Sur la régularité du flot de Ricci

Chih-Wei Chen

UNIVERSITÉ DE GRENOBLE

國立台灣大學

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JURY

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國立臺灣大學理學院數學所

與

科諾伯勒大學資訊科技與數學學校傅立葉研究所

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Doctoral Dissertation

論瑞奇流的正則性

On the regularity of the Ricci flow

陳志偉

Chih-Wei Chen

指導教授: 李瑩英 博士、杰哈. 貝松 博士

Advisors: Yng-Ing Lee and Gérard Besson

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誌謝

這論文能如期完成是一件神蹟。在十個月前, 絕大部分的成果是不存在的。當時我只證明第四章的最後一節以及附錄中的內容。當我把這部分的結果放上 ArXiv 之後,G. Carron 教授向我表示文章中的一個關鍵假設是本來就成立的, 他也教導我如何證明, 對此我由衷感謝。此後, 我爲了改善這些結果, 開始研究 Ricci 張量的微分, 並得到了第三章所呈現的估計。同時我也和 Thomas 一起解決了先前 Gérard 交給我們的問題 (第二章第一節所造的例子)。事實上這個問題已經困擾了我很久, 我們設想過一些方法, 後來幾乎是把這問題擱著了。直到今年一月赫然發現先前的方法全拼湊起來便可以得出我們想要的結果。爲此要感謝上帝。

整個博士求學階段, 全是靠李瑩英老師細心栽培, 即使最後兩年在法國的日子, 老師也不曾間斷對我的關心。回想剛入學之際, 老師雖認爲我基礎知識薄弱仍願意指導我, 是我今日得以學成畢業最重要的因素。一年級時, 老師鼓勵我和楊凱一起去西雅圖參加夏季學校, 那是我第一次離開台灣。這次經驗讓我更能開放地去嘗試與學習不同的事物。當年正逢林長壽院士開課講解 Ricci 流, 老師也有意讓我們學習這理論, 所以就在王慕道老師與崔茂培老師的教導下學習龐加萊猜想的證明。博三時, 我向老師提及想去法國唸書的事情, 老師的鼓勵與協助總是比我所期待的還多。在老師的推薦下, 我先到中國中山大學短期訪問朱熹平教授與陳兵龍教授, 在那裏學習了關於 Ricci soliton 的基本性質。兩位教授的熱情招待與分享是我研究路上的第一站。緊接著我就到法國, 在 Gérard Besson 的指導下學習。

礙於長久以來台法雙方有限的交流, 我們對彼此的學術圈都不是很熟悉。特別感謝 Gérard 在彼此陌生的情況下仍答應指導我做研究。謝謝 Gérard 常常支持我去參加各種學術活動。Gérard 在各方面都給予我正面的鼓勵, 讓我在遇到挫折時不致於失去信心。在法國的期間, 感謝許多教授們的照顧與幫忙, 諸如 P. Bérard、Z. Djadli、L. Bessières、L. Manivel、M. Simon、S. Gallot 與 C. Mantegazza, 還有許多同學們的陪伴, 諸如 Maxime、彬斌、Alix、Thomas、Marcos Petrucio 與

Vincent。

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感謝爸媽與姐姐對我的關心與付出，他們對我的教育是我得以成長的基礎。感謝我的妻子明慧爲我所做的以及所犧牲的一切，她和家人們的支持是我能夠完成這論文的重要因素。

摘要

本論文包含四章以及附錄。在第一章中, 我們從方程和幾何兩方面來探討瑞奇流的特性, 並介紹瑞奇流的演變與基本思想。我們試圖說明本論文的成果與瑞奇流的正則性之關聯。我們在第二章利用某類旋轉對稱流形來構造一個奇異點發生在無窮遠處的解。此外還利用佩雷曼的結果來討論遠古解的無塌陷性質。在第三章中, 我們證明在滿足比昂奇不等式的流形上, 瑞奇張量滿足施皖雄型態的一階微分估計。第四章將用來討論瑞奇孤立子的性質以及它們的分類問題。我們證明當擴張孤立子的曲率遞減階數超過二次時, 其無窮遠切錐爲歐氏空間。

關鍵詞: 瑞奇流、施皖雄估計、瑞奇孤立子。

Résumé

Cette thèse se compose de quatre chapîtres et une appendice. Le premier chapître est consacre à des idées fondamentales de la théorié du flot de Ricci, qui montre comment nos travaux sont reliés a l'histoire entière. Dans le deuxième chapître, nous construisons une solution du flot de Ricci sur une variété à symetrie de rotation de telle sorte qu'il reste un collecteur complet a l'heure maximale. Nous dérivons également le non-effondrement pour certaines solutions anciennes à proximité de leur temps maximal. Chacun de ces deux resultats sont liés à la régularité des limites des solutions. Dans le troisième chapître, nous montrons qu'une estimation de type Shi d'ordre un est valable pour tenseur de Ricci sur des variétés qui satisfont l'inégalité Bianchi faibles. Le dernier chapître s'interesse aux gradient solitons de Ricci qui sont en expansion. Nous discutons du problème de classification et montrons que chaque cône tangent à l'infini d'un soliton expansion à "fast-than-quadratic-decay" courbure doit être $\mathbb{R}^n$.

Mots clés: flot de Ricci, estimation de Shi, soliton de Ricci

Abstract

This thesis consists of four chapters and an appendix. The first chapter is dedicated to the fundamental ideas of the theory of Ricci flow, which shows how our works are connected to the whole story. In the second chapter, we construct a solution of Ricci flow on a rotationally symmetric manifold such that it remains a complete manifold at the maximal time. We also derive a noncollapsing property for certain ancient solutions near their maximal times. Both of these two results are related to the regularity of limits of solutions. In the third chapter, we show that a first order Shi-type estimate holds for Ricci tensor on manifolds which satisfy the weak Bianchi inequality. The last chapter is concerned with expanding gradient Ricci solitons. There we discuss the classification problem and show that every tangent cone at infinity of an expanding soliton with fast-than-quadratic-decay curvature must be $\mathbb{R}^n$.

Key words: Ricci flow, Shi's estimate, Ricci soliton.

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Chapter 1

Introduction

In this chapter, we give a short introduction to the theory of Ricci flow which shows how our works, being stated in the other chapters, are related to the whole theory. Although the Ricci flow has been well known by its great success on the geometrization problem, we do not sketch it around this theme. Instead, we provide here a more basic description concerning its equational and geometrical natures. This treatment reveals that there are many basic problems still unsolved, even not discussed. We hope that fruitful results would be discovered in the way towards these problems.

1 Ricci flow: from the analytic point of view

For a given manifold M and an initial metric g on M, the Ricci flow is an evolution of the metric defined by the following equation:

$$\frac{\partial}{\partial t} g = -2Ric(g),$$

where $Ric(g)$ is the Ricci curvature tensor of g.

Under different coordinates, the Ricci tensor can be expressed in terms of g in different forms. However, it is essentially a Laplacian of g. For example, when using the harmonic coordinate, which simplifies the Laplace-Beltrami operator Δ to be $g^{ij}\partial_i\partial_j$ on a neighborhood, the Ricci flow equation becomes

$$\frac{\partial}{\partial t} g = -2Ric(g) = \Delta g + Q(g, \partial g),$$

where Q is some quadratic function involving the determinant of g. This equation is a nonlinear parabolic equation, whose short-time existence was derived by R. S. Hamilton in [32], D. DeTurck in [26] and W.-X. Shi in [58]. In the viewpoint of differential equation, one can see that there exists a competition between the reaction and diffusion terms in the right hand side. This shows that in general we can not expect the flow to exist for all time unless the reaction term is trifling. Indeed, as Hamilton showed in [32], the reaction term in the evolution equation of the scalar curvature

$$\frac{\partial}{\partial t}R = \Delta R + 2|Ric|^2 \geq \Delta R + \frac{2}{n}R^2$$

forces the flow to become singular in some finite time provided that the initial compact manifold has positive scalar curvature. This argument, which depends on the classical maximum principle, is true for the compact case and is false on non-compact manifolds whose infimum of scalar curvature is realized in any finite region. For example, there exist several expanding Ricci solitons, which exist for all time, with decaying positive curvature operators.

Since the Ricci flow is an evolution equation of the metric *tensor*, the classical maximum principle for functions is not sufficient for many situations. In [33], Hamilton derived a maximum principle for tensors which is confirmed to be a crucial tool in the proof of Poincaré's conjecture and the differential sphere theorem. In the 90's, by using this, Hamilton [36, 37] proved that the least eigenvalue, if is negative and is bounded, of the Ricci tensor of a three-manifold has smaller and smaller measure comparing to the largest one along the Ricci flow. This phenomenon was also observed independently by T. Ivey [39] and is called Hamilton-Ivey pinching estimate. Dozen years later, B.-L. Chen [17] generalized this pinching estimate into a local version and got rid of the assumption of bounded curvature. When using this generalized estimate, we prefer to cite it as Hamilton-Ivey-Chen pinching theorem. On the other hand, using Hamilton's maximum principle and his idea of pinching set, C. Böhm and B. Wilking [4] proved that if the curvature operator is positive on a compact manifold M with dimension $n \geq 4$, then it converges to a constant along the normalized Ricci flow. The powerful method they used in the proof was adapted soon by S. Brendle and R. Schoen [5, 6] to achieve a proof of the sphere theorem. From these results, we have had a somewhat clear picture for the change of curvatures along the flow, although it is still far away from full understanding.

A natural further step is to study the derivative of curvatures. In [32], Hamilton considered the Ricci flow on compact three-dimensional manifolds with $Ric > 0$ and proved that for any $\alpha > 0$, there exists a constant C_α such that

$$|\nabla R| \le \alpha R^{\frac{3}{2}} + C_\alpha. \tag{1}$$

For the derivatives of curvature operator, one has Shi's estimate which was derived by W.-X. Shi in [58]. This estimate roughly says that, along the Ricci flow, all derivatives of the curvature operator are bounded provided that the curvature operator is bounded. The local version of Shi's estimate says that for any solution of Ricci flow, if $|Rm|$ is bounded in a parabolic region, then for any $m \ge 1$ we have bounds for $|\nabla^m Rm|$ in a slightly smaller region. In particular, such an a priori bound does not depend on the metric directly although it depends on the bound of $|Rm|$.

However, for the Ricci tensor, there is no result analogous to Hamilton's estimate for scalar curvature (1) or Shi's estimate for curvature operator. The crucial difficulty is that the evolution equation of Ricci tensor

$$\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} + 2R_{ikjl}R_{kl}$$

involves the full curvature operator. In Chapter 3 of this thesis, we show that if a Hamilton-type estimate holds for Ricci curvature, then a Shi-type estimate holds too. Indeed, we show that there exist constants θ_0 and $C = C_{n,\alpha,\beta}$ such that for any solution of Ricci flow, if $|Ric| \le K$ and

$$|\nabla Ric| \le \alpha K \sqrt{\frac{1}{r^2} + \frac{1}{t} + K} + \beta |\nabla R| \tag{2}$$

in a parabolic region $B_0(x, r) \times [0, t_0]$ where $r \le \sqrt{\frac{\theta_0}{K}}$ and $t_0 \le \frac{\theta_0}{K}$, then we have $|\nabla Ric| \le CK\sqrt{\frac{1}{r^2} + \frac{1}{t} + K}$ in $B_0\left(x, \frac{r}{\sqrt{2}}\right) \times [0, t_0]$. The condition (2) is named as *weak Bianchi inequality*. Although it looks like Hamilton's estimate (1) which holds for compact three-dimensional solutions with $Ric > 0$, we have not yet found out any solution on which the weak Bianchi inequality holds automatically.

The derivative of Ricci curvature is related to the curvature operator in many different ways. First, from the second Bianchi identity, we have $\nabla_k R_{ijkl} = \nabla_i R_{jl} - \nabla_j R_{il}$. It means that a bound of ∇Ric is also a bound of $div(Rm)$. Secondly, by the elliptic regularity and the equality $\Delta g = -Q(g, \partial g) - 2Ric(g)$ which holds on a local chart of harmonic coordinates, a bound of ∇Ric ensures that g has a $C^{2,\delta}$-bound. Hence we can derive a bound for Rm which depends on the bound of ∇Ric and the lower bound of injectivity radius. More detailed discussions about this can be found in Section 4, Chapter 3. Thirdly, we will see in the last chapter that radial sectional curvatures are dominated by ∇Ric on expanding solitons with fast-decay Ricci curvature.

The second aspect is concerned with the tension between the infinite-speed propagation and the locality. In classical p.d.e. theory, a parabolic equation tends to disperse its *heat* with infinite speed and hence loses its locality. In his monumental papers [48] and [49], however, G. Perelman proved that the Ricci flow possesses a kind of pseudo-locality. This is not saying that there exists no infinite-speed propagation in the Ricci flow. Instead, the Ricci flow sometimes do make the unbounded flat part to be curved instantly. We should say that it is not expectable in general that the locality or the infinite-speed propagation dominates the game. For example, let us imagine a non-compact ancient solution with at least one end which looks like an Euclidean cone. By using Perelman's noncollapsing theorem, we can prove that there does not exist any collapsing phenomenon outside a thick Euclidean end. (Detailed argument can be found in Chapter 2.) It can be seen that the thick end gives its *heat* to the other part of the manifold. This example reveals the complexity of the problem of pseudo-locality. (We remind the reader that Perelman has proved that this could not happen for ancient solutions whose curvature operator is nonnegative, because they must have zero asymptotic volume ratio. Moreover, by Hamilton-Ivey-Chen estimate, such example could exist only when $n \geq 4$.)

One may study the Ricci flow equation in the third aspect, that is, from the viewpoint of variational method. It is well-known that the Ricci flow is related to the variation of the integral of scalar curvature (cf. [32]). However, we have not known what significant geometric quantity is reduced along the flow except in the two-dimensional case, until Perelman proposed his two functionals for compact manifolds:

$$\mathcal{F}(g(t), f(t)) = \int (R + |\nabla f|^2) e^{-f} dvol$$

and

$$\mathcal{W}(g(t), f(t), t) = \int [-t(R + |\nabla f|^2) + f - n](4\pi t)^{-n/2} e^{-f} dvol,$$

where f is a potential function which satisfies a backward parabolic equation on M. He proved that both $\mathcal{F}(g(t), f(t))$ and $\mathcal{W}(g(t), f(t), t)$ are nondecreasing along the Ricci flow and their critical points are gradient Ricci solitons. With another nondecreasing quantity discovered by him, which is called reduced volume and not easy to be defined in one sentence, Perelman succeeded in proving the noncollapsing property of the Ricci flow. Perelman's method gives a new approach to discover more possible energy functionals which depends not only on the metric, but also on a potential function. He actually opened a mystic chapter for differential equations on the Ricci flow.

2 Ricci flow: from the geometric point of view

As we have seen in the first section, Ricci flow is a dialectical process between the reaction and the diffusion. These two corps contribute to the singular and regular phenomenons occurring along the flow respectively. We may ask several questions: when does the flow encounter singularities? how do them look like? and, how regular is a solution outside its singular portion? The simplest singularity that one could imagine is a pinching neck, as shown in Figure 1. This was constructed explicitly by S. Angenent and D. Knopf [2].

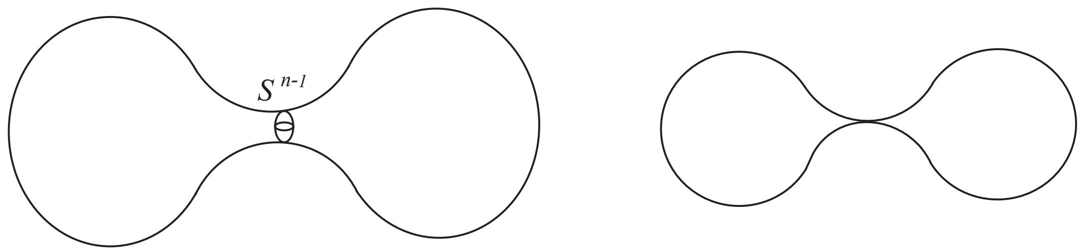

Figure 1: A pinching neck of the Ricci flow on S^n.

See also [59] for a construction of necks on complete non-compact manifolds by M. Simon. The following picture shows another singularity with different blow-up rate of its curvature, which was described by Hamilton and constructed explicitly by H.-L. Gu and X.-P. Zhu [31].

Since R. Schoen applied the blow-up method to study the Yamabe problem, mathematicians have known that certain geometric structure may hide inside the singularities. After

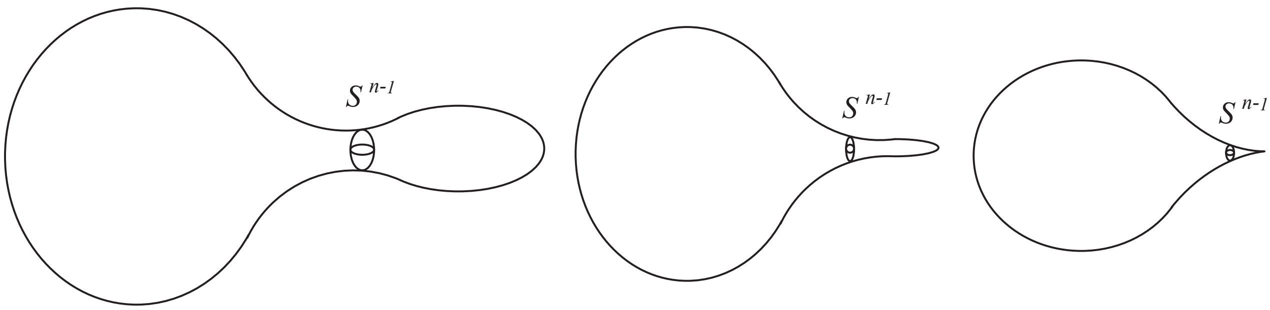

Figure 2: A type-II pinch of the Ricci flow on S^n.

renormalizing the solution along a sequence of space-time points which converges to a singular point, we can discover the *asymptotic shape* near the singular point. For example, the blow-up limit of a pinching neck as shown in Figure 1 is a cylinder. In order to realize this procedure, we need to consider the convergence of pieces of solutions. For compact solutions, a smooth (ancient) blow-up limit at a singularity exists provided that the curvature is under controlled. This is due to Shi's estimate (which controls the higher-order derivatives of curvature), Hamilton's compactness theorem (which is a local, space-time version of the classical convergence theorem for manifolds) and Perelman's noncollapsing theorem (which gives the injectivity radius estimate). These blow-up limits have been divided into three types by Hamilton [36] and each type can be modeled by certain class of solutions. Several subclasses have been observed to contain only Ricci solitons, due to Hamilton [34], H.-D. Cao [8], B.-L. Chen and X.-P. Zhu [18], N. Sesum [56] and A. Naber [45]. (For more details, we suggest the reader to consult a survey paper of H.-D. Cao [9].) Therefore, the classification of Ricci solitons are important in the singularity analysis. In Chapter 4, we will discuss the classification problem of expanding solitons and prove that expanding solitons with $\lim_{dist(O,x)\to\infty} |Sect| \cdot dist(O,x)^2 = 0$ must have $\mathbb{R}^n$ as its tangent cone at infinity. (Here we assume that the soliton is simply connected at infinity, has only one end and $n \geq 3$.)

For compact three-dimensional case, The study of singular/regular behavior is quite complete thanks to Hamilton's study on nonsingular solutions, Perelman's canonical neighborhood theorem and the thick-thin decomposition of surgical solutions. However, for complete non-compact solutions (even for the three-dimensional case), because the noncollapsing property is not as good as in the compact case, we do not have a clear description for the singular (or regular) part of solutions of the Ricci flow. For instance, one can imagine that one end of a horn-like manifold pinches along the Ricci flow at some finite time. In this case, the pinching part might be a bounded region, a half-line or a point at infinity, etc. In Chapter 2, we demonstrate an example constructed by T. Richard and the author in [22] which shows that the last case could happen in reality. That is, there exists a solution of the Ricci flow which converges to a complete manifold at its maximal time of existence.

On the other hand, there are some results about the behavior of curvatures when t approaches the maximal time of existence T. Hamilton [32] proved that in general cases Rm must blow up when $t \to T$. In other words, if the manifolds $(M, g(t))$ have uniformly bounded curvature for all t near T, then the flow can be deformed even more, passing the time T. N. Sesum [57] and L. Ma and L. Cheng [43] proved that if the manifold $(M, g(t))$ has uniformly bounded Ricci curvature for all t near T, then the curvature operator is also uniformly bounded. In many particular cases, for examples, the Kähler-Ricci flow or singularities of Type I, as shown by Z. Zhang [62], J. Enders, R. Müller and P. M. Topping [27] and N. Q. Le and N. Sesum [40], the boundedness of scalar curvature is enough to contribute a bound to the curvature operator. In the results of Sesum and Ma-Cheng, one can see that a (local) bound of Ricci curvature gives a (local) bound of curvature operator. However, the later bound depends on the solution itself. By using our Shi-type estimate mentioned in the previous section, we can derive an a priori bound of curvature operator which depends on the bound of Ricci curvature, the lower bound of injectivity radius and the coefficients α and β in the weak Bianchi inequality. (We conjecture that the coefficients in the weak Bianchi inequality only depend on the initial manifold.)

Now we talk about the global regularity of the Ricci flow. For a compact three-dimensional manifold with positive Ricci curvature, Hamilton [32] proved that the derivative of Ricci curvature in any order must converge to zero along the normalized Ricci flow. Later, C. Böhm and B. Wilking [4] used Hamilton's maximum principle for tensors to prove that both the traceless and Weyl parts of a curvature operator would be pinched along the normalized Ricci flow, provided that the initial compact manifold has 2-positive curvature operator. Their results show that Ricci flow tends to homogenize a given manifold, at least for positive curvature cases. In view of this, the weak Bianchi inequality might hold for certain initial manifolds because the traceless part of ∇Ric is probably controlled, or even pinched, by the traced part of ∇Ric, i.e. ∇R. If this is the case, then our Shi-type estimate can be used to study the regularity in a local portion.

For a complete non-compact manifold, we would like to understand the asymptotic behavior of it. In [36], Hamilton showed that the condition $Rm \to 0$ is preserved along the Ricci flow. In [44], L. Ma and X. Dai proved that a faster-than-quadratic-decay curvature is also preserved. In Chapter 4, we proved that if a non-flat expanding soliton M satisfies $\lim_{dist(O,x)\to\infty} |Sect| \cdot dist(O,x)^2 = 0$ and is simply connected at infinity, then each tangent cone at infinity of M is the Euclidean space $\mathbb{R}^n$. (Here we assume that the soliton has only one end and has dimension $n \geq 3$.) It is still unknown that such expanding soliton must be flat or not.

Chapter 2

Regularity of the limit of solution at the maximal time

Let $(M, g(t))$ be an n-dimensional solution of the Ricci flow which exists up to the maximal time T. In this chapter, we try to discover more regularity of the limit space (M, d_T). In Section 2, we demonstrate an example constructed by T. Richard and the author in [22], which shows that (M, d_T) could be a complete manifold. In Section 3, assuming that $(M, g(t))$ is an ancient solution with an ancient positively curved thick end, we prove that (M, d_T) is κ-noncollapsed in the limiting sense. That is, for all t near T, the manifolds $(M, g(t))$'s are κ-noncollapsed for the same $\kappa > 0$.

1 An example of regular limit

Definition 1. A *warped-product manifold* (M, g) is a topological manifold $M = \mathbb{R} \times N$ equipped with a warped-product metric $g \equiv g_{v,\varphi}(x, q) \equiv v^2(x)dx^2 + \varphi^2(x)\gamma(q)$ where (N, γ) is an $n-1$-dimensional manifold and $v, \varphi : \mathbb{R} \to \mathbb{R}$ are two positive functions. When (N, γ) is an Einstein manifold which satisfies that $Ric = k\gamma$ for some $k > 0$, we denote $g \equiv g_{v,\varphi,k}$.

A warped-product manifold is called *rotationally symmetric* if (N, γ) is the standard sphere $\mathbb{S}^{n-1}$ with constant sectional curvature 1. Because of the computable property which comes from the $SO(n)$-symmetry, it has been well studied for a long time. We recall here some formulae.

Proposition 1. *Let* $(M, g_{v,\varphi,n-2} = v^2dx^2 + \varphi^2\gamma = ds^2 + \varphi^2 g_{\mathbb{S}^{n-1}})$ *be an* n*-dimensional rotationally symmetric manifold. Then the radial(horizontal) and the spherical(vertical) sectional curvatures are*

$$K_0 = -\frac{\varphi_{ss}}{\varphi} \text{ and } K_1 = \frac{1-\varphi_s^2}{\varphi^2},$$

respectively. Moreover, the Ricci curvature and the scalar curvature are given by

$$Ric = (n-1)K_0 ds^2 + (K_0 + (n-2)K_1)\varphi^2 g_{\mathbb{S}^{n-1}}$$

and

$$R = (n-1)K_0 + (n-1)(K_0 + (n-2)K_1).$$

The Ricci flow on a warped-product manifold was used by M. Simon [59] to construct a neck-pinch singularity. The following class of manifolds, although not exactly the same, was introduced by Simon.

Definition 2. (Simon's class of warped-product manifolds) We say that an n-dimensional manifold (M, g) is *of Simon's class* $S(N, \gamma)$ if there exist two positive functions $v(x)$ and $\varphi(x)$ and an n-dimensional Einstein manifold (N, γ) such that
1. $g = g_{v,\varphi,k}(x, q) \equiv v^2(x)dx^2 + \varphi^2(x)\gamma(q)$ and (N, γ) satisfies that $Ric = k\gamma$ for some $k > 0$,
2. $Ric(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) \leq 0$, $Ric(V_q, V_q) \geq 0$ for all $V_q \in T_qN$ and for all $q \in N$, and
3. $\inf_{\mathbb{R}} \varphi(x) > 0$, $\inf_{\mathbb{R}} v(x) > 0$, $\sup_{\mathbb{R}} v(x) < \infty$ and $\sup_{\mathbb{R}} \left(\left|\frac{\partial^j}{\partial x^j} v(x)\right| + \left|\frac{\partial^j}{\partial x^j} \log \varphi(x)\right| \right) < \infty$ for all $j = 1, 2, \cdots$.

The following lemma is one of Simon's theorems which concerns about our usage in this section.

Lemma 1. (M. Simon, [59]) *Taking a manifold* $(M, g_{v,\varphi_0,k})$ *of Simon's class* $S(N, \gamma)$ *as an initial manifold, the Ricci flow* $(M, g(t))$ *exists up to a maximal time* $T > 0$ *and the manifold* $(M, g(t))$ *is still of Simon's class* $S(N, \gamma)$ *for each* $t \in [0, T)$*. Moreover, we have*

$$\varphi^2(x, t) \geq \varphi_0^2(x) - 2kt$$

on $[0, T)$*.*

Simon considered a kind of manifolds in Simon's class $S(N,\gamma,a,b,x_0)\subset S(N,\gamma)$, namely, manifolds $(M,g_{v,\varphi_0,k})$ which satisfy that $\varphi_0(x)\to\infty$ as $|x|\to\infty$ and $\varphi_0^2(x)\le a^2\rho^2(x)+b^2$ for some constants $0<a^2<\frac{k}{n-2}$ and $b>0$. He proved that, along the Ricci flow, such a manifold will develop singularities inside a compact region at its maximal time.

Here we consider another kind of manifolds of Simon's class, the cylindrical horns. We show that each of them stops at a finite-time along the Ricci flow, but no singularity occurs at the maximal time T, i.e. $(M,g(T))$ is a complete Riemannian manifold. Now we specify what cylindrical horn signifies.

Definition 3. The *standard k-cylinder* is the metric product manifold $\mathbb{R}\times\mathbb{S}_k^{n-1}$, where $\mathbb{S}_k^{n-1}$ is the $n-1$-sphere of constant sectional curvature $\frac{k}{n-2}$. A *k-cylindrical horn* is a manifold of Simon's class $S(N,\gamma)$ whose generating function $\varphi(x)$ is strictly decreasing and $\varphi(x)\to 1$ (but never achieves) as $x\to\infty$.

It is well-known that, along the Ricci flow, the standard k-cylinder collapses into a straight line when $t\to\frac{1}{2k}$. Indeed, it is also of Simon's class (with $\varphi_0\equiv 1$) and satisfies that $\varphi^2(x,t)=\varphi_0^2(x)-2kt$. Another known example is a capped half cylinder with nonnegative curvature, which was studied by B.-L. Chen and X.-P. Zhu in Appendix A in [19]. They showed that such manifold collapses to a half line at $t=\frac{1}{2k}$. For a k-cylindrical horn, it is natural to guess that it develops a singularity at $t=\frac{1}{2k}$. However, it is not obvious whether a cylindrical horn will collapse or not. Our aim is to show that a k-cylindrical horn will not collapse at $t=\frac{1}{2k}$.

Theorem 1. (C.-W. Chen and T. Richard [22]) *Consider a k-cylindrical horn $(M,g(0))$ and solve the Ricci flow equation on it until the maximal time T. Then $T=\frac{1}{2k}$ and $(M,g(T))$ is a complete smooth manifold.*

Proof. We first prove that the flow cannot exist for a longer time than $\frac{1}{2k}$.

Suppose in opposite that $T=\frac{1}{2k}+2\delta$ for some $\delta>0$. Let $\bar{t}=T-\delta>\frac{1}{2k}$ and consider a pointed sequence $(M,g(\bar{t}),x_i)$ with $x_i\to\infty$. Since $(M,g(t))$ is still in Simon's class, $\inf_{\mathbb{R}}\varphi>0$ implies that the curvature on $(M,g(t)),t\in(0,\bar{t}]$, is uniformly bounded and there exists a small r_0 such that the volumes of the cubes $B^{n-1}_{\frac{r_0}{10}}(x_i)\times[-\frac{r_0}{10},\frac{r_0}{10}]\subset B_{r_0}(x_i)$ have a uniform lower bound. Combine the curvature bound and the volume bound, by an estimate of S.-Y. Cheng, P. Li and S.-T. Yau [23] (or J. Cheeger, M. Gromov and M. Taylor [16], Theorem 4.7), we know that the injectivity radius of x_i are uniformly bounded from below.

By Hamilton's compactness theorem, $(M, g(t), x_i)$ converges to a solution of Ricci flow $(Y, h(t), y), t \in (0, \bar{t}]$ as $i \to \infty$. Especially, for any fixed t, $(M, g(t), x_i)$ converges to $(Y, h(t), y)$ in the Gromov-Hausdorff sense.

Moreover, since $(Y, h(t), y)$ satisfies the same curvature bound as $(M, g(t))$, $(Y, h(t))$ has uniformly bounded curvature on $(0, \bar{t})$. Therefore, we can apply Lemma 14.2 in Hamilton's '82 paper [32] to show that $h(t)$ uniformly converges to a continuous metric h_0 on M as t goes to 0 . This implies that the associated distances $d_{h(t)}$ uniformly converge to d_{h_0} on every compact subset of Y, and in particular, $(Y, h(t), y)$ converges to (Y, h_0, y) as t goes to 0 in the pointed Gromov-Hausdorff sense. Similarly, $(M, g(t), x_i)$ converges to $(M, g(0), x_i)$ as t goes to 0 in the same sense.

On the other hand, since $\varphi_0(x)$ decreases to 1 as $x \to \infty$, it is easy to check that $(M, g(0), x_i)$ converges to the standard k-cylinder (C, g_{cyl}, O) as $x_i \to \infty$ (after translating all x_i onto O).

Now we are ready to show that (Y, h_0, y) is isometric to (C, g_{cyl}, O). In order to do this, it is sufficient to prove that $d_{GH}((B_{h_0}(y, r), h_0), (B_{g_{cyl}}(x, r), g_{cyl})) = 0$ for any fixed $r > 0$.

For any $i \in \mathbb{N}$ and $t > 0$, the triangle inequality tells that the Gromov-Hausdorff distance between $(B_{h_0}(y, r), h_0)$ and $(B_{g_{cyl}}(x, r), g_{cyl})$ is less than or equal to

$$d_{GH}((B_{h_0}(y, r), h_0), (B_{h(t)}(y, r), h(t))) \tag{1}$$
$$+d_{GH}((B_{h(t)}(y, r), h(t)), (B_{g(t)}(x_i, r), g(t))) \tag{2}$$
$$+d_{GH}((B_{g(t)}(x_i, r), g(t)), (B_{g(0)}(x_i, r), g(0))) \tag{3}$$
$$+d_{GH}((B_{g(0)}(x_i, r), g(0)), (B_{g_{cyl}}(x, r), g_{cyl})). \tag{4}$$

Fix $\varepsilon > 0$, if i is big enough, then (2) and (4) are less than ε. Now with this particular i fixed, we can choose $t > 0$ small enough in order to have that (1) and (3) are less than ε.

This shows that $d_{GH}((B_{h_0}(y, r), h_0), (B_{g_{cyl}}(x, r), g_{cyl})) = 0$. So g_{cyl} and h_0 are isometric. By the uniqueness theorem for non-compact Ricci flows with bounded curvature, which was derived by B.-L. Chen and X.-P. Zhu in [20] and S.-Y. Hsu [38], $h(t)$ is isometric to the Ricci flow of g_{cyl} which exists for $t \in [0, \frac{1}{2k})$. This contradicts the fact that $h(t)$ exists for all $t \in [0, \bar{t} = \frac{1}{2k} + \delta)$.

Now we show that $T \geq \frac{1}{2k}$. Because $r_0(x) > 1$ and $r^2(x,t) \geq r_0^2(x) - 2kt$, $r(x,t)$ has a positive lower bound when $t < \frac{1}{2k}$ and is positive everywhere when $t = \frac{1}{2k}$. Therefore, $T \geq \frac{1}{2k}$ and $(M, g(\frac{1}{2k}))$ is a complete manifold. By using W.-X. Shi's estimate in [58], the smoothness follows from the uniform curvature bound on any compact domain when t is close to T.

□

2 Noncollapsing property of certain ancient solutions

In this section, we use two powerful theorems of Perelman (Theorem 8.2 and Corollary 11.6(b) in [48]) to study the noncollapsing property of ancient solutions.

First of all, we have to make clear the following two terminologies.

Definition 4. Let $(M, g(t))$ be a solution of the Ricci flow. We denote the *parabolic (metric) ball with base point* (x, t_0) as

$$\mathcal{P}(x, t_0, r) := \{(y,t) | dist_t(y,x) \leq r, t \in [t_0 - r^2, t_0]\}.$$

On the other hand, the union of topological region $B_{t_0}(x,r)$ in backward time

$$\bigcup_{t \in [t_0 - a, t_0]} B_{t_0}(x,r) \equiv B_{t_0}(x,r) \times [t_0 - a, t_0],$$

for some $a > 0$, is called a *(backward) parabolic region based on* $B_{t_0}(x,r)$.

When Perelman discussed the noncollapsing property, he used the notion *parabolic region*. However, in his pseudo-locality theorem, both the condition and the conclusion were concerned with *parabolic (metric) ball*. In order to combine his results together, we need the following lemma which says that a parabolic region is equivalent to a parabolic metric ball provided that Ric is uniformly bounded. This technical lemma has been observed and used implicitly by many people. For the reader's convenience, we write down the exact statement and give a proof here.

Lemma 2. *For any solution of the Ricci flow, if* $Ric \leq \frac{C}{r^2} g$ *on a parabolic ball* $\mathcal{P}(x, t_0, r)$ *with some constants* r *and* $C > 0$*, then the parabolic region* $B_{t_0}\left(x, \frac{e^{-C}}{2} r\right) \times [t_0 - r^2, t_0]$ *is contained in* $\mathcal{P}(x, t_0, r)$.

On the other hand, if $Ric \geq \frac{-C}{r^2} g$ on a parabolic region $B_{t_0}(x, r) \times [t_0 - r^2, t_0]$ with some constants r and $C > 0$, then the parabolic ball $\mathcal{P}(x, t_0, e^{-C} r)$ is contained in $B_{t_0}(x, r) \times [t_0 - r^2, t_0]$.

Proof. Consider $B_{t_0}\left(x, \frac{e^{-C}}{2} r\right)$ in $B_{t_0}(x, r)$. We want to show that $B_{t_0}\left(x, \frac{e^{-C}}{2} r\right) \times \{t\}$ is contained in $B_t(x, r)$ for all $t \in [t_0 - r^2, t_0]$. Let t^* be the first backward time that $B_{t_0}\left(x, \frac{e^{-C}}{2} r\right) \times \{t\}$ exceeds the metric ball $B_t(x, r)$, i.e.

$$t^* := \sup \left\{ t | \exists y \in B_{t_0}\left(x, \frac{e^{-C}}{2} r\right) \times \{t\} \text{ such that } dist_t(x, y) > r \right\}.$$

It is sufficient to show that $t^* \leq t_0 - r^2$. Suppose on the contrary that $t_0 - t^* < r^2$. Since $Ric \leq \frac{C}{r^2}$ in $B_{t_0}(x, \frac{e^{-C}}{2} r) \times \{t\}$ whenever $t \geq t^*$, we have

$$\frac{d}{dt} dist_t(x, y) = \int_0^{dist_t(x,y)} -Ric(\gamma', \gamma') ds \geq \frac{-C}{r^2} \cdot dist_t(x, y).$$

This inequality is valid for all $y \in B_{t_0}(x, r)$ and for all $t \in [t^*, t_0]$. Integrating both sides on the time interval $[t^*, t_0]$, we get $\log dist_{t_0}(x, y) - \log dist_{t^*}(x, y) \geq \frac{-C}{r^2}(t_0 - t^*)$, i.e.

$$dist_{t^*}(x, y) \leq dist_{t_0}(x, y) e^{\frac{C}{r^2}(t_0 - t^*)} < dist_{t_0}(x, y) e^C \leq \frac{1}{2} r.$$

This contradicts the definition of t^* because the distance function cannot jump from $\frac{1}{2} r$ to r at t^*.

On the other hand, if $Ric \geq \frac{-C}{r^2} g$ in $B_{t_0}(x, r) \times [t_0 - r^2, t_0]$, by similar calculation as above, we know that the backward radius decreases at most exponentially. Precisely, $B_{t_0}(x, r) \times [t_0 - r^2, t_0]$ contains $B_t(x, e^{-C} r)$ for all $t \in [t_0 - r^2, t_0]$. Hence it contains the parabolic ball $\mathcal{P}(x, t_0, e^{-C} r)$.

□

From this lemma, we can say that a parabolic ball is equivalent to a parabolic region provided that Ricci curvature is bounded from both sides. Now we introduce the definition of noncollapsing property which was defined by Perelman [48].

Definition 5. Let $\kappa > 0$ be a constant and $(M, g(t))$ be a complete solution of the Ricci flow. The metric $g(t_0)$ is said to be *κ-noncollapsed on scales less then r_0 at a point $x \in M$* if for any $r < r_0$, $|Rm| \leq \frac{1}{r^2}$ in $B_{t_0}(x, r) \times [t_0 - r^2, t_0]$ implies $Vol_{t_0}(B_{t_0}(x, r)) \geq \kappa r^n$. Moreover, an open subset Ω in M is said to be *κ-noncollapsed on scales less then r_0* if the metric is κ-noncollapsed on scales less then r_0 at every points $x \in \Omega$.

In [48], Perelman proved the following pseudo-locality theorem. This theorem says that, if both the volume ratio of $B_{t_0}(x, r_0)$ and the curvature on $\mathcal{P}(x, t_0, r_0)$ are bounded from below, then we have an upper bound for the curvature near (x, t_0) in $\mathcal{P}(x, t_0, r_0)$.

Proposition 2. (G. Perelman, Corollary 11.6(b) in [48])*For every $w > 0$, there exist positive constants $\tau_0 = \tau_0(w) < 1$ and $B = B(w, \tau_0) < \infty$ with the following property. On a solution of Ricci flow $(M, g(t))$, if $Rm \geq -\frac{1}{r_0^2}$ in $\mathcal{P}(x, t_0, r_0)$ and $Vol_{t_0}(B_{t_0}(x, r_0)) \geq wr_0^n$, then $|Rm| \leq \frac{B}{r_0^2}$ in $\mathcal{P}\left(x, t_0, \frac{\sqrt{\tau_0} r_0}{4}\right)$.*

From the conclusion of this theorem, we have a parabolic ball $\mathcal{P}\left(x, t_0, \frac{\sqrt{\tau_0} r_0}{4}\right)$ with bounded curvature $|Rm| \leq \frac{B}{r_0^2}$. By using the standard volume comparison with the curvature lower bound, the volume ratio $Vol_{t_0}\left(B_{t_0}\left(x, \frac{\sqrt{\tau_0} r_0}{4}\right)\right) / \left(\frac{\sqrt{\tau_0} r_0}{4}\right)^n$ has a lower bound w', which depends only on w. Hence we can use Lemma 2 and Perelman's noncollapsing theorem to prove a noncollapsing result for ancient solutions, namely, Theorem 2. We recall Perelman's noncollapsing theorem as follows.

Proposition 3. (G. Perelman, Theorem 8.2 in [48])*For any given constant $A > 0$, there exists another constant $\kappa = \kappa(A) > 0$ such that: if (M, g) is a solution of the Ricci flow with bounded curvature such that $|Rm| \leq \frac{1}{r_0^2}$ in $B_{t_0}(x, r_0) \times [t_0 - r_0^2, t_0]$ for some r_0 and $Vol_{t_0}(B_{t_0}(x, r_0)) \geq \frac{1}{A} r_0^n$, then $(B_{t_0}(x, Ar_0), g(t_0))$ is κ-noncollapsed on scales less then r_0.*

In order to simplify the statement of our theorem, we introduce the following definition.

Definition 6. Let E be an end of a manifold (M, g) which contains a reference point O. If E contains a sequence of remote balls $B(x_k, r_k), r_k = \frac{1}{2} dist(O, x_k) \to \infty$, which satisfies that $Vol(B(x_k, r_k)) \geq wr_k^n$ for all k and some $w > 0$, then E is called a *thick end (of scale w)*. Let $(M, g(t))$ be an ancient solution of the Ricci flow defined on $t \in (-\infty, 0]$ and $(M, g(0))$ contains a thick end E. As a subset of the topological manifold M, $E(t)$ is well-defined for all $t \in (-\infty, 0]$. If there exists $C > 0$ such that $Rm \geq -\frac{C}{r_k^2}$ in each parabolic region based on the remote balls, i.e. $B_0(x_k, r_k) \times [-r_k^2, 0]$, then we call $E(t)$ an *ancient thick end (of scale (C, w))*.

Remark 1. For any manifold M with $Ric \geq 0$, the asymptotic volume ratio is positive if and only if there exists a thick end.

The advantage of using the notion of sequence of remote balls, instead of the asymptotic volume ratio, is that we can study multi-ended solutions. Indeed, a solution with positive asymptotic volume ratio might contain some thick ends and some non-thick ends. Our theorem below tells that an ancient thick end forces the other (non-thick) ends to be noncollapsed.

Theorem 2. *Let $(M, g(t))$ be an ancient solution of the Ricci flow defined on $t \in (-\infty, 0]$ with bounded curvature. If it contains an ancient thick end of scale (C, w), then $(M, g(0))$ is κ-noncollapsed on all scales, where $\kappa > 0$ depends only on w, C and the dimension n.*

Proof. Let $B_0(x_k, r_k), r_k = \frac{1}{2}dist_0(O, x_k) \to \infty$, be the sequence of remote balls which satisfies that $Rm \geq -\frac{C}{r_k^2}$ in $B_0(x_k, r_k) \times [-r_k^2, 0]$ for all $t \in (-\infty, 0]$ and $Vol_0(B_0(x_k, r_k)) \geq wr_k^2$ for all k. By Lemma 2, the lower bound of curvature shows that each $B_0(x_k, r_k) \times [-r_k^2, 0]$ contains a smaller parabolic metric ball $\mathcal{P}(x_k, 0, e^{-C}r_k)$. By Perelman's pseudo-locality theorem, there exist $\mathcal{P}\left(x_k, 0, \frac{\sqrt{\tau_0}e^{-C}r_k}{4}\right) \subset \mathcal{P}(x_k, 0, e^{-C}r_k)$ such that $|Rm| \leq \frac{B}{\left(\frac{\sqrt{\tau_0}e^{-C}r_k}{4}\right)^2}$ in $\mathcal{P}\left(x_k, 0, \frac{\sqrt{\tau_0}e^{-C}r_k}{4}\right)$ and $Vol_{t_0}\left(B_0\left(x_k, \frac{\sqrt{\tau_0}e^{-C}r_k}{4}\right)\right) \geq w'\left(\frac{\sqrt{\tau_0}e^{-C}r_k}{4}\right)^n$, where the constants B and w' depend only on w, C and the dimension n. By Lemma 2 and the upper bound of curvature, we can find a smaller parabolic region $B_{t_0}\left(x_k, \frac{\sqrt{\tau_0}e^{-B-C}r_k}{8}\right) \times \left[t_0 - \left(\frac{\sqrt{\tau_0}e^{-B-C}r_k}{8}\right)^2, t_0\right]$ which is contained in $\mathcal{P}\left(x_k, 0, \frac{\sqrt{\tau_0}e^{-C}r_k}{4}\right)$. By using Perelman's noncollapsing theorem and taking a large A such that $\frac{1}{A} \leq w'$ and $A\frac{\sqrt{\tau_0}e^{-B-C}}{8} > 3$, we know that $B_0(x, 3r_k)$ is κ-noncollapsed on scales less then r_0. Since $r_k = \frac{1}{2}dist_0(O, x_k)$ and $r_k \to \infty$, $B_0(x, 3r_k)$ is an exhaustion family of the manifold M. Therefore, M is κ-noncollapsed. □

Remark 2. In [48], Perelman have proved that any non-flat ancient solution with bounded nonnegative curvature must have zero asymptotic volume ratio. Therefore, such solution cannot have an ancient thick end. For ancient solutions with $Ric \geq 0$, we do not know that it contains an ancient thick end or not.

The following corollary is an easy consequence of Theorem 2.

Corollary 1. *Let $(M, g(t))$ be an ancient solution of the Ricci flow defined on $t \in (-\infty, 0]$ with bounded curvature and E be a thick end, i.e. it contains a sequence of remote balls with* *bounded volume ratio. If* $Rm \geq 0$ *on* $E(t)$ *for all* $t \in (-\infty, 0]$, *then* $E(t)$ *is an ancient thick end. In particular, by Theorem 2,* $(M, g(0))$ *is* κ*-noncollapsed for some* $\kappa > 0$.

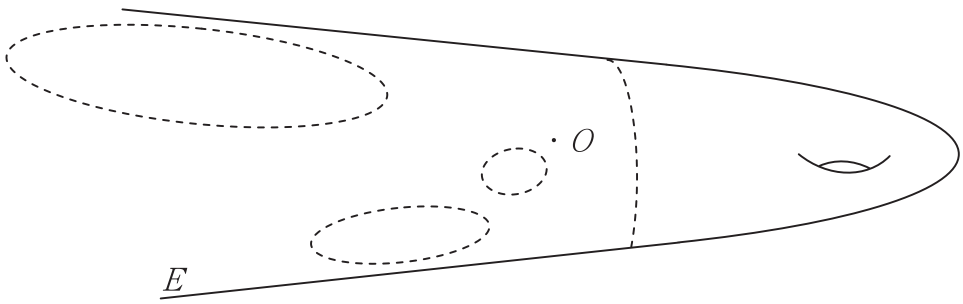

Figure 1: A manifold with one thick end.

For a compact solution, Perelman proved that every singularity is not locally collapsed. Our theorem provides a sufficient condition for attesting the noncollapsing property of complete non-compact ancient solutions. Intuitively speaking, an ancient thick end spreads its "heat" to the other part of the manifold. In this case, although the metric could become degenerate somewhere, it cannot be collapsed in the sense of Perelman.

Chapter 3

Derivative estimates of Ricci curvature

1 Introduction

For every Riemannian manifold, there always holds the second Bianchi identity

$$\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0.$$

Taking trace by g^{km} and g^{jl} successively, we get

$$\nabla_i R_{jl} - \nabla_j R_{il} + \nabla_k R_{ijlk} = 0$$

and the so-called traced second Bianchi identity

$$\nabla_i R = 2\nabla_j R_{ij}.$$

In view of this, we say a manifold satisfies the *strong Bianchi inequality* if it satisfies the pointwise norm estimate $|\nabla Ric| \leq \beta|\nabla R|$ for some $\beta > 0$. In this thesis, we consider a weaker condition as follows.

Definition 7. (i) A solution of the Ricci flow $(M, g(t)), t \in [0, T]$, is said to satisfy the *weak Bianchi inequality* if $|Ric| \leq K$ on $M \times [0, T]$ implies that $|\nabla Ric| \leq \alpha K t^{\frac{-1}{2}} + \beta|\nabla R|$ on $M \times [0, T]$ for some constants $\alpha, \beta \in \mathbb{R}_+$.

(ii) Let $(M, g(t)), t \in [0, T]$, be a solution of the Ricci flow and $B_0(p, r)$ be an open geodesic ball of $(M, g(0))$. The parabolic region $B_0(p, r) \times [0, T]$ is said to satisfy the *weak Bianchi* *inequality* if $|Ric| \leq K$ on $B_0(p,r) \times [0,T]$ implies that $|\nabla Ric| \leq \alpha K\sqrt{\frac{1}{r^2}+\frac{1}{t}+K} + \beta|\nabla R|$ on $B_0(p,r) \times [0,T]$ for some constants $\alpha, \beta \in \mathbb{R}_+$.

(When $t = 0$, we define $\frac{1}{t} = \infty$.)

Remark 3. One should notice that each time slice of a parabolic region $B_0(p,r) \times [0,T]$ is a fixed region on the topological manifold M while each time slice of a parabolic metric ball refers to a geodesic balls whose territory varies according to the distance function at that time, i.e. $B_t(p,r) = \{x \in M | dist_t(p,x) < r\}$.

Assuming that the weak Bianchi inequality holds, we can derive a first order Shi-type estimate for the Ricci flow. In [58], W.-X. Shi proved that all derivatives of the Riemannian curvature operator are bounded along the Ricci flow provided that the curvature operator is bounded. This is an important fact which provides the regularity portion in Hamilton's convergence theorem and is called Shi's estimate nowadays. If the boundedness condition of the full curvature operator is replaced by the one of Ricci curvature, then it seems that no Shi-type estimate can hold. However, using Hamilton's approach towards Shi's theorem [36], we can show that if $|Ric| \leq K$ and $|\nabla Ric| \leq \alpha K t^{\frac{-1}{2}} + \beta|\nabla R|$ on M, then $|\nabla Ric|^2 \leq CK^2t^{-1}$. In particular, if the derivatives of Ricci tensor are dominated by the derivatives of scalar curvature, then both of them are uniformly bounded. This first order Shi-type estimate also holds locally. In Section 3, we prove this local version and use it to show that if $|Ric| \leq K$, $inj(p) \geq w$ and $|\nabla Ric| \leq \alpha K\sqrt{\frac{1}{r^2}+\frac{1}{t}+K} + \beta|\nabla R|$ on a parabolic region $B_0(p,r) \times [0,T]$, then $|Rm| \leq K' = K'(K,w,n,\alpha,\beta)$ in a smaller parabolic region (with uniform size). We remind the reader that, as pointed out by Miles Simon to the author, our proof does not work for higher order derivatives of Ric even assuming the weak Bianchi inequality holds for all orders.

Note that the weak Bianchi inequality is quite looser than the strong one, because it allows $|\nabla Ric| \neq 0$ whenever $|\nabla R| = 0$. It has been known that a Riemannian manifold with $\nabla Ric = 0$ is either Einstein or reducible. In the last section, we also show that certain class of rotationally symmetric manifolds do satisfy the weak Bianchi inequality.

2 Global estimate

The following theorem says that a solution which satisfies the weak Bianchi inequality must have bounded derivative of Ricci curvature. For the reader's convenience, we write down the detailed assumption in the statement instead of using the term "weak Bianchi inequality".

Theorem 3. *(Global estimate) There exists a constant $C > 0$, depending only on β and n such that for every n-dimensional solution $(M, g(t))_{t\in[0,T)}$ of the Ricci flow, if the Ricci curvature and its derivative satisfies that $|Ric| \leq K$ and $|\nabla Ric| \leq \alpha K t^{\frac{-1}{2}} + \beta|\nabla R|$ for all $t \in [0, \frac{1}{K}] \subset [0, T)$, where K is a positive constant, then*

$$|\nabla Ric|^2 \leq CK^2 t^{-1}$$

for all $t \in [0, \frac{1}{K}]$.

Remark 4. When $t = 0$, we define $\frac{1}{t}$ to be ∞. Hence the aforementioned inequalities, which are concerned, hold trivially.

Proof. Since $\frac{\partial}{\partial t}R = \Delta R + 2|Ric|^2$, by $|Ric| \leq K$, we have

$$\frac{\partial}{\partial t}R^2 = 2R(\Delta R + 2|Ric|^2) = \Delta R^2 - 2|\nabla R|^2 + 4R \cdot |Ric|^2.$$

Moreover, using $|\nabla Ric| \leq \alpha K t^{\frac{-1}{2}} + \beta|\nabla R|$, we can derive

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla R|^2 &= 2 < \nabla R, \nabla(\Delta R + 2|Ric|^2) > -2Ric(\nabla R, \nabla R) \\
&\leq \Delta|\nabla R|^2 - 2|\nabla^2 R|^2 + 4|Ric| \cdot |\nabla R|^2 + 8|Ric| \cdot |\nabla Ric| \cdot |\nabla R| \\
&\leq \Delta|\nabla R|^2 + K(4 + 8\beta)|\nabla R|^2 + 8\alpha K^2 t^{\frac{-1}{2}} \cdot |\nabla R| \\
&\leq \Delta|\nabla R|^2 + K(4 + 8\beta)|\nabla R|^2 + \alpha^2 K^4 t^{-1} + 16|\nabla R|^2 \\
&= \Delta|\nabla R|^2 + (16 + K(4 + 8\beta))|\nabla R|^2 + \alpha^2 K^4 t^{-1}.
\end{aligned}$$

Let $F = t|\nabla R|^2 + AR^2$, where $A > 4\beta + 11$ is a constant. Since $t \leq 1/K$, one can derive

$$\begin{aligned}
\frac{\partial}{\partial t}F &= |\nabla R|^2 + t\frac{\partial}{\partial t}|\nabla R|^2 + A\frac{\partial}{\partial t}R^2 \\
&\leq \Delta F + (17 + (4 + 8\beta)Kt - 2A)|\nabla R|^2 + \alpha^2 K^3 + 4A|R| \cdot |Ric|^2 \\
&\leq \Delta F + (17 + (4 + 8\beta) - 2A)|\nabla R|^2 + \alpha^2 K^3 + 4A|R| \cdot K^2 \\
&\leq \Delta F + (4An + \alpha^2)K^3.
\end{aligned}$$

By comparing with the o.d.e. $\frac{\partial}{\partial t}\phi = (4An+\alpha^2)K^3$, we have

$$t|\nabla R|^2 \leq F(\cdot,t) \leq F(\cdot,0) + (4An+\alpha^2)K^3 t \leq An^2K^2 + (4An+\alpha^2)K^2.$$

Hence

$$|\nabla R|^2 \leq (An^2+4An+\alpha^2)K^2t^{-1}.$$

Moreover, by using the weak Bianchi inequality again, we have

$$|\nabla Ric|^2 \leq 2\alpha^2K^2t^{-1} + 2\beta^2(An^2+4An+\alpha^2)K^2t^{-1} \leq 2(\alpha^2+\beta^2(An^2+4An+\alpha^2))K^2t^{-1}.$$

□

This estimate gives the following uniqueness theorem. The proof is just combining our estimate and the uniqueness theorem of B.-L. Chen and X.-P. Zhu [20] or S.-Y. Hsu [38]. We omit the proof because it is essentially the same to the one of Corollary 3 in Section 4 of this chapter.

Corollary 2. *Let $(M, g_1(t))$ and $(M, g_2(t))$ be two solutions of the Ricci flow for $t \in [0,T]$ with the same initial manifold $(M, g(0))$. Suppose that both of them have bounded Ricci curvatures and satisfy the weak Bianchi inequality. If the injectivity radius are bounded from below on $(M, g_1(t))$ and $(M, g_2(t))$ for all $t \in [0,T]$, then $g_1 \equiv g_2$ on $M \times [0,T]$.*

3 Local estimate

Let $(M, g(t))$ be an n-dimensional solution of the Ricci flow on $t \in [0,T)$. In this section, we prove that our Shi-type estimate holds locally provided that the weak Bianchi inequality holds. To prove this, we need a cut-off function introduced by Hamilton in [36] as follows. One can find an explicit construction of such cut-off function in the paper of H.-D. Cao and X.-P. Zhu [12], p. 193.

Lemma 3. (R. S. Hamilton, [36]) *Given a geodesic ball $B_0(p,r) \subset (M, g(0))$, there exists a cut-off function φ such that supp$(\varphi) = \overline{B_0(p,r)}$, $\varphi = r$ in $B_0\left(p, \frac{r}{\sqrt{2}}\right)$, $0 \leq \varphi \leq r < Ar$, $|\nabla\varphi| \leq A$ and $|\nabla^2\varphi| \leq \frac{A}{nr}$ for some constant $A > 1$ depending only on n.*

Remark 5. Although the cut-off function can be trivially extended to $B_0(p,r) \times [0,t_0]$ for any $t_0 > 0$, the derivatives of it will change according to the evolution of the Riemannian connection.

Theorem 4. *(Local estimate) There exist positive constants θ_0 and C depending only on α, β and n such that for every solution $(M, g(t))$ of the Ricci flow, if $|Ric| \leq K$ and $|\nabla Ric| \leq \alpha K \left(\frac{1}{r^2} + \frac{1}{t} + K\right)^{\frac{1}{2}} + \beta|\nabla R|$ on $B_0(p, r) \times [0, t_0]$ for some $r \leq \sqrt{\theta_0/K}$ and $t_0 \leq \theta_0/K$, then*

$$|\nabla Ric|^2 \leq CK^2 \left(\frac{1}{r^2} + \frac{1}{t} + K\right)$$

on $B_0\left(p, \frac{r}{\sqrt{2}}\right) \times [0, t_0]$.

Remark 6. When $t = 0$, we define $\frac{1}{r^2} + \frac{1}{t} + K$ to be ∞. Hence the aforementioned inequalities, which are concerned, hold trivially.

Proof. Recall that

$$\frac{\partial}{\partial t} R^2 = \Delta R^2 - 2|\nabla R|^2 + 4R \cdot |Ric|^2$$

and

$$\frac{\partial}{\partial t}|\nabla R|^2 \leq \Delta|\nabla R|^2 - 2|\nabla^2 R|^2 + 4|Ric| \cdot |\nabla R|^2 + 8|Ric| \cdot |\nabla Ric| \cdot |\nabla R|.$$

Denoting $u = \frac{1}{r^2} + \frac{1}{t} + K$, by the assumptions and Yang's inequality, we have

$$\frac{\partial}{\partial t}|\nabla R|^2 \leq \Delta|\nabla R|^2 - 2|\nabla^2 R|^2 + C_1 K|\nabla R|^2 + C_1 K^3 u$$

for some constant $C_1 > 0$.

Let $S = (BK^2 + R^2) \cdot |\nabla R|^2$ where $B > \max\{n^2 + 4nC_1^{-1}, 32n^2\}$ is a constant. We derive

$$\begin{aligned}
\frac{\partial}{\partial t} S &= \frac{\partial}{\partial t} R^2 \cdot |\nabla R|^2 + (BK^2 + R^2)\frac{\partial}{\partial t}|\nabla R|^2 \\
&\leq (\Delta R^2 - 2|\nabla R|^2 + 4R \cdot |Ric|^2) \cdot |\nabla R|^2 \\
&\quad +(BK^2 + R^2)(\Delta|\nabla R|^2 - 2|\nabla^2 R|^2 + C_1 K|\nabla R|^2 + C_1 K^3 u) \\
&\leq \Delta S - 2\nabla R^2 \cdot \nabla|\nabla R|^2 - 2|\nabla R|^4 - 2(B + n^2)K^2|\nabla^2 R|^2 \\
&\quad +(C_1 B + C_1 n^2 + 4n)K^3|\nabla R|^2 + C_1(B + n^2)K^5 u \\
&\leq \Delta S - 2\nabla R^2 \cdot \nabla|\nabla R|^2 - 2|\nabla R|^4 - 2BK^2|\nabla^2 R|^2 \\
&\quad +2CBK^3|\nabla R|^2 + 2C_1 BK^5 u.
\end{aligned}$$

We want to control the bad terms $2\nabla R^2 \cdot \nabla|\nabla R|^2$, whose sign is unknown, and $2C_1 BK^3|\nabla R|^2$, which may not be bounded. Indeed, using the following two inequalities, they can be absorbed by the other two negative terms:

$$\left|2\nabla R^2 \cdot \nabla|\nabla R|^2\right| \leq 8nK|\nabla R|^2 \cdot \left|\nabla^2 R\right| \leq \frac{1}{2}|\nabla R|^4 + 32n^2 K^2 \left|\nabla^2 R\right|^2$$

and

$$2C_1BK^3|\nabla R|^2 \leq \frac{1}{2}|\nabla R|^4 + 2C_1^2B^2K^6 \leq \frac{1}{2}|\nabla R|^4 + \frac{2}{3}C_1^2B^2K^5u.$$

Since $B > 32n^2$, substituting these two inequalities into the evolution equation of S, we get

$$\frac{\partial}{\partial t}S \leq \Delta S - |\nabla R|^4 + C_2B^2K^5u \leq \Delta S - \frac{S^2}{4B^2K^4} + C_2B^2K^5u,$$

for some constant C_2. Consider $F = bSK^{-4}$ with some constant $0 < b \leq \min\left\{\frac{1}{4B^2K^4}, \frac{1}{C_2B^2K^4}\right\}$, one can derive

$$\frac{\partial}{\partial t}F \leq \Delta F - F^2 + u^2.$$

Let φ be the cut-off function as indicated in Lemma 3. By continuity, $|\nabla\varphi|^2 \leq 2A^2$ and $\varphi|\nabla^2\varphi| \leq \frac{2}{n}A^2$ holds on $B_0(p,r) \times [0, \theta_1/K]$ up to some $\theta_1 > 0$. Moreover, we have

Lemma 4. *Let $H = \frac{cA^2}{\varphi^2} + \frac{d}{t} + K$ for some constants $c > 17$ and $d > 3 + 2\theta_1$. Then $\frac{\partial}{\partial t}H > \Delta H - H^2 + u^2$ on $B_0(p,r) \times [0, \theta_1/K]$.*

By using the maximum principle, one can show that $H - F$ cannot vanish at any point in $B_0(p,r) \times [0, \theta_1/K]$. Hence $H - F > 0$ on $B_0(p,r) \times [0, \theta_2/K]$ for some $\theta_2 > \theta_1$. Combining with the following lemma, we can show that θ_1 has a uniform lower bound. i.e. θ_1 must be larger or equal to the uniform constant θ_0 described in the following lemma.

Lemma 5. (R. S. Hamilton, [36]) *There exists a constant θ_0 which depends only on α, β, n and φ such that if $|Ric| \leq K$ and $F \leq H$ on $B_0(p,r) \times [0, \theta/K]$ for some $\theta \leq \theta_0$ and $r \leq \sqrt{\theta/K}$, then $|\nabla\varphi|^2 \leq 2A^2$ and $\varphi|\nabla^2\varphi| \leq \frac{2}{n}A^2$ on $B_0(p,r) \times [0, \theta/K]$.*

Indeed, suppose in the contrary that $\theta_1 < \theta_0$, then this lemma tells us that the estimates of derivatives of φ hold for time beyond θ_1. This contradicts the definition of θ_1.

Therefore, $F < H$ on $B_0(p,r) \times [0, \theta_0/K]$. We conclude that

$$|\nabla R|^2 = \frac{FK^4}{b(BK^2 + R^2)} \leq \frac{K^4}{bBK^2}\left(\frac{(12 + 4\sqrt{n})A^2}{\varphi^2} + \frac{1}{t} + K\right) \leq CK^2\left(\frac{1}{\varphi^2} + \frac{1}{t} + K\right)$$

and

$$|\nabla Ric|^2 \leq \alpha^2K^2u + \beta^2|\nabla R|^2 \leq \alpha^2K^2u + CK^2\left(\frac{1}{\varphi^2} + \frac{1}{t} + K\right) \leq CK^2\left(\frac{1}{r^2} + \frac{1}{t} + K\right)$$

for some C depending only on α, β and n. □

Now we give proofs of Lemma 4 and 5.

Proof of Lemma 4. We show that $-\frac{\partial}{\partial t}H + \Delta H + u^2 < H^2$ by the following calculations. Using $|\nabla\varphi|^2 \le 2A^2$, $\varphi|\nabla^2\varphi| \le \frac{2}{n}A^2$ and $t \le \theta_1/K$.

$$\begin{aligned}
-\frac{\partial}{\partial t}H + \Delta H + u^2 &= \frac{d}{t^2} + cA^2\Delta\left(\frac{1}{\varphi^2}\right) + \left(\frac{1}{r^2} + \frac{1}{t} + K\right)^2 \\
&\le \frac{d}{t^2} + \frac{cA^2}{\varphi^4}(6|\nabla\varphi|^2 - 2\varphi\Delta\varphi) + \left(\frac{1}{r^2} + \frac{1}{t} + \frac{\theta_1}{t}\right)^2 \\
&\le \frac{d}{t^2} + \frac{cA^2}{\varphi^4}(12A^2 + 4A^2) + 2\left(\frac{1}{r^2}\right)^2 + 2\left(\frac{1+\theta_1}{t}\right)^2 \\
&\le \frac{16cA^4}{\varphi^4} + 2\left(\frac{A^2}{\varphi^2}\right)^2 + \frac{2(1+\theta_1)^2 + d}{t^2} \\
&= \frac{(16c+2)A^4}{\varphi^4} + \frac{2(1+\theta_1)^2 + d}{t^2}.
\end{aligned}$$

Choose $c > 17$ and $d > 3 + 2\theta_1$, then we have

$$-\frac{\partial}{\partial t}H + \Delta H + u^2 \le \frac{(16c+2)A^4}{\varphi^4} + \frac{2(1+\theta_1)^2 + d}{t^2} \le \left(\frac{cA^2}{\varphi^2}\right)^2 + \left(\frac{d}{t}\right)^2 \le H^2$$

□

Proof of Lemma 5. By definition, $\nabla\varphi = g^{ij}\varphi_j e_i = \varphi^i e_i$. Thus

$$\frac{\partial}{\partial t}|\nabla\varphi|^2 = \frac{\partial}{\partial t}(g^{ij}\varphi_i\varphi_j) = 2R_{pq}g^{ip}g^{jq}\varphi_i\varphi_j \le 2K|\nabla\varphi|^2$$

whenever $|Ric| \le K$. Therefore, $|\nabla\varphi|^2 \le A^2 e^{2Kt} \le 2A^2$ when $t \le \frac{\theta}{K}$ and $\theta \le \log\sqrt{2}$.

By using Uhlenbeck's orthonormal frame $\{E_a\}$ (cf. [33]), which satisfies $\frac{\partial}{\partial t}E_a^i = g^{ij}R_{jk}E_a^k$, one can derive

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla_a\nabla_b\varphi &= \frac{\partial}{\partial t}E_aE_b\varphi - \frac{\partial}{\partial t}(\Gamma_{ab}^c E_c\varphi) \\
&= R_{bc}\nabla_a\nabla_c\varphi + R_{da}\nabla_d\nabla_b\varphi - (\nabla_a R_{cb} + \nabla_b R_{ac} - \nabla_c R_{ab})E_c\varphi
\end{aligned}$$

Hence

$$\frac{\partial}{\partial t}\varphi|\nabla^2\varphi| = \varphi\frac{\partial}{\partial t}|\nabla^2\varphi| \le C\varphi(|Ric||\nabla^2\varphi| + |\nabla Ric||\nabla\varphi|).$$

By the assumption $F = b(BK^2 + R^2)K^{-4}|\nabla R|^2 \le H = \frac{cA^2}{\varphi^2} + \frac{d}{t} + K$ and the weak Bianchi inequality, we have

$$|\nabla R|^2 \le \frac{K^4}{b(BK^2 + R^2)}\left(\frac{cA^2}{\varphi^2} + \frac{d}{t} + K\right) \le \frac{K^2}{bB}\left(\frac{cA^2}{\varphi^2} + \frac{d+\theta_1}{t}\right)$$

and

$$|\nabla Ric| \leq \alpha K\sqrt{\frac{1}{r^2}+\frac{1}{t}+K}+\beta\frac{K}{\sqrt{bB}}\sqrt{\frac{cA^2}{\varphi^2}+\frac{d+\theta_1}{t}} \leq CK\left(\sqrt{\frac{cA^2}{\varphi^2}+\frac{d+\theta_1}{t}}\right),$$

where C depends on α, β, b and B. (Recall that $\varphi \leq r$ and $A > 1$. Moreover, from Lemma 4 above, $c > 17$ and $d > 3 + 2\theta_1$.) Hence

$$\frac{\partial}{\partial t}\varphi|\nabla^2\varphi| \leq CK\varphi|\nabla^2\varphi| + CK|\nabla\varphi|\sqrt{cA^2+\frac{(d+\theta_1)\varphi^2}{t}} \leq CK\varphi|\nabla^2\varphi| + A + \frac{r}{\sqrt{t}},$$

where C depends on c, α, β, b and B. By comparing with the o.d.e. $\frac{d}{dt}\phi = CK\left(\phi + A + \frac{r}{\sqrt{t}}\right)$, as Hamilton did in [36], one can show that

$$\varphi|\nabla^2\varphi| \leq e^{CKt}(A^2 + CK(At + 2r\sqrt{t}).$$

Therefore, when $r \leq \sqrt{\frac{\theta}{K}}$ and $t \leq \frac{\theta}{K}$ for some very small $\theta = \theta(A, \alpha, \beta, n) \leq \theta_1$, we have $\varphi|\nabla^2\varphi| \leq \frac{2}{n}A^2$.

□

4 Backward estimate and its application

In Theorem 4, we have a forward control on the derivatives of Ricci curvature. One should notice that $B_0(p, r)$, which indicates a topological region on M, is usually no longer a geodesic ball when $t > 0$. That is to say, there is no forward-backward symmetry in this estimate, hence we cannot simply replace $B_0(p, r)$ by $B_{t_0}(p, r)$ in the statement and keep the estimate holding. Moreover, one cannot go through a similar backward argument to gain a backward Shi-type estimate since we do not have a backward maximum principle.

The following theorem and its proof demonstrate that, by analyzing how the geodesic ball $B_{t_0}(p, r)$ deforms in the time interval $[t_0 - \varepsilon, t_0]$, $\varepsilon \leq \frac{\theta_0}{K}$, we can derive a Shi-type estimate of a backward parabolic region $B_{t_0}(p, r) \times [t_0 - \varepsilon, t_0]$.

Theorem 5. *(Local backward estimate) There exist positive constants θ_0 and C_{θ_0} depending only on α, β and n such that for every solution $(M, g(t))_{t\in[0,t_0]}$ of the Ricci flow, if $|Ric| \leq K$ and $|\nabla Ric| \leq \alpha K\left(\frac{1}{r^2}+\frac{1}{t-(t_0-\varepsilon)}+K\right)^{\frac{1}{2}}+\beta|\nabla R|$ on $B_{t_0}(p, r)\times[t_0-\varepsilon, t_0]$ for some $r \leq \sqrt{\theta_0/K}$*

and $\varepsilon \le \theta_0/K$*, then*

$$|\nabla Ric|^2 \le C_{\theta_0} K^2 \left(\frac{1}{r^2} + \frac{1}{t-(t_0-\varepsilon)} + K \right)$$

on $B_{t_0}\left(p, \frac{e^{-2\theta_0}}{\sqrt{2}} r\right) \times [t_0-\varepsilon, t_0]$.

Remark 7. This theorem has a scaling invariant form when $\varepsilon = r^2$.

Proof. Since $B_{t_0}(p,r)$ indicates an open set in M, it does not change its topology when we trace back in time. (Only the metric on it changes.) Since $Ric \ge -Kg$ on $B_{t_0}(p,r)\times[t_0-\varepsilon, t_0]$, for any point $q \in \partial B_{t_0}(p,r)$, the backward derivative of distance between p and q is

$$\frac{\partial}{\partial \tau} dist_t(p,q) = \int_0^{dist_t(p,q)} Ric \;\; ds \ge -K \cdot dist_t(p,q),$$

where $\tau = t_0 - t$. Hence $dist_{t_0-\tau}(p,q) \ge e^{-K\tau} dist_{t_0}(p,q)$ for all $\tau \le \varepsilon$. This shows that $B_{t_0-\tau}\left(p, e^{-K\varepsilon} r\right)$ is contained in $B_{t_0}(p,r) \times \{t_0-\tau\}$ for all $\tau \le \varepsilon$.

Now, using the weak Bianchi inequality, which holds on $B_{t_0-\tau}\left(p, e^{-K\varepsilon}r\right) \times [t_0-\varepsilon, t_0]$, and Theorem 4 with a time shifting, we have a forward estimate based on $B_{t_0-\varepsilon}\left(p, \frac{e^{-\theta_0}}{\sqrt{2}} r\right)$. Indeed,

$$\begin{aligned} |\nabla Ric|^2 &\le CK^2 \left(\frac{1}{(e^{-K\varepsilon} r)^2} + \frac{1}{t-(t_0-\varepsilon)} + K \right) \\ &= CK^2 \left(\frac{e^{2K\varepsilon}}{r^2} + \frac{1}{t-(t_0-\varepsilon)} + K \right) \\ &\le C_{\theta_0} K^2 \left(\frac{1}{r^2} + \frac{1}{t-(t_0-\varepsilon)} + K \right), \end{aligned}$$

because $\varepsilon \le \theta_0/K$.

Similarly, $Ric \le Kg$ and $\varepsilon \le \frac{\theta_0}{K}$ implies that $B_{t_0}\left(p, \frac{e^{-2\theta_0}}{\sqrt{2}} r\right) \times [t_0-\varepsilon, t_0]$ is contained in $B_{t_0-\varepsilon}\left(p, \frac{e^{-\theta_0}}{\sqrt{2}} r\right) \times [t_0-\varepsilon, t_0]$. □

Remark 8. The same proof shows that Shi's estimate (for the curvature operator) is valid for both time directions, which is well-known for the experts of the Ricci flow.

As we mentioned in Chapter 1, the uniform boundedness of Ricci curvature along a Ricci flow ensures that there exists a bound of curvature operator, by the works of N. Sesum [57], L. Ma and L. Cheng [43]. However, the bound of curvature operator in their theorems is not an a priori bound, i.e. it depends on the flow. By using our Shi-type estimate, we can derive a local estimate of curvature operator which only depends on the injectivity radius and the coefficients in the weak Bianchi inequality, instead of the Riemannian metric itself.

Corollary 3. *For any given constants $K, \alpha, \beta, w \leq 1$ and $n \geq 3$, there exists a constant K' which satisfies the following property. Let $(M^n, g(t))$ be a solution of the Ricci flow and $\theta_0 > 0$ be the constant as in Theorem 4. If $inj(p) \geq wr$, $|Ric| \leq K$ and $|\nabla Ric| \leq \alpha K \left(\frac{1}{r^2} + \frac{1}{t-(t_0-\varepsilon)} + K\right)^{\frac{1}{2}} + \beta|\nabla R|$ on $B_{t_0}(p, r) \times [t_0 - \varepsilon, t_0]$ for some $r \leq \sqrt{\theta_0/K}$ and $\varepsilon \leq \theta_0/K$, then there exists a constant $w' = \lambda w$, where $\lambda > 0$ depends on K and the dimension n, such that $|Rm| \leq K'$ on $B_{t_0}(p, \min\{Ar, w'r\})$, where $A = \frac{e^{-2\theta_0}}{\sqrt{2}}$.*

Proof. By the weak Bianchi inequality and our backward Shi-type estimate, we have

$$|\nabla Ric|^2 \leq C_{\theta_0} K^2 \left(\frac{1}{r^2} + \frac{1}{\varepsilon} + K\right) \leq C_{\theta_0} K^2 \left(\frac{K}{\theta_0} + \frac{1}{\varepsilon} + K\right)$$

in $B_{t_0}(p, Ar)$, where $A = \frac{e^{-2\theta_0}}{\sqrt{2}}$. Therefore, for any $\delta < 1$, Ric has a $C^{0,\delta}(B_{t_0}(p, Ar))$-bound which depends on $n, \alpha, \beta, \varepsilon, \theta_0$ and K (by Sobolev embedding theorem).

On the other hand, from the lower bound of injectivity radius and Lemma 2.2 of M. Anderson's paper [1], we know that for any $0 < \delta < 1$ and $C > 1$, there associates a neighborhood $B(p, w'r)$ which admits a harmonic coordinate such that $\|g\|_{C^{1,\delta}} \leq C + \frac{C}{w'^{\delta}}$, where $w' = \lambda w$ and $\lambda > 0$ depends on δ, C, K and the dimension n. Since $g \in C^{1,\delta}$, $Ric \in C^{0,\delta}$ and $\Delta g = -Q(g, \partial g) - 2Ric$ in $B_{t_0}(p, \min\{Ar, w'r\})$, we have $g \in C^{2,\delta}$ in $B_{t_0}(p, \min\{Ar, w'r\})$. Hence, $|Rm| \in C^{0,\delta}$ in $B_{t_0}(p, \min\{Ar, w'r\})$. In particular, for any chosen δ and C, Rm is uniformly bounded by a constant K' which depends on $n, w, \alpha, \beta, \varepsilon$ and K. □

5 Further discussions on the Bianchi inequalities

An important issue is to prove that our Bianchi-type inequalities are preserved under the Ricci flow with the same or smaller constants α and β. We have not yet achieved this goal, however, we suspect that this property holds for most general cases.

In this section, we only discuss in which cases the Bianchi inequalities shall hold on a fixed manifold. For general Riemannian manifolds, the derivative of Ricci tensor can be decomposed as follows.

Proposition 4. (R. S. Hamilton, the proof of Lemma 11.6 in [32]) *Given an n-dimensional manifold (M, g) with $n \geq 3$. Let $E_{ijk} = a(g_{ij}\nabla_k R + g_{ik}\nabla_j R) + bg_{jk}\nabla_i R$ where $a = \frac{n-2}{2n^2+2n-4}$ and $b = \frac{1}{2} - a(n+1)$. Then the decomposition $\nabla_i R_{jk} = E_{ijk} + F_{ijk}$ satisfies that $g^{ij}F_{ijk} = g^{jk}F_{ijk} = g^{ki}F_{ijk} = 0$ and $\langle E_{ijk}, F_{ijk}\rangle = 0$. In particular, we have*

$$|\nabla_i R_{jk}|^2 = |E_{ijk}|^2 + |F_{ijk}|^2$$

and

$$|E_{ijk}|^2 = \left[2(n+1)a^2 + 4ab + nb^2\right] |\nabla R|^2.$$

Remark 9. When $n = 3$, $a = \frac{1}{20}$, $b = \frac{3}{10}$ and $|E_{ijk}|^2 = \frac{7}{20}|\nabla R|^2$; when $n = 4$, $a = \frac{1}{18}$, $b = \frac{2}{9}$ and $|E_{ijk}|^2 = \frac{5}{18}|\nabla R|^2$.

From this proposition, we know that a manifold satisfies a time-independant weak Bianchi inequality if the trace-free part of ∇Ric is bounded, i.e. $|F_{ijk}| \leq C$.

On the other hand, we can compute explicitly on manifolds with rotationally symmetric metrics.

Theorem 6. *Let $(M, g), g = ds^2 + \varphi^2(s)g_{\mathbb{S}^{n-1}}$, be a rotationally symmetric n-dimensional manifold and $n \geq 3$. Denote the radial and spherical sectional curvatures as K_0 and K_1, respectively. Suppose that $\frac{\partial}{\partial s}K_0 \cdot \frac{\partial}{\partial s}K_1 \geq -\frac{C^2}{2(n-1)^2(n-2)}$ for some constant C. Then*

$$|\nabla Ric|^2 \leq C^2 + \frac{1}{2}|\nabla R|^2$$

on M.

Proof. It is well-known that for rotationally symmetric manifolds we have

$$Ric = (n-1)K_0 ds^2 + (K_0 + (n-2)K_1)\varphi^2 g_{\mathbb{S}^{n-1}}$$

and

$$R = (n-1)K_0 + (n-1)(K_0 + (n-2)K_1).$$

Hence

$$
\begin{aligned}
2|\nabla Ric|^2 &= 2(\nabla_1 R_{11})^2 + 2(\nabla_1 R_{jj})^2 \\
&= 2(n-1)^2 \left(\frac{\partial}{\partial s} K_0\right)^2 + 2(n-1)\left(\frac{\partial}{\partial s} K_0 + (n-2)\frac{\partial}{\partial s} K_1\right)^2 \\
&\leq 3(n-1)^2 \left(\frac{\partial}{\partial s} K_0\right)^2 + (n-1)^2\left(\frac{\partial}{\partial s} K_0 + (n-2)\frac{\partial}{\partial s} K_1\right)^2 \\
&\quad +2(n-1)^2(n-2)\left(\frac{\partial}{\partial s} K_0\right)\left(\frac{\partial}{\partial s} K_1\right) + C^2 \\
&= |\nabla R|^2 + 2C^2.
\end{aligned}
$$

□

Remark 10. In this theorem, we do not assume that $|Ric|$ is bounded by some constant. Hence, in particular, (M, g) satisfies the strong Bianchi inequality $|\nabla Ric| \leq \frac{1}{2}|\nabla R|$ whenever $\frac{\partial}{\partial s} K_0 \cdot \frac{\partial}{\partial s} K_1$ is nonnegative.

Chapter 4

Expanding gradient Ricci solitons

The Ricci solitons, which are generalizations of the Einstein manifolds, are important solutions to the Ricci flow. Besides the advantage of having explicit equations, they occur in the analysis of blow-up limits near singularities. Here we concentrate on expanding solitons whose Ricci curvature is not nonnegative. We remind the reader that all the closed expanding solitons are Einstein. So we are only interested in the complete non-compact case.

In this chapter, we introduce some basic properties of expanding solitons such as the estimates of the growth of the potential function f. In Section 2, we describe some recent developments on expanding solitons, including the lower bound estimate of asymptotic volume ratio derived by the author in [21]. We also study the topology of expanding solitons with $\lim_{s\to\infty} s^2 \cdot |Ric| = 0$ or $\lim_{s\to\infty} s^2 \cdot |Sect| = 0$. For the later case, we show further that the local volume ratio has a uniform lower bound. Therefore we derive that any tangent cone at infinity of such soliton is a flat cone. Moreover, for $n \geq 3$, this flat cone must be the Euclidean space $\mathbb{R}^n$. (Here we assume that the soliton has only one end and is simply connected at infinity.)

1 Definition and basic properties

Let us begin by defining the expanding gradient Ricci soliton.

Definition 8. Let (M, g) be a Riemannian manifold without boundary. If there exists a function f and a positive constant λ such that $R_{ij} + \nabla_i \nabla_j f = -\lambda g_{ij}$, then (M, g, f) is called an *expanding gradient Ricci soliton.*

It was known that an expanding soliton (M, g, f) can generate a self-similar expanding solution of the Ricci flow. That is, the evolution of the Riemannian metric is only a pull-back coupled with a rescaling: $g(t) = \rho(t)\varphi_t^* g(0) \equiv \rho(t)\varphi_t^* g$, where $\{\varphi_t\}$ is a one parameter family of diffeomorphisms. Moreover, one can show that $\rho(t) = Ct + 1$ and each time slice of this solution is an expanding soliton with $\lambda = \frac{C}{2Ct+2}$. Therefore, when studying the geometry of an expanding soliton, we can always assume that $\lambda = 1$.

Let (M, g, f) be an expanding gradient Ricci soliton, which satisfies $R_{ij} + \nabla_i \nabla_j f = -g_{ij}$ and R be the scalar curvature of (M, g, f). The following three lemmas are well-known.

Lemma 6. (R. S. Hamilton, [36]) *We have* $R + |\nabla f|^2 + 2f = C_1$ *for some constant* C_1 *which can be absorbed by* f.

Lemma 7. (R. S. Hamilton, [35]) *The time-independent Harnack quantity* $\Delta R - \langle \nabla R, \nabla f \rangle + 2(R + |Ric|^2)$ *vanishes on* (M, g, f).

Lemma 8. (B.-L. Chen, [17]) *We have* $R \geq -C_2$ *for some constant* $C_2 > 0$.

As we can see in Lemma 6, there is a normalization on the function f which is usually used to simplify proofs in many cases. Considering the derivative of f, we have the following a priori relation between ∇f and ∇R.

Theorem. (C.-W. Chen, [21])*If* $\nabla f(p) = 0$ *for some* $p \in M$, *then* $\nabla R(p) = 0$. *On the other hand, if* $\nabla R(p) = 0$ *and* $R(p) < -\frac{n-1+|Ric|^2}{2}$, *then* $\nabla f(p) = 0$.

Proof. If $\nabla f(p) = 0$, then we have $dR = 2Ric(\nabla f, \cdot) = 0$ at p.

On the other hand, suppose $\nabla R(p) = 0$ and $\nabla f(p) \neq 0$, we claim that $R(p) \geq -\frac{n-1+|Ric|^2}{2}$. Since $|\nabla f|$ is locally Lipschitz, by Lemma 6 above, we have

$$-2\nabla f = \nabla |\nabla f|^2 = 2|\nabla f| \cdot \nabla |\nabla f|.$$

By using the assumption $\nabla f(p) \neq 0$ and Kato's inequality $|\nabla |\nabla f|| \leq |Hess f|$, we can divide both sides by $|\nabla f|$ and get

$$|g + Ric|^2 = |Hess f|^2 \geq |\nabla |\nabla f||^2 = \left| \frac{\nabla f}{|\nabla f|} \right|^2 = 1,$$

i.e. $n + 2R + |Ric|^2 \geq 1$. Hence $R \geq -\frac{n-1+|Ric|^2}{2}$ at p. The statement of theorem follows by reduction to the absurd.

□

Remark 11. Similar computation holds for shrinking solitons.

Given a fixed point $O \in M$, we set $s = dist(O, x)$ and $\gamma(s)$ be a unit-speed minimizing geodesic connecting O and x, where $x \in M$ is chosen arbitrarily. We use the notation $'$ to denote the differentiation with respect to s along $\gamma(s)$. The following proposition, which seems to appear first time in the literature in [64], is an easy consequence of Lemmas 6 and 8.

Proposition 5. *For every expanding soliton (M, g, f), we have $|f'(x)| \leq |\nabla f(x)| \leq s + L(O)$, where $L(x) = \sqrt{C_1 + C_2 - 2f(x)} = \sqrt{C_2 + R(x) + |\nabla f(x)|^2}$. Moreover, when $Ric \geq 0$, we have $f'(x) \leq -s + f'(O)$.*

Proof. Since $-C_2 + |\nabla f|^2 + 2f \leq R + |\nabla f|^2 + 2f = C_1$, we have $|\nabla f| \leq \sqrt{C_1 + C_2 - 2f} = L$. Combining with $\nabla L = \frac{-\nabla f}{\sqrt{C_1+C_2-2f}}$, we have $|\nabla L| \leq 1$.

Integrating it from the point O to some point $x = \gamma(s)$ along γ, we have

$$L(x) - L(O) = \int_0^s L' \leq \int_0^s |\nabla L| \leq s.$$

Hence, $|f'(x)| \leq |\nabla f(x)| \leq L(x) \leq s + L(O)$.

When $Ric \geq 0$,

$$\int_0^s f'' \leq \int_0^s Ric(\gamma', \gamma') + \int_0^s f'' = -s$$

implies that $f'(x) \leq -s + f'(O)$. □

From this proposition, it is easy to see that for every expanding gradient Ricci soliton which has nonnegative Ricci curvature, the potential function $f(x)$ must decrease quadratically in s. The following theorem shows that this property holds for expanding solitons whose Ricci curvatures may be negative somewhere.

Theorem 7. (C.-W. Chen, [21]) *If $Ric \geq -Cs^{-\varepsilon} g, s \equiv dist(O, x)$, for some constant $\varepsilon < 1$ and some point $O \in M$, then f grows quadratically. In particular, if $|Ric| \leq Cs^{-\varepsilon}$, then* *there exists a point $p \in M$ and $C_3, C_4 > 0$ such that $|Ric| \le C_3 \cdot dist(p,x)^{-\varepsilon}$ and f satisfies*

$$-r\left(1+\frac{C_4}{r^{\varepsilon}}\right) \le f'(x) \le -r\left(1-\frac{C_4}{r^{\varepsilon}}\right),$$

where $r = dist(p,x)$. As a consequence, we have

$$-\frac{1}{2}r^2\left(1+\frac{C_5}{r^{\varepsilon}}\right) + f(p) \le f(x) \le -\frac{1}{2}r^2\left(1-\frac{C_5}{r^{\varepsilon}}\right) + f(p).$$

Proof. We only give a proof for the second case, i.e. $|Ric| \le Cs^{-\varepsilon}$ because the first one can be worked out by using the same calculation and Proposition 5. From

$$-C\int_0^s s^{-\varepsilon} + \int_0^s f'' \le \int_0^s Ric(\gamma',\gamma') + \int_0^s f'' = \int_0^s -1 \le C\int_0^s s^{-\varepsilon} + \int_0^s f'',$$

we have

$$-s - C\int_0^s s^{-\varepsilon} \le \int_0^s f'' \le -s + C\int_0^s s^{-\varepsilon}$$

and hence

$$-s\left(1+\frac{C_4}{s^{\varepsilon}}\right) + f'(O) \le f'(x) \le -s\left(1-\frac{C_4}{s^{\varepsilon}}\right) + f'(O).$$

In order to achieve the conclusion, it is enough to show that f has a critical point p (and then repeat the calculation above.) This can be observed by considering the geodesic sphere $\partial B_s(O)$ with s very large. Since $\nabla f \cdot \nabla s$ is negative on such sphere, ∇f must point inwards. So $\nabla f = 0$ at some point p inside the ball $B_s(O)$.

□

Remark 12. As long as $|\nabla f|$ grows linearly with leading coefficient 1, it is easy to see that $\left|\frac{|\nabla f|^{-2} - s^{-2}}{s^{-2}}\right| = \left|\frac{s^2}{|\nabla f|^2} - 1\right| \to 0$ as $s \to \infty$, i.e. $\left|\frac{1}{|\nabla f|^2} - \frac{1}{s^2}\right| \in o(s^{-2})$.

2 Recent development of expanding solitons

2.1 Why expanding solitons are important?

Comparing to the other cases, i.e. shrinking and steady solitons, the expanding ones are less studied. There are two obvious reasons: it is less related to the surgery of singularity and its curvatures possess less positivity than the other two solitons. However, we still have enough motivations to study expanding solitons. First, it is a special solution of the Ricci flow. By analyzing the property of expanding soliton, Hamilton found a metric Harnack quantity which vanishes on every expanding soliton and is nonnegative on all solutions [35]. Nowadays, people still try to discover more properties in this direction, for example, E. Cabezas-Rivas and P. Topping [7].

The second motivation is more geometric: expanding solitons can be used to smooth out a manifold with singularities. This idea is quite natural because when tracing back an expanding soliton along the flow, we get a singular initial space. However the results and techniques are not easy to be seen. For the interested reader, we suggest him to consult the results of F. Schulze and M. Simon [54], M. Feldman, T. Ilmanen and D. Knopf [28] and A. Futaki and M.-T. Wang [29]. We remind the reader that such problem was also discussed in the (Lagrangian) mean curvature flow, see for example the works of Y.-I. Lee and M.-T. Wang [41, 42].

The third motivation, as mentioned in the previous section, comes from the equation itself. Since a soliton is a generalization of an Einstein manifold, we would like to know what topological constraint shall be imposed to the manifold by the equation. For example, an expanding soliton (M, g, f) with $Ric \geq 0$ must be diffeomorphic to $\mathbb{R}^n$. Moreover, in [21], the author proved that if $Ric \geq -C \cdot s^{-\varepsilon} g$, then M must have finite fundamental group. We also proved that the ends of an expanding soliton with $\lim_{s\to\infty} s^2 \cdot |Rm| = 0$ are diffeomorphic to $\mathbb{R} \times \mathbb{S}^{n-1}/\Gamma$. When the condition is released to be $\lim_{s\to\infty} s^2 \cdot |Ric| = 0$, we show in the next section that the ends are diffeomorphic to $\mathbb{R} \times N^{n-1}$ by assuming that the strong Bianchi inequality holds, where N^{n-1} is an almost Einstein manifold with positive Ricci curvature. In the proof, we can see that such soliton has fast decay radial sectional curvatures. Moreover, if the injectivity grows linearly, then we can prove that $\lim_{s\to\infty} s^2 \cdot |Rm| = 0$ and hence $N^{n-1} = \mathbb{S}^{n-1}/\Gamma$.

If all radial sectional curvatures of (M, g, f) are zero and the scalar curvature is constant, then this soliton must be rigid. This was proved by P. Petersen and W. Wylie [51]. For other results about rigidity, one can consult results of G. Catino and C. Mantegazza [13] and S. Pigola, M. Rimoldi and A. G. Setti [53]. We will discuss the classification problem in the next section.

2.2 Classification problem of expanding solitons

In the three-dimensional case, the classification of shrinking solitons under some reasonable conditions leads to the performance of Perelman's surgery in [48]. These conditions have been verified to be superfluous in this decade by L. Ni and N. Wallach [47], A. Naber [45] and H.-D. Cao, B.-L. Chen and X.-P. Zhu [10]. That is, all three-dimensional shrinking solitons have been classified without any assumption. In the four dimensional case, A. Naber [45] proved that a complete non-compact shrinking soliton with bounded nonnegative curvature must be isometric to $\mathbb{R}^4$ or a finite quotient of $\mathbb{R}^2 \times \mathbb{S}^2$ or $\mathbb{R}^1 \times \mathbb{S}^3$. Furthermore, for dimensions greater or equal to 4, all locally conformally flat shrinking solitons are classified by L. Ni and N. Wallach [47], P. Petersen and W. Wylie [52], G. Catino and C. Mantegazza [13] and Z.-H. Zhang [63]. In the paper of Catino and Mantegazza, they also derived that every locally conformally flat steady soliton either is flat or is the Bryant soliton.

However, the situation for expanding soliton is much more unclear. Even for a locally conformally flat expanding soliton (M^n, g, f) with nonnegative Ricci curvature, we only know that it is either the (flat) Gaussian soliton or a warped-product manifold of $\mathbb{S}^{n-1}$ [13]. Indeed, there exist several non-trivial rotationally symmetric expanding solitons. A two-dimensional example was constructed by B. Chow, P. Lu and L. Ni [24]. For general dimensions, H.-D. Cao [8] constructed a family of expanding Kähler-Ricci solitons whose curvature may be either positive or negative. More recently, M. Feldman, T. Ilmanen and D. Knopf [28] also constructed a family of rotationally symmetric expanding solitons on total spaces of holomorphic line bundle over $\mathbb{CP}^{n-1}$. Therefore, to achieve the classification, we need to propose more admissible conditions.

In [18], B.-L. Chen and X.-P. Zhu proved that an expanding soliton cannot have $Sect \geq 0$ and positively ϵ-pinched Ricci curvature, i.e. $Ric \geq \epsilon R g$ and $R > 0$, when $n \geq 3$. Today it is still unknown that whether the condition on the sectional curvature can be removed or not. L. Ni posed this problem in [46] and proved that every expanding soliton with dimension $n \geq 3$ and positively ϵ-pinched Ricci curvature must have exponential decay Ricci curvature. (A proof can be found in the book of Chow, Lu and Ni [24].) Such decay rate is quite unusual for a Riemannian manifold with nonnegative Ricci curvature, because a classical gap theorem says that there exists no manifold which possesses faster-than-quadratic-decay curvature, nonnegative Ricci curvature and positive asymptotic volume ratio. (This theorem was derived by S. Bando, A. Kasue and H. Nakajima in [3]. They assumed that the manifold is simply-connected at infinity, has only one end and dimension $n \geq 3$.)

Ni's problem motivates us to study expanding solitons with fast curvature decay. Although we have not yet succeeded in solving Ni's problem, we can understand the asymptotic behavior of solitons with fast-decay *sectional* curvature by using Bando-Kasue-Nakajima's approach. We recall that in the proof of the classical gap theorem, Bando, Kasue and Nakajima first scale the manifold downward to a fix point to gain a limit metric space which is called a tangent cone at infinity. Since the asymptotic volume ratio is positive, by the nonnegativity of Ricci curvature and Bishop-Gromov volume comparison, one knows that all geodesic balls have a uniform lower bound of their volume ratios. Together with the decay rate of curvature, we know that the injectivity radius grows linearly. (cf. [23] or [16].) Now by classical convergence theory, this tangent cone is in fact a flat manifold except the vertex which might be singular. The last step, again using the nonnegativity of Ricci curvature and Bishop-Gromov volume comparison, is to conclude that the original manifold is flat.

Using this strategy, we can show that if a non-flat expanding Ricci soliton M satisfies $\lim_{dist(O,x)\to\infty} |Sect| \cdot dist(O,x)^2 = 0$ and is simply connected at infinity, then each tangent cone at infinity of M is isometric to the Euclidean space $\mathbb{R}^n$. (Here we assume that the soliton has only one end and has dimension $n \geq 3$.) The significance of our proof is that, without the nonnegativity of Ricci curvature, we can show that a uniform local volume lower bound holds automatically for solitons with faster-than-quadratic-decay curvature and, moreover, the flat tangent cone at infinity is a manifold. The remaining problem is how to conclude that the expanding soliton is isometric to the Euclidean space $\mathbb{R}^n$ (without assuming that Ricci curvature is nonnegative)? Actually, we suspend that this is impossible, i.e. we believe that there exists a non-trivial expanding soliton which has $\mathbb{R}^n$ as one of its tangent cone at infinity. Such soliton, if exists, cannot be Ricci nonnegative. We remind the reader that the examples of M. Feldman, T. Ilmanen and D. Knopf [28] are not Ricci nonnegative (as pointed out by L. Ni [46]), since their topology are not trivial.

2.3 Asymptotic volume ratio of expanding solitons

In [21], the author proved the following theorems, which are concerned with asymptotic volume ratio of nonsteady solitons.

Theorem 8. (C.-W. Chen, [21]) *Let (M, g, f) be a complete non-compact expanding gradient Ricci soliton with scalar curvature R. If there exists $O \in M$ such that $\frac{1}{Vol(B_s)} \int_{B_s} R \geq -Cs^{-\varepsilon}$, where $\varepsilon > 0$ is a constant, then $\liminf_{s\to\infty} \frac{Vol(B_s)}{s^n} \geq \eta$. Moreover, if we have $Ric \geq -Cs^{-\varepsilon} g$ and $\frac{1}{Vol(B_s)} \int_{B_s} R \leq Cs^{-\varepsilon}$, then*

$$C^{-1}s^n \leq Vol(B_s) \leq Cs^n$$

holds for all $s \geq A$, where A is a large constant.

Proof. Taking the trace of the soliton equation $R_{ij} + \nabla_i \nabla_j f = -g_{ij}$ and integrating it on B_s, we have

$$\begin{aligned} -nVol(B_s) &= \int_{B_s} R + \int_{B_s} \Delta f = \int_{B_s} R + \int_{\partial B_s} \nabla f \cdot \nabla s \geq \int_{B_s} R - \int_{\partial B_s} (s + L(O)) \\ &= \int_{B_s} R - (s + L(O))Area(\partial B_s) = \int_{B_s} R - (s + L(O))\frac{d}{ds}Vol(B_s). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d}{ds} \log Vol(B_s) &\geq \frac{1}{(s + L(O))Vol(B_s)} \int_{B_s} R + \frac{n}{s + L(O)} \\ &= \frac{1}{(s + L(O))Vol(B_s)} \int_{B_s} R + \frac{d}{ds} \log(s + L(O))^n \end{aligned}$$

$$\Rightarrow \quad \frac{d}{ds} \log \frac{Vol(B_s)}{(s + L(O))^n} \geq \frac{1}{(s + L(O))Vol(B_s)} \int_{B_s} R \geq \frac{-C}{(s + L(O))s^{\varepsilon}} \geq \frac{-C}{s^{1+\varepsilon}}$$

$$\Rightarrow \quad \log \frac{Vol(B_s)}{(s + L(O))^n} \geq \int_{\rho}^{s} \frac{-C}{s^{1+\varepsilon}} + \log \frac{Vol(B_\rho)}{(\rho + L(O))^n} = \frac{C}{\varepsilon}s^{-\varepsilon} - \frac{C}{\varepsilon}\rho^{-\varepsilon} + \log \frac{Vol(B_\rho)}{(\rho + L(O))^n}$$

for any positive constant $\rho < s$

$$\Rightarrow \quad \frac{Vol(B_s)}{(s + L(O))^n} \geq \left(e^{\frac{C}{\varepsilon}s^{-\varepsilon} - \frac{C}{\varepsilon}\rho^{-\varepsilon}}\right) \frac{Vol(B_\rho)}{(\rho + L(O))^n} \geq e^{-\frac{C}{\varepsilon}\rho^{-\varepsilon}} \cdot \frac{Vol(B_\rho)}{(\rho + L(O))^n}.$$

Hence,

$$\liminf_{s\to\infty} \frac{Vol(B_s)}{s^n} \geq e^{-\frac{C}{\varepsilon}\rho^{-\varepsilon}} \cdot \frac{Vol(B_\rho)}{(\rho + L(O))^n} \equiv \eta > 0.$$

For the reader's convenience, we write down the proof of the upper bound estimate although it is almost the same to the above one.

From the lower bound of the Ricci curvature and Theorem 7, we have $f'(x) \leq -s + C$ for s large enough. Together with the lower bound of the averaged scalar curvature, we have

$$-nVol(B_s) = \int_{B_s} R + \int_{\partial B_s} \nabla f \cdot \nabla s \leq Cs^{-\varepsilon}Vol(B_s) - (s - C)\frac{d}{ds}Vol(B_s)$$

$$\Rightarrow \frac{-n}{s-C} \leq \frac{C}{(s-C)s^{\varepsilon}} - \frac{d}{ds}\log Vol(B_s) \leq \frac{C}{s^{1+\frac{\varepsilon}{2}}} - \frac{d}{ds}\log Vol(B_s).$$

Hence we get a similar inequality $\frac{d}{ds}\log\frac{Vol(B_s)}{(s+C)^n} \leq \frac{C}{s^{1+\frac{\varepsilon}{2}}}$. The rest of the proof is easy to work out.

□

For shrinking gradient Ricci solitons, the same calculation gives the following theorem.

Theorem 9. (C.-W. Chen, [21])*Let (M, g, f) be a complete non-compact shrinking gradient Ricci soliton which satisfies $R_{ij} + \nabla_i\nabla_j f = g_{ij}$. If there exists $O \in M$ such that $\frac{1}{Vol(B_s)}\int_{B_s} R \leq Cs^a$, where a is a nonzero constant, then its volume ratio $\frac{Vol(B_s)}{s^n}$ is bounded from below by $C \cdot e^{\frac{-C}{a}s^a}$ for s large enough. When $\frac{1}{Vol(B_s)}\int_{B_s} R \leq \delta_1 < n$ (for s large enough), we have $Vol(B_s) \geq C \cdot s^{n-\delta_1}$ for s large enough. Similarly, $\frac{1}{Vol(B_s)}\int_{B_s} R \geq \delta_2 > 0$ implies $Vol(B_s) \leq C \cdot s^{n-\delta_2}$.*

Proof. Similar calculations as in the proof of Theorem 8, we have

$$nVol(B_s) = \int_{B_s} R + \int_{B_s} \Delta f = \int_{B_s} R + \int_{\partial B_s} \nabla f \cdot \nabla s.$$

By using the fact that $R > 0$ for all shrinking solitons (cf. Corollary 2.5 in [17]), one can show that $|\nabla f| \leq s + \sqrt{2f(0)}$ and hence

$$n = \frac{1}{Vol(B_s)}\int_{B_s} R + \frac{1}{Vol(B_s)}\int_{\partial B_s} \nabla f \cdot \nabla s \leq Cs^a + \frac{1}{Vol(B_s)}\left(s + \sqrt{2f(0)}\right)\frac{d}{ds}Vol(B_s)$$

whenever $\frac{1}{Vol(B_s)}\int_{B_s} R \leq Cs^a$, where a is a nonzero constant. Therefore,

$$\frac{d}{ds}\log Vol(B_s) \geq \frac{-Cs^a}{s+\sqrt{2f(0)}} + \frac{n}{s+\sqrt{2f(0)}} \geq -Cs^{-1+a} + \frac{d}{ds}\log(s+\sqrt{2f(0)})^n$$

implies that

$$\frac{Vol(B_s)}{s^n} \geq \frac{Vol(B_s)}{(s+\sqrt{2f(0)})^n} \geq e^{-\frac{C}{a}s^a} \cdot \frac{Vol(B_\rho)}{(\rho+\sqrt{2f(0)})^n} \cdot e^{\frac{c}{a}\rho^a},$$

for any $0 < \rho < s$. Thus we get $\frac{Vol(B_s)}{s^n} \geq C \cdot e^{\frac{-C}{a}s^a}$ for s large enough.

In the case that $\frac{1}{Vol(B_s)}\int_{B_s} R \leq \delta_1$, we have

$$n = \frac{1}{Vol(B_s)}\int_{B_s} R + \frac{1}{Vol(B_s)}\int_{\partial B_s} \nabla f \cdot \nabla s \leq \delta_1 + \left(s + \sqrt{2f(0)}\right)\frac{d}{ds}\log Vol(B_s).$$

Hence

$$\frac{d}{ds}\log Vol(B_s) \geq \frac{n-\delta_1}{s+\sqrt{2f(0)}} \Rightarrow \frac{d}{ds}\frac{Vol(B_s)}{(s+\sqrt{2f(0)})^{n-\delta_1}} \geq 0.$$

This monotonicity shows that

$$\frac{Vol(B_s)}{s^{n-\delta_1}} \geq \frac{Vol(B_s)}{(s+\sqrt{2f(0)})^{n-\delta_1}} \geq C \Rightarrow Vol(B_s) \geq C \cdot s^{n-\delta_1}.$$

An upper bound estimate holds similarly for $\frac{1}{Vol(B_s)}\int_{B_s} R \geq \delta_2$. The cruical difference is the usage of the fact $|\nabla f| \geq s - C$ for some constant $C > 0$. This can be seen in Cao-Zhou's paper [11], in the proof of Proposition 2.1. □

Some of the results in the theorem above were proved in different articles. A similar result to the case $a = 0$, i.e. the case $\frac{1}{Vol(B_s)}\int_{B_s} R \leq \delta_1 < n$, was proved by Cao and Zhou in [11]. The last statement concerning the sharp upper volume bound was proved by S. Zhang in [61].

Moreover, we remind the reader that Cao and Zhou [11] also proved that the upper bound $Vol(B_s) \leq C \cdot s^n$ always holds for *all* shrinking gradient Ricci solitons.

Recently, after the author posed the preprint [21] on the website ArXiv.org, B. Chow, P. Lu and B. Yang [25] derived a criterion for a shrinking soliton to have positive asymptotic volume ratio.

3 Topology of expanding solitons with fast curvature decay

In [21], the author proved the following theorem.

Theorem 10. (C.-W. Chen, [21]) *Let (M, g, f) be a complete non-compact expanding gradient soliton with $\lim_{s\to\infty} s^2 \cdot |Sect| = 0$. Then each end of M is diffeomorphic to $\mathbb{R} \times N^{n-1}$, where $N = \mathbb{S}^{n-1}/\Gamma$ is a metric quotient of the spherical space form.*

This result follows from the analysis on the level sets of the potential function f. Given an expanding soliton (M, g, f) with $\lim_{s\to\infty} s^2 \cdot |Sect| = 0$, we consider a level set $\Sigma_a := \{f = a\}$ of f. We remind the reader that in our case, f decays to $-\infty$ at the end of M. Hence the unit inner normal vector of Σ_a shall be $\frac{\nabla f}{|\nabla f|}$ and the second fundamental form of Σ_a is

$$II_{ij} = -\left\langle \nabla_{e_i} \frac{\nabla f}{|\nabla f|}, e_j \right\rangle = \left\langle -\frac{\nabla_{e_i} \nabla f}{|\nabla f|} + \frac{(e_i|\nabla f|)\nabla f}{|\nabla f|^2}, e_j \right\rangle = \frac{Hess(-f)_{ij}}{|\nabla f|} = \frac{R_{ij} + g_{ij}}{|\nabla f|},$$

for all $i, j = 1, \cdots, n-1$. By Gauss equation, we have

$$R_{ijij} = R^{\Sigma}_{ijij} - II_{ii}II_{jj} + II^2_{ij} = R^{\Sigma}_{ijij} - \frac{1}{|\nabla f|^2}(R_{ii} + g_{ii})(R_{jj} + g_{jj}).$$

Since $|Sect| \in o(s^{-2})$, we have $\left| R^{\Sigma}_{ijij} - \frac{1}{|\nabla f|^2} \right| \in o(s^{-2})$ and $\left| \frac{1}{|\nabla f|^2} - \frac{1}{s^2} \right| \in o(s^{-2})$ (cf. Remark 12 in section 1). That is, the level set of f has almost constant sectional curvatures when s becomes large. By the diffeomorphic sphere theorem (or by the local volume estimate in the next section and Cheeger's center of mass), we have verified the theorem above.

Now we consider an expanding soliton with $\lim_{s\to\infty} s^2 \cdot |Ric| = 0$. The previous procedure remains valid only if we can control the decay rate of radial sectional curvatures, i.e. $\left\langle R\left(E, \frac{\nabla f}{|\nabla f|}\right) E, \frac{\nabla f}{|\nabla f|}\right\rangle$. The following theorem shows that all radial sectional curvatures are controlled by $|\nabla Ric|$.

Theorem 11. *Let (M, g, f) be a complete non-compact expanding gradient Ricci soliton with $\lim_{s\to\infty} s^2 \cdot |Ric| = 0$. If it satisfies the strong Bianchi inequality, i.e. $|\nabla Ric| \leq \beta|\nabla R|$ for some $\beta > 0$, then each end of M is diffeomorphic to $\mathbb{R} \times N^{n-1}$, where N^{n-1} is some almost Einstein manifold with positive Ricci curvature. Moreover, if the injectivity radius grows linearly, then $N = \mathbb{S}^{n-1}/\Gamma$ is a metric quotient of the spherical space form.*

We remind the reader that even an almost Einstein manifold is simply-connected, its topology type is still undetermined. If we assume, furthermore, that N has nonnegative isotropic curvature, then it must be diffeomorphic to a symmetric space of compact type. This was proved by H. Seshadri in [55].

Proof. Let x be a point in $\Sigma_a := \{f = a\}$ and $\{x_i\}_{i=1}^{n-1}$ be the normal coordinates at $x \in \Sigma_a$ such that the second fundamental form II of $\Sigma_a \subset M$ is diagonalized at x. Since $\nabla f \neq 0$ in a neighborhood $U \subset M$ of x, the local diffeomorphisms $\varphi_t : \Sigma_a \cap U \to \Sigma_{a+t} \cap U, t \in (-\varepsilon, \varepsilon)$, generated by $\frac{\nabla f}{|\nabla f|^2}$ are locally well-defined. Thus we can construct a local coordinate chart $\{x_i\}_{i=1}^n$ near x such that $\frac{\partial}{\partial x_n} = \frac{\nabla f}{|\nabla f|}$. In particular, we have $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$ for all $i, j = 1, \cdots, n$. (These special coordinates are useful when studing submanifolds and we prefer to call them *subnormal coordinates* with respect to Σ_a.) One should notice that the integral curve of $\frac{\partial}{\partial x_n}$ is in general not a geodesic.

Using this coordinate chart, we can derive

$$\Gamma^i_{nn} = \left\langle \nabla_n \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_i} \right\rangle = \left\langle \frac{\partial}{\partial x_n}, \nabla_n \frac{\partial}{\partial x_i} \right\rangle = \left\langle \frac{\partial}{\partial x_n}, \nabla_i \frac{\partial}{\partial x_n} \right\rangle = \frac{\partial}{\partial x_i} \left| \frac{\partial}{\partial x_n} \right| = 0.$$

Furthermore, we have

$$\begin{aligned} \left\langle \nabla_i \nabla_n \nabla f, \frac{\partial}{\partial x_i} \right\rangle &= \frac{\partial}{\partial x_i} \left\langle \nabla_n \nabla f, \frac{\partial}{\partial x_i} \right\rangle - \left\langle \nabla_n \nabla f, \nabla_i \frac{\partial}{\partial x_i} \right\rangle \\ &= \frac{\partial}{\partial x_i} (\nabla_n \nabla_i f) - f_{nn} II_{ii} \\ &= -\frac{\partial}{\partial x_i} (R_{ni} + g_{ni}) - (R_{nn} + 1) II_{ii} \end{aligned}$$

and

$$\begin{aligned} \left\langle \nabla_n \nabla_i \nabla f, \frac{\partial}{\partial x_i} \right\rangle &= \frac{\partial}{\partial x_n} \left\langle \nabla_i \nabla f, \frac{\partial}{\partial x_i} \right\rangle - \left\langle \nabla_i \nabla f, \nabla_n \frac{\partial}{\partial x_i} \right\rangle \\ &= \frac{\partial}{\partial x_n} (\nabla_i \nabla_i f) - \left\langle \nabla_i \nabla f, -II_{ij} \frac{\partial}{\partial x_j} \right\rangle \\ &= -\frac{\partial}{\partial x_n} (R_{ii} + g_{ii}) - |\nabla f| |II_{ij}|^2 \end{aligned}$$

at x. We remind the reader that $|II_{ij}|^2 = \frac{|R_{ij} + g_{ij}|^2}{|\nabla f|^2} = \frac{|Ric|^2 + 2R + n}{|\nabla f|^2}$ decays as fast as $\frac{1}{|\nabla f|^2}$, i.e. $|II_{ij}|^2 \in o(s^{-2})$.

By using these two equalities, we derive

$$\begin{aligned} & \left| \left\langle R \left(\frac{\partial}{\partial x_i}, \nabla f \right) \nabla f, \frac{\partial}{\partial x_i} \right\rangle \right| \\ = \; & |\nabla f| \left| \left\langle \nabla_i \nabla_n \nabla f - \nabla_n \nabla_i \nabla f, \frac{\partial}{\partial x_i} \right\rangle \right| \\ = \; & |\nabla f| \left| -\frac{\partial}{\partial x_i} (R_{ni} + g_{ni}) - (R_{nn} + 1) II_{ii} + \frac{\partial}{\partial x_n} (R_{ii} + g_{ii}) + |\nabla f| |II_{ij}|^2 \right| \\ \leq \; & |\nabla f| \left(2|\nabla Ric| + (3n-2)|Ric||II| + 4|II| + |Ric||II| + |II| + |\nabla f||II|^2 \right). \end{aligned}$$

Since $|\nabla Ric| \leq \beta |\nabla R| \leq 2\beta |Ric||\nabla f| \in o(s^{-1})$, $|II| \in o(s^{-1})$ and $|\nabla f|$ grows linearly, we have $\left|\left\langle R\left(\frac{\partial}{\partial x_i},\nabla f\right)\nabla f,\frac{\partial}{\partial x_i}\right\rangle\right|\in o(1)$, i.e., $|R_{inin}|\in o(s^{-2})$. From Gauss equation, we have

$$
\begin{aligned}
R^{\Sigma}_{ii} &= \sum_{j=1}^{n-1} R^{\Sigma}_{ijij} \\
&= \sum_{j=1}^{n} R_{ijij} - R_{inin} + \sum_{j=1}^{n-1}\left(II_{ii}II_{jj} - II_{ij}^2\right) \\
&= R_{ii} - R_{inin} + II_{ii}\left(\frac{R - R_{nn} + n - 1}{|\nabla f|} - II_{ii}\right) \\
&= R_{ii} - R_{inin} + (R_{ii}+1)\left(\frac{R - R_{nn} + n - 1 - R_{ii} - 1}{|\nabla f|^2}\right) \\
&= R_{ii} - R_{inin} + \frac{n-2}{|\nabla f|^2} + o(s^{-4}).
\end{aligned}
$$

Hence

$$\left|R^{\Sigma}_{ii} - \frac{n-2}{|\nabla f|^2}\right| \in o(s^{-2}).$$

By Remark 12 in section 1, we know that $\left|\frac{1}{|\nabla f|^2} - \frac{1}{s^2}\right| \in o(s^{-2})$. After rescaling, the Ricci curvature of level sets of f is arbitrarily close to a constant as s is large enough. By Theorem 7, f is proper and strictly convex outside a compact set of M, hence each end of M is diffeomorphic to $\mathbb{R}\times N^{n-1}$, where N^{n-1} is an almost Einstein manifold.

Moreover, suppose the injectivity radius grows linearly, we can use Schauder estimate to prove that $|Sect|\in o(s^{-2})$ (cf. Corollary 3 in Chapter 3). Hence the statement holds as a corollary of Theorem 10. □

4 Geometry of expanding solitons with fast curvature decay

Theorem 12. (C.-W. Chen, [21]) *If a complete non-compact expanding gradient Ricci soliton* (M,g,f) *satisfies* $\lim_{s\to\infty} s^2\cdot|Sect| = 0$, *then we have*

$$Vol(B_r(x)) \geq Cr^n$$

for all $x\in M$ *and for all* $r>0$. *We also have* $Vol(B_r(x))\leq Cr^n$ *for all* $r\leq\frac{s}{2}$ *and for all* $x\neq O$. *Moreover, if its asymptotic volume ratio exists, then we have*

$$C^{-1}r^n \leq Vol(B_r(x)) \leq Cr^n$$

for all $x \in M$ and for all $r > 0$.

Proof. Step 1. We prove first that the lower bound estimate holds for all $x \neq O$ and $r = \frac{s}{2} := \frac{1}{2}dist(O, x)$. It suffices to show that, for s large enough, $B_{\frac{s}{2}}(x)$ contains a "cube" whose volume is at least $\bar{\delta}s^n$ for some $\bar{\delta}$ independent of x. Let $f(x) = a$, $\Sigma_a := \{f = a\}$ and $II = Hess(-f)/|\nabla f|$ be the second fundamental form of Σ_a. Since $\|Hess(-f) - g\| \leq |Ric| \in o(s^{-2})$ implies $\|II - \frac{g}{|\nabla f|}\| \in o(s^{-3})$, by Gauss equation and the fast decay of the curvature of M, we have $|Sect^{\Sigma} - \frac{1}{s^2}| \in o(s^{-2})$. Hence for s large enough, there exists an intrinsic ball $B^{\Sigma}_{\delta s}(x) \subset \Sigma_a$ such that $Vol^{\Sigma}(B^{\Sigma}_{\delta s}) \geq Cs^{n-1}$.

Furthermore, by using the one parameter family of diffeomorphisms $\{\varphi_t : \Sigma_a \to \Sigma_{a+t}\}_{t\in(-\frac{s}{10},\frac{s}{10})}$, which is generated by $\frac{\nabla f}{|\nabla f|^2}$, we know the cube $Cube := \{y \in M | y = \varphi_t(B^{\Sigma}_{\delta s}(x)), t \in (-\frac{s}{10}, \frac{s}{10})\}$ is contained in $B_{\frac{s}{2}}(x)$ and $Vol(Cube) \geq \bar{\delta}s^n$ for some constant $\bar{\delta}$ which is independent of x whenever s is large enough.

Step 2. For $r \leq \frac{s}{2}$, by using the Bishop-Gromov's comparison, we have

$$\begin{aligned} Vol(B_r(x)) &\geq \left(Vol_{\frac{-C}{s^2}}(B_r)/Vol_{\frac{-C}{s^2}}(B_{\frac{s}{2}})\right) \cdot Vol(B_{\frac{s}{2}}(x)) \\ &\geq \left(Vol_{\mathbb{R}^n}(B_r)/Vol_{\frac{-C}{s^2}}(B_{\frac{s}{2}})\right) \cdot \bar{\delta}s^n \\ &\geq Cr^n. \end{aligned}$$

The last inequality comes from

$$Vol_{\frac{-C}{s^2}}(B_s) = C\int_0^s \left(\frac{s}{\sqrt{C}}\sinh\left(\frac{\sqrt{C}}{s}t\right)\right)^{n-1} dt \leq Cs^n,$$

where $Vol_{\frac{-C}{s^2}}$ is the volume functional of the hyperbolic space with $Ric = \frac{-C}{s^2}$. Up to now, the lower bound estimate holds when x lies outside a large compact set, i.e. for s large enough. Inside the compact region, the estimate holds by the continuity of volume functional and the smoothness of manifold. That is, we have $Vol(B_r(x)) \geq Cr^n$ for all $x \neq O$ and $r \leq \frac{s}{2}$.

Step 3. Furthermore, such lower bound holds for all $B_r(O)$ because each of them contains a $B_{\frac{r}{4}}(x)$, where $dist(O, x) = \frac{r}{2}$. Similarly, since $B_r(x)$ contains $B_s(O)$ whenever $r > 2s$, we get the lower bound estimate for all $B_r(x)$ with $r > 2s$ (and $r \leq \frac{s}{2}$ as shown in Step 2). At last, it is easy to see that this estimate holds for all $r \in [\frac{s}{2}, 2s]$ because $Vol(B_r(x)) \geq Vol(B_s(x)) \geq Cs^n \geq \frac{C}{2^n}(2s)^n \geq \frac{C}{2^n}r^n$. Thus we complete the proof of the lower bound estimate for all balls in M.

Step 4. The upper bound can be derived by Bishop-Gromov's comparison. Since $Ric \geq -C \cdot s^{-2} g$ on $B_r(x)$ with $r \leq \frac{s}{2}$, as in the last inequality of Step 2, we have

$$Vol(B_r(x)) \leq Vol_{\frac{-C}{r^2}}(B_r) = C \int_0^r \left(\frac{r}{\sqrt{C}} \sinh \left(\frac{\sqrt{C}}{r} t \right) \right)^{n-1} dt \leq C r^n.$$

Therefore we have proved the first statement of this theorem. Now suppose that we can control the upper bound of the volume ratio at infinity, i.e., there exist two constants C and A such that $Vol(B_r(O)) \leq C r^n$ for all $s \geq A$. It is easy to see that, for all $r > \frac{s}{2}$, $B_r(x)$ is contained in $B_{3r}(O)$ and hence has an upper bound on its volume ratio. □

A tangent cone at infinity is a Cheeger-Gromov limit of a sequence of blow-down metrics with a fixing marked point. Since we have a uniform estimate of volume lower bound from Theorem 12, we can derive a lower bound of injectivity radius from the controlled sectional curvature (see [23] and [16].) In this section, we prove that every tangent cone at infinity of M is the Euclidean space $\mathbb{R}^n$ under some admissible conditions.

Theorem 13. (C.-W. Chen, [21]) *Let (M, g, f) be a complete non-compact expanding gradient soliton with $\lim_{s\to\infty} s^2 \cdot |Sect| = 0$. If M is simply connected at infinity, has only one end and has dimension $n \geq 3$, then every tangent cone at infinity of M is the Euclidean space $\mathbb{R}^n$.*

Remark 13. We remind the reader that a manifold M is said to be simply connected at infinity if for each compact subset $C \subset M$, there is a compact set $D \subset M$ containing C so that the induced map $\pi_1(M \setminus D) \to \pi_1(M \setminus C)$ is trivial. Geometrically speaking, any loop far from a compact set D can be contracted outside the compact set, no matter how large the set is.

Proof. Consider a tangent cone at infinity M^∞, which is a Gromov-Hausdorff limit of a sequence $(M, O, \widetilde{g}_k) := (M, O, \frac{1}{\lambda_k^2} g)$ with vertex O, where $\lambda_k \to \infty$ as $k \to \infty$. Here we use a tilde to emphasize that the metric is rescaled. Any arbitrary point $q \in M^\infty, q \neq O$ and $dist_\infty(O, q) = r_0$, is associated with a sequence $q_k \to q$, where $dist_k(O, q_k) = \lambda_k r_0 \to \infty$ as $k \to \infty$. By using our volume estimate in the previous section and the curvature bound, we know that there exists a subsequence of manifolds (M, g_k) which converges in $W^{2,p}_{loc}$-topology, for any $p < \infty$, to the limit space $(M^\infty, dist_\infty)$ except at the point O. Therefore, $(M^\infty \setminus \{O\}, g^\infty)$ is an incomplete Riemannian manifold.

Noting that $\left|\widetilde{\nabla}_i\widetilde{\nabla}_j f_k\right| = \left|(\widetilde{g}_k)_{ij} + \frac{1}{\lambda_k^2}(\widetilde{Ric_k})_{ij}\right|$, together with the estimates of the growth of f and ∇f which are stated in Section 2, we know that $f_k := \frac{-f}{\lambda_k^2}$ converges in $W^{2,p}_{loc}$-topology to a function f^∞ with $|\nabla f| = r$ on $M^\infty \setminus \{O\}$. Moreover, $\nabla^\infty\nabla^\infty f^\infty = g^\infty$ and $f^\infty(q) = \lim_{k\to\infty} \frac{-f}{\lambda_k^2}(q_k) = \frac{1}{2}r_0^2$. Since q was chosen arbitrarily, we have

$$f^\infty(x) = \frac{1}{2}r^2 \quad \text{and} \quad g^\infty = Hess\left(\frac{1}{2}r^2\right)$$

where $r(x) := dist_\infty(O,x)$ and $x \in M^\infty \setminus \{O\}$.

In [14], J. Cheeger and T. H. Colding have proven that $M^\infty \setminus \{O\}$ with $g^\infty = Hess\left(\frac{r^2}{2}\right)$ must be a warped product manifold and $g^\infty = dr^2 + kr^2\bar{g}$ for some $k > 0$, where $\bar{g}$ is the metric of $N := \{x \in M^\infty | r(x) = 1\}$. In order to prove that M^∞ is isometric to $\mathbb{R}^n$, we only need to show that N is the standard sphere with sectional curvature k. (Because the standard metric on $\mathbb{R}^n$ can be written as $g_{Eucl} = dr^2 + Cr^2 g_{\mathbb{S}^{n-1}(C)}$ for any given $C > 0$ and $g_{\mathbb{S}^{n-1}(C)}$ denotes the standard metric on sphere with constant sectional curvature C.)

Since $|\nabla r| \neq 0$, we can extend the normal coordinates $\{x^i\}_{i=2,\cdots,n}$ around $p \in N$ to be a local chart $\{r, x^i\}_{i=2,\cdots,n}$ in M such that

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & g_{n2} & \cdots & g_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & kr^2\bar{g}_{22} & \cdots & kr^2\bar{g}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & kr^2\bar{g}_{n2} & \cdots & kr^2\bar{g}_{nn} \end{pmatrix}.$$

(One can consult the proof of Theorem 11 for a construction of these *subnormal coordinates*.) Hence, for all $i, j = 2, \cdots, n$ and $i \neq j$, we have $\Gamma^r_{jj}(p) = -k$ and $\Gamma^r_{ij}(p) = 0$. Moreover, $\frac{\partial}{\partial x^j}(g(\frac{\partial}{\partial r}, \frac{\partial}{\partial x^j})) = 0$ implies that $\Gamma^j_{jr}(p) = -\frac{1}{k}\Gamma^r_{jj}(p) = 1$. When $n \geq 3$, we can compute the curvature of N at p by using

$$0 = R^i_{ijj} = \bar{R}^i_{ijj} + \Gamma^i_{ir}\Gamma^r_{jj} = \bar{R}^i_{ijj} - k.$$

By the assumption that M is simply connected at infinity, we know that N must be the standard sphere with all its sectional curvatures equal k.

□

Remark 14. When n is odd, by Synge's theorem (cf. Theorem 5.9 in [15]), we need not assume that M is simply connected at infinity.

As we mentioned in the beginning of this chapter, our theorem does not hold for $n = 2$. By using our proof, we can only show that the tangent cone at infinity has metric $g = dr^2 + Cr^2 d\theta^2$ for some indefinite constant $C > 0$.

The crucial part in the proof is to show that the vertex of the flat cone is in fact regular, i.e. it admits a completification of Riemannian metric. When a cone is not flat, it is harder to show that the vertex is regular. P. D. Smith and D. Yang have dealt with this problem (in a more general setting) in [60]. They showed that if the curvature blows up not so fast in a simply connected neighborhood B_ϵ of the vertex and there exists no closed geodesic $\gamma \subset B_\epsilon$ passing the vertex, then the vertex is regular. (The dimension is assumed to be greater than or equal to 3.)

Appendix: A study of injectivity radius on non-compact manifolds

In this appendix, we study the injectivity radius in a general setting. We propose a sufficient condition to guarantee that the injectivity radius grows linearly.

1 Introduction

The injectivity radius plays an important role in the research of differential geometry. A suitable lower bound estimate of it can prevent the manifolds, and even their rescaled subsequential limit, from collapses. Usually, we can derive such lower bound from the controlled sectional curvature and the volume of geodesic balls, see [23] and [16]. However, in this article, we would like to derive the lower bound by analyzing the geodesic loops directly. Our result is closely related to the famous work of D. Gromoll and W. Meyer [30] where they derived a global lower bound estimate of injectivity radius for manifolds with positive sectional curvature. Here we try to determine whether the injectivity radius grows linearly on a manifold under additional conditions. We indeed focus on the issue that how to determine a given complete non-compact manifold (M, g) is collapsed or not.

Because the word "collapse" may have various meanings in different texts, we have better to define it before going on.

Definition 9. A sequence of open geodesic balls $\{(B_{r_i}(x_i), g_i)\}_{i=1}^{\infty}$ is said to be a *collapsing sequence* if $|Sect_{g_i}| \leq \frac{C}{r_i^2}$ on $B_{r_i}(x_i)$ for some constant $C > 0$ and $\frac{inj(x_i)}{r_i} \to 0$ as $i \to \infty$, where $Sect_{g_i}$ denotes the sectional curvature of g_i and $inj(x_i)$ denotes the injectivity radius at x_i. A manifold (M, g) is said to be collapsed if there exists a collapsing sequence $\{(B_{r_i}(x_i), g)\}_{i=1}^{\infty}$

on it; otherwise, it is called *noncollapsed*.

In this article, a closed piecewise smooth geodesic is called a geodesic loop. When it is smooth at every point, we call it a smooth (geodesic) loop. We show that if (M, g) contains no smooth geodesic loops and satisfies the *non-accumulated property*, which is stated in Section 3, then it shall be noncollapsed. The statement here is different to our main theorem, Theorem 16 in Section 3, which concerns more about the asymptotic behavior of the manifold, however, the proof is the same. Indeed, we prove that the non-accumulated property cannot hold on a collapsing sequence. The reason that we write Theorem 16 in the present form is that we wish it could be used to study the asymptotic behavior of manifolds with fast curvature decay.

In the following section, we find out some manifolds which contain no smooth geodesic loop. In the last section, we use non-smooth geodesic loops to define the non-accumulated property and prove the main theorem.

2 Non-existence of smooth geodesic loops

Given a Riemannian manifold M and a point $O \in M$, we denote $s := dist(O, x)$ for any arbitrary point $x \in M$ and introduce the following condition.

Smooth Loop Condition. *There are constants c_0 and d_0 such that there exists no smooth geodesic loop passing through x with length less than $c_0 \cdot s$ when $s \geq d_0$.*

In the rest paragraphs of this section, we find out some Riemannian manifolds and Ricci solitons which satisfy the smooth loop condition. First, we show that the smooth loop condition is implied by the positivity of the sectional curvature. This theorem is essentially due to D. Gromoll and W. Meyer, Lemma 3 in [30].

Theorem 14. *Let M be a complete non-compact manifold with nonnegative sectional curvature and γ be a smooth loop on M. Then the sectional curvature cannot be positive everywhere on γ. In particular, if the sectional curvature is positive on all ends of M, then M satisfies the smooth loop condition.*

Proof. We argue by contradiction. Suppose that there exists a smooth geodesic loop γ such that $Sect(V_1, V_2) > 0$ for all vectors $V_1, V_2 \in T_qM$ with unit length and $q \in \gamma$. Since a loop is compact, we may assume that $Sect > \epsilon$ on the tubular neighborhood $\{x \in M | dist(x, \gamma) < \delta\}$ of γ, where ϵ and δ are two small positive constants. Choose a point $p \in M \setminus \gamma$ and consider the shortest geodesic σ from p to γ. By using the first variation on σ, we know that σ is perpendicular to γ. Furthermore, since σ realizes the distance between p and γ, the second variation of the length functional (with p fixed) must be nonnegative. Now we compute the second variation as follows and derive a contradiction.

Let $\widetilde{\gamma'}(t)$ be the parallel extension of γ' along $\sigma(t)$, $t \in [0, l]$. Consider the variational vector field $V(t) := (1 - \frac{t}{l})\widetilde{\gamma'}(t)$, we have

$$
\begin{aligned}
I(V, V) &= \int_0^l |\nabla_{\sigma'(t)} V(t)|^2 - \left(1 - \frac{t}{l}\right)^2 Sect\left(\sigma'(t), \widetilde{\gamma'}(t)\right) dt \\
&\leq \frac{1}{l} - \left(1 - \frac{\delta}{l}\right)^2 \int_0^\delta Sect\left(\sigma'(t), \widetilde{\gamma'}(t)\right) dt \\
&\leq \frac{1}{l} - \left(1 - \frac{\delta}{l}\right)^2 \delta\epsilon,
\end{aligned}
$$

where $Sect\left(\sigma'(t), \widetilde{\gamma'}(t)\right)$ is the sectional curvature on the two plane spanned by $\sigma'(t)$ and $\widetilde{\gamma'}(t)$. It is clear that $I(V, V)$ will become negative when we choose p to be far away from γ. □

The following theorem shows that the smooth loop condition holds on certain Ricci solitons. For readers who are not familiar to the Ricci flow, we note that a Ricci soliton is a manifold which satisfies that $R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}$ for some function $f : M \to \mathbb{R}$ and $\lambda \in \mathbb{R}$. It can be seen as a time slice of a self-similar solution of the Ricci flow.

Theorem 15. *Consider a gradient Ricci soliton M which satisfies $R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}$. Let $h : M \to \mathbb{R}$ be a nonnegative function such that $h(x) \to 0$ as $s \to \infty$. If one of the following three conditions holds:*

(i) $\lambda = 1$ and $Ric \leq h \cdot g$;

(ii) $\lambda = 0$ and $Ric > 0$;

(iii) $\lambda = -1$ and $Ric \geq -h \cdot g$,

then M contains no smooth geodesic loop outside a compact set K (K is empty for case (ii)). In particular, M satisfies the smooth loop condition.

Proof. Suppose that there is a smooth geodesic loop $\gamma \subset M \setminus B_s(O)$ whose length is denoted by l. Integrating the equation of soliton on γ, we have

$$\lambda l = \int_\gamma \lambda |\gamma'|^2 = \int_\gamma Ric(\gamma', \gamma') + \int_\gamma f'' = \int_\gamma Ric(\gamma', \gamma').$$

This contradicts all the three conditions. □

Remark 15. It is easy to see that this theorem holds for non-gradient solitons. On the other hand, the condition $Ric \leq h \cdot g$ (resp. $Ric \geq -h \cdot g$) can be replaced by $Ric < \lambda \cdot g$ (resp. $Ric > \lambda g$) on the ends of M. Note that the condition $Ric < \lambda \cdot g$ on a shrinking soliton is equivalent to the convexity of f.

3 Non-smooth loops and injectivity radius estimate

In this section, we introduce the non-accumulated condition on non-smooth geodesic loops and derive a lower bound estimate on the injectivity radius of certain manifolds, especially of Ricci solitons. Before moving into further discussion, we should recall some fundamental properties of the cut point. If $y \in M$ is a cut point of $x \in M$, then either y is conjugate to x or there exists a geodesic loop γ which passes through x and y. In the second case, γ is composed by two minimizing geodesics from x to y. If we assume that y is a nearest cut point of x, then the only possible singular point of γ is x (cf. Lemma 5.6 in [15]). We say that such y realizes the injectivity radius of x via γ. Hence the smooth loop condition means that, if the injectivity radius of a point x is small, then there exists another point y which either is conjugate to x or realizes the injectivity radius of x via a *non-smooth* γ.

In order to study nonsmooth geodesic loops, we develop the following notion: geodesic chains.

Definition 10. If a (finite) sequence of points $\{x^{(i)}\}_{i=0}^m \subset M$ satisfies that each $x^{(i)}$, $i = 1, \ldots, m$, realizes the injectivity radius of $x^{(i-1)}$ via some geodesic loop $\gamma^{(i)}$, then such points and loops together is called a *geodesic chain*. We denote it as $G(x^{(0)}, \ldots, x^{(m)})$.

A manifold M is said to satisfy the *non-accumulated property* if for all $D > 0$, there exists a positive integer n_0 such that $\frac{dist(x^{(0)}, x^{(n_0)})}{inj(x^{(0)})} > D$ for all $x^{(0)} \in M$ and all geodesic chains $G(x^{(0)}, \ldots, x^{(m)}) \subset M$ satisfying that $G(x^{(0)}, \ldots, x^{(m)}) \setminus B_{2D \cdot inj(x^{(0)})} \neq \phi$.

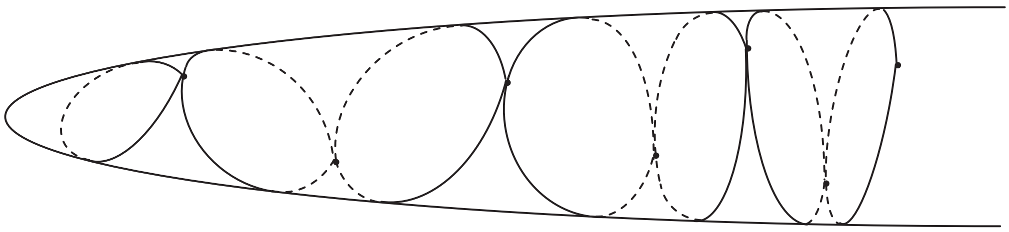

Figure 1: A cylinder-like end does not satisfy the non-accumulated property.

Theorem 16. *Let M be a complete Riemannian manifold satisfying $|Sect| \leq C \cdot s^{-2}$, where $s := dist(O, x)$. If M satisfies the smooth loop condition and the non-accumulated property, then there exists a constant $\delta > 0$ such that $inj(x) \geq \delta \cdot s$ for all $x \in M$.*

The first idea of the proof is to rescale the collapsing sequence of balls, if there exists, and get a flat complete limiting manifold. This argument was used before by others such as the proofs of Main Lemma 2.2 in Anderson's paper [1] and Theorem 5.42 in Chow-Lu-Ni's book [24]. The second step in our proof is to derive a contradiction from the observation of limiting closed geodesic under the non-accumulated condition. This step relies on the product structure of a complete flat manifold. Thus we know that there exists no collapsing sequence and the injectivity radius indeed grows linearly.

Proof. Let $q_k \in M$ and $\lambda_k := \frac{1}{2} dist(O, q_k) \to \infty$. For $x \in B_{\lambda_k}(q_k)$, we want to show that $inj(x) \geq \delta \cdot dist(x, \partial B_{\lambda_k}(q_k))$. If this is the case, then this theorem follows by taking $x = q_k$.

We argue by contradiction. Suppose that there exist $\delta_k \searrow 0$ and $x_k \in B_{\lambda_k}(q_k)$ such that $inj(x_k) = \delta_k \cdot d_k$, where $d_k := dist(x, \partial B_{\lambda_k}(q_k))$. Furthermore, we may assume that the function $F(y) := \frac{inj(y)}{dist(y, \partial B_{\lambda_k}(q_k))}$, $y \in B_{\lambda_k}(q_k)$, achieves its minimum at x_k. (Notice that F blows up on the boundary.) Hence

$$inj(y) = F(y) \cdot dist(y, \partial B_{\lambda_k}(q_k)) \geq F(x_k) \cdot \frac{1}{2} dist(x_k, \partial B_{\lambda_k}(q_k)) = \frac{1}{2} inj(x_k)$$

for all $y \in B_{\frac{1}{2}d_k}(x_k)$.

Let $\widetilde{g}_k := (\delta_k d_k)^{-2} g$ be the sequence of rescaled metrics which has $\widetilde{inj}(x_k) \equiv 1$. Consider the sequence of rescaled pointed geodesic balls $\left(\widetilde{B}_{\frac{1}{2\delta_k}}(x_k), x_k, \widetilde{g}_k\right)$ and its limit. Since $|\widetilde{Sect}| \leq C \cdot \lambda_k^{-2}\delta_k^2 d_k^2 \to 0$ and $\widetilde{inj} \geq \frac{1}{2}$ on $\widetilde{B}_{\frac{1}{2\delta_k}}(x_k)$, by using the harmonic coordinates, we know that the sequence converges to a complete flat manifold (B, x_∞, g_∞) in $C^{1,\sigma} \cap L^{2,p}$-topology (for all p and $\sigma \in (0,1)$.) For the usage of the harmonic coordinates, one can consult, for example, [1, 50].

Notice that the flat limit manifold B is non-compact because $diam\left(\widetilde{B}_{\frac{1}{2\delta_k}}\right) \to \infty$. So it might be $\mathbb{R}^{n-1} \times \mathbb{S}^1$ or $\mathbb{R}^{n-k} \times \mathbb{F}^k$, where $\mathbb{F}^k$ is a Bieberbach manifold. Furthermore, since the rescaling factor $(\delta_k d_k)^{-2}$ was chosen to make $\widetilde{inj}(x_k) = 1$ for all k, $inj(x_\infty) = 1$ implies that $B \neq \mathbb{R}^n$. For later use, we denote D as the diameter of one slice of B, that is, $\mathbb{S}^1$ or $\mathbb{F}^k$.

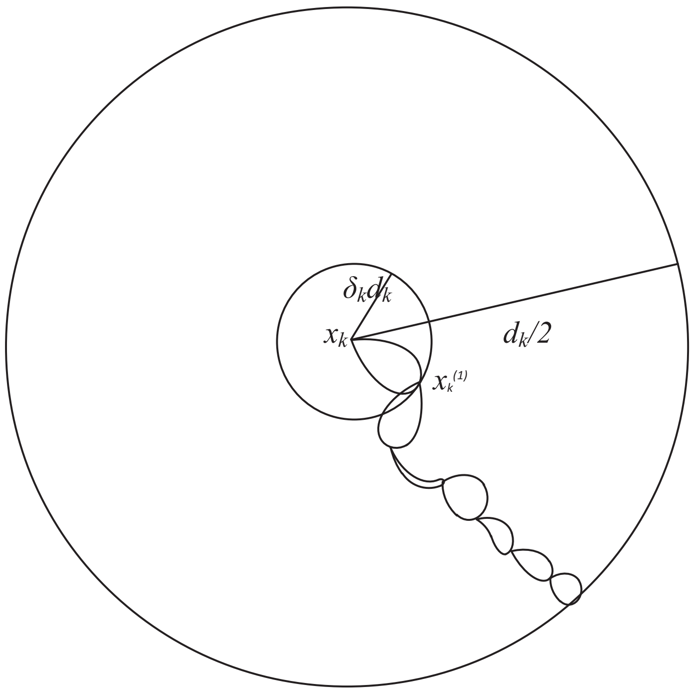

Figure 2: Construct a geodesic chain from each $x_k^{(0)} \equiv x_k$.

In the rest of the proof, we show that none of these cases can happen, hence a contradiction arises. Consider a point $x_k^{(1)}$ which realizes the injectivity radius of $x_k^{(0)} := x_k$. By the assumption on the sectional curvature, we know that $x_k^{(1)}$ cannot be conjugate to x_k. Therefore, there exists a geodesic loop $\gamma_k^{(1)}$ passing through $x_k^{(1)}$ and x_k with length $2\delta_k d_k$.

Since the loop must be smooth at $x_k^{(1)}$ (cf. Lemma 5.6 [15]), by our smooth loop condition, this loop is not smooth at x_k. This implies that $inj\left(x_k^{(1)}\right) < inj(x_k)$.

So we can find another point $x_k^{(2)}$ which realizes the injectivity radius of $x_k^{(1)}$. This process can continue until some point $x_k^{(m_k)}$ has its nearest cut point $x_k^{(m_k+1)} \notin B_{\frac{1}{2}d_k}(x_k)$. (Note that for every real number $D > 0$ and k large enough, there is an integer $m'_k < m_k$ such that $x_k^{(m'_k)} \notin B_{D\delta_k d_k}(x_k)$. A priori, m'_k depends on k. However, the non-accumulated condition acclaims that there exists a number n_0 such that $x_k^{(n_0)} \notin B_{D\delta_k d_k}(x_k)$ for all k. We shall use this in the next paragraph.)

Now, on each rescaled ball $\widetilde{B}_{\frac{1}{2\delta_k}}(x_k)$ we have a geodesic chain $G\left(x_k^{(0)} \equiv x_k, \ldots, x_k^{(m_k)}, \ldots\right)$. Exactly, we have a finite sequence of points $\left\{x_k^{(i)}\right\}_{i=0}^{m_k}$ and geodesic loops $\left\{\gamma_k^{(i)}\right\}_{i=1}^{m_k}$ with lengthes $\left|\gamma_k^{(i)}\right| \geq \frac{1}{2}, \forall i = 1, \ldots, m_k$. We want to take a subsequential limit of these chains into B (and derive a contradiction.) There are two possibilities: either there are two limit points $x_\infty^{(i-1)}$ and $x_\infty^{(i)}$ lying in different slices, or all the points accumulate to the same slice $\{x_\infty\} \times \mathbb{F}^k$ (or $\{x_\infty\} \times \mathbb{S}^1$.) By the non-accumulated condition, there is a limit point $x_\infty^{(n_0)}$ such that $dist(x_\infty^{(n_0)}, x_\infty) > 2D$ where D is the diameter of one slice of B. Hence the second case shall be ruled out.

The first case is also impossible. Indeed, if there exists a geodesic loop $\gamma_\infty^{(i)}$ which is not contained in the slice $\{x_\infty^{i-1}\} \times \mathbb{F}^k$ (or $\{x_\infty^{i-1}\} \times \mathbb{S}^1$) of B, then we can project it to get a strictly shorter geodesic loop which is contained in $\{x_\infty^{i-1}\} \times \mathbb{F}^k$ (or $\{x_\infty^{i-1}\} \times \mathbb{S}^1$). This contradicts the fact that $inj(x_\infty^{(i-1)}) = \frac{1}{2}\left|\gamma_\infty^{(i)}\right|$. □

Combining our results in the previous section, we can prove some noncollapsing properties on certain Ricci solitons.

Corollary 4. *Let M be a gradient Ricci soliton satisfying $R_{ij} + \nabla_i\nabla_j f = \lambda g_{ij}$ and the non-accumulated property. Suppose $|Sect| \leq \frac{C}{r^2}$ on $B_r(x) \subset M$ for some $x \in M$ and $r > 0$. If*

(i) $\lambda > 0$ and $Ric < \lambda \cdot g$ on $B_r(x)$,

(ii) $\lambda = 0$ and $Ric > 0$ on $B_r(x)$ or

(iii) $\lambda < 0$ and $Ric > \lambda \cdot g$ on $B_r(x)$,

then $inj(x) \geq \kappa r$ for some constant $\kappa > 0$.

Remark 16. In [45], A. Naber proved that every n-dimensional shrinking Ricci soliton with bounded curvature and $n \geq 2$ is κ-noncollapsed (in the sense of Perelman.) In [21], it was proved that every n-dimensional expanding Ricci soliton with $\lim_{dist(O,x)\to\infty} dist(O,x)^2 \cdot |Sect| = 0$ and $n \geq 3$ has linearly growing injectivity radius.

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