Automorphisms of right-angled Artin groups

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Automorphisms of right-angled Artin groups

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A mon oncle

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Résumé

Cette thèse a pour objet l'étude des automorphismes des groupes d'Artin à angles droits. Etant donné un graphe simple fini Γ, le groupe d'Artin à angles droits G_Γ associé à Γ est le groupe défini par la présentation dont les générateurs sont les sommets de Γ, et dont les relateurs sont les commutateurs $[v, w]$, où $\{v,w\}$ est une paire de sommets adjacents. Le premier chapitre est conçu comme une introduction générale à la théorie des groupes d'Artin à angles droits et de leurs automorphismes. Dans un deuxième chapitre, on démontre que tout sous-groupe sous-normal d'indice une puissance de p d'un groupe d'Artin à angles droits est résiduellement p-séparable. Comme application de ce résultat, on montre que tout groupe d'Artin à angles droits est résiduellement séparable dans la classe des groupes nilpotents sans torsion. Une autre application de ce résultat est que le groupe des automorphismes extérieurs d'un groupe d'Artin à angles droits est virtuellement résiduellement p-fini. On montre également que le groupe de Torelli d'un groupe d'Artin à angles droits est résiduellement nilpotent sans torsion, et, par suite, résiduellement p-fini et bi-ordonnable. Dans un troisième chapitre, on établit une présentation du sous-groupe $Conj(G_\Gamma)$ de $Aut(G_\Gamma)$ formé des automorphismes qui envoient chaque générateur sur un conjugué de lui-même.

Abstract

The purpose of this thesis is to study the automorphisms of right-angled Artin groups. Given a finite simplicial graph Γ, the right-angled Artin group G_Γ associated to Γ is the group defined by the presentation whose generators are the vertices of Γ, and whose relators are commutators of pairs of adjacent vertices. The first chapter is intended as a general introduction to the theory of right-angled Artin groups and their automorphisms. In a second chapter, we prove that every subnormal subgroup of p-power index in a right-angled Artin group is conjugacy p-separable. As an application, we prove that every right-angled Artin group is conjugacy separable in the class of torsion-free nilpotent groups. As another application, we prove that the outer automorphism group of a right-angled Artin group is virtually residually p-finite. We also prove that the Torelli group of a right-angled Artin group is residually torsion-free nilpotent, hence residually p-finite and bi-orderable. In a third chapter, we give a presentation of the subgroup $Conj(G_\Gamma)$ of $Aut(G_\Gamma)$ consisting of the automorphisms that send each generator to a conjugate of itself.

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Chapter 1

Introduction

A *right-angled Artin group* is a finitely generated group subject to the relations that some of the generators commute. At one extreme is the free group F_n of rank n (none of the generators commute). At the other extreme is the free abelian group $\mathbb{Z}^n$ (all of the generators commute). Thus, the automorphism group of a general right-angled Artin group interpolates between $Aut(F_n)$ and $GL_n(\mathbb{Z})$. Similarly, the outer automorphism group of a general right-angled Artin group interpolates between $Out(F_n)$ and $GL_n(\mathbb{Z})$, two groups that have been shown to share a large number of properties, but that have also been shown to differ in significant ways – especially with regard to the Tits alternative[1]. So one is naturally led to ask which properties shared by $Out(F_n)$ and $GL_n(\mathbb{Z})$ are in fact shared by the outer automorphism groups of all right-angled Artin groups. Another natural question to ask is to determine which properties depend on the shape of Γ and how they depend on it.

Right-angled Artin groups were first introduced by Baudisch in the 1970's under the name "semifree groups" (see [B1], [B2]). The study of right-angled Artin groups was developed by Droms in the 1980's under the name "graph groups" (see [Dr1], [Dr2], [Dr3]). They have been widely studied since that time (we refer to [C2] for a general survey of right-angled Artin groups). Although they have a very simple presentation, right-angled Artin groups turn out to be extremely interesting from both algebraic and geometric viewpoints. They contain many interesting subgroups, and have nice actions on CAT(0) cube complexes.

Right-angled Artin groups belong to the more general class of Artin

[1] The Tits alternative states that every subgroup is either virtually solvable or contains a non-abelian free group. Both $Out(F_n)$ and $GL_n(\mathbb{Z})$ satisfy the Tits alternative, however $Out(F_n)$ satisfies a stronger Tits alternative: every subgroup of $Out(F_n)$ is either virtually abelian or contains a non-abelian free group.

groups. Artin groups arise as natural generalizations of braids groups. So in order to put the theory of right-angled Artin groups in context, we begin with a very short introduction to braid groups.

1.1 From braid groups to Artin groups

Braid groups. The definition of braid group was introduced by Artin in 1925 (see [A]), although the braid automorphisms of free groups were first studied by Hurwitz in 1891 (see [Hu]). Braids play an important role in many areas of mathematics. The beauty of braids lies on their intrinsic geometric nature, and their close relations to other fundamental objects such as knots, mapping class groups, and configuration spaces.

Let $\mathcal{B}_n$ denote the braid group on n strands. An element of $\mathcal{B}_n$ can be represented by n interlaced strings attached at the top and bottom to n fixed points. Two braids are considered equal if one can be obtained from the other by moving the strings without moving the endpoints. Multiplication in the braid group is given by concatenation. The braid group $\mathcal{B}_n$ is generated by $n-1$ elements σ_1,...,σ_{n-1}, where σ_i is the braid represented in Figure 1.

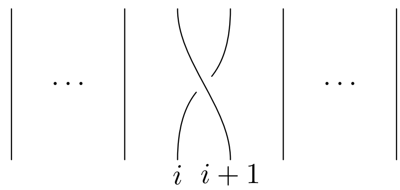


Figure 1. The braid σ_i

This gives rise to a well-known presentation of the braid group (due to Artin):

$$< \sigma_1,...,\sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } \mid i-j \mid \geq 2,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} >.$$

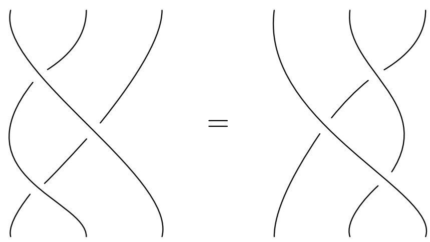


Figure 2. The braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

If we add to this presentation the relations $\sigma_i^2 = 1$, we obtain a presentation of the symmetric group S_n. Thus, there is a natural epimorphism $\mathcal{B}_n \to S_n$. Its kernel is called the *pure braid group on n strands* and is often denoted $\mathcal{PB}_n$. An element of $\mathcal{PB}_n$ is a braid whose strands begin and end at the same point. The group $\mathcal{PB}_n$ is a characteristic subgroup of $\mathcal{B}_n$ (a result also due to Artin).

The braid group $\mathcal{B}_n$ can be interpreted in terms of configuration spaces. Set

$$X_n = \{(z_1, ..., z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}.$$

The space X_n is called the *configuration space of n ordered points* in $\mathbb{C}$. The symmetric group S_n acts on X_n by permuting coordinates. The quotient space $Y_n = X_n/S_n$ is called the *configuration space of n unordered points* in $\mathbb{C}$. It was shown by Fox and Neuwirth (see [FN]) that

$$\mathcal{B}_n \simeq \pi_1(Y_n).$$

The braid group $\mathcal{B}_n$ can also be thought of as a subgroup of the automorphism group of the free group F_n on n generators x_1,...,x_n. More specifically, we say that an automorphism φ of F_n is a *braid automorphism* if all of the following hold:

(i) there exists a permutation $\pi \in S_n$ such that $\varphi(x_k)$ is conjugate in F_n to $x_{\pi(k)}$ for all $k \in \{1,...,n\}$,

(ii) we have $\varphi(x_1 \cdots x_n) = x_1 \cdots x_n$.

An example of braid automorphism is given by:

$$\tau_i(x_j) = \begin{cases} x_{i+1} & \text{if } j = i, \\ x_{i+1}^{-1} x_i x_{i+1} & \text{if } j = i+1, \\ x_j & \text{otherwise.} \end{cases}$$

The mapping $\sigma_i \mapsto \tau_i$ $(1 \leq i \leq n-1)$ defines an isomorphism from the braid group $\mathcal{B}_n$ to the group of braid automorphisms of F_n. This has as a consequence that the braid group is residually finite and Hopfian.

The braid group $\mathcal{B}_n$ is also isomorphic to the mapping class group of the n-punctured disk.

The automorphism group of the braid group was determined by Dyer and Grossman (see [DG]):

$$Out(\mathcal{B}_n) \simeq \mathbb{Z}/2\mathbb{Z}.$$

Artin groups. Artin groups were first introduced by Tits in 1966 (see [T]). They are sometimes called "generalized braid groups" or "Artin-Tits groups". They are defined by a braid-type presentation and are closely related to Coxeter groups.

A *Coxeter matrix* is a symmetric matrix $(m_{i,j})_{1\leq i,j\leq n}$ such that $m_{i,i}=1$ for all $i\in\{1,...,n\}$, $m_{i,j}\in\mathbb{N}\cup\{\infty\}$ and $m_{i,j}\geq 2$ for all i, $j\in\{1,...,n\}$ such that $i\neq j$. Given a Coxeter matrix $M=(m_{i,j})_{1\leq i,j\leq n}$, the *Artin group* of type M is defined by the presentation:

$$A(M) = < s_1,...,s_n \mid \underbrace{s_is_js_i...}_{m_{i,j}\text{ terms}} = \underbrace{s_js_is_j...}_{m_{i,j}\text{ terms}} \text{ for } m_{i,j}\neq\infty >,$$

Adding the relations $s_i^2=1$ yields a presentation of the *Coxeter group* of type M:

$$\begin{aligned} W(M) &= < s_1,...,s_n \mid s_i^2=1,\ \underbrace{s_is_js_i...}_{m_{i,j}\text{ terms}} = \underbrace{s_js_is_j...}_{m_{i,j}\text{ terms}} \text{ for } m_{i,j}\neq\infty >, \\ &= < s_1,...,s_n \mid s_i^2=1,\ (s_is_j)^{m_{i,j}}=1 \text{ for } m_{i,j}\neq\infty >. \end{aligned}$$

Let S denote the set $\{s_1,...,s_n\}$. The pair (A,S) is called an *Artin system*, and the pair (W,S) is called a *Coxeter system.* Every Coxeter system is associated to an Artin system, and reciprocally. By abuse of language, we will always assume that the set of generators is specified when we refer to an "Artin group" or "Coxeter group".

To each Coxeter matrix $M=(m_{i,j})_{1\leq i,j\leq n}$, one can associate a *Coxeter graph* Γ. The vertex set of Γ is the set $\{1,...,n\}$. There is an edge labelled $m_{i,j}$ between i and j whenever $m_{i,j}\geq 3$ or $m_{i,j}=\infty$. By convention, we omit the labels which are equal to 3. Note that the absence of an edge between two vertices indicates that the corresponding generators commute.

Every Coxeter group W admits a faithful representation as a reflection group, that is, a discrete group generated by a set of reflections of a finite-dimensional real vector space V, endowed with a certain bilinear form B. If W is finite, then the bilinear form B is definite positive, so that we can identify V with $\mathbb{R}^n$, endowed with the usual dot product. In this case, we have a finite hyperplane arrangement in $\mathbb{R}^n$:

$$\mathcal{A}=\{H_r \mid H_r \text{ is the fixed hyperplane of some reflection } r\in W\}.$$

It is known that W acts freely on the complement of $\mathcal{A}$. Complexifying this action, we get a free action of W on the complement $X_W=\mathbb{C}^n\setminus(\bigcup_r H_r+iH_r)$. The Artin group A associated to W is then isomorphic to the fundamental group of the quotient space $Y_W=X_W/W$.

We say that an Artin group is *irreducible* if its Coxeter graph is connected. Note that every Artin group is isomorphic to a direct product of irreducible Artin groups corresponding to the connected components of its Coxeter graph.

We say that an Artin group is of *finite type* (or *spherical*) if the associated Coxeter group is finite. We say that an Artin group is of *affine type* if the associated Coxeter group is affine. Irreducible finite and affine type Coxeter groups were completely classified (see, for example, [Bo]).

The irreducible finite type Coxeter groups consist of the three infinite families of Coxeter groups associated to the Coxeter graphs A_n, B_n, and D_n depicted in Figure 3, the dihedral groups Dih_n ($n \geq 5$), and six exceptional groups E_6, E_7, E_8, F_4, H_3, and H_4.

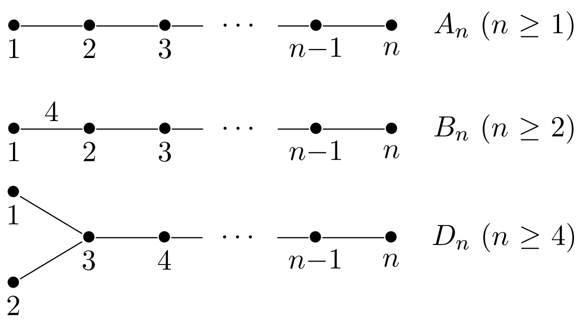


Figure 3. The Coxeter graphs A_n, B_n, and D_n.

Note that the Artin group of type A_n is nothing but the braid group B_{n+1}. Finite type Artin groups behave like braid groups, with which they share a large number of properties. In particular, they are known to be torsion-free (see [BS]), linear (see [CW]), and biautomatic – and hence to have solvable word and conjugacy problem – (see [C1]). The center of any finite type Artin group is infinite cyclic (see [BS]). For every finite type Artin group A, with associated Coxeter group W, the kernel of the natural epimorphism $A \to W$ is a characteristic subgroup of A (see [CP] for the irreducible case, and [FrP] for the non-irreducible case).

The automorphism groups of each of the Artin groups of finite type B_n, and affine type $\widetilde{A}_{n-1}$ and $\widetilde{C}_{n-1}$ were determined by Charney and Crisp (see [CC]):

Theorem 1.1.1. *Set $C_2 = \mathbb{Z}/2\mathbb{Z}$. Then for every $n \geq 3$, we have:*

$$Out(A(B_n)) \simeq (\mathbb{Z} \rtimes C_2) \times C_2,$$
$$Out(A(\widetilde{A}_{n-1})) \simeq Dih_n \times C_2,$$
$$Out(A(\widetilde{C}_{n-1})) \simeq S_3 \times C_2.$$

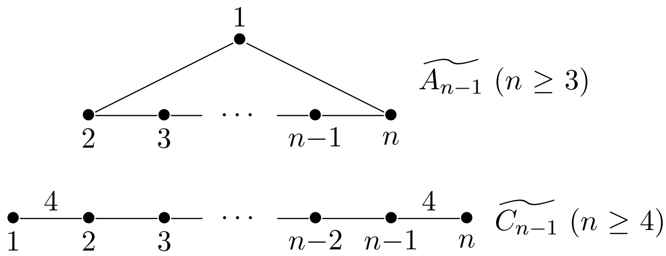


Figure 4. The Coxeter graphs $\widetilde{A_{n-1}}$ and $\widetilde{C_{n-1}}$.

1.2 Right-angled Artin groups

Definition and examples. From now on, we will concentrate on a variety of Artin groups known as right-angled Artin groups. A *right-angled Artin group* is an Artin group for which $m_{i,j} \in \{2,\infty\}$ for all i, $j \in \{1,...,n\}$ such that $i \neq j$. Right-angled Coxeter groups are defined similarly. The presentation of a right-angled Artin group can be specified by a finite simplicial graph, the *defining graph* Γ. The vertex set of Γ is the set $S = \{s_1,...,s_n\}$. There is an edge between s_i and s_j if and only if $m_{i,j} = 2$. Note that, unlike above, two generators commute if and only if they are connected by an edge. Conversely, every finite simplicial graph $\Gamma = (V,E)$ is the defining graph of a right-angled Artin group G_Γ, defined by the presentation:

$$G_\Gamma = \langle\ V \mid vw = wv,\ \forall\ \{v,w\} \in E\ \rangle.$$

Examples 1.2.1. At one extreme, Γ is a discrete graph, in which case G_Γ is the free group F_n of rank n. At the other extreme, Γ is a complete graph, in which case G_Γ is the free abelian group $\mathbb{Z}^n$. If Γ is the bipartite graph $K_{3,3}$ of the Figure 4, then G_Γ is the direct product of two free groups: $G_\Gamma = F_3 \times F_3$.

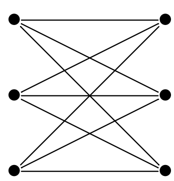


Figure 5. The bipartite graph $K_{3,3}$.

Whereas if Γ is the cyclic graph of length 5, then G_Γ cannot be decomposed as a direct or free product.

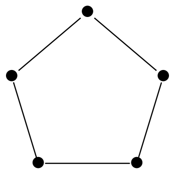


Figure 6. The cyclic graph of length 5.

Droms proved that two right-angled Artin groups G_Γ and G_Δ are isomorphic (as groups) if and only if their defining graphs Γ and Δ are isomorphic (as graphs) (see [Dr2]). Paris proved that the same result holds for finite type Artin groups (see [P1]).

Duchamp and Krob proved that right-angled Artin groups are residually torsion-free nilpotent (see [DK2]). It follows that right-angled Artin groups are bi-orderable and hence torsion-free.

Humphries proved that right-angled Artin groups are linear and hence residually finite (see [H]). More recently, Davis and Januszkiewicz proved that any right-angled Artin group embeds as a finite index subgroup of some right-angled Coxeter group (see [DJ]).

Metaftsis and Raptis proved that G_Γ is subgroup separable if and only if Γ contains neither a path of length 3 nor a square as a full subgraph (see [MR]).

Subgroups. If W is any subset of V, then the subgroup of G_Γ generated by W is naturally isomorphic to the right-angled Artin group $G_{\Gamma(W)}$ associated to the full subgraph of Γ spanned by W. This subgroup is called a *special subgroup* of G_Γ. Note that a subgroup of a right-angled Artin group is not, in general, a right-angled Artin group (see [Dr3]).

Right-angled Artin groups contain many interesting subgroups. Extending work of Droms, Servatius, and Servatius (see [DSS]), Crisp and Wiest proved that any surface group embeds in a right-angled Artin group, with three exceptions (see [CrW]). They also proved that graph braid groups embed in right-angled Artin groups. Bestvina and Brady constructed subgroups of right-angled Artin groups which are of type FP but are not finitely presented (see [BB]).

Right-angled Artin groups also arise as subgroups of other groups. It was conjectured by Tits and proved by Crisp and Paris that the squares of the generators of a general Artin group generate a right-angled Artin group (see [CP]). In a recent work, Koberda proved that, given a finite set of elements $\{f_1,...,f_k\}$ in the mapping class group $Mod_{g,n}$ of a surface, and under appropriate conditions, there exists an exponent N such that the subgroup of $Mod_{g,n}$ generated by $\{f_1^N,...,f_k^N\}$ is a right-angled Artin group (see [K]).

The word and conjugacy problem. Given a finitely presented group G, the *word problem* consists in finding an algorithm which decides whether two words represent the same element of G. The *conjugacy problem* asks for an algorithm which decides whether two words represent conjugate elements of G. Note that a solution to the conjugacy problem contains a solution to the word problem. A case where it is known that a group G has solvable word and conjugacy problem is when G is biautomatic (we refer to [ECHLPT] for the definition of biautomaticity).

Let v be a vertex of Γ. The *link* of v, denoted by $lk(v)$, is the subset of V consisting of all vertices that are adjacent to v. The *star* of v, denoted by $st(v)$, is $lk(v) \cup \{v\}$.

Let w be a word in $V^{\pm 1}$. The *support* of w, denoted by $supp(w)$, is the subset of V of all vertices v such that v or v^{-1} is a letter of w. A word w in $V^{\pm 1}$ is said to be *reduced* if it contains no subwords of the form $vw'v^{-1}$ or $v^{-1}w'v$ with $supp(w') \subset st(v)$. For a word w in $V^{\pm 1}$, we denote by $|w|$ the length of w. In her thesis (see [Gr]), Green proved that if two reduced words represent the same element of G_Γ, then they have same length and same support. Therefore, we can define the *length* of an element g of G_Γ, denoted by $|g|$, as the length of any reduced word representing g, and the *support* of g, denoted by $supp(g)$, as the support of any reduced word representing g. Green also proved that, for any $g \in G_\Gamma$, the set of vertices that appear as the initial letter of some reduced word representing g spans a clique (that is, a complete subgraph) Δ in Γ. This yields a procedure to write g in a normal form, as follows. First, write $g = w_0 g_1$, where $w_0 \in G_\Delta$ is of maximal length. By repeating this process, we obtain a decomposition of g into $g = w_0 w_1 \cdots w_k$, where each w_i belongs to a special subgroup G_{Δ_i} associated to a clique Δ_i in Γ. This normal form is unique up to the commutation relations in G_{Δ_i}. In [HM], Hermiller and Meier proved that this normal form gives rise to a biautomatic structure on G_Γ (see [V] for an alternative proof of biautomaticity).

We say that an element g of G_Γ is *cyclically reduced* if it can not be written vhv^{-1} or $v^{-1}hv$ with $v \in V$, and $|g| = |h| + 2$. In [Ser], Servatius proved that every element of G_Γ is conjugate to a unique (up to cyclic permutation) cyclically reduced element.

Centralizers in right-angled Artin groups were described by Servatius in [Ser]. Let g be an element of G_Γ. Let h be a cyclically reduced element conjugate to g. Let Γ_1,...,Γ_k be the connected components of the complement of $\Gamma(supp(h))$. Let $G_{\Gamma(supp(h))} = G_{\Gamma_1} \times \cdots \times G_{\Gamma_k}$ be the corresponding decomposition of $G_{\Gamma(supp(h))}$. Then h can be written uniquely (up to permutation of the factors) as a product $h = h_1^{\alpha_1} \cdots h_k^{\alpha_k}$, where $h_i \in G_{\Gamma_i}$, and $\alpha_i \in \mathbb{N}$ is maximal. Servatius proved that the centralizer $C(h)$ of h is given by:

$$C(h) = \langle h_1 \rangle \cdots \langle h_k \rangle \langle lk(supp(h)) \rangle,$$

where $lk(supp(h))$ is the set of vertices v such that $v \notin supp(h)$ and v is adjacent to every vertex in $supp(h)$. Hence, the centralizer of a vertex v is the special subgroup of G_Γ generated by $st(v)$, and the center $Z(G_\Gamma)$ of G_Γ is the special subgroup of G_Γ generated by $\bigcap_{v \in V} st(v)$.

We would like to end this section by mentioning a recent development in the theory of right-angled Artin groups. In a recent research announcement (see [Wi]), Wise claimed to have proved the following:

Theorem 1.2.2. *Let G be a hyperbolic group with a quasi-convex hierarchy. Then G contains a finite index subgroup that embeds as a quasiconvex subgroup of a right-angled Artin group.*

Theorem 1.2.2 implies in particular the Baumslag conjecture on the residual finiteness of one relator groups with torsion.

1.3 Automorphisms of right-angled Artin groups

The study of automorphism groups of free groups was initiated by Nielsen in 1924 (see [N]). They have been extensively studied since that time. Although Formanek and Procesi have shown that $Aut(F_n)$ is not linear for $n \geq 3$ (nor is $Out(F_n)$ for $n \geq 4$) (see [FP]), $Aut(F_n)$ and $GL_n(\mathbb{Z})$ have many properties in common. Automorphism groups of right-angled Artin groups that lie between these two extremes have received less attention.

Generators. Automorphisms of right-angled Artin groups were first studied by Servatius in 1989 (see [Ser]). Drawing on Nielsen automorphisms for free groups, Servatius defined four classes of automorphisms – namely, inversions, partial conjugations, transvections, and symmetries –, and conjectured that they generate $Aut(G_\Gamma)$. Servatius proved his conjecture for some classes of right-angled Artin groups – for example, when Γ is a tree. Thereafter Laurence proved the conjecture for a general right-angled Artin group in [L].

Let v, w be vertices of Γ. We use the notation $v \geq w$ to mean $lk(w) \subset st(v)$.

The Laurence-Servatius generators for $Aut(G_\Gamma)$ are defined as follows:

Inversions: Let $v \in V$. The automorphism ι_v that sends v to v^{-1} and fixes all other vertices is called an *inversion.*

Partial conjugations: Let $x = v^{\pm 1} \in V^{\pm 1}$, and Y be a non-empty union of connected components of $\Gamma \setminus st(v)$. The automorphism $c_{x,Y}$ that sends each vertex y in Y to $x^{-1}yx$ and fixes all vertices not in Y is called a *partial conjugation.*

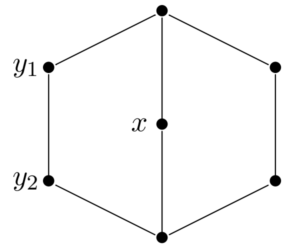


Figure 7. A partial conjugation $y_i \mapsto x^{-1}y_ix$.

Transvections: Let v, $w \in V$ be such that $v \geq w$. The automorphism $\tau_{v,w}$ that sends w to vw and fixes all other vertices is called a *transvection.*

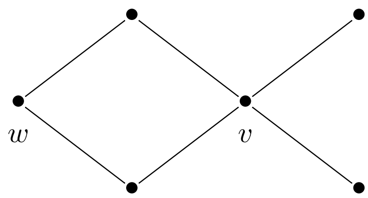


Figure 8. A transvection $w \mapsto vw$.

Symmetries: Let φ be an automorphism of the graph Γ. The automorphism ϕ given by $\phi(v) = \varphi(v)$ for all $v \in V$ is called a *symmetry.*

In [D1], Day extended the concept of Whitehead automorphism to arbitrary right-angled Artin groups, and gave a finite presentation for $Aut(G_\Gamma)$. This presentation will be described in detail in Chapter 3.

CAT(0) cube complexes. Recall that a *CAT(0) space* is a geodesic space in which each geodesic triangle Δ is "at least as thin" as a triangle in the Euclidean plane that has the same side lengths – such a triangle is called a comparison triangle. That is, the distance between any two points on Δ is less than or equal to the distance between the corresponding points on the comparison triangle. A *cube complex* is a complex built from cubes by identifying some subcubes. Associated to each right-angled Artin group is a CAT(0) cube complex S_Γ, constructed as follows. First, S_Γ has a single vertex, and a loop labelled v_i for each generator v_i, together with its inverse v_i^{-1}. For each edge $\{v_i,v_j\}$ in Γ, attach a 2-torus with boundary labelled $v_iv_jv_i^{-1}v_j^{-1}$. For each triangle in Γ with vertices v_i, v_j, v_k, attach a 3-torus whose faces are the tori that correspond to the three edges of the triangle. Continuing this process, for each k-clique in Γ, attach a k-torus. The resulting space S_Γ is called the *Salvetti complex* of G_Γ. It is clear that the fundamental group of S_Γ is G_Γ. Let $\tilde{S_\Gamma}$ be the universal cover of S_Γ. Charney and Davis proved that $\tilde{S_\Gamma}$ is a CAT(0) cube complex, and hence S_Γ is a $K(G_\Gamma, 1)$ space (see [CD]).

We define the *dimension* of G_Γ to be the dimension of $\tilde{S_\Gamma}$. Alternatively, the dimension of G_Γ is the size of a maximal clique in Γ, or the rank of a maximal abelian subgroup of G_Γ.

Recall that a group G is said to be *virtually $\mathcal{P}$* (where $\mathcal{P}$ is a group property) if there exists a finite index subgroup $H < G$ such that H has Property $\mathcal{P}$.

In [CV1], Charney and Vogtmann argued by induction on the dimension of G_Γ to prove the following:

Theorem 1.3.1. *$Out(G_\Gamma)$ is virtually torsion-free and has finite virtual cohomological dimension.*

Residual properties. Let $\mathcal{K}$ be a class of group. A group G is said to be *residually $\mathcal{K}$* if for all $g \in G\backslash\{1\}$, there exists a homomorphism φ from G to some group of $\mathcal{K}$ such that $\varphi(g) \neq 1$. If $\mathcal{K}$ is the class of all finite groups (respectively, the class of all finite p-groups), we say that G is *residually finite* (respectively, *residually p-finite*).

For a group G and for g, $h \in G$, we use the notation $g \sim h$ to mean that g and h are conjugate. A group G is said to be *conjugacy $\mathcal{K}$-separable* (or *conjugacy separable in the class $\mathcal{K}$*) if for all g, $h \in G$, either $g \sim h$, or there exists a homomorphism φ from G to some group of $\mathcal{K}$ such that $\varphi(g) \nsim \varphi(h)$. If $\mathcal{K}$ is the class of all finite groups (respectively, the class of all finite p-groups), we say that G is *conjugacy separable* (respectively, *conjugacy p-separable*). Clearly, if a group is conjugacy $\mathcal{K}$-separable, then it is residually $\mathcal{K}$.

Conjugacy separability has two main applications. One was discovered by Mal'cev in [M], where he proved that a finitely presented conjugacy separable group has solvable conjugacy problem.

The second application was discovered by Grossman in [Gro]. A classical theorem of Baumslag states that the automorphism group of a finitely generated residually finite group G is residually finite. However the outer automorphism group of G may not be residually finite. Grossman proved that if G is a finitely generated conjugacy separable group, then $Out(G)$ is residually finite provided that every conjugating automorphism of G is inner. (An automorphism φ of G is called *conjugating* if $\varphi(g) \sim g$ for all $g \in G$).

In [Mi], Minasyan proved that every finite index subgroup in a right-angled Artin group is conjugacy separable – that is, right-angled Artin groups are "hereditarily conjugacy separable". Using Grossman's result, he attained the following:

Theorem 1.3.2. *$Out(G_\Gamma)$ is residually finite.*

This result was obtained independently by Charney and Vogtmann in [CV2], where they also proved that, for a large class of graphs, $Out(G_\Gamma)$ satisfies the Tits alternative.

This thesis consists of two papers, written by the author during his doctorate at the Université de Bourgogne. They form Chapters 2 and 3, respectively.

Recall that a *subnormal subgroup* of a group G is a subgroup H of G such that there exists a finite sequence of subgroups of G:

$$H = H_0 < H_1 < ... < H_n = G,$$

such that H_i is normal in H_{i+1} for all $i \in \{0,...,n-1\}$. In Chapter 2, we introduce the following definition, which naturally generalizes that of [CZ]:

Definition 1.3.3. *Let G be a group. We say that G is hereditarily conjugacy p-separable if every subnormal subgroup of p-power index in G is conjugacy p-separable.*

Hereditary conjugacy p-separability is obviously stronger than conjugacy p-separability. Actually, we will show that a group G is hereditarily conjugacy p-separable if and only if it is conjugacy p-separable and satisfies the condition of the following definition:

Definition 1.3.4. *We say that a group G satisfies the p-centralizer condition if, for every normal subgroup H of p-power index in G, and for all $g \in G$, there exists a normal subgroup K of p-power index in G such that $K < H$, and:*

$$C_{G/K}(\varphi(g)) \subset \varphi(C_G(g)H),$$

where $\varphi : G \to G/K$ denotes the canonical projection.

The main theorem of Chapter 2 is the following:

Theorem 1.3.5. *G_Γ is hereditarily conjugacy p-separable.*

We will now discuss some applications of Theorem 1.3.5. The first application that we mention is an application of Theorem 1.3.5 to separability properties of G_Γ:

Corollary 1.3.6. *G_Γ is conjugacy separable in the class of torsion-free nilpotent groups.*

We now turn to applications of Theorem 1.3.5 to residual properties of $Out(G_\Gamma)$. Combining Theorem 1.3.5 with a result of Paris (see [P2]), we obtain the following:

Corollary 1.3.7. *$Out(G_\Gamma)$ is virtually residually p-finite.*

On the other hand, combining Theorem 1.3.5 with a result of Myasnikov (see [My]), we obtain the following:

Corollary 1.3.8. *$Out(G_\Gamma)$ is residually $\mathcal{K}$, where $\mathcal{K}$ is the class of all outer automorphism groups of finite p-groups.*

The natural action $Aut(G_\Gamma) \to GL_r(\mathbb{Z})$ of $Aut(G_\Gamma)$ on $H_1(G_\Gamma, \mathbb{Z})$ gives rise to a homomorphism $Out(G_\Gamma) \to GL_r(\mathbb{Z})$, whose kernel is called the *Torelli group* $\mathcal{T}(G_\Gamma)$ of G_Γ – by analogy with the Torelli group of a mapping class group. In Chapter 2, we combine well-known results of Bass-Lubotzsky (see [BL]) and Duchamp-Krob (see [DK1], [DK2]) with Theorem 1.3.5 to attain the following:

Theorem 1.3.9. *$\mathcal{T}(G_\Gamma)$ is residually torsion-free nilpotent.*

Hence $\mathcal{T}(G_\Gamma)$ is residually p-finite and bi-orderable.

Our proof of Theorem 1.3.5 is purely combinatorial. Some of the key tools we use here include HNN extensions and retractions.

We argue by induction on the rank of G_Γ to prove that G_Γ is conjugacy p-separable and satisfies the p centralizer condition. The key observation is that if G is a right-angled Artin group of rank r, then G can be written as an HNN extension of any of its special subgroups of rank $r-1$:

$$G = \langle\ H, t \mid t^{-1}kt = k, \forall\ k \in K\ \rangle.$$

To perform the inductive step, we need to pass from a homomorphism $\varphi : H \to P$ from H to a finite p-group P to a homomorphism $\overline{\varphi} : G \to Q$, where Q is the HNN extension of P relative to $\varphi(K)$:

$$Q = \langle\ P, \bar{t} \mid \bar{t}^{-1}\varphi(k)\bar{t} = \varphi(k), \forall\ k \in K\ \rangle.$$

The latter is an extension of a free group by a finite p-group, for which we establish the following:

Theorem 1.3.10. *Every extension of a free group by a finite p-group is conjugacy p-separable.*

Our proof follows that of Minasyan (see [Mi]), with the difference being that Minasyan approximated right-angled Artin groups by HNN extensions of finite groups (which are known to be virtually free) and then applied a theorem of Dyer, stating that virtually free groups are conjugacy separable (see [Dy1]). Theorem 1.3.10 is the analogue of Dyer's theorem for conjugacy p-separability and is the key technical lemma in the proof of Theorem 1.3.5.

In Chapter 3, we give a presentation of the subgroup of $Aut(G_\Gamma)$ consisting of the automorphisms of G_Γ that send each generator to a conjugate of itself, thus generalizing a result of McCool on *basis-conjugating* automorphisms of free groups.

Definition 1.3.11. *We say that an automorphism φ of G_Γ is vertex-conjugating if $\varphi(v) \sim v$ for all $v \in V$.*

Vertex-conjugating automorphisms were first introduced by Laurence in [L], where they are called conjugating. As one of the steps in the proof of Servatius' conjecture, Laurence proved that the set of vertex-conjugating automorphisms coincides with the subgroup $Conj(G_\Gamma)$ of $Aut(G_\Gamma)$ generated by the partial conjugations.

Set $L = V \cup V^{-1}$. For $x \in L$, we define the *vertex* of x, denoted by $v(x)$, to be the unique element of $V \cap \{x, x^{-1}\}$, and we set $lk_L(x) = lk(v(x)) \cup lk(v(x))^{-1}$, and $st_L(x) = st(v(x)) \cup st(v(x))^{-1}$.

The main theorem of Chapter 3 is the following:

Theorem 1.3.12. *The group $Conj(G_\Gamma)$ has a presentation with generators $c_{x,Y}$, for $x \in L$ and Y a non-empty union of connected components of $\Gamma - st(v(x))$, and relations:*

$$
\begin{gathered}
(c_{x,Y})^{-1} = c_{x^{-1},Y}, \\
c_{x,Y}c_{x,Z} = c_{x,Y\cup Z} \text{ if } Y \cap Z = \emptyset, \\
c_{x,Y}c_{y,Z} = c_{y,Z}c_{x,Y} \text{ if } v(x) \notin Z,\ v(y) \notin Y,\ x \neq y,\ y^{-1}, \text{ and at least one of} \\
Y \cap Z = \emptyset \text{ or } y \in lk_L(x) \text{ holds}, \\
\omega_y c_{x,Y} \omega_y^{-1} = c_{x,Y} \text{ if } v(y) \notin Y,\ x \neq y,\ y^{-1}.
\end{gathered}
$$

Our proof relies on geometric methods. Following McCool (see [Mc3]), we construct a finite, connected 2-complex K with fundamental group $Conj(G_\Gamma)$ $= \langle S \mid R \rangle$. An important observation is that every partial conjugation is a long-range Whitehead automorphism in the sense of [D1].

Note that the presentation given in the theorem of [Mc3] cannot be generalized to the right-angled Artin group case.

As a conclusion, we will mention a construction that generalizes both right-angled Artin and right-angled Coxeter groups. Let $\Gamma = (V,E)$ be a finite simplicial graph. Let $\{G_v\}_{v\in V}$ be a family of groups indexed by V.

The *graph product* $\mathcal{G}_\Gamma$ of $\{G_v\}_{v\in V}$ with respect to Γ is the quotient of the free product of the G_v ($v \in V$), obtained by adding the relations:

$[g_v, g_w] = 1$, for all $g_v \in G_v$, $g_w \in G_w$, whenever v and w are connected by an edge.

Note that graph products include right-angled Artin groups (when $G_v = \mathbb{Z}$ for all $v \in V$), as well as right-angled Coxeter groups (when $G_v = \mathbb{Z}/2\mathbb{Z}$ for all $v \in V$). Note also that if Γ is a discrete graph, then $\mathcal{G}_\Gamma$ is the free product of the G_v ($v \in V$), whereas if Γ is a complete graph, then $\mathcal{G}_\Gamma$ is the direct product of the G_v ($v \in V$). Graph products were first introduced and studied by Green in her Ph.D. thesis (see [Gr]). Further properties of graph products were investigated by Hermiller and Meier (see [HM]), and by Hsu and Wise (see [HW]). Extending the work of Laurence, Corredor and Gutierrez gave a set of generators for the automorphism group of a graph product of finitely generated abelian groups (see [CG]). In [CF], Charney and Farber considered graph products associated to random graphs. In a recent work, Antolin and Minasyan explored different Tits alternatives for graph products (see [AM]). This suggests a possible direction for future research.

Chapter 2

Conjugacy p-separability of right-angled Artin groups and applications

In this chapter, we prove that every subnormal subgroup of p-power index in a right-angled Artin group is conjugacy p-separable. As an application, we prove that every right-angled Artin group is conjugacy separable in the class of torsion-free nilpotent groups. As another application, we prove that the outer automorphism group of a right-angled Artin group is virtually residually p-finite. We also prove that the Torelli group of a right-angled Artin group is residually torsion-free nilpotent, hence residually p-finite and bi-orderable.

2.1 Introduction

Let $\Gamma = (V,E)$ be a finite simplicial graph, and let G_Γ be the *right-angled Artin group* associated to Γ:

$$G_\Gamma = \langle\ V \mid vw = wv,\ \forall\ \{v,w\} \in E\ \rangle.$$

The *rank* of G_Γ is by definition the number of vertices of Γ. A *special subgroup* of G_Γ is a subgroup generated by a subset W of the set of vertices V of Γ – it is naturally isomorphic to the right-angled Artin group $G_{\Gamma(W)}$, where $\Gamma(W)$ denotes the full subgraph of Γ spanned by W. Let v be a vertex of Γ. The *link* of v, denoted by $lk(v)$, is the subset of V consisting of all vertices that are adjacent to v. The *star* of v, denoted by $st(v)$, is $lk(v) \cup \{v\}$.

Let $\mathcal{K}$ be a class of group. A group G is said to be *residually* $\mathcal{K}$ if for all $g \in G \backslash \{1\}$, there exists a homomorphism φ from G to some group of $\mathcal{K}$ such that $\varphi(g) \neq 1$. Note that if $\mathcal{K}$ is the class of all finite groups, this notion reduces to residual finiteness.

For a group G and for g, $h \in G$, we use the notation $g \sim h$ to mean that g and h are conjugate. A group G is said to be *conjugacy $\mathcal{K}$-separable* (or *conjugacy separable in the class $\mathcal{K}$*) if for all g, $h \in G$, either $g \sim h$, or there exists a homomorphism φ from G to some group of $\mathcal{K}$ such that $\varphi(g) \nsim \varphi(h)$. Note that if $\mathcal{K}$ is the class of all finite groups, this notion reduces to conjugacy separability. Clearly, if a group is conjugacy $\mathcal{K}$-separable, then it is residually $\mathcal{K}$.

Our focus here is on conjugacy separability in the class of finite p-groups.

Let p be a prime number. If $\mathcal{K}$ is the class of all finite p-groups, then, instead of saying "G is residually $\mathcal{K}$", we shall say that G is *residually p-finite*. Note that this implies residually finite as well as residually nilpotent. Instead of saying "G is conjugacy $\mathcal{K}$-separable", we shall say that G is *conjugacy p-separable*. Following Ivanova (see [I]), we say that a subset S of a group G is *finitely p-separable* if for every $g \in G \setminus S$, there exists a homomorphism φ from G onto a finite p-group P such that $\varphi(g) \notin \varphi(S)$. Note that G is conjugacy p-separable if and only if every conjugacy class of G is finitely p-separable.

Examples of groups which are known to be conjugacy p-separable include free groups (see, for example, [LS]) and fundamental groups of oriented closed surfaces (see [P2]).

There is a connection between these notions and a topology on G, the "pro-p topology" on G. The *pro-p topology* on G is defined by taking the normal subgroups of p-power index in G as a basis of neighbourhoods of 1 (see [RZ]). Equipped with the pro-p topology, G becomes a topological group. Observe that G is Hausdorff if and only if it is residually p-finite. One can show that a subset S of G is closed in the pro-p topology on G if and only if it is finitely p-separable. Thus, G is conjugacy p-separable if and only if every conjugacy class of G is closed in the pro-p topology on G.

We have already noted that conjugacy separability can be seen as an "algebraic pendant" to the conjugacy problem. It is worth noting that this notion does not pass to finite index subgroups or finite extensions. In 1986, Gorjaga constructed an example of a (finitely generated) non conjugacy separable group possessing a conjugacy separable group of index 2 (see [G]). More recently, Chagas and Zalesskii constructed an example of a conjugacy separable group possessing a non conjugacy separable subgroup of finite index. This led them to introduce the notion of "hereditarily conjugacy separable group": a group whose finite index subgroups are conjugacy separable (see [CZ]). Thereafter Martino and Minasyan proved the stronger statement that for every integer n, there exists a finitely presented conjugacy separable group G possessing a non conjugacy separable subgroup H of index n (see [MM]). Further, G and H can be chosen in such a way that G is a subgroup of some right-angled Artin group, and both G and H have solvable conjugacy problem.

Recall that a *subnormal subgroup* of a group G is a subgroup H of G such that there exists a finite sequence of subgroups of G:

$$H = H_0 < H_1 < \ldots < H_n = G,$$

such that H_i is normal in H_{i+1} for all $i \in \{0,\ldots,n-1\}$.

A subgroup H of a group G is open in the pro-p topology on G if and only if it is subnormal of p-power index (see Appendix A). This leads us to the following definition, which naturally generalizes that of [CZ]:

Definition 2.1.1. *Let G be a group. We say that G is hereditarily conjugacy p-separable if every subnormal subgroup of p-power index in G is conjugacy p-separable.*

In [Mi], Minasyan proved that right-angled Artin groups are hereditarily conjugacy separable. The main theorem of Chapter 2 is the following:

Theorem 2.1.2. *Every right-angled Artin group is hereditarily conjugacy p-separable.*

We will now discuss some applications of Theorem 2.1.2. The first application that we mention is an application of Theorem 2.1.2 to separability properties of G_Γ:

Corollary 2.1.3. *Every right-angled Artin group is conjugacy separable in the class of torsion-free nilpotent groups.*

We now turn to applications of Theorem 2.1.2 to residual properties of $Out(G_\Gamma)$.

Let $\mathcal{P}$ be a group property. A group G is said to be *virtually* $\mathcal{P}$ if there exists a finite index subgroup $H < G$ such that H has Property $\mathcal{P}$. Combining Theorem 2.1.2 with a result of Paris (see [P2]), we obtain the following:

Corollary 2.1.4. *The outer automorphism group of a right-angled Artin group is virtually residually p-finite.*

On the other hand, combining Theorem 2.1.2 with a result of Myasnikov (see [My]), we obtain the following:

Corollary 2.1.5. *The outer automorphism group of a right-angled Artin group is residually $\mathcal{K}$, where $\mathcal{K}$ is the class of all outer automorphism groups of finite p-groups.*

The next application was suggested to the author by Ruth Charney and Luis Paris.

The natural action $Aut(G_\Gamma) \to GL_r(\mathbb{Z})$ of $Aut(G_\Gamma)$ on $H_1(G_\Gamma, \mathbb{Z})$ gives rise to a homomorphism $Out(G_\Gamma) \to GL_r(\mathbb{Z})$, whose kernel is called the *Torelli group* of G_Γ – by analogy with the Torelli group of a mapping class group. In Section 2.7, we combine well-known results of Bass-Lubotzky (see [BL]), and Duchamp-Krob (see [DK1], [DK2]) with Theorem 2.1.2 to attain the following:

Theorem 2.1.6. *The Torelli group of a right-angled Artin group is residually torsion-free nilpotent.*

Corollary 2.1.7. *The Torelli group of a right-angled Artin group is residually p-finite.*

Recall that a group G is said to be *bi-orderable* if it can be endowed with a total order $\leq$ such that if $g \leq h$, then $kg \leq kh$ and $gk \leq hk$ for all g, h, $k \in G$.

Corollary 2.1.8. *The Torelli group of a right-angled Artin group is bi-orderable.*

Our proof follows closely that of Minasyan (see [Mi]). Both proofs proceed by induction on the rank of G_Γ. The key observation is that a right-angled Artin group of rank r can be written as an HNN extension of any of its special subgroups of rank $r-1$. After passing to an HNN extension of a finite group (which is known to be virtually free), Minasyan applies a theorem of Dyer stating that virtually free groups are conjugacy separable (see [Dy1]).

This chapter is organized as follows. In Section 2.3, we introduce the p centralizer condition which is the analogue of the centralizer condition in [Mi], and we prove that a group is hereditarily conjugacy p-separable if and only if it is conjugacy p-separable and satisfies the p centralizer condition. In Section 2.4, we prove the following analogue of Dyer's theorem for conjugacy p-separability:

Theorem 2.1.9. *Every extension of a free group by a finite p-group is conjugacy p-separable.*

Section 2.5 deals with retractions that are key tools in the proof of our main theorem, which is the object of Section 2.6.

2.2 HNN extensions

In this section, we recall the definition and basic properties of HNN extensions (see [LS]).

Let H be a group. Then by the notation:

$$\langle\ H,\ s,\ldots \mid r,\ldots\ \rangle,$$

we mean the group defined by the presentation whose generators are the generators of H together with s,... and the relators of H together with r,...

Let H be a group. Let I be a set of indices. Let $\{K_i\}_{i\in I}$ and $\{L_i\}_{i\in I}$ be families of subgroups of H and let $\{\psi_i : K_i \to L_i\}_{i\in I}$ be a family of isomorphisms. The *HNN extension with base H, stable letters t_i $(i \in I)$, and associated subgroups K_i and L_i $(i \in I)$*, is the group defined by the presentation:

$$G = \langle\ H,\ t_i\ (i \in I) \mid t_i^{-1}k_it_i = \psi_i(k_i),\ \forall\ k_i \in K_i\ (i \in I)\ \rangle.$$

In particular, let H be a group, let K and L be subgroups of H and let ψ be an isomorphism from K to L. The *HNN extension of H relative to ψ* is the group defined by the presentation:

$$G = \langle\ H,\ t \mid t^{-1}kt = \psi(k),\ \forall\ k \in K\ \rangle.$$

From now on, we assume that $K = L$ and $\psi = id_K$. In this case, G is called the *HNN extension of H relative to K*:

$$G = \langle\ H,\ t \mid t^{-1}kt = k,\ \forall\ k \in K\ \rangle.$$

Every element of G can be written as a word $x_0t^{a_1}x_1\cdots t^{a_n}x_n$ $(n \geq 0$, $x_0,\ldots,x_n \in H$, $a_1,\ldots,a_n \in \mathbb{Z}\setminus\{0\})$. Following Minasyan (see [Mi]), we will say that the word $x_0t^{a_1}x_1\cdots t^{a_n}x_n$ is *reduced* if $x_0 \in H$, $x_1,\ldots,x_{n-1} \in H\setminus K$, and $x_n \in H$. Every element of G can be written as a reduced word. Note that our definition of a reduced word is stronger than the definition of a reduced word in [LS].

Lemma 2.2.1 (Britton's Lemma)**.** *If a word $x_0t^{a_1}x_1\cdots t^{a_n}x_n$ is reduced with $n \geq 1$, then $x_0t^{a_1}x_1\cdots t^{a_n}x_n \neq 1$.*

Proof: Proved in [LS] (see Theorem IV.2.1). □

Lemma 2.2.2. *If $x_0t^{a_1}x_1\cdots t^{a_n}x_n$ and $y_0t^{b_1}y_1\cdots t^{b_m}y_m$ are reduced words such that $x_0t^{a_1}x_1\cdots t^{a_n}x_n = y_0t^{b_1}y_1\cdots t^{b_m}y_m$, then $m = n$ and $a_i = b_i$ for all $i \in \{1,\ldots,n\}$.*

Proof: Proved in [LS] (see Lemma IV.2.3). □

A *cyclic permutation* of the word $t^{a_1}x_1\cdots t^{a_n}x_n$ is a word $t^{a_k}x_k\cdots t^{a_n}x_n$ $t^{a_1}x_1\cdots t^{a_{k-1}}x_{k-1}$ with $k \in \{1,\ldots,n\}$. A word $t^{a_1}x_1\cdots t^{a_n}x_n$ is said to be *cyclically reduced* if any cyclic permutation of $t^{a_1}x_1\cdots t^{a_n}x_n$ is reduced. Note that, if $t^{a_1}x_1\cdots t^{a_n}x_n$ is reduced and $n \geq 2$, then $t^{a_1}x_1\cdots t^{a_n}x_n$ is cyclically reduced if and only if $x_n \in H \setminus K$. Every element of G is conjugate to a cyclically reduced word.

Lemma 2.2.3 (Collins' Lemma)**.** *If $g = t^{a_1}x_1 \cdots t^{a_n}x_n$ $(n \geq 1)$ and $h = t^{b_1}y_1 \cdots t^{b_m}y_m$ $(m \geq 1)$ are cyclically reduced and conjugate, then there exists a cyclic permutation h^* of h and an element $\alpha \in K$ such that $g = \alpha h^* \alpha^{-1}$.*

Proof: Proved in [LS] (see Theorem IV.2.5). □

Remark 2.2.4. There exists a natural homomorphism $f : G \to H$, defined by $f(h) = h$ for all $h \in H$, and $f(t) = 1$.

Remark 2.2.5. Let P be a group and let $\varphi : H \to P$ be a homomorphism. Let Q be the HNN extension of P relative to $\varphi(K)$:

$$Q = \langle\ P, \bar{t} \mid \bar{t}^{-1}\varphi(k)\bar{t} = \varphi(k), \forall\ k \in K\ \rangle.$$

Then φ induces a homomorphism $\overline{\varphi} : G \to Q$, defined by $\overline{\varphi}(h) = \varphi(h)$ for all $h \in H$, and $\overline{\varphi}(t) = \bar{t}$.

Lemma 2.2.6. *With the notations of Remark 2.2.5, $ker(\overline{\varphi})$ is the normal closure of $ker(\varphi)$ in G.*

Proof: Proved in [Mi] (see Lemma 7.5). □

The following observation is the key in the proof of our main theorem:

Remark 2.2.7. Let G be a right-angled Artin group of rank r $(r \geq 1)$. Let H be a special subgroup of G of rank $r-1$. In other words, there is a partition of V, $V = W \cup \{t\}$, such that $H = \langle W \rangle$. Then G can be written as the HNN extension of H relative to the special subgroup $K = \langle lk(t) \rangle$ of H:

$$G = \langle\ H, t \mid t^{-1}kt = k, \forall\ k \in K\ \rangle.$$

2.3 Hereditary conjugacy p-separability and p-centralizer condition

We start with an observation that the reader has to keep in mind, because it will be used repeatedly in the rest of the chapter: if H and K are two normal subgroups of p-power index in a group G, then $H \cap K$ is a normal subgroup of p-power index in G.

The centralizer conditon was first introduced by Chagas and Zalesskii as a sufficient condition for a conjugacy separable group to be hereditarily conjugacy separable (see [CZ]). Thereafter Minasyan showed that this condition is also necessary; that is, a group is hereditarily conjugacy separable if and only if it is conjugacy separable and satisfies the centralizer condition (see [Mi]).

We make the following definition, which naturally generalizes that of [Mi]:

Definition 2.3.1. *We say that G satisfies the p centralizer condition (pCC) if, for every normal subgroup H of p-power index in G, and for all $g \in G$, there exists a normal subgroup K of p-power index in G such that $K < H$, and:*

$$C_{G/K}(\varphi(g)) \subset \varphi(C_G(g)H),$$

where $\varphi : G \to G/K$ denotes the canonical projection.

In Appendix B, it is shown that the p centralizer condition can be reformulated in terms of centralizers in the pro-p completion $G_{\widehat{p}}$ of G.

We shall show that a group G is hereditarily conjugacy p-separable if and only if it is conjugacy p-separable and satisfies the p centralizer condition (see Proposition 2.3.6). If H is a subgroup of G, and $g \in G$, we set $C_H(g) = \{h \in H \mid gh = hg\}$. For technical reasons, we have to introduce the following definitions:

Definition 2.3.2. *Let G be a group, H be a subgroup of G, and $g \in G$. We say that the pair (H, g) satisfies the p centralizer condition in G (pCC_G) if, for every normal subgroup K of p-power index in G, there exists a normal subgroup L of p-power index in G such that $L < K$, and:*

$$C_{\varphi(H)}(\varphi(g)) \subset \varphi(C_H(g)K),$$

where $\varphi : G \to G/L$ denotes the canonical projection. We say that H satisfies the p centralizer condition in G (pCC_G) if the pair (H, g) satisfies the p centralizer condition in G for all $g \in G$.

If G is a group, H is a subgroup of G, and $g \in G$, then we set: $g^H = \{\alpha g \alpha^{-1} \mid \alpha \in H\}$. In order to prove Proposition 2.3.6, we need the following statements, which are the analogues of some statements obtained in [Mi] (Lemma 3.4, Corollary 3.5, and Lemma 3.7 respectively):

Lemma 2.3.3. *Let G be a group, H be a subgroup of G, and $g \in G$. Suppose that the pair (G, g) satisfies pCC_G, and that g^G is finitely p-separable in G. If $C_G(g)H$ is finitely p-separable in G, then g^H is also finitely p-separable in G.*

Proof: Let $h \in G$ such that $h \notin g^H$. If $h \notin g^G$, then, since g^G is finitely p-separable in G, there exists a homomorphism φ from G onto a finite p-group P such that $\varphi(h) \notin \varphi(g^G)$. In particular, $\varphi(h) \notin \varphi(g^H)$. Thus we can assume that $h \in g^G$. Let $\alpha \in G$ be such that $h = \alpha g \alpha^{-1}$. Suppose that $C_G(g) \cap \alpha^{-1}H \neq \emptyset$. Let $k \in C_G(g) \cap \alpha^{-1}H$. Then $\alpha k \in H$, and $h = \alpha g \alpha^{-1} = \alpha k g (\alpha k)^{-1} \in g^H$ – which is a contradiction. Thus $C_G(g) \cap \alpha^{-1}H = \emptyset$, i.e. $\alpha^{-1} \notin C_G(g)H$. As $C_G(g)H$ is finitely p-separable in G, there exists a normal subgroup K of p-power index in G such that $\alpha^{-1} \notin C_G(g)HK$. Now the condition pCC_G implies that there exists a normal subgroup L of p-power index in G such that $L < K$, and:

$$C_{G/L}(\varphi(g)) \subset \varphi(C_G(g)K),$$

where $\varphi : G \to G/L$ denotes the canonical projection. We claim that $\varphi(h) \notin \varphi(g^H)$. Indeed, if there is $\beta \in H$ such that $\varphi(h) = \varphi(\beta g \beta^{-1})$, then $\varphi(\alpha^{-1}\beta)\varphi(g) = \varphi(\alpha^{-1}\beta)\varphi(\beta^{-1}h\beta) = \varphi(\alpha^{-1}h\alpha)\varphi(\alpha^{-1}\beta) = \varphi(g)\varphi(\alpha^{-1}\beta)$, i.e. $\varphi(\alpha^{-1}\beta) \in C_{G/L}(\varphi(g))$. But then $\varphi(\alpha^{-1}) \in C_{G/L}(\varphi(g))\varphi(H) \subset \varphi(C_G(g)K\, H)$. Hence $\alpha^{-1} \in C_G(g)HKL = C_G(g)HK$ (because $L < K$) – which is a contradiction. □

Corollary 2.3.4. *Let G be a conjugacy p-separable group satisfying pCC, and H be a subgroup of G such that $C_G(h)H$ is finitely p-separable in G for all $h \in H$. Then H is conjugacy p-separable. Moreover, for all $h \in H$, h^H is finitely p-separable in G.*

Proof: Let $h \in H$. Since G satisfies pCC, the pair (G, h) satisfies pCC_G. Since G is conjugacy p-separable, h^G is finitely p-separable in G. Lemma 2.3.3 now implies that h^H is finitely p-separable in G. Therefore h^H is finitely p-separable in H. □

Lemma 2.3.5. *Let G be a group, H be a subgroup of G, and $g \in G$. Let K be a normal subgroup of p-power index in G. If $g^{H\cap K}$ is finitely p-separable in G, then there exists a normal subgroup L of p-power index in G such that $L < K$, and:*

$$C_{\varphi(H)}(\varphi(g)) \subset \varphi(C_H(g)K),$$

where $\varphi : G \to G/L$ denotes the canonical projection.

Proof: Note that $H\cap K$ is of finite index n in H. Actually, $H\cap K$ is of p-power index in H (because $\frac{H}{H\cap K} \simeq \frac{KH}{K} < \frac{G}{K}$), but this is not needed here. There exist $\alpha_1,...,\alpha_n \in H$ such that $H = \sqcup_{i=1}^n \alpha_i(H \cap K)$. Up to renumbering, we can assume that there exists $l \in \{0,...,n\}$ such that $\alpha_i^{-1} g \alpha_i \in g^{H\cap K}$ for all $i \in \{1,...,l\}$ and $\alpha_i^{-1} g \alpha_i \notin g^{H\cap K}$ for all $i \in \{l+1,...,n\}$. By the assumptions, there exists a normal subgroup L of p-power index in G such that $\alpha_i^{-1} g \alpha_i \notin g^{H\cap K}L$ for all $i \in \{l+1,...,n\}$. Up to replacing L by $L\cap K$, we can assume that $L < K$. Let $\varphi : G \to G/L$ be the canonical projection. Let $\overline{h} \in C_{\varphi(H)}(\varphi(g))$. There exists $h \in H$ such that $\overline{h} = \varphi(h)$. There exist $i \in \{1,...,n\}$ and $k \in H\cap K$ such that $h = \alpha_i k$. We have $\varphi(h^{-1}gh) = \varphi(h)^{-1}\varphi(g)\varphi(h) = \varphi(g)$. Thus $h^{-1}gh \in gL$. But then $\alpha_i^{-1} g \alpha_i = kh^{-1}ghk^{-1} \in kgLk^{-1} = kgk^{-1}L \subset g^{H\cap K}L$. Therefore $i \leq l$. Then there exists $\beta \in H\cap K$ such that: $\alpha_i^{-1} g \alpha_i = \beta g \beta^{-1}$. This is to say that $\alpha_i\beta \in C_H(g)$, and then $h = \alpha_i k = (\alpha_i\beta)(\beta^{-1}k) \in C_H(g)(H \cap K) \subset C_H(g)K$. We have shown that $C_{\varphi(H)}(\varphi(g)) \subset \varphi(C_H(g)K)$. □

We are now ready to prove:

Proposition 2.3.6. *A group is hereditarily conjugacy p-separable if and only if it is conjugacy p-separable and satisfies pCC.*

Proof: Suppose that G is conjugacy p-separable and satisfies pCC. Let H be a subnormal subgroup of p-power index in G. Thus H is closed in the pro-p topology on G (because $G \setminus H = \cup\{gH \mid g \notin H\}$). Let $h \in H$. The set $C_G(h)H$ is a finite union of left cosets modulo H and thus is closed in the pro-p topology on G. Corollary 2.3.4 now implies that H is conjugacy p-separable. Therefore G is hereditarily conjugacy p-separable. Suppose now that G is hereditarily conjugacy p-separable. In particular, G is conjugacy p-separable. We shall show that G satisfies pCC. Let $g \in G$. Let K be a normal subgroup of p-power index in G. Let $H = K\langle g\rangle$. Since $K < H$, $[G : H]$ is a power of p. As $\frac{G}{K}$ is a finite p-group, every subgroup of it is subnormal. Thus H is subnormal in G. Therefore H is conjugacy p-separable. Note that $g^{G\cap K} = g^K = g^H \subset H$. As g^H is closed in the pro-p topology on H, it is closed in the pro-p topology on G, because the topology induced on H by the pro-p topology on G coincides with the pro-p topology on H (see, for example, [RZ2], Corollary 5.8). The result now follows from lemma 2.3.5. $\square$

2.4 Extensions of free groups by finite p-groups are conjugacy p-separable

We start with an observation that the reader has to keep in mind because it will be used repeatedly in the proof of Theorem 2.4.2: if $\varphi : G \to H$ is a homomorphism from a group G to a group H, whose kernel is torsion-free, then the restriction of φ to any finite subgroup of G is injective.

We need the following lemma:

Lemma 2.4.1. *Let $G = G_1 * \cdots * G_n$ be a free product of n conjugacy p-separable groups G_1,...,G_n. Let g, $h \in G \setminus \{1\}$ be two non-trivial elements of finite order in G such that $g \nsim h$. There exists a homomorphism φ from G onto a finite p-group P such that $\varphi(g) \nsim \varphi(h)$.*

Proof: Since g is of finite order in G, there exists $i \in \{1,...,n\}$ such that g is conjugate to an element of finite order in G_i. Thus we may assume that g belongs to G_i. Similarly, we may assume that there exists j in $\{1,...,n\}$ such that h belongs to G_j. Suppose that $i \neq j$. Let $\varphi : G_i \to P$ be a homomorphism from G_i onto a finite p-group P such that $\varphi(g) \neq 1$. Let $\widetilde{\varphi}$: $G \to P$ be the natural homomorphism extending φ. Then $\widetilde{\varphi}(g) \nsim \widetilde{\varphi}(h)$. Suppose that $i = j$. Then g and h are not conjugate in G_i – otherwise they would be conjugate in G. Since G_i is conjugacy p-separable, there exists a homomorphism $\varphi : G_i \to P$ from G_i onto a finite p-group P such that $\varphi(g)$

$\nsim \varphi(h)$. Let $\widetilde{\varphi} : G \to P$ be defined as above. We have $\widetilde{\varphi}(g) \nsim \widetilde{\varphi}(h)$. $\square$

In Section 2.4, by a graph, we mean a unoriented graph, possibly with loops or multiple edges.

Recall that a *graph of groups* is a connected graph $\Gamma = (V, E)$, together with a function $\mathcal{G}$ which assigns:

- to each vertex $v \in V$, a group G_v,
- and to each edge $e = \{v,w\} \in E$, a group G_e together with two injective homomorphisms $\alpha_e : G_e \to G_v$ and $\beta_e : G_e \to G_w$ – we are not assuming that $v \neq w$ –,

(see [Se], see also [Dy1]). The groups G_v $(v \in V)$ are called the *vertex groups* of Γ, the groups G_e $(e \in E)$ are called the *edge groups* of Γ. The monomorphisms α_e and β_e $(e \in E)$ are called the *edge monomorphisms.* The images of the edge groups under the edge monomorphisms are called the *edge subgroups.*

Choose disjoint presentations $G_v = \langle\ X_v \mid R_v\ \rangle$ for the vertex groups of Γ. Choose a maximal tree T in Γ. Assign a direction to each edge of Γ. Let $\{t_e \mid e \in E\}$ be a set in one-to-one correspondence with the set of edges of Γ, and disjoint from the X_v $(v \in V)$. The *fundamental group* of the above graph of groups Γ is the group G_Γ defined by the presentation whose generators are:

$$\begin{gathered} X_v \ (v \in V), \\ t_e \ (e \in E) \end{gathered}$$

(called vertex and edge generators respectively) and whose relations are:

$$\begin{gathered} R_v \ (v \in V), \\ t_e = 1 \ (e \in T), \\ t_e \alpha_e(g_e) t_e^{-1} = \beta_e(g_e), \forall\ g_e \in G_e \ (e \in E). \end{gathered}$$

(called vertex, tree, and edge relations respectively). One can prove that this is well-defined – that is, independent of our choice of T, etc. Note that it suffices to write the edge relations for g_e in a set of generators for G_e.

<u>Convention</u>: The groups G_v $(v \in V)$ and G_e $(e \in E)$ will be regarded as subgroups of G_Γ.

Let $\{\Gamma_i\}_{i \in I}$ be a collection of connected and pairwise disjoint subgraphs of Γ. We may define a graph of groups Γ^* from Γ *by contracting* Γ_i *to a point for all* $i \in I$, as follows. The graph Γ^* is obtained from Γ by contracting Γ_i to a point p_i for all $i \in I$. The function $\mathcal{G}^*$ is obtained from $\mathcal{G}$ by using the fundamental group of Γ_i for the vertex group at p_i, and by composing the edge monomorphisms of Γ by the natural inclusions of the vertex groups of Γ_i into the fundamental group of Γ_i, if necessary. Clearly, G_Γ is isomorphic to the fundamental group G_{Γ^*} of Γ^*.

If $\pi : G_\Gamma \to H$ is a homomorphism from G_Γ to a group H, such that the restriction of π to each edge subgroup of Γ is injective, then we may define a graph of groups Γ' from Γ *by composing with* π, as follows. The vertex set of Γ' is V, and the edge set of Γ' is E. The vertex groups of Γ' are the groups $G'_v = \pi(G_v)$ $(v \in V)$, and the edge groups of Γ' are the groups $G'_e = G_e$ $(e \in E)$. The edge monomorphisms are the monomorphisms $\alpha'_e = \pi \circ \alpha_e$ and $\beta'_e = \pi \circ \beta_e$ $(e \in E)$. Present G_Γ and $G_{\Gamma'}$ using the same symbols for edge generators and with the same choice of maximal tree. There exist two homomorphisms, $\pi_V : G_\Gamma \to G_{\Gamma'}$ and $\pi_E : G_{\Gamma'} \to H$ such that the diagram:

$$\begin{tikzcd} G_\Gamma \arrow[r, "\pi"] \arrow[d, "\pi_V"'] & H \\ G_{\Gamma'} \arrow[ur, "\pi_E"'] & \end{tikzcd}$$

commutes, and that the restriction of π_E to each vertex group of $G_{\Gamma'}$ is injective. The homomorphism π_V is given by:

$$(\pi_V)_{|G_v} = \pi_{|G_v}, \ \forall\ v \in V,$$
$$\pi_V(t_e) = t_e, \ \forall\ e \in E.$$

And the homomorphism π_E is given by:

$$(\pi_E)_{|G'_v} = (id_H)_{|G'_v}, \ \forall\ v \in V,$$
$$\pi_E(t_e) = \pi(t_e), \ \forall\ e \in E.$$

In [Dy1], Dyer proved that every extension of a free group by a finite group is conjugacy separable. The following theorem is the analogue of Dyer's theorem for conjugacy p-separability.

Theorem 2.4.2. *Every extension of a free group by a finite p-group is conjugacy p-separable.*

Proof: Our proof was inspired by that of Dyer (see [Dy1]). Let G be an extension of a free group by a finite p-group. In other words, there exists a short exact sequence:

$$1 \longrightarrow F \longrightarrow G \xrightarrow{\pi} P \longrightarrow 1\ ,$$

where F is a free group, and P is a finite p-group. Let $g \in G$. Let $h \in G$ such that $g \nsim h$.

Step 1: We show that we may assume that G satisfies a short exact sequence:

$$1 \longrightarrow F \longrightarrow G \xrightarrow{\pi} C_{p^n} \longrightarrow 1 \ ,$$

where F is a free group, $n \geq 1$, C_{p^n} denotes the cyclic group of order p^n, and $\pi(g) = \pi(h)$.

Since G is an extension of a free group by a finite p-group, G is residually p-finite by [Gru], Lemma 1.5. Therefore, if $g = 1$, then $g^G = \{1\}$ is finitely p-separable in G. On the other hand, if g is of infinite order in G, then g^G is finitely p-separable in G by [I], Proposition 5. Therefore we may assume that $g \neq 1$ and that g is of finite order in G. Similarly, we may assume that $h \neq 1$ and that h is of finite order in G. If $\pi(g)$ and $\pi(h)$ are not conjugate in P, we are done. Thus, up to replacing h by a conjugate of itself, we may assume that $\pi(g) = \pi(h)$. Since $ker(\pi) = F$ is torsion-free, g and h have the same order p^n $(n \in \mathbb{N}^*)$. Let $H = F\langle g \rangle$. Note that H is a subgroup of p-power index in G, and that g and h belong to H. As $\frac{G}{F} = P$ is a finite p-group, every subgroup of it is subnormal. Thus H is subnormal in G. Then we may replace G by H, by [I], Proposition 4[1], so as to assume that G satisfies the short exact sequence:

$$1 \longrightarrow F \longrightarrow G \xrightarrow{\pi} C_{p^n} \longrightarrow 1 \ .$$

Now, G is the fundamental group of a graph of groups Γ, whose vertex groups are all finite groups, by [S], Theorem. As $\pi_{|G_v}$ is injective for all $v \in V$, G_v is isomorphic to a subgroup of C_{p^n} for all $v \in V$. From now on, the groups G_v $(v \in V)$ will be regarded as subgroups of C_{p^n}.

Step 2: We show that we may assume that all edge groups are non-trivial, that if two different vertices are connected by an edge, then the corresponding edge group is a proper subgroup of C_{p^n}, and that g and h belong to two different vertex groups.

First, we show that we may assume that all edge groups are non-trivial. Indeed, Let Γ_0 be the subgraph of Γ whose vertices are all the vertices of Γ, and whose edges are the edges of Γ for which the edge group is non-trivial. Let Γ_1,...,Γ_r be the connected components of Γ_0. Let Γ^* be the graph of groups obtained from Γ by contracting Γ_i to a point for all $i \in \{1,...,r\}$. Let

[1] Strictly speaking, it follows from the proof of [I], Proposition 4, that, if there exists a homomorphism $\varphi : H \to P$ from H onto a finite p-group P such that $\varphi(g) \nsim \varphi(h)$, then there exists a homomorphism $\psi : G \to Q$ from G onto a finite p-group Q such that $\psi(g)$ $\nsim \psi(h)$. The exact statement of [I], Proposition 4, is slightly different.

T be a maximal tree of Γ^*. Then G is isomorphic to the fundamental group G^* of Γ^*. Observe that G^* is the free product of the free group on $\{t_e \mid e \notin T\}$ and the fundamental groups of the Γ_i ($i \in \{1,...,r\}$). Thus, it suffices to consider the case where $\Gamma = \Gamma_i$ ($i \in \{1,...,r\}$), by Lemma 2.4.1. Since each Γ_i ($i \in \{1,...,r\}$) is a graph of groups whose edge groups are all non-trivial, the first part of the assertion is proved.

Now, we show that we may assume that if two different vertices are connected by an edge, then the corresponding edge group is a proper subgroup of C_{p^n}. Indeed, let Γ_0 be the subgraph of Γ whose vertices are all the vertices of Γ, and whose edges are the edges of Γ for which the edge group is isomorphic to C_{p^n}. Let Γ_1,...,Γ_r be the connected components of Γ_0. Choose a maximal tree T_i in Γ_i, for all $i \in \{1,...,r\}$. Let Γ^* be the graph of groups obtained from Γ by contracting T_i to a point for all $i \in \{1,...,r\}$. Then G is isomorphic to the fundamental group G^* of Γ^*. Note that a vertex group of Γ^* is either a vertex group of Γ, or the fundamental group of T_i, for some $i \in \{1,...,r\}$, in which case it is isomorphic to C_{p^n} (because each T_i ($i \in \{1,...,r\}$) is a tree of groups whose vertex and edge groups are all equal to C_{p^n}). Thus, we may replace Γ by Γ^*, so that the second part of the assertion is proved.

Since g is of finite order in G, there exists a vertex v of Γ, an element g_0 of finite order in the vertex group G_v of v, and an element α of G such that $g = \alpha g_0 \alpha^{-1}$. Similarly, there exists a vertex w of Γ, an element h_0 of finite order in the vertex group G_w of w, and an element β of G such that $h = \beta h_0 \beta^{-1}$. As C_{p^n} is abelian, we have: $\pi(g_0) = \pi(h_0)$. Thus, up to replacing g by g_0 and h by h_0, we may assume that g belongs to G_v, and h belongs to G_w. Since $\pi_{|G_v}$ is injective, and $\pi(g) = \pi(h)$, we have $v \neq w$.

Step 3: We show that we may assume that Γ has exactly two vertices, and that all edges join these two vertices.

Indeed, choose a maximal tree T in Γ. There is a path P in T joining v to w. Choose an edge e on this path. Then $T \setminus \{e\}$ is the disjoint union of two trees, T_v and T_w – with $v \in T_v$ and $w \in T_w$. Let Γ_v be the full subgraph of Γ generated by the vertices of T_v, and Γ_w be the full subgraph of Γ generated by the vertices of T_w. Let Γ^* be the graph of groups obtained from Γ by contracting Γ_v to a point v^* and Γ_w to a point w^*. Observe that Γ^* has exactly two vertices and that all edges join these two vertices. The vertex groups of Γ^* are the fundamental groups of Γ_v and Γ_w, respectively. The edge groups of Γ^* are non-trivial proper subgroups of C_{p^n}. And G is isomorphic to the fundamental group G^* of Γ^*. Now, since the restriction of π to each edge subgroup of Γ^* is injective, we may define a graph of groups Γ' from Γ^* by composing with π, as described above. Denote by G' the fundamental group of Γ'. There exist two homomorphisms, $\pi_V : G \to G'$

and $\pi_E : G' \to C_{p^n}$, such that the diagram:

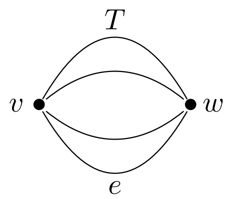
$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "\pi_V"'] & C_{p^n} \\ G' \arrow[ur, "\pi_E"'] & \end{tikzcd}$$

commutes, and that the restriction of π_E to each vertex group of Γ' is injective. Consequently, $ker(\pi_E)$ is free by [Se], II, 2.6., Lemma 8.

Set $g' = \pi_V(g)$, and $h' = \pi_V(h)$. As g' and h' have order p^n, the vertex groups of Γ' are equal to C_{p^n}. The edge groups of Γ' are non-trivial proper subgroups of C_{p^n}. Observe that g' and h' belong to two different vertex groups, and that g' (resp. h') is not conjugate to an element of one of the edge groups. Let e be an edge of Γ'. Then g' and h' are not conjugate in $G'_v *_{G'_e} G'_w$, by [MKS], Theorem 4.6. Observe that G' is an HNN extension (in the general sense) of $G'_v *_{G'_e} G'_w$ with stable letters t_a $(a \in E \setminus \{e\})$, and associated subgroups $\alpha'_a(G'_a)$ and $\beta'_a(G'_a)$ $(a \in E \setminus \{e\})$. Therefore g' and h' are not conjugate in G' (see, for example, [Dy2], Theorem 3). Thus, we may replace Γ by Γ', G by G', g by g', and h by h', so as to assume that Γ has two vertices and that all edges join these two vertices.

Step 4: We show that we may assume that Γ has at most two edges.

Suppose that Γ has more than two edges. Choose a maximal tree T in Γ – that is, an edge of Γ. Present $G_v = \langle g \mid g^{p^n} = 1\rangle$, $G_w = \langle h \mid h^{p^n} = 1\rangle$, and G as described above. Choose an edge $e \in E \setminus T$.



The edge relations corresponding to e can be reduced to the following:

$$t_e \alpha_e(g_e) t_e^{-1} = \beta_e(g_e),$$

where g_e is a generator of G_e. Let p^s be the order of G_e $(s \in \{1,...,n-1\})$. Then $\alpha_e(g_e)$ generates a subgroup of order p^s of G_v. But there exists a unique subgroup of order p^s in G_v; it is cyclic, generated by g^{p^r}, where $r = n - s$. Thus, up to replacing g_e by the preimage of g^{p^r} under α_e, we may assume that $\alpha_e(g_e) = g^{p^r}$. There exists $k \in \mathbb{N}$, such that p and k are coprime, and that $\beta_e(g_e) = h^{kp^r}$. Therefore the edge relation corresponding to e can be written:

$$t_e g^{p^r} t_e^{-1} = h^{kp^r},$$

where $r \in \{1,...,n-1\}$, $k \in \mathbb{N}$, and p and k are coprime. Now, since $\pi : G \to C_{p^n}$ satisfies $\pi(g) = \pi(h)$, we have: $\pi(g)^{p^r} = \pi(h)^{kp^r} = \pi(g)^{kp^r}$, and then $\pi(g)^{(k-1)p^r} = 1$ (in C_{p^n}). As $\pi(g)$ has order p^n in C_{p^n}, we deduce that p^{n-r} divides $k-1$. There exists $a \in \mathbb{Z}$ such that $k = ap^{n-r} + 1$. We conclude that the edge relation corresponding to e can be written:

$$t_e g^{p^r} t_e^{-1} = h^{p^r},$$

where $r \in \{1,...,n-1\}$.

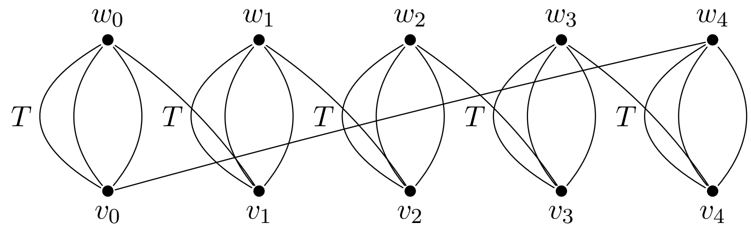
Let H be the normal subgroup of G generated by the elements:

$$g,\ h,\ t_a\ (a \in E \setminus \{e\}),\ t_e^p.$$

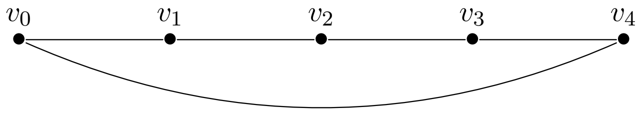
Then H has index p in G, and g and h belong to H. Thus we may replace G by H by [I], Proposition 4. Let G_0 be the fundamental group of the graph of groups $\Gamma \setminus \{e\}$. Set $G_0 = \langle X_0 \mid R_0 \rangle$, where the presentation is as fundamental group of the graph of groups $\Gamma \setminus \{e\}$. Set $G_i = t_e^i G_0 t_e^{-i} = \langle X_i \mid R_i \rangle$, for all $i \in \{1,...,p-1\}$. Clearly $\{1,t_e,...,t_e^{p-1}\}$ is a Schreier transversal for H in G. The Reidemeister-Schreier method yields the presentation:

$$H = \langle\ X_0,\ X_1,...,\ X_{p-1},\ u \mid R_0,\ R_1,...,\ R_{p-1},\ g_1^{p^r} = h_0^{p^r},\ g_2^{p^r} = h_1^{p^r},...,\ g_{p-1}^{p^r} = h_{p-2}^{p^r},\ u g_0^{p^r} u^{-1} = h_{p-1}^{p^r}\ \rangle,$$

where $u = t_e^p$, $g_i = t_e^i g t_e^{-i}$ ($i \in \{0,...,p-1\}$), and $h_j = t_e^j h t_e^{-j}$ ($j \in \{0,...,p-1\}$). Replace g by g_0, and h by h_1. Observe that H is the fundamental group of a graph of groups $\widetilde{\Gamma}$, as follows. The graph $\widetilde{\Gamma}$ has $2p$ vertices, say v_0, w_0, v_1, w_1,...,v_{p-1}, w_{p-1}, and $p\,|E|$ edges. Let $\widetilde{\Gamma}_i$ be the full subgraph of $\widetilde{\Gamma}$ generated by $\{v_i, w_i\}$ for all $i \in \{0,...,p-1\}$. Then $\widetilde{\Gamma}_i$ is isomorphic to $\Gamma \setminus \{e\}$. There is one edge joining w_0 to v_1, one edge joining w_1 to v_2,..., one edge joining w_{p-2} to v_{p-1}, and one edge joining v_0 to w_{p-1}, and the edge groups associated to these egdes are isomorphic to G_e. Note that g belongs to the vertex group of v_0 and h belongs to the vertex group of w_1.



Let Γ^* be the graph of groups obtained from $\widetilde{\Gamma}$ by contracting $\widetilde{\Gamma}_i$ to a point for all $i \in \{1,...,p-1\}$. Then G is isomorphic to the fundamental group of Γ^*. The graph Γ^* has p vertices, say v_0,...,v_{p-1}. There is one edge joining v_0 to v_1, one edge joining v_1 to v_2,..., one edge joining v_{p-2} to v_{p-1}, and one edge joining v_0 to v_{p-1}, and the edge groups associated to these edges are all isomorphic to G_e. Note that g belongs to the vertex group of v_0 and h belongs to the vertex group of v_1.



Let T be the maximal tree $T = v_0v_1 \cdots v_{p-2}v_{p-1}$. Then $T \setminus \{v_0v_1\}$ is the disjoint union of two trees : v_0 and $v_1v_2 \cdots v_{p-2}v_{p-1}$. Set $\Gamma_1^* = v_0$ and $\Gamma_2^* = v_1v_2 \cdots v_{p-2}v_{p-1}$. Let Λ be the graph of groups obtained from Γ^* by contracting Γ_i^* to a point for all $i \in \{1,2\}$. Let Λ' be the graph of groups obtained from Λ by composing with π. As in Step 3, we may replace Γ by Λ', so as to assume that Γ has two vertices and two edges joining these two vertices.

End of the proof: Present $G_v = \langle g \mid g^{p^n} = 1\rangle$, $G_w = \langle h \mid h^{p^n} = 1\rangle$, and G as described above. There are two cases:

Case 1: Γ has one edge.

In this case, G is an amalgamated product of two finite abelian p-groups. Since G is residually p-finite, G is conjugacy p-separable by [I], Theorem 2. Thus, there exists a homomorphism φ from G onto a finite p-group P such that $\varphi(g) \nsim \varphi(h)$.

Case 2: Γ has two edges.

We have:

$$G = \langle\ g,\ h,\ t \mid g^{p^n} = 1,\ h^{p^n} = 1,\ g^{p^r} = h^{p^r},\ tg^{p^s}t^{-1} = h^{p^s}\ \rangle,$$

where $r \in \{1,...,n-1\}$, $s \in \{1,...,n-1\}$. Let:

$$A = C_{p^n} \times \underbrace{C_{p^s} \times \cdots \times C_{p^s}}_{p^r - 1} \times C_{p^r}$$

Set $m = p^r + 1$. Present each factor of this product in the natural way, using generators x_1,...,x_m respectively. Let α be the automorphism of A defined by:

$$\begin{gathered}\alpha(x_1) = x_1x_2x_m \\ \alpha(x_i) = x_{i+1},\ \forall\ i \in \{2,...,m-2\} \\ \alpha(x_{m-1}) = (x_2 \cdots x_{m-1})^{-1} \\ \alpha(x_m) = x_m\end{gathered}$$

It is easily seen that α has order $m-1=p^r$. We have:

$$\begin{aligned} \alpha^0(x_1) &= x_1, \\ \alpha^1(x_1) &= x_1x_2x_m, \\ \alpha^2(x_1) &= x_1x_2x_3x_m^2, \\ &\dots \\ \alpha^{m-2}(x_1) &= x_1x_2x_3\cdots x_{m-1}x_m^{m-2}. \end{aligned}$$

Let $B = A \rtimes \langle\alpha\rangle$ be the semidirect product of A by $\langle\alpha\rangle$. Note that B is a finite p-group. Let $\varphi : G \to B$ be the homomorphism defined by:

$$\begin{aligned} \varphi(g) &= x_1, \\ \varphi(h) &= x_1x_m, \\ \varphi(t) &= \alpha. \end{aligned}$$

Observe that the conjugacy class of $\varphi(g)$ in B is $\varphi(g)^B = \{\alpha^k(x_1) \mid k \in \{0,...,m-2\}\}$. Thus, $\varphi(g)$ and $\varphi(h)$ are not conjugate in B. $\square$

Corollary 2.4.3. *Let P be a finite p-group. Let A be a subgroup of P. Let Q be the HNN extension of P relative to A:*

$$Q = \langle\, P,\, t \mid t^{-1}at = a,\, \forall\, a \in A \,\rangle.$$

Then Q is hereditarily conjugacy p-separable.

Proof: Let R be an arbitrary subgroup of Q. Let $f : Q \to P$ be the natural homomorphism. We have $ker(f) \cap P = \{1\}$. Therefore $ker(f)$ is free by [KS], Theorem 6. That is, Q is an extension of a free group by a finite p-group. Thus R is itself an extension of a free group by a finite p-group. Therefore R is conjugacy p-separable by Theorem 2.4.2. $\square$

Remark 2.4.4. It is known that a fundamental group of a graph of groups, whose vertex groups are all finite p-groups is residually p-finite if and only if it is an extension of a free group by a finite p-group (see, for example, [AF1], Lemma 3.3). Thus, as an immediate consequence of Theorem 2.4.2, we have that a fundamental group of a graph of groups whose vertex groups are all finite p-groups is conjugacy p-separable if and only if it is residually p-finite.

2.5 Retractions

In this section, we shall prove several results on retractions that will allow us to control the growth of the intersection of Lemma 2.6.5 in finite p-group quotients of G_Γ.

Definition 2.5.1. *Let G be a group, and H be a subgroup of G. We say that H is a retract of G if there exists a homomorphism $\rho_H : G \to G$ such that $\rho_H(G) = H$ and $\rho_H(h) = h$ for all $h \in H$. The homomorphism ρ_H is called a retraction of G onto H.*

Remark 2.5.2. If G is a right-angled Artin group, and $H = \langle W \rangle$ is a special subgroup of G, then H is a retract of G. A retraction of G onto H is given by:

$$\rho_H(v) = \begin{cases} v & \text{if } v \in W \\ 1 & \text{if } v \in V \setminus W \end{cases}$$

Lemma 2.5.3. *Let G be a group, and H be a subgroup of G. Suppose that H is a retract of G. Let ρ_H be a retraction of G onto H. Let N be a normal subgroup of G such that $\rho_H(N) \subset N$. Then ρ_H induces a retraction $\rho_{\overline{H}} : G/N \to G/N$ of G/N onto the canonical image $\overline{H}$ of H in G/N, defined by: $\rho_{\overline{H}}(gN) = \rho_H(g)N$ for all $gN \in G/N$.*

Proof : Proved in [Mi] (see Lemma 4.1). □

Remark 2.5.4. Let G be a group, and let H, H' be two subgroups of G. Suppose that H and H' are retracts of G and that the corresponding retractions, ρ_H and $\rho_{H'}$, commute. Then $\rho_H(H') = \rho_{H'}(H) = H \cap H'$. Moreover $H \cap H'$ is a retract of G. A retraction of G onto $H \cap H'$ is given by $\rho_{H \cap H'} = \rho_H \circ \rho_{H'} = \rho_{H'} \circ \rho_H$.

Proposition 2.5.5. *Let G be a group and H_1,...,H_n be n subgroups of G. Suppose that H_1,...,H_n are retracts of G and that the corresponding retractions pairwise commute. Then, for every normal subgroup K of p-power index in G, there exists a normal subgroup N of p-power index in G such that $N < K$ and $\rho_{H_i}(N) \subset N$ for all $i \in \{1,...,n\}$. Consequently, for every $i \in \{1,...,n\}$, the retraction ρ_{H_i} induces a retraction $\rho_{\overline{H_i}}$ of G/N onto the canonical image $\overline{H_i}$ of H_i in G/N.*

Proof: Proved in [Mi] (see Proposition 4.3 and Remark 4.4). □

Lemma 2.5.6. *Let G be a group, and let H, H' be two subgroups of G. Suppose that H and H' are retracts of G and that the corresponding retractions, ρ_H and $\rho_{H'}$, commute. Let N be a normal subgroup of G such that $\rho_H(N) \subset N$ and $\rho_{H'}(N) \subset N$. Then $\varphi(H \cap H') = \varphi(H) \cap \varphi(H')$, where $\varphi : G \to G/N$ denotes the canonical projection.*

Proof: Proved in [Mi] (see Lemma 4.5). □

The next statement is the analogue of Lemma 4.6 in [Mi]:

Corollary 2.5.7. *Let G be a group and H_1,...,H_n be n subgroups of G. Suppose that H_1,...,H_n are retracts of G and that the corresponding retractions ρ_{H_1},...,ρ_{H_n} pairwise commute. Then, for every normal subgroup K of p-power index in G, there exists a normal subgroup N of p-power index in G such that $N < K$ and $\rho_{H_i}(N) \subset N$, for all $i \in \{1,...,n\}$. Moreover, if $\varphi : G \to G/N$ denotes the canonical projection, then $\varphi(\bigcap_{i=1}^n H_i) = \bigcap_{i=1}^n \varphi(H_i)$.*

Proof: By Proposition 2.5.5, there exists a normal subgroup N of p-power index in G such that $N < K$ and $\rho_{H_i}(N) \subset N$ for all $i \in \{1,...,n\}$. We denote by $\varphi : G \to G/N$ the canonical projection. We argue by induction on $k \in \{1,...,n\}$ to prove that $\varphi(\bigcap_{i=1}^{k} H_i) = \bigcap_{i=1}^{k} \varphi(H_i)$. If $k = 1$, then the result is trivial. Thus we can assume that $k \geq 2$ and that the result has been proved for $k-1$. We set $H' = \bigcap_{i=1}^{k-1} H_i$. By Remark 2.5.4, H' is a retract of G. A retraction of G onto H' is given by $\rho_{H'} = \rho_{H_1} \circ ... \circ \rho_{H_{k-1}}$. We have:

$$\begin{aligned}\rho_{H'}(N) &= \rho_{H_1}(...(\rho_{H_{k-2}}(\rho_{H_{k-1}}(N)))) \\ &\subset \rho_{H_1}(...(\rho_{H_{k-2}}(N))) \\ &\quad ... \\ &\subset \rho_{H_1}(N) \\ &\subset N.\end{aligned}$$

The retractions $\rho_{H'}$ and ρ_{H_k} commute, so we can apply Lemma 2.5.6 to conclude that $\varphi(H' \cap H_k) = \varphi(H') \cap \varphi(H_k)$. By the induction hypothesis, $\varphi(H') = \bigcap_{i=1}^{k-1} \varphi(H_i)$. Finally $\varphi(\bigcap_{i=1}^{k} H_i) = \bigcap_{i=1}^{k} \varphi(H_i)$. □

In the following lemmas, G is a group, and A, B are two subgroups of G. We assume that A and B are retracts of G and that the corresponding retractions, ρ_A and ρ_B, commute.

Lemma 2.5.8. *Let x, $y \in G$. We set $\alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1})$ ($\in AxB$) and $\beta = \rho_A(\rho_B(y)y^{-1})y\rho_B(y^{-1})$ ($\in AyB$). Then the following are equivalent:*

1. *$y \in AxB$,*
2. *$\beta \in \alpha^{A\cap B}$.*

Proof: Proved in [Mi] (see Lemma 5.1). □

Lemma 2.5.9. *Let $x \in G$. We set $\alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1})$ ($\in AxB$) and $\gamma = \rho_A(\rho_B(x)x^{-1})$ ($\in A$). Then we have:*

$$A\cap xBx^{-1} = \gamma^{-1}C_{A\cap B}(\alpha)\gamma.$$

Proof: Proved in [Mi] (see Lemma 5.2). □

The next five statements are the analogues of some statements in [Mi] (Lemma 5.3, Corollary 5.4, Lemma 5.5, Lemma 5.6, and Lemma 5.7 respectively):

Lemma 2.5.10. *Let $x \in G$. We set: $\alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1})$ ($\in AxB$). If $\alpha^{A\cap B}$ is finitely p-separable in G, then AxB is also finitely p-separable in G.*

Proof: Let $y \in G$ such that $y \notin AxB$. We set $\beta = \rho_A(\rho_B(y)y^{-1})y\rho_B(y^{-1})$. By Lemma 2.5.8, we have $\beta \notin \alpha^{A\cap B}$. Since $\alpha^{A\cap B}$ is finitely p-separable in G, there exists a normal subgroup K of p-power index in G such that, if $\psi : G \to G/K$ denotes the canonical projection, we have: $\psi(\beta) \notin \psi(\alpha^{A\cap B}) = \psi(\alpha)^{\psi(A\cap B)}$. By Corollary 2.5.7, there exists a normal subgroup N of p-power index in G such that $N < K$, $\rho_A(N) \subset N$, $\rho_B(N) \subset N$ and, if $\varphi : G \to G/N$ denotes the canonical projection, then: $\varphi(A\cap B) = \varphi(A)\cap\varphi(B)$. Assume that $\varphi(\beta) \in \varphi(\alpha)^{\varphi(A\cap B)}$. Let $g \in A\cap B$ be such that $\varphi(\beta) = \varphi(g)\varphi(\alpha)\varphi(g)^{-1}$. Then $\beta \in g\alpha g^{-1}N$. Since $N < K$, we obtain $\beta \in g\alpha g^{-1}K$. But this contradicts the fact that $\psi(\beta) \notin \psi(\alpha)^{\psi(A\cap B)}$. Therefore we have: $\varphi(\beta) \notin \varphi(\alpha)^{\varphi(A\cap B)}$ i.e. $\varphi(\beta) \notin \varphi(\alpha)^{\varphi(A)\cap\varphi(B)}$. We set $\overline{A} = \varphi(A)$ and $\overline{B} = \varphi(B)$. By Lemma 2.5.3, ρ_A induces a retraction $\rho_{\overline{A}}$ of G/N onto $\overline{A}$ and ρ_B induces a retraction $\rho_{\overline{B}}$ of G/N onto $\overline{B}$. We set: $\overline{x} = \varphi(x)$ and $\overline{y} = \varphi(y)$. We have: $\varphi(\alpha) = \rho_{\overline{A}}(\rho_{\overline{B}}(\overline{x})\overline{x}^{-1})\overline{x}\rho_{\overline{B}}(\overline{x}^{-1})$ and $\varphi(\beta) = \rho_{\overline{A}}(\rho_{\overline{B}}(\overline{y})\overline{y}^{-1})\overline{y}\rho_{\overline{B}}(\overline{y}^{-1})$. By Lemma 2.5.8, we have $\overline{y} \notin \overline{A}\overline{x}\overline{B}$ i.e. $\varphi(y) \notin \varphi(AxB)$. □

Corollary 2.5.11. *Let G be a group, and A, B be two subgroups of G. Suppose that G is residually p-finite. If A and B are retracts of G, such that the corresponding retractions commute, then AB is finitely p-separable in G.*

Proof: We apply Lemma 2.5.10 to $x = 1$. □

Lemma 2.5.12. *Let G be a group, and A be a subgroup of G. Suppose that G is residually p-finite and that A is a retract of G. Then if a subset S of A is closed in the pro-p topology on A, it is also closed in the pro-p topology on G.*

Proof: We denote by $\overline{S}$ the closure of S in G – equipped with the pro-p topology. We shall show that $\overline{S} \subset S$. By Corollary 2.5.11, A is closed in G. Therefore $\overline{S} \subset A$. Let $a \in G\backslash S$. We can assume that $a \in A$. There exists a homomorphism ψ from A onto a finite p-group P such that $\psi(a) \notin \psi(S)$. We set: $\varphi = \psi \circ \rho_A$. We have: $\varphi(a) = \psi(a) \notin \psi(S) = \varphi(S)$. Then $a \notin \overline{S}$.□

Lemma 2.5.13. *Let $x \in G$. We set $\alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1})$. Suppose that the pair ($A\cap B$,α) satisfies the p-centralizer condition in G. Then, for every normal subgroup K of p-power index in G, there exists a normal subgroup N of p-power index in G such that $N < K$, $\rho_A(N) \subset N$, $\rho_B(N) \subset N$ and, if $\varphi : G \to G/N$ denotes the canonical projection, $\varphi(A)\cap\varphi(xBx^{-1}) \subset \varphi(A\cap xBx^{-1})\varphi(K)$.*

Proof: Let K be a normal subgroup of p-power index in G. We set $\gamma = \rho_A(\rho_B(x)x^{-1}) \in A$. By Lemma 2.5.9, we have: $A\cap xBx^{-1} = \gamma^{-1}C_{A\cap B}(\alpha)\gamma$. Since the pair ($A\cap B$,$\alpha$) satisfies pCC_G, there exists a normal subgroup L of p-power index in G such that $L < K$ and, if $\psi : G \to G/L$ denotes the canonical projection, $C_{\psi(A\cap B)}(\psi(\alpha)) \subset \psi(C_{A\cap B}(\alpha)K)$. This is equivalent to $\psi^{-1}(C_{\psi(A\cap B)}(\psi(\alpha))) \subset C_{A\cap B}(\alpha)K$. Indeed let $g \in \psi^{-1}(C_{\psi(A\cap B)}(\psi(\alpha)))$.

We have $\psi(g) \in C_{\psi(A\cap B)}(\psi(\alpha)) \subset \psi(C_{A\cap B}(\alpha)K)$. Then $g \in C_{A\cap B}(\alpha)KL \subset C_{A\cap B}(\alpha)K$ (because $L < K$). By Corollary 2.5.7, there exists a normal subgroup N of p-power index in G such that $N < L$, $\rho_A(N) \subset N$, $\rho_B(N) \subset N$ and, if $\varphi : G \to G/N$ denotes the canonical projection, $\varphi(A\cap B) = \varphi(A)\cap \varphi(B)$. We set $\overline{A} = \varphi(A)$, $\overline{B} = \varphi(B)$. By Lemma 2.5.3, ρ_A induces a retraction $\rho_{\overline{A}}$ of G/N onto $\overline{A}$, and ρ_B induces a retraction $\rho_{\overline{B}}$ of G/N onto $\overline{B}$. Obviously $\rho_{\overline{A}}$ and $\rho_{\overline{B}}$ commute. We set $\overline{x} = \varphi(x)$, $\overline{\alpha} = \rho_{\overline{A}}(\rho_{\overline{B}}(\overline{x})\overline{x}^{-1})\overline{x}\rho_{\overline{B}}(\overline{x}^{-1})$ $(\in G/N)$ and $\overline{\gamma} = \rho_{\overline{A}}(\rho_{\overline{B}}(\overline{x})\overline{x}^{-1})$ $(\in \overline{A})$. Observe that $\overline{\alpha} = \varphi(\alpha)$ and $\overline{\gamma} = \varphi(\gamma)$. Then, by Lemma 2.5.9, we have: $\overline{A}\cap\overline{x}\overline{B}\overline{x}^{-1} = \overline{\gamma}^{-1}C_{\overline{A}\cap\overline{B}}(\overline{\alpha})\overline{\gamma}$. Now, $\overline{A} \cap \overline{B} = \varphi(A\cap B)$. Thus:

$$\varphi^{-1}(\overline{A}\cap\overline{x}\overline{B}\overline{x}^{-1}) = \varphi^{-1}(\overline{\gamma}^{-1}C_{\varphi(A\cap B)}(\overline{\alpha})\overline{\gamma}) = \gamma^{-1}\varphi^{-1}(C_{\varphi(A\cap B)}(\overline{\alpha}))\gamma.$$

We claim that:

$$\varphi^{-1}(C_{\varphi(A\cap B)}(\varphi(\alpha))) \subset \psi^{-1}(C_{\psi(A\cap B)}(\psi(\alpha))).$$

Indeed let $g \in \varphi^{-1}(C_{\varphi(A\cap B)}(\varphi(\alpha)))$. We have $\varphi(g) \in \varphi(A\cap B)$ i.e. $g \in (A\cap B)N$, which implies $g \in (A\cap B)L$ i.e. $\psi(g) \in \psi(A\cap B)$; and $\varphi(g)\varphi(\alpha) = \varphi(\alpha)\varphi(g)$ i.e. $g\alpha g^{-1}\alpha^{-1} \in N$, which implies $g\alpha g^{-1}\alpha^{-1} \in L$ i.e. $\psi(g)\psi(\alpha) = \psi(\alpha)\psi(g)$. We deduce that:

$$\varphi^{-1}(C_{\varphi(A\cap B)}(\varphi(\alpha))) \subset C_{A\cap B}(\alpha)K,$$

and hence:

$$\varphi^{-1}(\overline{A}\cap\overline{x}\overline{B}\overline{x}^{-1}) \subset \gamma^{-1}C_{A\cap B}(\alpha)\gamma K = (A\cap xBx^{-1})K.$$

We conclude that:

$$\varphi(A)\cap\varphi(xBx^{-1}) \subset \varphi(A\cap xBx^{-1})\varphi(K).$$

□

Lemma 2.5.14. *Let $x, y \in G$. We set $C = xBx^{-1}$ $(< G)$ and $\alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1})$. If $\alpha^{A\cap B}$ and $y^{A\cap C}$ are finitely p-separable in G, and if the pair $(A\cap B, \alpha)$ satisfies pCC_G, then $C_A(y)C$ is finitely p-separable in G.*

Proof: Let $z \in G$ such that $z \notin C_A(y)C$. Suppose first that $z \notin AC$. Since $\alpha^{A\cap B}$ is finitely p-separable in G, AxB is finitely p-separable in G by Lemma 2.5.10. Therefore $AC = AxBx^{-1}$ is also finitely p-separable in G. Consequently there exists a normal subgroup N of p-power index in G such that $z \notin ACN$. We obviously have $z \notin C_A(y)CN$. Thus we can assume that $z \in AC$. Let $a \in A$, $c \in C$ be such that $z = ac$. Since $z \notin C_A(y)C$, $a^{-1}ya \notin y^{A\cap C}$. Indeed, if there is $g \in A\cap C$ such that $a^{-1}ya = gyg^{-1}$, then $(ag)^{-1}y(ag) = y$ i.e. $ag \in C_A(y)$. We obtain $a \in C_A(y)C$, and then $z \in C_A(y)C$ – which is a contradiction. Now $y^{A\cap C}$ is finitely p-separable in G. Then there exists a normal subgroup K of p-power index in G such that

$a^{-1}ya \notin y^{A\cap C}K$. By Lemma 2.5.13, there exists a normal subgroup N of p-power index in G such that $N < K$ and, if $\varphi : G \to G/N$ denotes the canonical projection, $\varphi(A) \cap \varphi(C) \subset \varphi(A \cap C)\varphi(K)$. For a subset S of G, we set $\overline{S} = \varphi(S)$. For an element g of G, we set $\overline{g} = \varphi(g)$. We have: $\overline{y}^{\overline{A}\cap\overline{C}} \subset \overline{y}^{\overline{A\cap C}.\overline{K}}$. Note that $\overline{K} \lhd G/N$. Then $\overline{y}^{\overline{A}\cap\overline{C}} \subset \overline{y}^{\overline{A\cap C}}\overline{K}$. Observe that $\overline{a}^{-1}\overline{y}\,\overline{a} \notin \overline{y}^{\overline{A\cap C}}\overline{K}$ – otherwise we would have $a^{-1}ya \in y^{A\cap C}KN$, and then $a^{-1}ya \in y^{A\cap C}K$ (because $N < K$). We deduce that $\overline{a}^{-1}\overline{y}\,\overline{a} \notin \overline{y}^{\overline{A}\cap\overline{C}}$. Now it suffices to show that $\varphi(z) \notin \varphi(C_A(y)C)$. Suppose the contrary. Let $a' \in C_A(y)$, $c' \in C$ be such that $\varphi(z) = \varphi(a'c')$. Then $\varphi(ac) = \varphi(a'c')$. Thus $\varphi(a'^{-1}a) = \varphi(c'c^{-1})$. We set $\overline{g} = \varphi(a'^{-1}a) = \varphi(c'c^{-1})$ $(\in \overline{A} \cap \overline{C})$. We have: $\varphi(z) = \varphi(a')\overline{g}\varphi(c)$ and $\overline{a} = \varphi(z)\varphi(c)^{-1} = \varphi(a')\overline{g}$. Then $\overline{a}^{-1}\overline{y}\,\overline{a} = \overline{g}^{-1}\varphi(a'^{-1}ya')\overline{g} = \overline{g}^{-1}\varphi(y)\overline{g} = \overline{g}^{-1}\overline{y}\,\overline{g} \in \overline{y}^{\overline{A}\cap\overline{C}}$ – a contradiction. We have shown that $C_A(y)C$ is finitely p-separable in G. □

2.6 Proof of the main theorem

We turn now to the proof that right-angled Artin groups are hereditarily conjugacy p-separable. We need the following theorem, which is due to Duchamp and Krob (see [DK2]).

Theorem 2.6.1. *Right-angled Artin groups are residually p-finite.*

(Note that the exact statement of [DK2], Theorem 2.3, is that right-angled Artin groups are residually torsion-free nilpotent; Theorem 2.6.1 then follows from [Gru], Theorem 2.1.) This theorem can also be proved using HNN extensions (see [Lo], Theorem 2.11).

Basically, Proposition 2.6.2 establishes the main result. Proposition 2.6.2.1 and Proposition 2.6.2.2 will be proved simultaneously by induction on the rank of G.

Proposition 2.6.2. *Let G be a right-angled Artin group.*

2.6.2.1 Every special subgroup S of G satisfies the p-centralizer condition in G (pCC_G).

2.6.2.2 For all $g \in G$ and for every special subgroup S of G, g^S is finitely p-separable in G.

From now on, we assume that G is a right-angled Artin group of rank r $(r \geq 1)$, and that H $\langle W \rangle$ is a special subgroup of G of rank $r-1$. Thus, G can be written as an HNN extension of H relative to the special subgroup $K = \langle lk(t) \rangle$ of H:

$$G = \langle\ H, t \mid t^{-1}kt = k, \forall\ k \in K\ \rangle.$$

Recall that H is a retract of G. A retraction of G onto H is given by:

$$\rho_H(v) = \begin{cases} v & \text{if } v \in W \\ 1 & \text{if } v \in V \setminus W \end{cases}$$

We also assume that:

- every special subgroup S of H satisfies the p-centralizer condition in H (pCC_H),
- for all $h \in H$ and for every special subgroup S of H, h^S is finitely p-separable in H.

The next results (Lemma 2.6.3 to Lemma 2.6.14) are preliminaries to the proof of Proposition 2.6.2.

In general, if A and B are subgroups of a group G, the image of the intersection of A and B under a homomorphism $\varphi : G \to H$ does not coincide with the intersection of the images of A and B in H. However, the p-centralizer condition and the above results on retractions will allow us to obtain the following lemma, which will be used to apply Minasyan's criterion for conjugacy in HNN extensions (see Lemma 2.6.5).

Lemma 2.6.3. *Let be given A_0, a conjugate of a special subgroup of H, $A_1,\ldots,A_n$, n special subgroups of H, and $\alpha,x_0,\ldots,x_n,y_1,\ldots,y_n$, 2(n+1) elements of H. Then, for every normal subgroup L of p-power index in H, there exists a normal subgroup N of p-power index in H such that $N < L$ and, if $\varphi : H \to H/N$ denotes the canonical projection, then:*

$$\overline{\alpha} C_{\overline{A_0}}(\overline{x_0}) \cap \bigcap_{i=1}^n \overline{x_i}\,\overline{A_i}\,\overline{y_i} \subset \varphi((\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^n x_i A_i y_i)L),$$

where $\overline{A_i} = \varphi(A_i)$ ($i \in \{0,\ldots,n\}$), $\overline{\alpha} = \varphi(\alpha)$, $\overline{x_j} = \varphi(x_j)$ ($j \in \{0,\ldots,n\}$), $\overline{y_k} = \varphi(y_k)$ ($k \in \{1,\ldots,n\}$).

Proof: Let L be a subgroup of p-power index in H. We argue by induction on n. Strictly speaking, the basis of our induction is n = 0 but we will need the case n = 1. By the assumptions, there exist a special subgroup A of H, and an element β of H such that $A_0 = \beta A \beta^{-1}$.

<u>n = 0</u>: We set $x = \beta^{-1} x_0 \beta$. The pair (A,x) satisfies pCC_H by the assumptions. There exists a normal subgroup N of p-power index in H such that $N < L$ and, if $\varphi : H \to H/N$ denotes the canonical projection, then $C_{\varphi(A)}(\varphi(x)) \subset \varphi(C_A(x)L)$. But $C_{A_0}(x_0) = \beta C_A(x) \beta^{-1}$. We deduce that: $\varphi(\alpha) C_{\varphi(A_0)}(\varphi(x_0)) \subset \varphi((\alpha C_{A_0}(x_0))L)$.

<u>n = 1</u>: There are two cases:

Case 1: $\alpha C_{A_0}(x_0) \cap x_1 A_1 y_1 = \emptyset$. This is equivalent to say that: $x_1 \notin \alpha C_{A_0}(x_0) y_1^{-1} A_1$. We set $B = (y_1\beta)^{-1} A_1 y_1 \beta$, so that we have: $x_1 \notin \alpha\beta(C_A(x)B)\beta^{-1}y_1^{-1}$. Now the intersection of conjugates of two special subgroups of H is a conjugate of a special subgroup of H (see [Mi], Lemma 6.5). Then $A \cap A_1$ is a conjugate of a special subgroup C of H. There exists $\gamma \in H$ such that $A \cap A_1 = \gamma C \gamma^{-1}$. Therefore if $h \in H$, $h^{A\cap A_1} = \gamma(\gamma^{-1}h\gamma)^C\gamma^{-1}$. Now $(\gamma^{-1}h\gamma)^C$ is finitely p-separable in H by the assumptions. We deduce that $h^{A\cap A_1}$ is finitely p-separable in H. With the same argument, $x^{A\cap B}$ is finitely p-separable in H. Now the pair $(A \cap A_1,h)$ satisfies pCC_H by the assumptions. We deduce that $C_A(x)B$ is finitely p-separable in H by Lemma 2.5.14. This implies that $\alpha C_{A_0}(x_0)y_1^{-1}A_1$ is finitely p-separable in H. There exists a normal subgroup M of p-power index in H such that $x_1 \notin \alpha C_{A_0}(x_0) y_1^{-1} A_1 M$. Up to replacing M by $M \cap L$, we can assume that $M < L$. Now the pair (A_0,x_0) satisfies pCC_H by the assumptions. There exists a normal subgroup N of p-power index in H such that $N < M$ and, if $\varphi : H \to H/N$ denotes the canonical projection, then $C_{\varphi(A_0)}(\varphi(x_0)) \subset \varphi(C_{A_0}(x_0)M)$, or, equivalently, $\varphi^{-1}(C_{\varphi(A_0)}(\varphi(x_0))) \subset C_{A_0}(x_0)M$. Then $\varphi^{-1}(\overline{\alpha} C_{\overline{A_0}}(\overline{x_0})\overline{y_1}^{-1}\overline{A_1}) \subset \alpha\varphi^{-1}(C_{\overline{A_0}}(\overline{x_0}))y_1^{-1}A_1 \subset \alpha C_{A_0}(x_0)y_1^{-1}A_1M$ (with the same notations as in the statement of the lemma). Therefore: $x_1 \notin \varphi^{-1}(\overline{\alpha} C_{\overline{A_0}}(\overline{x_0})\overline{y_1}^{-1}\overline{A_1})$. Finally: $\overline{\alpha} C_{\overline{A_0}}(\overline{x_0}) \cap \overline{x_1}\,\overline{A_1}\,\overline{y_1} = \emptyset$.

Case 2: $\alpha C_{A_0}(x_0) \cap x_1 A_1 y_1 \neq \emptyset$.

Remark 2.6.4. If G is a group, and H, K are two subgroups of G such that $aH \cap bKc \neq \emptyset$ – where a, b, $c \in G$ –, then for all $g \in aH \cap bKc$, we have $aH \cap bKc = g(H \cap c^{-1}Kc)$.

Choose $g \in \alpha C_{A_0}(x_0) \cap x_1 A_1 y_1$. By Remark 6.4, we have: $\alpha C_{A_0}(x_0) \cap x_1 A_1 y_1 = g(C_{A_0}(x_0) \cap y_1^{-1} A_1 y_1)$. We set $D = A_0 \cap y_1^{-1} A_1 y_1$. Then $\alpha C_{A_0}(x_0) \cap x_1 A_1 y_1 = g C_D(x_0)$. Now, D is a conjugate of a special subgroup E of H by [Mi], Lemma 6.5. There exists $\delta \in H$ such that $D = \delta E \delta^{-1}$. As above, the pair (D,x_0) satisfies pCC_H. There exists a normal subgroup M of p-power index in H such that $M < L$ and, if $\psi : H \to H/M$ denotes the canonical projection, we have: $C_{\psi(D)}(\psi(x_0)) \subset \psi(C_D(x_0)L)$. Now by Lemma 2.5.13, there exists a normal subgroup N of p-power index in H such that $N < M$ and, if $\varphi : H \to H/N$ denotes the canonical projection, then $\varphi(A) \cap \varphi((y_1\beta)^{-1}A_1y_1\beta) \subset \varphi(A \cap (y_1\beta)^{-1}A_1y_1\beta)\varphi(M)$. Therefore:

$$\begin{gathered}\overline{A_0} \cap \overline{y_1}^{-1}\overline{A_1}\,\overline{y_1} = \varphi(\beta A \beta^{-1}) \cap \varphi(y_1^{-1}A_1y_1) = \\ \varphi(\beta)(\varphi(A) \cap \varphi((y_1\beta)^{-1}A_1y_1\beta))\varphi(\beta^{-1}) \subset \\ \varphi(\beta)(\varphi(A \cap (y_1\beta)^{-1}A_1y_1\beta)\varphi(M))\varphi(\beta^{-1}) = \varphi(A_0 \cap y_1^{-1}A_1y_1)\varphi(M) = \\ \varphi(D)\varphi(M)\ (*),\end{gathered}$$

(with the same notations as in the statement of the lemma). We set $\overline{g} = \varphi(g)$. Note that $\overline{g} \in \overline{\alpha} C_{\overline{A_0}}(\overline{x_0}) \cap \overline{x_1}\,\overline{A_1}\,\overline{y_1}$. Therefore $\overline{\alpha} C_{\overline{A_0}}(\overline{x_0}) \cap \overline{x_1}\,\overline{A_1}\,\overline{y_1} = \overline{g}(C_{\overline{A_0}}(\overline{x_0}) \cap \overline{y_1}^{-1}\overline{A_1}\,\overline{y_1})$. Considering $(*)$, we obtain:

$$\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \overline{x_1}\,\overline{A_1}\,\overline{y_1} = \overline{g}C_{\overline{A_0}\cap\overline{y_1}^{-1}\overline{A_1}\,\overline{y_1}}(\overline{x_0}) \subset \overline{g}C_{\varphi(D)\varphi(M)}(\overline{x_0}).$$

Recall that $N < M$. Then $\psi : H \to H/M$ induces a homomorphism $\widetilde{\psi}$: $H/N \to H/M$ such that $\psi = \widetilde{\psi} \circ \varphi$. Note that $\widetilde{\psi}(\varphi(D)\varphi(M)) = \psi(D)$. Let $z \in C_{\varphi(D)\varphi(M)}(\overline{x_0})$. Then:

$$\widetilde{\psi}(z) \in C_{\psi(D)}(\psi(x_0)) \subset \psi(C_D(x_0)L) = \widetilde{\psi}(\varphi(C_D(x_0)L)).$$

Therefore $z \in \varphi(C_D(x_0)L)ker(\widetilde{\psi}) = \varphi(C_D(x_0)L)$ (because $ker(\widetilde{\psi}) = \varphi(M)$ $< \varphi(L)$). We deduce that $C_{\varphi(D)\varphi(M)}(\overline{x_0}) \subset \varphi(C_D(x_0)L)$. We conclude that

$$\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \overline{x_1}\,\overline{A_1}\,\overline{y_1} \subset \overline{g}\varphi(C_D(x_0)L) = \varphi(gC_D(x_0)L) = \\ \varphi((\alpha C_{A_0}(x_0) \cap x_1A_1y_1)L).$$

<u>Inductive step</u>: Suppose that $n \geq 1$ and that the result has been proved for $n-1$. Note that if $\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i = \emptyset$, then by the induction hypothesis, there exists a normal subgroup N of p-power index in H such that, if $\varphi : H \to H/N$ denotes the canonical projection, then

$$\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \bigcap_{i=1}^{n-1} \overline{x_i}\,\overline{A_i}\,\overline{y_i} \subset \varphi((\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i)L) = \emptyset.$$

Obviously:

$$\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \bigcap_{i=1}^{n} \overline{x_i}\,\overline{A_i}\,\overline{y_i} = \emptyset \subset \varphi((\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n} x_iA_iy_i)L)$$

Thus we can assume that $\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i \neq \emptyset$. Therefore $\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i = g(C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} y_i^{-1}A_iy_i)$ for some $g \in H$. We set $F = A_0 \cap \bigcap_{i=1}^{n-1} y_i^{-1}A_iy_i$. Again, F is a conjugate of a special subgroup of H by [Mi], Lemma 6.5. We have: $\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i = gC_F(x_0)$. Now, by the case n = 1, there exists a normal subgroup M of p-power index in H such that $M < L$ and, if $\psi : H \to H/M$ denotes the canonical projection, then:

$$\psi(g)C_{\psi(F)}(\psi(x_0)) \cap \psi(x_nA_ny_n) \subset \psi((gC_F(x_0) \cap x_nA_ny_n)L),$$

or, equivalently:

$$\psi^{-1}(\psi(g)C_{\psi(F)}(\psi(x_0)) \cap \psi(x_nA_ny_n)) \subset (gC_F(x_0) \cap x_nA_ny_n)L.$$

On the other hand, by the induction hypothesis, there exists a normal subgroup N of p-power index in H such that $N < M$ and, if $\varphi : H \to H/N$ denotes the canonical projection, then:

$$\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \bigcap_{i=1}^{n-1} \overline{x_i}\,\overline{A_i}\,\overline{y_i} \subset \varphi((\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i)M)$$

or, equivalently:

$$\varphi^{-1}(\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \bigcap_{i=1}^{n-1} \overline{x_i}\,\overline{A_i}\,\overline{y_i}) \subset (\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i)M$$

Thus we have:

$$\varphi^{-1}(\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \textstyle\bigcap_{i=1}^{n} \overline{x_i}\overline{A_i}\overline{y_i}) =$$
$$\varphi^{-1}(\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \textstyle\bigcap_{i=1}^{n-1} \overline{x_i}\overline{A_i}\overline{y_i}) \cap \varphi^{-1}(\overline{x_n}\overline{A_n}\overline{y_n}) \subset$$
$$(\alpha C_{A_0}(x_0) \cap \textstyle\bigcap_{i=1}^{n-1} x_iA_iy_i)M \cap x_nA_ny_nN = gC_F(x_0)M \cap x_nA_ny_nN.$$

Recall that $N < M$. Finally we have:

$$\varphi^{-1}(\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \textstyle\bigcap_{i=1}^{n} \overline{x_i}\overline{A_i}\overline{y_i}) \subset gC_F(x_0)M \cap x_nA_ny_nM \subset$$
$$\psi^{-1}(\psi(g)C_{\psi(F)}(\psi(x_0))) \cap \psi^{-1}(\psi(x_nA_ny_n)) =$$
$$\psi^{-1}(\psi(g)C_{\psi(F)}(\psi(x_0)) \cap \psi(x_nA_ny_n)) \subset (gC_F(x_0) \cap x_nA_ny_n)L =$$
$$(\alpha C_{A_0}(x_0) \cap \textstyle\bigcap_{i=1}^{n} x_iA_iy_i)L.$$

□

We need the following criterion for conjugacy in HNN extensions:

Lemma 2.6.5. *Let $G = \langle\, H, t \mid t^{-1}kt = k, \forall\, k \in K \,\rangle$ be an HNN extension. Let S be a subgroup of H. Let $g = x_0t^{a_1}x_1 \cdots t^{a_n}x_n$ ($n \geq 1$) and $h = y_0t^{b_1}y_1 \cdots t^{b_m}y_m$ be elements of G in reduced form. Then $h \in g^S$ if and only if all of the following conditions hold:*

1. *$m = n$ and $a_i = b_i$, for all $i \in \{1,\ldots,n\}$,*
2. *$y_0 \cdots y_n \in (x_0 \cdots x_n)^S$,*
3. *if $\alpha \in S$ satisfies $y_0 \cdots y_n = \alpha x_0 \cdots x_n\alpha^{-1}$, then:*

$$\alpha C_S(x_0 \cdots x_n) \cap y_0Kx_0^{-1} \cap (y_0y_1)K(x_0x_1)^{-1} \cap \cdots \cap$$
$$(y_0 \cdots y_{n-1})K(x_0 \cdots x_{n-1})^{-1} \neq \emptyset.$$

Proof: Proved in [Mi] (see Lemma 7.11). □

The following is the analogue of Lemma 6.8 in [Mi]:

Lemma 2.6.6. *Let S be a special subgroup of H. Let $g \in G \setminus H$. Let $h \in G \setminus g^S$. There exists a normal subgroup L of p-power index in H such that, if $\varphi : H \to P = H/L$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\varphi(K)$ and if $\overline{\varphi} : G \to Q$ denotes the homomorphism induced by φ, we have $\overline{\varphi}(h) \notin \overline{\varphi}(g)^{\overline{\varphi}(S)}$.*

Proof: Write $g = x_0t^{a_1}x_1 \cdots t^{a_n}x_n$ and $h = y_0t^{b_1}y_1 \cdots t^{b_m}y_m$ in reduced forms. We have $n \geq 1$ – as $g \notin H$.

Step 1: We assume that the first condition in Minasyan's criterion (see Lemma 2.6.5) is not satisfied by g and h.

It follows from Lemma 2.10 in [Lo], and Theorem 2.6.1 (see, alternatively, [Lo], Theorem 2.11) that the special subgroup K is closed in the pro-p topology on H. (Note that this can also be obtained by combining Corollary 2.5.11 and Theorem 2.6.1.) Thus, there exists a normal subgroup L of p-power index in H such that:

$$\forall\ i \in \{1,...,n-1\},\ x_i \notin KL\ (*),$$
$$\forall\ j \in \{1,...,m-1\},\ y_j \notin KL\ (**).$$

We denote by $\varphi : H \to P = H/L$ the canonical projection. If Q denotes the HNN extension of P relative to $\varphi(K)$:

$$Q = \langle\ P,\, \bar{t} \mid \bar{t}^{-1}\varphi(k)\bar{t} = \varphi(k),\, \forall\ k \in K\ \rangle,$$

and if $\overline{\varphi} : G \to Q$ denotes the homomorphism induced by φ – with $\overline{\varphi}_{|_H} = \varphi$ and $\overline{\varphi}(t) = \bar{t}$ –, then $\overline{\varphi}(g) = \overline{x_0}\bar{t}^{a_1}\overline{x_1}\cdots\bar{t}^{a_n}\overline{x_n}$ and $\overline{\varphi}(h) = \overline{y_0}\bar{t}^{b_1}\overline{y_1}\cdots\bar{t}^{b_m}\overline{y_m}$ are reduced forms in Q by $(*)$ and $(**)$ – where $\overline{x_i} = \overline{\varphi}(x_i)$ ($i \in \{0,...,n\}$) and $\overline{y_j} = \overline{\varphi}(y_j)$ ($j \in \{0,...,m\}$). But then the first condition in Minasyan's criterion will not hold for $\overline{\varphi}(g)$ and $\overline{\varphi}(h)$.

Conclusion of Step 1: We can assume that $m = n$ and $a_i = b_i$ for all $i \in \{1,...,n\}$.

Step 2: We assume that the second condition in Minasyan's criterion is not satisfied by g and h. We set $x = x_0 \cdots x_n$ and $y = y_0 \cdots y_n$. Thus $y \notin x^S$.

By the assumptions, x^S is finitely p-separable in H. Therefore there exists a homomorphism φ from H onto a finite p-group P such that $\varphi(y) \notin \varphi(x)^{\varphi(S)}$. Denote by Q the HNN extension of P relative to $\varphi(K)$, and by $\overline{\varphi} : G \to Q$ the homomorphism induced by φ. Now let $f : Q \to P$ be the natural homomorphism. We have:

$$f(\overline{\varphi}(g)) = f(\overline{x_0}\bar{t}^{a_1}\overline{x_1}\cdots\bar{t}^{a_n}\overline{x_n}) = \overline{x_0}\cdots\overline{x_n} = \varphi(x),$$
$$f(\overline{\varphi}(h)) = f(\overline{y_0}\bar{t}^{a_1}\overline{y_1}\cdots\bar{t}^{a_n}\overline{y_n}) = \overline{y_0}\cdots\overline{y_n} = \varphi(y)$$

(with the same notations as above). Since $\varphi(y) \notin \varphi(x)^{\varphi(S)}$, we see that $\overline{\varphi}(h) \notin \overline{\varphi}(g)^{\overline{\varphi}(S)}$.

Conclusion of Step 2: We can assume that $y \in x^S$. There exists $\alpha \in S$ such that $y = \alpha x \alpha^{-1}$.

End of the proof: Considering Minasyan's criterion, since $h \notin g^S$, we must have:

$$\alpha C_S(x_0 \cdots x_n) \cap y_0 K x_0^{-1} \cap (y_0 y_1) K (x_0 x_1)^{-1} \cap \cdots \cap$$
$$(y_0 \cdots y_{n-1}) K (x_0 \cdots x_{n-1})^{-1} = \emptyset.$$

As we noted above, K is closed in the pro-p topology on H; thus, there exists a normal subgroup L of p-power index in H such that:

$$\forall\, i \in \{1,...,n-1\},\ x_i \notin KL\ (*),$$
$$\forall\, j \in \{1,...,n-1\},\ y_j \notin KL\ (**).$$

Now by Lemma 2.6.3, there exists a normal subgroup N of p-power index in H such that $N < L$ and, if $\varphi : H \to P = H/N$ denotes the canonical projection, then:

$$\overline{\alpha} C_{\overline{S}}(\overline{x}) \cap \overline{y_0}\overline{K}\overline{x_0}^{-1} \cap \overline{y_0}\,\overline{y_1}\overline{K}(\overline{x_0}\,\overline{x_1})^{-1} \cap \cdots \cap \overline{y_0}\cdots\overline{y_{n-1}}\overline{K}(\overline{x_0}\cdots\overline{x_{n-1}})^{-1} \subset \varphi((\alpha C_S(x) \cap y_0 K x_0^{-1} \cap y_0 y_1 K(x_0 x_1)^{-1} \cap \cdots \cap y_0 \cdots y_{n-1} K(x_0 \cdots x_{n-1})^{-1})L) = \emptyset\ (***).$$

where $\overline{S} = \varphi(S)$, $\overline{\alpha} = \varphi(\alpha)$, $\overline{x} = \varphi(x)$, $\overline{x_i} = \varphi(x_i)$ $(i \in \{0,...,n\})$, $\overline{y_j} = \varphi(y_j)$ $(j \in \{0,...,n\})$. Let Q be the HNN extension of P relative to $\varphi(K)$ and let $\overline{\varphi}$: $G \to Q$ be the homomorphism induced by φ. Then, by $(*)$ and $(**)$, $\overline{\varphi}(g)$ $= \overline{x_0}\overline{t}^{a_1}\overline{x_1}\cdots\overline{t}^{a_n}\overline{x_n}$ and $\overline{\varphi}(h) = \overline{y_0}\overline{t}^{a_1}\overline{y_1}\cdots\overline{t}^{a_n}\overline{y_n}$ are reduced forms in Q. So, in view of $(***)$, we have $\overline{\varphi}(h) \notin \overline{\varphi}(g)^{\overline{\varphi}(S)}$. □

The following is the analogue of Lemma 8.8 in [Mi]:

Lemma 2.6.7. *Let $g_0 = t^{a_1}x_1\cdots t^{a_n}x_n$ $(n \geq 1)$ and $h_0 = t^{b_1}y_1\cdots t^{b_m}y_m$ be cyclically reduced elements of G. Let h_1,...,h_k be elements of G. If $h_i \notin g_0^K$ for all $i \in \{1,...,k\}$, then there exists a normal subgroup L of p-power index in H such that, if $\varphi : H \to P = H/L$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\varphi(K)$, and if $\overline{\varphi} : G \to Q$ denotes the homomorphism induced by φ, we have:*

1. *$\overline{\varphi}(g_0) = \overline{t}^{a_1}\overline{x_1}\cdots\overline{t}^{a_n}\overline{x_n}$ and $\overline{\varphi}(h_0) = \overline{t}^{b_1}\overline{y_1}\cdots\overline{t}^{b_m}\overline{y_m}$ are cyclically reduced in Q – where $\overline{x_i} = \overline{\varphi}(x_i)$ $(i \in \{1,...,n\})$ and $\overline{y_j} = \overline{\varphi}(y_j)$ $(j \in \{1,...,n\})$,*
2. *$\overline{\varphi}(h_i) \notin \overline{\varphi}(g_0)^{\overline{\varphi}(K)}$ for all $i \in \{1,...,k\}$.*

Proof: As we noted above, K is closed in the pro-p topology on H; thus, there exists a normal subgroup L_0 of p-power index in H such that:

$$\forall\, i \in \{1,...,n-1\},\ x_i \notin KL_0\ (*),$$
$$\forall\, j \in \{1,...,m-1\},\ y_j \notin KL_0\ (**).$$

Let $i \in \{1,...,k\}$. Since $h_i \notin g_0^K$, there exists a normal subgroup L_i of p-power index in H such that, if $\varphi_i : H \to P_i = H/L_i$ denotes the canonical projection, if Q_i denotes the HNN extension of P_i relative to $\varphi_i(K)$ and if $\overline{\varphi_i} : G \to Q_i$ denotes the homomorphism induced by φ_i, we have $\overline{\varphi_i}(h_i) \notin \overline{\varphi_i}(g_0)^{\overline{\varphi_i}(K)}$ – by Lemma 2.6.6. Set $L = L_0 \cap L_1 \cdots \cap L_k$. Let $\varphi : H \to P = H/L$ be the canonical projection, let Q be the HNN extension of P relative to $\varphi(K)$, and let $\overline{\varphi} : G \to Q$ be the homomorphism induced by φ. Since $L < L_0$, $\overline{\varphi}(g_0) = \overline{t}^{a_1}\overline{x_1}\cdots\overline{t}^{a_n}\overline{x_n}$ and $\overline{\varphi}(h_0) = \overline{t}^{b_1}\overline{y_1}\cdots\overline{t}^{b_m}\overline{y_m}$ are cyclically reduced in Q by $(*)$ and $(**)$ (with the same notations as in the statement of the lemma). As $L < L_i$ for all $i \in \{1,...,k\}$, we have $\overline{\varphi}(h_i) \notin \overline{\varphi}(g_0)^{\overline{\varphi}(K)}$ for all $i \in \{1,...,k\}$. □

Lemma 2.6.8. *Let $G = \langle\, H, t \mid t^{-1}kt = k, \forall\, k \in K \,\rangle$ be an HNN extension. Let S be a subgroup of H. Let $g = x_0t^{a_1}x_1\cdots t^{a_n}x_n$ be an element of G in reduced form ($n \geq 1$). Then:*

$$C_S(g) = C_S(x_0\cdots x_n) \cap x_0Kx_0^{-1} \cap (x_0x_1)K(x_0x_1)^{-1} \cap \cdots \cap (x_0\cdots x_{n-1})K(x_0\cdots x_{n-1})^{-1}.$$

Proof: Proved in [Mi] (see Lemma 7.12). □

The following is the analogue of Lemma 8.9 in [Mi]:

Lemma 2.6.9. *Let S be a special subgroup of H. Let L be a normal subgroup of p-power index in G, and let $g = x_0t^{a_1}x_1\cdots t^{a_n}x_n$ be an element of G in reduced form and not contained in H. Then there exists a normal subgroup N of p-power index of H such that, if $\varphi : H \to P = H/N$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\varphi(K)$, and if $\overline{\varphi} : G \to Q$ denotes the homomorphism induced by φ, we have:*

1. *$C_{\overline{\varphi}(S)}(\overline{\varphi}(g)) \subset \overline{\varphi}(C_S(g)L)$,*
2. *$ker(\varphi) = N < H \cap L$,*
3. *$ker(\overline{\varphi}) < L$.*

Proof: We have $n \geq 1$ – as $g \notin H$.

As we noted above, K is closed in the pro-p topology on H. Therefore there exists a normal subgroup M of p-power index in H such that:

$$\forall\, i \in \{1,...,n-1\},\ x_i \notin KM\ (*).$$

We set $L' = H \cap L$. Note that L' is a normal subgroup of p-power index in H. Thus, up to replacing M by $M \cap L'$, we can assume that $M < L'$. We set $x = x_0\cdots x_n$. We have:

$$C_S(g) = C_S(x) \cap x_0Kx_0^{-1} \cap (x_0x_1)K(x_0x_1)^{-1} \cap \cdots \cap (x_0\cdots x_{n-1})K(x_0\cdots x_{n-1})^{-1},$$

by Lemma 2.6.8. We denote by I the intersection in the right-hand side. By Lemma 2.6.3, there exists a normal subgroup N of p-power index in H such that $N < M$ and, if $\varphi : H \to P = H/N$ denotes the canonical projection, we have:

$$C_{\overline{S}}(\overline{x}) \cap \overline{x_0}\overline{K}\overline{x_0}^{-1} \cap \overline{x_0}\,\overline{x_1}\overline{K}(\overline{x_0}\,\overline{x_1})^{-1} \cap \cdots \cap \overline{x_0}\cdots\overline{x_{n-1}}\overline{K}(\overline{x_0}\cdots\overline{x_{n-1}})^{-1} \subset \varphi(IM)$$

where $\overline{S} = \varphi(S)$, $\overline{x} = \varphi(x)$, $\overline{x_i} = \varphi(x_i)$ ($i \in \{0,...,n-1\}$). We denote by J the intersection in the left-hand side. Let Q be the HNN extension of P relative to $\varphi(K)$, and let $\overline{\varphi} : G \to Q$ be the homomorphism induced by φ. Then $\overline{x_0}\overline{t}^{a_1}\overline{x_1}\cdots\overline{t}^{a_n}\overline{x_n}$ is a reduced form of $\overline{\varphi}(g)$ in Q by ($*$). But then $C_{\overline{\varphi}(S)}(\overline{\varphi}(g)) = J$ – by Lemma 2.6.8. Now $\varphi(M) < \varphi(L') = \overline{\varphi}(L') < \overline{\varphi}(L)$. Therefore:

$$C_{\overline{\varphi}(S)}(\overline{\varphi}(g)) = J \subset \varphi(IM) = \varphi(I)\varphi(M) \subset \overline{\varphi}(I)\overline{\varphi}(L) = \overline{\varphi}(C_S(g))\overline{\varphi}(L) = \overline{\varphi}(C_S(g)L).$$

Finally we remark that $ker(\varphi) = N < M < L' = H \cap L < L$. Since $ker(\overline{\varphi})$ is the normal closure of $ker(\varphi)$ in G, we conclude that $ker(\overline{\varphi}) < L$ (because L is normal in G). $\square$

A *prefix* of $t^{a_1}x_1 \cdots t^{a_n}x_n$ is an element of G of the form $t^{a_1}x_1 \cdots t^{a_k}x_k$ for some $k \in \{0,...,n\}$. We need the following result:

Proposition 2.6.10. *Let $G = \langle\ H,\ t \mid t^{-1}kt = k,\ \forall\ k \in K\ \rangle$ be an HNN extension. Let $g = t^{a_1}x_1 \cdots t^{a_n}x_n$ be a cyclically reduced element of G ($n \geq 1$). Let $\{p_1,...,p_{n+1}\}$ be the set of all prefixes of g – we are not assuming that $p_1,...,p_{n+1}$ are ordered. There are two cases:*

1. *if $x_n \in K$, then $n = 1$ and $C_G(g) = \langle t\rangle C_K(g)$.*
2. *if $x_n \in H \setminus K$, let $\{p_1,...,p_m\}$ be the set of prefixes of g satisfying $p_i^{-1}gp_i \in g^K$ ($m \in \{0,...,n+1\}$). For each $i \in \{1,...,m\}$, we choose $\alpha_i \in K$ such that $p_i^{-1}gp_i = \alpha_i^{-1}g\alpha_i$. We set $S = \{\alpha_i p_i^{-1} \mid i \in \{1,...,m\}\}$. Then $C_G(g) = C_K(g)\langle g\rangle\Omega$.*

Proof: Proved in [Mi] (see Proposition 7.8). $\square$

The following is the analogue of Lemma 8.10 in [Mi]:

Lemma 2.6.11. *Let L be a normal subgroup of p-power index in G. Let $g_0 = t^{a_1}x_1 \cdots t^{a_n}x_n$ ($n \geq 1$) be a cyclically reduced element of G. There exists a normal subgroup N of p-power index in H such that, if $\varphi : H \to P = H/N$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\varphi(K)$, and if $\overline{\varphi} : G \to Q$ denotes the homomorphism induced by φ, we have:*

1. *$C_Q(\overline{\varphi}(g_0)) \subset \overline{\varphi}(C_G(g_0)L)$,*
2. *$ker(\varphi) = N < H \cap L$,*
3. *$ker(\overline{\varphi}) < L$.*

Proof: Let $\{p_1,...,p_{n+1}\}$ be the set of all prefixes of g_0. Renumbering $p_1,...,p_{n+1}$, if necessary, we can assume that there exists $m \in \{1,...,n+1\}$ such that $p_i^{-1}g_0p_i \in g_0^K$ for all $i \in \{1,...,m\}$, and $p_i^{-1}g_0p_i \notin g_0^K$ for all $i \in \{m+1,...,n+1\}$. For each $i \in \{1,...,m\}$, we choose $\alpha_i \in K$ such that $p_i^{-1}g_0p_i = \alpha_i^{-1}g_0\alpha_i$. We set $S = \{\alpha_i p_i^{-1} \mid i \in \{1,...,m\}\}$. We set $h_i = p_i^{-1}g_0p_i$ for all $i \in \{m+1,...,n+1\}$. By Lemma 2.6.7, there exists a normal subgroup N_1 of p-power index in H such that, if $\varphi_1 : H \to P_1 = H/N_1$ denotes the canonical projection, if Q_1 denotes the HNN extension of P_1 relative to $\varphi_1(K)$, and if $\overline{\varphi_1} : G \to Q_1$ denotes the homomorphism induced by φ_1, then $\varphi_1(g_0)$ is cyclically reduced in Q_1, and $\overline{\varphi_1}(h_i) \notin \overline{\varphi_1}(g_0)^{\overline{\varphi_1}(K)}$ for all $i \in \{m+1,...,n+1\}$. On the other hand, by Lemma 2.6.9, there exists a normal subgroup N_2 of p-power index in H such that, if $\varphi_2 : H \to P_2 = H/N_2$ denotes the canonical projection, if Q_2 denotes the HNN extension of P_2 relative to $\varphi_2(K)$, and if $\overline{\varphi_2} : G \to Q_2$ denotes the homomorphism induced by φ_2, we have: $C_{\overline{\varphi_2}(K)}(\overline{\varphi_2}(g_0)) \subset \overline{\varphi_2}(C_K(g_0)L)$, $ker(\varphi_2) < H \cap L$, and $ker(\overline{\varphi_2}) < L$. Set $N = N_1 \cap N_2$. Let $\varphi : H \to P = H/N$ be the canonical projection, let Q be the HNN extension of P relative to $\varphi(K)$, and let $\overline{\varphi} : G \to Q$ be the homomorphism induced by φ. Since $N < N_1$, $\overline{\varphi}(g_0)$ is cyclically reduced in Q and $\overline{\varphi}(h_i) \notin \overline{\varphi}(g_0)^{\overline{\varphi}(K)}$ for all $i \in \{m+1,...,n+1\}$. On the other hand, since $N < N_2$, we have:

$$\overline{\varphi}^{-1}(C_{\overline{\varphi}(K)}(\overline{\varphi}(g_0))) \subset \overline{\varphi_2}^{-1}(C_{\overline{\varphi_2}(K)}(\overline{\varphi_2}(g_0))) \subset C_K(g_0)L \ (*).$$

There are two cases:

Case 1: $x_n \in K$. Then $n = 1$, $C_G(g_0) = \langle t\rangle C_K(g_0)$, and $C_Q(\overline{\varphi}(g_0)) = \langle \overline{t}\rangle C_{\varphi(K)}(\overline{\varphi}(g_0))$ – by Proposition 2.6.10. Now $(*)$ implies:

$$C_Q(\overline{\varphi}(g_0)) \subset \langle\overline{\varphi}(t)\rangle\overline{\varphi}(C_K(g_0)L) = \overline{\varphi}(\langle t\rangle C_K(g_0)L) = \overline{\varphi}(C_G(g_0)L).$$

Case 2: $x_n \in H \setminus K$. If $i \in \{1,...,m\}$, $\overline{\varphi}(p_i)^{-1}\overline{\varphi}(g_0)\overline{\varphi}(p_i) = \overline{\varphi}(p_i^{-1}g_0p_i) \in \overline{\varphi}(g_0)^{\varphi(K)}$ – because $p_i^{-1}g_0p_i \in g_0^K$ –, whereas if $i \in \{m+1,...,n+1\}$, $\overline{\varphi}(p_i)^{-1}$ $\overline{\varphi}(g_0)\overline{\varphi}(p_i) = \overline{\varphi}(h_i) \notin \overline{\varphi}(g_0)^{\varphi(K)}$. Therefore $\{\overline{\varphi}(p_1),...,\overline{\varphi}(p_m)\}$ is the set of all prefixes of $\overline{\varphi}(g_0)$ satisfying $\overline{\varphi}(p_i)^{-1}\overline{\varphi}(g_0)\overline{\varphi}(p_i) \in \overline{\varphi}(g_0)^{\varphi(K)}$. By Proposition 2.6.10, $C_G(g_0) = C_K(g_0)\langle g_0\rangle\Omega$, and $C_Q(\overline{\varphi}(g_0)) = C_{\varphi(K)}(\overline{\varphi}(g_0))\langle\overline{\varphi}(g_0)\rangle\overline{\Omega}$, where $\overline{\Omega} = \overline{\varphi}(\Omega) = \{\overline{\varphi}(\alpha_i)\overline{\varphi}(p_i)^{-1} \mid i \in \{1,...,m\}\}$. We deduce that:

$$C_Q(\overline{\varphi}(g_0)) \subset \overline{\varphi}(C_K(g_0)L)\langle\overline{\varphi}(g_0)\rangle\overline{\varphi}(\Omega) = \overline{\varphi}(C_K(g_0)L\langle g_0\rangle\Omega) = \overline{\varphi}(C_G(g_0)L).$$

□

Proposition 2.6.12. *Let G be a right-angled Artin group of rank r ($r \geq 1$). Let $g \in G$. If $g \neq 1$, then there exists a special subgroup H of rank $r-1$ of G such that $g \notin H^G$, where $H^G = \cup_{h\in H} h^G$.*

Proof: Proved in [Mi] (see Lemma 6.8). □

Lemma 2.6.13. *Every special subgroup S of G satisfies the p-centralizer condition in G (pCC_G).*

Proof: Let $g \in G$. Let L be a normal subgroup of p-power index in G. There are two cases:

<u>Case 1</u>: $S \neq G$.

Let H be a special subgroup of rank $r-1$ of G such that $S < H$. Then G can be written as an HNN extension of H, relative to a special subgroup K of H:

$$G = \langle\ H, t \mid t^{-1}kt = k, \forall\ k \in K\ \rangle.$$

We set $L' = H \cap L$. We note that L' is a normal subgroup of p-power index in H. There are two cases:

<u>Subcase 1</u>: $g \in H$. By the assumptions, the pair (S,g) satisfies the p-centralizer condition in H (pCC_H). There exists a normal subgroup M of p-power index in H such that $M < L'$ and, if $\psi : H \to P = H/M$ denotes the canonical projection, we have:

$$C_{\psi(S)}(\psi(g)) \subset \psi(C_S(g)L')\ (*).$$

We denote by $f : G \to H$ the natural homomorphism. We note that $f^{-1}(M)$ is a normal subgroup of p-power index in G (because $f^{-1}(M)$ is the kernel of the homomorphism $\psi \circ f$). Therefore, $N = L \cap f^{-1}(M)$ is a normal subgroup of p-power index in G. Moreover $N < L$ and $f(N) < M$. We denote by $\varphi : G \to Q = G/N$ the canonical projection. We observe that $ker(\psi) = M$, $ker(\varphi) = N$, $M < f^{-1}(M) \cap L \cap H = N \cap H$ and $N \cap H \subset f(N) < M$. Therefore $M = N \cap H$. Thus we can assume that $P < Q$ and $\varphi_{|_H} = \psi$. But then $\psi(L') = \varphi(L') \subset \varphi(L)$. Recall that $g \in H$ and $S < H$. Thus considering $(*)$, we obtain:

$$C_{\varphi(S)}(\varphi(g)) = C_{\psi(S)}(\psi(g)) \subset \psi(C_S(g))\psi(L') \subset \varphi(C_S(g))\varphi(L) = \varphi(C_S(g)L).$$

<u>Subcase 2</u>: $g \in G \setminus H$. Write $g = x_0 t^{a_1} x_1 \cdots t^{a_n} x_n$ in a reduced form ($n \geq 1$). Then, by Lemma 2.6.9, there exists a normal subgroup M of p-power index in H such that, if $\psi : H \to P = H/M$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\psi(K)$, and if $\overline{\psi}$: $G \to Q$ denotes the homomorphism induced by ψ, then: $C_{\overline{\psi}(S)}(\overline{\psi}(g)) \subset \overline{\psi}(C_S(g)L)$, $ker(\psi) < H \cap L$, and $ker(\overline{\psi}) < L$. We note that $\overline{\psi}(S) \cap \overline{\psi}(L)$ $= \psi(S) \cap \overline{\psi}(L) < P$ is finite. Since Q is residually p-finite, $\overline{\psi}(g)^{\overline{\psi}(S) \cap \overline{\psi}(L)}$ is finitely p-separable in Q. Therefore, by Lemma 2.3.5, there exists a normal subgroup N of p-power index in Q such that $N < \overline{\psi}(L)$ and, if $\chi : Q \to R$ $= Q/N$ denotes the canonical projection, then:

$$C_{\chi(\overline{\psi}(S))}(\chi(\overline{\psi}(g))) \subset \chi(C_{\overline{\psi}(S)}(\overline{\psi}(g))\overline{\psi}(L)).$$

We set $\varphi = \chi \circ \overline{\psi} : G \to R$. We have: $ker(\varphi) = \overline{\psi}^{-1}(ker(\chi)) = \overline{\psi}^{-1}(N) \subset \overline{\psi}^{-1}(\overline{\psi}(L)) = Lker(\overline{\psi})$. Now $ker(\overline{\psi}) < L$. Then $ker(\varphi) < L$. And:

$$C_{\varphi(S)}(\varphi(g)) = C_{\chi(\overline{\psi}(S))}(\chi(\overline{\psi}(g))) \subset \chi(C_{\overline{\psi}(S)}(\overline{\psi}(g))\overline{\psi}(L)) \subset \\ \chi(\overline{\psi}(C_S(g)L)\overline{\psi}(L)) = \varphi(C_S(g)L).$$

Case 2: $S = G$.

If $g = 1$, then the result is trivial. Thus we can assume that $g \neq 1$. Then, by Proposition 2.6.12, there exists a special subgroup H of rank $r-1$ of G such that $g \notin H^G$. As above, G can be written as an HNN extension of H relative to a special subgroup K of H:

$$G = \langle\ H, t \mid t^{-1}kt = k, \forall\ k \in K\ \rangle.$$

Let $g_0 = t^{a_1}x_1 \cdots t^{a_n}x_n$ be a cyclically reduced element in G conjugate to g. Choose $\alpha \in G$ such that $g = \alpha g_0 \alpha^{-1}$. Note that $g \notin H^G$ implies that $n \geq 1$. By Lemma 2.6.11, there exists a normal subgroup M of p-power index in H such that, if $\psi : H \to P = H/M$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\psi(K)$, and if $\overline{\psi} : G \to Q$ denotes the homomorphism induced by ψ, then: $C_Q(\overline{\psi}(g_0)) \subset \overline{\psi}(C_G(g_0)L)$, $ker(\psi) < H \cap L$, and $ker(\overline{\psi}) < L$. Now Q is hereditarily conjugacy p-separable by Corollary 2.4.3. Then Q satisfies the p-centralizer condition by Proposition 2.3.6. There exists a normal subgroup N of p-power index in Q such that $N < \overline{\psi}(L)$ and if $\chi : Q \to R = Q/N$ denotes the canonical projection, we have:

$$C_R(\chi(\overline{\psi}(g_0))) \subset \chi(C_Q(\overline{\psi}(g_0))\overline{\psi}(L)).$$

We set $\varphi = \chi \circ \overline{\psi} : G \to R$. As above, we have $ker(\varphi) = \overline{\psi}^{-1}(ker(\chi)) = \overline{\psi}^{-1}(N) \subset \overline{\psi}^{-1}(\overline{\psi}(L)) = Lker(\overline{\psi})$. Now $ker(\overline{\psi}) < L$. Then $ker(\varphi) < L$. And:

$$C_R(\varphi(g_0)) = C_{\varphi(G)}(\varphi(g_0)) = C_{\chi(\overline{\psi}(G))}(\chi(\overline{\psi}(g_0))) \subset \chi(C_{\overline{\psi}(G)}(\overline{\psi}(g_0))\overline{\psi}(L)) \subset \\ \chi(\overline{\psi}(C_G(g_0)L)\overline{\psi}(L)) = \varphi(C_G(g_0)L).$$

Finally:

$$\varphi(\alpha)C_R(\varphi(g_0))\varphi(\alpha)^{-1} \subset \varphi(\alpha)\varphi(C_G(g_0)L)\varphi(\alpha)^{-1}.$$

That is,

$$C_R(\varphi(g)) \subset \varphi(C_G(g)L).$$

$\square$

Lemma 2.6.14. *For every $g \in G$ and for every special subgroup S of G, g^S is finitely p-separable in G.*

Proof: There are two cases:

Case 1: $S \neq G$.

Let H be a special subgroup of rank $r-1$ of G such that $S < H$. As above, G can be written as an HNN extension of H relative to a special subgroup K of H:

$$G = \langle\ H,\, t \mid t^{-1}kt = k,\, \forall\ k \in K\ \rangle.$$

Let $g \in G$. There are two cases:

Subcase 1: $g \in H$. Then g^S is finitely p-separable in H by the assumptions. Since G is residually p-finite by Theorem 2.6.1, g^S is finitely p-separable in G by Lemma 2.5.12.

Subcase 2: $g \in G \setminus H$. Let $h \in G \setminus g^S$. By Lemma 2.6.6, there exists a normal subgroup L of p-power index in H such that, if $\psi : H \to P = H/L$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\psi(K)$ and if $\overline{\psi} : G \to Q$ denotes the homomorphism induced by ψ, then: $\overline{\psi}(h) \notin \overline{\psi}(g)^{\overline{\psi}(S)}$. Now $\overline{\psi}(S) = \psi(S) < P$ is finite and Q is residually p-finite. Then there exists a homomorphism $\chi : Q \to R$ from Q onto a finite p-group R such that $\chi(\overline{\psi}(h)) \notin \chi(\overline{\psi}(g)^{\overline{\psi}(S)})$. Thus the homomorphism $\varphi = \chi \circ \overline{\psi} : G \to R$ satisfies the condition $\varphi(h) \notin \varphi(g^S)$, as required.

Case 2: $S = G$.

Let $g \in G$. If $g = 1$, then, since G is residually p-finite by Theorem 2.6.1, $g^G = \{1\}$ is finitely p-separable in G. Thus we can assume that $g \neq 1$. Then, by Proposition 2.6.12, there exists a special subgroup H of rank $r-1$ of G such that $g \notin H^G$. As above, G can be written as an HNN extension of H relative to a special subgroup K of H:

$$G = \langle\ H,\, t \mid t^{-1}kt = k,\, \forall\ k \in K\ \rangle.$$

Let $h \in G \setminus g^G$. Let $g_0 = t^{a_1}x_1 \cdots t^{a_n}x_n$ and $h_0 = t^{b_1}y_1 \cdots t^{b_m}y_m$ be cyclically reduced elements of G conjugate to g and h respectively. Note that $g \notin H^G$ implies that $n \geq 1$. There are two cases:

Subcase 1: $h_0 \in H$. Then, by Lemma 2.6.7, there exists a normal subgroup L of p-power index in H such that, if $\psi : H \to P = H/L$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\psi(K)$, and if $\overline{\psi} : G \to Q$ denotes the homomorphism induced by ψ, then: $\overline{\psi}(g_0) = \overline{t}^{a_1}\overline{x_1} \cdots \overline{t}^{a_n}\overline{x_n}$ is cyclically reduced in Q – where $\overline{x_i} = \overline{\psi}(x_i)$ ($i \in \{1,...,n\}$). Since $n \geq 1$, we have: $\overline{\psi}(g_0) \notin P^Q = \overline{\psi}(H^G)$. Therefore $\overline{\psi}(g_0) \notin \overline{\psi}(h_0)^Q = \overline{\psi}(h_0^G) \subset \overline{\psi}(H^G)$. Now Q is conjugacy p-separable by Corollary 2.4.3. Then there exists a homomorphism χ from Q onto a finite p-group R

such that $\chi(\overline{\psi}(g_0)) \notin \chi(\overline{\psi}(h_0))^R$. Therefore $\chi(\overline{\psi}(g)) \notin \chi(\overline{\psi}(h))^R$. Thus the homomorphism $\varphi = \chi \circ \overline{\psi} : G \to R$ satisfies the condition $\varphi(h) \notin \varphi(g^S)$, as desired.

Subcase 2: $h_0 \in G \setminus H$. Let $\{h_1,...,h_m\}$ be the set of all cyclic permutations of h_0. Then, since $h \notin g^G$, we have: $h_i \notin g_0^G$ for all $i \in \{1,...,m\}$. Therefore, by Lemma 2.6.7, there exists a normal subgroup L of p-power index in H such that, if $\psi : H \to P = H/L$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\psi(K)$, and if $\overline{\psi} : G \to Q$ denotes the homomorphism induced by ψ, then: $\overline{\psi}(g_0) = \overline{t}^{a_1}\overline{x_1} \cdots \overline{t}^{a_n}\overline{x_n}$ and $\overline{\psi}(h_0) = \overline{t}^{b_1}\overline{y_1} \cdots \overline{t}^{b_m}\overline{y_m}$ are cyclically reduced in Q – where $\overline{x_i} = \overline{\psi}(x_i)$ ($i \in \{1,...,n\}$) and $\overline{y_j} = \overline{\psi}(y_j)$ ($j \in \{1,...,n\}$) – and $\overline{\psi}(h_i) \notin \overline{\psi}(g_0)^{\overline{\psi}(K)}$ for all $i \in \{1,...,m\}$. Consequently, by Lemma 2.2.3, $\overline{\psi}(g_0) \notin \overline{\psi}(h_0)^Q$. Now Q is conjugacy p-separable by Corollary 2.4.3. Then there exists a homomorphism χ from Q onto a finite p-group R such that: $\chi(\overline{\psi}(g_0)) \notin \chi(\overline{\psi}(h_0))^R$. Hence $\chi(\overline{\psi}(g)) \notin \chi(\overline{\psi}(h))^R$. Thus the homomorphism $\varphi = \chi \circ \overline{\psi} : G \to R$ satisfies the condition $\varphi(h) \notin \varphi(g^S)$, as required. □

Proof of Proposition 2.6.2: We argue by induction on the rank r of G. If $r = 0$, then the result is trivial. Thus we can assume that $r \geq 1$ and that the result has been proved for $1,...,r-1$. Now, Proposition 2.6.2.1 follows from Lemma 2.6.13, and Proposition 2.6.2.2 follows from Lemma 2.6.14. □

We are now ready to prove:

Theorem 2.6.15. *Every right-angled Artin group is hereditarily conjugacy p-separable.*

Proof: Let G be a right-angled Artin group. Let $g \in G$. Then g^G is finitely p-separable in G by Proposition 2.6.2.1. We deduce that G is conjugacy p-separable. On the other hand, G satisfies the p-centralizer condition by Proposition 2.6.2.2. We conclude that G is hereditarily conjugacy p-separable by Proposition 2.3.6. □

2.7 Applications

The first application that we mention is an application of our main theorem to separability properties of G_Γ.

For a group G, we denote by $(C^n(G))_{n\geq 1}$ the lower central series of G. Recall that $(C^n(G))_{n\geq 1}$ is defined inductively by $C^1(G) = G$, and $C^{n+1}(G) = [G, C^n(G)]$ for all $n \geq 1$.

Corollary 2.7.1. *Every right-angled Artin group is conjugacy separable in the class of torsion-free nilpotent groups.*

Proof: Let G be a right-angled Artin group. Let $g, h \in G$ such that $g \nsim h$. Let p be a prime number. Then G is conjugacy p-separable by Theorem 2.6.15. Thus, there exists a homomorphism φ from G onto a finite p-group P such that $\varphi(g) \nsim \varphi(h)$. Now, P is nilpotent. Therefore, there exists $n \geq 1$ such that $C^n(P) = \{1\}$. Let $\pi : G \to \frac{G}{C^n(G)}$ be the canonical projection. It follows from [DK2], Theorem 2.1, that for all $n \geq 1$, there exists $d_n \in \mathbb{N}$ such that:

$$\frac{C^n(G)}{C^{n+1}(G)} \simeq \mathbb{Z}^{d_n}.$$

Thus, an easy induction on n shows that $\frac{G}{C^n(G)}$ is torsion-free for all $n \geq 1$. Hence $\frac{G}{C^n(G)}$ is a torsion-free nilpotent group for all $n \geq 1$. Since $\varphi(C^n(G)) < C^n(P) = \{1\}$, φ induces a homomorphism $\widetilde{\varphi} : \frac{G}{C^n(G)} \to P$ such that $\varphi = \widetilde{\varphi} \circ \pi$. As $\varphi(g) \nsim \varphi(h)$, we have $\pi(g) \nsim \pi(h)$. $\square$

We now turn to applications of our main theorem to residual properties of $Out(G_\Gamma)$.

An automorphism φ of a group G is said to be *conjugating* if for every $g \in G$, $\varphi(g) \sim g$. We say that G has *Property A* if every conjugating automorphism of G is inner. The following proposition is due to Minasyan (see [Mi], Proposition 6.9):

Proposition 2.7.2. *Right-angled Artin groups have Property A.*

For a group G, we denote by $\mathcal{I}_p(G)$ the kernel of the natural homomorphism $Out(G) \to GL(H_1(G, \mathbb{F}_p))$ (where $\mathbb{F}_p$ denotes the finite field with p elements). The following theorem is due to Paris (see [P2], Theorem 2.5):

Theorem 2.7.3. *Let G be a finitely generated group. If G is conjugacy p-separable and has Property A, then $\mathcal{I}_p(G)$ is residually p-finite.*

Thus, combining Theorem 2.7.3 and Proposition 2.7.2 with Theorem 2.6.15, we obtain:

Corollary 2.7.4. *The outer automorphism group of a right-angled Artin group is virtually residually p-finite.*

The following theorem is due to Myasnikov (see [My], Theorem 1):

Theorem 2.7.5. *Let G be a finitely generated group. If G is conjugacy p-separable and has property A, then $Out(G)$ is residually $\mathcal{K}$, where $\mathcal{K}$ is the class of all outer automorphism groups of finite p-groups.*

Thus, combining Theorem 2.7.5 and Proposition 2.7.2 with Theorem 2.6.15, we obtain:

Corollary 2.7.6. *The outer automorphism group of a right-angled Artin group is residually $\mathcal{K}$, where $\mathcal{K}$ is the class of all outer automorphism groups of finite p-groups.*

In the remainder of this chapter, we prove Theorem 2.1.6. Let $G = G_\Gamma$ be a right-angled Artin group. Let r be the rank of G. We denote by $T(G)$ the kernel of the natural homomorphism $Aut(G) \to GL_r(\mathbb{Z})$, and by $\mathcal{T}(G)$ the kernel of the natural homomorphism $Out(G) \to GL_r(\mathbb{Z})$. Note that $\mathcal{T}(G) = T(G)/Inn(G)$. Day proved that $T(G)$ is finitely generated (see [D2], Theorem B). Therefore $\mathcal{T}(G)$ is finitely generated.

In order to prove Theorem 2.1.6, we have to introduce the notion of separating $\mathbb{Z}$-linear central filtration.

Recall that a *central filtration* on a group G is a sequence $(G_n)_{n\geq 1}$ of subgroups of G satisfying the conditions:

$$G_1 = G,$$
$$G_n > G_{n+1},$$
$$[G_m, G_n] < G_{m+n} \text{ for all } m,\, n \geq 1.$$

Let $\mathcal{F} = (G_n)_{n\geq 1}$ be a central filtration. Then the mapping $G \times G \to G$, $(x, y) \mapsto xyx^{-1}y^{-1}$ induces on:

$$\mathcal{L}_{\mathcal{F}}(G) = \bigoplus_{n\geq 1} \frac{G_n}{G_{n+1}}$$

a Lie bracket which makes $\mathcal{L}_{\mathcal{F}}(G)$ into a graded Lie $\mathbb{Z}$-algebra.

We say that $(G_n)_{n\geq 1}$ is a *separating filtration* if $\cap_{n\geq 1} G_n = \{1\}$. We say that $(G_n)_{n\geq 1}$ is $\mathbb{Z}$*-linear* if for all $n \geq 1$, the $\mathbb{Z}$-module $\frac{G_n}{G_{n+1}}$ is free of finite rank.

For a group G, we denote by $(C^n_{\mathbb{Z}}(G))_{n\geq 1}$ the sequence of subgroups of G defined inductively by $C^1_{\mathbb{Z}}(G) = G$, $[G, C^n_{\mathbb{Z}}(G)] < C^{n+1}_{\mathbb{Z}}(G)$, and $\frac{C^{n+1}_{\mathbb{Z}}(G)}{[G,C^n_{\mathbb{Z}}(G)]}$ is the torsion subgroup of $\frac{C^n_{\mathbb{Z}}(G)}{[G,C^n_{\mathbb{Z}}(G)]}$ for all $n \geq 1$.

Proposition 2.7.7. *For all m, $n \geq 1$, $[C^m_{\mathbb{Z}}(G), C^n_{\mathbb{Z}}(G)] < C^{m+n}_{\mathbb{Z}}(G)$.*

Proof: Proved in [BL] (see Proposition 7.2). □

Thus, $(C^n_{\mathbb{Z}}(G))_{n\geq 1}$ is a central filtration on G. We denote by $\mathcal{L}_{\mathbb{Z}}(G)$ the corresponding graded Lie $\mathbb{Z}$-algebra.

For a Lie algebra $\mathfrak{g}$, we denote by $Z(\mathfrak{g})$ the center of $\mathfrak{g}$. Let G be a group. For $n \geq 1$, we denote by A_n the kernel of the natural homomorphism $Aut(G) \to Aut(\frac{G}{C^{n+1}_{\mathbb{Z}}(G)})$. Let $\pi : Aut(G) \to Out(G)$ be the canonical projection. For $n \geq 1$, we set $B_n = \pi(G_n)$.

Theorem 2.7.8. *If G^{ab} is finitely generated, and $Z(\mathbb{F}_p \otimes \mathcal{L}_{\mathbb{Z}}(G)) = \{0\}$ for every prime number p, then $(B_n)_{n\geq 1}$ is a $\mathbb{Z}$-linear central filtration on B_1. Furthermore, $(B_n)_{n\geq 1}$ is separating if and only if G satisfies the condition:*

$(IN(G))$: For every $\varphi \in Aut(G)$, if φ induces an inner automorphism of $\frac{G}{C^n_{\mathbb{Z}}(G)}$ for all $n \geq 1$, then φ is inner.

Proof: Proved in [BL] (see Corollary 9.9). □

From now on, we assume that $G = G_\Gamma$ is a right-angled Artin group of rank r ($r \geq 1$). We shall show that G satisfies the conditions of Theorem 2.7.8. Since B_1 is precisely the Torelli group of G, Theorem 2.1.6 will then result from the following:

Theorem 2.7.9. *Let B be a group. Suppose that B admits a separating $\mathbb{Z}$-linear central filtration $(B_n)_{n\geq 1}$. Then B is residually torsion-free nilpotent.*

Proof: Proved in [BL] (see Theorem 6.1). □

We need to introduce the following notations. Let K be a commutative ring. We denote by M_Γ the monoid defined by the presentation:

$$M_\Gamma = \langle\ V \mid vw = wv, \forall\ \{v,w\} \in E\ \rangle,$$

by A_Γ the associative K-algebra of the monoid M_Γ, and by L_Γ the Lie K-algebra defined by the presentation:

$$L_\Gamma = \langle\ V \mid [v,w] = 0, \forall\ \{v,w\} \in E\ \rangle.$$

The following theorem is due to Duchamp and Krob (see [DK1], Corollary II.16):

Theorem 2.7.10. *The K-module L_Γ is free.*

Thus, by the Poincaré-Birkhoff-Witt theorem, L_Γ can be regarded as a Lie subalgebra of its enveloping algebra, for which Duchamp and Krob established the following (see [DK1], Corollary I.2):

Theorem 2.7.11. *The enveloping algebra of L_Γ is isomorphic to A_Γ.*

Furthermore, Duchamp and Krob proved the following (see [DK2], Theorem 2.1), which generalizes a well-known theorem of Magnus:

Theorem 2.7.12. *Suppose that $K = \mathbb{Z}$. The graded Lie $\mathbb{Z}$-algebra associated to the lower central series of G is isomorphic to L_Γ.*

Set $Z = \cap_{v\in V} star(v)$. It follows from Servatius' Centralizer Theorem (see [Ser], Theorem 1) that the center $Z(G)$ of G is the special subgroup of G generated by Z.

Lemma 2.7.13. *Suppose that $Z(G) = \{1\}$. Then $Z(L_\Gamma) = \{0\}$.*

Proof: Let $g \in Z(L_\Gamma)$. Suppose that $g \neq 0$. Let $v \in V$. We have $[g, v] = 0$ (in L_Γ). Now, L_Γ can be regarded as a Lie subalgebra of A_Γ by Theorem 2.7.10 and Theorem 2.7.11. Thus, we have $gv = vg$ (in A_Γ). Therefore g belongs to the subalgebra of A_Γ generated by $st(v)$ (see [KR]). Since v is arbitrary, this leads to a contradiction with our assumption. □

From now on, we assume that $K = \mathbb{Z}$. We now turn to prove:

Theorem 2.7.14. *The Torelli group of a right-angled Artin group is residually torsion-free nilpotent.*

Proof: Let H be the special subgroup of G generated by $V \setminus Z$. Note that $Z(H) = \{1\}$. We have: $G = H \times Z(G)$. First, we show that $\mathcal{T}(G) = \mathcal{T}(H)$. Let $\varphi : T(H) \to T(G)$ be the homomorphism defined by:

$$\varphi(\alpha)(h, k) = (\alpha(h), k)$$

for all $\alpha \in T(H)$, $h \in H$, $k \in Z(G)$. Clearly, φ is well-defined and injective. We shall show that φ is surjective. Let $\beta \in T(G)$. For $g \in G$, we set $\beta(g) = (\beta_1(g), \beta_2(g))$, where $\beta_1(g) \in H$ and $\beta_2(g) \in Z(G)$. Let $h \in H$. We denote by $\overline{h}$ the canonical image of h in H^{ab}. Note that the canonical image of h in $G^{ab} = H^{ab} \times Z(G)$ is $(\overline{h}, 1)$. Since $\beta \in T(G)$, we have: $(\overline{h}, 1) = (\overline{\beta_1(h)}, \beta_2(h))$, and then $\beta_2(h) = 1$. Let $k \in Z(G)$. Since $\beta(k)$ lies in the center of G, we have $\beta_1(k) = 1$. Note that the canonical image of k in G^{ab} is $(1, k)$. As $\beta \in T(G)$, we have $\beta_2(k) = k$. Finally, we have:

$$\beta(h, k) = (\beta_1(h), k),$$

for all $h \in H$ and $k \in Z(G)$. Applying the same argument to β^{-1}, we obtain that the restriction α of β_1 to H is an automorphism of H. Therefore $\beta = \varphi(\alpha)$. We deduce that φ is an isomorphism. Note that $\varphi(Inn(H)) = Inn(G)$. We conclude that $\mathcal{T}(G) = \mathcal{T}(H)$. Thus, up to replacing G by H, we can assume that $Z(G) = \{1\}$. As we noted above, $\frac{G}{C^n(G)}$ is torsion-free for all $n \geq 1$. Now, for all $n \geq 1$, $C^n(G) < C^n_{\mathbb{Z}}(G)$, and $\frac{C^n_{\mathbb{Z}}(G)}{C^n(G)}$ is the torsion subgroup of $\frac{G}{C^n(G)}$ by [BL], Proposition 7.2. It follows that $C^n_{\mathbb{Z}}(G) = C^n(G)$ for all $n \geq 1$, and that $\mathcal{L}_{\mathbb{Z}}(G) = L_\Gamma$ by Theorem 2.7.12. Since $Z(G) = \{1\}$, we have $Z(\mathbb{F}_p \otimes L_\Gamma) = \{0\}$ for every prime number p – by Lemma 2.7.13. We deduce that $(B_n)_{n \geq 1}$ is a $\mathbb{Z}$-linear central filtration on $\mathcal{T}(G)$ by Theorem 2.7.8. Now, let $\varphi \in Aut(G)$ such that φ induces an inner automorphism on $\frac{G}{C^n(G)}$ for all $n \geq 1$. Let $g \in G$. Suppose that $\varphi(g)$ and g are not conjugate in G. Then it follows from the proof of Theorem 2.7.1 that there exists $n \geq 1$ such that the canonical images of $\varphi(g)$ and g in $\frac{G}{C^n(G)}$ are not conjugate in $\frac{G}{C^n(G)}$ – contradicting our assumption. Thus φ is conjugating. Therefore

φ is inner by Proposition 2.7.2. We deduce that $(B_n)_{n\geq 1}$ is separating by Theorem 2.7.8. We conclude that $\mathcal{T}(G)$ is residually torsion-free nilpotent by Theorem 2.7.9. □

Corollary 2.7.15. *The Torelli group of a right-angled Artin group is residually p-finite.*

Proof: Since $\mathcal{T}(G)$ is finitely generated by [D2], Theorem B, and residually torsion-free nilpotent by Theorem 2.7.14, it is residually p-finite by [Gru], Theorem 2.1. □

It is known that residually torsion-free nilpotent groups are bi-orderable (see, for example, [CKM], Remark 2.6). Thus, Theorem 2.7.14 immediately yields:

Corollary 2.7.16. *The Torelli group of a right-angled Artin group is bi-orderable.*

Chapter 3

A finitely presented subgroup of the automorphism group of a right-angled Artin group

In this chapter, we give a presentation of the subgroup $Conj(G_\Gamma)$ of $Aut(G_\Gamma)$ consisting of the automorphisms that send each generator to a conjugate of itself. This generalizes a result of McCool on basis-conjugating automorphisms of free groups.

3.1 Introduction

In 1936, Whitehead proved what is now known as "Whitehead's theorem": there is an algorithm which, given two m-tuples $(u_1, ..., u_m)$ and $(v_1, ..., v_m)$ of elements of F_n, decides whether there exists an automorphism $\alpha \in Aut(F_n)$ such that $\alpha(u_i) = v_i$ for all $i \in \{1,...,m\}$ (see [W]). To this end, he introduced a set of transformations of F_n, now known as the "Whitehead automorphisms". Whitehead's proof used topological methods. In 1958, Rapaport gave an algebraic proof of Whitehead's theorem (see [R]), which was later simplified by Higgins and Lyndon (see [HL]). Using a refinement of the argument of Higgins and Lyndon, McCool obtained a finite presentation for $Aut(F_n)$, with the Whitehead automorphisms as generating set (see [Mc1]). McCool also proved that the stabilizer of an m-tuple of cyclic words in F_n is finitely presented (see [Mc2]). (A *cyclic word* in F_n can be thought of as the set of all cyclic permutations of a given cyclically reduced word.) Thereafter McCool obtained a finite presentation for the subgroup of $Aut(F_n)$ consisting of the automorphisms that send each generator to a conjugate of itself (see [Mc3]).

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a finite simplicial graph, and let G_Γ be the right-angled

Artin group associated to Γ:

$$G_\Gamma = \langle \ \mathcal{V} \mid vw = wv, \forall \ \{v,w\} \in \mathcal{E} \ \rangle.$$

Our focus here is on the automorphisms of G_Γ that satisfy the following definition:

Definition 3.1.1. *We say that an automorphism φ of G_Γ is vertex-conjugating if $\varphi(v)$ is conjugate to v for all $v \in \mathcal{V}$.*

Vertex-conjugating automorphisms were first introduced by Laurence in [L], where they are called conjugating. They also appear in the recent work of Duncan, Kazachkov and Remeslennikov (see [DKR], see also [DR]). As one of the steps in the proof of Servatius' conjecture, Laurence established the following (see [L]):

Proposition 3.1.2. *The set of vertex-conjugating automorphisms coincides with the subgroup $Conj(G_\Gamma)$ of $Aut(G_\Gamma)$ generated by the partial conjugations.*

Let S denote the set of all partial conjugations of G_Γ. In Section 3.3, we define a finite set R of relations satisfied by the elements of S. The main theorem of Chapter 3 is the following:

Theorem 3.1.3. *The group $Conj(G_\Gamma)$ has the presentation $\langle S \mid R \rangle$.*

In Section 3.3, we shall state a more precise version of Theorem 3.1.3 which yields an explicit finite presentation for $Conj(G_\Gamma)$ (see Theorem 3.3.1).

Our proof relies on geometric methods. Following arguments from McCool [Mc2], [Mc3], we construct a finite, connected 2-complex K with fundamental group $Conj(G_\Gamma) = \langle S \mid R \rangle$. An important observation is that every partial conjugation is a long-range Whitehead automorphism in the sense of [D1].

Note that we cannot hope for a generalization of the presentation given in the theorem of [Mc3] (see Remark 3.3.2 below).

3.2 Preliminaries

In this section, we fix our notations and review some notions that will be used in the proof of Theorem 3.1.3.

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a finite simplicial graph, and let G_Γ be the right-angled Artin group associated to Γ. Let v be a vertex of Γ. The *link* of v, denoted by $lk(v)$, is the subset of $\mathcal{V}$ consisting of all vertices that are adjacent to v. The *star* of v, denoted by $st(v)$, is $lk(v) \cup \{v\}$. We set $L = \mathcal{V} \cup \mathcal{V}^{-1}$. Let $x \in L$. The *vertex* of x, denoted by $v(x)$, is the unique element of $\mathcal{V} \cap \{x, x^{-1}\}$. We set $lk_L(x) = lk(v(x)) \cup lk(v(x))^{-1}$, and $st_L(x) = st(v(x)) \cup st(v(x))^{-1}$.

Let w be a word in $\mathcal{V} \cup \mathcal{V}^{-1}$. The *support* of w, denoted by $supp(w)$, is the subset of $\mathcal{V}$ of all vertices v such that v or v^{-1} is a letter of w. A word w in $\mathcal{V} \cup \mathcal{V}^{-1}$ is said to be *reduced* if it contains no subwords of the form $vw'v^{-1}$ or $v^{-1}w'v$ with $supp(w') \subset st(v)$. For a word w in $\mathcal{V} \cup \mathcal{V}^{-1}$, we denote by $|w|$ the length of w. The *length* of an element g of G_Γ is by definition the minimal length of any word representing g. Note that the length of g is equal to the length of any reduced word representing g. We say that an element g of G_Γ is *cyclically reduced* if it can not be written vhv^{-1} or $v^{-1}hv$ with $v \in \mathcal{V}$, and $|g| = |h| + 2$. By [Ser], Proposition 2, every element of G_Γ is conjugate to a unique (up to cyclic permutation) cyclically reduced element. The *length* of a conjugacy class is by definition the minimal length of any of its representative elements. Observe that the length of a conjugacy class is equal to the length of a cyclically reduced element representing it. For an n-tuple of conjugacy classes W, we define the *length* of W, denoted by $|W|$, as the sum of the lengths of its elements ($n \geq 1$).

Let v, w be vertices of Γ. We use the notation $v \geq w$ to mean $lk(w) \subset st(v)$. We use the notation $v \sim w$ to mean $v \geq w$ and $w \geq v$.

The Laurence-Servatius generators for $Aut(G_\Gamma)$ are defined as follows:

Inversions: Let $v \in \mathcal{V}$. The automorphism ι_v that sends v to v^{-1} and fixes all other vertices is called an *inversion.*

Partial conjugations: Let $x \in L$, and let Y be a non-empty union of connected components of $\Gamma \setminus st(v(x))$. The automorphism $c_{x,Y}$ that sends each vertex y in Y to $x^{-1}yx$ and fixes all vertices not in Y is called a *partial conjugation.*

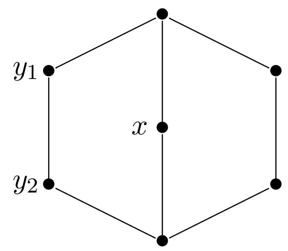


Figure 1. A partial conjugation $y_i \mapsto x^{-1}y_ix$.

Transvections: Let v, $w \in \mathcal{V}$ be such that $v \geq w$. The automorphism $\tau_{v,w}$ that sends w to vw and fixes all other vertices is called a *transvection.*

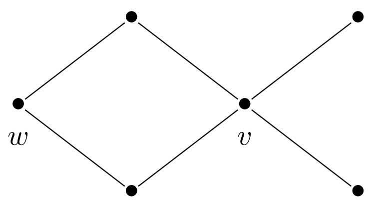


Figure 2. A transvection $w \mapsto vw$.

Symmetries: Let φ be an automorphism of the graph Γ. The automorphism ϕ given by $\phi(v) = \varphi(v)$ for all $v \in \mathcal{V}$ is called a *symmetry.*

A *Whitehead automorphism* is an automorphism $\alpha \in Aut(G_\Gamma)$ of one of the following two types:

Type 1: α restricted to $\mathcal{V} \cup \mathcal{V}^{-1}$ is a permutation of $\mathcal{V} \cup \mathcal{V}^{-1}$.

Type 2: There is an element $a \in L$, called the *multiplier* of α, such that $\alpha(a) = a$, and for each $x \in \mathcal{V}$, the element $\alpha(x)$ lies in $\{x, xa, a^{-1}x, a^{-1}xa\}$.

One can show that the set of type 1 Whitehead automorphisms is the subgroup of $Aut(G_\Gamma)$ generated by inversions and symmetries.

Following [D1], we say that a Whitehead automorphism α is *long-range* if α is of type 1 or if α is of type 2 and α fixes the vertices of $lk(v(a))$ (where a is the multiplier of α).

We denote by $\mathcal{W}$ the set of Whitehead automorphisms, by $\mathcal{W}_1$ the set of Whitehead automorphisms of type 1, and by $\mathcal{W}_2$ the set of Whitehead automorphisms of type 2. We also denote by $\mathcal{W}_\ell$ the set of long-range Whitehead automorphisms.

We use the following notation for type 2 Whitehead automorphisms. Let A be a subset of L, and let $a \in L$, such that $a \in A$ and $a^{-1} \notin A$. Provided that it exists, (A, a) denotes the automorphism given by:

$$(A, a)(a) = a,$$

and, for all $x \in \mathcal{V} \setminus \{v(a)\}$,

$$(A, a)(x) = \begin{cases} x & \text{if } x \notin A \text{ and } x^{-1} \notin A \\ xa & \text{if } x \in A \text{ and } x^{-1} \notin A \\ a^{-1}x & \text{if } x \notin A \text{ and } x^{-1} \in A \\ a^{-1}xa & \text{if } x \in A \text{ and } x^{-1} \in A \end{cases}$$

If A is a subset of L, we set $A^{-1} = \{a^{-1} \mid a \in A\}$. If A and B are subsets of L, and a is an element of L, we use the notations $A - B$ for $A \setminus B$, $A + B$ for $A \sqcup B$ (if $A \cap B = \emptyset$), $A - a$ for $A \setminus \{a\}$ and $A + a$ for $A \sqcup \{a\}$ (if $a \notin A$).

The following remark will be of particular importance in our proof:

Remark 3.2.1. Let $x \in L$, and let Y be a non-empty union of connected components of $\Gamma \setminus st(v(x))$. Set $A = Y \cup Y^{-1} \cup \{x\}$, and $a = x$. Then the Whitehead automorphism (A, a) is nothing but the partial conjugation $c_{x,Y}$. In particular, the Whitehead automorphism $(L - lk_L(a) - a^{-1}, a)$ is the inner automorphism ω_a induced by a. Note that there is not a unique way to write a partial conjugation as a type 2 Whitehead automorphism. More specifically, if $B \subset lk(v(a))$, then the Whitehead automorphisms (A, a) and $(A + B + B^{-1}, a)$ represent the same element of S.

In [D1], Day proved that $Aut(G_\Gamma)$ is generated by the Whitehead automorphisms, subject to the relations:

$$(A, a)^{-1} = (A - a + a^{-1}, a^{-1}), \tag{R1}$$

for $(A, a) \in \mathcal{W}_2$.

$$(A, a)(B, a) = (A \cup B, a), \tag{R2}$$

for (A, a), $(B, a) \in \mathcal{W}_2$ with $A \cap B = \{a\}$.

$$(B, b)(A, a)(B, b)^{-1} = (A, a), \tag{R3}$$

for (A, a), $(B, b) \in \mathcal{W}_2$ such that $a \notin B$, $a^{-1} \notin B$, $b \notin A$, $b^{-1} \notin A$, and at least one of (a) $A \cap B = \emptyset$ or (b) $b \in lk_L(a)$ holds.

$$(B, b)(A, a)(B, b)^{-1} = (A, a)(B - b + a, a), \tag{R4}$$

for (A, a), $(B, b) \in \mathcal{W}_2$ such that $a \notin B$, $a^{-1} \notin B$, $b \notin A$, $b^{-1} \in A$, and at least one of (a) $A \cap B = \emptyset$ or (b) $b \in lk_L(a)$ holds.

$$(A - a + a^{-1}, b)(A, a) = (A - b + b^{-1}, a)\sigma_{a,b}, \tag{R5}$$

for $(A, a) \in \mathcal{W}_2$, $b \in L$ such that $b \in A$, $b^{-1} \notin A$, $b \neq a$, and $v(b) \sim v(a)$. Here $\sigma_{a,b}$ denotes the type 1 Whitehead automorphism that sends a to b^{-1} and b to a, and fixes the other generators.

$$\sigma(A, a)\sigma^{-1} = (\sigma(A), \sigma(a)), \tag{R6}$$

for $(A, a) \in \mathcal{W}_2$, and $\sigma \in \mathcal{W}_1$.

$$\text{The entire multiplication table of } \mathcal{W}_1 \tag{R7}$$

– which forms a finite subgroup of $Aut(G_\Gamma)$.

$$(A, a) = (L - a^{-1}, a)(L - A, a^{-1}), \tag{R8}$$

for $(A,a) \in \mathcal{W}_2$.

$$(A,a)(L-b^{-1},b)(A,a)^{-1} = (L-b^{-1},b), \tag{R9}$$

for $(A,a) \in \mathcal{W}_2$, $b \in L$ such that $b \notin A$, $b^{-1} \notin A$.

$$(A,a)(L-b^{-1},b)(A,a)^{-1} = (L-a^{-1},a)(L-b^{-1},b), \tag{R10}$$

for $(A,a) \in \mathcal{W}_2$, $b \in L$ such that $b \in A$, $b^{-1} \notin A$, and $b \neq a$.

Note that the relation (R8) is a direct consequence of the relations (R1) and (R2).

In order to prove Theorem 3.1.3, we need to introduce the following technical definitions.

Let α, $\beta \in \mathcal{W}$, and let W be an n-tuple of conjugacy classes ($n \geq 1$). Following [D1], we say that α is a *peak* of $\beta\alpha$ with respect to W if:

$$|W| \leq |\alpha.W|,$$
$$|\beta\alpha.W| \leq |\alpha.W|,$$

and at least one of these inequalities is strict. Let α_1,...,$\alpha_k \in \mathcal{W}$ ($k \geq 1$). We say that α_i is a *peak* of the product $\alpha_k \cdots \alpha_1$ with respect to W if $1 \leq i < k$ and α_i is a peak of $\alpha_{i+1}\alpha_i$ with respect to $\alpha_{i-1} \cdots \alpha_1.W$. We say that the product $\alpha_k \cdots \alpha_1$ is *peak-reduced* with respect to W if it has no peaks with respect to W. The *height* of a peak α_i is $|\alpha_i \cdots \alpha_1.W|$.

3.3 Proof of the main theorem

In this section, we prove the following:

Theorem 3.3.1. *The group $Conj(G_\Gamma)$ has a presentation with generators $c_{x,Y}$, for $x \in L$ and Y a non-empty union of connected components of $\Gamma \setminus st(v(x))$, and relations:*

$$\begin{aligned}
(c_{x,Y})^{-1} &= c_{x^{-1},Y}, \\
c_{x,Y}c_{x,Z} &= c_{x,Y\cup Z} \text{ if } Y \cap Z = \emptyset, \\
c_{x,Y}c_{y,Z} &= c_{y,Z}c_{x,Y} \text{ if } v(x) \notin Z,\ v(y) \notin Y,\ x \neq y,\ y^{-1}, \text{ and at least one of } \\
&\qquad Y \cap Z = \emptyset \text{ or } y \in lk_L(x) \text{ holds,} \\
\omega_y c_{x,Y} \omega_y^{-1} &= c_{x,Y} \text{ if } v(y) \notin Y,\ x \neq y,\ y^{-1}.
\end{aligned}$$

Proof: Our proof is based on arguments developed by McCool in [Mc2] and [Mc3] (similar arguments were used in [D1]). Recall that S denotes the set of partial conjugations. Let R denote the set of relations given in the statement of Theorem 3.3.1. We shall construct a finite, connected 2-complex K with fundamental group $Conj(G_\Gamma) = \langle S \mid R \rangle$.

We identify a partial conjugation with any of its representatives in $\mathcal{W}_2$ (see Remark 3.2.1 above). Note that for every $(A, a) \in \mathcal{W}_2$, we have $(A, a) \in S$ if and only if $(A - a)^{-1} = A - a$.

Set $\mathcal{V} = \{v_1,...,v_n\}$ $(n \geq 1)$. Let W denote the n-tuple $(v_1,...,v_n)$.

The set of vertices $K^{(0)}$ of K is the set of n-tuples $\alpha.W$, where α ranges over the set $\mathcal{W}_1$ of type 1 Whitehead automorphisms. For any α, $\beta \in \mathcal{W}_1$, the vertices $\alpha.W$ and $\beta\alpha.W$ are joined by a directed edge $(\alpha.W, \beta\alpha.W; \beta)$ labelled β. Note that, at this stage, K is just the Cayley graph of $\mathcal{W}_1$. Next, for any $\alpha \in \mathcal{W}_1$, and $(A, a) \in S$, we add a loop $(\alpha.W, \alpha.W; (A, a))$ labelled (A, a) at $\alpha.W$. This defines the 1-skeleton $K^{(1)}$ of K.

We shall define the 2-cells of K. These 2-cells will derive from the relations (R1)-(R10) of [D1]. First, let K_1 be the 2-complex obtained by attaching 2-cells corresponding to the relations (R7) to $K^{(1)}$. Note that, if C is the 2-complex obtained from K_1 by deleting the loops $(\alpha.W, \alpha.W; (A, a))$ $(\alpha \in \mathcal{W}_1,\ (A, a) \in S)$, then C is just the Cayley complex of $\mathcal{W}_1$, and therefore is simply connected.

We now explore the relations (R1)-(R5) and (R8)-(R10) of [D1] to determine which of these will give rise to relations on the elements of S.

The relation (R1) will give rise to the following:

$$(A, a)^{-1} = (A - a + a^{-1}, a^{-1}), \tag{1}$$

for $(A, a) \in S$.

The relation (R2) will give rise to:

$$(A, a)(B, a) = (A \cup B, a), \tag{2}$$

for (A, a), $(B, a) \in S$, with $A \cap B = \{a\}$.

The relation (R3) will give rise to:

$$(A, a)(B, b) = (B, b)(A, a), \tag{3}$$

for (A, a), $(B, b) \in S$, such that $a \notin B$, $a^{-1} \notin B$, $b \notin A$, and $b^{-1} \notin A$, and at least one of (a) $A \cap B = \emptyset$ or (b) $b \in lk_L(a)$ holds.

From (R4), no relations arise. Indeed, suppose that (A, a), (B, b) are in S with $a^{-1} \notin B$, $b \notin A$, and $b^{-1} \in A$. Then $b^{-1} = a$ (because $(A - a)^{-1} = A - a$). But then $a^{-1} = b \in B$ – leading to a contradiction with our assumption on a.

From (R5), no relations arise (by the same argument as above).

From (R8), we obtain a relation which is a direct consequence of (1) and (2).

The relation (R9) will give rise to the following:

$$(A, a)(L - lk_L(b) - b^{-1}, b)(A, a)^{-1} = (L - lk_L(b) - b^{-1}, b), \tag{4}$$

for $(A, a) \in S$, and $b \in L$ such that $b \notin A$, and $b^{-1} \notin A$.

From (R10), no relations arise (by the same argument as above).

We rewrite the relations (1)-(4) in the form:

$$\sigma_k^{\varepsilon_k} \cdots \sigma_1^{\varepsilon_1} = 1,$$

where $\sigma_1,...,\sigma_k \in S$, and $\varepsilon_1,...,\varepsilon_k \in \{-1,1\}$. Let K_2 be the 2-complex obtained from K_1 by attaching 2-cells corresponding to the relations (1)-(4). Note that the boundary of each of these 2-cells has the form:

$$(\alpha.W, \alpha.W; \sigma_1)^{\varepsilon_1}(\alpha.W, \alpha.W; \sigma_2)^{\varepsilon_2} \cdots (\alpha.W, \alpha.W; \sigma_k)^{\varepsilon_k},$$

for $\alpha \in \mathcal{W}_1$.

Finally, the relations (R6) will give rise to the following:

$$\alpha(A, a)\alpha^{-1} = (\alpha(A), \alpha(a)), \tag{5}$$

for $(A, a) \in S$, and $\alpha \in \mathcal{W}_1$. Then K is obtained from K_2 by attaching 2-cells corresponding to the relations (5). Observe that the boundary of each of these 2-cells has the form:

$$(\beta.W, \beta.W; (\alpha(A), \alpha(a)))^{-1}(\beta.W, \alpha^{-1}\beta.W; \alpha)^{-1}(\alpha^{-1}\beta.W, \alpha^{-1}\beta.W; (A, a))$$
$$(\alpha^{-1}\beta.W, \beta.W; \alpha),$$

for $\beta \in \mathcal{W}_1$.

It remains to show that $\pi_1(K, W) = Conj(G_\Gamma) = \langle S \mid R \rangle$.

Let T be a maximal tree in the 1-skeleton $K^{(1)}$ of K. Note that T is in fact a maximal tree in the 1-skeleton $C^{(1)}$ of C (i.e., the Cayley graph of $\mathcal{W}_1$). We compute a presentation of $\pi_1(K, W)$ using T. For every vertex V of K, there exists a unique reduced path p_V from W to V in T. To each edge $(V_1, V_2; \alpha)$ of K, we associate the element of $\pi_1(K, W)$ represented by the loop $p_{V_1}(V_1, V_2; \alpha)p_{V_2}^{-1}$. We again denote this by $(V_1, V_2; \alpha)$. Evidently these elements generate $\pi_1(K, W)$. Now, since C is simply connected, we have

$$(\alpha.W, \beta\alpha.W; \beta) = 1 \ \text{ (in } \pi_1(K, W)), \tag{6}$$

for all α, $\beta \in \mathcal{W}_1$.

Let $\mathcal{P}$ be the set of combinatorial paths in the 1-skeleton $K^{(1)}$ of K. We define a map $\widehat{\varphi} : \mathcal{P} \to Aut(G_\Gamma)$ as follows. For an edge $e = (V_1, V_2; \alpha)$, we set $\widehat{\varphi}(e) = \alpha$, and for a path $p = e_k^{\varepsilon_k} \cdots e_1^{\varepsilon_1}$, we set $\widehat{\varphi(p)} = \widehat{\varphi}(e_k)^{\varepsilon_k} \cdots \widehat{\varphi}(e_1)^{\varepsilon_1}$. Clearly, if p_1 and p_2 are loops at W such that $p_1 \sim p_2$, then $\widehat{\varphi}(p_1) = \widehat{\varphi}(p_2)$. Hence, $\widehat{\varphi}$ induces a map $\varphi : \pi_1(K, W) \to Aut(G_\Gamma)$. It is easily seen that φ is a homomorphism. Then we see from (6) that φ maps $\pi_1(K, W)$ to $Conj(G_\Gamma)$. It follows immediately from the construction of K that $\varphi : \pi_1(K, W) \to Conj(G_\Gamma)$ is surjective. Thus, it suffices to show that φ is injective. Let p be a loop at W such that $\varphi(p) = 1$. We have to show that $p \sim 1$. Write $p = e_k^{\varepsilon_k} \cdots e_1^{\varepsilon_1}$, where $k \geq 1$ and $\varepsilon_i \in \{-1,1\}$ for all $i \in \{1,...,k\}$. Using the 2-cells arising from the relations (1), and the fact that $\mathcal{W}_1^{-1} = \mathcal{W}_1$, we can restrict our attention to the case where $p = e_k \cdots e_1$. Set $\alpha_i = \varphi(e_i)$ for all $i \in \{1,...,k\}$. Note that $\alpha_i \in S \cup \mathcal{W}_1 \subset \mathcal{W}_\ell$ for all $i \in \{1,...,k\}$.

Let Z be a tuple containing each conjugacy class of length 2 of G_Γ, each appearing once. We prove the following:

<u>Claim</u>: $p \sim e'_l \cdots e'_1$, such that, if we set $\alpha'_i = \varphi(e'_i)$ for all $i \in \{1,...,l\}$, then $\alpha'_i \in \mathcal{W}_1$ or $\alpha'_i \in \mathcal{W}_2 \cap Inn(G_\Gamma)$ for each $i \in \{1,...,l\}$.

First, we examine the case where $\alpha_k \cdots \alpha_1$ is peak-reduced with respect to Z. We claim that the sequence

$$|Z|,\ |\alpha_1.Z|,\ |\alpha_2\alpha_1.Z|,...,\ |\alpha_{k-1} \cdots \alpha_1.Z|,\ |\alpha_k \cdots \alpha_1.Z| = |Z|.$$

is a constant sequence. Suppose the contrary. By Lemma 5.2 in [D1], $|Z|$ is the least element of the set $\{|\alpha.Z| \mid \alpha \in \langle \mathcal{W}_\ell \rangle\}$. Hence we can find $i \in \{1,...,k-1\}$ such that we have

$$\begin{aligned} |\alpha_{i-1} \cdots \alpha_1.Z| &\leq |\alpha_i \cdots \alpha_1.Z|, \\ |\alpha_{i+1} \cdots \alpha_1.Z| &\leq |\alpha_i \cdots \alpha_1.Z|, \end{aligned}$$

and at least one of these inequalities is strict – which contradicts the fact that $\alpha_k \cdots \alpha_1$ is peak-reduced. Therefore we have

$$|\alpha_i \cdots \alpha_1.Z| = |Z|,$$

for all $i \in \{1,...,k\}$. We argue by induction on $i \in \{1,...,k\}$ to prove that $\alpha_i \cdots \alpha_1.Z$ is a tuple containing each conjugacy class of length 2 of G_Γ, each appearing once. The result holds for $i = 0$ by assumption. Suppose that $i \geq 1$, and that the result holds for $i - 1$. Observe that a type 1 Whitehead automorphism does not change the length of a conjugacy class. Thus, we can assume that α_i is a type 2 Whitehead automorphism. Since $|\alpha_i\alpha_{i-1} \cdots \alpha_1.Z| = |\alpha_{i-1} \cdots \alpha_1.Z|$, α_i is trivial, or an inner automorphism by [D1], Lemma 5.2. Thus, the result holds for i. In this case, p has already the desired form.

We now turn to prove the claim. We define:

$$h_p = \max\{|\alpha_i \cdots \alpha_1.Z| \mid i \in \{0,...,k\}\},$$

and:

$$N_p = |\{i \mid i \in \{0,...,k\} \text{ and } |\alpha_i \cdots \alpha_1.Z| = h_p\}|.$$

We argue by induction on h_p. The base of induction is $|Z|$ – the smallest possible value for h_p by [D1], Lemma 5.2. If $h_p = |Z|$, then the product $\alpha_k \cdots \alpha_1$ is peak-reduced and we are done. Thus, we can assume that $h_p > |Z|$ and that the result has been proved for all loops p' with $h_{p'} < h_p$. Let $i \in \{1,...,k\}$ be such that α_i is a peak of height h_p. An examination of the proof of Lemma 3.18 in [D1] shows that $e_{i+1}e_i \sim f_j \cdots f_1$ such that, if we set $\beta_\kappa = \varphi(f_\kappa)$ for all $\kappa \in \{1,...,j\}$, then:

$$|\beta_\kappa \cdots \beta_1 \alpha_{i-1} \cdots \alpha_1.Z| < |\alpha_i \alpha_{i-1} \cdots \alpha_1.Z|\,, \tag{7}$$

for all $\kappa \in \{1,...,j-1\}$. Therefore, we get $p \sim e_k \cdots e_{i+2} f_j \cdots f_1 e_{i-1} \cdots e_1 = p'$, and a new product $\alpha_k \cdots \alpha_{i+2} \beta_j \cdots \beta_1 \alpha_{i-1} \cdots \alpha_1$. We argue by induction on N_p. If $N_p = 1$, then (7) implies that $h_{p'} < h_p$ and we can apply the induction hypothesis on h_p. If $N_p \geq 2$, then (7) implies that $h_{p'} = h_p$ and $N_{p'} < N_p$, and we can apply the induction hypothesis on N_p. This proves the claim.

Hence, using the 2-cells arising from the relations (5), we obtain $p \sim h_s \cdots h_1 g_r \cdots g_1$, where, if we set $\gamma_i = \varphi(g_i)$ for all $i \in \{1,...,r\}$ and $\delta_j = \varphi(h_j)$ for all $j \in \{1,...,s\}$, then $\delta_i \in \mathcal{W}_1$ for all $i \in \{1,...,s\}$ and $\gamma_j \in \mathcal{W}_2 \cap Inn(G_\Gamma)$ for all $j \in \{1,...,r\}$. Using (6), we obtain $p \sim g_r \cdots g_1$. Set $\mathcal{Z} = \cap_{v \in \mathcal{V}} st(v)$. It follows from Servatius' Centralizer Theorem (see [Ser]) that the center $Z(G_\Gamma)$ of G_Γ is the special subgroup of G_Γ generated by $\mathcal{Z}$. Let Γ' be the full subgraph of Γ spanned by $\mathcal{V} \setminus \mathcal{Z}$. We have $G_{\Gamma'} \simeq Inn(G_\Gamma)$, where the isomorphism is given by $v \mapsto \omega_v$ (see, for example, [D1], Lemma 5.3). Write $\gamma_i = (L - lk_L(c_i) - c_i^{-1}, c_i)$, where $c_i \in \mathcal{V} \setminus \mathcal{Z} \cup (\mathcal{V} \setminus \mathcal{Z})^{-1}$ ($i \in \{1,...,r\}$). Since $\gamma_r \cdots \gamma_1 = 1$ (in $Inn(G_\Gamma)$), we have $c_r \cdots c_1 = 1$ (in $G_{\Gamma'}$). Therefore $c_r \cdots c_1$ is a product of conjugates of defining relators of G_Γ. Using the 2-cells corresponding to the relations (1) and (3)(b), we deduce that $p \sim 1$. We conclude that φ is injective, and thus $Conj(G_\Gamma) = \pi_1(K, W)$.

Now, using the 2-cells arising from the relations (5) (with $\alpha = \beta$), we obtain:

$$(\alpha.W, \alpha.W; (\alpha(A), \alpha(a))) = (\alpha.W, W; \alpha^{-1})(W, W; (A, a))(W, \alpha.W; \alpha),$$

and then, using (6),

$$(\alpha.W, \alpha.W; (\alpha(A), \alpha(a))) = (W, W; (A, a)),$$

for all $\alpha \in \mathcal{W}_1$, and $(A, a) \in S$. It then follows that $Conj(G_\Gamma)$ is generated by the $(W, W; (A, a))$, for $(A, a) \in S$. We identify $(W, W; (A, a))$ with (A, a) for all $(A, a) \in S$. Any relation in $Conj(G_\Gamma) = \pi_1(K, W)$ will come from the 2-cells of K. Then we see from (5) that these relations will result from the relations (1)-(4) above. It is easily seen that the relations (1)-(4) above are equivalent to those of R. We have shown that $Conj(G_\Gamma)$ has the presentation $\langle S \mid R \rangle$. $\square$

Remark 3.3.2. We cannot hope for a generalization of the presentation given in the theorem of [Mc3], since, in a general right-angled Artin group, the existence of one-term partial conjugations depends on the existence of domination relations between the vertices of Γ. (A *one-term partial conjugation* is a partial conjugation of the form $c_{x,\{y\}}$ with $x \geq y$.)

Remark 3.3.3. There is a slightly different definition of partial conjugation which requires that a partial conjugation $\gamma_{v,C}$ conjugates a single connected component C of $\Gamma - st(v)$ by v, where v is a vertex of Γ. With this definition, the first relation vanishes. The second relation has to be replaced by:

$$\gamma_{v,C}\gamma_{v,D} = \gamma_{v,D}\gamma_{v,C} \text{ if } C \cap D = \emptyset.$$

But then the fourth relation becomes less simple since, with this definition, an inner automorphism is not a partial conjugation but a product of partial conjugations (or their inverses).

Appendices

Appendix A

Let G be a group, and let H be a subgroup of G. Recall that the *normal core* of H, denoted by H_G, is defined to be the largest normal subgroup of G that is contained in H, i.e. $H_G = \cap_{g \in G} gHg^{-1}$. The following lemma is probably well-known, though it does not seem to be in the literature. We include a proof for completeness.

Lemma A.1. Let G be a group, and let H be a subgroup of G. Then H is open in the pro-p topology on G if and only if H is subnormal of p-power index.

Proof: If H is open in the pro-p topology on G, then it contains a normal subgroup K of p-power index in G. Thus $[G : H]$ is a power of p. As $\frac{G}{K}$ is a finite p-group, every subgroup of it is subnormal. Therefore H is subnormal in G.

Conversely, if H is a subnormal subgroup of p-power index in G, then $[G : H_G]$ is a power of p (see, for example, [AF2], Lemma 3.3). Thus H contains an open subgroup of G, and hence is open itself. $\square$

Appendix B

Let G be a group. The centralizer condition was originally introduced by Chagas and Zalesskii in terms of centralizers in the profinite completion $\widehat{G}$ of G (see [CZ]). It was then reformulated in terms of centralizers in the finite quotients of G by Minasyan in [Mi]. In Chapter 2, we introduced the p-centralizer condition, which naturally generalizes that of [Mi]. In this Appendix, we show that, if G is residually p-finite, the p-centralizer condition can be reformulated in terms of centralizers in the pro-p completion $G_{\widehat{p}}$ of G.

A *pro-p group* is an inverse limit, $G = \varprojlim_{i \in I} G_i$, of a surjective inverse system $\{G_i, \varphi_{i,j}, I\}$ of finite p-groups, where each G_i $(i \in I)$ is endowed with the discrete topology (see [RZ]). Let G be a group. In what follows, we assume that G is endowed with the pro-p topology. Let $\mathcal{N}$ denote the set of all normal subgroups of p-power index in G. Note that $\mathcal{N}$ is a directed partially ordered set with:

$$\forall\ (M,N) \in \mathcal{N}^2,\ M \prec N \Leftrightarrow M > N.$$

For M, N in $\mathcal{N}$ with $N \succ M$, let $\varphi_{N,M} : \frac{G}{N} \to \frac{G}{M}$ denote the natural epimorphism. Note that $\{\frac{G}{N}, \varphi_{N,M}, \mathcal{N}\}$ is a surjective inverse system. Its inverse limit, $\varprojlim_{N\in\mathcal{N}} \frac{G}{N}$, is called the *pro-p completion* of G and is usually denoted $G_{\widehat{p}}$. Note that $G_{\widehat{p}}$ is a pro-p group. There is a natural embedding of $G_{\widehat{p}}$ into the Cartesian product $\prod_{N\in\mathcal{N}} \frac{G}{N}$, where each $\frac{G}{N}$ ($N \in \mathcal{N}$) is endowed with the discrete topology, and $\prod_{N\in\mathcal{N}} \frac{G}{N}$ is endowed with the product topology. In what follows, we assume that $G_{\widehat{p}} < \prod_{N\in\mathcal{N}} \frac{G}{N}$.

From now on, we assume that G is residually p-finite. For $N \in \mathcal{N}$, let $\varphi_N : G \to \frac{G}{N}$ denote the canonical projection. Let $\varphi : G \to G_{\widehat{p}}$ be the map defined by $\varphi(g) = (\varphi_N(g))_{N\in\mathcal{N}}$ for all $g \in G$. Note that φ is a continuous homomorphism. Since G is residually p-finite, φ is injective. Therefore we can assume that $G < G_{\widehat{p}}$. For $N \in \mathcal{N}$, φ_N can be uniquely extended to a continuous homomorphism $\widehat{\varphi_N} : G_{\widehat{p}} \to \frac{G}{N}$ – $\widehat{\varphi_N}$ can be regarded as the restriction of the canonical projection from $\prod_{M\in\mathcal{N}} \frac{G}{M}$ to $\frac{G}{N}$.

For a subset S of $G_{\widehat{p}}$, we denote by $\overline{S}$ the closure of S in $G_{\widehat{p}}$. The following is the analogue of Proposition 12.1. in [Mi]:

Proposition B.1. Let G be a residually p-finite group, H be a subgroup of G, and $g \in G$. The following are equivalent:

1. The pair (H, g) satisfies the p-centralizer condition in G (pCC_G).

2. $\overline{C_H(g)} = C_{\overline{H}}(g)$.

Proof: Suppose that the pair (H, g) satisfies the p-centralizer condition in G. Consider any $h \in \overline{H}$ and suppose that $h \notin \overline{C_H(g)}$. We have $\overline{C_H(g)} = \varprojlim_{N\in\mathcal{N}} \varphi_N(C_H(g))$ (by [RZ], Corollary 1.1.8(b)). Thus, for $k \in \overline{H}$, we have the equivalence:

$$k \in \overline{C_H(g)} \Leftrightarrow \forall\ N \in \mathcal{N},\ \widehat{\varphi_N}(k) \in \varphi_N(C_H(g)).$$

Since $h \notin \overline{C_H(g)}$, there exists $N \in \mathcal{N}$ such that $\widehat{\varphi_N}(h) \notin \varphi_N(C_H(g))$. Since the pair (H, g) satisfies the p-centralizer condition in G, there exists a normal subgroup M of p-power index in G, such that $M < N$, and $C_{\varphi_M(H)}(\varphi_M(g)) \subset \varphi_M(C_H(g)N)$. Note that $\varphi_{M,N} \circ \widehat{\varphi_M} = \widehat{\varphi_N}$. Therefore if $\widehat{\varphi_M}(h) \in \varphi_M(C_H(g)N)$, then $\widehat{\varphi_N}(h) \in \varphi_{M,N}(\varphi_M(C_H(g)N)) = \varphi_N(C_H(g)N)$ $= \varphi_N(C_H(g))$ – which leads to a contradiction with our assumption. Then $\widehat{\varphi_M}(h) \notin \varphi_M(C_H(g)N)$. Thus $\widehat{\varphi_M}(h) \notin C_{\varphi_M(H)}(\varphi_M(g))$. We deduce that h $\notin C_{\overline{H}}(g)$. We conclude that $C_{\overline{H}}(g) \subset \overline{C_H(g)}$.

Conversely if $h \in \overline{C_H(g)}$, then there exists a net $(h_i)_{i\in I}$ in $C_H(g)$ which converges to h. Hence $h \in C_{\overline{H}}(g)$, and we are done.

Now suppose that $\overline{C_H(g)} = C_{\overline{H}}(g)$. Let $N \in \mathcal{N}$. Set $\mathcal{M} = \{M \in \mathcal{N} \mid M < N\}$. Note that $\mathcal{M}$ is a directed subset of $\mathcal{N}$. Suppose that for every $M \in \mathcal{M}$, there exists $x_M \in H$ such that $\varphi_M(x_M) \in C_{\varphi_M(H)}(\varphi_M(g)) \setminus \varphi_M(C_H(g)N)$. Since $(x_M)_{M\in\mathcal{M}}$ is a net in the compact space $G_{\widehat{p}}$, $(x_M)_{M\in\mathcal{M}}$ has a cluster point h ($\in \overline{H}$).

Let $M \in \mathcal{N}$. Set $L = M \cap N$ ($\in \mathcal{M}$). Since $\widehat{\varphi_L}$ is continuous and $\frac{G}{L}$ is discrete, there exists $K \in \mathcal{M}$ such that $K < L$, and $\varphi_L(x_K) = \widehat{\varphi_L}(h)$. By the assumptions, we have $\varphi_K(x_K) \in C_{\varphi_K(H)}(\varphi_K(g))$, which implies $\varphi_L(x_K) \in C_{\varphi_L(H)}(\varphi_L(g))$ – because $K < L$. Thus $\widehat{\varphi_L}(h) \in C_{\varphi_L(H)}(\varphi_L(g))$, which implies $\widehat{\varphi_M}(h) \in C_{\varphi_M(H)}(\varphi_M(g))$ – because $L < M$. This holds for all $M \in \mathcal{N}$, hence $h \in C_{\overline{H}}(g)$.

Now, we show that $h \notin \overline{C_H(g)}$. As above, there exists $K \in \mathcal{M}$ such that $\varphi_N(x_K) = \widehat{\varphi_N}(h)$. By the assumptions, $\varphi_K(x_K) \notin \varphi_K(C_H(g)N)$. Then $\varphi_N(x_K) \notin \varphi_N(C_H(g))$ – otherwise we would have $x_K \in C_H(g)N$. Thus $\widehat{\varphi_N}(h) \notin \varphi_N(\overline{C_H(g)})$, and therefore $h \notin \overline{C_H(g)}$. Hence we obtain $h \in C_{\overline{H}}(g) \setminus \overline{C_H(g)}$ – a contradiction. Finally there exists $M \in \mathcal{N}$ such that $C_{\varphi_M(H)}(\varphi_M(g)) \subset \varphi_M(C_H(g)N)$, and the pair (H, g) satisfies the p-centralizer condition in G. $\square$

Thus, for residually p-finite groups, the p-centralizer condition can be interpreted as follows:

Corollary B.2. Let G be a residually p-finite group. Then G satisfies the p-centralizer condition if and only if

$$\overline{C_G(g)} = C_{G_{\widehat{p}}}(g)$$

for all $g \in G$.

The following lemma is probably well-known:

Lemma B.3. Let G be a residually p-finite group. Then G is conjugacy p-separable if and only if

$$g^{G_{\widehat{p}}} \cap G = g^G$$

for all $g \in G$.

Proof: Suppose that G is conjugacy p-separable. Let $g \in G$. Consider any $h \in G$ and suppose that $h \notin g^G$. Since G is conjugacy p-separable, there exists $N \in \mathcal{N}$ such that $\varphi_N(g)$ and $\varphi_N(h)$ are not conjugate in $\frac{G}{N}$. Thus g and h are not conjugate in $G_{\widehat{p}}$. Therefore $g^{G_{\widehat{p}}} \cap G \subset g^G$. The reverse inclusion is obvious.

Now suppose that $g^{G_{\widehat{p}}} \cap G = g^G$ for all $g \in G$. Let $g \in G$. Let $h \in G$ such that $h \notin g^G$. By the assumptions, $h \notin g^{G_{\widehat{p}}}$. Suppose that $\varphi_N(h) \in \varphi_N(g^G)$ for all $N \in \mathcal{N}$. Since $\overline{g^G} = \varprojlim_{N \in \mathcal{N}} \varphi_N(g^G)$ (by [RZ], Corollary 1.1.8(b)), we have $h \in \overline{g^G}$. Thus there exists a net $(\alpha_i)_{i \in I}$ in G such that the net $(\alpha_i g \alpha_i^{-1})_{i \in I}$ converges to h. Since $G_{\widehat{p}}$ is compact, $(\alpha_i)_{i \in I}$ has a cluster point α ($\in G_{\widehat{p}}$). We obtain $h = \alpha g \alpha^{-1} \in g^{G_{\widehat{p}}}$ – a contradiction. Finally there exists $N \in \mathcal{N}$ such that $\varphi_N(h) \notin \varphi_N(g^G) = \varphi_N(g)^{\frac{G}{N}}$, and G is conjugacy p-separable. $\square$

In other words, if G is a residually p-finite group, then G is conjugacy p-separable if and only if two elements g and h of G that are conjugate in $G_{\widehat{p}}$ are conjugate in G.

Thus, for residually p-finite groups, hereditary conjugacy p-separability can be reformulated as follows:

Corollary B.4. Let G be a residually p-finite group. Then G is hereditarily conjugacy p-separable if and only if, for all $g \in G$, all of the following hold:

1. $g^{G_{\widehat{p}}} \cap G = g^G$,
2. $\overline{C_G(g)} = C_{G_{\widehat{p}}}(g)$.

Proof: This follows immediately from Corollary B.2, Lemma B.3, and Proposition 2.3.6. $\square$

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