On Unipotent Supports of Reductive Groups With a Disconnected Centre

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On Unipotent Supports of Reductive Groups With a Disconnected Centre

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Declaration

I hereby declare that this thesis has been composed by me and is based on work done by me and that this thesis has not been presented for assessment in any previous application for a degree, diploma or other similar award. I also declare that all sources of information have been specifically acknowledged and all quotations distinguished by quotation marks.

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Summary

Assume $\mathbf{G}$ is a connected reductive algebraic group defined over an algebraic closure of the finite field of prime order $p > 0$. Furthermore we assume $\mathbf{G}$ has an $\mathbb{F}_q$-rational structure $G = \mathbf{G}^F$ where $q = p^a$ is a power of p and $F : \mathbf{G} \to \mathbf{G}$ is the corresponding Frobenius endomorphism.

Through a great deal of work Lusztig, largely in **[Lus84a]** and **[Lus88]**, has managed to obtain a classification of the irreducible characters of G and their corresponding character degrees. To understand in general the values of irreducible characters of G Lusztig has developed a geometric theory of character sheaves which gives the framework to express links between the geometry of $\mathbf{G}$ and the characters of G, (see **[Lus85]**). Let $\mathrm{Irr}(G)$ denote the set of ordinary irreducible characters of G. In **[Lus92]** using character sheaves Lusztig has associated to every $\chi \in \mathrm{Irr}(G)$ a unique F-stable unipotent class of $\mathbf{G}$ called the *unipotent support* of χ, which we denote by $\mathcal{O}_\chi$. This was originally done under the assumption that p and q are sufficiently large, however the assumptions on p and q were later removed in **[Gec96]** and **[GM00]**. A consequence of the existence of unipotent supports is that we have a natural map

$$\mathrm{Irr}(G) \to \{F\text{-stable unipotent conjugacy classes of } \mathbf{G}\}$$

given by $\chi \mapsto \mathcal{O}_\chi$.

Lusztig has shown that for each $\chi \in \mathrm{Irr}(G)$ there is a well defined integer n_χ such that $n_\chi \cdot \chi(1)$ is a monic polynomial in q with integer coefficients. If $x \in \mathbf{G}$ then we write $A_{\mathbf{G}}(x)$ for the component group $C_{\mathbf{G}}(x)/C_{\mathbf{G}}(x)^\circ$. The following result was observed by Lusztig in **[Lus84a**, 13.4.4] and later verified by Lusztig and Hézard through a detailed case by case analysis, (see **[Lus09]** and **[Héz04]**).

Theorem (Lusztig, Hézard). *Assume p is a good prime for $\mathbf{G}$, $\mathbf{G}/Z(\mathbf{G})$ is simple, $Z(\mathbf{G})$ is connected and $\mathcal{O}$ is an F-stable unipotent class of $\mathbf{G}$. There exists a character $\chi \in \mathrm{Irr}(G)$ such that $\mathcal{O}_\chi = \mathcal{O}$ and $n_\chi = |A_{\mathbf{G}}(u)|$ for any $u \in \mathcal{O}$.*

The usefulness of characters satisfying the condition $n_\chi = |A_{\mathbf{G}}(u)|$ can be seen in the theory of *generalised Gelfand–Graev representations* (GGGRs). These are characters of G which are obtained by induction from subgroups of the unipotent radical of a Borel

subgroup of $\mathbf{G}$. In particular, if χ satisfies this technical condition then it severly limits the appearance of χ as a constituent of the Alvis–Curtis dual of a GGGR associated to a unipotent element in $\mathcal{O}_\chi^F$. The main result of this thesis is the following extension of the above theorem.

Theorem A. *Let* $\mathbf{G}$ *be a simple algebraic group and assume* p *is a good prime for* $\mathbf{G}$. *Given any F-stable unipotent class* $\mathcal{O}$ *of* $\mathbf{G}$ *there exists an irreducible character* $\chi \in \mathrm{Irr}(G)$ *such that* $\mathcal{O}_\chi = \mathcal{O}$ *and* $n_\chi = |A_{\mathbf{G}}(u)^F|$, *(where* $u \in \mathcal{O}^F$ *is a well-chosen representative).*

When $Z(\mathbf{G})$ is connected it is known that we can always choose a representative such that F acts trivially on $A_{\mathbf{G}}(u)$. However this is not the case when $Z(\mathbf{G})$ is disconnected. Therefore we must work with well-chosen representatives which are such that $|A_{\mathbf{G}}(u)|/|A_{\mathbf{G}}(u)^F| = |Z_{\mathbf{G}}(u)|/|Z_{\mathbf{G}}(u)^F|$ where $Z_{\mathbf{G}}(u)$ is the image of the centre in $A_{\mathbf{G}}(u)$. When $\mathbf{G}$ is adjoint $Z(\mathbf{G})$ is trivial and thus our theorem is a generalisation of that of Lusztig and Hézard.

The proof of this theorem involves a detailed case by case analysis in line with those of Lusztig and Hézard. Our main philosophy is to use an embedding of $\mathbf{G}$ into a connected reductive group with a connected centre and restriction of characters, (as in [**Lus88**]). The initial layout of the thesis is as follows:

- Chapter 1 contains the necessary background concerning the classification of finite reductive groups and the ordinary character theory of G.
- Chapter 2 contains material describing unipotent conjugacy classes of $\mathbf{G}$, in particular the order of the component groups $|A_{\mathbf{G}}(u)|$.
- Chapter 4 contains material concerning quasi-isolated semisimple elements.
- Chapters 3, 5 and 6 contain the case by case check forming the proof of Theorem A.

The remaining chapters of this thesis are concerned with applications of Theorem A. In [**GH08**] Geck and Hézard have shown that when $Z(\mathbf{G})$ is connected, (and assuming p and q are large), the following conjecture of Kawnaka's holds.

Conjecture (Kawanaka). *Assume* $\mathbf{G}$ *is a connected reductive algebraic group and* p *is a good prime for* $\mathbf{G}$. *The GGGRs of* G *form a* $\mathbb{Z}$*-basis for the* $\mathbb{Z}$*-module of all unipotently supported virtual characters of* G.

Note that we call a class function of G unipotently supported if its support as a function is contained in the set of unipotent elements of G. Our goal was to prove that Kawanaka's conjecture holds assuming p and q are large enough but this has not been possible, (see Remark 7.14). However we have been able to obtain the following extension of Geck and Hézard's result, which is proved in Chapter 7.

Theorem B. *Assume* p, q *are large enough and* $\mathbf{G}$ *is a simple algebraic group which is not a spin or half-spin group then Kawanaka's conjecture holds.*

It is known that there are two bases of the space of all unipotently supported class functions of G. One consists of the GGGRs of G and the other consists of the characteristic functions of unipotent conjugacy classes. In **[Lus92]** as a part of his work on unipotent supports he constructs explicitly the change of basis matrices for these two bases. Let us denote by $\mathcal{N}_{\mathbf{G}}$ the set of all pairs $\iota = (\mathcal{O}, \psi)$ where $\mathcal{O}$ is a unipotent class of $\mathbf{G}$ and ψ is an irreducible character of the component group $A_{\mathbf{G}}(u)$, (where $u \in \mathcal{O}$ is a representative).

The change of basis matrix contains in it certain unknown fourth roots of unity ζ'_ι which are associated to the pairs $\iota \in \mathcal{N}_{\mathbf{G}}$. These fourth roots of unity can be computed inductively from cuspidal pairs associated to Levi subgroups. In **[Gec99]** Geck gives an expression for the fourth root of unity associated to cuspidal pairs for the case where $\mathbf{G}$ is an adjoint simple group and p, q are sufficiently large, (in particular p is a good prime for $\mathbf{G}$). Adapting his argument, which is based on Theorem A, we obtain the final result of this thesis. This is proved in Chapter 8.

Theorem C. *Assume $\mathbf{G}_n$ is $\mathrm{Sp}_{2n}(\mathbb{K})$ or $\mathrm{SO}_{2n}(\mathbb{K})$ and p, q are sufficiently large, (in particular $p \neq 2$). If such a pair exists we denote by ι_0 the unique cuspidal pair in $\mathcal{N}_{\mathbf{G}_n}$. Let $\varepsilon \in \{\pm 1\}$ and $a \in \mathbb{N}$ be such that $p \equiv \varepsilon \pmod 4$ and $q = p^a$ then we have*

$$\zeta'_{\iota_0} = \begin{cases} \varepsilon^{\frac{an}{2}} & \text{if } n \text{ is even,} \\ (-1)^{an} & \text{if } n \text{ is odd and } \varepsilon = 1, \\ (-j)^{an} & \text{if } n \text{ is odd and } \varepsilon = -1. \end{cases}$$

Here j denotes a fixed primitive fourth root of unity in $\overline{\mathbb{Q}}_\ell$. It should be noted that Theorem C was already proved by Waldspurger in the context of **[Wal04]**. Here we merely give an alternative approach to obtaining these values. One would hope that a statement similar to that of Theorem C could be made for the case where $\mathbf{G}$ is a spin group. However the argument used to prove Theorem C fails in this case, (see the discussion in Section 8.3).

Contents

Chapter 1

Irreducible Characters of Finite Reductive Groups

Throughout $\mathbb{K} = \overline{\mathbb{F}_p}$ will be an algebraic closure of the finite field $\mathbb{F}_p$ of prime characteristic p. Recall that $\mathbb{K}$ is an algebraic group under addition, which we denote $\mathbb{K}^+$, and $\mathbb{K} \setminus \{0\}$ is an algebraic group under multiplication, which we denote $\mathbb{K}^\times$. Let $r = p^a$ be a power of p for some natural number $a > 0$ then we write $\mathbb{F}_r = \{x \in \mathbb{K} \mid x^r = x\}$ for the finite subfield of $\mathbb{K}$ of order r. We denote by $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ the set of non-zero natural numbers and by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$. As usual we denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of rational numbers, the field of real numbers and the field of complex numbers.

We will denote by $\overline{\mathbb{Q}}_\ell$ an algebraic closure of the field of ℓ-adic numbers for some prime ℓ distinct from p. We assume fixed once and for all an involutary automorphism $\overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell$, denoted $x \mapsto \overline{x}$, such that $\overline{\omega} = \omega^{-1}$ for every root of unity $\omega \in \overline{\mathbb{Q}}_\ell^\times$. Such an automorphism can be identified with complex conjugation under an isomorphism between $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$.

We fix now an isomorphism of groups $\imath : (\mathbb{Q}/\mathbb{Z})_{p'} \to \mathbb{K}^\times$ and an injective homomorphism of groups $\jmath : \mathbb{Q}/\mathbb{Z} \to \overline{\mathbb{Q}}_\ell^\times$. Note that if H is a group then we denote by $H_{p'}$ the subset consisting of all elements whose order is coprime to p. If H is abelian then $H_{p'}$ is a group and if H is also finite then the order of $H_{p'}$ is the largest divisor of the order of H which is coprime to p. The composition $\jmath \circ \imath^{-1}$ gives an injective homomorphism $\kappa : \mathbb{K}^\times \to \overline{\mathbb{Q}}_\ell^\times$.

1.1 The Classification of Semisimple Algebraic Groups

The main theorem of this thesis will involve a case by case check through the finite reductive groups which arise from simple algebraic groups. In this section we recall how reductive algebraic groups are classified and develop some conventions which we maintain throughout.

1.1.1 Root Data

Let X, $\check{X}$ be free abelian groups of the same finite rank and let Φ and $\check{\Phi}$ be finite subsets of X and $\check{X}$ respectively. As X, $\check{X}$ are abelian groups they are also $\mathbb{Z}$-modules hence we may form the tensor products of $\mathbb{Z}$-modules $\mathbb{R}X = \mathbb{R} \otimes_{\mathbb{Z}} X$ and $\mathbb{R}\check{X} = \mathbb{R} \otimes_{\mathbb{Z}} \check{X}$. The tensor products $\mathbb{R}X$ and $\mathbb{R}\check{X}$ are real vector spaces and we may identify X and $\check{X}$ with their natural images in these spaces. Typically we will suppress the tensors when writing elements of $\mathbb{R}X$ and $\mathbb{R}\check{X}$.

Definition 1.1. We say the quadruple $\Psi = (X, \Phi, \check{X}, \check{\Phi})$ is a *root datum* if the following conditions are satisfied.

- There exists a non-degenerate bilinear map $\langle -, - \rangle : X \times \check{X} \to \mathbb{Z}$ such that $\chi \mapsto \langle \chi, - \rangle$ and $\gamma \mapsto \langle -, \gamma \rangle$ give isomorphisms $X \to \mathrm{Hom}(\check{X}, \mathbb{Z})$ and $\check{X} \to \mathrm{Hom}(X, \mathbb{Z})$, (i.e. $\langle -, - \rangle$ is a *perfect pairing*).
- There exists a bijection $\Phi \to \check{\Phi}$ denoted by $\alpha \mapsto \check{\alpha}$, such that $\langle \alpha, \check{\alpha} \rangle = 2$.
- For every $\alpha \in \Phi$ the maps $s_\alpha : X \to X$ and $s_{\check{\alpha}} : \check{X} \to \check{X}$ defined by

$$\begin{aligned} s_\alpha(\chi) &= \chi - \langle \chi, \check{\alpha} \rangle \alpha && \text{for all } \chi \in X, \\ s_{\check{\alpha}}(\gamma) &= \gamma - \langle \alpha, \gamma \rangle \check{\alpha} && \text{for all } \gamma \in \check{X}, \end{aligned}$$

  are such that $s_\alpha(\Phi) = \Phi$ and $s_{\check{\alpha}}(\check{\Phi}) = \check{\Phi}$.

Assume now that $\Psi = (X, \Phi, \check{X}, \check{\Phi})$ is a root datum. We may easily extend the bilinear map in Definition 1.1 to a non-degenerate bilinear map $\mathbb{R}X \times \mathbb{R}\check{X} \to \mathbb{Z}$ by setting

$$\langle r_1\chi, r_2\gamma \rangle = r_1 r_2 \langle \chi, \gamma \rangle \qquad \text{for all } r_1\chi \in \mathbb{R}X \text{ and } r_2\gamma \in \mathbb{R}\check{X}.$$

Let $\mathbb{R}X_\Phi \subseteq \mathbb{R}X$ and $\mathbb{R}\check{X}_{\check{\Phi}} \subseteq \mathbb{R}\check{X}$ be the subspaces spanned by Φ and $\check{\Phi}$ respectively then Φ and $\check{\Phi}$ form crystallographic root systems of $\mathbb{R}X_\Phi$ and $\mathbb{R}\check{X}_{\check{\Phi}}$. Extending linearly we may consider the automorphisms s_α, $s_{\check{\alpha}}$ of X and $\check{X}$ to be elements of $\mathrm{GL}(\mathbb{R}X)$ and $\mathrm{GL}(\mathbb{R}\check{X})$, which are precisely the reflections associated to the elements of Φ and $\check{\Phi}$. Let W_Φ, $W_{\check{\Phi}}$ be the subgroups of $\mathrm{GL}(\mathbb{R}X)$, $\mathrm{GL}(\mathbb{R}\check{X})$ generated by these reflections then these are isomorphic Weyl groups and we call W_Φ the abstract Weyl group of Φ.

By [**Spr09**, §7.1.7] we may fix a W_Φ-invariant positive definite symmetric bilinear form $(\cdot, \cdot) : \mathbb{R}X \times \mathbb{R}X \to \mathbb{R}$ then the s_α are Euclidean reflections with respect to this metric. By [**Hum90**, §1.1] we have for each $\alpha \in \Phi$ that

$$s_\alpha(\chi) = \chi - \frac{2(\chi, \alpha)}{(\alpha, \alpha)}\alpha$$

for all $\chi \in \mathbb{R}X$, in particular $\langle -, \check{\alpha} \rangle = 2(\alpha, \alpha)^{-1}(-, \alpha)$. This form gives an identification of $\mathbb{R}\check{X}$ as the dual vector space of $\mathbb{R}X$. It follows that under this identification we have

$$\check{\alpha} = \frac{2\alpha}{(\alpha, \alpha)}, \tag{1.1}$$

(see for instance [**Bou02**, Ch. VI - §1 - no. 1 - Lemma 2]).

Let $\Psi' = (X', \Phi', \check{X}', \check{\Phi}')$ be another root datum and assume $\varphi \in \mathrm{Hom}(X', X)$ is a homomorphism of abelian groups. Precomposing with φ gives an induced homomorphism $\mathrm{Hom}(X, \mathbb{Z}) \to \mathrm{Hom}(X', \mathbb{Z})$. By identifying $\mathrm{Hom}(X, \mathbb{Z})$ and $\mathrm{Hom}(X', \mathbb{Z})$ with $\check{X}$ and $\check{X}'$, using the respective perfect pairings, we obtain a homomorphism $\langle -, \gamma \rangle \mapsto \langle -, \gamma \rangle \circ \varphi$. We then define $\check{\varphi} \in \mathrm{Hom}(\check{X}, \check{X}')$ to be the homomorphism such that $\langle -, \check{\varphi}(\gamma) \rangle = \langle -, \gamma \rangle \circ \varphi$ and call this the map *dual* to φ. Equivalently we have $\check{\varphi}$ is the unique homomorphism satisfying $\langle \varphi(\chi), \gamma \rangle = \langle \chi, \check{\varphi}(\gamma) \rangle$ for all $\chi \in X'$ and $\gamma \in \check{X}$. Similarly we have a duality map $\mathrm{Hom}(\check{X}, \check{X}') \to \mathrm{Hom}(X', X)$, also denoted $\varphi \mapsto \check{\varphi}$, and the composition of these dualities satisfies $\check{\check{\varphi}} = \varphi$. This implies that φ is an isomorphism if and only if $\check{\varphi}$ is an isomorphism.

Remark 1.2. It is easily seen that the automorphism $s_{\check{\alpha}}$ of $\check{X}$ in Definition 1.1 is the automorphism dual to s_α, thus we may often denote this by $\check{s}_\alpha$.

Definition 1.3. We say a triple $(\varphi, b, q) : \Psi' \to \Psi$ is an *isogeny of root data* if the following conditions hold:

- $\varphi : X' \to X$ and its dual $\check{\varphi} : \check{X} \to \check{X}'$ are injective homomorphisms of abelian groups.
- $b : \Phi \to \Phi'$ is a bijection and $q : \Phi \to \{p^a \mid a \in \mathbb{N}_0\}$ is a function such that $\varphi(b(\alpha)) = q(\alpha)\alpha$ and $\check{\varphi}(\widecheck{b(\alpha)}) = q(\alpha)\check{\alpha}$ for all $\alpha \in \Phi$, where either $p = 1$ or $p > 1$ is a prime.

We call (φ, b, q) a *central isogeny* if $q = 1$, i.e. $q(\alpha) = 1$ for all $\alpha \in \Phi$. The isogeny (φ, b, q) is an *isomorphism of root data* if φ is an isomorphism and $q = 1$, (note b must then be the restriction of φ^{-1} to Φ). We call $(\varphi, b, 1)$ an *automorphism of root data* if it is an isomorphism and $\Psi = \Psi'$. We will often denote an isogeny merely by φ.

Let us choose a positive system of roots $\Phi^+ \subset \Phi$ then this determines a unique system of simple roots $\Delta \subseteq \Phi^+$, (this follows from [**Hum90**, §1.3 - Theorem]). Let $\check{\Delta}$ be the image of Δ under the bijection $\Phi \to \check{\Phi}$ then by [**Bou02**, Ch. VI - §1 - no. 5 - Remark (5)] and Eq. (1.1) we have $\check{\Delta}$ is a system of simple roots for $\check{\Phi}$ hence determines a unique system of positive roots $\check{\Phi}^+ \subset \check{\Phi}$. We will denote by $\mathbb{S}$ the set of *simple reflections* $\{s_\alpha \mid \alpha \in \Delta\}$, which is a generating set for W_Φ such that $(W_\Phi, \mathbb{S})$ is a Coxeter system, (see [**Hum90**, §1.9 - Theorem]).

We call the following subspaces of $\mathbb{R}X$ and $\mathbb{R}\check{X}$

$$\Lambda = \{\chi \in \mathbb{R}X \mid \langle \chi, \check{\alpha} \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\},$$
$$\check{\Lambda} = \{\gamma \in \mathbb{R}\check{X} \mid \langle \alpha, \gamma \rangle \in \mathbb{Z} \text{ for all } \check{\alpha} \in \check{\Phi}\},$$

the *weight lattice* and *coweight lattice* of Ψ, (their elements are called weights and coweights respectively). To any simple root $\alpha \in \Delta$ we associate the unique element of the weight lattice $\varpi_\alpha \in \Lambda$ satisfying $\langle \varpi_\alpha, \check{\beta} \rangle = \delta_{\alpha,\beta}$ for all $\beta \in \Delta$, where $\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$ and 0 otherwise. Similarly we can associate a unique element of the coweight lattice $\check{\omega}_\alpha \in \check{\Lambda}$ satisfying $\langle \beta, \check{\omega}_\alpha \rangle = \delta_{\alpha,\beta}$. We call the sets $\Omega = \{\varpi_\alpha \mid \alpha \in \Delta\}$ and $\check{\Omega} = \{\check{\omega}_\alpha \mid \alpha \in \Delta\}$ the sets of *fundamental dominant weights* and *fundamental dominant coweights*.

With these notions in place we can consider the following sequence of subspace inclusions $\mathbb{Z}\Phi \subseteq X \subseteq \Lambda$ where $\mathbb{Z}\Phi$ is the $\mathbb{Z}$-span of the roots (similarly we have the inclusions $\mathbb{Z}\check{\Phi} \subseteq \check{X} \subseteq \check{\Lambda}$). The two quotient spaces

$$\Pi = \Lambda/\mathbb{Z}\Phi \qquad\qquad \check{\Pi} = \check{\Lambda}/\mathbb{Z}\check{\Phi}$$

have the structure of a finite abelian group and are isomorphic, (see [**Bou02**, Ch. VI - §1 - no. 9]). We call this group the *fundamental group* of Φ, (this should not be confused with Λ/X which is also called the fundamental group of Ψ).

1.1.2 Connected Reductive Algebraic Groups

Let $\mathbf{G}$ be a connected reductive algebraic group defined over $\mathbb{K}$, $\mathbf{T}_0 \leqslant \mathbf{G}$ a fixed maximal torus of $\mathbf{G}$ and $\mathbf{B}_0 \leqslant \mathbf{G}$ a fixed Borel subgroup containing $\mathbf{T}_0$. We define the root datum of $\mathbf{G}$ relative to $\mathbf{T}_0$ to be $\Psi(\mathbf{T}_0) = (X(\mathbf{T}_0), \Phi(\mathbf{T}_0), \check{X}(\mathbf{T}_0), \check{\Phi}(\mathbf{T}_0))$. Here $X(\mathbf{T}_0) = \mathrm{Hom}(\mathbf{T}_0, \mathbb{K}^\times)$ is the set of all homomorphisms from $\mathbf{T}_0$ to $\mathbb{K}^\times$ and $\check{X}(\mathbf{T}_0) = \mathrm{Hom}(\mathbb{K}^\times, \mathbf{T}_0)$ is the set of all homomorphisms from $\mathbb{K}^\times$ to $\mathbf{T}_0$. The sets $\Phi(\mathbf{T}_0) \subset X(\mathbf{T}_0)$ and $\check{\Phi}(\mathbf{T}_0) \subset \check{X}(\mathbf{T}_0)$ are the sets of roots and coroots which we will define below. When there is no ambiguity regarding $\mathbf{T}_0$ or $\mathbf{G}$ we denote this root datum simply by $\Psi = (X, \Phi, \check{X}, \check{\Phi})$. Before discussing the definition of the subsets Φ and $\check{\Phi}$ we first make the following remark regarding homomorphisms.

Remark 1.4. Let $\varphi : \mathbf{G}_1 \to \mathbf{G}_2$ be a map between two algebraic groups $\mathbf{G}_1$, $\mathbf{G}_2$. We will take the statement "φ is a homomorphism" to mean that φ is a homomorphism of algebraic groups, i.e. it is a homomorphism of abstract groups and a morphism of varieties. Furthermore we will take the statement "φ is an isomorphism" to mean that φ is a bijective homomorphism of algebraic groups whose inverse φ^{-1} is also a homomorphism of algebraic groups. If we wish to make clear that φ^{-1} is not a homomorphism of algebraic groups, i.e. φ is only an isomorphism of abstract groups, then we will say "φ is an abstract isomorphism". Such conventions will also be used for automorphisms. If φ is an abstract isomorphism we will say $\mathbf{G}_1$ and $\mathbf{G}_2$ are abstractly isomorphic.

Notation. If H is a group then given any two elements $g, h \in H$ we denote by ${}^g h$ the element ghg^{-1} and by h^g the element $g^{-1}hg$. Similarly if $K \leqslant H$ is a subgroup then we denote by ${}^g K$ the group $\{gkg^{-1} \mid k \in K\}$ and by K^g the group $\{g^{-1}kg \mid k \in K\}$. We denote by $\operatorname{Aut}(H)$ the group of all automorphisms of H, (if H is an algebraic group then these are assumed to be automorphisms of the algebraic group). If $g \in H$ we denote the corresponding inner automorphism by $\operatorname{inn} g$, this is the element of $\operatorname{Aut}(H)$ such that $(\operatorname{inn} g)(x) = {}^g x$ for all $x \in H$. The subgroup of all inner automorphisms is denoted by $\operatorname{Inn} H$, which is a normal subgroup of $\operatorname{Aut}(H)$. The quotient $\operatorname{Aut}(H)/\operatorname{Inn}(H)$ is denoted by $\operatorname{Out}(H)$ and is the group of all outer automorphisms of H. Note that if H is an algebraic group then $\operatorname{Aut}(H)$, $\operatorname{Inn}(H)$ and $\operatorname{Out}(H)$ have only the structure of an abstract group.

Let $\mathbf{X}$ be a connected one-dimensional unipotent subgroup of $\mathbf{G}$. Assume further that $\mathbf{X}$ is normalised by $\mathbf{T}_0$ then after fixing an isomorphism $x : \mathbb{K}^+ \to \mathbf{X}$ there exists an element $\alpha \in X$ such that $tx(c)t^{-1} = x(\alpha(t)c)$ for all $t \in \mathbf{T}_0$ and $c \in \mathbb{K}^+$. This follows from the fact that every automorphism of $\mathbb{K}^+$ is of the form $c \mapsto \lambda c$ for some $\lambda \in \mathbb{K}^\times$. We denote by $\mathbf{X}_\alpha$ the subgroup $\mathbf{X}$ and by x_α the automorphism x. The element α is called a *root* of $\mathbf{G}$ and $\mathbf{X}_\alpha$ its corresponding *root subgroup*. There are finitely many distinct choices for the group $\mathbf{X}$ and distinct choices give rise to distinct roots. The set $\Phi(\mathbf{T}_0)$ is then defined to be the finite set of all roots of $\mathbf{G}$. Recall that our chosen Borel subgroup $\mathbf{B}_0$ has a semidirect product decomposition $\mathbf{B}_0 = \mathbf{U}_0\mathbf{T}_0$ where $\mathbf{U}_0$ is the unipotent radical of $\mathbf{B}_0$. This determines a system of positive roots by specifying $\alpha \in \Phi^+$ if $\mathbf{X}_\alpha \leqslant \mathbf{U}_0$.

Assume $\chi \in X$ and $\gamma \in \check{X}$ then $\chi \circ \gamma \in \operatorname{Hom}(\mathbb{K}^\times, \mathbb{K}^\times)$ so must be of the form $\lambda \mapsto \lambda^{n_{\chi,\gamma}}$ for some unique $n_{\chi,\gamma} \in \mathbb{Z}$. Hence we define the pairing $\langle -, - \rangle$ by setting $\langle \chi, \gamma \rangle = n_{\chi,\gamma}$. Assume $\alpha \in \Phi$ is a root then by **[Spr09**, Lemma 7.1.8(i)**]** there exists a unique element $\check{\alpha} \in \check{X}$ such that $\langle \alpha, \check{\alpha} \rangle = 2$ and $s_\alpha(\chi) = \chi - \langle \chi, \check{\alpha} \rangle \alpha$. We call the element $\check{\alpha}$ the *coroot* of $\mathbf{G}$ associated to α and we define $\check{\Phi}(\mathbf{T}_0)$ to be the set of all such coroots. We call X and $\check{X}$ the *character* and *cocharacter groups* of $\mathbf{G}$ and define their group operations to be

$$\begin{aligned}
(\chi_1 + \chi_2)(t) &= \chi_1(t)\chi_2(t) && \text{for all } \chi_1, \chi_2 \in X \text{ and } t \in \mathbf{T}_0, \\
(\gamma_1 + \gamma_2)(\lambda) &= \gamma_1(\lambda)\gamma_2(\lambda) && \text{for all } \gamma_1, \gamma_2 \in \check{X} \text{ and } \lambda \in \mathbb{K}^\times.
\end{aligned}$$

Remark 1.5. Assume $\mathbf{T}$ is another maximal torus of $\mathbf{G}$ and let $\Psi(\mathbf{T})$ be the root datum of $\mathbf{G}$ with respect to $\mathbf{T}$. As all maximal tori are conjugate there exists an element $g \in \mathbf{G}$ such that $t \mapsto {}^g t$ gives an isomorphism $\mathbf{T}_0 \to \mathbf{T}$. It is readily checked that this naturally gives a corresponding isomorphism of root data $\Psi(\mathbf{T}) \to \Psi(\mathbf{T}_0)$, hence the isomorphism type of Ψ does not depend on the choice of $\mathbf{T}_0$.

We define the *Weyl group* of $\mathbf{G}$ relative to $\mathbf{T}_0$ to be the finite quotient group $\mathbf{W}(\mathbf{T}_0) = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ where $N_{\mathbf{G}}(\mathbf{T}_0)$ is the normaliser of $\mathbf{T}_0$ in $\mathbf{G}$. When there is no ambiguity regarding $\mathbf{T}_0$ we denote this group simply by $\mathbf{W}$. This group is finite because $N_{\mathbf{G}}(\mathbf{T}_0)^\circ = C_{\mathbf{G}}(\mathbf{T}_0)^\circ$ and, as $\mathbf{G}$ is reductive, we have $C_{\mathbf{G}}(\mathbf{T}_0) = C_{\mathbf{G}}(\mathbf{T}_0)^\circ = \mathbf{T}_0$, (see [**Spr09**, Corollary 3.2.9] and [**Spr09**, Corollary 7.6.4(ii)]). For each $w \in \mathbf{W}$ we assume fixed once and for all a representative $\dot{w}$ of w in $N_{\mathbf{G}}(\mathbf{T}_0)$ and given $t \in \mathbf{T}_0$ we define t^w to be the element $\dot{w}^{-1}t\dot{w}$. This gives an action of $\mathbf{W}$ on $\mathbf{T}_0$ which induces an action of $\mathbf{W}$ on the character and cocharacter groups given by

$$
\begin{aligned}
(w \cdot \chi)(t) &= \chi(t^w) && \text{for all } \chi \in X,\ w \in \mathbf{W},\ t \in \mathbf{T}_0,\\
(\check{w} \cdot \gamma)(t) &= \gamma(\lambda)^w && \text{for all } \gamma \in \check{X},\ w \in \mathbf{W},\ \lambda \in \mathbb{K}^\times.
\end{aligned}
$$

Note that we have $\langle w \cdot \chi, \gamma \rangle = \langle \chi, \check{w} \cdot \gamma \rangle$ hence the action of $\check{w}$ on $\check{X}$ is simply the dual to that of w on X.

The map $\chi \mapsto w \cdot \chi$ is an automorphism of X and extending linearly for each $w \in \mathbf{W}$ we obtain an element of $\mathrm{GL}(\mathbb{R}X_\Phi)$. By [**Hum75**, §26.3 - Theorem(b)] this element stabilises the set of roots thus we obtain a map $\mathbf{W} \to W_\Phi$ which by [**Hum75**, §27.1 - Theorem] is a group isomorphism. We denote again by $s_\alpha \in \mathbf{W}$ the element whose image in W_Φ is the reflection corresponding to the root $\alpha \in \Phi$. The set $\mathbb{S} = \{s_\alpha \mid \alpha \in \Delta\} \subseteq \mathbf{W}$ will denote the set of simple reflections, (if α_i is a simple root indexed by $i \in \mathbb{N}$ then we will often denote s_{α_i} simply by s_i). Recall that $\mathbf{W}$ is equipped with a function $\ell : \mathbf{W} \to \mathbb{N} \cup \{0\}$ known as the *length function*, (see [**Hum90**, §1.6]). This function is defined for any Coxeter system $(\mathbf{W}, \mathbb{S})$ and for any such system we will denote this function by ℓ.

The connected reductive group $\mathbf{G}$ is such that it is an almost direct product $\mathbf{G}' \cdot Z(\mathbf{G})^\circ$, where $\mathbf{G}'$ is the derived subgroup of $\mathbf{G}$ and $Z(\mathbf{G})^\circ$ is the connected component of the centre. We say a connected reductive algebraic group $\mathbf{H}$ is an *almost direct product* of connected reductive groups $\mathbf{H}_1$ and $\mathbf{H}_2$ if $\mathbf{H} = \mathbf{H}_1\mathbf{H}_2 = \mathbf{H}_2\mathbf{H}_1$ and $\mathbf{H}_1 \cap \mathbf{H}_2$ is finite, (note also that $\mathbf{H}_1 \cap \mathbf{H}_2$ must be contained in the centre of $\mathbf{H}$). Recall that $\mathbf{G}$ is called semisimple if $Z(\mathbf{G})^\circ = \{1\}$ or equivalently $\mathbb{R}X_\Phi = \mathbb{R}X$, (see [**Spr09**, Proposition 8.1.8]); clearly such a group is perfect. We say $\mathbf{G}$ is a *simple algebraic group* if it is connected and has no connected normal subgroups.

To discuss the classification of connected reductive algebraic groups we need to understand the relationship between morphisms of algebraic groups and their root data.

Definition 1.6. Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be two connected reductive algebraic groups then an *isogeny* $\pi : \mathbf{G}_1 \to \mathbf{G}_2$ is a surjective homomorphism with finite kernel. We say π is an *isotypic homomorphism* if $\mathrm{Ker}(\pi)$ is contained in $Z(\mathbf{G}_1)$ and $\mathrm{Im}(\pi)$ contains the derived subgroup of $\mathbf{G}_2$.

Remark 1.7. Assume $\mathbf{N} \leqslant \mathbf{G}$ is a finite normal subgroup of a connected algebraic group $\mathbf{G}$. For each $n \in \mathbf{N}$ the morphism of varieties $\varphi_n : \mathbf{G} \to \mathbf{N}$ given by $\varphi_n(g) = gng^{-1}$ has a finite and connected image. Therefore as $\varphi_n(1) = n$ we must have $\mathrm{Im}(\varphi_n) = \{n\}$ so $gng^{-1} = n$ for all $g \in \mathbf{G}$. This shows that $\mathbf{N}$ lies in the centre of $\mathbf{G}$. In particular for any isogeny $\pi : \mathbf{G}_1 \to \mathbf{G}_2$ we have $\mathrm{Ker}(\pi)$ is necessarily contained in $Z(\mathbf{G}_1)$.

As the names suggest there is a strong relationship between isogenies of algebraic groups and isogenies of root data. In particular we have the following.

Lemma 1.8 ([Ste99, pg. 370]). *Assume $\pi : \mathbf{G}_1 \to \mathbf{G}_2$ is an isogeny and define $\mathbf{T}_2 = \pi(\mathbf{T}_1)$ to be a maximal torus of $\mathbf{G}_2$ where $\mathbf{T}_1$ is a maximal torus of $\mathbf{G}_1$. The map $\varphi : X(\mathbf{T}_2) \to X(\mathbf{T}_1)$ given by $\varphi(\chi_2) = \chi_2 \circ \pi_{\mathbf{T}_1}$, where $\pi_{\mathbf{T}_1}$ is the restriction of π to $\mathbf{T}_1$, determines an isogeny of root data (φ, b, q). The following properties are satisfied by this relationship.*

(i) For each $\alpha \in \Phi(\mathbf{T}_1)$ there exists $d_\alpha \in \mathbb{K}^\times$ such that $\pi(x_\alpha(c)) = x_{b(\alpha)}(d_\alpha c^{q(\alpha)})$ for all $c \in \mathbb{K}^+$.

(ii) The restriction of π to $N_{\mathbf{G}_1}(\mathbf{T}_1)$ induces an isomorphism $\mathbf{W}(\mathbf{T}_1) \to \mathbf{W}(\mathbf{T}_2)$ such that $s_\alpha \mapsto s_{b(\alpha)}$ for all $\alpha \in \Phi(\mathbf{T}_1)$.

We call the isogeny of root data in Lemma 1.8 the isogeny *induced by* π. With this notation in mind the classification of connected reductive algebraic groups is encompassed by the following theorem.

Theorem 1.9 (Chevalley, [Spr09, Theorem 9.6.2 and 10.1.1]). *Assume $\mathbf{G}_i$ is a connected reductive algebraic group with corresponding root data $\Psi(\mathbf{T}_i)$, with $i \in \{1, 2\}$.*

(i) Assume there is an isomorphism of root data $\varphi : \Psi(\mathbf{T}_2) \to \Psi(\mathbf{T}_1)$ then there exists an isomorphism $\pi : \mathbf{G}_1 \to \mathbf{G}_2$ such that $\pi(\mathbf{T}_1) = \mathbf{T}_2$ and φ is induced by π. If π' is another such isomorphism then there exists $t \in \mathbf{T}_1$ such that $\pi' = \pi \circ \mathrm{inn}\, t$. Conversely for any $t' \in \mathbf{T}_1$ the map $\pi \circ \mathrm{inn}\, t'$ is such an isomorphism.

(ii) If Ψ' is a root datum then there exists a connected reductive algebraic group $\mathbf{G}$ with maximal torus $\mathbf{T}$ such that $\Psi(\mathbf{T})$ is isomorphic to Ψ'.

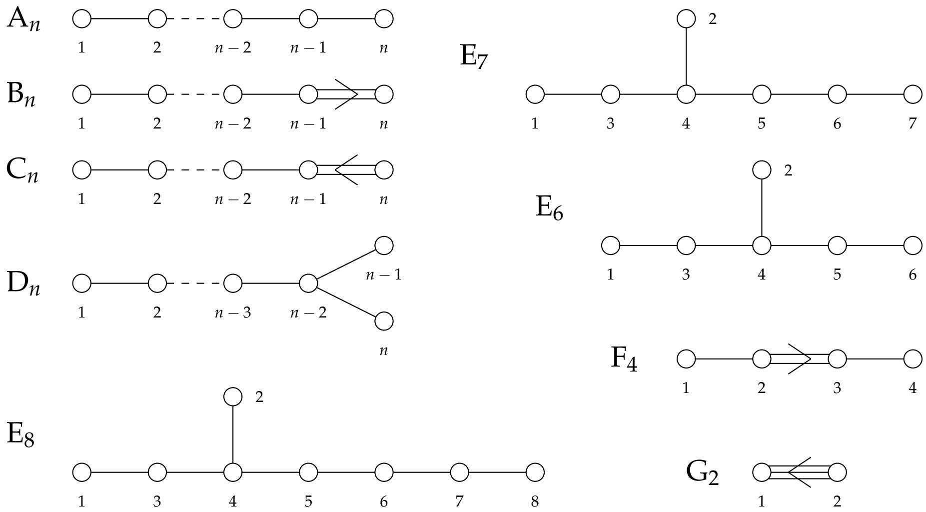


Figure 1.1: The Dynkin Diagrams of the Irreducible Crystallographic Root Systems

In particular, when $\mathbf{G}$ is semisimple Theorem 1.9 says that $\mathbf{G}$ is characterised by its root system Φ and the image of X in the fundamental group of Φ. When $\mathbf{G}$ is semisimple we can identify two extreme possibilities for the character group X. If $X = \mathbb{Z}\Phi$ we say $\mathbf{G}$ is *adjoint* and if $X = \Lambda$ we say $\mathbf{G}$ is *simply connected*. If $\mathbf{G}$ is adjoint then $\check{X} = \check{\Lambda}$ and if $\mathbf{G}$ is simply connected then $\check{X} = \mathbb{Z}\check{\Phi}$. Given a connected reductive algebraic group $\mathbf{G}$ we now fix a corresponding adjoint group $\mathbf{G}_{\mathrm{ad}}$ such that the root datum of $\mathbf{G}_{\mathrm{ad}}$ with respect to some maximal torus is isomorphic to $(\mathbb{Z}\Phi, \Phi, \check{\Lambda}, \check{\Phi})$. Similarly we fix a corresponding simply connected group $\mathbf{G}_{\mathrm{sc}}$ such that the root datum of $\mathbf{G}_{\mathrm{sc}}$ with respect to some maximal torus is isomorphic to $(\Lambda, \Phi, \mathbb{Z}\check{\Phi}, \check{\Phi})$.

Let us decompose the root system $\Phi = \Phi_1 \sqcup \Phi_2 \sqcup \cdots \sqcup \Phi_k$ into a disjoint union of irreducible subsystems Φ_i. As $\mathbf{G}$ is semisimple we have the simple roots Δ form a basis for the vector space $\mathbb{R}X$. Let $\Delta = \Delta_1 \sqcup \Delta_2 \sqcup \cdots \sqcup \Delta_k$ be a corresponding decomposition of the simple roots so that Δ_i is a set of simple roots for Φ_i. We obtain a corresponding direct product decomposition of the vector space $\mathbb{R}X = V_1 \times V_2 \times \cdots \times V_k$ where V_i is the subspace spanned by Δ_i. Also we have a corresponding decomposition of the weight lattice $\Lambda = \Lambda_1 \sqcup \Lambda_2 \sqcup \cdots \sqcup \Lambda_k$ where Λ_i is defined as above with respect to the irreducible root system Φ_i and the subspace V_i. We then clearly have the following isomorphism of finite abelian groups

$$\Pi \cong \Lambda_1/\mathbb{Z}\Phi_1 \times \Lambda_2/\mathbb{Z}\Phi_2 \times \cdots \times \Lambda_k/\mathbb{Z}\Phi_k.$$

There exist corresponding simple subgroups $\mathbf{G}_1, \ldots, \mathbf{G}_k \leqslant \mathbf{G}$ such that $\mathbf{G}$ is an almost direct product of these subgroups. The isomorphism type of each $\mathbf{G}_i$ is determined by Φ_i and the image of X in $\Lambda_i/\mathbb{Z}\Phi_i$. If $\mathbf{G}$ is simply connected then each $\mathbf{G}_i$ must be

Type	A_n	B_n	C_n	D_n (n even)	D_n (n odd)	E_6	E_7	E_8	F_4	G_2
Π	$\mathbb{Z}_{n+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	1	1	1
$\mathbf{G}_{\mathrm{sc}}$	SL_{n+1}	Spin_{2n+1}	Sp_{2n}	Spin_{2n}		–	–	–	–	–
$\mathbf{G}_{\mathrm{ad}}$	PGL_{n+1}	SO_{2n+1}	PCSp_{2n}	$\mathrm{P}(\mathrm{CO}^{\circ}_{2n})$		–	–	–	–	–

Table 1.1: The Fundamental Groups of the Irreducible Root Systems

simply connected and $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_k$ is a direct product, similarly if $\mathbf{G}$ is adjoint each $\mathbf{G}_i$ is adjoint and we have a direct product. The direct product follows in the adjoint case from the fact that the intersection of two simple subgroups is contained in the centre of $\mathbf{G}$ and in the simply connected case by [**Ste68b**, Corollary(a) – pg. 44].

We now assume $\mathbf{G}$ is simple so that the root system Φ is irreducible. The classification of irreducible crystallographic root systems is given by the Dynkin diagrams listed in Figure 1.1. In these diagrams the nodes correspond to simple roots such that a node enumerated by $i \in \mathbb{N}$ corresponds to the simple root $\alpha_i \in \Delta$. We will take the irreducible root systems to be defined as in [**Bou02**, Plates I - IX]. We say a group $\mathbf{G}$ is of a given type $\mathrm{A}, \dots, \mathrm{E}$ if its underlying root system is of that type.

In Table 1.1 we list the isomorphism type of the different fundamental groups for each irreducible root system. Note that if Φ is of type D_n then the isomorphism type of Π depends upon whether n is even or odd. For the root systems of type A_n, B_n, C_n and D_n we give realisations of the simply connected and adjoint groups as classical groups. In type D_n this realisation does not depend upon the isomorphism type of the fundamental group. For more information on these groups see [**Car93**, §1.11] or [**Car72**].

It is easy to see from the description of the fundamental groups that a simple group $\mathbf{G}$ is necessarily simply connected or adjoint unless $\mathbf{G}$ is of type A_n or D_n. We wish to consider how the non-simply connected and adjoint groups of type A_n and D_n are constructed. The following result gives us a relationship between the centre of a group and its fundamental group.

Lemma 1.10 ([Bon06, Proposition 4.1]). *Let $\mathbf{G}$ be a connected semisimple algebraic group. There exists a surjective homomorphism $\mathbb{Q} \otimes_{\mathbb{Z}} \check{X}(\mathbf{T}_0) \to \mathbf{T}_0$ which induces an isomorphism $(\check{\Lambda}/\check{X})_{p'} \to Z(\mathbf{G})$.*

Assume $\mathbf{G}$ is of type A_n and $(\check{\Lambda}/\check{X})_{p'} = \check{\Lambda}/\check{X}$ then $\mathbf{G}$ can be realised as the quotient of $\mathrm{SL}_{n+1}(\mathbb{K})$ by a subgroup of its centre. It can happen that the centre of the special linear group is trivial, (for example when $n+1$ is a power of p), then there are no non-trivial subgroups in the centre of the special linear group. In this situation there is not such a simple realisation of the algebraic group $\mathbf{G}$, however one could construct $\mathbf{G}$ as the image of $\mathrm{SL}_{n+1}(\mathbb{K})$ acting on an exterior power of its natural rational representation, (see [**Che05**, §20.3 - Théorème 2]).

Assume $\mathbf{G}$ is of type D_n then if n is odd there is only one possibility for $\mathbf{G}$ corresponding to the unique proper subgroup of the fundamental group. Namely $\mathbf{G}$ can be realised as a special orthogonal group SO_{2n}. Assume n is even then there are three unique subgroups of the fundamental group of order 2. One of these subgroups gives rise to a special orthogonal group SO_{2n} and the remaining two give rise to isomorphic groups, namely a half spin group HSpin_{2n}. We will make this more precise in Section 1.2.1.

1.2 Dual Groups and Central Isogenies

We return to our assumption that $\mathbf{G}$ is a connected reductive algebraic group. We define a *dual group* $\mathbf{G}^\star$ of $\mathbf{G}$ to be a connected reductive algebraic group such that the root datum of $\mathbf{G}^\star$ with respect to a maximal torus $\mathbf{T}_0^\star$ is isomorphic to $(\check{X}, \check{\Phi}, X, \Phi)$. By Theorem 1.9 the existence of such a group is assured and the isomorphism type of $\mathbf{G}^\star$ is unique. By [**Car93**, Proposition 4.2.2] we have $\mathbf{G}^\star$ is a dual group of $\mathbf{G}$ if there exists an isomorphism $\varphi : X(\mathbf{T}_0) \to \check{X}(\mathbf{T}_0^\star)$ such that $\varphi(\Phi(\mathbf{T}_0)) = \check{\Phi}(\mathbf{T}_0^\star)$ and φ is compatible with the perfect pairings of $\mathbf{G}$ and $\mathbf{G}^\star$. Similar to the discussion in Section 1.1.1 such an isomorphism gives rise to an isomorphism $\mathrm{Hom}(X(\mathbf{T}_0), \mathbb{Z}) \to \mathrm{Hom}(\check{X}(\mathbf{T}_0^\star), \mathbb{Z})$, hence an isomorphism $\check{X}(\mathbf{T}_0) \to X(\mathbf{T}_0^\star)$. We now assume that the pair $(\mathbf{G}^\star, \mathbf{T}_0^\star)$ and the isomorphism φ are fixed. When there is no ambiguity regarding $\mathbf{T}_0^\star$ we will denote the root datum of $\mathbf{G}^\star$ with respect to $\mathbf{T}_0^\star$ simply by $\Psi^\star = (X^\star, \Phi^\star, \check{X}^\star, \check{\Phi}^\star)$. Typically we denote by $\alpha^\star \in \Phi^\star$ and $\check{\alpha}^\star \in \check{\Phi}^\star$ the roots and coroots of $\mathbf{G}^\star$. We will see that the dual group plays an important role in the character theory of finite reductive groups.

We denote by $\mathbf{W}^\star = \mathbf{W}(\mathbf{T}_0^\star)$ the Weyl group of $\mathbf{G}^\star$ with respect to $\mathbf{T}_0^\star$ and $\mathbb{S}^\star$ its fixed set of simple reflections. Assume $\mu : X(\mathbf{T}_0) \to X(\mathbf{T}_0)$ is an endomorphism then define $\check{\mu}^\star : \check{X}(\mathbf{T}_0^\star) \to \check{X}(\mathbf{T}_0^\star)$ to be the endomorphism $\varphi \circ \mu \circ \varphi^{-1}$. Taking the dual of this endomorphism we obtain a resulting endomorphism $\mu^\star : X(\mathbf{T}_0^\star) \to X(\mathbf{T}_0^\star)$. This gives us maps $\mathrm{Hom}(X(\mathbf{T}_0), X(\mathbf{T}_0)) \to \mathrm{Hom}(X(\mathbf{T}_0^\star), X(\mathbf{T}_0^\star))$ and $\mathrm{Hom}(X(\mathbf{T}_0^\star), X(\mathbf{T}_0^\star)) \to \mathrm{Hom}(X(\mathbf{T}_0), X(\mathbf{T}_0))$ each denoted by $\mu \mapsto \mu^\star$ and satisfying $\mu^{\star\star} = \mu$. By [**Car93**, Proposition 4.2.3] the map $w \mapsto w^\star$ is an anti-isomorphism $\mathbf{W} \to \mathbf{W}^\star$ such that for each $\alpha \in \Phi$ we have $s_\alpha^\star = s_{\widetilde{\varphi(\alpha)}}$, (in other words sending $\mathbb{S}$ to $\mathbb{S}^\star$).

In the previous section we discussed that any isogeny of algebraic groups gives rise to an isogeny of the corresponding root data. The following theorem gives the converse, in particular it says that any isogeny of root data comes from an isogeny of corresponding algebraic groups.

Theorem 1.11 (Chevalley, [Ste99, 1.5]). *Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be two connected reductive algebraic groups with corresponding root data $\Psi(\mathbf{T}_1)$ and $\Psi(\mathbf{T}_2)$. Assume there is an isogeny of root data $(\varphi, b, q) : \Psi(\mathbf{T}_2) \to \Psi(\mathbf{T}_1)$ then there exists an isogeny $\pi : \mathbf{G}_1 \to \mathbf{G}_2$ such that (φ, b, q) is induced by π and $\pi(\mathbf{T}_1) = \mathbf{T}_2$. If π' is another such isogeny then there exists $t \in \mathbf{T}_1$ such that $\pi' = \pi \circ \operatorname{inn} t$. Conversely for any $t' \in \mathbf{T}_1$ the homomorphism $\pi \circ \operatorname{inn} t'$ is such an isogeny.*

Remark 1.12. Assume now that the group $\mathbf{G}_1$ in Theorem 1.11 is semisimple. By Lemma 1.8 for each $\alpha \in \Phi(\mathbf{T}_1)$ there exists $d_\alpha \in \mathbb{K}^\times$ such that

$$\pi(x_\alpha(c)) = x_{b(\alpha)}(d_\alpha c^{q(\alpha)})$$

for all $c \in \mathbb{K}^+$. Let $\Delta(\mathbf{T}_1)$ denote a simple system of roots in $\Phi(\mathbf{T}_1)$. As the set $\Delta(\mathbf{T}_1)$ is linearly independent in $X(\mathbf{T}_1)$ there exists $t \in \mathbf{T}_1$ such that $\alpha(t) = d_\alpha$ for all $\alpha \in \Delta(\mathbf{T}_1)$, (see **[Hum75**, §16.2 - Lemma C]). By Theorem 1.11 (φ, b, q) is the isogeny of root data induced by $\pi' = \pi \circ \operatorname{inn} t^{-1}$ and by construction satisfies

$$\pi'(x_\alpha(c)) = x_{b(\alpha)}(c^{q(\alpha)}) \text{ for all } \alpha \in \Delta(\mathbf{T}_1) \text{ and } c \in \mathbb{K}^+. \tag{1.2}$$

By **[Hum75**, §27.3 - Theorem] the group $\mathbf{G}_1$ is generated by the set $\{x_\alpha(c) \mid \alpha \in \Delta(\mathbf{T}_1), c \in \mathbb{K}^+\}$ so the isogeny π' is uniquely determined by the condition in Eq. (1.2). In particular we have shown that given an isogeny (φ, b, q) of root data there exists a unique isogeny π' satisfying Eq. (1.2).

It should be noted that the first part of Theorem 1.9 is a special case of Theorem 1.11. As a consequence of this theorem we may easily obtain the following result.

Lemma 1.13. *Assume $\mathbf{G}$ is semisimple then there exist central isogenies $\delta_{\mathrm{ad}} : \mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$ and $\delta_{\mathrm{sc}} : \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ called an* adjoint quotient *and* simply connected cover *of $\mathbf{G}$. Furthermore assume $\delta : \mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$ is any adjoint quotient of $\mathbf{G}$ then we have $\operatorname{Ker}(\delta) = Z(\mathbf{G})$.*

Proof. Firstly, by Theorem 1.9, to show the existence of these isogenies it is sufficient to work with root data isomorphic to those of $\mathbf{G}_{\mathrm{ad}}$ and $\mathbf{G}_{\mathrm{sc}}$. By Theorem 1.11 the existence of δ_{ad} and δ_{sc} are assured once there are isogenies of root data

$$(\mathbb{Z}\Phi, \Phi, \check{\Lambda}, \check{\Phi}) \to (X, \Phi, \check{X}, \check{\Phi}) \to (\Lambda, \Phi, \mathbb{Z}\check{\Phi}, \check{\Phi}).$$

However such isogenies are clearly given by $(\varphi_{\mathrm{ad}}, \mathrm{id}, 1)$ and $(\varphi_{\mathrm{sc}}, \mathrm{id}, 1)$ where $\varphi_{\mathrm{ad}} : \mathbb{Z}\Phi \to X$ and $\varphi_{\mathrm{sc}} : X \to \Lambda$ are the natural inclusion maps, (note the latter exists because $\mathbb{R}X_\Phi = \mathbb{R}X$). Now if $\delta : \mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$ is any adjoint quotient then $\delta(Z(\mathbf{G})) \subseteq Z(\mathbf{G}_{\mathrm{ad}})$ and $Z(\mathbf{G}_{\mathrm{ad}})$ is trivial by Lemma 1.10 so $\operatorname{Ker}(\delta) = Z(\mathbf{G})$. ■

Example 1.14. Assume $\mathbf{G}$ is the adjoint group $\mathrm{PGL}_n(\mathbb{K})$ then there exists a simply connected cover $\mathrm{SL}_n(\mathbb{K}) \to \mathrm{PGL}_n(\mathbb{K})$.

Assume now that $\mathbf{G}$ is semisimple. To $\mathbf{G}$ and $\mathbf{G}^\star$ we fix simply connected covers $\delta_{\mathrm{sc}} : \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ and $\delta^\star_{\mathrm{sc}} : \mathbf{G}^\star_{\mathrm{ad}} \to \mathbf{G}^\star$ and adjoint quotients $\delta_{\mathrm{ad}} : \mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$ and $\delta^\star_{\mathrm{ad}} : \mathbf{G}^\star \to \mathbf{G}^\star_{\mathrm{sc}}$. Furthermore we will assume that δ_{sc}, $\delta^\star_{\mathrm{sc}}$, δ_{ad} and $\delta^\star_{\mathrm{ad}}$ are the unique isogenies satisfying Eq. (1.2). The composition of these isogenies $\delta_{\mathrm{ad}} \circ \delta_{\mathrm{sc}}$ and $\delta^\star_{\mathrm{ad}} \circ \delta^\star_{\mathrm{sc}}$ give us simply connected coverings of the adjoint groups $\mathbf{G}_{\mathrm{ad}}$ and $\mathbf{G}^\star_{\mathrm{sc}}$. By Lemma 1.13 we have $\mathrm{Ker}(\delta_{\mathrm{ad}} \circ \delta_{\mathrm{sc}}) = Z(\mathbf{G}_{\mathrm{sc}})$, $\mathrm{Ker}(\delta^\star_{\mathrm{ad}} \circ \delta^\star_{\mathrm{sc}}) = Z(\mathbf{G}^\star_{\mathrm{ad}})$, $\mathrm{Ker}(\delta^\star_{\mathrm{ad}}) = Z(\mathbf{G}^\star)$ and $\mathrm{Ker}(\delta_{\mathrm{ad}}) = Z(\mathbf{G})$. Let $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_k$ be the decomposition of Φ into disjoint irreducible subsystems. If none of the Φ_i, with $1 \leqslant i \leqslant k$, are of type B_n or C_n then the root systems of $\mathbf{G}$ and $\mathbf{G}^\star$ are isomorphic, (see [**Bou02**, Ch. VI - §4 - no. 2 - Remark (2)]). If this is the case then we will have $\mathbf{G}^\star_{\mathrm{ad}}$ and $\mathbf{G}_{\mathrm{sc}}$, (resp. $\mathbf{G}^\star_{\mathrm{sc}}$ and $\mathbf{G}_{\mathrm{ad}}$), are isomorphic as they are simply connected, (resp. adjoint), groups of the same type. Therefore we may and will take $\mathbf{G}^\star_{\mathrm{ad}} = \mathbf{G}_{\mathrm{sc}}$ and $\mathbf{G}^\star_{\mathrm{sc}} = \mathbf{G}_{\mathrm{ad}}$. The isogenies of root data induced by $\delta_{\mathrm{ad}} \circ \delta_{\mathrm{sc}}$ and $\delta^\star_{\mathrm{ad}} \circ \delta^\star_{\mathrm{sc}}$ are the same as they are both adjoint quotients. As the isogenies δ_{sc}, $\delta^\star_{\mathrm{sc}}$, δ_{ad} and $\delta^\star_{\mathrm{ad}}$ were chosen to satisfy Eq. (1.2) we will have the compositions $\delta_{\mathrm{ad}} \circ \delta_{\mathrm{sc}}$ and $\delta^\star_{\mathrm{ad}} \circ \delta^\star_{\mathrm{sc}}$ both satisfy Eq. (1.2) so we must have $\delta_{\mathrm{ad}} \circ \delta_{\mathrm{sc}} = \delta^\star_{\mathrm{ad}} \circ \delta^\star_{\mathrm{sc}}$ as such an isogeny is unique.

Let us fix maximal tori and Borel subgroups $\mathbf{T}_{\mathrm{sc}} \leqslant \mathbf{B}_{\mathrm{sc}}$ of $\mathbf{G}_{\mathrm{sc}}$ then we may assume that $\mathbf{T}_0 \leqslant \mathbf{B}_0$ are the images of $\mathbf{T}_{\mathrm{sc}} \leqslant \mathbf{B}_{\mathrm{sc}}$ under the isogeny δ_{sc}. Furthermore we may define a maximal torus and Borel subgroup $\mathbf{T}_{\mathrm{ad}} \leqslant \mathbf{B}_{\mathrm{ad}}$ of $\mathbf{G}_{\mathrm{ad}}$ by letting these be the images of $\mathbf{T}_{\mathrm{sc}} \leqslant \mathbf{B}_{\mathrm{sc}}$ under the isogeny $\delta_{\mathrm{ad}} \circ \delta_{\mathrm{sc}}$. We now fix $\mathbf{T}^\star_{\mathrm{ad}}$ to be a maximal torus of $\mathbf{G}^\star_{\mathrm{ad}}$ and $\mathbf{B}^\star_{\mathrm{ad}}$ to be a Borel subgroup of $\mathbf{G}^\star_{\mathrm{ad}}$ containing $\mathbf{T}^\star_{\mathrm{ad}}$. We define $\mathbf{T}^\star_0 \leqslant \mathbf{B}^\star_0$ and $\mathbf{T}^\star_{\mathrm{sc}} \leqslant \mathbf{B}^\star_{\mathrm{sc}}$ to be the images of $\mathbf{T}^\star_{\mathrm{ad}} \leqslant \mathbf{B}^\star_{\mathrm{ad}}$ under the respective isogenies $\delta^\star_{\mathrm{sc}}$ and $\delta^\star_{\mathrm{ad}} \circ \delta^\star_{\mathrm{sc}}$. If none of the Φ_i, with $1 \leqslant i \leqslant k$, are of type B_n or C_n then we will assume $\mathbf{T}_{\mathrm{sc}} = \mathbf{T}^\star_{\mathrm{ad}}$ and $\mathbf{B}_{\mathrm{sc}} = \mathbf{B}^\star_{\mathrm{ad}}$.

By composing with the appropriate isogenies we obtain inclusion maps for the character and cocharacter groups. Identifying the character and cocharacter groups with their images in $X(\mathbf{T}_{\mathrm{sc}})$ and $\check{X}(\mathbf{T}_{\mathrm{ad}})$ we have the following inclusions

$$\begin{aligned} X(\mathbf{T}_{\mathrm{ad}}) \subseteq X(\mathbf{T}_0) \subseteq X(\mathbf{T}_{\mathrm{sc}}) &\qquad \check{X}(\mathbf{T}_{\mathrm{sc}}) \subseteq \check{X}(\mathbf{T}_0) \subseteq \check{X}(\mathbf{T}_{\mathrm{ad}}), \\ X(\mathbf{T}^\star_{\mathrm{sc}}) \subseteq X(\mathbf{T}^\star_0) \subseteq X(\mathbf{T}^\star_{\mathrm{ad}}) &\qquad \check{X}(\mathbf{T}^\star_{\mathrm{ad}}) \subseteq \check{X}(\mathbf{T}^\star_0) \subseteq \check{X}(\mathbf{T}^\star_{\mathrm{sc}}). \end{aligned}$$

Under these inclusions we obtain bijections between the sets of roots and coroots so we will denote by Φ, $\check{\Phi}$ the common set of roots and coroots of $\mathbf{G}_{\mathrm{ad}}$, $\mathbf{G}$ and $\mathbf{G}_{\mathrm{sc}}$; also we write $\Phi^\star$, $\check{\Phi}^\star$ for the common set of roots and coroots of $\mathbf{G}^\star_{\mathrm{sc}}$, $\mathbf{G}^\star$ and $\mathbf{G}^\star_{\mathrm{ad}}$. If none of the Φ_i, with $1 \leqslant i \leqslant k$, are of type B_n or C_n then we will have $X(\mathbf{T}_{\mathrm{ad}}) = X(\mathbf{T}^\star_{\mathrm{sc}})$ and $\check{X}(\mathbf{T}_{\mathrm{ad}}) = \check{X}(\mathbf{T}^\star_{\mathrm{sc}})$, which means we also have $\Phi = \Phi^\star$ and $\check{\Phi} = \check{\Phi}^\star$. The Weyl groups of $\mathbf{G}_{\mathrm{ad}}$ and $\mathbf{G}_{\mathrm{sc}}$ may be identified with $\mathbf{W}$ through the above isogenies. Similarly we will identify the Weyl groups of $\mathbf{G}^\star_{\mathrm{sc}}$ and $\mathbf{G}^\star_{\mathrm{ad}}$ with $\mathbf{W}^\star$ through the above isogenies.

Assume now that $\mathbf{G}$ is reductive and let $\mathbf{T}_0' \leqslant \mathbf{B}_0'$ be the maximal torus and Borel subgroup of the semisimple derived subgroup $\mathbf{G}'$ prescribed by the above process. We then assume that our fixed maximal torus and Borel subgroup of $\mathbf{G}$ are chosen such that $\mathbf{T}_0 = \mathbf{T}_0' \cdot Z(\mathbf{G})^\circ$ and $\mathbf{B}_0 = \mathbf{B}_0' \cdot Z(\mathbf{G})^\circ$. With these choices we have the images of $\mathbf{T}_0 \leqslant \mathbf{B}_0$ under the surjective homomorphism $\delta_{\mathrm{ad}} : \mathbf{G} \to \mathbf{G}/Z(\mathbf{G})^\circ \to \mathbf{G}_{\mathrm{ad}}$ are $\mathbf{T}_{\mathrm{ad}} \leqslant \mathbf{B}_{\mathrm{ad}}$, (where the latter homomorphism is just the adjoint quotient of the semisimple group $\mathbf{G}/Z(\mathbf{G})^\circ$). Thus we again obtain inclusions $X(\mathbf{T}_{\mathrm{ad}}) \subseteq X(\mathbf{T}_0)$ and $\check{X}(\mathbf{T}_{\mathrm{sc}}) \subseteq \check{X}(\mathbf{T}_0)$, hence also bijections between the sets of roots and coroots of $\mathbf{G}$ and $\mathbf{G}_{\mathrm{ad}}$. We may also identify the Weyl group of $\mathbf{G}$ with that of $\mathbf{G}_{\mathrm{ad}}$ through the isotypic homomorphism δ_{ad}.

1.2.1 Conventions For Simple Groups of Type D_n

When dealing with the simple algebraic groups of type D_n, with $n \geqslant 4$, we must be quite careful. In this section we assume $\mathbf{G}$ is such a group. The notational conventions we develop in this section for such groups shall be maintained throughout. We start by considering precisely the structure of the fundamental group Π. From [**Bou02**, Plate IV(VIII)] we have the fundamental group is $\Pi = \{\mathbb{Z}\Phi, \varpi_1 + \mathbb{Z}\Phi, \varpi_{n-1} + \mathbb{Z}\Phi, \varpi_n + \mathbb{Z}\Phi\}$, where these elements satisfy the relations

$$2\varpi_{n-1} + \mathbb{Z}\Phi = 2\varpi_n + \mathbb{Z}\Phi = \mathbb{Z}\Phi \qquad \varpi_{n-1} + \varpi_n + \mathbb{Z}\Phi = \varpi_1 + \mathbb{Z}\Phi$$

if n is even and

$$2\varpi_n + \mathbb{Z}\Phi = \varpi_1 + \mathbb{Z}\Phi \qquad 3\varpi_n + \mathbb{Z}\Phi = \varpi_{n-1} + \mathbb{Z}\Phi \qquad 4\varpi_n + \mathbb{Z}\Phi = \mathbb{Z}\Phi$$

if n is odd.

In [**Car81**, §7] Carter describes explicitly how the images of X in Π determine $\mathbf{G}$ up to isomorphism, (note that Carter's notation for the root system of type D_n is consistent with ours). If $n \equiv 1 \pmod 2$ then there is a unique proper subgroup of Π of order 2, namely the subgroup generated by $\varpi_1 + \mathbb{Z}\Phi$. If the image of X in Π is this subgroup then $\mathbf{G}$ is a special orthogonal group $\mathrm{SO}_{2n}(\mathbb{K})$. Assume now that $n \equiv 0 \pmod 2$ then there are three proper subgroups of Π of order 2, namely those generated by the elements $\varpi_1 + \mathbb{Z}\Phi$, $\varpi_{n-1} + \mathbb{Z}\Phi$ and $\varpi_n + \mathbb{Z}\Phi$. If the image of X is $\langle \varpi_1 + \mathbb{Z}\Phi \rangle$ then again $\mathbf{G}$ is a special orthogonal group $\mathrm{SO}_{2n}(\mathbb{K})$. If the image of X is either $\langle \varpi_{n-1} + \mathbb{Z}\Phi \rangle$ or $\langle \varpi_n + \mathbb{Z}\Phi \rangle$ then $\mathbf{G}$ is a half spin group $\mathrm{HSpin}_{2n}(\mathbb{K})$.

The problem arises here in the choice over the root datum of a half spin group. Note these groups are isomorphic because there exists an isomorphism of their root data which exchanges the weights ω_{n-1} and ω_n; this comes from the graph automorphism of order 2 on the Dynkin diagram of type D_n. We will now fix a choice of half spin group but it will be clear, because of this isomorphism, that the results we prove do not depend upon this choice.

> If $\mathbf{G}$ is a half spin group then we assume the image of X in the fundamental group Π is $\langle \varpi_n + \mathbb{Z}\Phi \rangle$.

Let us assume now that $\mathbf{G}$ is a half spin group and fix a basis $\{\chi_1, \dots, \chi_n\}$ of $\mathbb{R}X$ such that $\chi_1 = \varpi_n$ and $\chi_i = \alpha_i$ for $2 \leqslant i \leqslant n$. We write A for the change of basis matrix of $\mathbb{R}X$ sending the simple roots Δ to $\{\chi_1, \dots, \chi_n\}$. Such a matrix has the form

$$A = \left[\begin{array}{c|ccc} a_1 & a_2 & \cdots & a_n \\ \hline 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \end{array}\right],$$

where $(a_1, \dots, a_n) = (\frac{1}{2}, 1, \frac{3}{2}, 2, \dots, \frac{(n-2)}{2}, \frac{(n-2)}{4}, \frac{n}{4})$ and I_{n-1} is the $(n-1) \times (n-1)$ identity matrix.

To our chosen basis $\{\chi_1, \dots, \chi_n\}$ of $\mathbb{R}X$ we have a dual basis $\{\gamma_1, \dots, \gamma_n\}$ of $\mathbb{R}\check{X}$. These are dual in the sense that they satisfy the condition $\langle \chi_i, \gamma_j \rangle = \delta_{ij}$ for all $1 \leqslant i, j \leqslant n$. Let B be the change of basis matrix of $\mathbb{R}\check{X}$ sending the simple coroots $\check{\Delta}$ to $\{\gamma_1, \dots, \gamma_n\}$. For each $1 \leqslant i, j \leqslant n$ we may express χ_i and γ_j as

$$\chi_i = \sum_{k=1}^{n} A_{ik}\alpha_k \qquad \text{and} \qquad \gamma_j = \sum_{\ell=1}^{n} B_{j\ell}\check{\alpha}_\ell,$$

where A_{ik} denotes the entry of A in the ith row and kth column and $B_{j\ell}$ denotes the entry of B in the jth row and ℓth column. Substituting these expressions into the relation $\langle \chi_i, \gamma_j \rangle = \delta_{ij}$ we obtain

$$\delta_{ij} = \langle \sum_{k=1}^{n} A_{ik}\alpha_k, \sum_{\ell=1}^{n} B_{j\ell}\check{\alpha}_\ell \rangle = \sum_{k=1}^{n}\sum_{\ell=1}^{n} A_{ik}\langle \alpha_k, \check{\alpha}_\ell \rangle B_{j\ell} \tag{1.3}$$

In particular Eq. (1.3) shows that the matrices A and B satisfy the condition $ACB^T = I_n$ where $C = (\langle \alpha_i, \check{\alpha}_j \rangle)_{1 \leqslant i,j \leqslant n}$ is the *Cartan matrix* and I_n is the $n \times n$ identity matrix.

Let us now consider the root datum of the associated dual group $\mathbf{G}^\star$. If X has image $\langle \varpi_n + \mathbb{Z}\Phi \rangle$ in Π then for X to be isomorphic to $\check{X}^\star$, (and for such an isomorphism to preserve the pairing $\langle -, - \rangle$), we must have $\check{X}^\star$ has image $\langle \check{\varpi}_n + \mathbb{Z}\check{\Phi} \rangle$ in $\check{\Pi}$. Assume X is as above then we can easily calculate the image of $\check{X}$ in $\check{\Pi}$ as follows. Recall that the matrix B expresses the decomposition of the basis of $\mathbb{R}\check{X}$ in terms of the simple coroots. The Cartan matrix expresses the decomposition of the simple coroots in terms of the fundamental dominant coweights hence $BC = A^{-T}$ gives the decomposition of $\{\gamma_1, \dots, \gamma_n\}$ in terms of $\{\check{\omega}_1, \dots, \check{\omega}_n\}$. We easily determine that this matrix has the form

$$A^{-T} = \left[\begin{array}{c|ccc} a'_1 & 0 & \cdots & 0 \\ \hline a'_2 & & & \\ \vdots & & I_{n-1} & \\ a'_n & & & \end{array}\right],$$

where $(a'_1, \dots, a'_n) = (2, -2, -3, \dots, -(n-2), -\frac{(n-2)}{2}, -\frac{n}{2})$ and I_{n-1} is the $(n-1) \times (n-1)$ identity matrix.

The image of $\check{X}$ in $\check{\Pi}$ is determined by the image of γ_n in $\check{\Pi}$, i.e. the element $\frac{n}{2}\check{\omega}_1 + \check{\omega}_n + \mathbb{Z}\check{\Phi}$. Therefore we have the image of $\check{X}$ in $\check{\Pi}$ is $\langle \check{\omega}_n + \mathbb{Z}\check{\Phi} \rangle$ if $n \equiv 0 \pmod 4$ and $\langle \check{\omega}_{n-1} + \mathbb{Z}\check{\Phi} \rangle$ if $n \equiv 2 \pmod 4$. In particular, using the dual isomorphism $\check{X} \to X^\star$, we must have the image of $X^\star$ in Π is $\langle \varpi_n + \mathbb{Z}\Phi \rangle$ if $n \equiv 0 \pmod 4$ and $\langle \varpi_{n-1} + \mathbb{Z}\Phi \rangle$ if $n \equiv 2 \pmod 4$. As a consequence we see that the dual group of a half spin group is again isomorphic to a half spin group but its root datum depends upon n. Recall that if $p \neq 2$, by Lemma 1.10, we have an isomorphism $\check{\Pi} \cong Z(\mathbf{G}_{\mathrm{sc}})$. We now make the following convention regardless of the congruence of $n \pmod 2$.

Assume $\mathbf{G}$ is a simply connected group of type D_n, (and $p \neq 2$), then we denote the centre of $\mathbf{G}$ by $Z(\mathbf{G}) = \{1, \hat{z}_1, \hat{z}_{n-1}, \hat{z}_n\}$. We fix the notation such that $\check{\omega}_{n-1} + \mathbb{Z}\check{\Phi} \mapsto \hat{z}_{n-1}$ and $\check{\omega}_n + \mathbb{Z}\check{\Phi} \mapsto \hat{z}_n$ under the isomorphism given to us by Lemma 1.10. Note that the structure of the group $Z(\mathbf{G})$ will depend upon n.

Under this convention whenever $\mathbf{G}$ is a half spin group we will have $\mathrm{Ker}(\delta_{\mathrm{sc}}^\star) = \langle \hat{z}_n \rangle$, however $\mathrm{Ker}(\delta_{\mathrm{sc}})$ will be $\langle \hat{z}_n \rangle$ if $n \equiv 0 \pmod 4$ and $\langle \hat{z}_{n-1} \rangle$ if $n \equiv 2 \pmod 4$. Furthermore whenever $\mathbf{G}$ is a special orthogonal group we will have $\mathrm{Ker}(\delta_{\mathrm{sc}}^\star) = \mathrm{Ker}(\delta_{\mathrm{sc}}) = \langle \hat{z}_1 \rangle$.

1.3 Endomorphisms of Algebraic Groups

We return to the assumption that $\mathbf{G}$ is a connected reductive algebraic group. In this section we wish to discuss certain endomorphisms of $\mathbf{G}$. Our focus will be on generalised Frobenius endomorphisms and the group $\mathrm{Aut}(\mathbf{G})$.

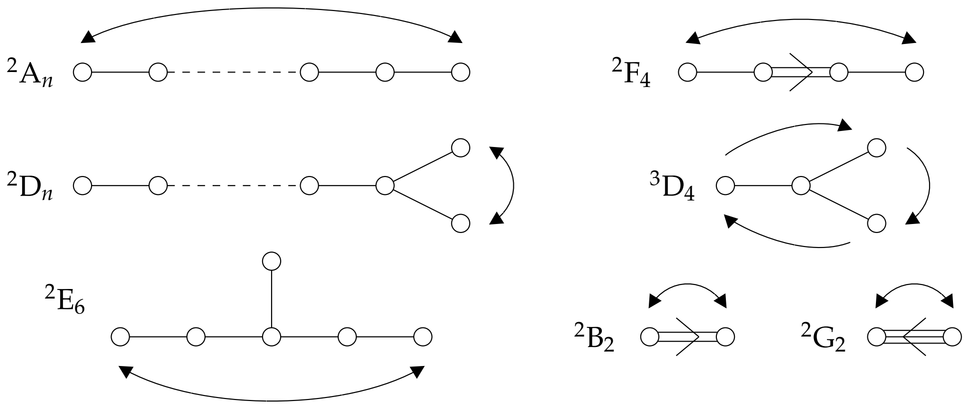


Figure 1.2: The Possible Graph Automorphisms of Dynkin Diagrams.

1.3.1 Automorphisms of G

Let γ be an automorphism of $\mathbf{G}$. Recall $\mathbf{G}$ is an almost direct product $\mathbf{G}'Z(\mathbf{G})^\circ$ where $\mathbf{G}'$ is the derived subgroup of $\mathbf{G}$. Clearly γ stabilises the subgroups $\mathbf{G}'$ and $Z(\mathbf{G})$ hence γ stabilises $Z(\mathbf{G})^\circ$. Therefore we can understand γ by its induced automorphisms of $\mathbf{G}'$ and $Z(\mathbf{G})^\circ$. The group $Z(\mathbf{G})^\circ$ is a torus and the automorphisms of such a group are given by the following result.

Lemma 1.15. *Assume* $\mathbf{T}$ *is a torus of rank n then* $\mathrm{Aut}(\mathbf{T})$ *is isomorphic to* $\mathrm{GL}_n(\mathbb{Z})$.

Proof. Firstly given an automorphism $\varphi \in \mathrm{Aut}(\mathbf{T})$ we have a unique induced automorphism $\varphi^* \in \mathrm{Aut}(\check{X}(\mathbf{T}))$ where $\check{X}(\mathbf{T}) = \mathrm{Hom}(\mathbb{K}^\times, \mathbf{T})$ and $\varphi^*(\gamma) = \varphi \circ \gamma$ for all $\gamma \in \check{X}(\mathbf{T})$. The map $\varphi \mapsto \varphi^*$ is then an injective homomorphism $\mathrm{Aut}(\mathbf{T}) \to \mathrm{Aut}(\check{X}(\mathbf{T}))$. By [**Car93**, Proposition 3.1.2(ii)] we have an isomorphism $\mathbb{K}^\times \otimes_{\mathbb{Z}} \check{X}(\mathbf{T}) \to \mathbf{T}$ given by $k \otimes \gamma \mapsto \gamma(k)$. Given any automorphism $\varphi \in \mathrm{Aut}(\check{X}(\mathbf{T}))$ we can extend this to an automorphism $\tilde{\varphi}$ of the tensor product $\mathbb{K}^\times \otimes_{\mathbb{Z}} \check{X}(\mathbf{T})$ by setting $\tilde{\varphi}(k \otimes \gamma) = k \otimes \varphi(\gamma)$ and extending linearly. Through the isomorphism we have $\tilde{\varphi}$ determines an element of $\mathrm{Aut}(\mathbf{T})$ and we have the induced automorphism $\tilde{\varphi}^*$ is precisely φ. In particular the map $\mathrm{Aut}(\mathbf{T}) \to \mathrm{Aut}(\check{X}(\mathbf{T}))$ is surjective so is an isomorphism. Now $\check{X}(\mathbf{T})$ is isomorphic to $\mathbb{Z}^n$ so we have $\mathrm{Aut}(\check{X}(\mathbf{T})) \cong \mathrm{Aut}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Z})$ as required. ∎

For instance the automorphism group of $\mathbb{K}^\times$ is $\mathrm{GL}_1(\mathbb{Z})$ which is simply a cyclic group of order two. The two automorphisms of $\mathbb{K}^\times$ are the trivial automorphism and the automorphism given by $x \mapsto x^{-1}$.

Let us now assume that $\mathbf{G}$ is semisimple. If $\gamma \in \mathrm{Aut}(\mathbf{G})$ is an automorphism then $\gamma(\mathbf{B}_0)$ is a Borel subgroup of $\mathbf{G}$. By [**Hum75**, §21.3 - Theorem] all Borel subgroups of $\mathbf{G}$ are conjugate so there exists an element $g \in \mathbf{G}$ such that $\mathrm{inn}\, g \circ \gamma$ stabilises $\mathbf{B}_0$. Similarly the two maximal tori $\mathbf{T}_0$ and $(\mathrm{inn}\, g \circ \gamma)(\mathbf{T}_0)$ are conjugate by an element $b \in \mathbf{B}_0$ so $\mathrm{inn}\, bg \circ \gamma$ stabilises both $\mathbf{B}_0$ and $\mathbf{T}_0$. In particular this shows that for any automorphism $\gamma \in \mathrm{Aut}(\mathbf{G})$ there exists $h \in \mathbf{G}$ and $\beta \in \mathrm{Aut}(\mathbf{G})$ such that $\gamma = \mathrm{inn}\, h \circ \beta$ and β stabilises both $\mathbf{B}_0$ and $\mathbf{T}_0$.

Let us assume now that γ is an automorphism stabilising both $\mathbf{B}_0$ and $\mathbf{T}_0$. By Lemma 1.8 γ induces an automorphism of root data of the form $(\varphi, b, 1)$. By Remark 1.12 there exists a unique automorphism $\beta \in \mathrm{Aut}(\mathbf{G})$ satisfying $\beta \circ x_\alpha = x_{b(\alpha)}$ for all $\alpha \in \Delta$ and by Theorem 1.9 there exists $t \in \mathbf{T}_0$ such that $\gamma = \mathrm{inn}\, t \circ \beta$. We call β a *graph automorphism* of $\mathbf{G}$. As there are only finitely many bijections $\Delta \to \Delta$ we see the group $\mathrm{Out}(\mathbf{G})$ is finite. We have thus shown the following classical result.

Theorem 1.16. *Assume* $\mathbf{G}$ *is semisimple then every automorphism* $\gamma \in \mathrm{Aut}(\mathbf{G})$ *can be expressed as a composition* $\mathrm{inn}\, g \circ \beta$*, where* β *is a graph automorphism of* $\mathbf{G}$.

Remark 1.17. Assume Φ is irreducible then $\mathbf{G}$ has a non-trivial graph automorphism only when $\mathbf{G}$ is of type A_n, D_n or E_6. The graph automorphisms in these cases are those given in Figure 1.2. Note that the diagram automorphisms of B_2, F_4 and G_2 do not come from automorphisms of $\mathbf{G}$ but from abstract automorphisms of $\mathbf{G}$, (see the discussion in [**Lie98**, §1]).

1.3.2 Generalised Frobenius Endomorphisms

We now fix a *generalised Frobenius endomorphism* $F : \mathbf{G} \to \mathbf{G}$. This is an isogeny of $\mathbf{G}$ such that for some $\delta \in \mathbb{N}$ we have F^δ is a *Frobenius endomorphism* giving $\mathbf{G}$ an $\mathbb{F}_{p^e}$-rational structure, (see [**Gec03**, Definition 4.1.1]). We define q to be $p^{e/\delta}$, this means that q is not necessarily a natural number and is in general a real number. Note that there is a unique minimal choice of δ such that F^δ is a Frobenius endomorphism.

By [**Gec03**, Example 4.3.3] we may and will assume that $\mathbf{T}_0$ and $\mathbf{B}_0$ are F-stable, i.e. $F(\mathbf{T}_0) = \mathbf{T}_0$ and $F(\mathbf{B}_0) = \mathbf{B}_0$. This is largely a consequence of the following fundamental result.

Theorem 1.18 (Lang–Steinberg, [Gec03, Theorem 4.1.12]). *Assume* $\mathbf{G}$ *is a connected reductive algebraic group and* $F : \mathbf{G} \to \mathbf{G}$ *is a generalised Frobenius endomorphism. The morphism* $\mathscr{L} : \mathbf{G} \to \mathbf{G}$ *given by* $\mathscr{L}(g) = g^{-1}F(g)$ *is surjective.*

We will refer to Theorem 1.18 as the Lang–Steinberg theorem and the map $\mathscr{L}$ as the *Lang–Steinberg map*. The isogeny F induces an isogeny of root data (φ, b, q) such that $b(\Phi^+) = \Phi^+$, hence $b(\Delta) = \Delta$. By Lemma 1.8 the isogeny F induces an automorphism of $\mathbf{W}$ stabilising $\mathbb{S}$, which we again denote by F.

The fixed point group $G = \mathbf{G}^F = \{g \in \mathbf{G} \mid F(g) = g\}$ is a finite subgroup of $\mathbf{G}$ which we call a *finite reductive group*. Given F we fix a generalised Frobenius endomorphism $F^\star : \mathbf{G}^\star \to \mathbf{G}^\star$ such that the triples $(\mathbf{G}, \mathbf{T}_0, F)$ and $(\mathbf{G}^\star, \mathbf{T}_0^\star, F^\star)$ are *in duality*, which means $\mathbf{T}_0^\star$ and $\mathbf{B}_0^\star$ are $F^\star$-stable and our isomorphism $\varphi : X(\mathbf{T}_0) \to \check{X}(\mathbf{T}_0^\star)$ can be chosen such that $\varphi \circ F = F^\star \circ \varphi$. We denote the associated fixed point group by $G^\star = \mathbf{G}^{\star F^\star}$.

Remark 1.19. Let q and $q^\star$ be the real numbers uniquely determined by F and $F^\star$, then $q = q^\star$ by [**Car93**, Proposition 4.4.3].

Example 1.20. We consider now the possibilities for F under the assumption that $\mathbf{G}$ is simple. By [**Ste68a**, 7.1(a)] we have F is an abstract automorphism of $\mathbf{G}$ and such automorphisms have been completely described by Steinberg in [**Ste68b**, Theorem 30 - pg. 158]. Let $r = p^a$ be a power of p for some $a \in \mathbb{N}$ then we define a corresponding isogeny $F_r : \mathbf{G} \to \mathbf{G}$ given by $F_r(x_\alpha(c)) = x_\alpha(c^r)$ for all $\alpha \in \Phi$ and $c \in \mathbb{K}^+$.

Assume we are in one of the following cases: $p = 2$ and $\mathbf{G}$ is of type B_2 or F_4, or $p = 3$ and $\mathbf{G}$ is of type G_2. In this situation we define an isogeny $\tau : \mathbf{G} \to \mathbf{G}$ such that

$$\tau(x_\alpha(c)) = x_{b(\alpha)}(\varepsilon_\alpha c^{q(\alpha)})$$

for all $\alpha \in \Phi$ and $c \in \mathbb{K}^+$. Here we have:

- $b : \Phi \to \Phi$ is the bijection arising from the bijection $b : \Delta \to \Delta$ corresponding to the diagram automorphism listed in Figure 1.2.
- $q(\alpha) = 1$ if α is a long root and $q(\alpha) = p$ if α is a short root.
- $\varepsilon_\alpha \in \{\pm 1\}$ for all $\alpha \in \Phi$ and $\varepsilon_\alpha = 1$ if $\pm\alpha \in \Delta$.

We call the isogeny τ an *exceptional graph automorphism* even though τ is only an abstract automorphism of $\mathbf{G}$. With this notation in place we now have F can be expressed as a composition $\operatorname{inn} g \circ F_r \circ \tau$ for some $g \in \mathbf{G}$, $r = p^a$ and τ a graph automorphism or exceptional graph automorphism. In fact, as F stabilises $\mathbf{T}_0$ and $\mathbf{B}_0$, we have $g \in N_{\mathbf{G}}(\mathbf{T}_0) \cap N_{\mathbf{G}}(\mathbf{B}_0) = \mathbf{T}_0$.

Assume $F' : \mathbf{G} \to \mathbf{G}$ is another generalised Frobenius endomorphism such that $F = \operatorname{inn} t \circ F'$ for some $t \in \mathbf{T}_0$ then $\mathbf{G}^F$ and $\mathbf{G}^{F'}$ are isomorphic as finite groups. For this reason we may assume that F is of the form $F_r \circ \tau$. Note by [**DM91**, Proposition 3.6(i)] when τ is a graph automorphism we have $F_r \circ \tau$ is a Frobenius endomorphism. If τ acts trivially on the root system of $\mathbf{G}$ then the resulting finite group G is called a *Chevalley group*. If τ is a non-trivial graph automorphism of $\mathbf{G}$ then the resulting finite group is called a *twisted group*.

Notation. If $\mathbf{G}$ is a closed subgroup of $\mathrm{GL}_n(\mathbb{K})$ then we will assume F_r denotes the restriction to $\mathbf{G}$ of the Frobenius endomorphism $F_r : \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K})$ given by $F_r(x_{ij}) = (x_{ij}^r)$. If $\mathbf{G}$ is one of the classical groups $\mathrm{SL}_n(\mathbb{K})$, $\mathrm{SO}_n(\mathbb{K})$ or $\mathrm{Sp}_n(\mathbb{K})$ with n even, as defined in [**Gec03**, Example 1.3.15], then $F_r(\mathbf{G}) \subseteq \mathbf{G}$ and the restriction of F_r to

G is a Frobenius endomorphism by [**Gec03**, Corollary 4.1.5]. As the centre of $\mathrm{GL}_n(\mathbb{K})$ is preserved by F_r we have F_r gives a Frobenius endomorphism of the quotient group $\mathrm{PGL}_n(\mathbb{K})$, which we again denote by F_r. By [**Gec03**, Proposition 3.4.6(b)] and [**Gec03**, §1.7.9] we may take $\mathbf{B}_0$, (resp. $\mathbf{T}_0$), to be the intersection of the subgroup of upper triangular matrices, (resp. diagonal matrices), in $\mathrm{GL}_n(\mathbb{K})$ with **G**. It is easily checked that F_r then has the desired effect on the root subgroups of any of the simple groups we have just considered.

Assume **G** is of type B_2, G_2 or F_4 and F induces the associated exceptional graph automorphism. In these cases the value q associated to F is not a natural number but a power of $p^{\frac{1}{2}}$ as F^2 is the smallest power of F which is a Frobenius endomorphism. The groups ${}^2\mathrm{F}_4(q^2)$ and ${}^2\mathrm{G}_2(q^2)$ are called the *Ree groups* and the groups ${}^2\mathrm{B}_2(q^2)$ are called the *Suzuki groups*. Note that there is also a finite sporadic simple group called the Suzuki group.

Remark 1.21. We must be somewhat careful when we consider the generalised Frobenius endomorphism $F_r \circ \tau$. Let $x \in \mathbf{G}$ then it can happen that $\tau(x) \neq x$ and $F_r(x) \neq x$ but $(F_r \circ \tau)(x) = x$. For example one sees this when **G** is the simply connected group of type E_6. Let $Z(\mathbf{G}) = \langle z \rangle \cong \mathbb{Z}_3$ then if τ is the graph automorphism of order 2 we have τ exchanges the elements z and z^2. However if $r \equiv 2 \pmod 3$ then we also have F_r exchanges the elements z and z^2. Hence $F_r \circ \tau$ fixes all the central elements.

Let S be any subset of **G** then we say S is *F-stable* if $F(S) = S$. If S is F-stable then we have F induces a map $F : S \to S$. We say an element $x \in S$ is *F-fixed* if $F(x) = x$ and we write S^F for the set of all F-fixed elements of S.

Notation. Throughout we will adopt the convention that a bold face character such as **G** denotes an algebraic group and the corresponding italicised character such as G denotes the associated fixed point group under a chosen endomorphism F. For instance we denote $\mathbf{T}_0^F$, $\mathbf{B}_0^F$ and $\mathbf{U}_0^F$ by T_0, B_0 and U_0 respectively.

Let us assume **G** is semisimple. We will need to understand the relationship between the isogenies discussed in Section 1.2 and the generalised Frobenius endomorphisms F and $F^\star$. Firstly let us recall the following definition.

Definition 1.22. Let $i = 1, 2$, we consider $\mathbf{G}_i$ a connected reductive algebraic group and $F_i : \mathbf{G}_i \to \mathbf{G}_i$ a generalised Frobenius endomorphism of $\mathbf{G}_i$. We assume the real number uniquely determined by F_1 and F_2 is the same and we call this common number q. Now let $S_i \subseteq \mathbf{G}_i$ be any subset such that $F_i(S_i) = S_i$ then we say a map $\varphi : S_1 \to S_2$ is *defined over* $\mathbb{F}_q$ if $\varphi \circ F_1 = F_2 \circ \varphi$.

Note that q is not necessarily a natural number so it does not make sense to speak of $\mathbb{F}_q$. However we consider this to be a generalisation of the definition "defined over $\mathbb{F}_q$"

for the case where F is a Frobenius endomorphism, (see for example [**Gec03**, Lemma 4.1.7]). Here $\mathbb{F}_q$ only denotes a finite field when q is a natural number. Throughout we will make the assumption that all isogenies are defined over $\mathbb{F}_q$.

If F is a generalised Frobenius endomorphism of $\mathbf{G}$ then by [**MT11**, Proposition 22.7] there exist generalised Frobenius endomorphisms F_{sc} of $\mathbf{G}_{\mathrm{sc}}$ and F_{ad} of $\mathbf{G}_{\mathrm{ad}}$ such that δ_{sc} and δ_{ad} are defined over $\mathbb{F}_q$. For notational convenience we will denote the generalised Frobenius endomorphisms F_{sc} and F_{ad} again by F. In a similar way we will denote by $F^\star$ a generalised Frobenius endomorphism of both $\mathbf{G}^\star_{\mathrm{sc}}$ and $\mathbf{G}^\star_{\mathrm{ad}}$ such that $\delta^\star_{\mathrm{sc}}$ and $\delta^\star_{\mathrm{ad}}$ are defined over $\mathbb{F}_q$. With this in mind we will have the triples $(\mathbf{G}_{\mathrm{sc}}, \mathbf{T}_{\mathrm{sc}}, F)$ and $(\mathbf{G}^\star_{\mathrm{sc}}, \mathbf{T}^\star_{\mathrm{sc}}, F^\star)$ will be in duality, similarly the triples $(\mathbf{G}_{\mathrm{ad}}, \mathbf{T}_{\mathrm{ad}}, F)$ and $(\mathbf{G}^\star_{\mathrm{ad}}, \mathbf{T}^\star_{\mathrm{ad}}, F^\star)$.

1.4 The Character Theory of Deligne and Lusztig

We wish to introduce certain class functions of the finite group G as were defined by Deligne and Lusztig. In general these class functions will not be characters but they are *virtual characters*, in other words a $\mathbb{Z}$-linear combination of irreducible characters of G. We will state some of the main results regarding these virtual characters without proof, however we will give references to proofs of all statements. All results in this section are due to either Deligne, Lusztig or Deligne and Lusztig.

Notation. Let H be a finite group then we denote the set of all $\overline{\mathbb{Q}}_\ell$-class functions by $\mathrm{Cent}(H)$, i.e. this is the $\overline{\mathbb{Q}}_\ell$-vector space $\{f : H \to \overline{\mathbb{Q}}_\ell \mid f(ghg^{-1}) = f(h) \text{ for all } g, h \in G\}$. We denote by $\mathrm{Irr}(H) \subset \mathrm{Cent}(H)$ the subset of all irreducible characters and by $\mathbb{Z}\,\mathrm{Irr}(H) \subseteq \mathrm{Cent}(H)$ the subset of all virtual characters, i.e. the $\mathbb{Z}$-span of $\mathrm{Irr}(H)$.

The vector space $\mathrm{Cent}(H)$ is an inner product space with respect to the product $\langle -, - \rangle_H : \mathrm{Cent}(H) \times \mathrm{Cent}(H) \to \overline{\mathbb{Q}}_\ell$ given by

$$\langle f, f' \rangle_H = \frac{1}{|H|} \sum_{h \in H} f(h)\overline{f'(h)},$$

for all $f, f' \in \mathrm{Cent}(H)$. We will often write $\langle -, - \rangle$ without the subscript if there is no confusion over the underlying finite group.

Recall that if H is any finite group and $K \leqslant H$ is a subgroup then we have two linear maps; the restriction map $\mathrm{Res}^H_K : \mathrm{Cent}(H) \to \mathrm{Cent}(K)$ and the induction map $\mathrm{Ind}^H_K : \mathrm{Cent}(K) \to \mathrm{Cent}(H)$. For $\chi \in \mathrm{Cent}(H)$ and $\psi \in \mathrm{Cent}(K)$ these are given by

$$(\mathrm{Res}^H_K)(\chi)(k) := \chi(k) \qquad \text{and} \qquad (\mathrm{Ind}^H_K)(\psi)(h) = \frac{1}{|K|} \sum_{x \in G} \dot{\psi}(xhx^{-1}),$$

for all $h \in H$ and $k \in K$, where $\dot{\psi}(y) = \psi(y)$ if $y \in K$ and 0 otherwise. Given any $x \in H$

and any class function $\psi \in \mathrm{Cent}(K)$ we define $\psi^x \in \mathrm{Cent}(K^x)$ and ${}^x\psi \in \mathrm{Cent}({}^xK)$ to be such that $\psi^x(k) = \psi({}^xk)$ and ${}^x\psi(k) = \psi(k^x)$ for all $k \in K$. Extended linearly this gives corresponding isomorphisms $\mathrm{Cent}(K) \to \mathrm{Cent}(K^x)$ and $\mathrm{Cent}(K) \to \mathrm{Cent}({}^xK)$ which are such that

$$\langle \psi, \chi \rangle_K = \langle \psi^x, \chi^x \rangle_{K^x} = \langle {}^x\psi, {}^x\chi \rangle_{{}^xK}$$

for all $\psi, \chi \in \mathrm{Cent}(K)$, (i.e. these isomorphisms are also isometries). If K is normal in H then this gives a left and right action of H on $\mathrm{Irr}(K)$. Note that as K fixes every character in $\mathrm{Irr}(K)$ we may consider these to be actions of the quotient group H/K without loss of generality.

Assume now that K is normal in H and there exists a subgroup $L \leqslant H$ such that $H = KL = LK$ and $K \cap L = \{1\}$ then we can consider the so called inflation map $\mathrm{Inf}_L^H : \mathrm{Cent}(L) \to \mathrm{Cent}(H)$, which is defined in the following way. Given $x \in H$ there exists a unique $k \in K$, $l \in L$ such that $x = kl$ therefore we define $(\mathrm{Inf}_L^H)(\psi)(x) = \psi(l)$. In other words we compose ψ with the natural projection map from $H \to L$.

Let $\mathbf{T}$ be any F-stable maximal torus of $\mathbf{G}$ and $\mathbf{B}$ any Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$. Recall we have a semidirect product decomposition $\mathbf{B} = \mathbf{UT}$, where $\mathbf{U}$ is the unipotent radical of the Borel. The set $\mathscr{L}^{-1}(\mathbf{U}) = \{g \in \mathbf{G} \mid g^{-1}F(g) \in \mathbf{U}\}$ is an algebraic subset of $\mathbf{G}$. As such this is an affine variety and a (G, T)-bimodule where the actions are given by left and right multiplication respectively. This is because for $g \in G$, $t \in T$ and for all $x \in \mathscr{L}^{-1}(\mathbf{U})$ we have

$$\begin{aligned}
\mathscr{L}(gx) &= x^{-1}g^{-1}F(g)F(x) = x^{-1}F(x) \in \mathbf{U}, \\
\mathscr{L}(xt) &= t^{-1}x^{-1}F(x)F(t) = t^{-1}x^{-1}F(x)t \in t^{-1}\mathbf{U}t = \mathbf{U}.
\end{aligned}$$

The direct product $G \times T$ acts on $\mathscr{L}^{-1}(\mathbf{U})$ as a finite group of automorphisms via the action $(g, t) \cdot x = gxt^{-1}$. These actions give us an induced action on the ith ℓ-adic cohomology group with compact support, which we denote by $H_c^i(\mathscr{L}^{-1}(\mathbf{U}), \overline{\mathbb{Q}}_\ell)$. Let $\theta \in \mathbb{Z}\,\mathrm{Irr}(T)$ be any virtual character then there exists a right T-submodule $H_c^i(\mathscr{L}^{-1}(\mathbf{U}), \overline{\mathbb{Q}}_\ell)_\theta$, (namely the θ-isotypic component), such that any $t \in T$ acts on $H_c^i(\mathscr{L}^{-1}(\mathbf{U}), \overline{\mathbb{Q}}_\ell)_\theta$ as multiplication by $\theta(t)$. This is then a left G-module.

Definition 1.23. Let $\mathbf{X}$ be any affine variety and H a finite group which acts on $\mathbf{X}$ as a group of automorphisms. We define

$$\mathcal{L}(h, \mathbf{X}) = \sum_{i \geqslant 0} (-1)^i \operatorname{tr}(h, H_c^i(\mathbf{X}, \overline{\mathbb{Q}}_\ell))$$

for any $h \in H$ to be the *Lefschetz number* of h acting on $\mathbf{X}$. If $\mathbf{T}$ is any F-stable maximal torus of $\mathbf{G}$ we define an associated *Deligne–Lusztig induction map* $R_{\mathbf{T}}^{\mathbf{G}} : \mathrm{Irr}(T) \to$

$\mathbb{Z}\,\mathrm{Irr}(G)$ such that for any character $\theta \in \mathrm{Irr}(T)$ we have

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) = \sum_{i\geqslant 0}(-1)^i \operatorname{tr}(g, H_c^i(\mathscr{L}^{-1}(\mathbf{U}), \overline{\mathbb{Q}}_\ell)_\theta),$$

for all $g \in G$. We call the virtual character $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ a *Deligne–Lusztig character* of G.

Remark 1.24. By linearity we can extend the map $R_{\mathbf{T}}^{\mathbf{G}}$ to give a map between $\mathrm{Cent}(T)$ and $\mathrm{Cent}(G)$.

At first sight the Deligne–Lusztig induction map $R_{\mathbf{T}}^{\mathbf{G}}$ appears to depend upon the choice of Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$. However it is a consequence of the Mackey formula that this is not the case, (see for example [**DM91**, Proposition 6.1]). The virtual characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ are a central tool in classifying the set $\mathrm{Irr}(G)$ of irreducible characters of G. We now give some of the most important properties regarding the virtual characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$.

Proposition 1.25 ([Car93, Theorem 7.2.4]**).** *Assume* $\mathbf{T}$ *is an* F*-stable maximal torus contained in an* F*-stable Borel subgroup* $\mathbf{B}$ *then* $R_{\mathbf{T}}^{\mathbf{G}} = \mathrm{Ind}_B^G \circ \mathrm{Inf}_T^B$.

Recall that if $\mathbf{B}$ is F-stable then B is a semidirect product $UT = TU$ so we can form the inflation map Inf_T^B. The map $\mathrm{Ind}_B^G \circ \mathrm{Inf}_T^B$ is known as *Harish-Chandra induction*. In particular, this proposition says that Deligne–Lusztig induction extends the notion of Harish-Chandra induction. For a recap on Harish-Chandra theory see [**DM91**, Chapter 6].

We would now like to consider a slightly better description for the virtual character $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$. In general the best one can achieve is the following mild rewording.

Proposition 1.26 (see [Car93, Proposition 7.2.3]**).** *For any* $g \in G$ *we have*

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) = \frac{1}{|T|}\sum_{t\in T}\theta(t^{-1})\mathcal{L}((g,t),\mathscr{L}^{-1}(\mathbf{U})).$$

However with this we can start to consider how much $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ depends upon the choice of $\mathbf{T}$ and θ. Before we do this we introduce the following notation.

Definition 1.27. Let $\nabla(\mathbf{G},F)$ denote the set of all pairs $(\mathbf{T},\theta)$ such that $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\theta \in \mathrm{Irr}(T)$. We say two pairs $(\mathbf{T}',\theta')$, $(\mathbf{T},\theta) \in \nabla(\mathbf{G},F)$ are *rationally conjugate* if there exists an element $x \in G$ such that $\mathbf{T}' = {}^x\mathbf{T}$ and $\theta' = {}^x\theta$. This defines an equivalence relation $\sim_G$ on $\nabla(\mathbf{G},F)$ and we denote the set of all equivalence classes by $\nabla(\mathbf{G},F)/G$.

Lemma 1.28. *If* $(\mathbf{T}',\theta') \sim_G (\mathbf{T},\theta)$ *then* $R_{\mathbf{T}'}^{\mathbf{G}}(\theta') = R_{\mathbf{T}}^{\mathbf{G}}(\theta)$.

Proof. Assume $x \in G$ is such that $\mathbf{T}' = {}^x\mathbf{T}$ and $\theta' = {}^x\theta$. Note that $\mathbf{T}'$ is contained in the Borel subgroup ${}^x\mathbf{B}$ with unipotent radical ${}^x\mathbf{U}$. Computing the Deligne–Lusztig character $R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ we see that for all $g \in G$ we have

$$\begin{aligned}
R_{\mathbf{T}'}^{\mathbf{G}}(\theta')(g) &= \frac{1}{|T'|}\sum_{t\in T'} \theta'(t^{-1})\mathcal{L}((g,t),\mathscr{L}^{-1}({}^x\mathbf{U})),\\
&= \frac{1}{|T|}\sum_{t\in T} \theta'({}^xt^{-1})\mathcal{L}((g,{}^xt),\mathscr{L}^{-1}({}^x\mathbf{U})),\\
&= \frac{1}{|T|}\sum_{t\in T} \theta(t^{-1})\mathcal{L}((g,t),\mathscr{L}^{-1}(\mathbf{U})),\\
&= R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g).
\end{aligned}$$

Note that the second equality holds because $T' = {}^xT$ and conjugate subgroups have the same order. Finally we note that the map $\varphi : \mathscr{L}^{-1}(\mathbf{U}) \to \mathscr{L}^{-1}({}^x\mathbf{U})$ given by $\varphi(h) = hx^{-1}$ is a bijective morphism of varieties satisfying $\varphi((g,t)\cdot h) = (g,{}^xt)\cdot\varphi(h)$ for all $(g,t) \in G \times T$, hence the third equality holds by [**Car93**, Property 7.1.5]. In particular $R_{\mathbf{T}'}^{\mathbf{G}}(\theta') = R_{\mathbf{T}}^{\mathbf{G}}(\theta)$, as required. ∎

Shortly we will see that the converse of Lemma 1.28 is also true. Although we cannot give a better description for $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ in general, we can give a considerably simpler description when θ is the trivial character. To do this we first need to consider how the F-stable maximal tori of $\mathbf{G}$ fall into conjugacy classes under the finite group G. Let $\mathfrak{T}$ denote the set of all F-stable maximal tori of $\mathbf{G}$. A maximal torus $\mathbf{T}$ of $\mathbf{G}$ is contained in $\mathfrak{T}$ if and only if there exists $g \in \mathbf{G}$ such that $\mathbf{T} = {}^g\mathbf{T}_0$ and $\mathscr{L}(g) \in N_{\mathbf{G}}(\mathbf{T}_0)$. The element $\mathscr{L}(g)$ naturally projects onto an element of the Weyl group $\mathbf{W}$ hence we have a map $\mathfrak{T} \to \mathbf{W}$.

Definition 1.29. Let H be a group and $\varphi : H \to H$ a homomorphism. We say two elements x, $y \in H$ are *φ-conjugate* if there exists $h \in H$ such that $x = h^{-1}y\varphi(h)$. This forms an equivalence relation on H and we call the equivalence classes the *φ-conjugacy classes* of H. We denote the set of all φ-conjugacy classes by $H^1(\varphi, H)$.

Lemma 1.30 (see [Car93, Proposition 3.3.3]**).** *The map $\mathfrak{T} \to \mathbf{W}$ determines a bijection $\mathfrak{T}/G \to H^1(F,\mathbf{W})$ where $\mathfrak{T}/G$ denotes the set of G-conjugacy classes of F-stable maximal tori of* $\mathbf{G}$.

Given an element $w \in \mathbf{W}$ we now fix a corresponding F-stable maximal torus $\mathbf{T}_w$ such that $\mathbf{T}_w = g\mathbf{T}_0g^{-1}$ for some $g \in \mathbf{G}$ satisfying $\mathscr{L}(g) = \dot{w}$. We say $\mathbf{T}_w$ is an F-stable maximal torus *obtained from $\mathbf{T}_0$ by twisting with w*. The corresponding fixed point group under F is given by

$$T_w = \{{}^gt \mid t \in \mathbf{T}_0 \text{ and } F({}^gt) = {}^gt\},$$

$$= \{{}^{g}t \mid t \in \mathbf{T}_0 \text{ and } F(t) = t^w\}$$

because $F(g) = g\dot{w}$. We define $\widehat{T}_w = {}^{g^{-1}}T_w$, which is the subgroup of $\mathbf{T}_0$ consisting of all elements satisfying $F(t) = t^w$.

Remark 1.31. Let us denote by $\mathfrak{T}^\star$ the set of all $F^\star$-stable maximal tori of $\mathbf{G}^\star$. We fix a representative $\dot{w}^\star \in N_{\mathbf{G}^\star}(\mathbf{T}_0^\star)$ for each element $w^\star \in \mathbf{W}^\star$. As above we can fix an $F^\star$-stable maximal torus $\mathbf{T}_{w^\star}^\star$ such that $\mathbf{T}_{w^\star}^\star = g^\star \mathbf{T}_0^\star g^{\star -1}$ for some $g^\star \in \mathbf{G}^\star$ satisfying $\mathscr{L}(g^\star) = \dot{w}^\star$, (where here $\mathscr{L}$ is applied in $\mathbf{G}^\star$). It is easily checked that the anti-isomorphism $\mathbf{W} \to \mathbf{W}^\star$ induces a bijection $H^1(F, \mathbf{W}) \to H^1(F^\star, \mathbf{W}^\star)$ which in turn induces a bijection $\mathfrak{T}/G \to \mathfrak{T}^\star/G^\star$ given by $\mathbf{T}_w \mapsto \mathbf{T}_{w^\star}^\star$. The triples $(\mathbf{G}, \mathbf{T}_w, F)$ and $(\mathbf{G}^\star, \mathbf{T}_{w'}^\star, F^\star)$ are in duality, in the sense of Section 1.2, if w and w' are in corresponding classes under the bijection $H^1(F, \mathbf{W}) \to H^1(F^\star, \mathbf{W}^\star)$. Given any maximal torus $\mathbf{T} \in \mathfrak{T}$ we denote by $\mathbf{T}^\star \in \mathfrak{T}^\star$ a maximal torus such that the corresponding classes in $\mathfrak{T}/G$ and $\mathfrak{T}^\star/G^\star$ are in bijective correspondence. We call $\mathbf{T}$ and $\mathbf{T}^\star$ *dual maximal tori.*

We aim now to introduce a variety which will give us a more intuitively simple construction of the character $R_{\mathbf{T}_w}^{\mathbf{G}}(1)$. Given $w \in \mathbf{W}$ we define the following algebraic subset of $\mathbf{G}$ associated to w

$$\mathscr{L}^{-1}(\dot{w}\mathbf{U}_0) = \{g \in \mathbf{G} \mid g^{-1}F(g) \in \dot{w}\mathbf{U}_0\}.$$

Let $x \in \mathscr{L}^{-1}(\dot{w}\mathbf{U}_0)$ and $t \in \widehat{T}_w$ then $xt \in \mathscr{L}^{-1}(\dot{w}\mathbf{U}_0)$ because

$$\mathscr{L}(xt) = t^{-1}\mathscr{L}(x)F(t) \in t^{-1}\dot{w}\mathbf{U}_0\dot{w}^{-1}t\dot{w} = \dot{w}(\dot{w}^{-1}t\dot{w})^{-1}\mathbf{U}_0(\dot{w}^{-1}t\dot{w}) = \dot{w}\mathbf{U}_0,$$

where the last equality follows from the fact that $\dot{w}^{-1}t\dot{w} \in \mathbf{T}_0 \leqslant N_{\mathbf{G}}(\mathbf{U}_0)$. In particular this shows that $\widehat{T}_w$ acts on $\mathscr{L}^{-1}(\dot{w}\mathbf{U}_0)$ by right multiplication. To each $w \in \mathbf{W}$ we can define the variety $\mathbf{X}_w = \mathscr{L}^{-1}(\dot{w}\mathbf{U}_0)/\widehat{T}_w$ to be the affine quotient by the finite group $\widehat{T}_w$, (which we can form by [**Gec03**, Proposition 2.5.10]). Clearly the finite group G acts on $\mathbf{X}_w$ by left multiplication because for any $x \in \mathbf{X}_w$ and $g \in G$ we have $\mathscr{L}(gx) = \mathscr{L}(x)$. We then have the following description for the Deligne–Lusztig character $R_{\mathbf{T}_w}^{\mathbf{G}}(1)$.

Proposition 1.32 (see [Gec03, Proposition 4.5.6]**).** *For all $w \in \mathbf{W}$ we have $R_{\mathbf{T}_w}^{\mathbf{G}}(1)(g) = \mathcal{L}(g, \mathbf{X}_w)$ for any $g \in G$.*

Remark 1.33. We consider now whether the construction given above depends upon our choice of representative $\dot{w}$ for w. Assume $\ddot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)$ is another representative for w and let $\mathbf{T}_w' = {}^{g'}\mathbf{T}_0$ for some $g' \in \mathbf{G}$ such that $\mathscr{L}(g') = \ddot{w}$. By Lemma 1.30 we have $\mathbf{T}_w$ and $\mathbf{T}_w'$ are conjugate under G so by Lemma 1.28 we have $R_{\mathbf{T}_w}^{\mathbf{G}}(1) = R_{\mathbf{T}_w'}^{\mathbf{G}}(1)$. In particular this says $\mathcal{L}(-, \mathbf{X}_w)$ does not depend upon our choice of $\dot{w}$.

We will now go on to state two of the most important theorems regarding Deligne–Lusztig characters, the first being a formula for computing their inner product. As a corollary of this we will see that the converse to Lemma 1.28 holds.

Theorem 1.34 (see [DM91, Corollary 11.15]). *Given two pairs $(\mathbf{T}, \theta)$, $(\mathbf{T}', \theta') \in \nabla(\mathbf{G}, F)$ we have the inner product of the corresponding Deligne–Lusztig characters is*

$$\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta')\rangle = \frac{1}{|T|}|\{n \in G \mid {}^n\mathbf{T} = \mathbf{T}' \text{ and } {}^n\theta = \theta'\}|.$$

Corollary 1.35. *We have two Deligne–Lusztig characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ and $R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ are orthogonal if $(\mathbf{T}, \theta) \not\sim_G (\mathbf{T}', \theta')$, in particular $R_{\mathbf{T}}^{\mathbf{G}}(\theta) = R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ if and only if $(\mathbf{T}, \theta) \sim_G (\mathbf{T}', \theta')$.*

Assume $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ and $R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ are orthogonal then as they are virtual characters they may still have irreducible constituents in common. However if $\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle = 1$ we must have $\pm R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ is an irreducible character of G. This naturally leads us to the following definition.

Definition 1.36. Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$ then we say $\theta \in \mathrm{Irr}(T)$ is in *general position* if no element of $N_{\mathbf{G}}(\mathbf{T})^F \setminus \mathbf{T}^F$ fixes θ.

Proposition 1.37 (see [Car93, Theorem 7.5.1 & Proposition 7.5.2]). *If $\theta \in \mathrm{Irr}(T_w)$ is an irreducible character then $(-1)^{\ell(w)} R_{\mathbf{T}_w}^{\mathbf{G}}(\theta)$ is a proper character of G and its degree is given by $(-1)^{\ell(w)} R_{\mathbf{T}_w}^{\mathbf{G}}(\theta)(1) = [G : T_w]_{p'}$.*

By $[G : T_w]$ we mean the finite group index of T_w in G. The subscript p' is to denote the largest divisor of the index which is coprime to p. We now wish to give the second important theorem regarding Deligne–Lusztig characters. However before we can do this we need to recast our labelling set for Deligne–Lusztig characters in terms of the dual group.

Definition 1.38. Let $\nabla^\star(\mathbf{G}, F)$ denote the set of all pairs $(\mathbf{T}^\star, s)$ such that $\mathbf{T}^\star$ is an $F^\star$-stable maximal torus of $\mathbf{G}^\star$ and $s \in T^\star$. We say two pairs $(\mathbf{T}^{\star\prime}, s')$, $(\mathbf{T}^\star, s) \in \nabla^\star(\mathbf{G}, F)$ are *rationally conjugate* if there exists an element $x \in G^\star$ such that $\mathbf{T}^{\star\prime} = {}^x\mathbf{T}^\star$ and $s' = {}^x s$. This defines an equivalence relation $\sim_{G^\star}$ on $\nabla^\star(\mathbf{G}, F)$ and we denote the set of all equivalence classes by $\nabla^\star(\mathbf{G}, F)/G^\star$.

There is a strong relationship between the equivalence classes $\nabla^\star(\mathbf{G}, F)/G^\star$ and $\nabla(\mathbf{G}, F)/G$, namely we have the following result.

Lemma 1.39 (see [DM91, Proposition 13.13]). *We have a bijective correspondence between the sets of equivalence classes $\nabla(\mathbf{G}, F)/G \to \nabla^\star(\mathbf{G}, F)/G^\star$ such that $(\mathbf{T}, 1) \mapsto (\mathbf{T}^\star, 1)$, where 1 denotes the trivial character of T or the identity in $G^\star$.*

Assume $(\mathbf{T}, \theta) \in \nabla(\mathbf{G}, F)$ corresponds to $(\mathbf{T}^\star, s)$ under the bijection in Lemma 1.39 then we may write $R^{\mathbf{G}}_{\mathbf{T}^\star}(s)$ for $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ without ambiguity. We can now state the second main result for Deligne–Lusztig characters.

Theorem 1.40 (see [Bon06, Théorème 11.8]). *Suppose $(\mathbf{T}^{\star\prime}, s')$ and $(\mathbf{T}^\star, s)$ determine different classes in $\nabla^\star(\mathbf{G}, F)/G^\star$ then $R^{\mathbf{G}}_{\mathbf{T}^{\star\prime}}(s')$ and $R^{\mathbf{G}}_{\mathbf{T}^\star}(s)$ have no irreducible constituent in common.*

In particular Theorem 1.40 and Proposition 1.37 tell us the following. Let $\{w_1, \dots, w_r\}$ be a set of representatives for the distinct F-conjugacy classes in $H^1(F, \mathbf{W})$. For each maximal torus $\mathbf{T}_{w_i}$ and irreducible character $\theta \in \mathrm{Irr}(T_{w_i})$ in general position we have $(-1)^{\ell(w_i)} R^{\mathbf{G}}_{\mathbf{T}_{w_i}}(\theta)$ will be an irreducible character of G and for distinct $1 \leqslant i, j \leqslant r$ we will have $(-1)^{\ell(w_i)} R^{\mathbf{G}}_{\mathbf{T}_{w_i}}(\theta) \neq (-1)^{\ell(w_j)} R^{\mathbf{G}}_{\mathbf{T}_{w_j}}(\theta)$. Asymptotically the irreducible characters arising as Deligne–Lusztig virtual characters are all the irreducible characters of G, (see [**Lus78**, Introduction]).

We now use the Deligne–Lusztig characters to give us an initial partitioning of the irreducible characters of G. Using Theorem 1.40, we have two Deligne–Lusztig characters $R^{\mathbf{G}}_{\mathbf{T}^{\star\prime}}(s')$ and $R^{\mathbf{G}}_{\mathbf{T}^\star}(s)$ have no irreducible constituents in common unless s', s are in the same $G^\star$-conjugacy class.

Definition 1.41. Let $[s] = [s]_{G^\star}$ be the $G^\star$-conjugacy class of semisimple elements containing $s \in G^\star$. We define the *Lusztig series* of G associated to $[s]$ to be the set

$$\mathcal{E}(G, s) = \mathcal{E}(G, [s]) = \{\chi \in \mathrm{Irr}(G) \mid \langle \chi, R^{\mathbf{G}}_{\mathbf{T}^\star}(s) \rangle \neq 0 \text{ for some } (\mathbf{T}^\star, s) \in \nabla^\star(\mathbf{G}, F)\}.$$

Remark 1.42. We have chosen here to define the Lusztig series to be the so called *rational Lusztig series*. However it should be noted that many people would define a Lusztig series to be the *geometric Lusztig series*. The distinction is made for historical reasons and is explained in detail in [**Bon06**, Chapitre 3].

By [**DM91**, Corollary 12.14] every irreducible character of G occurs in some Deligne–Lusztig character. Therefore by Theorem 1.40 the Lusztig series give a partition of the irreducible characters of G

$$\mathrm{Irr}(G) = \bigsqcup_{[s]} \mathcal{E}(G, s),$$

where $[s]$ runs over all $G^\star$-conjugacy classes of semisimple elements.

1.5 Unipotent Characters

Definition 1.43. An irreducible character $\chi \in \mathrm{Irr}(G)$ is called *unipotent* if it occurs in a Deligne–Lusztig character $R^{\mathbf{G}}_{\mathbf{T}}(1)$ or equivalently it is an element of $\mathcal{E}(G, 1)$.

In [**Lus84a**] Lusztig gives a method for parameterising the unipotent characters of any connected reductive algebraic group using combinatorial data from $\mathbf{W}$. To parameterise the unipotent characters it is enough to determine the decomposition of each $R_{\mathbf{T}}^{\mathbf{G}}(1)$ into its irreducible constituents. However Lusztig's approach is to decompose a related set of class functions known as *unipotent almost characters*, which are certain linear combinations of the Deligne–Lusztig characters $R_{\mathbf{T}}^{\mathbf{G}}(1)$.

We recall the following situation from [**Lus84a**, Chapter 3.1]. The generalised Frobenius endomorphism F induces an automorphism on the Weyl group $\mathbf{W}$ which stabilises the set $\mathbb{S}$. We define $\widetilde{\mathbf{W}}$ to be the semidirect product of $\mathbf{W}$ with the infinite cyclic group $\langle c \rangle$, where c is such that the identity $c \cdot w \cdot c^{-1} = F(w)$ holds for all $w \in \mathbf{W}$. As F acts on $\mathbf{W}$ we have an action of F on $\operatorname{Irr}(\mathbf{W})$ by composition, hence we have a set of F-fixed elements $\operatorname{Irr}(\mathbf{W})^F$.

Proposition 1.44 (Lusztig, [Lus84a, Proposition 3.2]**).** *If $\rho \in \operatorname{Irr}(\mathbf{W})^F$ then ρ can be extended to an irreducible character $\tilde{\rho} \in \operatorname{Irr}(\widetilde{\mathbf{W}})$, which factors through a finite quotient of $\widetilde{\mathbf{W}}$. Furthermore there are precisely two possible choices for $\tilde{\rho}$ which can be realised over $\mathbb{Q}$. One is obtained from the other by interchanging the action of c and $-c$.*

Definition 1.45. Let $\rho \in \operatorname{Irr}(\mathbf{W})^F$ and let $\tilde{\rho} \in \operatorname{Irr}(\widetilde{\mathbf{W}})$ be a fixed extension of ρ as in Proposition 1.44 then we define the class function

$$R_{\tilde{\rho}}^{\mathbf{G}} = \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \tilde{\rho}(c \cdot w) R_{\mathbf{T}_w}^{\mathbf{G}}(1),$$

to be the *unipotent almost character*, or simply the *almost character*, of G associated to ρ. Note that $R_{\tilde{\rho}}^{\mathbf{G}}$ is defined uniquely up to sign and, as $\tilde{\rho}$ is defined over $\mathbb{Q}$, we have $R_{\tilde{\rho}}^{\mathbf{G}}$ is a $\mathbb{Q}$-linear combination of irreducible characters of G.

Lusztig's idea was to relate the unipotent characters of G to the irreducible characters of the Weyl group using almost characters. He does this via the notion of families, whose definition was first given conjecturally in [**Lus79**].

Definition 1.46. We say two unipotent characters $\chi, \chi' \in \mathcal{E}(G, 1)$ lie in the same *family* if there exists a sequence $\chi = \chi^1, \dots, \chi^r = \chi'$ such that for each $1 \leqslant i \leqslant r$ there exists $\rho^i \in \operatorname{Irr}(\mathbf{W})^F$ such that

$$\langle R_{\tilde{\rho}^i}^{\mathbf{G}}, \chi^i \rangle \neq 0 \qquad \text{and} \qquad \langle R_{\tilde{\rho}^i}^{\mathbf{G}}, \chi^{i+1} \rangle \neq 0.$$

Likewise two F-fixed irreducible characters $\rho, \rho' \in \operatorname{Irr}(\mathbf{W})^F$ are in the same *family* if there exists a sequence $\rho = \rho^1, \dots, \rho^t = \rho'$ such that for each $1 \leqslant j \leqslant t$ there exists

$\chi^j \in \mathcal{E}(G,1)$ such that

$$\langle R^{\mathbf{G}}_{\tilde{\rho}^j}, \chi^j \rangle \neq 0 \qquad \text{and} \qquad \langle R^{\mathbf{G}}_{\tilde{\rho}^{j+1}}, \chi^j \rangle \neq 0.$$

It is clear from this definition that there is a natural bijective correspondence between families of unipotent characters and families of the Weyl group. If $\mathcal{F}$ is a family of unipotent characters then we write $\mathbf{W}(\mathcal{F})$ for the corresponding family of characters of the Weyl group. What Lusztig managed to do was to create an equivalent definition of families for the Weyl group based upon induction of characters from parabolic subgroups of $\mathbf{W}$ and numerical invariants of the associated Hecke algebra. We will discuss families of Weyl group characters in more detail in the next section.

We end this section with some results that determine the invariance of unipotent characters under automorphisms. To start with we will need the following result proved by Deligne and Lusztig concerning unipotent characters.

Lemma 1.47 (see [DM91, Proposition 13.20]). *Let $\overline{\mathbf{G}}$ be a connected reductive algebraic group and $\pi : \mathbf{G} \to \overline{\mathbf{G}}$ an isotypic morphism which is defined over $\mathbb{F}_q$. The induced map $\pi^* : \mathrm{Cent}(\overline{G}) \to \mathrm{Cent}(G)$ given by $\pi^*(\chi) = \chi \circ \pi$ gives a bijection between the unipotent characters of $\overline{G}$ and the unipotent characters of G.*

Let $\overline{\mathbf{G}}$ be the quotient group $\mathbf{G}/Z(\mathbf{G})^\circ$ and π the natural projection map, (so we are in the situation of Lemma 1.47). We will denote by $\mathrm{Aut}(\mathbf{G}, F)$ the set of all automorphisms of $\mathbf{G}$ which are defined over $\mathbb{F}_q$, i.e. the automorphisms of $\mathbf{G}$ which commute with F. Any automorphism in $\mathrm{Aut}(\mathbf{G}, F)$ will determine an automorphism of G so we have a homomorphism $\mathrm{Aut}(\mathbf{G}, F) \to \mathrm{Aut}(G)$, (by [**Bon06**, §6.A - Remarque] this is not injective in general). Let $\gamma \in \mathrm{Aut}(\mathbf{G}, F)$ then $\gamma(Z(\mathbf{G})^\circ) = Z(\mathbf{G})^\circ$ which means γ also determines an automorphism of $\overline{\mathbf{G}}$. The automorphism γ will act on the sets of irreducible characters $\mathrm{Irr}(\overline{G})$ and $\mathrm{Irr}(G)$ by composition. Using Lemma 1.47 we see that for any unipotent character $\chi \in \mathcal{E}(\overline{G},1)$ we have $\pi^*(\chi \circ \gamma) = \pi^*(\chi) \circ \gamma$ because $\pi \circ \gamma = \gamma \circ \pi$. In particular to understand the action of γ on $\mathcal{E}(G,1)$ it is sufficient to understand the action of γ on $\mathcal{E}(\overline{G},1)$, hence we will assume that $\mathbf{G}$ is semisimple until the end of this section.

Lemma 1.48. *Assume $\mathbf{G}$ is semisimple and $F : \mathbf{G} \to \mathbf{G}$ is a generalised Frobenius endomorphism of $\mathbf{G}$ stabilising $\mathbf{T}_0$ and $\mathbf{B}_0$. Every automorphism $\gamma \in \mathrm{Aut}(\mathbf{G}, F)$ can be expressed as a composition $\mathrm{inn}\, g \circ \mathrm{inn}\, s \circ \beta$, where $\beta \in \mathrm{Aut}(\mathbf{G}, F)$ is a graph automorphism of $\mathbf{G}$, $g \in G$ and $s \in \mathbf{T}_0$.*

Proof. By Theorem 1.16 we have $\gamma = \mathrm{inn}\, g \circ \beta$ where $g \in \mathbf{G}$ and β is a graph automorphism of $\mathbf{G}$. Denote by $(\varphi, b, 1)$ the automorphism of root data induced by β and (φ', b', q) the isogeny of root data induced by F. Using the same arguments as in Remark 1.12 we have by Lemma 1.8 and [**Hum75**, §16.2 - Lemma C] there exists $t \in \mathbf{T}_0$ such that

$$\begin{aligned}(F \circ \gamma)(x_\alpha(c)) &= F(\operatorname{inn} g(x_{b(\alpha)}(c))) = (\operatorname{inn} F(g)t)(x_{(b'\circ b)(\alpha)}(c^{q(b(\alpha))})),\\ (\gamma \circ F)(x_\alpha(c)) &= \gamma(\operatorname{inn} t(x_{b'(\alpha)}(c^{q(\alpha)}))) = (\operatorname{inn} g\beta(t))(x_{(b\circ b')(\alpha)}(c^{q(\alpha)})),\end{aligned}$$

for all $c \in \mathbb{K}^+$ and $\alpha \in \Delta$. As $F \circ \gamma = \gamma \circ F$ we have for all $\alpha \in \Delta$ and $c \in \mathbb{K}^+$ that

$$(\operatorname{inn} \beta(t)^{-1} g^{-1} F(g)t)(x_{(b'\circ b)(\alpha)}(c^{q(b(\alpha))})) = x_{(b\circ b')(\alpha)}(c^{q(\alpha)}).$$

Let h denote the element $\beta(t)^{-1}g^{-1}F(g)t$. Both F and β preserve $\mathbf{T}_0$ and $\mathbf{B}_0$ so $b' \circ b$ and $b \circ b'$ preserve Δ hence also Φ^+, in particular $\operatorname{inn} h$ preserves $\mathbf{B}_0$ and $\mathbf{T}_0$. As $\operatorname{inn} h$ stabilises $\mathbf{B}_0$ we must have $h \in \mathbf{B}_0$ because $N_{\mathbf{G}}(\mathbf{B}_0) = \mathbf{B}_0$, (see [**Hum75**, §23.1 - Theorem]). As $\operatorname{inn} h$ stabilises $\mathbf{T}_0$ we have $h \in N_{\mathbf{B}_0}(\mathbf{T}_0) = \mathbf{T}_0$, however it's clear that no element of $\mathbf{T}_0$ can affect the above automorphism if it's non-trivial. Therefore we must have $\operatorname{inn} h$ is trivial so $b' \circ b = b \circ b'$ and $q \circ b = q$. These latter requirements show that β commutes with F.

As the inner automorphism is trivial this implies

$$g^{-1}F(g) = \beta(t)t^{-1}.$$

By the Lang–Steinberg theorem, applied inside the connected group $\mathbf{T}_0$, there exists $s \in \mathbf{T}_0$ such that $g^{-1}F(g) = s^{-1}F(s) \Rightarrow F(gs^{-1}) = gs^{-1}$. Denote by h' the element $gs^{-1} \in G$ then we have $\gamma = \operatorname{inn} h' \circ \operatorname{inn} s \circ \beta$, which is of the required form. ■

Let $\widetilde{\operatorname{Inn}}(G)$ be the subgroup of $\operatorname{Aut}(\mathbf{G})$ consisting of all automorphisms of $\mathbf{G}$ induced by conjugating with an element of $G = \mathbf{G}^F$, (this is clearly contained in $\operatorname{Aut}(\mathbf{G}, F)$ and is also normal). Note that we have a natural surjective homomorphism $\widetilde{\operatorname{Inn}}(G) \to \operatorname{Inn}(G)$ given by restricting the automorphism to G. By [**Bon06**, Remarque 6.2] we have $Z(G) = Z(\mathbf{G})^F$ so any element centralising G also centralises $\mathbf{G}$, in particular we have $\widetilde{\operatorname{Inn}}(G) \to \operatorname{Inn}(G)$ is an isomorphism. We may then justifiably denote by $\operatorname{Out}(\mathbf{G}, F)$ the quotient group $\operatorname{Aut}(\mathbf{G}, F)/\operatorname{Inn} G$.

Recall we have a homomorphism $\operatorname{Aut}(\mathbf{G}, F) \to \operatorname{Aut}(G)$ given by restriction which induces a homomorphism $\operatorname{Out}(\mathbf{G}, F) \to \operatorname{Out}(G)$, (again by [**Bon06**, §6.A - Remarque] this is also not injective in general). The statement of Lemma 1.48 says that every element of $\operatorname{Out}(\mathbf{G}, F)$ can be expressed as a composition $\operatorname{inn} s \circ \beta$ where β is a graph automorphism of $\mathbf{G}$ and $s \in \mathbf{T}_0$. The outer automorphism $\operatorname{inn} s$ of G is called a *diagonal automorphism*.

We assume until the end of this section that $\gamma \in \mathrm{Aut}(\mathbf{G}, F)$ is a representative for an element $\gamma \in \mathrm{Out}(\mathbf{G}, F)$ such that γ stabilises $\mathbf{T}_0$ and $\mathbf{B}_0$. By Lemma 1.8 we have γ induces an automorphism of $\mathbf{W}$ stabilising $\mathbb{S}$, which we again denote by γ. With all of this in mind we can now prove the following concerning the action of outer automorphisms on Deligne–Lusztig characters.

Proposition 1.49. *For any $w \in \mathbf{W}$ we have the action of γ on the Deligne–Lusztig character $R^{\mathbf{G}}_{\mathbf{T}_w}(1)$ is given by*

$$R^{\mathbf{G}}_{\mathbf{T}_w}(1) \circ \gamma = R^{\mathbf{G}}_{\mathbf{T}_{\gamma^{-1}(w)}}(1).$$

Proof. As γ stabilises $\mathbf{T}_0$ and $\mathbf{B}_0$ we must also have γ stabilises $\mathbf{U}_0$. Let $g \in \mathscr{L}^{-1}(\dot{w}\mathbf{U}_0)$ then we have $\gamma^{-1}(g) \in \mathscr{L}^{-1}(\gamma^{-1}(\dot{w})\mathbf{U}_0)$ because

$$\mathscr{L}(\gamma^{-1}(g)) = \gamma^{-1}(g^{-1})F(\gamma^{-1}(g)) = \gamma^{-1}(g^{-1}F(g)) \in \gamma^{-1}(\dot{w}\mathbf{U}_0) = \gamma^{-1}(\dot{w})\mathbf{U}_0.$$

Therefore γ^{-1} restricts to an isomorphism $\mathscr{L}^{-1}(\dot{w}\mathbf{U}_0) \to \mathscr{L}^{-1}(\gamma^{-1}(\dot{w})\mathbf{U}_0)$. By the remark after Proposition 1.32 $\mathcal{L}(-, \mathbf{X}_w)$ does not depend upon our choice of $\dot{w}$ hence the result follows by [**Car93**, Property 7.1.5]. ■

Note that the proof of the above proposition is essentially part of the proof of [**DM91**, Proposition 13.20] with some extra care taken over the action of γ on $\mathbf{W}$. We now apply this result to the almost characters. Let $\rho \in \mathrm{Irr}(\mathbf{W})^F$ be an F-fixed irreducible character of the Weyl group and let $\tilde{\rho} \in \mathrm{Irr}(\widetilde{\mathbf{W}})$ be an extension of ρ to an irreducible character of $\widetilde{\mathbf{W}}$, (as in Proposition 1.44). Recall that $\gamma \in \mathrm{Aut}(\mathbf{G}, F)$ commutes with F then γ acts naturally on $\mathrm{Irr}(\mathbf{W})^F$ by composition and we can extend that to an action on $\mathrm{Irr}(\widetilde{\mathbf{W}})$ by letting $\tilde{\rho} \circ \gamma$ be such that $(\tilde{\rho} \circ \gamma)(c^i \cdot w) = \tilde{\rho}(c^i \cdot \gamma(w))$ for all $w \in \mathbf{W}$. It is clear that $\mathrm{Res}^{\widetilde{\mathbf{W}}}_{\mathbf{W}}(\tilde{\rho} \circ \gamma) = \rho \circ \gamma$ so $\tilde{\rho} \circ \gamma$ is an extension of $\rho \circ \gamma$ to $\widetilde{\mathbf{W}}$. By Proposition 1.44 we know $\tilde{\rho} \circ \gamma$ is one of two possible extensions of $\rho \circ \gamma$ which differ only by a change of sign.

Corollary 1.50. *For any $\rho \in \mathrm{Irr}(\mathbf{W})^F$ we have $R^{\mathbf{G}}_{\tilde{\rho}} \circ \gamma = R^{\mathbf{G}}_{\tilde{\rho} \circ \gamma}$.*

Proof. A simple application of Proposition 1.49 gives us

$$\begin{aligned} R^{\mathbf{G}}_{\tilde{\rho}} \circ \gamma &= \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \tilde{\rho}(c \cdot w)(R^{\mathbf{G}}_{\mathbf{T}_w}(1) \circ \gamma), \\ &= \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \tilde{\rho}(c \cdot w) R^{\mathbf{G}}_{\mathbf{T}_{\gamma^{-1}(w)}}(1), \\ &= \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \tilde{\rho}(c \cdot \gamma(w)) R^{\mathbf{G}}_{\mathbf{T}_w}(1), \\ &= R^{\mathbf{G}}_{\tilde{\rho} \circ \gamma}. \end{aligned}$$

■

This result will allow us to deal with γ acting on unipotent characters which are contained in a family of cardinality one because such characters are up to sign almost characters. If the family of unipotent characters has more than one element then we will use the following result of Digne and Michel, (whose proof uses Corollary 1.50 together with conditions which ensure the unicity of the parameterisation of unipotent characters).

Proposition 1.51 ([DM90, Proposition 6.6]). *Assume* $\mathbf{G}$ *is simple and* $\gamma \in \mathrm{Aut}(\mathbf{G}, F)$ *stabilises* $\mathbf{B}_0$ *and* $\mathbf{T}_0$. *If* $\mathcal{F} \subseteq \mathcal{E}(G, 1)$ *is a family of unipotent characters containing more than one element then* $\chi \circ \gamma = \chi$ *for all* $\chi \in \mathcal{F}$.

Remark 1.52. The statement in [**DM90**, Proposition 6.6] is actually for any isogeny $\gamma : \mathbf{G} \to \mathbf{G}$ commuting with F. However by [**Mal07**, Proposition 3.9] the statement of [**DM90**, Proposition 6.6] fails in the following situation: $\mathbf{G}$ is simple of type G_2, F is the Frobenius endomorphism F_r and γ is the exceptional graph automorphism in Figure 1.2, (note we necessarily assume $p = 3$). Here there are two principal series unipotent characters lying in a family of cardinality eight which are exchanged by the exceptional graph automorphism. Note that if $\mathbf{G}$ is of type B_2 or F_4, $p = 2$ and F is the Frobenius endomorphism F_r then the exceptional graph automorphisms do not commute with F as r is even. In particular these cases do not contradict [**DM90**, Proposition 6.6]. In spite of this one can readily check that the proof given in [**DM90**] does cover the weaker statement given in Proposition 1.51.

1.6 Combinatorics

In this section we introduce the combinatorics of partitions and symbols which will be used to parameterise: characters of Weyl groups, unipotent conjugacy classes and unipotent characters. The symbols we describe were first introduced by Lusztig in [**Lus77**, §3].

Recall that a *partition* is a (possibly empty) finite weakly increasing sequence of non-zero natural numbers $\lambda = (\lambda_1, \dots, \lambda_s)$. If λ is empty we define $|\lambda| \in \mathbb{N}$ to be 0, if not we define it to be the sum $\lambda_1 + \cdots + \lambda_s$. We say λ is a *partition of* n if $|\lambda| = n$ and we denote by $\mathcal{P}(n)$ the set of all such partitions. A *bipartition* is an ordered pair of partitions $(\pi; \tau)$, which is a *bipartition of* n if $|\pi| + |\tau| = n$. If π, respectively τ, is empty then we denote the resulting bipartition by $(-; \tau)$, respectively by $(\pi; -)$. We denote by $\mathcal{BP}(n)$ the set of all bipartitions of n.

We have a bijection $\mathcal{P}(n) \to \mathcal{P}(n)$ denoted by $\lambda \mapsto \lambda^*$, where λ^* is the partition *dual* to λ. This is defined in the following way. First let $\lambda' = (\lambda'_1, \dots, \lambda'_n)$ be the sequence obtained by setting $\lambda'_j = |\{\lambda_i \mid 1 \leqslant i \leqslant s \text{ and } \lambda_i \geqslant j\}|$ for all $1 \leqslant j \leqslant n$. We then define λ^* to be the partition of n obtained from λ' by removing all entries equal to zero.

By writing the sequence $k,\dots,k$ containing r terms as k^r we may express any partition λ as $(1^{r_1}, 2^{r_2}, 3^{r_3}, \dots)$ where $r_i \geqslant 0$ denotes the number of times i occurs in λ. If $\varepsilon \in \{0,1\}$ we define the subset $\mathcal{P}_\varepsilon(n)$ of $\mathcal{P}(n)$ to be all partitions λ such that $r_i \equiv 0 \pmod 2$ whenever $i \equiv \varepsilon \pmod 2$. If n is even then we say a partition $\lambda = (\lambda_1,\dots,\lambda_s) \in \mathcal{P}_0(n)$ is *degenerate* if every λ_i is even. In this situation we define $\overline{\mathcal{P}_0(n)}$ to be the set containing all non-degenerate partitions in $\mathcal{P}_0(n)$ together with two partitions λ_+ and λ_- for each degenerate partition $\lambda \in \mathcal{P}_0(n)$. Note that $\overline{\mathcal{P}_0(n)} = \mathcal{P}_0(n)$ unless $n \equiv 0 \pmod 4$. We say a bipartition $(\pi;\tau) \in \mathcal{BP}(n)$ is *degenerate* if $\pi = \tau$. In this situation we define $\overline{\mathcal{BP}(n)}$ to be the set of all non-degenerate bipartitions, with the ordering forgotten, together with two bipartitions $(\pi;\pi)_+$ and $(\pi;\pi)_-$ for each degenerate bipartition $(\pi;\pi) \in \mathcal{BP}(n)$.

Example 1.53. Consider the case $n = 4$ then we have

$$\begin{aligned}
\mathcal{P}(4) &= \{(4), (2^2), (1,3), (1^2,2), (1^4)\},\\
\mathcal{P}_0(4) &= \{(2^2), (1,3), (1^4)\},\\
\mathcal{P}_1(4) &= \{(4), (2^2), (1^2,2), (1^4)\}.
\end{aligned}$$

The set $\mathcal{P}_0(4)$ contains one degenerate partition, namely (2^2), hence we have $\overline{\mathcal{P}_0(4)}$ is the set $\{(2^2)_+, (2^2)_-, (1,3), (1^4)\}$. If $n = 2$ then the set of bipartitions is given by

$$\mathcal{BP}(2) = \{(-;1^2), (-;2), (1;1), (2;-), (1^2;-)\}.$$

There is only one degenerate bipartition of 2, namely $(1;1)$, hence we have $\overline{\mathcal{BP}(2)}$ is the set $\{(-;1^2), (-;2), (1;1)_+, (1;1)_-\}$.

A notion closely related to that of partitions is the notion of a *β-set* which is a (possibly empty) finite subset $B \subset \mathbb{N}_0$. If $B = \{\beta_1,\dots,\beta_s\}$ then we assume the elements of B are enumerated to form a strictly increasing sequence, i.e. $\beta_1 < \cdots < \beta_s$. We define the rank of B to be

$$\mathrm{rk}(B) = \begin{cases} \sum_{i=1}^{s} \beta_i - \binom{s}{2} & \text{if } B \neq \varnothing,\\ 0 & \text{otherwise,}\end{cases}$$

and denote by $\mathbb{X}_n$ the set of all β-sets of rank n. For any $t \geqslant 1$ we define a *shift* operation ${}^{+t} : \mathbb{X}_n \to \mathbb{X}_n$ given by

$$B^{+t} = \begin{cases} \{0,1,\dots,t-1,\beta_1+t,\dots,\beta_s+t\} & \text{if } B \neq \varnothing,\\ \{0,1,\dots,t-1\} & \text{otherwise.}\end{cases}$$

It is easy to check that $\mathrm{rk}(B) = \mathrm{rk}(B^{+t})$.

With the notion of shift we can define a partial ordering on $\mathbb{X}_n$ by setting $B \leqslant B'$ if there exists $t \geqslant 1$ such that $B^{+t} = B'$ or $B = B'$. This induces an equivalence relation $\sim$ on $\mathbb{X}_n$ by setting $B \sim B'$ if $B \leqslant B'$ or $B' \leqslant B$. We denote the resulting equivalence class containing B by $[B]$ and the set of all such equivalence classes by $\widetilde{\mathbb{X}}_n$. We call the elements of $\widetilde{\mathbb{X}}_n$ the *symbols of rank n*, where we unambiguously define the rank of $[B]$ to be $\mathrm{rk}([B]) := \mathrm{rk}(B)$.

It is clear that in each class $[B]$ there is a unique element $B^{\min}$ such that $B^{\min} \leqslant B'$ for all $B' \in [B]$ and we will typically represent the class $[B]$ by $[B^{\min}]$. Assume $[B] \in \widetilde{\mathbb{X}}_n$ is a class of β-sets and $B^{\min} = (\beta_1, \dots, \beta_s)$ its unique minimal element then we can define a partition $\lambda([B]) = (\lambda_1, \dots, \lambda_s)$ of n by setting $\lambda_i = \beta_i - (i-1)$; we call this the *partition associated to* $[B]$. Thus we have a map $\lambda : \widetilde{\mathbb{X}}_n \to \mathcal{P}(n)$ which is easily seen to be a bijection.

Example 1.54. Consider the case $n = 3$ then $\widetilde{\mathbb{X}}_3 = \{[1,2,3],[1,3],[3]\}$, where each class is represented by its unique minimal element. Applying the map λ to $\widetilde{\mathbb{X}}_3$ we have

$$\lambda([1,2,3]) = (1,1,1) \qquad \lambda([1,3]) = (1,2) \qquad \lambda([3]) = (3)$$

and clearly the set of partitions is given by $\mathcal{P}(3) = \{(1^3),(1,2),(3)\}$.

We call a *β-pair* an ordered pair $\Lambda = \binom{A}{B}$ where A and B are β-sets. We define the cardinality of a β-pair Λ to be $|\Lambda| = |A| + |B|$. If $\Lambda' = \binom{A'}{B'}$ is another β-pair then we take $\Lambda \equiv \Lambda'$ to mean $A \cup B = A' \cup B'$. If A, (resp. B), is empty then we denote the β-pair by $\binom{-}{B}$, (resp. $\binom{A}{-}$). We define the *defect* of the β-pair Λ to be $d(\Lambda) = |A| - |B|$ and its rank to be

$$\mathrm{rk}(\Lambda) = \mathrm{rk}(A) + \mathrm{rk}(B) + \left\lfloor \left(\frac{d(\Lambda)}{2}\right)^2 \right\rfloor$$

where, for any $x \in \mathbb{R}$, we have $\lfloor x \rfloor = \sup\{k \in \mathbb{N}_0 \mid k \leqslant x\}$. We denote by $\mathbb{Y}_n^d$ the set of all β-pairs of rank n and defect d. If $t \geqslant 1$ then the shift operation on β-sets determines a shift operation on β-pairs ${}^{+t} : \mathbb{Y}_n^d \to \mathbb{Y}_n^d$ given by

$$\Lambda^{+t} = \begin{pmatrix} A^{+t} \\ B^{+t} \end{pmatrix}.$$

As for β-sets the notion of shift induces a corresponding equivalence relation on β-pairs. We denote the set of equivalence classes under this relation by $\widetilde{\mathbb{Y}}_n^d$ and the equivalence class containing Λ by $[\Lambda]$. Equivalent β-pairs will have the same defect, hence the same rank, so we may define the defect and rank of $[\Lambda]$ to be that of Λ. We call the set of equivalence classes $\widetilde{\mathbb{Y}}_n^d$ the *symbols of rank n and defect d*. In each class $[\Lambda]$ there is a unique minimal element $\Lambda^{\min}$ with respect to shift and we will typically represent the class $[\Lambda]$ by $[\Lambda^{\min}]$.

Assume now that $A = \{\alpha_1, \ldots, \alpha_{s+k}\}$ and $B = \{\beta_1, \ldots, \beta_s\}$ with $k \in \{0, 1\}$. If $k = 0$ we say the β-pair $\binom{A}{B}$ is *special* if either of the following conditions hold

$$\alpha_1 \leqslant \beta_1 \leqslant \alpha_2 \leqslant \beta_2 \leqslant \cdots \leqslant \alpha_s \leqslant \beta_s,$$
$$\beta_1 \leqslant \alpha_1 \leqslant \beta_2 \leqslant \alpha_2 \leqslant \cdots \leqslant \beta_s \leqslant \alpha_s.$$

If $k = 1$ then we say the β-pair $\binom{A}{B}$ is *special* if

$$\alpha_1 \leqslant \beta_1 \leqslant \alpha_2 \leqslant \beta_2 \leqslant \cdots \leqslant \alpha_s \leqslant \beta_s \leqslant \alpha_{s+1}.$$

The definition of a special β-pair is invariant under shift so we may unambiguously define a symbol $[\Lambda]$ to be special if Λ is special. Note that if Λ has zero defect then being special is independent of the ordering of A and B.

Assume $[\Lambda] \in \widetilde{\mathbb{Y}}_n^d$ is a symbol of rank n and defect d and let $\Lambda^{\min} = \binom{A}{B}$ be the unique minimal element of $[\Lambda]$. We can define a bipartition $\lambda([\Lambda]) = (\lambda([A]); \lambda([B]))$ of $k = \mathrm{rk}([A]) + \mathrm{rk}([B])$, where $\lambda([A])$ and $\lambda([B])$ are the partitions associated to $[A]$ and $[B]$. Thus we have a map $\lambda : \widetilde{\mathbb{Y}}_n^d \to \mathcal{BP}(k)$ which gives rise to two bijections $\lambda : \widetilde{\mathbb{Y}}_n^1 \to \mathcal{BP}(n)$ and $\lambda : \widetilde{\mathbb{Y}}_n^0 \to \mathcal{BP}(n)$, which follows from the definition of rank for symbols and the case of β-sets.

Example 1.55. Consider the case where $n = 2$ then we have

$$\widetilde{\mathbb{Y}}_2^1 = \left\{ \begin{bmatrix} 0 & & 1 & & 2 \\ & 1 & & 2 & \end{bmatrix}, \begin{bmatrix} 0 & & 1 \\ & 2 & \end{bmatrix}, \begin{bmatrix} 0 & & 2 \\ & 1 & \end{bmatrix}, \begin{bmatrix} 1 & & 2 \\ & 0 & \end{bmatrix}, \begin{bmatrix} 2 \\ - \end{bmatrix} \right\}$$

Applying the map λ to the elements of $\widetilde{\mathbb{Y}}_2^1$ we see

$$\lambda\left(\begin{bmatrix} 0 & & 1 \\ & 2 & \end{bmatrix}\right) = (-; 2) \qquad \lambda\left(\begin{bmatrix} 0 & & 1 & & 2 \\ & 1 & & 2 & \end{bmatrix}\right) = (-; 1^2)$$
$$\lambda\left(\begin{bmatrix} 0 & & 2 \\ & 1 & \end{bmatrix}\right) = (1; 1) \qquad \lambda\left(\begin{bmatrix} 2 \\ - \end{bmatrix}\right) = (2; -)$$
$$\lambda\left(\begin{bmatrix} 1 & & 2 \\ & 0 & \end{bmatrix}\right) = (1^2; -).$$

and these bipartitions exhaust the set $\mathcal{BP}(2)$. There are three special symbols in the set $\widetilde{\mathbb{Y}}_2^1$, namely

$$\begin{bmatrix} 0 & & 1 & & 2 \\ & 1 & & 2 & \end{bmatrix} \qquad \begin{bmatrix} 0 & & 2 \\ & 1 & \end{bmatrix} \qquad \begin{bmatrix} 2 \\ - \end{bmatrix}.$$

Let us now consider the symbols of rank n and defect 0. A β-pair $\Lambda = \binom{A}{B} \in \mathbb{Y}_n^0$

and its corresponding symbol $[\Lambda]$ are called *degenerate* if $A = B$, (note that being degenerate is invariant under shift). We denote by $\overline{\mathbb{Y}}_n^0$ the set of all non-degenerate symbols, with the ordering forgotten, together with two symbols $[\Lambda]_+$ and $[\Lambda]_-$ for each degenerate symbol $[\Lambda] \in \widetilde{\mathbb{Y}}_n^0$. More generally if $d \neq 0$ we denote by $\overline{\mathbb{Y}}_n^d$ the set of all symbols $\widetilde{\mathbb{Y}}_n^d \cup \widetilde{\mathbb{Y}}_n^{-d}$ with the ordering forgotten. As any degenerate symbol is special we say the symbols $[\Lambda]_\pm$ are special symbols. The bijection $\lambda : \widetilde{\mathbb{Y}}_n^0 \to \mathcal{BP}(n)$ induces a bijection $\overline{\lambda} : \overline{\mathbb{Y}}_n^0 \to \overline{\mathcal{BP}(n)}$, where $\overline{\lambda}([\Lambda]_\pm) = \lambda([\Lambda])_\pm$ and $\overline{\lambda}$ agrees with λ on non-degenerate symbols.

Example 1.56. Consider the case where $n = 2$ then we have

$$\widetilde{\mathbb{Y}}_2^0 = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\},$$
$$\overline{\mathbb{Y}}_2^0 = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}_+, \begin{bmatrix} 1 \\ 1 \end{bmatrix}_- \right\}$$

Applying the map λ to the elements of $\widetilde{\mathbb{Y}}_2^0$ we see

$$\lambda\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = (-;2) \qquad \lambda\left(\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\right) = (-;1^2)$$
$$\lambda\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = (1;1) \qquad \lambda\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right) = (1^2;-)$$
$$\lambda\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = (2;-).$$

and these bipartitions exhaust the set $\mathcal{BP}(2)$. Similarly applying the map $\overline{\lambda}$ to the elements of $\overline{\mathbb{Y}}_2^0$ we see

$$\overline{\lambda}\left(\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\right) = (-;1^2) \qquad \overline{\lambda}\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = (-;2) \qquad \overline{\lambda}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}_\pm\right) = (1;1)_\pm.$$

and these bipartitions exhaust the set $\overline{\mathcal{BP}(2)}$. Finally we note that every symbol in $\overline{\mathbb{Y}}_2^0$ is special.

1.7 Characters of Weyl Groups

Let $\rho \in \mathrm{Irr}(\mathbf{W})$ be an irreducible character then Benson and Curtis have associated to ρ a unique polynomial $D_\rho(u) \in \mathbb{Q}[t]$ called the *generic degree polynomial*, (see **[BC72**, Definition 2.4]). This is defined in terms of the generic Hecke algebra. Using $D_\rho(u)$ we associate to ρ three integers a_ρ, b_ρ and f_ρ in the following way.

- a_ρ is the largest $i \geqslant 0$ such that u^i divides the generic degree polynomial $D_\rho(u)$.
- b_ρ is the smallest $i \geqslant 0$ such that ρ appears with non-zero multiplicity in the ith symmetric power of the reflection representation of $\mathbf{W}$. Equivalently this is the largest $i \geqslant 0$ such that u^i divides the *fake degree polynomial* $P_\rho(u)$ associated to ρ, see [**Car93**, pg. 367].
- f_ρ is such that $D_\rho(u) = f_\rho^{-1}u^{a_\rho} +$ higher order terms.

Lusztig introduces the notion of *truncated induction* for $\mathbf{W}$ based on the a and b values. We say a subgroup of $\mathbf{W}$ is a *reflection subgroup* if it is generated by reflections associated to a subset of the root system Φ.

Notation. Assume $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_k$ is an orthogonal decomposition as in Section 1.1.2 and $\Delta = \Delta_1 \sqcup \cdots \sqcup \Delta_k$ is a corresponding decomposition. As each root system Φ_i is irreducible it contains a unique lowest root which we denote by $\alpha_{i,0}$. Let $\tilde{\Delta}_i$ be the set $\Delta_i \cup \{\alpha_{i,0}\}$ then we define $\tilde{\Delta}$ to be the union $\tilde{\Delta}_1 \sqcup \cdots \sqcup \tilde{\Delta}_k$. We denote by $\mathbb{S}_0$ the set of reflections $\mathbb{S} \cup \{s_{\alpha_{1,0}}, \dots, s_{\alpha_{k,0}}\}$. If Φ is irreducible we will denote the unique lowest root by α_0 and the corresponding reflection by s_0.

If $\mathbf{W}_\mathbb{I}$ is a subgroup of $\mathbf{W}$ generated by a subset $\mathbb{I} \subseteq \mathbb{S}_0$ we call $\mathbf{W}_\mathbb{I}$ a *parahoric subgroup* of $\mathbf{W}$. If $\mathbb{I}$ is contained in $\mathbb{S}$ then we call $\mathbf{W}_\mathbb{I}$ a *parabolic subgroup* of $\mathbf{W}$. By [**DL11**, Corollary 1] every reflection subgroup of $\mathbf{W}$ is conjugate in $\mathbf{W}$ to a parahoric subgroup. Let $\mathbf{W}'$ be a reflection subgroup of $\mathbf{W}$ and ρ' an irreducible character of $\mathbf{W}'$. Consider the induced character $\mathrm{Ind}_{\mathbf{W}'}^{\mathbf{W}}(\rho') = \sum_{\psi \in \mathrm{Irr}(\mathbf{W})} n_{\rho',\psi}\psi$ then we define the truncated induced characters to be

$$J_{\mathbf{W}'}^{\mathbf{W}}(\rho') = \sum_{\psi} n_{\rho',\psi}\psi \qquad \text{summing over all } \psi \in \mathrm{Irr}(\mathbf{W}) \text{ such that } a_{\rho'} = a_\psi,$$

$$j_{\mathbf{W}'}^{\mathbf{W}}(\rho') = \sum_{\psi} n_{\rho',\psi}\psi \qquad \text{summing over all } \psi \in \mathrm{Irr}(\mathbf{W}) \text{ such that } b_{\rho'} = b_\psi.$$

Note that the character $j_{\mathbf{W}'}^{\mathbf{W}}(\rho')$ was first constructed by Macdonald and a more natural interpretation for this character can be found in [**Car93**, §11.2]. We can now describe Lusztig's definition of families of $\mathbf{W}$ following [**Lus84a**, §5.29].

Definition 1.57. A character ρ of $\mathbf{W}$ is called *constructible* if it satisfies the following conditions. If $\mathbf{W} = \{1\}$ then ρ is the trivial character. If $\mathbf{W} \neq \{1\}$ then there exists a proper parabolic subgroup $\mathbf{W}' < \mathbf{W}$ and a constructible character ρ' of $\mathbf{W}'$ such that either $\rho = J_{\mathbf{W}'}^{\mathbf{W}}(\rho')$ or $\rho = J_{\mathbf{W}'}^{\mathbf{W}}(\rho') \otimes \mathrm{sgn}$. Note that every irreducible character occurs as a constituent of a constructible character. A family of $\mathrm{Irr}(\mathbf{W})$ is an equivalence class under the relation generated by the condition "$\rho, \psi \in \mathrm{Irr}(\mathbf{W})$ both occur as constituents of some constructible character".

Remark 1.58. This definition of families in $\mathrm{Irr}(\mathbf{W})$ is independent of the associated finite reductive group and by the work in [**Lus84a**] it coincides with Definition 1.46. We will always write $\mathbf{W}(\mathcal{F})$ for a family of Weyl group characters but it may not be the case that there is a corresponding family of unipotent characters $\mathcal{F}$. Families of characters in $\mathrm{Irr}(\mathbf{W})^F$ are precisely given by the families $\mathbf{W}(\mathcal{F})$ such that $\mathbf{W}(\mathcal{F})^F = \mathbf{W}(\mathcal{F})$. This follows from the fact that the action of F on $\mathrm{Irr}(\mathbf{W})$ permutes the families of $\mathbf{W}$, (see Lemma 2.20 below).

Lusztig has observed that for every character $\rho \in \mathrm{Irr}(\mathbf{W})$ we always have $a_\rho \leqslant b_\rho$. We say an irreducible character is *special* if $a_\rho = b_\rho$. It was shown by Lusztig that every family in $\mathrm{Irr}(\mathbf{W})$ contains a unique special character. For simplicity, throughout the remainder of this section we make the following assumption.

We assume that the generalised Frobenius endomorphism F associated to $\mathbf{G}$ induces the identity on $\mathbf{W}$.

Note that the process we describe now is outlined in full in [**Lus84a**, Chapter 4]. Let $\mathcal{F} \subseteq \mathcal{E}(G,1)$ be a family of unipotent characters of G. To $\mathcal{F}$ we have an associated family $\mathbf{W}(\mathcal{F}) \subseteq \mathrm{Irr}(\mathbf{W})$ of irreducible characters of $\mathbf{W}$ with special character ρ. Also associated to $\mathcal{F}$ is a finite group $\mathcal{G}_\mathcal{F}$ which is determined in the following way. If the derived subgroup $\mathbf{G}'$ of $\mathbf{G}$ is simple and of classical type then $\mathcal{G}_\mathcal{F}$ is the elementary abelian 2-group of order f_ρ. If $\mathbf{G}'$ is simple of exceptional type then $\mathcal{G}_\mathcal{F}$ is isomorphic to a symmetric group $\mathfrak{S}_2$, $\mathfrak{S}_3$, $\mathfrak{S}_4$ or $\mathfrak{S}_5$ such that $|\mathcal{G}_\mathcal{F}| = f_\rho$.

Let us now describe $\mathcal{G}_\mathcal{F}$ in the situation where $\mathbf{G}'$ is not simple. If this is the case then $\mathbf{G}'$ is an almost direct product of a finite number of simple groups, say $\mathbf{G}_1, \dots, \mathbf{G}_k$. The family $\mathcal{F}$ will be a direct product $\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$, where $\mathcal{F}_i \subseteq \mathcal{E}(G_i,1)$ is a family of unipotent characters of the simple component, (this is primarily due to Lemma 1.47). The family of characters in the Weyl group will be a direct product $\mathbf{W}(\mathcal{F}) = \mathbf{W}(\mathcal{F}_1) \boxtimes \cdots \boxtimes \mathbf{W}(\mathcal{F}_k)$. Hence we define the finite group $\mathcal{G}_\mathcal{F}$ to be the direct product $\mathcal{G}_{\mathcal{F}_1} \times \cdots \times \mathcal{G}_{\mathcal{F}_k}$. Note that if $\rho_i \in \mathbf{W}(\mathcal{F}_i)$ is the special character of the family then we have $f_\rho = f_{\rho_1} \cdots f_{\rho_k}$, in particular $|\mathcal{G}_\mathcal{F}| = f_\rho$.

The finite group $\mathcal{G}_\mathcal{F}$ is an essential tool in determining the unipotent characters in the family $\mathcal{F}$. Lusztig associates to $\mathcal{G}_\mathcal{F}$ a set $\mathcal{M}(\mathcal{G}_\mathcal{F})$ consisting of pairs (x,σ) such that $x \in \mathcal{G}_\mathcal{F}$ and σ is an irreducible character of $C_{\mathcal{G}_\mathcal{F}}(x)$, chosen up to conjugacy. He then gives an embedding of $\mathbf{W}(\mathcal{F})$ into $\mathcal{M}(\mathcal{G}_\mathcal{F})$, which is given in the form of a list for simple exceptional groups. For the simple classical groups this is done by introducing a set of symbols $\mathcal{M}_\mathcal{F}$ as defined in Section 1.6, for which there is a natural embedding $\mathbf{W}(\mathcal{F}) \hookrightarrow \mathcal{M}_\mathcal{F}$, and then a bijection $\mathcal{M}_\mathcal{F} \to \mathcal{M}(\mathcal{G}_\mathcal{F})$.

We describe the construction of the sets $\mathcal{M}_{\mathcal{F}}$ below for each group of classical type and we assume that the unipotent characters for groups of classical type are parameterised by these sets. Note that Lusztig's parameterisation of the unipotent characters satisfies several amazing properties. The principal series unipotent characters are precisely those whose label in $\mathcal{M}(\mathcal{G}_{\mathcal{F}})$ is in the image of $\mathbf{W}(\mathcal{F})$. Using $\mathcal{M}(\mathcal{G}_{\mathcal{F}})$ together with a certain Fourier transform matrix Lusztig gives the multiplicities of the unipotent characters in each uniform unipotent almost character. By Proposition 1.37 the degrees of such characters are known, which gives the degrees of the unipotent characters by [**DM91**, Proposition 12.20].

In [**Lus84a**, Chapter 4] Lusztig gives lists of all the families of irreducible characters of exceptional irreducible Weyl groups and uses the symbols described in Section 1.6 to determine the families of classical irreducible Weyl groups. We now recall his methods for the groups of classical type.

1.7.1 Type A_{n-1}

If $\mathbf{G}$ is of type A_{n-1} then we identify $\mathbf{W}$ with the symmetric group $\mathfrak{S}_n$ such that, for $1 \leqslant i \leqslant n-1$, the simple reflection s_i corresponds to the transposition $(i, i+1)$. We have a bijection $\mathcal{P}(n) \to \mathrm{Irr}(\mathbf{W})$ denoted by $\pi \mapsto \rho_\pi$, (as described in [**Lus84a**, §4.4]), such that $\rho_{(n)}$ is the trivial character and $\rho_{(1^n)}$ is the sign character. By composing with the bijection $\widetilde{\mathbb{X}}_n \to \mathcal{P}(n)$ described in Section 1.6 we obtain a resulting bijection $\widetilde{\mathbb{X}}_n \to \mathrm{Irr}(\mathbf{W})$. We will denote the elements of $\mathrm{Irr}(\mathbf{W})$ by their corresponding symbol in $\widetilde{\mathbb{X}}_n$.

Lemma 1.59 (Lusztig, [Lus84a, §4.4]). *If $\mathbf{G}$ is of type A_n then every irreducible character of $\mathbf{W}$ is special and lies in its own family. Furthermore there is a bijection between irreducible characters of $\mathbf{W}$ and unipotent characters of G. In particular, the symbols $\widetilde{\mathbb{X}}_n$ parameterise the unipotent characters.*

1.7.2 Types B_n and C_n

If $n \geqslant 1$ we denote by $W_n \leqslant \mathfrak{S}(V)$ the subgroup of the symmetric group on the set $V = \{1, \dots, n, n', \dots, 1'\})$ which preserves the unordered pairs $\{(1,1'), \dots, (n,n')\}$. If $n = 0$ we take $W_n = \{1\}$. If $\mathbf{G}$ is of type B_n or C_n then we identify $\mathbf{W}$ with W_n such that, for $1 \leqslant i \leqslant n-1$, the simple reflection s_i corresponds to the product of transpositions $(i, i+1)((i+1)', i')$ and s_n corresponds to (n, n'). We have a bijection $\mathcal{BP}(n) \to \mathrm{Irr}(\mathbf{W})$ denoted by $(\pi; \tau) \mapsto \rho_{(\pi;\tau)}$, (as described in [**Lus84a**, §4.5]), such that $\rho_{(n;-)}$ is the trivial character and $\rho_{(-;1^n)}$ is the sign character. By composing with the bijection $\widetilde{\mathbb{Y}}_n^1 \to \mathcal{BP}(n)$ described in Section 1.6 we obtain a resulting bijection $\widetilde{\mathbb{Y}}_n^1 \to \mathrm{Irr}(\mathbf{W})$. We will denote the elements of $\mathrm{Irr}(\mathbf{W})$ by their corresponding symbol in $\widetilde{\mathbb{Y}}_n^1$.

Lemma 1.60 (Lusztig, [Lus84a, §4.5]). *An irreducible character $\rho \in \mathrm{Irr}(\mathbf{W})$ is special if and only if its corresponding symbol is special. Let $[\Lambda_1], [\Lambda_2] \in \widetilde{\mathbb{Y}}_n^1$ be two symbols with their representatives chosen such that $|\Lambda_1| = |\Lambda_2|$, then their corresponding characters lie in the same family if and only if $\Lambda_1 \equiv \Lambda_2$.*

Let $\mathbf{W}(\mathcal{F}) \subseteq \mathrm{Irr}(\mathbf{W})$ be a family of irreducible characters of $\mathbf{W}$. Assume $[Z] \in \widetilde{\mathbb{Y}}_n^1$ is the unique special symbol in $\mathbf{W}(\mathcal{F})$ and $Z \in \mathbb{Y}_n^1$ is a fixed representative of $[Z]$. We define $\mathcal{M}_{\mathcal{F}}$ to be the set of all symbols $[\Lambda] \in \cup_{d\equiv 1 \pmod 2} \widetilde{\mathbb{Y}}_n^d$ with positive defect such that $|\Lambda| = |Z|$ and $\Lambda \equiv Z$. Note that, using shifts, we can see the definition of $\mathcal{M}_{\mathcal{F}}$ does not depend upon our choice of representative for $[Z]$. The unipotent characters in the family $\mathcal{F} \subseteq \mathcal{E}(G,1)$ corresponding to $\mathbf{W}(\mathcal{F})$ are then in bijection with the set $\mathcal{M}_{\mathcal{F}}$.

Example 1.61. Assume $\mathbf{G}$ is simple of type B_2. Recall the set $\widetilde{\mathbb{Y}}_2^1$, which was described in Example 1.55. There are three families $\mathbf{W}(\mathcal{F}) \subseteq \mathrm{Irr}(\mathbf{W})$, namely

$$\left\{\begin{bmatrix} 0 & & 1 & & 2 \\ & 1 & & 2 & \end{bmatrix}\right\} \quad \left\{\begin{bmatrix} 0 & & 1 \\ & 2 & \end{bmatrix}, \begin{bmatrix} 0 & & 2 \\ & 1 & \end{bmatrix}, \begin{bmatrix} 1 & & 2 \\ & 0 & \end{bmatrix}\right\} \quad \left\{\begin{bmatrix} 2 \\ - \end{bmatrix}\right\}$$

If $|\mathbf{W}(\mathcal{F})| = 1$ then we have $\mathcal{M}_{\mathcal{F}}$ contains only the symbol parameterising the character in $\mathbf{W}(\mathcal{F})$, hence the corresponding family of unipotent characters contains only one character. Let us now consider the remaining family in $\mathrm{Irr}(\mathbf{W})$. In this case we have the set $\mathcal{M}_{\mathcal{F}}$ contains the four symbols

$$\begin{bmatrix} 1 & & 2 \\ & 0 & \end{bmatrix} \qquad \begin{bmatrix} 0 & & 2 \\ & 1 & \end{bmatrix} \qquad \begin{bmatrix} 0 & & 1 \\ & 2 & \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 \\ & - & \end{bmatrix}.$$

Therefore there are four unipotent characters of G in the corresponding family $\mathcal{F}$.

1.7.3 Type D_n

If $n \geqslant 2$ we denote by W_n' the subgroup $\mathfrak{A}(V) \cap W_n$, where $\mathfrak{A}(V) \leqslant \mathfrak{S}(V)$ is the alternating subgroup of the symmetric group and W_n is as in Section 1.7.2. If $n = 0$ we take $W_n' = \{1\}$. If $\mathbf{G}$ is of type D_n and $n \geqslant 2$ then we identify $\mathbf{W}$ with W_n' such that, for $1 \leqslant i \leqslant n-1$, the simple reflection s_i corresponds to the product of transpositions $(i, i+1)(i', (i+1)')$ and s_n corresponds to $(n-1, n')(n, (n-1)')$. We have a bijection $\overline{\mathcal{BP}(n)} \to \mathrm{Irr}(\mathbf{W})$ denoted by $(\pi;\tau) \mapsto \rho_{(\pi;\tau)}$, (as described in [**Lus84a**, §4.6]), such that $\rho_{(n;-)} = \rho_{(-;n)}$ is the trivial character and $\rho_{(-;1^n)} = \rho_{(1^n;-)}$ is the sign character. By composing with the bijection $\overline{\mathbb{Y}}_n^0 \to \overline{\mathcal{BP}(n)}$ described in Section 1.6 we obtain a resulting bijection $\overline{\mathbb{Y}}_n^0 \to \mathrm{Irr}(\mathbf{W})$. We will denote the elements of $\mathrm{Irr}(\mathbf{W})$ by their corresponding symbol in $\overline{\mathbb{Y}}_n^0$.

Lemma 1.62 (Lusztig, [Lus84a, §4.6]). *An irreducible character $\rho \in \mathrm{Irr}(\mathbf{W})$ is special if and only if its corresponding symbol is special. Let $[\Lambda_1], [\Lambda_2] \in \overline{\mathbb{Y}}_n^1$ be two non-degenerate symbols with their representatives chosen such that $|\Lambda_1| = |\Lambda_2|$ then their corresponding characters lie in the same family if and only if $\Lambda_1 \equiv \Lambda_2$. The character corresponding to any degenerate symbol lies in its own family.*

If $\mathbf{W}(\mathcal{F}) \subseteq \mathrm{Irr}(\mathbf{W})$ is a family of characters parameterised by degenerate symbols then $|\mathbf{W}(\mathcal{F})| = 1$ and we take $\mathcal{M}_{\mathcal{F}}$ to be the symbol parameterising the character in $\mathbf{W}(\mathcal{F})$. Assume now that $\mathbf{W}(\mathcal{F}) \subseteq \mathrm{Irr}(\mathbf{W})$ is a family of irreducible characters which is parameterised by non-degenerate symbols. Assume $[Z] \in \overline{\mathbb{Y}}_n^0$ is the unique special symbol in $\mathbf{W}(\mathcal{F})$ and $Z \in \mathbb{Y}_n^0$ is a fixed representative of $[Z]$. We define $\mathcal{M}_{\mathcal{F}}$ to be the set of all symbols $[\Lambda] \in \cup_{d \equiv 0 \pmod 4} \overline{\mathbb{Y}}_n^d$ such that $|\Lambda| = |Z|$ and $\Lambda \equiv Z$. Note that, using shifts, we can see the definition of $\mathcal{M}_{\mathcal{F}}$ does not depend upon our choice of representative for $[Z]$. The unipotent characters in the family $\mathcal{F} \subseteq \mathcal{E}(G, 1)$ corresponding to $\mathbf{W}(\mathcal{F})$ are then in bijection with the set $\mathcal{M}_{\mathcal{F}}$.

Example 1.63. Assume $\mathbf{G}$ is a simple group of type D_4. There are two degenerate bipartitions in $\mathcal{BP}(4)$, namely $(1^2; 1^2)$ and $(2; 2)$. The degenerate bipartitions $(1^2; 1^2)_+$ and $(1^2; 1^2)_-$ are in bijective correspondence with the degenerate symbols

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}_+ \qquad \text{and} \qquad \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}_- .$$

Every character in $\mathrm{Irr}(\mathbf{W})$ lies in its own family except for the three characters with corresponding symbols

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix},$$

which together form a family. The corresponding set $\mathcal{M}_{\mathcal{F}}$ contains the four symbols

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 & 3 \\ & - & & \end{bmatrix}.$$

Therefore there are four unipotent characters of G in the corresponding family $\mathcal{F}$.

1.8 Invariance of Unipotent Characters

Now we have discussed characters of Weyl groups in detail we can discuss precisely when a unipotent character is invariant under composition with an automorphism of $\mathbf{G}$ defined over $\mathbb{F}_q$.

Lemma 1.64. *Assume $\mathbf{G}$ is simple and $\gamma \in \mathrm{Out}(\mathbf{G}, F)$ is an automorphism of $\mathbf{G}$ commuting with F. If γ induces the identity on $\mathbf{W}$ then every unipotent character is fixed under compo-*

sition with γ. Assume $\mathbf{G}$ *is of type* A_n, D_n *or* E_6 *and* γ *acts on* $(\mathbf{W}, \mathbb{S})$ *as one of the graph automorphisms listed in Figure 1.2. Every unipotent character is fixed under composition with* γ *except in the following cases:*

- $\mathbf{G}$ *is of type* D_{2n}, γ *has order 2 and the character is parameterised by a degenerate symbol.*
- $\mathbf{G}$ *is of type* D_4, γ *has order 3 and the character is parameterised by one of the symbols*

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix}_{\pm} \quad \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}_{\pm} \quad \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}. \tag{1.4}$$

Proof. Let us start with the case where γ acts trivially on $\mathbf{W}$, hence γ acts trivially on $\mathrm{Irr}(\mathbf{W})$. Let χ be a unipotent character of G. If χ lies in a family which contains more than one character then it is fixed under composition with γ by Proposition 1.51. Hence let us assume that χ lies in a family of cardinality one then there exists a sign $\varepsilon_\rho \in \{\pm 1\}$ and a uniform almost character $R_{\tilde{\rho}}^{\mathbf{G}}$, for some $\rho \in \mathrm{Irr}(\mathbf{W})^F$, such that $\chi = \varepsilon_\rho R_{\tilde{\rho}}^{\mathbf{G}}$. By Corollary 1.50 we have

$$\chi \circ \gamma = \varepsilon_\rho R_{\tilde{\rho}}^{\mathbf{G}} \circ \gamma = \varepsilon_\rho R_{\tilde{\rho}\circ\gamma}^{\mathbf{G}} = \chi.$$

Note that we obtain the last equality in the following way. As $\tilde{\rho} \circ \gamma$ is an extension of $\rho \circ \gamma = \rho$ we have by Proposition 1.44 that $\tilde{\rho} \circ \gamma = \pm\tilde{\rho}$. Therefore $\varepsilon_\rho R_{\tilde{\rho}\circ\gamma}^{\mathbf{G}} = \pm\chi$ which implies $\varepsilon_\rho R_{\tilde{\rho}\circ\gamma}^{\mathbf{G}} = \chi$ as γ must take characters to characters.

We now assume that γ acts on $(\mathbf{W}, \mathbb{S})$ as one of the graph automorphisms listed in Figure 1.2. Note that because of Proposition 1.51 we need only consider the action of γ on unipotent characters lying in a family by themselves, hence on almost characters. If $\mathbf{G}$ is of type A_n, D_{2n+1} or E_6 then every irreducible character of the Weyl group is fixed under composition with γ, (this is mentioned in [**Lus84a**, §4.18 and 4.19]). The result then follows by the argument in the previous case.

If $\mathbf{G}$ is of type D_{2n} then an irreducible character of the Weyl group is fixed under composition with γ if and only if the symbol parameterising the character is non-degenerate, (see [**Lus84a**, §4.18 and 4.19]). If F does not act trivially on $\mathbf{W}$ then $[\Lambda]_\pm \notin \mathrm{Irr}(\mathbf{W})^F$ so the result clearly holds in this case. If F acts as the identity on $\mathbf{W}$ then $[\Lambda]_\pm \in \mathrm{Irr}(\mathbf{W})^F$ so the only unipotent characters not fixed under composition by γ are the characters given up to sign by the almost characters $R_{[\Lambda]_\pm}^{\mathbf{G}}$.

Finally if $\mathbf{G}$ is of type D_4 and γ is of order 3 then the only Weyl group characters not fixed under composition with γ are the characters listed in Eq. (1.4), (this is mentioned in [**Lus84a**, §4.19]). Therefore the statement is clear by the previous arguments. ∎

Remark 1.65. The above result was already known to Lusztig in [**Lus88**, §5] but was not stated in detail. This statement was explicitly made by Malle in [**Mal07**, Proposition 3.7]. His proof of this result, (which uses Hecke algebra techniques on the extended Weyl group $\widetilde{\mathbf{W}}$), can be found in [**Mal91**, §1].

Assume $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$ is a direct product of isomorphic simple groups and denote by F' a Frobenius endomorphism of the isomorphic groups $\mathbf{G}_1$ and $\mathbf{G}_2$. Let $F = F' \times F'$ so that $\mathbf{G}^F = \mathbf{G}_1^{F'} \times \mathbf{G}_2^{F'}$ is a direct product of isomorphic groups. Any unipotent character of G will be a direct product $\chi_1 \boxtimes \chi_2$ where χ_i is a unipotent character of G_i. Assume $\gamma \in \mathrm{Out}(\mathbf{G}, F)$ acts as an involutary graph automorphism permuting the two isomorphic simple groups, (i.e. $\gamma(\mathbf{G}_1) = \mathbf{G}_2$ and $\gamma(\mathbf{G}_2) = \mathbf{G}_1$). It is then clear that $\chi_1 \boxtimes \chi_2$ is fixed by γ if and only if $\chi_1 \circ \gamma = \chi_2$, in other words γ induces an isomorphism of the underlying representations.

1.9 Beyond Unipotent Characters

So far we have described part of Lusztig's method for determining the irreducible characters of G lying in the Lusztig series $\mathcal{E}(G, 1)$. However, we now show the importance of unipotent characters in determining the remaining Lusztig series. Let us first recall the following definition

Definition 1.66. A character $\chi \in \mathrm{Irr}(G)$ is called *semisimple* if its Alvis–Curtis dual is up to sign an irreducible constituent of a Gelfand–Graev character.

We won't expand this definition here but for more information see [**DM91**, §14]. In general semisimple characters as those characters whose degree is coprime to p. If the centre of $\mathbf{G}$ is connected then each Lusztig series $\mathcal{E}(G, s)$ contains a unique semisimple character. This is a consequence of the fact that, under these conditions, there is a unique Gelfand–Graev character which is the sum of all the regular characters of G, (see [**DM91**, Corollary 14.47]).

If $s \in G^\star$ then the restriction of $F^\star$ to $C_{\mathbf{G}^\star}(s)$ is a Frobenius endomorphism and we denote the corresponding fixed point group by $C_{G^\star}(s) = C_{\mathbf{G}^\star}(s)^{F^\star}$. When $Z(\mathbf{G})$ is connected we have that the centraliser of every semisimple element in $\mathbf{G}^\star$ is connected, (see [**Car93**, Theorem 4.5.9]). With this notation in place we can now state the following result of Lusztig which demonstrates the importance of unipotent characters.

Theorem 1.67 (Lusztig, [Lus84a, Main Theorem 4.23]). *Let $\mathbf{G}$ be a connected reductive algebraic group such that $Z(\mathbf{G})$ is connected. There exists a bijection*

$$\Psi_s : \mathcal{E}(G, s) \to \mathcal{E}(C_{G^\star}(s), 1)$$

such that $\langle \chi, R^{\mathbf{G}}_{\mathbf{T}^\star}(s)\rangle = \pm\langle \Psi_s(\chi), R^{C_{\mathbf{G}^\star}(s)}_{\mathbf{T}^\star}(1)\rangle$ *for all pairs* $(\mathbf{T}^\star, s) \in \nabla^\star(\mathbf{G}, F)$. *Furthermore if* $\chi_{\mathrm{ss}} \in \mathcal{E}(G, s)$ *is the unique semisimple character in the Lusztig series then the degree of* χ *is given by* $\chi(1) = \chi_{\mathrm{ss}}(1)\Psi_s(\chi)(1)$.

The techniques developed by Lusztig to parameterise unipotent characters are applicable to any connected reductive algebraic group. Hence when $Z(\mathbf{G})$ is connected we have a parameterisation of the characters in $\mathcal{E}(C_{G^\star}(s), 1)$ and can even determine their degrees. Note also that the sign $\pm$ in the statement of the theorem can be described explicitly using the notion of $\mathbb{F}_q$-rank. Theorem 1.67 then gives us a very explicit and powerful way to parameterise all irreducible characters of G and determine their degrees.

Remark 1.68. When $Z(\mathbf{G})$ is connected the bijection in Theorem 1.67 allows us to define a *family* of characters $\mathcal{F} \subseteq \mathcal{E}(G, s)$ to be the preimage of a family of characters in $\mathcal{E}(C_{G^\star}(s), 1)$. We will not need the notion of families of characters when $Z(\mathbf{G})$ is disconnected, which is a slightly more delicate issue.

If $Z(\mathbf{G})$ is no longer connected then it is not necessarily true that the centraliser of every semisimple element in the dual is connected.

Example 1.69. Assume $p \neq 2$ and take $\mathbf{G}$ to be $\mathrm{SL}_2(\mathbb{K})$ then we may take $\mathbf{G}^\star$ to be $\mathrm{PGL}_2(\mathbb{K})$. Let s be the image of the diagonal matrix $\mathrm{diag}(1, -1)$ in $\mathrm{PGL}_2(\mathbb{K})$. The centraliser $C_{\mathbf{G}^\star}(s)$ has two connected components and a set of coset representatives for $C_{\mathbf{G}^\star}(s)^\circ$ is given by $\{1, \dot{w}\}$ where w is the non-identity element of the Weyl group of $\mathbf{G}^\star$.

With this in mind if $Z(\mathbf{G})$ is disconnected we cannot expect Theorem 1.67 to hold as even the definition of unipotent character requires the group to be connected. However Lusztig has managed to use Clifford theory to obtain a generalisation of this result for groups with a disconnected centre. Following Lusztig, (see [**Lus88**, §7]), we make the following definition.

Definition 1.70. Let $\tilde{\mathbf{G}}$ be a connected reductive algebraic group and $\iota : \mathbf{G} \to \tilde{\mathbf{G}}$ a homomorphism defined over $\mathbb{F}_q$. We say ι is a *regular embedding* if:

- $\tilde{\mathbf{G}}$ has a connected centre,
- ι restricts to an isomorphism of $\mathbf{G}$ with a closed subgroup of $\tilde{\mathbf{G}}$
- and $\iota(\mathbf{G})$, $\tilde{\mathbf{G}}$ have the same derived subgroup.

From now on we fix a group $\tilde{\mathbf{G}}$ and a regular embedding $\iota : \mathbf{G} \to \tilde{\mathbf{G}}$. If $Z(\mathbf{G})$ is connected then we assume $\tilde{\mathbf{G}} = \mathbf{G}$ and $\iota(g) = g$ for all $g \in \mathbf{G}$.

If $\mathbf{G}$ is such that $Z(\mathbf{G})$ is disconnected then a general construction for a group $\tilde{\mathbf{G}}$ and a regular embedding ι was given by Deligne and Lusztig, (see [**DL76**, pg. 132] or [**Gec03**, Exercise 4.7.7]). As $\iota(\mathbf{G})$ contains the derived subgroup we must have it is a normal subgroup of $\tilde{\mathbf{G}}$. Let $\mathbf{G}^\star$ and $\tilde{\mathbf{G}}^\star$ be the dual groups of $\mathbf{G}$ and $\tilde{\mathbf{G}}$ respectively. We will constantly identify $\mathbf{G}$ with its image $\iota(\mathbf{G})$ in $\tilde{\mathbf{G}}$ without specific reference.

We define $\tilde{\mathbf{T}}_0$ and $\tilde{\mathbf{B}}_0$ to be the unique maximal torus and Borel subgroup of $\tilde{\mathbf{G}}$ such that $\mathbf{T}_0 = \mathbf{G} \cap \tilde{\mathbf{T}}_0$ and $\mathbf{B}_0 = \mathbf{G} \cap \tilde{\mathbf{B}}_0$. We assume F denotes a generalised Frobenius endomorphism of $\tilde{\mathbf{G}}$ such that $F \circ \iota = \iota \circ F$ then it is clear that $\tilde{\mathbf{T}}_0$ and $\tilde{\mathbf{B}}_0$ are F-stable. Let us fix a dual group $\tilde{\mathbf{G}}^\star$ of $\tilde{\mathbf{G}}$ with maximal torus $\tilde{\mathbf{T}}_0^\star$ such that the triples $(\tilde{\mathbf{G}}, \tilde{\mathbf{T}}_0, F)$ and $(\tilde{\mathbf{G}}^\star, \tilde{\mathbf{T}}_0^\star, F^\star)$ are in duality, (where $F^\star$ is an appropriate generalised Frobenius endomorphism of $\tilde{\mathbf{G}}^\star$).

Lemma 1.71. *The regular embedding ι induces a surjective morphism $\iota^\star : \tilde{\mathbf{G}}^\star \to \mathbf{G}^\star$ defined over $\mathbb{F}_q$ such that $\iota^\star(\tilde{\mathbf{T}}_0^\star) = \mathbf{T}_0^\star$, which is unique up to composition by an element of* $\operatorname{Inn} \tilde{\mathbf{T}}_0^\star$. *Furthermore* $\operatorname{Ker}(\iota^\star)$ *is contained in $Z(\tilde{\mathbf{G}}^\star)$.*

Proof. As the image under ι of $\mathbf{T}_0$ is a subtorus of $\tilde{\mathbf{T}}_0$ there exists a subtorus $\mathbf{S}$ of $\tilde{\mathbf{T}}_0$ such that $\tilde{\mathbf{T}}_0 = \mathbf{T}_0 \times \mathbf{S}$ is a direct product, (see [**DM91**, Proposition 0.5]). In particular $X(\tilde{\mathbf{T}}_0) = X(\mathbf{T}_0) \oplus X(\mathbf{S})$ is a direct sum so we have a surjective homomorphism of abelian groups $X(\tilde{\mathbf{T}}_0) \to X(\mathbf{T}_0)$ given by projection to $X(\mathbf{T}_0)$. By duality this gives a surjective homomorphism of abelian groups $\varphi : \check{X}(\tilde{\mathbf{T}}_0^\star) \to \check{X}(\mathbf{T}_0^\star)$.

By the duality between the character and cocharacter groups we obtain a homomorphism of abelian groups $\check{\varphi} : X(\mathbf{T}_0^\star) \to X(\tilde{\mathbf{T}}_0^\star)$ which we claim is injective. Assume $\check{\varphi}(\chi_1) = \check{\varphi}(\chi_2)$ then we have

$$\langle \check{\varphi}(\chi_1), \gamma \rangle = \langle \check{\varphi}(\chi_2), \gamma \rangle \Rightarrow \langle \chi_1, \varphi(\gamma) \rangle = \langle \chi_2, \varphi(\gamma) \rangle$$

for all $\gamma \in \check{X}(\tilde{\mathbf{T}}_0^\star)$. However as φ is surjective we have $\langle \chi_1, \gamma' \rangle = \langle \chi_2, \gamma' \rangle$ for all $\gamma' \in \check{X}(\mathbf{T}_0^\star)$ which implies $\chi_1 = \chi_2$. The existence and uniqueness of the surjective morphism follows from the remarks at the start of [**Ste99**, Chapter 5] which gives conditions for Theorem 1.11 to hold for arbitrary surjective morphisms. The statement regarding the kernel comes from the fact that the two groups are of the same type. ■

From now on we assume fixed a surjective endomorphism $\iota^\star : \tilde{\mathbf{G}}^\star \to \mathbf{G}^\star$. If $Z(\mathbf{G})$ is connected then we assume $\tilde{\mathbf{G}}^\star = \mathbf{G}^\star$ and $\iota^\star(g) = g$ for all $g \in \tilde{\mathbf{G}}^\star$.

Note that we may identify the Weyl groups of $\mathbf{G}$ and $\tilde{\mathbf{G}}$ through the embedding ι, similarly we may identify the Weyl groups of $\mathbf{G}^\star$ and $\tilde{\mathbf{G}}^\star$ through the surjection $\iota^\star$.

Example 1.72. Let $\mathbf{G}$ be $\mathrm{SL}_n(\mathbb{K})$ then we naturally have $\mathbf{G}$ is the derived subgroup of $\mathrm{GL}_n(\mathbb{K})$, hence we can take $\tilde{\mathbf{G}}$ to be $\mathrm{GL}_n(\mathbb{K})$. By [**DM91**, pg. 105 - Examples] we can take $\tilde{\mathbf{G}}^\star = \mathrm{GL}_n(\mathbb{K})$ and $\mathbf{G}^\star = \mathrm{PGL}_n(\mathbb{K})$, (this latter case follows from the fact that the dual of a simply connected group is always adjoint). Our morphisms are simply the obvious morphisms

$$\iota : \mathrm{SL}_n(\mathbb{K}) \hookrightarrow \mathrm{GL}_n(\mathbb{K}) \qquad\qquad \iota^\star : \mathrm{GL}_n(\mathbb{K}) \to \mathrm{PGL}_n(\mathbb{K}).$$

It is trivial to see that ι will satisfy all the conditions of a regular embedding.

If $s \in \mathbf{G}^\star$ is a semisimple element then we fix a semisimple element $\tilde{s} \in \tilde{\mathbf{G}}^\star$ such that $\iota^\star(\tilde{s}) = s$. The idea of Lusztig is to understand the series $\mathcal{E}(G,s)$ by restricting characters in the series $\mathcal{E}(\tilde{G},\tilde{s})$ to G. The most astounding part of Lusztig's result is that given any irreducible character of $\tilde{G}$ the restriction to G is multiplicity free, (see [**Lus88**, Proposition 10]).

Before stating Lusztig's theorem, we introduce some notation that we will maintain throughout. Assume $x \in \mathbf{G}$ then we denote the component group of the centraliser of x by $A_{\mathbf{G}}(x) := C_{\mathbf{G}}(x)/C_{\mathbf{G}}(x)^\circ$ and if $x \in G$ we denote its rational counter part by $A_G(x) = C_G(x)/C_G(x)^\circ$, where $C_G(x)^\circ = C_{\mathbf{G}}(x)^{\circ F}$. If $\mathbf{G}$ has an associated generalised Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ and $x \in G$ then $F : A_{\mathbf{G}}(x) \to A_{\mathbf{G}}(x)$ induces an automorphism of the finite group $A_{\mathbf{G}}(x)$ as an abstract group. Furthermore we have $A_{\mathbf{G}}(x)^F = A_G(x)$ by [**DM91**, Corollary 3.13].

Theorem 1.73 (Lusztig, [Lus88, Proposition 5.1]**).** *Let $\mathbf{G}$ be any connected reductive algebraic group then there exists a surjective map*

$$\Psi_s : \mathcal{E}(G,s) \to \mathcal{E}(C_{G^\star}(s)^\circ, 1)/A_{G^\star}(s)$$

with the following properties.

- *The fibres of Ψ_s are precisely the orbits of the action of $\tilde{G}/GZ(\tilde{G})$ on $\mathcal{E}(G,s)$.*
- *If $\Theta \in \mathcal{E}(C_{G^\star}(s)^\circ, 1)/A_{G^\star}(s)$ and Γ is the stabiliser in $A_{G^\star}(s)$ of an element in Θ, then the fibre $\Psi_s^{-1}(\Theta)$ has precisely $|\Gamma|$ elements.*
- *If $\chi \in \Psi_s^{-1}(\Theta)$ and $\mathbf{T}^\star$ is an $F^\star$-stable maximal torus of $\mathbf{G}^\star$ containing s then*

$$\langle \chi, R_{\mathbf{T}^\star}^{\mathbf{G}}(s)\rangle_G = \pm \sum_{\psi\in\Theta} \langle \psi, R_{\mathbf{T}^\star}^{C_{\mathbf{G}^\star}(s)^\circ}(1)\rangle_{C_{G^\star}(s)^\circ}$$

We now expand some of the statements of the theorem. As $\mathbf{G}$ is a normal subgroup of $\tilde{\mathbf{G}}$ we have G is a normal subgroup of $\tilde{G}$. The subgroup $GZ(\tilde{G})$ fixes every element of $\mathrm{Irr}(G)$ under the conjugation of $\tilde{G}$ on $\mathrm{Irr}(G)$ so we may consider this action to be an action of the quotient $\tilde{G}/GZ(\tilde{G})$ on $\mathrm{Irr}(G)$. It can be shown that this action preserves each Lusztig series $\mathcal{E}(G,s)$, (see for example [**Bon06**, 10.4]). This means we get a well defined action of the quotient on each Lusztig series.

We now consider the action on the Lusztig series of $C_{G^\star}(s)^\circ$. Again the connected component $C_{\mathbf{G}^\star}(s)^\circ$ is a normal subgroup of $C_{\mathbf{G}^\star}(s)$, which means $C_{G^\star}(s)^\circ$ is a normal subgroup of $C_{G^\star}(s)$. We have a natural action of $A_{G^\star}(s)$ on $\mathrm{Irr}(C_{G^\star}(s)^\circ)$ and using the same reference as before it can be checked that this action stabilises the set of unipotent characters. In particular we have an action of $A_{G^\star}(s)$ on $\mathcal{E}(C_{G^\star}(s)^\circ,1)$. We denote the orbits of this action by $\mathcal{E}(C_{G^\star}(s)^\circ,1)/A_{G^\star}(s)$.

The action of conjugation by an element of $C_{G^\star}(s)$ on $C_{\mathbf{G}^\star}(s)^\circ$ is an automorphism of $C_{\mathbf{G}^\star}(s)^\circ$ which commutes with $F^\star$ so is an element of $\mathrm{Aut}(C_{\mathbf{G}^\star}(s)^\circ,F^\star)$. Therefore we may apply Lemma 1.64 to understand explicitly the action of the component group $A_{G^\star}(s)$ on $\mathcal{E}(C_{G^\star}(s)^\circ,1)$. Following Lusztig, (see [**Lus88**, 12]), we make the following definition.

Definition 1.74. An irreducible character of $C_{G^\star}(s)$ is called *unipotent* if the restriction to $C_{G^\star}(s)^\circ$ is a sum of unipotent characters of $C_{G^\star}(s)^\circ$. We denote the set of all unipotent characters of $C_{G^\star}(s)$ by $\mathcal{E}(C_{G^\star}(s),1)$.

Corollary 1.75 (of Theorem 1.73). *For any connected reductive algebraic group* $\mathbf{G}$ *there exists a bijection*

$$\mathcal{E}(G,s) \to \mathcal{E}(C_{G^\star}(s),1)$$

for each Lusztig series of G.

We now consider Lusztig's result in terms of the Lusztig series $\mathcal{E}(\tilde{G},\tilde{s})$ and restriction of characters. We have the following sequence of bijections

$$\mathcal{E}(\tilde{G},\tilde{s}) \xrightarrow{\sim} \mathcal{E}(C_{\tilde{G}^\star}(\tilde{s}),1) \xrightarrow{\sim} \mathcal{E}(C_{G^\star}(s)^\circ,1) \tag{1.5}$$

The first bijection is given by Theorem 1.67 because $\tilde{\mathbf{G}}$ has a connected centre. The restriction of the morphism $\iota^\star$ to the centraliser $C_{\tilde{\mathbf{G}}^\star}(\tilde{s})$ clearly gives us an isotypic morphism $C_{\tilde{\mathbf{G}}^\star}(\tilde{s}) \to C_{\mathbf{G}^\star}(s)$, which we again denote by $\iota^\star$. However $C_{\tilde{\mathbf{G}}^\star}(\tilde{s})$ is connected so we must have $\iota^\star(C_{\tilde{\mathbf{G}}^\star}(\tilde{s})) = C_{\mathbf{G}^\star}(s)^\circ$, (see [**Gec03**, Proposition 2.2.14(b)]). By Lemma 1.47 composition with $\iota^\star$ induces the second bijection.

Let $\chi \in \mathcal{E}(G,s)$ then by Lusztig's result we know we can find a character $\psi \in \mathcal{E}(\tilde{G},\tilde{s})$ such that $\langle \chi, \mathrm{Res}_G^{\tilde{G}}(\psi)\rangle_G = 1$, (see for example [**Bon05**, Proposition 11.7(b)]). Under the bijections described in Eq. (1.5) we have an associated character $\bar{\psi} \in \mathcal{E}(C_{G^\star}(s)^\circ,1)$ so we have defined a map $\mathcal{E}(G,s) \to \mathcal{E}(C_{G^\star}(s)^\circ,1)$. We now ask how much freedom there is in the choice of ψ. Let $\psi_1, \psi_2 \in \mathcal{E}(\tilde{G}, \tilde{s})$ be two characters such that

$$\langle \chi, \mathrm{Res}_G^{\tilde{G}}(\psi_1) \rangle_G = \langle \chi, \mathrm{Res}_G^{\tilde{G}}(\psi_2) \rangle_G = 1$$

then the statement of Theorem 1.73 says that $\bar{\psi}_1$ and $\bar{\psi}_2$ will lie in the same orbit under the action of $A_{G^\star}(s)$.

We make one more remark regarding the compatibility of restriction of characters from $\tilde{G}$ and characters of $C_{G^\star}(s)$. Let $\mathbb{Z}\mathcal{E}(G,s)$ and $\mathbb{Z}\mathcal{E}(C_{G^\star}(s)^\circ, 1)$ denote the $\mathbb{Z}$-spans of $\mathcal{E}(G,s)$ and $\mathcal{E}(C_{G^\star}(s)^\circ, 1)$ respectively then we have the following maps

$$\begin{tikzcd}
\mathcal{E}(\tilde{G}, \tilde{s}) \arrow[d] \arrow[drr] & & \mathcal{E}(C_{G^\star}(s), 1) \arrow[d] \\
\mathbb{Z}\mathcal{E}(G,s) & & \mathbb{Z}\mathcal{E}(C_{G^\star}(s)^\circ, 1)
\end{tikzcd}$$

where the diagonal arrow is the bijection specified in Eq. (1.5) and the vertical arrows are given by restriction. Let $\psi \in \mathcal{E}(\tilde{G}, \tilde{s})$ be a character and $\bar{\psi} \in \mathcal{E}(C_{G^\star}(s)^\circ, 1)$ be the character in bijective correspondence with ψ. Then the number of distinct irreducible constituents in $\mathrm{Res}_G^{\tilde{G}}(\psi)$ is the same as the number of irreducible characters in $\mathcal{E}(C_{G^\star}(s), 1)$ whose restriction to $C_{G^\star}(s)^\circ$ contains $\bar{\psi}$.

Example 1.76. Assume $\mathbf{G}$ is $\mathrm{SL}_4(\mathbb{K})$, $p \neq 2$ and F is the Frobenius endomorphism F_r so that $G = \mathrm{SL}_4(q)$. We assume that $\tilde{\mathbf{G}} = \mathrm{GL}_4(\mathbb{K})$ with the natural embedding then we take $\tilde{\mathbf{G}}^\star = \mathrm{GL}_4(\mathbb{K})$ and $\mathbf{G}^\star = \mathrm{PGL}_4(\mathbb{K})$ with $\iota^\star$ such that $\iota^\star(g) = gZ(\mathrm{GL}_4(\mathbb{K}))$. We assume that $\mathbf{T}_0$ and $\mathbf{T}_0^\star$ are the standard maximal tori consisting of diagonal matrices. Consider the semisimple element $\tilde{s} = \mathrm{diag}(1,1,-1,-1) \in \mathrm{GL}_4(\mathbb{K})$ then $s = \iota^\star(\tilde{s})$ is the image in $\mathrm{PGL}_4(\mathbb{K})$.

The centraliser $C_{\tilde{\mathbf{G}}^\star}(\tilde{s})$ will be isomorphic to $\mathrm{GL}_2(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$ so the derived subgroup will be a semisimple group of type $\mathrm{A}_1\,\mathrm{A}_1$. Hence $C_{\mathbf{G}^\star}(s)^\circ$ will also be of semisimple type $\mathrm{A}_1\,\mathrm{A}_1$. We have $|A_{G^\star}(s)| = 2$ and a set of minimal length coset representatives is given by $\{1, \dot{w}^\star\}$ where $w^\star \in \mathbf{W}^\star$ is the product of the longest elements from $\mathbf{W}^\star$ and the Weyl group of $C_{\mathbf{G}^\star}(s)^\circ$ with respect to $\mathbf{T}_0^\star$, (which is a parabolic subgroup of $\mathbf{W}^\star$). The conjugation action of $\dot{w}^\star$ on $C_{\mathbf{G}^\star}(s)^\circ$ will be a graph automorphism which exchanges the two components of type A_1.

The Weyl group of type A_1 is isomorphic to the symmetric group $\mathfrak{S}_2$. There are only two irreducible characters of $\mathfrak{S}_2$, namely the trivial and sign character. By Section 1.7.1 we know there are two corresponding unipotent characters in a group of type A_1, in fact these will be the trivial and Steinberg character. Therefore we must have the unipotent characters in $\mathcal{E}(C_{G^\star}(s)^\circ, 1)$ are given by

$$1 \boxtimes 1 \qquad 1 \boxtimes \mathrm{St} \qquad \mathrm{St} \boxtimes 1 \qquad \mathrm{St} \boxtimes \mathrm{St},$$

where St denotes the Steinberg character and 1 denotes the trivial character. The orbits of these characters under the action of the $A_{G^\star}(s)$ are given by $\{1 \boxtimes 1\}$, $\{1 \boxtimes \mathrm{St}, \mathrm{St} \boxtimes 1\}$ and $\{\mathrm{St} \boxtimes \mathrm{St}\}$. By Theorem 1.73 we have $\mathcal{E}(G, s)$ contains five characters $\chi_1, \dots, \chi_5$ and the map Ψ_s is such that

$$\begin{tikzcd}
\chi_1 \arrow[dr] & & \chi_2 \arrow[dl] & & \chi_3 \arrow[d] & & \chi_4 \arrow[dr] & & \chi_5 \arrow[dl] \\
& \{1 \boxtimes 1\} & & & \{1 \boxtimes \mathrm{St}, \mathrm{St} \boxtimes 1\} & & & \{\mathrm{St} \boxtimes \mathrm{St}\} &
\end{tikzcd}$$

Chapter 2

Unipotent Supports and The Springer Correspondence

Assume $\mathbf{G}$ is a connected reductive algebraic group. We will denote by $\mathfrak{Cl}(\mathbf{G})$ the set of all $\mathbf{G}$-conjugacy classes and by $\mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})$ the subset of all unipotent conjugacy classes of $\mathbf{G}$. We then have the subset $\mathfrak{Cl}(\mathbf{G})^F$ of all F-stable conjugacy classes and the subset $\mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})^F = \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G}) \cap \mathfrak{Cl}(\mathbf{G})^F$ of all F-stable unipotent conjugacy classes. If the centre of $\mathbf{G}$ is connected then Lusztig, in [**Lus84a**, §13], gives a map $\mathrm{Irr}(G) \to \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})^F$. He later goes on to define the notion of a unipotent support for a character χ and shows that the unipotent conjugacy class he defines in [**Lus84a**, §13] is the unipotent support of χ. Lusztig's original method relies in an essential way upon the Springer correspondence, (between irreducible characters of a Weyl group and unipotent classes in an algebraic group), and truncated induction in the Weyl group.

In this chapter we will describe the method outlined by Lusztig in [**Lus84a**, §13]. We will also describe part of the Springer correspondence and $|A_{\mathbf{G}}(u)|$ whenever $\mathbf{G}$ is simple of type A_n, B_n, C_n, D_n, E_6 or E_7 and $u \in \mathbf{G}$ is a unipotent element. It will become clear at the end of Section 2.5 why we describe only this select information. The description of unipotent classes and the Springer correspondence is dependent upon whether p is a good or bad prime for $\mathbf{G}$. Recall that these are defined in the following way.

Definition 2.1. Assume $\mathbf{G}$ is a simple algebraic group and let $\alpha_0 = -\sum_{\alpha\in\Delta} m_\alpha \alpha$ be the unique lowest root in Φ, where $m_\alpha \in \mathbb{N}_0$. We say p is a *bad prime* for $\mathbf{G}$ if p divides m_α for some $\alpha \in \Delta$. Simply we say p is a *good prime* for $\mathbf{G}$ if it is not a bad prime. Assume $\mathbf{G}$ is any connected reductive algebraic group and let $\mathbf{G}_1 \cdots \mathbf{G}_r$ be the decomposition of the derived subgroup of $\mathbf{G}$ as an almost direct product of simple groups. We say p is a bad prime for $\mathbf{G}$ if it is a bad prime for some $\mathbf{G}_i$ and a good prime otherwise.

The description of the bad primes is easy to work out and for each simple type we describe these values in Table 2.1. Much of the information we gather in this section, (and some of the constructions we will consider), are invalid when p is a bad prime.

G	p
A_n	–
$\mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n$ $(n \geqslant 2)$	2
$\mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7$	2,3
E_8	2,3,5

Table 2.1: Bad Primes for Simple Algebraic Groups

Therefore we make the following assumption.

From now until the end of this thesis we assume that p is a good prime for $\mathbf{G}$.

2.1 Springer's Construction

Let us assume that $\mathbf{G}$ is a simple adjoint algebraic group which, for this time only, is defined over $\mathbb{C}$. There is a particularly amazing construction of irreducible representations of Weyl groups which was first introduced by Springer in [**Spr78**]. Although we will outline the process of [**Spr78**] it was shown by Lusztig, (see [**Lus84b**, §6]), that the Springer correspondence holds in any characteristic and for any connected reductive algebraic group isogenous to $\mathbf{G}$.

Given any unipotent element $u \in \mathbf{G}$ we have a variety $\mathfrak{B}_u^{\mathbf{G}}$ defined to be the collection of all Borel subgroups of $\mathbf{G}$ which contain u. The Weyl group does not act naturally on this variety but Springer has defined an action of $\mathbf{W}$ on the classical cohomology groups $H^i(\mathfrak{B}_u^{\mathbf{G}}, \mathbb{Q})$, (in other words the cohomology is taken with respect to the Zariski topology on the variety $\mathfrak{B}_u^{\mathbf{G}}$). Note that Springer works with algebraic groups defined over $\mathbb{C}$ so that he can work with classicial cohomology instead of having to work with ℓ-adic cohomology. Although the Weyl group does not act on $\mathfrak{B}_u^{\mathbf{G}}$ we do have a natural action of the centraliser $C_{\mathbf{G}}(u)$ on $\mathfrak{B}_u^{\mathbf{G}}$. This group acts on $\mathfrak{B}_u^{\mathbf{G}}$ by conjugation and the connected component acts trivially on each irreducible component of $\mathfrak{B}_u^{\mathbf{G}}$, (see [**Gec03**, Lemma 2.5.1(b)]). Hence the finite group $A_{\mathbf{G}}(u)$ acts on the set of irreducible components of $\mathfrak{B}_u^{\mathbf{G}}$.

Let $H^e(\mathfrak{B}_u^{\mathbf{G}}, \mathbb{Q})$ be the top non-vanishing cohomology group. Springer showed that the action of $\mathbf{W}$ he defined on $H^e(\mathfrak{B}_u^{\mathbf{G}}, \mathbb{Q})$ commutes with the action of the component group $A_{\mathbf{G}}(u)$. Hence we get an action of the direct product $A_{\mathbf{G}}(u) \times \mathbf{W}$ on $H^e(\mathfrak{B}_u^{\mathbf{G}}, \mathbb{Q})$, such that $H^e(\mathfrak{B}_u^{\mathbf{G}}, \mathbb{Q})$ is an $(A_{\mathbf{G}}(u), \mathbf{W})$-bimodule. Let $\psi \in \mathrm{Irr}(A_{\mathbf{G}}(u))$ be a character of $A_{\mathbf{G}}(u)$ and let $H^e(\mathfrak{B}_u^{\mathbf{G}}, \mathbb{Q})_\psi$ be the sum of all left $A_{\mathbf{G}}(u)$-submodules of $H^e(\mathfrak{B}_u^{\mathbf{G}}, \mathbb{Q})$ which afford the character ψ. Then $H^e(\mathfrak{B}_u^{\mathbf{G}}, \mathbb{Q})_\psi$ is naturally a right $\mathbf{W}$-module.

Let $\tilde{\rho}_{u,\psi}$ be the character of $\mathbf{W}$ afforded by $H^e(\mathfrak{B}_u^{\mathbf{G}}, \mathbb{Q})_\psi$. Springer's main result is that either $H^e(\mathfrak{B}_u^{\mathbf{G}}, \mathbb{Q})_\psi = 0$ or $\tilde{\rho}_{u,\psi} = \psi(1)\rho_{u,\psi}$ where $\rho_{u,\psi}$ is an irreducible character of $\mathbf{W}$. We call the character $\rho_{u,\psi}$ a *Springer character* of $\mathbf{W}$. By [**Spr78**, Theorem 1.13] every irreducible character of $\mathbf{W}$ is a Springer character and $\rho_{u_1,\psi_1} = \rho_{u_2,\psi_2}$ if and only if there exists $g \in \mathbf{G}$ such that ${}^g u_1 = u_2$ and ${}^g\psi_1 = \psi_2$. In particular we may parameterise $\mathrm{Irr}(\mathbf{W})$ by pairs $(\mathcal{O}, \psi)$, where $\mathcal{O}$ is a unipotent class of $\mathbf{G}$ and $\psi \in \mathrm{Irr}(A_{\mathbf{G}}(u))$ for some $u \in \mathcal{O}$. This parameterisation is what we refer to as the *Springer correspondence*. We must be careful to point out that it is not the case that every such pair $(\mathcal{O}, \psi)$ is needed to parameterise the irreducible characters of the Weyl group. Therefore the Springer correspondence gives us an injective map $\mathrm{Irr}(\mathbf{W}) \to \mathcal{N}_{\mathbf{G}}$ where $\mathcal{N}_{\mathbf{G}} = \{(\mathcal{O}, \psi) \mid \mathcal{O}$ is a unipotent class and $\psi \in \mathrm{Irr}(A_{\mathbf{G}}(u))$ for some $u \in \mathcal{O}\}$. An explanation for the missing pairs is given by Lusztig's description of the so called generalised Springer correspondence in [**Lus84b**].

The Springer correspondence has been described combinatorially by Shoji for all classical groups and F_4, by Springer for G_2 and by Alvis-Lusztig for E_6, E_7 and E_8. However in [**Lus84b**] Lusztig has given an alternative description of the Springer correspondence to Shoji for classical groups, which is also described in [**Car93**, §13.3].

Before we describe the Springer correspondence we make a few remarks regarding unipotent classes. Assume $\mathbf{G}$ is a connected reductive algebraic group then if p is a good prime for $\mathbf{G}$ the Bala–Carter theorem gives a classification of the unipotent classes of $\mathbf{G}$ which depends only upon the semisimple type of its derived subgroup, (see Theorem A.10). If $\mathbf{G}$ is a semisimple algebraic group then $\mathbf{G}_{\mathrm{sc}}$ and $\mathbf{G}_{\mathrm{ad}}$ have the same number of unipotent classes as $\mathbf{G}$ and are in bijection through the isogenies δ_{sc}, δ_{ad}. Therefore if $\mathcal{O}$ is a unipotent class of $\mathbf{G}$ we write $\mathcal{O}_{\mathrm{sc}}$ and $\mathcal{O}_{\mathrm{ad}}$ for the unipotent classes of $\mathbf{G}_{\mathrm{sc}}$ and $\mathbf{G}_{\mathrm{ad}}$ such that $\delta_{\mathrm{sc}}(\mathcal{O}_{\mathrm{sc}}) = \mathcal{O}$ and $\delta_{\mathrm{ad}}(\mathcal{O}) = \mathcal{O}_{\mathrm{ad}}$. Furthermore we fix representatives $u_{\mathrm{sc}} \in \mathcal{O}_{\mathrm{sc}}$, $u = \delta_{\mathrm{sc}}(u_{\mathrm{sc}}) \in \mathcal{O}$ and $u_{\mathrm{ad}} = (\delta_{\mathrm{ad}} \circ \delta_{\mathrm{sc}})(u_{\mathrm{sc}}) \in \mathcal{O}_{\mathrm{ad}}$ of all these classes.

We will be interested in the values $|A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})|$, $|A_{\mathbf{G}}(u)|$ and $|A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})|$. Although the classes are parameterised in the same way these values will be different in general. The unipotent classes of $\tilde{\mathbf{G}}_{\mathrm{sc}}$, $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{G}}_{\mathrm{ad}}$ are also in bijection with the unipotent classes of $\mathbf{G}_{\mathrm{sc}}$, $\mathbf{G}$ and $\mathbf{G}_{\mathrm{ad}}$ under the regular embeddings associated to each group. Therefore we write $\tilde{\mathcal{O}}_{\mathrm{sc}}$, $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{O}}_{\mathrm{ad}}$ for the unipotent classes of $\tilde{\mathbf{G}}_{\mathrm{sc}}$, $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{G}}_{\mathrm{ad}}$ which are respectively the images of $\mathcal{O}_{\mathrm{sc}}$, $\mathcal{O}$ and $\mathcal{O}_{\mathrm{ad}}$ under the regular embeddings. Finally we fix class representatives $\tilde{u}_{\mathrm{sc}} \in \tilde{\mathcal{O}}_{\mathrm{sc}}$, $\tilde{u} \in \tilde{\mathcal{O}}$ and $\tilde{u}_{\mathrm{ad}} \in \tilde{\mathcal{O}}_{\mathrm{ad}}$ to be the images of u_{sc}, u and u_{ad} under the respective regular embeddings. Note that we do not assume any compatibility between the elements $\tilde{u}_{\mathrm{sc}}$, $\tilde{u}$ and $\tilde{u}_{\mathrm{ad}}$.

Lemma 2.2. *Assume $\mathbf{G}$ is a connected reductive algebraic group with connected centre and let $\delta_{\mathrm{ad}} : \mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$ be an adjoint quotient of $\mathbf{G}$. Assume $u \in \mathbf{G}$ is a unipotent element and*

$u_{\mathrm{ad}} \in \mathbf{G}_{\mathrm{ad}}$ *is the image of* u *under* δ_{ad}*, then* $A_{\mathbf{G}}(u) \cong A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})$*.*

Proof. The restriction of δ_{ad} to the centraliser $C_{\mathbf{G}}(u)$ gives a homomorphism $C_{\mathbf{G}}(u) \to C_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})$ which induces a homomorphism $A_{\mathbf{G}}(u) \to A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})$. We have an injective homomorphism $Z(\mathbf{G}) \to C_{\mathbf{G}}(u)$ and, because $Z(\mathbf{G})$ is connected, the image of $Z(\mathbf{G})$ is contained in $C_{\mathbf{G}}(u)^{\circ}$. In particular the homomorphism $A_{\mathbf{G}}(u) \to A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})$ is injective because $\mathrm{Ker}(\delta_{\mathrm{ad}}) = Z(\mathbf{G})$ which implies these groups are isomorphic because they are finite. ■

From this lemma we see that

$$|A_{\tilde{\mathbf{G}}_{\mathrm{sc}}}(\tilde{u}_{\mathrm{sc}})| = |A_{\tilde{\mathbf{G}}}(\tilde{u})| = |A_{\tilde{\mathbf{G}}_{\mathrm{ad}}}(\tilde{u}_{\mathrm{ad}})| = |A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})|,$$

because each of these groups has a connected centre. In the following sections we aim to describe part of the Springer correspondence for most simple groups. Instead of describing the full Springer correspondence we will only be interested in the Springer characters of the form $\rho_{u,1}$. The Springer correspondence is described by Geck and Malle in **[GM00]** and we follow their description in what follows.

2.2 Description of the Springer Correspondence and $|A_{\mathbf{G}}(u)|$

In this section we assume $\mathbf{G}$ is a simple algebraic group of the type specified by the heading of the subsection. The values $|A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})|$, $|A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})|$ are well described and documented for each simple type. However the values $|A_{\mathbf{G}}(u)|$ for $\mathbf{G}$ of type A_n and D_n are not so well documented. We therefore describe here how to obtain a description of $|A_{\mathbf{G}}(u)|$ from a knowledge of $|A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})|$.

The restriction of δ_{sc} to $C_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})$ gives us a homomorphism $C_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}}) \to C_{\mathbf{G}}(u)$, which we again denote by δ_{sc}. As δ_{sc} is surjective given $g \in C_{\mathbf{G}}(u)$ there exists $h \in \mathbf{G}_{\mathrm{sc}}$ such that $\delta_{\mathrm{sc}}(h) = g$. Now $gug^{-1} = u$ implies that there exists $z \in \mathrm{Ker}(\delta_{\mathrm{sc}})$ such that $hu_{\mathrm{sc}}h^{-1} = u_{\mathrm{sc}}z$, however as u_{sc} is unipotent we must have $u_{\mathrm{sc}}z$ is unipotent. The element z is semisimple as it is contained in $\mathrm{Ker}(\delta_{\mathrm{sc}}) \leqslant \mathbf{T}_0$ and as it commutes with u_{sc} the element $u_{\mathrm{sc}}z$ is expressed in its unique Jordan decomposition. In particular $u_{\mathrm{sc}}z$ is unipotent if and only if $z = 1$, hence $h \in C_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})$ and the restriction of δ_{sc} to $C_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})$ is surjective. As δ_{sc} is a homomorphism of algebraic groups we have $\delta_{\mathrm{sc}}(C_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})^{\circ}) = C_{\mathbf{G}}(u)^{\circ}$ so this gives rise to an exact sequence

$$\mathrm{Ker}(\delta_{\mathrm{sc}}) \to A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}}) \to A_{\mathbf{G}}(u) \to 1.$$

Clearly this shows that to understand $A_{\mathbf{G}}(u)$ we need only understand the image of $\mathrm{Ker}(\delta_{\mathrm{sc}}) \leqslant Z(\mathbf{G}_{\mathrm{sc}})$ in $A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})$.

Remark 2.3. Assume $\mathbf{G}$ is a closed subgroup of $\mathrm{GL}_n(\mathbb{K})$ for some n. As any unipotent element $u \in \mathbf{G}$ is an element of a general linear group we may conjugate u to its Jordan normal form. The sizes of the Jordan blocks of the Jordan normal form of $\pi(u)$ then give a partition of n. In this way we can associate to every unipotent conjugacy class of $\mathbf{G}$ a partition n. If $\mathbf{G}$ is $\mathrm{GL}_n(\mathbb{K})$ itself then these partitions classify the unipotent conjugacy classes of $\mathbf{G}$.

2.2.1 Type A_{n-1}

Taking $\mathbf{G} = \mathrm{SL}_n(\mathbb{K})$ in Remark 2.3 gives a bijection $\mathcal{P}(n) \to \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})$ denoted by $\lambda \mapsto \mathcal{O}_\lambda$. The Springer correspondence in this case is particularly simple. Recall that we also have a bijection $\mathcal{P}(n) \to \mathrm{Irr}(\mathbf{W})$. Let $\lambda \in \mathcal{P}(n)$ be a partition then the Springer correspondence is such that $\rho_\lambda \mapsto (\mathcal{O}_\lambda, 1)$. Assume now that $u \in \mathcal{O}_\lambda$. Let us write the partition λ as $(\lambda_1, \dots, \lambda_s)$ and denote by $n_{p'}$ the largest divisor of n which is coprime to p. Then

$$|A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})| = \gcd(\lambda_1, \dots, \lambda_s, n_{p'}) \qquad\qquad |A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})| = 1.$$

The case of $\mathbf{G}_{\mathrm{sc}}$ is discussed in [**Lus84b**, §10.3] and the case of $\mathbf{G}_{\mathrm{ad}}$ is discussed in [**Car93**, §13.1].

It is well known that in a group of type A representatives for the component group $A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})$ can always be chosen to be elements of the centre. This is easily seen by taking $\mathbf{G}$ to be the adjoint group in the above discussion and noting that $A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})$ is trivial for any unipotent element $u_{\mathrm{ad}} \in \mathbf{G}_{\mathrm{ad}}$. We can understand the map $\mathrm{Ker}(\delta_{\mathrm{sc}}) \to A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})$ purely by numerical values as $Z(\mathbf{G}_{\mathrm{sc}})$ is cyclic. A non-trivial element $\hat{z} \in \mathrm{Ker}(\delta_{\mathrm{sc}})$ will determine a non-trivial element in $A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})$ if and only if the order of $\hat{z}$ divides the order of the component group. Therefore we have

$$|A_{\mathbf{G}}(u)| = |A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})| / \gcd(|\mathrm{Ker}(\delta_{\mathrm{sc}})|, |A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})|). \tag{2.1}$$

If $\mathbf{G}$ is simply connected then $\mathrm{Ker}(\delta_{\mathrm{sc}})$ is trivial and Eq. (2.1) tells us $|A_{\mathbf{G}}(u)| = |A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})|$. If $\mathbf{G}$ is adjoint then $\mathrm{Ker}(\delta_{\mathrm{sc}})$ is $Z(\mathbf{G}_{\mathrm{sc}})$ and the above says $|A_{\mathbf{G}}(u)| = 1$.

2.2.2 Type B_n

Taking $\mathbf{G} = \mathrm{SO}_{2n+1}(\mathbb{K})$ in Remark 2.3 gives a bijection $\mathcal{P}_0(2n+1) \to \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})$ denoted by $\lambda \mapsto \mathcal{O}_\lambda$. Let us write the partition $\lambda \in \mathcal{P}_0(2n+1)$ as $(\lambda_1, \lambda_2, \dots, \lambda_s)$, (note that s is necessarily odd). Following [**GM00**, §2.C] we partition λ into blocks of lengths one or two such that all odd λ_i lie in a block of length one and all even λ_i lie in a block of length two. Define a sequence $(c_1, c_2, \dots, c_s)$ by setting

$$\begin{cases} c_i := \frac{\lambda_i - 1}{2} + i - 1 & \text{if } \{\lambda_i\} \text{ is a block,} \\ c_i := c_{i+1} := \frac{\lambda_{i+1}}{2} + i - 1 & \text{if } \{\lambda_i, \lambda_{i+1}\} \text{ is a block.} \end{cases}$$

We now reorder the sequence such that $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_s$. We associate to the unipotent class $\mathcal{O}_\lambda$ the symbol

$$\mathrm{Spr}_{\mathrm{B}}(\mathcal{O}_\lambda) = \begin{bmatrix} c_1 & & c_3 & & \dots & & c_s \\ & c_2 & & \dots & & c_{s-1} & \end{bmatrix},$$

whose rank will vary depending upon λ. From $\mathrm{Spr}_{\mathrm{B}}(\mathcal{O}_\lambda)$ we define a bipartition $(\pi; \tau)$ of n in the following way. For each $1 \leqslant i \leqslant \frac{s+1}{2}$ let $\pi'_i = c_{2i-1} - 2(i-1)$ and for each $1 \leqslant i \leqslant \frac{s-1}{2}$ let $\tau'_i = c_{2i} - 2(i-1)$. Then $(\pi; \tau)$ is the bi-partition obtained from π', τ' by removing all zeros from the start of each sequence. The Springer correspondence is then given by the map $\rho_{(\pi;\tau)} \mapsto (\mathcal{O}_\lambda, 1)$.

Let $n_{\mathrm{B}}(u) = |\{i \in \mathbb{N} \mid i \text{ is odd and } r_i \neq 0\}|$ and $\kappa_{\mathrm{B}}(u) = \max\{r_i \mid i \text{ is odd}\}$ then

$$|A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})| = \begin{cases} 2^{n_{\mathrm{B}}(u_{\mathrm{sc}})} & \text{if } \kappa_{\mathrm{B}}(u_{\mathrm{sc}}) = 1, \\ 2^{n_{\mathrm{B}}(u_{\mathrm{sc}})-1} & \text{if } \kappa_{\mathrm{B}}(u_{\mathrm{sc}}) \geqslant 2. \end{cases} \qquad |A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})| = 2^{n_{\mathrm{B}}(u_{\mathrm{ad}})-1}.$$

Note that the case of $\mathbf{G}_{\mathrm{sc}}$ is discussed in **[Lus84b**, §14.3] and the case of $\mathbf{G}_{\mathrm{ad}}$ is discussed in **[Car93**, §13.1]. We comment here that in Carter's description of $|A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})|$ he treats separately the case $n_{\mathrm{B}}(u_{\mathrm{ad}}) = 0$, however we omit this as it cannot occur because λ partitions an odd number.

Example 2.4. Assume $\mathbf{G}$ is simple of type B_2. There are four unipotent classes of $\mathbf{G}$ which are parameterised by the partitions (5), $(1, 2^2)$, $(1^2, 3)$ and (1^5) respectively. Using the above algorithm we compute the bi-partition associated to each class.

Blocks of λ	$\mathrm{Spr}_{\mathrm{B}}(\mathcal{O}_\lambda)$	(π', τ')	(π, τ)	$n_{\mathrm{B}}(u_{\mathrm{sc}})$	$\kappa_{\mathrm{B}}(u_{\mathrm{sc}})$
$(\{5\})$	$\begin{bmatrix} 2 \\ - \end{bmatrix}$	$(2, -)$	$(2, -)$	1	1
$(\{1\}, \{2, 2\})$	$\begin{bmatrix} 0 & & 2 \\ & 2 & \end{bmatrix}$	$(00, 2)$	$(-, 2)$	1	1
$(\{1\}, \{1\}, \{3\})$	$\begin{bmatrix} 0 & & 3 \\ & 1 & \end{bmatrix}$	$(01, 1)$	$(1, 1)$	2	2
$(\{1\}, \{1\}, \{1\}, \{1\}, \{1\})$	$\begin{bmatrix} 0 & & 2 & & 4 \\ & 1 & & 3 & \end{bmatrix}$	$(000, 11)$	$(-, 11)$	1	5

2.2.3 Type C_n

Taking $\mathbf{G} = \mathrm{Sp}_{2n}(\mathbb{K})$ in Remark 2.3 gives a bijection $\mathcal{P}_1(2n) \to \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})$ denoted by $\lambda \mapsto \mathcal{O}_\lambda$. Let us write the partition $\lambda \in \mathcal{P}_1(2n)$ as $(\lambda_1, \lambda_2, \dots, \lambda_s)$. By possibly adding a zero to the beginning of $(\lambda_1, \lambda_2, \dots, \lambda_s)$ we can arrange this sequence such that $\lambda_1 = 0$ and s is odd. Following [**GM00**, §2.B] we partition the sequence $(\lambda_1, \lambda_2, \dots, \lambda_s)$ into blocks of lengths one or two such that all even λ_i lie in a block of length one and all odd λ_i lie in a block of length two. Define a sequence $(c_1, c_2, \dots, c_s)$ by setting

$$\begin{cases} c_i := \frac{\lambda_i}{2} + i - 1 & \text{if } \{\lambda_i\} \text{ is a block,} \\ c_i := c_{i+1} := \frac{\lambda_{i+1}+1}{2} + i - 1 & \text{if } \{\lambda_i, \lambda_{i+1}\} \text{ is a block.} \end{cases}$$

We now reorder the sequence such that $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_s$. We associate to the unipotent class $\mathcal{O}_\lambda$ the symbol

$$\mathrm{Spr}_{\mathrm{C}}(\mathcal{O}_\lambda) = \begin{bmatrix} c_1 & & c_3 & & \dots & & c_s \\ & c_2 & & \dots & & c_{s-1} & \end{bmatrix}.$$

whose rank will vary depending upon λ. From $\mathrm{Spr}_{\mathrm{C}}(\mathcal{O}_\lambda)$ we define a bipartition $(\pi; \tau)$ of n in the following way. For each $1 \leqslant i \leqslant \frac{s+1}{2}$ let $\pi_i' = c_{2i-1} - 2(i-1)$ and for each $1 \leqslant i \leqslant \frac{s-1}{2}$ let $\tau_i' = c_{2i} - 2i + 1$. Then $(\pi; \tau)$ is the bi-partition of n obtained from π', τ' by removing all zeros from the start of each sequence. The Springer correspondence is then given by the map $\rho_{(\pi;\tau)} \mapsto (\mathcal{O}_\lambda, 1)$

Let $n_{\mathrm{C}}(u) = |\{i \in \mathbb{N} \mid i \text{ is even and } r_i \neq 0\}|$ and $\delta_{\mathrm{C}}(u)$ be 1 if $r_i \neq 0$ for some even number $i \in \mathbb{N}$ and 0 otherwise. Then

$$|A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})| = 2^{n_{\mathrm{C}}(u_{\mathrm{sc}})} \qquad |A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})| = 2^{n_{\mathrm{C}}(u_{\mathrm{ad}}) - \delta_{\mathrm{C}}(u_{\mathrm{ad}})}.$$

Note that the case of $\mathbf{G}_{\mathrm{sc}}$ is discussed in [**Lus84b**, §10.4] and the case of $\mathbf{G}_{\mathrm{ad}}$ is discussed in [**Car93**, §13.1].

Example 2.5. Assume $\mathbf{G}$ is simple of type C_2. There are four unipotent classes of $\mathbf{G}$ which are parameterised by the partitions (4), $(1^2, 2)$, (1^4) and (2^2) respectively. Using the above algorithm we compute the bi-partition associated to each class.

Blocks of λ	$\mathrm{Spr}_{\mathrm{C}}(\mathcal{O}_\lambda)$	(π', τ')	(π, τ)	$n_{\mathrm{C}}(u)$	$\delta_{\mathrm{C}}(u)$
$(\{0\}, \{0\}, \{4\})$	$\begin{bmatrix} 0 & & 4 \\ & 1 & \end{bmatrix}$	$(02, 0)$	$(2, -)$	1	1
$(\{0\}, \{0\}, \{1,1\}, \{2\})$	$\begin{bmatrix} 0 & & 3 & & 5 \\ & 1 & & 3 & \end{bmatrix}$	$(011, 00)$	$(11, -)$	1	1

Blocks of λ	$\mathrm{Spr}_{\mathrm{C}}(\mathcal{O}_\lambda)$	(π', τ')	(π, τ)	$n_{\mathrm{C}}(u)$	$\delta_{\mathrm{C}}(u)$
$(\{0\}, \{1,1\}, \{1,1\})$	$\begin{bmatrix} 0 & & 2 & & 4 \\ & 2 & & 4 & \end{bmatrix}$	$(000, 11)$	$(-, 11)$	0	0
$(\{0\}, \{2\}, \{2\})$	$\begin{bmatrix} 0 & & 3 \\ & 2 & \end{bmatrix}$	$(01, 1)$	$(1, 1)$	1	0

2.2.4 Type D_n

Taking $\mathbf{G} = \mathrm{SO}_{2n}(\mathbb{K})$ in Remark 2.3 gives a bijection $\overline{\mathcal{P}_0(2n)} \to \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})$ denoted by $\lambda \mapsto \mathcal{O}_\lambda$. If $\lambda_\pm$ is a degenerate partition then we denote the associated class by $\mathcal{O}_\lambda^\pm$. We refer to conjugacy classes corresponding to degenerate partitions as *degenerate unipotent classes*. Let us write the partition $\lambda \in \overline{\mathcal{P}_0(2n)}$ as $(\lambda_1, \lambda_2, \dots, \lambda_s)$, (note that s is necessarily even). Following [**GM00**, §2.D] we partition the sequence $(\lambda_1, \lambda_2, \dots, \lambda_s)$ into blocks of lengths one or two such that all odd λ_i lie in a block of length 1 and all even λ_i lie in a block of length 2. Define a sequence $(c_1, c_2, \dots, c_s)$ by setting

$$\begin{cases} c_i = \frac{\lambda_i - 1}{2} + i - 1 & \text{if } \{\lambda_i\} \text{ is a block,} \\ c_i := c_{i+1} = \frac{\lambda_{i+1}}{2} + i - 1 & \text{if } \{\lambda_i, \lambda_{i+1}\} \text{ is a block.} \end{cases}$$

We now reorder the sequence such that $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_s$. We associate to the unipotent class $\mathcal{O}_\lambda$ the symbol

$$\mathrm{Spr}_{\mathrm{D}}(\mathcal{O}_\lambda) = \begin{bmatrix} c_1 & c_3 & \dots & c_{s-1} \\ c_2 & c_4 & \dots & c_s \end{bmatrix},$$

whose rank will vary depending upon λ. From $\mathrm{Spr}_{\mathrm{D}}(\mathcal{O}_\lambda)$ we define a bipartition $(\pi; \tau)$ of n in the following way. For each $1 \leqslant i \leqslant \frac{s}{2}$ let $\pi_i' = c_{2i-1} - 2(i-1)$ and $\tau_i' = c_{2i} - 2(i-1)$. Then $(\pi; \tau)$ is the bi-partition of n obtained from π', τ' by removing all zeros from the start of each sequence. The Springer correspondence is then given by the map $\rho_{(\pi;\tau)} \mapsto (\mathcal{O}_\lambda, 1)$.

The bipartition $(\pi; \tau)$ is degenerate if and only if λ is degenerate. We will now describe in detail, (following the idea in Carter [**Car93**, §13.3]), how to match the two degenerate unipotent classes to the corresponding Weyl group characters. In the following discussion we will freely use the notation and terminology introduced in Appendix A. Recall that degenerate partitions only occur in the case where $2n \equiv 0 \pmod 4$ so we may assume that n is even.

Let $\lambda \in \mathcal{P}_0(2n)$ be a degenerate partition then we can express this partition as

$(2\eta_1, 2\eta_1, \dots, 2\eta_{s'}, 2\eta_{s'})$, where $s' = s/2$ and $\eta_i \in \mathbb{N}$. We denote by η the sequence $(\eta_1, \dots, \eta_{s'})$ then η is a partition of $n/2$. If $\xi = (\xi_1, \dots, \xi_r) \in \mathcal{P}(n/2)$ then there are two $\mathbf{G}$-conjugacy classes of Levi subgroups with semisimple type $\mathrm{A}_{2\xi_1-1} \cdots \mathrm{A}_{2\xi_r-1}$. Two class representatives can be given by standard Levi subgroups and the two possibilities depend on whether the root α_n or α_{n-1} is contained in the root system of the standard Levi. We will denote $\mathbf{L}_\xi^+$ the Levi subgroup whose root system contains α_{n-1} and $\mathbf{L}_\xi^-$ the Levi subgroup whose root system contains α_n.

By Theorem A.10 if $\mathcal{O}$ is a unipotent class parameterised by the degenerate partition λ then either $\mathcal{O} \cap \mathbf{L}_\eta^+$ contains the regular unipotent class of $\mathbf{L}_\eta^+$ or $\mathcal{O} \cap \mathbf{L}_\eta^-$ contains the regular unipotent class of $\mathbf{L}_\eta^-$. Specifically this is because the partition λ gives the sizes of the Jordan blocks of a matrix representative for the class in a corresponding orthogonal group. We can now use this fact to distinguish the two degenerate unipotent classes $\mathcal{O}_\lambda^\pm$.

We fix $\mathcal{O}_\lambda^\pm$ to be the unipotent class such that $\mathcal{O}_\lambda^\pm \cap \mathbf{L}_\eta^\pm$ contains the regular unipotent class of $\mathbf{L}_\eta^\pm$. This uniquely distinguishes the two classes.

We now come to an interesting dichotomy, (which is the duality discussed by Spaltenstein in [**Spa82**, Chapitre III]). The way we have identified the two degenerate unipotent classes will allow us to compute the order of the component groups of their centraliser, however it will not allow us to compute the Springer correspondence. For this we must identify these classes as Richardson classes associated to their canonical parabolic subgroup and use a result of Lusztig–Spaltenstein to compute the Springer correspondence. In particular we must compute explicitly the weighted Dynkin diagram of these degenerate unipotent classes, which we do by fixing explicit class representatives.

Remark 2.6. The following discussion is only needed to prove Corollary 2.12. It became apparent after writing this that the statement of Corollary 2.12 is in fact shown by Spaltenstein in [**Spa82**, Proposition II.7.6]. Therefore the reader may feel free to ignore this discussion.

To compute the weighted Dynkin diagram of the degenerate classes we may work with any simple algebraic group $\mathbf{G}$ of type D_n. Therefore we take $\mathbf{G}$ to be the special orthogonal group $\mathrm{SO}_{2n}(\mathbb{K})$ which we define following [**Gec03**, 1.3.15]. Consider the following square matrix

$$J_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathrm{Mat}_n(\mathbb{K})$$

then $\mathrm{SO}_{2n}(\mathbb{K})$ is the set $\{X \in \mathrm{Mat}_{2n}(\mathbb{K}) \mid X^T J_{2n} X = J_{2n} \text{ and } \det(X) = 1\}$. We take $\mathbf{T}_0$ to be the maximal torus of all diagonal matrices and $\mathbf{B}_0$ to be the Borel subgroup consisting of all upper triangular matrices. Indexing the rows and columns of our matrices from top to bottom and left to right by $1, \dots, n, -n, \dots, -1$ any element $t \in \mathbf{T}_0$ will be of the form

$$t = \mathrm{diag}(t_1, \dots, t_n, t_{-n}, \dots, t_{-1}).$$

where $t_i \in \mathbb{K}^\times$ and $t_{-i} = t_i^{-1}$ for all $1 \leqslant i \leqslant n$.

For all $1 \leqslant i \leqslant n$ let $\varepsilon_i : \mathbf{T}_0 \to \mathbb{K}^\times$ be the homomorphism defined by $\varepsilon_i(t) = t_i$ for all $t \in \mathbf{T}_0$. We can describe the set of roots, positive roots and simple roots of $\mathbf{G}$ relative to $\mathbf{T}_0$ and $\mathbf{B}_0$ by

$$\begin{aligned} \Phi &= \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i < j \leqslant n\}, \\ \Phi^+ &= \{\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i < j \leqslant n\}, \\ \Delta &= \{\varepsilon_i - \varepsilon_{i+i}, \varepsilon_{n-1} + \varepsilon_n \mid 1 \leqslant i \leqslant n-1\}. \end{aligned}$$

We have the set of simple roots corresponds to the Dynkin diagram of type D_n given in Figure 1.1 by the labelling $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for all $1 \leqslant i \leqslant n-1$ and $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$. Given a positive root $\varepsilon_i \pm \varepsilon_j \in \Phi^+$ the corresponding root subgroup is given by

$$\mathbf{X}_{\varepsilon_i \pm \varepsilon_j} = \{I_{2n} + \kappa(E_{i,j} - E_{-j,-i}) \mid \kappa \in \mathbb{K}^+\},$$

where $E_{i,j} \in \mathrm{Mat}_{2n}(\mathbb{K})$ is the elementary matrix whose only non-zero entry is 1, which occurs in the ith row and jth column.

We let $\mathfrak{g}$ denote the Lie algebra of $\mathbf{G}$, which is $\mathfrak{so}_{2n}(\mathbb{K}) = \{X \in \mathrm{Mat}_{2n}(\mathbb{K}) \mid X^T J_{2n} + J_{2n} X = 0\}$ by [**Gec03**, Theorem 1.5.13]; we also denote by $\mathfrak{t}_0$ and $\mathfrak{b}_0$ the Lie algebras of $\mathbf{T}_0$ and $\mathbf{B}_0$. Differentiating the above gives us the corresponding combinatorial data for $\mathfrak{g}$ with respect to $\mathfrak{t}_0$ and $\mathfrak{b}_0$, (we will denote the roots in exactly the same way). As the weighted Dynkin diagram of a unipotent class depends only upon the characteristic being good, we may assume p is as large as we like for this computation. Using a Springer morphism it is also sufficient to determine representatives of the corresponding nilpotent orbits in $\mathfrak{g}$. To do this we will use the explicit standard triples given by Collingwood and McGovern in [**CM93**, §5.3]. There they work over $\mathbb{C}$ but assuming p is sufficiently large their results we use are equally applicable over $\mathbb{K}$.

Let us denote by $\widehat{\mathcal{O}}_\lambda^{\pm} \subset \mathfrak{g}$ the nilpotent orbits corresponding to the unipotent conjugacy classes $\mathcal{O}_\lambda^{\pm}$. We now describe the recipe given by Collingwood and McGovern for constructing standard triples of $\widehat{\mathcal{O}}_\lambda^{\pm}$, (see [**CM93**, Recipe 5.2.6]). Recall we rewrote the partition λ as $(2\eta_1, 2\eta_1, \ldots, 2\eta_{s'}, 2\eta_{s'})$. To each block $\{2\eta_i, 2\eta_i\}$ we fix a corresponding set of indices $\mathcal{C}_i = \{N_i + 1, \ldots, N_i + 2\eta_i\} \subseteq \{1, \ldots, n\}$, where $N_1 = 0$ and $N_i = 2(\eta_1 + \cdots + \eta_{i-1})$ for $i > 1$. Note that $\mathcal{C}_i \cap \mathcal{C}_j = \varnothing$ whenever $i \neq j$ and $\{1, \ldots, n\} = \cup_{i=1}^{s'} \mathcal{C}_i$, which follows from the fact that $(2\eta_1, \ldots, 2\eta_{s'})$ is a partition of n.

First we consider the case $\widehat{\mathcal{O}}_\lambda^{+}$. To the indexing set $\mathcal{C}_i$ we associate a corresponding set of positive roots $\mathcal{C}_i^+$ given by $\{\varepsilon_{N_i+1} - \varepsilon_{N_i+2}, \ldots, \varepsilon_{N_i+2\eta_i-1} - \varepsilon_{N_i+2\eta_i}\}$. We define the following element

$$h_{\mathcal{C}_i^+} = \sum_{k=1}^{2\eta_i} (2\eta_i - 2k + 1)(E_{N_i+k, N_i+k} - E_{-N_i-k, -N_i-k}) \in \mathfrak{t}_0$$

and take $h^+ = \sum_{i=1}^{s'} h_{\mathcal{C}_i^+}$. For any positive root $\alpha \in \Phi^+$ denote by $e_\alpha \in \mathfrak{g}$ a fixed element in the differential of the root subgroup $\mathbf{X}_\alpha$. We take $e^+ \in \mathfrak{b}_0$ to be $\sum_{i=1}^{s'} \sum_{\alpha \in \mathcal{C}_i^+} e_\alpha$ then there exists a nilpotent element f^+ such that $\{e^+, h^+, f^+\}$ is a standard triple for $\widehat{\mathcal{O}}_\lambda^{+}$.

Now we consider the case $\widehat{\mathcal{O}}_\lambda^{-}$. If $i \neq s'$ then to the indexing set $\mathcal{C}_i$ we define $\mathcal{C}_i^-$ to be the set of positive roots $\mathcal{C}_i^+$ defined above. If $i = s'$ we define $\mathcal{C}_i^-$ to be the set of positive roots $\{\varepsilon_{N_i+1} - \varepsilon_{N_i+2}, \ldots, \varepsilon_{N_i+2\eta_i-2} - \varepsilon_{N_i+2\eta_i-1}, \varepsilon_{N_i+2\eta_i-1} + \varepsilon_{N_i+2\eta_i}\}$. If $i \neq s'$ we define $h_{\mathcal{C}_i^-}$ to be the element $h_{\mathcal{C}_i^+}$ defined above and if $i = s'$ we define this element to be

$$h_{\mathcal{C}_i^-} = \sum_{k=1}^{2\eta_i-1} (2\eta_i - 2k + 1)(E_{N_i+k, N_i+k} - E_{-N_i-k, -N_i-k}) + (2\eta_i - 1)(E_{n,n} - E_{-n,-n}).$$

Note that we have only changed the sign of the coefficient of the final term. Take $h^- = \sum_{i=1}^{s'} h_{\mathcal{C}_i^-}$ and $e^- = \sum_{i=1}^{s'} \sum_{\alpha \in \mathcal{C}_i^-} e_\alpha$ then there exists a nilpotent element f^- such that $\{e^-, h^-, f^-\}$ is a standard triple for $\widehat{\mathcal{O}}_\lambda^{-}$.

Remark 2.7. We pause to make some comments before going on to compute an example of the above standard triples.

- The orthogonal form used to define $\mathfrak{so}_{2n}(\mathbb{K})$ in [**CM93**] is different from the one we use here, however the results are easily translated between the two realisations of $\mathfrak{so}_{2n}(\mathbb{K})$.
- Let $\mathfrak{l}_\eta^{\pm}$ be the Levi subalgebra of $\mathfrak{g}$ corresponding to $\mathbf{L}_\eta^{\pm}$. It can be verified, using the construction of standard triples in type A_n given in [**CM93**, Recipe

5.2.1], that the standard triples $\{e^{\pm}, h^{\pm}, f^{\pm}\}$ are representatives for the regular nilpotent orbit of $\mathfrak{l}_{\eta}^{\pm}$. In particular this shows that these really do correspond to the conjugacy classes $\mathcal{O}_{\lambda}^{\pm}$ as they have been defined above.

Example 2.8. Let us consider the case where $\mathbf{G}$ is $\mathrm{SO}_{12}(\mathbb{K})$ and $\lambda = (2^2, 4^2)$. We have two sets of indices $\mathcal{C}_1 = \{1,2\}$ and $\mathcal{C}_2 = \{3,4,5,6\}$. We have the corresponding sets of positive roots are given by $\mathcal{C}_1^+ = \mathcal{C}_1^- = \{\alpha_1, \alpha_2\}$ and $\mathcal{C}_2^+ = \{\alpha_3, \alpha_4, \alpha_5\}$ and $\mathcal{C}_2^- = \{\alpha_3, \alpha_4, \alpha_6\}$. The elements $h^{\pm}$ in the standard triples are given by

$$\begin{aligned} h^+ &= \mathrm{diag}(1,-1,3,1,-1,-3,3,1,-1,-3,1,-1), \\ h^- &= \mathrm{diag}(1,-1,3,1,-1,3,-3,1,-1,-3,1,-1). \end{aligned}$$

The reason that we have given these concrete constructions for standard triples in the nilpotent orbit is that we may now use the following result.

Lemma 2.9 (Collingwood–McGovern,[CM93, Lemma 5.3.5]). *We denote the weights of the diagram $\Delta(\widehat{\mathcal{O}}_{\lambda}^{\pm})$ associated to the nodes α_{n-1} and α_n by $(n_{\alpha_{n-1}}^{\pm}, n_{\alpha_n}^{\pm})$. We then have $(n_{\alpha_{n-1}}^{+}, n_{\alpha_n}^{+}) = (a,b)$ and $(n_{\alpha_{n-1}}^{-}, n_{\alpha_n}^{-}) = (b,a)$ where $b = 2-a$ and*

$$a = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4, \\ 2 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

To make the Springer correspondence precise we will need to know that the degenerate classes $\mathcal{O}_{\lambda}^{\pm}$ are Richardson classes. To do this we use the following result.

Proposition 2.10 ([Hum95, Proposition 7.9]). *Assume $\mathbf{G}$ is a connected reductive algebraic group and $\mathcal{O}$ is a unipotent class of $\mathbf{G}$. Let n_α be the weight of a root $\alpha \in \Delta$ on the weighted Dynkin diagram $\Delta(\mathcal{O})$. Assume $n_\alpha \neq 1$ for all $\alpha \in \Delta$ then $\mathcal{O}$ is the Richardson class of its canonical parabolic subgroup.*

Remark 2.11. The classes mentioned in Proposition 2.10 are called *even classes*. The reference in [**Hum95**] does not give a complete proof of the result, however he does outline a proof. The proofs of all statements used in the outline can be found in [**Car93**, Chapter 5].

Corollary 2.12. *If $n \equiv 0 \pmod 4$ then the unipotent classes $\mathcal{O}_{\lambda}^{\pm}$ are Richardson classes for parabolic subgroups with Levi complement $\mathbf{L}_{\eta^*}^{\pm}$. If $n \equiv 2 \pmod 4$ then the unipotent classes $\mathcal{O}_{\lambda}^{\pm}$ are Richardson classes for parabolic subgroups with Levi complement $\mathbf{L}_{\eta^*}^{\mp}$.*

Proof. To show this we must recall the construction of the weighted Dynkin diagram from [**CM93**, pg. 83]. For all λ_i, with $1 \leqslant i \leqslant s$, we consider the sequence $(\lambda_i - 1, \lambda_i - 3, \ldots, \lambda_i - (2\lambda_i - 1))$ containing λ_i entries. We write $\Omega = (\lambda_1 - 1, \ldots, \lambda_1 - (\lambda_1 -$

$1), \lambda_2 - 1, \dots)$ for the concatenation of all these sequences. Let $\Omega = (\omega_1, \dots, \omega_{2n})$ be a reordering of the sequence such that the entries are weakly decreasing, i.e. $\omega_1 \geqslant \omega_2 \geqslant \cdots \geqslant \omega_{2n}$. The weights of the Dynkin diagram are given by $n_{\alpha_i} = \omega_i - \omega_{i+1}$ for all $1 \leqslant i \leqslant n-2$ and by Lemma 2.9 when $i \in \{n-1, n\}$.

It is clear that the canonical Levi subgroup is a product of type A groups. The semisimple type of the canonical Levi subgroup does not depend upon the ordering of the weights $n_{\alpha_{n-1}}$ and n_{α_n}. Therefore we assume that $n_{\alpha_i} = \omega_i - \omega_{i+1}$ for $i \in \{n-1, n\}$. To determine the semisimple rank of these groups we need to determine the size of the blocks of consecutive weights on the Dynkin diagram that are equal to 0. However it is equivalent to determine the size of the blocks of consecutive ω_i's that are equal.

Note that every entry in Ω is an odd number of the form $\lambda_j - (2\ell + 1)$ with $0 \leqslant \ell \leqslant \lambda_j - 1$. Assume $1 \leqslant i \leqslant n$ then $\lambda_j - (2\ell+1) = 2i - 1 \Rightarrow \lambda_j = 2(i + \ell) \Rightarrow \lambda_j \geqslant 2i$. In particular for any $1 \leqslant i \leqslant n$ we have

$$\begin{aligned} |\{\omega_j \mid 1 \leqslant j \leqslant n \text{ and } \omega_j = 2i-1\}| &= |\{\lambda_j \mid 1 \leqslant j \leqslant s \text{ and } \lambda_j \geqslant 2i\}|, \\ &= 2|\{\eta_j \mid 1 \leqslant j \leqslant s/2 \text{ and } \eta_j \geqslant i\}|, \end{aligned}$$

however the last value is $2\eta'_i$ where $\eta' = (\eta'_1, \dots, \eta'_n)$ is as in the definition of the dual partition given in Section 1.6. Therefore the canonical Levi subgroup has semisimple type $\mathrm{A}_{2\eta^*_1 - 1} \cdots \mathrm{A}_{2\eta^*_r - 1}$ where $\eta^* = (\eta^*_1, \dots, \eta^*_r)$. The precise Levi subgroup is then determined by the weights $n_{\alpha_{n-1}}$ and n_{α_n} which are given in Lemma 2.9. ■

Assume $\mathbf{L}$ is a Levi subgroup of $\mathbf{G}$ contained in a parabolic subgroup $\mathbf{P}$ with unipotent radical $\mathbf{U_P}$. In **[LS79]** Lusztig and Spaltenstein have defined an induction map $\mathrm{Ind}_{\mathbf{L}}^{\mathbf{G}}$ taking a unipotent conjugacy class of $\mathbf{L}$ to a unipotent conjugacy class of $\mathbf{G}$, which is defined in the following way. If $\mathcal{O}$ is a unipotent class of $\mathbf{L}$ then $\mathrm{Ind}_{\mathbf{L}}^{\mathbf{G}}(\mathcal{O})$ is the unique unipotent conjugacy class of $\mathbf{G}$ such that $\mathrm{Ind}_{\mathbf{L}}^{\mathbf{G}}(\mathcal{O}) \cap \mathcal{O}\mathbf{U_P}$ is dense in $\mathcal{O}\mathbf{U_P}$. They show that this does not depend on the choice of $\mathbf{P}$ and depends only on the pair $(\mathbf{L}, \mathcal{O})$ up to $\mathbf{G}$ conjugacy. Hence we may assume that $\mathbf{L}$ is a Levi subgroup of $\mathbf{G}$ containing $\mathbf{T}_0$. The statements in **[LS79]** have some restrictions but these were removed in **[Lus84b]**.

Proposition 2.13 (Lusztig and Spaltenstein, [LS79, Theorem 3.5**]).** *Assume $\mathbf{G}$ is a connected reductive algebraic group and $\mathcal{O}$ is a unipotent conjugacy class of a Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ containing $\mathbf{T}_0$. Let $v \in \mathcal{O}$ and write $\rho_{v,1}^{\mathbf{W}(\mathbf{L})}$ for the Springer character of the Weyl group $\mathbf{W}(\mathbf{L})$ of $\mathbf{L}$, which is a parabolic subgroup of $\mathbf{W}$. Let $u \in \mathrm{Ind}_{\mathbf{L}}^{\mathbf{G}}(\mathcal{O})$ and write $\rho_{u,1}^{\mathbf{W}}$ for the Springer character of the Weyl group of $\mathbf{G}$. Then $\rho_{u,1}^{\mathbf{W}} = j_{\mathbf{W}(\mathbf{L})}^{\mathbf{W}}(\rho_{v,1}^{\mathbf{W}(\mathbf{L})})$.*

By **[LS79**, Proposition 19.(b)**]** we know that $\mathcal{O}_\lambda^{\pm}$ is $\mathrm{Ind}_{\mathbf{L}_{\eta^*}^{\pm}}^{\mathbf{G}}(\mathcal{O}_0)$ if $n \equiv 0 \pmod 4$ and

$\mathrm{Ind}_{\mathbf{L}_{\eta^*}^{\mp}}^{\mathbf{G}}(\mathcal{O}_0)$ if $n \equiv 2 \pmod 4$, where $\mathcal{O}_0$ denotes the trivial unipotent class. There is a restriction on [**LS79**, Proposition 19.(b)] that p is sufficiently large but this is only to ensure that unipotent classes are parameterised by their weighted Dynkin diagrams, which is known to hold in good characteristic. The Springer character of the trivial class is always the sign character, therefore the Springer characters of the degenerate unipotent classes are given by $j_{\mathbf{W}(\mathrm{A}_{\eta^*}^{\pm})}^{\mathbf{W}}(\mathrm{sgn}_{\mathbf{W}(\mathrm{A}_{\eta^*}^{\pm})})$, where $\mathbf{W}(\mathrm{A}_{\eta^*}^{\pm})$ is the Weyl group of $\mathbf{L}_{\eta^*}^{\pm}$ with respect to $\mathbf{T}_0$ and $\mathrm{sgn}_{\mathbf{W}(\mathrm{A}_{\eta^*}^{\pm})}$ is the sign character. It is commented by Carter in [**Car93**, §13.3] that these characters are the characters $\rho_{(\pi,\pi)_\pm}$ of $\mathbf{W}$, where $(\pi;\pi)$ is the degenerate bipartition obtained from $\mathrm{Spr}_{\mathrm{D}}(\mathcal{O}_\lambda^{\pm})$ at the start of this section.

> We now fix the notation $(\pi,\pi)_\pm$ to be such that $\rho_{(\pi,\pi)_\pm}$ is the induced character $j_{\mathbf{W}(\mathrm{A}_{\eta^*}^{\pm})}^{\mathbf{W}}(\mathrm{sgn}_{\mathbf{W}(\mathrm{A}_{\eta^*}^{\pm})})$. This uniquely distinguishes the two characters.

Now that we have concretely fixed the notation we can complete the description of the Springer correspondence. If $n \equiv 0 \pmod 4$ then the Springer character $\rho_{u^\pm,1}$ corresponding to the unipotent class $\mathcal{O}_\lambda^{\pm}$ is $\rho_{(\pi,\pi)_\pm}$. If $n \equiv 2 \pmod 4$ then the Springer character $\rho_{u^\pm,1}$ corresponding to the unipotent class $\mathcal{O}_\lambda^{\pm}$ is $\rho_{(\pi,\pi)_\mp}$.

If we had merely wanted to compute the Springer correspondence concretely then we would not have needed most of the above discussion. However the fruits of our labour are now revealed as we will be able to accurately compute the component groups $A_{\mathbf{G}}(u)$ for degenerate classes in half spin groups. We define the following three values to describe the orders of component groups.

$$\begin{aligned}
n_{\mathrm{D}}(u) &= |\{i \in \mathbb{N} \mid i \text{ is odd and } r_i \neq 0\}|,\\
\delta_{\mathrm{D}}(u) &= \begin{cases} 1 & \text{if there exists an odd number } i \in \mathbb{N} \text{ such that } r_i \equiv 1 \pmod 2,\\ 0 & \text{otherwise,}\end{cases}\\
\kappa_{\mathrm{D}}(u) &= \max\{r_i \mid i \text{ is odd}\}.
\end{aligned}$$

We have the following description for the simply connected case

$$|A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})| = \begin{cases} 2^{n_{\mathrm{D}}(u_{\mathrm{sc}})} & \text{if } n_{\mathrm{D}}(u_{\mathrm{sc}}) \geqslant 1, \kappa_{\mathrm{D}}(u_{\mathrm{sc}}) = 1,\\ 2^{n_{\mathrm{D}}(u_{\mathrm{sc}})-1} & \text{if } n_{\mathrm{D}}(u_{\mathrm{sc}}) \geqslant 1, \kappa_{\mathrm{D}}(u_{\mathrm{sc}}) \geqslant 2,\\ 2 & \text{if } n_{\mathrm{D}}(u_{\mathrm{sc}}) = 0,\end{cases}$$

and in the adjoint case

$$|A_{\mathbf{G}_{\text{ad}}}(u_{\text{ad}})| = \begin{cases} 2^{n_{\text{D}}(u_{\text{ad}})-1-\delta_{\text{D}}(u_{\text{ad}})} & \text{if } n_{\text{D}}(u_{\text{ad}}) \geqslant 1, \\ 1 & \text{if } n_{\text{D}}(u_{\text{ad}}) = 0. \end{cases}$$

Note that the case of $\mathbf{G}_{\text{sc}}$ is discussed in [**Lus84b**, §14.3] and the case of $\mathbf{G}_{\text{ad}}$ is discussed in [**Car93**, §13.1].

If $\mathbf{G}$ is neither simply connected nor adjoint then up to isomorphism we have two possibilities for $\mathbf{G}$, either it is isomorphic to a special orthogonal group or a half spin group. Assume that $\mathbf{G}$ is a special orthogonal group then

$$|A_{\mathbf{G}}(u)| = \begin{cases} 2^{n_{\text{D}}(u)-1} & \text{if } n_{\text{D}}(u) \geqslant 1, \\ 1 & \text{if } n_{\text{D}}(u) = 0. \end{cases}$$

This is described by Lusztig in [**Lus84b**, §10.6].

Finally we come to the case where $\mathbf{G}$ is a half spin group. Let $\bar{\mathbf{G}}$ be a special orthogonal group of type D_n then we have an isogeny $\mathbf{G}_{\text{sc}} \to \bar{\mathbf{G}}$ and we denote by $\bar{u}$ the image of u_{sc} under this isogeny. We discuss each class case by case:

- Assume $n_{\text{D}}(u) \geqslant 1$ and $\kappa_{\text{D}}(u) = 1$, which necessarily means $\delta_{\text{D}}(u) = 1$, then $|A_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})| = 4|A_{\mathbf{G}_{\text{ad}}}(u_{\text{ad}})|$. Hence we must have $Z(\mathbf{G}_{\text{sc}}) \cap C_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})^\circ = \{1\}$.
- Assume $n_{\text{D}}(u) \geqslant 1$, $\kappa_{\text{D}}(u) \geqslant 2$ and $\delta_{\text{D}}(u) = 0$ then $|A_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})| = |A_{\mathbf{G}_{\text{ad}}}(u_{\text{ad}})|$. Hence in this case we must have $Z(\mathbf{G}_{\text{sc}}) \leqslant C_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})^\circ$.
- Assume $n_{\text{D}}(u) \geqslant 1$, $\kappa_{\text{D}}(u) \geqslant 2$ and $\delta_{\text{D}}(u) = 1$ then $|A_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})| = |A_{\bar{\mathbf{G}}}(\bar{u})|$ and $|A_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})| = 2|A_{\mathbf{G}_{\text{ad}}}(u_{\text{ad}})|$. The first condition implies that $\hat{z}_1 \in C_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})^\circ$ and the second condition implies that either $\hat{z}_n \notin C_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})^\circ$ or $\hat{z}_{n-1} \notin C_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})^\circ$. However as $\hat{z}_1 \in C_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})^\circ$ we must have $\hat{z}_n C_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})^\circ = \hat{z}_{n-1} C_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})^\circ$, (because $\hat{z}_1 = \hat{z}_n \hat{z}_{n-1}$), which means $Z(\mathbf{G}_{\text{sc}}) \cap C_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})^\circ = \{1, \hat{z}_1\}$.
- Assume $n_{\text{D}}(u) = 0$ then $|A_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})| = 2$ and $|A_{\bar{\mathbf{G}}}(\bar{u})| = 1$. This implies that $\hat{z}_1$ determines the only non-trivial coset representative of $A_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})$. We have two possibilities: either $\hat{z}_1 C_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})^\circ = \hat{z}_n C_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})^\circ$, (which means $\hat{z}_{n-1} \in C_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})^\circ$), or $\hat{z}_1 C_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})^\circ = \hat{z}_{n-1} C_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})^\circ$, (which means $\hat{z}_n \in C_{\mathbf{G}_{\text{sc}}}(u_{\text{sc}})^\circ$). It is discussed by Lusztig, in [**Lus84b**, §14.3], that the two possibilities correspond to the two degenerate classes associated to the partition. In particular one of these classes is such that $|A_{\mathbf{G}}(u)| = 1$ and the other is such that $|A_{\mathbf{G}}(u)| = 2$.

Using the information above it's clear that whenever $n_{\mathrm{D}}(u) \neq 0$ we have

$$|A_{\mathbf{G}}(u)| = \begin{cases} 2^{n_{\mathrm{D}}(u)-1} & \text{if } n_{\mathrm{D}}(u) \geqslant 1, \kappa_{\mathrm{D}}(u) = 1, \\ 2^{n_{\mathrm{D}}(u)-1} & \text{if } n_{\mathrm{D}}(u) \geqslant 1, \kappa_{\mathrm{D}}(u) \geqslant 2 \text{ and } \delta_{\mathrm{D}}(u) = 0, \\ 2^{n_{\mathrm{D}}(u)-2} & \text{if } n_{\mathrm{D}}(u) \geqslant 1, \kappa_{\mathrm{D}}(u) \geqslant 2 \text{ and } \delta_{\mathrm{D}}(u) = 1. \end{cases}$$

We claim that the component group orders when $n_{\mathrm{D}}(u) = 0$ are given by

$$|A_{\mathbf{G}}(u)| = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 4 \text{ and } u \in \mathcal{O}_\lambda^+, \\ 1 & \text{if } n \equiv 0 \pmod 4 \text{ and } u \in \mathcal{O}_\lambda^-, \\ 1 & \text{if } n \equiv 2 \pmod 4 \text{ and } u \in \mathcal{O}_\lambda^+, \\ 2 & \text{if } n \equiv 2 \pmod 4 \text{ and } u \in \mathcal{O}_\lambda^-. \end{cases}$$

We now verify this claim. The intersection $\mathcal{O}_\lambda^\pm \cap \mathbf{L}_\eta^\pm$ contains the regular unipotent class of $\mathbf{L}_\eta^\pm$. Therefore take $u^\pm \in \mathcal{O}_\lambda^\pm$ to be such that it is a regular unipotent element of $\mathbf{L}_\eta^\pm$. We have a natural embedding $C_{\mathbf{L}_\eta^\pm}(u^\pm) \to C_{\mathbf{G}}(u^\pm)$, which induces an embedding $A_{\mathbf{L}_\eta^\pm}(u^\pm) \to A_{\mathbf{G}}(u^\pm)$. As $u^\pm$ is a regular unipotent element we have $A_{\mathbf{L}_\eta^\pm}(u^\pm) \cong \mathcal{Z}(\mathbf{L}_\eta^\pm)$ where $\mathcal{Z}(\mathbf{L}_\eta^\pm)$ denotes the component group $Z(\mathbf{L}_\eta^\pm)/Z(\mathbf{L}_\eta^\pm)^\circ$, (see for example the proof of [**DM91**, Proposition 14.24]). To determine whether $|A_{\mathbf{G}}(u^\pm)| = 2$ or 1 it is enough to determine when $|\mathcal{Z}(\mathbf{L}_\eta^\pm)| = 2$ or 1.

To do this calculation we will use a result of Digne–Lehrer–Michel. We have a natural embedding $Z(\mathbf{G}) \to Z(\mathbf{L}_\eta^\pm)$, which induces a surjective map $Z(\mathbf{G}) \to \mathcal{Z}(\mathbf{L}_\eta^\pm)$ by [**Bon06**, Proposition 4.2]. The kernel of this surjective map is given to us by the following result.

Proposition 2.14 (Digne–Lehrer–Michel, [Bon06, Proposition 4.5]**).** *Let $\mathbb{I} \subset \Delta$ be a set of simple roots and $\mathbf{L}_{\mathbb{I}}$ the standard Levi subgroup corresponding to $\mathbb{I}$. The kernel of the map $Z(\mathbf{G}) \to \mathcal{Z}(\mathbf{L}_{\mathbb{I}})$ is the image of $\langle \check{\varpi}_\alpha + \check{X} \mid \alpha \in \Delta \setminus \mathbb{I} \rangle$ under the isomorphism $(\check{\Lambda}/\check{X})_{p'} \cong Z(\mathbf{G})$ of Lemma 1.10.*

Recall from Section 1.2.1 that the image of $\check{X}$ in $\check{\Pi}$ depends upon the congruence of n (mod 4). We treat the two cases seperately.

- $n \equiv 0 \pmod 4$ then $\check{X} = \langle \check{\varpi}_n + \mathbb{Z}\check{\Phi} \rangle$. By Proposition 2.14 we have the kernel of the map $Z(\mathbf{G}) \to \mathcal{Z}(\mathbf{L}_\eta^\pm)$ is non-trivial whenever α_{n-1} is *not* in the root system of the Levi. If the kernel is non-trivial then the order of $\mathcal{Z}(\mathbf{L}_\eta^\pm)$ is 1. Hence we have $|\mathcal{Z}(\mathbf{L}_\eta^+)| = 2$ and $|\mathcal{Z}(\mathbf{L}_\eta^-)| = 1$.
- $n \equiv 2 \pmod 4$ then $\check{X} = \langle \check{\varpi}_{n-1} + \mathbb{Z}\check{\Phi} \rangle$. By Proposition 2.14 we have the kernel of the map $Z(\mathbf{G}) \to \mathcal{Z}(\mathbf{L}_\eta^\pm)$ is non-trivial whenever α_n is *not* in the root system of the Levi. If the kernel is non-trivial then the order of $\mathcal{Z}(\mathbf{L}_\eta^\pm)$ is 1. Hence we have $|\mathcal{Z}(\mathbf{L}_\eta^+)| = 1$ and $|\mathcal{Z}(\mathbf{L}_\eta^-)| = 2$.

This now verifies the statements regarding the component group orders.

Example 2.15. Assume $\mathbf{G}$ is simple of type D_3. There are five unipotent classes of $\mathbf{G}$ which are parameterised by the partitions (3^2), $(1,5)$, $(1^2,2^2)$, $(1^3,3)$ and (1^6) respectively. Using the above algorithm we compute the bi-partition associated to each class.

Blocks of λ	$\mathrm{Spr}_{\mathrm{D}}(\mathcal{O})$	(π',τ')	(π,τ)	$n_D(u)$	$\delta_D(u)$
$(\{3\},\{3\})$	$\begin{bmatrix}1\\2\end{bmatrix}$	$(1,2)$	$(1,2)$	1	0
$(\{1\},\{5\})$	$\begin{bmatrix}0\\3\end{bmatrix}$	$(0,3)$	$(-,3)$	2	1
$(\{1\},\{1\},\{2,2\})$	$\begin{bmatrix}0&3\\1&3\end{bmatrix}$	$(01,11)$	$(1,11)$	1	0
$(\{1\},\{1\},\{1\},\{3\})$	$\begin{bmatrix}0&2\\1&4\end{bmatrix}$	$(00,12)$	$(-,12)$	2	1
$(\{1\},\{1\},\{1\},\{1\},\{1\},\{1\})$	$\begin{bmatrix}0&2&4\\1&3&5\end{bmatrix}$	$(000,111)$	$(-,111)$	1	0

2.2.5 Type E_6

Recall that Theorem A.10 associates to a unipotent conjugacy class $\mathcal{O}$ a unique pair $(\mathbf{L},\mathbf{P}_{\mathbf{L}'})$, (up to $\mathbf{G}$-conjugacy), where $\mathbf{L} \leqslant \mathbf{G}$ is a Levi subgroup and $\mathbf{P}_{\mathbf{L}'}$ is a distinguished parabolic subgroup of the derived subgroup of the Levi $\mathbf{L}'$. We will refer to $\mathcal{O}$ using this labelling, which is described in [**Car93**, §13.1].

In Table 2.5 for each class $\mathcal{O}$ we list the corresponding Springer character $\rho_{u,1}$. We label the characters of $\mathbf{W}$ using the notation of [**Lus84a**, §4.11]. In the table we also give the values $a_{\rho_{u,1}}$, $b_{\rho_{u,1}}$ and $f_{\rho_{u,1}}$ associated to the Springer character $\rho_{u,1}$. The values $a_{\rho_{u,1}}$ and $b_{\rho_{u,1}}$ can be obtained from [**Héz04**, §1.8.3] and the value $f_{\rho_{u,1}}$ can be obtained from [**Car93**, §13.9]. We also list the values $|A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})|$ and $|A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})|$, which can be found in [**Car93**, §13.1] and [**Lus09**, §7.1].

$\mathcal{O}$	$\lvert A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})\rvert$	$\lvert A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})\rvert$	$a_{\rho_{u,1}}$	$b_{\rho_{u,1}}$	$f_{\rho_{u,1}}$	$\rho_{u,1}$	Special
1	1	1	36	36	1	$1_p'$	yes
A_1	1	1	25	25	1	$6_p'$	yes
$2\,\mathrm{A}_1$	1	1	20	20	1	$20_p'$	yes
$3\,\mathrm{A}_1$	1	1	15	16	2	$15_q'$	no

$\mathcal{O}$	$\lvert A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})\rvert$	$\lvert A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})\rvert$	$a_{\rho_{u,1}}$	$b_{\rho_{u,1}}$	$f_{\rho_{u,1}}$	$\rho_{u,1}$	Special
A_2	2	2	15	15	2	$30'_p$	yes
$\mathrm{A}_2+\mathrm{A}_1$	1	1	13	13	1	$64'_p$	yes
$2\,\mathrm{A}_2$	1	3	12	12	1	$24'_p$	yes
$\mathrm{A}_2+2\,\mathrm{A}_1$	1	1	11	11	1	$60'_p$	yes
A_3	1	1	10	10	1	$81'_p$	yes
$2\,\mathrm{A}_2+\mathrm{A}_1$	1	3	7	9	3	10_s	no
$\mathrm{A}_3+\mathrm{A}_1$	1	1	7	8	2	60_s	no
$\mathrm{D}_4(a_1)$	6	6	7	7	6	80_s	yes
A_4	1	1	6	6	1	81_p	yes
D_4	1	1	6	6	1	24_p	yes
$\mathrm{A}_4+\mathrm{A}_1$	1	1	5	5	1	60_p	yes
A_5	1	3	3	4	2	15_q	no
$\mathrm{D}_5(a_1)$	1	1	4	4	1	64_p	yes
$\mathrm{E}_6(a_3)$	2	6	3	3	2	30_p	yes
D_5	1	1	2	2	1	20_p	yes
$\mathrm{E}_6(a_1)$	1	3	1	1	1	6_p	yes
E_6	1	3	0	0	1	1_p	yes

Table 2.5: Part of the Springer Correspondence for E_6.

2.2.6 Type E_7

As in Section 2.2.5 we will refer to a unipotent class $\mathcal{O}$ by the pair in Theorem A.10. In Table 2.6 for each class $\mathcal{O}$ we list the corresponding Springer character $\rho_{u,1}$. We label the characters of $\mathbf{W}$ using the notation of [**Lus84a**, §4.12]. In the table we also give the values of $a_{\rho_{u,1}}$, $b_{\rho_{u,1}}$ and $f_{\rho_{u,1}}$ associated to the Springer character $\rho_{u,1}$. The values $a_{\rho_{u,1}}$ and $b_{\rho_{u,1}}$ can be obtained from [**Héz04**, §1.8.4] and the value $f_{\rho_{u,1}}$ can be obtained from [**Car93**, §13.9]. We also list the values $\lvert A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})\rvert$ and $\lvert A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})\rvert$, which can be found in [**Car93**, §13.1] and [**Lus09**, §7.1].

$\mathcal{O}$	$\lvert A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})\rvert$	$\lvert A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})\rvert$	$a_{\rho_{u,1}}$	$b_{\rho_{u,1}}$	$f_{\rho_{u,1}}$	$\rho_{u,1}$	Special
1	1	1	63	63	1	$1'_a$	yes
A_1	1	1	46	46	1	7_a	yes
$2\,\mathrm{A}_1$	1	1	37	37	1	$27'_a$	yes
$(3\,\mathrm{A}_1)''$	1	2	36	36	1	21_b	yes
$(3\,\mathrm{A}_1)'$	1	1	30	31	2	$35'_b$	no
A_2	2	2	30	30	2	56_a	yes
$4\,\mathrm{A}_1$	1	2	25	28	2	15_a	no
$\mathrm{A}_2+\mathrm{A}_1$	2	2	25	25	2	$120'_a$	yes

$\mathcal{O}$	$\lvert A_{\mathbf{G}_{\mathrm{ad}}}(u_{\mathrm{ad}})\rvert$	$\lvert A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})\rvert$	$a_{\rho_{u,1}}$	$b_{\rho_{u,1}}$	$f_{\rho_{u,1}}$	$\rho_{u,1}$	Special
$\mathrm{A}_2 + 2\,\mathrm{A}_1$	1	1	22	22	1	189_b	yes
A_3	1	1	21	21	1	$210'_a$	yes
$2\,\mathrm{A}_2$	1	1	21	21	1	$168'_a$	yes
$\mathrm{A}_2 + 3\,\mathrm{A}_1$	1	2	21	21	1	$105'_b$	yes
$(\mathrm{A}_3+\mathrm{A}_1)''$	1	2	20	20	1	189_c	yes
$2\,\mathrm{A}_2+\mathrm{A}_1$	1	1	16	18	3	70_a	no
$(\mathrm{A}_3+\mathrm{A}_1)'$	1	1	16	17	2	$280'_b$	no
$\mathrm{D}_4(a_1)$	6	6	16	16	6	315_a	yes
$\mathrm{A}_3 + 2\,\mathrm{A}_1$	1	2	15	16	2	216_a	no
D_4	1	1	15	15	1	$105'_c$	yes
$\mathrm{D}_4(a_1)+\mathrm{A}_1$	2	4	15	15	2	$405'_a$	yes
$\mathrm{A}_3+\mathrm{A}_2$	2	2	14	14	1	378_a	yes
A_4	2	2	13	13	2	$420'_a$	yes
$\mathrm{A}_3+\mathrm{A}_2+\mathrm{A}_1$	1	2	13	13	1	$210'_b$	yes
$(\mathrm{A}_5)''$	1	2	12	12	1	105_c	yes
$\mathrm{D}_4+\mathrm{A}_1$	1	2	10	12	2	84_a	no
$\mathrm{A}_4+\mathrm{A}_1$	2	2	11	11	2	$512'_a$	yes
$\mathrm{D}_5(a_1)$	2	2	10	10	2	420_a	yes
$\mathrm{A}_4+\mathrm{A}_2$	1	1	10	10	1	210_b	yes
$(\mathrm{A}_5)'$	1	1	8	9	2	$216'_a$	no
$\mathrm{A}_5+\mathrm{A}_1$	1	2	7	9	3	$70'_a$	no
$\mathrm{D}_5(a_1)+\mathrm{A}_1$	1	2	9	9	1	$378'_a$	yes
$\mathrm{D}_6(a_2)$	1	2	7	8	2	280_b	no
$\mathrm{E}_6(a_3)$	2	2	8	8	2	405_a	yes
D_5	1	1	7	7	1	$189'_c$	yes
$\mathrm{E}_7(a_5)$	6	12	7	7	6	$315'_a$	yes
A_6	1	1	6	6	1	105_b	yes
$\mathrm{D}_5+\mathrm{A}_1$	1	2	6	6	1	168_a	yes
$\mathrm{D}_6(a_1)$	1	2	6	6	1	210_a	yes
$\mathrm{E}_7(a_4)$	2	4	5	5	1	$189'_b$	yes
D_6	1	2	3	4	2	35_b	no
$\mathrm{E}_6(a_1)$	2	2	4	4	2	120_a	yes
E_6	1	1	3	3	1	$21'_b$	yes
$\mathrm{E}_7(a_3)$	2	4	3	3	2	$56'_a$	yes
$\mathrm{E}_7(a_2)$	1	2	2	2	1	27_a	yes
$\mathrm{E}_7(a_1)$	1	2	1	1	1	$7'_a$	yes
E_7	1	2	0	0	1	1_a	yes

Table 2.6: Part of the Springer Correspondence for E_7.

2.3 Unipotent Supports

We return to the assumption that $\mathbf{G}$ is a connected reductive algebraic group. Let χ be an irreducible character of G and let $\mathcal{O}$ be any F-stable unipotent class of $\mathbf{G}$. The fixed point set $\mathcal{O}^F$ is a union of G-conjugacy classes and we denote by $u_1, \ldots, u_r \in \mathcal{O}^F$ representatives for the G-classes contained in $\mathcal{O}^F$. In **[Lus80]** Lusztig poses the following problem.

Problem 1. *Show there exists a unique F-stable unipotent class $\mathcal{O}$ of $\mathbf{G}$ such that*

$$\sum_{g \in \mathcal{O}^F} \chi(g) = \sum_{j=1}^{r} [G : C_G(u_j)] \chi(u_j) \neq 0$$

and $\mathcal{O}$ has maximal dimension among all classes with this property.

It turns out that trying to work directly with $\sum_{g \in \mathcal{O}^F} \chi(g)$ is quite difficult so Lusztig introduces a different average value. He arrives at this after considering Problem 1 in the context of Kawanaka's theory of generalised Gelfand–Graev representations and his theory of character sheaves. Using the alternate average value he poses the following problem in **[Lus92]**.

Problem 2. *Show there exists a unique F-stable unipotent class $\mathcal{O}$ of $\mathbf{G}$ such that*

$$\sum_{j=1}^{r} [A_{\mathbf{G}}(u_j) : A_{\mathbf{G}}(u_j)^F] \chi(u_j) \neq 0$$

and $\mathcal{O}$ has maximal dimension among all classes with this property.

Assuming p and q are large then Lusztig gives a positive solution to Problem 2 in **[Lus92]**. In **[Gec96]** Geck weakens the conditions on p and q and shows that Problem 2 has a positive solution whenever p is a good prime for $\mathbf{G}$ and q is any power of p. He also shows that such a solution is also a solution for Problem 1, (see **[Gec96**, Theorem 1.4]). In **[GM00]** Geck and Malle give a positive solution for Problem 2 without any restrictions on p and q but it is not known if p is a bad prime that their solution of Problem 2 is a positive solution of Problem 1. With all of this in mind we may now make the following definition.

Definition 2.16. Assume $\mathbf{G}$ is a connected reductive algebraic group, F is a Frobenius endomorphism and $p > 0$ is any prime. Given an irreducible character $\chi \in \mathrm{Irr}(G)$ we denote by $\mathcal{O}_\chi$ the unique F-stable unipotent conjugacy class of maximal dimension satisfying

$$\sum_{j=1}^{r} [A_{\mathbf{G}}(u_j) : A_{\mathbf{G}}(u_j)^F] \chi(u_j) \neq 0.$$

We call $\mathcal{O}_\chi$ the *unipotent support* of χ.

Thus we have a well defined map $\mathrm{Irr}(G) \to \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})^F$ given by $\chi \mapsto \mathcal{O}_\chi$. On the other hand in [**Lus84a**, §13.3, §13.4], using the Springer correspondence, Lusztig describes a different map associating a unipotent class to an irreducible character. Note however that this is only valid under the assumption that p is a good prime for $\mathbf{G}$. We describe this process now by describing two maps $\Phi_{\mathbf{G}}$ and $\xi_{\mathbf{G}}$, which are defined by Lusztig in [**Lus84a**, §13.3, §13.4].

Let $s \in \mathbf{T}_0^\star$ be a semisimple element of the dual and consider the connected reductive group $C_{\mathbf{G}^\star}(s)^\circ$. Let $\mathbf{B}_s^\star \leqslant C_{\mathbf{G}^\star}(s)^\circ$ be a Borel subgroup containing $\mathbf{T}_0^\star \leqslant C_{\mathbf{G}^\star}(s)^\circ$, (note we cannot assume $\mathbf{B}_s^\star$ is $\mathbf{B}_0^\star$). By [**Car93**, Theorem 3.5.4] the group $C_{\mathbf{G}^\star}(s)^\circ$ is reductive and its root system with respect to $\mathbf{T}_0^\star$ is given by $\Phi(s) = \{\alpha^\star \in \Phi^\star \mid \alpha^\star(s) = 1\}$; this is then clearly an additively closed root subsystem of $\Phi^\star$. Let $\Phi^+(s)$ denote the system of positive roots determined by our choice of Borel subgroup $\mathbf{B}_s^\star$. We denote the Weyl group of $C_{\mathbf{G}^\star}(s)^\circ$ with respect to $\mathbf{T}_0^\star$ by $\mathbf{W}^\star(s)^\circ$. We then define

$$\mathbf{W}^\star(s) = \{w \in \mathbf{W}^\star \mid s^w = s\} \quad \text{and} \quad \mathcal{A}(s) = \{w \in \mathbf{W}^\star(s) \mid w \cdot \Phi^+(s) = \Phi^+(s)\}.$$

It is well known that $\mathbf{W}^\star(s) = \mathbf{W}^\star(s)^\circ \rtimes \mathcal{A}(s)$ and that we have an isomorphism $\mathcal{A}(s) \to A_{\mathbf{G}^\star}(s)$ given by $w \mapsto \dot{w}C_{\mathbf{G}^\star}(s)^\circ$, (see for example [**Bon05**, Proposition 1.3]). In particular $\mathbf{W}^\star(s)^\circ$ is a normal subgroup of $\mathbf{W}^\star(s)$ hence we have a natural action of $\mathcal{A}(s)$ on $\mathrm{Irr}(\mathbf{W}^\star(s)^\circ)$. If $\mathbf{G}$ has a connected centre then $\mathbf{W}^\star(s) = \mathbf{W}^\star(s)^\circ$ for all semisimple elements $s \in \mathbf{T}_0^\star$. The group $\mathbf{W}^\star(s)^\circ$ is conjugate to a parahoric subgroup of $\mathbf{W}^\star$ but in general $\mathbf{W}^\star(s)$ need not be a Coxeter group.

Remark 2.17. If $t \in \mathbf{G}^\star$ is any semisimple element then there exists $s \in \mathbf{T}_0^\star$ and $g \in \mathbf{G}^\star$ such that $t = {}^g s$. An easy group theoretic argument shows $C_{\mathbf{G}^\star}(t) = {}^g C_{\mathbf{G}^\star}(s)$ and indeed the group $\mathbf{B}_t^\star := {}^g\mathbf{B}_s^\star$ is a Borel subgroup of $C_{\mathbf{G}^\star}(t)^\circ$. In light of this we define $\mathbf{W}^\star(t) := {}^g\mathbf{W}^\star(s)$, $\mathbf{W}^\star(t)^\circ := {}^g\mathbf{W}^\star(s)^\circ$ and $\mathcal{A}(t) := {}^g\mathcal{A}(s)$. With these definitions we clearly have $\mathbf{W}^\star(t) = \mathbf{W}^\star(t)^\circ \rtimes \mathcal{A}(t)$ and an isomorphism $A_{\mathbf{G}^\star}(t) \to \mathcal{A}(t)$.

Recall that we have an anti-isomorphism $\mathbf{W} \to \mathbf{W}^\star$ sending $\mathbb{S}$ to $\mathbb{S}^\star$. We define $\mathbf{W}(s)$ and $\mathbf{W}(s)^\circ$ to be the preimages of $\mathbf{W}^\star(s)$ and $\mathbf{W}^\star(s)^\circ$ under this anti-isomorphism. If $\mathbf{W}^\star(\mathcal{F}) \subseteq \mathrm{Irr}(\mathbf{W}^\star(s)^\circ)$ is a family of characters then we define $\mathbf{W}(\mathcal{F})$ to be the corresponding family of characters under the identification of $\mathbf{W}^\star(s)^\circ$ and $\mathbf{W}(s)^\circ$. Note that $\mathbf{W}(s)^\circ$ is a reflection subgroup of $\mathbf{W}$ and we have the following result due to Lusztig.

Proposition 2.18 (Lusztig, [Lus84a, §13.3]). *Let $s \in \mathbf{T}_0^\star$ be a semisimple element and consider a family of characters $\mathbf{W}^\star(\mathcal{F}) \subseteq \mathrm{Irr}(\mathbf{W}^\star(s)^\circ)$. Let ρ be the unique special character of $\mathbf{W}(\mathcal{F})$ then there exists a unique unipotent class $\mathcal{O}$ such that $j_{\mathbf{W}(s)^\circ}^{\mathbf{W}}(\rho)$ is the Springer character $\rho_{u,1}$, where $u \in \mathcal{O}$.*

We define $\mathcal{S}_{\mathbf{G}}$ to be the following set of pairs $\{((s), \mathbf{W}^\star(\mathcal{F})) \mid (s)$ is a $\mathbf{G}^\star$-conjugacy class of semisimple elements and $\mathbf{W}^\star(\mathcal{F}) \subseteq \mathrm{Irr}(\mathbf{W}^\star(s)^\circ)$ is a family of characters $\}$. We will often denote a pair simply by $(s, \mathbf{W}^\star(\mathcal{F}))$, (with it implicit that s is taken up to $\mathbf{G}^\star$-conjugacy), and we will assume that the representative s is always an element of $\mathbf{T}_0^\star$. The map $\Phi_{\mathbf{G}} : \mathcal{S}_{\mathbf{G}} \to \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})$ is then defined as follows. Let $((s), \mathbf{W}^\star(\mathcal{F})) \in \mathcal{S}_{\mathbf{G}}$ and let $\rho \in \mathbf{W}(\mathcal{F})$ be the unique special character of the family. We define $\Phi_{\mathbf{G}}((s), \mathbf{W}^\star(\mathcal{F}))$ to be the unipotent class $\mathcal{O}$ such that $j_{\mathbf{W}(s)^\circ}^{\mathbf{W}}(\rho) = \rho_{u,1}$ for some $u \in \mathcal{O}$, (such a class exists by Proposition 2.18).

In the definition of $\Phi_{\mathbf{G}}$ we have identified a character of $\mathbf{W}^\star(s)^\circ$ with a character of $\mathbf{W}(s)^\circ$ then applied j-induction. However we could have first applied j-induction then identified the resulting character of $\mathbf{W}^\star$ with a character of $\mathbf{W}$. Note that this is not ambiguous as for each $s \in \mathbf{T}_0^\star$ the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Cent}(\mathbf{W}^\star) & \longrightarrow & \mathrm{Cent}(\mathbf{W}) \\ {\scriptstyle j_{\mathbf{W}^\star(s)^\circ}^{\mathbf{W}^\star}} \Big\uparrow & & \Big\uparrow {\scriptstyle j_{\mathbf{W}(s)^\circ}^{\mathbf{W}}} \\ \mathrm{Irr}(\mathbf{W}^\star(s)^\circ) & \longrightarrow & \mathrm{Irr}(\mathbf{W}(s)^\circ) \end{array}$$

where the horizontal arrows are the maps induced by the anti-isomorphism $\mathbf{W} \to \mathbf{W}^\star$.

Remark 2.19. Originally Lusztig defines $\Phi_{\mathbf{G}}$ to be a map between special conjugacy classes of $\mathbf{G}^\star$ and unipotent classes of $\mathbf{G}$. Let $\mathcal{C} \subseteq \mathbf{G}^\star$ be a conjugacy class of $\mathbf{G}^\star$ and g a representative of $\mathcal{C}$. This element has a Jordan decomposition $g = su = us$, where $s \in \mathbf{G}^\star$ is semisimple and $u \in C_{\mathbf{G}^\star}(s)^\circ$ is unipotent, (note it must be in the connected component by [**SS70**, Corollary 4.4]). After possibly replacing g by a conjugate we may assume $s \in \mathbf{T}_0^\star$. We say $\mathcal{C}$ is a *special class* if the Springer character $\rho_{u,1} \in \mathrm{Irr}(\mathbf{W}^\star(s)^\circ)$ is special. It is clear that there is a bijection between the set $\mathcal{S}_{\mathbf{G}}$ and the set of all special conjugacy classes.

We now define the map $\xi_{\mathbf{G}} : \mathrm{Irr}(G) \to \mathcal{S}_{\mathbf{G}}$ in the following way. Let $\chi \in \mathrm{Irr}(G)$ be a character then there exists a semisimple element $t \in G^\star$ such that $\chi \in \mathcal{E}(G, t)$. The image of χ under the map Ψ_t is an orbit of unipotent characters of $C_{G^\star}(t)^\circ$. Let $\psi \in \Psi_t(\chi)$ be any character in the orbit then this is contained in a unique family $\mathcal{F}$ of unipotent characters of $C_{G^\star}(t)^\circ$. In turn $\mathcal{F}$ determines a unique family $\mathbf{W}^\star(\mathcal{F})$ of irreducible characters of $\mathbf{W}^\star(t)^\circ$. We may assume $t = {}^g s$ for some $g \in \mathbf{G}^\star$ where $s \in \mathbf{T}_0^\star$ then we can define $\xi_{\mathbf{G}}(\chi)$ to be the pair $((s), \mathbf{W}^\star(\mathcal{F})^g) \in \mathcal{S}_{\mathbf{G}}$. The composition $\Phi_{\mathbf{G}} \circ \xi_{\mathbf{G}}$ gives us a map $\mathrm{Irr}(G) \to \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})$ and it is known that $(\Phi_{\mathbf{G}} \circ \xi_{\mathbf{G}})(\chi) = \mathcal{O}_\chi$. This was shown originally by Lusztig in [**Lus92**, Theorem 11.2] for the case of p, q large and by Geck in [**Gec96**, §5.4] for the case of p good, (and no restriction on q).

Note that if $C_{\mathbf{G}^\star}(t)$ is connected then $\Psi_t(\chi)$ contains only one character hence the family $\mathcal{F}$ is uniquely determined by χ. However if $C_{\mathbf{G}^\star}(t)$ is disconnected then $\Psi_t(\chi)$ may contain characters lying in different families, (see for instance Example 1.76). In particular the pair $\xi_{\mathbf{G}}(\chi) \in \mathcal{S}_{\mathbf{G}}$ depends upon the choice of family but we claim that the composition $\Phi_{\mathbf{G}} \circ \xi_{\mathbf{G}}$ does not depend upon this choice. For this we will need the following result concerning families.

Lemma 2.20. *Assume* $\mathbf{W}$ *is a finite Coxeter group with generating set* $\mathbb{S}$ *such that* $(\mathbf{W}, \mathbb{S})$ *is a Coxeter system. Let* γ *be an automorphism of* $\mathbf{W}$ *stabilising* $\mathbb{S}$ *then the natural action of* γ *on* $\mathrm{Irr}(\mathbf{W})$ *permutes the families of irreducible characters of* $\mathbf{W}$*.*

Proof. Let $\mathcal{H} := \mathcal{H}(\mathbf{W}, \mathbb{S})$ be the generic Hecke algebra over the Laurent polynomial ring $A := \mathbb{Z}[u, u^{-1}]$. The algebra $\mathcal{H}$ has a standard basis $\{T_w \mid w \in \mathbf{W}\}$ as an A-algebra which satisfies the multiplication relations given in [**Hum90**, §7.4], (note that by [**Lus03**, Chapter 4] these basis elements are invertible). The ring A admits an automorphism $^- : A \to A$ such that $\overline{u^n} = u^{-n}$ for all $n \in \mathbb{Z}$. By [**Lus03**, Lemma 4.2] this extends to a ring homomorphism of $\mathcal{H}$ by setting

$$\overline{\sum_{w \in \mathbf{W}} a_w T_w} = \sum_{w \in \mathbf{W}} \bar{a}_w T_{w^{-1}}^{-1}, \tag{2.2}$$

this automorphism is called the *bar involution*. It was shown in [**KL79**, Theorem 1.1] that there exists a unique basis $\{C_w \mid w \in \mathbf{W}\}$ of $\mathcal{H}$ such that $\overline{C}_w = C_w$ for all $w \in \mathbf{W}$, which we call the *Kazhdan–Lusztig basis* of $\mathcal{H}$. In particular Kazhdan and Lusztig show that

$$C_w = \sum_{y \leqslant w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(u^{-1}) u^{\ell(w)/2} u^{-\ell(y)} T_y, \tag{2.3}$$

where $\leqslant$ denotes the Bruhat ordering on $\mathbf{W}$, (as defined in [**Hum90**, §5.9]), and $P_{y,w}(u) \in \mathbb{Z}[u]$ is a *Kazhdan–Lusztig polynomial*.

The automorphism γ extends to an automorphism of the A-algebra $\mathcal{H}$ by setting $\gamma(T_w) = T_{\gamma(w)}$ for all $w \in \mathbf{W}$. It is easily seen using Eq. (2.2) that the automorphism γ commutes with the bar involution. In particular $\{\gamma(C_w) \mid w \in \mathbf{W}\}$ is a basis of $\mathcal{H}$ such that $\overline{\gamma(C_w)} = \gamma(C_w)$ hence this must be the Kazhdan–Lusztig basis. Before considering the basis element $\gamma(C_w)$ let us first recall the following. The automorphism γ stabilises $\mathbb{S}$ so must come from a permutation of the roots in the underlying root system of $\mathbf{W}$ hence γ stabilises the set of reflections of $\mathbf{W}$, (see [**Hum90**, §1.14 - Proposition]). In particular by the definition of the Bruhat ordering we have $\gamma(x) \leqslant \gamma(w) \Leftrightarrow x \leqslant w$ for all $x, w \in \mathbf{W}$. In other words γ is an automorphism of the Bruhat ordering and by [**Hum90**, §8.8] we have $\ell \circ \gamma = \ell$.

With all this in mind we have by the description in Eq. (2.3) that

$$\begin{aligned}\gamma(C_w) &= \sum_{y\leqslant w}(-1)^{\ell(w)-\ell(y)}P_{y,w}(u^{-1})u^{\ell(w)/2}u^{-\ell(y)}T_{\gamma(y)},\\ &= \sum_{\gamma^{-1}(z)\leqslant w}(-1)^{\ell(w)-\ell(\gamma^{-1}(z))}P_{y,w}(u^{-1})u^{\ell(w)/2}u^{-\ell(\gamma^{-1}(z))}T_z,\\ &= \sum_{z\leqslant\gamma(w)}(-1)^{\ell(\gamma(w))-\ell(z)}P_{y,w}(u^{-1})u^{\ell(\gamma(w))/2}u^{-\ell(z)}T_z.\end{aligned}$$

In particular it follows that $\gamma(C_w) = C_{\gamma(w)}$ and $P_{\gamma(y),\gamma(w)}(u) = P_{y,w}(u)$ for all $y, w \in \mathbf{W}$, (see for instance the similar remark made in [**Lus03**, §5.6]). Recall that Lusztig has given a decomposition of $\mathbf{W}$ into *two sided cells* which are defined in terms of multiplication of the Kazhdan–Lusztig basis, (see [**GJ11**, §1.6.1]). As in [**GJ11**, §1.6.2] each two-sided cell gives rise to a *cell module* of $\mathbf{W}$. One readily checks from the definitions and the above discussion that γ induces a permutation of the two sided cells of $\mathbf{W}$, hence a permutation of the corresponding cell modules of $\mathbf{W}$. By [**Lus84a**, Theorem 5.25] we have two characters $\chi, \chi' \in \mathrm{Irr}(\mathbf{W})$ are in the same family if and only if they appear as common irreducible constituents in the character of a cell module of a two-sided cell. Hence the families are in natural bijective correspondence with the two-sided cells of $\mathbf{W}$ and in this way we see that γ permutes the families of irreducible characters of $\mathbf{W}$. ■

Assume now that $\psi' \in \Psi_t(\chi)$ is another choice of irreducible character and $\mathcal{F}'$ is the unique family of unipotent characters of $C_{G^\star}(t)^\circ$ containing ψ'. By definition of Ψ_t there exists an element $h \in C_{G^\star}(t)$ such that $\psi' = \psi^h$. The element h determines an element $w \in \mathbf{W}^\star(t)$ and we may replace w by an element of the right coset $w\mathbf{W}^\star(t)^\circ$ inducing an automorphism of $\mathbf{W}^\star(t)^\circ$ stabilising the set of Coxeter generators determined by the Borel subgroup $\mathbf{B}_t^\star$. In particular w induces a Coxeter automorphism of $\mathbf{W}^\star(t)^\circ$ with respect to the Coxeter generators determined by $\mathbf{B}_t^\star$. By Lemma 2.20 the natural conjugation action of w on $\mathrm{Irr}(\mathbf{W}^\star(t)^\circ)$ permutes the families of irreducible characters. In particular, (recalling the definition of families of unipotent characters in Definition 1.46), we see that we must have $\mathcal{F}' = \mathcal{F}^h$ and $\mathbf{W}^\star(\mathcal{F}') = \mathbf{W}^\star(\mathcal{F})^w$. Therefore if $\rho' \in \mathbf{W}^\star(\mathcal{F}')$ is the unique special character we have $\rho' = \rho^w$ where ρ is the unique special character of $\mathbf{W}^\star(\mathcal{F})$. Hence we have $\Phi_{\mathbf{G}}((s), \mathbf{W}^\star(\mathcal{F})^g) = \Phi_{\mathbf{G}}((s), \mathbf{W}^\star(\mathcal{F}')^g)$ as $j_{\mathbf{W}^\star(s)^\circ}^{\mathbf{W}^\star}(\rho') = j_{\mathbf{W}^\star(s)^\circ}^{\mathbf{W}^\star}(\rho^w)$.

Remark 2.21. We could replace the set $\mathcal{S}_{\mathbf{G}}$ by the set of pairs where the family $\mathbf{W}^\star(\mathcal{F})$ of irreducible characters of $\mathbf{W}^\star(s)^\circ$ is replaced by an $\mathcal{A}(s)$-orbit of families. This would mean the map $\xi_{\mathbf{G}}$ would not depend upon the choice of character in $\Psi_t(\chi)$ for all $\chi \in \mathrm{Irr}(G)$. However in doing this we would no longer have a bijection between the set $\mathcal{S}_{\mathbf{G}}$ and the set of special conjugacy classes in the dual.

It is clear that the image of the map $\xi_{\mathbf{G}}$ will not in general be the whole of $\mathcal{S}_{\mathbf{G}}$. We would now like to give criteria which will identify which pairs in $\mathcal{S}_{\mathbf{G}}$ are in the image of $\xi_{\mathbf{G}}$. Assume $t \in G^\star$ is a semisimple element then there exists $s \in \mathbf{T}_0^\star$ and $g \in \mathbf{G}^\star$ such that $t = {}^g s$ and we may assume $\mathscr{L}(g) = \dot{w}$ for some $w \in \mathbf{W}^\star$. In particular $F^\star(s) = s^w$ and we may assume that w is chosen to be the unique element of minimal length in the right coset $w\mathbf{W}^\star(t)^\circ$, (see [**Lus84a**, Lemma 1.9(i)]). This element is characterised by the fact that w stabilises the Borel subgroup $\mathbf{B}_s^\star$ of $C_{\mathbf{G}^\star}(s)^\circ$, (i.e. w induces a Coxeter automorphism of $\mathbf{W}^\star(s)^\circ$ with respect to the Coxeter generators determined by $\mathbf{B}_s^\star$). Now $F^\star(C_{\mathbf{G}^\star}({}^g s)) = C_{\mathbf{G}^\star}({}^g s)$ because ${}^g s$ is $F^\star$-fixed which implies that $F^\star(C_{\mathbf{G}^\star}(s)) = C_{\mathbf{G}^\star}(s)^{g^{-1}F^\star(g)} = C_{\mathbf{G}^\star}(s)^{\dot{w}}$.

Definition 2.22. Assume $s \in \mathbf{T}_0^\star$ is a semisimple element such that $F^\star(s) = s^w$ for some $w \in \mathbf{W}^\star$. Assume w is the unique element of minimal length in the right coset $w\mathbf{W}^\star(s)^\circ$. We define $F_s^\star : C_{\mathbf{G}^\star}(s) \to C_{\mathbf{G}^\star}(s)$ to be the generalised Frobenius endomorphism, stabilising $\mathbf{T}_0^\star$ and $\mathbf{B}_s^\star$, given by $F_s^\star(h) = {}^{\dot{w}}F^\star(h)$ for all $h \in C_{\mathbf{G}^\star}(s)$.

We have an induced action of $F_s^\star$ on the group $\mathbf{W}^\star(s)$ given by $F_s^\star(x) = {}^w F^\star(x)$ for all $x \in \mathbf{W}^\star(s)$. This is an automorphism which restricts to an automorphism of $\mathbf{W}^\star(s)^\circ$ stabilising a set of Coxeter generators corresponding to our choice of $\mathbf{B}_s^\star$.

Remark 2.23. Assume $\tilde{s} \in \tilde{\mathbf{T}}_0^\star$ is a semisimple element such that $\iota^\star(\tilde{s}) = s$. If we again denote by $\dot{w}$ a representative of w in $N_{\tilde{\mathbf{G}}}(\tilde{\mathbf{T}}_0^\star)$ then we can similarly define a generalised Frobenius endomorphism $F_{\tilde{s}}^\star : C_{\tilde{\mathbf{G}}}(\tilde{s}) \to C_{\tilde{\mathbf{G}}}(\tilde{s})$ satisfying $\iota^\star \circ F_{\tilde{s}}^\star = F_s^\star \circ \iota^\star$. Taking $\tilde{\mathbf{B}}_s^\star$ to be the Borel subgroup of $C_{\tilde{\mathbf{G}}}(\tilde{s})$ such that $\iota^\star(\tilde{\mathbf{B}}_s^\star) = \mathbf{B}_s^\star$ we have $F_{\tilde{s}}^\star$ stabilises $\tilde{\mathbf{T}}_0^\star$ and $\tilde{\mathbf{B}}_s^\star$.

We have a bijection $\mathrm{Irr}(\mathbf{W}^\star(t)^\circ) \to \mathrm{Irr}(\mathbf{W}^\star(s)^\circ)$ given by $\rho \mapsto \rho^g$, which sends special characters to special characters, (note $\mathbf{W}^\star(t)^\circ$ is the Weyl group of $C_{\mathbf{G}^\star}(t)^\circ$ with respect to the maximal torus ${}^g\mathbf{T}_0^\star$). Inspecting the action of the Frobenius endomorphism we see that

$$(\rho \circ F^\star)({}^g x) = \rho({}^{F^\star(g)}F^\star(x)) = \rho({}^{gw}F^\star(x)) = \rho^g({}^w F^\star(x)) = (\rho^g \circ F_s^\star)(x)$$

for all $x \in \mathbf{W}^\star(s)^\circ$. In particular the character $\rho \in \mathrm{Irr}(\mathbf{W}^\star(t)^\circ)$ is $F^\star$-fixed if and only if the corresponding character $\rho^g \in \mathrm{Irr}(\mathbf{W}^\star(s)^\circ)$ is $F_s^\star$-fixed. Applying this to the situation where ρ is special we have the family $\mathbf{W}^\star(\mathcal{F})$ is $F^\star$-stable if and only if the corresponding family $\mathbf{W}^\star(\mathcal{F})^g$ of $\mathbf{W}^\star(s)^\circ$ is $F_s^\star$-stable.

Finally, given the above discussion, we wish to now identify the following subset of $\mathcal{S}_{\mathbf{G}}$ which is the image of the map $\xi_{\mathbf{G}}$.

Definition 2.24. A pair $(s, \mathbf{W}^\star(\mathcal{F})) \in \mathcal{S}_{\mathbf{G}}$ is called *F-stable* if the following conditions are satisfied.

- The semisimple conjugacy class (s) is $F^\star$-stable.
- The family $\mathbf{W}^\star(\mathcal{F})$ is invariant under composition with $F_s^\star$.

We will write $\mathcal{S}_{\mathbf{G}}^F$ for the subset of $\mathcal{S}_{\mathbf{G}}$ consisting of all F-stable pairs.

It is remarked by Hézard that a pair $(s, \mathbf{W}^\star(\mathcal{F})) \in \mathcal{S}_{\mathbf{G}}$ is F-stable if and only if its corresponding special class of $\mathbf{G}^\star$ is $F^\star$-stable, (see [**Héz04**, pg. 89]). It is commented by Lusztig in [**Lus84a**, §13.4] that if the special class is $F^\star$-stable then $\Phi_{\mathbf{G}}(s, \mathbf{W}^\star(\mathcal{F}))$ is F-stable. In particular by [**Lus84a**, 13.4.2] this gives us a surjective map $\mathrm{Irr}(G) \to \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})^F$.

We would like to use this construction of Lusztig to investigate the relationship between an irreducible character and its unipotent support. In particular we will investigate the relationship between certain numerical invariants associated to characters and classes. For conjugacy classes a natural numerical invariant is the order of the component group of the centraliser, which has been described above in detail. Natural invariants for irreducible characters come from the degree of the character, in fact we have the following result.

Theorem 2.25 (Lusztig). *Assume $\mathbf{G}$ is a connected reductive algebraic group and F is a Frobenius endomorphism. Let $\chi \in \mathrm{Irr}(G)$ be an irreducible character of G such that $\chi \in \mathcal{E}(G, s)$ and $\Psi_s(\chi)$ lies in the family $\mathcal{F} \subseteq \mathcal{E}(C_{G^\star}(s)^\circ, 1)$. Let $\mathbf{W}(\mathcal{F})$ be the family of Weyl group characters corresponding to $\mathcal{F}$ and let $\rho \in \mathbf{W}(\mathcal{F})$ be the unique special character. There exists a natural number n_χ such that*

$$n_\chi \cdot \chi(1) = q^{a_\rho} + \textit{higher powers of } q.$$

If $\mathbf{G}$ has a connected centre then this is given in [**Lus84a**, 4.26.3]. When the centre of $\mathbf{G}$ is not connected this follows from the fact that every character in a given fibre of Ψ_s has the same character degree. This result gives two natural invariants arising from the character degree, namely a_ρ and n_χ. It was shown by Geck and Malle, (see [**GM00**, Proposition 3.5]), that $a_\rho = \dim \mathfrak{B}_u^{\mathbf{G}}$ where $u \in \mathcal{O}_\chi$. What we will be interested in is the relationship between n_χ and the unipotent support of χ.

In [**Lus84a**, 13.4.2 to 13.4.5] Lusztig gives, without explicit proof, some properties of the map $\xi_{\mathbf{G}}$. We are most interested in [**Lus84a**, 13.4.4] which states that if $\mathcal{O}$ is an F-stable unipotent class of $\mathbf{G}$ then $|A_{\mathbf{G}}(u)| = \sup_\chi\{n_\chi\}$ where χ runs over all irreducible characters in the preimage of $\mathcal{O}$ under $\Phi_{\mathbf{G}} \circ \xi_{\mathbf{G}}$. We now recall a theorem which was proved by Hézard, which will be the starting point for the main focus of this thesis. However before we state the theorem we will need the following definition.

Definition 2.26. Let $\mathbf{G}$ be any connected reductive algebraic group and $s \in \mathbf{G}$ a semisimple element. We call s *isolated* if $C_{\mathbf{G}}(s)^\circ$ is not contained in a Levi complement of a proper parabolic subgroup of $\mathbf{G}$. Furthermore s is called *quasi-isolated* if $C_{\mathbf{G}}(s)$ is not contained in a Levi complement of a proper parabolic subgroup of $\mathbf{G}$.

Theorem 2.27 (Hézard, [Héz04, Théorème B]**).** *Let $\mathbf{G}$ be a connected reductive algebraic group such that the centre of $\mathbf{G}$ is connected, $\mathbf{G}/Z(\mathbf{G})$ is simple and p is a good prime for $\mathbf{G}$. Let $\mathcal{O}$ be an F-stable unipotent class of $\mathbf{G}$ then there exists an F-stable pair $((s), \mathbf{W}(\mathcal{F})) \in \mathcal{S}_{\mathbf{G}}^F$ such that the image of s under an adjoint quotient of $\mathbf{G}^\star$ is quasi-isolated and $|A_{\mathbf{G}}(u)| = |\mathcal{G}_{\mathcal{F}}|$ for any $u \in \mathcal{O}$.*

Remark 2.28. We make some brief remarks regarding Hézard's result.

- The requirement that $((s), \mathbf{W}(\mathcal{F}))$ is an F-stable pair means that to $\mathbf{W}(\mathcal{F})$ there is a corresponding family $\mathcal{F}$ of unipotent characters for $C_{G^\star}(t)$, where $t \in (s)$ is some $F^\star$-fixed element. If $\mathcal{F}$ is a family of unipotent characters for a finite reductive group G, (where F is a Frobenius endomorphism), there exists a character $\chi \in \mathcal{F}$ such that $n_\chi = |\mathcal{G}_{\mathcal{F}}|$. If $\mathbf{G}$ is a classical group then this is seen to be true of any character in the family as n_χ is constant on families. If $\mathbf{G}$ is an exceptional group then one can verify this easily by checking the tables of unipotent character degrees given in [**Car93**, §13.9]. Hence this result does verify Lusztig's claim in [**Lus84a**, 13.4.4].
- Our statement of Theorem 2.27 is not exactly the same as that in [**Héz04**] as there Hézard states that s is an isolated element but this statement is not always correct. For instance if $\mathbf{G}$ is of type C_n then Hézard occasionally uses a semisimple element whose centraliser is of type B_{n-1}. Such an element is not isolated as this is a proper Levi subgroup of the dual. To our knowledge in all other cases the semisimple element chosen by Hézard is indeed isolated.

2.4 Component Groups of Unipotent Elements

Assume $\mathcal{C}$ is an F-stable conjugacy class of $\mathbf{G}$ and let $x \in \mathcal{C}^F$ be an F-fixed class representative. We have an induced automorphism of the finite group $A_{\mathbf{G}}(x)$ which we again denote by F. Although the structure of $A_{\mathbf{G}}(x)$ does not depend upon our choice of representative the action of F will depend upon our choice of representative. In this section we wish to show that if $\mathbf{G}$ is simple we can make an optimal choice for x with respect to the action of F on $A_{\mathbf{G}}(x)$.

Remark 2.29. Let us show precisely why the group $A_{\mathbf{G}}(x)$ does not depend upon the choice of representative $x \in \mathcal{C}$. Assume $y \in \mathcal{C}$ is another class representative then there exists an element $g \in \mathbf{G}$ such that $x = {}^g y$. It is an easy group theoretic statement to verify that $C_{\mathbf{G}}(x) = {}^g C_{\mathbf{G}}(y)$. The map $C_{\mathbf{G}}(y) \to {}^g C_{\mathbf{G}}(y)$, given by conjugation, is a homomorphism of algebraic groups so we have $({}^{g}C_{\mathbf{G}}(y))^{\circ} = {}^{g}(C_{G}(y)^{\circ})$. Hence $A_{\mathbf{G}}(x) = {}^{g}A_{\mathbf{G}}(y)$, where ${}^{g}A_{\mathbf{G}}(y) = {}^{g}C_{\mathbf{G}}(y)/{}^{g}C_{\mathbf{G}}(y)^{\circ}$, which means conjugation by g induces an isomorphism $A_{\mathbf{G}}(x) \to A_{\mathbf{G}}(y)$.

In this section we show that in a simple algebraic group we can always pick representatives of a unipotent class such that they are *well chosen*. By this we mean the following

Definition 2.30. Assume $\mathbf{G}$ is a connected reductive algebraic group and let $\mathcal{O} \in \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})^F$ be an F-stable conjugacy class. We say a class representative $u \in \mathcal{O}^F$ is *well chosen* if $|A_{\mathbf{G}}(u)^F| = |Z_{\mathbf{G}}(u)^F||A_{\tilde{\mathbf{G}}}(\tilde{u})|$, where $Z_{\mathbf{G}}(u)$ is the image of the map $Z(\mathbf{G}) \to C_{\mathbf{G}}(u) \to A_{\mathbf{G}}(u)$.

Remark 2.31. It would be preferable if we could give a more structural definition than the one given above. Ideally one would make the definition to be such that u is well chosen if $A_{\mathbf{G}}(u) \setminus A_{\mathbf{G}}(u)^F \subseteq Z_{\mathbf{G}}(u)$. However this stronger condition fails for certain classes when $\mathbf{G}$ is simply connected of type D_n and F acts non-trivially on $Z(\mathbf{G})$, (but this is the only case where it does fail). Therefore we must settle for the weaker condition used above.

To show that such elements exist we will need the following easy lemmas and a thoerem about groups with a connected centre.

Lemma 2.32. *Let $\mathbf{G}$ be a connected reductive algebraic group and $\mathcal{C}$ an F-stable conjugacy class of $\mathbf{G}$ such that $A_{\mathbf{G}}(x)$ is abelian for some $x \in \mathcal{C}^F$. The following are then equivalent:*

(i) the number of G-conjugacy classes contained in $\mathcal{C}^F$ is $|A_{\mathbf{G}}(x)|$,

(ii) $A_{\mathbf{G}}(x)^F = A_{\mathbf{G}}(x)$,

(iii) $A_{\mathbf{G}}(x')^F = A_{\mathbf{G}}(x')$ for all $x' \in \mathcal{C}^F$.

Proof. By **[Gec03**, Theorem 4.3.5] we know the G-classes contained in $\mathcal{C}^F$ are in bijective correspondence with the F-conjugacy classes of $A_{\mathbf{G}}(x)$. Therefore the number of G-classes contained in $\mathcal{C}^F$ is the same as $|A_{\mathbf{G}}(x)|$ if and only if every F-conjugacy class of $A_{\mathbf{G}}(x)$ contains only one element. As $A_{\mathbf{G}}(x)$ is abelian this proves the equivalence of the first two statements. If (i) is true for $x \in \mathcal{C}^F$ then it is true for all $x' \in \mathcal{C}^F$ as $|A_{\mathbf{G}}(x)|$ is independent of the choice of x. Therefore this together with the equivalence of (i) and (ii) gives the equivalence of (i) and (iii). ∎

Lemma 2.33. *Let $\mathbf{G}$ be a connected reductive algebraic group and let $\mathcal{C}$ be an F-stable conjugacy class of $\mathbf{G}$. If the automorphism of $A_{\mathbf{G}}(x)$ induced by F, for some $x \in \mathcal{C}^F$, is an inner automorphism then there exists $x' \in \mathcal{C}^F$ such that $A_{\mathbf{G}}(x')^F = A_{\mathbf{G}}(x')$.*

Proof. Assume there exists an element $y \in C_{\mathbf{G}}(x)$ such that for all $zC_{\mathbf{G}}(x)^\circ \in A_{\mathbf{G}}(x)$ we have $F(zC_{\mathbf{G}}(x)^\circ) = y^{-1}zyC_{\mathbf{G}}(x)^\circ$. As $y \in \mathbf{G}$, by the Lang-Steinberg theorem, there exists $g \in \mathbf{G}$ such that $y = g^{-1}F(g) \Rightarrow F(g) = gy$. Clearly $A_{\mathbf{G}}({}^g x) = {}^g A_{\mathbf{G}}(x)$ so $gzC_{\mathbf{G}}(x)^\circ g^{-1} \in A_{\mathbf{G}}({}^g x)$. Furthermore we have

$$F(gzC_{\mathbf{G}}(x)^\circ g^{-1}) = gy(y^{-1}zy)C_{\mathbf{G}}(x)^\circ y^{-1}g^{-1} = gzC_{\mathbf{G}}(x)^\circ g^{-1},$$

so F acts trivially on $A_{\mathbf{G}}({}^g x)$. Taking $x' = {}^g x \in \mathcal{C}^F$ we have $A_{\mathbf{G}}(x')^F = A_{\mathbf{G}}(x')$ as required. Note x' is fixed by F because $y \in C_{\mathbf{G}}(x)$. ■

Theorem 2.34. *Let $\mathbf{G}$ be a connected reductive algebraic group with connected centre such that $\mathbf{G}/Z(\mathbf{G})$ is simple, then for any $\mathcal{O} \in \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})^F$ we have $A_{\mathbf{G}}(u)^F = A_{\mathbf{G}}(u)$ for some unipotent element $u \in \mathcal{O}^F$.*

Proof. The natural surjective morphism of algebraic groups $\pi : \mathbf{G} \to \mathbf{G}/Z(\mathbf{G})$ induces a bijection $\mathfrak{Cl}_{\mathrm{U}}(\mathbf{G}) \to \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G}/Z(\mathbf{G}))$. If $u \in \mathbf{G}$ is unipotent then the restriction of π to $C_{\mathbf{G}}(u)$ induces an isomorphism $A_{\mathbf{G}}(u) \cong A_{\mathbf{G}/Z(\mathbf{G})}(\pi(u))$, which is defined over $\mathbb{F}_q$. Therefore we may prove this result using any group $\mathbf{G}$ which has a connected centre and simple quotient $\mathbf{G}/Z(\mathbf{G})$.

The structure of the component groups $A_{\mathbf{G}}(u)$ have been determined on a case by case basis. In the case where $\mathbf{G}$ is an adjoint exceptional group it is known that $A_{\mathbf{G}}(u)$ is either trivial or isomorphic to a symmetric group $\mathfrak{S}_2$, $\mathfrak{S}_3$, $\mathfrak{S}_4$ or $\mathfrak{S}_5$, (see for example the tables in [**Car93**, §13.1]). Every automorphism of such a group is an inner automorphism, so the result holds by Lemma 2.33.

We now turn to the classical groups. The case of type A_n is trivial as all component groups are trivial. Assume $\mathbf{G}$ is either a symplectic or special orthogonal group and $\mathcal{O} \in \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})^F$ is a conjugacy class with representative $u \in \mathcal{O}^F$. The component group $A_{\mathbf{G}}(u)$ is an elementary abelian 2-group whose order is given in [**SS70**, IV - 2.26 and 2.27]. By inspecting the descriptions of $|A_{\mathbf{G}}(u)|$ we see by [**Wal63**, p. 38], (resp. [**Wal63**, p. 42]), for the symplectic groups, (resp. special orthogonal groups), that the number of G-classes contained in $\mathcal{O}^F$ is the same as $|A_{\mathbf{G}}(u)|$. Note the statement given in [**Wal63**] is for the orthogonal groups but the maximal number of G-classes contained in $\mathcal{O}^F$ is $|A_{\mathbf{G}}(u)|$ and there can only be less classes in the orthogonal group so this suffices. As $A_{\mathbf{G}}(u)$ is abelian we have $A_{\mathbf{G}}(u)^F = A_{\mathbf{G}}(u)$ by Lemma 2.32. As odd dimensional special orthogonal groups are adjoint groups of type B_n this case is covered.

Let $\mathbf{G}$ be an adjoint group of type C_n or D_n then there exists a symplectic or special orthogonal group $\bar{\mathbf{G}}$, (together with a Frobenius endomorphism $F : \bar{\mathbf{G}} \to \bar{\mathbf{G}}$), such that we have an isogeny $\pi : \bar{\mathbf{G}} \to \mathbf{G}$ which is defined over $\mathbb{F}_q$. Let $\bar{u}$ be a unipotent element of $\bar{\mathbf{G}}$ and set $u := \pi(\bar{u})$. The restriction of π to $C_{\bar{\mathbf{G}}}(\bar{u})$ induces a surjective morphism $A_{\bar{\mathbf{G}}}(\bar{u}) \to A_{\mathbf{G}}(u)$ defined over $\mathbb{F}_q$, therefore $A_{\bar{\mathbf{G}}}(\bar{u})^F = \mathrm{A}_{\bar{\mathbf{G}}}(\bar{u})$ implies $A_{\mathbf{G}}(u)^F = A_{\mathbf{G}}(u)$ so we're done. ■

Remark 2.35. The statement of Theorem 2.34 is not a new result. It follows in most cases from a much stronger statement concerning the existence of so called split elements, see [**Sho87**, Remark 5.1]. However, we include this here as it gives a simpler argument for this statement and also circumvents the caveat that split elements do not always exist in type E_8.

We want to discuss how to relate the component group $A_{\mathbf{G}}(u)$ to the component group $A_{\tilde{\mathbf{G}}}(\tilde{u})$, (for this we follow [**Gec96**, pg. 306]). Recall that we have a regular embedding $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ and a canonical quotient map $C_{\tilde{\mathbf{G}}}(\tilde{u}) \to A_{\tilde{\mathbf{G}}}(\tilde{u})$. The group $\tilde{\mathbf{G}}$ is an almost direct product $\tilde{\mathbf{G}} = \mathbf{G}Z(\tilde{\mathbf{G}})$ so we have an almost direct product $C_{\tilde{\mathbf{G}}}(\tilde{u}) = C_{\mathbf{G}}(u)Z(\tilde{\mathbf{G}})$. The restriction of ι to the centraliser $C_{\mathbf{G}}(u)$ induces a surjective map $A_{\mathbf{G}}(u) \to A_{\tilde{\mathbf{G}}}(\tilde{u})$.

Let us consider the kernel of this surjective map. It is clear that $C_{\tilde{\mathbf{G}}}(\tilde{u})^\circ = C_{\mathbf{G}}(u)^\circ Z(\tilde{\mathbf{G}})$ because $C_{\mathbf{G}}(u)^\circ Z(\tilde{\mathbf{G}}) \subseteq C_{\tilde{\mathbf{G}}}(\tilde{u})^\circ$ and the two sets have the same dimension. If $xC_{\mathbf{G}}(u)^\circ$ is non-identity element of $A_{\mathbf{G}}(u)$ in the kernel of the map then $\iota(x) \in C_{\tilde{\mathbf{G}}}(\tilde{u})^\circ$ but as $x \notin C_{\mathbf{G}}(u)^\circ$ we must have $\iota(x) \in Z(\tilde{\mathbf{G}})$. Therefore it is easy to see that the kernel of this map is the image of $Z(\mathbf{G})$ in $A_{\mathbf{G}}(u)$ so we obtain an exact sequence

$$Z(\mathbf{G}) \to A_{\mathbf{G}}(u) \to A_{\tilde{\mathbf{G}}}(\tilde{u}) \to 1. \tag{2.4}$$

As all morphisms are defined over $\mathbb{F}_q$ this induces a sequence

$$Z(\mathbf{G})^F \to A_{\mathbf{G}}(u)^F \to A_{\tilde{\mathbf{G}}}(\tilde{u}) \to 1, \tag{2.5}$$

where here we assume $\tilde{u}$ is chosen such that $A_{\tilde{\mathbf{G}}}(\tilde{u})^F = A_{\tilde{\mathbf{G}}}(\tilde{u})$, (which is possible by Theorem 2.34). Although the sequence in (2.4) is exact it is not necessarily the case that the sequence in (2.5) is exact as the map $A_{\mathbf{G}}(u)^F \to A_{\tilde{\mathbf{G}}}(\tilde{u})$ may not be surjective. If we can show that it is surjective then the existence of well-chosen elements will be assured.

There are two cases where this is trivial to show. If $|A_{\tilde{\mathbf{G}}}(\tilde{u})| = 1$ then it is obvious that such a map is surjective. The second case is when $|A_{\mathbf{G}}(u)| = |A_{\tilde{\mathbf{G}}}(\tilde{u})|$ because then we must have $A_{\mathbf{G}}(u) \cong A_{\tilde{\mathbf{G}}}(\tilde{u})$, which implies $A_{\mathbf{G}}(u)^F \cong A_{\tilde{\mathbf{G}}}(\tilde{u})^F$. In fact, we see from the descriptions of component group orders above that we are almost always in the situation of these two trivial cases. We deal with the remaining cases in exceptional groups in two lemmas.

Lemma 2.36. *Let $\mathbf{G}$ be a simply connected group of type E_6 and $\mathcal{O} \in \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})^F$ the unipotent* *class* $\mathrm{E}_6(a_3)$*. For any* $u \in \mathcal{O}^F$ *we have* $A_{\tilde{\mathbf{G}}}(\tilde{u})^F = A_{\tilde{\mathbf{G}}}(\tilde{u})$ *and* $A_{\mathbf{G}}(u)^F \to A_{\tilde{\mathbf{G}}}(\tilde{u})$ *is surjective.*

Proof. We see that $A_{\mathbf{G}}(u)$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_2$ as it is a finite group of order 6 with a non-trivial centre. The group $A_{\tilde{\mathbf{G}}}(\tilde{u})$ is abelian thus $A_{\tilde{\mathbf{G}}}(\tilde{u})^F = A_{\tilde{\mathbf{G}}}(\tilde{u})$ by Theorem 2.34 and Lemma 2.32. Let $Z(\mathbf{G}) = \langle z \rangle$ and $x \in A_{\mathbf{G}}(u)$ be the element of order 2 such that $A_{\mathbf{G}}(u) = \langle z, x \rangle$. As the action of F on $A_{\tilde{\mathbf{G}}}(\tilde{u})$ is trivial we must have $F(x) \in \{x, xz, xz^2\}$. However the element x is of order 2 but xz and xz^2 are of order 6 so $F(x) = x$. ■

Lemma 2.37. *Let* $\mathbf{G}$ *be a simply connected group of type* E_7 *and* $\mathcal{O} \in \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})^F$ *the unipotent class* $\mathrm{D}_4(a_1) + \mathrm{A}_1$*,* $\mathrm{E}_7(a_3)$*,* $\mathrm{E}_7(a_4)$ *or* $\mathrm{E}_7(a_5)$*. For some representative* $u \in \mathcal{O}^F$ *we have* $A_{\tilde{\mathbf{G}}}(\tilde{u})^F = A_{\tilde{\mathbf{G}}}(\tilde{u})$ *and* $A_{\mathbf{G}}(u)^F \to A_{\tilde{\mathbf{G}}}(\tilde{u})$ *is surjective.*

Proof. Let $u \in \mathcal{O}^F$ be an element such that its image $\tilde{u} \in \tilde{\mathcal{O}}^F$ has the property $A_{\tilde{\mathbf{G}}}(\tilde{u})^F = A_{\tilde{\mathbf{G}}}(\tilde{u})$, which is possible because of Theorem 2.34. We use the work of Mizuno [**Miz80**] to confirm that F only acts non-trivially on $Z_{\mathbf{G}}(u)$. By

(i) [**Miz80**, Lemma 30] for $\mathrm{D}_4(a_1) + \mathrm{A}_1$ – Mizuno label $\mathrm{D}_4(a_1) + \mathrm{A}_1$,

(ii) [**Miz80**, Lemma 13] for $\mathrm{E}_7(a_3)$ – Mizuno label $\mathrm{D}_6 + \mathrm{A}_1$,

(iii) [**Miz80**, Lemma 17] for $\mathrm{E}_7(a_4)$ – Mizuno label $\mathrm{D}_6(a_1) + \mathrm{A}_1$,

(iv) and [**Miz80**, Lemma 21(2)] for $\mathrm{E}_7(a_5)$ – Mizuno label $\mathrm{D}_6(a_2) + \mathrm{A}_1$,

we know the number of G-classes contained in $\mathcal{O}^F$ is $|A_{\mathbf{G}}(u)|$. If $\mathcal{O}$ is not $\mathrm{E}_7(a_5)$ then the group $A_{\mathbf{G}}(u)$ is abelian so the result follows by Lemma 2.32.

Assume $\mathcal{O}$ is the class $\mathrm{E}_7(a_5)$, by [**Miz80**, Table 9] we see $A_{\mathbf{G}}(u)$ is isomorphic to $\mathfrak{S}_3 \times \mathbb{Z}_2$. Let $Z(\mathbf{G}) = \langle z \rangle$, then z is a generator for the $\mathbb{Z}_2$ component of $A_{\mathbf{G}}(u)$. We know the action of F on the component group $A_{\tilde{\mathbf{G}}}(\tilde{u})$ is trivial so we need only show $F(x) \neq xz$ for all $x \in A_{\mathbf{G}}(u) \setminus Z_{\mathbf{G}}(u)$. If x is a 3-cycle of $A_{\mathbf{G}}(u)$ then we cannot have $F(x) = xz$ because x and xz are of different orders. If x is a 2-cycle such that $F(x) = xz$ then the number of F-conjugacy classes of $A_{\mathbf{G}}(u)$ would be less than 6 and this cannot possibly happen. ■

We now deal with the cases of classical type. Let $\mathbf{G}$ be a spin group, (of type B_n or D_n), and let $\mathcal{O} = \mathcal{O}_\lambda \in \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})^F$ be a unipotent class. There is only one non-trivial case to consider and this is the case where the partition λ parameterising $\mathcal{O}$ contains at least one odd number, ($n_{\mathrm{D}}(u_{\mathrm{sc}}) \geqslant 1$), and the maximum number of times any odd number occurs in λ is 1, ($\kappa_{\mathrm{B}}(u_{\mathrm{sc}}) = \kappa_{\mathrm{D}}(u_{\mathrm{sc}}) = 1$). Equivalently these are the classes such that $Z(\mathbf{G})$ embeds into $A_{\mathbf{G}}(u)$. We assume now that $\mathcal{O}$ is a class with such a partition. In [**Lus84b**, §14.3] Lusztig gives a description of the component group $A_{\mathbf{G}}(u)$, which we recall here.

First of all let $\bar{\mathbf{G}}$ be a special orthogonal group, (together with a Frobenius endomorphism $F : \bar{\mathbf{G}} \to \bar{\mathbf{G}}$), such that we have an isogeny $\pi : \mathbf{G} \to \bar{\mathbf{G}}$ defined over $\mathbb{F}_q$. Recall that the kernel of this isogeny is a central subgroup of order 2, which we denote $\mathrm{Ker}(\pi) = \{1, \vartheta\}$. Every non-trivial element of $Z(\mathbf{G})$ determines a non-trivial element of $A_{\mathbf{G}}(u)$, in particular ϑ determines a non-trivial element of $A_{\mathbf{G}}(u)$. Let $I := \{a_1, \dots, a_k\}$ be the set of odd numbers occuring in λ. We let S be the group generated by the elements $\vartheta, x_1, \dots, x_k$, which satisfy the relations

$$\vartheta^2 = 1 \qquad x_i^2 = \vartheta^{a_i(a_i-1)/2} \qquad x_i x_j = x_j x_i \vartheta \qquad \vartheta x_i = x_i \vartheta$$

for all $i \neq j$. The group $A_{\mathbf{G}}(u)$ is then isomorphic to the subgroup of S which consists of all elements that can be expressed as a word in an even number of the generators $x_1, \dots, x_k$. For the remainder of this discussion we will identify $A_{\mathbf{G}}(u)$ with its image in S. For all $2 \leqslant i \leqslant k$ let $y_i = x_1 x_i$ then for any $i, j \neq 1$ it is clear that either $x_i x_j = y_i y_j$ or $y_i y_j \vartheta$ depending upon whether $x_1^2 = \vartheta$ or 1. Hence the set $\{\vartheta, y_2, \dots, y_k\}$ forms a set of generators for $A_{\mathbf{G}}(u)$.

Consider the image $\bar{u} = \pi(u)$ of u in the special orthogonal group $\bar{\mathbf{G}}$. The map π induces a surjective map $A_{\mathbf{G}}(u) \to A_{\bar{\mathbf{G}}}(\bar{u})$ with kernel $\{1, \vartheta\}$. By Theorem 2.34 and Lemma 2.32 we know the action of F on $A_{\bar{\mathbf{G}}}(\bar{u})$ must be trivial for any class representative $u \in \mathcal{O}^F$. Therefore given any element $x \in A_{\mathbf{G}}(u)$ we have $F(x) \in \{x, x\vartheta\}$, however we always have $F(\vartheta) = \vartheta$. Any automorphism of $A_{\mathbf{G}}(u)$ is uniquely determined by its action on the generators so we break the study of F into two possible cases:

(a) $F(y_i) = y_i \vartheta$ for an even number of y_i's,

(b) $F(y_i) = y_i \vartheta$ for an odd number of y_i's.

In case (a) we claim F acts as an inner automorphism. We start by noticing that if i_1, i_2, j_1, j_2 are all distinct then we have $x_{i_1} x_{i_2}$ and $x_{j_1} x_{j_2}$ commute. Let σ_{ij} be the automorphism such that $\sigma_{ij}(y_i) = y_i \vartheta$ and $\sigma_{ij}(y_j) = y_j \vartheta$ but $\sigma_{ij}(y_\ell) = y_\ell$ whenever i, j, ℓ are all distinct. We claim that the automorphism σ_{ij} is an inner automorphism. Consider the element $x_i x_j \in A_{\mathbf{G}}(u)$ then we have

$$\begin{aligned}
y_i^{x_i x_j} &= x_j^{-1} x_i^{-1} (x_1 x_i) x_i x_j = x_j^{-1} (x_1 x_i) x_j \vartheta = x_1 x_i \vartheta = y_i \vartheta, \\
y_j^{x_i x_j} &= x_j^{-1} x_i^{-1} (x_1 x_j) x_i x_j = x_j^{-1} (x_1 x_j) x_j = x_1 (x_j^{-1} x_j) x_j \vartheta = x_1 x_j \vartheta = y_j \vartheta.
\end{aligned}$$

Furthermore by our remark we know $y_\ell^{x_i x_j} = y_\ell$ for any ℓ distinct from both i and j so the automorphism σ_{ij} is inner. As F must be a composition of automorphisms of the form σ_{ij} we have F is an inner automorphism.

We now consider case (b). Assume $\mathbf{G}$ is of type D_n then by composing with a sufficient number of inner automorphisms we may assume that F acts non-trivially on precisely one generator y_i. As F acts non-trivially only on y_i it will be true that $A_{\mathbf{G}}(u)^F$ is the subgroup $\langle\vartheta, y_2, \dots, y_{i-1}, y_{i+1}, \dots, y_k\rangle$, in particular $|A_{\mathbf{G}}(u)^F| = 2^{|I|-1}$. As $\mathbf{G}$ is of type D_n we have

$$|A_{\mathbf{G}}(u)^F|/|A_{\tilde{\mathbf{G}}}(\tilde{u})| = 2^{|I|-1}/2^{|I|-2} = 2. \tag{2.6}$$

We claim that we also have $|Z_{\mathbf{G}}(u)^F| = 2$. By the third paragraph of [**Lus08**, §3.7(a)] we know the centre of $A_{\mathbf{G}}(u)$ is given by $\{1, \vartheta, z, z\vartheta\}$ where we take z to be the element $x_1x_2\cdots x_{k-1}x_k$. Note that this means the centre of $A_{\mathbf{G}}(u)$ coincides with the image of $Z(\mathbf{G})$. We have the following expression of z in terms of our chosen generating set

$$z = x_1x_2\cdots x_k = y_2y_3\cdots y_k\vartheta^c$$

for some $c \in \{0,1\}$. Therefore we have $F(z) = z\vartheta$ and $F(z\vartheta) = z$ so $Z_{\mathbf{G}}(u)^F = \{1, \vartheta\}$ as required. Note that this case is the only case where $A_{\mathbf{G}}(u)^F \to A_{\tilde{\mathbf{G}}}(\tilde{u})$ fails to be surjective.

Assume now $\mathbf{G}$ is of type B_n. Recall that λ partitions an odd number so $k-1$ is even. By composing with a sufficient number of inner automorphisms let us assume that F acts on all but one generator non-trivially. Assume y_i is the generator such that $F(y_i) = y_i$ then it is easy to check that $y_j^{x_1x_i} = y_j\theta$ whenever $j \neq i$ and clearly $y_i^{x_1x_i} = y_i$. In particular F is an inner automorphism. With this in hand we can prove the following proposition.

Proposition 2.38. *Assume* $\mathbf{G}$ *is a simple algebraic group then any* F*-stable unipotent class* $\mathcal{O}$ *of* $\mathbf{G}$ *contains a well-chosen representative.*

Proof. If $\mathbf{G}$ is adjoint then the existence of well-chosen representatives follows from Theorem 2.34, hence we may assume $Z(\mathbf{G})$ is disconnected. We have already covered the two trivial cases where $|A_{\mathbf{G}}(u)| = |A_{\tilde{\mathbf{G}}}(\tilde{u})|$ or $|A_{\tilde{\mathbf{G}}}(\tilde{u})| = 1$, which covers the case of type A_n. If $\mathbf{G}$ is an exceptional group then the only cases left to consider are those covered by Lemma 2.36 and Lemma 2.37. Assume $\mathbf{G}$ is a symplectic group or a special orthogonal group of type D_n then the statement was proved as part of the proof of Theorem 2.34.

If $\mathbf{G}$ is a spin group then the result follows from the above discussion and Lemma 2.33. Finally assume $\mathbf{G}$ is a half spin group then we need only comment that case (b) above cannot happen because the Frobenius cannot act non-trivially on the centre of $\mathbf{G}_{\mathrm{sc}}$ as it must preserve $\mathrm{Ker}(\delta_{\mathrm{sc}})$. In other words the map $A_{\mathbf{G}}(u)^F \to A_{\tilde{\mathbf{G}}}(\tilde{u})$ is always surjective. ■

Before we end this section on component groups we gather slightly more information regarding component groups in spin groups. This will be useful for later applications.

Lemma 2.39. *Let* $\mathbf{G}$ *be a spin group of type* B_n *or* D_n *and* $\mathcal{O} \in \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})$ *a class such that* $\kappa_{\mathrm{B}}(u) = 1$ *or* $n_{\mathrm{D}}(u) \geqslant 1$ *and* $\kappa_{\mathrm{D}}(u) = 1$*. Recall from the above notation that* k *is the number of odd numbers occurring in the partition parameterising* $\mathcal{O}$*, (i.e.* $n_{\mathrm{B}}(u)$ *or* $n_{\mathrm{D}}(u)$*). Then* $\mathrm{Irr}(A_{\mathbf{G}}(u))$ *is comprised of* 2^{k-1} *linear characters and:*

- *one character of degree* $2^{\frac{k-1}{2}}$ *if* $\mathbf{G}$ *is of type* B_n
- *two characters of degree* $2^{\frac{k-2}{2}}$ *if* $\mathbf{G}$ *is of type* D_n*.*

Proof. Let us identify ϑ with its image in $A_{\mathbf{G}}(u)$. It's clear from the above description that $A_{\mathbf{G}}(u)/\langle\vartheta\rangle$ is abelian. In particular $\langle\vartheta\rangle$ is the derived subgroup of $A_{\mathbf{G}}(u)$. As the linear characters of a finite group H are in bijection with $\mathrm{Irr}(H/[H,H])$, (where $[H,H]$ is the derived subgroup of H), the statement regarding linear characters is clear.

To compute the number of remaining irreducible characters we compute the number of conjugacy classes of $A_{\mathbf{G}}(u)$. As ϑ is central in $A_{\mathbf{G}}(u)$ we need only determine whether an element $x \in A_{\mathbf{G}}(u)$ is conjugate to $x\vartheta$. We must have there are at least $2^{k-1}+1$ conjugacy classes in $A_{\mathbf{G}}(u)$. Now consider the element

$$y_1^{i_1}\cdots y_k^{i_k},$$

where $i_j \in \{0,1\}$ for each $1 \leqslant j \leqslant k$. Assume not all i_j are equal, then we may choose indices $1 \leqslant j_1, j_2 \leqslant k$ such that $i_{j_1} = 1$ and $i_{j_2} = 0$. From the above discussion we must have

$$x_{j_2}^{-1}x_{j_1}^{-1}(y_1^{i_1}\cdots y_k^{i_k})x_{j_1}x_{j_2} = y_1^{i_1}\cdots y_k^{i_k}\vartheta$$

Let us now assume that $\mathbf{G}$ is of type D_n. The cases where all i_j are equal correspond to central elements of $A_{\mathbf{G}}(u)$. Hence the above argument shows that there are $4 + \frac{1}{2}(2^k-4) = 2^{k-1}+2$ conjugacy classes in $A_{\mathbf{G}}(u)$. This means there are two non-linear characters $\chi_1, \chi_2 \in \mathrm{Irr}(A_{\mathbf{G}}(u))$ and their degrees must satisfy

$$2^{k-1} + \chi_1(1)^2 + \chi_2(1)^2 = 2^k \Rightarrow \chi_1(1)^2 + \chi_2(1)^2 = 2^{k-1} \Rightarrow \chi_1(1) = \chi_2(1) = 2^{\frac{k-2}{2}}.$$

Let us now assume that $\mathbf{G}$ is of type B_n. We claim the element $y_1\cdots y_k$ is conjugate to $y_1\cdots y_k\vartheta$. Assume that it wasn't then these elements would be central and by the above we would have there are two remaining irreducible characters of degree $2^{\frac{k-2}{2}}$. However k must be odd so these degrees wouldn't be integers, a contradiction. ■

2.5 Theorem A

Due to the results of the previous section, we make the following assumption on unipotent class representatives.

> If $\mathcal{O} \in \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})^F$ is an F-stable unipotent class then we assume our class representative of $\mathcal{O}$ to be well chosen.

The proof given by Hézard of Theorem 2.27 involves a detailed case by case analysis of each simple type. The first main goal of this thesis is to extend the work of Hézard to all simple groups. In particular our first aim is to prove the following theorem, (this is analogous to Hézard's theorem by Remark 2.28).

Theorem A. *Let* $\mathbf{G}$ *be a simple algebraic group and assume* p *is a good prime for* $\mathbf{G}$. *Given any class* $\mathcal{O} \in \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})^F$ *there exists an irreducible character* $\chi \in \mathrm{Irr}(G)$ *such that* $\mathcal{O}$ *is the unipotent support of* χ *and* $n_\chi = |A_{\mathbf{G}}(u)^F| = |Z_{\mathbf{G}}(u)^F||A_{\tilde{\mathbf{G}}}(\tilde{u})|$.

Proof. If $\mathbf{G}$ is adjoint then this follows from Theorem 2.27, hence we may assume $Z(\mathbf{G})$ is disconnected. Let $\mathcal{O}$ be an F-stable unipotent class of $\mathbf{G}$. If $\tilde{s} \in \tilde{\mathbf{T}}_0^\star$ is a semisimple element then we let $s \in \mathbf{T}_0^\star$ be the semisimple element such that $\iota^\star(\tilde{s}) = s$. If this is the case then $\iota^\star$ restricts to a surjective morphism $C_{\tilde{\mathbf{G}}^\star}(\tilde{s}) \to C_{\mathbf{G}^\star}(s)^\circ$ which induces an isomorphism $\mathbf{W}^\star(\tilde{s}) \to \mathbf{W}^\star(s)^\circ$, so we will constantly identify $\mathbf{W}^\star(\tilde{s})$ with $\mathbf{W}^\star(s)^\circ$. By precomposing with this isomorphism we will identify $\mathrm{Irr}(\mathbf{W}^\star(s)^\circ)$ with $\mathrm{Irr}(\mathbf{W}^\star(\tilde{s}))$.

Assume $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}})) \in \mathcal{S}_{\tilde{\mathbf{G}}}^F$ is an F-stable pair. We have generalised Frobenius endomorphisms $F_{\tilde{s}}^\star$ of $C_{\tilde{\mathbf{G}}^\star}(\tilde{s})$ and $F_s^\star$ of $C_{\mathbf{G}^\star}(s)^\circ$ such that $\iota^\star \circ F_{\tilde{s}}^\star = F_s^\star \circ \iota^\star$. We denote by $C_{\tilde{G}^\star}(\tilde{s})$ and $C_{G^\star}(s)^\circ$ the fixed point groups under these generalised Frobenius endomorphisms. To $\mathbf{W}^\star(\tilde{\mathcal{F}})$ we have a corresponding family of unipotent characters $\tilde{\mathcal{F}} \subseteq \mathcal{E}(C_{\tilde{G}^\star}(\tilde{s}), 1)$. Let us denote by $\bar{\psi} \in \mathcal{E}(C_{G^\star}(s)^\circ, 1)$ a unipotent character in a family $\mathcal{F}$ which is in bijective correspondence with $\tilde{\mathcal{F}}$. We now assume that this triple $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}), \bar{\psi})$ satisfies the following properties:

(P1) $n_{\bar{\psi}} = |A_{\tilde{\mathbf{G}}}(\tilde{u})|$.

(P2) $|\operatorname{Stab}_{A_{G^\star}(s)}(\bar{\psi})| = |Z_{\mathbf{G}}(u)^F|$.

(P3) $j_{\mathbf{W}(\tilde{s})}^{\mathbf{W}}(\rho)$ is the Springer character $\rho_{\tilde{u},1}$ so $\Phi_{\tilde{\mathbf{G}}}(\tilde{s}, \mathbf{W}(\tilde{\mathcal{F}})) = \tilde{\mathcal{O}}$.

(Note that here $A_{G^\star}(s)$ denotes the fixed point group $A_{\mathbf{G}^\star}(s)^{F_s^\star}$ with respect to the generalised Frobenius endomorphism $F_s^\star$).

Let $\tilde{t} = {}^{\tilde{g}}\tilde{s} \in \tilde{G}^\star$ be such that $\tilde{g} \in \tilde{\mathbf{G}}^\star$ is an element satisfying $\mathscr{L}(\tilde{g}) = \dot{w}$, we then take $t \in G^\star$ to be $\iota^\star(\tilde{t})$. We have an isomorphism $C_{\mathbf{G}^\star}(s) \to C_{\mathbf{G}^\star}(t)$ given by $h \mapsto {}^{g}h$, which induces a bijection $\mathcal{E}(C_{G^\star}(s)^\circ, 1) \to \mathcal{E}(C_{G^\star}(t)^\circ, 1)$ given by $\chi \mapsto {}^{g}\chi$. We denote by $\bar{\psi}'$ the image of $\bar{\psi}$ under this bijection. It is clear that $x \in C_{G^\star}(s)$ stabilises $\bar{\psi}$ if and only if ${}^{g}x \in C_{G^\star}(t)$ stabilises $\bar{\psi}'$ hence we have $|Z_{\mathbf{G}}(u)^F| = |\operatorname{Stab}_{A_{G^\star(s)}}(\bar{\psi})| = |\operatorname{Stab}_{A_{G^\star(t)}}(\bar{\psi}')|$.

Under the bijection given in Eq. (1.5) we have a character $\psi' \in \mathcal{E}(\tilde{G}, \tilde{t})$ corresponding to $\bar{\psi}' \in \mathcal{E}(C_{G^\star}(t)^\circ, 1)$ such that $\psi'(1) = \psi_{\mathrm{ss}}(1)\bar{\psi}'(1)$, where $\psi_{\mathrm{ss}} \in \mathcal{E}(\tilde{G}, \tilde{t})$ is the unique semisimple character. By [**Car93**, Theorem 8.4.8] we have $|\tilde{G}|_{p'} = |C_{\tilde{G}}(\tilde{t})|_{p'}\psi_{\mathrm{ss}}(1)$ and by the order formula for finite reductive groups, (see [**Car93**, pg. 75]), both $|\tilde{G}|_{p'}$ and $|C_{\tilde{G}}(\tilde{t})|_{p'}$ are monic polynomials in q. In particular $\psi_{\mathrm{ss}}(1)$ must also be a monic polynomial in q hence $n_{\psi'} = n_{\bar{\psi}'}$. By Property (P1) we have $n_{\psi'} = n_{\bar{\psi}'} = n_{\bar{\psi}} = |A_{\tilde{\mathbf{G}}}(\tilde{u})|$. By Property (P3) we have $\tilde{\mathcal{O}}$ is the unipotent support of ψ' and by Theorem 1.73 we have $\operatorname{Res}_G^{\tilde{G}}(\psi') = \chi_1 + \cdots + \chi_k$, where $\chi_i \in \mathcal{E}(G, t)$ and $\chi_i(1) = |\operatorname{Stab}_{A_{G^\star(t)}}(\bar{\psi})|^{-1}\psi'(1)$. For any $1 \leqslant i \leqslant k$ we have by Property (P2) and the above discussion that

$$n_{\chi_i} = |\operatorname{Stab}_{A_{G^\star(t)}}(\bar{\psi})| \cdot n_{\psi'} = |Z_{\mathbf{G}}(u)^F| \cdot |A_{\tilde{\mathbf{G}}}(\tilde{u})| = |A_{\mathbf{G}}(u)^F|.$$

By [**GM00**, Theorem 3.7] we know that $\mathcal{O}$ is the unipotent support of each χ_i hence any of the χ_i will be a solution to our thereom. ∎

Our proof of Theorem A will be complete once we have verified the following proposition.

Proposition A. *Assume* $\mathbf{G}$ *is a simple algebraic group with a disconnected centre,* p *is a good prime for* $\mathbf{G}$ *and* $\mathcal{O} \in \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})^F$. *There exists a triple* $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}), \bar{\psi})$ *as specified in the proof of Theorem A satisfying properties (P1) to (P3). The image of* $\iota^\star(\tilde{s})$ *under an adjoint quotient of* $\mathbf{G}^\star$ *is quasi-isolated. The following condition holds unless* $\mathbf{G}$ *is a spin/half spin group and* $A_{\mathbf{G}}(u)$ *is non-abelian:*

♣ $\mathfrak{X}_{\mathcal{F}} := \{\bar{\psi} \in \mathcal{F} \mid |\operatorname{Stab}_{A_{G^\star(s)}}(\bar{\psi})| \neq |Z_{\mathbf{G}}(u)^F|\} = \varnothing.$

The condition ♣ will be important for applications to generalised Gelfand–Graev representations. The proof of Proposition A will involve a case by case check through each simple type, along the same lines as Hézard. We make some comments regarding exactly which cases need to be considered.

- If $\mathbf{G}$ is simple of type G_2, F_4 or E_8 then the fundamental group is trivial so $\mathbf{G}$ is necessarily adjoint.
- If $\mathbf{G}$ is of type D_4 and F induces a graph automorphism of order three then G does not depend upon the isomorphism type of $\mathbf{G}$, hence we may assume that $\mathbf{G}$ is adjoint.

- The Suzuki and Ree groups are defined only in bad characteristic so we do not consider them here.

Therefore it is left to check Theorem A for the case where $\mathbf{G}$ is a non-adjoint simple group of type A_n, B_n, C_n, D_n, E_6 or E_7 and F induces a graph automorphism of order at most two. This explains the selective information we have gathered about the Springer correspondence.

We will also need to know which unipotent conjugacy classes are F-stable. To understand this we can use the following result.

Lemma 2.40. *Let $\mathbf{G}$ be a connected reductive algebraic group and $\mathcal{O} \in \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})$ a unipotent class. Assume p is a good prime then we have $\mathcal{O}$ is F-stable if and only if its weighted Dynkin diagram $\Delta(\mathcal{O})$ is invariant under the diagram automorphism induced by F.*

Proof. Let $\mathbf{G}'$ be a connected reductive algebraic group over $\mathbb{C}$ whose root datum, with respect to a fixed maximal torus $\mathbf{T}_0'$ and Borel subgroup $\mathbf{B}_0'$ is isomorphic to that of $\mathbf{G}$. In particular we have an isomorphism $X(\mathbf{T}_0) \to X(\mathbf{T}_0')$ and a bijection $b : \Phi(\mathbf{T}_0) \to \Phi(\mathbf{T}_0')$. Recall that we may decompose the Frobenius endomorphism F as a composition $F_r \circ \tau$, where τ is a graph automorphism of $\mathbf{G}$ and r is a power of p. We then have a graph automorphism τ' of $\mathbf{G}'$ such that $b \circ \tau = \tau' \circ b$, where we denote again by τ and τ' the bijections on the roots $\Phi(\mathbf{T}_0)$ and $\Phi(\mathbf{T}_0')$.

In [**Spa82**, Théorème III.5.2] Spaltenstein gives a map $\pi_{\mathbf{G}} : \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G}') \to \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})$, which is a bijection when p is good. By the discussion in [**GM00**, §4.C] we have a commutative diagram

$$\begin{array}{ccc}
\mathfrak{Cl}_{\mathrm{U}}(\mathbf{G}') & \xrightarrow{\pi_{\mathbf{G}}} & \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G}) \\
\big\downarrow \tau' & & \big\downarrow F \\
\mathfrak{Cl}_{\mathrm{U}}(\mathbf{G}') & \xrightarrow{\pi_{\mathbf{G}}} & \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})
\end{array}$$

Let $\mathcal{O}'$ be the preimage of $\mathcal{O}$ under the map $\pi_{\mathbf{G}}$. We need only show that $\mathcal{O}'$ is stable under τ' if and only if its weighted Dynkin diagram $\Delta(\mathcal{O}')$ is stable under the diagram automorphism induced by τ'.

As $\mathbf{G}'$ is a group over a field of characteristic zero we have a bijection between the unipotent conjugacy classes of $\mathbf{G}$ and the nilpotent orbits of its Lie algebra $\mathfrak{g}'$. We denote by $\hat{\mathcal{O}}'$ the nilpotent orbit corresponding to $\mathcal{O}'$ and $\{e, h, f\}$ a standard triple for $\hat{\mathcal{O}}'$, (such that $e \in \mathfrak{b}_0'$ and $h \in \mathfrak{t}_0'$). Differentiating τ' we obtain an automorphism of the Lie algebra, which we again denote by τ'. The corresponding automorphisms of $\mathbf{G}'$ and $\mathfrak{g}'$ commute with the bijection between unipotent classes and nilpotent orbits. In particular we have $\mathcal{O}'$ is τ'-stable if and only if $\hat{\mathcal{O}}'$ is τ'-stable. It is clear that $\{\tau'(e), \tau'(h), \tau'(f)\}$ will be a standard triple corresponding to a nilpotent orbit such that $\tau'(e) \in \mathfrak{b}'_0$ and $\tau'(h) \in \mathfrak{t}'_0$. The weighted Dynkin diagram corresponding to $\{\tau'(e), \tau'(h), \tau'(f)\}$ is $\Delta(\hat{\mathcal{O}}')$ if and only if $\tau'(e) \in \hat{\mathcal{O}}'$, which is true if and only if the orbit is τ'-stable. The only thing left to remark is that the weighted Dynkin diagram corresponding to $\{\tau'(e), \tau'(h), \tau'(f)\}$ is obtained from $\Delta(\hat{\mathcal{O}}')$ by applying the diagram automorphism induced from τ'. ■

Assume now that $\mathbf{G}$ is simple. The following can be verified using the information in [**Car93**, Chapter 13] together with Lemma 2.40. If $\mathbf{G}$ is of type A_n or E_6 the weighted Dynkin diagram of every unipotent class is invariant under the non-trivial graph automorphism, (see also [**CM93**, Lemma 3.6.5] for A_n). If $\mathbf{G}$ is of type D_n then the only unipotent classes not invariant under the graph automorphism of order 2 are the degenerate unipotent classes. This now settles the question of which unipotent classes we must prove Proposition A for.

Chapter 3

Proposition A: Type C_n

In this chapter we wish to prove Proposition A for the case where $\mathbf{G}$ is a simply connected group of type C_n ($n \geqslant 2$). Such a group and its dual $\mathbf{G}^\star$ can both be realised as classical matrix groups. For this case we will purposely work naively with the matrix representations of these groups. Apart from proving Proposition A we will make a slight digression and discuss some general results concerning the restriction of characters from $\tilde{G}$ to G in a given Lusztig series.

3.1 Basic Definitions

We follow [**Gec03**, 1.3.15 and 1.3.16] in defining the corresponding matrix groups. We define the *symplectic group* $\mathbf{G} = \mathrm{Sp}_{2n}(\mathbb{K})$ in the following way. Consider the square matrices

$$J_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathrm{Mat}_n(\mathbb{K}) \qquad \text{and} \qquad \Omega_n = \begin{bmatrix} & J_n \\ -J_n & \end{bmatrix} \in \mathrm{Mat}_{2n}(\mathbb{K}), \tag{3.1}$$

then $\mathrm{Sp}_{2n}(\mathbb{K}) = \{X \in \mathrm{Mat}_{2n}(\mathbb{K}) \mid X^T\Omega_n X = \Omega_n\}$. In other words $\mathrm{Sp}_{2n}(\mathbb{K})$ is the subgroup of $\mathrm{GL}_{2n}(\mathbb{K})$ consisting of those matrices which preserve our chosen symplectic matrix Ω_n. We take F to be the Frobenius endomorphism given by $F(x_{ij}) = (x_{ij}^q)$ for all $(x_{ij}) \in \mathbf{G}$. The corresponding finite reductive group G is the finite symplectic group $\mathrm{Sp}_{2n}(q)$. As p is a good prime for $\mathbf{G}$ we have by the classification of finite reductive groups, (see [**Car93**, §1.19]), that for any other choice of Frobenius endomorphism F' the resulting finite group $\mathbf{G}^{F'}$ is isomorphic to G.

We will index the rows and columns of our matrices in $\mathbf{G}$ from top to bottom and left to right by $1, \dots, n, -n, \dots, -1$. We take $\mathbf{T}_0$ to be the maximal torus of all diagonal matrices. Explicitly any element $t \in \mathbf{T}_0$ will be of the form

$$t = \mathrm{diag}(t_1, \dots, t_n, t_{-n}, \dots, t_{-1}), \tag{3.2}$$

where $t_i \in \mathbb{K}^\times$ and $t_{-i} = t_i^{-1}$ for each $1 \leqslant i \leqslant n$. The subgroup $\mathbf{T}_0$ is an F-stable maximal torus of $\mathbf{G}$. We take $\mathbf{B}_0$ to be the subgroup of all upper triangular matrices in $\mathbf{G}$, which is an F-stable Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}_0$. We take the dual group $\mathbf{G}^\star = \mathrm{SO}_{2n+1}(\mathbb{K})$ to be the odd-dimensional special orthogonal group defined to be $\mathrm{SO}_{2n+1}(\mathbb{K}) = \{X \in \mathrm{Mat}_{2n+1}(\mathbb{K}) \mid X^T J_{2n+1} X = J_{2n+1} \text{ and } \det(X) = 1\}$. The Frobenius endomorphism $F^\star$ is again given by $F^\star(x_{ij}) = (x_{ij}^q)$. The resulting finite reductive group $G^\star$ is the finite special orthogonal group $\mathrm{SO}_{2n+1}(q)$.

We will again index the rows and columns of our matrices in $\mathbf{G}^\star$ from top to bottom and left to right by $1, \dots, n, 0, -n, \dots, -1$. We take $\mathbf{T}_0^\star$ to again be the maximal torus of all diagonal matrices. Explicitly any element $s \in \mathbf{T}_0^\star$ will be of the form

$$s = \mathrm{diag}(s_1, \dots, s_n, 1, s_{-n}, \dots, s_{-1}), \tag{3.3}$$

where $s_i \in \mathbb{K}^\times$ and $s_{-i} = s_i^{-1}$ for each $1 \leqslant i \leqslant n$. The subgroup $\mathbf{T}_0^\star$ is clearly an $F^\star$-stable maximal torus of $\mathbf{G}^\star$. We take $\mathbf{B}_0^\star$ to be the subgroup of all upper triangular matrices in $\mathbf{G}^\star$, which is an $F^\star$-stable Borel subgroup of $\mathbf{G}^\star$ containing $\mathbf{T}_0^\star$. We may take our isomorphism $X(\mathbf{T}_0) \to \check{X}(\mathbf{T}_0^\star)$ to be such that $\omega_i \mapsto \check{\omega}_i^\star$ for all $1 \leqslant i \leqslant n$. As the fundamental dominant weights and coweights form bases for $X(\mathbf{T}_0)$ and $\check{X}(\mathbf{T}_0^\star)$ respectively this prescribes our isomorphism.

For $1 \leqslant i \leqslant n$ let $\varepsilon_i \in \mathrm{Hom}(\mathbf{T}_0^\star, \mathbb{K}^\times)$ be such that for all $s \in \mathbf{T}_0^\star$ we have $\varepsilon_i(s) = s_i$. The set of roots of $\mathbf{G}^\star$ with respect to $\mathbf{T}_0^\star$ is then given by

$$\begin{aligned}
\Phi(\mathbf{T}_0^\star) &= \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i \leqslant n,\ i < j\},\\
\Phi(\mathbf{T}_0^\star)^+ &= \{\varepsilon_i, \varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i \leqslant n,\ i < j\},\\
\Delta(\mathbf{T}_0^\star) &= \{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_n \mid 1 \leqslant i \leqslant n-1\}.
\end{aligned}$$

The simple roots $\Delta(\mathbf{T}_0^\star) = \{\alpha_1^\star, \dots, \alpha_n^\star\}$ correspond to the Dynkin diagram of type B_n given in Figure 1.1 by the labelling $\alpha_i^\star = \varepsilon_i - \varepsilon_{i+1}$ for all $1 \leqslant i \leqslant n-1$ and $\alpha_n^\star = \varepsilon_n$. The negative of the highest root is always given by $\alpha_0^\star = -(\varepsilon_1 + \varepsilon_2) = -(\alpha_1^\star + 2\alpha_2^\star + \cdots + 2\alpha_n^\star)$. For each simple root the associated 1-dimensional root subgroup $\mathbf{X}_\alpha \leqslant \mathbf{B}_0^\star$ is given by

$$\begin{aligned}
\mathbf{X}_{\alpha_n^\star} &= \left\{ I_{2n+1} + \kappa(E_{0,-n} - E_{n,0}) - \frac{1}{2}\kappa^2 E_{-n,n} \,\middle|\, \kappa \in \mathbb{K}^+ \right\},\\
\mathbf{X}_{\alpha_i^\star} &= \{I_{2n+1} + \kappa(E_{i,i+1} - E_{-(i+1),-i}) \mid \kappa \in \mathbb{K}^+\},
\end{aligned}$$

where $E_{k,\ell}$ is an elementary matrix in $\mathrm{Mat}_{2n+1}(\mathbb{K})$ whose only non-zero entry is 1 in the kth row and ℓth column.

Example 3.1. Assume $n = 2$ then $\mathbf{G}^\star = \mathrm{SO}_5(\mathbb{K})$ and this a simple adjoint algebraic group of type B_2. The root system contains eight roots and the positive system contains four roots. In particular we have

$$\Phi^\star(\mathbf{T}_0^\star)^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_1 + \varepsilon_2\}.$$

The simple roots are given by the set $\Delta(\mathbf{T}_0^\star) = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2\}$. The two root subgroups corresponding to the two simple roots have the form

$$\mathbf{X}_{\varepsilon_2} = \left\{ \begin{bmatrix} 1 & . & . & . & . \\ & 1 & \kappa & -\frac{1}{2}\kappa^2 & . \\ & & 1 & -\kappa & . \\ & & & 1 & . \\ & & & & 1 \end{bmatrix} \right\} \qquad \mathbf{X}_{\varepsilon_1 - \varepsilon_2} = \left\{ \begin{bmatrix} 1 & \kappa & . & . & . \\ & 1 & . & . & . \\ & & 1 & . & . \\ & & & 1 & -\kappa \\ & & & & 1 \end{bmatrix} \right\}$$

where $\kappa \in \mathbb{K}^+$ and a . represents a zero entry. It is easy to check that conjugating the matrices in these root subgroups with an element of $\mathbf{T}_0^\star$ induces the appropriate automorphism of $\mathbb{K}^+$ that gives rise to the simple roots.

Let us finally consider the Weyl group of $\mathbf{G}$ and $\mathbf{G}^\star$. We have $\mathbf{W} \cong \mathbf{W}^\star$ is isomorphic to the wreath product $\mathbb{Z}_2 \wr \mathfrak{S}_n$, which can be seen by noticing that $N_{\mathbf{G}}(\mathbf{T}_0)$ and $N_{\mathbf{G}^\star}(\mathbf{T}_0^\star)$ are the subgroups of matrices which permute the unordered pairs of indices $(1,-1), \dots, (n,-n)$ in the elements $t \in \mathbf{T}_0$ and $s \in \mathbf{T}_0^\star$ as defined in Eqs. (3.2) and (3.3). We wish to describe the action of the Weyl group on the maximal torus $\mathbf{T}_0^\star$ but it will be sufficient to describe the action of the simple reflections. Let $s \in \mathbf{T}_0^\star$ be as in Eq. (3.3) then we have the following descriptions:

- the simple reflection $s_{\alpha_i^\star} \in \mathbb{S}^\star$, for $1 \leqslant i \leqslant n-1$, interchanges s_i, s_{i+1} and s_i^{-1}, s_{i+1}^{-1},
- the reflection $s_{\alpha_n^\star} \in \mathbb{S}^\star$ interchanges s_n, s_n^{-1}.

3.2 Centralisers of Semisimple Elements in the Dual Group

In this section we discuss in more detail the structure of centralisers of semisimple elements. We give examples of these structural results in our dual group $\mathbf{G}^\star$. The main result in this direction comes from Steinberg.

Theorem 3.2 (Steinberg, [Car93, Theorem 3.5.3]). *Let $\mathbf{G}$ be a connected reductive algebraic group. For any semisimple element $t \in \mathbf{T}_0$ we have the centraliser and its connected* *component are given by*

$$C_{\mathbf{G}}(t) = \langle \mathbf{T}_0, \mathbf{X}_\alpha, \dot{w} \mid \alpha(t) = 1 \textit{ for } \alpha \in \Phi \textit{ and } s^w = s \textit{ for } w \in \mathbf{W} \rangle,$$
$$C_{\mathbf{G}}(t)^\circ = \langle \mathbf{T}_0, \mathbf{X}_\alpha \mid \alpha(t) = 1 \textit{ for } \alpha \in \Phi \rangle.$$

From the statement of Steinberg's thereom we can easily see the existence of the isomorphism $\mathcal{A}(s) \to A_{\mathbf{G}^\star}(s)$ mentioned in Section 2.3.

Example 3.3. Consider the case of $\mathbf{G}^\star = \mathrm{SO}_5(\mathbb{K})$. In certain cases it is possible to determine the structure of the centraliser of a semisimple element using entirely brute force methods. Let $s = \mathrm{diag}(\eta, \eta, 1, \eta^{-1}, \eta^{-1}) \in \mathbf{T}_0^\star$ be such that $\eta \neq \pm 1$. If $x = (x_{ij}) \in \mathrm{Mat}_5(\mathbb{K})$ is such that $xsx^{-1} = s$ then we have $xs = sx$ so

$$\begin{bmatrix} \eta x_{11} & \eta x_{12} & x_{13} & \eta^{-1}x_{14} & \eta^{-1}x_{15} \\ \eta x_{21} & \eta x_{22} & x_{23} & \eta^{-1}x_{24} & \eta^{-1}x_{25} \\ \eta x_{31} & \eta x_{32} & x_{33} & \eta^{-1}x_{34} & \eta^{-1}x_{35} \\ \eta x_{41} & \eta x_{42} & x_{43} & \eta^{-1}x_{44} & \eta^{-1}x_{45} \\ \eta x_{51} & \eta x_{52} & x_{53} & \eta^{-1}x_{54} & \eta^{-1}x_{55} \end{bmatrix} = \begin{bmatrix} \eta x_{11} & \eta x_{12} & \eta x_{13} & \eta x_{14} & \eta x_{15} \\ \eta x_{21} & \eta x_{22} & \eta x_{23} & \eta x_{24} & \eta x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ \eta^{-1}x_{41} & \eta^{-1}x_{42} & \eta^{-1}x_{43} & \eta^{-1}x_{44} & \eta^{-1}x_{45} \\ \eta^{-1}x_{51} & \eta^{-1}x_{52} & \eta^{-1}x_{53} & \eta^{-1}x_{54} & \eta^{-1}x_{55} \end{bmatrix},$$
$$\Rightarrow x = \begin{bmatrix} x_{11} & x_{12} & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 & 0 \\ 0 & 0 & x_{33} & 0 & 0 \\ 0 & 0 & 0 & x_{44} & x_{45} \\ 0 & 0 & 0 & x_{54} & x_{55} \end{bmatrix}.$$

We now wish to consider when such a matrix lies inside $\mathrm{SO}_5(\mathbb{K})$. We recall that x should satisfy the conditions $x^T J_5 x = J_5$ and $\det(x) = 1$. Considering x as a block matrix we have

$$\begin{bmatrix} A^T & 0 & 0 \\ 0 & x_{33} & 0 \\ 0 & 0 & B^T \end{bmatrix} \begin{bmatrix} 0 & 0 & J_2 \\ 0 & 1 & 0 \\ J_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} A & 0 & 0 \\ 0 & x_{33} & 0 \\ 0 & 0 & B \end{bmatrix} = \begin{bmatrix} 0 & 0 & A^T J_2 B \\ 0 & x_{33}^2 & 0 \\ B^T J_2 A & 0 & 0 \end{bmatrix}.$$

Therefore we must have $A^T J_2 B = J_2 \Rightarrow B = J_2(A^{-1})^T J_2$ and $x_{33}^2 = 1 \Rightarrow x_{33} = \pm 1$. The determinant condition gives us that $\det(x) = x_{33} = 1$ so we can describe the centraliser of s as

$$C_{\mathbf{G}^\star}(s) = \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & J_2(A^{-1})^T J_2 \end{bmatrix} \middle| A \in \mathrm{GL}_2(\mathbb{K}) \right\}.$$

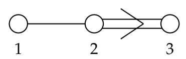
It is clear from this description of the centraliser that it is isomorphic to $\mathrm{GL}_2(\mathbb{K})$, so must be connected. In general this method of determining the structure of centralisers of semisimple elements is very ineffective. Indeed, even in $\mathrm{SO}_5(\mathbb{K})$, there are some cases where it is difficult to determine whether the centraliser is connected or disconnected.

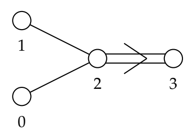
In [**Car81**] Carter explicitly determines the semisimple types of connected components of centralisers of semisimple elements in simple groups of classical type. Firstly, the semisimple type of the connected component of the centraliser of a semisimple element does not depend upon the choice of class representative. Hence we may assume without loss of generality, until the end of this section, that $s \in \mathbf{G}^\star$ is contained in our fixed maximal torus $\mathbf{T}_0^\star$. If this is the case then $\mathbf{T}_0^\star \leqslant C_{\mathbf{G}^\star}(s)^\circ$ and we may construct the root datum for $C_{\mathbf{G}^\star}(s)^\circ$ with respect to $\mathbf{T}_0^\star$. Let $\Phi(s)$ be the root system of $C_{\mathbf{G}^\star}(s)^\circ$ with respect to $\mathbf{T}_0^\star$, (as in Section 2.3), then this is an additively closed subsystem of $\Phi^\star$. Carter gives criteria for subsystems to come from the connected component of the centraliser of a semisimple element. By [**Goo07**, §2] up to $\mathbf{W}^\star$-conjugacy all possible additively closed root subsystems of $\Phi^\star$ can be obtained using the following algorithm of Borel–De Siebenthal, (originally stated in [**BDS49**]).

The Algorithm of Borel–De Siebenthal. Let Φ be a root system with irreducible components $\Phi_1, \dots, \Phi_r$. We assume $\Delta_i \subset \Phi_i$ is a simple system of roots for the irreducible root system Φ_i.

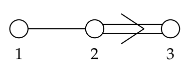
- Take a subset $\mathbb{I}$ of $\tilde{\Delta}$ such that $\mathbb{I} \cap \tilde{\Delta}_i \neq \varnothing$ for each $1 \leqslant i \leqslant r$.
- Generate a root subsystem $\overline{\Phi}$ of Φ using the set $\tilde{\Delta} \setminus \mathbb{I}$.
- Repeat the first two steps with $\Phi := \overline{\Phi}$ until all possibilities have been accounted for.

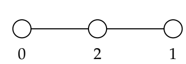
Example 3.4. Consider the irreducible root system Φ of type B_3. The Dynkin diagram for this root system and its extended Dynkin diagram, (corresponding to the set $\tilde{\Delta}$), are given by

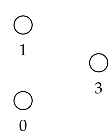
B_3 

$\tilde{B}_3$ 

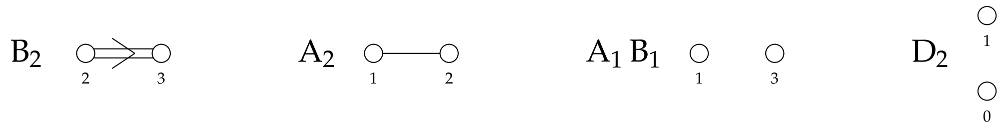
We give a list of $\mathbf{W}$-conjugacy classes of additively closed root subsystems of Φ by their Dynkin diagram. Note that roots of different lengths are not conjugate under $\mathbf{W}$. Therefore we write A_1 for a rank 1 root system containing a long root of Φ and B_1 for a rank 1 root system containing a short root of Φ. We will also write D_2 for a root system of type $A_1\,A_1$ involving the roots labelled by 0 and 1 to be in keeping with the notation we introduce below. If $|\mathbb{I}| = 1$ in the algorithm of Borel–De Siebenthal we have three possible diagrams.

B_3 

A_3 

$D_2\,B_1$ 

If $|\mathbb{I}| = 2$ then we have the following four possiblities



Finally if $|\mathbb{I}| = 3$ then we only have the two subsystems of type A_1 and B_1, examples of which are given by the subsets $\{\alpha_1\}$ and $\{\alpha_3\}$ respectively. We also have the case where $|\mathbb{I}| = 4$ and the diagram is empty, corresponding to the empty root system. This list then exhausts all possible **W**-conjugacy classes of additively closed subroot systems of Φ.

We now consider this discussion in the context of our dual group $\mathbf{G}^\star$. From the algorithm of Borel–De Siebenthal it is clear to see that the semisimple type of the connected component of the centraliser of a semisimple element of $\mathbf{G}^\star$ can only contain components of type A, D and B. In fact it can only contain one component of type B. Carter shows in [**Car81**, §4 - Table 1] that the connected component of the centraliser of a semisimple element in $\mathbf{G}^\star$ contains at most one component of type D.

Let $s \in \mathbf{T}_0^\star$ be a semisimple element such that s is as in Eq. (3.3). For each $\lambda \in \mathbb{K}^\times$ we denote by $m(\lambda)$ the cardinality of the set $N_\lambda := \{i \mid s_i = \lambda \text{ and } 1 \leqslant i \leqslant n\}$ and we denote by $\widetilde{\mathbb{K}}^\times$ the set $\mathbb{K}^\times \setminus \{\pm 1\}$. After possibly replacing s by a conjugate under $\mathbf{W}^\star$ we may assume

$$\lambda \in \{s_i \mid 1 \leqslant i \leqslant n\} \Rightarrow \lambda^{-1} \notin \{s_i \mid 1 \leqslant i \leqslant n\} \text{ for all } \lambda \in \widetilde{\mathbb{K}}^\times. \tag{3.4}$$

Let us now consider the indecomposable subsystems of $\Phi(s)$.

- Assume $\lambda \in \widetilde{\mathbb{K}}^\times$ then for each $i, j \in N_\lambda$ we have $(\varepsilon_i - \varepsilon_j)(s) = s_i s_j^{-1} = 1$. As $\lambda \neq 1$ we have $(\pm\varepsilon_i)(s) \neq 1$ for all $1 \leqslant i \leqslant n$. Assume $1 \leqslant i, j \leqslant n$ are such that $(\varepsilon_i + \varepsilon_j)(s) = s_i s_j = 1$ then $s_i = s_j^{-1}$ but by our assumption on s this implies $s_i = s_j = \pm 1$. In particular for each $\lambda \in \widetilde{\mathbb{K}}^\times$ the set $\{\varepsilon_i - \varepsilon_j \mid i, j \in N_\lambda \text{ and } i \neq j\}$ is an additively closed root subsystem of $\Phi(s)$ of type $\mathrm{A}_{m(\lambda)-1}$. Note that this set is simply the empty root subsystem if $m(\lambda) < 2$.
- For each $i, j \in N_{-1}$ we have $(\varepsilon_i - \varepsilon_j)(s) = (\varepsilon_i + \varepsilon_j)(s) = 1$ but clearly for each $i \in N_{-1}$ we have $(\pm\varepsilon_i)(s) \neq 1$. In particular the set $\{\pm\varepsilon_i \pm \varepsilon_j \mid i, j \in N_{-1} \text{ and } i \neq j\}$ is an additively closed root subsystem of $\Phi(s)$ of type $\mathrm{D}_{m(-1)}$. Note that this set is simply the empty set if $m(-1) < 2$. Finally the set $\{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid i, j \in N_1\}$ is an additively closed root subsystem of $\Phi(s)$ of type $\mathrm{B}_{m(1)}$.

Summarising this discussion we have for any $s \in \mathbf{T}_0^\star$ satisfying the assumption in Eq. (3.4) that the root system $\Phi(s)$ is the following disjoint union of irreducible subsystems.

$$\bigcup_{\lambda\in\widetilde{\mathbb{K}}^\times}\{\varepsilon_i-\varepsilon_j \mid i,j\in N_\lambda \text{ and } i\neq j\}\bigcup\{\pm\varepsilon_i\pm\varepsilon_j \mid i,j\in N_{-1} \text{ and } i\neq j\}$$
$$\bigcup\{\pm\varepsilon_i\pm\varepsilon_j, \pm\varepsilon_j \mid i,j\in N_1 \text{ and } i\neq j\}, \tag{3.5}$$

In particular $\Phi(s)$ is a root system of type $\prod_{\lambda\in\widetilde{\mathbb{K}}^\times} A_{m(\lambda)-1}\,\mathrm{D}_{m(-1)}\,\mathrm{B}_{m(1)}$ where A_{-1} and D_1 should be taken to be the empty root system. Note that the description given in Eq. (3.5) of $\Phi(s)$ is directly comparable with that given in [**Car81**, §3]. Using this description we can consider when a semisimple element of $\mathbf{G}^\star$ is isolated.

Remark 3.5. It is easily seen that any conjugate of an isolated, (resp. quasi-isolated), semisimple element is again isolated, (resp. quasi-isolated). As every semisimple element of $\mathbf{G}^\star$ is conjugate to an element of our chosen maximal torus $\mathbf{T}_0^\star$ we need only determine which elements of $\mathbf{T}_0^\star$ are isolated, (resp. quasi-isolated). Furthermore we may work with one element from each $\mathbf{W}^\star$-orbit of $\mathbf{T}_0^\star$. In particular we will work with an element $s\in\mathbf{T}_0^\star$ satisfying the assumption in Eq. (3.4) so that the root system $\Phi(s)$ is then given as in Eq. (3.5).

Lemma 3.6. *Let $s\in\mathbf{T}_0^\star$ be a semisimple element satisfying the assumption in Eq. (3.4) so that the root system $\Phi(s)$ of $C_{\mathbf{G}^\star}(s)^\circ$ is given as in Eq. (3.5). We have s is isolated if and only if $m(\lambda)=0$ for all $\lambda\in\widetilde{\mathbb{K}}^\times$ and $m(-1)\neq 1$.*

Proof. Assume that $m(\lambda)=0$ for all $\lambda\in\widetilde{\mathbb{K}}^\times$ then $\Phi(s)$ is a root system of type $\mathrm{D}_{m(-1)}\,\mathrm{B}_{m(1)}$. If $m(-1)=0$ then $\Phi(s)=\Phi^\star$ so clearly s is isolated as $C_{\mathbf{G}^\star}(s)^\circ=\mathbf{G}^\star$ and this is not a proper Levi subgroup. If $m(-1)=1$ then $\Phi(s)$ is a subsystem of type B_{n-1} hence $C_{\mathbf{G}^\star}(s)^\circ$ is a proper Levi subgroup. Assume now that $m(-1)\notin\{0,1\}$ then, after possibly replacing s by a $\mathbf{W}^\star$-conjugate, we can see that $\alpha_0=-(\varepsilon_1+\varepsilon_2)$ forms a simple root of $\Phi(s)$. In particular we cannot have $C_{\mathbf{G}^\star}(s)^\circ$ is contained in a proper Levi subgroup. Now assume $m(\lambda)\neq 0$ for some $\lambda\in\widetilde{\mathbb{K}}^\times$. It is easy to see that $\Phi(s)$ is contained in a proper root subsystem of type $\prod_{\lambda\in\widetilde{\mathbb{K}}^\times}\mathrm{A}_{m(\lambda)-1}\,\mathrm{B}_{m(-1)+m(1)}$. This corresponds to $C_{\mathbf{G}^\star}(s)^\circ$ lying in a proper Levi subgroup of $\mathbf{G}^\star$ so s is not isolated. ∎

We now consider for each semisimple element $s\in\mathbf{T}_0^\star$ the structure of the full centraliser $C_{\mathbf{G}^\star}(s)$. To do this we must consider what Weyl group elements commute with s.

Lemma 3.7. *Let $s\in\mathbf{T}_0^\star$ be a semisimple element satisfying the assumption in Eq. (3.4) so that the root system $\Phi(s)$ of $C_{\mathbf{G}^\star}(s)^\circ$ is given as in Eq. (3.5) then $\mathbf{W}^\star(s)=\mathbf{W}^\star(s)^\circ$ if and only if $m(-1)=0$. Furthermore $\mathbf{W}^\star(s)$ is a Coxeter group of type $\prod_{\lambda\in\widetilde{\mathbb{K}}^\times}\mathrm{A}_{m(\lambda)-1}\,\mathrm{B}_{m(-1)}\,\mathrm{B}_{m(1)}$.*

Proof. The set $\Delta(s)=\Phi(s)\cap\tilde{\Delta}^\star$ will be a system of simple roots for $\Phi(s)$. The group $\mathcal{A}(s)$ is then simply given by $\{w\in\mathbf{W}^\star(s)\mid w\cdot\Delta(s)=\Delta(s)\}$, which means we have an inclusion $\mathcal{A}(s)\subseteq\mathrm{Aut}_{\mathbf{W}^\star}(\Delta(s))=\{w\in\mathbf{W}^\star\mid w\cdot\Delta(s)=\Delta(s)\}$.

The group $\mathrm{Aut}_{\mathbf{W}^\star}(\Delta(s))$ is described by Carter in [**Car81**, §3 - Proposition 11]. In particular $\mathrm{Aut}_{\mathbf{W}^\star}(\Delta(s))$ consists of: all permutations of isomorphic components of type A and the graph automorphisms of order 2 of components of type A and D. Assume that $m(-1) = 0$ then we claim that no such automorphism can centralise s. Assume that $m(\lambda_1) = m(\lambda_2) \neq 0$ for some distinct $\lambda_1, \lambda_2 \in \widetilde{\mathbb{K}}^\times$ then the corresponding type A subsystems are of the same semisimple rank. Let $w \in \mathrm{Aut}_{\mathbf{W}^\star}(\Delta(s))$ be the element which exchanges the two corresponding subsystems of $\Phi(s)$. Such an element acts on s by exchanging the sets of entries $\{s_j \mid j \in N_{\lambda_1}\}$ and $\{s_j \mid j \in N_{\lambda_2}\}$ so cannot possibly centralise s.

Let us assume that $m(\lambda) \geqslant 2$ for some $\lambda \in \widetilde{\mathbb{K}}^\times$ then $\Phi(s)$ contains a subsystem of type $\mathrm{A}_{m(\lambda)-1}$ such that $m(\lambda) - 1 \geqslant 1$. Let $w \in \mathrm{Aut}_{\mathbf{W}^\star}(\Delta(s))$ be the element inducing the graph automorphism of order 2 on the component of type $\mathrm{A}_{m(\lambda)-1}$, (note that if $m(\lambda) = 2$ then this automorphism is simply the identity). This Weyl group element acts on the semisimple element s by exchanging s_i, (for any $i \in N_\lambda$), with s_j^{-1} for some uniquely determined $j \in N_\lambda$. Such an element clearly cannot centralise s as $s_i \neq s_j^{-1}$, (because s satisfies the assumption in Eq. (3.4)). As $m(-1) = 0$ we must have $\mathcal{A}(s)$ is trivial as required.

We now assume that $m(-1) > 0$. For all $j \in N_{-1}$ we have $s_j = -1$ so $\varepsilon_j(s) \neq 1$ and the reflection $s_{\varepsilon_j} \in \mathbf{W}^\star$ acts on s by exchanging the entries s_j and s_j^{-1} so $s_{\varepsilon_j} \in \mathbf{W}^\star(s)$. Let $j \in N_{-1}$ be the largest value in N_{-1} then s_{ε_j} is such that $s_{\varepsilon_j} \cdot (\varepsilon_j - \varepsilon_{j-1}) = -\varepsilon_j - \varepsilon_{j-1}$ hence induces the graph automorphism of the component of type $\mathrm{D}_{m(-1)}$. We have $\mathbf{W}^\star(s) = \langle \mathbf{W}^\star(s)^\circ, s_{\varepsilon_j} \rangle$ so is a group generated by reflections. It is clear that the extra reflection will complete the component of type $\mathrm{D}_{m(-1)}$ to a component of type $\mathrm{B}_{m(-1)}$ so $\mathbf{W}^\star(s)$ is always a Coxeter group. ■

Note that this is a reasonably exceptional case as $\mathbf{W}^\star(s)$ is not a Coxeter group in general. Using this we can now settle when an element is quasi-isolated.

Corollary 3.8. *The element s is quasi-isolated if and only if $m(\lambda) = 0$ for all $\lambda \in \widetilde{\mathbb{K}}^\times$.*

Proof. Assume $m(\lambda) \neq 0$ for some $\lambda \in \widetilde{\mathbb{K}}^\times$. As in the proof of Lemma 3.6 we can argue that $C_{\mathbf{G}^\star}(s)$ will be contained in a Levi of type $\prod_{\lambda \in \widetilde{\mathbb{K}}^\times} \mathrm{A}_{m(\lambda)-1} \mathrm{B}_{m(-1)+m(1)}$ as clearly the Weyl group of type $\mathrm{B}_{m(-1)+m(1)}$ will contain the reflection $s_{\varepsilon_j} \in \mathbf{W}^\star(s) \setminus \mathbf{W}^\star(s)^\circ$. Therefore we have s is not quasi-isolated.

Assume $m(\lambda) = 0$ for all $\lambda \in \widetilde{\mathbb{K}}^\times$ then by Lemma 3.6 if $m(-1) \neq 1$ we already know s is isolated hence quasi-isolated so we need only consider the case when $m(-1) = 1$. In this case $\Phi(s)$ is of type B_{n-1} and the only root subsystem of $\Phi^\star$ properly containing the root subsystem of type B_{n-1} is $\Phi^\star$ itself. As $\mathbf{W}^\star(s) \neq \mathbf{W}^\star(s)^\circ$ we must have $C_{\mathbf{G}^\star}(s)$ is not contained in any proper Levi subgroup hence s is quasi-isolated. ■

Using Lemma 3.7 it is easy to determine what happens to a character of $\tilde{G}$ upon restriction to G, (recall that in the following $\iota^\star$ denotes the surjective morphism $\tilde{\mathbf{G}}^\star \to \mathbf{G}^\star$). First let us note that if $s \in \mathbf{T}_0^\star$ is a semisimple element then $2m(-1)$ and $2m(1)+1$ give the multiplicities of -1 and 1 as eigenvalues of s. With this in mind we will cast the following results in terms of eigenvalues.

Proposition 3.9. *Let $\psi \in \mathcal{E}(\tilde{G}, \tilde{s})$ be an irreducible character of $\tilde{G}$ for some semisimple element $\tilde{s} \in \tilde{G}^\star$. Assume $\iota^\star(\tilde{s})$ has no eigenvalue equal to -1 then $\mathrm{Res}_G^{\tilde{G}}(\psi)$ is an irreducible character of G.*

Proof. By Lemma 3.7 we know that the centraliser of s in $\mathbf{G}^\star$ is connected if and only if it has no eigenvalue equal to -1. Therefore by Theorem 1.73 we know that every restricted character in the given Lusztig series is irreducible. ■

In fact we can be very specific about precisely which characters will split and which characters will remain irreducible upon restriction.

Proposition 3.10. *Let $\psi \in \mathcal{E}(\tilde{G}, \tilde{s})$ be an irreducible character of $\tilde{G}$ for some semisimple element $\tilde{s} \in \tilde{G}^\star$ and assume $\iota^\star(\tilde{s})$ has -1 an eigenvalue. Let $[\Lambda_{\mathrm{D}}]$ be the symbol of the irreducible character of the component of type* D *appearing in ψ then $\mathrm{Res}_G^{\tilde{G}}(\psi)$ is the sum of two irreducible characters of G if $[\Lambda_D]$ is non-degenerate and irreducible otherwise.*

Proof. By Lemma 3.7 we know that the centraliser of s in $\mathbf{G}^\star$ is disconnected. We will denote by $\bar{\psi} \in \mathcal{E}(C_{G^\star}(s)^\circ, 1)$ the unipotent character determined by ψ under the bijection in Eq. (1.5). By Theorem 1.73 we can determine the number of irreducible components of $\mathrm{Res}_G^{\tilde{G}}(\psi)$ by inspecting the order of $\mathrm{Stab}_{A_{G^\star}(s)}(\bar{\psi})$. First let us note that $A_{\mathbf{G}^\star}(s)$ is at most of order two so $F^\star$ must act trivially, in other words $A_{\mathbf{G}^\star}(s)^{F^\star} = A_{\mathbf{G}^\star}(s)$. Therefore to determine the stabiliser we need only determine the action of the full centraliser on the component of type D. By the proof of Lemma 3.7 we see that the non-identity element of $A_{G^\star}(s)$ induces the graph automorphism of order 2 on the component of type D. Finally by Lemma 1.64 we know the stabiliser is $A_{G^\star}(s)$ if $[\Lambda_D]$ is non-degenerate and trivial if $[\Lambda_D]$ is degenerate. ■

3.3 Proof of Proposition A

To prove this result we will use the following result of Hézard.

Proposition 3.11 (Hézard, [Héz04, §4.2.3]). *Let $\tilde{\mathcal{O}}$ be an F-stable unipotent class of $\tilde{\mathbf{G}}$ such that*

$$\mathrm{Spr}_{\mathrm{C}}(\tilde{\mathcal{O}}) = \begin{bmatrix} a_1 & & a_2 & & \cdots & & a_{m+1} \\ & b_1 & & \cdots & & b_m & \end{bmatrix}$$

We have two cases which we must consider separately.

- $\delta_{\mathrm{C}}(\tilde{u}) = 0$. *We construct a symbol* $[\Lambda]$ *by letting*

$$\Lambda = \begin{pmatrix} a_1 & & a_2 - 1 & & \cdots & & a_{m+1} - m \\ & b_1 - 1 & & \cdots & & b_m - m & \end{pmatrix}.$$

Then $[\Lambda]$ *parameterises a special irreducible character of* $\mathbf{W}$. *Let* $\mathbf{W}(\tilde{\mathcal{F}}) \subseteq \mathrm{Irr}(\mathbf{W})$ *be the unique family of irreducible characters containing* $[\Lambda]$ *then* $|\mathcal{G}_{\tilde{\mathcal{F}}}| = |A_{\tilde{\mathbf{G}}}(\tilde{u})|$ *and* $[\Lambda]$ *is the Springer character* $\rho_{\tilde{u},1}$. *It is clear that* $(\tilde{1}, \tilde{\mathcal{F}}) \in \mathcal{S}^F_{\tilde{\mathbf{G}}}$.

- $\delta_{\mathrm{C}}(\tilde{u}) = 1$. *We define two pairs of sequences* (α, β) *and* (γ, δ) *in the following way*

$$\begin{cases} \alpha_i = a_{i+1} - \lfloor a_{i+1}/2 \rfloor - 1 & \text{for } 1 \leqslant i \leqslant m \\ \beta_i = b_i - \lfloor b_i/2 \rfloor - 1 & \text{for } 1 \leqslant i \leqslant m \end{cases} \qquad \begin{cases} \gamma_i = \lfloor a_i/2 \rfloor & \text{for } 1 \leqslant i \leqslant m+1, \\ \delta_i = \lfloor b_i/2 \rfloor & \text{for } 1 \leqslant i \leqslant m. \end{cases}$$

We construct two symbols $[\Lambda_{\mathrm{D}}]$, $[\Lambda_{\mathrm{B}}]$ *by setting*

$$\Lambda_{\mathrm{D}} = \begin{pmatrix} \alpha_1 & \cdots & \alpha_m \\ \beta_1 & \cdots & \beta_m \end{pmatrix} \qquad \text{and} \qquad \Lambda_{\mathrm{B}} = \begin{pmatrix} \gamma_1 & & \gamma_2 & & \cdots & & \gamma_{m+1} \\ & \delta_1 & & \cdots & & \delta_m & \end{pmatrix}.$$

The product $[\Lambda_{\mathrm{D}}] \boxtimes [\Lambda_{\mathrm{B}}]$ *is a special irreducible character of a Weyl group of type* $\mathrm{D}_\mu \, \mathrm{B}_\nu$, *where* μ, $\nu \geqslant 1$ *are integers determined by* $[\Lambda_{\mathrm{D}}]$ *and* $[\Lambda_{\mathrm{B}}]$ *such that* $\mu + \nu = n$. *Furthermore* $[\Lambda_{\mathrm{D}}]$ *is a non-degenerate symbol. There exists a semisimple element* $\tilde{s} \in \tilde{\mathbf{T}}_0^\star$, *lying in an* $F^\star$*-stable* $\mathbf{W}^\star$*-orbit, such that* $s = \iota^\star(\tilde{s})$ *is the element*

$$\mathrm{diag}(\underbrace{-1, \dots, -1}_{\mu}, \underbrace{1, \dots, 1}_{2\nu+1}, \underbrace{-1, \dots, -1}_{\mu}) \in \mathrm{SO}_{2n+1}(\mathbb{K}).$$

The centraliser of $\tilde{s}$ *is connected and is of type* $\mathrm{D}_\mu \, \mathrm{B}_\nu$. *Let* $\mathbf{W}(\tilde{\mathcal{F}}) \subseteq \mathrm{Irr}(\mathbf{W}(\tilde{s}))$ *be the family determined by* $[\Lambda_{\mathrm{D}}] \boxtimes [\Lambda_{\mathrm{B}}]$ *then* $|\mathcal{G}_{\tilde{\mathcal{F}}}| = |A_{\tilde{\mathbf{G}}}(\tilde{u})|$ *and the induced caracter* $j^{\mathbf{W}}_{\mathbf{W}(\tilde{s})}([\Lambda_{\mathrm{D}}] \boxtimes [\Lambda_{\mathrm{B}}])$ *is the Springer character* $\rho_{\tilde{u},1}$. *Finally* $(\tilde{s}, \tilde{\mathcal{F}}) \in \mathcal{S}^F_{\tilde{\mathbf{G}}}$ *is an F-stable pair.*

Remark 3.12. As the question of F-stability in Proposition 3.11 is important for Proposition A we briefly recap Hézard's argument. In [**Héz04**, Remarque 4.4] Hézard explains that the automorphism $F^\star_{\tilde{s}}$ of $\mathbf{W}^\star(\tilde{s})$ cannot exchange the two components of type D and B. This does not mean that $F^\star_{\tilde{s}}$ always acts trivially on $\mathbf{W}^\star(\tilde{s})$ as it may induce an automorphism of order 2 on the component of type D. However as $[\Lambda_{\mathrm{D}}]$ is non-degenerate we will have $\mathbf{W}^\star(\tilde{\mathcal{F}})$ is invariant under $F^\star_{\tilde{s}}$.

Let $\mathcal{O}$ be an F-stable unipotent class of $\mathbf{G}$ and $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}})) \in \mathcal{S}_{\tilde{\mathbf{G}}}$ be the pair prescribed by Proposition 3.11 for $\tilde{\mathcal{O}}$. We take $\bar{\psi} \in \mathcal{E}(C_{G^\star}(s)^\circ, 1)$ to be the image of a character in $\tilde{\mathcal{F}}$, under the bijection given in Eq. (1.5), such that $n_{\bar{\psi}} = |\mathcal{G}_{\tilde{\mathcal{F}}}|$. We claim that the triple $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}), \bar{\psi})$ is a solution to Proposition A. Noting that $\iota^\star(\tilde{s})$ is clearly quasi-isolated by Corollary 3.8 we see that it is sufficient to show that Property (P2) holds as the remaining statements are all contained in Hézards result. We treat the two cases separately.

- $\delta_{\mathrm{C}}(u) = 0$. By Section 2.2.3 we know $|Z_{\mathbf{G}}(u)| = 1$ hence we need to show that $|\operatorname{Stab}_{A_{G^\star}(s)}(\bar{\psi})| = |Z_{\mathbf{G}}(u)^F| = 1$. However this is clear as $\tilde{s}$ is the identity so $A_{\mathbf{G}^\star}(s)$ is trivial.
- $\delta_{\mathrm{C}}(u) = 1$. By Section 2.2.3 we know $|Z_{\mathbf{G}}(u)| = 2$ therefore every non-trivial element of the centre determines a non-trivial element of $A_{\mathbf{G}}(u)$. Clearly F acts trivially on $Z(\mathbf{G})$ so we need to show that $|\operatorname{Stab}_{A_{G^\star}(s)}(\bar{\psi})| = |Z_{\mathbf{G}}(u)^F| = 2$. It is easy to see that s is $F^\star$-fixed and $F^\star$ acts trivially on $\mathbf{W}^\star$ so $A_{\mathbf{G}^\star}(s)^{F^\star} = A_{\mathbf{G}^\star}(s)$. We see that $[\Lambda_D]$ is non-degenerate so this is a consequence of Proposition 3.10.

For the validity of condition ♣ we notice the following.

- When $\delta_{\mathrm{C}}(u) = 0$ the chosen semisimple element $\tilde{s}$ is the identity. In particular $C_{\mathbf{G}^\star}(s) = \mathbf{G}^\star$ is connected and every character in $\mathcal{F}$ has a trivial stabiliser.
- When $\delta_{\mathrm{C}}(u) = 1$ then any character in $\mathcal{F}$ is parameterised by symbols $[\Lambda_1] \boxtimes [\Lambda_2]$ such that $[\Lambda_1]$ is not degenerate, (because $[\Lambda_D]$ is not degenerate). Therefore we have the stabiliser of any character in $\mathcal{F}$ is of order 2.

In other words we see that $\mathfrak{X}_{\mathcal{F}} = \varnothing$. This now completes the proof of Proposition A for the case where $\mathbf{G}$ is simple of type C_n.

Chapter 4

Quasi-Isolated Semisimple Elements

It is clear from the statement of Proposition A that the necessary semisimple elements in $\mathcal{S}_{\mathbf{G}}^{F}$ come from quasi-isolated semisimple elements in adjoint groups. A classification of such semisimple elements was completed by Bonnafé in [**Bon05**]. In this chapter we will focus on when these classes are F-stable. Furthermore we must deal with certain rationality questions regarding semisimple classes in a simply connected covering which surject onto quasi-isolated classes.

The results we prove in this chapter will be for $\mathbf{G}$ but in reality we will apply them in the dual group. We use the notation of the group $\mathbf{G}$ for notational convenience. The exception to this will be in Sections 4.2 and 4.5 where we prove results concerning the relationship between $\mathbf{G}$ and $\mathbf{G}^{\star}$.

4.1 Bonnafé's Classification

Throughout this chapter we will always assume, unless otherwise stated, that $\mathbf{G}$ is a simple algebraic group. We start by describing Bonnafé's classification of quasi-isolated semisimple elements, (see [**Bon05**, Table 2]). Bonnafé's classification works in all characteristics but the assumption that p is a good prime is not restrictive. This is because when p is a bad prime it may only happen that certain classes are no longer quasi-isolated. For example, if $\mathbf{G}$ is adjoint of type B_n, C_n or D_n and $p = 2$ then the only quasi-isolated semisimple class is the identity, (see [**Bon05**, Example 4.8]).

Notation. Assume $\mathbf{G}$ is a connected reductive algebraic group and $\mathbf{T} \leqslant \mathbf{G}$ is a maximal torus. Recall that we have an isomorphism $\mathbb{K}^{\times} \otimes_{\mathbb{Z}} \check{X}(\mathbf{T}) \to \mathbf{T}$ given by $k \otimes \gamma \mapsto \gamma(k)$. Using the isomorphism $\imath : (\mathbb{Q}/\mathbb{Z})_{p'} \to \mathbb{K}^{\times}$ we obtain an isomorphism $\imath_{\mathbf{T}} : (\mathbb{Q}/\mathbb{Z})_{p'} \otimes_{\mathbb{Z}} \check{X}(\mathbf{T}) \to \mathbf{T}$ given by $\imath_{\mathbf{T}}(r \otimes \gamma) = \gamma(\imath(r))$. We have an action of F on $(\mathbb{Q}/\mathbb{Z})_{p'} \otimes_{\mathbb{Z}} \check{X}(\mathbf{T})$ given by $F(r \otimes \gamma) = r \otimes F(\gamma)$ which is compatible with the action of F on $\mathbf{T}$.

As $\mathbf{G}_{\mathrm{ad}}$ is adjoint the character group $X(\mathbf{T}_{\mathrm{ad}})$ may be identified with the root lattice $\mathbb{Z}\Phi$. Furthermore the cocharacter group $\check{X}(\mathbf{T}_{\mathrm{ad}})$ can be identified with the coweight lattice, which means we can naturally consider all fundamental dominant coweights $\check{\omega}_\alpha \in \check{\Omega}$ to be elements of $\check{X}(\mathbf{T}_{\mathrm{ad}})$. Let $\mathcal{A} := \mathrm{Aut}_{\mathbf{W}}(\tilde{\Delta}) = \{x \in \mathbf{W} \mid x(\tilde{\Delta}) = \tilde{\Delta}\} \leqslant \mathbf{W}$ be the automorphism group of the extended Dynkin diagram in $\mathbf{W}$ and let $\mathcal{Q}(\mathbf{G}_{\mathrm{ad}})$ denote the set of subsets $\Sigma \subset \tilde{\Delta}$ such that the stabiliser of Σ in $\mathcal{A}$ acts transitively on Σ. We then have the following theorem of Bonnafé.

Theorem 4.1 (Bonnafé, [Bon05, Theorem 5.1]). *Let $\Sigma \in \mathcal{Q}(\mathbf{G}_{\mathrm{ad}})$ and define an element $t_\Sigma \in \mathbf{T}_{\mathrm{ad}}$ by setting*

$$t_\Sigma = \imath_{\mathbf{T}_{\mathrm{ad}}}\left(\sum_{\alpha\in\Sigma} \frac{1}{m_\alpha|\Sigma|} \otimes \check{\omega}_\alpha\right),$$

where $\check{\omega}_\alpha \in \check{\Omega}$, (and m_α is as in Definition 2.1). Following the conventions of Bonnafé we let $\check{\omega}_{\alpha_0} = 0$ and $m_{\alpha_0} = 1$, (see [Bon05, §3.B]). The following then hold:

- *the map $\Sigma \mapsto t_\Sigma$ induces a bijection between the set of orbits of $\mathcal{A}$ acting on $\mathcal{Q}(\mathbf{G}_{\mathrm{ad}})$ and the set of conjugacy classes of quasi-isolated semisimple elements in $\mathbf{G}_{\mathrm{ad}}$.*
- *for any $\Sigma \in \mathcal{Q}(\mathbf{G}_{\mathrm{ad}})$ we have:*
  - $\mathbf{W}(t_\Sigma)^\circ = \langle s_\alpha \in \mathbb{S}_0 \mid \alpha \in \tilde{\Delta} - \Sigma\rangle$;
  - $\mathcal{A}(t_\Sigma) = \{x\mathbf{W}(t_\Sigma)^\circ \mid x \in \mathcal{A}$ *and* $x \cdot \Sigma = \Sigma\}$.

An important aspect of Bonnafé's theorem is that he determines the structure of $A_{\mathbf{G}}(t_\Sigma)$, which is important to us in verifying the validity of Property (P2). In Tables 4.1 and 4.2 we reproduce Bonnafé's classification of quasi-isolated semisimple elements in classical and exceptional adjoint algebraic groups, as found in [**Bon05**, Tables 2 and 3]. In the case of G_2, F_4 and E_8 the notion of isolated and quasi-isolated semisimple elements coincide as the adjoint and simply connected groups coincide.

Although it will not be a question that concerns us we make the following comment regarding quasi-isolated semisimple elements. If $s \in \mathbf{G}$ is quasi-isolated then $\delta_{\mathrm{ad}}(s) \in \mathbf{G}_{\mathrm{ad}}$ is quasi-isolated. Furthermore s is isolated if and only if $\delta_{\mathrm{ad}}(s)$ is isolated, (see [**Bon05**, Proposition 2.3]). So to determine classes of quasi-isolated semisimple elements in simple algebraic groups it is enough to determine such classes in adjoint groups.

We have described the group $\mathcal{A}$ of induced graph automorphisms in Appendix B. Hence it is easy for us to know exactly the structure and actions of $A_{\mathbf{G}}(t_\Sigma)$. In Table 4.1 we have indicated conditions on n for such a class of elements to exist. Finally we note that in the original table of Bonnafé the class representative for the class corresponding to $\{\alpha_{n/2}\}$ in D_n is denoted as having order 4. However it is clear that this element is in fact of order 2 as it is isolated, (see [**Bon05**, Proposition 5.5]).

$\mathbf{G}_{\mathrm{ad}}$	Σ	$m_\alpha\lvert\Sigma\rvert$	$C_{\mathbf{G}_{\mathrm{ad}}}(t_\Sigma)^\circ$	$\lvert A_{\mathbf{G}_{\mathrm{ad}}}(t_\Sigma)\rvert$	Isolated?
A_n	$\{\alpha_{j(n+1)/d} \mid 0 \leqslant j \leqslant d-1\}$ $d \mid n+1$ and $p \nmid d$	d	$(\mathrm{A}_{(n+1-d)/d})^d$	d	$d=1$
B_n	$\{\alpha_0\}$	1	B_n	1	yes
	$\{\alpha_0,\alpha_1\}$	2	B_{n-1}	2	no
	$\{\alpha_d\}$	2	$\mathrm{D}_d\,\mathrm{B}_{n-d}$	2	yes
C_n	$\{\alpha_0\}$	1	C_n	1	yes
	$\{\alpha_d\}$, $1 \leqslant d < n/2$	2	$\mathrm{C}_d\,\mathrm{C}_{n-d}$	1	yes
	$\{\alpha_{n/2}\}$, (only if $2 \mid n$)	2	$\mathrm{C}_{n/2}\,\mathrm{C}_{n/2}$	2	yes
	$\{\alpha_0,\alpha_n\}$	2	A_{n-1}	2	no
	$\{\alpha_d,\alpha_{n-d}\}$, $1 \leqslant d < n/2$	4	$\mathrm{C}_d\,\mathrm{A}_{n-2d-1}\,\mathrm{C}_d$	2	no
D_n	$\{\alpha_0\}$	1	D_n	1	yes
	$\{\alpha_d\}$, $1 \leqslant d < n/2$	2	$\mathrm{D}_d\,\mathrm{D}_{n-d}$	2	yes
	$\{\alpha_{n/2}\}$, (only if $2 \mid n$)	2	$\mathrm{D}_{n/2}\,\mathrm{D}_{n/2}$	4	yes
	$\{\alpha_d,\alpha_{n-d}\}$, $2 \leqslant d < n/2$	4	$\mathrm{D}_d\,\mathrm{A}_{n-2d-1}\,\mathrm{D}_d$	4	no
	$\{\alpha_0,\alpha_1,\alpha_{n-1},\alpha_n\}$	4	A_{n-3}	4	no
	$\{\alpha_0,\alpha_1\}$	2	D_{n-1}	2	no
	$\{\alpha_0,\alpha_{n-1}\}$, (only if $2 \mid n$)	2	A_{n-1}	2	no
	$\{\alpha_0,\alpha_n\}$, (only if $2 \mid n$)	2	A_{n-1}	2	no

Table 4.1: Classes of Quasi-Isolated Semisimple Elements in Classical Groups

4.2 Component Groups of Semisimple Elements

In this section we will be interested in $|A_{\mathbf{G}^\star}(s)|$ where $s \in \mathbf{T}_0^\star$ is a semisimple element such that $\bar{s} = \delta_{\mathrm{ad}}^\star(s) \in \mathbf{T}_{\mathrm{sc}}^\star$ is quasi-isolated. An expression for this is already obtained by Bonnafé in [**Bon05**, Proposition 3.14(b)], however we wish to determine a slightly different description which will make numerical comparisons between component groups of unipotent elements in $\mathbf{G}$ simpler. If $\mathbf{G}^\star$ is simply connected then we know that the centraliser of every semisimple element is connected, (by the classical work of Steinberg), so this value will always be 1. It suffices therefore to only consider simple groups which are neither simply connected nor adjoint. These groups can only occur in types A_n and D_n so we may assume that $\mathbf{G}$ is such a simple group.

Let $\hat{s} \in \mathbf{T}_{\mathrm{ad}}^\star$ be such that $\delta_{\mathrm{sc}}^\star(\hat{s}) = s \in \mathbf{T}_0^\star$. Following Bonnafé [**Bon05**, §2.B] we define two homorphisms $\omega_{\bar{s}} : C_{\mathbf{G}_{\mathrm{sc}}^\star}(\bar{s}) \to Z(\mathbf{G}_{\mathrm{ad}}^\star)$ and $\omega_s : C_{\mathbf{G}^\star}(s) \to Z(\mathbf{G}_{\mathrm{ad}}^\star)$ by setting $\omega_{\bar{s}}(\bar{x}) = [\hat{s},\hat{x}]$ and $\omega_s(y) = [\hat{s},\hat{y}]$, where $\hat{x}$, $\hat{y}$ are such that $(\delta_{\mathrm{ad}}^\star \circ \delta_{\mathrm{sc}}^\star)(\hat{x}) = \bar{x}$ and $\delta_{\mathrm{sc}}^\star(\hat{y}) = y$. We recall the following result of Bonnafé

Lemma 4.2 (Bonnafé, [Bon05, Corollary 2.8]**).** *The homomorphisms $\omega_{\bar{s}}$, ω_s induce embeddings $\tilde{\omega}_{\bar{s}} : A_{\mathbf{G}_{\mathrm{sc}}^\star}(\bar{s}) \to Z(\mathbf{G}_{\mathrm{ad}}^\star)$ and $\tilde{\omega}_s : A_{\mathbf{G}^\star}(s) \to Z(\mathbf{G}_{\mathrm{ad}}^\star)$. Their respective images*

$\mathbf{G}_{\mathrm{ad}}$	Σ	p	$C_{\mathbf{G}_{\mathrm{ad}}}(t_\Sigma)^\circ$	$\lvert A_{\mathbf{G}}(t_\Sigma)\rvert$	Isolated?
	$\{\alpha_0\}$		E_6	1	yes
	$\{\alpha_2\}$	2	$\mathrm{A}_5\,\mathrm{A}_1$	1	yes
E_6	$\{\alpha_4\}$	3	$\mathrm{A}_2\,\mathrm{A}_2\,\mathrm{A}_2$	3	yes
	$\{\alpha_0,\alpha_1,\alpha_6\}$	3	D_4	3	no
	$\{\alpha_2,\alpha_3,\alpha_5\}$	$2,3$	$\mathrm{A}_1\,\mathrm{A}_1\,\mathrm{A}_1\,\mathrm{A}_1$	3	no
	$\{\alpha_0\}$		E_7	1	yes
	$\{\alpha_1\}$	2	$\mathrm{A}_1\,\mathrm{D}_6$	1	yes
	$\{\alpha_2\}$	2	A_7	2	yes
E_7	$\{\alpha_3\}$	3	$\mathrm{A}_2\,\mathrm{A}_5$	1	yes
	$\{\alpha_4\}$	2	$\mathrm{A}_3\,\mathrm{A}_3\,\mathrm{A}_1$	2	yes
	$\{\alpha_0,\alpha_7\}$	2	E_6	2	no
	$\{\alpha_1,\alpha_6\}$	2	$\mathrm{D}_4\,\mathrm{A}_1\,\mathrm{A}_1$	2	no
	$\{\alpha_3,\alpha_5\}$	$2,3$	$\mathrm{A}_2\,\mathrm{A}_2\,\mathrm{A}_2$	2	no
	$\{\alpha_0\}$		E_8	1	yes
	$\{\alpha_1\}$	2	D_8	1	yes
	$\{\alpha_2\}$	3	A_8	1	yes
	$\{\alpha_3\}$	2	$\mathrm{A}_1\,\mathrm{A}_7$	1	yes
E_8	$\{\alpha_4\}$	$2,3$	$\mathrm{A}_2\,\mathrm{A}_1\,\mathrm{A}_5$	1	yes
	$\{\alpha_5\}$	5	$\mathrm{A}_4\,\mathrm{A}_4$	1	yes
	$\{\alpha_6\}$	2	$\mathrm{D}_5\,\mathrm{A}_3$	1	yes
	$\{\alpha_7\}$	3	$\mathrm{E}_6\,\mathrm{A}_2$	1	yes
	$\{\alpha_8\}$	2	$\mathrm{E}_7\,\mathrm{A}_1$	1	yes
	$\{\alpha_0\}$		F_4	1	yes
	$\{\alpha_1\}$	2	$\mathrm{A}_1\,\mathrm{C}_3$	1	yes
F_4	$\{\alpha_2\}$	3	$\mathrm{A}_2\,\mathrm{A}_2$	1	yes
	$\{\alpha_3\}$	2	$\mathrm{A}_3\,\mathrm{A}_1$	1	yes
	$\{\alpha_4\}$	2	B_4	1	yes
	$\{\alpha_0\}$		G_2	1	yes
G_2	$\{\alpha_1\}$	2	$\mathrm{A}_1\,\mathrm{A}_1$	1	yes
	$\{\alpha_2\}$	3	A_2	1	yes

Table 4.2: Quasi-Isolated Semisimple Elements in Exceptional Groups

are given by

$$\begin{aligned}
\mathrm{Im}(\tilde{\omega}_{\bar{s}}) &= \{\hat{z} \in Z(\mathbf{G}^\star_{\mathrm{ad}}) \mid \hat{s} \textit{ and } \hat{s}\hat{z} \textit{ are conjugate in } \mathbf{G}^\star_{\mathrm{ad}}\},\\
\mathrm{Im}(\tilde{\omega}_{s}) &= \{\hat{z} \in \mathrm{Ker}(\delta^\star_{\mathrm{sc}}) \mid \hat{s} \textit{ and } \hat{s}\hat{z} \textit{ are conjugate in } \mathbf{G}^\star_{\mathrm{ad}}\}.
\end{aligned}$$

It is easily checked that we have $\omega_{\bar{s}} \circ F^\star_{\bar{s}} = F^\star \circ \omega_{\bar{s}}$ and $\omega_s \circ F^\star_s = F^\star \circ \omega_s$. From this lemma we see that $A_{\mathbf{G}^\star_{\mathrm{sc}}}(\bar{s}) \cong \mathrm{Im}(\tilde{\omega}_{\bar{s}})$ and $A_{\mathbf{G}^\star}(s) \cong \mathrm{Im}(\tilde{\omega}_s)$ so to determine $|A_{\mathbf{G}^\star}(s)|$ we need only determine $|\,\mathrm{Im}(\tilde{\omega}_{\bar{s}}) \cap \mathrm{Ker}(\delta^\star_{\mathrm{sc}})|$.

In later chapters we will want to compare $|A_{\mathbf{G}^\star}(s)|$ with $|A_{\mathbf{G}}(u)|$ for some unipotent element $u \in \mathbf{G}$. We now take the time to prove some small results which will facilitate this. First we will need the following standard result on φ-conjugacy classes of a finite abelian group.

Lemma 4.3 ([Bon05, Exemple 1.1]). *Assume H is a finite abelian group and $\varphi : H \to H$ is a homomorphism of H then $H^1(\varphi, H) = H/(\varphi - 1)H$, where $\varphi - 1 : H \to H$ is the homomorphism $(\varphi - 1)(h) = h^{-1}\varphi(h)$. In particular $|H^1(\varphi, H)| = |H^\varphi|$, where H^φ denotes the subgroup of φ-fixed points.*

Proof. If H is abelian, we have u, v are φ-conjugate if and only if there exists $h \in H$ such that $uh^{-1}\varphi(h) = v$. Therefore $H^1(\varphi, H)$ is given by the quotient group $H/(\varphi - 1)H$. Finally it is clear that $\mathrm{Ker}(\varphi - 1) = H^\varphi$ so $|H|/|H^\varphi| = |(\varphi - 1)H|$, which implies $|H^1(\varphi, H)| = |H/(\varphi - 1)H| = |H^\varphi|$. ∎

Lemma 4.4. *Assume $\mathbf{G}$ is a connected semisimple algebraic group then the groups $\mathrm{Ker}(\delta_{\mathrm{sc}})$ and $\mathrm{Irr}(Z(\mathbf{G}^\star))$ are isomorphic and this isomorphism is defined over $\mathbb{F}_q$.*

Proof. The restriction of the isogeny δ_{sc} to the maximal tori $\mathbf{T}_{\mathrm{sc}} \to \mathbf{T}_0$ gives rise to an injective homomorphism $\check{X}(\mathbf{T}_{\mathrm{sc}}) \to \check{X}(\mathbf{T}_0)$ by precomposing with δ_{sc}. By [**Bon06**, Proposition 1.11] this induces a natural isomorphism $(\check{X}(\mathbf{T}_0)/\check{X}(\mathbf{T}_{\mathrm{sc}}))_{p'} \cong \mathrm{Ker}(\delta_{\mathrm{sc}})$, where we identify $\check{X}(\mathbf{T}_{\mathrm{sc}})$ with its image in $\check{X}(\mathbf{T}_0)$. Using duality this gives rise to an isomorphism

$$(X(\mathbf{T}_0^\star)/X(\mathbf{T}_{\mathrm{sc}}^\star))_{p'} \cong (\check{X}(\mathbf{T}_0)/\check{X}(\mathbf{T}_{\mathrm{sc}}))_{p'} \cong \mathrm{Ker}(\delta_{\mathrm{sc}}).$$

Recall that $X(\mathbf{T}_{\mathrm{sc}}^\star)$ can be identified with $\mathbb{Z}\Phi^\star$ so by [**Bon06**, Proposition 4.1] we have a natural isomorphism $X(Z(\mathbf{G})) \cong (X(\mathbf{T}_0^\star)/X(\mathbf{T}_{\mathrm{sc}}^\star))_{p'}$. The morphism $X(Z(\mathbf{G}^\star)) \to \mathrm{Irr}(Z(\mathbf{G}^\star))$, given by $\chi \mapsto \kappa \circ \chi$ is an isomorphism of finite abelian groups. Finally, checking the statements in [**Bon06**], we can see that all morphisms are defined over $\mathbb{F}_q$. ∎

Corollary 4.5. *Assume $\mathbf{G}$ is a connected semisimple algebraic group then $|\mathrm{Ker}(\delta_{\mathrm{sc}})^F| = |Z(\mathbf{G}^\star)^{F^\star}|$.*

Proof. From Lemma 4.4 we see that $\mathrm{Ker}(\delta_{\mathrm{sc}})^F \cong \mathrm{Irr}(Z(\mathbf{G}^\star))^{F^\star}$ because the isomorphism is defined over $\mathbb{F}_q$. The group $\mathrm{Irr}(Z(\mathbf{G}^\star))^{F^\star}$ is canonically isomorphic to the group $\mathrm{Irr}(H^1(F^\star, Z(\mathbf{G}^\star)))$ because χ is an element of $\mathrm{Irr}(Z(\mathbf{G}^\star))^{F^\star}$ if and only if $\chi(F^\star(z)) = \chi(z)$ for all $z \in Z(\mathbf{G}^\star)$. On the other hand this is true if and only if $\chi(z^{-1}F^\star(z)) = \chi(1)$ for all $z \in Z(\mathbf{G}^\star)$, which means $\mathrm{Ker}(\chi) = (F^\star - 1)Z(\mathbf{G}^\star)$. Using Lemma 4.3 we see

$$|\mathrm{Ker}(\delta_{\mathrm{sc}})^F| = |\mathrm{Irr}(Z(\mathbf{G}^\star))^{F^\star}| = |H^1(F^\star, Z(\mathbf{G}^\star))| = |Z(\mathbf{G}^\star)^{F^\star}|.$$

To obtain the second equality we have used the fact that $H^1(F^\star, Z(\mathbf{G}^\star))$ has the same order as its character group because it is a finite abelian group. ■

We finally end this discussion on component groups with a particularly useful observation relating fixed point groups.

Lemma 4.6. *Assume* $\mathbf{G}$ *is a semisimple algebraic group with a cyclic centre and* $\mathcal{O}$ *is an* F*-stable unipotent class. Let* $s \in \mathbf{T}_0^\star$ *be a semisimple element such that* $|A_{\mathbf{G}^\star}(s)| = |Z_{\mathbf{G}}(u)|$ *then* $|A_{G^\star}(s)| = |Z_{\mathbf{G}}(u)^F|$*, (where here* $A_{G^\star}(s) = A_{\mathbf{G}^\star}(s)^{F_s^\star}$*).*

Proof. By Lemma 4.2 we have $A_{\mathbf{G}^\star}(s)$ is isomorphic to a subgroup of $\mathrm{Ker}(\delta_{\mathrm{sc}}^\star)$ and it is clear that $Z_{\mathbf{G}}(u)$ is isomorphic to a subgroup of $Z(\mathbf{G})$, (furthermore these isomorphisms are defined over $\mathbb{F}_q$). We therefore reduce the question to the following: if $K \leqslant \mathrm{Ker}(\delta_{\mathrm{sc}}^\star)$ and $L \leqslant Z(\mathbf{G})$ are subgroups with common order $d := |K| = |L|$ is $|K^{F^\star}| = |L^F|$?

By Lemma 4.4 we have $|\mathrm{Ker}(\delta_{\mathrm{sc}}^\star)| = |Z(\mathbf{G})| = N$, hence these groups are isomorphic to a cyclic group of order N. We may assume that $1 \leqslant n, m \leqslant N$ are such that $F^\star(x) = x^n$ and $F(y) = y^m$, where x is a generator for $\mathrm{Ker}(\delta_{\mathrm{sc}}^\star)$ and y is a generator for $Z(\mathbf{G})$. Taking $\lambda = N/d$ it is not difficult to see that $|K^{F^\star}| = \lambda \cdot \gcd(n-1, d)$ and $|L^F| = \lambda \cdot \gcd(m-1, d)$. To show that these groups have the same order it is enough to show that $\gcd(n-1, d) = \gcd(m-1, d)$. However, because $\gcd(n-1, d) = \gcd(n-1, d, N) = \gcd(n-1, N)$, (similarly for $\gcd(m-1, d)$), it is sufficient to show that $\gcd(n-1, N) = \gcd(m-1, N)$. On the other hand this is just a restatement of Corollary 4.5, so we're done. ■

4.2.1 Groups of Type A_n

If $\mathbf{G}$ is a group of type A_n then $Z(\mathbf{G}_{\mathrm{ad}}^\star)$ is a cyclic group so this simplifies trying to understand $|A_{\mathbf{G}^\star}(s)|$. We know $|A_{\mathbf{G}_{\mathrm{sc}}^\star}(\bar{s})| = |\mathrm{Im}(\tilde{\omega}_{\bar{s}})|$ in particular, as $\mathrm{Ker}(\delta_{\mathrm{sc}}^\star)$ is cyclic, we know $z \in \mathrm{Im}(\tilde{\omega}_{\bar{s}}) \cap \mathrm{Ker}(\delta_{\mathrm{sc}}^\star)$ if and only if the order of z divides $|\mathrm{Im}(\tilde{\omega}_{\bar{s}})|$ and $|\mathrm{Ker}(\delta_{\mathrm{sc}}^\star)|$. Hence it is easy to see that we have

$$|A_{\mathbf{G}^\star}(s)| = \gcd(|A_{\mathbf{G}_{\mathrm{sc}}^\star}(\bar{s})|, |\mathrm{Ker}(\delta_{\mathrm{sc}}^\star)|).$$

Assume $u \in \mathbf{G}$ is a unipotent element then $d := |A_{\mathbf{G}}(u)|$ is a divisor of $|A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})|$. By the description of $|A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})|$ given in Section 2.2.1 we have d is a divisor of $n+1$ and $p \nmid d$. Therefore there exists a semisimple element $s \in \mathbf{T}_0^\star$ such that $\bar{s} = \delta_{\mathrm{ad}}(s)$ is quasi-isolated and $|A_{\mathbf{G}_{\mathrm{sc}}^\star}(\bar{s})| = d$. As $A_{\mathbf{G}}(u) = Z_{\mathbf{G}}(u)$ we have d divides $|Z(\mathbf{G})| = |\mathrm{Ker}(\delta_{\mathrm{sc}}^\star)|$ so $|A_{\mathbf{G}^\star}(s)| = d$.

What we have shown here is that for each unipotent conjugacy class $\mathcal{O}$ of $\mathbf{G}$ there exists a semisimple element $s \in \mathbf{T}_0^\star$ such that $|A_{\mathbf{G}}(u)| = |A_{\mathbf{G}^\star}(s)|$.

4.2.2 Groups of Type D_n

If $\mathbf{G}$ is a simple group of type D_n such that $n \equiv 1 \pmod 2$ then $\mathbf{G}$ must be isomorphic to a special orthogonal group. The kernel $\mathrm{Ker}(\delta_{\mathrm{sc}}^\star)$ will be the unique subgroup of order 2 in $Z(\mathbf{G}_{\mathrm{sc}})$ so

$$|A_{\mathbf{G}^\star}(s)| = \begin{cases} 1 & \text{if } |A_{\mathbf{G}_{\mathrm{ad}}}(\bar{s})| = 1, \\ 2 & \text{if } |A_{\mathbf{G}_{\mathrm{ad}}}(\bar{s})| \geqslant 2. \end{cases}$$

If $n \equiv 0 \pmod 2$ then $\mathbf{G}$ is either isomorphic to a special orthogonal group or a half spin group. It is clear that if $|A_{\mathbf{G}_{\mathrm{ad}}}(\bar{s})| = 1$ or 4 then we will respectively have $|A_{\mathbf{G}^\star}(s)| = 1$ or 2. The problem now arises when $|A_{\mathbf{G}_{\mathrm{ad}}}(\bar{s})| = 2$. Assume $\bar{s}$ is a quasi-isolated semisimple element with this property then in Table 4.3 we describe the orders of $|A_{\mathbf{G}^\star}(s)|$ depending upon whether $\mathbf{G}$ is a special orthogonal group or a half spin group.

To determine the information in Table 4.3 one has to only check which element of $\mathcal{A}$ stabilises Σ then see if the corresponding element of $Z(\mathbf{G}_{\mathrm{sc}})$ lies in $\mathrm{Ker}(\delta_{\mathrm{sc}}^\star)$. The correspondence between $\mathcal{A}$ and $Z(\mathbf{G}_{\mathrm{sc}})$ for groups of type D is outlined in detail at the end of Appendix B.

Σ	$C_{\mathbf{G}_{\mathrm{ad}}^\star}(\bar{s})^\circ$	$\mathrm{SO}_{2n}(\mathbb{K})$	$\mathrm{HSpin}_{2n}(\mathbb{K})$
$\{\alpha_d\}, 2 \leqslant d < n/2$	$\mathrm{D}_d\,\mathrm{D}_{n-d}$	2	1
$\{\alpha_{n/2}\}$, (only if $2 \mid n$)	$\mathrm{D}_{n/2}\,\mathrm{D}_{n/2}$	2	2
$\{\alpha_0, \alpha_1\}$	D_{n-1}	2	1
$\{\alpha_0, \alpha_{n-1}\}$	A_{n-1}	1	1
$\{\alpha_0, \alpha_n\}$	A_{n-1}	1	2

Table 4.3: Component Group Orders in Groups of Type D_n

In the table we have indicated the value $|A_{\mathbf{G}^\star}(s)|$ assuming that $\bar{s} = t_\Sigma$ for the specified set $\Sigma \in \mathcal{Q}(\mathbf{G}_{\mathrm{ad}})$.

4.3 F-stability of Classes in Adjoint Groups

In the remainder of this chapter we wish to address two issues. Firstly we wish to show that, modulo some exceptions, every class of quasi-isolated semisimple elements in a simple adjoint algebraic group is F-stable. Secondly we wish to show that, if $\mathcal{C}$ is an F-stable class of quasi-isolated semisimple elements of $\mathbf{G}_{\mathrm{ad}}$ then there exists an F-stable class of $\mathbf{G}_{\mathrm{sc}}$ whose image under $\delta_{\mathrm{ad}} \circ \delta_{\mathrm{sc}}$ is $\mathcal{C}$.

Proposition 4.7. *Let $\mathbf{G}_{\mathrm{ad}}$ be a simple adjoint algebraic group of classical type and F a Frobenius endomorphism written as $F_r \circ \tau$ where τ is a graph automorphism of $\mathbf{G}_{\mathrm{ad}}$. Given any set $\Sigma \in \mathcal{Q}(\mathbf{G}_{\mathrm{ad}})$ we have t_Σ is conjugate to $F(t_\Sigma)$ unless:*

- *$\mathbf{G}_{\mathrm{ad}}$ is of type D_n, the graph automorphism τ is of order 2 and Σ is $\{\alpha_0, \alpha_{n-1}\}$ or $\{\alpha_0, \alpha_n\}$.*
- *$\mathbf{G}_{\mathrm{ad}}$ is of type D_4, the graph automorphism τ is of order 3 and $\Sigma = \{\alpha_0, \alpha_1\}$, $\{\alpha_0, \alpha_3\}$ or $\{\alpha_0, \alpha_4\}$.*

In particular, except for those mentioned above, the conjugacy class containing t_Σ is F-stable.

Proof. Let $\mathcal{C}_\Sigma$ be the class of quasi-isolated semisimple elements of $\mathbf{G}_{\mathrm{ad}}$ such that $t_\Sigma \in \mathcal{C}_\Sigma$. Furthermore let $y_\Sigma = \imath_{\mathbf{T}_{\mathrm{ad}}}^{-1}(t_\Sigma) = \sum_{\alpha\in\Sigma} 1/m_\alpha|\Sigma| \otimes \check{\omega}_\alpha \in (\mathbb{Q}/\mathbb{Z})_{p'} \otimes \check{X}(\mathbf{T}_{\mathrm{ad}})$ and recall that m_α is constant on Σ. Using the fact that the tensor product is taken over $\mathbb{Z}$ we have the action of F on y_Σ is given by $F(y_\Sigma) = \sum_{\alpha\in\Sigma} r/m_\alpha|\Sigma| \otimes \check{\omega}_{\tau(\alpha)}$. To show $\mathcal{C}_\Sigma$ is F-stable we need only show that y_Σ and $F(y_\Sigma)$ lie in the same $\mathbf{W}$-orbit.

- If $\Sigma = \{\alpha_0\}$ then $y_\Sigma = 0$, which is always F-fixed.
- Let $\mathbf{G}_{\mathrm{ad}}$ be of type A_n and assume τ is trivial. In this instance it will be much more transparent to work with a concrete matrix realisation of $\mathbf{G}_{\mathrm{ad}}$, namely $\mathrm{PGL}_{n+1}(\mathbb{K})$. Let d be a divisor of $n+1$ and Σ the corresponding subset of the roots. Following Bonnafé we define a matrix $J_d = \mathrm{diag}(1, \eta_d, \eta_d^2, \dots, \eta_d^{d-1}) \in \mathrm{GL}_d(\mathbb{K})$, where η_d is a primitive dth root of unity in $\mathbb{K}$. Let $\tilde{s}_\Sigma = I_{n+1/d} \otimes J_d \in \mathrm{GL}_{n+1}(\mathbb{K})$ be the Kronecker product of the matrices, where $I_{n+1/d} \in \mathrm{GL}_{n+1/d}(\mathbb{K})$ is the identity matrix. Considering the standard quotient map $\pi : \mathrm{GL}_{n+1}(\mathbb{K}) \to \mathrm{PGL}_{n+1}(\mathbb{K})$ we have $s_\Sigma = \pi(\tilde{s}_\Sigma)$ is a representative in $\mathbf{G}_{\mathrm{ad}}$ of the class parameterised by Σ. The action of the Frobenius is given by $F(\tilde{s}_\Sigma) = I_{n+1/d} \otimes \mathrm{diag}(1, \eta_d^q, \eta_d^{2q}, \dots, \eta_d^{q(d-1)})$. As q and d are coprime we have $\eta_d^q = \eta_d^i$ for some $1 \leqslant i \leqslant d-1$ so the entries $\eta_d^q, \dots, \eta_d^{q(d-1)}$ are just a permutation of $\eta_d, \dots, \eta_d^{d-1}$. There is clearly an element of the Weyl group $w_d \in \mathbf{W}$ such that $F(\tilde{s}_\Sigma) = \tilde{s}_\Sigma^{w_d}$. If $\tilde{\mathcal{C}}_\Sigma$ is the conjugacy class of $\mathrm{GL}_{n+1}(\mathbb{K})$ containing $\tilde{s}_\Sigma$ then $\pi(\tilde{\mathcal{C}}_\Sigma) = \mathcal{C}_\Sigma$ and $F(\tilde{\mathcal{C}}_\Sigma) = \tilde{\mathcal{C}}_\Sigma$. As π is defined over $\mathbb{F}_q$ we have $\pi(F(\tilde{\mathcal{C}}_\Sigma)) = \pi(\tilde{\mathcal{C}}_\Sigma) \Rightarrow F(\pi(\tilde{\mathcal{C}}_\Sigma)) = F(\mathcal{C}_\Sigma) = \mathcal{C}_\Sigma$ so $\mathcal{C}_\Sigma$ is an F-stable class.
- Let $\mathbf{G}_{\mathrm{ad}}$ be of type A_n and assume τ is of order 2. The map τ acts on the simple roots by sending $\alpha_k \mapsto \alpha_{n+1-k}$ for all $1 \leqslant k \leqslant n$. Furthermore it is such that $\tau(\alpha_0) = \alpha_0$. The roots in Σ are all of the form $\alpha_{j(n+1)/d}$ for some $0 \leqslant j \leqslant d-1$, where d is a divisor of $n+1$ as in the previous case. If $j \neq 0$ then we have $\tau(\alpha_{j(n+1)/d}) = \alpha_{(d-j)(n+1)/d}$ so it is clear that τ preserves the set Σ. Hence y_Σ and $F(y_\Sigma)$ are in the same $\mathbf{W}$-orbit.
- Let $\mathbf{G}_{\mathrm{ad}}$ be of type B_n, C_n or D_n and assume τ is trivial. If $m_\alpha|\Sigma| = 2$ then $F(y_\Sigma) = y_\Sigma$ because q is odd hence $q/2 = 1/2 \in \mathbb{Q}/\mathbb{Z}$.

Assume $m_\alpha|\Sigma| = 4$ then $\mathbf{G}_{\mathrm{ad}}$ must be of type C_n or D_n. If $q \equiv 1 \pmod 4$ then $q/4 = 1/4 \in \mathbb{Q}/\mathbb{Z}$ and $F(y_\Sigma) = y_\Sigma$. If $q \equiv 3 \pmod 4$ then $q/4 = 3/4 = -1/4 \in \mathbb{Q}/\mathbb{Z}$ so $F(y_\Sigma) = -y_\Sigma$. If $\mathbf{G}_{\mathrm{ad}}$ is of type C_n or it is of type D_n and $n \equiv 0 \pmod 2$ then the longest element $w_0 \in \mathbf{W}$ acts on the coweights by -1, (see **[Bou02**, Plates II - IV (XI)]), therefore $F(y_\Sigma)$ and y_Σ are conjugate by w_0. If $\mathbf{G}_{\mathrm{ad}}$ is of type D_n and n is odd then the longest element $w_0 \in \mathbf{W}$ acts on the coroots as $-\varepsilon$ where ε is such that $\varepsilon(\check{\varpi}_i) = \check{\varpi}_i$ for all $1 \leqslant i \leqslant n-2$ and ε exchanges $\check{\varpi}_{n-1}$ and $\check{\varpi}_n$, (see [**Bou02**, Plate IV (XI)]). All subsets Σ considered here are stable under ε so $F(y_\Sigma)$ and y_Σ are conjugate by w_0.

- Let $\mathbf{G}_{\mathrm{ad}}$ be of type D_n and assume τ is of order 2. It is clear that all subsets Σ are stable under τ, except for $\Sigma = \{\alpha_0, \alpha_n\}$ or $\{\alpha_0, \alpha_{n-1}\}$. As τ cannot be induced by an element of $\mathbf{W}$ we cannot have $F(t_\Sigma)$ conjugate to t_Σ in these two cases.
- Let $\mathbf{G}_{\mathrm{ad}}$ be of type D_4 and assume τ is of order 3. The possible subsets $\Sigma \subset \tilde{\Delta}$ are $\{\alpha_0\}$, $\{\alpha_2\}$, $\{\alpha_0, \alpha_1, \alpha_3, \alpha_4\}$, $\{\alpha_0, \alpha_1\}$, $\{\alpha_0, \alpha_3\}$ and $\{\alpha_0, \alpha_4\}$. The first three sets are the only ones stable under τ. As τ cannot be induced by an element of $\mathbf{W}$ we cannot have $F(t_\Sigma)$ conjugate to t_Σ in these remaining cases. ■

If $\mathbf{G}_{\mathrm{ad}}$ is simple and of exceptional type then we can compute the F-stability of the semisimple classes using the development version of CHEVIE maintained by Jean Michel at [**Mic11**]. This uses the implementation of semisimple elements in [**GHL$^+$96**], which was done by Bonnafé and Michel as a part of their computational proof of the Mackey formula, (see [**BM11**]). In Appendix D we give the code for a program which will verify the F-stability of any class of quasi-isolated semisimple elements. For example if $\mathbf{G}_{\mathrm{ad}}$ is of type D_n ($n \geqslant 4$) such that $n \equiv 0 \pmod 2$ then it can be used to check the failure of the classes labelled by $\{\alpha_0, \alpha_{n-1}\}$ and $\{\alpha_0, \alpha_n\}$ to be F-stable. Using this program we get the following proposition.

Proposition 4.8. *Let $\mathbf{G}_{\mathrm{ad}}$ be a simple adjoint algebraic group of exceptional type then any class of quasi-isolated semisimple elements in $\mathbf{G}_{\mathrm{ad}}$ is F-stable.*

4.4 F-stability of Classes in Reductive Groups

Assume $\mathbf{G}$ is a connected reductive algebraic group such that $\mathbf{G}/Z(\mathbf{G})$ is simple and $\delta_{\mathrm{ad}} : \mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$ is an adjoint quotient which is defined over $\mathbb{F}_q$. We would like to positively answer the following question: Given an F-stable class $\mathcal{C}_{\mathrm{ad}}$ of quasi-isolated semisimple elements in $\mathbf{G}_{\mathrm{ad}}$ does there exist an F-stable class $\mathcal{C}$ in $\mathbf{G}$ such that $\delta_{\mathrm{ad}}(\mathcal{C}) = \mathcal{C}_{\mathrm{ad}}$?

Let $\mathbf{G}'$ be the simple derived subgroup of $\mathbf{G}$ and assume $\delta'_{\mathrm{ad}} : \mathbf{G}' \to \mathbf{G}_{\mathrm{ad}}$ is an adjoint quotient of $\mathbf{G}'$. Assume there exists an F-stable class $\mathcal{C}'$ of $\mathbf{G}'$ such that $\delta'_{\mathrm{ad}}(\mathcal{C}') = \mathcal{C}_{\mathrm{ad}}$, then such a class would be a solution to our question. This is clear as $\mathbf{G}$ is an almost direct product of $\mathbf{G}'$ and $Z(\mathbf{G})$. Let $\delta'_{\mathrm{sc}} : \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}'$ be a simply connected cover of $\mathbf{G}'$ and assume that $\mathcal{C}_{\mathrm{sc}} \in \mathfrak{Cl}(\mathbf{G}_{\mathrm{sc}})^F$ is an F-stable class of semisimple elements such that $(\delta'_{\mathrm{ad}} \circ \delta'_{\mathrm{sc}})(\mathcal{C}_{\mathrm{sc}}) = \mathcal{C}_{\mathrm{ad}}$ then $\mathcal{C}' = \delta'_{\mathrm{sc}}(\mathcal{C}_{\mathrm{sc}})$ is a class as required. Following this we may assume without loss of generality that $\mathbf{G}$ is a simple simply connected algebraic group.

Example 4.9. We give an example to illustrate the subtlety of the question just posed. Let $\mathbf{G} = \mathrm{SL}_2(\mathbb{K})$ and $\mathbf{G}_{\mathrm{ad}} = \mathrm{PGL}_2(\mathbb{K})$ and take $F = F_q$ such that $G = \mathrm{SL}_2(q)$ and $G_{\mathrm{ad}} = \mathrm{PGL}_2(q)$. We have a natural adjoint quotient $\delta_{\mathrm{ad}} : \mathrm{SL}_2(\mathbb{K}) \to \mathrm{PGL}_2(\mathbb{K})$ which is defined over $\mathbb{F}_q$. By [**Car93**, Theorem 3.7.6(i)] we know the number of F-stable semisimple classes in both $\mathbf{G}$ and $\mathbf{G}_{\mathrm{ad}}$ is q. However it is clear that the central conjugacy classes $\{I_2\}$ and $\{-I_2\}$ in $\mathrm{SL}_2(\mathbb{K})$ both map to the identity class in $\mathrm{PGL}_2(\mathbb{K})$. Hence there must exist at least one F-stable class in $\mathrm{PGL}_2(\mathbb{K})$ which is not the image of any F-stable semisimple class of $\mathrm{SL}_2(\mathbb{K})$. In fact what we are really noticing here is that the restriction of δ_{ad} to $\mathrm{SL}_2(q)$ fails to be surjective.

Proposition 4.10. *Assume $\mathbf{G}$ is a connected simple simply connected algebraic group. Let $\mathcal{C}_{\mathrm{ad}} \in \mathfrak{Cl}(\mathbf{G}_{\mathrm{ad}})^F$ be an F-stable class of quasi-isolated semisimple elements then there exists an F-stable class of semisimple elements $\mathcal{C} \in \mathfrak{Cl}(\mathbf{G})^F$ such that $\delta_{\mathrm{ad}}(\mathcal{C}) = \mathcal{C}_{\mathrm{ad}}$.*

Proof. We keep the notational conventions specified in the proof of Proposition 4.7.

- Assume $\mathcal{C}_{\mathrm{ad}} = \{1\}$ then clearly $\mathcal{C} = \{1\}$ satisfies our conditions.
- We will deal with a lot of cases by using the following argument inspired by Lusztig and Bonnafé, (see [**Lus88**, §8] and [**Bon05**, §2.B]). Let $s \in \mathcal{C}_{\mathrm{ad}}$ and choose $\hat{s} \in \mathbf{G}$ such that $\delta_{\mathrm{ad}}(\hat{s}) = s$. If we can show $\hat{s}$ is conjugate to $F(\hat{s})$ then the class containing $\hat{s}$ is F-stable and its image under δ_{ad} is $\mathcal{C}_{\mathrm{ad}}$. Note that we have $\delta_{\mathrm{ad}}(F(\hat{s})) = s$ so we must have $F(\hat{s}) = \hat{s}\hat{z}$ for some $\hat{z} \in Z(\mathbf{G})$. By Lemma 4.2 we know $A_{\mathbf{G}_{\mathrm{ad}}}(s)$ is isomorphic to the group $\{\hat{z} \in Z(\mathbf{G}) \mid \hat{s}\hat{z}$ is conjugate to $\hat{s}$ in $\mathbf{G}\}$, hence whenever $|A_{\mathbf{G}_{\mathrm{ad}}}(s)| = |Z(\mathbf{G})|$ we're done.
- Let $\mathbf{G}_{\mathrm{ad}}$ be $\mathrm{PGL}_{n+1}(\mathbb{K})$. Consider the matrix $\tilde{s}_\Sigma = I_{(n+1)/d} \otimes J_d \in \mathrm{GL}_{n+1}(\mathbb{K})$ specified in the proof of Proposition 4.7; then we already know that this lies in an F-stable conjugacy class of $\mathrm{GL}_{n+1}(\mathbb{K})$. We have $\det(\tilde{s}_\Sigma) = \pm 1$, (with -1 only when d is even and $(n+1)/d$ is odd), so it is clear that $\hat{s}_\Sigma = \det(\tilde{s}_\Sigma)\tilde{s}_\Sigma \in \mathrm{SL}_{n+1}(\mathbb{K})$, which is a simply connected group of type A_n. Applying the Frobenius we see that $F(\hat{s}_\Sigma) = \det(\tilde{s}_\Sigma)F(\tilde{s}_\Sigma)$ but we know $F(\tilde{s}_\Sigma)$ and $\tilde{s}_\Sigma$ are conjugate by an element of $\mathbf{W}$ so clearly $F(\hat{s}_\Sigma)$ and $\hat{s}_\Sigma$ are conjugate. Therefore we can take $\mathcal{C}$ to be the conjugacy class containing $\hat{s}_\Sigma$.

To prove the remaining cases we will consider the following argument. Recall that we have $\check{X}(\mathbf{T}_0) \subseteq \check{X}(\mathbf{T}_{\mathrm{ad}})$ and $\check{X}(\mathbf{T}_0)$ can be identified with the coroot lattice $\mathbb{Z}\check{\Phi}$. Consider the element $y_\Sigma \in (\mathbb{Q}/\mathbb{Z})_{p'} \otimes \check{X}(\mathbf{T}_{\mathrm{ad}})$ then we can express this as a sum of the simple coroots. We then have an element $\hat{y}_\Sigma = \sum_{\alpha\in\Delta} r_\alpha \otimes \check{\alpha}$ for some $r_\alpha \in (\mathbb{Q}/\mathbb{Z})_{p'}$, which gives us a representative $\imath_{\mathbf{T}_0}(\hat{y}_\Sigma)$ of the preimage $\delta_{\mathrm{ad}}^{-1}(\imath_{\mathbf{T}_{\mathrm{ad}}}(y_\Sigma))$. We then argue that $\hat{y}_\Sigma$ is conjugate to $F(\hat{y}_\Sigma)$ so we can take $\mathcal{C}$ to be the class containing $\hat{y}_\Sigma$.

- Let $\mathbf{G}_{\mathrm{ad}}$ be of type C_n then we need only consider the case where $\Sigma = \{\alpha_d\}$ for $1 \leqslant d \leqslant n-1$. Recall the coroots and coweights are a root system of type B_n. Using the information in [**Bou02**, Plate II(VI)] we see that $\check{\omega}_d$ lies in the coroot lattice so r_α is either 1 or $\frac{1}{2}$ for all $\alpha \in \Delta$. By the arguments used in the proof of Proposition 4.7 it is clear that $\hat{y}_\Sigma$ is F-fixed.
- Let $\mathbf{G}_{\mathrm{ad}}$ be of type D_n and let Σ be such that $m_\alpha|\Sigma| = 2$ then $F(y_\Sigma) = y_\Sigma$. Using the information in [**Bou02**, Plate IV(VI)] we see r_α is either 1, $\frac{1}{2}$, $\frac{1}{4}$ or $\frac{3}{4}$ for all $\alpha \in \Delta$. Furthermore $r_{\alpha_{n-1}} = r_{\alpha_n}$ if $\Sigma \neq \{\alpha_0, \alpha_{n-1}\}$ or $\{\alpha_0, \alpha_n\}$. Again by the arguments used in the proof of Proposition 4.7 we see $\hat{y}_\Sigma$ and $F(\hat{y}_\Sigma)$ are conjugate.
- Let $\mathbf{G}_{\mathrm{ad}}$ be of type E_6. We need only consider the case where $\Sigma = \{\alpha_2\}$. Using the information in [**Bou02**, Plate V(VI)] we see

$$\hat{y}_{\{\alpha_2\}} = \frac{1}{2} \otimes \check{\alpha}_1 + \frac{1}{2} \otimes \check{\alpha}_4 + \frac{1}{2} \otimes \check{\alpha}_6.$$

  This is clearly F-fixed.
- Let $\mathbf{G}_{\mathrm{ad}}$ be of type E_7. We need only consider the cases where $\Sigma = \{\alpha_1\}$ or $\{\alpha_3\}$. Using the information in [**Bou02**, Plate VI(VI)] we see

$$\begin{aligned}\hat{y}_{\{\alpha_1\}} &= \frac{1}{2} \otimes \check{\alpha}_3 + \frac{1}{2} \otimes \check{\alpha}_5 + \frac{1}{2} \otimes \check{\alpha}_7,\\ \hat{y}_{\{\check{\alpha}_3\}} &= \frac{1}{3} \otimes \check{\alpha}_2 + \frac{2}{3} \otimes \check{\alpha}_4 + \frac{1}{3} \otimes \check{\alpha}_6 + \frac{2}{3} \otimes \check{\alpha}_7.\end{aligned}$$

  It is clear that the element $\hat{y}_{\{\alpha_1\}}$ is F-fixed and if $q \equiv 1 \pmod 3$ then $\hat{y}_{\{\alpha_3\}}$ is F-fixed. If $q \equiv 2 \pmod 3$ then $F(\hat{y}_{\{\alpha_3\}}) = -\hat{y}_{\{\alpha_3\}}$ but by [**Bou02**, Plates VI(XI)] we see that the longest element $w_0 \in \mathbf{W}$ acts on the coroots by -1. Hence $\hat{y}_{\{\alpha_3\}} = \check{w}_0 \cdot F(\hat{y}_{\{\alpha_3\}})$ so we're done. ∎

If $\mathbf{G}$ is of type C or D it will be useful to know that we can choose class representatives with a specific action of F. In particular we record the following corollary for later, which follows immediately from the proofs of Propositions 4.7 and 4.10.

Corollary 4.11. *Assume $\mathbf{G}$ is simply connected of type C_n or D_n and $\Sigma = \{\alpha_d\} \in \mathcal{Q}(\mathbf{G}_{\mathrm{ad}})$ is a subset where $1 \leqslant d \leqslant n-1$. If $\mathbf{G}$ is of type C_n there exists an element $s \in \mathbf{T}_0$ such that $\delta_{\mathrm{ad}}(s) = t_\Sigma$ and $F(s) = s$. If $\mathbf{G}$ is of type D_n there exists an element $s \in \mathbf{T}_0$ such that $\delta_{\mathrm{ad}}(s) = t_\Sigma$ and $F(s) = s$ or $F(s) = s^{w_0}$, (where $w_0 \in \mathbf{W}$ is the longest element).*

4.5 Remarks on Fixing Class Representatives

We assume that $\mathbf{G}$ is a connected simple algebraic group. We will now fix representatives, which we assume chosen once and for all, for conjugacy classes surjecting onto quasi-isolated classes. Let $\Sigma \in \mathcal{Q}(\mathbf{G}_{\mathrm{sc}}^\star)$ be a set of roots and let $t_\Sigma \in \mathbf{T}_{\mathrm{sc}}^\star$ be its corresponding semisimple element. We assume that the class containing t_Σ is $F^\star$-stable.

Let $\tilde{\mathbf{G}}'$ be the derived subgroup of $\tilde{\mathbf{G}}^\star$ and denote by $\delta'_{\mathrm{sc}} : \mathbf{G}_{\mathrm{ad}}^\star \to \tilde{\mathbf{G}}'$ and $\delta'_{\mathrm{ad}} : \tilde{\mathbf{G}}' \to \mathbf{G}_{\mathrm{sc}}^\star$ a simply connected cover and adjoint quotient of $\tilde{\mathbf{G}}'$. By Proposition 4.10 we may fix a semisimple element $s_{\mathrm{sc}} \in \mathbf{T}_{\mathrm{ad}}^\star$ such that the conjugacy class of $\mathbf{G}_{\mathrm{ad}}^\star$ containing s_{sc} is $F^\star$-stable and $(\delta'_{\mathrm{ad}} \circ \delta'_{\mathrm{sc}})(s_{\mathrm{sc}}) = t_\Sigma$, (this also satisfies the conditions of Corollary 4.11). We then define the semisimple element $\tilde{s} \in \tilde{\mathbf{G}}^\star$ to be the image of $\delta'_{\mathrm{sc}}(s_{\mathrm{sc}})$ under the natural embedding $\tilde{\mathbf{G}}' \to \tilde{\mathbf{G}}^\star$.

Recall that by duality we have a surjective morphism $\iota^\star : \tilde{\mathbf{G}}^\star \to \mathbf{G}^\star$, which restricts to an isogeny $\tilde{\mathbf{G}}' \to \mathbf{G}^\star$ defined over $\mathbb{F}_q$. We define $s \in \mathbf{T}_0^\star$ to be the image of $\tilde{s}$ under $\iota^\star$. This element lies in an $F^\star$-stable conjugacy class of $\mathbf{G}^\star$ and the image $\delta_{\mathrm{ad}}^\star(s)$ in $\mathbf{G}_{\mathrm{sc}}^\star$ will be t_Σ. By defining the representatives in this way we have s, $\tilde{s}$ and t_Σ are all conjugate to their image under the Frobenius by the same element of the Weyl group, (where here we identify the Weyl groups through the appropriate morphisms). In particular we will be able to uniformly describe the automorphism induced by the generalised Frobenius endomorphism on the component groups of their centralisers.

Chapter 5

Proposition A: Types A_n, B_n, E_6 and E_7

In this chapter we will prove Proposition A for the case where $\mathbf{G}$ is a non-adjoint simple group of type A_n ($n \geqslant 1$), B_n ($n \geqslant 2$), E_6 or E_7. In Chapter 3 the choices made by Hézard were sufficient to prove Proposition A for the case of type C_n. However, in general, it will be necessary to use results of Hézard together with the more extensive results of Lusztig contained in [**Lus09**]. In [**Lus09**] Lusztig works with algebraic groups defined over $\mathbb{C}$ but he comments, (see [**Lus09**, §1.8]), that his results hold whenever p is a very good prime for $\mathbf{G}$. By this we mean the following.

Definition 5.1. Let $\mathbf{G}$ be a simple algebraic group. If $\mathbf{G}$ is of type A_n then $\mathbf{G}$ is a very good prime for $\mathbf{G}$ if p does not divide $n+1$. Otherwise p is very good for $\mathbf{G}$ if it is good for $\mathbf{G}$. Assume $\mathbf{G}$ is any connected reductive algebraic group and let $\mathbf{G}_1 \cdots \mathbf{G}_r$ be the decomposition of the derived subgroup of $\mathbf{G}$ as an almost direct product of simple groups. Then p is a *very good prime* for $\mathbf{G}$ if it is very good for each $\mathbf{G}_i$.

We will discuss Lusztig's results in type A and show that, for our concerns, these work equally well in type A in any characteristic. To make stating the results of Lusztig slightly more convenient we adopt the following convention. Consider a subset $\mathbb{I} \subseteq \mathbb{S}_0$ then we have a corresponding parahoric subgroup $\mathbf{W}_{\mathbb{I}}$ of $\mathbf{W}$ generated by $\mathbb{I}$. Let $\rho \in \mathrm{Irr}(\mathbf{W}_{\mathbb{I}})$ be an irreducible character of $\mathbf{W}_{\mathbb{I}}$ and let $\mathcal{A}_{\mathbb{I}}$ be the set of all elements of $\mathcal{A}$ which normalise $\mathbf{W}_{\mathbb{I}}$. We will write $\mathrm{Stab}_{\mathcal{A}}(\rho)$ to denote the subgroup of $\mathcal{A}_{\mathbb{I}}$ which stabilises the pair $(\rho, \mathbf{W}_{\mathbb{I}})$ under the natural conjugation action of $\mathcal{A}_{\mathbb{I}}$ on $\mathbf{W}_{\mathbb{I}}$.

5.1 Type A_n

Let μ, $\nu \geqslant 1$ be integers such that $\mu + \nu = n - 1$ then we denote by $\mathbf{W}(\mathrm{A}_\mu\,\mathrm{A}_\nu)$ the parabolic subgroup of $\mathbf{W}$ generated by the reflections $\mathbb{S} \setminus \{s_{\mu+1}\}$, which is naturally identified with $\mathfrak{S}_{\mu+1} \times \mathfrak{S}_{\nu+1}$. Lusztig has combinatorially described the j-induction from $\mathbf{W}(\mathrm{A}_\mu\,\mathrm{A}_\nu)$ to $\mathbf{W}$ in the following way.

Lemma 5.2 (Lusztig, [Lus09, §3.1(b)]). *Let* $[\Lambda_\mu] \boxtimes [\Lambda_\nu] \in \mathrm{Irr}(\mathbf{W}(\mathrm{A}_\mu\,\mathrm{A}_\nu))$ *be a character. Applying an appropriate amount of shifts we may assume that the representatives* $\Lambda_\mu = \{\lambda_1, \dots, \lambda_s\}$ *and* $\Lambda_\nu = \{\mu_1, \dots, \mu_s\}$ *of the symbols are of the same cardinality. The induced character is then given by*

$$j_{\mathbf{W}(\mathrm{A}_\mu\,\mathrm{A}_\nu)}^{\mathbf{W}}([\Lambda_\mu] \boxtimes [\Lambda_\nu]) = [\lambda_1 + \mu_1, \lambda_2 + \mu_2 - 1, \dots, \lambda_s + \mu_s - (s-1)].$$

As j-induction is transitive this gives us a way to describe the j-induction of characters from any parabolic subgroup of $\mathbf{W}$.

Example 5.3. Consider the case of A_6 and the parabolic subgroup $\mathbf{W}(\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_1)$ generated by the set of reflections $\mathbb{S} \setminus \{s_2, s_5\}$. The sign characters of $\mathfrak{S}_2$ and $\mathfrak{S}_3$ are given by the symbols $[1,2] = [0,2,3]$ and $[1,2,3]$ respectively. We have $\mathbf{W}(\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_1)$ is naturally a parabolic subgroup of the parabolic subgroup $\mathbf{W}(\mathrm{A}_4\,\mathrm{A}_1)$ generated by the set of reflections $\mathbb{S} \setminus \{s_5\}$. Applying Lemma 5.2 we have

$$j_{\mathbf{W}(\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_1)}^{\mathbf{W}(\mathrm{A}_4\,\mathrm{A}_1)}([0,2,3] \boxtimes [1,2,3] \boxtimes [0,2,3]) = [1,3,4] \boxtimes [0,2,3].$$

Again applying Lemma 5.2 we have

$$j_{\mathbf{W}(\mathrm{A}_4\,\mathrm{A}_1)}^{\mathbf{W}}([1,3,4] \boxtimes [0,2,3]) = [1,4,5].$$

The partition associated to the symbol $[1,4,5]$ is $\lambda([1,4,5]) = (1,3,3)$. This parameterises an irreducible character of $\mathbf{W}$ and we can verify the j-induction is correct by using the j-induction tables contained in **[GHL⁺96]**.

With this result it will be simple to prove Proposition A. Let $\mathcal{O}$ be a unipotent class of $\mathbf{G}$ parameterised by a partition $\pi = (\pi_1, \dots, \pi_s)$ of $n+1$. Let $d = |A_{\mathbf{G}}(u)|$ and recall that $d \mid \pi_i$ for each $1 \leqslant i \leqslant s$, (as d divides $|A_{\mathbf{G}_{\mathrm{sc}}}(u_{\mathrm{sc}})|$). We can form a new partition $\pi' = (\pi'_1, \dots, \pi'_s)$ of $(n+1)/d$, where $\pi'_i = \pi_i/d$ for all $1 \leqslant i \leqslant s$. We denote by $\mathbf{W}(\mathrm{A}_{(n+1-d)/d}{}^d)$ the parabolic subgroup of $\mathbf{W}$ generated by the reflections $\mathbb{S} \setminus \{s_{j(n+1)/d} \mid 0 \leqslant j \leqslant d-1\}$. The partition π' parameterises an irreducible character of a Weyl group of type $\mathrm{A}_{(n+1-d)/d}$ and has an associated symbol $[\Lambda_{\pi'}] \in \widetilde{\mathbb{X}}_{(n+1)/d}$ such that $\lambda([\Lambda_{\pi'}]) = \pi'$.

Proposition 5.4 (Lusztig, [Lus09, §3.2]). *Let* $\boxtimes^d[\Lambda_{\pi'}]$ *denote the product of the character* $[\Lambda_{\pi'}]$ *containing d copies then* $\boxtimes^d[\Lambda_{\pi'}] \in \mathrm{Irr}(\mathbf{W}(\mathrm{A}_{(n+1-d)/d}{}^d))$ *is an irreducible character of the parabolic subgroup. This character is such that* $\mathrm{Stab}_{\mathcal{A}}(\boxtimes^d[\Lambda_{\pi'}]) = \langle \sigma^{(n+1)/d} \rangle$ *and* $j_{\mathbf{W}(\mathrm{A}_{(n+1-d)/d}{}^d)}^{\mathbf{W}}(\boxtimes^d[\Lambda_{\pi'}])$ *is the Springer character* $\rho_{\tilde{u},1}$.

Proof. Let $\mathbf{W}'$ denote the parabolic subgroup $\mathbf{W}(\mathrm{A}_{(n+1-d)/d}{}^d)$, then the assertion concerning the stabiliser is clear as a generator for the subgroup of $\mathcal{A}$ normalising $\mathbf{W}'$

acts by cyclically permuting the isomorphic irreducible components. Using transitivity and Lemma 5.2 precisely $d-1$ times we have

$$\begin{aligned}
j_{\mathbf{W}'}^{\mathbf{W}}(\boxtimes^d[\Lambda_{\pi'}]) &= [d\pi_1', d(\pi_2'+1)-(d-1), \dots, d(\pi_s'+(s-1))-(d-1)(s-1)],\\
&= [d\pi_1', d\pi_2'+1, \dots, d\pi_s'+(s-1)],\\
&= [\pi_1, \pi_2+1, \dots, \pi_s+(s-1)].
\end{aligned}$$

Clearly the partition associated to this symbol is the partition π, hence this character is the Springer character $\rho_{\tilde{u},1}$ by the description of the Springer correspondence given in Section 2.2.1. ■

By the results in Chapter 4 there exists a semisimple element $\tilde{s} \in \tilde{\mathbf{T}}_0^\star$ which lies in an $F^\star$-stable conjugacy class such that $\mathbf{W}(\tilde{s}) = \mathbf{W}(\mathrm{A}_{(n+1-d)/d}{}^d)$. Let $\mathbf{W}^\star(\tilde{\mathcal{F}}) \subseteq \mathrm{Irr}(\mathbf{W}^\star(\tilde{s}))$ be the family determined by the special character $\boxtimes^d[\Lambda_{\pi'}]$ described in Proposition 5.4. As $\mathbf{W}^\star(\tilde{s})$ is a direct product of groups of type A we have $\mathbf{W}^\star(\tilde{\mathcal{F}})$ contains only the character $\boxtimes^d[\Lambda_{\pi'}]$. This character is clearly invariant under all graph automorphisms, so $\mathbf{W}^\star(\tilde{\mathcal{F}})$ must be invariant under $F_{\tilde{s}}^\star$ and $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}))$ is an F-stable pair.

We claim that the triple $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}), \tilde{\psi})$, where $\tilde{s}$ and $\mathbf{W}^\star(\tilde{\mathcal{F}})$ are as above and $\tilde{\psi}$ is the unipotent character parameterised by $\boxtimes^d[\Lambda_{\pi'}]$, satisfies properties (P1) to (P3). Note that as we are dealing with products of groups of type A we always have $n_{\tilde{\psi}} = 1$. Furthermore by Section 2.2.1 we know $|A_{\tilde{\mathbf{G}}}(\tilde{u})| = 1$ so Property (P1) holds. From Sections 2.2.1 and 4.2.1 we see that $|A_{\mathbf{G}^\star}(s)| = |Z_{\mathbf{G}}(u)|$ and by the results in Section 1.8 we know $\tilde{\psi}$ is invariant under all graph automorphisms, hence by Lemma 4.6 we have Property (P2) holds. Finally Property (P3) is given to us by Proposition 5.4.

The condition ♣ is trivially true as $\tilde{\psi}$ is the only character in $\mathcal{F}$, so $|\mathfrak{X}_{\mathcal{F}}| = 0$.

5.2 Type B_n

Let $\mathcal{O}$ be an F-stable unipotent class of $\mathbf{G}$. There are two separate cases to consider here which arise from the numerical value $\kappa_{\mathrm{B}}(u)$ introduced in Section 2.2.2. Throughout each subsection we assume the conditions of the heading. Let $\mu, \nu \in \mathbb{N}$ then if $\mu + \nu = n$ we denote by $\mathbf{W}(\mathrm{C}_\mu\,\mathrm{C}_\nu)$ the parahoric subgroup generated by the reflections $\mathbb{S}_0 \setminus \{s_\mu\}$. If $2\mu+\nu = n$ we denote by $\mathbf{W}(\mathrm{C}_\mu\,\mathrm{A}_\nu\,\mathrm{C}_\mu)$ the parahoric subgroup of $\mathbf{W}$ generated by the reflections $\mathbb{S}_0 \setminus \{s_\mu, s_{n-\mu}\}$.

5.2.1 The Case $\kappa_{\mathrm{B}}(u) \geqslant 2$

For this case we use the following result of Hézard. Note that a similar statement is proved by Lusztig in [**Lus09**, §4.12], however in a less direct way. We use Hézard's result as it will provide an easier way to prove Property (P2).

Proposition 5.5 (Hézard, [Héz04, §4.2.2]). *Let $\tilde{\mathcal{O}}$ be an F-stable unipotent class of $\tilde{\mathbf{G}}$ such that $\kappa_{\mathrm{B}}(\tilde{u}) \geqslant 2$ and*

$$\mathrm{Spr}_{\mathrm{B}}(\tilde{\mathcal{O}}) = \begin{bmatrix} a_1 & & a_2 & & \cdots & & a_{m+1} \\ & b_1 & & \cdots & & b_m & \end{bmatrix}$$

We define two pairs of sequences (α, β) and (γ, δ) in the following way

$$\begin{cases} \alpha_i = \lfloor a_i/2 \rfloor & \text{for } 1 \leqslant i \leqslant m+1 \\ \beta_i = \lfloor b_i/2 \rfloor & \text{for } 1 \leqslant i \leqslant m \end{cases} \qquad \begin{cases} \gamma_i = a_i - \lfloor a_i/2 \rfloor & \text{for } 1 \leqslant i \leqslant m+1, \\ \delta_i = b_i - \lfloor b_i/2 \rfloor & \text{for } 1 \leqslant i \leqslant m. \end{cases}$$

We construct two symbols $[\Lambda_1]$, $[\Lambda_2]$ by letting

$$\Lambda_1 = \begin{pmatrix} \alpha_1 & & \alpha_2 & & \cdots & & \alpha_{m+1} \\ & \beta_1 & & \cdots & & \beta_m & \end{pmatrix},$$
$$\Lambda_2 = \begin{pmatrix} \gamma_1 & & \gamma_2 & & \cdots & & \gamma_{m+1} \\ & \delta_1 & & \cdots & & \delta_m & \end{pmatrix}.$$

The product $[\Lambda_1] \boxtimes [\Lambda_2]$ is a special irreducible character of the parahoric subgroup $\mathbf{W}(\mathrm{C}_\mu \, \mathrm{C}_\nu)$, where μ, ν are integers determined by $[\Lambda_1]$ and $[\Lambda_2]$ such that $\mu + \nu = n$. Let $\mathbf{W}(\tilde{\mathcal{F}}) \subseteq \mathrm{Irr}(\mathbf{W}(\mathrm{C}_\mu \, \mathrm{C}_\nu))$ be the family of characters determined by $[\Lambda_1] \boxtimes [\Lambda_2]$ then $|\mathcal{G}_{\tilde{\mathcal{F}}}| = |A_{\tilde{\mathbf{G}}}(\tilde{u})|$ and the induced character $j_{\mathbf{W}(\mathrm{C}_\mu \, \mathrm{C}_\nu)}^{\mathbf{W}}([\Lambda_1] \boxtimes [\Lambda_2])$ is the Springer character $\rho_{\tilde{u},1}$.

Remark 5.6. Hézard proves more than what we have stated here. However we do not wish to use his proof of the existence of an F-stable semisimple element as we instead favour our results obtained from Bonnafé's work.

By the results in Chapter 4 there exists a semisimple element $\tilde{s} \in \tilde{\mathbf{T}}_0^\star$ which is $F^\star$-fixed, (see Corollary 4.11), and such that $\mathbf{W}(\tilde{s}) = \mathbf{W}(\mathrm{C}_\mu \, \mathrm{C}_\nu)$. Let $[\Lambda_1] \boxtimes [\Lambda_2]$ be the special character prescribed in Proposition 5.5 and let $\mathbf{W}^\star(\tilde{\mathcal{F}}) \subseteq \mathrm{Irr}(\mathbf{W}^\star(\tilde{s}))$ be the corresponding family, then $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}))$ is an F-stable pair. The family of unipotent characters $\mathcal{F}$ will be a product $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ determined by the special characters $[\Lambda_1]$ and $[\Lambda_2]$. Let $\bar{\psi}_1 \in \mathcal{F}_1$ and $\bar{\psi}_2 \in \mathcal{F}_2$ be such that $n_{\bar{\psi}_1} = |\mathcal{G}_{\mathcal{F}_1}|$ and $n_{\bar{\psi}_2} = |\mathcal{G}_{\mathcal{F}_2}|$. We take $\bar{\psi}$ to be the product $\bar{\psi}_1 \boxtimes \bar{\psi}_2$ and we claim that $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}), \bar{\psi})$ is a triple satisfying properties (P1) to (P3). Clearly Property (P1) holds because $n_{\bar{\psi}} = n_{\bar{\psi}_1} n_{\bar{\psi}_2} = |\mathcal{G}_{\mathcal{F}_1}||\mathcal{G}_{\mathcal{F}_2}| = |\mathcal{G}_{\tilde{\mathcal{F}}}|$ and $|\mathcal{G}_{\tilde{\mathcal{F}}}| = |A_{\tilde{\mathbf{G}}}(\tilde{u})|$ by Proposition 5.5. Also Property (P3) holds by Proposition 5.5.

Recall from Section 2.2.3 that we have $|Z_{\mathbf{G}}(u)| = 1$ hence to show Property (P2) is true we must prove that $|\operatorname{Stab}_{A_{G^\star}(s)}(\bar{\psi})| = 1$. If $\mu \neq \nu$ then this is trivially true as the centraliser is connected so $|A_{\mathbf{G}^\star}(s)| = 1$. In this situation it is clear that ♣ holds as every character in $\mathcal{F}$ has to have a trivial stabiliser, in particular they have the same stabiliser.

Assume $\mu = \nu$ then $A_{\mathbf{G}^\star}(s)$ has order 2 and the non-trivial coset representative acts by exchanging the two components of type C. If we can show that $\bar{\psi}_1 \neq \bar{\psi}_2$ then these characters cannot be exchanged by the non-trivial graph automorphism and we will have $\mathrm{Stab}_{A_{G^\star}(s)}(\bar{\psi})$ is trivial. In this case we must have $n \equiv 0 \pmod 2$ and it will be enough to show that $[\Lambda_1] \neq [\Lambda_2]$ so the families $\mathcal{F}_1$ and $\mathcal{F}_2$ are distinct. Assume for a contradiction that $[\Lambda_1] = [\Lambda_2]$ then every entry of $\mathrm{Spr}_{\mathrm{B}}(\tilde{\mathcal{O}})$ must be even because $a_i = 2\lfloor a_i/2 \rfloor$ and $b_i = 2\lfloor b_i/2 \rfloor$. Let $\lambda = (\lambda_1, \dots, \lambda_s)$ be the partition parametrising $\tilde{\mathcal{O}}$ then as $\kappa_{\mathrm{B}}(\tilde{\mathcal{O}}) \geqslant 2$ there exists an i such that $\lambda_i = \lambda_{i+1}$ are odd parts of λ. By the construction given in Section 2.2.2 we have $c_{i+1} = c_i + 1$, where the c_i are the corresponding entries of the Springer symbol. Hence at least one of those entries must be odd so we're done. This argument also shows that every character in $\mathcal{F}$ has trivial stabiliser in $A_{G^\star}(s)$, which means $|\mathfrak{X}_{\mathcal{F}}| = 0$ so ♣ holds.

5.2.2 The Case $\kappa_{\mathrm{B}}(u) = 1$

Proposition 5.7 (Lusztig, [Lus09, §4.10]). *Let $\tilde{\mathcal{O}}$ be a unipotent class with $\kappa_{\mathrm{B}}(\tilde{u}) = 1$. There exist $\mu, \nu \in \mathbb{N}$, such that $2\mu + \nu = n$, and a special irreducible character $[\Lambda_1] \boxtimes [\Lambda_2] \boxtimes [\Lambda_1] \in \mathrm{Irr}(\mathbf{W}(\mathrm{C}_\mu\,\mathrm{A}_\nu\,\mathrm{C}_\mu))$, (which can be constructed inductively from $\mathrm{Spr}_{\mathrm{B}}(\tilde{\mathcal{O}})$), such that:*

- $f_{[\Lambda_1]\boxtimes[\Lambda_2]\boxtimes[\Lambda_1]} = |A_{\tilde{\mathbf{G}}}(\tilde{\mathcal{O}})|$,
- $\mathrm{Stab}_{\mathcal{A}}([\Lambda_1] \boxtimes [\Lambda_2] \boxtimes [\Lambda_1]) = \mathcal{A}$,
- $j_{\mathbf{W}(\mathrm{C}_\mu\,\mathrm{A}_\nu\,\mathrm{C}_\mu)}^{\mathbf{W}}([\Lambda_1] \boxtimes [\Lambda_2] \boxtimes [\Lambda_1])$ *is the Springer character $\rho_{\tilde{u},1}$.*

By the results in Chapter 4 there exists a semisimple element $\tilde{s} \in \tilde{\mathbf{T}}_0^\star$ which lies in an $F^\star$-stable conjugacy class such that $\mathbf{W}(\tilde{s}) = \mathbf{W}(\mathrm{C}_\mu\,\mathrm{A}_\nu\,\mathrm{C}_\mu)$. Let $\mathbf{W}^\star(\tilde{\mathcal{F}}) \subseteq \mathrm{Irr}(\mathbf{W}^\star(\tilde{s}))$ be the family of characters determined by the special character $[\Lambda_1] \boxtimes [\Lambda_2] \boxtimes [\Lambda_1]$ prescribed by Proposition 5.7. The character $[\Lambda_2]$ will be invariant under the graph automorphism of A_ν and as the components of type C_μ have the same character we have $[\Lambda_1] \boxtimes [\Lambda_2] \boxtimes [\Lambda_1]$ is fixed under all graph automorphisms. Therefore $\mathbf{W}^\star(\mathcal{F})$ must be invariant under $F_{\tilde{s}}^\star$ and $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}))$ is an F-stable pair.

The family of unipotent characters $\mathcal{F}$ will be a product $\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_1$, where the families $\mathcal{F}_1$ and $\mathcal{F}_2$ are determined by the special Weyl group characters $[\Lambda_1]$ and $[\Lambda_2]$. Let $\bar{\psi}_1 \in \mathcal{F}_1$ and $\bar{\psi}_2 \in \mathcal{F}_2$ be such that $n_{\bar{\psi}_1} = |\mathcal{G}_{\mathcal{F}_1}|$ and $n_{\bar{\psi}_2} = |\mathcal{G}_{\mathcal{F}_2}|$. We take $\bar{\psi}$ to be the product $\bar{\psi}_1 \boxtimes \bar{\psi}_2 \boxtimes \bar{\psi}_1$ and claim that $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}), \bar{\psi})$ is a triple satisfying properties (P1) to (P3).

Clearly Property (P1) holds because $n_{\bar{\psi}} = n_{\bar{\psi}_1} n_{\bar{\psi}_2} n_{\bar{\psi}_1} = |\mathcal{G}_{\mathcal{F}_1}||\mathcal{G}_{\mathcal{F}_2}||\mathcal{G}_{\mathcal{F}_1}| = |\mathcal{G}_{\tilde{\mathcal{F}}}|$ and $|\mathcal{G}_{\tilde{\mathcal{F}}}| = |A_{\tilde{\mathbf{G}}}(\tilde{u})|$ by Proposition 5.7. Recall from Section 2.2.3 that $|Z_{\mathbf{G}}(u)| = 2$ so it is clear that we have $|A_{\mathbf{G}^\star}(s)| = |Z_{\mathbf{G}}(u)|$. We know $\bar{\psi}$ is invariant under all possible graph automorphisms so Property (P2) follows from Lemma 4.6. Finally it is clear that Property (P3) holds by Proposition 5.7.

Every character $\bar{\psi} \in \mathcal{F}$ is parametrised by symbols $[\Lambda_1] \boxtimes [\Lambda_2] \boxtimes [\Lambda_3]$ and we know that $|\operatorname{Stab}_{A_{G^\star}(s)}(\bar{\psi})| = 2$ if and only if $[\Lambda_1] = [\Lambda_3]$. It is then clear that $|\mathfrak{X}_{\mathcal{F}}| = |\mathcal{F}_1|^2 - |\mathcal{F}_1|$ so in particular if $|\mathcal{F}_1| \neq 1$ then condition ♣ does not hold. However we note that $A_{\mathbf{G}}(u)$ is not abelian in this case.

5.3 Type E$_6$

In the following we recall the results of Lusztig given in [**Lus09**, §7.1]. In the table we list for each unipotent class, (labelled as in [**Car93**, §13.1]), a parahoric subgroup of $\mathbf{W}^\star$, (which is $\mathbf{W}^\star(\tilde{s})$ for some semisimple element $\tilde{s} \in \tilde{\mathbf{T}}_0^\star$), and a special irreducible character $\rho \in \operatorname{Irr}(\mathbf{W}^\star(\tilde{s}))$. Note that we have listed the parahoric subgroups only by their type and this should be interpreted in the following way. For each type listed in Table 5.1 there is a corresponding set of roots $\Sigma \subseteq \tilde{\Delta}^\star$ listed in Table 4.2. We take the parahoric subgroup of that given type to be the subgroup generated by the set of reflections $\mathbb{S}_0^\star \setminus \{s_\alpha \mid \alpha \in \Sigma\}$.

Let $\mathbf{W}^\star(\tilde{s}) = \mathbf{W}_1^\star \times \cdots \times \mathbf{W}_r^\star$ be a decomposition into irreducible Weyl groups, then we have a corresponding decomposition of the family $\mathbf{W}^\star(\tilde{\mathcal{F}}) = \mathbf{W}^\star(\tilde{\mathcal{F}}_1) \times \cdots \times \mathbf{W}^\star(\tilde{\mathcal{F}}_r)$ determined by ρ. The family $\mathcal{F}$ of unipotent characters of $C_{G^\star}(s)^\circ$ in bijective correspondence with the family $\tilde{\mathcal{F}}$ will then decompose as a corresponding product of families $\mathcal{F} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_r$. We choose unipotent characters $\bar{\psi}_i \in \mathcal{F}_i$ such that $\bar{\psi}_i = |\mathcal{G}_{\mathcal{F}_i}| = |\mathcal{G}_{\tilde{\mathcal{F}}_i}|$ for each $1 \leqslant i \leqslant r$. We also make the extra assumption that if $\mathbf{W}_i^\star \cong \mathbf{W}_j^\star$ for some $i \neq j$ and $\rho_i = \rho_j$ then we take $\bar{\psi}_i = \bar{\psi}_j$. Take $\bar{\psi}$ to be the product $\bar{\psi}_1 \boxtimes \cdots \boxtimes \bar{\psi}_r$ then we claim the triple $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}), \bar{\psi})$ satisfies properties (P1) to (P3).

Note that we have used the same choices as Lusztig in [**Lus09**, §7.1] except for the unipotent classes labelled $\mathrm{A}_3 + \mathrm{A}_1$ and E_6 as here Lusztig did not choose the Weyl group of some quasi-isolated semisimple element.

$\mathcal{O}$	$\lvert A_{\tilde{\mathbf{G}}}(\tilde{u})\rvert$	$\lvert A_{\mathbf{G}}(u)\rvert$	$\mathbf{W}^\star(\tilde{s})$	ρ
1	1	1	E_6	$1_p'$
A_1	1	1	E_6	$6_p'$
$2\,\mathrm{A}_1$	1	1	E_6	$20_p'$
$3\,\mathrm{A}_1$	1	1	$\mathrm{A}_5\,\mathrm{A}_1$	$[123456] \boxtimes [12]$
A_2	2	2	E_6	$30_p'$
$\mathrm{A}_2 + \mathrm{A}_1$	1	1	E_6	$64_p'$

$\mathcal{O}$	$\lvert A_{\tilde{\mathbf{G}}}(\tilde{u})\rvert$	$\lvert A_{\mathbf{G}}(u)\rvert$	$\mathbf{W}^\star(\tilde{s})$	ρ
$2\,\mathrm{A}_2$	1	3	D_4	$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$
$\mathrm{A}_2 + 2\,\mathrm{A}_1$	1	1	E_6	$60_p'$
A_3	1	1	E_6	$81_p'$
$2\,\mathrm{A}_2 + \mathrm{A}_1$	1	3	$\mathrm{A}_2\,\mathrm{A}_2\,\mathrm{A}_2$	$[123] \boxtimes [123] \boxtimes [123]$
$\mathrm{A}_3 + \mathrm{A}_1$	1	1	$\mathrm{A}_5\,\mathrm{A}_1$	$[1245] \boxtimes [12]$
$\mathrm{D}_4(a_1)$	6	6	E_6	80_s
A_4	1	1	E_6	81_p
D_4	1	1	E_6	24_p
$\mathrm{A}_4 + \mathrm{A}_1$	1	1	E_6	60_p
A_5	1	3	$\mathrm{A}_1\,\mathrm{A}_1\,\mathrm{A}_1\,\mathrm{A}_1$	$[12] \boxtimes [12] \boxtimes [12] \boxtimes [12]$
$\mathrm{D}_5(a_1)$	1	1	E_6	64_p
$\mathrm{E}_6(a_3)$	2	6	D_4	$\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$
D_5	1	1	E_6	20_p
$\mathrm{E}_6(a_1)$	1	3	D_4	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$
E_6	1	3	$\mathrm{A}_2\,\mathrm{A}_2\,\mathrm{A}_2$	$[3] \boxtimes [3] \boxtimes [3]$

Table 5.1: Proof of Proposition A for E_6.

In the table we have listed characters of $\mathbf{W}$ using the labelling described in **[Lus84a**, §4.11] and characters of groups of classical type by their symbols. Before we can explain why properties (P1) to (P3) are satisfied we must first explain why $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}))$ is an F-stable pair. By the results in Chapter 4 there exists a semisimple element $\tilde{s} \in \tilde{\mathbf{T}}_0^\star$ which lies in an $F^\star$-stable conjugacy class such that $\mathbf{W}^\star(\tilde{s})$ is one of the parahoric subgroups listed in Table 5.1. We need only show that $\mathbf{W}^\star(\tilde{\mathcal{F}})$ is invariant under $F_{\tilde{s}}^\star$. If an irreducible component $\mathbf{W}_i^\star$ of $\mathbf{W}^\star(\tilde{s})$ is of type D_4 then the corresponding character ρ_i is not degenerate and not one of the characters listed in Eq. (1.4). Furthermore whenever $\mathbf{W}_i^\star \cong \mathbf{W}_j^\star$ for some $j \neq i$ we always have $\rho_i = \rho_j$. Therefore the special character is fixed under any graph automorphism hence $\mathbf{W}^\star(\tilde{\mathcal{F}})$ is invariant under $F_{\tilde{s}}^\star$ so we're done.

(P1) We claim that for each character ρ we have $f_\rho = |A_{\tilde{\mathbf{G}}}(\tilde{u})|$ hence $n_{\tilde{\psi}} = |\mathcal{G}_{\tilde{\mathcal{F}}}| =$

$|A_{\tilde{\mathbf{G}}}(\tilde{u})|$. Firstly recall that if $\rho \in \mathrm{Irr}(\mathbf{W}^\star(\tilde{s}))$ is a product of special characters $\rho_1 \boxtimes \cdots \boxtimes \rho_r$ then $f_\rho = f_{\rho_1 \boxtimes \cdots \boxtimes \rho_r} = f_{\rho_1} \cdots f_{\rho_r}$. Therefore we need only determine the information for the irreducible constituents. If the component is of type E_6 then the information can be read off Table 2.5. In a group of type A we always have $f_\rho = 1$ for all characters. Finally in type D_4 we have $f_\rho = 1$ if ρ is the only character in its family and $f_\rho = 2$ if the character lies in a family containing three characters, (note the families were computed in Example 1.63).

(P2) Reading from Tables 4.2 and 5.1 we see $|A_{\mathbf{G}^\star}(s)| = |Z_{\mathbf{G}}(u)|$. By Lemma 1.64 it is clear that the character $\bar{\psi}$ will be invariant under any graph automorphism induced by an element of $A_{\mathbf{G}^\star}(s)$ so we will have $\mathrm{Stab}_{A_{G^\star}(s)}(\bar{\psi}) = A_{G^\star}(s)$. Therefore by Lemma 4.6 we have Property (P2) holds.

(P3) If $\mathbf{W}^\star(\tilde{s})$ is of type E_6 then this is clear from the description of the Springer correspondence given in Table 2.5. In the remaining cases this can be verified using the j-induction tables, (which can be computed with [**GHL**$^+$**96**]), together with Table 2.5. The labelling of characters we choose here is different to that of [**GHL**$^+$**96**] so for convenience we give a dictionary to make checking this easier, (see Table C.1).

We now consider the validity of the condition ♣. If $|A_{\tilde{\mathbf{G}}}(\tilde{u})| = 1$ then $|\mathcal{F}| = 1$ so it is obvious that $|\mathfrak{X}_{\mathcal{F}}| = 0$. If $\mathcal{O}$ is the class A_2 or $\mathrm{D}_4(a_1)$ then $|\mathfrak{X}_{\mathcal{F}}| = 0$ because $C_{\mathbf{G}^\star}(s)$ is connected so clearly $|\,\mathrm{Stab}_{A_{G^\star}(s)}(\bar{\psi})| = 1$ for all $\bar{\psi} \in \mathcal{F}$. Finally we need now only consider when $\mathcal{O}$ is the class $\mathrm{E}_6(a_3)$. However by Lemma 1.64 we know all unipotent characters in the family are invariant under all graph automorphisms so we're done.

5.4 Type E_7

In the following we recall the results of Lusztig given in [**Lus09**, §7.1]. In the table we list for each unipotent class, (labelled as in [**Car93**, §13.1]), a parahoric subgroup of $\mathbf{W}^\star$, (which is $\mathbf{W}^\star(\tilde{s})$ for some semisimple element $\tilde{s} \in \tilde{\mathbf{T}}_0^\star$), and a special irreducible character $\rho \in \mathrm{Irr}(\mathbf{W}^\star(\tilde{s}))$. Note that we have listed the parahoric subgroups only by their type and this should be interpreted in the following way. For each type listed in Table 5.2 there is a corresponding set of roots $\Sigma \subseteq \tilde{\Delta}^\star$ listed in Table 4.2. We take the parahoric subgroup of that given type to be the subgroup generated by the set of reflections $\mathbb{S}_0^\star \setminus \{s_\alpha \mid \alpha \in \Sigma\}$.

Let $\mathbf{W}^\star(\tilde{s}) = \mathbf{W}_1^\star \times \cdots \times \mathbf{W}_r^\star$ be a decomposition into irreducible Weyl groups, then we have a corresponding decomposition of the family $\mathbf{W}^\star(\tilde{\mathcal{F}}) = \mathbf{W}^\star(\tilde{\mathcal{F}}_1) \times \cdots \times \mathbf{W}^\star(\tilde{\mathcal{F}}_r)$ determined by ρ. The family $\mathcal{F}$ of unipotent characters of $C_{G^\star}(s)^\circ$ in bijective correspondence with the family $\tilde{\mathcal{F}}$ will then decompose as a corresponding product of families $\mathcal{F} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_r$. We choose unipotent characters $\bar{\psi}_i \in \mathcal{F}_i$ such that

$\bar{\psi}_i = |\mathcal{G}_{\tilde{\mathcal{F}}_i}|$ for each $1 \leqslant i \leqslant r$. We also make the extra assumption that if $\mathbf{W}_i^\star \cong \mathbf{W}_j^\star$ for some $i \neq j$ and $\rho_i = \rho_j$ then we take $\bar{\psi}_i = \bar{\psi}_j$. Take $\bar{\psi}$ to be the product $\bar{\psi}_1 \boxtimes \cdots \boxtimes \bar{\psi}_r$ then we claim the triple $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}), \bar{\psi})$ satisfies properties (P1) to (P3).

Note that we have used the same choices as Lusztig in [**Lus09**, §7.1] except for the unipotent classes labelled A_4, $(A_5)''$ and E_7 as here Lusztig did not choose the Weyl group of some quasi-isolated semisimple element.

$\mathcal{O}$	$\lvert A_{\tilde{\mathbf{G}}}(\tilde{u})\rvert$	$\lvert A_{\mathbf{G}}(u)\rvert$	$\mathbf{W}^\star(\tilde{s})$	ρ
1	1	1	E_7	$1_a'$
A_1	1	1	E_7	7_a
$2\,\mathrm{A}_1$	1	1	E_7	$27_a'$
$(3\,\mathrm{A}_1)''$	1	2	E_6	$1_p'$
$(3\,\mathrm{A}_1)'$	1	1	$\mathrm{A}_1\,\mathrm{D}_6$	$[12] \boxtimes \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$
A_2	2	2	E_7	56_a
$4\,\mathrm{A}_1$	1	2	A_7	$[12345678]$
$\mathrm{A}_2 + \mathrm{A}_1$	2	2	E_7	$120_a'$
$\mathrm{A}_2 + 2\,\mathrm{A}_1$	1	1	E_7	189_b
A_3	1	1	E_7	$210_a'$
$2\,\mathrm{A}_2$	1	1	E_7	$168_a'$
$\mathrm{A}_2 + 3\,\mathrm{A}_1$	1	2	A_7	$[1234568]$
$(\mathrm{A}_3 + \mathrm{A}_1)''$	1	2	E_6	$20_p'$
$2\,\mathrm{A}_2 + \mathrm{A}_1$	1	1	$\mathrm{A}_2\,\mathrm{A}_5$	$[123] \boxtimes [123456]$
$(\mathrm{A}_3 + \mathrm{A}_1)'$	1	1	$\mathrm{A}_1\,\mathrm{D}_6$	$[12] \boxtimes \begin{bmatrix} 0 & 1 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$
$\mathrm{D}_4(a_1)$	6	6	E_7	315_a
$\mathrm{A}_3 + 2\,\mathrm{A}_1$	1	2	A_7	$[123467]$
D_4	1	1	$\mathrm{A}_2\,\mathrm{A}_5$	$[3] \boxtimes [123456]$
$\mathrm{D}_4(a_1) + \mathrm{A}_1$	2	4	E_6	$30_p'$
$\mathrm{A}_3 + \mathrm{A}_2$	2	2	$\mathrm{A}_1\,\mathrm{D}_6$	$[12] \boxtimes \begin{bmatrix} 0 & 1 & 2 & 4 \\ 1 & 2 & 3 & 5 \end{bmatrix}$
A_4	2	2	$\mathrm{A}_1\,\mathrm{D}_6$	$[2] \boxtimes \begin{bmatrix} 0 & 1 & 2 & 4 \\ 1 & 2 & 3 & 5 \end{bmatrix}$

$\mathcal{O}$	$\lvert A_{\tilde{\mathbf{G}}}(\tilde{u})\rvert$	$\lvert A_{\mathbf{G}}(u)\rvert$	$\mathbf{W}^{\star}(\tilde{s})$	ρ
$A_3 + A_2 + A_1$	1	2	$A_3\,A_3\,A_1$	$[1234] \boxtimes [1234] \boxtimes [12]$
$(A_5)''$	1	2	$D_4\,A_1\,A_1$	$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \boxtimes [2] \boxtimes [2]$
$D_4 + A_1$	1	2	A_7	$[2345]$
$A_4 + A_1$	2	2	E_7	$512'_a$
$D_5(a_1)$	2	2	E_7	420_a
$A_4 + A_2$	1	1	E_7	210_b
$(A_5)'$	1	1	$A_1 \times D_6$	$[12] \boxtimes \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$
$A_5 + A_1$	1	2	$A_2\,A_2\,A_2$	$[123] \boxtimes [123] \boxtimes [123]$
$D_5(a_1) + A_1$	1	2	A_7	$[1346]$
$D_6(a_2)$	1	2	A_7	$[1256]$
$E_6(a_3)$	2	2	E_7	405_a
D_5	1	1	E_7	$189'_c$
$E_7(a_5)$	6	12	E_6	80_s
A_6	1	1	E_7	105_b
$D_5 + A_1$	1	2	A_7	$[236]$
$D_6(a_1)$	1	2	A_7	$[1238]$
$E_7(a_4)$	2	4	$D_4\,A_1\,A_1$	$\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \boxtimes [12] \boxtimes [12]$
D_6	1	2	A_7	$[4567]$
$E_6(a_1)$	2	2	E_7	120_a
E_6	1	1	E_7	$21'_b$
$E_7(a_3)$	2	4	E_6	30_p
$E_7(a_2)$	1	2	E_6	20_p
$E_7(a_1)$	1	2	E_6	6_p
E_7	1	2	A_7	$[8]$

Table 5.2: Proof of Proposition A for E_7.

In the table we have listed characters of $\mathbf{W}$ and $\mathbf{W}(E_6)$ using the labelling described in [**Lus84a**, §4.11] and characters of groups of classical type by their symbols. Before we can explain why properties (P1) to (P3) are satisfied we must first explain why $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}))$ is an F-stable pair. By the results in Chapter 4 there exists a semisimple element $\tilde{s} \in \tilde{\mathbf{T}}_0^\star$ which lies in an $F^\star$-stable conjugacy class such that $\mathbf{W}^\star(\tilde{s})$ is one of the parahoric subgroups listed in Table 5.2. We need only show that $\mathbf{W}^\star(\tilde{\mathcal{F}})$ is invariant under $F_{\tilde{s}}^\star$. If an irreducible component $\mathbf{W}_i^\star$ of $\mathbf{W}^\star(\tilde{s})$ is of type D then the corresponding character ρ_i is not degenerate and is not one of the characters listed in Eq. (1.4). Furthermore whenever $\mathbf{W}_i^\star \cong \mathbf{W}_j^\star$ for some $j \neq i$ we always have $\rho_i = \rho_j$. Therefore the special character is fixed under any graph automorphism hence $\mathbf{W}^\star(\tilde{\mathcal{F}})$ is invariant under $F_{\tilde{s}}^\star$ so we're done.

(P1) As in the case of E_6 we only need to verify information for irreducible constituents. If $\mathbf{W}^\star(\tilde{s})$ is of type E_6 or E_7 then the information about f_ρ can be read off Tables 2.5 and 2.6. In type A we always have $f_\rho = 1$ for all characters. Finally in types D_4 or D_6 we have $f_\rho = 1$ if ρ is the only character in its family and $f_\rho = 2$ if the character lies in a family containing three characters.

(P2) Reading from Tables 4.2 and 5.2 we see $|A_{\mathbf{G}^\star}(s)| = |Z_{\mathbf{G}}(u)|$. By Lemma 1.64 it is clear that the character $\bar{\psi}$ will be invariant under any graph automorphism induced by an element of $A_{\mathbf{G}^\star}(s)$, so we will have $\mathrm{Stab}_{A_{G^\star}(s)}(\bar{\psi}) = A_{G^\star}(s)$. Therefore by Lemma 4.6 we have Property (P2) holds.

(P3) If $\mathbf{W}^\star(\tilde{s})$ is of type E_7 then this is clear from the description of the Springer correspondence given in Table 2.6. In the remaining cases this can be verified using the j-induction tables, (which can be computed with [**GHL**$^+$**96**]), together with Table 2.6. The labelling of characters we choose here is different to that of [**GHL**$^+$**96**] so for convenience we give a dictionary to make checking this easier, (see Table C.2).

We now consider the validity of the condition ♣. If $|A_{\tilde{\mathbf{G}}}(\tilde{u})| = 1$ then $|\mathcal{F}| = 1$ so it is obvious that $|\mathfrak{X}_{\mathcal{F}}| = 0$. Assume $\mathcal{O}$ is such that $|A_{\tilde{\mathbf{G}}}(\tilde{u})| = |A_{\mathbf{G}}(u)|$ then $|\mathfrak{X}_{\mathcal{F}}| = 0$ because $C_{\mathbf{G}^\star}(s)$ is connected so clearly $|\,\mathrm{Stab}_{A_{G^\star}(s)}(\bar{\psi})| = 1$ for all $\bar{\psi} \in \mathcal{F}$. Finally we need now only consider when $\mathcal{O}$ is the class $\mathrm{D}_4(a_1) + \mathrm{A}_1$, $\mathrm{E}_7(a_3)$, $\mathrm{E}_7(a_4)$ or $\mathrm{E}_7(a_5)$. However using Lemma 1.64 we see all characters in the family $\mathcal{F}$ are invariant under all graph automorphisms so we're done.

Chapter 6

Proposition A: Type D_n

In this chapter we will prove Proposition A for the case where $\mathbf{G}$ is a non-adjoint simple group of type D_n ($n \geqslant 4$), which will complete our proof of Proposition A. Again we will use both the results obtained by Lusztig in **[Lus09]** and Hézard in **[Héz04]**. Let $\mathcal{O}$ be an F-stable unipotent class of $\mathbf{G}$. There are four separate cases to consider here, which arise from the numerical values $n_{\mathrm{D}}(u)$, $\kappa_{\mathrm{D}}(u)$ and $\delta_{\mathrm{D}}(u)$ introduced in Section 2.2.4. Throughout each subsection we assume the conditions of the heading.

Remark 6.1. The subgroup of $\mathbf{W}$ which we denote $\mathcal{A}$ is denoted Ω by Lusztig. We now describe how Lusztig's chosen generators for Ω can be identified with our chosen generators for $\mathcal{A}$. Lusztig fixes two generators ω_1 and ω_2 of Ω such that: if n is even ω_1 maps $i \mapsto (n+1-i)'$ and $i' \mapsto n+1-i$ for all $1 \leqslant i \leqslant n$, if n is odd then ω_1 maps $i \mapsto (n+1-i)'$ and $i' \mapsto n+1-i$ for all $1 \leqslant i \leqslant n-1$ and maps $n \mapsto 1$ and $n' \mapsto 1'$, finally ω_2 maps $i \mapsto i$ for all $2 \leqslant i \leqslant n-1$ and interchanges 1 with $1'$ and n with n'. From this description it is not difficult to check that ω_2 is identified with σ_1. Furthermore if $n \equiv 0 \pmod 2$ then ω_1 is identified with σ_{n-1} and if $n \equiv 1 \pmod 2$ then ω_1 is identified with σ_n. This now removes any difficulty in verifying our statements of Lusztig's results.

Let $\mu, \nu \in \mathbb{N}$ then if $\mu + \nu = n$ and $\mu, \nu \geqslant 2$ we denote by $\mathbf{W}(\mathrm{D}_\mu \, \mathrm{D}_\nu)$ the parahoric subgroup generated by the reflections $\mathbb{S}_0 \setminus \{s_\mu\}$. If $\mu = 1$ we denote by $\mathbf{W}(\mathrm{D}_\mu \, \mathrm{D}_\nu)$ the parahoric subgroup generated by the reflections $\mathbb{S}_0 \setminus \{s_0, s_1\}$, similarly if $\nu = 1$ we take the set $\mathbb{S}_0 \setminus \{s_{n-1}, s_n\}$. If $2\mu + \nu = n$ and $\mu \geqslant 2$ we denote by $\mathbf{W}(\mathrm{D}_\mu \, \mathrm{A}_\nu \, \mathrm{D}_\mu)$ the parahoric subgroup generated by the reflections $\mathbb{S}_0 \setminus \{s_\mu, s_{n-\mu}\}$. If $\mu = 1$ then we denote by $\mathbf{W}(\mathrm{D}_\mu \, \mathrm{A}_\nu \, \mathrm{D}_\mu)$ the parabolic subgroup generated by the reflections $\mathbb{S}_0 \setminus \{s_0, s_1, s_{n-1}, s_n\}$. Finally we denote by $\mathbf{W}(\mathrm{A}_{n-1}^+)$ the parabolic subgroup generated by the reflections $\mathbb{S}_0 \setminus \{s_0, s_n\}$ and by $\mathbf{W}(\mathrm{A}_{n-1}^-)$ the parabolic subgroup generated by the reflections $\mathbb{S}_0 \setminus \{s_0, s_{n-1}\}$, (this is in keeping with the notation introduced in Section 2.2.4).

6.1 The case $n_{\mathrm{D}}(u) \geqslant 1$, $\kappa_{\mathrm{D}}(u) \geqslant 2$ and $\delta_{\mathrm{D}}(u) = 0$

For this case we will use the following result of Hézard as it will provide an easier way to prove Property (P2).

Proposition 6.2 (Hézard, [Héz04, §4.2.4]). *Let $\tilde{\mathcal{O}}$ be an F-stable unipotent class of $\tilde{\mathbf{G}}$ such that $\delta_{\mathrm{D}}(\tilde{u}) = 0$ and*

$$\mathrm{Spr}_{\mathrm{D}}(\tilde{\mathcal{O}}) = \begin{bmatrix} a_1 & \cdots & a_m \\ b_1 & \cdots & b_m \end{bmatrix}$$

We construct a symbol $[\Lambda]$ by letting

$$\Lambda = \begin{pmatrix} a_1 & a_2 - 1 & \cdots & a_m - (m-1) \\ b_1 & b_2 - 1 & \cdots & b_m - (m-1) \end{pmatrix},$$

which parameterises a special irreducible character of $\mathbf{W}$. Let $\mathbf{W}(\tilde{\mathcal{F}}) \subseteq \mathrm{Irr}(\mathbf{W})$ be the unique family of irreducible characters determined by $[\Lambda]$ then $|\mathcal{G}_{\tilde{\mathcal{F}}}| = |A_{\tilde{\mathbf{G}}}(\tilde{u})|$ and the character $[\Lambda]$ is the Springer character $\rho_{\tilde{u},1}$.

Let $[\Lambda]$ be the special character prescribed by Proposition 6.2 and $\mathbf{W}^\star(\tilde{\mathcal{F}}) \subseteq \mathrm{Irr}(\mathbf{W}^\star)$ the corresponding family of characters. To show that $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}))$, where $\tilde{s}$ is the identity in $\tilde{\mathbf{T}}_0^\star$, is an F-stable pair we need only show that $[\Lambda]$ is not degenerate. Let $\lambda = (\lambda_1, \dots, \lambda_s)$ be the partition parameterising the unipotent class $\tilde{\mathcal{O}}$. Under the assumed conditions we have every number occurring in λ occurs an even number of times and at least one odd number occurs in λ. Hence there exists an entry λ_i, with i odd, such that $\lambda_i = \lambda_{i+1}$ but by the construction given in Section 2.2.4 this would mean the corresponding entries of the Springer symbol are such that $c_{i+1} = c_i + 1$. Therefore the symbol $[\Lambda]$ cannot be degenerate in this case.

Let $\bar{\psi} \in \mathcal{F}$ be such that $n_{\bar{\psi}} = |\mathcal{G}_{\mathcal{F}}|$. By Proposition 6.2 we clearly have Properties (P1) and (P3) hold. From Section 2.2.4 we see $|Z_{\mathbf{G}}(u)| = 1$ regardless of the isomorphism type of $\mathbf{G}$ and trivially $|\mathrm{Stab}_{A_{G^\star}(s)}(\bar{\psi})| = |Z_{\mathbf{G}}(u)^F| = 1$ because $C_{\mathbf{G}^\star}(s) = \mathbf{G}^\star$ is connected, which means Property (P2) holds. Clearly the stabiliser in $A_{G^\star}(s)$ of every character in $\mathcal{F}$ is trivial, which means ♣ holds.

6.2 The case $n_{\mathrm{D}}(u) \geqslant 1$, $\kappa_{\mathrm{D}}(u) \geqslant 2$ and $\delta_{\mathrm{D}}(u) = 1$

Proposition 6.3 (Lusztig, [Lus09, §6.14]). *Let $\tilde{\mathcal{O}}$ be a unipotent class with $n_{\mathrm{D}}(\tilde{u}) \geqslant 1$, $\kappa_{\mathrm{D}}(\tilde{u}) \geqslant 2$ and $\delta_{\mathrm{D}}(\tilde{u}) = 1$. There exists $\mu, \nu \in \mathbb{N}$ such that $\mu + \nu = n$ and a special character $[\Lambda_1] \boxtimes [\Lambda_2] \in \mathrm{Irr}(\mathbf{W}(\mathrm{D}_\mu \, \mathrm{D}_\nu))$, (which can be constructed inductively from $\mathrm{Spr}_{\mathrm{D}}(\tilde{\mathcal{O}})$), such that:*

- $f_{[\Lambda_1]\boxtimes[\Lambda_2]} = |A_{\tilde{\mathbf{G}}}(\tilde{\mathcal{O}})|$,

- $\mathrm{Stab}_{\mathcal{A}}([\Lambda_1] \boxtimes [\Lambda_2]) = \langle \sigma_1 \rangle$
- $j^{\mathbf{W}}_{\mathbf{W}(\mathrm{D}_\mu\,\mathrm{D}_\nu)}([\Lambda_1] \boxtimes [\Lambda_2])$ *is the Springer character* $\rho_{\tilde{u},1}$.

By the results in Chapter 4 there exists a semisimple element $\tilde{s} \in \tilde{\mathbf{T}}_0^\star$ such that $F^\star(\tilde{s}) = \tilde{s}$ or $F^\star(\tilde{s}) = \tilde{s}^{w_0}$, (see Corollary 4.11), and $\mathbf{W}(\tilde{s}) = \mathbf{W}(\mathrm{D}_\mu\,\mathrm{D}_\nu)$, (recall that in Section 2.3 we defined $\mathbf{W}(\tilde{s})$ to be the inverse image of $\mathbf{W}^\star(\tilde{s})$ under the anti-isomorphism $\mathbf{W} \to \mathbf{W}^\star$). Let $[\Lambda_1] \boxtimes [\Lambda_2]$ be the special character prescribed by Proposition 6.3 and $\mathbf{W}^\star(\tilde{\mathcal{F}}) \subseteq \mathrm{Irr}(\mathbf{W}^\star(\tilde{s}))$ the associated family of characters. We claim $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}))$ is an F-stable pair. If $F^\star(\tilde{s}) = \tilde{s}$ then $F^\star_{\tilde{s}} = F^\star$ so this is obvious and there is nothing to prove. If $\mu \neq \nu$ then the only possible graph automorphisms are the automorphisms of order 2 on the irreducible components. However clearly $[\Lambda_1]$ and $[\Lambda_2]$ are not degenerate because, by Proposition 6.3, they are invariant under the graph automorphism induced by σ_1. Therefore it is clear that the family $\mathbf{W}^\star(\tilde{\mathcal{F}})$ will be invariant under $F^\star_{\tilde{s}}$ so we need only consider the case where $\mu = \nu$ and $F^\star(\tilde{s}) = \tilde{s}^{w_0}$.

To prove this we consider the unique element $w_1 \in w_0\mathbf{W}^\star(\tilde{s})$ of minimal length in the right coset. We can write $w_1 = w_0 y$ for some $y \in \mathbf{W}^\star(\tilde{s})$ and w_1 has the property that it preserves the subset of S_0 generating the parahoric subgroup $\mathbf{W}^\star(\tilde{s})$. The element w_0 will act by sending every root to its negative, so y must do the same. However there is only one element of $\mathbf{W}^\star(\tilde{s})$ with such a property and that is the longest element. Hence the graph automorphism induced on $\mathbf{W}^\star(\tilde{s})$ by w_1 is a product of the graph automorphisms induced by the longest elements but these automorphisms won't permute the two irreducible constituents. Therefore we must have $\mathbf{W}^\star(\tilde{\mathcal{F}})$ is invariant under $F^\star_{\tilde{s}}$.

Let us recall that $\mathcal{F}$ denotes the family of unipotent characters of $C_{G^\star}(s)^\circ$ in bijective correspondence with the family $\tilde{\mathcal{F}}$. The family $\mathcal{F}$ will be a product $\mathcal{F}_1 \boxtimes \mathcal{F}_2$, where the families $\mathcal{F}_1$ and $\mathcal{F}_2$ are determined by the special characters $[\Lambda_1]$ and $[\Lambda_2]$. Let $\bar{\psi}_1 \in \mathcal{F}_1$ and $\bar{\psi}_2 \in \mathcal{F}_2$ be such that $n_{\bar{\psi}_1} = |\mathcal{G}_{\mathcal{F}_1}|$ and $n_{\bar{\psi}_2} = |\mathcal{G}_{\mathcal{F}_2}|$. We take $\bar{\psi}$ to be the product $\bar{\psi}_1 \boxtimes \bar{\psi}_2$ then we claim $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}), \bar{\psi})$ is a triple satisfying properties (P1) to (P3). Clearly Property (P1) holds because $n_{\bar{\psi}} = n_{\bar{\psi}_1} n_{\bar{\psi}_2} = |\mathcal{G}_{\mathcal{F}_1}||\mathcal{G}_{\mathcal{F}_2}| = |\mathcal{G}_{\tilde{\mathcal{F}}}|$ and $|\mathcal{G}_{\tilde{\mathcal{F}}}| = |A_{\tilde{\mathbf{G}}}(\tilde{u})|$ by Proposition 6.3. It is also clear that Property (P3) holds by Proposition 6.3.

We now concern ourselves with Property (P2), which we must consider for each isomorphism type. Assume first that $\mathbf{G}$ is either simply connected or isomorphic to a special orthogonal group then by the results in Section 2.2.4 we have $|Z_{\mathbf{G}}(u)| = 2$. If $\mathbf{G}$ is isomorphic to a special orthogonal group then $|A_{\mathbf{G}^\star}(s)| = |Z_{\mathbf{G}}(u)|$, (see Table 4.3). The non-trivial element of $A_{\mathbf{G}^\star}(s)$ induces the graph automorphism $\sigma_1 \in \mathcal{A}$, therefore $\mathrm{Stab}_{A_{G^\star}(s)}(\bar{\psi}) = A_{G^\star}(s)$ and Property (P2) follows from Lemma 4.6.

If $\mathbf{G}$ is simply connected then $Z(\mathbf{G})$ is not necessarily cylic, which means we cannot use Lemma 4.6 so we must argue directly. As $\mathrm{Stab}_{\mathcal{A}}([\Lambda_1] \boxtimes [\Lambda_2]) \neq \mathcal{A}$ we must have $[\Lambda_1] \neq [\Lambda_2]$ so $|\mathrm{Stab}_{A_{G^\star}(s)}(\bar{\psi})| \leqslant 2$ hence we need only show that $A_{\mathbf{G}^\star}(s)^{F^\star_{\tilde{s}}} = A_{\mathbf{G}^\star}(s)$ and $Z_{\mathbf{G}}(u)^F = Z_{\mathbf{G}}(u)$. If F acts trivially on $Z(\mathbf{G})$ then $F^\star$ acts trivially on $\mathrm{Ker}(\delta^\star_{\mathrm{sc}})$ so everything is fixed. Assume F acts non-trivially on the centre then it acts by exchanging $\hat{z}_{n-1}$, $\hat{z}_n$ hence we will have $Z(\mathbf{G})^F = \langle \hat{z}_1 \rangle = \mathrm{Ker}(\delta^\star_{\mathrm{sc}})^{F^\star}$, (where here we've identified $Z(\mathbf{G})$ with $Z(\mathbf{G}^\star_{\mathrm{ad}})$). It is clear that $A_{\mathbf{G}^\star}(s)$ is isomorphic to the subgroup $\langle \hat{z}_1 \rangle$ so $A_{\mathbf{G}^\star}(s)^{F^\star_{\tilde{s}}} = A_{\mathbf{G}^\star}(s)$. Recall from Section 2.2.4 that $Z_{\mathbf{G}}(u)$ is given by $C_{\mathbf{G}}(u)^\circ$ and $\hat{z}_{n-1}C_{\mathbf{G}}(u)^\circ = \hat{z}_n C_{\mathbf{G}}(u)^\circ$ so we also have $Z_{\mathbf{G}}(u)^F = Z_{\mathbf{G}}(u)$.

Assume now that $\mathbf{G}$ is isomorphic to a half spin group then by the results in Section 2.2.4 we have $|Z_{\mathbf{G}}(u)| = 1$. If $\mu \neq \nu$ then $|A_{\mathbf{G}^\star}(s)| = 1$ and the result is clear. If $\mu = \nu$ then $|A_{\mathbf{G}^\star}(s)| = 2$ however the graph automorphism induced by the non-trivial element of $A_{\mathbf{G}^\star}(s)$ exchanges the two components of type $\mathrm{D}_{n/2}$. As was mentioned above $[\Lambda_1] \neq [\Lambda_2]$ so $\mathrm{Stab}_{A_{G^\star}(s)}(\bar{\psi})$ is trivial.

As $[\Lambda_1]$, $[\Lambda_2]$ are not degenerate this is true of all characters in the families $\mathcal{F}_1$ and $\mathcal{F}_2$. In particular the stabiliser of any character in the family $\mathcal{F}$ has the same order as $\mathrm{Stab}_{A_{G^\star}(s)}(\bar{\psi})$ so condition ♣ holds.

6.3 The case $n_{\mathrm{D}}(u) \geqslant 1$, $\kappa_{\mathrm{D}}(u) = 1$

Note that in this case we necessarily have $\delta_{\mathrm{D}}(u) = 1$.

Proposition 6.4 (Lusztig, [Lus09, §6.11]). *Let $\tilde{\mathcal{O}}$ be a unipotent class with $n_{\mathrm{D}}(\tilde{\mathcal{O}}) \geqslant 1$, $\kappa_{\mathrm{D}}(\tilde{u}) = 1$ and $\delta_{\mathrm{D}}(\tilde{u}) = 1$. There exists $\mu \in \mathbb{N}$ and $\nu \in \mathbb{N}_0$ such that $2\mu + \nu = n$ and a special character $[\Lambda_1] \boxtimes [\Lambda_2] \boxtimes [\Lambda_1] \in \mathrm{Irr}(\mathbf{W}(\mathrm{D}_\mu\,\mathrm{A}_\nu\,\mathrm{D}_\mu))$, (which can be constructed inductively from $\mathrm{Spr}_{\mathrm{D}}(\tilde{\mathcal{O}})$), such that:*

- $f_{[\Lambda_1]\boxtimes[\Lambda_2]\boxtimes[\Lambda_1]} = |A_{\tilde{\mathbf{G}}}(\tilde{\mathcal{O}})|$,
- $\mathrm{Stab}_{\mathcal{A}}([\Lambda_1] \boxtimes [\Lambda_2] \boxtimes [\Lambda_1]) = \mathcal{A}$
- $j^{\mathbf{W}}_{\mathbf{W}(\mathrm{D}_\mu\,\mathrm{A}_\nu\,\mathrm{D}_\mu)}([\Lambda_1] \boxtimes [\Lambda_2] \boxtimes [\Lambda_1])$ *is the Springer character $\rho_{\tilde{u},1}$.*

By the results in Chapter 4 there exists a semisimple element $\tilde{s} \in \tilde{\mathbf{T}}^\star_0$ which lies in an $F^\star$-stable conjugacy class such that $\mathbf{W}^\star(\tilde{s})$ is identified with the $\mathbf{W}(\mathrm{D}_\mu\,\mathrm{A}_\nu\,\mathrm{D}_\mu)$ under the anti-isomorphism $\mathbf{W} \to \mathbf{W}^\star$. Let $[\Lambda_1] \boxtimes [\Lambda_2] \boxtimes [\Lambda_1]$ be the special character prescribed by Proposition 6.4 and $\mathbf{W}^\star(\tilde{\mathcal{F}}) \subseteq \mathrm{Irr}(\mathbf{W}^\star(\tilde{s}))$ its associated family of characters then the pair $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}))$ is F-stable. This is clear because we must have $[\Lambda_1] \boxtimes [\Lambda_2] \boxtimes [\Lambda_1]$ is invariant under all graph automorphisms by the statement regarding the stabiliser in Proposition 6.4 so $\mathbf{W}^\star(\tilde{\mathcal{F}})$ will be invariant under $F^\star_{\tilde{s}}$.

The family of unipotent characters $\mathcal{F}$ will be a product $\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_1$ where the families $\mathcal{F}_1$ and $\mathcal{F}_2$ are determined by the special characters $[\Lambda_1]$ and $[\Lambda_2]$. Let $\bar{\psi}_1 \in \mathcal{F}_1$ and $\bar{\psi}_2 \in \mathcal{F}_2$ be such that $n_{\bar{\psi}_1} = |\mathcal{G}_{\mathcal{F}_1}|$ and $n_{\bar{\psi}_2} = |\mathcal{G}_{\mathcal{F}_2}|$. We take $\bar{\psi}$ to be the product $\bar{\psi}_1 \boxtimes \bar{\psi}_2 \boxtimes \bar{\psi}_1$ and we claim that $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}), \bar{\psi})$ is a triple satisfying properties (P1) to (P3). Clearly Property (P1) holds because $n_{\bar{\psi}} = n_{\bar{\psi}_1} n_{\bar{\psi}_2} n_{\bar{\psi}_1} = |\mathcal{G}_{\mathcal{F}_1}||\mathcal{G}_{\mathcal{F}_2}||\mathcal{G}_{\mathcal{F}_1}| = |\mathcal{G}_{\tilde{\mathcal{F}}}|$ and $|\mathcal{G}_{\tilde{\mathcal{F}}}| = |A_{\tilde{\mathbf{G}}}(\tilde{u})|$ by Proposition 6.4. It is also clear that Property (P3) holds by Proposition 6.4.

We now concern ourselves with Property (P2), which we must consider for each isomorphism type. Assume $\mathbf{G}$ is simply connected then by the results in Section 2.2.4 we have $|Z_{\mathbf{G}}(u)| = 4$ and by the results in Chapter 4 we have $|A_{\mathbf{G}^\star}(s)| = |Z_{\mathbf{G}}(u)|$. In particular we have $Z_{\mathbf{G}}(u) \cong Z(\mathbf{G})$ and $A_{\mathbf{G}^\star}(s) \cong \mathrm{Ker}(\delta_{\mathrm{sc}}^\star)$ so Property (P2) holds by Corollary 4.5 and the fact that $\bar{\psi}$ is invariant under all graph automorphisms. Assume $\mathbf{G}$ is isomorphic to a special orthogonal or half spin group then $|Z_{\mathbf{G}}(u)| = 2$ and $|A_{\mathbf{G}^\star}(s)| = |Z_{\mathbf{G}}(u)|$. Again we have $Z_{\mathbf{G}}(u) \cong Z(\mathbf{G})$ and $A_{\mathbf{G}^\star}(s) \cong \mathrm{Ker}(\delta_{\mathrm{sc}}^\star)$ so Property (P2) holds by Corollary 4.5.

Every character $\bar{\psi} \in \mathcal{F}$ is parameterised by symbols $[\Lambda_1] \boxtimes [\Lambda_2] \boxtimes [\Lambda_3]$, where neither $[\Lambda_1]$ nor $[\Lambda_3]$ are degenerate and we know $|\mathrm{Stab}_{A_{G^\star}(s)}(\bar{\psi})| = |Z_{\mathbf{G}}(u)^F|$ if and only if $[\Lambda_1] = [\Lambda_3]$. Hence it is clear that $|\mathfrak{X}_{\mathcal{F}}| = |\mathcal{F}_1|^2 - |\mathcal{F}_1|$ so in particular if $|\mathcal{F}_1| \neq 1$ then condition ♣ does not hold. However we note that here $A_{\mathbf{G}}(u)$ is not abelian.

6.4 The case $n_{\mathrm{D}}(u) = 0$

Recall that if $n_{\mathrm{D}}(u) = 0$ then $n \equiv 0 \pmod 2$ and $\mathcal{O}$ is necessarily a degenerate unipotent class. By Lemma 2.40 if we express F as a composition $F_r \circ \tau$ then we must have τ is a trivial graph automorphism. In particular we may assume that F is the Frobenius endomorphism F_q.

Proposition 6.5 (Lusztig, [Lus09, §6.13]). *Let $\tilde{\mathcal{O}}$ be a degenerate unipotent class. There exists a parabolic subgroup $\mathbf{W}'$, which is either $\mathbf{W}(\mathrm{A}_{n-1}^+)$ or $\mathbf{W}(\mathrm{A}_{n-1}^-)$, and a special irreducible character $[\Lambda] \in \mathrm{Irr}(\mathbf{W}')$, (which can be constructed inductively from $\mathrm{Spr}_{\mathrm{D}}(\tilde{\mathcal{O}})$), such that:*

- $f_{[\Lambda]} = |A_{\tilde{\mathbf{G}}}(\tilde{u})|$,
- $\mathrm{Stab}_{\mathcal{A}}([\Lambda]) = \begin{cases} \langle \sigma_n \rangle & \text{if } [\Lambda] \in \mathrm{Irr}(\mathbf{W}(\mathrm{A}_{n-1}^+)), \\ \langle \sigma_{n-1} \rangle & \text{if } [\Lambda] \in \mathrm{Irr}(\mathbf{W}(\mathrm{A}_{n-1}^-)), \end{cases}$
- $j_{\mathbf{W}'}^{\mathbf{W}}([\Lambda])$ *is the Springer character* $\rho_{\tilde{u},1}$.

Remark 6.6. In **[Lus09**, §6.3**]** Lusztig describes two ways to construct a parabolic subgroup of $\mathbf{W}$ which is of type A_{n-1} and denotes these by $W_0' \times S_n^{(\lambda)} \times W_0'$ with $\lambda = 0$ or 3. It is easily checked that the parabolic subgroup with $\lambda = 0$ corresponds to

$\mathbf{W}(\mathrm{A}_{n-1}^{+})$ and the parabolic subgroup with $\lambda = 3$ corresponds to $\mathbf{W}(\mathrm{A}_{n-1}^{-})$. Unless $\mathbf{G}$ is isomorphic to a half spin group the choice of parabolic subgroup is immaterial to our result. However when $\mathbf{G}$ is isomorphic to a half-spin group this choice is important and we will explain below that it can be made concrete by the transitivity of j-induction.

By the results in Chapter 4 there exists a semisimple element $\tilde{s} \in \tilde{\mathbf{T}}_0^{\star}$ which lies in an $F^{\star}$-stable conjugacy class such that $\mathbf{W}(\tilde{s}) = \mathbf{W}(\mathrm{A}_{n-1}^{\pm})$. Let $[\Lambda]$ be the special character prescribed by Proposition 6.5 and let $\mathbf{W}^{\star}(\tilde{\mathcal{F}}) \subseteq \mathrm{Irr}(\mathbf{W}^{\star}(\tilde{s}))$ be its associated family of characters then the pair $(\tilde{s}, \mathbf{W}^{\star}(\tilde{\mathcal{F}}))$ is F-stable. This is clear because $\mathbf{W}^{\star}(\tilde{\mathcal{F}})$ contains only the character $[\Lambda]$ and any irreducible character of a Weyl group of type A is invariant under all graph automorphisms.

The corresponding family $\mathcal{F}$ will contain only one irreducible character $\bar{\psi}$ and it necessarily satisfies $n_{\bar{\psi}} = |\mathcal{G}_{\tilde{\mathcal{F}}}|$. We claim the triple $(\tilde{s}, \mathbf{W}^{\star}(\tilde{\mathcal{F}}), \bar{\psi})$ satisfies properties (P1) to (P3). Clearly Property (P1) holds because $n_{\bar{\psi}} = |\mathcal{G}_{\tilde{\mathcal{F}}}|$ and $|\mathcal{G}_{\tilde{\mathcal{F}}}| = |A_{\tilde{\mathbf{G}}}(\tilde{u})|$ by Proposition 6.5. It is also clear that Property (P3) holds by Proposition 6.5.

We now concern ourselves with Property (P2), which we must consider for each isomorphism type. Assume $\mathbf{G}$ is simply connected then by the results in Section 2.2.4 we have $|Z_{\mathbf{G}}(u)| = 2$ and by the results in Chapter 4 we have $|A_{\mathbf{G}^{\star}}(s)| = 2$, so we at least have $|A_{\mathbf{G}^{\star}}(s)| = |Z_{\mathbf{G}}(u)|$. The Frobenius endomorphism F_q cannot exchange the elements $\hat{z}_{n-1}$ and $\hat{z}_n$ because these are both elements of order 2 and q is odd, it then follows that $Z(\mathbf{G})^F = Z(\mathbf{G})$ and $\mathrm{Ker}(\delta_{\mathrm{sc}}^{\star})^{F^{\star}} = \mathrm{Ker}(\delta_{\mathrm{sc}}^{\star})$. Hence $|A_{\mathbf{G}^{\star}}(s)^{F_s^{\star}}| = |Z_{\mathbf{G}}(u)^F|$ and by Proposition 6.5 we have $\mathrm{Stab}_{A_{G^{\star}}(s)}(\bar{\psi}) = A_{G^{\star}}(s)$ so Property (P2) holds. If $\mathbf{G}$ is isomorphic to a special orthogonal group then $|Z_{\mathbf{G}}(u)| = |A_{\mathbf{G}^{\star}}(s)| = 1$ so Property (P2) clearly holds.

Finally we deal with the case where $\mathbf{G}$ is isomorphic to a half spin group. We will freely use the notation introduced in Section 2.2.4 with regards to degenerate unipotent classes. We must first clarify the choice over the parabolic subgroup in Proposition 6.5. The subgroup $\mathbf{W}(\mathrm{A}_{\eta^*}^{\pm})$ is a parabolic subgroup of $\mathbf{W}(\mathrm{A}_{n-1}^{\pm})$ so by the transitivity of j-induction we have

$$j_{\mathbf{W}(\mathrm{A}_{\eta^*}^{\pm})}^{\mathbf{W}}(\mathrm{sgn}_{\mathbf{W}(\mathrm{A}_{\eta^*}^{\pm})}) = j_{\mathbf{W}(\mathrm{A}_{n-1}^{\pm})}^{\mathbf{W}}(j_{\mathbf{W}(\mathrm{A}_{\eta^*}^{\pm})}^{\mathbf{W}(\mathrm{A}_{n-1}^{\pm})}(\mathrm{sgn}_{\mathbf{W}(\mathrm{A}_{\eta^*}^{\pm})})).$$

Let $\mathbf{W}(\mathrm{A}_{\eta^*}')$ denote either $\mathbf{W}(\mathrm{A}_{\eta^*}^{+})$ or $\mathbf{W}(\mathrm{A}_{\eta^*}^{-})$. By this remark on the transitivity of j-induction we must have the parabolic subgroup $\mathbf{W}'$ in Section 2.2.4 contains the parabolic subgroup $\mathbf{W}(\mathrm{A}_{\eta^*}')$ for which $j_{\mathbf{W}(\mathrm{A}_{\eta^*}')}^{\mathbf{W}}(\mathrm{sgn}_{\mathbf{W}(\mathrm{A}_{\eta^*}')})$ is the Springer character. As the Springer correspondence depends upon n we consider this in two cases.

Notation. We denote by $\mathbf{W}^{\star}(\mathrm{A}_{n-1}^{+})$ the parabolic subgroup of $\mathbf{W}^{\star}$ generated by the reflections $\mathbb{S}^\star \setminus \{s_n^\star\}$ and by $\mathbf{W}^\star(\mathrm{A}_{n-1}^-)$ the parabolic subgroup of $\mathbf{W}^\star$ generated by the reflections $\mathbb{S}^\star \setminus \{s_{n-1}^\star\}$. Similarly we denote by $\mathbf{W}^\star(\mathrm{A}_{\eta*}^\pm)$ the appropriate parabolic subgroup of $\mathbf{W}^\star(\mathrm{A}_{n-1}^\pm)$.

Assume $n \equiv 0 \pmod 4$ then the Springer character $\rho_{\tilde{u},1}$ is the induced character $j_{\mathbf{W}(\mathrm{A}_{n-1}^\pm)}^{\mathbf{W}}(\mathrm{sgn}_{\mathbf{W}(\mathrm{A}_{n-1}^\pm)})$. Under the duality described in Section 1.2.1 we see that the simple reflection s_i is sent to the simple reflection $s_i^\star$, for all $1 \leqslant i \leqslant n$, under the isomorphism $\mathbf{W} \to \mathbf{W}^\star$. Hence under this isomorphism the subgroup $\mathbf{W}(\mathrm{A}_{n-1}^\pm)$ is sent to the subgroup $\mathbf{W}^\star(\mathrm{A}_{n-1}^\pm)$. In particular the element $\tilde{s} \in \tilde{\mathbf{T}}_0^\star$ is such that $\mathbf{W}^\star(\tilde{s}) = \mathbf{W}^\star(\mathrm{A}_{n-1}^\pm)$. From the information in Section 2.2.4 and table 4.3 we see that we always have $|A_{\mathbf{G}^\star}(s)| = |Z_{\mathbf{G}}(u)|$, therefore Property (P2) holds as $\mathrm{Stab}_{A_{G^\star}(s)}(\bar{\psi}) = A_{G^\star}(s)$ and the Frobenius acts trivially on $A_{\mathbf{G}^\star}(s)$ and $Z_{\mathbf{G}}(u)$.

Assume $n \equiv 2 \pmod 4$ then the Springer character $\rho_{\tilde{u}^\pm,1}$ is the induced character $j_{\mathbf{W}(\mathrm{A}_{\eta*}^\mp)}^{\mathbf{W}}(\mathrm{sgn}_{\mathbf{W}(\mathrm{A}_{\eta*}^\mp)})$. Under the duality described in Section 1.2.1 we see that the simple reflection s_i is sent to the simple reflection $s_i^\star$, for all $1 \leqslant i \leqslant n-2$, under the isomorphism $\mathbf{W} \to \mathbf{W}^\star$. However s_{n-1} is sent to $s_n^\star$ and s_n is sent to $s_{n-1}^\star$, hence under this isomorphism the subgroup $\mathbf{W}(\mathrm{A}_{n-1}^\mp)$ is sent to the subgroup $\mathbf{W}^\star(\mathrm{A}_{n-1}^\pm)$. In particular the element $\tilde{s} \in \tilde{\mathbf{T}}_0^\star$ is such that $\mathbf{W}^\star(\tilde{s}) = \mathbf{W}^\star(\mathrm{A}_{n-1}^\pm)$. From the information in Section 2.2.4 and table 4.3 we see that we always have $|A_{\mathbf{G}^\star}(s)| = |Z_{\mathbf{G}}(u)|$, therefore Property (P2) holds as $\mathrm{Stab}_{A_{G^\star}(s)}(\bar{\psi}) = A_{G^\star}(s)$ and the Frobenius acts trivially on $A_{\mathbf{G}^\star}(s)$ and $Z_{\mathbf{G}}(u)$.

As $|\mathcal{F}| = 1$ it is obvious that $|\mathfrak{X}_{\mathcal{F}}| = 0$ so ♣ holds.

Chapter 7

Kawanaka's Conjecture

Remark 7.1. In this chapter and the next we will constantly use the triple in Proposition A. For checking Proposition A it was convenient for us to always have our semisimple elements lie in our fixed maximal torus. This came at the expense of us having to work with the generalised Frobenius endomorphism $F^{\star}_{\tilde{s}}$. From this point forward we will assume that the semisimple elements lie in $\tilde{G}^{\star}$, which we may do by replacing $\tilde{s}$ by a geometric conjugate as in the proof of Theorem A.

7.1 Generalised Gelfand–Graev Representations

Let $\mathbf{G}$ be any connected reductive algebraic group, p a good prime for $\mathbf{G}$ and F a Frobenius endomorphism of $\mathbf{G}$. If $u \in G$ is a unipotent element of the finite reductive group $\mathbf{G}^F$ then Kawanaka has associated to u a so called generalised Gelfand–Graev representation, (or GGGR for short), which we denote Γ_u. This is a representation of G whose construction depends only on the G-conjugacy class of u. These representations are such that Γ_1 is the regular representation, (where $1 \in G$ is the identity), and Γ_u is a Gelfand–Graev representation if u is a regular unipotent element.

The general definition of GGGRs was first given by Kawanaka in [**Kaw86**, §3.1] but for the purposes of this discussion we will follow the equivalent definition given by Geck in [**Gec04**, §2]. Note that in [**Gec04**] after writing F as a composition $F_r \circ \tau$ it is assumed that the automorphism τ is trivial but it is clear that this condition is unnecessary for the definition of a GGGR.

Let $\mathcal{O} \in \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})^F$ be an F-stable unipotent class then, by Lemma 2.40, the corresponding weighted Dynkin diagram $\Delta(\mathcal{O})$ is stable under the action of F on the roots of $\mathbf{G}$. Recall that in Appendix A we associated to $\Delta(\mathcal{O})$ an additive function $\eta : \Phi \to \mathbb{Z}$.

For each $i \in \mathbb{N}$ we define a variety

$$\mathbf{U}_{\eta,i} = \prod_{\substack{\alpha\in\Phi^+ \\ \eta(\alpha)\geqslant i}} \mathbf{X}_\alpha \subseteq \mathbf{U}_0,$$

then the canonical parabolic subgroup, (as defined in Appendix A.1), associated to $\mathcal{O}$ has a Levi decompositon $\mathbf{P}_\eta = \mathbf{U}_{\eta,1}\mathbf{L}_\eta$. By [**Kaw86**, Theorem 2.1.1(i)] the class $\mathcal{O}$ is the unique F-stable unipotent class such that $\mathcal{O} \cap \mathbf{U}_{\eta,2}$ is dense in $\mathbf{U}_{\eta,2}$. Furthermore $\mathcal{O} \cap \mathbf{U}_{\eta,2}$ is a single $\mathbf{P}_\eta$-conjugacy class which is invariant under right multiplication by elements from $\mathbf{U}_{\eta,3}$.

Let $\mathfrak{g}$ be the corresponding Lie algebra of $\mathbf{G}$ defined over $\mathbb{K}$, then $\mathfrak{g}$ also has an $\mathbb{F}_q$-rational structure with an associated Frobenius map $F : \mathfrak{g} \to \mathfrak{g}$, (for example this is the differential of the Frobenius map on $\mathbf{G}$). Recall that $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{t}_0 \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ and for each $\alpha \in \Phi$ we fix an element $e_\alpha \in \mathfrak{g}_\alpha$ such that $\mathfrak{g}_\alpha = \mathbb{K}e_\alpha$, (note that the maximal toral subalgebra $\mathfrak{t}_0$ is such that $F(\mathfrak{t}_0) = \mathfrak{t}_0$). We write $x \mapsto x^*$ for an opposition $\mathbb{F}_q$-automorphism of $\mathfrak{g}$, i.e. this is an automorphism such that $\mathfrak{t}_0^* = \mathfrak{t}_0$ and $e_\alpha^* \in \mathbb{F}_q e_{-\alpha}$ for all $\alpha \in \Phi$, (such an automorphism exists by [**Car72**, §4.2 - pg. 56]). We define $c_\alpha := \kappa(e_\alpha^*, e_\alpha)$, where $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ is a non-degenerate $\mathbf{G}$-invariant associative bilinear form, (for example the Killing form).

Definition 7.2. We fix a non-trivial additive character $\chi_1 : \mathbb{F}_p^+ \to \overline{\mathbb{Q}}_\ell^\times$, where here $\mathbb{F}_p$ is viewed as a group under addition. If r is a power of p then we denote by $\chi_r : \mathbb{F}_r^+ \to \overline{\mathbb{Q}}_\ell^\times$ the non-trivial character $\chi_1 \circ \mathrm{Tr}_{\mathbb{F}_r/\mathbb{F}_p}$, (where $\mathrm{Tr}_{\mathbb{F}_r/\mathbb{F}_p} : \mathbb{F}_r \to \mathbb{F}_p$ is the field trace). Consider a unipotent element $u \in \mathcal{O} \cap U_{\eta,2}$ then write

$$u \in \Big(\prod_{\substack{\alpha\in\Phi^+ \\ \eta(\alpha)=2}} x_\alpha(r_\alpha) \Big) \cdot U_{\eta,3} \text{ where } r_\alpha \in \mathbb{F}_q.$$

Using this notation we define a map $\varphi_u : U_{\eta,2} \to \overline{\mathbb{Q}}_\ell^\times$ by

$$\varphi_u\Big(\prod_{\substack{\alpha\in\Phi^+ \\ \eta(\alpha)\geqslant 2}} x_\alpha(v_\alpha) \Big) = \chi_q\Big(\sum_{\substack{\alpha\in\Phi^+ \\ \eta(\alpha)=2}} c_\alpha r_\alpha v_\alpha \Big) \text{ where } v_\alpha \in \mathbb{F}_q.$$

The map φ_u is a group homomorphism so a linear character of $U_{\eta,2}$. By [**Kaw86**, Lemma 3.1.12] we have

$$\gamma_u = [U_{\eta,1} : U_{\eta,2}]^{-1/2} \operatorname{Ind}_{U_{\eta,2}}^G(\varphi_u).$$

is a character of G. The *generalised Gelfand–Graev representation* Γ_u associated to u is the representation of G admitting γ_u as its character.

Remark 7.3. By [**Kaw86**, Lemma 3.1.12] we know $[U_{\eta,1} : U_{\eta,2}]$ is an even power of q

so the square root exists. By [**Gec04**, Remark 2.2] if $u_1, u_2 \in \mathcal{O} \cap U_{\eta,2}$ are G-conjugate then $\gamma_{u_1} = \gamma_{u_2}$. By the same remark the intersection $\mathcal{O} \cap U_{\eta,2}$ contains a representative for each G-conjugacy class contained in $\mathcal{O}^F$. In particular we may assume that a well chosen representative of $\mathcal{O}$ lies in $U_{\eta,2}$ as any G-conjugate of a well-chosen representative is well chosen.

If **G** is a connected reductive algebraic group with a disconnected centre then we will want to relate characters of GGGRs of $\tilde{G}$ with characters of GGGRs of G. We start by making the following obvious observation.

Lemma 7.4. *Assume* **G** *has a disconnected centre. Let $\tilde{\mathcal{O}}$ be a $\tilde{G}$-conjugacy class of unipotent elements and let $\mathcal{O}_i$, for $1 \leqslant i \leqslant d$, be the G-conjugacy classes such that $\tilde{\mathcal{O}} \cap G = \mathcal{O}_1 \sqcup \cdots \sqcup \mathcal{O}_d$. Let $u_i \in \mathcal{O}_i$ be class representatives then $\tilde{\gamma}_{u_i} = \mathrm{Ind}_G^{\tilde{G}}(\gamma_{u_i})$ for all $1 \leqslant i \leqslant d$, where $\tilde{\gamma}_{u_i}$ is the character of the GGGR of $\tilde{G}$ associated to $u_i \in \tilde{\mathcal{O}}$.*

Proof. This is clear from the definition of the character of a GGGR as $U_{\eta,2}$ is a subgroup of $\tilde{G}$ and induction is transitive. ■

If $u \in \tilde{G}$ is a unipotent element then u is also a unipotent element of G^F. We will denote the character of $\tilde{G}$ admitted by the GGGR associated to u by $\tilde{\gamma}_u$ and the character of G admitted by the GGGR associated to u by γ_u. We will need another observation regarding the natural conjugation action of $\tilde{G}$ on a GGGR. The group $\tilde{G}$ is the product $G \cdot \tilde{T}_0$, therefore we may assume that any left transversal of G in $\tilde{G}$ is contained in $\tilde{T}_0$. In particular, if $\psi \in \mathrm{Cent}(G)$ is a class function of G then for any $g \in \tilde{G}$ there exists $t \in \tilde{T}_0$ such that $\psi^g = \psi^t$.

Proposition 7.5 (Geck, [Gec93, Proposition 2.2]). *For any unipotent element $u \in U_0$ and any $t \in \tilde{T}_0$ we have $\gamma_u^t = \gamma_{tut^{-1}}$.*

Proof. Let us adopt the notation of Definition 7.2. Recall that $x_\alpha(r_\alpha)$ is a unipotent element in $\tilde{G}$ and the action of t by conjugation on this element is given by $tx_\alpha(r_\alpha)t^{-1} = x_\alpha(\alpha(t)r_\alpha)$. It is clear that

$$tut^{-1} \in \Big(\prod_{\substack{\alpha\in\Phi^+ \\ \eta(\alpha)=2}} x_\alpha(\alpha(t)r_\alpha) \Big) \cdot U_{\eta,3}.$$

Inspecting the map $\varphi_{tut^{-1}}$ we see for all $v_\alpha \in \mathbb{F}_q$ that

$$\begin{aligned} \varphi_{tut^{-1}}\Big(\prod_{\substack{\alpha\in\Phi^+ \\ \eta(\alpha)\geqslant 2}} x_\alpha(v_\alpha) \Big) &= \chi_q\Big(\sum_{\substack{\alpha\in\Phi^+ \\ \eta(\alpha)=2}} c_\alpha(\alpha(t)r_\alpha)v_\alpha \Big), \\ &= \chi_q\Big(\sum_{\substack{\alpha\in\Phi^+ \\ \eta(\alpha)=2}} c_\alpha r_\alpha(\alpha(t)v_\alpha) \Big), \end{aligned}$$

$$= \varphi_u\Big(\prod_{\substack{\alpha\in\Phi^+\\ \eta(\alpha)\geqslant 2}} x_\alpha(\alpha(t)v_\alpha)\Big),$$
$$= \varphi_u^t\Big(\prod_{\substack{\alpha\in\Phi^+\\ \eta(\alpha)\geqslant 2}} x_\alpha(v_\alpha)\Big).$$

Using the definition of an induced character and that t normalises G we see

$$\mathrm{Ind}_{U_{\eta,2}}^G(\varphi_{tut^{-1}}) = \mathrm{Ind}_{U_{\eta,2}}^G(\varphi_u^t) = \mathrm{Ind}_{U_{\eta,2}}^G(\varphi_u)^t.$$

By the definition of GGGRs we now have $\gamma_{tut^{-1}} = \gamma_u^t$ as required. ■

The focus of this chapter is the following conjecture that was made by Kawanaka concerning the GGGRs associated to a finite reductive group. We will denote the unipotent variety of $\mathbf{G}$ by $\mathcal{U}$, in other words $\mathcal{U}$ is the subvariety of $\mathbf{G}$ containing all unipotent elements of $\mathbf{G}$.

Definition 7.6. Let $\mathbf{G}$ be a connected reductive algebraic group and F a generalised Frobenius endomorphism of $\mathbf{G}$. We say a complex class function ψ of G is *unipotently supported* if $\psi(x) \neq 0$ implies $x \in \mathcal{U}^F$. In other words, the support of ψ as a function is contained in $\mathcal{U}^F$. We define $\mathrm{Cent}_{\mathrm{U}}(\mathbf{G})$ to be the subspace of $\mathrm{Cent}(\mathbf{G})$ consisting of all unipotently supported class functions.

Conjecture 7.7 (Kawanaka, [Kaw85, 3.3.1]). *Assume* $\mathbf{G}$ *is a connected reductive algebraic group and* p *is a good prime for* $\mathbf{G}$. *Let* $u_1, \dots, u_r$ *be representatives for the unipotent conjugacy classes of* G *then the set* $\{\gamma_{u_i} \mid 1 \leqslant i \leqslant r\}$ *forms a* $\mathbb{Z}$*-basis for the* $\mathbb{Z}$*-module of all unipotently supported virtual characters of* G.

Assuming $\mathbf{G}$ has a connected centre and p, q are large enough then this was proved by Geck and Hézard in [**GH08**, Theorem 4.5]. Here p, q large enough means that the results of [**Lus92**] are true, (hence also the results of [**GH08**]). In particular by [**Lus92**, 1.3] this means p is large enough that the maps $\exp : \mathcal{N} \to \mathcal{U}$ and $\log : \mathcal{U} \to \mathcal{N}$ define inverse bijections between the unipotent variety of $\mathbf{G}$ and the nilpotent variety of $\mathfrak{g}$. It would be sufficient in all cases to assume $p > 3(h-1)$, where h is the Coxeter number of $\mathbf{G}$. The bound on q comes from [**Lus90**, Theorem 1.14], which is a fixed constant depending only on the root system of $\mathbf{G}$.

We assume from now until the end of this chapter that p, q are large enough in the sense just defined.

Our aim in this chapter is to show that Kawanaka's conjecture is true for certain simple groups with a disconnected centre. To do this we will follow the same line of argument as in [**GH08**, §4]. In particular the focus will be on the following observation.

Lemma 7.8 (Geck & Hézard, [GH08, Lemma 4.2]). *Let $u_1, \dots, u_d$ be representatives for the G-conjugacy classes of unipotent elements. Assume that there exist virtual characters $\chi_1, \dots, \chi_d \in \mathbb{Z}\,\mathrm{Irr}(G)$ such that the matrix of scalar products $(\langle \chi_i, \gamma_{u_j} \rangle_G)_{1 \leqslant i,j \leqslant d}$ is invertible over $\mathbb{Z}$ then Conjecture 7.7 holds.*

7.2 The Case of Abelian Component Groups

In this section we will prove a result which is crucial in dealing with almost all cases. However before we can do this we must gather together some results regarding Alvis–Curtis duality.

Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{G}$ such that $\mathbf{P}$ is a parabolic subgroup containing $\mathbf{L}$ and $\mathbf{P} = \mathbf{U_L} \rtimes \mathbf{L}$, where $\mathbf{U_L}$ is the unipotent radical of the parabolic. Furthermore assume that $\mathbf{P}$ and $\mathbf{L}$ are F-stable, (hence $\mathbf{U_L}$ is also F-stable). Let $\phi \in \mathrm{Cent}(L)$ be a class function of the Levi subgroup then define $R_{\mathbf{L}}^{\mathbf{G}} : \mathrm{Cent}(L) \to \mathrm{Cent}(G)$ to be the linear map $R_{\mathbf{L}}^{\mathbf{G}}(\phi) = (\mathrm{Ind}_P^G \circ \mathrm{Inf}_L^P)(\phi)$. The map $R_{\mathbf{L}}^{\mathbf{G}}$ is a *Harish-Chandra induction map* for the Levi subgroup $\mathbf{L}$ and is a generalisation of the construction given in Proposition 1.25. There is also a corresponding *Harish-Chandra restriction map* denoted ${}^*R_{\mathbf{L}}^{\mathbf{G}} : \mathrm{Cent}(G) \to \mathrm{Cent}(L)$. These are obtained from a pair of adjoint functors between the $\overline{\mathbb{Q}}_\ell$-module categories of G and L, in particular they satisfy Frobenius reciprocity. It is a consequence of the Mackey formula, (as in Section 1.4), that these maps depend only upon the choice of $\mathbf{L}$ and not on the choice of $\mathbf{P}$, (see for example [**DM91**, Proposition 6.1]).

Definition 7.9 ([DM91, Definition 8.8]). Assume $\mathbf{G}$ is a connected reductive algebraic group and F is a generalised Frobenius endomorphism of $\mathbf{G}$. We define a map $D_{\mathbf{G}} : \mathrm{Cent}(G) \to \mathrm{Cent}(G)$ to be

$$D_{\mathbf{G}} = \sum_{\mathbf{P} \geqslant \mathbf{B}_0} (-1)^{\mathrm{rk}(\mathbf{P})} R_{\mathbf{L}}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}}^{\mathbf{G}}$$

where the summation is over the set of all F-stable parabolic subgroups of $\mathbf{G}$ containing $\mathbf{B}_0$ and $\mathbf{L}$ is any Levi subgroup of $\mathbf{P}$. Note $\mathrm{rk}(\mathbf{P})$ denotes the semisimple $\mathbb{F}_q$-rank of $\mathbf{P}$, (for a definition see [**DM91**, Definition 8.6]). We call $D_{\mathbf{G}}$ the *Alvis–Curtis duality map*.

From [**DM91**, Corollary 8.14 and Proposition 8.10] we know $D_{\mathbf{G}}$ is an isometry of $\mathrm{Cent}(G)$ and $D_{\mathbf{G}} \circ D_{\mathbf{G}}$ is the identity. Hence, for any irreducible character $\chi \in \mathrm{Irr}(G)$ there exists a sign $\varepsilon_\chi \in \{1, -1\}$ such that $\varepsilon_\chi D_{\mathbf{G}}(\chi) \in \mathrm{Irr}(G)$. For any $\chi \in \mathrm{Irr}(G)$ we denote by χ^* the irreducible character $\varepsilon_\chi D_{\mathbf{G}}(\chi)$.

Lemma 7.10. *Let* $\mathbf{G}$ *be a connected reductive algebraic group with a disconnected centre and* F *a generalised Frobenius endomorphism of* $\mathbf{G}$*. Then*

$$\mathrm{Res}_G^{\tilde{G}} \circ D_{\tilde{\mathbf{G}}} = D_{\mathbf{G}} \circ \mathrm{Res}_G^{\tilde{G}}.$$

Proof. By [**DM91**, Proposition 2.2] if $\tilde{\mathbf{H}}$ is a Borel, parabolic or Levi subgroup of $\tilde{\mathbf{G}}$ then $\mathbf{H} := \tilde{\mathbf{H}} \cap \mathbf{G}$ is also such a subgroup of $\mathbf{G}$ and the map $\tilde{\mathbf{H}} \to \mathbf{H}$ gives a bijection between the sets of such subgroups. Identifying these groups in this way and using the statements in [**Bon06**, Proposition 10.10] we have

$$\begin{aligned}
\mathrm{Res}_G^{\tilde{G}} \circ D_{\tilde{\mathbf{G}}} &= \sum_{\tilde{\mathbf{P}} \geqslant \tilde{\mathbf{B}}} (-1)^{\mathrm{rk}(\tilde{\mathbf{P}})} (\mathrm{Res}_G^{\tilde{G}} \circ R_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}) \circ {}^*R_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}, \\
&= \sum_{\tilde{\mathbf{P}} \geqslant \tilde{\mathbf{B}}} (-1)^{\mathrm{rk}(\tilde{\mathbf{P}})} R_{\mathbf{L}}^{\mathbf{G}} \circ (\mathrm{Res}_L^{\tilde{L}} \circ {}^*R_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}), \\
&= \sum_{\mathbf{P} \geqslant \mathbf{B}} (-1)^{r(\mathbf{P})} (R_{\mathbf{L}}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}}^{\mathbf{G}}) \circ \mathrm{Res}_G^{\tilde{G}}, \\
&= D_{\mathbf{G}} \circ \mathrm{Res}_G^{\tilde{G}}.
\end{aligned}$$

To obtain the third equality we have used the fact that $\mathbf{P}$ and $\tilde{\mathbf{P}}$ will have isomorphic derived subgroups, which implies $\mathrm{rk}(\mathbf{P}) = \mathrm{rk}(\tilde{\mathbf{P}})$. ■

Corollary 7.11. *Let* $\tilde{\chi} \in \mathrm{Irr}(\tilde{G})$ *and* $\chi_i \in \mathrm{Irr}(G)$*, for* $1 \leqslant i \leqslant k$*, be such that* $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}) = \chi_1 + \cdots + \chi_k$*. Then* $\varepsilon_{\tilde{\chi}} = \varepsilon_{\chi_i}$ *for all* $1 \leqslant i \leqslant k$*, in particular* $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}^*) = \chi_1^* + \cdots + \chi_k^*$*.*

Proof. We know $\varepsilon_{\tilde{\chi}} D_{\tilde{\mathbf{G}}}(\tilde{\chi}) \in \mathrm{Irr}(\tilde{G})$ so it is a character of $\tilde{G}$. This means

$$\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}^*) = \varepsilon_{\tilde{\chi}}(\mathrm{Res}_G^{\tilde{G}} \circ D_{\tilde{\mathbf{G}}})(\tilde{\chi}) = \varepsilon_{\tilde{\chi}}(D_{\mathbf{G}} \circ \mathrm{Res}_G^{\tilde{G}})(\tilde{\chi}) = \varepsilon_{\tilde{\chi}} D_{\mathbf{G}}(\chi_1) + \cdots + \varepsilon_{\tilde{\chi}} D_{\mathbf{G}}(\chi_k)$$

is a character of G, where the second equality is obtained by Lemma 7.10. As it is a character all coefficients of irreducible constituents must be positive therefore we cannot have $\varepsilon_{\chi_i} \neq \varepsilon_{\tilde{\chi}}$ for any $1 \leqslant i \leqslant k$. The final statement is clear by definition. ■

From now until the end of this chapter we assume that $\mathbf{G}$ is a simple algebraic group with a disconnected centre.

We temporarily fix the following notation. If $\tilde{\mathcal{O}}$ is an F-stable unipotent class of $\tilde{\mathbf{G}}$ then we denote by $\tilde{\mathcal{O}}_i$, for $1 \leqslant i \leqslant d$, the $\tilde{G}$-conjugacy classes such that $\tilde{\mathcal{O}}^F = \tilde{\mathcal{O}}_1 \sqcup \cdots \sqcup \tilde{\mathcal{O}}_d$. For each $\tilde{\mathcal{O}}_i$ we write $\mathcal{O}_{i,j}$, for $1 \leqslant j \leqslant k_i$ (where k_i is a number depending upon the class $\tilde{\mathcal{O}}_i$), for the G-conjugacy classes such that $\tilde{\mathcal{O}}_i \cap G = \mathcal{O}_{i,1} \sqcup \cdots \sqcup \mathcal{O}_{i,k_i}$. Finally we fix G-conjugacy class representatives $u_{i,j} \in \mathcal{O}_{i,j}$ for each $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant k_i$.

Proposition 7.12. *Let $\mathcal{O}$ be an F-stable unipotent class of $\mathbf{G}$ and assume $A_{\mathbf{G}}(u)$ is abelian. Consider the triple $(\tilde{s}, \mathbf{W}^{\star}(\tilde{\mathcal{F}}), \bar{\psi})$ prescribed by Proposition A. There exist irreducible characters $\chi_{i,j} \in \mathcal{E}(G,s)$, (which are constituents of restrictions from $\tilde{\mathcal{F}}$), such that $\langle \chi_{i,j}^*, \gamma_{u_{x,y}} \rangle = \delta_{i,x}\delta_{j,y}$, for $1 \leqslant i, x \leqslant d$ and $1 \leqslant j, y \leqslant k_i$. Furthermore $k_i = |Z_{\mathbf{G}}(u)^F|$ for all i.*

Proof. Recall that $\tilde{u} \in \tilde{\mathcal{O}}^F$ is a class representative such that $A_{\tilde{\mathbf{G}}}(\tilde{u})^F = A_{\tilde{\mathbf{G}}}(\tilde{u})$. The $\tilde{G}$-classes in $\tilde{\mathcal{O}}^F$ are in bijection with the F-conjugacy classes in $A_{\tilde{\mathbf{G}}}(\tilde{u})$ therefore $d = |A_{\tilde{\mathbf{G}}}(\tilde{u})|$ by Lemma 4.3. As $A_{\mathbf{G}}(u)$ is also abelian we have the number of G-conjugacy classes contained in $\mathcal{O}$ is $|H^1(F, A_{\mathbf{G}}(u))| = |A_{\mathbf{G}}(u)^F| = \sum_{i=1}^{d} k_i$.

By [**GH08**, Proposition 4.3], which we can use as Property (P1) and Property (P3) hold, there exist irreducible characters $\tilde{\chi}_{1,1}, \ldots, \tilde{\chi}_{d,1} \in \tilde{\mathcal{F}}$ such that $\langle \tilde{\chi}_{i,1}^*, \tilde{\gamma}_{u_{x,1}} \rangle = \delta_{i,x}$ for all $1 \leqslant i, x \leqslant d$. Using Lemma 7.4 followed by Frobenius reciprocity we see

$$\delta_{i,x} = \langle \tilde{\chi}_{i,1}^*, \tilde{\gamma}_{u_{x,1}} \rangle_{\tilde{G}} = \langle \tilde{\chi}_{i,1}^*, \mathrm{Ind}_G^{\tilde{G}}(\gamma_{u_{x,1}}) \rangle_{\tilde{G}} = \langle \mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}_{i,1}^*), \gamma_{u_{x,1}} \rangle_G.$$

As $A_{\mathbf{G}}(u)$ is abelian we know ♣ holds, in particular using Corollary 7.11 we know the restriction has the following decomposition into irreducible characters

$$\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}_{i,1}^*) = \chi_{i,1}^* + \cdots + \chi_{i,|Z_{\mathbf{G}}(u)^F|}^* \Rightarrow \delta_{i,x} = \sum_{j=1}^{|Z_{\mathbf{G}}(u)^F|} \langle \chi_{i,j}^*, \gamma_{u_{x,1}} \rangle.$$

Without loss of generality we may assume the labelling to be such that $\langle \chi_{i,j}^*, \gamma_{u_{x,1}} \rangle = \delta_{i,x}\delta_{j,1}$. We will write $\mathrm{Stab}_{\tilde{G}}(\chi_{i,1}^*)$ for the stabiliser of $\chi_{i,1}^*$ in $\tilde{G}$ under the natural conjugation action of $\tilde{G}$ on $\mathrm{Irr}(G)$. By Clifford theory and the remark before Proposition 7.5 there exists a set $\{t_{i,1}, \ldots, t_{i,|Z_{\mathbf{G}}(u)^F|}\} \subseteq \tilde{T}_0$, (which can be completed to form a left transversal of $\mathrm{Stab}_{\tilde{G}}(\chi_{i,1}^*)$ in $\tilde{G}$), such that $\chi_{i,j}^* := {\chi_{i,1}^*}^{t_{i,j}}$ satisfies the condition $\chi_{i,j}^* = \chi_{i,k}^*$ if and only if $j = k$. We assume for convenience that $t_{i,1}$ is the identity for all i. As conjugation by elements of $\tilde{G}$ is an isometry of the space of all class functions

$$\delta_{i,x} = \langle {\chi_{i,1}^*}^{t_{i,j}}, \gamma_{u_{x,1}}^{t_{i,j}} \rangle = \langle \chi_{i,j}^*, \gamma_{t_{i,j}(u_{x,1})t_{i,j}^{-1}} \rangle.$$

We claim that given distinct indices j, k we cannot have $t_{i,j}(u_{x,1})t_{i,j}^{-1}$ and $t_{i,k}(u_{x,1})t_{i,k}^{-1}$ are in the same G-conjugacy class. If they were in the same class then we would have $\gamma_{u_{x,1}}^{t_{i,j}} = \gamma_{u_{x,1}}^{t_{i,k}}$ by Proposition 7.5, which would mean

$$\delta_{i,x} = \langle {\chi_{i,1}^*}^{t_{i,k}}, \gamma_{u_{x,1}}^{t_{i,j}} \rangle = \langle {\chi_{i,1}^*}^{t_{i,k}t_{i,j}^{-1}}, \gamma_{u_{x,1}} \rangle.$$

However of all the components of $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}_{i,1}^*)$ only $\chi_{i,1}^*$ satisfies this property. Hence we would have $\chi_{i,1}^* = {\chi_{i,1}^*}^{t_{i,k}t_{i,j}^{-1}}$, which would imply $t_{i,k}$ and $t_{i,j}$ have the same image in the quotient $\tilde{G}/\mathrm{Stab}_{\tilde{G}}(\chi_{i,1}^*)$ but this is a contradiction.

This argument shows that there are at least $|Z_{\mathbf{G}}(u)^F|$ conjugacy classes contained in $\tilde{\mathcal{O}}_i \cap G$ for each $1 \leqslant i \leqslant d$, in other words $|Z_{\mathbf{G}}(u)^F| \leqslant k_i$. Conversely this clearly gives us an inequality

$$|A_{\mathbf{G}}(u)^F| = |Z_{\mathbf{G}}(u)^F||A_{\tilde{\mathbf{G}}}(\tilde{u})| = d|Z_{\mathbf{G}}(u)^F| \leqslant \sum_{i=1}^{d} k_i = |A_{\mathbf{G}}(u)^F|$$

so we must have $k_i = |Z_{\mathbf{G}}(u)^F|$ for all i. By this argument we may now redefine our class representatives to be such that, for all $1 \leqslant x \leqslant d$ and $1 \leqslant y \leqslant |Z_{\mathbf{G}}(u)^F|$, we have $u_{x,y} := t_{x,y}u_{x,1}t_{x,y}^{-1}$. With all this in mind the statement of the proposition is now simple. Indeed we first see that

$$\langle \chi_{i,j}^*, \gamma_{u_{x,y}} \rangle = \langle {\chi_{i,1}^*}^{t_{i,j}}, \gamma_{u_{x,1}}^{t_{x,y}} \rangle = \langle {\chi_{i,1}^*}^{t_{i,j}t_{x,y}^{-1}}, \gamma_{u_{x,1}} \rangle$$

but it is clear that $\langle {\chi_{i,1}^*}^{t_{i,j}t_{x,y}^{-1}}, \mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}_{i,1}^*) \rangle = 1$, which tells us the above inner product is 0 unless $i = x$. Now assume $i = x$ then $\langle {\chi_{i,1}^*}^{t_{i,j}t_{i,y}^{-1}}, \gamma_{u_{i,1}} \rangle = 1$ if and only if ${\chi_{i,1}^*}^{t_{i,j}t_{i,y}^{-1}} = \chi_{i,1}^*$, which is true if and only if $j = y$ so we're done. ■

For convenience we restate the above proposition with slightly less cumbersome notation.

Corollary 7.13. *Let $\mathcal{O}$ be an F-stable unipotent class of $\mathbf{G}$ and assume $A_{\mathbf{G}}(u)$ is abelian. Let $d' := |A_{\mathbf{G}}(u)^F|$ and denote by $\mathcal{O}_i$ the G-conjugacy classes such that $\mathcal{O}^F = \mathcal{O}_1 \sqcup \cdots \sqcup \mathcal{O}_{d'}$ with class representatives $u_i \in \mathcal{O}_i$. Consider the triple $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}), \bar{\psi})$ prescribed by Proposition A then there exist irreducible characters $\chi_i \in \mathcal{E}(G,s)$, (which are constituents of restrictions from $\tilde{\mathcal{F}}$), such that $\langle \chi_i^*, \gamma_{u_j} \rangle = \delta_{i,j}$, for all $1 \leqslant i,j \leqslant d'$.*

Remark 7.14. Let us relax the condition that $A_{\mathbf{G}}(u)$ is abelian in the above result but instead assume $A_{\tilde{\mathbf{G}}}(\tilde{u})$ is abelian. In particular let $\mathbf{G}$ be a spin / half spin group and u be such that $|Z_{\mathbf{G}}(u)| = |Z(\mathbf{G})|$ then we recall we are in the following situation. The family $\tilde{\mathcal{F}}$ described in the proof of Proposition 7.12 has order m^2, where m is a power of 2. The group $A_{\tilde{\mathbf{G}}}(\tilde{u})$ is abelian so we may use [**GH08**, Proposition 4.3], as in the above proof, to obtain m characters in $\tilde{\mathcal{F}}$ satisfying the inner product condition. In this situation ♣ fails and in fact there are *only* m characters in $\tilde{\mathcal{F}}$ whose restriction to G contains the correct number of irreducible constituents. The problem is that it is not clear that the characters provided by the result of Geck and Hézard coincide with the characters which have the correct restriction from $\tilde{G}$ to G. This is the key obstruction to showing Kawanaka's conjecture holds for spin / half spin groups.

One way to fix this problem would be to make sure that the character sheaves used in the proof of [**GH08**, Corollary 3.5] can be chosen to have the same labelling as the irreducible characters in $\tilde{\mathcal{F}}$ whose restriction to G is not irreducible. The author

$n_{\tilde{\chi}}$	6	6	3	3	3	3	2	2
$D_{\tilde{\mathbf{G}}}(\tilde{\gamma}_{u_1})$	1	1	2	.	.	.	.	.
$D_{\tilde{\mathbf{G}}}(\tilde{\gamma}_{u_2})$	.	.	.	1	1	1	.	.
$D_{\tilde{\mathbf{G}}}(\tilde{\gamma}_{u_3})$	.	.	.	.	.	.	1	1

Table 7.1: Multiplicities in the case $A_{\tilde{\mathbf{G}}}(\tilde{u}) \cong \mathfrak{S}_3$

considered this in the case where $\mathbf{G}$ is a spin group of type B_n by trying to adapt the explicit results obtained in [**Lus86**]. However it seems that trying to describe which character sheaves have non-zero restriction to the unipotent class is significantly complicated.

7.3 Proof of Theorem B

The result in the previous section will be crucial in determining Kawanaka's conjecture in simple groups of classical type. However in exceptional types we know that there exist unipotent classes $\mathcal{O}$ such that the associated component group $A_{\mathbf{G}}(u)$ is non-abelian. In particular these are the classes $\mathrm{D}_4(a_1)$ in E_6 and $\mathrm{D}_4(a_1)$, $\mathrm{E}_7(a_5)$ in E_7. In all of these cases we have the associated component group $A_{\tilde{\mathbf{G}}}(\tilde{u})$ is isomorphic to the symmetric group $\mathfrak{S}_3$. To deal with these cases we will use the following result of Geck.

Proposition 7.15 (Geck, [Gec99, Proposition 6.7]**).** *Let $\mathcal{O}$ be an F-stable class of $\mathbf{G}$ such that $A_{\tilde{\mathbf{G}}}(\tilde{u}) \cong \mathfrak{S}_3$ and let $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}), \bar{\psi})$ be the triple prescribed by Proposition A. Write $\tilde{\mathcal{O}}_1$, $\tilde{\mathcal{O}}_2$ and $\tilde{\mathcal{O}}_3$ for the $\tilde{G}$-conjugacy classes such that $\tilde{\mathcal{O}}^F = \tilde{\mathcal{O}}_1 \sqcup \tilde{\mathcal{O}}_2 \sqcup \tilde{\mathcal{O}}_3$ and fix representatives $u_i \in \tilde{\mathcal{O}}_i \cap G$. The values in Table 7.1 describe the matrix of multiplicities between the irreducible characters in $\mathcal{E}(\tilde{G}, \tilde{s})$ and the Alvis–Curtis duals of the characters of the GGGRs associated to $\tilde{\mathcal{O}}$ (where . stands for 0).*

Remark 7.16.

- Because $A_{\tilde{\mathbf{G}}}(\tilde{u}) \cong \mathfrak{S}_3$ and $A_{\tilde{\mathbf{G}}}(\tilde{u})^F = A_{\tilde{\mathbf{G}}}(\tilde{u})$ we have $\tilde{\mathcal{O}}$ contains precisely three $\tilde{G}$-conjugacy classes. We refer to a character solely by its n_χ value.
- In Geck's proof of the above result he assumes that the graph automorphism induced by F is trivial but with a minor remark in the proof of [**Héz04**, Proposition 6.7] Hézard shows that this also holds in the case of twisted E_6

Let us keep the assumptions and notation of Proposition 7.15. We can find three characters $\tilde{\chi}_1$, $\tilde{\chi}_2$, $\tilde{\chi}_3 \in \mathrm{Irr}(\tilde{G})$ such that $\langle D_{\tilde{\mathbf{G}}}(\tilde{\chi}_i), \tilde{\gamma}_{u_j}\rangle = \langle \tilde{\chi}_i, D_{\tilde{\mathbf{G}}}(\tilde{\gamma}_{u_j})\rangle = \delta_{i,j}$. In particular $\varepsilon_{\tilde{\chi}} = 1$ for all $\tilde{\chi} \in \mathcal{E}(\tilde{G}, \tilde{s})$, which means $\langle \tilde{\chi}_i^*, \tilde{\gamma}_{u_j}\rangle = \delta_{i,j}$. Let $\mathcal{O}$ be the class labelled $D_4(a_1)$ in E_6 or E_7, then $A_{\tilde{\mathbf{G}}}(\tilde{u}) \cong A_{\mathbf{G}}(u)$. Hence $\mathcal{O}_i := \tilde{\mathcal{O}}_i \cap G$ is a single

G-conjugacy class and $\mathcal{O}^F = \mathcal{O}_1 \sqcup \mathcal{O}_2 \sqcup \mathcal{O}_3$. As ♣ holds we know $\chi_i := \mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}_i)$ is irreducible, in particular for each $1 \leqslant i, j \leqslant 3$ we have $\langle \chi_i^*, \gamma_{u_j} \rangle = \delta_{i,j}$.

Assume $\mathcal{O}$ is the class $\mathrm{E}_7(a_5)$ in E_7. By the proof of Lemma 2.37 we know that there are three F-conjugacy classes in $A_{\mathbf{G}}(u)$. We know ♣ holds for this class so for each $1 \leqslant i \leqslant 3$ we have $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}_i) = \chi_{i,1} + \chi_{i,2}$, where $\chi_{i,j} \in \mathrm{Irr}(G)$. Using the techniques in the proof of Proposition 7.12 it is easy to see that each of the three $\tilde{G}$-conjugacy classes $\tilde{\mathcal{O}}_i$ is such that $\tilde{\mathcal{O}}_i \cap G = \mathcal{O}_{i,1} \sqcup \mathcal{O}_{i,2}$, where $\mathcal{O}_{i,j}$ is a G-conjugacy class for $1 \leqslant j \leqslant 2$. Let us choose class representatives $u_{x,y} \in \mathcal{O}_{x,y}$ for each $1 \leqslant x \leqslant 3$ and $1 \leqslant y \leqslant 2$ then the following relation holds $\langle \chi_{i,j}^*, \gamma_{u_{x,y}} \rangle = \delta_{i,x}\delta_{j,y}$. In particular we have the following corollary.

Corollary 7.17. *Let $\mathcal{O}$ be an F-stable unipotent class of $\mathbf{G}$ and assume $A_{\tilde{\mathbf{G}}}(\tilde{u}) \cong \mathfrak{S}_3$. Let us write $\mathcal{O}_i$, for $1 \leqslant i \leqslant d'$, for the G-conjugacy classes such that $\mathcal{O}^F = \mathcal{O}_1 \sqcup \cdots \sqcup \mathcal{O}_{d'}$ and let $u_i \in \mathcal{O}_i$ denote a class representative. Consider the triple $(\tilde{s}, \mathbf{W}^\star(\tilde{\mathcal{F}}), \bar{\psi})$ prescribed by Proposition A then there exist irreducible characters $\chi_i \in \mathcal{E}(G, s)$, (which are constituents of restrictions from $\tilde{\mathcal{F}}$), such that $\langle \chi_i^*, \gamma_{u_j} \rangle = \delta_{i,j}$ for all $1 \leqslant i, j \leqslant d'$.*

We have now gathered all the preliminary information that we need to prove the following theorem. Note that its proof is identical to that of [**GH08**, Theorem 4.5].

Theorem B. *Assume p, q are large enough and $\mathbf{G}$ is a simple algebraic group, which is not a spin or half-spin group then Conjecture 7.7 holds.*

Proof. We may assume $\mathbf{G}$ is not adjoint as the adjoint simple groups are covered by the result of Geck and Hézard, (see [**GH08**, Theorem 4.5]). Let $\mathcal{O}_1, \dots, \mathcal{O}_N$ be the N distinct F-stable unipotent classes of $\mathbf{G}$, where we choose the numbering to be such that $\dim(\mathcal{O}_i) \leqslant \dim(\mathcal{O}_{i+1})$ for each $1 \leqslant i \leqslant N-1$. Let $(\tilde{s}_i, \mathbf{W}(\tilde{\mathcal{F}}_i), \bar{\psi}_i)$ be the triple prescribed by Proposition A for each $\mathcal{O}_i$.

For each $1 \leqslant i \leqslant N$ we write $\mathcal{O}_{i,1}, \dots, \mathcal{O}_{i,k_i}$ for the G-conjugacy classes such that $\mathcal{O}_i^F = \mathcal{O}_{i,1} \sqcup \cdots \sqcup \mathcal{O}_{i,k_i}$. By Corollaries 7.13 and 7.17 we can find irreducible characters $\{\chi_{i,1}, \dots, \chi_{i,k_i}\} \subseteq \mathcal{E}(G, s_i)$ such that $\langle \chi_{i,j}^*, \gamma_{u_{i,y}} \rangle = \delta_{j,y}$. In particular the matrix of multiplicities $(\langle \chi_{i,j}^*, \gamma_{u_{x,y}} \rangle)$, (where we have $1 \leqslant i, x \leqslant N$, $1 \leqslant j \leqslant k_i$, $1 \leqslant y \leqslant k_x$), is a square block matrix with identity matrices on the diagonal. We claim that all blocks in the lower triangular part of this matrix are zero. If this were true then the matrix of multiplicites would be invertible over $\mathbb{Z}$ and by Lemma 7.8 Kawanaka's conjecture would hold.

Therefore to show Kawanaka's conjecture holds we need only prove the following statement

$$\langle \chi_{i,j}^*, \gamma_{u_{x,y}} \rangle = 0 \quad \text{if} \quad i < x.$$

Recall from the definition that $\langle \chi_{i,j}^*, \gamma_{u_{x,y}} \rangle = \varepsilon_{\chi_{i,j}} \langle \chi_{i,j}, D_{\mathbf{G}}(\gamma_{u_{x,y}}) \rangle$ so by definition of the scalar product we have

$$\langle \chi_{i,j}, D_{\mathbf{G}}(\gamma_{u_{x,y}}) \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{i,j}(g) \overline{D_{\mathbf{G}}(\gamma_{u_{x,y}})(g)}.$$

We will now try and compute this sum. Using the property stated in [**Gec99**, 2.4(c)] we know $D_{\mathbf{G}}(\gamma_{u_{x,y}})(g) \neq 0$ if and only if $g \in \mathcal{U}^F$, hence it is enough to let this sum run over $\mathcal{U}^F$. Assume $g \in \mathcal{U}^F$ and, if possible, that the corresponding term in the above sum is non-zero. Let $\mathcal{O}$ denote the **G**-conjugacy class containing g. By [**Lus92**, Theorem 11.2(iv)] we have $\chi_{i,j}(g) \neq 0$ implies $\dim(\mathcal{O}) \leqslant \dim(\mathcal{O}_i)$ with equality if and only if $\mathcal{O} = \mathcal{O}_i$. However if $D_{\mathbf{G}}(\gamma_{u_{x,y}})(g) \neq 0$ then by [**Gec96**, 2.3(a)] we know $\mathcal{O}_x \subseteq \mathcal{O}$, which implies $\dim(\mathcal{O}_x) \leqslant \dim(\mathcal{O})$. As we ordered the unipotent classes so that their dimensions were weakly increasing we have a sequence of inequalities

$$\dim(\mathcal{O}) \leqslant \dim(\mathcal{O}_i) \leqslant \dim(\mathcal{O}_x) \leqslant \dim(\mathcal{O}) \Rightarrow \dim(\mathcal{O}) = \dim(\mathcal{O}_i) = \dim(\mathcal{O}_x).$$

By the uniqueness of a unipotent support we have $\mathcal{O}_x \subseteq \mathcal{O}_i$ so these classes must be equal as they have the same dimension, which is a contradiction. In particular the above sum must be zero. ∎

Remark 7.18. In [**GH08**, Proposition 4.6] Geck and Hézard give a characterisation of GGGRs in terms of character values. This characterisation follows as a formal consequence of Kawanaka's conjecture so this now also holds for simple groups which are not a spin / half spin group.

Chapter 8

Computing Some Fourth Roots of Unity

In this chapter we would like to use Theorem A together with the techniques of [**Gec99**, §3] to compute certain fourth roots of unity that arise in connection to GGGRs.

8.1 Some Preliminaries

Before we begin we must first recall the setup of [**Gec99**]. Let $\iota = (\mathcal{O}_\iota, \psi_\iota) \in \mathcal{N}_{\mathbf{G}}$ be a pair then we say ι is F-stable if $F(\mathcal{O}_\iota) = \mathcal{O}_\iota$ and $\psi_\iota \circ F = \psi_\iota$. We denote the subset of F-stable pairs by $\mathcal{N}_{\mathbf{G}}^F \subseteq \mathcal{N}_{\mathbf{G}}$.

8.1.1 A Basis for the Space of Unipotently Supported Class Functions

Assume now that ι is an F-stable pair. As ψ_ι is invariant under the action of F we may extend it to a character of the semidirect product $A_{\mathbf{G}}(u) \rtimes \langle F \rangle$, where F acts as a cyclic group of automorphisms. Let $\tilde{\psi}_\iota$ be a fixed choice of such an extension. For each $x \in A_{\mathbf{G}}(u)$ we write u_x for an element of $\mathcal{O}_\iota^F$ obtained by twisting u with the element $x \in A_{\mathbf{G}}(u)$. We then define

$$Y_\iota(g) = \begin{cases} \tilde{\psi}_\iota(xF) & \text{if } g = u_x \text{ for some } x \in A_{\mathbf{G}}(u), \\ 0 & \text{otherwise} \end{cases}$$

for all $g \in G$. The function Y_ι is an element of $\mathrm{Cent}(G)$ and the set $\mathcal{Y} = \{Y_\iota \mid \iota \in \mathcal{N}_{\mathbf{G}}^F\}$ forms a basis for the subspace $\mathrm{Cent}_{\mathrm{U}}(G)$ of $\mathrm{Cent}(G)$.

Recall that in [**Lus84b**] Lusztig has associated to every pair $\iota \in \mathcal{N}_{\mathbf{G}}$ a unique, (up to $\mathbf{G}$-conjugacy), pair $(\mathbf{L}_\iota, v_\iota)$ where $\mathbf{L}_\iota$ is a Levi subgroup of $\mathbf{G}$ and $v_\iota \in \mathcal{N}_{\mathbf{L}_\iota}$ is a *cuspidal pair*. This is known as the generalised Springer correspondence. Lusztig has shown that we have a disjoint union

$$\mathcal{N}_{\mathbf{G}} = \bigsqcup_{(\mathbf{L},v)} \mathscr{I}(\mathbf{L},v) \qquad \text{where} \qquad \mathscr{I}(\mathbf{L},v) = \{\iota \in \mathcal{N}_{\mathbf{G}} \mid (\mathbf{L}_\iota, v_\iota) = (\mathbf{L},v)\}.$$

We call $\mathscr{I}(\mathbf{L},v)$ a *block* of $\mathcal{N}_{\mathbf{G}}$. Note that a pair $\iota \in \mathcal{N}_{\mathbf{G}}$ is cuspidal if and only if it lies in a block of size 1. To each pair $\iota \in \mathcal{N}_{\mathbf{G}}$ we also assign the following value

$$b_\iota = \frac{1}{2}(\dim \mathbf{G} - \dim \mathcal{O}_\iota - \dim Z(\mathbf{L}_\iota)^\circ).$$

8.1.2 GGGRs and Fourth Roots of Unity

We have already seen that $\mathcal{Y}$ is a basis of $\mathrm{Cent}_{\mathrm{U}}(G)$ and we wish to consider another basis of $\mathrm{Cent}_{\mathrm{U}}(G)$ constructed using GGGRs. Let $\mathcal{O}$ be an F-stable unipotent class of G and $u_1, \dots, u_d \in \mathcal{O}^F$ class representatives for the G-conjugacy classes contained in $\mathcal{O}^F$. We then define, as in [**Lus92**, 7.5], for any pair $\iota \in \mathcal{N}_{\mathbf{G}}^F$ with $\mathcal{O}_\iota = \mathcal{O}$ the class function

$$\gamma_\iota = \sum_{r=1}^{d} [A_{\mathbf{G}}(u_r) : A_{\mathbf{G}}(u_r)^F] Y_\iota(u_r) \gamma_{u_r}.$$

Note γ_{u_r} is the GGGR of G determined by the G-conjugacy class containing u_r. The set $\mathcal{X} = \{\gamma_\iota \mid \iota \in \mathcal{N}_{\mathbf{G}}^F\}$ is also a basis for the subspace $\mathrm{Cent}_{\mathrm{U}}(G)$.

In [**Lus92**, Theorem 7.3] Lusztig explicitly constructs the change of basis from $\mathcal{X}$ to $\mathcal{Y}$. The expression of the function $\gamma_\iota \in \mathcal{X}$ in terms of elements of $\mathcal{Y}$ involves an unknown fourth root of unity ζ_ι. This root of unity is defined in [**Lus92**, Proposition 7.2] and it is shown there that $\zeta_\iota = \zeta_{\iota'}$ whenever ι, ι' belong to the same block.

Following [**Lus92**, 8.4] to each ι we define $\delta_\iota = (-1)^{\mathrm{rank}(\mathbf{L}_\iota / Z(\mathbf{L}_\iota))}$ and $\zeta'_\iota = \delta_\iota \zeta_\iota^{-1}$. As δ_ι and ζ_ι only depend on the block containing ι the same must be true for ζ'_ι. Moreover we have the following more precise statement

Lemma 8.1 (Lusztig, [Lus92, Proposition 7.2]**).** *Let $\mathscr{I}(\mathbf{L},v)$ be a block of $\mathcal{N}_{\mathbf{G}}$ and assume that $v \in \mathcal{N}_{\mathbf{L}}^F$. If ζ'_v is the fourth root of unity associated to v in $\mathbf{L}^F$ then $\zeta'_\iota = \zeta'_v$ for all $\iota \in \mathscr{I}(\mathbf{L},v)$.*

Note that there is one extreme case where we know the value ζ'_ι. From the definitions it is clear that if $\mathbf{L}_\iota$ is a maximal torus then $\zeta'_\iota = 1$.

Following Geck we take the following statements as axioms upon which the remaining arguments will be built.

(A1) Let $\iota, \iota' \in \mathcal{N}_{\mathbf{G}}^F$ be such that $\mathcal{O}_\iota = \mathcal{O} = \mathcal{O}_{\iota'}$ then $\langle D_{\mathbf{G}}(\gamma_\iota), Y_{\iota'} \rangle_G = |A_{\mathbf{G}}(u)| \zeta'_\iota q^{-b_\iota} \delta_{\iota,\iota'}$.

(A2) If $\mathrm{D}_{\mathbf{G}}(\gamma_\iota)(x) \neq 0$ for some $x \in G$ then $\langle x \rangle \leqslant \mathcal{O}_\iota$ where $\langle x \rangle$ is the G-conjugacy class containing x.

Under the assumption that p and q are large enough Axiom (A1) is known to be true by [**Gec94**, Lemma 3.5] and Axiom (A2) is known to be true by [**Gec96**, 2.3(a)]. It is hoped that these axioms are true with only mild restrictions on p and q, (for example p a good prime).

From now until the end of this chapter we assume that **G** is a simple algebraic group with a disconnected centre.

Our main focus in this chapter is to show that using Theorem A we can obtain statements like [**Gec99**, Theorem 3.8] for symplectic and even dimensional special orthogonal groups. To do this we need the following easy extensions of results of Geck.

Proposition 8.2. *Let $\mathcal{O} \in \mathfrak{Cl}_{\mathrm{U}}(\mathbf{G})^F$ be a unipotent class and consider the triple $(\tilde{s}, \mathbf{W}(\tilde{\mathcal{F}}), \bar{\psi})$ prescribed by Proposition A. Let d', (resp. d), be the number of $\tilde{G}$, (resp. G), conjugacy classes contained in $\tilde{\mathcal{O}}^F$. There exists a character $\chi \in \mathcal{E}(G,s)$, (which occurs in the restriction of a character in $\tilde{\mathcal{F}}$), and an index $1 \leqslant x \leqslant d$ such that*

$$\langle \chi^*, \gamma_{u_y} \rangle = \delta_{x,y} \qquad \textit{and} \qquad \langle \chi^*, \gamma_\iota \rangle = \frac{|Z_{\mathbf{G}}(u)|}{|Z_{\mathbf{G}}(u)^F|} \cdot \overline{Y_\iota(u_x)},$$

for all $\iota \in \mathcal{N}_{\mathbf{G}}^F$ with $\mathcal{O}_\iota = \mathcal{O}$.

Proof. The proof of this result is almost identical in nature to that of Proposition 7.12. Let us adopt the notational conventions introduced immediately before the statement of Proposition 7.12. Let $\tilde{\chi} \in \mathcal{E}(\tilde{G}, \tilde{s})$ be the character corresponding to $\bar{\psi}$ under the Jordan decomposition of characters then by [**Gec99**, Proposition 3.1] there exists an index $1 \leqslant x \leqslant d'$ such that $\langle \tilde{\chi}^*, \tilde{\gamma}_{\tilde{u}_y} \rangle_{\tilde{G}^F} = \delta_{x,y}$ for all $1 \leqslant y \leqslant d'$.

Using the same argument as in Proposition 7.12 we see there is a unique irreducible constituent χ in $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi})$ satisfying $\langle \chi^*, \gamma_{u_y} \rangle_G = \delta_{x,y}$ for all $1 \leqslant y \leqslant d$. By [**Gec99**, 2.8(b)]

$$\langle \chi^*, \gamma_{(\mathcal{O},1)} \rangle_G = \frac{|A_{\mathbf{G}}(u)|}{n_\chi} \Rightarrow \langle \chi^*, \gamma_{(\mathcal{O},1)} \rangle_G = \frac{|A_{\mathbf{G}}(u)|}{|A_{\mathbf{G}}(u)^F|} = \frac{|Z_{\mathbf{G}}(u)|}{|Z_{\mathbf{G}}(u)^F|}.$$

From the definition of γ_ι we conclude

$$\langle \chi^*, \gamma_\iota \rangle_G = \left\langle \chi^*, \sum_{r=1}^{d} \frac{|A_{\mathbf{G}}(u)|}{|A_{\mathbf{G}}(u_r)^F|} Y_i(u_r) \gamma_{u_r} \right\rangle_G = \frac{|Z_{\mathbf{G}}(u)|}{|Z_{\mathbf{G}}(u)^F|} \cdot \overline{Y_i(u_x)}$$

because χ only has non-zero multiplicity in γ_{u_x}. ■

Corollary 8.3. *Let $\mathcal{O}$, χ and x be as in Proposition 8.2. Let $u \in \mathcal{O}$ be a representative and assume either $A_{\mathbf{G}}(u)$ is abelian or $A_{\mathbf{G}}(u)$ is non-abelian but $A_{\mathbf{G}}(u)^F = A_{\mathbf{G}}(u)$ then*

$$\varepsilon_\chi \cdot \chi(u_x) = \frac{1}{|A_{\mathbf{G}}(u)^F|} \sum_\iota \zeta'_\iota q^{b_\iota} \psi_\iota(1)^2, \tag{8.1}$$

where the sum is taken over all $\iota \in \mathcal{N}_{\mathbf{G}}^F$ with $\mathcal{O}_\iota = \mathcal{O}$. In particular the expression on the right hand side is an algebraic integer.

Proof. From [**Gec99**, Corollary 2.6] we get

$$\begin{aligned}
\chi(u_x) &= \frac{1}{|A_{\mathbf{G}}(u)|} \sum_\iota \zeta'_\iota q^{b_\iota} Y_\iota(u_x) \langle \chi, D_{\mathbf{G}}(\gamma_\iota)\rangle, \\
&= \varepsilon_\chi \cdot \frac{1}{|Z_{\mathbf{G}}(u)||A_{\tilde{\mathbf{G}}}(\tilde{u})|} \sum_\iota \zeta'_\iota q^{b_\iota} Y_\iota(u_x) \langle \chi^*, \gamma_\iota\rangle, \\
&= \varepsilon_\chi \cdot \frac{|Z_{\mathbf{G}}(u)|}{|Z_{\mathbf{G}}(u)^F||Z_{\mathbf{G}}(u)||A_{\tilde{\mathbf{G}}}(\tilde{u})|} \sum_\iota \zeta'_\iota q^{b_\iota} Y_\iota(u_x)\overline{Y_\iota(u_x)}, \\
&= \varepsilon_\chi \cdot \frac{1}{|Z_{\mathbf{G}}(u)^F||A_{\tilde{\mathbf{G}}}(\tilde{u})|} \sum_\iota \zeta'_\iota q^{b_\iota} Y_\iota(u_x)\overline{Y_\iota(u_x)}, \\
&= \varepsilon_\chi \cdot \frac{1}{|A_{\mathbf{G}}(u)^F|} \sum_\iota \zeta'_\iota q^{b_\iota} Y_\iota(u_x)\overline{Y_\iota(u_x)}.
\end{aligned}$$

Assume first that $A_{\mathbf{G}}(u)$ is abelian and recall $Y_\iota(u_x) = \tilde{\psi}_\iota(xF)$. As $\tilde{\psi}_\iota$ will be a linear character every character value will be a root of unity, in particular $Y_\iota(u_x)\overline{Y_\iota(u_x)} = 1 = \tilde{\psi}_\iota(1) = \psi_\iota(1)^2$.

We assume now that $A_{\mathbf{G}}(u)$ is non-abelian and $A_{\mathbf{G}}(u)^F = A_{\mathbf{G}}(u)$. Using the arguments in [**Gec99**, Proposition 3.1] we see the condition $n_\chi = |A_{\mathbf{G}}(u)|$ implies $A_{\mathbf{G}}(u_x)^F = A_{\mathbf{G}}(u_x)$. Let $g \in \mathbf{G}$ be such that $u_x = gug^{-1}$, (i.e. $g^{-1}F(g) \mapsto x \in A_{\mathbf{G}}(u)$), then the component group has the form $A_{\mathbf{G}}(u_x) = \{gyC_{\mathbf{G}}(u)^\circ g^{-1} \mid y \in A_{\mathbf{G}}(u)\}$. If every element of $A_{\mathbf{G}}(u_x)$ is fixed by F then

$$F(gy)C_{\mathbf{G}}(u)^\circ F(g)^{-1} = gyC_{\mathbf{G}}(u)^\circ g^{-1} \Rightarrow (g^{-1}F(g))yC_{\mathbf{G}}(u)^\circ (g^{-1}F(g))^{-1} = yC_{\mathbf{G}}(u)^\circ$$

for all $y \in A_{\mathbf{G}}(u)$. The image of $g^{-1}F(g)$ in $A_{\mathbf{G}}(u)$ is x, which implies that x is in the centre of $A_{\mathbf{G}}(u)$. This ensures $\psi_\iota(x) = \psi_\iota(1)$ because x must be in the kernel of every character so again $Y_\iota(u_x)\overline{Y_\iota(u_x)} = \psi_\iota(x)^2 = \psi_\iota(1)^2$. ■

8.2 The Special Orthogonal and Symplectic Groups

We want to adapt the argument of [**Gec99**, Theorem 3.8] to symplectic groups and even dimensional special orthogonal groups. Recall that we assume p is a good prime, in particular $p \neq 2$. Our computation of the fourth root of unity will use a result of

Group	Condition	Partition	$\lvert A_{\mathbf{G}}(u)\rvert$
Sp_{2n}	$n = 1 + \cdots + k$	$(2, 4, 6, \dots, 2k)$	2^k
SO_{2n}	$n = 2k^2$	$(1, 3, 5, \dots, 4k-1)$	2^{2k-1}

Table 8.1: Conditions for the existence of cuspidal pairs.

Digne–Lehrer–Michel using Gauss sums which depends upon two choices, the first being a choice of primitive fourth root of unity in $\overline{\mathbb{Q}}_\ell$. We denote by $\tilde{\jmath} : \mathbb{Q} \to \overline{\mathbb{Q}}_\ell^\times$ the composition of $\jmath : \mathbb{Q}/\mathbb{Z} \to \overline{\mathbb{Q}}_\ell^\times$ with the natural quotient map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. We now define j to be $\tilde{\jmath}(1/4)$, which is a primitive fourth root of unity in $\overline{\mathbb{Q}}_\ell$.

The second choice we need to make is of a square root of p in $\overline{\mathbb{Q}}_\ell$, which we will now do following [**Bon06**, §36]. Assume that r is a power of p then in Definition 7.2 we fixed a non-trivial additive character $\chi_r : \mathbb{F}_r \to \mathbb{F}_p$. We denote by $\theta_r : \mathbb{F}_r^\times \to \overline{\mathbb{Q}}_\ell^\times$ the unique linear character of degree 2 and define the associated Gauss sum to be

$$\mathcal{G}_r(\theta_r) = \sum_{x \in \mathbb{F}_r^\times} \theta_r(x)\chi_r(x)$$

We denote by $p^{\frac{1}{2}}$ our fixed square root of p in $\overline{\mathbb{Q}}_\ell$, which is chosen in the following way

$$p^{\frac{1}{2}} = \begin{cases} \mathcal{G}_1(\theta_1) & \text{if } p \equiv 1 \pmod 4, \\ j^{-1}\mathcal{G}_1(\theta_1) & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Given any $a \in \mathbb{Z}$ we denote by $p^{\frac{a}{2}}$ the term $(p^{\frac{1}{2}})^a$.

In Table 8.1 we have listed the conditions for the existence of a cuspidal pair in special orthogonal and symplectic groups. This information has been adapted from [**Lus84b**] and we see that in any given case there is only one cuspidal pair. The argument we will employ is based on induction and for this to work we will need to know ζ'_{ι_0} where ι_0 is the unique cuspidal pair of $\mathbf{G} = \mathrm{SL}_2(\mathbb{K})$. However this can be deduced from a result of Digne, Lehrer and Michel.

Lemma 8.4 (Digne–Lehrer–Michel). *Assume* $\mathbf{G} = \mathrm{SL}_2(\mathbb{K})$ *where* $p \neq 2$ *and* $q = p^a$. *Let* ι_0 *be the unique cuspidal pair of* $\mathbf{G}$ *then*

$$\zeta'_{\iota_0} = \begin{cases} (-1)^a & \text{if } p \equiv 1 \pmod 4, \\ (-j)^a & \text{if } p \equiv -1 \pmod 4. \end{cases}$$

Proof. Taking $n = e = 2$ in [**DLM97**, Proposition 2.8] and using [**Bon06**, 36.3] for the computation of the Gauss sum $\mathcal{G}_r(\theta_r)$ we have

$$\zeta_{\iota_0} = \begin{cases} (-1)^{a-1} & \text{if } p \equiv 1 \pmod 4, \\ (-1)^{a-1}j^a & \text{if } p \equiv -1 \pmod 4. \end{cases}$$

As $\delta_{\iota_0} = -1$ the statement clearly follows by the definition of ζ'_{ι_0}. ■

Remark 8.5. It should be noted that the statement of [**DLM97**, Proposition 2.8] depends upon the validity of Axiom (A1). However as the full character table of $\mathrm{SL}_2(q)$ is known, this could be computed without this result.

We now turn to the final piece of information we will need regarding the symplectic groups, which involves a simple counting argument. Let $\iota_0 \in \mathcal{N}_{\mathbf{G}}$ be a cuspidal pair then we define $X_{\iota_0} = \{\iota \in \mathcal{N}_{\mathbf{G}}^F \mid \mathcal{O}_\iota = \mathcal{O}_{\iota_0}\}$ where we recall that $\iota = (\mathcal{O}_\iota, \psi_\iota)$.

Lemma 8.6. *Let $\mathbf{G} = \mathrm{Sp}_{2n}(\mathbb{K})$ and assume $n = 1+2+\cdots+k$ for some $k \geqslant 1$. Under this assumption there exists a unique cuspidal pair ι_0 in $\mathcal{N}_{\mathbf{G}}$ and in this situation we have*

$$|\{\iota \in X_{\iota_0} \mid m_\iota \text{ is even}\}| = |\{\iota \in X_{\iota_0} \mid m_\iota \text{ is odd}\}| = 2^{k-1},$$

where $m_\iota = \mathrm{rank}(\mathbf{L}_\iota/Z(\mathbf{L}_\iota))$.

Proof. By [**Lus84b**, 11.6.1] the elements of X_{ι_0} are parameterised by unordered β-pairs $\Lambda = \binom{A}{B}$ such that $0 \notin B$ and $|\Lambda|$ is odd. More specifically we have the following, which follows from [**Lus84b**, Corollary 12.4]. Let $d_{\iota_0} = -k$ if k is odd and $k+1$ if k is even then d_{ι_0} is an odd number satisfying $n = \frac{1}{2}d_{\iota_0}(d_{\iota_0}-1)$. If k is even we have a bijection $\iota \mapsto \binom{A_\iota}{B_\iota}$ between X_{ι_0} and the set of all unordered β-pairs satisfying

- $A_\iota \subseteq \{0,2,4,\dots,2d_{\iota_0}-2\}$,
- $B_\iota \subseteq \{2,4,\dots,2d_{\iota_0}-2\}$,
- $A_\iota \cap B_\iota = \varnothing$ and $A_\iota \cup B_\iota = \{0,2,4,\dots,2d_{\iota_0}-2\}$.

In particular this bijection is such that $\iota_0 \mapsto (\{0,2,4\dots,2d_{\iota_0}-2\},\varnothing)$. If k is odd we again have a bijection $\iota \mapsto \binom{A_\iota}{B_\iota}$ between X_{ι_0} and the set of all unordered β-pairs satisfying

- $A_\iota, B_\iota \subseteq \{1,3,5,\dots,1-2(d_{\iota_0}+1)\}$,
- $A_\iota \cap B_\iota = \varnothing$ and $A_\iota \cup B_\iota = \{1,3,5,\dots,1-2(d_{i_0}+1)\}$.

In particular this bijection is such $\iota_0 \mapsto (\varnothing,\{1,3,5,\dots,1-2(d_{\iota_0}+1)\})$. We denote the defect of $\binom{A_\iota}{B_\iota}$ by d_ι. In both cases we have $\binom{A_{\iota_0}}{B_{\iota_0}}$ is the unique β-pair of defect d_{ι_0} satisfying the above conditions.

The defect of the β-pair is important as it is related to the value m_ι in the following way. For each $\iota \in X_{\iota_0}$ let k_ι be $d_\iota - 1$ if $d_\iota \geqslant 1$ and $-d_\iota$ if $d_\iota \leqslant -1$, (note that $k_{\iota_0} = k$). The rank m_ι is then given by

$$m_\iota = \frac{1}{2}k_\iota(k_\iota + 1).$$

We can now consider the number of pairs $\iota \in X_{\iota_0}$ which have m_ι even and the number of pairs which have m_ι odd. Assume $k_\iota = 2y_\iota$ is even then it is not difficult to see that $m_\iota \equiv y_\iota \pmod 2$. Similarly if $k_\iota = 2y_\iota + 1$ is odd then it is not difficult to see that $m_\iota \equiv y_\iota + 1 \pmod 2$.

Assume $k = 2y$ is even then for each $0 \leqslant x \leqslant k$ there are precisely $\binom{k}{x}$ subsets $B_\iota \subseteq \{2, 4, \dots, 2d_{\iota_0} - 2\}$ such that $|B_\iota| = x$. If $x \leqslant y$ then the β-pairs $\binom{A_\iota}{B_\iota}$ associated to these subsets all have defect $d_\iota = k + 1 - 2x = 2(y - x) + 1 \geqslant 1$. Similarly if $x > y$ then the β-pairs have defect $d_\iota = 2x - (k - 1) = 2(x - y) + 1 \leqslant -1$. By the above we always have $m_\iota \equiv y - x \pmod 2$. The result then follows from the basic fact of binomial coefficients that

$$\sum_{x \text{ even}} \binom{k}{x} = \sum_{x \text{ odd}} \binom{k}{x} = \sum_{z=0}^{k-1} \binom{k-1}{z} = 2^{k-1}. \tag{8.2}$$

Assume $k = 2y + 1$ is odd then for each $0 \leqslant x \leqslant k$ there are precisely $\binom{k}{x}$ subsets $A_\iota \subseteq \{1, 3, \dots, 1 - 2(d_{\iota_0} + 1)\}$ such that $|A_\iota| = x$. The symbols (A_ι, B_ι) associated to these subsets all have defect $d_\iota = 2x - k = 2(x - y) - 1$. By the above we always have $m_\iota \equiv y - x + 1 \pmod 2$. The result again follows from Eq. (8.2). ∎

Remark 8.7. It should be noted that some of the statements of [**Lus84b**, Corollary 12.4] are not quite correct. In particular the description of the symbol corresponding to the cuspidal pair in the case k is odd. This was corrected by Shoji in [**Sho97**, Remark 5.8] and we have used the corrected statement above, (we take this opportunity to thank Frank Lübeck for this reference).

We have now gathered all the preliminary information that we need to prove the following result.

Theorem C. *Assume $\mathbf{G}_n$ is $\mathrm{Sp}_{2n}(\mathbb{K})$ or $\mathrm{SO}_{2n}(\mathbb{K})$ and $p \neq 2$. If such a pair exists we assume ι_0 is the unique cuspidal pair in $\mathcal{N}_{\mathbf{G}_n}$. Let $\varepsilon \in \{\pm 1\}$ be such that $p \equiv \varepsilon \pmod 4$, then we have*

$$\zeta'_{\iota_0} = \begin{cases} \varepsilon^{\frac{an}{2}} & \text{if } n \text{ is even,} \\ (-1)^{an} & \text{if } n \text{ is odd and } \varepsilon = 1, \\ (-j)^{an} & \text{if } n \text{ is odd and } \varepsilon = -1. \end{cases}$$

Proof. The proof of this statement is an adaptation of the proof of [**Gec99**, Theorem

3.8], which is a proof by induction on n. The proof is identical for the case of special orthogonal groups but requires slightly more work for the case of symplectic groups. Note we will let k be the appropriate value as described in Table 8.1. If $n = 0$ then $\mathbf{G}_n$ is a torus and the formula is trivially true. If $n = 1$ then $\mathbf{G}_n = \mathrm{Sp}_2(\mathbb{K}) \cong \mathrm{SL}_2(\mathbb{K})$ and we can see that the formula coincides with that given by Lemma 8.4. If $n = 2$ then $\mathbf{G}_n = \mathrm{SO}_4(\mathbb{K})$ and the simply connected covering $\mathbf{G}_{\mathrm{sc}}$ of $\mathbf{G}_n$ is isomorphic to $\mathrm{SL}_2(\mathbb{K}) \times \mathrm{SL}_2(\mathbb{K})$. By [**Lus84b**, §10.1] ι_0 is the image of the direct product $v_0 \times v_0$, where v_0 is the unique cuspidal pair of $\mathrm{SL}_2(\mathbb{K})$. There are two distinct rational structures on $\mathbf{G}_{\mathrm{sc}}$ either $\mathbf{G}_{\mathrm{sc}}^F = \mathrm{SL}_2(q) \times \mathrm{SL}_2(q)$ or $\mathbf{G}_{\mathrm{sc}}^F = \mathrm{SL}_2(q^2)$. However in both cases we have $\zeta'_{\iota_0} = (\zeta'_{v_0})^2$ and by Lemma 8.4 we can see the formula is valid.

Now we assume that $n \geqslant 3$ and the statement is true for all $\mathbf{G}_m$ with $m < n$. If $\iota \in X_{\iota_0}$ then by [**Lus84b**, §10.4 and §10.6] we have $\mathbf{L}_\iota/Z(\mathbf{L}_\iota)^\circ$ is isomorphic to $\mathbf{G}_{m_\iota}$ for some $m_\iota \leqslant n$. Let N be the number of positive roots of $\mathbf{G}$ then as $\operatorname{rank}\mathbf{G} = \operatorname{rank}\mathbf{L}_\iota$ for all ι we can express b_ι in the following way

$$\begin{aligned} b_\iota &= \frac{1}{2}(\dim\mathbf{G} - \dim\mathcal{O}_\iota - \dim Z(\mathbf{L}_\iota)^\circ), \\ &= \frac{1}{2}(2N + \operatorname{rank}\mathbf{G} - \dim\mathcal{O}_\iota - \dim Z(\mathbf{L}_\iota)^\circ), \\ &= \frac{1}{2}(2N - \dim\mathcal{O}_\iota) + \frac{1}{2}(\operatorname{rank}\mathbf{L}_\iota - \dim Z(\mathbf{L}_\iota)^\circ), \\ &= \dim\mathfrak{B}_u^{\mathbf{G}} + \frac{m_\iota}{2}, \end{aligned}$$

where u is a representative of the class $\mathcal{O}_\iota$. Note that we have used the dimension formula given in [**Car93**, Theorem 5.10.1] to obtain the last equality.

We now consider the criterion given to us from Corollary 8.3. Recall that u is a well-chosen element of a symplectic or special orthogonal group, so in particular $A_{\mathbf{G}}(u)^F = A_{\mathbf{G}}(u)$. Furthermore the component group $A_{\mathbf{G}}(u)$ is abelian which means $\psi_\iota(1) = 1$ for all ι. Using the information we have gathered we may rewrite the sum in Corollary 8.3 in the following way

$$\sum_\iota \zeta'_\iota q^{b_\iota}\psi_\iota(1)^2 = \zeta'_{\iota_0} q^{\dim\mathfrak{B}_u^{\mathbf{G}} + \frac{n}{2}} + \sum_{\iota\neq\iota_0} \zeta'_\iota q^{\dim\mathfrak{B}_u^{\mathbf{G}} + \frac{m_\iota}{2}} = q^{\dim\mathfrak{B}_u^{\mathbf{G}}}\left(\zeta'_{\iota_0} q^{\frac{n}{2}} + \sum_{\iota\neq\iota_0}\zeta'_\iota q^{\frac{m_\iota}{2}}\right).$$

The statement of Corollary 8.3 says that $|A_{\mathbf{G}}(u)|$ divides the expression on the right hand side in the ring of algebraic integers. The order of the component group is a power of 2 therefore as q is a power of an odd prime we must have

$$|A_{\mathbf{G}}(u)| \quad \text{divides} \quad \zeta'_{\iota_0} p^{\frac{an}{2}} + \sum_{\iota\in X_{\iota_0}\setminus\{\iota_0\}} \zeta'_\iota p^{\frac{am_\iota}{2}} \tag{8.3}$$

in the ring of algebraic integers. As $n \geqslant 3$ we have $|A_{\mathbf{G}}(u)| \geqslant 4$ so 4 divides the expression on the right which means we may consider the expression modulo 4. However we must be careful only to reduce integer powers of p modulo 4 and never rational powers.

We wish to consider the possibilities for the terms in the sum modulo 4. Let us assume $\iota \in X_{\iota_0} \setminus \{\iota_0\}$ then reducing only integer powers of p modulo 4 we see

$$\zeta'_\iota p^{\frac{am_\iota}{2}} \equiv \begin{cases} 1 \pmod 4 & \text{if } m_\iota \text{ is even,} \\ (-1)^a p^{\frac{a}{2}} \pmod 4 & \text{if } m_\iota \text{ is odd and } \varepsilon = 1, \\ (-j)^a p^{\frac{a}{2}} \pmod 4 & \text{if } m_\iota \text{ is odd and } \varepsilon = -1. \end{cases}$$

Let us assume $\mathbf{G}_n = \mathrm{SO}_{2n}(\mathbb{K})$. From Table 8.1 we see m_ι is even for all $\iota \in X_{\iota_0}$ then (8.3) tells us

$$4 \quad \text{divides} \quad \zeta'_{\iota_0} p^{\frac{an}{2}} - 1 \tag{8.4}$$

in the ring of algebraic integers. Multiplying the term on the right by the same term with $-$ exchanged by $+$ we have

$$4 \quad \text{divides} \quad (\zeta'_{\iota_0})^2 p^{an} - 1.$$

All the terms on the right are integers so we must have 4 divides this term in $\mathbb{Z}$. In other words $(\zeta'_{\iota_0})^2 p^{an} - 1 \equiv 0 \pmod 4 \Rightarrow (\zeta'_{\iota_0})^2 \equiv 1 \pmod 4$ as n is even, hence $\zeta'_{\iota_0} = \pm 1$. Returning to (8.4) we see that all the values in the expression are integers, so it must be true that 4 divides this term in $\mathbb{Z}$. In particular $\zeta'_{\iota_0} \varepsilon^{\frac{an}{2}} \equiv 1 \pmod 4$, which provides the desired formula.

Now assume $\mathbf{G}_n = \mathrm{Sp}_{2n}(\mathbb{K})$. Using Lemma 8.6 we have the sum in (8.3) can be expressed as

$$\zeta'_{\iota_0} p^{\frac{an}{2}} + \sum_{\iota \in X_{\iota_0} \setminus \{\iota_0\}} \zeta'_\iota p^{\frac{am_\iota}{2}} = \begin{cases} \zeta'_{\iota_0} p^{\frac{an}{2}} + (2^{k-1} - 1) + 2^{k-1} \eta^a p^{\frac{a}{2}} & \text{if } n \text{ is even,} \\ \zeta'_{\iota_0} p^{\frac{an}{2}} + 2^{k-1} + (2^{k-1} - 1) \eta^a p^{\frac{a}{2}} & \text{if } n \text{ is odd.} \end{cases} \tag{8.5}$$

where $\eta = -1$ or $-j$ depending upon the congruence of p modulo 4. We would like to eliminate the rational powers of p. Assume for the moment that $k > 2$ then 4 divides 2^{k-1} so we may simplify extensively the expressions in (8.5). Let us consider the following products

$$(\zeta'_{\iota_0} p^{\frac{an}{2}} - 1)(\zeta'_{\iota_0} p^{\frac{an}{2}} + 1) \qquad \text{and} \qquad (\zeta'_{\iota_0} p^{\frac{an}{2}} - \eta^a p^{\frac{a}{2}})(\zeta'_{\iota_0} p^{\frac{an}{2}} + \eta^a p^{\frac{a}{2}})$$

where the left comes from the case n is even and the right comes from the case n is odd. By (8.3) and (8.5) we have 4 divides these products in the ring of algebraic integers. Expanding the brackets we see all terms in the expression are in $\mathbb{Z}$, hence 4 must divide these expressions in $\mathbb{Z}$. Reducing modulo 4 we get $(\zeta'_{\iota_0})^2 p^{an} - 1 \equiv 0 \pmod 4$, (because $\eta^2 = \varepsilon$ and $\varepsilon^a p^a \equiv 1 \pmod 4$), in particular $(\zeta'_{\iota_0})^2 \equiv \varepsilon^{an} \pmod 4$.

If n is even then the argument is identical to the above argument for $\mathrm{SO}_{2n}(\mathbb{K})$. Therefore let us assume n is odd, then $(\zeta'_{\iota_0})^2 \equiv \varepsilon^a \pmod 4$ hence $\zeta_{\iota_0} = \pm\eta^a$ where η is as in (8.5). Returning to (8.3) and (8.5) we see

$$4 \quad \text{divides} \quad \zeta'_{\iota_0} p^{\frac{an}{2}} - \eta^a p^{\frac{a}{2}}$$

in the ring of algebraic integers. Assume $n = 2x + 1$ then we have $p^{\frac{an}{2}} = p^{ax} p^{\frac{a}{2}} \equiv \varepsilon^{ax} p^{\frac{a}{2}} \pmod 4$. This implies

$$4 \quad \text{divides} \quad (\zeta'_{\iota_0} \varepsilon^{ax} - \eta^a) p^{\frac{a}{2}}$$

in the ring of algebraic integers. Assume for a contradiction that $\zeta'_{\iota_0} = -\varepsilon^{ax}\eta^a$ then this says 4 divides $-2\eta^a p^{\frac{a}{2}}$ in the ring of algebraic integers. In particular there exists an algebraic integer y such that $2\eta^a p^{\frac{a}{2}} = 4y \Rightarrow \eta^a p^{\frac{a}{2}} = 2y$. Squaring this expression we see $\eta^{2a} p^a = 4y^2$ but $\eta^{2a} p^a \in \mathbb{Z}$ and as 2 divides the right hand side we must have 2 divides p^a in $\mathbb{Z}$ but this is impossible as p is odd. Therefore $\zeta'_{\iota_0} = \varepsilon^{ax}\eta^a$ and the result follows by noticing $\eta^{an} = \eta^{2ax}\eta^a = \varepsilon^{ax}\eta^a$.

To finish the proof we must deal with the case where $k = 2$, in other words $n = 3$ and $\mathbf{G}_n = \mathrm{Sp}_6(\mathbb{K})$. In this case $2^{k-1} = 2$, so returning to (8.5) we see

$$\zeta'_{\iota_0} p^{\frac{an}{2}} + \sum_{\iota \in X_{\iota_0} \setminus \{\iota_0\}} \zeta'_\iota p^{\frac{am_\iota}{2}} = \zeta'_{\iota_0} p^{\frac{an}{2}} + 2 + \eta^a p^{\frac{a}{2}}. \tag{8.6}$$

Again we consider the product

$$(\zeta'_{\iota_0} p^{\frac{an}{2}} + 2 + \eta^a p^{\frac{a}{2}})(\zeta'_{\iota_0} p^{\frac{an}{2}} - 2 - \eta^a p^{\frac{a}{2}}) = (\zeta'_{\iota_0})^2 p^{an} - 4 - 4\eta^a p^{\frac{a}{2}} - \eta^{2a} p^a.$$

Reducing this modulo 4 we see $(\zeta'_{\iota_0})^2 \equiv \varepsilon^a \pmod 4$, hence $\zeta_{\iota_0} = \pm\eta^a$ as before. Again we write $n = 2x + 1$ then returning to (8.6) and reducing modulo 4 we have

$$4 \qquad \text{divides} \qquad 2 + (\zeta'_{\iota_0} \varepsilon^{ax} + \eta^a) p^{\frac{a}{2}}$$

in the ring of algebraic integers. If $\zeta'_{\iota_0} = -\varepsilon^{ax}\eta^a$ then the above statement says 4 divides 2 in the ring of integers, a contradiction. Hence we must have $\zeta'_{\iota_0} = \varepsilon^{ax}\eta^a$ and as above this is η^{an} as required. ∎

Remark 8.8. It should be noted that the fourth root of unity given above was already computed by Waldspurger in the context of **[Wal04]**. The above result merely gives an alternative proof for these values.

8.3 Failure of the Argument in Other Cases

One would hope that, assuming p is a good prime, the above argument may work in other cases of groups with a disconnected centre. However in all other cases the argument is doomed to failure for various reasons, which we detail below.

- Assume $\mathbf{G}$ is simple of type A_n then if $\iota_0 \in \mathcal{N}_{\mathbf{G}}$ is cuspidal we must have $\mathcal{O}_{\iota_0}$ is the regular unipotent class. In particular $|A_{\mathbf{G}}(u)| = |Z(\mathbf{G})|$, which is a divisor of $n+1$. In this case the divisibility criterion in Eq. (8.1) will clearly not give us enough information to determine ζ'_{ι_0}. However by work of Digne, Lehrer and Michel we already have good information in the case of type A_n, (see [**DLM97**, Proposition 2.8]).
- Assume $\mathbf{G} = \mathrm{Spin}_{2n}(\mathbb{K})$ is a spin group and F acts trivially on $Z(\mathbf{G})$. We have an isogeny $\pi : \mathrm{Spin}_{2n}(\mathbb{K}) \to \mathrm{SO}_{2n}(\mathbb{K})$. Let us denote the kernel of π by $\mathrm{Ker}(\pi) = \{1, \vartheta\}$. Assume $n = 1 + \cdots + k$ then there exists a cuspidal pair $\iota_0 \in \mathcal{N}_{\mathbf{G}}$ such that $\psi_{\iota_0}(\vartheta) \neq 1$, where here we identify ϑ with its image in $A_{\mathbf{G}}(u)$. In particular this cuspidal pair does not correspond to a cuspidal pair in $\mathrm{SO}_{2n}(\mathbb{K})$, (under the correspondence described in [**Lus84b**, §10.1]).

  Under this condition we see that ψ_{ι_0} cannot be a linear character of $A_{\mathbf{G}}(u)$ because every linear character has ϑ in its kernel. Hence $\psi_{\iota_0}(1)^2 = 2^{k-2}$, which means when $k \geqslant 4$ and we reduce the sum in Eq. (8.1) modulo 4 the term containing ζ'_{ι_0} will go to zero.
- Assume $\mathbf{G}$ is of type E_6 then there are two cuspidal pairs in $\mathcal{N}_{\mathbf{G}}$ corresponding to the two characters of $Z(\mathbf{G})$. However the two cuspidal pairs ι_1, ι_2 are such that $\mathcal{O}_{\iota_1} = \mathcal{O}_{\iota_2}$. Hence both ζ'_{ι_1}, ζ'_{ι_2} appear in the sum contained in Eq. (8.1) so it will not be possible to distinguish them. Note that if F acts non-trivially on $Z(\mathbf{G})$ then these cuspidal pairs are not in $\mathcal{N}_{\mathbf{G}}^F$.
- Finally assume $\mathbf{G}$ is of type E_7. We can run the argument without issues until the end, where we are left with a divisibility criterion in the ring of algebraic integers which is ineffectual.

Appendix A

Unipotent Conjugacy Classes

In this appendix we consider some of the ideas behind the classification of unipotent conjugacy classes in algebraic groups. We discuss some of the aspects of Dynkin–Kostant theory and the idea of a weighted Dynkin diagram. The topics covered in this appendix are covered in more detail in [**CM93**, Chapter 3], [**Car93**, Chapter 5] and [**Hum95**, Chapter 7].

Recall that the classification of unipotent classes for connected reductive algebraic groups does not depend upon the centre of the algebraic group. Therefore let $\mathbf{G}$ be a simple algebraic group of adjoint type and $\mathfrak{g}$ its associated Lie algebra with Lie bracket $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. We let $\mathcal{N}$ denote the *nilpotent cone* of $\mathfrak{g}$, (i.e. the subset of all nilpotent elements in $\mathfrak{g}$), and let $\mathcal{U}$ denote the unipotent variety of $\mathbf{G}$, (i.e. the subset of all unipotent elements of $\mathbf{G}$).

Recall that for every $x \in \mathbf{G}$ we have defined a conjugation morphism $\operatorname{inn} x : \mathbf{G} \to \mathbf{G}$. As $\operatorname{Inn} x$ is an automorphism of $\mathbf{G}$ its differential, which we denote by $\operatorname{Ad} x$, is an automorphism of $\mathfrak{g}$. Hence we have a homomorphism $\operatorname{Ad} : \mathbf{G} \to \operatorname{Aut}(\mathfrak{g})$. This is not to be confused with the adjoint homomorphism $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ given by $(\operatorname{ad} x)(y) = [x, y]$ for all $x, y \in \mathfrak{g}$ (by [**Hum75**, §10.4 - Theorem] this is the differential of Ad).

Springer has shown that if p is a very good prime for $\mathbf{G}$ then there is a bijective morphism of varieties $\phi : \mathcal{U} \to \mathcal{N}$ such that $\phi \circ \operatorname{inn} x = \operatorname{Ad} x \circ \phi$ for all $x \in \mathbf{G}$. Therefore if $\mathcal{U}/\mathbf{G}$ denotes the set of unipotent conjugacy classes and $\mathcal{N}/\operatorname{Ad}\mathbf{G}$ denotes the orbits of $\mathcal{N}$ under the action of $\operatorname{Ad}\mathbf{G}$ we have ϕ induces a bijection $\mathcal{U}/\mathbf{G} \to \mathcal{N}/\operatorname{Ad}\mathbf{G}$. The main idea in understanding $\mathcal{U}/\mathbf{G}$ is to instead try and understand $\mathcal{N}/\operatorname{Ad}\mathbf{G}$, which is in some sense easier.

If we let $\mathfrak{t}_0$ be the Lie algebra of $\mathbf{T}_0$ and $\mathfrak{b}_0$ be the Lie algebra of $\mathbf{B}_0$ then $\mathfrak{t}_0 \subseteq \mathfrak{b}_0$ is a Cartan subalgebra and Borel subalgebra of $\mathfrak{g}$. Differentiating the roots Φ gives the roots of the Lie algebra with respect to $\mathfrak{t}_0$. We sloppily denote the set of roots for $\mathbf{G}$

and $\mathfrak{g}$ using the same notation.

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Lie subalgebra which is isomorphic to $\mathfrak{sl}_2(\mathbb{K})$ then there exists a subset $\{e, h, f\} \subseteq \mathfrak{g}$, which spans $\mathfrak{h}$ and satisfies the bracket relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. We call $\{e, h, f\}$ a *standard triple* for $\mathfrak{h}$. We wish to try and reduce the problem of classifying nilpotent orbits in the Lie algebra to classifying orbits of subalgebras isomorphic to $\mathfrak{sl}_2(\mathbb{K})$. In other words let $\mathcal{H} = \{\mathfrak{h} \subseteq \mathfrak{g} \mid \mathfrak{h} \cong \mathfrak{sl}_2(\mathbb{K})\}$ be the collection of all subalgebras isomorphic to $\mathfrak{sl}_2(\mathbb{K})$ then we want to show that the map

$$\Omega : \mathcal{H}/\operatorname{Ad}\mathbf{G} \to \mathcal{N}/\operatorname{Ad}\mathbf{G} \qquad\qquad \Omega(\{e, h, f\}) = e$$

is a bijection. Note $\mathcal{H}/\operatorname{Ad}\mathbf{G}$ and $\mathcal{N}/\operatorname{Ad}\mathbf{G}$ denote the $\operatorname{Ad}\mathbf{G}$-orbits of $\mathcal{H}$ and $\mathcal{N}$ and $\{e, h, f\}$ is a standard triple for an algebra in $\mathcal{H}$. The surjectivity of this map comes from the Jacobson–Morozov theorem.

Theorem A.1 (Jacobson–Morozov, [Car93, Theorem 5.3.2]). *Let $\mathbf{G}$ be a simple algebraic group and p be a good prime for $\mathbf{G}$. Let $e \in \mathfrak{g}$ be a non-zero nilpotent element with $(\operatorname{ad} e)^m = 0$ where $m \leqslant p - 2$. There exists a subalgebra $\mathfrak{h} \in \mathcal{H}$ with elements $f, h \in \mathfrak{h}$ such that $\{e, h, f\}$ is a standard triple for $\mathfrak{h}$.*

An alternative way to view the Jacobson–Morozov theorem is the following: given a nilpotent element $e \in \mathfrak{g}$ satisfying the condition of the theorem we can always find a homomorphism $\varphi_e : \mathfrak{sl}_2(\mathbb{K}) \to \mathfrak{g}$ such that $\varphi_e(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]) = e$. The injectivity comes from the following result.

Proposition A.2 ([Car93, Proposition 5.5.10]). *Assume $p > 3(h-1)$, (where h is the Coxeter number of $\mathbf{G}$), and let $e \in \mathfrak{g}$ be a non-zero nilpotent element. Given any two standard triples (e, h_1, f_1), (e, h_2, f_2) there exists an element $g \in \mathbf{G}$ such that $(\operatorname{Ad} g)(e) = e$, $(\operatorname{Ad} g)(h_1) = h_2$ and $(\operatorname{Ad} g)(f_1) = f_2$.*

By [**Car93**, Proposition 5.5.2] we know the conditions of Theorem A.1 are satisfied for all nilpotent elements of $\mathfrak{g}$ whenever $p > 2h$. Therefore, by the above two results, we know Ω is a bijection whenever $p > 3(h-1)$ and $p > 2h$. However if $h \neq 2$ then $3(h-1) \geqslant 2h$ so the condition $p > 3(h-1)$ is sufficient. If $h = 2$ then $3(h-1) = 3$ and $2h = 4$, however if $p > 3(h-1) = 3$ then p must be at least 5 so $p > 2h = 4$. Hence the condition $p > 3(h-1)$ is sufficient for the bijection to hold.

From now until the end of this appendix we assume $p > 3(h-1)$, where h is the Coxeter number of $\mathbf{G}$.

A.1 Weighted Dynkin Diagrams

The bijection Ω says that we can reduce the problem of classifying $\mathcal{N}/\operatorname{Ad}\mathbf{G}$ to that of classifying $\mathcal{H}/\operatorname{Ad}\mathbf{G}$ but this is still not quite good enough. What we would like is a way to describe an orbit $\mathcal{O} \in \mathcal{N}/\operatorname{Ad}\mathbf{G}$ combinatorially. Such a combinatorial description of the orbits comes from the notion of a weighted Dynkin diagram.

Definition A.3. We define a *weighted Dynkin diagram* to be a Dynkin diagram with a number from the set $\{0,1,2\}$ attached to each node.

We describe how we can attach a weighted Dynkin diagram to any non-zero nilpotent orbit $\mathcal{O}$. Let $e \in \mathcal{O}$ and $\mathfrak{h} \in \mathcal{H}$ be a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{K})$, with associated standard triple $\{e,h,f\}$ such that $\Omega(\{e,h,f\}) = e$. We can consider the Lie algebra $\mathfrak{g}$ as an $\mathfrak{h}$-module under the adjoint action of $\operatorname{ad}\mathfrak{h}$ on $\mathfrak{g}$. Due to the restriction on p by [**Car93**, Theorem 5.4.8] we have $\mathfrak{g}$ is semisimple as an $\mathfrak{h}$-module so decomposes into a direct sum of simple $\mathfrak{h}$-modules.

If M is a simple $\mathfrak{h}$-module then the eigenvalues of the map $h : M \to M$ lie in $\mathbb{Z}$, (see [**Car93**, Proposition 5.5.3]). As $\mathfrak{g}$ is a direct sum of simple $\mathfrak{h}$-modules we have the eigenvalues of the map $\operatorname{ad}h : \mathfrak{g} \to \mathfrak{g}$ also lie in $\mathbb{Z}$. In particular we get a direct sum decomposition of $\mathfrak{g}$ into weight spaces

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i) \qquad \text{where} \qquad \mathfrak{g}(i) := \{x \in \mathfrak{g} \mid (\operatorname{ad}h)(x) = [h,x] = ix\}.$$

As h is semisimple we can find a Cartan subalgebra $\mathfrak{t}_h$ of $\mathfrak{g}$ such that $h \in \mathfrak{t}_h$. However as all maximal tori of $\mathbf{G}$ are conjugate there exists an element $g \in \mathbf{G}$ such that $(\operatorname{Ad}g)(\mathfrak{t}_h) = \mathfrak{t}_0$. Therefore replacing $\{e,h,f\}$ by $\{(\operatorname{Ad}g)(e),(\operatorname{Ad}g)(h),(\operatorname{Ad}g)(f)\}$ we can assume h lies in $\mathfrak{t}_0$. Note replacing e by $(\operatorname{Ad}g)(e)$ clearly doesn't affect the orbit $\mathcal{O}$. Furthermore, there exists an element $g' \in N_{\mathbf{G}}(\mathbf{T}_0)$ such that $(\operatorname{Ad}g')(h)$ is in the fundamental domain of the Weyl group acting on $\mathfrak{t}_0$. If the standard triple for $\mathcal{O}$ is chosen in this way we call it a *fundamental standard triple* for $\mathcal{O}$.

Recall that with respect to the Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{g}$ we have the following Cartan decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{t}_0 \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha \qquad \text{where} \qquad \mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid (\operatorname{ad}t)(x) = \alpha(t)x \text{ for all } t \in \mathfrak{t}\}.$$

The root space $\mathfrak{g}_\alpha$ is 1-dimensional so we can assume $\mathfrak{g}_\alpha = \mathbb{K}e_\alpha$ for some $e_\alpha \in \mathfrak{g}_\alpha$. By the decomposition of $\mathfrak{g}$ with respect to $\operatorname{ad}h$ we have e_α lies in $\mathfrak{g}(i)$ for some $i \in \mathbb{Z}$ so we may define $n_\alpha \in \mathbb{Z}$ to be such that $(\operatorname{ad}h)e_\alpha = n_\alpha e_\alpha$ for each $\alpha \in \Phi$.

Proposition A.4 ([Car93, Lemma 5.6.5 and Proposition 5.6.6]**).** *Assuming that the standard triple $\{e,h,f\}$ is fundamental we have for any simple root $\alpha \in \Delta$ that $n_\alpha \in \{0,1,2\}$.*

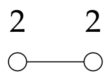
Therefore to $\mathcal{O}$ we associate the weighted Dynkin diagram $\Delta(\mathcal{O})$ whose nodes, indexed by simple roots $\alpha \in \Delta$, are labelled by the corresponding values n_α. If $\mathcal{O}$ is the zero nilpotent orbit then we do the same except we take $n_\alpha = 0$ for each $\alpha \in \Delta$.

Example A.5. Consider $\mathbf{G} = \mathrm{SL}_3(\mathbb{K})$ then the associated Lie algebra is $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{K})$, (see for instance [**Gec03**, Example 1.5.8]). Take $\mathbf{T}_0 \leqslant \mathbf{B}_0$ to be the standard maximal torus of diagonal matrices and Borel subgroup of upper triangular matrices. We consider the following nilpotent element $e \in \mathfrak{g}$ and corresponding elements $h, f \in \mathfrak{g}$

$$e = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad h = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad f = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

The triple $\{e, h, f\}$ is a standard triple containing the nilpotent element $e \in \mathfrak{g}$. The roots of $\mathfrak{sl}_3(\mathbb{K})$ are given by $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2 - \varepsilon_3$ where $\varepsilon_i(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) = \lambda_i$ for $1 \leqslant i \leqslant 3$. It is clear that $\alpha_1(h) = \alpha_2(h) = 2$ so h is in the fundamental domain of $\mathbf{W}$ acting on $\mathfrak{t}_0$, in particular the standard triple is fundamental.

Let e_{α_1} and e_{α_2} be the standard basis elements for the root spaces $\mathfrak{g}_{\alpha_1}$ and $\mathfrak{g}_{\alpha_2}$. By the very definition of the root space we have $(\mathrm{ad}\, h)(e_{\alpha_1}) = 2e_{\alpha_1}$ and $(\mathrm{ad}\, h)(e_{\alpha_2}) = 2e_{\alpha_2}$ so $n_{\alpha_1} = n_{\alpha_2} = 2$. It is clear to see that the weighted Dynkin diagram of the nilpotent orbit containing e is given by

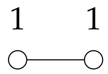


This makes sense as the orbit containing e is the regular nilpotent orbit. In a group of type A_{n-1} we know every node of the weighted Dynkin diagram of the regular unipotent class has weight 2, (see [**CM93**, Example 4.1.9]).

Now consider the following nilpotent element $e' \in \mathfrak{g}$ and corresponding elements $h', f' \in \mathfrak{g}$

$$e' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad h' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad f' = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The set $\{e', h', f'\}$ is a standard triple containing the nilpotent element $e' \in \mathfrak{g}$. We have $\alpha_2(h') = -1$ therefore h' does not lie in the fundamental chamber. The Weyl group is the symmetric group $\mathfrak{S}_3$, which acts on the Cartan subalgebra $\mathfrak{t}_0$ by permuting the diagonal entries. We can take a conjugate standard triple $\{e'', h'', f''\}$ such that $h'' = \mathrm{diag}(1, 0, -1)$ then h'' is in the fundamental chamber because $\alpha_1(h'') = \alpha_2(h'') = 1$. In particular we have $\{e'', h'', f''\}$ is a fundamental standard triple for the orbit containing e. As above we have $n_{\alpha_1} = n_{\alpha_2} = 1$ so the weighted Dynkin diagram of the nilpotent orbit containing e' is given by



These two diagrams together with the zero diagram give all the weighted Dynkin diagrams of nilpotent orbits in $\mathfrak{g}$.

Theorem A.6 ([Car93, Proposition 5.6.7 and 5.6.8]). *Let $\mathcal{O}$, $\mathcal{O}' \in \mathcal{N} / \operatorname{Ad} \mathbf{G}$ be nilpotent orbits. The weighted Dynkin diagram $\Delta(\mathcal{O})$ does not depend upon the choice of nilpotent element $e \in \mathcal{O}$ used to define it. Furthermore we have $\Delta(\mathcal{O}) = \Delta(\mathcal{O}') \Leftrightarrow \mathcal{O} = \mathcal{O}'$.*

Let $\mathcal{D} = \{\Delta(\mathcal{O}) \mid \mathcal{O} \in \mathcal{N} / \operatorname{Ad} \mathbf{G}\}$ be the set of all weighted Dynkin diagrams associated to nilpotent orbits. The above theorem tells us that the map $\Delta : \mathcal{N} / \operatorname{Ad} G \to \mathcal{D}$ given by $\mathcal{O} \mapsto \Delta(\mathcal{O})$ is a bijection.

To finish our discussion of weighted Dynkin diagrams we consider how to associate to a class $\mathcal{O}$ a canonical parabolic and Levi subgroup using the Dynkin diagram $\Delta(\mathcal{O})$. Given $\Delta(\mathcal{O})$ we define an additive function $\eta : \Phi \to \mathbb{Z}$ in the following way. Let $\alpha \in \Phi$ be a root and $\alpha = \sum_{\beta \in \Delta} x_\beta \beta$ be its expression as a sum of simple roots. We define $\eta(\alpha) = \sum_{\beta \in \Delta} x_\beta n_\beta$, where n_β are the weights on the diagram $\Delta(\mathcal{O})$. Then we define two subgroups

$$\begin{aligned}
\mathbf{P}_\eta &= \langle \mathbf{T}_0, \mathbf{X}_\alpha \mid \alpha \in \Phi \text{ and } \eta(\alpha) \geqslant 0 \rangle, \\
\mathbf{L}_\eta &= \langle \mathbf{T}_0, \mathbf{X}_\alpha \mid \alpha \in \Phi \text{ and } \eta(\alpha) = 0 \rangle,
\end{aligned}$$

where $\mathbf{X}_\alpha$ is the root subgroup associated to $\alpha \in \Phi$. The group $\mathbf{P}_\eta$ is a parabolic subgroup of $\mathbf{G}$ because it contains $\mathbf{X}_\alpha$ for all positive roots $\alpha \in \Phi^+$ hence contains a Borel subgroup of $\mathbf{G}$. The group $\mathbf{L}_\eta \leqslant \mathbf{P}_\eta$ is a Levi complement of $\mathbf{P}_\eta$ because for all roots $\alpha \in \Phi$ we have $\eta(\alpha) = 0 \Leftrightarrow \eta(-\alpha) = 0$. We call $\mathbf{P}_\eta$, (resp. $\mathbf{L}_\eta$), the *canonical parabolic subgroup*, (resp. *canonical Levi subgroup*), associated to the class $\mathcal{O}$.

Although the weighted Dynkin diagrams are a good combinatorial object to classify the nilpotent orbits there is a small problem. It is very difficult to say whether a given weighted Dynkin diagram is the diagram of some nilpotent orbit. In fact it is known that very few weighted Dynkin diagrams arise as diagrams associated to nilpotent orbits. Even in Example A.5 we see that there are 9 possible weighted Dynkin diagrams but only 3 nilpotent orbits. In general this problem is, in some sense, solved by the Bala–Carter theorem.

A.2 The Bala–Carter Theorem

The idea of the Bala–Carter theorem is to try and identify a unipotent class by its intersection with Levi subgroups. The Bala–Carter theorem uses, in a recursive fashion, an idea originally attributed to Richardson.

Theorem A.7 (Richardson, [Car93, Theorem 5.2.1]). *Let $\mathbf{G}$ be a connected reductive algebraic group and $\mathbf{P} \leqslant \mathbf{G}$ a parabolic subgroup with unipotent radical $\mathbf{U_P} \leqslant \mathbf{P}$. There exists a unique unipotent conjugacy class $\mathcal{O} \in \mathcal{U}/\mathbf{G}$ such that $\mathcal{O} \cap \mathbf{U_P}$ is an open dense subset of $\mathbf{U_P}$. Furthermore $\mathcal{O} \cap \mathbf{U_P}$ is a single orbit under the conjugation action of $\mathbf{P}$ on $\mathbf{U_P}$.*

Richardson's theorem then tells us that to every parabolic subgroup of $\mathbf{G}$ we can associate a unique unipotent conjugacy class of $\mathbf{G}$ and this class determines a single orbit in the unipotent radical of the parabolic. The theorem of Richardson leads us to the following definition.

Definition A.8. Let $\mathbf{G}$ be a connected reductive algebraic group and $\mathcal{O}$ a unipotent class of $\mathbf{G}$. We say $\mathcal{O}$ is a *Richardson class* if there exists a parabolic subgroup $\mathbf{P} \leqslant \mathbf{G}$ with unipotent radical $\mathbf{U_P} \leqslant \mathbf{P}$ such that $\mathcal{O} \cap \mathbf{U_P}$ is an open dense subset of $\mathbf{U_P}$.

Upon reading Richardson's theorem one may think that it is possible to classify unipotent classes by parabolic subgroups. However it is known that not every unipotent class of a reductive algebraic group is a Richardson class but that this is the case in type A. What we wish to do is identify a special type of parabolic subgroup for the Bala–Carter classification.

Definition A.9. Let $\mathbf{G}$ be a connected reductive algebraic group with parabolic subgroup $\mathbf{P} \leqslant \mathbf{G}$. Let $\mathbf{P} = \mathbf{L_P U_P}$ be a Levi decomposition of the parabolic subgroup then we write $\mathbf{U}_\mathbf{P}'$ for the derived subgroup of $\mathbf{U_P}$. We say $\mathbf{P}$ is a *distinguished parabolic subgroup* if $\dim \mathbf{L_P} = \dim \mathbf{U_P}/\mathbf{U}_\mathbf{P}'$.

With this definition we can now state the Bala–Carter classification theorem for unipotent classes of $\mathbf{G}$.

Theorem A.10 (Bala-Cater, [Car93, Theorem 5.9.6]). *Let $\mathbf{G}$ be a simple algebraic group of adjoint type and assume $p > 3(h-1)$. There is a bijective map between $\mathcal{U}/\mathbf{G}$ and $\mathbf{G}$-conjugacy classes of pairs $(\mathbf{L}, \mathbf{P}_{\mathbf{L}'})$ where $\mathbf{L} \leqslant \mathbf{G}$ is a Levi subgroup of $\mathbf{G}$ and $\mathbf{P}_{\mathbf{L}'}$ is a distinguished parabolic subgroup of the derived subgroup of the Levi $\mathbf{L}'$. The unipotent class $\mathcal{O} \in \mathcal{U}/\mathbf{G}$ corresponding to a pair $(\mathbf{L}, \mathbf{P}_{\mathbf{L}'})$ is such that $\mathcal{O} \cap \mathbf{L}'$ is a Richardson class associated to $\mathbf{P}_{\mathbf{L}'}$.*

Example A.11. Assume $\mathbf{G}$ is simple of type A_n. By the classification of distinguished parabolic subgroups given in [Car93, §5.9] we know that a parabolic subgroup of $\mathbf{G}$ is distinguished if and only if it is a Borel subgroup. Therefore the Bala–Carter theorem says that there is a bijection between unipotent classes of $\mathbf{G}$ and conjugacy classes of Levi subgroups. This bijection is such that if $\lambda = (\lambda_1, \dots, \lambda_s)$ is a partition of $n+1$ recording the sizes of the Jordan blocks of a representative of the class $\mathcal{O}$ under an embedding $\mathbf{G} \to \mathrm{GL}_{n+1}(\mathbb{K})$ then the corresponding Levi subgroup has semisimple type $\mathrm{A}_{\lambda_1-1} \times \cdots \times \mathrm{A}_{\lambda_s-1}$.

Let us return to the situation of Example A.5. There are three conjugacy classes of Levi subgroups of $\mathbf{G}$. These have representatives: $\mathbf{G}$ itself, the maximal torus $\mathbf{T}_0$ and the subgroup

$$\left\{ \begin{bmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & \star \end{bmatrix} Z \right\},$$

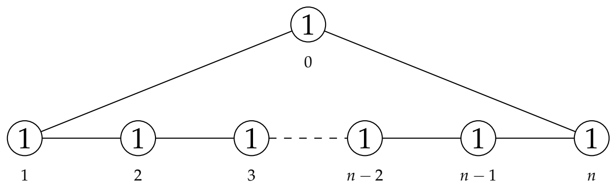
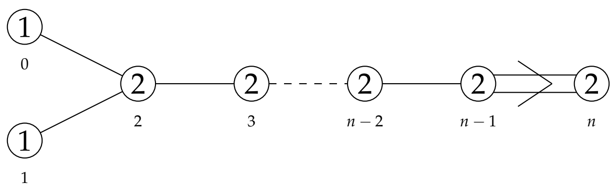
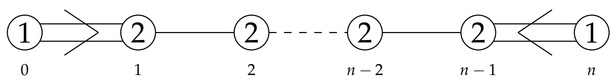
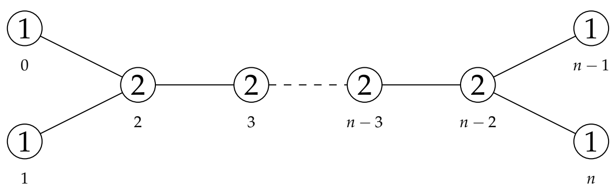
(where $\star$ denotes an appropriate matrix entry and Z denotes the centre of $\mathrm{GL}_3(\mathbb{K})$). The trivial unipotent class clearly corresponds to $\mathbf{T}_0$ and the regular unipotent class corresponds to $\mathbf{G}$. Therefore there is only one class remaining and this corresponds to the Levi subgroup outlined above.

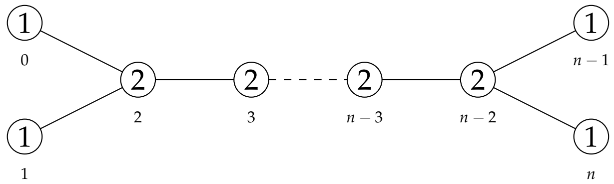
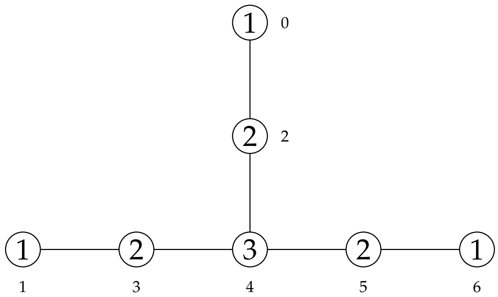
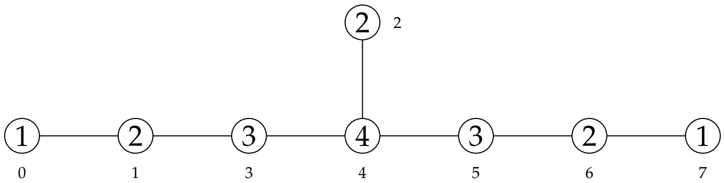
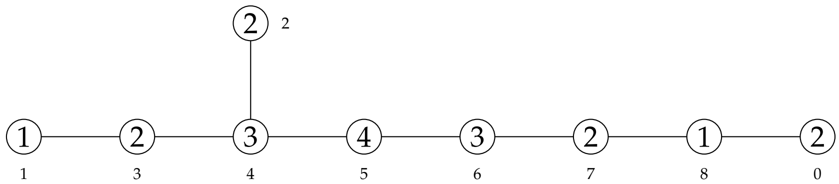
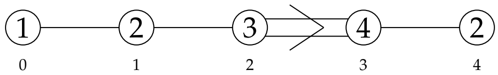
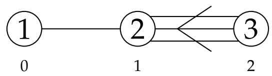
We end by discussing the restriction on p. Pommerening has shown, by an indepth case by case analysis, that the Bala Carter theorem is true whenever p is a good prime, (see [**Car95**, §5.11]). However it should also be mentioned that Premet in [**Pre03**] has given a completely case free proof of the Bala–Carter theorem in good characteristic. It is shown in [**Kaw86**, Theorem 2.1.1] that the classification by weighted Dynkin diagrams also holds when p is a good prime. Therefore we may assume that the results mentioned in this appendix hold in good characteristic.

Appendix B

Affine Dynkin Diagrams

In this appendix we reproduce a table of Bonnafé, (see [**Bon05**, Table 1]), listing the affine Dynkin diagrams of each simple root system. The numbers underneath the nodes of each diagram represent the indices for the simple roots α_i, where α_0 is the negative of the highest root. The number inside a node represents the value m_α associated to the simple root. In this table we also list the isomorphism type of $\mathcal{A} = \mathrm{Aut}_{\mathbf{W}}(\tilde{\Delta})$; we describe the action of $\mathcal{A}$ on the roots explicitly below.

Type	$\tilde{\Delta}$	$\mathcal{A}$
A_n		$\mathbb{Z}_{n+1}$
B_n		$\mathbb{Z}_2$
C_n		$\mathbb{Z}_2$
D_n (n even)		$\mathbb{Z}_2 \times \mathbb{Z}_2$

Type	$\tilde{\Delta}$	$\mathcal{A}$
D_n (n odd)		$\mathbb{Z}_4$
E_6		$\mathbb{Z}_3$
E_7		$\mathbb{Z}_2$
E_8		1
F_4		1
G_2		1

It will be useful to know how the group $\mathcal{A}$ acts on the root system of each simple type. We recall this information from [**Bou02**, Plates I-IX(XII)] and denote a set of simple roots for each type by $\{\alpha_j\}$.

Type A_n The single generator σ acts by mapping $\alpha_n \mapsto \alpha_0$ and $\alpha_j \mapsto \alpha_{j+1}$ for all $0 \leqslant j \leqslant n-1$.

Type B_n The single generator σ acts by exchanging α_0 and α_1 and fixes all α_j for $2 \leqslant j \leqslant n$.

Type C_n The single generator σ acts by mapping $\alpha_j \mapsto \alpha_{n-j}$ for all $0 \leqslant j \leqslant n$.

Type D_n We denote the elements of $\mathcal{A}$ by the set $\{1, \sigma_1, \sigma_{n-1}, \sigma_n\}$. If $n \equiv 0 \pmod 2$ then the element σ_{n-1} acts by exchanging the elements in the sets $\{\alpha_0, \alpha_{n-1}\}$, $\{\alpha_1, \alpha_n\}$, $\{\alpha_j, \alpha_{n-j}\}$ for each $2 \leqslant j \leqslant n-2$. The element σ_n acts by exchanging α_j with α_{n-j} for all $0 \leqslant j \leqslant n$. Furthermore $\mathcal{A}$ is generated by σ_{n-1} and σ_n.

If $n \equiv 1 \pmod 2$ then the element σ_n acts by mapping $\alpha_0 \mapsto \alpha_n \mapsto \alpha_1 \mapsto \alpha_{n-1} \mapsto \alpha_0$ and exchanges α_j with α_{n-j} for $2 \leqslant j \leqslant n-2$. Furthermore $\mathcal{A}$ is generated by σ_n.

The element σ_1 always acts by exchanging the elements in the sets $\{\alpha_0, \alpha_1\}$, $\{\alpha_{n-1}, \alpha_n\}$ and fixes α_j for all $2 \leqslant j \leqslant n-2$.

Type E_6 The single generator σ acts by mapping $\alpha_0 \mapsto \alpha_6 \mapsto \alpha_1 \mapsto \alpha_0$ and $\alpha_2 \mapsto \alpha_5 \mapsto \alpha_3 \mapsto \alpha_2$ and fixes α_4.

Type E_7 The single generator σ exchanges the elements in the sets $\{\alpha_0, \alpha_7\}$, $\{\alpha_1, \alpha_6\}$, $\{\alpha_3, \alpha_5\}$ and fixes the roots α_4 and α_8.

Recall from [**Bon05**, 3.7] that we have an isomorphism $\mathcal{A} \to \check{\Pi}$. If $\mathbf{G}$ is a connected simply connected simple algebraic group and p is a very good prime for $\mathbf{G}$ then by Lemma 1.10 we also have an isomorphism $\check{\Pi} \to Z(\mathbf{G})$. By composing these isomorphisms we have an isomorphism $\mathcal{A} \to Z(\mathbf{G})$. We wish to describe this isomorphism especially in the case where $\mathbf{G}$ is of type D.

We describe the isomorphism $\mathcal{A} \to \check{\Pi}$ following [**Bon05**, §3.B]. Let $\alpha_j \in \tilde{\Delta}$ be a root for some j then we denote by Δ_j the set $\Delta \setminus \{\alpha_j\}$. We write $\Phi_j \subseteq \Phi$ for the parabolic subsystem generated by the set Δ_j and $\mathbf{W}_j = \langle w_\alpha \mid \alpha \in \Delta_j \rangle$ the corresponding parabolic subgroup of $\mathbf{W}$. Let $\Phi_j^+ = \Phi_j \cap \Phi^+$ be a system of positive roots for Φ_j then we denote by $w_j \in \mathbf{W}_j$ the unique element such that $w_j(\Phi_j^+) = -\Phi_j^+$, (i.e. the longest word in $\mathbf{W}_j$). Define $x_j = w_j w_0 \in \mathbf{W}$ then $\mathcal{A} = \{x_j \mid m_{\alpha_j} = 1\}$ by [**Bon05**, §3.5]. The isomorphism $\mathcal{A} \to \check{\Pi}$ is then given by $x_j \mapsto \check{\varpi}_j + \mathbb{Z}\check{\Phi}$.

We consider what this means for a simply connected group of type D_n. From the diagrams above we see that in type D_n we have $\mathcal{A} = \{x_0, x_1, x_{n-1}, x_n\}$. It is clear from the description that x_0 is the identity. If j is $n-1$ or n then it is easy to determine the action of x_j because the longest word in $\mathbf{W}_j$ will induce the unique non-trivial graph automorphism on the root system Δ_j of type A_{n-1}. Furthermore the longest element $w_0 \in \mathbf{W}$ will induce no graph automorphism if n is even and the unique graph automorphism of order 2 if n is odd. Comparing with Section 1.2.1 we see that we have chosen the labelling such that $\sigma_j \mapsto \hat{z}_j$, (for $j \in \{1, n-1, n\}$), under the isomorphism $\mathcal{A} \to Z(\mathbf{G})$.

Appendix C

Dictionaries For j-Induction

In this appendix we give two tables which are designed to make the task of verifying the j-induction for E_6 and E_7 easier. For each parahoric subgroup $\mathbf{W}(\tilde{s})$ we list the characters $\rho \in \mathrm{Irr}(\mathbf{W}(\tilde{s}))$ and $j_{\mathbf{W}(\tilde{s})}^{\mathbf{W}}(\rho)$ in the notation of this thesis and the notation used in CHEVIE.

Thesis			CHEVIE	
$\mathbf{W}(\tilde{s})$	ρ	$j_{\mathbf{W}(\tilde{s})}^{\mathbf{W}}(\rho)$	ρ	$j_{\mathbf{W}(\tilde{s})}^{\mathbf{W}}(\rho)$
$A_5\,A_1$	$[123456] \boxtimes [12]$	$15'_q$	$111111, 11$	$\phi_{15,16}$
$A_5\,A_1$	$[1245] \boxtimes [12]$	60_s	$2211, 11$	$\phi_{60,8}$
$A_2\,A_2\,A_2$	$[123] \boxtimes [123] \boxtimes [123]$	10_s	$111, 111, 111$	$\phi_{10,9}$
$A_2\,A_2\,A_2$	$[3] \boxtimes [3] \boxtimes [3]$	1_p	$3, 3, 3$	$\phi_{1,0}$
$A_1\,A_1\,A_1\,A_1$	$[12] \boxtimes [12] \boxtimes [12] \boxtimes [12]$	15_q	$11, 11, 11, 11$	$\phi_{15,4}$
D_4	$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$	$24'_p$	$.1111$	$\phi_{24,12}$
D_4	$\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$	30_p	1.21	$\phi_{30,3}$
D_4	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$	6_p	1.3	$\phi_{6,1}$

Table C.1: Dictionary For Type E_6.

Thesis			CHEVIE	
$\mathbf{W}(\tilde{s})$	ρ	$j_{\mathbf{W}(\tilde{s})}^{\mathbf{W}}(\rho)$	ρ	$j_{\mathbf{W}(\tilde{s})}^{\mathbf{W}}(\rho)$
E_6	6_p	$7'_a$	$\phi_{6,1}$	$\phi_{7,1}$
E_6	20_p	27_a	$\phi_{20,2}$	$\phi_{27,2}$
E_6	30_p	$56'_a$	$\phi_{30,3}$	$\phi_{56,3}$

Thesis			CHEVIE	
$\mathbf{W}(\tilde{s})$	ρ	$j_{\mathbf{W}(\tilde{s})}^{\mathbf{W}}(\rho)$	ρ	$j_{\mathbf{W}(\tilde{s})}^{\mathbf{W}}(\rho)$
E_6	80_s	$315'_a$	$\phi_{80,7}$	$\phi_{315,7}$
E_6	$30'_p$	$405'_a$	$\phi_{30,15}$	$\phi_{405,15}$
E_6	$20'_p$	189_c	$\phi_{20,20}$	$\phi_{189,20}$
E_6	$1'_p$	21_b	$\phi_{1,36}$	$\phi_{21,36}$
A_7	[8]	1_a	8	$\phi_{1,0}$
A_7	[4567]	35_b	44	$\phi_{35,4}$
A_7	[1238]	210_a	5111	$\phi_{210,6}$
A_7	[236]	168_a	422	$\phi_{168,6}$
A_7	[1256]	280_b	3311	$\phi_{280,8}$
A_7	[1346]	$378'_a$	3221	$\phi_{378,9}$
A_7	[2345]	84_a	2222	$\phi_{84,12}$
A_7	[123467]	216_a	221111	$\phi_{216,16}$
A_7	[1234568]	$105'_b$	2111111	$\phi_{105,21}$
A_7	[12345678]	15_a	11111111	$\phi_{15,28}$
$A_1\,D_6$	$[12] \boxtimes \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$	$216'_a$	$11.22, 11$	$\phi_{216,9}$
$A_1\,D_6$	$[12] \boxtimes \begin{bmatrix} 0 & 1 & 2 & 4 \\ 1 & 2 & 3 & 5 \end{bmatrix}$	378_a	$1.2111, 11$	$\phi_{378,14}$
$A_1\,D_6$	$[12] \boxtimes \begin{bmatrix} 0 & 1 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$	280_b	$11.1111, 11$	$\phi_{280,17}$
$A_1\,D_6$	$[12] \boxtimes \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$	$35'_b$	$.111111, 11$	$\phi_{35,31}$
$A_1\,D_6$	$[2] \boxtimes \begin{bmatrix} 0 & 1 & 2 & 4 \\ 1 & 2 & 3 & 5 \end{bmatrix}$	$420'_a$	$1.2111, 2$	$\phi_{420,13}$
$A_2\,A_5$	$[123] \boxtimes [123456]$	70_a	$111, 111111$	$\phi_{70,18}$
$A_2\,A_5$	$[3] \boxtimes [123456]$	$105'_c$	$3, 111111$	$\phi_{70,18}$
$A_2\,A_2\,A_2$	$[123] \boxtimes [123] \boxtimes [123]$	$70'_a$	$111, 111, 111$	$\phi_{70,9}$
$A_3\,A_3\,A_1$	$[1234] \boxtimes [1234] \boxtimes [12]$	$210'_b$	$1111, 1111, 11$	$\phi_{210,13}$

Thesis			CHEVIE	
$\mathbf{W}(\tilde{s})$	ρ	$j_{\mathbf{W}(\tilde{s})}^{\mathbf{W}}(\rho)$	ρ	$j_{\mathbf{W}(\tilde{s})}^{\mathbf{W}}(\rho)$
$A_1\,D_4\,A_1$	$[12] \boxtimes \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \boxtimes [12]$	$189'_b$	$11, 1.21, 11$	$\phi_{189,5}$
$A_1\,D_4\,A_1$	$[2] \boxtimes \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \boxtimes [2]$	105_c	$.1111, 2, 2$	$\phi_{105,12}$

Table C.2: Dictionary For Type E_7.

Appendix D

A Small Program For GAP

The following is the code for a program used to calculate the F-stability of conjugacy classes of semisimple elements in Chapter 4. This program requires the development version of CHEVIE as maintained by Jean Michel, (see [**Mic11**]).

```
ReflectionOrbit:=function(W,p)
  local quasi1,quasi,vec,denom,orbit,result,list,rep,w0,fact,test,F,i,j,k,t,
  s,s1,s0;
  quasi1:=QuasiIsolatedRepresentatives(W);
  F:=CoxeterCoset(W,p);
  w0:=LongestCoxeterElement(W);

  denom:=Set([]);
  vec:=[];
  test:=true;
  rep:=[];
  orbit:=[];
  fact:=[];
  list:=[];
  quasi:=[];
  result:=[];

  for i in [1..Length(quasi1)] do
    if Set(quasi1[i].v) <> [0] then
      Add(quasi,quasi1[i]);
    fi;
  od;

  for i in [1..Length(quasi)] do
    vec:=quasi[i].v;
    rep[i]:=[];
```

```
  for j in [1..Length(vec)] do
    if Denominator(vec[j]) <> 1 then
      denom[i]:=Denominator(vec[j]);
    fi;
  od;
  fact[i]:=Set(Factors(denom[i]));
  if denom[i]=2 then
    for k in [1..W.rank] do
    if vec[k] = 0 then
      list[k]:=0;
    else
      list[k]:=1/2;
    fi;
    od;
    Add(rep[i],Frobenius(F)(SemisimpleElement(W,list)).v);
  else
    for j in [2..denom[i]-1] do
      for k in fact[i] do
        if Mod(j,k)=0 then
          test:=false;
        fi;
      od;
      if test=true then
        for k in [1..W.rank] do
        if vec[k]= 0 then
          list[k]:=0;
        else
          list[k]:=j/denom[i];
        fi;
        od;
        Add(rep[i],Frobenius(F)(SemisimpleElement(W,list)).v);
      else
        test:=true;
      fi;
    od;
    rep[i]:=Set(rep[i]);
  fi;
od;

for i in [1..Length(quasi)] do
s:=quasi[i];
s1:=s^Identity(W);
RemoveSet(rep[i],s1.v);
```

```
  s0:=s^w0;
  RemoveSet(rep[i],s0.v);
  result[i]:=[s,true];
  if Length(rep[i]) <> 0 then
    orbit[i]:=Orbit(W,s);
    for j in [1..Length(orbit[i])] do
      orbit[i][j]:=orbit[i][j].v;
    od;
    j:=1;
    repeat
      t:=rep[i][j];
      if t in orbit[i] then
        result[i][2]:=true;
      else
        result[i][2]:=false;
      fi;
      j:=j+1;
    until result[i][2] = false or j = Length(rep[i])+1;
  fi;
  od;
  return result;
end;
```

We explain briefly how this program works. It first creates a list of representatives for the non identity conjugacy classes of quasi-isolated semismple elements in the algebraic group corresponding to the Coxeter group **W**, which it stores in `quasi`. For every class representative it works out every possible image of the semisimple element under the action of the Frobenius endomorphism specified by the permutation of the simple roots `p`, which it stores in `rep`. Finally it checks for every class representative in `quasi` that every image of that class representative is contained in the orbit under the action of the Weyl group. The program then returns a list of lists such that each sublist contains a quasi-isolated representative and the answer true / false.

Example D.1. We give some examples of how to use this program and the corresponding output of GAP. We calculate the F-stability of orbits in $\mathrm{D}_6(q)$, ${}^2\mathrm{D}_6(q^2)$, for $p \neq 2$, and $\mathrm{E}_8(q)$ for $p \neq 2, 3, 5$.

```
gap> ReflectionOrbit(CoxeterGroup("D",6),());
[ [ <0,0,0,0,0,1/2>, true ], [ <0,0,0,0,1/2,0>, true ],
  [ <0,0,0,1/2,0,0>, true ], [ <0,0,1/4,0,1/4,0>, true ],
  [ <0,1/2,0,0,0,0>, true ], [ <1/4,1/4,0,0,0,1/4>, true ],
  [ <1/2,0,0,0,0,0>, true ] ]
```

```
gap> ReflectionOrbit(CoxeterGroup("D",6),(1,2));
[ [ <0,0,0,0,0,1/2>, true ], [ <0,0,0,0,1/2,0>, true ],
  [ <0,0,0,1/2,0,0>, true ], [ <0,0,1/4,0,1/4,0>, true ],
  [ <0,1/2,0,0,0,0>, false ], [ <1/4,1/4,0,0,0,1/4>, true ],
  [ <1/2,0,0,0,0,0>, false ] ]
gap> ReflectionOrbit(CoxeterGroup("E",8),());
[ [ <0,0,0,0,0,0,0,1/2>, true ], [ <0,0,0,0,0,0,1/3,0>, true ],
  [ <0,0,0,0,0,1/4,0,0>, true ], [ <0,0,0,0,1/5,0,0,0>, true ],
  [ <0,0,0,1/6,0,0,0,0>, true ], [ <0,0,1/4,0,0,0,0,0>, true ],
  [ <0,1/3,0,0,0,0,0,0>, true ], [ <1/2,0,0,0,0,0,0,0>, true ] ]
```

The program is best left overnight to do the calculation for E_8.

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