Lyapunov functions : a weak KAM approach

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Résumé

Cette thèse est divisée en trois parties. Dans une première partie, on donne une description nouvelle des points récurrents par chaînes d'un système dynamique comme ensemble d'Aubry projeté d'une barrière ultramétrique. Cette approche permet de munir l'ensemble des composantes transitives par chaînes d'une structure d'espace ultramétrique expliquant leur topologie totalement discontinue, et de retrouver un théorème célèbre de Charles Conley concernant l'existence de fonctions de Lyapunov décroissant strictement le long des orbites non-récurrentes par chaînes. Dans une deuxième partie, on développe une théorie d'Aubry-Mather pour les homéomorphismes d'un espace métrique compact. On introduit dans ce cadre un ensemble d'Aubry métrique, puis topologique, ainsi qu'un ensemble de Mañé. Ces notions, plus fines que la récurrence par chaînes, permettent de mieux comprendre les fonctions de Lyapunov d'un tel système dynamique. Dans une dernière partie, on montre un résultat général de densité de certains contre-exemples au théorème de Sard pour lesquels l'ensemble des points critiques est un arc topologique et on donne des applications dynamiques de ce résultat. Celles-ci sont liées à des problèmes d'unicité, à constantes près, des solutions KAM faibles (ou solutions de viscosité) de certaines équations d'Hamilton-Jacobi.

Abstract

This thesis is divided into three parts. In the first part, we give a new description of chain-recurrence using an ultrametric barrier. This barrier allows to endow the space of chain-transitive components with an ultrametric structure, explaining its topology and leading to the famous result of Charles Conley about Lyapunov function decreasing along non chain-recurrent orbits. Most of the results, first given in the setting of a continuous map on a compact metric space are then generalised to multivalued map on arbitrary separable metric spaces. In the second part, we develop an Aubry-Mather theory for a homeomorphism on a compact metric space. In this setting, we introduce metric and topological Aubry set and Mañé set, allowing a better understanding of Lyapunov functions arising in such a dynamical system. In the last part, we prove a general density result for some counterexamples of Sard's theorem for which the set of critical points is a topological arc and we give applications to dynamics.

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Introduction

Lorsque l'on cherche à étudier le comportement d'un système dynamique, il peut être extrêmement intéressant de construire ou d'exhiber des fonctions possèdant une certaine monotonie le long des orbites de ce système. De telles fonctions sont appelées, de manière générale, des *fonctions de Lyapunov*. Celles-ci tirent leur nom du mathématicien russe A.M Lyapunov, qui les introduit pour la première fois à la fin du $19^{\text{ième}}$ siècle, pour étudier la stabilité de certains points d'équilibres d'équations différentielles. Les sous-niveaux d'une fonction de Lyapunov fournissent en effet des ensembles semi-invariants pour la dynamique, permettant de confiner les orbites et de détecter des propriétés qualitatives importantes comme l'attraction, la non-errance ou encore la récurrence par chaînes. Réciproquement, une information sur la dynamique impose à son tour un certain nombre de contraintes aux fonctions de Lyapunov. De telles considérations permettent par exemple d'aborder certains problèmes d'unicité en théorie des équations aux dérivées partielles.

Dans cette thèse, nous montrons comment des idées issues de la théorie KAM faible permettent de construire des fonctions de Lyapunov et de préciser leurs interactions avec la dynamique. Bien qu'aucune connaissance préalable en théorie KAM faible ne soit nécessaire à sa lecture, un aperçu général de certaines grandes notions de cette théorie aidera à apprécier les nombreuses analogies qui ont fondé ces travaux : c'est le but de la première partie de cette introduction. Les deux parties restantes sont consacrées à une présentation des résultats obtenus durant cette thèse.

1 Théorie KAM faible et barrière de Peierls

Considérons une variété compacte M, de classe C^∞ et sans bord. On notera par (x, v) un point du fibré tangent TM, avec $x \in M$ et $v \in T_xM$, l'espace tangent à M en x. La projection canonique $\pi : TM \to M$ est alors donnée par $(x, v) \mapsto x$. On notera similairement par (x, p) un point du fibré cotangent T^*M, où $p \in T_x^*M$ est une forme linéaire sur T_xM.

On supposera dans la suite que $L : TM \to \mathbb{R}$ est un lagrangien de *Tonelli*, c'est à dire un fonction de classe C^2, superlinéaire et strictement convexe dans les fibres. Les extrémales de L définissent alors un flot ϕ_t sur TM appelé flot d'*Euler-Lagrange*. On notera $H : T^*M \to \mathbb{R}$ le hamiltonien associé à L par dualité convexe i.e défini pour tout $(x, p) \in T^*M$ par

$$H(x, p) = \sup_{v \in T_xM} \{p(v) - L(x, v)\}.$$

Sous les hypothèses de Tonelli, le supremum est atteint en l'unique $v \in T_xM$ tel que $p = \partial L(x, v)/\partial v$ et la *transformation de Legendre* $\mathcal{L} : TM \to T^*M$ définie par

$(x,v) \mapsto (x, \partial L(x,v)/\partial v)$ est un difféomorphisme conjugant le flot ϕ_t au flot hamiltonien ϕ_t^* de H. Nous renvoyons aux premiers chapitres de [10] pour une présentation détaillée de ces résultats.

Lorsque l'on étudie la dynamique d'un tel lagrangien de Tonelli, but premier de la théorie KAM faible, il est naturel de rechercher des ensembles invariants par le flot d'Euler-Lagrange ϕ_t. Une méthode consiste alors à résoudre l'équation d'*Hamilton-Jacobi*

$$H(x, d_x u) = c, \quad c \in \mathbb{R},$$

puisque toute solution $u : M \to \mathbb{R}$ de classe C^1 de cette équation fournit un graphe invariant, voir [10, Chapitre IV] :

Théorème 1.1 (Hamilton-Jacobi)**.** *Si $u : M \to \mathbb{R}$ est une solution C^1 de l'équation d'Hamilton-Jacobi*

$$H(x, d_x u) = c, \quad c \in \mathbb{R},$$

alors l'ensemble $\mathrm{Graph}(du) = \{(x, d_x u) \mid x \in M\}$ *est invariant par le flot hamiltonien de H.*

Malheureusement, comme expliqué dans [6], de telles solutions existent rarement. On est alors amené, pour obtenir des résulats d'existence généraux, a développer une notion plus faible de solution : c'est un des objets de la théorie KAM faible.

1.1 Sous-solutions de l'équation d'Hamilton-Jacobi et solutions KAM faibles

Supposons que $u : M \to \mathbb{R}$ soit une solution de classe C^1 de l'équation d'Hamilton-Jacobi $H(x, d_x u) = c$ et considérons une courbe $\gamma : [a,b] \to M$, $a \leq b$, continue et C^1 par morceaux. L'inégalité

$$d_x u(v) \leq L(x,v) + H(x, d_x u),$$

satisfaite pour tout $(x,v) \in TM$, conduit alors par intégration à l'inégalité suivante :

$$u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(s), \dot\gamma(s)) ds + c(b-a).$$

Une fonction $u : M \to \mathbb{R}$ satisfaisant cette inégalité pour toute courbe $\gamma : [a,b] \to M$, $a \leq b$, continue et C^1 par morceaux est dite *dominée* par $L + c$. Une courbe $\gamma : [a,b] \to \mathbb{R}$ est alors dite (u, L, c)*-calibrée* lorsque l'inégalité précédente est une égalité. De manière plus générale, une courbe $\gamma : I \to \mathbb{R}$ définie sur un intervalle non-compact I de $\mathbb{R}$ sera dite (u, L, c)-calibrée si sa restriction à tout sous-intervalle compact de I est (u, L, c)-calibrée.

Comme remarqué par A.Fathi [10], la notion de domination ne requiert aucune hypothèse de différentiabilité sur u et peut être utilisée comme définition de *sous-solution* continue de l'équation d'Hamilton-Jacobi. Cette définition coincide en fait avec la notion de *sous-solution de viscosité*, introduite par Crandall et Lions. On montre alors qu'il existe une constante $c(H) \in \mathbb{R}$ telle que l'équation d'Hamilton-Jacobi $H(x, d_x u) = c$ n'admette aucune sous-solution pour $c < c(H)$ mais possède des sous-solutions pour $c \geq c(H)$. La constante $c(H)$ est appelée *valeur critique* *de Mañé* et les sous-solutions $u : M \to \mathbb{R}$ de $H(x, d_x u) = c(H)$ sont appelées *sous-solutions critiques*. On peut montrer que ces sous-solutions sont localement lipschitziennes donc presque partout différentiable par le théorème de Rademacher. De plus, si $x \in M$ est un point de différentiabilité d'une sous-solution critique $u : M \to \mathbb{R}$, on a

$$H(x, d_x u) \leq c(H).$$

Une sous-solution critique $u : M \to \mathbb{R}$ est dite *stricte* en un point $x \in M$ s'il existe une constante $c < c(H)$ et un voisinage U de x tels que, pour presque tout $y \in U$,

$$H(y, d_y u) \leq c.$$

A.Fathi et A.Siconolfi ont montré, voir [14], qu'il existait toujours une sous-solution critique C^1 à l'équation d'Hamilton-Jacobi, stricte en dehors d'un fermé optimal $\mathcal{A}$ de M, défini dans la section suivante. Plus récemment, P.Bernard [5] a même montré qu'une telle solution pouvait être choisie de classe $C^{1,1}$.

Les *solutions KAM faibles* sont des sous-solutions critiques particulières possèdant de nombreuses courbes calibrantes. Elles sont définies par la propriété additionnelle suivante : pour tout $x \in M$, il existe une courbe $(u, L, c(H))$-calibrée $\gamma_x :]-\infty, 0] \to M$, de classe C^1, telle que $\gamma_x(0) = x$. Ces solutions correspondent aux points fixes du semi-groupe de Lax-Oleinik et leur existence est assurée par le théorème KAM faible de Fathi, voir [10]. On peut en fait montrer que les solutions KAM faibles coincident avec la notion usuelle de *solution de viscosité* de l'équation d'Hamilton-Jacobi $H(x, d_x u) = c(H)$. De plus, la constante critique $c(H)$ est l'unique constante c telle que l'équation d'Hamilton-Jacobi $H(x, d_x u) = c$ admette de telles solutions KAM faibles. Nous renvoyons le lecteur à [10] pour plus de détails.

1.2 Ensemble d'Aubry et ensemble de Mañé

Si $u : M \to \mathbb{R}$ est une sous-solution critique, c'est à dire si u est dominée par $L + c(H)$ et si $\gamma : \mathbb{R} \to M$ est une courbe $(u, L, c(H))$-calibrée, alors γ est nécessairement une extrémale de L. A toute sous-solution critique $u : M \to \mathbb{R}$, on associe ainsi le sous-ensemble $\tilde{\mathcal{I}}(u)$ de TM défini par

$$\tilde{\mathcal{I}}(u) = \{(x, v) \in TM \mid \gamma_{(x,v)} \text{ est } (u, L, c(H))\text{-calibrée}\}$$

où $\gamma_{(x,v)} : \mathbb{R} \to M$ est la projection sur M de l'orbite du flot d'Euler-Lagrange passant par le point $(x, v) \in TM$ au temps $t = 0$. Le résultat suivant, voir [10, Chapitre 5], est une généralisation C^0 du théorème d'Hamilton-Jacobi et explique l'intérêt des sous-solutions strictes dans l'étude de la dynamique du flot d'Euler-Lagrange.

Théorème 1.2. *L'ensemble $\tilde{\mathcal{I}}(u)$ est non vide et invariant par le flot d'Euler-Lagrange. De plus, si $(x, v) \in \tilde{\mathcal{I}}(u)$, la fonction u est différentiable en x et on a*

$$(x, v) = \mathcal{L}^{-1}(x, d_x u) \text{ et } H(x, d_x u) = c(H).$$

En particulier, l'ensemble $\tilde{\mathcal{I}}(u)$ et un graphe au dessus de sa projection $\mathcal{I}(u)$ sur M. De plus, ce graphe est localement bi-lipschitzien.

L'*ensemble d'Aubry* est le sous-ensemble $\tilde{\mathcal{A}}$ de TM défini par

$$\tilde{\mathcal{A}} = \bigcap_{u \in \mathcal{SS}} \tilde{\mathcal{I}}(u),$$

l'intersection étant prise sur l'ensemble $\mathcal{SS}$ des sous-solutions critiques de l'équation d'Hamilton-Jacobi. C'est un graphe bi-lipschitzien au dessus de sa projection, invariant par le flot d'Euler-Lagrange. Important dans la compréhension des systèmes dynamiques lagrangiens, il a été découvert indépendament par Aubry et Mather en 1982 dans le cas des twist maps, puis dans toute sa généralité par Mather en 1988. L'*ensemble d'Aubry projeté*, noté $\mathcal{A}$, est la projection sur M de l'ensemble d'Aubry $\tilde{\mathcal{A}}$. Il vérifie les propriétés suivantes, voir [14] :

(i) il n'existe aucune sous-solution critique $u : M \to \mathbb{R}$ stricte en un point $x \in \mathcal{A}$,

(ii) il existe une sous-solution critique $u : M \to \mathbb{R}$ qui est stricte en dehors de $\mathcal{A}$.

De plus, les solutions KAM faibles sont entièrement définies par leur restriction à l'ensemble d'Aubry projeté, puisque l'on a le résultat suivant, voir [10, Chapitre V] ou [11, théorème 1] :

Théorème 1.3. *Deux solutions KAM faibles qui coïncident sur $\mathcal{A}$ coïncident sur M.*

L'*ensemble de Mañé* est défini par

$$\tilde{\mathcal{N}} = \bigcup_{u \in \mathcal{SS}} \tilde{\mathcal{I}}(u).$$

Comme l'ensemble d'Aubry, l'ensemble de Mañé est un compact invariant par le flot d'Euler-Lagrange. De plus, on a le résulat dynamique suivant :

Théorème 1.4 (Mañé)**.** *Chaque point du compact invariant $\tilde{\mathcal{A}}$ est récurrent par chaîne pour la restriction $\phi_t|_{\tilde{\mathcal{A}}}$ du flot d'Euler-Lagrange et le compact invariant $\tilde{\mathcal{N}}$ est transitif par chaînes pour la restriction $\phi_t|_{\tilde{\mathcal{N}}}$ du flot d'Euler-Lagrange.*

1.3 La barrière de Peierls

On peut donner une définition alternative de l'ensemble d'Aubry projeté reposant sur la *barrière de Peierls*, introduite par Mather [21, page 1372]. Pour $t > 0$ on définit, suivant Mather, la fonction $h_t : M \times M \to \mathbb{R}$ par

$$h_t(x, y) = \inf \int_0^t L(\gamma(s), \dot{\gamma}(s)) ds$$

où l'infimum est pris sur les courbes $\gamma : [0, t] \to M$, continues et C^1 par morceaux, vérifiant $\gamma(0) = x$, $\gamma(t) = y$. La barrière de Peierls est l'application $h : M \times M \to \mathbb{R}$ définie par

$$h(x, y) = \liminf_{t \to +\infty} \{h_t(x, y) + c(H)t\}.$$

C'est une application lipschitzienne, vérifiant les propriétés suivantes, voir [10, Chapitre V] :

(i) pour tout $x \in M$,
$$h(x,x) \geq 0,$$

(ii) pour tout $x, y, z \in M$,
$$h(x,y) \leq h(x,z) + h(z,y),$$

(iii) on a
$$\mathcal{A} = \{x \in M \mid h(x,x) = 0\}.$$

Les propriétés (i) à (iii) permettent de définir, toujours suivant Mather [21, page 1370], une pseudo-métrique δ_M sur $\mathcal{A}$, définie par
$$\delta_M(x,y) = h(x,y) + h(y,x).$$

L'ensemble quotient $\mathcal{A}_M$ obtenu en identifiant les points de $\mathcal{A}$ à distance nulle pour δ_M forme un espace métrique $(\mathcal{A}_M, \delta_M)$ appelé *quotient de Mather*. La fonction δ_M est la *pseudo-métrique de Mather*.

La barrière de Peierls entretient également un rapport étroit avec les sous-solutions critiques de l'équation d'Hamilton Jacobi, au travers des deux propriétés suivantes, voir [10, Chapitre V] ou [14] :

(a) si $u : M \to \mathbb{R}$ est une sous-solution critique de l'équation d'Hamilton-Jacobi, alors pour tout $x, y \in M$,
$$u(y) - u(x) \leq h(x,y).$$

(b) pour tout $x \in M$, la fonction
$$h(x,\cdot) : M \to \mathbb{R}$$
est une solution KAM faible de l'équation d'Hamilton-Jacobi.

Ces deux propriétés mènent à une *formule de représentation* de la pseudo-métrique de Mather à l'aide des sous-solutions critiques, voir [13, Lemme 2.7] : pour tout $x, y \in \mathcal{A}$,
$$\delta_M(x,y) = \max_{u_1,u_2 \in \mathcal{SS}} \{(u_1 - u_2)(y) - (u_1 - u_2)(x)\}.$$

Associée au théorème 1.3, cette formule fournit un critère élégant d'unicité, à constante près, des solutions KAM faibles, voir [13, Proposition 4.4] :

Proposition 1.5. *Les propositions suivantes sont équivalentes :*

(i) *deux solutions KAM faibles diffèrent d'une constante,*

(ii) *le quotient de Mather $(\mathcal{A}_M, \delta_M)$ est trivial i.e. réduit à un point.*

De plus, si l'une de ces conditions est vérifiées, on a $\tilde{\mathcal{A}} = \tilde{\mathcal{N}}$.

2 Une approche KAM faible de la théorie des fonctions de Lyapunov

Il existe de nombreuses définitions des fonctions de Lyapunov, variant d'un contexte à l'autre. Toutes ont cependant en commun la propriété de décroitre le long des orbites d'un système dynamique sous-jacent. Cette propriété simple ne requiert en particulier aucune hypothèse de différentiabilité. Nous adopterons donc la définition suivante. Soit (X, d) un espace métrique compact et $h : X \to X$ (resp. $(\varphi_t)_{t\in\mathbb{R}} : X \to X$) un homéomorphisme de X (resp. un flot continu sur X.) On appelle *fonction de Lyapunov* pour h (resp. pour φ_t) toute fonction continue $\theta : X \to \mathbb{R}$ vérifiant $\theta \circ h \leq \theta$ (resp. $\theta \circ \varphi_t \leq \theta$, pour tout $t \geq 0$.) Notez que les fonctions constantes sont des fonctions de Lyapunov, qualifiées de *triviales*. Dans la suite, nous nous concentrerons essentiellement sur le cas des homéomorphismes, puisque la plupart des résultats analogues concernant les flots peuvent s'obtenir en considérant leurs temps 1.

2.1 Ensemble neutre d'une fonction de Lyapunov et théorème de Conley

Une fonction de Lyapunov θ est utile si l'on peut donner une description à priori de son *ensemble neutre*, défini par

$$N(\theta) = \{x \in X \mid \theta(h(x)) = \theta(x)\}.$$

Cet ensemble n'est jamais vide puisqu'il contient les points x où θ atteint son minimum. De plus, on a les inclusions suivantes

$$\mathrm{Fix}(h) \subset \mathrm{Per}(h) \subset \Omega(h) \subset N(\theta).$$

Une constante $\alpha \in \mathbb{R}$ est appelée *valeur neutre* de θ si

$$\theta^{-1}(\{\alpha\}) \cap N(\theta) = \emptyset.$$

Si α n'est pas une valeur neutre de θ, le fermé F_α de X défini par

$$F_\alpha = \{x \in X \mid \theta(x) \leq \alpha\}$$

est envoyé dans son intérieur par h. Cette propriété permet de montrer le critère important suivant :

Lemme 2.1. *Si les valeurs neutres de θ sont d'intérieur vide dans $\mathbb{R}$ alors on a l'inclusion*

$$\mathcal{R}(h) \subset N(\theta)$$

où $\mathcal{R}(h)$ désigne l'ensemble des points récurrents par chaînes de h.

Dans sa démarche de classification des flots sur un espace métrique compact, voir [8], Conley démontre alors le résultat suivant, voir [7, Chapter II, Section 6.4] :

Théorème 2.2 (Conley)**.** *Il existe une fonction de Lyapunov $\theta : X \to \mathbb{R}$ pour h, dont l'ensemble des valeurs neutres est d'intérieur vide dans $\mathbb{R}$, satisfaisant*

$$N(\theta) = \mathcal{R}(h).$$

De plus, la fonction θ est constante sur chaque composante transitive par chaînes et prend des valeurs différentes sur des composantes différentes.

Ce théorème montre en particulier que l'ensemble des composantes transitives par chaînes de h, muni de la topologie quotient, est totalement discontinu. La dynamique obtenue en collapsant les composantes transitives par chaînes ressemble alors à la dynamique du flot gradient d'une fonction de Morse puisqu'elle vérifie les propriétés suivantes :

(i) il existe une fonction de Lyapunov strictement décroissante le long des orbites non-constantes,

(ii) l'ensemble des points fixes coïncide avec l'ensemble des points récurrents par chaînes, qui est totalement discontinu.

L'étude de la dynamique de h peut donc se décomposer une partie "gradient" et une partie récurrente par chaînes : c'était la motivation initiale de Conley.

Dans le premier chapitre de cette thèse, on donne une démonstration du théorème de Conley basée sur l'utilisation d'une technique de barrière inspirée de la théorie KAM faible. Il est en effet possible de voir les points récurrents par chaînes de h comme ensemble d'Aubry (projeté) associé à la *barrière de Conley* $S : X \times X \to \mathbb{R}_+$ définie par

$$S(x, y) = \inf \left\{ \max_{i=0,..,n-1} d(h(x_i), x_{i+1}) \right\},$$

l'infimum étant pris sur toutes les suites finies $\{x_0, .., x_n\}$, $n \geq 1$, de points de X satisfaisant $x_0 = x$, $x_n = y$,

$$\mathcal{R}(h) = \{x \in X \mid S(x, x) = 0\}.$$

L'inégalité ultramétrique satisfaite par S

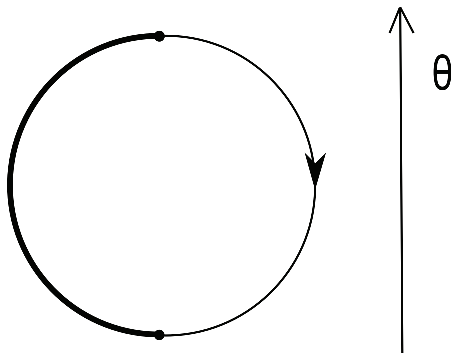
$$S(x, y) \leq \max\{S(x, z), S(z, y)\}$$

permet alors de munir, à la manière de la pseudo-métrique de Mather, l'ensemble des composantes transitives par chaînes de h d'une structure d'espace ultramétrique, expliquant sa topologie totalement discontinue. De plus, les fonctions $S(x, \cdot)$, $x \in X$, sont des fonctions de Lyapunov pour h qui, associées à l'ultramétricité, permettent de retrouver l'énoncé de Conley. Ces résultats sont ensuite étendus au cas plus général d'une application multivaluée sur un espace métrique séparable localement compact, permettant ainsi de retrouver certaines généralisations initialement dues à Hurley [17].

2.2 Théorie d'Aubry-Mather d'un homéomorphisme

Lorsque l'on retire l'hypothèse d'intérieur vide sur les valeurs neutres d'une fonction de Lyapunov, il est tout à fait possible que l'inclusion $\mathcal{R}(h) \subset N(\theta)$ ne soit pas satisfaite, même si la fonction θ est extrêmement régulière, comme le montre l'exemple suivant.

Exemple 2.3. La flèche indique la direction de la dynamique, les points gras étant fixés.



Sur cet exemple, la fonction hauteur θ est une fonction de Lyapunov C^∞. L'ensemble neutre de θ est réduit au demi-cercle de points fixes mais tous les points sont récurrents par chaînes. Notez que la fonction θ possède bien tout un intervalle de valeurs neutres.

Dans la deuxième partie de cette thèse, on introduit un compact invariant $\mathcal{A}_d(h)$ de l'espace métrique (X, d), dépendant uniquement de h et de la métrique d, optimal au sens suivant :

(i) toute fonction de Lyapunov $\theta : X \to \mathbb{R}$ lipschitzienne pour d satisfait $\mathcal{A}_d(h) \subset N(\theta)$,

(ii) il existe une fonction de Lyapunov $\theta : X \to \mathbb{R}$ lipschitzienne pour d satisfaisant $\mathcal{A}_d(h) = N(\theta)$.

Cet *ensemble d'Aubry métrique* pour h est le point de départ d'une théorie d'Aubry-Mather pour les homéomorphismes. Sa définition repose encore une fois sur l'utilisation d'une barrière, ici définie par

$$L_d(x, y) = \inf \left\{ \sum_{i=0}^{n-1} d(h(x_i), x_{i+1}) \right\},$$

l'infimum étant pris sur toutes les suites finies $\{x_0, .., x_n\}$, $n \geq 1$, de X vérifiant $x_0 = x$ et $x_n = y$,

$$\mathcal{A}_d(h) = \{x \in X \mid L_d(x, x) = 0\}.$$

Puisque L_d est positive et satisfait l'inégalité triangulaire

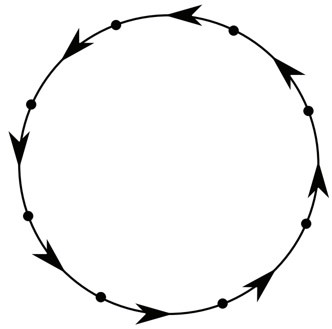
$$L_d(x, y) \leq L_d(x, z) + L_d(z, y),$$

la fonction symétrique $L_d^*(x, y) = L_d(x, y) + L_d(y, x)$ définit une pseudo-métrique sur $\mathcal{A}_d(h)$. L'espace métrique quotient $(\mathcal{M}_d(h), L_d^*)$ obtenu en identifiant les points à distance nulle pour L_d^* est appelé *d-quotient de Mather* de h. On a alors le résultat suivant :

Théorème 2.4. *Il existe une fonction de Lyapunov lipschitzienne non-triviale θ : $(X, d) \to \mathbb{R}$ pour h si et seulement si $\mathcal{M}_d(h)$ est non-trivial i.e. est non réduit à un point.*

Ce résulat est intéressant, puisqu'il permet en de construire des fonctions de Lyapunov non-triviales pour des dynamiques récurrentes par chaînes, comme dans l'example 2.3, ce qui n'était pas le cas du théorème de Conley. De plus, l'aspect métrique permet de distinguer certaines dynamiques identiques d'un point de vue topologique.

Exemple 2.5. On se place sur $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ muni de la métrique plate usuelle. Soit $K \subset \mathbb{T}$ un ensemble de Cantor et soit $\varphi : \mathbb{T} \to [0, +\infty[$ une fonction C^∞ telle que $\varphi^{-1}(0) = K$. On note h le temps 1 du flot du champs de vecteur $X(x) = \varphi(x)\frac{\partial}{\partial x}$.



Si la mesure de Lebesgue de K est nulle, alors $\mathcal{A}_d(h) = \mathbb{T}$ et $\mathcal{M}_d(h)$ est réduit à un point. Si la mesure de Lebesgue de K n'est pas nulle, alors $\mathcal{A}_d(h) = K$ et $\mathcal{M}_d(h)$ est homéomorphe à K. Notez que dans les deux cas, tous les points sont récurrents par chaînes et qu'il n'y a qu'une seule composante transitive par chaînes.

Puisque toute fonction continue de X dans $\mathbb{R}$ est lipschitzienne pour une métrique adaptée, ces résultats permettent également de décrire l'ensemble $\mathrm{Aus}(h)$ des *points récurrents généralisés* de h, introduit par Auslander dans [3] et défini par la propriété suivante :

(i) pour toute fonction de Lyapunov $\theta : X \to \mathbb{R}$, on a $\mathrm{Aus}(h) \subset N(\theta)$,

(ii) il existe une fonction de Lyapunov $\theta : X \to \mathbb{R}$ avec $\mathrm{Aus}(h) = N(\theta)$.

Plus précisément, on a le théorème suivant :

Théorème 2.6. *On a*

$$\mathrm{Aus}(h) = \bigcap_d \mathcal{A}_d(h).$$

l'intersection étant prises sur toutes les métriques d compatibles avec la topologie de X.

L'ensemble des points récurrents généralisés d'Auslander joue donc le rôle d'*ensemble d'Aubry topologique* de h et le théorème 2.6 suggère la définition suivante de l'*ensemble de Mañé* :

$$\tilde{\mathcal{N}}(h) = \bigcup_d \mathcal{A}_d(h).$$

On démontre alors le résultat suivant :

Théorème 2.7. *On a*

$$\tilde{\mathcal{N}}(h) = \mathrm{Fix}(h) \cup \mathcal{R}(h_{|X \setminus \mathrm{int}(\mathrm{Fix(h)})}),$$

où $\mathcal{R}(h_{|X \setminus \mathrm{int}(\mathrm{Fix(h)})})$ désigne l'ensemble des points récurrents par chaînes de la restriction de h à $X \setminus \mathrm{int}(\mathrm{Fix(h)})$.

3 Fonctions de Lyapunov en théorie KAM faible : l'exemple des lagrangiens de Mañé

Des fonctions de Lyapunov apparaissent naturellement en théorie KAM faible. Elles permettent de mieux comprendre la dynamique du flot d'Euler-Lagrange et de s'intéresser à des problèmes d'unicité des solutions KAM faible de l'équation d'Hamilton-Jacobi. Nous invitons le lecteur intéressé à consulter [13, Section 4] pour une présentation détaillée. Nous nous consacrons ici au cas particulier des lagrangiens de Mañé.

3.1 Lagrangiens de Mañé

On considère une variété riemannienne compacte connexe (M, g), de classe C^∞ et sans bord, sur laquelle est définie un champ de vecteur X de classe C^k, $k \geq 2$. La norme d'un élément $v \in T_xM$ relativement à la métrique g sera notée $||v||_x$ et le flot de X sur M sera noté φ_t. On notera $\pi : TM \to M$ la projection canonique de TM sur M.

Il existe une manière naturelle d'inclure la dynamique de X dans une dynamique lagrangienne en considérant le lagrangien $L_X : TM \to \mathbb{R}$ défini par

$$L_X(x, v) = \frac{1}{2}||v - X(x)||_x^2.$$

On appelle L_X le *lagrangien de Mañé* de X. C'est un lagrangien de classe C^k, $k \geq 2$, satisfaisant les hypothèses de Tonelli. On notera ϕ_t son flot. L'ensemble d'Aubry associé à L_X sera noté $\tilde{\mathcal{A}}_X$ et l'ensemble d'Aubry projeté correspondant sera noté $\mathcal{A}_X$. L'ensemble de Mañé associé à L_X sera lui noté $\tilde{\mathcal{N}}_X$. On a alors le résultat suivant, voir [13, proposition 4.13] :

Proposition 3.1. *L'hamiltonien $H_X : T^*M \to \mathbb{R}$ associé à L_X est donné par*

$$H_X(x, p) = \frac{1}{2}||p||_x^2 + p(X(x))$$

et les constantes sont solutions de l'équation d'Hamilton-Jacobi

$$H_X(x, d_xu) = 0.$$

La constante critique $c(H_X)$ est donc nulle. De plus on a

$$\tilde{\mathcal{I}}(0) = \mathrm{Graph}(X) = \{(x, X(x)) \mid x \in X\},$$

et la restriction $\phi_t|_{\tilde{\mathcal{I}}(0)}$ du flot d'Euler-Lagrange à l'ensemble invariant $\tilde{\mathcal{I}}(0)$ est conjugué (par $\pi|_{\tilde{\mathcal{I}}(0)}$) au flot φ_t de X sur M.

Puisque tous les points de $\tilde{\mathcal{A}}_X$ sont récurrent par chaînes pour la restriction $\phi_t|_{\tilde{\mathcal{A}}_X}$ du flot d'Euler-Lagrange à $\tilde{\mathcal{A}}_X$, on déduit de la proposition précédente l'inclusion générale suivante :

Proposition 3.2. *On a*

$$\mathcal{A}_X \subset \mathcal{R}(X),$$

où $\mathcal{R}(X)$ désigne l'ensemble des points récurrents par chaîne du flot de X sur M.

3.2 Propriété de Lyapunov des sous-solutions critiques

Si $u : M \to \mathbb{R}$ est une sous-solution critique de $H_X(x, d_x u) = 0$ on a, pour tout chemin $\gamma : [a, b] \to M$, $a \leq b$, de classe C^1,

$$u(\gamma(b)) - u(\gamma(a)) \leq \frac{1}{2} \int_a^b ||\dot{\gamma}(s) - X(\gamma(s))||^2_{\gamma(s)} ds.$$

Appliquée au flot φ_t de X sur M, cette propriété montre donc que pour tout $t \geq 0$ et tout $x \in M$,

$$u(\varphi_t(x)) - u(x) \leq 0.$$

Toute sous-solution critique u de l'équation $H_X(x, d_x u) = 0$ fournit donc une fonction de Lyapunov pour le flot φ_t de X sur M. De plus, si $x \in X$ est un point de différentiabilité de u, on a $H_X(x, d_x u) \leq 0$ i.e.

$$\frac{1}{2} ||d_x u||^2_x + d_x u(X(x)) \leq 0.$$

En particuier, $X \cdot u(x) = d_x u(X(x)) \leq 0$ avec égalité si et seulement si $d_x u = 0$. On a donc la proposition suivante :

Proposition 3.3 (Propriété de Lyapunov)**.** *Si $u : M \to \mathbb{R}$ est une sous-solution critique de $H_X(x, d_x u) = 0$ alors u est une fonction de Lyapunov pour le flot φ_t de X. De plus, si x est un point de différentiabilité de u on a $X \cdot u(x) \leq 0$ avec égalité si et seulement si $d_x u = 0$. Dans ce cas, on a $H_X(x, d_x u) = 0$.*

Corollaire 3.4. *Si $u : M \to \mathbb{R}$ est une sous-solution critique C^1 de l'équation d'Hamilton-Jacobi*

$$H_X(x, d_x u) = 0$$

alors pour tout $x \in M$ on a $X \cdot u(x) \leq 0$ avec égalité si et seulement si $d_x u = 0$.

Le champ de vecteur X se comporte donc comme un *gradient* pour toute sous-solution stricte de l'équation $H_X(x, d_x u) = 0$.

3.3 Condition de disconnection de Mather

On dira que le lagrangien L_X satisfait la *condition de disconnection de Mather*, voir [13, Section 4] si, pour toute sous-solution critique u et v de l'équation $H_X(x, d_x u) = 0$, l'image $(u - v)(\mathcal{A}_X)$ est d'intérieur vide dans $\mathbb{R}$. Si L_X satisfait la condition de disconnection de Mather, toute sous-solution critique u de l'équation $H_X(x, d_x u) = 0$ vérifie en particulier que l'image $u(\mathcal{A}_X)$ est d'intérieur vide dans $\mathbb{R}$ puisque 0 est une sous-solution critique de cette même équation.

Considérons alors une sous-solution critique $u : M \to \mathbb{R}$ de classe C^1, stricte en dehors de $\mathcal{A}_X$ i.e. vérifiant $H_X(x, d_x u) \leq 0$ avec égalité si et seulement si $x \in \mathcal{A}_X$. Une telle sous-solution existe d'après les travaux de A.Fathi et A.Siconolfi [14, Théorème 1.3]. Cette sous-solution fournit alors, d'après la proposition 3.3, une fonction de Lyapunov pour le temps 1 du flot de X sur M, dont l'ensemble neutre $N(u)$ est contenu dans l'ensemble d'Aubry projeté $\mathcal{A}_X$. Si L_X satisfait la condition de disconnection de Mather, l'image $u(N(u))$ est alors d'intérieur vide et d'après le lemme 2.1, on a $\mathcal{R}(\varphi_1) = \mathcal{R}(X) \subset \mathcal{A}_X$. Ceci montre le résulat suivant :

Proposition 3.5. *Si L_X satisfait la condition de disconnection de Mather, on a*

$$\mathcal{A}_X = \mathcal{R}(X).$$

Si le flot de X sur M est récurrent par chaînes, i.e. $\mathcal{R}(X) = M$, et si L_X satisfait la condition de disconnection de Mather, toute sous-solution critique $u : M \to \mathbb{R}$ est donc nécessairement constante sur M puisque M est connexe et l'image $u(M)$ est alors totalement discontinue. En particulier, si L_X satisfait la condition de disconnection de Mather et si le flot de M est récurrent par chaînes, les seules solutions KAM faibles de l'équation $H_X(x, d_x u) = 0$ sont les constantes. Réciproquement, si les seules solutions KAM faibles de l'équation $H_X(x, d_x u) = 0$ sont les constantes, on deduit de la proposition 2 que $\tilde{\mathcal{A}}_X = \tilde{\mathcal{N}}$. Puisque $\tilde{\mathcal{A}}_X \subset \tilde{\mathcal{I}}(0) \subset \tilde{\mathcal{N}}$ on a donc $\tilde{\mathcal{A}}_X = \tilde{\mathcal{I}}(0)$. Tous les points de $\tilde{\mathcal{A}}_X$ étant récurrent par chaînes pour la restriction $\phi_t|_{\tilde{\mathcal{A}}_X}$ du flot d'Euler-Lagrange à $\tilde{\mathcal{A}}_X$, on déduit de la propositon 5.1 que $\mathcal{R}(X) = M$. On a donc le théorème suivant, voir [13, Lemme 4.14] :

Théorème 3.6. *Si L_X satisfait la condition de disconnection de Mather, les propriétés suivantes sont vérifiées :*

(i) *l'ensemble d'Aubry projecté $\mathcal{A}_X$ est l'ensemble des points récurrents par chaînes du flot de X sur M,*

(ii) *les constantes sont les seules solutions KAM faibles de l'équation*

$$H_X(x, d_x u) = 0$$

si et seulement si le flot de X sur M est récurrent par chaînes.

En s'intéressant à la mesure de Hausdorff 1-dimensionnelle du quotient de Mather de $\tilde{\mathcal{A}}_X$, il est possible de montrer, voir [13], que L_X satisfait la condition de disconnection de Mather dès que $\mathrm{Dim}(M) = 1, 2$ et X est de classe C^2 ou $\mathrm{Dim}(M) = 3$ et X est de classe $C^{k,1}$ avec $k \geq 3$.

3.4 Fonctions dont l'ensemble des points critiques est un arc

Le théorème précédent répond donc, sous certaines conditions, à deux questions générales soulevées par A.Fathi dans [13] :

Questions. Etant donné un lagrangien de Mañé $L_X : TM \to \mathbb{R}$ associé à un champs de vecteur X de classe C^k, $k \geq 2$, sur une variété riemannienne compact connexe M,

(1) l'ensemble des points récurrents par chaînes du flot de X sur M coïncide-t-il avec l'ensemble d'Aubry projeté $\mathcal{A}_X$?

(2) peut on donner une condition sur la dynamique de X assurant que les seules solutions KAM faibles soient les constantes ?

Si L_X ne satisfait pas la condition de disconnection de Mather il est possible, en utilisant des contres-exemples de Whitney au théorème de Sard, de construire des exemples pour lesquels $\mathcal{A}_X \neq \mathcal{R}(X)$, voir [13, Section 4.4]. La réponse à la première question est donc négative en général. On ne sait toujours pas cependant si la condition $\mathcal{R}(X) = M$ permet d'affirmer, sans l'hypothèse de disconnection de Mather, que les seules solutions KAM faibles soient les constantes. La troisième partie de cette thèse est ainsi motivée par la question sous-jacente suivante :

(Q) Est il possible de trouver un champs de vecteur C^k, $k \geq 2$, sur une variété riemannienne compacte connexe M, tel que tous les points de M soient récurrents par chaînes pour le flot de X, bien qu'il existe une solution KAM faible $v : M \to \mathbb{R}$ non-constante à l'équation d'Hamilton-Jacobi $H_X(x, d_x u) = 0$?

Supposons qu'une telle solution $v : M \to \mathbb{R}$ existe. Puisque deux solutions KAM faibles coïncidant sur $\mathcal{A}_X$ coïncident sur M et que les constantes sont des solutions KAM faibles, la fonction v n'est pas constante sur $\mathcal{A}_X$. D'après les résultats de [14], il existe une sous-solution critique $u : M \to \mathbb{R}$ de classe C^1, coïncidant avec v sur l'ensemble d'Aubry projeté $\mathcal{A}_X$. La fonction u est donc une fonction C^1 non-constante sur M satisfaisant, d'après le corollaire 3.4 : pour tout $x \in M$, $X \cdot u(x) \leq 0$, avec égalité si et seulement si x est un point critique de u. On est donc amené à se poser la question intermédiaire suivante :

(Q') Est il possible de trouver un champs de vecteur C^k, $k \geq 2$, sur une variété compacte connexe M, tel que tous les points de M soient récurrents par chaînes pour le flot de X, bien qu'il existe une fonction non-constante $u : M \to \mathbb{R}$ de classe C^1, telle que

  (i) pour tout $x \in M$, on a $X \cdot u(x) \leq 0$,

  (ii) étant donné $x \in M$, on a $X \cdot u(x) = 0$ si et seulement si $d_x u = 0$.

L'existence d'un tel couple (X, u) n'est pas du tout évidente. La fonction u doit en particulier violer les conclusions du théorème de Sard. En effet, les propriétés (i) et (ii) impliquent que u est une fonction de Lyapunov pour le temps 1 du flot de X, telle que $N(u) \subset \mathrm{Crit}(u)$. Si les valeurs critiques de u étaient d'intérieur vide, on aurait par le lemme 2.1

$$M = \mathcal{R}(X) \subset N(u) \subset \mathrm{Crit}(u)$$

et la fonction u serait nécessairement constante sur la variété connexe M. Il est cependant possible, en utilisant un type particulier de contres exemples au théorème de Sard, de montrer le résultat suivant :

Théorème 3.7. *Soit M une variété C^∞ compacte connexe, sans bord avec* $\mathrm{Dim}(M) \geq 2$. *Il existe un champs de vecteur X sur M, de classe C^∞, et une fonction non-constante $u : M \to \mathbb{R}$ de classe C^1 tels que*

(i) *tous les points de M sont récurrents par chaînes pour le flot de X,*

(ii) *pour tout $x \in M$, on a $X \cdot u(x) \leq 0$ avec égalité si et seulement si $d_x u = 0$.*

Comme expliqué dans la troisième partie de cette thèse, il suffit pour démontrer ce théorème de construire une fonction C^1 non-constante sur M dont l'ensemble des points critiques est un connexe de M. Une telle fonction est bien évidement surprenante puisque les minimums et les maximums d'une fonction sont habituellement imaginés appartenir à des composantes distinctes des points critiques. L'existence d'une telle fonction est pourtant une conséquence du théorème général suivant, démontré dans cette thèse à l'aide d'outils d'altération développés par Körner [20] :

Théorème 3.8. *Soit M une variété C^∞ compacte connexe, sans bord avec* $\mathrm{Dim}(M) \geq 2$. *L'ensemble des fonctions $f \in C^1(M, \mathbb{R})$ dont l'ensemble des points critiques est un arc i.e. est homéomorphe au segment* $[0,1]$, *est dense dans* $C^0(M, \mathbb{R})$.

Un tel énoncé incite donc à répondre par la négative à la question (Q) même s'il ne fournit pas de contre exemple à proprement dit. On conjecture qu'il est possible de trouver, dès que $\mathrm{Dim}(M) \geq n \geq 2$, une fonction non-constante $u \in C^{n-1}(M, \mathbb{R})$ dont l'ensemble des points critiques est connexe. Pour $n \geq 4$, une telle fonction u fournirait effectivement un contre exemple. En effet, si X est le champs de vecteur $-\frac{1}{2}\operatorname{grad}_g(u)$, alors X est de classe C^2, tous les point de M sont récurrents par chaînes pour le flot de X et u est une solution KAM faible non-constante de l'équation d'Hamilton-Jacobi $H_X(x, d_x u) = 0$.

Chapter 1

Conley barriers and their applications

1 Introduction

The purpose of this paper is to shed a different light on chain-recurrence for dynamical systems on arbitrary separable metric space. The initial work of Conley [7] describes the structure of chain-recurrent points in terms of attractors of f and their basins of attraction. It is in line with the theory of dynamical systems done in the last fifty years, see for example [25]. The work of Conley is surveyed by Hurley [17, 18] where it is extended to the settings of arbitrary separable metric space. Moreover, in this work Hurley constructs a type of Lyapunov function which gives a good insight in the structure of chain-recurrent points. Here is a statement.

Theorem 1.1. *Let X be a separable metric space and f be a continuous map from X to itself. Then there exists a continuous function $\phi : X \longrightarrow \mathbb{R}$ such that*

(i) *The function ϕ is nonincreasing along orbits of f and is decreasing along orbits of non chain-recurrent points.*

(ii) *The function ϕ takes on distinct values on distinct chain-transitive components and sends the set of chain-recurrent points in a subset of the Cantor middle-third set.*

The point of view taken in this paper is different and is inspired by the work of Fathi [10] in Weak KAM theory. We will associate a cost to chains in order to construct a barrier function, called a Conley barrier. Here are its main properties.

Theorem 1.2. *Let X be a compact metric space and f be a continuous map from X to itself. Then there exists a continuous function*

$$S : X \times X \longrightarrow \mathbb{R}_+$$

such that

(i) *For every $(x, y) \in X^2$, we have $S(x, y) = 0$ if and only if for every $\varepsilon > 0$ there exists an ε-chain from x to y.*

(ii) *For every $(x, y, z) \in X^3$, we have $S(x, y) \leq \max(S(x, z), S(z, y))$,* The existence of such a barrier allows to describe chain-recurrence only in terms of continuous functions and introduced an ultrametric structure on the set of chain-transitive components. This will lead to similar results as Hurley's ones, at least in the case of a separable locally compact metric space. For the sake of clarity, the first part of this paper is devoted to the compact case. Nevertheless, the compactness assumption is not essential to obtain a Conley barrier. This is the object of the second section. Moreover, we will deal with compactum-valued maps since this does not raise any new difficulty. Finally we highlight the link between chain-recurrence for the identity map on X and topological properties of X.

2 The compact case

2.1 Definitions and background

Throughout this section (X, d) will denote a compact metric space and f a continuous map from X to itself.

Definition 2.1. Let $(x, y) \in X^2$ and $\varepsilon > 0$. An *ε-chain* for f from x to y is a finite sequence $(x_0 = x, ..., x_n = y)$, $n \geq 1$, of X such that

$$\forall i \in \{0, .., n-1\},\ d(f(x_i), x_{i+1}) < \varepsilon.$$

A point x in X is called *chain-recurrent* if for every $\varepsilon > 0$ there exists an ε-chain from x to x. We denote by $\mathcal{R}(f)$ the set of *chain-recurrent points* of f. We define an equivalence relation $\backsim$ on the set $\mathcal{R}(f)$ by $x \backsim y$ if and only if for every $\varepsilon > 0$ there are ε-chains from x to y and from y to x. The equivalence classes are called the *chain-transitive components* of f and the associated quotient space is denoted by $\mathcal{R}(f)/\backsim$.

It would be straightforward to verify that these notions are topological and do not depend on the metric d on X. In fact, it will be made clear in section 3. We now describe the main object of this paper.

Definition 2.2. Let X be a compact metric space and f be a continuous map from X to itself. A *Conley barrier* for f is a continuous function

$$S : X \times X \longrightarrow \mathbb{R}_+$$

with the properties that

(i) *For every $(x, y) \in X^2$, we have $S(x, y) = 0$ if and only for every $\varepsilon > 0$ there exists an ε-chain from x to y.*

(ii) *For every $(x, y, z) \in X^3$, we have $S(x, y) \leq \max(S(x, z), S(z, y))$.*

With respect to property (i) any Conley barrier is in fact a barrier for chain-recurrence. The following simple lemma will be used many time.

Lemma 2.3. *For every $x \in X$, we have $S(x, f(x)) = 0$.*

Proof. For every $x \in X$ and $\varepsilon > 0$, the chain $(x, f(x))$ is always an ε-chain from x to $f(x)$. Thus we have $S(x, f(x)) = 0$ everywhere on X. □

As stated in the following theorem, we can always find a Conley barrier for dynamical systems on compact metric space.

Theorem 2.4. *Let X be a compact metric space and f be a continuous map from X to itself. Then there exists a Conley barrier for f.*

Proof. The proof of this theorem will be done in section 2.4. □

Corollary 2.5. *The set $\mathcal{R}(f)$ is a closed subset of X.*

Proof. It follows from property (i) of definition 2.2 that $\mathcal{R}(f) = \{x \in X,\ S(x,x) = 0\}$. Since S is continuous, this set is a closed subset of X. □

Proposition 2.6. *The subset $\mathcal{R}(f)$ and the chain-transitive components are stable under f.*

Proof. First, we will show that

$$\forall x \in \mathcal{R}(f), S(f(x), x) = 0.$$

Let $x \in \mathcal{R}(f)$. If $f(x) = x$, there is nothing to prove. Therefore, we can assume that $d(f(x), x) > 0$. Let $\varepsilon > 0$ and consider $\eta > 0$ such that $\eta < \min(d(f(x), x), \frac{\varepsilon}{2})$. Since x is chain-recurrent, there exists a η-chain $(x_0 = x, ..., x_m = x)$ from x to x. The condition $\eta < d(f(x), x)$ forces $m \geq 2$. By continuity of f, reducing even more η if necessary, we can also assume that $f(B(f(x), \eta)) \subset B(f^2(x), \frac{\varepsilon}{2})$. The chain $(f(x), x_2, ..., x_m = x)$ is then an ε-chain from $f(x)$ to x. Since ε is arbitrary, it follows that $S(f(x), x) = 0$.

Now if $x \in \mathcal{R}(f)$ then $S(f(x), f(x)) \leq \max(S(f(x), x), S(x, f(x))) = 0$ by lemma 2.3. Thus $S(f(x), f(x)) = 0$ and $f(x) \in \mathcal{R}(f)$. Moreover, since $S(x, f(x)) = S(f(x), x) = 0$ the points x and $f(x)$ are in the same chain-transitive component. Thus the subset $\mathcal{R}(f)$ and the chain-transitive components are stable under f.

□

Before making S explicit, we are going to develop two consequences: an ultrametric distance on the set of chain-transitive components, and the existence of Lyapunov functions for f.

2.2 An ultrametric distance on the space of chain-transitive components

Pseudo-distance In this section, we recall some general facts about pseudo-distances. They will be used to endow the space of chain-transitive components with an ultrametric distance.

Definition 2.7. A *pseudo-distance* on a space E is a function

$$d : E \times E \longrightarrow \mathbb{R}_+$$

such that

(i) *For every $x \in E$, we have $d(x, x) = 0$.*

(ii) *For every $x, y, z \in E$, we have $d(x, y) \leq d(x, z) + d(z, y)$.*

(iii) *For every $x, y \in E$, we have $d(x, y) = d(y, x)$.*

Let d be a pseudo-distance on E. We define an equivalence relation $\mathcal{R}$ on E by

$$x\mathcal{R}y \iff d(x, y) = 0.$$

We denote by $E/\mathcal{R}$ the set of associated equivalence classes. The following lemma is well-known so we omit its proof.

Lemma 2.8. *The pseudo-distance d induces a distance $\overline{d}$ on the quotient space $E/\mathcal{R}$. Moreover, if the space E is endowed with a topology making d continuous, then the quotient topology is finer than the topology defined by the metric $\overline{d}$.*

Remark 2.9. In the lemma above, if the pseudo-distance d satisfies the stronger *ultrametric* inequality

$$d(x, y) \leq \max(d(x, z), d(z, y))$$

then the distance $\overline{d}$ inherits of the same property and thus defines an *ultrametric distance* on the quotient space $E/\mathcal{R}$.

Ultrametric distance induced by a Conley barrier on the set of chain-transitive components The existence of a Conley barrier leads to the existence of a non-trivial ultrametric distance on the set of chain-transitive components. To see this, let us remark that the equivalence relation $\backsim$ defined on the set of chain-transitive components can be formulated in the following way

$$x \backsim y \iff \max(S(x, y), S(y, x)) = 0.$$

The quantity

$$\Delta(x, y) := \max(S(x, y), S(y, x))$$

is a symmetric expression in x and y and inherits of the ultrametric inequality satisfied by S. Thus, on the subset $\mathcal{R}(f) = \{x \in X,\ \Delta(x, x) = 0\}$ the function Δ is satisfying all axioms of an ultrametric pseudo-distance. As described in the previous section, it naturally induces an ultrametric distance $\overline{\Delta}$ on the quotient space $\mathcal{R}(f)/\backsim$, i.e. on the space of chain-transitive components.

Corollary 2.10. *Let X be a compact metric space and f be a continuous map from X to itself. Then the set of chain-transitive components with the quotient topology is a compact ultrametric space. We can take as a metric any ultrametric distance induced by a Conley barrier for f. In particular, this set is totally disconnected and Hausdorff.*

Proof. The set of chain-recurrent points is closed in X and hence compact. Since the canonical projection

$$\mathcal{R}(f) \xrightarrow{p} (\mathcal{R}(f)/\backsim, \textit{quotient topology})$$

is continuous, the space $(\mathcal{R}(f)/\backsim, \textit{quotient topology})$ is also compact.

Let $\overline{\Delta}$ introduced above be an ultrametric distance induced by a Conley barrier on the set of chain-transitive components of f. Since Δ is continuous, it follows from lemma 2.8 that the quotient topology is finer than the ultrametric topology induced by $\overline{\Delta}$. Thus, in the following diagram the identity map

$$(\mathcal{R}(f)/\backsim, \textit{quotient topology}) \xrightarrow{Id} \left(\mathcal{R}(f)/\backsim,\ \overline{\Delta}\right)$$

is a continuous bijection. Since the metric space $\left(\mathcal{R}(f)/\backsim, \overline{\Delta}\right)$ is Hausdorff, the same goes for $(\mathcal{R}(f)/\backsim, \textit{quotient topology})$. This set is thus a compact Hausdorff space. The identity map is then an homeomorphism and both topologies are the same. Since for an ultrametric distance every open ball is also closed, the set of chain-transitive components is totally disconnected. □

2.3 Lyapunov functions

Definitions. We can use a Conley barrier to construct different types of Lyapunov functions for f. The following definition is used by Hurley, see [17, 18]. For general recalls about Hausdorff dimension, see [16].

Definition 2.11. A *strict* Lyapunov function for f is a continuous function $\varphi : X \longrightarrow \mathbb{R}$ such that

(i) *For every $x \in X$, we have $\varphi(f(x)) \leq \varphi(x)$.*

(ii) *For every $x \in X \setminus \mathcal{R}(f)$, we have $\varphi(f(x)) < \varphi(x)$.*

A strict Lyapunov function is said to be *complete* if it satisfies the following additional property

(i') *The function φ is constant on each chain-transitive component, takes on distinct values on distinct chain-transitive components and sends the subset $\mathcal{R}(f)$ into a subset of $\mathbb{R}$ whose Hausdorff dimension is zero.*

Our construction of Lyapunov functions will use a particular kind of functions, called *sub-solutions* of S. Here is the definition.

Definition 2.12. Let S be a Conley barrier for f. A *sub-solution* for S is a continuous function

$$u : X \longrightarrow \mathbb{R}$$

such that

$$\forall (x, y) \in X^2,\ u(y) - u(x) \leq S(x, y).$$

A sub-solution is said to be *strict* if the inequality is strict as soon as x is not chain-recurrent for f.

Lemma 2.13. *Any sub-solution for S is nonincreasing along orbits of f and any strict sub-solution is decreasing along orbits of non chain-recurrent points. Thus any strict sub-solution for S is a strict Lyapunov function for f.*

Proof. The proof follows from definitions and lemma 2.3. □

The following lemma gives a fundamental example of sub-solutions.

Lemma 2.14. *For every z in X, the function*

$$S_z: \begin{array}{rcl} X & \longrightarrow & \mathbb{R} \\ x & \longmapsto & S(z,x) \end{array}$$

is a sub-solution for S.

Proof. Since a Conley barrier satisfies an ultrametric inequality, it also satisfies the triangle inequality. Thus for every x, y in X we have

$$S(z,y) \leq S(z,x) + S(x,y)$$

which yields the wanted inequality. □

Strict Lyapunov functions. We now construct a strict Lyapunov function for f. We will see later how sub-solutions of the type S_x can in fact be used to construct a complete Lyapunov function for f.

Theorem 2.15. *Let X be a compact metric space and f be a continuous map from X to itself. There exists a sequence $(x_i)_{i\in\mathbb{N}}$ of points of X and a sequence $(\eta_i)_{i\in\mathbb{N}}$ of positive reals such that the series*

$$\varphi = \sum_{i\in\mathbb{N}} \eta_i S_{x_i}$$

is a strict sub-solution for S, and thus a strict Lyapunov function for f.

Proof. Since the metric space X is compact, it is separable. Let $(x_i)_{i\in\mathbb{N}}$ be a dense sequence in X and $(\eta_i)_{i\in\mathbb{N}}$ be a sequence of positive reals such that $\sum_{i\in\mathbb{N}} \eta_i = 1$. The continuous function S is bounded on the compact set $X \times X$. Thus, the condition $\sum_{i\in\mathbb{N}} \eta_i = 1$ insures that the series $\sum_{i\in\mathbb{N}} \eta_i S_{x_i}$ converges uniformly on X. Hence, it defines a continuous function φ on X. Moreover, the function φ is a sub-solution since a convex combination of sub-solutions is still a sub-solution. Now suppose that $x \in X$ is not chain-recurrent. Then we have $S(x,x) > 0$ and thus $S(x,y) - S(x,x) < S(x,y)$. By density of the $(x_i)_{i\in\mathbb{N}}$ and continuity of S, we can find an integer $j \in \mathbb{N}$ such that $S(x_j,y) - S(x_j,x) < S(x,y)$. Since the functions S_{x_i} are sub-solutions, we always have

$$\forall i \in \mathbb{N},\ S(x_i,y) - S(x_i,x) \leq S(x,y)$$

it follows that

$$\begin{aligned} \varphi(y) - \varphi(x) &= \sum_{i\in\mathbb{N}} \eta_i (S(x_i,y) - S(x_i,x)) \\ &< \sum_{i\in\mathbb{N}} \eta_i S(x,y) = S(x,y) \end{aligned}$$

Thus the function φ is a strict sub-solution for S and hence, a strict Lyapunov function for f. □

Complete Lyapunov function The construction of a complete Lyapunov function for f relies on the underlying ultrametric structure of the set of chain-transitive components. It strongly limits values taken by the sub-solutions $S_x, x \in X$ and will lead to functions with images of finite cardinality. The following lemma and corollary are thus fundamental.

Lemma 2.16. *For every $x \in X$, the function S_x is constant in the neighborhood of each point of the set $\mathcal{R}(f) \setminus \{S(x,\cdot) = 0\}$.*

Proof. Let $x \in X$ and $y \in \mathcal{R}(f)$ be such that $S(x,y) > 0$. Consider the open subset $U_{x,y}$ of X

$$U_{x,y} = \{S(y,\cdot) - S(x,y) < 0\} \cap \{S(\cdot,y) - S(x,\cdot) < 0\}.$$

Since $y \in \mathcal{R}(f)$ we have $S(y,y) = 0$ and thus $y \in U_{x,y}$. If $z \in U_{x,y}$ we have

$$S(x,z) \leq \max(S(x,y), S(y,z)) = S(x,y),$$

$$S(x,y) \leq \max(S(x,z), S(z,y)) = S(x,z).$$

Thus $S(x,z) = S(x,y)$ and S_x is constant on $U_{x,y}$. □

Corollary 2.17. *For every $x \in X$, the set $\{S(x,y),\ y \in \mathcal{R}(f)\}$ is countable. Moreover, the only possible accumulation point is zero. In particular, for every $\varepsilon > 0$, the function $\theta_\varepsilon \circ S_x$ where $\theta_\varepsilon(t) := \max(t - \varepsilon, 0)$ takes a finite number of values on $\mathcal{R}(f)$.*

Proof. Let $(x_i)_{i\in\mathbb{N}}$ be a dense sequence in X. Let $x \in X$. At each point of $\mathcal{R}(f)$, the function S_x is either 0 or constant in a neighborhood of that point. Thus, the set $\{S(x,y),\ y \in \mathcal{R}(f)\}$ is included in the set $\{S(x,x_j),\ j \in \mathbb{N}\} \cup \{0\}$ and hence is countable.

Now let α be an accumulation point of the set $\{S(x,y),\ y \in \mathcal{R}(f)\}$. There exists a sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathcal{R}(f)$ such that the sequence $(S(x,y_n))_{n\in\mathbb{N}}$ admits α as a limit with $S(x,y_n) \neq \alpha$, for every $n \in \mathbb{N}$. By compactness of X, we can suppose that y_n admits a limit $y \in X$. Since the set $\mathcal{R}(f)$ is closed, we have $y \in \mathcal{R}(f)$ and the continuity of S implies that $\alpha = S(x,y)$. If α is non zero then S_x would be constant in the neighborhood of y. This would contradicts the fact that for every $n \in \mathbb{N}, S(x,y_n) \neq \alpha$. Thus α is zero.

□

We can now prove the existence of a complete Lyapunov function for f.

Theorem 2.18. *Let X be a compact metric space and f be a continuous map from X to itself. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X and a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals such that the series*

$$\varphi = \sum_{n\in\mathbb{N}} \varepsilon_n \theta_{\frac{1}{n+1}} \circ S_{x_n}$$

defines a complete Lyapunov function for f.

Proof. For this proof, we will use lemma 5.1 of our Appendix. Let $(x_n)_{n\in\mathbb{N}}$ be a dense sequence in X. Repeating each x_n infinitely many times, we can suppose without lost of generality that for every $k \in \mathbb{N}$ the sequence $(x_n)_{n\geq k}$ is still dense in X. It easily follows from the ultrametric inequality satisfied by S and the definition of the relation $\backsim$ on the space of chain-recurrent points

$$x \backsim y \Longleftrightarrow \max(S(x,y), S(y,x)) = 0$$

that the functions S_x for x in X are constant on each chain-transitive components. It follows from corollary 4.3 that for every x in X and every $\varepsilon > 0$ the function

$$\theta_\varepsilon \circ S_x : X \longrightarrow \mathbb{R}$$

induces a function $\overline{\theta_\varepsilon \circ S_x}$ on the set of chain-transitive components with an image of finite cardinality. We will now apply lemma 5.1 of the Appendix to the space $A = \mathcal{R}(f)/\backsim$ together with the family $\left(\overline{\theta_{\frac{1}{n+1}} \circ S_{x_n}}\right)_{n\in\mathbb{N}}$. We just have to prove that this family separates chain-transitive components. If x and y are in distinct chain-transitive components, we have for example $S(x,y) > 0$. Since $S(x,x) = 0$, the continuity of S and the density of the $(x_n)_{n\geq k}$ for every $k \in \mathbb{N}$, implies that we can find an integer $n \in \mathbb{N}$ such that $0 \leq S(x_n, x) < S(x_n, y) - \frac{1}{n+1}$. Hence we have $\overline{\theta_{\frac{1}{n+1}} \circ S_{x_n}}(x) \neq \overline{\theta_{\frac{1}{n+1}} \circ S_{x_n}}(y)$. We conclude similarly if $S(y,x) > 0$.

Thus, lemma 5.1 furnishes a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals such that the series $\sum_{n\in\mathbb{N}} \varepsilon_n \overline{\theta_{\frac{1}{n+1}} \circ S_{x_n}}$ converges on $\mathcal{R}(f)/\backsim$, separates points of $\mathcal{R}(f)/\backsim$ and has an image in $\mathbb{R}$ whose Hausdorff dimension is zero. Each continuous functions $\theta_{\frac{1}{n+1}} \circ S_{x_n}$ is bounded on the compact set X. Since the positive reals $(\varepsilon_n)_{n\in\mathbb{N}}$ can be chosen arbitrarily small, we can also suppose that the non-negative series

$$\varphi = \sum_{n\in\mathbb{N}} \varepsilon_n \theta_{\frac{1}{n+1}} \circ S_{x_n}$$

converges uniformly on X. The fact that the series $\sum_{n\in\mathbb{N}} \varepsilon_n \overline{\theta_{\frac{1}{n+1}} \circ S_{x_n}}$ separates points of $\mathcal{R}(f)/\backsim$ and has an image in $\mathbb{R}$ whose Hausdorff dimension vanishes precisely means that the function φ takes on distinct values on distinct chain-transitive components and sends $\mathcal{R}(f)$ in a subset of $\mathbb{R}$ whose Hausdorff dimension is zero.

To complete the proof, we just have to show that φ is nonincreasing along orbits of f and decreasing along orbits of non chain-recurrent points. The first part is true since for every $x \in X$ the sub-solution S_x is nonincreasing along orbits of f and each θ_ε is monotonous. Now if $x \in X \setminus \mathcal{R}(f)$, we have $S(x,x) > 0$. Since $S(x, f(x)) = 0$ and $(x_n)_{n\geq k}$ is dense for every $k \in \mathbb{N}$, we can find $n \in \mathbb{N}$ such that

$$0 \leq S(x_n, f(x)) < S(x_n, x) - \frac{1}{n+1}.$$

Thus we have $\theta_{\frac{1}{n+1}} \circ S_{x_n}(f(x)) < \theta_{\frac{1}{n+1}} \circ S_{x_n}(x)$ so that $\varphi(f(x)) < \varphi(x)$. □

2.4 Conley barrier

We now come to the construction of a Conley barrier. As a cost for chain, we will consider the maximum of the size of the different jumps. This leads to the following.

Definition 2.19. For every $(x,y) \in X^2$, we set

$$S(x,y) := \inf \left\{ \max_{i\in\{0,..,n-1\}} d(f(x_i), x_{i+1}) \mid n \geq 1,\ x_0 = x, ..., x_n = y \right\}.$$

We now prove that the function S is a Conley barrier for f.

Lemma 2.20. *The function S satisfies the barrier property: for every (x,y) in X^2 we have $S(x,y) = 0$ if and only if for every $\varepsilon > 0$ there exists an ε-chain from x to y.*

Proof. The property becomes clear with the following equivalent definition of S

$$S(x,y) = \inf\{\varepsilon > 0 \mid \text{there exists an } \varepsilon\text{-chain from } x \text{ to } y\}.$$

□

Lemma 2.21. *The function S satisfies the ultrametric inequality*

$$\forall (x,y,z) \in X^3,\ S(x,y) \leq \max(S(x,z), S(z,y)).$$

Proof. Let $x,y,z \in X$ and $(x_0 = x, ..., x_n = z)$, $(z_0 = z, ..., z_m = y)$ be two chains from x to z and from z to y. The concatenated chain provides a chain $(y_0 = x, ..., y_{m+n+1} = y)$ from x to y and thus

$$\begin{aligned} S(x,y) &\leq \max_{j\in\{0,..,m+n\}} d(f(y_j), y_{j+1}) \\ &\leq \max\left(\max_{i\in\{0,..,n-1\}} d(f(x_i), x_{i+1}), \max_{j\in\{0,..,m-1\}} d(f(z_j), z_{j+1}) \right). \end{aligned}$$

The result follows by taking the infimum on chains from x to z and then on chains from z to y. □

Lemma 2.22. *The function S is continuous.*

Proof. Let $x, x', y, y' \in X$. If $(x_0 = x, ..., x_n = y)$ is a chain from x to y, the chain $(\widetilde{x}_0, ..., \widetilde{x}_n)$ obtained by replacing $x_n = y$ by y' is a chain from x to y' such that

$$\begin{aligned} \max_{i\in\{0,..,n-1\}} d(f(\widetilde{x}_i), \widetilde{x}_{i+1}) &\leq \max_{i\in\{0,..,n-1\}} d(f(x_i), x_{i+1}) \\ &\quad + |d(f(x_{n-1}), y) - d(f(x_{n-1}), y')| \\ &\leq \max_{i\in\{0,..,n-1\}} d(f(x_i), x_{i+1}) + d(y,y'). \end{aligned}$$

Hence we get

$$S(x,y') \leq \max_{i\in\{0,..,n-1\}} d(f(\widetilde{x}_i), \widetilde{x}_{i+1}) \leq \max_{i\in\{0,..,n-1\}} d(f(x_i), x_{i+1}) + d(y,y').$$

Taking the infimum on chains $(x_0, ..., x_n)$ from x to y we get

$$S(x,y') \leq S(x,y) + d(y,y').$$

Similarly, replacing $x_0 = x$ by x' we have

$$S(x', y) \leq S(x, y) + d(f(x), f(x')).$$

Exchanging role played by x, x' and y, y', we thus get

$$\begin{aligned} |S(x, y') - S(x, y)| &\leq d(y, y'), \\ |S(x', y) - S(x, y)| &\leq d(f(x), f(x')). \end{aligned}$$

It follows that

$$\begin{aligned} |S(x, y) - S(x', y')| &\leq |S(x, y) - S(x', y)| + |S(x', y) + S(x', y')| \\ &\leq d(f(x), f(x')) + d(y, y') \end{aligned}$$

and the continuity of S now follows from the continuity of f. □

Remark 2.23. This last proof shows that every function $S_x = S(x, \cdot)$ is 1-Lipschitzian. It follows that our Lyapunov functions are also Lipschitzian.

3 General construction

We would like to remove the compactness assumption made on X and to cover the case of compactum-valued maps, i.e. maps with values in the set $\Gamma(X)$ of nonempty compact subsets of X. In fact, as we will see, the existence of a Conley barrier only requires the separability of the ambient metric space.

3.1 Hausdorff metric and compactum-valued map

We briefly recall the definition of the Hausdorff topology on $\Gamma(X)$. For more details, see [22].

Definition 3.1. Let (X, d) be a metric space. If K and K' are two compact subsets of X, we define

$$\mathcal{D}_d(K, K') = \inf\{\varepsilon > 0 \mid K' \subset V_\varepsilon^d(K) \text{ and } K \subset V_\varepsilon^d(K')\}$$

where $V_\varepsilon^d(K) = \{x \in X, d(x, K) < \varepsilon\}$.

Proposition 3.2. *The function $\mathcal{D}_d$ is a distance on the set $\Gamma(X)$ of compact subsets of X. The topology it defines does not depend on the metric d used. It is called the Hausdorff topology on $\Gamma(X)$.*

Proof. The fact that the function $\mathcal{D}_d$ is a distance is clear. It does not depends on the metric used since the convergence of a sequence K_n to K can be expressed in a purely topological way. Indeed, the compactness of K implies that $\mathcal{D}_d(K_n, K) \to 0$ as $n \to +\infty$ if and only if

(i) For every neighborhood V of K there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $K_n \subset V$.

(ii) For every x in K there is a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in K_n$ such that $x_n \to x$ as $n \to +\infty$.

□

Definition 3.3. A *compactum-valued* map is a map from X to $\Gamma(X)$. It is said to be continuous if it is continuous for the Hausdorff topology on $\Gamma(X)$.

3.2 Chain-recurrence on arbitrary separable metric space

In the settings of a noncompact metric space, the notion of chain-recurrence is usually defined using the set $\mathcal{P}$ of continuous functions from X to $\mathbb{R}_+^*$ instead of constants $\varepsilon > 0$. We thus keep topological invariance, see [17]. The notion of $\mathcal{U}$-chain now introduced gives a powerful way to avoid using this set $\mathcal{P}$ and emphasizes the fact that the notion of chain-recurrence is a purely topological one.

Definition 3.4. Let $\mathcal{U}$ be an open covering of X. For $A \subset X$ we set

$$\mathcal{S}t(A,\mathcal{U}) = \bigcup_{\substack{U \in \mathcal{U} \\ A \cap U \neq \emptyset}} U.$$

An open covering $\mathcal{V}$ of X is called an *open refinement* of $\mathcal{U}$ and is denoted by $\mathcal{V} \propto \mathcal{U}$ if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$. An *open barycentric refinement* of $\mathcal{U}$ is an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that

$$\{\mathcal{S}t\left(\{x\}, \mathcal{V}\right), x \in X\} \propto \mathcal{U}\,.$$

Proposition 3.5. *In a metric space X, any open covering of X admits an open barycentric refinement.*

Proof. See for example [9, Chapter VIII, theorem 3.5]. □

Remark 3.6. The notion of barycentric refinement will be used to generalize arguments involving triangular inequalities.

Definition 3.7. Let (X, d) be a metric space and $f : X \longrightarrow \Gamma(X)$ be a compactum-valued map. Given an open covering $\mathcal{U}$ of X and (x, y) in X^2, a $\mathcal{U}$*-chain* from x to y for f is a sequence $(x_0 = x, ..., x_n = y), n \geq 1$, of X such that

$$\forall i \in \{0, .., n-1\},\ x_{i+1} \in \mathcal{S}t(f(x_i),\mathcal{U}).$$

We define similarly the set $\mathcal{R}(f)$ of chain-recurrent points, i.e. of points of X such that for every open covering $\mathcal{U}$ of X there exists a $\mathcal{U}$-chain from x back to x. The chain-transitive components are similarly defined using the equivalence relation $\backsim$ on $\mathcal{R}(f)$ given by $x \backsim y$ if and only if for every open covering $\mathcal{U}$ of X there exists $\mathcal{U}$-chains from x to y and from y to x. Two points x and y in X will be said to be *f-separated* by $\mathcal{U}$ if there exists no $\mathcal{U}$-chain for f from x to y.

Remark 3.8. Any continuous map $f : X \longrightarrow X$ can be seen as a continuous compactum-valued map since singletons are compact. Then, the previous definition just reduces to a sequence $(x_0 = x, ..., x_n = y), n \geq 1$, of X such that

$$\forall i \in \{0, .., n-1\},\ \exists U \in \mathcal{U},\ \begin{cases} f(x_i) \in U, \\ x_{i+1} \in U. \end{cases}$$

3.3 Chain-recurrence adapted distance

From now on, f will denote a continuous compactum-valued map on a separable metric space (X,d). Our purpose is to construct a distance δ on X which allows to define chain-recurrence in the same way as in the compact case. We will follow a scheme given essentially in the work of Hurley, see [17, 18].

Definition 3.9. A metric δ on X is said to be *chain-recurrence adapted* for f if it defines the topology of X and if for every x and y in X the following assertions are equivalent:

(i) *For every open covering* $\mathcal{U}$ *of* X*, there exists a* $\mathcal{U}$*-chain from* x *to* y*.*

(ii) *For every number* $\varepsilon > 0$*, there exists an* ε*-chain for* δ *from* x *to* y*.*

Remark 3.10. In the compactum-valued case, an ε-chain for δ is defined similarly with $\delta(f(x_i), x_{i+1})$ the distance from the point x_{i+1} to the compact subset $f(x_i)$.

A central point in the construction of a chain-recurrence adapted distance is to show that the elements of the set

$$E = \left\{ (x,y) \in X \times X \mid \begin{array}{l} \textit{there exists an open covering } \mathcal{U} \textit{ of } X \\ \textit{which } f\textit{-separates } x \textit{ and } y \end{array} \right\}$$

can be obtained from a countable family of open coverings of X.

Lemma 3.11. *If the metric space* (X,d) *is separable then there exists a countable family* $(\mathcal{U}_l)_{l\in\mathbb{N}}$ *of open coverings of* X *such that for every* $(x,y) \in E$ *there exists an open covering* $\mathcal{U}_k$ *in* $(\mathcal{U}_l)_{l\in\mathbb{N}}$ *that* f*-separates* x *from* y*.*

Remark 3.12. Such a family will be called a f*-separating* family.

Proof. Let $(x,y) \in E$ and $\mathcal{U}_{x,y}$ be an open covering of X which f-separates x from y. We will show that there are open neighborhoods $W_{x,y}$ of x and $W'_{x,y}$ of y and an open covering $\mathcal{V}_{x,y}$ of X which f-separates every point of $W_{x,y}$ from every point of $W'_{x,y}$.

Let $\widetilde{\mathcal{V}}_{x,y}$ be an open barycentric refinement of the open covering $\mathcal{U}_{x,y}$. The compact subset $f(x)$ is included into the open subset $\mathcal{S}t(f(x), \widetilde{\mathcal{V}}_{x,y})$. Thus by continuity of f, we can find a neighborhood $W_{x,y}$ of x such that

$$\forall x' \in W_{x,y},\ f(x') \subset \mathcal{S}t(f(x), \widetilde{\mathcal{V}}_{x,y}).$$

We first show that the open covering $\widetilde{\mathcal{V}}_{x,y}$ f-separates every point of $W_{x,y}$ from y. Let us suppose that for some $x' \in W_{x,y}$ there exists a $\widetilde{\mathcal{V}}_{x,y}$-chain $(x_0 = x', x_1, ..., x_n = y)$ from x' to y. Since $x_1 \in \mathcal{S}t(f(x'), \widetilde{\mathcal{V}}_{x,y})$ we can find $V_1 \in \widetilde{\mathcal{V}}_{x,y}$ such that

$$\begin{cases} V_1 \cap f(x') \neq \emptyset, \\ x_1 \in V_1. \end{cases}$$

Since $V_1 \cap f(x') \neq \emptyset$ and $f(x') \subset \mathcal{S}t(f(x), \widetilde{\mathcal{V}}_{x,y})$, we can find $V_2 \in \widetilde{\mathcal{V}}_{x,y}$ such that

$$\begin{cases} V_2 \cap f(x) \neq \emptyset, \\ V_1 \cap V_2 \neq \emptyset. \end{cases}$$

Now, since $V_1, V_2 \in \widetilde{\mathcal{V}}_{x,y}$, $V_1 \cap V_2 \neq \emptyset$ and $\widetilde{\mathcal{V}}_{x,y}$ is an open barycentric refinement of $\mathcal{U}_{x,y}$, we can find $U \in \mathcal{U}_{x,y}$ such that $V_1 \cup V_2 \subset U$. But then we have $x_1 \in U$ and $U \cap f(x) \neq \emptyset$, i.e. $x_1 \in \mathcal{S}t(f(x), \mathcal{U}_{x,y})$. Since the open covering $\widetilde{\mathcal{V}}_{x,y}$ is a fortiori an open refinement of $\mathcal{U}_{x,y}$, the chain $(x_0 = x, x_1, ..., x_n = y)$ is thus a $\mathcal{U}_{x,y}$-chain from x to y, which is absurd. Thus the open covering $\widetilde{\mathcal{V}}_{x,y}$ f-separates every point of $W_{x,y}$ from y.

Now let $\mathcal{V}_{x,y}$ be an open barycentric refinement of $\widetilde{\mathcal{V}}_{x,y}$. Let $W'_{x,y}$ be any open set of $\mathcal{V}_{x,y}$ containing y. Since $\mathcal{V}_{x,y}$ is an open barycentric refinement of $\widetilde{\mathcal{V}}_{x,y}$ and $y \in W'_{x,y} \in \mathcal{V}_{x,y}$, a similar proof shows that if $(x_0, x_1, ..., x_n)$ is a $\mathcal{V}_{x,y}$-chain starting in $W_{x,y}$ and ending in $W'_{x,y}$ then the chain $(x_0, ..., x_{n-1}, y)$ is a $\widetilde{\mathcal{V}}_{x,y}$-chain starting in $W_{x,y}$ and ending at y. Since the open covering $\widetilde{\mathcal{V}}_{x,y}$ f-separates every point of $W_{x,y}$ from y, we conclude that the open covering $\mathcal{V}_{x,y}$ f-separates every point of $W_{x,y}$ from every point of $W'_{x,y}$.

In particular, we have shown that the subset E of $X \times X$ is open. The space X being metric and separable, the same goes for E which thus satisfies the Lindelöf property. We can thus extract from the open covering $\left\{W_{x,y} \times W'_{x,y},\ (x,y) \in E\right\}$ of E a countable sub-covering $\left(W_{x_i,y_i} \times W'_{x_i,y_i}\right)_{i \in \mathbb{N}}$. The family of associated open coverings $(\mathcal{V}_{x_i,y_i})_{i \in \mathbb{N}}$ provides the wanted countable family.

□

We will apply the following well-known lemma to the family $(\mathcal{U}_l)_{l \in \mathbb{N}}$ of open coverings furnished by the previous lemma to obtain the desired chain-recurrence adapted distance.

Lemma 3.13. *Given a countable family $(\mathcal{U}_l)_{l \in \mathbb{N}}$ of open coverings of X, there exists a metric δ on X that defines the topology of X and such that*

$$\forall l \in \mathbb{N},\ \left\{B_\delta\left(x, \frac{1}{2^l}\right),\ x \in X\right\} \propto \mathcal{U}_l.$$

Proof. Let $l \in \mathbb{N}$. Since any metric space is paracompact, we can find a partition of unity $\left(\varphi^l_U\right)_{U \in \mathcal{U}_l}$ subordinate to $\mathcal{U}_l$ such that the supports of the φ^l_U form a neighborhood finite closed covering of X, see [9, Chapter VIII]. For any open set U in $\mathcal{U}_l$ we set

$$\psi^l_U(x) := \frac{\varphi^l_U(x)}{\sup\limits_{U' \in \mathcal{U}_l} \varphi^l_{U'}(x)}.$$

The function ψ^l_U is well defined since the supports of the $\left(\varphi^l_U\right)_{U \in \mathcal{U}_l}$ form a locally finite family and is continuous since the φ^l_U are. Moreover, we have $0 \leq \psi^l_U \leq 1$ and thus the series $\sum_{l \in \mathbb{N}} \frac{1}{2^l} \max_{U \in \mathcal{U}_l} \left|\psi^l_U(x) - \psi^l_U(y)\right|$ converges uniformly and defines a continuous function on $X \times X$.

We then define

$$\delta(x,y) := d(x,y) + \sum_{l \in \mathbb{N}} \frac{1}{2^l} \max_{U \in \mathcal{U}_l} \left|\psi^l_U(x) - \psi^l_U(y)\right|.$$

The function δ is a distance. Let us show that it induces the topology of X. Since $d \leq \delta$, if $x_n \longrightarrow x$ for δ then $x_n \longrightarrow x$ for d. Conversely, if $x_n \longrightarrow x$ for d then by continuity of the function

$$(x,y) \longmapsto \sum_{l\in\mathbb{N}} \frac{1}{2^l} \max_{U\in\mathcal{U}_l} \left|\psi_U^l(x) - \psi_U^l(y)\right|$$

we have $x_n \longrightarrow x$ for δ.

We now show the refinement property. Let $l \in \mathbb{N}$ and $x \in X$. Since the supports of the $\left(\varphi_U^l\right)_{U\in\mathcal{U}_l}$ form a locally finite family there exists $U_x \in \mathcal{U}_l$ such that $\varphi_{U_x}^l(x) = \sup\limits_{U'\in\mathcal{U}_l} \varphi_U^l(x)$. We then have $\psi_{U_x}^l(x) = 1$. But then, for $y \in B_\delta(x, \frac{1}{2^l})$ we have

$$\frac{1}{2^l}\left|1 - \psi_{U_x}^l(y)\right| \leq \frac{1}{2^l} \max_{U\in\mathcal{U}_l} \left|\psi_U^l(x) - \psi_U^l(y)\right| \leq \delta(x,y) < \frac{1}{2^l}.$$

Thus, we have $\left|1 - \psi_{U_x}^l(y)\right| < 1$ and necessarily $\psi_{U_x}^l(y) > 0$, hence $y \in U_x$. Thus $B_\delta(x, \frac{1}{2^l}) \subset U_x \in \mathcal{U}_l$ and the lemma is proved. □

We can now prove the following theorem.

Theorem 3.14. *Let X be a separable metric space and $f : X \longrightarrow \Gamma(X)$ be a continuous map. Then there exists a chain-recurrence adapted distance for f on X.*

Proof. We apply the previous lemma to the f-separating family $(\mathcal{U}_l)_{l\in\mathbb{N}}$ of lemma 4.7 to obtain a distance δ on X that defines the topology of X. Let us prove that this distance is chain-recurrence adapted. For x, y in X we have to prove that the following assertions are equivalent

(i) For every open covering $\mathcal{U}$ of X, there exists a $\mathcal{U}$-chain from x to y.

(ii) For every number $\varepsilon > 0$, there exists an ε-chain for δ from x to y.

Let us suppose (i). The open coverings $\left\{B_\delta(x', \frac{\varepsilon}{2}),\ x' \in X\right\}$, $\varepsilon > 0$, provides by triangle inequality ε-chains for δ from x to y. Since ε is arbitrary, it shows (ii). Conversely, let us suppose (ii). For every $l \in \mathbb{N}$ we have

$$\left\{B_\delta(x', \frac{1}{2^l}),\ x' \in X\right\} \propto \mathcal{U}_l.$$

Thus, every $\frac{1}{2^l}$-chain for δ from x to y is in fact a $\mathcal{U}_l$-chain from x to y. Since the family $(\mathcal{U}_l)_{l\in\mathbb{N}}$ is a f-separating one, it shows (i). □

3.4 Conley barrier

In the setting of a noncompact metric space, we define what a Conley barrier is using the notion of $\mathcal{U}$-chains.

Definition 3.15. Let X be a metric space and $f : X \longrightarrow \Gamma(X)$ be a continuous map. A *Conley barrier* for f is a continuous function

$$S : X \times X \longrightarrow \mathbb{R}_+$$

with the properties that

(i) *For every* $(x, y) \in X^2$, $S(x, y) = 0$ *if and only if for every open covering* $\mathcal{U}$ *of* X *there exists a* $\mathcal{U}$*-chain for* f *from* x *to* y.

(ii) *For every* $(x, y, z) \in X^3$, *we have* $S(x, y) \leq \max(S(x, z), S(z, y))$.

As in the compact case, we will show the following theorem.

Theorem 3.16. *If* X *is a separable metric space and* $f : X \longrightarrow \Gamma(X)$ *is a continuous map then there exists a Conley barrier for* f.

Proof. According to theorem 3.14, there exists a chain-recurrence adapted distance δ on X for f. Since chain properties are fully described using the metric δ, it is enough to construct a continuous function S such that

(1) For every $(x, y) \in X^2$, we have $S(x, y) = 0$ if and only if for every $\varepsilon > 0$ there exists an ε-chain for δ from x to y.

(2) For every $(x, y, z) \in X^3$, we have $S(x, y) \leq \max(S(x, z), S(z, y))$.

The only difference with the compact case is that f is now a compactum-valued map. For every $(x, y) \in X^2$, we thus define similarly S as

$$S(x, y) := \inf \left\{ \max_{i \in \{0,..n-1\}} \delta(f(x_i), x_{i+1}) \mid n \geq 1,\ x_0 = x, ..., x_n = y \right\}.$$

The distance from $f(x_i)$ to x_{i+1} being understood as the distance of the point x_{i+1} to the compact set $f(x_i)$. A similar proof than in the compact case then shows that

$$|S(x, y) - S(x', y')| \leq \delta(y, y') + \mathcal{D}_\delta(f(x), f(x')).$$

Thus the function S inherits of the continuity of f. The proofs of properties (1) and (2) can now be readily adapted. □

3.5 Ultrametric distance induced on the space of chain-transitive components

The fact that a Conley barrier induces an ultrametric distance on the set of chain-transitive components does not use compactness of X. Thus, the constructions of section 2.2 can be readily adapted. In particular, any Conley barrier furnishes an ultrametric distance on the set of chain-transitive components of f and the induced ultrametric topology is coarser than the quotient topology. Thus, we have the following.

Theorem 3.17. *Let* X *be a separable metric space and* $f : X \longrightarrow \Gamma(X)$ *be a continuous map. Then the set of chain-transitive components of* f *is Hausdorff and totally disconnected.*

Nevertheless, contrary to the compact case, the ultrametric topology induced by a Conley barrier may differ from the quotient topology. A counterexample is given in section 4.2.

3.6 Lyapunov functions

Definitions. In the case of a compactum-valued map, the definitions of Lyapunov functions need to be slightly modified.

Definition 3.18. Given a metric space X and a continuous map $f : X \longrightarrow \Gamma(X)$, a *strict Lyapunov function* for f is a continuous function $\varphi : X \longrightarrow \mathbb{R}$ such that

(i) *For every x in X and every y in $f(x)$, we have $\varphi(y) \leq \varphi(x)$.*

(ii) *For every x in $X \setminus \mathcal{R}(f)$ and every y in $f(x)$, we have $\varphi(y) < \varphi(x)$.*

A strict Lyapunov function is said to be *complete* if it satisfies the following additional property

(i') *The function φ is constant on each chain-transitive component, takes on distinct values on distinct chain-transitive components and sends the subset $\mathcal{R}(f)$ into a subset of $\mathbb{R}$ whose Hausdorff dimension is zero.*

The notion of *sub-solution* for a Conley barrier S is similarly defined. Moreover, proofs of lemma 2.13 and 2.14 are unchanged.

Strict Lyapunov function. Our construction of a strict Lyapunov function for f is still based on sub-solutions of the type S_x for $x \in X$. The existence of a uniform bound for S is there replaced by the following lemma.

Lemma 3.19. *There is a countable open covering $(U_n)_{n\in\mathbb{N}}$ of X such that for every $x \in X$ and for every $n \in \mathbb{N}$, the function S_x is bounded on U_n.*

Proof. Let $x \in X$. By continuity of S_x, there is an open neighborhood U_x of x such that $S(x, \cdot)$ is bounded on U_x. For $x' \in X$ we have

$$\forall y \in U_x,\ S(x', y) \leq \max(S(x', x), S(x, y)).$$

Thus the function $S(x', \cdot)$ is also bounded on U_x. Since the metric space X is separable, it is Lindelöf . Hence, a countable sub-covering of the open covering $\{U_x, x \in X\}$ of X provides the wanted covering. □

Corollary 3.20. *For every sequence $(x_i)_{i\in\mathbb{N}}$ of X, there exists a sequence $(\eta_i)_{i\in\mathbb{N}}$ of positive reals such that the non-negative series $\sum_{i\in\mathbb{N}} \eta_i S_{x_i}$ converges uniformly in the neighborhood of each points of X.*

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be an open covering of X furnished by the previous lemma. Each function $S_{x_i}, i \in \mathbb{N}$, is bounded on U_0. Thus, there is a sequence $(\rho_i^0)_{i\in\mathbb{N}}$ of positive reals such that the series $\sum_{i\in\mathbb{N}} \rho_i^0 S_{x_i}$ converges uniformly on U_0. Similarly, there is a sequence $(\rho_i^1)_{i\in\mathbb{N}}$ of positive reals such that the series $\sum_{i\in\mathbb{N}} \rho_i^1 S_{x_i}$ converges uniformly on U_1. Moreover, reducing the ρ_i^1 if necessary, we can also suppose that $\rho_i^1 < \rho_i^0$.

We thus construct using induction sequences $(\rho_i^k)_{i\in\mathbb{N}}$, for k in $\mathbb{N}$, such that $0 < \rho_i^{k+1} < \rho_i^k$ and the series $\sum_{i\in\mathbb{N}} \rho_i^k S_{x_i}$ converges uniformly on U_k. These both conditions then imply that the series $\sum_{i\in\mathbb{N}} \eta_i S_{x_i}$, with $\eta_i = \rho_i^i$, converges uniformly on each $U_k, k \in \mathbb{N}$. The result follows since $(U_n)_{n\in\mathbb{N}}$ is an open covering of X. □

Remark 3.21. If we define instead η_i by $\min(\rho_i^i, \frac{1}{2^{i+1}})$, we can also assume that the series $\sum_{i\geq 1}\eta_i$ converges and belongs to $]0,1[$. Thus, changing η_0 in $1-\sum_{i\geq 1}\eta_i$, we can suppose without lost of generality that $\sum_{i\in\mathbb{N}}\eta_i = 1$.

We can now prove the following theorem.

Theorem 3.22. *Let X be a separable metric space and $f : X \longrightarrow \Gamma(X)$ be a continuous map. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ of points of X and a sequence $(\eta_n)_{n\in\mathbb{N}}$ of positive reals such that the series*

$$\varphi = \sum_{n\in\mathbb{N}} \eta_n S_{x_n}$$

is a strict sub-solution for S and thus a strict Lyapunov function for f.

Proof. As in the compact case, let us choose a dense sequence $(x_i)_{i\in\mathbb{N}}$ of X. Let $(\eta_i)_{i\in\mathbb{N}}$ be the associated sequence given by corollary 3.20. Thanks to remark 3.21, we can suppose that $\sum_{i\in\mathbb{N}}\eta_i = 1$. The same proof as in the compact case then shows that the function $\varphi = \sum_{i\in\mathbb{N}}\eta_i S_{x_i}$ is a strict sub-solution for S and thus a strict Lyapunov function for f. □

Complete Lyapunov function. If we had an hypothesis of local compactness, the same tools as in section 2.3 can be used to construct a complete Lyapunov function. In particular, the proof of the following lemma did not use any compactness and is still valid.

Lemma 3.23. *Let X be a separable metric space. For every $x \in X$, the function S_x is constant in the neighborhood of each point of the set $\mathcal{R}(f) \setminus \{S(x,\cdot) = 0\}$.*

Corollary 3.24. *Let X be a separable metric space. For every compact subset K of X and for every x in X, the set $\{S(x,y),\ y \in \mathcal{R}(f)\cap K\}$ is countable and the only possible accumulation point is zero. In particular, for every $\varepsilon > 0$, the function $\theta_\varepsilon \circ S_x$ where $\theta_\varepsilon(t) := \max(t-\varepsilon, 0)$ takes a finite number of values on $\mathcal{R}(f)\cap K$.*

Proof. The proof is the same as proof of corollary 4.3 once the set $\mathcal{R}(f)$ has been replaced by $\mathcal{R}(f)\cap K$. □

Theorem 3.25. *Let X be a locally compact and separable metric space and $f : X \longrightarrow \Gamma(X)$ be a continuous map. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ of points of X and a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals such that the series*

$$\varphi = \sum_{n\in\mathbb{N}} \varepsilon_n \theta_{\frac{1}{n+1}} \circ S_{x_n}$$

defines a complete Lyapunov function for f.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a dense sequence in X. Without lost of generality, we can suppose that for every $k \in \mathbb{N}$ the sequence $(x_n)_{n\geq k}$ is still dense in X. Since X is locally compact, metric and separable, there exist a family $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X such that $X = \cup_{n\in\mathbb{N}} K_n$ and for every $n \in \mathbb{N}$, we have $K_n \subset \mathring{K}_{n+1}$. For every $n \in \mathbb{N}$, each function $\theta_{\frac{1}{k+1}} \circ S_{x_k}, k \in \mathbb{N}$, is bounded on the compact set

K_n. Using a diagonal process, we can find a sequence $(\eta_n)_{n\in\mathbb{N}}$ of positive reals such that the series $\sum_{n\in\mathbb{N}} \eta_n \theta_{\frac{1}{n+1}} \circ S_{x_n}$ converges uniformly on each K_n and thus defines a continuous function on X.

As in the compact case, the functions $\theta_\varepsilon \circ S_x$ are constant on chain-transitive components and induce functions $\overline{\theta_\varepsilon \circ S_x}$ on the quotient space $\mathcal{R}(f)/\backsim$. Thanks to corollary 3.24, for every $(k,n) \in \mathbb{N}^2$, the function $\theta_{\frac{1}{k+1}} \circ S_{x_k}$ takes a finite number of values on $\mathcal{R}(f) \cap K_n$. We will now use lemma 5.1 with the set $A = \mathcal{R}(f)/\backsim$, the family $(\overline{\theta_{\frac{1}{n+1}} \circ S_{x_n}})_{n\in\mathbb{N}}$ and $A_n = p(K_n)$ where p denotes the canonical projection from $\mathcal{R}(f)$ onto $\mathcal{R}(f)/\backsim$. As in the compact case, we easily verify that for every $k \in \mathbb{N}$ the family $(\overline{\theta_{\frac{1}{n+1}} \circ S_{x_n}})_{n\geq k}$ separates points of $\mathcal{R}(f)/\backsim$. Thus lemma 5.1 furnishes a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals such that the series $\sum_{n\in\mathbb{N}} \varepsilon_n \overline{\theta_{\frac{1}{n+1}} \circ S_{x_n}}$ converges on $\mathcal{R}(f)/\backsim$, separates points of $\mathcal{R}(f)/\backsim$ and has an image of zero Hausdorff dimension in $\mathbb{R}$. Since the positive reals $(\varepsilon_n)_{n\in\mathbb{N}}$ can be chosen arbitrarily small, we can also assume that for every $n \in \mathbb{N}$ we have $\varepsilon_n < \eta_n$. Hence, the function

$$\varphi = \sum_{n\in\mathbb{N}} \varepsilon_n \theta_{\frac{1}{n+1}} \circ S_{x_n}$$

converges uniformly on each $K_n, n \in \mathbb{N}$, and thus defines a continuous function on X. It is constant on each chain-transitive component, takes on distinct values on distinct chain-transitive components and sends $\mathcal{R}(f)$ in a subset of $\mathbb{R}$ whose Hausdorff dimension is zero. The rest of the proof is now similar to the compact case. □

4 The case $f = Id_X$

In the particular case $f = Id_X$, a $\mathcal{U}$-chain from x to y just corresponds to a sequence $(U_i)_{0\leq i\leq n}$ of open sets of the open covering $\mathcal{U}$ such that

$$x \in U_0,\ y \in U_n,\ \forall i \in \{0,..,n-1\},\ U_i \cap U_{i+1} \neq \emptyset.$$

In particular, a Conley barrier associated to the identity is symmetric. Chain-recurrence properties are then linked with the topology of X.

4.1 The quasicomponents

Definition 4.1. Let X be a topological space. Two points x and y of X are said to be *separated* in X if the space X can be split into two disjoint open sets U and V containing respectively x and y.

The relation *not being separated* defines an equivalence relation on X. The associated equivalence classes are called the *quasicomponents* of X. Two point x and y lie in the same quasicomponent if and only if every open and closed subset of X containing x or y contains both x and y. Thus, the quasicomponent of a point x coincides with the intersection of open and closed subsets of X that contain x. In particular, the connected component of x is included into the quasicomponent of x.

Remark 4.2. In a compact space, the connected component of a point x coincides with the quasicomponents of x, see [16, Chapter II]. Nevertheless, even if the space is locally compact, quasicomponents may be larger than connected components. See for example the counterexample of nested rectangle in [26].

The quasicomponents are essentially characterized by a Conley barrier associated to the identity, as shown in the following result.

Lemma 4.3. *Let X be a separable metric space. Then the quasicomponents of X coincide with the chain-transitive components of Id_X.*

Proof. We have to show that two point x and y are separated in X if and only if there exists an open covering $\mathcal{U}$ of X that Id_X-separates x from y. Let us suppose that for every open covering $\mathcal{U}$ of X, there is a $\mathcal{U}$-chain for the identity map from x to y. If x and y where separated in X say by U and V, the open covering $\{U, V\}$ would leads to a contradiction. Conversely, let us suppose that there is an open covering $\mathcal{U}$ of X such that there is no $\mathcal{U}$-chain for the identity map from x to y. Let $U \in \mathcal{U}$ be an open set such that $x \in U$. We consider the set

$$O = \bigcup_{n\in\mathbb{N}} St^n(U,\mathcal{U}) \quad \text{where} \quad St^n(U,\mathcal{U}) = \underbrace{St(...St}_{n\ times}(U,\mathcal{U})..,\mathcal{U}).$$

The set O is open and we claim that the same is true for $X\setminus O$. Indeed, let $z \in X\setminus O$. If we denote by V an element of $\mathcal{U}$ such that $z \in V$, then $V \subset X \setminus O$. Moreover we have $y \in X \setminus O$ since there is no $\mathcal{U}$-chain from x to y for Id_X. The points x and y are thus separated by the open subsets O and $X \setminus O$. □

We then deduce the following corollary.

Corollary 4.4. *Let X be a separable metric space and S be a Conley barrier for the identity map on X. Then two points x and y of X are in the same quasicomponent if and only if $S(x, y) = 0$.*

If the metric space X is compact, the quasicomponents and the connected components of X coincide. We thus obtain the following known result, which follows from corollary 4.0.12.

Theorem 4.5. *Let X be a compact metric space. Then the set of connected components of X is an ultrametric space.*

If some quasicomponent fail to be compact, the ultrametric topology induced by a Conley barrier may be strictly coarser than the quotient topology. Such an example is studied in the next section.

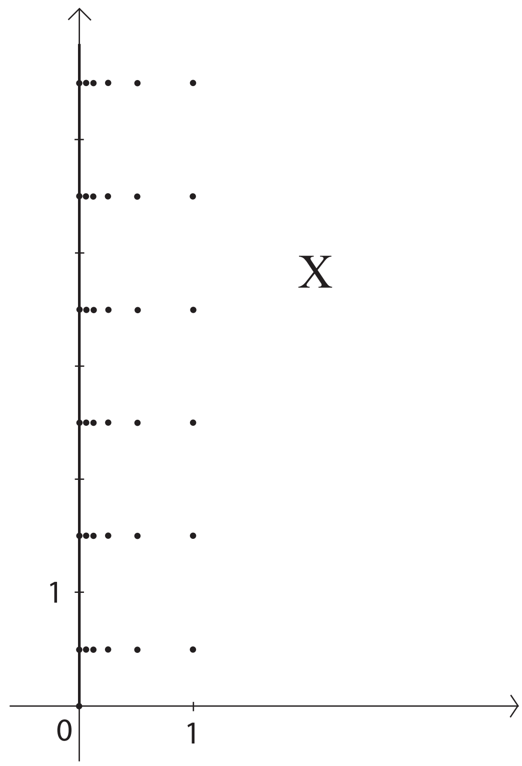
4.2 A counterexample

We consider the plane $\mathbb{R}^2$ and for $k \in \mathbb{N}$ we set

$$\mathcal{D} = \{(0, y), y \geq 0\},$$

$$A_k = \left\{\left(\frac{1}{n}, k + \frac{1}{2}\right),\ n \geq 1\right\},$$

$$X = \left(\bigcup_{k\in\mathbb{N}} A_k\right) \bigcup \mathcal{D}.$$



We endow the space X with the Euclidean topology inherited from $\mathbb{R}^2$. The space X thus obtained is a closed subset of $\mathbb{R}$ and hence is locally compact.

Lemma 4.6. *For every countable family $(V_i)_{i\in\mathbb{N}}$ of open sets of $\mathbb{R}^2$ containing $\mathcal{D}$, there is an open set V of $\mathbb{R}^2$ containing $\mathcal{D}$ and such that*

$$\forall i \in \mathbb{N},\ X \cap V_i \not\subseteq X \cap V.$$

Proof. We first construct a sequence $(U_k)_{k\in\mathbb{N}}$ of open sets of $\mathbb{R}^2$ such that

(i) *For every $k \in \mathbb{N}$, $\{0\} \times [k, k+1] \subset U_k$.*

(ii) *For every $k \neq l$, $U_k \cap A_l = \emptyset$.*

(iii) *For every $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}^*$ such that $\left(\frac{1}{n_k}, k + \frac{1}{2}\right) \in V_k \setminus U_k$.*

To insure the first two points, it is enough to choose U_k contained in the strip

$$\left\{(x, y)\,, x \in \mathbb{R}, k - \frac{1}{4} < y < k + \frac{5}{4}\right\} \supset \{0\} \times [k, k+1].$$

For the last point, we notice that the point $\left(0, k + \frac{1}{2}\right)$ lies in $V_k \cap \bar{A}_k$. Thus there is an integer $n_k > 0$ such that $\left(\frac{1}{n_k}, k + \frac{1}{2}\right) \in V_k$. We thus set

$$U_k = \left\{(x, y) \in \mathbb{R}^2 \mid x < \frac{1}{n_k},\ k - \frac{1}{4} < y < k + \frac{5}{4}\right\}.$$

From (i), the open set $V = \bigcup_{k\in\mathbb{N}} U_k$ contains $\mathcal{D}$. Now let $i \in \mathbb{N}$. By construction we have

$$\left(\frac{1}{n_i}, i + \frac{1}{2}\right) \notin U_i$$

and from (ii) we have

$$\forall l \neq i,\ \left(\frac{1}{n_i}, i + \frac{1}{2}\right) \notin U_l$$

Thus $\left(\frac{1}{n_i}, i + \frac{1}{2}\right) \notin X \cap V$ while $\left(\frac{1}{n_i}, i + \frac{1}{2}\right) \in X \cap V_i$. We thus have

$$X \cap V_i \not\subseteq X \cap V$$

as assumed. □

Corollary 4.7. *The set of quasicomponents of the metric space X defined above is not metrizable. Hence, the topology induced by a Conley barrier for Id_X on the set of quasicomponents is strictly coarser than the quotient topology.*

Proof. The quasicomponents of X are the half line $\mathcal{D}$ and the singletons $\left(\frac{1}{n}, k + \frac{1}{2}\right)_{n\geq 1, k>0}$. We will show that $\mathcal{D}$ does not admit any countable basis of open neighborhoods in the quotient topology.

Otherwise, let $(\widetilde{O_i})_{i\in\mathbb{N}}$ be such a basis. The inverse images by the canonical projection p provide a family $(O_i)_{i\in\mathbb{N}}$ of open sets of X that contain $\mathcal{D}$. Thus there is a family $(V_i)_{i\in\mathbb{N}}$ of open set of $\mathbb{R}^2$ containing $\mathcal{D}$ and such that $O_i = V_i \cap X = p^{-1}(\widetilde{O_i})$. According to lemma 4.6, there is an open set V of $\mathbb{R}^2$ containing $\mathcal{D}$ such that for every $i \in \mathbb{N}$, $X \cap V_i \not\subseteq X \cap V$. Since V contains $\mathcal{D}$ and since the quasicomponents of $X \setminus \mathcal{D}$ are reduced to singletons, we have $p^{-1}(p(V)) = V \cap X$. Thus the set $p(V)$ is an open set that contains $\mathcal{D}$. But for every $i \in \mathbb{N}$ the set $p^{-1}(\widetilde{O_i}) = O_i = X \cap V_i$ is not include in $X \cap V$. Thus $\widetilde{O_i} \not\subseteq p(V)$ and this contradicts the fact that $(\widetilde{O_i})_{i\in\mathbb{N}}$ is a basis of open neighborhoods of $\mathcal{D}$ in the quotient. □

4.3 Totally separated space

We can now also answer the following question: under which conditions are chain-transitive components of Id_X reduced to singletons ?

Definition 4.8. A topological space X is said to be

(i) *totally disconnected* if connected components of X are reduced to singletons.

(ii) *totally separated* if two distinct points of X can always be separated.

(iii) *of dimension 0* if every point of X have a basis of open sets with empty boundary.

We always have (iii) ⇒ (ii) ⇒ (i) and if X is a locally compact space, these notions coincide. In the general setting, they may be different, see [16, Chapter II].

Proposition 4.9. *Let X be a separable metric space. Then the chain-transitive components associated to the identity are reduced to singletons if and only if X is totally separated.*

Proof. It is corollary 4.4. □

5 Appendix

5.1 Function series and Hausdorff dimension

In this section, we develop some general facts about the Hausdorff dimension of images of some particular function series. They are used to construct complete Lyapunov functions for f in section 2.3 and 3.6.

Throughout this section, $(f_i)_{i\in\mathbb{N}}$ will denote a family of real valued functions on a set A, such that either

(i) *For every $i \in \mathbb{N}$, the set $f_i(A)$ is finite.*

(ii) *The family $(f_i)_{i\in\mathbb{N}}$ separates points of A, i.e. for each a, b in A with $a \neq b$, there exists an f_i such that $f_i(a) \neq f_i(b)$.*

or

(i) $A = \cup_{n\in\mathbb{N}} A_n$.

(ii) *For every $(k, n) \in \mathbb{N}^2$, the set $f_k(A_n)$ is finite.*

(iii) *For every $k \in \mathbb{N}$, the family $(f_k)_{k\geq n}$ separates points of A.*

Lemma 5.1. *In both cases, there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of arbitrarily small positive reals such that the series $\sum_{n\in\mathbb{N}} \varepsilon_n f_n$ converges on A, separates points of A and has an image of zero Hausdorff dimension in $\mathbb{R}$.*

Proof. We begin with the second case. Considering sets $\tilde{A}_n = \cup_{k\leq n} A_k$ instead of A_n, we can suppose that

$$\forall n \in \mathbb{N}, A_n \subset A_{n+1}.$$

Since for every $(k, n) \in \mathbb{N}^2$ the set $f_k(A_n)$ is finite, we can construct using induction a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals such that

(i) $\varepsilon_0 > 0$,

(ii) $\forall n \in \mathbb{N}, \displaystyle\sum_{k\geq n+1} \varepsilon_k \max_{A_n} |f_k| < \frac{1}{2}\eta_n,$

(iii) $\forall n \in \mathbb{N}, \displaystyle\sum_{k\geq n+1} \varepsilon_k \max_{A_n} |f_k| < e^{-n\nu_n},$

where

$$\nu_n = \mathrm{Card}\left(\sum_{k=0}^{n} \varepsilon_k f_k(A_n)\right)$$

and η_n is the minimum of the distance between two distinct points of the finite set $\sum_{k=0}^{n} \varepsilon_k f_k(A_n)$. If this image is reduced to a single point, we just set $\eta_n = 1$. Note that the $(\varepsilon_n)_{n\in\mathbb{N}}$ can be chosen arbitrarily small.

Property (iii) implies that the series $\sum_{n\in\mathbb{N}} \varepsilon_n f_n$ converges uniformly on each A_n and thus converges on A. Now, let $a, b \in A$ be two distinct points of A. Since $A = \cup_{n\in\mathbb{N}} A_n$ and $A_n \subset A_{n+1}$, we can choose n large enough so that a and b lie in A_n. If $\sum_{k\leq n} \varepsilon_k f_k(a) \neq \sum_{k\leq n} \varepsilon_k f_k(b)$ then by property (ii) we have $\sum_{k\in\mathbb{N}} \varepsilon_k f_k(a) \neq$

$\sum_{k\in\mathbb{N}} \varepsilon_k f_k(b)$. Otherwise, by hypothesis the family $(f_k)_{k\geq n+1}$ separates points of A, thus there is a first $n_0 \geq n+1$ such that $f_{n_0}(a) \neq f_{n_0}(b)$. Hence we have $\sum_{k\leq n_0} \varepsilon_k f_k(a) \neq \sum_{k\leq n_0} \varepsilon_k f_k(b)$. Since $a, b \in A_n \subset A_{n_0}$, we can conclude similarly. Thus the series $\sum_{n\in\mathbb{N}} \varepsilon_n f_n$ separates points of A.

Let us now prove that the set $\sum_{n\in\mathbb{N}} \varepsilon_n f_n(A)$ is a subset of $\mathbb{R}$ whose Hausdorff dimension is zero. Since this property is stable under countable union, it is enough to show that for every $n \in \mathbb{N}$ the set $\sum_{k\in\mathbb{N}} \varepsilon_k f_k(A_n)$ has a zero Hausdorff dimension in $\mathbb{R}$. Let $n \in \mathbb{N}$. We write

$$\sum_{k\in\mathbb{N}} \varepsilon_k f_k(A_n) = \sum_{k\leq n} \varepsilon_k f_k(A_n) + \sum_{k\geq n+1} \varepsilon_k f_k(A_n).$$

By property (iii), the subset $\sum_{k\in\mathbb{N}} \varepsilon_k f_k(A_n)$ can be covered by ν_n balls of radius $e^{-n\nu_n}$. Since for every $l \in \mathbb{N}$, $A_n \subset A_{n+l}$, we conclude that the subset $\sum_{k\in\mathbb{N}} \varepsilon_k f_k(A_n)$ can be covered by ν_{n+l} balls of radius $e^{-(n+l)\nu_{n+l}}$. Since

$$\forall \rho > 0,\ \nu_{n+l}(e^{-(n+l)\nu_{n+l}})^\rho \to 0,\ (l \to +\infty)$$

the subset $\sum_{k\in\mathbb{N}} \varepsilon_k f_k(A_n)$ has a zero Hausdorff dimension.

For the first case of the lemma, we take $A_n = A$ for every $n \in \mathbb{N}$ and we construct similarly a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals. If a, b are two distinct points of A then by property (ii) there is a first $n \in \mathbb{N}$ such that $\sum_{k\leq n} \varepsilon_k f_k(a) \neq \sum_{k\leq n} \varepsilon_k f_k(b)$. Then, by construction of the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ we have $\sum_{k\in\mathbb{N}} \varepsilon_k f_k(a) \neq \sum_{k\in\mathbb{N}} \varepsilon_k f_k(b)$. The end of the proof is now similar. □

5.2 On the equivalence of chain-recurrence definitions

In this section, we give another definition of chain-recurrence which is used by Hurley in [17] and we prove that it is equivalent to the $\mathcal{U}$-chain approach. Throughout this section, (X, d) will denote a separable metric space and f a continuous map from X to itself. We will denote by $\mathcal{P}$ the set of continuous functions $\varepsilon : X \longrightarrow \mathbb{R}_+^*$. The set $\mathcal{P}$ is introduced by Hurley in [17] in order to keep topological invariance.

Definition 5.2. Let $x, y \in X$ and $\varepsilon \in \mathcal{P}$. An *$\varepsilon$-chain* for f from x to y is a finite sequence $(x_0 = x, ..., x_n = y), n \geq 1$, of X such that

$$\forall i \in \{0, .., n-1\},\ d(f(x_i), x_{i+1}) < \varepsilon(f(x_i)).$$

Remark 5.3. If X is compact, we only need to use constant $\varepsilon > 0$ instead of elements of $\mathcal{P}$ since any continuous function reaches its minimum on X. Definition 5.2 is thus a generalization of the compact case one.

As shown in the following proposition, this definition leads us to the same notion of chain-recurrence than definition 5.2.

Proposition 5.4. *Let $x, y \in X$. The following assertions are equivalent:*

(i) *For every $\varepsilon \in \mathcal{P}$, there is an ε-chain from x to y.*

(ii) *For every open covering $\mathcal{U}$ of X, there is an $\mathcal{U}$-chain from x to y.*

Proof. Let $\mathcal{U}$ be an open covering of X. A metric space is paracompact so there is a locally finite refinement $\tilde{\mathcal{U}}$ of $\mathcal{U}$. For $U \in \tilde{\mathcal{U}}$ let

$$\varepsilon_U(x) = \frac{d\,(x, X\backslash U)}{2} \quad \text{and} \quad \varepsilon(x) = \max_{U \in \widetilde{\mathcal{U}}} \varepsilon_U(x)$$

with the convention that $d(x, \emptyset) = 1$. The function ε is well defined and continuous since the open covering $\tilde{\mathcal{U}}$ is locally finite and each ε_U is continuous. Moreover, this function is positive everywhere on X since $\widetilde{\mathcal{U}}$ is an open covering of X. For an open set $U \in \widetilde{\mathcal{U}}$ that realizes the maximum in the definition of $\varepsilon(x)$, we have $B_d(x, \varepsilon(x)) \subset U$. Thus

$$\{B_d(x, \varepsilon(x)), x \in X)\} \propto \tilde{\mathcal{U}} \propto \mathcal{U}$$

and every ε-chain from x to y provides a $\mathcal{U}$-chain from x to y. It shows that (i) $\Rightarrow$ (ii).

Conversely, let $\varepsilon \in \mathcal{P}$. Then for every $x \in X$, there is an open neighborhood U_x of x such that for every $x' \in U_x$ we have $\varepsilon(x') > \frac{\varepsilon(x)}{2}$. Reducing U_x, we can also suppose that $U_x \subset B_d(x, \varepsilon(x))$. We then consider the open covering $\mathcal{U} = \{U_x,\ x \in X\}$ of X. Let $(x_0 = x, x_1, ..., x_{n-1}, x_n = y)$ be a $\mathcal{U}$-chain from x to y. For every $i \in \{0, ..., n-1\}$ there is $z_i \in X$ such that $f(x_i)$ and x_{i+1} lie in $U_{z_i} \in \mathcal{U}$. Since $U_{z_i} \subset B_d(z_i, \varepsilon(z_i))$ we have $d(f(x_i), x_{i+1}) \leq d(f(x_i), z_i) + d(z_i, x_{i+1}) \leq 2\varepsilon(z_i) < 4\varepsilon(f(x_i))$. The chain $(x, x_1, ..., x_{n-1}, y)$ is thus a 4ε-chain from x to y. It shows that (ii) $\Rightarrow$ (i). $\square$

Chapter 2

The Aubry-Mather theory of a homeomorphism

In collaboration with Albert Fathi

1 Neutral set of a Lyapunov function

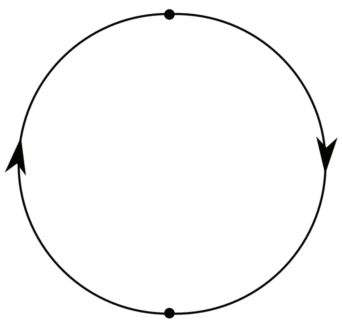
Throughout this paper, (X, d) will denote a compact metric space and h will denote a homeomorphism of X. A *Lyapunov function* for h is a continuous real-valued function $\theta : X \to \mathbb{R}$ such that $\theta \circ h \leq \theta$ i.e. the function θ is non-increasing along orbits of h. Constant functions are example of *trivial* Lyapunov functions. Given a Lyapunov function θ for h, we will say that a point $x \in X$ is a *neutral point* of θ if $\theta(h(x)) = \theta(x)$. We denote by $N(\theta)$ the set of neutral points of θ, that is,

$$N(\theta) = \{x \in X \mid \theta(h(x)) = \theta(x)\}.$$

We define the *neutral values* of θ as the images under θ of neutral points. Notice that the neutral set of a Lyapunov function θ is never empty since minimums of θ are neutral points. The terminology of critical points rather than neutral points is sometime used but it may cause confusion. Indeed, when the function θ turns out to be differentiable, the set of critical points $\{x \in X \mid d_x\theta = 0\}$ and of critical values $\theta(\{x \in X \mid d_x\theta = 0\})$ of θ as a differentiable function do not coincide with neutral points and neutral values, see example 1.3.

A Lyapunov function is useful if we can get an a priori description of its neutral set. Of course, the neutral set of a Lyapunov function always contains fixed points of h as well as periodic points or even non-wandering points $\Omega(h)$. Nevertheless, these inclusions may be strict.

Example 1.1. The arrows indicates the direction from x to $h(x)$ and the bold points are fixed.



The non-wandering points are the two fixed points but any Lyapunov function must be trivial and therefore admits the whole circle as neutral set.

A famous theorem of Conley [7, Chapter II, Section 6.4] asserts that we can always find a Lyapunov function θ for h such that the neutral set of θ coincides with the chain recurrent set of h.

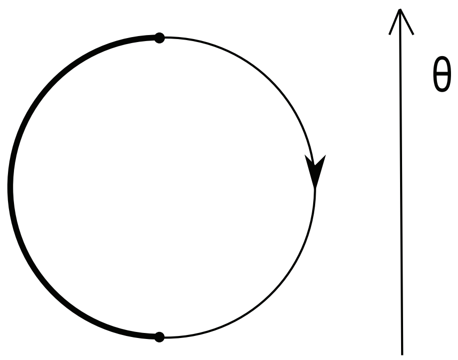
Theorem 1.2 (Conley)**.** *There is a Lyapunov function $\theta : X \to \mathbb{R}$ with*

$$N(\theta) = \mathcal{R}(h)$$

and such that the neutral values of θ are nowhere dense. Moreover, the function θ is constant on every chain transitive components and takes on different values on different chain transitive components.

This result will turn out to be sharp because of the hypothesis made on the neutral values, see corollary 2.3. Nevertheless, in a general way, there is no relation between the neutral set of a Lyapunov function θ and the chain recurrent set of h. In particular, the assumption $\mathcal{R}(h) \subset N(\theta)$ is in general wrong if the neutral values of θ do not have empty interior, even if the function θ is extremely regular.

Example 1.3. Here, every point is chain recurrent while the height function θ is a C^∞ Lyapunov function for which neutral points coincide only with the half circle made of fixed points.



Notice that the neutral values of the height function is then a whole non-trivial closed segment.

Section 2 is precisely devoted to the study of the (not so well known) link between chain-recurrence and neutral set of a Lyapunov function. As suggested above, the topology of the neutral values will be the relevant factor.

The main purpose of this paper is then to give sharp description of the neutral set of a Lipschitzian Lyapunov function in a general way. More precisely, we are going to construct a closed invariant subset $\mathcal{A}_d(h)$ of X such that

(i) any Lyapunov function θ which is Lipschitzian for the metric d satisfies $\mathcal{A}_d(h) \subset N(\theta)$,

(ii) there is a Lipschitzian Lyapunov function θ for which $\mathcal{A}_d(h) = N(\theta)$.

This goal is achieved in Section 4 with theorem 4.4. Since any Lyapunov function is Lipschitzian for an appropriate metric, these results also yield to a sharp description of the neutral set of any continuous Lyapunov function. This is the object of Section 5 and theorem 5.2. In particular, we recover the notion of *generalized recurrence* introduced by Auslander in [3]. The definition of the set $\mathcal{A}_d(h)$ is given in Section 3 and relies on a *barrier*, that is a continuous function

$$L_d : X \times X \to [0, +\infty[$$

satisfying the triangular inequality with a dynamical meaning in terms of recurrence. A similar approach was already used in chapter 1, see also [24], to give a new description of chain-recurrence and recover Conley's theorem with an appropriate barrier satisfying a stronger ultrametric inequality.

2 Chain-recurrence and neutral set

We recall from chapter 1 some facts about chain-recurrence. Let $x, y \in X$ and let $\varepsilon > 0$. An *ε-chain* for h from x to y is a finite sequence $\{x_0, ..., x_n\}$ in X such that $x_0 = x$, $x_n = y$ and, for every $i \in \{0, .., n-1\}$

$$d(h(x_i), x_{i+1}) < \varepsilon.$$

We define a closed transitive relation $\mathcal{P}(h)$ on X in the following way: $(x, y) \in \mathcal{P}(h)$ if and only if, for every $\varepsilon > 0$, there is an ε-chain for h from x to y. A point $x \in X$ is said to be *chain recurrent* for h if $(x, x) \in \mathcal{P}(h)$. We denote by $\mathcal{R}(h)$ the set of chain recurrent points of h. The relation $\mathcal{P}(h)$ becomes a preorder once restricted to $\mathcal{R}(h)$ and therefore induces an equivalence relation on $\mathcal{R}(h)$ in the following way: $x \sim y$ if and only if $(x, y) \in \mathcal{P}(h)$ and $(y, x) \in \mathcal{P}(h)$. The corresponding equivalence classes are called the *chain transitive components* of h. They are closed invariant subsets of X. Notice that previous definitions do not depend on the metric d since X is assumed to be compact. Recall that if S denotes the *Conley barrier* associated to h i.e.

$$S_d(x, y) = \inf\{\max_{i=0,..,n-1} d(h(x_i), x_{i+1}),\ n \geq 1,\ x_0 = x, .., x_n = y\},$$

then we have

$$(x, y) \in \mathcal{P}(h) \Leftrightarrow S_d(x, y) = 0$$

and in particular

$$\mathcal{R}(h) = \{x \in X \mid S_d(x, x) = 0\}.$$

Lemma 2.1. *Let $\theta : X \to \mathbb{R}$ be a Lyapunov function for h and let $(x, y) \in \mathcal{P}(h)$. If $\theta(y) \geq \theta(x)$ then*

$$[\theta(x), \theta(y)] \subset \theta(N(\theta) \cap I_{x,y}),$$

where

$$I_{x,y} = \{z \in X \mid (x, z) \in \mathcal{P}(h),\ (z, y) \in \mathcal{P}(h)\}.$$

Proof. Let $(x, y) \in \mathcal{P}(h)$ and assume that $\theta(y) \geq \theta(x)$. Let $t \in [\theta(x), \theta(y)]$. Since $(x, y) \in \mathcal{P}(h)$, for every $\varepsilon > 0$, we can find an ε-chain $\{x_0^\varepsilon, .., x_{n_\varepsilon}^\varepsilon\}$ from x to y. Since $\theta(x) \leq t \leq \theta(y)$, there is $k_\varepsilon \in \{0, .., n_\varepsilon - 1\}$ such that

$$\theta(x_{k_\varepsilon}) \leq t \leq \theta(x_{k_\varepsilon+1}). \tag{2.0.1}$$

Moreover, we have

$$d(h(x_{k_\varepsilon}), x_{k_\varepsilon+1}) < \varepsilon. \tag{2.0.2}$$

Let ω_θ be a modulus of continuity of θ. Since θ is a Lyapunov function for h, we get

$$\theta(x_{k_\varepsilon+1}) \leq \theta(h(x_{k_\varepsilon})) + \omega_\theta(\varepsilon) \leq \theta(x_{k_\varepsilon}) + \omega_\theta(\varepsilon). \tag{2.0.3}$$

Let (x_∞, y_∞) be an accumulation point of the the family $(x_{k_\varepsilon}, x_{k_\varepsilon+1})_{\varepsilon>0}$ as $\varepsilon \to 0$. Passing to the limit in 2.0.1, 2.0.2 and 2.0.3, we get

$$\theta(x_\infty) \leq t \leq \theta(y_\infty), \quad h(x_\infty) = y_\infty, \quad \theta(y_\infty) \leq \theta(x_\infty).$$

Hence we have $\theta(x_\infty) = \theta(y_\infty) = \theta(h(x_\infty)) = t$ and thus $x_\infty \in N(\theta)$. Moreover, we have $x_\infty \in I_{x,y}$ as seen by considering chains $\{x_0^\varepsilon, .., x_{k_\varepsilon-1}^\varepsilon, x_\infty\}$ and $\{x_\infty, x_{k_\varepsilon+1}^\varepsilon, .., x_{n_\varepsilon}^\varepsilon\}$ when $\varepsilon \to 0$. □

Lemma 2.1 directly leads to the following proposition.

Proposition 2.2. *Let $\theta : X \to \mathbb{R}$ be a Lyapunov function for h and assume that $\theta(N(\theta))$ is totally disconnected. Then θ is non-decreasing with respect to $\mathcal{P}(h)$ i.e.*

$$(x, y) \in \mathcal{P}(h) \Rightarrow \theta(x) \leq \theta(y).$$

Any function $\theta : X \to \mathbb{R}$ which is non-decreasing with respect to $\mathcal{P}(h)$ is constant on every chain transitive component of h. Since these components are invariant and partition $\mathcal{R}(h)$, we deduce the following corollary.

Corollary 2.3. *Let $\theta : X \to \mathbb{R}$ be a Lyapunov function for h and assume that $\theta(N(\theta))$ is totally disconnected. Then*

$$\mathcal{R}(h) \subset N(\theta).$$

Lemma 2.1 also leads to the following general result.

Proposition 2.4. *Let $\theta : X \to \mathbb{R}$ be a Lyapunov function for h and let C be a chain transitive component of h. Then $\theta(C)$ is an interval and $\theta(C) = \theta(N(\theta) \cap C)$.*

Proof. Let $t, t' \in \theta(C)$ with $t \leq t'$ and let $x, y \in C$ be such that $\theta(x) = t$ and $\theta(y) = t'$. Since C is a chain transitive component of h, we have $C = I_{x,y}$ and we deduce from lemma 2.1 and $t \leq t$ that $[t, t'] \subset \theta(N(\theta) \cap C)$. Hence $\theta(C)$ is an interval and $\theta(C) \subset \theta(N(\theta) \cap C)$. The result follows. □

Remark 2.5. The fact that $\theta(C)$ is an interval is well known in the settings of flows because chain transitive component are then connected, see [8, Theorem 3.6D].

3 The d-Mather barrier and its Aubry set

In the sequel, a *chain* will denote a finite sequence $\{x_0, .., x_n\}$, $n \geq 1$, of points of X. The integer n is called the *length* of the chain. A chain $\{x_0, .., x_n\}$ is said to go from x to y if $x_0 = x$ and $x_n = y$. We will denote by $\mathcal{C}(x, y)$ the set of chains from x to y. We define the *d-defect* of a chain $C = \{x_0, .., x_n\}$ by

$$l_d(C) = \sum_{i=0}^{n-1} d(h(x_i), x_{i+1}).$$

The *d-Mather barrier* is the function

$$L_d : X \times X \to [0, +\infty[$$

defined by

$$L_d(x, y) = \inf\{C \in \mathcal{C}(x, y),\ l_d(C)\}.$$

Main properties of the d-Mather barrier are gathered in the following proposition.

Proposition 3.1. *The d-Mather barrier satisfies the following properties*

(i) *for every x, y, z in X we have*

$$L_d(x, y) \leq L_d(x, z) + L_d(z, y),$$

(ii) *for every x in X we have*

$$L_d(x, h(x)) = 0,$$

(iii) *for a given x in X we have*

$$L_d(x, x) = 0 \Leftrightarrow L_d(h(x), x) = 0 \Leftrightarrow L_d(h(x), h(x)) = 0,$$

(iv) *for every x, y, z in X we have*

$$|L_d(x, y) - L_d(x, z)| \leq d(y, z),$$

and

$$|L_d(x, y) - L_d(z, y)| \leq d(h(x), h(z)).$$

In particular, the d-Mather barrier is continuous.

Proof. Let $x, y, z \in X$. A chain from x to z and a chain from z to y can always be concatenated to obtain a chain from x to y. Triangular inequality (i) is a consequence of this remark. Property (ii) is straightforward by considering the chain $\{x, h(x)\}$ from x to $h(x)$. To prove property (iv), let C be a chain from x to y. The chain $\tilde{C}$ obtained by changing the last term of C into z is then a chain from x to z such that $l_d(\tilde{C}) \leq l_d(C) + d(y, z)$. Hence we get $L_d(x, z) \leq l_d(C) + d(y, z)$. The first part of property (iv) follows by taking the infimum on chains C from x to y. The second part is proved similarly. It remains to prove property (iii). Let ω be a modulus of continuity of h and let $C = \{x_0, .., x_n\}$ be a chain from x to x. Concatenating the chain C with itself if needed, we can assume that $n \geq 2$. The chain $\hat{C} = \{h(x), x_2, .., x_n\}$ is then a chain from $h(x)$ to x such that $l_d(\hat{C}) \leq l_d(C) + \omega(l_d(C))$. Hence, if $L_d(x, x) = 0$ then $L_d(h(x), x) = 0$. Conversely if $L_d(h(x), x) = 0$ we have $0 \leq L_d(x, x) \leq L_d(x, h(x)) + L_d(h(x), x) = 0$ and $L_d(x, x) = 0$. We prove similarly that $L^d(x, x) = 0$ if and only if $L_d(x, h^{-1}(x)) = 0$, which leads to property (iii). □

The *d-Aubry set* of h is the subset $\mathcal{A}_d(h)$ of X defined by

$$\mathcal{A}_d(h) = \{x \in X \mid L_d(x,x) = 0\}.$$

It follows from proposition 3.0.5 that the d-Aubry set is a closed invariant subset of X. Moreover, since $0 \le S_d \le L_d$ we have

$$\mathcal{A}_d(h) \subset \mathcal{R}(h).$$

Since the d-Mather barrier is non-negative and satifies the triangular inequality, we define a closed preorder $\preceq_d$ on $\mathcal{A}_d(h)$ in the following way

$$y \preceq_d x \Leftrightarrow L_d(x,y) = 0.$$

The preorder $\preceq_d$ naturally induces an equivalence relation $\sim_d$ on $\mathcal{A}_d(h)$ by $x \sim_d y$ if and only if $x \preceq_d y$ and $y \preceq_d x$. The equivalence classes of $\sim_d$ are called the *d-Mather classes* of h. It follows from proposition 3.0.5 that they are closed invariant subsets of X. Moreover, they form a partition of $\mathcal{A}_d(h)$. The quotient space is called the *d-Mather quotient* of h and will by denoted by $\mathcal{M}_d(h)$. The function

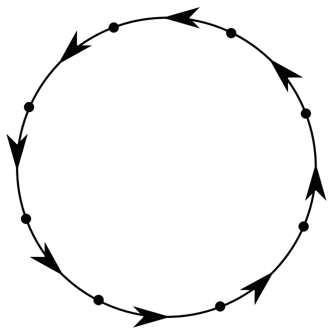
$$L_d^*(x,y) = \max\{L_d(x,y), L_d(y,x)\}$$

then induces a metric on $\mathcal{M}_d(h)$, which defines the quotient topology. Moreover, the canonical projection

$$\pi_d : (\mathcal{A}_d(h), d) \to (\mathcal{M}_d(h), L_d^*)$$

is 1-Lipschitzian. Indeed, for every $x, y \in \mathcal{A}_d(h)$ we have $|L_d(x,y)| = |L_d(x,y) - L_d(x,x)| \le d(x,y)$.

Example 3.2. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ endowed with the usual flat metric d. Let $K \subset \mathbb{T}$ be a Cantor set and let $\varphi : \mathbb{T} \to [0, +\infty[$ be a C^∞ function such that $\varphi^{-1}(0) = K$. Let h be the time-one map of the flow of the vector field $X(x) = \varphi(x)\frac{\partial}{\partial x}$.



If K has vanishing Lebesgue measure then $\mathcal{A}_d(h) = \mathbb{T}$ and $\mathcal{M}_d(h)$ is reduced to a point. If K has non-vanishing Lebesgue measure then $\mathcal{A}_d(h) = K$ and $\mathcal{M}_d(h)$ is homeomorphic to K. Notice that in both cases every point is chain recurrent and there is only one chain transitive component.

4 L_d-domination and Lipschitzian Lyapunov function

A function $u : X \to \mathbb{R}$ is said to be *(K,L_d)-dominated*, $K \geq 0$, or *L_d-dominated* for short, if for every x, y in X we have

$$u(y) - u(x) \leq KL_d(x, y).$$

The function u is said to be *strict* at a point $x \in X$ if the inequality is strict for every $y \in X$. The function u is then said to be strict on a subset $A \subset X$ if u is strict at every point $x \in A$. Notice that an L_d-dominated function cannot be strict at a point $x \in \mathcal{A}_d(h)$ where $L_d(x, x) = 0$. Moreover, we have the following.

Proposition 4.1. *There is a Lipschitzian L_d-dominated function $u : (X, d) \to \mathbb{R}$ which is strict outside $\mathcal{A}_d(h)$.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a dense sequence in X. We set

$$u(x) = \sum_{n\in\mathbb{N}} \frac{1}{2^n} L_d(x_n, x).$$

The function u is well defined and continuous because L_d is bounded on the compact set $X \times X$. Using triangular inequality of L_d we get, for every $x, y \in X$

$$\begin{aligned} u(y) - u(x) &= \sum_{n\in\mathbb{N}} \frac{1}{2^n}(L_d(x_n, y) - L_d(x_n, x)), \\ &\leq \sum_{n\in\mathbb{N}} \frac{1}{2^n} L_d(x, y), \\ &\leq 2L_d(x, y). \end{aligned}$$

Hence the function u is $(2, L_d)$-dominated. It is also 2-Lipschitzian because for every $n \in \mathbb{N}$ we have $|L_d(x_n, y) - L_d(x_n, x)| \leq d(x, y)$. It remains to show that u is strict outside the Aubry set $\mathcal{A}_d(h)$. Let $x \in X \setminus \mathcal{A}_d(h)$. We then have $L_d(x, x) > 0$. By density of the sequence $(x_n)_{n\in\mathbb{N}}$, there is at least one $n \in \mathbb{N}$ such that $L_d(x_n, y) - L_d(x_n, x) < L_d(x, y)$ and thus $u(y) - u(x) < 2L_d(x, y)$. □

Since any L_d-dominated function is non-decreasing with respect to the preorder $\preceq_d$, it is constant on every d-Mather classes. The following proposition is then straightforward.

Proposition 4.2. *Any (K, L_d)-dominated function $u : X \to \mathbb{R}$ is constant on every d-Mather classe and induces a K-Lipschitzian function on the d-Mather quotient $(\mathcal{M}_d(h), L_d^*)$.*

The link between L_d-domination and Lyapunov functions is given in the following fundamental lemma.

Lemma 4.3. *Any continuous L_d-dominated function is a Lyapunov function for h. Conversely, any K-Lipschitzian Lyapunov function for h is (K, L_d)-dominated.*

Proof. Let $u : X \to \mathbb{R}$ be a continuous L_d-dominated function. Then, for some $K \geq 0$ and every $x \in X$, we have $u(h(x)) - u(x) \leq KL_d(x, h(x)) = 0$. Since u is continuous, it is a Lyapunov function for h. Conversely, let $\theta : (X, d) \to \mathbb{R}$ be a K-Lipschitzian Lyapunov function for h. Let $x, y \in X$ and let $C = \{x_0, .., x_n\}$ be a chain from x to y. We then have

$$\begin{aligned} \theta(x_{i+1}) - \theta(x_i) &\leq \theta(x_{i+1}) - \theta(h(x_i)), \\ &\leq Kd(h(x_i), x_{i+1}). \end{aligned}$$

If we sum these inequalities for $i = 0$ to $n-1$, we get $\theta(y) - \theta(x) \leq Kl_d(C)$. Taking infimum on chains C from x to y then leads to the desired result. □

Theorem 4.4. *Any Lipschitzian Lyapunov function $\theta : (X, d) \to \mathbb{R}$ satisfies*

$$\mathcal{A}_d(h) \subset N(\theta).$$

Moreover, there is a Lipschitzian Lyapunov function $\theta : (X, d) \to \mathbb{R}$ such that

$$\mathcal{A}_d(h) = N(\theta).$$

Proof. Any Lipschitzian Lyapunov function $\theta : (X, d) \to \mathbb{R}$ is L_d-dominated by lemma 4.3. Hence, it is constant on every d-Mather class by proposition 4.2. Since these classes form a partition of $\mathcal{A}_d(h)$ and are invariant by h, we have $\mathcal{A}_d(h) \subset N(\theta)$. Let $u : (X, d) \to \mathbb{R}$ be a Lipschitzian (K, L_d)-dominated function which is strict outside $\mathcal{A}_d(h)$. Such a function exists by proposition 4.1 and is a Lipschitzian Lyapunov function for h by lemma 4.3. Moreover, for $x \in X \setminus \mathcal{A}_d(h)$ we have

$$u(h(x)) - u(x) < KL_d(x, h(x)) = 0.$$

Hence $N(u) \subset \mathcal{A}_d(h)$ and thus $N(u) = \mathcal{A}_d(h)$. □

Corollary 4.5. *We have*

$$\mathcal{A}_d(h) = \bigcap_{\theta \in \mathcal{L}_d(h)} N(\theta)$$

where $\mathcal{L}_d(h)$ denotes the set of Lipschitzian Lyapunov function $\theta : (X, d) \to \mathbb{R}$ for h. Moreover we have

$$\mathcal{A}_d(h) \subset \mathrm{Fix}(h) \cup \mathcal{R}\left(h_{|X \setminus \mathrm{int}(\mathrm{Fix}(h))}\right).$$

Proof. First part of the corollary follows from theorem 4.4. To prove the second part, it suffices to find a Lipschitzian Lyapunov function $\theta : (X, d) \to \mathbb{R}$ such that

$$N(\theta) = \mathrm{Fix}(h) \cup \mathcal{R}\left(h_{|X \setminus \mathrm{int}(\mathrm{Fix(h)})}\right).$$

By Conley's theorem, there is a Lyapunov function

$$\theta : (X \setminus \mathrm{int}(\mathrm{Fix(h)}), d) \to \mathbb{R}$$

for $h_{|X \setminus \mathrm{int}(\mathrm{Fix(h)})}$ such that

$$N(\theta) = \mathcal{R}\left(h_{|X \setminus \mathrm{int}(\mathrm{Fix(h)})}\right).$$

In fact, Conley's construction can easily be done in the realm of Lipschitzian functions and there is no loss of generality to assume that θ is Lipschitzian. We can also refer to [24, Theorem 2.15, Remark 2.23]. The function θ can then be *extended* to the whole of X to a Lipschitzian function $\theta : (X, d) \to \mathbb{R}$ by

$$\theta(x) = \inf_{y \in X \setminus \mathrm{int}(\mathrm{Fix}(h))} \theta(y) + \mathrm{Lip}(\theta) d(x, y).$$

Notice that the function θ is still a Lyapunov function for h on X because the condition $\theta \circ h \leq \theta$ is automatically satisfied on the subset $\mathrm{int}(\mathrm{Fix}(h))$. Moreover, we have

$$N(\theta) = \mathrm{Fix}(h) \cup \mathcal{R}\left(h_{|X \setminus \mathrm{int}(\mathrm{Fix(h)})}\right),$$

as desired. $\square$

Proposition 4.6. *For every $x, y \in \mathcal{A}_d(h)$, we have*

$$L_d(x, y) = \sup_{\theta \in \mathcal{L}^1_d(h)} \theta(y) - \theta(x),$$
$$L^*_d(x, y) = \sup_{\theta \in \mathcal{L}^1_d(h)} |\theta(y) - \theta(x)|.$$

where $\mathcal{L}^1_d(h)$ is the set of 1-Lipschitzian Lyapunov functions $\theta : (X, d) \to \mathbb{R}$ for h.

Proof. Let $x, y \in \mathcal{A}_d(h)$ and let $\theta \in \mathrm{Lip}^1_d(X, \mathbb{R})$. It follows from lemma 4.3 that θ is $(1, L_d)$-dominated and thus $\theta(y) - \theta(x) \leq L_d(x, y)$. Hence, we have

$$L_d(x, y) \geq \sup_{\theta \in \mathrm{Lip}^1_d(X, \mathbb{R})} \theta(y) - \theta(x).$$

For the converse inequality, consider the function $\theta_x = L_d(x, \cdot)$. It is a 1-Lipschitzian $(1, L_d)$-dominated function for which $L_d(x, y) = \theta_x(y) - \theta_x(x)$ because $L_d(x, x) = 0$. The equality $L^*_d(x, y) = \sup_{\theta \in \mathcal{L}^1_d(h)} |\theta(y) - \theta(x)|$ is proved similarly, using the fact that L_d is non-negative. $\square$

Proposition 4.2 and lemma 4.3 then lead to the following corollary.

Corollary 4.7. *Let $\mathcal{L}_d(h)$ be the set of Lipschitzian Lyapunov functions $\theta : (X, d) \to \mathbb{R}$ for h. Any function $\theta \in \mathcal{L}_d(h)$ is constant on every d-Mather classes and induces a Lipschitzian function $\bar{\theta}$ on $(\mathcal{M}_d(h), L^*_d)$. Moreover, the family of functions $\{\bar{\theta} \mid \theta \in \mathcal{L}_d(h)\}$ separates points of $(\mathcal{M}_d(h), L^*_d)$.*

We thus obtain a criteria for the existence of non-trivial Lipschitzian Lyapunov function in terms of the d-Mather quotient.

Theorem 4.8. *The only Lipschitzian Lyapunov functions $\theta : (X, d) \to \mathbb{R}$ for h are the constants if and only if the d-Mather quotient $\mathcal{M}_d(h)$ is trivial i.e. is reduced to a point. In that case, we have $\mathcal{A}_d(h) = X$.*

Proof. If every Lipschitzian Lyapunov function for h is constant then $\mathcal{M}_d(h)$ is reduced to a point by the previous corollary. Conversely, suppose that $\mathcal{M}_d(h)$ is reduced to point. The d-Aubry set $\mathcal{A}_d(h)$ is then made of a single d-Mather class and any Lipschitzian Lyapunov function for h is constant on $\mathcal{A}_d(h)$. Let $\theta : X \to \mathbb{R}$ be a Lipschitzian Lyapunov function for h. Since the alpha and omega limit set of every $x \in X$ are contained in $\mathcal{A}_d(h)$ where θ is constant and θ is a Lyapunov function for h, then θ is constant on X. Last part of the statement follows from theorem 4.4. $\square$

Example 4.9. We come back to example 3.2. If K has vanishing Lebesgue measure, we have $\mathcal{A}_d(h) = \mathbb{T}$ and there is only one d-Mather classe. Hence, any Lipschitzian Lyapunov function for h must be constant. If K has non-vanishing Lebesgue measure, we can find $\beta > 0$ such that

$$\beta \lambda_{Leb}(K) - \lambda_{Leb}(\mathbb{T} \setminus K) = 0.$$

The function

$$\theta(t) = \int_0^t (\beta \chi_K(s) - \chi_{\mathbb{T}\setminus K}(s)) ds$$

then induces a Lipschitzian Lyapunov function for h on $\mathbb{T}$ such that

$$N(\theta) = K = \mathcal{A}_d(h).$$

Notice that the function $Id - \theta$ is nothing else than a so-called *devil staircase*. Now let x, y be two distinct points of K. The subset $\mathbb{T} \setminus \{x, y\}$ is made of two non-trivial segments I_1 and I_2 and one of them, say I_1, must satisfies

$$\lambda_{Leb}(I_1 \cap K) > 0.$$

We set $K_1 = K \cap I_1$. Let $\alpha > 0$ be such that

$$\alpha \lambda_{Leb}(K_1) - \lambda_{Leb}(\mathbb{T} \setminus K_1) = 0.$$

The function defined by

$$\psi(t) = \int_0^t (\alpha \chi_{K_1}(s) - \chi_{\mathbb{T}\setminus K_1}(s)) ds$$

is then a Lipschitzian Lyapunov function for h such that $\psi(x) \neq \psi(y)$. The d-Mather classes of h are then reduced to singletons and $\mathcal{M}_d(h)$ is homeomorphic to K.

Remark 4.10. Contrary to Conley's theorem, we cannot assume that the function θ given by theorem 4.4 separates d-Mather classes i.e. induces a one-to-one map $\bar{\theta}$ on $\mathcal{M}_d(h)$. In that case, the function $\bar{\theta}$ would induce a homeomorphism between $\mathcal{M}_d(h)$ and the neutral values $\theta(N(\theta))$ of θ. But these set might have different topologies. In the previous example for instance, when K has non-vanishing Lebesgue measure, the d-Mather quotient of h is homeomorphic to K and hence is totally disconnected. Nevertheless, the neutral values of θ cannot be totally disconnected because every point is chain reccurent, see corollary 2.3.

5 Auslander set and Mañé set

Let $\mathcal{D}$ be set of all metric compatible with the topology of X. The *Auslander set* of h is the subset $\mathrm{Aus}(h)$ of X defined by

$$\mathrm{Aus}(h) = \bigcap_{d \in \mathcal{D}} \mathcal{A}_d(h).$$

The *Mañé set* of h is the subset $\tilde{\mathcal{N}}(h)$ of X defined by

$$\tilde{\mathcal{N}}(h) = \bigcup_{d \in \mathcal{D}} \mathcal{A}_d(h).$$

In analogy with Aubry-Mather theory, we should have called $\mathrm{Aus}(h)$ the *topological Aubry set* of h but this set was already discovered by Auslander in the 1960's, see [3]. He called it the *generalized recurrent set* and gave a description of it in terms of orbit prolongations, via transfinite induction. This set also appears in the book of Akin [1, Chapter I] as the *generalized nonwandering set* of h. The definition is different and uses the smallest closed and transitive relation containing the graph of h. A unified approach can be found in [2].

Theorem 5.1. *We have*

$$\tilde{\mathcal{N}}(h) = \mathrm{Fix}(h) \cup \mathcal{R}\left(h_{|X \setminus \mathrm{int}(\mathrm{Fix(h)})}\right).$$

Proof. The proof of his result is rather long and technical. We postpone it to Appendix. □

There is a natural partition of the Auslander set $\mathrm{Aus}(h)$ by equivalence classes of the relation $\sim_h$ given by: $x \sim_h y$ if and only if $x \sim_d y$ for every metric $d \in \mathcal{D}$. The equivalence classes of the relation $\sim_h$ are called the *Mather classes* of h. They are closed invariant subsets of X. The corresponding quotient space $\mathrm{Aus}(h)/\sim_h$ is called the *Mather quotient* of h and will be denoted by $\mathcal{M}(h)$.

Theorem 5.2. *Let $\mathcal{L}(h)$ be the set of Lyapunov functions for h. The family $\{N(\theta) \mid \theta \in \mathcal{L}(h)\}$ is stable by (finite or infinite) intersection and we have*

$$\mathrm{Aus}(h) = \bigcap_{\theta \in \mathcal{L}(h)} N(\theta).$$

In particular, there is $\theta \in \mathcal{L}(h)$ such that $N(\theta) = \mathrm{Aus}(h)$.

Proof. Let $\mathcal{F} \subset \mathcal{L}(h)$ be a non-empty subset of $\mathcal{L}(h)$. Since any compact metric space is second countable there is, see [9, theorem 6.3], an at most countable family $(d_n)_{n \in \mathbb{N}}$ in $\mathcal{F}$ such that

$$\bigcap_{\theta \in \mathcal{F}} N(\theta) = \bigcap_{n \in \mathbb{N}} N(\theta_n).$$

For every $\lambda > 0$ and $\theta \in \mathcal{L}(h)$ we have $\lambda\theta \in \mathcal{L}(h)$ and $N(\lambda\theta) = N(\theta)$. Hence, we can assume that the family $(\theta_n)_{n \in \mathbb{N}}$, is equi-bounded. The function

$$\theta = \sum_{n \in \mathbb{N}} \frac{1}{2^n} \theta_n$$

then satisfies $\theta \in \mathcal{L}(h)$ and $N(\theta) = \cap_{n \in \mathbb{N}} N(\theta_n) = \cap_{\theta \in \mathcal{F}} N(\theta)$. This shows that the family $\{N(\theta) \mid \theta \in \mathcal{L}(h)\}$ is stable by intersection. It follows from corollary 4.5 that

$$\mathrm{Aus}(h) = \bigcap_{d \in \mathcal{D}} \mathcal{A}_d(h) = \bigcap_{d \in \mathcal{D}} \bigcap_{\theta \in \mathcal{L}_d(h)} N(\theta).$$

Since any continuous Lyapunov function $\theta \in \mathcal{L}(h)$ is Lipschitzian for the compatible metric

$$d_\theta(x, y) = d(x, y) + |\theta(y) - \theta(x)|$$

we deduce that

$$\mathrm{Aus}(h) = \bigcap_{\theta \in \mathcal{L}(h)} N(\theta).$$

□

Proposition 5.3. *Let $\mathcal{L}(h)$ be the set of Lyapunov functions for h. Any function $\theta \in \mathcal{L}(h)$ is constant on every Mather class of h and therefore induces a continuous function $\bar{\theta}$ on the Mather quotient $\mathcal{M}(h)$. Moreover, the family of functions $\{\bar{\theta} \mid \theta \in \mathcal{L}(h)\}$ separates points of $\mathcal{M}(h)$.*

Proof. Let $\theta : X \to \mathbb{R}$ be a Lyapunov function for h. The function θ is Lipschitzian for the compatible metric

$$d_\theta(x, y) = d(x, y) + |\theta(x) - \theta(y)|$$

Hence, the function θ is constant on every d_θ-Mather class and is therefore constant on every Mather class. Let $[x], [y]$ be two distinct Mather classes. For some metric δ on X, the δ-Mather classes of x and y are then different and by corollary 4.7, there is Lipschitzian (and hence continuous) Lyapunov function $\theta : (X, \delta) \to \mathbb{R}$ such that $\theta(x) \neq \theta(y)$. □

Example 5.4. We consider again example 3.2. If K has non-vanishing Lebesgue measure, we have $\mathcal{A}_d(h) = K = \mathrm{Fix}(h) \subset \mathrm{Aus}(h)$ and thus $\mathcal{A}(h) = K$. Moreover, Mather classes of h are then reduced to singletons because it is already the case of d-Mather classes. Hence $\mathcal{M}(h)$ is homeomorphic to K. If K has vanishing Lebesgue measure, that dynamical system is topologicaly conjugated to the case $\lambda_{Leb}(K) > 0$ and same conclusions hold. In particular, we can always find a *continuous* Lyapunov function θ for h such that $N(\theta) = K$, even if K has vanishing Lebesgue measure. One can get also a direct construction of θ by using an appropriate devil's staircase.

6 Appendix

This section is devoted to the proof of theorem 5.1. The proof relies on the fact that any compact metric space can be embedded into a real infinite dimensional Hilbert space and on a contraction lemma. In the following, $(H, ||\cdot||)$ will denote a real infinite dimensional Hilbert space and $\mathrm{Isol}(X)$ will denote the set of isolated points of X. Since the final result we want to prove is obvious when $X = \mathrm{Isol}(X)$, we will suppose that $X \setminus \mathrm{Isol}(X)$ is not empty.

6.1 A contraction lemma

Lemma 6.1. *Let $a, b \in H$ and let U be an open neighborhood of the closed segment $[a, b]$. There is a C^∞ diffeomorphism φ of H with* $\mathrm{Supp}(\varphi) \subset U$ *such that both φ and φ^{-1} are Lipschitzian and $\varphi(a) = b$.*

Proof. Without loss of generality, we can suppose that $a = 0$. If $b = a = 0$, the identity map will do the job. Otherwise, let F be the orthogonal complement of the vector space spanned by b

$$H = F \overset{\perp}{\oplus} \mathbb{R}b.$$

A point of H will then be denoted by (x, s), $x \in F$, $s \in \mathbb{R}$. Let B_ε, $\varepsilon > 0$, denotes the open ball in F of radius ε centered in 0. Let $\varepsilon > 0$ small enough such that

$$B_\varepsilon \times]-\varepsilon, 1+\varepsilon[\subset U.$$

Let $\psi : \mathbb{R} \to [0,1]$ be a C^∞ function with support in $]-\varepsilon, 1+\varepsilon[$ and such that $\psi_{|[0,1]} = 1$. We denote by Φ the flow on $\mathbb{R}$ of the differential equation

$$\dot{\gamma}(t) = \psi(\gamma(t))$$

that is

$$\begin{cases} \frac{\partial}{\partial t}\Phi(t,s) = \psi(\Phi(t,s)), \\ \Phi(0,s) = s. \end{cases}$$

The flow Φ is defined for every time because the function ψ has compact support. Moreover, since $\psi_{|[0,1]} = 1$ we have $\Phi(1,0) = 1$. Let $\rho : \mathbb{R} \to [0,1]$ be a C^∞ function with support in $]-\varepsilon, \varepsilon[$ and such that $\rho(0) = 1$. We set $g(x) = \rho(||x||^2)$. The function g is C^∞ and $\mathrm{Supp}(g) \subset B_\varepsilon$. We set

$$\varphi(x, s) = (x, \Phi(g(x), s)).$$

The map φ is then a diffeomorphim of H such that

$$\mathrm{Supp}(\varphi) \subset B_\varepsilon \times]-\varepsilon, 1+\varepsilon[\subset U$$

and φ sends $a = (0,0)$ to $b = (0,1)$. Moreover, since $\varphi^{-1}(x,s) = (x, \Phi(-g(x), s))$, both diffeomorphisms φ and φ^{-1} are lipschitzian as g and Φ. $\square$

Lemma 6.2 (Contraction lemma)**.** *Let $\{x_k, y_k\}$, $k = 1, .., r$, be pairs of points of H such that $\{x_1, .., x_r\}$ are pairwise disjoint and*

$$\{x_1, .., x_r\} \cap \{y_1, .., y_r\} = \emptyset.$$

Let F be a finite subset of H such that

$$\{x_1, .., x_r\} \cap F = \emptyset.$$

Let $\varepsilon > 0$. Suppose that we have for every k in $\{1, .., r\}$,

$$||x_k - y_k|| < \varepsilon.$$

Then for every $0 < \delta < \varepsilon$ we can find a C^∞ diffeomorphism φ of H such that $\varphi_{|F} = \mathrm{Id}_{|F}$ and for every k in $\{1, .., r\}$,

$$||\varphi(x_k) - \varphi(y_k)|| < \delta.$$

Moreover, we can suppose that both φ and φ^{-1} are Lipschitzian and

$$||\varphi - \mathrm{Id}\,||_\infty < \varepsilon, \; ||\varphi^{-1} - \mathrm{Id}\,||_\infty < \varepsilon.$$

Proof. Let E be the vector space spanned by $F \cup \{x_1, .., x_r\} \cup \{y_1, .., y_r\}$. Since H is infinite dimensional, we can find a linearly independent family $\{v_1, .., v_r\}$ orthogonal to E. For $\eta > 0$ small enough, the family $\{\tilde{y}_1, .., \tilde{y}_k\}$ defined by

$$\tilde{y}_i = y_i + \eta v_i$$

is then made of pairwise disjoint points of H such that

$$\{\tilde{y}_1, .., \tilde{y}_k\} \cap (F \cup \{x_1, .., x_r\} \cup \{y_1, .., y_r\}) = \emptyset$$

and for every $k \in \{1, .., r\}$,

$$||\tilde{y}_k - y_k|| < \delta, \; ||\tilde{y}_k - x_k|| < \varepsilon.$$

Moreover, the closed segments $[x_k, \tilde{y}_k]$, $k \in \{1, .., r\}$, are disjoint and neither meet F nor $\{y_1, .., y_r\}$. Let $U_1, .., U_r$ be disjoint open neighborhoods of the segments $[x_1, \tilde{y}_1], .., [x_r, \tilde{y}_r]$ such that, for every $k \in \{1, .., r\}$, we have

$$U_k \cap (F \cup \{y_1, .., y_r\}) = \emptyset.$$

Since $||x_k - \tilde{y}_k|| < \varepsilon$, we can also suppose that every open subset U_k has diameter less than ε. By the previous lemma, there are C^∞ diffeomorphisms $\varphi_1, .., \varphi_r$ such that, for every $k \in \{1, .., r\}$, both φ_k and φ_k^{-1} are Lipschitzian, $\mathrm{Supp}(\varphi_k) \subset U_k$ and $\varphi_k(x_k) = \tilde{y}_k$. We then set

$$\varphi = \varphi_r \circ ... \circ \varphi_1.$$

Since supports of the diffeomorphisms φ_k are disjoint and do not meet the set $F \cup \{y_1, .., y_r\}$, we have

$$\varphi_{|F} = Id_{|F}$$

and for every $k \in \{1, .., r\}$,

$$\varphi(x_k) = \tilde{y}_k, \; \varphi(y_k) = y_k.$$

Moreover, since supports of the diffeomorphisms φ_k are disjoint and have diameter less than ε, we have

$$||\varphi - Id||_\infty < \varepsilon, \; ||\varphi^{-1} - Id|| < \varepsilon.$$

Last, since every diffeomorphism φ_k (resp. φ_k^{-1}) is Lipschitzian, so is φ (resp. φ^{-1}). □

Remark 6.3. The case where H is finite dimensionnal is well-known and slightly more involved, see [23], or [25] for the easier $\dim(H) \geq 3$ case.

6.2 Proof of theorem 5.1

We define the *essential points* $\mathcal{E}(C)$ of a chain $C = \{x_0, .., x_n\}$ of X by

$$\mathcal{E}(C) = \{x_{k+1} \in C, \; 0 \leq k \leq n-1 \mid x_{k+1} \neq h(x_k)\}.$$

Lemma 6.4. *Let F be a finite subset of X and let d be a metric on X defining the topology of X. There is $\varepsilon(F) > 0$ such that*

$$\forall x \in F \cap \mathrm{Isol}(X), \; d(x, X \setminus \{x\}) \geq \varepsilon(F) > 0.$$

Proof. If $F \cap \mathrm{Isol}(X) = \emptyset$, any positive $\varepsilon(F)$ will be fine. Otherwise, the set $F \cap \mathrm{Isol}(X)$ consists in a finite number of isolated points of X. Take $\varepsilon(F) > 0$ such that

$$\forall x \in F \cap \mathrm{Isol}(X),\ B_d(x, \varepsilon(F)) = \{x\}.$$

□

Lemma 6.5. *Let $\eta > 0$ and let d be a metric on X defining the topology of X. There is $\varepsilon(\eta) > 0$ such that every essential point x of any $\varepsilon(\eta)$-chain for d satisfies $d(x, X \setminus \mathrm{Isol}(X)) < \eta$.*

Proof. Suppose the contrary. We can then find a sequence $(C_r)_{r\in\mathbb{N}}$ of ε_r-chains for d with $\varepsilon_r \to 0$ as $r \to +\infty$ and essential points $x_{k_r} \in C_r$ such that

$$d(x_{k_r}, X \setminus \mathrm{Isol}(X)) \geq \eta.$$

By compactness of X, we can assume that $x_{k_r} \to x_\infty$ as $r \to +\infty$. We then have

$$d(x_\infty, X \setminus \mathrm{Isol}(X)) \geq \eta.$$

Hence, the point x_∞ does not belong to the closed set $X \setminus \mathrm{Isol}(X)$ i.e. x_∞ is an isolated point of X. In particular, the converging sequence of essential points $(x_{k_r})_{r\in\mathbb{N}}$ is eventually stationary to x_∞. Hence, for r large enough we have

$$0 < d(x_{k_r}, h(x_{k_r-1})) = d(x_\infty, h(x_{k_r-1})) < \varepsilon_r$$

and we deduce from $\varepsilon_r \to 0$ that the point x_∞ is not isolated in X, which is a contradiction.

□

Proposition 6.6. *Let F be a finite subset of X and let $x \in \mathcal{R}\left(h_{|X\setminus \mathrm{int}(\mathrm{Fix(h)})}\right)$. Let d be a metric on X defining the topology of X. For any $\varepsilon > 0$, there is an ε-chain $C = \{x_0, .., x_n\}$ for d such that*

(i) *essential points of C are pairwise disjoint,*

(ii) *we have $\mathcal{E}(C) \cap (F \cup h(C)) = \emptyset$,*

(iii) *we have $x_0 = x$ and $d(x_n, x) < \varepsilon$.*

Proof. Let $x \in \mathcal{R}\left(h_{|X\setminus \mathrm{int}(\mathrm{Fix(h)})}\right)$. The set

$$Y = X \setminus \mathrm{int}(\mathrm{Fix(h)})$$

is a compact metric space and the restriction $h_{|Y}$ induces an homeomorphism of Y such that $\mathrm{Fix}(h_{|Y})$ has no interior in Y. Hence, working on the metric space Y instead, we can suppose that $x \in \mathcal{R}(h)$ and that $\mathrm{Fix}(h)$ has no interior. Let $\varepsilon(F) > 0$ given by lemma 6.4. Let $\varepsilon > 0$. The homeomorphism h is uniformly continuous on X. Thus, there is $\eta > 0$ such that

$$\sup_{d(x,y)<\eta} d(h(x), h(y)) < \min\left(\frac{\varepsilon}{3}, \frac{\varepsilon(F)}{2}\right).$$

Moreover, we can suppose that

$$0 < \eta < \frac{\varepsilon}{3}.$$

Let $\varepsilon(\eta) > 0$ given by lemma 6.5. Let $\rho > 0$ such that

$$0 < \rho < \min\left(\eta, \varepsilon(\eta), \frac{\varepsilon(F)}{2}\right).$$

Since the point x is chain recurrent, there is a ρ-chain $C = \{x_0, .., x_n\}$ from x to x for d. Reducing the chain if necessary, we can suppose that

$$\forall p, q \in \{0, .., n-1\},\ p \neq q \Rightarrow x_p \neq x_q. \tag{6.2.1}$$

Let $x_{k+1} \in C$, $k \in \{0, .., n-1\}$. If $x_{k+1} \in \mathcal{E}(C)$, then by lemma 6.5 we can find a point $y_{k+1} \in X \setminus \mathrm{Isol}(X)$ such that

$$0 < d(x_{k+1}, y_{k+1}) < \eta.$$

The existence of y_{k+1} also obviously holds if $x_{k+1} \in X \setminus \mathrm{Isol}(X)$. We thus define a new chain $\tilde{C} = \{\tilde{x}_0, .., \tilde{x}_n\}$ from x to x in the following way:

$$\tilde{x}_0 = x_0 = x$$

and for every $k \in \{0, .., n-1\}$,

(1) if $x_{k+1} \in \mathcal{E}(C)$ or if $x_{k+1} \in X \setminus \mathrm{Isol}(X)$ then

$$\tilde{x}_{k+1} = y_{k+1} \in X \setminus \mathrm{Isol}(X) \text{ and } d(x_{k+1}, y_{k+1}) < \eta,$$

(2) if $x_{k+1} \notin \mathcal{E}(C)$ and $x_{k+1} \in \mathrm{Isol}(X)$ then

$$\tilde{x}_{k+1} = x_{k+1}.$$

Moreover, since points y_{k+1} are not isolated and $\mathrm{Fix}(h)$ has no interior, they can be chosen such that

$$y_{k+1} \notin F \cup h(\tilde{C}) \cup h^{-1}(\tilde{C}). \tag{6.2.2}$$

First, notice that for every $k \in \{0, .., n\}$ we have

$$d(\tilde{x}_k, x_k) < \eta < \frac{\varepsilon}{3}$$

and in particular $d(\tilde{x}_n, x) < \varepsilon$. Moreover, for every $k \in \{0, .., n-1\}$ we have

$$\begin{aligned} d(h(\tilde{x}_k), \tilde{x}_{k+1}) &\leq d(h(\tilde{x}_k), h(x_k)) + d(h(x_k), x_{k+1}) \\ &\quad + d(x_{k+1}, \tilde{x}_{k+1}), \\ &< \sup_{d(x,y)<\eta} d(h(x), h(y)) + \rho + \eta < \varepsilon. \end{aligned}$$

Thus, the chain $\tilde{C}$ is an ε-chain for d satisfying property (iii). We now claim that the chain $\tilde{C}$ satisfies property (ii). Let $\tilde{x}_{k+1} \in \mathcal{E}(\tilde{C})$. We distinguish two cases. If $\tilde{x}_{k+1}$

is coming from case (1) i.e $\tilde{x}_{k+1} = y_{k+1}$ then by 6.2.2 we do have $\tilde{x}_{k+1} \notin F \cup h(\tilde{C})$. Otherwise, we are in case (2) i.e $\tilde{x}_{k+1} = x_{k+1}$, and

$$\tilde{x}_{k+1} \in \mathrm{Isol}(X)$$

and

$$h(x_k) = x_{k+1}.$$

We then have

$$\begin{aligned} d(\tilde{x}_{k+1}, h(\tilde{x}_k)) &\leq d(h(x_k), h(\tilde{x}_k)), \\ &\leq \sup_{d(x,y)<\eta} d(h(x), h(y)) < \varepsilon(F). \end{aligned}$$

Remember that $\tilde{x}_{k+1} \in \mathcal{E}(\tilde{C})$ hence $h(\tilde{x}_k) \neq \tilde{x}_{k+1}$. Since $\tilde{x}_{k+1} \in \mathrm{Isol}(X)$, we deduce from lemma 6.4 that

$$\tilde{x}_{k+1} \notin F.$$

Now, suppose that $\tilde{x}_{k+1} = h(\tilde{x}_p) \in h(\tilde{C})$. Since $h(x_k) = x_{k+1} = \tilde{x}_{k+1}$, the injectivity of h implies that $\tilde{x}_p = x_k$. Since $h(\tilde{x}_k) \neq \tilde{x}_{k+1}$, the same argument implies that

$$\tilde{x}_k \neq x_k.$$

Now since $\tilde{x}_p \in h^{-1}(\tilde{C})$, we get from 6.2.2 that $\tilde{x}_p = x_p$. Thus we have $x_p = x_k$. Since $k \in \{0, .., n-1\}$, we deduce from 6.2.1 that either $p = k$ or $k = 0$ and $p = n$. If $p = k$ then the equality $\tilde{x}_p = x_p$ contradicts the fact that $\tilde{x}_k \neq x_k$. If $k = 0$ then the equality $\tilde{x}_0 = x_0$ contradicts again $\tilde{x}_k \neq x_k$. In both cases we obtain a contradiction and thus $\tilde{x}_{k+1} \notin h(\tilde{C})$. Hence property (ii) is satisfied. Now, reducing the chain if necessary, we can assume that points of $\mathcal{E}(\tilde{C})$ are pairwise disjoint, so that property (i) holds. ☐

Proof of theorem 5.1. The inclusion

$$\tilde{\mathcal{N}}(h) \subset \mathrm{Fix}(h) \cup \mathcal{R}\left(h_{|X\setminus \mathrm{int(Fix(h))}}\right)$$

follows from corollary 4.5. Conversely, let $x \in \mathrm{Fix}(h) \cup \mathcal{R}\left(h_{|X\setminus \mathrm{int(Fix(h))}}\right)$. If $x \in \mathrm{Fix}(h)$ then of course $x \in \tilde{\mathcal{N}}(h)$. Hence, we will suppose that $x \in \mathcal{R}\left(h_{|X\setminus \mathrm{int(Fix(h))}}\right)$. Any compact metric space can be embedded into the Hilbert's cube, see [16, Theorem V.4]. Hence, there is no loss of generality to assume that X is a subspace of an infinite real dimensional Hilbert space $(H, ||\cdot||)$. Using induction, we will construct a sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of X and a sequence $(\phi_n)_{n\in\mathbb{N}}$ of Lipschitzian diffeomorphisms of H such that, for every $n \in \mathbb{N}$,

(i) $F_n \subset F_{n+1}$,

(ii) $\phi_{n+1|F_n} = \phi_{n|F_n}$,

(iii) $||\phi_{n+1} - \phi_n||_\infty \leq \frac{1}{2^{n+1}}$ and $||\phi_{n+1}^{-1} - \phi_n^{-1}||_\infty \leq \frac{1}{2^{n+1}}$,

(iv) for every $n \geq 1$, there is a chain $\{x_0^n = x, .., x_{l_n}^n\}$ in F_n such that

$$\begin{cases} \displaystyle\sum_{k=0}^{l_n-1} ||\phi_n(x_{k+1}^n) - \phi_n(h(x_k^n))|| \leq \frac{1}{2^n}, \\ ||\phi_n(x_{l_n}^n) - \phi_n(x)|| \leq \dfrac{1}{2^n}, \end{cases}$$

(v) both diffeomorphisms ϕ_n and ϕ_n^{-1} are Lipschitzian.

We set $F_0 = \{x\}$, $\phi_0 = Id$. Suppose that the subsets F_i and the diffeomorphisms ϕ_i have been constructed for $i = 0, .., n$. We consider the metric

$$d(x, y) = ||\phi_n(x) - \phi_n(y)||$$

on X. Fix $0 < \varepsilon < \frac{1}{2^{n+1}}$. By previous proposition used with the metric d, there is a chain

$$C_{n+1} = \{x_0^{n+1} = x, .., x_{l_{n+1}}^{n+1}\}$$

in X with the properties that

(1) $\mathcal{E}(C_{n+1}) \cap (F_n \cup h(C_{n+1})) = \emptyset$,

(2) Points of $\mathcal{E}(C_{n+1})$ are pairwise disjoint,

(3) for every $k \in \{0, .., l_{n+1} - 1\}$ we have

$$||\phi_n(x_{k+1}^{n+1}) - \phi_n(h(x_k^{n+1}))|| < \varepsilon,$$

(4) $||\phi_n(x_{l_{n+1}}^{n+1}) - \phi_n(x)|| < \varepsilon$.

Using lemma 6.2 with the pairs

$$\left\{\phi_n(h(x_k^{n+1})), \phi_n(x_{k+1}^{n+1})\right\}, x_{k+1}^{n+1} \in \mathcal{E}(C_{n+1}),$$

and the finite set $\phi_n(F_n)$, we can find a diffeomorphism φ of H such that both φ and φ^{-1} are Lipschitzian and

(a) for every $x_{k+1}^{n+1} \in \mathcal{E}(C^{n+1})$,

$$||\varphi\left(\phi_n\left(x_{k+1}^{n+1}\right)\right) - \varphi\left(\phi_n\left(h\left(x_k^{n+1}\right)\right)\right)|| \leq \frac{\varepsilon}{l_{n+1}}.$$

(b) $\varphi_{|\phi_n(F_n)} = Id_{|\phi_n(F_n)}$,

(c) $||\varphi - Id||_\infty < \varepsilon$ and $||\varphi^{-1} - Id||_\infty < \varepsilon$.

We set

$$\phi_{n+1} = \varphi \circ \phi_n,$$

and

$$F_{n+1} = F_n \cup C_{n+1}.$$

We then have, for $\varepsilon > 0$ small enough,

1. $\sum_{k=0}^{l_{n+1}-1} ||\phi_{n+1}(x_{k+1}^{n+1}) - \phi_{n+1}(h(x_k^{n+1}))|| \leq l_{n+1}\frac{\varepsilon}{l_{n+1}} < \frac{1}{2^{n+1}}$,
2. $||\phi_{n+1} - \phi_n||_\infty \leq ||\varphi - Id||_\infty < \varepsilon < \frac{1}{2^{n+1}}$,
3. $||\phi_{n+1}^{-1} - \phi_n^{-1}||_\infty \leq \mathrm{Lip}(\phi_n^{-1})||\varphi^{-1} - Id||_\infty \leq \varepsilon\,\mathrm{Lip}(\phi_n^{-1}) < \frac{1}{2^{n+1}}$,

4.
$$\begin{aligned}||\phi_{n+1}(x^{n+1}_{l_{n+1}})-\phi_{n+1}(x)|| &\leq ||\phi_{n+1}(x^{n+1}_{l_{n+1}})-\phi_n(x^{n+1}_{l_{n+1}})|| \\ &\quad +||\phi_n(x^{n+1}_{l_{n+1}})-\phi_n(x)|| \\ &\quad +||\phi_n(x)-\phi_{n+1}(x)||, \\ &\leq 3\varepsilon < \frac{1}{2^{n+1}},\end{aligned}$$

5. $\phi_{n+1|F_n} = \phi_{n|F_n}$,

6. both diffeomorphisms ϕ_{n+1} and ϕ^{-1}_{n+1} are Lipschitzian.

This complete the induction step and proves the existence of the families $(F_n)_{n\in\mathbb{N}}$ and $(\phi_n)_{n\in\mathbb{N}}$. By property (iii), the maps

$$\phi(x) = \lim_{n\to+\infty}\phi_n(x),$$
$$\phi^{-1}(x) = \lim_{n\to+\infty}\phi^{-1}_n(x),$$

are well defined. Since the convergences are uniform, they are both continuous and reciprocal one of each others. The map ϕ is thus an homeomorphism of H. Moreover, it follows from properties (i), (ii) and (iv) that for every $n\geq 1$, the chain $\{x^n_0 = x, .., x^n_{l_n}\}$ satisfies

$$\sum_{k=0}^{l_n-1}||\phi(x^n_{k+1})-\phi(h(x^n_k))|| = \sum_{k=0}^{l_n-1}||\phi_n(x^n_{k+1})-\phi_n(h(x^n_k))|| \leq \frac{1}{2^n}$$

and
$$||\phi(x_{l_n})-\phi(x)|| = ||\phi_n(x^n_{l_n})-\phi_n(x)|| \leq \frac{1}{2^n}.$$

Thus the chain obtained by changing x_{l_n} into x is a chain from x to x satisfying

$$\sum_{k=0}^{l_n-1}||\phi(x^n_{k+1})-\phi(h(x^n_k))|| \leq \frac{1}{2^{n-1}}.$$

Since $n\geq 1$ is arbitrary we have $x\in\mathcal{A}_\delta(h)$ for the compatible metric

$$\delta(x,y) = ||\phi(x)-\phi(y)||.$$

and $x\in\tilde{\mathcal{N}}(h)$. □

Chapter 3

Functions whose set of critical points is an arc

1 Introduction

Let M be a C^∞ connected closed manifold. For every $k \in \mathbb{N}$, we denote by $C^k(M, \mathbb{R})$ the set of C^k real-valued functions on M, endowed with the usual C^k-topology. In this paper, we are interested in the set $\mathcal{J} \subset C^1(M, \mathbb{R})$ of C^1 real-valued functions f on M satisfying

(i) the subset $\mathrm{Crit}(f)$ is an arc i.e. is homeomorphic to the unit interval $I = [0, 1]$,

(ii) the function $f_{|\,\mathrm{Crit}(f)}$ is nowhere locally constant on $\mathrm{Crit}(f)$.

The non-emptiness of $\mathcal{J}$ is not at all obvious and will follow from the main result of this paper. This fact is surprising because minima and maxima of any function $f \in \mathcal{J}$ are then connected by an arc of critical points and there is no other critical point. Actually, any function $f \in \mathcal{J}$ provides a so-called Whitney example i.e. a function that is not constant along an arc of critical points, and therefore violates conclusions of Sard's theorem. The first such example is due to Whitney [27] who constructed a C^1 real-valued function f on $\mathbb{R}^2$ together with an arc γ of critical points of f such that $f(\gamma(0)) \neq f(\gamma(1))$. Modern approach of this result can be found in [15]. Nevertheless, the function constructed by Whitney might have additionnal critical points outside the arc γ. In particular, there is no reason for the set of critical points of f to be connected. However, we shall prove the following.

Theorem 1.1. *Let M be a C^∞ connected closed manifold with $\dim(M) \geq 2$. Let $\mathcal{J} \subset C^1(M, \mathbb{R})$ be the set of C^1 real-valued functions on M such that*

(i) *the subset $\mathrm{Crit}(f)$ is an arc i.e. is homeomorphic to the unit interval $I = [0, 1]$,*

(ii) *the function $f_{|\,\mathrm{Crit}(f)}$ is nowhere locally constant on $\mathrm{Crit}(f)$.*

Then $\mathcal{J}$ is dense in $C^0(M, \mathbb{R})$.

As mentioned above, it follows from Sard's theorem that any function $f \in \mathcal{J}$ is at most $C^{\dim(M)-1}$. Notice also that the theorem becomes false if we replace $C^0(M, \mathbb{R})$

with $C^1(M, \mathbb{R})$ in the statement. Indeed, being a regular value is an open condition in the C^1-topology and extrema of any Morse function, for instance, are separated by a regular value. The proof of theorem 1.1, which will take the subsequent three sections of this paper, is based on the very flexible tools developed by Körner in [20]. Applications in dynamics are given in the last section.

2 δ-Zig-zags

We use the standard euclidean norm $||\cdot||$ on $\mathbb{R}^n$. The closed line segment joining two points a and b in $\mathbb{R}^n$ will be denoted by $[a, b]$ and the unit interval of the real line will be denoted by $I = [0, 1]$. A *path* in $\mathbb{R}^n$ will denote a continuous map $\gamma : I \to \mathbb{R}^n$. A path γ is said to be a *polygonal path* if we can find a subdivision $t_0 = 0 < t_1 < ... < t_N = 1$ of I such that the restriction of γ to every subinterval $[t_i, t_{i+1}]$ is an affine map. Such a subdivision is then called *adapted to* γ. The minimal integer N appearing among all adapted subdivisions will be called the *number of segments of* γ. A *polygonal arc* is a one-to-one polygonal path. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function and let δ be a positive real. A polygonal arc γ is said to be a *δ-zig-zag* for g if we can find a subdivision $t_0 = 0 < t_1 < ... < t_N = 1$ of I adapted to γ and a sequence $H_0, .., H_{N-1}$ of hyperplanes of $\mathbb{R}^n$ such that for every $i \in \{0, .., N-1\}$ the following two properties hold

(1) we have $\gamma(t_{i+1}) - \gamma(t_i) \in H_i$,

(2) for every $t \in [t_i, t_{i+1}]$ and for every $v \in H_i$ we have

$$|d_{\gamma(t)}g(v)| \leq \delta ||v||.$$

This section is devoted to the construction of δ-zig-zags. The underlying heuristic idea is that "we can always climb a steep hill by cutting a zig-zag into the hillside" see [20]. The following lemma is a reformulation of [20, Lemma 3.1] in a high dimensional setting.

Lemma 2.1. *Let $[a, b] \subset \mathbb{R}^n$, $n \geq 2$, be a closed line segment and let $g : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. Given any neighbourhood U of $[a, b]$ and any $\delta > 0$, we can find a sequence of points $a_0 = a, a_1, .., a_{2N} = b$ in U and a sequence $H_0, .., H_{2N-1}$ of hyperplanes of $\mathbb{R}^n$ such that for every $i \in \{0, .., 2N-1\}$*

(i) *we have $a_{i+1} - a_i \in H_i$,*

(ii) *for every $x \in [a_i, a_{i+1}]$ and for every $v \in H_i$ we have*

$$|d_x g(v)| \leq \delta ||v||.$$

Proof. Let $N > 0$ to be chosen later. For $k \in \{0, .., N\}$ we set

$$a_{2k} = a + \frac{k}{N}(b - a)$$

and for $k \in \{0, .., N-1\}$ we set

$$m_k = \frac{a_{2k+2} - a_{2k}}{2}.$$

Since $\dim(\operatorname{Ker} d_{m_k} g) \geq n-1$ and $n \geq 2$ we have $\dim(\operatorname{Ker} d_{m_k} g) \geq 1$. Hence, we can find $u_k \in \operatorname{Ker} d_{m_k} g$ such that $||u_k|| = 1$. We then set

$$a_{2k+1} = m_k + \frac{b-a}{2\sqrt{N}} u_k.$$

Note that $a_{2k} \in U$ for N large enough. Since $\dim(\operatorname{Ker} d_{m_k} g) \geq n-1$, we can find two vector subspaces F_{2k} and F_{2k+1} of the vector space $\operatorname{Ker} d_{m_k} g$ such that the following sums

$$H_{2k} = \operatorname{Vect}_{\mathbb{R}}(a_{2k+1} - a_{2k}) \overset{\perp}{\oplus} F_{2k},$$
$$H_{2k+1} = \operatorname{Vect}_{\mathbb{R}}(a_{2k+2} - a_{2k+1}) \overset{\perp}{\oplus} F_{2k+1},$$

are orthogonal and define two hyperplanes of $\mathbb{R}^n$ containing respectively the vectors $a_{2k+1} - a_{2k}$ and $a_{2k+2} - a_{2k+1}$. We now prove that property (ii) is satisfied for N large enough. Let $x \in [a_{2k}, a_{2k+1}]$ and let

$$v = u + w \in \operatorname{Vect}_{\mathbb{R}}(a_{2k+1} - a_{2k}) \overset{\perp}{\oplus} F_{2k} = H_{2k}.$$

We have

$$d_x g(v) = d_{m_k} g(v) + d_x g(v) - d_{m_k} g(v)$$

and since $||u_k|| = 1$ and $x \in [a_{2k}, a_{2k+1}]$ we have

$$\begin{aligned} ||x - m_k|| &\leq ||a_{2k} - m_k|| + ||a_{2k+1} - m_k||, \\ &\leq \frac{b-a}{2N} + \frac{b-a}{2\sqrt{N}}. \end{aligned}$$

Thus

$$|d_x g(v)| \leq |d_{m_k} g(v)| + M(N)||v||$$

where

$$M(N) = \sup_{||x-y|| \leq \frac{b-a}{2N} + \frac{b-a}{2\sqrt{N}}, y \in [a,b]} ||d_x g - d_y g||.$$

The uniform continuity of the map dg in a neighbourhood of $[a, b]$ then implies that for N large enough we have $M(N) \leq \frac{\delta}{2}$ and then

$$|d_x g(v)| \leq |d_{m_k} g(v)| + \frac{\delta}{2}||v||.$$

It thus remains to prove that for N large enough we have

$$|d_{m_k} g(v)| \leq \frac{\delta}{2}||v||.$$

Since $v = u + w \in \operatorname{Vect}(a_{2k+1} - a_{2k}) \overset{\perp}{\oplus} F_{2k}$ and $F_{2k} \subset \operatorname{Ker} d_{m_k} g$ we have

$$d_{m_k} g(v) = d_{m_k} g(u).$$

Since the sum is orthogonal we have $||u|| \leq ||v||$ so it just suffices to show that for N large enough we have

$$|d_{m_k} g(u)| \leq \frac{\delta}{2}||u||.$$

Since $u \in \mathrm{Vect}_{\mathbb{R}}(a_{2k+1} - a_{2k})$ it suffices to show, using homogeneity, that for N large enough we have

$$|d_{m_k} g(a_{2k+1} - a_{2k})| \leq \frac{\delta}{2} ||a_{2k+1} - a_{2k}||.$$

But

$$\begin{aligned} a_{2k+1} - a_{2k} &= m_k - a_{2k} + a_{2k+1} - m_k, \\ &= m_k - a_{2k} + \frac{b-a}{2\sqrt{N}} u_k, \end{aligned} \tag{2.0.1}$$

with $u_k \in \mathrm{Ker}\, d_{m_k} g$. Thus

$$\begin{aligned} |d_{m_k} g(a_{2k+1} - a_{2k})| &= |d_{m_k} g(m_k - a_{2k})|, \\ &\leq \left(\sup_{[a,b]} ||dg|| \right) ||m_k - a_{2k}||. \end{aligned}$$

Since $||m_k - a_{2k}|| = \frac{b-a}{2N}$, we have for N large enough

$$\begin{aligned} \left(\sup_{[a,b]} ||dg|| \right) ||m_k - a_{2k}|| &= \sup_{[a,b]} ||dg|| \frac{b-a}{2N}, \\ &\leq \frac{\delta}{2} \left(\frac{b-a}{2\sqrt{N}} - \frac{b-a}{2N} \right) \end{aligned}$$

Now using 2.0.1 and $||u_k|| = 1$ we have

$$\begin{aligned} ||a_{2k+1} - a_{2k}|| &\geq \frac{b-a}{2\sqrt{N}} - ||m_k - a_{2k}||, \\ &\geq \frac{b-a}{2\sqrt{N}} - \frac{b-a}{2N}. \end{aligned}$$

Hence for N large enough we have

$$|d_{m_k} g(a_{2k+1} - a_{2k})| \leq \frac{\delta}{2} ||a_{2k+1} - a_{2k}||$$

as desired. To sum up, we have shown that for N large enough, for every $x \in [a_{2k}, a_{2k+1}]$ and for every $v \in H_{2k}$ we have

$$|d_x g(v)| \leq \delta ||v||.$$

Exchanging the role played by a_{2k} and a_{2k+2}, the same proof shows the corresponding result for the segment $[a_{2k+1}, a_{2k+2}]$ with the hyperplane H_{2k+1}. $\square$

Corollary 2.2. *Let $[a,b] \subset \mathbb{R}^n$, $n \geq 2$, be a closed line segment not reduced to a point and let $g : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. Given any neighbourhood U of $[a,b]$ and any $\delta > 0$, we can find a δ-zig-zag γ for g such that $\gamma \subset U$, $\gamma(0) = a$ and $\gamma(1) = b$.*

Proof. Let $\varepsilon > 0$. Since $[a,b]$ is a convex subset of $\mathbb{R}^n$, the open neighbourhood $\mathcal{V}_\epsilon([a,b])$ of $[a,b]$ defined by

$$\mathcal{V}_\epsilon([a,b]) = \{x \in \mathbb{R}^n \mid d(x, [a,b]) < \varepsilon\}$$

is convex and included in U for ε small enough. Hence, we can assume that U is a convex neighbourhood of $[a, b]$. Let $a_0 = a, a_1.., a_{2N} = b$ and $H_0, .., H_{2N-1}$ given by the previous lemma. For $k = 0, .., 2N$ we set $t_k = \frac{k}{2N}$ and we set $\gamma_1(t_k) = a_k$. We now define γ_1 to be affine on every subinterval $[t_k, t_{k+1}]$. Since U is convex, we have $\gamma_1 \subset U$. The path γ_1 now satisfies all the desired properties except that γ_1 may fail to be one-to-one. Let consider a polygonal path $\gamma \subset \gamma_1$ from a to b with a minimal number of segments. Since $a \neq b$, the path γ can be one-to-one parametrized by I. Since for a sufficiently fine subdivision $t'_0 = 0 < t'_1 < .. < t'_r = 1$ of I every segment $\gamma([t'_i, t'_{i+1}])$ is included in some initial segment $[a_{k_i}, a_{k_i+1}]$, the hyperplanes $H_0, ..., H_{2N-1}$ can be used to check that the path γ does furnish a δ-zig-zag for g. □

Results of this section are summed up in the following corollary.

Corollary 2.3. *Let $g : \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, be a C^1 function and let γ be a polygonal arc. Given any $\delta > 0$, any $\varepsilon > 0$ and any neighbourhood U of γ, we can find a δ-zig-zag $\beta \subset U$ for g such that $\beta(0) = \gamma(0), \beta(1) = \gamma(1)$ and, for every t in I*

$$||\beta(t) - \gamma(t)|| < \varepsilon.$$

Moreover, we can suppose that $\beta \cap \mathrm{Crit}(g) = \gamma \cap \mathrm{Crit}(g)$.

Proof. Let $t_0 = 0 < t_1 < .. < t_N = 1$ be a subdivision of I adapted to γ. Considering a finer subdivision if necessary, we can suppose that every segment $\gamma([t_k, t_{k+1}])$ has a diameter less than ε. Since γ is one-to-one, we can find open convex neighbourhoods $U_0, ..., U_{N-1}$ of segments $\gamma([t_0, t_1]), ..., \gamma([t_{N-1}, t_N])$ such that the diameter of every U_k is less than ε and

$$U_l \cap U_m = \emptyset \quad \text{for } |l - m| > 1. \tag{2.0.2}$$

Using the uniform continuity of the map dg in a neighbourhood of γ, we can assume, reducing ε if necessary, that

$$||dg_{|U_k}|| < \delta \quad \text{whenever} \quad U_k \cap \mathrm{Crit}(g) \neq \emptyset. \tag{2.0.3}$$

By the previous lemma, we can find a δ-zig-zag β_k in U_k from $\gamma(t_k)$ to $\gamma(t_{k+1})$ for every $k \in \{0, .., N-1\}$. Note that by 2.0.3, we can assume that

$$\beta_k = \gamma([t_k, t_{k+1}]) \quad \text{whenever} \quad U_k \cap \mathrm{Crit}(g) \neq \emptyset \tag{2.0.4}$$

because in that case, any hyperplane containing the vector $\gamma(t_{k+1}) - \gamma(t_k)$ can be used to check that β_k is a δ-zig-zag for g. We define a polygonal path β from $\gamma(0)$ to $\gamma(1)$ in the following way: starting with $k = 0$, follow the δ-zig-zag β_k until it crosses the δ-zig-zag β_{k+1} and then follow β_{k+1} from this crossing point until it crosses β_{k+2}, etc... It follows from 2.0.2 and from the injectivity of every δ-zig-zag β_k that the path β is one-to-one. It is then a δ-zig-zag for g as a concatenation of δ-zig-zags for g. Since every neighbourhood U_k has a diameter less than ε, we can parametrize the δ-zig-zag β by the unit interval in such a way that $||\beta(t) - \gamma(t)|| < \varepsilon$ for every t in I. Reducind ε if necessary, condition $\beta \subset U$ will now follow. It thus remains to prove that $\beta \cap \mathrm{Crit}(g) = \gamma \cap \mathrm{Crit}(g)$. According to 2.0.4, a point of the δ-zig-zag β is either contained in some neighbourhood U_k with $U_k \cap \mathrm{Crit}(g) = \emptyset$ or is a point of the original path γ. Hence we have $\beta \cap \mathrm{Crit}(g) \subset \gamma \cap \mathrm{Crit}(g)$. Now let x be a possible point of $\gamma \cap \mathrm{Crit}(g)$. Let $r \in \{0, .., N\}$ be such that $x \in \gamma([t_r, t_{r+1}])$ and let $t_x \in [t_r, t_{r+1}]$ be such that $x = \gamma(t_x)$. Since $x \in \mathrm{Crit}(g)$, it follows from 2.0.4 that we have $\beta_r = \gamma([t_r, t_{r+1}])$. Since the δ-zig-zag β_r is followed for a while during the construction of the δ-zig-zag β, we can find t_- and t_+ with $t_r \leq t_- \leq t_+ \leq t_{r+1}$ such that $\beta \cap \beta_r = \gamma([t_-, t_+])$. To conclude the proof, it suffices to show that $t_- \leq t_x \leq t_+$. We prove that $t_- \leq t_x$, the proof of $t_x \leq t_+$ being similar. If $r = 0$ we have $t_- = t_r$ and there is noting to prove. If $r \geq 1$ and $U_{r-1} \cap \mathrm{Crit}(g) \neq \emptyset$, it follows from 2.0.4 that $\beta_{r-1} = \gamma_{|[t_{r-1}, t_r]}$ and again $t_- = t_r$. So we can suppose that $r \geq 1$ and $U_{r-1} \cap \mathrm{Crit}(g) = \emptyset$. Since both points $\gamma(t_r)$ and $\gamma(t_-)$ lie in U_{r-1} and since U_{r-1} is convex, we deduce that the segment $\gamma([t_r, t_-])$ is included in U_{r-1}. If $t_- > t_x$ then $x = \gamma(t_x) \in U_{r-1}$, which contradicts $U_{r-1} \cap \mathrm{Crit}(g) = \emptyset$.

□

3 Alteration lemma

As we will now see, it is possible to alter a function in a neighbourhood of a δ-zig-zag to obtain a function with a small differential in a neighbourhood of this δ-zig-zag. The key alteration lemma 3.1 is essentially taken from [20, Lemma 3.3]. The difference with [20, Lemma 3.3] is that it can be done without adding or removing any critical point. In the sequel, the support of a real-valued function F will be denoted by $\mathrm{Supp}(F)$ and the partial derivatives of a C^1 function $h : \mathbb{R}^n \to \mathbb{R}$ will be denoted by $\partial_k h$, $k = 1, .., n$. We recall that the closed line segment between two points a, b of $\mathbb{R}^n$ is denoted by $[a, b]$.

Lemma 3.1 (Alteration lemma). *Let $g : \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, be a C^1 function and let $[a, b]$ be a closed line segment in $\mathbb{R}^n$ not reduced to a point. Suppose that we can find an hyperplane H of $\mathbb{R}^n$ and $\delta > 0$ such that $b - a \in H$ and, for every $x \in [a, b]$ and every $v \in H$, we have*

$$|d_x g(v)| \leq \delta ||v||. \tag{3.0.5}$$

Then given any $\eta > 0$ and any neighbourhood U of $[a, b]$, we can find a C^∞ function $h : \mathbb{R}^n \to \mathbb{R}$ such that

(i) *we have* $\mathrm{Supp}(h) \subset U$ *and* $||h||_\infty < \eta$,

(ii) *for every $x \in [a, b]$, we have $||d_x(g + h)|| < \delta + \eta$,*

(iii) *we have* $\mathrm{Crit}(g + h) = \mathrm{Crit}(g)$,

(iv) *for every $x \in \mathbb{R}^n$ and every $u \in \mathbb{R}^n$, we have*

$$|d_x(g + h)(u)| < |d_x g(u)| + (\delta + \eta)||u||.$$

Proof. Without loss of generality, we can assume that

$$H = \{x \in \mathbb{R}^n \mid x_n = 0\} \simeq \mathbb{R}^{n-1}$$

and that $a = (0, .., 0)$, $b = (1, 0, .., 0)$. For $r > 0$ small enough, the compact subset

$$C_r = [-r, 1 + r] \times [-r, r]^{n-1}$$

is included in U. Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1$ and

$$\varepsilon \left(\sup_{C_r} ||dg|| \right) < \frac{\eta}{3}. \tag{3.0.6}$$

We define two compact subsets of H by

$$K_1 = \left\{ (x_1, .., x_{n-1}, 0) \in C_r,\ |\partial_n g(x_1, .., x_{n-1}, 0)| \geq \frac{\eta}{3} \right\},$$

$$K_2 = \left\{ (x_1, 0, .., 0),\ 0 \leq x_1 \leq 1,\ |\partial_n g(x_1, 0, .., 0)| \geq \frac{\eta}{2} \right\}.$$

Note that if $K_2 = \emptyset$ then conditions (i) to (iv) are satisfied for $h = 0$. So we can suppose that $K_2 \neq \emptyset$. As subsets of H, we then have $K_2 \subset \mathring{K}_1$ and we can find a C^∞ function $\psi : H \to I$ such that $\mathrm{Supp}(\psi) \subset K_1$ and $\psi_{|K_2} = 1$. Let $F : \mathbb{R} \to I$ be a C^∞ compactly supported function such that $F'(0) = 1$ and $-\varepsilon \leq F' \leq 1$. Let $G : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function such that

$$||dG - dg||_\infty < \frac{\eta}{3}\varepsilon.$$

Let $R > 0$ to be chosen later. We set

$$h(x) = -(1-\varepsilon)\psi(x_1, .., x_{n-1})\partial_n G(x_1, .., x_{n-1}, 0)\frac{1}{R}F(Rx_n).$$

We will show that properties (i) to (iv) are satisfied provided only that R is large enough. Since F is compactly supported, we can find a constant $M > 0$ such that $\mathrm{Supp}(F) \subset [-M, M]$ and we have

$$\mathrm{Supp}(h) \subset \mathrm{Supp}(\psi) \times \left[-\frac{M}{R}, \frac{M}{R} \right]. \tag{3.0.7}$$

Hence, for R large enough, we get $\mathrm{Supp}(h) \subset C_r \subset U$. Moreover, since the function $\psi\partial_n G$ is compactly supported and $||F||_\infty \leq 1$ we have $||h||_\infty < \eta$ for R large enough and property (i) is satisfied whenever R is large enough. Similarly, the quantities $||\partial_1 h||_\infty, .., ||\partial_{n-1} h||_\infty$ can by made arbitrarly small for R large enough. We now evaluate the partial derivative $\partial_n h$ at a point $x \in \mathbb{R}^n$. We have

$$\partial_n h(x) = -(1-\varepsilon)\psi(x_1, .., x_{n-1})\partial_n G(x_1, .., x_{n-1}, 0)F'(Rx_n).$$

Since $||(1-\varepsilon)\psi F'||_\infty \leq 1$, we get for every $x \in \mathbb{R}^n$

$$\begin{aligned} \partial_n h(x) &= -(1-\varepsilon)\psi(x_1, .., x_{n-1})\partial_n g(x_1, .., x_{n-1}, 0)F'(Rx_n) \\ &\quad +\Delta_1(x) \end{aligned} \tag{3.0.8}$$

with

$$|\Delta_1(x)| \leq ||dG - dg||_\infty < \frac{\eta}{3}\varepsilon.$$

We can now prove property (ii). Let $x = (x_1, 0, .., 0) \in \mathbb{R}^n$ with $0 \leq x_1 \leq 1$. Since $F'(0) = 1$, we get from 3.0.8 that

$$\partial_n(g+h)(x_1, 0, .., 0) = \partial_n g(x_1, 0, .., 0)[1 - (1-\varepsilon)\psi(x_1, 0, .., 0)] + \Delta_1(x).$$

We distinguish two cases. If $(x_1, 0, .., 0) \notin K_2$ then $|\partial_n g(x_1, 0, .., 0)| < \frac{\eta}{2}$ and since $0 \leq \psi \leq 1$ and $|\Delta_1(x_1, 0, .., 0)| < \frac{\eta}{3}$ we have

$$|\partial_n(g + h)(x_1, 0, .., 0)| < \frac{5}{6}\eta.$$

Otherwise, we have $(x_1, 0, .., 0) \in K_2$ and $\psi(x_1, 0, .., 0) = 1$. Thus

$$\partial_n(g + h)(x_1, 0, .., 0) = \varepsilon\partial_n g(x_1, 0, .., 0) + \Delta_1(x_1, 0, .., 0).$$

Since ε has been chosen such that

$$\varepsilon \sup_{0 \leq x_1 \leq 1} |\partial_n g(x_1, 0, .., 0)| < \frac{\eta}{3}$$

and $|\Delta_1(x_1, 0, .., 0)| \leq ||dG - dg||_\infty < \frac{\eta}{3}$, we get

$$|\partial_n(g + h)(x_1, 0, .., 0)| < \frac{2}{3}\eta.$$

In both cases, since the first $n - 1$ partial derivatives of h can be made arbitrarly small for R large enough, we deduce from 3.0.5 that, for R large enough

$$||d_{(x_1,0,..,0)}(g + h)|| < \delta + \eta$$

and property (ii) is satisfied. Let us now prove property (iv). We set

$$C(R) = \sup_{x \in \mathrm{Supp}(\psi) \times \left[-\frac{M}{R}, \frac{M}{R}\right]} |\partial_n g(x) - \partial_n g(x_1, .., x_{n-1}, 0)|.$$

Since $\partial_n g$ is continuous, we have

$$\lim_{R \to +\infty} C(R) = 0.$$

Note that property (iv) is straightforward when $x \notin \mathrm{Supp}(h)$. Hence, we can suppose by 3.0.7 that $x \in \mathrm{Supp}(\psi) \times \left[-\frac{M}{R}, \frac{M}{R}\right]$. We then get from 3.0.8 and $||(1-\varepsilon)\psi F'||_\infty \leq 1$ that

$$\partial_n h(x) = -(1 - \varepsilon)\psi(x_1, .., x_{n-1})\partial_n g(x) F'(Rx_n) + \Delta_2(x)$$

with

$$|\Delta_2(x)| \leq ||dG - dg||_\infty + C(R).$$

Since $-\varepsilon \leq F' \leq 1$ and $0 \leq \psi \leq 1$ we have $-\varepsilon \leq (1 - \varepsilon)\psi F' \leq 1 - \varepsilon$ and thus

$$\partial_n h(x) = -\alpha(x)\partial_n g(x) + \Delta_2(x) \tag{3.0.9}$$

with

$$-\varepsilon \leq \alpha(x) \leq 1 - \varepsilon.$$

Let $u = (u_1, .., u_n) \in \mathbb{R}^n$. Since the first $n-1$ sup norms of the partial derivatives of h can be made arbitrarly small for R large enough, we have from 3.0.9, for R large enough

$$\begin{aligned}
|d_x(g+h)(u)| &\leq \left|\sum_{i=1}^{n-1} \partial_i g(x) u_i + (1-\alpha(x))\partial_n g(x) u_n\right| \\
&\quad + \left(\frac{\eta}{4} + |\Delta_2(x)|\right) ||u|| \\
&\leq \left|(1-\alpha(x)) d_x g(u) + \alpha(x) \sum_{i=1}^{n-1} \partial_i g(x) u_i\right| \\
&\quad + \left(\frac{\eta}{4} + |\Delta_2(x)|\right) ||u||.
\end{aligned}$$

Since $|1-\alpha(x)| \leq 1+\varepsilon$, $|\alpha(x)| \leq 1$ and $|\Delta_2(x)| \leq \frac{\eta}{3} + C(R)$, we get from 3.0.5 and 3.0.6 that for R large enough

$$|d_x(g+h)(u)| \leq |d_x g(u)| + (\delta+\eta)||u||$$

which is property (iv). It now remains to prove property (iii). It suffices to show that for a given $x \in \mathrm{Supp}(h)$ we have $d_x g = 0$ if and only if $d_x(g+h) = 0$. In fact we will prove that for R large enough, we have

$$\mathrm{Crit}(g) \cap \mathrm{Supp}(h) = \mathrm{Crit}(g+h) \cap \mathrm{Supp}(h) = \emptyset.$$

Let $x \in \mathrm{Supp}(h)$. Since $\mathrm{Supp}(\psi) \subset K_1$, we get from 3.0.7 that $(x_1, .., x_{n-1}, 0) \in K_1$ and thus

$$|\partial_n g(x_1, .., x_{n-1}, 0)| \geq \frac{\eta}{3}.$$

But then, for R large enough

$$|\partial_n g(x)| \geq \frac{\eta}{3} - C(R) > 0$$

and $\mathrm{Crit}(g) \cap \mathrm{Supp}(h) = \emptyset$. Now, using 3.0.9 and $|1-\alpha(x)| \geq \varepsilon$ we get, for R large enough and $x \in \mathrm{Supp}(h)$

$$\begin{aligned}
|\partial_n(g+h)(x)| &= |(1-\alpha(x))\partial_n g(x) + \Delta_2(x)| \\
&\geq \frac{\eta}{3}\varepsilon - |\Delta_2(x)| \\
&\geq \frac{\eta}{3}\varepsilon - ||dG - dg||_\infty - C(R) > 0
\end{aligned}$$

because G has been chosen such that $\frac{\eta}{3}\varepsilon - ||dG - dg||_\infty > 0$. Hence for R large enough we have $\mathrm{Crit}(g+h) \cap \mathrm{Supp}(h) = \emptyset$, as desired. □

Following Körner, alteration lemma leads to the following corollary.

Corollary 3.2. *Let $g : \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, be a C^1 function and let γ be a δ-zig-zag for g. Then given any $\eta > 0$ and any open neighbourhood U of γ, we can find a C^∞ function $h : \mathbb{R}^n \to \mathbb{R}$ such that*

(i) *we have* $\mathrm{Supp}(h) \subset U$ *and* $||h||_\infty < \eta$.

(ii) *for every $t \in I$, we have $||d_{\gamma(t)}(g+h)|| < 3\delta + \eta$,*

(iii) *we have* $\mathrm{Crit}(g+h) = \mathrm{Crit}(g)$,

(iv) *for every* x *in* $\mathbb{R}^n$ *we have* $||d_x(g+h)|| < ||d_x g|| + 3\delta + \eta$.

Proof. Since γ is a δ-zig-zag for g, we can find a subdivision $t_0 = 0 < t_1 < .. < t_N = 1$ of I adapted to γ and a sequence $H_0, .., H_{N-1}$ of hyperplans of $\mathbb{R}^n$ such that for every $i \in \{0, .., N-1\}$ we have

(a) $\gamma(t_{i+1}) - \gamma(t_i) \in H_i$,

(b) for every $x \in [\gamma(t_i), \gamma(t_{i+1})]$ and every $v \in H_i$,

$$|d_x g(v)| \leq \delta ||v||.$$

We can then apply the alteration lemma in sufficiently small disjoint neighbourhoods of the disjoint segments $[\gamma(t_i), \gamma(t_{i+1})]$, $i \in \{0, .., N-1\} \cap 2\mathbb{Z}$, to get a C^∞ function h_1 such that

(1) we have $\mathrm{Supp}(h_1) \subset U$ and $||h_1||_\infty < \frac{\eta}{3}$,

(2) for every $i \in \{0, .., N-1\} \cap 2\mathbb{Z}$ and every $x \in [\gamma(t_i), \gamma(t_{i+1})]$,

$$||d_x(g+h_1)|| < \delta + \frac{\eta}{3},$$

(3) we have $\mathrm{Crit}(g+h_1) = \mathrm{Crit}(g)$,

(4) for every $x \in \mathbb{R}^n$ and every $u \in \mathbb{R}^n$,

$$|d_x(g+h_1)(u)| < |d_x g(u)| + \left(\delta + \frac{\eta}{3}\right) ||u||.$$

This last condition implies that the path γ is still a $\left(2\delta + \frac{\eta}{3}\right)$-zig-zag for $g + h_1$. We can thus apply again the alteration lemma in neighbourhoods of the disjoint segments $[\gamma(t_i), \gamma(t_{i+1})]$, $i \in \{0, .., N-1\} \cap (2\mathbb{Z}+1)$ with the function $g + h_1$ to get a C^∞ function h_2 such that

(1') we have $\mathrm{Supp}(h_2) \subset U$ and $||h_2||_\infty < \frac{\eta}{3}$,

(2') for every $i \in \{0, .., N-1\} \cap (2\mathbb{Z}+1)$ and every $x \in [\gamma(t_i), \gamma(t_{i+1})]$,

$$||d_x(g+h_1+h_2)|| < 2\left(\delta + \frac{\eta}{3}\right),$$

(3') we have $\mathrm{Crit}(g+h_1+h_2) = \mathrm{Crit}(g+h_1)$,

(4') for every $x \in \mathbb{R}^n$ and every $u \in \mathbb{R}^n$,

$$|d_x(g+h_1+h_2)(u)| < |d_x(g+h_1)(u)| + 2\left(\delta + \frac{\eta}{3}\right) ||u||.$$

The function $h = h_1 + h_2$ now satisfies (i) and (iii).Property (iv) follows from (4) and (4′) and property (ii) follows from (2), (2′) and (4′).

□

4 Proof of the main result

In this section, results of sections 2 and 3 are gathered to prove the main theorem, which will follow from the next proposition.

Proposition 4.1. *Let $g : \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, be a C^1 function and let γ be a polygonal arc in $\mathbb{R}^n$. Given any $\varepsilon > 0$ and any neighbourhood U of γ, we can find a C^1 function $h : \mathbb{R}^n \to \mathbb{R}$ and an arc β such that*

(i) *we have $\beta(0) = \gamma(0)$, $\beta(1) = \gamma(1)$ and for every $t \in I$*

$$||\beta(t) - \gamma(t)|| < \varepsilon,$$

(ii) *we have $\mathrm{Supp}(h) \subset U$ and $||h||_\infty < \varepsilon$,*

(iii) *we have $\mathrm{Crit}(g + h) = \mathrm{Crit}(g) \cup \beta$ and $\mathrm{Crit}(g) \cap \gamma \subset \beta$,*

Moreover, if the subset $\mathrm{Crit}(g)$ is finite, we can assume that the function $g + h$ is nowhere locally constant along β.

Proof. We set $E = (C^0(I, \mathbb{R}^n), || \cdot ||_\infty)$ and $F = (C^0_c(\mathbb{R}^n, \mathbb{R}), || \cdot ||_\infty)$. Let $(J_k)_{k \geq 1}$ be a countable family of nonempty open intervals of I such that, for every nonempty open interval $J \subset I$, there is $k \geq 1$ with $J_k \subset J$. For every $k \geq 1$, we define

$$O_k = \{\gamma \in E \mid \min_{t,t' \in I \setminus J_k} ||\gamma(t) - \gamma(t')|| > 0\},$$

and

$$W_k = \{(\gamma, h) \in E \times F \mid \text{the function } (g + h) \circ \gamma \text{ is not constant on } J_k\}.$$

The subset O_k is an open subset of E and the subset W_k is an open subset of $E \times F$. Moreover, a path $\gamma \in E$ is an arc if and only if $\gamma \in \cap_{k \geq 1} O_k$ and the function $g + h$ is nowhere locally constant along γ if and only if $(\gamma, h) \in \cap_{k \geq 1} W_k$.

Using induction, we will construct C^1 functions $(g_k)_{k \in \mathbb{N}}$, polygonal arcs $(\gamma_k)_{k \in \mathbb{N}}$ and positive reals $(\varepsilon_k)_{k \in \mathbb{N}}$ such that $g_0 = g, \gamma_0 = \gamma$, $\varepsilon_0 = \frac{\varepsilon}{2}$ and, for some constant $C > 0$ and every $k \in \mathbb{N}$,

(1) $\gamma_{k+1}(0) = \gamma_k(0)$, $\gamma_{k+1}(1) = \gamma_k(1)$ and $||\gamma_{k+1} - \gamma_k||_\infty < \varepsilon_k$,

(2) $||g_{k+1} - g_k||_\infty < \varepsilon_k$ and $||dg_{k+1} - dg_k||_\infty < C\varepsilon_k$,

(3) for every $t \in I$, $||d_{\gamma_k(t)} g_k|| < C\varepsilon_k$,

(4) $\mathrm{Supp}(g_{k+1} - g_k) \subset \mathcal{V}_{2\varepsilon_k}(\gamma_k)$,

(5) $\mathrm{Crit}(g_{k+1}) = \mathrm{Crit}(g_k)$ and $\mathrm{Crit}(g_{k+1}) \cap \gamma_{k+1} = \mathrm{Crit}(g_k) \cap \gamma_k$,

(6) $0 < 2\varepsilon_{k+1} < \varepsilon_k$,

(7) $B_E(\gamma_{k+1}, 2\varepsilon_{k+1}) \subset O_{k+1}$.

Moreover, if the subset $\mathrm{Crit}(g)$ is finite, we can add the following condition

(7') $B_E(\gamma_{k+1}, 2\varepsilon_{k+1}) \times B_F(g_{k+1} - g, 2\varepsilon_{k+1}) \subset W_{k+1}$.

Let $C > 0$ be a constant such that

$$\sup_{t\in I} ||d_{\gamma(t)}g|| < C\frac{\varepsilon}{2}.$$

Property (3) is then satisfied for $k = 0$. Suppose that the function g_j, the arc γ_j and the real ε_j have been constructed for $j = 0, .., k$. Since g_k is C^1, it follows from (3) that we can find a neighbourhood U_k of γ_k such that $U_k \subset \mathcal{V}_{2\varepsilon_k}(\gamma_k)$ and

$$\sup_{x\in U_k} ||d_x g_k|| < C\varepsilon_k. \tag{4.0.10}$$

Let $\delta > 0$ to be chosen later. By corollary 2.3, we can find a δ-zig-zag $\gamma_{k+1} \subset U_k$ for g_k satisfying property (1) and such that

$$\mathrm{Crit}(g_k) \cap \gamma_{k+1} = \mathrm{Crit}(g_k) \cap \gamma_k. \tag{4.0.11}$$

Let $\eta > 0$ to be chosen later. By corollary 3.2, we can find a C^∞ function h_k such that

(a) $\mathrm{Supp}(h_k) \subset U_k$ and $||h_k||_\infty < \eta$,

(b) for every $t \in I$, $||d_{\gamma_{k+1}(t)}(g_k + h_k)|| < 3\delta + \eta$,

(c) $\mathrm{Crit}(g_k + h_k) = \mathrm{Crit}(g_k)$,

(d) for every $x \in \mathbb{R}^n$, $||d_x(g_k + h_k)|| < ||d_x g_k|| + 3\delta + \eta$.

We set $g_{k+1} = g_k + h_k$. The function $g_{k+1} - g$ is compactly supported and we claim that, if the subset $\mathrm{Crit}(g)$ is finite, we can assume $(\gamma_{k+1}, g_{k+1} - g) \in W_{k+1}$. Indeed, suppose that the function $g_{k+1} \circ \gamma_{k+1}$ is constant on J_k. Since γ_{k+1} is an arc and the subset $\mathrm{Crit}(g)$ is finite, we can find $t \in J_k$ such that $\gamma_{k+1}(t) \notin \mathrm{Crit}(g)$. Since $\mathrm{Crit}(g) = \mathrm{Crit}(g_k)$, we can then modify the function h_k in a neighbourhood of $\gamma_{k+1}(t)$ by adding a C^∞ compactly supported bump function, arbitrarly small in the C^1 topology, such that the function $g_{k+1} \circ \gamma_{k+1}$ is no more constant on J_k, without changing properties (a) to (d).

Since γ_{k+1} is an arc we also have $\gamma_{k+1} \in O_{k+1}$ and since O_{k+1} and W_{k+1} are both open subsets, we can then find ε_{k+1} satisfying properties (6),(7) and (7′). Property (3) follows from (b) for δ and η small enough. Property (4) follows from (a) and $U_k \subset \mathcal{V}_{2\varepsilon_k}(\gamma_k)$. Property (5) follows from (c) and 4.0.11. Property (2) follows from (a) and (d) together with 4.0.10 for δ and η small enough. This finishes the induction step.

It follows from (1) and (6) that the path γ_k converges in E to a path β satisfying $\beta(0) = \gamma(0)$, $\beta(1) = \gamma(1)$ and, for every $k \in \mathbb{N}$

$$||\beta - \gamma_k||_\infty \leq \sum_{n=k}^{+\infty} \varepsilon_n < 2\varepsilon_k. \tag{4.0.12}$$

Property (i) is then satisfied by the choice of γ_0 and ε_0. It follows from (2) and (6) that the sequence $g_k - g = \sum_{j=0}^{k}(g_{j+1} - g_j)$ converges in the C^1-topology to a C^1 function h satisfying, for every $k \in \mathbb{N}$

$$||h - (g_k - g)||_\infty \leq \sum_{n=k}^{\infty} \varepsilon_n < 2\varepsilon_k. \tag{4.0.13}$$

In particular, it follows from the choice of g_0 and ε_0 that $||h||_\infty < \varepsilon$. Moreover, it follows from $||\gamma_{k+1} - \gamma_k||_\infty < \varepsilon_k$ and (6) that

$$\mathcal{V}_{2\varepsilon_{k+1}}(\gamma_{k+1}) \subset \mathcal{V}_{2\varepsilon_k}(\gamma_k). \tag{4.0.14}$$

Therefore, we get from (4) that $\mathrm{Supp}(h) \subset \mathcal{V}_{2\varepsilon_0}(\gamma_0) = \mathcal{V}_\varepsilon(\gamma)$ and, reducing ε is necessary, property (ii) is satisfied. Since g_k converges in the C^1-topology to $g + h$ and γ_k converges to β in the C^0-topology, we deduce from (3) and $\varepsilon_k \to 0$ that $\beta \subset \mathrm{Crit}(g + h)$. Moreover, it follows from 4.0.12 and $\varepsilon_k \to 0$ that

$$\beta = \bigcap_{k \in \mathbb{N}} \mathcal{V}_{2\varepsilon_k}(\gamma_k)$$

Hence, it follows from (4) and 4.0.14 that the sequence g_k is eventually stationary in the neighbourhood of every point $x \notin \beta$. We then deduce from the first part of (5) that $\mathrm{Crit}(g + h) = \mathrm{Crit}(g) \cup \beta$. The second part of (5) implies that every arc γ_k contains the subset $\mathrm{Crit}(g_0) \cap \gamma_0 = \mathrm{Crit}(g) \cap \gamma$. Since γ_k converges to β in the C^0-topology, we thus get $\mathrm{Crit}(g) \cap \gamma \subset \beta$ and property (iii) holds. Now, we get from 4.0.12 and (7) that $\beta \in \cap_{k \geq 1} O_k$ i.e β is an arc. Moreover, if the subset $\mathrm{Crit}(g)$ is finite, we get from 4.0.13 and (7′) that $(\beta, h) \in \cap_{k \geq 1} W_k$, i.e. the function $g + h$ is nowhere locally constant along β. □

We can now prove the main result. We recall the statement.

Theorem 4.2. *Let M be a C^∞ connected closed manifold with $\dim(M) \geq 2$. Let $\mathcal{J} \subset C^1(M, \mathbb{R})$ be the set of C^1 real-valued functions on M such that*

(i) *the subset $\mathrm{Crit}(f)$ is an arc i.e. is homeomorphic to the unit interval $I = [0, 1]$,*

(ii) *the function $f_{|\,\mathrm{Crit}(f)}$ is nowhere locally constant on $\mathrm{Crit}(f)$.*

Then $\mathcal{J}$ is dense in $C^0(M, \mathbb{R})$.

Proof. Let $g \in C^0(M, \mathbb{R})$. Since Morse functions are dense in $C^0(M, \mathbb{R})$, we can assume that g is a Morse function and hence has a finite number of critical points $c_1, .., c_r$, $r \geq 1$. Since M is connected we can assume, pushing these points via isotopies if needed, that they all belong to the same coordinate chart (V, φ) with φ a C^∞ diffeomorphism such that $\varphi(V) = \mathbb{R}^n$. Pushing again the points $\varphi(c_1), .., \varphi(c_r)$ if needed, we can even assume that $\varphi(c_k) = (k, 0, .., 0)$ for $k = 1, .., r$. Thus, there is an obvious polygonal arc from $\varphi(c_1)$ to $\varphi(c_r)$, namely $\gamma(t) = (1 - t)\varphi(c_1) + t\varphi(c_r)$. Choose U any compact neighbourhood of the arc γ. By the previous proposition, we can find a C^1 function $h : \mathbb{R}^n \to \mathbb{R}$, arbitrary small in the C^0-topology, with $\mathrm{Supp}(h) \subset U$, and an arc $\beta \subset U$ such that $\mathrm{Crit}(g \circ \varphi^{-1} + h) = \mathrm{Crit}(g \circ \varphi^{-1}) \cup \beta$ and $\mathrm{Crit}(g \circ \varphi^{-1}) \cap \gamma = \{\varphi(c_1), .., \varphi(c_r)\} \subset \beta$. Thus we have $\mathrm{Crit}(g \circ \varphi^{-1} + h) = \beta$.

Moreover, we can assume that the function $g \circ \varphi^{-1} + h$ is nowhere locally constant along β because the subset $\mathrm{Crit}(g \circ \varphi^{-1})$ is finite. Since the function h is compactly supported, the function $f = g + h \circ \varphi$, defined on V, can be extended by g to the whole manifold M and satisfies $\mathrm{Crit}(f) = \varphi^{-1}(\beta)$ and f is nowhere locally constant along $\varphi^{-1}(\beta)$. Moreover, the function f can be made arbitrary close to g in the C^0-topology because h can be taken arbitrary small in the C^0-topology. □

5 Applications in dynamics

Throughout this section, the reader will be supposed to be familiar with the notion of chain-recurrence for a flow. Good references are [1,19]. Let M be a C^∞ connected closed manifold and let X be a C^∞ vector field on M generating a complete flow Φ_X. A function $u : M \longrightarrow \mathbb{R}$ is said to be a *strong Lyapunov function* for X if it is C^1 and satisfies the following two properties

(i) *for every x in M, we have $d_x u(X(x)) \leq 0$,*

(ii) *for a given x in M, we have $d_x u(X(x)) = 0$ if and only if $d_x u = 0$.*

Note that in that case, the vector field X appears as *gradientlike* for u. Moreover, it follows from property (ii) that any strong Lyapunov function u for X is a Lyapunov function for the time-one map of ϕ_X such that $N(u) \subset \mathrm{Crit}(u)$. Hence, we get from corollary 2.3 of chapter 2 the following result, see also [19, Proposition 4].

Proposition 5.1. *Let u be a strong Lyapunov function for X. If the regular values of u are dense in $\mathbb{R}$, then every chain-recurrent point of Φ_X is a critical point of u.*

Hence, if every point of M is chain-recurrent under the flow of X and if a strong Lyapunov function u satisfies conclusion of Sard's theorem then it has to be constant on M. In particular, the following results of Bates [4] shows that u is constant whenever it is $C^{\dim(M)-1,1}$.

Theorem 5.2 (Bates)**.** *Let n, m be positive integers with $n > m$ and $k = n - m + 1$. If $f \in C^{k-1,1}(\mathbb{R}^n, \mathbb{R}^m)$ then the set of critical values of f has Hausdorff m-measure zero.*

Nevertheless, we will see that if the density of the regular values fails, this conclusion may be wrong. In fact, theorem 1.1 can be used to construct counterexamples on any C^∞ closed and connected manifold M with $\dim M \geq 2$, as stated in the following result.

Theorem 5.3. *Let M be a C^∞ closed connected manifold with $\dim(M) \geq 2$. There is a C^∞ vector field X on M and a non-constant C^1 strong Lyapunov function u for X such that every point of M is chain-recurrent under the flow of X.*

Remark 5.4. Such examples show in a dramatic way that the problem of regularizing Lyapunov functions cannot be solved only through Wilson's smoothing techniques [28] but requires additional assumptions on the neutral values. Note that such weaker examples have already been constructed in [19] and [13, Section 4.4].

In the sequel, we will denote by $Z(X)$ the set of points x in M such that $X(x) = 0$. Theorem 5.3 is then a direct consequence of the following two lemmas and theorem 1.1.

Lemma 5.5. *Let $u : M \to \mathbb{R}$ be a C^1 function. There is a C^∞ vector field X on M such that u is a strong Lyapunov function for X and* $\mathrm{Crit}(u) = Z(X)$.

Proof. Let g be any Riemannian metric on M. We will denote by $\langle \cdot, \rangle_x$ (resp. $|| \cdot ||_x$) the scalar product (resp. the norm) induced by g on each tangent space T_xM of M. We first defined a vector field X_1 on M by

$$X_1 = -\operatorname{grad} u$$

where the gradient is defined with respect to the Riemannian metric g. The vector field X_1 has then the desired properties except that X_1 is only continuous. Let X_2 be a C^∞ vector field defined on the open subset $U = M \setminus \mathrm{Crit}(u)$ of M such that for every $x \in U$ we have

$$|d_xu(X_2(x)) - d_xu(X_1(x))| < \frac{1}{2}|d_xu(X_1(x))| = \frac{1}{2}|| \operatorname{grad} u(x)||_x^2.$$

This condition implies that $d_xu(X_2(x)) < 0$ everywhere on U. By a classical result, see for example [12], there is a C^∞ function $\varphi : M \to [0, +\infty[$ such that $\varphi^{-1}(0) = \mathrm{Crit}(u)$ and $(\varphi_{|U})X_2$ can be extended by 0 to a C^∞ vector field X on M. It is then easy to check that the vector field X satisfies all the desired properties. □

Lemma 5.6. *Let u be a strong Lyapunov function for X. If the subset* $\mathrm{Crit}(u)$ *is connected and if we have* $\mathrm{Crit}(u) = Z(X)$ *then every point of M is chain-recurrent under the flow of X.*

Proof. Let x be a point of M. The usual omega limit and alpha limit sets of x with respect to the flow Φ_X will be denoted by $\omega_X(x)$ and $\alpha_X(x)$. By compactness of M, the subset $\omega_X(x)$ is nonempty. Let y be a point of $\omega_X(x)$. Since u is a strong Lyapunov function for X, the function $t \mapsto u(\Phi_X(t, x))$ is nonincreasing. Being bounded from below, it admits a limit $l_x \in \mathbb{R}$ when t tends to $+\infty$. The continuity of u shows that $u(y) = l_x$. Since the subset $\omega_X(x)$ is invariant by the flow and y is arbitrary, we deduce that $u(\Phi_X(t, y)) = l_x$ for every $t \in \mathbb{R}$. Hence $d_yu(X(y)) = 0$ and $d_yu = 0$. Thus $\omega_X(x) \subseteq \mathrm{Crit}(u)$. A similar proof shows that $\alpha_X(x) \subseteq \mathrm{Crit(u)}$. If $\mathrm{Crit}(u) = Z(X)$ then every critical point of u is fixed by the flow. Thus, if $\mathrm{Crit}(u)$ is connected, it is chain-transitive under the flow of X, see [19, Lemma 10]. Hence, for every x in M, the subsets $\omega_X(x)$ and $\alpha_X(x)$ intersect a common chain-transitive subset of M. This implies that every point of M is chain-recurrent under the flow of X. □

Remark 5.7. If the subset $\mathrm{Crit}(u)$ is an arc along which the function u is nowhere locally constant then, for every $x \in M$, the subsets $\omega_X(x)$ and $\alpha_X(x)$ are reduced to points because they are connected subsets of $\mathrm{Crit}(u)$ on which u is constant. Hence, the trajectory $\phi_X(t, x)$ of every point $x \in M$ converges both in the past and in the future.

Remark 5.8. It would be very interesting to construct more regular examples of functions f with a connected set of critical points. They would lead to the existence of nonconstant weak KAM solutions for the Mañé Lagrangian associated to the vector field $X = -\frac{1}{2}\operatorname{grad}(f)$, such that every points of M is chain-recurrent under the flow of X. Compare with [13, section 4.3, lemma 4.14].

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