Équations aux dérivées partielles stochastiques de type parabolique avec un potentiel singulier

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Résumé

Nous nous intéressons dans cette thèse à l'étude de trois dynamiques en dimension infinie, liées à des problèmes d'interface aléatoire. Il s'agira de résoudre une équation aux dérivées partielles stochastiques paraboliques avec différents potentiels singuliers. Trois types de potentiel sont étudiés, dans un premier temps nous considérons l'équation de la chaleur stochastique avec un potentiel convexe sur $\mathbb{R}^d$, correspondant à l'évolution d'une corde aléatoire dans un ensemble convexe $O \subset \mathbb{R}^d$ et se réfléchissant sur le bord de O. La mesure de réflexion , vue comme la fonctionnelle additive d'un processus de Hunt, est étudiée au travers de sa mesure de Revuz. L'unicité trajectorielle et l'existence d'une solution forte continue sont prouvées. Pour cela nous utilisons des résultats récents sur la convergence étroite de processus de Markov avec une mesure invariante log-concave.

Nous étudions ensuite l'équation de la chaleur avec un bruit blanc espace-temps, et un potentiel singulier faisant apparaître un temps local en espace. Cette fois le processus de Markov étudié possède une mesure invariante de type mesure de Gibbs mais avec un potentiel non convexe. L'existence d'une solution est prouvée, ainsi que la convergence, vers une solution stationnaire, d'une suite d'approximation, construite par projections sur des espaces de dimension finie. une étude du semi-groupe permet d'obtenir des solutions non-stationnaires

Nous combinons enfin les deux précédents modèles. L'existence d'une solution stationnaire est prouvée ainsi que la convergence d'un schéma d'approximation comme précédemment.

Mots clés:

Formules d'intégration par parties, équations aux dérivées partielles stochastiques, temps locaux, formes de Dirichlet, processus de Markov, fonctionnelles additives, Mosco convergence

Abstract

Stochastic Partial Differential Equations of Parabolic Type with Singular potential

This thesis deals with some topics linked with interface model, ours aim is to find solution of some SPDE of parabolic type with singular potential. Firstly We study the motion of a random string in a convex domain O in $\mathbb{R}^d$, namely the solution of a vector-valued stochastic heat equation, confined in the closure of O and reflected at the boundary of O. We study the structure of the reflection measure by computing its Revuz measure in terms of an infinite dimensional integration by parts formula. We prove extistence and uniqueness of a strong solution. Our method exploits recent results on weak convergence of Markov processes with log-concave invariant measures.

Secondly We consider a stochastic heat equation driven by a space-time white noise and with a singular drift, where a local-time in space appears. The process we study has an explicit invariant measure of Gibbs type, with a non-convex potential. We obtain existence of a Markov solution, which is associated with an explicit Dirichlet form. Moreover we study approximations of the stationary solution by means of a regularization of the singular drift or by a finite-dimensional projection.

Finaly, we extend the previous methods for a SPDE in which the two types of singularity appear

Keywords:

Integration by parts formulas, Stochastic partial differential equation, local times, Dirichlet forms, Markov processes, additive functionnal, Mosco convergence

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CHAPTER 1

INTRODUCTION

In this thesis we want to study a class of stochastic processes in infinite dimension, namely of *stochastic partial differential equations* (SPDEs) with singular drift coefficients. We consider several non-linear perturbations of a stochastic heat equation driven by a space-time white noise with a one-dimensional space variable.

We treat three different such perturbations. In chapter 3 we study a SPDE whose solution takes values in a convex domain in $\mathbb{R}^d$ and is reflected at the boundary of this domain; we call the solution a *random string*. In chapter 4 we deal with a SPDE whose solution takes real values and feels the effect of one or several sharp interfaces, in analogy with the classical skew Brownian motion in one dimension, and we call this SPDE a *skew stochastic heat equation*. In chapter 5 we combine the two previous non-linearities and consider a skew SPDE with reflection.

These non-linearities can be written as derivatives of non-continuous functions in the sense of distributions, and for this reason the classical theories for existence and uniqueness in the case of Lipschitz coefficients do not apply. On the other hand, we restrict to a class of equations which share an important structural property, namely they are associated with a symmetric Dirichlet form of gradient type. In other words, the non-linearities we add to our equations are (formally) the gradient of a scalar functional (the *potential*) in the Hilbert space $H := L^2(0,1)$ (or $L^2(0,1;\mathbb{R}^d)$ in the case of chapter 3). This makes a number of important technical tools available, like an explicit invariant measure and the Itô-Fukushima stochastic calculus.

The three equations we consider are presented in increasing order of difficulty. The random string in a convex domain is associated with a convex potential, and this allows to obtain a complete well-posedness theory for all initial conditions in H; in particular, we have pathwise uniqueness and existence of strong solutions.

In the two skew equations, with or without reflection, the convexity of the potential is lost and we can obtain only weaker results, in particular pathwise uniqueness is out of reach. As a substitute for uniqueness, we show that the solutions we construct are natural, in the sense that they are the limit in law of processes solving smooth approximations of the equation, and such approximating equations have unique solutions. Moreover, we construct also finite-dimensional approximations of the equations and show convergence in law of stationary solutions. The main tool for such limit results is the Mosco convergence of the associated Dirichlet forms, in the version where the invariant measures can also vary, recently developed by several researchers.

At the beginning of this thesis, in chapter 2 we have collected the main (classical) definitions and results which are important for us.

1.1 Stochastic heat equations and gradient systems

All equations we consider in this thesis can be interpreted as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \theta^2} - \dfrac{1}{2} f'(u) + \dot{W}, \\ \\ u(t,0) = u(t,1) = 0 \\ \\ u(0,\theta) = u_0(\theta), \qquad \theta \in [0,1] \end{cases} \tag{1.1.1}$$

where $\dot{W}$ is a space-time white noise, see section 2.2.6 below, and $f : \mathbb{R} \mapsto \mathbb{R}$ (in chapter 3 it is rather $f : \mathbb{R}^d \mapsto \mathbb{R}$) is a (bounded) function which will have different degrees of regularity: smooth, or convex, or only with bounded variation. A very important fact for us is that equation (1.1.1) is a gradient system in the Hilbert space $H := L^2(0,1)$ (or $H : L^2(0,1;\mathbb{R}^d)$ in the case of chapter 3). Indeed, if we introduce the *potential* $F : H \mapsto \mathbb{R}$ defined by

$$F(x) := \int_0^1 f(x(\theta))\, d\theta, \qquad x \in H,$$

then for all $h \in H$, denoting by $\langle \cdot, \cdot \rangle$ the canonical scalar product in $L^2(0,1)$,

$$\langle \nabla F(x), h \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(F(x+\varepsilon h) - F(x)) = \int_0^1 f'(x_\theta)\, h_\theta\, d\theta = \langle f'(x), h \rangle$$

i.e. $\nabla F(x) = f'(x_\cdot)$ and equation (1.1.1) can formally be written as the gradient system

$$du = -\frac{1}{2}\nabla V(u)\, dt + dW, \qquad V(x) := \int_0^1 (\dot{x})^2 d\theta + F(x),$$

where $(W_t, t \geq 0)$ is a cylindrical Wiener process in H, see section 2.2.7 below. Then, in [17, Chapter 12] it is proved that we have an expression for the invariant measure of (1.1.1)

$$\nu(dx) = \frac{1}{\int e^{-F} d\mu}\, e^{-F(x)}\, \mu(dx),$$

where μ is the law of the Brownian bridge $(\beta_r, r \in [0,1])$, with $\beta_0 = \beta_1 = 0$. Is moreover proved in [17, Chapter 12] that the process $(u(t,\cdot), t \geq 0)$ is associated with (the closure of) the following Dirichlet form

$$\mathcal{D}(\varphi, \psi) := \frac{1}{2}\int_H \langle \nabla\varphi, \nabla\psi \rangle\, d\nu$$

in $L^2(\nu)$, see section 2.1 below for the main definitions on the theory of Dirichlet forms.

These results yield the possibility of using a number of tools: the Fukushima stochastic calculus [27], the Lyons-Zheng decomposition (see section 2.1.6 below), some powerful a priori estimates on stationary processes (see Lemma 4.6.1 below), the theory of Mosco-convergence (see section 2.4), a number of integration by parts formulae in infinite dimension, etc. We also recall that reaction-diffusion SPDEs like (4.1.4) have been extensively studied, see for instance [13] and the references thereins.

1.2 A stochastic string in a convex domain

In this chapter we want to prove well-posedness of stochastic partial differential equations driven by space-white noise and reflected on the boundary of a convex region of $\mathbb{R}^d$. More precisely, we consider a convex open domain O in $\mathbb{R}^d$ with a smooth boundary ∂O and a proper l.s.c. convex function $\varphi : \overline{O} \mapsto \mathbb{R}$, and we study solutions (u, η) of the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \theta^2} + n(u(t,\theta)) \cdot \eta(t,\theta) - \dfrac{1}{2}\partial\varphi_0(u(t,\theta)) + \dot{W}(t,\theta) \\ u(0,\theta) = x(\theta), \quad u(t,0) = a, \ u(t,1) = b \\ u(t,\theta) \in \overline{O}, \ \eta \geq 0, \ \eta(\{(t,\theta)\,|\,u(t,\theta) \notin \partial O\}) = 0 \end{cases} \tag{1.2.1}$$

where $u \in C\left([0,T] \times [0,1]; \overline{O}\right)$ and η is a locally finite positive measure on $]0,T] \times [0,1]$; moreover $a, b \in O$ are some fixed points, $\dot{W}$ is a vector of d independent copies of a space-time white noise and for all $y \in \partial O$ we denote by $n(y)$ the inner normal vector at y to the boundary ∂O; finally, $\partial\varphi_0 : O \mapsto \mathbb{R}^d$ is the element of minimal norm in the subdifferential of ϕ and the initial condition $x : [0,1] \mapsto \overline{O}$ is continuous.

Solutions (u, η) of equation (1.2.1) are such that u takes values in the convex closed set $\overline{O}$ and evolve as solutions of a standard SPDE in the interior O, while the reflection measure η pushes $u(t,\theta)$ along the inner normal vector $n(u(t,\theta))$, whenever $u(t,\theta)$ hits the boundary. The condition $\eta(\{(t,\theta)\,|\,u(t,\theta) \notin \partial O\}) = 0$ means that the reflection term acts only when it is necessary, i.e. only when $u(t,\theta) \in \partial O$.

This kind of equations has been considered, in the case of O being an interval in $\mathbb{R}$, in a number of papers, like [51, 21, 29, 64, 14, 19, 18, 35], as a natural extension of the classical theory of stochastic differential inclusions in finite dimension to an infinite-dimensional setting. Moreover, such equations arise naturally as scaling limit of discrete interface models, see e.g. [29]. However, the finite dimensional situation is very well understood, see [12], while in infinite dimension only particular cases can be treated, often with *ad hoc* arguments.

The main results we want to prove in this chapter are summarized in the following

Theorem 1.2.1.

1. *For all* $x \in C\left([0,1];\overline{O}\right)$*, the problem* (1.2.1) *enjoys pathwise uniqueness of weak solutions and existence of a strong solution.*

2. *Setting* $U : H \mapsto \mathbb{R}$

$$U(x) := \int_0^1 \varphi(x_\theta)\, d\theta, \quad x \in H, \qquad \nu(dx) := \frac{1}{Z}\, \mathbb{1}_K(x)\, \exp(-U(x))\, \mu(dx)$$

and the bilinear form

$$\mathcal{E}(F,G) := \frac{1}{2}\int \langle \nabla F, \nabla G\rangle\, d\nu, \qquad F, G \in C_b^1(H),$$

then $\mathcal{E}$ *is closable in* $L^2(\nu)$ *and the closure is a Dirichlet form, associated with the solution to* (1.2.1).

3. *The Markov semigroup* $(P_t)_{t\geq 0}$ *associated with the Dirichlet form* $(\mathcal{E}, D(\mathcal{E}))$ *is Strong Feller, i.e. for any bounded Borel* $\varphi : H \mapsto \mathbb{R}$

$$|P_t\varphi(x) - P_t\varphi(y)| \;\leq\; \frac{\|\varphi\|_\infty}{\sqrt{t}}\, \|x-y\|_H, \qquad x, y \in H,\ t > 0.$$

4. *A.s. the reflection measure* η *is supported by a Borel set* $\mathcal{S} \subset]0,+\infty[\, \times [0,1]$*, i.e.* $\eta(\mathcal{S}^c) = 0$*, such that for all* $s \geq 0$*, the section* $\{\theta \in [0,1] : (s,\theta) \in \mathcal{S}\}$ *has cardinality* 0 *or* 1*. Moreover, if* $r(s) \in \mathcal{S} \cap (\{s\} \times [0,1])$ *then*

$$u(s, r(s)) \in \partial O, \qquad u(s,\theta) \notin \partial O,\ \forall \theta \in [0,1] \setminus \{r(s)\}.$$

All previous papers on SPDEs with reflection deal with versions of (1.2.1) where u takes real values, with one or two barriers (one above, one below the solution). This article seems to be the first to tackle the problem of a random string u confined in a convex region in $\mathbb{R}^d$. This case is not a trivial generalization of the one-dimensional one. Indeed, in one dimension the reflection term in (1.2.1) has a definite sign if there is only one barrier, and is the difference of two positive terms acting on disjoint supports, if there are two barriers. This makes it easy to obtain estimates on the total variation of the reflection term. This structure is lost in the case of a convex region in $\mathbb{R}^d$, since the positive measure η is multiplied by the normal vector n at the boundary, which moves in the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$. See the beginning of section 3.7 below for a more precise discussion.

In the same spirit, we recall that most of the first papers on this topic make essential use of monotonicity properties of equation (1.2.1), related with *the maximum principle* satisfied by the second derivative and with the existence of a unique barrier. However more recent works have shown that monotonicity properties are not so essential: for instance a fourth-order operator, without maximum principle, replaces the second derivative in [19, 18, 35], and two barriers in $\mathbb{R}$ are considered in [23, 54, 18].

This chapter makes use of an approach based on Dirichlet forms, infinite dimensional integration by parts formulae, and, crucially, a recent result on stability of Fokker-Planck equations associated with log-concave reference measures, see Theorem 3.4.2 below. This stability result, developed in [3] using recent advances in the

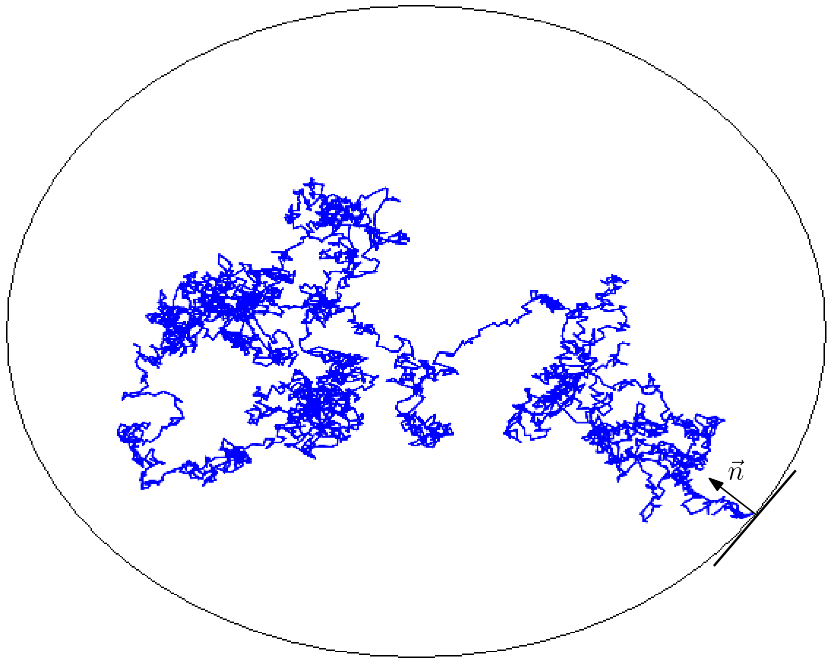

Figure 1.1: A random string of theorem 1.2.1 above

theory of optimal transport, yields convergence of approximating equations to the solution to (1.2.1), replacing the monotonicity properties used e.g. in [51]. We recall that a probability measure ν is *log-concave* if for all pairs of open sets $A, B \subset H$ we have:

$$\log \nu((1-t)A + tB) \geq (1-t)\log \nu(A) + t \log \nu(B)$$

where $(1-t)A + tB := \{(1-t)a + tb \,|\, a \in A, b \in B\}$ for $t \in [0,1]$. A Gaussian measure μ is always log-concave, and a probability measure given by $e^{-U} d\mu$ with U convex is also log-concave, see for instance [2, Theorem 9.4.11].

Another important tool is an integration by parts formula with respect to the law of a Brownian bridge conditioned to stay in the domain O, proved in [38], extending the first formula of this kind, which appeared in [64]. See formula (3.6.1) below and the discussion therein.

We also want to mention that a similar equation, written in the abstract form of a *stochastic differential inclusion*

$$dX_t + (AX_t + N_K(X_t))dt \ni dW_t, \qquad X_0 = x \tag{1.2.2}$$

has been considered in [6], where $A : D(A) \subset H \mapsto H$ is a self-adjoint positive definite operator in a Hilbert space H, $K \subset H$ is a closed convex subset with regular boundary, $N_K(y)$ is the normal cone to K at y and W is a cylindrical Wiener process in H. The authors of [6] assume crucially that K has *non-empty interior* in H. Our equation (1.2.1) could be interpreted as an example of (1.2.2) in the framework of [6], where in our case $H = L^2([0,1];\mathbb{R}^d)$ and

$$K := \left\{ x \in L^2([0,1];\mathbb{R}^d) : x_\theta \in \overline{O} \quad \text{for all } \theta \in [0,1] \right\}.$$

However, in the topology of $L^2([0,1];\mathbb{R}^d)$, K has empty interior and therefore the approach of [6] does not work in our case. Moreover, our results are somewhat stronger than those of [6], which only deal with the generator and the Dirichlet form rather than with existence and uniqueness of solutions of the SPDE, as we do.

The content of this chapter has been published in [8].

1.3 A skew stochastic heat equation

In this chapter we consider a stochastic partial differential equation whose solution $(u(t,x), t \geq 0, x \in [0,1])$ takes real values and which is of the form (1.1.1), where however the function f is bounded with bounded variation, but could have jumps.

Let us consider for instance the case $f(y) = \alpha\,\mathbb{1}_{(y\geq 0)}$ with $\alpha > 0$. In this case the derivative of f in the sense of distributions is a multiple of a Dirac mass, and therefore it is not clear how to define $f'(u)$ in (1.1.1). This class of equations is however interesting, since the potential becomes in this case

$$F(x) = \int_0^1 f(x_\theta)\,d\theta = \alpha \int_0^1 \mathbb{1}_{(x_\theta \geq 0)}\,d\theta$$

and the associated invariant measure $\nu(dx) = \dfrac{1}{\int e^{-F} d\mu}\, e^{-F(x)}\,\mu(dx)$ shows that the system has a preference for the negative values of the solution and it is clearly of interest to study such a situation.

The first problem is to give a meaning to the nonlinearity $f'(u)$ in (1.1.1). This is done via an occupation times formula *in space*. Indeed, let us notice that, by integrating in $d\theta$ the non-linearity $f'(u)$ multiplied by a test function $h \in C_c^2(0,1)$, by an application of the occupation times formula (see e.g. [58][VI.1.6, VI.1.15]) for the process $(u(t,\theta), \theta \in [0,1])$, $t > 0$ fixed,

$$\int_0^1 f'(u(t,\theta))\,h_\theta\,d\theta = \int_{\mathbb{R}} da\, f'(a) \int_0^1 h_r\,\ell_t^a(dr) = -\int_{\mathbb{R}} f(da) \int_0^1 h'_r\,\ell_t^a(r)\,dr$$

where in the last equality we have performed an integrating by parts. Here $(\ell_t^a(r), a \in \mathbb{R}, t > 0, r \in [0,1])$ is the family of local times at a of the process $(u(t,\theta), \theta \in [0,1])$, $t > 0$ fixed, accumulated over $[0,r]$.

We notice that several remarks are in order for this formal discussion. First of all, it is not clear whether the process $(u(t,\theta), \theta \in [0,1])$ is a semi-martingale with respect to some filtration, so that existence of local times and validity of an occupation times formula are not trivial. Moreover, even if these points are resolved, in the occupation times formula one should in fact integrate with respect to the quadratic covariation $\langle u, u\rangle_\theta$ of u in space, and we are here implicitly assuming that this is equal to θ: why should this be the case? In fact, a part of the work to be done consists in justifying (in a different way) the claim that the occupation times formula above makes sense.

We note that for smooth f, an important property of the reaction-diffusion equation (1.1.1) is the Strong-Feller property, see e.g. (3.4.4) below, namely the transition semigroup maps bounded Borel functions on the state space into continuous functions (for positive time $t > 0$). A well-known consequence is that the law of the solution at time $t > 0$ is absolutely continuous w.r.t. the invariant measure. For (1.1.1) the invariant measure is equivalent to the law of a Brownian bridge, namely a semimartingale, whose quadratic covariation is moreover equal to θ. This seems to justify our claim, at least morally.

However, in our situation, f is not smooth, and the Strong-Feller property can not be proved with the techniques available in the literature. One of the novelties of chapter 4 is to show that the law of the solution at time $t > 0$ is absolutely continuous w.r.t. the invariant measure, using a comparison argument between eigenvalues of infinitesimal generators in different L^2 spaces (see Proposition 4.2.7 below). This is important and useful in the Fukushima theory of Dirichlet forms, see e.g. [27, Theorem 4.1.2 and formula (4.2.9)].

Using an integration by parts formula and the Fukushima stochastic calculus, we show that the following SPDE

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \theta^2} - \dfrac{1}{2}\displaystyle\int_{\mathbb{R}} f(da)\frac{\partial}{\partial \theta}\ell^a_{t,\theta} + \dot{W}, \\ u(t,0) = u(t,1) = 0, \\ u(0,\theta) = u_0(\theta), \quad \theta \in [0,1] \end{cases} \tag{1.3.1}$$

has a weak solution, which is a Markov process associated with a Dirichlet form of gradient type. More precisely, the main results we want to prove in this chapter are summarized in the following

Theorem 1.3.1.

1. *Setting $F : H \mapsto \mathbb{R}$*

$$F(x) := \int_0^1 f(x_\theta)\, d\theta, \quad x \in H, \qquad \nu(dx) := \frac{1}{Z}\exp(-F(x))\,\mu(dx)$$

and the bilinear form

$$\mathcal{E}(F,G) := \frac{1}{2}\int \langle \nabla F, \nabla G\rangle\, d\nu, \qquad F,G \in C^1_b(H),$$

then $\mathcal{E}$ is closable in $L^2(\nu)$ and the closure is a Dirichlet form.

2. *The Markov process associated with $(\mathcal{E}, D(\mathcal{E}))$ is a weak solution to* (1.3.1)*. In particular for all initial conditions, almost surely, for a.e. t there exists a bi-continuous family of local times $[0,1] \ni (r,a) \mapsto \ell^a_{t,r}$ of $(u_t(\theta), \theta \in [0,1])$.*

3. *The Markov semigroup $(P_t)_{t\geq 0}$ associated with $(\mathcal{E}, D(\mathcal{E}))$ enjoys the absolute-continuity condition, namely there exists a measurable kernel $(p_t(x,dy), t \geq 0, x \in H)$ such that for all $\varphi \in L^2(\nu)$, $\nu-$a.s. $P_t\varphi = \displaystyle\int \varphi(y)\, p_t(\cdot, dy)$ and $p_t(x,\cdot) \ll \nu(\cdot)$ for all $t > 0$.*

4. *If $f_n \in C_b^1(\mathbb{R})$ and $f_n \to f$ locally uniformly outside the discontinuity set of f, then solutions to* (1.1.1) *with f_n replaced by f converge in the sense of finite-dimensional distributions to our solution to* (1.3.1).

5. *A stationary process, given by a finite-dimensional collection of interacting skew Brownian motions, converges in the sense of finite-dimensional distributions to our stationary solution to* (1.3.1).

An important difference between this chapter and the previous one, is that in this case we can not prove a pathwise-uniqueness result for the solution. The drift-coefficient is too singular for the standard techniques to apply, and to our knowledge there is no result on uniqueness for this kind of equations in the literature. However, we show that the process we construct, which will be a weak solution to our equation (1.3.1), is canonical. Indeed, we construct smooth approximations of (1.3.1), by considering drift coefficients f_n' with f_n very regular, and we show that the solutions to such regularized equations converge (in law or in the sense of finite-dimensional distributions) to the solution to (1.3.1) that we have constructed.

This convergence result is obtained via the classical technique of Mosco-convergen -ce [48], recently extended by Kuwae and Shioya [43] and Andres-von Renesse [4]. This theory gives a criterion for convergence of resolvent operators of Dirichlet forms, the classical setting being valid when all Dirichlet forms are defined in the same L^2 space, while the recent extensions concern cases where also the reference L^2 spaces change (and "converge"), see section 2.4 below.

We also deal with finite-dimensional approximations of equation (1.3.1). Indeed, we project (in a sense to be made precise) the equation onto finite-dimensional subspaces of H, we construct associated Markov dynamics and Dirichlet forms, and we show Mosco-convergence and convergence in distribution of (stationary) processes to our stationary of (1.3.1). Interestingly, the finite-dimensional dynamics we find are interacting skew Brownian motions, see [45] and sections 4.1.1-4.5.1 below. This, together with the analogies pointed out in section 4.1.2 below, justify the name of *skew heat equation* that we have chosen for (1.3.1).

We remark that the invariant measure ν in this setting is not log-concave, and the approach based on optimal transport given in [3] can not be used in order to prove convergence of approximations. Mosco-convergence is a valid alternative in this situation, also rather simpler to prove, although it yields somewhat weaker results. In fact, using [3] one can obtain a uniform estimate on the relative entropy of the approximating transition semigroups with respect to the associated invariant measures, which we can not prove here.

The content of this paper has appeared in [9].

1.4 A skew reflected heat equation

In this chapter we combine the two non-linearities presented in the two previous chapters. Namely, we want to consider the following SPDE in one space-dimension driven by space-time white noise with reflection (e.g. at 0) and with a "skew" drift coefficient of the form $f'(u)$ with a bounded f with bounded variation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \theta^2} + \eta(t,\theta) - \dfrac{1}{2}\displaystyle\int_{\mathbb{R}} f(da)\frac{\partial}{\partial \theta}\ell^a_{t,\theta} + \dot{W}(t,\theta) \\ u(0,\theta) = x(\theta), \quad u(t,0) = b,\ u(t,1) = b \\ u(t,\theta) \geq 0,\ \eta \geq 0,\ \eta(\{(t,\theta) \,|\, u(t,\theta) \neq 0\}) = 0 \end{cases} \tag{1.4.1}$$

where $b > 0$, $(\ell^a_{t,\theta}, \theta \in [0,1])$ is the family of local times at $a \in \mathbb{R}$ accumulated over $[0,\theta]$ by the process $(u(t,r), r \in [0,1])$, and η is a locally finite non-negative measure.

Again, one has to show that the above equation makes senses, as for equation (1.3.1). However, a few additional difficulties arise, because of the presence of the reflection measure. The main results we want to prove in this chapter are summarized in the following

Theorem 1.4.1.

1. *Setting $F : H \mapsto \mathbb{R}$*

$$F(x) := \int_0^1 f(x_\theta)\, d\theta, \quad x \in H, \qquad \zeta(dx) := \frac{1}{Z}\exp(-F(x))\,\mu^b(dx)$$

where μ^b is the law of the Brownian bridge $(\beta_r, r \in [0,1])$, with $\beta_0 = \beta_1 = b$, and the bilinear form

$$\mathcal{E}(F,G) := \frac{1}{2}\int \langle \nabla F, \nabla G\rangle\, d\zeta, \qquad F, G \in C^1_b(H),$$

then $\mathcal{E}$ is closable in $L^2(\zeta)$ and the closure is a Dirichlet form.

2. *The Markov process associated with $(\mathcal{E}, D(\mathcal{E}))$ is a weak solution to* (1.4.1). *In particular for all initial conditions, almost surely, for a.e. t there exists a bi-continuous family of local times $[0,1] \ni (r,a) \mapsto \ell^a_{t,r}$ of $(u_t(\theta), \theta \in [0,1])$.*

3. *If $f_n \in C^1_b(\mathbb{R})$ and $f_n \to f$ locally uniformly outside the discontinuity set of f, then stationary solutions to* (1.4.1) *with f replaced by f_n converge in the sense of finite-dimensional distributions to our solution to* (1.4.1).

4. *A stationary process, given by a finite-dimensional collection of interacting reflected skew Brownian motions, converges in the sense of finite-dimensional distributions to our stationary solution to* (1.4.1).

It is immediately clear that the results we obtain on equation (1.4.1) are somewhat weaker than those obtained for equation (1.3.1). First of all, we lose the *absolute-continuity condition* on the transition semigroup $(P_t)_{t\geq 0}$, which means in particular that only solutions outside sets of null capacity can be obtained and handled: that is why we restrict to stationary solutions, which are anyway well motivated and allow to avoid taking care of such exceptional sets. The reason for this loss is that we can not estimate the Dirichlet form from below with the Gaussian Dirichlet form of the linear stochastic heat equation, because of the indicator function $\mathbb{1}_K$ which appears in the density of ζ with respect to μ.

Moreover we need to consider boundary conditions for u at 0 and 1 equal to $b > 0$, in particular different from the reflection level at 0. It is well known that the probability that the infimum of a standard Brownian bridge from 0 to 0 be greater than $-b$ is equal to $1 - \exp(-2b^2)$, and it is 0 if $b = 0$. Therefore, the conditioning which defines the probability measure ν makes sense for $b > 0$ and then has to be justified with a limit in law as $b \downarrow 0$. This can be done in many situations but fails here.

In this situation, as in the previous chapter, the invariant measure ζ is not log-concave, and again the theory of [3] based on optimal transport can not be used. In fact, even Mosco-convergence is less effective for (1.4.1) than for (1.3.1), because of the presence of a boundary term in the integration by parts formula with respect to the invariant measure ζ. For this reason, we need to develop an intermediate convergence result, in order to handle the case of reflected SPDEs with a smooth drift f, in particular for the convergence of finite-dimensional approximations. This is done in Theorem 5.6.2 at the end of the last chapter, and seems to be a result of independent interest.

1.5 A motivation from pinned random surfaces

In the previous chapters we have considered stochastic dynamics whose invariant measures are given by a non log-concave probability measure. There is one important example that remains out of reach for the moment, the case of the law of a reflecting Brownian motion and the convergence of stationary dynamics associated with a pinned interface model near a wall.

We recall a pinning model which has been considered in particular in [20]. For $\varepsilon \geq 0$, $N \in \mathbb{N} = \{1, 2, \ldots\}$, consider the probability measures $\mathbf{P}^{\mathrm{f}}_{\varepsilon,N}$ and $\mathbf{P}^{\mathrm{c}}_{\varepsilon,N}$ on $\mathbb{R}_+^N := [0,\infty)^N$:

$$\mathbf{P}^a_{\varepsilon,N}(dx) := \frac{1}{Z^a_{\varepsilon,N}} \exp(-H^a_N(x)) \prod_{i=1}^{N} \left(dx_i^+ + \varepsilon\delta_0(dx_i)\right) \qquad (1.5.1)$$

where dx^+ is the Lebesgue measure on $\mathbb{R}_+$, $\delta_0(\cdot)$ is the Dirac measure concentrated in 0, $Z^a_{\varepsilon,N}$ is the normalizing constant, a is a label that may stand for f (*free*) or c (*constrained*), and for $x = (x_1, \ldots, x_N) \in \mathbb{R}_+^N$:

$$H^{\mathrm{f}}_N(x) := \sum_{i=0}^{N-1} V(x_{i+1} - x_i), \qquad x_0 := 0,$$

$$H^{\mathrm{c}}_N(x) := \sum_{i=0}^{N} V(x_{i+1} - x_i), \qquad x_0 := x_{N+1} := 0,$$

where $V : \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\}$ is such that $\exp(-V(\cdot))$ is continuous, $V(0) < \infty$ and

$$\kappa := \int_{\mathbb{R}} e^{-V(y)}\, dy < \infty.$$

Here $(x_0, \ldots, x_N) \in [0, +\infty[^{N+1}$ is interpreted as the profile of a one-dimensional interface separating two regions.

If we introduce the IID sequence $(Y_i)_i$ such that Y_i is a continuous random variable with density $f_{Y_i}(\cdot) = \exp(-V(\cdot))/\kappa$. The law of $(Y_i)_i$ is denoted by $\mathbf{P}$. If moreover we set $S_0 := 0$ and $S_n := Y_1 + \cdots + Y_n$, then $\mathbf{P}^{\mathrm{f}}_{0,N}$ is just the law of $(S_1, \ldots, S_N)$, under the non–negativity constraint $\{S_1 \geq 0, S_2 \geq 0, \ldots, S_N \geq 0\}$. On the other hand $\mathbf{P}^{\mathrm{c}}_{0,N}$ is the law of the same random vector under the further constraint $S_{N+1} = 0$.

If $\varepsilon > 0$, then every trajectory $(S_1, \ldots, S_N) \in [0, +\infty[^N$ has under a reward/penal -ty given by ε^k, where k is the number of $i \in \{1, \ldots, N\}$ such that $S_k = 0$. It is clear that for small $\varepsilon > 0$ the interface has under $\mathbf{P}^a_{\varepsilon,N}$

1.5.1 Scaling limits

Let us define the map $X^N : \mathbb{R}^N \mapsto C([0,1])$:

$$X^N_t(x) = \frac{x_{\lfloor Nt \rfloor}}{\sigma N^{1/2}} + (Nt - \lfloor Nt \rfloor)\, \frac{x_{\lfloor Nt \rfloor + 1} - x_{\lfloor Nt \rfloor}}{\sigma N^{1/2}}, \qquad t \in [0,1], \tag{1.5.2}$$

where $\lfloor Nt \rfloor$ denotes the integer part of Nt, and set for $a = \mathrm{f}, \mathrm{c}$:

$$Q^a_{\varepsilon,N} := \mathbf{P}^a_{\varepsilon,N} \circ (X^N)^{-1}. \tag{1.5.3}$$

We will need a notation for

- The Brownian motion $(B_\tau)_{\tau \in [0,1]}$;
- The Brownian bridge $(\beta_\tau)_{\tau \in [0,1]}$ between 0 and 0;
- The Brownian motion *conditioned to be non-negative on* $[0,1]$ or, more precisely, the Brownian meander $(m_\tau)_{\tau \in [0,1]}$, see [58];
- The Brownian bridge *conditioned to be non-negative on* $[0,1]$ or, more precisely, the normalized Brownian excursion $(e_\tau)_{\tau \in [0,1]}$, also known as the Bessel bridge of dimension 3 between 0 and 0, see [58] .

The main result in [20], later refined in [11], is the following

Theorem 1.5.1. *Assume that*

$$\sigma^2 := \mathbf{E}\left[|Y_1|^2\right] < \infty \quad \text{and} \quad \mathbf{E}\left[Y_1\right] = 0 \tag{1.5.4}$$

and that there exists an n such that $\sup f_{S_n}(\cdot) < \infty$. *Then $\varepsilon_c > 0$ and both the free and constrained models undergo a wetting (or delocalization) transition at $\varepsilon = \varepsilon_c$. In particular*

1. *(The subcritical regime.) If $\varepsilon \in [0, \varepsilon_c)$ then*

- $Q^{\mathrm{f}}_{\varepsilon,N}$ *converges weakly in* $C([0,1])$ *to the law of* m*;*
- $Q^{\mathrm{c}}_{\varepsilon,N}$ *converges weakly in* $C([0,1])$ *to the law of* e*.*

2. *(The critical regime.)*
   - $Q^{\mathrm{f}}_{\varepsilon_c,N}$ *converges weakly in* $C([0,1])$ *to the law of a reflecting Brownian motion, namely* $|B|$*;*
   - $Q^{\mathrm{c}}_{\varepsilon_c,N}$ *converges weakly in* $C([0,1])$ *to the law of a reflecting Brownian bridge, namely* $|\beta|$*.*

3. *(The supercritical regime). If* $\varepsilon \in (\varepsilon_c, \infty)$ *then* $Q^a_{\varepsilon,N}$ *converges weakly in* $C([0,1])$ *to the measure concentrated on the constant function taking the value zero.*

These results characterize the Brownian scaling of the model, including the critical scaling.

1.5.2 Stochastic dynamics

Let us now suppose that $V(\cdot)$ is sufficiently smooth and let us consider a random dynamics being reversible with respect to the probability law $Q^a_{\varepsilon,N}$ in (1.5.3) below. This can by done by defining a Dirichlet form of gradient type in $L^2(Q^a_{\varepsilon,N})$, as it is done in the following chapters. This is not particularly difficult, and it turns out that the stochastic differential equation associated this this Dirichlet form is a family of interacting *sticky Brownian motions*.

In the spirit of [31][section 15.2], under a suitable scaling limit compatible with (1.5.2), one expects to see as $N \to +\infty$ in the subcritical or critical regime of Theorem 1.5.1 a SPDE whose invariant measure is, respectively, a Brownian meander/excursion, a reflecting Brownian motion/bridge. Whereas the former is well understood, see [64], the latter is still a mistery.

Indeed, it is simple to define a Dirichlet form of gradient type with respect to the law of a reflecting Brownian motion, but it is much more difficult to give a meaning to the associated stochastic dynamics. An integration by parts formula has been written in [66], giving a first hint of what the associated SPDE should look like, but no rigorous meaning has been obtained yet for that expression. Perhaps the rough approach [36] will yield a stronger technique, but this has not been attempted yet for this class of equations.

1.5.3 The link with this thesis

The reason why the aforementioned problem is a motivation for this thesis, is that the probability measures $\mathbf{P}^a_{\varepsilon,N}$ in (1.5.1) and $Q^a_{\varepsilon,N}$ in (1.5.3) are not log-concave, and we hope that some of the techniques of the chapters 4 and 5 could be extended to give some results in this direction.

The link becomes even more apparent if we note that the Dirac mass δ_0 in (1.5.1) can be approximated by measures

$$b_n\, e^{a_n\, \mathbb{1}_{[0,1/n]}(x)}\, dx^+, \qquad b_n, a_n > 0, \quad b_n \to 0, \;\; \frac{b_n e^{a_n}}{n} \to 1, \quad n \to +\infty.$$

Therefore, if we define

$$\tilde{\mathbf{P}}^a_{\varepsilon,N}(dx) := \frac{1}{Z^a_{\varepsilon,N}} \exp(-H^a_N(x)) \prod_{i=1}^N \left(1 + \varepsilon\, b_n\, e^{a_n \, \mathbb{1}_{[0,1/n]}(x)}\right) dx_i^+$$

$\tilde{\mathbf{P}}^a_{\varepsilon,N}$ is an approximation of $\mathbf{P}^a_{\varepsilon,N}$. Moreover the function

$$\begin{aligned} f(x) &:= -\log\left(1 + \varepsilon\, b_n\, e^{a_n \, \mathbb{1}_{[0,1/n]}(x)}\right) \\ &= -\log\left(1 + \varepsilon\, b_n\, e^{a_n}\right) \mathbb{1}_{[0,1/n]}(x) - \log\left(1 + \varepsilon\, b_n\right) \mathbb{1}_{\mathbb{R}\setminus[0,1/n]}(x) \end{aligned}$$

is in the class of non-linear drifts f that we can treat in chapters 4 and 5. Therefore, we can replace interacting sticky Brownian motions with interacting skew Brownian motions and still have a reasonable approximation of our system.

Unfortunately our approach does not allow f to change in n in such a singular way, and indeed the law of the reflecting Brownian motion is not covered by our results. However, we hope that in a near future the ideas of this thesis will give a contribution to the solution of this long-standing problem.

CHAPTER 2

PRELIMINARY MATERIAL

In this chapter we collect some definitions and notations which will be used throughout the thesis. The material presented here is classical and we give references to the relevant literature in each section.

2.1 Dirichlet forms of gradient type

The theory of Dirichlet forms is the main tool of this thesis. The classical references are [27] for finite-dimensional (locally compact) state spaces and [46] for general state spaces.

2.1.1 Definition and examples

Definition 2.1.1. Let $(H, \mathcal{B}, \gamma)$ a σ-finite measure space. If $D \subset L^2(\gamma)$ is a dense linear space and $\mathcal{E} : D \times D \mapsto \mathbb{R}$ a symmetric bilinear function such that $\mathcal{E}(u, u) \geq 0$ for all $u \in D$, then we call $(\mathcal{E}, D)$ a *non-negative symmetric bilinear form*, or simply a *form*. We define the scalar product on D

$$\mathcal{E}_1(u, v) := \int u\, v\, d\gamma + \mathcal{E}(u, v), \qquad u, v \in D.$$

We say that

1. $(\mathcal{E}, D)$ is *closed* in $L^2(\gamma)$ if D is complete w.r.t. $\mathcal{E}_1$, i.e. if for any sequence $(u_n) \subset D$ which is Cauchy w.r.t. $\mathcal{E}_1$ there exists $u \in D$ such that $\mathcal{E}_1(u_n - u, u_n - u) \to 0$.

2. $(\mathcal{E}, D)$ is a *closable form* in $L^2(\gamma)$ if, for any $(u_n) \subset D$ which is Cauchy w.r.t. $\mathcal{E}_1$ and converges to 0 in $L^2(\gamma)$, we have that u_n converges to 0 w.r.t. $\mathcal{E}_1$. In other words, closability means that if $\|u_n\|_{L^2(\gamma)} \to 0$ and $\mathcal{E}(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to +\infty$, then $\mathcal{E}(u_n, u_n) \to 0$.

Remark 2.1.2. If $(\mathcal{E}, D)$ is a *closable form* in $L^2(\gamma)$, then there exists a unique closed form $(\overline{\mathcal{E}}, \overline{D})$ in $L^2(\gamma)$ such that

1. $D \subset \overline{D}$ and D is dense in $\overline{D}$ w.r.t. $\overline{\mathcal{E}}_1$

2. $\overline{\mathcal{E}}(u,v) = \mathcal{E}(u,v)$, for all $u, v \in D$.

$(\overline{\mathcal{E}}, \overline{D})$ is called the closure of $(\mathcal{E}, D)$ in $L^2(\gamma)$ and it is customary to denote it by $(\mathcal{E}, D(\mathcal{E}))$. Notice that, for any form $(\mathcal{E}, D)$, the space $(D, \mathcal{E}_1)$ has an abstract completion $(\overline{D}, \overline{\mathcal{E}}_1)$. A form is closable if and only if $\overline{D}$ has an *injection* in $L^2(\gamma)$ which extends continuously the canonical immersion $i : D \mapsto L^2(\gamma)$ given by the identity map. Lack of closability means that there exists a sequence $(u_n) \subset D$ such that $u_n \to 0$ in $L^2(\gamma)$ and $u_n \to v \in \overline{D}\backslash\{0\}$ w.r.t. $\overline{\mathcal{E}}_1$.

The proof of closability of $(\mathcal{E}, D)$ is a not very exciting but necessary technical step. A useful criterion is the following

Lemma 2.1.3. *Let $(\mathcal{E}, D)$ be a non-negative symmetric bilinear form. If for all $(u_n) \subset D$ such that $u_n \to 0$ in $L^2(\gamma)$ we have $\mathcal{E}(u_n, v) \to 0$ for all $v \in D$, then $(\mathcal{E}, D)$ is closable.*

The closedness of a given form in H can be read on its quadratic functionnal only. We have the following characterisation

Lemma 2.1.4. *A form $\mathcal{E}$ si closed in H if and only if the quadratic functional $\mathcal{E}$ is lower semi-continuous*

Our main example is the following: we set

$$(T, \mathcal{B}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)), \qquad D := C_c^2(\mathbb{R}^d), \qquad \mathcal{E}(u,v) := \frac{1}{2}\int \langle \nabla u, \nabla v \rangle \, d\gamma,$$

where $C_c^2(\mathbb{R}^d)$ is the space of functions in $C^2(\mathbb{R}^d)$ with compact support and γ is a Borel measure on $\mathbb{R}^d$. When the form $(\mathcal{E}, D)$ of this example is closable, then we call its closure $(\mathcal{E}, D(\mathcal{E}))$ a *Dirichlet form.*

Example 2.1.5. Consider the case of γ equal to the Lebesgue measure on $\mathbb{R}$. Then we have

$$\mathcal{E}_1(u,u) = \|u\|_{L^2}^2 + \|u'\|_{L^2}^2 =: \|u\|_{H^1}^2, \qquad D := C_c^1(\mathbb{R}).$$

Let $(f_n) \subset D$ be a Cauchy sequence for $\|\cdot\|_{H^1}$. Then (f_n) and (f_n') are Cauchy in L^2; since L^2 is complete, there exist f and g in L^2 such that $f_n \to f$ and $f_n' \to g$ in L^2. In order to say that $g = f'$ in a weak sense, we have to prove that g only depends on f and not on the particular sequence (f_n); in other words, if any other sequence $\hat{f}_n \subset D$ converges to f in L^2 and $\hat{f}_n'$ converges to h in L^2, then h must be equal to g. By taking the difference $f_n - \hat{f}_n$, this is equivalent to say that for any sequence $u_n \subset D$ converging to 0 in L^2 such that u_n' converges to w in L^2 we must have $w = 0$.

Notice that closability of $(\mathcal{E}, D)$ is equivalent to closability of the linear operator $\nabla : C_c^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ in the norm of $L^2(\mathbb{R})$. Moreover, the space $D(\mathcal{E})$ is the classical Sobolev space $W^{1,2}(\mathbb{R}) = H^1(\mathbb{R})$ of all functions in $L^2(\mathbb{R})$ such that the (distributional) first derivative belongs to $L^2(\mathbb{R})$.

2.1.2 Generator and resolvent

We fix throughout this section a closed form $(\mathcal{E}, D(\mathcal{E}))$ in $L^2(\gamma)$. We define for $\lambda > 0$:

$$\mathcal{E}_\lambda(u,v) := \lambda \int u\, v\, d\gamma + \mathcal{E}(u,v), \qquad u, v \in D(\mathcal{E}).$$

Proposition 2.1.6. *For all $\lambda > 0$ and $f \in L^2(\gamma)$, there exists a unique $v \in D(\mathcal{E})$ such that*

$$\mathcal{E}_\lambda(v,g) = \int f\, g\, d\gamma, \qquad \forall\, g \in D(\mathcal{E}).$$

We denote $v = R_\lambda f$. Moreover

1. *For all $\lambda > 0$ and $f \in L^2(\gamma)$, $\lambda \|R_\lambda f\|_{L^2(\gamma)} \le \|f\|_{L^2(\gamma)}$*
2. *The bounded operator $R_\lambda : L^2(\gamma) \mapsto L^2(\gamma)$ is symmetric and injective in $L^2(\gamma)$.*
3. *For all $\alpha, \beta > 0$:*

$$R_\alpha - R_\beta = -(\alpha - \beta) R_\alpha R_\beta = -(\alpha - \beta) R_\beta R_\alpha. \tag{2.1.1}$$

The family of operators $(R_\lambda)_{\lambda > 0}$ in $L^2(\gamma)$ is called the Resolvent family *associated with $(\mathcal{E}, D(\mathcal{E}))$.*

Proposition 2.1.7. *There exists a unique operator $L : D(L) \subset L^2(\gamma) \mapsto L^2(\gamma)$ such that, for all $\lambda > 0$, $R_\lambda(L^2(\gamma)) = D(L)$ and*

$$(\lambda - L)\, R_\lambda f = f, \quad \forall\, f \in L^2(\gamma), \qquad R_\lambda\, (\lambda - L) f = f, \quad \forall\, f \in D(L). \tag{2.1.2}$$

We write $R_\lambda = (\lambda - L)^{-1}$. Moreover $(L, D(L))$ is self-adjoint in $L^2(\gamma)$, $D(L) \subset D(\mathcal{E})$ and

$$\mathcal{E}(u,v) = -\int u\, Lv\, d\gamma = -\int v\, Lu\, d\gamma, \qquad \forall\, u, v \in D(L). \tag{2.1.3}$$

The operator $(L, D(L))$ is called the generator *of $(\mathcal{E}, D(\mathcal{E}))$, while $(R_\lambda)_{\lambda > 0}$ is called the* Resolvent family *of $(L, D(L))$.*

We have the easy

Proposition 2.1.8. *The domain $D(L)$ of L is given by*

$$D(L) = \{ f \in D(\mathcal{E}) : \text{the map } D(\mathcal{E}) \ni g \mapsto \mathcal{E}(f,g) \text{ is continuous w.r.t. } \mathcal{E}_1 \}.$$

Finally we can define the so-called Dirichlet form which is likely to be associated with a Markov process

Definition 2.1.9. *A (symmetric) Dirichlet form is a (symmetric) form on $L^2(H, \gamma)$, which is closed and Markovian ie for each $\epsilon > 0$, there exists a real function ϕ_ϵ on $\mathbb{R}$, such that:*

- $\phi_\epsilon(t) = t$, $\forall t \in [0,1]$
- $-\epsilon \le \phi_\epsilon(t) \le 1 + \epsilon$, $\forall t \in \mathbb{R}$

- $0 \leq \phi_\epsilon(t') - \phi_\epsilon(t) \leq t' - t$, *whenever* $t < t'$

- $\forall u \in \mathcal{D}(\mathcal{E})$, *we have* $\phi_\epsilon(u) \in \mathcal{D}(\mathcal{E})$ *and* $\mathcal{E}(\phi_\epsilon(u), \phi_\epsilon(u)) \leq \mathcal{E}(u, u)$

In fact Dirichlet form can own more property close of the structure of its process associated, such as path properties, we enumerate this properties below

Definition 2.1.10. *1. $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ possesses a core $\mathcal{C} \subset L^2(H, \gamma)$, iff $\mathcal{C}$ is a subset of $\mathcal{D}(\mathcal{E}) \cap C_0(H)$ such that $\mathcal{C}$ is $\mathcal{E}_1$ dense in $\mathcal{D}(\mathcal{E})$ and dense in $C_0(H)$ with uniform norm. In this case $\mathcal{E}$ is said to be regular*

2. *We say that $\mathcal{E}$ possesses the local property if for all $u, v \in \mathcal{D}(\mathcal{E})$, such that $supp[u] \cap supp[v] = \emptyset$ imply $\mathcal{E}(u, v) = 0$*

it is proved in [46] that the local property is equivalent with the continuity of the path up to to the lifetime. To finish we introduce the infinite dimensional case in the following definition

Definition 2.1.11. *A symmetric Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is quasi-regular on $L^2(\gamma)$ if:*

1. *If there exist a familly of compact sets $(F_k)_k$ such that $\lim_k Cap(F_k) = 0$*

2. *There is a $\mathcal{E}_1$ dense subset of $\mathcal{D}(\mathcal{E})$ whose elements have $\mathcal{E}$-quasi-continuous version. u is $\mathcal{E}$-quasi-continuous if there is an increasing sequence of close sets $(E_k)_k$ such that, $\forall n$ the restiction of u on E_k is continuous and $\lim_k Cap(E_k) = 0$*

3. *There exist $(un)_n \in \mathcal{D}(\mathcal{E})$ having $\mathcal{E}$-quasi-continuous version $(\tilde{u}_n)_n$ and a $\mathcal{E}$-exceptional set N such that $\{\tilde{u}_n | n \in \mathbb{N}\}$ separate the point of $H \backslash N$*

For the definition of $Cap(\cdot)$ see the paragraph 3.7.1 chapter 3 below. These properties are a subsitute of the local compacteness, of course a regular Dirichlet form on a locally compact separable metric space is quasi-regular.

2.1.3 The process associated with $\mathcal{E}$

Definition 2.1.12. Let γ be a Borel measure on $\mathbb{R}^d$ and $(\mathcal{E}, D(\mathcal{E}))$ a Dirichlet form in $L^2(\gamma)$. A semigroup $(P_t)_{t \geq 0}$ in $L^2(\gamma)$, such that $[0, \infty) \ni t \mapsto P_t f \in L^2(\gamma)$ is continuous for all $f \in L^2(\gamma)$, is associated with $(\mathcal{E}, D(\mathcal{E}))$ if there exists $\lambda_0 > 0$ s.t.

$$R_\lambda f = \int_0^\infty e^{-\lambda t} \, P_t f \, dt, \qquad \forall \ \lambda > \lambda_0.$$

In particular, the resolvent family is the Laplace transform in time of the semigroup. A Markov process X in $\mathbb{R}^d$ is associated with $(\mathcal{E}, D(\mathcal{E}))$ if

$$\mathbb{E}[f(X_t(x))] = P_t f(x), \quad \gamma\text{-a.e. } x, \qquad \forall \ f \in C_b(\mathbb{R}^d), \quad t \geq 0.$$

Remark 2.1.13. Recall that

$$\int_0^\infty e^{-\alpha t} \, dt = \frac{1}{\alpha}, \qquad \forall \ \alpha > 0.$$

Since, in the setting of the previous definition,

$$\int_0^\infty e^{-\lambda t}\, P_t f\, dt = R_\lambda f = (\lambda - L)^{-1}, \qquad \forall\ \lambda > 0,$$

then it is customary to write that $P_t = e^{tL}$. This assertion can be made precise, for instance by means of spectral theory.

Remark 2.1.14. If a process X is associated with $(\mathcal{E}, D(\mathcal{E}))$ in $L^2(\gamma)$, then the transition semigroup (P_t) is symmetric in $L^2(\gamma)$. Indeed, R_λ is symmetric and this property transfers to (P_t) by the injectivity of the Laplace transform.

Suppose now that moreover the constant functions belong to $D(\mathcal{E})$; in this case the constant function equal to 1 is in $L^2(\gamma)$, which is possible if and only if γ is a finite measure. Then $\gamma(\cdot)/\gamma(\mathbb{R}^d)$ is an invariant reversible finite measure for X. Indeed we have for all $f \in L^2(\gamma)$

$$\mathcal{E}_\lambda(R_\lambda f, 1) = \int \lambda\, R_\lambda f\, d\gamma = \int f\, d\gamma, \qquad \forall\ \lambda > 0,$$

and again by injectivity of the Laplace transform we have $\displaystyle\int P_t f\, d\gamma = \int f\, d\gamma$, $t \geq 0$.

2.1.4 Examples in finite dimension

The standard Brownian motion

Let now $X_t(x) = x + B_t$ and $\gamma = \mathcal{L}_d$, Lebesgue measure in $\mathbb{R}^d$. Define

$$D := C_c^2(\mathbb{R}^d), \qquad \mathcal{E}(u, v) := \frac{1}{2}\int \langle \nabla u, \nabla v\rangle\, d\mathcal{L}_d.$$

Also in this case we have an integration by parts formula

$$\frac{1}{2}\int \langle \nabla u, \nabla v\rangle\, d\mathcal{L}_d = -\frac{1}{2}\int u\, \Delta v\, d\mathcal{L}_d, \qquad \forall\ u, v \in D.$$

Moreover it is well known that the function $u(t, x) = \mathbb{E}[\varphi(x + B_t)]$, $t \geq 0$, $x \in \mathbb{R}^d$, $\varphi \in C_b(\mathbb{R}^d)$, satisfies

$$u(t, x) = \varphi(x) + \int_0^t \frac{1}{2}\Delta u(s, x)\, ds, \qquad \forall\ t \geq 0,\ x \in \mathbb{R}^d.$$

We can therefore repeat most of the above considerations, obtaining that $(\mathcal{E}, D)$ is closable and B is associated with the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. Notice that $D(\mathcal{E}) = H^1(\mathbb{R}^d)$ and $\mathcal{E}$ is also known as the Dirichlet integral, from which the name Dirichlet form comes.

Clearly $\mathcal{L}_d$ is not a finite measure and $1 \notin L^2(\mathcal{L}_d)$. However, $\mathcal{L}_d$ is still a (not finite) invariant measure for B; in fact, it is even reversible, in the sense that the transition semigroup is symmetric in $L^2(\mathcal{L}_d)$.

Ornstein-Uhlenbeck processes

Let A be a symmetric matrix in $\mathbb{R}^d$, with strictly negative eigenvalues $(-\lambda_i)_{i=1,\dots,d}$. Set $U(x) := -\langle Ax, x\rangle$, $x \in \mathbb{R}^d$. Then $D^2U = -2A$ and $2\min\{\lambda_1, \dots, \lambda_d\}I \le D^2U \le 2\max\{\lambda_1, \dots, \lambda_d\}I$. By the above results, the Ornstein-Uhlenbeck process, unique solution of

$$X_t(x) = x + \int_0^t AX_s(x)\,ds + W_t, \qquad t \ge 0,\ x \in \mathbb{R}^d,$$

is the process associated with the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$, closure of the form

$$\frac{1}{2}\int \langle \nabla u, \nabla v\rangle\,d\mu, \qquad u, v \in C_b^2(\mathbb{R}^d)$$

in $L^2(\mu)$, where

$$\mu(dx) = \frac{1}{Z}\exp\left(\langle Ax, x\rangle\right)dx = \mathcal{N}(0, (-2A)^{-1})(dx). \tag{2.1.4}$$

Therefore μ is the unique invariant probability measure of X and it is moreover reversible.

Notice also that A can be diagonalized, i.e. there exists a matrix U such that $U^*U = UU^* = I$ and U^*AU =diag$(-\lambda_1, \dots, -\lambda_d)$. Setting $\hat{X} := U^*X$, $\hat{x} := U^*x$, $\hat{W} := U^*W$, we have that $\hat{W}$ has the same law as W and

$$\hat{X}_t(\hat{x}) = \hat{x} + \int_0^t \mathrm{diag}(-\lambda_1, \dots, -\lambda_d)\hat{X}_s(\hat{x})\,ds + \hat{W}_t, \qquad t \ge 0,\ \hat{x} \in \mathbb{R}^d.$$

In particular, setting $\hat{x}_t^i := \langle \hat{X}_t, e_i\rangle$ and $\hat{w}_t^i := \langle \hat{W}_t, e_i\rangle$, where (e_i) is a basis of $\mathbb{R}^d$ such that $U^*AUe_i = -\lambda_i e_i$, we obtain

$$\hat{x}_t^i = \hat{x}^i - \lambda_i \int_0^t \hat{x}_s^i\,ds + \hat{w}_t^i, \qquad t \ge 0,\ i = 1, \dots, d.$$

Since the $(\hat{w}^i)$ are independent, so are the $(\hat{x}^i)$. Therefore, X can be constructed as a linear function of a vector of independent one-dimensional O.-U. processes, each with invariant measure $\mathcal{N}(0, (2\lambda_i)^{-1})$.

Reflecting BM

We consider now the set $[0, \infty[$, endowed with the Lebesgue measure, and the form

$$D := C_c^1([0, \infty[), \qquad \mathcal{E}_{\mathrm{Neu}}(u, v) := \frac{1}{2}\int_0^\infty \langle \nabla u, \nabla v\rangle\,dx.$$

Notice that D has no "boundary condition" at 0. We want to prove closability in $L^2([0, \infty[)$. We notice that, as in the previous subsection, we can "embed" the form $(\mathcal{E}, D_{\mathrm{Neu}})$ in another closed form; indeed, we set

$$D_{\mathrm{even}} := \{f \in C_c(\mathbb{R}),\ f(-x) = f(x) \quad \forall\, x \in \mathbb{R}, \quad f_{|[0,\infty[} \in C_c^1([0, \infty[)\},$$

and

$$\mathcal{E}(u,v) := \frac{1}{2}\int_{\mathbb{R}} \langle \nabla u, \nabla v\rangle \, dx$$

noting that u' is continuous and bounded on $\mathbb{R}\backslash\{0\}$, with a jump at 0 if $\lim_{x\to 0+} u'(x) \neq 0$. If we endow $\mathbb{R}$ with the Lebesgue measure, then there is an isomorphism between $(\mathcal{E}_{\mathrm{Neu},1}, D)$ and $(\mathcal{E}_1, D_{\mathrm{even}})$. Since we know that $(\mathcal{E}_1, D_{\mathrm{even}})$ is closable by the example of the standard Brownian Motion, then $(\mathcal{E}_{\mathrm{Neu}}, D)$ is also closable. We obtain a Dirichlet form $(\mathcal{E}_{\mathrm{Neu}}, D(\mathcal{E}_{\mathrm{Neu}}))$.

We notice the important integration by parts formula:

$$\int_0^\infty \varphi'(x)\, dx = -\varphi(0), \qquad \forall\, \varphi \in C_c^1([0,\infty[),$$

which yields

$$\mathcal{E}_{\mathrm{Neu}}(u,v) = -\frac{1}{2}\int_0^\infty u''\, v\, dx - \frac{1}{2}\, u'(0)\, v(0), \qquad \forall\, u,v \in C_c^1([0,\infty[).$$

One can prove from this formula and Proposition 2.1.8 that

$$D(L_{\mathrm{Neu}}) = \{u \in H^2([0,\infty[) : u'(0) = 0\}, \qquad L_{\mathrm{Neu}} u = \frac{1}{2}\, u''.$$

The condition $u'(0) = 0$ is called a *homogeneous Neumann boundary condition*.

Set now $C_{b,\mathrm{even}}(\mathbb{R}) := \{f \in C_b(\mathbb{R}) : \ f(-x) = f(x) \quad \forall\, x \in \mathbb{R}\}$ and $H^1_{\mathrm{even}}(\mathbb{R}) := \{f \in H^1(\mathbb{R}) : \ f(-x) = f(x) \quad \forall\, x \in \mathbb{R}\}$ and notice that the definition makes sense, since $H^1(\mathbb{R}) \subset C(\mathbb{R})$, and therefore $H^1_{\mathrm{even}}(\mathbb{R}) = H^1(\mathbb{R}) \cap C_{b,\mathrm{even}}(\mathbb{R})$. It is easy to see that the semigroup of the Brownian Motion leaves $C_{b,\mathrm{even}}(\mathbb{R})$ invariant:

$$\begin{aligned} P_t^{BM} f(-x) &= \int \mathcal{N}(-x,t)(dy)\ f(y) = \int \mathcal{N}(0,t)(dy)\ f(y-x) \\ &= \int \mathcal{N}(0,t)(dy)\ f(-y+x) = \int \mathcal{N}(0,t)(dy)\ f(y+x) \\ &= \int \mathcal{N}(x,t)(dy)\ f(y) = P_t^{BM} f(x), \end{aligned}$$

for all $x \in \mathbb{R}$ and $f \in C_{b,\mathrm{even}}(\mathbb{R})$. In particular, the resolvent family of BM leaves invariant both $H^1(\mathbb{R})$ and $C_{b,\mathrm{even}}(\mathbb{R})$ and therefore their intersection $H^1_{\mathrm{even}}(\mathbb{R})$. We recall that for all $f \in L^2(\mathbb{R})$

$$\mathcal{E}_\lambda(R_\lambda^{BM} f, g) = \int_{\mathbb{R}} f\, g\, dx, \qquad \forall\, g \in H^1(\mathbb{R}),\ \lambda > 0.$$

We obtain, by restriction to $[0,\infty[$ and to $g \in C^1_{c,\mathrm{even}}(\mathbb{R})$, since

$$\int_{\mathbb{R}} u\, v\, dx = 2\int_0^\infty u\, v\, dx, \qquad \forall\, u,v \in C_c(\mathbb{R}) \cap C_{b,\mathrm{even}}(\mathbb{R}),$$

that

$$\mathcal{E}_{\mathrm{Neu},\lambda}(R_\lambda^{BM} f, g) = \int_0^\infty f\, g\, dx, \qquad \forall\, g \in C_c^1([0,\infty[),\ \lambda > 0.$$

In particular, by injectivity of the Laplace tranform, the semigroup (P_t^{Neu}) associated with $(\mathcal{E}_{\mathrm{Neu}}, D(\mathcal{E}_{\mathrm{Neu}}))$ has the representation

$$P_t^{\mathrm{Neu}} f(x) = \mathbb{E}[f(|x+B_t|)], \qquad \forall\, x \in [0,\infty[,\ t \geq 0,\ f \in C_c(\mathbb{R}).$$

In particular, $P_t^{\mathrm{Neu}} f$ is the restriction to $[0,\infty[$ of $P_t^{BM} \hat{f}$, where $\hat{f}(x) := f(|x|)$, $x \in \mathbb{R}$. Since $P_t^{BM} \hat{f}$ is smooth and even, we obtain that its derivative w.r.t. x at 0 is 0, i.e. $P_t^{\mathrm{Neu}} f$ satisfies the homogeneous Neumann boundary condition.

We have now the classical Skorohod Lemma, see e.g. [58, VI.2.1].

Lemma 2.1.15 (Skorohod)**.** *For any continuous function $a : [0,\infty[\mapsto \mathbb{R}$ such that $a(0) \geq 0$ there exists a unique pair (x,ℓ) of continuous functions from $[0,\infty[$ to $[0,\infty[$ such that*

1. *ℓ is monotone non-decreasing and $\ell(0) = 0$,*
2. *$\displaystyle\int_0^\infty x_t\, d\ell_t = 0$,*
3. *$x_t = a_t + \ell_t$, $t \geq 0$.*

Moreover, we have the explicit representation

$$\ell_t = \sup_{s \leq t} (a_s)^-, \qquad x_t = a_t + \sup_{s \leq t} (a_s)^-, \qquad t \geq 0.$$

We apply this lemma to $a_t := x + B_t$, where $x \geq 0$ and B is a standard BM. Then we obtain that there exists a unique pair of continuous processes $(X_t(x), L_t)_{t \geq 0}$ such that $X \geq 0$, $L_0 = 0$ and L is monotone non-decreasing, and

$$X_t(x) = x + B_t + L_t, \qquad \int_0^\infty X_t\, dL_t = 0.$$

This condition is equivalent to saying that the measure dL_t supported by the set $\{t : X_t(x) = 0\}$.

The process $X_t(x)$ is called the *reflecting BM at* 0. Indeed, as long as $X_t > 0$ we have $dL_t = 0$ and therefore $dX_t = dB_t$; however, when $X_t(x) = 0$, then dL_t can be non-zero and gives a kick to the process, keeping it positive. The condition $\displaystyle\int_0^\infty X_t\, dL_t = 0$ means that the reflection term dL_t acts only when the process hits the obstacle 0.

We now define for any $f \in C_b^2([0,\infty[)$: $u(t,x) := P_t^{\mathrm{Neu}} f(x)$. We know that u is smooth and satisfies the homogeneous Neumann boundary condition. We consider as usual the process $[0,T] \ni t \mapsto u(T-t, X_t(x))$ and we obtain by the Itô formula for semimartingales

$$\begin{aligned} u(T-t, X_t(x)) - u(T,x) = {} & \int_0^t \left(-\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \right) (T-s, X_s(x))\, ds \\ & + \int_0^t \frac{\partial u}{\partial x}(T-s, X_s(x))\, dB_s + \int_0^t \frac{\partial u}{\partial x}(T-s, X_s(x))\, dL_s. \end{aligned}$$

By the above considerations

$$\int_0^t \frac{\partial u}{\partial x}(T-s, X_s(x))\, dL_s = \int_0^t \frac{\partial u}{\partial x}(T-s, 0)\, dL_s = 0.$$

We obtain, taking expectation and letting $t = T$

$$u(T, x) = \mathbb{E}[f(X_t(x))].$$

Concerning the invariant measure of X, in this case we have that the constant function 1 does not belong to $L^2(0,\infty)$. However, the representation of the semigroup of X in terms of the semigroup of BM shows that the Lebesgue measure on $[0,\infty[$ is an invariant and reversible measure for X. It is interesting to notice that we have the representation

$$1_{[0,\infty[}(x)\, dx = e^{-U(x)}\, dx, \qquad U(x) := \begin{cases} 0, & x \geq 0, \\ +\infty, & x < 0 \end{cases}$$

The function U is discontinuous but convex.

2.1.5 Hunt process, definition and properties

This part deals with the construction of Hunt process for certain class of Dirichlet forms. Suppose now that H is a polish space (Lusin is possible) then we saw in the previous part each Dirichlet forms associated to a Markov process have its generator (resp. resolvent, and semi-group) which coincide with the generator (resp. resolvent, and semi-group) of the associated process. In this part we recall the existence of such process, following [27] and [46]. Let us start with some definition before stating the main theorem.

Definition 2.1.16. *Let $M := (\Omega, \mathcal{F}, (X_t)_{t\in[0,\infty]}, (\mathbb{P}_x)_{x\in H_\Delta})$ be a family of stochastic processes with state space $(H_\Delta, \mathcal{B}_\Delta)$. Where $(H, \mathcal{B}(H))$ is a polish space and Δ be an extra point, therefore we denote by $H_\Delta := H \cup \{\Delta\}$ and $\mathcal{B}_\Delta := \mathcal{B}(H) \cup \{b \cup \{\Delta\} : b \in \mathcal{B}(H)\}$. M is a Markov process on $(H, \mathcal{B}(H))$ if*

(i) For each $x \in H_\Delta$, $(\Omega, \mathcal{F}, (X_t)_{t\in[0,\infty]}, \mathbb{P}_x)$ is a stochastic process with state space $(H_\Delta, \mathcal{B}_\Delta)$ and time parameter set $[0, +\infty]$

(ii) the map $x \mapsto \mathbb{P}_x(X_t \in b)$ is $\mathcal{B}(H)$-measurable for each $t \geq 0$ and $b \in \mathcal{B}(H)$

(iii) There is an admissible filtration $(\mathcal{F}_t)_t$ such that

$$\mathbb{P}(X_{t+s} \in b | \mathcal{F}_t) = \mathbb{P}_{X_t}(X_s \in b), \mathbb{P}_x - as \tag{2.1.5}$$

for any $x \in H$, $s, t \geq 0$, and $b \in \mathcal{B}(H)$

(iv) $\mathbb{P}_\Delta(X_t = \Delta) = 1$, $t \geq 0$

(v) Finally, if $\mathbb{P}_x(X_0) = 1$ $\forall x \in H$, the Markov Process M is called normal

Definition 2.1.17. *The process M has the strong Markov property iff* *(i)* $(\mathcal{F}_t)_t$ *is right continuous*

(ii) $\mathbb{P}_\mu(X_{\sigma+s} \in b|\mathcal{F}_\sigma) = \mathbb{P}_{X_\sigma}(X_s \in b)$, $\mathbb{P}_\mu$ *a.s.* $\mu \in \mathcal{P}(H_\Delta)$, $b \in \mathcal{B}_\Delta$, $s \geq 0$ *for any stopping time* σ

Finally we have:

Definition 2.1.18. *M is a Hunt process iff M is a strong Markov process with* $(\mathcal{F}_t)_t$

(i) $X_\infty(w) = \Delta$, *for all* $w \in \Omega$

(ii) $X_t(w) = \Delta$, $\forall t \geq \xi(w)$, *where* $\xi(w) := \inf\{t \geq 0 | X_t(w) = \Delta\}$

(iii) for each $t \in [0, +\infty]$, *there is a map* θ_t *from* Ω *to* Ω *such that* $X_s(\theta_t(w)) = X_{t+s}(w)$, $s \geq 0$

(iv) for each $w \in \Omega$, *the sample path* $t \mapsto X_t(w)$ *is right continuous on* $[0, +\infty) \to H_\Delta$ *and has left limit on* $(0, +\infty) \to H_\Delta$

(v) M *is quasi-left continuous on* $(0, +\infty)$, *i.e. for any stopping time* $\sigma_n \uparrow \sigma$

$$\mathbb{P}_\mu(\lim_n X_{\sigma_n} = X_\sigma, \sigma < +\infty) = \mathbb{P}_\mu(\sigma < +\infty) \tag{2.1.6}$$

$\mu \in \mathcal{P}(H_\Delta)$

A normal strong Markov process satisfying only (iv) is called a *Right* process. If the condition (iv) of definition 2.1.18 and quasi-left are weakened by shortening the time interval $(0, +\infty)$ into $(0, \xi(w))$, then M is said to be a standard process. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E})$ a symmetric Dirichlet form with semigroup $(P_t)_t$, let M be a right process with transition semigroup $(p_t)_t$. So M is properly associated to $\mathcal{E}$ if for all measurable functions $f \in L^2(\gamma)$, $p_t f$ is $\mathcal{E}$-quasi-continuous measurable version of $P_t f$, for all $t > 0$. Sometimes it is the only tractable notion of uniqueness. So we can now state the fundamental theorem from [46].

Theorem 2.1.19. *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a quasi-regular symmetric Dirichlet form on $L^2(H, \gamma)$. Then there exists a Right (in fact a m-tight special standard see [46] p. 92) process M which is properly associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Moreover if H is locally compact and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a regular symmetric Dirichlet then M is a Hunt process.*

Let $(H^*, \mathcal{B}^*)$ be a measurable space and let $i : H \to H^*$ be a $\mathcal{B}/\mathcal{B}^*$-measurable map. Let $\mu^* = \mu \circ i^{-1}$ the image measure of μ under i, let $I : L^2(H^*, \mu^*) \to L^2(H, \mu)$ such that $I(u^*) = \tilde{u^*} \circ i$ where $\tilde{u^*}$ is a $\mathcal{B}^*$- measurable μ-version of u^*. Define

$$\begin{aligned} \mathcal{D}(\mathcal{E}^*) &:= \{u^* \in L^2(H^*, \mu^*) | I(u^*) \in \mathcal{D}(\mathcal{E})\} \\ \mathcal{E}^*(u^*, v^*) &:= \mathcal{E}(I(u^*), I(v^*)) \qquad u^*, \quad v^* \in \mathcal{D}(\mathcal{E}^*) \end{aligned} \tag{2.1.7}$$

Definition 2.1.20. *Let* $M = (\Omega, \mathcal{F}, \mathcal{F}_t, (\mathbb{P}_x)_{x \in H_\Delta})$ *be a process. We define its trivial extension* $\bar{M} = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, (\bar{\mathbb{P}}_x)_{x \in E_\Delta})$ *with state space* E *in such a way that each*

$x \in N := E \backslash S$ is a trap for M. More precisely, add N to Ω as an extra set and define

$$\begin{aligned}
\bar{\Omega} &:= \Omega \cup N \\
\bar{\mathcal{F}} &:= \{B \cup B_0 | B \in \mathcal{F}, B_0 \in \mathcal{B}(N)\} \\
\bar{\xi}(\omega) &:= \begin{cases} \xi(\omega, & \text{if } \omega \in \Omega \\ \\ +\infty, & \text{if} \omega \in N. \end{cases} \\
\bar{X}_t(\omega) &:= \begin{cases} X_t(\omega), & \text{if } \omega \in \Omega \\ \\ \omega, & \text{if } \omega \in N. \end{cases} \\
\bar{\mathbb{P}}_x(B) &:= \begin{cases} \mathbb{P}_x(B \cap \Omega), & \text{if } x \in H_\Delta \\ \\ \delta_x(B), & \text{if } x \in E \backslash H_\Delta. \end{cases}
\end{aligned}$$

Theorem 2.1.21. *Let $M = (\Omega, \mathcal{F}, X_t, \mathbb{P}_z, z \in H_\Delta)$ be a right process properly associated with the quasi-regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(H, \mu)$. Then there exists $N \subset E$, N $\mathcal{E}$-exceptional, such that $S := H \backslash N$ is M-invariant and if $\bar{M}$ is a trivial extension to $H^*$of M restricted to S, then $\bar{M}$ is a Hunt process properly associated with the regular Dirichlet form $\mathcal{E}^*, \mathcal{D}(\mathcal{E}^*))$ on $L^2(H^*, \mu^*)$, where H^*_Δ is taken as the one point compactification of H^*.*

The previous theorem extends a result of [27], where the existence of a Hunt process is proved in locally compact state spaces, for regular Dirichlet forms. This local compactification method is done in such way that we can transfert results obtained in locally compact framework to a more general situation.

2.1.6 The Lyons-Zheng decomposition

Let us first describe the heuristic idea. Let X be a reversible process in $\mathbb{R}^d$, i.e. for all $T > 0$, setting $\tilde{X}_t := X_{T-t}$, $(X_t)_{t\in[0,T]}$ and $(\tilde{X}_t)_{t\in[0,T]}$ have the same distribution. We suppose e.g. that X solves the SDE

$$dX_t = dW_t - \nabla U(X_t)\, dt,$$

where $U : \mathbb{R}^d \mapsto \mathbb{R}$ is a smooth function. Let f be in the domain of the infinitesimal generator $\mathcal{L}$ of X; then for $t \in [0, T]$:

$$f(X_t) = f(X_0) + \int_0^t \langle \nabla f(X_s), dW_s \rangle + \int_0^t \mathcal{L} f(X_s)\, ds$$

On the other hand by reversibility of X there is a Brownian motion $\tilde{W}$ such that

$$f(\tilde{X}_t) = f(\tilde{X}_0) + \int_0^t \langle \nabla f(\tilde{X}_s), d\tilde{W}_s \rangle + \int_0^t \mathcal{L} f(\tilde{X}_s)\, ds.$$

By computing $f(\tilde{X}_t)$ at time $T-t$ and T we obtain

$$
\begin{aligned}
f(X_t) &= f(X_T) + \int_0^{T-t} \langle \nabla f(\tilde{X}_s), d\tilde{W}_s \rangle + \int_0^{T-t} \mathcal{L} f(X_{T-s})\, ds, \\
f(X_0) &= f(X_T) + \int_0^{T} \langle \nabla f(\tilde{X}_s), d\tilde{W}_s \rangle + \int_0^{T} \mathcal{L} f(X_{T-s})\, ds.
\end{aligned}
$$

We denote by M (respectively $\tilde{M}$) the martingale with respect to the filtration $F_t := \sigma(X_s, s \leq t)$ (resp. $\tilde{F}_t := \sigma(\tilde{X}_s, s \leq t)$)

$$
M_t := \int_0^t \langle \nabla f(X_s), dW_s \rangle, \qquad \tilde{M}_t := \int_0^t \langle \nabla f(\tilde{X}_s), d\tilde{W}_s \rangle.
$$

We obtain by a subtraction

$$
\begin{aligned}
f(X_t) - f(X_0) &= -\int_{T-t}^{T} \mathcal{L} f(X_{T-s})\, ds + \tilde{M}_{T-t} - \tilde{M}_T \\
&= -\int_0^{t} \mathcal{L} f(X_s)\, ds + \tilde{M}_{T-t} - \tilde{M}_T
\end{aligned}
$$

so that

$$
f(X_t) - f(X_0) = \frac{1}{2}(M_t + \tilde{M}_{T-t} - \tilde{M}_T).
$$

This is the so-called Lyons-Zheng decomposition. This equality is still true when the function f is in the domain of the Dirichlet form of X, in particular we obtain that

$$
\langle X_t - X_0, h \rangle = \frac{1}{2}(M_t + \tilde{M}_{T-t} - \tilde{M}_T)
$$

where M and $\tilde{M}$ are martingales with quadratic variation:

$$
\langle M \rangle_t = \langle \tilde{M} \rangle_t = t\|h\|^2.
$$

This remark will be very useful later, since it will allow to obtain tightness estimates by using Bukholder-Davis-Gundy inequalities. Here is a general theorem for the Lyons-Zheng decomposition [27, Th. 5.7.1]:

Theorem 2.1.22. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form with strong local property on $L^2(H, \mu)$, we suppose moreover that the corresponding diffusion $(\Omega_T, F_T, X_t, \mathbb{P}_x)$ is conservative, i.e. $p_t 1 = 1$ for all $x \in H$. Then for all $u \in \mathcal{F}_{loc}$ there is a martingale $M^{[u]}$ such that*

$$
\tilde{u}(X_t) - \tilde{u}(X_0) = \frac{1}{2}(M_t^{[u]} + M_{T-t}^{[u]} \circ r_T - M_T^{[u]} \circ r_T) \tag{2.1.8}
$$

where r_T is the time reversal operator and $\tilde{u}$ a quasi-continuous version of u

2.2 White noises

2.2.1 Gaussian measures

Let $a \in \mathbb{R}^d$ and Q a symmetric $d \times d$ matrix with positive eigenvalues. Then we can define $A := -\dfrac{1}{2}Q^{-1}$, a symmetric matrix with negative eigenvalues. The Gaussian probability measure $\mathcal{N}(a, Q)$ on $\mathbb{R}^d$ is defined by

$$\begin{aligned}\mathcal{N}(a,Q)(dx) &= \frac{1}{\sqrt{(2\pi)^d \det Q}} \exp\left(-\frac{1}{2}\langle Q^{-1}(x-a), x-a\rangle\right) dx \\ &= \frac{1}{Z}\exp\left(\langle A(x-a), x-a\rangle\right) dx\end{aligned}$$

and we have the important formula for the Fourier transform of $\mathcal{N}(a, Q)$: if $X \sim \mathcal{N}(a, Q)$ then

$$\mathbb{E}\left(e^{i\langle X,h\rangle}\right) = \int e^{i\langle x,h\rangle}\,\mathcal{N}(a,Q)(dx) = \exp\left(i\langle a,h\rangle - \frac{1}{2}\langle Qh,h\rangle\right) \tag{2.2.1}$$

from which, by derivation, one obtains for all $h, k \in \mathbb{R}^d$

$$\mathbb{E}(\langle X,h\rangle) = \langle a,h\rangle, \qquad \mathrm{Cov}(\langle X,h\rangle,\langle X,k\rangle) = \mathbb{E}(\langle X-a,h\rangle\langle X-a,k\rangle) = \langle Qh,k\rangle.$$

Then a is the mean and Q the covariance operator of X. Moreover one obtains the main property of a Gaussian family $\{X_i\}_{i\in I}$: if $J, K \subset I$, $J \cap I = \emptyset$, and $\mathrm{Cov}(X_j, K_k)$ for all $j \in J$ and $k \in K$, then $\{X_j\}_{j\in J}$ and $\{X_k\}_{k\in K}$ are independent. In particular, if $\{e_1, \ldots, e_d\}$ are eigenfunctions of Q with respective eigenvalues $\{\lambda_1, \ldots, \lambda_d\}$, then the variables $\{\langle X, e_j\rangle, j = 1, \ldots, n\}$ are independent and $\langle X, e_j\rangle \sim \mathcal{N}(\langle a, e_j\rangle, \lambda_j)$.

In the same way, let $(H, \langle .,.\rangle)$ be a separable Hilbert, and Q be a symmetric non negative trace-class operator. Let $\mathcal{B}(H)$ the Borel set of H, i.e. the smallest σ-field of subsets of H containing all set of the form $\{x \in H | \langle x, y\rangle \leq \alpha\}$ with $y \in H$ and $\alpha \in \mathbb{R}$. Then the Bochner theorem provides the existence of a probability measure $\mathbb{P}$ whose Fourier transform is given by (2.2.1) where a is called the mean, Q is a trace class operator, called the covariance operator.

2.2.2 White noises

Let $\mathcal{H}$ be a separable Hilbert space.

Proposition 2.2.1. *There exists a process $(W(h), h \in \mathcal{H})$ such that $h \mapsto W(h)$ is linear, $W(h)$ is a centered real Gaussian random variable and*

$$\mathbb{E}(W(h)\,W(k)) = \langle h,k\rangle_{\mathcal{H}}, \qquad \forall\ h,k \in \mathcal{H}.$$

Proof. Let $(Z_i)_i$ be a i.i.d. sequence of real standard Gaussian variables and $(h_i)_i$ a complete orthonormal system in $\mathcal{H}$ and set

$$W^n(h) := \sum_{i=1}^{n} \langle h, h_i\rangle_{\mathcal{H}}\, Z_i, \qquad h \in \mathcal{H}.$$

Then it is easy to see that

$$\mathbb{E}(W^n(h)\, W^n(k)) = \sum_{i=1}^{n} \langle h, h_i\rangle_{\mathcal{H}} \langle k, h_i\rangle_{\mathcal{H}}, \qquad \forall\, h, k \in \mathcal{H}.$$

Moreover for all $n < m$

$$\mathbb{E}\left((W^n(h) - W^m(h))^2\right) = \sum_{i=n+1}^{m} \langle h, h_i\rangle^2 \to 0$$

as $n, m \to +\infty$. The conclusion is standard. $\square$

Notice that the application $\mathcal{H} \ni h \mapsto W(h)$ is an isomorphism of Hilbert spaces between $\mathcal{H}$ and a space of Gaussian random variables.

Let now $(T, \mathcal{B}, m)$ be a separable measurable space, with m a σ-finite measure without atoms. We apply Proposition 2.2.1 to $\mathcal{H} := L^2(T, \mathcal{B}, m)$. The process $(W(h), h \in \mathcal{H})$ is called a *Gaussian white noise* over $(T, \mathcal{B}, m)$.

If $A \in \mathcal{B}$ and $m(A) < +\infty$, then $1_A \in \mathcal{H}$ and we denote $W(A) := W(1_A)$. If $A, B \in \mathcal{B}$ with $m(A) + m(B) < +\infty$ then

$$\mathbb{E}(W(A)\, W(B)) = m(A \cap B).$$

In particular, if $m(A \cap B) = 0$, then $\{W(A'), A' \subseteq A\}$ and $\{W(B'), B' \subseteq B\}$ are independent.

It is customary to use the notation

$$W(A) = \int_A W(dt), \qquad W(h) = \int_T h(t)\, W(dt).$$

We have the important property, which follows immediately from the fact that $W : L^2(T, m) \mapsto L^2(\Omega)$ is an isometry:

Proposition 2.2.2. *If $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ is such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $m(\cup_n A_n) < +\infty$, then*

$$\lim_{n\to+\infty} \sum_{i=0}^{n} W(A_i) = W(\cup_n A_n) \qquad \text{in } L^2.$$

Moreover, since $A_i \cap A_j = \emptyset$ for $i \neq j$, then the sequence $(W(A_n))_n$ is independent and, since all variables $W(A_n)$ are centered, the sequence is orthogonal in L^2.

Notice however that $W(dt)$ is, in general, *not* a signed measure, as this notation might suggest; indeed, the process $h \mapsto W(h)$ does not always admit a modification such that $W(h)$ is defined on the same set of probability 1 for all h: see Remark 2.2.3 below; however, it is possible to interpret $W(dt)$ as a (random) distribution in the sense of Schwarz, see subsection 2.2.10.

2.2.3 Finite dimensional white noise

Let us consider first the easiest case: $T = \{1, \dots, d\}$ and m is the counting measure. In this case $L^2(T, \mathcal{B}, m) = \mathbb{R}^d$ and the white noise $W(h)$ can be realized as $W(h) = \langle W, h\rangle_{\mathbb{R}^d}$, where $W \sim \mathcal{N}(0, I)$.

2.2.4 Brownian motion

Let now $T = \mathbb{R}$ endowed with the Borel σ-algebra and the Lebesgue measure λ_1. Then for any choice of two intervals $[a, b]$ and $[c, d]$ in $\mathbb{R}$

$$\mathbb{E}(W([a,b])\, W([c,d])) = \lambda_1([a,b] \cap [c,d]).$$

Then the process

$$\hat{W}_t := \begin{cases} W([0,t]), & t \geq 0, \\ \\ W([t,0]), & t < 0. \end{cases}$$

is a two-sided standard Brownian motion, and $W(dt)$ is simply called *white noise* over $\mathbb{R}$. In particular, the process $(W([0,t]),\, t \geq 0)$ is a standard BM.

Remark 2.2.3. In this case we can see very clearly why in general a white noise can no be written as a signed measure. If was the case, this would imply that the BM $(W([0,t]),\, t \geq 0)$ has a.s. paths with bounded variation, which is notoriously false.

2.2.5 Multi-dimensional Brownian motion

Let now $T = \mathbb{R} \times \{1, \ldots, d\}$ endowed with the Borel σ-algebra and the measure $\lambda_1 \otimes m$ where m is the counting measure.

Then for any choice of two intervals $[a, b]$ and $[c, d]$ in $\mathbb{R}$ and for any $i, j \in \{1, \ldots, d\}$

$$\mathbb{E}(W([a,b] \times \{i\})\, W([c,d] \times \{j\})) = \lambda_1([a,b] \cap [c,d])\, \mathbb{1}_{(i=j)}.$$

Then the process $(\hat{W}^1_t, \ldots, \hat{W}^d_t)$, defined by

$$\hat{W}^i_t := \begin{cases} W([0,t] \times \{i\}), & t \geq 0, \\ \\ W([t,0] \times \{i\}), & t < 0. \end{cases}$$

is a two-sided standard Brownian motion and $W(dt)$ is simply called *white noise* over $\mathbb{R}$. In particular, the process $(W^1([0,t]), \ldots, W^d([0,t]))_{t \geq 0}$ is a standard BM in $\mathbb{R}^d$.

2.2.6 Brownian sheet

If $T = \mathbb{R}^2$ endowed with the Borel σ-algebra and the Lebesgue measure λ_2, then

$$\mathbb{E}(W([0,t] \times [0,t'])\; W([0,s] \times [0,s'])) = (t \wedge t')\; (s \wedge s'), \qquad t, t', s, s' \geq 0.$$

The process $(W(t,s) := W([0,t] \times [0,s]), t, s \geq 0)$ is called a *Brownian sheet* and $W(dt, ds)$ a *space-time white noise*. One can also use the notations

$$W(dt, ds) = \frac{\partial^2 W}{\partial t \partial s} = \dot{W}(t,s).$$

Notice that the same construction can be done if $T = \mathbb{R}^d$: this gives a space-time white noise with a d-dimensional space variable.

2.2.7 Cylindrical Brownian motion

Let H be any separable Hilbert space and $(e_i)_{i\geq 1}$ a complete orthonormal basis of H. Let us consider a sequence of independent standard real Brownian motions $(w_t^i, t \geq 0)_i$. We set for all $n \in \mathbb{N}$:

$$W_t^n := \sum_{i=1}^{n} w_t^i \, e_i, \qquad \langle W_t^n, h \rangle = \sum_{i=1}^{n} w_t^i \, \langle h, e_i \rangle, \qquad t \geq 0.$$

Now, for all $h \in H$ we have for $n < m$

$$\mathbb{E}\left((\langle W_t^n, h\rangle - \langle W_t^m, h\rangle)^2 \right) = \mathbb{E}\left(\left(\sum_{i=n+1}^{m} w_t^i \, \langle h, e_i \rangle \right)^2 \right) \leq \sum_{i=n+1}^{m} \langle h, e_i \rangle^2 \to 0$$

as $n, m \to \infty$, since $\displaystyle\sum_i \langle h, e_i\rangle^2 < +\infty$. Therefore, for all $t \geq 0$ the series

$$\langle W_t, h \rangle = \sum_{i=1}^{\infty} w_t^i \, \langle h, e_i \rangle.$$

converges in $L^2(\mathbb{P})$. Notice that for all $h, k \in H$ and $s, t \geq 0$ we have

$$\begin{aligned} \mathbb{E}\left(\langle W_t, h \rangle \, \langle W_s, k \rangle \right) &= \mathbb{E}\left(\sum_{i,j=1}^{\infty} w_t^i \, w_s^j \, \langle h, e_i \rangle \, \langle k, e_j \rangle \right) = \sum_{i=1}^{\infty} \mathbb{E}\left(w_t^i \, w_s^i \right) \, \langle h, e_i \rangle \, \langle k, e_i \rangle \\ &= t \wedge s \, \langle h, k \rangle. \end{aligned} \tag{2.2.2}$$

Formally, the series

$$W_t := \sum_{i=1}^{\infty} \langle W_t, e_i \rangle \, e_i = \sum_{i=1}^{\infty} w_t^i \, e_i, \qquad t \geq 0$$

defines a Brownian motion in H. However, this series does not define a H-valued variable. In fact, it can be seen that $\mathbb{P}(W_t \in H) = 0$; one easily notes that

$$\mathbb{E}\left(\|W_t\|_H^2 \right) = \sum_{i=1}^{\infty} \mathbb{E}\left((w_t^i)^2 \right) = \sum_{i=1}^{\infty} t = +\infty, \quad t > 0.$$

Since W_t is not well defined in H, but $\langle W_t, h \rangle$ is for all $h \in H$, the process $(\langle W_t, h \rangle, h \in H)$ is called a *cylindrical Brownian motion*.

2.2.8 Fourier construction of space-time white noise

Let now $H := L^2(0,1)$. Then (2.2.2) becomes

$$\mathbb{E}\left(\langle W_s, h \rangle \, \langle W_t, k \rangle \right) = s \wedge t \int_0^1 h_x \, k_x \, dx, \qquad \forall \, h, k \in H.$$

In particular, if $1_{[0,y]}$ and $1_{[0,z]}$ denote the indicator functions of two intervals $[0,y]$ and respectively $[0,z]$ in $[0,1]$, then for all $s,t \geq 0$

$$\mathbb{E}\left(\langle W_s, 1_{[0,y]}\rangle \langle W_t, 1_{[0,z]}\rangle\right) = s \wedge t \,\langle 1_{[0,y]}, 1_{[0,z]}\rangle = (s \wedge t)\ (y \wedge z).$$

Therefore, the process $(\langle W_t, 1_{[0,s]}\rangle, t, s \geq 0)$ is a Brownian sheet. Therefore we can use the representation in terms of the space-time white noise:

$$\langle W_t, h\rangle = \int_0^t \int_0^1 h(x)\, W(ds, dx).$$

2.2.9 A physicist's description

Let us start from the white noise in 1 dimension. If $(W_t, t \geq 0)$ is a standard real BM, then the classical formula

$$\mathbb{E}(W_t\, W_s) = t \wedge s, \qquad t, s \geq 0,$$

can be interpreted by saying that

$$\mathbb{E}\left(\dot{W}_t\, \dot{W}_s\right) = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\, t \wedge s = \frac{\partial}{\partial t}\, 1_{[s,+\infty[}(t) = \delta(t-s)$$

where $\delta(t)$ is the Dirac mass at 0. Since $\delta(t-s) = 0$ if $t \neq s$ and $(\dot{W}_t, t \geq 0)$ is a Gaussian process, then $\dot{W}_t$ and $\dot{W}_s$ are independent for $t \neq s$. In the case of the Brownian sheet, we have analogously

$$\mathbb{E}(W(t,s)\, W(t',s')) = (t \wedge t')\,(s \wedge s'), \qquad t, t', s, s' \geq 0,$$

and therefore

$$\mathbb{E}(W(dt, ds)\, W(dt', ds')) = \mathbb{E}\left(\dot{W}(t,s)\, \dot{W}(t',s')\right) = \delta(t-t')\,\delta(s-s').$$

Then, $\dot{W}(t,s)$ and $\dot{W}(t',s')$ are independent, unless $(t,s) = (t',s')$.

The Fourier representation of the space-time white noise reads

$$\dot{W}(t,x) := \frac{\partial}{\partial t}\, W_t(x) = \sum_{i=1}^{\infty} \frac{dw_t^i}{dt}\, e_i(x),$$

where $(e_i)_i$ is *any* complete orthonormal system in $L^2(\mathbb{R}_+, dx)$.

2.2.10 Random distribution

Another possible interpretation of the white noise on $\mathbb{R}^d$ is the random distribution viewpoint. Notice first that the covariance structure implies, e.g. if $s \leq s'$, that

$$\mathbb{E}(|W(t,s) - W(t',s')|^2) = ts + t's' - 2(t \wedge t')\,(s \wedge s') = |t-t'|s + t'|s-s'|.$$

Since $(W(t,s) - W(t',s'))$ is a Gaussian r.v. then there exists a constant $C_{m,T}$ such that

$$\mathbb{E}(|W(t,s) - W(t',s')|^{2m}) \leq C_{m,T}(|t-t'|^m + |s-s'|^m), \qquad \forall\, t, t', s, s' \in [0,T].$$

Therefore by the Kolmogorov criterion the process $(W(s,t), s,t \geq 0)$ has an a.s. continuous modification. The same holds for $(W(t_1,\dots,t_d), t_1,\dots,t_d \geq 0)$. Now, if $\varphi \in C_c^\infty(\mathbb{R}^d)$, then

$$W(\varphi) = \int_{\mathbb{R}^d} \varphi(x)\, W(dx) = (-1)^d \int_{\mathbb{R}^d} \frac{\partial^d \varphi}{\partial x_1 \cdots \partial x_d}(x)\, W([0,x])\, dx,$$

where $[0,x] := [0,x_1] \times \cdots \times [0,x_d]$. This expression gives a *measurable modification* $C_c^\infty(\mathbb{R}^d) \ni \varphi \mapsto W(\varphi)(\omega)$, for $\mathbb{P}$-a.e. ω, of the white noise.

2.3 The stochastic heat equation

Let us now give some results on the stochastic heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \dot{W}, \\[2ex] u(t,0) = u(t,1) = 0 \\[2ex] u(0,x) = u_0(x), \qquad x \in [0,1] \end{cases} \tag{2.3.1}$$

which has already appeared in (4.1.4) above. We recall that $\dot{W}(t,x)$ is a space-time white-noise as in section 2.2.6 above.

For the general theory of SPDEs, see [15, 16].

2.3.1 The deterministic heat equation

Let us start from the heat equation without noise:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2}, \\[2ex] v(t,0) = v(t,1) = 0 \\[2ex] v(0,x) = v_0(x), \qquad x \in [0,1] \end{cases} \tag{2.3.2}$$

where $v_0 \in L^2(0,1)$. We set for all $k \geq 1$:

$$e_k(x) := \sqrt{2}\, \sin(k\pi x), \qquad x \in [0,1]. \tag{2.3.3}$$

We recall that $\{e_k\}_{k\geq 1}$ is a complete orthonormal basis of $L^2(0,1)$. Notice that $\{e_k\}_{k\geq 1}$ is a complete basis of eigenvectors of the second derivative with homogeneous Dirichlet boundary conditions:

$$\frac{d^2}{dx^2} e_k = -(\pi k)^2 e_k, \qquad e_k(0) = e_k(1) = 0, \qquad k \geq 1.$$

Setting

$$D(A) = \left\{ f \in L^2(0,1) : \sum_{k\geq 1} k^4 \langle e_k, f\rangle^2_{L^2(0,1)} < +\infty \right\}.$$

$$Af = -\sum_{k\geq 1} \frac{(k\pi)^2}{2} \langle e_k, f\rangle \, e_k, \qquad f \in \ D(A),$$

we obtain a closed operator in $L^2(0,1)$ extending $\dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}$ on $C^2_c(0,1)$. The solution of the heat equation (2.3.2) is therefore

$$v(t,x) = \sum_{k\geq 1} e^{-t\frac{(k\pi)^2}{2}} \langle e_k, v_0\rangle \, e_k(x), \qquad t > 0, \ x \in [0,1].$$

Since $|e_k(x)| \leq \sqrt{2}$ and $\sum_{k\geq 1} e^{-t\frac{(k\pi)^2}{2}} k^m < +\infty$ for all $m \in \mathbb{N}$, the above series converges uniformly on $[0,1]$ together with all its partial derivatives in t and x. One can write more compactly, using the semigroup notation,

$$v_t := v(t,\cdot) = e^{tA} v_0, \qquad t \geq 0.$$

2.3.2 Fourier expansion of (2.3.1)

Let us consider the scalar product of both terms of (2.3.1) and e_k. Setting $u^k_t := \langle u(t,\cdot), e_k\rangle$ we obtain

$$\begin{cases} du^k = -\dfrac{(k\pi)^2}{2}\, u^k \, dt + dW^k_t, \\ u^k_0 = \langle u_0, e_k\rangle \end{cases}$$

where

$$W^k_t := \int_{[0,t]\times[0,1]} e_k(x)\, W(ds,dx).$$

It is easy to see that $(W^k_t, t \geq 0)_{k\geq 1}$ is an independent sequence of Brownian motions.

Setting $\lambda_k = (\pi k)^2/2$, we obtain that $(u^k_t, t \geq 0)_{k\geq 1}$ is an independent family of Ornstein-Uhlenbeck processes, i.e.

$$u^k_t = e^{-\lambda_k t} u^k_0 + \int_0^t e^{-\lambda_k (t-s)}\, dW^k_s, \qquad t \geq 0, \tag{2.3.4}$$

or, equivalently,

$$u^k_t = u^k_0 - \lambda_k \int_0^t u^k_s \, ds + W^k_t, \qquad t \geq 0, \tag{2.3.5}$$

An important remark is the following:

$$\sum_k \frac{1}{\lambda_k} = \sum_k \frac{2}{(\pi k)^2} < +\infty.$$

Since $u^k_t \sim \mathcal{N}\left(e^{-\lambda_k t} u^k_0, \dfrac{1}{2\lambda_k}(1 - e^{-2\lambda_k t})\right)$, then

$$\mathbb{E}\left(\left\| \sum_{k=n+1}^m u^k_t \, e_k \right\|^2 \right) = \sum_{k=n+1}^m \left[e^{-2\lambda_k t} \left(u^k_0\right)^2 + \frac{1}{2\lambda_k}(1 - e^{-2\lambda_k t})\right] \to 0$$

as $n, m \to +\infty$. Therefore the series

$$u_t := \sum_{k=1}^{+\infty} u_t^k \, e_k$$

converges in $L^2(\Omega; L^2(0,1))$ to a well-defined r.v. u_t taking values in $L^2(0,1)$. Formula (2.3.4) becomes

$$u_t = e^{tA} u_0 + \int_0^t e^{(t-s)A} \, dW_s, \qquad u_0 \in L^2(0,1), \; t \geq 0, \tag{2.3.6}$$

while formula (2.3.5) becomes

$$\langle u_t, h \rangle = \langle u_0, h \rangle + \frac{1}{2} \int_0^t \langle u_s, h'' \rangle \, ds + \langle W_t, h \rangle, \qquad t \geq 0, \; h \in D(A), \tag{2.3.7}$$

which can be interpreted as a weak formulation of

$$du = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \, dt + dW.$$

2.3.3 Path continuity

Until now we have considered u_t as a $L^2(0,1)$-valued random variable. In fact, almost sure continuity of $(u(t,x) : t \geq 0, x \in [0,1])$ is a fundamental property of SPDEs with one space-dimension and space-time white noise; in particular this is crucial for the definition of SPDEs with reflection in the following chapters.

We start by noting that if $x \in [0,1]$ is fixed, then

$$u_n(t,x) := \sum_{k=1}^{n} u_t^k \, e_k(x) = \sum_{k=1}^{n} \langle u_t, e_k \rangle \, e_k(x) \in \mathbb{R}$$

is well defined for all $n \geq 1$, $t \geq 0$. Let us suppose that $u_0 = 0$, so that $u_0^k := 0$ for all $k \geq 1$. Then

$$\begin{aligned} \mathbb{E}\left(|u_n(t,x) - u_m(t,x)|^2\right) &= \sum_{k=n+1}^{m} \mathbb{E}\left(\left(u_t^k\right)^2\right) e_k^2(x) \leq \sum_{k=n+1}^{m} \int_0^t e^{-2\lambda_k(t-s)} \, ds \\ &= \sum_{k=n+1}^{m} \frac{1 - e^{-2\lambda_k t}}{2\lambda_k} \to 0 \end{aligned}$$

as $n, m \to +\infty$. Therefore, there exists a well defined stochastic process $(u(t,x), t \geq 0, x \in [0,1])$, limit in $L^2(\mathbb{P})$ of $(u_n(t,x), t \geq 0, x \in [0,1])$ as $n \to \infty$.

Notice that $(u(t,x) - u(s,y))$ is a real Gaussian variable with 0 mean; in order to estimate its moments, it is enough to compute the second one, i.e. it is enough to prove that for some constant C

$$\mathbb{E}\left(|u(t,x) - u(s,y)|^2\right) \leq C \left(|t-s|^{1/2} + |x-y|\right), \quad \forall \, t, s \geq 0, \quad x, y \in [0,1].$$

Since $(u(t,x)-u(s,y))$ is the limit in $L^2(\mathbb{P})$ of $(u_n(t,x)-u_n(s,y))$ as $n\to\infty$, then it is enough to estimate the variance of $(u_n(t,x)-u_n(s,y))$ uniformly in n. First we have

$$|u_n(t,x)-u_n(s,y)|^2 \le 2\,|u_n(t,x)-u_n(s,x)|^2 + 2\,|u_n(s,x)-u_n(s,y)|^2.$$

Now:

$$\begin{aligned}
\mathbb{E}\left(|u_n(s,x)-u_n(s,y)|^2\right) &= \mathbb{E}\left(\left|\sum_{k=1}^n u_s^k\,(e_k(x)-e_k(y))\right|^2\right) \\
&= \sum_{k=1}^n \frac{1-e^{-2\lambda_k s}}{2\lambda_k}(e_k(x)-e_k(y))^2 \le \sum_{k=1}^n \frac{1\wedge(|x-y|\,k)^2}{k^2} \\
&\le 1\wedge|x-y| + \int_1^\infty \frac{1\wedge(|x-y|\,k)^2}{k^2}\,dk \le 3\,|x-y|.
\end{aligned}$$

With similar computations:

$$\begin{aligned}
&\mathbb{E}\left(|u_n(t,x)-u_n(s,x)|^2\right) \\
&= \sum_{k=1}^n e_k^2(x)\left[(1-e^{-\lambda_k(t-s)})^2\frac{1-e^{-2\lambda_k s}}{2\lambda_k} + e^{-2\lambda_k s}\frac{1-e^{-2\lambda_k(t-s)}}{2\lambda_k}\right] \\
&\le 2\sum_{k=1}^n \frac{1\wedge(|t-s|\,k^2)}{k^2} \le 2\left(1\wedge|t-s| + \int_1^\infty \frac{1\wedge(|t-s|\,k^2)}{k^2}\,dk\right) \\
&\le 6\sqrt{|t-s|}.
\end{aligned}$$

We have used the fact that

$$(1-e^{-\lambda_k(t-s)})^2 \le (1\wedge[\lambda_k(t-s)])^2 \le 1\wedge[\lambda_k(t-s)].$$

Passing to the limit in $n\to\infty$, we have.

Lemma 2.3.1. *For all $m\in\mathbb{N}$ there exists a constant $C_m<+\infty$ such that*

$$\mathbb{E}\left(|u(t,x)-u(s,y)|^{2m}\right) \le C_m\left(|t-s|^{m/2}+|x-y|^m\right), \quad \forall\, t,s\ge 0, \quad x,y\in[0,1].$$

By the Kolmogorov criterion, we obtain that there exists an a.s. continuous modification of v, that we call again v, such that in particular for all $\varepsilon\in\,]0,1[$ and $T<+\infty$

$$\sup_{x,y\in[0,1],\,t,s\in[0,T]} \frac{|u(t,x)-u(s,y)|}{|t-s|^{\frac{1-\varepsilon}{4}}+|x-y|^{\frac{1-\varepsilon}{2}}} < +\infty, \qquad \text{a.s.}$$

Proposition 2.3.2. *There exists an a.s. continuous stochastic process $(u(t,x),t\ge 0,x\in[0,1])$ such that for all $t\ge 0$, $u(t,\cdot)=u_t$ in $L^2(0,1)$, a.s.*

Finally, continuity of $u_0=(u_0(x),x\in[0,1])$ implies continuity of $(e^{tA}u_0(x),t\ge 0,x\in[0,1])$, while if u_0 is merely in $L^2(0,1)$, then we have continuity of $(e^{tA}u_0(x),t>0,x\in[0,1])$.

2.3.4 The invariant measure

Since u_t can be written as a sequence of independent O.U. processes, it is easy to extend properties from the single processes to u_t. For instance, the unique probability invariant measure of the sequence is necessarily $\otimes_{k=1}^{+\infty} \mu_{\lambda_k}$, which means that the only probability invariant measure of $(u_t)_{t\geq 0}$ is the distribution of

$$\beta := \sum_{k=1}^{+\infty} \frac{1}{\pi k} \, e_k \, Z_k \in L^2(0,1),$$

where $(Z_k)_{k\geq 1}$ is an i.i.d. sequence of $\mathcal{N}(0,1)$ variables. Notice that

$$\mathbb{E}\left(\beta_x \, \beta_y\right) = \sum_{k=1}^{+\infty} \frac{1}{(\pi k)^2} \, e_k(x) \, e_k(y), \qquad x, y \in [0,1].$$

Is it possible to compute explicitly this covariance function? Notice that for $h \in L^2(0,1)$ we have

$$f := (-2A)^{-1} h = \sum_{k=1}^{+\infty} \frac{1}{2\lambda_k} \langle h, e_k \rangle \, e_k = \sum_{k=1}^{+\infty} \frac{1}{(\pi k)^2} \langle h, e_k \rangle \, e_k.$$

Moreover, f is the (unique) solution of the equation

$$\begin{cases} -\dfrac{d^2 f}{dx^2} = h, \\ \\ f(0) = f(1) = 0, \end{cases}$$

A trite computation yields

$$f(x) = \int_0^1 (x \wedge y - xy) h(y) \, dy, \qquad x \in [0,1].$$

Therefore

$$\mathbb{E}\left(\beta_x \, \beta_y\right) = \sum_{k=1}^{+\infty} \frac{1}{(\pi k)^2} \, e_k(x) \, e_k(y) = x \wedge y - xy, \qquad x, y \in [0,1],$$

which is the covariance function of a Brownian bridge on $[0,1]$: $\beta_x := B_x - xB_1$, where B is a BM.

2.3.5 The Dirichlet form

Proposition 2.3.3. *The solution of* (2.3.1) *is the Markov process in* $L^2(0,1)$ *associated with the Dirichlet form closure of*

$$\mathcal{E}(u,v) = \frac{1}{2} \int_H \langle \nabla u, \nabla v \rangle_H \, d\mu, \qquad u, v \in C_b^1(H).$$

It is well-known that the invariant measure μ is the law of a Brownian bridge $(\beta_x)_{x\in[0,1]}$; moreover the covariance function

$$\mathbb{E}(\beta_x\,\beta_y) = x\wedge y - xy, \qquad x,y\in[0,1]$$

is associated with the elliptic equation

$$\begin{cases} -\dfrac{d^2 f}{dx^2} = h, \\ \\ f(0)=f(1)=0, \end{cases}$$

since we have $f(x) = \displaystyle\int_0^1 (x\wedge y - xy)\,h(y)\,dy$. Recall now the explicit formula (2.1.4) for the density of a Gaussian measure in $\mathbb{R}^d$ with mean 0 and covariance $Q=(-2A)^{-1}$. In our case, we have just proven that $Qh=f$ and therefore

$$-2Af = -\frac{d^2 f}{dx^2}, \qquad \langle Af, f\rangle = -\frac{1}{2}\int_0^1 (f')^2 dx, \qquad f(0)=f(1)=0.$$

This yields the formal expression for the law of the invariant measure of (2.3.1)

$$\mu(df) = \frac{1}{2}\exp(-U(f))\,df, \qquad U(f) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 (f')^2 dx, & \text{if } f(0)=f(1)=0, \\ \\ +\infty & \text{otherwise} \end{cases}$$

It is easy to see that

$$\langle \nabla U(f), \alpha\rangle = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(U(f+\varepsilon\alpha)-U(f)) = \int_0^1 f'\alpha'\,dx = -\int_0^1 f''\alpha\,dx = \langle -f'', \alpha\rangle$$

i.e. $\nabla U(f) = -f''$ and equation (2.3.1) can be written as a gradient system in $L^2(0,1)$

$$du = -\frac{1}{2}\nabla U(u)\,dt + dW.$$

2.3.6 The stochastic convolution

There are a lot of ways to define the stochastic convolution see [61] The process Z defined in (2.3.6) above is called the *stochastic convolution*. We recall that the function

$$g_t(x,y) := \sum_{k=1}^{\infty} e^{-t(\pi k)^2/2}\, e_k(x)\, e_k(y), \qquad t>0,\ x,y\in[0,1], \tag{2.3.8}$$

where e_k is as in (2.3.3), is C^∞ for $(t,x,y)\in\,]0,+\infty[\times[0,1]\times[0,1]$, and is called the *fundamental solution* of the heat equation on $[0,1]$ with Dirichlet boundary condition. Indeed, g satisfies

$$\begin{cases} \dfrac{\partial g}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 g}{\partial x^2}, \\ \\ g_t(0,y) = g_t(1,y) = 0, \\ \\ g_0(x,y) = \delta(x-y). \end{cases} \tag{2.3.9}$$

Indeed, for all $t > 0$ the series in (2.3.8) converges uniformly with all partial derivatives w.r.t. x and by (2.3.3), for $t > 0$

$$\begin{aligned} \frac{\partial g}{\partial t}(t,x) &= \frac{\partial}{\partial t}\sum_{k=1}^{\infty} e^{-t(\pi k)^2/2}\, e_k(x)\, e_k(y) = -\sum_{k=1}^{\infty} \frac{(\pi k)^2}{2}\, e^{-t(\pi k)^2/2}\, e_k(x)\, e_k(y) \\ &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\sum_{k=1}^{\infty} e^{-t(\pi k)^2/2}\, e_k(x)\, e_k(y) = \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x) \end{aligned}$$

and for $t = 0$ and any $f, g \in L^2(0,1)$

$$\begin{aligned} &\int_{[0,1]^2} f(y)\, g(x)\, g_0(x,y)\, dx\, dy = \int_{[0,1]^2} f(y)\, g(x)\, \sum_{k=1}^{\infty} e_k(x)\, e_k(y)\, dx\, dy \\ &= \sum_{k=1}^{\infty} \langle g, e_k\rangle\, \langle f, e_k\rangle = \langle f, g\rangle = \int_{[0,1]} f(x)\, g(x)\, dx = \int_{[0,1]^2} f(y)\, g(x)\, \delta(x-y)\, dx\, dy. \end{aligned}$$

If we use the fundamental solution g of the heat equation defined in (2.3.8) and the space-time white noise representation $W(ds, dy)$ in terms of the Brownian sheet, then from (2.3.6) we obtain yet another expression for z,

$$z(t,x) = \int_0^1 g_t(x,y)\, u_0(y)\, dy + \int_0^t \int_0^1 g_{t-s}(x,y)\, W(ds, dy).$$

2.4 Mosco convergence

In chapters 4 and 5 we prove convergence in law of stationary processes associated with a sequence of Dirichlet forms. To this aim, we use the technique of Mosco-convergence in order to obtain convergence of the transition semigroups.

We recall that Mosco [48] introduced a version of the classical Γ-convergence for bilinear forms, defined on the same Hilbert space, which turns out to be equivalent to the convergence of the associated resolvent operators. The original motivation was to study homogeneization of partial differential equations with oscillating coefficients in a so-called composite medium.

Later, Kuwae and Shioya have [43] extended the study of Mosco convergence for Dirichlet forms defined on different Hilbert spaces. More generally the authors present a systematic and functional analytic framework of some topologies on the set of spectral structures, covering, for example, the behaviour of the spectrum of

Laplacian operator under a pertubation of the topology of the underlying Riemannian manifold.

Recently, Andres and von Renesse [4] have given a stronger version of Kuwae and Shioya's theory, with the aim of constructing a system of interacting two sided Bessel processes and showing that the associated empirical process converges to an infinite-dimensional stochastic process (the Wasserstein Diffusion).

Let us now recall the main definition and results of this theory, which will play an important rôle in the final chapters of this thesis. Let $\mathcal{E}_n^\beta$ be the family of forms defined on the whole $\mathbb{H}_n$ by

$$\mathcal{E}^\beta(u,v) := \beta\langle u - \beta R_\beta u, v\rangle_n \, u, v \in \mathbb{H}_n \tag{2.4.1}$$

Where R_β is the resolvent. These family is called the Deny-Yosida approximation. We have

$$\begin{aligned} \mathcal{D}(\mathcal{E}_n) &= \left\{ u \in \mathbb{H}_n \,\middle|\, \lim_{\beta\to+\infty} \mathcal{E}_n^\beta(u,u) < +\infty \right\} \\ \mathcal{E}_n(u,u) &= \lim_{\beta\to+\infty} \mathcal{E}_n^\beta(u,u) \end{aligned} \tag{2.4.2}$$

the last limit is non-decreasing as $\beta \to +\infty$. Moereover we have

$$\begin{aligned} \mathcal{E}_n^\beta(u,u) &= \min_{v\in\mathbb{H}_n} \left\{ \mathcal{E}_n(v,v) + \beta\|u-v\|_n^2 \right\} \\ &= \mathcal{E}_n(\beta R_\beta u, \beta R_\beta u) + \beta\|u - \beta R u\|_n \end{aligned} \tag{2.4.3}$$

Definition 2.4.1. *A sequence of Hilbert spaces $\mathbb{H}_n$ converges to a hilbert $\mathbb{H}$ if there is a family of linear maps $\{\Phi_n : \mathbb{H} \to \mathbb{H}_n\}$ such that:*

$$\lim_{n\to+\infty} \|\Phi_n(x)\|_{\mathbb{H}_n} = \|x\|_{\mathbb{H}}, \quad x \in \mathbb{H} \tag{2.4.4}$$

A sequence $(x_n)_n$, $x_n \in \mathbb{H}_n$, converges strongly *to a vector $x \in \mathbb{H}$ if there exists a sequence $(\tilde{x}_n)_n$ in $\mathbb{H}$ such that $\tilde{x}_n \to x$ in $\mathbb{H}$ and*

$$\lim_{n\to+\infty} \overline{\lim_{m\to+\infty}} \|\Phi_m(\tilde{x}_n) - x_m\|_{\mathbb{H}_m} = 0 \tag{2.4.5}$$

and $(x_n)_n$ converge weakly *to x if*

$$\lim_{n\to+\infty} \langle x_n, z_n\rangle_{\mathbb{H}_n} = \langle x, z\rangle_{\mathbb{H}} \tag{2.4.6}$$

for any $z \in \mathbb{H}$ and sequence $(z_n)_n$, $z_n \in \mathbb{H}_n$, such that $z_n \to z$ strongly.

Now we can give the definition of Mosco-convergence of Dirichlet forms. This concept is useful for our purposes, since it was proved in [43] to imply the convergence in a strong sense of the associated resolvents and semigroups.

Definition 2.4.2. *If $\mathcal{E}^n$ is a quadratic form on $\mathbb{H}_n$, then $\mathcal{E}^n$ Mosco-converges to the quadratic form $\mathcal{E}$ on $\mathbb{H}$ if the two following conditions hold:*

Mosco I. For any sequence $x_n \in \mathbb{H}_n$, converging weakly to $x \in \mathbb{H}$,

$$\mathcal{E}(x,x) \le \underline{\lim}_{n\to+\infty} \mathcal{E}^n(x_n,x_n). \tag{2.4.7}$$

Mosco II. For any $x \in \mathbb{H}$, there is a sequence $x_n \in \mathbb{H}_n$ converging strongly to $x \in \mathbb{H}$ such that

$$\mathcal{E}(x,x) = \lim_{n\to+\infty} \mathcal{E}^n(x_n,x_n). \tag{2.4.8}$$

Now we can give the following

Definition 2.4.3. *We say that a sequence of bounded operarors $(B_n)_n$ on $\mathbb{H}_n$, converges strongly to an operator B on $\mathbb{H}$, if $\mathbb{H}_n \ni B_n u_n \to Bu \in \mathbb{H}$ strongly for all sequence $u_n \in \mathbb{H}_n$ converging strongly to $u \in \mathbb{H}$.*

Then Kuwae and Shioya have proved in [43] the following equivalence between Mosco convergence and strong convergence of the associated resolvent operators.

Theorem 2.4.4 (Mosco [48], Kuwae and Shioya [43])**.** *The sequence $(R^n_\beta)_n$ converges strongly for all $\beta > 0$ to R_β, if and only if Mosco convergence holds*

Proof.

- We first prove that strong convergence of the resolvent implies Mosco convergence. Let u in H and u_n (resp. $\tilde{u}_n$) which converges weakly (resp. strongly) to u. Then we have

$$\begin{aligned} \mathcal{E}_n(u_n) &\geq \mathcal{E}^\beta_n(u_n) \\ &\geq \mathcal{E}^\beta_n(\tilde{u}_n) + 2\beta\langle \tilde{u}_n - \beta R^n_\beta \tilde{u}_n, v_n - \tilde{u}_n\rangle_n \\ \liminf_n \mathcal{E}(u_n) &\geq \mathcal{E}^\beta(u),\ \beta > 0 \end{aligned} \tag{2.4.9}$$

  so we get Mosco 1 when $\beta \to +\infty$. Let $(u_n)_n$ converges strongly to u then

$$\begin{aligned} \mathcal{E}(u,u) &\geq \lim_\beta \mathcal{E}^\beta(u,u) \\ &\geq \lim_\beta \lim_n \mathcal{E}^\beta_n(u_n,u_n) \end{aligned} \tag{2.4.10}$$

  By a diagonal argument we can choose a sequence (β_n) such that

$$\mathcal{E}(u,u) \geq \lim_n \mathcal{E}^{\beta_n}_n(u_n,u_n)$$

  suppose now that $v_n = \beta_n R^n_{\beta_n} u_n$ where $(v_n)_n$, then

$$\mathcal{E}^{\beta_n}_n(u_n,u_n) = \mathcal{E}_n(v_n,v_n) + \beta_n\|u_n - v_n\|^2_n \tag{2.4.11}$$

  we have the result taking the limsup.

- We suppose now that Mosco convergence holds. We have to prove that for every $u \in H$ and every sequence $(u_n)_n$ tending strongly to u, the sequence $v_n := R^n_\beta u_n$ converges strongly to $v := R_\beta u$. The norm of R^n_β is bounded by β^{-1}, there is a subsequence of $(v_n)_n$, still detoned by $(v_n)_n$ which converges weakly to some $\tilde{v} \in H$ (lemma 2.2 of [43]). By Mosco II for $w \in H$ there is a sequence $(w_n)_n$ such that $\mathcal{E}^n(w_n,w_n) \to \mathcal{E}(w,w)$, because v_n is a minimizers of $w \mapsto \mathcal{E}^n(w,w) + \beta\langle w,w\rangle_n - 2\langle w,u_n\rangle_n$.

$$\begin{aligned} \mathcal{E}^n(v_n,v_n) + \beta\langle v_n,v_n\rangle_n - 2\langle v_n,u_n\rangle_n &\leq \mathcal{E}^n(w_n,w_n) \\ &\quad +\beta\langle w_n,w_n\rangle_n - 2\langle w_n,u_n\rangle_n \end{aligned} \tag{2.4.12}$$

By Mosco I we have

$$\mathcal{E}(\tilde{v}, \tilde{v}) + \beta\langle\tilde{v}, \tilde{v}\rangle - 2\langle\tilde{v}, u\rangle \leq \mathcal{E}(w, w) + \beta\langle w, w\rangle - 2\langle w, u\rangle$$

So $\tilde{v} = v$, and weak convergence of the resolvent holds. Let $w \in \mathcal{D}(\mathcal{E})$ and $w_n \to w$ strongly such that $\mathcal{E}^n(w_n, w_n) \to \mathcal{E}(w, w)$ then the resolvent inequality for R^n_β yields

$$\mathcal{E}^n(v_n, v_n) + \beta\|v_n - u_n/\beta\|_n^2 \leq \mathcal{E}^n(w_n, w_n) + \beta\|w_n - u_n/\beta\|_n^2 \qquad (2.4.13)$$

Taking the limit for $n \to +\infty$,

$$\limsup_n \beta\|v_n - u_n/\beta\|_n^2 \leq \mathcal{E}(w, w) - \mathcal{E}(v, v) + \beta\|w - u/\beta\|^2$$

We may let $w \to u \in \mathcal{D}(E)$

$$\limsup_n \|v_n - u_n/\beta\|_n^2 \leq \|v - u/\beta\|^2$$

Due to the lower semicontinuity of the norm we have the equality $\lim_n \|v_n - u_n/\beta\|_n^2 = \|v - u/\beta\|^2$. Weak convergence plus convergence in norms provide the strong convergence (lemma 2.3 of [43]).

□

Theorem 2.4.5 (Andres and von Renesse [4]). *Mosco I and Mosco II are equivalent to Mosco I and the following hypothesis, noted Mosco II':*

Mosco II' there is a core $K \subset \mathcal{D}(\mathcal{E})$ such that for all $u \in K$, there is a sequence $(u_n)_n$ with $u_n \in \mathcal{D}(\mathcal{E}^n)$ which converges strongly to u such that $\mathcal{E}(u, u) = \lim_n \mathcal{E}^n(u_n, u_n)$

Proof. We saw previously that the strong convergence of the resolvent provides Mosco I and Mosco II, it is obvious that Mosco II' follows from Mosco II, indeed if $u \in \mathcal{D}(\mathcal{E})$ and $(u_n)_n$ as in Mosco II then there is $m \in \mathbb{N}$ such that u_n must be in $\mathcal{D}(\mathcal{E}^n)$, for $n \geq m$.

To prove the reciprocal, we remark that one could reproduce the second part of the previous proof in taking $w \in K$ in the inequalities (2.4.12) and (2.4.13) and use the core property of K. □

CHAPTER 3

REFLECTED RANDOM STRING IN A CONVEX DOMAIN

3.1 Introduction

In this chapter we want to prove well-posedness of stochastic partial differential equations driven by space-white noise and reflected on the boundary of a convex region of $\mathbb{R}^d$. More precisely, we consider a convex open domain O in $\mathbb{R}^d$ with a smooth boundary ∂O and a proper l.s.c. convex function $\varphi : \overline{O} \mapsto \mathbb{R}$, and we study solutions (u, η) of the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \theta^2} + n(u(t,\theta)) \cdot \eta(t,\theta) - \dfrac{1}{2}\partial\varphi_0(u(t,\theta)) + \dot{W}(t,\theta) \\ u(0,\theta) = x(\theta), \quad u(t,0) = a, \ u(t,1) = b \\ u(t,\theta) \in \overline{O}, \ \eta \geq 0, \ \eta(\{(t,\theta) \,|\, u(t,\theta) \notin \partial O\}) = 0 \end{cases} \tag{3.1.1}$$

where $u \in C\left([0,T] \times [0,1]; \overline{O}\right)$ and η is a locally finite positive measure on $]0,T] \times [0,1]$; moreover $a, b \in O$ are some fixed points, $\dot{W}$ is a vector of d independent copies of a space-time white noise and for all $y \in \partial O$ we denote by $n(y)$ the inner normal vector at y to the boundary ∂O; finally, $\partial\varphi_0 : O \mapsto \mathbb{R}^d$ is the element of minimal norm in the subdifferential of ϕ and the initial condition $x : [0,1] \mapsto \overline{O}$ is continuous.

Solutions $u(t,\theta)$ of equation (3.1.1) take values in the convex closed set $\overline{O}$ and evolve as solutions of a standard SPDE in the interior O, while the reflection measure η pushes $u(t,\theta)$ along the inner normal vector $n(u(t,\theta))$, whenever $u(t,\theta)$ hits the boundary. The condition $\eta(\{(t,\theta) \,|\, u(t,\theta) \notin \partial O\}) = 0$ means that the reflection term acts only when it is necessary, i.e. only when $u(t,\theta) \in \partial O$.

The chapter is organized as follows. In section 2 we give a precise definition of solutions to equation (3.1.1), together with some notation. In section 4 we introduce the approximating equation and recall the stability results already mentioned above. In section 5 we prove path continuity of the candidate solution. In section 6 we state the integration by parts formula we need. In section 7 we prove existence of weak solutions of equation (3.1.1), and in section 8 pathwise uniqueness and existence of strong solutions. Finally, in section 9 we prove some properties of the reflection measure η.

The content of this chapter has been published in [8].

3.2 Notations and setting

We first discuss the notion of solution of (3.1.1). We consider a convex l.s.c. $\varphi : \overline{O} \mapsto [0, +\infty]$ such that $\varphi < +\infty$ on O. We denote by $D(\varphi) := \{\varphi < +\infty\}$ the *domain* of φ and by $\partial\varphi$ the *subdifferential* of φ:

$$\partial\varphi(y) := \left\{ z \in \mathbb{R}^d : \varphi(w) \geq \varphi(y) + \langle z, w - y\rangle, \ \forall\, w \in \overline{O} \right\}, \qquad y \in D(\varphi).$$

The set $\partial\varphi(y)$ is non-empty, closed and convex in $\mathbb{R}^d$, and therefore it has a unique element of minimal norm, that we call $\partial_0\varphi(y)$. Notice that we do not assume smoothness of $y \mapsto \partial_0\varphi(y)$. We can also allow $\partial_0\varphi(y)$ to blow up as $y \to \partial O$, but not too fast. Indeed, throughout the chapter we assume that $\partial_0\varphi : D(\varphi) \mapsto \mathbb{R}^d$ satisfies

$$\int_O |\partial_0\varphi(y)|^2 \, dy < +\infty \tag{3.2.1}$$

where dy denotes the Lebesgue measure on O. This assumption is not optimal, see Remark 3.2.6 below, but already covers interesting cases, like logarithmic divergences or polynomial divergences with small exponent, see [18] or [65] for related studies in convex subsets of $\mathbb{R}$.

For two vectors $a, b \in \mathbb{R}^d$, we denote by $a \cdot b$ their canonical scalar product. We consider the Hilbert space $H := L^2([0, 1]; \mathbb{R}^d)$, endowed with the canonical scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$,

$$\langle h, k\rangle := \int_0^1 h(\theta) \cdot k(\theta)\, d\theta, \qquad \|h\|^2 := \langle h, h\rangle, \qquad h, k \in H.$$

Definition 3.2.1. *Let $x \in C\left([0, 1]; \overline{O}\right)$. An adapted triple (u, η, W), defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$, is a weak solution of* (3.1.1) *if*

- *a.s. $u \in C(]0, T] \times [0, 1]; \overline{O})$ and $\mathbb{E}[\|u_t - x\|^2] \to 0$ as $t \downarrow 0$*
- *a.s. η is a positive measure on $]0, T] \times [0, 1]$ such that $\eta([\varepsilon, T] \times [0, 1]) < +\infty$ for all $0 < \varepsilon \leq T$*
- *a.s. the function $(t, \theta) \mapsto |\partial_0\varphi(u(t, \theta))|$ is in $L^1_{\mathrm{loc}}([\varepsilon, T]\times]0, 1[)$ for all $0 < \varepsilon \leq T$*
- *$W = (W^1, \ldots, W^d)$ is a vector of d independent copies of a Brownian sheet*
- *for all $h \in C^2_c((0, 1); \mathbb{R}^d)$ and $0 < \varepsilon \leq t$*

$$\begin{aligned}\langle u_t - u_\varepsilon, h\rangle = \frac{1}{2}\int_\varepsilon^t \langle h'', u_s\rangle\, ds + \int_\varepsilon^t \int_0^1 h(\theta) \cdot n(u(s, \theta))\, \eta(ds, d\theta) \\ - \frac{1}{2}\int_\varepsilon^t \langle h, \partial_0\phi(u_s)\rangle\, ds + \int_\varepsilon^t \int_0^1 h(\theta)\, W(ds, d\theta)\end{aligned} \tag{3.2.2}$$

- *a.s. the support of* η *is contained in* $\{(t,\theta) : u(t,\theta) \in \partial O\}$*, i.e.*

$$\eta(\{(t,\theta) \,|\, u(t,\theta) \notin \partial O\}) = 0. \tag{3.2.3}$$

A weak solution (u,η,W) *is said to be a strong solution if* (u,η) *is adapted to the natural filtration of* W*.*

We say that pathwise uniqueness holds for equation (3.1.1) if any two weak solutions (u^1,η^1,W) and (u^2,η^2,Z) coincide. In this article we want to prove the following result:

Theorem 3.2.2. *For all* $x \in C\left([0,1];\overline{O}\right)$*, the problem* (3.1.1) *enjoys pathwise uniqueness of weak solutions and existence of a strong solution.*

Next, we want to study some properties of the reflection measure η. We recall that its support is contained in the contact set, i.e. in the set $\{(t,\theta) : u(t,\theta) \in \partial O\}$. The next result shows that η is concentrated on a subset $\mathcal{S}$ of the contact set, such that each section $\mathcal{S} \cap (\{s\} \times [0,1])$, $s \geq 0$, contains at most one point. Moreover, $u(s,\cdot)$ hits the boundary ∂O at this point and not elsewhere.

Theorem 3.2.3. *A.s. the reflection measure* η *is supported by a Borel set* $\mathcal{S} \subset]0,+\infty[\times[0,1]$*, i.e.* $\eta(\mathcal{S}^c) = 0$*, such that for all* $s \geq 0$*, the section* $\{\theta \in [0,1] : (s,\theta) \in \mathcal{S}\}$ *has cardinality* 0 *or* 1*. Moreover, if* $r(s) \in \mathcal{S} \cap (\{s\} \times [0,1])$ *then*

$$u(s,r(s)) \in \partial O, \qquad u(s,\theta) \notin \partial O,\ \forall \theta \in [0,1] \setminus \{r(s)\}.$$

This property is analogous to that discovered in [64] for reflected SPDEs in $[0,+\infty)$. We recall that in this one-dimensional setting, sections of the contact set have been studied in detail in [14]. It would be very interesting to prove the same kind of results in our multi-dimensional setting.

3.2.1 Notations

We fix now some notations which will be used throughout the chapter. Let $E := H^{[0,\infty)}$ and define the canonical process $X_t : E \mapsto H$, $t \geq 0$, $X_t(e) := e(t)$, and the associated natural filtration

$$\mathcal{F}^0_\infty \ := \ \sigma\{X_s,\, s \in [0,\infty)\}, \quad \mathcal{F}^0_t \ := \ \sigma\{X_s,\, s \in [0,t]\}, \quad t \in [0,+\infty].$$

We denote by μ the law of the Brownian bridge from a to b in $\mathbb{R}^d$. Let us define

$$K := \{x \in L^2([0,1];\mathbb{R}^d) : x_\theta \in \overline{O},\ \text{for all } \theta \in [0,1]\}$$

and for all $x \in H = L^2([0,1];\mathbb{R}^d)$ we define $U : H \mapsto [0,+\infty]$ as follows

$$U(x) := \begin{cases} \displaystyle\int_0^1 \varphi(x_\theta)\, d\theta, & \text{if } x \in K \\ +\infty, & \text{otherwise.} \end{cases}$$

Lemma 3.2.4. *The probability measure ν on K*

$$\nu(dx) := \frac{1}{Z}\exp(-U(x))\,\mu(dx). \tag{3.2.4}$$

is well defined, i.e. $\mu(K) > 0$ *and* $Z := \mu(e^{-U}) \in]0,1]$.

We note that U is l.s.c. and convex. For the next Lemma, see [10, Chapter 2].

Lemma 3.2.5 (Yosida approximation)**.** *Let* $\Phi : \mathbb{R}^d \mapsto \mathbb{R} \cup \{+\infty\}$ *be convex lower semi-continuous, and* $\partial\Phi$ *be the subdifferential of* Φ*. Set for* $n \in \mathbb{N}$

$$\Phi_n(x) := \inf_{y\in\mathbb{R}^d}\left\{\Phi(y) + n\,\|x-y\|^2\right\}, \qquad x \in \mathbb{R}^d.$$

Then

1. $\partial\Phi_n$ *is* n*-Lipschitz continuous*
2. $\forall y \in D(\Phi)$, $\Phi_n(y) \uparrow \Phi(y)$, *as* $n \uparrow +\infty$
3. $\forall y \in D(\partial\Phi_n)$,

$$\lim_{n\to+\infty}\partial\Phi_n(y) = \partial_0\Phi(y), \text{ and } |\partial\Phi_n(y)| \uparrow |\partial_0\Phi(y)| \text{ as } n \to +\infty$$

Then we define $U_n : H \mapsto [0,+\infty)$ as follows

$$U_n(x) := \int_0^1 \Phi_n(x_\theta)\,d\theta, \quad \text{if } x \in H. \tag{3.2.5}$$

Remark 3.2.6. The assumption (3.2.1) on ϕ is far from optimal. In fact, our approach covers a more general class of non-linearity; indeed, the proof we give below yields Theorems 3.2.2 and 3.2.3 under the assumption

$$\int\int_\delta^{1-\delta} |\partial_0\varphi(x_\theta)|^2\,d\theta\,\nu(dx) < +\infty, \qquad \forall\delta \in (0,1/2), \tag{3.2.6}$$

see Lemma 3.7.2 below.

Finally, we need to introduce some function spaces. We denote by $C_b(H)$ the Banach space of all $\varphi : H \mapsto \mathbb{R}$ being bounded and continuous in the norm of H, endowed with the norm $\|\varphi\|_\infty := \sup|\varphi|$. Moreover we denote by $\mathcal{FC}^1$ the set of all functions F of the form

$$F(w) = f(\langle l_1, w\rangle, \ldots, \langle l_n, w\rangle), \qquad w \in H, \tag{3.2.7}$$

with $n \in \mathbb{N}$, $l_i \in L^2(0,1)$ and $f \in C_b^1(\mathbb{R}^n)$.

3.3 The approximating equation

Let us introduce the convex function $\Phi : \mathbb{R}^d \mapsto \mathbb{R} \cup \{+\infty\}$

$$\Phi(y) := \begin{cases} \varphi(y), & \text{if } y \in \overline{O} \\ \\ +\infty, & \text{otherwise} \end{cases} \tag{3.3.1}$$

and its Yosida approximation Φ_n, defined as in Lemma 3.2.5. We introduce the SPDE

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u_n}{\partial \theta^2} - \dfrac{1}{2}\partial\Phi_n(u_n(t,\theta)) + \dot{W}(t,\theta) \\ \\ u_n(0,\theta) = x(\theta),\ u_n(t,0) = a,\ u_n(t,1) = b \end{cases} \tag{3.3.2}$$

By Lemma 3.2.5, $\partial\Phi_n$ is Lipschitz continuous, and therefore it is a classical result that for any $x \in H$ equation (3.3.2) has a unique solution u_n, which is moreover a.s. continuous on $]0,\infty[\,\times[0,1]$.

Equation (3.3.2) is a natural approximation of equation (3.1.1) and one expects u_n to converge to u in some sense as $n \to \infty$. A convergence in law indeed holds and follows from a general result proven in [3], see the discussion in Theorem 3.4.2 below.

We denote by $\mathbb{P}^n_x$ the law on $E = H^{[0,\infty)}$ of $(u^n(t,\cdot))_{t\geq 0}$, solution of (3.3.2). We also define the probability measure

$$\nu_n(dx) := \frac{1}{Z_n}\exp(-U_n(x))\,\mu(dx), \qquad U_n(x) := \int_0^1 \Phi_n(x_\theta)\,d\theta, \tag{3.3.3}$$

and the symmetric bilinear form $(\mathcal{E}^n, \mathcal{FC}^1)$

$$\mathcal{E}^n(F,G) := \frac{1}{2}\int \langle \nabla F, \nabla G\rangle\,d\nu_n, \qquad F,G \in \mathcal{FC}^1. \tag{3.3.4}$$

We denote by $(P^n_t)_{t\geq 0}$ the transition semigroup associated to equation (3.3.2):

$$P^n_t\varphi(x) := \mathbb{E}^n_x(\varphi(X_t)), \qquad \forall\,\varphi \in C_b(H),\ x \in H,\ t \geq 0,$$

and the associated resolvent

$$R^n_\lambda\varphi(x) := \int_0^\infty e^{-\lambda t}\,\mathbb{E}^n_x\left[\varphi(X_t)\right]dt, \qquad x \in H,\ \lambda > 0.$$

The following result is well known, see [46] and [16].

Theorem 3.3.1.

1. *$(\mathcal{E}^n, \mathcal{FC}^1)$ is closable in $L^2(\nu_n)$: we denote by $(\mathcal{E}^n, D(\mathcal{E}^n))$ the closure.*
2. *$(\mathbb{P}^n_x)_{x\in H}$ is a Markov process, associated with the Dirichlet form $(\mathcal{E}^n, D(\mathcal{E}^n))$ in $L^2(\nu_n)$, i.e. for all $\lambda > 0$ and $\varphi \in L^2(\nu_n)$, $R^n_\lambda\varphi \in D(\mathcal{E}^n)$ and:*
$$\lambda\int_H R^n_\lambda\varphi\,\psi\,d\nu_n + \mathcal{E}^n(R^n_\lambda\varphi,\psi) \;=\; \int_H \varphi\,\psi\,d\nu_n, \quad \forall\psi \in D(\mathcal{E}^n).$$
3. *ν_n is the unique invariant probability measure of $(P^n_t)_{t\geq 0}$. Moreover, $(P^n_t)_{t\geq 0}$ is symmetric with respect to ν_n.*

We recall an important property of equation (3.3.2): the associated transition semigroup $(P^n_t)_{t\geq 0}$ is *Strong Feller*, i.e. P^n_t maps bounded Borel functions into bounded continuous functions for all $t > 0$. Indeed, P^n_t satisfies for any bounded Borel $\varphi : H \mapsto \mathbb{R}$

$$|P^n_t\varphi(x) - P^n_t\varphi(y)| \;\leq\; \frac{\|\varphi\|_\infty}{\sqrt{t}}\,\|x-y\|_H, \qquad x,y \in H,\ t > 0, \tag{3.3.5}$$

see [13, Proposition 4.4.4].

3.4 The Dirichlet Form

One of the main tools of this chapter is the Dirichlet form associated with equation (3.1.1). Recall the definition (3.2.4) of the probability measure ν. Notice that μ is Gaussian and U is convex. It follows that ν is *log-concave*, i.e. for all pairs of open sets $A, B \subset H$ we have:

$$\log \nu((1-t)A + tB) \geq (1-t)\log \nu(A) + t \log \nu(B)$$

where $(1-t)A + tB := \{(1-t)a + tb \,|\, a \in A, b \in B\}$ for $t \in [0,1]$, see for instance [2, Theorem 9.4.11]. Notice that ν_n defined in (3.3.3) above is also log-concave for the same reason.

Let us consider now the bilinear form

$$\mathcal{E}(F,G) := \frac{1}{2}\int \langle \nabla F, \nabla G \rangle \, d\nu, \qquad F, G \in \mathcal{FC}^1. \tag{3.4.1}$$

Then by [3, Theorem 1.2]

Theorem 3.4.1. *In the previous setting we have:*

1. *The bilinear form $(\mathcal{E}, \mathcal{FC}^1)$ is closable in $L^2(\mu)$ and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form.*
2. *There is a Markov family $(\mathbb{P}_x)_{x\in K}$ of probability measures on the canonical path space $(K^{[0,+\infty[}, \mathcal{F}, (\mathcal{F}_t), (X_t)_{t\geq 0})$ associated with $\mathcal{E}$.*
3. *for all $x \in K$, $\mathbb{P}_x$-a.s. $(X_t)_{t>0}$ is continuous in H and $\mathbb{E}_x[\|X_t - x\|^2] \to 0$ as $t \to 0$.*

Let us remark that clearly, since $U_n \uparrow U$, we have

$$\nu_n \rightharpoonup \nu. \tag{3.4.2}$$

A look at the Dirichlet forms (3.3.4) and (3.4.1) suggests that the laws of the associated processes could also converge. In general this is false, and a number of papers have been devoted to this problem, see for instance [43] and [41]. However, it turns out that, in the setting of Dirichlet forms of the form (3.3.4) with log-concave reference measures, (3.4.2) does imply convergence in law of the associated Markov processes. This general *stability property* is one of the main results of [3]. By (3.4.2) and [3, Theorem 1.5] we have that

Theorem 3.4.2 (Stability)**.** *For any sequence $x_n \in H$ converging to $x \in K$, we have that*

(a) *$\mathbb{P}^n_{x_n} \to \mathbb{P}_x$ weakly in $H^{[0,+\infty[}$ as $n \to \infty$,*

(b) *for all $0 < \varepsilon \leq T < +\infty$, $\mathbb{P}^n_{x_n} \to \mathbb{P}_x$ weakly in $C([\varepsilon, T]; H_w)$,*

(c) *for all $0 \leq T < +\infty$, $\mathbb{P}^n_{\nu_n} \to \mathbb{P}_\nu$ weakly in $C([0,T]; H_w)$,*

where H_w is H endowed with the weak topology and

$$\mathbb{P}^n_{\nu_n} = \int \mathbb{P}^n_y \, \nu_n(dy), \qquad \mathbb{P}_\nu = \int \mathbb{P}_y \, \nu(dy).$$

This stability result will be very useful to prove several properties of the solution to (3.1.1). We notice that, by point (a) of Theorem 3.4.2,

$$\lim_{n\to\infty} P_t^n\varphi(x) = P_t\varphi(x) := \mathbb{E}_x(\varphi(X_t)), \qquad \forall\, t>0,\ x\in K,\ \varphi\in C_b(H). \tag{3.4.3}$$

This already allows to draw an important consequence of Thorem 3.4.2.

Proposition 3.4.3.

- *The Markov semigroup $(P_t)_{t\geq 0}$ associated with the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is Strong Feller, i.e. for any bounded Borel $\varphi : H \mapsto \mathbb{R}$*

$$|P_t\varphi(x) - P_t\varphi(y)| \,\leq\, \frac{\|\varphi\|_\infty}{\sqrt{t}}\, \|x-y\|_H, \qquad x,y\in H,\ t>0. \tag{3.4.4}$$

- *The Markov process $(\mathbb{P}_x)_{x\in K}$ associated with $(\mathcal{E}, D(\mathcal{E}))$ satisfies the absolute continuity condition: the transition probability $p_t(x,\cdot) = \mathbb{P}_x(X_t\in\cdot)$ is absolutely continuous w.r.t. the invariant measure ν*

$$p_t(x,\cdot) \ll \nu(\cdot), \qquad x\in H,\ t>0. \tag{3.4.5}$$

Proof. The first point follows from (3.3.5) and the weak convergence result of Theorem 3.4.2-(a). The second claim follows from the first. Indeed, let $A\subset H$ be a Borel set with $\nu(A)=0$. Then for all $t>0$, by invariance $\nu(P_t\mathbb{1}_A) = \nu(A) = 0$, i.e. $P_t\mathbb{1}_A(x) = 0$ for ν-a.e. $x\in H$. But by the Strong Feller property $P_t\mathbb{1}_A$ is continuous on K, therefore we obtain that $P_t\mathbb{1}_A(x) = 0$ for all x in the support of ν, which coincides with K. □

Finally, we give a result on existence and uniqueness of invariant measures of (3.1.1).

Proposition 3.4.4. *There exists a unique invariant probability measure of the Markov semigroup $(P_t)_{t\geq 0}$, and it is equal to ν.*

Proof. It is well known that ν_n is an invariant probability measure of the Markov semigroup $(P_t^n)_{t\geq 0}$. The weak convergence of ν_n to ν, the convergence formula (3.4.3) of P_t^n to P_t and the Strong Feller property, uniform in n, of P_t^n allow to show that ν is invariant for $(P_t^n)_{t\geq 0}$.

To prove uniqueness, we use a coupling argument. Let m^1 and m^2 be two invariant probability measures for $(P_t)_{t\geq 0}$ and let q_1 and q_2 be K-valued random variables, such that the law of q^i is m^i and $\{q^1, q^2, W\}$ is an independent family. Let u_i^n the solution of equation (3.3.2) with $u_i^n(0,\cdot) = q_i$, $i=1,2$. Setting $v := u_1^n(t,\cdot) - u_2^n(t,\cdot)$, we have:

$$\frac{d}{dt}\|v\|^2 \,=\, -\|v'\|^2 - \langle u_1^n(t,\cdot) - u_2^n(t,\cdot), \partial\Phi_n(u_1^n(t,\cdot)) - \partial\Phi_n(u_2^n(t,\cdot))\rangle \leq -\pi^2\|v\|^2$$

since $\langle p-q, \partial\Phi_n(p) - \partial\Phi_n(q)\rangle \geq 0$ by convexity of Φ_n. Therefore for all n

$$\|u_1^n(t,\cdot) - u_2^n(t,\cdot)\| \,\leq\, e^{-\pi^2 t/2}\|q_1 - q_2\|, \quad \forall t\geq 0.$$

Passing to the limit as $n \to \infty$, we obtain by Theorem 3.4.2 that (u_1^n, u_2^n, W) converges in law as $n \to \infty$ to (u_1, u_2, W), where (u_i, W) is a weak solution of (3.1.1) with $u_i(0, \cdot) = q_i$, $i = 1, 2$. Then we obtain that

$$\|u_1(t, \cdot) - u_2(t, \cdot)\| \le e^{-\pi^2 t/2} \|q_1 - q_2\|, \quad \forall t \ge 0.$$

Since the law of $u_i(t, \cdot)$ is equal to m^i for all $t \ge 0$, this implies $m^1 = m^2$. □

3.4.1 Wasserstein gradient flows

In this paragraph we recall the results of [3] which yield Theorems 3.4.1 and 3.4.2. Roughly speaking, in [3] and references cited therein, the authors study Markov processes whose dynamics is given by a stochastic differential equation with non-linear drift. The non-linearities are gradients of convex functionals, their works is based on the interpretation of a Fokker-Planck equation as the steepest descent flow of the relative entropy functional in the space of probability measures endowed with the Wassestein distance. Let u, v be two probability measures on H, the relative entropy functional with respect v is:

$$\mathcal{H}(u|v) := \begin{cases} \displaystyle\int_H \frac{du}{dv} \log \frac{du}{dv}\, dv, & u \ll v \\ +\infty & \text{otherwise} \end{cases} \tag{3.4.6}$$

where $\dfrac{du}{dv}$ is the Radon-Nikodym derivative. Let $\mathcal{M}(u, v)$ be the set of all coupling between u and v on $H \times H$. The Wasserstein distance is defined by:

$$W_2(u, v) := \inf \left\{ \left(\int_{H \times H} \|x - y\|^2 d\mu(x, y) \right)^{\frac{1}{2}} : \mu \in \mathcal{M}(u, v) \right\}.$$

The set $\mathcal{P}_2(H) := \left\{ u \in \mathcal{P}(H) : \int_H \|x\|^2 \, du < +\infty \right\}$ endowed with this distance is a complete and separable metric space whose convergence implies weak convergence. If v is log-concave the relative entropy $\mathcal{H}(\cdot \,|\, v)$ enjoys a crucial convexity property in terms of Wasserstein distance, named the Strongly Displacement convexity property. Indeed, for all u^0, u^1, and $\nu \in \mathcal{P}_2(H)$ there is a curve $(u_t)_{t \in [0,1]}$ such that $u_0 = u^0$ and $u_1 = u^1$ and $\forall t \in [0, 1]$

$$\begin{cases} W_2^2(u_t, \nu) \le (1 - t) W_2^2(u^0, \nu) + t W_2^2(u^1, \nu) - (1 - t) t W_2^2(u^1, u^0), \\ \mathcal{H}(u_t|v) \le (1 - t)\mathcal{H}(u^0|v) + t\mathcal{H}(u^1|v) \end{cases}$$

The Fokker-Planck equation, associated to the SPDE with a convex potential, has an interpretation as a varational inequation in the Wasserstein space $\mathcal{P}_2(H)$. We illustrate this remark in a simple setting. Let $(X_t^x)_t$ be the solution of the following equation

$$dX_t^x = -\nabla V(X_t^x) dt + dB_t, \qquad X_0^x = x \tag{3.4.7}$$

in $\mathbb{R}^k$. If ν_t^x is the law of X_t^x, and $u_t := \displaystyle\int \nu_t^x du^0$, then the Itô formula implies that $(u_t)_t$ solves the Fokker-Planck equation in the sense of distributions:

$$\frac{du_t}{dt} = \Delta u_t + \nabla \cdot (\nabla V u_t).$$

It is proved in [3] that the solution $(u_t)_t$ of (3.4.1) solves, still in the sense of distributions, the following variational evolution inequation:

$$\frac{1}{2}\frac{d}{dt}W_2^2(u_t, m) + \mathcal{H}(u_t|v) \leq \mathcal{H}(m|v), \qquad \forall \nu \in \mathcal{P}_2(H). \tag{3.4.8}$$

This equation can be interpreted as a weak formulation of the differential inclusion

$$\frac{du_t}{dt} \in -\nabla\mathcal{H}(u_t|v),$$

where $\nabla\mathcal{H}(\cdot\,|\,v)$ is the sub-differential of $\mathcal{H}(\cdot\,|\,v)$. This is the steepest descent interpretation of the Fokker-Planck equation. The solution of equation(3.4.8) is called a gradient flow associated with $\mathcal{H}(\cdot, v)$, the distance W_2 and the initial data u_0. Conversely, the gradient flow of (3.4.8), with potential $\mathcal{H}(\cdot\,|\,\nu)$ and initial datum δ_x is the probability kernel of the Markov processes associated with the Diriclet form (3.4.1) (see [3, theorem 7.3]).

Let $(v_n)_n$ be a sequence in $\mathcal{P}(H)$ of long-concave probabitlity on H weakly converging to v. Let H_n be the smallest closed affine subspace of H containing the support of v_n, for all n, and satisfying:

$$c\|h\|_H \leq \|h\|_{H_n} \leq \|h\|_H \quad \forall h \in H_n, \qquad n \in \mathbb{N} \tag{3.4.9}$$

for some constant $c > 0$. We denote, moreover, by $(u_t^n)_t$ the gradient flows associated with $\mathcal{H}(\cdot, v_n)$ in $\mathcal{P}_2(H_n)$, the distance W_{2,H_n}, and with initial data u_0^n. So it is proved in [3, theorem 6.1]

Theorem 3.4.5 (Stability of Gradient flows)**.** *In the above setting, if u_0^n converges to u_0 in $\mathcal{P}_2(H)$, then, for all $t \geq 0$, $u_t^n \to u_t$ in $\mathcal{P}_2(H)$, where $(u_t)_t$ is the gradient flows associated with $\mathcal{H}(\cdot|v)$ and initial datum u_0.*

3.5 Continuity properties of X

From the general theory of [3] and Theorem 3.4.1 above, one obtains only relatively mild continuity path properties of X, namely continuity in t with values in $L^2(0,1)$. However, for the contact condition (3.2.3) to make sense, we need $u(t,\cdot) = X_t$ to be jointly continuous, since we need to evaluate u at points $(t,\theta) \in [0,T] \times [0,1]$. This is the content of the main result of this section

Proposition 3.5.1. *For any $x \in K$, there exists a modification of $u(t,\cdot) = X_t$ which is $\mathbb{P}_x$-a.s. continuous on $]0,+\infty[\,\times[0,1]$ and such that $\mathbb{E}_x[\|u_t(\cdot) - x\|^2] \to 0$ as $t \to 0$.*

We start by proving continuity of stationary solutions of (3.1.1). To this aim, we are going to use the approximating equations (3.3.2) and the convergence result of Theorem 3.4.2-(c) for the stationary solutions. In particular, we are going to prove tightness of $(\mathbb{P}^n_{\nu_n})_n$ in $C([0,1]\times[0,T])$.

Lemma 3.5.2. *The sequence $(\mathbb{P}^n_{\nu_n})_n$ is tight in $C([0,1]\times[0,T])$.*

Proof. We follow the proof of Lemma 5.2 in [19]. We first recall a result of [26, Th. 7.2 ch 3]. Let (P,d) be a Polish space, and let $(X_\alpha)_\alpha$ be a family of processes with sample paths in $C([0,T];P)$. Then the laws of $(X_\alpha)_\alpha$ are relatively compact if and only if the following two conditions hold:

1. For every $\eta>0$ and rational $t\in[0,T]$, there is a compact set $\Gamma^t_\eta\subset P$ such that:
$$\inf_\alpha \mathbb{P}\left(X_\alpha\in\Gamma^t_\eta\right)\geq 1-\eta \tag{3.5.1}$$

2. For every $\eta,\epsilon>0$ and $T>0$, there is $\delta>0$ such that
$$\sup_\alpha \mathbb{P}\left(w(X_\alpha,\delta,T)\geq\epsilon\right)\leq\eta \tag{3.5.2}$$

where $w(\omega,\delta,T):=\sup\{d(\omega(r),\omega(s)) : r,s\in[0,T],\,|r-s|\leq\delta\}$ is the modulus of continuity in $C([0,T];P)$.

We introduce the space $H^{-1}(0,1)$, completion of $L^2(0,1)$ w.r.t. the norm:

$$\|f\|^2_{-1} := \sum_{k=1}^\infty k^{-2}\,|\langle f,e_k\rangle_{L^2(0,1)}|^2$$

where $e_k(r):=\sqrt{2}\sin(\pi kr)$, $r\in[0,1]$, $k\geq1$, are the eigenvectors of the second derivative with homogeneous Dirichlet boundary conditions at $\{0,1\}$. Recall that $L^2(0,1)=H$, in our notation. We denote by κ the Hilbert-Schmidt norm of the inclusion $H\to H^{-1}(0,1)$, which by definition is equal in our case to

$$\kappa=\sum_{k\geq1}k^{-2}<+\infty.$$

We claim that for all $p>1$ there exists $C_p\in(0,\infty)$, independent of n, such that:

$$\left(\mathbb{E}\left[\|X^n_t-X^n_s\|^p_{H^{-1}(0,1)}\right]\right)^{\frac{1}{p}}\leq C_p\,|t-s|^{\frac12},\qquad t,s\geq0. \tag{3.5.3}$$

To prove (3.5.3), we fix $n>0$ and $T>0$ and use the Lyons-Zheng decomposition, see Theorem 2.1.22 above and e.g. [27, Th. 5.7.1], to write for $t\in[0,T]$ and $h\in H$:

$$\langle h,X^n_t-X^n_0\rangle_H = \frac12 M_t-\frac12(N_T-N_{T-t}),$$

where M, respectively N, is a martingale w.r.t. the natural filtration of X^n, respectively of $(X^n_{T-t},\ t\in[0,T])$. Moreover, the quadratic variations are both equal to: $\langle M\rangle_t=\langle N\rangle_t=t\cdot\|h\|^2_H$. By the Burkholder-Davis-Gundy inequality we can find

$c_p \in (0,\infty)$ for all $p>1$ such that: $(\mathbb{E}\left[|\langle X^n_t - X^n_s, e_k\rangle|^p\right])^{\frac{1}{p}} \leq c_p\,|t-s|^{\frac{1}{2}}$, $t,s\in[0,T]$, and therefore

$$\left(\mathbb{E}\left[\|X^n_t - X^n_s\|^p_{H^{-1}(0,1)}\right]\right)^{\frac{1}{p}} \leq \sum_{k\geq 1} k^{-2}\left(\mathbb{E}\left[|\langle X^n_t - X^n_s, e_k\rangle|^p\right]\right)^{\frac{1}{p}}$$

$$\leq c_p \sum_{k\geq 1} k^{-2}|t-s|^{\frac{1}{2}}\|e_k\|^2_{L^2(0,1)} \leq c_p\,\kappa\,|t-s|^{\frac{1}{2}}, \quad t,s\in[0,T],$$

and (3.5.3) is proved. Let us introduce now the norm $\|\cdot\|_{W^{\eta,r}(0,1)}$ for $\eta>0$, $r\geq 1$

$$\|x\|^r_{W^{\eta,r}(0,1)} = \int_0^1 |x_s|^r ds + \int_0^1\int_0^1 \frac{|x_s - x_t|^r}{|s-t|^{r\eta+1}}\,dt\,ds.$$

By stationarity

$$\left(\mathbb{E}\left[\|X^n_t - X^n_s\|^p_{W^{\eta,r}(0,1)}\right]\right)^{\frac{1}{p}} \leq \left(\mathbb{E}\left[\|X^n_t\|^p_{W^{\eta,r}(0,1)}\right]\right)^{\frac{1}{p}} + \left(\mathbb{E}\left[\|X^n_s\|^p_{W^{\eta,r}(0,1)}\right]\right)^{\frac{1}{p}}$$

$$= 2\left(\int_H \|x\|^p_{W^{\eta,r}(0,1)}\,d\nu_n\right)^{\frac{1}{p}} \leq\, c\left(\int_H \|x\|^p_{W^{\eta,r}(0,1)}\,d\mu\right)^{\frac{1}{p}} \tag{3.5.4}$$

since $U \geq U_n \geq 0$, where $c = Z^{-1/p}$. If $r>p\geq 1$ the Jensen inequality for a concave function gives us, for $\eta\in(0,1/2)$,

$$\left(\mathbb{E}\left(\|\beta\|^p_{W^{\eta,r}(0,1)}\right)\right)^{\frac{r}{p}} \leq \mathbb{E}\left(\|\beta\|^p_r + \int_0^1\int_0^1 \frac{|\beta_s-\beta_t|^r}{|s-t|^{r\eta+1}}\,dt\,ds\right)$$

$$\leq 1 + \int_0^1\int_0^1 |s-t|^{r(\frac{1}{2}-\eta)-1}\,dt\,ds < +\infty.$$

The latter term is finite since μ is the law of a Brownian bridge. Let us now fix any $\eta\in(0,1/2)$ and $\gamma\in(0,1)$ such that

$$\gamma > \frac{1}{1+\frac{2}{3}\eta}.$$

From this it follows that $\alpha := \gamma\eta - (1-\gamma) > 0$ and therefore, if $r>0$ is such that

$$r > \max\left\{\frac{2}{1-\gamma}, \frac{1}{\eta - \frac{3}{2}\frac{1-\gamma}{\gamma}}\right\},$$

then we obtain that

$$\frac{r}{2}(1-\gamma) > 1, \qquad \frac{1}{d} := \gamma\frac{1}{r} + (1-\gamma)\frac{1}{2} < \alpha.$$

Then by interpolation, see [1, Chapter 7],

$$\left(\mathbb{E}\left[\|X^n_t - X^n_s\|^p_{W^{\alpha,d}(0,1)}\right]\right)^{\frac{1}{p}} \leq$$

$$\leq \left(\mathbb{E}\left[\|X^n_t - X^n_s\|^p_{W^{\eta,r}(0,1)}\right]\right)^{\frac{\gamma}{p}}\left(\mathbb{E}\left[\|X^n_t - X^n_s\|^p_{H^{-1}(0,1)}\right]\right)^{\frac{1-\gamma}{p}}.$$

Since $\alpha d > 1$, there exists $\beta > 0$ such that $(\alpha-\beta)d > 1$. By the Sobolev embedding, $C^\beta([0,1]) \subset W^{\alpha,d}(0,1)$. Since $\frac{r}{2}(1-\gamma) > 1$, there is $1 < p < r$ such that $\frac{p}{2}(1-\gamma) = 1+\zeta > 1$, and by (3.5.3) and (3.5.4), we find that

$$\left(\mathbb{E}\left[\|X_t^n - X_s^n\|^p_{C^\beta([0,1])}\right]\right) \le \tilde{c}\,|t-s|^{\frac{1-\gamma}{2}p}.$$

We consider now, as Polish space (P,d), the Banach space $C^\beta([0,1])$. By Kolmogorov's criterion, see e.g. [58, Thm. I.2.1], we obtain that a.s. $w(X^n,\delta,T) \le C\,\delta^{\frac{\zeta}{2p}}$, with $C \in L^p$. Therefore by the Markov inequality, if $\epsilon > 0$

$$\mathbb{P}\left(w(X^n,\delta,T) \ge \epsilon\right) \le \mathbb{E}\left[C^p\right]\,\delta^{\frac{\zeta}{2}}\epsilon^{-p},$$

and (3.5.2) follows for δ small enough.

Finally, since for all $t \ge 0$ the law of X_t^n is ν_n, which converges as $n \to \infty$ weakly in $C([0,1])$, tightness of the laws of $(X^n)_{n>0}$ in $C([0,T]\times[0,1])$ and therefore (3.5.1) follow. □

Proof of Proposition 3.5.1. By Theorem 3.4.2, we have $\mathbb{P}^n_{\nu_n} \rightharpoonup \mathbb{P}_\nu$ in $C_b([0,T];H_w)$ and by Lemma 3.5.2 the sequence $(\mathbb{P}^n_{\nu_n})_n$ is tight in $C([0,T]\times[0,1])$, so that letting $n \to +\infty$ we obtain $\mathbb{P}_\nu(C([0,T]\times[0,1])) = 1$. Now, we want to prove that $\mathbb{P}_x(C(]0,T]\times[0,1])) = 1$ for all $x \in K$. Let $\varepsilon > 0$. By (3.4.5), $\mathbb{P}_x \ll \mathbb{P}_\nu$ over the σ-algebra $\sigma\{X_s, s \ge \varepsilon\}$. Therefore $\mathbb{P}_x(C([\varepsilon,T]\times[0,1])) = 1$ for all $\varepsilon > 0$. □

3.6 An integration by parts formula

An important tool in the construction of a solution to equation (3.1.1) is the following integration by parts formula on the law μ of the Brownian bridge on the set K of trajectories contained in $\overline{O}$, proved in [38, Theorem 1.1]:

$$\int_K \partial_h F\,d\mu = \int_K h'\cdot wF\,d\mu - \int_{\partial O}\sigma(dy)\,\mathbb{E}_{a,y,b}\left[n(y)\cdot h(S_w)\,F(w)\right]\,\lambda(y) \qquad (3.6.1)$$

where $F \in \mathcal{FC}^1$, $h'\cdot w$ is the stochastic integral with respect w ($= -\langle h'', w\rangle$, if $h'' \in L^2(0,1)$), and

1. h is in the Cameron-Martin space of μ

$$H^1_0 = \left\{h \in C^0 \,|\, h_0 = h_1 = 0,\ h_t = \int_0^t \dot{h}_s\,ds,\ \dot{h} \in L^2(0,1)\right\}$$

2. $\mathbb{P}_{a,y,b}$ is the law of two independent Brownian motions put together back to back at their first exit time of O, across y. More precisely, let B and $\hat{B}$ be two independent Brownian motion such that $B_0 = a$ and $\hat{B}_0 = b$. Let $\tau(B)$ and $\tau(\hat{B})$ be the first exit times from O of B and $\hat{B}$ respectively. Conditionally on $\tau(B)+\tau(\hat{B}) = 1$, $B_{\tau(B)} = y$ and $\hat{B}_{\tau(\hat{B})} = y$, define the process X by

$$X_t = \begin{cases} B_t & 0 \le t \le \tau(B) \\ \hat{B}_{\tau(B)+\tau(\hat{B})-t}, & \tau(B) \le t \le \tau(B)+\tau(\hat{B}) \end{cases}$$

Then X has the law $\mathbb{P}_{a,y,b}$. For $w \in C([0,1];\overline{O})$ we denote by S_w the first time at which $w_{S_w} \in \partial O$, if there is any:

$$S_w := \inf\{s \in [0,1] : w_s \in \partial O\}, \qquad \inf \emptyset := 0.$$

Then $w_{S_w} = y$ for $\mathbb{P}_{a,y,b}$-a.e. w.

3. σ is the surface measure on ∂O, n_y is the inward normal vector

4. $(p_t(x,y))_{t>0,x,y\in O}$ is the fundamental solution to the Cauchy problem

$$\frac{\partial}{\partial t} - \frac{1}{2}\Delta = 0$$

where Δ is the Laplace operator on O with homogeneous Dirichlet boundary conditions at ∂O, and

$$\lambda(y) := \frac{1}{2p_1(a,b)} \int_0^1 \frac{\partial}{\partial n_y} p_u(a,y) \frac{\partial}{\partial n_y} p_{1-u}(b,y)\, du, \qquad y \in \partial O.$$

To prove (3.6.1), the author of [38] assumes that $\partial\Omega$ is smooth, and in particular that

1. for each $t > 0$ and $y \in O$, $p_t(\cdot, y)$ is C^1 up to the boundary
2. the restriction to ∂O of harmonic functions on O, and C^1 up to the boundary, are dense in $C(\partial O)$

see [38, Remarks 1.1 and 1.2]. Under this assumption the law of $(\tau(B), B_{\tau(B)})$ is given for $a \in O$ by

$$\mathbb{P}_a(\tau(B) \in dt, B_{\tau(B)} \in dy) = \frac{1}{2} \frac{\partial}{\partial n_y} p_t(a,y)\, \sigma(dy)\, dt,$$

where $\partial/\partial n_y$ denotes the normal derivative at $y \in \partial O$, see [38, formula (1.4)].

We want to deduce from (3.6.1) the following integration by parts formula for ν. We set for all bounded Borel $F : H \mapsto \mathbb{R}$

$$\int F(w)\, \Sigma(y, dw) := \frac{1}{Z}\, \mathbb{E}_{a,y,b}\left[F(w)\, e^{-U(w)}\right]\, \lambda(y). \tag{3.6.2}$$

Proposition 3.6.1. *For all $F \in \mathcal{FC}^1$*

$$\begin{aligned} \int \partial_h F\, d\nu = &- \int \langle h'', x\rangle\, F\, d\nu + \int \langle h, \partial_0 \varphi\rangle F\, d\nu \\ &- \int_{\partial O} \sigma(dy) \int n(y) \cdot h(S_w)\, F(w)\, \Sigma(y, dw) \end{aligned} \tag{3.6.3}$$

Proof. If $F \in \mathcal{FC}^1$, then we apply (3.6.1) with the function Fe^{-U_n}, where U_n is defined in (3.2.5), and we obtain

$$\begin{aligned} \int_K \partial_h F\, e^{-U_n}\, d\mu = &- \int_K \langle h'', x\rangle F\, e^{-U_n}\, d\mu + \int_K \langle h, \partial \Phi_n\rangle F\, e^{-U_n}\, d\mu \\ &- \int_{\partial O} \sigma(dy)\, \mathbb{E}_{a,y,b}\left[n(y) \cdot h(S)\, F\, e^{-U_n}\right] \lambda(y). \end{aligned}$$

The dominated convergence theorem and Lemma 3.2.5 provide the desired result. □

3.7 Existence of weak solutions

By Theorem 3.4.1 we have a Markov process associated with the Dirichlet form $\mathcal{E}$ defined by (3.4.1), but we still have to show that it is a solution of (3.1.1). In particular, we have the process X, namely the function u, but not the reflection measure η. The aim of this section is to construct η and obtain a weak solution of equation (3.1.1), in particular to prove the following

Proposition 3.7.1. *For all $x \in K$ there exists a weak solution (u, η, W) of equation (3.1.1).*

We are going to use Fukushima's theory [27] and in particular the powerful correspondence between positive continuous additive functionals (PCAF) and *smooth* measures, i.e. positive measures which do not charge sets with zero capacity. This theory is the content of [27, Chapters 4 and 5], to which we refer for all details.

We explain now why construction of a solution of (3.1.1) is not trivial, despite all information we already have. Since the main difficulty comes from the reflection term, let us suppose for simplicity that $\varphi \equiv 0$ and therefore, recalling the definition (3.3.1) of Φ, we have $\Phi \equiv 0$ on $\overline{O}$ and $\Phi \equiv +\infty$ on $\mathbb{R}^d \setminus \overline{O}$. Then the Yosida approximation Φ_n of Φ is equal to

$$\Phi_n(y) = nd^2(y, \overline{O}) := n \inf_{z \in \overline{O}} \|y - z\|^2, \qquad y \in \mathbb{R}^d$$

and its differential is $\partial \Phi_n(y) = 2nd(y, \overline{O}) \dfrac{y - p(y)}{|y - p(y)|}$, where $p(y) \in \overline{O}$ minimizes the distance from y, i.e. $d(y, \overline{O}) = \|y - p(y)\|$. Therefore, (3.3.2) becomes

$$\frac{\partial u_n}{\partial t} = \frac{1}{2} \frac{\partial^2 u_n}{\partial \theta^2} - nd(u_n, \overline{O}) \frac{u_n - p(u_n)}{|u_n - p(u_n)|} + \dot{W}.$$

By Theorem 3.4.2, we already know that u_n converges weakly to a process u. In all papers on reflected SPDEs with real values, one uses at some point that if $O \subset \mathbb{R}$ is an interval, then $\dfrac{y - p(y)}{|y - p(y)|}$ belongs to $\{\pm 1\}$ and is therefore locally constant. In other words one can decompose the non-linearity

$$nd(u_n, \overline{O}) \frac{u_n - p(u_n)}{|u_n - p(u_n)|} = \eta_n^+ - \eta_n^-$$

where $\eta_n^+, \eta_n^- \geq 0$ have well separated supports by the continuity of u_n. Moreover, it is not too difficult to obtain bounds on the total variation η_n^+, η_n^-, which yield tightness and therefore convergence of η_n^+, η_n^- as $n \to \infty$, as has been done in a number of papers, see [51, 21, 23, 19, 35, 18] among others.

On the other hand, if $u_n \in \mathbb{R}^d$, then such a decomposition becomes impossible, since the vector $\dfrac{y - p(y)}{|y - p(y)|}$ varies continuously in $\mathbb{S}^{d-1}$ and the process

$$t \mapsto L_n(t) := \int_0^t \left[nd(u_n, \overline{O}) \frac{u_n - p(u_n)}{|u_n - p(u_n)|} \right] (s, \theta) \, ds$$

has no definite sign. Therefore, convergence of u_n yields some form of convergence of L_n to a process L, but, without control on the total variation of L_n, we cannot even guarantee that L has bounded variation, a necessary condition if we want to obtain a measure η in equation (3.1.1). This is the main reason why the approaches available in the literature do not work in our case.

3.7.1 Dirichlet forms and Additive Functionals

We recall here the basics of potential theory which are needed in what follows, referring to [27] and [46] for all proofs. By Theorem 3.4.1, the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ has an associated Markov process, which is also a Hunt process. Therefore, by [46, Theorem IV.5.1], the Dirichlet form is *quasi-regular*, i.e. it can be embedded into a regular Dirichlet form; in particular, the classical theory of [27] can be applied. Moreover, the important *absolute continuity condition* (3.4.5) allows in the end to get rid of exceptional sets: see for instance [27, Theorem 4.1.2 and formula (4.2.9)].

We denote by $\mathcal{F}_\infty^\lambda$ (resp. $\mathcal{F}_t^\lambda$) the completion of $\mathcal{F}_\infty^0$ (resp. completion of $\mathcal{F}_t^0$ in $\mathcal{F}_\infty^\lambda$) with respect to $\mathbb{P}_\lambda$ and we set $\mathcal{F}_\infty := \cap_{\lambda\in\mathcal{P}(K)} \mathcal{F}_\infty^\lambda$, $\mathcal{F}_t := \cap_{\lambda\in\mathcal{P}(K)} \mathcal{F}_t^\lambda$, where $\mathcal{P}(K)$ is the set of all Borel probability measures on K.

Capacity

Let A be an open subset of H, we define by $\mathcal{L}_A := \{u \in D(\mathcal{E}) : u \geq 1,\ \nu\text{-a.e. on } A\}$. Then we set

$$\mathrm{Cap}(A) = \begin{cases} \inf\limits_{u\in\mathcal{L}_A} \mathcal{E}_1(u,u), & \mathcal{L}_A \neq \emptyset, \\ +\infty & \mathcal{L}_A = \emptyset, \end{cases}$$

where $\mathcal{E}_1$ is the inner product on $D(\mathcal{E})$ defines as follow

$$\mathcal{E}_1(u,v) = \mathcal{E}(u,v) + \int_H u(x)\, v(x)\, d\nu, \quad u, v \in D(\mathcal{E}).$$

For any set $A \subset H$ we let

$$\mathrm{Cap}(A) = \inf_{B \text{ open}, A\subset B\subset H} \mathrm{Cap}(B)$$

A set $N \subset H$ is *exceptional* if $\mathrm{Cap}(N) = 0$.

Additive functionals

By a Continuous Additive Functional (CAF) of X, we mean a family of functions $A_t : E \mapsto \mathbb{R}^+$, $t \geq 0$, such that:

(A.1) $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted

(A.2) There exists a set $\Lambda \in \mathcal{F}_\infty$ and a set $N \subset K$ with $\mathrm{Cap}(N) = 0$ such that $\mathbb{P}_x(\Lambda) = 1$ for all $x \in K \setminus N$, $\theta_t(\Lambda) \subseteq \Lambda$ for all $t \geq 0$, and for all $\omega \in \Lambda$: $t \mapsto A_t(\omega)$ is continuous, $A_0(\omega) = 0$ and for all $t, s \geq 0$:

$$A_{t+s}(\omega) \ = \ A_s(\omega) + A_t(\theta_s\omega),$$

where $(\theta_s)_{s\geq 0}$ is the time-translation semigroup on E.

Moreover, by a Positive Continuous Additive Functional (PCAF) of X we mean a CAF of X such that:

(A.3) For all $\omega \in \Lambda$: $t \mapsto A_t(\omega)$ is non-decreasing.

Two CAFs A^1 and A^2 are said to be equivalent if

$$\mathbb{P}_x\left(A_t^1 = A_t^2\right) \,=\, 1, \quad \forall t > 0,\ \forall x \in K \setminus N.$$

If A is a linear combination of PCAFs of X, the Revuz-measure of A is a Borel signed measure Σ on K such that:

$$\int_K \varphi \, d\Sigma \,=\, \int_K \mathbb{E}_x\left[\int_0^1 \varphi(X_t)\, dA_t\right] \nu(dx), \quad \forall \varphi \in C_b(K).$$

The Fukushima decomposition

Let $h \in C_0^2((0,1);\mathbb{R}^d)$, and set $U : K \mapsto \mathbb{R}$, $U(x) := \langle x, h\rangle$. By Theorem 3.4.1, the Dirichlet Form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular. Therefore we can apply the Fukushima decomposition, as it is stated in Theorem VI.2.5 in [46], p. 180: for any $U \in \mathrm{Lip}(H) \subset D(\mathcal{E})$, we have that there exist an exceptional set N, a Martingale Additive Functional of finite energy $M^{[U]}$ and a Continuous Additive Functional of zero energy $N^{[U]}$, such that for all $x \in K \setminus N$:

$$U(X_t) - U(X_0) \,=\, M_t^{[U]} + N_t^{[U]}, \quad t \geq 0,\ \mathbb{P}_x - \text{a.s.} \tag{3.7.1}$$

Smooth measures

We recall now the notion of smoothness for a positive Borel measure Σ on H, see [27, page 80]. A positive Borel measure Σ is *smooth* if

1. Σ charges no set of zero capacity
2. there exists an increasing sequence of closed sets $\{F_k\}$ such that $\Sigma(F_n) < \infty$, for all n and $\lim\limits_{n\to\infty} \mathrm{Cap}(K - F_n) = 0$ for all compact set K.

By definition, a signed measure Σ on H is smooth if its total variation measure $|\Sigma|$ is smooth. That happens if and only if $\Sigma = \Sigma^1 - \Sigma^2$, where Σ^1 and Σ^2 are positive smooth measures, obtained from Σ by applying the Jordan decomposition (see [27, page 221]).

We recall a definition from [27, Section 2.2]. We say that a positive Radon measure Σ on H is *of finite energy* if for some constant $C > 0$

$$\int |v| \, d\Sigma \leq C\sqrt{\mathcal{E}_1(v,v)}, \qquad \forall\, v \in D(\mathcal{E}) \cap C_b(H). \tag{3.7.2}$$

If (3.7.2) holds, then there exists an element $U_1\Sigma$ such that

$$\mathcal{E}_1(U_1\Sigma, v) = \int_H v \, d\Sigma, \qquad \forall\, v \in D(\mathcal{E}) \cap C_b(H).$$

Moreover, by [27, Lemma 2.2.3], all measures of finite energy are smooth.

Finally; by [27, Theorem 5.1.4], if Σ is a positive smooth measure, then there exists a PCAF $(A_t)_{t\geq 0}$, unique up to equivalence, with Revuz measure equal to Σ.

3.7.2 The non-linearity

We prove first that a.s. the function $(t,\theta) \mapsto |\partial_0\varphi(u(t,\theta))|$ is in $L^1_{\mathrm{loc}}([0,T]\times]0,1[)$ for all $T \geq 0$. We start by the following

Lemma 3.7.2. *For all $\delta \in (0,1/2)$*

$$\int \nu(dx) \left(\int_\delta^{1-\delta} |\partial_0\varphi(x_\theta)|\, d\theta \right)^2 < +\infty.$$

Proof. We have for all $\theta \in [\delta, 1-\delta]$, by the definition (3.2.4) of ν

$$\begin{aligned}\int \nu(dx)\, |\partial_0\varphi(x_\theta)|^2 \leq \frac{1}{Z} \int \mu(dx)\, |\partial_0\varphi(x_\theta)|^2\, \mathbb{1}_{\{x_\theta \in \overline{O}\}} \\ = \frac{1}{C(\theta)} \int_O |\partial_0\varphi(z)|^2\, e^{-\frac{|z|^2}{2\theta(1-\theta)}}\, dz \\ \leq \frac{1}{C_\delta} \int_O |\partial_0\varphi(z)|^2\, dz < +\infty\end{aligned}$$

by (3.2.1). Since this quantity does not depend on $\theta \in [\delta, 1-\delta]$, we have the desired result by Hölder's inequality:

$$\int \nu(dx) \left(\int_\delta^{1-\delta} |\partial_0\varphi(x_\theta)|\, d\theta \right)^2 \leq \int \nu(dx) \int_\delta^{1-\delta} |\partial_0\varphi(x_\theta)|^2\, d\theta < +\infty.$$

$\square$

Now we obtain that

Proposition 3.7.3. *The functional*

$$C_t := \int_0^t \int_\delta^{1-\delta} |\partial_0\varphi(u(s,\theta))|\, d\theta\, ds, \qquad t \geq 0,$$

is a well-defined PCAF of X. In particular, the function $(t,\theta) \mapsto |\partial_0\varphi(u(t,\theta))|$ is in $L^1_{\mathrm{loc}}([0,T]\times]0,1[)$ for all $T \geq 0$, $\mathbb{P}_x$-a.s. for all $x \in K \setminus N$ for some $N \subset K$ with $\mathrm{Cap}(N) = 0$.

Proof. Setting

$$F : H \mapsto [0,+\infty], \qquad F(w) := \int_\delta^{1-\delta} |\partial_0\varphi(w_\theta)|\, d\theta,$$

then by Lemma 3.7.2 $F \in L^2(\nu)$ and moreover we can write $C_t = \int_0^t F(X_s)\, ds$, $t \geq 0$. Denoting

$$R_1F(x) := \int_0^\infty e^{-t}\, \mathbb{E}_x\left[F(X_t)\right] dt, \qquad x \in K,$$

then it is well known that

$$\mathcal{E}_1(R_1F, v) = \int_H F\, v\, d\nu, \quad \forall v \in D(\mathcal{E}),$$

and therefore (3.7.2) holds. Then, $F\, d\nu$ is smooth and the associated PCAF is $(C_t)_{t\geq 0}$. $\square$

3.7.3 The reflection measure

We are going to apply (3.7.1) to $U^h(x) := \langle x, h\rangle$, $x \in H$, with $h \in C_c^2((0,1);\mathbb{R}^d)$. Clearly $U^h \in \mathrm{Lip}(H) \subset D(\mathcal{E})$. Our aim is to prove the following

Proposition 3.7.4. *There are an exceptional set N and a unique measure $\eta(ds, d\theta)$ on $[0,+\infty[\times[0,1]$ such that for all $x \in K \setminus N$, $\mathbb{P}_x$-a.s. for all $t \geq 0$*

$$\begin{aligned} &N_t^{[U^h]} = \\ &= \int_0^t \int_0^1 h_\theta \cdot n(u(s,\theta))\, \eta(ds, d\theta) + \frac{1}{2}\int_0^t \langle h'', u_s\rangle ds - \frac{1}{2}\int_0^t \langle h, \partial_0\varphi(u_s)\rangle ds \end{aligned} \tag{3.7.3}$$

where $h \in C_c^\infty((0,1);\mathbb{R}^d)$, and $\mathrm{Supp}(\eta) \subset \{(t,\theta) \,|\, u(t,\theta) \in \partial O\}$.

The main tools of the proof are the integration by parts formula (3.6.3) and a number of results from the theory of Dirichlet forms in [27]. We start by noticing that, by applying (3.7.1) to $U^h(x) := \langle x, h\rangle$, $x \in H$, we obtain, recalling the definition (3.6.2) of $\Sigma(y, dw)$:

Lemma 3.7.5. *The process $N^{[U^h]}$ is a linear combination of PCAFs of X, and its Revuz measure is $\dfrac{1}{2}\Sigma^h$, where*

$$\Sigma^h(dw) := (\langle w, h''\rangle - \langle \partial_0\phi(w), h\rangle)\cdot\nu(dw) + \int_{\partial O} \sigma(dy)\, n(y)\cdot h(S_w)\,\Sigma(y, dw). \tag{3.7.4}$$

Proof. The integration by parts formula (3.6.1) can be rewritten as

$$\mathcal{E}(U^h, v) = \frac{1}{2}\int v(w)\,\Sigma^h(dw), \qquad \forall\, v \in D(\mathcal{E}) \cap C_b(H).$$

By [27, Corollary 5.4.1], this implies that $\frac{1}{2}\Sigma^h$ is the Revuz measure of $N^{[U^h]}$ and that Σ^h is a smooth signed measure. By [27, Theorem 5.4.2], this implies that $N^{[U^h]}$ is $\mathbb{P}_x$-a.s. a bounded variation process for all $x \in K$; moreover, the Jordan decomposition $\Sigma^h = \Sigma_1^h - \Sigma_2^h$, with Σ_i^h positive measures concentrated on disjoint sets, corresponds to a decomposition $N^{[U^h]} = N_1^h - N_2^h$ with N_i^h a PCAF of X with Revuz measure $\frac{1}{2}\Sigma_i^h$.

Lemma 3.7.6. *For all $h \in C_c^2((0,1);\mathbb{R}^d)$, the total variation measure $|\Sigma^h|$ of Σ^h is equal to*

$$|\Sigma^h|(dw) = |\langle w, h''\rangle - \langle \partial_0\phi(w), h\rangle| \cdot \nu(dw) \,+ \int_{\partial O} \sigma(dy)\; |n(y)\cdot h(S_w)|\; \Sigma(y, dw).$$

Proof. Now, we can notice that $\nu(dw)$ and $\displaystyle\int_{\partial O} \sigma(dy)\,\Sigma(y, dw)$ are mutually singular, since the former measure is concentrated on trajectories not hitting the boundary ∂O, and the latter on trajectories hitting ∂O. Therefore

$$|\Sigma^h|(dw) = |\langle w, h''\rangle - \langle \partial_0\phi(w), h\rangle| \cdot \nu(dw) \,+ \left|\int_{\partial O} \sigma(dy)\, n(y)\cdot h(S_\cdot)\,\Sigma(y, \cdot)\right|(dw).$$

Now, by considering the sets $A := \{w : n(w(S_w)) \cdot h(S_w) \geq 0\}$ and $B = \{w : n(w(S_w)) \cdot h(S_w) < 0\}$, we can see that

$$\left| \int_{\partial O} \sigma(dy)\, n(y) \cdot h(S_\cdot)\, \Sigma(y, \cdot) \right| (dw) = \int_{\partial O} \sigma(dy)\, |n(y) \cdot h(S_w)|\, \Sigma(y, dw)$$

and we have the desired result. □

By definition, the total variation measure $|\Sigma^h|$ is smooth, and therefore so is the measure

$$\int_{\partial O} \sigma(dy)\, |n(y) \cdot h(S_w)|\, \Sigma(y, dw),$$

since it is non-negative and bounded above by $|\Sigma^h|$, for any $h \in C_c^2((0,1); \mathbb{R}^d)$. Let us now consider a non-negative $g \in C_c^2(0,1)$ and a basis $\{e_1, \ldots, e_d\}$ of $\mathbb{R}^d$. Then the measure

$$\Lambda^g(dw) := \sum_{i=1}^{d} \int_{\partial O} \sigma(dy)\, g(S_w)\, |n(y) \cdot e_i|\, \Sigma(y, dw)$$

is smooth since it is sum of smooth measures. For any interval $I \Subset (0,1)$ we can set

$$\Gamma^I(dw) := \int_{\partial O} \sigma(dy)\, \mathbb{1}_I(S_w)\, \Sigma(y, dw). \tag{3.7.5}$$

Let $\kappa := \min_{z \in \mathbb{S}^{d-1}} \sum_{i=1}^{d} |z \cdot e_i|$. By compactness, $\kappa > 0$ and therefore, for $g \in C_c^2(0,1)$ such that $g \geq \mathbb{1}_I$, we obtain $0 \leq \Gamma^I \leq \Lambda^g/\kappa$. Hence, Γ^I is smooth.

In particular, if $\{I_n\}_n$ is any countable partition of $(0,1)$ in intervals $I_n \Subset (0,1)$, then we obtain that the finite measure

$$\Gamma(dw) := \Gamma^1(dw) = \sum_n \Gamma^{I_n}(dw) \tag{3.7.6}$$

is also smooth and finite by its explicit expression. Now, for any $g \in C([0,1])$, the measure

$$\Gamma^g(dw) := \int_{\partial O} \sigma(dy)\, g(S_w)\, \Sigma(y, dw),$$

is also smooth, since $|g| \leq \|g\|_\infty 1$ implies $0 \leq |\Gamma^g| \leq \|g\|_\infty \Gamma^1$. By [27, Theorem 5.1.4], there exists a PCAF $(A_t^g)_{t \geq 0}$, unique up to equivalence, with Revuz measure equal to Γ^g, for any $g \in C([0,1])$. At the same time, for any interval $I \subseteq [0,1]$, there exists a PCAF $(A_t^I)_{t \geq 0}$, unique up to equivalence, with Revuz measure equal to Γ^I. Moreover

$$|A_t^g| \leq \|g\|_\infty A_t^1, \qquad \forall\, t \geq 0, \tag{3.7.7}$$

since the positive finite measure $(\|g\|_\infty \cdot \Gamma^1 - \Gamma^g)(dx)$ is finite and smooth and is therefore the Revuz measure of a PCAF, so that we can conclude by the linearity of the Revuz correspondence.

We want now to prove that there exists a finite positive measure η on $[0,T]\times[0,1]$ such that

$$A_T^g = \int_{[0,T]\times[0,1]} g(\theta)\,\eta(ds,d\theta), \qquad \forall T\geq 0,\ g\in C([0,1]). \tag{3.7.8}$$

Let $(g_n)_n$ be a dense sequence in $C([0,1])$. By the Revuz correspondence we have $A^{g_n+g_m} = A^{g_m}+A^{g_n}$. Let $\Lambda = \bigcap_{n,m}\{A^{g_n+g_m} = A^{g_m}+A^{g_n}\}$, so that $\mathbb{P}_x(\Lambda)=1$ for all $x\in K\setminus N$, where N is an exceptional set. By (3.7.7), we obtain that the map $C([0,1])\ni g\mapsto A_T^g$ is linear, continuous and if $g\geq 0$ then $A_T^g\geq 0$. Then by the Riesz representation theorem there exists a Radon measure $A_T(d\theta)$ on $[0,1]$, such that

$$A_T^g = \int_{[0,1]} g(\theta)\,A_T(d\theta), \qquad \forall\, g\in C([0,1]).$$

Moreover, $A_t(d\theta)$ satisfies $0\leq A_t(d\theta)\leq A_T(d\theta)$ for $0\leq t\leq T$. Therefore $A_t\ll A_T$. By the Radon-Nykodim theorem we have

$$A_t(B) = \int_B C_t(y) A_T(dy)$$

where $C_t\in L^1(A_T(d\theta))$.

Now, the problem is that $C_t(y)$ is defined for A_T-a.e. y, and the set of definition might depend on t. We must show that it is possible to find a version of $(C_t)_{0\leq t\leq T}$ defined on the same set of full A_T-measure.

We claim that for A_T-a.e. θ, $t\mapsto C_t(\theta)$ is equal to a càdlàg function. Indeed, let $(q_n)_n$ be a dense sequence in $[0,T]$ and set

$$\Lambda := \bigcap_n \left\{\theta : C_{q_n}(\theta)\leq C_t(\theta),\ \forall\, t\in (q_m)_m,\ q_n\leq t,\ C_{q_n}(\theta) = \lim_{s\in(q_m)_m\downarrow q_n} C_s(\theta)\right\}.$$

Notice that $A_T(\Lambda^c)=0$. $C.$ is dA a.e well defined. We denote by $\tilde{C}.(\cdot)$ the function defined on $[0,T]\times[0,1]$ by

$$\tilde{C}_t(\theta) := \lim_{s\in(q_n)\downarrow t} C_s(\theta), \quad (t,\theta)\in [0,T[\,\times\Lambda, \qquad \tilde{C}_T := C_T,$$

and $\tilde{C}_t(\theta):=0$ if $t<T$ and $\theta\notin\Lambda$. By continuity of $t\mapsto A_{0,t}(B)$, we obtain that

$$A_t(B) = \int_B \tilde{C}_t(y)\,A_T(dy).$$

Moreover $\tilde{C}.(\theta)$ is càdlàg and non-decreasing and measurable, so that there exists a measurable kernel $(\gamma_y(B),\ y\in[0,1],\ B\in\mathcal{B}([0,T]))$, such that $\tilde{C}_t(y)-\tilde{C}_s(y) = \gamma_y(]s,t])$ and therefore

$$\begin{aligned} A_t(B)-A_s(B) &= \int_B \left(\tilde{C}_t(y)-\tilde{C}_s(y)\right) A_T(dy) \\ &= \int_B \gamma_y(]s,t])\,A_T(dy), \qquad t,s\in[0,T]. \end{aligned}$$

Therefore, the measure $\eta(ds, dy) := \gamma_y(ds)\, A_T(dy)$ on $[0,T]\times[0,1]$ satisfies (3.7.8).

Now we have to show that the measure η satisfies $\mathrm{Supp}(\eta) \subset \{(t,\theta)\,|\,u(t,\theta)\in \partial O\}$ and (3.7.4). We set

$$F : C\left([0,1];\overline{O}\right) \mapsto \mathbb{R}, \quad F(w) := \mathbb{1}_{\{w(\theta)\notin\partial O,\ \forall\,\theta\in[0,1]\}}$$

and

$$L_t := \int_0^t F(X_s)\,\eta(ds\times[0,1]), \qquad t\geq 0.$$

Then, by [27, Theorem 5.1.3], $(L_t)_{t\geq 0}$ is a PCAF of X with Revuz measure given by $\dfrac{1}{2} f(w)\cdot\Gamma(dw)$, see (3.7.6). On the other hand, $\Gamma(\{w : w(\theta)\notin\partial O,\ \forall\,\theta\in[0,1]\}) = 0$ by the very definition of Γ; indeed, $\Sigma(y,dw)$ is the law of a process which visits a.s. $y\in\partial O$ at same time in $[0,1]$.

Therefore, by the one-to-one correspondence between PCAFs and positive smooth measures, see [27, Theorem 5.1.3], we conclude that $f(w)\cdot\Gamma(dw)\equiv 0$ and therefore $L\equiv 0$. Thus, for $\eta(ds\times[0,1])$-a.e. s, $u(s,\cdot)$ visits ∂O at some $\theta\in(0,1)$, and in particular $S(u(s,\cdot))$, i.e. the smallest such θ, is in $(0,1)$.

Let us now notice that, again by [27, Theorem 5.1.3] and by (3.7.8), for any bounded Borel $G : C\left([0,1];\overline{O}\right)\mapsto\mathbb{R}$ and for any bounded Borel $g:[0,1]\mapsto\mathbb{R}$, the process

$$t\mapsto \int_0^t G(X_s)\,dA^g_s = \int_{[0,t]\times[0,1]} G(X_s)\,g(\theta)\,\eta(ds,d\theta)$$

is a PCAF of X with Revuz measure $\dfrac{1}{2}G(w)g(S_w)\,\Gamma(dw)$. Therefore for any bounded Borel $G : C\left([0,1];\overline{O}\right)\times[0,1]\mapsto\mathbb{R}$, the process

$$t\mapsto \int_{[0,t]\times[0,1]} G(X_s,\theta)\,\eta(ds,d\theta)$$

is a PCAF of X with Revuz measure $\dfrac{1}{2}G(w,S_w)\,\Gamma(dw)$. In particular, if we choose $G(w,\theta) := \mathbb{1}_{\{w(\theta)\notin\partial O\}}$, then the process

$$t\mapsto \int_{[0,t]\times[0,1]} \mathbb{1}_{\{u(s,\theta)\notin\partial O\}}\,\eta(ds,d\theta)$$

is a PCAF of X with Revuz measure $\dfrac{1}{2}\mathbb{1}_{\{w(S_w)\notin\partial O\}}\Gamma(dw)\equiv 0$.

Therefore, by the one-to-one correspondence between PCAFs and positive smooth measures, see again [27, Theorem 5.1.3], we conclude that $\eta(\{(s,\theta) : u(s,\theta)\notin\partial O\}) = 0$, i.e. $\mathrm{Supp}(\eta)\subset\{(s,\theta)\,|\,u(s,\theta)\in\partial O\}$.

It remains to show (3.7.4). We recall that, by (3.7.8), for all Borel $I\subseteq[0,1]$, the process $t\mapsto\eta([0,t]\times I)$ is a PCAF of X with Revuz measure $\dfrac{1}{2}\Gamma^I$, see (3.7.5). Now, it is enough to notice that the CAF in the right hand side of (3.7.4) has Revuz measure $\dfrac{1}{2}\Sigma^h$, given by (3.7.4). Since $N^{[U^h]}$ has the same Revuz measure, then by the one-to-one correspondence between PCAFs and positive smooth measures, $N^{[U^h]}$ and the CAF in the right hand side of (3.7.4) are equivalent. $\square$

3.7.4 Identification of the noise term

We deal now with the identification of $M^{[U^h]}$ with the integral of h with respect to a space-time white noise.

Proposition 3.7.7. *There exists a Brownian sheet* $(W(t,\theta),\, t \geq 0, \theta \in [0,1])$*, such that*

$$M_t^{[U^h]} = \int_0^t \int_0^1 h_\theta \, W(ds, d\theta), \qquad h \in H. \tag{3.7.9}$$

Proof. We recall that, for $U \in D(\mathcal{E})$, the process $M^{[U]}$ is a continuous martingale, whose quadratic variation $(\langle M^{[U]} \rangle_t)_{t \geq 0}$ is a PCAF of X with Revuz measure $\mu_{\langle M^{[U]} \rangle}$ given by the formula

$$\int f \, d\mu_{\langle M^{[U]} \rangle} = 2\mathcal{E}(Uf, U) - \mathcal{E}(U^2, f), \qquad \forall\, f \in D(\mathcal{E}) \cap C_b(H), \tag{3.7.10}$$

see [27, Theorem 5.2.3]. Now, if we apply this formula to $U^h(x) = \langle x, h \rangle$, then we obtain

$$\int f \, d\mu_{\langle M^{[U^h]} \rangle} = \|h\|^2 \int f \, d\nu, \qquad \forall\, f \in D(\mathcal{E}) \cap C_b(H).$$

Therefore, the quadratic variation $\langle M^{[U^h]} \rangle_t$ is equal to $\|h\|^2 t$ for all $t \geq 0$, and, by Lévy's Theorem, $(M^{[U^h]} \cdot \|h\|^{-1})_{t \geq 0}$ is a Brownian motion. Moreover, the parallelogram law, if $h_1, h_2 \in H$ and $\langle h_1, h_2 \rangle = 0$, then the quadratic covariation between $M^{[U^{h_1}]}$ and $M^{[U^{h_2}]}$ is equal to

$$\langle M^{[U^{h_1}]}, M^{[U^{h_2}]} \rangle_t = t \, \langle h_1, h_2 \rangle, \qquad t \geq 0.$$

Therefore, $(M_t^{[U^h]}, t \geq 0, h \in H)$ is a Gaussian process with covariance structure

$$\mathbb{E}_x \left(M_t^{[U^{h_1}]} \, M_s^{[U^{h_2}]} \right) = s \wedge t \, \langle h_1, h_2 \rangle.$$

If we define $W(t,\theta) := M_t^{[U^h]}$ with $h := \mathbb{1}_{[0,\theta]}$, $t \geq 0$, $\theta \in [0,1]$, then W is the desired Brownian sheet. □

3.7.5 From $K \setminus N$ to K

We have so far proved existence of an exceptional set N such that for all $x \in K \setminus N$ there is a weak solution of equation (3.1.1). We show now how to construct a weak solution for $x \in N$.

Let $x \in N$. By the absolute continuity relation (3.4.5), we have that $\mathbb{P}_x$-a.s. $X_\varepsilon \in K \setminus N$ for $\varepsilon > 0$, since $\nu(K \setminus N) = 1$. Therefore, we can set for all $\omega \in E$ and $0 < \varepsilon \leq s \leq t$

$$\eta^\varepsilon([s,t] \times I)(\omega) := \eta([s-\varepsilon, t-\varepsilon] \times I)(\theta_\varepsilon \omega),$$

where $(\theta_t)_{t \geq 0}$ is the time-translation operator of E. Then $\varepsilon \mapsto \eta^\varepsilon([s,t] \times I)$ is monotone non-increasing, since

$$\eta^\varepsilon([s,t] \times I)(\omega) - \eta^\delta([s,t] \times I)(\omega) = \eta^{\delta-\varepsilon}([s,t] \times I)(\theta_\varepsilon \omega), \qquad 0 < \varepsilon < \delta.$$

As $\varepsilon \downarrow 0$, we obtain existence of a σ-finite measure $\eta(ds, d\theta)$ on $]0,T] \times [0,1]$, which satisfies the required properties. A similar argument works for the non-linear part. The proof of Proposition 3.7.1 is concluded.

3.8 Pathwise uniqueness and strong solutions

We prove that equation (3.1.1) has a pathwise unique solution. This follows the lines of [51]. By a Yamada-Watanabe type result from [42], pathwise uniqueness and existence of weak solutions imply existence and uniqueness of strong solutions and uniqueness in law.

3.8.1 Pathwise uniqueness

Proposition 3.8.1. *Pathwise uniqueness holds for equation* (3.1.1)*.*

Proof. Let (u^1, η^1, W) and (u^2, η^2, W) be two weak solutions of (3.1.1), we denote

$$z := u^1 - u^2, \qquad \pi(ds, d\theta) = n(u^1(s,\theta)) \cdot \eta^1(ds, d\theta) - n(u^2(s,\theta)) \cdot \eta^2(ds, d\theta),$$

so for $h \in C_c^2((0,1) \times [0,T]; \mathbb{R}^d)$ and $0 < \varepsilon \leq T$, denoting $\partial_s = \dfrac{\partial}{\partial s}$ and $\partial_\theta^2 = \dfrac{\partial^2}{\partial \theta^2}$,

$$\begin{aligned} \langle h_T, z_T \rangle - \langle h_\varepsilon, z_\varepsilon \rangle = \frac{1}{2} \int_\varepsilon^T \langle h'', z_s \rangle \, ds - \frac{1}{2} \int_\varepsilon^T \langle h_s, \partial_0 \phi(u_s^1) - \partial_0 \phi(u_s^2) \rangle ds \\ + \int_\varepsilon^T \int_0^1 h(s,\theta) \cdot \pi(ds, d\theta) + \int_\varepsilon^T \langle \partial_s h_s, z_s \rangle ds. \end{aligned} \tag{3.8.1}$$

Let ζ be an infinitely differentiable even function, with support contained in $[-1,1]$, such that $\int_{[-1,1]} \zeta(x) dx = 1$ and $\sum_{i,j} \zeta(x_i - x_j) y_i y_j \geq 0$ for any $(x_i)_{i \leq n}$ and $(y_i)_{i \leq n} \in \mathbb{R}^n$, $n \in \mathbb{N}$. Let ψ be an infinitely differentiable function with compact support, we consider now the function $h_{n,m}$ defined by

$$h_{n,m} := ((z\psi) * \zeta_{n,m}) \psi$$

where $\zeta_n(x) := n\zeta(nx)$ and $\zeta_{n,m}(t,\theta) := \zeta_n(t)\zeta_m(\theta)$. We will study the asymptotic behaviour of each term in (3.8.1) substituting h by $h_{n,m}$. First we have

$$\lim_{n,m} \langle h_{n,m}(t), z(t) \rangle = \|z(t)\, \psi\|^2.$$

Next

$$\int_\varepsilon^T \langle \partial_s h_{n,m}(s), z(s) \rangle \, ds = \int_\varepsilon^T \int_{(t-1/n)^+}^{t+1/n} \zeta_n'(t-s)\, \Gamma_m(s,t)\, ds\, dt$$

where Γ_m is a symmetric function of (s,t), defined by

$$\Gamma_m(s,t) := \int_0^1 \int_0^1 z(s,\theta) \cdot z(t,\upsilon)\, \psi(\theta)\, \zeta_m(\upsilon - \theta)\, \psi(\upsilon)\, d\theta\, d\upsilon.$$

As $\zeta'(s) = -\zeta'(-s)$ the integral

$$\int_\varepsilon^T \int_{\max(t-1/n,\varepsilon)}^{\min(t+1/n,T)} \zeta_n'(t-s)\, \Gamma_m(s,t)\, ds\, dt = \int_{[\varepsilon,T]^2} \mathbb{1}_{\{|t-s| \leq 1/n\}}\, \zeta_n'(t-s)\, \Gamma_m(s,t)\, ds\, dt$$

vanishes. Therefore if $1/n \le \varepsilon$ then as $n \to +\infty$

$$\left| \int_{\varepsilon}^{T} \langle \partial_s h_{n,m}(s), z(s) \rangle \, ds \right| \le \left| \int_{T-1/n}^{T} dt \int_{T}^{t+1/n} ds \, \zeta_n'(t-s) \, \Gamma_m(s,t) \right|$$
$$+ \left| \int_{\varepsilon}^{\varepsilon+1/n} dt \int_{t-1/n}^{\varepsilon} ds \, \zeta_n'(t-s) \, \Gamma_m(s,t) \right| \le \frac{K}{n} \to 0.$$

Now, because of the properties of (u_i, η_i)

$$\lim_{n,m} \int_{\varepsilon}^{T} \int_{0}^{1} h_{n,m}(s,\theta) \, \pi(ds, d\theta) = \int_{\varepsilon}^{T} \int_{0}^{1} \psi(\theta) \, z(s,\theta) \cdot \pi(ds, d\theta)$$
$$= - \int_{\varepsilon}^{T} \int_{0}^{1} \psi^2(\theta) \left\{ u^2(s,\theta) \cdot n(u^1(s,\theta)) \, \eta^1(ds, d\theta) + u^1(s,\theta) \cdot n(u^2(s,\theta)) \, \eta^2(ds, d\theta) \right\}$$
$$\le 0.$$

By the convexity of ϕ we have

$$\lim_{n,m} \int_{\varepsilon}^{T} \langle h_{n,m}(s), \partial_0 \phi(u_s^1) - \partial_0 \phi(u_s^2) \rangle \, ds \ge 0.$$

For the last term, we notice that $\int_{\varepsilon}^{T} \langle \partial_\theta^2 h_{n,m}, z_s \rangle ds \to \int_{\varepsilon}^{T} \langle \partial_\theta^2 h_n, z_s \rangle ds$ when $m \to \infty$. We first suppose that z is smooth, integrating by parts $\langle \partial_\theta^2 h_n(s), z(s) \rangle$ we obtain

$$\langle \partial_\theta^2 h_n(s), z(s) \rangle \le \langle (z\psi) * \zeta_n, \psi'' \, z(s) \rangle + \langle (z\psi') * \zeta_n, \psi' \, z(s) \rangle.$$

Moreover we obtain the same inequality for z approximating z with smooth functions. As a result

$$\liminf_{n} \lim_{m} \int_{\varepsilon}^{T} \langle \partial_\theta^2 h_{n,m}, z_s \rangle ds \le \frac{1}{2} \int_{\varepsilon}^{T} \int_{0}^{1} |z_s|^2 \, (\psi^2)'' ds$$

Finally, we have obtained

$$\int_{0}^{1} \left(z^2(T,\theta) - z^2(\varepsilon,\theta) \right) \psi^2(\theta) \, d\theta \le \frac{1}{2} \int_{\varepsilon}^{T} \int_{0}^{1} z^2(s,\theta) \, (\psi^2)''(\theta) \, ds \, d\theta$$

and letting $\varepsilon \to 0$

$$\int_{0}^{1} z^2(T,\theta) \, \psi^2(\theta) \, d\theta \le \frac{1}{2} \int_{0}^{T} \int_{0}^{1} z^2(s,\theta) \, (\psi^2)''(\theta) \, ds \, d\theta.$$

The rest of the proof consists of choosing a judicious expression for ψ, which can be done as at the end of the proof of uniqueness in [51]. Finally, we obtain that that $z \equiv 0$ and $\eta^1 = \eta^2$. □

3.8.2 Strong solutions

Until now we have dealt with weak solutions. Now we show that all weak solutions are in fact strong.

We first recall some results of [42], where a general version of the Yamada-Watanabe and Engelbert results relating existence and uniqueness of strong and weak solutions for stochastic equations is given.

For Itô equations, Yamada and Watanabe [62] proved that weak existence and strong uniqueness imply strong existence and weak uniqueness. Engelbert [24] extended this result to a somewhat more general class of equations and gave a converse where the roles of existence and uniqueness are reversed, that is, weak uniqueness, in the sense that the joint distribution of $\mathcal{X}$ and W (such as below) is uniquely determined, and strong existence imply strong uniqueness. Let us start with some definitions.

Let S_1 and S_2 be Polish spaces, and let $\Gamma : S_1 \times S_2 \to \mathbb{R}$ be a Borel measurable function, let $\mathcal{Y}$ be an S_2-valued random variable with distribution ν. We are interested in solutions of the equation

$$\Gamma(\mathcal{X}, \mathcal{Y}) = 0 \quad a.s., \quad \mathcal{L}(\mathcal{Y}) = \nu \tag{3.8.2}$$

where $(\mathcal{X}, \mathcal{Y})$ is a couple of random variables with values in $S_1 \times S_2$. We say that $(\mathcal{X}, \mathcal{Y})$ is a solution for (Γ, ν) if (3.8.2) holds. Being a solution of (3.8.2) is a property of the joint distribution of $(\mathcal{X}, \mathcal{Y})$. A measure μ on $S_1 \times S_2$ is a joint solution measure if $\mu(S_1 \times \cdot) = \nu(\cdot)$ and

$$\int_{S_1 \times S_2} |\Gamma(x, y)| \, d\mu(x, y) = 0. \tag{3.8.3}$$

Let $\mathcal{P}(S_1 \times S_2)$ be the space of probability measure on $S_1 \times S_2$, we denote by $\mathcal{S}_{\Gamma,\nu}$ the set of all joint solutions.

Definition 3.8.2. *A solution $(\mathcal{X}, \mathcal{Y})$ for (Γ, ν) is a strong solution if there exists a Borel measurable function $F : S_1 \to S_2$ such that $\mathcal{X} = F(\mathcal{Y})$ a.s.*

Definition 3.8.3. *Pointwise uniqueness holds for* (3.8.2) *holds if $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{Y}$ defined on the same probability space with $\mu_{\mathcal{X}_1,\mathcal{Y}}, \mu_{\mathcal{X}_2,\mathcal{Y}} \in \mathcal{S}_{\Gamma,\nu}$ implies $\mathcal{X}_1 = \mathcal{X}_2$ a.s.*

Engelbert, in [24], introduces a weaker notion which is aqnalogous to:

Definition 3.8.4. *For $\mu \in \mathcal{S}_{\Gamma,\nu}$, μ-pointwise uniqueness holds if $\mathcal{X}_1$, $\mathcal{X}_2$, cY defined on the same probability space with $\mu_{\mathcal{X}_1,\mathcal{Y}} = \mu_{\mathcal{X}_1,\mathcal{Y}} = \mu$ implies $\mathcal{X}_1 = \mathcal{X}_2$ a.s.*

We can exhibit the following result which corresponds to lemma 2.7 of Kurtz [42]

Lemma 3.8.5. *If $\mu \in \mathcal{S}_{\Gamma,\nu}$ and μ-pointwise uniqueness holds, then μ is the joint distribution for a strong solution. Moreover if there is a $\mu \in \mathcal{S}_{\Gamma,\nu}$ that does not correspond to a strong solution, then pointwise uniqueness does not hold.*

Remark 3.8.6. *If we drop any mention of the equation* (3.8.2) *and simply require that $\mathcal{S}_{\Gamma,\nu}$ is a convex subset of $\mathcal{P}(S_1 \times S_2)$ such that $\mu \in \mathcal{S}_{\Gamma,\nu}$ implies $\mu(S_1 \times \cdot) = \nu$. If moreover one say that $(\mathcal{X}, \mathcal{Y})$ is a solution for (Γ, ν) if $\mu_{(\mathcal{X},\mathcal{Y})} \in \mathcal{S}_{\Gamma,\nu}$, then all the definitions make sense and the previous results holds.*

Let us now consider two Polish spaces E_1 and E_2,let $S_i := \mathbb{D}_{E_i}[0, +\infty)$, be the Skorohod space of cadlag E_i-valued functions, and let $\mathcal{Y}$ be a r.v. in $\mathbb{D}_{E_2}[0, +\infty)$. We denote by $\mathcal{F}_t^{\mathcal{Y}}$ the σ-algebra $\sigma(\mathcal{Y}_s; s \le t)$

Definition 3.8.7. *A process $\mathcal{X}$ in $\mathbb{D}_{E_1}[0,+\infty)$ is compatible with $\mathcal{Y}$ if for each $t \geq 0$ and $h \in \mathcal{B}(\mathbb{D}_{E_2}[0,+\infty)$,*

$$\mathbb{E}[h(\mathcal{Y})|\mathcal{F}_t^{\mathcal{X},\mathcal{Y}}] = \mathbb{E}[h(\mathcal{Y})|\mathcal{F}_t^{\mathcal{Y}}] \quad a.s. \tag{3.8.4}$$

We have the following result:

Lemma 3.8.8. *1. $\mathcal{X}$ is compatible with $\mathcal{Y}$ if and only if every $(\mathcal{F}_t^{\mathcal{Y}})_t$-martingale, is an $(\mathcal{F}_t^{\mathcal{X},\mathcal{Y}})_t$-martingale.*

2. If $\mathcal{Y}$ has independent increments, then $\mathcal{X}$ is compatible with $\mathcal{Y}$ if and only if for each $t \geq 0$, $(\mathcal{Y}_{t+s} - \mathcal{Y}_t)_s$ is independent of $\mathcal{F}_t^{\mathcal{X},\mathcal{Y}}$.

3. If $\mathcal{X}$ is compatible with $\mathcal{Y}$. If $\mathcal{X} = F(\mathcal{Y})$ for some measurable $F : S_2 \to S_1$, then $\mathcal{X}$ is adapted to $\mathcal{Y}$.

We denote $\mathcal{S}_{\Gamma,\nu}^c$ the convex subset of $\mathcal{P}(S_1 \times S_2)$ such that for all $\mu \in \mathcal{S}_{\Gamma,\nu}^c$ fullfils the constraints in Γ, $\mu(S_1 \times \cdot) = \nu$, and if $(\mathcal{X},\mathcal{Y})$ has law μ, then $\mathcal{X}$ is compatible with $\mathcal{Y}$. To take into into account the compatibility requirement, we change the definition of pointwise uniqueness

Definition 3.8.9. *Let $\mathcal{X}_1$, $\mathcal{X}_2$, and $\mathcal{Y}$ be defined on the same probability space. Let $\mathcal{X}_1$, $\mathcal{X}_2$, be S_1-valued and $\mathcal{Y}$ be S_2-valued. $(\mathcal{X}_1,\mathcal{X}_2)$ are jointly compatible with $\mathcal{Y}$ if for all $t \geq 0$ and $f \in \mathcal{B}(S_2)$*

$$\mathbb{E}[f(\mathcal{Y})|\mathcal{F}_t^{\mathcal{X}_1,\mathcal{X}_2,\mathcal{Y}}] = \mathbb{E}[f(\mathcal{Y})|\mathcal{F}_t^{\mathcal{Y}}] \tag{3.8.5}$$

pointwise uniqueness holds for compatible solutions of (Γ,ν), if for every triple processes $(\mathcal{X}_1,\mathcal{X}_2,\mathcal{Y})$ defined on the same sample space such that $\mu_{\mathcal{X}_1,\mathcal{Y}}$, $\mu_{\mathcal{X}_1,\mathcal{Y}} \in \mathcal{S}_{\Gamma,\nu}^c$ and $(\mathcal{X}_1,\mathcal{X}_2)$ is jointly compatible with $\mathcal{Y}$, $\mathcal{X}_1 = \mathcal{X}_2$ a.s.

Finally we have:

Theorem 3.8.10. *If $\mu \in \mathcal{S}_{\Gamma,\nu}^c$. Then μ-pointwise uniqueness holds if and only if the solution corresponding to μ is strong.*

We recall now that a weak solution of equation (3.1.1) is given by a triple (u,η,W).

Lemma 3.8.11. *We set $\mathcal{X} := (u,\eta)$ and $\mathcal{Y} := W$. Thus defined, $\mathcal{X}$ is compatible with $\mathcal{Y}$.*

Proof. Indeed $\mathcal{Y}$ is adapted to $(\mathcal{F}_t^{\mathcal{X}})_t$ by equation (3.1.1). Now we recall that for all $h \in L^2((0,1))$, $\langle h, W_t\rangle$ and $\langle h, \eta_t\rangle$ are additive functionals of X. As $M_t^h := \langle h, W_t\rangle$ is a $(\mathcal{F}_t^{\mathcal{X}})_t$-martingale with quadratic variation equals to $\|h\|^2 t$, $(M_{\cdot+t}^h - M_t^h)\|h\|^{-1}$ is a Brownian motion independent of $\mathcal{F}_t^{\mathcal{X}}$. $\mathcal{Y}$, as process valued in the Sobolev space $H^{-1}((0,1))$, has independent increments with respect $(\mathcal{F}_t^{\mathcal{X}})_t$. One can conclude by lemma 3.8.8. $\square$

In the notation of [42], equation (3.1.1) can be interpreted as a relation our constraint Γ with on $S_1 \times S_2$ a Borel function defined on the product of two Polish spaces S_1 and S_2, for which *pathwise* (or *pointwise*) uniqueness holds by Proposition 3.8.1. Therefore, by [42, Lemma 3.5], reminded above any weak solution of (3.1.1) is also strong. This concludes the proof of Theorem 3.2.2.

3.9 The reflection measure

We want now to prove Theorem 3.2.3, following the approach of [64]. Let $I \subseteq [0,1]$ be a Borel set. Denote by ψ_I the indicator function of the set $\{x \in K : x(\theta) \notin \partial O, \forall \theta \in [0,1] \backslash I\}$. The key point is the following formula: for all $F \in C_b(H)$

$$\int_{\partial O} \sigma(dy) \int \psi_I(w)\, F(w)\; \Sigma(y,dw) = \int_{\partial O} \sigma(dy) \int F(w)\, \mathbb{1}_I(S_w)\, \Sigma(y,dw). \quad (3.9.1)$$

By the definition of ψ_I, this follows because $\Sigma(y,dw)$-a.s. S_w is the only $\theta \in [0,1]$ such that $w_\theta \in \partial O$. Let $A_t := \eta([0,t] \times [0,1])$, $t \geq 0$. We consider the following PCAF of X:

$$(\psi_I \cdot A)_t \;:=\; \int_0^t \psi_I(X_s)\, dA_s, \quad t \geq 0.$$

Its Revuz measure is

$$\frac{1}{2}\, \psi_I(w) \int_{\partial O} \sigma(dy)\, \Sigma(y,dw).$$

In particular, by (3.9.1):

$$\int_K \mathbb{E}_x \left[\int_0^1 [F\psi_I]\,(X_s)\, dA_s \right] \nu^F(dx)$$

$$= \frac{1}{2} \int_{\partial O} \sigma(dy) \int [F\psi_I]\,(w)\, \Sigma(y,dw) \;=\; \frac{1}{2} \int_{\partial O} \sigma(dy) \int F(w)\, \mathbb{1}_I(S_w)\, \Sigma(y,dw)$$

which is the Revuz measure of A^{1_I}, see (3.7.8). By Theorem 5.1.6 in [27], we obtain that $\psi_I \cdot A$ and A^{1_I} are in fact equivalent as PCAFs of X, i.e. for all $x \in K$:

$$\eta([0,t],I) \;=\; \int_0^t \psi_I(X_s)\, \eta(ds,[0,1]) \quad \forall t \geq 0,\ \mathbb{P}_x\text{−a.s.} \quad (3.9.2)$$

Fix $x \in K$. We consider regular conditional distributions $(t,J) \mapsto \gamma(t,J)$ of η on $[0,\infty) \times [0,1]$, w.r.t. the Borel map $(t,\theta) \mapsto t$, where $t \geq 0$, $J \subseteq [0,1]$ Borel. In other words, we obtain a σ-finite measurable kernel $(t,J) \mapsto \gamma(t,J)$ such that:

$$\eta([t,T],J) \;=\; \int_t^T \gamma(s,J)\, \eta(ds,[0,1]) \quad (3.9.3)$$

for all $J \subset [0,1]$ and $0 \leq t \leq T < \infty$. By (3.9.2) and (3.9.3) there exists a measurable set $S \subseteq \mathbb{R}^+$ such that a.s.:

$$\eta\left(\left[\mathbb{R}^+ \backslash S\right] \times [0,1]\right) \;=\; 0, \quad \text{and for all } s \in S:\ \gamma(s,[0,1]) \;>\; 0,$$

$$\gamma(s,[a_n,b_n]) \;=\; \psi_{[a_n,b_n]}(X_s), \quad \forall a_n, b_n \in \mathbb{Q} \cap [0,1]. \quad (3.9.4)$$

Notice that, since ψ_I is an indicator function, the right hand side of (3.9.4) can assume only the values 0 and 1. Therefore the measure $I \mapsto \gamma(s,I)$ takes only the values 0 and 1 on all intervals I with rational extremes in $[0,1]$, and the value 1 is assumed, since $\gamma(s,[0,1]) > 0$. Then $\gamma(s,\cdot)$ is a Dirac mass at some point $r(s) \in [0,1]$.

Let now $s \in S$ and $q_n, p_n \in \mathbb{Q}$, such that $q_n \uparrow r(s)$, $p_n \downarrow r(s)$. Set $I_n := [q_n, p_n]$: then

$$1 = \gamma(s, I_n) = \psi_{I_n}(X_s),$$

which, by the definition of ψ_{I_n}, means $u(s,\theta) \notin \partial O$ for all $\theta \in [0,1] \backslash I_n$; moreover

$$0 = \gamma(s, [0,1] \backslash \{r(s)\}) = \psi_{[0,1]\backslash\{r(s)\}}(X_s),$$

so that $u(s, r(s)) \in \partial O$. Therefore, $r(s)$ is the unique $\theta \in [0,1]$ such that $u(s,\theta) \in \partial O$. Finally, since the support of η is contained in $\{(t,\theta) : u(t,\theta) \in \partial O\}$ and a.s.

$$(S \times [0,1]) \cap \{(t,\theta) : u(t,\theta) \in \partial O\} \; = \; \{(s, r(s)) : s \in S\} := \mathcal{S},$$

then $\eta\left((\mathbb{R}^+ \times [0,1]) \backslash \mathcal{S}\right) \; = \; 0$. This concludes the proof of Theorem 3.2.3.

CHAPTER 4

A SKEW STOCHASTIC HEAT EQUATION

In this chapter we introduce a Stochastic PDE with a possibly non-continuous and non-convex potential. In particular, we want to consider a potential which (mildly) favors positive over negative values of the solutions, in analogy with a well-known one-dimensional process known in the literature as the skew Brownian motion, see section 4.1.1 below.

4.1 Introduction

4.1.1 The skew Brownian motion

Consider the following stochastic differential equation in $\mathbb{R}$:

$$X_t = X_0 + B_t + \beta \tilde{L}_t^0, \qquad t \geq 0, \tag{4.1.1}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}$, and $(\tilde{L}_t^0)_{t\geq 0}$ is the symmetric local time at 0 of the process $(X_t)_{t\geq 0}$, namely

$$\tilde{L}_t^0 = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{(|X_s| \leq \varepsilon)} \, ds. \tag{4.1.2}$$

Harrison and Shepp [39] have proved that equation (4.1.1)-(4.1.2) has a unique solution iff $|\beta| \leq 1$ and there is no solution if $|\beta| > 1$. In the former case, the process $(X_t)_{t\geq 0}$ has the law of the *skew Brownian motion* with parameter $\alpha = (1+\beta)/2$, i.e. a Brownian motion whose excursions are chosen to be positive, respectively negative, independently of each other, and each with probability α, resp. $1-\alpha$.

In this chapter we want to introduce a stochastic heat equation which has some analogy with (4.1.1)-(4.1.2). Let us also note that an invariant measure for $(X_t)_{t\geq 0}$ is given by

$$m_\alpha(dx) = (1-\alpha)\mathbb{1}_{(x>0)} \, dx + \alpha \mathbb{1}_{(x<0)} \, dx = C \exp(-c\, \mathbb{1}_{(x>0)}(x)) \, dx,$$

where c, C are constants depending on α. Moreover $(X_t)_{t\geq 0}$ is associated with the

Dirichlet form in $L^2(m_\alpha)$

$$E(u,v) := \frac{1}{2}\int_{\mathbb{R}} u'v'\,dm_\alpha.$$

For more on the skew BM, we refer to [39, 45], and Exercices III.1.16, VI.1.25, VI.2.24, VII.1.23, X.2.24, XII.2.16 in [58]. See also sections 4.5.1 and 4.5.2 below.

4.1.2 A skew SPDE

In this chapter we want to study a *skew stochastic heat equation*, namely the stochastic partial diffential equation (SPDE)

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\alpha}{2}\dfrac{\partial}{\partial \theta}\ell^0_\theta + \dot{W}, \\ u(t,0) = u(t,1) = 0, \\ u(0,\theta) = u_0(\theta), \quad \theta \in [0,1] \end{cases} \tag{4.1.3}$$

where $(\ell^a_{t,\theta}, \theta \in [0,1])$ is the family of local times at $a \in \mathbb{R}$ accumulated over $[0,\theta]$ by the process $(u(t,\theta), \theta \in [0,1])$, $W(t,\theta)$ is a Brownian sheet over $[0,+\infty[\times[0,1]$ and $\dot{W}(t,\theta)$ is therefore a space-time white-noise and $u_0 \in L^2(0,1)$. In fact, we consider a more general version of equation (4.1.3), see (3.1.1) below.

We recall that the *stochastic heat equation* is given by

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 v}{\partial \theta^2} + \dot{W}, \\ v(t,0) = v(t,1) = 0 \\ v(0,\theta) = u_0(\theta), \qquad x \in [0,1] \end{cases} \tag{4.1.4}$$

The process $(v_t, t \geq 0)$ is an-infinite dimensional Ornstein-Uhlenbeck process and it is associated with the Dirichlet form

$$\mathcal{E}^0(\varphi,\psi) := \frac{1}{2}\int_H \langle \nabla\varphi, \nabla\psi\rangle\, d\mu,$$

in $L^2(\mu)$, where $H := L^2(0,1)$, ∇ is the Fréchet gradient on H and μ is the law of a standard Brownian bridge from 0 to 0 over $[0,1]$, see [16].

Equation (4.1.3) is naturally associated with a perturbation of $\mathcal{E}^0$, defined by means of the probability measure on H

$$\nu(dx) := \frac{1}{Z}\exp\left(-\alpha\int_0^1 \mathbb{1}_{(x_s>0)}\,ds\right)\mu(dx),$$

with $\alpha \in \mathbb{R}$, and of the Dirichlet form

$$\mathcal{E}(\varphi,\psi) := \frac{1}{2}\int_H \langle \nabla\varphi, \nabla\psi\rangle\, d\nu, \tag{4.1.5}$$

in $L^2(\nu)$. Equation (4.1.3) is therefore a natural infinite-dimensional version of (4.1.1): indeed, its invariant measure ν favors paths over $[0,1]$ which spend more time in the positive axis than in the negative one. The definition and construction of this process are non-trivial, for several reasons.

First, the local-time term plays the role of a very singular drift, which furthermore lacks any dissipativity property; this makes a well-posedness result difficult to expect. Secondly, the explicit invariant measure ν is not *log-concave*, a condition which would insure a number of nice properties of the Dirichlet form $\mathcal{E}$ and of the associated Markov process, see e.g. [3] and section 4.2.1 below.

In particular, the process is not Strong-Feller, or at least a proof of this property is out of our reach, see [13] for a host of examples and consequences of this nice continuity property. We are at least able to prove something weaker, namely the *absolute continuity* of the transition semigroup w.r.t. the invariant measure ν, see Proposition 4.2.5 below; our proof of this technical step seems to be new and of independent interest.

We also consider two different regularizations of equation (3.1.1): first we approximate f with a sequence of smooth functions; then we consider finite-dimensional projections (without regularizing f). In both cases we prove convergence in law of the associated stationary processes. The main technical tool is the Γ-convergence (or, in this context, the *Mosco-convergence*) of a sequence of Dirichlet forms with underlying Hilbert space depending on n. This notion has been introduced by Kuwae and Shioya in [43] as a generalization of the original idea of Mosco [48] and later developed by Kolesnikov in [41] for finite-dimensional and a particular class of infinite-dimensional problems. Our approach has been largely inspired by the recent work of Andres and von Renesse, see [4, 5].

4.1.3 Main results

We start by giving the main definition. We consider a bounded function $f : \mathbb{R} \mapsto \mathbb{R}$ with bounded variation and we want to study the following equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \theta^2} - \dfrac{1}{2}\displaystyle\int_{\mathbb{R}} f(da)\frac{\partial}{\partial \theta}\ell^a_{t,\theta} + \dot{W}, \\[2ex] u(t,0) = u(t,1) = 0, \\[2ex] u(0,\theta) = u_0(\theta), \quad \theta \in [0,1] \end{cases} \tag{4.1.6}$$

where $(\ell^a_{t,\theta}, \theta \in [0,1])$ is the family of local times at $a \in \mathbb{R}$ accumulated over $[0,\theta]$ by the process $(u(t,\theta), \theta \in [0,1])$.

Definition 4.1.1. *Let $x \in L^2(0,1)$. An adapted process u, defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$, is a weak solution of (4.1.6) if*

- *a.s. $u \in C(]0,T] \times [0,1])$ and $\mathbb{E}[\|u_t - x\|^2] \to 0$ as $t \downarrow 0$*
- *a.s. for dt-a.e. t the process $(u(t,\theta), \theta \in [0,1])$ has a family of local times*

$\mathbb{R}^+ \times [0,1] \ni (t,\theta) \mapsto \ell^a_{t,\theta}$, $a \in \mathbb{R}$, such that

$$\int_0^\theta g(u(t,r))\,dr = \int_{\mathbb{R}} g(a)\,\ell^a_{t,\theta}\,da, \qquad \theta \in [0,1],\ t \geq 0,$$

for all bounded Borel $g : \mathbb{R} \mapsto \mathbb{R}$.

- *there is a Brownian sheet W such that for all $h \in C^2_c((0,1))$ and $0 < \varepsilon \leq t$*

$$\begin{aligned}\langle u_t - u_\varepsilon, h\rangle = \frac{1}{2}\int_\varepsilon^t \langle h'', u_s\rangle_{L^2(0,1)}\,ds + \frac{1}{2}\int_\varepsilon^t \int_{\mathbb{R}} f(da)\int_0^1 h'(\theta)\,\ell^a_{s,\theta}\,d\theta\,ds \\ + \int_\varepsilon^t \int_0^1 h(\theta)\,W(ds,d\theta)\end{aligned} \tag{4.1.7}$$

A Brownian sheet is a Gaussian process $W = \{W(t,\theta) : (t,\theta) \in \mathbb{R}^2_+\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, such that $\{W(t,\theta) : \theta \in \mathbb{R}_+\}$ is $\mathcal{F}_t$-measurable for all $t \geq 0$, with zero mean and covariance function

$$\mathbb{E}\left[W(t,\theta)W(t',\theta')\right] = (t \wedge t')(\theta \wedge \theta'), \qquad t,\theta,t',\theta' \in \mathbb{R}_+.$$

In section 4.2 we study the Dirichlet form $\mathcal{E}$ defined by (4.1.5), proving in particular that it satisfies the absolute continuity condition, namely the resolvent operators have kernels which admit a density with respect to the reference measure ν. In section 4.3 we show that the Markov process associated with $\mathcal{E}$ is a weak solution of (4.1.6). Altough for general f a uniqueness result for solutions to (4.1.6) seems to be out of reach, the process we construct is somewhat canonical, since it is associated with the Dirichlet form $\mathcal{E}$ and moreover it is obtained as the limit of natural regularization/discretization procedures, as shown in sections 4.4, respectively 4.5. Indeed, in section 4.4 we regularize the nonlinearity f and show that the (stationary) solutions to the approximated equations converge to the stationary solution of (4.1.6). In section 4.5 we show convergence of finite-dimensional processes, obtained via a space-discretization, to the solution of (4.1.6).

4.1.4 Motivations

There is an extensive literature on reaction-diffusion stochastic partial differential equations of the form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial \theta^2} - \frac{1}{2}f'(u) + \dot{W}, \qquad t \geq 0,\ \theta \in [0,1],$$

see for instance the monography by Cerrai [13]; note that by the occupation times formula, for smooth f this equation is equivalent to (4.1.6). This kind of equation has also been used as a model for fluctuations of effective interface models, see [31]. However, in order to give a sense to the above equation, it is typically assumed that f is smooth or convex. In this chapter we study this equation in the case where f is neither convex nor necessarily smooth and can even have jumps.

One of the motivations of this work is given by the problem of extending the results of [29] on convergence of fluctuations of a stochastic interface near a hard wall to a non log-concave situation. In particular, it is a long standing problem to prove the same result as in [29] for a critical pinning model, see e.g. [20], where the invariant measure converges in the limit to the law of a reflecting Brownian motion. Such a situation is highly non log-convex and the techniques developed for instance in [3] do not apply. In this chapter we show that the Mosco-(Γ-)convergence is an effective tool also in this context.

4.1.5 Notations

We consider the Hilbert space $H = L^2(0,1)$ endowed with the canonical scalar product

$$\langle h,k\rangle_H := \int_0^1 h_\theta\, k_\theta\, d\theta, \qquad \|h\|^2 := \langle h,h\rangle, \qquad h,k \in H.$$

$$C_0 := C_0(0,1) := \{c : [0,1] \mapsto \mathbb{R} \ \text{ continuous}, \ c(0) = c(1) = 0\},$$

$$A : D(A) \subset H \mapsto H, \quad D(A) := W^{2,2} \cap W_0^{1,2}(0,1), \quad A := \frac{1}{2}\frac{d^2}{d\theta^2}.$$

We introduce the following function spaces:

- We denote by $C_b(H)$ the space of all $\varphi : H \mapsto \mathbb{R}$ being bounded and uniformly continuous in the norm of H. We let $\|\varphi\|_\infty := \sup|\varphi|$. Then $(C_b(H), \|\cdot\|_\infty)$ is a Banach space.
- We denote by $\mathrm{Exp}_A(H)$ the linear span of $\{1, \cos(\langle\cdot,h\rangle), \sin(\langle\cdot,h\rangle) : h \in D(A)\}$.
- The space $\mathrm{Lip}(H)$ is the set of all $\varphi \in C_b(H)$ such that:

$$\|\varphi\|_{\mathrm{Lip}} \ := \ \|\varphi\|_\infty \ + \ \sup_{x\neq y} \frac{|\varphi(x) - \varphi(y)|}{\|x-y\|} \ < \ \infty.$$

- The space $C_b^1(H)$ is defined as the set of all Fréchet-differentiable $\varphi \in C_b(H)$, with continuous bounded gradient $\nabla\varphi : H \mapsto H$.

We sometimes write: $m(\varphi)$ for $\displaystyle\int_H \varphi\, dm$, $\varphi \in C_b(H)$.

4.2 The Dirichlet form $\mathcal{E}$

In this section we give a detailed study of the Dirichlet form $\mathcal{E}$, proving in particular that it satisfies the *absolute continuity property*, see Proposition 4.2.5 below.

4.2.1 A non-log-concave probability measure

Let $\beta = (\beta_\theta, \theta \in [0,1])$ be a standard Brownian bridge and let us denote its law by μ. Then μ is a Gaussian measure on the Hilbert space $H = L^2(0,1)$. We consider a bounded function $f : \mathbb{R} \mapsto \mathbb{R}$ with bounded variation and we define $F : H \mapsto \mathbb{R}$:

$$F(x) := \int_0^1 f(x_r)\, dr, \qquad x \in H.$$

We define the probability measure on H

$$\nu(dx) = \frac{1}{Z}\exp(-F(x))\,\mu(dx), \qquad Z := \int \exp(-F)\,d\mu. \tag{4.2.1}$$

where Z is normalizing constant. Note that f is not assumed to be convex, and therefore ν is in general not log-concave, see [3]. Finally we have clearly

$$\frac{1}{C}\|\cdot\|^2_{L^2(\mu)} \le \|\cdot\|^2_{L^2(\nu)} \le C\|\cdot\|^2_{L^2(\mu)} \tag{4.2.2}$$

for some constant $C > 0$, since f is bounded.

4.2.2 The Gaussian Dirichlet Form

We define now

$$\mathcal{E}^0(\varphi,\psi) := \frac{1}{2}\int_H \langle\nabla\varphi,\nabla\psi\rangle\,d\mu, \qquad \forall\varphi,\psi\in C^1_b(H).$$

Then it is well known that the symmetric positive bilinear form $(\mathcal{E}^0, \mathrm{Exp}_A(H))$ is closable in $L^2(\mu)$, see e.g. [17]: we denote by $(\mathcal{E}^0, D(\mathcal{E}^0))$ the closure. We recall that μ, law of a standard Brownian bridge β, has covariance $Q := (-2A)^{-1}$, a compact operator on H which can be diagonalized as follows:

$$Qh = \sum_{k=1}^{\infty}\lambda_k\,\langle h, e_k\rangle_H\,e_k, \qquad h\in H,$$

where

$$\lambda_k := \frac{1}{(\pi k)^2}, \qquad e_k(x) := \sqrt{2}\,\sin(k\pi x), \qquad x\in[0,1],\ k\in\mathbb{N}^*.$$

It is well known that the Markov process defined by (4.1.4), i.e. the solution of the stochastic heat equation, is associated with the Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ in $L^2(\mu)$. This process is Gaussian and can be written down explicitly as a stochastic convolution. We recall the following result from [17]:

Proposition 4.2.1. *Let $\Gamma := \{\gamma : \mathbb{N}^* \mapsto \mathbb{N} : \sum_k \gamma_k < +\infty\}$. Then there exists a complete orthonormal system $(H_\gamma)_{\gamma\in\Gamma}$ in $L^2(\mu)$ such that*

$$\mathcal{E}^0(\varphi,\varphi) = \sum_{\gamma\in\Gamma}\Lambda_\gamma\,\langle\varphi, H_\gamma\rangle^2_{L^2(\mu)}, \qquad \forall\varphi\in D(\mathcal{E}^0),$$

where Λ_γ is given by

$$\Lambda_\gamma := \sum_{k\in\mathbb{N}^*}\gamma_k\,\lambda_k^{-1}. \tag{4.2.3}$$

In particular, the embedding $D(\mathcal{E}^0) \mapsto L^2(\mu)$ is compact.

It follows that $(H_\gamma)_{\gamma\in\Gamma}$ is a c.o.s. of eigenvalues of the Ornstein-Uhlenbeck operator associated with $\mathcal{E}^0$. We denote by $(P_t^0)_{t\geq 0}$ the associated semigroup in $L^2(\mu)$, which can be of course written as

$$P_t^0\varphi = \sum_{\gamma\in\Gamma} e^{-\Lambda_\gamma t} \langle \varphi, H_\gamma\rangle_{L^2(\mu)} H_\gamma, \qquad \forall \varphi \in L^2(\mu).$$

Then we have the following

Proposition 4.2.2. *For all $t > 0$ the operator $P_t^0 : L^2(\mu) \mapsto L^2(\mu)$ is Hilbert-Schmidt, i.e.*

$$\sum_{\gamma\in\Gamma} e^{-2\Lambda_\gamma t} = \prod_{k=1}^{\infty} \frac{1}{1-e^{-2t\pi^2 k^2}} < +\infty, \qquad t > 0. \tag{4.2.4}$$

In particular, the series

$$p_t^0(x,y) := \sum_{\gamma\in\Gamma} e^{-\Lambda_\gamma t} H_\gamma(x)\, H_\gamma(y)$$

converges in $L^2(\mu\otimes\mu)$ and yields an integral representation of P_t^0:

$$P_t^0\varphi(x) = \int \varphi(y)\, p_t^0(x,y)\, \mu(dy), \qquad \mu\text{−a.e. } x, \ \forall\, \varphi \in L^2(\mu).$$

Proof. Let us define C_n, for $n \in \mathbb{N}$, as the number of $\gamma \in \Gamma$ such that $\sum_k \gamma_k\, k^2 = n$. Then

$$\sum_{\gamma\in\Gamma} e^{-2\Lambda_\gamma t} = \sum_{\gamma\in\Gamma} \sum_{n=0}^{\infty} \mathbb{1}_{(\Lambda_\gamma = n)}\, e^{-2\Lambda_\gamma t} = \sum_{n=0}^{\infty} C_n e^{-2\pi^2 t\, n}.$$

Now, by a classical formula due to Euler, the generating function of the sequence $(C_n)_{n\geq 0}$ is given by

$$\chi(r) := \sum_{n=0}^{\infty} C_n r^n = \prod_{k=1}^{\infty} \frac{1}{1-r^{k^2}}, \qquad |r| < 1.$$

The infinite product converges, since by taking the logarithm

$$-\log\left(1 - r^{k^2}\right) \sim r^{k^2}, \qquad k \to +\infty, \qquad |r| < 1,$$

which is a summable sequence. By choosing $r = e^{-2t\pi^2}$, the first claim follows. The rest is a trivial consequence of this result. □

From (4.2.4) one can obtain the following

Proposition 4.2.3. *The embedding $D(\mathcal{E}^0) \mapsto L^2(\mu)$ is not Hilbert-Schmidt.*

Proof. The embedding $D(\mathcal{E}^0) \mapsto L^2(\mu)$ is Hilbert-Schmidt if and only if

$$\sum_{\gamma\in\Gamma\setminus\{0\}} \frac{1}{\Lambda_\gamma} < +\infty.$$

Again we can write

$$\sum_{\gamma\in\Gamma\setminus\{0\}} \frac{1}{\Lambda_\gamma} = \sum_{\gamma\in\Gamma}\sum_{n=1}^{\infty} \mathbb{1}_{(\Lambda_\gamma=n)} \frac{1}{\Lambda_\gamma} = \sum_{n=1}^{\infty} \frac{C_n}{n}.$$

Now, using the generating function χ of the sequence C_n we obtain

$$\sum_{n=1}^{\infty} \frac{C_n}{n} = \int_0^1 dr \sum_{n=1}^{\infty} C_n r^{n-1} = \int_0^1 \frac{\chi(r)-1}{r}\, dr,$$

since $C_0 = 1$. The latter integral converges near 0, but it diverges near 1, since $\chi(r) \geq (1-r)^{-1}$. Therefore the above sum is infinite. □

4.2.3 The Dirichlet form associated with (4.1.6)

We define the symmetric positive bilinear form

$$\mathcal{E}(\varphi,\psi) := \frac{1}{2}\int_H \langle\nabla\varphi,\nabla\psi\rangle\, d\nu, \qquad \forall\,\varphi,\psi\in C_b^1(H).$$

Let us set $\mathcal{K} := \mathrm{Exp}_A(H)$.

Lemma 4.2.4. *The symmetric positive bilinear form $(\mathcal{E},\mathcal{K})$ is closable in $L^2(\nu)$. We denote by $(\mathcal{E}, D(\mathcal{E}))$ the closure.*

Proof. By (4.2.2) we have that

$$\frac{1}{C}\,\mathcal{E}_1^0 \leq \mathcal{E}_1 \leq C\,\mathcal{E}_1^0. \tag{4.2.5}$$

Closability of $(\mathcal{E}^0,\mathcal{K})$ yields immediately the result. □

4.2.4 Absolute continuity

Let $(P_t)_{t\geq 0}$ be the semigroup associated with the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ in $L^2(\nu)$. We denote by $R_\lambda := \displaystyle\int_0^\infty e^{-\lambda t}\, P_t\, dt$, $\lambda > 0$, the resolvent family of $(P_t)_{t\geq 0}$. In this section we want to prove the following

Proposition 4.2.5. *There exists a measurable kernel $(\rho_\lambda(x,dy), \lambda > 0, x\in H)$ such that*

$$R_\lambda\varphi(x) = \int \varphi(y)\,\rho_\lambda(x,dy), \qquad \nu\text{−a.e. } x,\ \forall\,\varphi\in L^2(\nu),$$

and such that for all $\lambda > 0$ and for all $x\in H$ we have $\rho_\lambda(x,dy) \ll \nu(dy)$.

We are going to use the following result, see [22, pp. 1543].

Theorem 4.2.6 (Minimax principle)**.** *Let $(T,\mathcal{D}(T))$ a self-adjoint linear operator on the separable Hilbert space $\mathbb{H}$ such that $T\geq 0$ and $(\lambda - T)^{-1}$ is a compact operator* *for some* $\lambda > 0$*. We denote by* $\mathcal{S}^n$ *the family of* n*-dimensional subspace of* $\mathbb{H}$*, and for* $n \geq 1$ *we let* λ_n *the number defined as follows*

$$\lambda_n := \sup_{G \in \mathcal{S}^n} \inf_{u \in (G \cap D(T)) \setminus \{0\}} \frac{\langle u, Tu \rangle_{\mathbb{H}}}{\langle u, u \rangle_{\mathbb{H}}}. \tag{4.2.6}$$

Then there exists a complete orthonormal system $(\psi_n)_{n \geq 1}$ *such that*

$$T\,\psi_n = \lambda_n\,\psi_n, \qquad n \geq 1.$$

In other words, the sequence $(\lambda_n)_{n \geq 1}$ *is the non-decreasing enumeration of the eigenvalues of* T*, each repeated a number of times equal to its multiplicity. Moreover the* sup *in* (4.2.6) *is attained for* G *equal to the span of* $\{\psi_1, \dots, \psi_n\}$*.*

With the help of Theorem 4.2.6, we can first prove the following

Proposition 4.2.7. *The operator* $P_t : L^2(\nu) \mapsto L^2(\nu)$ *is Hilbert-Schmidt and there exists a function* $p_t \in L^2(\nu \otimes \nu)$ *such that*

$$P_t\varphi(x) = \int \varphi(y)\,p_t(x,y)\,\nu(dy), \qquad \nu\text{–a.e. } x, \ \forall\,\varphi \in L^2(\nu).$$

Proof. We recall that an analogous result has been proved in Proposition 4.2.2 for the semigroup $(P_t^0)_{t \geq 0}$ associated with the Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ in $L^2(\mu)$. Now we want to deduce the same result for $(P_t)_{t \geq 0}$.

We apply first Theorem 4.2.6 to the Ornstein-Uhlenbeck operator L^0 associated with $(\mathcal{E}^0, D(\mathcal{E}^0))$ in $L^2(\mu)$. Since $R_1^0 := (1 - L^0)^{-1}$ maps $L^2(\mu)$ into $D(\mathcal{E}^0)$ and the embedding $D(\mathcal{E}^0) \mapsto L^2(\mu)$ is compact by Proposition 4.2.3, then R_1^0 is compact and also symmetric since $\mathcal{E}^0$ is symmetric. By Proposition 4.2.3, the spectrum of $(-L^0)$ is pure point, its eigenvalues are $(\Lambda_\gamma)_{\gamma \in \Gamma}$ and the associated eigenvectors are the c.o.s. $(H_\gamma)_{\gamma \in \Gamma}$. If we call $(\delta_n^0)_{n \geq 1}$ the non-decreasing enumeration of $(\Lambda_\gamma)_{\gamma \in \Gamma}$, then by the above result we obtain that

$$\delta_n^0 := \sup_{G \in \mathcal{S}^n} \inf_{u \in (G \cap D(L^0)) \setminus \{0\}} \frac{\mathcal{E}^0(u,u)}{\langle u, u \rangle_{L^2(\mu)}}.$$

In fact, since the sup above is attained for G equal to the span of $\{\psi_1, \dots, \psi_n\} \subseteq D(\mathcal{E}^0)$, then we can also write

$$\delta_n^0 = \sup_{G \in \mathcal{S}^n} \inf_{u \in (G \cap D(\mathcal{E}^0)) \setminus \{0\}} \frac{\mathcal{E}^0(u,u)}{\langle u, u \rangle_{L^2(\mu)}}.$$

In the same way, setting

$$\delta_n := \sup_{G \in \mathcal{S}^n} \inf_{u \in (G \cap D(\mathcal{E})) \setminus \{0\}} \frac{\mathcal{E}(u,u)}{\langle u, u \rangle_{L^2(\nu)}},$$

then $(\delta_n)_{n \geq 1}$ is the non-decreasing enumeration of the eigenvalues of $(-L) : D(L) \subset L^2(\nu) \mapsto L^2(\nu)$. Now, by (4.2.2) and (4.2.5), we obtain that

$$\frac{1}{C}\delta_n^0 \leq \delta_n \leq C\delta_n^0, \qquad n \geq 1.$$

Therefore for $t > 0$

$$\sum_n e^{-2t\delta_n} \leq \sum_n e^{-2t\frac{1}{C}\delta_n^0}$$

and the latter sum is finite by (4.2.4). Therefore $P_t : L^2(\nu) \mapsto L^2(\nu)$ is Hilbert-Schmidt, symmetric and non-negative. Then Proposition 4.2.7 follows from a well-known characterization of operators with such properties. □

Proof of Proposition 4.2.5. In [27, Theorem 7.2.1] it is proved that there exist a set of zero capacity N and a measurable Markov kernel $(p_t(x, dy), t \geq 0, x \in N^c)$ on N^c, such that the function $x \mapsto \displaystyle\int \varphi(y)\, p_t(x, dy)$ is ν-a.s. equal to $P_t\varphi$ and quasi-continuous on N^c for all $t, > 0$. By quasi-continuity we want to say that there is a sequence of nondecreasing closed set $(F_n)_n$, with no isolated point, such that the previous map, restricted on F_n, is continuous for all $t > 0$ and $N^c = \cup_n F_n$ (see [27] p.67). By Proposition 4.2.7, for ν-a.e. x we have $p_t(x, dy) = p_t(x, y)\, \nu(dy)$, with $p_t \in L^2(\nu \otimes \nu)$ and $p_t \geq 0$, $\nu \otimes \nu$-almost surely. It follows that the kernel $\rho_\lambda(x, dy)$ representing the resolvent operator $R_\lambda := \displaystyle\int_0^\infty e^{-\lambda t}\, P_t\, dt$ is in fact given for ν-a.e. x by $\rho_\lambda(x, dy) = \rho_\lambda(x, y)\, \nu(dy)$, where for $\nu \otimes \nu$-a.e. (x, y)

$$\rho_\lambda(x, y) := \int_0^{+\infty} e^{-\lambda t}\, p_t(x, y)\, dt.$$

Moreover $R_\lambda\varphi$ is continuous on N^c for all $\varphi \in L^2(\nu)$. This allows to prove that $\rho_\lambda(x, dy) \ll \nu(dy)$ for all $x \in N$: indeed, if B is a measurable set such that $\nu(B) = 0$, then $\rho_\lambda(x, B) = 0$ for ν-a.e. x and therefore, by density and continuity, for all $x \in N^c$. As in [27], we can set $\rho_\lambda(x, dy) \equiv 0$ for all $x \in N$, and the proof is complete. □

4.3 Existence of a solution

In this section we want to prove the following

Proposition 4.3.1. *The Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular and the associated Markov process is a weak solution of equation* (4.1.6)*.*

4.3.1 The associated Markov process

We have first the following

Lemma 4.3.2. *The Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular.*

Proof. By (4.2.5) and by [46, Definition IV.3.1], quasi-regularity of $(\mathcal{E}, D(\mathcal{E}))$ follows from quasi-regularity of $(\mathcal{E}^0, D(\mathcal{E}^0))$, which in turns follows from the fact that this Dirichlet form is associated with the solution to the stochastic heat equation (4.1.4). □

By [46, Theorem IV.3.5], quasi-regularity implies existence of a Markov process associated with $(\mathcal{E}, D(\mathcal{E}))$.

Existence of local times

Proposition 4.3.3. *Almost surely, for a.e. t there exists a bi-continuous family of local times $[0,1] \ni (r,a) \mapsto \ell^a_{t,r}$ of $(u_t(\theta), \theta \in [0,1])$.*

Proof. Let us recall that ν is equivalent to the law μ of $(\beta_r, r \in [0,1])$, where β is a Brownian bridge over $[0,1]$. Since β is a semi-martingale, for μ-a.e. x there exists a family of local times ℓ^a_r such that

$$\int_0^r g(x_s)\, ds = \int_{\mathbb{R}} g(a)\, \ell^a_r\, da, \qquad r \in [0,1],$$

and the map $[0,1] \times \mathbb{R} \ni (r,a) \mapsto \ell^a_r \in \mathbb{R}$ is continuous. In particular, setting

$$S := \{w \in C([0,1]) : w \text{ has a bi-continuous family of local times } (\ell^a_r)_{(r,a) \in [0,1]\times\mathbb{R}}\},$$

then $\nu(S) = 1$ and therefore

$$\mathbb{E}_x \left[\int_0^t \mathbb{1}_{(u_s \in S^c)}\, ds \right] = \int_0^t \mathbb{P}_x(u_s \in S^c)\, ds = \int_0^t p_s(x, S^c)\, ds = 0$$

since the law of $(u_s(\theta), \theta \in [0,1])$ by Proposition 4.2.7 is absolutely continuous w.r.t. ν. Therefore, the time spent by $(u_s, s \geq 0)$ in S^c is a.s. equal to 0. □

We need now an integration by parts formula on the Dirichlet form $\mathcal{E}$. We recall the definitions

$$F(x) := \int_0^1 f(x_r)\, dr, \qquad \rho(x) := \exp(-F(x)), \qquad x \in H,$$

where $f : \mathbb{R} \mapsto \mathbb{R}$ is a bounded function with bounded variation.

Proposition 4.3.4. *For any $h \in D(A)$ and $\varphi \in C^1_b(H)$*

$$\mathbb{E}[\rho(\beta)\, \partial_h \varphi(\beta)] = \mathbb{E}\left[\rho(\beta)\, \varphi(\beta) \left(-\langle h'', \beta \rangle + \int_{\mathbb{R}\times[0,1]} f(da)\, h_r\, \ell^a(dr) \right)\right]. \tag{4.3.1}$$

Proof. Let $h \in D(A)$ and $\varepsilon \in \mathbb{R}$, by the occupation time formula:

$$\begin{aligned} F(\beta + \varepsilon h) &= \int_0^1 f(\beta_r + \varepsilon h_r)\, dr = \int_{\mathbb{R}} \int_0^1 f(a + \varepsilon h_r)\, \ell^a(dr)\, da \\ &= \int_{\mathbb{R}\times\mathbb{R}\times[0,1]} da\, f(da)\, \ell^a(dr)\, \mathbb{1}_{(a \geq s - \varepsilon h_r)} \qquad \text{a.s.} \end{aligned}$$

where $(\ell^a(r), a \in \mathbb{R}, r \in [0,1])$ is the local times family of β. Therefore

$$\frac{d}{d\varepsilon} F(\beta + \varepsilon h)\Big|_{\varepsilon=0} = -\int_{\mathbb{R}\times[0,1]} f(da)\, h_r\, \ell^a(dr).$$

Then by using the Cameron-Martin formula

$$\mathbb{E}[\rho(\beta)\, \varphi(\beta + \varepsilon h)] = \mathbb{E}[\rho(\beta - \varepsilon h)\, \varphi(\beta)\, \exp\left(-\varepsilon \langle h'', \beta \rangle - \|h\|^2 \varepsilon^2/2\right)]$$

and by differentiating w.r.t. ε at $\varepsilon = 0$ we conclude. □

We want now to show that the process associated with $\mathcal{E}$ satisfies (4.1.6). We are going to apply (3.7.1) to $U^h(x) := \langle x, h\rangle$, $x \in H$, with $h \in C_c^2((0,1);\mathbb{R}^d)$. Clearly $U^h \in \mathrm{Lip}(H) \subset D(\mathcal{E})$. Our aim is to prove the following

Proposition 4.3.5. *There is an exceptional set N such that for all $x \in H \setminus N$, $\mathbb{P}_x$-a.s. for all $t \geq 0$*

$$N_t^{[U^h]} = \frac{1}{2}\int_0^t \langle h'', u_s\rangle\, ds + \frac{1}{2}\int_{]0,t]\times[0,1]}\int_{\mathbb{R}} f(da)\, h_r'\, \ell^a_{s,r}\, ds\, dr \tag{4.3.2}$$

where a.s. for all $s > 0$

$$-\int_{[0,1]}\int_{\mathbb{R}} h_r'\, \varphi(a)\, \ell^a_{s,r}\, dr = \int_0^1 h_r\, \varphi(u_s(r))\, dr, \qquad \forall\, \varphi \in C_b(\mathbb{R}).$$

Proof. The main tools of the proof are the integration by parts formula (4.3.1) and a number of results from the theory of Dirichlet forms in [27]. We start by applying (3.7.1) to $U^h(x) := \langle x, h\rangle$, $x \in H$. By approximation and linearity we can assume that $h \in D(A)$, $h'' \geq 0$ and therefore $h \geq 0$ as well. The process $N^{[U^h]}$ is a CAF of X, and its Revuz measure is $\frac{1}{2}\,\Sigma^h$, where

$$\Sigma^h(dw) := \left(\langle w, h''\rangle - \int_{\mathbb{R}\times[0,1]} f(da)\, h_r\, d\ell^a_r\right)\nu(dw) \tag{4.3.3}$$

and ℓ^a_r is the bi-continuous family of local times of the Brownian bridge. Remark that we have the estimate

$$\mathbb{E}\left(\left(\int_{\mathbb{R}\times[0,1]} f(da)\, h_r\, d\ell^a_r\right)^2\right) < +\infty$$

since $f(da)$ has globally bounded variation, h is bounded and ℓ^a_1 is in L^p for any $p \geq 1$.

By linearity, it is enough to consider the case $h \geq 0$. Then the measurable function $\Phi(w) := \int_{[0,1]\times\mathbb{R}} h_r\, d\ell^a_r\, f(da)$ is non-negative, and $\Phi\, d\nu$ is a measure with finite energy, since

$$\int |v|\, \Phi\, d\nu \leq \|\Phi\|_{L^2(\nu)}\, \|v\|_{L^2(\nu)} \leq \|\Phi\|_{L^2(\nu)}\, \sqrt{\mathcal{E}_1(v,v)}, \qquad \forall\, v \in D(\mathcal{E}) \cap C_b(H),$$

see (3.7.2) above. In particular, $\Phi\, d\nu$ is a smooth measure. By theorem 5.1.3 of [27], there is an associated PCAF, denoted by N_t. Notice that the process

$$N_t^n := \int_0^t (\Phi \wedge n)(X_s)\, ds$$

is a well defined PCAF with Revuz measure $\Phi \wedge n\, d\nu$ and $N_t^n \leq N_t$, since $N_t - N_t^n$ is a CAF with a non-negative Revuz measure. By monotone convergence we find for all non-negative $\varphi \in C_b(H)$

$$\begin{aligned}\int_H \varphi\,\Phi\,d\nu = \lim_n \int_H \varphi\,\Phi \wedge n\,d\nu &= \lim_n \mathbb{E}_\nu\left[\int_0^1 \varphi(X_t)\,(\Phi\wedge n)(X_t)\,dt\right] \\ &= \mathbb{E}_\nu\left[\int_0^1 \varphi(X_t)\,\Phi(X_t)\,dt\right].\end{aligned}$$

Therefore, $t \mapsto \int_0^t \Phi(X_s)\,ds$ is a PCAF with Revuz measure $\Phi\,d\nu$ and must therefore be equivalent to $t \mapsto N_t$. $\square$

4.3.2 Identification of the noise term

We deal now with the identification of $M^{[U^h]}$ with the integral of h with respect to a space-time white noise.

Proposition 4.3.6. *There exists a Brownian sheet* $(W(t,\theta),\, t \geq 0, \theta \in [0,1])$*, such that*

$$M_t^{[U^h]} = \int_0^t \int_0^1 h_\theta\,W(ds,d\theta), \qquad h \in H. \tag{4.3.4}$$

Proof. We recall that, for $U \in D(\mathcal{E})$, the process $M^{[U]}$ is a continuous martingale, whose quadratic variation $(\langle M^{[U]}\rangle_t)_{t\geq 0}$ is a PCAF of X with Revuz measure $\mu_{\langle M^{[U]}\rangle}$ given by the formula

$$\int f\,d\mu_{\langle M^{[U]}\rangle} = 2\mathcal{E}(Uf,U) - \mathcal{E}(U^2,f), \qquad \forall\, f \in D(\mathcal{E}) \cap C_b(H), \tag{4.3.5}$$

see [27, Theorem 5.2.3]. Now, if we apply this formula to $U^h(x) = \langle x, h\rangle$, then we obtain

$$\int f\,d\mu_{\langle M^{[U^h]}\rangle} = \|h\|^2 \int f\,d\nu, \qquad \forall\, f \in D(\mathcal{E}) \cap C_b(H).$$

Therefore, the quadratic variation $\langle M^{[U^h]}\rangle_t$ is equal to $\|h\|^2 t$ for all $t \geq 0$, and, by Lévy's Theorem, $(M^{[U^h]} \cdot \|h\|^{-1})_{t\geq 0}$ is a Brownian motion. Moreover, the parallelogram law, if $h_1, h_2 \in H$ and $\langle h_1, h_2\rangle = 0$, then the quadratic covariation between $M^{[U^{h_1}]}$ and $M^{[U^{h_2}]}$ is equal to

$$\langle M^{[U^{h_1}]}, M^{[U^{h_2}]}\rangle_t = t\,\langle h_1, h_2\rangle, \qquad t \geq 0.$$

Therefore, $(M_t^{[U^h]}, t \geq 0, h \in H)$ is a Gaussian process with covariance structure

$$\mathbb{E}_x\left(M_t^{[U^{h_1}]}\,M_s^{[U^{h_2}]}\right) = s \wedge t\,\langle h_1, h_2\rangle.$$

If we define $W(t,\theta) := M_t^{[U^h]}$ with $h := 1_{[0,\theta]}$, $t \geq 0$, $\theta \in [0,1]$, then W is the desired Brownian sheet. $\square$

Proof of Proposition 4.3.1. Quasi-regularity has been proved in Lemma 4.3.2. First we apply the Fukushima decomposition (3.7.1) to the function $U_h(x) := \langle x, h\rangle$ and identify the terms using propositions 4.3.6 and 4.3.5 and the above results. It remains to prove that the process $(X_t)_{t\geq 0}$ satisfies the desired continuity properties. To this aim, we use the result of Lemma 4.6.1 below. We notice that for any $\eta \in (0,1/2)$ and $p>1$

$$\frac{1}{C}\int_H \|x\|^p_{W^{\eta,p}(0,1)}\,\nu(dx) \leq \mathbb{E}\left(\|\beta\|^p_{W^{\eta,p}(0,1)}\right) \leq \mathbb{E}\left(|\beta_r|^p + \int_0^1\int_0^1 \frac{|\beta_s-\beta_t|^p}{|s-t|^{p\eta+1}}\,dt\,ds\right)$$
$$\leq 1 + \int_0^1\int_0^1 |s-t|^{p(\frac{1}{2}-\eta)-1}\,dt\,ds < +\infty.$$

Then by Lemma 4.6.1 and by Kolmogorov's criterion in the Polish space $C^\beta([0,1])$ we obtain that under $\mathbb{P}_\nu$ the coordinate process has a modification in $C([0,T]\times[0,1])$ for all $T>0$.

Finally, in order to prove continuity of a non-stationary solution, we use the absolute-continuity property of proposition 4.2.5. Let us consider the set $C := C([0,1])$ endowed with the uniform topology. Let $S \subset]0,+\infty[$ be countable and satisfying $\varepsilon := \inf S > 0$ and $\sup S < \infty$, and define $B_S \subset C^{]0,+\infty[}$ as

$$B_S := \left\{\omega \in C^{]0,+\infty[} : \text{ the restriction of } \omega \text{ to } S \text{ is uniformly continuous}\right\},$$

then we know that $\mathbb{P}_\nu(B_S)=1$, i.e. $\mathbb{P}_x(B_S)=1$ for ν-a.e. x. For all $x\in N^c$, where N is exceptional, the law of X_ε under $\mathbb{P}_x$ is absolutely continuous w.r.t. ν for all $\varepsilon>0$. Then $\mathbb{P}_{X_\varepsilon}(B_{S-\varepsilon})=1$, $\mathbb{P}_x$-almost surely. Taking expectations, and using the Markov property, we get $\mathbb{P}_x(B_S)=1$. Arguing as in [59, Lemma 2.1.2] we obtain that $\mathbb{P}^*_x\left(C(]0,+\infty[;C)\right)=1$, where $\mathbb{P}^*_\nu$ denotes the outer measure.

□

4.4 Convergence of regularized equations

In this section we consider a smooth approximation f_n of f and and we study convergence in law of u^n to u, where

$$\begin{cases} \dfrac{\partial u^n}{\partial t} = \dfrac{1}{2}\,\dfrac{\partial^2 u^n}{\partial\theta^2} - \dfrac{1}{2}\,f_n'(u^n) + \dot{W}, \\[2ex] u^n(t,0)=u^n(t,1)=0, \\[2ex] u^n(0,\theta)=u^n_0(\theta), \quad \theta\in[0,1]. \end{cases} \tag{4.4.1}$$

By a Γ-convergence technique, we shall prove convergence in law of the stationary processes.

Since f is bounded and with bounded variation, then it is continuous outside a countable set Δ_f. Moreover we can find a sequence of smooth functions $f_n:\mathbb{R}\mapsto\mathbb{R}$ such that

1. $(f_n)_n$ is uniformly bounded
2. $f_n\to f$ as $n\to+\infty$ locally uniformly in $\mathbb{R}\setminus\Delta_f$.

We define the probability measure on H

$$\nu_n(dx) = \frac{1}{Z_n} \exp(-F_n(x))\, \mu(dx), \qquad Z_n := \int \exp(-F_n)\, d\mu, \tag{4.4.2}$$

where Z_n is a normalizing constant. Again, ν_n is not necessarily log-concave, see [3]. Setting

$$\rho_0 := 1, \quad \rho_n := \frac{d\nu_n}{d\mu}, \quad n \geq 1, \qquad \rho := \frac{d\nu}{d\mu},$$

we find that $0 < c \leq \rho_n \leq C < +\infty$ and $0 < c \leq \rho \leq C < +\infty$ on H, since f_n and f are bounded for all $n \in \mathbb{N}$. We have then the simple

Lemma 4.4.1. *There is a canonical identification between the Hilbert spaces $L^2(\nu)$ and $L^2(\nu_n)$ for all $n \in \mathbb{N}$ and for positive constants c, C*

$$\frac{c}{C} \| \cdot \|^2_{L^2(\nu)} \leq \| \cdot \|^2_{L^2(\nu_n)} \leq \frac{C}{c} \| \cdot \|^2_{L^2(\nu)}. \tag{4.4.3}$$

Proof. This is obvious since $0 < c \leq \rho_n \leq C < +\infty$ and $0 < c \leq \rho \leq C < +\infty$. □

In particular we can consider $L^2(\nu_n)$ as being a copy of $L^2(\nu)$ endowed with a different norm $\| \cdot \|_{L^2(\nu_n)}$. We shall use this notation below.

We define the symmetric positive bilinear form

$$\mathcal{E}^n(\varphi, \psi) := \frac{1}{2} \int_H \langle \nabla\varphi, \nabla\psi \rangle \, d\nu_n, \qquad \forall \varphi, \psi \in C^1_b(H),$$

Let us set $\mathcal{K} := \mathrm{Exp}_A(H)$.

Lemma 4.4.2. *The symmetric positive bilinear forms $(\mathcal{E}^n, \mathcal{K})$ is closable in $L^2(\nu_n)$. We denote by $(\mathcal{E}^n, D(\mathcal{E}^n))$ the closure.*

Proof. The proof is identical to that of Lemma 4.2.4. □

We recall that the Dirichlet form $(\mathcal{E}^n, D(\mathcal{E}^n))$ is associated with the solution of equation (4.4.1), see e.g. [17].

4.4.1 Convergence of Hilbert spaces

We recall now the following definition, given by Kuwae and Shioya in [43].

Definition 4.4.3. *A sequence of Hilbert spaces $\mathbb{H}_n$ converges to a hilbert $\mathbb{H}$ if there is a family of linear maps $\{\Phi_n : \mathbb{H} \to \mathbb{H}_n\}$ such that:*

$$\lim_{n\to+\infty} \|\Phi_n(x)\|_{\mathbb{H}_n} = \|x\|_{\mathbb{H}}, \quad x \in \mathbb{H} \tag{4.4.4}$$

A sequence $(x_n)_n$, $x_n \in \mathbb{H}_n$, converges strongly *to a vector $x \in \mathbb{H}$ if there exists a sequence $(\tilde{x}_n)_n$ in $\mathbb{H}$ such that $\tilde{x}_n \to x$ in $\mathbb{H}$ and*

$$\lim_{n\to+\infty} \overline{\lim_{m\to+\infty}} \|\Phi_m(\tilde{x}_n) - x_m\|_{\mathbb{H}_m} = 0 \tag{4.4.5}$$

and $(x_n)_n$ converge weakly *to x if*

$$\lim_{n\to+\infty} \langle x_n, z_n \rangle_{\mathbb{H}_n} = \langle x, z \rangle_{\mathbb{H}} \tag{4.4.6}$$

for any $z \in \mathbb{H}$ and sequence $(z_n)_n$, $z_n \in \mathbb{H}_n$, such that $z_n \to z$ strongly.

Lemma 4.4.4.

1. *The sequence of Hilbert spaces $L^2(\nu_n)$ converges to $L^2(\nu)$, by choosing Φ_n equal to the natural identification of equivalence classes in $L^2(\nu_n)$ and $L^2(\nu)$.*

2. *$u_n \in L^2(\nu_n)$ converges strongly to $u \in L^2(\nu)$ if and only if $u_n \to u$ in $L^2(\nu)$.*

3. *$u_n \in L^2(\nu_n)$ converges weakly to $u \in L^2(\nu)$ if and only if $u_n \to u$ weakly in $L^2(\nu)$.*

Proof. 1. We have to prove that for all $x \in L^2(\nu)$ we have $\|x\|_{L^2(\nu_n)} \to \|x\|_{L^2(\nu)}$ as $n \to \infty$. Since e^{-F_n}/Z_n converges a.s. to e^{-F}/Z and it is uniformly bounded, then the result follows by dominated convergence.

2. Let $(u_n)_n$ converges strongly to $u \in L^2(\nu)$ so there is a sequence $(\tilde{u}_n)_n$ in $L^2(\nu)$ tending to u in $L^2(\nu)$ such that:

$$\lim_n \overline{\lim_m} \|\tilde{u}_n - u_m\|_{L^2(\nu)_m} = 0. \tag{4.4.7}$$

Then we have:

$$\overline{\lim_m} \|u - u_m\|_{L^2(\nu)} \le \lim_n \|u - \tilde{u}_n\|_{L^2(\nu)} + \frac{C}{c} \lim_n \overline{\lim_m} \|u_m - \tilde{u}_n\|_{L^2(\nu_m)} = 0,$$

so that $u_n \to u$ in $L^2(\nu)$. Conversely, if $u_n \to u$ in $L^2(\nu)$ then we can consider $\tilde{u}_n = u$ for all $n \in \mathbb{N}$ and (4.4.7) holds.

3. Let $u_n \in L^2(\nu_n)$ be a sequence which converges weakly to $u \in L^2(\nu)$, i.e. for all $v \in L^2(\nu)$ and any sequence $v_n \in L^2(\nu_n)$ strongly convergent to v

$$\langle u_n, v_n \rangle_{L^2(\nu_n)} \to \langle u, v \rangle_{L^2(\nu)}, \quad n \to +\infty.$$

Let $v_n := v \cdot \rho \cdot \rho_n^{-1}$, then by the dominated convergence theorem $\|v_n - v\|_{L^2(\nu)} \to 0$ and by the previous point $v_n \in L^2(\nu_n)$ converges strongly to v. So we have

$$\langle u_n, v \rangle_{L^2(\nu)} = \langle u_n, v_n \rangle_{L^2(\nu_n)} \to \langle u, v \rangle_{L^2(\nu)}, \quad n \to +\infty.$$

Viceversa, let us suppose that for all $v \in L^2(\nu)$ we have $\langle u_n, v \rangle_{L^2(\nu)} \to \langle u, v \rangle_{L^2(\nu)}$ and let us consider any sequence $v_n \in L^2(\nu_n)$ strongly convergent to v. Setting $w_n := v_n \cdot \rho_n \cdot \rho^{-1}$, by dominated convergence $\|w_n - v\|_{L^2(\nu)} \to 0$ and therefore $\langle u_n, v_n \rangle_{L^2(\nu_n)} = \langle u_n, w_n \rangle_{L^2(\nu)} \to \langle u, v \rangle_{L^2(\nu)}$ and the proof is finished.

□

4.4.2 Convergence of Dirichlet Forms

Now we can give the definition of Mosco-convergence of Dirichlet forms. This concept is useful for our purposes, since it was proved in [43] to imply the convergence in a strong sense of the associated resolvents and semigroups.

Definition 4.4.5. *If $\mathcal{E}^n$ is a quadratic form on $\mathbb{H}_n$, then $\mathcal{E}^n$ Mosco-converges to the quadratic form $\mathcal{E}$ on $\mathbb{H}$ if the two following conditions hold:*

Mosco I. *For any sequence $x_n \in \mathbb{H}_n$, converging weakly to $x \in \mathbb{H}$,*

$$\mathcal{E}(x,x) \leq \varliminf_{n\to+\infty} \mathcal{E}^n(x_n,x_n). \tag{4.4.8}$$

Mosco II. *For any $x \in \mathbb{H}$, there is a sequence $x_n \in \mathbb{H}_n$ converging strongly to $x \in \mathbb{H}$ such that*

$$\mathcal{E}(x,x) = \lim_{n\to+\infty} \mathcal{E}^n(x_n,x_n). \tag{4.4.9}$$

We recall that a sequence of bounded operarors $(B_n)_n$ on $\mathbb{H}_n$, converges strongly to an operator B on $\mathbb{H}$, if $\mathbb{H}_n \ni B_n u_n \to Bu \in \mathbb{H}$ strongly for all sequence $u_n \in \mathbb{H}_n$ converging strongly to $u \in \mathbb{H}$, see definition 2.4.3. Then Kuwae and Shioya have proved in [43] the following equivalence between Mosco convergence and strong convergence of the associated resolvent operators.

4.4.3 Mosco convergence

Proposition 4.4.6. *The Dirichlet form $\mathcal{E}^n$ Mosco-converges to $\mathcal{E}$ on $L^2(\nu)$.*

Proof. The proof of the condition Mosco II is trivial in our case; indeed, for all $x \in D(\mathcal{E})$, we set $x_n := x \in D(\mathcal{E}^n)$ for all $n \in \mathbb{N}$; by dominated convergence $\mathcal{E}(x,x) = \lim_n \mathcal{E}^n(x,x)$. If $x \notin D(\mathcal{E})$, then again $x_n := x \notin D(\mathcal{E}^n)$ satisfies $\mathcal{E}(x,x) = \lim_n \mathcal{E}^n(x,x) = +\infty$.

Let us prove now condition Mosco I. We first assume that $u \in \mathcal{D}(\mathcal{E})$. By the integration by parts formula (4.3.1) we have for any $v \in \mathcal{K} = \mathrm{Exp}_A(H)$

$$2\mathcal{E}(u,v) = -\int_H u \cdot \mathrm{Tr}(D^2 v)\, d\nu + \int_H u \left(\langle \cdot, A\nabla v\rangle_H - \int_{\mathbb{R}\times[0,1]} f(da)\, \nabla_r v\, \ell^a(dr) \right) d\nu.$$

Let $u_n \in L^2(\nu_n)$ a sequence converging weakly to u, then we know from Theorem 4.4.4 that $u_n \to u$ weakly in $L^2(\nu)$. By the compactness of the embedding $D(\mathcal{E}^0) \mapsto L^2(\mu)$ proved in Proposition 4.2.3, $u_n \to u$ strongly in $L^2(\nu)$. By linearity it is enough to consider $v(x) = \exp(i\langle h, x\rangle_H)$, $h \in D(A)$, $x \in H$. Notice that $\nabla v = i\, v\, h$. Then we can write

$$\int_{\mathbb{R}\times[0,1]} f(da)\, \nabla_r v(\beta)\, \ell^a(dr) = i\, v(\beta) \int_{\mathbb{R}\times[0,1]} f(da)\, h_r\, \ell^a(dr).$$

Moreover by the occupation times formula

$$\langle \nabla v(\beta), f_n'(\beta)\rangle_H = i\, v(\beta) \int_0^1 h_r\, f_n'(\beta_r)\, dr = i\, v(\beta) \int_{\mathbb{R}\times[0,1]} h_r\, \ell^a(dr)\, f_n'(a)\, da.$$

Since $f_n'(a)\, da \rightharpoonup f(da)$, by dominated convergence we obtain

$$2\mathcal{E}(u,v) = \lim_{n\to\infty} \left(-\int_H u^n \cdot \mathrm{Tr}(D^2 v)\, d\nu^n + \int_H u^n (\langle x, A\nabla v\rangle_H + \langle \nabla v, f_n'\rangle)\, d\nu^n \right).$$

We can suppose that each u^n is in $\mathcal{D}(\mathcal{E}^n)$ (else $\mathcal{E}^n(u^n,u^n) = +\infty$) so we have for any $v \in \mathcal{K} \setminus \{0\}$

$$\varliminf_{n\to+\infty} \Big(\mathcal{E}^n(u^n,u^n)\Big)^{1/2} \geq \varliminf_{n\to+\infty} \frac{\mathcal{E}^n(u^n,v)}{\sqrt{\mathcal{E}^n(v,v)}} = \frac{\mathcal{E}(u,v)}{\sqrt{\mathcal{E}(v,v)}}$$

and by considering the sup over v we obtain the desired result.

Suppose now that $u \notin \mathcal{D}(\mathcal{E})$ and let $L^2(\nu_n) \ni u^n \to u \in L^2(\nu)$ weakly, then we know from Theorem 4.4.4 that $u_n \to u$ weakly in $L^2(\nu)$. By the compactness of the embedding $D(\mathcal{E}^0) \mapsto L^2(\mu)$ proved in Proposition 4.2.3, $u_n \to u$ strongly in $L^2(\nu)$. If $\liminf_{n\to\infty} \mathcal{E}^n(u^n,u^n) < +\infty$, then we also have $\liminf_{n\to\infty} \mathcal{E}(u^n,u^n) < +\infty$. But since $\mathcal{E}$ is lower semi-continuous in $L^2(\nu)$, then $\mathcal{E}(u,u) < +\infty$, which is absurd since we assumed that $u \notin \mathcal{D}(\mathcal{E})$. □

4.4.4 Convergence of stationary solutions

We denote by $\mathbb{P}^n_{\nu_n}$ the law of the stationary solution of (4.4.1) and by $\mathbb{P}_\nu$ the law of the Markov process associated with $\mathcal{E}$ and started with law ν. We have the following convergence result

Proposition 4.4.7. *The sequence $\mathbb{P}^n_{\nu_n}$ converges weakly to $\mathbb{P}_\nu$ in $C([0,T]\times[0,1])$.*

Proof. Let us first prove convergence of finite-dimensional distributions, i.e.

$$\lim_{n\to+\infty} \mathbb{E}^n_{\nu_n}(f(X_{t_1},\dots,X_{t_m})) = \mathbb{E}_\nu(f(X_{t_1},\dots,X_{t_m})),$$

for all $f \in C((C([0,1])^m)$. The Mosco convergence of the Dirichlet forms $\mathcal{E}^n$ provides the strong convergence of the semi-group and, by the Markov property, the convergence of the finite dimensional laws. Indeed let f be in $C((C([0,1])^m)$ of the form $f(x_1,\dots,x_m) = f_1(x_1)\cdot\ldots\cdot f_m(x_m)$ then

$$\begin{aligned}&P^n_{t_1}(f_1 \cdot P^n_{t_2-t_1}(f_2\cdot\ldots(f_{m-1}P^n_{t_m-t_{m-1}}f_m)\ldots))\\ &\to P_{t_1}(f_1 \cdot P_{t_2-t_1}(f_2\cdot\ldots(f_{m-1}P_{t_m-t_{m-1}}f_m)\ldots)), \qquad \text{strongly.}\end{aligned}$$

Then by the Markov property

$$\begin{aligned}&\mathbb{E}^n_{\nu_n}(f(X_{t_1},\dots,X_{t_m})) = \langle 1, P^n_{t_1}(f_1 \cdot P^n_{t_2-t_1}(f_2\cdot\ldots(f_{m-1}P^n_{t_m-t_{m-1}}f_m)\ldots))\rangle_{H_n}\\ &\to \langle 1, P_{t_1}(f_1 \cdot P_{t_2-t_1}(f_2\cdot\ldots(f_{m-1}P_{t_m-t_{m-1}}f_m)\ldots))\rangle_{H} = \mathbb{E}_\nu(f(X_{t_1},\dots,X_{t_m})).\end{aligned}$$

We need now to prove tightness in $C([0,T]\times[0,1])$. We first recall a result of [26, Th. 7.2 ch 3]. Let (P,d) be a Polish space, and let $(X_\alpha)_\alpha$ be a family of processes with sample paths in $C([0,T];P)$. Then the laws of $(X_\alpha)_\alpha$ are relatively compact if and only if the following two conditions hold:

1. For every $\eta > 0$ and rational $t \in [0,T]$, there is a compact set $\Gamma^t_\eta \subset P$ such that:
$$\inf_\alpha \mathbb{P}\left(X_\alpha \in \Gamma^t_\eta\right) \geq 1-\eta \tag{4.4.10}$$

2. For every $\eta, \epsilon > 0$ and $T > 0$, there is $\delta > 0$ such that

$$\sup_{\alpha} \mathbb{P}\left(w(X_\alpha, \delta, T) \geq \epsilon\right) \leq \eta \tag{4.4.11}$$

where $w(\omega, \delta, T) := \sup\{d(\omega(r), \omega(s)) : r, s \in [0,T], |r-s| \leq \delta\}$ is the modulus of continuity in $C([0,T]; P)$.

We consider now, as Polish space (P, d), the Banach space $C^\theta([0,1])$. Since $\mathbb{P}^n_{\nu_n}$ is stationary, (4.4.10) is reduced to a condition on ν_n. In fact we have

$$\left(\int_H \|x\|^p_{W^{\eta,p}(0,1)}\, d\nu_n\right)^{\frac{1}{p}} \leq \left(\frac{C}{c}\int_H \|x\|^p_{W^{\eta,p}(0,1)}\, d\mu\right)^{\frac{1}{p}}.$$

Now, since the Brownian bridge $(\beta_r)_{r\in[0,1]}$ is a Gaussian process with covariance function $r \wedge s - rs$, then

$$\begin{aligned}\mathbb{E}\left(\|\beta\|^p_{W^{\eta,p}(0,1)}\right) &\leq \mathbb{E}\left(\|\beta\|^p_p + \int_0^1\int_0^1 \frac{|\beta_s - \beta_t|^p}{|s-t|^{p\eta+1}}\, dt\, ds\right) \\ &\leq C_p\left(1 + \int_0^1\int_0^1 |s-t|^{p(\frac{1}{2}-\eta)-1}\, dt\, ds\right) < +\infty.\end{aligned}$$

For any $\eta < 1/2$, $\theta < \eta$ and $p > 1/(\eta - \theta)$ we have by the Sobolev embedding Theorem that $W^{\eta,p}(0,1) \subset C^\theta([0,1])$ with continuous embedding, so that

$$\sup_n \int_H \|x\|^p_{C^\theta([0,1])}\, d\nu_n < \infty.$$

By Lemma 4.6.1 below we obtain existence of a constant K independent of n such that

$$\mathbb{E}^n_{\nu_n}\left[\|X_t - X_s\|^p_{C^\theta([0,1])}\right] \leq K\,|t-s|^\xi, \qquad \forall\, n \geq 1,\ t, s \in [0,T].$$

By Kolmogorov's criterion, see [58, Thm. I.2.1], we obtain that a.s. $w(X^n, \delta, T) \leq C\,\delta^{\frac{1-\xi}{2p}}$, with $C \in L^p$. Therefore by the Markov inequality, if $\epsilon > 0$

$$\mathbb{P}\left(w(X^n, \delta, T) \geq \epsilon\right) \leq \mathbb{E}\left[C^p\right]\, \delta^{\frac{1-\xi}{2}} \epsilon^{-p},$$

and (4.4.11) follows for δ small enough. □

4.5 Convergence of finite dimensional approximations

From now on we turn our attention to another problem: convergence in law of finite dimensional approximations of equation (4.1.6). We want to project, in a sense to be made precise, (4.1.6) onto an equation in a finite dimensional subspace of $H := L^2(0,1)$. To be more precise, we consider the space H_n of functions in $L^2(0,1)$ which are constant on each interval $[(i-1)2^{-n}, i2^{-n}[$, $i = 1, \ldots, 2^n$ and we endow H_n with the scalar product inherited from H.

Notice that H_n is a linear closed subspace of $L^2(0,1)$, so that there exists a unique orthogonal projector $P_n : L^2(0,1) \mapsto H_n$, given explicitly by

$$P_n x := 2^n \sum_{i=0}^{2^n-1} \mathbb{1}_{[i2^{-n},(i+1)2^{-n}[}\, \langle \mathbb{1}_{[i2^{-n},(i+1)2^{-n}[}, x\rangle. \tag{4.5.1}$$

We call μ_n the law of $P_n\beta$; then μ_n is a Gaussian law on H with zero mean and non-degenerate covariance operator P_nQP_n, where Q is the covariance operator of μ, which has been studied in detail in section 4.2.2. In what follows we write

$$P_nQP_n = (-2A_n)^{-1}, \qquad A_n : H_n \mapsto H_n.$$

We also define π_n as

$$\pi_n(dx) = \frac{1}{Z_n}\,\exp(-F(x))\,\mu_n(dx) = \frac{1}{Z_n}\exp\left(-\frac{1}{2^n}\sum_{i=0}^{2^n-1} f(x(i))\right)\mu_n(dx). \quad (4.5.2)$$

where $Z_n := \mu_n(\exp(-F))$ is a normalization constant.

Then, a natural approximation of $\mathcal{E}$ defined on H_n is given by the following symmetric bilinear non-negative form

$$\Lambda^n(u,v) := \frac{1}{2}\int \langle \nabla u, \nabla v\rangle_{H_n}\, d\pi_n, \quad u,v \in C_b^1(H_n) \qquad (4.5.3)$$

with reference measure π_n. Then we have

$$\Lambda^n(u,v) = \frac{1}{2}\int \langle \nabla(u\circ P_n), \nabla(v\circ P_n)\rangle_H \,\frac{1}{Z_n}\,\exp(-F\circ P_n)\,d\mu, \quad u,v\in C_b^1(H_n). \qquad (4.5.4)$$

We write

$$f(y) = f_0(y) + \sum_{j=1}^{k}\alpha_j\,\mathbb{1}_{(y\le y_j)}, \qquad y\in\mathbb{R} \qquad (4.5.5)$$

where f_0 is smooth and bounded and $\alpha_j, y_j \in \mathbb{R}$. Clearly, f has a jump in each y_j of respective size α_j. We have the following integration by parts formula

$$\begin{aligned}\int \partial_h\varphi\,d\pi_n = &-\int \varphi\,\langle x, A_n h\rangle\,\pi_n(dx) + \int\varphi(x)\,2^{-n}\sum_{i=0}^{2^n-1} h_i\,f_0'(x(i))\,\pi_n(dx)\\ &-\int\varphi(x)\sum_{i=0}^{2^n-1} h_i\sum_j 2\,\frac{1-e^{-\alpha_j\,2^{-n}}}{1+e^{-\alpha_j\,2^{-n}}}\,\pi_n(dx\,;\,x(i)=y_j),\end{aligned} \qquad (4.5.6)$$

where we use the notation

$$\pi_n(A\,;\,x(i)=y_j) := \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\,\pi_n(A\cap\{|x(i)-y_j|\le\varepsilon\}).$$

This suggests that the associated dynamic solves the stochastic differential equation

$$dX^i = \frac{1}{2}\left((A_nX)^i - f_0'(X^i)\right)dt + \sum_j \frac{1-e^{-\alpha_j\,2^{-n}}}{1+e^{-\alpha_j\,2^{-n}}}\,d\tilde{\ell}_t^{i,y_j} + 2^{-\frac{n}{2}}\,dw_t^i \qquad (4.5.7)$$

where $(\tilde{\ell}_t^{i,a}, t\ge 0)$ is the symmetric local time of $(X^i(t), t\ge 0)$ at a.

4.5.1 Skew Brownian motion

Let $(X_t)_{t\geq 0}$ be the skew Brownian motion defined in (4.1.1) with $|\beta| < 1$ and $\tilde{L}^0$ equal to the symmetric local time. If $(L^a_t)_{t\geq 0, a\in\mathbb{R}}$ is the right-continuous version of the family of local times of the semimartingale $(X_t)_{t\geq 0}$ defined by the Tanaka formula [58, Theorem VI.1.2]:

$$\begin{aligned} (X_t - a)^+ &= (X_0 - a)^+ + \int_0^t \mathbb{1}_{(X_s > a)}\, dX_s + \frac{1}{2}\, L^a_t \\ (X_t - a)^- &= (X_0 - a)^- - \int_0^t \mathbb{1}_{(X_s \leq a)}\, dX_s + \frac{1}{2}\, L^a_t \end{aligned} \tag{4.5.8}$$

then a simple computation shows that in this case $(1+\beta)\tilde{L}^0_t = L^0_t$. Indeed, by the occupation times formula

$$\tilde{L^0_t} = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{(|X_s|\leq\varepsilon)}\, ds = \frac{L^0_t + L^{0-}_t}{2},$$

see also [58, VI.1.25]. On the other hand, by the same formula for $a < 0$ we have

$$\begin{aligned} (X_t)^+ &= (X_0 - a)^+ + \int_0^t \mathbb{1}_{(X_s > 0)}\, dB_s + \frac{1}{2}\, L^0_t \\ (X_t - a)^- &= (X_0 - a)^- - \int_0^t \mathbb{1}_{(X_s \leq a)}\, dB_s - \beta\tilde{L}^0_t + \frac{1}{2}\, L^a_t \end{aligned}$$

taking the difference of the two formulae above and letting $a \uparrow 0$, we obtain

$$\frac{L^0_t - L^{0-}_t}{2} = \beta\, \tilde{L}^0_t$$

(see also [58, Theorem VI.1.7]). Then we obtain

$$L^0_t = (1+\beta)\, \tilde{L}^0_t.$$

This allows to compute the scale function of X; indeed, setting $\gamma := \dfrac{1+\beta}{2}$ and

$$s(x) := \begin{cases} (1-\gamma)x, & x \geq 0 \\ \gamma x, & x < 0, \end{cases}$$

then we can see that

$$s(X_t) = s(X_0) + \int_0^t \left((1-\gamma)\mathbb{1}_{(X_s>0)} + \gamma \mathbb{1}_{(X_s\leq 0)} \right) dB_s,$$

with the different local times canceling out. (In fact we recall that the scale function is defined up to an affine transformation).

Lemma 4.5.1. *The process $(X_t)_{t\geq 0}$ is associated with the Dirichlet form*

$$D(u) := \frac{1}{2} \int_{\mathbb{R}} (\dot{u})^2 \, \exp(-\alpha \mathbb{1}_{]-\infty,0]})\, dx$$

in $L^2(\exp(-\alpha\mathbb{1}_{]-\infty,0]})\, dx)$, where $\alpha \in \mathbb{R}$ is defined by $\dfrac{1-e^{-\alpha}}{1+e^{-\alpha}} = \beta \in]-1,+1[$.

Proof. The form $(D, C_b^1(\mathbb{R}))$ is closable in $L^2(\exp(-\alpha \mathbb{1}_{]-\infty,0]})\,dx)$ since it is equivalent to the standard Dirichlet forms associated with the Brownian motion. By the same argument, the closure of $(D, C_b^1(\mathbb{R}))$ is regular and therefore there exists an associated Hunt process $(X_t)_{t\geq 0}$. We want now to prove that this process is a weak solution of (4.1.1). The following integration by parts formula

$$\begin{aligned}\int \varphi' \exp(-\alpha \mathbb{1}_{]-\infty,0]})\,dx &= -\,(1-e^{-\alpha})\,\varphi(0)\\ &= 2\,\frac{1-e^{-\alpha}}{1+e^{-\alpha}} \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \varphi\, \exp(-\alpha \mathbb{1}_{]-\infty,0]})\,dx,\end{aligned}$$

together with the Fukushima decomposition, shows that X_t is a semimartingale and that it satisfies (4.1.1) for quasi-every initial point $X_0 = x$, i.e. for all x outside a set N of null capacity. However, we can in fact choose $N = \emptyset$ by noting that the transition semigroup of the skew Brownian motion with $-1 \leq \beta \leq 1$ has an explicit Markov transition density with respect to the Lebesgue measure (see III.1.16, VII.1.23, XII.2.16 in [58]). Therefore X satisfies the *absolute continuity assumption* and we can use [27, Theorem 4.1.2 and formula (4.2.9)].

□

Theorem 4.5.2. *The form* Λ^n*, defined in* (4.5.3)*, is a regular Dirichlet form in* $L^2(\pi_n)$*, and the associated Markov process is a weak solution of* (4.5.7)*. Moreover such solution is unique in law.*

Proof. As in the proof of Lemma 4.5.1, Λ^n is a regular Dirichlet form with the strong local property because it is equivalent to the Dirichlet form of a finite dimensional Ornstein-Uhlenbeck process. So by [27] there is a continuous Hunt process associated to Λ^n.

By the integration by parts formula (4.5.6) and the Fukushima decomposition, the Hunt process associated with Λ^n has the following property: the process $(\langle h, X_t\rangle)_{t\geq 0}$ is a semi-martingale

$$\langle h, X_t^n\rangle - \langle h, X_0^n\rangle = M_t^h + N_t^h \tag{4.5.9}$$

and the Revuz measure of the bounded-variation CAF N^h is

$$\Sigma^h(dx) = \frac{1}{2}\langle A_n x - f_0'(x), h\rangle\, \pi_n(dx) + \sum_{i=0}^{2^n-1} h_i \sum_j \frac{1-e^{-\alpha_j 2^{-n}}}{1+e^{-\alpha_j 2^{-n}}}\, \pi_n(dx\,;\, x(i) = y_j). \tag{4.5.10}$$

Because of the structure of Σ^h, the process N^h can be written as

$$N_t^h = \int_0^t \frac{1}{2}\langle A_n X_s - f_0'(X_s), h\rangle\, ds + \sum_{i=0}^{2^n-1} h_i \sum_j \frac{1-e^{-\alpha_j 2^{-n}}}{1+e^{-\alpha_j 2^{-n}}}\, \tilde{\ell}_t^{i,y_j},$$

where $\tilde{\ell}_t^{i,y_j}$ is adapted to the natural filtration of $(X_t, t \geq 0)$. We want now to show that in fact $\tilde{\ell}_t^{i,y_j}$ is adapted to the natural filtration of $(X_t^i, t \geq 0)$. Since X_t^i is a semimartingale, by Tanaka's formula

$$|X_t^i - y_j| = |X_0^i - y_j| + \int_0^t \operatorname{sign}(X_s^i - y_j)\, dX_s^i + L_t^{y_j}(X^i) \tag{4.5.11}$$

where $L^{y_j}(X^i)$ is the local time of X_t^i at y_j. Since $|\langle e_i,\,\cdot\rangle - y_j| \in \Lambda^n$, then $L^{y_j}(X^i)$ is an additive functional of X. Now we can compute the Revuz measure of $L^{y_j}(X^i)$, using theorem 5.4.2 of [27]. With an integration by parts formula we see that for all φ smooth enough:

$$\Lambda^n(|\langle e_i,\cdot\rangle - y_j|,\varphi) = \frac{1}{2}\int \operatorname{sign}(x_i - y_j)\,\partial_i\varphi(x)\,d\pi_n$$
$$= -\frac{1}{2}\int \operatorname{sign}(x_i - y_j)\,\left((A_n x)^i - f_0'(x_i)\right)\,\varphi(x)\,d\pi_n - \int \varphi(x)\,\pi_n(dx; x(i) = y_j).$$

By comparison with (4.5.11), we see that $\pi_n(dx; x(i) = y_j)$ is the Revuz measure of $t \mapsto L_t^{y_j}(X^i)$ and therefore by (4.5.10) the processes $(L_t^{y_j}(X^i), t \geq 0)$ and $(\ell_t^{i,y_j}, t \geq 0)$ are equal up to a multiplicative constant.

We want now to prove uniqueness in law for (4.5.7). We define the exponential martingale

$$M_t := \exp\left(-2^{\frac{n}{2}-1}\int_0^t \langle A_n X_s - f_0'(X_s), dw_s\rangle - 2^{n-3}\int_0^t \|A_n X_s - f_0'(X_s)\|^2 ds\right).$$

Then under the probability measure $M_T\cdot\mathbb{P}_x$, by the Girsanov theorem the canonical process is a solution in law of

$$dX^i = \sum_j \frac{1 - e^{-\alpha_j\,2^{-n}}}{1 + e^{-\alpha_j\,2^{-n}}}\,d\tilde{\ell}_t^{i,y_j} + 2^{-\frac{n}{2}}d\hat{w}_t^i, \qquad t \in [0,T],$$

where the Brownian motions $(\hat{w}_t^i, t \geq 0)_i$ are independent; therefore we have reduced to an independent vector of skew-Brownian motions and uniqueness in law holds for such processes by the pathwise uniqueness proved by Harrison and Shepp in [39].

Moreover, by the property recalled in the proof of Lemma 4.5.1, the transition semigroup of the skew-Brownian motion satisfies the absolute continuity condition and therefore all the above statements are true for all initial conditions. $\square$

Excursion construction of the skew Brownian motion

To conclude this part we just give the excursion construction of the skew Brownian motion. We denote by $(\mathbb{W},\mathcal{F},\mathbb{P})$ the Wiener space and by $U \subset \mathbb{W}$ the space of all functions w such that $+\infty > R(w) > 0$, where $R(w) := \inf\{t > 0 | w(t) = 0\}$. $\mathcal{U}$ is the trace of the Borel σ-field . Then the excursion process $(e_s, s > 0)$, for the Brownian motion, is defined as follow:

$$e_s(w) : r \mapsto B_{\tau_{s-}(w)+r}(w)1_{(r\leq\tau_s(w)-\tau_{s-}(w))} \tag{4.5.12}$$

if $\tau_s(w) - \tau_{s-}(w) > 0$. Note that $\tau_s = \sum_{u\leq s} R(e_u)$ and $\tau_{s-} = \sum_{u<s} R(e_u)$, L_t is defined as the inverse of τ_t. we prove now the the process

$$X_t^\alpha(w) = Y_{\tau_{s-}(w)}|e_s(t - \tau_{s-}(w)); w)|,\; t \in [\tau_{s-}(w), \tau_s(w)[\tag{4.5.13}$$

is a skew brownian motion. $(Y_a)_a$ is countable family of independent Bernoulli random variables with parameter α indexed by a denumerable set.

Proposition 4.5.3. *The processus X^α is a Markov process, with transition semigroup:*

$$\begin{aligned} p_t(0,y) &= (1+\beta)\, g_t(y)\, \mathbb{1}_{(y>0)} + (1-\beta)\, g_t(y)\, \mathbb{1}_{(y<0)} \\ p_t(x,y) &= \mathbb{1}_{(x>0)}[\, g_t(y-x) + \beta\, g_t(x+y)\, \mathbb{1}_{(y>0)} + (1-\beta)\, g_t(y-x)\, \mathbb{1}_{(y<0)}] \\ &\quad + \mathbb{1}_{(x<0)}[\, g_t(y-x) - \beta\, g_t(x+y)\, \mathbb{1}_{(y<0)} + (1+\beta)\, g_t(y-x)\, \mathbb{1}_{(y>0)}], \end{aligned}$$

where $g_t(\cdot)$ is the density of the Gaussian law $\mathcal{N}(0,t)$.

See exercice III.1.16 and exercice XII.2.16 in [58].

4.5.2 Scale function and speed measure

In this chapter we calculate the speed function of the process

$$X_t = X_0 + B_t + \int \tilde{l}_t^a(x) f'(da) \tag{4.5.14}$$

where $(\tilde{l}_t^a)_{t,a}$ is the symmetric local time of X, and f is the function

$$f(y) = \sum_{j=1}^{k} \beta_j\, 1_{(y \le y_j)}, \qquad y \in \mathbb{R} \tag{4.5.15}$$

with $|\beta_j| < 1$ and $y_1 < \cdots < y_k < y_{k+1} := +\infty$; we define a continuous piecewise linear function $s : \mathbb{R} \mapsto \mathbb{R}$, such that $s(x) = x$ for $x < y_1$, s' is well-defined and constant on $]y_j, y_{j+1}[$, $j = 1, \ldots, k$, and satisfies

$$s'_+(y_j) = s'_-(y_j)\, \frac{1-\beta_j}{1+\beta_j}, \qquad j = 1, \ldots, k, \tag{4.5.16}$$

where s'_+ and s'_- denote respectively the right and left derivative of s. Then by a calculation similar to that performed in section 4.5.1 we can see that

$$s(X_t) = s(X_0) + \int_0^t s'(X_u)\, dB_u, \qquad t \ge 0,$$

i.e. s is the scale function of s, where s' is defined by $\dfrac{s'_+ + s'_-}{2}$ (we recall again that the scale is unique up to an affine transformation). Indeed, by the Itô-Tanaka formula applied to the semimartingale X we obtain

$$\begin{aligned} s(X_t) &= s(X_0) + \int_0^t s'(X_s)\, dX_s + \frac{1}{2} \int_{\mathbb{R}} \tilde{l}_t^a\, s''(da) \\ &= s(X_0) + \int_0^t s'(X_s)\, dB_s + \int_{\mathbb{R}} s'(a)\, \tilde{l}_t^a\, f'(da) + \frac{1}{2} \int_{\mathbb{R}} \tilde{l}_t^a\, s''(da), \end{aligned}$$

see [58, VI.1.25]. By (4.5.16), $\beta_j(s'_+(y_i) + s'_-(y_i)) = s'_-(y_i) - s'_+(y_i)$ and therefore for all $t > 0$

$$\begin{aligned} \int_{\mathbb{R}} s'(a)\, \tilde{l}_t^a\, f'(da) &= \sum_1^k \beta_j\, \frac{s'_+ + s'_-}{2}(y_j)\, \tilde{l}_t^{y_j} = \sum_1^k \frac{s'_- - s'_+}{2}(y_j)\, \tilde{l}_t^{y_j} \\ &= -\frac{1}{2} \int_{\mathbb{R}} \tilde{l}_t^a\, s''(da) \end{aligned}$$

and therefore

$$s(X_t) = s(X_0) + \int_0^t s'(X_s)\, dB_s, \qquad t \geq 0.$$

4.5.3 Convergence of the Hilbert spaces

Proposition 4.5.4. *The sequence of Hilbert spaces $(L^2(\pi_n))_n$ converges to $L^2(\nu)$ in the sense of Definition 4.4.3.*

Proof. According to Definition 4.4.3, we have first to define a map $\Phi_n : L^2(\nu) \mapsto L^2(\pi_n)$. We consider now the Borel σ-field $\mathcal{B}$ on $L^2(0,1)$, completed with all μ-null sets (we use the same notation for the completed σ-field).

Setting $\bar{\beta} := P_n\beta$, let us introduce the filtration $\mathcal{F}_n := \sigma(\bar{\beta}_{i2^{-n}},\, i = 1,\dots,2^n)$ and the linear map $\Phi_n : L^2(\mu) \mapsto L^2(\mu_n)$ defined as follows: $\Phi_n(\varphi) = \varphi_n$, where

$$\varphi_n(\bar{\beta}_{i2^{-n}},\, i = 1,\dots,2^{-n}) = \mathbb{E}(\varphi(\beta) \,|\, \mathcal{F}_n). \tag{4.5.17}$$

Then φ_n is well defined μ_n-a.e. For any $\varphi \in L^2(\mu)$ the sequence $(\varphi_n)_n$ is a martingale bounded in $L^2(\mu)$, therefore converging a.s. and in $L^2(\mu)$. Now, since $L^2(\mu) \equiv L^2(\nu)$ and $L^2(\mu_n) \equiv L^2(\pi_n)$ with equivalence of norms (uniformly in n), then the map Φ_n is still well defined and $\sup_n \|\varphi_n\|_{L^2(\pi_n)} < +\infty$ for all $\varphi \in L^2(\nu)$. We have to prove that $\|\varphi_n\|_{L^2(\pi_n)} \to \|\varphi\|_{L^2(\nu)}$ as $n \to +\infty$.

We first prove that $F(\bar{\beta}^n)$ converges a.s. to $F(\beta)$, where $\beta^n := \bar{\beta}_{\lfloor r2^n \rfloor}$, $r \in [0,1]$. We have that

$$F(\beta^n) = 2^n \sum_{i=1}^{2^n-1} f(\beta_{i2^{-n}}) = \int_0^1 f(\beta_{\lfloor r2^n \rfloor})\, dr.$$

Now by dominated convergence it is enough to prove that a.s. $f(\beta^n_r) \underset{n\to+\infty}{\to} f(\beta_r)$ for a.e. $r \in [0,1]$. By (4.5.5), f is continuous outside the finite set $\Delta_f = \{y_j\}$. For all $a \in \mathbb{R}$, a.s. $\{r \in [0,1] : \beta_r = a\}$ is a compact set with zero Lebesgue measure and therefore a.s. $U := \{r \in [0,1] : \beta_r \in \Delta_f\}$ also has zero Lebesgue measure. Therefore for all $r \in [0,1] \setminus U$, $f(\beta^n_r) \underset{n\to+\infty}{\to} f(\beta_r)$ and by dominated convergence $F(\beta^n)$ converges a.s. to $F(\beta)$. In particular, by dominated convergence $Z_n = \mu_n(e^{-F}) = \mathbb{E}(e^{-F(\bar{\beta}^n)})$ converges to $Z = \mathbb{E}(e^{-F(\beta)})$.

Now, let us prove that $\|\varphi_n\|_{L^2(\pi_n)} \to \|\varphi\|_{L^2(\nu)}$. Since $Z_n \underset{n\to+\infty}{\to} Z$, we have to prove that

$$\mathbb{E}\left(\varphi_n^2(\bar{\beta}^n)\, e^{-F(\bar{\beta}^n)}\right) \underset{n\to+\infty}{\to} \mathbb{E}\left(\varphi^2(\beta)\, e^{-F(\beta)}\right).$$

We have shown above that $\varphi_n(\bar{\beta}^n)$ converges to $\varphi(\beta)$ in L^2. Therefore $(\varphi_n^2(\beta^n))_n$ is uniformly integrable and so is also $(\varphi_n^2(\bar{\beta}^n)\, e^{-F(\bar{\beta}^n)})_n$, since $(e^{-F(\bar{\beta}^n)})_n$ is bounded in L^∞. We can then conclude since a u.i. sequence converging a.s. converges in L^1. □

4.5.4 Mosco convergence

We want now to prove that Λ^n Mosco converges to $\mathcal{E}$. In [4, Thm. 3.5], Andres and von Renesse have proved that Theorem 2.4.4 still holds if one replaces the condition

Mosco II with the following condition *Mosco II'*.

Definition 4.5.5 (*Mosco II'*)**.** *There is a core $\mathcal{K} \subset \mathcal{D}(\mathcal{E})$ such that for any $x \in K$ there exists a sequence $x_n \in \mathcal{D}(\Lambda^n)$ converging strongly to x and such that $\mathcal{E}(x,x) = \lim_{n\to+\infty} \Lambda^n(x_n, x_n)$.*

Theorem 4.5.6. *The Dirichlet form Λ^n Mosco-converges to Λ as $n \to +\infty$.*

Lemma 4.5.7. *Let $u_n \in L^2(\pi_n)$ be a sequence which converges weakly to $u \in L^2(\nu)$, and such that $\liminf_n \Lambda^n(u_n, u_n) < +\infty$, then there is a subsequence of $(u_n \circ P_n)_n$ converging to u in $L^2(\nu)$.*

Proof. By passing to a subsequence, we can suppose that $\limsup_n \Lambda^n(u_n, u_n) < +\infty$. By (4.5.4), we have that $\mathcal{E}^0(u_n \circ P_n, u_n \circ P_n) \le C\Lambda^n(u_n, u_n)$, for some constant $C > 0$, and therefore $\limsup_n \mathcal{E}^0(u_n \circ P_n, u_n \circ P_n) < +\infty$. By Proposition 4.2.1, the inclusion $D(\mathcal{E}^0) \subseteq L^2(\nu)$ is compact, so that we can extract a subsequence $v_{n_k} := u_{n_k} \circ P_{n_k}$ converging in $L^2(\nu)$. This subsequence $v_{n_k} \in L^2(\pi_{n_k})$ converges strongly to $u \in L^2(\nu)$, since $\Phi_n(u_n \circ P_n) = u_n$, by the definition of Φ_n given in the proof of Proposition 4.5.4. $\square$

Proof of Theorem 4.5.6. Let us consider the following regularization of f: we fix a function $\rho : \mathbb{R} \mapsto \mathbb{R}$ such that $\rho(x) = 1$ for all $x \le 0$, $\rho(x) = 0$ for all $x \ge 1$, ρ is monotone non-increasing and twice continuously differentiable on $\mathbb{R}$ with $0 \le \rho' \le 1$; then we set

$$f_n(y) = f_0(y) + \sum_{j=1}^{k} \alpha_j\, \rho(n(y - y_j) + \mathbb{1}_{(\alpha_j<0)}), \qquad y \in \mathbb{R}. \tag{4.5.18}$$

Notice that $f_n \downarrow f$ pointwise as $n \uparrow +\infty$. Now we define the measure

$$\tilde{\pi}_n(dx) = \frac{1}{Z_n} \exp(-F_n(x))\, \mu_n(dx) = \frac{1}{Z_n} \exp\left(-\frac{1}{2^n} \sum_{i=1}^{2^n-1} f_n(x(i2^{-n}))\right) \mu_n(dx);$$

note that $\tilde{\pi}_n$ is not normalized to be a probability measure, in fact $\tilde{\pi}_n \le \pi_n$ since $f_n \ge f$. We also define the Dirichlet form

$$\tilde{\Lambda}^n(\varphi, \psi) := \frac{1}{2} \int \langle \nabla\varphi, \nabla\psi \rangle \, d\tilde{\pi}_n, \qquad \forall\, \varphi, \psi \in D(\Lambda^n).$$

The form $\tilde{\Lambda}^n$ is clearly equivalent to Λ^n on $D(\Lambda^n)$. Moreover $\tilde{\Lambda}^n(u,u) \le \Lambda^n(u,u)$ for all $u \in D(\Lambda^n)$.

Let us show first condition Mosco II'. For $v \in \mathcal{K} := \mathrm{Exp}_A(H)$, we have that

$$v(w) = \sum_{m=1}^{k} \lambda_k \exp(i\langle w, h_m \rangle)$$

and we can suppose that $v \ne 0$. We set $v_n := v_{|H_n}$. Then it is easy to see that v_n converges strongly to v; indeed, setting $\tilde{v}_n := v \circ P_n$, we have $\Phi_m(\tilde{v}_n) = v_n$ for $m \ge n$ by construction; therefore

$$\|\Phi_m(\tilde{v}_n) - v_m\|_{L^2(\pi_m)} = \|v_n - v_m\|_{L^2(\pi_m)} \le C\|v \circ P_n - v \circ P_m\|_{L^2(\mu)},$$

which tends to 0 as $m \to +\infty$ and then $n \to +\infty$. Moreover

$$\Lambda^n(v_n, v_n) = \frac{1}{2} \int \|P_n \nabla v\|_H^2 \, d\pi_n \to \mathcal{E}(v, v),$$

so that Mosco II' holds.

Let us prove now Mosco I. Let $u_n \in L^2(\pi_n)$ be a sequence converging weakly to $u \in L^2(\nu)$; we can suppose that $u \in \mathcal{D}(\mathcal{E})$ and that $\liminf_n \Lambda^n(u_n, u_n) < +\infty$; then by lemma 4.5.7, up to passing a subsequence, we can suppose that $u_n \to u$ strongly.

Since $\tilde{\Lambda}^n \le \Lambda^n$, we have

$$\liminf_{n\to\infty} \Lambda^n(u_n, u_n) \ge \liminf_{n\to\infty} \tilde{\Lambda}^n(u_n, u_n).$$

Now for any $v_n \in D(\Lambda^n)$

$$\tilde{\Lambda}^n(u_n, u_n) \ge \frac{\left(\tilde{\Lambda}^n(u_n, v_n)\right)^2}{\tilde{\Lambda}^n(v_n, v_n)}. \tag{4.5.19}$$

Suppose that $v \ne 0$ and $v \in \mathrm{Exp}_A(H)$ is a linear combination of exponential functions. We set $v_n := v_{|H_n}$. Then arguing as above we have $\tilde{\Lambda}^n(v_n, v_n) \to \mathcal{E}(v, v)$. Now we prove that $\tilde{\Lambda}^n(u_n, v_n) \to \mathcal{E}(u, v)$. By linearity, we can suppose that $v = \exp(i\langle \cdot, h\rangle)$. Integrating by parts we see that

$$\begin{aligned} 2\,\tilde{\Lambda}^n(u_n, v_n) = -i \int u_n(x)\, v_n(x)\, \langle A_n x - \, f_n'(x), P_n h\rangle \, \pi_n(dx). \\ + \|P_n h\|_H^2 \int u_n v_n \, d\pi_n \end{aligned}$$

the last term converges easily. The claim follows if we prove that

$$\int u_n(x)\, v_n(x)\, \langle n\rho'(n(x-y)), P_n h\rangle \, \pi_n(dx) \to \int u(x)\, v(x)\, \langle \ell^y_\cdot, h\rangle \, \nu(dx).$$

Note that, with the notation $\beta^n = P_n \beta$,

$$\int \varphi(x)\, \langle n\rho'(n(x-y)), h\rangle \, \pi_n(dx) = \mathbb{E}(\varphi(\beta^n)\, \langle n\rho'(n(\beta^n - y)), h\rangle).$$

Now

$$|\langle n\rho'(n(\beta^n - y)) - n\rho'(n(\beta - y)), P_n h\rangle| \le n \sup_{|r-s|\le 2^{-n}} |\beta_r - \beta_s| \, \|h\|_\infty.$$

Moreover, if h has support in $[\varepsilon, 1-\varepsilon]$, then

$$\begin{aligned} &\left| \langle n\rho'(n(\beta - y)), h_n\rangle - \int_0^1 h_n \, d\ell^y \right| = \left| \int h_n(r) \left(\int n\rho'(n(a-y))\, (\ell^a - \ell^y)(dr)\, da \right) \right| \\ &= \left| \int_\varepsilon^{1-\varepsilon} h_n'(r) \left(\int n\rho'(n(a-y))\, (\ell^a(r) - \ell^y(r))\, da \right) dr \right| \\ &\le \|h'\| \sup_{|a-y|\le 1/n} \sup_{r\in[\varepsilon, 1-\varepsilon]} |\ell^a(r) - \ell^y(r)|. \end{aligned}$$

We want now to show that these quantities converge to 0 in L^2 as $n \to +\infty$. Indeed, since $(\beta_{1-r}, r \in [0,1])$ has the same law as $(\beta_r, r \in [0,1])$, we can write

$$\begin{aligned}
&\mathbb{E}\left(\sup_{|r-s|\le 2^{-n}} |\beta_r - \beta_s|^2\right) \le 2\,\mathbb{E}\left(\sup_{|r-s|\le 2^{-n}, r,s\le\frac{3}{4}} |\beta_r - \beta_s|^2\right) \\
&= 2\,\mathbb{E}\left(\sup_{|r-s|\le 2^{-n},\, r,s\le\frac{3}{4}} |B_r - B_s|^2\,\frac{p_{1/4}(B_{3/4})}{p_1(0)}\right) \le 4\,\mathbb{E}\left(\sup_{|r-s|\le 2^{-n},\, r,s\le\frac{3}{4}} |B_r - B_s|^2\right) \\
&\le C(2^{-n})^{1/2}
\end{aligned}$$

by Kolmogorov's continuity criterion for the standard Brownian motion $(B_r)_{r\ge 0}$. For the other term, we also reduce to a known result on the local time $(\ell_t^a)_{a\in\mathbb{R},t\ge 0}$ of Brownian motion:

$$\begin{aligned}
&\mathbb{E}\left(\sup_{|a-y|\le 1/n}\,\sup_{r\in[\varepsilon,1-\varepsilon]} |\ell^a(r) - \ell^y(r)|^2\right) \\
&= \mathbb{E}\left(\sup_{|a-y|\le 1/n}\,\sup_{r\in[\varepsilon,1-\varepsilon]} |\ell^a(r) - \ell^y(r)|^2\,\frac{p_\varepsilon(B_{1-\varepsilon})}{p_1(0)}\right) \\
&\le \varepsilon^{-1/2}\,\mathbb{E}\left(\sup_{|a-y|\le 1/n}\,\sup_{r\in[\varepsilon,1-\varepsilon]} |\ell^a(r) - \ell^y(r)|^2\right) \le C(1/n)^{1/2},
\end{aligned}$$

see [58] p.225-226. It only remains to prove that

$$\lim_n \int u_n\, v_n\, \langle A_n x - f_0'(x), P_n h\rangle\, \pi_n(dx) = \int u\, \left(\langle x, Ah\rangle - \langle f_0'(x), h\rangle\right) \nu(dx). \quad (4.5.20)$$

The term containing $f_0'(x)$ gives no difficulty; as for $\int u_n\, v_n\, \langle \cdot, A^n P_n h\rangle\, d\pi_n$, we have

$$\int u_n\, v_n\, \langle \cdot, A^n P_n h\rangle\, d\pi_n = \frac{1}{Z_n}\int u_n\, v_n\, \langle \cdot, A^n P_n h\rangle\, e^{-F_n}\, d\mu_n.$$

Now, notice that by an integration by part formula, we have for all $g \in C_b^1(H)$

$$\int g\, \langle \cdot, A^n P_n h\rangle\, d\mu_n = \int \partial_{P_n h} g\, d\mu_n.$$

Moreover

$$\int \langle \cdot, A^n P_n h\rangle^2 d\mu_n = \|P_n h\|^2 \le \|h\|^2.$$

Therefore, the linear functional $L^2(\mu) \ni g \mapsto \int g\, \langle P_n\cdot, A^n P_n h\rangle\, d\mu$ is uniformly bounded in n and converges on $C_b^1(H)$, a dense subset in $L^2(\mu)$. By a density argument, this sequence of functionals converges weakly in $L^2(\mu)$.

We recall now that $L^2(\pi_n) \ni u_n$ converges strongly to $u \in L^2(\nu)$. We want to show that $(u_n v_n e^{-F_n}) \circ P_n \to u v e^{-F}$ in $L^2(\mu)$. Indeed by lemma 4.5.7, from any subsequence of $(u_n \circ P_n)_n$ we can extract a sub-subsequence converging to u in $L^2(\nu)$

and ν-almost surely. On the other hand $(v_n e^{-F_n}) \circ P_n$ converges pointwise to ve^{-F} and $((v_n e^{-F_n}) \circ P_n)_n$ is uniformly bounded, so we conclude with the dominated convergence theorem. Therefore, we obtain that

$$\lim_n \int u_n v_n e^{-F_n} \langle \cdot, A^n P_n h\rangle \, d\mu_n = \int uve^{-F} \langle \cdot, APh\rangle \, d\mu, \tag{4.5.21}$$

and (4.5.20) is proved.

Finally we prove that if $\liminf_n \Lambda^n(u_n, u_n) < +\infty$, then $u \in \mathcal{D}(\mathcal{E})$. Indeed for all $u_n \in \mathcal{D}(\Lambda^n)$ we have $u_n \circ P_n \in \mathcal{D}(\mathcal{E})$, moreover $(u_n)_n$ converges weakly to u then $(u_n \circ P_n)_n$ converges weakly to u in $L^2(\nu)$; then, as at the end of the proof of Proposition 4.4.6, by the compact injection of $\mathcal{D}(\mathcal{E})$ in $L^2(\nu)$ we have that $u \in \mathcal{D}(\mathcal{E})$, which ends the proof. □

Remark 4.5.8 (some extension)**.** *Using an monotone argument as in chapter 5 to reach Mosco I, the last results still true (by approximation with step function) if f is bounded, with bounded variation. The finite dimensional diffusion, in the interface approximation, was studied in [44]. Let $\phi \in C_b^1(\mathbb{R})$, we have the following integration by part formula*

$$\int_{\mathbb{R}} \phi' e^{-f} dx = \int_{\mathbb{R}} \phi e^{-f} \, df \tag{4.5.22}$$

The Dirichlet form $\Lambda(u,v) = 1/2 \int u'v'e^{-f} dx$ is associated to a SDE of the form

$$X_t = X_0 + w_t + \int_{\mathbb{R}} \tilde{l}_t^a dm(a) \tag{4.5.23}$$

Where the support of m is the support of df. If $|m(\{x\})| < 1$ for all $x \in \mathbb{R}$, pathwise uniqueness holds.

4.5.5 Convergence in law of stationary processes

We denote now by $(\mathbb{Q}^n_{\pi_n})_n$ the law of the stationary solution of equation (4.5.7) started with initial law π_n. We want to prove a convergence result for $(\mathbb{Q}^n_{\pi_n})_n$ to $\mathbb{P}_\nu$, the stationary solution to equation (4.1.6). We define the space $H^{-1}(0,1)$ as the completion of $L^2(0,1)$ with respect to the Hilbertian norm

$$\|x\|^2_{H^{-1}(0,1)} := \int_0^1 d\theta \, \langle x, \mathbb{1}_{[0,\theta]}\rangle^2_{L^2(0,1)},$$

and the linear isometry $J : H^{-1}(0,1) \mapsto L^2(0,1)$ given by the closure of

$$H^{-1}(0,1) \supset L^2(0,1) \ni x \mapsto Jx := \langle x, \mathbb{1}_{[0,\cdot]}\rangle_{L^2(0,1)}.$$

Lemma 4.5.9. *The sequence $\mathbb{Q}^n_{\pi_n}$ converges weakly to $\mathbb{P}_\nu$ in $C([0,T]; H^{-1}(0,1))$.*

Proof. We define $\mathbb{S}_n := \mathbb{Q}^n_{\pi_n} \circ J^{-1}$, i.e. the law of $(JX^n_t)_{t\geq 0}$, where X^n_t has law $\mathbb{Q}^n_{\pi_n}$. Since J maps $L^2(0,1)$ continuously into $H^1(0,1)$, we obtain that ${\pi_n}^n \circ J^{-1}$

satisfies condition (4.6.1) below. Therefore by Lemma 4.6.1 below, $(\mathbb{S}_n)_n$ is tight in $C([0,T]\times[0,1])$ and therefore $(\mathbb{Q}^n_{\pi_n})_n$ is tight in $C([0,T];H^{-1}(0,1))$.

Let us now prove convergence of finite dimensional distributions. As in the proof of Proposition 4.4.7, let $f\in C_b(H^m)$ of the form $f(x_1,\ldots,x_m)=f_1(x_1)\cdots f_m(x_m)$. By the Markov property, it is enough to prove that

$$\begin{aligned}&P^n_{t_1}(f_1\cdot P^n_{t_2-t_1}(f_2\cdot\ldots(f_{m-1}P^n_{t_m-t_{m-1}}f_m)\ldots))\\ &\to P_{t_1}(f_1\cdot P_{t_2-t_1}(f_2\cdot\ldots(f_{m-1}P_{t_m-t_{m-1}}f_m)\ldots)),\qquad\text{strongly.}\end{aligned}$$

Arguing by recurrence, we only need to prove that, if $L^2(\pi_n)\ni v_n\to v\in L^2(\nu)$ strongly, and $g\in C_b(H)$, then $g\cdot v_n$ converges strongly to $g\cdot v$. Recalling that Φ_m is a conditional expectation and using the notation of definition 4.4.3 and (4.5.1), we have

$$\begin{aligned}&\|\Phi_m(g\cdot\tilde v_n)-g\cdot v_m\|_{\mathbb{H}_m}\\ &\le\|\Phi_m(g\cdot\tilde v_n-g\circ P_m\cdot\tilde v_n)\|_{\mathbb{H}_m}+\|g\cdot(\Phi_m(\tilde v_n)-v_m)\|_{\mathbb{H}_m}\\ &\le\|(g-g\circ P_m)\tilde v_n\|_{\mathbb{H}}+\|g\|_\infty\|\Phi_m(\tilde v_n)-v_m\|_{\mathbb{H}_m}.\end{aligned}$$

Since the conditional expectation is a contraction in L^2 and $g\circ P_m$ converges almost surely to g if $m\to+\infty$. Then we obtain the convergence in law of the finite dimensional laws . □

4.6 A priori estimate

We prove in this section an estimate which has been used above to prove tightness properties in $C([0,T]\times[0,1])$. We consider here a probability measure γ on H and Dirichlet form $(\mathbb{D},D(\mathbb{D}))$ in $L^2(\gamma)$ such that $C^1_b(H)$ is a core of $\mathbb{D}$ and

$$\mathbb{D}(u,v)=\frac{1}{2}\int\langle\nabla u,\nabla v\rangle\,d\gamma,\qquad\forall\,u,v\in C^1_b(H).$$

Let us define for $\eta\in\,]0,1[$ and $r\ge1$ the norm $\|\cdot\|_{W^{\eta,r}(0,1)}$, given by

$$\|x\|^r_{W^{\eta,r}(0,1)}=\int_0^1|x_s|^r ds+\int_0^1\int_0^1\frac{|x_s-x_t|^r}{|s-t|^{r\eta+1}}\,dt\,ds.$$

Then we have the following

Lemma 4.6.1. *Let $(X_t)_{t\ge0}$ be the stationary Markov process associated with $\mathbb{D}$, i.e. such that the law of X_0 is γ. Suppose that there exist $\eta\in\,]0,1[$, $\zeta>0$ and $p>1$ such that*

$$\zeta>\frac{1}{1+\frac{2}{3}\eta},\qquad p>\max\left\{\frac{2}{1-\zeta},\frac{1}{\eta-\frac{3}{2}\frac{1-\zeta}{\zeta}}\right\},$$

and

$$\int_H\|x\|^p_{W^{\eta,p}(0,1)}\,\gamma(dx)=C_{\eta,p}<+\infty.\tag{4.6.1}$$

Then there exist $\theta\in\,]0,1[$, $\xi>1$ and $K>0$, all depending only on (η,ζ,p), such that

$$\mathbb{E}\left[\|X_t-X_s\|^p_{C^\theta([0,1])}\right]\;\le\;K\,|t-s|^\xi.$$

Proof. We follow the proof of Lemma 5.2 in [19]. We introduce first the space $H^{-1}(0,1)$, completion of $L^2(0,1)$ w.r.t. the norm:

$$\|f\|_{-1}^2 := \sum_{k=1}^{\infty} k^{-2} \, |\langle f, e_k \rangle_{L^2(0,1)}|^2$$

where $e_k(r) := \sqrt{2}\sin(\pi k r)$, $r \in [0,1]$, $k \geq 1$, are the eigenvectors of the second derivative with homogeneous Dirichlet boundary conditions at $\{0,1\}$. Recall that $L^2(0,1) = H$, in our notation. We denote by κ the Hilbert-Schmidt norm of the inclusion $H \to H^{-1}(0,1)$, which by definition is equal in our case to

$$\kappa = \sum_{k \geq 1} k^{-2} < +\infty.$$

We claim that for all $p > 1$ there exists $C_p \in (0,\infty)$, depending only on p, such that

$$\left(\mathbb{E}\left[\|X_t - X_s\|_{H^{-1}(0,1)}^p \right] \right)^{\frac{1}{p}} \leq C_p \, \kappa \, |t-s|^{\frac{1}{2}}, \qquad t, s \geq 0. \tag{4.6.2}$$

To prove (4.6.2), we fix $T > 0$ and use the Lyons-Zheng decomposition, see e.g. [27, Th. 5.7.1], to write for $t \in [0,T]$ and $h \in H$:

$$\langle h, X_t - X_0 \rangle_H = \frac{1}{2} M_t - \frac{1}{2} (N_T - N_{T-t}),$$

where M, respectively N, is a martingale w.r.t. the natural filtration of X, respectively of $(X_{T-t},\ t \in [0,T])$. Moreover, the quadratic variations are both equal to: $\langle M \rangle_t = \langle N \rangle_t = t \cdot \|h\|_H^2$. By the Burkholder-Davis-Gundy inequality we can find $c_p \in (0,\infty)$ for all $p > 1$ such that: $(\mathbb{E}[|\langle X_t - X_s, e_k \rangle|^p])^{\frac{1}{p}} \leq c_p \, |t-s|^{\frac{1}{2}}$, $t, s \in [0,T]$, and therefore

$$\begin{aligned}
\left(\mathbb{E}\left[\|X_t - X_s\|_{H^{-1}(0,1)}^p \right] \right)^{\frac{1}{p}} &\leq \sum_{k \geq 1} k^{-2} \left(\mathbb{E}\left[|\langle X_t - X_s, e_k \rangle|^p \right] \right)^{\frac{1}{p}} \\
&\leq c_p \sum_{k \geq 1} k^{-2} |t-s|^{\frac{1}{2}} \|e_k\|_{L^2(0,1)}^2 \leq c_p \, \kappa \, |t-s|^{\frac{1}{2}}, \quad t, s \in [0,T],
\end{aligned}$$

and (4.6.2) is proved. By stationarity

$$\begin{aligned}
\left(\mathbb{E}\left[\|X_t - X_s\|_{W^{\eta,p}(0,1)}^p \right] \right)^{\frac{1}{p}} &\leq \left(\mathbb{E}\left[\|X_t\|_{W^{\eta,p}(0,1)}^p \right] \right)^{\frac{1}{p}} + \left(\mathbb{E}\left[\|X_s\|_{W^{\eta,p}(0,1)}^p \right] \right)^{\frac{1}{p}} \\
&= 2 \left(\int_H \|x\|_{W^{\eta,p}(0,1)}^p \, d\gamma \right)^{\frac{1}{p}} = 2 \, (C_{\eta,p})^{1/p}.
\end{aligned} \tag{4.6.3}$$

By the assumption on ζ and p it follows that $\alpha := \zeta\eta - (1-\zeta) > 0$ and

$$\frac{p}{2}(1-\zeta) > 1, \qquad \frac{1}{d} := \zeta \frac{1}{p} + (1-\zeta)\frac{1}{2} < \alpha.$$

Then by interpolation, see [1, Chapter 7],

$$\left(\mathbb{E}\left[\|X_t - X_s\|^p_{W^{\alpha,d}(0,1)}\right]\right)^{\frac{1}{p}} \leq$$

$$\leq \left(\mathbb{E}\left[\|X_t - X_s\|^p_{W^{\eta,p}(0,1)}\right]\right)^{\frac{\zeta}{p}} \left(\mathbb{E}\left[\|X_t - X_s\|^p_{H^{-1}(0,1)}\right]\right)^{\frac{1-\zeta}{p}} .$$

Since $\alpha d > 1$, there exists $\theta > 0$ such that $(\alpha - \theta)d > 1$. By the Sobolev embedding, $W^{\alpha,d}(0,1) \subset C^{\theta}([0,1])$ with continuous embedding. Then we find that

$$\mathbb{E}\left[\|X_t - X_s\|^p_{C^{\theta}([0,1])}\right] \leq K\,|t-s|^{\xi}$$

with $\xi := \dfrac{p}{2}(1-\zeta) > 1$ and K a constant depending only on (η, ζ, p). □

4.7 Polar sets

We conclude this study by a property, we obtain for the skew stochastic heat equation, due to (4.2.5) and the absolute continuity of the semi-group, indeed we saw that $\mathcal{E}$ and $\mathcal{E}^0$ have the same sets with null capacity. We now recall three different notions of exceptional sets in the Hunt probabilistic potential theory: that of polar set, semi-polar set and set with potential zero.

1. A set N is polar if there is a set $\tilde{N}$ which is nearly Borel such that $\mathbb{P}_x(\sigma_N < \infty) = 0$ for all $x \in H$, where $\sigma_N = \inf\{t > 0 : X_t \in N\}$.
2. A set N is semi-polar if there a sequence of thin sets $(A_n)_n$ such that $N \subset \cup_n A_n$. By thin set we mean that for each A_n there is a nearly Borel set $\tilde{A}_n$ such that $\tilde{A}^r_n = \emptyset$, where $\tilde{A}^r_n = \emptyset$ is the set of all regular point of $\tilde{A}_n$.
3. A universally Borel set N has zero potential if for all $x \in H$, $R^{\alpha}(x, A) = 0$.

A set $N \subset H$ is exceptional if there is a nearly Borel set $\tilde{N} \supset N$ such that $\mathbb{P}_{\nu}(\sigma_{\tilde{N}} < \infty) = 0$. A set N is properly exceptional if N is nearly Borel, $\nu(N) = 0$ and $H \backslash N$ is invariant for X. We have the following theorem (see [27] p.137).

Theorem 4.7.1. *Let N a subset of H.*

1. *If N is exceptional, then N is contained in a properly exceptional set B. B can be taken Borel.*
2. *Assume that any compact set is of finite capacity. Then a set $N \subset H$ is exceptional if and only if $Cap(N) = 0$.*

The following theorem gives a criterion for a set to be polar.

Theorem 4.7.2. *The following conditions are equivalent:*

1. *A set is polar if and if it is exceptional*
2. *$R^{\alpha}(x, .)$ is absolute continuous with respect to ν for each $\alpha > 0$ and $x \in H$*

3. $p_t(x, .)$ is absolutely continuous with respect to ν for all $t > 0$ and $x \in H$

Finally, we can enunciate a interesting property of the solution of (4.1.6). Indeed theorems 4.7.1 and 4.7.2 say that the polar sets are the sets with zero capacity, moreover we have seen that the Dirichlet forms $\mathcal{E}^0$ and $\mathcal{E}$ are equivalent in 4.2.5 so:

Corollary 4.7.3. *The solution X of* (4.1.6) *has the same Polar sets as the infinite dimensional Ornstein-Uhlenbeck process* (4.1.4).

CHAPTER 5

THE REFLECTED SKEW SPDE

5.1 Introduction

In this chapter we want to combine the two kinds of non-linearity of the previous two chapters, namely we want to study the following reflected skew heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \theta^2} + \eta(t,\theta) - \dfrac{1}{2}\displaystyle\int_{\mathbb{R}} f(da)\dfrac{\partial}{\partial \theta}\ell^a_{t,\theta} + \dot{W}(t,\theta) \\ u(0,\theta) = x(\theta), \quad u(t,0) = b,\ u(t,1) = b \\ u(t,\theta) \geq 0,\ \eta \geq 0,\ \eta(\{(t,\theta) \,|\, u(t,\theta) \neq 0\}) = 0 \end{cases} \tag{5.1.1}$$

where u takes values in $[0,+\infty[$, $b > 0$, $(\ell^a_{t,\theta}, \theta \in [0,1])$ is the family of local times at $a \in \mathbb{R}$ accumulated over $[0,\theta]$ by the process $(u(t,r), r \in [0,1])$, and η is a locally finite measure. Moreover we suppose that $f : \mathbb{R} \mapsto \mathbb{R}$ has the form

$$f(y) = f_0(y) + \sum_{j=1}^{k} \alpha_j \, 1_{(y \leq y_j)}, \qquad y \in \mathbb{R} \tag{5.1.2}$$

where f_0 is continuous and bounded with its first and second derivative and $\alpha_j, y_j \in \mathbb{R}$. Clearly, f has a jump in each y_j of respective size α_j (as previously, using a monotone argument, some fact in this chapter still true if f is a bounded function with bounded variation).

5.1.1 Definition of stationary solutions

We start by giving the main definition.

Definition 5.1.1. *Let $x \in L^2(0,1)$. An adapted process u, defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$, is a weak stationary solution of* (5.1.1) *if*

- *a.s. $u \in C([0,T] \times [0,1])$ and $u(t,0) = u(t,1) = b$ for all $t \geq 0$*

- $(u(t+T,x))_{t\geq 0, x\in[0,1]}$ *has the same law as* $(u(t,x))_{t\geq 0, x\in[0,1]}$ *for all* $T \geq 0$.
- *a.s. for dt-a.e. t the process* $(u(t,r), r \in [0,1])$ *has a family of local times* $[0,1]\times\mathbb{R} \ni (r,t) \mapsto \ell^a_{t,\theta}$, $a \in \mathbb{R}$, *such that*

$$\int_0^\theta g(u(t,r))\,dr = \int_{\mathbb{R}} g(a)\,\ell^a_{t,\theta}\,da, \qquad \theta \in [0,1],\ t \geq 0,$$

for all bounded Borel $g : \mathbb{R} \mapsto \mathbb{R}$.

- *there is a Brownian sheet* W *such that for all* $h \in C^2_c((0,1))$ *and* $t \geq 0$

$$\begin{aligned}\langle u_t - u_0, h\rangle = \frac{1}{2}\int_0^t \langle h'', u_s\rangle_{L^2(0,1)}\,ds + \frac{1}{2}\int_0^t \int_{\mathbb{R}} f(da)\int_0^1 h'(\theta)\,\ell^a_{s,\theta}\,d\theta\,ds \\ + \int_0^t\int_0^1 h(\theta)\,W(ds,d\theta) + \int_0^t\int_0^1 h(\theta)\,\eta(ds,d\theta)\end{aligned} \tag{5.1.3}$$

The techniques we adopt are a combination of those of chapters 2 and 3.

5.1.2 The invariant measure

We consider now the one dimensional reflected SPDE studied in [51], and consider now the maps $F : H \to \mathbb{R}$ such that

$$F(x) := \int_0^1 f(x_\theta)\,d\theta, \qquad x \in H.$$

We denote by q_b the law on Let K be the set defined by

$$K := \{x \in L^2([0,1]) : x_\theta \geq 0, \text{ for all } \theta \in [0,1]\}$$

and q_b is the law of the Bessel bridge of dimension 3 from b to b over $[0,1]$. If β denotes a Brownian bridge from 0 to 0 over $[0,1]$, we call μ^b the law of $(b+\beta)$. Then it is well known that, for $b > 0$, the law of $(b+\beta)$ conditioned on $\{b+\beta_r \geq 0, \forall\, r \in [0,1]\}$, is equal to q_b, the law of the Bessel bridge of dimension 3 from b to b over $[0,1]$. Namely, we have

$$q_b = \mu^b(\,\cdot\,|\,K), \qquad b > 0. \tag{5.1.4}$$

The formula makes sense for $b > 0$ since in this case $\mu^b(K) = 1 - \exp(-2b^2) > 0$. For the rest of the paper we fix $b > 0$. Then we can define the probability measure ζ on K

$$\zeta(dx) := \frac{1}{Z}\exp(-F(x))\,q_b(dx). \tag{5.1.5}$$

We recall the integration by part formula for q_b, proved in [64]. Let $h \in C^2_c(0,1)$, then

$$\int \partial_h F\,dq_b = -\int F(w)\,\langle h'', w\rangle\,dq_b - \int_0^1 dr\,h_t\,\alpha(b,r)\int F\,dq^r_b \tag{5.1.6}$$

where $q_b^r(dw)$ is the law of q_b conditioned on $\{w_r = 0\}$ and

$$\alpha(b,r) = \frac{1}{\sqrt{2\pi r^3(1-r)^3}}\,\frac{2b^2}{1-e^{-2b^2}}\,e^{-b^2/(2r(1-r))}, \qquad b>0,\ r\in]0,1[.$$

We also fix a sequence f_n of smooth bounded functions such that $f_n \downarrow f$ and we define

$$F_n(x) := \int_0^1 f_n(x_\theta)\,d\theta, \qquad \zeta_n(dx) := \frac{1}{Z}\,\exp(-F_n(x))\,q_b(dx).$$

Theorem 5.1.2. *Let φ be $C_b^1(H)$, and $h \in C_c^2(0,1)$. Then we have the following integration by parts formula:*

$$\begin{aligned}\int_K \partial_h\varphi\,d\zeta = &-\int_K \varphi(w)\left(\langle h'', w\rangle d\zeta - \int_{\mathbb{R}} f(da)\int_0^1 h_r\,d\ell_r^a\right)\zeta(dw) \\ &-\frac{1}{Z}\int_0^1 h_r\,\alpha(b,r)\int_K \varphi\,e^{-F}dq_b^r.\end{aligned} \tag{5.1.7}$$

Proof. The desired result follows easily by (5.1.6), arguing as in the proof of Proposition 3.6.1 □

Let us consider now the closure in $L^2(\zeta)$ of the bilinear form

$$\mathcal{E}(F,G) := \frac{1}{2}\int \langle \nabla F, \nabla G\rangle\,d\zeta, \qquad F,G\in C_b^1(H). \tag{5.1.8}$$

Since f is assumed to be bounded, the norm associated with this bilinear form in $L^2(\zeta)$ is equivalent to the analogous norm associated with the case $f\equiv 0$, i.e. with the Dirichlet form of the reflected SPDE studied in [64]. By this equivalence results, arguing as in Lemmas 4.2.4 and 4.3.2 above we obtain

Lemma 5.1.3. *In the previous setting we have:*

The bilinear form $(\mathcal{E}, C_b^1(H))$ is closable in $L^2(\zeta)$ and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form.

In particular we obtain by [46] existence of a Markov process properly associated with $(\mathcal{E}, D(\mathcal{E}))$.

5.2 Existence of a stationary solution

In this section we follow the technique of section 4.3 above. However in this setting we do not have for equation (5.1.1) the absolute-continuity relation proved in Proposition 4.2.7 for equation (4.1.6) above. Therefore the results will be limited to a stationary solution, unlike what we had in section 4.3.

5.2.1 Existence of the Local Time

Let us recall that ζ is absolutely continuous with respect to the law q_b of $(e_r, r\in[0,1])$, where e is a Bessel bridge of dimension 3 over $[0,1]$ from b to b. Since e is a semi-martingale, for q_b-a.e. x there exists a measurable family of local times ℓ^a_r such that

$$\int_0^r g(x_s)\, ds = \int_{\mathbb{R}} g(a)\, \ell^a_r\, da, \qquad r \in [0,1],$$

In particular, setting

$$C_L := \{w \in C([0,1]) : w \text{ has a measurable family of local times } (\ell^a_r)_{(r,a)\in[0,1]}\},$$

then $\zeta(C_L) = 1$. Then we have the following

Proposition 5.2.1. *$\mathbb{P}_\zeta$-almost surely, for a.e. t there exists a measurable family of local times $[0,1] \ni (r,a) \mapsto \ell^a_t(r)$ of $(u_t(\theta), \theta \in [0,1])$.*

Proof. Let $A \subset H$ a measurable set with $\zeta(A) = 0$. Then

$$\mathbb{E}_\zeta \left[\int_0^t \mathbb{1}_{(u_s \in A)}\, ds \right] = \int_0^t \mathbb{P}_\zeta(u_s \in A)\, ds = t\, \zeta(A) = 0.$$

Therefore, the time spent by $(u_s, s \geq 0)$ in A is a.s. equal to 0. In order to conclude, we choose A as the complement of the set of trajectories with the desired property. □

We want now to show that the process associated with $\mathcal{E}$ satisfies (5.1.1). We are going to apply (3.7.1) to $U^h(x) := \langle x, h\rangle$, $x \in H$, with $h \in C^2_c((0,1); \mathbb{R}^d)$. Clearly $U^h \in \mathrm{Lip}(H) \subset D(\mathcal{E})$. Our aim is to prove the following

Proposition 5.2.2. *There is a unique measure $\eta(ds, d\theta)$ on $[0, +\infty[\times [0,1]$ such that $\mathbb{P}_\zeta$-a.s. for all $t \geq 0$*

$$N^{[U^h]}_t = -\int_0^t G^h(u_s) ds + \int_0^t \int_0^1 h_\theta \eta(ds, d\theta) + \frac{1}{2} \int_0^t \langle h'', u_s\rangle ds \tag{5.2.1}$$

where $h \in C^\infty_c((0,1); \mathbb{R}^d)$, and G^h is the map $w \mapsto \int_{\mathbb{R}} \int_0^1 h_\theta\, dl^a_\theta(w)\, f(da)$ defined ζ-almost surely by Proposition 5.2.1. And $\mathrm{Supp}(\eta) \subset \{(t,\theta) \,|\, u(t,\theta) \in \partial O\}$

Proof. The main tools of the proof are the integration by parts formula (5.1.7) and a number of results from the theory of Dirichlet forms in [27]. We start by applying (3.7.1) to $U^h(x) := \langle x, h\rangle$, $x \in H$. By approximation and linearity we can assume that $h \in D(A)$, $h'' \geq 0$ and therefore $h \geq 0$ as well. The process $N^{[U^h]}$ is a CAF of X, and its Revuz measure is $\frac{1}{2}\Sigma^h$, where, in the notations introduced in section 5.1.2

$$\begin{aligned} \Sigma^h(dw) \quad &:= \quad \left(\langle w, h''\rangle - \int_{\mathbb{R}\times[0,1]} f(da)\, h_r\, l^a(dr) \right) \zeta(dw) \\ &\quad + \int_0^1 dr\, \alpha(b,r) h_r\, \frac{1}{Z}\, \exp(-F(w))\, q^r_b(dw) \\ &=: \quad \Sigma^h_1(dw) + \Sigma^h_2(dw) \end{aligned}$$

where $\Sigma_1^h(dw)$ and $\Sigma_2^h(dw)$ are mutually singular measures, with $\Sigma_2^h \geq 0$. Arguing as in the proof of Proposition 3.7.4 we can prove that there exists a non-negative measure $\eta(dt, dx)$ with the desired properties. Moreover by Tanaka's formula we have the estimate

$$\mathbb{E}_\zeta\left(\left(\int_{\mathbb{R}}\int_0^1 h_\theta \, dl_\theta^a(w)\, f(da)\right)^2\right) < +\infty$$

since $\zeta \ll \mu^b$, $f(da)$ has globally bounded variation, h is bounded and l_1^a is in $L^p(\mu^b)$ for any $p \geq 1$. Then we can conclude arguing as in the proof of Proposition 4.3.5. □

We now state the existence result of weak solution for (5.1.1).

Proposition 5.2.3. *The stationary Markov process associated with $\mathcal{E}$ is a weak solution of* (5.1.1).

Proof. Continuity of almost all paths follows from the a priori estimate of Lemma 4.6.1 and by analogs of (4.4.10) and (4.4.11). We argue as in section 4.3 above, by using the Fukushima decomposition, Proposition 5.2.2 and by identifying the noise term with a cylindrical Brownian motion as in Proposition 4.3.6 above. □

Remark 5.2.4. *There is an ipp when $b = 0$, see [64], so the previous treatment works for null Dirichlet boundary condition. Nevertheless we cannot prove at the moment the Mosco convergence in this case.*

5.3 The approximating SPDE

As in the proof of the integration by part formula (5.1.7) we consider the approximation of F by a sequence of functionals F_n defined as follow:

$$F_n(x) := \int_0^1 f_n(x_s)\, ds,$$

where as before the function f_n is bounded, continuously differentiable and $f_n \downarrow f$ pointwise, see (4.5.18). The approximating dynamics

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \theta^2} + \eta(t,\theta) - \dfrac{1}{2} f_n'(u) + \dot{W}(t,\theta) \\[2ex] u(0,\theta) = x(\theta), \quad u(t,0) = b, \; u(t,1) = b \\[2ex] u(t,\theta) \geq 0, \; \eta \geq 0, \; \eta(\{(t,\theta)\,|\,u(t,\theta) \neq 0\}) = 0 \end{cases} \tag{5.3.1}$$

This equation has been solved in [51] and [21] where existence of a strong solution which has continuous path, for a solution which vanishes at the boundary.

This dynamic has a closed Dirichlet forms which is equivalent to the one of the reflected SPDE corresponding to $f \equiv 0$.

5.3.1 Mosco convergence of stationary approximating SPDEs

In the setting of definition 4.4.3, we can now see that the Hilbert spaces $L^2(\zeta_n)$ converge to $L^2(\zeta)$. More precisely, analogously to Lemma 4.4.4 (and with identical proof) we have the following

Lemma 5.3.1.

1. *The sequence of Hilbert spaces $L^2(\zeta_n)$ converges to $L^2(\zeta)$, by choosing Φ_n equal to the natural identification of equivalence classes in $L^2(\zeta_n)$ and $L^2(\zeta)$.*
2. *$u_n \in L^2(\zeta_n)$ converges strongly to $u \in L^2(\zeta)$ if and only if $u_n \to u$ in $L^2(\zeta)$.*
3. *$u_n \in L^2(\zeta_n)$ converges weakly to $u \in L^2(\zeta)$ if and only if $u_n \to u$ weakly in $L^2(\zeta)$.*

Let us consider now the closure in $L^2(\zeta_n)$ of the bilinear form

$$\mathcal{E}^n(F,G) := \frac{1}{2}\int \langle \nabla F, \nabla G\rangle \, d\zeta_n, \qquad F, G \in C_b^1(H). \tag{5.3.2}$$

We now prove Mosco convergence of $\mathcal{E}_n$ to $\mathcal{E}$ in the sense defined in section 4.4.2.

Proposition 5.3.2. *The Dirichlet form $\mathcal{E}^n$ Mosco-converges to $\mathcal{E}$ on $L^2(\zeta)$.*

Proof. The proof of the condition Mosco II is trivial in our case; indeed, for all $x \in D(\mathcal{E})$, we set $x_n := x \in D(\mathcal{E}^n)$ for all $n \in \mathbb{N}$; by dominated convergence $\mathcal{E}(x,x) = \lim_n \mathcal{E}^n(x,x)$. If $x \notin D(\mathcal{E})$, then again $x_n := x \notin D(\mathcal{E}^n)$ satisfies $\mathcal{E}(x,x) = \lim_n \mathcal{E}^n(x,x) = +\infty$.

Let us prove now condition Mosco I, by following the simple monotonicity argument in the proof of Theorem 4.5.6 above. Recall that $f_n \downarrow f$ pointwise as $n \uparrow +\infty$. Now we define the measure

$$\tilde{\zeta}_n(dx) = \frac{1}{Z}\exp(-F_n(x))\, q_b(dx), \tag{5.3.3}$$

where $Z = \displaystyle\int \exp(-F)\, d\zeta$ is the normalization constant of ζ in (5.1.5); in particular $\tilde{\zeta}_n$ is not normalized to be a probability measure, in fact $\tilde{\zeta}_n \le \zeta_n$ since $f_n \ge f$. We also define the Dirichlet form

$$\tilde{\mathcal{E}}^n(\varphi,\psi) := \frac{1}{2}\int \langle \nabla\varphi, \nabla\psi\rangle \, d\tilde{\zeta}_n, \qquad \forall\, \varphi, \psi \in D(\mathcal{E}^n). \tag{5.3.4}$$

The form $\tilde{\mathcal{E}}^n$ is clearly equivalent to $\mathcal{E}^n$ on $D(\mathcal{E}^n)$. Moreover $\tilde{\mathcal{E}}^n(u,u) \le \mathcal{E}^n(u,u)$ for all $u \in D(\mathcal{E}^n)$.

Let now $u_n \in L^2(\zeta_n)$ be a sequence converging weakly to $u \in L^2(\zeta)$; we can suppose that $u \in \mathcal{D}(\mathcal{E})$ and that $\liminf_n \mathcal{E}^n(u_n,u_n) < +\infty$; then by lemma 4.5.7, up to passing a subsequence, we can suppose that $u_n \to u$ strongly.

Since $\tilde{\mathcal{E}}^n \le \mathcal{E}^n$, we have for $m \le n$

$$\liminf_{n\to\infty} \mathcal{E}^n(u_n,u_n) \ge \liminf_{n\to\infty} \tilde{\mathcal{E}}^m(u_n,u_n) \ge \tilde{\mathcal{E}}^m(u,u).$$

Therefore, letting $m \to +\infty$ we obtain by monotone convergence

$$\liminf_{n\to\infty} \mathcal{E}^n(u_n, u_n) \geq \lim_{m\to\infty} \tilde{\mathcal{E}}^m(u, u) = \mathcal{E}(u, u),$$

i.e. the condition Mosco I. □

5.3.2 Convergence in law of stationary approximating SPDEs

Arguing as in section 4.5.5, we obtain the following

Proposition 5.3.3. *The stationary solutions of* (5.3.1) *converge in law to the stationary solution of* (5.1.1).

5.4 Approximation by a finite-dimensional SDE

From now on we turn our attention to another problem: convergence in law of finite dimensional approximations of equation (5.1.1). As in section 4.5 above, we want to project, in a sense to be made precise, (5.1.1) onto an equation in a finite dimensional subspace of $H := L^2(0, 1)$. Again, we consider the space H_n of functions in $L^2(0, 1)$ which are constant on each interval $[(i-1)2^{-n}, i2^{-n}[$, $i = 1, \ldots, 2^n$ and we endow H_n with the scalar product inherited from H. We recall that there exists a unique orthogonal projector $P_n : L^2(0, 1) \mapsto H_n$, given explicitly by

$$P_n x := 2^n \sum_{i=0}^{2^n-1} 1_{[i2^{-n},(i+1)2^{-n}[} \langle 1_{[i2^{-n},(i+1)2^{-n}[}, x \rangle.$$

Let again β denote a Brownian bridge from 0 to 0 over $[0, 1]$. Then it is well known that, for $b > 0$, the law of $(b + \beta)$ conditioned on $\{b + \beta_r > 0, \forall\, r \in [0, 1]\}$, is equal to q_b, the law of the Bessel bridge of dimension 3 from b to b over $[0, 1]$. Moreover, as $b \downarrow 0$ we obtain that q_0 is the law of the Brownian bridge conditioned to be non-negative over $[0, 1]$.

We generalize the notations of section 4.5 above: we call μ_n^b the law of $P_n(b + \beta)$ for $b \geq 0$; then μ_n^b is a Gaussian law on H with mean equal to the constant function b and non-degenerate covariance operator $P_n Q P_n$, where Q is the covariance operator of μ. In what follows we write

$$P_n Q P_n = (-2A_n)^{-1}, \qquad A_n : H_n \mapsto H_n.$$

We also define ξ_n as

$$\xi_n(dx) = \frac{1}{Z_n} \mathbb{1}_K(x) \exp(-F(x)) \, \mu_n^b(dx), \qquad b \geq 0. \tag{5.4.1}$$

where Z_n is a normalization constant and $K := \{x \in L^2(0, 1) : x \geq 0\}$. Note that $\xi_n(dx) = \pi_n(b + dx \,|\, K)$, where π_n is defined as in (4.5.2).

A natural approximation of $\mathcal{E}$ defined on H_n is given by the following symmetric bilinear non-negative form

$$\Lambda^n(u, v) := \frac{1}{2} \int \langle \nabla u, \nabla v \rangle_{H_n} \, d\xi_n, \quad u, v \in C_b^1(H_n) \tag{5.4.2}$$

with reference measure ξ_n. Then we have

$$\Lambda^n(u,v) = \frac{1}{2}\int \langle \nabla(u\circ P_n), \nabla(v\circ P_n)\rangle_H \,\frac{1}{Z_n}\,\exp(-F\circ P_n)\,d\mu^b, \quad u,v\in C_b^1(H_n). \tag{5.4.3}$$

We have the following integration by parts formula

$$\begin{aligned}\int \partial_h\psi\,d\xi_n = &-\int \psi\,\langle x, A_n h\rangle\,\xi_n(dx) + \int \psi(x)\,2^{-n}\sum_{i=0}^{2^n-1} h_i\,f_0'(x_i)\,\xi_n(dx)\\ &-\int \psi(x)\sum_{i=0}^{2^n-1} h_i\,\sum_j 2\,\frac{1-e^{-\alpha_j\,2^{-n}}}{1+e^{-\alpha_j\,2^{-n}}}\,\xi_n(dx\,;\,x(i)=y_j),\\ &-\int \psi(x)\sum_{i=0}^{2^n-1} h_i\,\xi_n(dx\,;\,x(i)=0)\end{aligned} \tag{5.4.4}$$

where we use the notation

$$\xi_n(A\,;\,x(i)=y_j) := \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\,\xi_n(A\cap\{|x(i)-y_j|\le\varepsilon\}).$$

This shows that the dynamics associated with Λ^n solves the stochastic differential equation

$$dX_t^i = \frac{1}{2}\left((A_nX)^i - f_0'(X^i)\right)dt + \sum_j \frac{1-e^{-\alpha_j\,2^{-n}}}{1+e^{-\alpha_j\,2^{-n}}}\,d\tilde{\ell}_t^{i,y_j} + d\ell_t^{i,0} + 2^{-\frac{n}{2}}\,dw_t^i \tag{5.4.5}$$

where $(\tilde{\ell}_t^{i,a}, t\ge 0)$ is the symmetric local time of $(X^i(t), t\ge 0)$ at a, and moreover

$$X^i \ge 0, \qquad d\ell_t^{i,0}\ge 0, \qquad \int_0^\infty X_t^i\,d\ell_t^{i,0} = 0.$$

Then $(X_t^i)_i$ is a vector of interacting Reflected skew Brownian motions.

5.4.1 Reflected Skew Brownian motion

We want here to study the following SDE in $\mathbb{R}$: for all $t\ge 0$

$$X_t = X_0 + B_t + \beta\,\tilde{l}_t^1 + l_t, \qquad X_t\ge 0, \qquad dl_t\ge 0, \qquad \int_0^t X_s\,dl_s = 0, \tag{5.4.6}$$

where $|\beta|<1$, $(B_t)_{t\ge 0}$ is a standard Brownian motion in $\mathbb{R}$ and $\tilde{l}_t^a$ is the symmetric local time of $(X_t)_{t\ge 0}$ at $a\in\mathbb{R}$. Then we have the

Proposition 5.4.1. *Equation* (5.4.6) *satisfies existence and uniqueness in law of the pair* $(X_t, B_t)_{t\ge 0}$ *for all* $X_0\ge 0$*. Moreover* $(2l_t)_{t\ge 0}$ *is the local time of* $(X_t)_{t\ge 0}$ *at* 0.

Proof. By the Itô-Tanaka formula we have

$$X_t = (X_t)^+ = (X_0)^+ + \int_0^t \mathbb{1}_{(X_s>0)}\,dX_s + \frac{1}{2}\,L_t^0 =$$

Let us set $\gamma = (\beta+1)/2$ and

$$s(x) = \begin{cases} (1-\gamma)(x-1)+\gamma, & x \in [1,+\infty[\\ \gamma x, & x \in [0,1[. \end{cases} \tag{5.4.7}$$

Applying the symmetric Itô-Tanaka formula, we have arguing as in section 4.5.1

$$s(X_t) = s(X_0) + \int_0^t s'(X_s)\,dB_s + \gamma\, l_t, \qquad t \geq 0,$$

where

$$s'(x) := \frac{s'_+ + s'_-}{2}(x) = (1-\gamma)\mathbb{1}_{(x>1)} + \frac{1}{2}\mathbb{1}_{(x=1)} + \gamma\mathbb{1}_{(x<1)}$$

is the symmetric derivative. Then, setting $Y_t := s(X_t)$ and

$$f(x) := s' \circ s^{-1}(y) = \begin{cases} 1-\gamma, & y \in]\gamma,+\infty[\\ 1/2, & y = \gamma \\ \gamma, & y \in [0,\gamma[\end{cases} \tag{5.4.8}$$

we have that Y satisfies the SDE

$$Y_t = Y_0 + \int_0^t f(Y_s)\,dB_s + \gamma l_t, \tag{5.4.9}$$

with the additional constraint

$$Y_t \geq 0, \qquad dl_t \geq 0, \qquad \int_0^t Y_s\,dl_s = 0.$$

Note now that f is bounded below by a positive constant. Let A_t be the time change defined as follow:

$$C_t := \int_0^t f^2(Y_s)\,ds, \qquad C_{A_t} = t, \qquad t \geq 0,$$

then $W_t := \displaystyle\int_0^{A_t} f(Y_s)\,dB_s$ is a Brownian motion, and setting $L_t := \gamma\,\ell_{A_t}$ we have

$$Z_t := Y_{A_t} = Z_0 + W_t + L_t, \qquad Z_t \geq 0, \qquad dL_t \geq 0, \qquad \int_0^t Z_s\,dL_s = 0. \tag{5.4.10}$$

Therefore, the law of the process $(Z_t, W_t)_{t\geq 0}$ is determined by the Skorokhod Lemma, see lemma 2.1.15 below. Since

$$A'_t = \frac{1}{C'_{A_t}} = \frac{1}{f^2(Z_t)}, \qquad t > 0,$$

then

$$A_t = \int_0^t f^{-2}(Z_s)\,ds, \qquad A_{C_t} = t, \qquad t > 0,$$

and therefore $X_t = s^{-1}(Z_{C_t})$, $t \geq 0$. This gives existence and uniqueness in law of $(X_t, B_t)_{t\geq 0}$.

□

Lemma 5.4.2. *The process $(X_t)_{t\geq 0}$ is associated with the Dirichlet form*

$$D(u) := \frac{1}{2}\int_{\mathbb{R}^+} (\dot{u})^2 \exp(-\alpha 1_{[1,+\infty[})\, dx$$

in $L^2(\exp(-\alpha 1_{[1,+\infty[}) 1_{\mathbb{R}^+}\, dx)$, where $\alpha \in \mathbb{R}$ is defined by $\dfrac{1-e^{-\alpha}}{1+e^{-\alpha}} = \beta$.

Proof. The form $(D, C_b^1(\mathbb{R}^+))$ is closable in $L^2(\exp(-\alpha 1_{[1,+\infty[}) 1_{\mathbb{R}^+}\, dx)$ since it is equivalent to the standard Dirichlet forms associated with the reflected Brownian motion. By the same argument, the closure of $(D, C_b^1(\mathbb{R}^+))$ is regular and therefore there exists an associated Hunt process $(X_t)_{t\geq 0}$. We want now to prove that this process is a weak solution of (5.4.6). The following integration by parts formula

$$\int_{\mathbb{R}^0} \psi' \exp(-\alpha 1_{[1,+\infty[})\, dx = (1-e^{-\alpha})\, \psi(1) - \psi(0)$$

together with the Fukushima decomposition, shows that X_t is a semimartingale and that it satisfies (5.4.6) for quasi-every initial point $X_0 = x$, i.e. for all x outside a set N of null capacity. □

Theorem 5.4.3. *The form Λ^n, defined in* (5.4.2)*, is a regular Dirichlet form in $L^2(\xi_n)$, and the associated Markov process is a weak solution of* (5.4.5)*. Moreover such solution is unique in law.*

Proof. As in the proof of Lemma 5.4.2, Λ^n is a regular Dirichlet form with the strong local property because it is equivalent to the Dirichlet form of a finite dimensional Ornstein-Uhlenbeck process. So by [27] there is a continuous Hunt process associated to Λ^n.

By the integration by parts formula (5.4.4) and the Fukushima decomposition, the Hunt process associated with Λ^n has the following property: the process $(\langle h, X_t\rangle)_{t\geq 0}$ is a semi-martingale

$$\langle h, X_t^n\rangle - \langle h, X_0^n\rangle = M_t^h + N_t^h \tag{5.4.11}$$

and the Revuz measure of the bounded-variation CAF N^h is

$$\begin{aligned}\Sigma^h(dx) = \frac{1}{2}\langle A_n x - f_0'(x), h\rangle\, \xi_n(dx) + \sum_{i=0}^{2^n-1} h_i \sum_j \frac{1-e^{-\alpha_j 2^{-n}}}{1+e^{-\alpha_j 2^{-n}}}\, \xi_n(dx\,;\, x(i) = y_j)\\ + \sum_{i=0}^{2^n-1} h_i\, \xi_n(dx\,;\, x(i) = 0).\end{aligned} \tag{5.4.12}$$

Because of the structure of Σ^h, the process N^h can be written as

$$\begin{aligned}N_t^h = \int_0^t \frac{1}{2}\langle A_n X_s - f_0'(X_s), h\rangle\, ds + \sum_{i=0}^{2^n-1} h_i \sum_j \frac{1-e^{-\alpha_j 2^{-n}}}{1+e^{-\alpha_j 2^{-n}}}\, \tilde{\ell}_t^{i,y_j}\\ + \sum_{i=0}^{2^n-1} h_i\, \ell_t^{i,0}\end{aligned}$$

where $\widetilde{\ell}_t^{i,y_j}$ and $\ell_t^{i,0}$ are adapted to the natural filtration of $(X_t, t \geq 0)$. We want now to show that in fact $\widetilde{\ell}_t^{i,y_j}$ is adapted to the natural filtration of $(X_t^i, t \geq 0)$, we pose in the sequel $y_0 = 0$. Since X_t^i is a semimartingale, by Tanaka's formula

$$|X_t^i - y_j| = |X_0^i - y_j| + \int_0^t \operatorname{sign}(X_s^i - y_j)\, dX_s^i + L_t^{y_j}(X^i) \tag{5.4.13}$$

where $L^{y_j}(X^i)$ is the local time of X_t^i at y_j. Since $|\langle e_i, \cdot\rangle - y_j| \in \Lambda^n$, then $L^{y_j}(X^i)$ is an additive functional of X. Now we can compute the Revuz measure of $L^{y_j}(X^i)$, using theorem 5.4.2 of [27]. With an integration by parts formula we see that for all φ smooth enough:

$$\begin{aligned}
&\Lambda^n(|\langle e_i, \cdot\rangle - y_j|, \psi) = \frac{1}{2}\int \operatorname{sign}(x_i - y_j)\, \partial_i \psi(x)\, d\xi_n \\
&= -\frac{1}{2}\int \operatorname{sign}(x_i - y_j)\, \left((A_n x)^i - f_0'(x_i)\right)\, \psi(x)\, d\xi_n \\
&- \int \psi(x)\, \xi_n(dx; x(i) = y_j) + \int \psi(x)\, \xi_n(dx; x(i) = 0).
\end{aligned}$$

By comparison with (5.4.13), we see that $\xi_n(dx; x(i) = y_j)$ is the Revuz measure of $t \mapsto L_t^{y_j}(X^i)$ and therefore by (5.4.12) the processes $(L_t^{y_j}(X^i), t \geq 0)$ and $(\widetilde{\ell}_t^{i,y_j}, t \geq 0)$ are equal up to a multiplicative constant.

We want now to prove uniqueness in law for (5.4.5). We define the exponential martingale

$$M_t := \exp\left(-2^{n/2-1}\int_0^t \langle A_n X_s - f_0'(X_s), dw_s\rangle - 2^{n-3}\int_0^t \|A_n X_s - f_0'(X_s)\|^2 ds\right).$$

Then under the probability measure $M_T \cdot \mathbb{P}_x$, by the Girsanov theorem the canonical process is a solution in law of

$$dX^i = \sum_j \frac{1 - e^{-\alpha_j\, 2^{-n}}}{1 + e^{-\alpha_j\, 2^{-n}}}\, d\widetilde{\ell}_t^{i,y_j} + d\ell_t^{i,0} + 2^{-n/2} d\hat{w}_t^i, \qquad t \in [0, T],$$

where the Brownian motions $(\hat{w}_t^i, t \geq 0)_i$ are independent; therefore we have reduced to an independent vector of skew-Brownian motions and uniqueness in law holds for such processes by the pathwise uniqueness proved below.

Moreover, by the property recalled in the proof of Lemma 5.4.2, the transition semigroup of the reflected skew-Brownian motion satisfies the absolute continuity condition and therefore all the above statements are true for all initial conditions. □

5.5 Convergence of finite-dimensional approximations

We prove here the Mosco convergence for the skew reflected heat equation. In this situation the technique adopted in the previous chapter does not work directly. The additional difficulty arises when we integrate by parts in the proof of Mosco I, since we obtain now a boundary term containing an integral with respect to $\xi_n(dx\,;\,x(i)=0)$, see (5.4.12) above. Since in Mosco I we have sequences u_n converging weakly (or even strongly) in $L^2(\xi_n)$, and since the measure $\xi_n(dx\,;\,x(i)=0)$ is singular with respect to ξ_n, we have no way of controlling the limit of the integrals.

We are therefore going to use a different method to prove convergence of the regularized versions of (5.4.5), i.e. for the equations where f is replaced by f_m for fixed m, where reflection at zero remains unchanged. Therefore prove first convergence in law of a finite dimensional approximation of equation (5.3.1), and then we exploit this result to prove Mosco convergence of the finite dimensional approximations to the skew reflected heat equation. In order to perform the first step, we use a result based on semi-convexity of the finite dimensional approximation of equation (5.3.1). An abstract result is proved in Theorem 5.6.2 below, here we shall verify the assumptions of this result and use it.

5.5.1 A further approximation

We introduce the notation

$$\xi_{n,m}(dx) = \frac{1}{Z}\,\mathbb{1}_K(x)\,\exp(-F_m(x))\,\mu_n^b(dx), \qquad b>0.$$

Here $Z = \displaystyle\int \exp(-F)\,d\zeta$ is the normalization constant of $\zeta(dx)$ defined in (5.1.5) above, and therefore $\xi_{n,m}$ is not necessarily a probability measure. In fact, for $m \geq n$ we have $F_n \geq F_m$ and therefore $\xi_{n,m} \geq \xi_n = \xi_{n,n}$.

We first prove the following convergence of Hilbert spaces

Proposition 5.5.1. *The sequence of Hilbert spaces $(L^2(\xi_n))_n$ converges to $L^2(\zeta)$ in the sense of Definition 4.4.3. Moreover, for fixed $m \in \mathbb{N}$, the sequence of Hilbert spaces $(L^2(\xi_{n,m}))_n$ converges to $L^2(\tilde{\zeta}_m)$ (see the definition of $\tilde{\zeta}_m$ in (5.3.3) above).*

Proof. Let first $b>0$. We want to use the result of Proposition 4.5.4 above. Indeed, $L^2(\xi_n)$ is in fact isometric to a subspace of $L^2(\pi_n)$, where we use the notation (4.5.2). The isometry is defined by setting $L^2(\xi_n) \ni \varphi \mapsto C\,\varphi(\cdot+b)\,\mathbb{1}_K(\cdot+b) \in L^2(\pi_n)$, where $C := 1/\sqrt{Z_n}$. As $n \to +\infty$, we obtain by the convergence of $L^2(\pi_n)$ to $L^2(\nu)$ the convergence of $L^2(\xi_n)$ to $L^2(\zeta)$. The same argument yields the second assertion. $\square$

Let $(\Lambda^{n,m}, D(\Lambda^{n,m}))$ be the Dirichlet form in $L^2(\xi_{n,m})$ associated with the reference measure $\xi_{n,m}$

$$\Lambda^{n,m}(u,v) := \frac{1}{2}\int \langle \nabla u, \nabla v\rangle_{H_n}\,d\xi_{n,m}, \quad u,v \in C_b^1(H_n)$$

Then we have

Proposition 5.5.2. *For fixed $m \in \mathbb{N}$, the sequence of Dirichlet forms $(\Lambda^{n,m})_{n\in\mathbb{N}}$ Mosco converges in $L^2(\xi_{n,m})$ as $n \to +\infty$.*

Proof. We are going to use the result of Theorem 5.6.2 below. We must therefore check that Hypothesis 5.6.1 below holds.

Arguing as in (5.4.5), we can see that the dynamics associated with $\Lambda^{n,m}$ in $L^2(\xi_{n,m})$ is

$$dX_t^i = \frac{1}{2}\left((A_n X)^i - f_m'(X^i)\right) dt + d\ell_t^{i,0} + 2^{-\frac{n}{2}}\, dw_t^i,$$

with $X^i \geq 0$, $d\ell_t^{i,0} \geq 0$ and $\displaystyle\int_0^\infty X_t^i\, d\ell_t^{i,0} = 0$.

Since f_m is smooth with bounded second derivative, then it is easy to see that this is a semiconvex gradient system in H_n, i.e. a gradient system with a potential whose Hessian is bounded from below. Then, it is easy to see by Itô's formula that for some constant $c \geq 0$

$$d\,\|X_t(x) - X_t(y)\|^2 \leq c\,\|X_t(x) - X_t(y)\|^2\, dt$$

i.e. $\|X_t(x) - X_t(y)\| \leq e^{ct}\|x - y\|$, where of course $X_0(x) = x$ and $X_0(y) = y$. Then (5.6.1) below holds. The points (2), (3), (4) and (6) of Hypothesis 5.6.1 below are easy to check in our situation. We must now check the assumptions (5) of Hypothesis 5.6.1.

We consider now $h \geq 0$ and we write the integration by parts formula

$$\begin{aligned}\int \partial_h \psi\, d\xi_{n,m} = &- \int \psi \left(\langle x, A_n h\rangle - 2^{-n}\sum_{i=0}^{2^n-1} h_i\, f_m'(x_i)\right) \xi_{n,m}(dx) \\ &- \int \psi(x) \sum_{i=0}^{2^n-1} h_i\, \xi_{n,m}(dx\,;\, x(i) = 0) =: -\int \psi\, d\Sigma_{n,m}^h,\end{aligned} \tag{5.5.1}$$

we call $\sigma_{n,m}^h$ the non-negative measure

$$\sigma_{n,m}^h := \sum_{i=0}^{2^n-1} h_i\, \xi_{n,m}(dx\,;\, x(i) = 0)$$

and $\gamma_{n,m}^h$ the signed measure

$$\left(\langle x, A_n h\rangle - 2^{-n}\sum_{i=0}^{2^n-1} h_i\, f_m'(x_i)\right) \xi_{n,m}(dx) = \Sigma_{n,m}^h(dx) - \sigma_{n,m}^h(dx).$$

By arguing as in (4.5.21) above, we obtain that $(\gamma_{n,m}^h)_{n\in\mathbb{N}}$ satisfy (5.6.3)-(5.6.4) below as $n \to +\infty$. Then, it is enough to prove that $(\sigma_{n,m}^h)_{n\in\mathbb{N}}$ satisfies the same properties as $n \to +\infty$. By the integration by parts formula (5.5.1) and the result for $(\Sigma_{n,m}^h - \sigma_{n,m}^h)_{n\in\mathbb{N}}$, we have that $\displaystyle\int \psi\, d\sigma_{n,m}^h$ converges as $n \to +\infty$ for all $\psi \in C_b^1(H)$. Moreover $\sigma_{n,m}^h$ is a non-negative measure, and therefore its total-variation is simply equal to the integral of the constant function equal to 1, which belongs to $C_b^1(H)$. Arguing again as in the proof of Theorem 4.5.6 above, we conclude that $(\sigma_{n,m}^h)_{n\in\mathbb{N}}$ is convergent and therefore tight in H.

By Theorem 5.6.2, we have $P_t^{n,m}\varphi(x_n) \to P_t^m\varphi(x)$ as $n \to +\infty$ for all $\varphi \in C_b(H)$, where $(P_t^{n,m})_{t\geq 0}$ (respectively $(P_t^m)_{t\geq 0}$) is the semigroup associated with $\Lambda^{n,m}$ (resp. Λ^m) and $K_n \ni x_n \to x \in K$.

We denote by $(R_\lambda^{n,m})_{\lambda>0}$ (resp $(\tilde{R}_\lambda^m)_{\lambda>0}$) the resolvent family of $\Lambda^{n,m}$ (resp $\tilde{\mathcal{E}}^m$). Now, we have to prove that for all $\lambda > 0$ the resolvent operator $R_\lambda^{n,m}$ converges strongly to $\tilde{R}_\lambda^m$ as $n \to +\infty$ in the sense of Definition 2.4.3, namely that for all sequence $(f_n)_n \in L^2(\xi_{n,m})$ converging strongly to $f \in L^2(\tilde{\zeta}_m)$, then $R_\lambda^{n,m} f_n$ converges strongly to $R_\lambda^m f$.

Following Proposition 4.5.4 above and the notations introduced therein, we can define a map $\Psi_n : L^2(\mu^b) \mapsto L^2(\mu_n^b)$ defined as follows:

$$\Psi_n(\varphi) = \varphi_n, \qquad \varphi_n(b + \bar{\beta}_{i2^{-n}},\, i = 1, \ldots, 2^{-n}) = \mathbb{E}(\varphi(b + \beta) \,|\, \mathcal{F}_n).$$

Let us denote for simplicity of notations $\mathbb{H}_n := L^2(\xi_{n,m})$, $\mathbb{H} := L^2(\tilde{\zeta}_m)$, $R^n := R_\lambda^{n,m}$ and $R := R_\lambda^m$. We know from Proposition 5.5.1 that $\mathbb{H}_n$ converges to $\mathbb{H}$, the map $\Phi_n : \mathbb{H} \mapsto \mathbb{H}_n$ which appears in definition 2.4.1 above being equal to $\Phi_n(f) := \Psi_n(f\, \mathbb{1}_K)\, \mathbb{1}_K$.

Let $f_n \in \mathbb{H}_n$ converging strongly to $f \in \mathbb{H}$; then we have to prove that $R^n f_n$ converges strongly to Rf. Namely, we have to prove that there is a sequence $(p_n)_n \subset \mathbb{H}$ such that $p_n \to Rf$ in $\mathbb{H}$ and:

$$\lim_n \limsup_m \|\Phi_m(p_n) - R^m f_m\|_{\mathbb{H}_m} = 0. \tag{5.5.2}$$

As f_n converges strongly, there is a sequence $\tilde{f}_n$ in $\mathbb{H}$ such that $\tilde{f}_n \to f$ in $\mathbb{H}$ and

$$\lim_n \limsup_m \|\Phi_m(\tilde{f}_n) - f_m\|_{\mathbb{H}_m} = 0.$$

Let us show that $f_n \circ P_n \to f$ in $\mathbb{H}$ and

$$\lim_n \limsup_m \|\Phi_m(f_n \circ P_n) - f_m\|_{\mathbb{H}_m} = 0,$$

where P_n is the projection defined in (4.5.1). The latter formula is clear, since, for $m \geq n$ we have $\Phi_m(f_n \circ P_n) = f_n$, by the definition of Ψ_m above. Now it is enough to see that $f_n \circ P_n \to f$ in $\mathbb{H}$. Let us extend by convention all functions defined on K setting them equal to 0 on $H \setminus K$. Then

$$\begin{aligned}
&\|f - f_n \circ P_n\|_{\mathbb{H}} \leq C\, \|f - f_n \circ P_n\|_{L^2(\mu^b)} \\
&\leq\ C\, \|f - \Phi_n(\tilde{f}_m) \circ P_n\|_{L^2(\mu^b)} + C\, \|\Phi_n(\tilde{f}_m) \circ P_n - f_n \circ P_n\|_{L^2(\mu^b)}.
\end{aligned}$$

Now, by the definition of Ψ_n above we have that $(\Psi_n(\tilde{f}_m) \circ P_n)_n$ is a martingale and it converges to $\tilde{f}_m$ in $L^2(\mu^b)$ as $n \to +\infty$. On the other hand

$$\|\Phi_n(\tilde{f}_m) \circ P_n - f_n \circ P_n\|_{L^2(\mu^b)} \leq C\, \|\Phi_n(\tilde{f}_m) - f_n\|_{\mathbb{H}_n},$$

so that we find

$$\limsup_n \|f - f_n \circ P_n\|_{\mathbb{H}} \leq \lim_m \|f - \tilde{f}_m\|_{\mathbb{H}} + \lim_m \limsup_n \|\Phi_n(\tilde{f}_m) - f_n\|_{\mathbb{H}_n} = 0.$$

We set therefore $p_n := R(f_n \circ P_n)$. We must now prove that $p_n \to Rf$ in $\mathbb{H}$ and that (5.5.2) holds.

Suppose that $g \in \mathrm{Lip}(H)$; then $(R^n g) \circ P_n$ converges to R in $\mathbb{H}$ as $n \to +\infty$. Indeed the convergence holds pointwise on H, as a consequence of theorem 5.6.2 below. Moreover the sequence $((R^n g) \circ P_n)_n$ is bounded by $\|g\|_\infty$, so that we can conclude with the dominated convergence theorem.

Let $p_k = \tilde{R}^m_\lambda (f_k \circ P_k)$, so we have, using theorem 5.6.2 and the Jensen inequality

$$\begin{aligned}\|\Phi_n(p_k) - R^n f_n\|_{\mathbb{H}_n} &\le \|\Phi_n(p_k) - R^n g\|_{\mathbb{H}_n} + \|R^n f_n - R^n g\|_{\mathbb{H}_n} \\ &\le C_n \|\Phi_n(p_k) \circ P_n - (R^n g) \circ P_n\|_{L^2(\mu^b)} + \|f_n - g\|_{\mathbb{H}_n}.\end{aligned}$$

By theorem 5.6.2 and by $g \in \mathrm{Lip}(H)$ we have

$$\|\Phi_n(p_k) \circ P_n - (R^n g) \circ P_n\|_{L^2(\mu^b)} \to \|p_k - Rg\|_{L^2(\mu^b)} \le C \|f_k \circ P_k - g\|_{\mathbb{H}}.$$

In the same way

$$\|f_n - g\|_{\mathbb{H}_n} \le C \,\|(f_n - g) \circ P_n\|_{L^2(\mu^b)} \to C\|f - g\|_{\mathbb{H}},\ n \to +\infty.$$

A density argument provides the claim. By Theorem 2.4.4 above on the equivalence between strong convergence of resolvent families and Mosco convergence of Dirichlet forms, we obtain the desired result. □

5.5.2 Mosco convergence

Proposition 5.5.3. *The Dirichlet form Λ^n Mosco-converges to $\mathcal{E}$ on $L^2(\zeta)$.*

Proof. The proof of the condition Mosco II is trivial in our case; indeed, for all $x \in D(\mathcal{E})$, we set $x_n := x \in D(\Lambda^n)$ for all $n \in \mathbb{N}$; by dominated convergence $\mathcal{E}(x, x) = \lim_n \Lambda^n(x, x)$. If $x \notin D(\mathcal{E})$, then again $x_n := x \notin D(\mathcal{E}^n)$ satisfies $\mathcal{E}(x, x) = \lim_n \Lambda^n(x, x) = +\infty$.

Let us prove now condition Mosco I, by following the simple monotonicity argument in the proof of Theorem 4.5.6 above. Let $u_n \in L^2(\xi_n)$ be a sequence converging weakly to $u \in L^2(\zeta)$; we can suppose that $u \in \mathcal{D}(\mathcal{E})$ and that $\liminf_n \Lambda^n(u_n, u_n) < +\infty$; then by lemma 4.5.7, up to passing a subsequence, we can suppose that $u_n \to u$ strongly.

Since $\Lambda^{n,m} \le \Lambda^n$, we have for $m \le n$

$$\liminf_{n\to\infty} \Lambda^n(u_n, u_n) \ge \liminf_{n\to\infty} \Lambda^{n,m}(u_n, u_n) \ge \tilde{\mathcal{E}}^m(u, u).$$

Therefore, letting $m \to +\infty$ we obtain by monotone convergence

$$\liminf_{n\to\infty} \Lambda^n(u_n, u_n) \ge \lim_{m\to\infty} \tilde{\mathcal{E}}^m(u, u) = \mathcal{E}(u, u),$$

i.e. the condition Mosco I. □

5.5.3 Convergence in law of stationary processes

We denote now by $(\mathbb{Q}^n_{\xi_n})_n$ the law of the stationary solution of equation (4.5.7) started with initial law ξ_n. Arguing as in section 4.5.5, we want to prove a convergence result for $(\mathbb{Q}^n_{\xi_n})_n$ to $\mathbb{P}_\zeta$, the stationary solution to equation (5.1.1). We define the space $H^{-1}(0,1)$ as the completion of $L^2(0,1)$ with respect to the Hilbertian norm

$$\|x\|^2_{H^{-1}(0,1)} := \int_0^1 d\theta \, \langle x, 1_{[0,\theta]}\rangle^2_{L^2(0,1)},$$

and the linear isometry $J : H^{-1}(0,1) \mapsto L^2(0,1)$ given by the closure of

$$H^{-1}(0,1) \supset L^2(0,1) \ni x \mapsto Jx := \langle x, 1_{[0,\cdot]}\rangle_{L^2(0,1)}.$$

Lemma 5.5.4. *The sequence $\mathbb{Q}^n_{\xi_n}$ converges weakly to $\mathbb{P}_\zeta$ in $C([0,T]; H^{-1}(0,1))$.*

The proof is identical to that of Lemma 4.5.9.

5.6 Convergence for semi-convex systems

In this section we prove a convergence result which is needed in the proof of Proposition 5.5.2 above. We consider a family of Markov processes $(X^n_t)_{t\geq 0}$, such that X^n takes values in a convex subset K_n of a closed affine subspace $H_n \subset H$, each endowed with the scalar product $\langle .,. \rangle_H$ inherited from H. For all n, ν^n is on H_n with support $K_n \subset H_n$. We assume that

Hypothesis 5.6.1. There exists $c > 0$ such that for all $n \in \mathbb{N}$:

1. The transition semigroup $(P^n_t)_{t\geq 0}$ of X^n acts on $\mathrm{Lip}(K_n)$ and for all $\varphi \in \mathrm{Lip}(K_n)$:
$$|P^n_t\varphi(x) - P^n_t\varphi(y)| \leq e^{ct} \, [\varphi]_{\mathrm{Lip}(K_n)} \, \|x-y\|_H, \qquad x, y \in K_n, \ t \geq 0. \quad (5.6.1)$$
2. The following bilinear form is closable:
$$\mathcal{E}^n(\varphi,\psi) := \frac{1}{2}\int \langle \nabla_{H_n}\varphi, \nabla_{H_n}\psi\rangle_{H_n} \, d\nu^n, \qquad \forall\, \varphi,\psi \in C^1_b(H_n).$$
and the closure $(\mathcal{E}^n, D(\mathcal{E}^n))$ in $L^2(\nu^n)$ is a Dirichlet form with associated semigroup $(P^n_t)_{t\geq 0}$.
3. For all h in a dense subset $D_n \subset H_n$ there exists a finite signed measure Σ^n_h on H_n such that:
$$\int \partial_h \varphi \, d\nu^n = -\int \varphi \, d\Sigma^n_h, \qquad \forall\, \varphi \in C^1_b(H_n). \quad (5.6.2)$$
4. ν^n converges weakly on H to a probability measure ν, with convex topological support $K \subseteq H$
5. For all h in a dense subset $D \subset H$ there is a sequence $h_n \in D_n$ with $h_n \to h$ in H and there exists a finite signed measure Σ_h on H and a sequence of compact sets $(J_m)_m$ in H such that:
$$\lim_{n\to\infty} \int \varphi \, d\Sigma^n_{h_n} = \int \varphi \, d\Sigma_h, \qquad \forall\, \varphi \in C_b(H), \quad (5.6.3)$$
$$\left|\Sigma^n_{h_n}\right|(H \backslash J_m) \leq \frac{1}{m}. \qquad \forall\, n, m \in \mathbb{N}. \quad (5.6.4)$$

6. The following bilinear form is closable:

$$\mathcal{E}(\varphi,\psi) := \frac{1}{2}\int \langle \nabla_H\varphi, \nabla_H\psi\rangle_H \, d\nu, \qquad \forall\ \varphi,\psi \in \mathrm{Exp}_D(H).$$

and the closure $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form with associated semigroup $(P_t)_{t\geq 0}$. Moreover $\mathrm{Lip}(H) \subset D(\mathcal{E})$ and $\mathcal{E}(\varphi,\varphi) \leq [\varphi]^2_{\mathrm{Lip}(H)}$.

By (5.6.2) and (5.6.3), we have the following integration by parts formula for ν:

$$\int \partial_h\varphi \, d\nu = -\int \varphi \, d\Sigma_h, \qquad \forall\ h \in D, \quad \forall\ \varphi \in C_b^1(H), \tag{5.6.5}$$

We define for all $\varphi \in C_b(K_n)$ the resolvent of X^n:

$$R^n_\lambda\varphi(x) := \int_0^\infty e^{-\lambda t}\, P^n_t\varphi(x)\, dt, \qquad x \in K_n,\ \lambda > 0.$$

Under hypothesis 5.6.1 we have:

Theorem 5.6.2. *Suppose that Hypothesis 5.6.1 hold. Then for all $\phi \in C_b(H)$ and $x \in K$: $R^n_\lambda\phi(x_n) \to R_\lambda\phi(x)$ as $n \to \infty$.*

We describe the idea of the proof: for $\varphi \in \mathrm{Lip}(H)$, by the uniform Feller property (5.6.1) we have that $R^n_\lambda\phi$ is an equicontinuous and equibounded family. By Arzelà-Ascoli's Theorem we can extract converging subsequences on compact sets with large mass with respect to ν^n and Σ^n_h. Now we consider the formula which characterizes $R^n_\lambda\phi$ for $\lambda > 0$:

$$\lambda \int R^n_\lambda\varphi\ \psi\, d\nu^n + \mathcal{E}^n(R^n_\lambda\varphi,\, \psi) = \int \psi\,\varphi\, d\nu^n, \qquad \forall\ \psi \in D(\mathcal{E}^n).$$

We would like to pass to the limit, but $\mathcal{E}^n$ contains the gradient of $R^n_\lambda\varphi$. However, if $\psi = \exp(i\langle\cdot,h\rangle) \in \mathrm{Exp}_D(H)$, with $i^2 = -1$, then we can use the integration by parts formula (5.6.2) and write:

$$\mathcal{E}^n(R^n_\lambda\varphi,\, \psi) = -i\int R^n_\lambda\varphi\ \psi\, d\Sigma^n_h \to -i\int F\,\psi\, d\Sigma_h, \quad n \to \infty,$$

where F is a pointwise limit of $(R^n_\lambda\varphi)_n$. Using (5.6.5), the latter expression is equal to $\mathcal{E}(F,\psi)$, i.e. we obtain:

$$\lambda\int F\,\psi\, d\nu + \mathcal{E}(F,\, \psi) = \int \psi\,\varphi\, d\nu, \qquad \forall\ \psi \in \mathrm{Exp}_D(H),$$

which is very close to characterize F as the λ-resolvent of $\mathcal{E}$ in $L^2(\nu)$ applied to φ. The proof of Theorem 5.6.2 makes these arguments rigorous.

In the following proofs we use a number of times, often without further mention, the following easily proven fact.

Lemma 5.6.3. *Let E be a Polish space, $(M_n : n \in \mathbb{N}\cup\{\infty\})$ a sequence of finite signed measures on E and $(\varphi_n : n \in \mathbb{N}\cup\{\infty\})$ a sequence of functions on E, such that:*

1. *for all* $\varphi \in C_b(E)$:
$$\lim_{n\to\infty} \int \varphi \, dM_n \,=\, \int \varphi \, dM_\infty$$

2. *there exists a sequence of compact sets* $(J_m)_m$ *in* E *such that:*
$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} \, |M_n| \, (E \backslash J_m) \,=\, 0.$$

3. $(\varphi_n : n \in \mathbb{N} \cup \{\infty\})$ *is equi-bounded and equi-continuous*

4. φ_n *converges pointwise to* φ_∞ *on* $\cup_m J_m$

Then:
$$\lim_{n\to\infty} \int_E \varphi_n \, dM_n \,=\, \int_E \varphi_\infty \, dM_\infty.$$

Proof. We notice that by Arzelà-Ascoli's Theorem, φ_n converges uniformly to φ on J_m for all $m \in \mathbb{N}$. Moreover, by the Banach-Steinhaus Theorem the norms of the functionals $C_b(E) \ni \varphi \mapsto \displaystyle\int_E \varphi \, dM_n$ are bounded, therefore $|M_n|(E) \le C < \infty$ for all $n \in \mathbb{N}$. Then:
$$\left| \int_E \varphi_n \, dM_n \,-\, \int_E \varphi_\infty \, dM_\infty \right| \le \left| \int_E (\varphi_n - \varphi_\infty) \; dM_n \right| + \left| \int_E \varphi_\infty \; (dM_n - dM_\infty) \right|$$
and the second term in the right hand side tends to 0 by our first assumption. Now:
$$\begin{aligned} \left| \int_E (\varphi_n - \varphi_\infty) \; dM_n \right| &\le \int_{J_m} |\varphi_n - \varphi_\infty| \, d|M_n| + \int_{E\backslash J_m} |\varphi_n - \varphi_\infty| \, d|M_n| \\ &\le \sup_{J_m} \, |\varphi_n - \varphi_\infty| \; C \,+\, \|\varphi_n - \varphi_\infty\|_\infty \, |M_n| \, (E\backslash J_m) \,. \end{aligned}$$
Taking the limsup as $n \to \infty$ and then letting $m \to \infty$ we have the claim. □

Proof of Theorem 5.6.2. We divide the proof in several steps.

Step 1. We recall that $\Pi_{K_n} : H \mapsto K_n$ is 1-Lipschitz in H, and therefore, by (5.6.1) for all $\lambda > c$
$$\|(R^n_\lambda \psi) \circ \Pi_{K_n}\|_\infty \,\le\, \|\psi\|_\infty, \quad [(R^n_\lambda \psi) \circ \Pi_{K_n}]_{\mathrm{Lip}(H)} \,\le\, \frac{1}{\lambda - c} \, [\psi]_{\mathrm{Lip}(H)}, \quad \forall\, \psi \in C^1_b(H). \tag{5.6.6}$$
Fix $\psi \in C^1_b(H)$. Let $(n_j)_j$ be any sequence in $\mathbb{N}$ and $(x_k)_k$ a countable dense set in H. With a diagonal procedure, we can find a subsequence $(m_i)_i$ and a function $F : \{x_k, k \in \mathbb{N}\} \mapsto \mathbb{R}$ such that $R^n_\lambda \psi(\Pi_{K_n}(x_k)) \to F(x_k)$ as $n = m_i \to \infty$ for all $k \in \mathbb{N}$. By (5.6.6), F is Lipschitz on $\{x_k, k \in \mathbb{N}\}$ and therefore can be extended to a function in $\Psi_{\lambda,\psi} \in \mathrm{Lip}(H)$ and:
$$\Psi_{\lambda,\psi}(x) \,=\, \lim_{i\to\infty} R^{m_i}_\lambda \psi(\Pi_{K_{m_i}}(x)) \qquad \forall\, x \in H. \tag{5.6.7}$$
Finally, by a diagonal procedure, we can suppose that such limit holds along the same subsequence for all $\lambda \in \mathbb{N}$. Notice that in fact we are going to prove that the

limit exists as $n \to \infty$ for all $\lambda > c$. We define $\Delta := \mathrm{Span}\{\Psi_{\lambda,\psi} : \psi \in C_b^1(H),\, \lambda \in \mathbb{N}\} \subset \mathrm{Lip}(H)$.

Step 2. We would like to apply the integration by parts formula (5.6.5) to $\Psi_{\lambda,\psi}$, which is not in $C_b^1(H)$ but only in $\mathrm{Lip}(H)$. However, notice that for all $\varphi, \Phi \in C_b^1(H)$:

$$\int \varphi\, \partial_h \Phi \, d\nu \;=\; -\int \Phi\, \partial_h \varphi \, d\nu \;-\; \int \varphi\, \Phi \, d\Sigma_h, \qquad \forall\; h \in D. \tag{5.6.8}$$

If now $\Phi \in \mathrm{Lip}(H)$, then there exists a sequence $(\Phi_m)_m \subset C_b^1(H)$ such that:

$$\lim_m \Phi_m(x) \;=\; \Phi(x), \quad \forall\; x \in H, \qquad \|\Phi_m\|_\infty + [\Phi_m]_{\mathrm{Lip}(H)} \;\le\; \|\Phi\|_\infty + [\Phi]_{\mathrm{Lip}(H)}.$$

By (5.6.8) we have that $\partial_h \Phi_m$ converges weakly in $L^2(\nu)$ to an element of $L^\infty(\nu)$ that we call $\partial_h \Phi$ and with this definition (5.6.5) holds for all $\Phi \in \mathrm{Lip}(H)$. Moreover, we obtain in this way that $\nabla_H \Phi \in L^\infty(H, \nu; H)$ is well defined and:

$$\mathcal{E}(\Phi, \Phi) \;\le\; \liminf_m \, \mathcal{E}(\Phi_m, \Phi_m) \;\le\; \liminf_m \, [\Phi_m]^2_{\mathrm{Lip}(H)} \;\le\; [\Phi]^2_{\mathrm{Lip}(H)}.$$

Moreover, for all $\Phi \in \mathrm{Lip}(H)$ it is possible to find a multi-sequence $(\Phi_M)_M \subset \mathrm{Exp}_D(H)$, where $M = (m_1, \ldots, m_5) \in \mathbb{N}^5$, such that Φ_M converges to Φ pointwise and:

$$\sup_M \left(\|\Phi_M\|_\infty + [\Phi_M]_{\mathrm{Lip}(H)} \right) \;<\; \infty, \qquad \lim_M \, \mathcal{E}(\Phi_M, \Psi) \;=\; \mathcal{E}(\Phi, \Psi), \quad \forall\; \Psi \in \mathrm{Lip}(H), \tag{5.6.9}$$

where $\lim_M$ means that we let $m_1 \to \infty$, then $m_2 \to \infty$ and so on until $m_5 \to \infty$ (see [17, Proposition 11.2.10] for similar results).

Step 3. We want to prove now that for all $\lambda \in \mathbb{N}$ and $\Psi_{\lambda,\psi}$ as in the first step:

$$\mathcal{E}_\lambda(\Psi_{\lambda,\psi}, v) \;:=\; \lambda \int \Psi_{\lambda,\psi}\, v \, d\nu + \mathcal{E}(\Psi_{\lambda,\psi}, v) \;=\; \int \psi\, v \, d\nu, \qquad \forall\, v \in \Delta. \tag{5.6.10}$$

First we prove (5.6.10) for $v \in \mathrm{Exp}_D(H)$. Fix $h \in D$ and $h_n \in D_n$ as in Hypothesis 5.6.1 and set:

$$\varphi_n(k) \;:=\; \exp(i \langle h_n, \Pi_{H_n} k \rangle_{H_n}), \qquad \varphi(k) \;:=\; \exp(i\langle h, k\rangle_H), \qquad k \in H,$$

where $i \in \mathbb{C}$ with $i^2 = -1$ and Π_{H_n} denotes the orthogonal projection from H to H_n. By Hypothesis 5.6.1: $\|k - \Pi_{H_n} k\|_H \to 0$ for all $k \in H$. Indeed, this is true for all $k \in D$ since there is a sequence $k_n \in H_n$ such that $k_n \to k$ and by density of D in H we conclude, since Π_{H_n} is 1-Lipschitz continuous in H. Therefore $\varphi_n(k) \to \varphi(k)$ for all $k \in H$.

Since R^n_λ is the resolvent operator associated with $\mathcal{E}^n$:

$$\mathcal{E}^n_\lambda(R^n_\lambda \psi, \varphi_n) \;:=\; \lambda \int R^n_\lambda \psi \, \varphi_n \, d\nu^n + \mathcal{E}^n(R^n_\lambda \psi, \varphi_n) \;=\; \int \psi\, \varphi_n \, d\nu^n.$$

Notice that $\nabla_{H_n} \varphi_n = i\, h_n\, \varphi_n$. Then, by the integration by parts formula (5.6.2):

$$2\, \mathcal{E}^n(R^n_\lambda \psi, \varphi_n) \;=\; \int R^n_\lambda \psi \; \|h_n\|^2_{H_n} \, \varphi_n \, d\nu^n \;-\, i \int R^n_\lambda \psi \, \varphi_n \, d\Sigma^n_{h_n}.$$

Since $\nu^n \rightharpoonup \nu$ and $\Sigma^n_{h_n} \rightharpoonup \Sigma_h$ as $n \to \infty$, by (5.6.3) and Lemma 5.6.3:

$$\lim_{n\to\infty} \int g \, \|h_n\|^2_{H_n} \varphi_n \, d\nu^n \, - i \int g \, \varphi_n \, d\Sigma^n_{h_n} = \int g \, \|h\|^2_H \, \varphi \, d\nu - i \int g \, \varphi \; d\Sigma_h,$$

for all $g \in C_b(H)$. The crucial fact is now the following: by (5.6.3)-(5.6.4), (5.6.6) and Lemma 5.6.3, we can substitute g with $R^n_\lambda \psi$ in the last formula and prove that:

$$\lim_{i\to\infty} \int R^{m_i}_\lambda \psi \, \varphi_{m_i} \, d\Sigma^{m_i}_{h_{m_i}} \; = \; \int \Psi_{\lambda,\psi} \, \varphi \; d\Sigma_h. \tag{5.6.11}$$

In particular we obtain:

$$\int \psi \, \varphi \, d\nu \; = \lim_{i\to\infty} \int \psi \, \varphi_{m_i} \, d\nu^{m_i} \; = \; \int \Psi_{\lambda,\psi} \left(\lambda + \frac{1}{2} \|h\|^2 \right) \varphi \, d\nu - i \, \frac{1}{2} \int \Psi_{\lambda,\psi} \, \varphi \; d\Sigma_h$$

andthe last expression is equal to $\mathcal{E}_\lambda(\Psi_{\lambda,\psi}, \varphi)$ by the integration by parts formula (5.6.5), i.e. we have proven (5.6.10) for $v = \varphi$. By linearity we obtain (5.6.10) for all $v \in \mathrm{Exp}_D(H)$. By (5.6.9) we obtain (5.6.10) for all $v \in \Delta$.

Step 4. Finally, we want to show that $(\mathcal{E}, D(\mathcal{E}))$ coincides with $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$ constructed in the previous step. To this aim we show first that $D(\tilde{\mathcal{E}})$ contains all Lipschitz functions on K and in particular $\mathrm{Exp}_D(H)$.

Consider $\psi \in \mathrm{Lip}(H) \subset D(\mathcal{E}^n)$: by the general theory of Dirichlet Forms,

$$\psi \in D(\tilde{\mathcal{E}}) \iff \sup_{\lambda > 0} \int \lambda \, (\psi - \lambda \tilde{R}_\lambda \psi) \; \psi \; d\nu \; < \; \infty.$$

By (5.6.1) we have:

$$\int \lambda \, (\psi - \lambda R^n_\lambda \psi) \; \psi \; d\nu^n \; = \; \mathcal{E}^n(\lambda R^n_\lambda \psi, \psi) \; \le \; [\psi]^2_{\mathrm{Lip}(H)},$$

so that letting $n \to \infty$:

$$\int \lambda \, (\psi - \lambda \tilde{R}_\lambda \psi) \; \psi \; d\nu \; \le \; [\psi]^2_{\mathrm{Lip}(H)},$$

and therefore $\mathrm{Lip}(H) \subset D(\tilde{\mathcal{E}})$. Since by construction $\Delta \subset \mathrm{Lip}(H)$, then the closure of $(\mathcal{E}, \mathrm{Lip}(H))$ is $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$. Now, in order to prove the density of $\mathrm{Exp}_D(H)$ in $D(\tilde{\mathcal{E}})$, we remark that the density with respect to the norm-topology is equivalent to the density in the weak topology, which follows from (5.6.9) and from the density of $\mathrm{Lip}(H)$ in $D(\tilde{\mathcal{E}})$.

Notice that the limit Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ does not depend on the subsequences $(n_j)_j$ and $(m_i)_i$ chosen in step 1, since it is the closure of $(\mathcal{E}, \mathrm{Exp}_D(H))$. Then $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}})) = (\mathcal{E}, D(\mathcal{E}))$ is the Dirichlet form we wanted to construct and $R_\lambda = \tilde{R}_\lambda$ is the associated resolvent operator. In particular the limit in (5.6.7) does not depend on the subsequence $(m_i)_i$ and

$$R_\lambda \psi(x) \; = \; \lim_{n\to\infty} R^n_\lambda \psi(\Pi_{K_n}(x)) \qquad \forall x \in K, \;\; \lambda \in \mathbb{N}.$$

We can now repeat the argument of step 1 and step 3 and obtain that the latter formula holds for all $\lambda > c$. $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet Form, because R^n_λ is given by a Markovian kernel, so that R_λ is also Markovian and the result follows from Theorem 4.4 of [46]. The Feller property follows from (5.6.1). By the density of $\mathrm{Lip}(H)$ in $C_b(H)$, $R^n_\lambda\psi(\Pi_K(x))$ converges to $R_\lambda\psi(x)$ for all $\psi \in C_b(H)$. $\square$

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