1.

Proof of Theorem 2.6.1 First note that we can define the process of exchanges $\bar{\pi} := \frac{1}{2}(\pi + \widetilde{\pi})$ and the linear production function $\bar{L} := \frac{1}{2}(L + \widetilde{L})$ such that $(\bar{\pi}, \bar{L})$ dominates (π, L) and $\mathbf{NA}^r(\bar{\pi}, \bar{L})$ holds. We denote $\bar{K}$ the set-valued process generated by $\bar{\pi}$ as in equation (2.2.2).

1. Assume $\mathbf{NA}^r(\bar{\pi}, \bar{L})$. Put $\mathfrak{X}_t^1(T) := \mathfrak{X}_t^{\bar{\pi},\bar{L}}(T) \cap L^1(\mathbb{R}^d, \mathcal{F}_T)$ and note that $\mathfrak{X}_t^1(T)$ is closed and convex, see Proposition 2.6.1. Since $\mathfrak{X}_t^{\bar{\pi}\bar{L}}(T) \subset \mathfrak{X}_t^{\widetilde{\pi},\widetilde{L}}(T)$, condition (2.6.1) holds for $(\bar{\pi}, \bar{L})$ so the Hahn-Banach separation theorem allows us to find, for every $\phi \in L^1(\mathbb{R}^d_+ \setminus \{0\}, \mathcal{F}_T)$, $\eta_\phi \in L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ such that $\mathbb{E}\left[\eta'_\phi V_T\right] < \mathbb{E}\left[\eta'_\phi \phi\right]$ for all $V_T \in \mathfrak{X}_t^1(T)$. Since $(\xi \mathbf{1}_{t=T}, 0)$ is an admissible strategy for $\xi \in -\bar{K}_T$, $\eta_\phi \in L^0(\bar{K}_T^*, \mathcal{F}_T)$ for any ϕ. By the argument in proofs of Lemma 4 and Corollary 1 in [Kabanov 03], we can find

η such that $\mathbb{E}[\eta' V_T] \le 0$ for all $V_T \in \mathfrak{X}^1_t(T)$ with $\eta \neq 0$ $\mathbb{P}-$a.s. Set $Z_s = \mathbb{E}[\eta|\mathcal{F}_s]$. Z is a martingale and $L^1(-\bar{K}_s, \mathcal{F}_s) \subset \mathfrak{X}^1_t(T)$ for $s \ge t$ implies that $Z_s \in \bar{K}^*_s \subset \mathrm{ri}(K^*_s)$. As $(\bar{L}_{s+1} - I_d)\beta_s \in \mathfrak{X}^1_t(T)$ for $t \le s < T$ and every $\beta_s \in L^\infty(\mathbb{R}^d_+, \mathcal{F}_s)$ we then have, by taking conditional expectation, $\mathbb{E}\left[Z'_{s+1}(\bar{L}_{s+1} - I_d)\beta_s|\mathcal{F}_s\right] \le 0$. Since $\bar{L}$ dominates L, $\mathbb{E}\left[Z'_{s+1}(L_{s+1} - I_d)\beta_s|\mathcal{F}_s\right] < 0$ for $\beta_s \neq 0$. Thus $Z \in \mathcal{M}^T_t(\mathrm{ri}(K^*)) \cap \mathcal{L}^T_t(\mathrm{int}\mathbb{R}^d_-)$.

2. Let $Z \in \mathcal{M}^T_0(\mathrm{ri}(K^*)) \cap \mathcal{L}^T_0(\mathrm{int}(\mathbb{R}^d_-))$ and, for all $t \le T$, set

$$\widetilde{K}_t(\omega) := \left\{ x \in \mathbb{R}^d \ : (Z_t(\omega))' x \ge 0 \right\}.$$

As $Z_t \in \mathrm{ri}(K^*_t)$ we have $K_t \subset \widetilde{K}_t$ and $K_t \backslash K^0_t \subset \mathrm{ri}(\widetilde{K}_t)$, for all $0 \le t \le T$. Since $\mathbb{E}[Z'_t(L_t - I_d)|\mathcal{F}_{t-1}] \in \mathrm{int}\mathbb{R}^d_-$, we can find $\widetilde{L}_t$ such that $(\widetilde{L}_t - L_t)\mathbb{R}^d_+ \subset \mathrm{ri}(K_t)$ and since $L_t - I$ is a cone, $\mathbb{E}\left[Z'_t(\widetilde{L}_t - I_d)|\mathcal{F}_{t-1}\right] \in \mathrm{int}\mathbb{R}^d_-$, for all $1 \le t \le T$. Take now $\eta := V^{\xi,\beta}_T \in \mathfrak{X}^{\widetilde{K},\widetilde{L}}_0 \cap L^0(\mathbb{R}^d_+, \mathcal{F})$. Then $V^{\xi',\beta}_T = 0$ with ξ' defined through $\xi'_t = \xi_t - \eta \mathbf{1}_{t=T} \in -\widetilde{K}_t$. By taking conditional expectation of $Z'_T V^{\xi',\beta}_T$, we get

$$\mathbb{E}\left[\sum_{t=0}^{T} Z'_t \xi'_t + \mathbf{1}_{t<T}\mathbb{E}\left[Z'_{t+1}(\widetilde{L}_{t+1} - I_d)|\mathcal{F}_t\right]\beta_t\right] = 0.$$

Since $\mathbf{1}_{t<T}\mathbb{E}\left[Z'_{t+1}(\widetilde{L}_{t+1} - I_d)|\mathcal{F}_t\right]\beta_t \le 0$ and $Z_t \in \mathrm{ri}(K^*_t)$ for $0 \le t \le T$, we immediately have $\xi'_t \in \widetilde{K}^0_t$. This implies $\eta \in -K_T$ and property (2.6.1). □

Chapitre 3

Conditional sure profit condition in continuous time

3.1 Introduction

This present chapter intends to push forward the study in Chapter 2 by extending the framework to continuous time market models. As in Chapter 2, the reasoning is the following. In the framework of purely financial portfolios, Arbitrage Pricing Theory ensures by a no-arbitrage condition a closedness property for the set of attainable terminal wealth for self financing portfolios. This key property has direct applications. It provides a dual formulation, expressed by the existence of an equivalent martingale measure for pricing purposes, see Chapter 1. In our particular framework, if the financial market runs as usual, production is not bound up with any particular economical condition : it is an idiosyncratic action of the agent. We thus propose in this chapter a general parametric constraint upon the production possibilities of the agent in order to apply arbitrage pricing techniques. In practice, the additional condition is calibrated to market data and the producer's activity. In theory, this condition implies the closedness property of the set of attainable terminal positions, as it is sought in the purely financial case. This property allows to display many financial techniques, such as risk measures or portfolio optimization. The purpose of this note is to demonstrate and apply the undermentioned super-replication theorem for the investor-producer. Let $\mathfrak{X}_0^R(T)$ denote the set of possible portfolio outcomes at time T that the investor-producer can reach starting from 0 at time 0. Let $\mathcal{M}$ be the set of *pricing measures* for the financial market model, we thus show afterwards the following result :

Theorem 3.1.1. *Let H be a contingent claim (see Definition 3.2.2 shortly after). Then*

$$H \in \mathfrak{X}_0^R(T) \iff \mathbb{E}\left[Z'_T H\right] \le \alpha_0^R(Z),\ \forall Z \in \mathcal{M}$$

where $\alpha_0^R(Z) := \sup\left\{\mathbb{E}\left[Z'_T V_T\right]\ :\ V_T \in \mathfrak{X}_0^R(T)\right\}$ *is the support function of* $Z \in \mathcal{M}$ *on* $\mathfrak{X}_0^R(T)$.

The above theorem has a usual interpretation. A (properly defined) contingent claim is super replicable with a strategy starting from nothing at time 0 if and only if the expectation with respect to a pricing measure $Z \in \mathcal{M}$ always verifies a given bounding condition. The chapter is thus structured around that theorem as follows. In Section 3.2, we introduce properly the objects $\mathfrak{X}_0^R(T)$, $\mathcal{M}$ and H. In Section 3.3, we propose the economical condition under which Theorem 3.1.1 holds. In Section 3.4, we give an application of Theorem 3.1.1. Section 3.5 is dedicated to the proof of Theorem 3.1.1.
We shall make a few distinctions with the last chapter. In the latter, we propose to extend the no-arbitrage of second kind condition of Rásonyi [Rásonyi 10] to portfolios augmented by a linear production system. A condition for general production functions has then been introduced using the extended condition in order to allow marginal arbitrages for reasonable levels of production. In the present chapter, we propose an alternative condition which has a close economical interpretation : the *conditional sure profit* condition. Contrary to the *no marginal arbitrage condition* of Chapter 2, it deals directly with general production possibilities and avoids to introduce a linear production system. This is the contribution of Section 3.3. We also focus on investors-producers with specific means of production. Production possibilities are in discrete time as before but we additionally assume concavity and boundedness of the production function. In counterpart, our framework encompasses continuous time financial market models with and without transaction costs. This is the contribution of Section 3.2. The contribution of Section 3.4 is to apply Theorem 3.1.1 in situation. We put a price on a power futures contract for an electricity producer endowed with a simple mean of production.

3.2 The framework

We first introduce the financial possibilities of the agent. We consider an abstract setting allowing to deal with a very large class of market models. This is mainly inspired by [Denis 11b]. To illustrate our framework, we provide two examples in Section 3.2.4. We then introduce production possibilities for the investor.

Preamble. Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a continuous-time filtered stochastic basis on a finite time interval $[0, T]$ satisfying the usual conditions. We assume without loss of generality that $\mathcal{F}_0$ is trivial and $\mathcal{F}_{T^-} = \mathcal{F}_T$. For any $0 \le t \le T$, let $\mathcal{T}$ denote the family of stopping times taking values in $[0,T]$ $\mathbb{P}$-almost surely. From now on, we consider a pair of set-valued $\mathbb{F}$-adapted process K and K^* such that $K_t(\omega)$ is a proper convex closed cone of $\mathbb{R}^d$ including $\mathbb{R}^d_+$ for all $t \in [0,T]$ $\mathbb{P}-$ a.s. The process K^* is defined by

$$K_t^*(\omega) := \left\{ y \in \mathbb{R}^d_+ \ : \ xy \ge 0, \ \forall x \in K_t(\omega) \right\} . \tag{3.2.1}$$

Since $K_t(\omega)$ is proper, its dual $K_t^*(\omega) \neq \{0\}$ for all $t \in [0,T]$ $\mathbb{P}-$ a.s. In the literature on markets with transaction costs, K_t usually stands for the solvency region at time t, and $-K_t$ for the set of possible trades at time t, see [Kabanov 09] and the reference therein. In practice, K and K^* are given by the market model we consider, see the examples of Sections 3.2.4 and 3.4. We use here the process K to introduce a partial order on $\mathbb{R}^d$ at any stopping time in $\mathcal{T}$.

Definition 3.2.1. *Let $\tau \in \mathcal{T}$. For $(\xi, \kappa) \in L^0(\mathbb{R}^{2d}, \mathcal{F}_\tau)$, $\xi \succeq_\tau -\kappa$ if and only if $\xi + \kappa \in L^0(K_\tau, \mathcal{F}_\tau)$.*

Definition 3.2.2. *A contingent claim is a random variable $H \in L^0(\mathbb{R}^d, \mathcal{F}_T)$ such that $H \succeq_T -\kappa$ for some $\kappa \in \mathbb{R}^d_+$.*

3.2.1 The set of financial positions

We consider a financial market on $[0,T]$ with d assets. The market also includes the prices of commodity entering the production process, e.g., fuel or raw materials. The agent we consider has the possibility to trade on this market by starting a portfolio strategy at any time $\rho \in \mathcal{T}$. The financial possibilities of the agent are then represented by a family of sets of wealth processes denoted $(\mathfrak{X}^0_\rho)_{\rho \in \mathcal{T}}$. The superscript 0 stands for *no production*, or *pure financial*.

Definition 3.2.3. *For any $\rho \in \mathcal{T}$, the set $\mathfrak{X}^0_\rho$ is a set of $\mathbb{F}$-adapted d-dimensional processes ξ defined on $[0,T]$ such that $\xi_t = 0$ $\mathbb{P}-$ a.s. for all $t \in [0,\rho)$. We denote by $\mathfrak{X}^0_\rho(T) := \left\{ \xi_T \ : \ \xi \in \mathfrak{X}^0_\rho \right\}$ the corresponding set of attainable financial positions at time T.*

We do not give more details on what a financial strategy is. In all the considered examples, it will denote a self financing portfolio value as commonly defined in Arbitrage Pricing Theory. The multidimensional setting is justified by models of financial portfolios in markets with proportional transaction costs, see [Kabanov 09]. In that case, portfolio are expressed in physical units of assets. Just note that we implicitly assume that the initial wealth of the agent does not influence his financial possibilities, so that a portfolio generically starts with a null wealth in our setting.

Assumption 3.2.1. *For any $\rho \in \mathcal{T}$, the set $\mathfrak{X}^0_\rho(T)$ has the following properties :*

(i) Convexity *: $\mathfrak{X}^0_\rho(T)$ is a convex subset of $L^0(\mathbb{R}^d, \mathcal{F}_T)$ containing* 0.

(ii) Liquidation possibilities *: $\mathfrak{X}^0_\rho(T) - L^\infty(K_s, \mathcal{F}_s) \subseteq \mathfrak{X}^0_\rho(T), \quad \forall s \in [\rho, T]$ $\mathbb{P} - a.s.$*

(iii) Concatenation *: $\mathfrak{X}^0_\rho(T) = \left\{\xi_\sigma + \zeta_T \ : \ (\xi, \zeta) \in \mathfrak{X}^0_\rho \times \mathfrak{X}^0_\sigma, \text{ for any } \sigma \in \mathcal{T} \text{ s.t. } \sigma \geq \rho\right\}$.*

The convexity property holds in most of market models, see [Kabanov 09]. Assumption 3.2.1.(ii) means that whatever the financial position of the agent is, it is always possible for him to throw away a non-negative quantity of assets at any time, or to do an arbitrarily large transfer of assets allowed by the cone $-K_s$. This last possibility is again made for models of markets with convex transaction costs. Finally, the concatenation property also holds in most of market models and sets the additive structure of portfolio processes over time. Note that Assumption (3.2.1) (i) and (iii) imply that $\mathfrak{X}^0_\rho(T) \subset \mathfrak{X}^0_\tau(T)$ for any $(\rho, \tau) \in \mathcal{T}^2$ such that $\rho \geq \tau$.

3.2.2 Absence of arbitrage in the financial market

As for any investor on a financial market, we assume that our investor-producer cannot find an arbitrage opportunity. We elaborate below this condition by relying on the core result of Arbitrage Pricing Theory, which resides in the following fact, see Chapter 1. Formally, when the financial market prices are represented by a process S, the no-arbitrage property for the market holds if and only if there exists a stochastic deflator, i.e., a strictly positive martingale Γ such that the process $Z := \Gamma S$ is a martingale. The process Z can then be seen as the shadow price or fair price of assets. We assume that such a process Z exists by introducing the following

Definition 3.2.4. *Let $\mathcal{M}$ be the set of $\mathbb{F}$-adapted martingales Z on $[0, T]$ taking values in K^*, with strictly positive components, such that*

$$\sup\left\{\mathbb{E}\left[Z'_T\xi_T\right] \ : \ \xi \in \mathfrak{X}^0_0 \ \text{ and } \ \exists\kappa \in \mathbb{R}^d_+ \text{ such that } \forall\tau \in \mathcal{T}, \xi_\tau \succeq_\tau -\kappa\right\} < +\infty\ . \quad (3.2.2)$$

In condition (3.2.2), we apply the pricing measure Z to the subset of $\mathfrak{X}^0_0(T)$ comprising financial wealth processes with a finite credit line κ. We need this basic concept of portfolio admissibility to define $\mathcal{M}$ properly. We will extend admissibility of wealth processes in the next section. Definition 3.2.4 needs more comment. If the set $\mathfrak{X}^0_0(T)$ is a cone, the left hand of (3.2.2) is null for any $Z \in \mathcal{M}$, according to Assumption 3.2.1 (i). In the general non conical case, see Section 3.2.4, the support function in equation (3.2.2) might be positive, justifying the more general condition. If it is equal to 0 then, for any $Z \in \mathcal{M}$ and any $\xi \in \mathfrak{X}^0_0$ with a finite credit line, according to Assumption 3.2.1 (iii), $Z'\xi$

is a supermartingale. We then meet the common no arbitrage condition, see especially Section 3.2.4 below. We thus express absence of arbitrage on the financial market by the following assumption.

Assumption 3.2.2. $\mathcal{M} \neq \emptyset$.

Note that defining $\mathcal{M}$ as above is tailor-made for separation arguments, see the proof of Theorem 3.1.1 and comments following Theorem 1.2.1.

3.2.3 Admissible portfolios and closedness property

If $d = 1$, a financial position ξ_t is naturally solvable if $\xi_t \geq 0$ $\mathbb{P}-$a.s. In the general setting with $d \geq 1$, we use the partial order on $\mathbb{R}^d$ induced by the process K. Defining solvency allows to define admissibility which is central in continuous time : the closedness property concerns the subset of $\mathfrak{X}_0^0(T)$ constituted of admissible portfolios, see [Delbaen 94, Campi 06, Denis 11b, Denis 11a] and the various definitions provided therein. From a financial point of view, it imposes realistic constraints on portfolios and avoids doubling strategies. Here, we use a definition close to the one proposed in [Campi 06].

Definition 3.2.5. *For some constant vector $\kappa \in \mathbb{R}_+^d$, a portfolio $\xi \in \mathfrak{X}_0^0$ is said to be* κ-admissible *if $Z_\tau' \xi_\tau \geq -Z_\tau' \kappa$ for all $\tau \in \mathcal{T}$ and all $Z \in \mathcal{M}$, and $\xi_T \succeq_T -\kappa$.*

Given $\mathcal{M} \neq \emptyset$, the concept of admissibility allows to consider a wider class of terminal wealths than those considered in equation (3.2.2). According to Definition 3.2.4, a wealth process ξ is κ-admissible in the sense of Definition 3.2.5 if ξ verifies $\xi_\tau \succeq_\tau -\kappa$ for all $\tau \in \mathcal{T}$ and some $\kappa \in \mathbb{R}_+^d$. The reciprocal is not always true, and is the object of the so-called **B** assumption investigated in [Denis 11b]. We can finally define the set of admissible elements of $\mathfrak{X}_t^0$:

Definition 3.2.6. *We define $\mathfrak{X}_{t,adm}^0 := \left\{\xi \in \mathfrak{X}_t^0,\ \xi \text{ is } \kappa\text{-admissible for some } \kappa \in \mathbb{R}_+^d\right\}$, and $\mathfrak{X}_{t,adm}^0(T) := \left\{\xi_T \ :\ \xi \in \mathfrak{X}_{t,adm}^0\right\}$.*

The closedness property will be assigned to the sets $\mathfrak{X}_{t,adm}^0(T)$, and is conveyed under the following technical and standing assumption :

Assumption 3.2.3. *For $t \in [0, T]$, let $(\xi^n)_{n\geq 1} \subset \mathfrak{X}_{t,adm}^0$ be a sequence of admissible portfolios such that $\xi_T^n \succeq_T -\kappa$ for some $\kappa \in \mathbb{R}_+^d$ and all $n \geq 1$. Then there exists a sequence $(\zeta^n)_{n\geq 1} \subset \mathfrak{X}_{t,adm}^0$ constructed as a convex combination (with strictly positive weights) of $(\xi^n)_{n\geq 1}$, i.e., $\zeta^n \in \mathrm{conv}(\xi^k)_{k\geq n}$, such that ζ_T^n converges a.s. to some $\zeta_T^\infty \in \mathfrak{X}_t^0(T)$ with n.*

The above assumption calls for the notion of Fatou-convergence. Recall that a sequence of random variables is Fatou-convergent if it is bounded by below and almost surely convergent. According to Assumption 3.2.1 (i), $\mathfrak{X}^0_{t,adm}(T)$ is a convex set, which ensures that the new sequence lies in the set. In Arbitrage Pricing Theory, the Fatou-closedness of $\mathfrak{X}^0_{0,adm}(T)$ often relies on a convergence lemma. Schachermayer [Schachermayer 92] introduced the version of Komlos Lemma provided by Theorem 1.2.2 in Chapter 1, and which is fundamental in [Delbaen 94], while Campi and Schachermayer [Campi 06] proposed another version for markets with proportional transaction costs, see Theorem 1.2.4 in Chapter 1. Assumption 3.2.3 expresses a synthesis of this result, see Sections 3.2.4 and 3.4 for applications.

3.2.4 Illustration of the framework by examples of financial markets

We illustrate here the theoretical framework. We treat two examples, based on [Delbaen 94, Delbaen 95] and [Pennanen 10, Kabanov 03] respectively. In section 3.4, we also apply our results to a continuous time market with càdlàg price processes and proportional transaction costs, as studied in [Campi 06].

A multidimensional frictionless market in continuous time

Consider a filtered stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on $[0,T]$, satisfying the usual assumptions. Let S be a locally bounded $(0,\infty)^d$-valued $\mathbb{F}$-adapted càdlàg semimartingale, representing the price process of d risky assets. We suppose the existence of a non risky asset which is taken constant on $[0,T]$ without loss of generality. Let Θ be the set of $\mathbb{F}$-predictable S-integrable processes and Π the set of $\mathbb{F}$-predictable increasing processes on $[0,T]$. We define, for all $\rho \in \mathcal{T}$

$$\mathfrak{X}^0_\rho := \left\{ \xi = (\xi^1, 0, \ldots, 0) \ : \ \xi^1_s = \int_\rho^s \vartheta_u . dS_u - (\ell_s - \ell_{\rho^-}) \ : \ (\vartheta, \ell) \in \Theta \times \Pi, \ s \in [\rho, T] \right\} .$$

Observe that the set $\mathfrak{X}^0_\rho(T)$ is a convex cone of random variables taking values in $\mathbb{R} \times \{0\}^{d-1}$ $\mathbb{P}-$a.s.. It also contains 0. The set Θ defines the financial strategies. The set Π represents possible liquidation or consumption in the portfolio. The introduction of the latter ensures Assumption (3.2.1) (ii), but does not infer on the mathematical treatment of [Delbaen 94] where Π is not considered. The set $\mathfrak{X}^0_0(T)$ also verifies Assumption (3.2.1) (i) and (iii).

In this context, Delbaen and Schachermayer introduced the No Free Lunch with Vanishing Risk condition (NFLVR) and proved that it is equivalent to

$$\mathcal{Q} := \{\mathbb{Q} \sim \mathbb{P} \text{ such that } S \text{ is a } \mathbb{Q}-\text{local martingale}\} \neq \emptyset \quad \text{(Theorem 1.1 in [Delbaen 94])} .$$

To relate the NFLVR condition to Definition 3.2.4, we define $\mathcal{M}$ as the set of $\mathbb{P}$-equivalent local martingale measure processes $\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_\cdot}$ for $\mathbb{Q} \in \mathcal{Q}$. If S is a locally bounded martingale, elements of $\mathfrak{X}_0^0$ are local supermartingales. We now apply Definition 3.2.5 of admissibility. We take without ambiguity $\widehat{K} = \widehat{K}^* = \mathbb{R}_+^d$. As a consequence, a portfolio $\xi \in \mathfrak{X}_{0,adm}^0$ is κ-admissible only if $\xi_t^1 \geq -\kappa$ for all $t \in [0,T]$, and we retrieve the definition of admissibility of [Delbaen 94]. Therefore, any admissible portfolio is a true supermartingale under $\mathbb{Q} \in \mathcal{Q}$.

By Theorem 4.2 in [Delbaen 94], NFLVR implies that the subset $\mathfrak{X}_{0,adm}^{0,\star}(T)$ of $\mathfrak{X}_{0,adm}^0(T)$ composed of wealths with no consumption, i.e., with $\ell_T = 0$, is Fatou-closed. The proof uses the following convergence property : for any 1-admissible sequence $\xi^n \in \mathfrak{X}_0^{0,\star}$ (defined similarly by portofolio processes without consumption), it is possible to find $\zeta^n \in \mathrm{conv}(\xi^k)_{k\geq n}$ such that ζ^n converges in the semimartingale topology (Lemmata 4.10 and 4.11 in [Delbaen 94]). Hence, ζ_T^n Fatou-converges in $\mathfrak{X}_0^{0,\star}(T)$. This can be easily extended to $\mathfrak{X}_\tau^{0,\star}(T)$ for any $\tau \in \mathcal{T}$ and for any bound of admissibility. In this case, Definition 3.2.5 and the martingale property of ξ^1 imply uniform admissibility in the sense of [Delbaen 94]. Assumption 3.2.3 then holds in this context. The closedness property extends to $\mathfrak{X}_{0,adm}^0(T)$ without difficulty by applying Proposition 3.4 in [?] with the actual definition of admissibility.

A physical market with convex transaction costs in discrete time

Let $(t_i)_{0\leq i \leq N} \subset [0,T]$ be an increasing sequence of deterministic times with $t_N = T$. Let us consider the discrete filtration $\mathbb{G} := (\mathcal{F}_{t_i})_{0\leq i\leq N}$. Here, the market is modelled by a $\mathbb{G}$-adapted sequence $C = (C_{t_i})_{0\leq i\leq N}$ of closed-valued mappings $C_{t_i} : \Omega \mapsto \mathbb{R}^d$ with $\mathbb{R}_-^d \subset C_{t_i}(\omega)$ and $C_{t_i}(\omega)$ convex for every $0 \leq i \leq N$ and $\omega \in \Omega$. We define the recession cones $C_t^\infty(\omega) = \bigcap_{\alpha>0} \alpha C_t(\omega)$ and their dual cones $C_t^{\infty,*}(\omega) = \left\{y \in \mathbb{R}^d \ : \ xy \geq 0, \ \forall x \in C_t^\infty(\omega)\right\}$, see also [Pennanen 10] for a freestanding definition

This setting has been introduced in [Pennanen 10] to model markets with convex transaction costs, such as currency markets with illiquidity costs, in discrete time. Every financial position is labelled in physical units of the d assets, and the sets C_{t_i} denote the possible self financing changes of position at time t_i, so that

$$\mathfrak{X}_{t_i}^0(T) := \left\{ \sum_{k=i}^{N} \xi_{t_k} \ : \ \xi_{t_k} \in L^0(C_{t_k}, \mathcal{F}_{t_k}), \ \forall i \leq k \leq N \right\} \text{ for all } 0 \leq i \leq N \ .$$

In this context, Assumption 3.2.1 trivially holds. If $C_{t_i}(\omega)$ is a cone in $\mathbb{R}^d$ for all $0 \leq i \leq N$ and $\omega \in \Omega$, i.e., $C = C^\infty$, we retrieve a market with proportional transaction costs as described in [Kabanov 03]. In the latter, Kabanov and al. show that the Fundamental Theorem of Asset Pricing can be expressed with respect to the *robust no-arbitrage* property, see [Kabanov 03] for a definition. This condition is equivalent to the existence of a martingale process Z such that $Z_{t_i} \in L^\infty(\mathrm{ri}(C^{\infty,*}_{t_i}), \mathcal{F}_{t_i})$, where $\mathrm{ri}(C^{\infty,*}_{t_i})$ denotes the relative interior of $C^{\infty,*}_{t_i}$. The super replication theorem, see Lemma 3.3.2 in [Kabanov 09], allows $\mathcal{M}$ given by Definition 3.2.4 to be characterized by such elements Z. In that case, the reader can see that C^∞ replaces our conventional cone process K.

As mentioned in [Pennanen 10], the case of general convex transaction costs leads to two possible definitions of arbitrage. One of them is based on the recession cone. Following the terminology of [Pennanen 10], the market represented by C satisfies the *robust no-scalable arbitrage* property if C^∞ satisfies the robust no-arbitrage property. This definition implies that arbitrages might exist, but they are limited for elements of $\mathfrak{X}^0_0(T)$ and even not possible for the recession cone. Pennanen and Penner [Pennanen 10] proved that the set $\mathfrak{X}^0_0(T)$ is closed in probability under this condition. Hence, it is Fatou-closed. The convergence result used in this context is a different argument than the one of Assumption 3.2.3. However, the latter can be applied, see Chapter 2 in which Assumption 3.2.3 has been applied in a very similar context. The notion of admissibility can also be avoided in the discrete time case.

3.2.5 Addition of production possibilities

The previous introduction of a financial market comes from the possibility to interpret the available assets on the market as raw material or saleable goods for a producer. Therefore, we model the production as a function transforming a consumption of the d assets in a new wealth in $\mathbb{R}^d$. Other observations from the situation of an electricity provider lead to our upcoming setting. On a deregulated electricity market, power is provided with respect to an hourly time grid. Production control can thus be fairly approximated by a discrete time framework. We also introduce a delay in the control, as a physical constraint in the production process. See [Kallrath 09] for a monograph illustrating these concerns.

Definition 3.2.7. *Let $(t_i)_{0\le i\le N} \subset [0,T]$ be a deterministic collection of strictly increasing times. We then define a* production regime *as an element β in $\mathcal{B}$, where*

$$\mathcal{B} := \left\{ (\beta_{t_i})_{0\le i<N} \ : \ \beta_{t_i} \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t_i}),\ 0 \le i < N \right\} .$$

A production function *is then a collection of maps $R := (R_{t_i})_{0<i\le N}$ such that for $0 < i \le N$, R_{t_i} is a $\mathcal{F}_{t_i}$-measurable map from $\mathbb{R}^d_+$ to $\mathbb{R}^d$, so that $R_{t_i}(\beta_{t_{i-1}}) \in L^0(\mathbb{R}^d, \mathcal{F}_{t_i})$ for $\beta_{t_{i-1}} \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{i-1}})$.*

Without loss of generality, it is also possible to consider an increasing sequence of stopping times in $\mathcal{T}$ instead of the $(t_i)_{0\leq i\leq N}$. The set $\mathcal{B}$ can also be defined via sequences $(\beta_{t_i})_{0\leq i<N}$ such that β_{t_i} takes values in a convex closed subset of $\mathbb{R}^d_+$. The proofs in section 3.5 would be identical and we refrain from doing this. Notice also that it has no mathematical cost to consider separate times of injection and times of production, i.e., a non-decreasing sequence $\{t_0, s_0, t_1, s_1, \dots, t_N, s_N\} \subset [0,T]$ with $t_i < s_i$, $(t_i)_{0\leq i<N}$ and $(s_i)_{0<i\leq N}$ allowing to define $\mathcal{B}$ and R respectively. As invoked in the introduction, we add fundamental assumptions on the production function.

Assumption 3.2.4. *The production function has the three following properties :*
(i) Concavity *: for all* $0 < i \leq N$*, for all* $(\beta^1, \beta^2) \in L^0(\mathbb{R}^{2d}_+, \mathcal{F}_{t_{i-1}})$ *and* $\lambda \in L^0([0,1], \mathcal{F}_{t_{i-1}})$*,*

$$R_{t_i}(\lambda\beta^1 + (1-\lambda)\beta^2) - \lambda R_{t_i}(\beta^1) - (1-\lambda)R_{t_i}(\beta^2) \in \mathbb{R}^d_+ \ \mathbb{P}-a.s.$$

(ii) Boundedness *: there exists a constant* $\mathfrak{K} \in \mathbb{R}^d_+$ *such that for all* $0 < i \leq N$*,*

$$\mathfrak{K} - |R_{t_i}(\beta) - \beta| \in \mathbb{R}^d_+ \ \mathbb{P}-a.s. \ , \quad \textit{for all } \beta \in \mathbb{R}^d_+ \ .$$

(iii) Continuity *: For any* $0 < i \leq N$*, we have that* $\lim_{\beta^n \to \beta^0} R_{t_i}(\beta^n) = R_{t_i}(\beta^0)$ *.*

These assumptions are fundamental for the continuous time setting. Assumption 3.2.4 (i) keeps the convexity property for the set $\mathfrak{X}^R_0(T)$, see Proposition 3.5.1 in the proofs section. Assumption 3.2.4 (ii) does not only ensure the admissibility of investment-production portfolios when we add production. From the economical point of view, it affirms that the net production income is bounded, which forbids infinite profits. It thus provides a realistic framework for physical production systems. Finally, Assumption 3.2.4.(iii) is a technical assumption in order to use Assumption 3.2.3. It is only needed to ensures upper semicontinuity on the boundary of $\mathbb{R}^d_+$, since continuity comes from (i) inside the domain. See Theorem 2.2.2 in Chapter 2, where convexity is not needed and upper semicontinuity is sufficient. Notice that concavity and the upper bound $\mathfrak{K}$ for the production incomes are given with respect to $\mathbb{R}^d_+$ and not K. This is a useful artefact in the proofs, but also a meaningful expression of a physical bound of production, which has nothing to do with a financial model.

With Assumption 3.2.4, it is possible to fairly approximate a generation asset, see Section 3.4.

Definition 3.2.8. *The set of investment-production wealth processes starting at time* t *is denoted* $\mathfrak{X}^R_t$ *and is given by*

$$\left\{ V \ : \ V_s := \xi_s + \sum_{i=1}^{N} R_{t_i}(\beta_{t_{i-1}} \mathbb{1}_{\{t_{i-1}\geq t\}}) \mathbb{1}_{\{t_i \leq s\}} - \beta_{t_{i-1}} \mathbb{1}_{\{t \leq t_{i-1} \leq s\}}, \ (\xi, \beta) \in \mathfrak{X}^0_{t,adm} \times \mathcal{B} \right\} .$$

The set of terminal possible outcomes for the investor-producer is given by $\mathfrak{X}_t^R(T) := \{V_T \ : \ V \in \mathfrak{X}_t^R\}$.

The agent manages his production system as follows. Assume that he starts an investment-production strategy at time t. On one hand, he performs a financial strategy given by $\xi \in \mathfrak{X}_{t,adm}^0$. On the other hand, he can decide to put a quantity of assets $\beta_{t_{i-1}}$ at time t_i into the production system if $t_{i-1} \geq t$. The latter returns a position $R_{t_i}(\beta_{t_{i-1}})$ labelled in assets at time t_i. At this time, the agent also decides the regime of production β_{t_i} for the next step of time, and so on until time reaches t_N.

The generalization to continuous time controls raises mathematical difficulties. When coming to a continuous time control, we have to make a distinction between the continuous and the discontinuous part of the control, i.e., between a regime of production as a rate and an instantaneous consumption of assets put in the production system. This natural distinction has already been observed for liquidity matters in financial markets, see [Cetin 06]. This implies a separate treatment of consumption in the function R. With a continuous control and as in [Cetin 06] the production becomes a linear function of that control, which is very restrictive and similar to the polyhedral cone setting of markets with proportional transaction costs. With a discontinuous control, non linearity can appear but we face two difficulties. If the number of discontinuities is bounded, it is easy to see that the set of controls is not convex. On the contrary, if it is not bounded, the set is not closed. This problem typically appears in impulse control problems and is not easy to overcome, see Chapter 7 in [Oksendal 05]. We ought to focus on that difficulty in future research.

3.3 The conditional sure profit condition

In the situation of our agent, even if we accept no arbitrage on the financial market, there is no economical justification for the interdiction of profits coming from the production. This is the reason why the concept of *no marginal arbitrage for high production regime* has been introduced in Chapter 2 (**NMA** for short). The **NMA** condition expresses the possibility to make sure profits coming from the production possibilities, but that marginally tend to zero if the production regime β is pushed toward infinity. This condition relied on an affine bound for the production function, introducing then an auxiliary linear production function for which sure profits are forbidden. We propose another parametric condition based on the idea of possibly making solvable profits for a small regime of production. It is stronger than **NMA** under Assumption 3.2.4, see Remark 2.2.3 in Chapter 2, but we express directly the new condition with the production function.

Definition 3.3.1. *We say that there are only* conditional sure profits *for the production function R,* **CSP***(R) holds for short, if there exists $C > 0$ such that for all $0 \le k < N$ and for all $(\xi, \beta) \in \mathfrak{X}^0_{t_k,adm} \times \mathcal{B}$ we have :*

$$\xi_T + \sum_{i=k}^{N-1} R_{t_{i+1}}(\beta_{t_i}) - \beta_{t_i} \succeq_T \sum_{i=k}^{N-1} R_{t_{i+1}}(0) \ \mathbb{P} - a.s. \quad \Longrightarrow \quad \|\beta_{t_i}\| \le C \text{ for } k \le i < N \ .$$

The condition **CSP**(R) thus reads as follows. Since we do not specify portfolios by an initial holding, we can focus on portfolios starting at any time before T with any initial position. If the agent starts an investment-production strategy at an intermediary date $t \in (t_{k-1}, t_k]$ for some k (whatever his initial position is at t), then he can start his production at index k. The condition **CSP**(R) assess that he can do better than the strategy $(0, 0) \in \mathfrak{X}^0_0 \times \mathcal{B}$ only if the regime of production is bounded by C. On a purely financial market, a possible interpretation of the absence of arbitrage is that there is no strategy better than the null strategy $\mathbb{P}$ − a.s. **CSP**(R) is a transposition of this interpretation to production-investment portfolios, where doing nothing means that the agent is subject to fixed costs expressed by $R(0)$. There is no argument against the possibility for an industrial producer to make sure profits, if we put apart the fixed cost of his installation. It is however unrealistic to assume that his production system is not subject to some risks if the regime of production is pushed too high. A comprehensive economical interpretation is available in the previous chapter.

The terminology **CSP**(R) refers to the *no sure profit* property introduced by Rasonyi [Rásonyi 10] (which became the *no sure gain in liquidation value* condition in the final version), since it is formulated in a very similar way and expresses the interdiction for sure profit if some condition is not fulfilled. The **CSP**(R) property is indeed very flexible. It is possible to change the condition "$\|\beta_{t_i}\| \le C$ for $k \le i \le N$" by any restriction implying that :

"There exists a value $c_i \in (0, +\infty)$ such that $\|\beta_{t_i}\| \ne c_i$ for all $0 \le i < N$ ".

This can convey the condition that the regime of production shall be null or greater than a threshold to allow profits, or observe a more precise condition on its components as long as it also constrains the norm of β. Posing **CSP**(R) implies that the closedness property on the financial market alone transmits to the market with production possibilities. Theorem 3.1.1 given in introduction then follows as a corollary to the following proposition.

Proposition 3.3.1. *The set $\mathfrak{X}^R_0(T)$ is Fatou-closed under* **CSP***(R).*

Notice that **CSP**(R) does not have to hold for a specific value of C. As a consequence, Theorem 3.1.1 does not depend on the form of **CSP**(R). This reduces the importance of the chosen form for the condition, since the super-replication price is independent from it.

3.4 Application to the pricing of a power future contract

We illustrate Theorem 3.1.1 by an application to an electricity producer endowed with a generation system converting a raw material, e.g. fuel, into electricity and who has the possibility to trade that asset on a market. We address here the question of a possible price of a term contract a producer can propose on power when he takes into account his generation asset. We assume that the financial market is submitted to proportional transaction costs. For this reason, we place ourselves in the financial framework developed by Campi and Schachermayer [Campi 06].

3.4.1 The financial market

We consider a financial market on $[0, T]$ composed of two assets, cash and fuel, which are indexed by 1 and 2 respectively. The market is represented by a so-called bid-ask process π, see [Campi 06] for a general definition.

Assumption 3.4.1. *The process* $\pi = (\pi_t^{12}, \pi_t^{21})_t$ *is a* $(0, +\infty)^2$*-valued* $\mathbb{F}$*-adapted càdlàg process verifying efficient frictions, i.e.,*

$$\pi_t^{12} \times \pi_t^{21} > 1 \text{ for all } t \in [0, T] \ \mathbb{P} - a.s.$$

Here π_t^{12} denotes at time t the quantity of cash necessary to obtain and $(\pi_t^{21})^{-1}$ denotes the quantity of cash that can be obtained by selling one unit of fuel. The efficient frictions assumption conveys the presence of positive transaction costs. The process π generates a set-valued random process which defines the solvency region :

$$K_t(\omega) := \text{cone}(e^1, e^2, \pi_t^{12}(\omega)e^1 - e^2, \pi_t^{21}(\omega)e^2 - e^1) \quad \forall (t, \omega) \in [0, T] \times \Omega \ .$$

Here (e^1, e^2) is the canonical base of $\mathbb{R}^2$. The process K is $\mathbb{F}$-adapted and closed convex cone-valued. It provides the partial order on $\mathbb{R}^2$ of Definition 3.2.1.

Assumption 3.4.2. *Every* $\xi \in \mathfrak{X}_0^0$ *is a làdlàg* $\mathbb{R}^2$*-valued* $\mathbb{F}$*-predictable process with finite variation verifying, for every* $(\sigma, \tau) \in \mathcal{T}_{[0,T]}^2$ *with* $\sigma < \tau$,

$$(\xi_\tau - \xi_\sigma)(\omega) \in \overline{\text{conv}} \left(\bigcup_{\sigma(\omega) \le u \le \tau(\omega)} -K_u(\omega) \right) \ ,$$

the bar denoting the closure in $\mathbb{R}^d$.

Assumption 3.4.2 implies Assumption 3.2.1. Admissible portfolios are defined via Definitions 3.2.5 and 3.2.1.

Corollary 3.4.1. *Every* $Z \in \mathcal{M}$ *is a* $\mathbb{R}^2_+$*-valued martingale verifying* $(\pi^{21}_t)^{-1} \leq Z^1_t/Z^2_t \leq \pi^{12}_t$ $\mathbb{P} - a.s.$ *and :*
– *for all* $\sigma \in \mathcal{T}$, $(\pi^{21}_\sigma)^{-1} < Z^1_\sigma/Z^2_\sigma < \pi^{12}_\sigma$ *;*
– *for all predictable* $\sigma \in \mathcal{T}$, $(\pi^{21}_{\sigma^-})^{-1} < Z^1_{\sigma^-}/Z^2_{\sigma^-} < \pi^{12}_{\sigma^-}$.

Proof The market model is conical, so that $\alpha^0_0(Z) := \sup\left\{\mathbb{E}\left[Z'_T V_T\right] \ : \ V \in \mathfrak{X}^0_{0,adm}\right\} = 0$, for all $Z \in \mathcal{M}$. The fact that Definition 3.2.4 corresponds to these elements Z follows from the construction of K and is a part of the proof of Theorem 4.1 in [Campi 06]. □

Under the assumption that $\mathcal{M} \neq \emptyset$, $Z\xi$ is a supermartingale for all $Z \in \mathcal{M}$ and admissible $\xi \in \mathfrak{X}^0_0$, see Lemma 2.8 in [Campi 06]. Finally, Assumption 3.2.3 is given by Proposition 3.4 in [Campi 06]. For a comprehensive introduction of all these objects, we refer to [Campi 06].

3.4.2 The generation asset

We suppose that the agent possesses a thermal plant allowing to produce electricity out of fuel on a fixed period of time. The electricity spot price is determined per hour, so that we define the calendar of production as $(t_i)_{0\leq i\leq N} \subset [0, T]$, where N represents the number of generation actions for each hour of the fixed period. At time t_i, the agent puts a quantity $\beta_{t_i} = (\beta^1_{t_i}, \beta^2_{t_i})$ of assets in the plant. The production system transforms at time t_{i+1} the quantity $\beta^2_{t_i}$ of fuel, given a fixed heat rate $q_{i+1} \in \mathbb{R}_+$, into a quantity $q_{i+1}\beta^2_{t_i}$ of electricity (in MWh). The producer has a limited capacity of injection of fuel given by a threshold $\Delta_{i+1} \in L^\infty(\mathbb{R}_+, \mathcal{F}_{t_{i+1}})$. This implies that any additional quantity over Δ_{i+1} of fuel injected in the process will be redirected to storage facilities, i.e., as fuel in the portfolio. The electricity is immediately sold on the market via the hourly spot price. On most of electricity markets, the spot price is legally bounded. It can also happen to be negative. It is thus given by $P_{i+1} \in L^\infty(\mathbb{R}, \mathcal{F}_{t_{i+1}})$. For a given time t_{i+1}, the agent is subject to a fixed cost γ_{i+1} in cash. The agent also faces a cost in fuel in order to maintain the plant activity. This is given by a supposedly non-positive increasing concave function c_{i+1} on $[0, \Delta_{i+1}]$ such that $c'_{i+1}(\Delta_{i+1}) \geq 1$, where c'_{i+1} represents the left derivative. Altogether, we propose the following.

Assumption 3.4.3. *The production function is given by*

$$R_{t_{i+1}}(\beta_{t_i}) = (R^1_{t_{i+1}}(\beta_{t_i}), R^2_{t_{i+1}}(\beta_{t_i}))$$

for $0 \le i < N$*, where*

$$\left\{\begin{array}{lcl} R^1_{t_{i+1}}((\beta^1_{t_i},\beta^2_{t_i})) & = & P_{i+1}q_{i+1}\min(\beta^2_{t_i},\Delta_{i+1}) - \gamma_{i+1} + \beta^1_{t_i} \\ R^2_{t_{i+1}}((\beta^1_{t_i},\beta^2_{t_i})) & = & c_{i+1}(\min(\beta^2_{t_i},\Delta_{i+1})) + \max(\beta^2_{t_i} - \Delta_{i+1}, 0) \end{array}\right. .$$

We can constraint $\beta^1_{t_i}$ to be null at every time t_i without any loss of generality. Indeed $R_{t_i}(\beta_{t_{i-1}}) - \beta_{t_{i-1}}$ does not depend on $\beta^1_{t_{i-1}}$ for any i.

Corollary 3.4.2. *Assumption 3.2.4 holds under Assumption 3.4.3.*

Proof For each i, $R_{t_{i+1}}$ verifies Assumption 3.2.4 (ii) :

$$|R^1_{t_{i+1}}((\beta^1_{t_i},\beta^2_{t_i})) - \beta^1_{t_i}| \le |P_{i+1}q_{i+1}\Delta_{i+1}| + |\gamma_{i+1}| \in L^\infty(\mathbb{R},\mathcal{F}_{t_{i+1}})$$

and

$$|R^2_{t_{i+1}}((\beta^1_{t_i},\beta^2_{t_i})) - \beta^2_{t_i}| \le \max(|c_{i+1}(0)|, |c_{i+1}(\Delta_{i+1}) - \Delta_{i+1}|) \in L^\infty(\mathbb{R}_+,\mathcal{F}_{t_{i+1}}) .$$

Notice that since c_{i+1} is concave with $c'_{i+1}(\Delta_{i+1}) \ge 1$, the function $R^2_{t_{i+1}}$ is clearly concave. The function R is then concave in each component with respect to the usual order, so that Assumption 3.2.4 (i) holds with the partial order induced by K. It is also continuous, so that Assumption 3.2.4 (iii) holds. □

3.4.3 Super replication price of a power futures contract

We now fix a condition provided by the agent in order to apply Definition 3.3.1. For example suppose that the agent knows at time t_i that by producing under a typical regime C (a given threshold of fuel to put in his system) and selling the production at the market price, he can refund the quantity of fuel needed to produce. It is a conceivable phenomenon on the electricity spot market. Since the electricity spot price is actually an increasing function of the total amount of electricity produced by the participants, the agent can sell a small quantity of electricity at high price if the total production is high. He can then partially or totally recover his fixed cost and even make sure profit. The constant C can depend on external factors of the model, such as the level of aggregated demand of electricity.

Assumption 3.4.4. *We assume that there exists* $C > 0$ *such that*

$$R^1_{t_{i+1}}(\beta^2_{t_i}) + \gamma_{i+1} \ge (\pi^{12}_{t_{i+1}})^{-1}(R^2_{t_{i+1}}(\beta^2_{t_i}) - \beta^2_{t_i} - c_{i+1}(0))\ \mathbb{P} - a.s. \Longrightarrow \beta^2_{t_i} \le C \quad (3.4.1)$$

Assumption 3.4.4 compares the marginal return provided by selling produced energy to the cost of production converted into cash. It is a reformulation of **CSP**(R) in the transaction costs framework : here, an immediate transfer $\xi_{t_{i+1}}$ of quantity $R^1_{t_{i+1}}(\beta^2_{t_i})$ of asset 1 brought in asset 2 gives $\xi_{t_{i+1}} + R_{t_{i+1}}(\beta_{t_i}) \succeq R_{t_{i+1}}(0)$. Thus, **CSP**(R) condition holds under Assumption 3.4.4. The latter implies that the set $\mathfrak{X}^R_{0,adm}(T)$ is Fatou-closed, so that we can apply Theorem 3.1.1.

Now we consider the following contingent claim. We denote by $F(x)$ the price of a power futures contract with physical delivery. Buying this contract at time 0 provides a fixed power x (in MW) for N consecutive hours of a fixed period. Here, the N hours correspond to the $(t_i)_{1\leq i\leq N}$. Theorem 3.1.1 can be immediately applied to obtain the price at which the investor-producer can sell the contract.

Corollary 3.4.3. *The price of the futures contract for the investor-producer is given by*

$$F(x) = \sup_{Z\in\mathcal{M}} \left(\frac{1}{Z^1_0} \mathbb{E}\left[\sum_{i=1}^N Z^1_{t_i} P_i x \right] - \alpha^R_0(Z) \right)$$

where the support function $\alpha^R_0(Z)$ is defined by

$$\sup_{\beta\in\mathcal{B}} \mathbb{E}\left[\sum_{i=1}^N Z^1_{t_i} \left(P_i q_i \min(\beta^2_{t_{i-1}}, \Delta_i) - \gamma_i \right) + Z^2_{t_i} \left(c_i(\min(\beta^2_{t_{i-1}}, \Delta_i)) - \min(\beta^2_{t_{i-1}}, \Delta_i) \right) \right] .$$

Theorem 3.1.1 then ensures the existence of a wealth process, involving a financial strategy starting with wealth $F(x)$ and production activities, such that his terminal position is solvent $\mathbb{P}-$ a.s..

3.5 Proofs

3.5.1 Proof of Proposition 3.3.1

We define a collection of sets

$$\widetilde{\mathfrak{X}}^k_t := \left\{ V \ : \ \begin{array}{c} V_s := \xi_s + \sum_{i=1}^k R_{t_{N+1-i}}(\beta_{t_{N-i}})\mathbb{1}_{\{t_{N+1-i}\leq s\}} - \beta_{t_{N-i}}\mathbb{1}_{\{t_{N-i}\leq s\}}, \\ \text{for some } (\xi,\beta) \in \mathfrak{X}^0_{t,adm} \times \mathcal{B} \end{array} \right\}$$

and $\widetilde{\mathfrak{X}}^k_t(T) := \left\{ V_T \ : \ V \in \widetilde{\mathfrak{X}}^k_t \right\}$ for $t\in[0,T]$ and $0\leq k\leq N$, with the convention that

$$\sum_{i=1}^0 R_{t_{N+1-i}}(\beta_{t_{N-i}}) - \beta_{t_{N-i}} = 0 .$$

Note thus that $\widetilde{\mathfrak{X}}^0_t(T)$ corresponds precisely to the set $\mathfrak{X}^0_{t,adm}(T)$. We are conducted by the following guideline. According to Assumption 3.2.3, $\widetilde{\mathfrak{X}}^0_{t_N}(T)$ is Fatou closed. We then proceed by induction in two steps : we first show that $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$ is closed if $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ is closed. Then we prove that $\widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$ is closed if $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$ is closed.

Proposition 3.5.1. *For all $0 \leq k \leq N$, the set $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ is convex.*

Proof This is a consequence of Assumption 3.2.4 (i). Indeed take (ξ^1, β^1) and (ξ^2, β^2) in $\mathfrak{X}^0_{t_{N-k},adm} \times \mathcal{B}$ and $\lambda \in [0,1]$. Take $(\kappa^1, \kappa^2) \in \mathbb{R}^{2d}_+$ the respective bounds of admissibility for ξ^1 and ξ^2. Note that $\lambda\xi^1 + (1-\lambda)\xi^2$ is clearly $(\lambda\kappa^1 + (1-\lambda)\kappa^2)$-admissible since K is a cone-valued process. By Assumption 3.2.4 (i), there exists $(\ell_{t_{N+1-i}})_{1\leq i\leq k}$ with $\ell_{t_{N+1-i}} \in L^0(\mathbb{R}^d_-, \mathcal{F}_{t_{N+1-i}})$ such that for $1 \leq i \leq k$,

$$R_{t_{N+1-i}}(\lambda\beta^1_{t_{N-i}} + (1-\lambda)\beta^2_{t_{N-i}}) + \ell_{t_{N+1-i}} = \lambda R_{t_{N+1-i}}(\beta^1_{t_{N-i}}) + (1-\lambda)R_{t_{N+1-i}}(\beta^2_{t_{N-i}}) \ .$$

Notice that $\mathbb{R}^d_- \subset K_t$ for any $t \in [0,T]$, so that $\ell_{t_{N+1-i}} \in L^0(-K_{t_{N+1-i}}, \mathcal{F}_{t_{N+1-i}})$. We will use this fact throughout the proof. Notice also that, according to Assumption 3.2.4 (ii), each $\ell_{t_{N+1-i}}$ is bounded by below by $2\mathfrak{K}$ for $1 \leq i \leq k$, where $\mathfrak{K}$ is the bound of net production incomes. By relation (3.2.1) and the above fact, $\lambda\xi^1_T + (1-\lambda)\xi^2_T + \sum_{i=1}^k \ell_{t_{N+1-i}} \in \mathfrak{X}^0_{t_{N-k},adm}(T)$. Assembling the parts gives the proposition. □

Proposition 3.5.2. *If $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ is Fatou-closed, then the same holds for $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$.*

Proof Let $(V^n_T)_{n\geq1} \subset \widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$ be a sequence such that V^n_T Fatou-converges to some V^0_T. Let $(\xi^n)_{n\geq1} \subset \mathfrak{X}^0_{t_{N-(k+1)},adm}$ and $(\beta^n_{t_{N-i}})_{1\leq i\leq k, n\geq1}$ with $(\beta^n_{t_{N-i}})_{n\geq1} \subset L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{N-i}})$ for $1 \leq i \leq k$, and $\kappa \in \mathbb{R}^d_+$, such that

$$V^n_T = \xi^n_T + \sum_{i=1}^k R_{t_{N+1-i}}(\beta^n_{t_{N-i}}) - \beta^n_{t_{N-i}} \succeq_T -\kappa \quad \forall n \geq 1 \ .$$

According to Assumption 3.2.4 (ii), and since $\mathbb{R}^d_+ \subset K_T$, we have that for any $n \geq 1$,

$$-k\mathfrak{K} \preceq_T \sum_{i=1}^k R_{t_{N+1-i}}(\beta^n_{t_{N-i}}) - \beta^n_{t_{N-i}} =: \widehat{V}^n_T \in \widetilde{\mathfrak{X}}^k_{t_{N-k}}(T) \ .$$

Due to Assumption 3.2.4 (ii) also, we have that $\xi^n_T \succeq_T -(\kappa + k\mathfrak{K})$ for all $n \geq 1$. According to Assumption 3.2.3, we can then find a sequence of convex combinations $\widetilde{\xi}^n$ of ξ^n, $\widetilde{\xi}^n \in \mathrm{conv}(\xi^m)_{m\geq n}$, such that $\widetilde{\xi}^n_T$ Fatou-converges to some $\widetilde{\xi}^0_T \in \mathfrak{X}^0_{t_{N-(k+1)},adm}(T)$. The convergence of $\widetilde{\xi}^n_T$ implies, by using the same convex weights, that there exists a sequence $(\widetilde{V}^n_T)_{n\geq1}$ of convex combinations of $\widehat{V}^m_T$, $m \geq n$, converging $\mathbb{P}-$ a.s. to some $\widetilde{V}^0_T$. By

Proposition 3.5.1 above, the sequence $(\widetilde{V}_T^n)_{n\geq 1}$ lies in $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$. Recall that it is also bounded by below. Since $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ is Fatou-closed, $\widetilde{V}_T^0 \in \widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ and moreover, $\widetilde{V}_T^0$ is of the form $\sum_{i=1}^k R_{t_{N+1-i}}(\beta^0_{t_{N-i}}) - \beta^0_{t_{N-i}} + \ell^0_{t_{N+1-i}}$ for some $\beta^0 \in \mathcal{B}$ and $(\ell^0_{t_{N+1-i}})_{1\leq i\leq k}$ with $\ell^0_{t_{N+1-i}} \in L^\infty(-K_{t_{N+1-i}}, \mathcal{F}_{t_{N+1-i}})$ for $1 \leq i \leq k$. This is due to Assumption 3.2.4 (i)-(ii). If we let $(\lambda_m)_{m\geq n}$ be the above convex weights, we can always write for $1 \leq i \leq k$ and $n \geq 1$

$$\sum_{m\geq n} \lambda_m \left(R_{t_{N+1-i}}(\beta^m_{t_{N-i}}) - \beta^m_{t_{N-i}} \right) = R_{t_{N+1-i}}(\sum_{m\geq n} \lambda_m \beta^m_{t_{N-i}}) - \sum_{m\geq n} \lambda_m \beta^m_{t_{N-i}} + \ell^n_{t_{N+1-i}} \, .$$

The sets $L^0(-K_{t_{N+1-i}}, \mathcal{F}_{t_{N+1-i}})$ and $L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{N-i}})$ are closed convex cones for $1 \leq i \leq k$, so that $\ell^n_{t_{N+1-i}}$ and $\sum_{m\geq n} \lambda_m \beta^m_{t_{N-i}}$ and their possible limits stay in those sets respectively. From the boundedness condition of Assumption 3.2.4 (ii), the vectors $\ell^n_{t_{N+1-i}}$ are uniformly bounded by below by $2\mathfrak{K}$ for any $1 \leq i \leq k$ and $n \geq 1$, and so are $\ell^0_{t_{N+1-i}}$ for $1 \leq i \leq k$. According to (3.2.1), $\widetilde{\xi}_T^n + \sum_{i=1}^k \ell^0_{t_{N+1-i}} \in \mathfrak{X}^0_{t_{N-(k+1)},adm}(T)$. We then have that $\widetilde{\xi}_T^n + \widetilde{V}_T^N$ converges to $\widetilde{\xi}_T^0 + \widetilde{V}_T^0 = V_T^0 \in \widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$. □

Proposition 3.5.3. *If* $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$ *is Fatou-closed, then the same holds for* $\widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$.

Proof Let $(V_T^n)_{n\geq 1} \subset \widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$ such that there exists $\kappa \in \mathbb{R}^d_+$ verifying $V_T^n \succeq_T -\kappa$ for $n \geq 1$, and V_T^n converges $\mathbb{P}-$a.s. toward $V_T \in L^0(\mathbb{R}^d, \mathcal{F}_T)$ when n goes to infinity. We let $(\bar{V}_T^n, \bar{\beta}^n)_{n\geq 1} \subset \widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T) \times L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{N-(k+1)}})$ be such that $V_T^n = \bar{V}_T^n + R_{t_{N-k}}(\bar{\beta}^n) - \bar{\beta}^n$. Define $\eta^n = |\bar{\beta}^n|$ and the $\mathcal{F}_{t_{N-(k+1)}}$-measurable set $E := \{\limsup_{n\to\infty} \eta^n < +\infty\}$. We consider two cases.

1. First assume that $E = \Omega$. Then $(\bar{\beta}^n)_{n\geq 1}$ is $\mathbb{P}-$ a.s. uniformly bounded. According to Theorem 1.2.5 of Chapter 1, we can find a $\mathcal{F}_{t_{N-(k+1)}}$-measurable random subsequence of $(\bar{\beta}^n)_{n\geq 1}$, still indexed by n for sake of clarity, which converges $\mathbb{P}-$ a.s. to some $\bar{\beta}^0 \in L^\infty(\mathbb{R}^d_+, \mathcal{F}_{t_{N-(k+1)}})$. By Assumption 3.2.4 (iii), $R_{t_{N-k}}(\bar{\beta}^n)$ converges to $R_{t_{N-k}}(\bar{\beta}^0)$, Recall that $\bar{V}_T^n \succeq -\kappa - \mathfrak{K}$ for $n \geq 1$. Since it is $\mathbb{P}$-almost surely convergent to $V_T - R_{t_{N-k}}(\bar{\beta}^0) + \bar{\beta}^0 =: \bar{V}_T^0$ and that $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$ is Fatou-closed, the limit $\bar{V}_T^0$ lies in that set. This implies that $V_T \in \widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$.

2. Assume now that $\mathbb{P}[E^c] > 0$. Since E^c is $\mathcal{F}_{t_{N-(k+1)}}$-measurable, we argue conditionally to that set and suppose without loss of generality that $E^c = \Omega$. We then know that there exists a $\mathcal{F}_{t_{N-(k+1)}}$-measurable subsequence of $(\eta^n)_{n\geq 1}$ converging $\mathbb{P}$-almost surely to infinity with n by an argument similar to the one of Theorem 1.2.5. We overwrite n by the index of this subsequence. We write V_T^n as follows :

$$V_T^n = \xi_T^n + R_{t_{N-k}}(\beta^n_{t_{N-(k+1)}}) - \beta^n_{t_{N-(k+1)}} + \sum_{i=1}^k R_{t_{N+1-i}}(\beta^n_{t_{N-i}}) - \beta^n_{t_{N-i}} \, , \tag{3.5.1}$$

with $(\xi^n)_{n\geq 1} \subset \mathfrak{X}^0_{t_{N-(k+1)},adm}$ and $(\beta^n_{t_{N-i}})_{1\leq i\leq k+1, n\geq 1}$ with $(\beta^n_{t_{N-i}})_{n\geq 1} \subset L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{N-i}})$ for $1 \leq i \leq k+1$, and with the natural convention that for all $n \geq 1$, $\beta^n_{t_{N-(k+1)}} = \bar{\beta}^n$. We then define

$$(\widetilde{V}^n_T, \widetilde{\xi}^n_T, \widetilde{\beta}^n_{t_{N-(k+1)}}, \ldots, \widetilde{\beta}^n_{t_N}) := \frac{2\|C\|}{1+\eta^n}(V^n_T, \xi^n_T, \beta^n_{t_{N-(k+1)}}, \ldots, \beta^n_{t_N}) \ . \tag{3.5.2}$$

Now that $(\widetilde{\beta}^n_{t_{N-(k+1)}})_{n\geq 1}$ is a bounded sequence, we can extract a random subsequence, still indexed by n, such that $(\widetilde{\beta}^n_{t_{N-(k+1)}})_{n\geq 1}$ converges $\mathbb{P}-$ a.s. toward some $\beta^0_{t_{N-(k+1)}}$ in $L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{N-(k+1)}})$. Notice for later that $\|\widetilde{\beta}^n_{t_{N-(k+1)}}\|$ converges to $\|\beta^0_{t_{N-(k+1)}}\| = 2\|C\|$. It is clear that Assumption 3.2.4 (i) allows to write

$$\frac{2\|C\|}{1+\eta^n}\left(R_{t_{N+1-i}}(\beta^n_{t_{N-i}}) - \beta^n_{t_{N-i}}\right) = \\ R_{t_{N+1-i}}(\widetilde{\beta}^n_{t_{N-i}}) - \widetilde{\beta}^n_{t_{N-i}} - \left(1 - \frac{2\|C\|}{1+\eta^n}\right) R_{t_{N+1-i}}(0) + \ell^n_{t_{N+1-i}} \ , \tag{3.5.3}$$

with $(\ell^n_{t_{N+1-i}})_{n\geq 1} \subset L^\infty(-K_{t_{N+1-i}}, \mathcal{F}_{t_{N+1-i}})$ for $1 \leq i \leq k+1$. Note that, according to Assumption 3.2.4 (iii), the particular case $i = k+1$ gives

$$\lim_{n\uparrow\infty} R_{t_{N-k}}(\widetilde{\beta}^n_{t_{N-(k+1)}}) - \widetilde{\beta}^n_{t_{N-(k+1)}} = R_{t_{N-k}}(\beta^0_{t_{N-(k+1)}}) - \beta^0_{t_{N-(k+1)}} \ . \tag{3.5.4}$$

The general case $i \leq k$ follows from Assumption 3.2.4 (ii) applied to equation (3.5.3) : the left hand term converges to 0 and $(1 - \frac{2\|C\|}{1+\eta^n})$ converges to 1, so that

$$\lim_{n\uparrow\infty} R_{t_{N+1-i}}(\widetilde{\beta}^n_{t_{N-i}}) - \widetilde{\beta}^n_{t_{N-i}} + \ell^n_{t_{N+1-i}} = R_{t_{N+1-i}}(0) \ . \tag{3.5.5}$$

By construction of the subsequence, the convexity of $\mathfrak{X}^0_{t_{N-(k+1)},adm}(T)$ and the belonging of 0 to that set, $\widetilde{\xi}^n_T \in \mathfrak{X}^0_{t_{N-(k+1)},adm}(T)$. By using property of Assumption 3.2.1 (ii) and since the sequence $(\ell^n_{t_{N+1-i}})_{n\geq 1}$ is uniformly bounded for any $1 \leq i \leq k+1$, see proof of Proposition 3.5.2 above, we define

$$\widehat{V}^n_T := \widetilde{\xi}^n_T + \ell^n_{t_{N-k}} + \sum_{i=1}^{k}\left(R_{t_{N+1-i}}(\widetilde{\beta}^n_{t_{N-i}}) - \widetilde{\beta}^n_{t_{N-i}} + \ell^n_{t_{N+1-i}}\right) \in \widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T) \ ,$$

which converges by definition and equations (3.5.4) and (3.5.5) to $\widehat{V}^0_T$ such that

$$\widehat{V}^0_T + R_{t_{N-k}}(\beta^0_{t_{N-(k+1)}}) - \beta^0_{t_{N-(k+1)}} \succeq_T \sum_{i=1}^{k+1} R_{t_{N+1-i}}(0) \ . \tag{3.5.6}$$

Notice also that by Assumption 3.2.4 (ii), for all $n \geq 1$

$$\widehat{V}^n_T = \widetilde{V}^n_T - R_{t_{N-k}}(\widetilde{\beta}^n_{t_{N-(k+1)}}) + \widetilde{\beta}^n_{t_{N-(k+1)}} + \sum_{i=1}^{k+1}\left(1 - \frac{2\|C\|}{1+\eta^n}\right) R_{t_{N+1-i}}(0) \succeq_T -(\kappa + (k+1)\mathfrak{K}) \ .$$

By Fatou-closedness of $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$, we finally obtain that $\widehat{V}^0_T + R_{t_{N-k}}(\beta^0_{t_{N-(k+1)}}) - \beta^0_{t_{N-(k+1)}} \in \widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$. By equation (3.5.6) and **CSP**(R), $\|\beta^0_{t_{N-(k+1)}}\| \leq C$ but by construction, $\|\beta^0_{t_{N-(k+1)}}\| = 2\|C\|$, so that we fall on a contradiction. The case **2.** is not possible. □

Remark that the flexibility of the **CSP**(R) condition is reflected in the construction in equation (3.5.2) used in the last lines of the proof of Proposition 3.5.3. The choice of a good norm for $\widetilde{\beta}$ can indeed vary according to the condition we aim at. Following Propositions 3.5.2 and 3.5.3, $\widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$ is Fatou-closed if $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ is Fatou-closed. Proposition 3.5.2 is used a last time to pass from the closedness of $\widetilde{\mathfrak{X}}^N_{t_0}(T)$ to the closedness of $\mathfrak{X}^R_0(T)$.

3.5.2 Proof of Theorem 3.1.1

Proof The "⇒" sense is obvious. To prove the "⇐" sense, we take $H \in L^0(\mathbb{R}^d, \mathcal{F}_T)$ such that $H \succeq -\kappa$ for some $\kappa \in \mathbb{R}^d_+$ and such that $\mathbb{E}[ZH] \leq \alpha^R_0(Z)$ for all $Z \in \mathcal{M}$ and $H \notin \mathfrak{X}^R_0(T)$, and work toward a contradiction. Let $(H^n)_{n\geq 1}$ be the sequence defined by $H^n := H\mathbb{1}_{\{\|H\|\leq n\}} - \kappa\mathbb{1}_{\{\|H\|>n\}}$. By Proposition 3.3.1, $\mathfrak{X}^R_0(T)$ is Fatou-closed, so by Theorem 1.2.7 in Chapter 1, $\mathfrak{X}^R_0(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ is weak*-closed. Since $H \notin \mathfrak{X}^R_0(T)$, there exists k large enough such that $H^k \notin \mathfrak{X}^R_0(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ but, because any $Z \in \mathcal{M}$ has positive components, still satisfies

$$\mathbb{E}\left[Z'_T H^k\right] \leq \alpha^R_0(Z) := \sup\left\{\mathbb{E}\left[Z'_T V_T\right] \;:\; V_T \in \mathfrak{X}^R_0(T)\right\} \quad \text{for all } Z \in \mathcal{M}. \tag{3.5.7}$$

By Proposition 3.5.1, the set $\mathfrak{X}^R_0(T)$ is convex, so that we deduce from the Hahn-Banach theorem that we can find $z \in L^1(\mathbb{R}^d, \mathcal{F}_T)$ such that

$$\sup\left\{\mathbb{E}\left[z'V_T\right] \;:\; V_T \in \mathfrak{X}^R_0(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)\right\} < \mathbb{E}\left[z'H^k\right] < +\infty. \tag{3.5.8}$$

We define $\widetilde{Z}$ by $\widetilde{Z}_t = \mathbb{E}[z|\mathcal{F}_t]$. By using the same argument as in the end of the proof of Proposition 2.3.2 in the previous chapter, we have that $\mathfrak{X}^R_0(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ is dense in $\mathfrak{X}^R_0(T)$ and so that the left hand term of equation (3.5.8) is precisely $\alpha^R_0(\widetilde{Z})$. The process $\widetilde{Z}$ is a non negative martingale and since

$$\left(\mathfrak{X}^R_0(T) - L^\infty(K_t, \mathcal{F}_t)\right) \subset \left(\mathfrak{X}^R_0(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)\right) \; \forall t \in [0, T],$$

we have $\widetilde{Z}_t \in L^1(K^*_t, \mathcal{F}_t)$. The contrary would make the left term of equation (3.5.8) equal to $+\infty$ for suitable sequences $(\xi^m)_{m\geq 1} \subset \mathfrak{X}^R_0$ (see the proof of Proposition 2.3.2 in

Chapter 2). By using the same arguments as above, and since $\mathfrak{X}^0_{0,adm}(T)$ is Fatou-closed too, we have that $\mathfrak{X}^0_{0,adm}(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ is dense in $\mathfrak{X}^0_{0,adm}(T)$. This implies that

$$\begin{aligned}
\alpha_0^0(\widetilde{Z}) &:= \sup\left\{\mathbb{E}\left[\widetilde{Z}'_T V_T\right] \ : \ V_T \in \mathfrak{X}^0_{0,adm}(T)\right\} \\
&= \sup\left\{\mathbb{E}\left[\widetilde{Z}'_T V_T\right] \ : \ V_T \in \mathfrak{X}^0_{0,adm}(T) \cap L^\infty(R^d, \mathcal{F}_T)\right\} \\
&\geq \sup\left\{\mathbb{E}\left[\widetilde{Z}'_T V_T\right] \ : \ V \in \mathfrak{X}^0_0 \text{ and } V_\tau \succeq_\tau -\kappa \text{ for all } \tau \in \mathcal{T}, \text{ for some } \kappa \in \mathbb{R}^d_+\right\}
\end{aligned}$$

Moreover, according to Assumption 3.2.4 (ii), $\xi_T + \sum_{i=1}^N R_{t_i}(0) \in \mathfrak{X}^R_0(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ for any $\xi_T \in \mathfrak{X}^0_{0,ad\frac{1}{2}}(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$, so that

$$\begin{aligned}
\alpha_0^0(\widetilde{Z}) - N\widetilde{Z}'_0\mathfrak{K} &\leq \alpha_0^0(\widetilde{Z}) + \mathbb{E}\left[\widetilde{Z}'_T \sum_{i=1}^N R_{t_i}(0)\right] \\
&\leq \sup\left\{\mathbb{E}\left[z' V_T\right] \ : \ V_T \in \mathfrak{X}^R_0(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)\right\}
\end{aligned}$$

and then $\alpha_0^0(\widetilde{Z})$ is finite according to equation (3.5.8). Take $Z \in \mathcal{M}$. Then there exists $\varepsilon > 0$ small enough such that, by taking $\check{Z} = \varepsilon Z + (1-\varepsilon)\widetilde{Z}$,

$$\alpha_0^R(\check{Z}) \leq \varepsilon\alpha_0^R(Z) + (1-\varepsilon)\alpha_0^R(\widetilde{Z}) < \varepsilon\mathbb{E}\left[Z'_T H^k\right] + (1-\varepsilon)\mathbb{E}\left[\widetilde{Z}'_T H^k\right] = \mathbb{E}\left[\check{Z}'_T H^k\right] \ .$$

It is easy to see that $\check{Z} \in \mathcal{M}$, so that the above inequality contradicts (3.5.7). □

Conclusion of Part 1

Producers still being the majority of participants of a deregulated electricity market, it seems fair to study their situation, and integrate the financial possibilities to production outcomes. This is why, in the first part of the thesis, we propose to extend arbitrage pricing methodology to an agent having production possibilities.

In Chapter 2, we proposed an exhausted framework, in discrete time, of financial market with proportional transaction costs for an agent with delayed production control. The production function is rather general in that case. We provide a parametric economical condition which forbids marginal profits asymptotically. Associated with the no-arbitrage of second kind condition, it provides a fundamental theorem of asset pricing based on measurable selection arguments. The closedness property of the set of terminal wealth is thus not needed and is a corollary of the FTAP. Consequent results are provided for applications : we prove several versions of the super-hedging theorem and provide existence in a simple utility maximization problem.

Chapter 3 represents an attempt of extension of this work. Considering a general production function (unbounded, not concave) seems difficult since continuous time models of financial markets mostly rely on Fatou-convergence in convex sets. We then consider a proper financial market model which encompasses most known models, and then add concave production possibilities at discrete time dates. The provided economical condition is more flexible than the previous one, so that it can hold without much difficulty. Note that, as in the first case, the economical condition does not impact the super-hedging theorem, which relies on the closedness property and the martingale selector only.

This theoretical approach is an autonomous proposition for the construction of a pricing rule for specific agents. Even though it conveys a very large class of models, the results are rather difficult to put in practice for an electricity provider. The second part of the Thesis proposes a more practical approach of this problem.

Première partie

Risk pricing and hedging in electricity market

Abstract

The objective of this part is to present two applications of mathematical finance to Electricity derivative pricing. Both chapters propose a treatment of market incompleteness by means of a specific martingale measure. The first one is an attempt to model electricity spot prices and the corresponding forward contracts by relying on the underlying fuels markets, thus avoiding the electricity non-storability restriction. The structural aspect of the model and the source of incompleteness come from the fact that the electricity spot prices depend on the dynamics of electricity demand and random available capacity of each production mean, which are unhedgeable risk factors. We then use the minimal martingale measure of Föllmer and Schweizer [Föllmer 91] to obtain explicit formulae, and we finally propose calibration and estimation procedures, with results on French market data. The second chapter is a practical application of the stochastic target approach with target in expectation introduced in [Bouchard 09]. It is called up to overpass the problem of granularity of the Electricity prices term structure, introduced as a half-complete market setting. Along the lines of the original paper, we use the convex conjugate of the value function to highlight an explicit formulation based on an equivalent martingale measure. For the general semi-complete market case, we propose a numerical solution of the problem and apply it to the pricing and hedging of an European option on non-existent futures contract.

Keywords : electricity prices ; martingale measure ; structural model ; incomplete market ; forward contracts ; stochastic target ; Monte Carlo simulation ; risk measure.

Note

The content of chapter 4 is based on an article written in collaboration with René Aïd, Luciano Campi and Nizar Touzi and published in *The International Journal Of Theoretical and Applied Finance* (Vol. 12, No. 7, pp. 925-947) in 2009. The second chapter of this part is partially inspired by a paper in preparation with Ludovic Moreau, Nadia Oudjane and Alexandre Tamisier.

Chapitre 4

A structural risk-neutral model of electricity prices

4.1 Introduction

In securities markets, the following relationship between spot and forward prices of a given security holds :

$$F(t,T) = S_t e^{r(T-t)}, \quad t \leq T.$$

As usual, T is the maturity of the forward contract, S_t is the spot price at t and r is the interest rate which is assumed constant for simplicity. We also assumed no dividends. The no-arbitrage arguments usually used to prove such an equality lie heavily upon the fact that securities are storable with zero costs. For storable commodities (oil, soy beans, silver...), the former relation has been extended by including storage costs and and an unobservable variable, the convenience yield (see Schwartz [Schwartz 97], [Routledge 00], and Geman [Geman 07], sec. 3.7). But, when one considers electricity markets (see Burger and al. [Burger 08] or Geman and Roncoroni [Geman 02] for an exhaustive description), such a property does not hold anymore : Once purchased, the electricity has to be consumed, so that the above relation does not make sense. This remark has long been recognized in electricity markets literature (see, e.g., Clewlow & Strickland [Clewlow 00]) but has not prevented the development of many electricity spot price models in the Black & Scholes framework [Benth 03, Benth 07a, Benth 07b, Benth 08, Burger 04, Cartea 05, Eydeland 02] (see Benth [Benth 07b] or [Ventosa 05] for a survey of the literature).

Nevertheless, the fact that electricity is not a storable good is not enough to claim that no relation holds between spot and forward prices and that no arbitrage relations constraint the term structure of the electricity prices, except the constraints coming from overlaping forward contracts. Indeed, one could argue that even if electricity cannot be stored, the fuels that are used to produce electricity can. To see that this observation leads to constraints on the term structure of electricity prices, let us consider a fictitious economy in which power is produced by a single technology - coal thermal units with the same efficiency - and that the electricity spot market is competitive. Then, the electricity price should satisfy the following relation :

$$F_e(t,T) = q_c F_c(t,T), \quad t \leq T,$$

where the subscript e stands for electricity, c stands for coal, and q_c denotes the heat rate. If there is $t < T$ such that $F_e(t,T) > q_c F_c(t,T)$, then one can at time t sell a forward on electricity at $F_e(t,T)$ and buy q_c coal forward at $F_c(t,T)$ and, at time T, sell q_c coal at $S_c(T)$, buy electricity at $S_e(T) = q_c S_c(T)$. One can check that this strategy provides a positive benefit. Moreover, the opposite relation can be obtained by a similar arbitrage. Here, in this fictitious economy, the important feature is not that electricity can be produced by coal, but that the relation between spot prices of coal and electricity is known. Furthermore, it extends directly to the forward prices.

In real economies, similar no-arbitrage relations between electricity and fuels prices can not be identified so easily. The reason for this is that electricity can be produced out of many technologies with many different efficiency levels : Coal plants more or less ancient, fuel plants, nuclear plants, hydro, solar and windfarms, and so on. Generally, the electricity spot prices is considered to be the day-ahead hourly markets. At that time horizon, any producer will perform an ordering of its production means on the basis of their production costs. This operation is refered to a unit commitment problem and one can find a huge literature on this optimization problem in power systems literature (see Batut and Renaud [Batut 92] and Dentcheva et al. [Dentcheva 97] for examples). Depending on the market fuels prices and on the state of power system (demand, outages, inflows, wind and so forth), this ordering may vary through time. Hence, when the forward contract is being signed, the ordering at the contract maturity is not known.

The objective of this chapter is to build a model for electricity spot prices and the corresponding forward contracts, which relies on the underlying fuels markets, thus avoiding the non-storability restriction. The structural aspect of our model comes from the fact that the electricity spot prices depend on the dynamic of the electricity demand at the maturity T, and on the random available capacity of each production means. Our model allows to explain, in a stylized fact, how the different fuels prices together with the demand combine to produce electricity prices. This modeling methodology allows to transfer

to electricity prices the risk-neutral probabilities of the fuels market, under a certain independence hypothesis (see Assumption 4.2.2). Moreover, the model produces, by nature, the well-known peaks observed on electricity market data. In our model, spikes occur when the producer has to switch from one technology to the next lowest cost available one. And, the dynamics of the demand process explains this switching process. Then, one easily understands that the spikes result from a high level of the demand process which forces the producer to use a more expensive technology.
Our model is close to Barlow's model [Barlow 02], since the electricity spot price is defined as an equilibrium between demand and production. But, in our model, the stack curve is described by the different available capacities and not a single parametrized curve. Moreover, this model shares some ideas with Fleten and Lemming forward curve reconstruction method [Fleten 03]. But, whereas the authors methodology relies on an external structural model provided by the SINTEF, our methodology does not require such inputs.

This chapter is structured in the following way : Section 4.2 is devoted to the description of the model; Section 4.4 describes the relation between the futures prices; Section 4.5 presents the model on a case with only two fuels; Section 4.6 presents numerical results showing the potential of the model on the two technologies case of the preceeding section ; and, Section 4.7 provides some research perspectives and recent improvements.

4.2 The Model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space sufficiently rich to support all the processes we will introduce throughout this paper. Let (W^0, W) be an $(n+1)$-dimensional standard Wiener process with $W = (W^1, \dots, W^n)$, $n \geq 1$. In the sequel, we will distinguish between the filtration $\mathcal{F}^0 = (\mathcal{F}^0_t)$ generated by W^0 and the filtration $\mathcal{F}^W = (\mathcal{F}^W_t)$ generated by the n-dimensional Wiener process W.

4.2.1 Commodities market

We consider a market where agents can trade $n \geq 1$ commodities and purchase electricity. We consider only commodities that can be used to produce electricity. For $i = 1, \dots n$, S^i_t denotes the price of the quantity of commodity i necessary to produce 1 KWh of electricity and is assumed to follow the following SDE :

$$dS^i_t = S^i_t \left(\mu^i_t dt + \sum_{j=1}^{n} \sigma^{ij}_t dW^j_t \right), \quad t \geq 0, \tag{4.2.1}$$

where μ^i and σ^{ij} are $\mathcal{F}^W$-adapted processes suitably integrable (see Assumption 4.2.1). We also assume that the market contains a riskless asset with price process

$$S_t^0 = e^{\int_0^t r_u du}, \ t \geq 0,$$

where the instantaneous interest rate $(r_t)_{t\geq 0}$ is an $\mathcal{F}^W$-adapted non-negative process such that $\int_0^t r_u du$ is finite a.s. for every $t \geq 0$. As a consequence, (r_t) is independent of the Brownian motion W^0. We will frequently used the notation $\widetilde{X}_t := X_t/S_t^0$ for any process (X_t). We make the following standard assumption (see, e.g. Karatzas [Karatzas 97], Section 5.6).

Assumption 4.2.1. *The volatility matrix $\sigma_t = (\sigma_t^{ij})_{1\leq i,j\leq n}$ is invertible and both matrices σ and σ^{-1} are bounded uniformly on $[0, T^*] \times \Omega$. Finally, let θ denote the market price of risk, i.e.*

$$\theta_t := \sigma_t^{-1}[\mu_t - r_t \mathbf{1}_n], t \geq 0,$$

where $\mathbf{1}_n$ is the n-dimensional vector with all unit entries. We assume that such a process θ satisfies the so-called Novikov condition

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^{T^*} ||\theta_t||^2 dt\right\}\right] < \infty \ a.s.$$

Remark 4.2.1. *Imposing the Novikov condition on the commodities market price of risk ensures that the minimal martingale measure we will use for pricing in Section 4.4 is well defined. The reader is referred to Section 5.6 in Karatzas's book [Karatzas 97].*

4.2.2 Market demand for electricity

We model the electricity market demand by a real-valued continuous process $D = (D_t)_{t\geq 0}$ adapted to the filtration $\mathcal{F}^0 = (\mathcal{F}_t^0)$ generated by the Brownian motion W^0. Observe that, under our assumptions, the processes S^i $(i = 0, \dots, n)$ are independent under $\mathbb{P}$ of the demand process D. To be more precise, the process D models the whole electricity demand of a given geographical area (e.g. U.K., Switzerland, Italy and so on). With that respect, it must be strictly positive. Nevertheless, in Section 4.6 where empirical analysis is performed, to reduce the number of possible technologies, it is more convenient to use a *residual demand.* A residual demand is the whole demand less the production of some generation assets (like nuclear power, run of the river hydrolic plants, wind farms). It is clear that the residual demand can be negative.

4.2.3 Electricity spot prices

We denote by P_t the electricity spot price at time t. At any time t, the electricity producer can choose among the n commodities which is the most convenient to produce electricity at that particular moment and the electricity spot price will be proportional to the spot price of the chosen commodity. We recall that the proportionality constant is already included in the definition of each S^i so that, if at time t the producer chooses commodity i then $P_t = S_t^i$, $1 \le i \le n$.

How does the electricity producer choose the most convenient commodity to use ? For each $i = 1, \ldots, n$, we denote $\Delta_t^i > 0$ the given capacity of the i-th technology for electricity production at time t. (Δ_t^i) is a stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and assumed independent of (W^0, W). We denote $\mathcal{F}^\Delta = (\mathcal{F}_t^\Delta)$ its filtration. Moreover, we assume that each Δ_t^i takes values in $[m_i, M_i]$ where $0 \le m_i < M_i$ are the minimal and the maximal capacity of i-th technology, both values being known to the producer. In reality, the producer fills capacity constraints, so as to deal with demand variability, security conditions and failures risk. Thus, in order to represent capacity management and partial technology failures, the production capacity is considered as a stochastic process on its own filtration.

For every given $(t, \omega) \in \mathbb{R}_+ \times \Omega$, the producer performs an ordering of the commodities from the cheapest to the most expensive. The ordered commodities prices are denoted by

$$S_t^{(1)}(\omega) \le \cdots \le S_t^{(n)}(\omega).$$

This order induces a permutation over the index set $\{1, \ldots, n\}$ denoted by

$$\pi_t = \{\pi_t(1), \ldots, \pi_t(n)\} .$$

Notice that π_t defined an $\mathcal{F}^W$-adapted stochastic process, and we follow the usual probabilistic notation omitting its dependence on ω. Given a commodities order π_t at time t, we set

$$I_k^{\pi_t}(t) := \left[\sum_{i=1}^{k-1} \Delta_t^{\pi_t(i)}, \sum_{i=1}^{k} \Delta_t^{\pi_t(i)} \right), \quad 1 \le k \le n,$$

with the convention $\sum_{i=1}^{0} \equiv 0$.

For the the sake of simplicity, we will assume from now on that the electricity market is competitive and we will not take into account the short term constraints on generation assets as well as start-up costs. Hence, the electricity spot price is equal the cost of the last production unit used in the stack curve (marginal unit). Thus, if the market demand at time t for electricity D_t belongs to the interval $I_k^{\pi_t}(t)$, the last unit of electricity is produced by means of technology $\pi_t(k)$, when available. Otherwise, it is produced with the next one with respect to the time-t order π_t. This translates into the following formula :

$$P_t = \sum_{i=1}^{n} S_t^{(i)} \mathbb{1}_{\{D_t \in I_i^{\pi_t}(t)\}}, \quad t \geq 0. \tag{4.2.2}$$

Let $T^* > 0$ be a given finite horizon, in the sequel we will work on the finite time interval $[0, T^*]$. Typically, all maturities and delivery dates of forward contracts we will consider in the sequel, will always belong to the time interval $[0, T^*]$.

Assumption 4.2.2. *Let $\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{F}_t^W \vee \mathcal{F}_t^\Delta$, $t \in [0, T^*]$, be the market filtration. There exists an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ defined on $\mathcal{F}_{T^*}$, such that the discounted commodities prices $\tilde{S} = (\tilde{S}^1, \ldots, \tilde{S}^n)$ (i.e. without electricity) are local $\mathbb{Q}$-martingales with respect to $(\mathcal{F}_t)$.*

This hypothesis is equivalent to assuming absence of arbitrage in the fuels market [Delbaen 94]. Notice that we are not making this assumption on the electricity market, as announced in the introduction. Thanks to relation (4.2.2), any electricity derivative can be viewed as a basket option on fuels. Hence, Assumption 4.2.2 allows us to properly apply the usual risk neutral machinery to price electricity derivatives.

4.3 The choice of an equivalent martingale measure

The market of commodities *and* electricity is clearly incomplete, due to the presence of additional unhedgeable randomness source W^0 driving electricity demand's dynamics D. Thus, in order to price derivatives on electricity we have to choose an equivalent martingale measure among infinitely many to use as a pricing measure. One possible choice is the following. Let $\mathbb{Q} := \mathbb{Q}^{min}$ denote the minimal martingale measure introduced in Chapter 1, initially proposed by Föllmer and Schweizer [Föllmer 91], i.e.

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{-\int_0^{T^*} \theta_u' dW_u - \frac{1}{2}\int_0^{T^*} ||\theta_u||^2 du\right\} \tag{4.3.1}$$

where we recall that $\theta_t = \sigma_t^{-1}(\mu_t - r_t \mathbf{1}_n)$ is the market price of risk for the commodities market $(S^1, \ldots, S^n)$. This form follows Theorem 1.1.1 in Chapter 1 Notice that, due to Assumption 4.2.1, such a measure is well defined, i.e. (4.3.1) defines a probability measure on $\mathcal{F}_{T^*}$, which is equivalent to the objective measure $\mathbb{P}$.

Remark 4.3.1. *It can be easily checked that under $\mathbb{Q}$ the laws of processes W^0 and Δ^i ($1 \leq i \leq n$) are the same as under the objective probability $\mathbb{P}$ and the independence between the filtrations $\mathcal{F}^0$, $\mathcal{F}^\Delta$ and $\mathcal{F}^W$ is preserved under $\mathbb{Q}$.*

A justification for that particular choice of pricing measure, along the lines of Remark 4.3.1, is that Q minimizes the relative entropy $\mathcal{H}(.|\mathbb{P})$ defined by

$$\mathcal{H}(\mathbb{P}'|\mathbb{P}) = \int_\Omega log(\frac{d\mathbb{P}'}{d\mathbb{P}})d\mathbb{P}' \ .$$

One can then see $\mathbb{Q}$ as the closest equivalent martingale measure for $\widetilde{S}$ to the objective measure $\mathbb{P}$, given this criteria. This recall Theorem 1.1.2 in Chapter 1 along with the comments in the corresponding section.

The measure $\mathbb{Q}$ will be used as pricing measure in the rest of the chapter. This is a core assumption. Indeed, if one refers to [Schweizer 01], such a measure $\mathbb{Q}$ is related to locally risk minimization procedure, in the sense that, given a contingent claim H with some maturity $T^* > 0$, $\mathbb{E}^{\mathbb{Q}}[\exp(-\int_0^T r_s ds)H]$ is the minimum price allowing an agent to approximately (and locally in L^2) hedge the claim. Namely, (H_0, ϕ) is a local risk minimization strategy if and only if H admits a *Föllmer-Schweizer* decomposition

$$H = H_0 + \int_0^{T^*} \phi_t' d\widetilde{S}_t + L_T^H$$

where L^H is a $\mathbb{P}$-martingale bounded in $L^2(\mathbb{P})$ orthogonal to S. This strategy is in fact uniquely determined and defined by the minimal martingale measure (see Theorem 3.14 in [Föllmer 91]). The expectation of H under $\mathbb{Q}$ is one of the infinitely possible no-arbitrage prices of H, but it is precisely the initial wealth allowing to hedge the hedgeable part of H, i.e. the part depending on commodities.

Under such a probability $\mathbb{Q}$, commodities prices S^i, $1 \leq i \leq n$, satisfy the SDEs

$$dS_t^i = S_t^i \left(r_t dt + \sum_{j=1}^d \sigma_t^{i,j} d\widetilde{W}_t^j \right), \quad S_0^i > 0,$$

whose solutions are given by

$$S_t^i = S_0^i \exp\left\{ \int_0^t \left(r_u - \frac{1}{2}||\sigma_u^i||^2 \right) du + \int_0^t \sigma_u^{i\prime} d\widetilde{W}_u \right\}, \quad t \geq 0,$$

where $\widetilde{W} = (\widetilde{W}^1, \ldots, \widetilde{W}^d)$ is a n-dimensional Brownian motion under $\mathbb{Q}$, and $\sigma^i = (\sigma^{i,1}, \ldots, \sigma^{i,n})$.

Remark 4.3.2. *Notice that including storage costs c^i and convenience yields δ^i changes only the drifts coefficients in commodities dynamics from r_t to $r_t + c_i - \delta_i$.*

4.4 Electricity forward prices

We now consider a so-called forward contract on electricity with maturity $T_1 > 0$ and delivery period $[T_1, T_2]$ for $T_1 < T_2 \leq T^*$, i.e. a contract defined by the payoff

$$(T_2 - T_1)^{-1} \int_{T_1}^{T_2} P_T dT \tag{4.4.1}$$

at the maturity T_1, whose time-t price $F_t(T_1, T_2)$ is to be paid at T_1.
The following observation is crucial : according to formula 4.2.2, the payoff (4.4.1) can be expressed in terms of the fuels prices, so that in our model the forward contract on electricity can be viewed as a forward contract on fuels and since the classical no-arbitrage theory makes sense on the fuels market, it can also be used to price electricity derivatives such as (4.4.1). In other terms, our production-based structural model relating electricity and fuels prices allows us to transfer the whole no-arbitrage classical approach from fuels to electricity market, so overcoming the non-storability issue.
By Assumption 4.2.2 and classical result on forward pricing (see [Björk 04], Chapter 26), it immediately follows that :

$$F_t(T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \frac{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du} P_T\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\right]} dT, \tag{4.4.2}$$

$\mathbb{E}_t^{\mathbb{Q}}$ denoting the conditional $\mathbb{Q}$-expectation given market's filtration $\mathcal{F}_t$, for $t \geq 0$.
Let $T \in [T_1, T_2]$. It is convenient for the next calculations to introduce the forward measure $\mathbb{Q}_T$ defined by the density

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}} := \frac{e^{-\int_t^T r_u du}}{B_t(T)} \quad \text{on } \mathcal{F}_T^W,$$

where

$$B_t(T) := \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\right]$$

is the time-t price of a zero-coupon bond with maturity T. Then :

$$\begin{aligned} F_t(T_1, T_2) &= \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \mathbb{E}^{\mathbb{Q}_T}\left[P_T | \mathcal{F}_t\right] dT & (4.4.3) \\ &= \sum_{i=1}^{n} \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \mathbb{E}^{\mathbb{Q}_T}\left[S_T^{(i)} \mathbb{1}_{\{D_T \in I_i^{\pi_T}(T)\}} | \mathcal{F}_t\right] dT. & (4.4.4) \end{aligned}$$

We denote by Π_n the set of all permutations over the index set $\{1, \dots, n\}$. Let $\pi \in \Pi_n$ be a given non-random permutation. Under the assumption $S_t^i \in L^1(\mathbb{Q}_t)$ for any $t \geq 0$

and $1 \le i \le n$, we can define the following changes of probability on $\mathcal{F}_T^W$:

$$\frac{d\mathbb{Q}_T^i}{d\mathbb{Q}_T} = \frac{S_T^i}{\mathbb{E}^{\mathbb{Q}_T}[S_T^i]}, \quad 1 \le i \le n, T \le T^*.$$

Proposition 4.4.1. *If our model assumptions hold and if $S_T^i \in L^1(\mathbb{Q}_T)$ for all $T \in [T_1, T_2]$ and $1 \le i \le n$, we have*

$$F_t(T_1,T_2) = \frac{1}{T_2-T_1}\sum_{i=1}^{n}\sum_{\pi\in\Pi_n}\int_{T_1}^{T_2} F_t^{\pi(i)}(T)\mathbb{Q}_T^{\pi(i)}[\pi_T=\pi|\mathcal{F}_t^W]\mathbb{Q}_T[D_T\in I_i^{\pi}(T)|\mathcal{F}_t^{0,\Delta}]dT, \tag{4.4.5}$$

for $t \in [0, T_1]$, where $F_t^i(T)$ denotes the price at time t of forward contract on the i-th commodity with maturity T and $\mathcal{F}_t^{0,\Delta}$ is the natural filtration generated by both W^0 and Δ.

Proof Notice first that

$$F_t(T_1,T_2) = \frac{1}{T_2-T_1}\int_{T_1}^{T_2} F_t(T)dT,$$

where $F_t(T) = \mathbb{E}^{\mathbb{Q}_T}[P_T|\mathcal{F}_t]$ can be interpreted as the t-price of a forward contract with maturity T and instantaneous delivery at maturity. By the definition of electricity forward price $F_t(T)$, we have

$$\begin{aligned} F_t(T) &= \sum_{i=1}^{n}\mathbb{E}^{\mathbb{Q}_T}\left[S_T^{(i)}\mathbb{1}_{\{D_T\in I_i^{\pi_T}(T)\}}|\mathcal{F}_t\right] \\ &= \sum_{i=1}^{n}\sum_{\pi\in\Pi_n}\mathbb{E}^{\mathbb{Q}_T}\left[S_T^{\pi(i)}\mathbb{1}_{\{D_T\in I_i^{\pi}(T)\}}\mathbb{1}_{\{\pi_T=\pi\}}|\mathcal{F}_t\right]. \end{aligned}$$

If we use the mutual (conditional) independence between W, W^0 and Δ as in Remark 4.3.1, we get

$$F_t(T) = \sum_{i=1}^{n}\sum_{\pi\in\Pi_n}\mathbb{E}^{\mathbb{Q}_T}\left[S_T^{\pi(i)}\mathbb{1}_{\{\pi_T=\pi\}}|\mathcal{F}_t^W\right]\mathbb{Q}_T[D_T\in I_i^{\pi}(T)|\mathcal{F}_t^{0,\Delta}].$$

Using the change of probability $d\mathbb{Q}_T^{\pi(i)}/d\mathbb{Q}_T$ yields

$$\mathbb{E}^{\mathbb{Q}_T}\left[S_T^{\pi(i)}\mathbb{1}_{\{\pi_T=\pi\}}|\mathcal{F}_t^W\right] = F_t^{\pi(i)}(T)\mathbb{Q}_T^{\pi(i)}[\pi_T=\pi|\mathcal{F}_t^W],$$

so giving, after integrating between T_1 and T_2 and dividing by $T_2 - T_1$, the announced formula. □

The main formula (4.4.5) provides a formal expression to the current intuition of electricity market players that the forward prices are expected to be equal to a weighted average of forward fuels prices. Such weights are determined by the crossing of the expected demand with the expected stack curve of technologies. We will see in Section 4.6 that this model is able to explain the spikes of electricity. Nonetheless, we can already observe that the main formula reproduces the stylized fact that the paths of electricity forward prices are much smoother than those of spot prices. This is due to the averaging effect of the conditional expectation on the indicator functions appearing in formula (4.2.2), even in the degenerate case when the delivery period reduces to a singleton.
In the next section, we will perform some explicit computations of the conditional probabilities involved in the previous formula for electricity forward prices, under more specific assumptions on prices and demand dynamics.

4.5 A model with two technologies and constant coefficients

In order to push further the explicit calculations, we assume now that the combustibles volatilities are constant, i.e. $\sigma_t^{i,j} = \sigma^{i,j}$ for some constant numbers $\sigma^{i,j} > 0$, $1 \leq i, j \leq n$, and that the interest rate is constant $r_t = r > 0$. Under the latter simplification, the forward-neutral measures $\mathbb{Q}_T$ all coincide with the minimal martingale measure $\mathbb{Q} = \mathbb{Q}^{min}$. Similar closed-form expressions can be obtained by assuming a Gaussian Heath-Jarrow-Morton model for the yield curve. Let us assume from now on that only two technologies are available, i.e. $n = 2$.

4.5.1 Dynamics of capacity processes Δ^i

In order to get explicit formulae for forward prices we have to specify the dynamics of capacity processes Δ^i for the i-th technology. We assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supports four (independent) standard Poisson processes $N_t^{1,u}, N_t^{1,d}, N_t^{2,u}$ and $N_t^{2,d}$ with constant intensities $\lambda_1^u, \lambda_1^d, \lambda_2^u, \lambda_2^d > 0$ and we assume that each Δ^i follows

$$d\Delta_t^i = (m_i - M_i)\mathbb{1}_{\{(\Delta_t^i = M_i)\}} dN_t^{i,d} + (M_i - m_i)\mathbb{1}_{\{(\Delta_t^i = m_i)\}} dN_t^{i,u}, \quad \Delta_0^i = M_i \,. \quad (4.5.1)$$

Remark 4.5.1. *Basically we are assuming that each capacity i can take only two values $M_i > m_i$ and it switches from m_i to M_i (resp. from M_i to m_i) when the process $N^{i,u}$ (resp. $N^{i,d}$) jumps. Each capacity evolves independently of each other. At $t = 0$ both technologies have maximal capacity M_i. The fact that the intensities of upside and downside jumps of Δ^i are not necessarily equal introduces a skewness in the probability of being at capacity M_i or m_i.*

Let T be any time in the delivery period $[T_1, T_2]$. First observe that, since Δ is independent of W^0 and its law is invariant under the probability change from $\mathbb{P}$ to $\mathbb{Q} = \mathbb{Q}_T$ as in Remark 4.3.1, we have $\mathbb{Q}_T[\Delta_T^{\pi(1)} = x_1 | \mathcal{F}_t^{0,\Delta}] = \mathbb{P}[\Delta_T^{\pi(1)} = x_1 | \Delta_t]$ as well as

$$\mathbb{Q}_T[\Delta_T^{\pi(1)} = x_1, \Delta_T^{\pi(2)} = x_2 | \mathcal{F}_t^{0,\Delta}] = \mathbb{P}[\Delta_T^{\pi(1)} = x_1, \Delta_T^{\pi(2)} = x_2 | \Delta_t]$$

for $x_1 \in \{m_1, M_1\}$ and $x_2 \in \{m_2, M_2\}$.
As a consequence of the previous assumption on the dynamics of capacities Δ^i, the conditional probabilities $\mathbb{Q}_T[D_T \in I_k^\pi(T) | \mathcal{F}_t^{0,\Delta}]$ appearing in the main formula (4.4.5) can be decomposed as follows

$$\begin{aligned}\mathbb{Q}_T[D_T \in I_1^\pi(T) | \mathcal{F}_t^{0,\Delta}] =& \mathbb{Q}_T\left[D_T \leq \Delta_T^{\pi(1)} | \mathcal{F}_t^{0,\Delta}\right] \\ =& \mathbb{P}[\Delta_T^{\pi(1)} = m_1 | \mathcal{F}_t^{\Delta}] \mathbb{Q}_T\left[D_T \leq m_1 | \mathcal{F}_t^0\right] \\ &+ \mathbb{P}[\Delta_T^{\pi(1)} = M_1 | \mathcal{F}_t^{\Delta}] \mathbb{Q}_T\left[D_T \leq M_1 | \mathcal{F}_t^0\right] .\end{aligned}$$

A similar decomposition for $\mathbb{Q}_T[D_T \in I_2^\pi(T) | \mathcal{F}_t^{0,\Delta}]$ holds too. It is clear now that the building blocks appearing in such formulae are the probabilities $\mathbb{P}[\Delta_T^k = x | \Delta_t^k]$ and $\mathbb{Q}_T\left[D_T \leq y | \mathcal{F}_t^0\right]$.

Proposition 4.5.1. *We have the following :*

$$\mathbb{P}[\Delta_T^k = M_k | \Delta_t^k = M_k] = \frac{\lambda_k^d}{\lambda_k^d + \lambda_k^u}(1 - e^{-(\lambda_k^d + \lambda_k^u)(T-t)}), \quad k = 1, 2 . \tag{4.5.2}$$

Proof For the sake of simplicity, we will drop in the proof the index k from the notation, that is we will write Δ_T for Δ_T^k, M for M_k, and so on.
Let τ^d be the last jump time of the process N_t^d before T, i.e. $\tau^d = \sup\{t \in [0, T] : \Delta N_t^d = 1\}$ with the convention that $\sup \emptyset = 0$. Notice that on the event $\{\tau^d > 0\}$ we have $\{\Delta_T = m\} = \{N_{\tau^d}^u = N_T^u\}$. On the other hand, on the set $\{\tau^d = 0\}$ the process Δ has no jump downwards over the time interval $[0, T]$, so that $\mathbb{P}(\Delta_T = m, \tau^d = 0 | \Delta_0 = M) = 0$. Using the independence between N^d and N^u and the stationarity of N^u, one has

$$\begin{aligned}\mathbb{P}[\Delta_T = m | \Delta_0 = M] =& \mathbb{E}[\mathbb{P}(N_{\tau^d}^u = N_T^u | \tau^d) \mathbb{1}_{\{\tau^d > 0\}}] \\ =& \mathbb{E}[\mathbb{P}(N_{T-\tau^d}^u = 0 | T - \tau^d) \mathbb{1}_{\{T-\tau^d < T\}}] \\ =& \mathbb{E}[e^{-\lambda^u (T-\tau^d)} \mathbb{1}_{\{T-\tau^d < T\}}].\end{aligned}$$

By the time-reversal property of the standard Poisson process, the process $(N_T^d - N_{(T-t)-}^d)_{t \geq 0}$ as the same law as $(N_t^d)_{t \geq 0}$. Then the random variable $T - \tau^d$ has the same law as $T_1^d \wedge T$, where T_1^d is the first jump time of $(N_t^d)_{t\geq 0}$. We recall that T_1 has exponential law with parameter λ^d. Thus we have

$$\begin{aligned}\mathbb{P}[\Delta_T = m|\Delta_0 = M] =& \mathbb{E}[e^{-\lambda^u(T_1^d\wedge T)}\mathbb{1}_{\{T_1^d<T\}}] = \mathbb{E}[e^{-\lambda^u T_1^d}\mathbb{1}_{\{T_1^d<T\}}] \\ =& \frac{\lambda^d}{\lambda^d+\lambda^u}(1-e^{-(\lambda^d+\lambda^u)T})\end{aligned}$$

The result of the proposition follows by stationarity. □

Using the same arguments one can retrieve immediately $\mathbb{P}[\Delta_T^k = x|\mathcal{F}_t^\Delta]$ for $x = M_k, m_k$ and $k = 1, 2$.

4.5.2 Dynamics of the electricity demand D

We also assume that the residual demand is defined by the a mean-reverting Ornstein-Uhlenbeck process.

$$dD_t = a(b(t) - D_t)dt + \delta dW_t^0 \quad , D_0 \in \mathbb{R}\ . \tag{4.5.3}$$

It is well-known that this process has a positive probability to be negative. Nonetheless, in the empirical study, it will be applied to a residual demand, which can be negative (see Section 4.2). Parameters a and δ are supposed to be strictly positive constants, and we define a long-term mean $b(t)$ which can vary with time, to incorporate annual seasonal effects as in [Barlow 02] :

$$b(t) = b_0 + b_1\cos(2\pi t - b_2) - \frac{2\pi}{a}\sin(2\pi t - b_2) \quad ,$$

where b_0, b_1 and b_2 are (positive) constants. We set $\tilde{b}(t) = b_0 + b_1\cos(2\pi t - b_2)$. In our case, we obtain explicit formulae for $\mathbb{Q}[D_T \leq x_1|\mathcal{F}_t^0]$ and $\mathbb{Q}[x_1 < D_T \leq x_1 + x_2|\mathcal{F}_t^0]$, for any $0 \leq t \leq T$ and $x_1, x_2 \in \mathbb{R}$, given by

$$\mathbb{Q}[D_T \leq x_1|\mathcal{F}_t^0] = \Phi\left(\frac{x_1 - \tilde{b}(T) - (D_t - \tilde{b}(t))e^{-a(T-t)}}{\delta\sqrt{\frac{1}{2a}\left(1 - e^{-2a(T-t)}\right)}}\right) \tag{4.5.4}$$

$$\mathbb{Q}[x_1 < D_T \leq x_1 + x_2|\mathcal{F}_t^0] = \mathbb{Q}[D_T \leq x_1 + x_2|\mathcal{F}_t^0] - \mathbb{Q}[D_T \leq x_1|\mathcal{F}_t^0] \tag{4.5.5}$$

where Φ denotes the cumulative distribution function of an $\mathcal{N}(0, 1)$ random variable.

4.5.3 Forward prices

Let $T \in [T_1, T_2]$. The next step consists in computing the law of the couple (S_T^1, S_T^2) under each probability $\mathbb{Q}_T^{\pi(i)}$ for any permutation $\pi \in \Pi_2$ and any $i = 1, 2$, in order to get an explicit expression for the conditional probability $\mathbb{Q}_T[\pi_T = \pi | \mathcal{F}_t^W] = \mathbb{Q}[\pi_T = \pi | \mathcal{F}_t^W]$ appearing in formula (4.4.5). It can be easily done in this setting by using multidimensional Girsanov's theorem (see, e.g., Karatzas and Shreve's book [Karatzas 91], Theorem 5.1 in Chapter 3). Indeed, if we denote σ^i the 2-dimensional vector $(\sigma^{i,1}, \sigma^{i,2})$ and we set

$$Z_t^i := \left.\frac{d\mathbb{Q}_T^i}{d\mathbb{Q}}\right|_{\mathcal{F}_t^W},$$

we get that

$$Z_t^i = \exp\left\{\sigma^i \cdot \widetilde{W}_t - \frac{1}{2}||\sigma^i||^2 t\right\}, \quad t \in [0, T].$$

A simple application of Girsanov's theorem provides the following $\mathbb{Q}_T^i$-dynamics of each price process S^j for $j = 1, 2$:

$$S_t^j = S_0^j \exp\left\{\left(r - \frac{1}{2}||\sigma^j||^2 + \sigma^j \cdot \sigma^i\right) t + \sigma^j \cdot \widehat{W}_t\right\}, \quad t \in [0, T],$$

where $\widehat{W} = (\widehat{W}^1, \widehat{W}^2)$ is a 2-dimensional Brownian motion under $\mathbb{Q}_T^i$. Altogether, previous computations ensure the following result :

Proposition 4.5.2. *Let $T_2 > T_1 > 0$. Under our model assumptions, the price at time t of an electricity forward contract with maturity T_1 and delivery period $[T_1, T_2]$, denoted by $F_t(T_1, T_2)$, is given by the following formula :*

$$F_t(T_1, T_2) = \sum_{\pi \in \Pi_2} \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} (A_1(t, T) + A_2(t, T)) dT, \tag{4.5.6}$$

where

$$A_1(t, T) := \sum_{\{x_1 = m_{\pi(1)}, M_{\pi(1)}\}} F_t^{\pi(1)}(T) \mathbb{Q}_T^{\pi(1)}[\pi_T = \pi | \mathcal{F}_t^W] \mathbb{P}[\Delta_T^{\pi(1)} = x_1 | \Delta_t] \mathbb{Q}[D_T \le x_1 | \mathcal{F}_t^0]$$

$$A_2(t, T) := \sum_{\substack{\{x_1 = m_{\pi(1)}, M_{\pi(1)};\\ x_2 = m_{\pi(2)}, M_{\pi(2)}\}}} F_t^{\pi(2)}(T) \mathbb{Q}_T^{\pi(2)}[\pi_T = \pi | \mathcal{F}_t^W] \mathbb{P}[\Delta_T^{\pi(1)} = x_1, \Delta_T^{\pi(2)} = x_2 | \Delta_t] \times \mathbb{Q}[x_1 < D_T \le x_1 + x_2 | \mathcal{F}_t^0]$$

where, for any $\pi \in \Pi_2$ and $i = 1, 2$, the conditional probabilities $\mathbb{Q}[D_T \le x_1 | \mathcal{F}_t^0]$ and $\mathbb{Q}[x_1 < D_T \le x_1 + x_2 | \mathcal{F}_t^0]$ are given by (4.5.4) and (4.5.5), and

$$\mathbb{Q}_T^{\pi(i)}[\pi_T = \pi | \mathcal{F}_t^W] = 1 - \Phi(m(t)/\gamma(t)),$$

where $m(t)$ and $\gamma(t)$ are defined as follows :

$$m(t) = \ln \frac{S_t^{\pi(1)}}{S_t^{\pi(2)}} - \left(\frac{1}{2} ||\sigma^{\pi(1)} - \sigma^{\pi(2)}||^2 - (\sigma^{\pi(1)} - \sigma^{\pi(2)}) \cdot \sigma^{\pi(i)} \right) (T - t)$$
$$\gamma^2(t) = ||\sigma^{\pi(1)} - \sigma^{\pi(2)}||^2 (T - t).$$

Proof It suffices to combine the different formulae obtained in this section and observe that for any $\pi \in \Pi_2$ and $i = 1, 2$ we have

$$\mathbb{Q}_T^{\pi(i)}[\pi_T = \pi | \mathcal{F}_t^0] = \mathbb{Q}_T^{\pi(i)}[S_T^{\pi(1)} \leq S_T^{\pi(2)} | \mathcal{F}_T^W] = \mathbb{Q}_T^{\pi(i)}[X \leq 0 | \mathcal{F}_t^W]$$

where $X := \ln(S_T^{\pi(1)}/S_T^{\pi(2)})$. Under $\mathbb{Q}_T^{\pi(i)}$,

$$X = \ln \frac{S_t^{\pi(1)}}{S_t^{\pi(2)}} + \sum_{j=1}^{2} (\sigma^{\pi(1),j} - \sigma^{\pi(2),j})(\widehat{W}_T^j - \widehat{W}_t^j)$$
$$- \sum_{j=1}^{2} \left(\frac{1}{2}((\sigma^{\pi(1),j})^2 - (\sigma^{\pi(2),j})^2) - (\sigma^{\pi(1),j} - \sigma^{\pi(2),j})\sigma^{\pi(i),j} \right) (T - t).$$

Thus, conditioned to $\mathcal{F}_t^W$, the random variable X is normal with mean $m(t)$ and variance $\gamma^2(t)$, where

$$m(t) = \ln \frac{S_t^{\pi(1)}}{S_t^{\pi(2)}} - \sum_{j=1}^{2} \left(\frac{1}{2}((\sigma^{\pi(1),j})^2 - (\sigma^{\pi(2),j})^2) - (\sigma^{\pi(1),j} - \sigma^{\pi(2),j})\sigma^{\pi(i),j} \right) (T - t)$$

and

$$\gamma^2(t) = \sum_{j=1}^{2} (\sigma^{\pi(1),j} - \sigma^{\pi(2),j})^2 (T - t).$$

Notice that only the mean $m(t)$ depends on $\pi(i)$. Finally, we have

$$\begin{aligned}\mathbb{Q}_T^{\pi(i)}[\pi_T = \pi | \mathcal{F}_t^W] =& \mathbb{Q}_T^{\pi(i)}[X \leq 0 | \mathcal{F}_t^W] \\ =& \mathbb{Q}_T^{\pi(i)}[(X - m(t))/\gamma(t) \leq -m(t)/\gamma(t) | \mathcal{F}_t^W] \\ =& \Phi(-m(t)/\gamma(t)) = 1 - \Phi(m(t)/\gamma(t)),\end{aligned}$$

where Φ is the c.d.f. of a standard gaussian random variable. □

4.6 Numerical results

To provide a coherent and tractable framework for numerical examples, we follow the two fuels model of the previous section and we push further the simplification.

4.6.1 Data choice

We test the model on the French deregulated power market. The data cover the period going from January, 1st, 2007 to December, 31st, 2008. For the demand process, we used the data provided by the French TSO, RTE [1], on its web site. The hourly demand can be retrieved. The two technologies we have chosen are natural gas plants and fuel combustion turbines. They are known to frequently determine the spot price during peaking hours, since they are the most expensive ones. Moreover, a decomposition of the production is provided by RTE for each type of generation asset (nuclear, hydrolic plants, coal and gas, fuels, peak). Hence, it allowed us to deduce the residual demand addressed to gas and fuels technologies by substracting the nuclear and hydrolic production to the demand. Since these two technologies are setting the price during peaking hour, we focused our analysis on one particular hour of the day. We have chosen the 12th hour, which is usualy the first peaking hour of the day (the next one being 19th hour). The electricity spot and futures prices are provided by EEX French Power Futures Market, previously Powernext. The CO_2 prices are provided by PointCarbon data. For fuels and gas prices, we used Platt's data. Gas prices are quoted in GBP and fuels prices en USD. We used the daily exchange rate to convert USD to EUR.

4.6.2 Reconstruction of S_t^1 and S_t^2

In our model, we need to rebuild the spot prices of the two technologies S_t^1 and S_t^2. To tackle with the problem of aggregating the numerous gas and fuel power plants into only two technologies, we used the information provided by the French Ministry of Industry on electricity production costs [2]. It gives an average heat rate for each techology. We use also an average emission rate for CO_2 emissions of each technology. Furthermore, for fuel power plants production costs, one need to take into account the transportation cost from ARA zone the location of the plants. We used an average fixed cost. Thus, we obtain the following expressions for the prices of the two technologies.

$$\begin{cases} S_t^1 = 101.08 \cdot S_t^g + 0.49 \cdot S_t^{co_2} \\ S_t^2 = 0.38 \cdot S_t^f + 0.88 \cdot S_t^{co_2} + 13.44 \end{cases}$$

where S^g, S^f and S^{co_2} denote respectively gas price (€/therm) and fuel and carbon emission prices (€/ton).

1. RTE : www.rte-france.fr
2. Ministère de l'Industrie et des Finances, www.energie.minefi.gouv.fr/energie/electric/f1e_elec.htm, see "Les coûts de référence de la production électrique"

Remark 4.6.1. *One can observe on historical data that the ordering between the two technologies never changes. Fuel combustion turbines are known to be more expensive than gas plants. If the technologies prices follow the dynamics given by (4.2.1), the probability to have different orders* $\pi(t) \in \Pi$ *can be positive. Nevertheless, for a reasonable choice of parameters, this probability can be made sufficiently small. Hence, we make the approximation that* $\forall t, \mathbb{P}(S_t^1 < S_t^2) = 1$.

4.6.3 Estimation of electricity demand

We recall that the demand process is not affected by the minimal martingale measure, and that its respective filtration is independent of prices evolution. Thus, it is perfectly consistent to conduct a statistical estimation of the demand dynamics. The demand process given by expression (4.5.3) is then estimated via the Maximum Likelihood Principle. Let's remind that the demand process is given by :

$$D_t = \tilde{b}(t) + X_t = b_0 + b_1 \cos(2\pi t - b_2) + X_t$$

where X_t is an Ornstein Uhlenbeck process with a known Likelihood expression (see [Ait-Sahalia 02], sec. 5). For a discrete sample $(D_{t_1}, \ldots, D_{t_n})$ observed at fixed times with a constant time grid $\delta := (t_i - t_{i-1}), i = 1 \ldots n$, an expression of the Likelihood is given by

$$\mathcal{L}(b_0, b_1, b_2, a, \delta, D_{t_1}, \ldots, D_{t_n}) = \frac{1}{(\sqrt{2\pi v})^n} \exp\Big(-\frac{1}{2v} \textstyle\sum_{i=1}^{n-1} \big((D_{t_{i+1}} - \tilde{b}(t_{i+1})) - e^{a\Delta t}(D_{t_i} - \tilde{b}(t_i))\big)^2 \Big),$$

where $v = \delta^2 \frac{e^{2a\Delta t} - 1}{2a}$ and $\tilde{b}(t)$ is defined above. We numerically maximize this expression to obtain an estimation for the set of parameters. We then test the hypothesis that each parameter is null and finally obtain the set given in Table 4.1. The parameter $\hat{b}_2$ is not significantly different from 0 with threshold 99 %, thus it is fixed null.

$\hat{b}_0$	$\hat{b}_1$	$\hat{b}_2$	$\hat{a}$	$\hat{\delta}$
4814	905	0	87.55	17256

TABLE 4.1 – Parameters estimation for the demand process.

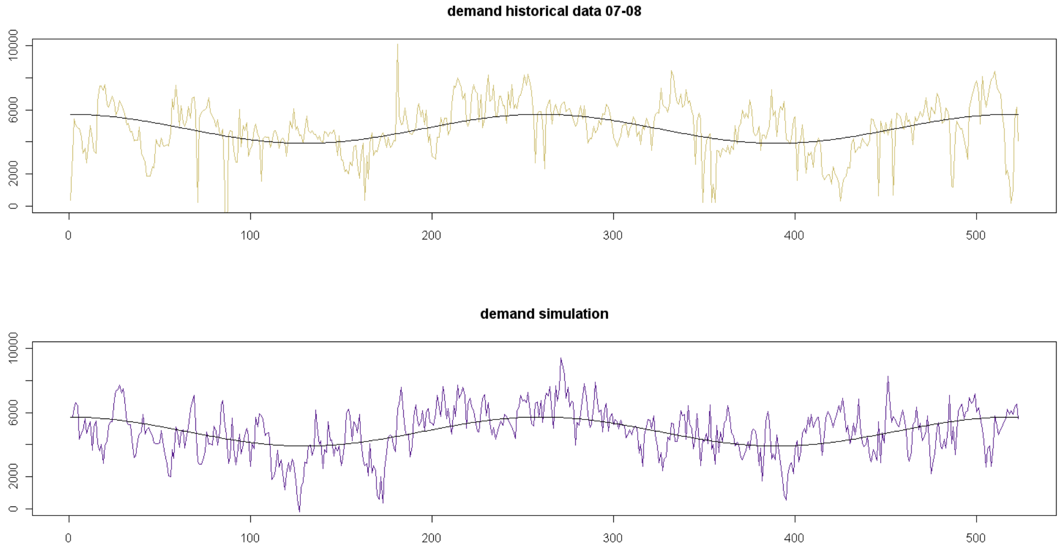


FIGURE 4.1 – Midday daily demand (day-ahead peakload demand from 01/01/2007 to 31/12/2008, RTE) and simulation with fitted parameters. Coordinates=(time in days, demand in MW). In black line, we showed the long trend $\tilde{b}(t)$.

4.6.4 Estimation of capacity process

For two technologies, the implementation of formula (4.2.2) is very simple. We define the following variables :

$$R^1 = \min(D_t^+, \Delta_t^1), \quad R^2 = \min((D_t - \Delta_t^1)^+, \Delta_t^2),$$

where D_t is *here* the sum of residual demands for the two technologies. The electricity spot price is defined by the following rule : If R^2 is positive, then we take $P = S^2$, and if it is null, $P = S^1$. However, in our context of a raw approximation of the electricity spot market, the application of this rule to estimate the capacity process Δ^1 and Δ^2 would lead to the opinion that only the second technology (the most expensive one) is being used. Hence, to take into account all the complexity of the short-term bidding process involving production constraints (start-up cost, ramp constraints, minimal runtime...), we introduce a threshold $\bar{\Delta}^1$ such that the price is given by the second technology althought $R^1 = \bar{\Delta}^1 < \Delta^1$.
Noting that the inequality on R^1 is equivalent to $R^2 > (\Delta^1 - \bar{\Delta}^1)$, the threshold $\bar{\Delta}^1$ is

obtained by solving the following program :

$$\min_{x>0} \sum_{i=1}^{n} \mathcal{R}\Big(P_{t_i} - S^1_{t_i} 1\!\!1_{\{R^2_{t_i} \leq x\}} - S^2_{t_i} 1\!\!1_{\{R^2_{t_i} > x\}}\Big).$$

The function $\mathcal{R}$ is a risk criterion : we tested two cases, the L_1 and the L_2 norms. The absolute error (L_1) showed a global minimum and the quadratic error (L_2) showed a local minimum on a reasonable interval (very high price peaks disturb the convergence). Thus, we use the L_1 criterion to determine that the intermediate parameter $\Delta^1 - \bar{\Delta}^1$ equals 610 MW. Eventually, we have new values for $(D_t - \Delta^1_t) 1\!\!1_{\{D_t > \Delta^1_t\}}$ and since we know exactly when $P_t = S^i_t$, for $i = 1, 2$, the computation of values taken by the model on historical data is straightforward (see Figure 4.2).

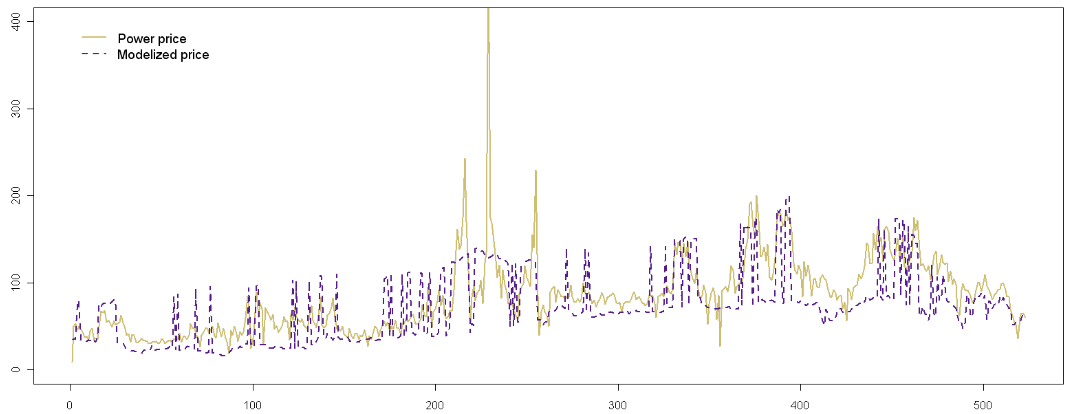


FIGURE 4.2 – Midday daily prices and model fitted on historical data (POWERNEXT® day-ahead peakload prices from 01/01/2007 to 31/12/2008). Coordinates=(time in days, prices in euros).

Finally, we can estimate parameters for the capacity process Δ^1_t as $D_t = R^1_t + R^2_t$ is available. Theoretically, capacity thresholds m_i and M_i are structural and are known to producers. But, since they vary over time due to maintenance scheduling and weather conditions, we estimate their constant counterparts. Moreover, we had to deal with the fact that in our model Δ^1 does take two values. Thus, we proceeded in two steps. First, we filtered the data to define a Δ^1_t taking only two values. Second, we estimated on that filtered time serie the free parameters λ^u_1 and λ^d_1.
The capacity process Δ^1 is partially hidden, since it is observed only if $D_t > \Delta^1_t$. Thus, we suppose that we observe data at discrete times t_i, and we calibrate the capacity

levels by minimizing the quadratic error between the series $(\Delta^1_{t_i}\mathbb{1}_{\left\{D_{t_i}>\Delta^1_{t_i}\right\}})_{i=1\dots n}$ and two constant values, taking into account the two following structural constraints :

$$M_1 \geq \sup_{t\in[0,T],D_t\leq\Delta^1_t} D_t \quad ; \quad m_1 \geq \inf_{t\in[0,T],D_t>\Delta^1_t} D_t.$$

Solving this calibration problem, we deduce the transformed serie $\widetilde{\Delta}^1$ which takes two values :

$$\widetilde{\Delta}_{t_i} = m_1\mathbb{1}_{\{|\Delta_{t_i}-m_1|<|\Delta_{t_i}-M_1|\}} + M_1\mathbb{1}_{\{|\Delta_{t_i}-m_1|\geq|\Delta_{t_i}-M_1|\}}, \quad i=1\dots n.$$

On that series, we estimate λ^u_1 and λ^d_1 by observing the series $(\widetilde{\Delta}^1_{t_i}\mathbb{1}_{\left\{D_{t_i}>\widetilde{\Delta}^1_{t_i}\right\}})_{i=1\dots n}$. We denote $(t_{k(i)})_{i=1\dots n}$ the subgrid of the discrete times where $t_{k(i)}$ is the last time before t_i when we observe $(\Delta^1_{t_i})_{i=1\dots n}$. Then, by the Bayes rule and the independence between D_t and $\widetilde{\Delta}^1_t$, the probability $\mathbb{Q}\left[\widetilde{\Delta}^1_{t_i}=x|D_{t_i}>\widetilde{\Delta}^1_{t_i},\widetilde{\Delta}^1_{t_{k(i)}}\right]$ for $i=1\dots n$ is :

$$\mathbb{Q}_i\left[x\right] := \mathbb{Q}\left[\widetilde{\Delta}^1_{t_i}=x|D_{t_i}>\widetilde{\Delta}_{t_i},\widetilde{\Delta}^1_{t_{k(i)}}\right] = \frac{\mathbb{P}\left[\widetilde{\Delta}^1_{t_i}=x|\widetilde{\Delta}^1_{t_{k(i)}}\right]\mathbb{Q}\left[D_{t_i}>x\right]}{\mathbb{Q}\left[D_{t_i}>\widetilde{\Delta}^1_{t_i}|\widetilde{\Delta}^1_{t_{k(i)}}\right]}.$$

If follows that :

$$\mathbb{Q}_i\left[x\right] \equiv \frac{\mathbb{P}\left[\widetilde{\Delta}^1_{t_i}=x|\widetilde{\Delta}^1_{t_{k(i)}}\right]\mathbb{Q}\left[D_{t_i}>x\right]}{\mathbb{P}\left[\widetilde{\Delta}^1_{t_i}=M_1|\widetilde{\Delta}^1_{t_{k(i)}}\right]\mathbb{Q}\left[D_{t_i}>M_1\right]+\mathbb{P}\left[\widetilde{\Delta}^1_{t_i}=m_1|\widetilde{\Delta}^1_{t_{k(i)}}\right]\mathbb{Q}\left[D_{t_i}>m_1\right]}.$$

An expression of the Likelihood for the given sample is :

$$\mathcal{L}(\lambda^u_1,\lambda^d_1,\widetilde{\Delta}_{t_1},\dots,\widetilde{\Delta}_{t_n},D_{t_1},\dots,D_{t_n}) = \\ \prod_{i=1}^n\left(\mathbb{Q}_i\left[x\right]^{\mathbb{1}_{\left\{\widetilde{\Delta}^1_{t_i}=x\right\}}}\left(1-\mathbb{Q}_i\left[x\right]\right)^{\left(1-\mathbb{1}_{\left\{\widetilde{\Delta}^1_{t_i}=x\right\}}\right)}\right)^{\mathbb{1}_{\left\{D_{t_i}>\widetilde{\Delta}^1_{t_i}\right\}}}.$$

We maximize this expression to obtain intensity parameters. The parameters values of the capacity process are summarized in Table 4.2. We notice that $\lambda^u_1>\lambda^d_1$ means that $\mathbb{P}[\widetilde{\Delta}^1_T=M_1]>\mathbb{P}[\widetilde{\Delta}^1_T=m_1]$ for a sufficiently long maturity T.

M_1 (MW)	m_1 (MW)	λ^u_1 (y^{-1})	λ^d_1 (y^{-1})
5708	4292	34.78	24.89

TABLE 4.2 – Parameters for the capacity process. Unit in parenthesis.

4.6.5 A comparison with a naive econometric model

To evaluate the benefit of adding the demand and production capacity to the modeling process, we compare it to a simple econometric approach. We propose the alternative linear model :

$$P_t = \alpha_0 + \alpha_1 S_t^1 + \alpha_2 S_t^2 + \epsilon_t, \tag{4.6.1}$$

where ϵ_t is a Gaussian white noise. And, we compare the linear model (4.6.1) with our structural model where we added free linear parameters and also a Gaussian noise to facilitate the comparison :

$$P_t = \beta_0 + \sum_{i=1,2} \beta_i S_t^i \mathbb{1}_{\{D_t \in I_i^{\pi_t}(t)\}} + \epsilon_t.$$

In both cases, we estimated the parameters using a quadratic loss minimization. The Table 4.3 as well as Figure 4.3 shows that there is a positive gain to add demand and production capacity dynamics to the electricity spot price modeling.

Price	Corr	MaxE	MAE	MSE	MPE
Linear model	0.756	406.96	18.35	919.53	23.734%
Structural Model	0.702	385.23	17.54	786.20	23.956%

TABLE 4.3 – Model comparison. Corr := correlation with historical price ; MaxE := maximum error ; MAE := mean absolute error ; MSE := mean square error ; MPE=Mean percentage error. Errors are calculated w.r.t. historical data (POWERNEXT® day-ahead prices from 01/01/2007 to 31/12/2008).

4.6.6 Forward prices computation

Following the approximation given in Remark 4.6.1, in our two technologies case, the expression (4.4.5) writes :

$$F_t(T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \sum_{x_1 = m_1, M_1} \mathbb{P}[\Delta_T^1 = x_1 | \Delta_t] \big(F_t^2(T) + (F_t^1(T) - F_t^2(T))(\mathbb{Q}[D_T \leq x_1 | \mathcal{F}_t^0]) \big) dT. \tag{4.6.2}$$

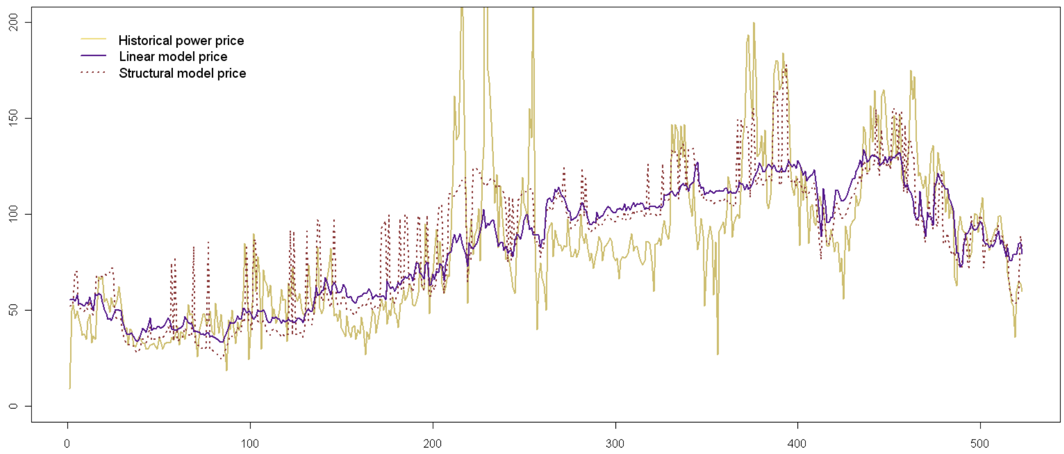


FIGURE 4.3 – Prices and econometric estimation of our model and a linear model (POWERNEXT® day-ahead prices from 01/01/2007 to 31/12/2008). Coordinates=(time in days, prices in euros).

We do not have forward prices $F_t^i(T)$ at our disposal but only swap prices, i.e., values of $\frac{1}{T_2-T_1}\int_{T_1}^{T_2} F_t^i(T)dT$ for delivery periods $[T_1, T_2]$. Nevertheless, we make the approximation that :

$$F_t^i(T) \approx \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} F_t^i(T)dT, \quad T \in [T_1, T_2] .$$

This approximation can be considered rough for forward gas prices, since the spot market has daily granularity ; but, for fuel prices, it is quite reasonable since spot prices are limited to a value per month.

We calibrate the spot price model on the former period, till June 2008, and then back-test it on future prices from July 2008 to February 2009. On that sufficiently wide interval, we can focus on two assets : the two quarters ahead and three quarter ahead futures, covering Spring 2009 (April, May, June) and Summer 2009 (July, August and September). The results are illustrated on Figure 4.4 and Figure 4.5. We observe that, as expected, the predicted price overestimates the real price. Indeed, we estimated the model on high peak hours of each day, which is over the mean price most of the time. However we observe strong correlation between predicted and historical price as shown in Table 4.4.

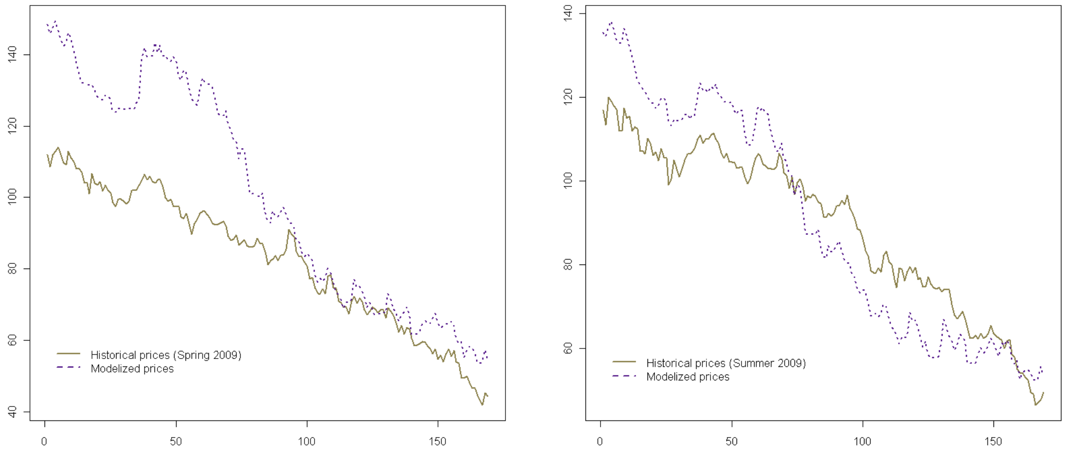


FIGURE 4.4 – Forward prices : model anticipations and market data (POWERNEXT® Future prices on peak load from 01/07/2008 to 27/02/2009, 169 obs.). Left = Spring 2009 ; right = Summer 2009. Coordinates=(time in days, prices in Euro).

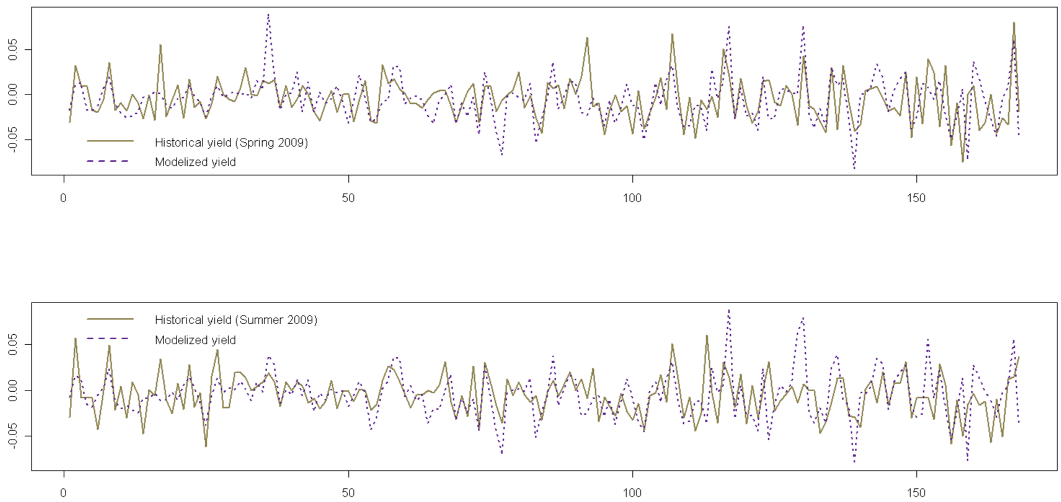


FIGURE 4.5 – Forward yields : model anticipations and market data (POWERNEXT® Future yields on peak load from 01/07/2008 to 27/02/2009, 169 obs.). Up = Spring 2009 ; down = Summer 2009. Coordinates=(time in days, yields in percentage).

Asset	Corr	$\mathbb{E}[\Delta F_t(T_1, T_2)]$	$\mathbb{V}[\Delta F_t(T_1, T_2)]$	ME	MSE	MPE
Spring 09	0.958	-0.582 (-0.403)	2.409 (1.840)	49.624	851.981	28.297%
Summer 09	0.939	-0.505 (-0.402)	2.174 (2.014)	30.928	213.484	12.695%

TABLE 4.4 – Model anticipations results. Corr = correlation with historical price ; $\mathbb{E}$ = yield mean (in parenthesis the real asset value) ; $\mathbb{V}$ = yield variance ; ME = maximum price error ; MAE = mean absolute error ; MSE = mean squared error ; MPE = mean percentage error. Errors are calculated w.r.t. historical data.

4.6.7 Calibration on forward prices

The model gives two relations between power price and commodities prices. As we estimated the parameters on spot prices, we can now do the same on forward prices. Using formula (4.6.2), and under the previous assumptions on the prices $F_t^i(T)$, $i = 1, 2$, the model can be calibrated directly on forward prices. However, given the great number of parameters, we must assess that a part of them is already known to solve the identification problem : The capacity levels M_1 and m_1, and the parameters of the demand process D_t are now fixed. Thus, the probability $\mathbb{P}\left[\Delta_T^1 = x|\Delta_t\right]$ for $x = m_1, M_1$, which is integrated on the period $[T_1, T_2]$, is the only free variable. The goal is to calibrate numerically this variable on the following expression :

$$F_t(T_1, T_2) = f^1(\lambda, \Delta_t, D_t)F_t^1(T_1, T_2) + (1 - f^1(\lambda, \Delta_t, D_t))F_t^2(T_1, T_2)$$

where

$$f^1(\lambda, \Delta_t, D_t) = \sum_{x=m_1,M_1} \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \mathbb{P}\left[\Delta_T^1 = x|\Delta_t^1\right] \mathbb{Q}\left[D_T = x|D_t\right] dT.$$

These expressions depend on Δ_t and D_t via the formulae (4.5.4) and (4.5.2). Thus, $f^1(\lambda, \Delta_t, D_t)$ actually depends on t in an explicit manner. We can make a few approximations for an easier computation. Indeed, calibration is made difficult due to the fact that $e^{-(\lambda_1^d+\lambda_1^u)(T-t)}$ is very small when $T \gg t$. Hence, if $T \gg t$ or the parameter λ (relation (4.5.2)) and the parameter a (relation (4.5.4)) are large enough, we can make the following approximations : $\mathbb{P}[\Delta_T = x|\Delta_t] \cong \lim_{T\uparrow\infty} \mathbb{P}[\Delta_T = x]$ and $\mathbb{Q}[D_T > x|D_t] \cong \lim_{T\uparrow\infty} \mathbb{Q}[D_T > x]$. Then, the calibration is equivalent to a linear model estimation under constraints, whose coefficients are $f^1(\lambda)$ and $1 - f^1(\lambda)$.
Under that approximation, we obtain $\mathbb{P}[\Delta_T = M_1]$ and $\mathbb{P}[\Delta_T = m]$, which give the expected failure probabilities for the cheapest technology on the delivery period $[T_1, T_2]$. The computation gives a sound result for calibration on Summer 2009 Future price ($\mathbb{P}[\Delta_T = M_1] = 0.865$), but not for Spring 2009 Future, which is clearly overestimated.

We explain this drawback by the fact that we used the two most expensive technologies to price electricity.

4.6.8 Spot price simulations

This structural model can be easily improved to provide simulation trajectories with high spikes. If the residual demand D_t is negative, it corresponds to the case when nuclear power is being the marginal unit of the system. Its cost is well-known to be constant over time ($\cong$ 15€/MWh). On the hother hand, if the residual demand D_t exceeds the total capacity $\Delta_t^1 + \Delta_t^2$ of our two technologies, it corresponds to situations when electricity has to be imported. In the French market, which is a structural exporter, it corresponds to tension on the system and electricity is bought at high cost. This high cost is arbitrarily fixed to a constant value (500€/MWh). In order to simulate the commodities prices, we quickly estimate on our first sample of data (January 2007 to December 2008) the multivariate diffusion process given by the relation (4.2.1). The Figure 4.6.8 shows that this simple device makes visible price spikes.

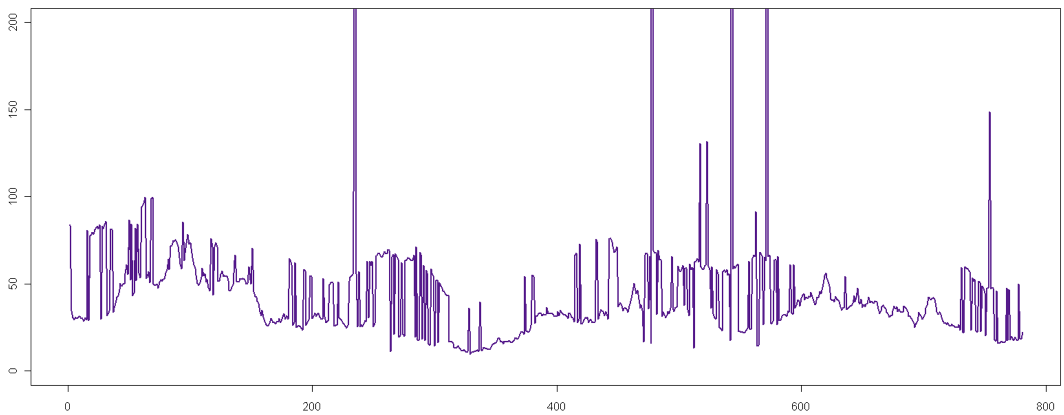


FIGURE 4.6 – Spot price simulation. Parameters calibrated on the period 01/2007 - 12/2008. We use two thresholds for very high price peaks (when $D_t > 8500$MWh, the price is fixed to 500€) and low demand prices (when $D_t < 0$MWh, the price is fixed to 15€). The process is simulated on 780 points (3 years). Coordinates=(time in days,price in Euro).

4.7 Conclusion and perspectives

Going back to the supposed storable fuels, the model presented in this part provides a possible solution to the question of the suitable risk-neutral probability for electricity prices dynamics. This first model should be considered more like a methodology than a definitive model for electricity spot and forward prices. Indeed, it has recently been improved in [Aid 10] to incorporate several features. We provide here some of this improvements in order to illustrate the potential.

The scarcity function

First of all, the authors introduce a scarcity function depending on available total capacity $C_t^{max} - D_t = \sum_{i=1}^n \Delta_t^i - D_t$:

$$g \; : \; x \longmapsto \min(M, \frac{\gamma}{x^\alpha})\mathbb{1}_{\{x>0\}} + M_{x\leq 0}$$

where γ, M and α are positive parameters. Obviously, M represents the maximum price on the market and (γ, α) are parameters of speed to achieve this bound. This function aims at indicate the tension in the system due to scarcity, since margin capacity seems to be a better state variable to capture electricity prices spikes than demand, see [Cartea 08]. The price is then affected in the following way :

$$P_t = g(C_t^{max} - D_t) \sum_{i=1}^n S_t^{(i)} \mathbb{1}_{\left\{D_t \in I_k^{\pi_t}(t)\right\}} \, .$$

After an estimation procedure, the authors show a real improvment in fitting historical data. It has also been taken into account in partial derivatives of the price with respect to capacity evolution.

Electricity Future as hedging instrument

The main result of the paper is the use of electricity products available on the futures market in order to price and hedge commonly exchanged derivatives, i.e., spread options and European options on electricity forwards, that depends on electricity and fuel prices, but also the demand level. Prices are computed under the minimal EMM $\mathbb{Q}^{min}$, and will correspond to the initial wealth allowing for approximately replicate an option in a local risk minimization sense. The hedging strategy is composed of forward contracts on electricity and forward contracts on fuel. In order to develop the expressions, it is necessary to use the Galtchouk-Kunita-Watanabe decomposition of the claim H under

$\mathbb{Q}$:

$$H = \mathbb{E}^{\mathbb{Q}}[H] + \int_0^T \xi_t' dF_t(T^*) + \int_0^T \xi_t^{e'} dF_t^e(T*) + L_T^H$$

where L_T^H is the terminal value of a $\mathbb{Q}$-martingale orthogonal to $F(T^*)$ and $F^e(T^*)$, representing the unhedgeable risk. This allows to obtain the hedging strategy (ξ, ξ^e) and numerical results, and conclude on an important pattern of hedging results :

- Far from maturity T^* (until two weeks before it), the partial hedge is very good. Indeed for large values of $T^* - t$, some coefficients are almost constants. Then the electricity futures are only driven by the commodities, and behave like a basket of assets.
- Close to maturity, the partial hedge is almost useless. The demand starts to drive electricity prices, and the unhedgeable risk becomes overwhelming.

Conclusion of Chapter 4

We conclude this part by expecting that this direction of research will be continued, since it presents very rich preliminary results and carries out many expectations.

First, the spot price model now fits well the very specific patterns of electricity spot prices. Then, the calibration of the model and the statistical estimation of its parameters do not raise great difficulties. Finally, the methodology is very general, and allows for a wide number of variations : one can change the supposed competitive market equilibrium on the spot market to take into account strategic bidding, or extend the spot model to a multizonal framework, where electricity is exchanged between different market places with different spot prices.

Several other directions can be drawn from here. In spite of some refinements in the dynamics, forward prices and even option prices are quasi-explicitly computable. As it has been investigated in [Aid 10], this model can be used to price and hedge contingent claims on electricity. Since it is based on an aggregated generation behaviour with respect to commodities, the model can also enable to assess the problem of optimal timing of investment in generation assets. We hope that many of these points will be investigated in the future.

Chapitre 5

Hedging electricity options with controlled loss

5.1 Introduction

This chapter is dedicated to a pricing problem in electricity future markets. We consider the situation of an agent endowed with a financial derivative on a futures which is not yet quoted. This situation is of practical concern for traders on realistic power futures markets. As long as electricity is assumed to be non-storable, the term structure of electricity prices cannot be derived from arbitrage arguments. In order to ensure sufficient liquidity for futures contracts (hereafter, futures), only a small but meaningful number of maturities and delivery periods are available for market participants. Futures are indeed contracts that deliver a certain amount of energy for a fixed remote date, but also over a specific time period. They are more commonly referred as swap contracts in the financial literature, see [Benth 07b]. It is thus possible to hold an option on such a contract which has not yet appeared in the market.
In practice, an intuitive way for hedging the option is to deploy a cross-hedging strategy with quoted assets, see [Verschuere 03, Eichhorn 05, Lindell 09]. Since futures are classical financial assets, we are allowed to put in place a dynamic strategy with them. Moreover, we will see in Section 5.4.1 that arbitrage arguments are still valid by a game of covering periods between contracts. Hence, we will introduce a simple model which takes into account a structural correlation for two futures prices, and an additional independent risk factor. Two observations then emerge. The first one is that it is possible to apply arbitrage pricing methods to existing contracts, and suppose reasonably that the resulting sub-market is complete. The second one is that the independent risk factor is unhedgeable,

and that the initial problem implies an incomplete market setting. We will thus introduce a special case of incompleteness denoted *semi-completeness*, in reference to [Becherer 01]. As in the previous chapter, the incomplete market setting is the occasion to introduce a pricing criterion. Following [Föllmer 00], we want to put in place a strategy which tolerates a given threshold of loss. For this purpose, we use the stochastic target formulation of [Bouchard 09]. By this way, we try to determine the risk premium associated to a fixed level of expected loss. In the complete market setting, Bouchard and al. [Bouchard 09] provide an explicit formulation of the risk premium in a Markovian setting. By using the historical probability setting, the stochastic target approach with controlled expected loss will naturally takes into account the exogenous risk factor in the expectation. With this face-lifting phenomena, we will be able to retrieve the complete market setting from the semi-complete one.

As a preliminary, we provide a slight generalization of the quantile hedging problem provided in [Bouchard 09] to controlled loss. This is a simple rewriting which partially follows [Moreau 11, Bouchard 11a]. Our contribution then holds in three steps. We first provide a reformulation of the explicit results of [Bouchard 09] in the semi-complete market setting. We then propose a general numerical algorithm for stochastic target problems based on probabilistic formulation of the control and the associated PDE. We finally apply our results to the financial problem described above.

The rest of the chapter is guided as follows. Section 5.2 introduces the theory of stochastic target with expectation criterion. We recall the general results of [Bouchard 09] for a loss function, and provide the explicit formulation with the previously described generalization in Section 5.2.2. In Section 5.3, we introduce the semi-complete setting as an extension of the complete market case target problem. Since our resolution leads to a numerical problem, we introduce in Section 5.3.2 the numerical algorithm to solve the target problem in a very general form. Section 5.4 proposes the model and the application of the previous methods to the evaluation of risk induced by the holding of a European option on a not-yet-quoted futures.

5.2 The stochastic target problem with controlled loss

This section intends to reformulate the central results of Bouchard and al. [Bouchard 09], and their applications in section 4 of the latter, with some modifications that one can find in [Moreau 11] and [Bouchard 11a]. In [Bouchard 09], the authors developed in a rather synthetic and powerful way an application of the stochastic target problem with controlled loss to quantile hedging. After introducing the problem and notations, we reformulate the application to a general loss function in a complete market setting.

5.2.1 General framework

State space and control

Let T be a finite horizon and $\Omega = \mathcal{C}([0,T];\mathbb{R}^d)$ be the space of continuous paths on $[0,T]$. Let $\mathbb{P}$ be the Wiener measure on Ω and W the canonical process $W_t(\omega) = \omega_t$. The process W is a d-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathbb{F} = \{\mathcal{F}_t,\ 0 \le t \le T\}$ the $\mathbb{P}$-augmentation of the filtration generated by W. For every $t \in [0,T]$, we set $\mathbb{F}^t := (\mathcal{F}^t_s)_{s \ge 0}$, where $\mathcal{F}^t_s$ is the completion of $\sigma(W_r - W_t,\ t \le r \le s \vee t)$ by null sets of $\mathcal{F}$. We introduce a family $\mathcal{U}$ of $\mathbb{F}$-progressively measurable processes $\nu \in L^2([0,T] \times \Omega)$ taking values in U, a bounded closed subset of $\mathbb{R}^d$. We fix a constant $\kappa \ge 0$, and we denote $\mathbf{Z} := (0,\infty) \times [-\kappa, \infty)$. For $t \in [0,T]$, $z := (x,y) \in \mathbf{Z}$ and $\nu \in \mathcal{U}$, we define $Z^\nu_{t,z} := (X^\nu_{t,x}, Y^\nu_{t,z})$ as the $(0,\infty)^d \times \mathbb{R}$-valued unique strong solution of the stochastic differential equation :

$$\begin{cases} dX^\nu_{t,x}(r) = \mu(X^\nu_{t,x}(r), \nu_r)dr + \sigma(X^\nu_{t,x}(r), \nu_r)dW_r \\ dY^\nu_{t,x,y}(r) = \mu_Y(Z^\nu_{t,z}(r), \nu_r)dr + \sigma_Y(Z^\nu_{t,z}(r), \nu_r) \cdot dW_r \end{cases}, \quad t \le r \le T,$$

satisfying the initial condition $Z^\nu_{t,z}(t) = (X^\nu_{t,x}(t), Y^\nu_{t,z}(t)) = (x,y)$. Here, we assume that

$$(\mu_Y, \sigma_Y) \ : \ \mathbb{R}^d \times \mathbb{R} \times U \to \mathbb{R} \times \mathbb{R}^d \quad \text{and} \quad (\mu, \sigma) \ : \ \mathbb{R}^d \times U \to \mathbb{R}^d \times \mathbb{M}^d$$

are locally Lipschitz functions satisfying

$$|\mu_Y(z,u)| + |\mu(x,u)| + |\sigma_Y(z,u)| + |\sigma(x,u)| \le K(z)(1 + |u|)$$

for a locally bounded map K.
For a fixed time $t \in [0,T]$ and $z \in \mathbf{Z}$, we now reduce the set of studied controls to $\mathcal{U}_{t,z} \subset \mathcal{U}$ composed of $\mathbb{F}^t$-progressively measurable controls ν such that

$$Y^\nu_{t,x,y}(r) \ge -\kappa \quad \forall r \in [t,T] \, .$$

This means that $Z^\nu_{t,z}(s) \in \mathbf{Z}$ for all $s \in [t,T]$ and $\nu \in \mathcal{U}_{t,z}$.
In our context, $X^\nu_{t,x}(.)$ is the price process of d risky assets, $Y^\nu_{t,z}(.)$ is the value of a financial portfolio process, given under a rather general form. Note that the price process X is possibly influenced by the control ν. The set $\mathcal{U}_{t,z}$ denotes the set of controls ν independent from the state space before t and satisfying a finite credit line κ for $Y^\nu_{t,x,y}$ at any considered time until the terminal date T.

The target problem

We introduce a loss function, given by $\Psi : \mathbb{R}_+^d \times \mathbb{R} \longrightarrow \mathbb{R}_-$, which is assumed to verify

Assumption 5.2.1. *We assume the following :*
(i) the map $y \mapsto \Psi(x,y)$ is non decreasing, and right continuous for all $x \in \mathbb{R}_+^d$;
(ii) the map $z \mapsto \Psi(z)$ is uppersemicontinuous on $\mathbf{Z}$.
(iii) Ψ has a polynomial growth, i.e., there exists $C > 0$ and $k \in \mathbb{N}$ such that

$$|\Psi(z)| \leq C(1+|z|^k) \ , \quad \forall z \in (0,\infty)^d \times \mathbb{R} \ ; \tag{5.2.1}$$

(iv) for any $(t,z) \in [0,T] \times \mathbf{Z}$ and any $\nu \in \mathcal{U}_{t,z}$, $\mathbb{E}\left[|\Psi(Z^\nu_{t,z}(T))|^2\right] < \infty$;
(v) $0 \in E := \overline{\mathrm{conv}}(\Psi(\mathbf{Z})) \subset \mathbb{R}_-$. ($E$ is the closed convex hull of the image of Ψ.)

See [Bouchard 09, Moreau 11, Bouchard 11a] for a detailed use of these assumptions. We will sometimes write $\Psi(x,y)$ for $\Psi(z)$, and abusing of this notation, we denote by $p \mapsto \Psi^{-1}(x,p) := \inf\{y \geq -\kappa \ : \ \Psi(x,y) \geq p\}$ the general inverse function in y. We then denote by $\odot\Psi^{-1}$ the convex hull of Ψ^{-1} in p, meaning the greatest convex function in p under Ψ^{-1}. By the convexity property, $p \mapsto \odot\Psi^{-1}(x,p)$ is continuous on $\mathrm{int}(E)$.

Definition 5.2.1. *Given initial condition of the state process (t,x) and a threshold $p \leq 0$, the value function of the stochastic target problem with controlled loss is defined by*

$$v(t,x,p) := \inf\left\{y \geq -\kappa \ : \ \mathbb{E}\left[\Psi(Z^\nu_{t,z}(T))\right] \geq p \quad \textit{for some } \nu \in \mathcal{U}_{t,z}\right\} \ . \tag{5.2.2}$$

The problem consists in finding, at each date t and for the state of the market x, the minimal amount of wealth ensuring to reach, *in expectation*, a given risk criterion at terminal date T. This criterion limit is defined by a threshold p. The key idea for solving this problem is to augment the dimension of the state process to retrieve a stochastic target problem in the conventional form first provided by [Soner 02a]. It is based on the martingale representation theorem in the Brownian motion framework. We introduce the stochastic process $P^\alpha_{t,p}$ verifying the following SDE :

$$P^\alpha_{t,p}(s) = p + \int_t^s P_{t,p}(u)\alpha_u \cdot dW_u, \quad \forall t \leq s \leq T \tag{5.2.3}$$

where α is an additional control. We introduce for this purpose, and for a fixed $(t,p) \in [0,T] \times \mathbb{R}_-$, the set $\mathcal{A}_{t,p}$ of $\mathbb{F}^t$-progressively measurable processes α taking values in $\mathbb{R}^d$ such that $\alpha P^\alpha_{t,p} \in L^2([0,T] \times \Omega)$ and that $P^\alpha_{t,p}$ is a square integrable martingale taking values in E. Proposition 3.1 in [Bouchard 09] then states that we are able to write

$$v(t,x,p) = \inf\left\{y \geq -\kappa : \Psi(Z^\nu_{t,z}(T)) \geq P^\alpha_{t,p}(T) \ \text{ for some } (\nu,\alpha) \in \mathcal{U}_{t,z} \times \mathcal{A}_{t,p}\right\} \ . \tag{5.2.4}$$

Remark 5.2.1. *To retrieve the target in probability, we shall replace the loss function by* $\Psi(x,y) = -\mathbb{1}_{\{y<G(x)\}}$*, where* $G(X^{\nu}_{t,x}(T))$ *is a given contingent claim. Then, for* $p \in [-1,0]$*,*

$$v(t,x,p) = \inf\left\{y \geq -\kappa \ : \ \mathbb{P}\left[Y^{\nu}_{t,x,y}(T) \geq G(X^{\nu}_{t,x}(T))\right] \geq 1+p \quad \text{for some } \nu \in \mathcal{U}_t\right\} .$$

In this case, the process $P^{\alpha}_{t,p}$ *lives in* $[0,1]$*, which also introduces new boundary conditions for* $p \in \{0,1\}$*. In our setting,* $P^{\alpha}_{t,p}$ *evolves in* $\mathbb{R}_-$*, providing only one endpoint to study.*

Viscosity property

The mathematical difficulty comes from the fact that the control (ν,α) takes values in $\bar{U} := U \times \mathbb{R}^d$. Whereas the set U is taken bounded to provide a regular control problem, the addition of α leads inevitably to an unbounded domain. This leads to a singular stochastic target problem, which is tackled with the introduction of semi-limit relaxation of the dynamic programming equation. In the stochastic target problem without state constraint, Soner and Touzi [Soner 02a] introduced the geometric dynamic programming principle (GDPP) allowing to derive the PDE characterization. It has been extended in the general case in [Bouchard 10]. Here, note that if $\alpha \in L^2([0,T]\times\Omega)$ is an unbounded control leading to $\Psi(Z^{\nu}_{t,z}(T)) \geq P^{\alpha}_{t,p}(T)$, it is always possible to find $\bar{\alpha} \in \mathcal{A}_{t,p}$ verifying the same property. This is provided by

Lemma 5.2.1. *Fix* $(t,z,p) \in [0,T]\times\mathbf{Z}\times\mathbb{R}_-$*. Assume that there exists* $\nu \in \mathcal{U}_{t,z}$ *and a* $\mathbb{F}^t$*-progressively measurable process* α *taking values in* $\mathbb{R}^d$ *such that* $\Psi(Z^{\nu}_{t,z}(T)) \geq P^{\alpha}_{t,p}(T)$ $\mathbb{P}-a.s.$*. Then there exists* $\bar{\alpha} \in \mathcal{A}_{t,p}$ *such that* $\Psi(Z^{\nu}_{t,z}(T)) \geq P^{\bar{\alpha}}_{t,p}(T)$ $\mathbb{P}-a.s.$

Proof According to dynamics (5.2.3), $P^{\alpha}_{t,p}$ is a submartingale. Thus, $\mathbb{E}\left[\Psi(Z^{\nu}_{t,z}(T))\right] \geq p$. According to Assumption 5.2.1.(iii), the martingale representation theorem implies the existence of a square-integrable martingale $P^{\bar{\alpha}}_{t,p}$, with $P^{\bar{\alpha}}_{t,p}(t) = p$ and

$$\Psi(Z^{\nu}_{t,z}(T)) - P^{\bar{\alpha}}_{t,p}(T) = \mathbb{E}\left[\Psi(Z^{\nu}_{t,z}(T))\right] - p \geq 0 \quad \mathbb{P}-\text{a.s.}$$

Since $E \subset \mathbb{R}_-$, we can choose it to follow dynamics (5.2.3). This implies that $\bar{\alpha}$ is a real-valued $\mathbb{F}^t$-progressively measurable process such that $\bar{\alpha}P^{\bar{\alpha}}_{t,p} \in L^2([0,T]\times\Omega)$, and $\bar{\alpha} \in \mathcal{A}_{t,p}$. □

Lemma 5.2.1 echoes Standing Assumption 4 and Remark 6 in [Bouchard 11a]. This statement is missing in [Bouchard 09] and necessary to use the GDPP. We provide one side of the principle used to derive the supersolution.

Theorem 5.2.1. (GDP1) *Fix $(t,z,p) \in [0,T] \times \mathbf{Z} \times \mathbb{R}_-$ such that $y > v(t,x,p)$ and a family of stopping times $\{\theta^{\nu,\alpha} \ : \ (\nu,\alpha) \in \mathcal{U}_{t,z} \times \mathcal{A}_{t,p}\}$. Then there exists $(\nu,\alpha) \in \mathcal{U}_{t,z} \times \mathcal{A}_{t,p}$ such that $Y^{\nu}_{t,x,y}(\theta^{\nu,\alpha}) \geq v(\theta^{\nu,\alpha}, X^{\nu}_{t,x}(\theta^{\nu,\alpha}), P^{\alpha}_{t,p}(\theta^{\nu,\alpha}))$ and $Y^{\nu}_{t,x,y}(s \wedge \theta^{\nu,\alpha}) \geq -\kappa$ for all $s \in [t,T]$ $\mathbb{P}-a.s.$*

In what follows, we introduce only the supersolution property for v_*. We define the latter function by

$$v_*(t,x,p) := \liminf_{B \ni (t',x',p') \to (t,x,p)} v(t',x',p')$$

where B denotes an open subset of $[0,T] \times (0,\infty)^d \times \mathbb{R}_-$ with $(t,x,p) \in \mathrm{cl}(B)$. Notice that v_* is finite under the following

Assumption 5.2.2. *We assume that v is locally bounded on $[0,T) \times (0,\infty)^d \times \mathbb{R}_-$.*

Recall that $\bar{U} := U \times \mathbb{R}^d$. For $(u,a) \in \bar{U}$, set

$$\bar{\mu}(x,u) := \begin{pmatrix} \mu(x,u) \\ 0 \end{pmatrix}, \quad \bar{\sigma}(x,p,u,a) := \begin{pmatrix} \sigma(x,u) \\ a^T p \end{pmatrix}$$

Since $\bar{U}$ is unbounded, we introduce $\bar{F}^*(\Theta) := \limsup_{\varepsilon \searrow 0, \Theta' \to \Theta} \bar{F}_\varepsilon(\Theta')$ where, for $\varepsilon \geq 0$ and $\Theta = (x,p,y,q,A) \in \mathbb{R}^d_+ \times \mathbb{R}_- \times \mathbb{R} \times \mathbb{R}^{d+1} \times \mathbb{S}^{d+1}$,

$$\bar{F}_\varepsilon(\Theta) := \sup_{(u,a) \in \bar{\mathcal{N}}_\varepsilon(\Theta)} \left\{ \mu_Y(z,u) - \bar{\mu}(x,u) \cdot q - \frac{1}{2} \mathrm{Tr} \left[\bar{\sigma}\bar{\sigma}^T(x,p,u,a) A \right] \right\} \tag{5.2.5}$$

and

$$\bar{\mathcal{N}}_\varepsilon(\Theta) := \left\{ (u,a) \in \bar{U} \ : \ |\sigma_Y(z,u) - \bar{\sigma}(x,p,u,a)^T q| \leq \varepsilon \right\} .$$

We adopt the convention $\sup \emptyset = -\infty$ and $\bar{F}^* \varphi = \bar{F}^*(x,p,\varphi(x,p), D\varphi(t,x), H\varphi(t,x))$ with $D\varphi$ and $H\varphi$ being the gradient and Hessian matrix of a function φ respectively. We hence formulate the supersolution property of Theorem 2 in [Bouchard 11a]. Note that the constraint $v \geq -\kappa$ only appears in the subsolution property. This implies that we retrieve the formulation of Theorem 2.1 and Corollary 3.1 in [Bouchard 09].

Theorem 5.2.2. *The function v_* is a viscosity supersolution of $-\frac{\partial \varphi}{\partial t} + \bar{F}^* \varphi \geq 0$ on $[0,T) \times (0,\infty)^d \times (-\infty, 0)$.*

For the supersolution property on the terminal boundary, we introduce

$$\bar{\mathbf{N}}(x,p,y,q) := \left\{ |\sigma_Y(z,u) - \bar{\sigma}(x,p,u,a)^T q| \ : \ (u,a) \in \bar{U} \right\}$$

and the operator $\bar{\delta}(x,p,y,q) := \mathrm{dist}(0, \bar{\mathbf{N}}^c) - \mathrm{dist}(0, \bar{\mathbf{N}})$ together with its upper-semi-continuous envelop $\bar{\delta}^*$. We write $\bar{\delta}^* \varphi(x,p) = \bar{\delta}^*(x,p,\varphi(x,p), D\varphi(x,p))$ for a function φ

of (x,p). The latter operator is introduced to deal with the possible discontinuities of v. We are able now to recall the viscosity property of v_* at the terminal condition originally given by Theorem 2.2 of [Bouchard 09]. See also [Moreau 11] for the jump-diffusion case with Ψ defined as above. In our case however, the controls in $\mathcal{U}_{t,z}$ are bounded so that the coefficients of $Z^{\nu}_{t,z}$ are Lipschitz continuous uniformly in the control variable. This implies, together with Assumption 5.2.1.(iii), that Proposition 3.2.(i) in [Moreau 11] holds. We can then write

Theorem 5.2.3. *The function* $(x,p) \in (0,\infty)^d \times (-\infty,0) \mapsto v_*(T,x,p)$ *is a viscosity supersolution of* $\min\left\{(v_*(T,.) - \odot\Psi^{-1})\mathbf{1}_{\left\{\bar{F}^* v_*(T,.)<\infty\right\}},\ \bar{\delta}^* v_*(T,.)\right\} \geq 0$ *on* $(0,\infty)^d \times (-\infty,0)$.

The condition $\bar{F}^* v_*(T,.) < \infty$ is useless in most examples, but is still necessary in general since α is unbounded.
For the state constraint $p = 0$, we have by definition

$$V(t,x) := v(t,x,0) = \inf\left\{y \geq -\kappa\ :\ \Psi(Z^{\nu}_{t,z}(T)) = 0 \quad \text{for some } \nu \in \mathcal{U}_{t,z}\right\}\ . \tag{5.2.6}$$

and as above, the boundary condition has to be stated in general under a weak form. We introduce

$$V_*(t,x) := \liminf_{B\ni(t',x')\to(t,x)} v(t',x',0) \quad \text{and} \quad F^*(\Theta) := \limsup_{\varepsilon\searrow 0,\Theta'\to\Theta} F_\varepsilon(\Theta')$$

where, for $\Theta = (x,y,q,A) \in \mathbb{R}^d_+ \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ and $\varepsilon \geq 0$, $F_\varepsilon(\Theta)$ is the operator defined by

$$F_\varepsilon(\Theta) := \sup\left\{\mu_Y(z,u) - \mu(x,u)\cdot q - \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^T(x,u)A\right]\ :\ u \in \mathcal{N}_\varepsilon(x,y,q)\right\}$$

with $\mathcal{N}_\varepsilon(x,y,q) := \left\{u \in U\ :\ |\sigma_Y(z,u) - \sigma(x,u)^T q| \leq \varepsilon\right\}$. We also redefine in the same way δ^* by replacing $\bar{\sigma}$ with σ. We still abuse notation by writing $F^*\varphi(x)$ instead of $F^*(x,\varphi(x),D\varphi(x),H\varphi(x))$ for a function φ of x. We invoke Theorem 3.1 of [Bouchard 09] in our case. Note that the boundary condition of Proposition 3.2.(i) in [Moreau 11] is still valid for $p = 0$.

Theorem 5.2.4. *Assume that for all compact subset* $A \subset \mathbb{R}^d_+ \times [-\kappa,\infty) \times \mathbb{R}^d \times \mathbb{S}^d$*, there exists* $C > 0$ *such that* $F_\varepsilon(\Theta) \leq C(1+\varepsilon^2)$ *for all* $\varepsilon \geq$ *and all* $\Theta \in A$*. Then* V_* *is a viscosity supersolution of*

$$\begin{cases} -\partial_t V_* + F^* V_* \geq 0 & \text{on } [0,T) \times (0,\infty)^d \\ \min\left\{(V_* - \odot\Psi^{-1}(.,0))\mathbf{1}_{\{F^*V_*<\infty\}},\ \delta^* V_*\right\} \geq 0 & \text{on } \{T\} \times (0,\infty)^d \end{cases}$$

By definition (5.2.6) $v^*(.,0) \le V^*$, the upper star standing for the upper-semicontinuous version. Under a comparison assumption, $V^* = V_* = v_*(.,0) = v^*(.,0)$, see Theorem 3.1 in [Bouchard 09]. The above PDE characterization thus holds for $v(.,0)$ and the operator F_ε can be seen as the simplification of $\bar{F}_\varepsilon$ with $p = 0$.

5.2.2 The complete market case

In the specific case of complete market, we are able, via Fenchel duality arguments, to provide a quasi-explicit solution to (5.2.2). This has been done for the quantile hedging problem in [Bouchard 09], and we adapt the exact same arguments to the loss problem. The proof is given in Section 5.6 for self-countenance of the thesis. We consider the following dynamics :

$$\begin{cases} dX_{t,x}(r) = \mu(r, X_{t,x}(r))dr + \sigma(r, X_{t,x}(r))dW_r \\ dY^{\nu}_{t,x,y}(r) = \nu_r \cdot dX_{t,x}(r) \end{cases}, \quad t \le r \le T, \tag{5.2.7}$$

where μ and σ are Lipschitz continuous functions. This implies that $\mu_Y(z,u) = u\mu(x)$ and $\sigma_Y(z,u) = u\sigma(x)$ are uniformly Lipschitz in $u \in U$ and define a unique strong solution for $Y^{\nu}_{t,z}$. Here, $X_{t,x}$ is the actualized price process of d risky assets, not affected by the control ν, and $Y^{\nu}_{t,z}$ is the actualized value of a self-financing portfolio which is composed at each instant s of ν^i_s shares of the i-th risky asset, for $1 \le i \le d$. The actualization implies the usual reduction of the interest rate to zero. In order to avoid arbitrage possibilities, we assume that $\sigma(t,x)$ is invertible for all $(t,x) \in [0,T] \times \mathbb{R}^+$ and by denoting $\theta(t,x) = \sigma^{-1}(t,x)\mu(t,x)$, we assume

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^+} |\theta(t,x)| < \infty \,.$$

Definition 5.2.2. *We denote by $\mathbb{Q}_{t,x}$ the $\mathbb{P}$-equivalent martingale measure defined by*

$$\frac{d\mathbb{Q}_{t,x}}{d\mathbb{P}} = \exp\left\{ -\int_t^T \theta(s, X_{t,x}(s)) \cdot dW_s - \frac{1}{2}\int_t^T |\theta(s, X_{t,x}(s))|^2 ds \right\} \,.$$

According to Assumption 5.2.1.(iii) and (v), the stochastic target problem $V(t,x)$ corresponds to the super-hedging problem in complete market of a contingent claim $\Psi^{-1}(X_{t,x}(T),0)$. Note that if moreover $x \mapsto \Psi^{-1}(x,0)$ is a Lipschitz continuous payoff function, V is also continuous and given by $V(t,x) = \mathbb{E}^{\mathbb{Q}_{t,x}}\left[\Psi^{-1}(X_{t,x}(T),0)\right]$. In the general case, according to Theorem 5.2.4, V_* is a supersolution on $[0,T) \times (0,\infty)^d$ of the Black-Scholes equation

$$-\frac{\partial \varphi}{\partial t}(t,x) - \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma' H\varphi(t,x)\right] \ge 0 \,.$$

Following the application provided in [Bouchard 09], we are able to explicitly compute $v(t,x,p)$ for $p < 0$.

Corollary 5.2.1. *For $(t,x,p) \in [0,T) \times (0,\infty)^d \times \mathrm{int}(E)$, the problem* (5.2.2) *has a regular solution given by*

$$v(t,x,p) = \mathbb{E}^{\mathbb{Q}_{t,x}}\left[\odot\Psi^{-1}(X_{t,x}(T), J(X_{t,x}(T), Q_{t,x,\bar{q}}(T)))\right]$$

where $J(x,q) := \arg\sup_{p\in\mathbb{R}_-}\left\{pq - \odot\Psi^{-1}(x,p)\right\}$ *and* $Q_{t,x,\bar{q}}$ *is a strong solution of :*

$$Q_{t,x,\bar{q}}(t) = \bar{q} \quad \text{and} \quad dQ_{t,x,\bar{q}}(s) = Q_{t,x,\bar{q}}(s)\theta(s, X_{t,x}(s))dW_s^{\mathbb{Q}_{t,x}}$$

with $W^{\mathbb{Q}_{t,x}} = W + \int_t^\cdot \theta(s, X_{t,x}(s))ds$ *and* $\bar{q}$ *such that* $\mathbb{E}^{\mathbb{Q}_{t,x}}\left[J(X_{t,x}(T), Q_{t,x,\bar{q}}(T)\right] = p$ *.*

Proof is provided in Section 5.6. With additional assumptions, it is possible to retrieve the financial strategy ν by using the Itô representation of v provided by Corollary 5.2.1. Fix $(t,x,p) \in [0,T] \times (0,\infty)^d \times \mathrm{int}(E)$ and $q(t,x,p) =: \bar{q} \in (0,\infty)$ as in Corollary 5.2.1. Since $(X_{t,x}, Q_{t,x,\bar{q}})$ is a $\mathbb{F}$-Markovian process,

$$\begin{aligned} v(t,x,p) &= \mathbb{E}^{\mathbb{Q}_{t,x}}\left[\odot\Psi^{-1}(X_{t,x}(T), J(X_{t,x}(T), Q_{0,x,\bar{q}}(T)))|\mathcal{F}_t\right] \\ &=: \bar{v}(x,\bar{q})\ . \end{aligned}$$

Since $(X_{t,x}, Q_{t,x,\bar{q}})$ is a martingale under $\mathbb{Q}_{t,x}$, by assuming $\bar{v}$ is regular and applying Itô's formula,

$$d\bar{v}(x,\bar{q}) = \frac{\partial\bar{v}}{\partial x}(x,\bar{q})dX_{t,x}(t) + \frac{\partial\bar{v}}{\partial q}(x,\bar{q})dQ_{t,x,\bar{q}}(t)\ .$$

Now expressing $dW_s^{\mathbb{Q}_{t,x}}$ with respect to $dX_{t,x}(s)$, we obtain

$$d\bar{v}(x,\bar{q}) = \left(\frac{\partial\bar{v}}{\partial x} + \sigma^{-1}\theta Q_{0,x,\bar{q}}(t)\frac{\partial\bar{v}}{\partial q}\right)(x,\bar{q})dX_{t,x}(t)$$

which allows to deduce the optimal dynamic strategy.

Remark 5.2.2. *The formula in Corollary 5.2.1 can take a more explicit form in numerous cases. Consider a convex non-decreasing non-negative loss function ℓ on $\mathbb{R}$ with polynomial growth and a Lipschitz continuous payoff function g. We introduce*

$$(x,y) \mapsto \Psi(x,y) := -\ell\left(g(x) - y\right)\ .$$

This case contains our approach in the application of Section 5.4. It corresponds to the appreciation of losses induced by the holding of an European option with payoff $g(X_{t,x}(T))$. In that case, Assumption 5.2.1 holds and if the inverse of ℓ can be properly defined, it is convex and differentiable in p, providing the form $\Psi^{-1}(x,p) := g(x) - \ell^{-1}(-p)$. We can thus replace $J(x,q)$ by $(\frac{\partial\ell^{-1}}{\partial p})^{-1}(q)$. In that case, Corollary 5.2.1 reveals a more explicit form, with the hedging price of the claim $\mathbb{E}^{\mathbb{Q}_{t,x}}[g(X_{t,x}(T))]$ and an additional term involving only the variable p. This term corresponds to the penalty term in the dual expression of the acceptance set of a risk measure, see [Föllmer 06], if ℓ can be considered as one.

5.3 Extension to the semi-complete market framework

The above problem is a formulation in the stochastic control theory of a question raised by Föllmer and Leukert in [Föllmer 00] in financial mathematics. While the approach of Föllmer and Leukert encompasses a general semimartingale setting in incomplete market, we were compelled in the last section with Markovian price processes in a complete market. In the stochastic target formulation, the incomplete market framework cannot be expressed as the non-uniqueness of equivalent martingale measures. By avoiding arbitrage pricing arguments, we also avoid a general formulation of incomplete market. However, it is always possible to express the target problem with non-hedgeable sources of risk or state constraints. It appears that in some explicit cases of incompleteness, such as random volatility, it is not possible to retrieve the linear PDE for the convex conjugate of v. There, a direct approach must be undertaken, appealing to comparison arguments. We can nevertheless extend the problem (5.2.2) to very specific incomplete market cases.

In this section, we adapt and extend in a simple way the stochastic target problem in order to solve the loss hedging problem on electricity futures. We try to provide a general setting, since other examples seem to benefit from this framework. We start with the theoretical extension to a certain type of non-Brownian filtrations. In a second time, we propose a purely numerical resolution of the non-linear PDE.

5.3.1 The semi-complete market framework

Notations and problem

Recall that $\Omega = C([0,T],\mathbb{R}^d)$ is the space of continuous paths, $\mathbb{P}$ is the Wiener measure on Ω and $W_t(\omega) = \omega_t$. The filtrations $\mathbb{F}$ and $\mathbb{F}^t$, for $t \in [0,T]$, are defined as before. We consider an additional space $(\Omega^\lambda, \mathcal{G}, \mathbb{P}^\lambda)$, and a random variable $\Lambda \in L^1(L,\mathcal{G})$ where L is a metric separable subset of $\mathbb{R}^k$ for $k \in \mathbb{N}$. We then consider the product space

$$(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}) := (\Omega \times \Omega^\lambda, \mathcal{F} \times \mathcal{G}, \mathbb{P} \times \mathbb{P}^\lambda) .$$

We fix a time $t_0 \in [0,T]$. We then construct an augmented filtration $\widetilde{\mathbb{F}} := (\widetilde{\mathcal{F}}_t)_t$ on $\widetilde{\Omega}$, such that $\widetilde{\mathcal{F}}_t = \mathcal{F}_t \vee \{\emptyset, \Omega^\lambda\}$ on $[0,t_0)$ and $\widetilde{\mathcal{F}}_t = \mathcal{F}_t \vee \mathcal{G}$ on $[t_0,T]$. As in the previous case, we define $\widetilde{\mathbb{F}}^t$ the filtration generated by the increments from t of the Brownian motion and the realisation of the variable Λ. We write it as

$$\widetilde{\mathcal{F}}^t_s := \sigma\{W_r - W_t \ : \ t \le r \le s \vee t\} \vee \sigma\{\Lambda \text{ if } s \ge t_0 > t\} , \quad 0 \le t \le T .$$

For each $(t,z) \in [0,T] \times \mathbf{Z}$, we naturally extend $\mathcal{U}_{t,z}$ (resp. $\mathcal{A}_{t,p}$) to the set $\widetilde{\mathcal{U}}_{t,z}$ (resp. $\widetilde{\mathcal{A}}_{t,p}$) of $\widetilde{\mathbb{F}}^t$-adapted controls ν (resp. controls α). We assume that $Z^{\widetilde{\nu}}_{t,z}$ satisfies dynamics (5.2.7) where $\nu \in \mathcal{U}_{t,z}$ is replaced by $\widetilde{\nu} \in \widetilde{\mathcal{U}}_{t,z}$. We also assume that $|\theta(t,x)|$ is uniformly bounded in (t,x), implying that the market represented by $X_{t,x}$ is complete. We introduce an extended loss function $\widetilde{\Psi}:\ \mathbb{R}^d_+ \times \mathbb{R} \times L \longrightarrow \mathbb{R}_-$ which verifies

Assumption 5.3.1. *We assume that*
(i) the map $z \mapsto \widetilde{\Psi}(z,\lambda)$ verifies Assumption 5.2.1 uniformly in $\lambda \in L$, ;
(ii) for any $(t,z,\lambda) \in [0,T] \times \mathbf{Z} \times L$, the map $\widetilde{\nu} \in L^2([0,T]\times\Omega) \mapsto \widetilde{\Psi}(Z^{\widetilde{\nu}}_{t,z}(T),\lambda))$ is lower-semicontinuous (in $L^2(\mathbb{R}_-,\mathcal{F}_T)$) ;
(iii) for any $(t,z) \in [0,T] \times \mathbf{Z}$, $\mathbb{E}\left[|\widetilde{\Psi}(Z^{\widetilde{\nu}}_{t,z}(T),\Lambda)|\right]$ is bounded uniformly in $\widetilde{\nu} \in \mathcal{U}_{t,z}$;
(iv) $0 \in \widetilde{E}(\lambda) := \overline{\mathrm{conv}}(\widetilde{\Psi}((0,\infty)^d,[-\kappa,\infty),\lambda))$ for all $\lambda \in L$.

Note that Assumption 5.3.1.(ii) holds if $y \mapsto \widetilde{\Psi}(x,y,\lambda)$ is continuous. Indeed, the map $\widetilde{\nu} \mapsto Z^{\widetilde{\nu}}_{t,z}(T)$ is continuous and $\widetilde{\Psi}$ is right-continuous and non-decreasing in y according to Assumption 5.3.1.(i). We denote $\widetilde{\Psi}^{-1}$ the general inverse in y, $\odot\widetilde{\Psi}^{-1}$ the convex hull of $\widetilde{\Psi}^{-1}$ in p and $\widetilde{E} := \overline{\mathrm{conv}}(\widetilde{\Psi}((0,\infty)^d,[-\kappa,\infty),L))$.

The extension of problem (5.2.2) to the semi-complete market framework can then be formulated depending of the date t. When $t \geq t_0$, the event $\{\Lambda = \lambda\}$ is known and the projection of $\Omega \times \{\Lambda = \lambda\}$ on Ω corresponds to the Brownian framework. Every control in $\widetilde{\mathcal{U}}_{t,z}$ has a $\mathbb{F}^t$-progressively measurable version in $\mathcal{U}_{t,z}$. The geometric dynamic programming principle of Theorem 5.2.1 holds and, conditionally to $\{\Lambda = \lambda\}$, the financial market is complete so that we can apply the results of Section 5.2.2. If $t \geq t_0$, there is thus no ambiguity. For $t < t_0$, this cannot be written as well. By construction, we have that formally $\widetilde{\mathcal{F}}_s = \mathcal{F}_s$ for $0 \leq s \leq t$, but we do not have $\widetilde{\mathbb{F}}^t = \mathbb{F}^t$. This does not affect the right to apply the GDPP, but the market is no longer complete. We then write

Definition 5.3.1. *For any $(t,x,p,\lambda) \in [t_0,T] \times (0,\infty)^d \times \mathbb{R}_- \times L$, the value function of the semi-complete market problem is*

$$\widetilde{v}(t,x,p,\lambda) := \inf\left\{y \geq -\kappa \ :\ \mathbb{E}\left[\widetilde{\Psi}(Z^{\nu}_{t,z}(T),\lambda)\right] \geq p \text{ for some } \nu \in \mathcal{U}_{t,z}\right\} . \qquad (5.3.1)$$

For any $(t,x,p,\lambda) \in [0,t_0) \times (0,\infty)^d \times \mathbb{R}_- \times L$, the value function of the semi-complete market problem is

$$\widetilde{v}(t,x,p) := \inf\left\{y \geq -\kappa \ :\ \mathbb{E}\left[\widetilde{\Psi}(Z^{\widetilde{\nu}}_{t,z}(T),\Lambda)\right] \geq p \text{ for some } \widetilde{\nu} \in \widetilde{\mathcal{U}}_t\right\} \qquad (5.3.2)$$

where Λ is the $\widetilde{\mathcal{F}}_{t_0}$-measurable random variable defined previously.

Two problems appear with formulation (5.3.2), First, the problem reduction based on Proposition 3.1 in [Bouchard 09] shall be done with respect to the filtration $\widetilde{\mathbb{F}}$ and the new process $P^{\alpha}_{t,p}$ then jumps at time t_0. The problem falls in the framework of [Moreau 11] where explicitness is considerably more difficult to reach. Second, the new optimal control $\widetilde{\nu}$ is to be taken in $L^2([0,T] \times \widetilde{\Omega})$ which closely depends on both the law of Λ and the dependence of $\widetilde{\Psi}$ in λ. Hereafter, we provide a reduced formulation of the problem for $t < t_0$ in order to retrieve the complete market framework. To motivate the general framework, we illustrate the previous framework by a simple example.

Example 5.3.1 (Insurancial risk). *Take $t_0 = T$. Consider a variable Λ taking values in $\{0,1\}$. In this simple setting, Λ can represent an idiosyncratic risk as mortality or longevity.s Let us denote g a Lipschitz continuous payoff function and the loss function*

$$\Psi(x,y) = -(g(x) - y)\mathbb{1}_{\{g(x)>y\}}\mathbb{1}_{\{\Lambda=1\}}$$

which corresponds to the simple loss due to the holding of a contingent claim with payoff $g(X_{t,x}(T))$, but conditionally to the event $\{\Lambda = 1\}$. The hedging portfolio $Y^{\widetilde{\nu}}_{t,x,y}$ will thus depend on the values taken by Λ. The objective is then to evaluate the risk associated to each situation and, according to $\mathbb{P}^{\lambda}[\Lambda = 0]$ and $\mathbb{P}^{\lambda}[\Lambda = 1]$, propose the minimal capital y necessary to satisfy the level p of loss in expectation.
This situation can be considered for one client and multiplied in order to provide a framework for insurancial risk premium valuation. If the clients i are characterized by a variable Λ_i independent of the Λ_j, $j \neq i$, it is tempting to use a diversification rule and apply the law of large numbers to obtain a mean price for every client. In our context, we can directly use the law of $\Lambda := (\Lambda_1, \dots, \Lambda_n)$ for a finite $n \in \mathbb{N}$ and provide a premium for a finite number of clients without using asymptotic reasoning. For a recent view on diversification in insurance with financial hedging, see [Bouchard 11b].

Intermediary condition and piecewise problem

To overcome the difficulty, we are guided by the following financial argument. Considering $(t,y) \in [0,t_0) \times [-\kappa, \infty)$ and a strategy $\nu \in \mathcal{U}_{t,z}$, we arrive at time t_0 to the wealth $Y^{\nu}_{t,x,y}(t_0)$. At the apparition of the exogenous risk factor Λ, the portfolio $Y^{\nu}_{t,x,y}(t_0)$ can be greater or smaller than $\widetilde{v}(t_0, X_{t,x}(t_0), P^{\alpha}_{t,p}(t_0), \lambda)$ for some $\alpha \in \mathcal{A}_{t,p}$, depending on the value λ taken by Λ. In any case, the optimal behaviour is to maximize the value of $\mathbb{E}\left[\widetilde{\Psi}(X_{t,x}(T), Y^{\widetilde{\nu}}_{Y^{\nu}_{t,x,y,p}(t_0)}(T), \lambda)\right]$ with an optimal control $\widetilde{\nu} \in \mathcal{U}_{t_0, Z^{\nu}_{t,z}(t_0)}$. We thus pass from a stochastic target problem to an optimal control formulation, and both are carried out in the Brownian framework. We introduce the following face-lifted target function

$$\Xi(z) := \int_{\Omega^{\lambda}} \sup_{\nu \in \mathcal{U}_{t_0,z}} \mathbb{E}\left[\widetilde{\Psi}(Z^{\nu}_{t_0,z}(T), \Lambda(\omega^{\lambda})\right] d\mathbb{P}^{\lambda}(\omega^{\lambda}) \tag{5.3.3}$$

which is a Borel-measurable well-defined deterministic function.
The following result provides, in a weak form, the equivalence between the ill-formulation of equation (5.3.2) and the formulation of equation (5.3.4).

Proposition 5.3.1. *We set w the function defined by*

$$w(t,x,p) := \inf\left\{y \geq -\kappa \ : \ \mathbb{E}\left[\Xi(Z^{\nu}_{t,z}(t_0))\right] \geq p \text{ for some } \nu \in \mathcal{U}_{t,z}\right\} \tag{5.3.4}$$

for $(t,x,p) \in [0,t_0) \times (0,\infty) \times (-\infty,0)$. Then,
1. *For $(t,x,p) \in [0,t_0) \times (0,\infty) \times (-\infty,0)$, we have $w(t,x,p) \leq \widetilde{v}(t,x,p)$.*
2.(a) *Assume that $\lambda \mapsto \widetilde{\Psi}(x,y,\lambda)$ is continuous on L for any $(x,y) \in \mathbf{Z}$. Then for any $\delta > 0$ and $(t,x,p) \in [0,t_0) \times (0,\infty)^d \times (-\infty,0)$, we have*

$$\widetilde{v}(t,x,p-\delta) \leq w(t,x,p) \ . \tag{5.3.5}$$

2.(b) *Assume that Λ takes a countable number of values. Then for any $\delta > 0$ and $(t,x,p) \in [0,t_0) \times (0,\infty)^d \times (-\infty,0)$, equation (5.3.5) holds.*

Proof is given in Section 5.6. If the value function $\widetilde{v}$ is left-continuous in p, the equality holds. With this proposition, we are able to use the dynamic programming equation on $[0,t_0)$ with the terminal condition given by Ξ at time t_0, and controls ν having a $\mathbb{F}^t$-progressively measurable version. Following Proposition 5.3.1, one switch from a stochastic target problem on $[t,t_0)$ to an optimal control problem at time t_0. In fact, it is possible to link the optimal control problem on $[t_0,T]$ to a stochastic target formulation by an equivalence result given in [Bouchard 12].

Lemma 5.3.1. *Let us introduce the function*

$$\widetilde{v}^{-1}(t_0,x,y,\lambda) := \sup\left\{p \ : \ \widetilde{v}(t_0,x,p,\lambda) \leq y\right\} \ . \tag{5.3.6}$$

for $(x,y,\lambda) \in \mathbf{Z} \times L$. Then $\Xi(x,y) = \int_{\Omega^\lambda} \widetilde{v}^{-1}(t_0,x,y,\Lambda(\omega^\lambda))d\mathbb{P}^\lambda(\omega^\lambda)$.

Proof This is a direct application of Lemma 2.1 in [Bouchard 12]. □

One can easily extend the above framework to a finite sequence of deterministic times $(t_i)_{i\leq m}$. As we retrieve a problem in a standard form, we can apply recursively Proposition 5.3.1. The problem, as above, becomes a piecewise stochastic target problem, with a new condition on each interval $[t_i,t_{i+1}]$, $1 \leq i \leq m$. This framework can be applied to random changes such as dividends. We can also easily extend the approach we will follow for the granularity problem in the electricity futures market.

The above formulation holds for $p < 0$. The case $p = 0$ case is secondary in our context, and we appeal to [Moreau 11] for details in that framework.

Remark 5.3.1. *Let us justify the semi-complete market terminology. According to formulation (5.3.4), the intermediary target does not depend any more on the variable* Λ*. If we consider* Λ *as an external risk factor unlinked to the market, then the setting of this section has the explicitly interpretation of keeping the financial market complete and to allow for the construction of Section 5.2.2. In the general probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$*, there are several* $\widetilde{\mathbb{P}}$*-equivalent martingale measures. The hidden assumption is thus that for any* $\widetilde{\mathbb{P}}$*-equivalent martingale measures* $\mathbb{Q}^*$ *and* $\mathbb{Q}'$*,* $\mathbb{E}\left[d\mathbb{Q}^*/d\mathbb{P}|\mathcal{F}_T\right] = \mathbb{E}\left[d\mathbb{Q}'/d\mathbb{P}|\mathcal{F}_T\right]$*, see Remarks 2.2, 2.3 in [Bouchard 11b].*

5.3.2 Numerical resolution of the Stochastic Target problem

The function Ξ given in Proposition 5.3.1 is not explicit in most cases. The expectation formulation of Corollary 5.2.1 could then be exploited via a numerical approximation of Ξ and its derivatives. In the case when Ξ possesses the sufficient properties to apply Corollary 5.2.1, we obtain directly the value function and the dynamic strategy. If these properties no not hold, we are obliged to tackle the problem via another approach.

This is why in this section, we propose another approach which consists in solving the non linear PDE (5.2.5). This latter approach is more expensive in term of computation time than the numerical approximation of Corollary 5.2.1, but it has the advantage to be more general and applicable to a wide range of control problems. We propose here a Monte Carlo method based on Howard fixed point algorithm, see [Bokanowski 09], and the expectation formulation provided by the Feynman-Kac formula.

Expectation formulation

Following Proposition 5.3.1, it is still possible to apply Theorems 5.2.2 and 5.2.3, and obtain a viscosity property of the value function w for $t \in [0, t_0)$. In this section, we will apply a formal reasoning with extra simplifications. We assume for sake of simplicity that W is a one dimension Brownian Motion, so that $d = 1$, $\sigma(s, x) = \sigma > 0$ and $\mu(s, x) = \mu \in \mathbb{R}$ for all $(s, x) \in [t, t_0] \times \mathbb{R}_+$. The following strong assumptions allow to work directly on the value function w. The approach via test functions and the general framework in which the following holds true is clearly beyond the scope of this study.

Assumption 5.3.2. *We assume here that*
(i) the function w *is in* $\mathcal{C}^{1,2,2}([0, t_0) \times (0, \infty)^d \times \mathbb{R}_-)$ *;*
(ii) the function $(x, p) \mapsto \Xi^{-1}(x, p)$ *is convex in* p *and defined and* $\mathcal{C}^{1,3}$ *on* $(0, \infty)^d \times \mathbb{R}_-$ *;*
(iii) there exists $(\nu^*, \alpha^*) \in \widetilde{\mathcal{U}}_t \times \widetilde{\mathcal{A}}_t$ *such that the supremum is reached in equation (5.2.5).*

Let us explain more Assumption 5.3.2.(iii). It appears that $(\nu^*, \alpha^*) \in \bar{\mathcal{N}}_0(t,x,p,y,Dw,Hw)$ implies that, for some control $\alpha \in \mathcal{A}_{t,p}$ and processes $(X_{t,x}, P^{\alpha}_{t,p})$ taking the value (x',p') at time $t' \in [t,T]$, the process α^* at time t' shall take the value

$$a^* = (p\frac{\partial^2 w}{\partial p^2})^{-1}\left(\theta\frac{\partial w}{\partial p} - \sigma x\frac{\partial w}{\partial x\partial p}\right)(t',x',p') \ , \tag{5.3.7}$$

See the proof of Corollary 5.2.1 for details. Coming back to the non-linear PDE (5.2.5), w is thus assumed to be a regular solution on $[0,t_0) \times (0,\infty)^d \times (-\infty,0)$ of

$$\begin{cases} -\dfrac{\partial w}{\partial t} - \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 w}{\partial x^2} - \alpha^* p(x\sigma\dfrac{\partial^2 w}{\partial x\partial p} - \theta\dfrac{\partial w}{\partial p}) - \dfrac{1}{2}\alpha^{*2}p^2\dfrac{\partial^2 w}{\partial p^2} = 0 \ , \\ w(t_0,x,p) = \Xi^{-1}(x,p) \ . \end{cases} \tag{5.3.8}$$

Without going into the details of comparison given in Section 5.2.2, we can formally express $w(t,x,p)$ in terms of a conditional expectation by means of the Feynman-Kac formula :

$$w(t,x,p) = \mathbb{E}[\Xi^{-1}(X_{t,x}(t_0), P^{\alpha^*}_{t,x,p}(t_0))] \ . \tag{5.3.9}$$

with dynamics

$$\begin{cases} dX_{t,x}(s) &= X_{t,x}(s)\sigma(s,X_{t,x}(s))dW(s) \\ dP^{\alpha^*}_{t,p}(s) &= P_{t,p}(s)\alpha^*(s)(dW(s) + \theta(s,X_{t,x}(s))ds) \ . \end{cases} \tag{5.3.10}$$

The idea is then to use formulation (5.3.9) together with tangent process techniques to provide the different derivatives appearing in formula (5.3.7) This makes appear a fixed point which is exploited in the following discrete version of the control problem.

Discrete time approximated problem

Let us consider a regular mesh $(t =: \tau_0, \tau_1, \cdots, \tau_N := t_0)$, with $\delta := \tau_{k+1} - \tau_k$. Let $\tilde{\alpha}_k^{N-1} := (\tilde{\alpha}_q)_{q=k,\cdots,N-1}$ be a sequence of real valued functions defined on $\mathbb{R}_+ \times \mathbb{R}_-$. We define an associated sequence of random variables $(\tilde{X}_k, \tilde{P}_k^{\tilde{\alpha}_0^{k-1}})_{0\le k\le N}$ such that

$$\begin{cases} \tilde{X}_k &= X_{\tau_0,x}(\tau_k) \\ \tilde{P}_k^{\tilde{\alpha}_0^{k-1}} &= \tilde{P}_{k-1}^{\tilde{\alpha}_0^{k-2}} \exp\left\{\tilde{\alpha}_{k-1}\left((\theta - \frac{1}{2}\tilde{\alpha}_{k-1})\delta + \left(W_{\tau_k} - W_{\tau_{k-1}}\right)\right)\right\} \\ \tilde{P}_{\tau_0} &= p \ , \end{cases} \tag{5.3.11}$$

where to simplify notations we have omitted to state explicitly the following relation $\tilde{\alpha}_{k-1} = \tilde{\alpha}_{k-1}\big(\tilde{X}_{k-1}, \tilde{P}_{k-1}^{\tilde{\alpha}_0^{k-2}}\big)$. Observe in particular that $(\tilde{X}, \tilde{P})$ is a Markov chain. In the general case where σ and θ are functions defined on $[0,T] \times (0,\infty)^d$, one will replace dynamics (5.3.11) with an Euler scheme of dynamic (5.3.10), and shall do the following computations in regard of the new dynamics. We skip that technical part here.

Consider now the value function $\widetilde{w}$ of the problem (5.2.4) under dynamics (5.3.11). With the latter notations, the terminal condition writes $\widetilde{w}(\tau_N, x, p) := w(\tau_N, x, p) = \Xi^{-1}(x, p)$. According to Assumption 5.3.2, its partial derivatives are properly defined. Assume that formulation (5.3.9) holds too for this new problem, for some $\tau_k \leq \tau_N$ and a control process α^* of the form

$$\alpha^*(s) = \sum_{k \leq q < N} \widehat{\alpha}_q \mathbf{1}_{\tau_q \leq s < \tau_{q+1}} \ , \ \tau_k \leq s \leq \tau_N \ , \tag{5.3.12}$$

where $\widehat{\alpha}_q$ are real-valued functions. We can then write the alternative value function as a conditional expectation as follows

$$\widetilde{w}(\tau_k, x, p) := \mathbb{E}[\widetilde{w}(\tau_N, \widetilde{X}_{\tau_k,x}(\tau_N), \widetilde{P}_{\tau_k,p}^{\widehat{\alpha}_{k-1}^{N-1}}(\tau_N))] \ . \tag{5.3.13}$$

Recall that $\widehat{\alpha}_q$ is a function of (x, p) for $k \leq q < N$. The question is thus how to determine the value function $\widetilde{w}$ at time τ_{k-1}. A theoretical analysis ensuring the convergence of $\widetilde{w}$ toward w with δ going to 0 is beyond the scope of the present work.

One step optimization

In this paragraph, we fix $k \leq N$ and we suppose that the functions $\frac{\partial \widetilde{w}}{\partial p}(\tau_k, .)$, $\frac{\partial^2 \widetilde{w}}{\partial x \partial p}(\tau_k, .)$, and $\frac{\partial^2 \widetilde{w}}{\partial p^2}(\tau_k, .)$ are known and well-defined on $\mathbb{R}_+ \times \mathbb{R}_-$. We introduce the real valued functions $\overline{W^k}_p$, $\overline{W^k}_{xp}$ and $\overline{W^k}_{pp}$ defined on $\mathbb{R}_+ \times \mathbb{R}_-$ by

$$\overline{W}_p^k(.) = p\frac{\partial \widetilde{w}}{\partial p}(\tau_k, .) \ , \ \overline{W}_{xp}^k(.) = xp\frac{\partial^2 \widetilde{w}}{\partial x \partial p}(\tau_k, .) \ \text{ and } \ \overline{W}_{pp}^k(.) = p^2\frac{\partial^2 \widetilde{w}}{\partial p^2}(\tau_k, .) \ . \tag{5.3.14}$$

Definition 5.3.2. *Let $\widetilde{\alpha}_{k-1}$ be a real-valued function on $\mathbb{R}_+ \times \mathbb{R}_-$, we introduce a Markov kernel operator $M_k^{\widetilde{\alpha}_{k-1}}$ as follows. For any bounded measurable function $\varphi : \mathbb{R}_+ \times \mathbb{R}_- \to \mathbb{R}$, we set the conditional expectation transition kernel*

$$(x, p) \mapsto (M_k^{\tilde{\alpha}_{k-1}}\varphi)(x, p) = \int M_k^{\tilde{\alpha}_{k-1}}(x, p, du)\varphi(u) \ ,$$

such that $(M_k^{\tilde{\alpha}_{k-1}}\varphi)(x, p) = \mathbb{E}\left[\varphi\big(\widetilde{X}_{\tau_{k-1},x}(\tau_k), P_{\tau_{k-1},p}^{\widetilde{\alpha}_{k1}}(\tau_k)\big)\right]$. (For any such test function φ, note that $(M_k^{\tilde{\alpha}_{k-1}}\varphi)$ is again a bounded measurable function.)

According to equation (5.3.13) and dynamics (5.3.11), the optimal function $\widehat{\alpha}_{k-1}$ shall verify

$$\widetilde{w}(\tau_{k-1}, x, p) = M_k^{\widehat{\alpha}_{k-1}} \widetilde{w}(\tau_k, .)(x, p)$$

Additionally, using an Envelope argument and a tangent process approach (see [Broadie 96]) one can informally obtain

$$\begin{aligned} p\frac{\partial \widetilde{w}}{\partial p}(\tau_{k-1},x,p) &= M_k^{\widehat{\alpha}_{k-1}}\overline{W}_p^k(x,p) \\ xp\frac{\partial^2 \widetilde{w}}{\partial x \partial p}(\tau_{k-1},x,p) &= M_k^{\widehat{\alpha}_{k-1}}\overline{W}_{xp}^k(x,p) \\ pp\frac{\partial^2 \widetilde{w}}{\partial p^2}(\tau_{k-1},x,p) &= M_k^{\widehat{\alpha}_{k-1}}\overline{W}_{pp}^k(x,p) \end{aligned} \tag{5.3.15}$$

Following formulation (5.3.7), we are now in position to define an operator T_k on functions $\tilde{\alpha}_{k-1}$ on $\mathbb{R}_+ \times \mathbb{R}_-$ which associates a real valued function defined on $\mathbb{R}_+ \times \mathbb{R}_-$:

$$T_k(\tilde{\alpha}_{k-1}) = \frac{M_k^{\tilde{\alpha}_{k-1}}\left(\theta \overline{W}_p^k - \sigma \overline{W}_{xp}^k\right)}{M_k^{\tilde{\alpha}_{k-1}}\overline{W}_{pp}^k} \ . \tag{5.3.16}$$

The operator is related to (5.3.7) by the following relation. If equation (5.3.12) holds at time τ_{k-1}, then $\widehat{\alpha}_{k-1}$ defines a fixed point for T_k, i.e.,

$$T_k(\widehat{\alpha}_{k-1}) = \widehat{\alpha}_{k-1} \ . \tag{5.3.17}$$

To ensure the convergence of a fixed point algorithm, we shall verify the contraction properties of the operator T_k. We propose to illustrate this property below under some specific sufficient assumptions. The study of minimal assumptions for the property to hold is beyond the scope of this analysis.

Property 5.3.1. *Assume that the functions $\overline{W}_p^k$, $\overline{W}_{xp}^k$, $\overline{W}_{pp}^k$ are bounded functions. Assume that $|\overline{W}_{pp}^k(x,p)| > \varepsilon$ for all $(x,p) \in \mathbb{R}_+ \times \mathbb{R}_-$ and some $\varepsilon > 0$. Assume moreover that the functions*

$$\overline{W}_{ppp}^k(x,p) := p^3\frac{\partial^3 \widetilde{w}}{\partial p^3}(\tau_k,x,p) \quad \text{and} \quad \overline{W}_{xpp}^k(x,p) := xp^2\frac{\partial^3 \widetilde{w}}{\partial x \partial p^2}(\tau_k,x,p)$$

are bounded. Let $\widetilde{\alpha}_{k-1}$ and $\widetilde{\alpha}'_{k-1}$ be two bounded real valued functions on $\mathbb{R}_+ \times \mathbb{R}_-$. Then there exists $\bar{A} > 0$ such that

$$\|T_k(\widetilde{\alpha}_{k-1}) - T_k(\widetilde{\alpha}'_{k-1})\|_\infty \le \sqrt{\delta}\bar{A}\|\widetilde{\alpha}_{k-1} - \widetilde{\alpha}'_{k-1}\|_\infty \ .$$

Proof Notice that for all bounded function α, the real $T_k(\alpha)(x,p)$ does not depend on the whole function α but only on $\alpha(x,p)$. We thus can define a real valued function R_k defined on $\mathbb{R}$ such that for all function α,

$$R_k(\alpha(x,p)) := T_k(\alpha)(x,p) \ . \tag{5.3.18}$$

Differentiating R_k we obtain

$$\frac{dR_k}{d\alpha}(\alpha(x,p)) = \frac{\sqrt{\delta}\Big[\theta A_p + (\theta - 2T_k(\alpha))A_{pp} - T_k(\alpha)A_{ppp} - \sigma(A_{xp} + A_{xpp})\Big]}{M_k^\alpha \frac{\partial^2 \widetilde{w}}{\partial p^2}(\tau_k, .)}(x,p) \tag{5.3.19}$$

where, for $\epsilon = \frac{W_{\tau_k} - W_{\tau_{k-1}}}{\sqrt{\delta}} - (\theta + \alpha)\sqrt{\delta}$, we introduced the functions

$$A_p = M_k^\alpha \epsilon \overline{W}_p^k,\ A_{xp} = M_k^\alpha \epsilon \overline{W}_{xp}^k,\ A_{pp} = M_k^\alpha \epsilon \overline{W}_{pp}^k,\ A_{xpp} = M_k^\alpha \epsilon \overline{W}_{xpp}^k\ , A_{ppp} = M_k^\alpha \epsilon \overline{W}_{ppp}^k\ .$$

Using Cauchy-Schwartz inequality and the fact that the various derivatives are uniformly bounded, we can bound uniformly in (x,p) the real valued functions A_p, A_{xp}, A_{pp}, A_{xpp} and A_{ppp}. This is also the case for $T_k(\alpha)$. Then, there exists a positive function $(x,p) \mapsto A(x,p)$ such that $R'(\alpha(x,p)) < \varepsilon^{-1}\sqrt{\delta}A(x,p)$. Since the derivatives of $\widetilde{w}(\tau_k, .)$ are uniformly bounded in (x,p), the function A is bounded and there exists a finite positive constant $\bar{A} \geq A(x,p)$ such that

$$\begin{aligned} \|T_k(\widetilde{\alpha}_{k-1}) - T_k(\widetilde{\alpha}'_{k-1})\|_\infty &\leq \sup_{(x,p)\in(0,\infty)\times\mathbb{R}_-} |R_k(\widetilde{\alpha}_{k-1}(x,p)) - R_k(\widetilde{\alpha}'_{k-1}(x,p))| \\ &\leq \sqrt{\delta}A(x,p)\,|\widetilde{\alpha}_{k-1}(x,p) - \widetilde{\alpha}'_{k-1}(x,p)| \\ &\leq \sqrt{\delta}\bar{A}\|\widetilde{\alpha}_{k-1} - \widetilde{\alpha}'_{k-1}\|_\infty\ . \end{aligned}$$

We thus deduce the contraction property for T in the L^∞ norm. □

Algorithm

We apply this technique recursively until $k = 0$, in order to obtain a piecewise constant time continuous process defined along (5.3.12). We first have the partial derivatives of $\widetilde{w}(\tau_N, .)$ from the terminal condition. In the generic case $k < N$, computing the partial derivatives of $w(\tau_k, .)$ requires to compute conditional expectation represented by the kernel operator M_k^α in order to obtain (5.3.16). This is done by a regression approach on a fixed grid given by $(\widetilde{X}_{k,i}, \widetilde{P}_{k,i}^{\alpha_{k-1}})_{k=1,\cdots,n}^{i=1,\cdots M}$ of M trajectories simulated independently for a given choice of $(\alpha_{k-1})_{k=1\ldots N}$.

The computation of the optimal control of equation (5.3.12) is made backward since (5.3.13) is assumed to hold on $[\tau_k, \tau_N]$ for a given k. Assume that an approximation of the functions $(\widehat{\alpha}_0, \cdots, \widehat{\alpha}_{N-1})$ satisfying the fixed point (5.3.17), denoted $(\widetilde{\alpha}_0, \cdots, \widetilde{\alpha}_{N-1})$ shall found for a tolerance parameter $\epsilon > 0$. We provide an initialization of α_{k-1}, say $-\theta$, at any step $0 < k \leq N$. The program consists in starting with step A$(N-1)$ in the following algorithm :

A(k) : `SET` $a := \alpha_k$; `GOTO` B(k, a) ;
B(k, a) : 1. `SET` $a' := T_{k+1}(a)$;
2. `IF` $|a' - a| \leq \epsilon$
`THEN SET` $\widetilde{\alpha}_k := a'$.
`IF` $k = 0$ `THEN STOP` ;
`ELSE GOTO` A($k - 1$) ;
`ELSE GOTO` B(k, a') ;

By using a cumulated approximation for the function $w(\tau_k, .)$ and its derivatives, the numerical error grows rapidly with the number of steps n. On the contrary, the convergence of $\widehat{\alpha}$ towards α^* and the error due to the discrete time approximation is controlled with δ. To validate the algorithm we proceed in Section 5.5.2 to a comparison between the explicit formula of Corollary 5.2.1 and the value provided by the algorithm.

5.4 Application to a cascading strategy for controlling loss on electricity Futures market

We develop here an application of the latter work. Following commodity markets, maturities and delivery periods are based on calendar dates, meaning that delivery starts on Monday for week period, the first day of a month for month or quarter-long delivery periods and the first day of the year for year futures. Therefore, the availability of a futures depends on its delivery period and the remoteness of its maturity, as illustrated on Figure 5.1. It is referred in the industry as a cascading rule or the granularity problem of the term structure. When a period splits and the related contracts appear, an arbitrage structure is taken in consideration in order to fix prices. We explain this structure and introduce hereafter a simple model which takes it into account. We then apply the above results to this situation.

5.4.1 Model and structural correlation

We consider the situation of an agent which is endowed with an option, a payoff function on a monthly period futures with expiration at time T. The option is defined by a function g which is Lipschitz continuous. This underlying asset X^{M_i} for $1 \leq i \leq 3$ (in Euro per MWh) appears at time $t_0 \in [0, T]$, and before its apparition, the quarterly period futures with price X^Q covering the same period is available on $[0, t_0]$. Here, i takes value between 1 and 3 for the respective first, second and last month of the quarter. In

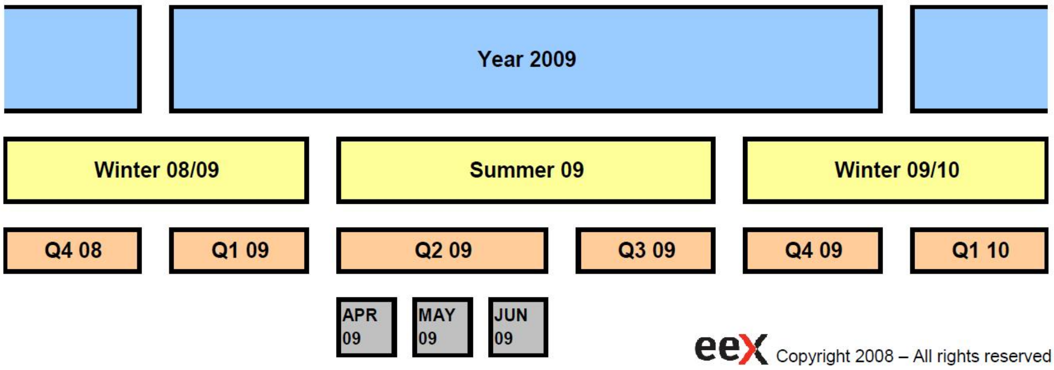


FIGURE 5.1 – Cascading rule of futures price decomposition on the French EEX Power Derivatives Market.

reality, the contracts are quoted together for a certain time, so that, in the case that the three months of the quarter are available, the following structural relation holds at each time during the period of common quotation :

$$X^Q(t) = \frac{1}{\sum_{i=1}^{3} N_i} \sum_{i=1}^{3} N_i X^{M_i}(t) \; .$$

Here, N_i is the number of hours in the period covered by contract X^{M_i}. This fact induces a structural correlation between monthly and quarterly futures prices, and motivates the following model.

We know focus on a particular monthly period asset and avoid the notation i. We consider the complete filtered space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, \widetilde{\mathbb{P}})$. Let us assume that the prices have the following dynamics :

$$\begin{cases} X^Q_{t,x_Q}(r) = x_Q + \int_0^r \mu_Q X_{t,x}(s)ds + \int_0^r \sigma_Q X_{t,x}(s)dW^Q_s & \text{for all } r \in [0,T] \\ X^M_{t_0,x_M}(r) = x_M + \int_{t_0}^r \mu_M X_{t,x}(s)ds + \int_{t_0}^r \sigma_M X_{t,x}(s)dW^M_s & \text{for all } r \in [t_0,T] \end{cases} \tag{5.4.1}$$

where (W^Q, W^M) is a $\widetilde{\mathbb{F}}$-adapted Brownian motion with possible correlation. We also assume without loss of generality that $(\mu_Q, \sigma_Q) = (\mu_M, \sigma_M) =: (\mu, \sigma) \in \mathbb{R} \times (0, \infty)$. We consider the portfolio as defined in Section 5.2.2. We also consider the problem of partial lower moment loss, where Ψ is defined by

$$\Psi(x,y) = -\frac{1}{n}((g(x) - y)\mathbb{1}_{\{g(x) \geq y\}})^n$$

for $n \in \mathbb{N}^*$. The problem is then to find the smallest initial capital y at time t such that

$$\mathbb{E}\left[-\frac{1}{n}\left(g(X_{t_0,x_M}(T)) - Y^{\nu}_{t,x,y}(T)\right)^n \mathbb{1}_{\{(g(X_{t_0,x_M}(T))\geq Y^{\nu}_{t,x,y}(T)\}})|(X_{t_0,x_M}(t), X_{0,x_Q}(t))\right] \geq p$$

for some $p \leq 0$ and some control $\widetilde{\nu} \in \widetilde{\mathcal{U}}_t$. It is clear that x_M is supposed to be a measurable only after t_0, so that the problem is not well-posed, at least before t_0. It is easy to notice that the dynamics can be reduced to a one dimension problem by the following reasoning. On the period $[t_0, T]$, the monthly asset is available, and the market is considered to be completed. We can then adapt the notations to retrieve the semi-complete setting. Consider the new Brownian motion W defined on $[0, T]$ by $W := (W^Q_{.\wedge t_0} + (W^M - W^Q)_{t\vee t_0})$, and the filtration $\mathbb{F}$ defined by $\mathcal{F}_t := \sigma(W_s,\ 0 \leq s \leq t)$. We then define the $\mathbb{F}$-adapted price process X by

$$X_{0,x}(t) = x_Q + \int_0^t \mu X_{0,x_Q}(s)ds + \int_0^t \sigma X_{0,x_Q}(s)dW_s\ , \quad t \in [0,T]$$

with $x = x_Q$. We then introduce the random variable

$$\Lambda := \frac{X^M_{t_0,x_M}(t_0)}{X^Q_{0,x_Q}(t_0)} = \frac{x_M}{X_{0,x_Q}(t_0)}$$

and the σ-algebra $\mathcal{G} := \sigma(\Lambda)$. The price process X can then be seen as the price of the quarterly futures on $[0, t_0]$, and for a given value λ taken by Λ, λX is the process describing the price of the monthly futures on $[t_0, T]$. The variable Λ can be seen as a shaping factor, giving a weight for the impact of the average price of electricity on the considered monthly period M into the quarterly period Q. If the considered month contains N_i hours and the two other months contain N_j and N_k hours respectively, and if we assume that the electricity price is always non negative, we have for $t \geq t_0$

$$0 \leq X^M_{t,x_M}(t) \leq \frac{N_i + N_j + N_k}{N_i} X^Q_{t,x_Q}(t)\ ,$$

justifying the following

Assumption 5.4.1. *We assume that Λ takes it values in $L := [0, L_{max}]$ with $L_{max} := \frac{N_i+N_j+N_k}{N_i}$, and that $\mathcal{G}$ is independent of $\mathcal{F}_{t_0}$.*

From now on, we assume that $N_i = N_j = N_k$ so that $L_{max} = 3$. It is clear that by defining the filtration $\widetilde{\mathbb{F}}$ by $\widetilde{\mathcal{F}}_t = \mathcal{F}_t$ on $[0, t_0)$ and $\widetilde{\mathcal{F}}_t = \mathcal{F}_t \vee \mathcal{G}$ for $t \in [t_0, T]$, we retrieve the information :

$$\widetilde{\mathcal{F}}_t := \sigma(X^Q_{0,x_Q}(s \wedge t_0), X^M_{t_0,x_M}(s)\mathbb{1}_{\{s\geq t_0\}},\ 0 \leq s \leq t), \quad 0 \leq t \leq T\ .$$

We can finally write $\widetilde{\Psi}(x,y,\lambda) = \Psi(\lambda x, y)$ and Problem 5.3.1 becomes the following pricing problem with controlled loss :

$$\widetilde{v}(0,x_Q,p) = \inf\Big\{ y\in\mathbb{R}\ :\ \exists\nu\in\mathcal{U}_t \text{ s.t.} \\ \mathbb{E}\left[-\tfrac{1}{n}\left(g(\Lambda X_{0,x_Q}(T)) - Y^{\nu}_{0,x_Q,y}(T)\right)^n \mathbb{1}_{\left\{(g(\Lambda X_{0,x_Q}(T))\geq Y^{\nu}_{0,x_Q,y}(T))\right\}}\right] \geq p \Big\}\ .$$

5.4.2 The complete market case for $t \geq t_0$

For $t \geq t_0$, the problem has an explicit solution provided by Corollary 5.2.1. It is easy to see that once Λ is known (taking the value λ), the quarterly period asset X^Q is useless on this period. Corollary 5.2.1 then rewrites as so.

Corollary 5.4.1. *For all* $(t,x,p,\lambda)\in[t_0,T]\times(0,\infty)\times\mathbb{R}\times L$,

$$\widetilde{v}(t,x,p,\lambda) = \mathbb{E}^{\mathbb{Q}_{t,x}}\left[g(\lambda X_{0,x_Q}(T))|X_{0,x_Q}(t)=x\right] - (-np)^{1/n}\exp\left\{\frac{\theta^2}{2(n-1)}(T-t)\right\}\ .$$

For sake of simplicity, we introduce the complete market valuation function $(t,x)\mapsto C(t,x)$ and $x\mapsto C_0(x)$ such that

$$C(t,x) = \mathbb{E}^{\mathbb{Q}_{t,x}}[g(X_{t_0,x_M}(T))|X^M_{t_0,x_M}(t)=x]\ ,\quad \text{and}\quad C_0(x) = C(t_0,x)\ . \tag{5.4.2}$$

Since g is a Lipschitz continuous function, $C\in\mathcal{C}^{1,2}([t_0,T]\times(0,\infty)^d)$. According to Corollary 5.4.1, the partial derivatives of $\widetilde{v}$ exist. We thus recall the expressions of the optimal controls α^* and ν^* in terms of the derivatives of $\widetilde{v}$, see the proof of Corollary 5.2.1 :

$$\left\{\begin{array}{rcl} \alpha^*(t,x,p) &=& \left(\dfrac{\partial^2\widetilde{v}}{\partial p^2}\right)^{-1}\left(\theta\dfrac{\partial\widetilde{v}}{\partial p} - x\sigma\dfrac{\partial^2\widetilde{v}}{\partial x\partial p}\right)(t,x,p) \\ \nu^*(t,x,p) &=& \left(\dfrac{\partial\widetilde{v}}{\partial x} + \dfrac{\alpha}{x\sigma}\dfrac{\partial\widetilde{v}}{\partial p}\right)(t,x,p) \end{array}\right. \tag{5.4.3}$$

According to this formula, it is possible to compute the partial derivatives of $\widetilde{v}$ with respect to the ones of C. We have :

$$\begin{cases} \frac{\partial \widetilde{v}}{\partial t}(t,x,p) &=& \frac{\partial C}{\partial t}(t,x) + \frac{\theta^2}{2(n-1)}\exp\left\{\frac{\theta^2}{2(n-1)}(T-t)\right\} \\ \frac{\partial \widetilde{v}}{\partial x}(t,x,p) &=& \frac{\partial C}{\partial x}(t,x) \\ \frac{\partial^2 \widetilde{v}}{\partial x^2}(t,x,p) &=& \frac{\partial^2 C}{\partial x^2}(t,x) \\ \frac{\partial \widetilde{v}}{\partial p}(t,x,p) &=& \exp\left\{\frac{1-n}{n}\log(-np) + \frac{\theta^2}{2(n-1)}(T-t)\right\} \ . \end{cases}$$

One can then notice that the sensibility with respect to the asset price is the same as in the Black-Scholes framework. Comparing to equation (5.4.3), the strategy in the complete market case consists in superhedging $g(\lambda X_{t,x}(T))$ and superhedging the threshold $P^{\alpha}_{t,p}(T)$ with a correcting term.

5.4.3 Numerical procedure for $t = t_0$

The intermediary condition

According to the results of Section 5.3.1, we have to compute the face-lifted function Ξ at time t_0. For a fixed value $\lambda \in L$, we obtain an explicit formula for $\widetilde{v}_0^{-1}(x,y,\lambda)$ defined in equation (5.3.6),

$$\widetilde{v}_0^{-1}(x,y,\lambda) = -\frac{1}{n}(C_0(\lambda x) - y)^n \exp\left\{-\frac{\theta^2 n}{2(n-1)}(T-t_0)\right\} \mathbb{1}_{\{C_0(x)\geq y\}} \ ,$$

but for a general expression of the law of Λ, we have to compute numerically the value of Ξ at time t_0 :

$$\Xi(x,y) = \exp\left\{-\frac{\theta^2 n}{2(n-1)}(T-t_0)\right\} \int_{\Omega^\lambda} -\frac{1}{n}\big(C_0(\Lambda(\omega^\lambda)x) - y\big)^n \mathbb{1}_{\left\{C_0(\Lambda(\omega^\lambda)x)\geq y\right\}} d\mathbb{P}^\lambda(\omega^\lambda) \ . \tag{5.4.4}$$

Following Proposition 5.3.1 and equation (5.3.8), we need to compute the inverse of the above function and its partial derivatives. The value $w(t_0^-,x,p)$ is precisely given by $w(t_0^-,x,p) = \Xi^{-1}(x,p)$. Let us introduce the real valued functions j_k and j_k^Δ for

$k \in \{n-1, n-2\}$ such that for any positive real values x and p,

$$\begin{cases} j_k(x,p) &= \displaystyle\int_{\Omega^\lambda} \big(C_0(\Lambda(\omega^\lambda)x) - \Xi^{-1}(x,p)\big)_+^k d\mathbb{P}^\lambda(\omega^\lambda) \\ j_k^\Delta(x,p) &= \displaystyle\int_{\Omega^\lambda} \Lambda(\omega^\lambda)\frac{\partial C_0}{\partial x}(\Lambda(\omega^\lambda)x)\big(C_0(\Lambda(\omega^\lambda)x) - \Xi^{-1}(x,p)\big)_+^k d\mathbb{P}^\lambda(\omega^\lambda) \ . \end{cases} \tag{5.4.5}$$

By simple calculations, we obtain the following expressions for the derivatives of the intermediary condition Ξ^{-1} :

$$\begin{cases} \dfrac{\partial \Xi^{-1}}{\partial x}(x,p) &= \dfrac{j_{n-1}^\Delta}{j_{n-1}}(x,p) \\ \dfrac{\partial \Xi^{-1}}{\partial p}(x,p) &= -\exp\left\{\dfrac{\theta^2 n}{2(n-1)}(T-t_0)\right\}\dfrac{1}{j_{n-1}}(x,p) \\ \dfrac{\partial^2 \Xi^{-1}}{\partial p^2}(x,p) &= \left[(n-1)\dfrac{j_{n-2}}{j_{n-1}}\left(\dfrac{\partial \Xi^{-1}}{\partial p}\right)^2\right](x,p) \\ \dfrac{\partial^2 \Xi^{-1}}{\partial x \partial p}(x,p) &= \left[(n-1)\dfrac{\partial \Xi^{-1}}{\partial p}\dfrac{\big(j_{n-1}^\Delta j_{n-2} - j_{n-2}^\Delta j_{n-1}\big)}{\big(j_{n-1}\big)^2}\right](x,p) \ . \end{cases}$$

By computing numerically the functions j_k and j_k^Δ for $k \in \{n-1, n-2\}$, we are able to obtain the above derivatives. Once we obtain these functions, we apply the numerical procedure of Section 5.3.2 to obtain $w(t,x,p)$ for $t < t_0$.

Controls and threshold $P^\alpha_{t,p}(t_0)$

Injecting these expressions in equations (5.4.3), we deduce the following formula for the optimal controls of our stochastic target problem at time t_0 :

$$\begin{cases} \alpha^*(t_0,x,p) &= \dfrac{\theta}{n-1}\dfrac{j_{n-1}^2}{j_{n-2}}(t_0,x,p)\exp\left\{-\dfrac{\theta^2 n}{2(n-1)}(T-t_0)\right\} \\ \nu^*(t_0,x,p) &= \dfrac{j_{n-2}^\Delta}{j_{n-2}}(t_0,x,p) + \dfrac{\theta}{x\sigma(n-1)}\dfrac{j_{n-1}}{j_{n-2}}(t_0,x,p) \ . \end{cases} \tag{5.4.6}$$

The resulting strategy is obtained by averaging, among the possible values taken by Λ, the *a posteriori* (knowing $\Lambda = \lambda$) optimal strategies, i.e. $\lambda\Delta_0(\lambda x)$, weighted by a measure of the associated risk given by the probability measure

$$\mathcal{R}_{x,p}(\omega^\lambda)d\mathbb{P}^\lambda(\omega^\lambda) = \frac{(C_0(\Lambda(\omega^\lambda)x) - \Xi^{-1}(x,p))_+^{n-2}}{j_{n-2}(x,p)}d\mathbb{P}^\lambda(\omega^\lambda) \ .$$

In other words, if λ corresponds to a risky situation (w.r.t. to the loss function), then the corresponding a posteriori strategy $\lambda\Delta_0(\lambda x)$ will be highly weighted, whereas if λ corresponds to a riskless situation then the corresponding a posteriori optimal strategy $\lambda\Delta_0(\lambda x)$ will weakly (or not at all) contribute to the a priori optimal strategy (for which the event $\{\Lambda = \lambda\}$ is unknown).

We can also derive the dynamic of $P_{t_0^-,p}$ at the time of the jump using the following condition

$$\widetilde{v}(t_0, x, P_{t_0^-,p}(t_0), \lambda) = \widetilde{v}(t_0^-, x, p) \ ,$$

$$v(t_0, \lambda x, P_{t_0^-,p}(t_0)) \quad = \quad C(t_0, \lambda x) - (-nP_{t_0^-,p}(t_0))^{1/n} e^{\frac{\theta^2}{2(n-1)}(T-t_0)} \ ,$$

which yields

$$-P_{t_0^-,p}(t_0) = \frac{1}{n} e^{-\frac{n}{2(n-1)}\theta^2(T-t_0)} \left(C(t_0, \lambda x) - v(t_0^-, x, p)\right)_+^n . \qquad (5.4.7)$$

5.4.4 The Black-Scholes benchmark

To evaluate the performance of the novel approach, we introduce a simple benchmark based on the Black-Scholes strategy. We consider the following trading strategy. Before the apparition of the month futures, the agent can naively use an estimation of the futures price at time $t < t_0$, which is given by

$$\overline{X}(t) := \mathbb{E}\left[\Lambda X_{0,x_Q}(t_0) | \widetilde{\mathcal{F}}_t\right] = \bar{\lambda} X_{0,x_Q}(t) e^{(\mu - \frac{\sigma^2}{2})(t_0 - t)} \ ,$$

where $\bar{\lambda} = \int_{\Omega^\lambda} \Lambda(\omega^\lambda) d\mathbb{P}^\lambda(\omega^\lambda)$. If the price of the appearing asset indeed takes this value, then it suffices to constitute a portfolio starting with value $C_0(\bar{\lambda} X_{0,x_Q}(t_0))$ at time t_0 and apply the super-hedging strategy to cancel the risk of the claim $g(\bar{\lambda} X_{0,x_Q}(T))$. Then, the Black-Scholes price of a contingent claim $g(\Lambda X_{0,x_Q}(T))$ is given by $\mathbb{E}^{\mathbb{Q}_{0,x_Q}}[g(\bar{\lambda} X_{0,x_Q}(T))]$. Before t_0, the agent uses the quarter asset to hedge the option, and since we assume that the market represented by X_{0,x_Q} is complete on $[0, t_0)$, the agent reaches almost surely the wealth $\mathbb{E}^{\mathbb{Q}_{0,x_Q}}[g(\bar{\lambda} X_{0,x_Q}(T)) | X_{0,x_Q}(t_0)]$ at time t_0.

In the call option case where $g(x) = (x - K)^+$ for some $K > 0$, it is easy to retrieve with the above methodology the Black-Scholes price of the option at time t_0, here C_{BS} :

$$\begin{aligned} C_{BS}(t, X^M, K) \quad &:= \mathbb{E}^{\mathbb{Q}_{t,x_0}}[(X^M(T) - K)^+ | X^M_{t_0,x_M}(t) = x_0] \\ &= \mathbb{E}^{\mathbb{Q}_{t,x_0}}[(\bar{\lambda} X_{0,x_Q}(T) - K)^+ | X_{0,x_Q}(t) = x_0/\bar{\lambda}] \\ &= \bar{\lambda}\mathbb{E}^{\mathbb{Q}_{t,x_0}}[(X_{0,x_Q}(T) - K/\bar{\lambda})^+ | X_{0,x_Q}(t) = x_0/\bar{\lambda}] \\ &= \bar{\lambda} C_{BS}(t, X^Q, K/\bar{\lambda}) \ . \end{aligned}$$

The delta-hedging strategy on the quarter contract follows immediately.
If the value of λ falls out, the agent still proceeds to the delta-hedging strategy and propagates his misestimation till time T. The theoretical terminal hedging error is thus given by :

$$\varepsilon_T = C(t_0\bar{\lambda}x) - C(t_0, \lambda x) \tag{5.4.8}$$

By averaging the value λ, the agent naively expects to average his losses around 0. This is however not true since the super-hedging price of the claim is not a linear function of the asset price.

5.5 Numerical results

5.5.1 Parameters estimation and sources of error

To provide a realistic framework, we refer to historical data. This allows to propose a model for L and Λ, and values for parameters (μ, σ).

The available data designate daily quotations of futures contracts prices on the French Power Market. Market data are provided by EEX, and represent here a period from October 2004 to March 2001, i.e., 78 Month delivery futures during their whole quotation period and the respective Quarter contracts covering them. Each Month contract has a different lifetime depending on its order in the quarter. Indeed, every month contract appears at least 3 months ahead but, by absence of arbitrage opportunity, the last month appears with the second one. We present here some statistical characteristics of $\Lambda = X^M(t_0)/X^Q(t_0)$, and the parameters of the dynamics of equation (5.4.1). The parameters μ and σ are computed on the aggregated returns of month futures and quarter futures. with μ including the actualization since we assumed that the interest rate is null.

$\mathbb{E}[\lambda]$	$\mathbb{V}[\lambda]$	μ	σ	n
1.0012	0.081	0	0.28	2

TABLE 5.1 – Parameters estimation

To estimate the law of Λ we first notice that we are looking for a probability law on a bounded interval. We then assume the following.

Assumption 5.5.1. *The law of* Λ *is given by* $\frac{1}{3}\Lambda \sim \beta(a, b)$, *with* $(a, b) = (114, 227)$.

We estimate the parameters on realizations and we provide on Figure 5.2 the non-parametric historical density and the estimated parametric density.

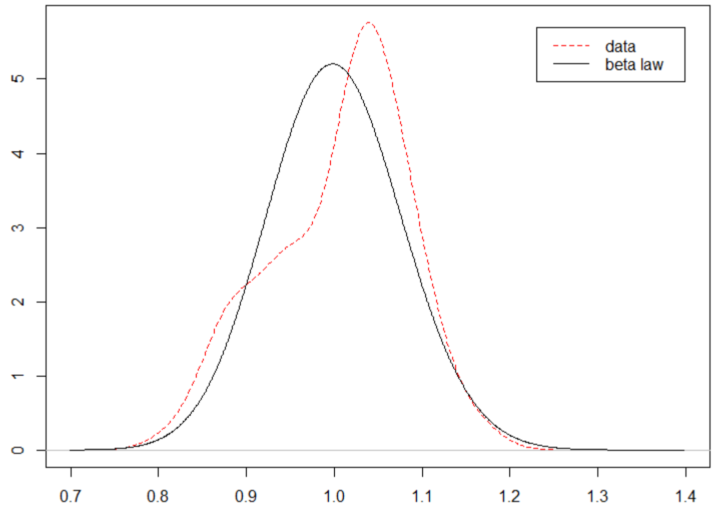
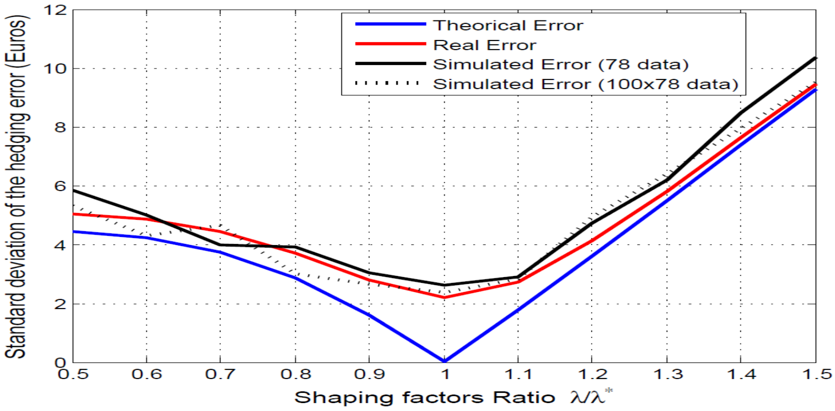


FIGURE 5.2 – Kernel density estimation versus parametrized Beta law.

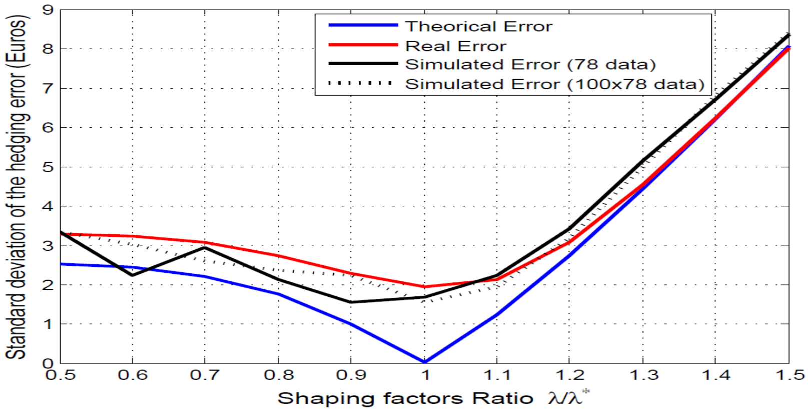
We first shall evaluate if we can significantly compare the two approaches. In order to do so, wes study the possible sources of error, both on real and simulated data, in the applied hedging strategy. This has another implicit motivation : we argue that a wrong estimation of Λ can impact significantly the hedging error if we proceed to the naive delta-hedging strategy. This preliminary task takes the following form.

We consider the hedging of a call option of payoff $g(\lambda X_{t,x_M}(T)) = (\lambda X_{t,x_M}(T) - K)_+$, with three possible values of K. We apply the Black-Scholes strategy with different prior estimations of Λ. We assume that the trader can make an error of 50% around the real value in the worst case. We thus use the same time grid as data format for price simulations. We also provide the hedging error of the Black-Scholes strategy for the same number of simulations as provided by data series, and for a greater number of trajectories.

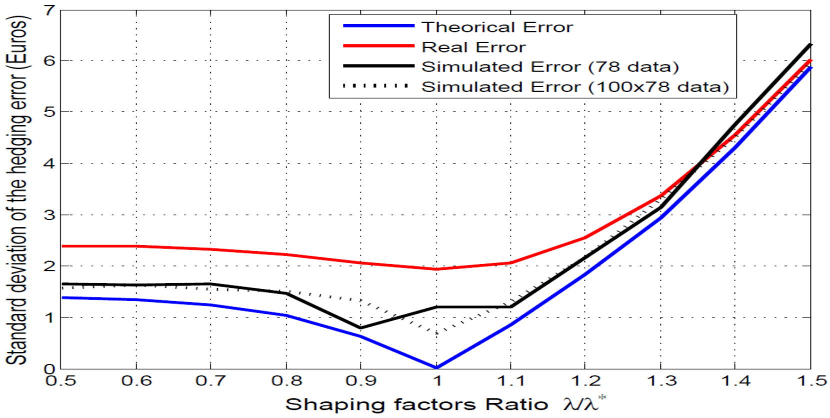
By comparing hedging error on real data and similar simulated trajectories we test the model error with respect to real price dynamics. If this error is small, we make the comparison of hedging error between a small and a great number of simulations, providing the error which is due to lack of data. Finally, for a great number of simulations of dynamics (5.4.1), the hedging error should reasonably converge to the error of discretization of the Black-Scholes strategy. We then compare the hedging error of the simulated discrete strategy with the theoretical hedging error given by equation (5.4.8). We sum up the results in Figure 5.3.



(a) $K = 0,8X(t_0)$



(b) $K = X(t_0)$



(c) $K = 1.2X(t_0)$

FIGURE 5.3 – Cumulated errors from the estimation (ε_T, blue), discrete strategy (dotted), lack of price trajectories (black) and difference between model and real dynamics (red) for 3 strikes K. Error is given in Euros (to compare with the initial Black-Scholes price).

5.5.2 Numerical validation

In this paragraph, we are interested in testing the performance of the algorithm described above in a simple case where an exact benchmark solution is known. We consider the problem of controlled loss in the complete market framework, with the solution given by Corollary 5.4.1. Recall that we obtain the following explicit expression for the martingale $P^{\alpha}_{t,p}$ initialized at time $t \geq t_0$ and the function v for any $s \in [t, T)$,

$$\begin{cases} v(s, X_{t,x}(s), P^{\alpha}_{t,x}(s)) = C(s, X_{t,x}(s)) - (-nP^{\alpha}_{t,p}(s))^{1/n} \exp\left\{\frac{\theta^2}{2(n-1)}(T-s)\right\} \\ P^{\alpha}_{t,p}(s) = p\left(\frac{X_{t,x}(s)}{x}\right)^{-\frac{n}{n-1}\frac{\mu}{\sigma^2}} \exp\left\{\frac{n^2(\theta^2-\mu)}{2(n-1)}(s-t)\right\} \ , \end{cases} \tag{5.5.1}$$

where $C(s, x)$ denotes the Black-Scholes value at time $s \in [t, T)$ of the call option maturing at time T knowing that the underlying asset price $X(s) = x$. Observe that $P^{\alpha}_{t,p}(s)$ can be expressed as a function of $X_{t,x}(s)$ i.e. $P_{t,p}(s) = p(t, s, X_{t,x}(s))$. Hence we analyse the performance of our algorithm by observing its ability to approximate the one dimensional real valued function u_s such that

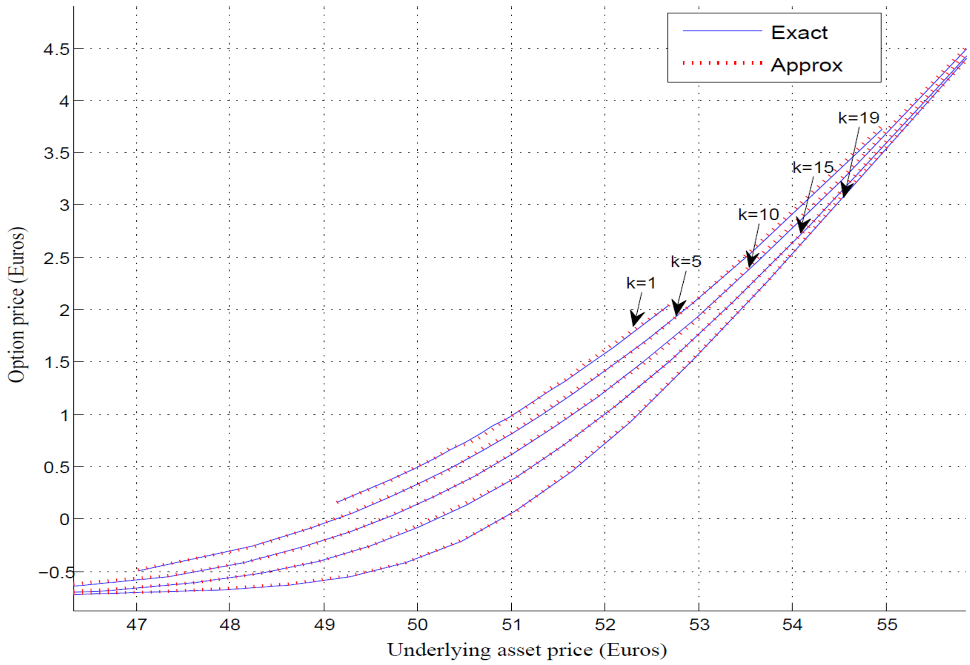
$$u_s(x) = v(s, x, p(t, s, x)) \ . \tag{5.5.2}$$

In our simulations, we consider the following parameters.

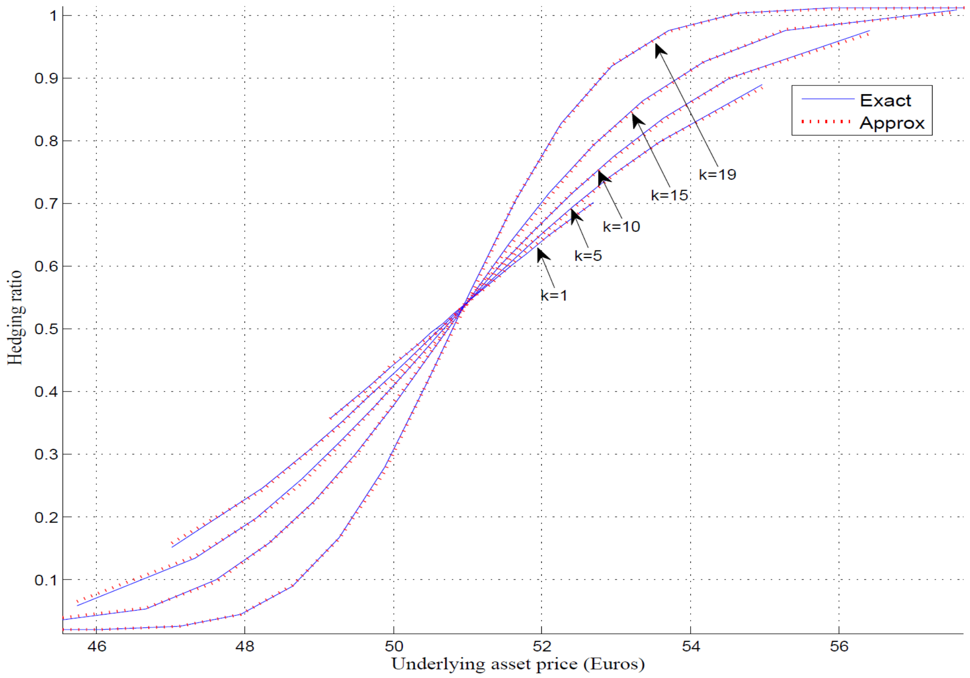
$T-t$	x	K	p	$\widehat{\sigma}$	$\widehat{\mu}$
20/250	50	52.5	-0.1	0.28	0.1

TABLE 5.2 – Parameters for simulation and explicit solution. Time is given in trading years (250 days), x and strike K in Euro, and p in Euro^2, σ and μ in annual percentage.

We have performed our algorithm with $M = 10^5$ particles to estimate at each step of time the conditional expectations and a time discretization mesh $\tau_0 = 0, \cdots \tau_k, \cdots \tau_n = T$ with a time step $\delta = \tau_{k+1} - \tau_k = 1/250$. In our tests, it appeared that the fixed point algorithm implemented at each step of time has always converged in no more than three iterations, for any precision $\epsilon > 0$. We have represented on Figure 5.4 the value of $u_s(x)$ with respect to x computed by the explicit formula and the numerical algorithm. We also provide the value of the control ν at the initial date to illustrate the convergence of derivatives too.



(a) Value function $x \mapsto u_s(x)$ with $p = -0,1$ Euros and $t = k$.



(b) Optimal strategy $x \mapsto \nu(t, x, p)$ for $p = -0,1$ Euros and $t = k$.

FIGURE 5.4 – Comparison between our numerical approach and the solution provided by the explicit formula of (5.2.1).

5.5.3 Comparison for a call option with simulations

We consider the loss approach (hereafter denoted *shortfall risk*, or SR) and the benchmark strategy (hereafter *Black-Scholes*, or BS) upon a call option. The aim of this section is to compare in simulations the performances of the Naive hedging provided by Black-Scholes delta-hedging assuming that $\widetilde{\mathbb{P}}[\Lambda = \bar{\lambda}] = 1$ against the performances of the shortfall risk taking into account the uncertainty on Λ and allowing for a limited expected loss p in the hedging strategy. For each approach, we implement the associated hedging strategy on i.i.d. $M_{hedge} = 10000$ simulated price paths. For each path we compute both hedging errors. Then we compute by Monte Carlo approximation (on these i.i.d. $M_{hedge} = 10000$ simulations) the expected loss associated to the Black-Scholes approach and the shortfall risk hedge. Of course, in our experiments, the trading strategies are not implemented continuously but at each trading day. As a consequence, the resulting hedging errors may differ from the theoretical time continuous setting.

- For the option, we compare several strike possibilities : $K = \rho\bar{\lambda}x_Q$ with

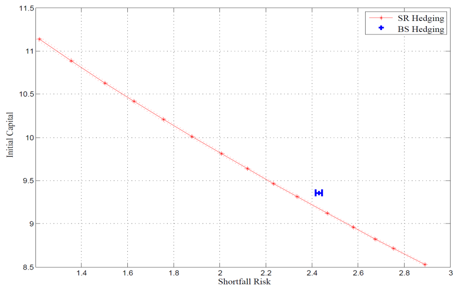
$$\rho \in \{0.85; 0.9; 0.95; 1; 1.05, 1.1; 1.15; 1.2\} \ .$$

- The loss function is the partial moment loss function of Section 5.4.1 with $n = 2$, i.e.,

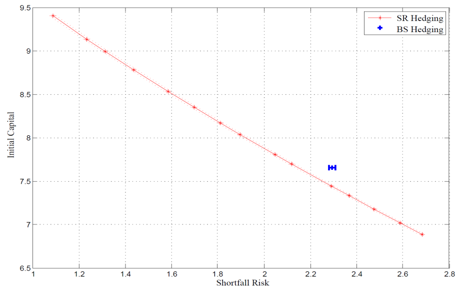
$$\widetilde{\Psi}(x, y, \lambda) = -\frac{1}{2}\left[((\lambda x - K)_+ - y)_+\right]^2$$

  and the threshold p varies enough to evaluate its impact.
- The variable Λ is given by a law $\beta(a, b)$ with the parameters (a, b) of Section 5.5.1.
- The Black-Scholes strategy is here given with the a priori known expectation, i.e., $\bar{\lambda} = \int_{\Omega^\lambda} \Lambda(\omega^\lambda) d\mathbb{P}^\lambda(\omega^\lambda)$.
- We use the same time grid as real data : strategies are daily frequent. We simulate the strategy on 10000 trajectories.

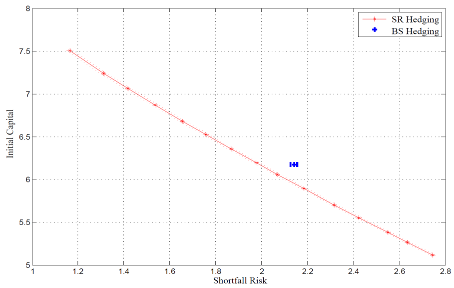
We take the square root of the obtained error in order to express it in euros. This justifies the terminology *shortfall*, which is a monetary homogeneous quantity. Figure 5.5 sums up the simulations and compares, for the different values of K, the value function in function of p. Figure 5.6 provides a comparison between the two approaches for another criterion : the conditional Value-at-Risk, or expected shortfall.



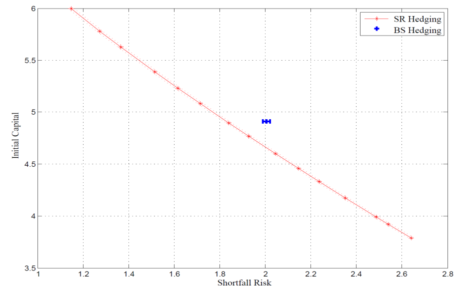
(a) $K = 0,85X(t_0)$



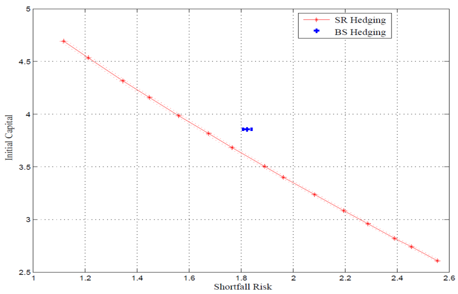
(b) $K = 0,90X(t_0)$



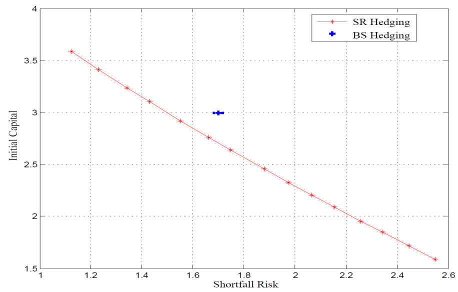
(c) $K = 0,95X(t_0)$



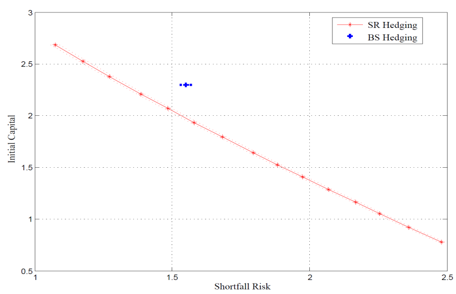
(d) $K = X(t_0)$



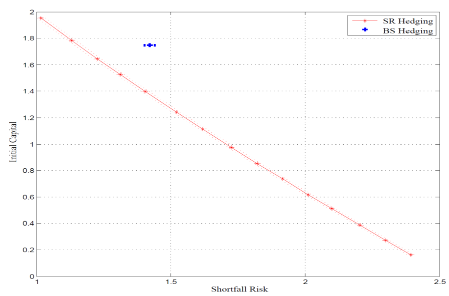
(e) $K = 1,05X(t_0)$



(f) $K = 1,10X(t_0)$

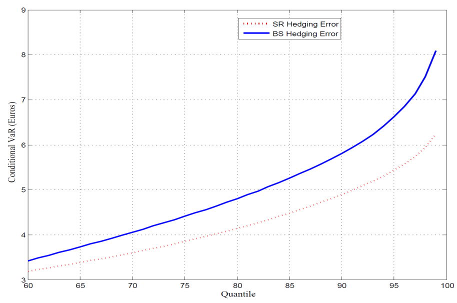


(g) $K = 1,15X(t_0)$

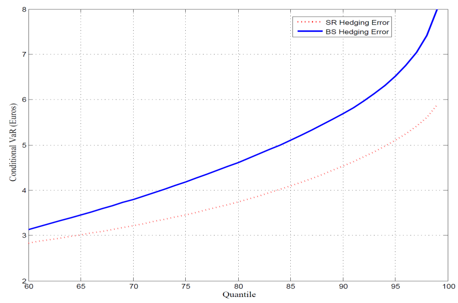


(h) $K = 1,15X(t_0)$

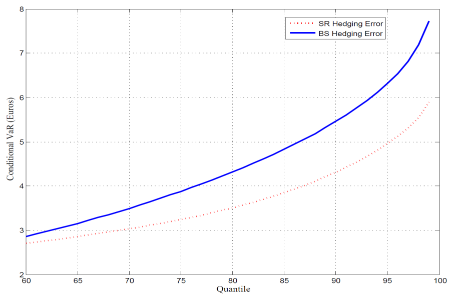
FIGURE 5.5 – Initial capital w.r.t. the associated Shortfall Risk of the Black-Scholes strategy (blue) and the Shortfall strategy (red) computed on $M_{hedge} = 10000$ simulated trajectories with 95 % confidence interval (in dotted lines). Six strikes K.



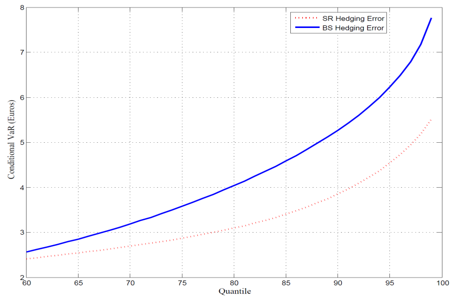
(a) $K = 0,85X(t_0)$



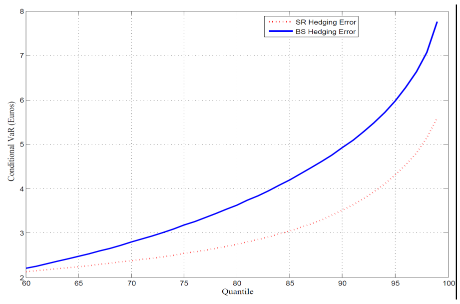
(b) $K = 0,90X(t_0)$



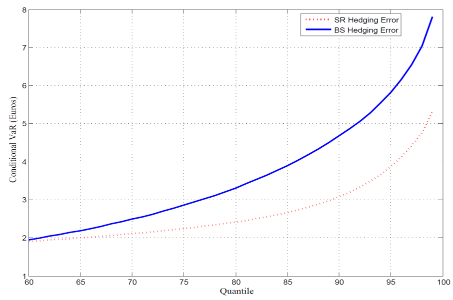
(c) $K = 0,95X(t_0)$



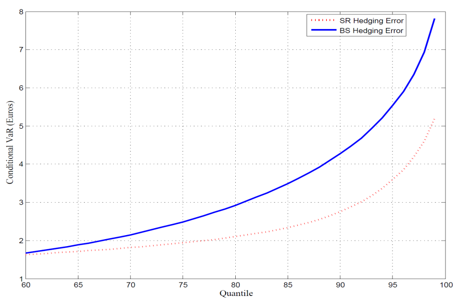
(d) $K = X(t_0)$



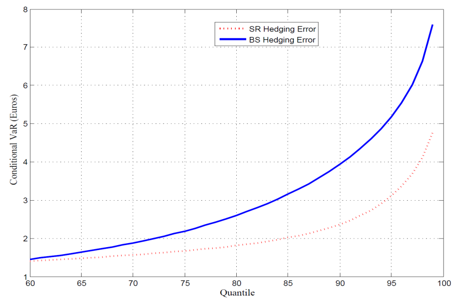
(e) $K = 1,05X(t_0)$



(f) $K = 1,10X(t_0)$



(g) $K = 1,15X(t_0)$



(h) $K = 1,15X(t_0)$

FIGURE 5.6 – CVaR value w.r.t. the quantile level of the the Black-Scholes strategy (blue) and the Shortfall strategy (red) computed on $M_{hedge} = 10000$ trajectories. Six strikes K

5.6 Proofs

5.6.1 Proof of Corollary 5.2.1

Proof We introduce the Legendre-Fenchel transform of v_* in p :

$$u(t,x,q) := \sup_{p\in\mathbb{R}_-} \{pq - v_*(t,x,p)\} \quad \forall (t,x,p) \in [0,T] \times \mathbb{R}^d_+ \times \mathbb{R}_+ \tag{5.6.1}$$

The function $q \mapsto u(.,q)$ is convex and upper-semicontinuous on $[0,T] \times (0,\infty)^d \times (0,\infty)$. Let φ be a smooth function with bounded derivatives, such that (t_0,x_0,q_0) is a local maximizer of $u-\varphi$ with $(u-\varphi)(t_0,x_0,q_0) = 0$. The function $q \mapsto \varphi(.,q)$ is convex, but as in the proof of section 4 in [Bouchard 09], a standard penalization of φ shows that we can take without loss of generality φ strictly convex with quadratic growth.
The Legendre-Fenchel transform of φ with respect to q,

$$\widetilde{\varphi}(t,x,p) := \sup_{q\in\mathbb{R}_+} \{qp - \varphi(t,x,q)\}$$

is then a strictly convex function of p and smooth on its domain. According to (5.6.1) and the quadratic growth of φ, there exists $p_0 \in \mathbb{R}_-$ such that for the fixed q_0,

$$p_0q_0 - v_*(t_0,x_0,p_0) = u(t_0,x_0,q_0) = \varphi(t_0,x_0,q_0) = \sup_{p\in\mathbb{R}_-} \{pq_0 - \widetilde{\varphi}(t_0,x_0,p)\}$$

which, by taking the left and right sides of the above equation, implies that (t_0,x_0,p_0) is a local minimizer of $v_* - \widetilde{\varphi}$ and $(v_* - \widetilde{\varphi})(t_0,x_0,p_0) = 0$. It comes from the above definition that $p_0 = J(t_0,x_0,q_0)$ where $q \mapsto J(.,q)$ is the inverse of $q \mapsto \frac{\partial\widetilde{\varphi}}{\partial p}(.,q)$, which exists by strict monotony of the last function. Since the volatility σ is invertible, $\bar{\mathcal{N}}_0(x_0,p_0,\frac{\partial\widetilde{\varphi}}{\partial x}(t_0,x_0,p_0),\frac{\partial\widetilde{\varphi}}{\partial p}(t_0,x_0,p_0)) \neq \emptyset$ and is composed of elements of the form

$$\left(\left[(\sigma^T)^{-1}(\sigma^T\frac{\partial\widetilde{\varphi}}{\partial x} + ap\frac{\partial\widetilde{\varphi}}{\partial p})\right](t_0,x_0,p_0), a\right), \quad a \in \mathbb{R}\ .$$

According to Theorem 5.2.2, $\widetilde{\varphi}$ is thus a supersolution in (t_0,x_0,p_0) to the dynamic programming equation :

$$-\frac{\partial\widetilde{\varphi}}{\partial t} - \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^T\frac{\partial^2\widetilde{\varphi}}{\partial x^2}\right] - \inf_{a\in\mathbb{R}}\left\{-ap_0(\theta(x_0)\frac{\partial\widetilde{\varphi}}{\partial p} - \sigma(x_0)\frac{\partial^2\widetilde{\varphi}}{\partial p\partial x}) + \frac{1}{2}(ap_0)^2\frac{\partial^2\widetilde{\varphi}}{\partial p^2}\right\} \geq 0 \tag{5.6.2}$$

where we recall that $\theta(x_0) = \sigma^{-1}\mu(x_0)$. Note that in the special case $p_0 = 0$, we retrieve the Black-Scholes equation. This is also a consequence of Theorem 5.2.4. Since $\frac{\partial^2\widetilde{\varphi}}{\partial p^2}(t_0,x_0,p_0) > 0$, the infimum in the above equation is reached for

$$a = -\frac{\left(\sigma\frac{\partial^2\widetilde{\varphi}}{\partial p\partial x} - \theta\frac{\partial\widetilde{\varphi}}{\partial p}\right)}{p_0\frac{\partial^2\widetilde{\varphi}}{\partial p^2}}(t_0,x_0,p_0)\ , \tag{5.6.3}$$

providing at point (t_0, x_0, p_0)

$$-\frac{\partial \widetilde{\varphi}}{\partial t} - \frac{1}{2}\text{Tr}\left[\sigma\sigma^T \frac{\partial^2 \widetilde{\varphi}}{\partial x^2}\right] + \frac{\left(\theta\frac{\partial \widetilde{\varphi}}{\partial p} - \sigma\frac{\partial^2 \widetilde{\varphi}}{\partial p \partial x}\right)^2}{2\frac{\partial^2 \widetilde{\varphi}}{\partial p^2}} \geq 0 \ .$$

Now observe that, according to (5.6.1), $\frac{\partial \widetilde{\varphi}}{\partial p} = q$, $\frac{\partial \widetilde{\varphi}}{\partial t} = -\frac{\partial \varphi}{\partial t}$, $\frac{\partial \widetilde{\varphi}}{\partial x} = -\frac{\partial \varphi}{\partial x}$, $\frac{\partial^2 \widetilde{\varphi}}{\partial x^2} = -\frac{\partial^2 \varphi}{\partial x^2} + \frac{(\frac{\partial^2 \varphi}{\partial x \partial q})^T \frac{\partial^2 \varphi}{\partial x \partial q}}{\frac{\partial^2 \varphi}{\partial q^2}}$, $\frac{\partial^2 \widetilde{\varphi}}{\partial p^2} = \frac{\partial q^2}{\partial^2 \varphi}$ and $\frac{\partial^2 \widetilde{\varphi}}{\partial p \partial x} = -\frac{\frac{\partial^2 \varphi}{\partial x \partial q}}{\frac{\partial^2 \varphi}{\partial q^2}}$, so that φ now verifies at (t_0, x_0, q_0) :

$$-\frac{\partial \varphi}{\partial t} - \sigma\theta\frac{\partial^2 \varphi}{\partial x \partial q} - \frac{1}{2}(\text{Tr}\left[\sigma\sigma^T \frac{\partial^2 \varphi}{\partial x^2}\right] + |\theta|^2 q^2 \frac{\partial^2 \varphi}{\partial q^2}) \leq 0 \ . \tag{5.6.4}$$

This implies that u is a viscosity subsolution of (5.6.4) on $[0,T) \times (0,\infty)^d \times (0,\infty)$. The terminal condition is given by the definition of u and Theorem 5.2.3 :

$$u(T,x,q) = \sup_{p \in \mathbb{R}_-} \{pq - v_*(T,x,p)\} = \sup_{p \in \mathbb{R}_-} \left\{pq - \odot\Psi^{-1}(x,p)\right\} =: U(x,q)$$

Let $\bar{u}$ be the function defined by

$$\bar{u}(t,x,q) = \mathbb{E}^{\mathbb{Q}_{t,x}}\left[U(X_{t,x}(T), Q_{t,x,q}(T))\right]$$

for

$$\begin{cases} X_{t,x}(s) = x + \int_t^s \sigma(X_{t,x}(u))dW_u^{\mathbb{Q}_{t,x}} \\ Q_{t,x,q}(s) = q + \int_t^s Q_{t,x,q}(u)\theta(X_{t,x}(u)) \cdot dW_u^{\mathbb{Q}_{t,x}} \end{cases} , \quad s \in [t,T]$$

where $\mathbb{Q}_{t,x}$ is a $\mathbb{P}$-equivalent measure such that $d\mathbb{P}/d\mathbb{Q}_{t,x} = Q_{t,x,1}$. According to the Feynman-Kac formula, $\bar{u}$ is a supersolution to equation (5.6.4). Let use define

$$J(x,q) = \arg\sup_{p \in \mathbb{R}_-} \left\{pq - \odot\Psi^{-1}(x,p)\right\} \ .$$

Notice that since Ψ is non-decreasing in y but bounded by above, and that $q > 0$, $p \mapsto pq - \odot\Psi^{-1}(x,p)$ is coercive and $J(x,q)$ is well-defined. One can also see that the image of $J(x,(0,\infty))$ is $(-\infty,0)$ for any $x \in (0,\infty)^d$. Notice also that since $\odot\Psi^{-1}(x,p)$ is differentiable in p on $\text{int}(E)$, J corresponds to the inverse of $\frac{\partial \odot\Psi^{-1}}{\partial p}$ in (x,q). According to what was just said, we can then introduce the function $\bar{q}(t,x,p)$ for which

$$\mathbb{E}^{\mathbb{Q}_{t,x}}\left[J(X_{t,x}(T), Q_{t,x,\bar{q}(t,x,p)}(T))Q_{t,x,1}(T)\right] = p \ .$$

Therefore, we have

$$\begin{aligned} v(t,x,p) &\geq v_*(t,x,p) \\ &\geq \sup_{q>0}\{qp - \bar{u}(t,x,q)\} \\ &\geq p\bar{q}(t,x,p) - \mathbb{E}^{\mathbb{Q}_{t,x}}\left[U(X_{t,x}(T), Q_{t,x,\bar{q}(t,x,p)}(T))\right] \\ &\geq \bar{q}(t,x,p)(p - \mathbb{E}^{\mathbb{Q}_{t,x}}\left[J(X_{t,x}(T), Q_{t,x,\bar{q}(t,x,p)})Q_{t,x,1}\right]) \\ &\qquad\qquad + \mathbb{E}^{\mathbb{Q}_{t,x}}\left[\odot\Psi^{-1}(X_{t,x}(T), J(X_{t,x}(T), Q_{t,x,\bar{q}(t,x,p)}))\right] \\ &\geq \mathbb{E}^{\mathbb{Q}_{t,x}}\left[\odot\Psi^{-1}(X_{t,x}(T), J(X_{t,x}(T), Q_{t,x,\bar{q}(t,x,p)}))\right] \\ &=: y(t,x,p)\ . \end{aligned}$$

By the martingale representation theorem, there exists $\nu \in \mathcal{U}_t$ such that

$$Y^{\nu}_{t,x,y(t,x,p)}(T) = \odot\Psi^{-1}(X_{t,x}(T), J(X_{t,x}(T), Q_{t,x,\bar{q}(t,x,p)}(T)))$$

which implies that

$$\begin{aligned} \mathbb{E}\left[\Psi(X_{t,x}(T), Y^{\nu}_{t,x,y}(T))\right] &\geq \mathbb{E}\left[J(X_{t,x}(T), Q_{t,x,\bar{q}(t,x,p)}))\right] \\ &= \mathbb{E}^{\mathbb{Q}_{t,x}}\left[J(X_{t,x}(T), Q_{t,x,\bar{q}(t,x,p)})Q_{t,x,1}\right] = p \end{aligned}$$

and therefore, $y(t,x,p) \geq v(t,x,p)$. □

5.6.2 Proof of Proposition 5.3.1

W start with an application of the fundamental measurable selection theorem, see Example 2.4 in [Rieder 78] and Theorem 1.2.9 in Chapter 1.

Theorem 5.6.1. *Fix $(z,\lambda) \in \mathbf{Z} \times L$. Fix $\varepsilon > 0$. Then there exists a measurable function $z \mapsto \nu^{\lambda}(z)$ on $\mathbf{Z}$ such that $\nu^{\lambda}(z) \in \mathcal{U}_{t_0,z}$ and*

$$\mathbb{E}\left[\widetilde{\Psi}(Z^{\nu^{\lambda}(z)}_{t_0,z}(T), \lambda)\right] \geq \sup_{\widetilde{\nu}\in\mathcal{U}_{t_0,z}} \mathbb{E}\left[\widetilde{\Psi}(Z^{\widetilde{\nu}}_{t_0,z}(T), \lambda)\right] - \varepsilon \quad \forall z \in \mathbf{Z}\ .$$

Proof The set $L^2([0,T]\times\Omega^1)$ being a separable metric space, the class

$$\mathcal{L} := \left\{ C \in \mathcal{B}(\mathbf{Z}) \otimes \mathcal{B}(\mathcal{U}) \ : \ \begin{array}{l} (a)\ C(x) \text{ is complete for } x \in pC \text{ and} \\ (b)\ p(C \cap \mathbf{Z} \times K) \in \mathcal{B}(\mathbf{Z}) \text{ for all compactum } K \in \mathcal{U} \end{array} \right\}$$

is a selection class for $(\mathcal{B}(\mathbf{Z}), \mathcal{B}(\mathcal{U}))$, where $pC := \{\nu \in \mathcal{U}\ :\ (z,\nu) \in C\}$ and $C(z) := \{\nu \in \mathcal{U}\ :\ (z,\nu) \in C\}$ for all set $C \subset \mathbf{Z} \times \mathcal{U}$. We thus prove below assumptions (i) and (ii) f Corollary 3.2 in [Rieder 78] to obtain the desired result.
(i). Define $D := \{(z,\nu)\ :\ z \in \mathbf{Z}, \nu \in \mathcal{U}_{t_0,z}\}$. Define $\mathcal{U}_{t_0}$ the subset of $\mathcal{U}$ composed of $\mathbb{F}^{t_0}$-progressively measurable processes. Note that U being closed, $\mathcal{U}_{t_0}$ is closed in $L^2([t_0,T]\times$

Ω^1). According to dynamics 5.2.7, the map $\nu \mapsto Z^{\nu}_{t_0,z}(s)$ is continuous for any $t \le s \le T$ and any $z \in \mathbf{Z}$. For a fixed $s \in [t,T]$, the set $K(z,s) := \left\{\nu \in \mathcal{U}_{t_0} \ : \ Z^{\nu}_{t_0,z}(s) \ge -\kappa\right\}$ is then closed in $\mathcal{U}$. The countable intersection $\bigcap_{s\in[t_0,T]\cap\mathbb{Q}} K(z,s)$ is also closed. By continuity of $Z^{\nu}_{t,z}(.)$, $\mathcal{U}_{t_0,z} = \bigcap_{s\in[t_0,T]\cap\mathbb{Q}} K(z,s)$ so that $\mathcal{U}_{t_0,z}$ is a closed subset of $L^2([t_0,T]\times\Omega^1)$ for all $z \in \mathbf{Z}$. By the Riesz-Fischer theorem, it is complete as a closed subspace of a complete space, and (a) holds for D. Since $\mathbf{Z}$ does not depend on $\mathcal{U}_{t_0,z}$ for any $z \in \mathbf{Z}$, (b) holds for D. Thus, $D \in \mathcal{L}$.
(ii). For $c \in \mathbb{R}$, define $U_c := \left\{(z,\nu) \in D \ : \ \mathbb{E}\left[\widetilde{\Psi}(Z^{\nu}_{t,z}(T),\lambda)\right] \ge c\right\}$. Fix $z \in pU_c$. Then $U_c(z) := \left\{\nu \in \mathcal{U}_{t_0,z} \ : \ \mathbb{E}\left[\widetilde{\Psi}(Z^{\nu}_{t,z}(T),\lambda)\right] \ge c\right\}$ is closed according to Assumption 5.3.1.(ii). As for D, $U_c(z)$ is complete and (a) holds for U_c. For any compact $K \in \mathcal{U}_{t_0}$, the set $\left\{z \in \mathbf{Z} \ : \ \mathbb{E}\left[\widetilde{\Psi}(Z^{\nu}_{t,z}(T),\lambda\right] \ge c \text{ for some } \nu \in \mathcal{U}_{t_0,z} \cap K\right\}$ is a Borel set, and (b) holds for U_c. This comes from the fact that $z \mapsto \widetilde{\Psi}(Z^{\nu}_{t,z}(T),\lambda)$ is upper-semicontinuous for any $(\nu,\lambda) \in \mathcal{U}_{t_0} \times L$, recall Assumption 5.2.1.(i) and (ii). We thus have that $U_c \in \mathcal{L}$ for any $c \in \mathbb{R}$. □

We now come to the proof of Proposition 5.3.1.

Proof Along the proof, for $(t,x,p) \in [0,t_0) \times (0,\infty)^d \times \mathbb{R}_-$, we denote

$$A(t,x,p) := \left\{y \ge -\kappa \ : \ \exists \widetilde{\nu} \in \widetilde{\mathcal{U}}_{t,z} \text{ s.t. } \mathbb{E}\left[\widetilde{\Psi}(Z^{\widetilde{\nu}}_{t,x,y}(T),\Lambda)\right] \ge p\right\}$$

and

$$B(t,x,p) := \left\{y \ge -\kappa \ : \ \exists \nu \in \mathcal{U}_{t,z} \text{ s.t. } \mathbb{E}\left[\Xi(Z^{\nu}_{t,x,y}(t_0))\right] \ge p\right\}$$

with

$$\Xi(z) := \int_{\Omega^\lambda} \sup_{\nu\in\mathcal{U}_{t_0,z}} \mathbb{E}\left[\widetilde{\Psi}(Z^{\nu}_{t_0,z}(T),\Lambda(\omega^\lambda))|\mathcal{F}_{t_0}\right] d\mathbb{P}^\lambda(\omega^\lambda)\ .$$

Fix $(t,x,p) \in [0,t_0) \times (0,\infty)^d \times \mathbb{R}_-$. We denote $\Omega^0 := \{(\omega_{s\wedge t_0})_{0\le s\le T} \ : \ \omega \in \Omega\}$ and $\Omega^1 := \{(\omega_s - \omega_{s\wedge t_0})_{0\le s\le T} \ : \ \omega \in \Omega\}$ the respective spaces of trajectories on $[t,t_0]$ and the trajectories on $[t_0,T]$ shifted to 0 at t_0. According to the properties of the Brownian paths, one can rewrite a natural bijection between $\widetilde{\omega} \in \widetilde{\Omega}$ and $(\omega^0,\omega^\lambda,\omega^1) \in \Omega^0\times\Omega^\lambda\times\Omega^1$. If we consider the space $(\widetilde{\Omega},\widetilde{\mathcal{F}}^t_T,\mathbb{P})$, we then naturally change it for the space $(\Omega^0\times\Omega^\lambda\times\Omega^1,\mathcal{F}^t_{t_0}\otimes\mathcal{G}\otimes\mathcal{F}^{t_0}_T,\mathbb{P}^0\times\mathbb{P}^\lambda\times\mathbb{P}^1)$, where $\mathbb{P}^0,\mathbb{P}^\lambda$ and $\mathbb{P}^1$ denote the marginal laws on each probability subspace.
1. Take now $y \in A(t,x,p)$. Then there exists $\widetilde{\nu} \in \widetilde{\mathcal{U}}_{t,z}$ such that $\mathbb{E}\left[\widetilde{\Psi}(Z^{\widetilde{\nu}}_{t,z}(T),\Lambda)\right] \ge p$. Notice that $(Z^{\widetilde{\nu}}_{t,z},\widetilde{\nu})$ is $\widetilde{\mathbb{F}}^t$-progressively measurable and $\widetilde{\Psi}(Z^{\widetilde{\nu}}_{t,x,y}(T),\Lambda)$ is $\widetilde{\mathcal{F}}^t_T$-measurable. This means that $(Z^{\widetilde{\nu}}_{t,z},\widetilde{\nu})(\omega^0,\omega^\lambda,\omega^1)(s) = (Z^{\widetilde{\nu}}_{t,z},\widetilde{\nu})(\omega^0)(s)$ for any $s \in [t,t_0)$. The notation hods at time t_0 for Z by continuity. According to the above change of notations, and by

the flow property (see [Soner 02a] for details) we can write

$$\mathbb{E}\left[\widetilde{\Psi}(Z^{\widetilde{\nu}}_{t,x,y}(T),\Lambda)\right] = \iiint_{\Omega^0\times\Omega^\lambda\times\Omega^1} \widetilde{\Psi}(Z^{\widetilde{\nu}(\omega^0,\omega^\lambda,\omega^1)}_{t,x,y}(\omega^1)(T),\Lambda(\omega^\lambda))d\mathbb{P}^0(\omega^0)d\mathbb{P}^\lambda(\omega^\lambda)d\mathbb{P}^1(\omega^1)$$

$$= \iiint_{\Omega^0\times\Omega^\lambda\times\Omega^1} \widetilde{\Psi}(Z^{\widetilde{\nu}(\omega^0,\omega^\lambda,\omega^1)}_{t_0,Z^{\widetilde{\nu}(\omega^0)}_{t,x,y}(\omega^0)(t_0)}(\omega^1)(T),\Lambda(\omega^\lambda))d\mathbb{P}^0(\omega^0)d\mathbb{P}^\lambda(\omega^\lambda)d\mathbb{P}^1(\omega^1)\ .$$

Note that we omit the dependence of Z in ω^λ for $s\in[t_0,T]$, which holds only via $\widetilde{\nu}$. For any fixed $(\omega^0,\omega^\lambda)\in\Omega^0\times\Omega^\lambda$, the control $\widetilde{\nu}(\omega^0,\omega^\lambda,.)(.)$ is a $\mathbb{F}^{t_0}$-progressively measurable process in $L^2([t_0,T]\times\Omega^1)$. Thus, $\widetilde{\nu}(\omega^0,\omega^\lambda,.)$ belongs to $\mathcal{U}_{t_0,Z^{\widetilde{\nu}}_{t,z}(\omega^0)(t_0)}$ for a.e. ω^0, since $Z^{\nu(\omega^0,\omega^\lambda,.)}_{t,z}(s)\geq -\kappa$ for a.e. $\omega^\lambda\in\Omega^\lambda$ and $s\in[t_0,T]$. This implies that for a fixed ω^0,

$$\int_{\Omega^1}\widetilde{\Psi}(Z^{\widetilde{\nu}(\omega^0,\omega^\lambda,\omega^1)}_{t_0,Z^{\widetilde{\nu}(\omega^0)}_{t,x,y}(\omega^0)(t_0)}(\omega^1)(T),\Lambda(\omega^\lambda))d\mathbb{P}^1(\omega^1)$$
$$\leq \sup_{\nu\in\mathcal{U}_{t_0,Z^{\widetilde{\nu}}_{t,z}(\omega^0)(t_0)}}\int_{\Omega^1}\widetilde{\Psi}(Z^{\nu(\omega^1)}_{t_0,Z^{\widetilde{\nu}(\omega^0)}_{t,x,y}(\omega^0)(t_0)}(\omega^1)(T),\Lambda(\omega^\lambda))d\mathbb{P}^1(\omega^1)$$

and by integrating on Ω^λ,

$$\int_{\Omega^\lambda}\int_{\Omega^1}\widetilde{\Psi}(Z^{\widetilde{\nu}(\omega^0,\omega^\lambda,\omega^1)}_{t_0,Z^{\widetilde{\nu}(\omega^0)}_{t,x,y}(\omega^0)(t_0)}(\omega^1)(T),\Lambda(\omega^\lambda))d\mathbb{P}^1(\omega^1)d\mathbb{P}^\lambda(\omega^\lambda)$$
$$\leq \int_{\Omega^\lambda}\sup_{\nu\in\mathcal{U}_{t_0,Z^{\widetilde{\nu}}_{t,z}(\omega^0)(t_0)}}\int_{\Omega^1}\widetilde{\Psi}(Z^{\nu(\omega^1)}_{t_0,Z^{\widetilde{\nu}(\omega^0)}_{t,x,y}(\omega^0)(t_0)}(\omega^1)(T),\Lambda(\omega^\lambda))d\mathbb{P}^1(\omega^1)d\mathbb{P}^\lambda(\omega^\lambda)$$
$$= \Xi(Z^{\widetilde{\nu}(\omega^0)}_{t,x,y}(\omega^0)(t_0))\ .$$

Recalling that $\mathbb{E}\left[\widetilde{\Psi}(Z^{\widetilde{\nu}}_{t,z}(T),\Lambda)\right]\geq p$, we integrate on Ω^0 the above result and rewrite it as $\mathbb{E}\left[\Xi(Z^{\widetilde{\nu}}_{t,x,y}(t_0))\right]\geq p$ with $\widetilde{\nu}\in\widetilde{\mathcal{U}}_t$. The control $\widetilde{\nu}$ has a $\mathbb{F}^t$-progressively measurable version in $\mathcal{U}_t$, such that $Z^{\widetilde{\nu}}_{t,x,y}(t_0)=Z^{\nu}_{t,x,y}(t_0)$ $\mathbb{P}^0$-a.s. Thus $y\in B(t,x,p)$, meaning that $w(t,x,p)\leq\widetilde{v}(t,x,p)$.

2.(a). Take now $y\in B(t,x,p)$. There exists $(\nu,\alpha)\in\mathcal{U}_{t,z}\times\mathcal{A}_{t,p}$ such that

$$\Xi(Z^\nu_{t,z}(t_0))\geq P^\alpha_{t,p}(t_0)\ \ \mathbb{P}^0-\text{a.s.}$$

Fix $\lambda\in L$. According to Theorem 5.6.1 above, there exists a $\mathcal{F}_{t_0}$-measurable selector ν^λ of $\mathbf{Z}$, such that $\nu^\lambda(z)\in\mathcal{U}_{t_0,z}$ and $\nu^\lambda\in L^2([t_0,T]\times\Omega^1)$, such that

$$\mathbb{E}\left[\widetilde{\Psi}(Z^{\nu^\lambda}_{t_0,Z^\nu_{t,z}(t_0)}(T),\lambda)|\mathcal{F}_{t_0}\right]\geq \sup_{\widehat{\nu}\in\mathcal{U}_{t_0,Z^\nu_{t,z}(t_0)}}\mathbb{E}\left[\widetilde{\Psi}(Z^{\widehat{\nu}}_{t_0,Z^\nu_{t,z}(t_0)}(T),\lambda)|\mathcal{F}_{t_0}\right]-\varepsilon\ .$$

We then proceed as in the proof of Theorem 2.4 in [Bouchard 11c]. By continuity of $\widetilde{\Psi}$ in λ, and following Lemma 2.1 in [Bouchard 11c], there exists an open ball $B(\lambda)$ of centre

λ and radius $\eta > 0$ (which size depends on λ and ε) in L such that

$$\sup_{\widehat{\nu}\in\mathcal{U}_{t_0,Z^{\nu}_{t,z}(t_0)}} \mathbb{E}\left[\widetilde{\Psi}(Z^{\widehat{\nu}}_{t_0,Z^{\nu}_{t,z}(t_0)}(T),\lambda')|\mathcal{F}_{t_0}\right] \leq \sup_{\widehat{\nu}\in\mathcal{U}_{t_0,Z^{\nu}_{t,z}(t_0)}} \mathbb{E}\left[\widetilde{\Psi}(Z^{\widehat{\nu}}_{t_0,Z^{\nu}_{t,z}(t_0)}(T),\lambda)|\mathcal{F}_{t_0}\right] + \varepsilon$$

and

$$\mathbb{E}\left[\widetilde{\Psi}(Z^{\nu^{\lambda}}_{t_0,Z^{\nu}_{t,z}(t_0)}(T),\lambda')|\mathcal{F}_{t_0}\right] \geq \mathbb{E}\left[\widetilde{\Psi}(Z^{\nu^{\lambda}}_{t_0,Z^{\nu}_{t,z}(t_0)}(T),\lambda)|\mathcal{F}_{t_0}\right] - \varepsilon$$

for all $\lambda' \in B(\lambda)$. The set $\{B(\lambda)\ :\ \lambda \in L\}$ forms an open cover of L. Since L is metric separable, it has the Lindelöf property, and there exists a countable sequence $(\lambda_i)_{i\geq 1} \subset L$ such that $\{B(\lambda_i)\}_{i\geq 1}$ forms a cover of L. We set $\nu^i := \nu^{\lambda_i}$, $B_i := B(\lambda_i)$ and a measurable partition $(C_i)_{i\geq 1}$ of $\bigcup_{i\geq 1} B_i$ defined by

$$C_1 := B_1 \ \text{ and } \ C_{i+1} := B_{i+1} \setminus \cup_{1\leq j\leq i} B_i,\ i \geq 1\ .$$

Since $C_i \subset B(\lambda_i)$, we have for all $\lambda' \in C_i$:

$$\mathbb{E}\left[\widetilde{\Psi}(Z^{\nu^i}_{t_0,Z^{\nu}_{t,z}(t_0)}(T),\lambda')|\mathcal{F}_{t_0}\right] \geq \sup_{\widehat{\nu}\in\mathcal{U}_{t_0,Z^{\nu}_{t,z}(t_0)}} \mathbb{E}\left[\widetilde{\Psi}(Z^{\widehat{\nu}}_{t_0,Z^{\nu}_{t,z}(t_0)}(T),\lambda')|\mathcal{F}_{t_0}\right] - 3\varepsilon\ .$$

Now let $\Gamma_i := \{\Lambda \in C_i\} \subset \Omega^{\lambda}$ be a $\widetilde{\mathcal{F}}_{t_0}$-measurable set for any $i \geq 1$, and $\Gamma(k) := \bigcup_{1\leq i\leq k} \Gamma_i$ for any $k \in \mathbb{N}$. Since $C_i \cap C_k = \emptyset$ for all $i \neq j$, $\Gamma_i \cap \Gamma_j = \emptyset$ for all $i \neq j$. We then consider for all $k \in \mathbb{N}$ the control $\nu(k) = \sum_{i=1}^{k} \nu_i \mathbf{1}_{\Gamma_i}$. Note that $\nu(k)$ is in $L^2([t_0,T]\times\Omega^1)$ and thus $\nu(k) \in \widetilde{\mathcal{U}}_{t_0,Z^{\nu}_{t,z}(t_0)}$ for every fixed k. We then have

$$\mathbb{E}\left[\widetilde{\Psi}(Z^{\nu(k)}_{t_0,Z^{\nu}_{t,x,y}(t_0)}(T),\Lambda)|\mathcal{F}_{t_0}\right]\mathbf{1}_{\Gamma(k)} \geq \sup_{\widehat{\nu}\in\mathcal{U}_{t_0,Z^{\nu}_{t,z}(t_0)}} \mathbb{E}\left[\widetilde{\Psi}(Z^{\widehat{\nu}}_{t_0,Z^{\nu}_{t,z}(t_0)}(T),\Lambda)|\mathcal{F}_{t_0}\right]\mathbf{1}_{\Gamma(k)} - 3\varepsilon\ .$$

According to Assumption 5.3.1.(i) and (iii), usual estimates provide (for any $\lambda \in L$)

$$\mathbb{E}\left[\widetilde{\Psi}(Z^{\nu(k)}_{t_0,Z^{\nu}_{t,z}(t_0)}(T),\lambda)|\mathcal{F}_{t_0}\right] \geq \mathbb{E}\left[\widetilde{\Psi}(X_{t,x},-\kappa,\lambda)|\mathcal{F}_{t_0}\right] \geq -C(1+(|x|+\kappa)^k)$$

for some $C > 0$. Then, $\lim_k \Gamma(k) = \Omega^{\lambda}$ implies that there exists k large enough such that

$$-\varepsilon \leq \int_{\Omega^{\lambda}} \mathbb{E}\left[\widetilde{\Psi}(Z^{\nu(k)}_{t_0,Z^{\nu}_{t,z}(t_0)}(T),\Lambda(\omega^{\lambda})|\mathcal{F}_{t_0}\right]\mathbf{1}_{\Omega^{\lambda}\setminus\Gamma(k)}(\omega^{\lambda})d\mathbb{P}^{\lambda}(\omega^{\lambda}) \leq 0 \ \ \mathbb{P}^0\text{-a.s.}$$

This implies that for the same k,

$$\begin{aligned}\mathbb{E}\left[\widetilde{\Psi}(Z^{\nu(k)}_{t_0,Z^{\nu}_{t,x,y}(t_0)}(T),\Lambda)|\mathcal{F}_{t_0}\right] &\geq \mathbb{E}\left[\widetilde{\Psi}(Z^{\nu(k)}_{t_0,Z^{\nu}_{t,x,y}(t_0)}(T),\Lambda)\mathbf{1}_{\Gamma(k)}|\mathcal{F}_{t_0}\right] - \varepsilon \quad && \mathbb{P}^0\text{-a.s.}\\ &\geq \Xi(Z^{\nu}_{t,x,y}(t_0)) - 4\varepsilon && \mathbb{P}^0\text{-a.s.}\\ &\geq P^{\alpha}_{t,p}(t_0) - 4\varepsilon && \mathbb{P}^0\text{-a.s.}\end{aligned}$$

We clearly can find $\alpha' \in \mathcal{A}_t$ such that $P^{\alpha}_{t,p}(t_0) - 4\varepsilon \geq P^{\alpha'}_{t,p-4\varepsilon}(t_0)$, $\mathbb{P}^0$-a.s. The control $\widetilde{\nu} := \nu \mathbf{1}_{[t,t_0)} + \nu(k)\mathbf{1}_{[t_0,T]}$ is in $\widetilde{\mathcal{U}}_t$ by the concatenation property and thus $Z^{\nu(k)}_{t_0, Z^{\nu}_{t,x,y}(t_0)}(T) = Z^{\widetilde{\nu}}_{t,x,y}(T)$ by the flow property. Taking the expectation of the above inequality provides

$$\mathbb{E}\left[\widetilde{\Psi}(Z^{\widetilde{\nu}}_{t,x,y}(T), \Lambda)\right] \geq \mathbb{E}\left[P^{\alpha'}_{t,p-4\varepsilon}(t_0)\right] = p - 4\varepsilon \ \mathbb{P} - \text{a.s.}$$

Thus $y \in A(t,x,p-4\varepsilon)$, meaning that $w(t,x,p) \geq \widetilde{v}(t,x,p-4\varepsilon)$ (with arbitrary $\varepsilon > 0$).

2.(b). We follow the proof of 2.(a). except that we directly have the countable sequence $(\lambda_i)_{i\geq 1}$ without the covering argument. □

Conclusion of Part 2

By their fundamental characteristics, deregulated electricity markets forbid the classic Black-Scholes approach or any variation based on a complete market setting in order to price and hedge price risk.

In Chapter 4, we have seen that futures contract are not bonded only to the Spot price of electricity together with arbitrage arguments. It is essential to consider a complex and structural link between electricity prices and associated raw material prices. The structural model that we propose allows for semi-explicit formulation of forward prices and, by a fair approximation, of futures contract prices. In counterpart, it involves estimation and calibration of the electricity supply curve with data that are not purely financial. Followed by [Aid 10] and [Carmona 11], this class of model appears to be promising for derivative pricing purposes.

In Chapter 5, we consider futures as financial assets, but we avoid to reconstruct the term structure with arbitrage arguments, as in [Fleten 03] and [Hinz 05]. We face here the structural impossibility to reconstruct missing contracts which are used as the underlying for derivative claims. This incomplete market setting is tackled via a stochastic control approach. We propose a numerical application of the stochastic target with controlled loss approach, using complete market methods to obtain semi-explicit expectation formulations. The approach proved its efficiency on simulated and real data. It appears that our framework can be used for other financial problems. We also introduced a high performance method for the resolution of the non-linear PDE associated to the control problem. This heuristic method shall be deeply studied in a forthcoming future.

Giving a loss function and a threshold, the stochastic target approach is, not surprisingly, an efficient strategy in the described situation. Fixing p as the expected loss produced with the Black-Scholes strategy, we reduce the initial wealth needed to satisfy this criterion. From an equivalent point of view, it significantly reduces the given criterion if we start with the same wealth as in the naive Black-Scholes case. The resulting price of risk is also robust in the sense that, if we compare the two strategy in regard of a second risk measure (the conditional Value-at-Risk), we also have better performance.

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Résumé :
Cette thèse traite de la valorisation de produits dérivés du prix de l'électricité. Dans la première partie, nous nous intéressons à la valorisation par absence d'opportunité d'arbitrage de portefeuilles incluant la possibilité de transformation d'actifs par le biais d'un système de production, sur des marchés en temps discret avec coûts de transaction proportionnels. Nous proposons une condition qui nous permet de démontrer la propriété fondamentale de fermeture pour l'ensemble des portefeuilles atteignables, et donc l'existence d'un portefeuille optimal ou un théorème de sur-réplication. Nous continuons l'approche avec fonction de production en temps discret sur un marché en temps continu avec ou sans frictions. Dans le seconde partie, nous présentons une classe de modèles faisant apparaître un lien structurel entre le coût de production d'électricité et les matières premières nécessaires à sa production. Nous obtenons une formule explicite pour le prix de l'électricité spot, puis la mesure martingale minimale fournit un prix pour les contrats futures minimisant le risque quadratique de couverture. Nous spécifions le modèle pour obtenir des formules analytiques et des méthodes de calibration et d'estimation statistique des paramètres dans le cas où le prix spot dépend de deux combustibles. Dans un second temps, nous suivons la méthodologie initiée par Bouchard et *al.* (2009) pour l'évaluation de la prime de risque liée à un produit dérivé sur futures non disponible. Utilisant des résultats de dualité, nous étendons l'étude au cas d'un marché semi-complet, en proposant une réduction du problème et une méthode numérique pour traiter l'EDP non linéaire.

Abstract:
This Ph.D. dissertation deals with the pricing of derivatives on electricity price. The first part is a theoretical extension of Arbitrage Pricing Theory: we assess the problem of pricing contingent claims when the financial agent has the possibility to transform assets by means of production possibilities. We propose a specific concept of arbitrage for such portfolios in discrete time for markets with proportional transaction costs. This allows to show the closedness property, portfolio optimization problem or a super-hedging theorem. We then study such portfolios with financial possibilities in continuous time, with or without frictions. We apply these results to the pricing of futures contract on electricity. In the second part we introduce a class of models allowing to link the electricity spot price with its production cost by a structural relationship. We specify a two combustibles model with possible breakdown. It provides explicit formulae allowing to fit several pattern of electricity spot prices. Using the minimal martingale measure, we explicit an arbitrage price for futures contracts minimizing a quadratic risk criterion. We then specify the model to obtain explicit formulae, calibration methods and statistical estimation of parameters. We address in a second time the question of the risk premium associated to the holding of a European option upon a non-yet available futures contract. We essentially apply the ideas of Bouchard and *al.* (2009) to the semi-complete market framework and propose numerical procedures to obtain the risk premium associated to a given loss function.